As per New Syllabus
2020 -21

Engineering Mathematics-III

Computer Engineering and
Informatiom Technology Engineering

S. E. Semester - III

G. V. Kumbhojkar

P. Jamnadas LLP.

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A Text Book of **"Engineering Mathematics - III"** for S. E. Semester - III (Computer Engineering and Information Technology Engineering) of **"University of Mumbai"** written according to the Latest and Revised Syllabus effective from June 2020.

ENGINEERING MATHEMATICS - III

Up-to-date First Edition

Computer Engineering and Information Technology Engineering

By

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M.Sc.

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Preface

I am pleased to place this up-to-date First Edition of **Engineering Mathematics - III** in the hands of Second Year (Computer Engineering and Information Technology Engineering) Students of University of Mumbai in particular and other Indian Universities in general. The book is written strictly according to the syllabus of ***Engineering Mathematics - III (Computer Engineering and Information Technology Engineering) of Mumbai University*** in force from 2020-21. The book is based on my earlier books Applied Mathematics III and IV and is modified according to the new syllabus. It contains a number of additional examples–solved and unsolved–taken mostly from recent examination papers. I have taken this opportunity to rewrite some old articles in some chapters so as to make the discussion complete and easier to understand. Also I have taken care to remove the printing mistakes that were brought to my notice by my colleagues.

The book is an outcome of over thirty-five years of teaching of Mathematics to students of Engineering and Science, of writing over thirty books on Mathematics and statistics for degree and diploma courses, of careful study of the syllabus of University of Mumbai, of clear understanding of the nature of question papers of more than twenty years of Examinations of **all branches** of Engineering of University of Mumbai. I have also made extensive use of question papers of the past twenty years of Pune, Shivaji and other Indian Universities.

Mathematics, no doubt, is a tool in the hands of an engineer. But to make it effective, it is imperative for him to know clearly how the tool works. This, in turn, requires clear understanding of the concepts and methods of higher mathematics. A teacher of Engineering Mathematics has to strike a golden mean between rigorous mathematical proofs of the theorems, which often tend to be tedious, and mere applications of these theorems to engineering problems which tend to be difficult to grasp. With this difficulty in mind, an attempt has been made in this book to explain the theory through illustrations, diagrams and examples whenever it was found sufficient for understanding the concepts involved. Some concepts and theorems are discussed through examples, and rigorous mathematical proofs are given where it was found unavoidable. In short, every care has been taken to see that unnecessary material is not presented and necessary material is not left out, so that the book becomes student-friendly.

To make the matter easy for understanding, main topics have been divided into sub-topics, and illustrative examples with complete solutions are given after theoretical discussion of each topic. Whenever possible, alternative methods of solving problems are given. These examples are mostly taken from the university question papers and some are newly constructed according to the need. I have also given sufficient number of examples for practice after each sub-topic. This will help students to know the nature of problems they are expected to face and the methods of solving such problems. In addition to this, I have given a large number of miscellaneous examples with complete solutions at the end of each chapter, and an equally large number of examples in the exercise for practice with sufficient hints. These examples are properly graded and carefully classified so that a student can have sufficient drill-work requiring a particular technique. In short, the book is so designed that it becomes complete in all respects and meets the demands of the students of both average and above-average calibre. It also

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fulfils the expectations of the university syllabus in letter and spirit.

I am extremely and sincerely thankful to Prof. A. N. Nakra; M.Sc. P. G. (BARC); M.Sc. (Nuclear Engg.) Canada. (Formerly Senior Scientist, BARC), Prof. Nelson Periera, M.Sc. (Pure Mathematics) and Prof. A. S. Desai for their continuous assistance, valuable guidance and constant encouragement in writing the whole series of books for Mumbai University. I also take this opportunity to express my sincere thanks to Prof. (Dr.) A. V. Dubewar, Prof. A. B. Pawar, Prof. Mrs. S. Hegade, Prof. M. A. Rafik, Prof. Mrs. Seema Latkar, Prof. A. V. Deshmukh, Prof. Uday Kashid, Prof. Rachana Desai and Prof. N. R. Dasre for their help and appreciation of my books. My thanks are also due to Shri Y. B. Thorat for his prompt help.

I take this opportunity to express my sincere thanks and sense of gratitude to Late Parimal J. Shah (M.A.), Dr. Prabha P. Shah (M.Sc., Ph.D.) and Shri Rushabh P. Shah (B. Com.) who wholeheartedly accepted the responsibility of publishing the entire series. I also thank the printer for printing nicely and in time the book and the whole series. I am thankful to Shri. Rajan Bhate and other members of the staff of P. Jamnadas LLP. for sparing no efforts to see that the books reach the students and teachers in time and for their cooperation and assistance in production and distribution of the series. My thanks are also due to Shri Anil M. Vhatkar of Mahalaxmi D.T.P. Center for neat, careful and swift type-setting and to Shri. Kaushal Kulkarni of Creative Concepts for beautiful cover design.

I hope that the students and teachers of Engineering Mathematics of Mumbai and other Universities will appreciate my efforts and will receive this edition enthusiastically. Any suggestions for enhancing the utility of the book and for removing errors that might have gone unnoticed will be gratefully acknowledged.

15th August, 2020

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Syllabus
Engineering Mathematics - III

S. E. Semester - III

(University of Mumbai - Effective From 2020-21)

Common to Computer Engineering and Information Technology Engineering

Module 1 : Laplace Transform **(6 hrs.)**

1.1 Definition of Laplace transform, Condition of Existence of Laplace transform.

1.2 Laplace Transform (L) of Standard Functions like e^{at}, sin at, cos at, sin h at, cos h at and t^n, where $n \geq 0$.

1.3 **Properties of Laplace Transform :** Linearity, First Shifting Theorem, Second Shifting Theorem, Change of Scale, Multiplication by t, Division by t, Laplace Transform of derivatives and integrals (Properties without proof).

1.4 Evaluation of real improper integrals by using Laplace Transformation.

Self Learning Topics : Laplace Transform : Periodic functions, Heaviside's Unit Step function, Dirac Delta Function, Special functions (Error and Bessel)

Module 2 : Inverse Laplace Transform **(6 hrs.)**

2.1 Definition of Inverse Laplace Transform, Linearity property, Inverse Laplace Transform of standard functions, Inverse Laplace transform using derivatives.

2.2 Partial fractions method to find Inverse Laplace transform.

2.3 Inverse Laplace transform using Convolution theorem (without proof).

Self Learning Topics : Applications to solve initial and boundary value problems involving ordinary differential equations.

Module 3 : Fourier Series **(6 hrs.)**

3.1 Dirichlet's conditions, Definition of Fourier series and Parseval's Identity (without proof).

3.2 Fourier series of periodic function with period 2π and $2l$.

3.3 Fourier series of even and odd functions

3.4 Half range sine and cosine series.

Self Learning Topics : Orthogonal and orthonormal set of functions, Complex form of Fourier Series, Fourier Transforms.

Module 4 : Complex Variables **(6 hrs.)**

4.1 Function $f(z)$ of complex variable, Limit, Continuity and differentiability of $f(z)$, Analytic function, Necessary and sufficient conditions for $f(z)$ to be analytic (without proof)

4.2 Cauchy-Riemann equations in Cartesian coordinates (without proof).

4.3 Milne-Thomson method to determine analytic function $f(z)$ when real part (u) or Imaginary part (v) or its combination ($u + v$ or $u - v$) is given.

4.4 Harmonic function, Harmonic conjugate and Orthogonal trajectories.

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Self Learning Topics : Conformal mapping, Linear, Bilinear mapping, Cross ratio, Fixed points and Standard transformations.

Module 5 : Statistical Techniques **(6 hrs.)**

5.1 Karl Pearson's coefficient of correlation (r).

5.2 Spearman's Rank correlation coefficient (R) (with repeated and non-repeated ranks)

5.3 Lines of regression.

5.4 Fitting of first and second degree curves.

Self Learning Topics : Covariance, Fitting of exponential curve.

Module 6 : Probability **(6 hrs.)**

6.1 Definition and basics of probability, Conditional probability.

6.2 Total Probability theorem and Bayes' theorem.

6.3 Discrete and continuous random variable with probability distribution and probability density function.

6.4 Expectation, Variance, Moment generating function, Raw and central moments upto 4th order.

Self Learning Topics : Skewness and Kurtosis of distribution (data).

☆☆☆

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Covariance, Fitting of exponential curve

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Total **586**

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✦ Salient Features of this Book ✦

1. Covers entire syllabus of Engineering Mathematics - III of Mumbai University. (Computer Engineering and Information Technology Engineering)
2. Large Number of solved examples.
3. Sufficient examples for self-study.
4. Solved examples as well as examples in exercises are classified according to their types.
5. Printing mistakes and also errors of commission and omission are carefully removed. Hopefully there is no mistake in the book.
6. **Contains solutions of the examples from Mumbai University Examinations (Computer Engineering and Information Technology Engineering) upto November 2019.**

CHAPTER 1

Laplace Transforms - I

1. Introduction

By using a particular type of definite integral as an operator a new function can be defined. One such operator is called **Laplace Transform**. Laplace transform changes a function of one variable denoted by t into a function of another variable denoted by s.

Pierre-Simon Laplace (1749 - 1827)

Laplace was a well-known French mathematician and astronomer whose work contributed greatly to the development of mathematical astronomy and statistics. He summarised the work of his predecessors in five volumes titled 'Me'canique Ce'leste' (Celestial Mechanics). He formulated Laplace's equation, pioneered and developed Laplace transforms, used in many branches of science. The Laplacian differential operator is named after him. He was one of the first scientists who postulated the existence of black holes. He became a count of the First French Empire in 1806. He is remembered as one of the greatest scientist of all time and is called **Newton Of France.**

2. Definition

If $f(t)$ is a function of t satisfying certain conditions, then the definite integral

$$\boxed{\Phi(s) = \int_0^\infty e^{-st} \cdot f(t)\,dt} \qquad \text{..... (1)}$$

when it exists, is called the **Laplace Transform** of $f(t)$ and is written as $L[f(t)]$. Thus,

$$\boxed{L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt} \qquad \text{..... (2)}$$

Laplace transform exists if the integral on the right hand side is convergent.

There is one to one correspondence between $f(t)$ which **is a function of t** and $\Phi(s)$ **which is a function of s.**

Note ....

Some authors use $F(s)$ in place of $\Phi(s)$ and p in place of s.

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Sufficient Conditions For Existence of Laplace Transforms

We first learn two concepts *viz.* (i) piecewise continuity and (ii) exponential order.

(a) Piecewise continuity : A function $f(t)$ is said to be piecewise continuous in an interval if the interval can be divided into a finite number of subintervals such that in each of these subintervals (i) $f(t)$ is continuous and (ii) the limits of $f(t)$ as t tends to the end points are finite.

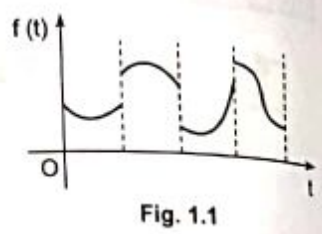

Fig. 1.1

In the adjoining figure we have shown a piecewise continuous function.

(b) Exponential order : A function $f(t)$ is said to be of exponential order for $t > T$ if we can find constant M and a such that $|f(t)| \le M e^{at}$ for $t > T$.

For example, $f(t) = t^2$ is a function of exponential order because $|f(t)| = |t^2| < e^{3t}$ for all positive t. Similarly t, e^{-2t}, $\cos t$ are all of exponential order because $|t| < e^t$, $|e^{-2t}| < e^t$, $|\cos t| < e^t$.

Theorem : If $f(t)$ is piecewise continuous in every finite interval $[0, T]$ and is of exponential order for $t > T$ then $L[f(t)]$ exists for $s > a$.

It should be noted carefully that the above conditions are only sufficient (and not necessary). If the conditions are not satisfied $L[f(t)]$ may still exist. For example, $L(t^{-1/2}) = \sqrt{\frac{\pi}{s}}$ [See Example, on page 1-6] exists eventhough $t^{-1/2}$ is not continuous in $0 \le t \le T$. [$\because$ $t^{-1/2} = \frac{1}{\sqrt{t}} \to \infty$ as $t \to 0$]

Example 1 : Find the Laplace transform of $f(t)$, where

$f(t) = t$, for $0 < t < 4$ and $f(t) = 5$, for $t > 4$. **(M.U. 2006)**

Sol. : $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^4 e^{-st} \cdot f(t)\,dt + \int_4^\infty e^{-st} \cdot f(t)\,dt$

$$= \int_0^4 e^{-st} \cdot t\,dt + \int_4^\infty e^{-st} \cdot 5\,dt$$

$$= \left[(t) \cdot \left(\frac{e^{-st}}{-s}\right) - (1) \cdot \left(\frac{e^{-st}}{s^2}\right)\right]_0^4 + 5\left[\frac{e^{-st}}{-s}\right]_4^\infty$$

$$= -\frac{4}{s} \cdot e^{-4s} - \frac{1}{s^2} \cdot e^{-4s} + \frac{1}{s^2} + \frac{5}{s} \cdot e^{-4s}$$

$$= \frac{1}{s^2} + \left(\frac{1}{s} - \frac{1}{s^2}\right) \cdot e^{-4s}.$$

Example 2 : Find the Laplace transform of $f(t)$, where

(1) $f(t) = a$, $0 < t < b$ and $f(t) = 0$, $t > b$.

(2) $f(t) = \cos(t - \alpha)$, $t > \alpha$ and $f(t) = 0$, $t < \alpha$.

Sol. : (1) $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^b e^{-st} \cdot f(t)\,dt + \int_b^\infty e^{-st} \cdot f(t)\,dt$

$$= \int_0^b e^{-st} \cdot a\,dt + \int_b^\infty e^{-st} \cdot 0\,dt = a\int_0^b e^{-st}dt = a\left[\frac{-e^{-st}}{s}\right]_0^b = \frac{a}{s}(1 - e^{-bt})$$

(2) $L[f(t)] = \int_0^\infty e^{-st} \cdot f(t)\,dt = \int_0^\alpha 0 \cdot dt + \int_\alpha^\infty e^{-st} \cdot \cos(t - \alpha)\,dt$

But $\int e^{ax} \cos bx\,dx = \frac{1}{a^2 + b^2} e^{ax}(a \cos bx + b \sin bx)$ ..... **(A)**

$$\therefore \quad L[f(t)] = \left[\frac{1}{s^2 + 1} \cdot e^{-st}\{-s\cos(t - \alpha) + \sin(t - \alpha)\}\right]_\alpha^\infty$$

$$= \frac{se^{-\alpha s}}{s^2 + 1}$$ [See also Ex. (i), page 1-27]

Example 3 : Find the Laplace transform of $f(t) = \cos t$, for $0 < t < \pi$ and $f(t) = \sin t$, for $t > \pi$. **(M.U. 1993, 2002, 05, 09)**

Sol. : $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\pi e^{-st} \cos t\,dt + \int_\pi^\infty e^{-st} \sin t\,dt.$

But $\int e^{ax} \cos bx\,dx = \frac{1}{(a^2 + b^2)} \cdot e^{ax}(a \cos bx + b \sin bx)$ and

$\int e^{ax} \sin bx\,dx = \frac{1}{(a^2 + b^2)} e^{ax}(a \sin bx - b \cos bx)$ ..... **(A)**

$$\therefore \quad L f(t) = \frac{1}{s^2 + 1} \cdot \left[e^{-st}(-s\cos t + \sin t)\right]_0^\pi + \frac{1}{s^2 + 1} \cdot \left[e^{-st}(-s\sin t - \cos t)\right]_\pi^\infty$$

$$= \frac{1}{s^2 + 1} \cdot \left[e^{-s\pi}(s) - (-s)\right] + \frac{1}{s^2 + 1} \cdot \left[-e^{-s\pi}\right]$$

$$= \frac{1}{s^2 + 1} \cdot \left[s + (s - 1)e^{-s\pi}\right]$$

Example 4 : If $L[f(t)] = \Phi(s)$ and if

$$g(t) = \begin{cases} 0, & 0 < t < a \\ f(t - a), & t > a \end{cases}, \text{ then } L[g(t)] = e^{-as}\Phi(s).$$ **(M.U. 2007)**

Sol. : $L[g(t)] = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$

$$= \int_0^a 0 \cdot dt + \int_a^\infty e^{-st} f(t - a)\,dt = \int_a^\infty e^{-st} f(t - a)\,dt$$

Now, put $t - a = u$ *i.e.* $t = a + u$, $dt = du$.

$$\therefore \quad L[g(t)] = \int_0^\infty e^{-s(a+u)} f(u)\,du = e^{-as}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt = e^{-as} L[f(t)] = e^{-as}\Phi(s)$$

[$\because$ The definite integral does not depend upon the letter used.]

EXERCISE - I

Find the Laplace transform of $f(t)$, where,

1. $f(t) = 3$, $0 < t < 5$; $f(t) = 0$, $t > 5$.
2. $f(t) = (t - 1)^2$, $t > 1$; $f(t) = 0$, $0 < t < 1$.

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3. $f(t) = (t-1)^3,\ t > 1$; $f(t) = 0,\ 0 < t < 1$.
4. $f(t) = t,\ 0 < t < 1/2$; $f(t) = t - 1,\ (1/2) < t < 1$; $f(t) = 0,\ t > 1$. (M.U. 2003)
5. $f(t) = \sin 2t,\ 0 < t < \pi$; $f(t) = 0,\ t > \pi$. (M.U. 2007)
6. $f(t) = 0,\ 0 \le t \le 1$; $f(t) = t,\ 1 < t < 2$, $f(t) = 0,\ t > 2$.
7. $f(t) = t,\ 0 < t < 3$; $f(t) = 6,\ t > 3$. (M.U. 2006)
8. $f(t) = \cos t,\ 0 < t < 2\pi$; $f(t) = 0,\ t > 2\pi$. (M.U. 2002, 04)
9. $f(t) = t^2,\ 0 < t < 1$; $f(t) = 1,\ t > 1$. (M.U. 2003)
10. $f(t) = (t-a)^3,\ t > a$; $f(t) = 0,\ t < a$.
11. $f(t) = t,\ 0 < t < a$; $f(t) = b,\ t > a$.
12. $f(t) = 0,\ 0 < t < \pi$; $f(t) = \sin^2(t-\pi),\ t > \pi$. (M.U. 2003)

[**Ans.** : (1) $\frac{3}{s}(1-e^{-5s})$, (2) $\frac{2e^{-s}}{s^3}$, (3) $\frac{6}{s^4}e^{-s}$, (4) $\frac{1}{s^2} - \frac{e^{-s}}{s^2} - \frac{e^{-s/2}}{s}$,

(5) $\frac{2(1-e^{-\pi s})}{s^2+4}$, (6) $\left(\frac{1}{s^2}+\frac{1}{s}\right)\cdot e^{-s} - \left(\frac{1}{s^2}+\frac{2}{s}\right)\cdot e^{-2s}$,

(7) $\frac{1}{s^2} + \left(\frac{3}{s} - \frac{1}{s^2}\right)e^{-3s}$, (8) $(1-e^{-2\pi s})\frac{s}{s^2+1}$,

(9) $\frac{1}{s}(1-e^{-s}) - 2\frac{e^{-s}}{s^2} + \frac{2}{s^3}(1-e^{-s})$, (10) $\frac{6}{s^4}e^{-as}$,

(11) $\frac{1}{s^2} + \left[\frac{(b-a)}{s} - \frac{1}{s^2}\right]e^{-as}$, (12) $\frac{e^{-\pi s}}{2}\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$]

(**Hint** : For integration, use generalised rule of integration by parts.)

3. Linearity Property

If k_1 and k_2 are constants then,

$$\boxed{L[k_1 f_1(t) + k_2 f_2(t)] = k_1 L[f_1(t)] + k_2 L[f_2(t)]} \quad \text{..... (3)}$$

The result can be easily proved by using the above definition.

4. Laplace Transforms of Standard Functions

By using the definition, we find below Laplace Transforms of some standard functions.

$$\boxed{L(k) = \frac{k}{s}} \quad \text{..... (3A)}$$

Proof : We have $L(k) = \int_0^\infty e^{-st}\, k\, dt = k\int_0^\infty e^{-st}\, dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{k}{-s}[0-1] = \frac{k}{s}$$

(1) $$\boxed{L(e^{at}) = \frac{1}{s-a}\ (s > a)} \quad \text{..... (4)}$$

Proof : We have $L(e^{at}) = \int_0^\infty e^{-st}\cdot e^{at}\, dt = \int_0^\infty e^{-(s-a)t}\cdot dt$

$$= \frac{1}{-(s-a)}\left[e^{-(s-a)t}\right]_0^\infty = \frac{1}{s-a} \quad \text{..... (A)}$$

Cor. 1 : If $a = 0$, $(s > 0)$ $\boxed{L(1) = \frac{1}{s}}$ ..... (5)

Cor. 2 : Changing the sign of a, $\boxed{L(e^{-at}) = \frac{1}{s+a}}$ ..... (6)

Cor. 3 : Since $c^a = e^{a\log c}$, $(c > 0,\ s > a\log c)$

$$\boxed{L(c^{at}) = \frac{1}{s - a\log c}}$$

$$\boxed{L(2^{3t}) = \frac{1}{s - 3\log 2}} \quad \text{(M.U. 2008)}$$

Note ....

Note that the integral in (A) is defined only if $(s-a) > 0$ *i.e.* $s > a$.

(2) $\boxed{L(\sin at) = \frac{a}{s^2+a^2}}$ and $\boxed{L(\cos at) = \frac{s}{s^2+a^2}\ (s > 0)}$ ..... (7)

Proof : Consider, $L(\cos at + i\sin at) = L(e^{iat}) = \frac{1}{s-ai}$ by (i) $= \frac{s+ai}{s^2+a^2}$

Equating real and imaginary parts we get (4).

Note ....

The above transforms (4) can also be obtained directly by using the definition.

(3) $\boxed{L(\sinh at) = \frac{a}{s^2-a^2}}$ and $\boxed{L(\cosh at) = \frac{s}{s^2-a^2}\ (s > |a|)}$ ..... (8)

Proof : We have, $L(\sinh t) = L\left(\frac{e^{at}-e^{-at}}{2}\right)$

$$\therefore\ L(\sinh t) = \frac{1}{2}\left\{L(e^{at}) - L(e^{-at})\right\}$$

$$= \frac{1}{2}\left\{\frac{1}{s-a} - \frac{1}{s+a}\right\} = \frac{a}{s^2-a^2} \quad \text{[By (4)]}$$

Similarly, we can prove $L(\cosh t) = \frac{s}{s^2-a^2}$

(4) $\boxed{L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}}$ $[(n+1) > 0$ and $s > 0]$ ..... (9)

Proof : We have, $L(t^n) = \int_0^\infty e^{-st}\cdot t^n\, dt$. Put $z = st$, $dz = s\, dt$.

$$= \int_0^\infty e^{-z}\cdot\frac{z^n}{s^n}\cdot\frac{dz}{s} = \frac{\Gamma(n+1)}{s^{n+1}}$$

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Example : Find $L\left[\frac{1}{\sqrt{\pi t}}\right]$.

Sol. : We have $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{\pi}} L\frac{1}{\sqrt{t}} = \frac{1}{\sqrt{\pi}} L(t^{-1/2})$

$$\therefore \quad L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{\pi}} \cdot \frac{\overline{|(-1/2)+1}}{s^{(-1/2)+1}} = \frac{1}{\sqrt{\pi}} \cdot \frac{\overline{|1/2}}{s^{1/2}}$$

$$= \frac{1}{\sqrt{\pi}} \cdot \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{\sqrt{s}} \qquad \left[\because \overline{\left|\frac{1}{2}\right.} = \sqrt{\pi}\right]$$

Cor. 4 : If n is a positive integer, $\overline{|n+1} = n!$

$$\boxed{L(t^n) = \frac{n!}{s^{n+1}} \text{ if } n \text{ is a +ve integer.}} \qquad \text{..... (10)}$$

e.g. $L(1) = \frac{1}{s}$, $L(t) = \frac{1}{s^2}$, $L(t^2) = \frac{2}{s^3}$, $L(t^3) = \frac{6}{s^4}$

(5) $\boxed{L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}}$ **(M.U. 1996, 97, 2003, 05, 09)** ..... (11)

Proof : We know that $erf\, t = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx$

$$\therefore \quad erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}dx = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\right]dx$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5(2!)} - \frac{x^7}{7(3!)} + \ldots\right]_0^{\sqrt{t}}$$

$$= \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5(2!)} - \frac{t^{7/2}}{7(3!)} + \ldots\right]$$

Taking Laplace transform of both sides and using $L(t^n) = \frac{\overline{|n+1}}{s^{n+1}}$

$$L(erf\sqrt{t}) = \frac{2}{\sqrt{\pi}}\left[\frac{\overline{|3/2}}{s^{3/2}} - \frac{1}{3}\frac{\overline{|5/2}}{s^{5/2}} + \frac{1}{5(2!)}\frac{\overline{|7/2}}{s^{7/2}} + \ldots\right]$$

But $\overline{|3/2} = (1/2)\overline{|1/2} = (1/2)\sqrt{\pi}$

$\overline{|5/2} = (3/2)(\overline{|3/2}) = (3/2)(1/2)\sqrt{\pi}$ etc.

$$\therefore \quad L(erf\sqrt{t}) = \frac{1}{s^{3/2}}\left[1 - \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{s^2} - \frac{1}{2}\cdot\frac{3}{4}\cdot\frac{5}{6}\cdot\frac{1}{s^3} + \ldots\right]$$

$$= \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2} = \frac{1}{s^{3/2}}\frac{\sqrt{s}}{\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}$$

(We shall obtain this result again in Ex. 3 on page 1-56.)

(6) $\boxed{L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}[\sqrt{s+1}+1]}}$ ..... (12)

Proof : We know that $erf\, x + erf_c\, x = 1$

$\therefore \; erf_c\, x = 1 - erf\, x \qquad \therefore \; erf_c\sqrt{t} = 1 - erf\sqrt{t}$

Taking Laplace transforms of both sides,

$$L[erf_c\sqrt{t}] = L(1) - L[erf\sqrt{t}] = \frac{1}{s} - \frac{1}{s\sqrt{s+1}} = \frac{\sqrt{s+1}-1}{s\sqrt{s+1}}$$

$$= \frac{\sqrt{s+1}-1}{s\sqrt{s+1}} \cdot \frac{\sqrt{s+1}+1}{\sqrt{s+1}+1} \qquad \text{[Rationalising the numerator]}$$

$$= \frac{s+1-1}{s\sqrt{s+1}[\sqrt{s+1}+1]} = \frac{1}{\sqrt{s+1}[\sqrt{s+1}+1]}$$

Note ....

For existence of the above Laplace transforms the conditions given in the brackets are necessary.

List of Transforms

(1) $L(e^{at}) = \frac{1}{s-a}$	(2) $L(e^{-at}) = \frac{1}{s+a}$	(3) $L(\sin at) = \frac{a}{s^2+a^2}$
(4) $L(\cos at) = \frac{s}{s^2+a^2}$	(5) $L(\sinh at) = \frac{a}{s^2-a^2}$	(6) $L(\cosh at) = \frac{s}{s^2-a^2}$
(7) $L(t^n) = \frac{\overline{|n+1}}{s^n}$	(8) $L(t^n) = \frac{n!}{s^n}$ if n is a +ve integer	
(9) $L(1) = \frac{1}{s}$	(10) $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$	(11) $L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}[\sqrt{s+1}+1]}$

EXERCISE - II

Write down the Laplace transforms of the following.

1. t^2, 2. $t^{3/2}$, 3. t^4, 4. $t^{1/2}$, 5. 1, 6. e^{2t}, 7. e^{-4t},
8. $\sin 2t$, 9. $\cos 3t$, 10. $\sin 5t$, 11. $\cos t$, 12. 10^{2t}, 13. $\sinh 3t$, 14. $\cosh 2t$,
15. 5^{3t}, 16. $1/\sqrt{\pi t}$.

[Ans. : Answers not given for obvious reason.]

Example 1 : Find the Laplace transforms of

(i) $4t^2 + \sin 3t + e^{2t}$ (ii) $(\sin 2t - \cos 2t)^2$ (iii) $\cosh^2 4t$ **(M.U. 2004)**

Sol. : (i) $L[4t^2 + \sin 3t + e^{2t}] = 4L(t^2) + L(\sin 3t) + L(e^{2t})$

$$= 4\cdot\frac{2}{s^3} + \frac{3}{s^2+3^2} + \frac{1}{s-2}$$

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(ii) $L[(\sin 2t - \cos 2t)^2] = L(1 - 2\sin 2t \cos 2t)$

$$= L(1 - \sin 4t) = L(1) - L(\sin 4t) = \frac{1}{s} - \frac{4}{s^2 + 4^2}$$

(iii) $L[\cosh^2 4t] = L[\frac{1}{2}(1 + \cosh 8t)] = \frac{1}{2}[L(1) + L(\cosh 8t)] = \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 - 8^2}\right]$

Example 2 : Find the Laplace transforms of

(i) $\sin(\omega t + \alpha)$. (ii) $t^2 - e^{-2t} + \cosh^2 3t$.

Sol. : (i) $L[\sin(\omega t + \alpha)] = L[\sin \omega t \cos \alpha + \cos \omega t \sin \alpha]$

$$= \cos \alpha\, L(\sin \omega t) + \sin \alpha\, L(\cos \omega t)$$

$$= \cos \alpha \cdot \frac{\omega}{s^2 + \omega^2} + \sin \alpha \cdot \frac{s}{s^2 + \omega^2}.$$

(ii) $L[t^2 - e^{-2t} + \cosh^2 3t] = L[t^2] - L[e^{-2t}] + L\frac{1}{2}[1 + \cosh 6t]$

$$= \frac{2}{s^3} - \frac{1}{s+2} + \frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2 - 6^2}\right]$$

Example 3 : Find the Laplace transform of the following.

(i) $\sin^5 t$ **(M.U. 2003, 05, 07, 10, 12)** (ii) $\sinh^5 t$

(iii) $\cos t \cos 2t \cos 3t$ **(M.U. 1995, 2003)** (iv) $(\cosh t - \sinh t)^n$

Sol. : (i) First we find the expression for $\sin^5 t$. (Refer to Applied Mathematics I by the same author.)

Let $x = \cos t + i \sin t$ $\therefore \frac{1}{x} = \cos t - i \sin t$

$\therefore (2i \sin t)^5 = \left(x - \frac{1}{x}\right)^5$ [See Ex. 3, § 4, page 2-25 of App. Maths - I]

$$= x^5 - 5x^4 \cdot \frac{1}{x} + 10x^3 \cdot \frac{1}{x^2} - 10x^2 \cdot \frac{1}{x^4} + 5x\frac{1}{x^4} - \frac{1}{x^5}$$

$$\therefore 32 i^5 \sin^5 t = \left(x^5 - \frac{1}{x^5}\right) - 5\left(x^3 - \frac{1}{x^3}\right) + 10\left(x - \frac{1}{x}\right)$$

$$= 2i \sin 5t - 5(2i \sin 3t) + 10(2i \sin t)$$

$$\therefore \sin^5 t = \frac{1}{16}(\sin 5t - 5\sin 3t + 10 \sin t)$$

$$\therefore L(\sin^5 t) = \frac{1}{16}[L(\sin 5t) - 5L\sin(3t) + 10 L \sin t]$$

$$= \frac{1}{16}\left[\frac{5}{s^2 + 25} - 5 \cdot \frac{3}{s^2 + 9} + 10 \cdot \frac{1}{s^2 + 1}\right] \quad \text{.....(1)}$$

$$= \frac{5}{16}\left[\frac{1}{s^2 + 25} - \frac{3}{s^2 + 9} + \frac{2}{s^2 + 1}\right] = \frac{5}{16}\left[\frac{s^2 + 9 - 3s^2 - 75}{(s^2 + 25)(s^2 + 9)} + \frac{2}{s^2 + 1}\right]$$

$$= \frac{5}{16}\left[\frac{-2s^2 - 66}{(s^2 + 25)(s^2 + 9)} + \frac{2}{s^2 + 1}\right] = \frac{5}{8}\left[\frac{-s^2 - 33}{(s^2 + 25)(s^2 + 9)} + \frac{1}{s^2 + 1}\right]$$

$$= \frac{5}{8}\left[\frac{-s^4 - s^2 - 33s^2 - 33 + s^4 + 34s^2 + 225}{(s^2 + 1)(s^2 + 9)(s^2 + 25)}\right]$$

$$= \frac{5}{8} \cdot \frac{192}{(s^2 + 1)(s^2 + 9)(s^2 + 25)} = \frac{120}{(s^2 + 1)(s^2 + 9)(s^2 + 25)}$$

$$\therefore L(\sin^5 t) = \frac{5!}{(s^2 + 1)(s^2 + 9)(s^2 + 25)}$$

Aliter : $\sin^5 t = \left(\frac{e^{it} - e^{-it}}{2i}\right)^5$

$$= \frac{1}{32i}\left[e^{5it} - 5^{4it} \cdot e^{-it} + 10 \cdot e^{3it} \cdot e^{-2it} - 10 e^{2it} \cdot e^{-3it} + 5 e^{it} \cdot e^{-4it} - e^{-5it}\right]$$

$$= \frac{1}{32i}\left[(e^{5it} - e^{-5it}) - 5(e^{3it} - e^{-3it}) + 10(e^{it} - e^{-it})\right]$$

$$= \frac{1}{16}\left[\left(\frac{e^{5it} - e^{-5it}}{2i}\right) - 5\left(\frac{e^{3it} - e^{-3it}}{2i}\right) + 10\left(\frac{e^{it} - e^{-it}}{2i}\right)\right]$$

$$\therefore \sin^5 t = \frac{1}{16}[\sin 5t - 5\sin 3t + 10 \sin t]$$

$$\therefore L[\sin^5 t] = \frac{1}{16}[L\sin 5t - 5L\sin 3t + 10 L \sin t]$$

$$= \frac{1}{16}\left[\frac{5}{s^2 + 5^2} - 5 \cdot \frac{3}{s^2 + 3^2} + 10 \cdot \frac{1}{s^2 + 1}\right]$$

Same as (1).

(ii) We have, $\sinh^5 t = \left(\frac{e^t - e^{-t}}{2}\right)^5$

$$= \frac{1}{32}\left[e^{5t} - 5e^{4t} \cdot e^{-t} + 10 e^{3t} \cdot e^{-2t} - 10 e^{2t} \cdot e^{-3t} + 5 e^{t} \cdot e^{-4t} - e^{-5t}\right]$$

$$\therefore \sinh^5 t = \frac{2}{32}\left[\left(\frac{e^{5t} - e^{-5t}}{2}\right) - 5\left(\frac{e^{3t} - e^{-3t}}{2}\right) + 10\left(\frac{e^{t} - e^{-t}}{2}\right)\right]$$

$$= \frac{1}{16}[\sinh 5t - 5\sinh 3t + 10 \sinh t]$$

$$\therefore L[\sinh^5 t] = \frac{1}{16}[L\sinh 5t - 5\sinh 3t + 10 \sinh t]$$

$$= \frac{1}{16}\left[\frac{5}{s^2 - 25} - \frac{15}{s^2 - 9} + \frac{10}{s^2 - 1}\right] = \frac{5}{16}\left[\frac{1}{s^2 - 25} - \frac{3}{s^2 - 9} + \frac{2}{s^2 - 1}\right]$$

$$= \frac{5}{16}\left[\frac{s^2 - 9 - 3s^2 + 75}{(s^2 - 25)(s^2 - 9)} + \frac{2}{s^2 - 1}\right] = \frac{5}{16}\left[\frac{-2s^2 + 66}{(s^2 - 25)(s^2 - 9)} + \frac{2}{s^2 - 1}\right]$$

$$= \frac{5}{16}\left[\frac{-2s^4 + 66s^2 + 2s^2 - 66 + 2s^4 - 68s^2 + 450}{(s^2 - 1)(s^2 - 9)(s^2 - 25)}\right]$$

$$= \frac{120}{(s^2 - 1)(s^2 - 9)(s^2 - 25)} = \frac{5!}{(s^2 - 1)(s^2 - 9)(s^2 - 25)}$$

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(iii) $L(\cos t \cos 2t \cos 3t) = L\left[\frac{1}{2}(\cos 3t + \cos t)\cos 3t\right]$

$$= \frac{1}{2}L\left[\cos^2 3t + \cos 3t \cos t\right] = \frac{1}{2}L\left[\frac{1}{2}(1+\cos 6t) + \frac{1}{2}\{\cos 4t + \cos 2t\}\right]$$

$$= \frac{1}{4}L\left[1 + \cos 2t + \cos 4t + \cos 6t\right]$$

$$= \frac{1}{4}\left[L(1) + L(\cos 2t) + L(\cos 4t) + L(\cos 6t)\right]$$

$$= \frac{1}{4}\left[\frac{1}{s} + \frac{s}{s^2+2^2} + \frac{s}{s^2+4^2} + \frac{s}{s^2+6^2}\right]$$

(iv) $(\cosh t - \sinh t)^n = \left(\frac{e^t + e^{-t}}{2} - \frac{e^t - e^{-t}}{2}\right)^n = (e^{-t})^n = e^{-nt}$

$\therefore\ L[(\cosh t - \sinh t)^n] = L(e^{-nt}) = \frac{1}{s+n}$

EXERCISE - III

Find the Laplace transforms of

1. $(t^2 + a)^2$.
2. $2t^3 + \cos 4t + e^{-2t}$.
3. $e^{2t} + 4t^3 - \sin 2t \cos 3t$
4. $\cos(wt + \beta)$. **(M.U. 1999)**
5. $\cos^2 2bt$.
6. $e^t + \sin 2t \cdot \sin 3t$.
7. $3t^2 + e^{-t} + \sin^3 2t$.
8. $\cos^3 2t$.
9. $\sinh^3 3t$.
10. $\cosh at - \cos bt$.
11. $\sinh at - \sin bt$.
12. $\cosh^3 2t$.
13. $\cos^5 t$.
14. $\cosh^5 t$.
15. $\cos^4 t$.
16. $\sin^4 t$ **(M.U. 2002, 03)**
17. $\cosh^4 t$ **(M.U. 2004)**
18. $(\cosh + \sinh ht)^n$
19. $\frac{1+2t}{\sqrt{t}}$
20. $\sqrt{1+\sin t}$ **(M.U. 2004, 08)**
21. $(\sqrt{t} - 1)^2$ **(M.U. 2004)**
22. $\left(\sqrt{t} \pm \frac{1}{\sqrt{t}}\right)^3$ **(M.U. 2007)**

[**Ans. :** (1) $\frac{a^2s^4 + 4as^2 + 24}{s^5}$. (2) $2 \cdot \frac{3!}{s^4} + \frac{s}{s^2+4^2} + \frac{1}{s+2}$.

(3) $\frac{1}{s-2} + 4 \cdot \frac{3!}{s^4} - \frac{1}{2}\left[\frac{5}{s^2+5^2} - \frac{1}{s^2+1^2}\right]$. (4) $\cos\beta \cdot \frac{s}{s^2+w^2} - \sin\beta \cdot \frac{w}{s^2+w^2}$.

(5) $\frac{1}{2}\left[\frac{1}{s} + \frac{s}{s^2+16b^2}\right]$. (6) $\frac{1}{s-1} + \frac{1}{2}\left[\frac{s}{s^2+1^2} - \frac{s}{s^2+5^2}\right]$.

(7) $3 \cdot \frac{2!}{s^3} + \frac{1}{s+1} + \frac{3}{4} \cdot \frac{2}{s^2+2^2} - \frac{1}{4} \cdot \frac{6}{s^2+6^2}$. (8) $\frac{s(s^2+28)}{(s^2+36)(s^2+4)}$.

(9) $\frac{162}{(s^2-81)(s^2-9)}$. (10) $\frac{s}{s^2-a^2} - \frac{s}{s^2+b^2}$. (11) $\frac{a}{s^2-a^2} - \frac{b}{s^2+b^2}$.

(12) $\frac{s(s^2-28)}{(s^2-36)(s^2-4)}$. (13) $\frac{s}{16}\left[\frac{1}{s^2+25} + \frac{5}{s^2+9} + \frac{10}{s^2+1}\right]$.

(14) $\frac{s}{16}\left[\frac{1}{s^2-25} + \frac{5}{s^2-9} + \frac{10}{s^2-1}\right]$. (15) $\frac{1}{8}\left[\frac{s}{s^2-16} + \frac{4s}{s^2-4} + \frac{6}{s}\right]$

(16) $\frac{1}{8}\left[\frac{3}{s} - \frac{4s}{s^2+4} + \frac{s}{s^2+16}\right]$. (17) $\frac{1}{8}\left[\frac{3}{s} + \frac{4s}{s^2-4} + \frac{s}{s^2-16}\right]$

(18) $\frac{1}{s-n}$. (19) $\sqrt{\frac{\pi}{s}}\left(1 + \frac{1}{s}\right)$ (20) $\frac{s}{s^2+(1/2)^2} + \frac{1/2}{s^2+(1/2)^2}$

(21) $\frac{1}{s^2} - \frac{\sqrt{\pi}}{s^{3/2}} + \frac{1}{s}$. (22) $\frac{\Gamma(5/2)}{s^{5/2}} \pm \frac{3\Gamma(3/2)}{s^{3/2}} + \frac{3\Gamma(1/2)}{s^{1/2}} - \frac{\Gamma(-1/2)}{s^{-1/2}}$.]

Miscellaneous Examples

Example 1 : Find the Laplace transform of the following.

(i) $\sin\sqrt{t}$ **(M.U. 1996, 2013)** (ii) $\frac{\cos\sqrt{t}}{\sqrt{t}}$ **(M.U. 2004, 09)**

Sol. : (i) Since, $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots$; $\sin\sqrt{t} = t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \ldots\ldots$

$\therefore\ L\left[\sin\sqrt{t}\right] = L(t^{1/2}) - \frac{1}{3!}L(t^{3/2}) + \frac{1}{5!}L(t^{5/2}) + \ldots\ldots$

But $L(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ and $\Gamma(n) = n\Gamma(n-1)$, $\Gamma(1/2) = \sqrt{\pi}$

$$\therefore\ L\left[\sin\sqrt{t}\right] = \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!} \cdot \frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!} \cdot \frac{\Gamma(7/2)}{s^{7/2}} - \ldots\ldots$$

$$= \frac{(1/2)\Gamma(1/2)}{s^{3/2}} - \frac{1}{3!} \cdot \frac{(3/2)(1/2)\Gamma(1/2)}{s^{5/2}} + \frac{1}{5!} \cdot \frac{(5/2)(3/2)(1/2)\Gamma(1/2)}{s^{7/2}}$$

$$= \frac{\Gamma(1/2)}{2s^{3/2}}\left[1 - \left(\frac{1}{2^2 \cdot s}\right) + \frac{1}{2!}\left(\frac{1}{2^2 \cdot s}\right)^2 - \ldots\ldots\right] \quad [\because \Gamma(n) = n\Gamma(n-1)]$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-1/(4s)} \quad [\because \Gamma(1/2) = \sqrt{\pi}]$$

(ii) We know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$

$\therefore\ \cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} + \ldots\ldots$; $\frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \ldots\ldots$

$\therefore\ L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \frac{\Gamma(1/2)}{s^{1/2}} - \frac{1}{2!} \cdot \frac{\Gamma(3/2)}{s^{3/2}} + \frac{1}{4!} \cdot \frac{\Gamma(5/2)}{s^{5/2}} - \frac{1}{6!} \cdot \frac{\Gamma(7/2)}{s^{7/2}} + \ldots\ldots$

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$$\therefore\ L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right)=\frac{\overline{|1/2}}{s^{1/2}}-\frac{1}{2!}\cdot\frac{(1/2)\overline{|1/2}}{s^{3/2}}+\frac{1}{4!}\cdot\frac{(3/2)(1/2)\overline{|1/2}}{s^{5/2}}-\frac{1}{6!}\cdot\frac{(5/2)(3/2)(1/2)\overline{|1/2}}{5^{7/2}}+\ldots\ldots$$

$$=\frac{\sqrt{\pi}}{\sqrt{s}}\left[1-\frac{1}{4s}+\frac{1}{2!(4s)^2}-\frac{1}{3!(4s)^3}+\ldots\ldots\right]=\sqrt{\frac{\pi}{s}}\cdot e^{-1/(4s)}.$$

Example 2 : If $J_o(t)=\sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{t}{2}\right)^{2r}$, find $L[J_o(t)]$. **(M.U. 2010, 1**

Sol. : By data $J_o(t)=1-\left(\frac{t}{2}\right)^2+\frac{1}{(2!)^2}\left(\frac{t}{2}\right)^4-\frac{1}{(3!)^2}\left(\frac{t}{2}\right)^6+\ldots\ldots$

$$=1-\frac{t^2}{2^2}+\frac{t^4}{2^2\cdot4^2}-\frac{t^6}{2^2\cdot4^2\cdot6^2}+\ldots\ldots$$

$$\therefore\ L[J_o(t)]=L(1)-\frac{1}{2^2}L(t^2)+\frac{1}{2^2\cdot4^2}\cdot L(t^4)-\frac{1}{2^2\cdot4^2\cdot6^2}\cdot L(t^6)-\ldots\ldots$$

$$=\frac{1}{s}-\frac{1}{2^2}\cdot\frac{2!}{s^3}+\frac{1}{2^2\cdot4^2}\cdot\frac{4!}{s^5}-\frac{1}{2^2\cdot4^2\cdot6^2}\cdot\frac{6!}{s^7}+\ldots\ldots$$

$$=\frac{1}{s}\left[1-\frac{1}{2}\left(\frac{1}{s^2}\right)+\frac{1}{2}\cdot\frac{3}{4}\left(\frac{1}{s^2}\right)^2-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\left(\frac{1}{s^2}\right)^3+\ldots\ldots\right]$$

$$=\frac{1}{s}\left(1+\frac{1}{s^2}\right)^{-1/2}=\frac{1}{\sqrt{1+s^2}}.$$

Example 3 : If $J_o(t)=\frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta$, prove that $L[J_o(t)]=\frac{1}{\sqrt{s^2+1}}$.

Hence, evaluate $\frac{1}{\pi}\int_0^{\infty}e^{-t}\left[\int_0^{\pi}\cos t\cos\theta\,d\theta\right]dt$. **(M.U. 2003, 0**

Sol. : We have $J_o(t)=\frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta=\frac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta$

Taking Laplace transforms of both sides,

$$L[J_o(t)]=\frac{2}{\pi}\int_0^{\infty}e^{-st}\left[\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta\right]dt$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^{\infty}e^{-st}\cos(t\cos\theta)\,d\theta\right]dt$$

$$=\frac{2}{\pi}\int_0^{\pi/2}[L\cos(t\cos\theta)]\,dt=\frac{2}{\pi}\int_0^{\pi/2}\frac{s}{s^2+\cos^2\theta}\,d\theta$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{s^2\sec^2\theta+1}\,d\theta=\frac{2}{\pi}\int_0^{\pi/2}\frac{\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}\,d\theta$$

Put $s\tan\theta=t$ $\quad\therefore\ s\sec^2\theta\,d\theta=dt$

When $\theta=0$, $t=0$; when $\theta=\pi/2$, $t=\infty$.

$$\therefore\ L[J_o(t)]=\frac{2}{\pi}\int_0^{\infty}\frac{dt}{t^2+(s^2+1)}=\frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\cdot\left[\tan^{-1}\left(\frac{t}{\sqrt{s^2+1}}\right)\right]_0^{\infty}$$

$$=\frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\left[\frac{\pi}{2}-0\right]=\frac{1}{\sqrt{s^2+1}}$$

By definition of Laplace transform this means

$$\int_0^{\infty}e^{-st}\cdot J_o(t)\,dt=\frac{1}{\sqrt{s^2+1}}$$

i.e. $$\int_0^{\infty}e^{-st}\cdot\left[\frac{1}{\pi}\int_0^{\pi}\cos(t\cos\theta)\,d\theta\right]dt=\frac{1}{\sqrt{s^2+1}}$$

Putting $s=1$, we get

$$\frac{1}{\pi}\int_0^{\infty}e^{-t}\left[\int_0^{\pi}\cos(t\cos\theta)\,d\theta\right]dt=\frac{1}{\sqrt{1+1}}=\frac{1}{\sqrt{2}}$$

Example 4 : If $J_o(t)=\frac{1}{\pi}\int_0^{\pi}\cos(t\sin\theta)\,d\theta$, prove that $L[J_o(t)]=\frac{1}{\sqrt{s^2+1}}$.

Sol. : Prove it yourself on the above lines.

5. Evaluation of the Integral $\int_0^{\infty}e^{-at}f(t)\,dt$

We have earlier remarked that Laplace transforms are highly useful to find certain integrals which cannot be evaluated by usual methods. Consider the following integral.

Find $\int_0^{\infty}e^{-at}f(t)\,dt$

To find $\int_0^{\infty}e^{-st}f(t)\,dt$, we first find $L\,f(t)\,dt$. Say it is $\Phi(s)$.

This means, $$\int_0^{\infty}e^{-st}f(t)\,dt=\Phi(s) \qquad \text{..... (1)}$$

Comparing with the given integral we see that we have to replace s by a.

Replacing s by a in (1), we get $\int_0^{\infty}e^{-at}f(t)\,dt=\Phi(a)$

For example, by definition,

$$\int_0^{\infty}e^{-st}\sin t\,dt=\frac{1}{s^2+1^2}\qquad\therefore\ \int_0^{\infty}e^{-3t}\sin t\,dt=\frac{1}{10}\ \text{for } s=3.$$

Also $$\int_0^{\infty}e^{-st}\cos 3t\,dt=\frac{s}{s^2+9}\qquad\therefore\ \int_0^{\infty}e^{-2t}\cos 3t\,dt=\frac{2}{13}\ \text{for } s=2.$$

Thus, Laplace transforms can be used to evaluate some integrals of the above type and of some more types very easily. (See also pages 1-24, 1-35, 1-43, 1-65)

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Example 1 : Evaluate $\int_0^\infty e^{-2t}\sin^2 2t\,dt$.

Sol. : We shall first obtain Laplace transform of $\sin^2 2t$.

Now, $L(\sin^2 2t) = L\left[\frac{1}{2}(1-\cos 4t)\right]$

$$= \frac{1}{2}[L(1) - L(\cos 4t)] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4^2}\right]$$

This means $\int_0^\infty e^{-st}\sin^2 2t\,dt = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$

Now, putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\sin^2 2t\,dt = \frac{1}{2}\left[\frac{1}{2} - \frac{2}{2^2+4^2}\right] = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{10}\right] = \frac{1}{5}.$$

Example 2 : Evaluate $\int_0^\infty e^{-2t}\cdot\sin^3 t\,dt$. **(M.U. 1985, 2002, 09)**

Sol. : We have $L(\sin^3 t) = L\left[\frac{3}{4}\cdot(\sin t) - \frac{1}{4}\cdot(\sin 3t)\right]$

$$= \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9}$$

This means $\int_0^\infty e^{-st}\sin^3 t\,dt = \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9}$

Putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\sin^3 t\,dt = \frac{3}{4}\cdot\frac{1}{4+1} - \frac{1}{4}\cdot\frac{3}{4+9} = \frac{6}{65}.$$

Example 3 : Evaluate $\int_0^\infty e^{-2t}t^5\cosh t\,dt$. **(M.U. 2016)**

Sol. : Consider $L(t^5\cosh t)$

$$L[t^5\cosh t] = L\left[t^5\left(\frac{e^t+e^{-t}}{2}\right)\right] = \frac{1}{2}\left[L(e^t t^5) + L(e^{-t}t^5)\right]$$

$$= \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right] = \frac{5!}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

This means,

$$\int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{5!}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

Now, put $s = 2$,

$$\therefore \int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{5!}{2}\left[\frac{1}{(2-1)^6} + \frac{1}{(2+1)^6}\right] = 60\left[1 + \frac{1}{3^6}\right]$$

Alternatively :

$$\int_0^\infty e^{-st}t^5\cosh t\,dt = \int_0^\infty e^{-2t}t^5\left(\frac{e^t+e^{-t}}{2}\right)dt$$

$$\therefore \int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{1}{2}\int_0^\infty (e^{-t}t^5 + e^{-3t}t^5)\,dt$$

Consider $L(t^5) = \frac{5!}{s^6}$.

This means, $\int_0^\infty e^{-st}t^5\,dt = \frac{5!}{s^6}$ ..... (A)

Hence, we get

$$\int_0^\infty e^{-st}t^5\cosh t\,dt = \frac{1}{2}\int_0^\infty e^{-t}t^5\,dt + \frac{1}{2}\int_0^\infty e^{-3t}t^5\,dt$$

$$= \frac{1}{2}\cdot\frac{5!}{16} + \frac{1}{2}\cdot\frac{5!}{3^6}$$

$$= \frac{5!}{2}\left[1 + \frac{1}{3^6}\right] \quad [\text{Putting } s = 1, s = 3]$$

$$= 60\left[1 + \frac{1}{3^6}\right]$$

Example 4 : Evaluate $\int_0^\infty e^{-2t}\cosh^5 t\,dt$.

Sol. : We have $\cosh^5 x = \left(\frac{e^x+e^{-x}}{2}\right)^5$

$$= \frac{1}{32}\left[e^{5x} + 5e^{4x}\cdot e^{-x} + 10e^{3x}\cdot e^{-2x} + 10e^{2x}\cdot e^{-3x} + 5e^x\cdot e^{-4x} + e^{-5x}\right]$$

$$= \frac{2}{32}\left[\left(\frac{e^{5x}+e^{-5x}}{2}\right) + 5\left(\frac{e^{3x}+e^{-3x}}{2}\right) + 10\left(\frac{e^x+e^{-x}}{2}\right)\right]$$

$$= \frac{1}{16}[\cosh 5x + 5\cosh 3x + 10\cosh x]$$

$$L(\cosh^5 t) = \frac{1}{16}[L(\cosh 5t) + 5L(\cosh 3t) + 10L(\cosh t)]$$

$$= \frac{1}{16}\left[\frac{s}{s^2-25} + 5\cdot\frac{s}{s^2-9} + 10\cdot\frac{s}{s^2-1}\right]$$

This means $\int_0^\infty e^{-st}\cosh^5 t\,dt = \frac{1}{16}\left[\frac{s}{s^2-25} + 5\cdot\frac{s}{s^2-9} + 10\cdot\frac{s}{s^2-1}\right]$

Now putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\cosh^5 t\,dt = \frac{1}{16}\left[\frac{2}{4-25} + \frac{10}{4-9} + \frac{20}{4-1}\right]$$

$$\therefore \int_0^\infty e^{-2t}\cosh^5 t\,dt = \frac{1}{16}\left[-\frac{2}{21} - \frac{10}{5} + \frac{20}{3}\right] = \frac{1}{16}\left[\frac{32}{7}\right] = \frac{2}{7}.$$

Example 5 : Find $L(\text{erf}\sqrt{t})$ and hence, obtain $\int_0^\infty \text{erf}\sqrt{t}\cdot e^{-t}\,dt$. **(M.U. 2000)**

Sol. : We have proved in (11), page 1-6 that $L(\text{erf}\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$.

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This means $\int_0^\infty e^{-st}\, erf\sqrt{t}\, dt = \frac{1}{s\sqrt{s+1}}$

Now, putting $s = 1$, we get $\int_0^\infty e^{-t} erf\sqrt{t}\, dt = \frac{1}{1\sqrt{1+1}} = \frac{1}{\sqrt{2}}$.

Example 6 : Find $\int_0^\infty e^{-t}\, erf_c\sqrt{t}\, dt$. (M.U. 19

Sol. : By (12), page 1-7, we have

$$L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}.$$

By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$$

Now putting $s = 1$, we get

$$\int_0^\infty e^{-t}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{2}(\sqrt{2}+1)} = \frac{1}{\sqrt{6}}.$$

Example 7 : Find $\int_0^\infty erf_c\sqrt{t}\, dt$.

Sol. : As seen above $L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$.

This means $\int_0^\infty e^{-st} erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$

Now, putting $s = 0$, we get $\int_0^\infty erf_c\sqrt{t}\, dt = \frac{1}{2}$.

Example 8 : If $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)dt = \frac{1}{4}$, then find α.

Sol. : Now, $L[\sin(t+\alpha)\cos(t-\alpha)] = \frac{1}{2}[L(\sin 2t) + \sin 2\alpha \cdot L(1)]$

$$= \frac{1}{2}\cdot\left[\frac{2}{s^2+4} + \sin 2\alpha \cdot \frac{1}{s}\right]$$

This means $\int_0^\infty e^{-st}\sin(t+\alpha)\cos(t-\alpha)dt = \frac{1}{2}\cdot\left[\frac{2}{s^2+4} + \sin 2\alpha\cdot\frac{1}{s}\right]$

Now, putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)dt = \frac{1}{2}\left[\frac{2}{4+4} + \frac{1}{2}\sin 2\alpha\right] = \frac{1}{8} + \frac{1}{4}\sin 2\alpha$$

But this is equal to $\frac{1}{4}$.

$\therefore \frac{1}{8} + \frac{1}{4}\sin 2\alpha = \frac{1}{4}$ $\therefore \frac{1}{4}\sin 2\alpha = \frac{1}{4} - \frac{1}{8} = \frac{1}{8}$

$\therefore \sin 2\alpha = \frac{1}{2}$ $\therefore 2\alpha = \frac{\pi}{6}$ $\therefore \alpha = \frac{\pi}{12}$.

EXERCISE - IV

(A) Evaluate by Laplace transform,

1. $\int_0^\infty e^{-3t}\cdot\cos^2 t\, dt$, 2. $\int_0^\infty e^{-4t}\cdot\cosh^3 t\, dt$, 3. $\int_0^\infty e^{-5t}\cdot\sinh^3 dt$,
4. $\int_0^\infty e^{-3t} t^5\, dt$, 5. $\int_0^\infty e^{-4t}\cdot\sin^3 t\, dt$, 6. $\int_0^\infty e^{-3t}\cdot\cos^3 t\, dt$. **(M.U. 2003)**

(B) If $\int_0^\infty e^{-2t}\sin(t+\alpha)\cos(t-\alpha)dt = \frac{3}{8}$, then find α. **(M.U. 2004, 06, 07, 09, 14)**

(C) Evaluate the following :

1. $\int_0^\infty e^{-8t}\, erf\sqrt{t}\, dt$ 2. $\int_0^\infty e^{-3t}\, erf_c\sqrt{t}\, dt$

[**Ans :** **(A)** (1) $\frac{11}{39}$. (2) $\frac{12}{35}$. (3) $\frac{1}{64}$. (4) $\frac{40}{243}$. (5) $\frac{6}{425}$. (6) $\frac{4}{15}$. **(B)** $\alpha = \frac{\pi}{4}$
(C) (1) $\frac{1}{24}$. (2) $\frac{1}{6}$.]

6. Change of Scale Property

If $L[f(t)] = \Phi(s)$, then $\boxed{L[f(at)] = \frac{1}{a}\Phi\left(\frac{s}{a}\right)}$ **(M.U. 2002, 06, 09)** ..... **(13)**

Proof : We have $L[f(at)] = \int_0^\infty e^{-st} f(at)dt$. Now, put $u = at$.

$$= \int_0^\infty e^{-s(u/a)} f(u)\cdot\frac{du}{a} = \frac{1}{a}\int_0^\infty e^{-(s/a)u} f(u)du = \frac{1}{a}\Phi\left(\frac{s}{a}\right).$$

e.g. (i) If $L f(t) = \frac{2s}{s^2+4}$, then $L[f(2t)] = \frac{1}{2}\left[\frac{2(s/2)}{(s/2)^2+4}\right] = \frac{2s}{s^2+16}$.

(ii) If $L f(t) = \frac{s^2-s}{s^2+3s+5}$, then $L[f(3t)] = \frac{1}{3}\left[\frac{(s/3)^2-(s/3)}{(s/3)^2+3(s/3)+5}\right] = \frac{1}{3}\left[\frac{s^2-3s}{s^2+9s+45}\right]$

Example 1 : (i) If $L f(t) = \frac{1}{s}e^{-1/s}$, find $L f(3t)$. (ii) If $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$, find $L\left(\sqrt{\frac{\pi}{t}}\right)$.

(iii) If $L(erf\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$, find $L(erf\, 3\sqrt{t})$. (iv) If $L J_o(t) = \frac{1}{\sqrt{s^2+1}}$, find $L J_o(at)$.

(M.U. 2003)

Sol. : We have if $L f(t) = \Phi(s)$, then $L f(at) = \frac{1}{a}\Phi\left(\frac{s}{a}\right)$.

(i) Since, $L f(t) = \frac{1}{s}e^{-1/s}$, $L f(3t) = \frac{1}{3}\cdot\frac{1}{(s/3)}\cdot e^{-1/(s/3)}$

$\therefore L f(3t) = \frac{1}{s}\cdot e^{-3/s}$.

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(ii) Since, $L\left(\frac{1}{\sqrt{\pi t}}\right) = \frac{1}{\sqrt{s}}$, $L\left(\sqrt{\frac{\pi}{t}}\right) = L\left(\pi\sqrt{\frac{1}{\pi t}}\right)$

$$\therefore L\left(\sqrt{\frac{\pi}{t}}\right) = \frac{1}{\pi\sqrt{s/\pi}} = \sqrt{\frac{\pi}{s}}$$

(iii) Since, $L(\text{erf}\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$ $\quad\therefore L(\text{erf } 3\sqrt{t}) = L(\text{erf}\sqrt{9t})$

$$\therefore L(\text{erf } 3\sqrt{t}) = \frac{1}{3}\cdot\frac{1}{(s/3)\sqrt{(s/9)+1}} = \frac{3}{s\sqrt{s+9}}$$

(iv) Since, $L J_0(t) = \frac{1}{\sqrt{s^2+1}}$, $\quad\therefore L J_0(at) = \frac{1}{a}\cdot\frac{1}{\sqrt{(s^2/a^2)+1}} = \frac{1}{\sqrt{s^2+a^2}}$.

Example 2 : Find $L\{\text{erf}\sqrt{t}\}$ and then evaluate $\int_0^\infty e^{-5t}\,\text{erf}(2\sqrt{t})\,dt$. **(M.U. 2008)**

Sol. : We have proved (11), page 1-6 that $L\{\text{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$.

But $L\{f(at)\} = \frac{1}{a}\Phi\left(\frac{s}{a}\right)$

$$\therefore L\{\text{erf}(2\sqrt{t})\} = L\{\text{erf}.(\sqrt{4t})\} = \frac{1}{4}\cdot\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}}$$

$$\therefore \int_0^\infty e^{-st}\cdot\text{erf}(2\sqrt{t})\,dt = \frac{2}{s\sqrt{s+4}}$$

Putting $s = 5$, $\quad\int_0^\infty e^{-5t}\text{erf}(2\sqrt{t})\,dt = \frac{2}{5\sqrt{5+4}} = \frac{2}{15}$.

EXERCISE - V

1. If $L f(t) = \frac{2}{s^3}e^{-s}$, find $L f(2t)$. [**Ans. :** $\frac{8}{s^3}e^{-s/2}$]
2. If $L\,\text{erf}_c(\sqrt{t}) = \frac{1}{\sqrt{s+1}+1}$, find $L\,\text{erf}_c(2\sqrt{t})$. [**Ans. :** $\frac{1}{2\sqrt{s+4}+4}$]
3. If $L f(t) = \log\left(\frac{s+3}{s+1}\right)$, find $L f(2t)$. [**Ans. :** $\log\sqrt{\frac{s+6}{s+2}}$]
4. If $L\left[\sin\sqrt{t}\right] = \frac{\sqrt{\pi}}{2s\sqrt{s}}\cdot e^{-1/(4s)}$, find $L\left[\sin 2\sqrt{t}\right]$. **(M.U. 2004, 06, 09)** [**Ans. :** $\frac{\sqrt{\pi}}{s\sqrt{s}}\cdot e^{-1/s}$]
5. If $L[f(t)] = \frac{8(s-3)}{(s^2-6s+25)^2}$, find $L[f(2t)]$. **(M.U. 2005)** [**Ans. :** $\frac{s-6}{(s^2-12s+100)^2}$]
6. If $L[f(t)] = \frac{20-4s}{s^2-4s+20}$, find $L[f(3t)]$. **(M.U. 2009)** [**Ans. :** $\frac{60-4s}{s^2-12s+180}$]

7. First Shifting Theorem

If $L[f(t)] = \Phi(s)$, then $\boxed{L[e^{-at}f(t)] = \Phi(s+a)}$ ..... **(14)**

(M.U. 1997, 2003, 06, 09, 11)

Proof : We have

$$L[e^{-at}f(t)] = \int_0^\infty e^{-st}\left\{e^{-at}f(t)\right\}dt = \int_0^\infty e^{-(s+a)t}f(t)\,dt = \Phi(s+a)$$

e.g. $L[\sin at] = \frac{a}{s^2+a^2}$ $\quad\therefore L\left[e^{-bt}\sin at\right] = \frac{a}{(s+b)^2+a^2}$

Cor. : Changing sign of a, we get,

If $L[f(t)] = \Phi(s)$, then $\boxed{L\left[e^{at}f(t)\right] = \Phi(s-a)}$

1.	$\because$ $L(\sin at) = \frac{a}{s^2+a^2}$,	$L(e^{bt}\sin at) = \frac{a}{(s-b)^2+a^2}$	
2.	$\because$ $L(\cos at) = \frac{s}{s^2+a^2}$,	$L(e^{bt}\cos at) = \frac{s-b}{(s-b)^2+a^2}$	
3.	$\because$ $L(\sin h\,at) = \frac{a}{s^2-a^2}$,	$L(e^{bt}\sin h\,at) = \frac{a}{(s-b)^2-a^2}$	
4.	$\because$ $L(\cos h\,at) = \frac{s}{s^2-a^2}$,	$L(e^{bt}\cos h\,at) = \frac{s-b}{(s-b)^2-a^2}$	
5.	$\because$ $L(t^n) = \frac{n!}{s^{n+1}}$,	$L(e^{bt}\cdot t^n) = \frac{n!}{(s-b)^{n+1}}$	(n is an integer)
6.	$\because$ $L(t^n) = \frac{\lfloor n+1}{s^{n+1}}$,	$L(e^{bt}\cdot t^n) = \frac{\lfloor n+1}{(s-b)^{n+1}}$	

EXERCISE - VI

Write down the Laplace transforms of the following.

1. $e^{3t}\sin 2t$, 2. $e^{-2t}\cos 3t$, 3. $e^{-t}\sin 4t$, 4. $e^{t}\cos t$,
5. $e^{2t}\sin h\,t$, 6. $e^{-t}\cos h\,2t$, 7. $e^{-2t}\sin h\,4t$, 8. $e^{-3t}\cos h\,3t$,
9. $e^{-3t}t^4$, 10. $e^{2t}t^{3/2}$, 11. $e^{-t}\sqrt{t}$, 12. $e^{t}\cos 3t$ **(M.U. 2003)**

[**Ans. :** Answers not given for obvious reason.]

Example 1 : If $L[f(t)] = \frac{s}{s^2+s+4}$, find $L[e^{-3t}f(2t)]$. **(M.U. 2014)**

Sol. : By change of scale property,

$$L[f(2t)] = \frac{1}{2}\cdot\frac{(s/2)}{(s/2)^2+(s/2)+4} = \frac{s}{s^2+2s+16}$$

By first shifting theorem,

$$L[e^{-3t}f(2t)] = \frac{(s+3)}{(s+3)^2+2(s+3)+16} = \frac{s+3}{s^2+8s+31}.$$

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Example 2 : Find the Laplace transform of (i) $e^{4t}\sin^3 t$, (ii) $\cosh 2t \cos 2t$. **(M.U. 2013)**

Sol. : (i) $L[\sin^3 t] = L\left[\frac{1}{4}(3\sin t - \sin 3t)\right] = \frac{3}{4}L(\sin t) - \frac{1}{4}L(\sin 3t)$

$$\therefore\ L[\sin^3 t] = \frac{3}{4}\cdot\frac{1}{s^2+1} - \frac{1}{4}\cdot\frac{3}{s^2+9}.$$

By first shifting theorem,

$$L\left[e^{4t}\sin^3 t\right] = \frac{3}{4}\cdot\frac{1}{(s-4)^2+1} - \frac{1}{4}\cdot\frac{3}{(s-4)^2+9}$$

$$= \frac{3}{4}\left[\frac{1}{s^2-8s+17} - \frac{1}{s^2-8s+25}\right]$$

$$= \frac{6}{(s^2-8s+17)(s^2-8s+25)}.$$

(ii) $L[\cosh 2t\cdot\cos 2t] = L\left[\frac{1}{2}\left(e^{2t}+e^{-2t}\right)\cos 2t\right] = \frac{1}{2}\left[L(e^{2t}\cos 2t) + L(e^{-2t}\cos 2t)\right]$

But $L\cos 2t = \frac{s}{s^2+4}$. $\therefore$ By shifting theorem,

$$L[\cosh 2t\cdot\cos 2t] = \frac{1}{2}\left[\frac{s-2}{(s-2)^2+4} + \frac{s+2}{(s+2)^2+4}\right] = \frac{s^3}{s^4+64}.$$

Example 3 : Find $L[(t^2\sinh t)^2]$.

Sol. : We have $(t^2\sinh t)^2 = t^4\left(\frac{e^t-e^{-t}}{2}\right)^2 = \frac{t^4}{4}[e^{2t}-2+e^{-2t}]$

$$\therefore\ L[(t^2\sinh t)^2] = L\left[\frac{t^4}{4}(e^{2t}-2+e^{-2t})\right] = \frac{1}{4}\left[L(e^{2t}t^4) - 2L(t^4) + L(e^{-2t}t^4)\right]$$

But $L[t^4] = \frac{4!}{s^5}$.

$$\therefore\ L[(t^2\sinh t)^2] = \frac{1}{4}\left[\frac{4!}{(s-2)^5} - 2\cdot\frac{4!}{s^5} + \frac{4!}{(s+2)^5}\right] = 6\left[\frac{1}{(s-2)^5} - \frac{2}{s^5} + \frac{1}{(s+2)^2}\right]$$

Example 4 : Find $L\left[e^t\left(1+\sqrt{t}\right)^4\right]$.

Sol. : We have

$$L\left(1+\sqrt{t}\right)^4 = L\left[1+4\sqrt{t}+6\left(\sqrt{t}\right)^2+4\left(\sqrt{t}\right)^3+\left(\sqrt{t}\right)^4\right] \quad \text{[By Binomial Theorem]}$$

$$= L\left[1+4\sqrt{t}+6t+4t^{3/2}+t^2\right]$$

$$= \frac{1}{s} + \frac{4\Gamma(3/2)}{s^{3/2}} + \frac{6\Gamma(2)}{s^2} + \frac{4\Gamma(5/2)}{s^{5/2}} + \frac{\Gamma(3)}{s^3}$$

$$= \frac{1}{s} + \frac{4(1/2)\Gamma(1/2)}{s^{3/2}} + 6\cdot\frac{1}{s^2} + \frac{4(3/2)(1/2)\Gamma(1/2)}{s^{5/2}} + \frac{2}{s^2}$$

(For Gamma functions refer to Applied Mathematics - II)

$$\therefore\ L\left[\left(1+\sqrt{t}\right)^4\right] = \frac{1}{s} + \frac{2\sqrt{\pi}}{s^{3/2}} + \frac{6}{s^2} + 3\cdot\frac{\sqrt{\pi}}{s^{5/2}} + \frac{2}{s^2}$$

Now, by first shifting theorem,

$$L\left[e^t\left(1+\sqrt{t}\right)^4\right] = \frac{1}{s-1} + \frac{2\sqrt{\pi}}{(s-1)^{3/2}} + \frac{6}{(s-1)^2} + \frac{3\sqrt{\pi}}{(s-1)^{5/2}} + \frac{2}{(s-1)^2}$$

Example 5 : Find Laplace transform of $\sinh at \sin at$. **(M.U. 2003, 05, 09)**

Sol. : We have

$$\sinh at \sin at = \left(\frac{e^{at}-e^{-at}}{2}\right)\sin at = \frac{1}{2}\left[e^{at}\sin at - e^{-at}\sin at\right]$$

Now, $L(\sin at) = \frac{a}{s^2+a^2}$.

By first Shifting Theorem,

$$L(e^{at}\sin at) = \frac{a}{(s-a)^2+a^2} = \frac{a}{s^2-2as+2a^2}$$

$$L(e^{-at}\sin at) = \frac{a}{(s+a)^2+a^2} = \frac{a}{s^2+2as+2a^2}$$

$$\therefore\ L(\sinh at\sin at) = \frac{1}{2}\left[L(e^{at}\sin at) - L(e^{-at}\sin at)\right]$$

$$\therefore\ L(\sinh at\sin at) = \frac{1}{2}\cdot\left[\frac{a}{s^2-2as+2a^2} - \frac{a}{s^2+2as+2a^2}\right]$$

$$= \frac{a}{2}\cdot\left[\frac{4as}{(s^2-2as+2a^2)(s^2+2as+2a^2)}\right]$$

$$= \frac{2a^2s}{(s^2+2a^2-2as)(s^2+2a^2+2as)} = \frac{2a^2s}{s^4+4a^4}$$

Example 6 : Show that $L[\sinh(t/2)\sin(t/2)] = \frac{2s}{4s^4+1}$.

Sol. : Putting $a = 1/2$ in the above example, we get

$$L[\sinh(t/2)\sin(t/2)] = \frac{2(1/4)s}{s^4+4\cdot(1/2)^4} = \frac{2s}{4s^4+1}$$

Or solve it independently.

Example 7 : Show that $L\left[\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}t}{2}\right)\right] = \frac{\sqrt{3}}{2}\cdot\frac{s}{(s^4+s^2+1)}$.

(M.U. 1993, 2002, 03, 09)

Sol. : We have, $\sinh(t/2)\cdot\sin(\sqrt{3}t/2) = \left(\frac{e^{t/2}-e^{-t/2}}{2}\right)\cdot\sin\frac{\sqrt{3}}{2}t$

Now $L\sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{s^2+(3/4)}$.

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By first shifting theorem,

$$\therefore\ L e^{t/2} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{[s-(1/2)]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1-s}$$

and $$L e^{-t/2} \cdot \sin\left(\frac{\sqrt{3}}{2}t\right) = \frac{\sqrt{3}/2}{[s+(1/2)]^2+3/4} = \frac{\sqrt{3}/2}{s^2+1+s}$$

$$\therefore\ L\left[\sinh\left(\frac{t}{2}\right)\sin\left(\frac{\sqrt{3}t}{2}\right)\right] = \frac{1}{2}\left[\frac{\sqrt{3}/2}{(s^2+1)-s} - \frac{\sqrt{3}/2}{(s^2+1)+s}\right]$$

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\left[\frac{(s^2+1+s)-(s^2+1-s)}{(s^2+1)^2-s^2}\right]$$

$$= \frac{\sqrt{3}}{2}\cdot\frac{s}{s^4+s^2+1}.$$

Example 8 : Find the Laplace transform of the following.

(i) $\dfrac{\cos 2t \sin t}{e^t}$, **(M.U. 1993, 2003, 04, 10, 18, 19)** (ii) $e^{-4t}\sinh t \sin t$, **(M.U. 1995)**

(iii) $e^t \sin 2t \sin 3t$, **(M.U. 1997)** (iv) $e^{-3t}\cosh 5t \sin 4t$. **(M.U. 1998)**

Sol. : (i) $\cos 2t \sin t = \frac{1}{2}\cdot 2\sin t \cos 2t = \frac{1}{2}[\sin(1+2)t + \sin(1-2)t]$

$$= \frac{1}{2}[\sin 3t - \sin t]$$

$$\therefore\ L(\cos 2t \sin t) = \frac{1}{2}[L(\sin 3t) - L(\sin t)] = \frac{1}{2}\cdot\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right]$$

Now by first shifting theorem,

$$L\left[e^{-t}(\cos 2t \sin t)\right] = \frac{1}{2}\left[\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\frac{3}{s^2+2s+10} - \frac{1}{s^2+2s+2}\right]$$

$$= \frac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$$

(ii) $$e^{-4t}\sinh t \sin t = e^{-4t}\cdot\left(\frac{e^t-e^{-t}}{2}\right)\sin t = \frac{1}{2}e^{-3t}\sin t - \frac{1}{2}e^{-5t}\sin t$$

$$\therefore\ L\left(e^{-4t}\sinh t \sin t\right) = \frac{1}{2}L\left(e^{-3t}\sin t\right) - \frac{1}{2}L\left(e^{-5t}\sin t\right)$$

$$= \frac{1}{2}\cdot\frac{1}{(s+3)^2+1^2} - \frac{1}{2}\cdot\frac{1}{(s+5)^2+1^2}$$

$$= \frac{1}{2}\left[\frac{s^2+10s+26-s^2-6s-10}{(s^2+6s+10)(s^2+10s+26)}\right]$$

$$= \frac{2(s+4)}{(s^2+6s+10)(s^2+10s+26)}.$$

In some of the following examples, we need the following trigonometric results.

1. $\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$
2. $\cos A \sin B = \frac{1}{2}[\sin(A+B) - \sin(A-B)]$
3. $\cos A \cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$
4. $\sin A \sin B = -\frac{1}{2}[\cos(A+B) - \cos(A-B)]$

(iii) $$\sin 2t \sin 3t = \frac{1}{2}\cdot 2\sin 3t \sin 2t = -\frac{1}{2}[\cos 5t - \cos t]$$

$$\therefore\ L(\sin 2t \sin 3t) = -\frac{1}{2}[L(\cos 5t) - L(\cos t)] = -\frac{1}{2}\left[\frac{s}{s^2+5^2} - \frac{s}{s^2+1^2}\right]$$

Now by first shifting theorem,

$$L\left[e^t \sin 2t \sin 3t\right] = -\frac{1}{2}\left[\frac{s-1}{(s-1)^2+5^2} - \frac{s-1}{(s-1)^2+1^2}\right]$$

$$= \frac{1}{2}\left[\frac{s-1}{s^2-2s+26} - \frac{s-1}{s^2-2s+2}\right]$$

$$= \frac{12(s-1)}{(s^2-2s+2)(s^2-2s+26)}$$

(iv) We have

$$e^{-3t}\cosh 5t \sin 4t = e^{-3t}\left(\frac{e^{5t}+e^{-5t}}{2}\right)\sin 4t = \frac{1}{2}\left(e^{2t}+e^{-8t}\right)\sin 4t$$

Now $L(\sin 4t) = \dfrac{4}{s^2+16}$

$\therefore$ By first shifting theorem,

$$L\left[e^{-3t}\cosh 5t \sin 4t\right] = \frac{1}{2}L\left[(e^{2t}+e^{-8t})\sin 4t\right] = \frac{1}{2}\left[\frac{4}{(s-2)^2+16} + \frac{4}{(s+8)^2+16}\right]$$

$$= \frac{4(s^2+6s+50)}{(s^2-4s+20)(s^2+16s+80)}$$

Example 9 : Find Laplace transform of $\sin 2t \cos t \cosh 2t$. **(M.U. 1994, 2004, 11, 14)**

Sol. : $$\sin 2t \cos t = \frac{1}{2}\cdot 2\sin 2t \cos t = \frac{1}{2}[\sin 3t + \sin t]$$

$$\cosh 2t = \frac{e^{2t}+e^{-2t}}{2}$$

$$\therefore\ \sin 2t \cos t \cosh 2t = \frac{1}{2}(e^{2t}+e^{-2t})(\sin 3t + \sin t) \quad \text{..... (1)}$$

$$\therefore\ L\sin 3t = \frac{3}{s^2+9}$$

By first shifting theorem,

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$$\therefore\ L\left[e^{2t}\sin 3t\right] = \frac{3}{(s-2)^2+9},\quad L\left[e^{-2t}\sin 3t\right] = \frac{3}{(s+2)^2+9}$$

$$\therefore\ L\left(e^{2t}\sin 3t\right) + L\left(e^{-2t}\sin 3t\right) = 3\left[\frac{1}{(s-2)^2+9} + \frac{1}{(s+2)^2+9}\right]$$

$$= \frac{3\cdot 2(s^2+13)}{s^4+10s^2+13^2}$$

Now $\sin t = \dfrac{1}{s^2+1}$

By first shifting theorem,

$$\therefore\ L(e^{2t}\sin t) = \frac{1}{(s-2)^2+1},\quad L(e^{-2t}\sin t) = \frac{1}{(s+2)^2+1}$$

$$\therefore\ L(e^{2t}\sin t) + L(e^{-2t}\sin t) = \frac{2(s^2+5)}{s^4-6s^2+5^2}$$

From (1), (2) and (3), we get

$$L[\sin 2t\cos t\cosh 2t] = \frac{3(s^2+13)}{s^4+10s^2+13^2} + \frac{s^2+5}{s^4-6s^2+5^2}$$

8. Evaluation of the Integral $\int_0^\infty e^{-at}\cdot e^{bt} f(t)\,dt$

To find $\int_0^\infty e^{-st}\cdot e^{bt} f(t)\,dt$ we first find $L[e^{bt} f(t)]$, say $\Phi(s)$.

