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1] Let a probability space (Ω, F, μ), a natural number $n \in N$, a function $X_0 : (\Omega, F) \to (\mathbb{R}^n, B_{\mathbb{R}^n})$, and a function $f : (\mathbb{R}^n, B_{\mathbb{R}^n}) \to (\mathbb{R}, B_{\mathbb{R}})$ be given. Suppose that

- $\xi : (\Omega, F) \to (\mathbb{R}, B_{\mathbb{R}})$ satisfies $\int_\Omega \xi \, d\mu = 1$ and
- $\mu(\{\omega \in \Omega : X_0(\omega) \in B\}) = \delta_x B$ for every $B \in B_{\mathbb{R}^n}$ where $x \in \mathbb{R}^n$ is given.

Prove that :- $\int_\Omega f(X_0(\omega)) \, \xi(\omega) \, d\mu(\omega) = f(x)$

$\Rightarrow$ $\int_\Omega f(X_0(\omega)) \, \xi(\omega) \, d\mu(\omega)$

= Integration by parts

$$\int uv \, dx = u \int v \, dx - \int u' \left(\int v \, dx \right) dx$$

$$= f(X_0(\omega)) \int_\Omega \xi(\omega) \, d\mu(\omega) - \int f'(X_0(\omega)) \left(\int \xi(\omega) \, d\mu(\omega) \right) d\mu(\omega)$$

$$= f(X_0(\omega)) - \int f'(X_0(\omega)) \, d\mu(\omega)$$

$f'(X_0(\omega)) = \dfrac{df(X_0(\omega))}{d\mu(\omega)} = 0$ as measure of the set is Dirac function.

$$\therefore \int_\Omega f(X_0(\omega)) \, \xi(\omega) \, d\mu(\omega) = f(x)$$

2] Let M be a metrizable space, and let $g : M \to \mathbb{R}^*$ be a lower semi-continuous function. Show that g is Borel measurable.

$\Rightarrow$ If the set $\{x \in M : g(x) \leq r\}$ is closed for every $r \in \mathbb{R}$, then g is called lower semi-continuous. $\rightarrow$ (Remark 3.4 ; class notes)

A set being closed means that its complement is a member of the topology of M.

i.e $\{x : g(x) > r\}$ is open for $r \in \mathbb{R}$

Since every open set is a Borel set, all lower semi-continuous functions are Borel measurable.

3] Let S and C be non-empty Borel Spaces, and let $N \in \mathbb{N}$ be given. Define the space Ω by $\Omega := (S \times C)^N \times S$, and define the function $X_t : \Omega \to S$ by $X_t(\omega) := x_t$ for every $\omega = (x_0, u_0, \ldots, x_{N-1}, u_{N-1}, x_N) \in \Omega$. Show that X_t is Borel measurable.

$\Rightarrow$ To show X_t is Borel measurable,

$$X_t^{-1}(B) = \{\omega : X_t(\omega) \in B\} \in \Omega \quad \forall B \in S$$

It is given that

$$X_t(\omega) = x_t \text{ for every } \omega \in \Omega$$

$$\therefore X_t(\omega) = x_t \in B \text{ for every } B \in S$$

Hence X_t is Borel measurable.

4] Let a metrizable space M be given. The Dirac measure in P_M concentrated at $x \in M$ is defined by

$$\delta_x(B) := I_B(x), \quad B \in B_M$$

where $I_B(x)$ is the indicator function of B evaluated at point x. Since M is metrizable, the map $l : M \to P_M$ defined by $l(x) := \delta_x$ is a homeomorphism; in particular, the map l is (weakly) continuous.

Let S, C be Borel spaces, and let $k_t : S \to C$ be a Borel measurable function.

Show that the function $\varphi_t : S \to P_C$ defined by $\varphi_t(z) := \delta_{k_t}(z)$ is Borel measurable. Consider the stochastic kernel $\pi(\cdot|z) := \delta_{k_t}(z)$. Explain why π_t is a Borel measurable stochastic kernel on C given S.

$\Rightarrow$ A deterministic Markov control policy is characterized by a sequence $\kappa := (k_0, k_1, \ldots, k_{N-1})$ of Borel measurable functions $K_t : S \to C$.

If $x_t \in S$ is the current state, then $\delta_{k_t}(x_t) \in P_C$ is the distribution of realizations of U_t. $\rightarrow$ [class notes 3.17]

Since $k_t : S \to C$ is Borel measurable, S is also a metrizable space.

$\therefore \varphi_t : S \to P_C$ is a homeomorphism ;

$\therefore \varphi_t(z) := \delta_{k_t}(z)$ is (weakly) continuous.

As all continuous functions are Borel measurable, $\varphi_t(z) : S \to P_C$ is also Borel measurable.

- For the stochastic kernel $\pi_t(\cdot|z) := \delta_{k_t}(z)$

C and S are metrizable spaces. By definition, a Borel-meas. stochastic kernel $\pi_t(\cdot|z)$ on C given S is a family of elements of P_C parametrized by elements of S, where the map $\varphi : S \to P_C$ is Borel measurable. $\rightarrow$ [Def. 3.12, class notes]

Hence $\pi_t(\cdot|z)$ is a stochastic kernel on C given S.

