CS2040S
Data Structures and Algorithms

(e-learning edition)

More shortest paths!

Roadmap

Shortest Paths

- Special Case: Tree
- Special Case: Non-negative weights (Dijkstra's)
- Special Case: Directed Acyclic Graphs

Roadmap

Shortest Paths

- Special Case: Tree
- Special Case: Non-negative weights (Dijkstra's)
- Special Case: Directed Acyclic Graphs

Edsger W. Dijkstra

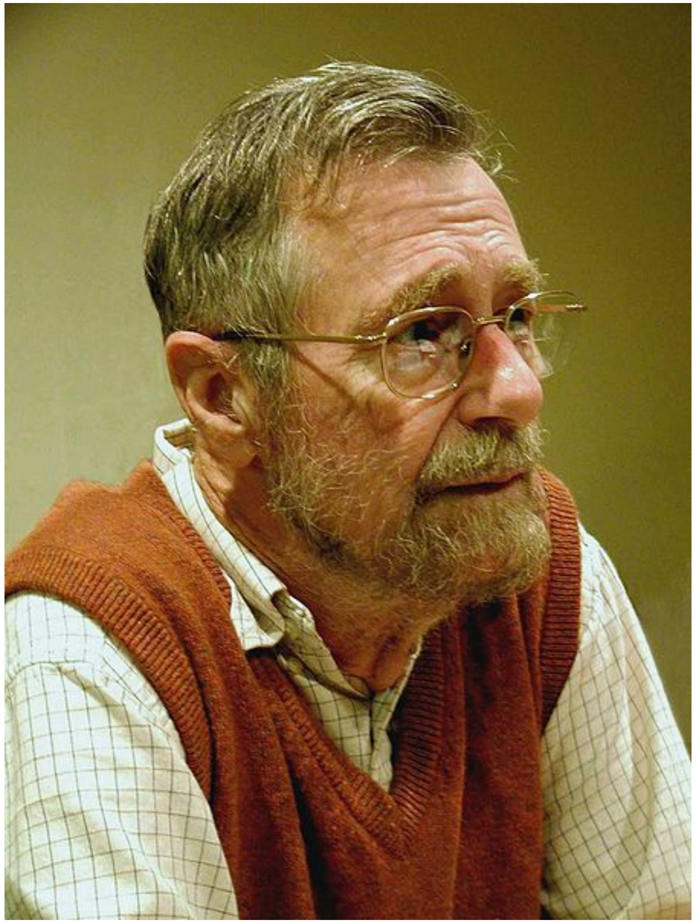
"Computer science is no more about computers than astronomy is about telescopes."

"The question of whether a computer can think is no more interesting than the question of whether a submarine can swim."

"There should be no such thing as boring mathematics."

"Elegance is not a dispensable luxury but a factor that decides between success and failure."

"Simplicity is prerequisite for reliability."



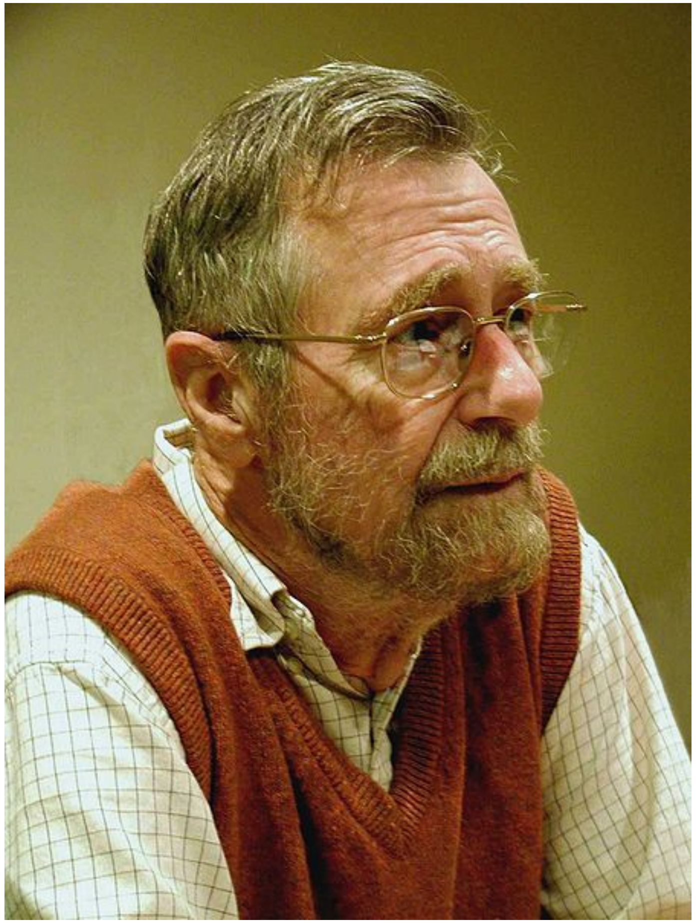
Edsger W. Dijkstra

“It is practically impossible to teach good programming to students that have had a prior exposure to BASIC: as potential programmers they are mentally mutilated beyond hope of regeneration.”

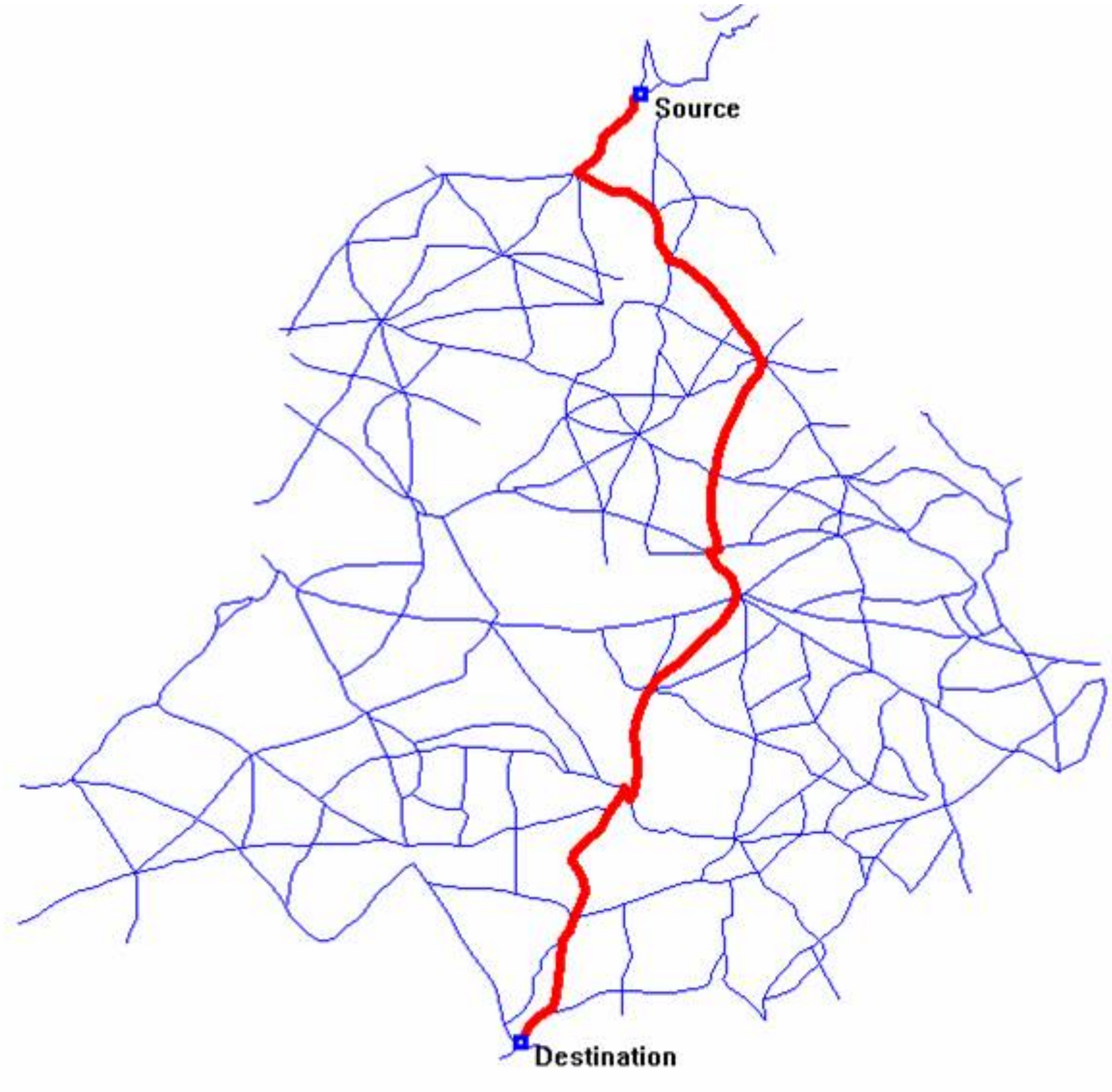
“The use of COBOL cripples the mind; its teaching should, therefore, be regarded as a criminal offense.”

“APL is a mistake, carried through to perfection. It is the language of the future for the programming techniques of the past: it creates a new generation of coding bums.”

“Object-oriented programming is an exceptionally bad idea which could only have originated in California.”



SHORTEST PATHS
(ON WEIGHTED GRAPHS)



Shortest Path Problem

Basic question: find the shortest path!

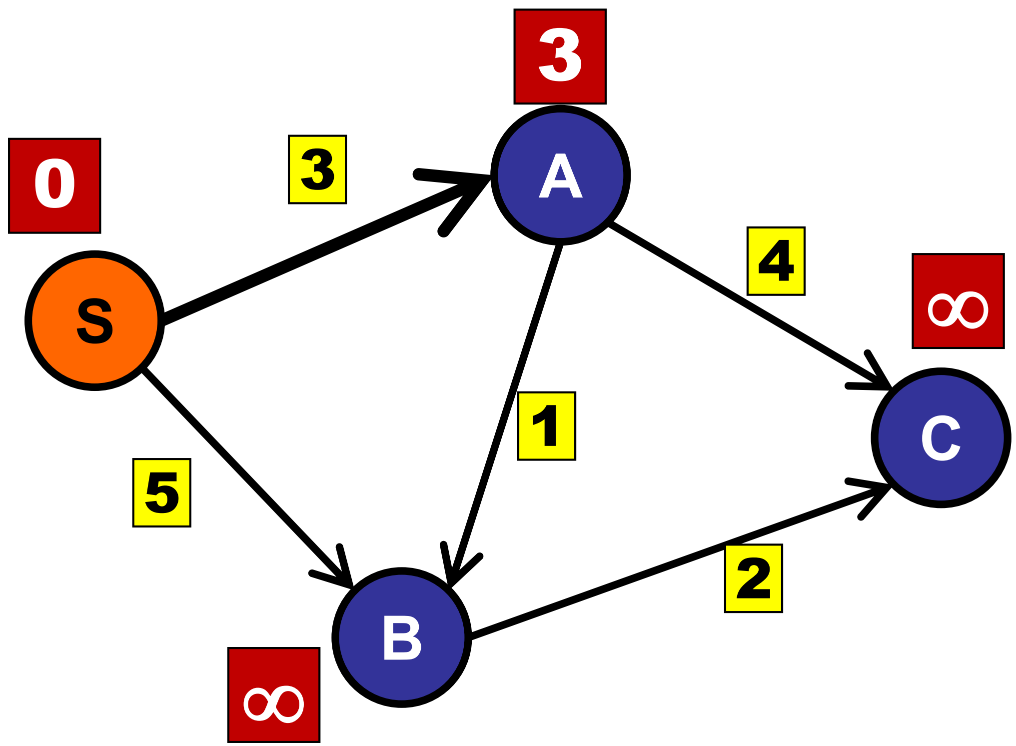
- Source-to-destination: one vertex to another
- Single source: one vertex to every other
- All pairs: between all pairs of vertices

Variants:

- Edge weights: non-negative, arbitrary, Euclidean, …
- Cycles: cyclic, acyclic, no negative cycles

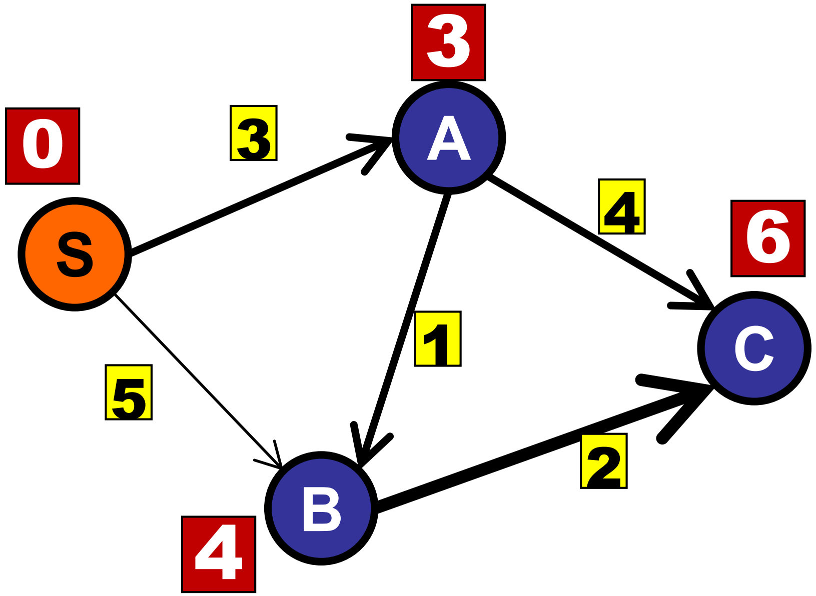
Shortest Paths

```
relax(int u, int v){
    if (dist[v] > dist[u] + weight(u,v))
        dist[v] = dist[u] + weight(u,v);
}
```



Bellman-Ford

```
n = V.length;
for (i=0; i<n; i++)
    for (Edge e : graph)
            relax(e)
```



Does Bellman-Ford always work in graphs with negative weights?

1. Yes
2. No
3. I forget

Bellman-Ford Summary

Basic idea:

- Repeat |V| times: relax every edge
- Stop when "converges".
- O(VE) time.

Special issues:

- If negative weight-cycle: impossible.
- Use Bellman-Ford to detect negative weight cycle.
- If all weights are the same, use BFS.

Today

Key idea:

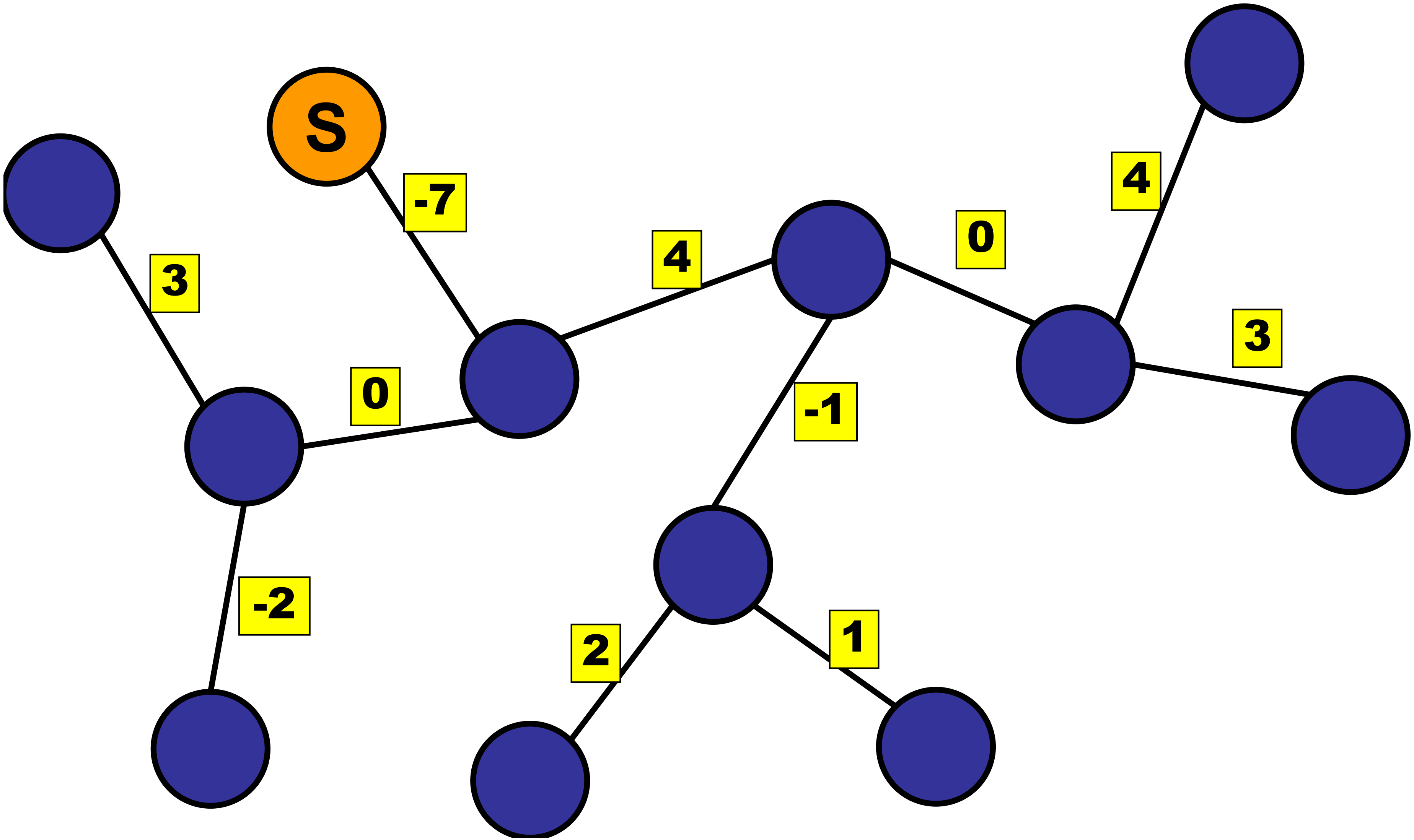
Relax the edges in the “right” order.

Only relax each edge once:

– O(E) cost (for relaxation step).

Special Case: Tree

Undirected, weighted



Aside: Trees, Redefined

What is an (undirected) tree?

- A graph with no cycles is an (undirected) tree.

What is a *rooted* tree?

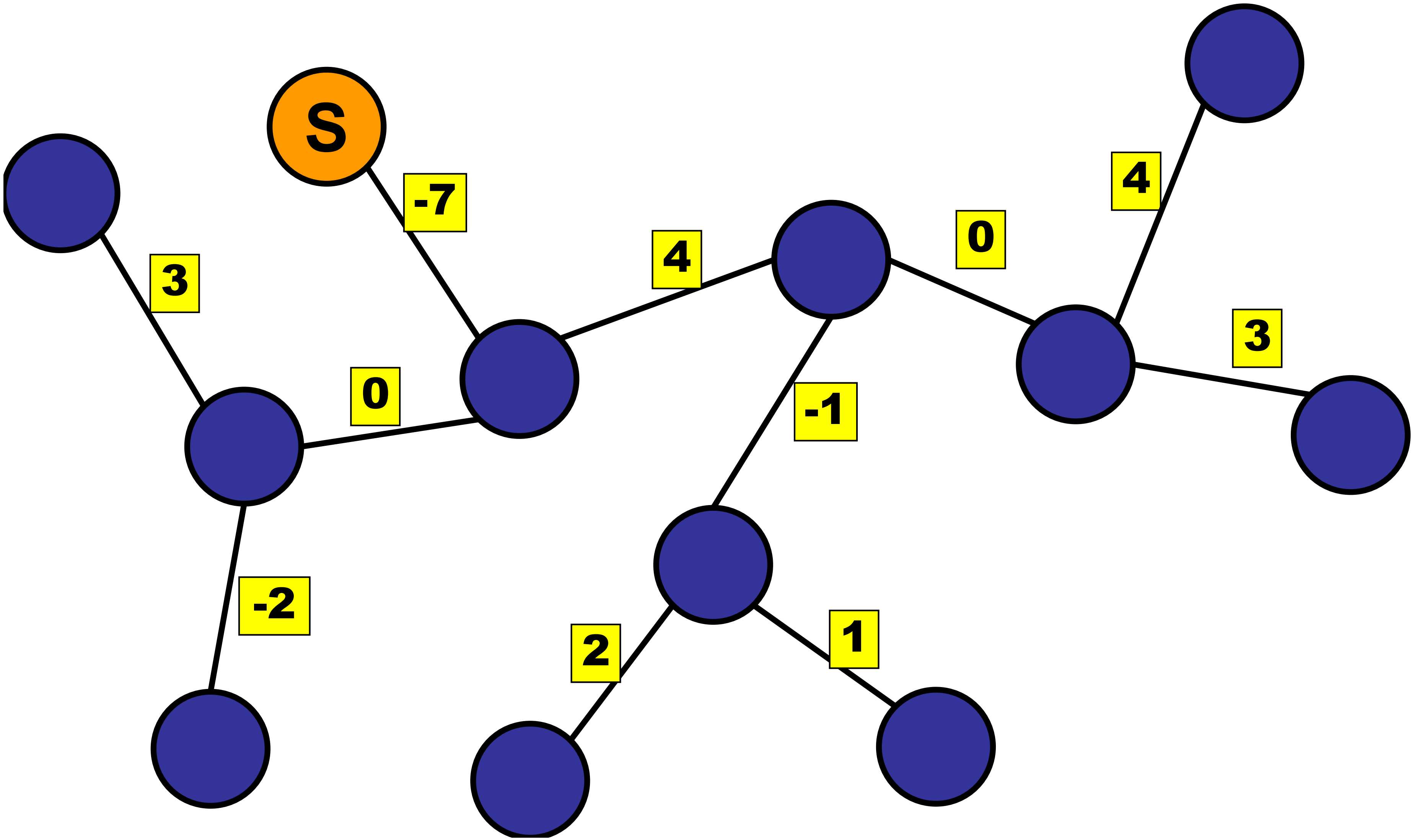
- A tree with a special designated root note.

Our previous (recursive) definition of a *tree*:

- A node with zero, one, or more sub-trees.
- I.e., a *rooted* tree.

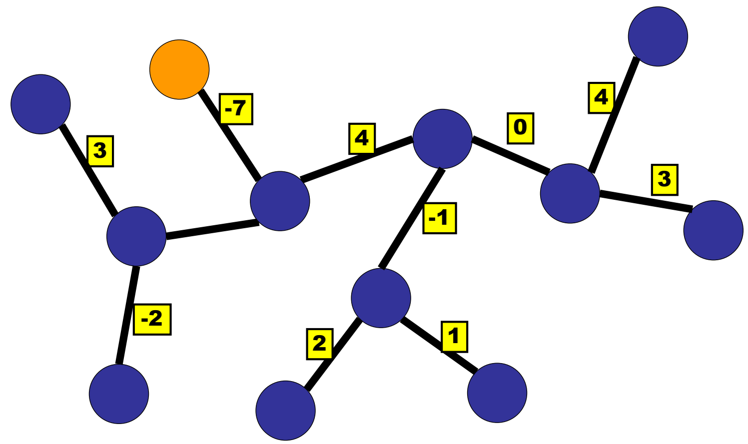
Special Case: Tree

Undirected, weighted



Which algorithm is best for checking if a graph is a tree?

1. BFS
2. DFS
3. Bellman Ford
4. Topological Sort
5. Dijkstra's Algorithm



Aside: Tree Checking

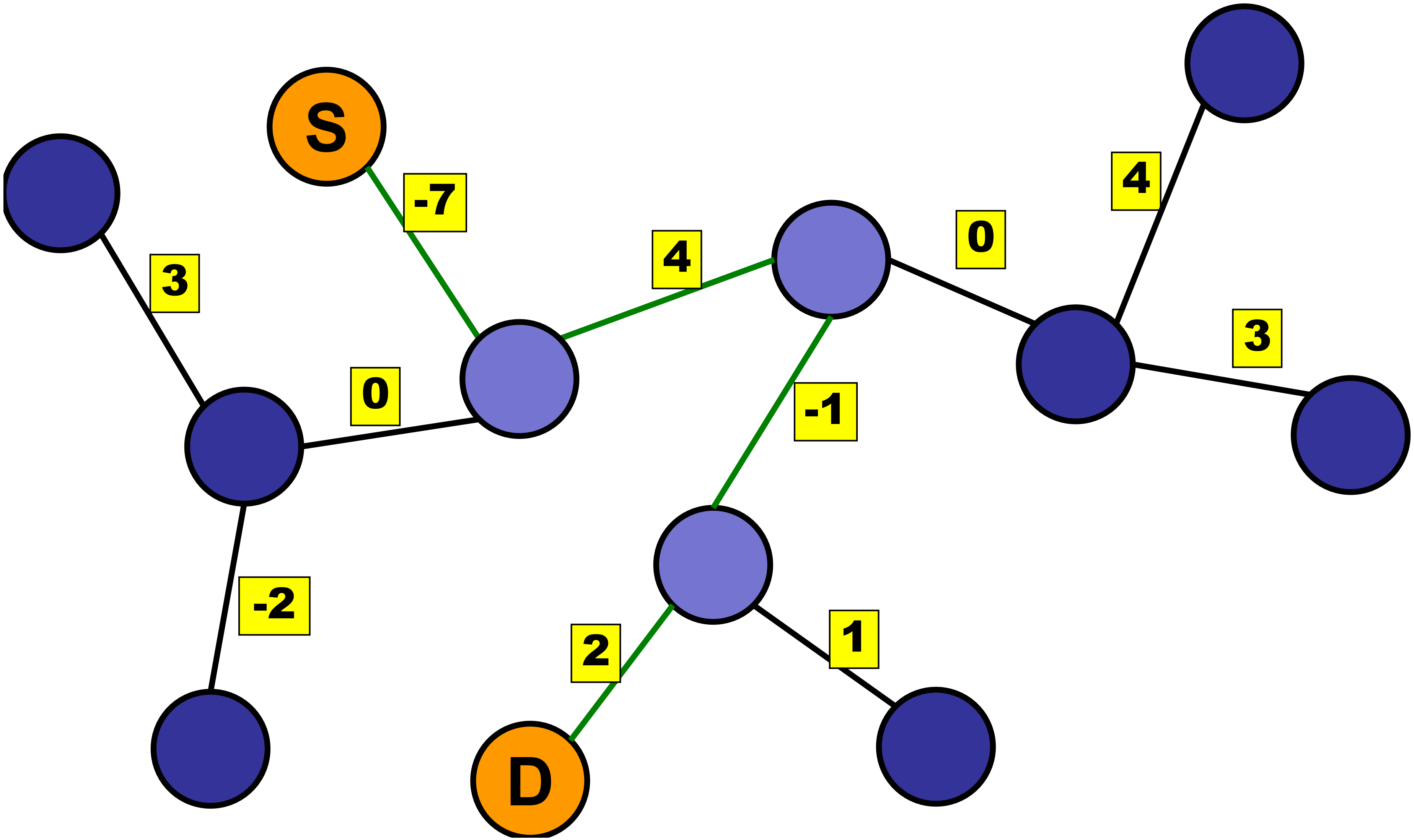
If it is connected…

If it is disconnected…

If it is directed…

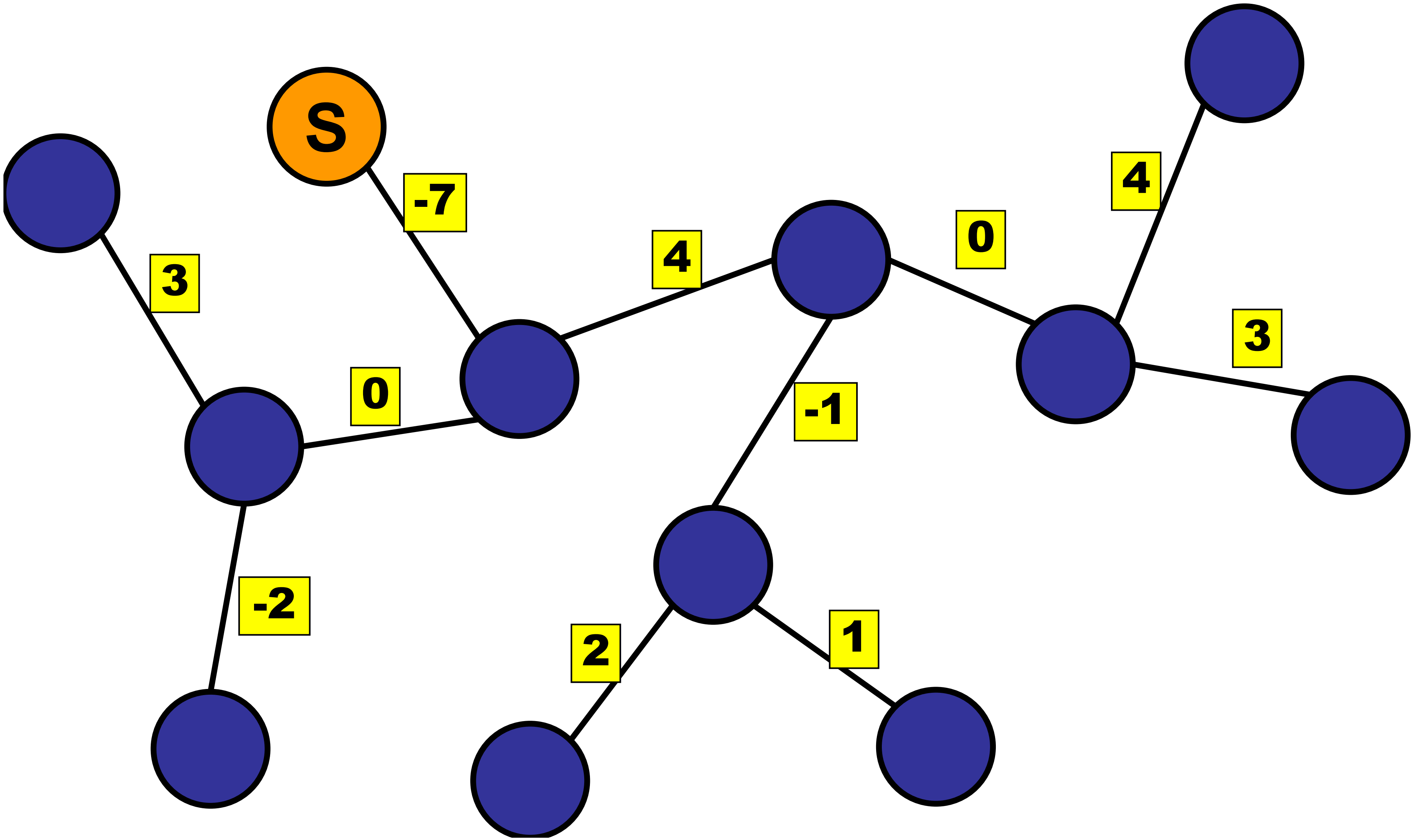
Special Case: Tree

source-to-destination: only one possible path!



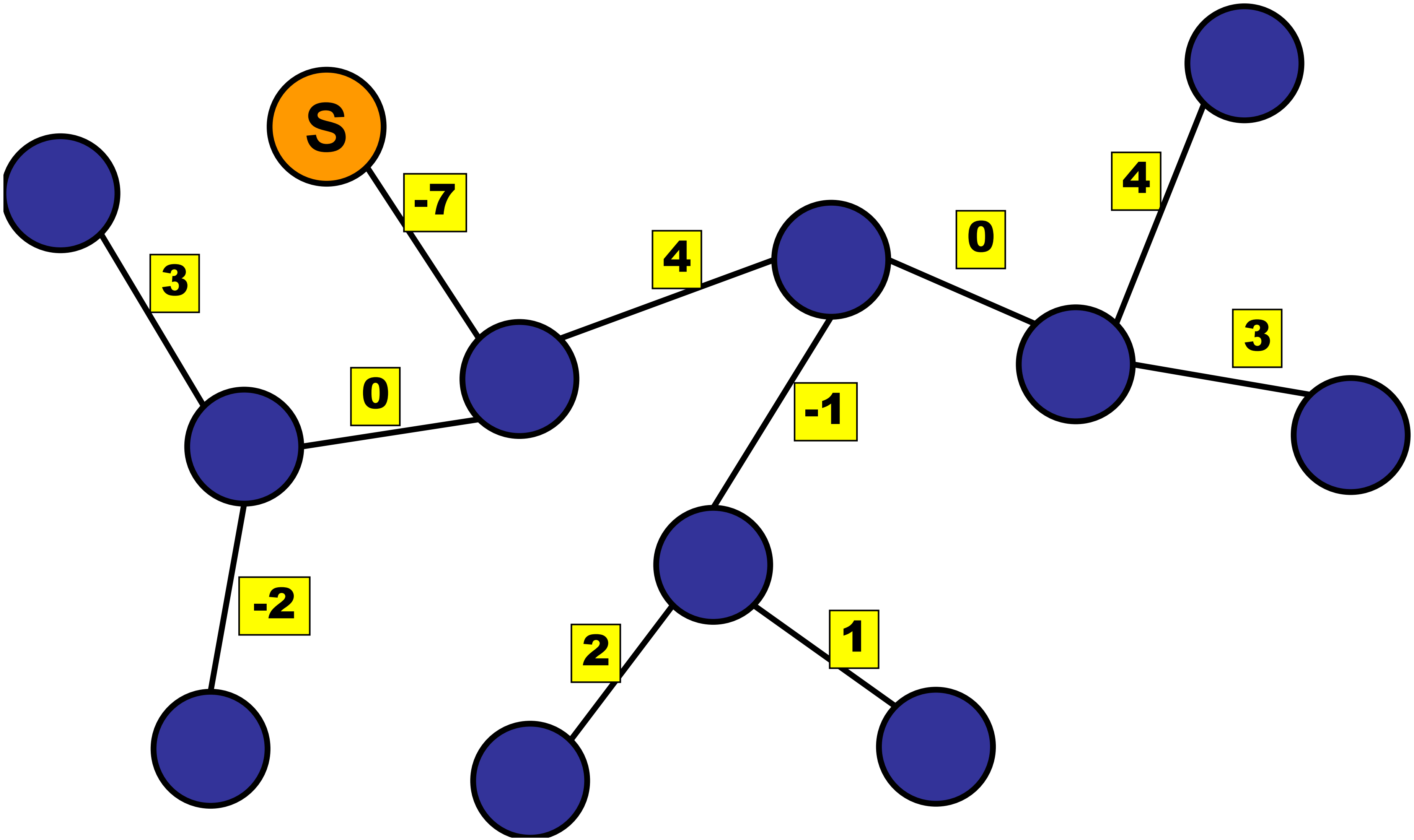
Special Case: Tree

source-to-all: what order to relax?



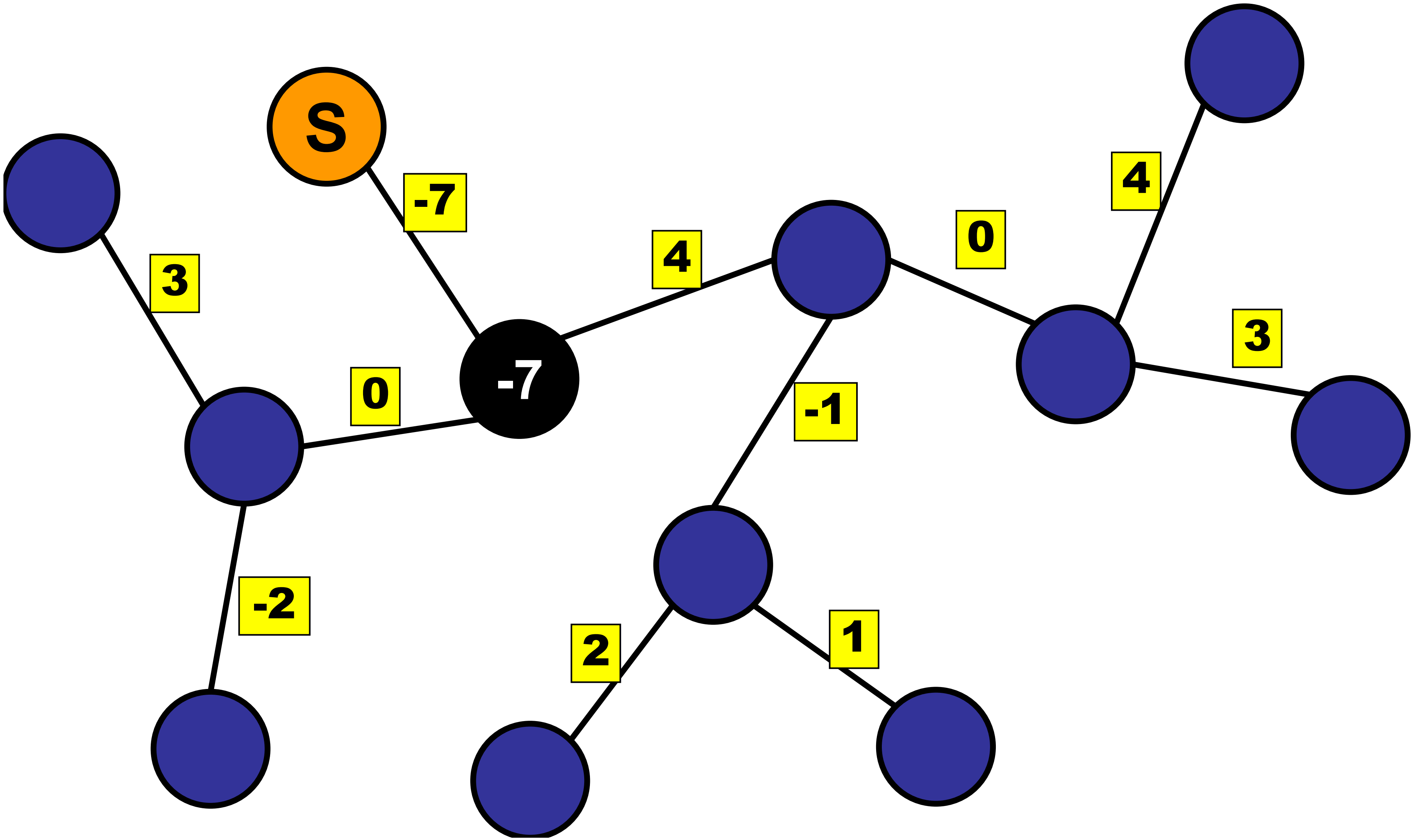
Special Case: Tree

Relax edges in (BFS or DFS order).