This means $\int_0^\infty e^{-st}\cdot e^{bt} f(t)\,dt = \Phi(s)$

Now, we put $s = a$ in both sides.

Example 1 : Evaluate $\int_0^\infty e^{-t}\sinh 2t\sin 3t\,dt$.

Sol. : Now, $L(\sin 3t) = \dfrac{3}{s^2+9}$. But $\sinh 2t = \dfrac{e^{2t}-e^{-2t}}{2}$

$$\therefore\ L\sinh 2t\cdot\sin 3t = \frac{1}{2}L\left[(e^{2t}-e^{-2t})\cdot\sin 3t\right] = \frac{1}{2}\left\{L(e^{2t}\sin 3t) - L(e^{-2t}\sin 3t)\right\}$$

By first shifting theorem,

$$L\sinh 2t\cdot\sin 3t = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right]$$

This means, $\displaystyle\int_0^\infty e^{-st}\sinh 2t\sin 3t\,dt = \frac{1}{2}\left[\frac{3}{(s-2)^2+9} - \frac{3}{(s+2)^2+9}\right]$

Putting $s = 1$, $\displaystyle\int_0^\infty e^{-t}\sinh 2t\sin 3t\,dt = \frac{1}{2}\left[\frac{3}{(1-2)^2+9} - \frac{3}{(1+2)^2+9}\right]$

$$= \frac{1}{2}\left[\frac{3}{1+9} - \frac{3}{9+9}\right] = \frac{1}{2}\left[\frac{3}{10} - \frac{3}{18}\right] = \frac{1}{15}.$$

Alternatively : We have to find

$$\int_0^\infty e^{-t}\left(\frac{e^{2t}-e^{-2t}}{2}\right)\sin 3t\,dt = \frac{1}{2}\left[\int_0^\infty e^{t}\sin 3t\,dt - \int_0^\infty e^{-3t}\sin 3t\,dt\right]$$

But $L\sin 3t = \dfrac{3}{s^2+9}$. This means, $\displaystyle\int_0^\infty e^{-st}\sin 3t\,dt = \frac{3}{s^2+9}$

Putting $s = -1$, we get $\displaystyle\int_0^\infty e^{t}\sin 3t\,dt = \frac{3}{1+9} = \frac{3}{10}$

Putting $s = 3$, we get $\displaystyle\int_0^\infty e^{-3t}\sin 3t\,dt = \frac{3}{9+9} = \frac{3}{18} = \frac{1}{6}$

$\therefore$ Required integral, $I = \dfrac{1}{2}\left(\dfrac{3}{10} - \dfrac{1}{6}\right) = \dfrac{1}{15}$.

Example 2 : Evaluate $\int_0^\infty e^{-t}\sin\frac{t}{2}\sinh\frac{\sqrt{3}}{2}t\,dt$. **(M.U. 2012)**

Sol. : Now $L\left(\sin\dfrac{t}{2}\right) = \dfrac{1/2}{s^2+(1/4)}$. $\sinh\dfrac{\sqrt{3}}{2}t = \dfrac{e^{(\sqrt{3}/2)t} - e^{-(\sqrt{3}/2)t}}{2}$

$$L\sinh\left(\frac{\sqrt{3}}{2}t\right)\sin\frac{t}{2} = \frac{1}{2}\left[L\left\{e^{(\sqrt{3}/2)t}\sin\frac{t}{2} - e^{-(\sqrt{3}/2)t}\sin\frac{t}{2}\right\}\right]$$

By first shifting theorem,

$$L\sinh\left(\frac{\sqrt{3}}{2}t\right)\sin\frac{t}{2} = \frac{1}{2}\left[\frac{1/2}{[s-(\sqrt{3}/2)]^2+(1/4)} - \frac{1/2}{[s+(\sqrt{3}/2)]^2+(1/4)}\right]$$

$$\therefore\ \int_0^\infty e^{-st}\sin\frac{t}{2}\sinh\frac{\sqrt{3}}{2}t\,dt = \frac{1}{2}\left[\frac{1/2}{[s-(\sqrt{3}/2)]^2+(1/4)} - \frac{1/2}{[s+(\sqrt{3}/2)]^2+(1/4)}\right]$$

Putting $s = 1$,

$$\therefore\ \int_0^\infty e^{-t}\sin\frac{t}{2}\sinh\frac{\sqrt{3}}{2}t\,dt = \frac{1}{2}\left[\frac{1/2}{[1-(\sqrt{3}/2)]^2+(1/4)} - \frac{1/2}{[1+(\sqrt{3}/2)]^2+(1/4)}\right]$$

$$= \frac{1}{4}\left[\frac{1}{1-\sqrt{3}+1} - \frac{1}{1+\sqrt{3}+1}\right] = \frac{1}{4}\left[\frac{1}{2-\sqrt{3}} - \frac{1}{2+\sqrt{3}}\right]$$

$$= \frac{1}{4}\left[\frac{2+\sqrt{3}-2+\sqrt{3}}{4-3}\right] = \frac{\sqrt{3}}{2}.$$

Alternatively : We have to find

$$\int_0^\infty e^{-t}\left(\frac{e^{(\sqrt{3}/2)t} - e^{-(\sqrt{3}/2)t}}{2}\right)\sin\frac{t}{2}\,dt = \frac{1}{2}\int_0^\infty\left[e^{-(1-\sqrt{3}/2)t} - e^{-(1+\sqrt{3}/2)t}\right]\cdot\sin\frac{t}{2}\,dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(1-\sqrt{3}/2)t}\cdot\sin\frac{t}{2}\,dt - \int_0^\infty e^{-(1+\sqrt{3}/2)t}\cdot\sin\frac{t}{2}\,dt\right]$$

But $L\left(\sin\dfrac{t}{2}\right) = \dfrac{1/2}{s^2+(1/4)} = \dfrac{2}{4s^2+1}$

That means, $\displaystyle\int_0^\infty e^{-st}\cdot\sin\frac{t}{2}\,dt = \frac{2}{4s^2+1}$

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Putting $s = 1 - \frac{\sqrt{3}}{2}$, we get

$$\int_0^\infty e^{-(1-\sqrt{3}/2)t} \cdot \sin\frac{t}{2}\,dt = \frac{2}{4(1-\sqrt{3}/2)^2+1} = \frac{2}{4[1-\sqrt{3}+(3/4)]+1}$$

$$= \frac{2}{4-4\sqrt{3}+3+1} = \frac{2}{8-4\sqrt{3}} = \frac{1}{4-2\sqrt{3}}$$

Similarly, putting $s = 1 + \frac{\sqrt{3}}{2}$, we get

$$\int_0^\infty e^{-(1+\sqrt{3}/2)t} \cdot \sin\frac{t}{2}\,dt = \frac{2}{4[1+\sqrt{3}+(3/4)]+1} = \frac{2}{8+4\sqrt{3}} = \frac{1}{4+2\sqrt{3}}$$

$\therefore$ Required integral,

$$I = \frac{1}{2}\left[\frac{1}{4-2\sqrt{3}} - \frac{1}{4+2\sqrt{3}}\right] = \frac{1}{2}\left[\frac{4\sqrt{3}}{16-12}\right] = \frac{\sqrt{3}}{2}.$$

EXERCISE - VII

(A) Find the Laplace transform of

1. $\cos at \cdot \sin h\, at$
2. $e^t \sin 4t \cdot \cos 2t$
3. $e^{-2t}(2\cos 3t - 3\sin 3t)$.
4. $\cos h\, at \sin at$
5. $\cos 3t \cdot \cos h\, 4t$
6. $t^n e^{-at} + \sin^3 t$
7. $\cos h\, at \cos at$
8. $t^2 - 3t + 5 + e^{2t} t^2$
9. $\cos h\, at \cos bt$
10. $e^{4t} t^{3/2}$
11. $e^{-4t} \cos ht \sin t$ **(M.U. 1995)**
12. $e^{-3t} \cos h\, 4t \sin 3t$ **(M.U. 1999)**
13. $e^{2t} \cos 2t \cos t$ **(M.U. 2002, 03)**
14. $e^{2t}(1+t)^2$ **(M.U. 2002, 03)**
15. $e^{2t} \sin^4 t$ **(M.U. 2007)**
16. $e^{-2t} \cos ht \sin t$ **(M.U. 2003, 06)**
17. $e^{-t} \sin^2 t$ **(M.U. 2009)**

[Ans. : (1) $\frac{a(s^2-2a^2)}{s^4+4a^4}$, (2) $\frac{3}{(s-1)^2+6^2} + \frac{1}{(s-1)^2+2^2}$, (3) $\frac{2s-5}{s^2+4s+13}$,

(4) $\frac{a(s^2+2a^2)}{s^4+2a^4}$, (5) $\frac{s(s^2-7)}{s^4+25^2-14s^2}$, (6) $\frac{\lfloor n+1}{(s+a)^{n+1}} - \frac{6}{(s^2+1)(s^2+9)}$,

(7) $\frac{s^3}{s^4+4a^4}$, (8) $\frac{5s^2-3s+2}{s^3} + \frac{2}{(s-2)^3}$, (9) $\frac{s[s^2+(b^2-a^2)]}{s^4+2(b^2-a^2)s^2+(a^2+b^2)^2}$,

(10) $\frac{\lfloor 5/2}{(s-4)^{5/2}}$, (11) $\frac{s^2+8s+18}{(s^2+6s+10)(s^2+10s+26)}$,

(12) $\frac{3(s^2+6s+34)}{(s^2-2s+10)(s^2+14s+58)}$, (13) $\frac{(s-2)(s^2-4s+9)}{(s^2-4s+13)(s^2-4s+5)}$

(14) $\frac{1}{s-2} + \frac{2}{(s-2)^2} + \frac{2}{(s-2)^3}$. (15) $\frac{1}{8}\left[\frac{3}{s-2} - \frac{4(s-2)}{s^2-4s+8} + \frac{s-2}{s^2-8s+20}\right]$

(16) $\frac{1}{2}\left[\frac{1}{s^2+2s+2} + \frac{1}{s^2+6s+10}\right]$ (17) $\frac{1}{2}\left[\frac{1}{s+1} - \frac{s+1}{s^2+2s+5}\right]$

(B) 1. If $L[f(t)] = \frac{s}{s^2+s+4}$, find $L[e^{-3t} f(2t)]$. **(M.U. 2003) [Ans. :** $\frac{s+3}{s^2+8s+10}$]

2. If $L[f(t)] = \frac{1}{s(s^2+1)}$, find $L[e^{-t} \cdot f(2t)]$. **(M.U. 2007) [Ans. :** $\frac{4}{(s+1)(s^2+2s+5)}$]

(C) Evaluate the following integrals.

1. $\int_0^\infty e^{-t}(t^2 - 3t + 5 + e^{2t} t^2)\,dt$ 2. $\int_0^\infty e^{-2t} \cos h \sin t\,dt$

3. $\int_0^\infty e^{-t} \cos ht \cos 2t\,dt$ **[Ans. : (1) 2, (2)** $\frac{3}{10}$, **(3)** $\frac{1}{8}$]

9. Second Shifting Theorem

If $L[f(t)] = \Phi(s)$ and $g(t) = f(t-a)$ when $t > a$ and $g(t) = 0$ when $t < a$, then prove that

$$\boxed{L[g(t)] = e^{-as}\,\Phi(s)}$$ **(M.U. 1995, 2002, 11)** ..... **(15)**

Proof : By definition of Laplace transform,

$$L\{g(t)\} = \int_0^\infty e^{-st} g(t)\,dt = \int_0^a e^{-st} g(t)\,dt + \int_a^\infty e^{-st} g(t)\,dt$$

$$= \int_0^a e^{-st} \cdot 0\,dt + \int_a^\infty e^{-st} f(t-a)\,dt$$

Now put $t - a = p$ $\therefore$ $dt = dp$

$$\therefore\ L\{g(t)\} = \int_0^\infty e^{-s(a+p)} f(p)\,dp = e^{-as}\int_0^\infty e^{-sp} f(p)\,dp$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt = e^{-as} L[f(t)] = e^{-as}\,\Phi(s)$$

Example : Using second shifting theorem find,

(i) $L[f(t)]$ where $f(t) = \cos(t-\alpha)$, $t > \alpha$ and $f(t) = 0$, $t < \alpha$.

(ii) $L[f(t)]$ where $f(t) = e^{t-k}$, $t > k$ and $f(t) = 0$, $t < k$.

(iii) $L[f(t)]$ where $f(t) = \sin(t - \pi/3)$, $t > \pi/3$ and $f(t) = 0$, $t < \pi/3$.

Sol. : (i) We have $L \cos t = \frac{s}{s^2+1}$. Hence, by second shifting theorem,

$$L\cos(t-\alpha) = e^{-\alpha s} \cdot \frac{s}{s^2+1}.$$

Note ....

We have obtained this result in Ex. 2(2) on page 1-2.

(ii) We have $L(e^t) = \frac{1}{s-1}$. Hence, by second shifting theorem,

$$L(e^{t-k}) = e^{-ks}\frac{1}{s-1}.$$

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(iii) We have $L \sin t = \dfrac{1}{s^2+1}$. Hence, by second shifting theorem

$$L \sin(t - \pi/3) = e^{-\pi s/3} \cdot \frac{1}{s^2+1}.$$

Remark ....

Solve the above examples (ii) and (iii) by using the definition of $L[f(t)]$.

EXERCISE - VIII

Using second shifting theorem find $L f(t)$ where,

1. $f(t) = \cos[t - (2\pi/3)]$, $t > 2\pi/3$ and $f(t) = 0$, $t < 2\pi/3$.
2. $f(t) = \sin[t - (2\pi/3)]$, $t > 2\pi/3$ and $f(t) = 0$, $t < 2\pi/3$.
3. $f(t) = (t-1)^3$, $t > 1$ and $f(t) = 0$, $t < 1$.
4. $f(t) = (t-2)^2$, $t > 2$ and $f(t) = 0$, $t < 2$.
5. $f(t) = (t-1)^4$, $t > 1$ and $f(t) = 0$, $t < 1$.

[**Ans. :** (1) $e^{-2\pi s/3} \cdot \dfrac{s}{s^2+1}$ (2) $e^{-2\pi s/3} \cdot \dfrac{1}{s^2+1}$ (3) $e^{-s} \cdot \dfrac{3!}{s^4}$

(4) $e^{-2s} \cdot \dfrac{2}{s^3}$ (5) $e^{-s} \cdot \dfrac{24}{s^5}$.]

10. Effect of Multiplication by t

If $L[f(t)] = \Phi(s)$, then $\boxed{L[t^n f(t)] = (-1)^n \dfrac{d^n}{ds^n}\Phi(s)}$ **(M.U. 2010, 11, 12)** ..... (16)

Proof : We shall prove this property by the method of induction.

Step 1 : Let $\Phi(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$

Differentiating both sides w.r.t. s and applying the rule of differentiation under integral sign

$$\Phi'(s) = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} f(t)\,dt\right] = -\int_0^\infty e^{-st}\, t f(t)\,dt = -L[t f(t)]$$

$$\therefore \quad L[t f(t)] = (-1)\frac{d}{ds}\Phi(s) \qquad \text{..... (1)}$$

Thus, the rule is true for $n = 1$.

Step 2 : Now we assume the rule is true for $n = m$ and prove that it is true for $n = m + 1$. *i.e.* we assume that

$$L[t^m f(t)] = (-1)^m \frac{d^m}{ds^m}\Phi(s)$$

$$\therefore \quad (-1)^m \frac{d^m}{ds^m}\Phi(s) = L[t^m f(t)] = \int_0^\infty e^{-st}\, t^m f(t)\,dt$$

Differentiating both sides w.r.t. s and applying the rule of differentiation under the integral sign

$$(-1)^m \frac{d^{m+1}}{ds^{m+1}}\Phi(s) = \int_0^\infty \frac{\partial}{\partial s}\left[e^{-st} \cdot t^m f(t)\,dt\right]$$

$$\therefore \quad (-1)^m \frac{d^{m+1}}{ds^{m+1}}\Phi(s) = -\int_0^\infty e^{-st} \cdot t^{m+1} f(t)\,dt = -L[t^{m+1} f(t)]$$

$$\therefore \quad L\left[t^{m+1} f(t)\right] = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}}\Phi(s)$$

Thus, if the property is true form $n = m$ then it is true for $n = m + 1$.

Step 3 : Since, the property is true for $n = 1$, it is by step (3) true for $n = 1 + 1 = 2$. Since, it is true for $n = 2$, by step (3) again it is true for $n = 2 + 1 = 3$. Hence, it is true for any value of n.

Note ....

In particular, if $L[f(t)] = \Phi(s)$, then $\boldsymbol{L[t f(t)] = -\Phi'(s)}$, $\boldsymbol{L[t^2 f(t)] = \Phi''(s)}$.

Example 1 : Find the Laplace transforms of

(i) $t e^{at}$ (ii) $t e^{-t} \cosh 2t$ **(M.U. 2003, 15)** (iii) $t \sin at$

(iv) $(1 + t e^{-t})^3$ **(M.U. 1997, 2009, 19)** (v) $t^2 \sin at$

(vi) $t e^{-4t} \cdot \sin 3t$ (vii) $t^2 e^{-t} \sin 4t$. **(M.U. 2008)**

Sol. : (i) $L(e^{at}) = \dfrac{1}{s-a}$ $\quad \therefore \quad L(t e^{at}) = -\dfrac{d}{ds}\left(\dfrac{1}{s-a}\right) = \dfrac{1}{(s-a)^2}$

(ii) $e^{-t}\cosh 2t = e^{-t} \cdot \dfrac{e^{2t} + e^{-2t}}{2} = \dfrac{e^{t} + e^{-3t}}{2}$

$$\therefore \quad L(e^{-t}\cosh 2t) = \frac{1}{2} \cdot L(e^{t} + e^{-3t}) = \frac{1}{2}\left[\frac{1}{s-1} + \frac{1}{s+3}\right]$$

$$\therefore \quad L(t e^{t}\cosh 2t) = -\frac{d}{ds}\left[\frac{1}{2}\left(\frac{1}{s-1} + \frac{1}{s+3}\right)\right] = \frac{1}{2}\left[\frac{1}{(s-1)^2} + \frac{1}{(s+3)^2}\right].$$

(iii) $L(t \sin at) = (-1) \cdot \dfrac{d}{ds}[L(\sin at)] = -\dfrac{d}{ds}\left(\dfrac{a}{s^2+a^2}\right) = +\dfrac{2as}{(s^2+a^2)^2}$

(iv) $L\left[(1+te^{-t})^3\right] = L\left[1 + 3te^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t}\right]$

$$= L(1) + 3L(t e^{-t}) + 3L(t^2 e^{-2t}) + L(t^3 e^{-3t})$$

$$= L(1) + 3(-1)\frac{d}{ds}\left[L(e^{-t})\right] + 3(-1)^2 \frac{d^2}{ds}\left[L(e^{-2t})\right] + (-1)^3 \frac{d^3}{ds^3}\left[L(e^{-3t})\right]$$

$$= \frac{1}{s} - 3\frac{d}{ds}\left(\frac{1}{s+1}\right) + 3\frac{d^2}{ds^2}\left(\frac{1}{s+2}\right) - \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{s} + \frac{3}{(s+1)^2} + \frac{6}{(s+2)^3} + \frac{6}{(s+3)^4}$$

Alternatively we can obtain the Laplace transform of $(1 + t e^{-t})^3$ by using first shifting theorem.

$$L[(1+te^{-t})^3] = L[1 + 3te^{-t} + 3t^2 e^{-2t} + t^3 e^{-3t}]$$

$$= L[1] + 3L[e^{-t} t] + 3L[e^{-2t} \cdot t^2] + L[e^{-3t} t^3]$$

But $L[t^n] = \dfrac{n!}{s^{n+1}}$ and $L[e^{bt} t^n] = \dfrac{n!}{(s-b)^{n+1}}$

$$\therefore \quad L[(1+te^{-t})^3] = \frac{1}{s} + 3 \cdot \frac{1}{(s+1)^2} + 3 \cdot \frac{2!}{(s+2)^3} + \frac{3!}{(s+3)^4}.$$

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(v) $L(t^2 \sin at) = (-1)^2 \dfrac{d^2}{ds^2} L(\sin at) = \dfrac{d^2}{ds^2}\left(\dfrac{a}{s^2+a^2}\right) = \dfrac{d}{ds}\left[\dfrac{-2as}{(s^2+a^2)^2}\right]$

$$= -\frac{(s^2+a^2)^2 \cdot 2a - 2as \cdot 2(s^2+a^2) \cdot 2s}{(s^2+a^2)^4} = \frac{2a(3s^2-a^2)}{(s^2+a^2)^3}$$

(vi) $L\left[t \cdot e^{-4t} \sin 3t\right] = (-1)\dfrac{d}{ds} L\left[e^{-4t} \sin 3t\right]$

$$= -\frac{d}{ds}\left[\frac{3}{(s+4)^2+9}\right] \qquad \text{[By shifting theorem]}$$

$$= -\frac{d}{ds}\left[\frac{3}{s^2+8s+25}\right] = \frac{6(s+4)}{(s^2+8s+25)^2}$$

(vii) $L\left[t^2 \cdot e^{-t} \sin 4t\right] = (-1)^2 \cdot \dfrac{d^2}{ds^2} L\left[e^{-t} \sin 4t\right]$

$$= \frac{d^2}{ds^2}\left[\frac{4}{(s+1)^2+16}\right] \qquad \text{[By shifting theorem]}$$

$$= \frac{d^2}{ds^2}\left[\frac{4}{s^2+2s+17}\right] = \frac{8(3s^2+6s-13)}{(s^2+2s+17)^3}.$$

Example 2 : Find the Laplace transforms of the following.

(i) $t \sin^3 t$ **(M.U. 1994, 2015)** (ii) $t \sin 2t \cos ht$ **(M.U. 1995)**

(iii) $t \cos^2 t$ **(M.U. 1993, 2012)** (iv) $t^5 \cos ht$ **(M.U. 1996)**

(v) $t e^{3t} \sin 4t$ **(M.U. 1997, 2016, 19)** (vi) $t^n e^{at}$.

Sol. : (i) $\sin 3t = 3 \sin t - 4 \sin^3 t$

$$L\left[\sin^3 t\right] = \frac{1}{4}\left[L(3 \sin t) - L(\sin 3t)\right] = \frac{1}{4} \cdot \left[\frac{3}{s^2+1} - \frac{3}{s^2+9}\right]$$

$$\therefore \; L\left[t \sin^3 t\right] = -\frac{3}{4}\frac{d}{ds}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9}\right] = -\frac{3}{4}\left[-\frac{2s}{(s^2+1)^2} + \frac{2s}{(s^2+9)^2}\right]$$

$$= \frac{3s}{2}\left[\frac{1}{(s^2+1)^2} - \frac{1}{(s^2+9)^2}\right] = \frac{3s}{2}\left[\frac{s^4+18s^2+81-s^4-2s^2-1}{(s^2+1)^2(s^2+9)^2}\right]$$

$$= \frac{3s}{2} \cdot \frac{16(s+5)}{(s^2+1)^2(s^2+9)^2} = 24 \cdot \frac{s(s+5)}{(s^2+1)^2(s^2+9)^2}$$

(ii) Since $\cos ht = \dfrac{e^t + e^{-t}}{2}$, $\sin 2t \cos ht = \dfrac{e^t + e^{-t}}{2} \cdot \sin 2t$

Now $t \sin 2t \cos ht = (-1)\dfrac{d}{ds} L(\sin 2t \cos ht) = (-1)\dfrac{d}{ds}\left[\dfrac{1}{2}\left(e^t \sin 2t + e^{-t} \sin 2t\right)\right]$

$$= -\frac{1}{2}\frac{d}{ds}\left[\frac{2}{(s-1)^2+2^2} + \frac{2}{(s+1)^2+2^2}\right]$$

$$= \frac{2s-2}{(s^2-2s+5)^2} + \frac{2s+2}{(s^2+2s+5)^2}$$

$$\therefore \; t \sin 2t \cos ht = 2\left[\frac{s-1}{(s^2-2s+5)^2} + \frac{s+1}{(s^2+2s+5)^2}\right]$$

(iii) $L(t \cos^2 t) = Lt\left(\dfrac{1+\cos 2t}{2}\right) = \dfrac{1}{2}L(t) + \dfrac{1}{2}L(t \cos 2t)$

$$= \frac{1}{2} \cdot \frac{1}{s^2} - \frac{1}{2} \cdot \frac{d}{ds} L(\cos 2t) = \frac{1}{2} \cdot \frac{1}{s^2} - \frac{1}{2}\frac{d}{ds} \cdot \frac{s}{s^2+2^2}$$

$$= \frac{1}{2s^2} - \frac{1}{2}\left[\frac{s^2+2^2-s \cdot 2s}{(s^2+2^2)^2}\right] = \frac{1}{2s^2} + \frac{1}{2} \cdot \frac{s^2-2^2}{(s^2+2^2)^2}$$

Similarly, $L(t \sin^2 t) = \dfrac{1}{2s^2} - \dfrac{1}{2}\left[\dfrac{s^2-2^2}{(s^2+2^2)^2}\right]$

(iv) $L(t^5 \cos ht) = Lt^5\left(\dfrac{e^t + e^{-t}}{2}\right) = \dfrac{1}{2}Lt^5(e^t) + \dfrac{1}{2}Lt^5(e^{-t})$

$$= \frac{1}{2} \cdot (-1)^5 \cdot \frac{d^5}{ds^5}\left(\frac{1}{s-1}\right) + \frac{1}{2} \cdot (-1)^5 \cdot \frac{d^5}{ds^5}\left(\frac{1}{s+1}\right)$$

But $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = -\dfrac{1}{x^2}$, $\dfrac{d^2}{dx^2}\left(\dfrac{1}{x}\right) = \dfrac{2}{x^3}$, $\dfrac{d^3}{dx^3}\left(\dfrac{1}{x}\right) = -\dfrac{2 \cdot 3}{x^4}$,

$\dfrac{d^4}{dx^4}\left(\dfrac{1}{x}\right) = \dfrac{2 \cdot 3 \cdot 4}{x^5} = \dfrac{4!}{x^5}$, $\dfrac{d^5}{dx^5}\left(\dfrac{1}{x}\right) = -\dfrac{5!}{x^6}$.

$$\therefore \; L(t^5 \cos ht) = -\frac{1}{2} \cdot \frac{(-5!)}{(s-1)^6} - \frac{1}{2} \cdot \frac{(-5!)}{(s+1)^6} = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$$

(v) $L(\sin 4t) = \dfrac{4}{s^2+4^2}$ $\quad\therefore$ By first shifting theorem

$$L\left[e^{3t} \sin 4t\right] = \frac{4}{(s-3)^2+4^2} = \frac{4}{s^2-6s+25}$$

$$\therefore \; L\left[t e^{3t} \sin 4t\right] = -\frac{d}{ds}\left(\frac{4}{s^2-6s+25}\right) = \frac{4(2s-6)}{(s^2-6s+25)^2} = \frac{8(s-3)}{(s^2-6s+25)^2}.$$

(vi) $L e^{at} = \dfrac{1}{s-a}$

$$L\left[t^n e^{at}\right] = (-1)^n \cdot \frac{d^n}{ds^n} \cdot \frac{1}{(s-a)} = (-1)^n \cdot \frac{n!}{(s-a)^{n+1}}$$

Example 3 : Find the Laplace transforms of the following.

(i) $t e^{3t} \sin t$ **(M.U. 1998, 2002, 13, 17)** (ii) $t\sqrt{1+\sin t}$ **(M.U. 1999, 2000, 02, 07, 13, 15, 17)**

(iii) $t^3 \cos t$ **(M.U. 2014)** (iv) $t e^{3t} erf \sqrt{t}$. **(M.U. 2000)** (v) $t\left(\dfrac{\sin t}{e^t}\right)^2$ **(M.U. 2004)**

Sol. : (i) We have $L \sin t = \dfrac{1}{s^2+1}$

$$\therefore \; L(e^{3t} \sin t) = \frac{1}{(s-3)^2+1} = \frac{1}{s^2-6s+10} \qquad \text{[By first shifting theorem]}$$

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$$\therefore\ L(te^{3t}\sin t) = -\frac{d}{ds}\left(\frac{1}{s^2-6s+10}\right) = \frac{2s-6}{(s^2-6s+10)^2}.$$

(ii) We have

$$\sqrt{1+\sin t} = \sqrt{\left[\sin^2(t/2)+\cos^2(t/2)+2\sin(t/2)\cos(t/2)\right]}$$

$$= \sqrt{\left[\sin(t/2)+\cos(t/2)\right]^2} = \sin(t/2)+\cos(t/2)$$

$$\therefore\ L\sqrt{1+\sin t} = L\left[\sin(t/2)+\cos(t/2)\right] = \frac{1/2}{s^2+(1/2)^2}+\frac{s}{s^2+(1/2)^2}$$

$$= \frac{1}{2}\cdot\frac{4}{(4s^2+1)}+\frac{4s}{(4s^2+1)} = \frac{4s+2}{(4s^2+1)} = \frac{2(2s+1)}{(4s^2+1)}$$

$$\therefore\ L\left[t\sqrt{1+\sin t}\right] = -\frac{d}{ds}\left[\frac{2(2s+1)}{(4s^2+1)}\right] = -2\left[\frac{(4s^2+1)2-(2s+1)8s}{(4s^2+1)^2}\right]$$

$$= -2\frac{[-8s^2-8s+2]}{(4s^2+1)^2} = 4\frac{(4s^2+4s-1)}{(4s^2+1)^2}$$

(iii) $L\cos t = \dfrac{s}{s^2+1}$

$$\therefore\ \frac{d}{ds}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)\cdot 1 - s\cdot 2s}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

$$\frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{(s^2+1)^2(-2s)-(1-s^2)\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}$$

$$\therefore\ \frac{d^2}{ds^2}\left(\frac{s}{s^2+1}\right) = \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3} = \frac{2s^3-6s}{(s^2+1)^3}$$

$$\frac{d^3}{ds^3}\left(\frac{d}{s^2+1}\right) = \frac{(s^2+1)^3(6s^2-6)-(2s^3-6s)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}$$

$$= \frac{6\left[(s^2+1)(s^2-1)-(2s^3-6s)s\right]}{(s^2+1)^4} = -\frac{6(s^4-6s^2+1)}{(s^2+1)^4}$$

$$\therefore\ L\left[t^3\cos t\right] = \frac{6(s^4-6s^2+1)}{(s^2+1)^4}.$$

(iv) We have $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$ $\quad\therefore\ L(e^{3t}erf\sqrt{t}) = \dfrac{1}{(s-3)\sqrt{s-2}}$

$$L(te^{2t}erf\sqrt{t}) = -\frac{d}{ds}\left[\frac{1}{(s-3)\sqrt{s-2}}\right] = -\left[-\frac{1}{(s-3)^2(s-2)}\frac{d}{ds}\left\{(s-3)\sqrt{s-2}\right\}\right]$$

$$= \frac{1}{(s-3)^2(s-2)}\left[(s-3)\frac{1}{2\sqrt{s-2}}+\sqrt{s-2}\right]$$

$$= \frac{1}{(s-3)^2(s-2)}\cdot\frac{(s-3+2s-4)}{2\sqrt{s-2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}$$

(v) We have

$$t\left(\frac{\sin t}{e^t}\right)^2 = t\cdot e^{-2t}\sin^2 t = t\cdot e^{-2t}\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\cdot t\cdot e^{-2t}\left[1-\cos 2t\right]$$

Now $L(1-\cos 2t) = L(1)-L(\cos 2t) = \dfrac{1}{s}-\dfrac{s}{s^2+2^2}$

By first shifting theorem,

$$L\{e^{-2t}(1-\cos 2t)\} = \frac{1}{s+2}-\frac{s+2}{(s+2)^2+2^2}$$

$$\therefore\ L\{e^{-2t}(1-\cos 2t)\} = \frac{1}{s+2}-\frac{s+2}{s^2+4s+8}$$

$$\therefore\ L\left\{\frac{1}{2}\cdot t\cdot e^{-2t}(1-\cos 2t)\right\} = -\frac{1}{2}\frac{d}{ds}\left\{\frac{1}{s+2}-\frac{s+2}{s^2+4s+8}\right\}$$

$$= -\frac{1}{2}\left[-\frac{1}{(s+2)^2}-\frac{(s^2+4s+8)\cdot 1-(s+2)(2s+4)}{(s^2+4s+8)^2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s+2)^2}+\frac{s^2+4s}{(s^2+4s+8)^2}\right]$$

Example 4 : If $L[erf\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$, find

(i) $L[t\,erf\,2\sqrt{t}]$ **(M.U. 2003, 06)** (ii) $L[t\,e^{3t}erf\sqrt{t}]$

Sol. : (i) By change of scale property if $L[f(t)] = \Phi(s)$, then $L[f(at)] = \dfrac{1}{a}\Phi\left(\dfrac{s}{a}\right)$.

Since, $L\left[erf\sqrt{t}\right] = \dfrac{1}{s\sqrt{s+1}}$,

$$L\left[erf\,2\sqrt{t}\right] = L\left[erf\sqrt{4t}\right] = \frac{1}{4}\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}} = \Phi(s)$$

(We have found $L\left[erf\sqrt{t}\right]$ in (11), page 1-6.)

By the effect of multiplication of t

$$L[t\,f(t)] = -\Phi'(s) = -\frac{d}{ds}\left[2(s^3+4s^2)^{-1/2}\right]$$

$$\therefore\ L[t\,f(t)] = -2\cdot\left(-\frac{1}{2}\right)(s^3+4s^2)^{-3/2}(3s^2+8s)$$

$$\therefore\ L\left[t\,erf\,2\sqrt{t}\right] = \frac{s(3s+8)}{s^3(s+4)^{3/2}} = \frac{3s+8}{s^2(s+4)^{3/2}}$$

(ii) Since, $L\left(erf\sqrt{t}\right) = \dfrac{1}{s\sqrt{s+1}}$

$$L\left[e^{3t}\cdot erf\sqrt{t}\right] = \frac{1}{(s-3)\sqrt{(s-3)+1}} = \frac{1}{(s-3)\sqrt{s-2}}$$

$$= -\frac{d}{dt}\left[\frac{1}{(s-3)\sqrt{s-2}}\right] = \frac{1}{(s-3)^2(s-2)}\frac{d}{dt}\left[(s-3)\sqrt{s-2}\right]$$

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$$\therefore\ L\left[t e^{3t}\cdot erf\sqrt{t}\right]=\frac{1}{(s-3)^2(s-2)}\frac{d}{dt}\left[(s-3)\sqrt{s-2}\right]$$

$$=\frac{1}{(s-3)^2(s-2)}\left[(s-3)\cdot\frac{1}{2\sqrt{s-2}}+\sqrt{s-2}\cdot 1\right]$$

$$=\frac{1}{2(s-3)^2(s-2)^{3/2}}[(s-3)+2(s-2)]$$

$$\therefore\ L\left[t e^{3t}\cdot erf\sqrt{t}\right]=\frac{3s-7}{2(s-3)^2(s-2)^{3/2}}.$$

Example 5 : Find $L[(t+e^{-t}+\sin t)^2]$. **(M.U. 2012)**

Sol. : We have

$$(t+e^{-t}+\sin t)^2=t^2+e^{-2t}+\sin^2 t+2te^{-t}+2t\sin t+2e^{-t}\sin t$$

$$\therefore\ L[(t+e^{-t}+\sin t)^2]=L(t^2)+L(e^{-2t})+\frac{1}{2}[L(1)-L(\cos 2t)]$$
$$+2[L(e^{-t}t)+L(t\sin t)+L(e^{-t}\sin t)]$$

$$\because\ L(t^2)=\frac{2}{s^3},\ L(e^{-2t})=\frac{1}{s+2},\ L(1)=\frac{1}{s},$$

$$L[\cos 2t]=\frac{s}{s^2+2^2},\ L(t)=\frac{1}{s^2},\ L(e^{-t}t)=\frac{1}{(s+1)^2},$$

$$L(t\sin t)=-\frac{d}{ds}\left[\frac{1}{s^2+1}\right]=\frac{2s}{(s^2+1)^2},\ L(e^{-t}\sin t)=\frac{1}{(s+1)^2+1}$$

$$\therefore\ L[(t+e^{-t}+\sin t)^2]=\frac{2}{s^3}+\frac{1}{s+2}+\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+2^2}\right]$$
$$+2\left[\frac{1}{(s+1)^2}+\frac{2s}{(s^2+1)^2}+\frac{1}{s^2+2s+2}\right].$$

EXERCISE - IX

(A) 1. If $L[f(t)]=\dfrac{s+3}{s^2+s+1}$, find $L[t f(2t)]$. **(M.U. 2004)** [**Ans. :** $\dfrac{s^2+12s+8}{(s^2+2s+4)^2}$]

2. If $L\left[erf\sqrt{t}\right]=\dfrac{1}{s\sqrt{s+1}}$, find $L\left[t\ erf\ 3\sqrt{t}\right]$. **(M.U. 2002, 11)** [**Ans. :** $\dfrac{9}{2}\cdot\dfrac{s+6}{s^2(s+9)^{3/2}}$]

(B) Find the Laplace transforms of the following.

1. $t^2\sin 3t$, 2. $t\cos^3 t$, 3. $te^{-2t}\sin t$, 4. $te^{3t}\sin 2t$, **(M.U. 2005)**
5. $te^{3t}\sin 3t$, **(M.U. 2004)** 6. $te^{-3t}\cos 2t$ 7. $te^{3t}\sin^2 t$, 8. $t(2\sin 3t+e^{2t})$,
9. $t(2\sin 3t-3\cos 3t)$, 10. $te^{2t}(\cos t-\sin t)$,
11. $\frac{1}{4}e^{-2t}(2t\cos 2t+\sin 2t)$, 12. $te^{t}\sin 2t\cos t$, **(M.U. 2003)**
13. $t\sqrt{1-\sin t}$, **(M.U. 2000)** 14. $te^{-2t}\sinh 4t$. **(M.U. 2000, 06)**
15. $te^{3t}\sinh 2t$ **(M.U. 2003)** 16. $t\ erf\ 2\sqrt{t}$

17. $t\sqrt{1+\sin 2t}$ **(M.U. 2003, 05, 07, 13)**
18. $t\cos(wt-\alpha)$ **(M.U. 2003)** 19. $(t+\sin 2t)^2$ **(M.U. 2004)**
20. $(t\sinh 2t)^2$ **(M.U. 2003)** 21. $te^{3t}\sinh 2t$ **(M.U. 2003)**
22. $te^{-3t}\cos 2t\cos 3t$ **(M.U. 2014)** 23. $te^{-t}\cosh 2t$ **(M.U. 2015)**

[**Ans. :** (1) $18\dfrac{(s^2-3)}{(s^2+9)^3}$, (2) $\dfrac{1}{4}\left[\dfrac{s^2-9}{(s^2+9)^2}+\dfrac{3(s^2+1)}{(s^2+1)^2}\right]$, (3) $\dfrac{2(s+2)}{(s^2+4s+5)^2}$,

(4) $4\cdot\dfrac{s-3}{(s^2-6s+13)^2}$, (5) $6\cdot\dfrac{(s-3)}{(s^2-6s+18)^2}$, (6) $\dfrac{s^2+6s+5}{(s^2+6s+13)^2}$,

(7) $\dfrac{1}{2}\left[\dfrac{1}{(s-3)^2}-\dfrac{(s^2-6s+5)}{(s^2-6s+13)^2}\right]$, (8) $\dfrac{12s}{(s^2+9)^2}+\dfrac{1}{(s-2)^2}$, (9) $\dfrac{3(9+4s-s^2)}{(s^2+9)^2}$,

(10) $\dfrac{s^2-6s+7}{(s^2-4s+5)^2}$, (11) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$, (12) $\dfrac{3(s-1)}{(s^2-2s+10)^2}+\dfrac{s-1}{(s^2-2s+2)}$,

(13) $4\cdot\dfrac{(1+4s-4s^2)}{(4s^2+1)^2}$, (14) $\dfrac{8(s+2)}{(s^2+4s-12)^2}$, (15) $\dfrac{4(s-3)}{(s^2-6s+5)^2}$

(16) $\dfrac{3s+8}{s^2(s+4)^{3/2}}$, (17) $\dfrac{s^2+2s-1}{(s^2+1)^2}$, (18) $\dfrac{(s^2-w^2)\cos\alpha+2ws\sin\alpha}{(s^2+w^2)^2}$,

(19) $\dfrac{2}{s^3}+\dfrac{8s}{(s^2+2^2)^2}+\dfrac{1}{2s}-\dfrac{s}{2(s^2+4^2)}$, (20) $\dfrac{1}{2}\left[\dfrac{1}{(s-4)^3}-\dfrac{2}{s^3}+\dfrac{1}{(s+4)^3}\right]$,

(21) $\dfrac{1}{2}\left[\dfrac{1}{(s-5)^2}-\dfrac{1}{(s-1)^2}\right]$ (22) $\dfrac{1}{2}\left[\dfrac{s^2+6s-16}{(s^2+6s+34)^2}+\dfrac{s^2+6s+8}{(s^2+6s+10)^2}\right]$ (23) $\dfrac{s^2+2s+5}{(s^2+2s-3)^2}$]

11. Evaluation of the Integral $\int_0^\infty e^{-at}\,t\,f(t)\,dt$

To find the above integral, we first find $L[t f(t)]$, say $\Phi(s)$.

This means, $\int_0^\infty e^{-st}\cdot t f(t)\,dt=\Phi(s)$.

Now, we put $s=a$.

Example 1 : Evaluate $\int_0^\infty e^{-3t}t\sin t\cdot dt$. **(M.U. 2002, 05)**

Sol. : Consider $L[t\sin t]=(-1)\dfrac{d}{ds}[L(\sin t)]=-\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right)=\dfrac{2s}{(s^2+1)^2}$

That means, $\int_0^\infty e^{-st}t\sin t\,dt=\dfrac{2s}{(s^2+1)^2}$.

Putting $s=3$, $\int_0^\infty e^{-3t}\cdot t\sin t\cdot dt=\dfrac{6}{(9+1)^2}=\dfrac{3}{50}$.

Example 2 : Evaluate $\int_0^\infty e^{-t}(t\sqrt{1+\sin t})\cdot dt$. **(M.U. 2016)**

Sol. : We have proved in Ex. 3 (ii), page 1-31 that

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$$L(t\sqrt{1+\sin t}) = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

By definition of Laplace transform this means,

$$\int_0^\infty e^{-st}[t\sqrt{1+\sin t}]\,dt = \frac{4(4s^2+4s-1)}{(4s^2+1)^2}$$

Putting $s = 1$ on both sides, we get

$$\int_0^\infty e^{-t}(t\sqrt{1+\sin t})\cdot dt = \frac{4(4+4-1)}{(4+1)^2} = \frac{28}{25}.$$

Example 3 : Evaluate $\int_0^\infty e^{-3t}\, t\cos t\cdot dt$. **(M.U. 2004, 06, 15)**

Sol. : Consider $L[t\cos t] = -\frac{d}{ds}L(\cos t) = -\frac{d}{ds}\left[\frac{s}{s^2+1}\right]$

$$= -\left[\frac{(s^2+1)-s\cdot 2s}{(s^2+1)^2}\right] = \frac{s^2-1}{(s^2+1)^3}$$

That means, $\int_0^\infty e^{-st}\, t\cos t\, dt = \frac{s^2-1}{(s^2+1)^3}$

Putting $s = 3$, $\int_0^\infty e^{-3t}\, t\cos t\cdot dt = \frac{8}{100} = \frac{2}{25}$.

Example 4 : Evaluate $\int_0^\infty e^{-t}\, t^3 \sin t\, dt$. **(M.U. 1996)**

Sol. : Consider $L[t^3\sin t] = (-1)^3\frac{d^3}{ds^3}L(\sin t) = (-1)\frac{d^3}{ds^3}\left[\frac{1}{s^2+1}\right]$

Now $\frac{d}{ds}\left[\frac{1}{s^2+1}\right] = -\frac{2s}{(s^2+1)^2}$

$$\frac{d}{ds}\left[-\frac{2s}{(s^2+1)^2}\right] = -2\left[\frac{(s^2+1)^2\cdot 1 - s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{(s^2+1)-4s^2}{(s^2+1)^3}\right] = -2\frac{(-3s^2+1)}{(s^2+1)^3} = 2\frac{(3s^2-1)}{(s^2+1)^3}$$

$$\frac{d}{ds}\left[2\frac{(3s^2-1)}{(s^2+1)^3}\right] = 2\left[\frac{(s^2+1)^3\cdot 6s - (3s^2-1)\cdot 3(s^2+1)^2\cdot 2s}{(s^2+1)^6}\right]$$

$$= 2\left[\frac{6s(s^2+1)-6s(3s^2-1)}{(s^2+1)^4}\right] = -24s\cdot\frac{(s^2-1)}{(s^2+1)^4}$$

This means, $\int_0^\infty e^{-st}\, t^3\sin t\, dt = 24s\cdot\frac{(s^2-1)}{(s^2+1)^4}$

To find the value of the given integral we put $s = 1$.

$\therefore \int_0^\infty e^{-t}\, t^3\sin t\, dt = 0.$

Example 5 : If $L[J_o(t)] = \frac{1}{\sqrt{1+s^2}}$, prove that

(i) $\int_0^\infty J_o(t)\,dt = 1.$ (ii) $L[t\,J_o(at)] = \frac{s}{(s^2+a^2)^{3/2}}.$

(iii) $L[e^{-bt}J_o(at)] = \frac{1}{\sqrt{(s+b)^2+a^2}}.$ **(M.U. 1998)**

(iv) $\int_0^\infty t\,e^{-3t}J_o(4t)\,dt = \frac{3}{125}.$ **(M.U. 1996, 2003)**

Sol. : (i) By definition of Laplace transform since $L[J_o(t)] = \frac{1}{\sqrt{1+s^2}}$,

$$\int_0^\infty e^{-st}J_o(t)\,dt = \frac{1}{\sqrt{1+s^2}}.$$

Putting $s = 0$, we get $\int_0^\infty J_o(t)\,dt = \frac{1}{\sqrt{1+0}} = \frac{1}{1} = 1.$

(ii) Since $J_o(t) = \frac{1}{\sqrt{1+s^2}}$, by change of scale property (§ 6 page 1-17).

$$J_o(at) = \frac{1}{a}\cdot\frac{1}{\sqrt{1+(s/a)^2}} = \frac{1}{\sqrt{s^2+a^2}}$$

$\therefore\ L[t\,J_o(at)] = (-1)\frac{d}{ds}\left(\frac{1}{\sqrt{s^2+a^2}}\right)$ [By § 10 (16) page 1-28]

$$= (-1)\left(-\frac{1}{2}\right)(s^2+a^2)^{-3/2}\cdot 2s = \frac{s}{(s^2+a^2)^{3/2}}$$

(iii) Since $J_o(at) = \frac{1}{\sqrt{s^2+a^2}}$, by first shifting theorem (§ 7 page 1-19).

$$L[e^{-bt}J_o(at)] = \frac{1}{\sqrt{(s+b)^2+a^2}}.$$

(iv) Now putting $a = 4$ in (ii) above $L[t\,J_o(4t)] = \frac{s}{(s^2+16)^{3/2}}.$

By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\, t\,J_o(4t)\,dt = \frac{s}{(s^2+16)^{3/2}}$$

Putting $s = 3$, we get $\int_0^\infty e^{-3t}\, t\,J_o(4t)\,dt = \frac{3}{(9+16)^{3/2}} = \frac{3}{125}.$

Example 6 : Evaluate $\int_0^\infty \frac{t^2\sin 3t}{e^{2t}}\,dt$. **(M.U. 2002, 07, 09)**

Sol. : Consider $\int_0^\infty e^{-st}\, t^2\sin 3t\, dt = L(t^2\sin 3t)$

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$$\therefore \int_0^\infty e^{-st} t^2 \sin 3t\, dt = (-1)^2 \frac{d^2}{ds^2}[L(\sin 3t)] = \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= \frac{d}{ds}\left(-\frac{3\cdot 2s}{(s^2+9)^2}\right) = -6\left[\frac{(s^2+9)^2\cdot 1 - s\cdot 2(s^2+9)\cdot 2s}{(s^2+9)^4}\right]$$

$$= -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right] = \frac{-6\cdot(-3s^2+9)}{(s^2+9)^3} = \frac{18(s^2-3)}{(s^2+9)^3}.$$

This means, $\int_0^\infty e^{-st} t^2 \sin 3t\, dt = \dfrac{18(s^2-3)}{(s^2+9)^3}$

Now putting $s = 2$, we get

$$\int_0^\infty e^{-2t} t^2 \sin 3t\, dt = \frac{18(4-3)}{(4+9)^3} = \frac{18}{2197}.$$

Example 7 : Given $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$, evaluate $\int_0^\infty t\cdot e^{-t} erf(\sqrt{t})\, dt$. **(M.U. 2009)**

Sol. : Since $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$,

$$L(t\, erf\sqrt{t}) = -\frac{d}{ds}\left(\frac{1}{s\sqrt{s+1}}\right) = -\frac{d}{ds}[(s^2+s)^{-1/2}]$$

$$\therefore L(t\, erf\sqrt{t}) = -\left[-\frac{1}{2}(s^2+s)^{-3/2}(2s+1)\right] = \frac{1}{2}(s^2+s)^{-3/2}(2s+1)$$

$$\therefore \int_0^\infty e^{-st}\cdot t\cdot erf(\sqrt{t})\, dt = \frac{1}{2}(s^2+s)^{-3/2}(2s+1)$$

Now, put $s = 1$,

$$\therefore \int_0^\infty e^{-t}\cdot t\cdot erf(\sqrt{t})\, dt = \frac{1}{2}(1+1)^{-3/2}(2+1) = \frac{1}{2\cdot 2\sqrt{2}}\cdot(3) = \frac{3}{4\sqrt{2}}.\ ?$$

EXERCISE - X

(A) Evaluate the following using Laplace transform.

1. $\int_0^\infty e^{-2t} t^3 \sin t\, dt$. **(M.U. 2004)**
2. $\int_0^\infty e^{-2t} t \cos t\, dt$.
3. $\int_0^\infty e^{-2t} t \sin^2 t\, dt$.
4. $\int_0^\infty e^{-3t} t^2 \sinh 2t\, dt$.

[**Ans. :** (1) $\dfrac{144}{625}$, (2) $\dfrac{3}{25}$, (3) $\dfrac{1}{8}$, (4) $\dfrac{124}{125}$.]

(B) If $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$, then evaluate the following

1. $\int_0^\infty e^{-t} J_0(t)\, dt$.
2. $\int_0^\infty e^{-4t} t\, J_0(3t)\, dt$.

[**Ans.** (1) $\dfrac{1}{\sqrt{2}}$, (2) $\dfrac{4}{125}$.]

(C) If $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$, evaluate $\int_0^\infty e^{-2t} erf(2\sqrt{t})\, dt$. **(M.U. 2003)** [**Ans. :** $1/\sqrt{6}$]

12. Effect of Division by t

If $L[f(t)] = \Phi(s)$, then $\boxed{L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \Phi(s)\, ds}$ **(M.U. 1994, 2005, 08)** ..... **(17)**

Proof : We have $\Phi(s) = \int_0^\infty e^{-st} f(t)\, dt$

Integrating both sides w.r.t. s between the limits s, ∞ and changing the order of integration on r.h.s.

$$\int_s^\infty \Phi(s)\, ds = \int_0^\infty \left[\int_s^\infty e^{-st} f(t)\, ds\right] dt = \int_0^\infty \left[\frac{e^{-st}}{-t} f(t)\right]_s^\infty dt$$

$$= \int_s^\infty e^{-st}\frac{f(t)}{t}\, dt = L\left[\frac{1}{t}f(t)\right]$$

Example 1 : Find the Laplace transform of the following.

(i) $\dfrac{1}{t}(1-\cos t)$ **(M.U. 1995, 2004)** (ii) $\dfrac{1}{t}e^{-t}\sin t$ **(M.U. 2003)**

(iii) $\dfrac{1}{t}(e^{-at} - e^{-bt})$ **(M.U. 1997)**

Sol. : (i) $L[1-\cos t] = L(1) - L(\cos t) = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

By (17), we get

$$L\left[\frac{1}{t}(1-\cos t)\right] = \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+1}\right] ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log(s^2+1) - \log s^2\right]_s^\infty = -\frac{1}{2}\left[\log\frac{s^2+1}{s^2}\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log\left(1+\frac{1}{s^2}\right)\right]_s^\infty \qquad \textbf{[Note this carefully]}$$

$$= -\frac{1}{2}\left[\log 1 - \log\left(1+\frac{1}{s^2}\right)\right] = \frac{1}{2}\log\left(1+\frac{1}{s^2}\right) = \frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

(ii) $L(e^{-t}\sin t) = \dfrac{1}{(s+1)^2+1}$

By (17), we get

$$L\left[\frac{1}{t}(e^{-t}\sin t)\right] = \int_s^\infty \frac{1}{(s+1)^2+1}\, ds = \left[\tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}(s+1) = \cot^{-1}(s+1)$$

(iii) $L(e^{-at} - e^{-bt}) = \dfrac{1}{s+a} - \dfrac{1}{s+b}$

By (17), we get

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$$L\left[\frac{1}{t}\left(e^{-at}-e^{-bt}\right)\right]=\int_s^\infty\left[\frac{1}{s+a}-\frac{1}{s+b}\right]ds=\left[\log(s+a)-\log(s+b)\right]_s^\infty$$

$$=\left[\log\left\{\frac{s+a}{s+b}\right\}\right]_s^\infty=-\log\left(\frac{s+a}{s+b}\right)=\log\left(\frac{s+b}{s+a}\right)$$

Example 2 : Find the Laplace transform of the following.

(i) $\dfrac{\sin^2 2t}{t}$ **(M.U. 1993, 2016)** (ii) $\dfrac{1-\cos t}{t^2}$ **(S.U. 2011)**

(iii) $\dfrac{e^{-2t}\sin 2t\cosh t}{t}$ **(M.U. 1996, 2003)**

Sol. : (i) $\sin^2 2t=\dfrac{1-\cos 4t}{2}$.

$$\therefore\ L\sin^2 2t=L\frac{1}{2}(1-\cos 4t)=\frac{1}{2}\left[L(1)-L(\cos 4t)\right]=\frac{1}{2}\left[\frac{1}{s}-\frac{s}{s^2+4^2}\right]$$

By (17), we get

$$L\left[\frac{\sin^2 2t}{t}\right]=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+4^2}\right]ds=\frac{1}{2}\left[\log s-\frac{1}{2}\log(s^2+4^2)\right]_s^\infty$$

$$=-\frac{1}{4}\left[\log(s^2+4^2)-\log s^2\right]_s^\infty=-\frac{1}{4}\left[\log\left(\frac{s^2+4^2}{s^2}\right)\right]_s^\infty=\frac{1}{4}\log\left(\frac{s^2+4^2}{s^2}\right).$$

(ii) $L[1-\cos t]=L(1)-L(\cos t)=\dfrac{1}{s}-\dfrac{s}{s^2+1}$

By (17), we get

$$L\left(\frac{1-\cos t}{t}\right)=\int_s^\infty\left[\frac{1}{s}-\frac{s}{s^2+1}\right]ds=\left[\log s-\frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right)\right]_s^\infty=-\frac{1}{2}\log\frac{s^2}{s^2+1}=\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)$$

By (17) again, we get

$$L\left(\frac{1-\cos t}{t^2}\right)=\int_s^\infty\frac{1}{2}\log\left(\frac{s^2+1}{s^2}\right)ds$$

Integrating by parts

$$L\left(\frac{1-\cos t}{t^2}\right)=\frac{1}{2}\left[\log\left(\frac{s^2+1}{s^2}\right)\cdot s-\int s\cdot\left(\frac{s^2}{s^2+1}\right)\left(\frac{s^2\cdot 2s-(s^2+1)2s}{s^4}\right)ds\right]_s^\infty$$

$$=\frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right)+2\int\frac{ds}{s^2+1}\right]_s^\infty=\frac{1}{2}\left[s\log\left(\frac{s^2+1}{s^2}\right)+2\tan^{-1}s\right]_s^\infty$$

$$\therefore\ L\left(\frac{1-\cos t}{t^2}\right)=\frac{1}{2}\left[0+2\cdot\frac{\pi}{2}-s\log\left(\frac{s^2+1}{s^2}\right)-2\tan^{-1}s\right]$$

$$=\frac{\pi}{2}-\frac{s}{2}\log\left(\frac{s^2+1}{s^2}\right)-\tan^{-1}s.$$

(iii) We have, $e^{-2t}\sin 2t\cosh t=e^{-2t}\sin 2t\,\dfrac{(e^t+e^{-t})}{2}=\dfrac{1}{2}\left(e^{-t}\sin 2t+e^{-3t}\sin 2t\right)$

$$\therefore\ L\left[e^{-2t}\sin 2t\cosh t\right]=\frac{1}{2}L\left(e^{-t}\sin 2t+e^{-3t}\sin 2t\right)$$

But $L\sin 2t=\dfrac{2}{s^2+2^2}$

$$\therefore\ L\left(e^{-t}\sin 2t\right)=\frac{2}{(s+1)^2+2^2}=\frac{2}{s^2+2s+5}$$

$$L\left(e^{-3t}\sin 2t\right)=\frac{2}{(s+3)^2+2^2}=\frac{2}{s^2+6s+13}$$

$$\therefore\ L\left[e^{-2t}\sin 2t\cosh t\right]=\frac{1}{s^2+2s+5}+\frac{1}{s^2+6s+13}$$

$$\therefore\ L\left[\frac{e^{-2t}\sin 2t\cosh t}{t}\right]=\int_s^\infty\left(\frac{1}{s^2+2s+5}+\frac{1}{s^2+6s+13}\right)ds$$

$$=\int_s^\infty\left[\frac{1}{(s+1)^2+2^2}+\frac{1}{(s+3)^2+2^2}\right]ds$$

$$=\left[\frac{1}{2}\tan^{-1}\frac{s+1}{2}+\frac{1}{2}\tan^{-1}\frac{s+3}{2}\right]_s^\infty$$

$$=\frac{1}{2}\cdot\left(\frac{\pi}{2}-\tan^{-1}\frac{s+1}{2}\right)+\frac{1}{2}\left(\frac{\pi}{2}-\tan^{-1}\frac{s+3}{2}\right)$$

$$=\frac{\pi}{2}-\frac{1}{2}\tan^{-1}\left(\frac{s+1}{2}\right)-\frac{1}{2}\tan^{-1}\left(\frac{s+3}{2}\right).$$

Example 3 : Find the Laplace transform of $\dfrac{\sin at}{t}$. Does Laplace transform of $\dfrac{\cos at}{t}$ exist ? **(M.U. 2015, 18)**

Sol. : Consider $f(t)=\sin at$ $\quad\therefore\ L[f(t)]=\dfrac{a}{s^2+a^2}=\Phi(s)$

$$\therefore\ L\left[\frac{\sin at}{t}\right]=\int_s^\infty\Phi(s)\,ds \qquad \text{[By (17), page 1-39]}$$

$$=\int_s^\infty\frac{a}{s^2+a^2}ds=\left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$=\frac{\pi}{2}-\tan^{-1}\frac{s}{a}=\cot^{-1}\frac{s}{a}.$$

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Now consider, $f(t) = \cos at$ $\quad \therefore \; L[f(t)] = \dfrac{s}{s^2+a^2} = \Phi(s)$

$$\therefore \; L\left[\frac{\cos at}{t}\right] = \int_s^\infty \Phi(s)\,ds = \int_s^\infty \frac{s}{s^2+a^2}\,ds = \frac{1}{2}\left[\log(s^2+a^2)\right]_s^\infty$$

Since $\log(s^2+a^2)$ is infinite when $s \to \infty$, $L\left[\dfrac{\cos at}{t}\right]$ does not exist.

Example 4 : Find $L\left[\dfrac{\sin^2 t}{t^2}\right]$. **(M.U. 2005)**

Sol. :
$$L(\sin^2 t) = L\left[\frac{1-\cos 2t}{2}\right] = \frac{1}{2}\left[L(1) - L\cos 2t\right] = \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right]$$

$$\therefore \; L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right]ds = \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty = -\frac{1}{4}\log\left(\frac{s^2}{s^2+4}\right) = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$$

$$\therefore \; L\left[\frac{\sin^2 t}{t^2}\right] = \int_s^\infty \frac{1}{4}\cdot\log\left(\frac{s^2+4}{s^2}\right)ds$$

Integrating by parts,

$$L\left[\frac{\sin^2 t}{t^2}\right] = \frac{1}{4}\left[\log\left(\frac{s^2+4}{s^2}\right)\cdot s - \int s\cdot\frac{s^2}{s^2+4}\left(\frac{s^2\cdot 2s - (s^2+4)\cdot 2s}{s^4}\right)ds\right]_s^\infty$$

$$= \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 8\int\frac{ds}{s^2+4}\right]_s^\infty = \frac{1}{4}\left[s\log\left(\frac{s^2+4}{s^2}\right) + 2\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty$$

$$= \frac{1}{4}\left[0 + 2\cdot\frac{\pi}{2} - s\log\left(\frac{s^2+4}{s^2}\right) - 2\tan^{-1}\frac{s}{2}\right]$$

$$\therefore \; L\left[\frac{\sin^2 t}{t^2}\right] = \frac{\pi}{4} - \frac{s}{4}\log\left(\frac{s^2+4}{s^2}\right) - \frac{1}{2}\cdot\tan^{-1}\left(\frac{s}{2}\right).$$

EXERCISE - XI

Find the Laplace transform of the following.

1. $\dfrac{\sin t}{t}$ **(M.U. 1994, 2015)**
2. $\dfrac{1}{t}[1-\cos at]$
3. $\dfrac{1}{t}[e^{-t}\sin at]$ **(M.U. 2013)**
4. $\dfrac{1}{t}[1-e^{2t}]$
5. $\dfrac{1}{t}[e^{-3t}\sin 2t]$
6. $\dfrac{1}{t}[\sin^2 t]$ **(M.U. 2003, 04)**
7. $\dfrac{1}{t}[e^{2t}\sin^3 t]$ **(M.U. 2006)**
8. $\dfrac{1}{t}[\cos at - \cos bt]$ **(M.U. 2010, 13)**
9. $\dfrac{1}{t}[1-\cos 3t]$
10. $\dfrac{1}{t}[e^{-2t}\sin 3t]$
11. $\dfrac{1}{t}(\sin^3 t)$
12. $\dfrac{1}{t}[e^{-t}\sin^3 t]$

13. $\dfrac{e^{2t}\sin t}{t}$ **(M.U. 1999)**
14. $\dfrac{\cos h\,2t\sin 2t}{t}$ **(M.U. 1999, 2009, 10)**
15. $\dfrac{2\sin t\sin 2t}{t}$ **(M.U. 2004, 11)**
16. $\dfrac{\sin t\sin 5t}{t}$ **(M.U. 2003, 07, 08)**
17. $\dfrac{(1-\cos 2t)}{t}$ **(M.U. 2003)**
18. $\dfrac{\sin h\,at}{t}$ **(M.U. 2004, 05)**

[**Ans. :** (1) $\cot^{-1}s$, (2) $\frac{1}{2}\log\frac{(s^2+a^2)}{s^2}$, (3) $\cot^{-1}\left(\frac{s+1}{a}\right)$, (4) $\log\left(\frac{s-2}{s}\right)$,

(5) $\cot^{-1}\left(\frac{s+3}{2}\right)$, (6) $\frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$, (7) $\frac{3}{4}\cot^{-1}(s-2) - \frac{1}{4}\cot^{-1}\left(\frac{s-2}{3}\right)$,

(8) $\frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$, (9) $\frac{1}{2}\log\left(\frac{s^2+9}{s^2}\right)$, (10) $\cot^{-1}\left(\frac{s+2}{3}\right)$,

(11) $\frac{1}{4}\left[3\cot^{-1}s - \cot^{-1}\frac{s}{3}\right]$, (12) $\frac{1}{4}\left[3\cot^{-1}(s+1) - \cot^{-1}\left(\frac{s+1}{3}\right)\right]$,

(13) $\cot^{-1}(s-2)$, (14) $\frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{s-2}{2}\right) - \frac{1}{2}\tan^{-1}\left(\frac{s+2}{2}\right)$,

(15) $\frac{1}{2}\log\left(\frac{s^2+9}{s^2+1}\right)$, (16) $\frac{1}{4}\log\left(\frac{s^2+36}{s^2+16}\right)$,

(17) $\frac{1}{2}\log\left(\frac{s^2+4}{s^2}\right)$, (18) $\frac{1}{2}\log\left(\frac{s+a}{s-a}\right)$.]

13. Evaluation of the Integral $\int_0^\infty e^{-at}\dfrac{f(t)}{t}dt$

To evaluate the integral $\int_0^\infty e^{-at}\dfrac{f(t)}{t}dt$, we first find $L\left[\dfrac{f(t)}{t}\right]$, say $\Phi(s)$.

This means, $\int_0^\infty e^{-st}\dfrac{f(t)}{t}dt = \Phi(s)$

Now, we put $s = a$ on both sides.

Example 1 : Evaluate $\int_0^\infty \dfrac{e^{-t}\sin t}{t}dt$. **(M.U. 2003, 08)**

Sol. : We have proved in Ex. 3, page 1-41 that $L\left[\dfrac{\sin at}{t}\right] = \cot^{-1}\left(\dfrac{s}{a}\right)$.

This means $\int_0^\infty e^{-st}\cdot\dfrac{\sin at}{t}dt = \cot^{-1}\dfrac{s}{a}$

Now, put $s = 1$, $a = 1$, then $\int_0^\infty \dfrac{e^{-t}\sin t}{t}dt = \cot^{-1}(1) = \dfrac{\pi}{4}$.

Example 2 : Show that $\int_0^\infty \dfrac{\sin at}{t}dt = \dfrac{\pi}{2}$. **(M.U. 1997, 2010)**

Sol. : We have proved in Ex. 3, page 1-41 that $L\left[\dfrac{\sin at}{t}\right] = \cot^{-1}\left(\dfrac{s}{a}\right)$.

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This means $\int_0^\infty e^{-st}\left(\frac{\sin at}{t}\right)dt = \cot^{-1}\frac{s}{a}$

Now put $s = 0$. $\therefore \int_0^\infty \frac{\sin at}{t}dt = \cot^{-1}(0) = \frac{\pi}{2}$.

Example 3 : Evaluate $\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}dt$. **(M.U. 2007)**

Sol. : Consider $f(t) = e^{-at} - e^{-bt}$ $\therefore L[f(t)] = \frac{1}{s+a} - \frac{1}{s+b}$

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty\left(\frac{1}{s+a} - \frac{1}{s+b}\right)ds = \left[\log(s+a) - \log(s+b)\right]_s^\infty$$

$$\therefore L\left[\frac{1}{t}f(t)\right] = -\left[\log\left(\frac{s+b}{s+a}\right)\right]_s^\infty = -\left[\log\frac{1+(b/s)}{1+(a/s)}\right]_s^\infty$$

$$= -\left[\log 1 - \log\frac{1+(b/s)}{1+(a/s)}\right] = \log\left(\frac{s+b}{s+a}\right)$$

This means $\int_0^\infty e^{-st}\left(\frac{e^{-at} - e^{-bt}}{t}\right)dt = \log\left(\frac{s+b}{s+a}\right)$

Putting $s = 0$, $\int_0^\infty\left(\frac{e^{-at} - e^{-bt}}{t}\right)dt = \log\frac{b}{a}$.

Example 4 : Evaluate $\int_0^\infty \frac{\cos at - \cos bt}{t}dt$. **(M.U. 2004, 07, 2013, 19)**

Sol. : Consider $f(t) = \cos at - \cos bt$ $\therefore L[f(t)] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty\left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+a^2) - \log(s^2+b^2)\right]_s^\infty$$

$$= -\frac{1}{2}\left[\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]_s^\infty = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right) \quad \text{..... (1)}$$

By definition of Laplace transform the equation (1) means

$$\int_0^\infty e^{-st}\left(\frac{\cos at - \cos bt}{t}\right)dt = \frac{1}{2}\log\left(\frac{s^2+b^2}{s^2+a^2}\right)$$

Putting $s = 0$, $\int_0^\infty \frac{\cos at - \cos bt}{t}dt = \frac{1}{2}\log\frac{b^2}{a^2} = \log\frac{b}{a}$.

Cor. : Putting $s = 1$, we get

$$\int_0^\infty e^{-t}\left(\frac{\cos at - \cos bt}{t}\right)dt = \frac{1}{2}\log\left(\frac{b^2+1}{a^2+1}\right)$$ **(M.U. 2005, 09)**

Example 5 : Evaluate $\int_0^\infty e^{-t}\left(\frac{\cos 3t - \cos 2t}{t}\right)dt$. **(M.U. 2005, 07, 15)**

Sol. : Consider $f(t) = \cos 3t - \cos 2t$ $\therefore L[f(t)] = \frac{s}{s^2+9} - \frac{s}{s^2+4}$

$$\therefore L\left[\frac{1}{t}f(t)\right] = \int_s^\infty\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)ds$$

$$= \frac{1}{2}\left[\log(s^2+9) - \log(s^2+4)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2+9}{s^2+4}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{1+(9/s^2)}{1+(4/s^2)}\right)\right]_s^\infty = \frac{1}{2}\left[\log 1 - \log\left(\frac{1+(9/s^2)}{1+(4/s^2)}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2+9}{s^2+4}\right) = \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

This means,

$$\int_0^\infty e^{-st}\left(\frac{\cos 3t - \cos 2t}{t}\right)dt = \frac{1}{2}\log\left(\frac{s^2+4}{s^2+9}\right)$$

Putting $s = 1$, we get

$$\int_0^\infty e^{-t}\left(\frac{\cos 3t - \cos 2t}{t}\right)dt = \frac{1}{2}\log\left(\frac{5}{10}\right) = \frac{1}{2}\log\left(\frac{1}{2}\right)$$

Example 6 : Evaluate $\int_0^\infty e^{-st}\cdot\frac{\sin^2(at/2)}{t}\cdot dt$.

Sol. : We have $\sin^2\left(\frac{at}{2}\right) = \frac{1-\cos at}{2}$

$$\therefore L\left(\sin^2\frac{at}{2}\right) = \frac{1}{2}L(1) - \frac{1}{2}L(\cos at) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{s}{(s^2+a^2)}$$

$$\therefore L\left(\frac{\sin^2(at/2)}{t}\right) = \int_s^\infty \frac{1}{2}\cdot\frac{1}{s}ds - \int_s^\infty \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2s}{s^2+a^2}ds$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2+a^2)\right]_s^\infty = \frac{1}{4}\left[\log\frac{s^2}{s^2+a^2}\right]_s^\infty$$

$$= -\frac{1}{4}\log\left(\frac{s^2}{s^2+a^2}\right) = \frac{1}{4}\log\left(\frac{s^2+a^2}{s^2}\right)$$

This means $\int_0^\infty e^{-st}\cdot\frac{\sin^2(at/2)}{t}dt = \frac{1}{4}\log\left(\frac{s^2+a^2}{s^2}\right)$.

Example 7 : Prove that $\int_0^\infty e^{-t}\cdot\frac{\sin^2 t}{t}dt = \frac{1}{4}\log 5$. **(M.U. 1999, 2000, 03, 04, 05, 09, 18, 19)**

Sol. : In the above example put $s = 1$, $a = 2$.

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$$\therefore \int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t} dt = \frac{1}{4}\log\left(\frac{1+4}{1}\right) = \frac{1}{4}\log 5$$

Or independently, consider $\sin^2 t = \dfrac{1-\cos 2t}{2}$.

$$\therefore L\sin^2 t = \frac{1}{2}L(1) - \frac{1}{2}L(\cos 2t) = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\cdot\frac{s}{s^2+4}$$

$$\therefore L\left(\frac{\sin^2 t}{t}\right) = \int_s^\infty \frac{1}{2}\cdot\frac{1}{s} ds - \int_s^\infty \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{2s}{s^2+4} ds$$

$$= \left[\frac{1}{2}\log s - \frac{1}{4}\log(s^2+4)\right]_s^\infty = \frac{1}{4}\left[\log\frac{s^2}{s^2+4}\right]_s^\infty$$

$$= -\frac{1}{4}\log\frac{s^2}{s^2+4} = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right).$$

This means $\displaystyle\int_0^\infty e^{-st} \cdot \frac{\sin^2 t}{t} dt = \frac{1}{4}\log\left(\frac{s^2+4}{s^2}\right)$

Now put $s = 1$, $\displaystyle\therefore \int_0^\infty e^{-t} \cdot \frac{\sin^2 t}{t} dt = \frac{1}{4}\log 5.$

Example 8 : Prove that $\displaystyle\int_0^\infty e^{-st} \cdot \frac{\sin t \sinh at}{t} = \frac{1}{2}\tan^{-1}\left[\frac{2a}{1+(s^2-a^2)}\right].$

Sol. : We have, $\sin t \sinh at = \left(\dfrac{e^{at}-e^{-at}}{2}\right)\sin t$. Now $L\sin t = \dfrac{1}{s^2+1}$

$$\therefore L\left(e^{at}\sin t\right) = \frac{1}{(s-a)^2+1}, \quad L\left(e^{-at}\sin t\right) = \frac{1}{(s+a)^2+1}$$

$$\therefore L\sin t \sinh at = \frac{1}{2}\left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right]$$

$$\therefore L\left(\frac{\sin t \sinh at}{t}\right) = \frac{1}{2}\int_s^\infty \left[\frac{1}{(s-a)^2+1} - \frac{1}{(s+a)^2+1}\right] ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s-a) - \tan^{-1}(s+a)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-a)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+a)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+a) - \tan^{-1}(s-a)\right]$$

Now let $\tan^{-1}(s+a) = \alpha$, $\tan^{-1}(s-a) = \beta$

$\therefore \tan\alpha = s+a$, $\tan\beta = s-a$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{(s+a)-(s-a)}{1+(s^2-a^2)} = \frac{2a}{1+(s^2-a^2)}.$$

$$\therefore \alpha - \beta = \tan^{-1}\frac{2a}{1+(s^2-a^2)}$$

$$\therefore \tan^{-1}(s+a) - \tan^{-1}(s-a) = \tan^{-1}\frac{2a}{1+(s^2-a^2)}$$

$$\therefore L\left(\frac{\sin t \sinh at}{t}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right)$$

This means $\displaystyle\int_0^\infty e^{-st} \cdot \frac{\sin t \sinh at}{t} dt = \frac{1}{2}\tan^{-1}\left(\frac{2a}{1+(s^2-a^2)}\right).$

Example 9 : Evaluate $\displaystyle\int_0^\infty e^{-2t}\sinh t \frac{\sin t}{t} dt.$ **(M.U. 2002, 16)**

Sol. : We have $\sinh t \sin t = \left(\dfrac{e^t - e^{-t}}{2}\right)\sin t$

Now, $L(\sin t) = \dfrac{1}{s^2+1}$

$$\therefore L(e^t \sin t) = \frac{1}{(s-1)^2+1} \quad \text{and} \quad L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$$

$$\therefore L(\sinh t \sin t) = \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore L\left[\frac{\sinh t \cdot \sin t}{t}\right] = \frac{1}{2}\left[\int_0^\infty \frac{ds}{(s-1)^2+1} - \frac{ds}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s-1) - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-1)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+1)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right]$$

But $[\tan^{-1}(s+1) - \tan^{-1}(s-1)] = \tan^{-1}\left(\dfrac{2}{s^2}\right)$

[Let $\tan^{-1}(s+1) = \alpha$ and $\tan^{-1}(s-1) = \beta$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta} = \frac{(s+1)-(s-1)}{1+(s^2-1)} = \frac{2}{s^2}$$

$$\therefore \alpha - \beta = \tan^{-1}\left(\frac{2}{s^2}\right).]$$

$$\therefore L\left[\frac{\sinh t \sin t}{t}\right] = \frac{1}{2}\tan^{-1}\left(\frac{2}{s^2}\right)$$

This means,

$$\int_0^\infty e^{-st} \cdot \frac{\sinh t \sin t}{t} dt = \frac{1}{2}\tan^{-1}\left(\frac{2}{s^2}\right)$$

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Putting $s = 2$, we get

$$\int_0^\infty e^{-2t} \cdot \frac{\sinh t \sin t}{t} dt = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right).$$

Example 10 : Prove that $\int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} = \frac{\pi}{8}$. **(M.U. 2002, 05, 07, 11, 16)**

Sol. : In the above example put $s = \sqrt{2}$, $a = 1$.

$$\therefore \int_0^\infty e^{-\sqrt{2}t} \frac{\sin t \sinh t}{t} dt = \frac{1}{2}\tan^{-1}\left(\frac{2}{1+(2-1)}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2}{2}\right)$$

$$= \frac{1}{2}\tan^{-1} 1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}$$

Or independently consider again,

$$\sin t \sinh t = \left(\frac{e^t - e^{-t}}{2}\right)\sin t.$$

Now, $L \sin t = \frac{1}{t^2+1}$

$$\therefore L(e^t \sin t) = \frac{1}{(s-1)^2+1}, \qquad L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$$

$$\therefore L \sin t \sinh t = \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore L\left(\frac{\sin t \sinh t}{t}\right) = \frac{1}{2}\int_s^\infty \left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right] ds$$

$$= \frac{1}{2}\left[\tan^{-1}(s-1) - \tan^{-1}(s+1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\left\{\frac{\pi}{2} - \tan^{-1}(s-1)\right\} - \left\{\frac{\pi}{2} - \tan^{-1}(s+1)\right\}\right]$$

$$= \frac{1}{2}\left[\tan^{-1}(s+1) - \tan^{-1}(s-1)\right].$$

Now let $\tan^{-1}(s+1) = \alpha$, $\tan^{-1}(s-1) = \beta$.

$$\therefore \tan\alpha = s+1, \quad \tan\beta = s-1$$

$$\therefore \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\alpha} = \frac{(s+1)-(s-1)}{1+(s^2-1)} = \frac{2}{1+(s^2-1)} = \frac{2}{s^2}$$

$$\therefore \alpha - \beta = \tan^{-1}\left(\frac{2}{s^2}\right) \qquad \therefore \tan^{-1}(s+1) - \tan^{-1}(s-1) = \tan^{-1}\frac{2}{s^2}$$

$$\therefore L\left(\frac{\sin t \sinh t}{t}\right) = \frac{1}{2}\tan^{-1}\frac{2}{s^2}$$

This means $\int_0^\infty e^{-st} \cdot \frac{\sin t \sinh t}{t} = \frac{1}{2}\tan^{-1}\frac{2}{s^2}$

Now put $s = \sqrt{2}$

$$\therefore \int_0^\infty e^{-\sqrt{2}t}\left(\frac{\sin t \sinh t}{t}\right) = \frac{1}{2}\tan^{-1}\frac{2}{2} = \frac{1}{2}\tan^{-1} 1 = \frac{1}{2}\cdot\frac{\pi}{4} = \frac{\pi}{8}.$$

Example 11 : Prove that $\int_0^\infty e^{-st}\left(\frac{\sin at + \sin bt}{t}\right) dt = \pi - \tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$.

