Optimal Transport-based Coverage Control (OTCC) for Swarm Robot Systems

Introduction to Swarm Robot Systems and Coverage Control

- **Swarm robot systems** consist of many mobile robots that work cooperatively to perform given tasks. They are notable for their strong **environmental adaptability** and **high fault tolerance** compared to single-robot systems.
- A fundamental task for these systems is **coverage control**. This involves robots autonomously moving and repositioning themselves to approximate a given spatial distribution.
- Applications for coverage control range from the **optimal placement of sensor networks** to rescue operations during disasters.
- Various methods have been proposed for coverage tasks, including potential-function-based control, probability-based control, and broadcast-based control.

Voronoi Tessellation-based Coverage Control (VTCC)

- Among existing methods, the **Voronoi tessellation-based coverage control (VTCC)** method, proposed by Cortes et al., is a **seminal and widely used** approach.
- In VTCC, the coverage area is **divided into Voronoi regions**, with each region assigned to a robot.
- The robot then moves towards the **center of gravity** of its assigned Voronoi region.
- The cost function for the entire robot swarm decreases over time, ensuring that robots are eventually **appropriately scattered** throughout the coverage area.
- VTCC is commonly used due to its **mathematical guarantee of stability**, along with the simplicity and scalability of its algorithm.

Optimal Transport Problem

- Coverage control can be interpreted as the problem of **transporting a given discrete distribution to approximate a target continuous distribution**. This is known as the **optimal transport problem**.
- The optimal transport problem's mathematical properties and numerical solutions have been extensively investigated.

- Recent applications of optimal transport extend to areas like **machine learning, image processing, and natural language processing**.
- A key concept in optimal transport is the **Kantorovich problem**, which seeks a simultaneous probability density function $p(x, y)$ to minimize a cost function $C_K(p) = \int_{R^d \times R^d} \frac{1}{2}\|x-y\|^2 p(x, y)\, dx\, dy$. The solution to this problem is denoted as $W(\rho_0, \rho_T)$, called the **Wasserstein metric**, which measures the distance between two distributions.
- The **Kantorovich dual problem** involves maximizing a cost function $C_{KD}(\phi, \psi) = \int_{R^d} \phi(x)\rho_0(x)\, dx + \int_{R^d} \psi(y)\rho_T(y)\, dy$, subject to a condition $\phi(x) + \psi(y) \leq \frac{1}{2}\|x - y\|^2$.
- **Strong duality** holds between the Kantorovich problem and its dual, meaning $C_K(p^*) = C_{KD}(\phi^*, \psi^*)$. This allows replacing the problem of finding p with finding ϕ and ψ.

Proposed Optimal Transport-based Coverage Control (OTCC) Method

- The study formulates coverage control as an **optimal transport problem** to propose a novel technique called the **optimal transport-based coverage control (OTCC) method**.
- Unlike existing optimal transport-based multi-agent control methods, OTCC considers **gradient flows for the cost function of the optimal transport problem**.
- OTCC aims to minimize the Wasserstein distance $W(\rho(x, t), \rho_T(x))$ between the robot distribution $\rho(x, t) = \frac{1}{n}\sum_i \delta(x - x_i(t))$ and the target distribution $\rho_T(x)$.
- This minimization problem is transformed into a **min-max problem**: $\min_x \max_\phi F(x, \phi)$.
- The OTCC method defines the robot dynamics based on two control laws:
  - Robot velocity: $u_i(t) = -k(x_i(t) - \tilde{b}_i(t))$.
  - Evolution of ϕ: $\dot{\phi}_i(t) = k'(\frac{1}{n} - \tilde{a}_i(t))$.
  - Here, k, k' are feedback gains, $\tilde{a}_i(t) = \int_{V_{\phi_i}(x)} \rho_T(y)\, dy$, and $\tilde{b}_i(t) = \frac{1}{\tilde{a}_i(t)} \int_{V_{\phi_i}(x)} y\rho_T(y)\, dy$.
- The regions $V_{\phi_i}(x)$ are called **Laguerre regions**. These regions are defined by $y \in R^d$ such that $\frac{1}{2}\|x_i - y\|^2 - \phi_i \leq \frac{1}{2}\|x_j - y\|^2 - \phi_j$ for all $j \neq i$.
- The key contributions of the OTCC formulation are:
  - The **VTCC cost function is shown to be a special case** of the Kantorovich dual problem's cost function.

  - A **new control law is derived as gradient flows** of the Kantorovich dual problem's cost function.
  - A **sufficient condition for Lyapunov stability** is provided.
  - **Numerical analysis** demonstrates that OTCC reaches a closer state to the global optimum than VTCC.

