ASEN 5044 Statistical Estimation for Dynamical Systems

1. Consider an aircraft moving in a plane with constant speed (i.e. magnitude of velocity) and turning with a constant angular rate. Such a model is often used by air traffic control tracking algorithms to describe an aircraft executing coordinated turns. Given 2D inertial position variables $\xi(t)$ (East position) and $\eta(t)$ (North position), the equations of motion are

$$\ddot{\xi} = -\Omega\dot{\eta}$$
$$\ddot{\eta} = \Omega\dot{\xi}$$

where Ω is the constant angular rate, such that $\Omega > 0$ implies a counterclockwise turn. Using the state representation $x(t) = [\xi, \dot{\xi}, \eta, \dot{\eta}]^T$, it can be shown that

$$e^{A\Delta t} = \begin{bmatrix} 1 & \frac{\sin(\Omega\Delta t)}{\Omega} & 0 & -\frac{1-\cos(\Omega\Delta t)}{\Omega} \\ 0 & \cos(\Omega\Delta t) & 0 & -\sin(\Omega\Delta t) \\ 0 & \frac{1-\cos(\Omega\Delta t)}{\Omega} & 1 & \frac{\sin(\Omega\Delta t)}{\Omega} \\ 0 & \sin(\Omega\Delta t) & 0 & \cos(\Omega\Delta t) \end{bmatrix}$$

where A is the CT LTI state matrix for this system. Do the following parts (a)-(c):

(a) Find A **(ignore measurements y and disregard process noise for now)**. What is the DT LTI model for this system if we are ignoring the process noise?

(b) For the following questions (i)-(ii), assume a sampling time Δt=0.5 s, $\Omega = 0.045$ rad/s, and an initial DT state uncertainty $x(0) \sim \mathcal{N}(\mu(0), P(0))$ at $k = 0$. **Again, disregard process noise and measurements for now** (we will revisit this problem again with process noise and measurements included later on, in the final HW 8 assignment).

i. Derive an expression for predicted DT state mean $\mu(k) = E[x(k)]$ and state covariance $P(k) = E[(x(k) - \mu(k))(x(k) - \mu(k))^T]$ for an arbitrary number of steps $k = N$ into the future, starting from $k = 0$. Be sure to express your answer express in terms of N, $\mu(0)$, and $P(0)$, and the DT state transition matrix F.

ii. Assume

$$\mu(0) = [0\ m, 85\cos(\pi/4)\ m/s, 0\ m, -85\sin(\pi/4)\ m/s]^T$$
$$P_a(0) = \text{diag}([10\ m^2, 2\ (m/s)^2, 10\ m^2, 2\ (m/s)^2]$$

(where the diag operator puts the input arguments into the corresponding diagonal entries of a square matrix). Perform a 150 step calculation of $\mu(k)$ and $P(k)$ for $k = 1, 2, ..., 150$ starting from these initial conditions. Plot the following in separate plots:

- each state element of $\mu(k)$ versus time, along with $\pm 2\sigma$ (2 standard deviations) upper/lower bounds showing the uncertainty for each state at each time k (these can be obtained by looking at the appropriate multiples of the square root of the corresponding diagonal entry of $P(k)$, since each diagonal entry is the variance of some state).
- only the positive 2σ values versus time for each state at time k.

Be sure to label all axes with appropriate units, and comment on the results.

(c) Consider now computing the probability of collision for two aircraft turning independently at the same altitude at the same time, where the initial position and velocity of the aircraft are unknown. Let x_a denote the state of aircraft a and let x_b denote the state of aircraft b. A 'collision' will simply be modeled as an event where both aircraft occupy a bounded rectangular region of airspace at the same. That is, if $r_a(k) = [\xi_a(k), \eta_a(k)]^T$ and $r_b(k) = [\xi_b(k), \eta_b(k)]^T$, a collision occurs at time k whenever $r_c(k) \equiv r_a(k) - r_b(k)$ lies inside the region R defined by $\Delta\xi(k) \in [-\xi_R, \xi_R]$ and $\Delta\eta(k) \in [-\eta_R, \eta_R]$, where $\Delta\xi(k) \equiv \xi_a(k) - \xi_b(k)$, $\Delta\eta(k) \equiv \eta_a(k) - \eta_b(k)$, and ξ_R and η_R are known constants.

At time $k = 0$, suppose both aircraft states are independent and normally distributed, with $x_a(0) \sim \mathcal{N}(\mu_a(0), P_a(0)$ and $x_b(0) \sim \mathcal{N}(\mu_b(0), P_b(0))$. Also assume that Δt=0.5 s, $\Omega_a = 0.045$ rad/s and $\Omega_b = -0.045$ rad/s (i.e. the aircraft turn in opposite directions at the same rate). **Again, ignore process noise and measurements.**

i. Derive the expression for the mean and covariance parameters of the pdf $p(r_c(k))$ at any time k, given that $r_c(k) \sim \mathcal{N}(\mu_{r_c}(k), P_{r_c}(k))$ (i.e. the pdf for $r_c(k)$ must be a multivariate Gaussian). Your answers should be in terms of F_a and F_b (the STMs for each aircraft), as well as $\mu_a(0), \mu_b(0)$ and $P_a(0), P_b(0)$. (**Hint:** in case you've forgotten, the concept of marginalizing multivariate Gaussian pdfs was covered earlier in the course, and will be very useful to apply here.)

ii. Use the result from part i. above to derive an integral expression for the probability of collision at any time k.

iii. Plot the probability of collision vs. time using a simulation run of 150 secs starting from the following initial conditions

$$\begin{aligned}
\mu_a(0) &= [0\ m, 85\cos(\pi/4)\ m/s, 0\ m, -85\sin(\pi/4)\ m/s]^T \\
P_a(0) &= \mathrm{diag}([10\ m^2, 4\ (m/s)^2, 10\ m^2, 4\ (m/s)^2], \\
\mu_b(0) &= [3200\ m, 85\cos(\pi/4)\ m/s, 3200\ m, -85\sin(\pi/4)\ m/s]^T \\
P_b(0) &= \mathrm{diag}([11\ m^2, 3.5\ (m/s)^2, 11\ m^2, 3.5\ (m/s)^2],
\end{aligned}$$

and using $\xi_R = 100$ m, $\eta_R = 100$ m. Be sure to label axes with appropriate units, and comment on the results. In particular, at what time steps are the vehicles in greatest danger of colliding? Also, what happens to the probability of collision as time increases?What explains this behavior? (Hint: Matlab's `mvncdf.m` command is useful here – read the documentation for this function to learn more about it).