Numerical Methods Term Paper Report

Efficient Chebyshev Pseudospectral Methods for Viscous Burgers' Equation in One and Two Space Dimensions

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1 Introduction

Numerical methods plays a crucial role in solving complex mathematical problems that cannot be solved analytically. In physics, especially in field such as *fluid mechanics* one encounter nonlinear partial differential equations that can be challenging due to its inherent complexity and nonlinearity. Methods such as *Finite difference method, Finite element methods, Spectral methods* are very commonly used numerical methods to find approximate solutions to nonlinear PDEs such as Navier-Stokes, Euler equations, Orr-Sommerfeld equations, Burgers' equation etc. Numerical methods discretize the spatial and temporal domains of the equation, allowing for the approximation of the solution at discrete points. This paper is interested in Burgers' equation. They propose an efficient and accurate numerical method for one and two dimensional nonlinear viscous Burgers' equations. The method is based on Chebyshev collocation method in space and fourth-order Runge-Kutta method in time. The paper consist of discussing the method and solving for various parameters and intial conditions for Burger's equation. So for the term paper , I have solved few of the exercises discussed in the paper and compare it with FDM.

2 Burgers' Equations

2.1 One-dimensional nonlinear viscous Burgers'equation

$$u_t + \alpha u u_x = \nu u_{xx},\ x \in [a,b],\ t \geq 0 \tag{1}$$

where α is a given constant , u is the velocity of the fluid and ν is the viscosity. The initial and boundary conditions are given by

$$u(x,0) = f(x),\ x \in [a,b]$$
$$u(a,t) = f_1(t),\ u(b,t) = f_2(t)\ t \geq 0$$

2.2 One-dimensional nonlinear coupled viscous Burgers' equations

$$u_t + \alpha_1 u u_x + \alpha_2 (uv)_x = \nu u_{xx}\ x \in [a,b] \tag{2}$$
$$v_t + \beta_1 v v_x + \beta_2 (uv)_x = \nu v_{xx}\ x \in [a,b] \tag{3}$$

where $alpha_1$, α_2 , β_1 and β_2 are given constants. The boundary conditions and initial condition are assumed to have the following forms

$$u(x,0) = f(x),\ v(x,0) = g(x),\ x \in [a,b]$$
$$u(a,t) = f_1(t),\ v(a,t) = g_1(t),\ u(b,t) = f_2(t), v(b,t) = g_2(t)\ t \geq 0$$

2.3 Two-dimensional nonlinear Burgers' equation

Consider rectangular domain $\Omega = [a_1,b_1] \times [a_2,b_2] \subseteq \mathbb{R}$. The two-dimensional nonlinear Burgers' equation is given by

$$u_t + \alpha u u_x + \beta u u_y = \nu(u_{xx} + u_{yy}),\ (x,y) \in \Omega,\ t \geq 0 \tag{4}$$

where α and β are given constants. The IC and Dirichlet BC on the boundary $\partial\Omega$ of Ω is of the following form

$$u(x,y,0) = f(x,y),\ (x,y) \in \Omega$$
$$u(x,y,t) = f_1(x,y,t),\ (x,y) \in \partial\Omega,\ t \geq 0$$

2.4 Two-dimensional nonlinear coupled Burgers' equation

The two-dimensional nonlinear coupled Burgers' equation is given by

$$u_t + \alpha_1 u u_x + \alpha_2 v u_y = \nu(u_{xx} + u_{yy}),\ (x,y) \in \Omega,\ t \geq 0 \tag{5}$$
$$v_t + \beta_1 u v_x + \beta_2 v v_y = \nu(v_{xx} + v_{yy}),\ (x,y) \in \Omega,\ t \geq 0 \tag{6}$$

This equation is subjected to the following form of side conditions

$$u(x,y,0) = f(x,y),\ v(x,y,0) = g(x,y),\ (x,y) \in \Omega$$
$$u(x,y,t) = f_1(x,y,t),\ v(x,y,t) = f_2(x,y,t),\ (x,y) \in \partial\Omega,\ t \geq 0$$

The exact solutions of these equations takes the form of infinite Fourier series. But as the nonlinear term dominates (ν is small), the solutions cannot capture the nonlinearity.Thus the solution converges very slowly for small values of ν or large values of $Re = 1/\nu$. This results in shock waves or sharp discontinuities in the solution regardless of the initial condition. And at this regime, solving numerically is also challenging. FDM, cubic-spline collocation methods etc were used to solve Burgers' equation upto some values of viscosity ν. But this paper suggest that the Chebyshev collocation method offers a robust and efficient method for solving Burgers' equations with large value of the Reynolds numbers.

