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THINK BIG WE DO

Maximum Likelihood Estimator For Object Localization and Parameter Estimation

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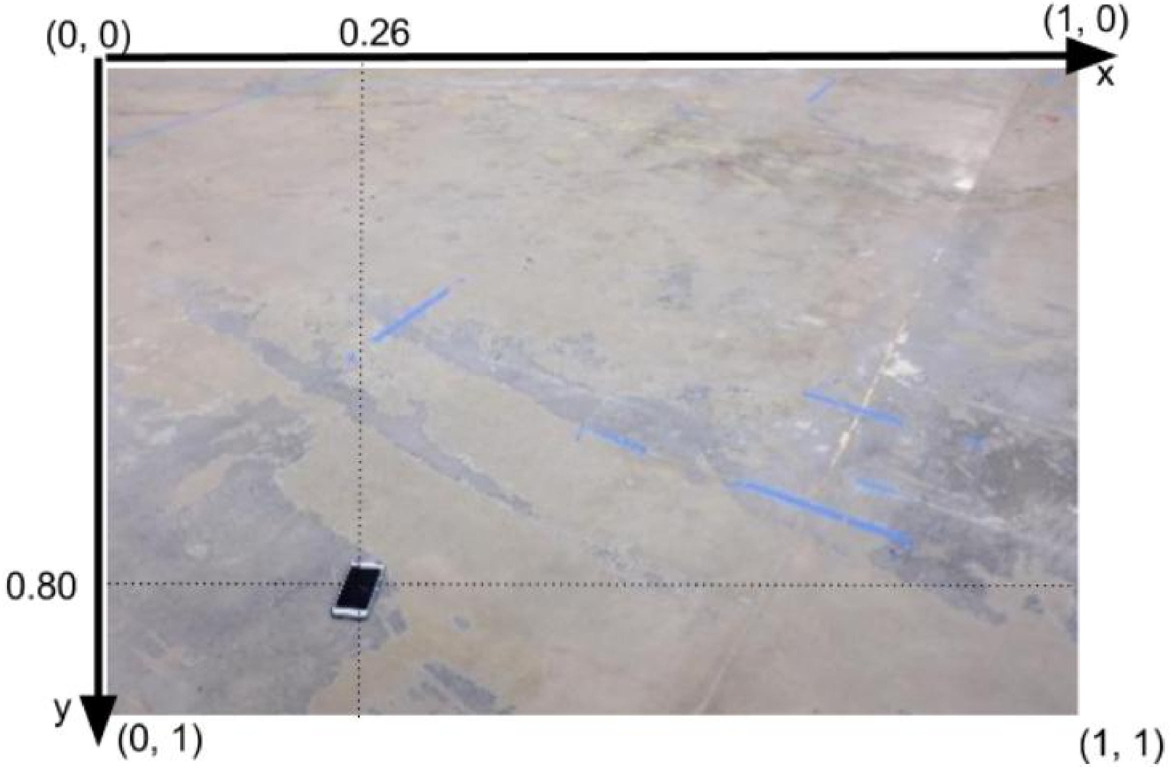
5/2/2018

Motivation

From a computer vision project：
Predict the center position of a smart phone

Machine Learning vs Estimation Theory ?

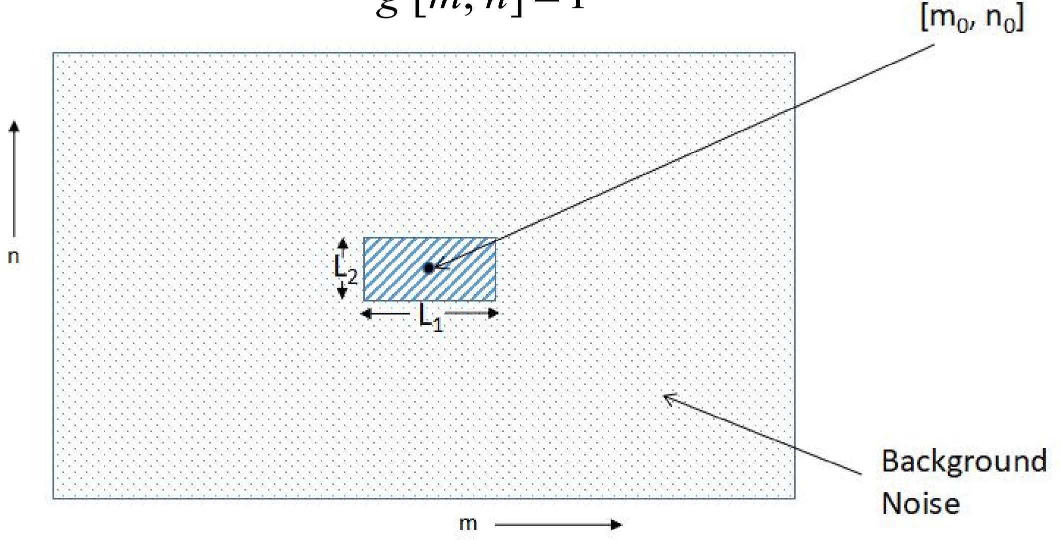


Task Description

Assume a gray-scale image with $M \times N$ pixels, $x[m, n]$, where $m = 0, 1, \ldots, M-1$ $n = 0, 1, \ldots, N-1$. An object of size is given by $L_1 \times L_2$, L_1, L_2 are known.