Sol. : We have $L(\sin at) = \frac{a}{s^2+a^2}$, $L(\sin bt) = \frac{b}{s^2+b^2}$

$$\therefore L\left(\frac{\sin at + \sin bt}{t}\right) = \int_s^\infty \frac{a}{s^2+a^2} ds + \int_s^\infty \frac{b}{s^2+b^2} ds = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty + \left[\tan^{-1}\frac{s}{b}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\frac{s}{a} + \frac{\pi}{2} - \tan^{-1}\frac{s}{b} = \pi - \left(\tan^{-1}\frac{s}{a} + \tan^{-1}\frac{s}{b}\right)$$

Now let $\tan^{-1}\frac{s}{a} = \alpha$, $\tan^{-1}\frac{s}{b} = \beta$ $\quad\therefore \tan\alpha = \frac{s}{a}$, $\tan\beta = \frac{s}{b}$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{(s/a)+(s/b)}{1-(s/a)(s/b)} = \frac{s(a+b)}{ab-s^2}$$

$$\alpha + \beta = \tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$$

$$\therefore L\left(\frac{\sin at + \sin bt}{t}\right) = \pi - \tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$$

This means $\int_0^\infty e^{-st} \cdot \left(\frac{\sin at + \sin bt}{t}\right) dt = \pi - \tan^{-1}\left[\frac{s(a+b)}{ab-s^2}\right]$.

Example 12 : Evaluate $\int_0^\infty e^{-2t} \cdot \frac{\cos 2t \sin 3t}{t} dt$. **(M.U. 2015)**

Sol. : We have

$$L(\sin 3t \cos 2t) = L\frac{1}{2}[\sin 5t + \sin t] = \frac{1}{2}\left[\frac{5}{s^2+5^2} + \frac{1}{s^2+1}\right]$$

$$\therefore L\left[\frac{\sin 3t \cos 2t}{t}\right] = \frac{1}{2}\int_s^\infty \left[\left(\frac{5}{s^2+5^2}\right) + \left(\frac{1}{s^2+1}\right)\right] ds$$

$$= \frac{1}{2}\left[\tan^{-1}\left(\frac{s}{5}\right) + \tan^{-1}\left(\frac{s}{1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{5}\right) + \frac{\pi}{2} - \tan^{-1} s\right]$$

$$= \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{s}{5}\right) - \frac{1}{2}\tan^{-1} s$$

This means,

$$\int_0^\infty e^{-st} \cdot \frac{\cos 2t \sin 3t}{t} dt = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{s}{5}\right) - \frac{1}{2}\tan^{-1} s$$

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Putting $s = 2$, we get

$$\int_0^\infty e^{-2t} \cdot \frac{\cos 2t \sin 3t}{t} dt = \frac{\pi}{2} - \frac{1}{2}\tan^{-1}\left(\frac{2}{5}\right) - \frac{1}{2}\tan^{-1} 2$$

$$= \frac{\pi}{2} - \frac{1}{2}\left[\tan^{-1}\left(\frac{2}{5}\right) + \tan^{-1} 2\right]$$

Now, let $\tan^{-1}\frac{2}{5} = \alpha$, $\tan^{-1} 2 = \beta$.

$$\therefore \quad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{(2/5) + 2}{1 - (2/5)\cdot 2} = 12$$

$$\therefore \quad \alpha + \beta = \tan^{-1} 12.$$

$$\therefore \quad I = \frac{\pi}{2} - \frac{1}{2}\tan^{-1} 12.$$

[In Ex. 11 above, put $a = 5$ and $b = 1$ and $s = 2$ after dividing by 2.]

Example 13 : Find the Laplace transform of $\dfrac{e^{-at} - \cos at}{t}$.

Hence, evaluate $\displaystyle\int_0^\infty \frac{e^{-t} - \cos t}{t e^{4t}} dt$. **(M.U. 2004)**

Sol. : We have $L(e^{-at}) = \dfrac{1}{s+a}$ and $L(\cos at) = \dfrac{s}{s^2 + a^2}$

$$\therefore \quad L[e^{-at} - \cos at] = \frac{1}{s+a} - \frac{s}{s^2 + a^2}$$

$$\therefore \quad L\left[\frac{e^{-at} - \cos at}{t}\right] = \int_s^\infty \left(\frac{1}{s+a} - \frac{s}{s^2 + a^2}\right) ds = \left[\log(s+a) - \frac{1}{2}\log(s^2 + a^2)\right]_s^\infty$$

$$= \left[\log\left(\frac{s+a}{\sqrt{s^2 + a^2}}\right)\right]_s^\infty = \left[\log\left(\frac{1 + (a/s)}{\sqrt{1 + (a^2/s^2)}}\right)\right]_s^\infty$$

$$= \log 1 - \log\left(\frac{1 + (a/s)}{\sqrt{1 + (a^2/s^2)}}\right) = \log\left(\frac{\sqrt{s^2 + a^2}}{s + a}\right)$$

This means, $\displaystyle\int_0^\infty e^{-st}\left(\frac{e^{-at} - \cos at}{t}\right) dt = \log\left(\frac{\sqrt{s^2 + a^2}}{s + a}\right)$ ..... (1)

Now, $\displaystyle\int_0^\infty \frac{e^{-t} - \cos t}{t e^{4t}} dt = \int_0^\infty e^{-4t}\left(\frac{e^{-t} - \cos t}{t}\right) dt$

Putting $s = 4$ and $a = 1$ in (1), we get

$$\int_0^\infty \frac{e^{-t} - \cos t}{t e^{4t}} dt = \log\left(\frac{\sqrt{16 + 1}}{4 + 1}\right) = \log\frac{\sqrt{17}}{5}.$$

Example 14 : Prove that $\displaystyle\int_0^\infty \left(\frac{\sin 2t + \sin 3t}{t e^t}\right) dt = \frac{3\pi}{4}$. **(M.U. 1995, 2003, 06, 12)**

Sol. : In Ex. 11 above put $s = 1$, $a = 2$, $b = 3$.

$$\therefore \quad \int_0^\infty e^{-t} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left[\frac{(1)(5)}{6 - 1}\right] = \pi - \tan^{-1}\left(\frac{5}{5}\right)$$

$$= \pi - \tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Or independently, we have,

$$L(\sin 2t) = \frac{2}{s^2 + 4}, \quad L(\sin 3t) = \frac{3}{s^2 + 9}$$

$$\therefore \quad L\left(\frac{\sin 2t + \sin 3t}{t}\right) = \int_s^\infty \left[\frac{2}{(s^2 + 4)} + \frac{3}{(s^2 + 9)}\right] ds = \left[\tan^{-1}\left(\frac{s}{2}\right)\right]_s^\infty + \left[\tan^{-1}\left(\frac{s}{3}\right)\right]_s^\infty$$

$$= \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{2}\right) + \left(\frac{\pi}{2} - \tan^{-1}\frac{s}{3}\right) = \pi - \left(\tan^{-1}\frac{s}{2} + \tan^{-1}\frac{s}{3}\right)$$

Now let $\tan^{-1}\frac{s}{2} = \alpha$, $\tan^{-1}\frac{s}{3} = \beta$ $\quad \therefore \quad \frac{s}{2} = \tan\alpha$, $\frac{s}{3} = \tan\beta$

$$\therefore \quad \tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \frac{(s/2) + (s/3)}{1 - (s/2)(s/3)} = \frac{5s}{6 - s^2}$$

$$\therefore \quad \alpha + \beta = \tan^{-1}\left(\frac{5s}{6 - s^2}\right)$$

$$\therefore \quad L\left(\frac{\sin 2t + \sin 3t}{t}\right) = \pi - \tan^{-1}\left(\frac{5s}{6 - s^2}\right)$$

This means $\displaystyle\int_0^\infty e^{-st} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left(\frac{5s}{6 - s^2}\right)$

Now put $s = 1$,

$$\int_0^\infty e^{-t} \cdot \left(\frac{\sin 2t + \sin 3t}{t}\right) dt = \pi - \tan^{-1}\left(\frac{5}{6 - 1}\right) = \pi - \tan^{-1}\left(\frac{5}{5}\right)$$

$$= \pi - \tan^{-1}(1) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

Example 15 : Evaluate $\displaystyle\int_0^\infty t e^{-t^2} erf(t) dt$. **(M.U. 2009)**

Sol. : Putting $t^2 = y$, $2t\,dt = dy$.

$$\int_0^\infty t e^{-t^2} erf(t) dt = \int_0^\infty e^{-y} erf(\sqrt{y}) \frac{dy}{2} \quad \text{..... (1)}$$

But $L\left[erf(\sqrt{t})\right] = \dfrac{1}{s\sqrt{s+1}}$. This means, $\displaystyle\int_0^\infty e^{-st} erf(\sqrt{t}) = \frac{1}{s\sqrt{s+1}}$.

Putting $s = 1$, $\displaystyle\int_0^\infty e^{-t} erf(\sqrt{t}) = \frac{1}{\sqrt{2}}$

Hence, from (1) $\displaystyle\int_0^\infty t e^{-t^2} erf(t) dt = \frac{1}{2\sqrt{2}}$.

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EXERCISE - XII

Evaluate the following integrals by using Laplace transform.

1. $\int_0^\infty e^{-t} \cdot \frac{\sin 3t}{t} dt$ **(M.U. 2014)**
2. $\int_0^\infty \frac{e^{-2t} - e^{-3t}}{t} dt$
3. $\int_0^\infty \frac{e^{-t} - e^{-3t}}{t} dt$ **(M.U. 1997, 2002, 04)**
4. $\int_0^\infty \frac{\cos 4t - \cos 3t}{t} dt$
5. $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$ **(M.U. 2005, 07, 08, 13, 14)**
6. $\int_0^\infty \frac{\sin 2t}{t} dt$ **(M.U. 2010)**
7. $\int_0^\infty \frac{e^{-t} \sin t}{t} dt$ **(M.U. 2004, 05)**
8. $\int_0^\infty \frac{\sin 3t + \sin 2t}{t e^t} dt$ **(M.U. 2012)**
9. $\int_0^\infty e^{-t} \cdot \frac{(1 - \cos 2t)}{2t} dt$
10. $\int_0^\infty \frac{e^{-t} \cdot \sin \sqrt{3} \cdot t}{t} dt$ **(M.U. 2003)**
11. $\int_0^\infty e^{-2t} \cdot \frac{\sinh t}{t} dt$ **(M.U. 2008)**

[Ans. : (1) $\cot^{-1}\left(\frac{1}{3}\right)$, (2) $\log\frac{3}{2}$, (3) $\log 3$, (4) $\log\frac{3}{4}$, (5) $\log\frac{2}{3}$, (6) $\frac{\pi}{2}$, (7) $\frac{\pi}{4}$, (8) $\frac{3\pi}{4}$, (9) $\frac{1}{4}\log 5$, (10) $\frac{\pi}{3}$, (11) $\frac{1}{2}\log 3$.

14. Laplace Transforms of Derivatives

$$\boxed{Lf'(t) = -f(0) + sLf(t)} \quad \text{(M.U. 2004, 13)} \qquad \ldots\ldots\ldots\ldots (18)$$

Proof : By definition of Laplace transform,

$$Lf'(t) = \int_0^\infty e^{-st} f'(t)\,dt. \qquad \text{Integrating by parts}$$

$$= \left[e^{-st} f(t)\right]_0^\infty - \int_0^\infty (-s) e^{-st} f(t)\,dt$$

$$\therefore \; Lf'(t) = -f(0) + s\int_0^\infty e^{-st} f(t)\,dt = -f(0) + sLf(t) \qquad \ldots\ldots\ldots\ldots (1)$$

Applying (1) again,

$$Lf''(t) = -f'(0) + s[Lf'(t)]$$
$$= -f'(0) + s[-f(0) + sLf(t)]$$

$$\therefore \; Lf''(t) = -f'(0) - sf(0) + s^2 Lf(t) \qquad \ldots\ldots\ldots\ldots (2)$$

Further, $Lf'''(t) = -f''(0) - sf'(0) - s^2 f(0) + s^3 Lf(t)$ $\ldots\ldots\ldots\ldots (3)$

In general

$$Lf^n(t) = -f^{n-1}(0) - s \cdot f^{n-2}(0) - s^2 \cdot f^{n-3}(0) + \ldots\ldots + s^n Lf(t)$$

Cor. 1 : If $f(0) = f'(0) = f''(0) = \ldots\ldots = 0$, we get

$$\boxed{\begin{aligned} &Lf'(t) = s \cdot Lf(t), \quad Lf''(t) = s^2 Lf(t), \\ &L^3 f'''(t) = s^3 Lf(t) \ldots\ldots Lf^n(t) = s^n Lf(t) \end{aligned}} \qquad \ldots\ldots\ldots\ldots (4)$$

Note ....

These results are going to be highly useful to solve differential equations.

Example 1 : Given $f(t) = t + 1$, $0 \le t \le 2$ and $f(t) = 3$, $t > 2$, find $L[f(t)]$, $L[f'(t)]$ and $L[f''(t)]$ **(M.U. 2003, 04, 05, 06)**

Sol. : By definition $L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$.

$$\therefore \; L[f(t)] = \int_0^2 e^{-st}(t+1)\,dt + \int_2^\infty e^{-st} \cdot 3\,dt$$

$$= \left[(t+1) \cdot \left(\frac{e^{-st}}{-s}\right) - (1) \cdot \frac{e^{-st}}{s^2}\right]_0^2 + 3\left[\frac{e^{-st}}{-s}\right]_2^\infty$$

$$= \frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right)$$

Now, as shown above, $L[f'(t)] = -f(0) + sL[f(t)]$

But by data $f(0) = 1$,

$$\therefore \; L[f'(t)] = -1 + s\left[\frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-2s}\right)\right] = -1 + 1 + \frac{1}{s}\left(1 - e^{-2s}\right) = \frac{1}{s}\left(1 - e^{-2s}\right)$$

Further, $L[f''(t)] = s^2 L[f(t)] - s[f(0)] - f'(0)$

$$= s^2\left[\frac{1}{s} + \frac{1}{s^2}(1 - e^{-2s})\right] - s - 1$$

$$= s + 1 - e^{-2s} - s - 1 = -e^{-2s}.$$

$[\because f(t) = t + 1, f'(t) = 1 \quad \therefore f(0) = 1, f'(0) = 1]$

Example 2 : Given $f(t) = \frac{\sin t}{t}$, find $L[f'(t)]$. **(M.U. 2002, 13)**

Sol. : Let $L \sin t = \frac{1}{s^2 + 1}$

$$\therefore \; L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2 + 1} = \left[\tan^{-1} s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1} s = \cot^{-1} s$$

But $L[f'(t)] = sLf(t) - f(0) = s\cot^{-1} s - 1$.

Example 3 : Given $L(\sin\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{3/2}} \cdot e^{-(1/4s)}$, prove that $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \sqrt{\frac{\pi}{s}} \cdot e^{-(1/4s)}$. **(M.U. 2009)**

Sol. : Let $f(t) = \sin\sqrt{t}$ $\quad \therefore \; L[f'(t)] = L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right]$

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Also $f(0) = \sin 0 = 0$

$\therefore$ By (4) above, $L[f'(t)] = s \cdot [L f(t)]$

$$\therefore \quad L\left[\frac{\cos\sqrt{t}}{2\sqrt{t}}\right] = s \cdot \frac{\sqrt{\pi}}{2 \cdot s^{3/2}} e^{-(1/4s)} \qquad \therefore \quad L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = \sqrt{\frac{\pi}{s}} \cdot e^{-(1/4s)}.$$

Example 4 : Find Laplace transform of $\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)$. **(M.U. 2003)**

Sol. : We have $L(1-\cos 2t) = L(1) - L(\cos 2t) = \frac{1}{s} - \frac{s}{s^2+2^2}$

$$\therefore \quad L\left(\frac{1-\cos 2t}{t}\right) = \int_s^\infty \Phi(s)\,ds \qquad \text{[By (17), page 1-39]}$$

$$= \int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+2^2}\right] ds = \left[\log s - \frac{1}{2}\log(s^2+2^2)\right]_s^\infty$$

$$\therefore \quad L\left(\frac{1-\cos 2t}{t}\right) = \left[\log\frac{s}{\sqrt{s^2+2^2}}\right] = \left[0 - \log\frac{s}{\sqrt{s^2+2^2}}\right] = \log\left(\frac{\sqrt{s^2+2^2}}{s}\right)$$

But $L[f'(t)] = s\,L[f(t)] - f(0)$ where $f(t) = \frac{1-\cos 2t}{t} = \frac{2\sin^2 t}{t}$

$$\therefore \quad f(0) = 2\sin t\left(\frac{\sin t}{t}\right) = 0$$

$$\therefore \quad L\left[\frac{d}{dt}\left(\frac{1-\cos 2t}{t}\right)\right] = s\log\left(\frac{\sqrt{s^2+2^2}}{s}\right)$$

Alternatively : We may differentiate $\left(\frac{1-\cos 2t}{t}\right)$ w.r.t. t and then obtain the Laplace transform of the derivative.

Example 5 : Find $L\left[\frac{d}{dt}\left(\frac{\sin 3t}{t}\right)\right]$. **(M.U. 2014)**

Sol. : $L(\sin 3t) = \frac{3}{s^2+9}$

$$\therefore \quad L\left[\frac{\sin 3t}{t}\right] = 3\int_s^\infty \frac{ds}{s^2+9} = 3 \cdot \frac{1}{3}\left[\tan^{-1}\frac{s}{3}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{3}\right) = \cot^{-1}\left(\frac{s}{3}\right).$$

$\therefore \quad L[f'(t)] = -f(0) - s\,L[f(t)]$ where $f(t) = \frac{\sin 3t}{t}$.

Now, $\lim_{t\to 0}\left(\frac{\sin 3t}{t}\right) = 3 \cdot \lim_{t\to 0}\left(\frac{\sin 3t}{3t}\right) = 3$

$$\therefore \quad L[f'(t)] = -3 - s\cot^{-1}\left(\frac{s}{3}\right)$$

Example 6 : If $L(t\sin\omega t) = \frac{2\omega s}{(s^2+\omega^2)}$, find $L[\omega t\cos\omega t + \sin\omega t]$. **(M.U. 2011)**

Sol. : Let $f(t) = t\sin\omega t \quad \therefore \quad f'(t) = \sin\omega t + \omega t\cos\omega t$.

Further since $f(0) = 0$ and $f'(0) = 0$.

By the above result (4), page 1-53, $\quad L[f'(t)] = s\,L[f(t)]$

$$\therefore \quad L[\omega t\cos\omega t + \sin\omega t] = s \cdot \frac{2\omega s}{(s^2+\omega^2)^2} = \frac{2\omega s^2}{(s^2+\omega^2)^2}.$$

EXERCISE - XIII

1. Given $f(t) = 3,\ 0 \le t < 5,\ f(t) = 0,\ t > 5$. Find $L[f(t)]$ and also $L[f'(t)]$.
2. Given $f(t) = t,\ 0 \le t < 3,\ f(t) = 6,\ t > 3$. Find $L[f(t)]$ and also $L[f'(t)]$. **(M.U. 2004)**
3. Find $L\left[\frac{d}{dt}\left(\frac{\sin t}{t}\right)\right]$. **(M.U. 2002)**

[**Ans. :** (1) $\frac{3}{s}\left[1-e^{-5s}\right],\ \frac{3}{s}\left(1-s-e^{-5s}\right)$.

(2) $\frac{1}{s^2} + e^{-3s}\left(\frac{3}{s} - \frac{1}{s^2}\right),\ \frac{1}{s} + e^{-3s}\left(3 - \frac{1}{s}\right)$. (3) $s\cot^{-1}s - 1$]

15. Laplace Transforms of Integrals

If $L[f(t)] = \Phi(s)$, then $\boxed{L\int_0^t f(u)\,du = \frac{1}{s}\Phi(s)}$ **(M.U. 1996, 2013)** ..... (19)

Proof : By definition,

$$L\left[\int_0^t f(u)\,du\right] = \int_0^\infty e^{-st}\left[\int_0^t f(u)\,du\right]dt \qquad \text{..... (1)}$$

$$= \left[\int_0^t f(u)\,du \cdot \left(\frac{-e^{-st}}{s}\right)\right]_0^\infty - \int_0^\infty \left[\left(\frac{-e^{-st}}{s}\right)\frac{d}{dt}\int_0^t f(u)\,du\right]$$

[By integration by parts]

But $\frac{d}{dt}\int_0^t f(u)\,du = f(t)$

$$\therefore \quad L\left[\int_0^t f(u)\,du\right] = \int_0^\infty \frac{1}{s} \cdot e^{-st} f(t)\,dt = \frac{1}{s} \cdot L[f(t)] = \frac{1}{s}\Phi(s)$$

Since $\Phi(s) = L[f(t)]$, we have

$$L\int_0^t f(u)\,du = \frac{1}{s}L\{f(t)\}$$

Corollary : The above result can be generalised as follows.

$$\boxed{L\left[\int_0^t\int_0^t \ldots\int_0^t f(u)(du)^n\right] = \frac{1}{s^n}L[f(t)]}$$

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Example 1 : Find the Laplace transform of

(i) $\int_0^t \sin 2u\, du$ (ii) $\int_0^t u \cosh u \cdot du$. **(M.U. 1998, 99)**

Sol. : (i) Since $L \sin 2t = \dfrac{2}{s^2+4} = \Phi(s)$, say

$$\therefore\ L\int_0^t \sin 2u\, du = \frac{1}{s}\Phi(s) = \frac{2}{s(s^2+4)}$$

(ii) Now, $L \cos ht = \dfrac{s}{s^2-a^2}$ $\quad\therefore\ Lt \cos ht = -\dfrac{d}{ds}\left[\dfrac{s}{s^2-a^2}\right]$ [By (16) § 10, page 1-28]

$$\therefore\ Lt \cos ht = \frac{s^2+a^2}{(s^2-a^2)^2} = \Phi(s), \text{ say}$$

$$\therefore\ L\int_0^t u \cos hu\, du = \frac{1}{s}\Phi(s) = \frac{s^2+a^2}{s(s^2-a^2)^2}.$$

Example 2 : Find the Laplace transform of $\int_0^t \sin u \cos 2u\, du$. **(M.U. 201**

Sol. : We have $\sin u \cos 2u = \dfrac{1}{2}\cdot 2\sin u \cos 2u = \dfrac{1}{2}[\sin 3u - \sin u]$

Now, $L\left[\dfrac{1}{2}(\sin 3u)\right] = \dfrac{1}{2}\cdot\dfrac{3}{s^2+9}$ and $L\left[\dfrac{1}{2}(\sin u)\right] = \dfrac{1}{2}\cdot\dfrac{1}{s^2+1}$

$$\therefore\ L\left[\int_0^t \sin u \cos 2u\, du\right] = \frac{1}{s}\cdot\Phi(s) = \frac{1}{2s}\left[\frac{3}{s^2+9} - \frac{1}{s^2+1}\right].$$

Example 3 : Find the Laplace transform of the following.

(i) $erf\sqrt{t}$ **(M.U. 1996, 97, 2003, 06)**

(ii) $erf\sqrt{2t}$ and evaluate $\int_0^\infty erf(2\sqrt{t})\cdot e^{-5t}\, dt$. **(M.U. 2005)**

(iii) $e^{3t} erf\sqrt{t}$ **(M.U. 1998)**

(iv) $e^{-3t} erf\sqrt{t}$ **(M.U. 1997, 99)**

Sol. : (i) We have $erf\, x = \dfrac{2}{\sqrt{\pi}}\int_0^x e^{-u^2} du$. Hence, $erf\sqrt{t} = \dfrac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-u^2} du$

Now put $u^2 = v$ $\quad\therefore\ u = \sqrt{v}$ $\quad\therefore\ du = \dfrac{1}{2\sqrt{v}} dv$

When $u = 0$, $v = 0$; When $u = \sqrt{t}$, $v = t$

$$\therefore\ erf\sqrt{t} = \frac{2}{\sqrt{\pi}}\int_0^t e^{-v}\cdot\frac{1}{2\sqrt{v}} dv = \frac{1}{\sqrt{\pi}}\int_0^t e^{-v} v^{-1/2} dv$$

By (9), page 1-5.

$$L(v^{-1/2}) = \frac{\Gamma 1/2}{s^{1/2}} = \frac{\sqrt{\pi}}{\sqrt{s}} \quad\therefore\ L\left(e^{-v} v^{-1/2}\right) = \frac{\sqrt{\pi}}{\sqrt{s+1}}$$ [By first shifting theorem]

$$\therefore\ L\int_0^t e^{-v} v^{-1/2} dv = \frac{\sqrt{\pi}}{s\sqrt{s+1}}$$

$$\therefore\ L\, erf\sqrt{t} = \frac{1}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{s\sqrt{s+1}} = \frac{1}{s\sqrt{s+1}}.$$

[See also (11), page 1-6]

(ii) Now $erf\, 2\sqrt{t} = erf\sqrt{4t}$. But $erf\sqrt{t} = \dfrac{1}{s\sqrt{s+1}}$

Hence, by (13), § 6 page 1-17,

$$erf\sqrt{4t} = \frac{1}{4}\frac{1}{(s/4)\sqrt{(s/4)+1}} = \frac{2}{s\sqrt{s+4}} \qquad \therefore\ erf\, 2\sqrt{t} = \frac{2}{s\sqrt{s+4}}$$

By definition of Laplace transform

$$\int_0^\infty e^{-st}\, erf\, 2\sqrt{t}\, dt = \frac{2}{s\sqrt{s+4}}$$

Put $s = 5$, $\int_0^\infty e^{-5t}\, erf\, 2\sqrt{t}\, dt = \dfrac{2}{5\sqrt{5+4}} = \dfrac{2}{15}$.

(iii) Since, $L(erf\sqrt{t}) = \dfrac{1}{s\sqrt{s+1}}$

$$L\left(e^{3t} erf\sqrt{t}\right) = \frac{1}{(s-3)\sqrt{s-3+1}} = \frac{1}{(s-3)\sqrt{s-2}}$$ [By (14) § 7, page 1-19]

(iv) Similarly, $L\left(e^{-3t} erf\sqrt{t}\right) = \dfrac{1}{(s+3)\sqrt{s+3+1}} = \dfrac{1}{(s+3)\sqrt{s+4}}$.

Example 4 : Find $\int_0^\infty e^{-t} erf\sqrt{t}\, dt$. **(M.U. 1997)**

Sol. : As proved above $L[erf\sqrt{t}] = \dfrac{1}{s\sqrt{s+1}}$

This means $\int_0^\infty e^{-st}\cdot erf\sqrt{t}\, dt = \dfrac{1}{s\sqrt{s+1}}$

Now put $s = 1$, $\quad\therefore\ \int_0^\infty e^{-t}\cdot erf\sqrt{t}\, dt = \dfrac{1}{\sqrt{2}}$

Example 5 : Find $L e^{3t}\cdot t\, erf\sqrt{t}$. **(M.U. 2000)**

Sol. :
$$L\, t\, erf\sqrt{t} = -\frac{d}{ds}\left(\frac{1}{s\sqrt{s+1}}\right) = -\left[-\frac{1}{s^2(s+1)}\left\{\frac{1}{2\sqrt{s+1}} + \sqrt{s+1}\right\}\right]$$

$$\therefore\ L\, t\, erf\sqrt{t} = \frac{1}{s^2(s+1)}\cdot\frac{s+2(s+1)}{2\sqrt{s+1}} = \frac{3s+2}{2s^2(s+1)^{3/2}}$$

$$\therefore\ L e^{3t}\cdot t\cdot erf\sqrt{t} = \frac{3(s-3)+2}{2(s-3)^2(s-3+1)^{3/2}} = \frac{3s-7}{2(s-3)^2(s-2)^{3/2}}.$$

Example 6 : Find $\int_0^\infty e^{-t}\, erf_c\sqrt{t}\, dt$. **(M.U. 1996)**

Sol. : By (12), page 1-7, we have

$$L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}.$$

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By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$$

Put $s = 1$, we get $\int_0^\infty e^{-t}\, erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{2}(\sqrt{2}+1)} = \frac{1}{\sqrt{6}}$.

Example 7 : Find $\int_0^\infty erf_c\sqrt{t}\, dt$.

Sol. : As proved above $L[erf_c\sqrt{t}] = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$.

This means $\int_0^\infty e^{-st} erf_c\sqrt{t}\, dt = \frac{1}{\sqrt{s+1}(\sqrt{s+1}+1)}$

Putting $s = 0$, we get $\int_0^\infty erf_c\sqrt{t}\, dt = \frac{1}{2}$.

Example 8 : Find $L\left[\int_0^t\int_0^t\int_0^t t\sin t\,(dt)^3\right]$. **(M.U. 2005)**

Sol. : By the corollary

$$L\left[\int_0^t\int_0^t\int_0^t t\sin t\,(dt)^3\right] = \frac{1}{s^3}L[t\sin t]$$

But by Ex. (1) (iii), page 1-29,

$$L[t\sin t] = \frac{2s}{(s^2+1)^2}$$

$$\therefore\ L\left[\int_0^t\int_0^t\int_0^t t\sin t\,(dt)^3\right] = \frac{1}{s^3}\cdot\frac{2s}{(s^2+1)^2} = \frac{2}{s^2(s^2+1)^2}.$$

Example 9 : Find the Laplace transform of the following.

(i) $\int_0^t u\cos^2 u\, du$. **(M.U. 1995, 2010)** (ii) $\int_0^t u e^{-3u}\cos^2 2u\, du$. **(M.U. 1993)**

(iii) $\int_0^t u^{-1}e^{-u}\sin u\, du$. **(M.U. 1996)** (iv) $\int_0^t \frac{1-e^{au}}{u}du$. **(M.U. 1997, 2003)**

(v) $\int_0^t \frac{\sin u}{u}du$. **(M.U. 1997, 2002)** (vi) $\int_t^\infty \frac{\cos u}{u}du$. **(M.U. 1997)**

(vii) $\int_0^t u^2\sin u\, du$. **(M.U. 2003)** (viii) $\int_0^t e^{-2u}\cos^2 u\, du$ **(M.U. 2008)**

Sol. : (i) $\cos^2 u = \frac{1+\cos 2u}{2}$

$$L\cos^2 u = L\frac{1}{2}(1+\cos 2u) = \frac{1}{2}L(1) + \frac{1}{2}L\cos 2u = \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+2^2}$$

$$L(u\cos^2 u) = -\frac{d}{ds}\cdot\left(\frac{1}{2}\cdot\frac{1}{s}\right) - \frac{d}{ds}\cdot\frac{1}{2}\cdot\left(\frac{s}{s^2+2^2}\right)$$

$$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\frac{(s^2+2^2)\cdot 1 - s\cdot 2s}{(s^2+2^2)^2}$$ [By (16) § 10, page 1-28]

$$\therefore\ L(u\cos^2 u) = \frac{1}{2s^2} + \frac{1}{2}\cdot\frac{s^2-2^2}{(s^2+2^2)^2} = \Phi(s), \text{ say}$$

$$\therefore\ L\int_0^t u\cos^2 u\, du = \frac{1}{s}\cdot\Phi(s) = \frac{1}{2s^3} + \frac{1}{2}\cdot\frac{s^2-2^2}{s(s^2+2^2)^2}$$ [By (19) § 15, page 1-55]

(ii) $\cos^2 2u = \frac{1+\cos 4u}{2}$

$$L\cos^2 2u = L\frac{1}{2}(1+\cos 4u) = \frac{1}{2}L(1) + \frac{1}{2}L(\cos 4u)$$

$$= \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+4^2}$$

$$L e^{-3u}\cos^2 2u = \frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{s+3}{(s+3)^2+4^2}$$

$$= \frac{1}{2}\cdot\frac{1}{s+3} + \frac{1}{2}\cdot\frac{s+3}{s^2+6s+25}$$

$$L u e^{-3u}\cos^2 2u = -\frac{d}{ds}\left[\frac{1}{2}\cdot\frac{1}{s+3}\right] - \frac{d}{ds}\left[\frac{1}{2}\cdot\frac{s+3}{s^2+6s+25}\right]$$ [By (16) § 10, page 1-28]

$$= \frac{1}{2}\cdot\frac{1}{(s+3)^2} - \frac{1}{2}\left[\frac{(s^2+6s+25)-(s+3)(2s+6)}{(s^2+6s+25)^2}\right]$$

$$= \frac{1}{2(s+3)^2} + \frac{1}{2}\cdot\frac{s^2+6s-7}{(s^2+6s+25)^2} = \Phi(s), \text{ say}$$

$$\therefore\ L\int_0^t u e^{-3u}\cos^2 2u\, du = \frac{1}{s}\Phi(s)$$ [By (19) § 15, page 1-55]

$$= \frac{1}{2s(s+3)^2} + \frac{1}{2}\cdot\frac{s^2+6s-7}{s(s^2+6s+25)^2}$$

(iii) $L\sin u = \frac{1}{s^2+1}$; $L e^{-u}\sin u = \frac{1}{(s+1)^2+1}$ [By shifting theorem]

$$\therefore\ L\left(\frac{1}{u}e^{-u}\sin u\right) = \int_s^\infty \frac{ds}{(s+1)^2+1}$$ [By (17) § 12, page 1-39]

$$= \left[\tan^{-1}(s+1)\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}(s+1)$$

$$= \cot^{-1}(s+1) = \Phi(s), \text{ say}$$

$$\therefore\ L\int_0^t \frac{e^{-u}\sin u}{u}du = \frac{1}{s}\Phi(s) = \frac{1}{s}\cot^{-1}(s+1).$$

(iv) $L(1-e^{au}) = L(1) - L(e^{au}) = \frac{1}{s} - \frac{1}{s-a}$

$$\therefore\ L\left(\frac{1-e^{au}}{u}\right) = \int_s^\infty\left[\frac{1}{s} - \frac{1}{s-a}\right]ds = \left[\log\frac{s}{s-a}\right]_s^\infty$$

$$= \left[0 - \log\frac{s}{s-a}\right] = \log\left(\frac{s-a}{s}\right)$$

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$$\therefore\ L\int_0^t\left(\frac{1-e^u}{u}\right)du = \frac{1}{s}\Phi(s) = \frac{1}{s}\log\left(\frac{s-a}{s}\right).$$

(v) $$L\sin u = \frac{1}{s^2+1}$$

$$\frac{L\sin u}{u} = \int_s^\infty \Phi(s)\,ds = \int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$$\therefore\ L\int_0^t \frac{\sin u}{u}du = \frac{1}{s}\Phi(s) = \frac{1}{s}\cot^{-1}s.$$

(vi) Let $f(t) = \int_t^\infty \frac{\cos u}{u}du$

Now put $u = vt$, $du = t\,dv$

When $u = t$, $v = 1$; when $u \to \infty$, $v \to \infty$

$$\therefore\ f(t) = \int_1^\infty \frac{\cos vt}{vt}\cdot t\,dv = \int_1^\infty \frac{\cos vt}{v}dv$$

$$\therefore\ L[f(t)] = \int_0^\infty e^{-st}\left[\int_1^\infty \frac{\cos vt}{v}dv\right]dt = \int_1^\infty \frac{dv}{v}\int_0^\infty e^{-st}\cos vt\,dt$$

$$= \int_1^\infty \frac{dv}{v}L[\cos vt]\,dt = \int_1^\infty \frac{s\,dv}{v(s^2+v^2)} = \frac{1}{s}\int_1^\infty\left[\frac{1}{v} - \frac{v}{s^2+v^2}\right]dv$$

$$= \frac{1}{s}\left[\log v - \frac{1}{2}\log(s^2+v^2)\right]_1^\infty = \frac{1}{2s}\left[\log\frac{v^2}{s^2+v^2}\right]_1^\infty$$

$$= \frac{1}{2s}\left[0 - \log\frac{1}{s^2+1}\right] = \frac{1}{2s}\log(s^2+1)$$

(vii) $$L(\sin u) = \frac{1}{s^2+1}$$

$$L(u^2\sin u) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2+1}\right) = \frac{d}{ds}\left(-\frac{2s}{(s^2+1)^2}\right)$$

$$= -2\frac{d}{ds}\cdot\left[\frac{s}{(s^2+1)^2}\right] = -2\left[\frac{(s^2+1)^2\cdot 1 - s\cdot 2(s^2+1)\cdot 2s}{(s^2+1)^4}\right]$$

$$= -2\left[\frac{s^2+1-4s^2}{(s^2+1)^3}\right] = -2\cdot\frac{(1-3s^2)}{(s^2+1)^3}$$

$$\therefore\ L\left[\int_0^t u^2\sin u\,du\right] = -\frac{2}{s}\cdot\frac{(1-3s^2)}{(s^2+1)^3}.$$

(viii) $$\cos^2 u = \frac{1+\cos 2u}{2}$$

$$L\cos^2 u = L\left[\frac{1}{2}(1+\cos 2u)\right] = \frac{1}{2}L(1) + \frac{1}{2}L\cos 2u$$

$$\therefore\ L\cos^2 u = \frac{1}{2}\cdot\frac{1}{s} + \frac{1}{2}\cdot\frac{s}{s^2+2^2}$$

$$L(e^{-2u}\cos^2 u) = \frac{1}{2}\cdot\frac{1}{(s+2)} + \frac{1}{2}\cdot\frac{s+2}{(s+2)^2+2^2} = \frac{1}{2}\cdot\frac{1}{(s+2)} + \frac{1}{2}\cdot\frac{s+2}{s^2+4s+8}$$

$$L\int_0^t e^{-2u}\cos^2 u\,du = \frac{1}{s}\Phi(s) = \frac{1}{2s(s+2)} + \frac{1}{2}\cdot\frac{s+2}{s(s^2+4s+8)}.$$

Example 10 : Find the Laplace transforms of the following.

(i) $t\int_0^t e^{-4u}\sin 3u\,du$

(ii) $e^{-t}\int_0^t \frac{\sin u}{u}du$ **(M.U. 2015)**

(iii) $t^{-1}\int_0^t e^{-u}\sin u\,du$

(iv) $e^{-4t}\int_0^t u\sin 3u\,du$ **(M.U. 2003, 04, 15, 16)**

(v) $e^{-t}\int_0^t e^u\cos h\,u\,du$ **(M.U. 2012)**

(vi) $\cos ht\int_0^t e^u\cos hu\,du$ **(M.U. 1996, 2008, 09)**

(vii) $t\int_0^t e^u\sin u\,du$ **(M.U. 2008)**

Sol. : (i) We have $L(\sin 3u) = \frac{3}{s^2+9}$ [By (7), page 1-5]

$$\therefore\ L(e^{-4u}\sin 3u) = \frac{3}{(s+4)^2+9}$$ [By (14), page 1-19]

$$\therefore\ L\left[\int_0^t e^{-4u}\sin 3u\,du\right] = \frac{1}{s}\cdot\frac{3}{(s+4)^2+9}$$ [By (19), page 1-55]

$$\therefore\ L\left[t\int_0^t e^{-4u}\sin 3u\,du\right] = (-1)\frac{d}{ds}\left[\frac{3}{s^3+8s^2+25s}\right]$$ [By (16), page 1-28]

$$= \frac{3(3s^2+16s+25)}{(s^3+8s^2+25s)^2}$$

(ii) As proved in (v) of the above Ex. 8,

$$L\left[\int_0^t \frac{\sin u}{u}du\right] = \frac{1}{s}\cdot\cot^{-1}s$$

$$\therefore\ L\left[e^{-t}\int_0^t \frac{\sin u}{u}du\right] = \frac{1}{(s+1)}\cot(s+1)$$ [By (14), page 1-19]

(iii) We have $L(\sin u) = \frac{1}{s^2+1}$ [By (7), page 1-5]

$$L(e^{-u}\sin u) = \frac{1}{(s+1)^2+1}$$ [By (14), page 1-19]

$$\therefore\ L\left[\int_0^t e^{-u}\sin u\,du\right] = \frac{1}{s}\cdot\frac{1}{s^2+2s+2}$$ [By (19), page 1-55]

$$\therefore\ L\left[\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right] = \int_s^\infty \frac{ds}{s(s^2+2s+2)}$$ [By (17), page 1-39]

$$= \frac{1}{2}\int_s^\infty\left[\frac{1}{s} - \frac{s+2}{s^2+2s+2}\right]ds$$ [By partial fractions]

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$$\therefore\ L\left[\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right]=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{1}{2}\cdot\frac{2(s+2)}{(s^2+2s+2)}\right]ds$$

$$=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{1}{2}\cdot\frac{2s+2}{(s^2+2s+2)}-\frac{2}{2(s^2+2s+2)}\right]ds$$

$$=\frac{1}{2}\int_s^\infty\left[\frac{1}{s}-\frac{1}{2}\cdot\frac{2s+2}{s^2+2s+2}-\frac{1}{(s+1)^2+1}\right]ds$$

$$=\frac{1}{2}\left[\log s-\frac{1}{2}\log(s^2+2s+2)-\tan^{-1}(s+1)\right]_s^\infty$$

$$=\frac{1}{2}\left[\log\frac{s}{\sqrt{s^2+2s+2}}-\tan^{-1}(s+1)\right]_s^\infty$$

$$=\frac{1}{2}\left[\left(\log 1-\frac{\pi}{2}\right)-\left(\log\frac{s}{\sqrt{s^2+2s+2}}-\tan^{-1}(s+1)\right)\right]$$

$$=\frac{1}{2}\left[\log\frac{\sqrt{s^2+2s+2}}{s}-\left(\frac{\pi}{2}-\tan^{-1}(s+1)\right)\right]$$

$$\therefore\ L\left[\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right]=\left[\frac{1}{4}\log\left(\frac{s^2+2s+2}{s^2}\right)-\frac{1}{2}\cot^{-1}(s+1)\right]$$

(iv) We have $L(\sin 3u)=\dfrac{3}{s^2+9}$ [By (7), page 1-5]

$\therefore\ L(u\sin 3u)=-\dfrac{d}{ds}\left(\dfrac{3}{s^2+9}\right)=\dfrac{3\cdot 2s}{(s^2+9)^2}$ [By (16), page 1-28]

$\therefore\ L\left[\int_0^t u\sin 3u\,du\right]=\dfrac{1}{s}\cdot\dfrac{6s}{(s^2+9)^2}=\dfrac{6}{(s^2+9)^2}$ [By (19), page 1-55]

$\therefore\ L\left[e^{-4t}\int_0^t u\sin 3u\,du\right]=\dfrac{6}{[(s+4)^2+9]^2}=\dfrac{6}{(s^2+8s+25)^2}$ [By (14), page 1-19]

(v) We have $L(\cosh u)=\dfrac{s}{s^2-1}$ [By (8), page 1-5]

$L(e^u\cosh u)=\dfrac{s-1}{(s-1)^2-1}=\dfrac{s-1}{s^2-2s+1-1}=\dfrac{s-1}{s(s-2)}$ [By (14), page 1-19]

$\therefore\ L\left[\int_0^t e^u\cosh u\,du\right]=\dfrac{1}{s}\cdot\dfrac{s-1}{s(s-2)}=\dfrac{s-1}{s^2(s-2)}$ [By (19), page 1-55]

$\therefore\ L\left[e^{-t}\int_0^t e^u\cosh u\,du\right]=\dfrac{(s+1)-1}{(s+1)^2[(s+1)-2]}$ [By (14), page 1-19]

$=\dfrac{s}{(s+1)^2(s-1)}$

(vi) We have $L(\cosh u)=\dfrac{s}{s^2-1}$ [By (8), page 1-5]

$\therefore\ L(e^u\cosh u)=\dfrac{s-1}{(s-1)^2-1}$ [By (14), page 1-19]

$\therefore\ L(e^u\cosh u)=\dfrac{s-1}{s^2-2s+1-1}=\dfrac{s-1}{s(s-2)}$

$\therefore\ L\left[\int_0^t e^u\cosh u\,du\right]=\dfrac{1}{s}\cdot\dfrac{s-1}{s(s-2)}=\dfrac{s-1}{s^2(s-2)}$ [By (19), page 1-55]

$$\therefore\ L\left[\cosh t\cdot\int_0^t e^u\cosh u\,du\right]=L\left[\frac{e^t+e^{-t}}{2}\int_0^t e^u\cosh u\,du\right]$$

$$=\frac{1}{2}\left[L\left\{e^t\int_0^t e^u\cosh u\,du\right\}+L\left\{e^{-t}\int_0^t e^u\cosh u\,du\right\}\right]$$

$$=\frac{1}{2}\left[\frac{(s-1)-1}{(s-1)^2(s-1-2)}+\frac{(s+1)-1}{(s+1)^2(s+1-2)}\right]\quad\text{[By (14), page 1-19]}$$

$$=\frac{1}{2}\left[\frac{s-2}{(s-1)^2(s-3)}+\frac{s}{(s+1)^2(s-1)}\right]$$

(vii) We have $L(\sin u)=\dfrac{1}{s^2+1}$

$L[e^u\sin u]=\dfrac{1}{(s-1)^2+1}=\dfrac{1}{s^2-2s+2}$ [By (14), page 1-19]

$\therefore\ L\left[\int_0^t e^u\sin u\,du\right]=\dfrac{1}{s}\cdot\dfrac{1}{s^2-2s+2}$ [By (19), page 1-55]

$\therefore\ L\left[t\int_0^t e^u\sin u\,du\right]=-\dfrac{d}{ds}\left[\dfrac{1}{s^3-2s^2+2s}\right]=\dfrac{3s^2-4s+2}{(s^3-2s^2+2s)^2}$ [By (16), page 1-28]

Example 11 : Find the Laplace transform of $\cosh t\int_0^t e^u\sin hu\,du$. **(M.U. 2015)**

Sol. : We have $L[\sin hu]=\dfrac{1}{s^2-1}$

$\therefore\ L[e^u\sin hu]=\dfrac{1}{(s-1)^2-1}=\dfrac{1}{s^2-2s+1-1}=\dfrac{1}{s^2-2s}=\dfrac{1}{s(s-2)}$

$\therefore\ L\left[\int_0^t e^u\sin hu\,du\right]=\dfrac{1}{s}\cdot\dfrac{1}{s(s-2)}=\dfrac{1}{s^2(s-2)}$

$$\therefore\ L\left[\cosh t\int_0^t e^u\sin hu\,du\right]=L\left[\left(\frac{e^t+e^{-t}}{2}\right)\int_0^t e^u\sin hu\,du\right]$$

$$=\frac{1}{2}\left[L\left\{e^t\int_0^t e^u\sin u\,du\right\}+L\left\{e^{-t}\int_0^t\sin u\,du\right\}\right]$$

$$=\frac{1}{2}\left[\frac{1}{(s-1)^2(s-1-2)}+\frac{1}{(s+1)^2(s+1-2)}\right]\quad\text{[By (14), page 1-19]}$$

$$=\frac{1}{2}\left[\frac{1}{(s-1)^2(s-3)}+\frac{1}{(s+1)^2(s-1)}\right]$$

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Example 12 : Find the Laplace transform of $t\int_0^t e^{-2u}\cos^2 u\,du$. **(M.U. 2013)**

Sol. : We have obtained in Ex. 9 (viii), page 1-58

$$L\left[\int_0^t e^{-2u}\cos^2 u\,du\right]=\frac{1}{2s(s+2)}+\frac{1}{2}\cdot\frac{s+2}{s(s^2+4s+8)}$$

$\therefore$ By (16), page 1-28

$$L\left[t\int_0^t e^{-2u}\cos^2 u\,du\right]=(-1)\cdot\frac{d}{ds}[\Phi(s)]$$

$$=-\frac{1}{2}\left[-\frac{1}{s^2(s+2)^2}[s\cdot 1+(s+2)\cdot 1]\right]$$

$$-\frac{1}{2}\left[\frac{s(s^2+4s+8)\cdot 1-(s+2)(3s^2+8s+8)}{s^2(s^2+4s+8)^2}\right]$$

$$=\frac{s+1}{s^2(s+2)^2}+\frac{s^3+5s^2+8s+8}{s^2(s^2+4s+8)^2}$$

Note ....

Note that in all these examples we have used more than one rules to find the Laplace transform. We start with the Laplace transform of the innermost function and arrive at the Laplace transform of the given function step by step.

EXERCISE - XIV

(A) Find the Laplace transform of

1. $\int_0^t e^{-3u}\sin 4u\,du$
2. $\int_0^t \frac{1}{u}\sin 3u\,du$
3. $\int_0^t ue^{-3u}\sin 4u\,du$ **(M.U. 2002, 12)**
4. $\int_0^t \frac{1-e^{-8u}}{u}du$ **(M.U. 2003)**

[Ans. : (1) $\frac{1}{s}\cdot\frac{4}{s^2+6s+25}$, **(2)** $\frac{1}{s}\cot^{-1}s$, **(3)** $\frac{8(s+3)}{s(s^2+6s+25)^2}$, **(4)** $\frac{1}{s}\log\left(\frac{s+a}{s}\right)$ **]**

(B) Find the Laplace transform of

1. $\int_0^t e^{-u}u^4du$
2. $\int_0^t e^{-u}\cos u\,du$
3. $\int_0^t \frac{1+e^{-u}}{u}du$
4. $\int_0^t \frac{e^{-u}\sin u}{u}du$ **(M.U. 1997, 2003, 09, 19)**
5. $e^{-3t}\int_0^t u\sin 3u\,du$ **(M.U. 2006, 07, 08, 10)**
6. $\int_0^t e^u\cdot\frac{\sin 4u}{u}du$ **(M.U. 2000)**
7. $\int_0^t u\cdot e^{-2u}\sin 3u\,du$ **(M.U. 2002, 2018)**
8. $\int_0^t u\cdot e^{-3u}\sin^2 u\,du$ **(M.U. 2004)**
9. $\int_0^t u\cosh au\,du$

[Ans. : (1) $\frac{4!}{s(s+1)^5}$, **(2)** $\frac{1}{s}\cdot\frac{s+1}{(s^2+2s+2)}$, **(3)** $\frac{1}{s}\log\left[s\left(\frac{s+1}{s}\right)\right]$, **(4)** $\frac{1}{s}\cot^{-1}(s+1)$,

(5) $\frac{6}{(s^2+6s+18)^2}$, **(6)** $\frac{1}{s}\cdot\cot^{-1}\left(\frac{s+1}{4}\right)$, **(7)** $\frac{1}{s}\cdot\frac{6(s+2)}{(s^2+4s+13)^2}$,

(8) $\frac{1}{2s}\left[\frac{1}{(s+3)^2}+\frac{s^2+6s+5}{(s^2+6s+13)^2}\right]$ **(9)** $\frac{s^2+a^2}{s(s^2-a^2)^2}$. **]**

(C) Find the following.

1. $\int_0^\infty e^{-8t}erf\sqrt{t}\,dt$
2. $\int_0^\infty e^{-3t}erf_c\sqrt{t}\,dt$

[Ans. : (1) $\frac{1}{24}$, **(2)** $\frac{1}{6}$ **]**

(D) Find **1.** $L\left[\int_0^t\int_0^t t\cdot\sin t\,(dt)^2\right]$ **2.** $L\left[\int_0^t\int_0^t\int_0^t \cos 2t\,(dt)^3\right]$

[Ans. : (1) $\frac{2}{s(s^2+1)^2}$, **(2)** $\frac{1}{s^2(s^2+4)}$ **]**

16. Evaluation of the Integral $\int_0^\infty e^{-at}\left(\int_0^t e^{-u}f(u)\,du\right)dt$

As in the earlier cases, § 5, page 1-13; § 8, page 1-24; § 11, page 1-35, § 13, page 1-43, we can find the value of certain definite integrals, using the Laplace transform. This is illustrated in the following examples.

To find the above integral we first find $L\left[\int_0^t e^{-u}f(u)\,du\right]$, say $\Phi(s)$.

By definition of Laplace transform this means $\int_0^\infty e^{-st}\left[\int_0^t e^{-u}f(u)\,du\right]=\Phi(s)$.

Now, we put $s=a$, on both sides.

Example 1 : Use Laplace transform to evaluate $\int_0^\infty e^{-t}\int_0^t \frac{\sin u}{u}du\,dt$. **(M.U. 2004, 13, 19)**

Sol. : We have $L(\sin u)=\frac{1}{s^2+1}$

$$L\left(\frac{\sin u}{u}\right)=\int_s^\infty\frac{ds}{s^2+1}=\left[\tan^{-1}s\right]_s^\infty=\frac{\pi}{2}-\tan^{-1}s=\cot^{-1}s$$

$$\therefore\ L\left[\int_0^t\frac{\sin u}{u}du\right]=\frac{1}{s}\cot^{-1}s$$

But this means $\int_0^\infty e^{-st}\left[\int_0^t\frac{\sin u}{u}du\right]dt=\frac{1}{s}\cot^{-1}s$

Put $s=1$, $\therefore\ \int_0^\infty e^{-t}\int_0^t\frac{\sin u}{u}du\,dt=\cot^{-1}1=\frac{\pi}{4}$.

Example 2 : Evaluate the following integral by using Laplace transforms.

(i) $\int_0^\infty e^{-2t}\left(\int_0^t\frac{e^{-u}\sin u}{u}du\right)dt$ (ii) $\int_0^\infty e^{-t}\left(\int_0^t u^2\sin hu\cos hu\,du\right)dt$ **(M.U. 2003, 11, 19)**

Sol. : (i) As seen in Ex. 9(iii) above, page 1-58.

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$$L\left[\int_0^t \frac{e^{-u}\sin u}{u}\,du\right] = \frac{1}{s}\cot^{-1}(s+1)$$

By definition of Laplace transform this means

$$\int_0^\infty e^{-st}\left[\int_0^t \frac{e^{-u}\sin u}{u}\,du\right]dt = \frac{1}{s}\cdot\cot^{-1}(s+1)$$

Putting $s = 2$, we get

$$\int_0^\infty e^{-2t}\left[\int_0^t \frac{e^{-u}\sin u}{u}\,du\right]dt = \frac{1}{2}\cot^{-1}(3).$$

(ii) We have $L(\sin hu\cos hu) = L\left(\frac{1}{2}\sin h\,2u\right) = \frac{1}{2}\cdot\frac{2}{s^2+2^2} = \frac{1}{s^2+4}$

$$\therefore\ L(u^2\sin hu\cos hu) = (-1)^2\frac{d^2}{ds^2}\left(\frac{1}{s^2+4}\right) = \frac{d}{ds}\left(-\frac{2s}{(s^2+4)^2}\right)$$

$$= -2\left[\frac{(s^2+4)^2 - s\cdot 2(s^2+4)\cdot 2s}{(s^2+4)^4}\right] = -2\left[\frac{s^2+4-4s^2}{(s^2+4)^3}\right]$$

$$\therefore\ L(u^2\sin hu\cos hu) = 2\cdot\frac{(3s^2-4)}{(s^2+4)^3} = \Phi(s) \text{ say}$$

$$\therefore\ L\left[\int_0^t u^2\sin hu\cos hu\,du\right] = \frac{1}{s}\cdot\Phi(s) = \frac{2}{s}\cdot\frac{(3s^2-4)}{(s^2+4)^3}$$ [By (19), page 1-55]

By definition of Laplace transform, this means,

$$\int_0^\infty e^{-st}\left[\int_0^t u^2\sin hu\cos hu\,du\right]dt = \frac{2}{s}\frac{(3s^2-4)}{(s^2+4)^3}$$

Putting $s = 1$, we get

$$\int_0^\infty e^{-t}\left[\int_0^t u^2\sin hu\cos hu\,du\right]dt = \frac{2}{1}\cdot\frac{(3\cdot 1-4)}{(1+4)^3} = -\frac{2}{125}$$

EXERCISE - XV

Evaluate the following integrals by using Laplace transforms.

1. $\int_s^\infty e^{-t}\left(\int_0^t u\cos^2 u\,du\right)dt$
2. $\int_0^\infty e^{-2t}\left[\int_0^t\left(\frac{1-e^{-u}}{u}\right)du\right]dt$
3. $\int_0^\infty e^{-4t}\left(\cos ht\int_0^t e^u\cos hu\,du\right)dt$
4. $\int_0^\infty e^{-t}\left(t\int_0^t e^{-4u}\cos u\,du\right)dt$ **(M.U. 2019)**
5. $\int_0^\infty e^{-t}\left(\frac{1}{t}\int_0^t e^{-u}\sin u\,du\right)dt$

[**Ans. :** **(1)** 11 / 25, **(2)** (1 / 2) log (3 / 2), **(3)** 31 / 225, **(4)** 77 / 338, **(5)** (1 / 4) log 5 – (1 / 2) $\cot^{-1}$ (2)]

EXERCISE - XVI

Theory

1. Define Laplace transform of $f(t)$, $t > 0$. **(M.U. 1995, 2002)**
   Also state the conditions for its existence. **(M.U. 2005)**
2. Define Laplace transform of a function of t and state the rule of change of scale with one example. **(M.U. 2002, 04)**
3. Find (I) $L(\sin t)$ **(M.U. 2003)** (ii) $L(\cos t)$
   (III) $L(e^{at})$ (Iv) $L(t^n)$
4. State and prove first shifting theorem. Hence, find $L\{e^{2t}\cos t\cos 2t\}$. **(M.U. 2003, 09)**
   [**Ans. :** $\frac{(s-2)(s^2-4s+9)}{(s^2-4s+13)(s^2-4s+5)}$]
5. State and prove change of scale property of Laplace Transform. **(M.U. 2004, 09)**
6. If $L[f(t)] = \Phi(s)$, prove that $L\left[e^{-at}f(t)\right] = \Phi(s+a)$. **(M.U. 1995)**
7. If $g(t) = f(t-a)$ $t > a$
   $= 0$ $t < a$
   and $L[f(t)] = \Phi(s)$, prove that $L[g(t)] = e^{-as}\Phi(s)$. **(M.U. 1995)**
8. If $L[f(t)] = \Phi(s)$, prove that $L\left[t^n f(t)\right] = (-1)^n\frac{d^n}{ds^n}\Phi(s)$. **(M.U. 1997, 2005)**
9. State and prove First Shifting Theorem. **(M.U. 1997, 99, 2003)**
10. If $L[f(t)] = \Phi(s)$, prove that $L[f(at)] = \frac{1}{a}\Phi\left(\frac{s}{a}\right)$.
11. If $L[f(t)] = \Phi(s)$, prove that $L[t\,f(t)] = -\Phi'(s)$. **(M.U. 1993)**
12. If $L[f(t)] = \Phi(s)$, prove that $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \Phi(s)\,ds$. **(M.U. 1994, 98, 2000)**
13. Prove that $L[f'(t)] = -f(0) + s\,L\,f(t)$.
14. If $L[f(t)] = \Phi(s)$, prove that $L\int_0^t f(u)\,du = \frac{1}{s}\Phi(s)$. **(M.U. 1996)**
15. State and prove second shifting theorem. **(M.U. 2002)**
16. If $J_0(t) = 1 - \frac{t^2}{2^2} + \frac{t^4}{2^2\cdot 4^2} - \frac{t^6}{2^2\cdot 4^2\cdot 6^2} + \ldots$, prove that $L[J_0(t)] = \frac{1}{\sqrt{s^2+1}}$.
17. If $L[f(t)]$ is known and $f(0) = f'(0) = \ldots = f^n(0) = 0$, write the formula for $L[f^n(t)]$. **(M.U. 2011)**

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CHAPTER 2

Laplace Transforms - II

1. Introduction

In this chapter we shall study the methods of obtaining inverse Laplace transform. We shall further study Laplace transforms of some special functions. And then we shall apply the Laplace transforms to solve differential equations and boundary value problems.

2. Inverse Laplace Transforms

Definition : If $L[f(t)] = \Phi(s) = \int_0^\infty e^{-st} f(t)\,dt$ then $f(t)$ is called the **inverse Laplace transform** of $\Phi(s)$ and can be denoted as $L^{-1}\Phi(s) = f(t)$.

We get the following inverse transforms of some standard functions from this definition.

Table of Inverse Transforms

(i) $L(1) = \frac{1}{s}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{1}{s}\right) = 1}$ ..... (1)

(ii) $L(e^{-at}) = \frac{1}{s+a}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}}$ ..... (2)

(iii) $L(e^{at}) = \frac{1}{s-a}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{1}{s-a}\right) = e^{at}}$ ..... (3)

(iv) $L(t^{n-1}) = \frac{|\overline{n}}{s^n}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{|\overline{n}}}$ ..... (4)

(v) $L(t^{n-1}) = \frac{(n-1)!}{s^n}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}}$ ..... (5)

if n is a positive integer.

e.g. $L^{-1}\left(\frac{1}{s}\right) = 1,\ L^{-1}\left(\frac{1}{s^2}\right) = t,\ L^{-1}\left(\frac{1}{s^3}\right) = \frac{t^2}{2}.$

(vi) $L(\sin at) = \frac{a}{s^2+a^2}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at}$ ..... (6)

(vii) $L(\cos at) = \frac{s}{s^2+a^2}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at}$ ..... (7)

(viii) $L(\sinh at) = \frac{a}{s^2-a^2}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at}$ ..... (8)

(ix) $L(\cosh at) = \frac{s}{s^2-a^2}$ $\quad\therefore\quad \boxed{L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at}$ ..... (9)

EXERCISE - I

Write down the inverse Laplace transforms of the following.

1. $\frac{1}{s^5}$. 2. $\frac{1}{s^{3/2}}$. 3. $\frac{1}{s^{1/2}}$. 4. $\frac{1}{s^6}$. 5. $\frac{1}{s^2+9}$. 6. $\frac{1}{s^2+25}$. 7. $\frac{1}{s-4}$. 8. $\frac{1}{s+4}$.
9. $\frac{2s}{s^2+4}$. **(M.U. 2004)** 10. $\frac{s}{s^2+16}$. 11. $\frac{1}{s^2-1}$. 12. $\frac{1}{s^2-4}$. 13. $\frac{s}{s^2-4}$.
14. $\frac{s}{s^2-36}$. 15. $\frac{1}{s}$. **[Ans. :** Not given for obvious reason. **]**

Example 1 : Find $L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2$. **(M.U. 2011)**

Sol. : We have

$$L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2 = L^{-1}\left(\frac{1-2\sqrt{s}+s}{s^4}\right) = L^{-1}\left(\frac{1}{s^4}\right) - 2L^{-1}\left(\frac{1}{s^{7/2}}\right) + L^{-1}\left(\frac{1}{s^3}\right)$$

But $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{|n}$ or $\frac{t^{n-1}}{(n-1)!}$

$$\therefore\ L^{-1}\left(\frac{1-\sqrt{s}}{s^2}\right)^2 = \frac{t^3}{3!} - 2\cdot\frac{t^{5/2}}{|7/2} + \frac{t^2}{2!} = \frac{t^3}{6} - \frac{16}{15\sqrt{\pi}}\cdot t^{5/2} + \frac{t^2}{2}.$$

Example 2 : If $L[f(t)] = \frac{s+2}{s^2+2}$, find $L[f'(t)]$. **(M.U. 2003)**

Sol. : We have obtained $L[f'(t)]$ and $L[f''(t)]$ on page 1-58, when $f(t)$ was given as a function of t. Here, we are given $L[f(t)]$ in terms of s.

$$\because\ L[f(t)] = \frac{s+2}{s^2+2} \quad \therefore\ f(t) = L^{-1}\left[\frac{s+2}{s^2+2}\right]$$

$$\therefore\ f(t) = L^{-1}\left[\frac{s}{s^2+2}\right] + L^{-1}\left[\frac{2}{s^2+2}\right] = \cos\sqrt{2}\,t + 2\cdot\frac{1}{\sqrt{2}}\sin\sqrt{2}\cdot t$$

Putting $t = 0$, $f(0) = \cos 0 = 1$.

Now, $L[f'(t)] = -f(0) + s\,L[f(t)]$ **[By (18), page 1-58]**

$$\therefore\ L[f'(t)] = -1 + s\cdot\frac{s+2}{s^2+2} = \frac{s^2+2s}{s^2+2} - 1 = \frac{2s-2}{s^2+2} = \frac{2(s-1)}{s^2+2}.$$

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Example 3 : If $L[f(t)] = \dfrac{s+3}{s^2+4}$, find $L[f'(t)]$. [**Ans. :** $\dfrac{3s-4}{s^2+4}$]

Sol. : Do it yourself.

Example 4 : Prove that $L^{-1}\left[\dfrac{1}{s}\cos\dfrac{1}{s}\right] = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \ldots\ldots$

Sol. : We know that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots\ldots$$

$$\therefore \cos\frac{1}{s} = 1 - \frac{1}{2!}\cdot\frac{1}{s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} - \frac{1}{6!}\cdot\frac{1}{s^6} + \ldots\ldots$$

$$\therefore \frac{1}{s}\cos\frac{1}{s} = \frac{1}{s} - \frac{1}{2!}\cdot\frac{1}{s^3} + \frac{1}{4!}\cdot\frac{1}{s^5} - \frac{1}{6!}\cdot\frac{1}{s^7} + \ldots\ldots$$

$$\therefore L^{-1}\left[\frac{1}{s}\cos\frac{1}{s}\right] = L^{-1}\left[\frac{1}{s} - \frac{1}{2!}\cdot\frac{1}{s^3} + \frac{1}{4!}\cdot\frac{1}{5^5} - \frac{1}{6}\cdot\frac{1}{s^7} + \ldots\ldots\right]$$

$$= 1 - \frac{1}{2!}\cdot\frac{t^2}{2!} + \frac{1}{4!}\cdot\frac{t^4}{4!} - \frac{1}{6!}\cdot\frac{t^6}{6!}$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots\ldots$$

3. Methods of Obtaining Inverse Laplace Transforms

There are various methods of obtaining inverse Laplace transform. The choice of the method depends upon the nature of the problem. Sometimes an example can be solved by more than one method. You will find below that some solved problems are repeated but note that they are solved by different methods. You are advised to find which method you find more easy.

(a) Use of Standard Results

Using the standard results (given on page 2-1), inverse Laplace transforms of some simple functions can be obtained as illustrated below :

Example : Find the inverse Laplace transform of (i) $\dfrac{2}{s} + \dfrac{1}{s^3} + \dfrac{1}{s+4}$, (ii) $\dfrac{3s+4}{s^2+16}$.

Sol. : (i) From the above table, we get,

$$L^{-1}\left[\frac{2}{s} + \frac{1}{s^3} + \frac{1}{s+4}\right] = 2L^{-1}\left(\frac{1}{s}\right) + L^{-1}\left(\frac{1}{s^3}\right) + L^{-1}\left(\frac{1}{s+4}\right)$$

$$= 2 + \frac{t^2}{2} + e^{-4t}$$

(ii) $L^{-1}\left(\dfrac{3s+4}{s^2+16}\right) = 3L^{-1}\left(\dfrac{s}{s^2+4^2}\right) + L^{-1}\left(\dfrac{4}{s^2+4^2}\right)$

$$= 3\cdot\cos 4t + \sin 4t$$

EXERCISE - II

Find the inverse Laplace transform of

1. $\dfrac{3+2s+s^2}{s^3}$, 2. $\dfrac{2s+3}{s^2+9}$, 3. $\dfrac{3s+5}{9s^2-25}$, 4. $\dfrac{1}{4s+5}$,

5. $\dfrac{1}{4s-5}$, 6. $\dfrac{4s+15}{16s^2-25}$ 7. $\dfrac{(s^2-1)^2}{s^5}$, 8. $\dfrac{s+3}{s^2+4}$

[**Ans. :** (1) $3\cdot\dfrac{t^2}{2} + 2\cdot t + 1$, (2) $2\cos 3t + \sin 3t$, (3) $\dfrac{1}{3}\cosh\left(\dfrac{5}{3}\right)t + \dfrac{1}{3}\sinh\left(\dfrac{5}{3}\right)t$, (4) $\dfrac{1}{4}e^{-(5/4)t}$,

(5) $\dfrac{1}{4}e^{(5/4)t}$, (6) $\dfrac{1}{4}\cosh\left[\dfrac{5}{4}\cdot t\right] + \dfrac{3}{4}\sinh\left[\dfrac{5}{4}\cdot t\right]$, (7) $1 - t^2 + \dfrac{t^4}{24}$, (8) $\cos 2t + \dfrac{3}{2}\sin 2t$.]

(b) Use of Shifting Theorem

We know that,

if $L[f(t)] = \Phi(s)$, then $Le^{-at}f(t) = \Phi(s+a)$.

This means if $f(t) = L^{-1}\Phi(s)$ then $L^{-1}[\Phi(s+a)] = e^{-at}f(t)$.

i.e. $\boxed{L^{-1}[\Phi(s+a)] = e^{-at}L^{-1}\Phi(s)}$

e.g. $L^{-1}\dfrac{1}{(s+2)^2} = e^{-2t}\cdot L^{-1}\dfrac{1}{s^2} = e^{-2t}\cdot t$

$$L^{-1}\left[\frac{(s+1)}{(s+1)^2+3^2}\right] = e^{-t}L^{-1}\frac{s}{s^2+3^2} = e^{-t}\cdot\cos 3t$$

Note the following results

(i) $L^{-1}\dfrac{1}{(s-b)^n} = e^{bt}L^{-1}\dfrac{1}{s^n} = e^{bt}\cdot\dfrac{t^{n-1}}{|\overline{n}}$ ..... (10)

(ii) $L^{-1}\dfrac{1}{(s-b)^n} = e^{bt}L^{-1}\dfrac{1}{s^n} = e^{bt}\cdot\dfrac{t^{n-1}}{(n-1)!}$ ..... (11)

(if n is an integer)

(iii) $L^{-1}\dfrac{1}{(s-b)^2+a^2} = e^{bt}L^{-1}\dfrac{1}{s^2+a^2} = \dfrac{e^{bt}}{a}\sin at$ ..... (12)

(iv) $L^{-1}\dfrac{(s-b)}{(s-b)^2+a^2} = e^{bt}L^{-1}\dfrac{s}{s^2+a^2} = e^{bt}\cos at$ ..... (13)

(v) $L^{-1}\dfrac{1}{(s-b)^2-a^2} = e^{bt}L^{-1}\dfrac{1}{s^2-a^2} = \dfrac{e^{bt}}{a}\sinh at$ ..... (14)

(vi) $L^{-1}\dfrac{(s-b)}{(s-b)^2-a^2} = e^{bt}L^{-1}\dfrac{s}{s^2-a^2} = e^{bt}\cosh at$ ..... (15)

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EXERCISE - III

Write down the inverse Laplace transform of the following.

1. $\frac{1}{s-2}$, 2. $\frac{1}{s+3}$, 3. $\frac{1}{(s-2)^{3/2}}$, 4. $\frac{1}{(s+3)^2}$, 5. $\frac{1}{(s-2)^3}$, 6. $\frac{1}{(s+5)^{5/2}}$,

7. $\frac{1}{(s-2)^2+2^2}$, 8. $\frac{(s-3)}{(s-3)^2+2^2}$, 9. $\frac{1}{(s-3)^2+4^2}$, 10. $\frac{(s-4)}{(s-4)^2+5^2}$,

11. $\frac{1}{(s-2)^2-2^2}$, 12. $\frac{(s-3)}{(s-3)^2-2^2}$, 13. $\frac{1}{(s-3)^2-4^2}$, 14. $\frac{(s-4)}{(s-4)^2-5^2}$.

[**Ans. :** Answers not given for obvious reason]

Example 1 : Find the inverse Laplace transform of $\frac{s}{(s-2)^6}$. **(M.U. 2014)**

Sol. :
$$L^{-1}\frac{s}{(s-2)^6} = L^{-1}\left[\frac{(s-2)+2}{(s-2)^6}\right] = L^{-1}\left[\frac{1}{(s-2)^5}+\frac{2}{(s-2)^6}\right]$$
$$= L^{-1}\frac{1}{(s-2)^5} + 2L^{-1}\frac{1}{(s-2)^6} = e^{2t}L^{-1}\frac{1}{s^5} + 2e^{2t}L^{-1}\frac{1}{s^6}$$
$$= e^{2t}\left[\frac{t^4}{4!}\right] + 2e^{2t}\left[\frac{t^5}{5!}\right] = e^{2t}\left[\frac{t^4}{4!}+2\frac{t^5}{5!}\right].$$

Example 2 : Find the inverse Laplace transform of $\frac{1}{\sqrt{2s+1}}$. **(M.U. 2019)**

Sol. :
$$L^{-1}\left[\frac{1}{\sqrt{2s+1}}\right] = L^{-1}\left[\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{s+(1/2)}}\right]$$
$$= \frac{1}{\sqrt{2}}e^{-(1/2)t}\cdot L^{-1}\left[\frac{1}{\sqrt{s}}\right]$$
$$= \frac{1}{\sqrt{2}}e^{-t/2}\cdot\frac{t^{-1/2}}{|1/2}$$
$$= \frac{1}{\sqrt{2\pi}}\cdot e^{-t/2}\cdot t^{-1/2}$$

Example 3 : Find the inverse Laplace transform of

(i) $\frac{s+2}{s^2-4s+13}$ **(M.U. 2016)** (ii) $\frac{4s+12}{s^2+8s+12}$ **(M.U. 2002, 03)**

Sol. : (i)
$$L^{-1}\left[\frac{s+2}{s^2-4s+13}\right] = L^{-1}\left[\frac{s+2}{(s-2)^2+3^2}\right] = L^{-1}\left[\frac{(s-2)+4}{(s-2)^2+3^2}\right]$$
$$= e^{2t}L^{-1}\left[\frac{s+4}{s^2+3^2}\right] = e^{2t}L^{-1}\left(\frac{s}{s^2+3^2}\right) + 4\cdot e^{2t}L^{-1}\left(\frac{1}{s^2+3^2}\right)$$
$$= e^{2t}\cdot\cos 3t + \frac{4}{3}e^{2t}\sin 3t$$

(ii)
$$L^{-1}\left[\frac{4s+12}{s^2+8s+12}\right] = L^{-1}\left[\frac{4(s+4)-2^2}{(s+4)^2-2^2}\right]$$
$$= L^{-1}\left[\frac{4(s+4)}{(s+4)^2-2^2}\right] - L^{-1}\left[\frac{2^2}{(s+4)^2-2^2}\right]$$
$$= 4e^{-4t}L^{-1}\left[\frac{s}{s^2-2^2}\right] - 4e^{-4t}L^{-1}\left[\frac{1}{s^2-2^2}\right]$$
$$= 4e^{-4t}\cosh 2t - 4e^{-4t}\cdot\frac{1}{4}\sinh 2t$$
$$= e^{-4t}(4\cosh 2t - \sinh 2t)$$

Example 4 : Find :

(i) $L^{-1}\left[\frac{s+2}{s^2+4s+7}\right]$ **(M.U. 1993, 2008)** (ii) $L^{-1}\left[\frac{2s+3}{s^2+2s+2}\right]$. **(M.U. 1994)** (iii) $L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right]$. **(M.U. 1999, 2003)**

(iv) $L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right]$ (v) $L^{-1}\left[\frac{s+2}{s^2-2s+17}\right]$ **(M.U. 2016)**

Sol. : (i)
$$L^{-1}\left[\frac{s+2}{s^2+4s+7}\right] = L^{-1}\left[\frac{(s+2)}{(s+2)^2+(\sqrt{3})^2}\right]$$
$$= e^{-2t}L^{-1}\frac{s}{s^2+(\sqrt{3})^2} = e^{-2t}\cdot\cos\sqrt{3}\,t.$$

(ii)
$$L^{-1}\left[\frac{2s+3}{s^2+2s+2}\right] = L^{-1}\left[\frac{2(s+1)+1}{(s+1)^2+1}\right]$$
$$= 2L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] + L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$
$$= 2e^{-t}L^{-1}\left[\frac{s}{s^2+1}\right] + e^{-t}L^{-1}\left[\frac{1}{s^2+1}\right]$$
$$= 2e^{-t}\cos t + e^{-t}\cdot\sin t = e^{-t}[2\cos t + \sin t].$$

(iii)
$$L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = L^{-1}\left[\frac{3(s-1)+10}{(s-1)^2-2^2}\right]$$
$$= 3L^{-1}\left[\frac{s-1}{(s-1)^2-(2)^2}\right] + 10L^{-1}\frac{1}{(s-1)^2-(2)^2}$$
$$\therefore\ L^{-1}\left[\frac{3s+7}{s^2-2s-3}\right] = 3e^{t}L^{-1}\left[\frac{s}{s^2-(2)^2}\right] + 10e^{t}L^{-1}\left[\frac{1}{s^2-(2)^2}\right]$$
$$= 3e^{t}\cosh 2t + \frac{10}{2}e^{t}\sinh 2t$$
$$= e^{t}(3\cosh 2t + 5\sinh 2t).$$

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(iv) $$L^{-1}\left[\frac{2s-1}{s^2+4s+29}\right] = L^{-1}\left[\frac{2s+4-5}{s^2+4s+4+25}\right]$$

$$= L^{-1}\left[\frac{2(s+2)}{(s+2)^2+5^2} - \frac{5}{(s+2)^2+5^2}\right]$$

$$= 2L^{-1}\left[\frac{s+2}{(s+2)^2+5^2}\right] - 5L^{-1}\left[\frac{1}{(s+2)^2+5^2}\right]$$

$$= 2e^{-2t}L^{-1}\left[\frac{s}{s^2+5^2}\right] - 5e^{-2t}\cdot L^{-1}\left[\frac{1}{s^2+5^2}\right]$$

$$= 2e^{-2t}\cos 5t - 5e^{-2t}\cdot\frac{1}{5}\sin 5t$$

$$= 2e^{-2t}\cos 5t - e^{-2t}\sin 5t$$

(v) $$L^{-1}\left[\frac{s+2}{s^2-2s+17}\right] = L^{-1}\left[\frac{(s-1)+3}{s^2-2s+1+16}\right] = L^{-1}\left[\frac{(s-1)+3}{(s-1)^2+4^2}\right]$$

$$= L^{-1}\left[\frac{s-1}{(s-1)^2+4^2}\right] + L^{-1}\left[\frac{3}{(s-1)^2+4^2}\right]$$

$$= e^t\cdot L^{-1}\left(\frac{s}{s^2+4^2}\right) + e^t\cdot 3\cdot L^{-1}\left(\frac{1}{s^2+4^2}\right)$$

$$= e^t\cos 4t + \frac{3}{4}e^t\sin 4t$$

EXERCISE - IV

Find the inverse Laplace transform of

1. $\dfrac{6s-4}{s^2-4s+20}$. **(M.U. 2005)**
2. $\dfrac{3s-7}{s^2-6s+8}$.
3. $\dfrac{1}{(s-3)^3}$.
4. $\dfrac{1}{(s+2)^4}$.
5. $\dfrac{1}{(s+1)^2} + \dfrac{s-2}{s^2-4s+5} + \dfrac{s-2}{s^3-4s+3}$.
6. $\dfrac{s}{s^2+2s+2}$ **(M.U. 2004)**
7. $\dfrac{s}{(2s+1)^2}$ **(M.U. 2003)**

[Ans. : (1) $6\cdot e^{2t}\cos 4t + 2e^{2t}\sin 4t$, (2) $3\cdot e^{3t}\cdot\cosh t + 2\cdot e^{3t}\sinh t$, (3) $\frac{1}{2}e^{3t}t^2$, (4) $\frac{1}{3!}e^{-2t}\cdot t^3$, (5) $e^{-t}\cdot t + e^{2t}\cos t + e^{2t}\cosh t$, (6) $e^{-t}(\cos t - \sin t)$, (7) $[e^{-t/2}(2-t)]/8$. **]**

(c) Method of Partial Fractions

Whenever possible we express the given function $\Phi(s)$ into the sum of linear or quadratic partial fractions as $\Phi(s) = \dfrac{A}{(s+a)^r} + \dfrac{Bs+C}{(s^2+a^2)^r}$ and then use the standard results given above to find L^{-1}.

