Software Project

EE25BTECH11007- Aniket

(1) Summary of Strang's SVD video

Let $A \in \mathbb{R}^{m\times n}$. The singular value decomposition (SVD) factors any matrix into

$$A = U\Sigma V^\top, \qquad U \in \mathbb{R}^{m\times m},\ V \in \mathbb{R}^{n\times n} \text{ orthonormal},\ \Sigma = \operatorname{diag}(\sigma_1 \geq \cdots \geq \sigma_r > 0),$$

where $r = \operatorname{rank}(A)$. Geometrically, V rotates (or reflects) coordinates to the directions of action of A, Σ stretches by the nonnegative singular values, and U rotates to the output axes. Strang emphasizes:

- $A^\top A$ is symmetric positive semidefinite. Its eigenpairs satisfy $A^\top A\, v_i = \sigma_i^2 v_i$; thus $\sigma_i = \sqrt{\lambda_i(A^\top A)}$ and $u_i = \frac{1}{\sigma_i} A v_i$.
- Formula Given

$$A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^\top$$

(2) One-sided Jacobi SVD: math and pseudocode

GENERAL IDEA: Orthogonalize the columns of A by applying Givens rotationson the right so that $A^\top A$ becomes (nearly) diagonal **without forming it**. For a column pair (p, q), form their Gram matrix

$$G = \begin{bmatrix} \alpha & \gamma \\ \gamma & \beta \end{bmatrix}, \quad \alpha = \langle a_p, a_p\rangle,\ \beta = \langle a_q, a_q\rangle,\ \gamma = \langle a_p, a_q\rangle.$$

Choose a plane rotation

$$R = \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \qquad t = \frac{\beta-\alpha}{2\gamma}, \quad \tau = \operatorname{sign}(t)/(|t| + \sqrt{1+t^2}), \quad c = \frac{1}{\sqrt{1+\tau^2}},\ s = c\,\tau,$$

that diagonalizes $R^\top G R$. Apply this **on the right** to the two columns:

$$\begin{bmatrix} a_p & a_q \end{bmatrix} \leftarrow \begin{bmatrix} a_p & a_q \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}, \qquad V \leftarrow V \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \text{ on columns } (p, q).$$

After enough sweeps over all pairs (p, q), the off-diagonal entries of $A^\top A$ are tiny, so the columns $\{a_j\}$ are mutually orthogonal. Then

$$\sigma_j = \|a_j\|_2, \qquad u_j = \frac{a_j}{\sigma_j}, \qquad \text{and the columns of } V \text{ are the } v_j.$$

Constructing U, Σ, V from one–sided Jacobi and ordering

After the final sweep, the method has applied a right orthogonal transform V so that

$$A_{\text{final}} \;=\; A\,V \;=\; [\,a_1'\ a_2'\ \ldots\ a_n'\,],$$

whose columns are (numerically) mutually orthogonal.

a) Column–norms–squared vector and ordering.: Compute

$$d_j = \|a'_j\|_2^2, \qquad \mathbf{d} = [d_1, \dots, d_n].$$

Let π be a permutation that sorts $\mathbf{d}$ in **descending** order: $d_{\pi(1)} \geq d_{\pi(2)} \geq \cdots \geq d_{\pi(n)}$, and let Π be the associated permutation matrix. Permute the columns of A_{final} and V simultaneously:

$$\widetilde{A} = A_{\text{final}}\Pi, \qquad \widetilde{V} = V\Pi.$$

b) Forming Σ, U, and (optionally) truncation.: Define singular values by

$$\sigma_j = \|\widetilde{a}_j\|_2 = \sqrt{d_{\pi(j)}} \quad (j = 1, \dots, n),$$

drop any j with $\sigma_j \approx 0$, and set

$$\Sigma = \operatorname{diag}(\sigma_1, \dots, \sigma_r), \qquad U = [\,\widetilde{a}_1/\sigma_1 \ \cdots \ \widetilde{a}_r/\sigma_r\,], \qquad V = \widetilde{V}.$$

Because the columns of $\widetilde{A}$ are orthogonal, $U^\top U = I_r$. The SVD is

$$A = U\Sigma V^\top.$$

For the truncated SVD, keep only the first k indices:

$$U_k = [\,\widetilde{a}_1/\sigma_1 \ \cdots \ \widetilde{a}_k/\sigma_k\,], \quad \Sigma_k = \operatorname{diag}(\sigma_1, \dots, \sigma_k), \quad V_k = \widetilde{V}(:, 1\!:\!k).$$

c) Pseudocode (one-sided Jacobi, truncated to top k).:

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1) Input: $A \in \mathbb{R}^{m\times n}$, target $k$, tolerance tol, max sweeps S.
2) Set $V \leftarrow I_n$.
3) For $s = 1, \dots, S$ (a “sweep”):
  a) For all pairs $1 \leq p < q \leq n$:
    i) $\alpha \leftarrow \langle a_p, a_p\rangle$, $\beta \leftarrow \langle a_q, a_q\rangle$, $\gamma \leftarrow \langle a_p, a_q\rangle$.
    ii) If $|\gamma|$ is small relative to $\sqrt{\alpha\beta}$, continue.
    iii) Compute $(c, s)$ using the formulas above.
    iv) Update the two columns of $A$: $(a_p, a_q) \leftarrow (a_p, a_q)\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$.
    v) Accumulate in $V$: $(v_p, v_q) \leftarrow (v_p, v_q)\begin{bmatrix} c & s \\ -s & c \end{bmatrix}$.
4) If convergence test satisfied, break.
5) Compute $\sigma_j = \|a_j\|_2$. Sort indices by decreasing $\sigma_j$.
6) Output $U_k = [a_{j_1}/\sigma_{j_1} \ \cdots \ a_{j_k}/\sigma_{j_k}]$, $\Sigma_k = \operatorname{diag}(\sigma_{j_1}, \dots, \sigma_{j_k})$, $V_k = [v_{j_1} \ \cdots \ v_{j_k}]$.
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(3) Algorithm comparison and choice

Golub–Reinsch (bidiagonalization + QR). Householder reductions drive A to bidiagonal form in $O(mn^2)$ (for $m \geq n$), then a specialized QR iteration diagonalizes it in $O(n^3)$. This is the standard high-performance approach (LAPACK), very fast and robust, but the implementation from scratch is longer and uses Level-2/3 BLAS-style kernels.

Randomized SVD (RSVD). Projects A to a low-dimensional subspace via random sketching, then computes a small SVD. Excellent for very large, sparse, or data-streaming settings; accuracy depends on oversampling/power iters. Requires more moving parts (random matrices, QR/LU) and careful parameter choices.

Power/Block Orthogonal Iteration. Iteratively applies A and $A^\top$ with reorthogonalization (e.g., MGS). Simple and effective for leading k, but convergence slows when singular values are clustered; maintaining orthogonality needs care.

One-sided Jacobi (used here).

- **Pros:** conceptually clear (implicitly diagonalizes $A^\top A$), extremely good orthogonality of U, V, embarrassingly parallel over column pairs, and easy to truncate—after convergence, just sort column norms. Implementation is compact and uses only dot products and plane rotations (good for a from-scratch C codebase with no LAPACK/BLAS).
- **Cons:** higher operation count per level of accuracy than Golub–Reinsch; per-sweep cost $O(mn^2)$ with nontrivial constants, so full SVD on large square images is slower.

Why this choice. For this assignment's goals (clarity, numerical stability, minimal dependencies, and easy top-k reconstruction for images), one-sided Jacobi is a great fit. It delivers highly orthogonal singular vectors and accurate singular values; truncation to different k just selects the largest column norms at the end, and the code stays short and dependency-free.

(4) Discussion of trade–offs and reflections on implementation choice

What I optimized for

My priorities were: (i) **orthogonality and numerical clarity** (easy to explain and verify), (ii) **few dependencies** (no BLAS/LAPACK), and (iii) **simple truncation to top-k** for image compression. The one–sided Jacobi method aligns well with all three.

Accuracy vs. speed

- **Accuracy/orthogonality.** Jacobi implicitly diagonalizes $A^\top A$ by right–rotations, yielding very small off–diagonals and highly orthogonal U, V in practice. This is excellent for downstream tasks (e.g., stability of $U_k \Sigma_k V_k^\top$ and error analysis).
- **Runtime.** A sweep processes all $\binom{n}{2}$ column pairs and touches m rows $\Rightarrow$ about $O(mn^2)$ work per sweep (for $m \geq n$). Convergence usually needs multiple sweeps. For square images, this is slower than bidiagonalization + QR (Golub–Reinsch), which attains near–optimal constants in tuned libraries.
- **When it shines.** Small–to–medium n (typical lab images), when code simplicity and robustness matter more than peak speed; when exact orthogonality is valued (teaching, diagnostics).