$$S[m, n] = AR_\theta(g[m - m_0, n - n_0])$$

$$g[m, n] = 1$$



A is an unknown amplitude, $[m_0, n_0]$ is an unknown center position

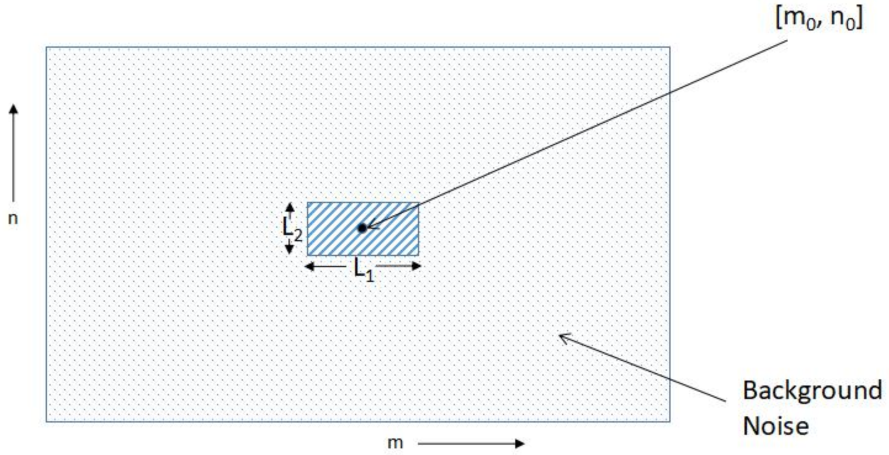
Task Description

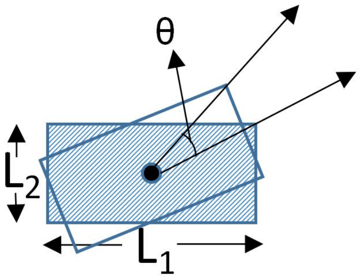
R_θ is a rotation operator.

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The background noise is given by: $\omega[m,n] = \mu + u[m,n]$. Where $u[m,n]$ is white Gaussian noise sample with variance σ^2. Assume μ and σ^2 are unknown.

$$x[m,n] = AR_\theta(g[m-m_0, n-n_0]) + \mu + u[m,n]$$





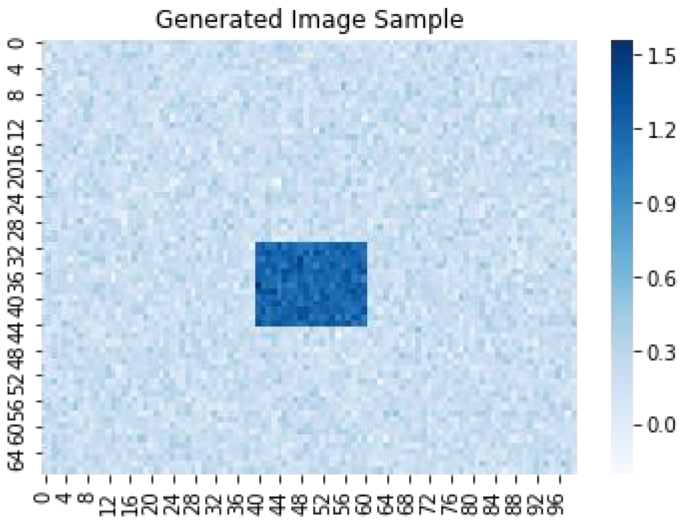
Sample Image Generation

No rotation:

Known： M=100, N=68, L_1=20, L_2=12

Need to estimate: μ=0.2, A=1, σ^2 =0.01, m_0=50, n_0=38

$$g[m-m_0, n-n_0] = \begin{cases} 1 & |m-m_0| < \frac{L_1}{2} \,\&\, |n-n_0| < \frac{L_2}{2} \\ 0 & otherwise \end{cases}$$



Case 1

No rotation:

Known: M=100, N=68, L_1=20, L_2=12, μ=0.2, A=1, $\sigma^2 = 0.01$

Need to estimate: m_0, n_0

$$P(x[m,n]) = \frac{1}{(2\pi\sigma^2)^{\frac{MN}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(x[m,n]-g[m-m_0,n-n_0]-\mu)^2}$$

According to MLE, we need to minimize:

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(x[m,n]-g[m-m_0,n-n_0]-\mu)^2$$

Grid Search ⇨ $\hat{m}_0, \ \hat{n}_0$

Real Center	(50, 38)
Estimated Center	(50, 38)