Example 1 : Find inverse Laplace transform of

(i) $\dfrac{s^2+16s-24}{s^4+20s^2+64}$ **(M.U. 2005)**

(ii) $\dfrac{s+29}{(s+4)(s^2+9)}$ **(M.U. 1999, 2009, 15)**

(iii) $\dfrac{s+2}{(s+3)(s+1)^3}$ **(M.U. 2005, 10)**

(iv) $\dfrac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)}$ **(M.U. 2005)**

(v) $\dfrac{s}{(s^2+1)(s^2+4)(s^2+9)}$

(vi) $\dfrac{s+2}{(s^2+4s+8)(s^2+4s+13)}$ **(M.U. 2007)**

Sol. : (i) $$\Phi(s) = \frac{s^2+16s-24}{s^4+20s^2+64} = \frac{s^2+16s-24}{(s^2+4)(s^2+16)}$$

Let $$\Phi(s) = \frac{as+b}{s^2+4} + \frac{cs+d}{s^2+16}$$

$$\therefore\ s^2+16s-24 = (as+b)(s^2+16) + (cs+d)(s^2+4)$$

$$= (a+c)s^3 + (b+d)s^2 + (16a+4c)s + (16b+4d)$$

Equating the coefficients of like powers of s we get, $a+c=0$, $b+d=1$, $16a+4c=16$, $16b+4d=-24$.

$$\therefore\ a = 4/3,\ c = -4/3,\ b = -7/3,\ d = 10/3$$

$$\therefore\ \Phi(s) = \frac{1}{3}\cdot\frac{4s-7}{s^2+4} - \frac{1}{3}\cdot\frac{4s-10}{s^2+16}$$

$$\therefore\ L^{-1}[\Phi(s)] = \frac{4}{3}L^{-1}\left(\frac{s}{s^2+4}\right) - \frac{7}{3}L^{-1}\left(\frac{1}{s^2+4}\right) - \frac{4}{3}L^{-1}\left(\frac{s}{s^2+16}\right) + \frac{10}{3}L^{-1}\left(\frac{1}{s^2+16}\right)$$

$$= \frac{4}{3}\cos 2t - \frac{7}{3}\cdot\frac{1}{2}\sin 2t - \frac{4}{3}\cdot\cos 4t + \frac{10}{3}\cdot\frac{1}{4}\sin 4t$$

(ii) $$\Phi(s) = \frac{s+29}{(s+4)(s^2+9)} = \frac{a}{s+4} + \frac{bs+c}{s^2+9}, \text{ say}$$

$$\therefore\ s+29 = a(s^2+9) + (bs+c)(s+4)$$

$$\therefore\ s+29 = (a+b)s^2 + (c+4b)s + (9a+4c).$$

Equating the coefficients of like powers of s, we get,

$$\therefore\ a+b=0,\ c+4b=1,\ 9a+4c=29 \qquad \therefore\ a=1,\ b=-1,\ c=5.$$

$$\therefore\ \Phi(s) = \frac{1}{s+4} - \frac{s-5}{s^2+9}$$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left(\frac{1}{s+4}\right) - L^{-1}\left(\frac{s}{s^2+9}\right) + 5L^{-1}\left(\frac{1}{s^2+9}\right) = e^{-4t} - \cos 3t + 5\cdot\frac{1}{3}\sin 3t$$

(iii) $$\Phi(s) = \frac{s+2}{(s+3)(s+1)^3}$$

$$= \frac{1}{8}\cdot\frac{1}{(s+3)} - \frac{1}{8}\cdot\frac{1}{(s+1)} + \frac{1}{4}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{(s+1)^3}$$

(after expressing the function in partial fractions)

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$$\therefore\ L^{-1}[\Phi(s)] = \frac{1}{8}\cdot L^{-1}\left(\frac{1}{s+3}\right) - \frac{1}{8}\cdot L^{-1}\left(\frac{1}{s+1}\right) + \frac{1}{4}\cdot L^{-1}\frac{1}{(s+1)^2} + \frac{1}{2}\cdot L^{-1}\frac{1}{(s+1)^3}$$

But $L^{-1}\left[\frac{1}{(s+b)^{n+1}}\right] = \frac{1}{n!}\cdot e^{-bt}\cdot t^n$

$$\therefore\ L^{-1}[\Phi(s)] = \frac{1}{8}\cdot e^{-3t} - \frac{1}{8}\cdot e^{-t} + \frac{1}{4}\cdot\frac{1}{1!}e^{-t}\cdot t + \frac{1}{2}\cdot\frac{1}{2!}e^{-t}\cdot t^2$$

$$= \frac{1}{8}\left[(2t^2+2t-1)e^{-t} + e^{-3t}\right]$$

(iv) $$\Phi(s) = \frac{s^2+2s-4}{(s^2+2s+5)(s^2+2s+2)} = \frac{as+b}{(s^2+2s+5)} + \frac{cs+d}{(s^2+2s+2)}$$

Simplifying and equating like powers of s on both sides, we get, $a = 0,\ b = 3,\ c = 0,\ d = -2$

$$\therefore\ \Phi(s) = \frac{3}{s^2+2s+5} - \frac{2}{s^2+2s+2} = \frac{3}{(s+1)^2+2^2} - \frac{2}{(s+1)^2+1^2}$$

$$\therefore\ L^{-1}\Phi(s) = 3L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right] - 2\cdot L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right]$$

$$= 3e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right] - 2e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= 3e^{-t}\cdot\frac{1}{2}\sin 2t - 2e^{t}\cdot 1\cdot\sin t$$

(v) Let us first consider

$$\frac{1}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1/24}{s^2+1} - \frac{1/15}{s^2+4} + \frac{1/40}{s^2+9}$$ **[Note this]**

Now, $$\frac{s}{(s^2+1)(s^2+4)(s^2+9)} = \frac{1}{24}\cdot\frac{s}{s^2+1} - \frac{1}{15}\cdot\frac{s}{s^2+4} + \frac{1}{40}\cdot\frac{s}{s^2+9}$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2+1)(s^2+4)(s^2+9)}\right] = \frac{1}{24}L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{1}{15}L^{-1}\left(\frac{s}{s^2+4}\right) + \frac{1}{40}L^{-1}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{1}{24}\cos t - \frac{1}{15}\cos 2t + \frac{1}{40}\cos 3t$$

(vi) $$\frac{s+2}{(s^2+4s+8)(s^2+4s+13)} = \frac{s+2}{[(s+2)^2+2^2][(s+2)^2+3^2]}$$

$$\therefore\ L^{-1}\frac{s+2}{(s^2+4s+8)(s^2+4s+13)} = L^{-1}\frac{1}{5}\left[\frac{s+2}{(s+2)^2+2^2} - \frac{s+2}{(s+2)^2+3^2}\right]$$

$$= \frac{1}{5}\left[e^{-2t}L^{-1}\frac{s}{s^2+2^2} - e^{-2t}L^{-1}\frac{s}{s^2+3^2}\right] = \frac{e^{-2t}}{5}(\cos 2t - \cos 3t)$$

Example 2 : Find the inverse Laplace transform of the following.

(i) $\frac{3s+1}{(s+1)(s^2+2)}$ **(M.U. 1995, 2010)**

(ii) $\frac{5s^2-15s-11}{(s+1)(s-2)^2}$ **(M.U. 1995, 2007, 11)**

(iii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ **(M.U. 1996, 98, 2000, 03)**

(iv) $\frac{s}{(s^2+a^2)(s^2+b^2)}$ **(M.U. 2004)**

(v) $\frac{5s^2+8s-1}{(s+3)(s^2+1)}$ **(M.U. 1996, 2002, 07)**

(vi) $\frac{2s}{s^4+4}$ **(M.U. 1997, 2002, 10, 15)**

(vii) $\frac{s+2}{s^2(s+3)}$ **(M.U. 1997, 2003)**

(viii) $\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$ **(M.U. 2003, 04, 05, 08)**

(ix) $\frac{s}{s^4+4a^4}$ **(M.U. 2004, 08)**

Sol. : (i) Let $\frac{3s+1}{(s+1)(s^2+2)} = \frac{a}{s+1} + \frac{bs+c}{s^2+2}$

$$\therefore\ 3s+1 = a(s^2+2) + (s+1)(bs+c) = (a+b)s^2 + (b+c)s + 2a + c.$$

Equating the coefficients of like powers of s.

$$\therefore\ a+b=0,\ b+c=3,\ 2a+c=1$$

Solving these equations, we get $a = -\frac{2}{3},\ b = \frac{2}{3},\ c = \frac{7}{3}.$

$$\therefore\ L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right] = -\frac{2}{3}L^{-1}\frac{1}{s+1} + \frac{2}{3}L^{-1}\frac{s}{s^2+2} + \frac{7}{3}L^{-1}\frac{1}{s^2+2}$$

$$= -\left(\frac{2}{3}\right)e^{-t}L^{-1}\frac{1}{s} + \frac{2}{3}L^{-1}\frac{s}{s^2+(\sqrt{2})^2} + \frac{7}{3}L^{-1}\frac{1}{s^2+(\sqrt{2})^2}$$

$$= -\left(\frac{2}{3}\right)e^{-t}\cdot 1 + \frac{2}{3}\cos\sqrt{2}\,t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}\,t.$$

(ii) Let $\frac{5s^2-15s-11}{(s+1)(s-2)^2} = \frac{a}{s+1} + \frac{b}{s-2} + \frac{c}{(s-2)^2}$

$$\therefore\ 5s^2-15s-11 = a(s-2)^2 + b(s+1)(s-2) + c(s+1)$$

Equating the coefficient of like power of s

$$a+b=5,\ -4a-b-c=-15,\ 4a-2b+c=-11 \qquad \therefore\ a=1,\ b=4,\ c=-7.$$

$$\therefore\ L^{-1}\left[\frac{5s^2-15s-11}{(s+1)(s-2)^2}\right] = L^{-1}\frac{1}{s+1} + 4L^{-1}\frac{1}{s-2} - 7L^{-1}\frac{1}{(s-2)^2}$$

$$= e^{-t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} - 7e^{2t}L^{-1}\frac{1}{s^2}$$

$$= e^{-t} + 4e^{2t} - 7e^{2t}t.$$

(iii) Let $s^2 = x \quad \therefore\ \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{x}{(x+a^2)(x+b^2)}$ **[Note this]**

Let $\frac{x}{(x+a^2)(x+b^2)} = \frac{A}{x+a^2} + \frac{B}{x+b^2} \quad \therefore\ x = A(x+b^2) + B(x+a^2)$

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Put $x = -a^2$ $\quad \therefore\ -a^2 = A(-a + B^2)$ $\quad \therefore\ A = a^2/(a^2 - b^2)$

Put $x = -b^2$ $\quad \therefore\ -b^2 = B(-b^2 + a^2)$ $\quad \therefore\ B = -b^2/(a^2 - b^2)$

$$\therefore\ \frac{s^2}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a^2-b^2}\left[\frac{a^2}{s^2+a^2} - \frac{b^2}{s^2+b^2}\right]$$

$$\therefore\ L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = \frac{1}{a^2-b^2}\left[L^{-1}\left(\frac{a^2}{s^2+a^2}\right) - L^{-1}\left(\frac{b^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[a^2 \cdot L^{-1}\left(\frac{1}{s^2+a^2}\right) - b^2 \cdot L^{-1}\left(\frac{1}{s^2+b^2}\right)\right]$$

$$= \frac{1}{a^2-b^2}\left[\frac{a^2}{a} \cdot \sin at - b^2 \cdot \frac{1}{b}\sin bt\right]$$

$$= \frac{1}{a^2-b^2}(a \sin at - b \sin bt)$$

[For another method see Ex. 3 (ii), page 2-20]

(iv) Let us first consider

$$\frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{b^2-a^2}\left[\frac{1}{s+a^2} - \frac{1}{s^2+b^2}\right]$$ **[Note this]**

$$\therefore\ L^{-1}\frac{s}{(s^2+a^2)(s^2+b^2)} = L^{-1}\frac{1}{b^2-a^2}\left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{(b^2-a^2)}\left[L^{-1}\frac{s}{s^2+a^2} - L^{-1}\frac{s}{s^2+b^2}\right]$$

$$= \frac{1}{b^2-a^2}(\cos at - \cos bt)$$

[For another method see Ex. 3 (iii), page 2-20]

(v) Let $\dfrac{5s^2+8s-1}{(s+3)(s^2+1)} = \dfrac{a}{s+3} + \dfrac{bs+c}{s^2+1}$

$$\therefore\ 5s^2 + 8s - 1 = a(s^2+1) + (bs+c)(s+3)$$

$$= (a+b)s^2 + (3b+c)s + (a+3c)$$

Equating the coefficients of like powers of s, we get,

$a + b = 5,\ 3b + c = 8,\ a + 3c = -1$

$\therefore\ a = 2,\ b = 3,\ c = -1$

$$\therefore\ L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = L^{-1}\left[\frac{2}{s+3} + 3 \cdot \frac{s}{s^2+1} - \frac{1}{s^2+1}\right]$$

$$\therefore\ L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = 2L^{-1}\frac{1}{s+3} + 3L^{-1}\frac{s}{s^2+1} - L^{-1}\frac{1}{s^2+1}$$

$$= 2e^{-3t} + 3\cos t - \sin t$$

(vi) $$\frac{2s}{s^4+4} = \frac{2s}{(s^4+4s^2+4)-4s^2} = \frac{2s}{(s^2+2)^2-(2s)^2}$$ **[Note this]**

$$= \frac{1}{2}\left[\frac{1}{(s^2-2s+2)} - \frac{1}{(s^2+2s+2)}\right]$$

$$= \frac{1}{2}\left[\frac{1}{(s-1)^2+1} - \frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ L^{-1}\frac{2s}{s^4+4} = \frac{1}{2}\left[L^{-1}\frac{1}{(s-1)^2+1} - L^{-1}\frac{1}{(s+1)^2+1}\right]$$

$$= \frac{1}{2}\left[e^t L^{-1}\frac{1}{s^2+1} - e^{-t}L^{-1}\frac{1}{s^2+1}\right]$$

$$\therefore\ L^{-1}\frac{2s}{s^4+4} = \frac{1}{2}\left[e^t \sin t - e^{-t}\sin t\right] = \sin t\left(\frac{e^t - e^{-t}}{2}\right) = \sin t \sinh t.$$

(vii) Let $\dfrac{s+2}{s^2(s+3)} = \dfrac{a}{s} + \dfrac{b}{s^2} + \dfrac{c}{s+3}$

$$\therefore\ s + 2 = as(s+3) + b(s+3) + cs^2$$

Putting $s = 0,\ 2 = 3b$; putting $s = -3,\ -1 = 9c$.

Equating the coefficients of s^2, $a + c = 0$. $\therefore\ a = 1/9$.

$$\therefore\ L^{-1}\frac{s+2}{s^2(s+3)} = \frac{1}{9}L^{-1}\left(\frac{1}{s}\right) + \frac{2}{3}L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{9}L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \frac{1}{9}(1) + \frac{2}{3}t - \frac{1}{9}e^{-3t} = \frac{1}{9}(1 + 6t - e^{-3t})$$

(viii) $$\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{(s+1)^2+2}{\left[(s+1)^2+2^2\right]\left[(s+1)^2+1^2\right]}$$ **[Note this]**

$$\therefore\ L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = e^{-t}L^{-1}\frac{s^2+2}{(s^2+4)(s^2+1)}$$

Let $s^2 = x$ and $\dfrac{x+2}{(x+4)(x+1)} = \dfrac{a}{x+4} + \dfrac{b}{x+1}$ **[Note this]**

$$\therefore\ x + 2 = a(x+1) + b(x+4)$$

When $x = -1,\ 1 = 3b$; when $x = -4,\ -2 = -3a$.

$$\therefore\ \frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2}{3} \cdot \frac{1}{s^2+4} + \frac{1}{3} \cdot \frac{1}{s^2+1}$$

$$\therefore\ L^{-1}\frac{s^2+2}{(s^2+4)(s^2+1)} = \frac{2}{3}L^{-1}\frac{1}{s^2+4} + \frac{1}{3}L^{-1}\frac{1}{s^2+1} = \frac{2}{3} \cdot \frac{1}{2}\sin 2t + \frac{1}{3}\sin t$$

$$\therefore\ L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}\right] = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

(For another method see Ex. 4 (iii), page 2-23.)

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(ix) We have, by completing the square, [Note this]

$$\frac{s}{s^4+4a^4} = \frac{s}{(s^4+4s^2a^2+4a^4)-(4s^2a^2)}$$

$$\therefore \frac{s}{s^4+4a^4} = \frac{s}{(s^2+2a^2+2as)(s^2+2a^2-2as)}$$

First consider that (by inspection)

$$\frac{s}{s^4+4a^4} = \frac{1}{4a}\left[\frac{1}{s^2+2a^2-2as} - \frac{1}{s^2+2a^2+2as}\right]$$

$$\therefore L^{-1}\frac{s}{s^4+4a^4} = \frac{1}{4a}\left\{L^{-1}\left[\frac{1}{(s-a)^2+a^2}\right] - L^{-1}\left[\frac{1}{(s+a)^2+a^2}\right]\right\}$$

$$= \frac{1}{4a}\left[e^{at}L^{-1}\left(\frac{1}{s^2+a^2}\right) - e^{-at}L^{-1}\left(\frac{1}{s^2+a^2}\right)\right]$$

$$\therefore L^{-1}\frac{s}{s^4+4a^4} = \frac{1}{4a}\left[e^{at}\frac{\sin at}{a} - e^{-at}\frac{\sin at}{a}\right]$$

$$= \frac{\sin at}{2a^2}\left[\frac{e^{at}-e^{-at}}{2}\right] = \frac{1}{2a^2}\sin at \sinh at.$$

EXERCISE - V

Find the inverse Laplace transform of

1. $\frac{s+1}{s^2-4}$ (M.U. 2013)
2. $\frac{2s^2-4}{(s+1)(s-2)(s-3)}$
3. $\frac{s+4}{(s^2-1)(s+1)}$ (M.U. 2003)
4. $\frac{s^2+1}{s^3+3s^2+2s}$ (M.U. 2012)
5. $\frac{s}{(s^2+5s+6)}$
6. $\frac{3s+7}{s^2-2s-3}$ (M.U. 2002, 10)
7. $\frac{2s^2-6s+5}{s^3-6s^2+11s-6}$ (M.U. 2003, 05, 19)
8. $\frac{s^2+10s+13}{(s-1)(s^2-5s+6)}$
9. $\frac{1}{s^2(s+1)}$
10. $\frac{1}{s^2(s+2)}$
11. $\frac{s^2}{(s+4)^3}$
12. $\frac{s^2}{(s+1)^3}$ (M.U. 2011)
13. $\frac{1}{s^2(s+3)^2}$
14. $\frac{1}{(s-2)(s+2)^2}$
15. $\frac{4s+4}{(s-1)^2(s+2)}$
16. $\frac{2s+3}{(s+1)^2(s+2)}$
17. *$\frac{1}{(s-2)^4(s+3)}$ (M.U. 2011)
18. $\frac{1}{s(s^2-a^2)}$
19. $\frac{2s^2-1}{(s^2+1)(s^2+4)}$ (M.U. 2011)
20. $\frac{s}{(s^2+1)(s^2+4)}$ (M.U. 2013)
21. $\frac{s+29}{(s+4)(s^2+9)}$ (M.U. 2007, 13)
22. $\frac{5s+3}{(s-1)(s^2+2s+5)}$ (M.U. 2002, 06)
23. *$\frac{11s^2-2s+5}{2s^3-3s^2-3s+2}$
24. $\frac{s}{(s-3)(s^2+4)}$
25. *$\frac{s^2}{(s^2+1)(s^2+4)}$
26. *$\frac{2s-1}{(s^4+s^2+1)}$ (M.U. 2007)
27. $\frac{s^2}{(s^2+9)(s^2+4)}$

28. $\frac{1}{(s-2)^2(s+3)}$ (M.U. 2003)
29. *$\frac{1}{s^3+1}$ (M.U. 2003)
30. $\frac{1}{s^3(s-1)}$ (M.U. 2003, 14)
31. $\frac{s}{(s+1)^2(s^2+1)}$ (M.U. 2004)
32. *$\frac{s+2}{(s+3)(s+1)^3}$ (M.U. 2005)
33. *$\frac{a(s^2+2a^2)}{s^4+4a^4}$ (M.U. 2009)
34. *$\frac{s^2}{(s+a)^3}$ (M.U. 2005)

(* Star indicates harder problems.)

[Ans. : (1) $\frac{1}{4}[e^{-2t}+3e^{2t}]$, (2) $-\frac{1}{6}e^{-t}-\frac{4}{3}e^{2t}+\frac{7}{2}e^{3t}$, (3) $\frac{5}{4}e^{t}-\frac{5}{4}e^{-t}-\frac{3}{2}te^{-t}$

(4) $\frac{1}{2}-2e^{-t}+\frac{5}{2}e^{-2t}$, (5) $3e^{-3t}-2e^{-2t}$, (6) $4e^{3t}-e^{-t}$,

(7) $\frac{1}{2}e^{t}-e^{2t}+\frac{5}{2}e^{3t}$, (8) $12e^{t}-37e^{2t}+26e^{3t}$, (9) $-1+t+e^{-t}$,

(10) $-\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}$, (11) $e^{-4t}(1-8t+8t)$, (12) $e^{-t}\left(1-2t+\frac{t^2}{2}\right)$,

(13) $\frac{1}{27}(-2+3t+2e^{-3t}+3t^2e^{-3t})$, (14) $\frac{1}{16}\left(e^{2t}-e^{-2t}-4te^{-2t}\right)$,

(15) $\frac{4}{9}(e^{t}-e^{-2t}+6te^{t})$, (16) $e^{-t}-e^{-2t}+te^{-t}$,

(17) $\frac{e^{2t}}{6}\left[\frac{t^3}{5}-\frac{3}{25}t^2+\frac{6}{125}t-\frac{6}{625}\right]+\frac{1}{625}e^{-3t}$,

(18) $\frac{1}{2a^2}[e^{at}+e^{-at}-2]$, (19) $\frac{3}{2}\cdot\sin 2t-\sin t$, (20) $\frac{1}{3}(\cos t-\cos 2t)$,

(21) $e^{-4t}-\cos 3t+\frac{5}{3}\sin 3t$, (22) $e^{t}-e^{-t}\cos 2t+\frac{3}{2}e^{-t}\sin 2t$,

(23) $2e^{-t}+5e^{2t}-\frac{3}{2}e^{t/2}$, (24) $\frac{3}{13}e^{3t}-\frac{3}{13}\cos 2t+\frac{2}{13}\sin 2t$,

(25) $-\frac{1}{3}\sin t+\frac{2}{3}\sin 2t$,

(26) $\frac{1}{2}e^{t/2}\cos\frac{\sqrt{3}}{2}t+\frac{\sqrt{3}}{2}e^{t/2}\sin\frac{\sqrt{3}}{2}t-\frac{1}{2}e^{-t/2}\cos\frac{\sqrt{3}}{2}t-\frac{5}{2\sqrt{3}}e^{-t/2}\sin\frac{\sqrt{3}}{2}t$

(27) $\frac{1}{5}(3\sin 3t-2\sin 2t)$, (28) $-\frac{1}{25}e^{2t}+\frac{t}{5}e^{2t}+\frac{1}{25}e^{-3t}$,

(29) $\frac{1}{3}e^{-t}-\frac{e^{t/2}}{3}\cdot\cos\left(\frac{\sqrt{3}}{2}\cdot t\right)+\frac{e^{t/2}}{\sqrt{3}}\cdot\sin\left(\frac{\sqrt{3}}{2}\cdot t\right)$,

(30) $e^{-t}-\left[1+t+\frac{t^2}{2}\right]$, (31) $\frac{1}{2}\left[\sin t-t\cdot e^{-t}\right]$,

(32) $\frac{1}{8}e^{-3t}-\frac{1}{8}e^{-t}+\frac{1}{4}t\cdot e^{-t}+\frac{1}{4}t^2\cdot e^{-t}$, (33) $\sin at \cos h\, at$.

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(34) $e^{-at}\left[1-2at+\frac{a^2t^2}{2}\right]$

(d) Use of by Convolution Theorem

Definition : If $f_1(t)$ and $f_2(t)$ are two functions then the following integral

$$\int_0^t f_1(u)f_2(t-u)\,du$$

is called the convolution (= twisting, coiling, winding together) of $f_1(t)$ and $f_2(t)$ and is denoted by $f_1(t) * f_2(t)$. Thus,

$$f_1(t) * f_2(t) = \int_0^t f_1(t)f_2(t-u)\,du.$$

Theorem : Let $L\,f_1(t) = \Phi_1(s)$ and $L\,f_2(t) = \Phi_2(s)$, then

$$\boxed{L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t f_1(u)\cdot f_2(t-u)\,du} \qquad \text{..... (16)}$$

where, $f_1(t) = L^{-1}\Phi_1(s)$ and $f_2(t) = L^{-1}\Phi_2(s)$

Proof. : Let $\Phi(t) = \int_0^t f_1(u)\cdot f_2(t-u)\,du$.

$$\therefore\quad L[\Phi(t)] = \int_0^\infty e^{-st}\,\Phi(t)\,dt = \int_0^\infty e^{-st}\int_0^t f_1(u)\cdot f_2(t-u)\,du\,dt$$

$$= \int_0^\infty\int_0^t e^{-st} f_1(u)\cdot f_2(t-u)\,du\,dt$$

On the r.h.s. integration is first carried out w.r.t. u and then w.r.t. t. Now, we change the order of integration. We integrate first with respect to t and then w.r.t. u. But now t varies from u to ∞ and then u varies from 0 to ∞.

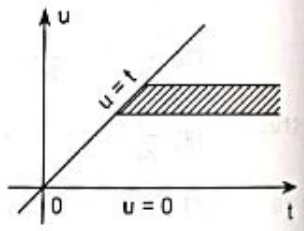

Fig. 2.1

$$\therefore\quad L[\Phi(t)] = \int_0^\infty\int_u^\infty e^{-st} f_1(u)f_2(t-u)\,dt\,du$$

$$= \int_0^\infty\int_u^\infty e^{-su}e^{-s(t-u)} f_1(u)f_2(t-u)\,dt\,du$$

$$= \int_0^\infty e^{-su} f_1(u)\,du\int_u^\infty e^{-s(t-u)} f_2(t-u)\,dt$$

Putting $t-u = p$, $dt = dp$

$$\therefore\quad L[\Phi(t)] = \int_0^\infty e^{-su} f_1(u)\,du\int_0^\infty e^{-sp} f_2(p)\,dp$$

Since for definite integral, the variable used does not matter, we replace u by t and p by t.

$$\therefore\quad L[\Phi(t)] = \int_0^\infty e^{-st} f_1(t)\,dt\cdot\int_0^\infty e^{-st} f_2(t)\,dt$$

$$\therefore\quad L\Phi(t) = L\,f_1(t)\cdot L\,f_2(t) \qquad \therefore\quad L\Phi(t) = \Phi_1(s)\cdot\Phi_2(s)$$

$$\therefore\quad L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \Phi(t) = \int_0^t f_1(u)\cdot f_2(t-u)\,du$$

Cor. : If $\Phi_1(s) = \Phi(s)$ and $\Phi_2(s) = \frac{1}{s}$ then $f_2(t) = L^{-1}\Phi_2(s) = 1$.

$$\therefore\quad \boxed{L^{-1}\left[\frac{1}{s}\Phi(s)\right] = \int_0^t f_1(u)\,du = \int_0^t L^{-1}[\Phi(s)]\,du} \qquad \text{..... (16A)}$$

Notes ....

1. Taking Laplace transforms of both sides of (16), we get

$$\Phi_1(s)\Phi_2(s) = L\left[\int_0^t f_1(u)f_2(t-u)\,du\right]$$

But $\Phi_1(s) = L\,[f_1(t)]$ and $\Phi_2(s) = L\,[f_2(t)]$

$$\therefore\quad L\left[\int_0^t f_1(u)\cdot f_2(t-u)\,du\right] = L[f_1(t)]\cdot L[f_2(t)]$$

This means the Laplace transform of the convolution of two functions is equal to the product of the Laplace transforms of the two functions.

2. The above result given in the corollary is obtained independently in § (f) (18), page 2-35.

Procedure of Applying Convolution Theorem

To find $L^{-1}\Phi_1(s)\cdot\Phi_2(s)$:-

1. Find $L^{-1}\Phi_1(s) = f_1(u)$, say, putting u in place of t.
2. Find $L^{-1}\Phi_2(s) = f_2(u)$, say, putting u in place of t.
3. Find $L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t f_1(u)f_2(t-u)\,du$.

Example 1 : Find the inverse Laplace Transforms by using convolution theorem.

(i) $\frac{1}{s(s+a)}$, (ii) $\frac{1}{s(s+a)^2}$, (iii) $\frac{1}{s(s^2+a^2)}$, (iv) $\frac{1}{s(s^2-a^2)}$.

Sol. : In all these examples put $\Phi_2(s) = \frac{1}{s}$, then by the corollary

$$\boxed{L^{-1}\left[\frac{1}{s}\cdot\Phi(s)\right] = \int_0^t\left[L^{-1}\Phi(s)\right]du}$$

(i) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+a}\right] = \int_0^t\left[L^{-1}\left(\frac{1}{s+a}\right)\right]du = \int_0^t e^{-au}\,du = \left[-\frac{e^{-au}}{a}\right]_0^t$$

$$= -\left[\frac{e^{-at}-1}{a}\right] = \frac{1-e^{-at}}{a}.$$

(ii) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s+a)^2}\right] = \int_0^t\left[L^{-1}\frac{1}{(s+a)^2}\right]du = \int_0^t e^{-au}\left[L^{-1}\left(\frac{1}{s^2}\right)\right]du$$

$$= \int_0^t e^{-au}\cdot u\,du = \int_0^t u\cdot e^{-au}\,du = \left[u\cdot\frac{e^{-au}}{-a} - \frac{e^{-au}}{(-a)^2}\cdot 1\right]_0^t$$

$$\therefore\quad L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s+a)^2}\right] = \left[-\frac{te^{-at}}{a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}\right] = \frac{1}{a^2}\left[1-(1+at)e^{-at}\right]$$

(iii) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s^2+a^2)}\right] = \int_0^t\left[L^{-1}\frac{1}{s^2+a^2}\right]du = \int_0^t\left(\frac{\sin au}{a}\right)du$$

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$$\therefore\ L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s^2+a^2)}\right] = -\frac{1}{a^2}\left[\cos au\right]_0^t = -\frac{1}{a^2}\left[\cos at - 1\right] = \frac{1-\cos at}{a^2}$$

(iv) $$L^{-1}\left[\frac{1}{s}\cdot\frac{1}{(s^2-a^2)}\right] = \int_0^t\left[L^{-1}\frac{1}{s^2-a^2}\right]du = \int_0^t \frac{\sinh au}{a}du$$

$$= \frac{1}{a^2}\left[\cosh au\right]_0^t = \frac{1}{a^2}\left[\cosh at - 1\right].$$

EXERCISE - VI

Find the inverse of the following functions using convolution theorem.

(i) $\frac{1}{s(s^2+4)}$. (ii) $\frac{1}{s(s+2)}$. (iii) $\frac{1}{s(s^2-9)}$. (iv) $\frac{1}{s(s+4)^2}$.

[**Ans.** : (i) $\frac{1}{4}(1-\cos 2t)$. (ii) $\frac{e^{2t}-1}{2}$. (iii) $\frac{1}{9}\left[\cosh 3t - 1\right]$. (iv) $\frac{1}{16}\left[1-(1+4t)e^{-4t}\right]$.]

Example 2 : Find the inverse Laplace Transforms by using convolution theorem

(i) $\frac{s^2}{(s^2+a^2)^2}$ **(M.U. 1995, 2000, 02, 07)**

(ii) $\frac{s^2}{(s^2-a^2)^2}$ **(M.U. 2002, 03, 09, 14)**

(iii) $\frac{1}{(s-a)(s+a)^2}$ **(M.U. 2019)**

(iv) $\frac{s}{(s^2+a^2)^2}$ **(M.U. 2000, 03, 11, 12, 13)**

(v) $\frac{s}{(s^2-a^2)^2}$ **(M.U. 2003, 15, 18)**

(vi) $\frac{1}{(s-1)(s^2+4)}$ **(M.U. 2014)**

Sol. : (i) Let $\Phi_1(s) = \frac{s}{s^2+a^2}$, $\Phi_2(s) = \frac{s}{s^2+a^2}$.

$$L^{-1}\Phi_1(s) = L^{-1}\frac{s}{s^2+a^2} = \cos at,\quad L^{-1}\Phi_2(s) = \cos at$$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\left(\frac{s}{s^2+a^2}\right)\right] = \int_0^t \cos au\cdot\cos a(t-u)du$$

$$= \frac{1}{2}\int_0^t[\cos at + \cos a(2u-t)]du = \frac{1}{2}\left[u\cos at + \frac{1}{2a}\sin a(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{2a}\sin at + \frac{1}{2a}\sin at\right] = \frac{1}{2a}\left[\sin at + at\cos at\right]$$

(ii) Let $\Phi_1(s) = \frac{s}{s^2-a^2}$, $\Phi_2(s) = \frac{s}{s^2-a^2}$

$$\therefore\ L^{-1}\Phi_1(s) = L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at,\qquad L^{-1}\Phi_2(s) = \cosh at$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t \cosh au\cdot\cosh a(t-u)du = \frac{1}{2}\int_0^t[\cosh at + \cosh a(2u-t)]du$$

$$\therefore\ L^{-1}[\Phi(s)] = \frac{1}{2}\left[u\cosh at + \frac{1}{2a}\sinh a(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cosh at + \frac{1}{2a}\sinh at + \frac{1}{2a}\sinh at\right]$$

$$= \frac{1}{2a}\left[\sinh at + at\cosh at\right]$$

(iii) Let $\Phi_1 = \frac{1}{s-a}$, $\Phi_2(s) = \frac{1}{(s+a)^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = L^{-1}\left[\frac{1}{s-a}\right] = e^{at}.$$

$$L^{-1}[\Phi_2(s)] = L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at}L^{-1}\left(\frac{1}{s^2}\right) = e^{-at}\cdot t$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^{au}\cdot e^{-a(t-u)}\cdot(t-u)du$$

$$= \int_0^t e^{2au-at}\cdot(t-u)du = e^{-at}\int_0^t e^{2au}(t-u)du$$

$$= e^{-at}\left[(t-u)\cdot\frac{e^{2au}}{2a} - \frac{e^{2au}}{4a^2}(-1)\right]_0^t = e^{-at}\left[(t-u)\cdot\frac{e^{2au}}{2a} + \frac{e^{2au}}{4a^2}\right]_0^t$$

$$= e^{-at}\left[0 + \frac{e^{2at}}{4a^2} - t\cdot\frac{e^0}{2a} - \frac{e^0}{4a^2}\right] = \frac{e^{-at}}{4a^2}\left[e^{2at} - 2at - 1\right]$$

$$= \frac{1}{4a^2}\left[e^{at} - 2at\cdot e^{-at} - e^{-at}\right]$$

(iv) $$L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos au,\quad L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin au$$

$$\therefore\ L^{-1}\left[\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right] = \int_0^t \cos au\cdot\frac{1}{a}\sin a(t-u)du$$

$$= \frac{1}{2a}\int_0^t[\sin at + \sin a(t-2u)]du$$

$$= \frac{1}{2a}\left[u\sin at + \frac{1}{2a}\cos a(t-2u)\right]_0^t = \frac{1}{2a}t\sin at.$$

(v) $$L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh au,\quad L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh au$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2-a^2)}\cdot\frac{1}{(s^2-a^2)}\right] = \int_0^t \cosh au\cdot\frac{1}{a}\sinh a(t-u)du$$

$$\therefore\ L^{-1}\left[\frac{s}{(s^2-a^2)}\cdot\frac{1}{(s^2-a^2)}\right] = \frac{1}{2a}\int_0^t[\sinh at + a\sinh a(t-2u)]du$$

$$= \frac{1}{2a}\left[u\sinh at - \frac{1}{2a}\cosh a(t-2u)\right]_0^t = \frac{1}{2a}\cdot t\cdot\sinh at.$$

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(vi) Let $\Phi_1(s) = \frac{1}{s-1}$, $\Phi_2(s) = \frac{1}{s^2+4}$

$\therefore\ L^{-1}[\Phi_1(s)] = e^t$, $L^{-1}[\Phi_2(s)] = \frac{1}{2}\sin 2t$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^u \cdot \frac{1}{2}\sin 2(t-u)\,du = -\frac{1}{2}\int_0^t e^u \sin 2(u-t)\,dt$$

$$= -\frac{1}{2}\left[\frac{1}{5}\cdot\left\{e^u(\sin 2(u-t) - 2\cos 2(u-t))\right\}\right]_0^t$$

$$= -\frac{1}{10}\left[e^t(0-2) - e^0(-\sin 2t - 2\cos 2t)\right]$$

$$= -\frac{1}{10}\left[-2e^t + \sin 2t + 2\cos 2t\right]$$

$$= \frac{1}{5}e^t - \frac{1}{10}\sin 2t - \frac{1}{5}\cos 2t$$

EXERCISE - VII

Find the inverse Laplace Transforms of the following functions by using convolution theorem.

1. $\frac{1}{(s-2)(s+2)^2}$ (M.U. 1994, 2006)
2. $\frac{1}{(s-3)(s+3)^2}$
3. $\frac{s^2}{(s^2+2^2)^2}$ (M.U. 1998, 99, 2004, 12)
4. $\frac{s^2}{(s^2+9)^2}$
5. $\frac{s}{(s^2+1)^2}$ (M.U. 2004)
6. $\frac{s}{(s^2+4)^2}$
7. $\frac{s}{s^4+8s^2+16}$ (M.U. 2003, 13, 15)

[**Ans.** : (1) $\frac{1}{16}[e^{2t} - 4te^{-2t} - e^{-2t}]$, (2) $\frac{1}{36}[e^{3t} - 6te^{-3t} - e^{-3t}]$, (3) $\frac{1}{4}(\sin 2t + 2t\cos 2t)$, (4) $\frac{1}{6}(\sin 3t + 3t\cos 3t)$, (5) $\frac{1}{2}t\sin t$, (6) $\frac{1}{4}t\sin 2t$, (7) $\frac{1}{4}t\sin 2t$.]

Example 3 : Find inverse Laplace transforms of the following by using convolution theorem.

(i) $\frac{1}{s^2(s+a)^2}$ (M.U. 2014, 19)

(ii) $\frac{s^2}{(s^2+a^2)(s^2+b^2)}$ (M.U. 2004, 07, 11, 13, 19)

(iii) $\frac{s}{(s^2+a^2)(s^2+b^2)}$ (M.U. 2003, 09, 10, 17)

(iv) $\frac{1}{(s-a)(s-b)}$

(v) $\frac{1}{(s^2+a^2)(s^2+b^2)}$

(vi) $\frac{1}{(s+a)}\cdot\frac{1}{(s+b)^2}$ (M.U. 2014)

Sol. : (i) Let $\Phi_1(s) = \frac{1}{(s+a)^2}$, $\Phi_2(s) = \frac{1}{s^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = L^{-1}\left[\frac{1}{(s+a)^2}\right] = e^{-at}\cdot L^{-1}\left(\frac{1}{s^2}\right) = e^{-at}\cdot t$$

$$L^{-1}[\Phi_2(s)] = L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^{-au}\cdot u\cdot(t-u)\,du = t\int_0^t e^{-au}u\,du - \int_0^t e^{-au}u^2\,du$$

$$= t\left[u\cdot\frac{e^{-au}}{-a} - \frac{e^{-au}}{a^2}\cdot 1\right]_0^t - \left[u^2\cdot\frac{e^{-au}}{-a} - \frac{e^{-au}}{a^2}\cdot 2u + \frac{e^{-au}}{-a^3}\cdot 2\right]_0^t$$

$$= t\left[\frac{t\cdot e^{-at}}{-a} - \frac{e^{-at}}{a^2} + \frac{1}{a^2}\right] - \left[t^2\cdot\frac{e^{-at}}{-a} - \frac{2te^{-at}}{a^2} - \frac{2e^{-at}}{a^3} + \frac{2}{a^3}\right]$$

$$= -t^2\cdot\frac{e^{-at}}{a} - \frac{te^{-at}}{a^2} + \frac{t}{a^2} + \frac{t^2e^{-at}}{a} + \frac{2te^{-at}}{a^2} + \frac{2e^{-at}}{a^3} - \frac{2}{a^3}$$

$$= \frac{te^{-at}}{a^2} + \frac{2e^{-at}}{a^3} + \frac{t}{a^2} - \frac{2}{a^3}$$

$$= \frac{1}{a^3}\left[e^{-at}\cdot at + 2e^{-at} + at - 2\right]$$

(ii) Let $\Phi_1(s) = \frac{s}{s^2+a^2}$, $\Phi_2(s) = \frac{s}{s^2+b^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at;\qquad L^{-1}[\Phi_2(s)] = L^{-1}\left(\frac{s}{s^2+b^2}\right) = \cos bt$$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}\right] = \int_0^t \cos au\cdot\cos b(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \cos[(a-b)u+bt] + \cos[(a+b)u-bt]\,du$$

$$= \frac{1}{2}\left[\frac{\sin[(a-b)u+bt]}{a-b} + \frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin at}{a-b} + \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} + \frac{\sin bt}{a+b}\right]$$

$$= \frac{1}{2}\left[\frac{(a+b+a-b)\sin at}{a^2-b^2} + \frac{(a-b-a-b)\sin bt}{a^2-b^2}\right]$$

$$= \frac{1}{2}\left[\frac{2a\sin at - 2b\sin bt}{a^2-b^2}\right] = \frac{a\sin at - b\sin bt}{a^2-b^2}. \qquad \text{..... (A)}$$

[For another method see Ex. 2 (iii), page 2-10.]

(iii) Let $\Phi_1(s) = \frac{s}{s^2+a^2}$, $\Phi_2(s) = \frac{1}{s^2+b^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = \cos at,\quad L^{-1}[\Phi_2(s)] = \frac{1}{b}\sin bt$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t \cos au\cdot\frac{1}{b}\sin b(t-u)\,du$$

$$= \frac{1}{2b}\int_0^t \{\sin[(a-b)u+bt] - \sin[(a+b)u-bt]\}\,du$$

$$= \frac{1}{2b}\left[-\frac{\cos[(a-b)u+bt]}{a-b} + \frac{\cos[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= \frac{1}{2b}\left[-\frac{\cos at}{a-b} + \frac{\cos at}{a+b} + \frac{\cos bt}{a-b} - \frac{\cos bt}{a+b}\right]$$

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$$\therefore\ L^{-1}[\Phi(s)] = \frac{1}{2b}\left[\frac{(a-b-a-b)\cos at}{a^2-b^2} + \frac{(a+b-a+b)\cos bt}{a^2-b^2}\right]$$

$$= \frac{1}{2b}\left[\frac{-2b\cos bt + 2b\cos bt}{a^2-b^2}\right] = \frac{\cos bt - \cos at}{a^2-b^2}.$$

[For another method see Ex. 2 (iv), page 2-10.]

(iv) Let $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$, $L^{-1}\left(\frac{1}{s-b}\right) = e^{bt}$

$$\therefore\ L^{-1}\left[\frac{1}{(s-a)}\cdot\frac{1}{(s-b)}\right] = \int_0^t e^{au}\cdot e^{b(t-u)}\,du = e^{bt}\int_0^t e^{(a-b)u}\,du$$

$$= e^{bt}\cdot\left[\frac{e^{(a-b)u}}{a-b}\right]_0^t = \frac{e^{bt}}{a-b}\left[e^{(a-b)t}-1\right] = \frac{e^{at}-e^{bt}}{a-b}.$$

[You can also use the method of partial fractions.

$$\because\ \frac{1}{(s-a)(s-b)} = \frac{1}{(a-b)}\left[\frac{1}{s-a} - \frac{1}{s-b}\right]\ ]$$

(v) Let $\Phi_1(s) = \frac{1}{s^2+a^2}$, $\Phi_2(s) = \frac{1}{s^2+b^2}$

$$\therefore\ L^{-1}[\Phi_1(s)] = L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at; \qquad L^{-1}[\Phi_2(s)] = L^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{1}{b}\sin bt$$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\frac{1}{s^2+a^2}\cdot\frac{1}{s^2+b^2}\right] = \int_0^t \frac{1}{a}\sin au\cdot\frac{1}{b}\sin b(t-u)\,du$$

$$= \frac{1}{ab}\int_0^t \sin au\cdot\sin b(t-u)\,du$$

$$= -\frac{1}{2ab}\int_0^t \{\cos[a-b]u + bt - \cos[(a+b)u - bt]\}\,du$$

$$= -\frac{1}{2ab}\left[\frac{\sin[(a-b)u+bt]}{a-b} - \frac{\sin[(a+b)u-bt]}{a+b}\right]_0^t$$

$$= -\frac{1}{2ab}\left[\frac{\sin at}{a-b} - \frac{\sin at}{a+b} - \frac{\sin bt}{a-b} - \frac{\sin bt}{a+b}\right]$$

$$= -\frac{1}{2ab}\left[2b\cdot\frac{\sin at}{a^2-b^2} - \frac{2a\sin bt}{a^2-b^2}\right] = \frac{a\sin bt - b\sin at}{ab(a^2-b^2)}.$$

[You can also use the method of partial fractions.

$$\because\ \frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{(a^2-b^2)}\left[\frac{1}{s^2+b^2} - \frac{1}{s^2+a^2}\right]\ ]$$

(vi) Let $\Phi_1(s) = \frac{1}{(s+b)^2}$ and $\Phi_2(s) = \frac{1}{s+a}$

$$\therefore\ L^{-1}[\Phi_1(s)] = L^{-1}\frac{1}{(s+b)^2} = e^{-bt}\cdot L^{-1}\left(\frac{1}{s^2}\right) = e^{-bt}\,t$$

$$L^{-1}[\Phi_2(s)] = L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^{-bu}\cdot u\cdot e^{-a(t-u)}\,du = e^{-at}\int_0^t u\cdot e^{(a-b)u}\,du$$

$$= e^{-at}\left[u\cdot\frac{e^{(a-b)u}}{a-b} - \int\frac{e^{(a-b)u}}{(a-b)}\cdot 1\cdot du\right]_0^t$$

$$= e^{-at}\left[\frac{u\cdot e^{(a-b)u}}{a-b} - \frac{e^{(a-b)u}}{(a-b)^2}\right]_0^t$$

$$= e^{-at}\left[\left(\frac{t\cdot e^{(a-b)t}}{(a-b)} - \frac{e^{(a-b)t}}{(a-b)^2}\right) - \left(0 - \frac{1}{(a-b)^2}\right)\right]$$

$$= e^{-at}\left[\frac{1}{(a-b)^2} + \frac{t\cdot e^{(a-b)t}}{(a-b)} - \frac{e^{(a-b)t}}{(a-b)^2}\right]$$

$$= \frac{1}{(a-b)^2}\left[e^{-at} + (a-b)e^{-bt}t - e^{-bt}\right]$$

EXERCISE - VIII

Find the inverse Laplace Transforms of the following functions by using convolution theorem.

1. $\frac{1}{s^2(s+1)^2}$ **(M.U. 1997, 99, 2014)**
2. $\frac{s^2}{(s^2+1)(s^2+4)}$ **(M.U. 2004, 06)**
3. $\frac{s}{(s^2+4)(s^2+1)}$ **(M.U. 1995, 2002, 03)**
4. $\frac{1}{(s-3)(s-1)}$
5. $\frac{1}{(s+3)(s-1)}$
6. $\frac{s^2}{(s^2+9)(s^2+4)}$ **(M.U. 2019)**
7. $\frac{1}{(s^2+9)(s^2+4)}$
8. $\frac{1}{(s^2+1)(s^2+9)}$
9. $\frac{1}{(s^2+1)(s^2+4)}$
10. $\frac{1}{s^2-s-6}$ **(M.U. 2004)**
11. $\frac{16}{(s-2)(s+2)}$ **(M.U. 2009)**
12. $\frac{1}{(s^2+1)^2}$ **(M.U. 2011)**
13. $\frac{1}{(s-3)(s+4)^2}$ **(M.U. 2014)**
14. $\frac{1}{s-2}\cdot\frac{1}{(s+2)^2}$ **(M.U. 2003)**

[**Ans. :** **(1)** $te^{-t} + 2e^{-t} + t - 2$, **(2)** $\frac{1}{3}(2\sin 2t - \sin t)$, **(3)** $\frac{1}{3}(\cos t - \cos 2t)$,
(4) $\frac{e^{3t}-e^{t}}{2}$, **(5)** $\frac{e^{t}-e^{-3t}}{4}$, **(6)** $\frac{1}{5}(3\sin 3t - 2\sin 2t)$,
(7) $\frac{3\sin 2t - 2\sin 3t}{30}$, **(8)** $\frac{3\sin t - \sin 3t}{24}$, **(9)** $\frac{2\sin t - \sin 2t}{6}$,
(10) $\frac{e^{3t}-e^{-2t}}{5}$, **(11)** $4(e^{2t}-e^{-2t})$, **(12)** $\frac{1}{2}(\sin t - t\cos t)$.]
(13) $\frac{1}{49}[e^{-3t} - 7te^{-4t} - e^{-4t}]$ **(14)** $\frac{1}{16}[e^{2t} - 4te^{-2t} - e^{-2t}]$

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Example 4 : Find the inverse of the following by using convolution theorem.

(i) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ **(M.U. 2004, 07)** (ii) $\dfrac{1}{(s+3)(s^2+2s+2)}$ **(M.U. 2004)** (iii) $\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$ **(M.U. 1996, 2004, 07, 09)**

Sol. : (i) $L^{-1}\left\{\dfrac{(s+2)^2}{(s^2+4s+8)^2}\right\} = L^{-1}\left\{\dfrac{(s+2)^2}{(s+2)^2+2^2}\right\} = e^{-2t}L^{-1}\dfrac{s^2}{(s^2+2^2)^2}$

$$\therefore\ L^{-1}\left[\frac{s}{s^2+2^2}\right] = \cos 2t$$

$$\therefore\ L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right] = \int_0^t \cos 2u \cdot \cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos(4u-2t)]\,du = \frac{1}{2}\left[u\cos 2t + \frac{1}{4}\sin(4u-2t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right] = \frac{1}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right]$$

$$\therefore\ L^{-1}\left[\frac{(s+2)^2}{(s^2+4s+8)^2}\right] = \frac{e^{-2t}}{2}\left[t\cos 2t + \frac{1}{2}\sin 2t\right] = \frac{e^{-2t}}{4}[2t\cos 2t + \sin 2t]$$

(ii) $L^{-1}\left[\dfrac{1}{(s+3)(s^2+2s+2)}\right] = L^{-1}\left[\dfrac{1}{[(s+1)+2][(s+1)^2+1]}\right]$

$$= e^{-t}L^{-1}\left[\frac{1}{(s+2)(s^2+1)}\right]$$

Let $\Phi_1(s) = \dfrac{1}{s^2+1}$ and $\Phi_2(s) = \dfrac{1}{s+2}$

$\therefore\ L^{-1}[\Phi_1(s)] = \sin at$ and $L^{-1}[\Phi_2(s)] = e^{-2t}$

$$\therefore\ L^{-1}[\Phi_1(s)\cdot\Phi_2(s)] = \int_0^t \sin u \cdot e^{-2(t-u)}\,du$$

$$= \int_0^t \sin u \cdot e^{-2t}\cdot e^{2u}\,du = e^{-2t}\int_0^t e^{2u}\cdot \sin u\,du$$

$$= e^{-2t}\cdot\frac{1}{4+1}\left[e^{2u}(2\sin u - \cos u)\right]_0^t$$

$$= \frac{1}{5}e^{-2t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$\therefore\ L^{-1}\left[\frac{1}{(s+3)(s^2+2s+2)}\right] = e^{-t}\cdot L^{-1}[\Phi_1(s)\Phi_2(s)]$$

$$= e^{-t}\cdot\frac{1}{5}\cdot e^{-2t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\cdot e^{-3t}\left[e^{2t}(2\sin t - \cos t) + 1\right]$$

$$= \frac{1}{5}\left[e^{-t}(2\sin t - \cos t) + e^{-3t}\right]$$

(iii) $L^{-1}\dfrac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = L^{-1}\dfrac{(s+1)^2+2}{[(s+1)^2+1][(s+1)^2+4]}$

$$= e^{-t}L^{-1}\frac{s^2+2}{(s^2+1)(s^2+4)}$$

$$= e^{-t}\cdot L^{-1}\frac{s^2}{(s^2+1)(s^2+4)} + e^{-t}L^{-1}\left[\frac{2}{(s^2+1)(s^2+4)}\right]$$

$$= e^{-t}\cdot L^{-1}\frac{s^2}{(s^2+1)(s^2+4)} + \frac{2e^{-t}}{3}\cdot L^{-1}\left[\frac{1}{(s^2+1)} - \frac{1}{(s^2+4)}\right]$$

By the result obtained above in (A), page 2-20, putting $a = 2$, $b = 1$, we get

$$= \frac{e^{-t}}{3}[2\sin 2t - \sin t] + \frac{2}{3}e^{-t}\left[\sin t - \frac{1}{2}\sin 2t\right]$$

$$= \frac{e^{-t}}{3}[\sin 2t + \sin t]$$

[See that we have solved this example by another method in Ex. 2 (viii), page 2-12.]

Example 5 : Using convolution theorem find the inverse Laplace transform of the following.

(i) $\dfrac{1}{(s-2)^4(s+3)}$ **(M.U. 1993, 98, 99, 2004, 07, 16, 19)** (ii) $\dfrac{1}{(s^2+4s+13)^2}$ **(M.U. 2002, 08, 12, 13, 19)**

(iii) $\dfrac{s+3}{(s^2+6s+10)^2}$ (iv) $\dfrac{s+2}{(s^2+4s+5)^2}$ **(M.U. 2013)** (v) $\dfrac{s^2+5}{(s^2+4s+13)^2}$ **(M.U. 2014)**

Sol. : (i) Let $\Phi_1(s) = \dfrac{1}{s+3}$, $\Phi_2(s) = \dfrac{1}{(s-2)^4}$

$$\therefore\ L^{-1}\Phi_1(s) = e^{-3t},\quad L^{-1}\Phi_2(s) = e^{2t}L^{-1}\frac{1}{s^4} = e^{2t}\cdot\frac{t^3}{6}$$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t e^{-3u}\cdot e^{2(t-u)}\cdot\frac{(t-u)^3}{6}\,du$$

$$= \int_0^t e^{(2t-5u)}\cdot\frac{(t-u)^3}{6}\,du = e^{2t}\int_0^t e^{-5u}\,\frac{(t-u)^3}{6}\,du$$

$$= e^{2t}\left[\frac{(t-u)^3}{6}\left(\frac{-e^{-5u}}{5}\right) - \frac{(t-u)^2}{2}(-1)\left(\frac{e^{-5u}}{25}\right) + (t-u)(+1)\left(\frac{-e^{-5u}}{125}\right) - (-1)\left(\frac{e^{-5u}}{625}\right)\right]_0^t$$

$$= e^{2t}\left[\frac{e^{-5t}}{625} - \left\{-\frac{t^3}{30} + \frac{t^2}{50} - \frac{t}{125} + \frac{1}{625}\right\}\right]$$

$$= \frac{e^{-3t}}{625} - e^{2t}\left[\frac{1}{625} - \frac{t}{125} + \frac{t^2}{50} - \frac{t^3}{30}\right]$$

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(ii) $L^{-1}\left[\dfrac{1}{(s^2+4s+13)^2}\right] = L^{-1}\left[\dfrac{1}{(s+2)^2+3^2}\right]^2 = e^{-2t}\cdot L^{-1}\left[\dfrac{1}{(s^2+3^2)}\right]^2$

We find $L^{-1}\left[\dfrac{1}{(s^2+3^2)^2}\right]$ by convolution theorem.

Let $\Phi_1(s) = \dfrac{1}{s^2+3^2}$ and $\Phi_2(s) = \dfrac{1}{s^2+3^2}$ and $\Phi(s) = \dfrac{1}{(s^2+3^2)^2}$

$\therefore\ L^{-1}[\Phi_1(s)] = \dfrac{1}{3}\sin 3t$ and $L^{-1}[\Phi_2(s)] = \dfrac{1}{3}\sin 3t$

$$\therefore\ L^{-1}[\Phi(s)] = \frac{1}{9}\int_0^t \sin 3u\cdot\sin 3(t-u)\,du$$

$$= -\frac{1}{18}\int_0^t [\cos 3t - \cos(6u-3t)]\,du$$

$$= -\frac{1}{18}\left[u\cos 3t - \frac{\sin(6u-3t)}{6}\right]_0^t$$

$$= -\frac{1}{18}\left[t\cos 3t - \frac{\sin 3t}{6} - \frac{\sin 3t}{6}\right] = \frac{1}{18}\left[\frac{\sin 3t}{3} - t\cos 3t\right]$$

$$\therefore\ L^{-1}\left[\frac{1}{(s^2+4s+13)^2}\right] = \frac{e^{-2t}}{18}\left[\frac{\sin 3t}{3} - t\cos t3t\right]$$

(iii) $L^{-1}\dfrac{s+3}{(s^2+6s+10)^2} = L^{-1}\dfrac{s+3}{[(s+3)^2+1^2]^2} = e^{-3t}\cdot L^{-1}\dfrac{s}{(s^2+1^2)^2}$ [By first shifting theorem]

Now, putting $a = 1$, in Ex. 2 (iv), page 2-17 or independently, we get

$$L^{-1}\left[\frac{s+3}{(s^2+6s+10)^2}\right] = e^{-3t}\cdot\frac{1}{2}\cdot t\sin t$$

(iv) $L^{-1}\left[\dfrac{s+2}{(s^2+4s+5)^2}\right] = L^{-1}\left[\dfrac{s+2}{(s^2+4s+4+1)^2}\right]$

$$= L^{-1}\left\{\frac{s+2}{[(s+2)^2+1]^2}\right\} = e^{-2t}\cdot L^{-1}\frac{s}{(s^2+1)^2}$$

Now, putting $a = 1$ in Ex. 2 (iv), page 2-17 or independently, we get

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right] = e^{-2t}\cdot\frac{1}{2}t\sin t.$$

(v) $L^{-1}\left[\dfrac{s^2+5}{(s^2+4s+13)^2}\right] = L^{-1}\left[\dfrac{(s^2+4s+13)-(4s+8)}{(s^2+4s+13)^2}\right]$

$$= L^{-1}\left[\frac{1}{s^2+4s+13}\right] - 4L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$$

$$= L^{-1}\left[\frac{1}{(s+2)^2+3^2}\right] - 4L^{-1}\left[\frac{s+2}{[(s+2)^2+3^2]^2}\right]$$

$$L^{-1}\left[\frac{s^2+5}{(s^2+4s+13)^2}\right] = e^{-2t}\cdot L^{-1}\left[\frac{1}{s^2+3^2}\right] - 4e^{-2t}\cdot L^{-1}\left[\frac{s}{(s^2+3^2)^2}\right]$$

Now, putting $a = 3$ in Ex. 2 (iv), page 2-17

$$= e^{-2t}\cdot\frac{1}{3}\sin 3t - 4e^{-2t}t\cdot\frac{1}{2\cdot 3}\sin 3t$$

$$= \frac{e^{-2t}}{3}(1-2t)\sin 3t.$$

Example 6 : Find the inverse Laplace transform of the following by convolution theorem.

(i) $\dfrac{1}{s\sqrt{s+4}}$ **(M.U. 2002, 05)** (ii) $\dfrac{s^2+s}{(s^2+1)(s^2+2s+2)}$ **(M.U. 2002, 19)** (iii) $\dfrac{(s+2)^2}{(s^2+4s+8)^2}$ **(M.U. 2007, 09, 16)**

(iv) $\dfrac{(s+3)^2}{(s^2+6s+5)^2}$ **(M.U. 2003, 12)**

Sol. : (i) $L^{-1}\dfrac{1}{(s+4)^{1/2}} = e^{-4u}\cdot L^{-1}\dfrac{1}{s^{1/2}} = e^{-4u}\cdot\dfrac{u^{-1/2}}{|1/2}$. And $L^{-1}\dfrac{1}{s} = 1$

$$\therefore\ L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right] = \int_0^t \frac{e^{-4u}\cdot u^{-1/2}}{\sqrt{\pi}}\cdot 1\cdot du$$

Put $4u = x^2$, $du = \dfrac{x\,dx}{2}$, $\sqrt{u} = x/2$

$$\therefore\ L^{-1}\left[\frac{1}{s}\cdot\frac{1}{\sqrt{s+4}}\right] = \int_0^{2\sqrt{t}}\frac{e^{-x^2}}{\sqrt{\pi}}dx = \frac{1}{2}erf(2\sqrt{t}) \qquad \left[\because erf(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx\right]$$

[For another method see Ex. 1 (v), page 2-35.]

(ii) Let $\Phi_1(s) = \dfrac{s+1}{s^2+2s+2}$ and $\Phi_2(s) = \dfrac{s}{s^2+1}$

$$L^{-1}\frac{s+1}{(s+1)^2+1} = e^{-u}L^{-1}\frac{s}{s^2+1} = e^{-u}\cos u, \qquad L^{-1}\frac{s}{s^2+1} = \cos u$$

$$\therefore\ L^{-1}\left[\frac{s+1}{(s+1)^2+1}\cdot\frac{s}{s^2+1}\right] = \int_0^t e^{-u}\cos u\cdot\cos(t-u)\,du$$

$$= \frac{1}{2}\int_0^t e^{-u}[\cos(2u-t)+\cos t]\,du$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-u}[-\cos(2u-t)+2\sin(2u-t)] - e^{-u}\cos t\right]_0^t$$

$$= \frac{1}{2}\left[\frac{1}{5}e^{-t}\{-\cos t+2\sin t\} - e^{-t}\cos t - \frac{1}{5}\{-\cos t-2\sin t\} + \cos t\right]$$

$$= \frac{1}{10}\left[e^{-t}(2\sin t - 6\cos t) + (2\sin t + 6\cos t)\right]$$

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(iii) $L^{-1}\dfrac{(s+2)^2}{(s^2+4s+8)^2} = L^{-1}\dfrac{(s+2)^2}{[(s+2)^2+2^2]^2} = e^{-2t}\,L^{-1}\dfrac{s^2}{(s^2+2^2)^2}$

Let $\Phi_1(s) = \dfrac{s}{s^2+2^2}$, $\Phi_2(s) = \dfrac{s}{s^2+2^2}$

$\therefore\ L^{-1}\Phi_1(s) = L^{-1}\dfrac{s}{s^2+2^2} = \cos 2t,\quad L^{-1}\Phi_2(s) = \cos 2t$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\left(\frac{s}{s^2+2^2}\right)\left(\frac{s}{s^2+2^2}\right)\right] = \int_0^t \cos 2u \cdot \cos 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos 2t + \cos 2(2u-t)]\,du$$

$$= \frac{1}{2}\left[u\cos 2t + \frac{1}{4}\sin 2(2u-t)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos 2t + \frac{1}{4}\sin 2t + \frac{1}{4}\sin 2t\right]$$

$$= \frac{1}{4}[\sin 2t + 2t\cos 2t]$$

$$\therefore\ L^{-1}\frac{(s+2)^2}{(s^2+4s+8)^2} = \frac{1}{4}e^{-2t}[\sin 2t + 2t\cos 2t]$$

(iv) $L^{-1}\dfrac{(s+3)^2}{(s^2+6s+5)^2} = L^{-1}\dfrac{(s+3)^2}{[(s+3)^2-2^2]^2} = e^{-3t}\cdot L^{-1}\dfrac{s^2}{(s^2-2^2)^2}$ [By first shifting theorem]

We find $L^{-1}\dfrac{s^2}{(s^2-2^2)^2}$ by convolution theorem.

Let $\Phi_1(s) = \dfrac{s}{s^2-2^2}$, $\Phi_2(s) = \dfrac{s}{s^2-2^2}$.

$\therefore\ L^{-1}\Phi_1(s) = \cosh 2u,\quad L^{-1}\Phi_2(s) = \cosh 2u$

$$\therefore\ L^{-1}[\Phi(s)] = \int_0^t \cosh 2u \cdot \cosh 2(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cosh 2t + \cosh 2(2u-t)]\,du$$

$$= \frac{1}{2}\left[u\cosh 2t + \frac{\sinh 2(2u-t)}{4}\right]_0^t$$

$$= \frac{1}{2}\left[t\cosh 2t + \frac{1}{4}\sinh 2t + \frac{1}{4}\sinh 2t\right]$$

$$= \frac{1}{4}[2t\cosh 2t + \sinh 2t]$$

$$\therefore\ L^{-1}\frac{(s+3)^2}{(s^2+6s+5)^2} = \frac{1}{4}e^{-3t}[2t\cosh 2t + \sinh 2t]$$

Example 7 : Using convolution theorem, find the inverse Laplace transform of

$$\frac{s}{s^4+8s^2+16}.$$ **(M.U. 2003, 15)**

Sol. : We have $\Phi = \dfrac{s}{s^4+8s^2+16} = \dfrac{s}{(s^2+4)^2} = \dfrac{s}{s^2+4}\cdot\dfrac{1}{s^2+4}$

Let $\Phi_1(s) = \dfrac{s}{s^2+4}$ and $\Phi_2(s) = \dfrac{1}{s^2+4}$.

$\therefore\ L^{-1}\Phi_1(s) = \cos 24,\quad L^{-1}\Phi_2(s) = \dfrac{1}{2}\sin 2u$

$$\therefore\ \Phi(s) = \int_0^t \cos 2u \cdot \frac{1}{2}\sin 2(t-u)\,du = \frac{1}{4}\int_0^t [\sin 2t - \sin(4u-2t)]\,du$$

$$= \frac{1}{4}\left[\sin 2t \cdot u + \frac{\cos(4u-2t)}{4}\right]_0^t$$

$$= \frac{1}{4}\left[t\cdot\sin 2t + \frac{\cos 2t}{4} - \frac{\cos 2t}{4}\right] = \frac{1}{4}t\sin 2t.$$

(Verify the answer from Ex. 1 (iii), page 1-29.)

EXERCISE - IX

Using convolution theorem find the inverse Laplace transform of,

1. $\dfrac{(s+3)^2}{(s^2+6s+18)^2}$
2. $\dfrac{(s+5)^2}{(s^2+10s+16)^2}$ **(M.U. 2006, 19)**
3. $\dfrac{1}{(s-3)(s+4)^2}$
4. $\dfrac{1}{(s-2)}\cdot\dfrac{1}{(s+2)^2}$ **(M.U. 2003)**
5. $\dfrac{s+3}{(s^2+6s+13)^2}$ **(M.U. 2004, 06)**
6. $\dfrac{(s-1)^2}{(s^2-2s+5)^2}$ **(M.U. 2006, 14)**
7. $\dfrac{s+1}{(s^2+2s+2)^2}$ **(M.U. 2003)**

[**Ans. :** (1) $\dfrac{1}{6}e^{-3t}\cdot[\sin 3t + 3t\cos 3t]$, (2) $\dfrac{1}{6}e^{-5t}\cdot[\sinh 3t + 3t\cosh 3t]$,

(3) $\dfrac{1}{49}\left[e^{3t} - e^{-4t} - 7te^{-4t}\right]$, (4) $\dfrac{1}{16}(e^{2t} - 4e^{-2t} - e^{-2t})$,

(5) $\dfrac{1}{4}e^{-3t}\cdot t\sin 2t$, (6) $\dfrac{e^t}{4}[\sin 2t + 2t\cos 2t]$,

(7) $e^{-t}\cdot\dfrac{1}{2}\cdot t\sin t$.]

(e) Use of differentiation of $\Phi(s)$

We know that [(16) § 10, page 1-28] if $L[f(t)] = \Phi(s)$, then $-\Phi'(s) = L[t f(t)]$.

i.e., if $L\left[\dfrac{-1}{t}f(t)\right] = \Phi(s)$, then $\Phi'(s) = L[f(t)]$.

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Hence, if $L^{-1}[\Phi'(s)] = f(t)$, then $L^{-1}[\Phi(s)] = -\frac{1}{t} f(t)$.

i.e., $$\boxed{L^{-1}\Phi(s) = -\frac{1}{t} L^{-1}[\Phi'(s)] \quad i.e. \quad L^{-1}[\Phi'(s)] = -t\, L^{-1}\Phi(s)} \qquad \text{..... (17)}$$

Remarks ....

1. The result can be extended further.
$$L^{-1}\Phi(s) = \frac{1}{t^2} L^{-1}[\Phi''(s)] \quad \text{or} \quad L^{-1}[\Phi''(s)] = t^2 \cdot L^{-1}[\Phi(s)]$$
2. These results can be profitably used to find $L^{-1}[F(s)]$ if we know $L^{-1}[\Phi'(s)]$ *i.e.*, if $\Phi'(s)$ comes out to be a standard result. [See Ex. 1].

Note ....

The above result is particularly useful when
$\Phi(s) = \log f(s)$ or $\Phi(s) = \tan^{-1} f(s)$ or $\Phi(s) = \tan h^{-1} f(s)$. See Ex. 1 and 2 below.

Example 1 : Find inverse Laplace transform of,

(i) $\log\left(\frac{s+a}{s+b}\right)$ **(M.U. 1993, 96, 13, 19)**

(ii) $\log\left(1+\frac{a^2}{s^2}\right)$ **(M.U. 1999, 2002, 05)**

(iii) $\frac{1}{s}\log\left(1+\frac{1}{s^2}\right)$ **(M.U. 1996, 2016)**

(iv) $\log\left[\frac{s^2+a^2}{\sqrt{s+b}}\right]$ **(M.U. 2003, 04, 06, 14)**

(v) $\log\left(\frac{s^2+a^2}{s^2+b^2}\right)$ **(M.U. 2007, 08, 13, 19)**

(vi) $2\tan h^{-1} s$ **(M.U. 2003, 05, 14)**

(vii) $s\log\left(\frac{s+1}{s-1}\right)$ **(M.U. 2016)**

(viii) $\log\left(\frac{s^2+1}{s(s+1)}\right)$ **(M.U. 2008)**

Sol. : We have $L^{-1}[\Phi(s)] = -\frac{1}{t} L^{-1}[\Phi'(s)]$

(i) $$\therefore\ L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left\{\log\left(\frac{s+a}{s+b}\right)\right\}\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\{\log(s+a) - \log(s+b)\}\right]$$
$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right] = -\frac{1}{t}(e^{-at} - e^{-bt})$$ [See Ex. 1 (iii), page 1-39]

(ii) $$L^{-1}\left[\log\left(1+\frac{a^2}{s^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(1+\frac{a^2}{s^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{a^2+s^2}{s^2}\right)\right]$$
$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\{\log(a^2+s^2) - \log s^2\}\right] = -\frac{1}{t} L^{-1}\left[\frac{2s}{a^2+s^2} - \frac{2}{s}\right]$$
$$= -\frac{1}{t}[2\cos at - 2\cdot 1] = \frac{2}{t}[1 - \cos at]$$ [See Ex. 1 (i), page 1-39]

(iii) $$L^{-1}\log\left(1+\frac{1}{s^2}\right) = -\frac{1}{u} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+1}{s^2}\right)\right]$$ [By (17) above]

$$= -\frac{1}{u} L^{-1}\left[\frac{d}{ds}\{\log(s^2+1) - \log s^2\}\right]$$
$$\therefore\ L^{-1}\log\left(1+\frac{1}{s^2}\right) = -\frac{1}{u} L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right] = -\frac{2}{u}(\cos u - 1) \quad \text{And } L^{-1}\frac{1}{s} = 1$$
$$\therefore\ L^{-1}\left[\frac{1}{s}\cdot\log\left(1+\frac{1}{s^2}\right)\right] = \int_0^t -\frac{2}{u}(\cos u - 1)\cdot 1\cdot du$$ [By Cor., page 2-15]

(**Note** that the result is expressed in terms of the integral only.)

(iv) $$L^{-1}\left[\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{s^2+a^2}{\sqrt{s+b}}\right)\right]$$
$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left\{\log(s^2+a^2) - \frac{1}{2}\log(s+b)\right\}\right]$$
$$= -\frac{1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{1}{2}\cdot\frac{1}{s+b}\right]$$
$$= -\frac{1}{t}\left[2\cos at - \frac{1}{2}e^{-bt}\right] = \frac{1}{t}\left[\frac{1}{2}e^{bt} - 2\cos at\right]$$

(v) $$L^{-1}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\left\{\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right\}\right]$$
$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\{\log(s^2+a^2) - \log(s^2+b^2)\}\right]$$
$$= -\frac{1}{t} L^{-1}\left[\frac{2s}{s^2+a^2} - \frac{2s}{s^2+b^2}\right]$$
$$= -\frac{1}{t}[2\cos at - 2\cos bt] = \frac{2}{t}(\cos bt - \cos at)$$

(See Ex. 8 of Exercise - XI, page 1-42.)

(vi) $$L^{-1}(2\tan h^{-1}s) = L^{-1}\left[2\cdot\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right] = L^{-1}\left[\log\left(\frac{1+s}{1-s}\right)\right]$$
$$= -\frac{1}{t} L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right)\right] = -\frac{1}{t} L^{-1}\left[\frac{1}{1+s} + \frac{1}{1-s}\right]$$
$$= -\frac{1}{t} L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = -\frac{1}{t}\left[e^{-t} - e^{t}\right]$$
$$= \frac{2}{t}\left(\frac{e^{t} - e^{-t}}{2}\right) = \frac{2}{t}\sin ht$$

(vii) $$L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right] = -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}s\log\left(\frac{s+1}{s-1}\right)\right]$$
$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\{s\log(s+1) - s\log(s-1)\}\right]$$
$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{s}{s+1} + \log(s+1) - \frac{s}{s-1} - \log(s-1)\right]$$

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$$\therefore\ L^{-1}\left[s\log\left(\frac{s+1}{s-1}\right)\right]=-\frac{1}{t}\cdot L^{-1}\left[\frac{s}{s+1}-\frac{s}{s-1}\right]-\frac{1}{t}\cdot L^{-1}[\log(s+1)-\log(s-1)]$$

$$=-\frac{1}{t}\cdot L^{-1}\left(-\frac{2s}{s^2-1}\right)-\frac{1}{t}\cdot\left(-\frac{1}{t}\right)\cdot L^{-1}\left\{\frac{d}{ds}[\log(s+1)-\log(s-1)]\right\}$$

$$=\frac{2}{t}\cdot L^{-1}\left(\frac{s}{s^2-1}\right)+\frac{1}{t^2}\cdot L^{-1}\left[\frac{1}{s+1}-\frac{1}{s-1}\right]$$

Note ....