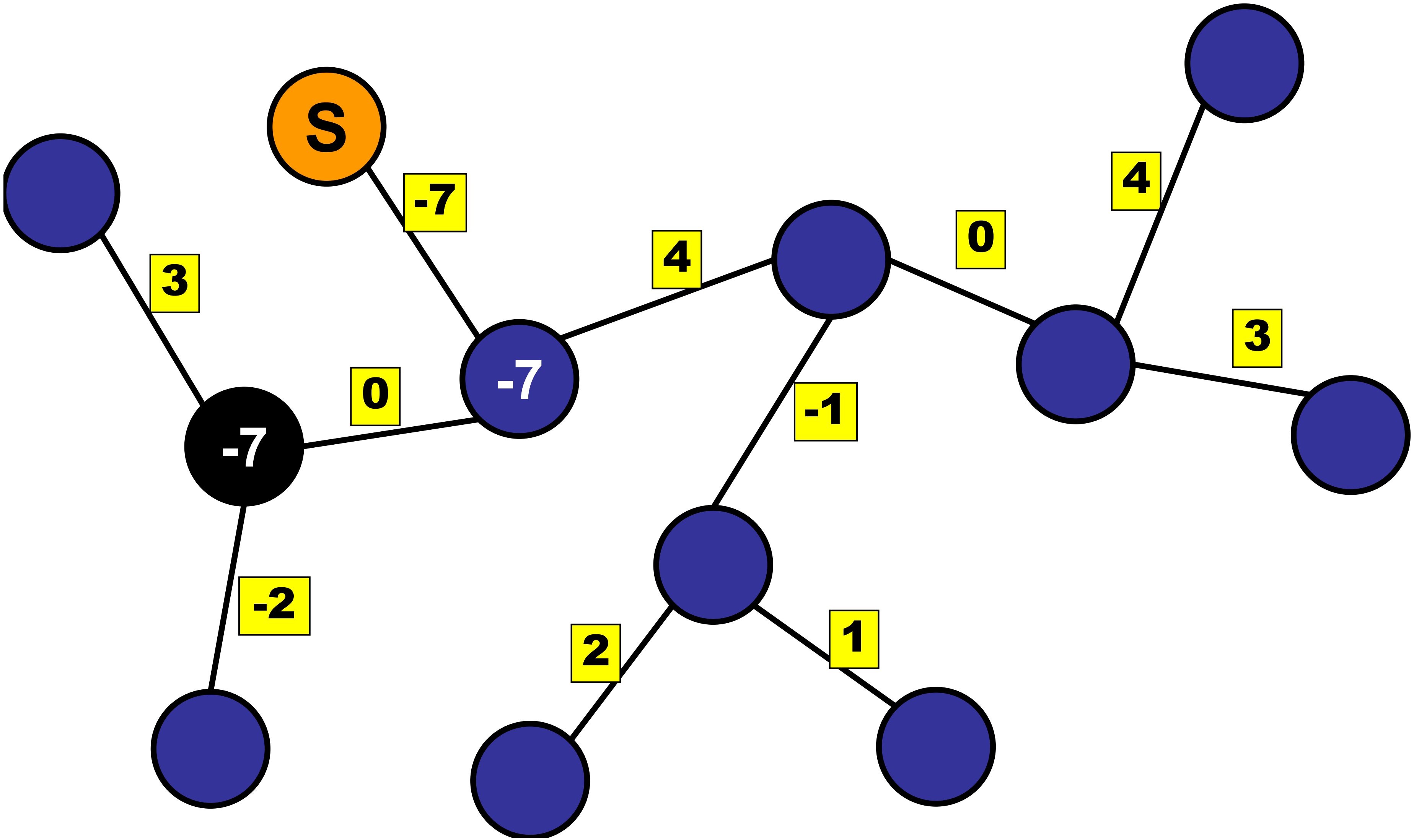
Special Case: Tree

Relax edges in DFS order.



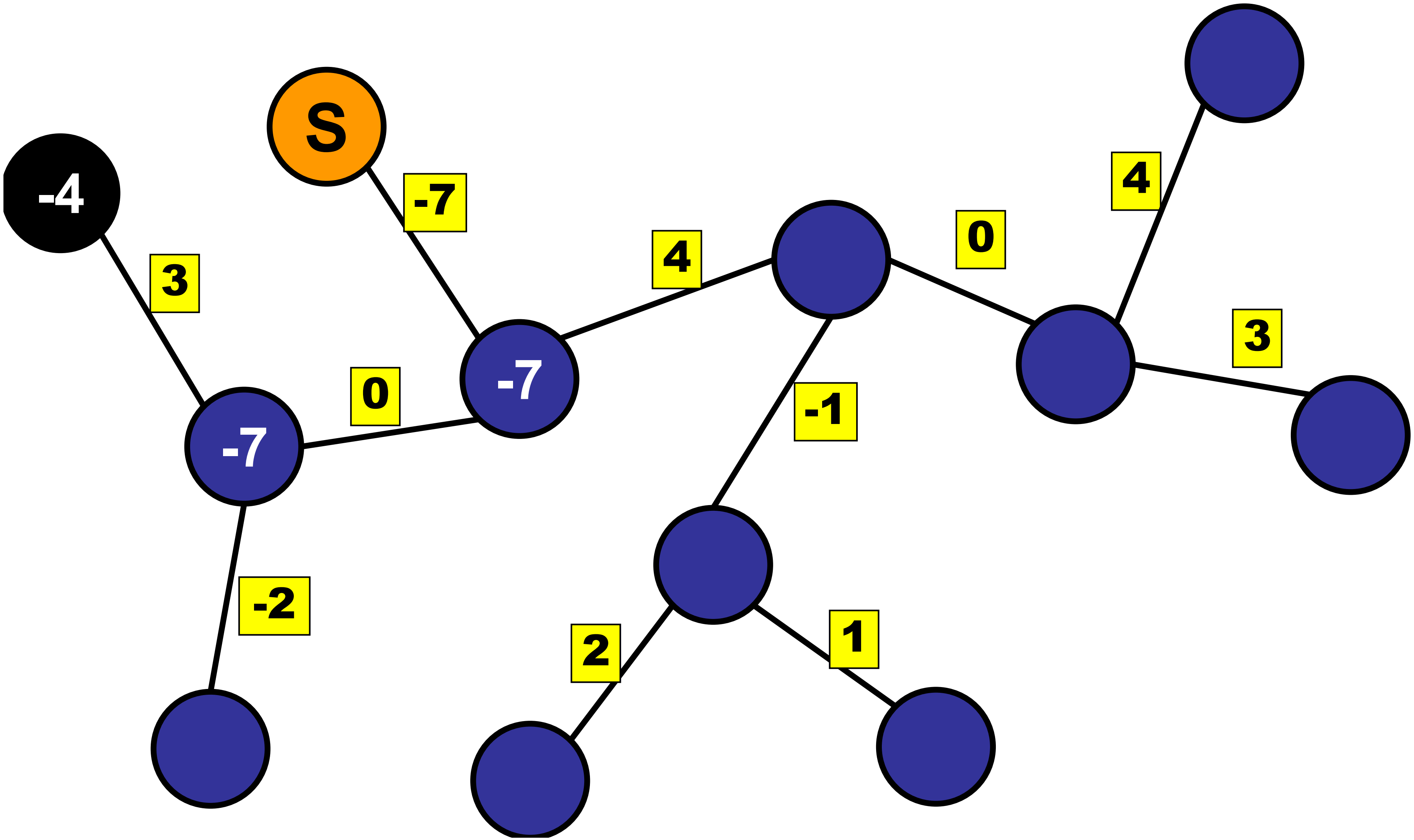
Special Case: Tree

Relax edges in DFS order.



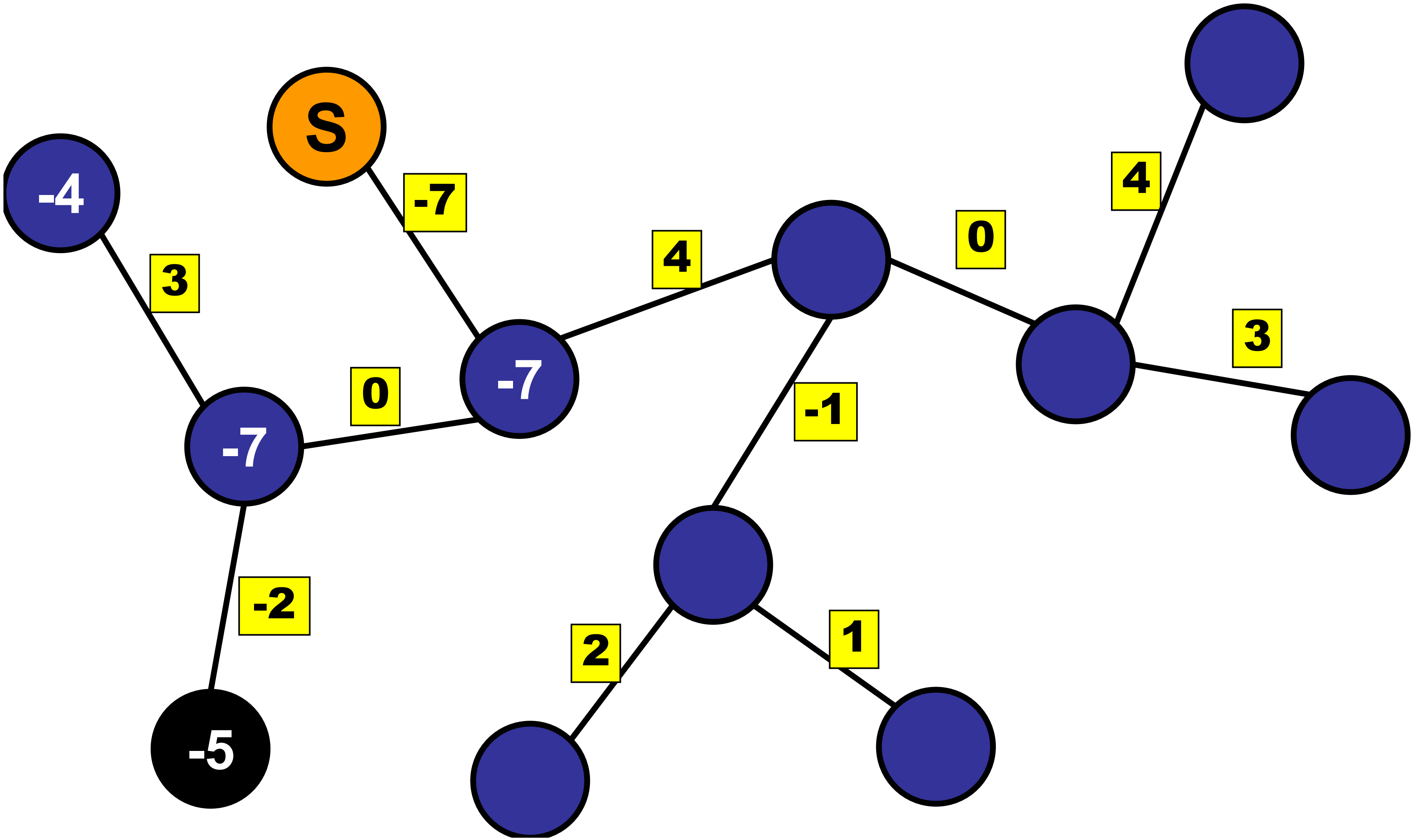
Special Case: Tree

Relax edges in DFS order.



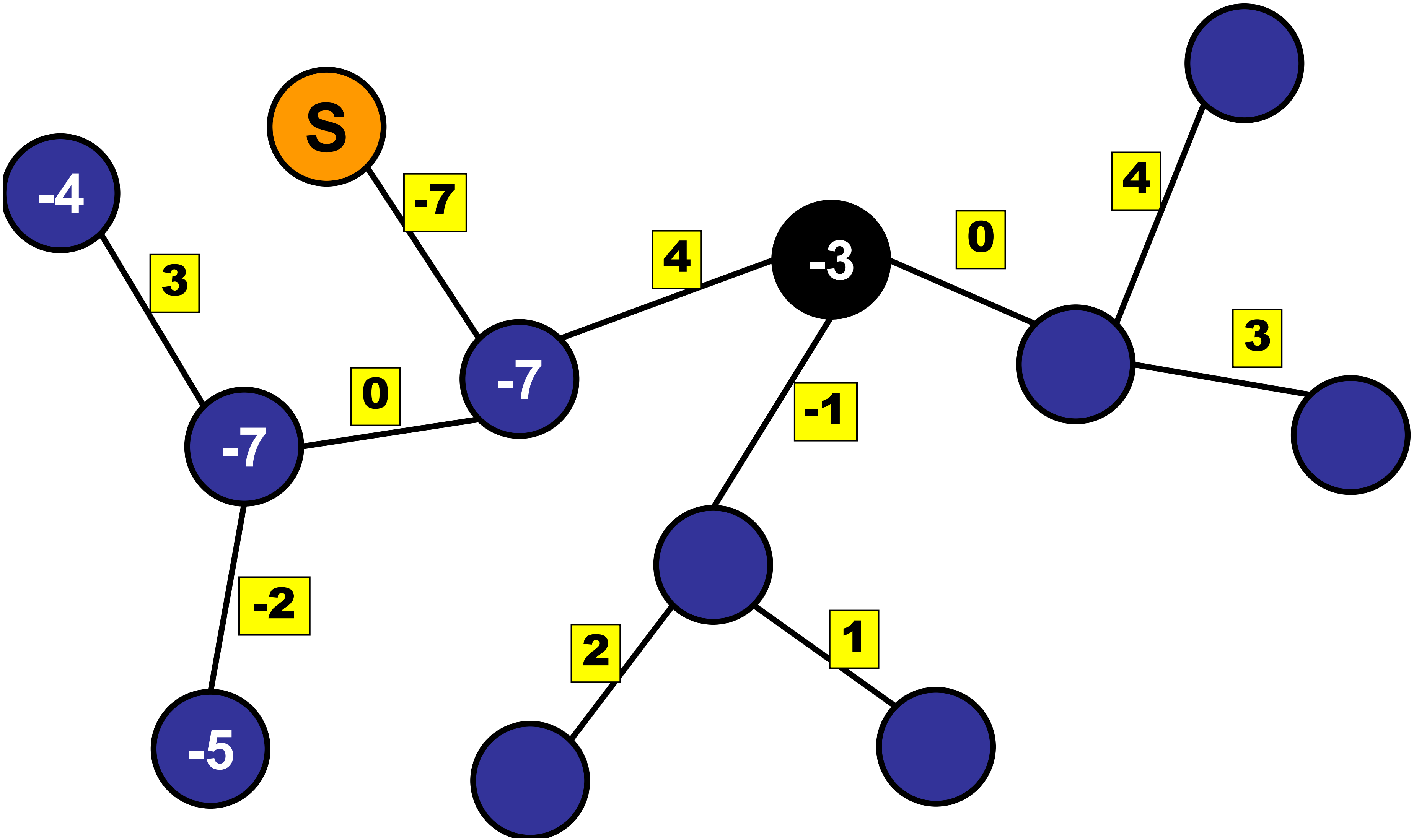
Special Case: Tree

Relax edges in DFS order.



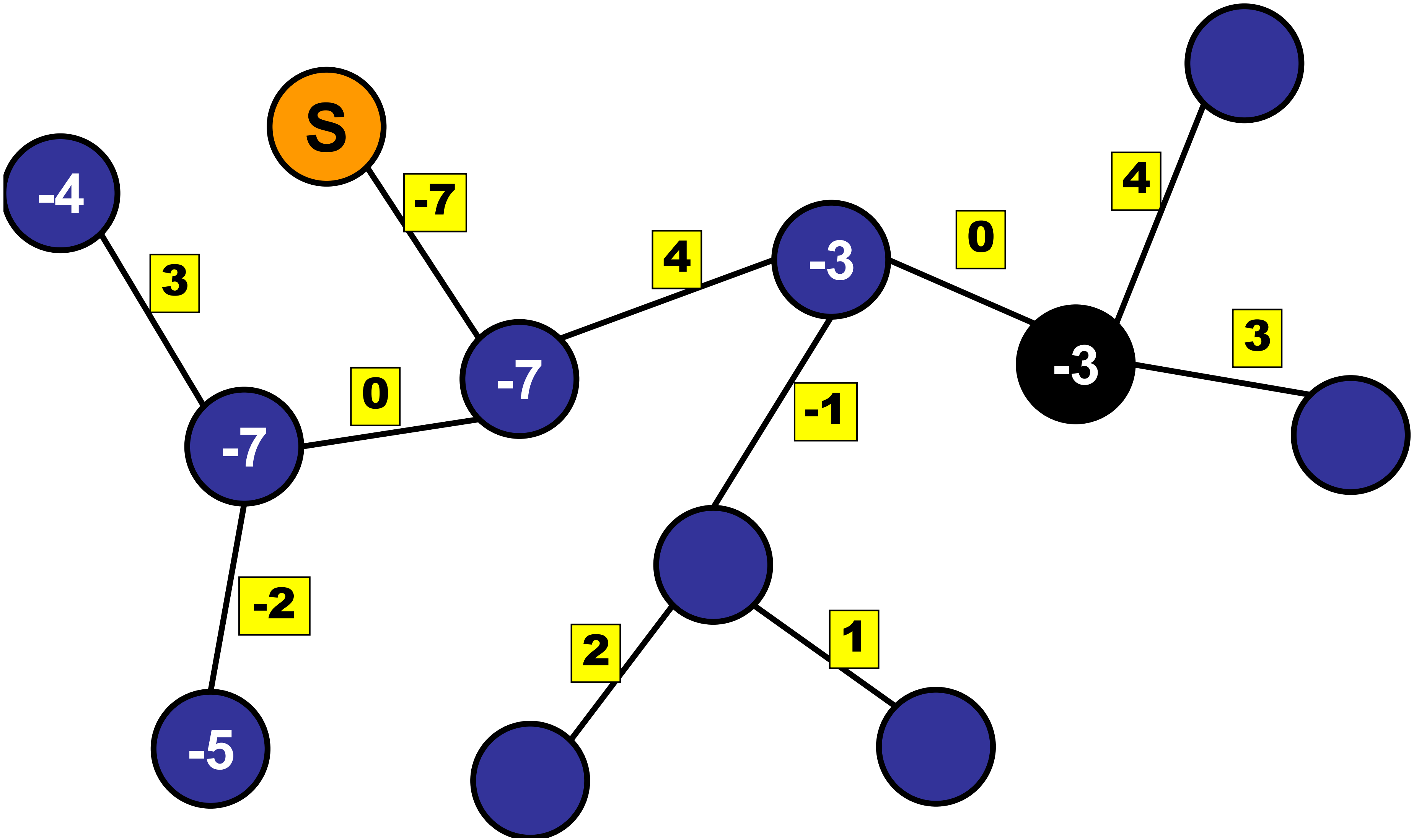
Special Case: Tree

Relax edges in DFS order.



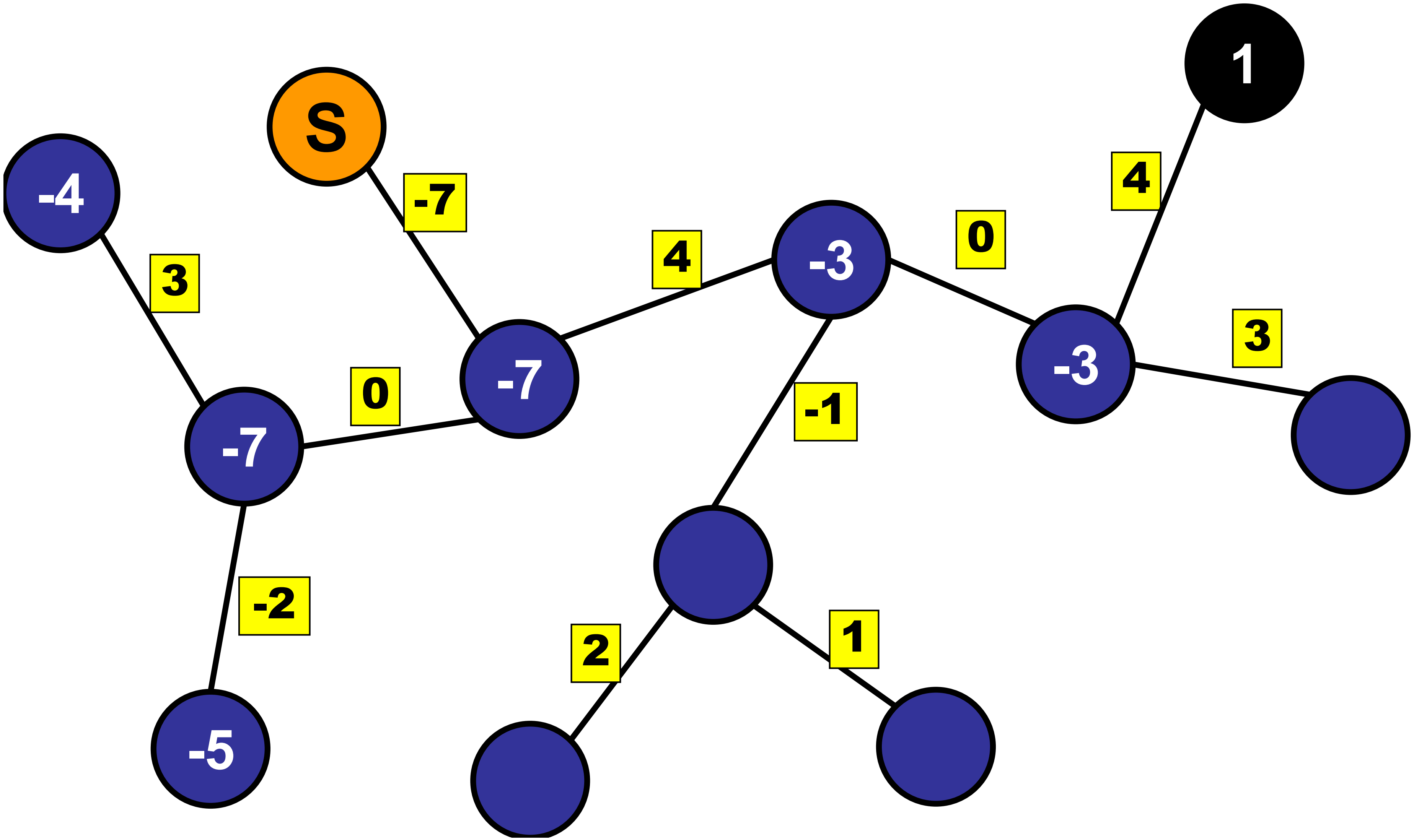
Special Case: Tree

Relax edges in DFS order.



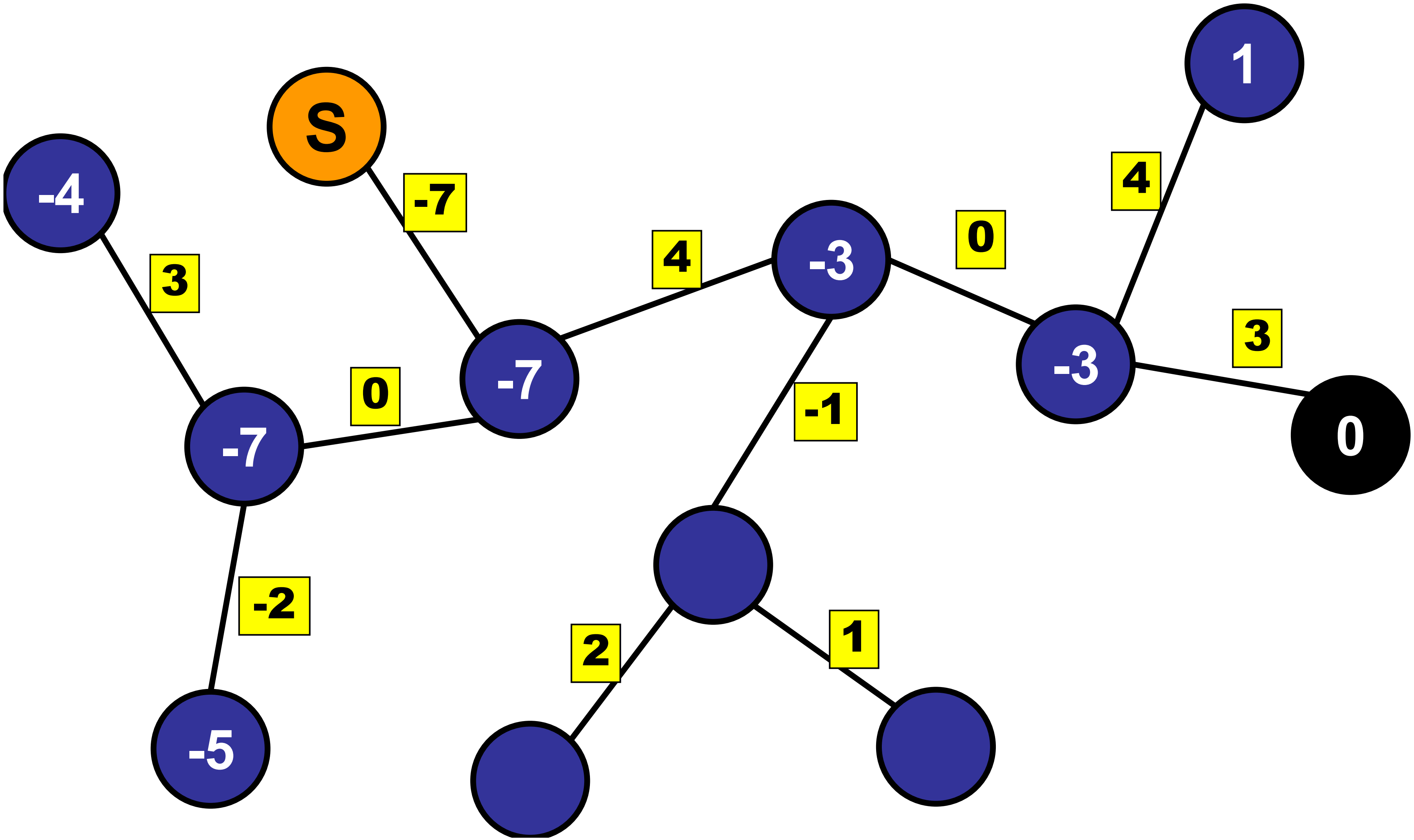
Special Case: Tree

Relax edges in DFS order.



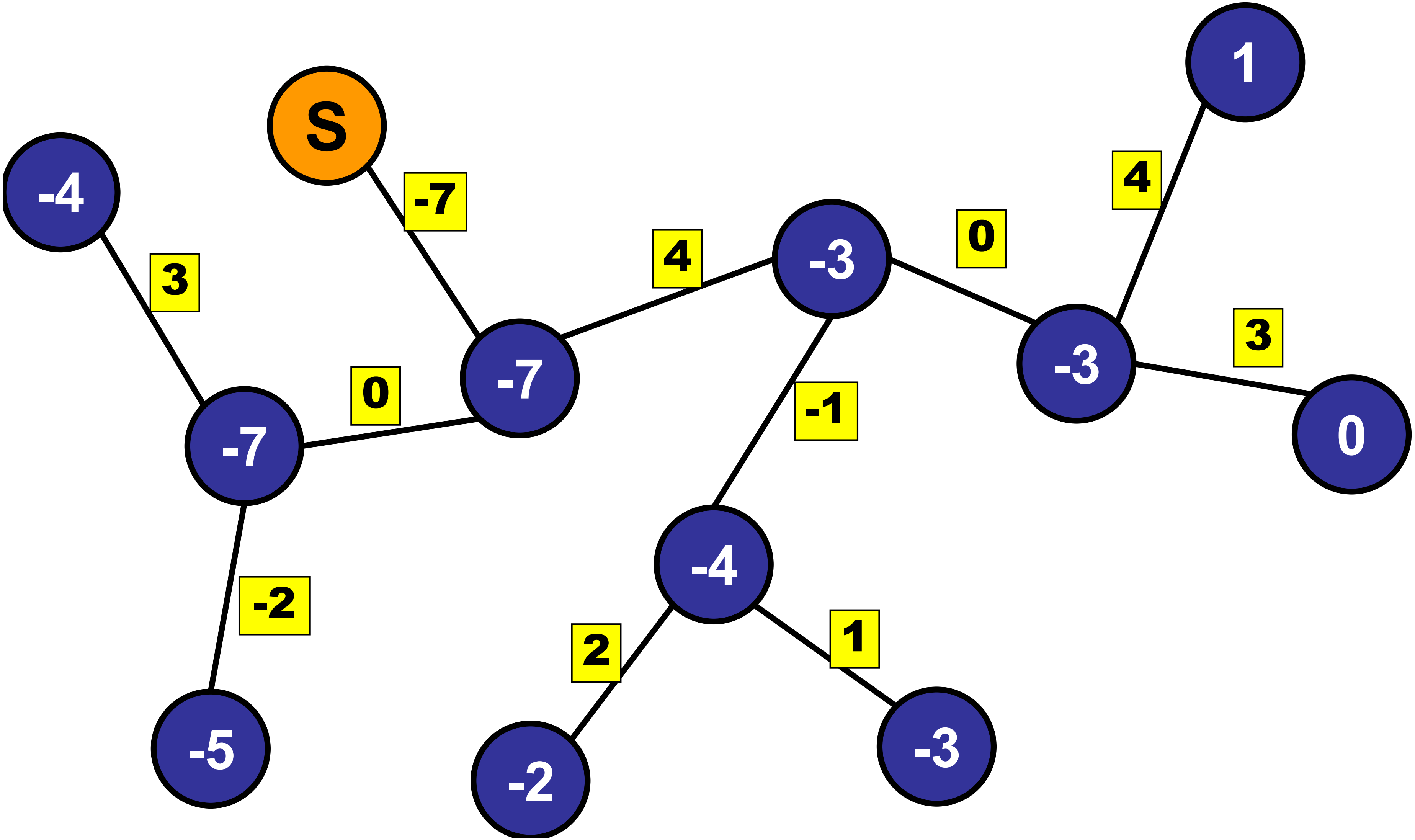
Special Case: Tree

Relax edges in DFS order.



Special Case: Tree

Relax edges in DFS order.



Special Case: Tree

Basic idea:

- Perform DFS or BFS
- Relax each edge the first time you see it.
- O(V) time.

Assumptions:

- Weighted edges
- Positive or negative weights
- Undirected tree

Why is the running time O(V)?

1. You only need to explore 1 outgoing edge for each vertex.
2. DFS/BFS run in O(V) time on a graph.
3. There are only O(V) edges in a tree.
4. It is not O(V): you need to explore every edge!
5. I'm confused.

Special Case: Tree

Basic idea:

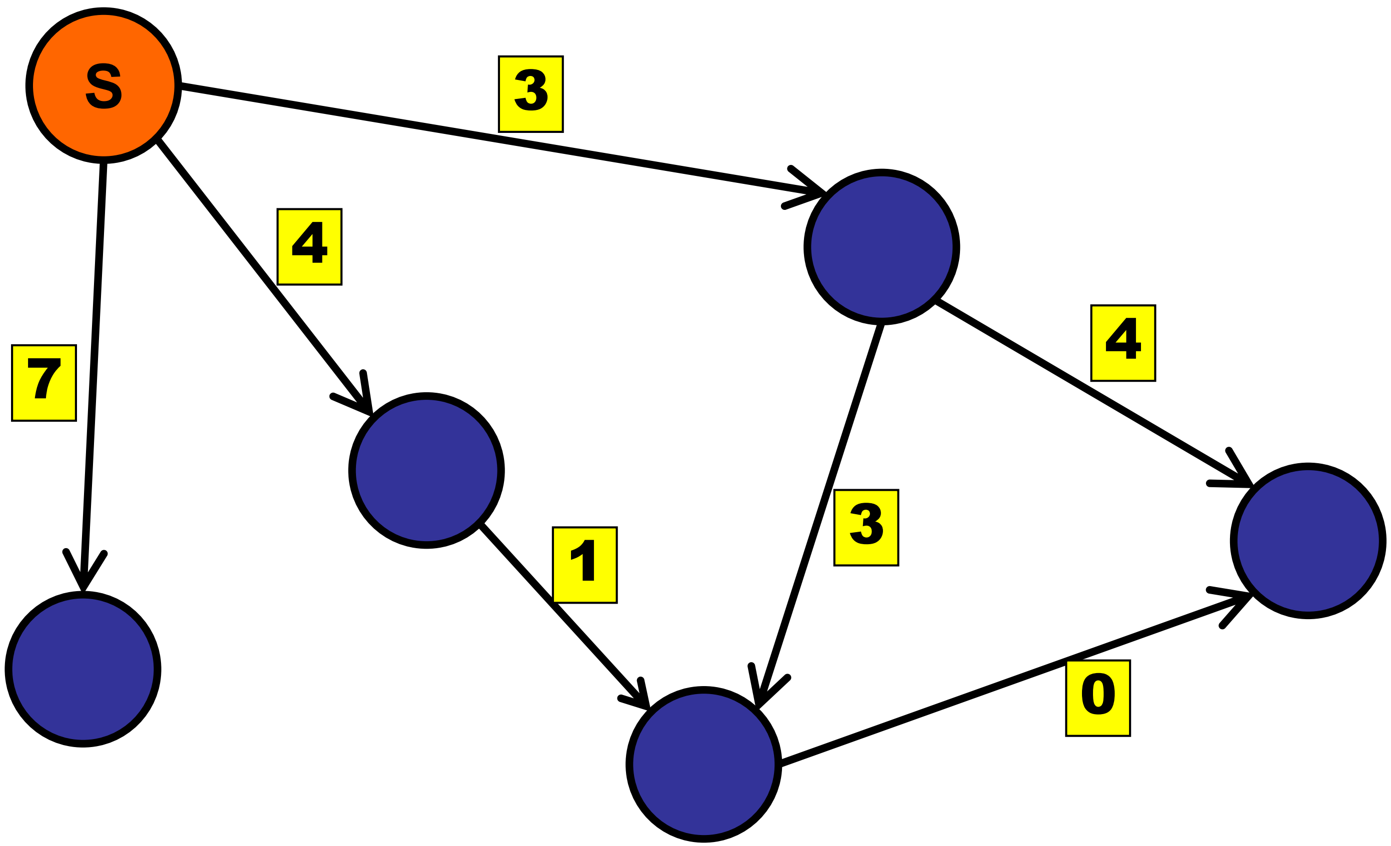
- Perform DFS or BFS
- Relax each edge the first time you see it.
- $O(V)$ time.

Assumptions:

- Weighted edges
- Positive or negative weights
- Undirected tree

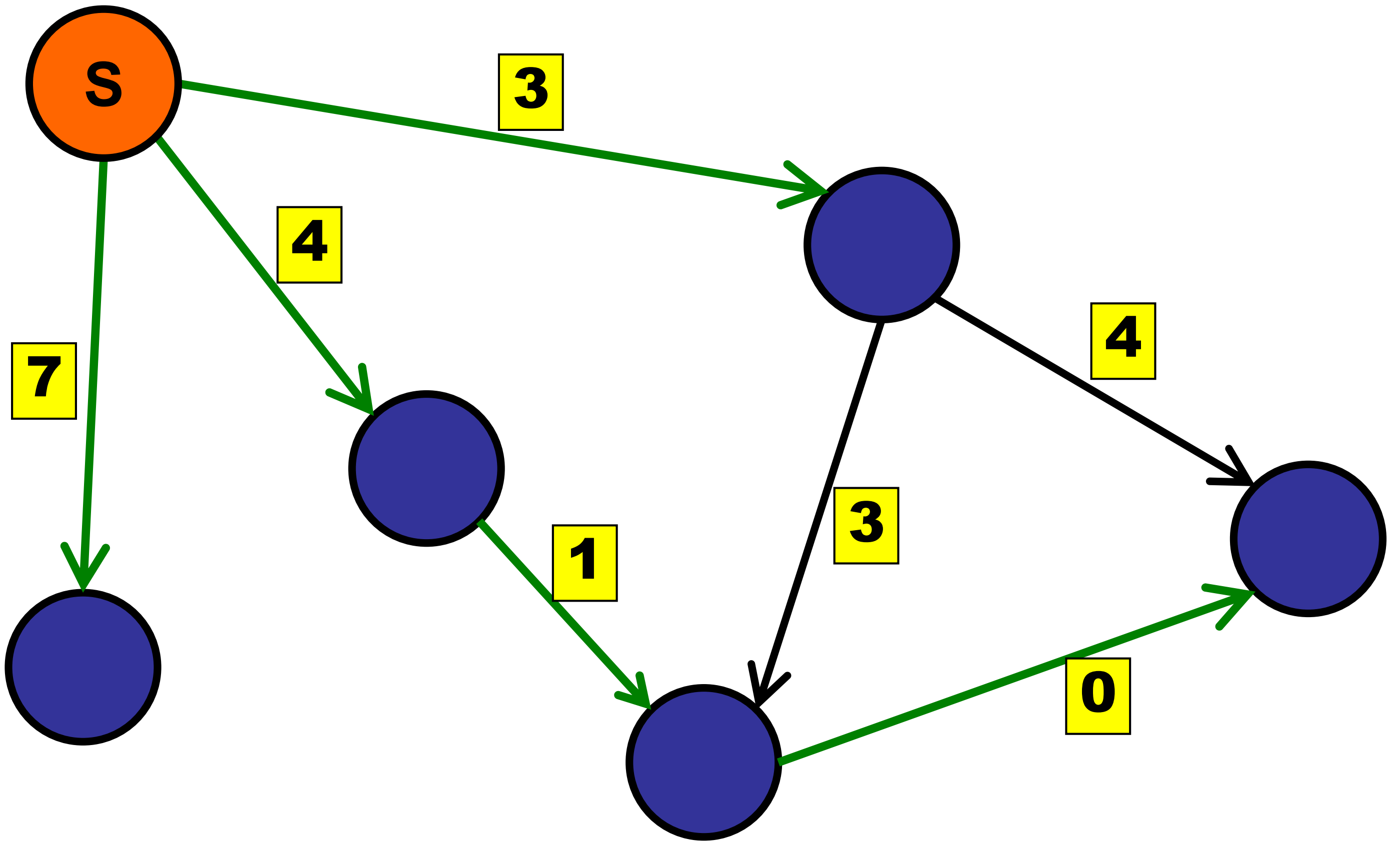
General Graph

Non-negative edges



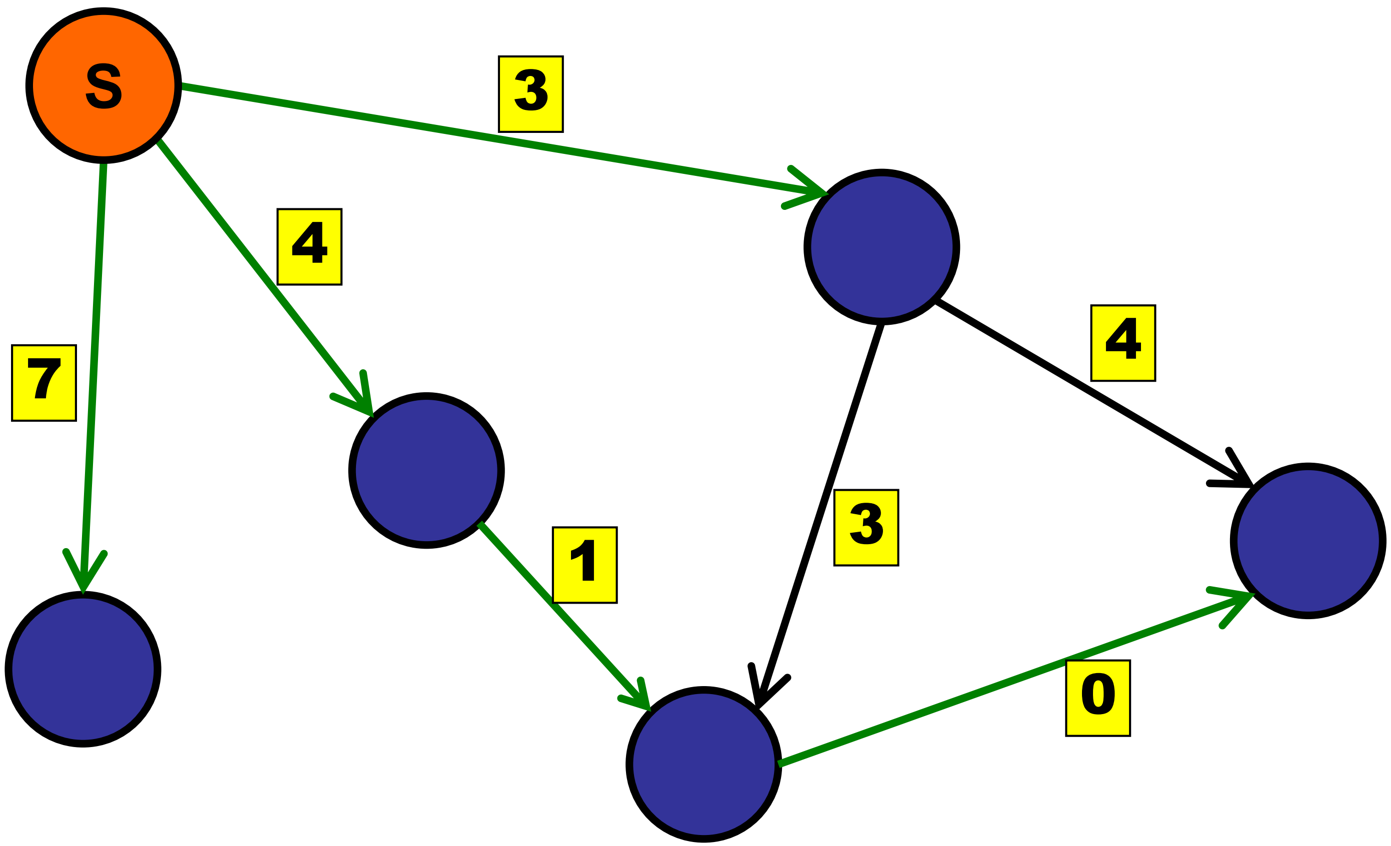
Shortest Path Tree

For every node: add 1 shortest path to the tree.