Memory, implementation complexity, and portability

- **Memory.** I store A and accumulate V; U is formed at the end by normalizing columns of A. Peak memory is $O(mn + n^2)$ (if all of V is kept). For **truncated** output, I can retain only the k most energetic columns after sorting, reducing storage to $k(m+n+1)$ numbers.

- **Complexity of code.** The core is dot products and 2×2 Givens rotations, so the implementation is compact, branch–light, and dependency–free. This also makes it easy to reason about correctness and add instrumentation (off–diagonal norms, sweep counters).
- **Portability.** Uses only standard C and `math.h`; no platform–specific intrinsics. That meets the assignment's "barebones" requirement.

Convergence control and tuning

- **Stopping rules.** I use a relative orthogonality test

$$\max_{p<q} \frac{|\langle a_p, a_q \rangle|}{\|a_p\|_2 \, \|a_q\|_2} \le \text{tol},$$

  or a cap on the number of sweeps. Tight tolerances improve orthogonality but increase runtime.
- **Pair ordering.** I used cyclic (p, q) pairs. Greedy/adaptive schedules (pick the largest $|\langle a_p, a_q \rangle|$) can reduce sweeps but add overhead.
- **Scaling.** Column–wise scaling (or simple pre–normalization) can help when columns have very different norms, but I avoided extra passes to keep code minimal.

Numerical stability

- **Pros.** Jacobi never forms $A^\top A$ explicitly, avoiding the squaring of condition numbers that can hurt eigen–based methods. Givens rotations are norm–preserving, which limits error growth and preserves column energies.
- **Cons.** Very clustered singular values can slow convergence (more sweeps). In such cases, Golub–Reinsch or block power methods with robust reorthogonalization may reach the same accuracy faster.

When I would choose a different method

- **Large square/tall problems with strict time budgets:** Prefer Golub–Reinsch (LAPACK) for full SVD, or **randomized SVD** / block power iterations for top-k only.
- **Very large sparse matrices / streaming data:** Randomized methods (with power iterations) are usually superior in wall time and memory.