$L^{-1}\left(\frac{s}{s+1}\right)$ and $L^{-1}\left(\frac{s}{s-1}\right)$ do not exist. Hence, we have combined these two terms as above.

$$=\frac{2}{t}\cosh t+\frac{1}{t^2}(e^{-t}-e^{t})=\frac{2}{t}\cosh t-\frac{2}{t^2}\left(\frac{e^{t}-e^{-t}}{2}\right)$$

$$=\frac{2}{t}\cosh t-\frac{2}{t^2}\sinh t$$

(viii) $$L^{-1}\log\left[\frac{s^2+1}{s(s+1)}\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left\{\log\frac{s^2+1}{s(s+1)}\right\}\right]$$

$$=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\{\log(s^2+1)-\log s-\log(s+1)\}\right]$$

$$=-\frac{1}{t}L^{-1}\left[\frac{2s}{s^2+1}-\frac{1}{s}-\frac{1}{s+1}\right]=-\frac{1}{t}\left[2L^{-1}\frac{s}{s^2+1}-L^{-1}\frac{1}{s}-L^{-1}\frac{1}{s+1}\right]$$

$$=-\frac{1}{t}[2\cos t-1-e^{-t}]$$

Example 2 : Find inverse Laplace transform of

(i) $\tan^{-1}\left(\frac{2}{s^2}\right)$ **(M.U. 1995, 2004, 05, 06, 07)** (ii) $\cot^{-1} as$

(iii) $\tan^{-1}\left(\frac{a}{s}\right)$ **(M.U. 2003, 04, 2007, 12, 13, 19)**

(iv) $\cot^{-1}(s+1)$ **(M.U. 2006)** (v) $\tan^{-1}\left(\frac{s+a}{b}\right)$ **(M.U. 2004, 05, 15, 19)**

Sol. : We have $L^{-1}[\Phi(s)]=-\frac{1}{t}L^{-1}[\Phi'(s)]$

(i) $$\therefore\ L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{d}{ds}\left(\tan^{-1}\frac{2}{s^2}\right)\right]=-\frac{1}{t}L^{-1}\left[\frac{1}{1+(4/s^4)}\left(-\frac{4}{s^3}\right)\right]$$

$$=-\frac{1}{t}L^{-1}\left[-\frac{4s}{s^4+4}\right]=\frac{4}{t}L^{-1}\left[\frac{s}{s^2+4}\right]=\frac{4}{t}\cdot L^{-1}\left[\frac{s}{(s^2+2)^2-(2s)^2}\right]$$

$$=\frac{4}{t}\cdot\frac{1}{4}L^{-1}\left[\frac{1}{s^2-2s+2}-\frac{1}{s^2+2s+2}\right]$$

$$=\frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s-1)^2+1}-\frac{1}{(s+1)^2+1}\right]$$

$$\therefore\ L^{-1}\left[\tan^{-1}\left(\frac{2}{s^2}\right)\right]=\frac{1}{t}\cdot\left[e^{t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right)-e^{-t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right)\right]=\frac{1}{t}[e^{t}\sin t-e^{-t}\sin t]$$

$$=\frac{2\sin t}{t}\left(\frac{e^{t}-e^{-t}}{2}\right)=\frac{2\sin t\sinh t}{t}$$

(ii) $$L^{-1}[\cot^{-1}(as)]=-\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\cot^{-1}(as)\right]=-\frac{1}{t}\cdot L^{-1}\left[\frac{-a}{1+a^2s^2}\right]=\frac{a}{t}\cdot L^{-1}\left(\frac{1}{1+a^2s^2}\right)$$

$$=\frac{a}{t}\cdot L^{-1}\left[\frac{1}{a^2}\cdot\frac{1}{s^2+(1/a)^2}\right]=\frac{1}{t}\cdot L^{-1}\left[\frac{1/a}{s^2+(1/a)^2}\right]=\frac{1}{t}\sin\frac{t}{a}.$$

(iii) $$L^{-1}\left[\tan^{-1}\left(\frac{a}{s}\right)\right]=-\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{a}{s}\right)\right]=-\frac{1}{t}\cdot L^{-1}\left[\frac{1}{1+(a/s)^2}\left(-\frac{a}{s^2}\right)\right]$$

$$=\frac{a}{t}\cdot L^{-1}\left(\frac{1}{s^2+a^2}\right)=\frac{a}{t}\cdot\frac{1}{a}\sin at=\frac{1}{t}\sin at$$

(iv) $$L^{-1}[\cot^{-1}(s+1)]=-\frac{1}{t}\frac{d}{ds}[\cot^{-1}(s+1)]=-\frac{1}{t}\cdot L^{-1}\left[\frac{-1}{1+(s+1)^2}\right]$$

$$=\frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s+1)^2+1}\right]=\frac{1}{t}\cdot e^{-t}\cdot L^{-1}\left(\frac{1}{s^2+1}\right)=\frac{1}{t}e^{-t}\sin t.$$

(v) $$L^{-1}\left[\tan^{-1}\left(\frac{s+a}{b}\right)\right]=-\frac{1}{t}\cdot L^{-1}\left[\frac{d}{ds}\tan^{-1}\left(\frac{s+a}{b}\right)\right]$$

$$=-\frac{1}{t}\cdot L^{-1}\left[\frac{1}{1+[(s+a)/b]^2}\cdot\frac{1}{b}\right]=-\frac{1}{t}\cdot L^{-1}\left[\frac{b}{(s+a)^2+b^2}\right]$$

$$=-\frac{1}{t}\cdot e^{-at}\cdot L^{-1}\left(\frac{b}{s^2+b^2}\right)=-\frac{1}{t}\cdot e^{-at}\sin bt.$$

EXERCISE - X

Find the inverse Laplace transform of,

1. $\log\left[\frac{s^2-4}{(s-3)^2}\right]$ **(M.U. 2003)**
2. $\log\left[1+\frac{4}{s^2}\right]$
3. $\frac{1}{2}\log\left[\frac{s-1}{s+1}\right]$
4. $\log\left[1+\frac{1}{s^2}\right]$ **(M.U. 2004)**
5. $\frac{1}{2}\log\left(\frac{s^2+1}{s^2+4}\right)$
6. $\frac{1}{2}\log\left(1-\frac{a^2}{s^2}\right)$ **(M.U. 1999)**
7. $\tan^{-1}(s+1)$ **(M.U. 2003)**
*8. $\cot^{-1}\left(\frac{2}{s^2}\right)$
9. $\log\sqrt{\frac{s^2+1}{s^2}}$ **(M.U. 1998, 2003)**
10. $\log\left[\frac{s^2+a^2}{(s+b)^2}\right]$ **(M.U. 2003, 04)**
11. $\log\left(1+\frac{a}{s}\right)$ **(M.U. 2003)**
12. $\tan^{-1}\frac{1}{s}$ **(M.U. 2003)**
13. $\cot^{-1} s$ **(M.U. 1996, 98, 2011)**
14. $\log\sqrt{\frac{s^2+a^2}{s^2}}$ **(M.U. 1995, 98, 2004)**
15. $\log\left(1-\frac{1}{s^2}\right)$ **(M.U. 1997)**
16. $\tan^{-1}\left(\frac{s}{2}\right)$ **(M.U. 2004)**

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[**Ans. :** (1) $\frac{2}{t}\left[e^{3t} - \cosh 2t\right]$, (2) $\frac{2}{t}[1 - \cos 2t]$, (3) $-\frac{1}{t}\sinh t$, (4) $\frac{2}{t}(1 - \cos t)$,

(5) $\frac{1}{t}(\cos 2t - \cos t)$, (6) $\frac{1}{t}(1 - \cosh at)$, (7) $-\frac{1}{t}e^{-t}\sin t$, (8) $-\frac{2}{t}[\sin t \sinh t]$,

(9) $\frac{1}{t}(1 - \cos t)$, (10) $\frac{2}{t}(e^{-bt} - \cos at)$, (11) $\frac{1}{t}(1 - e^{-at})$, (12) $\frac{1}{t}\sin 2t$,

(13) $-\frac{1}{t}\sin t$, (14) $\frac{1}{t}(1 - \cos at)$, (15) $\frac{2}{t}(1 - \cosh ht)$, (16) $-\frac{1}{t}\sin 2t$.]

Example 1 : Using convolution theorem, prove that

(i) $L^{-1}\left[\frac{1}{s}\log\left(a + \frac{b}{s^2}\right)\right] = \int_0^t \frac{2}{u}\left[1 - \cos\left(\frac{b}{a}\right)u\right]du.$ **(M.U. 1994)**

(ii) $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right] = \int_0^t \frac{e^{-bu} - e^{-au}}{u}du.$ **(M.U. 2004, 13)**

Sol. : We first note that both the functions are of the form $\Phi_1(s) = \frac{1}{s}\log f(s)$ and as such we shall use (17), page 2-29. [See note, page 2-29]

(i) By (17), page 2-29, we have

$$L^{-1}[\Phi(s)] = -\frac{1}{t}\cdot L^{-1}[\Phi'(s)] \quad \text{where } \Phi(s) = \log f(s).$$

$$\therefore\ L^{-1}\left[\log\left(a + \frac{b}{s^2}\right)\right] = -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\log\left(a + \frac{b}{s^2}\right)\right] \qquad \text{[By (17), page 2-29]}$$

$$= -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\log\left(\frac{as^2 + b}{s^2}\right)\right] = -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}[\log(as^2 + b) - \log s^2]$$

$$= -\frac{1}{t}\cdot L^{-1}\left(\frac{2as}{as^2 + b} - \frac{2s}{s^2}\right) = -\frac{2}{t}\cdot L^{-1}\left[\frac{s}{s^2 + (b/a)} - \frac{1}{s}\right]$$

$$= -\frac{2}{t}\left[\cos\left(\frac{b}{a}\cdot t\right) - 1\right] = \frac{2}{t}\left[1 - \cos\left(\frac{b}{a}\right)t\right]$$

By cor. (16A) of convolution theorem, page 2-15

$$L^{-1}\left[\frac{1}{s}\log\left(a + \frac{b}{s^2}\right)\right] = \int_0^t \frac{2}{u}\left[1 - \cos\left(\frac{b}{a}\right)u\right]du$$

(ii) As above, we have

$$L^{-1}\left[\log\left(\frac{s+a}{s+b}\right)\right] = -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\log\left(\frac{s+a}{s+b}\right)\right] \qquad \text{[By (17), page 2-29]}$$

$$= -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}[\log(s+a) - \log(s+b)] = -\frac{1}{t}\cdot L^{-1}\left[\frac{1}{s+a} - \frac{1}{s+b}\right]$$

$$= -\frac{1}{t}(e^{-at} - e^{-bt}) = \frac{1}{t}(e^{-bt} - e^{-at})$$

By cor. (16A) of convolution theorem, page 2-15

$$L^{-1}\left[\frac{1}{s}\log\left(\frac{s+a}{s+b}\right)\right] = \int_0^t \frac{1}{u}(e^{-bu} - e^{-au})du.$$

Example 2 : Using convolution theorem, prove that

(i) $L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{2}{s}\right] = \int_0^t \frac{1}{u}\cdot\sin 2u\,du$ (ii) $L^{-1}\left[\frac{1}{s}\cot^{-1}(s+1)\right] = \int_0^t \frac{1}{u}\cdot e^{-u}\sin u\,du$

(iii) $L^{-1}\frac{1}{s}\tanh^{-1} s = \int_0^t \frac{\sinh t}{t}dt$

Sol. : We first note that both the functions are of the form $\Phi_1(s) = \frac{1}{s}\tan^{-1} f(s)$ or $\frac{1}{s}\cos^{-1} f(s)$ and as such we shall use (17), page 2-29. [See note, page 2-29]

(i) As in Example 2 (iii), page 2-31, we have

$$L^{-1}[\Phi(s)] = -\frac{1}{t}\cdot L^{-1}[\Phi'(s)] \quad \text{where, } \Phi(s) = \tan^{-1}\left(\frac{2}{s}\right).$$

$$\therefore\ L^{-1}\left[\tan^{-1}\left(\frac{2}{s}\right)\right] = -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\tan^{-1}\left(\frac{2}{s}\right)\right] \qquad \text{[By (17), page 2-29]}$$

$$= -\frac{1}{t}\cdot L^{-1}\left[\frac{1}{1 + (4/s^2)}\cdot\left(-\frac{2}{s^2}\right)\right]$$

$$= \frac{1}{t}\cdot L^{-1}\left[\frac{2}{s^2 + 4}\right] = \frac{1}{t}\sin 2t$$

By cor. (16A) of convolution theorem, page 2-15.

$$L^{-1}\left[\frac{1}{s}\tan^{-1}\left(\frac{2}{s}\right)\right] = \int_0^t \frac{1}{u}\sin 2u\,du.$$

(ii) As above,

$$L^{-1}[\cot^{-1}(s+1)] = -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}[\cot^{-1}(s+1)] \qquad \text{[By (17), page 2-29]}$$

$$= -\frac{1}{t}\cdot L^{-1}\frac{d}{ds}\left[\frac{\pi}{2} - \tan^{-1}(s+1)\right]$$

$$= -\frac{1}{t}\cdot L^{-1}\left[-\frac{1}{1 + (s+1)^2}\right] = \frac{1}{t}\cdot L^{-1}\left[\frac{1}{(s+1)^2 + 1}\right]$$

$$= \frac{1}{t}\cdot e^{-t}\cdot L^{-1}\left(\frac{1}{s^2 + 1}\right) = \frac{1}{t}\cdot e^{-t}\sin t.$$

By cor. (16A) of convolution theorem, page 2-15.

$$L^{-1}\left[\frac{1}{s}\cot^{-1}(s+1)\right] = \int_0^t \frac{1}{u}\cdot e^{-u}\sin u\,du.$$

(iii) Now, $L^{-1}(\tanh^{-1} s) = L^{-1}\left[\frac{1}{2}\log\left(\frac{1+s}{1-s}\right)\right]$

$$= -\frac{1}{2t}\cdot L^{-1}\left[\frac{d}{ds}\log\left(\frac{1+s}{1-s}\right)\right] = -\frac{1}{2t}\cdot L^{-1}\left[\frac{1}{1+s} + \frac{1}{1-s}\right] \qquad \text{[By (17), page 2-29]}$$

$$= -\frac{1}{2t}\cdot L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right] = -\frac{1}{2t}\left[e^{-t} - e^{t}\right]$$

$$= \frac{1}{t}\left[\frac{e^{t} - e^{-t}}{2}\right] = \frac{\sinh t}{t}$$

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By cor. (16A) of convolution theorem, page 2-15.

$$\therefore\ L^{-1}\left(\frac{1}{s}\tan h^{-1}s\right)=\int_0^t \frac{\sin h\, t}{t}\,dt$$

EXERCISE - XI

Using convolution theorem prove that,

1. $L^{-1}\left[\frac{1}{s}\log\left(\frac{s+3}{s+4}\right)\right]=\int_0^t \frac{e^{-4u}-e^{-3u}}{u}\,du.$ **(M.U. 2008)**

2. $L^{-1}\left[\frac{1}{s}\tan^{-1}\frac{a}{s}\right]=\int_0^t \frac{1}{u}\cdot\sin au\,du.$

3. $L^{-1}\left[\frac{1}{s}\cot^{-1}s\right]=\int_0^t \frac{1}{u}\cdot\sin u\,du.$

4. $L^{-1}\left[\frac{1}{s}\cdot\log\left(\frac{s+1}{s+2}\right)\right]=\int_0^t\left(\frac{e^{-2u}-e^{-u}}{u}\right)du$ **(M.U. 2004)**

5. $L^{-1}\left[\frac{1}{s}\cdot\log\left(\frac{s+2}{s+3}\right)\right]=\int_0^t\left(\frac{e^{-3u}-e^{-2u}}{u}\right)du$ **(M.U. 2001)**

6. $L^{-1}\left[\frac{1}{s}\log\left(1+\frac{2}{s^2}\right)\right]=\int_0^t \frac{2}{u}[1-\cos 2u]\,du$

7. $L^{-1}\left[\frac{1}{s}\tan^{-1}\left(\frac{s+a}{b}\right)\right]=\int_0^t -\frac{1}{u}e^{-au}\sin bu\,du$ **(M.U. 2006)**

(f) Use of integration of $\Phi(s)$ (Division by s)

We have proved in § 15, page 1-55 that if $L[f(t)]=\Phi(s)$ then $L\int_0^t f(u)\,du=\frac{1}{s}\Phi(s)$.

Taking the inverse Laplace transform of both sides.

$$\int_0^t f(u)\,du=L^{-1}\frac{1}{s}\Phi(s).\qquad \text{But } f(u)=L^{-1}\Phi(s)$$

$$\therefore\ \boxed{L^{-1}\frac{1}{s}\Phi(s)=\int_0^t L^{-1}\Phi(s)\,ds}\qquad\text{..... (18)}$$

Note ....

See that we have obtained this result as a corollary to convolution theorem in (16A) on page 2-16.

Example 1 : Find : (i) $L^{-1}\left[\frac{1}{s(s^2+4)}\right]$ (ii) $L^{-1}\frac{1}{s^2(s+1)}$

(iii) $L^{-1}\frac{1}{s^3(s^2+a^2)}$ (iv) $L^{-1}\frac{s^2-a^2-s^3}{s^3(s^2-a^2)}$ (v) $L^{-1}\frac{1}{s\sqrt{s+4}}$

(M.U. 1993) **(M.U. 2002, 09)**

Sol. : (i) $L^{-1}\left[\frac{1}{s(s^2+4)}\right]=\int_0^t L^{-1}\left[\frac{1}{(s^2+4)}\right]\cdot du=\int_0^t\frac{1}{2}\cdot\sin 2u\,du$

$$=\frac{1}{2}\left[\frac{-\cos 2u}{2}\right]_0^t=\frac{1}{4}[1-\cos 2t]$$

(ii) $$L^{-1}\left[\frac{1}{s^2(s+1)}\right]=L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s(s+1)}\right]=\int_0^t L^{-1}\left[\frac{1}{s(s+1)}\right]\cdot du$$

$$=\int_0^t L^{-1}\left[\frac{1}{s}-\frac{1}{s+1}\right]du=\int_0^t L^{-1}\left[\frac{1}{s}\right]du-\int_0^t L^{-1}\left[\frac{1}{s+1}\right]du$$

$$=\int_0^t 1\cdot dt-\int_0^t e^{-u}\,du=[u]_0^t+\left[e^{-u}\right]_0^t=t+e^{-t}-1.$$

Aliter : We can use (18), page 2-35 repeatedly.

$$\therefore\ L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s+1}\right]=\int_0^t L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s+1}\right]du=\int_0^t\int_0^t L^{-1}\left(\frac{1}{s+1}\right)(du)^2$$

$$\therefore\ L^{-1}\left[\frac{1}{s^2}\cdot\frac{1}{s+1}\right]=\int_0^t\int_0^t(e^{-u})(du)^2=\int_0^t\left[-e^{-u}\right]_0^t du=\int_0^t(1-e^{-u})\,du$$

$$=\left[u+e^{-u}\right]_0^t=t+e^{-t}-1.$$

(iii) $$L^{-1}\left[\frac{1}{s^3(s^2+a^2)}\right]=L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2(s^2+a^2)}\right]=\int_0^t L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]du$$

$$=\frac{1}{a^2}\int_0^t L^{-1}\left[\frac{1}{s^2}-\frac{1}{s^2+a^2}\right]du=\frac{1}{a^2}\int_0^t\left[u-\frac{1}{a}\sin au\right]du$$

$$=\frac{1}{a^2}\left[\left\{\frac{u^2}{2}\right\}_0^t+\frac{1}{a^2}\{\cos au\}_0^t\right]=\frac{1}{a^2}\left[\frac{t^2}{2}\right]+\frac{1}{a^4}[\cos at-1]$$

Aliter : We may also use (18), page 2-35 repeatedly.

$$\therefore\ L^{-1}\left[\frac{1}{s^3}\cdot\frac{1}{(s^2+a^2)}\right]=L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s^2(s^2+a^2)}\right]=\int_0^t L^{-1}\left[\frac{1}{s^2(s^2+a^2)}\right]du$$

$$=\int_0^t L^{-1}\left[\frac{1}{s}\cdot\frac{1}{s(s^2+a^2)}\right]du=\int_0^t\int_0^t L^{-1}\left[\frac{1}{s(s^2+a^2)}\right](du)^2$$

$$=\int_0^t\int_0^t\int_0^t L^{-1}\frac{1}{(s^2+a^2)}(du)^3=\int_0^t\int_0^t\int_0^t\frac{1}{a}\sin(au)(du)^3$$

$$=\frac{1}{a}\int_0^t\int_0^t\left[-\frac{\cos au}{a}\right]_0^t(du)^2=\frac{1}{a^2}\int_0^t\int_0^t(1-\cos au)(du)^2$$

$$=\frac{1}{a^2}\int_0^t\left[u-\frac{\sin au}{a}\right]_0^t du=\frac{1}{a^2}\int_0^t\left(u-\frac{\sin au}{a}\right)du$$

$$=\frac{1}{a^2}\left[\frac{u^2}{2}+\frac{\cos au}{a^2}\right]_0^t=\frac{1}{a^2}\left[\frac{t^2}{2}+\frac{\cos at}{a^2}-\frac{1}{a^2}\right]$$

$$=\frac{1}{a^2}\left[\frac{t^2}{2}\right]+\frac{1}{a^4}(\cos at-1)$$

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(iv) $$L^{-1}\frac{s^2-a^2-s^3}{s^3(s^2-a^2)} = L^{-1}\left[\frac{1}{s}\cdot\frac{s^2-a^2-s^3}{s^2(s^2-a^2)}\right] = \int_0^t L^{-1}\left[\frac{s^2-a^2-s^3}{s^2(s^2-a^2)}\right]du$$

$$\therefore\ L^{-1}\frac{s^2-a^2-s^3}{s^3(s^2-a^2)} = \int_0^t L^{-1}\left[\frac{1}{s^2}-\frac{s}{s^2-a^2}\right]du = \int_0^t u\,du - \int_0^t \cosh au\,du$$

$$= \left[\frac{u^2}{2}\right]_0^t - \left[\frac{\sinh au}{a}\right]_0^t = \frac{t^2}{2}-\frac{1}{a}\sinh at.$$

(v) $$L^{-1}\frac{1}{s\sqrt{s+4}} = \int_0^t L^{-1}\frac{1}{\sqrt{s+4}}du = \int_0^t e^{-4u}L^{-1}\left(\frac{1}{\sqrt{s}}\right)du$$

$$= \int_0^t e^{-4u}\cdot\frac{u^{-1/2}}{|1/2}du = \frac{1}{\sqrt{\pi}}\int_0^t e^{-4u}u^{-1/2}du$$

Now put $4u = x^2$, $2du = x\,dx$, $\sqrt{u} = \frac{x}{2}$.

$$\therefore\ L^{-1}\frac{1}{s\sqrt{s+4}} = \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}\cdot\frac{2}{x}\cdot\frac{x}{2}dx = \frac{1}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}dx$$

$$= \frac{1}{2}\cdot\frac{2}{\sqrt{\pi}}\int_0^{2\sqrt{t}} e^{-x^2}du = \frac{1}{2}erf\,(2\sqrt{t}) \qquad \left[\because\ \frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}dx = erf\,(t)\right]$$

[For another method see Ex. 6 (i), page 2-26.]

EXERCISE - XII

Find : **1.** $L^{-1}\left[\frac{1}{s(s^2+9)}\right]$ **2.** $L^{-1}\left[\frac{1}{s^2(s+2)}\right]$ **3.** $L^{-1}\left[\frac{1}{s^3(s^2+1)}\right]$ **(M.U. 1994)**

4. $L^{-1}\frac{1}{s(s^2-a^2)}$ **5.** $L^{-1}\frac{54}{s^3(s-3)}$ **(M.U. 2003)**

[**Ans. :** **(1)** $\frac{1}{9}[1-\cos 3t]$, **(2)** $\frac{1}{4}\left[2t+e^{-2t}-1\right]$, **(3)** $\frac{t^2}{2}+\cos t-1$,

(4) $\frac{1}{a}[\cosh at-1]$, **(5)** $[-2-6t-9t^2+2e^{3t}]$.]

Miscellaneous Examples

Theorem : From (17), page 1-39, we get the following corollary easily.

If $L^{-1}[\Phi(s)] = f(t)$, then $L^{-1}\left[\int_s^\infty \Phi(s)\,ds\right] = \frac{1}{t}f(t)$.

Example 1 : Find $L^{-1}\left[\int_s^\infty\left(\frac{u}{u^2+a^2}-\frac{u}{u^2+b^2}\right)du\right]$. **(M.U. 2009)**

Sol. : Let $\Phi(s) = \frac{s}{s^2+a^2}-\frac{s}{u^2+b^2}$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\frac{s}{s^2+a^2}-\frac{s}{s^2+b^2}\right] = \cos at-\cos bt$$

Hence, by the above corollary,

$$L^{-1}\left[\int_s^\infty\left(\frac{u}{u^2+a^2}-\frac{u}{u^2+b^2}\right)du\right] = \frac{\cos at-\cos bt}{t}$$

Example 2 : Find $\int_0^\infty \cos(tx^2)\,dx$ and hence, find $\int_0^\infty \cos x^2\cdot dx$. **(M.U. 2010)**

Sol. : Let $f(t) = \int_0^\infty \cos(tx^2)\,dx$

$$\therefore\ Lf(t) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty e^{-st}\int_0^\infty[\cos(tx^2)\,dx]\,dt$$

$$\therefore\ Lf(t) = \int_0^\infty\left[\int_0^\infty e^{-st}\cos(tx^2)\,dt\right]dx = \int_0^\infty[L\cos(tx^2)]\,dx = \int_0^\infty\frac{s}{s^2+x^4}dx$$

Now put $x = \sqrt{s\tan\theta}$ $\quad\therefore\ dx = \frac{s\cdot\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore\ L[f(t)] = \int_0^{\pi/2}\frac{s}{s^2+s^2\tan^2\theta}\cdot\frac{s\cdot\sec^2\theta\,d\theta}{2\sqrt{s\tan\theta}}$$

$$= \int_0^{\pi/2}\frac{1}{2\sqrt{s\tan\theta}}d\theta = \frac{1}{2\sqrt{s}}\int_0^{\pi/2}(\sin\theta)^{-1/2}(\cos\theta)^{1/2}d\theta$$

$$= \frac{1}{2\sqrt{s}}\frac{|\overline{1/4}\,|\overline{3/4}}{2\cdot 1!}\left[\int_0^{\pi/2}\sin^p\theta\cos^p\theta\,d\theta = \frac{|\overline{(p+1)/2}\,|\overline{(q+1)/2}}{2|\overline{(p+q+2)/2}}\right]$$

$$= \frac{1}{2\sqrt{s}}\cdot\frac{\sqrt{2}\cdot\pi}{2} \qquad \left[\because\ |\overline{1/4}\,|\overline{3/4} = \sqrt{2}\cdot\pi\right]$$

$$\therefore\ L(f(t) = \frac{\pi}{2\sqrt{2}\sqrt{s}}$$

$$\therefore\ f(t) = \frac{\pi}{2\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2\sqrt{2}}\cdot\frac{t^{-1/2}}{|1/2} = \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{t}}$$

Now put $t = 1$, $\quad\therefore\ \int_0^\infty \cos x^2dx = \frac{\sqrt{\pi}}{2\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

Example 3 : Find $\int_0^\infty \sin(tx^2)$ and hence, find $\int_0^\infty \sin x^2\,dx$. **(M.U. 2003)**

Sol. : Let $f(t) = \int_0^\infty \sin(tx^2)\,dx$

$$\therefore\ Lf(t) = \int_0^\infty e^{-st}f(t)\,dt = \int_0^\infty e^{-st}\left[\int_0^\infty \sin(tx^2)\,dx\right]dt$$

$$= \int_0^\infty\left[\int_0^\infty e^{-st}\sin(tx^2)\,dt\right]dx = \int_0^\infty[L\sin(tx^2)]\,dx = \int_0^\infty\frac{x^2}{s^2+x^4}dx$$

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Now put $x = \sqrt{s\tan\theta}$, $dx = \dfrac{s\cdot\sec^2\theta\, d\theta}{2\sqrt{s\tan\theta}}$

$$\therefore\ Lf(t) = \int_0^{\pi/2} \frac{s\tan\theta}{s^2 + s^2\tan^2\theta}\cdot\frac{s\cdot\sec^2\theta}{2\sqrt{s\tan\theta}}\cdot d\theta$$

$$= \frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sqrt{\tan\theta}\, d\theta = \frac{1}{2\sqrt{s}}\int_0^{\pi/2}\sin^{1/2}\theta\cos^{-1/2}\theta\, d\theta$$

$$= \frac{1}{2\sqrt{s}}\frac{\Gamma(3/4)\,\Gamma(1/4)}{2\cdot 1} = \frac{1}{2\sqrt{s}}\frac{\sqrt{2}\cdot\pi}{2} = \frac{\pi}{2\sqrt{2}}\cdot\frac{1}{\sqrt{s}}$$

$$\therefore\ f(t) = \frac{\pi}{2\sqrt{2}}L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2\sqrt{2}}\cdot\frac{t^{-1/2}}{\Gamma(1/2)} = \frac{\sqrt{\pi}}{2\sqrt{2}}\cdot\frac{1}{\sqrt{t}}$$

Now put $t = 1$, $\quad\therefore\ \int_0^\infty \sin x^2 dx = \dfrac{\sqrt{\pi}}{2\sqrt{2}} = \dfrac{1}{2}\sqrt{\dfrac{\pi}{2}}$.

Example 4 : Find $\int_0^\infty e^{-tx^2}dx$ and hence, find $\int_0^\infty e^{-x^2}dx$. **(M.U. 2004)**

Sol. : Let $f(t) = \int_0^\infty e^{-tx^2}dx$

$$\therefore\ Lf(t) = \int_0^\infty e^{-st}\left[\int_0^\infty e^{-tx^2}dx\right]dt = \int_0^\infty\left[\int_0^\infty e^{-st}\cdot e^{-tx^2}dt\right]dx$$

$$= \int_0^\infty\left[L(e^{-tx^2})\right]dx = \int_0^\infty\frac{dx}{s+x^2} \qquad \left[\because\ Le^{-at} = \frac{1}{s+a}\right]$$

$$= \left[\frac{1}{\sqrt{s}}\tan^{-1}\frac{x}{\sqrt{s}}\right]_0^\infty = \frac{\pi}{2\sqrt{s}}$$

$$\therefore\ f(t) = \frac{\pi}{2}L^{-1}\left(\frac{1}{\sqrt{s}}\right) = \frac{\pi}{2}\frac{t^{-1/2}}{\Gamma(1/2)} = \frac{\pi}{2}\frac{1}{\sqrt{\pi}\sqrt{t}} = \frac{1}{2}\sqrt{\frac{\pi}{t}}.$$

Now, put $t = 1$, $\quad\therefore\ \int_0^\infty e^{-x^2}dx = \dfrac{\sqrt{\pi}}{2}$.

Example 5 : If $J_0(t) = \dfrac{1}{\pi}\int_0^\pi\cos(t\cos\theta)\,d\theta$, prove that $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$. **(M.U. 2003)**

Sol. : We have $J_0(t) = \dfrac{1}{\pi}\int_0^\pi\cos(t\cos\theta)\,d\theta = \dfrac{2}{\pi}\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta$

Taking Laplace transforms of both sides.

$$L[J_0(t)] = \frac{2}{\pi}\int_0^\infty e^{-st}\left[\int_0^{\pi/2}\cos(t\cos\theta)\,d\theta\right]dt$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\int_0^\infty e^{-st}\cos(t\cos\theta)\,dt\right]d\theta = \frac{2}{\pi}\int_0^{\pi/2}\left[L\cos(t\cos\theta)\right]d\theta$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left[\frac{s}{s^2+\cos^2\theta}\right]d\theta = \frac{2}{\pi}\int_0^{\pi/2}\frac{s\sec^2\theta}{s^2\sec^2\theta+1}d\theta$$

$$\therefore\ L[J_0(t)] = \frac{2}{\pi}\int_0^{\pi/2}\frac{s\sec^2\theta}{(s^2+1)+s^2\tan^2\theta}d\theta$$

Put $s\tan\theta = t \quad\therefore\ s\sec^2\theta\, d\theta = dt$

$$\therefore\ L[J_0(t)] = \frac{2}{\pi}\int_0^\infty\frac{dt}{t^2+(s^2+1)} = \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\left[\tan^{-1}\left(\frac{t}{\sqrt{s^2+1}}\right)\right]_0^\infty$$

$$= \frac{2}{\pi}\cdot\frac{1}{\sqrt{s^2+1}}\cdot\frac{\pi}{2} = \frac{1}{\sqrt{s^2+1}}.$$

Example 6 : If $J_0(t) = \dfrac{1}{\pi}\int_0^\pi\cos(t\sin\theta)\,d\theta$, prove that $L[J_0(t)] = \dfrac{1}{\sqrt{s^2+1}}$. **(M.U. 2003)**

Sol. : Do it yourself.

Example 7 : Prove that $L^{-1}\left[\dfrac{1}{s}\cos\dfrac{1}{s}\right] = 1 - \dfrac{t^2}{(2!)^2} + \dfrac{t^4}{(4!)^2} - \dfrac{t^6}{(6!)^2} + \ldots\ldots$

Sol. : We know that $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \ldots\ldots$

$$\therefore\ \cos\left(\frac{1}{s}\right) = 1 - \frac{1}{2!\,s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} - \frac{1}{6!}\cdot\frac{1}{s^6} + \ldots\ldots$$

$$\therefore\ \frac{1}{s}\left\{\cos\frac{1}{s}\right\} = \frac{1}{s}\left\{1 - \frac{1}{2!\,s^2} + \frac{1}{4!}\cdot\frac{1}{s^4} - \frac{1}{6!}\cdot\frac{1}{s^6} + \ldots\ldots\right\}$$

$$\therefore\ L^{-1}\left\{\frac{1}{s}\cos\frac{1}{s}\right\} = L^{-1}\frac{1}{s}\left\{1 - \frac{1}{2!\,s^2} + \frac{1}{4!\,s^4} - \frac{1}{6!\,s^6} + \ldots\ldots\right\}$$

$$= L^{-1}\left\{1 - \frac{1}{2!\,s^3} + \frac{1}{4!\,s^5} - \frac{1}{6!\,s^7} + \ldots\ldots\right\}$$

$$= 1 - \frac{1}{2!}\cdot\frac{t^2}{2!} + \frac{1}{4!}\cdot\frac{t^4}{4!} - \frac{1}{6!}\cdot\frac{t^6}{6!} + \ldots\ldots$$

$$= 1 - \frac{t^2}{(2!)^2} + \frac{t^4}{(4!)^2} - \frac{t^6}{(6!)^2} + \ldots\ldots$$

EXERCISE - XIII

Theory

1. State and prove convolution theorem. **(M.U. 2003)**
2. State convolution theorem and deduce that
$$L^{-1}\left[\frac{1}{s}\Phi(s)\right] = \int_0^t f(u)\,du \quad \text{where, } L[f(u)] = \Phi(s).$$ **(M.U. 2002)**
3. Prove that $L\,\delta(t-a) = e^{-as}$.

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4. Prove that $L[f(t-a)H(t-a)] = e^{-as} f(s)$ where $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$ (M.U. 2003)

5. Find $L[\delta(t-a)]$ where, $\delta(t-a) \begin{cases} = 0 & \text{for } t < a \\ = 1/\varepsilon & \text{for } a < t < a+\varepsilon \\ = 0 & \text{for } t > a+\varepsilon \end{cases}$. (M.U. 1998)

6. If $f(t-a) \cdot H(t-a) = 0 \quad t < a$
$= f(t-a) \quad t > a$, then prove that

$L[f(t-a) \cdot H(t-a)] = e^{-as} L[f(t)] = e^{-as}\Phi(s)$ where, $\Phi(s) = L[f(t)]$. (M.U. 1996, 2004)

7. If $f(t)$ is a periodic function of period a, prove that

$$L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt.$$ (M.U. 1996, 97, 2003, 07, 08)

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CHAPTER 3

Fourier Series

1. Introduction

The series

$$a_0 + (a_1 \cos x + b_1 \sin x) + (a_2 \cos 2x + b_2 \sin 2x) + \ldots + (a_n \cos nx + b_n \sin nx) + \ldots$$

or briefly $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where, all a's and b's are constants is called a **trigonometric series.**

The function $f(x)$ represented by the above series is a periodic function of period 2π since, on changing x to $2\pi + x$, we get the same series.

Any periodic function $f(x)$ of period 2π which satisfies certain conditions known as Dirichlet's conditions can be expressed in the form of the above series valid for all values of x in any interval c to $c + 2\pi$ of length 2π. The expansion of $f(x)$ in the form of the above series is called **Fourier Series.**

Joseph Fourier (1768 - 1830)

Jean Baptiste Joseph Fourier was a great French mathematician and physicist. He is best remembered for pioneering Fourier Series, Fourier Transforms. He went to Egypt with Napolean Bonapararte in 1798 and was made governor of Lower Egypt. He contributed several mathematical papers to Cairo Institute founded by Napolean. He was permanent secretary of French Academy of Sciences. In 1830 he was elected to Royal Swedish Academy Of Sciences.

2. Dirichlet's Conditions

A function $f(x)$ defined in the interval $c_1 \le x \le c_2$ can be expressed as Fourier Series if in the interval,

(i) $f(x)$ and its integrals are finite and single valued,

(ii) $f(x)$ has discontinuities, finite in number,

(iii) $f(x)$ has finite number of maxima and minima.

These conditions are known as **Dirichlet's conditions.**

For example, $\sin^{-1} x$ cannot be expressed as Fourier Series since, it is not a single valued function.

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Also tan x cannot be expressed as Fourier Series in $(0, 2\pi)$ since, tan x is infinite at $x = \frac{\pi}{2}\left(\text{and } \frac{3\pi}{2}\right)$ which is a point on interval.

But cos αx can be expanded as a Fourier Series since, it satisfies the above conditions.

Johann Peter Gustav Lejeune Dirichlet (1805 - 1859)

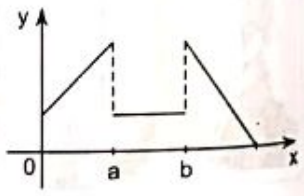
He was a great German mathematician. Poisson and Fourier were his doctoral advisers and Kronecker, Cantor, Dedekind, Riemann were among his many doctoral students. He showed a strong interest in mathematics before he was 12 and convinced his father to allow him to study mathematics. His father wanted him to be a merchant. His first original research giving a proof of Fermat's last theorem for $n = 5$ brought him immediate fame. He was then appointed as a lecturer at the French Academy of Sciences at the age of 20 in 1825, before he had a degree. He got his Ph.D. in 1827. He was in close contact with other great mathematicians, Jacobi, Liouville, etc. He died in Göttingen, Germnay. His brain is preserved in the department of Physiology at the university of Göttingen along with the brain of Gauss.

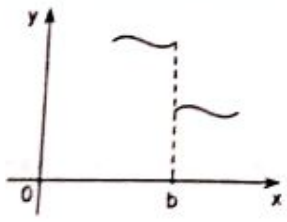
Dirichlet's main interest was Number Theory and Analysis. He was a member of several academies such as Prussian Academy of Sciences, French Academy of Sciences, Royal Swedish Academy of Sciences, Royal Belgian Academy of Sciences, etc. There are several theorems named after Dirichlet viz. Dirichlet's Approximation Theorem, Dirichlet's Unit Theorem, Dirichlet's Integral Theorem, Dirichlet's Convolution, Dirichlet's Series, Dirichlet's Function, Dirichlet's Pigeon hole Principle and many many others.

The Dirichlet crater on the Moon and the 11665 Dirichlet asteroid are named after him.

(a) Functions with discontinuities

A function which has finite disconti-nuities can be expanded as a Fourier Series. The graph of a function as shown in the adjoining figure may consist of a finite number of disjointed curves given by different equations.





If $x = c$ is a point of finite discontinuity both the limits of $f(x)$ as $x \to c^-$ and as $x \to c^+$ exist. At such a point Fourier Series gives the value of $f(x)$ at $x = c$ as the arithmetic mean of these two limits

i.e. $f(c) = \frac{1}{2}\left[\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right]$

Notes ....

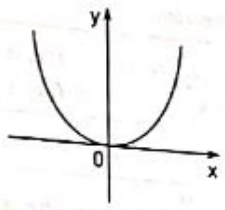
(i) A function $f(x)$ is said to be **periodic** with period a if $f(a + x) = f(a)$ e.g. sin x, cos x are periodic with period 2π.

(ii) (a) A function $f(x)$ is said to be **even** if $f(-x) = f(x)$ e.g. cos x, sec x, x^2 are even functions. Graphically, an even function is **symmetrical about the y-axis**.

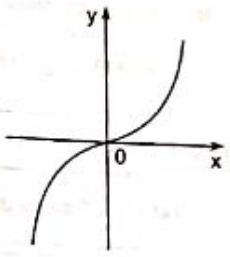
(b) A function $f(x)$ is said to be **odd** if $f(-x) = -f(x)$ e.g. sin x, cosec x, x^3 are odd functions.

Geometrically, an odd function is **symmetrical about the origin.**

(c) A function can be neither even nor odd e.g. e^x, 10^x, $x^2 - x$, are neither odd nor even.



Even Function



Odd Function

(iii) We also need the following results, where n is an integer

(i) $\sin n\pi = 0$ { (ii) $\sin 2n\pi = 0$ } (iii) $\cos n\pi = (-1)^n$ (iv) $\cos 2n\pi = 1$

(v) $\cos (n \pm 1)\pi = \cos n\pi \cos \pi \mp \sin n\pi \sin \pi = -\cos n\pi$

(vi) $\sin (n \pm 1)\pi = \sin n\pi \cos \pi \pm \cos n\pi \sin \pi = 0$.

(vii) $\sin (2n\pi + x) = \sin x$

(viii) $\cos (2n\pi + x) = \cos x$

(iv) We also need the following results

$2 \sin A \cos B = \sin (A + B) + \sin (A - B)$

$2 \cos A \sin B = \sin (A + B) - \sin (A - B)$

$2 \cos A \cos B = \cos (A + B) + \cos (A - B)$

$-2 \sin A \sin B = \cos (A + B) - \cos (A - B)$.

(v) **Properties of Even and Odd Functions**

We state below the properties of even and odd functions.

(i) The sum or difference of two (or more) even functions is even.

(ii) The sum or difference of two (or more) odd functions is odd.

(iii) The product of two even functions is even.

e.g., if $f(x) = x^2 + 1$ and $g(x) = x^4$, then

$f(x) \cdot g(x) = (x^2 + 1)x^4 = x^6 + x^4$ is even.

(iv) The product of two odd functions is even.

e.g., if $f(x) = x^3 + x$ and $g(x) = x$, then

$f(x) \cdot g(x) = (x^3 + x)x = x^4 + x^2$ is even.

(v) The product of an even function and an odd function is odd.

e.g., if $f(x) = x^4 + x^2$ and $g(x) = x$, then

$f(x) \cdot g(x) = (x^4 + x^2)x = x^5 + x^3$ is odd.

If $\phi(x) = f(x) \cdot g(x)$ then $\phi(x)$ will be even or odd according to the following table.

$f(x)$	E	E	O	O
$g(x)$	E	O	E	O
$\phi(x) = f(x) \cdot g(x)$	E	O	O	E

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(vi) We need the following theorem

Theorem $\boxed{\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(x) \text{ is even.} \quad = 0 \text{ if } f(x) \text{ is odd.}}$ ..... (1)

(vii) We also need the following two integrals often in Fourier series.

(a) $\boxed{\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]}$ ..... (2)

(b) $\boxed{\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]}$ ..... (3)

(viii) We also need the following results

(a) $\boxed{\int_c^{c+2\pi}\cos nx\,dx = \left[\frac{\sin nx}{n}\right]_c^{c+2\pi} = 0 \quad (n \neq 0)}$ ..... (4)

(b) $\boxed{\int_c^{c+2\pi}\sin nx\,dx = \left[-\frac{\cos nx}{n}\right]_c^{c+2\pi} = 0 \quad (n \neq 0)}$ ..... (5)

(c) $\int_c^{c+2\pi}\sin mx\cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi}\left[\sin(m+n)x + \sin(m-n)x\right]dx$

$$= \frac{1}{2}\left[-\frac{\cos(m+n)x}{m+n} - \frac{\cos(m-n)x}{m-n}\right]_c^{c+2\pi} = 0 \quad (m \neq n)$$

If $m = n$, $\int_c^{c+2\pi}\sin nx\cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi}2\sin nx\cos nx\,dx$

$$= \frac{1}{2}\int_c^{c+2\pi}\sin 2nx\,dx = \frac{1}{2}\left[-\frac{\cos 2nx}{2n}\right]_c^{c+2\pi} = 0$$

Hence, $\boxed{\int_c^{c+2\pi}\sin mx\cos nx\,dx = 0 \text{ for all } m, n}$ ..... (6)

(d) $\int_c^{c+2\pi}\cos mx\cos nx\,dx = \frac{1}{2}\int_c^{c+2\pi}\left[\cos(m+n)x + \cos(m-n)x\right]dx$

$$= \frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{(m-n)}\right]_c^{c+2\pi} = 0 \quad (m \neq n)$$

If $m = n$, $\int_c^{c+2\pi}\cos mx\cos nx\,dx = \int_c^{c+2\pi}\cos^2 nx\,dx$

$$= \int_c^{c+2\pi}\left(\frac{1+\cos 2nx}{2}\right)dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \pi$$

Hence, $\boxed{\int_c^{c+2\pi}\cos mx\cos nx\,dx = \begin{cases}0 & \text{if } m \neq n\\ \pi & \text{if } m = n\end{cases}}$ ..... (7)

(e) $\int_c^{c+2\pi}\sin mx\sin nx\,dx = -\frac{1}{2}\int_c^{c+2\pi}\left[\cos(m+n)x - \cos(m-n)x\right]dx$

$$= -\frac{1}{2}\left[\frac{\sin(m+n)x}{m+n} - \frac{\sin(m-n)x}{m-n}\right]_c^{c+2\pi} = 0 \quad (m \neq n)$$

If $m = n$, $\int_c^{c+2\pi}\sin mx\sin nx\,dx = \int_c^{c+2\pi}\sin^2 nx\,dx$

$$= \int_c^{c+2\pi}\left(\frac{1-\cos 2nx}{2}\right)dx = \frac{1}{2}\left[x - \frac{\sin 2nx}{2n}\right]_c^{c+2\pi} = \pi$$

Hence, $\boxed{\int_c^{c+2\pi}\sin mx\sin nx\,dx = \begin{cases}0 & \text{if } m \neq n\\ \pi & \text{if } m = n\end{cases}}$ ..... (8)

(ix) Putting $c = -\pi$ in the above results, we get the following results.

(A) $\int_{-\pi}^{\pi}\cos nx\,dx = 0, \quad \int_{-\pi}^{\pi}\sin nx\,dx = 0$ ..... (9)

(B) $\int_{-\pi}^{\pi}\sin mx\cos nx\,dx = 0$ for all m, n ..... (10)

(C) $\int_{-\pi}^{\pi}\cos mx\cos nx\,dx = 0$ if $m \neq n$ ..... (11)
$= \pi$ if $m = n$

(D) $\int_{-\pi}^{\pi}\sin mx\sin nx\,dx = 0$ if $m \neq n$ ..... (12)
$= \pi$ if $m = n$

3. Determination of Fourier Coefficients (Euler's Formulae)

Let $f(x)$ be a periodic function of period 2π which can be represented in the interval $(c, c+2\pi)$ by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx \quad \text{..... (1)}$$

We assume that the above series is uniformly convergent and can be integrated term by term in the given interval.

Integrating both sides of (1) from c to $c+2\pi$, we get

$$\int_c^{c+2\pi} f(x)\,dx = \int_c^{c+2\pi} a_0\,dx + \int_c^{c+2\pi}\sum a_n\cos nx\,dx + \int_c^{c+2\pi}\sum b_n\sin nx\,dx$$

$$= a_0\left[x\right]_c^{c+2\pi} + 0 + 0 = a_0(c+2\pi-c) = 2a_0\pi$$

$$\therefore \boxed{a_0 = \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx} \quad \text{..... (2)}$$

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Leonhard Euler (1707 - 1783)

One of the great mathematicians of Switzerland. His father wanted him to become a pastor (a priest). But Bernoulli persuaded his father to allow his son to pursue mathematics. He studied under his fellow countryman, mathematician Bernoulli and had published his first paper when he was 18. The word function first suggested by Leibnitz was generalised further by Bernoulli and Euler. Euler is supposed to be the most prolific mathematical writer in history. He has written a number of text books which are known for his clarity, detail and completeness. Although he had lost his eye-sight for the last 17 years of his life, he did not allow his work to be hampered because all the formulae from trigonometry and analysis (and many poems including the entire Latin epic-Aeneid) were on the tip of his tongue.

Now, multiply (1) by $\cos nx$ and integrate from c to $c + 2\pi$.

$$\therefore \int_c^{c+2\pi} f(x)\cos nx\,dx = a_0\int_c^{c+2\pi}\cos nx\,dx + \int_c^{c+2\pi}(\Sigma a_n \cos nx)\cos nx\,dx + \int_c^{c+2\pi}(\Sigma b_n \sin x)\cos nx\,dx$$

$$= 0 + a_n\pi + 0$$

$$\therefore \quad \boxed{a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx} \qquad \text{..... (3)}$$

Now, multiply (1) by $\sin nx$ and integrate from c to $c + 2\pi$.

$$\therefore \int_c^{c+2\pi} f(x)\sin nx\,dx = a_0\int_c^{c+2\pi}\sin nx\,dx + \int_c^{c+2\pi}(\Sigma a_n \cos nx)\sin nx\,dx + \int_c^{c+2\pi}(\Sigma b_n \sin nx)\sin nx\,dx$$

$$= 0 + 0 + b_n\pi$$

$$\therefore \quad \boxed{b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx} \qquad \text{..... (4)}$$

Thus, we have the following results

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_c^{c+2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \end{aligned}} \qquad \text{..... (5)}$$

Cor 1 : If $c = 0$ *i.e.* if the interval is $(0, 2\pi)$, then the Fourier series of $f(x)$ is given by $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx \end{aligned}} \qquad \text{..... (6)}$$

Cor. 2 : If $c = -\pi$ *i.e.* if the interval is $(-\pi, \pi)$, then the Fourier series is given by $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, where

$$\boxed{\begin{aligned} a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx \end{aligned}} \qquad \text{..... (7)}$$

4. Parseval's Identity in $(c, c + 2l)$

If $f(x)$ converges uniformly in $(c, c + 2l)$, then

$$\boxed{\frac{1}{2l}\int_c^{c+2l}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2)} \qquad \text{..... (A)}$$

This is known as **Parseval's Identity** for the function $f(x)$ in the interval $(c, c + 2l)$.

Proof : We shall see later that the Fourier Series for $f(x)$ in $(c, c + 2l)$ is given by (See (A), page 3-43).

$$f(x) = a_0 + \Sigma a_n \frac{\cos n\pi x}{l} + \Sigma b_n \frac{\sin n\pi x}{l} \qquad \text{..... (1)}$$

Now, multiplying both sides of (1) by $f(x)$ and integrating term by term from c to $c + 2l$, we get,

$$\int_c^{c+2l}[f(x)]^2\,dx = a_0\int_c^{c+2l} f(x)\,dx + \int_c^{c+2l}\Sigma a_n f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_c^{c+2l}\Sigma b_n f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad \text{..... (2)}$$

But $\int_c^{c+2l} f(x)\,dx = 2l\,a_0$ [See (3), page 3-43]

$$\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = a_n l$$

$$\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = b_n l$$

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Putting these values in (2), we get,

$$\int_c^{c+2l}[f(x)]^2dx = 2l\,a_0^2 + \Sigma a_n^2 l + \Sigma b_n^2 l$$

$$\therefore \quad \frac{1}{2l}\int_c^{c+2l}[f(x)]^2dx = a_0^2 + \frac{1}{2}\Sigma\left(a_n^2 + b_n^2\right)$$

Cor 1 : If $l = \pi$ *i.e.* if the interval is $(c, c + 2\pi)$ and $f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, [See (1) on the previous page], then

$$\frac{1}{2\pi}\int_c^{c+2\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

(This is Parseval's identity for $f(x)$ in $(c, c + 2\pi)$. We have obtained this result independently below.)

Cor 2 : If $c = 0$ in the above corollary 1 *i.e.* if the interval is $(0, 2\pi)$

and $\quad f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$

then $\quad \frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

Cor 3 : If $c = -\pi$ in the above corollary 1 *i.e.* if the interval is $(-\pi, \pi)$

and $\quad f(x) = a_0 + \Sigma a_n \cos nx + \Sigma b_n \sin nx$, then

then $\quad \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

Cor 4 : If $c = 0$ in the above identity (A) *i.e.* if the interval is $(0, 2l)$

and $\quad f(x) = a_0 + \Sigma a_n \cos\frac{n\pi x}{l} + \Sigma b_n \sin\frac{n\pi x}{l}$

then $\quad \frac{1}{2l}\int_0^{2l}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

Cor. 5 : If $c = -l$ in the above identity (A) *i.e.* if the interval is $(-l, l)$

and $\quad f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) + \Sigma b_n \sin\left(\frac{n\pi x}{l}\right)$

then $\quad \frac{1}{2l}\int_{-l}^{l}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$

(You are advised to derive this result independently.)

Cor. 6 : If the half range cosine series [See pages 3-62 and 3-63] in $(0, \pi)$ for $f(x)$ is

$f(x) = a_0 + \Sigma a_n \cos nx$,

then $\quad \frac{1}{\pi}\int_0^{\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$

Cor. 7 : If the half range sine series in $(0, \pi)$ for $f(x)$ is $\quad f(x) = \Sigma b_n \sin nx$

then $\quad \frac{1}{\pi}\int_0^{\pi}[f(x)]^2dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\ldots\infty\right]$

Cor. 8 : If the half range cosine series in $(0, l)$ for $f(x)$ is

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right)$$

then $\quad \frac{1}{l}\int_0^{l}[f(x)]^2dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$

(We shall derive this result independently below.)

Cor. 9 : If the half range sine series in $(0, l)$ for $f(x)$ is

$$f(x) = \Sigma b_n \sin\frac{n\pi x}{c}$$

then $\quad \frac{1}{l}\int_0^{l}[f(x)]^2dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\ldots\infty\right]$

(a) Parseval's Identity in $(c, c + 2\pi)$

If $f(x)$ converges uniformly in $(c, c + 2\pi)$, then

$$\boxed{\frac{1}{2\pi}\int_c^{c+2\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)}$$

This is known as Parseval's Identity for the function $f(x)$ in the interval $(c, c + 2\pi)$.

Proof : We know that the Fourier series for $f(x)$ in $(c, c + 2\pi)$ is given by

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos x + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots\ldots (1)$$

Now, multiplying both sides of (1) by $f(x)$ and integrating term by term from c to $c + 2\pi$, we get,

$$\int_c^{c+2\pi}[f(x)]^2dx = a_0\int_c^{c+2\pi}f(x)\,dx + \int_c^{c+2\pi}\Sigma a_n f(x)\cos nx\,dx$$

$$+ \int_c^{c+2\pi}\Sigma b_n f(x)\sin nx\,dx \qquad \ldots\ldots\ldots\ldots (2)$$

But $\quad \int_c^{c+2\pi}f(x)\,dx = 2\pi a_0 \qquad$ [By (2), page 3-5]

$\int_c^{c+2\pi}f(x)\cos nx\,dx = a_n\pi \qquad$ [By (3), page 3-6]

$\int_c^{c+2\pi}f(x)\sin nx\,dx = b_n\pi \qquad$ [(By (4), page 3-6]

Putting these values is (2), we get,

$$\int_c^{c+2\pi}[f(x)]^2\,dx = 2\pi a_0^2 + \Sigma a_n^2\pi + \Sigma b_n^2\pi$$

$$\therefore \quad \frac{1}{2\pi}\int_c^{c+2\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\Sigma\left(a_n^2 + b_n^2\right)$$

Cor. : Putting $c = -\pi$, we get

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2dx = a_0^2 + \frac{1}{2}\Sigma\left(a_n^2 + b_n^2\right).$$

(b) Parseval's Identity for half range cosine series in $(0, l)$ *

If $f(x)$ can be expanded as a half range cosine series in $(0, l)$ as

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right)$$

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then $\frac{1}{l}\int_0^l [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$

Proof : We know that the half range cosine series for $f(x)$ in the interval $(0, l)$ is given by (See page 3-62).

$$f(x) = a_0 + \Sigma a_n \cos\left(\frac{n\pi x}{l}\right) \qquad \text{[See § 13, page 3-62]}$$

Now, multiply both sides by $f(x)$ and integrate term by term from 0 to l.

$$\therefore \int_0^l [f(x)]^2 dx = a_0 \int_0^l [f(x)] dx + \int_0^l \Sigma a_n f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

But $\int_0^l f(x) dx = a_0 l$ [Page 3-63]

$$\int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx = \frac{a_n l}{2} \qquad \text{[Page 3-63]}$$

$$\therefore \int_0^l [f(x)]^2 dx = a_0^2 l + \frac{1}{2}\Sigma a_n^2 l$$

$$\therefore \frac{1}{l}\int_0^l [f(x)]^2 dx = a_0^2 + \frac{1}{2}\left[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty\right]$$

Cor. : Putting $l = \pi$

$$\frac{1}{\pi}\int_0^\pi [f(x)]^2 dx = a_0^2 + \frac{1}{2}[a_1^2 + a_2^2 + a_3^2 + \ldots\ldots\infty]$$

(● Proofs are not expected.)

5. Generalised Rule of Integration by Parts

If u, v are two functions (dashes denote the derivatives and suffixes denote the integrals), then

$$\int uv\, dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\ldots$$

In words the rule states that, to find the integral of the product : Integral = first term × integral of he second – then write the derivative of the first term × integral of the second and repeat the procedure taking alternatively positive and negative signs. The rule is highly useful especially when the first term is a positive integral power of x and the second term can be easily integrated.

6. Fourier Series in (0, 2π)

Example 1 : Find a Fourier Series to represent $f(x) = x^2$ in $(0, 2\pi)$ and hence, deduce that

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots$$

(M.U. 2004, 09, 10, 12, 18, 19)

Sol. : Let $x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ in $(0, 2\pi)$ ..... (1)

Then, $a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x) dx = \frac{1}{2\pi}\int_0^{2\pi} x^2 dx = \frac{1}{2\pi}\left[\frac{x^3}{3}\right]_0^{2\pi} = \frac{4\pi^2}{3}$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \cos nx\, dx$$

(By the generalised rule of integration by parts)

$$a_n = \frac{1}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - 2x\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi} = \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right] = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\, dx$$

$$= \frac{1}{\pi}\left[x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left\{-\frac{4\pi^2}{n} + \frac{2}{n^3}\right\} - \left\{\frac{2}{n^3}\right\}\right] = \frac{4\pi}{n}$$

Putting these values in (1),

$$x^2 = \frac{4\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx - 4\pi\sum_{n=1}^{\infty}\frac{1}{n}\sin nx$$

$$\therefore x^2 = \frac{4\pi^2}{3} + 4\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right] - 4\pi\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]$$

Now, put $x = \pi$,

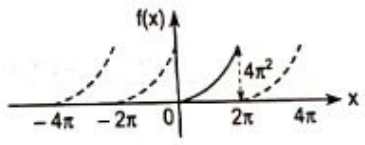
$$\therefore \pi^2 = \frac{4\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \ldots\right]$$

$$-\frac{\pi^2}{3} = -4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

Now, the given function, $y = x^2$ is a parabola with vertex at the origin and opening upwards. Further when $x = 0$, $y = 0$ and when $x = 2\pi$, $y = 4\pi^2$.

The graph of the above function is shown in the figure.



Example 2 : Find the Fourier Series of the function $f(x) = e^{-x}$, $0 < x < 2\pi$ and $f(x + 2\pi) = f(x)$. **(M.U. 2005)**

Hence, deduce the value of $\sum_{n=2}^{\infty}\frac{(-1)^n}{n^2+1}$.

Further derive the series for cosec $h\,\pi$.

Sol. : Let $f(x) = e^{-x} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

$$\therefore a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x) dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-x} dx = \frac{1}{2\pi}\left[-e^{-x}\right]_0^{2\pi} = \frac{1}{2\pi}\left(-e^{-2\pi} + 1\right) = \frac{1 - e^{-2\pi}}{2\pi}$$

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$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\cos nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{-x}(-\cos nx + n\sin nx)\right]_0^{2\pi} \qquad \text{[See (3), page 3-4]}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\cos 2n\pi + n\sin 2n\pi) - e^0(-\cos 0 + n\sin 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\cdot\left[e^{-2\pi}(-1) - (-1)\right] = \frac{1-e^{-2\pi}}{\pi(1+n^2)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} e^{-x}\sin nx\,dx$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{-x}(-\sin nx - n\cos nx\right]_0^{2\pi} \qquad \text{[See (2), page 3-4]}$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-\sin 2n\pi - n\cos 2n\pi) - e^0(-\sin 0 - n\cos 0)\right]$$

$$= \frac{1}{\pi(1+n^2)}\left[e^{-2\pi}(-n) - 1(-n)\right] = \frac{n}{\pi(1+n^2)}(1-e^{-2\pi})$$

Putting the values of a_0, a_n, b_n in (1), we get

$$e^{-x} = \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos nx + \sum_{n=1}^{\infty}\frac{n}{1+n^2}\sin nx\right] \quad \text{..... (2)}$$

$$= \frac{1-e^{-2\pi}}{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{1}{1+n^2}(\cos nx + n\sin nx)\right]$$

Putting $x = \pi$ in (2), we get,

$$e^{-\pi} = \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{1+n^2}\cos n\pi\right] \qquad (\because \sin n\pi = 0)$$

$$= \frac{1-e^{-2\pi}}{2\pi} + \frac{(1-e^{-2\pi})}{\pi}\left[-\frac{1}{2} + \sum_{n=2}^{\infty}\frac{1}{1+n^2}(-1)^n\right]$$

$$\therefore\ e^{-\pi} = \frac{1-e^{-2\pi}}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} \qquad \therefore\ \frac{\pi}{e^{\pi}(1-e^{-2\pi})} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore\ \frac{\pi}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} \qquad \therefore\ \frac{\pi}{2}\cdot\frac{2}{e^{\pi}-e^{-\pi}} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore\ \frac{\pi}{2}\cdot\frac{1}{\sinh\pi} = \sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2}$$

$$\therefore\ \operatorname{cosec h}\pi = \frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n}{1+n^2} = \frac{2}{\pi}\left[\frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \ldots\right]$$

Example 3 : Obtain the Fourier expansion of $f(x) = \left(\frac{\pi - x}{2}\right)^2$ in the interval $0 \le x \le 2\pi$ and $f(x + 2\pi) = f(x)$. **(M.U. 2002, 04, 11, 12, 13, 19)**

Also deduce that

(i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$ **(M.U. 2014)** (ii) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$ **(M.U. 2016)**

(iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$ **(M.U. 2003)** (iv) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$ **(M.U. 2016)**

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

$$\therefore\ a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\,dx$$

$$\therefore\ a_0 = \frac{1}{8\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} = -\frac{1}{24\pi}[-\pi^3 - \pi^3] = \frac{\pi^2}{12} \quad \text{..... (A)}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\cos nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\cos nx}{n^2}\right) + 2(-1)(-1)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{2\pi}$$

(By generalised rule of integration by parts.)

$$= \frac{1}{4\pi}\left[\left(0 + 2\pi\frac{\cos 2n\pi}{n^2} - 0\right) - \left(0 - \frac{2\pi}{n^2} - 0\right)\right]$$

$$\therefore\ a_n = \frac{1}{4\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right] = \frac{1}{n^2} \qquad [\because \cos 2n\pi = 1] \quad \text{..... (B)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\frac{(\pi - x)^2}{4}\cdot\sin nx\,dx$$

$$= \frac{1}{4\pi}\left[(\pi - x)^2\left(-\frac{\cos nx}{n}\right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2}\right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{4\pi}\left[\left(-\frac{\pi^2\cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3}\right) - \left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3}\right)\right]$$

$$\therefore\ b_n = \frac{1}{4\pi}\left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3}\right] = 0 \quad \text{..... (C)}$$

Putting these values in (1), we get

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

$$= \frac{\pi^2}{12} + \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots \quad \text{..... (2)}$$

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(i) Now, put $x = 0$ in (2).

$$\therefore \quad \frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \ldots\ldots\ldots\ldots (3)$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$$

(ii) Again, put $x = \pi$ in (2).

$$\therefore \quad 0 = \frac{\pi^2}{12} - \frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} + \ldots \qquad \ldots\ldots\ldots\ldots (4)$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

(iii) To get the result (iii) add (3) and (4).

$$\therefore \quad \frac{\pi^2}{6} + \frac{\pi^2}{12} = \frac{2}{1^2} + \frac{2}{3^2} + \frac{2}{5^2} + \frac{2}{7^2} + \ldots$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$$

(iv) To derive the last result we use Parseval's identity. We know that by Parseval's identity in $(0, 2\pi)$

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2) \qquad \ldots\ldots\ldots\ldots (5)$$

Now, $$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)^4 dx$$

$$= \frac{1}{32\pi}\int_0^{2\pi}[\pi^4 - 4\pi^3 x + 6\pi^2 x^2 - 4\pi x^3 + x^4]\,dx$$

$$= \frac{1}{32\pi}\left[\pi^4 x - 2\pi^3 x^2 + 2\pi^2 x^3 - \pi x^4 + \frac{x^5}{5}\right]_0^{2\pi}$$

$$= \frac{1}{32\pi}\left[2\pi^5 - 8\pi^5 + 16\pi^5 - 16\pi^5 + \frac{32\pi^5}{5}\right]$$

$$= \frac{1}{32\pi}\cdot\frac{2\pi^5}{5} = \frac{\pi^4}{80}$$

Hence, by (5) using (A), (B) and (C)

$$\frac{\pi^4}{80} = \frac{\pi^4}{144} + \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\pi^4\left(\frac{1}{80} - \frac{1}{144}\right) = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\pi^4\left(\frac{9-5}{720}\right) = \frac{\pi^4}{180} = \frac{1}{2}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots\right)$$

$$\therefore \quad \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$$

Example 4 : Expand $f(x) = x \sin x$ in the interval $0 \le x \le 2\pi$. **(M.U. 2001, 02, 03, 05, 10)**

Deduce that $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$.

Sol. : Let $$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots\ldots\ldots\ldots (1)$$

$$\therefore \quad a_0 = \int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}(x \sin x)\cdot dx$$

$$= \frac{1}{2\pi}\left[(x)(-\cos x) - (-\sin x)(1)\right]_0^{2\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{2\pi}[(-2\pi + 0) - (0 + 0)] = -1$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx\,dx \qquad \ldots\ldots\ldots\ldots (2)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin x \cos nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\cos nx \sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{2\pi}\left[x\left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(2\pi)\left\{-\frac{\cos 2(n+1)\pi}{(n+1)} + \frac{\cos 2(n-1)\pi}{(n-1)}\right\} - 0\right] \qquad \text{[By § 5, page 3-10]}$$

$$= -\frac{1}{n+1} + \frac{1}{n-1} = \frac{2}{n^2 - 1} \quad \text{if } n \ne 1$$

If $n = 1$, the above method fails.

Putting $n = 1$ in (2), we get,

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin x \cos x\,dx = \frac{1}{2\pi}\int_0^{2\pi} x \cdot \sin 2x\,dx$$

$$= \frac{1}{2\pi}\left[x\left(-\frac{\cos 2x}{2}\right) - (1)\left(-\frac{\sin 2x}{4}\right)\right]_0^{2\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{2\pi}\left[2\pi\left(-\frac{\cos 4\pi}{2}\right) - 0\right] = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx\,dx \qquad \ldots\ldots\ldots\ldots (3)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin x \sin nx\,dx = \frac{1}{2\pi}\int_0^{2\pi} x \cdot 2\sin nx \sin x\,dx$$

$$= -\frac{1}{2\pi}\int_0^{2\pi} x\,[\cos(n+1)x - \cos(n-1)x]\,dx \qquad \text{[By § 5, page 3-10]}$$

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$$\therefore\ b_n = -\frac{1}{2\pi}\left[x\left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^{2\pi}$$

$$= -\frac{1}{2\pi}\left[-(1)\left\{-\frac{\cos 2(n+1)\pi}{(n+1)^2} + \frac{\cos 2(n-1)\pi}{(n-1)^2}\right\} + (1)\left\{-\frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right\}\right]$$

$$= -\frac{1}{2\pi}\left[\frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]$$

$$= 0 \text{ if } n \neq 1$$

If $n = 1$, the above method fails.

Putting $n = 1$ in (3), we gat,

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x\, dx = \frac{1}{\pi}\int_0^{2\pi} x \sin^2 x\, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1 - \cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\left\{2\pi(2\pi - 0) - \left(\frac{4\pi^2}{2} + \frac{1}{4}\right)\right\} - \left(0 - \frac{1}{4}\right)\right] = \frac{1}{2\pi}\left[2\pi^2\right] = \pi$$

Putting these values in (1),

$$x \sin x = -1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2}{n^2 - 1}\cos nx + \pi \sin x$$

Deduction : Putting $x = 0$, we get $\frac{3}{4} = \sum_{n=2}^{\infty} \frac{1}{n^2 - 1}$.

Example 5 : Find the Fourier expansion for $f(x) = \sqrt{1 - \cos x}$ in $(0, 2\pi)$. **(M.U. 2010, 13, 19)**

Hence, deduce that $\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4n^2 - 1}$. **(M.U. 1994, 99, 2005, 06, 09)**

Sol. : Let $f(x) = \sqrt{1 - \cos x} = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

Here, $f(x) = \sqrt{2} \cdot \sin(x/2)$

$$\therefore\ a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} \sqrt{2} \cdot \sin\frac{x}{2}\, dx$$

$$= \frac{1}{\sqrt{2} \cdot \pi}\left[-2\cos\frac{x}{2}\right]_0^{2\pi} = \frac{1}{\sqrt{2} \cdot \pi}[-2(-1 - 1)] = \frac{4}{\sqrt{2} \cdot \pi} = \frac{2\sqrt{2}}{\pi}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2} \cdot \sin\frac{x}{2} \cdot \cos nx\, dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{1}{2} + n\right)x + \sin\left(\frac{1}{2} - n\right)x\right] dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(\frac{1 + 2n}{2}\right)x + \sin\left(\frac{1 - 2n}{2}\right)x\right] dx$$

$$\therefore\ a_n = \frac{\sqrt{2}}{2\pi}\left[-\frac{2}{1 + 2n}\cos\left(\frac{1 + 2n}{2}\right)x - \frac{2}{1 - 2n}\cos\left(\frac{1 - 2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[-\frac{1}{2n + 1}\cos\left(\frac{2n + 1}{2}\right)x + \frac{1}{2n - 1}\cos\left(\frac{2n - 1}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{\pi}\left[\frac{2}{2n + 1} - \frac{2}{2n - 1}\right] = -\frac{4\sqrt{2}}{\pi(4n^2 - 1)}$$

$[\because \cos(2n + 1)\pi = -1$ and $\cos(2n - 1)\pi = -1]$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \sqrt{2} \sin\frac{x}{2} \sin nx\, dx$$

$$= \frac{-\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{1}{2} + n\right)x - \cos\left(\frac{1}{2} - n\right)x\right] dx$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{2}{1 + 2n}\sin\left(\frac{1 + 2n}{2}\right)x - \frac{2}{1 - 2n}\sin\left(\frac{1 - 2n}{2}\right)x\right]_0^{2\pi}$$

$$= \frac{-\sqrt{2}}{2\pi}\left[\frac{1}{2n + 1}\sin\left(\frac{2n + 1}{2}\right)x - \frac{1}{2n - 1}\sin\left(\frac{2n - 1}{2}\right)x\right]_0^{2\pi}$$

$$= 0$$

$\therefore$ Putting these valued in (1)

$$\sqrt{1 - \cos x} = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} \cdot \cos nx$$

Putting $x = 0$, $\frac{1}{2} = \sum_{1}^{\infty} \frac{1}{4n^2 - 1}$.

Example 6 : Find the Fourier expansion of $\cos px$ where p is not an integer in $(0, 2\pi)$. Hence, deduce that

$$\pi \operatorname{cosec} \pi x = \frac{1}{p} + \sum_{n=1}^{\infty} (-1)^n\left[\frac{1}{p + n} + \frac{1}{p - n}\right]$$ **(M.U. 2013)**

Also deduce that $\pi \cot 2\pi p = \frac{1}{2p} + p\sum_{n=1}^{\infty} \frac{1}{p^2 - n^2}$.

Sol. : Let $\cos px = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

$$\therefore\ a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{2\pi}\int_0^{2\pi} \cos px\, dx$$

$$= \frac{1}{2\pi}\left[\frac{\sin px}{p}\right]_0^{2\pi} = \frac{1}{2\pi} \cdot \frac{\sin 2p\pi}{p} = \frac{\sin 2p\pi}{2\pi p}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} \cos px \cos nx\, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\cos px \cos nx\, dx = \frac{1}{2\pi}\int_0^{2\pi}[\cos(p + n)x + \cos(p - n)x]\, dx$$

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$$\therefore \quad a_n = \frac{1}{2\pi}\left[\frac{\sin(p+n)x}{p+n} + \frac{\sin(p-n)x}{p-n}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{\sin 2\pi(p+n)}{p+n} + \frac{\sin 2\pi(p-n)}{p-n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{\sin 2p\pi}{p+n} + \frac{\sin 2p\pi}{p-n}\right] = \frac{1}{2\pi}\cdot \sin 2p\pi\left[\frac{1}{p+n} + \frac{1}{p-n}\right]$$

$$= \frac{p}{\pi}\cdot\frac{\sin 2p\pi}{p^2-n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\cos px\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} 2\sin nx\cos px\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left[\sin(n+p)x + \sin(n-p)x\right]dx$$

$$= \frac{1}{2\pi}\left[-\frac{\cos(n+p)x}{n+p} - \frac{\cos(n-p)x}{n-p}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2\pi(n+p)}{n+p} - \frac{\cos 2\pi(n-p)}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{\cos 2p\pi}{n+p} - \frac{\cos 2p\pi}{n-p} + \frac{1}{n+p} + \frac{1}{n-p}\right]$$

$$= \frac{1}{2\pi}\left[\frac{1-\cos 2p\pi}{n+p} - \frac{1-\cos 2p\pi}{n-p}\right]$$

$$= \frac{1-\cos 2p\pi}{2\pi}\left[\frac{1}{n+p} + \frac{1}{n-p}\right] = \frac{1-\cos 2p\pi}{2\pi}\cdot\frac{2n}{n^2-p^2}$$

$$\therefore \quad b_n = -\frac{n}{\pi}\frac{(1-\cos 2p\pi)}{p^2-n^2}$$

Putting these values in (1), we get,

$$\cos px = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin nx}{p^2-n^2} \quad \text{..... (2)}$$

To deduce the first result, put $x = \pi$ in (2).

$$\therefore \quad \cos p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos n\pi}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin n\pi}{p^2-n^2}$$

But since, n is a positive integer, $\sin n\pi = 0$ and $\cos n\pi = (-1)^n$.

$$\therefore \quad \cos p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$= \frac{2\sin p\pi\cos p\pi}{2p\pi} + \frac{2p\sin p\pi\cos p\pi}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$\therefore \quad \pi\,\mathrm{cosec}\,p\pi = \frac{1}{p} + 2p\sum_{n=1}^{\infty}\frac{(-1)^n}{p^2-n^2}$$

$$= \frac{1}{p} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{1}{p+n} + \frac{1}{p-n}\right] \quad \text{[By partial fractions]}$$

To deduce the second result put $x = 2\pi$ in (2).

$$\therefore \quad \cos 2p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{\cos 2n\pi}{p^2-n^2} - \frac{(1-\cos 2p\pi)}{\pi}\sum_{n=1}^{\infty}\frac{n\sin 2n\pi}{p^2-n^2}$$

But, since, n is a positive integer, $\sin 2n\pi = 0$ and $\cos 2n\pi = 1$.

$$\therefore \quad \cos 2p\pi = \frac{\sin 2p\pi}{2p\pi} + \frac{p\sin 2p\pi}{\pi}\sum_{n=1}^{\infty}\frac{1}{(p^2-n^2)}$$

Dividing by $\sin 2p\pi$ throughout,

$$\therefore \quad \pi\cot 2p\pi = \frac{1}{2p} + p\sum_{n=1}^{\infty}\frac{1}{(p^2-n^2)}$$

Example 7 : Find the Fourier series for $f(x) = \dfrac{3x^2 - 6x\pi + 2\pi^2}{12}$ in $(0, 2\pi)$.

Hence, deduce that $\dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \ldots\ldots$ **(M.U. 2001, 04, 07, 11, 13)**

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\cos nx$

$$\therefore \quad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi}\int_0^{2\pi}\frac{(3x^2-6x\pi+2\pi^2)}{12}dx$$

$$= \frac{1}{24\pi}\left[x^3 - 3x^2\pi + 2\pi^2 x\right]_0^{2\pi} = \frac{1}{24\pi}\left[8\pi^3 - 12\pi^3 + 4\pi^3\right] = 0$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\cos nx\,dx$$

$$= \frac{1}{12\pi}\left[3\left\{x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right\}\right.$$

$$\left. - 6\pi\left\{x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\} + 2\pi^2\left(\frac{\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{12\pi}\left[3\left\{0 + \frac{4\pi}{n^2} + 0\right\} - 6\pi\left\{0 + \frac{1}{n^2}\right\} + 2\pi^2(0) - \left\{0 - 6\pi\left(\frac{1}{n^2}\right)\right\}\right]$$

$$= \frac{1}{n^2} \quad \text{[By § 5, page 3-10]}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi}\left(\frac{3x^2-6x\pi+2\pi^2}{12}\right)\sin nx\,dx$$

$$= \frac{1}{12\pi}\left[3\left\{x^2\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + 2(1)\left(\frac{\cos nx}{n^3}\right)\right\}\right.$$

$$\left. - 6\pi\left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right\} + 2\pi^2\left(-\frac{\cos nx}{n}\right)\right]_0^{2\pi}$$

[By § 5, page 3-10]

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$$\therefore\ b_n = \frac{1}{12\pi}\left[3\left\{-\frac{4\pi^2}{n}+\frac{2}{n^3}\right\} - 6\pi\left\{-\frac{2\pi}{n}\right\} - \frac{2\pi^2}{n} - \left\{3\left(\frac{2}{n^3}\right) - \frac{2\pi^2}{n}\right\}\right]$$

$$= 0$$

$$\therefore\ f(x) = \sum a_n \cos nx = \sum \frac{1}{n^2}\cos nx$$

$$\therefore\ \frac{3x^2 - 6x\pi + 2\pi^2}{12} = \frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \ldots$$

Putting $x = 0$, we get,

$$\frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots \qquad \therefore\ \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 8 : Find Fourier Series for $f(x)$ in $(0, 2\pi)$ where **(M.U. 2004, 06, 11)**

$$f(x) = \begin{cases} x, & 0 < x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$$

Hence, deduce that $\frac{\pi^2}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_0^{\pi} x\,dx + \frac{1}{2\pi}\int_{\pi}^{2\pi}(2\pi - x)\,dx = \frac{1}{2\pi}\left[\frac{x^2}{2}\right]_0^{\pi} + \frac{1}{2\pi}\left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{\pi^2}{2}\right] + \frac{1}{2\pi}\left[4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$\therefore\ a_0 = \frac{1}{2\pi}\cdot\pi^2 = \frac{\pi}{2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

$$a_n = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$\therefore\ a_n = \frac{1}{\pi\cdot n^2}\left[(-1)^n - 1\right] - \frac{1}{\pi\cdot n^2}\left[1 - (-1)^n\right] = \frac{-2}{\pi\cdot n^2}\left[1 - (-1)^n\right] \quad \text{[By § 5, page 3-10]}$$

$$\therefore\ a_1 = -\frac{4}{1^2},\ a_2 = 0,\ a_3 = -\frac{4}{3^2},\ a_4 = 0,\ a_5 = -\frac{4}{5^2}, \ldots\ldots \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(B)}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi} + \frac{1}{\pi}\left[(2\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$\therefore\ b_n = \frac{-1}{\pi n}\left[\pi(-1)^n\right] + \frac{1}{\pi n}\left[\pi(-1)^n\right] = 0 \qquad \text{[By § 5, page 3-10]}$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x$$

To deduce the required result we use the Parseval's identity

$$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum\left(a_n^2 + b_n^2\right) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Now, $$\frac{1}{2\pi}\int_0^{2\pi}[f(x)]^2 dx = \frac{1}{2\pi}\left[\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi}(2\pi - x)^2 dx\right]$$

$$= \frac{1}{2\pi}\left[\int_0^{\pi} x^2 dx + \int_{\pi}^{2\pi}(4\pi^2 - 4\pi x + x^2)\,dx\right]$$

$$= \frac{2}{2\pi}\left\{\left[\frac{x^3}{3}\right]_0^{\pi} + \left[4\pi^2 x - 2\pi x^2 + \frac{x^3}{3}\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{\pi^3}{3} + 8\pi^3 - 8\pi^3 + \frac{8\pi^3}{3} - 4\pi^3 + 2\pi^3 - \frac{\pi^3}{3}\right\}$$

$$= \frac{1}{2\pi}\cdot\frac{2\pi^3}{3} = \frac{\pi^2}{3}$$

Hence, by (1), using (A) and (B), we get by using the values of a_0, a_n and b_n.

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{1}{2}\cdot\frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^2}{12} = \frac{8}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$$

EXERCISE - I

Find the Fourier Series for the following functions.

1. $f(x) = e^{\alpha x}$, $(\alpha \ne 0)$ in $(0, 2\pi)$. $\left[\textbf{Ans. :}\ f(x) = \frac{1}{\pi}(e^{2\alpha\pi} - 1)\left[\frac{1}{2\alpha} + \sum_{n=1}^{\infty}\frac{\cos nx - n\sin nx}{\alpha^2 + n^2}\right]\right]$

2. $f(x) = e^x$ in $(0, 2\pi)$. $\left[\textbf{Ans. :}\ f(x) = \frac{1}{\pi}\cdot(e^{2\pi} - 1)\left[\frac{1}{2} + \sum_{n=1}^{\infty}\frac{\cos nx - n\sin nx}{1 + n^2}\right]\right]$

3. $f(x) = x$ in $(0, 2\pi)$. **(M.U. 2003, 12)** $\left[\textbf{Ans. :}\ f(x) = \pi - 2\sum_{n=1}^{\infty}\frac{\sin nx}{n}\right]$

4. $f(x) = \frac{1}{2}(\pi - x)$ in $(0, 2\pi)$. Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

(M.U. 2007, 11, 15) $\left[\textbf{Ans. :}\ f(x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n}.\ \text{Then put } x = \frac{\pi}{2}\right]$

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5. $f(x) = \begin{cases} \sin x, & 0 \le x \le \pi \\ 0, & \pi \le x \le 2\pi \end{cases}$

Hence, deduce that $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots$ or $1 = \sum_{n=1}^{\infty} \frac{2}{(2n-1)(2n+1)}$.

$$\left[\text{Ans. : } f(x) = \frac{1}{\pi} - \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{4n^2-1}\cos 2nx + \frac{1}{2}\sin x\right]$$

6. $f(x) = \begin{cases} 1, & 0 \le x \le \pi \\ 2 - \frac{\pi}{x}, & \pi \le x \le 2\pi \end{cases}$ **(M.U. 2010)**

$$\left[\text{Ans. : } f(x) = \frac{3}{4} - \frac{2}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right] + \frac{1}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \ldots\right]\right]$$

7. $f(x) = \begin{cases} mx, & 0 \le x \le \pi \\ 2m\pi - mx, & \pi \le x \le 2\pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{m\pi}{2} - \frac{4m}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]\right]$$

8. $f(x) = \begin{cases} a, & 0 < x < \pi \\ -a, & \pi < x < 2\pi \end{cases}$. Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$$\left[\text{Ans. : } f(x) = \frac{4a}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right]\right]$$

(For deduction put $x = \pi/2$).