Shortest Path Tree

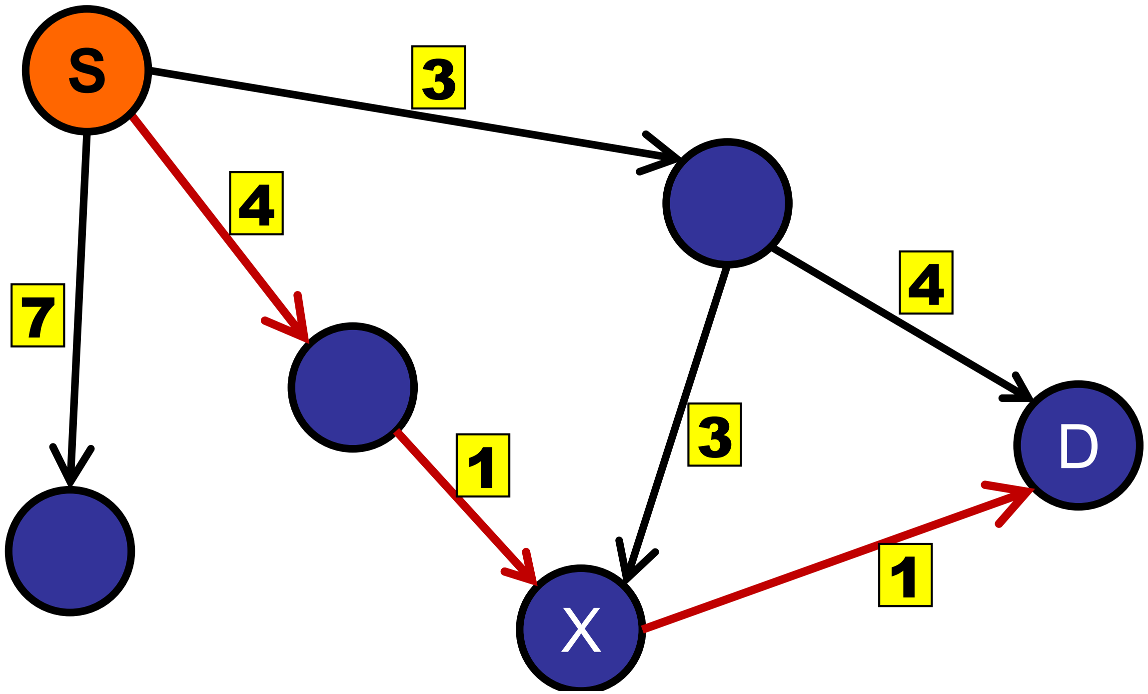
Why are there no cycles?



Shortest Path Tree

Key property:

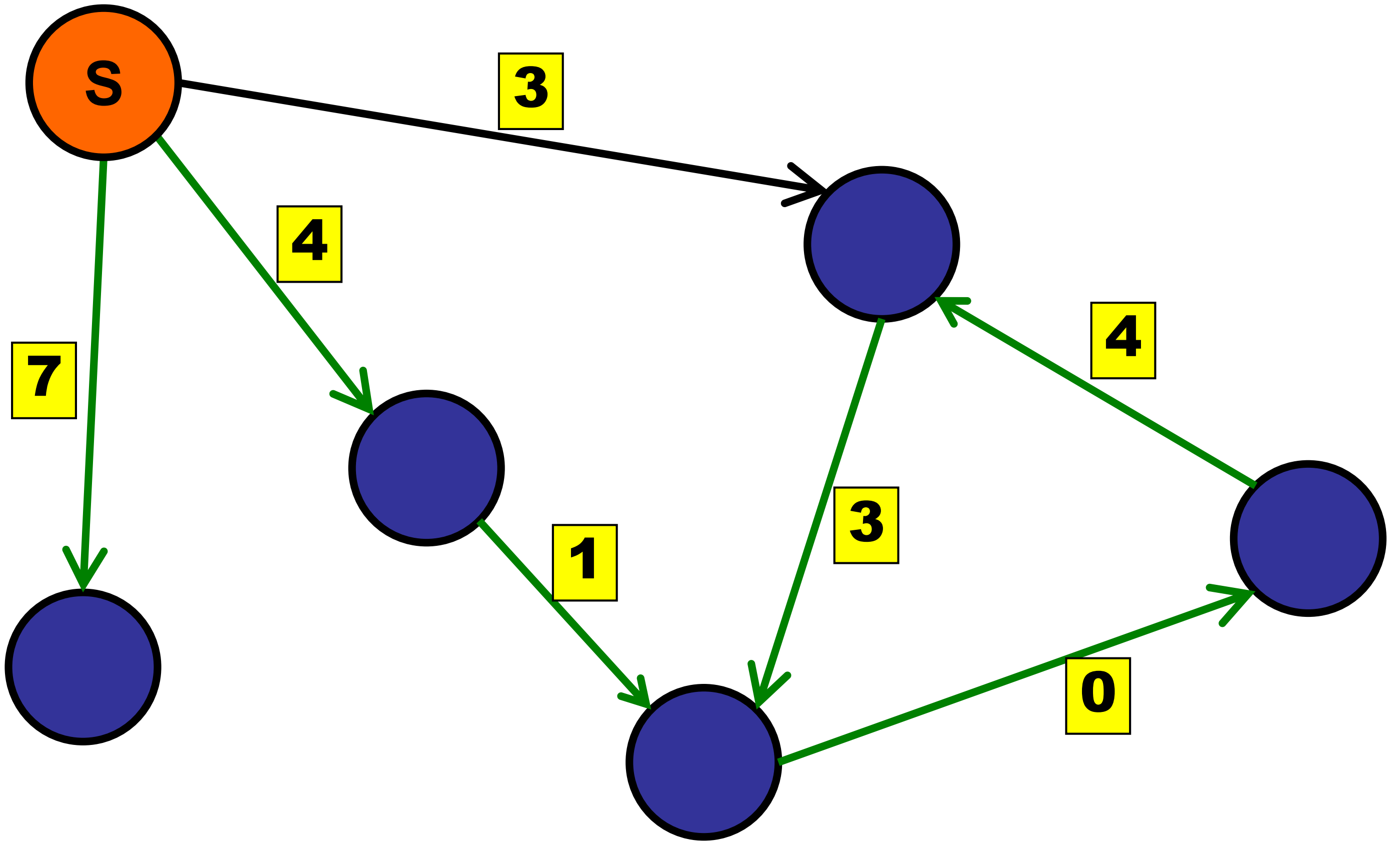
If P is the shortest path from S to D, and if P goes through X, then P is also the shortest path from S to X (and from X to D).



Shortest Path Tree

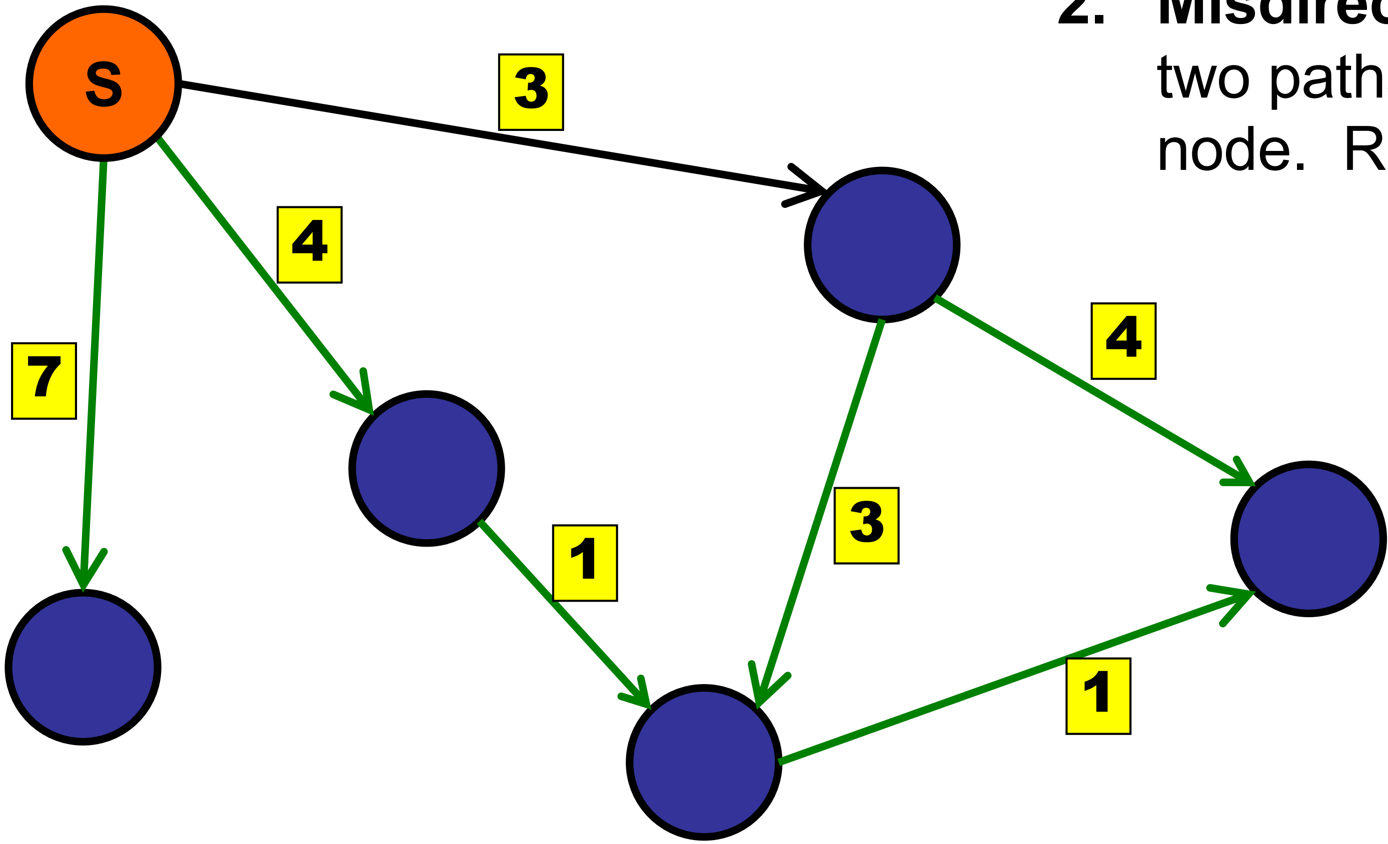
Why are there no cycles?

1. **Directed cycle**:
   remove one edge to get shorter paths.



Shortest Path Tree

Why are there no cycles?

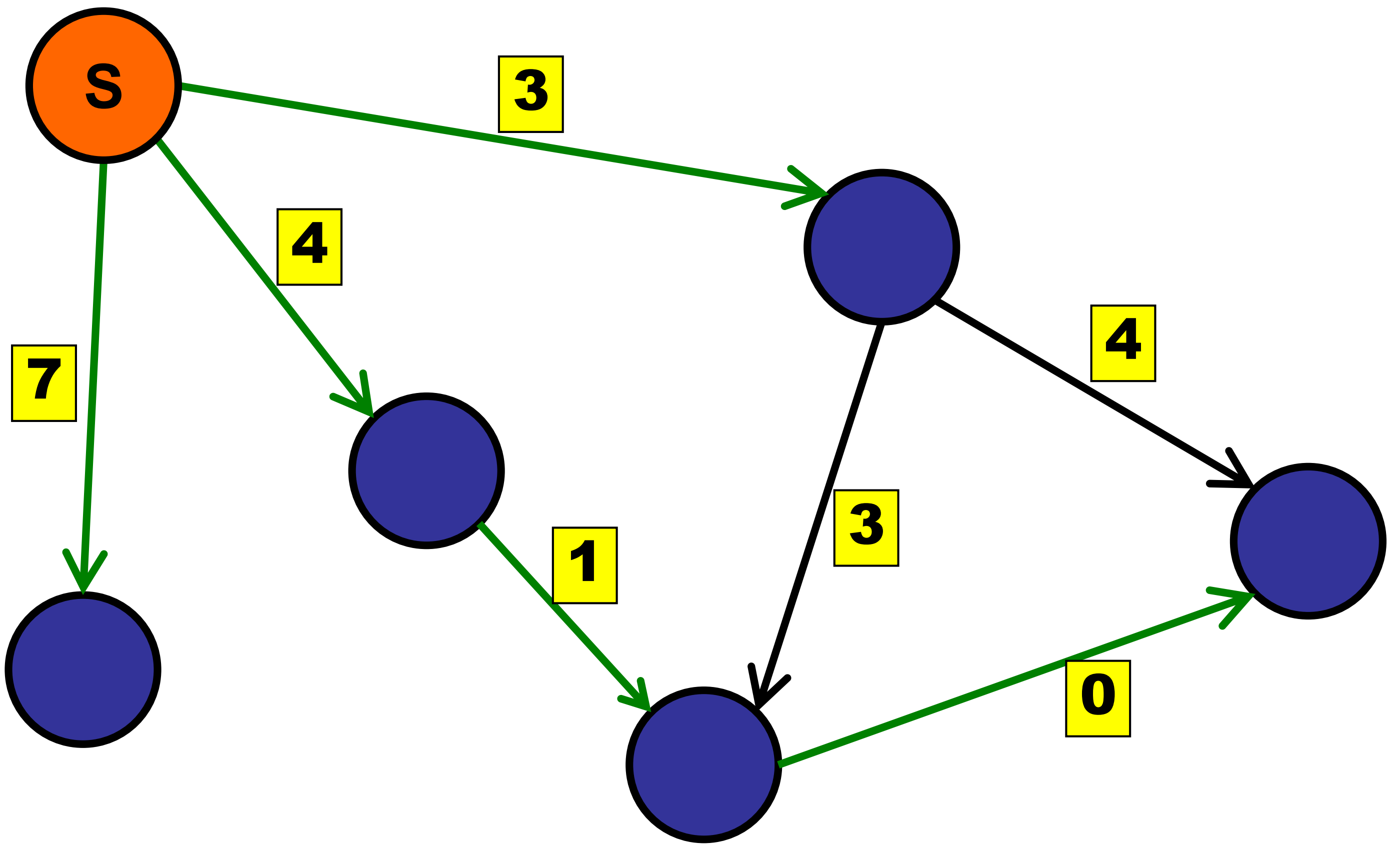


1. **Directed cycle**: remove one edge to get shorter paths.

2. **Misdirected cycle**: two paths to some node. Remove one.

Shortest Path Tree

No cycles in the shortest path tree.



Today

Key idea:

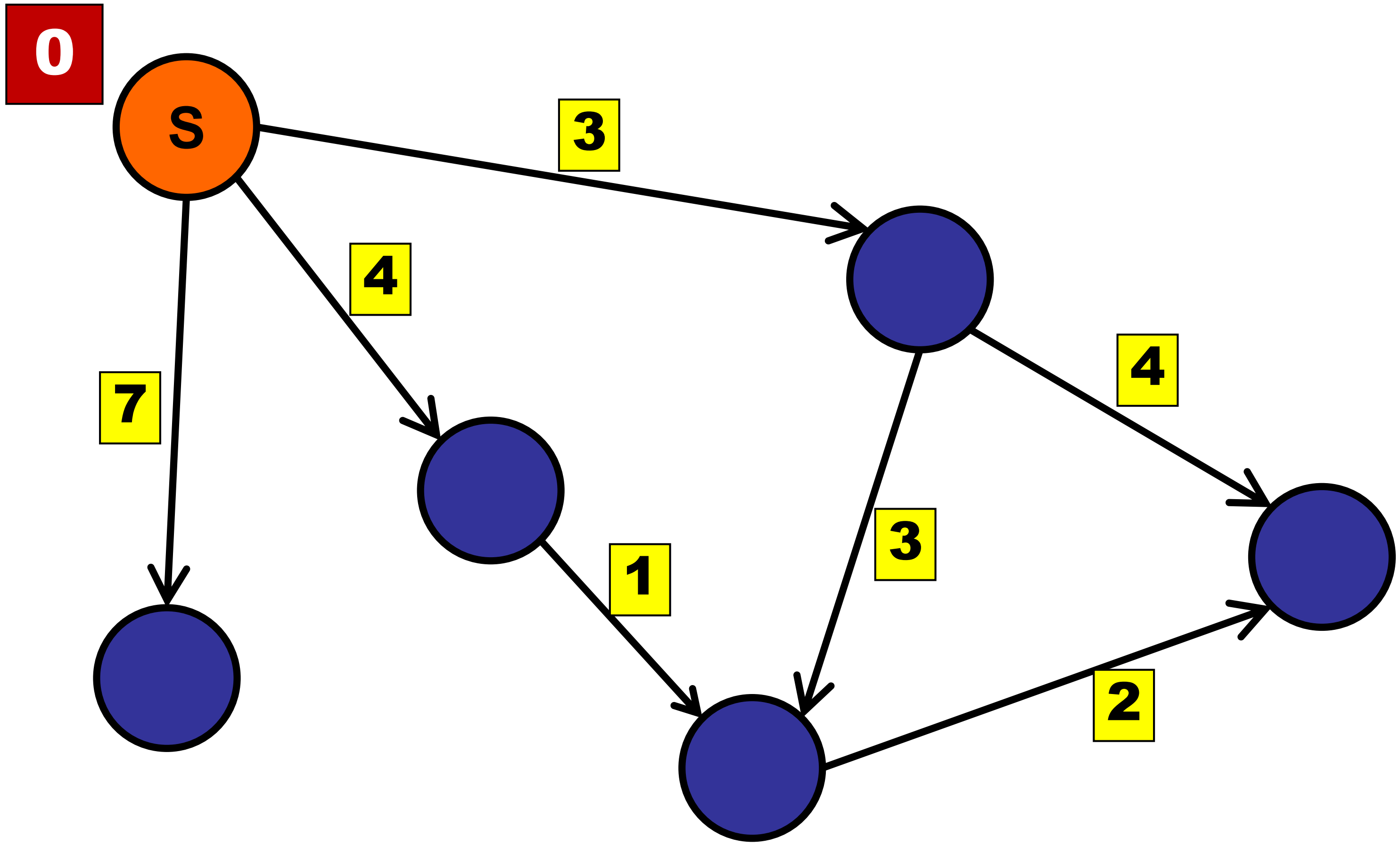
Relax the edges in the “right” order.

Only relax each edge once:

– O(E) cost (for relaxation step).

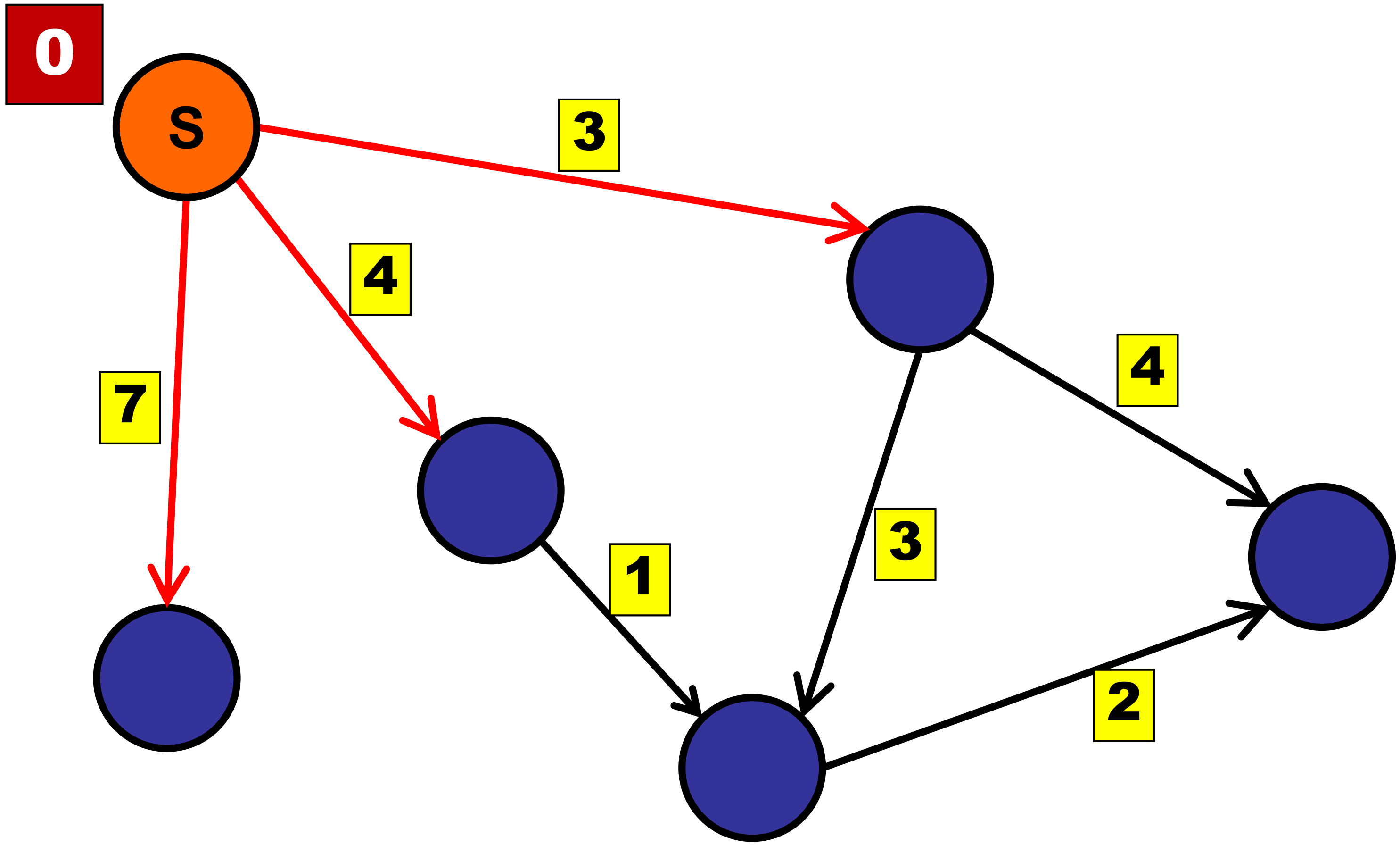
Dijkstra's Algorithm (First Try)

Relax shortest edge first



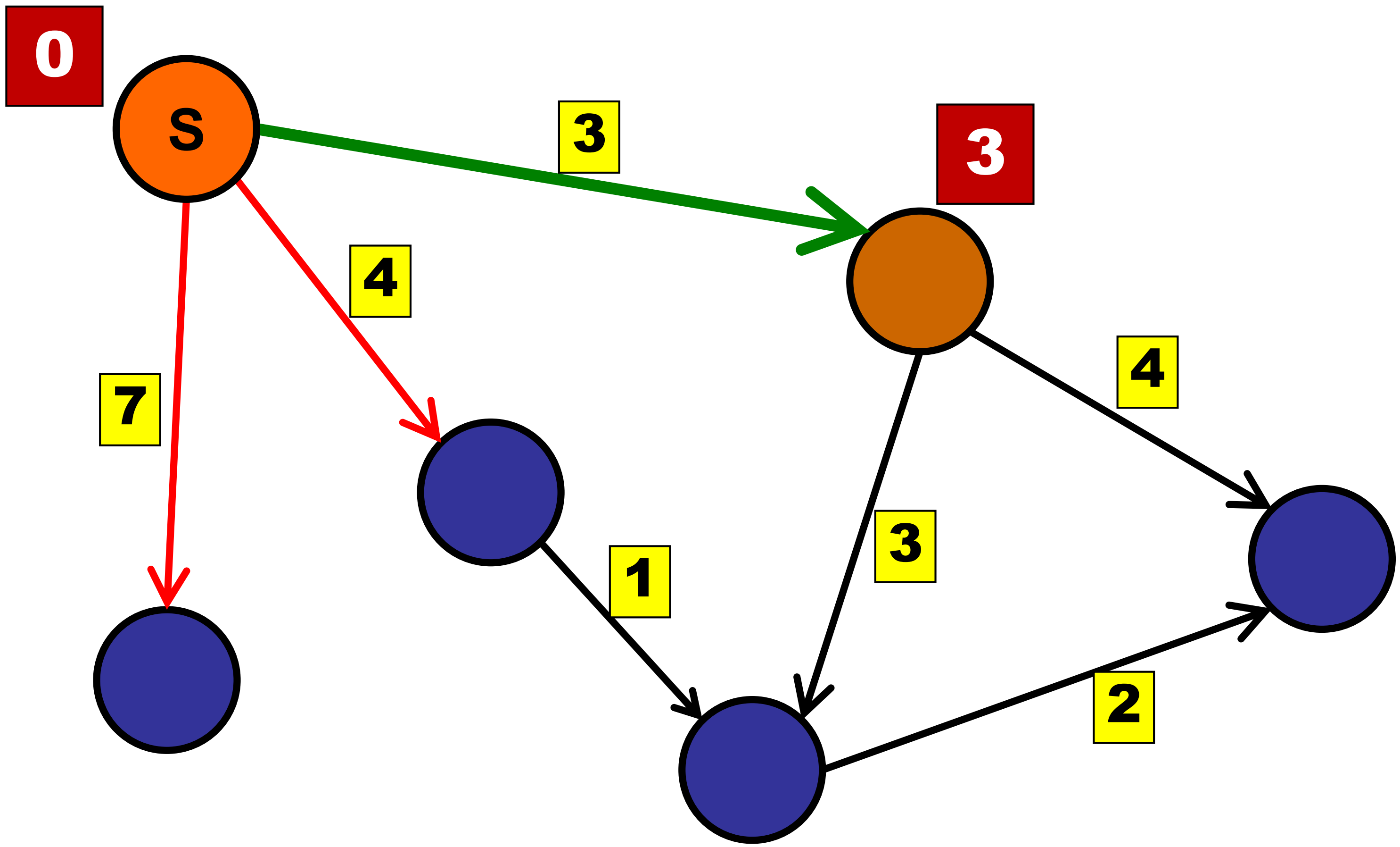
Dijkstra's Algorithm (First Try)

Relax shortest edge first



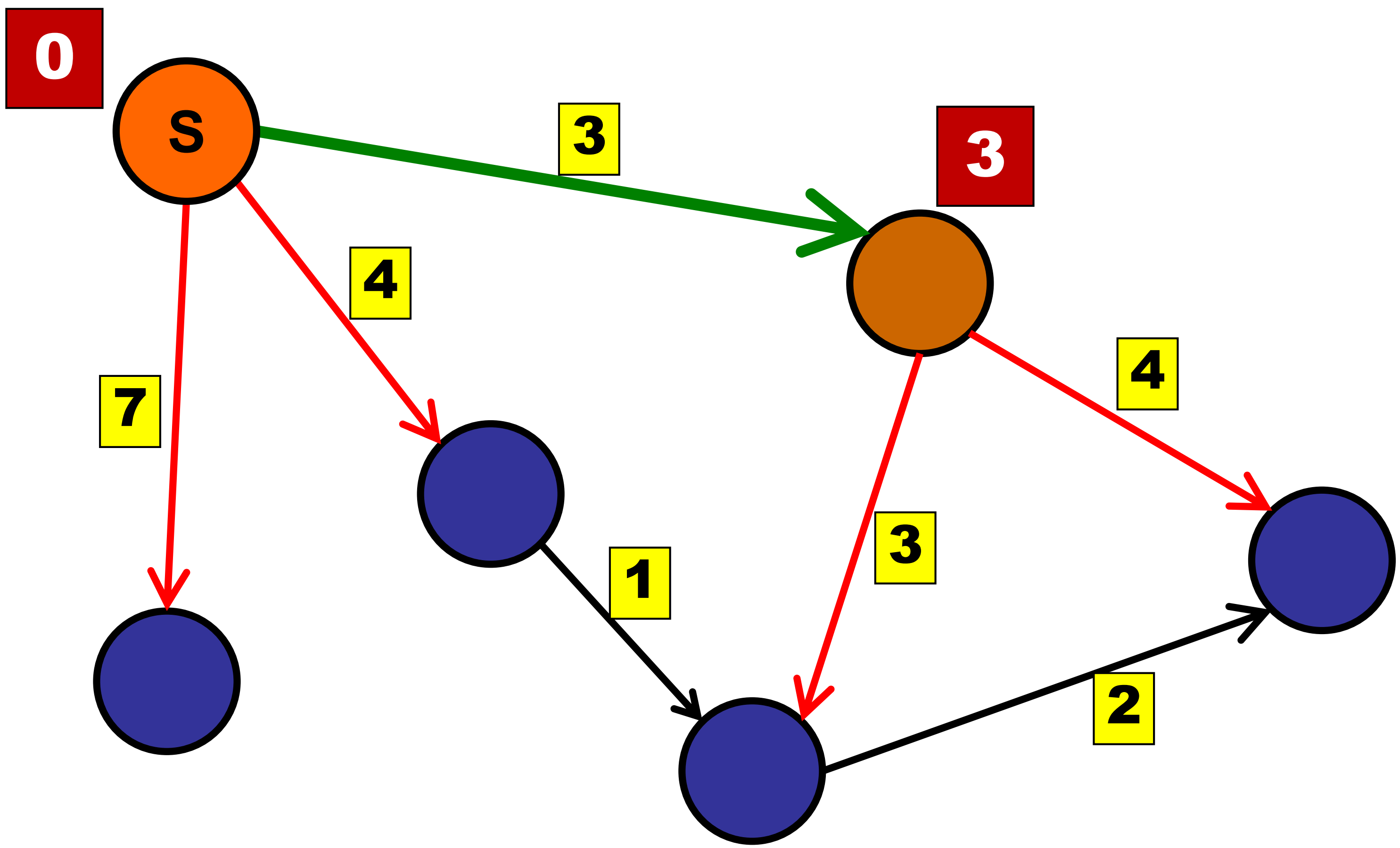
Dijkstra's Algorithm (First Try)

Relax shortest edge first



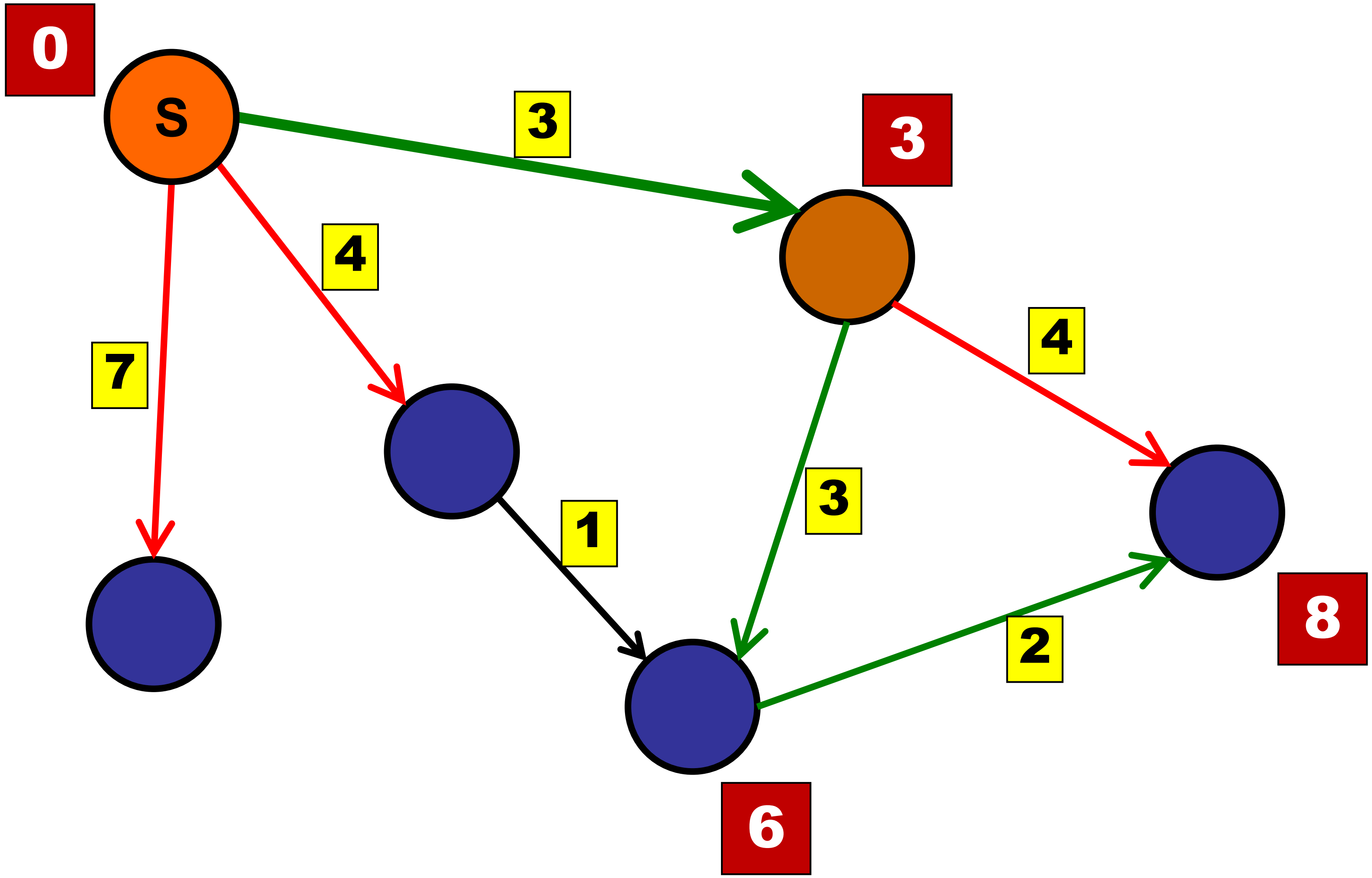
Dijkstra’s Algorithm (First Try)

Relax shortest edge first



Dijkstra’s Algorithm (Failed Try)

Oops....