Case 2

No rotation:
Known: M=100, N=68, L_1=20, L_2=12, μ=0.2, A=1,
Unknown: m_0, n_0 , σ^2

$$\ln P(x[m,n]) = \ln \frac{1}{(2\pi\sigma^2)^{\frac{MN}{2}}} - \frac{1}{2\sigma^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n]-g[m-m_0,n-n_0]-\mu\right)^2$$

Grid Search ⇨ $\hat{m}_0,\ \hat{n}_0$ $\xrightarrow{\text{Minimize}}$ $\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n]-g[m-m_0,n-n_0]-\mu\right)^2$

$$\frac{\partial \ln P}{\partial \sigma^2} = \frac{(2\pi\sigma^2)^{\frac{MN}{2}}\left(-\frac{MN}{2}\right)}{(2\pi\sigma^2)^{\frac{MN}{2}+1}}\cdot 2\pi + \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n]-g[m-\hat{m}_0,n-\hat{n}_0]-\mu\right)^2}{2\sigma^2}$$

$$= -\frac{MN}{2\sigma^2} + \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n]-g[m-\hat{m}_0,n-\hat{n}_0]-\mu\right)^2}{2\sigma^2} = 0$$

Case 2

No rotation:

Known： M=100, N=68, L_1=20, L_2=12, μ=0.2, A=1,

Unknown: m_0, n_0 , σ^2

$$\hat{\sigma}^2 = \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n] - g[m-\hat{m}_0, n-\hat{n}_0] - \mu\right)^2}{MN}$$

The simulation results are:

	Center Position	σ^2
Real Value	(50, 38)	0.01
Estimated Value	(50, 38)	0.0099

Case 3

No rotation:
Known： M=100, N=68, L_1=20, L_2=12, μ=0.2,
Unknown: m_0, n_0 ,σ^2, A

$$\ln P(x[m,n]) = \ln \frac{1}{(2\pi\sigma^2)^{\frac{MN}{2}}} - \frac{1}{2\sigma^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n] - Ag[m-m_0, n-n_0] - \mu\right)^2$$

$$\frac{\partial \ln P}{\partial A} = \frac{1}{\sigma^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n] - Ag[m-m_0, n-n_0] - \mu\right)\cdot g[m-m_0, n-n_0]$$

$$= \frac{1}{\sigma^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left((x[m,n]-\mu)g[m-m_0, n-n_0] - Ag[m-m_0, n-n_0]g[m-m_0, n-n_0]\right)$$

$$= 0$$

$$A = \frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left((x[m,n]-\mu)\cdot g[m-m_0, n-n_0]\right)}{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(g[m-m_0, n-n_0]\cdot g[m-m_0, n-n_0]\right)}$$

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Case 3

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No rotation:

Known: M=100, N=68, L_1=20, L_2=12, μ=0.2,

Unknown: m_0, n_0, σ^2, A

$$A=\frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left((x[m,n]-\mu)\cdot g[m-m_0,n-n_0]\right)}{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(g[m-m_0,n-n_0]\cdot g[m-m_0,n-n_0]\right)} \rightarrow \hat{A}$$

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n]-Ag[m-m_0,n-n_0]-\mu\right)^2 \rightarrow$$

Grid Search

$$\hat{m}_0,\ \hat{n}_0$$

$$\hat{\sigma}^2=\frac{\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}\left(x[m,n]-\hat{A}g[m-\hat{m}_0,n-\hat{n}_0]-\mu\right)^2}{MN}$$

	Center	σ²	A
Real Value	(50, 38)	0.01	1
Estimated Value	(50, 38)	0.0099	1.002

Case 4

No rotation:
Known: $M=100$, $N=68$, $L_1=20$, $L_2=12$,
Unknown: m_0, n_0, σ^2, A, μ

$$P(x[m,n]) = \frac{1}{(2\pi\sigma^2)^{\frac{MN}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(x[m,n]-Ag[m-m_0,n-n_0]-\mu)^2}$$

We can write it in a linear model representation:

$$P(\mathbf{x},\Theta) = \frac{1}{(2\pi\sigma^2)^{\frac{MN}{2}}} e^{-\frac{1}{2\sigma^2}(\mathbf{x}-\mathbf{H}\Theta)^T(x-\mathbf{H}\Theta)}$$

Obviously , to get the MLE, we need to minimize :

$$(\mathbf{x}-\mathbf{H}\boldsymbol{\Theta})^T(\mathbf{x}-\mathbf{H}\boldsymbol{\Theta})$$

Case 4

No rotation:
Known: M=100, N=68, L_1=20, L_2=12,
Unknown: m_0, n_0, σ^2, A, μ

$$(\mathbf{x}-\mathbf{H}\boldsymbol{\Theta})^T(\mathbf{x}-\mathbf{H}\boldsymbol{\Theta})$$