3 Chebyshev Collocation Method

Spectral method is found to be a powerful numerical tool for solving differential equations because of their high order of accuracy. The error for this method decays exponentially. In FDM, derivatives are approximated by finite difference quotients at discrete grid points that are usually equally spaced. The unknown functions are approximated at these discrete points for the solutions. While in spectral method, the function is represented by an infinite expansion of appropriate basis function; most commonly used basis being Fourier basis. Fourier basis are advised for problems with periodic boundary conditions. Thus FDM talks about local expansion while spectral method talks about global expansion of the function expressed in cosine and sine functions which spans the entire domain. Spectral method offers high accuracy. One very useful method that is rooted in spectral method is Chebyshev spectral collocation method (also called Pseudospectral method) which harness the properties of Chebyshev polynomials.

3.1 Chebyshev polynomials

Chebyshev polynomials are two sequences of polynomials related to sine and cosine functions notated as $T_n(x)$ and $U_n(x)$. $T_n(x)$ is called the Chebyshev polynomials of the first kind and the latter is called the second kind. For the interest of this paper, lets discuss about the first kind only.

$$T_n(x) = \cos(n\cos^{-1}x) \tag{7}$$

The roota of this polynomial is called the *Chebyshev-Gauss-Lobatto points*

$$x_j = \cos\left(\frac{\pi j}{N}\right) \tag{8}$$

- Recurrence relation
$$T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x) \tag{9}$$

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$$T_k(\pm 1) = (\pm)^k \tag{10}$$
$$T_k'(\pm 1) = (\pm 1)^k k^2 \tag{11}$$

- Orthogonality condition
$$\int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}dx = \begin{cases} \frac{\pi}{2}\delta_{mn} \\ \pi,\ m=0 \end{cases} \tag{12}$$

The key idea is to force the differential equation to be solved at the collocation points. Thereby we can also reduce the maximum interpolation error as the interpolation error is minimum at the collocation points. Thus the solution to the differential equation can be expressed in terms of Chbyshev polynomials as

$$u(x) = \sum_{n}^{N} u_n T_n(x) \tag{13}$$

Using the orthogonal property and recurssion relation, we can replace the derivatives $\frac{d}{dy}$ by a matrix D whose elements are as follows.

$$D_N^{(1)}(k,j) = \begin{cases} \frac{c_k(-1)^{k+j}}{c_j(y_k-y_j} & 0 \le k,j \le N,\ \ k \ne j \\ -\frac{y_k}{2(1-y_k^2)} & 1 \le k \le N-1,\ \ k=j \\ \frac{2N^2+1}{6} & k=0,\ j=0 \\ -\frac{2N^2+1}{6} & k=N,\ j=N \end{cases} \tag{14}$$

where,
$c_0 = c_N = 2, \quad c_j = 1 \quad \text{with} \quad 1 \le j \le N-1$

The discretized differentiation matrix in 2D have to be seen in *Tensor product* grid. Here the grid space are independently constructed in each direction. If the solution matrix takes the form as follows where the first n elements are from the first grid row keeping y constant and so on. 1

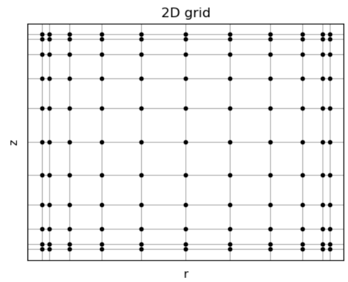
4 Result

In this section, I'm only displaying the result for coupled-nonlinear Burgers' equation given by (2) on $[-\pi,\pi]\times[0,T]$ with $T=1$, $\nu=1$, $\alpha_1=\beta_1=-2$, $\alpha_2=\beta_2=1$ and with initial condition $u(x,0) = v(x,0) = sin(x)$ and boundary conditions $u(-\pi,t)=u(\pi,t)=v(-\pi,t)=v(\pi,t)=0,\ t\in[0,1]$