(4) Reconstructed Image

1)EINSTEIN
2) GLOBE
3) GREYSCALE

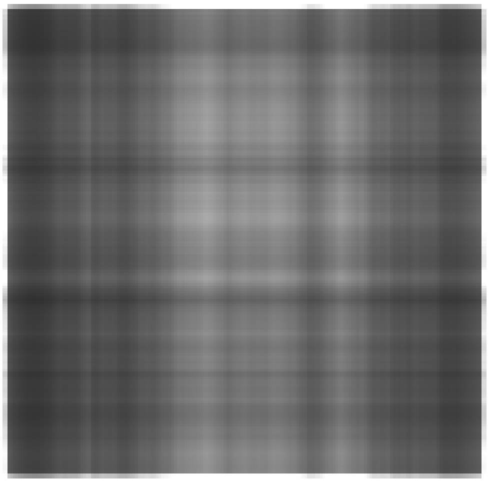


Fig. 1: Caption

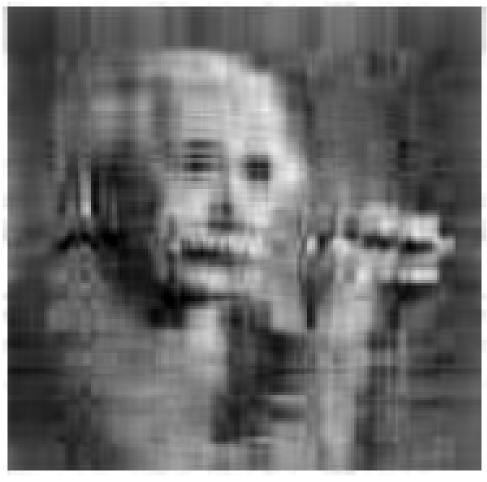


Fig. 2: Caption

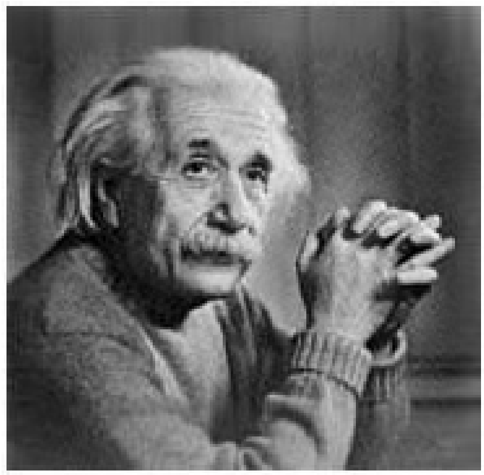
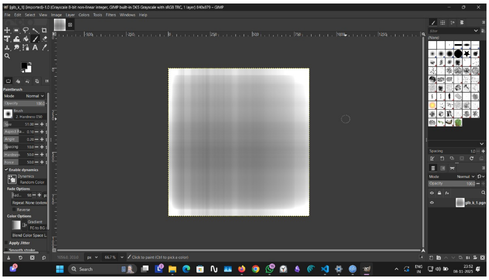
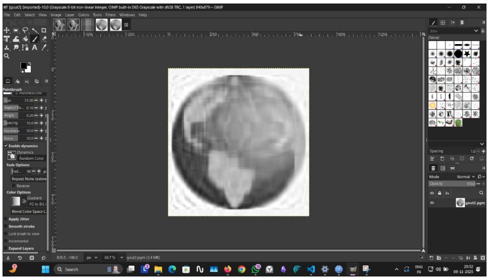


Fig. 3: Caption



[glb_k_1] (imported)-1.0 (Grayscale 8-bit non-linear integer, GIMP built-in D65 Grayscale with sRGB TRC, 1 layer) 840x879 – GIMP

Fig. 4: Caption



[gout2] (imported)-10.0 (Grayscale 8-bit non-linear integer, GIMP built-in D65 Grayscale with sRGB TRC, 1 layer) 840x879 – GIMP

Fig. 5: Caption

Fig. 6: Caption

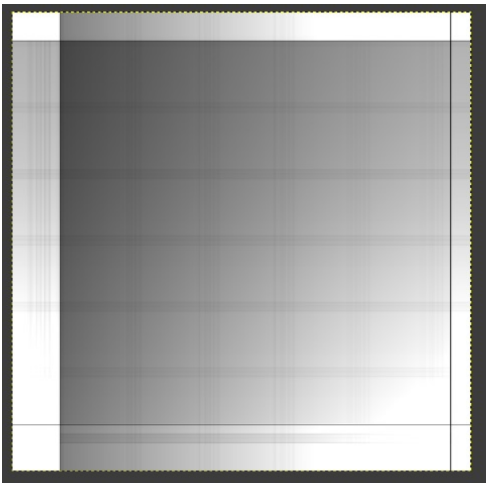


Fig. 7: Caption

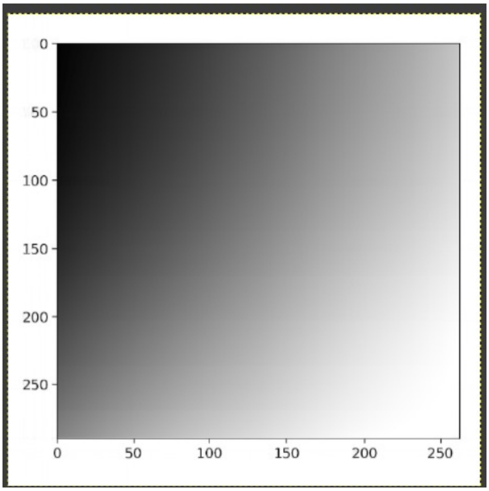


Fig. 8: Caption

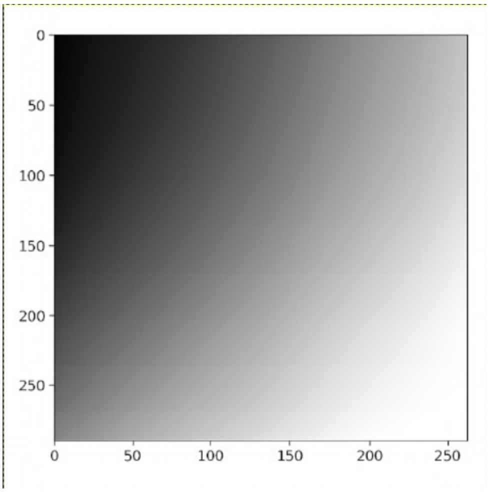


Fig. 9: Caption