Where we flatten 2 dimensional $\mathbf{x}[m,n]$ to a one dimensional vector,

$$\mathbf{x}=[\,x[0,0],x[1,0],\ldots,x[m,0],x[0,1],\ldots,x[m,1],\ldots,x[m,n]\,]^T$$
$$\mathbf{g}=[\,g[0,0],g[1,0],\ldots,g[m,0],g[0,1],\ldots,g[m,1],\ldots,g[m,n]\,]^T$$

$$\mathbf{H}=[\mathbf{g},\mathbf{1}] \qquad \Theta=[A,\mu]^T$$

where $\mathbf{1}$ represent a vector of all ones with a length of $M\times N$:

According to the MLE solution to the general linear model:

$$\Theta=(\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$$

Case 4

No rotation:
Known: M=100, N=68, L_1=20, L_2=12,
Unknown: m_0, n_0, σ^2, A, μ

$$\Theta = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x} \longleftarrow \mathbf{H}$$

$$(\mathbf{x} - \mathbf{H}\Theta)^T(\mathbf{x} - \mathbf{H}\Theta) \Rightarrow \boxed{\text{Grid Search}} \Rightarrow \hat{m}_0,\ \hat{n}_0$$

$$\hat{\sigma}^2 = \frac{(\mathbf{x} - \hat{\mathbf{H}}\hat{\Theta})^T(\mathbf{x} - \hat{\mathbf{H}}\hat{\Theta})}{MN}$$

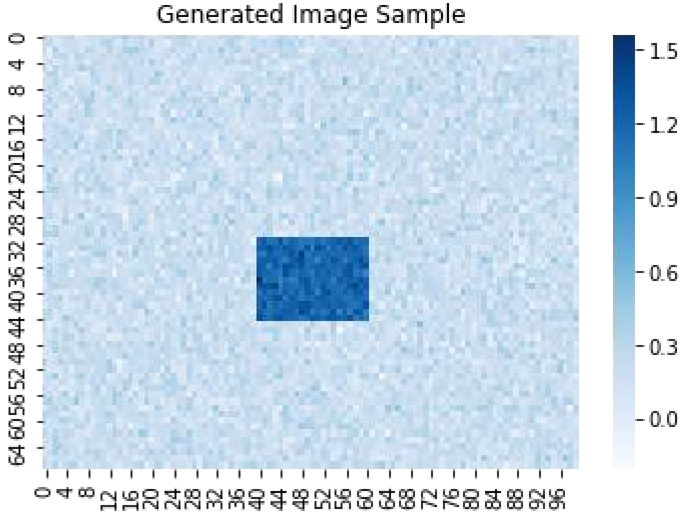
The simulation results are:

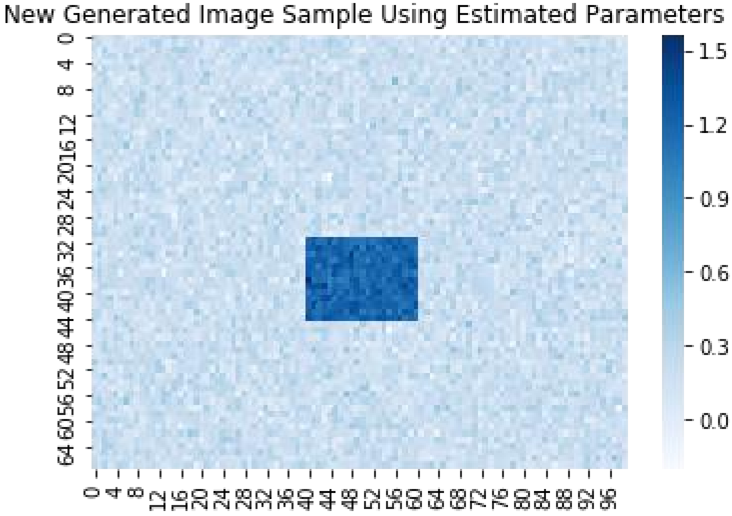
	Center	σ^2	A	μ
Real	(50, 38)	0.01	1	0.2
Estimated	(50, 38)	0.0099	1.003	0.1989

Case 4

No rotation:
Known: $M=100$, $N=68$, $L_1=20$, $L_2=12$,
Unknown: m_0, n_0, σ^2, A, μ

Compare with original image:





	Center	σ^2	A	μ
Real	(50, 38)	0.01	1	0.2
Estimated	(50, 38)	0.0099	1.003	0.1989

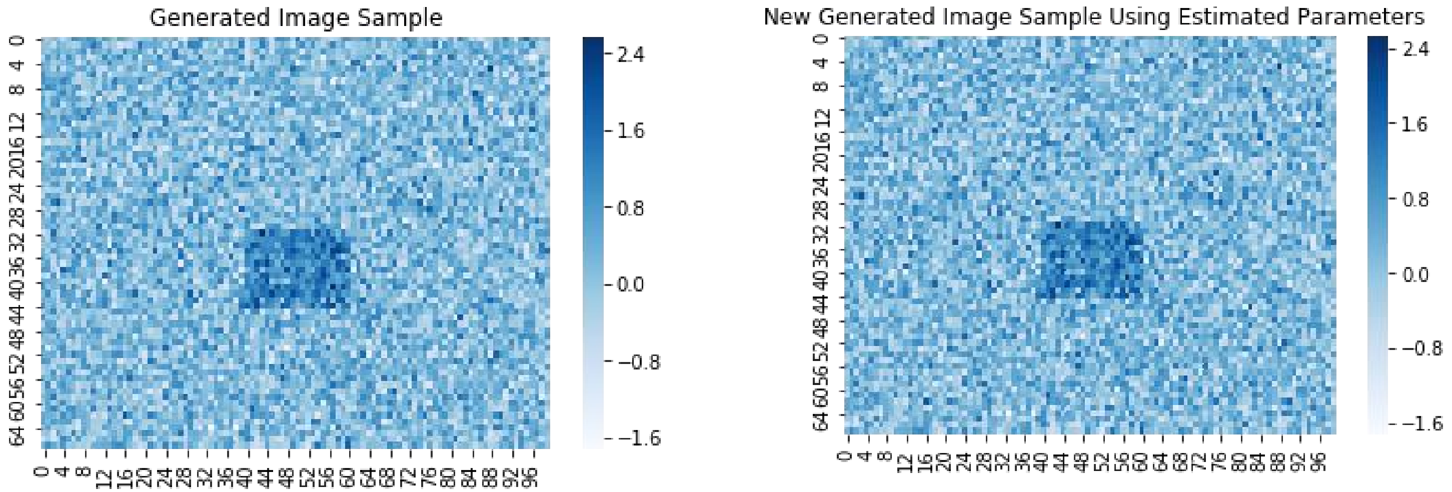
Case 4

No rotation:

Known: $M=100$, $N=68$, $L_1=20$, $L_2=12$,

Unknown: m_0, n_0, $\sigma^2 = 0.3$, A, μ

Compare with original image:

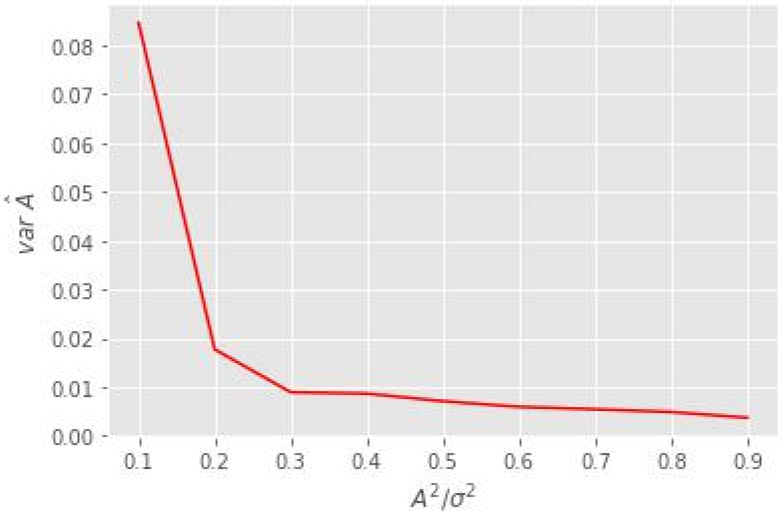


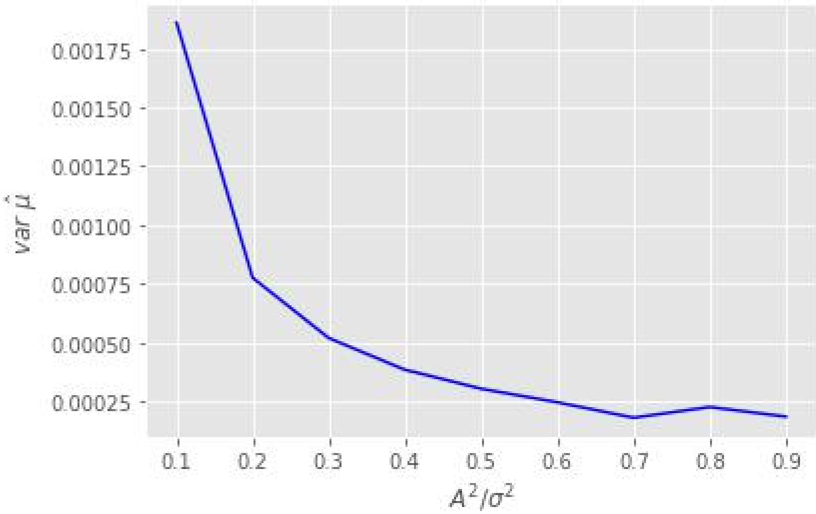
	Center	σ^2	A	μ
Real	(50, 38)	0.3	1	0.2
Estimated	(50, 38)	0.2977	0.9680	0.1960

Case 4

No rotation:
Known： M=100, N=68, L_1=20, L_2=12,
Unknown: m_0, n_0 ,σ^2 , A, μ

Change σ^2 , show the variance curve of $\hat{A}$ and $\hat{\mu}$:

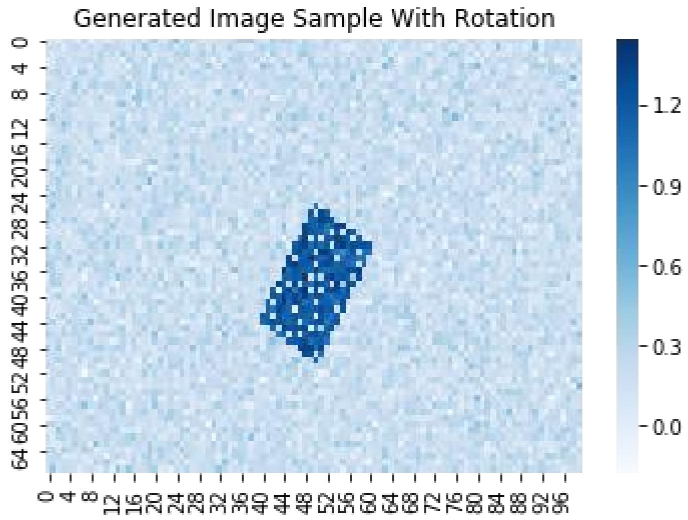




Case 5

Rotation:
Known: $M=100$, $N=68$, $L_1=20$, $L_2=12$,
Unknown: m_0, n_0, σ^2, A, μ, θ

Rotation: anti-closewise 60 degree



Case 5

Rotation:
Known： M=100, N=68, L_1=20, L_2=12,
Unknown: m_0, n_0 ,σ^2, A, μ, θ

$$P(x[m,n]) = \frac{1}{(2\pi\sigma^2)^{\frac{MN}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(x[m,n]-AR_\theta(g[m-m_0,n-n_0])-\mu)^2}$$

To get the MLE, we need to minimize :

$$\sum_{m=0}^{M-1}\sum_{n=0}^{N-1}(x[m,n]-AR_\theta(g[m-m0,n-n0])-\mu)^2$$

Here we also transform it to a general linear model representation:

$$(\mathbf{x}-\mathbf{H}\boldsymbol{\Theta})^T(\mathbf{x}-\mathbf{H}\boldsymbol{\Theta})$$

Case 5

Rotation:
Known： M=100, N=68, L_1=20, L_2=12,
Unknown: m_0, n_0 ,σ^2, A, μ, θ

$$(\mathbf{x} - \mathbf{H}\boldsymbol{\Theta})^T(\mathbf{x} - \mathbf{H}\boldsymbol{\Theta})$$

where :

$$\mathbf{x} = [\, x[0,0], x[1,0], \ldots, x[m,0], x[0,1], \ldots, x[m,1], \ldots, x[m,n] \,]^T$$

$$\mathbf{R}_\theta \mathbf{g} = [\, R_\theta(g[0,0]), R_\theta(g[1,0]), \ldots, R_\theta(g[m,0]), R_\theta(g[0,1]), \ldots, R_\theta(g[m,1]), \ldots, R_\theta(g[m,n])]^T$$

$$\mathbf{H} = [\mathbf{R}_\theta \mathbf{g}, \mathbf{1}] \qquad \Theta = [A, \mu]^T$$

According to the MLE solution to the general linear model:

$$\Theta = (\mathbf{H}^T \mathbf{H})^{-1} \mathbf{H}^T \mathbf{x}$$

Case 5

Rotation:

Known: M=100, N=68, L_1=20, L_2=12,

Unknown: m_0, n_0, σ^2, A, μ, θ

$$\Theta = (\mathbf{H}^T\mathbf{H})^{-1}\mathbf{H}^T\mathbf{x}$$

$$\mathbf{H}$$

$$(\mathbf{x}-\mathbf{H}\Theta)^T(\mathbf{x}-\mathbf{H}\Theta)$$

Grid Search

$$\hat{m}_0,\ \hat{n}_0$$

$$\hat{\sigma}^2 = \frac{(\mathbf{x}-\hat{\mathbf{H}}\hat{\Theta})^T(\mathbf{x}-\hat{\mathbf{H}}\hat{\Theta})}{MN}$$