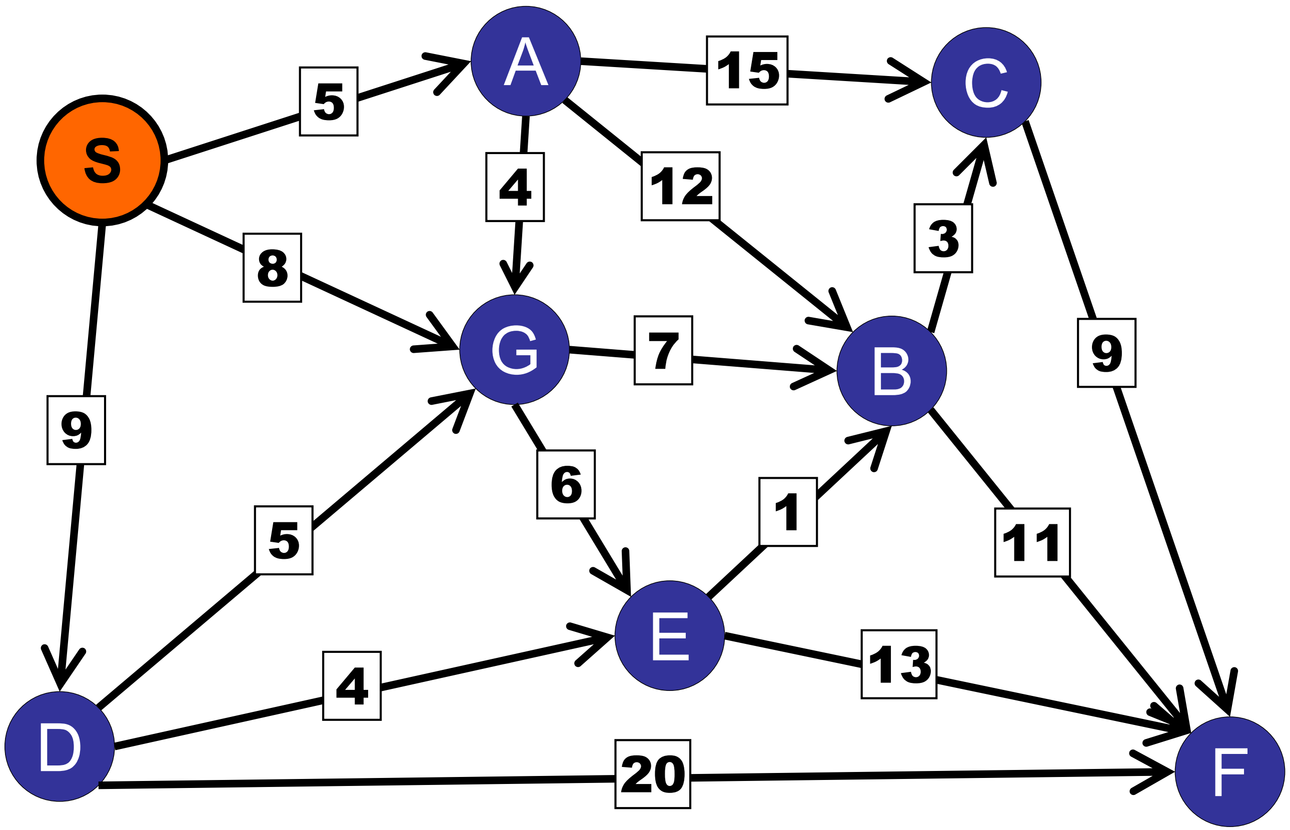


Dijkstra’s Algorithm

Basic idea:

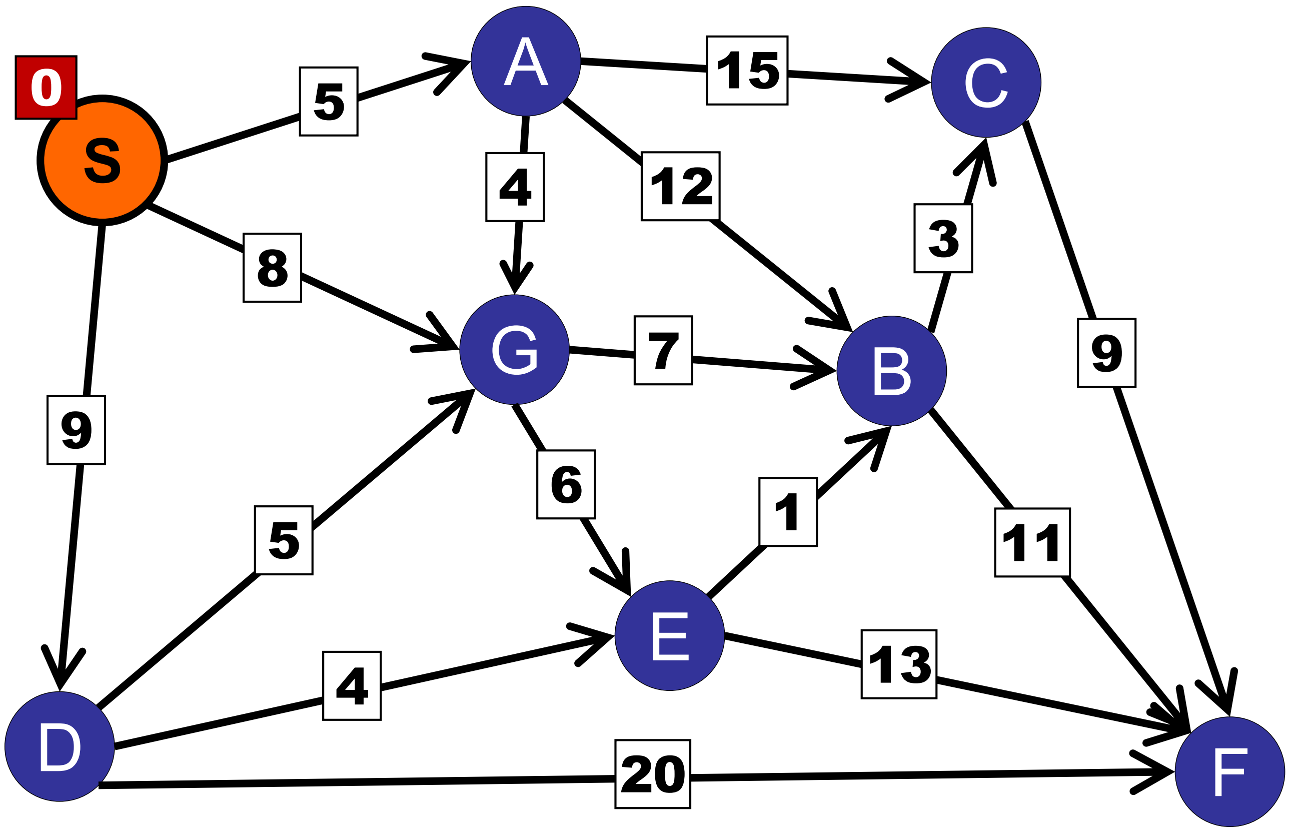
- Maintain distance estimate for every node.
- Begin with empty shortest-path-tree.
- Repeat:
  - Consider vertex with minimum estimate.
  - Add vertex to shortest-path-tree.
  - Relax all outgoing edges.

Shortest Paths



Dijkstra's Algorithm

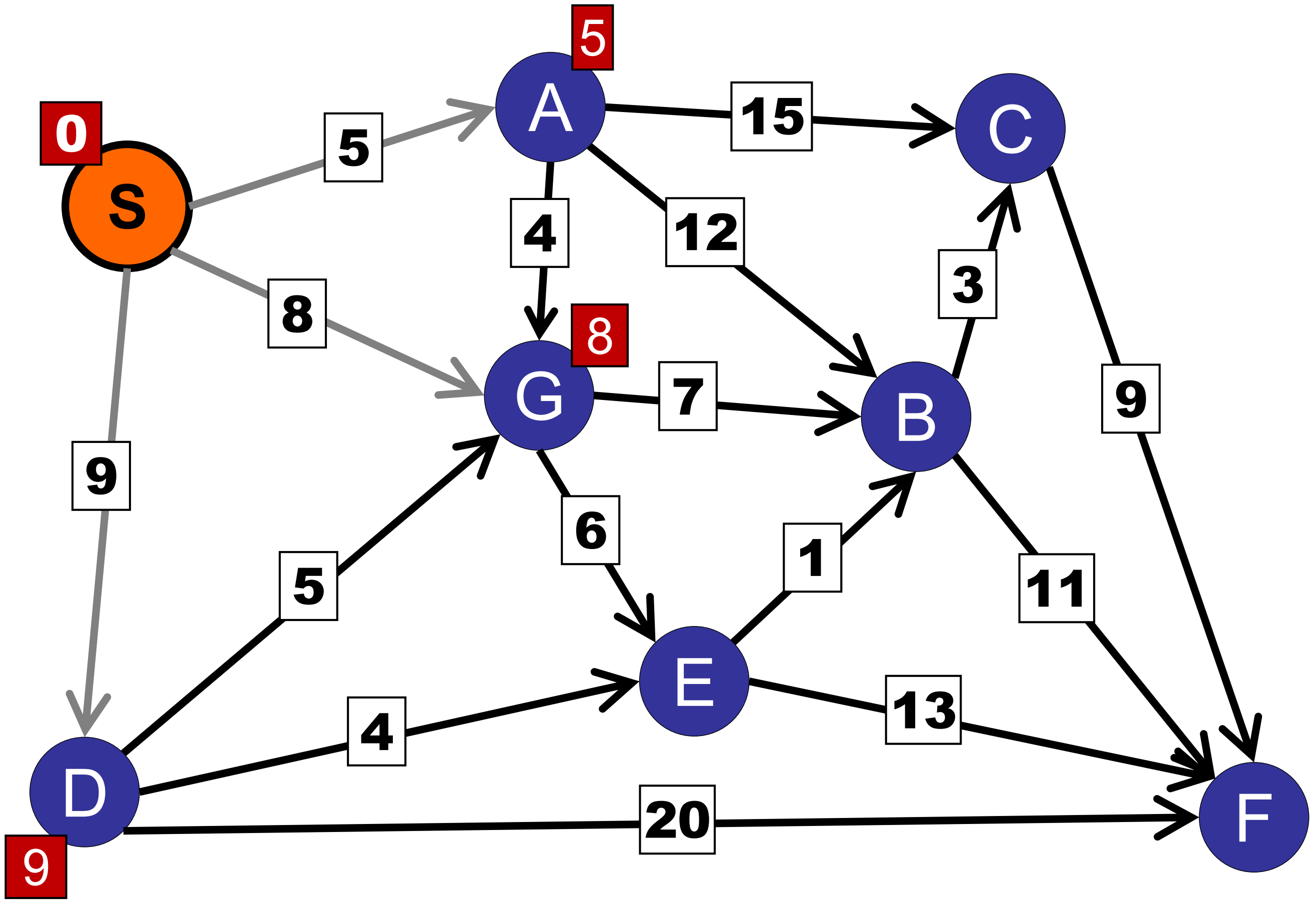
Step 1: Add source



Vertex	Dist.
S	0

Dijkstra's Algorithm

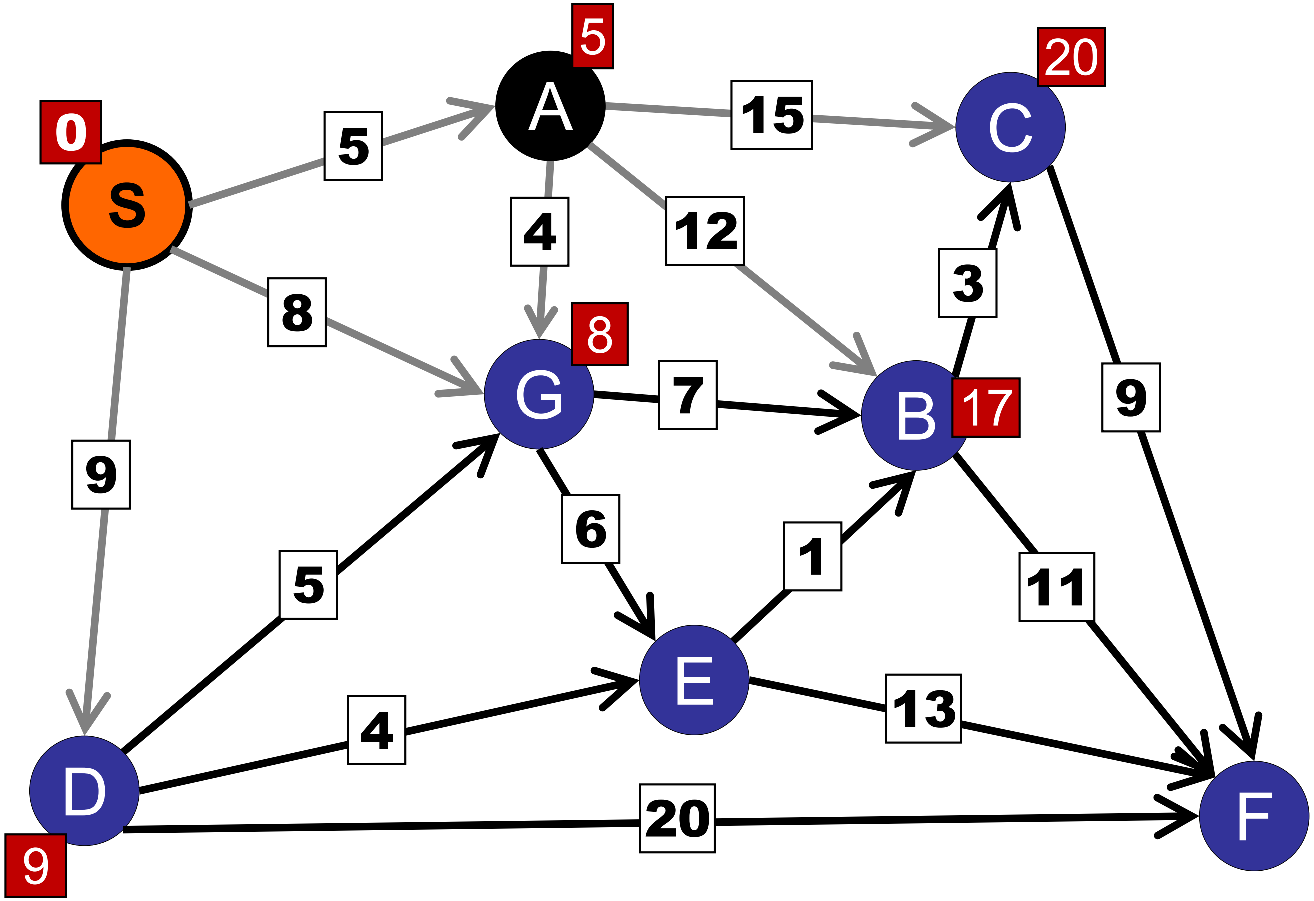
Step 2: Remove S and relax.



Vertex	Dist.
A	5
G	8
D	9

Dijkstra's Algorithm

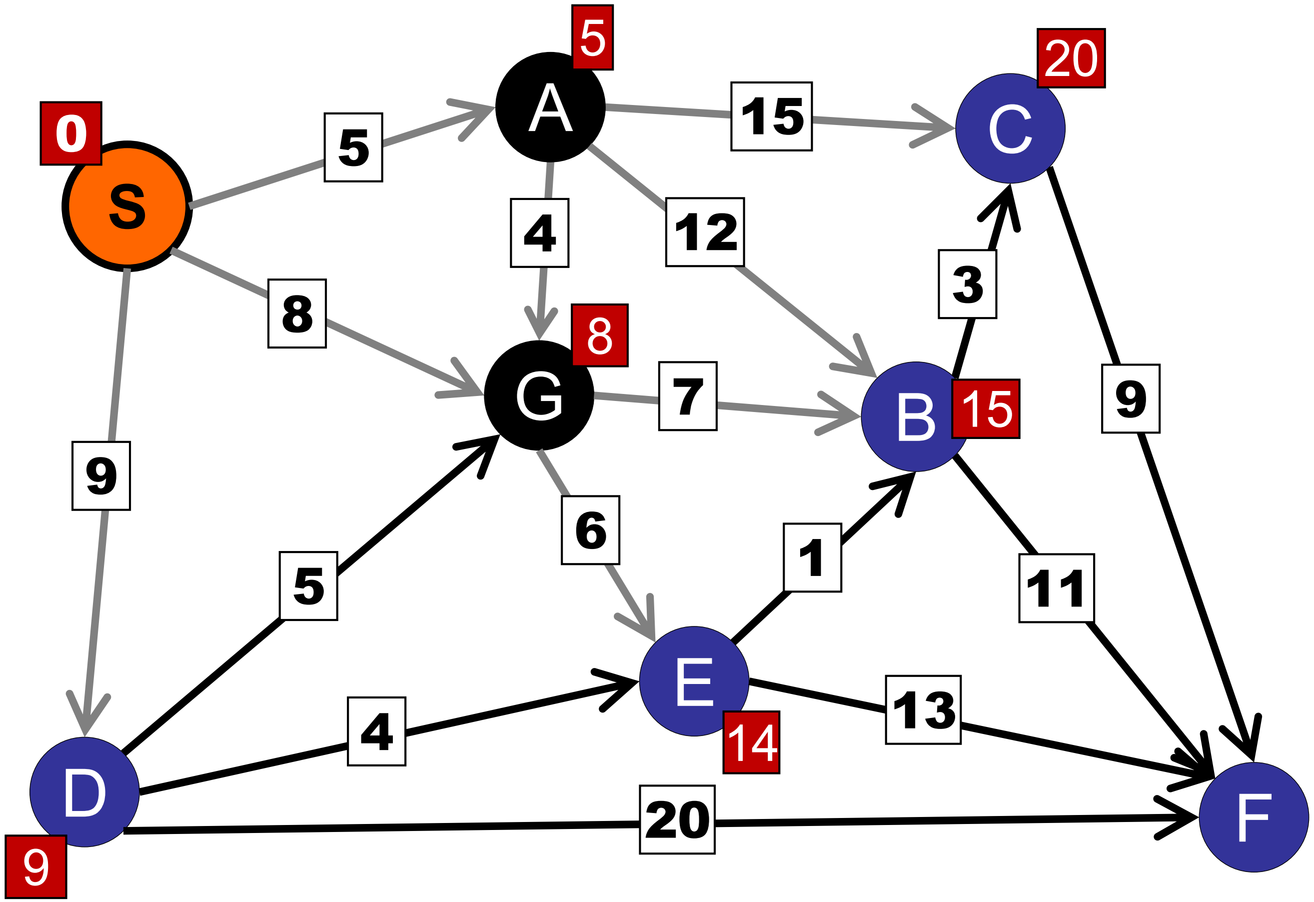
Step 3: Remove A and relax.



Vertex	Dist.
G	**8**
D	9
B	**17**
C	**20**

Dijkstra's Algorithm

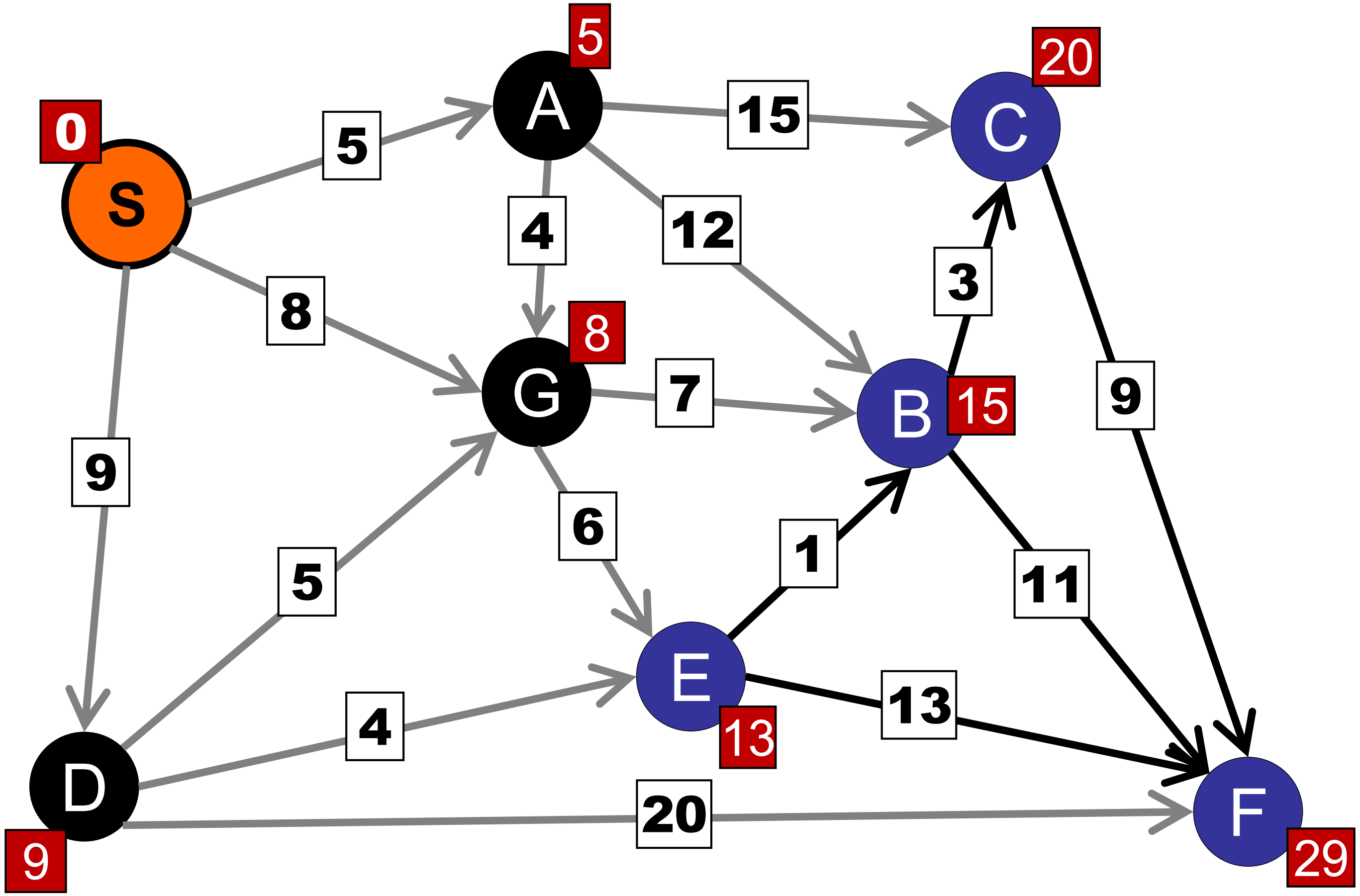
Step 4: Remove G and relax.



Vertex	Dist.
D	9
E	**14**
B	**15**
C	20

Dijkstra's Algorithm

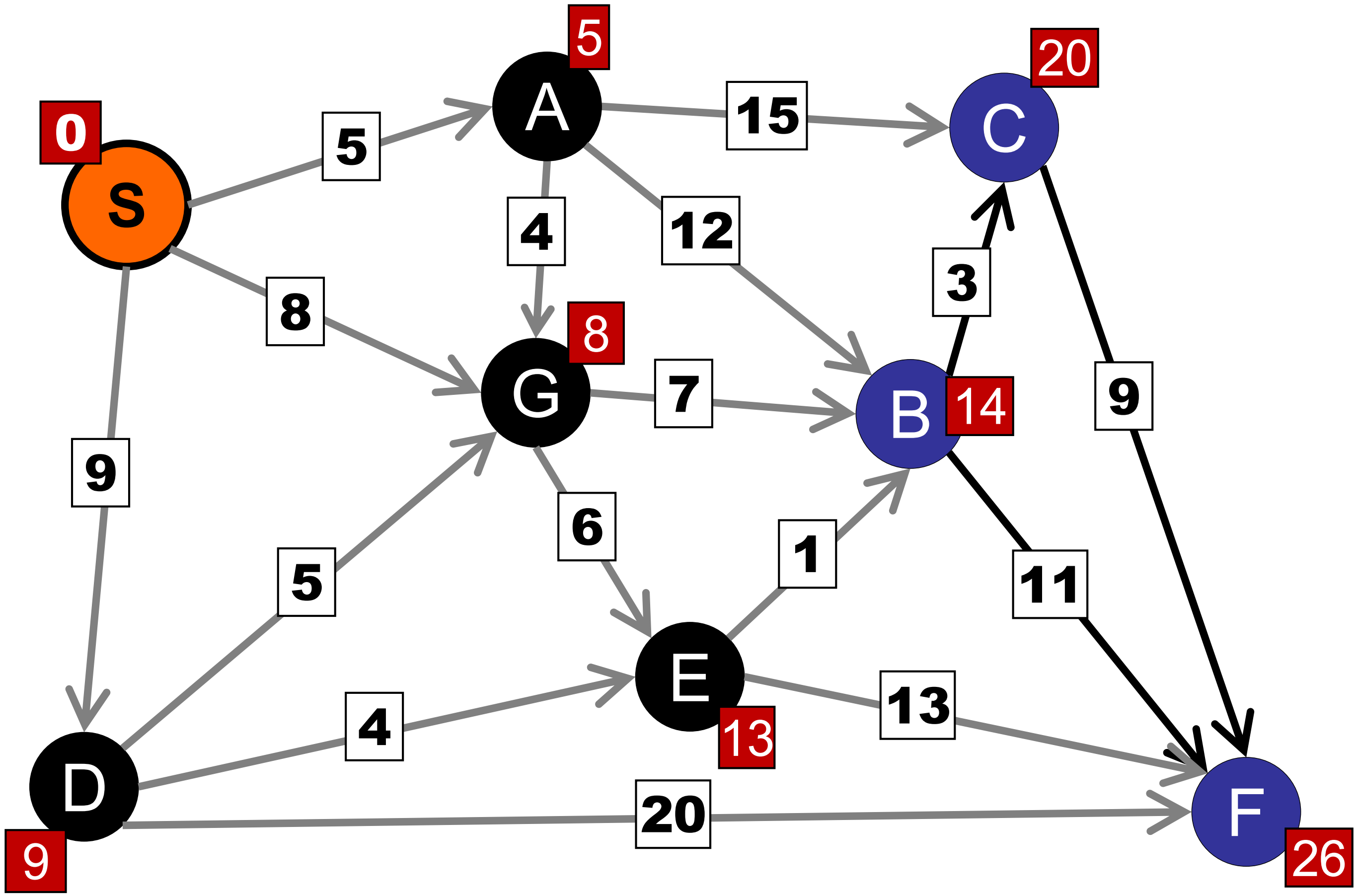
Step 5: Remove D and relax.



Vertex	Dist.
E	**13**
B	15
C	20
F	**29**

Dijkstra's Algorithm

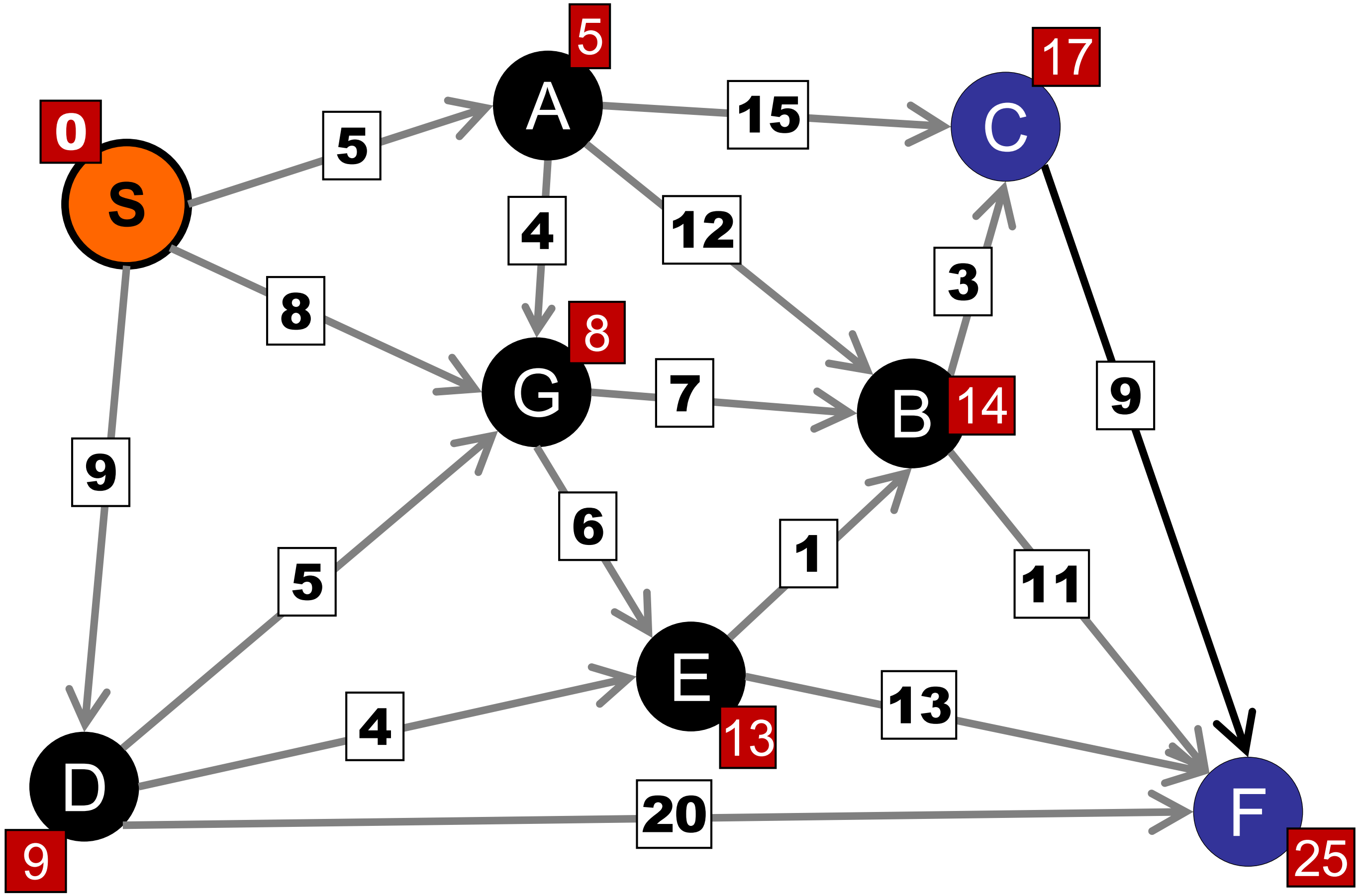
Step 5: Remove E and relax.



Vertex	Dist.
B	**14**
C	20
F	**26**

Dijkstra's Algorithm

Step 5: Remove B and relax.

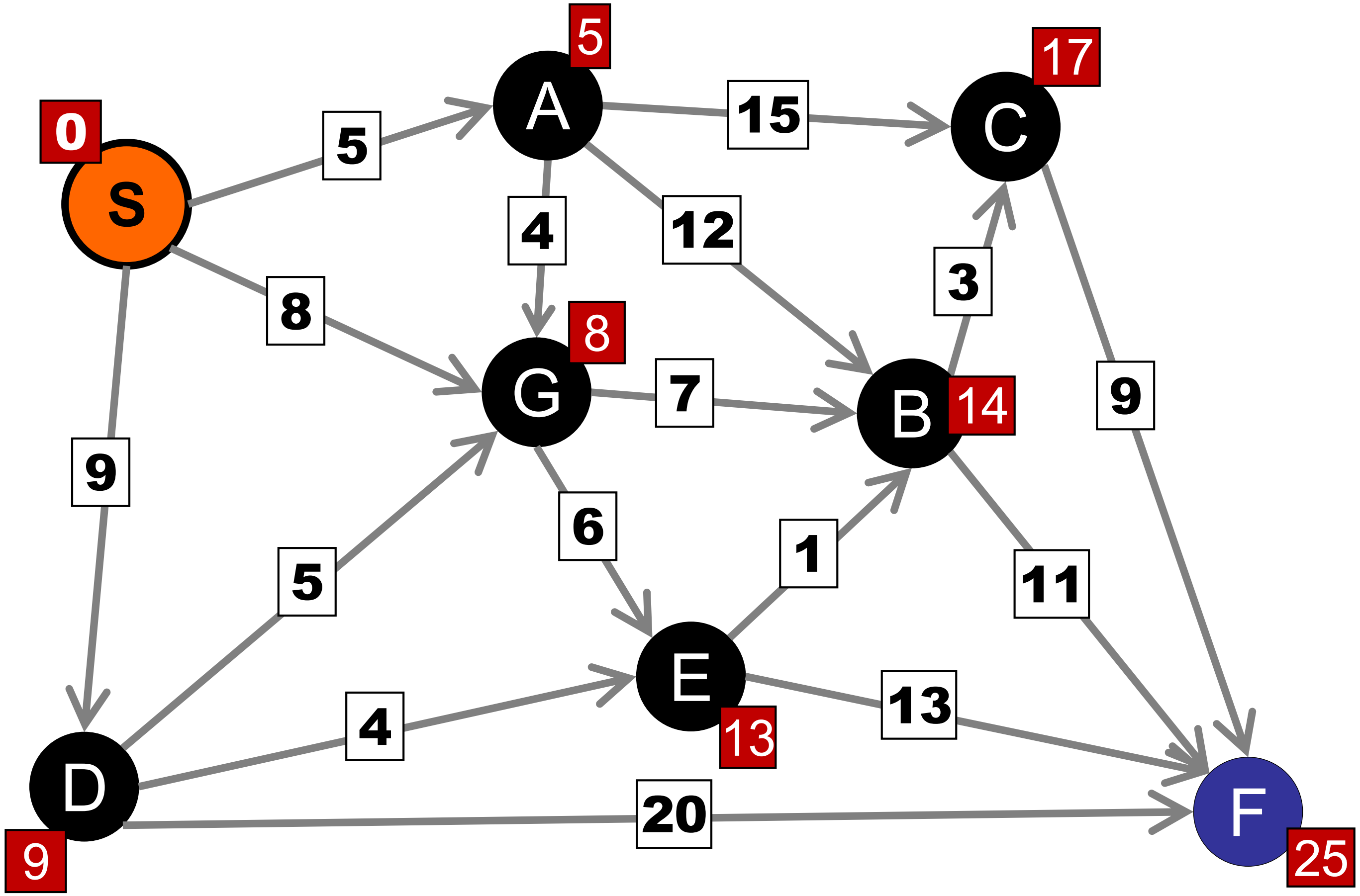


Vertex	Dist.
C	20
F	25

Dijkstra's Algorithm

Step 5: Remove C and relax.

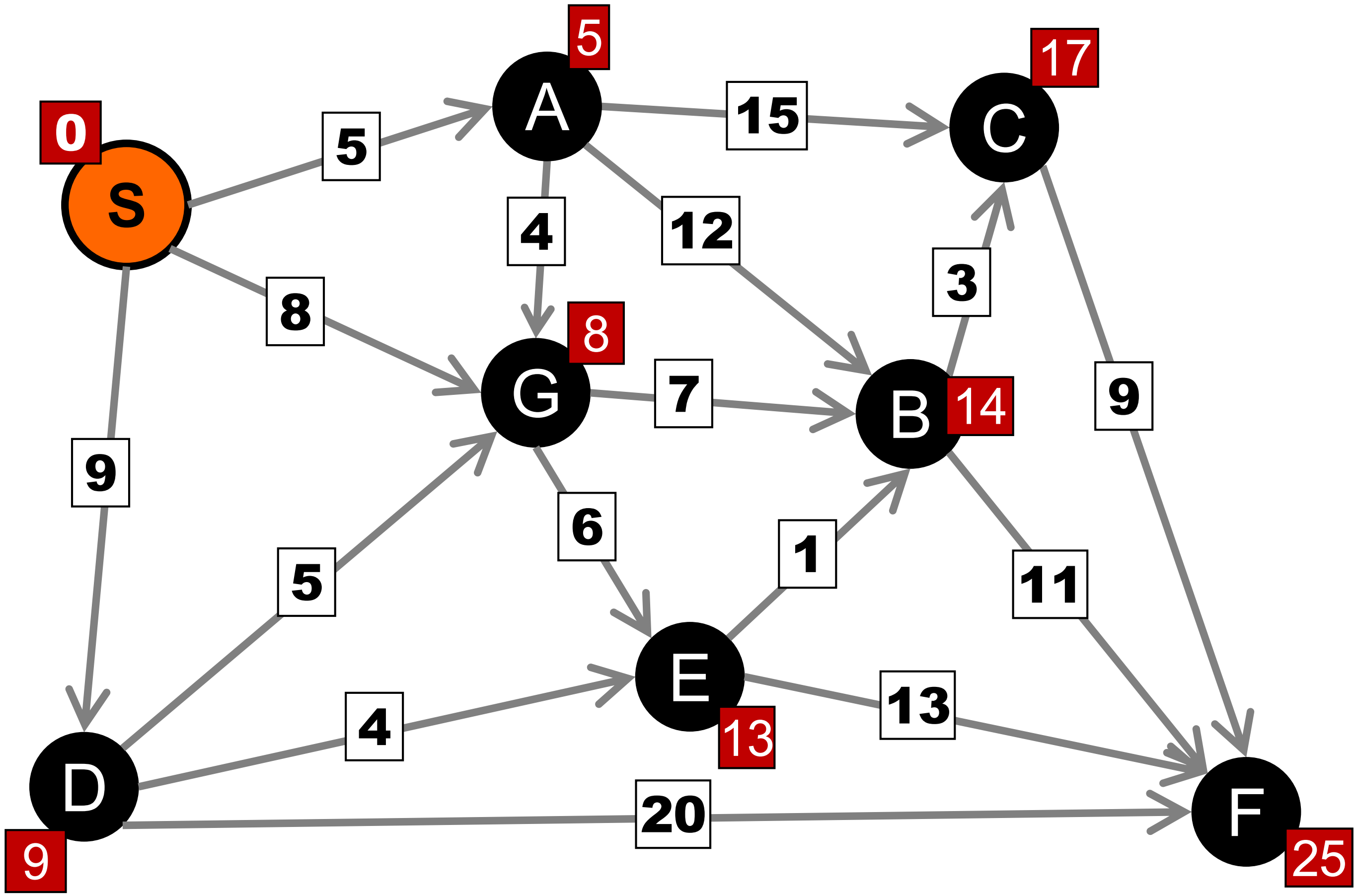
Vertex	Dist.
F	25



Dijkstra's Algorithm

Step 5: Remove F and relax.

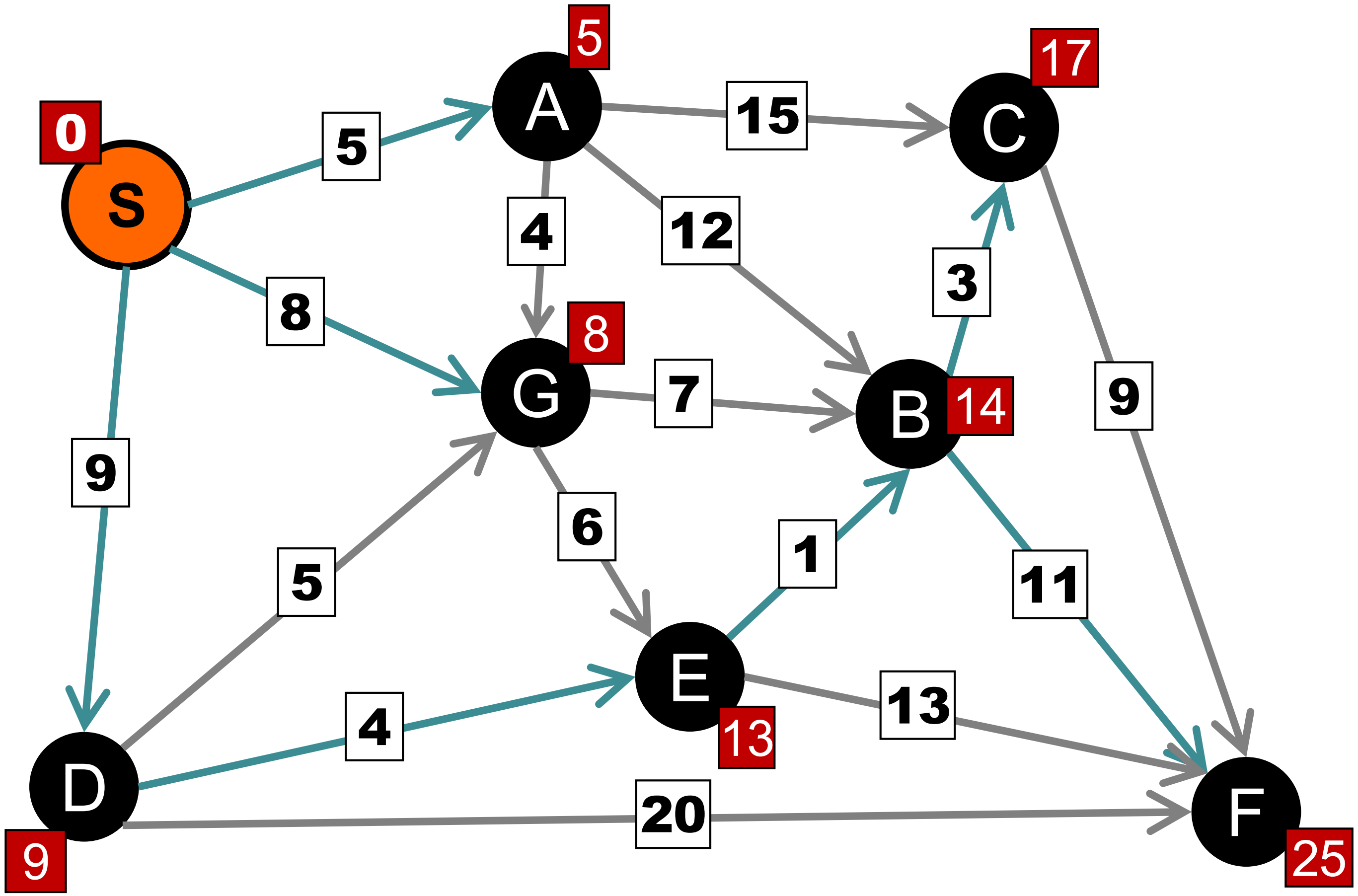
Vertex	Dist.