9. $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$ $\left[\text{Ans. : } f(x) = \frac{\pi}{2} - \frac{4}{5}\left[\frac{\cos x}{1} + \frac{\cos 3x}{3} + \frac{\cos 5x}{5} + \ldots\right]\right]$

10. $f(x) = \begin{cases} (x-\pi)^2, & 0 < x < \pi \\ \pi^2, & \pi < x < 2\pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{2}{3}\pi^2 + 2\left[\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \ldots\right] - 2\left[\left(\pi + \frac{2}{\pi}\right)\cos x + \left(\frac{\pi}{3} + \frac{2}{9\pi}\right)\cos 3x + \ldots\right]\right]$$

11. $f(x) = \begin{cases} 1, & 0 < x < \pi \\ 2, & \pi < x < 2\pi \end{cases}$. Hence, deduce that, $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$$\left[\text{Ans. : } f(x) = \frac{3}{2} - \frac{2}{\pi}\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\right)\right]$$

(For deduction put $x = \pi/2$).

12. $f(x) = \begin{cases} -\pi, & 0 < x < \pi \\ x - \pi, & \pi < x < 2\pi \end{cases}$

State the value of the series at $x = \pi$ and hence, show that $\frac{\pi^2}{8} = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

$$\left[\text{Ans. : } f(x) = -\frac{\pi}{4} + \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] + \left[-3\sin x - \frac{1}{2}\sin 2x - \sin 3x + \ldots\right]\right.$$

$f(\pi)\frac{-\pi + 0}{2} = -\frac{\pi}{2}$. Then put $x = \pi$.]

13. If $f(x) = 2x$, $0 \le x \le 2\pi$, find the Fourier series of $f(x)$. Also find a_4 and b_{10}. **(M.U. 2002)**

$$\left[\text{Ans. : } f(x) = 2\pi - 4\sum_{n=1}^{\infty} \frac{\sin nx}{n};\ a_4 = 0,\ b_{10} = -0.4\right]$$

7. Fourier Expansion of $f(x)$ in the Interval $(-\pi, \pi)$

Example 1 : Obtain the Fourier expansion of e^x in $-\pi < x < \pi$.

Sol. : Let $e^x = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x dx = \frac{1}{2\pi}\left[e^x\right]_{-\pi}^{\pi} = \frac{1}{2\pi}\left[e^x - e^{-\pi}\right] = \frac{1}{\pi}\sinh \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \cos nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{1+n^2}\cdot e^x(\cos nx + n\sin nx)\right]_{-\pi}^{\pi}$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{1}{a^2+b^2}\cdot e^{ax}(a\cos bx + b\sin bx)\right]$$

$$\therefore\ a_n = \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right] = \frac{2\cos n\pi}{\pi(1+n^2)}\cdot\frac{e^{\pi} - e^{-\pi}}{2}$$

$$= \frac{2\cos n\pi}{\pi(1+n^2)}\cdot\sinh \pi = \frac{(-1)^n}{\pi(1+n^2)}\cdot 2\sinh \pi$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin x\,dx$$

$$= \frac{1}{\pi}\left[\frac{1}{1+n^2}\cdot e^x(\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$\left[\therefore \int e^{ax}\sin bx\,dx = \frac{1}{a^2+b^2}\cdot e^{ax}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{\pi}\cdot\frac{1}{1+n^2}\left[-e^{\pi}\cdot n\cos n\pi + n e^{-\pi}\cos n\pi\right]$$

$$= \frac{-2\cos n\pi}{\pi(1+n^2)}\left[\frac{e^{\pi} - e^{-\pi}}{2}\right] = -\frac{2n\cos n\pi}{\pi(1+n^2)}\cdot\sinh \pi$$

$$\therefore\ b_n = -(-1)^n\cdot\frac{n}{\pi(1+n^2)}\cdot 2\sinh \pi$$

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$$\therefore \quad e^x = \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} + (-1)^n \cdot \sum_{n=1}^{\infty}\frac{1}{(1+n^2)}\cos nx - (-1)^n \cdot \sum_{n=1}^{\infty}\frac{n}{(1+n^2)}\sin nx\right]$$

$$= \frac{2\sinh\pi}{\pi}\left[\frac{1}{2} - \frac{1}{2}\cdot\cos x + \frac{1}{5}\cdot\cos 2x - \frac{1}{10}\cdot\cos 3x + \ldots + \frac{1}{2}\cdot\sin x - \frac{2}{5}\cdot\sin 2x + \frac{3}{10}\cdot\sin 3x - \ldots\right]$$

Example 2 : Find the Fourier Series for the periodic function

$$f(x) = \begin{cases} -\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$$

State the value of $f(x)$ at $x = 0$ and hence, deduce that

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$ **(M.U. 1996, 2003, 10)**

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0} -\pi \cdot dx + \int_0^{\pi} x\,dx\right]$$

$$= \frac{1}{2\pi}\left[-\pi\left\{x\right\}_{-\pi}^{0} + \left\{\frac{x^2}{2}\right\}_0^{\pi}\right] = \frac{1}{2\pi}\left[-\pi^2 + \frac{\pi^2}{2}\right] = -\frac{\pi}{4}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_0^{\pi} x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(\frac{\sin nx}{n}\right)_{-\pi}^{0} + \left\{x\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi}\right] \quad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{\pi}\left[-\pi(0-0) + \left\{\pi(0) + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] = \frac{1}{\pi n^2}\left[(-1)^n - 1\right] \quad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_0^{\pi} x\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[-\pi\left(-\frac{\cos nx}{n}\right)_{-\pi}^{0} + \left\{x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n}\right)\right\}_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\pi\frac{(1-\cos n\pi)}{n} + \left\{\pi\left(-\frac{\cos n\pi}{n}\right) - 0\right\}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi}{n}(1-\cos n\pi) - \frac{\pi}{n}\cos n\pi\right]$$

$$= \frac{1}{n}[1 - 2\cos n\pi] = \frac{1}{n}\left[1 - 2(-1)^n\right]$$

Putting these values in (1),

$$f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos nx + \sum_{n=1}^{\infty}\frac{1 - 2(-1)^n}{n}\sin nx$$

$$f(x) = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] + \left[3\cdot\frac{\sin x}{1} - \frac{1}{2}\sin 2x + \sin 3x - \ldots\right] \quad \ldots\ldots (2)$$

Now, $f(x)$ is discontinuous at $x = 0$. At a point of discontinuity $x = c$,

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right] \qquad \therefore \quad f(0) = \frac{1}{2}[-\pi + 0] = -\frac{\pi}{2}$$

Hence, putting $x = 0$ in (2),

$$-\frac{\pi}{2} = -\frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Example 3 : Find the Fourier Series for $f(x) = \begin{cases} \cos x, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$. **(M.U. 2016)**

Sol. : Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\cos x\,dx + \int_0^{\pi}\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\sin x\right\}_{-\pi}^{0} + \left\{-\cos x\right\}_0^{\pi}\right] = \frac{1}{2\pi}[(0) + (1+1)] = \frac{1}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\cos nx\,dx + \int_0^{\pi}\sin x\cos nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0} 2\cos nx\cos x\,dx + \int_2^{\pi} 2\cos nx\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}[\cos(n+1)x + \cos(n-1)x]\,dx + \int_0^{\pi}[\sin(n+1)x - \sin(n-1)x]\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{-\pi}^{0} + \left\{-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left\{0\right\} + \left\{-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right\}\right] \text{ if } n \neq 1$$

$$\therefore \quad a_n = \begin{cases} 0 & \text{if } n \text{ is odd and} \neq 1 \\ -\dfrac{2}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$$

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When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\cos x\,dx + \int_{0}^{\pi}\sin x\cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\left(\frac{1+\cos 2x}{2}\right)dx + \int_{0}^{\pi}\frac{\sin 2x}{2}dx\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right)_{-\pi}^{0} + \left(-\frac{\cos 2x}{4}\right)_{0}^{\pi}\right]$$

$$\therefore\ a_1 = \frac{1}{\pi}\left[\frac{1}{2}\left\{\left(0-(-\pi-0)\right)-\left(\frac{1}{4}-\frac{1}{4}\right)\right\}\right] = \frac{1}{2}$$

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\sin nx\,dx + \int_{0}^{\pi}\sin x\sin nx\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}2\sin nx\cos x\,dx + \int_{0}^{\pi}2\sin nx\sin x\,dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}[\sin(n+1)x+\sin(n-1)x]dx - \int_{0}^{\pi}[\cos(n+1)x-\cos(n-1)x]dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{\cos(n+1)x}{n+1}-\frac{\cos(n-1)x}{n-1}\right\}_{-\pi}^{0} - \left\{\frac{\sin(n+1)x}{n+1}-\frac{\sin(n-1)x}{n-1}\right\}_{0}^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[\left\{-\frac{1}{n+1}-\frac{1}{n-1}+\frac{\cos(n+1)\pi}{n+1}+\frac{\cos(n-1)\pi}{n-1}\right\}-0\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1+\cos n\pi}{n+1}-\frac{1+\cos n\pi}{n-1}\right] = -\frac{n}{\pi}\frac{(1+\cos n\pi)}{n^2-1} \text{ if } n \neq 1$$

$$\therefore\ b_n = \begin{cases} 0 & \text{if } n \text{ is odd and} \neq 1 \\ -\dfrac{2n}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin x\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\cos x\sin x\,dx + \int_{0}^{\pi}\sin^2 x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0}\frac{\sin 2x}{2}dx + \int_{0}^{\pi}\frac{1-\cos 2x}{2}dx\right]$$

$$= \frac{1}{\pi}\left[\left\{-\frac{\cos 2x}{2}\right\}_{-\pi}^{0} + \frac{1}{2}\left\{x-\frac{\sin 2x}{2}\right\}_{0}^{\pi}\right]$$

$$\therefore\ b_1 = \frac{1}{\pi}\left[-\left(\frac{1}{2}-\frac{1}{2}\right)+\frac{1}{2}(\pi-0)\right] = \frac{1}{2}$$

Putting these values in

$$f(x) = a_0 + a_1\cos x + [a_2\cos 2x + a_4\cos 4x + \ldots] + b_1\sin x + [b_2\sin 2x + b_4\sin 4x + \ldots]$$

$$= \frac{1}{\pi} + \frac{1}{2}\cos x - \frac{2}{\pi}\left[\frac{\cos 2x}{3}+\frac{\cos 4x}{15}+\ldots\right] + \frac{1}{2}\sin x - \frac{2}{\pi}\left[\frac{2\sin 2x}{3}+\frac{4\sin 4x}{15}+\ldots\right]$$

$$= \frac{1}{\pi} + \frac{1}{2}(\cos x+\sin x) + \frac{2}{\pi}\left[\sum_{n=1}^{\infty}\frac{1}{(1-4n^2)}\cos 2nx + \sum_{n=1}^{\infty}\frac{2n}{(1-4n^2)}\sin 2nx\right]$$

EXERCISE - II

Find Fourier expansion of

1. $f(x) = e^{ax}$ in $(-\pi, \pi)$. $\left[\textbf{Ans.}: e^{ax} = \dfrac{\sinh a\pi}{a\pi} + \dfrac{2\sinh a\pi}{\pi}\sum\dfrac{(-1)^n}{a^2+n^2}(a\cos nx - n\sin nx)\right]$

2. $f(x) = e^{-ax}$ in $(-\pi, \pi)$ and hence, show that $\dfrac{\pi}{\sinh a\pi} = \dfrac{1}{a} + \sum\dfrac{(-1)^n\cdot 2a}{n^2+a^2}$.

$$\left[\textbf{Ans.}: f(x) = \frac{2\sinh a\pi}{\pi}\left[\frac{1}{2a} - \frac{a\cos x}{a^2+1} + \frac{a\cos 2x}{a^2+2^2} - \ldots + \frac{\sin x}{a^2+1} - \frac{\sin 2x}{a^2+2^2} + \frac{\sin 3x}{a^2+3^2} - \ldots\right]\right]$$

3. $f(x) = \begin{cases} 1/2, & -\pi < x < 0 \\ x/\pi, & 0 < x < \pi \end{cases}$. Hence deduce that $\dfrac{\pi^2}{8} = \sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^2}$. **(M.U. 2002)**

$$\left[\textbf{Ans.}: f(x) = \frac{1}{2} - \frac{2}{\pi^2}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2} - \frac{1}{2}\sum_{n=1}^{\infty}\frac{\sin 2nx}{n}\right]$$

4. $f(x) = \begin{cases} x+\pi, & 0 \le x \le \pi \\ -x-\pi, & -\pi \le x \le 0 \end{cases}$ **(M.U. 2003)**

$$\left[\textbf{Ans.}: f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\pi}{(2n-1)^2} + 4\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}\right]$$

5. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}$ and $f(x+2\pi) = f(x)$.

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$; $\dfrac{\pi}{4} = \dfrac{1}{1} - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

$$\left[\textbf{Ans.}: f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n+1)^2}\cdot\cos(2n+1)x - \sum\frac{(-1)^n}{n}\sin nx\right]$$

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6. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Hence, deduce that (i) $\frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \ldots = \frac{1}{2}$

(ii) $\frac{1}{4}(\pi - 2) = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \ldots$ **(M.U. 2000, 03, 04, 09, 11)**

$$\left[\text{Ans. : } f(x) = \frac{1}{\pi} + \frac{\sin x}{2} - \frac{2}{\pi}\left[\frac{\cos 2x}{4\cdot 1^2 - 1} + \frac{\cos 4x}{4\cdot 2^2 - 1} + \ldots\right]\right]$$

7. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ 0, & 0 < x < \pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] - \left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \frac{1}{4}\sin 4x + \ldots\right]\right]$$

(M.U. 2003)

8. $f(x) = \begin{cases} x - \pi, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

$$\left[\text{Ans. : } f(x) = -\frac{\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\sin 5x + \ldots\right] + 4\left[\frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots\right]\right]$$

(M.U. 2009, 14)

9. $f(x) = \begin{cases} 0, & -\pi \le x \le 0 \\ x^2, & 0 \le x \le \pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{\pi^2}{6} - 2\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right] + \frac{2}{\pi}\left[\frac{1}{1^3}\sin x - \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x - \ldots\right] + \pi\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\right] - \frac{1}{\pi}\left[\frac{1}{1^3}\sin x + \frac{1}{2^3}\sin 2x + \frac{1}{3^3}\sin 3x + \ldots\right]\right]$$

8. Even and Odd Functions in $(-\pi, \pi)$

(i) Even function in $(-\pi, \pi)$: If $f(x)$ is even in the interval $(-\pi, \pi)$, then,

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{2\pi}\int_0^{\pi} f(x)\,dx \qquad [\because f(x) \text{ is even}]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \qquad [\because f(x)\cos nx \text{ is even}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = 0 \qquad [\because f(x)\sin nx \text{ is odd}]$$

(ii) Odd function in $(-\pi, \pi)$: If $f(x)$ is odd in the interval $(-\pi, \pi)$, then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = 0 \qquad [\because f(x) \text{ is odd}]$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = 0 \qquad [\because f(x)\cos nx \text{ is odd}]$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx \qquad [\because f(x)\sin nx \text{ is even}]$$

Thus, we have,

If $f(x)$ is even in $(-\pi, \pi)$

$$\boxed{a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad b_n = 0} \qquad \ldots\ldots\ldots\ldots\ldots \text{(A)}$$

If $f(x)$ is odd in $(-\pi, \pi)$

$$\boxed{a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx} \qquad \ldots\ldots\ldots\ldots\ldots \text{(B)}$$

Example 1 : Find the Fourier series of $f(x) = e^{-|x|}$ in the interval $(-\pi, \pi)$.

Sol. : We have $f(x) = e^{-|x|}$.

$\therefore \quad f(-x) = e^{-|-x|} = e^{-|x|} = f(x)$

Hence, $f(x)$ is an even function and hence $b_n = 0$.

Further, $f(x) = \begin{cases} e^{x}, & -\pi < x < 0 \\ e^{-x}, & 0 < x < \pi \end{cases}$

Now, using (A) above,

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx \text{ and } b_n = 0.$$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-x}\,dx = \frac{1}{\pi}\left[-e^{-x}\right]_0^{\pi} = \frac{1}{\pi}(1 - e^{-\pi})$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} e^{-x}\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\frac{e^{-x}}{1+n^2}(-\cos nx + n\sin nx)\right]_0^{\pi} \qquad [\text{By 7(3), page 3-4}]$$

$$= \frac{2}{\pi(1+n^2)}\left[-e^{-\pi}\cos n\pi - (-1)\right]$$

$$\therefore \quad a_n = \frac{2}{\pi(1+n^2)}\left[1 - (-1)^n e^{-\pi}\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$\therefore \quad f(x) = \frac{1}{\pi}(1 - e^{-\pi}) + \frac{2}{\pi}\sum_{1}^{\infty}\left[\frac{1 - (-1)^n e^{-\pi}}{1+n^2}\right]\cdot\cos nx$$

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Example 2 : Find the Fourier expansion of $f(x) = x^2$, $-\pi \le x \le \pi$, and hence, prove that

(i) $\frac{\pi^2}{6} = \sum_1^{\infty} \frac{1}{n^2}$, (ii) $\frac{\pi^2}{12} = \sum_1^{\infty} \frac{(-1)^{n+1}}{n^2}$, (iii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

(M.U. 2011, 13, 14, 19)

Sol. : Here $f(x)$ is an even function because $f(-x) = (-x)^2 = x^2 = f(x)$ $[\therefore b_n = 0]$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ ..... (1)

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2 dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 - (2\pi)\left(-\frac{\cos n\pi}{n^2}\right) + 0\right\} - \left\{0\right\}\right] = \frac{4}{n^2}\cos n\pi = \frac{4}{n^2}(-1)^n$$

Putting these values in (1).

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \quad \text{..... (2)}$$

(i) For the first deduction put $x = \pi$ in (2).

$$\therefore \quad \pi^2 = \frac{\pi^2}{3} + 4\sum \frac{(-1)^n}{n^2}\cos n\pi$$

$$\pi - \frac{\pi^2}{3} = 4\sum \frac{(-1)^n(-1)^n}{n^2} = 4\sum \frac{(-1)^{2n}}{n^2}$$

$$\therefore \quad \frac{2\pi^2}{3}\cdot\frac{1}{4} = \sum \frac{(-1)^{2n}}{n^2} \qquad \therefore \quad \frac{\pi^2}{6} = \sum \frac{1}{n^2}$$

(ii) For the second deduction put $x = 0$ in (1).

$$\therefore \quad 0 = \frac{\pi^2}{3} + 4\sum \frac{(-1)^n}{n^2}\cdot 1 \qquad \therefore \quad -\frac{\pi^2}{3} = 4\sum \frac{(-1)^n}{n^2}$$

$$\therefore \quad \frac{\pi^2}{12} = (-1)\sum \frac{(-1)^n}{n^2} \qquad \therefore \quad \frac{\pi^2}{12} = \sum \frac{(-1)^{n+1}}{n^2}$$

(iii) For the third deduction, add the above two results.

$$\therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \ldots; \qquad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \ldots$$

$$\therefore \quad \frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right) \qquad \therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Cor. Assuming the validity of term by term differentiation of the series for x^2, find the series for x and hence, deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Sol. : We have

$$x^2 = \frac{\pi^2}{3} + 4\sum_1^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\sin 3x + \ldots\ldots\right]$$

Differentiating w.r.t. x

$$2x = 4\left[\frac{1}{1^2}\sin x - \frac{2}{2^2}\sin 2x + \frac{3}{3^2}\sin 3x - \ldots\ldots\right]$$

$$\therefore \quad x = 2\left[\frac{1}{1}\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \ldots\ldots\right]$$

Putting $x = \frac{\pi}{2}$, we get, $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\ldots$

Example 3 : It is given that for $-\pi < x < \pi$,

$$x^2 = \frac{\pi^2}{3} + 4\sum(-1)^n \cdot \frac{\cos nx}{n^2}.$$

By using Parseval's identiyly prove that $\sum \frac{1}{n^4} = \frac{\pi^4}{90}$. **(M.U. 2004, 11)**

Sol. : By Pareseval's identity

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} [f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2)$$

Here, $f(x) = x^2$, $a_0 = \frac{\pi^2}{3}$, $a_n = \frac{4(-1)^n}{n^2}$, $b_n = 0$.

$$\therefore \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} x^4\,dx = \frac{1}{2\pi}\left[\frac{x^5}{5}\right]_{-\pi}^{\pi} = \frac{\pi^4}{5}$$

$$\therefore \quad \frac{\pi^4}{5} = \frac{\pi^4}{9} + \frac{1}{2}\cdot 16\left[\frac{1}{1^4} + \frac{1}{2^4} + \ldots\ldots\right]$$

$$\therefore \quad \pi^4\left(\frac{1}{5} - \frac{1}{9}\right) = 8\left[\frac{1}{1^4} + \frac{1}{2^4} + \ldots\ldots\right]$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\ldots$$

Example 4 : Obtain Fourier expansion of $f(x) = |\cos x|$ in $(-\pi, \pi)$. **(M.U. 2015)**

Sol. : Here $f(x)$ is an even function because $f(-x) = |\cos(-x)| = f(x)$. $[\therefore b_n = 0]$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ ..... (1)

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$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} |\cos x|\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} \cos x\,dx - \int_{\pi/2}^{\pi} \cos x \cdot dx\right]$$

[$\because$ $|\cos x| = \cos x$, for $0 < x < \pi/2$

and $|\cos x| = -\cos x$, for $\pi/2 < x < \pi$.]

$$= \frac{1}{\pi}\left[\{\sin x\}_0^{\pi/2} - \{\sin x\}_{\pi/2}^{\pi}\right]$$

$$\therefore \quad a_0 = \frac{1}{\pi}[\{1-0\} - \{0-1\}] = \frac{2}{\pi} \qquad \text{.....(2)}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} \cos x \cos nx\,dx - \int_{\pi/2}^{\pi} \cos x \cos nx\,dx\right] \qquad \text{.....(3)}$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} 2\cos nx \cos x\,dx - \int_{\pi/2}^{\pi} 2\cos nx \cos x\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi/2} \{\cos(n+1)x + \cos(n-1)x\}\,dx - \int_{\pi/2}^{\pi} \{\cos(n+1)x + \cos(n-1)x\}\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_0^{\pi/2} - \left\{\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} - 0\right\} - \left\{0 - \frac{\sin(n+1)\pi/2}{n+1} - \frac{\sin(n-1)\pi/2}{n-1}\right\}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos n(\pi/2)}{n+1} - \frac{\cos n(\pi/2)}{n-1}\right]$$

$$\therefore \quad a_n = \begin{cases} -\dfrac{4}{\pi(n^2-1)}\cos\left(\dfrac{n\pi}{2}\right) & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd and } n \neq 1 \end{cases} \qquad \text{.....(4)}$$

To find a_1, we put $n = 1$ in (3)

$$\therefore \quad a_1 = \frac{2}{\pi}\left[\int_0^{\pi/2} \cos^2 x\,dx - \int_{\pi/2}^{\pi} \cos^2 x\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x + \frac{\sin 2x}{2}\right\}_0^{\pi/2} - \left\{x + \frac{\sin 2x}{2}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi}{2}\right\} - \left\{\pi - \frac{\pi}{2}\right\}\right] = 0 \qquad \text{.....(5)}$$

Putting these values from (2) and (4) in (1), we get,

$$|\cos x| = \frac{2}{\pi} + \frac{4}{\pi}\left[\frac{\cos 2x}{3} - \frac{\cos 4x}{15} + \ldots\ldots\right]$$

Example 5 : Obtain Fourier Series for the function $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \dfrac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$

Deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots.$ **(M.U. 1995, 97, 2001, 05, 07, 08)**

Sol. : Here $f(x)$ is an even function because

$$f(-x) = \begin{cases} 1 - \dfrac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \dfrac{2x}{\pi}, & 0 \le -x \le \pi \end{cases} = \begin{cases} 1 - \dfrac{2x}{\pi}, & \pi \ge x \ge 0 \quad i.e.,\ 0 \le x \le \pi \\ 1 + \dfrac{2x}{\pi}, & 0 \ge x \ge -\pi \quad i.e.,\ -\pi \le x \le 0 \end{cases}$$

$$= f(x) \quad [\therefore b_n = 0]$$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$\therefore \quad a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cdot dx = \frac{1}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^{\pi} = \frac{1}{\pi}[\pi - \pi] = 0$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(-\frac{2}{\pi}\right)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{\pi}\left[\left\{0 - \frac{2\cos m\pi}{\pi n^2}\right\} - \left\{-\frac{2}{\pi n^2}\right\}\right] = \frac{4}{\pi^2 n^2}[1 - \cos m\pi]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{8}{\pi^2 n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad f(x) = \frac{8}{\pi^2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\ldots\right]$$

Now by putting $x = 0$ in the above result, we can get, $[\because f(0) = 1]$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Example 6 : Obtain the Fourier expansion of

$$f(x) = \begin{cases} \cos x, & -\pi < x < 0 \\ -\cos x, & 0 < x < \pi \end{cases} \text{ and } f(x + 2\pi) = f(x).$$

Sol. : Here $f(x)$ is an odd function because

$$f(-x) = \begin{cases} \cos(-x), & -\pi < -x < 0 \\ -\cos(-x), & 0 < -x < \pi \end{cases} = \begin{cases} \cos x, & \pi > x > 0 \quad i.e.,\ 0 < x < \pi \\ -\cos x, & 0 > x > -\pi \quad i.e.,\ -\pi < x < 0 \end{cases}$$

$$= -f(x) \quad [\therefore a_n = 0]$$

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Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

Now $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin nx\,dx$ ..... (2)

$$= -\frac{1}{\pi}\int_0^{\pi} 2\sin nx\cos x\,dx = -\frac{1}{\pi}\int_0^{\pi}[\sin(n+1)x + \sin(n-1)x]\,dx$$

$$= -\frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} - \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= -\frac{1}{\pi}\left[\frac{1+\cos n\pi}{n+1} + \frac{1+\cos n\pi}{n-1}\right]$$

$$= -\frac{2n}{\pi}\frac{(1+\cos n\pi)}{n^2-1} \quad \text{if } n \neq 1$$

$$\therefore\ b_n = \begin{cases} 0 & \text{if } n \text{ is odd and } \neq 1 \\ -\dfrac{4n}{\pi(n^2-1)} & \text{if } n \text{ is even} \end{cases}$$

Now putting $n = 1$ in (2)

$$b_1 = \frac{2}{\pi}\int_0^{\pi}(-\cos x)\sin x\,dx = -\frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx = -\frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

Putting these values in (1),

$$\therefore\ f(x) = -\frac{4}{\pi}\left[\frac{2}{3}\sin 2x + \frac{4}{15}\sin 4x + \frac{6}{35}\sin 6x + \ldots\ldots\right] = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{2n}{4n^2-1}\sin 2nx$$

Example 7 : Obtain Fourier series for

$$f(x) = \begin{cases} x + \dfrac{\pi}{2}, & -\pi < x < 0 \\ \dfrac{\pi}{2} - x, & 0 < x < \pi \end{cases}$$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **(M.U. 2004, 08, 09, 12, 19)**

Also deduce that $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots$ **(M.U. 1999, 2002)**

Sol. : Clearly $f(x)$ is an even function

$$f(-x) = \begin{cases} -x + \dfrac{\pi}{2}, & -\pi < -x < 0 \\ \dfrac{\pi}{2} + x, & 0 < -x < \pi \end{cases} = \begin{cases} \dfrac{\pi}{2} - x, & \pi > x > 0 \quad i.e.,\ 0 < x < \pi \\ \dfrac{\pi}{2} + x, & 0 < x < -\pi \quad i.e.,\ -\pi < x < 0 \end{cases}$$

$$= f(x) \quad [\therefore b_n = 0]$$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx = \frac{1}{\pi}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^{\pi} = 0 \quad \ldots\ldots (1)$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - (-1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi} \quad \text{[By § 5, page 3-10]}$$

$$\therefore\ a_n = \frac{2}{\pi}\left[-(-1)^n\cdot\frac{1}{n^2} + \frac{1}{n^2}\right] = \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$\therefore\ f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots\right] \quad \ldots\ldots (2)$$

(i) Now $f(x)$ is discontinuous at $x = 0$.

But at a point of discontinuity $x = c$.

$$f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right] \qquad \therefore\ f(0) = \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi}{2}$$

$\therefore$ Putting $x = 0$, in (2),

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right] \qquad \therefore\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

(ii) We now use Parseval's identity in $(-\pi, \pi)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = a_0^2 + \frac{1}{2}\sum(a_n^2 + b_n^2) \quad \ldots\ldots (3)$$

$$\therefore\ \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2\,dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(x + \frac{\pi}{2}\right)^2 dx + \int_0^{\pi}\left(\frac{\pi}{2} - x\right)^2 dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(x^2 + \pi x + \frac{\pi^2}{4}\right)dx + \int_0^{\pi}\left(\frac{\pi^2}{4} - \pi x + x^2\right)dx\right]$$

$$= \frac{1}{2\pi}\left[\left\{\frac{x^3}{3} + \frac{\pi x^2}{2} + \frac{\pi^2 x}{4}\right\}_{-\pi}^{0} + \left\{\frac{\pi^2 x}{4} - \frac{\pi x^2}{2} + \frac{x^3}{3}\right\}_0^{\pi}\right]$$

$$= \frac{1}{2\pi}\left[0 - \left\{-\frac{\pi^3}{3} + \frac{\pi^3}{2} - \frac{\pi^3}{4}\right\} + \left\{\frac{\pi^3}{4} - \frac{\pi^3}{2} + \frac{\pi^3}{3}\right\} - 0\right]$$

$$= \frac{2\pi^3}{2\pi}\left[\frac{1}{4} - \frac{1}{2} + \frac{1}{3}\right] = \frac{\pi^2}{12}$$

Hence, from (1), (2) and (3),

$$\frac{\pi^2}{12} = \frac{1}{2}\cdot\frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots\right]$$

$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots\ldots$$

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Example 8 : Prove that $\sinh ax = \frac{2}{\pi}\sinh a\pi\left[\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{n^2+a^2}\sin nx\right]$ **(M.U. 1998)**

Sol. : Here $f(x)$ is an odd function because

$$f(-x) = \sinh(-ax) = \frac{e^{-ax}-e^{ax}}{2}$$

$$= -\frac{e^{ax}-e^{-ax}}{2} = -\sinh ax = -f(x) \quad [\therefore a_n = 0]$$

$$\therefore \text{ Let } f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sinh ax \sin nx\,dx$$

To find b_n, we use the result

$$\cos(n-ia)x = \cos nx \cos i\,ax + \sin nx \sin i\,ax$$

$$\therefore \quad \cos(n-ia)x = \cos nx \cosh ax + i\sin nx \sinh ax.$$

$$[\because \cos ix = \cosh x,\ \sin ix = i\sinh x]$$

Thus, $\sin nx \sinh ax$ = Imaginary Part (I.P.) of $\cos(n-ia)x$

$$\text{Hence, } b_n = \frac{2}{\pi}\int_0^{\pi}\sin nx \sinh ax\,dx = \frac{2}{\pi}\text{I.P.}\int_0^{\pi}\cos(n-ia)x\,dx$$

$$= \frac{2}{\pi}\text{I.P.}\left[\frac{\sin(n-ia)x}{n-ia}\right]_0^{\pi} = \frac{2}{\pi}\text{I.P.}\frac{\sin(n-ia)\pi}{(n-ia)}$$

$$= \frac{2}{\pi}\text{I.P.}(n+ia)\frac{\sin(n-ia)\pi}{n^2-i^2a^2} = \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{n^2+a^2}\sin(n\pi - ia\pi)$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{(n^2+a^2)}[\sin n\pi\cos ia\pi - \cos n\pi \sin ia\pi]$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{n^2+a^2}[-i\cos n\pi \sinh a\pi]$$

$$= \frac{2}{\pi}\text{I.P.}\frac{(n+ia)}{n^2+a^2}(-i)(-1)^n\sinh a\pi = \frac{2n}{(n^2+a^2)\pi}(-1)^{n+1}\sinh a\pi$$

Alternatively $b_n = \frac{2}{\pi}\int_0^{\pi}\left(\frac{e^{ax}-e^{-ax}}{2}\right)\sin nx\,dx$

$$= \frac{1}{\pi}\left[\frac{e^{ax}}{a^2+n^2}(a\sin x - n\cos nx) - \frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{1}{\pi(a^2+n^2)}\left[\{e^{a\pi}(0-n\cos n\pi) - e^{-a\pi}(0-n\cos n\pi)\} - \{(0-n)-(0-n)\}\right]$$

$$= \frac{n}{\pi(n^2+a^2)}\left[e^{a\pi}(-1)^{n+1} - e^{-a\pi}(-1)^{n+1}\right]$$

$$\therefore \quad b_n = \frac{n}{(n^2+a^2)\pi}\cdot(-1)^{n+1}\left[\frac{e^{a\pi}-e^{-a\pi}}{2}\right]$$

$$\therefore \quad b_n = \frac{2n}{(n^2+a^2)\pi}\cdot(-1)^{n+1}\sinh a\pi \text{ as before.}$$

$$\therefore \quad f(x) = \sum b_n \sin nx = \frac{2}{\pi}\sinh a\pi\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\cdot n}{n^2+a^2}\sin nx.$$

Example 9 : Obtain Fourier Series of $x\cos x$ in $(-\pi, \pi)$. **(M.U. 2003, 09, 16, 19)**

Sol. : Since, $f(-x) = (-x)\cos(-x) = -x\cos x$

$= -f(x)$, $f(x)$ is an odd function $[\therefore a_n = 0]$

Therefore, let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x \sin nx\,dx \quad \text{..... (1)}$$

$$= \frac{1}{\pi}\int_0^{\pi} x\cdot 2\sin nx\cos x\,dx = \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x + \sin(n-1)x]dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x\sin(n+1)x\,dx + \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

$$= \frac{1}{\pi}\left\{\left[x\cdot\frac{-\cos(n+1)x}{n+1} - (1)\cdot\frac{-\sin(n+1)x}{(n+1)^2}\right]_0^{\pi} + \left[x\cdot\frac{-\cos(n-1)x}{n-1} - (1)\cdot\frac{-\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n+1}\cdot\cos(n+1)\pi - \frac{\pi}{n-1}\cos(n-1)\pi\right] \quad [\text{By § 5, page 3-10}]$$

$$= \frac{(-1)^n}{n+1} + \frac{(-1)^n}{n-1} \quad [\because \cos(n\pi \pm \pi) = -\cos n\pi]$$

$$= (-1)^n\left[\frac{1}{n+1} + \frac{1}{n-1}\right] = (-1)^n\cdot\frac{2n}{n^2-1} \text{ if } n \neq 1$$

For $n = 1$, we put $n = 1$ in (1).

$$\therefore \quad b_1 = \frac{2}{\pi}\int_0^{\pi} x\cos x\sin x\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\,dx$$

$$= \frac{1}{\pi}\left[x\cdot\frac{-\cos 2x}{2} - (1)\cdot\frac{-\sin 2x}{4}\right]_0^{\pi} = \frac{1}{\pi}\left[\left(-\frac{\pi}{2}\right)\right] = -\frac{1}{2}$$

$$\text{Hence, } x\cos x = -\frac{1}{2}\sin x + 2\sum_{n=2}^{\infty}(-1)^n\cdot\frac{n}{n^2-1}\sin nx.$$

Example 10 : Find the Fourier expansion of $f(x) = x + x^2$ when $-\pi \le x \le \pi$ and $f(x+2\pi) = f(x)$. **(M.U. 2016)**

Hence, deduce that (i) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ **(M.U. 2011)**

(ii) $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **(M.U. 1996, 2001)**

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Sol. : We first note that $f(x) = x + x^2$ is the sum of the odd function $f_1(x) = x$ and the even function $f_2(x) = x^2$.

Hence, Fourier expansion of $f(x)$ is the sum of the Fourier expansions of $f_1(x)$ and $f_2(x)$.

Now, since $f_1(x) = x$ is an odd function. $[\because a_n = 0]$

Let $f_1(x) = x = \sum_{n=1}^{\infty} b_n \sin nx$ ..... (1)

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - (1)\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{\pi(-1)\frac{(-1)^n}{n} - 0\right\} - \left\{0\right\}\right] = \frac{2(-1)^{n+1}}{n}$$

$$\therefore\ f_1(x) = x = 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\cdot\sin nx \qquad \text{..... (2)}$$

Now, since $f_2(x) = x^2$ is an even function. $[\because b_n = 0]$

Let $f_2(x) = x^2 = a_0 + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\cdot\cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(-\frac{\cos nx}{n^2}\right) + (2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^{\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{\pi}\left[\left\{0 + 2\pi\frac{\cos n\pi}{n^2} - 0\right\} - \left\{0\right\}\right] = 4\cdot\frac{(-1)^n}{n^2}$$

$$\therefore\ f_2(x) = x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx \qquad \text{..... (3)}$$

Form (2) and (3)

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx + 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

For deduction, we consider the series of x^2 only. We have obtained in (3).

$$x^2 = \frac{\pi^2}{3} + 4\sum\frac{(-1)^n}{n^2}\cos nx \qquad \text{..... (4)}$$

(i) Putting $x = \pi$ in (4)

$$\pi^2 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \ldots\ldots\right]$$

$$\frac{2\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots\right]; \qquad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots \qquad \text{..... (5)}$$

(ii) Putting $x = 0$ in (4), $0 = \frac{\pi^2}{3} + 4\left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\ldots\right]$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\ldots \qquad \text{..... (6)}$$

(iii) Adding (5) and (6), $\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots\right)$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots$$

Example 11 : Prove that $\frac{x}{12}(\pi - x)(2\pi - x) = \sum_{n=1}^{\infty}\frac{\sin nx}{n^3}$ where $0 \le x \le 2\pi$.

Sol. : If we put $x = z + \pi$ then when $x = 0$, $z = -\pi$ and when $x = 2\pi$, $z = \pi$. Thus, the interval changes from 0 to 2π to $-\pi$ to π.

$$F(z) = \frac{(z+\pi)}{12}\cdot(\pi - z - \pi)\cdot(2\pi - z - \pi)$$

$$= \frac{(z+\pi)}{12}\cdot(-z)\cdot(\pi - z) = \frac{z}{12}\cdot(z^2 - \pi^2)$$

Now $F(-z) = -\frac{z}{12}\cdot(z^2 - \pi^2) = -F(z)$

$\therefore$ $F(z)$ is an odd function ($\therefore$ $a_n = 0$). Let $F(z) = \sum b_n \sin nz$

$$\therefore\ b_n = \frac{2}{\pi}\int_0^{\pi} F(z)\sin nz\,dz = \frac{2}{\pi}\int_0^{\pi}\frac{1}{12}\cdot(z^3 - \pi^2 z)\sin nz\,dz$$

$$\therefore\ b_n = \frac{1}{6\pi}\left[(z^3 - \pi^2 z)\left(-\frac{\cos nz}{n}\right) - (3z^2 - \pi^2)\left(-\frac{\sin nz}{n^2}\right)\right.$$

$$\left. + (6z)\left(\frac{\cos nz}{n^3}\right) - 6(1)\left(\frac{\sin nz}{n^4}\right)\right]_0^{\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{6\pi}\left[\left\{0 - 0 + \frac{6\pi\cos n\pi}{n^3} - 0\right\} - \left\{0\right\}\right] = \frac{\cos n\pi}{n^3} = \frac{(-1)^n}{n^3}$$

$$\therefore\ F(z) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin nz$$

Putting $z = x - \pi$, we get

$$f(x) = \frac{x}{12}(\pi - x)(2\pi - x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n(x - \pi)$$

$$= \sum\frac{(-1)^n}{n^3}(-1)\sin(n\pi - nx) = \sum\frac{(-1)^n}{n^3}(-1)\{\sin n\pi\cos nx - \cos n\pi\sin nx\}$$

$$= \sum\frac{(-1)^n}{n^3}(-1)\{0 - (-1)^n\sin nx\} = \sum_{n=1}^{\infty}\frac{1}{n^3}\sin nx.$$

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EXERCISE - III

Find the Fourier Series for $f(x)$ where

1. $f(x) = \cos px\ (-\pi, \pi)$, where p is not an integer. Hence, prove that (M.U. 2011)

$$\cot p\pi = \frac{2p}{\pi}\left[\frac{1}{2p^2} - \frac{1}{p^2-1^2} + \frac{1}{p^2-2^2} - \frac{1}{p^2-3^2} + \ldots\right]$$

And deduce that $\cos\theta = \frac{1}{\theta} - \sum_{n=1}^{\infty} \frac{2\theta}{n^2\pi^2 - \theta^2}$.

Also deduce that $\frac{1}{2} - \frac{\pi\sqrt{3}}{18} = \frac{1}{9.1^2-1} + \frac{1}{9.2^2-1} + \frac{1}{9.3^2-1} + \ldots$ (M.U. 1993, 96, 2014)

$$\left[\text{Ans. : } \cos px = \frac{2p\sin p\pi}{\pi}\left[\frac{1}{2p^2} - \frac{\cos x}{p^2-1^2} + \frac{\cos 2x}{p^2-2^2} - \frac{\cos 3x}{p^2-3^2} + \ldots\right]\right.$$

Now put $x = \pi$, $\therefore \cot p\pi = \frac{2p}{\pi}\left[\frac{1}{2p^2} + \frac{1}{p^2-1^2} + \frac{1}{p^2-2^2} + \ldots\right]$

Now, put $p\theta = \pi$ $\therefore \cot\theta = \frac{1}{\theta} - 2\theta\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2-\theta^2}$

Now put $p = \frac{1}{3}$, $\left.\frac{1}{\sqrt{3}} = \frac{2}{3\pi}\left[\frac{9}{2} + \frac{9}{1-9\cdot1^2} + \frac{9}{1-9\cdot2^2} + \frac{9}{1-9\cdot3^2} + \ldots\right]\right]$

2. $f(x) = |\sin x|$ in $(-\pi, \pi)$. (M.U. 2003, 04, 07, 08, 13)

$$\left[\text{Ans. : } f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right]\right]$$

3. $f(x) = \sqrt{1-\cos x}$ in $(-\pi, \pi)$ and hence, deduce that $\frac{1}{2} = \sum_{n=1}^{\infty}\frac{1}{4n^2-1}$.

$$\left[\text{Ans. : } f(x)\frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum\frac{1}{4n^2-1}\cdot\cos nx.\ \text{Then put } x = 0\right]$$

4. $f(x) = |x|$ in $(-\pi, \pi)$. Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots$

(M.U. 2004, 14, 17)

$$\left(\text{Hint : } f(x) = \begin{cases} -x, & -\pi \le x \le 0 \\ x, & 0 \le x \le \pi \end{cases}\right) \quad \left[\text{Ans. : } \frac{\pi}{2} - \frac{4}{2}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]\right]$$

5. $f(x) = \sin ax,\ -\pi < x < \pi$ (M.U. 1997, 2004, 13)

$$\left[\text{Ans. : } f(x) = \frac{2\sin a\pi}{\pi}\left[\frac{\sin x}{1^2-a^2} - \frac{2\sin 2x}{2^2-a^2} + \frac{3\sin 3x}{3^2-a^2} - \ldots\right]\right]$$

6. $f(x) = x^3$, in $(-\pi, \pi)$ (M.U. 2009, 19) $\left[\text{Ans. : } f(x) = \sum_{n=1}^{\infty}(-1)^n\cdot\left(\frac{12}{n^3} - \frac{2\pi^2}{n}\right)\sin nx\right]$

7. $f(x) = x\sin x$ in $(-\pi, \pi)$. Hence, deduce that

$$\frac{\pi-2}{4} = \frac{1}{1\cdot3} - \frac{1}{3\cdot5} + \frac{1}{5\cdot7} - \frac{1}{7\cdot9} + \ldots$$ (M.U. 2000, 02, 06)

$$\left[\text{Ans. : } f(x) = 1 - \frac{1}{2}\cos x - 2\cdot\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2-1}\cos nx\right.$$

i.e. $f(x) = 1 - \frac{1}{2}\cos x - \frac{2}{3}\cos 2x + \frac{2}{8}\cos 3x - \frac{2}{15}\cos 4x + \ldots$

Now, put $x = \pi/2$.]

8. $f(x) = \sin px$ in $(-\pi, \pi)$. (M.U. 1997, 2004)

$$\left[\text{Ans. : } f(x) = \frac{2\sin p\pi}{\pi}\left(\sum_{n=1}^{\infty}(-1)^{n-1}\cdot\frac{n}{n^2-p^2}\cdot\sin nx\right)\right.$$ where p is not an integer]

9. $f(x) = \cos h\,px,\ -\pi < x < \pi$. $\left[\text{Ans. : } f(x) = \frac{2a\sin h\,p\pi}{\pi}\left[\frac{1}{2p^2} + \sum\frac{(-1)^n\cdot\cos nx}{n^2+a^2}\right]\right]$

(Hint : Refer to solved Ex. 6. $a_n = \frac{2}{\pi}\text{R.P.}\int_0^{\pi}\cos(n+ia)x\,dx,\ b_n = 0$.)

10. $f(x) = x - x^2,\ -\pi < x < \pi$. Hence, deduce that, $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$

(M.U. 1998, 2003) $\left[\text{Ans. : } f(x) = -\frac{\pi^2}{3} - 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos nx - 2\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin nx\right]$

11. $f(x) = \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}},\ -\pi < x < \pi$ (M.U. 2003)

(Hint : $f(x) = \frac{\sin h\,ax}{\sin h\,a\pi}$. See Ex. 6) $\left[\text{Ans. : } f(x) = \frac{2}{\pi}\sum\frac{(-1)^{n+1}\cdot n}{n^2+a^2}\cdot\sin nx\right]$

12. $f(x) = \frac{\pi^2}{12} - \frac{x^2}{4}$ in $(-\pi, \pi)$ (M.U. 2011)

$$\left[\text{Ans. : } f(x) = \frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \ldots\right]$$

13. (a) $f(x) = \begin{cases} \pi + x, & -\pi < x < 0 \\ \pi - x, & 0 < x < \pi \end{cases}$ (M.U. 2003)

$$\left[\text{Ans. : } f(x) = \frac{\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right]\right]$$

(b) $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x < 2\pi \end{cases}$ (M.U. 2003, 04)

$$\left(\text{Hint : Put } x = \pi + z \text{ then } f(x) = \begin{cases} \pi + z, & -\pi \le z \le 0 \\ \pi - z, & 0 \le z \le \pi \end{cases}\right)$$

$$\left[\text{Ans. : } \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots\right)\right]$$

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14. $f(x) = \begin{cases} -x, & -\pi < x < 0 \\ x, & 0 < x < \pi \end{cases}$ (M.U. 2003, 04)

$$\left[\text{Ans. : } f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)x\right]$$

15. $f(x) = \frac{x(\pi^2 - x^2)}{12}, \ -\pi < x < \pi.$ (M.U. 2004)

$$\left[\text{Ans. : } f(x) = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} - \ldots\ldots\right]$$

16. $f(x) = \begin{cases} 0, & -\pi < x < 0 \\ \sin x, & 0 < x < \pi \end{cases}$

Also deduce that $\frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \ldots\ldots = \frac{1}{2}$

and $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \ldots\ldots = \frac{\pi - 2}{4}.$ (M.U. 2004, 08)

17. $f(x) = \frac{x(\pi - x)(\pi + x)}{12}$ in $(-\pi, \pi)$. Hence, find $\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\ldots$ (M.U. 2004)

$$\left[\text{Ans. : } f(x) = \sum\frac{(-1)^{n+1}}{n^3}\sin nx. \text{ Then put } x = \frac{\pi}{2}\right]$$

9. Fourier Series in $(c, c + 2l)$

In many engineering problems the period of expansion of a function is not 2π but say $2l$. To apply previous-theory to this interval we change the interval from c to $c + 2l$ to c to $c + 2\pi$ by changing the variable x to z as explained below.

Let $f(x)$ be defined in the interval c to $c + 2l$.

To transform the interval from $2l$ to 2π, we put $x = \frac{lz}{\pi}$ or $z = \frac{\pi x}{l}$.

Now when $x = c$, $z = \frac{\pi c}{l} = d$ say and when $x = c + 2l$, $z = \frac{\pi(c + 2l)}{l} = \frac{\pi c}{l} + 2\pi = d + 2\pi$.

Thus, the function $f(x)$ of period $2l$ in the interval $(c, c + 2l)$ is transformed into the function $f\left(\frac{lz}{\pi}\right) = F(z)$ say of period 2π in the interval $(d, d + 2\pi)$.

Hence, we can write,

$$f(x) = f\left(\frac{lz}{\pi}\right) = F(z) = a_0 + \sum a_n \cos nz + \sum b_n \sin nz \quad \ldots\ldots\ldots\ldots (1)$$

where,

$$a_0 = \frac{1}{2\pi}\int_d^{d+2\pi} F(z)\,dz$$

$$a_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\cos nz\,dz$$

$$b_n = \frac{1}{\pi}\int_d^{d+2\pi} F(z)\sin nz\,dz \quad \ldots\ldots\ldots\ldots (2)$$

Now, in these expressions we make inverse substitution $z = \frac{\pi x}{l}$, $dz = \frac{\pi}{l}dx$, when $z = d$, $x = c$ and when $z = d + 2\pi$, $x = c + 2l$ (as seen above).

Hence, (2) gives

$$\left.\begin{aligned} a_0 &= \frac{1}{2l}\int_c^{c+2l} f(x)\,dx \\ a_n &= \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx \\ b_n &= \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots (3)$$

And (1) becomes

$$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l} \quad \ldots\ldots\ldots\ldots (A)$$

Cor. 1 : If $c = 0$ *i.e.* if the interval is 0 to $2l$ then putting $c = 0$ in (3), the constants are given by

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Cor. 2 : If $c = -l$ *i.e.* if the interval is $-l$ to l, then putting $c = -l$ in (3), the constants are given by

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

10. Fourier Series in the Interval $(0, 2l)$

Example 1 : Find the Fourier expansion of x^2 in $(0, a)$. (M.U. 2004, 05, 07, 19)

Hence, deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots\ldots$

Sol. : Here $2l = a$ $\therefore$ $l = a/2$.

Let $x^2 = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

i.e. $x^2 = a_0 + \sum a_n \cos\frac{2n\pi x}{a} + \sum b_n \sin\frac{2n\pi x}{a}$ $\ldots\ldots\ldots\ldots (1)$

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$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{a}\int_0^a x^2 dx = \frac{1}{a}\left[\frac{x^3}{3}\right]_0^3 = \frac{a^2}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{a}\int_0^a x^2\cdot\cos\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[x^2\left(\frac{a}{2n\pi}\cdot\sin\frac{2n\pi x}{a}\right) - (2x)\left(-\frac{a^2}{4n^2\pi^2}\cdot\cos\frac{2n\pi x}{a}\right) + (2)\left(-\frac{a^3}{8n^3\pi^3}\cdot\sin\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\left\{0 + (2a)\left(\frac{a^2}{4n^2\pi^2}\right) + 0\right\} - \left\{0\right\}\right] = \frac{a^2}{n^2\pi^2} \qquad \text{[By § 5, page 3-10]}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{a}\int_0^a x^2\cdot\sin\frac{2n\pi x}{a}\,dx$$

$$= \frac{2}{a}\left[x^2\left(-\frac{a}{2n\pi}\cdot\cos\frac{2n\pi x}{a}\right) - (2x)\left(-\frac{a^2}{4n^2\pi^2}\cdot\sin\frac{2n\pi x}{a}\right) + 2\left(\frac{a^3}{8n^3\pi^3}\cdot\cos\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \frac{2}{a}\left[\left\{a^2\left(-\frac{a}{2n\pi}\right) - 0 + 2\left(\frac{a^2}{8n^3\pi^3}\right)\right\} - \left\{0 - 0 + 2\left(\frac{a^3}{8n^3\pi^3}\right)\right\}\right]$$

$$= \frac{2}{a}\left[-\frac{a^3}{2n\pi}\right] = -\frac{a^2}{n\pi}$$

Putting these values in (1),

$$\therefore\ x^2 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{2\pi x}{a} + \frac{1}{2^2}\cos\frac{4\pi x}{a} + \ldots\right] - \frac{a^2}{\pi}\left[\frac{1}{1}\sin\frac{2\pi x}{a} + \frac{1}{2}\sin\frac{4\pi x}{a} + \ldots\right]$$

For deduction put $x = 0$ and $x = a$.

$$\therefore\ 0 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad i.e. \quad -\frac{1}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \ldots\ldots\text{(i)}$$

and $$a^2 = \frac{a^2}{3} + \frac{a^2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \quad i.e. \quad \frac{2}{3} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \ldots\ldots\text{(ii)}$$

Adding (i) and (ii), we get,

$$\frac{1}{3} = \frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \therefore\ \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 2 : Find the Fourier expansion of $f(x) = 2x - x^2$, $0 \le x \le 3$ whose period is 3. Also plot the graph of the function. **(M.U. 2005, 13)**

Sol. : Here period $2l = 3$ $\quad\therefore\ l = 3/2$.

Let $$f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$$

$$= a_0 + \sum a_n \cos\frac{2n\pi x}{3} + \sum b_n \sin\frac{2n\pi x}{3} \qquad \ldots\ldots\text{(1)}$$

Now $$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{3}\int_0^3 (2x - x^2)\,dx = \frac{1}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{1}{3}\left[9 - \frac{27}{3}\right] = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{(3/2)}\int_0^3 (2x - x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\cdot\left(\frac{3}{2n\pi}\cdot\sin\frac{2n\pi x}{3}\right)\right.$$

$$\left. -(2 - 2x)\left(-\frac{9}{4n^2\pi^2}\cdot\cos\frac{2n\pi x}{3}\right) + (-2)\left(-\frac{27}{8n^3\pi^3}\cdot\sin\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left\{0 - 4\cdot\frac{9}{4n^2\pi^2}\cos 2n\pi + 0\right\} - \left\{0 + 2\cdot\frac{9}{4n^2\pi^2} + 0\right\}\right]$$

$$= \frac{2}{3}\cdot\frac{9}{4n^2\pi^2}[-4 - 2] = -\frac{9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{(3/2)}\int_0^3 (2x - x^2)\cdot\sin\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left[(2x - x^2)\cdot\left(-\frac{3}{2n\pi}\cdot\cos\frac{2n\pi x}{3}\right)\right.$$

$$\left. -(2 - 2x)\left(-\frac{9}{4n^2\pi^2}\cdot\sin\frac{2n\pi x}{3}\right) + (-2)\left(\frac{27}{8n^3\pi^3}\cdot\cos\frac{2n\pi x}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left\{\frac{9}{2n\pi} - 0 - \frac{27}{4n^3\pi^3}\right\} - \left\{0 - 0 - \frac{27}{4n^3\pi^3}\right\}\right]$$

$$\therefore\ b_n = \frac{2}{3}\left[\frac{9}{2n\pi}\right] = \frac{3}{n\pi}.$$

Putting these values in (1),

$$f(x) = 0 + \sum\left(-\frac{9}{n^2\pi^2}\right)\cos\frac{2n\pi x}{3} + \sum\frac{3}{n\pi}\sin\frac{2n\pi x}{3}$$

$$= -\frac{9}{\pi^2}\sum\frac{1}{n^2}\cos\frac{2n\pi x}{3} + \frac{3}{\pi}\sum\frac{1}{n}\sin\frac{2n\pi x}{3}$$

For the graph we see that $y = 2x - x^2$ is a parabola.

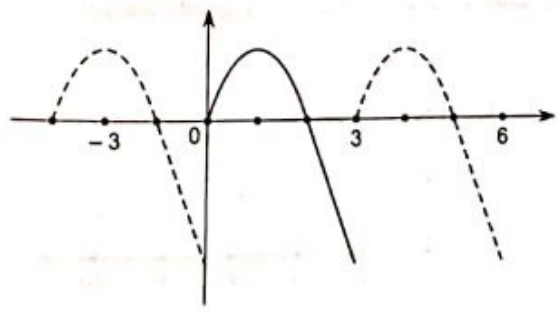
Now $y - 1 = -1 + 2x - x^2 = -(x - 1)^2$

$\therefore\ Y = -X^2$, where $Y = y - 1$ and $X = x - 1$. It opens downwards.

When $x = 0$, $y = 0$; when $x = 2$, $y = 0$, When $x = 3$, $y = -3$.

Since $f(x)$ is periodic with period 3, the graph repeats at 3, 6, ...

Thus, we get, the following graph.



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Example 3 : Find the Fourier expansion of $f(x) = 4 - x^2$ in the interval (0, 2). Graph the function and also state the values of the series for $x = 0, 1, 2, 10, 11$. **(M.U. 2002, 04)**

Hence, deduce that $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ **(M.U. 2004, 14)**

Sol. : Here $2l = 2 \quad \therefore \; l = 1.$

Let $f(x) = a_0 + \sum a_n \cos\frac{m\pi x}{l} + \sum b_n \sin\frac{m\pi x}{l}$

i.e. $f(x) = a_0 + \sum a_n \cos m\pi x + \sum b_n \sin m\pi x$ ..... (1)

$$\therefore \; a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\int_0^2 (4 - x^2)\,dx$$

$$= \frac{1}{2}\left[4x - \frac{x^3}{3}\right]_0^2 = \frac{1}{2}\left[8 - \frac{8}{3}\right] = \frac{8}{3}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{m\pi x}{l}\,dx = \int_0^2 (4 - x^2)\cos m\pi x\,dx$$

$$= \left[(4 - x^2)\left(\frac{\sin m\pi x}{m\pi}\right) - (-2x)\left(-\frac{\cos m\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3}\right)\right]_0^2$$

$$= \left[\left\{0 - \frac{4}{n^2\pi^2} + 0\right\} - \left\{0 - 0 - 0\right\}\right] = -\frac{4}{n^2\pi^2} \qquad \text{[By § 5, page 3-10]}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{m\pi x}{l}\,dx = \int_0^2 (4 - x^2)\sin m\pi x\,dx$$

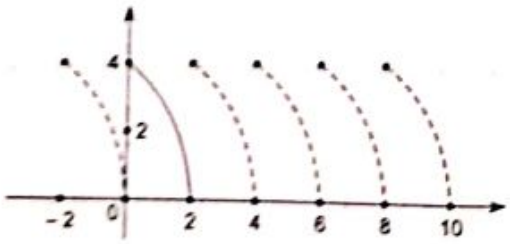
$$= \left[(4 - x^2)\left(-\frac{\cos m\pi x}{m\pi}\right) - (-2x)\left(-\frac{\sin m\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos m\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$\therefore \; b_n = \left[\left\{0 - 0 - \frac{2}{n^3\pi^3}\right\} - \left\{-\frac{4}{m\pi} - 0 - \frac{2}{n^3\pi^3}\right\}\right] = \frac{4}{m\pi}.$$

$$\therefore \; f(x) = 4 - x^2 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \ldots\right]$$
$$+ \frac{4}{\pi}\left[\frac{1}{1}\sin\pi x + \frac{1}{2}\sin 2\pi x + \frac{1}{3}\sin 3\pi x + \ldots\right]$$

Now $y = 4 - x^2$ *i.e.* $y - 4 = -x^2$ *i.e.* $Y = -X^2$, where $Y = y - 4$ and $X = x$ is a parabola with vertex at (0, 4) and opening downwards as shown below.

When $x = 0$, $y = 4$; when $x = 2$, $y = 0$. Since $f(x)$ is periodic with period 2, the graph repeats at 2, 4, 6, ...



Since, $f(x)$ is discontinuous at $x = 0, 2, 4, 6 \ldots$, we find its value as follows.

$$f(c) = \frac{\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)}{2}$$

Now, for $c = 0$, from the graph

$$f(0) = \frac{\lim_{x \to 0^-} f(x) + \lim_{x \to 0^+} f(x)}{2} = \frac{0 + 4}{2} = 2$$

$$f(2) = \frac{\lim_{x \to 2^-} f(x) + \lim_{x \to 2^+} f(x)}{2} = \frac{4 + 0}{2} = 2$$

Similarly, $f(4) = f(6) = f(8) = f(10) \ldots\ldots = 2$, since $f(x)$ is periodic with period 2.

Now, at $x = 1$, the function is continuous $\quad \therefore \; f(1) = 4 - (1) = 3$

Also at $x = 11$, the function is continuous $\quad \therefore \; f(11) = 3.$

Thus, we have $f(1) = 3$, $f(2) = 2$, $f(10) = 2$, $f(11) = 3$.

For deduction put $x = 0$ and $x = 2$.

$$\therefore \; 4 - 0 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$$

i.e. $\frac{1}{3} = -\frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$ ..... (i)

and $4 - 4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$

i.e. $-\frac{2}{3} = -\frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right]$ ..... (ii)

Adding (i) and (ii), we get,

$$-\frac{1}{3} = -\frac{2}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right] \qquad \therefore \; \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

Example 4 : Expand $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ period 2 into a Fourier Series.

(M.U. 1998, 2002, 14)

Sol. : Here $2l = 2 \quad \therefore \; l = 1$

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

$$a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 0\,dx\right] = \frac{1}{2}\left[\frac{\pi x^2}{2}\right]_0^1 = \frac{\pi}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{1}\left[\int_0^1 \pi x\cos\frac{n\pi x}{1}\,dx + \int_1^2 0 \cdot \cos\frac{n\pi x}{l}\,dx\right]$$

$$= \left[\pi x\left(\frac{1}{m\pi}\sin m\pi x\right) - (\pi)\left(-\frac{1}{n^2\pi^2}\cos m\pi x\right)\right]_0^1 \qquad \text{[By § 5, page 3-10]}$$

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$$\therefore \quad a_n = \pi \cdot \frac{1}{n^2\pi^2}\left[(-1)^n - 1\right] = -\frac{1}{\pi}\left[\frac{1-(-1)^n}{n^2}\right]$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{1}\left[\int_0^1 \pi x \sin\frac{n\pi x}{1}\,dx + \int_1^2 0\cdot\sin\frac{n\pi x}{1}\,dx\right]$$

$$= \left[\pi x\left(-\frac{1}{n\pi}\cos n\pi x\right) - (\pi)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1 = -\frac{1}{n}(-1)^n \qquad \text{[By § 5, page 3-10]}$$

$$\therefore \quad f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right]$$

$$-\left[-\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} - \frac{\sin 3\pi x}{3} + \ldots\right]$$

$$= \frac{\pi}{4} - \frac{2}{\pi}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5\pi x + \ldots\right]$$

$$+\left[\frac{1}{1}\cdot\sin\pi x - \frac{1}{2}\cdot\sin 2\pi x + \frac{1}{3}\cdot\sin 3\pi x - \ldots\right]$$

Example 5 : If $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2-x), & 1 \le x \le 2 \end{cases}$ with period 2, show that

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x \qquad \textbf{(M.U. 2003, 04, 07, 09, 12, 14)}$$

Sol. : Here $2l = 2 \quad \therefore \; l = 1$

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

i.e. $f(x) = a_0 + \sum a_n \cos n\pi x + \sum b_n \sin n\pi x$ ..... (1)

$$\therefore \quad a_0 = \frac{1}{2l}\int_0^{2l} f(x)\,dx = \frac{1}{2}\left[\int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx\right]$$

$$= \frac{1}{2}\left[\pi\left\{\frac{x^2}{2}\right\}_0^1 + \pi\left\{2x - \frac{x^2}{2}\right\}_1^2\right]$$

$$= \frac{\pi}{2}\left[\left\{\frac{1}{2}\right\} + \left\{4 - 2 - 2 + \frac{1}{2}\right\}\right] = \frac{\pi}{2}.$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \int_0^1 \pi x\cdot\cos n\pi x\,dx + \int_1^2 \pi(2-x)\cos n\pi x\,dx$$

$$= \pi\left[\left\{x\left(\frac{\sin n\pi x}{n\pi}\right) - (1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\left\{\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right\} + \left\{-\frac{1}{n^2\pi^2} + \frac{\cos n\pi}{n^2\pi^2}\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$\therefore \quad a_n = \frac{2\pi}{n^2\pi^2}\left[\cos n\pi - 1\right] = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ -\dfrac{4}{n^2\pi} & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^1 \pi x\cdot\sin n\pi x\,dx + \int_1^2 \pi(2-x)\sin n\pi x\,dx$$

$$= \pi\left[\left\{x\left(-\frac{\cos n\pi x}{n\pi}\right) - (1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_0^1 + \left\{(2-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right\}_1^2\right]$$

$$= \pi\left[\left\{-\frac{\cos n\pi}{n\pi}\right\} + \left\{+\frac{\cos n\pi}{n\pi}\right\}\right] = 0$$

Putting these values in (1)

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right]$$

$$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}\cos(2n+1)\pi x$$

EXERCISE - IV

Obtain the Fourier expansions of the following functions

1. $f(x) = 2 - \frac{x^2}{2}$ in $0 \le x \le 2$ **(M.U. 2003)**

$$\left[\textbf{Ans. :}\; f(x) = \frac{4}{3} - \frac{2}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{n^2}\cos n\pi x + \frac{2}{\pi}\cdot\sum\frac{1}{n}\cdot\sin n\pi x\right]$$

2. $f(x) = \frac{a}{2} - x, \; 0 < x < a$

$$\left[\textbf{Ans. :}\; f(x) = \frac{a}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\cdot\sin\frac{2n\pi x}{a}\right]$$

3. $f(x) = l - x, \quad 0 < x < l$
   $\quad\;\; = 0, \quad\quad l < x < 2l.$

Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$

$$\left[\textbf{Ans. :}\; f(x) = l\left[\frac{1}{4} + \frac{2}{\pi^2}\left(1 + \cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right)\right.\right.$$

$$\left.\left. + \frac{1}{\pi}\left(\frac{1}{l}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\right)\right]$$

Then put $x = l/2$.]

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4. $f(x) = 1, \quad 0 < x < 1$
$\quad\;\; = x, \quad 1 < x < 2$ **(M.U. 2007)**

$$\left[\textbf{Ans.}: f(x) = \frac{5}{4} - \frac{4}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} - \frac{1}{4^2}\cos\frac{4\pi x}{2} + \ldots\right]\right]$$

5. $f(x) = c, \quad 0 < x < a$
$\quad\;\; = 0, \quad a < x < l$

$$\left[\textbf{Ans.}: f(x) = \frac{ca}{l} + \frac{2c}{\pi}\left[\sin\left(\frac{\pi a}{l}\right)\cos\left(\frac{\pi x}{l}\right) + \frac{1}{2}\sin\left(\frac{2\pi a}{l}\right)\cos\left(\frac{2\pi x}{l}\right) + \ldots\right]\right]$$

6. (a) $f(x) = \begin{cases} kx, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 1998)** (b) $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ **(M.U. 2002)**

$$\left[\textbf{Ans.}: \text{(a) } f(x) = \frac{k}{4} - \frac{2k}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \ldots\right) + \frac{k}{\pi}\left(\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \frac{\sin 3\pi x}{3} + \ldots\right)\right.$$

(b) In (a) put $k = \pi$.

$$\left. f(x) = \frac{\pi}{4} - \frac{2}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \ldots\right) + \left(\frac{\sin\pi x}{1} + \frac{\sin 2\pi x}{2} + \ldots\right)\right]$$

7. $f(x) = \begin{cases} x, & 0 < x < 1 \\ 1 - x, & 1 < x < 2 \end{cases}$ **(M.U. 2013)**

$$\left[\textbf{Ans.}: f(x) = -\frac{4}{\pi^2}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right) + \frac{2}{\pi}\left(\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \ldots\right)\right]$$

8. $f(x) = \begin{cases} x, & 0 < x < c/2 \\ c - x, & c/2 < x < c \end{cases}$

$$\left[\textbf{Ans.}: f(x) = \frac{4c}{\pi^2}\left[\frac{1}{1^2}\sin\left(\frac{\pi x}{c}\right) - \frac{1}{3^2}\sin\left(\frac{3\pi x}{c}\right) + \frac{1}{5^2}\sin\left(\frac{5\pi x}{c}\right) - \ldots\right]\right]$$

9. $f(x) = \begin{cases} \pi x, & 0 < x < 1 \\ 0, & x = 1 \\ \pi(x - 2), & 1 < x < 2 \end{cases}$

Hence, show that $\frac{\pi}{4} = \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7}$. **(M.U. 1997, 2002, 03, 08)**

$$\left[\textbf{Ans.}: f(x) = \frac{\pi}{4} + 2\sum_{n=1}^{\infty}(-1)^{n+1}\sin n\pi x. \quad \text{Then put } x = \frac{1}{2}\right]$$

10. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 \le x \le 2 \\ 0, & x > 2 \end{cases}$

(**Hint :** In example 8 above put $c = 2$)

$$\left[\textbf{Ans.}: f(x) = \frac{8}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} - \ldots\right)\right]$$

11. $f(x) = \begin{cases} -x, & 0 < x < l \\ 0, & l \le x < 2l \end{cases}$

Hence, deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

$$\left[\textbf{Ans.}: f(x) = -\frac{l}{4} + \frac{2l}{\pi^2}\left[\frac{1}{1}\cos\frac{\pi x}{l} + \frac{1}{3}\cos\frac{3\pi x}{l} + \ldots\right] - \frac{l}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} - \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l}\right]\right.$$

Then put $x = 0$.]

12. $f(x) = \begin{cases} \pi x, & 0 \le x \le 1 \\ \pi(2 - x), & 1 \le x \le 2 \end{cases}$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$. **(M.U. 2004, 07)**

$$\left[\textbf{Ans.}: f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum\left\{\frac{1}{(2n-1)^2}\cos(2n-1)\pi x\right\}\right]$$

13. $f(x) = \begin{cases} 3kx/l, & 0 < x < (l/3) \\ 3k(l - 2x)/l, & (l/3) < x < (2l/3) \\ 3k(x - l)/l, & (2l/3) < x < l \end{cases}$ **(M.U. 2003)**

$$\left[\textbf{Ans.}: \frac{9k}{\pi^2}\sum\frac{1}{n^2}\sin\frac{2n\pi}{3}\cdot\sin\frac{2n\pi x}{l}\right]$$

11. Fourier Series in the Interval $(-l, l)$

Example 1 : Find the Fourier expansion of $f(x) = e^{ax}$ in $(-l, l)$.

Sol. : Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$ ..... (1)

$$\therefore \quad a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2l}\int_{-l}^{l} e^{ax}\,dx = \frac{1}{2l}\left[\frac{e^{ax}}{a}\right]_{-l}^{l}$$

$$= \frac{1}{2al}\left[e^{al} - e^{-al}\right] = \frac{1}{al}\left[\frac{e^{al} - e^{-al}}{2}\right] = \frac{\sinh al}{al}$$

$$\therefore \quad a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} e^{ax}\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{1}{a^2 + n^2\pi^2/l^2}\left\{e^{ax}\left(a\cos\frac{n\pi x}{l} + \frac{n\pi}{l}\sin\frac{n\pi x}{l}\right)\right\}\right]_{-l}^{l} \quad \text{[By (3), page 3-4]}$$

$$= \frac{l}{a^2l^2 + n^2\pi^2}\left[e^{al}(a\cos n\pi + 0) - e^{-al}(a\cos n\pi + 0)\right]$$

$$= \frac{al}{a^2l^2 + n^2\pi^2}\left[e^{al}(-1)^n - e^{-al}(-1)^n\right] \quad [\because \cos n\pi = (-1)^n]$$

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$$\therefore \quad a_n = \frac{2al(-1)^n}{a^2l^2+n^2\pi^2}\left[\frac{e^{al}-e^{-al}}{2}\right] = \frac{2al(-1)^n}{a^2l^2+n^2\pi^2}\sinh al.$$

$$\therefore \quad b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{l}\int_{-l}^{l} e^{ax}\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\left[\frac{1}{a^2+n^2\pi^2/l^2}\left\{e^{ax}\left(a\sin\frac{n\pi x}{l}-\frac{n\pi}{l}\cos\frac{n\pi x}{l}\right)\right\}\right]_{-l}^{l} \qquad \text{[By (2), page 3-4]}$$

$$= \frac{l}{a^2l^2+n^2\pi^2}\left[e^{al}\left(0-\frac{n\pi}{l}\cos n\pi\right)-e^{-al}\left(0-\frac{n\pi}{l}\cos n\pi\right)\right]$$

$$= \frac{l}{a^2l^2+n^2\pi^2}\left[e^{al}\left(0-\frac{n\pi}{l}(-1)^n\right)-e^{-al}\left(0-\frac{n\pi}{l}(-1)^n\right)\right] \qquad [\because \cos np = (-1)^n]$$

$$= \frac{l}{a^2l^2+n^2\pi^2}\cdot\frac{n\pi}{l}\cdot(-1)^{n+1}(e^{al}-e^{-al})$$

$$= \frac{2n\pi(-1)^{n+1}}{a^2l^2+n^2\pi^2}\cdot\left(\frac{e^{al}-e^{-al}}{2}\right) = \frac{2n\pi(-1)^{n+1}}{a^2l^2+n^2\pi^2}\cdot\sinh al.$$

Putting these values in (1).

$$f(x) = \frac{\sinh al}{al} + \sum\frac{2al(-1)^n}{a^2l^2+n^2\pi^2}\sinh al\cos\frac{n\pi x}{l}$$

$$+\sum\frac{2n\pi(-1)^{n+1}}{a^2l^2+n^2\pi^2}\cdot\sinh al\sin\frac{n\pi x}{l}$$

$$= \frac{\sinh al}{al} + 2al\sinh al\sum\frac{(-1)^n}{a^2l^2+n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$+2\pi\sinh al\sum\frac{(-1)^{n+1}}{a^2l^2+n^2\pi^2}\sin\frac{n\pi x}{l}.$$

Example 2 : Find the Fourier expansion of $f(x) = \begin{cases} 2, & -2 < x < 0 \\ x, & 0 < x < 2 \end{cases}$ **(M.U. 2009)**

Sol. : Comparing the interval $(-2, 2)$ with $(-l, l)$ we find that here $l = 2$.

Let $f(x) = a_0 + \sum a_n\cos\frac{n\pi x}{l} + \sum b_n\sin\frac{n\pi x}{l}$

$$= a_0 + \sum a_n\cos\frac{n\pi x}{2} + \sum b_n\sin\frac{n\pi x}{2} \qquad \text{..... (1)}$$

$$\therefore \quad a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{4}\int_{-2}^{2} f(x)\,dx = \frac{1}{4}\left[\int_{-2}^{0} 2\,dx + \int_{0}^{2} x\,dx\right]$$

$$= \frac{1}{4}\left[\left\{2x\right\}_{-2}^{0} + \left\{\frac{x^2}{2}\right\}_{0}^{2}\right] = \frac{1}{4}[\{0+4\}+\{2-0\}] = \frac{3}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\cos\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\cos\frac{n\pi x}{2}\,dx\right]$$

$$\therefore \quad a_n = \frac{1}{2}\left[\left\{2\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right\}_{-2}^{0} + \left\{(x)\cdot\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right)-(1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right\}_{0}^{2}\right]$$

$$= \frac{1}{2}\left[\left\{0-0\right\}+\left\{0+\frac{4}{n^2\pi^2}\cos n\pi - \frac{4}{n^2\pi^2}\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{n^2\pi^2}\left\{\cos n\pi - 1\right\} = \begin{cases} 0, & \text{when } n \text{ is even} \\ -4, & \text{when } n \text{ is odd} \end{cases} \qquad [\because \cos n\pi = (-1)^n]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}\,dx$$

$$= \frac{1}{2}\left[\int_{-2}^{0} 2\sin\frac{n\pi x}{2}\,dx + \int_{0}^{2} x\sin\frac{n\pi x}{2}\,dx\right]$$

$$= \frac{1}{2}\left[\left\{2\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\right\}_{-2}^{0} + \left\{(x)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)-(1)\left(-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\right\}_{0}^{2}\right]$$

$$= \frac{1}{2}\left[\left\{-\frac{4}{n\pi}+\frac{4}{n\pi}\cos n\pi\right\}+\left\{(2)\left(-\frac{2}{n\pi}\cos n\pi\right)+0\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{2}\left\{-\frac{4}{n\pi}+\frac{4}{n\pi}\cos n\pi - \frac{4}{n\pi}\cos n\pi\right\} = -\frac{2}{n\pi}$$

Putting these values in (1)

$$f(x) = \frac{3}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2}+\frac{1}{3^2}\cos\frac{3\pi x}{2}+\frac{1}{5^2}\cos\frac{5\pi x}{2}+\ldots\ldots\right]$$

$$-\frac{2}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{2}+\frac{1}{2}\sin\frac{2\pi x}{2}+\frac{1}{3}\sin\frac{3\pi x}{2}+\ldots\ldots\right]$$

Example 3 : Expand $f(x) = \begin{cases} 0, & -c < x < 0 \\ a, & 0 < x < c \end{cases}$ in a Fourier series of period $2a$. **(M.U. 2002)**

Sol. : Let $f(x) = a_0 + \sum a_n\cos\frac{n\pi x}{l} + \sum b_n\sin\frac{n\pi x}{l}$ and $l = c$ ..... (1)

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{1}{2c}\int_{-c}^{c} f(x)\,dx = \frac{1}{2c}\left[\int_{-c}^{0} 0\cdot dx + \int_{0}^{c} a\,dx\right]$$

$$= \frac{1}{2c}\left[0+\left[ax\right]_0^c\right] = \frac{1}{2c}\cdot ac = \frac{a}{2}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{c}\left[\int_{-c}^{0} 0\cdot\cos\frac{n\pi x}{c}\,dx + \int_{0}^{c} a\cos\frac{n\pi x}{c}\,dx\right]$$

$$= \frac{1}{c}\left[a\left(\frac{c}{n\pi}\sin\frac{n\pi x}{c}\right)\right]_0^c = \frac{a}{n\pi}[0-0] = 0$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi x}{l}\,dx = \frac{1}{c}\left[\int_{-c}^{0} 0\cdot\sin\frac{n\pi x}{c}\,dx + \int_{0}^{c} a\sin\frac{n\pi x}{c}\,dx\right]$$

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$$\therefore \quad b_n = \frac{1}{c}\left[a\left(-\frac{c}{n\pi}\right)\cos\frac{n\pi x}{c}\right]_0^c$$

$$= -\frac{a}{n\pi}\left\{\cos n\pi - 1\right\} = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{2a}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

Putting these values in (1)

$$f(x) = \frac{a}{2} + \frac{2a}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{c} + \frac{1}{3}\sin\frac{3\pi x}{c} + \frac{1}{5}\sin\frac{5\pi x}{c} + \ldots\right]$$

Example 4 : A sinusoidal voltage $E \sin \omega x$ is passed through a half-wave rectifier which clips of the wave and the resulting function is given by

$$f(x) = \begin{cases} 0, & -\dfrac{\pi}{\omega} < x < 0 \\ E\sin\omega x, & 0 < x < \dfrac{\pi}{\omega} \end{cases}$$

with period $\frac{2\pi}{\omega}$. Find the Fourier expansion of $f(x)$.

