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1] Let Λ be a non-empty set. Show that the collection of all subsets of Λ is a topology on Λ.

$\Rightarrow$ Let $\Lambda = \{a, b, c\}$
Here, Λ has 3 elements, $\therefore 2^3$ distinct subsets.
They are :-
$S_1 = \emptyset$, $S_2 = \{a\}$, $S_3 = \{b\}$, $S_4 = \{c\}$
$S_5 = \{a, b\}$, $S_6 = \{b, c\}$, $S_7 = \{a, c\}$, $S_8 = \{a, b, c\}$

Let τ contain all the subsets of Λ
$\therefore S_{1 \dots 8} \in \tau$

According to the definition of topology :-
τ is a topology of Λ if :-
1) $\emptyset, \Lambda$ in τ
2) τ is closed under arbitrary unions and finite intersections.

As τ satisfies all the properties, a collection of subsets of Λ is a topology on Λ.

2] Let Ω be a non-empty set and let B be a non-empty strictly proper subset of Ω. Show that $F = \{\emptyset, \Omega, B, \Omega \setminus B\}$ is a sigma algebra on Ω.

$\Rightarrow$ For a non-empty set Ω, a sigma algebra F is a collection of subsets of Ω with the following properties :-
1) $\Omega \in F$
2) closed under countable unions and complements

We are given that :-
1) Ω is a non-empty set
2) $B \subset \Omega$
3) $F = \{\emptyset, \Omega, B, \Omega \setminus B\}$

As $\Omega \in F$, it satisfies the first condition of sigma algebra.
$\emptyset, \Omega, B, \Omega \setminus B$ are subsets of $\Omega \Rightarrow (B \subseteq \Omega)$
If we take the complements of all the elements,
$\emptyset^c : \Omega$
$\Omega^c : \emptyset$
$B^c : \Omega \setminus B$
$(\Omega \setminus B)^c = B$

$\therefore F = \{\emptyset, \Omega, B, \Omega \setminus B\}$ is closed under countable unions and complements.

Hence, F is a sigma algebra on Ω.

3] Let (Ω, F) and (Ω', F') be measurable spaces, and suppose that μ is a measure on F. Let $f : \Omega \to \Omega'$ be a function that is not measurable wrt F and F'. Now, let $B \in F'$ be given. What is the issue with the expression :- $\mu(\{\omega \in \Omega : f(\omega) \in B\})$?

$\Rightarrow$ As $f : \Omega \to \Omega'$ is not measurable wrt F and F',

we cannot say that

$$f^{-1}(B) := \{\omega \in \Omega : f(\omega) \in B\} \in F$$

As μ is a measure on F

$\mu(\{\omega \in \Omega : f(\omega) \in B\})$ may not exist.

4] Consider the experiment of tossing a coin, where there are two possibilities: heads or tails. We would like to represent the experiment in measure theoric terms. We consider $\Omega = \{0, 1\}$ and $F = \{\emptyset, \{0\}, \{1\}, \Omega\}$, the set of all subsets of Ω. 0 denotes heads, 1 denotes tails. We define the map $\mu : F \to [0, \infty]$ by

$$\mu(B) := \begin{cases} 1/2, & \text{if } B = \{0\} \text{ or } B = \{1\} \\ 1, & \text{if } B = \Omega \\ 0, & \text{if } B = \emptyset \end{cases}$$

To show that μ is a probability measure on F.
Show that a function $X : \Omega \to \mathbb{R}$ is $(F, B_{\mathbb{R}})$ - measurable.

$\Rightarrow$ $\mu : F \to [0, \infty]$ satisfies :-
i) If $A_i \in F$ are disjoint then $\mu(\bigcup A_i) = \sum_{i=1}^{\infty} \mu(A_i)$

$\therefore$ Considering the disjoint elements of F
$A = \{\emptyset, \Omega\}$ as $(\Omega = \{0, 1\})$

$$\therefore \mu(\bigcup A_i) = \sum_{i=1}^{\infty} \mu(A_i)$$
$$= \mu(\emptyset) + \mu(\Omega)$$
$$= 0 + 1$$
$$= 1$$

$\therefore \mu$ is a probability measure on F.

- Ω and $\mathbb{R}$ are topological spaces.