Dijkstra’s Algorithm

Done

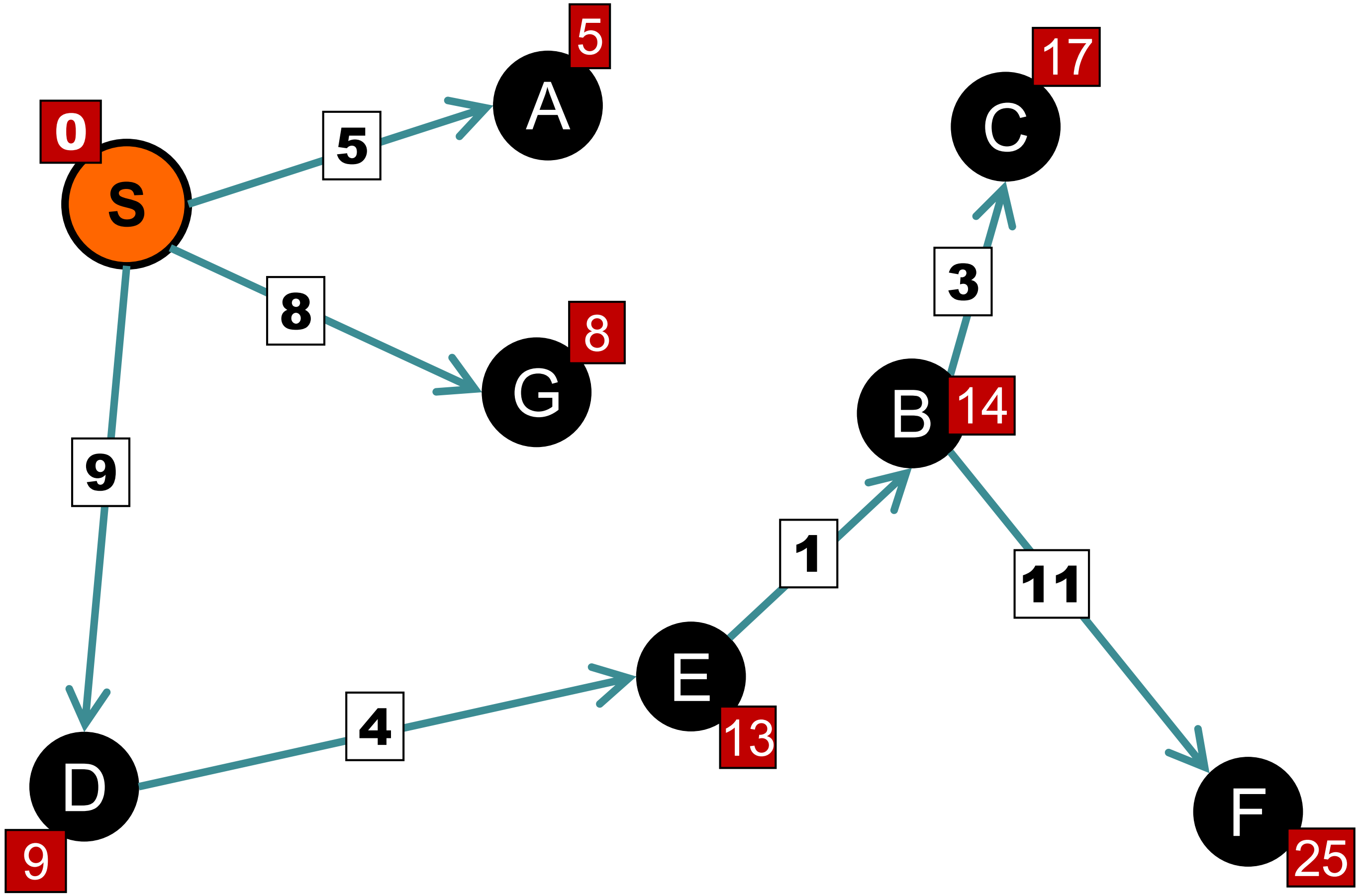
Vertex	Dist.



Dijkstra's Algorithm

Shortest Path Tree

Vertex	Dist.



What data structure to store vertices/distances?

1. Array
2. Linked list
3. Stack
4. Queue
5. AVL Tree
6. Huh?

Vertex	Dist.
B	**14**
C	20
F	**26**

Abstract Data Type

Priority Queue

interface	**IPriorityQueue<Key, Priority>**	
void	insert(Key k, Priority p)	*insert k with priority p*
Data	extractMin()	*remove key with minimum priority*
void	decreaseKey(Key k, Priority p)	*reduce the priority of key k to priority p*
boolean	contains(Key k)	*does the priority queue contain key k?*
boolean	isEmpty()	*is the priority queue empty?*

Notes:

Assume data items are unique.

```
public Dijkstra{
    private Graph G;
    private IPriorityQueue pq = new PriQueue();
    private double[] distTo;

    searchPath(int start) {
        pq.insert(start, 0.0);
        distTo = new double[G.size()];
        Arrays.fill(distTo, INFTY);
        distTo[start] = 0;
        while (!pq.isEmpty()) {
            int w = pq.deleteMin();
            for (Edge e : G[w].nbrList)
                relax(e);
        }
    }
```

Dijkstra’s Algorithm

```
relax(Edge e) {
    int v = e.from();
    int w = e.to();
    double weight = e.weight();
    if (distTo[w] > distTo[v] + weight) {
        distTo[w] = distTo[v] + weight;
        parent[w] = v;
        if (pq.contains(w))
            pq.decreaseKey(w, distTo[w]);
        else
            pq.insert(w, distTo[w]);
    }
}
```

Abstract Data Type

Priority Queue

interface	**IPriorityQueue<Key, Priority>**	
`void`	`insert(Key k, Priority p)`	*insert k with priority p*
`Data`	`extractMin()`	*remove key with minimum priority*
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`boolean`	`contains(Key k)`	*does the priority queue contain key k?*
`boolean`	`isEmpty()`	*is the priority queue empty?*

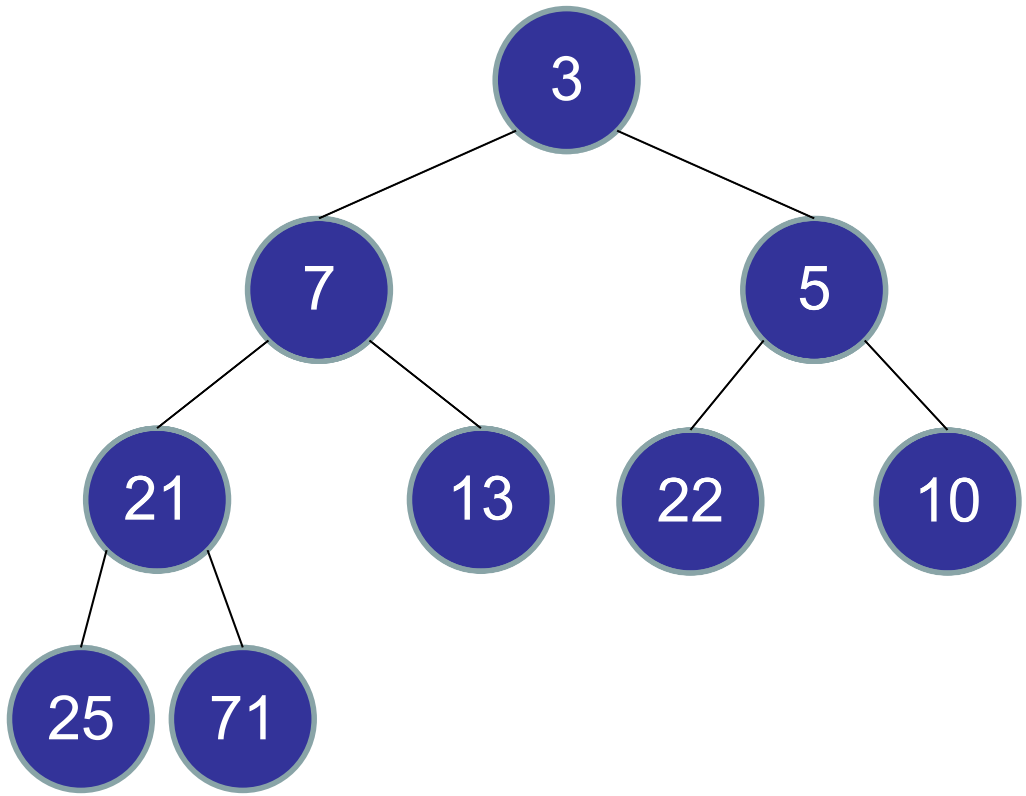
Notes:

Assume data items are unique.

Priority Queue

Binary Heap

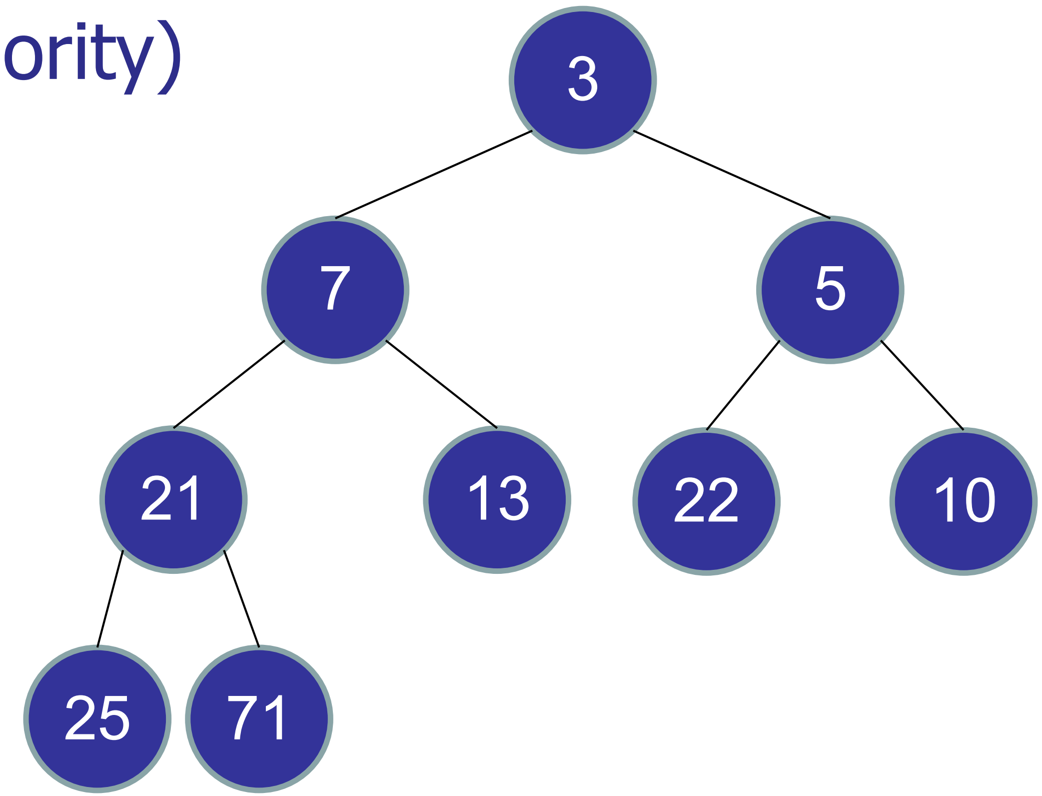
- Complete binary tree
- deleteMin: O(log n)
  - remove root
  - swap leaf to root
  - bubble down
- insert: O(log n)
  - add new leaf
  - bubble up



Priority Queue

Binary Heap

- How do we find a key? **(Hint: not a search tree!)**
- contains(key)
- decreaseKey(key, priority)



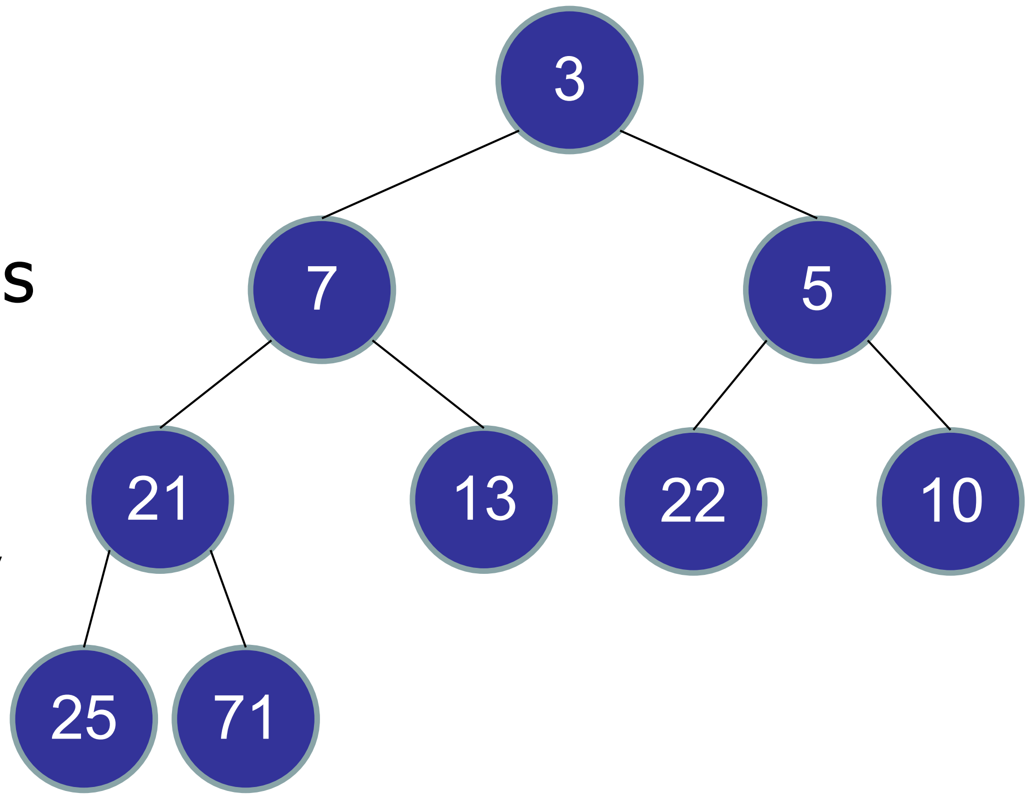
Priority Queue

Binary Heap

– decreaseKey(key, priority): O(logn)

– Hash Table:

- Map keys to locations in the binary tree.
- Update hash table whenever the binary tree changes.

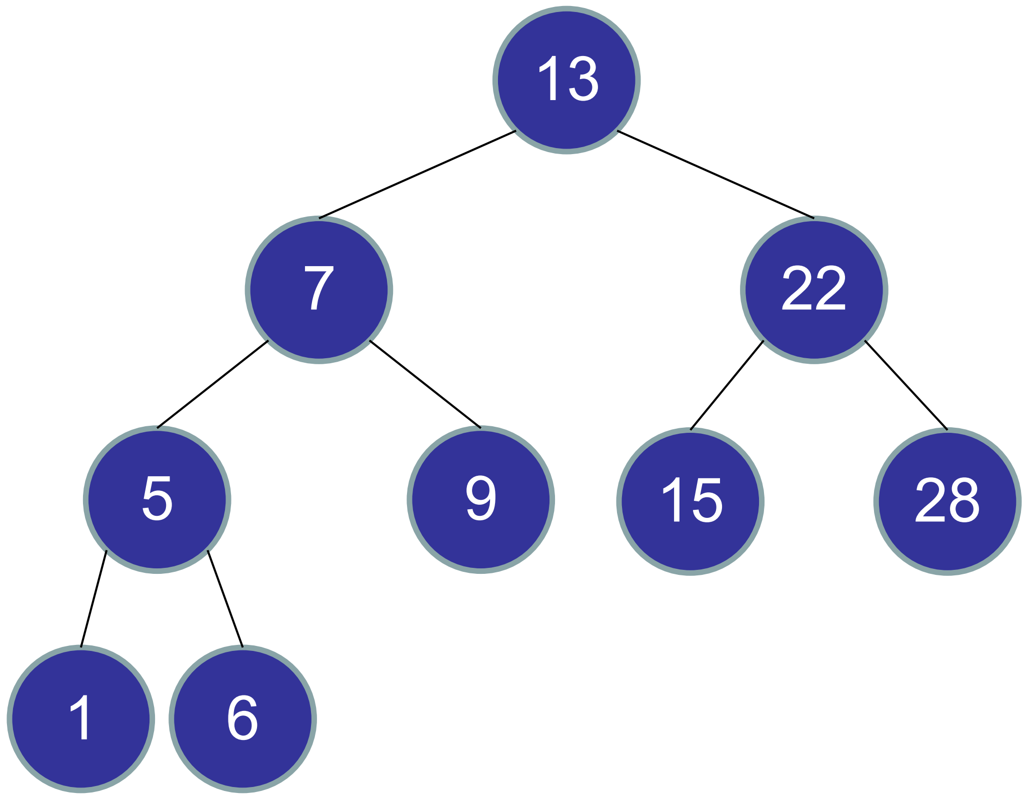


0	1	2	3	4	5	6	7	8	[9]	10	11
	3	7	6	21	13	22	10	25	71		

Priority Queue

AVL Tree

- Indexed by: priority
- Existing operations:
  - deleteMin()
  - insert(key, priority)

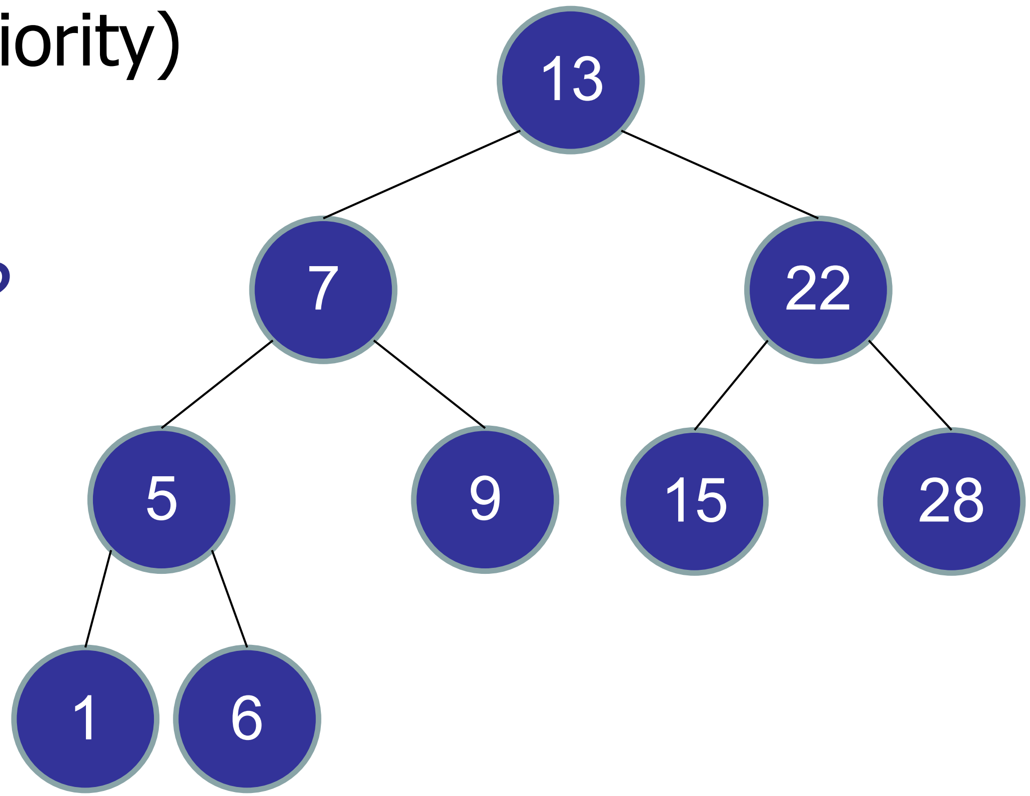


Priority Queue

AVL Tree

- Other operations:
  - contains()
  - decreaseKey(key, priority)

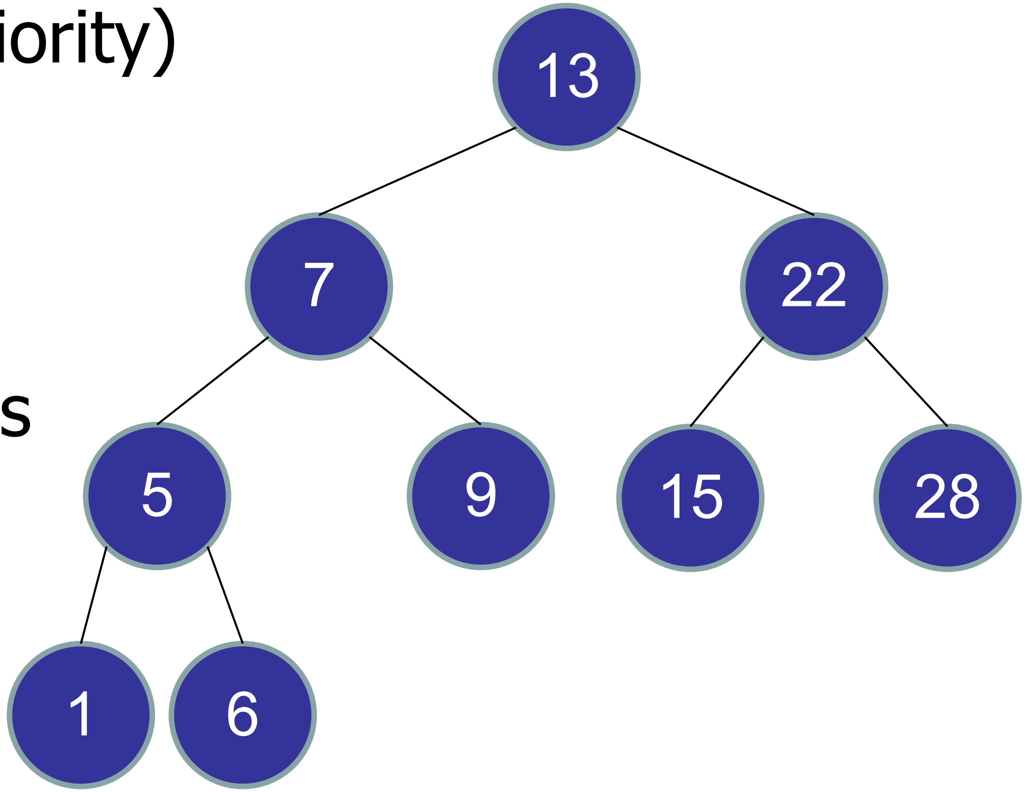
- How to find a vertex?



Priority Queue

AVL Tree

- Other operations:
  - contains()
  - decreaseKey(key, priority)
- Hash Table:
  - Map keys to locations in the binary tree.
  - Update hash table whenever the binary tree changes.



Dijkstra's Algorithm

Priority Queue by AVL tree:

- insert(key, priority): O(log n)
- deleteMin(): O(log n)
- decreaseKey(key, priority): O(log n)
- contains(key): O(1)

What is the running time of Dijkstra's Algorithm, using an AVL tree Priority Queue?

1. $O(V + E)$
2. $O(E \log V)$
3. $O(V \log V)$
4. $O(V^2)$
5. $O(VE)$
6. None of the above

```
public Dijkstra{
    private Graph G;
    private MinPriQueue pq = new MinPriQueue();
    private double[] distTo;

    searchPath(int start) {
        pq.insert(start, 0.0);
        distTo = new double[G.size()];
        Arrays.fill(distTo, INFTY);
        distTo[start] = 0;
        while (!pq.isEmpty()) {
            int w = pq.deleteMin();
            for (Edge e : G[w].nbrList)
                relax(e);
        }
    }
```

How many times?

How many times?

Dijkstra’s Algorithm

```
relax(Edge e) {
    int v = e.from();
    int w = e.to();
    double weight = e.weight();
    if (distTo[w] > distTo[v] + weight) {
        distTo[w] = distTo[v] + weight;
        parent[w] = v;
        if (pq.contains(w))
            pq.decreaseKey(w, distTo[w]);
        else
            pq.insert(w, distTo[w]);
    }
}
```

Dijkstra’s Algorithm

Analysis:

- insert / deleteMin: |V| times each
  - Each node is added to the priority queue **once**.
- relax / decreaseKey: |E| times
  - Each edge is relaxed once.
- Priority queue operations: O(log V)
- Total: O((V+E)log V) = O(E log V)

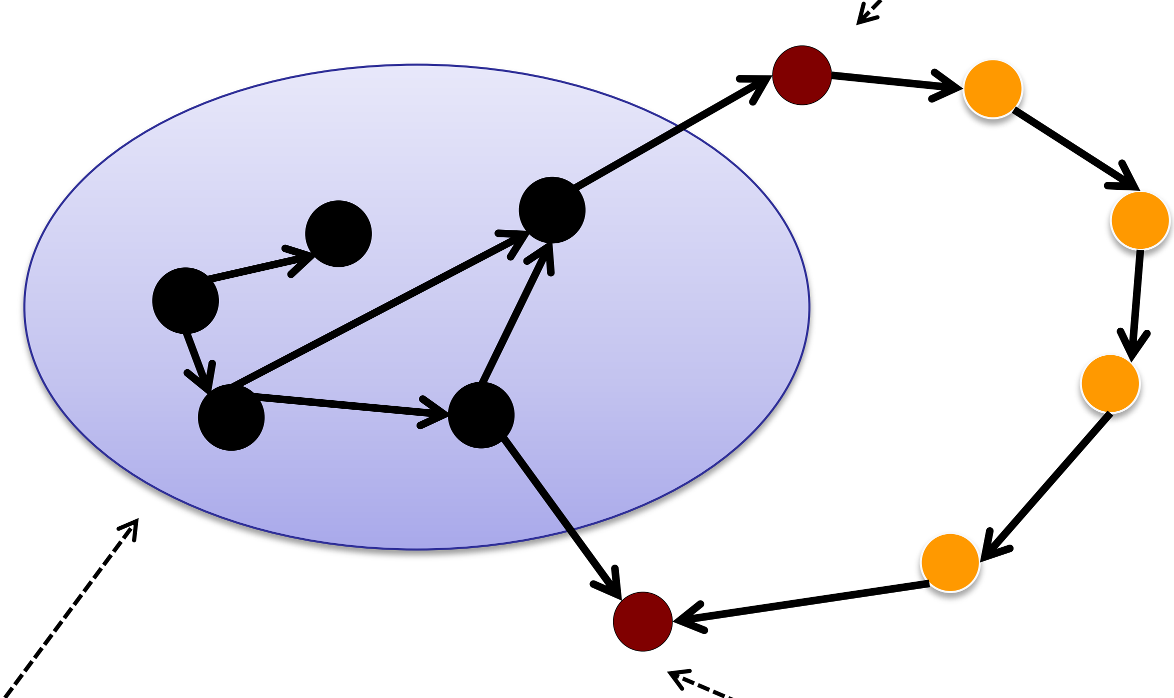
Dijkstra's Algorithm

Why does it work?

Dijkstra's Algorithm

fringe vertices:
neighbor of a
finished vertex.

Every edge crossing the boundary has been relaxed.



other vertices:
no known
estimate

finished vertices:
distance is accurate.

fringe vertices: in priority queue
neighbor of a finished vertex.

Dijkstra's Algorithm

Proof by induction:

- Every "finished" vertex has correct estimate.
- Initially: only "finished" vertex is start.

Dijkstra’s Algorithm

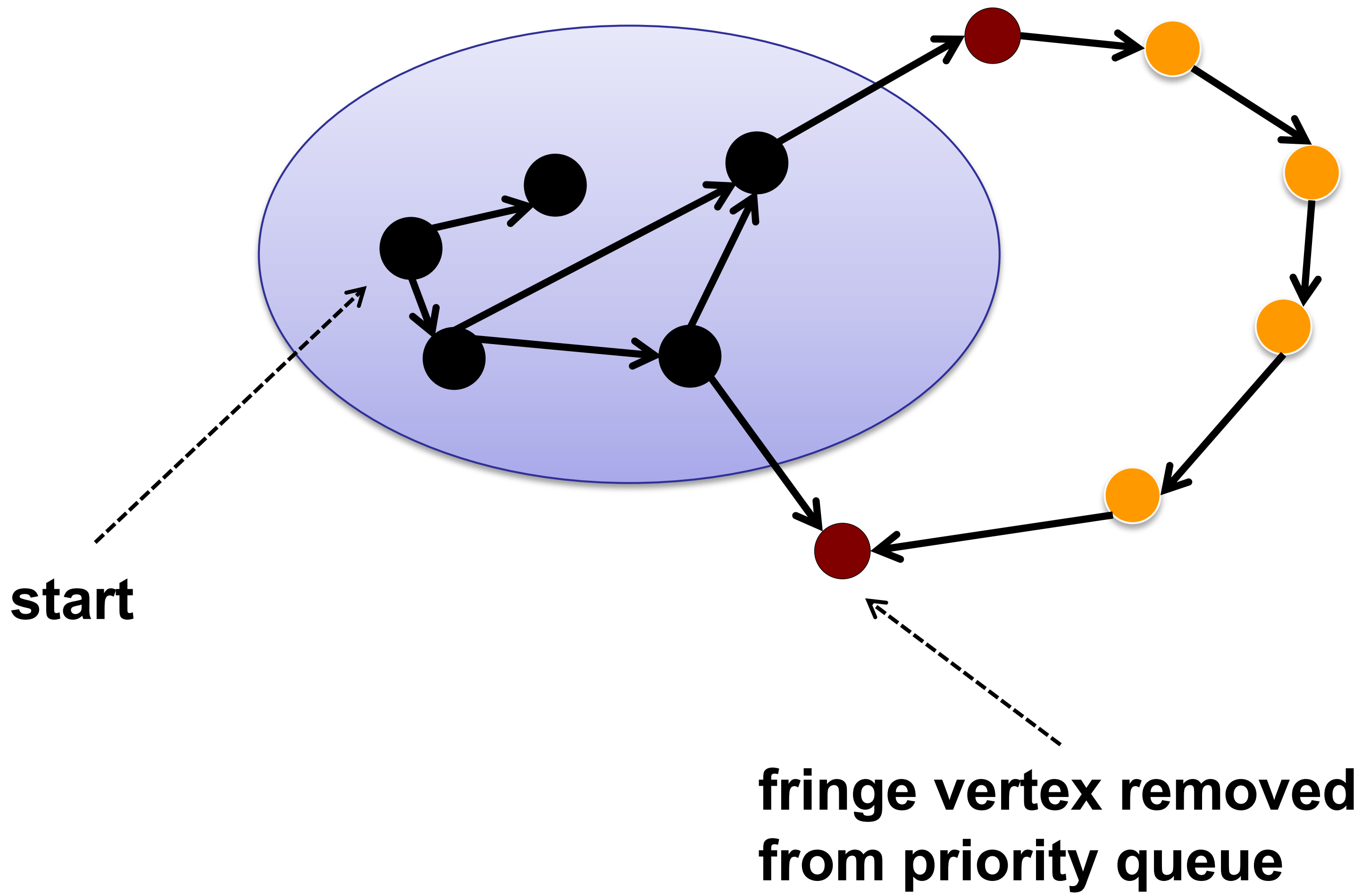
Proof by induction:

- Every “finished” vertex has correct estimate.
- Initially: only “finished” vertex is start.

- Inductive step:
  - Remove vertex from priority queue.
  - Relax its edges.
  - Add it to finished.
  - **Claim: it has a correct estimate.**

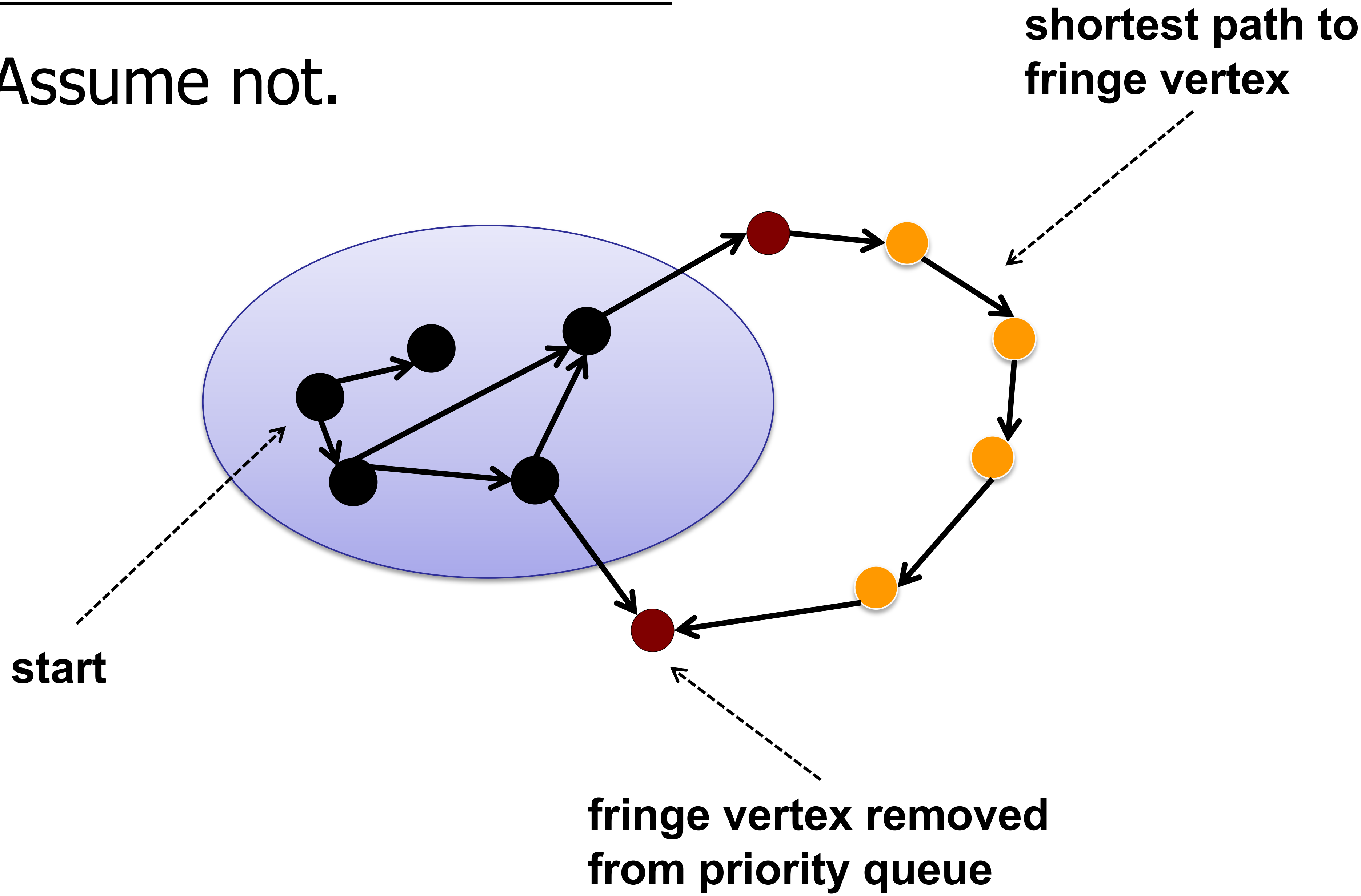
Dijkstra's Algorithm

Assume not.



Dijkstra's Algorithm

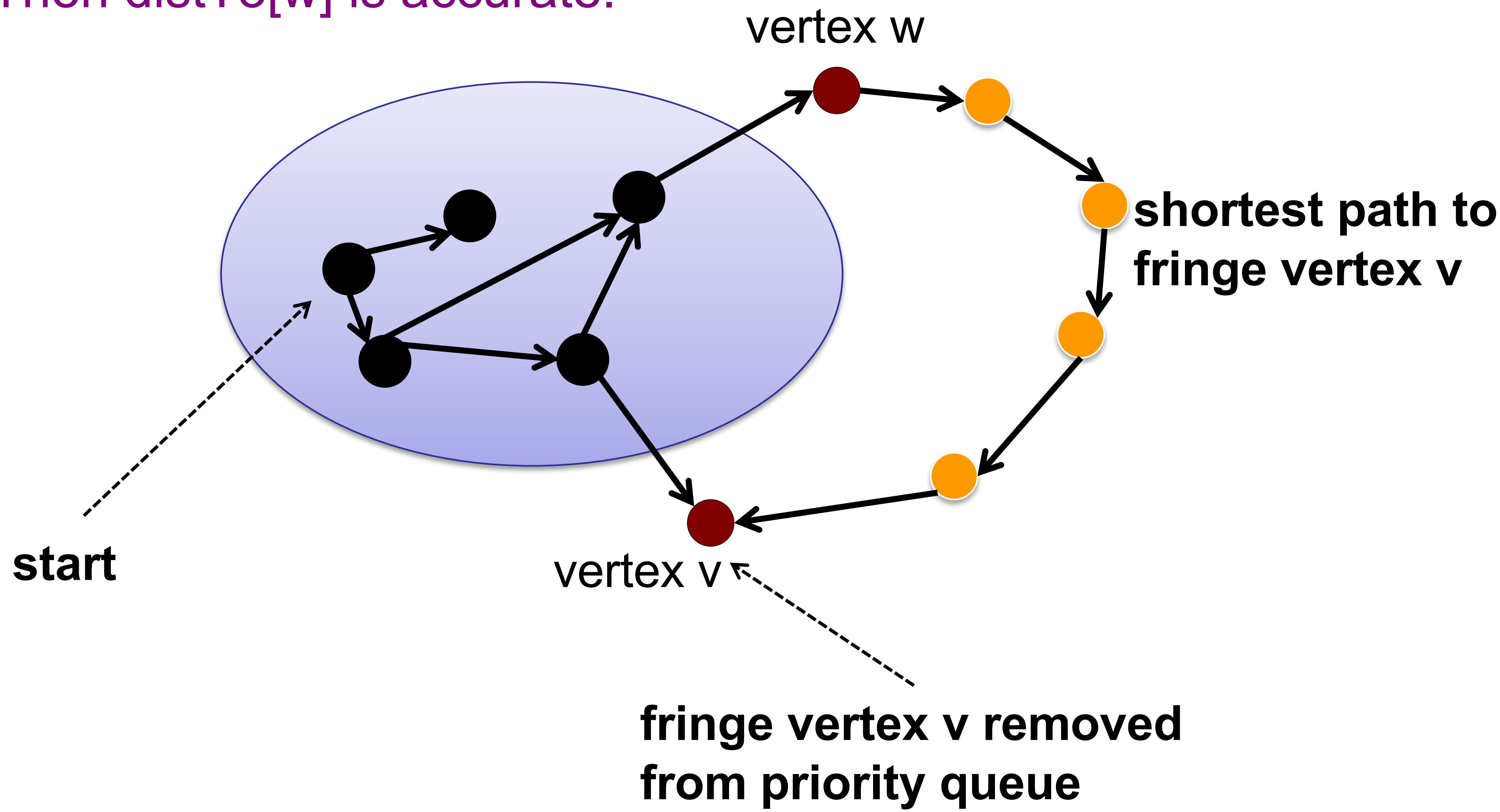
Assume not.