Sol. : Here $2l = \frac{2\pi}{\omega} \quad \therefore \; l = \frac{\pi}{\omega}$

Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} + \sum b_n \sin\frac{n\pi x}{l}$

$$= a_0 + \sum a_n \cos n\omega x + \sum b_n \sin n\omega x \qquad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)dx = \frac{1}{(2\pi/\omega)}\left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_0^{\pi/\omega} E\sin\omega x\, dx\right]$$

$$= \frac{\omega}{2\pi}\left[-\frac{E\cos\omega x}{\omega}\right]_0^{\pi/\omega} = \frac{-E}{2\pi}[-1-1] = \frac{E}{\pi}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(x)\cos n\omega x\, dx$$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_0^{\pi/\omega} E\sin\omega x \cdot \cos n\omega x\, dx\right]$$

$$= \frac{E\omega}{2\pi}\int_0^{\pi/\omega}\{\sin(1+n)\omega x + \sin(1-n)\omega x\}dx$$

$$= \frac{E\omega}{2\pi}\left[\left\{-\frac{\cos(1+n)\omega x}{(1+n)\omega} - \frac{\cos(1-n)\omega x}{(1-n)\omega}\right\}_0^{\pi/\omega}\right]$$

$$= \frac{E\omega}{2\pi}\left[-\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right]$$

$$= \frac{E}{2\pi}\left[\frac{\cos n\pi}{1+n} + \frac{\cos n\pi}{1-n} + \frac{1}{1+n} + \frac{1}{1-n}\right] \qquad [\because \cos(1\pm n)\pi = -\cos n\pi]$$

$$\therefore \quad a_n = \frac{E}{2\pi}(1+\cos n\pi)\left[\frac{1}{1+n} + \frac{1}{1-n}\right] = \frac{E}{2\pi}\left[1+(-1)^n\right]\cdot\frac{2}{1-n^2}$$

$$= \begin{cases} \dfrac{2E}{\pi}\cdot\dfrac{1}{1-n^2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd and } n \neq 1 \end{cases}$$

When $n = 1$, we have

$$a_1 = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{\pi x}{c}dx = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(x)\cos\omega x\, dx$$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_0^{\pi/\omega} E\sin\omega x \cdot \cos\omega x\, dx\right]$$

$$= \frac{\omega E}{2\pi}\int_0^{\pi/\omega}\sin 2\omega x\, dx = \frac{\omega E}{2\pi}\left[-\frac{\cos 2\omega x}{2\pi}\right]_0^{\pi/\omega} = -\frac{\omega E}{2\pi}[1-1] = 0$$

Now, $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx = \frac{\omega}{\pi}\int_{-l}^{l} f(x)\sin n\omega x\, dx$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_0^{\pi/\omega} E\sin\omega x \sin n\omega x\, dx\right]$$

$$= -\frac{E\omega}{2\pi}\int_0^{\pi/\omega}[\cos(1+n)\omega x - \cos(1-n)\omega x]dx$$

$$= -\frac{E\omega}{2\pi}\left[\frac{\sin(1+n)\omega x}{(1+n)\omega} - \frac{\sin(1-n)\omega x}{(1-n)\omega}\right]_0^{\pi/\omega}$$

$$= -\frac{E}{2\pi}\left[\frac{\sin(1+n)\pi}{1+n} - \frac{\sin(1-n)\pi}{1-n} - 0\right]$$

$$= 0 \quad \text{and} \quad n \neq 1 \qquad [\because \sin(1\pm n)\pi = 0]$$

When $n = 1$, we have

$$b_1 = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi x}{l}dx = \frac{\omega}{\pi}\int_{-\pi/\omega}^{\pi/\omega} f(x)\sin\omega x\, dx$$

$$= \frac{\omega}{\pi}\left[\int_{-\pi/\omega}^{0} 0 \cdot dx + \int_0^{\pi/\omega} E\sin\omega x \cdot \sin\omega x\, dx\right]$$

$$= \frac{\omega E}{\pi}\int_0^{\pi/\omega}\sin^2\omega x\, dx = \frac{\omega E}{\pi}\int_0^{\pi/\omega}\left(\frac{1-\cos 2\omega x}{2}\right)dx$$

$$= \frac{\omega E}{2\pi}\left[x - \frac{\sin 2\omega x}{2}\right]_0^{\pi/\omega} = \frac{\omega E}{2\pi}\left[\frac{\pi}{\omega}\right] = \frac{E}{2}$$

Hence, putting these values in (1),

$$f(x) = \frac{E}{\pi} - \frac{2E}{\pi}\left[\frac{1}{(2^2-1)}\cos 2\omega x + \frac{1}{(4^2-1)}\cos 4\omega x + \frac{1}{(6^2-1)}\cos 6\omega x + \ldots\right] + \frac{E}{2}\sin\omega x$$

$$= \frac{E}{\pi} - \frac{2E}{\pi}\sum_{n=1}^{\infty}\frac{1}{(4n^2-1)}\cos 2n\omega x + \frac{E}{2}\sin\omega x$$

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EXERCISE - V

Find the Fourier Series of the following functions.

1. $f(x) = \begin{cases} 0, & -5 < x < 0 \\ 3, & 0 < x < 5 \end{cases}$ period 10. **(M.U. 2003)**

$$\left[\textbf{Ans.}: f(x) = \frac{3}{2} + \frac{6}{\pi}\left[\frac{1}{1}\cdot\sin\frac{\pi x}{5} + \frac{1}{3}\cdot\sin\frac{3\pi x}{5} + \frac{1}{5}\cdot\sin\frac{5\pi x}{5} + \ldots\right]\right]$$

(**Hint** : See solved Ex. 3 above. Here $a = 3$, $c = 5$.)

2. $f(x) = \begin{cases} 2x, & 0 \le x < 3 \\ 0, & -3 < x < 0 \end{cases}$ period 6.

$$\left[\textbf{Ans.}: f(x) = \frac{3}{2} - \frac{12}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3} + \frac{1}{3^2}\cos\frac{3\pi x}{3} + \frac{1}{5^2}\cos\frac{5\pi x}{3} + \ldots\right] + \frac{1}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{3} - \frac{1}{2}\sin\frac{2\pi x}{3} + \frac{1}{3}\sin\frac{3\pi x}{3} - \ldots\right]\right]$$

3. $f(x) = \begin{cases} x, & -1 < x < 0 \\ x+2, & 0 < x < 1 \end{cases}$ period 2. $\left[\textbf{Ans.}: f(x) = 1 + \frac{2}{\pi}\sum_{n=1}^{\infty}\left[1 - 2(-1)^n\right]\sin n\pi x\right]$

(M.U. 2004, 19)

4. $f(x) = \begin{cases} l, & -l < x < 0 \\ x, & 0 < x < l \end{cases}$ $\left[\textbf{Ans.}: f(x) = \frac{3l}{4} - \frac{2l}{\pi^2}\left[\frac{1}{1^2}\cos\left(\frac{\pi x}{l}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{l}\right) + \ldots\right] - \frac{1}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \ldots\right]\right]$

5. $f(x) = \begin{cases} -x, & -1 < x < 0 \\ x, & 0 < x < 1 \end{cases}$ **(M.U. 2004, 10)** $\left[\textbf{Ans.}: 1 - \frac{4}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos n\pi x\right]$

6. $f(x) = |x|$, $-2 < x < 2$. Hence, deduct that $\sum\frac{1}{(2n-1)^4} = \frac{\pi^4}{96}$. **(M.U. 2001, 03, 10, 19)**

$$\left[\textbf{Ans.}: 1 - \frac{8}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)\pi x}{2}\right]\right]$$

12. Even and Odd Functions in the Interval $(-l, l)$

(i) **Even function in $(-l, l)$** : If $f(x)$ is even in the interval $(-l, l)$ then,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = \frac{2}{2l}\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l f(x)\,dx \qquad [\because f(x) \text{ is even}]$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx \qquad \left[\because f(x)\cos\frac{n\pi x}{l} \text{ is even}\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = 0 \qquad \left[\because f(x)\sin\frac{n\pi x}{l} \text{ is odd}\right]$$

(ii) **Odd function in $(-l, l)$** : If $f(x)$ is odd in the interval $(-l, l)$ then,

$$a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx = 0 \qquad [\because f(x) \text{ is odd}]$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\,dx = 0 \qquad \left[\because f(x)\cos\frac{n\pi x}{l} \text{ is odd}\right]$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \qquad \left[\because f(x)\sin\frac{n\pi x}{l} \text{ is even}\right]$$

Thus, we have

If $f(x)$ is even in $(-l, l)$

$$\boxed{a_0 = \frac{1}{l}\int_0^l f(x)\,dx,\ a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx,\ b_n = 0} \qquad \ldots\ldots\ldots\ldots \textbf{(A)}$$

If $f(x)$ is odd in $(-l, l)$

$$\boxed{a_0 = 0,\ a_n = 0,\ b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx} \qquad \ldots\ldots\ldots\ldots \textbf{(B)}$$

Example 6 : Obtain the Fourier expansion of x^2 from $x = -l$ to $x = l$ and hence, deduce that

(i) $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^2} + \ldots$ (ii) $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \ldots$

(M.U. 1995, 2004, 06)

Sol. : Since, $f(x) = x^2$ is an even function, we have $b_n = 0$.

Now, $$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l x^2\,dx = \frac{1}{l}\left[\frac{x^3}{3}\right]_0^l = \frac{l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l x^2\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[x^2\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\frac{2l^3}{n^2\pi^2}(-1)^n\right] = (-1)^n\frac{4l^2}{n^2\pi^2} \qquad [\text{By § 5, page 3-10}]$$

$$\therefore\ x^2 = \frac{l^2}{3} + \sum(-1)^n\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

$$\therefore\ x^2 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} - \frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right] \qquad \ldots\ldots\ldots\ldots (1)$$

Now, put $x = 0$, in (1) we get

$$\therefore\ 0 = \frac{l^2}{3} - \frac{4l^2}{\pi^2}\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots\right]$$

$$\therefore\ \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots$$

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(ii) Putting $x = l$, in (1) we get

$$\frac{2l^2}{3} = -\frac{4l^2}{\pi^2}\left[-\frac{1}{1^2}-\frac{1}{2^2}-\frac{1}{3^2}-\frac{1}{4^2}-\ldots\ldots\right]$$

$$\therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\ldots\ldots$$

Example 2 : Find the Fourier series of $f(x) = x\,|\,x\,|$ in $(-1, 1)$.

Sol. : We have $f(x) = x\,|\,x\,|$.

$\therefore \quad f(-x) = -x\,|-x\,| = -x\,|\,x\,| = -f(x) \qquad \therefore \; f(x)$ is an odd function.

Now, using (B) of the previous page,

$$a_0 = 0, \; a_n = 0 \text{ and } b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$$

Now, $f(x) = \begin{cases} -x^2, & -1 < x < 0 \\ x^2, & 0 < x < 1 \end{cases}$

$$\therefore \quad b_n = \frac{2}{1}\int_0^1 x^2 \sin n\pi x\,dx$$

$$= 2\left[x^2\left(-\frac{\cos n\pi x}{n\pi}\right) - 2x\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1 \qquad \text{[By § 5, page 3-10]}$$

$$= 2\left[-\frac{\cos n\pi}{n\pi} + \frac{2\cos n\pi}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]$$

$$= 2\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$\therefore \quad f(x) = 2\sum\left[-\frac{(-1)^n}{n\pi} + \frac{2(-1)^n}{n^3\pi^3} - \frac{2}{n^3\pi^3}\right]\sin n\pi x.$$

Example 3 : Find the Fourier Series for $f(x) = 1 - x^2$ in $(-1, 1)$. **(M.U. 2004, 10, 12, 19)**

Sol. : We have $f(-x) = 1 - (-x^2) = 1 - x^2 = f(x)$. Hence, $f(x)$ is even.

Comparing $(-l, l)$ with $(-1, 1)$ we get $l = 1$

$$\text{Now, } a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \int_0^1 (1-x^2)\,dx = \left[x - \frac{x^3}{3}\right]_0^1 = 1 - \frac{1}{3} = \frac{2}{3}$$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = 2\int_0^1 (1-x^2)\cos n\pi x\,dx$$

$$= 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (-2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[-\frac{2\cos n\pi}{n^2\pi^2}\right] = \frac{-4(-1)^n}{n^2\pi^2} \qquad \text{[By § 5, page 3-10]}$$

$$\text{Hence, } f(x) = \frac{2}{3} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi x.$$

Example 4 : Obtain the Fourier Series for $f(x)$, where $f(x) = \begin{cases} -c, & -a < x < 0 \\ c, & 0 < x < a \end{cases}$.

Sol. : We have $f(-x) = \begin{cases} -c, & -a < -x < 0 \\ c, & 0 < -x < a \end{cases} = \begin{cases} -c, & a > x > 0 \\ c, & 0 > x > -a \end{cases}$

$$= \begin{cases} c, & -a < x < 0 \\ -c, & 0 < x < a \end{cases} = -\begin{cases} -c, & -a < x < 0 \\ c, & 0 < x < a \end{cases} = -f(x)$$

Hence, $f(x)$ is odd function

$$\therefore \quad a_0 = 0, \qquad a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{a}\int_0^a c\cdot\sin\left(\frac{n\pi x}{a}\right)dx$$

$$= \frac{2c}{a}\left[-\frac{a}{n\pi}\cos\frac{n\pi x}{a}\right]_0^a = \frac{2c}{a}\left(-\frac{a}{n\pi}\right)\left[(-1)^n - 1\right]$$

$$= -\frac{2c}{n\pi}\left[(-1)^n - 1\right] = \begin{cases} 0 & \text{when } n \text{ is even} \\ \dfrac{4c}{n\pi} & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore \quad f(x) = \frac{4c}{\pi}\left[\frac{1}{1}\sin\frac{\pi x}{a} + \frac{1}{3}\sin\frac{3\pi x}{a} + \frac{1}{5}\sin\frac{5\pi x}{a} + \ldots\ldots\right]$$

Example 5 : Obtain Fourier expansion of $\sin ax$ in the interval $-l < x < l$, where a is not an integer. **(M.U. 2002)**

Sol. : Here $f(-x) = \sin(-ax) = -\sin ax = -f(x)$. Hence, $f(x)$ is an odd function. $[\therefore \; a_n = 0]$

$$\text{Let } \sin ax = \sum b_n \sin\frac{n\pi x}{l}$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l} = \frac{2}{l}\int_0^l \sin ax\sin\frac{n\pi x}{l}\,dx$$

$$= -\frac{1}{l}\int_0^l\left[\cos\left(a + \frac{n\pi}{l}\right)x - \cos\left(a - \frac{n\pi}{l}\right)x\right]dx$$

$$= -\frac{1}{l}\left[\frac{l}{al + n\pi}\sin\left(a + \frac{n\pi}{l}\right)x - \frac{l}{al - n\pi}\sin\left(a - \frac{n\pi}{l}\right)x\right]_0^l$$

$$= -\left[\frac{l}{al + n\pi}\sin(al + n\pi) - \frac{l}{al - n\pi}\sin(al - n\pi)\right]$$

$$= -\left[\frac{\sin al\cos n\pi}{al + n\pi} - \frac{\sin al\cos n\pi}{al - n\pi}\right]$$

$$= \frac{(-2n\pi)}{(a^2l^2 - n^2\pi^2)}\sin al\cos n\pi \qquad [\because \sin n\pi = 0]$$

$$= -\frac{2n\pi(-1)^n}{n^2\pi^2 - a^2l^2}\sin al \qquad [\because \cos n\pi = (-1)^n]$$

$$\therefore \quad f(x) = 2\pi\sin al\sum\frac{(-n)(-1)^n}{n^2\pi^2 - a^2l^2}\sin\frac{n\pi x}{l}$$

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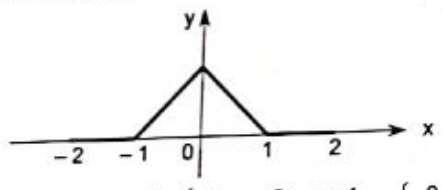
$$\therefore \quad f(x) = 2\pi \sin al\left[\frac{1}{(1^2\pi^2 - a^2l^2)}\sin\frac{\pi x}{l} - \frac{2}{(2^2\pi^2 - a^2l^2)}\sin\frac{2\pi x}{l} + \frac{3}{(3^2\pi^2 - a^2l^2)}\sin\frac{3\pi x}{l} - \ldots\right]$$

Example 6 : Find the Fourier expansion of $f(x) = \begin{cases} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

(M.U. 1993, 2002, 03, 13)

Sol. : The graph of the function is shown below.



Further $f(-x) = \begin{cases} 0, & -2 < -x < -1 \\ 1-x, & -1 < -x < 0 \\ 1+x, & 0 < -x < 1 \\ 0, & 1 < -x < 2 \end{cases} = \begin{cases} 0, & 2 > x > 1 \\ 1-x, & 1 > x > 0 \\ 1+x, & 0 > x > -1 \\ 0, & -1 > x > -2 \end{cases} = \begin{cases} 0, & -2 < x < -1 \\ 1+x, & -1 < x < 0 \\ 1-x, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases} = f(x)$

$\therefore$ $f(x)$ is an even function.

$\therefore$ $b_n = 0$ and $l = 2$.

$$\therefore \quad f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{l} = a_0 + \sum a_n \cos\frac{n\pi x}{2}$$

$$\therefore \quad a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{2}\int_0^2 f(x)\,dx$$

$$= \frac{1}{2}\left[\int_0^1 (1-x)\,dx + \int_1^2 0\cdot dx\right] = \frac{1}{2}\left[x - \frac{x^2}{2}\right]_0^1 = \frac{1}{4}.$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 (1-x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 0\cdot dx$$

$$= \left[(1-x)\left(\frac{2}{n\pi}\sin\frac{n\pi x}{2}\right) - (-1)\left(-\frac{4}{n^2\pi^2}\cos\frac{n\pi x}{2}\right)\right]_0^1 \qquad \text{[By § 5, page 3-10]}$$

$$\therefore \quad a_n = \left[\left(0 - \frac{4}{n^2\pi^2}\cos\frac{n\pi}{2}\right) - \left(0 - \frac{4}{n^2\pi^2}\right)\right] = \frac{4}{n^2\pi^2}\left(1 - \cos\frac{n\pi}{2}\right)$$

$$\therefore \quad f(x) = \frac{1}{4} + \frac{4}{\pi^2}\sum\frac{1}{n^2}\left(1 - \cos\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

Example 7 : Find Fourier expansion for $f(x) = x - x^2$, $-1 < x < 1$. **(M.U. 2005, 09, 10, 14)**

Sol. : The given function is the difference between odd and even functions which can be written as

$$f(x) = f_1(x) - f_2(x)$$

where, $f_1(x) = x$, is an odd function, and

$f_2(x) = x^2$, is an even function.

Here $l = 1$.

Now, for $f_1(x) = x$ which is odd, $a_n = 0$ and

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx = \frac{2}{1}\int_0^1 x\sin n\pi x\,dx$$

$$= 2\left[x\left(-\frac{1}{n\pi}\cos n\pi x\right) - (1)\left(-\frac{1}{n^2\pi^2}\sin n\pi x\right)\right]_0^1$$

$$= 2\left[-\frac{1}{n\pi}(-1)^n\right] = -\frac{2}{n\pi}(-1)^n \qquad \text{[By § 5, page 3-10]}$$

Further, for $f_2(x) = x^2$ which is even, $b_n = 0$.

$$a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{1}\int_0^1 x^2 dx = \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{c}\,dx = 2\int_0^1 x^2\cos n\pi x\,dx$$

$$= 2\left[x^2\left(\frac{1}{n\pi}\sin n\pi x\right) - 2x\left(-\frac{1}{n^2\pi^2}\cos n\pi x\right) + 2\left(-\frac{1}{n^3\pi^3}\sin n\pi x\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}(-1)^n \qquad \text{[By § 5, page 3-10]}$$

$$\therefore \quad f(x) = f_1(x) - f_2(x) = -f_2(x) + f_1(x)$$

$$\therefore \quad f(x) = -\frac{1}{3} - \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum\frac{(-1)^n}{n}\sin n\pi x$$

EXERCISE - VI

Obtain Fourier Series for the following functions.

1. $f(x) = a^2 - x^2$ in $(-a, a)$ **(M.U. 2004)**

$$\left[\textbf{Ans. :}\ f(x) = \frac{2a^2}{3} + \frac{4a^2}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{a} - \frac{1}{2^2}\cos\frac{2\pi x}{a} + \frac{1}{3^2}\cos\frac{3\pi x}{a} - \ldots\right]\right]$$

2. $f(x) = \sin 2x$ in $(-l, l)$. **(Hint : Put $a = 2$ in the solved ex. 4 above.)**

$$\left[\textbf{Ans. :}\ f(x) = 2\pi\sin 2l\left[\frac{1}{\pi^2 - 2^2l^2}\sin\frac{\pi x}{l} - \frac{2}{2^2\pi^2 - 2^2l^2}\sin\frac{2\pi x}{l} + \ldots\right]\right]$$

3. $f(x) = \begin{cases} -\sin\dfrac{\pi x}{c}, & -c < x < 0 \\ \sin\dfrac{\pi x}{c}, & 0 < x < c \end{cases}$ $\left[\textbf{Ans. :}\ f(x) = \dfrac{2}{\pi} - \dfrac{4}{\pi}\left[\dfrac{\cos 2x}{2^2 - 1} + \dfrac{\cos 4x}{4^2 - 1} + \ldots\right]\right]$

(M.U. 1999)

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(Hint : Put $l = 1$ in the solved ex. 1 above.)

4. $f(x) = x^2, \quad -1 < x < 1.$

$$\left[\text{Ans. : } f(x) = \frac{1}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos \pi x - \frac{1}{2^2}\cos 2\pi x + \frac{1}{3^2}\cos 3\pi x + \ldots\right]\right]$$

5. $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$ **(M.U. 2003)** [Ans. : See Ex. 3, page 3-58]

6. $f(x) = 9 - x^2$ in $(-3, 3)$. **(M.U. 2002, 15)**

(Hint : Put $a = 3$ in the above example 1 of this exercise.)

$$\left[\text{Ans. : } f(x) = 6 + \frac{36}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{3} - \frac{1}{2^2}\cos\frac{2\pi x}{3} + \frac{1}{3^2}\cos\frac{3\pi x}{3} - \ldots\right]\right]$$

7. $f(x) = x + x^2$ in $(-1, 1)$ **(M.U. 1994)**

$$\left[\text{Ans. : } f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum\frac{(-1)^n}{n^2}\cos n\pi x - \frac{2}{\pi}\sum\frac{(-1)^n}{n}\sin n\pi x\right]$$

8. $f(x) = x - x^3$ in $(-1, 1)$. **(M.U. 2001)** $\left[\text{Ans. : } f(x) = -\frac{12}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi x}{n^3}\right]$

9. $f(x) = x^2 - 2, \quad -2 \le x \le 2$ **(M.U. 2000, 04, 05)**

$$\left[\text{Ans. : } f(x) = -\frac{2}{3} - \frac{16}{\pi^2}\left[\cos\frac{\pi x}{2} - \frac{1}{4}\cos \pi x + \frac{1}{9}\cos\frac{3\pi x}{2} \ldots\right]\right]$$

10. $f(x) = \begin{cases} 0, & -2 < x < -1 \\ k, & -1 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$ $\left[\text{Ans. : } f(x) = \frac{k}{2} + \frac{2k}{\pi}\left[\frac{1}{1}\cos\frac{\pi x}{2} - \frac{1}{3}\cos\frac{2\pi x}{2} + \frac{1}{5}\cos\frac{5x}{2} - \ldots\right]\right]$

(M.U. 1996)

11. $f(x) = e^{-x}$ $(-a, a)$. **(M.U. 2003)** $\left[\text{Ans. : } f(x) = \frac{\sin ha}{a} + 2a\sin ha\sum\frac{(-1)^n}{a^2 + n^2\pi^2}\cos\frac{n\pi x}{a}\right.$

$$\left. + 2\pi\sin ha\sum\frac{(-1)^{n+1}\cdot n}{a^2 + n^2\pi^2}\sin\frac{n\pi x}{a}\right]$$

12. $f(x) = \begin{cases} a(x - l), & -l < x < 0 \\ a(l + x), & 0 < x < l \end{cases}$. Hence, deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$.

$$\left[\text{Ans. : } f(x) = \frac{2al}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 - 2(-1)^n\right]\sin\frac{n\pi x}{l}. \text{ Then put } x = \frac{l}{2}.\right]$$

3. Half-Range Series

We have seen in § 11, page 3-51 that in $(-l, l)$, $f(x)$ can be expanded in Fourier Series as

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right) + \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

ere $a_0 = \frac{1}{2l}\int_{-l}^{l} f(x)\,dx$, $a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$ and $b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

Further by the properties of even function if $f(x)$ is even in $(-l, l)$ then

$$a_0 = \frac{1}{2l}\cdot 2\int_0^l f(x)\,dx = \frac{1}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{1}{l}\cdot 2\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

and $b_n = 0$

Thus, if we imagine an even function in $(-l, l)$ *i.e.* if we define $f(x)$ in $(0, l)$ and take its reflection in the y-axis so that $f(x)$ becomes even in $(-l, l)$ then Fourier Series of $f(x)$ is given by

$$f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$$

where,

$$\boxed{a_0 = \frac{1}{l}\int_0^l f(x)\,dx \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx \text{ and } b_n = 0}$$

Such a series is called **half range cosine series**. Clearly this expansion does not contain sine terms.

Similarly, if we define $f(x)$ in $(0, l)$ and take its reflection in the x-axis so that $f(x)$ becomes odd in $(-l, l)$ then by the properties of odd function, $f(x)$ is given by

$$f(x) = \sum b_n \sin\left(\frac{n\pi x}{l}\right)$$

where,

$$\boxed{b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \quad a_0 = 0 \text{ and } a_n = 0}$$

Such a series is called **half-range sine series**. Clearly this expansion does not contain cosine terms.

Example 1 : Find a cosine series of period 2π to represent $\sin x$ in $0 \le x \le \pi$.

(M.U. 1999, 2003, 04, 05, 10, 14)

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \qquad [\because l = \pi]$

$$a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\sin x\,dx = \frac{1}{\pi}[-\cos x]_0^{\pi} = \frac{2}{\pi}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\sin x\cos nx\,dx \qquad \ldots\ldots\ldots\ldots (1)$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} - \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

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$$\therefore \quad a_n = \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left(-\frac{2}{n^2-1}\right) - \frac{2}{n^2-1}\right] = -\frac{2}{\pi(n^2-1)}[(-1)^n + 1]$$

$= 0$ if n is odd and $n \neq 1$.

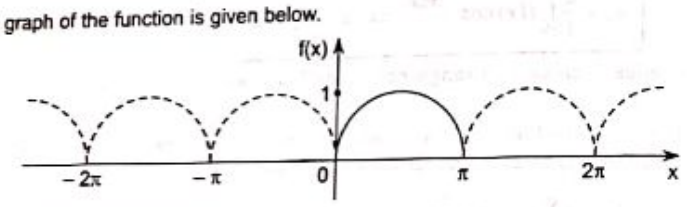
$\therefore \quad a_n = -\dfrac{4}{\pi(n^2-1)}$ if n is even.

$\therefore$ If $n = 1$ from (1), we get

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} \sin 2x\, dx = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = 0$$

$$\therefore \quad f(x) = \sin x = \frac{2}{\pi} - \frac{4}{\pi}\left(\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \ldots\right) \qquad \ldots\ldots\ldots\ldots (2)$$

The graph of the function is given below.



Cor. 1 : Hence, deduce that

$$\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots = \frac{\pi^2 - 8}{16}$$ **(M.U. 1996, 2011, 19)**

Sol. : By Paresval's identity (Cor. 6, page 3-8), we have

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\, dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots\right]$$

$$\therefore \quad \frac{1}{\pi}\int_0^{\pi} \sin^2 x\, dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots\right]$$

Now, $\dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin^2 x\, dx = \frac{1}{\pi}\int_0^{\pi}\left(\frac{1-\cos 2x}{2}\right)dx = \frac{1}{2\pi}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi} = \frac{1}{2\pi}[\pi] = \frac{1}{2}$

$$\therefore \quad \frac{1}{2} = \frac{1}{2}\left[2 \cdot \frac{4}{\pi^2} + \frac{16}{\pi^2}\left\{\frac{1}{3^2} + \frac{1}{15^2} + \frac{1}{35^2} + \ldots\right\}\right]$$

$$1 = \frac{8}{\pi^2} + \frac{16}{\pi^2}\left[\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots\right]$$

$$1 - \frac{8}{\pi^2} = \frac{16}{\pi^2}\left[\frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots\right]$$

$$\frac{\pi^2 - 8}{16} = \frac{1}{1^2 \cdot 3^2} + \frac{1}{3^2 \cdot 5^2} + \frac{1}{5^2 \cdot 7^2} + \ldots$$

Cor. 2 : From the above expansion deduce that

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$ **(M.U. 2005)**

Sol. : The series (2) can be written as $\quad \sin x = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n)^2 - 1}\cos 2nx$

Now, putting $x = \dfrac{\pi}{2}$, we get $\quad 1 = \dfrac{2}{\pi} - \dfrac{4}{\pi}\displaystyle\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)\cos n\pi$

$$\therefore \quad \frac{\pi}{2} = 1 - 2\sum_{n=1}^{\infty} \frac{1}{2}\left(\frac{1}{2n-1} - \frac{1}{2n+1}\right)(-1)^n$$

$$= 1 - 2\left[\frac{1}{2}\left\{-\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) - \ldots\right\}\right]$$

$$= 1 - \left[-\left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) - \left(\frac{1}{5} - \frac{1}{7}\right) + \left(\frac{1}{7} - \frac{1}{9}\right) + \ldots\right]$$

$$= 1 + \left[1 - \frac{1}{3} - \frac{1}{3} + \frac{1}{5} + \frac{1}{5} - \frac{1}{7} - \frac{1}{7} + \ldots\right] = 2 \cdot \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Cor. 3 : From the above expansion deduce that $\dfrac{1}{1 \cdot 3} + \dfrac{1}{3 \cdot 5} + \dfrac{1}{5 \cdot 7} + \ldots = \dfrac{1}{2}$.

(M.U. 1997, 2014)

Sol. : Putting $x = 0$ in (2) we get,

$$-\frac{2}{\pi} = -\frac{4}{\pi}\left[\frac{1}{3} + \frac{1}{15} + \frac{1}{35} + \ldots\right] \qquad \therefore \quad \frac{1}{2} = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \ldots$$

Example 2 : Obtain half range sine series for $f(x)$ when

$$f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$$ **(M.U. 2014)**

Hence, find the sum of $\displaystyle\sum_{(2n-1)}^{\infty} \frac{1}{n^4}$. **(M.U. 2002, 03, 07, 08, 11, 12, 14)**

Hence, deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \ldots$

Sol. : Let $f(x) = \sum b_n \sin nx \qquad [\because\ l = \pi]$

$$\therefore \quad b_n = \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x)\sin nx\, dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)(1)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right)(-1)\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\left\{-\frac{\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2} - 0 - 0\right\}\right.$$

$$\left. + \left\{0 - 0 + \frac{\pi}{2}\frac{\cos(n\pi/2)}{n} + \frac{\sin(n\pi/2)}{n^2}\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{4}{\pi} \cdot \frac{\sin(n\pi/2)}{n^2}$$

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$$\therefore \quad b_1 = \frac{4}{\pi}\cdot\frac{1}{1^2},\ b_2 = 0,\ b_3 = -\frac{4}{\pi}\cdot\frac{1}{3^2},\ b_4 = 0.....$$

$$\therefore \quad f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin x - \frac{1}{3^2}\sin 3x + \frac{1}{5^2}\sin 5x - \frac{1}{7^2}\sin 7x + ...\right] \qquad .....(1)$$

By Parseval's identity (Cor. 7, page 3-8)

$$\frac{1}{\pi}\int_0^{\pi}[f(x)]^2\,dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + ...\infty\right]$$

$$\therefore \quad \frac{1}{\pi}\left[\int_0^{\pi/2} x^2 dx + \int_{\pi/2}^{\pi}(\pi - x)^2\,dx\right] = \frac{1}{2}\left[b_1^2 + b_2^2 + ...\infty\right]$$

Now, $$\frac{1}{\pi}\left[\int_0^{\pi/2} x^2\,dx + \int_{\pi/2}^{\pi}(\pi^2 - 2\pi x + x^2)\,dx\right] = \frac{1}{\pi}\left[\left\{\frac{x^3}{3}\right\}_0^{\pi/2} + \left\{\pi^2 x - \pi x^2 + \frac{x^3}{3}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\pi^3}{24} - 0\right\} + \left\{\left(\pi^3 - \pi^3 + \frac{\pi^3}{3}\right) - \left(\frac{\pi^3}{2} - \frac{\pi^3}{4} + \frac{\pi^3}{24}\right)\right\}\right]$$

$$= \frac{1}{\pi}\cdot\frac{\pi^3}{12} = \frac{\pi^2}{12}$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{2}\cdot\left[\frac{16}{\pi^2}\cdot\frac{1}{1^4} + \frac{16}{\pi^2}\cdot\frac{1}{3^4} + \frac{16}{\pi^2}\cdot\frac{1}{5^4} + ...\right] \qquad \text{[From (1)]}$$

$$\therefore \quad \frac{\pi}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ...$$

For deduction put $x = \pi/2$ in (1).

$$\therefore \quad \frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...\right] \qquad \therefore \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + ...$$

Example 3 : Find half range cosine series for $f(x) = x$, $0 < x < 2$. Using Parseval's identity, deduce that

(i) $\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ...$ **(M.U. 2001, 02, 05, 06, 08, 09, 13, 17, 18)**

(ii) $\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...$

Sol. : Let $f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$. Here, $l = 2$.

$$\therefore \quad a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{2}\int_0^2 x\,dx = \frac{1}{2}\left[\frac{x^2}{2}\right]_0^2 = 1$$

$$= \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx = \frac{2}{2}\int_0^2 x\cos\frac{n\pi x}{2}\,dx$$

$$= \left[x\,\frac{\sin(n\pi x/2)}{n\pi/2} + \frac{\cos(n\pi x/2)}{n^2\pi^2/2^2}\cdot 1\right]_0^2 \qquad \text{[By § 5, page 3-10]}$$

$$\therefore \quad a_0 = \left[2\cdot(0) + \frac{\cos n\pi}{n^2\pi^2/2^2} - 0 - \frac{1}{n^2\pi^2/2^2}\right] = \frac{[(-1)^n - 1]}{n^2\pi^2/2^2}$$

$$= \begin{cases} -4\cdot\dfrac{2}{n^2\pi^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore \quad x = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + ...\right]$$

(i) By Parseval's identity (Cor. 8, page 3-8)

$$\frac{1}{l}\int_0^l [f(x)]^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + ...\right]$$

$$\therefore \quad \text{l.h.s.} = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2 = \frac{4}{3}$$

$$\therefore \quad \frac{4}{3} = \frac{1}{2}\left[2 + \frac{64}{\pi^4}\left\{\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ...\right\}\right]$$

$$\frac{8}{3} - 2 = \frac{64}{\pi^4}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ...\right] \qquad \therefore \quad \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ...$$

(ii) Let $$S = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + ...$$

$$= \left(\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + ...\right) + \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + ...\right)$$

$$= \left(\frac{\pi^4}{96}\right) + \frac{1}{2^4}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + ...\right)$$

$$\therefore \quad S = \frac{\pi^4}{96} + \frac{S}{16} \qquad \therefore \quad S = \frac{\pi^4}{90}.$$

Example 4 : Find half range sine series for $x \sin x$ in $(0, \pi)$ and hence, deduce that

$$\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + ... \qquad \textbf{(M.U. 2004, 16)}$$

Sol. : Let $f(x) = \sum b_n \sin nx \qquad [\because\ l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin x\sin nx\,dx \qquad .....(1)$$

$$= -\frac{1}{\pi}\int_0^{\pi} x\left[\cos(n+1)x - \cos(n-1)x\right]dx$$

$$= -\frac{1}{\pi}\left[x\left\{\frac{\sin(n+1)x}{n+1} - \frac{\sin(n-1)x}{n-1}\right\} - (1)\left\{-\frac{\cos(n+1)x}{(n+1)^2} + \frac{\cos(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= -\frac{1}{\pi}\left[\pi\left\{\frac{\sin(n+1)\pi}{n+1} - \frac{\sin(n-1)\pi}{n-1}\right\} - \left\{-\frac{\cos(n+1)\pi}{(n+1)^2} + \frac{\cos(n-1)\pi}{(n-1)^2}\right\}\right.$$

$$\left. - 0 - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right] \text{ if } n \neq 1 \qquad \text{[By § 5, page 3-10]}$$

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Now, $\sin(n \pm 1)\pi = 0$ and $\cos(n \pm 1)\pi = -\cos n\pi$.

$$= -\frac{1}{\pi}\left[-\frac{\cos n\pi}{(n+1)^2} + \frac{\cos n\pi}{(n-1)^2} - \frac{1}{(n+1)^2} + \frac{1}{(n-1)^2}\right]$$

$$= -\frac{1}{\pi}\begin{cases} 0 & \text{when } n \text{ is odd and } n \neq 1 \\ 2\left\{\dfrac{1}{(n-1)^2} - \dfrac{1}{(n+1)^2}\right\} & \text{when } n \text{ is even} \end{cases}$$

When $n = 1$ from (1),

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\sin^2 x\,dx = \frac{2}{\pi}\int_0^{\pi} x\,\frac{(1-\cos 2x)}{2}\,dx$$

$$= \frac{1}{\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - (1)\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{1}{\pi}\left[\pi^2 - 0 - \frac{\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{\pi}{2}$$

$$\therefore\ f(x) = x\sin x$$

$$= \frac{\pi}{2}\sin x - \frac{2}{\pi}\left[\left(\frac{1}{1^2} - \frac{1}{3^2}\right)\sin 2x + \left(\frac{1}{3^2} - \frac{1}{5^2}\right)\sin 4x + \left(\frac{1}{5^2} - \frac{1}{7^2}\right)\sin 6x + \ldots\right]$$

$$= \frac{\pi}{2}\sin x + \frac{2}{\pi}\left[\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\sin 2x + \left(\frac{1}{5^2} - \frac{1}{3^2}\right)\sin 4x + \left(\frac{1}{7^2} - \frac{1}{5^2}\right)\sin 6x + \ldots\right]$$

For deduction put $x = \frac{\pi}{4}$.

$$\therefore\ \frac{\pi}{4}\sin\frac{\pi}{4} = \frac{\pi}{2}\frac{1}{\sqrt{2}} + \frac{2}{\pi}\left[\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\cdot 1 + \left(\frac{1}{5^2} - \frac{1}{3^2}\right)\cdot 0 + \left(\frac{1}{7^2} - \frac{1}{5^2}\right)\cdot(-1) + \ldots\right]$$

$$\frac{\pi}{4}\cdot\frac{1}{\sqrt{2}} = \frac{\pi}{2}\cdot\frac{1}{\sqrt{2}} + \frac{2}{\pi}\left[\frac{1}{3^2} - \frac{1}{1^2} + \frac{1}{7^2} - \frac{1}{5^2} + \ldots\right]$$

$$-\frac{\pi}{4\sqrt{2}} = \frac{2}{\pi}\left[\frac{1}{3^2} - \frac{1}{1^2} + \frac{1}{7^2} - \frac{1}{5^2} + \ldots\right]$$

$$\frac{\pi^2}{8\sqrt{2}} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \ldots$$

Example 5 : In $(0, \pi)$ show that $x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x \ldots\right]$

(M.U. 2002, 12, 19)

Sol. : Let $f(x) = \sum b_n \sin nx \qquad [\because l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\,dx$$

$$= \frac{2}{\pi}\left[(x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{\sin nx}{n^2}\right) + (2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi} \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{n}(-\cos n\pi) - 0 + \frac{2}{n^3}(\cos n\pi) - \frac{2}{n^3}\right]$$

$$= \begin{cases} \dfrac{2}{\pi}\left[\dfrac{\pi^2}{n} - \dfrac{4}{n^3}\right] & \text{if } n \text{ is odd} \\ \dfrac{2}{\pi}\left[-\dfrac{\pi^2}{n}\right] & \text{if } n \text{ is even} \end{cases}$$

$$\therefore\ x^2 = \frac{2}{\pi}\left[\left(\frac{\pi^2}{1} - \frac{4}{1^3}\right)\sin x - \frac{\pi^2}{2}\sin 2x + \left(\frac{\pi^2}{3} - \frac{4}{3^3}\right)\sin 3x - \ldots + \ldots\right]$$

Example 6 : Expand $f(x) = \begin{cases} kx, & 0 < x < l/2 \\ k(l-x), & l/2 < x < l \end{cases}$ into half range cosine series.

Deduce the sum of the series $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ **(M.U. 1998, 2003, 15)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos\left(\frac{n\pi x}{l}\right)$

$$\therefore\ a_0 = \frac{1}{l}\int_0^l f(x)\,dx = \frac{1}{l}\left[\int_0^{l/2} kx\,dx + \int_{l/2}^l k(l-x)\,dx\right]$$

$$= \frac{1}{l}\left\{\left[k\frac{x^2}{2}\right]_0^{l/2} + \left[klx - k\frac{x^2}{2}\right]_{l/2}^l\right\}$$

$$= \frac{1}{l}\left[\frac{kl^2}{8} + kl^2 - \frac{kl^2}{2} - \frac{kl^2}{2} + \frac{kl^2}{8}\right]$$

$$= \frac{1}{l}\cdot\frac{kl^2}{4} = \frac{kl}{4}$$

$$a_n = \frac{2}{l}\left[\int_0^{l/2} kx\cos\frac{n\pi x}{l}\,dx + \int_{l/2}^l k(l-x)\cos\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{l}\left[\left\{kx\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - k\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_0^{l/2}\right.$$

$$\left. + \left\{k(l-x)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (-k)\left(\frac{-l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right)\right\}_{l/2}^l\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{l}\left[\frac{kl}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2} - \frac{kl^2}{n^2\pi^2} - \frac{kl^2}{n^2\pi^2}\cos n\pi\right.$$

$$\left. - \frac{kl}{2}\cdot\frac{l}{n\pi}\sin\frac{n\pi}{2} + \frac{kl^2}{n^2\pi^2}\cos\frac{n\pi}{2}\right]$$

$$a_1 = 0,\ a_2 = -\frac{8kl}{2^2\pi^2},\ a_3 = 0,\ a_4 = 0,\ a_5 = 0,\ a_6 = -\frac{8kl}{6^2\pi^2}, \ldots a_{10} = -\frac{8kl}{10^2\pi^2}$$

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$$\therefore\ f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]$$

Now, put $x = l/2$,

$$\therefore\ \frac{kl}{2} - \frac{kl}{4} = \frac{8kl}{\pi^2}\left[\frac{1}{2^2} + \frac{1}{6^2} + \frac{1}{10^2} + \ldots\right] \qquad \therefore\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

Example 7 : Find half range cosine series for

$$f(x) = \begin{cases} 1, & 0 < x < (a/2) \\ -1, & (a/2) < x < a \end{cases}$$

Sol. : Let $f(x) = a_0 + \sum a_n \cos\frac{n\pi x}{a} \quad [\because\ l = a]$

$$a_0 = \frac{1}{a}\left[\int_0^{a/2} dx - \int_{a/2}^{a} dx\right] = \frac{1}{a} + \left[\{x\}_0^{a/2} - \{x\}_{a/2}^{a}\right] = \frac{1}{a}\left[\frac{a}{2} - a + \frac{a}{2}\right] = 0$$

$$= \frac{2}{a}\left[\int_0^{a/2}\cos\frac{n\pi x}{a}dx - \int_{a/2}^{a}\cos\frac{n\pi x}{a}dx\right]$$

$$= \frac{2}{a}\left[\left\{\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right\}_0^{a/2} - \left\{\frac{a}{n\pi}\sin\frac{n\pi x}{a}\right\}_{a/2}^{a}\right]$$

$$= \frac{2}{a}\left[\frac{a}{n\pi}\sin\frac{n\pi}{2} - \frac{a}{n\pi}\sin n\pi + \frac{a}{n\pi}\sin\frac{n\pi}{2}\right] = \frac{2}{n\pi}\left[2\sin\frac{n\pi}{2}\right]$$

$$\therefore\ a_1 = \frac{4}{\pi},\quad a_3 = -\frac{4}{3\pi},\quad a_5 = \frac{4}{5\pi}$$

$$a_2 = 0,\quad a_4 = 0,\quad a_6 = 0 \ldots\ldots$$

$$\therefore\ f(x) = \frac{4}{\pi}\left[\frac{1}{1}\cos\frac{\pi x}{a} - \frac{1}{3}\cos\frac{\pi x}{a} + \frac{1}{5}\cos\frac{5\pi x}{a}\ldots\right]$$

Example 8 : If $f(x) = \begin{cases} \dfrac{x}{a}, & \text{for } 0 < x < a \\ \dfrac{l-x}{l-a}, & \text{for } a < x < l \end{cases}$

prove that $f(x) = \dfrac{2l^2}{a(l-a)\pi^2}\sum\dfrac{1}{n^2}\sin\dfrac{n\pi a}{l}\sin\dfrac{n\pi x}{l}$.

Sol. : We have to obtain half range sine series for $f(x)$.

Let $\quad f(x) = \sum b_n \sin\dfrac{n\pi x}{l}$

$$b_n = \frac{2}{l}\left[\int_0^a \frac{x}{a}\sin\frac{n\pi x}{l}dx + \int_a^l \frac{l-x}{l-a}\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{2}{l}\left[\left\{\frac{x}{a}\cdot\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{1}{a}\right)\cdot\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right\}_0^a\right.$$

$$\left. + \left\{\left(\frac{l-x}{l-a}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-1}{l-a}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right\}_a^l\right] \quad \text{[By § 5, page 3-10]}$$

$$\therefore\ b_n = \frac{2}{l}\left[-\frac{l}{n\pi}\cos\frac{n\pi a}{l} + \frac{l^2}{an^2\pi^2}\sin\frac{n\pi a}{l} + \frac{l}{n\pi}\cos\frac{n\pi a}{l} + \frac{l^2}{(l-a)n^2\pi^2}\sin\frac{n\pi a}{l}\right]$$

$$= \frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\left[\frac{1}{a} + \frac{1}{l-a}\right] = \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi a}{l}\left(\frac{1}{a(l-a)}\right)$$

$$\therefore\ b_n = \frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\cdot\left[\frac{1}{a} + \frac{1}{l-a}\right] = \frac{2l}{n^2\pi^2}\sin\frac{n\pi a}{l}\cdot\left(\frac{l}{a(l-a)}\right)$$

$$\therefore\ f(x) = \frac{2l^2}{a(l-a)\pi^2}\sum\frac{1}{n^2}\sin\frac{n\pi a}{l}\sin\frac{n\pi x}{l}$$

Example 9 : Obtain the expansion of $f(x) = x(\pi - x)$, $0 < x < \pi$ as a half-range cosine series. Hence, show that

(i) $\sum_1^\infty \dfrac{1}{n^2} = \dfrac{\pi^2}{6}$, (ii) $\sum_1^\infty \dfrac{(-1)^{n+1}}{n^2} = \dfrac{\pi^2}{12}$,

(iii) $\sum_1^\infty \dfrac{1}{n^4} = \dfrac{\pi^4}{90}$ **(M.U. 1993, 96, 99, 2005, 09, 12, 14, 19)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \quad [\because\ l = \pi]$

$$a_0 = \frac{1}{\pi}\int_0^\pi f(x)dx = \frac{1}{\pi}\int_0^\pi(\pi x - x^2)dx = \frac{1}{\pi}\left[\frac{\pi x^2}{2} - \frac{x^3}{3}\right]_0^\pi = \frac{\pi^2}{6}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^\pi(\pi x - x^2)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[(\pi x - x^2)\cdot\frac{\sin nx}{n} - (\pi - 2x)\cdot\left(-\frac{\cos nx}{n^2}\right) + (-2)\left(-\frac{\sin nx}{n^3}\right)\right]_0^\pi$$

$$\therefore\ a_n = -2\left(\frac{1+\cos n\pi}{n^2}\right) = \begin{cases} 0 & \text{when } n \text{ is odd} \\ -4/n^2 & \text{when } n \text{ is odd} \end{cases} \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ x(\pi - x) = \frac{\pi^2}{6} - 4\left[\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \frac{\cos 6x}{6^2} + \ldots\right]$$

$$= \frac{\pi^2}{6} - \left[\frac{1}{1^2}\cos 2x + \frac{1}{2^2}\cos 4x + \frac{1}{3^2}\cos 6x + \ldots\right]$$

(i) Now, put $x = 0$,

$$\therefore\ 0 = \frac{\pi^2}{6} - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\right) \qquad \therefore\ \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$$

(ii) Again put $x = \dfrac{\pi}{2}$,

$$\therefore\ \frac{\pi^2}{4} = \frac{\pi^2}{6} - \left[-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \ldots\right]$$

$$\therefore\ \frac{\pi^2}{4} - \frac{\pi^2}{6} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots \qquad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \ldots$$

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(iii) Now, by Parseval's identity (Cor. 6, page 3-8).

$$\frac{1}{\pi}\int_0^{\pi} f(x)^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots\right]$$

$$\therefore \quad \frac{1}{\pi}\int_0^{\pi} x^2(\pi - x)^2\,dx = \frac{1}{2}\left[2a_0^2 + a_1^2 + a_2^2 + a_3^2 + \ldots\right]$$

Now, $\frac{1}{\pi}\int_0^{\pi} x^2(\pi - x)^2\,dx = \frac{1}{\pi}\int_0^{\pi}(\pi^2x^2 - 2\pi x^3 + x^4)\,dx$

$$= \frac{1}{\pi}\left[\pi^2\frac{x^3}{3} - 2\pi\frac{x^4}{4} + \frac{x^5}{5}\right]_0^{\pi} = \frac{1}{\pi}\cdot\frac{\pi^5}{30} = \frac{\pi^4}{30}$$

$$\therefore \quad \frac{\pi^4}{30} = \frac{1}{2}\left[2\cdot\frac{\pi^4}{36} + \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots\right]$$

$$\frac{\pi^4}{15} - \frac{\pi^4}{18} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots$$

$$\frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots$$

Example 10 : Obtain half-range sine series in $(0, \pi)$ for $x(\pi - x)$. **(M.U. 2015)**

Hence, find the value of $\sum \frac{(-1)^n}{(2n-1)^3}$. **(M.U. 1996, 2013, 14)**

Sol. : Let $f(x) = \sum b_n \sin nx$

$$\therefore \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$$

$$= \frac{2}{\pi}\left[x(\pi - x)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left\{0 - 0 - \frac{2}{n^3}\cos(n\pi)\right\} - \left\{0 - 0 - \frac{2}{n^3}\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{4}{\pi}\left[\frac{1 - \cos n\pi}{n^3}\right] = \frac{4}{\pi n^3}\left[1 - (-1)^n\right] \qquad [\because \cos n\pi = (-1)^n]$$

$$\therefore \quad b_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \dfrac{8}{\pi n^3} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad x(\pi - x) = \frac{8}{\pi}\left[\frac{1}{1^3}\sin \pi x + \frac{1}{3^3}\sin 3\pi x + \frac{1}{5^3}\sin 5x + \ldots\right]$$

Now, put $x = \frac{\pi}{2}$,

$$\therefore \quad \frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{8}{\pi}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots\right]$$

$$\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

Example 11 : Show that $\cos x = \frac{8}{\pi}\sum_{m=1}^{\infty} \frac{m}{4m^2 - 1}\sin 2mx$ if $0 < x < \pi$ **(M.U. 2002, 03, 04)**

Sol. : We have to obtain half-range sine series for $\cos x$

Let $\cos x = \sum b_n \sin nx \qquad [\because l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\cos x \sin nx\,dx \qquad \ldots\ldots\ldots\ldots (1)$$

$$= \frac{1}{\pi}\int_0^{\pi}[\sin(1+n)x - \sin(1-n)x]\,dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos \pi x}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right] \qquad [\because \cos(n \pm 1)\pi = -\cos n\pi]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1}\right](1 + \cos n\pi) = \frac{1}{\pi}\cdot\frac{2n}{n^2 - 1}[1 + (-1)^n]$$

$$= \begin{cases} 0, & \text{if } n \text{ is odd and } n \neq 1 \\ \dfrac{1}{\pi}\cdot\dfrac{4n}{n^2 - 1}, & \text{is } n \text{ is even} \end{cases}$$

When $n = 1$, from (1), we get,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x\,dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x\,dx$$

$$\therefore \quad b_1 = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = -\frac{1}{2\pi}[1 - 1] = 0$$

$$\therefore \quad \cos x = \frac{4}{\pi}\left[\frac{2}{2^2 - 1}\sin 2x + \frac{4}{4^2 - 1}\sin 4x + \frac{6}{6^2 - 1}\sin 6x + \ldots\right]$$

$$= \frac{8}{\pi}\left[\frac{1}{2^2 - 1}\sin 2x + \frac{2}{4^2 - 1}\sin 4x + \frac{3}{6^2 - 1}\sin 6x + \ldots\right]$$

$$= \frac{8}{\pi}\sum \frac{m}{4m^2 - 1}\sin 2m\pi.$$

Example 12 : Expand $f(x) = lx - x^2$, $0 < x < l$ in a half-range (i) cosine series, (ii) sine series. **(M.U. 2013, 14, 16, 19)**

Hence, from sine series deduce that

(i) $\frac{\pi^3}{32} = 1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$ **(M.U. 1994, 2003, 05, 06, 07, 08)**

(ii) $\frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \frac{1}{7^6} + \ldots$ (iii) $\frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$

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Sol. : (i) Cosine Series

Let $f(x) = a_0 + \sum a_n \cos \frac{n\pi x}{l}$

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left[l\frac{x^2}{2} - \frac{x^3}{3}\right]_0^l = \frac{1}{l}\cdot\frac{l^3}{6} = \frac{l^2}{6}$$

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(\frac{1}{n\pi}\sin\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - l\cdot\frac{l^2}{n^2\pi^2}\cos n\pi - 0\right\} - \left\{0 + l\cdot\frac{l^2}{n^2\pi^2} + 0\right\}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= -\frac{2}{l}\cdot\frac{l^3}{n^2\pi^2}[\cos n\pi + 1] = -\frac{2l^2}{n^2\pi^2}[(-1)^n + 1]$$

$$\therefore\ a_n = \begin{cases} 0, & \text{if } n \text{ is odd} \\ -\dfrac{4l^2}{n^2\pi^2}, & \text{if } n \text{ is even} \end{cases}$$

$$\therefore\ f(x) = lx - x^2 = \frac{l^2}{6} - \frac{4l^2}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{4^2}\cos\frac{4\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \ldots\right]$$

(ii) Sine series

Let $f(x) = \sum b_n \sin\frac{n\pi x}{l}$

$$b_n = \frac{2}{l}\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{l}\left[(lx - x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left[\left\{0 - 0 - \frac{2l^3}{n^3\pi^3}\cos n\pi\right\} - \left\{0 - 0 - \frac{2l^3}{n^3\pi^3}\right\}\right] = \frac{2}{l}\cdot\frac{2l^3}{n^3\pi^3}(-\cos n\pi + 1)$$

$$= \begin{cases} \dfrac{8l^2}{n^3\pi^3}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \qquad \text{[By § 5, page 3-10]}$$

$$\therefore\ f(x) = lx - x^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3}\sin\frac{\pi x}{l} + \frac{1}{3^3}\sin\frac{3\pi x}{l} + \frac{1}{5^3}\sin\frac{5\pi x}{l} + \ldots\right]$$

(i) For deduction, we put, $x = \frac{l}{2}$.

$$\therefore\ l\cdot\left(\frac{l}{2}\right) - \left(\frac{l}{2}\right)^2 = \frac{8l^2}{\pi^3}\left[\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \ldots\right] \qquad \therefore\ \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$$

(ii) By corollary 9, page 3-9

$$\frac{1}{l}\int_0^l [f(x)]^2\,dx = \frac{1}{2}\left[b_1^2 + b_2^2 + b_3^2 + \ldots\infty\right]$$

$$\therefore\ \frac{1}{l}\int_0^l [lx - x^2]^2\,dx = \frac{1}{2}\left(\frac{8l^2}{\pi^3}\right)^2\left[\left(\frac{1}{1^3}\right)^2 + \left(\frac{1}{3^3}\right)^2 + \left(\frac{1}{5^3}\right)^2 + \ldots\right]$$

$$\frac{1}{l}\int_0^l (l^2x^2 - 2lx^3 + x^4)\,dx = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\frac{1}{l}\left[l^2\frac{x^3}{3} - 2l\frac{x^4}{4} + \frac{x^5}{5}\right]_0^l = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right]$$

$$\therefore\ \frac{1}{30}l^4 = \frac{32l^4}{\pi^6}\left[\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right] \qquad \therefore\ \frac{\pi^6}{960} = \frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots$$

(iii) Let $S = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \ldots$

$$= \left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots\right) + \left(\frac{1}{2^6} + \frac{1}{4^6} + \frac{1}{6^6} + \ldots\right)$$

$$\therefore\ S = \frac{\pi^6}{960} + \frac{1}{2^6}\left(\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} \ldots\right) = \frac{\pi^6}{960} + \frac{1}{64}S.$$

$$\therefore\ S - \frac{S}{64} = \frac{\pi^6}{960} \qquad \therefore\ \frac{63}{64}S = \frac{\pi^6}{960} \qquad \therefore\ S = \frac{\pi^6}{960}\cdot\frac{64}{63} = \frac{\pi^6}{945}$$

$$\therefore\ \frac{\pi^6}{945} = \frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots$$

Example 13 : Find half range cosine series for $f(x) = \begin{cases} x, & 0 < x < (\pi/2) \\ \pi - x, & (\pi/2) < x < \pi \end{cases}$. **(M.U. 2002)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos nx \qquad [\because\ l = \pi]$

$$\therefore\ a_0 = \frac{1}{\pi}\int_0^\pi f(x)\,dx = \frac{1}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^\pi (\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{x^2}{2}\right\}_0^{\pi/2} + \left\{\pi x - \frac{x^2}{2}\right\}_{\pi/2}^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right] = \frac{\pi}{4}$$

$$a_n = \frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^\pi (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{x\cdot\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right\}_0^{\pi/2} + \left\{(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right\}_{\pi/2}^\pi\right]$$

[By § 5, page 3-10]

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$$a_n = \frac{2}{\pi}\left[\left\{\frac{\pi}{2}\cdot\frac{1}{n}\cdot\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2}\right\} + \left\{-\frac{1}{n^2}\cos n\pi - \frac{\pi}{2}\cdot\frac{1}{n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right\}\right]$$

$$= \frac{2}{\pi n^2}\left[2\cos\frac{n\pi}{2} - \cos n\pi - 1\right]$$

$$\therefore\ a_1 = 0,\ a_2 = \frac{2}{\pi\cdot 2^2}[2(-1)-(+1)-1] = -\frac{8}{\pi\cdot 2^2}$$

$$a_3 = 0,\ a_4 = 0,\ a_5 = 0$$

$$a_6 = \frac{2}{\pi\cdot 6^2}[2(-1)-(+1)-1] = -\frac{8}{\pi\cdot 6^2}$$

$$a_7 = 0,\ a_8 = 0,\ a_9 = 0$$

$$a_{10} = \frac{2}{\pi\cdot 10^2}[2(-1)-(+1)-1] = -\frac{8}{\pi\cdot 10^2}$$

$$\therefore\ f(x) = \frac{\pi}{4} - \frac{8}{\pi}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \ldots\right]$$

Example 14 : Obtain half range sine series to represent

$$f(x) = \begin{cases} \dfrac{2x}{3}, & 0 \le x \le \dfrac{\pi}{3} \\ \dfrac{\pi - x}{3}, & \dfrac{\pi}{3} \le x \le \pi \end{cases}$$

Sol. : Let $f(x) = \sum b_n \sin mx \qquad [\because\ l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin n\pi x\,dx = \frac{2}{\pi}\left[\int_0^{\pi/3}\frac{2x}{3}\sin n\pi x\,dx + \int_{\pi/3}^{\pi}\left(\frac{\pi - x}{3}\right)\sin n\pi x\,dx\right]$$

$$= \frac{2}{3\pi}\left[\int_0^{\pi/3} 2x\sin n\pi x\,dx + \int_{\pi/3}^{\pi}(\pi - x)\sin n\pi x\,dx\right]$$

$$= \frac{2}{3\pi}\left[\left\{(2x)\left(-\frac{\cos n\pi x}{n}\right) - (2)\left(-\frac{\sin n\pi x}{n^2}\right)\right\}_0^{\pi/3} + \left\{(\pi - x)\left(-\frac{\cos n\pi x}{n}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2}\right)\right\}_{\pi/3}^{\pi}\right] \qquad \text{[By § 5, page 3-10]}$$

$$= \frac{2}{3\pi}\left[\left\{\left(\frac{2\pi}{3}\right)\left(-\frac{1}{n}\cos\frac{n\pi}{3}\right) + \frac{2}{n^2}\sin\frac{n\pi}{3}\right\} + \left\{0 + \frac{2\pi}{3}\left(\frac{1}{n}\cos\frac{n\pi}{3}\right) + \frac{1}{n^2}\sin\frac{n\pi}{3}\right\}\right]$$

$$= \frac{2}{3\pi}\cdot\frac{3}{n^2}\sin\frac{n\pi}{3} = \frac{2}{\pi}\cdot\frac{1}{n^2}\sin\frac{n\pi}{3}$$

$$\therefore\ f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{3}\sin nx$$

$$\therefore\ f(x) = \frac{2}{\pi}\left[\frac{1}{1^2}\cdot\frac{\sqrt{3}}{2}\sin x + \frac{1}{2^2}\frac{\sqrt{3}}{2}\cdot\cos 2x - \frac{1}{4^2}\cdot\frac{\sqrt{3}}{2}\sin 4x - \frac{1}{5^2}\cdot\frac{\sqrt{3}}{2}\sin 5x + \ldots\right]$$

$$\therefore\ f(x) = \frac{\sqrt{3}}{\pi}\left[\frac{1}{1^2}\sin x + \frac{1}{2^2}\sin 2x - \frac{1}{4^2}\sin 4x - \frac{1}{5^2}\sin 5x + \ldots\right]$$

Example 15 : Find half-range sine series for $f(x) = \frac{\pi}{4}$ in $(0, \pi)$. Hence, show that

(i) $\frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots$

(ii) $\frac{\pi}{8}x(\pi - x) = \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots$

(iii) $\frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots$

Sol. : Let $f(x) = \sum b_n \sin nx$

$$\therefore\ b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{\pi}{4}\sin nx\,dx$$

$$= \frac{1}{2}\left[-\frac{\cos nx}{n}\right]_0^{\pi} = -\frac{1}{2n}[\cos n\pi - 1] = \begin{cases} 1/n & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\therefore\ \frac{\pi}{4} = \frac{1}{1}\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots \qquad \ldots\ldots (1)$$

(i) Now, integrate both sides of (1) w.r.t. x from 0 to x.

$$\therefore\ \frac{\pi}{4}\int_0^x dx = \frac{1}{1}\int_0^x \sin dx + \frac{1}{3}\int_0^x \sin 3x\,dx + \ldots$$

$$\frac{\pi}{4}[x]_0^x = \frac{1}{1}[-\cos x]_0^x + \frac{1}{3}\left[-\frac{\cos 3x}{3}\right]_0^x + \ldots$$

$$\frac{\pi x}{4} = \frac{1}{1^2}(1 - \cos x) + \frac{1}{3^2}(1 - \cos 3x) + \ldots \qquad \ldots\ldots (2)$$

Now, put $x = \frac{\pi}{2}$ in (2)

$$\therefore\ \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots \qquad \ldots\ldots (3)$$

Now, subtract (2) from (3)

$$\therefore\ \frac{\pi^2}{8} - \frac{\pi x}{4} = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots$$

$$\therefore\ \frac{\pi}{4}\left(\frac{\pi}{2} - x\right) = \frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \ldots \qquad \ldots\ldots (4)$$

(ii) Now, integrate (4) w.r.t. x from 0 to x,

$$\frac{\pi}{4}\int_0^x\left(\frac{\pi}{2} - x\right)dx = \frac{1}{1^2}\int_0^x \cos x\,dx + \frac{1}{3^2}\int_0^x \cos 3x\,dx + \ldots$$

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$$\frac{\pi}{4}\left[\frac{\pi}{2}x - \frac{x^2}{2}\right]_0^x = \frac{1}{1^2}\left[\frac{\sin x}{1}\right]_0^x + \frac{1}{3^2}\left[\frac{\sin 3x}{3}\right]_0^x + \ldots$$

$$\therefore \quad \frac{\pi}{8}x(\pi - x) = \frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \ldots \qquad \ldots\ldots\ldots\ldots (5)$$

(iii) Now, put $x = \frac{\pi}{2}$ in (5).

$$\therefore \quad \frac{\pi^3}{32} = \frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \ldots \qquad \ldots\ldots\ldots\ldots (6)$$

Example 16 : Prove that in the interval $0 < x < \pi$,

$$\frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\left[\frac{\sin x}{a^2+1} - \frac{2\sin 2x}{a^2+4} + \frac{3\sin 3x}{a^2+9} - \ldots\right]$$ **(M.U. 1996, 2003, 05, 08, 11)**

Sol. : Let $f(x) = e^{ax} - e^{-ax}$, $0 < x < \pi$. $[\because l = \pi]$

Since, we want $f(x)$ to be expanded in sine terms only.

Let $f(x) = \sum b_n \sin nx$

$$b_n = \int_0^\pi f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^\pi (e^{ax} - e^{-ax})\sin nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^\pi e^{ax}\sin nx\,dx - \int_0^\pi e^{-ax}\sin nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left\{\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx)\right\}_0^\pi - \left\{\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right\}_0^\pi\right]$$

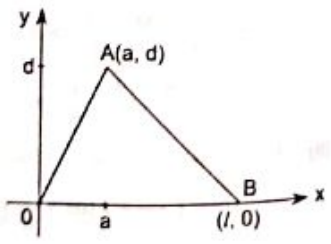
$$= \frac{2}{\pi}\left[\left\{\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{n}{a^2+n^2}\right\} - \left\{\frac{e^{-a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{n}{a^2+n^2}\right\}\right]$$

$$\therefore \quad b_n = \frac{2}{\pi}\cdot\frac{e^{a\pi} - e^{-a\pi}}{a^2+n^2}(-n\cos n\pi) \qquad \text{[By (2), page 3-4]}$$

$$\therefore \quad \frac{e^{ax} - e^{-ax}}{e^{a\pi} - e^{-a\pi}} = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}(-n\cos n\pi)\sin nx$$

$$= \frac{2}{\pi}\left[\frac{1}{a^2+1}\sin x - \frac{2}{a^2+2^2}\sin 2x + \frac{3}{a^2+3^2}\sin 3x - \ldots\right]$$

Example 17 : Find half range sine series for $f(x)$ given by the given graph in the interval $(0, l)$.



Sol. : Since, A is (a, d). The equation of OA is $\frac{y-0}{0-d} = \frac{x-0}{0-a}$.

$$\therefore \quad y = \frac{d}{a}\cdot x$$

Similarly, the equation of AB is $\frac{y-d}{d-0} = \frac{x-a}{a-l}$.

$$\therefore \quad y = d + \frac{d(x-a)}{a-l}$$

$$\therefore \quad y = \frac{da - dl + dx - da}{a-l} = \frac{d}{a-l}(x-l)$$

$$\therefore \quad f(x) = \begin{cases} \frac{d}{a}\cdot x, & 0 \le x \le a \\ \frac{d}{a-l}(x-l), & a \le x \le l \end{cases}$$

Let $f(x) = \sum b_n \sin\frac{n\pi x}{l}$

Now, refer to the solved example 8 above where $d = 1$.

Following the same steps we can obtain.

$$f(x) = \frac{2dl^2}{\pi^2\cdot a\cdot(l-a)}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi a}{l}\cdot\sin\frac{n\pi x}{l}$$

Example 18 : Find half-range cosine series for $f(x) = e^x$, $0 < x < 1$. **(M.U. 2013, 15)**

Sol. : Let $f(x) = a_0 + \sum a_n \cos n\pi x \quad [\because l = 1]$

$$\therefore \quad a_0 = \frac{1}{1}\int_0^1 e^x dx = \left[e^x\right]_0^1 = e - 1$$

$$= \frac{2}{1}\int_0^1 e^x \cos n\pi x\,dx = 2\left[\frac{e^x}{1+n^2\pi^2}(\cos n\pi x + n\pi\sin n\pi x)\right]_0^1$$

$$= 2\left[\frac{e}{1+n^2\pi^2}(-1)^n - \frac{1}{1+n^2\pi^2}\right]_0^1 \qquad \text{[By (3), page 3-4]}$$

$$\therefore \quad e^x = (e-1) + 2\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2}\left[e(-1)^n - 1\right]\cos n\pi x$$

$$= (e-1) + 2\left[-\frac{1}{1+1^2\cdot\pi^2}(e+1)\cdot\cos\pi x + \frac{1}{1+2^2\cdot\pi^2}(e-1)\cos 2\pi x - \frac{1}{1+3^2\cdot\pi^2}(e+1)\cos 3\pi x + \ldots\right]$$

EXERCISE - VII

(A) **1.** Obtain half range cosine series for **(M.U. 1994, 96)**

(i) $f(x) = x$ in $0 < x < 2$.

(ii) $f(x) = x$ in $0 < x < 1$.

Ans. : (i) $f(x) = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]$

(ii) $f(x) = \frac{1}{2} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{3^2}\cos 3\pi x + \frac{1}{5^2}\cos 5x + \ldots\right]$

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2. Obtain cosine series for the function $(mx + c)$ in the interval $0 \le x \le p$ and hence, deduce that

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

$$\left[\text{Ans. : } f(x) = \left(\frac{mp}{2} + c\right) - \frac{4mp}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{p} + \frac{1}{3^2}\cos\frac{3\pi x}{p} + \frac{1}{5^2}\cos\frac{5\pi x}{p} + \ldots\right]\right.$$

Then put $x = 0$.

By putting $p = 2$, $m = 1$, $c = 0$, you get ex. 1(i), by putting $p = 1$, $m = 1$, $c = 0$ you get ex. 1(ii).]

3. Obtain half range cosine series for

$$f(x) = \begin{cases} 1, & 0 < x < (c/2) \\ 1 - x, & (c/2) < x < c \end{cases}$$

$$\left[\text{Ans. : } f(x) = \left(1 - \frac{3c}{8}\right) + \frac{2c}{\pi^2}\left[(\pi + 1)\cos\frac{\pi x}{c} - \frac{1}{2}\cos\frac{2\pi x}{c} + \frac{1}{9}\left(1 - \frac{3\pi}{2}\right)\cos\frac{3\pi x}{c} + \ldots\right]\right]$$

4. Obtain half-range sine series for $f(x) = x(2 - x)$ in $0 < x < 2$ and hence, deduce that

$$\sum \frac{1}{n^6} = \frac{\pi^6}{945}.$$ **(M.U. 2003, 14)**

(Put $l = 2$ in the solved Ex. 12, page 3-73.)

5. Obtain half range sine series for $f(x) = \begin{cases} x, & 0 < x < a \\ a, & a < x < \pi - a \\ \pi - x, & \pi - a < x < \pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\sin a \sin x + \frac{1}{3^2}\sin 3a \sin 3x + \frac{1}{5^2}\sin 5a \sin 5x + \ldots\right]\right]$$

6. Obtain half range sine series for $f(x) = \begin{cases} \pi/3, & 0 \le x < (\pi/3) \\ 0, & (\pi/3) \le x < (2\pi/3) \\ -\pi/3, & (2\pi/3) \le x \le \pi \end{cases}$

$$\left[\text{Ans. : } f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{10}\sin 10x + \ldots\right]$$

7. Find half range cosine and sine series for $f(x)$, where

$$f(x) = \begin{cases} \pi/3, & 0 \le x \le (\pi/3) \\ 0, & (\pi/3) \le x \le (2\pi/3) \\ -\pi/3, & (2\pi/3) \le x \le \pi \end{cases}$$

$$\left[\text{Ans. : (i) } f(x) = \frac{2}{\sqrt{3}}\left[\frac{1}{1}\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \ldots\right]\right.$$

$$\left.\text{(ii) } f(x) = \sin 2x + \frac{1}{2}\sin 4x + \frac{1}{10}\sin 10x + \ldots\right]$$

8. Obtain half range cosine series for the above function.

$$\left[\text{Ans. : } f(x) = \frac{2}{\sqrt{3}}\left[\frac{1}{1}\cos x - \frac{1}{5}\cos 5x + \frac{1}{7}\cos 7x - \ldots\right]\right]$$

9. Obtain half range sine series for

$$f(x) = \begin{cases} (1/4) - x, & 0 < x < (1/2) \\ x - (3/4), & (1/2) < x < 1 \end{cases}$$ **(M.U. 1993, 2001)**

$$\left[\text{Ans. : } f(x) = \left(\frac{1}{\pi} - \frac{4}{\pi^2}\right)\sin \pi x + \left(\frac{1}{3\pi} - \frac{4}{3^2\pi^2}\right)\sin 3\pi x + \left(\frac{1}{5\pi} - \frac{4}{5^2\pi^2}\right)\sin 5\pi x + \ldots\right]$$

10. Show that in the interval $0 < x < \pi$,

$$x \sin x = 1 - \frac{1}{2}\cos x - 2\left[\frac{\cos 2x}{1 \cdot 3} - \frac{\cos 3x}{2 \cdot 4} + \frac{\cos 4x}{3 \cdot 5} - \ldots\right]$$ **(M.U. 1995, 97, 2004)**

11. Find half range cosine series for $f(x) = a\left\{1 - \frac{x}{l}\right\}$, $0 < x < l$. **(M.U. 2007)**

$$\left[\text{Ans. : } f(x) = \frac{a}{2} + \frac{4a}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{l} + \frac{1}{3^2}\cos\frac{3\pi x}{l} + \ldots\right]\right]$$

12. Find half range sine series for the above function.

$$\left[\text{Ans. : } f(x) = \frac{2a}{\pi}\left[\sin\frac{\pi x}{l} + \frac{1}{2}\sin\frac{2\pi x}{l} + \frac{1}{3}\sin\frac{3\pi x}{l} + \ldots\right]\right]$$

13. Obtain half-range cosine series for $f(x) = x$ in $0 < x < l$.

Hence, deduce that (i) $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$

(ii) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \ldots = \frac{\pi^4}{90}$ (iii) $\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots = \frac{\pi^4}{1440}$ **(M.U. 2001)**

14. Find half range sine series for

$$f(x) = \begin{cases} x, & 0 \le x \le (l/2) \\ l - x, & (l/2) \le x \le l \end{cases} \qquad \left[\text{Ans. : } f(x) = \frac{4l}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{1}{n^2} \cdot \sin\frac{n\pi}{2}\right) \cdot \sin\frac{n\pi x}{l}\right]$$

15. Obtain half-range cosine series for $f(x) = \begin{cases} kx, & 0 \le x \le (l/2) \\ k(l - x), & (l/2) \le x \le 1 \end{cases}$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots = \frac{\pi^2}{8}$ **(M.U. 2001, 03)**

and $\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \ldots = \frac{\pi^4}{96}$. **(M.U. 2004)**

$$\left[\text{Ans. : } f(x) = \frac{kl}{4} - \frac{8kl}{\pi^2}\left[\frac{1}{2^2}\cos\frac{2\pi x}{l} + \frac{1}{6^2}\cos\frac{6\pi x}{l} + \frac{1}{10^2}\cos\frac{10\pi x}{l} + \ldots\right]\right.$$

Then, put $x = l/2$.]

16. Find half range sine series of period $2l$ for $f(x) = \begin{cases} \frac{2x}{l}, & 0 \le x \le \frac{l}{2} \\ \frac{2}{l}(l - x), & \frac{l}{2} \le x \le l. \end{cases}$

(M.U. 1998, 2002, 04, 13) $\left[\text{Ans. : } f(x) = \frac{8}{\pi^2}\sum \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}\right]$

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17. Obtain half range sine series for $f(x) = x^2$ in $0 < x < 3$. **(M.U. 2013)**
[See solved Ex. 1, page 3-43 and evaluate b_n for $a = 3$.]

18. Obtain half-range sine series for $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \end{cases}$

Hence, deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \ldots = \frac{\pi^2}{8}$. **(M.U. 2007)**

$$\left[\textbf{Ans. : } f(x) = \frac{8}{\pi^2}\left[\frac{1}{1^2}\sin\frac{\pi x}{2} - \frac{1}{3^2}\sin\frac{3\pi x}{2} + \frac{1}{5^2}\sin\frac{5\pi x}{2} \ldots\right]\right]$$

19. Obtain half range sine series for $f(x) = \pi x - x^2$ in $(0, \pi)$ and use Parseval's identity to deduce that

$$\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{3^6} + \ldots = \frac{\pi^6}{980}.$$ **(M.U. 2006)**

20. Obtain half range sine series in $(0, \pi)$ for $f(x) = x(\pi - x)$ and show that $\sum \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$.

