ASEN 5044 Statistical Estimation for Dynamical Systems
Homework 8

1. ***LINEAR KF IMPLEMENTATION AND ANALYSIS***: Consider again the aircraft coordinated turning problem from Homework 7. Assuming the same model for the equations of motion and aircraft state vector $x = [\xi, \dot{\xi}, \eta, \dot{\eta}]$, do parts (a)-(c).

(a) Assume that the CT LTI dynamics for two aircraft A and B are now augmented to include process noise for accelerations due to directional wind disturbances modeled by vector AWGN processes $\tilde{w}_A(t), \tilde{w}_B(t) \in \mathbb{R}^2$, given by

$$\Gamma_A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E[\tilde{w}_A(t)] = 0,\ E[\tilde{w}_A(t)\tilde{w}_A^T(\tau)] = W \cdot \delta(t-\tau),$$

$$\Gamma_B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad E[\tilde{w}_B(t)] = 0,\ E[\tilde{w}_B(t)\tilde{w}_B^T(\tau)] = W \cdot \delta(t-\tau),$$

where the intensity covariance matrix W is the same for both A and B,

$$W = q_w \cdot \begin{bmatrix} 2 & 0.05 \\ 0.05 & 0.5 \end{bmatrix},$$

and has appropriate units for acceleration.

Specify the full DT LTI stochastic dynamics models for each aircraft, i.e. provide the corresponding (F_A, Q_A) and (F_B, Q_B) matrices for both aircraft (show your work). Assume that $q_w = 10$ (m/s)^2, $\Delta T = 0.5$ sec, $\Omega_A = 0.045$ rad/s and $\Omega_B = -0.045$ rad/s.

(b) In the following parts, you should fix the Matlab random number seed to 100 – in more recent versions of Matlab, this is accomplished using the `rng(100)` command. If not using Matlab, you should fix the random number seed for your programming environment to some constant initial value, to ensure reproduceable results between successive code refinements and runs.

(b.i) A ground tracking station monitors Aircraft A, and converts 3D range and bearing data into 2D 'pseudo-measurements' $y_A(k)$ with the following DT measurement model

$$y_A(k) = Hx_A(k) + v_A(k), \quad E[v_A(k)] = 0,\ E[v_A(k)v_A^T(j)] = R_A\delta(i,j),$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \ R_A = \begin{bmatrix} 20 & 0.05 \\ 0.05 & 20 \end{bmatrix},$$

where R_A has units of m^2.

The ground truth state history for A is contained in the file `hw8problem1data.mat` (in the array 'xasingle_truth' – the first column contains $x_A(0)$, and subsequent columns contain $x_A(k)$ for $k \geq 1$). Simulate a series of noisy measurements $y_A(k)$ for $k \geq 1$ up to 100 secs and store the results in an array. *Provide a plot of the components of your simulated $y_A(k)$ data vs. time for the first 20 seconds.*

(b.ii) Implement a Kalman filter to estimate aircraft A's state at each time step $k \geq 1$ of the simulated measurements you generated in part b.i. Initialize your filter estimate with the following state mean and covariance at time $k = 0$:

$$\mu_A(0) = [0\ m, 85\cos(\pi/4)\ m/s, 0\ m, -85\sin(\pi/4)\ m/s]^T$$
$$P_A(0) = 900 \cdot \text{diag}([10\ m^2, 2\ (m/s)^2, 10\ m^2, 2\ (m/s)^2]),$$

Provide plots for each component of the estimated state error vs. time, along with estimated 2σ error bounds. Comment on your results; in particular, is your KF output more certain about some states than others? If so, explain why.

(c) Consider now estimating the states of both aircraft A and B, whose true state histories are given in `hw8problem1data.mat` in the arrays 'xadouble_truth' and 'xbdouble_truth' (these contain $x_A(0)$ and $x_B(0)$ in their first columns, respectively, and subsequent columns contain $x_A(k)$ and $x_B(k)$ for $k \geq 1$). Assume that the initial state uncertainties for each aircraft at time $k = 0$ are

$$\mu_A(0) = [0\ m, 85\cos(\pi/4)\ m/s, 0\ m, -85\sin(\pi/4)\ m/s]^T$$
$$P_A(0) = 900 \cdot \text{diag}([10\ m^2, 2\ (m/s)^2, 10\ m^2, 2\ (m/s)^2]),$$
$$\mu_B(0) = [3200\ m, 85\cos(\pi/4)\ m/s, 3200\ m, -85\sin(\pi/4)\ m/s]^T$$
$$P_B(0) = 900 \cdot \text{diag}([11\ m^2, 4\ (m/s)^2, 11\ m^2, 4\ (m/s)^2]),$$

(c.i) Suppose the tracking station can only directly sense one aircraft at a time, and thus cannot sense B while it senses A. However, a transponder between A and B provides a noisy measurement $y_D(k)$ of the difference in their 2D positions, $r_A = [\zeta_A, \eta_A]^T$ and $r_B = [\zeta_B, \eta_B]^T$,

$$y_D(k) = r_A(k) - r_B(k) + v_D(k),$$
$$E[v_D(k)] = 0, \quad E[v_D(k)v_D^T(j)] = R_D\delta(i,j), \quad R_d = \begin{bmatrix} 10 & 0.15 \\ 0.15 & 10 \end{bmatrix},$$

where R_D has units of m^2. First, simulate a series of noisy transponder measurements $y_D(k)$ between both aircraft for $k \geq 1$, as well as a new set of ground measurements to aircraft A, $y'_A(k)$ (using the same noise statistics as before), and stack these on top of each other in a new data array for augmented measurements $y_S(k) = [y'_A(k), y_D(k)]^T$. Then, implement a new KF to estimate the joint augmented aircraft states $x_S(k) = [x_A(k), x_B(k)]^T$ at each time step $k \geq 1$. (**Hint:** you will first need to *carefully* define new F, Q, H, R, and $P(0)$ matrices for x_S to capture the combined state uncertainties and measurements – the `blkdiag` command will be useful here. **Be sure to explain how you got these matrices**). *Provide plots only of the position errors for each aircraft vs. time.*

(c.ii) Repeat part c.i if only the transponder measurements are now available, i.e. if $y_S(k) = y_D(k)$ for all time $k \geq 1$. Comment on your results – in particular, how is this different from the results obtained in part d.i? What explains this?

(c.iii) Explain what is so interesting about the structures of the covariance matrices produced by the KFs in c.i and c.ii, compared to the covariance matrices that would be produced for $x_S(k)$ under pure prediction updates (i.e. with dynamic propagation steps only and no measurement updates taking place in the KF whatsoever)