Dijkstra's Algorithm

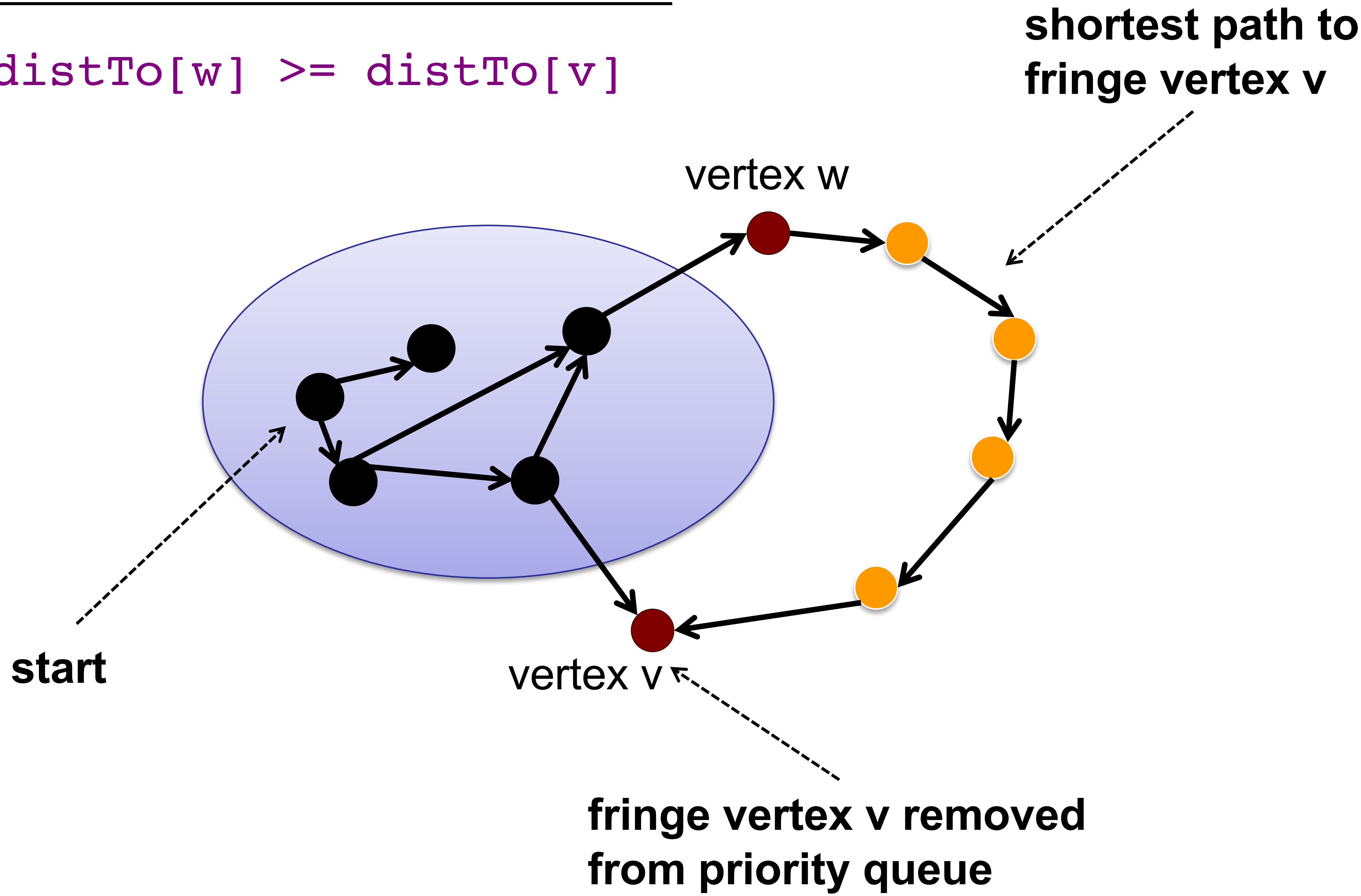
If P is shortest path to v, then prefix of P is shortest path to w.

Then distTo[w] is accurate.



Dijkstra's Algorithm

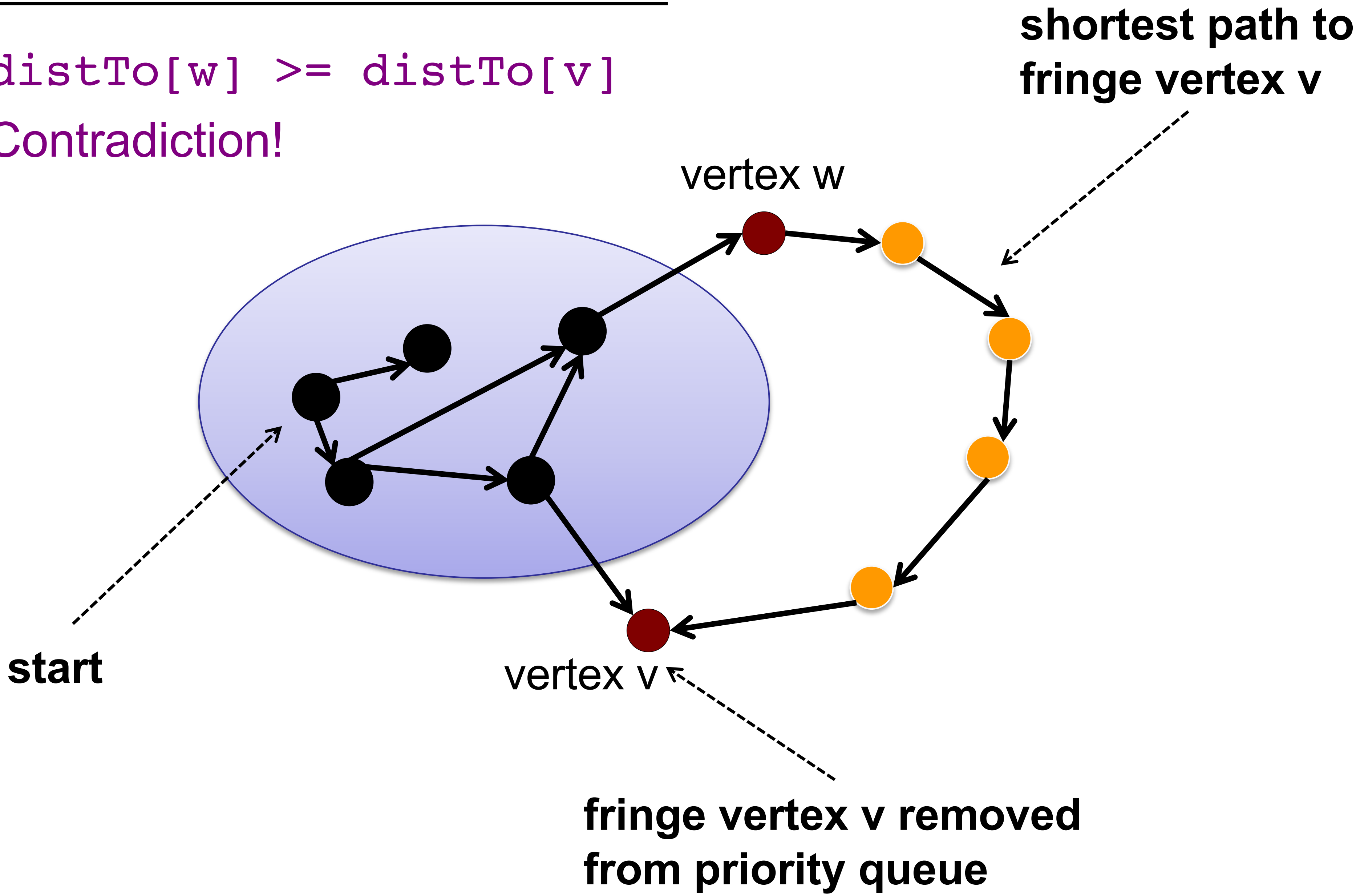
```
distTo[w] >= distTo[v]
```



Dijkstra's Algorithm

`distTo[w] >= distTo[v]`

Contradiction!



Dijkstra’s Algorithm

Proof by induction:

- Every “finished” vertex has correct estimate.
- Initially: only “finished” vertex is start.

- Inductive step:
  - Remove vertex from priority queue.
  - Relax its edges.
  - Add it to finished.
  - **Claim: it has a correct estimate.**

Dijkstra’s Algorithm

```
relax(Edge e) {
    int v = e.from();
    int w = e.to();
    double weight = e.weight();
    if (distTo[w] > distTo[v] + weight) {
        distTo[w] = distTo[v] + weight;
        parent[w] = v;
        if (pq.contains(w))
            pq.decreaseKey(w, distTo[w]);
        else
            pq.insert(w, distTo[w]);
    }
}
```

Dijkstra’s Algorithm

Analysis:

- insert / deleteMin: |V| times each
  - Each node is added to the priority queue **once**.
- decreaseKey: |E| times
  - Each edge is relaxed once.
- Priority queue operations: O(log V)
- Total: O((V+E)log V) = O(E log V)

Source-to-Destination Dijkstra

Can we stop as soon as we dequeue the destination?

1. Yes.
2. Only if the graph is sparse.
3. No.

Dijkstra's Algorithm

Source-to-Destination:

- What if you stop the first time you dequeue the destination?
- Recall:
  - a vertex is "finished" when it is dequeued
  - if the destination is finished, then stop

Dijkstra Summary

Basic idea:

- Maintain distance estimates.
- Repeat:
  - Find unfinished vertex with smallest estimate.
  - Relax all outgoing edges.
  - Mark vertex finished.

- O(E log V) time (with AVL tree).

Dijkstra’s Performance

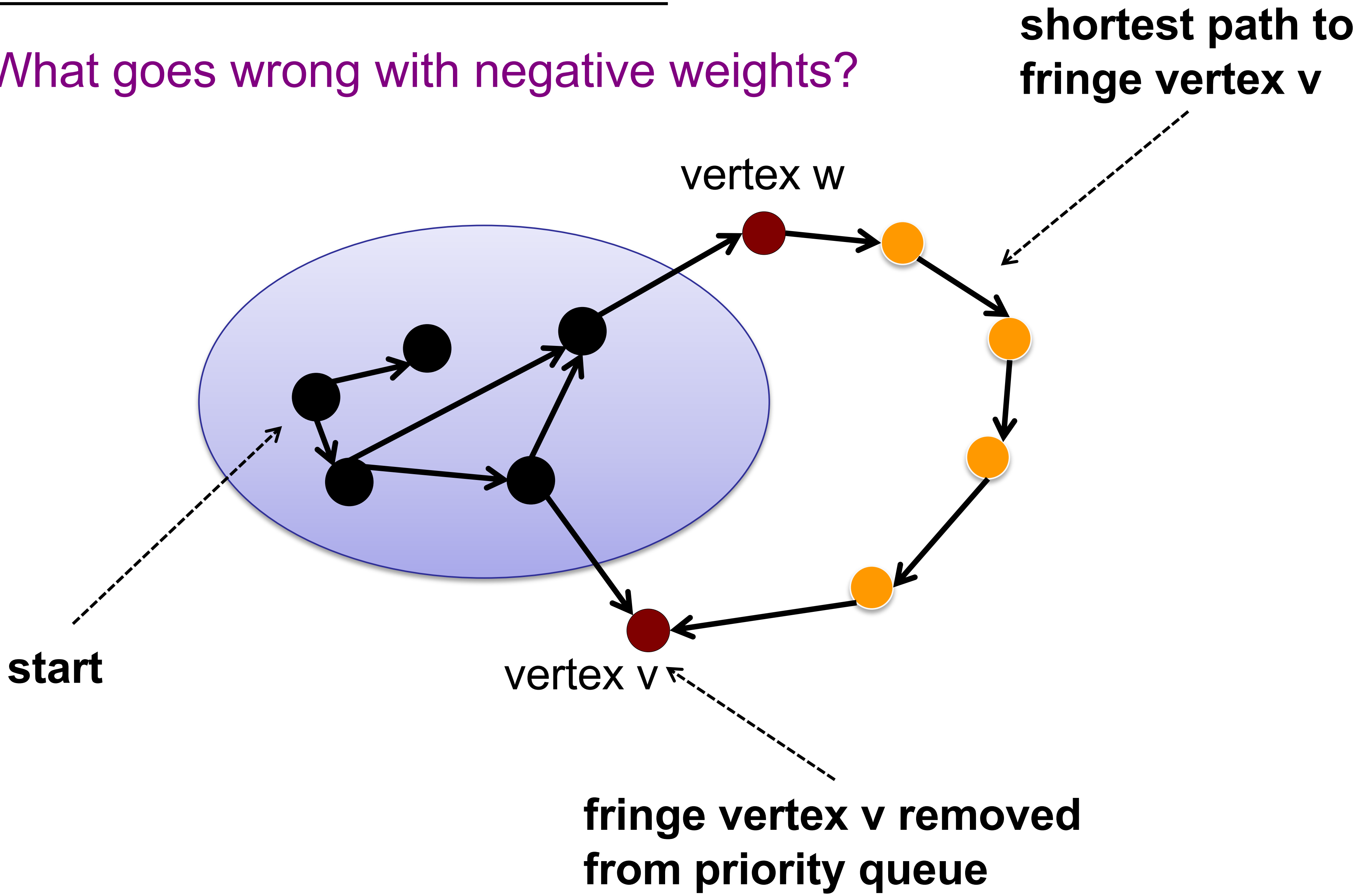
PQ Implementation	insert	deleteMin	decreaseKey	Total
Array	1	V	1	**$O(V^2)$**
AVL Tree	log V	log V	log V	**O(E log V)**
d-way Heap	$d\log_d V$	$d\log_d V$	$\log_d V$	**$O(E\log_{E/V} V)$**
Fibonacci Heap	1	log V	1	**O(E + V log V)**

Dijkstra Summary

Edges with negative weights?

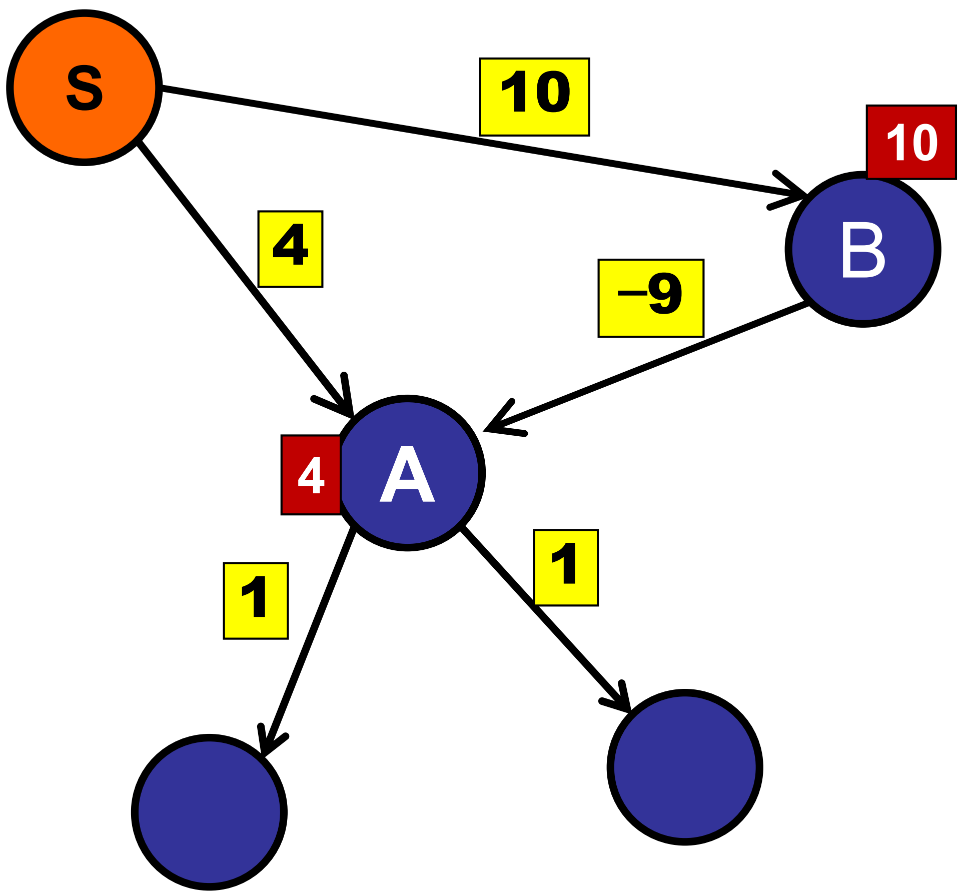
Dijkstra's Algorithm

What goes wrong with negative weights?



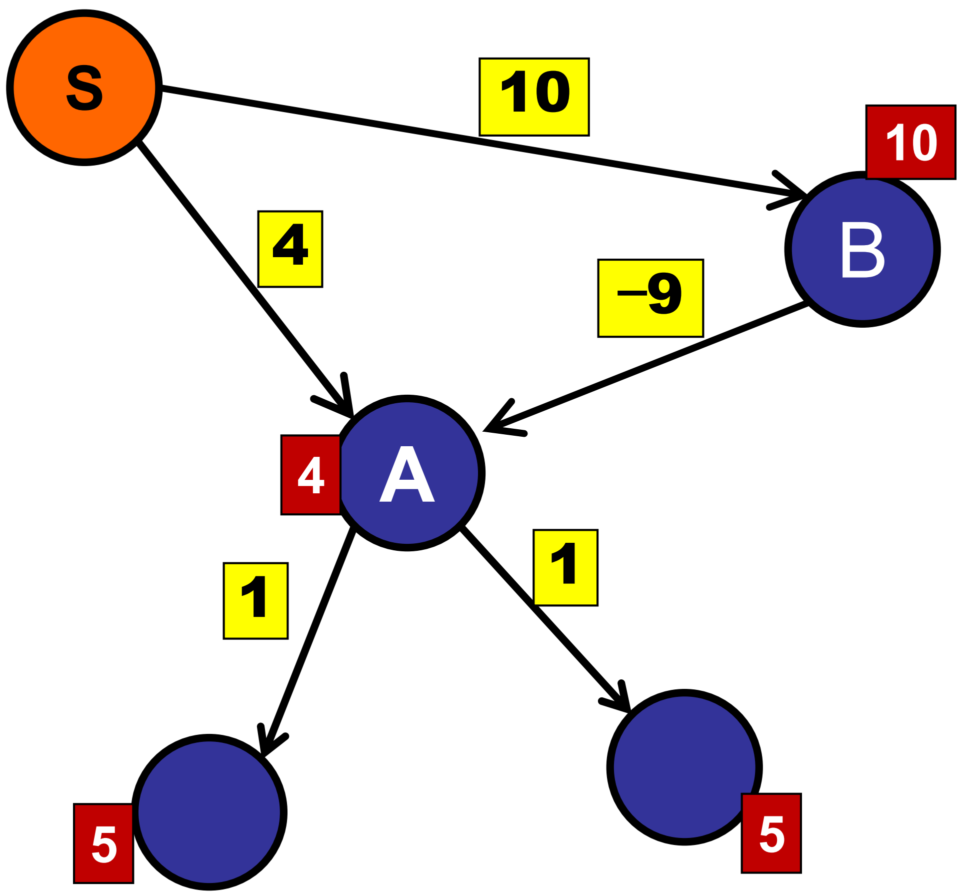
Dijkstra's Algorithm

Edges with negative weights?



Dijkstra's Algorithm

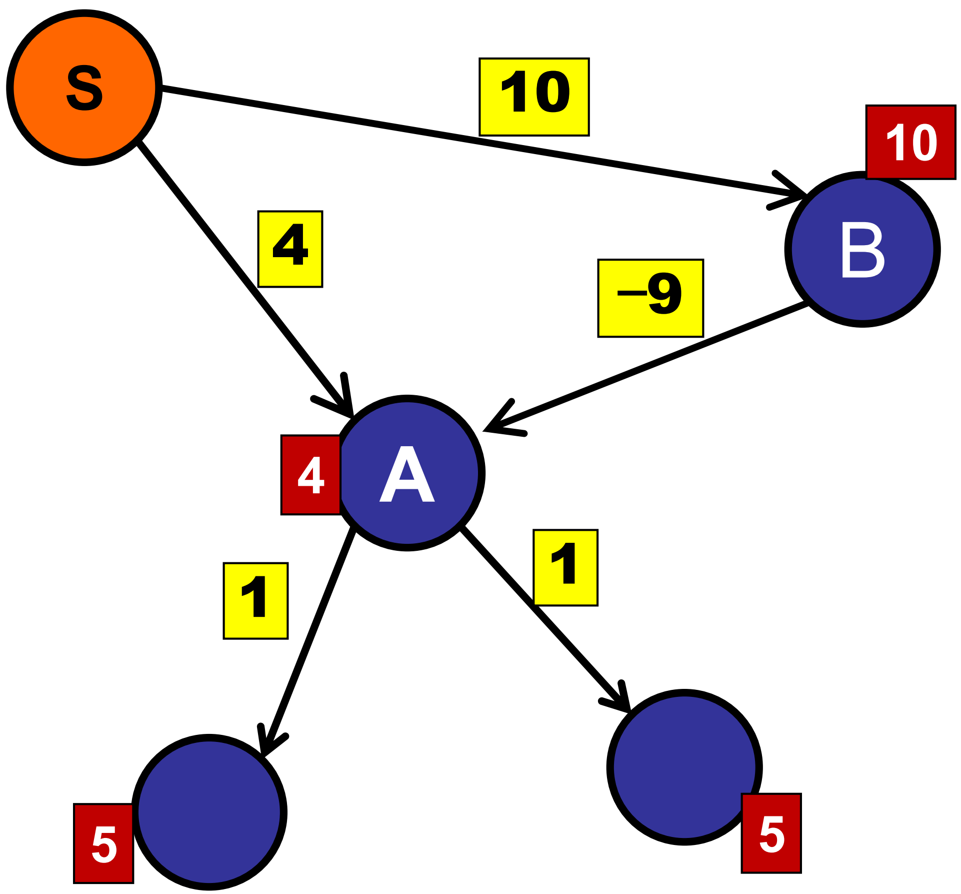
Edges with negative weights?



Step 1: Remove A.
Relax A.
Mark A done.

Dijkstra's Algorithm

Edges with negative weights?



Step 1: Remove A.
Relax A.
Mark A done.

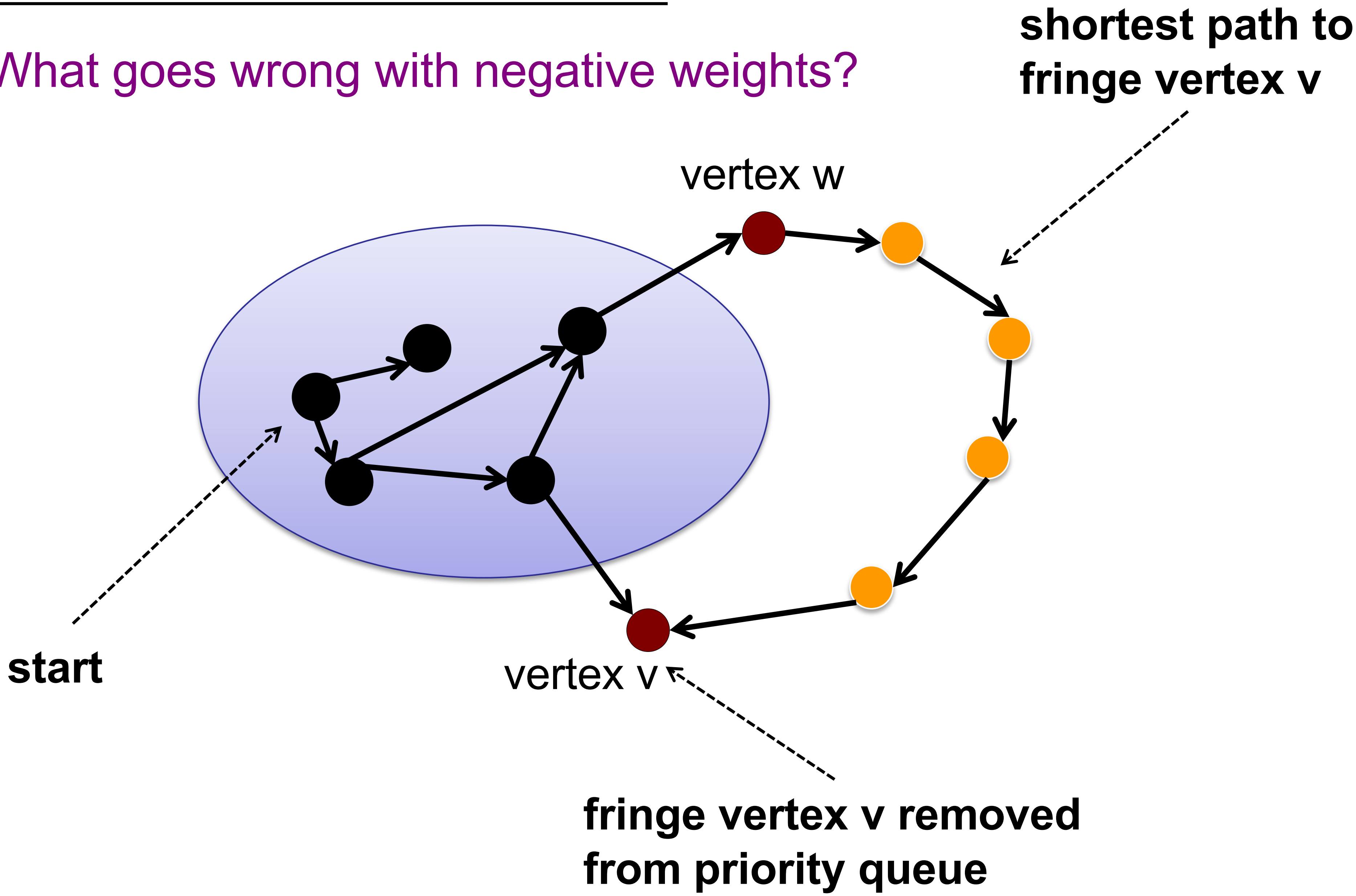
…

Step 4: Remove B.
Relax B.
Mark B done.

Oops: We need to update A.

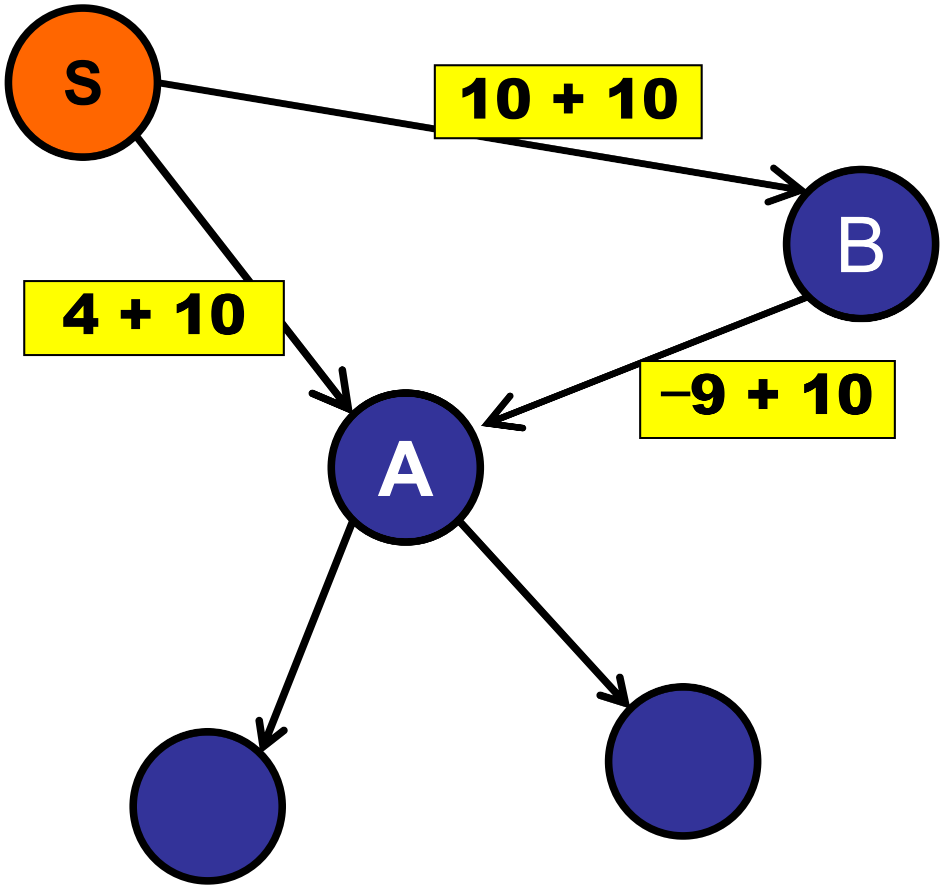
Dijkstra’s Algorithm

What goes wrong with negative weights?



Dijkstra's Algorithm

Can we reweight? e.g.: weight += 10

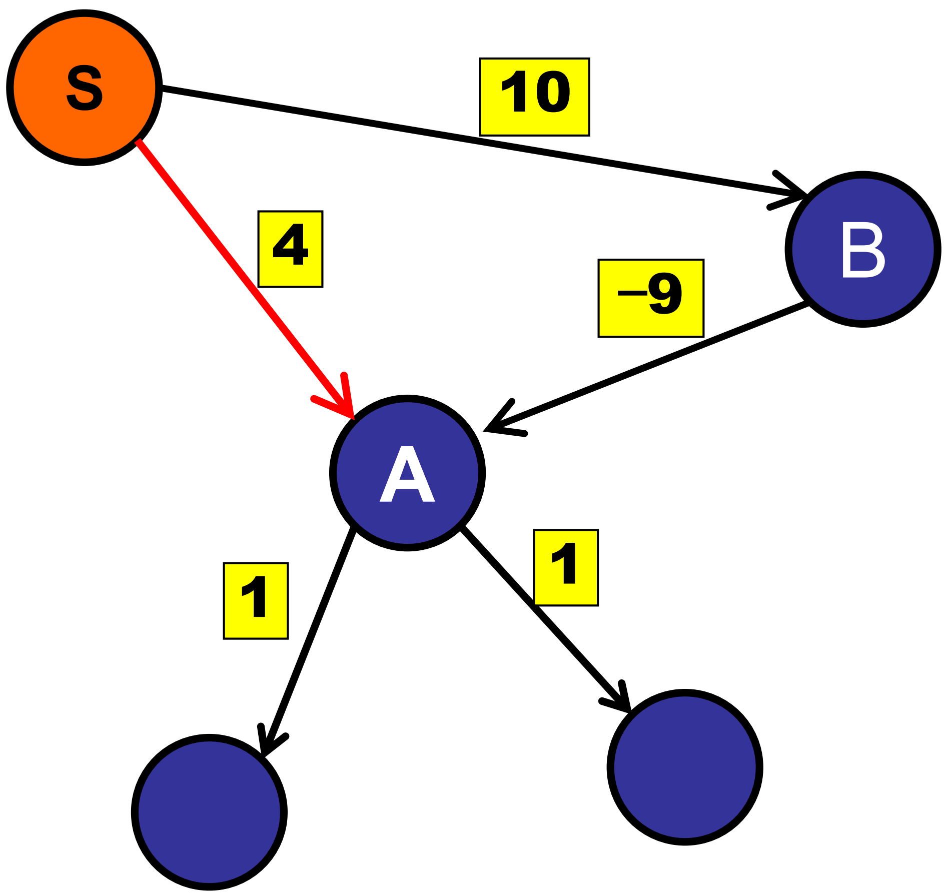


Can we reweight the graph?

1. Yes.
2. Only if there are no negative weight cycles.
3. No.

Dijkstra's Algorithm

Can we reweight?

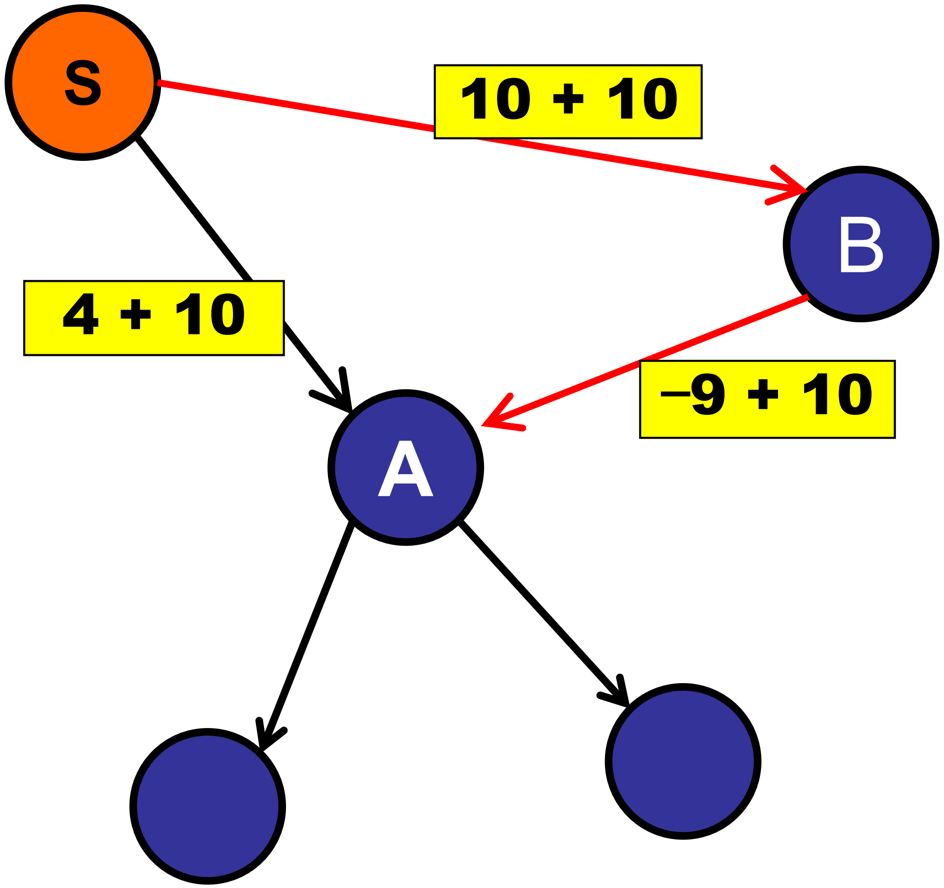


Path S-B-A: 1

Path S-A: 4

Dijkstra's Algorithm

Can we reweight?



Path S-B-A: 21

Path S-A: 14

Dijkstra Summary

Basic idea:

- Maintain distance estimates.
- Repeat:
  - Find unfinished vertex with smallest estimate.
  - Relax all outgoing edges.
  - Mark vertex finished.

- O(E log V) time (with AVL tree Priority Queue).

- No negative weight edges!

Dijkstra Comparison

Same algorithm:

- Maintain a set of explored vertices.
- Add vertices to the explored set by following edges that go from a vertex in the explored set to a vertex outside the explored set.
  - BFS: Take edge from vertex that was discovered **least** recently.
  - DFS: Take edge from vertex that was discovered **most** recently.
  - Dijkstra's: Take edge from vertex that is **closest** to source.

Dijkstra Comparison

Same algorithm:

- Maintain a set of explored vertices.
- Add vertices to the explored set by following edges that go from a vertex in the explored set to a vertex outside the explored set.

  - BFS: Use queue.
  - DFS: Use stack.
  - Dijkstra's: Use priority queue.

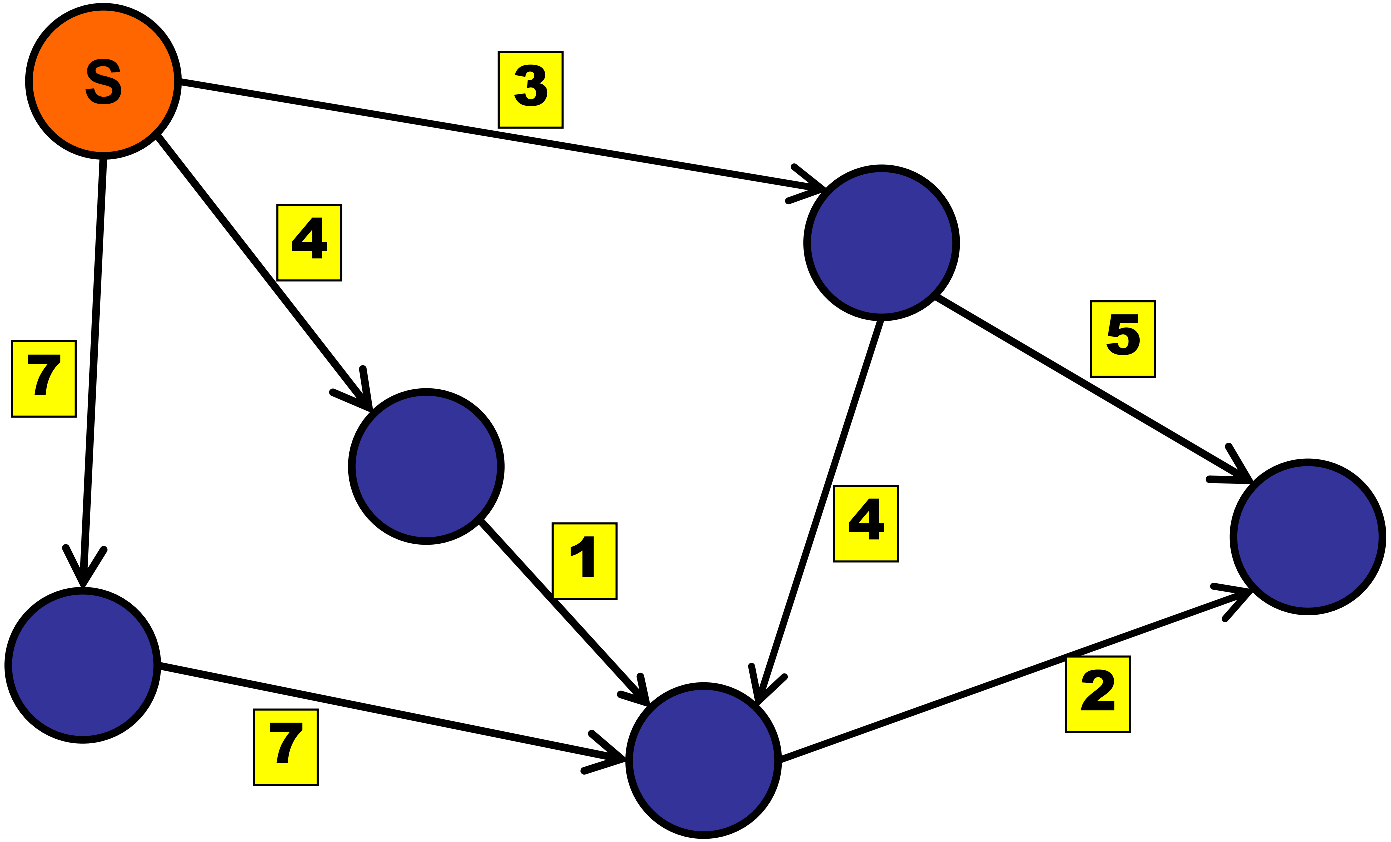
Roadmap

Shortest Paths

- Special Case: Tree
- Special Case: Non-negative weights (Dijkstra’s)
- Special Case: Directed Acyclic Graphs

Shortest Paths

Acyclic Graph: Suppose the graph has no cycles.