The exact solution to the given equation is given by

$$u(x,t) = v(x,t) = e^{-t}\sin(x),\ x\in[-\pi,\pi],\ t\in[0,T]$$

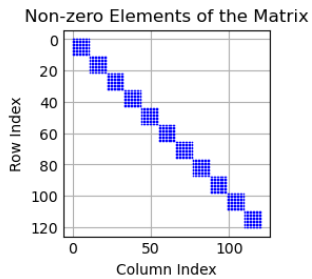
The solution with $N=20$ and $dt=10^{-3}$ is given by the figure 2:



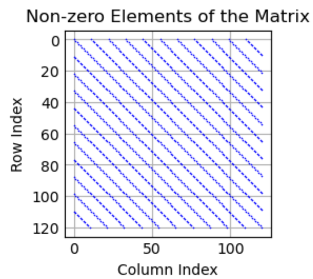
(a) Tensor product grid

$$\begin{pmatrix} u_{11} \\ u_{12} \\ \vdots \\ u_{1m} \\ u_{21} \\ \vdots \\ u_{2m} \\ \vdots \\ u_{n1} \\ u_{n2} \\ \vdots \\ u_{nm} \end{pmatrix}$$

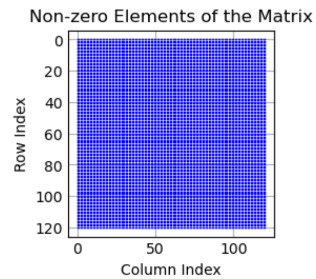
(b) solution matrix u in 2D



(c) $\frac{\partial^2}{\partial x^2}$

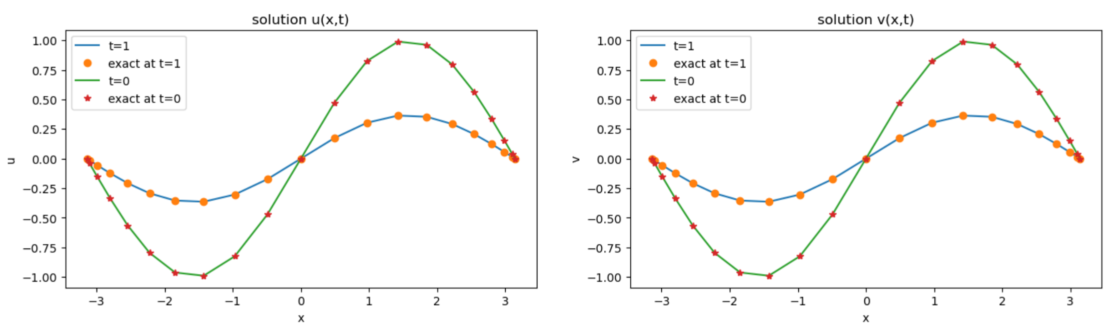


(d) $\frac{\partial^2}{\partial y^2}$

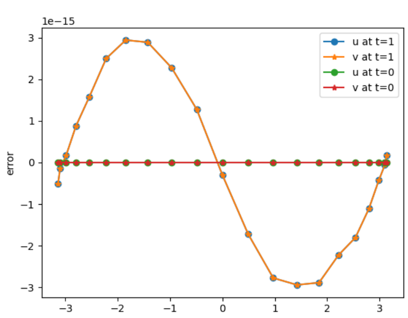


(e) $\frac{\partial^2}{\partial x \partial y}$

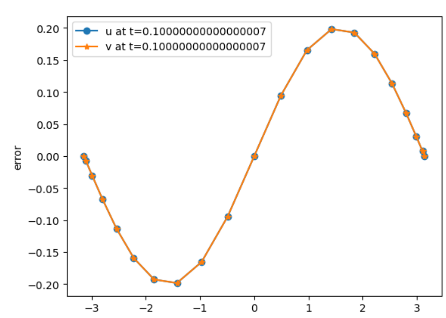
Figure 1: *blue dots represents the non-zero elements



(a) solution $u(x,t)$ and $v(x,t)$



(b) error



(c) error

Figure 2: Solution and error for nonlinear coupled Burgers' equation