Comparison and Advantages of OTCC over VTCC

- The VTCC method is shown to be a **special case of the proposed OTCC**.
- When the variable ϕ in OTCC is set to $\phi(t) \equiv 0$, the OTCC control laws (Eqs. 17, 18) align with the VTCC control laws (Eqs. 30, 31, 32), and their respective cost functions (Eqs. 12 and 28) coincide.
- This indicates that VTCC is a gradient flow that realizes transport, but it is **not optimal from the aspect of optimal transport** because its cost function is not maximized for ϕ at each time.
- The proposed OTCC overcomes this limitation, leading to **better control performance**.
- **Laguerre regions versus Voronoi regions:**
  - In both, the boundary between regions of neighboring robots i and j is perpendicular to the line connecting them.
  - In a Voronoi region, this boundary is a **bisector**.
  - In a Laguerre region, if ϕ is not zero, the boundary is **not a bisector**. If ϕ_i is larger than ϕ_j, the boundary shifts towards robot j, allowing robot i to acquire a larger area.
- **Improved performance and escaping local optima:**
  - The higher performance of OTCC stems from its **stricter equilibrium condition** compared to VTCC, making robots less likely to be trapped in a stationary point.
  - In VTCC, equilibrium is met when each robot is at its region's center of gravity ($x_i = b_i$). Once this is satisfied, robots stop moving.
  - OTCC adds an **additional condition**: the weighted area of each region must be equal ($\tilde{a}_i = 1/n$ in Eq. 33). Robots continue to move until this condition is met, even if the former condition is satisfied. This prevents robots from getting trapped at a stationary point prematurely.
- **Numerical Experiments:**

- In a one-dimensional case with 40 robots and a normal distribution target, OTCC **reproduces the target distribution more closely** over time than VTCC. VTCC left many robots in a suboptimal region, and the mean value of the distribution did not approach the target center.
- Quantitatively, the steady-state cost function value for VTCC was 1.36×10^{-3}, while for OTCC, it was 1.08×10^{-3}, indicating **better control performance** for OTCC.
- In a two-dimensional case with 25 robots targeting a bimodal distribution, OTCC better reproduced the target shape, with more robots moving to both distribution centers compared to VTCC.
- The cost function value in the steady state for VTCC was 9.70×10^{-1}, while for OTCC it was 8.84×10^{-1}, again showing **better reproduction** in OTCC.

Stability Analysis of OTCC

- A **Lyapunov stability analysis** of the controlled system using OTCC was performed.
- The system equilibrium (x^*, ϕ^*) for OTCC is characterized by $x_i^* = \tilde{b}_i$ and $\phi_i^* \in \{\phi \mid \tilde{a}_i = 1/n\}$.
- **Theorem 1** states that the equilibrium point (x^*, ϕ^*) is **Lyapunov stable** if a specific matrix $H(x, \phi^*)$ (defined in Eq. 35) is **positive definite** in a neighborhood of x^* (Eq. 34).
- This stability proof relies on showing that the function F (Eq. 12) is **convex-concave** around the equilibrium (x^*, ϕ^*) and that (x^*, ϕ^*) is a **saddle point** of F.
- **Theorem 2** provides a more specific condition for Lyapunov stability in the **one-dimensional case** ($d = 1$). It states that stability is guaranteed if the inequality $h_i(x^*, \phi^*) < 1$ (Eq. 40) holds for all robots i, where h_i is a function involving robot positions and ϕ values (Eq. 41).
- Numerical analysis for the one-dimensional case showed that the maximum value of the left-hand side of condition (40) was $5.69 \times 10^{-1} (< 1)$, confirming that the **stationary point was Lyapunov stable**.

Scalability of OTCC

- Under suitable conditions, the proposed OTCC algorithm is **scalable**.
- Each robot primarily needs information from its **neighboring robots** (those in adjacent Laguerre regions, N_ϕ^i) to compute its control laws.

- The algorithm is feasible if each robot's measurement range R_i is sufficient to obtain information from these neighbors, and this range does not significantly increase in common scenarios where robots are distributed and $|\phi_i - \phi_j|$ is small relative to the workspace size.

Conclusion

The Optimal Transport-based Coverage Control (OTCC) method is proposed as an improvement to the existing Voronoi tessellation-based coverage control (VTCC) method. OTCC is derived from the Kantorovich dual problem and generalizes VTCC, viewing it as a special case when $\phi(t) \equiv 0$. The study provides conditions for Lyapunov stability of the controlled system. Numerical calculations consistently demonstrate that OTCC is more effective at **escaping local optima** and achieving **better control performance** compared to VTCC. Future research plans include extending OTCC's applicability to a more general class of systems.