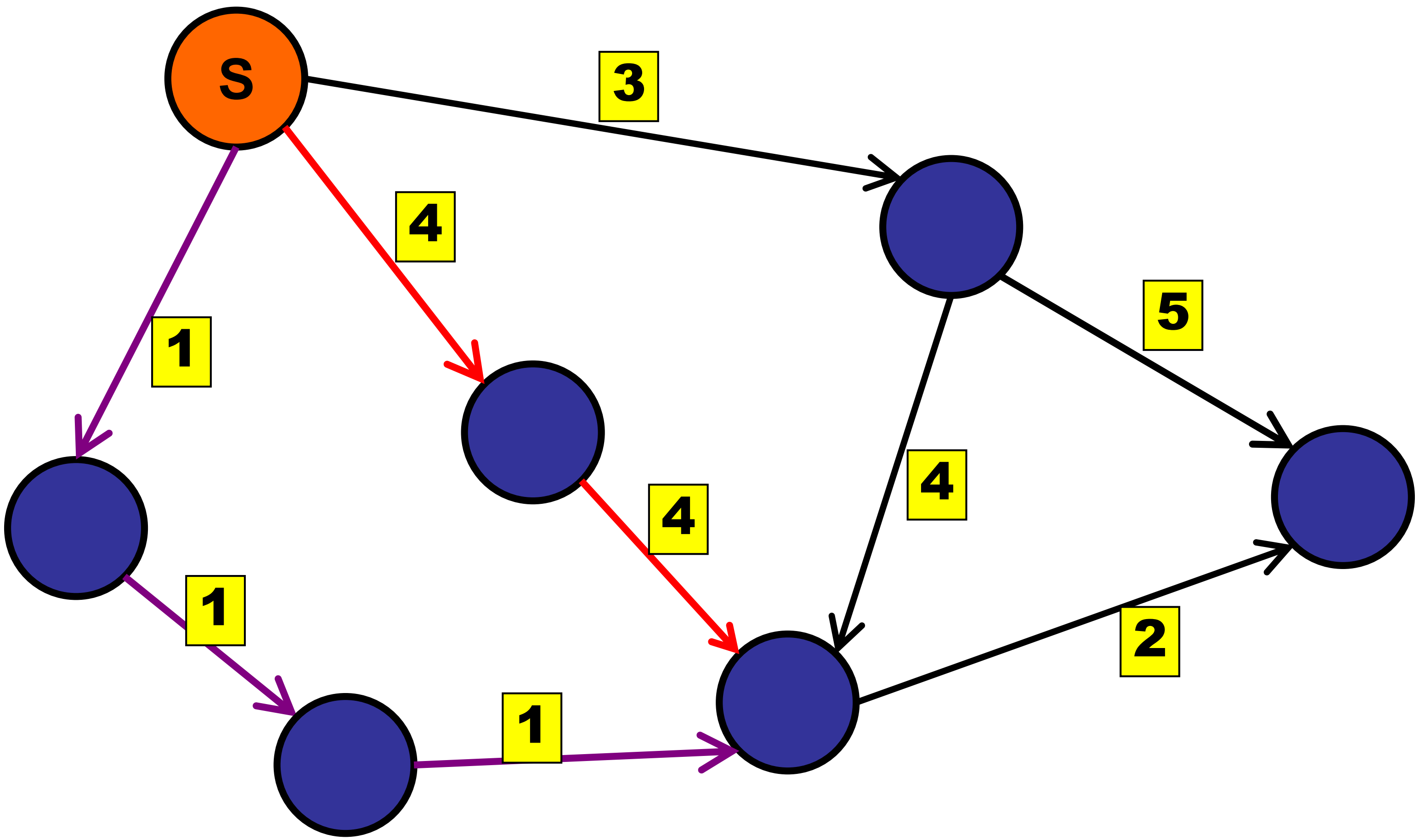


What order should we relax the nodes?

1. BFS
2. DFS pre-order
3. DFS post-order
4. Shortest edge
5. Longest edge
6. Other

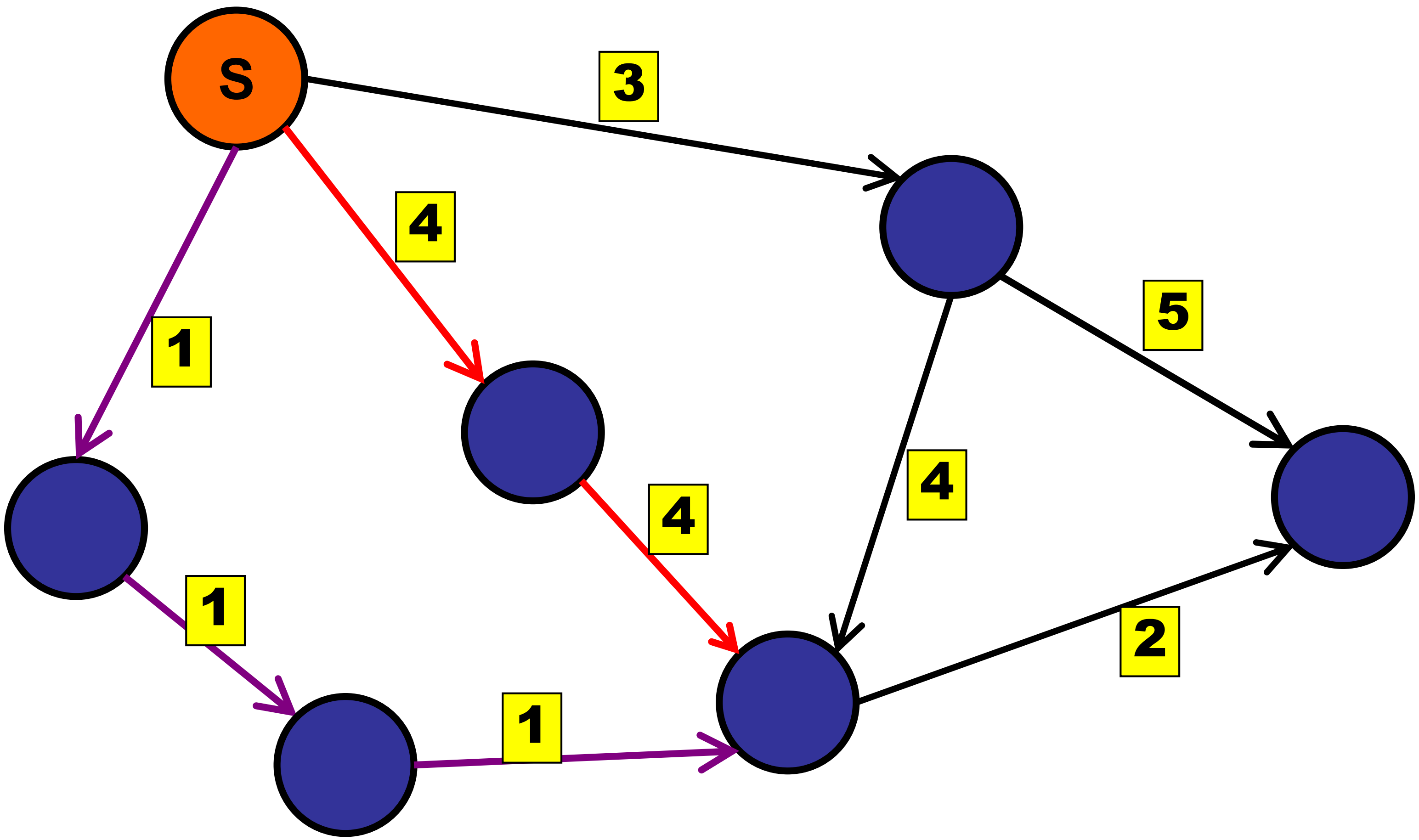
Shortest Paths

Acyclic Graph: Not BFS.



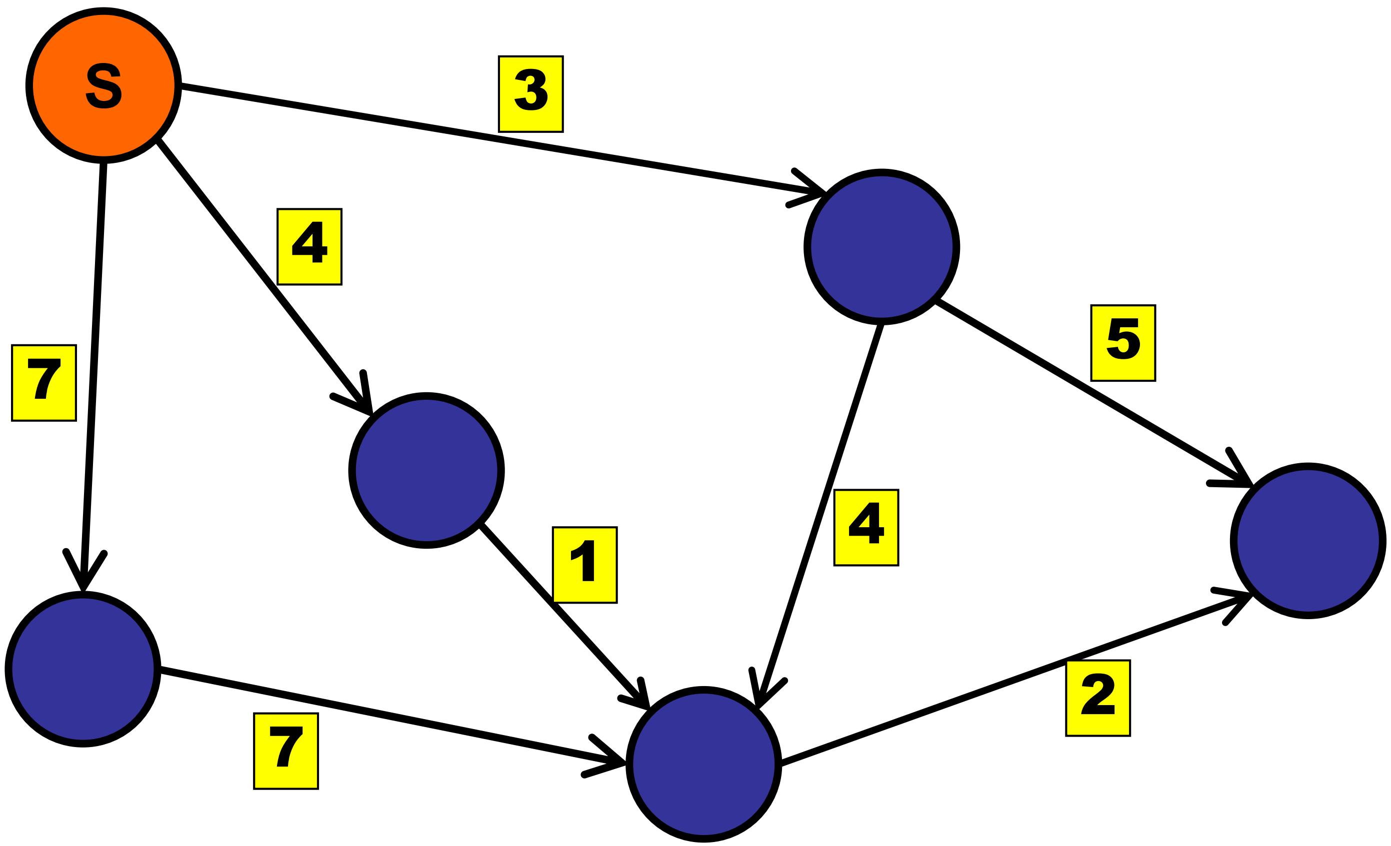
Shortest Paths

Acyclic Graph: Not DFS-preorder.



Shortest Paths

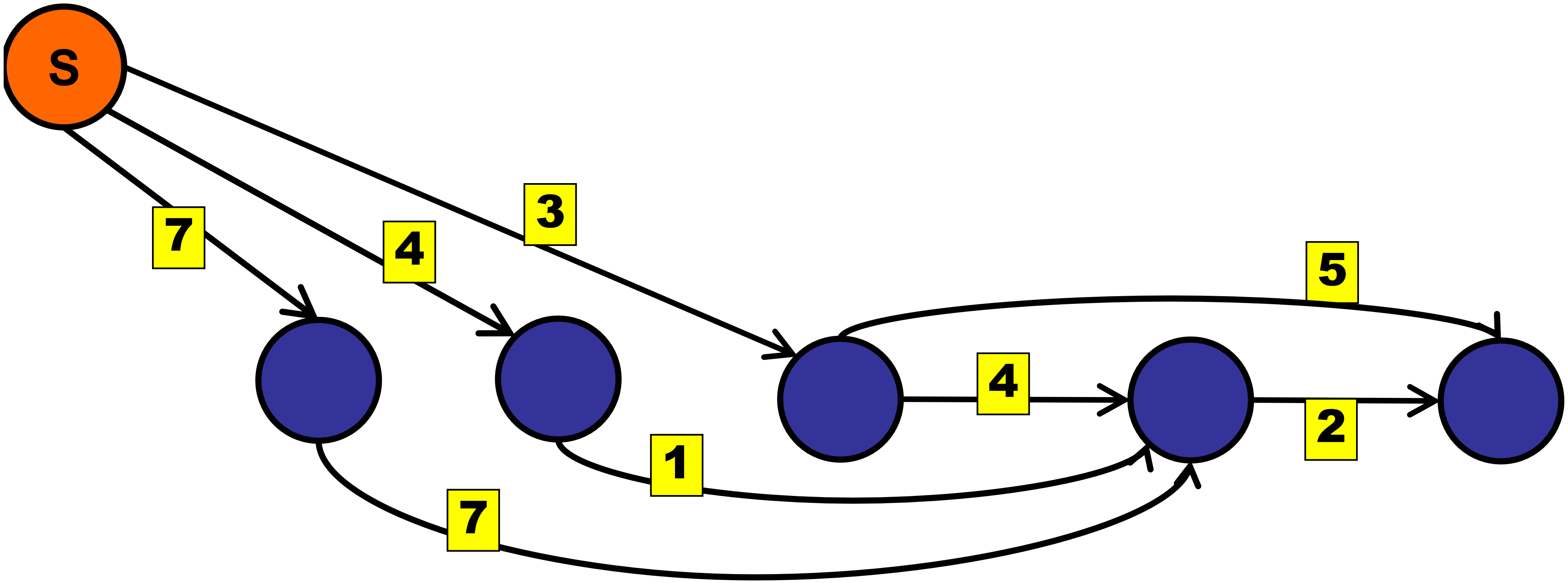
Acyclic Graph: has no cycles.



Shortest Paths

Acyclic Graph: has no cycles.

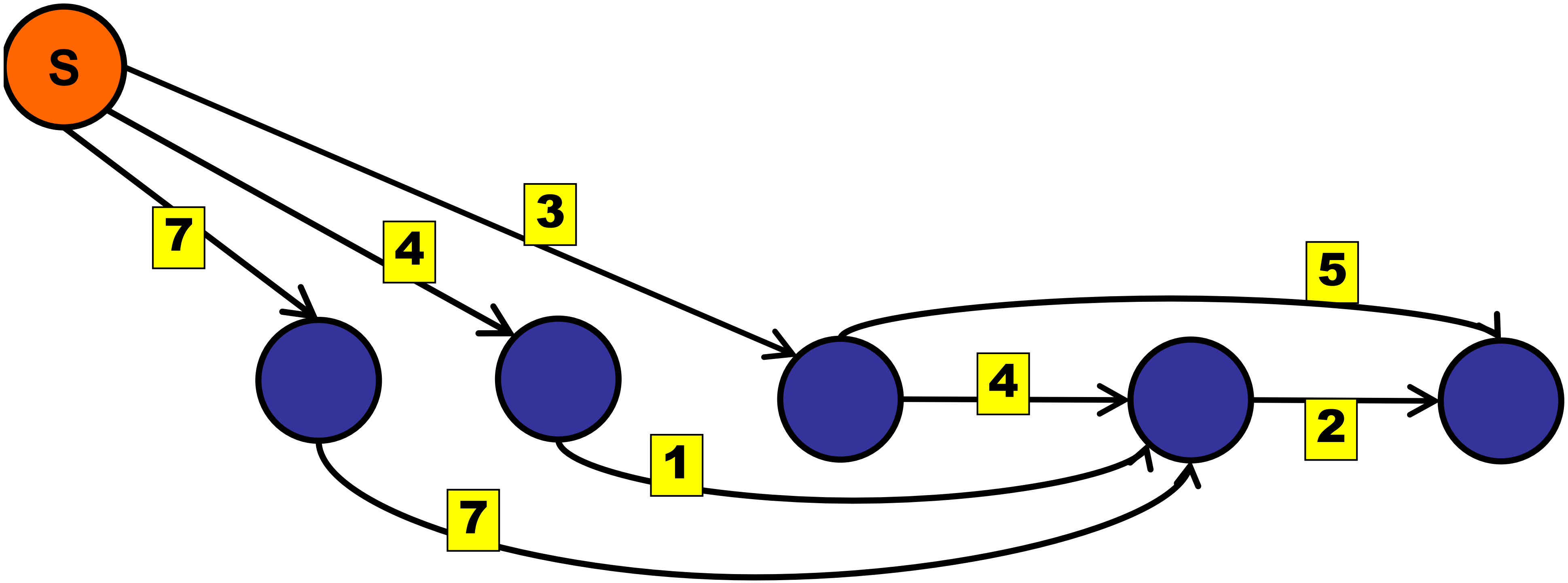
1. Topological sort



Shortest Paths

Acyclic Graph: has no cycles.

1. Topological sort
2. Relax in order.



Shortest Paths

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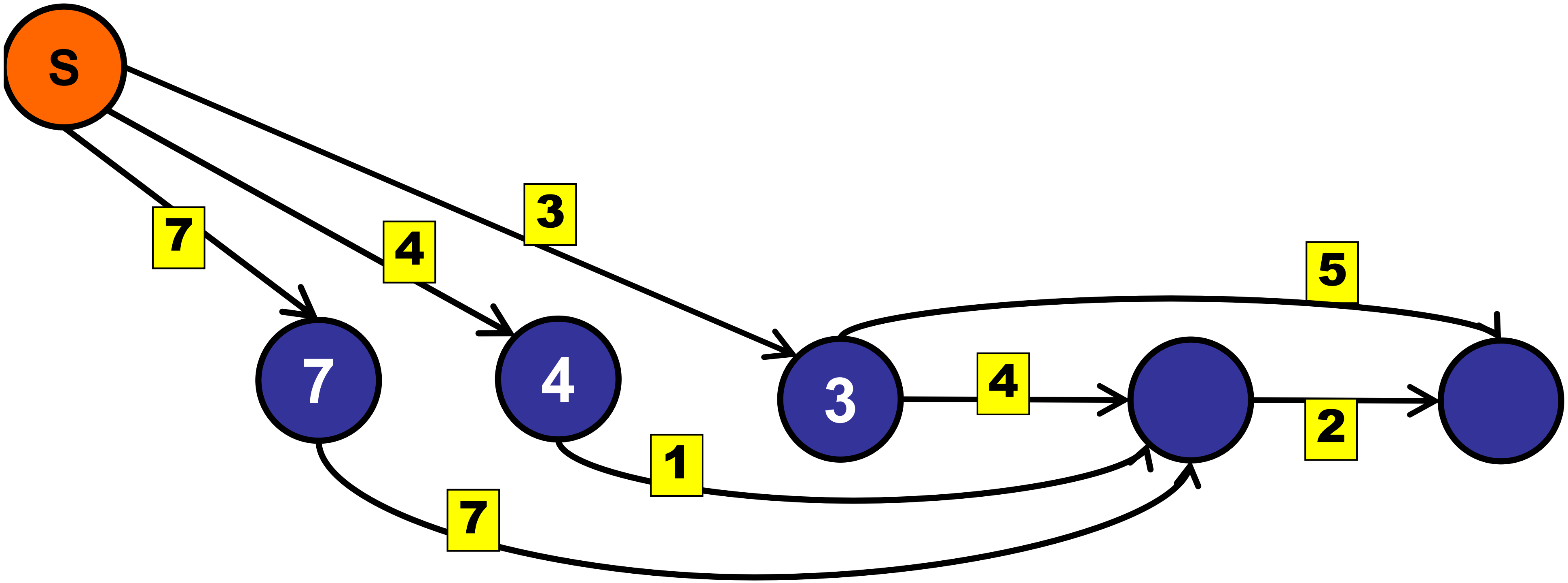

Shortest Paths

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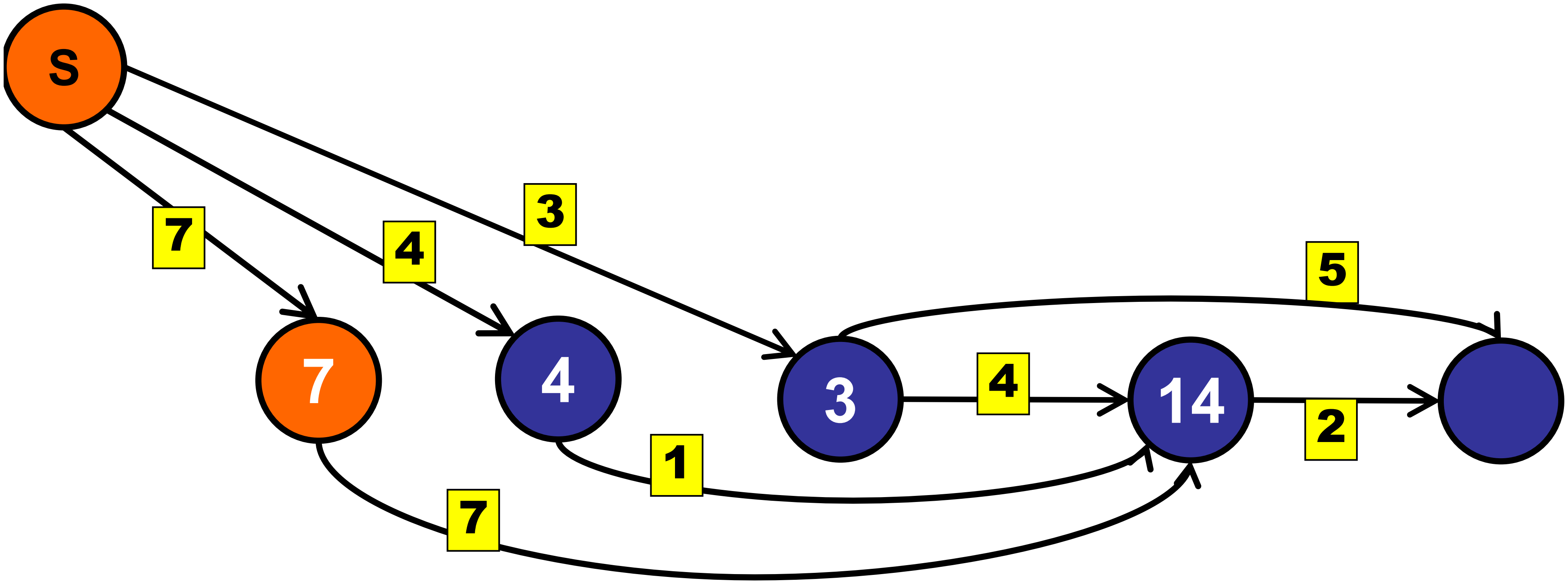

Shortest Paths

Acyclic Graph: has no cycles.

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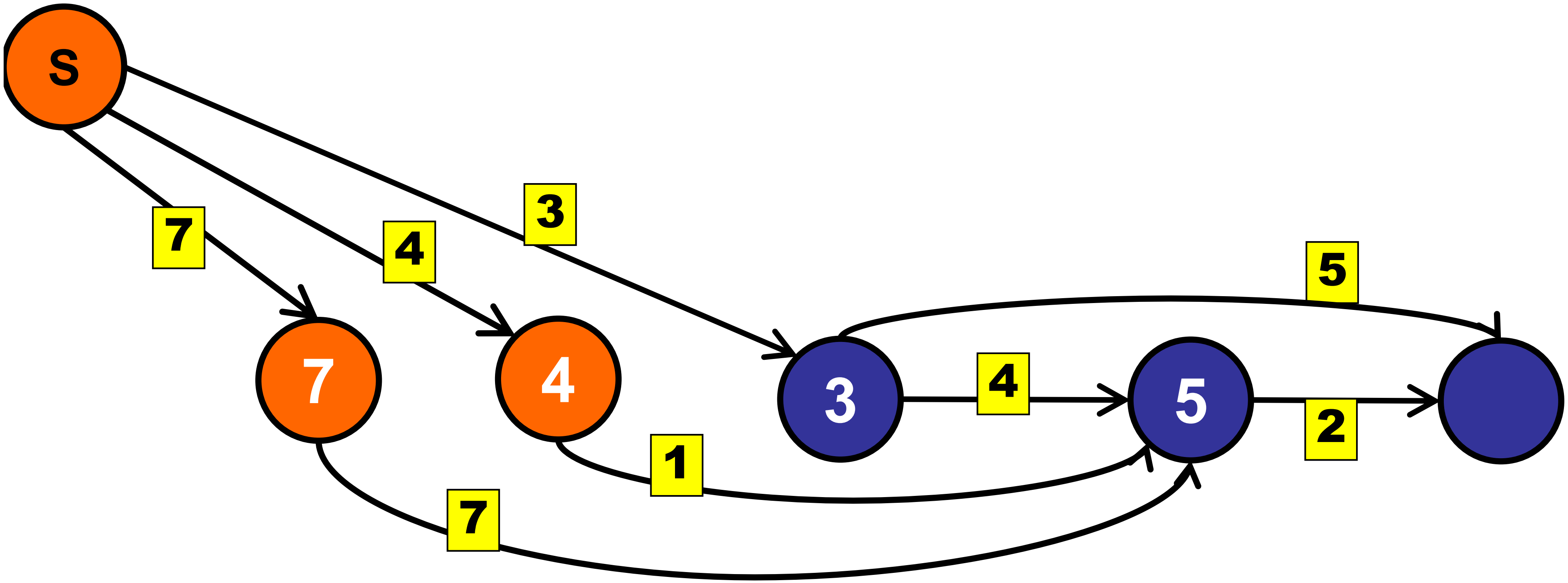

Shortest Paths

Acyclic Graph: has no cycles.

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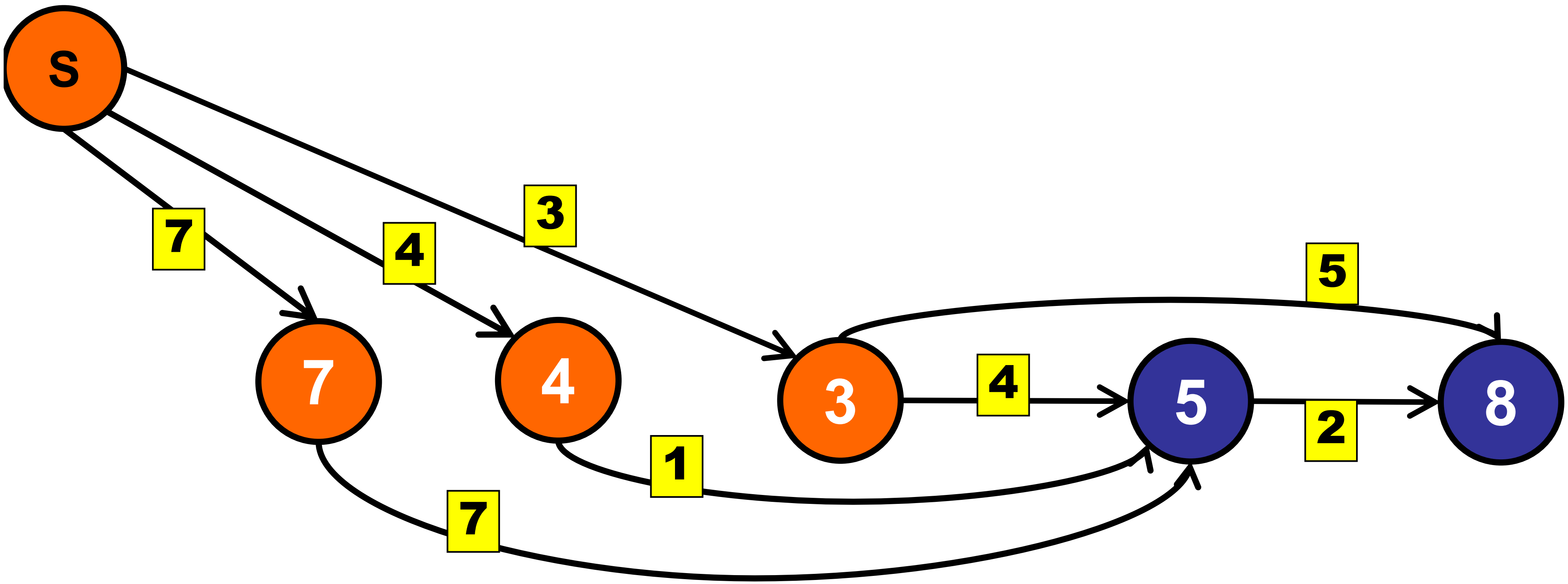

Shortest Paths

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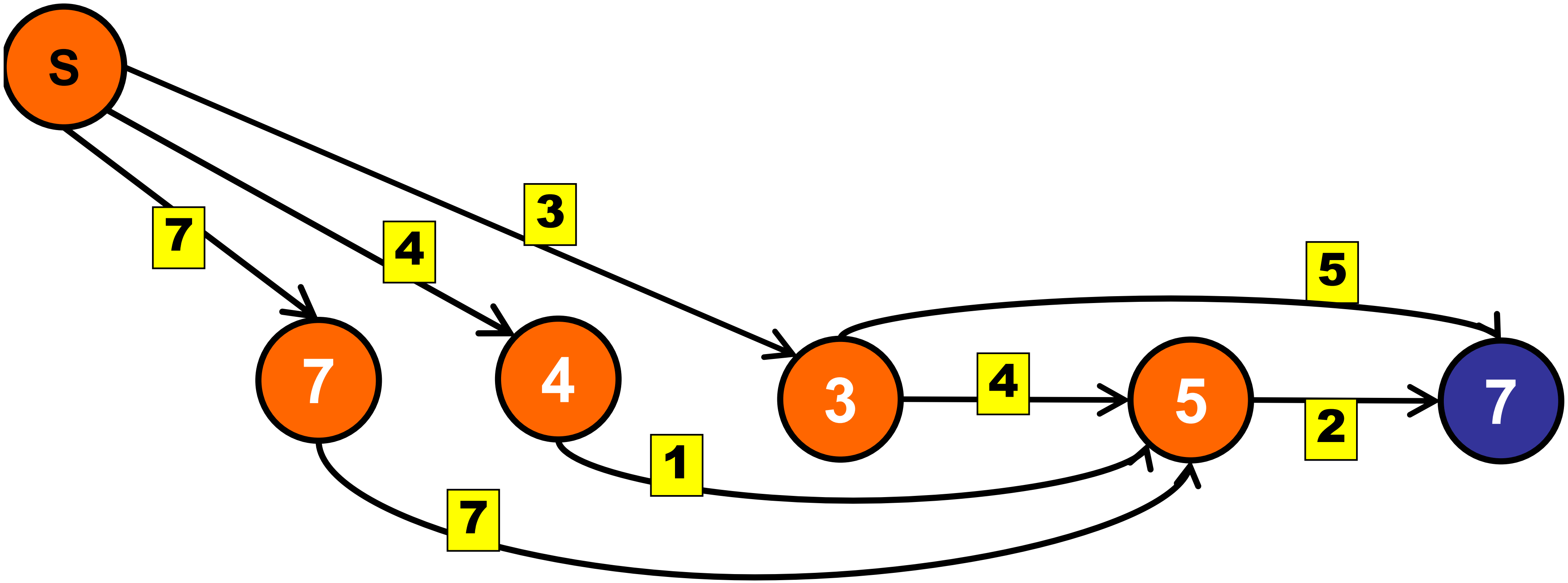

Shortest Paths

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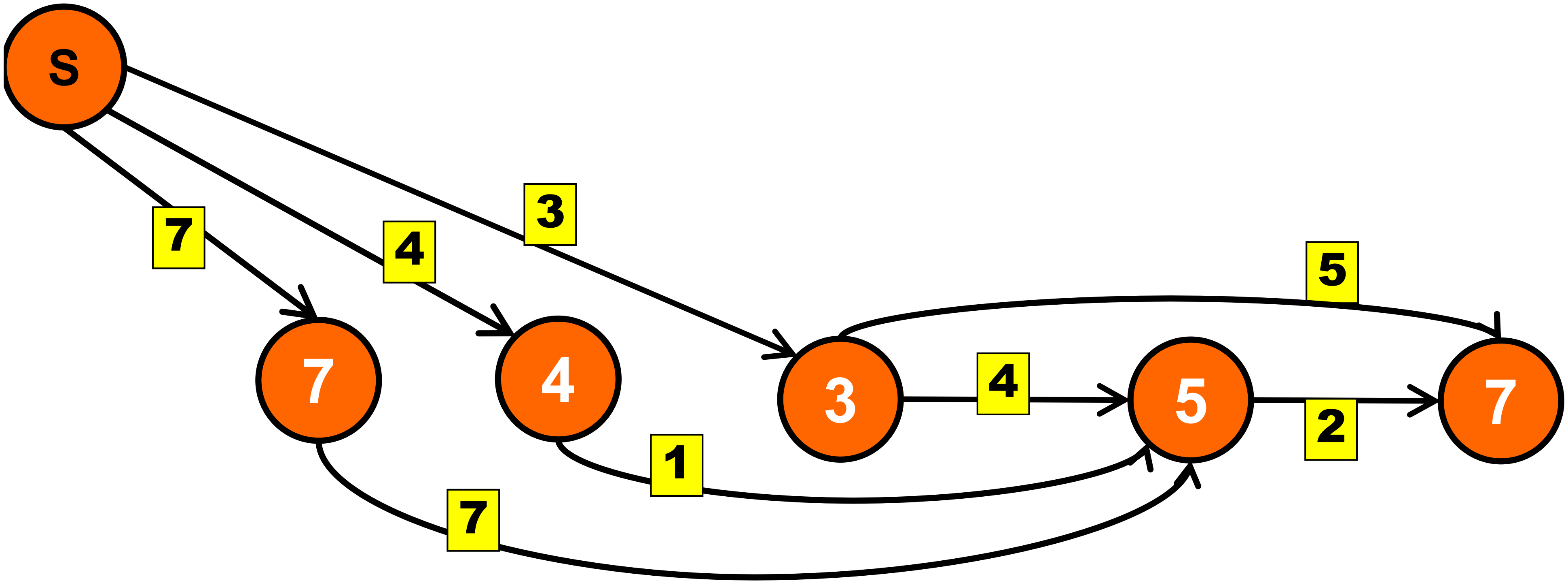
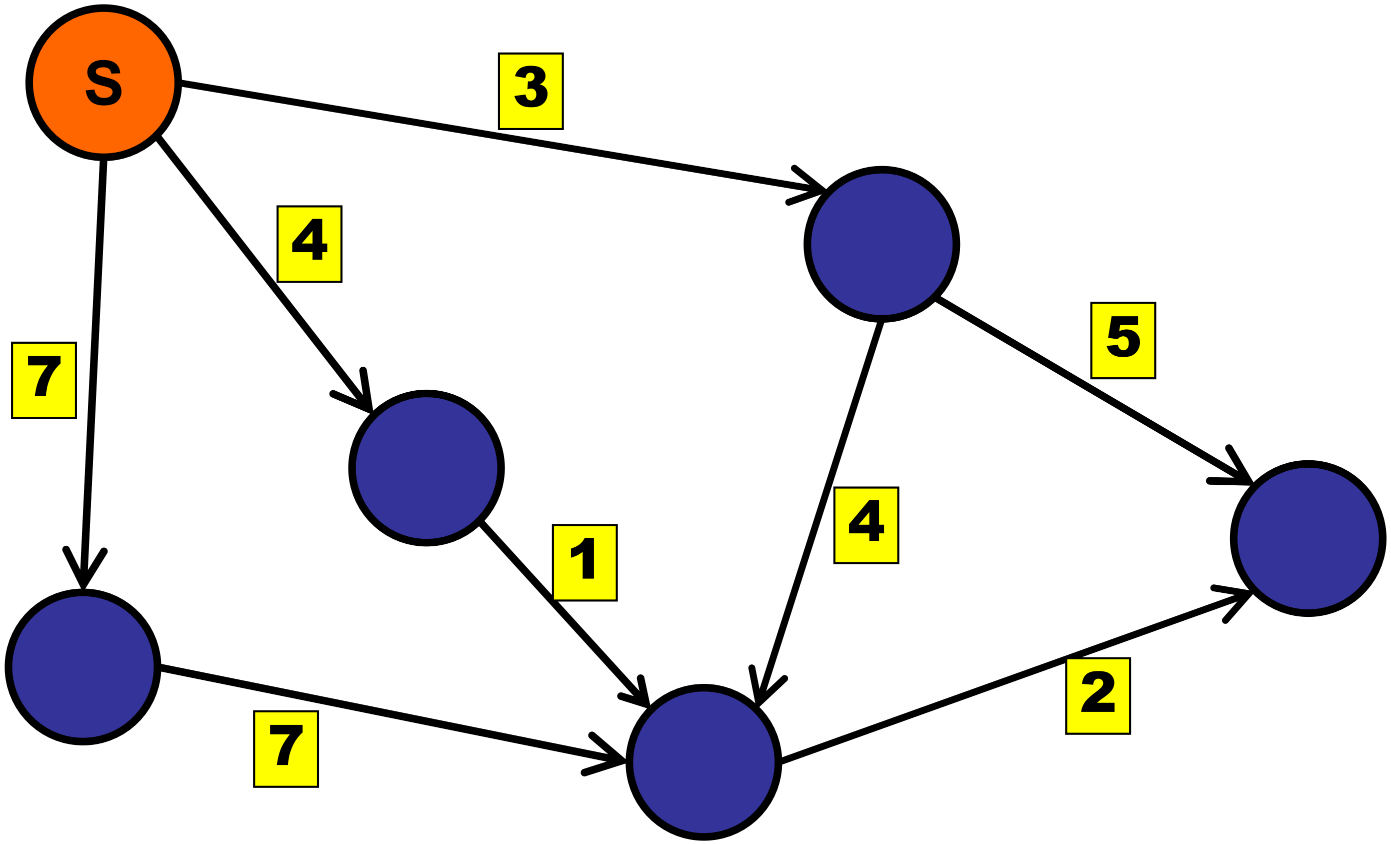


Shortest Paths

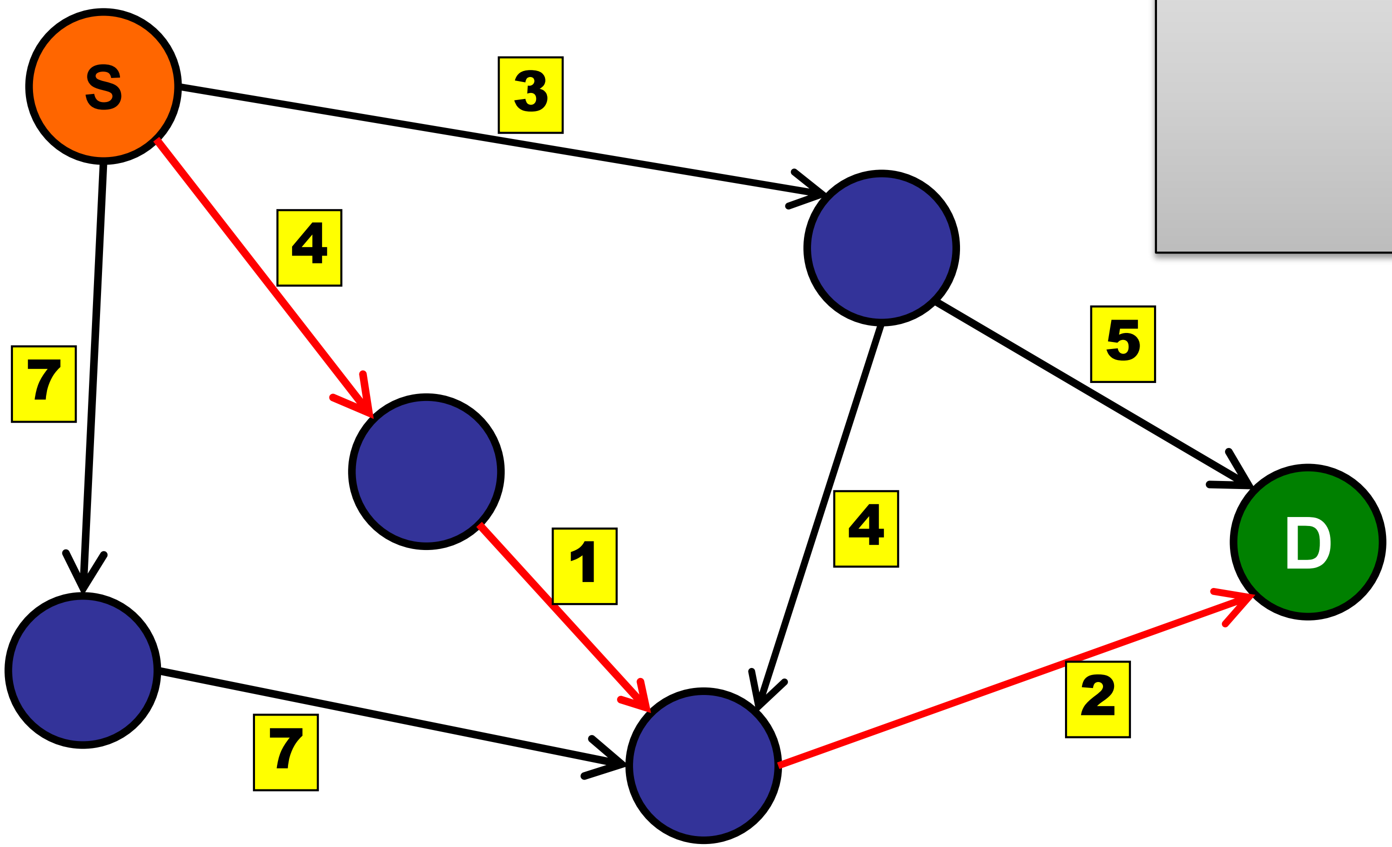
Acyclic Graph: Why topological order?