5] Suppose S and C are Borel spaces. Let π_0 be a Borel measurable stochastic kernel on C given S, and let q_0 be a Borel measurable stochastic kernel on S given $S \times C$. Let $x \in S$ be given, and consider the probability measure $P_{1,x}^\pi \in P_S$ defined by

$$P_{1,x}^\pi(\underline{S}) := \int_S \int_C q_0(\underline{S} | x_0, u_0) \, \pi_0(du_0 | x_0) \, \delta_x(dx_0), \quad \underline{S} \in B_S$$

Show that for any non-negative Borel-measurable function $g : S \to \mathbb{R}^*$, the following equality holds:

$$\int_S g(x_1) \, dP_{1,x}^\pi(x_1) = \int_S \int_C \int_S g(x_1) \, q_0(dx_1 | x_0, u_0) \, \pi_0(du_0 | x_0) \, \delta_x(dx_0)$$

$\Rightarrow$ For $x \in S$ and $\pi = (\pi_0, \pi_1, \ldots, \pi_{N-1}) \in \Pi$,

$$P_x^\pi(\underline{S_0} \times \underline{C_0} \times \underline{S_1} \times \underline{C_1} \times \ldots \times \underline{C_{N-1}} \times \underline{S_{N-1}})$$

$$= \int_{\underline{S_0}} \int_{\underline{C_0}} \int_{\underline{S_1}} \cdots \int_{\underline{C_{N-1}}} q_N(\underline{S_N} | x_{N-1}, u_{N-1}) \, \pi_{N-1}(du_{N-1} | x_{N-1}) \ldots q_0(dx_1 | x_0, u_0) \, \pi_0(du_0 | x_0) \, \delta_x(dx_0)$$

$\rightarrow$ [46, class notes]

$$\therefore E_x^\pi(Z) := \int_\Omega Z \, dP_x^\pi$$

$$= \int_S \int_C \int_S \cdots \int_C \int_S Z(x_0, u_0, \ldots, x_N) \, q_N(dx_N | x_{N-1}, u_{N-1}) \, \pi_{N-1}(du_{N-1} | x_{N-1}) \cdots q_0(dx_1 | x_0, u_0) \, \pi_0(du_0 | x_0) \, \delta_x(dx_0)$$

$\rightarrow$ [Eq. 47, class notes]

$$\therefore \int_S g(x_1) \, dP_{1,x}^\pi(x_1) = \int_S \int_C \int_S g(x_1) \, q_0(dx_1 | x_0, u_0) \, \pi_0(du_0 | x_0) \, \delta_x(dx_0)$$

6] Let (Ω, F, μ) be a probability space and let $X : (\Omega, F) \to (\mathbb{R}^n, B_{\mathbb{R}^n})$ be given, where $n \in N$. The random vector X is defined by

$$X(\omega) := [X_1(\omega) \ \ldots \ X_n(\omega)]^T, \quad \omega \in \Omega$$

In particular, $X_i : (\Omega, F) \to (\mathbb{R}, B_{\mathbb{R}})$ for every i. Suppose that the expectation of X is finite, i.e.

$$E(X) := [E(X_1) \ \ldots \ E(X_n)]^T \in \mathbb{R}^n$$

where $E(X_i) := \int_\Omega X_i \, d\mu$ for every i. Let $M \in \mathbb{R}^{n \times n}$ be symmetric positive definite matrix, that is, $M = M^T$ and $y^T M y > 0$ for every non-zero $y \in \mathbb{R}^n$. Let G be a sub sigma algebra of F, and consider the conditional expectation of X given G,

$$\hat{X} := E(X | G) := [E(X_1 | G) \ \ldots \ E(X_n | G)]^T$$

Explain why $\hat{X}^T M \hat{X}$ need not to be measurable. Provide a random object $\hat{Z}$ such that $\hat{Z} = \hat{X}$ a.e and $\hat{Z}^T M \hat{Z}$ is a $(G, B_{\mathbb{R}^*})$ - measurable function.

7] Let $x \in S$ be given. Show that the following statements :-

$\varphi_t^\pi \geq v_t$ a.e. $[P_{t,x}^\pi]$, for every $\pi \in \Pi$, for every $t \in T \cup \{N\}$

$\varphi_t^{\pi^*} = v_t$ a.e. $[P_{t,x}^{\pi^*}]$ for every $t \in T \cup \{N\}$

imply the desired optimality result :

$$v_0(x) = \inf_{\pi \in \Pi} E_x^\pi(Y) = E_x^{\pi^*}(Y)$$

$\Rightarrow$ $P_{0,x}^\pi = \delta_x$ $\Rightarrow$ [Lemma 3.29, class notes]

$\varphi_0^\pi(x) = E_x^\pi(Y)$ $\Rightarrow$ [Th. 3.30, Notes]

We define $\pi^* := (\pi_0^*, \pi_1^*, \ldots, \pi_{N-1}^*)$ where $\pi_t^*(\cdot | x_t) := \delta_{k_t^*}(x_t) \in P_C$ for every $x_t \in S$

As per statement 2: $\varphi_t^{\pi^*} = v_t$

$$v_N = c_N = \varphi_N^\pi$$

$$\therefore v_0(x) = \varphi_0^{\pi^*}(x) = E_x^{\pi^*}(Y)$$

The infimum is defined as the greatest lower bound of a set.

$\therefore$ As per statement 1 : $\varphi_t^\pi \geq v_t$

it implies that

$$v_0(x) = E_x^{\pi^*}(Y) = \inf_{\pi \in \Pi} E_x^\pi(Y)$$