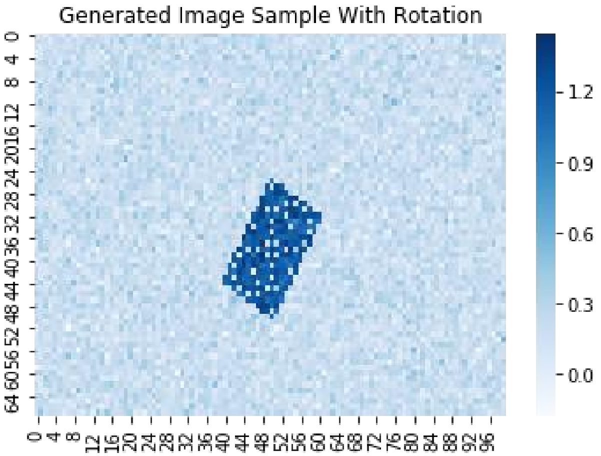
$$\hat{\theta}$$

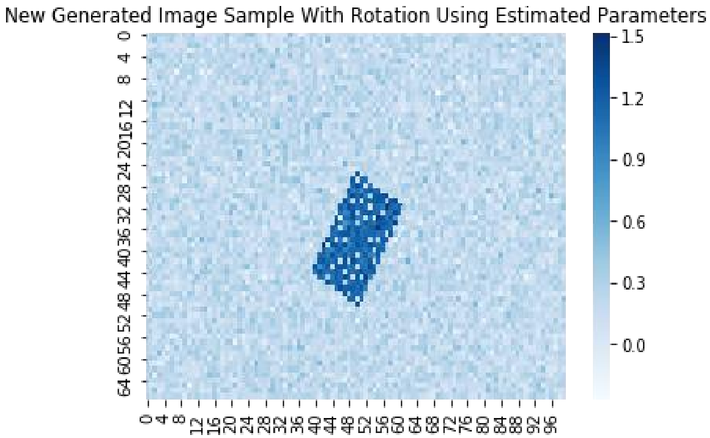
Grid Search

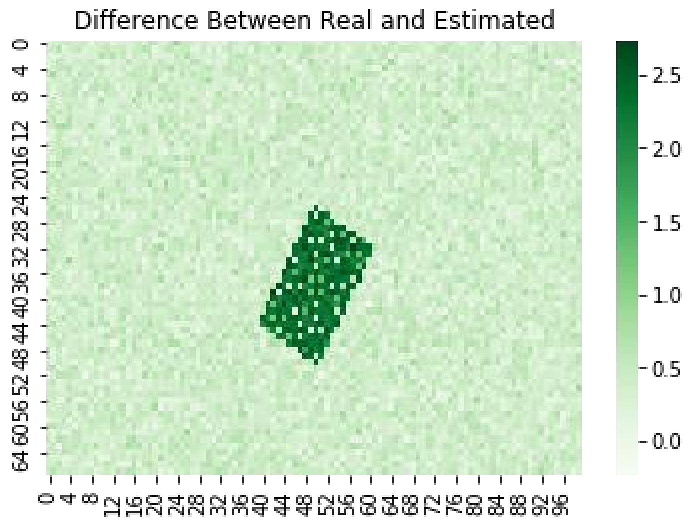
	(m_0, n_0)	σ^2	A	μ	θ
Real	(50, 38)	0.01	1	0.2	60°
Estimated	(50, 38)	0.0099	0.9452	0.1987	59.4°

Case 5

Compare with original image:







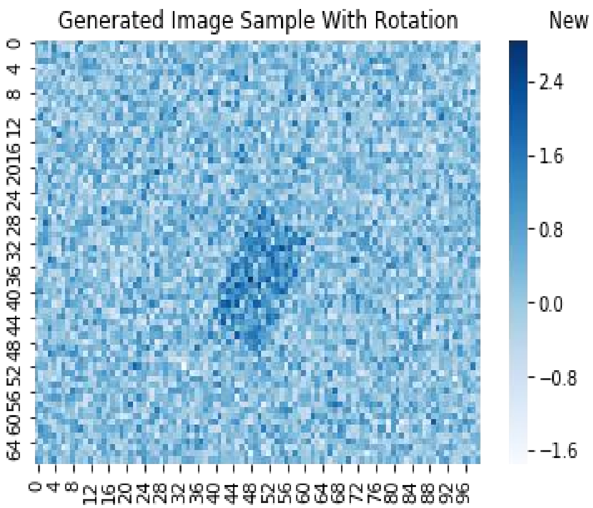
Case 5

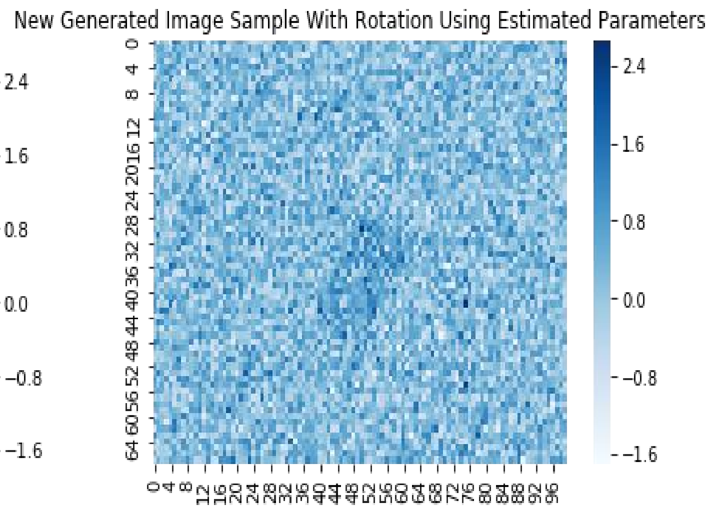
Rotation:

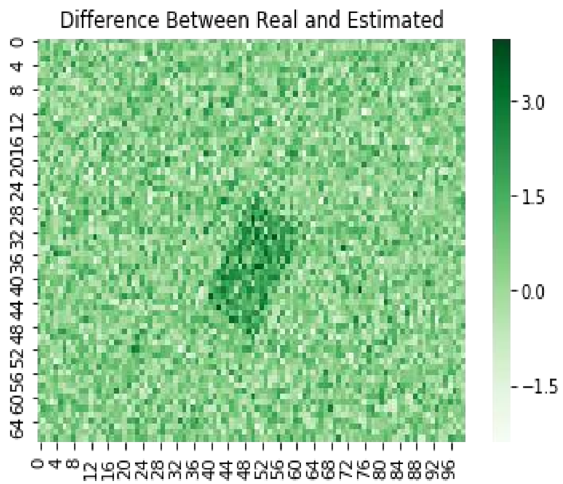
Known: M=100, N=68, L_1=20, L_2=12,

Unknown: m_0, n_0, σ^2, A, μ, θ

Let $\sigma^2 = 0.3$:





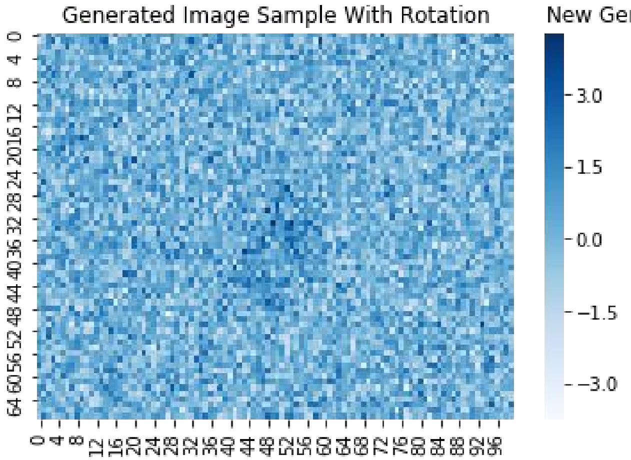


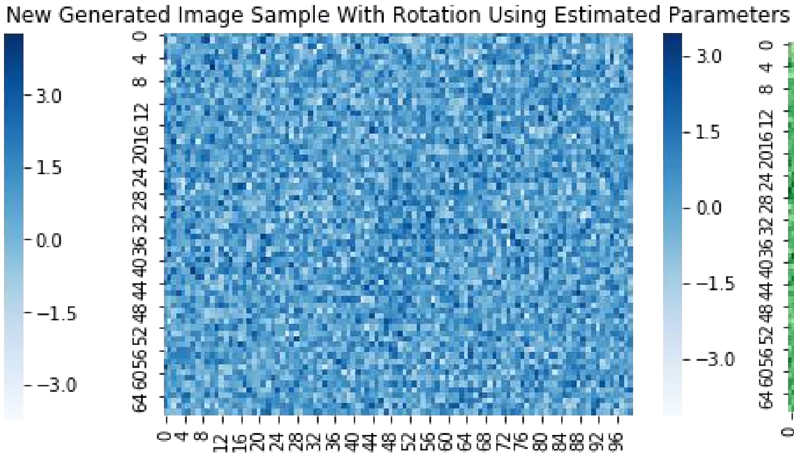
	(m_0, n_0)	σ^2	A	μ	θ
Real	(50, 38)	0.3	1	0.2	60°
Estimated	(50, 38)	0.29	0.476	0.1973	59.4°

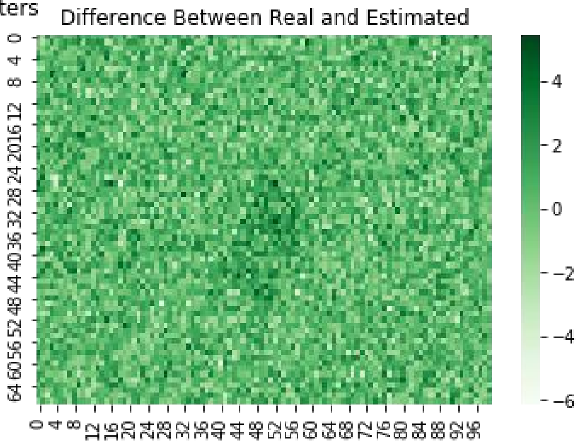
Case 5

Rotation:
Known: $M=100$, $N=68$, $L_1=20$, $L_2=12$,
Unknown: m_0, n_0, σ^2, A, μ, θ

Let $\sigma^2 = 1$:







	(m_0, n_0)	σ^2	A	μ	θ
Real	(50, 38)	1	1	0.2	60°
Estimated	(50, 38)	0.99	0.517	0.1735	61.28°

Compare with Machine Learning:

Advantages:

1. Give a clear picture of the process of estimation
2. Free from training, perform well on small amount of samples
3. Better accuracy
4. Provide more information of the image

Disadvantages:

1. Need strong assumptions on the background noise model
2. Prior knowledge about the object

Thank you!
Questions and suggestions?