Shortest Paths

Acyclic Graph: Why topological order?

Fix S-D shortest path.

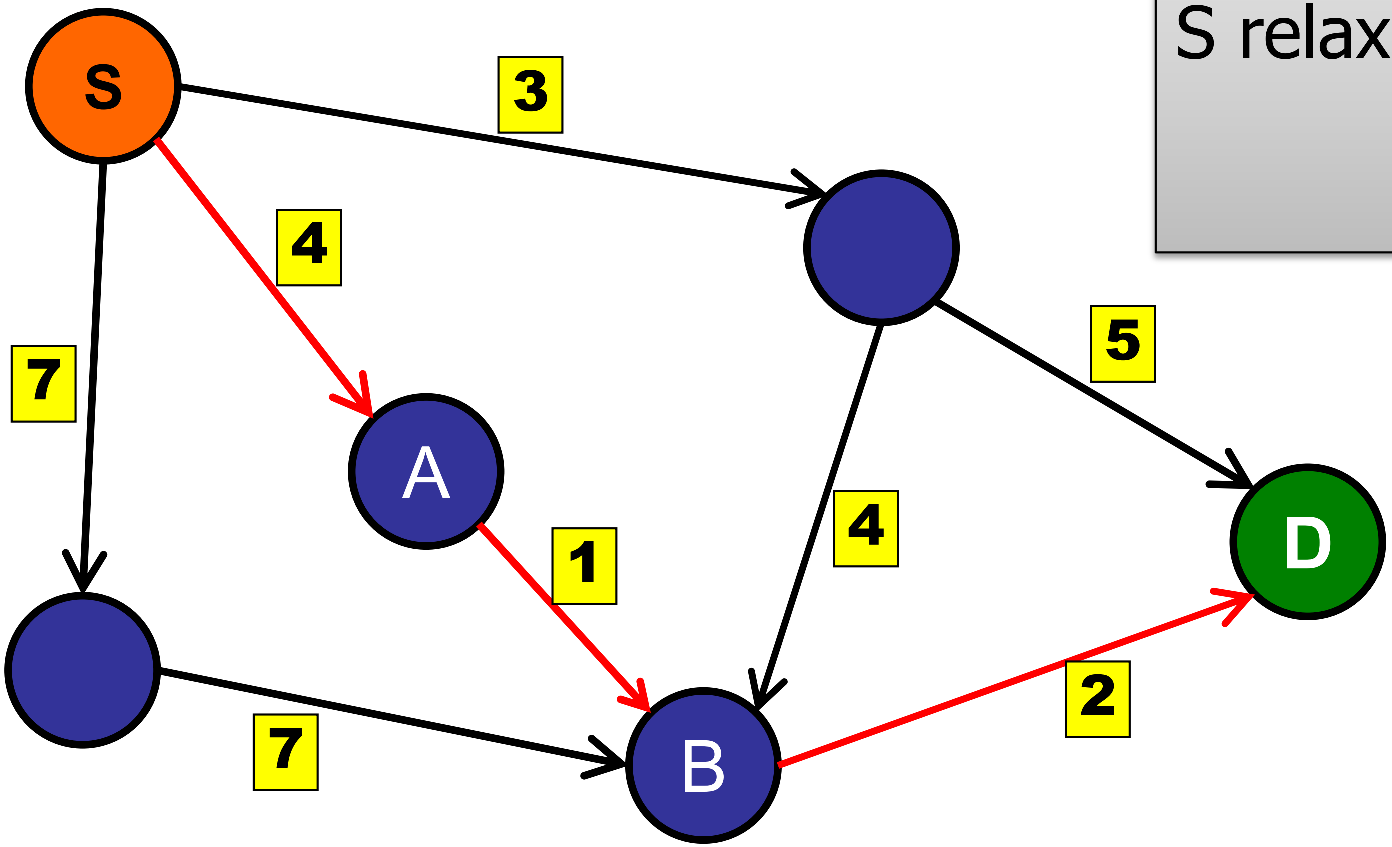


Shortest Paths

Acyclic Graph: Why topological order?

Fix S-D shortest path.

S relaxed before A.

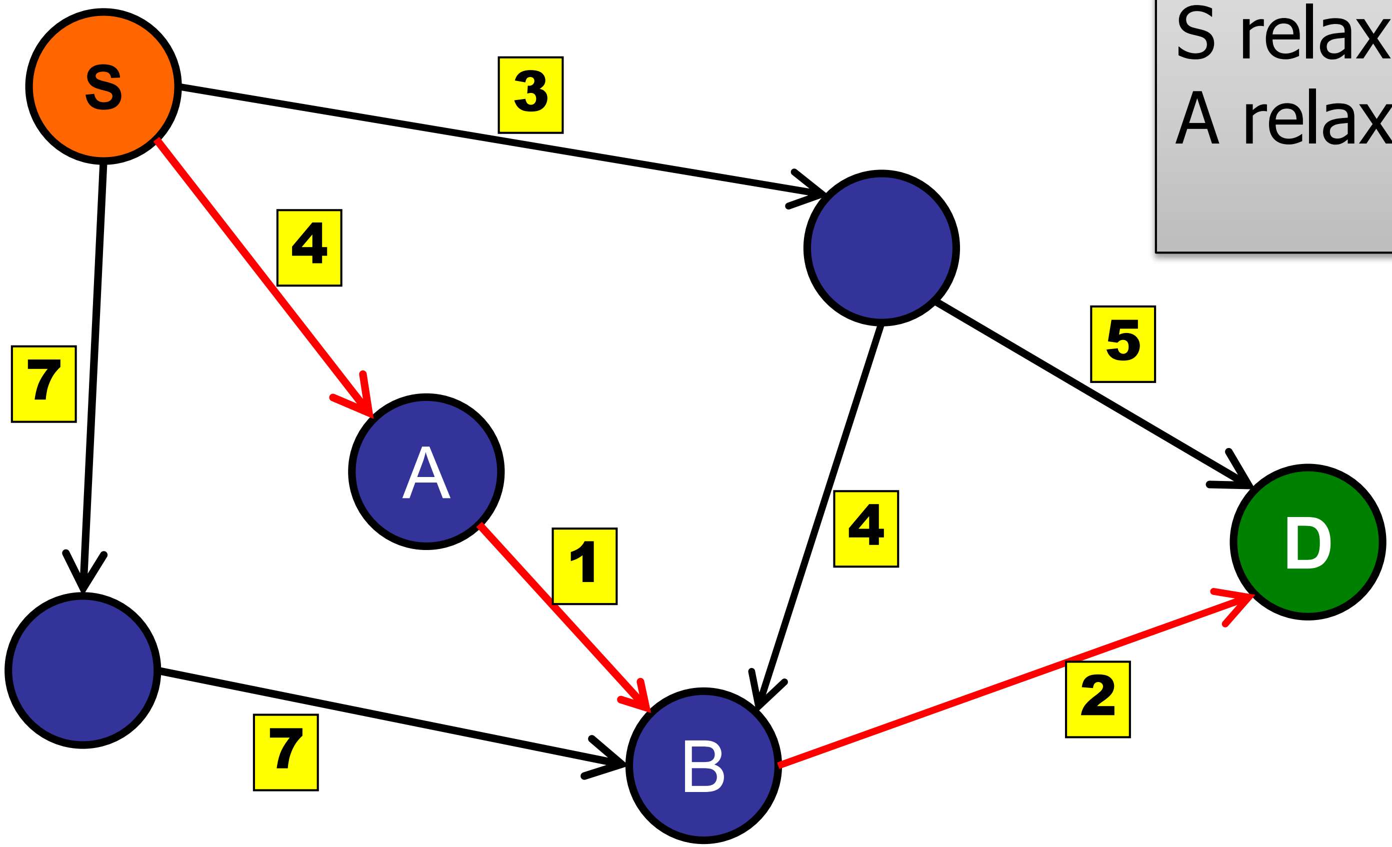


Shortest Paths

Acyclic Graph: Why topological order?

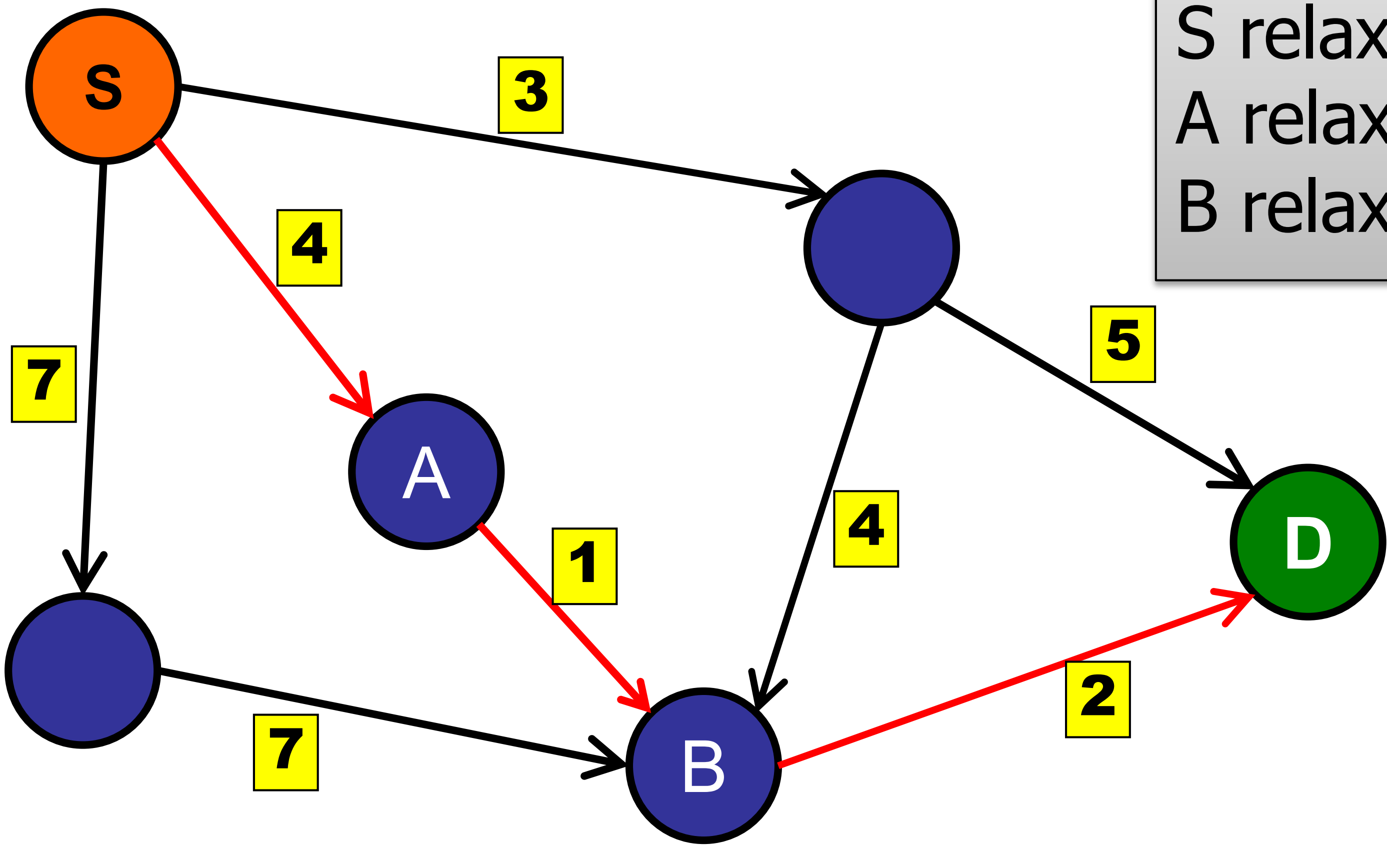
Fix S-D shortest path.

S relaxed before A.
A relaxed before B.



Shortest Paths

Acyclic Graph: Why topological order?

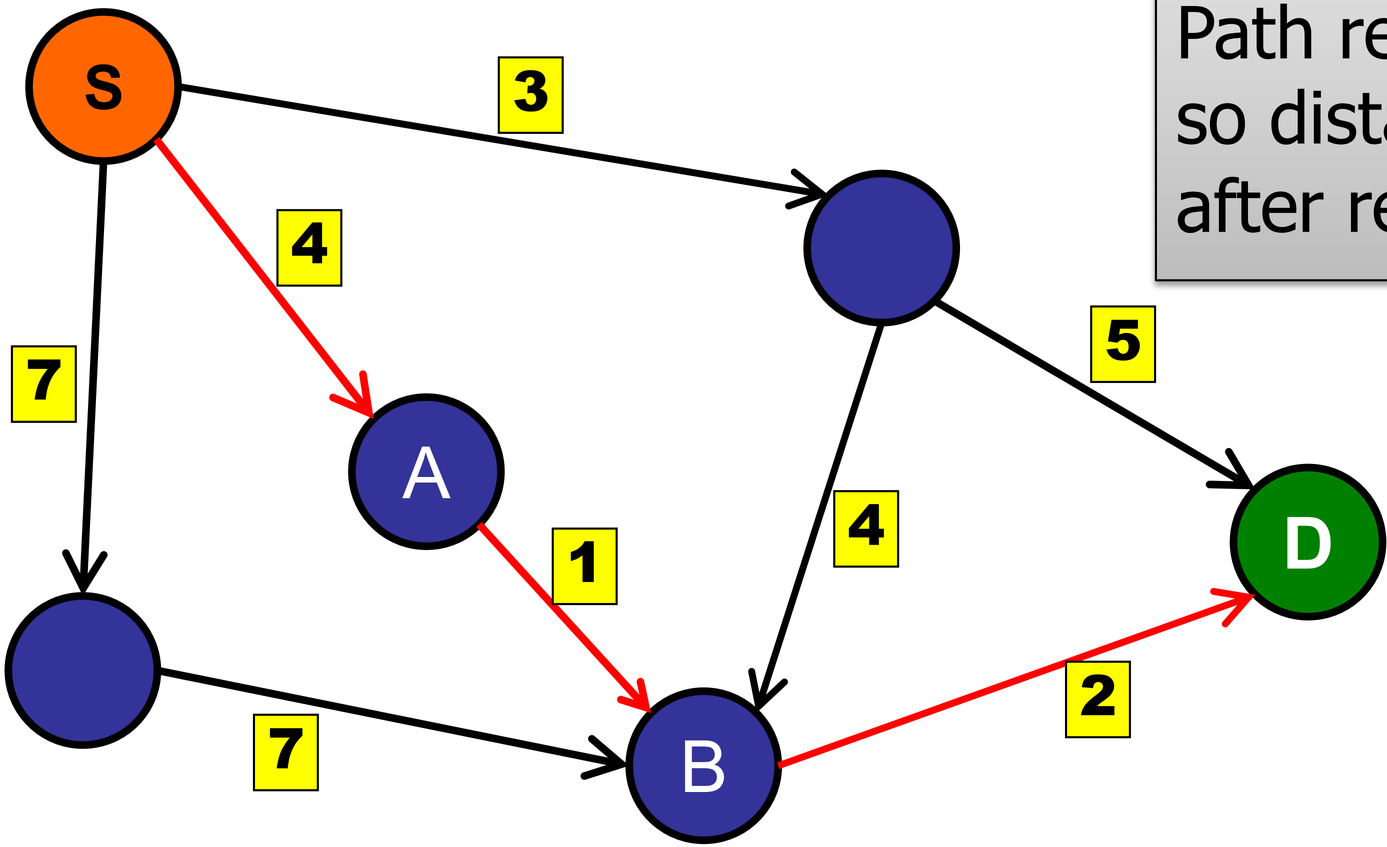


Fix S-D shortest path.

S relaxed before A.
A relaxed before B.
B relaxed before D.

Shortest Paths

Acyclic Graph: Why topological order?



Fix S-D shortest path.

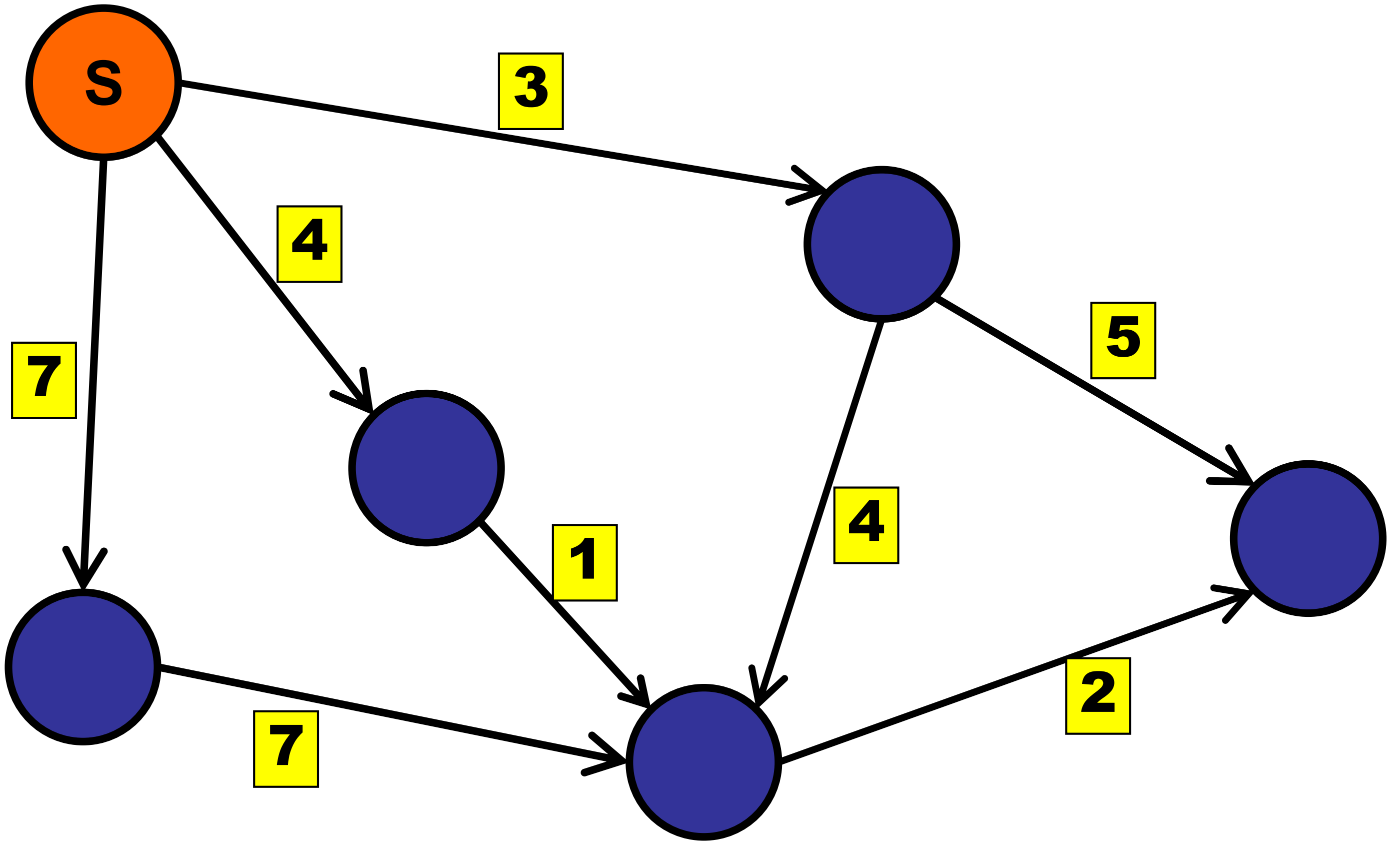
Path relaxed in-order, so distance is correct after relaxation.

What is the running time of shortest paths on a DAG?

1. $O(V)$
2. $O(E)$
3. $O(V^2)$
4. $O(E \log V)$
5. $O(V \log E)$
6. $O(VE)$

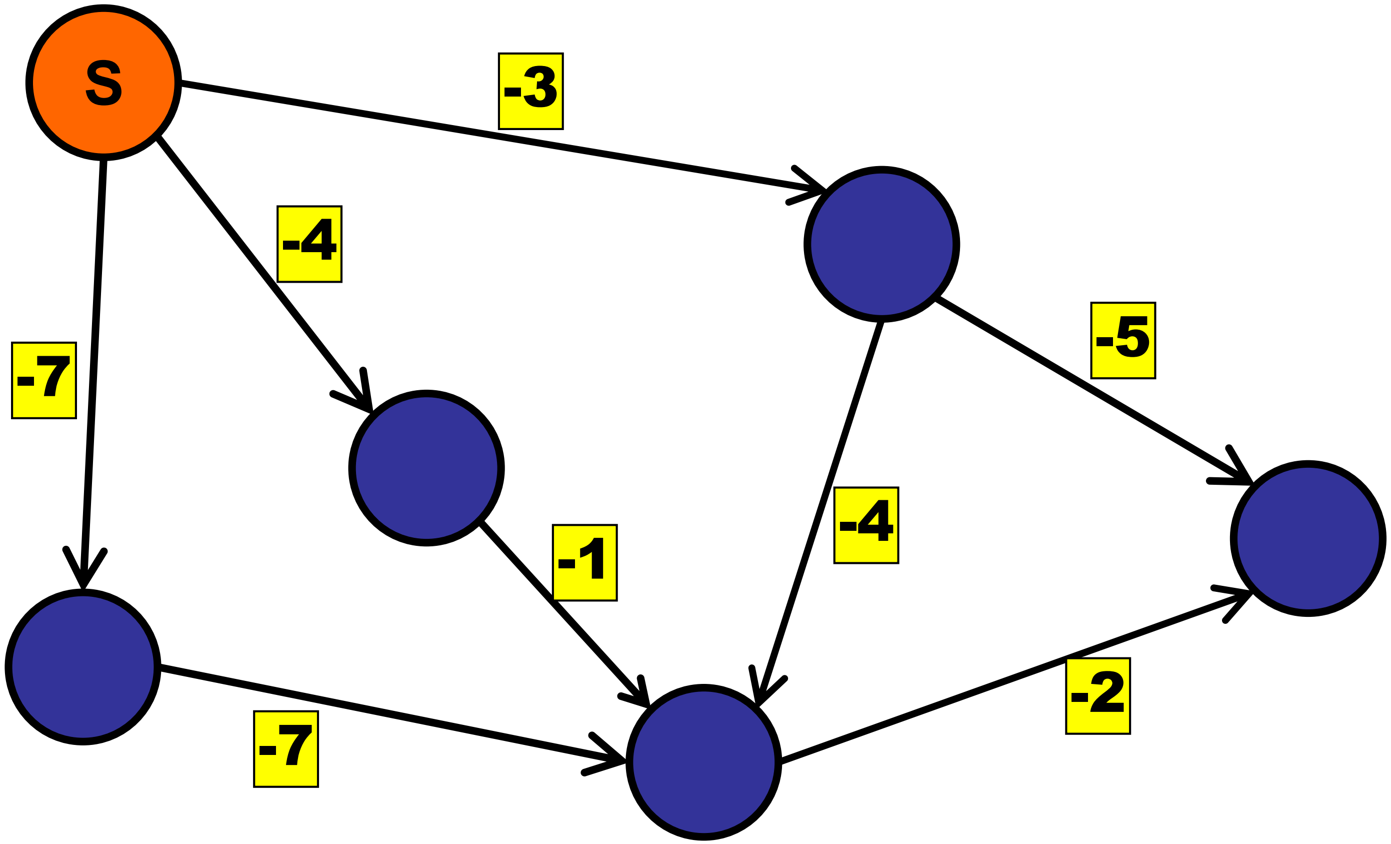
Longest Paths

Acyclic Graph: Any ideas?



Longest Paths

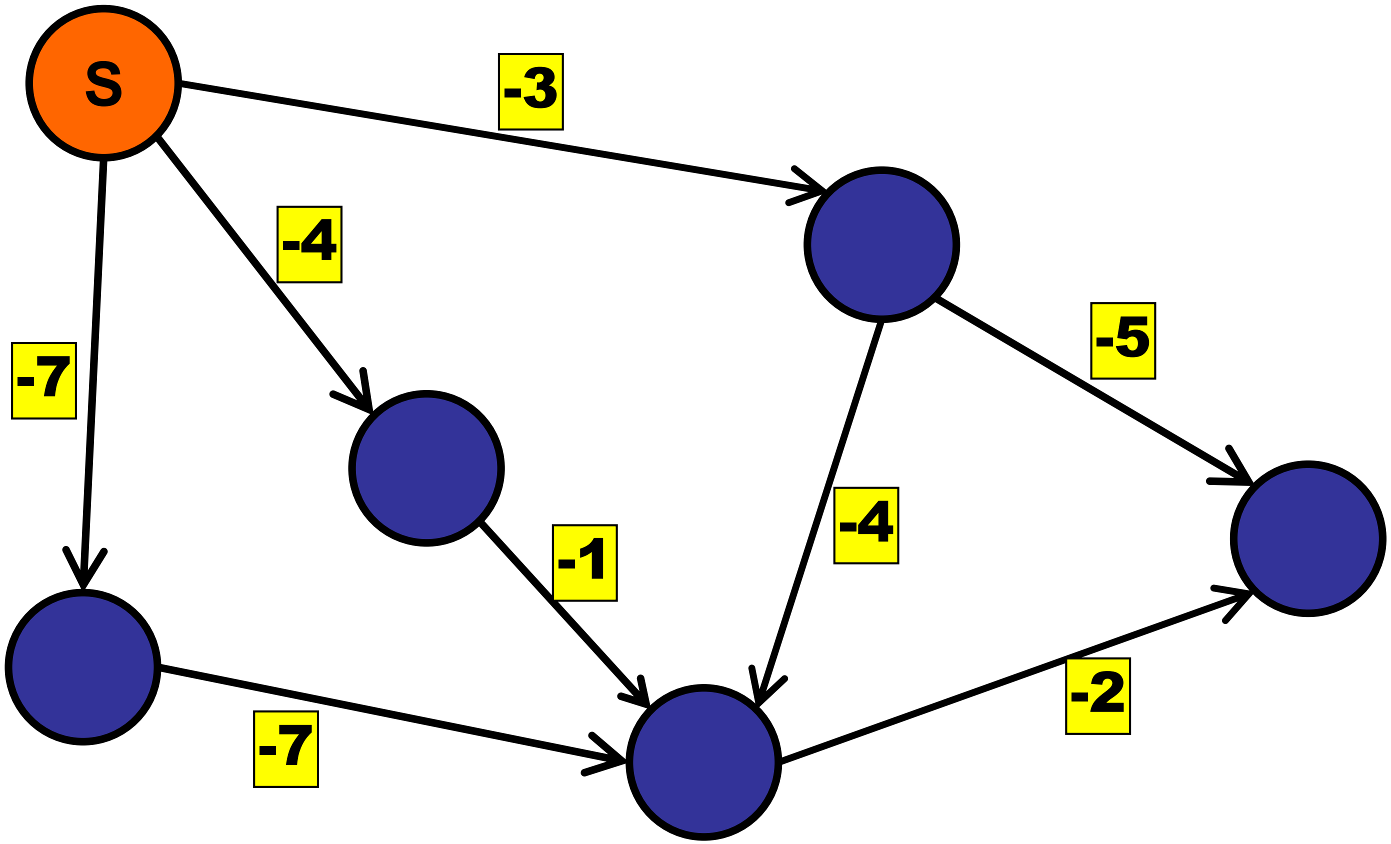
Acyclic Graph: Negate the edges!



Longest Paths

Acyclic Graph:

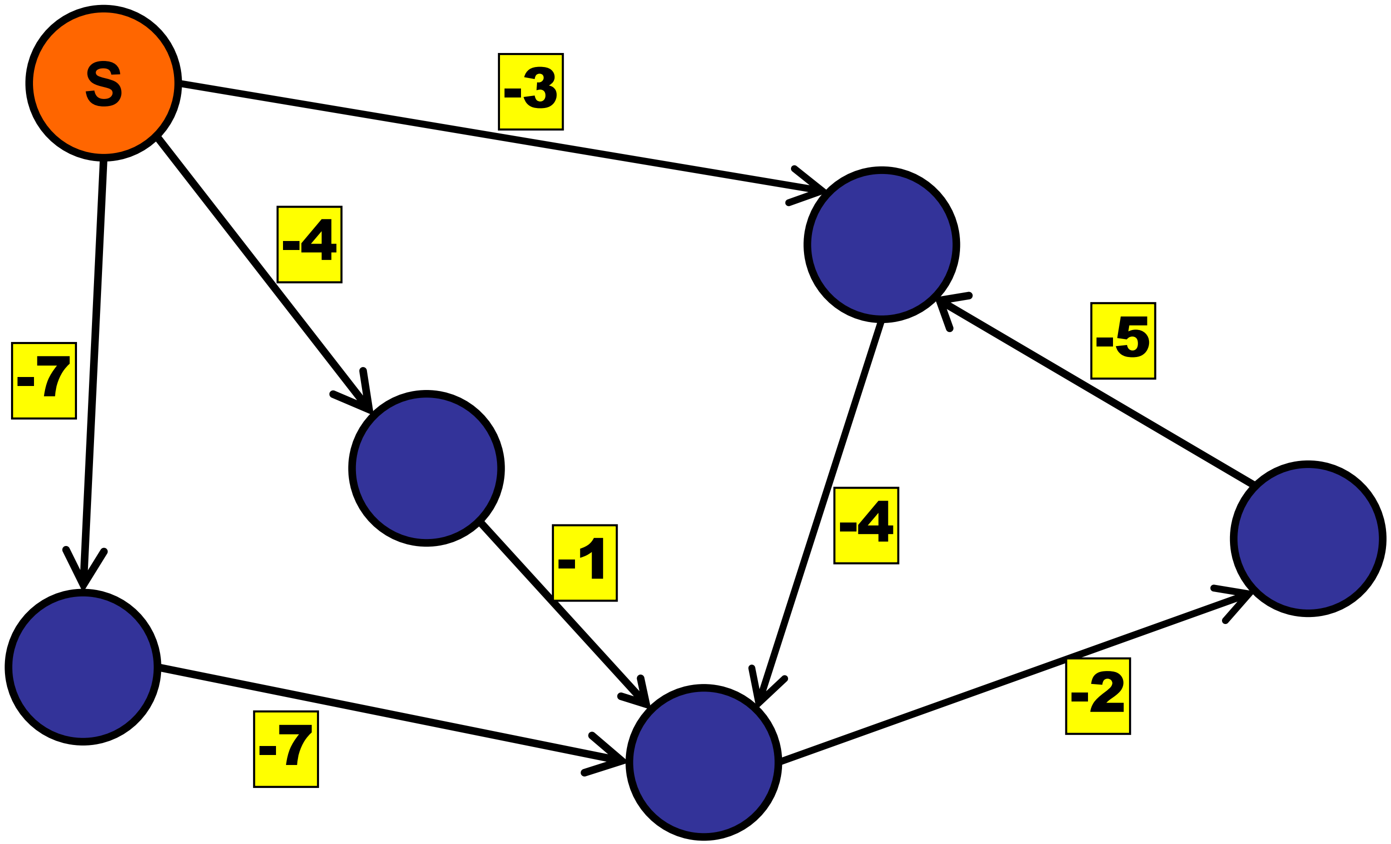
shortest path in negated=longest path in regular



Longest Paths

General (cyclic) Graph: (positive weights)

Can we use the same trick?

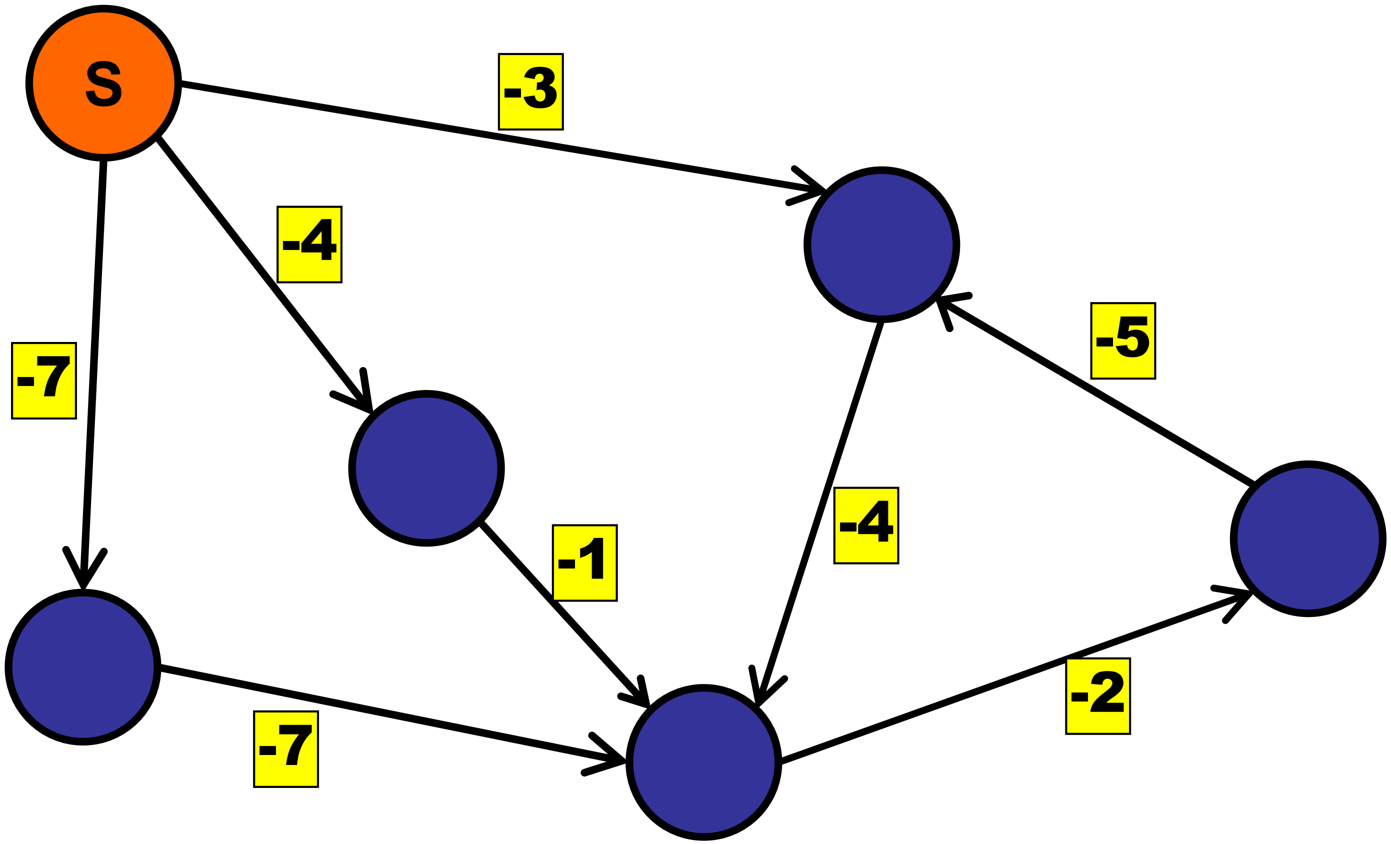


Longest Paths

General (cyclic) Graph: (positive weights)

Can we use the same trick? NO

Negative weight cycles!



Longest Path

Directed Acyclic Graph:

- Solvable efficiently using topological sort

General (cyclic) Graphs:

- NP-Hard
- Reduction from Hamiltonian Path:
  - If you could find the longest simple path, then you could decide if there is a path that visits every vertex.
  - Any polynomial time algorithm for longest path thus implies a polynomial time algorithm for HAMPATH.

Summary

Shortest Paths

- Special Case: Tree
- Special Case: Non-negative weights (Dijkstra’s)
- Special Case: Directed Acyclic Graphs