2D Bose gas

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Having in mind the renormalization argument reported in the previous paper, we are going to attempt to obtain the self-consistent equation for the qualitative behaviour of chemical potential for 2D Bose gas.

In the manner similar to the 3D case, we start with introducing g_2, which for 2D case is

$$g_2(E|p) = -\frac{2\pi}{\log\left(d\sqrt{\frac{1}{4}p^2 - E}\right)} \tag{1}$$

Here d is the interaction parameter that for quasi-2D systems is linked to the scattering length by

$$d = Cl_\perp \exp\left[\sqrt{\frac{\pi}{2}}\frac{l_\perp}{a}\right]$$

here C is a numerical constant.

Let us consider the back-of-the envelope estimation of the potential behaviour with only the 2-body term self-consistent equation

$$\mu = -\frac{4\pi n_0}{\log(d^2 2\mu)} + ... \tag{2}$$

that can also be presented as

$$\log(d^2 2\mu) = -\frac{4\pi n_0}{\mu} \tag{3}$$

$$d^2 2\mu = \exp\left(-\frac{4\pi n_0}{\mu}\right) \tag{4}$$

$$\mu_1 = \frac{B_2}{2}\exp\left(-\frac{4\pi n_0}{\mu}\right) \tag{5}$$

We have a transendental equation, let us try to understand its behaviour.

In the limit of high $B_2 \gg n_0$ and low gas concentration (n_0 has dimensions of k^2 in 2D), we have

$$\mu = B_2\left(1 - \frac{4\pi n_0}{\mu}\right)$$

The solution for this second order polynomial is ($\mu = 0$ aside)

$$\mu = 4\pi n_0\left(1 + \frac{2}{B_2}4\pi n_0\right) + o\left(\frac{1}{B_2}\right), B_2 \to \infty$$

$$\mu = \frac{B_2}{2} - 4\pi n_0 + o\left(1\right), B_2 \to \infty$$

Another way to consider this limit is

$$\frac{4\pi n_0}{\mu} = 1 - \sum_{n\in N}\left(\frac{4\pi n_0}{B_2}\right)^n \tag{6}$$

We have discussed before the g_3 form for 2D case

$$g_3 = \frac{16\pi^2}{\log^2(d\sqrt{2\mu})}\int_0^\Lambda \frac{dkk}{\{2\mu + k^2\}}\frac{G_3(k)}{\log\left(\frac{3}{4}k^2d^2 + 3\mu d^2\right)}$$

with $G_3(p)$ being the solution of the following integral equation

$$G_3(p) = 4\int_0^\Lambda \frac{dkk}{\log\left(\frac{3}{4}k^2d^2 + 3\mu d^2\right)}\left\{\frac{1}{2\mu + k^2} + G_3(k)\right\}\left\{(3\mu + k^2 + p^2)^2 - p^2k^2\right\}^{-\frac{1}{2}}$$

The self-consistent system of equations that sums up our approach is then

$$\mu = n_0 g_2(n_0, -\mu) + n_0^2 g_3(n_0, -\mu)$$

$$n = n(n_0, -\mu)$$

Let us see what the numerical implementation yeilds us

1 Lowest-order terms

$$\mu = n_0 g_2 + 2n_0^2 g_2^3 \int \frac{d^2k}{(2\pi)^2} \frac{1}{\left(k^2 + 2\mu\right)^2}$$

Introducing notations

$$y = \frac{\mu}{n_0}, x = \frac{n_0}{B_2}$$

$$z = \frac{\mu}{n} = y\frac{n_0}{n}, s = \frac{n}{B_2} = x\frac{n}{n_0}$$

$$\frac{n}{n_0} = 1 + \frac{2\pi}{y \log^2(2xy)}$$

we get

$$y^2 \log^3(2xy) + 4\pi y \log^2(2xy) + (4\pi)^2 = 0$$

$$z = \frac{y}{1 + \frac{2\pi}{y \log^2(2xy)}}$$

$$s = x\left[1 + \frac{2\pi}{y \log^2(2xy)}\right]$$

Dropping the last term gives us the following lowest-order expansion terms

$$y = \frac{4\pi}{\log\left[\frac{1}{8\pi x}\right]} \left\{1 - \frac{\log\left[\log\left(\frac{1}{8\pi x}\right)\right]}{\log\left[\frac{1}{8\pi x}\right]}\right\} + o(\text{ higher terms }), x \to 0$$

or ,introducing $t = \frac{1}{\log\left[\frac{1}{8\pi x}\right]}$

$$y = 4\pi t\left\{1 + t\log t\right\} + o(t\log(t)), t \to 0$$

$$t = \frac{y}{4\pi} \sum_{N=0,1..} \left\{\frac{y}{4\pi} \log\left[\frac{4\pi}{y}\right]\right\}^N$$