Further $\mathbb{R}$ is a Borel space $\Rightarrow$ [Any countable set Λ with discrete topology is a Borel space]

By definition, if Ω and $\mathbb{R}$ are topological spaces, $F = F$, $F' = B_{\mathbb{R}}$ then:
$X : \Omega \to \mathbb{R}$ is $(F, B_{\mathbb{R}})$ measurable.

5] Let (Ω, F) be a measurable space, and let $f : (\Omega, F) \to (\mathbb{R}, B_{\mathbb{R}})$ be given. Show that $f : (\Omega, F) \to (\mathbb{R}^*, B_{\mathbb{R}^*})$ holds as well.

$\Rightarrow$ $f : \Omega \to \Omega'$ is measurable wrt F and F' iff for all $a \in F'$
$$f^{-1}(B) := \{\omega \in \Omega : f(\omega) \leq a\} \in F$$

$\therefore$ For $f : (\Omega, F) \to (\mathbb{R}^*, B_{\mathbb{R}^*})$ to be measurable

$$\{\omega : f(\omega) \leq a\} \in F \quad \text{for all } a \in \mathbb{R}^*.$$

$\therefore f$ extended real valued function on (Ω, F) is measurable iff :-
1) $f^{-1}(\{\infty\}) \in F$
2) $f^{-1}(\{-\infty\}) \in F$
3) $f^{-1}((a, b]) \in F \quad \forall a \in \mathbb{R}$

$$\therefore f^{-1}(\{-\infty\}) = \bigcap_{n \in \mathbb{Z}} \{\omega : f(\omega) \leq n\} \in F$$

$$f^{-1}(\{\infty\}) = \bigcap_{n \in \mathbb{Z}} (\Omega \setminus \{\omega : f(\omega) \leq n\}) \in F$$

$$f^{-1}((a, b]) = \{\omega : f(\omega) \leq b\} \setminus \{\omega : f(\omega) \leq a\} \in F$$

6] Let (Ω, F, μ) be a measure space, where $\mu(F) < \infty$, and let $f : (\Omega, F) \to (\mathbb{R}^*, B_{\mathbb{R}^*})$ be given. If f is bounded below almost everywhere wrt μ, show that the integral $\int_\Omega f d\mu$ exists.

$\Rightarrow$ $$\int_\Omega f \, d\mu := \int_\Omega f(\omega) \mu d(\omega)$$

$$:= \int_\Omega \underbrace{\max(f(\omega), 0)}_{f+\omega} \mu(d\omega) - \int_\Omega \underbrace{\max(-f(\omega), 0)}_{f-\omega} \mu(d\omega)$$

As f is bounded below wrt Ω, the integral will never take the form $\infty - \infty$.

Hence $\int_\Omega f d\mu$ exists as long as f is bounded below wrt μ.

7] Let a probability space (Ω, F, μ), a natural number $n \in \mathbb{N}$, and a function $X_0 : (\Omega, F) \to (\mathbb{R}^n, B_{\mathbb{R}^n})$ be given. The Dirac measure on $B_{\mathbb{R}^n}$ concentrated at $x \in \mathbb{R}^n$ is defined by:

$$\delta_x(B) := \begin{cases} 1, & \text{if } x \in B \\ 0, & \text{if } x \in B^c \end{cases}$$

for every $B \in B_{\mathbb{R}^n}$. Note that $\delta_x(B) = I_B(x)$, the indicator function of B at the point x.
Suppose that
$$\mu(\{\omega \in \Omega : X_0(\omega) \in B\}) = \delta_x(B), \ B \in B_{\mathbb{R}^n}$$

Prove that $X_0 = x$ almost everywhere wrt μ.

$\Rightarrow$ $$\mu(\{\omega \in \Omega : X_0(\omega) \in B\}) = \delta_x(B) = I_B(x)$$

$X_0 : \Omega \to \mathbb{R}^n$ is measurable wrt F, $B_{\mathbb{R}^n}$

if $x \in B$, $\delta_x(B) = 1 = \mu(\{\omega \in \Omega : X_0(\omega) \in B\})$
if $x \in B^c$, $\delta_x(B) = 0 = \mu(\{\omega \in \Omega : X_0(\omega) \in B\})$

As μ is a probability measure
$\mu(\{\omega \in \Omega : X_0(\omega) \in B\}) = 1$

But this is only true for all $X_0 = X$.