(M.U. 2005) [**Ans. :** For the first part, see solved Ex. 10, page 3-72]

21. Find half range cosine series for $\cos \lambda x$ in $(0, \pi)$ where λ is not an integer and hence, show that

$$\pi \cot \pi\lambda = \frac{1}{\lambda} + \sum_{n=1}^{\infty} \frac{2\lambda}{\lambda^2 - n^2}. \qquad \left[\textbf{Ans. : } \cos \lambda x = \frac{\sin \lambda\pi}{\lambda\pi} + \frac{2\lambda \sin \lambda\pi}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{\lambda^2 - n^2} \cos nx\right]$$

22. Find half range sine series for $f(x) = \begin{cases} x, & 0 \le x \le 2 \\ 4 - x, & 2 \le x \le 4 \end{cases}$

(**Hint :** Put $l = 4$ in the above example number 11.)

$$\left[\textbf{Ans. : } f(x) = \frac{16}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{1}{n^2} \sin\frac{n\pi}{2}\right) \cdot \sin\frac{n\pi x}{4}\right]$$

23. Find half range cosine series for

$f(x) = 1, \quad 0 \le x \le 1$

$\quad = x, \quad 1 \le x \le 2.$

$$\left[\textbf{Ans. : } f(x) = \frac{5}{4} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} - \frac{1}{2^2}\cos\frac{2\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \frac{1}{5^2}\cos\frac{5\pi x}{2} + \ldots\right]\right]$$

24. Obtain half range cosine series for $f(x) = x - x^2$ for $0 \le x \le 1$. **(M.U. 1994)**
(**Hint :** See solved example 11 above.)

$$\left[\textbf{Ans. : } f(x) = \frac{1}{6} - \frac{4}{\pi^2}\left[\frac{1}{2^2}\cos 2\pi x + \frac{1}{4^2}\cos 4\pi x + \ldots\right]\right]$$

25. Obtain half-range cosine series for $f(x) = c - x, \quad 0 < x < c$.

$$\left[\textbf{Ans. : } f(x) = \frac{c}{2} + \frac{4c}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{c} + \frac{1}{3^2}\cos\frac{3\pi x}{c} + \ldots\right]\right]$$

26. Obtain cosine series for $f(x) = \frac{1}{2}(\pi - x)\sin x$ in $(0, \pi)$.

$$\left[\textbf{Ans. : } f(x) = \frac{1}{2} + \frac{1}{4}\cos x - \frac{1}{3}\cos 2x + \frac{1}{8}\cos 3x - \ldots\right]$$

27. Obtain half range sine series for

$f(x) = 1, \quad 0 < x < \frac{1}{2}$

$\quad = 0, \quad \frac{1}{2} < x < 1.$

$$\left[\textbf{Ans. : } f(x) = \frac{4}{\pi}\sum \frac{1}{n}\sin^2\frac{m\pi}{4}\sin m\pi x\right]$$

28. Obtain half-range cosine series for $f(x) = x \sin x$ in $(0, \pi)$. **(M.U. 2004)**

$$\left[\textbf{Ans. : } f(x) = 1 - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2 - 1}\cos nx\right]$$

29. Obtain half range cosine series for

$f(x) = c, \quad 0 < x < a$

$\quad = 0, \quad a < x < b$

$$\left[\textbf{Ans. : } f(x) = \frac{ac}{b} + \frac{2c}{\pi}\left[\sin\frac{\pi a}{b}\cos\frac{\pi x}{b} + \frac{1}{2}\sin\frac{2\pi a}{b}\cos\frac{2\pi x}{b} + \ldots\right]\right]$$

30. Obtain sine series for

$f(x) = mx, \quad 0 < x \le \frac{\pi}{2}$ **(M.U. 2004)**

$\quad = m(\pi - x), \quad \frac{\pi}{2} \le x < \pi$

$$\left[\textbf{Ans. : } f(x) = \frac{4m}{\pi}\left[\frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \ldots\right]\right]$$

31. Obtain half range cosine series for $f(x) = \sin\left(\frac{\pi x}{l}\right)$ in $0 < x < l$. **(M.U. 2000, 02, 04, 09)**

$$\left[\textbf{Ans. : } f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{1}{1 \cdot 3}\cos\frac{2\pi x}{l} + \frac{1}{3 \cdot 5}\cos\frac{4\pi x}{l} + \ldots\right]\right]$$

32. Obtain half-range cosine series for $f(x) = (x - 1)^2$ in $0 < x < 1$. **(M.U. 2002, 03, 09)**

Hence, find $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$.

[**Ans. :** $f(x) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2}$. Now, put $x = 0$ and then $x = 1$]

33. Obtain half-range sine series for $f(x) = \pi x - x^2$ in $(0, \pi)$ and hence deduce that

(i) $\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \ldots = \frac{\pi^2}{960}$ (ii) $\frac{1}{1^6} + \frac{1}{2^6} + \frac{1}{4^6} + \ldots = \frac{\pi^2}{945}$.

$$\left[\textbf{Ans. : } f(x) = \frac{8}{\pi}\left[\frac{\sin x}{1^3} + \frac{\sin 3x}{3^3} + \frac{\sin 5x}{5^3} + \ldots\right]\right]$$

34. Obtain half-range sine series for $f(x)$ given by

$$f(x) = \begin{cases} kx, & 0 < x < (l/2) \\ k(l - x), & (l/2) < x < l \end{cases}$$

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Hence, deduce that $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\ldots\ldots=\frac{\pi^4}{96}$.

$$\left[\text{Ans. : } f(x)=\frac{4kl}{\pi^2}\left[\frac{\sin(\pi x/l)}{1^2}-\frac{\sin(3\pi x/l)}{3^2}+\frac{\sin(5\pi x/l)}{5^2}+\ldots\right]\right]$$

35. Obtain cosine series for $f(x)=\frac{\sin \pi x}{l}$, $0<x<l$. **(M.U. 2009)**

(B) 1. If $x^2=\frac{4l^2}{3}+\frac{4l^2}{\pi^2}\sum\frac{1}{n^2}\cos\left(\frac{n\pi x}{l}\right)-\frac{4l^2}{\pi}\sum\frac{1}{n}\sin\frac{n\pi x}{l}$

in $0<x<2l$, find the sum of the series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots\ldots$ **(M.U. 2003)**

[Ans. : $\pi^2/6$]

2. If $\left(\frac{\pi-x}{2}\right)^2=\frac{\pi^2}{12}+\sum\frac{1}{n^2}\cos nx$ in $0<x<2\pi$, find the sum of the series $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\ldots$

[Ans. : $\frac{\pi^4}{90}$]

3. If $f(x)=\begin{cases}x+\pi/2, & -\pi<x<0\\ \pi/2-x, & 0<x<\pi\end{cases}$ has the Fourier expansion

$f(x)=\frac{4}{\pi}\left[\frac{1}{1^2}\cos x+\frac{1}{3^2}\cos 3x+\ldots\ldots\right]$, prove that $\frac{\pi^2}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\ldots\ldots$

4. If the Fourier series of the function $f(x)=|x|$ in $(2,-2)$ is given by

$f(x)=1-\frac{8}{\pi^2}\sum\frac{1}{(2n-1)^2}\cos\left[\frac{(2n-1)}{2}\pi x\right]$, show that $\sum\frac{1}{(2n-1)^4}=\frac{\pi^4}{96}$. **(M.U. 2003)**

5. If $\sin x=\frac{2}{\pi}-\frac{4}{\pi}\left(\cos\frac{2x}{3}+\cos\frac{4x}{15}+\cos\frac{6x}{35}+\ldots\ldots\right)$ in $0\le x\le\pi$, find the sum of the series

$\frac{1}{1^2\cdot 3^2}+\frac{1}{3^2\cdot 5^2}+\frac{1}{5^2\cdot 7^2}+\ldots\ldots$ [Ans. : $\frac{\pi^2-8}{16}$]

6. If $x-x^2=\frac{8}{\pi^3}\left[\frac{1}{1^3}\sin\pi x+\frac{1}{3^3}\sin 3\pi x+\frac{1}{5^3}\sin 5\pi x+\ldots\right]$ in $0<x<1$,

find the sum of the series $\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\ldots\ldots$ [Ans. : $\frac{\pi^6}{945}$]

7. If $x^2=\frac{\pi}{3}+4\sum(-1)^n\cdot\frac{\cos nx}{n^2}$ for $-\pi<x<\pi$, prove that $\sum\frac{1}{n^4}=\frac{\pi^4}{90}$. **(M.U. 2003)**

EXERCISE - VIII

Theory

1. Define Fourier Series and evaluate the Fourier Constants. **(M.U. 1999)**
2. State Parseval's identity for a function $f(x)$ in $(-l, l)$. **(M.U. 1998, 2003)**

3. Derive Euler's formulae.
4. State and prove Parseval's identity for the function $f(x)$ in the interval $(c, c+2\pi)$. **(M.U. 1997)**
5. Define Fourier Series and State Dirichlet's Conditions. **(M.U. 1993)**
6. State and prove Parseval's identity for half range cosine series for $f(x)$ in the interval $(0, l)$. **(M.U. 2004)**
7. State Parseval's identity for a function $f(x)$ in $(-\pi, \pi)$. **(M.U. 2000, 04)**

Short Answer Questions

1. If $f(x)=4-x^2$, $0<x<2$ and is periodic with period 2, find $f(x)$ at $x=0, 2, 4, 6, \ldots\ldots$
2. Show that the Fourier series of

$f(x)=\begin{cases}4-x, & 3<x<4\\ x-4, & 4<x<5\end{cases}$ has no sine terms. **(M.U. 2003)**

3. Show that the Fourier series of

$f(x)=\begin{cases}0, & -5<x<0\\ 3, & 0<x<5\end{cases}$ with period 10 has no cosine terms.

4. Show that the Fourier series of

$f(x)=\begin{cases}-x, & -1<x<0\\ x, & 0<x<1\end{cases}$ has no sine terms.

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CHAPTER 4

Complex Variables

1. Introduction

In this chapter we are going to study functions of a complex variable $z = x + iy$ where x and y are real. We shall consider how to differentiate a function of a complex variable, the meaning of analytic functions of a complex variable and the conditions of analyticity viz. Cauchy-Riemann equations.

2. Definition of A Complex Function

If by a rule or set of rules we can find one or more complex numbers w for every z $(= x + iy)$ in a given domain, we say that **w is a function of z** and denote it as

$$w = f(z)$$

Since, both z and w are complex quantities the function is called a complex function.

If for a given z there corresponds one and only one w then the function is called **single valued function.** For example $w = z^2$ is a single valued function. If on the other hand if for a given z there correspond two or more values of w, then the function is called **multiple valued function.** For example $w = \sqrt[6]{z}$ is a multiple valued function. We shall consider single valued functions only.

Since, $z = x + iy$, $w = f(z)$ can be put in the form $w = u(x, y) + i\,v(x, y)$ where, u and v are functions of x and y. Thus, we can write

$$\boldsymbol{w = u(x, y) + iv(x, y)}$$

For example, if $w = z^2 + 2z + 3$ then

$$w = (x + iy)^2 + 2(x + iy) + 3 = x^2 + 2i\,xy - y^2 + 2x + 2iy + 3$$
$$= (x^2 - y^2 + 2x + 3) + i(2xy + 2y) = u(x, y) + iv(x, y)$$

3. Z-plane and W-plane

We know that real function $y = f(x)$ can be represented by a curve on x-y plane. But $w = f(z)$ *i.e.* $w = u + iv$ where, x and y are real, involves four variables x, y, u and v. Hence, we cannot represent $w = f(z)$ on a single plane. However, we can have sufficient idea of a function $w = f(z)$ if we use two planes, one z-plane on which we plot a point (x, y) and another w-plane on which we plot the corresponding point (u, v).

For example, consider a simple function $w = z^2$ *i.e.* $w = (x + iy)^2 = (x^2 - y^2) + 2ixy$. Here, $u = x^2 - y^2$, $v = 2xy$. If we have a point $P(2, 1)$ on the z-plane then since, $u = 4 - 1 = 3$ and $v = 4$, the corresponding point on the w-plane will be $P'(3, 4)$. As the point P traces a curve C in the z-plane the corresponding point P' will trace another curve C' on the w-plane. Thus, a curve C in the z-plane is transformed into another curve C' in the w-plane. We shall consider in more details this type of transformation of curves in z-plane into the curves in w-plane in the next chapter.

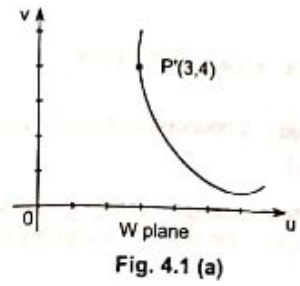

Fig. 4.1 (a)

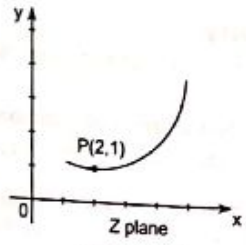

Fig. 4.1 (b)

4. Neighbourhood of A point $P(z_0)$

Consider, the inequality $|z - z_0| < \epsilon$ *i.e.* consider a circle with centre at z_0 and radius ϵ. The inequality clearly defines a region of all points lying within the circle $|z - z_0| = \epsilon$ including the point z_0 but excluding the points on the boundary of the circle. The circular region $|z - z_0| < \epsilon$ is called a **neighbourhood** of the point z_0. It is clear that as the number ϵ becomes smaller and smaller, the neighbourhood also becomes smaller and smaller.

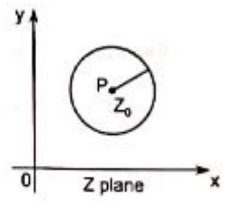

Fig. 4.2

5. Limit of A Function

The limit of a function of a complex variable $w = f(z)$ is defined on the lines of the definition of the limit of a function of a real variable.

Definition : Let $w = f(z)$ be a single valued function of z defined in a bounded and closed domain D and let z approach z_0 along any path in D. Given a positive number ϵ, however small (but not zero) if we can find another small positive number δ such that $|f(z) - w_0| < \epsilon$ for all z for which $0 < |z - z_0| < \delta$ then we say that w_0 is the limit of $f(z)$ as z tends to z_0 and denote it is

$$\lim_{z \to z_0} f(z) = w_0$$

(Note that the point z may approach z_0, along any path and the limit does not depend upon the path.)

Limits in terms of Real and Imaginary Parts : Let $f(z) = u + iv$ where, u and v are functions of real variables x, y. Let $z = x + iy$, $w_0 = u_0 + iv_0$, $z_0 = x_0 + iy_0$. Let $\lim_{z \to z_0} f(z) = w_0$.

Then by the above definition of the limit

$$|f(z) - w_0| < \epsilon \qquad \text{for } 0 < |z - z_0| < \delta$$

i.e. $|(u + iv) - (u_0 + iv_0)| < \epsilon$ for $0 < |(x + iy) - (x_0 + iy_0)| < \delta$.

i.e. $|(u - u_0) + i(v - v_0)| < \epsilon$ for $|(x - x_0) + i(y - y_0)| < \delta$.

This means, in terms of real functions

$$\lim_{\substack{x \to x_0 \\ y \to y_0}} u = u_0 \quad \text{and} \quad \lim_{\substack{x \to x_0 \\ y \to y_0}} v = v_0$$

$$\therefore \lim_{z \to z_0} f(z) = w_0 = u_0 + iv_0 = \lim_{\substack{x \to x_0 \\ y \to y_0}} u + i \lim_{\substack{x \to x_0 \\ y \to y_0}} v$$

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6. Continuity

Continuity of a function of a complex variable is defined exactly as the continuity of a function of a real variable.

Let $w = f(z)$ be a single valued function defined in a bounded and closed domain D. $w = f(z)$ is said to be continuous at $z = z_0$ if $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

Continuity in terms of Real and Imaginary Parts : Using the results obtained in § 5, it can be easily shown that if $w = f(z)$ is continuous at $z = z_0$ then the real and imaginary parts u and v are separately continuous at $z_0 = x_0 + i y_0$

i.e. $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} u = u_0$ and $\lim\limits_{\substack{x \to x_0 \\ y \to y_0}} v = v_0$

Example 1 : Discuss the continuity of the following functions at the given points

(i) $\dfrac{\bar{z}}{z}$ at $z = 0$ (ii) $\dfrac{z^2}{z^4 + 3z^2 + 1}$ at $z = e^{i\pi/4}$

Sol. : (i) Let $f(z) = \dfrac{\bar{z}}{z} = \dfrac{x - iy}{x + iy}$ ..... (i)

$$\therefore \; f(z) = \frac{x - iy}{x + iy} \cdot \frac{x - iy}{x - iy} = \frac{x^2 - y^2 - 2ixy}{x^2 + y^2} = \frac{x^2 - y^2}{x^2 + y^2} - 2i\frac{xy}{x^2 + y^2}$$

Let $u = \dfrac{x^2 - y^2}{x^2 + y^2}$, $v = -\dfrac{2xy}{x^2 + y^2}$

Let $y = kx$, $\therefore \; u = \dfrac{1 - k^2}{1 + k^2}$, $v = -\dfrac{2k}{1 + k^2}$

$\lim\limits_{\substack{x \to 0 \\ y \to 0}} u = \lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{x^2 - y^2}{x^2 + y^2} = \lim\limits_{x \to 0} \dfrac{1 - k^2}{1 + k^2} = \dfrac{1 - k^2}{1 + k^2}$ which depends upon k *i.e.* on the path

$\lim\limits_{\substack{x \to 0 \\ y \to 0}} v = \lim\limits_{\substack{x \to 0 \\ y \to 0}} \dfrac{-2xy}{x^2 + y^2} = \lim\limits_{x \to 0} -\dfrac{2k}{1 + k^2} = -\dfrac{2k}{1 + k^2}$ which depends upon k *i.e.* on the path.

Hence, $f(z)$ is not continuous at $z = 0$.

Alternative Method

We shall find the limits along the x-axis and along the y-axis.

Along the x-axis, $z = x + iy = x + i(0) = x$

$\therefore \; \lim\limits_{z \to 0} \dfrac{\bar{z}}{z} = \lim\limits_{x \to 0} \dfrac{x}{x} = 1$ from (i) ..... (ii)

Along the y-axis, $z = x + iy = 0 + iy = iy$

$\therefore \; \lim\limits_{z \to 0} \dfrac{\bar{z}}{z} = \lim\limits_{y \to 0} -\dfrac{iy}{iy} = -1$ from (i) ..... (iii)

From (i) and (ii), we see that the two limits along two different paths are different. Hence, the limit does not exist. Hence, $f(z)$ is not continuous at $z = 0$.

(ii) Let $f(z) = \dfrac{z^2}{z^4 + 3z^2 + 1}$

When $z = e^{i\pi/4}$, $z^2 = e^{i\pi/2} = \cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2} = i$

$z^4 = e^{i\pi} = \cos\pi + i\sin\pi = -1$

$\therefore$ At $z = e^{i\pi/4}$, $f(z) = \dfrac{i}{-1 + 3i + 1} = \dfrac{i}{3i} = \dfrac{1}{3}$

Now $\lim\limits_{z \to e^{i\pi/4}} = \dfrac{i}{-1 + 3i + 1} = \dfrac{i}{3i} = \dfrac{1}{3}$. Since, $\lim\limits_{z \to e^{i\pi/4}} f(z) = f(e^{i\pi/4})$

The function is continuous at $z = e^{i\pi/4}$.

Example 2 : Find the following limits

(i) $\lim\limits_{z \to i} \dfrac{z^2 + 1}{z^6 + 1}$ (ii) $\lim\limits_{z \to (1+i)} \left(\dfrac{z - (1+i)}{z^2 - 2z + z}\right)^3$

Sol. : (i) As $z \to i$, $z^2 \to -1$

$$\therefore \; \lim_{z \to i} \frac{z^2 + 1}{(z^2 + 1)(z^4 - z^2 + 1)} = \lim_{z^2 \to -1} \frac{1}{z^4 - z^2 + 1} = \frac{1}{3}.$$

(ii) Now, $z^2 - 2z + 2 = (z^2 - 2z + 1) + 1 = (z - 1)^2 - (i)^2$

$= (z - 1 + i)(z - 1 - i)$

$$\therefore \; \lim_{z \to (1+i)} \left\{\frac{z - (1+i)}{[z - (1-i)][z - (1+i)]}\right\}^3 = \lim_{z \to (1+i)} \left\{\frac{1}{z - (1-i)}\right\}^3$$

$$= \left\{\frac{1}{1 + i - 1 + i}\right\}^2 = \left\{\frac{1}{2i}\right\}^3 = -\frac{1}{8i}.$$

EXERCISE - I

1. Find $\lim\limits_{z \to 3i} (3x + iy^2)$.
2. Find $\lim\limits_{z \to 1} \dfrac{z^3 - 1}{z - 1}$.
3. Find $\lim\limits_{z \to i} \dfrac{z^2 + 1}{z - i}$.
4. Find $\lim\limits_{z \to i} \dfrac{z^3 + i}{z - i}$.
5. Show that $\lim\limits_{z \to 0} \dfrac{xy}{x^2 + y^2}$ does not exist.
6. Show that $\lim\limits_{z \to 0} \dfrac{x^2 y}{x^4 + y^2}$ does not exist. (**Hint :** Put $y = kx^2$)

[Ans. : (1) $9i$, (2) 3, (3) $2i$, (4) -3]

7. Differentiability

Definition : Let $w = f(z)$ be a single valued function of z defined in domain D. $f(z)$ is said to be differentiable at any point z if

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$$\lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

is unique as $\delta z \to 0$ along any path of the domain D.

Example 1 : Find whether the following functions are differentiable.

(i) z^3 at $z = i$ (ii) $\cos z$ at $z = i$

Sol. : (i) Let $f(z) = z^3$ and $z_0 = i$

$$\therefore \; f(z_0) = i^3 = -i$$

$$\therefore \; f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to i} \frac{z^3 - i^3}{z - i} = \lim_{z \to i} \frac{(z-i)(z^2 + iz + i^2)}{(z-i)}$$

$$= \lim_{z \to i} z^2 + iz + i^2 = 3i^2 = -3$$

$\therefore$ $f(z)$ is differentiable at $z = i$.

(ii) Let $f(z) = \cos z$ and $z_0 = i$.

$$\therefore \; f(z_0) = \cos i$$

$$\therefore \; f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to i} \frac{\cos z - \cos i}{z - i}$$

$$= \lim_{z \to i} \frac{-2\sin\left(\frac{z+i}{2}\right)\sin\left(\frac{z-i}{2}\right)}{z - i} = \lim_{z \to i} -2\sin\left(\frac{z+i}{2}\right)\frac{\sin\left(\frac{z-i}{2}\right)}{\left(\frac{z-i}{2}\right)} \cdot \frac{1}{2}$$

$$= -\sin i.$$

$\therefore$ $f(z)$ is differentiable at $z = i$.

Example 2 : Prove that the function $|z|^2$ is continuous everywhere but nowhere differentiable except at the origin.

Sol. : Let $f(z) = |z|^2 = |x + iy|^2 = x^2 + y^2$

Since, $x^2 + y^2$ is continuous everywhere, $f(z)$ is also continuous everywhere.

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z_0 + \delta z) - f(z_0)}{\delta z} = \lim_{\delta z \to 0} \frac{|z_0 + \delta z|^2 - |z_0|^2}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{(z_0 + \delta z)(\bar{z}_0 + \delta \bar{z}) - z_0 \bar{z}_0}{\delta z} = \lim_{\delta z \to 0} \frac{z_0 \delta \bar{z} + \bar{z}_0 \delta z + \delta z \delta \bar{z}}{\delta z}$$

$$= \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \bar{z}_0 + \delta \bar{z} \qquad \text{..... (i)}$$

(i) When δz is real : Then $\delta y = 0$ and $\delta \bar{z} = \delta z = \delta x$. As $\delta z \to 0$, $\delta x \to 0$.

$$\therefore \; f'(z_0) = \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \bar{z}_0 + \delta \bar{z} = \lim_{\delta x \to 0} (z_0 + \bar{z}_0 + \delta x) = z_0 + \bar{z}_0$$

(ii) When dz is imaginary : Then $dx = 0$ and $\delta z = i\,\delta y$, $\delta \bar{z} = -i\,\delta y$. As $\delta z \to 0$, $\delta y \to 0$.

$$\therefore \; f'(z_0) = \lim_{\delta z \to 0} z_0 \frac{\delta \bar{z}}{\delta z} + \bar{z}_0 + \delta \bar{z} = \lim_{\delta x \to 0} z_0 \left(-\frac{i\,\delta y}{i\,\delta y}\right) + \bar{z}_0 - i\,\delta y$$

$$= -z_0 + \bar{z}_0$$

Since, the two limits are different along two different paths except at $z = 0$, $f'(z_0)$ does not exist anywhere except at $z = 0$.

Hence, $f(z)$ is not differentiable anywhere except at $z = 0$.

8. Analytic Functions

If a single valued function $w = f(z)$ is defined and differentiable at each point of a domain D then it is called analytic or regular or holomorphic function of z in the domain D.

A function is said to be analytic at a point if it has a derivative at that point and in some neighbourhood of that point.

If a function ceases to be analytic at a point of the domain then the point is called a **singular point.**

9. Cauchy-Riemann Equations in Cartesian Coordinates

Theorem : The necessary and sufficient conditions for a continuous one valued function

$$w = f(z) = u(x, y) + i\,v(x, y)$$

to be analytic in a region R are (i) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are continuous functions of x and y in a region R and

(ii) $\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \; (i.e.\; u_x = v_y \text{ and } u_y = -v_x)}$

at each point of R.

The conditions (ii) are known as **Cauchy-Riemann equations** or briefly **C-R equations.**

Augustin Louis (Baron de) Cauchy (1789 - 1857)

A French mathematician of great repute who contributed to various branches of mathematics. He wanted to be an engineer but because of poor health he was advised to pursue mathematics. His mathematical work began in 1811 when he gave briliant solutions to some difficult problems of that time. In the next 35 years he published 700 papers in various branches of mathematics. He is supposed to have initiated the era of modern analysis.

Bernhard Riemann (1826 - 1866)

Bernhard Riemann was a great German mathematician who made lasting contributions to analysis and differential geometry. Riemann exhibited exceptional mathematical skills at early age. He joined University of Göttingen to study mathematics and at this university he first met Carl Friedrich Gauss. In 1847 Riemann moved to Berlin where Jacobi, Dirichlet and Steiner were teaching. He founded the field of Riemannian geometry which was used by Einstein in general theory of relativity. He was the first to suggest dimensions higher than three or four. Riemann made major contributions to real analysis. He defined Riemann integral by means of Riemann sums. He introduced Riemann-zeta function. He is also known for Riemannian metric, Riemannian geometry, Riemannian curvature tensor.

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Proof : (a) The conditions are necessary

Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic at every point of a region R. Then dw/dz exists uniquely at every point of R.

Let δx and δy be the increments in x, y. Let $\delta u, \delta v, \delta w$ be the corresponding increments in u, v, w respectively. Now,

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \lim_{\delta z \to 0} \frac{(u+\delta u) + i(v+\delta v) - (u+iv)}{\delta z}$$

$$= \lim_{\delta z \to 0} \frac{\delta u + i\,\delta v}{\delta z} = \lim_{\delta z \to 0} \left(\frac{\delta u}{\delta z} + i\frac{\delta v}{\delta z}\right)$$

As $w = f(z)$ is analytic in R, the above limit exists independent of the path along which $\delta z \to 0$. Since, $\delta z = \delta x + i\,\delta y$, the limit is independent of the path along which $\delta x \to 0$ and $\delta y \to 0$.

First consider the path ($QP'P$) first parallel to the x-axis *i.e.* $\delta z \to 0$ such that $\delta y = 0$ and $\delta z = \delta x$ and then $\delta x \to 0$.

$$\therefore\ f'(z) = \lim_{\delta x \to 0} \left(\frac{\delta u}{\delta x} + i\frac{\delta v}{\delta x}\right) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \qquad \text{..... (i)}$$

Now let $\delta z \to 0$ along the path ($QP''P$) first parallel to the y-axis *i.e.* $\delta z \to 0$ such that $\delta x = 0$ and $\delta z = i\,\delta y$ and then $\delta y \to 0$.

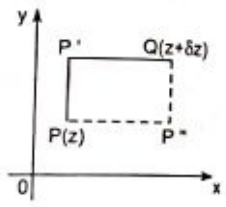

Fig. 4.3

$$\therefore\ f'(z) = \lim_{\delta y \to 0} \left(\frac{\delta u}{i\,\delta y} + i\frac{\delta v}{i\,\delta y}\right)$$

$$\therefore\ f'(z) = \frac{1}{i}\cdot\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \left[\because \frac{1}{i} = \frac{i}{i^2} = -i\right]$$

$$= -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} \qquad \text{..... (ii)}$$

For existence of $f'(z)$ (i) and (ii) must be equal.

From (i) and (ii) we get,

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

Equating the real and imaginary parts

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

This proves that for a function to be analytic, the **Cauchy-Riemann equations are necessary.**

(b) The conditions are sufficient

Let $f(z) = u(x, y) + iv(x, y)$ be a single valued function possessing continuous partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ at every point in the region R and satisfying the conditions

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

We have to show that $f'(z)$ exists at every point of R.

By Taylor's theorem for functions of two variables, omitting the second and higher degree terms in x and y, we get,

$$f(z+\delta z) = u(x+\delta x,\ y+\delta y) + iv(x+\delta x,\ y+\delta y)$$

$$= \left[u(x,y) + \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y\right)\right] + i\left[v(x,y) + \left(\frac{\partial v}{\partial x}\delta x + \frac{\partial v}{\partial y}\delta y\right)\right]$$

$$= [u(x,y) + iv(x,y)] + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$= f(z) + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

$$\therefore\ f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\delta y$$

Using the C-R equations (in the second bracket)

$$f(z+\delta z) - f(z) = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right)\delta y$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta x + \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)i\,\delta y \qquad [\because i^2 = -1]$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)(\delta x + i\,\delta y)$$

$$= \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\delta z \qquad [\because \delta x + i\,\delta y = \delta z]$$

$$\therefore\ \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\therefore\ f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

Hence, $f'(z)$ exists as $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial x}$ exist.

$\therefore$ $f(z)$ is analytic.

$\therefore$ The conditions are sufficient.

Notes ....

1. The Cauchy-Riemann equations are only **necessary conditions** for a function to be analytic. This means even if Cauchy-Riemann equations are satisfied the function need not be analytic at that point (see Examples 10 and 11, page 4-14)

2. When $f(z)$ is analytic, its derivative is given by any one of the following expressions

$$\boxed{\begin{aligned} f'(z) &= u_x + i\,v_x\ ;\ f'(z) = v_y + i\,v_x \\ f'(z) &= u_x - i\,u_y\ ;\ f'(z) = v_y - i\,u_y \end{aligned}}$$

3. **If $f(z)$ is analytic then it can be differentiated in usual manner.**

e.g. if $f(z) = z^2$ then $f'(z) = 2z$

if $f(z) = \sin z$ then $f'(z) = \cos z$

4. If $f(z) = f(x+iy) = u + iv$ and $f(z)$ is analytic then the functions u and v of real variables x and y are called **conjugate functions.**

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Example 1 : Prove that an analytic function with its derivative zero is constant.

Sol. : Let $f(z) = u + iv$ be the given analytic function whose derivative is zero.

$\therefore \quad f'(z) = u_x + iv_x = 0 \qquad \therefore \quad u_x = 0, v_x = 0$

But $f(z)$ is analytic. Hence, C-R equations are satisfied

$\therefore \quad u_x = v_y$ and $u_y = -v_x \qquad \therefore \quad v_y = 0, u_y = 0$

As $u_x = 0, u_y = 0 \;\therefore\; u$ = a constant and $v_x = 0, v_y = 0 \;\therefore\; v$ = a constant.

$\therefore \quad f(z) = u + iv$ = a constant.

Example 2 : If $f(z)$ and $\overline{f(z)}$ are both analytic, prove that $f(z)$ is constant.

(M.U. 1993, 2003, 06)

Sol. : Let $f(z) = u + iv$ then $\overline{f(z)} = u - iv = u + i(-v)$

Since, $f(z)$ is analytic $u_x = v_y$ and $u_y = -v_x$ C-R equations.

Since, $\overline{f(z)}$ is analytic $u_x = (-v_y)$ and $u_y = -(-v_x)$ C-R equation.

Adding $u_x = v_y$ and $u_x = -v_y$, we get $u_x = 0$.

Adding $u_y = -v_x$ and $u_y = v_x$, we get $u_y = 0$.

Since, $u_x = 0$ and $u_y = 0$, u = a constant.

Similarly by subtraction we can prove that $v_x = 0$ and $v_y = 0$. $\therefore$ v = a constant.

Hence, $f(z) = u + iv$ = a constant.

Example 3 : If $f(z)$ is an analytic function, show that $\dfrac{\partial f}{\partial \bar{z}} = 0$. **(M.U. 1996)**

Sol. : Since, $z = x + iy$, $\bar{z} = x - iy$.

$\therefore \quad x = \dfrac{1}{2}(z + \bar{z})$ and $y = \dfrac{1}{2i}(z - \bar{z})$.

Let $f(z) = u + iv$

$$\therefore \quad \frac{\partial f}{\partial \bar{z}} = \frac{\partial}{\partial \bar{z}}(u + iv) = \left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right) + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}\right)$$

$$= \left[\frac{\partial u}{\partial x}\cdot\frac{1}{2} + \frac{\partial u}{\partial y}\left(-\frac{1}{2i}\right)\right] + i\left[\frac{\partial v}{\partial x}\cdot\frac{1}{2} + \frac{\partial v}{\partial y}\left(-\frac{1}{2i}\right)\right]$$

$$= \frac{1}{2}u_x - \frac{1}{2i}u_y + \frac{i}{2}v_x - \frac{1}{2}v_y$$

$$= \frac{1}{2}u_x + \frac{i}{2}u_y + \frac{i}{2}v_x - \frac{1}{2}v_y$$

But since, $f(z)$ is analytic, $u_x = v_y$ and $u_y = -v_x$.

$$\therefore \quad \frac{\partial f}{\partial \bar{z}} = \frac{1}{2}v_y - \frac{i}{2}v_x + \frac{i}{2}v_x - \frac{1}{2}v_y = 0$$

Restatement

The above theorem can also be stated as follows.

"Every analytic function $w = u + iv$ can be expressed as a function of z only".

Notes ....

1. We have proved that if $f(z)$ is analytic then $\partial f / \partial \bar{z} = 0$. This means if $f(z)$ is analytic, $f(z)$ is free from $\bar{z}$. In other words if $f(z)$ is analytic, it can be reconstructed in z *i.e.* in $x + iy$ only.

2. It can be easily shown that usual rules of differentiating sums, products, quotients of real functions are applicable to analytic functions of complex variable.

The formulae for differentiating elementary complex functions are identical with the corresponding formulae in calculus of real variables.

For example, $\dfrac{d}{dz}(z^n) = n z^{n-1}$, $\dfrac{d}{dz}(e^z) = e^z$, $\dfrac{d}{dz}(\log z) = \dfrac{1}{z}$,

$\dfrac{d}{dz}(\sin z) = \cos z$, $\dfrac{d}{dz}(\sin hz) = \cos hz$.

In general, if the differential coefficient of $f(x)$ is $\phi(x)$, we can assume that if $f(z)$ is analytic, the differential coefficient of $f(z)$ is $\phi(z)$.

3. Since integration is inverse of differentiation a result similar to the one quoted in note 2 holds for integration also.

e.g. $\int z^n dz = \dfrac{z^{n+1}}{n+1}$, $\int e^z dz = e^z$ etc.

Derivatives of Elementary Functions

1. $\dfrac{d}{dz}(c) = 0$
2. $\dfrac{d}{dz}(z^n) = n z^{n-1}$
3. $\dfrac{d}{dz}(e^z) = e^z$
4. $\dfrac{d}{dz}(a^z) = a^z \log a$
5. $\dfrac{d}{dz}(\sin z) = \cos z$
6. $\dfrac{d}{dz}(\cos z) = -\sin z$
7. $\dfrac{d}{dz}(\tan z) = \sec^2 z$
8. $\dfrac{d}{dz}(\cot z) = -\operatorname{cosec}^2 z$
9. $\dfrac{d}{dz}(\sec z) = \sec z \tan z$
10. $\dfrac{d}{dz}(\operatorname{cosec} z) = -\operatorname{cosec} z \cot z$
11. $\dfrac{d}{dz}(\log z) = \dfrac{1}{z}$
12. $\dfrac{d}{dz}(\log_a z) = \dfrac{1}{z \log_e a}$
13. $\dfrac{d}{dz}(\sin^{-1} z) = \dfrac{1}{\sqrt{1 - z^2}}$
14. $\dfrac{d}{dz}(\cos^{-1} z) = -\dfrac{1}{\sqrt{1 - z^2}}$
15. $\dfrac{d}{dz}(\tan^{-1} z) = \dfrac{1}{1 + z^2}$
16. $\dfrac{d}{dz}(\cot^{-1} z) = -\dfrac{1}{1 + z^2}$
17. $\dfrac{d}{dz}(\sec^{-1} z) = \dfrac{1}{z\sqrt{z^2 - 1}}$
18. $\dfrac{d}{dz}(\operatorname{cosec}^{-1} z) = \dfrac{-1}{z\sqrt{z^2 - 1}}$
19. $\dfrac{d}{dz}(\sin hz) = \cos hz$
20. $\dfrac{d}{dz}(\cos hz) = \sin hz$
21. $\dfrac{d}{dz}(\tan hz) = \sec h^2 z$
22. $\dfrac{d}{dz}(\cot hz) = -\operatorname{cosec} h^2 z$
23. $\dfrac{d}{dz}(\sec hz) = -\sec hz \tan hz$
24. $\dfrac{d}{dz}(\operatorname{cosec} hz) = -\operatorname{cosec} hz \coth$

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25. $\frac{d}{dz}(\sinh^{-1}z) = \frac{1}{\sqrt{1+z^2}}$

26. $\frac{d}{dz}(\cosh^{-1}z) = \frac{1}{\sqrt{z^2-1}}$

27. $\frac{d}{dz}(\tanh^{-1}z) = \frac{1}{1-z^2}$

28. $\frac{d}{dz}(\coth^{-1}z) = \frac{1}{z^2-1}$

29. $\frac{d}{dz}(\sec h^{-1}z) = \frac{-1}{z\sqrt{1-z^2}}$

30. $\frac{d}{dz}(\operatorname{cosec} h^{-1}z) = \frac{-1}{z\sqrt{z^2+1}}$

Example 4 : If $f(z)$ is an analytic and $|f(z)|$ is constant, prove that $f(z)$ is constant. **(M.U. 1999, 2002, 03, 05, 08, 09)**

Sol. : Let $f(z) = u + iv$ but $|f(z)| = C$. $\therefore\ u^2 + v^2 = C^2$

Differentiating it partially w.r.t. x, $uu_x + vv_x = 0$

Differentiating it partially w.r.t. y, $uu_y + vv_y = 0$

Since, $f(z)$ is analytic $u_x = v_y$ and $u_y = -v_x$

$\therefore\ uu_x - vu_y = 0$ and $uu_y + vu_x = 0$

Eliminating u_y, $(u^2 + v^2)u_x$ $\therefore\ C^2u_x = 0$ $\therefore\ u_x = 0$

Similarly, we can show that $u_y = 0$, $v_x = 0$, $v_y = 0$

Since, $f(z)$ is analytic. $f'(z) = u_x + iv_x = 0$ $\therefore\ f(z) =$ constant.

Restatement

(i) The above theorem can also be restated as

"If $f(z)$ is an analytic function with constant modulus then, prove that $f(z)$ is constant." **(M.U. 1994, 99)**

(ii) A regular function of constant magnitude is constant. **(M.U. 2005)**

Example 5 : If $f(z)$ is analytic and if the amplitude of $f(z)$ is constant, prove that $f(z)$ is constant. **(M.U. 2003)**

Sol. : Let $f(z) = u + iv$. Since its amplitude $= \tan^{-1}(v/u)$ is constant c say, we have

$$\tan^{-1}\frac{v}{u} = c \quad \therefore\ \frac{v}{u} = \tan c$$

Differentiating this w.r.t. x and y,

$$\frac{uv_x - vu_x}{u^2} = 0 \text{ and } \frac{uv_y - vu_y}{u} = 0$$

$\therefore\ uv_x - vu_x = 0$ and $uv_y - vu_y = 0$

Since $f(z)$ is analytic, $u_x = v_y$ and $u_y = -v_x$.

$\therefore\ -uu_y - vu_x = 0$ ..... (1) and $uu_x - vu_y = 0$ ..... (2)

Multiply the first by u and second by v and add.

$\therefore\ (-u^2 - v^2)u_y = 0$ $\therefore\ u_y = 0$

Multiply the first by v and second by u and subtract

$\therefore\ (-v^2 - u^2)u_x = 0$ $\therefore\ u_x = 0$

But $u_x = v_y$ and $u_y = -v_x$ $\therefore\ v_y = 0$ and $v_x = 0$

Since, all four partial derivatives of u, v are zero, u and v are constants.

$\therefore\ f(z)$ is constant.

Example 6 : If $f(z) = u + iv$ is an analytic function and (i) u = constant or (ii) v = constant then $f(z)$ is constant

Sol. : If u is constant $u_x = 0$, $u_y = 0$.

But $f'(z) = u_x + iv_x$

$= u_x - iu_y$ (By C-R equations)

$= 0$ (By data)

$\therefore\ f(z) =$ constant.

Similarly we can prove the other part.

Note ....

From Examples 1, 2, 4, 5 and 6, we find that an **analytic function** $f(z)$ is constant if (i) $f'(z) = 0$ or (ii) $\overline{f(z)}$ is also analytic or (iii) modulus of $f(z)$ is constant or (iv) amplitude of $f(z)$ is constant or (v) real part is constant or (vi) imaginary part is constant.

Example 7 : Show that the following functions are analytic and find their derivatives.

(i) e^z, **(ii)** $\sin hz$ **(M.U. 1999, 2015)**

Sol. : (i) $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i\sin y)$

$\therefore\ u = e^x\cos y,\ v = e^x\sin y$

$u_x = e^x\cos y,\ u_y = -e^x\sin y;\quad v_x = e^x\sin y,\ v_y = e^x\cos y$

$\therefore\ u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x, v_y are continuous and Cauchy-Riemann equations are satisfied.

Hence, e^z is analytic.

Now, $f'(z) = u_x + iv_x = e^x\cos y + ie^x\sin y = e^x(\cos y + i\sin y)$

$= e^x \cdot e^{iy} = e^{x+iy} = e^z$.

(ii) $f(z) = \sin h\, z = \sin h(x + iy)$

$= \sin h\, x\cos h\, iy + \cos h\, x\sin h\, iy$

$= \sin h\, x\cos y + i\cos h\, x\sin y$

$\therefore\ u = \sin h\, x\cos y,\quad v = \cos h\, x\sin y$

$u_x = \cos h\, x\cos y,\ u_y = -\sin h\, x\sin y$

$v_x = \sin h\, x\sin y,\quad v_y = \cos h\, x\cos y$

$\therefore\ u_x = v_y$ and $u_y = -v_x$

Further, u_x, u_y, v_x, v_y are continuous and Cauchy-Riemann equations are satisfied.

Hence, $\sin h\, z$ is analytic.

Now, $f'(z) = u_x + iv_x$

$= \cos hx\cos y + i\sin hx\sin y$

$= \cos hx\cos hiy + \sin hx\sin hiy$

$= \cos h(x + iy) = \cos hz$.

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Example 8 : Show that the following functions are analytic and find their derivatives.

(i) $f(z) = z^3$ (ii) $f(z) = ze^z$ **(M.U. 2016)** (iii) $f(z) = \sin z$. **(M.U. 2003)**

Sol : (i) We have $f(z) = z^3 = (x + iy)^3$

$\therefore\ f(z) = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore\ u = x^3 - 3xy^2,\ v = 3x^2y - y^3$

$\therefore\ \dfrac{\partial u}{\partial x} = 3x^2 - 3y^2,\ \dfrac{\partial v}{\partial x} = 6xy,\ \dfrac{\partial u}{\partial y} = -6xy,\ \dfrac{\partial v}{\partial y} = 3x^2 - 3y^2$

$\therefore\ u_x = v_y$ and $u_y = -v_x$ $\therefore\ f(z) = z^3$ is analytic and can be differentiated as usual.

$\therefore\ f(z) = 3z^2$.

(ii) We have $f(z) = ze^z = (x + iy)\,e^{x+iy}$

$\therefore\ f(z) = (x + iy)\,e^x(\cos y + i\sin y)$

$\therefore\ u = e^x(x\cos y - y\sin y),\ v = e^x(x\sin y + y\cos y)$

$\dfrac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y,\quad \dfrac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$,

$\dfrac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x\sin y,\quad \dfrac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y)$.

$\therefore\ u_x = v_y$ and $u_y = -v_x$

$\therefore\ f(z) = ze^z$ is analytic and can be differentiated as usual.

$\therefore\ f'(z) = ze^z + e^z = e^z(z + 1)$.

(iii) $f(z) = \sin z = \sin(x + iy) = \sin x\cos iy + \cos x\sin iy$

$= \sin x\cosh y + i\cos x\sinh y$

$\therefore\ u = \sin x\cosh y,\ v = \cos x\sinh y$

$\dfrac{\partial u}{\partial x} = \cos x\cosh y,\ \dfrac{\partial v}{\partial x} = -\sin x\sinh y,\ \dfrac{\partial u}{\partial y} = \sin x\sinh y,\ \dfrac{\partial v}{\partial y} = \cos x\cosh y$

$\therefore\ u_x = v_y$ and $u_y = -v_x$

$\therefore\ f(z) = \sin z$ is analytic and can be differentiated as usual.

$\therefore\ f'(z) = \cos z$.

Example 9 : If $f(z)$ is equal to (a) $\bar{z}$ (b) $2x + ixy^2$, show that $f'(z)$ does not exist. **(M.U. 2002)**

Sol. : (a) $f(z) = \bar{z} = x - iy$ $\therefore\ u = x,\ v = -y$ $\therefore\ u_x = 1,\ u_y = 0\ ;\ v_y = -1,\ v_x = 0$

Since, $u_x \neq v_y$ Cauchy-Riemann equations are not satisfied and $f'(z)$ does not exist.

Alternatively

$$f'(z) = \lim_{\delta z \to 0} \frac{\overline{z + \delta z} - \bar{z}}{\delta z}$$

$$\therefore\ f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\overline{(x + iy + \delta x + i\delta y)} - \overline{(x + iy)}}{\delta x + i\delta y}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{x - iy + \delta x - i\delta y - x + iy}{\delta x + i\delta y} = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{\delta x - i\delta y}{\delta x + i\delta y}$$

If $\delta y = 0$, the required limit $= \lim\limits_{\delta x \to 0} \dfrac{\delta x}{\delta x} = 1$

If $\delta x = 0$, the required limit $= \lim\limits_{\delta y \to 0} -\dfrac{\delta y}{\delta y} = -1$

Since, the two limits are different $f'(z)$ does not exist.

(b) $f(z) = 2x + ixy^2$ $\therefore\ u = 2x,\ v = xy^2$

$u_x = 2,\ u_y = 0,\ v_x = y^2,\ v_y = 2xy$

Since, $u_x \neq v_y$ and $u_y \neq v_x$. Cauchy-Riemann equations are not satisfied and hence, $f'(z)$ does not exist.

Example 10 : Show that $f(z) = z\bar{z} = |z|^2$ satisfies Cauchy-Riemann equations at $z = 0$ and yet is not analytic anywhere. **(M.U. 2004)**

Sol. : $f(z) = |z|^2 = x^2 + y^2$ $\therefore\ u = x^2 + y^2,\ v = 0$

$\therefore\ \dfrac{\partial u}{\partial x} = 2x,\ \dfrac{\partial u}{\partial y} = 2y,\ \dfrac{\partial v}{\partial x} = 0$ and $\dfrac{\partial v}{\partial y} = 0$

Hence, $u_x = v_y = 0$ and $u_y = -v_x = 0$ when $x = 0$ and $y = 0$.

Thus, C-R equations are satisfied at $z = 0$.

The partial derivatives u_x, u_y, v_x, v_y are also continuous everywhere.

Thus, $f'(z) = |z|^2$ is differentiable **only** at $z = 0$ but at no other point. There is no neighbourhood of $z = 0$ in which the conditions of analyticity are satisfied. Hence, $f(z)$ is **not analytic** anywhere.

Example 11 : Show that $f(z) = \sqrt{|xy|}$ is not analytic at the origin although Cauchy-Riemann equations are satisfied at that point. **(M.U. 2005, 08)**

Sol. : Let $f(z) = u + iv$ so that $u = \sqrt{|xy|}$ and $v = 0$

Now, $\dfrac{\partial u}{\partial x} = \lim\limits_{x \to 0} \dfrac{u(x, 0) - u(0, 0)}{x} = \dfrac{0 - 0}{x} = 0$

$\dfrac{\partial u}{\partial y} = \lim\limits_{y \to 0} \dfrac{u(0, y) - u(0, 0)}{y} = \dfrac{0 - 0}{y} = 0$

Clearly since, $v = 0$, $\dfrac{\partial v}{\partial x} = 0$, $\dfrac{\partial v}{\partial y} = 0$ at zero.

$\therefore$ Cauchy-Riemann equations are satisfied at $z = 0$.

Now $f'(0) = \lim\limits_{z \to 0} \dfrac{f(z) - f(0)}{z} = \lim\limits_{z \to 0} \dfrac{\sqrt{|xy|} - 0}{x + iy}$

Let $z \to 0$ along $y = mx$

$$\therefore\ f'(0) = \lim_{x \to 0} \frac{\sqrt{|mx^2|}}{x(1 + im)} = \lim_{x \to 0} \frac{x\sqrt{|m|}}{x(1 + im)} = \frac{\sqrt{m}}{1 + im}$$

Since, the limit depends upon m, $f'(0)$ does not exist.

Note ....

The above function satisfies Cauchy-Riemann equations and yet is not analytic at $z = 0$. This is because for analyticity in addition to C-R equations its four partial derivatives $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$

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must be continuous at that point. The value of u_x at $x = 0$ is zero. We shall now find lim of u_x as $x \to 0$.

Now, if $x > 0$, $\delta x > 0$ $\frac{\partial u}{\partial x} = \frac{\sqrt{|y|}}{2\sqrt{|x|}}$, let $y = mx$

$$\therefore \lim_{x \to 0} u_x = \lim_{x \to 0} \frac{\sqrt{|mx|}}{2\sqrt{|x|}} = \frac{|m|}{2}$$

$\therefore$ Thus, the limit depends upon the path and hence, u_x is not continuous at $x = 0$. Similarly u_y, v_x, v_y are not continuous at $x = 0$. Hence, $f(x)$ is **not** analytic even though C-R equations are satisfied.

Example 12 : Prove that $f(z) = (x^3 - 3xy^2 + 2xy) + i(3x^2y - x^2 + y^2 - y^3)$ is analytic and find $f'(z)$ and $f(z)$ in terms of z. **(M.U. 2004)**

Sol. : We have $u = x^3 - 3xy^2 + 2xy$, $v = 3x^2y - x^2 + y^2 - y^3$

$$\therefore \frac{\partial u}{\partial x} = 3x^2 - 3y^2 + 2y, \quad \frac{\partial u}{\partial y} = -6xy + 2x$$

$$\frac{\partial v}{\partial x} = 6xy - 2x, \quad \frac{\partial v}{\partial y} = 3x^2 + 2y - 3y^2$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Also partial derivatives are continuous. $\therefore$ $f(z)$ is analytic.

Now, $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (3x^2 - 3y^2 + 2y) + i(6xy - 2x)$

$$\therefore f'(z) = 3(x^2 + 2i\,xy - y^2) - 2i(x + iy)$$
$$= 3(x^2 + 2i\,xy + i^2y^2) - 2i(x + iy) = 3z^2 - 2iz.$$

Or by Milne-Thompson method (See page 4-29), putting $x = z$, $y = 0$, we get $f'(z) = 3z^2 - 2iz$.

$\therefore$ By integration, $f(z) = z^3 - iz^2 + k$.

Example 13 : Show that $w = \frac{x}{x^2 + y^2} - \frac{iy}{x^2 + y^2}$ is an analytic function and find $\frac{dw}{dz}$ in terms of z. **(M.U. 2003, 07)**

Sol. : Since, $u = \frac{x}{x^2 + y^2}$, $\frac{\partial u}{\partial x} = u_x = \frac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$

$$\frac{\partial u}{\partial y} = u_y = -\frac{x \cdot 2y}{(x^2 + y^2)^2}$$

$$v = -\frac{y}{x^2 + y^2} \quad \therefore \frac{\partial v}{\partial x} = v_x = +\frac{y \cdot 2x}{(x^2 + y^2)^2}$$

$$\frac{\partial v}{\partial y} = v_y = -\frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{-x^2 + y^2}{(x^2 + y^2)^2}$$

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

Further u_x, u_y, v_x and v_y are continuous functions except at $z = x + iy = 0$ *i.e.* $(x = 0, y = 0)$. w is analytic everywhere except at $z = 0$.

$$\therefore \frac{dw}{dz} = u_x + iv_x = \frac{-x^2 + y^2}{(x^2 + y^2)^2} + i \cdot \frac{2xy}{(x^2 + y^2)^2} \quad \text{..... (i)}$$

$$= -\frac{(x^2 - 2ixy - y^2)}{(x^2 + y^2)^2} = -\frac{(x^2 - 2ixy + i^2y^2)}{(x^2 - i^2y^2)^2}$$

$$= -\frac{(x - iy)^2}{(x - iy)^2(x + iy)^2} = -\frac{1}{(x + iy)^2} = -\frac{1}{z^2}$$

[Or to find $\frac{dw}{dz}$ in terms of z, use Milne-Thomson method (See page 4-29) and put $x = z$, $y = 0$ in (i) $\therefore$ $\frac{dw}{dz} = -\frac{1}{z^2}$.]

Example 14 : Show that $f(z) = \frac{\bar{z}}{|\bar{z}|^2}$, $|z| \neq 0$ is analytic and find $f'(z)$. **(M.U. 2013)**

Sol. : We have $f(z) = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$

Now, proceed as in the above example.

Example 15 : Is $f(z) = \frac{z}{\bar{z}}$ analytic? **(M.U. 2016)**

Sol. : We have

$$f(z) = \frac{z}{\bar{z}} = \frac{x + iy}{x - iy} = \frac{(x + iy)}{(x - iy)} \cdot \frac{(x + iy)}{(x + iy)}$$

$$= \frac{(x + iy)^2}{x^2 + y^2} = \frac{x^2 + 2i\,xy + y^2}{x^2 + y^2}$$

$$= \frac{x^2 + y^2}{x^2 + y^2} + i \cdot \frac{2xy}{x^2 + y^2}$$

$\therefore$ $u = 1$ and $v = \frac{2xy}{x^2 + y^2}$.

Now, $\frac{\partial u}{\partial x} = 0$, $\frac{\partial u}{\partial y} = 0$. But $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are not zero.

Hence, $f(z)$ is not analytic.

Example 16 : Show that $f(z) = \frac{xy^2(x + iy)}{x^2 + y^4}$ when $z \neq 0$

$= 0$ when $z = 0$ is not differentiable at $z = 0$.

Sol. : By definition,

$$f'(0) = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0} = \lim_{z \to 0} \frac{xy^2(x + iy) - 0}{(x^2 + y^4)(x + iy)} = \lim_{z \to 0} \frac{xy^2}{x^2 + y^4}$$

Now, let $z \to 0$ along path $y^2 = mx$

$$\therefore f'(0) = \lim_{x \to 0} \frac{x \cdot mx}{x^2 + m^2x^2} = \lim_{x \to 0} \frac{x^2}{x^2} \cdot \left(\frac{m}{1 + m^2}\right) = \frac{m}{1 + m^2}$$

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Since, the derivative depends upon the path, $f'(0)$ does not exist and hence, the function is not differentiable at $z = 0$.

Example 17 : Prove that the function defined by

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} \quad \text{when } z \neq 0$$

$$= 0 \quad \text{when } z = 0$$

is (i) continuous ; (ii) Cauchy-Riemann equations are satisfied at the origin and yet (iii) $f'(0)$ does not exist. **(M.U. 2002, 13)**

Sol. : We have $u = \dfrac{x^3 - y^3}{x^2 + y^2}$, $v = \dfrac{x^3 + y^3}{x^2 + y^2}$

(i) When $z \neq 0$, $x \neq 0$, $y \neq 0$

Hence, u, v being rational functions of x and y are continuous and consequently $f(z)$ is continuous when $z \neq 0$

To test the continuity at $z = 0$ we put $x = r\cos\theta$, $y = r\sin\theta$

$$\therefore \quad u = r(\cos^3\theta - \sin^3\theta), \quad v = r(\cos^3\theta + \sin^3\theta)$$

When $z \to 0$, $r \to 0$

$$\therefore \quad \lim_{z\to 0} u = \lim_{r\to 0} r(\cos^3\theta - \sin^3\theta) = 0$$

$$\lim_{z\to 0} v = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta) = 0$$

$$\therefore \quad \lim_{z\to 0} f(z) = 0 = f(0) \text{ by data}$$

$\therefore$ $f(z)$ is continuous at $z = 0$. $\therefore$ $f(z)$ is continuous everywhere.

(ii) Now $\dfrac{\partial u}{\partial x} = \lim\limits_{x\to 0} \dfrac{u(x,0) - u(0,0)}{x} = \lim\limits_{x\to 0} \dfrac{x - 0}{x} = 1$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{-y - 0}{y} = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y - 0}{y} = 1$$

$\therefore$ $u_x = v_y = 1$ and $u_y = -v_x = -1$

Hence, C-R equations are satisfied at the origin.

(iii) Now, $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z) - f(0)}{z} = \lim\limits_{z\to 0} \dfrac{(x^3 - y^3) + i(x^3 + y^3) - 0}{(x^2 + y^2)(x + iy)}$

Let $z \to 0$ along the line $y = mx$

$$\therefore \quad f'(0) = \lim_{x\to 0} \frac{x^3(1 - m^3) + ix^3(1 + m^3)}{x^2(1 + m^2)\,x(1 + im)} = \frac{(1 - m^3) + i(1 + m^3)}{(1 + m^2)(1 + im)}$$

The limit depends upon the path,

$\therefore$ $f'(0)$ does not exist.

Note ....

The above function is continuous at $z = 0$; Cauchy-Riemann equations are satisfied at $z = 0$ and yet the function is not analytic at $z = 0$. This is so because for analyticity in addition to C-R equations the four partial derivatives must be continuous at that point.

Now we have proved above that the value of u_x at $x = 0$ is one, we prove below that $\partial u / \partial x$ is not continuous at $z = 0$

$$u = \frac{x^3 - y^3}{x^2 + y^2}, \quad u_x = \frac{(x^2 + y^2)3x^2 - (x^3 - y^3)\cdot 2x}{(x^2 + y^2)^2} = \frac{x^4 + 3x^2y^2 + 2xy^3}{(x^2 + y^2)^2}$$

Now, let $y = mx$, $\lim\limits_{x\to 0} u_x = \lim\limits_{x\to 0} \dfrac{x^4 + 3m^2x^4 + 2m^3x^4}{x^4(1 + m^2)^2} = \dfrac{1 + 3m^2 + 2m^3}{(1 + m^2)^2}$

The limit depends upon the path chosen and hence, is not continuous. Similarly other partial derivatives can be shown to be not continuous. Hence, the function is **not** analytic even though C-R equations are satisfied.

Example 18 : Show that the following function

$$f(z) = \frac{x^2y^5(x + iy)}{x^4 + y^{10}}, \quad z \neq 0$$

$$= 0 \quad z = 0$$

is not analytic at the origin although Cauchy-Riemann equations are satisfied.

Sol. : We have $u = \dfrac{x^3y^5}{x^4 + y^{10}}$, $v = \dfrac{x^2y^6}{x^4 + y^{10}}$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{0 - 0}{x} = 0$$

Similarly, $\dfrac{\partial u}{\partial y} = 0$, $\dfrac{\partial v}{\partial x} = 0$, $\dfrac{\partial v}{\partial y} = 0$

Hence, Cauchy-Riemann equations are satisfied.

Now $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z) - f(0)}{z - 0} = \lim\limits_{z\to 0} \left[\dfrac{x^2y^5(x + iy)}{x^4 + y^{10}} - 0\right] \Big/ (x + iy)$

$$= \lim_{z\to 0} \frac{x^2y^5}{x^4 + y^{10}} \qquad \text{..... (A)}$$

Let $z \to 0$ along the line $y = mx$

$$\therefore \quad f'(0) = \lim_{x\to 0} \frac{x^7m^5}{x^4[1 + m^{10}x^6]} = \lim_{x\to 0} \frac{x^3m^5}{1 + m^{10}x^6} = 0$$

Now let $z \to 0$ along $y^5 = x^2$ then from (A), we get,

$$\therefore \quad f'(0) = \lim_{x\to 0} \frac{x^4}{2x^4} = \frac{1}{2}$$

Since, the limit depends upon the path, $f(z)$ is not analytic at $z = 0$.

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Example 19 : Find k such that $\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{kx}{y}$ is analytic. **(M.U. 2004, 07, 16, 19)**

Sol. : Let $f(z)=\frac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{kx}{y}$

$$\therefore\ u=\frac{1}{2}\log(x^2+y^2),\ v=\tan^{-1}\frac{kx}{y}$$

$$\therefore\ u_x=\frac{x}{x^2+y^2},\quad u_y=\frac{y}{x^2+y^2}$$

$$v_x=\frac{1}{1+\frac{k^2x^2}{y^2}}\cdot\frac{k}{y}=\frac{ky}{k^2x^2+y^2};\qquad v_y=\frac{1}{1+\frac{k^2x^2}{y^2}}\cdot\left(-\frac{kx}{y^2}\right)=-\frac{kx}{k^2x^2+y^2}$$

Since, the function is given to be analytic C-R equations are satisfied.

$$\therefore\ u_x=v_y\ \text{ and }\ u_y=-v_x$$

$$\therefore\ \frac{x}{x^2+y^2}=-\frac{kx}{k^2x^2+y^2},\quad \frac{y}{x^2+y^2}=-\frac{ky}{k^2x^2+y^2}$$

which are satisfied when $k=-1$.

Example 20 : Determine the constants a, b, c, d if

$f(z)=x^2+2axy+by^2+i(cx^2+2dxy+y^2)$ is analytic. **(M.U. 1998, 2010, 11, 13, 19)**

Sol. : We have $f(z)=u+iv$

and $u=x^2+2axy+by^2$; $v=cx^2+2dxy+y^2$

$$\therefore\ u_x=2x+2ay,\ u_y=2ax+2by$$

$$v_x=2cx+2dy,\ v_y=2dx+2y$$

Since, $f(z)$ is analytic, Cauchy-Riemann Equations are satisfied.

$$\therefore\ u_x=v_y\ \text{ and }\ u_y=-v_x$$

$$\therefore\ 2x+2ay=2dx+2y\quad\text{and}\quad 2ax+2by=-2cx-2dy$$

Equating the coefficients of x and y, we get $a=1, d=1$ and $a=-c, b=-d$

$$\therefore\ a=1,\ b=-1,\ c=-1,\ d=1.$$

Example 21 : Find the constants a, b, c, d, e, if

$f(z)=(ax^4+bx^2y^2+cy^4+dx^2-2y^2)+i(4x^3y-exy^3+4xy)$ is analytic.

(M.U. 2002, 03, 07, 08, 09, 12, 13, 16)

Sol. : We have $f(z)=u+iv$

and $u=ax^4+bx^2y^2+cy^4+dx^2-2y^2$

and $v=4x^3y-exy^3+4xy$

$$\therefore\ u_x=4ax^3+2bxy^2+2dx;\qquad u_y=2bx^2y+4cy^3-4y.$$

And $v_x=12x^2y-ey^3+4y$; $\qquad v_y=4x^3-3exy^2+4x.$

Since $f(z)$ is analytic

Cauchy-Riemann equations are satisfied.

$$\therefore\ u_x=v_y\quad\text{and}\quad u_y=-v_x$$

$$\therefore\ 4ax^3+2bxy^2+2dx=4x^3-3exy^2+4x$$

and $2bx^2y+4cy^3-4y=-12x^2y+ey^3-4y$

$\therefore$ Equating the coefficients of like powers of x and y, we get

$4a=4,\quad 2b=-3e\quad\therefore\ 2d=4$ and $2b=-12,\quad 4c=e.$

Hence, we get $a=1,\ d=2,\ b=-6$

$\therefore\ -12=-3e\quad\therefore\ e=4$ and $4c=4\quad\therefore\ c=1.$

Thus, we have $a=1,\ b=-6,\ c=1,\ d=2,\ e=4.$

Example 22 : Find the values of z for which the following functions are not analytic.

(i) $z=e^{-v}(\cos u+i\sin u)$ (ii) $z=\sin hu\cos v+i\cos hu\sin v.$

Sol. : (i) We have $z=e^{-v}(\cos u+i\sin u)=e^{-v}e^{iu}$

$$\therefore\ z=e^{-v+iu}=e^{i^2v+iu}=e^{i(u+iv)}=e^{iw}\quad\text{where } w=u+iv.$$

$$\therefore\ iw=\log z\quad\therefore\ w=\frac{1}{i}\log z\quad\therefore\ \frac{dw}{dz}=\frac{1}{i}\cdot\frac{1}{z}$$

$\therefore$ w is not analytic at $z=0$.

(ii) We have $z=\sin hu\cos v+i\cos hu\sin v$

But $\cos v=\cos h\,iv$ and $i\sin v=\sin h\,iv$

$$\therefore\ z=\sin hu\cos h\,iv+\cos hu\sin h\,iv$$

$$=\sin h\,(u+iv)$$

$$=\sin hw\quad\text{where } w+iv$$

$$\therefore\ w=\sin h^{-1}z=\log(z+\sqrt{z^2+1})$$

$$\therefore\ \frac{dw}{dz}=\frac{1}{z+\sqrt{z^2+1}}\left(1+\frac{1}{\sqrt{z^2+1}}\right)=\frac{1}{\sqrt{z^2+1}}$$

$\therefore$ w is not analytic when $\sqrt{z^2+1}=0$ *i.e.* $z^2=-1$ *i.e.* $z=\pm i$.

Example 23 : If $f(z)=u+iv$ is analytic in R show that

(i) $\begin{vmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{vmatrix}=|f'(z)|^2$ **(M.U. 2004, 06, 07, 09)**

(ii) $\left[\frac{\partial|f(z)|}{\partial x}\right]^2+\left[\frac{\partial|f(z)|}{\partial y}\right]^2=|f'(z)|^2$ **(M.U. 1995, 97, 2009, 10)**

Sol. : First we note that [see note (2) page 4-8]

if $f(z)\ u+iv,\qquad f'(z)=\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}$ ..... (i)

(i) $$\text{l.h.s.}=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

But by C-R equations $u_x=v_y,\ u_y=-v_x$

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$$\therefore \text{ l.h.s.} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

From (i) r.h.s $= |f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

Hence, the result.

(ii) Now, $|f(z)| = \sqrt{u^2 + v^2}$

$$\therefore \frac{\partial}{\partial x}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)$$

$$\frac{\partial}{\partial y}|f(z)| = \frac{1}{2\sqrt{u^2+v^2}}\left(2u\frac{\partial u}{\partial y} + 2v\frac{\partial v}{\partial y}\right)$$

$$\therefore \text{ l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u^2\left(\frac{\partial u}{\partial y}\right)^2 + v^2\left(\frac{\partial v}{\partial y}\right)^2 + 2uv\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}\right]$$

Using C-R equations

$$\text{l.h.s.} = \frac{1}{u^2+v^2}\left[u^2\left(\frac{\partial u}{\partial x}\right)^2 + v^2\left(\frac{\partial v}{\partial x}\right)^2 + 2uv\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u^2\left(\frac{\partial v}{\partial x}\right)^2 + v^2\left(\frac{\partial u}{\partial x}\right)^2 - 2uv\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial x}\right]$$

$$= \frac{1}{(u^2+v^2)}(u^2+v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 = |f'(z)|^2 = \text{r.h.s.}$$

EXERCISE - II

1. Show that the following functions are not analytic.
   (i) $f(z) = \bar{z}$ (ii) $f(z) = z\,|z|$.
2. Determine whether the following functions are analytic and if so find their derivatives
   (i) $\cos h\, z$ (ii) $\cos z$ (iiii) $\frac{1}{z}$ (iv) $z^2 + z$ (v) $z^2 - \bar{z}$
   (vi) $e^x(\cos y - i \sin y)$ (vii) $\frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$
   (viii) $e^{-x}(\cos y - i\sin y)$ (ix) $x^2 - y^2 + 2ixy$ **(M.U. 2004, 14)**
   (x) $(x^3 - 3xy^2 + 3x) + i(3x^2y - y^3 + 3y)$ **(M.U. 2004)**
   (xi) $z\,e^{2z}$ **(M.U. 2003, 04, 05, 11)**

[Ans. : (i) Yes, (ii) Yes, (iii) Yes except at $z = 0$, (iv) Yes, (v) No, (vi) No, (vii) Yes ; $\log z + C$, (viii) Yes, (ix) Yes, (x) Yes, (xi) Yes.]

3. If $f(z) = \frac{x^3y(y - ix)}{x^6 + y^2}$, $z \neq 0$, $f(0) = 0$, prove that $\frac{f(z) - f(0)}{z - 0} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in **any manner.**

   (**Hint :** Putting $y = mx$ $\lim_{x\to 0}\frac{f(z)-f(0)}{z-0} = 0$. But putting $y = x^3$ lim is $-i/2$)

4. Prove that $f(z) = \frac{x^2(1+i) - y^2(1-i)}{x+y}$, $z \neq 0$
   $f(0) = 0$
   is not analytic although Cauchy-Riemann equations are satisfied at $z = 0$.
5. Prove that $f(z) = e^{-z^{-4}}$, $z \neq 0$
   $f(0) = 0$
   is not analytic although Cauchy-Riemann equations are satisfied at $z = 0$.
6. Prove that the following function
   $$f(z) = \frac{x^3y^5(x+iy)}{x^6 + y^{10}}, \quad z \neq 0$$
   $f(0) = 0$
   is not analytic at the origin although Cauchy-Riemann equations are satisfied.
   (**Hint :** Put $x^3 = y^5$)
7. Find the values of z for which the following function is not analytic
   $z = \sin u \cos hv + i \cos u \sin hv$ **[Ans. : $z = \pm 1$]**
8. If $f(z) = \frac{xy^2(x+iy)}{x^2+y^4}$, $z \neq 0$, $f(0) = 0$ prove that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in **any manner.**

   (**Hint :** Put $y = mx$ and then put $y^2 = x$)
9. Prove that the function $f(z) = e^{2z}$ is analytic and find $f'(z)$.

   (**Hint :** $u = e^{2x}\cos 2y$, $v = e^{2x}\sin 2y$, $f'(z) = 2e^{2z}$)
10. Show that $u + iv = \frac{x - iy}{x - iy + a}$, $a \neq 0$ is not analytic but $u - iv$ is analytic.

    (**Hint :** Let $f(z) = u + iv = \frac{\bar{z}}{\bar{z}+a} = \frac{\bar{z}+a-a}{\bar{z}+a} = 1 - \frac{a}{\bar{z}+a}$

    $\Phi(z) = u - iv = \frac{z}{z+a} = \frac{z+a-a}{z+a} = 1 - \frac{a}{z+a}$

    Now try to find $f'(z)$ and $\Phi'(z)$ on the lines of alternative method of Ex. (9) (a), page 4-13.)
11. If $u + iv$ is analytic prove that $v - iu$ and $-v + iu$ are analytic. Show further that $v + iu$ will be analytic if $u + iv$ is a constant.
12. Find the constants a, b, c, d, e if
    (i) $f(z) = (ax^3 + bxy^2 + 3x^2 + cy^2 + x) + i(dx^2y - 2y^3 + exy + y)$ **(M.U. 2006, 08, 19)**
    (ii) $f(z) = (ax^4 + bx^2y^2 + cy^4 + dx^2 - 2y^2) + i(4x^3y - exy^3 + 4xy)$
    are analytic. **(M.U. 2002, 03, 07, 08, 09, 12, 13)**

    [Ans. : (i) $a = 2, b = -6, c = -3, d = 6, e = 6$; (ii) $a = 1, b = -6, c = 1, d = 2, e = 4$]

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13. Show that the following function is not analytic at the origin although Cauchy-Riemann equations are satisfied

$$f(z) = \frac{xy(y-ix)}{x^2+y^2}, \quad z \neq 0$$

$$f(0) = 0$$

(M.U. 2004)

(**Hint :** $f'(0) = \dfrac{f(z)-f(0)}{z-0} = \lim\limits_{z\to 0} \dfrac{xy(y-ix)/(x^2+y^2)-0}{(x+iy)-0}$

$$= \lim_{z\to 0} \frac{xy(y-ix)}{(x^2+y^2)(x+iy)} = \lim_{z\to 0} -\frac{ixy(x+iy)}{(x^2+y^2)(x+iy)}$$

$$= \lim_{z\to 0} -\frac{ixy}{x^2+y^2}.$$

Now put $y = mx$.)

10. Cauchy-Riemann Equations in Polar Coordinates **(M.U. 2010)**

Let (r, θ) be the polar coordinates of a point whose Cartesian coordinates are (x, y).

$\therefore \quad x = r\cos\theta, \quad y = r\sin\theta,$

$$z = x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

Let $f(z) = u + iv$ be the given function

$$\therefore \quad f(z) = u + iv = f(re^{i\theta}) \qquad \text{..... (1)}$$

Differentiating (1) partially w.r.t. r,

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = f'(re^{i\theta})\cdot e^{i\theta} \qquad \text{..... (2)}$$

Differentiating (1) partially w.r.t. θ,

$$\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta} = f'(re^{i\theta})\cdot re^{i\theta}\cdot i = ir\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad \text{[By (2)]}$$

$$= -r\frac{\partial v}{\partial r} + ir\frac{\partial u}{\partial r}$$

Equating real and imaginary parts $\dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r}$ and $\dfrac{\partial v}{\partial \theta} = r\dfrac{\partial u}{\partial r}$

or $\boxed{\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \text{ and } \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \quad \text{i.e.} \quad u_r = \dfrac{1}{r}v_\theta \text{ and } u_\theta = -rv_r}$

Note ....

From (2), we get an important result.

$$f'(re^{i\theta}) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \qquad \therefore \quad \boxed{f'(z) = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)}$$

Example 1 : If $w = \log z$ determine whether w is analytic and find $\dfrac{dw}{dz}$.

Sol. : Let $w = u + iv = \log(z) = \log(re^{i\theta}) = \log r + i\theta \quad \therefore u = \log r, v = \theta$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}, \frac{\partial u}{\partial \theta} = 0 \; ; \frac{\partial v}{\partial r} = 0, \frac{\partial v}{\partial \theta} = 1.$$

$$\therefore \quad u_r = \frac{1}{r}\cdot 1 = \frac{1}{r}v_\theta \text{ and } u_\theta = 0 = -rv_r$$

Hence, C-R equations are satisfied when $r \neq 0$.

Partial derivatives are continuous except at $r = 0$ *i.e.* at $z = 0$

Now $f'(z) = \dfrac{d}{dz}(\log z) = e^{-i\theta}\left(\dfrac{\partial u}{\partial r} + i\dfrac{\partial u}{\partial r}\right) = e^{-i\theta}\left(\dfrac{1}{r}\right) = \dfrac{1}{re^{i\theta}} = \dfrac{1}{z}.$

Example 2 : Find p if $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$ is analytic. **(M.U. 1998, 08, 10)**

Sol. : Let $w = f(z) = u + iv = r^2\cos 2\theta + ir^2\sin p\theta$

$$\therefore \quad u = r^2\cos 2\theta, \quad v = r^2\sin p\theta$$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta; \quad \frac{\partial v}{\partial r} = 2r\sin p\theta, \quad \frac{\partial v}{\partial \theta} = pr^2\cos p\theta$$

Since, $f(z)$ is analytic. $\dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta}$ and $u_\theta = -rv_r$

The first relation gives, $2r\cos 2\theta = \dfrac{1}{r}\cdot pr^2\cos p\theta \quad \therefore p = 2$

And the second realtion also gives, $-2r^2\sin 2\theta = -2r^2\sin p\theta \quad \therefore p = 2.$

Example 3 : Find the value of k if $f(z) = r^3\cos k\theta + ir^k\sin 3\theta$ is analytic.

Sol. : Left to you. **[Ans. : $k = 3$]**

Example 4 : Is $f(z) = \dfrac{z}{\bar{z}}$ analytic ? **(M.U. 2004)**

Sol. : Put $z = re^{i\theta}$ and use C-R equations in polar form. No.

Example 5 : If $w = z^n$ find $\dfrac{dw}{dz}$.

Sol. : Let $z = re^{i\theta} \quad \therefore z^n = r^n e^{in\theta}$

$$\therefore \quad z^n = r^n(\cos n\theta + i\sin n\theta) \qquad \therefore \quad u = r^n\cos n\theta, \; v = r^n\sin n\theta.$$

$$\frac{\partial u}{\partial r} = nr^{n-1}\cos n\theta, \quad \frac{\partial u}{\partial \theta} = -r^n\cdot n\cdot\sin n\theta$$

$$\frac{\partial v}{\partial r} = nr^{n-1}\sin n\theta, \quad \frac{\partial v}{\partial \theta} = nr^n\cos n\theta$$

$$\therefore \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r}.$$

Also partial derivatives are continuous. Hence, w is analytic.

$$\therefore \quad \frac{dw}{dz} = e^{-i\theta}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) = e^{-i\theta}(nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta)$$

$$= nr^{n-1}\cdot e^{-i\theta}(\cos n\theta + i\sin n\theta) = nr^{n-1}\cdot e^{-i\theta}\cdot e^{in\theta}$$

$$= nr^{n-1}\cdot e^{i(n-1)\theta} = n(re^{i\theta})^{n-1} = nz^{n-1}.$$

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Example 6 : Show that $f(z) = \frac{1}{r^2}[\cos 2\theta - i \sin 2\theta]$ is analytic. **(M.U. 2019)**

Sol. : Left to you.

Example 7 : Using Cauchy-Riemann equations in polar form prove that

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Sol. : We know that Cauchy-Riemann equations in polar form are

$$u_r = \frac{1}{r} v_\theta \quad \text{.....(i)}$$

and $$u_\theta = -r v_r \quad \text{.....(ii)}$$

Differentiating (i) w.r.t. r, we get,

$$\frac{\partial^2 u}{\partial r^2} = -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} \quad \text{.....(iii)}$$

Differentiating (ii) w.r.t. θ, we get,

$$\frac{\partial^2 u}{\partial \theta^2} = -r\frac{\partial^2 v}{\partial \theta \partial r} \quad \text{.....(iv)}$$

Now, using (iii) and (iv), we get,

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \left(-\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r}\right) + \frac{1}{r}\cdot\frac{1}{r}\frac{\partial v}{\partial \theta} - \frac{1}{r^2}\cdot r\frac{\partial^2 v}{\partial \theta \partial r}$$

$$= -\frac{1}{r^2}\frac{\partial v}{\partial \theta} + \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} + \frac{1}{r^2}\frac{\partial v}{\partial \theta} - \frac{1}{r}\frac{\partial^2 v}{\partial \theta \partial r} = 0$$

Note ....

The equation $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ is called Laplace's equation in **Cartesian Form** and the equation $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$ is called Laplace's equation in **Polar Form.**

11. Harmonic Functions

Any function of x, y which has continuous partial derivatives of the first and second order and satisfies Laplace's equation $\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$ is called a **Harmonic Function.**

Theorem : The real and imaginary parts u, v of an analytic function $f(z) = u + iv$ are harmonic functions. **(M.U. 2003, 13)**

Proof : Since, $f(z)$ is an analytic function in some region of the z-plane, u, v satisfy C-R equations.

$$\therefore \quad u_x = v_y \text{ and } u_y = -v_x \quad \text{.....(A)}$$

Differentiating the first w.r.t. x and second w.r.t. y, we get,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} \text{ and } \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Assuming $\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial x \partial y}$ and adding the above results we get,

$$\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$$

Similarly differentiating the equations in (A) with respect to y and x respectively, we can show that

$$\boxed{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}$$

Hence, the result.

Notes ....

1. In other words the above theorem states that if $f(z) = u + iv$ is analytic, then its real and imaginary parts u, v satisfy Laplace equation.

$$\boxed{\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0}$$

2. The above theorem states that if $f(z) = u + iv$ is analytic then u and v satisfy Laplace's equation *i.e.* u and v are harmonic functions. But, the converse is **not** true. If u and v are any two functions satisfying Laplace's equation then $u + iv$ need not be analytic. [See Ex. 7 of § 13 page 4-47].

Theorem : If $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic then the real and imaginary parts u, v are harmonic.

Proof : You can prove it easily by using Cauchy-Riemann equations in polar form. Prove it.

Note ....

3. In other words the above theorem states that if $f(re^{i\theta}) = u(r, \theta) + iv(r, \theta)$ is analytic then the real and imaginary parts u, v satisfy Laplace equation in polar form

$$\boxed{\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0}$$

Example 1 : Show that a harmonic function satisfies the differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

Sol. : If u is a harmonic function then $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ .....(i)

Now, $x = \frac{1}{2}(z + \bar{z})$ and $y = \frac{1}{2i}(z - \bar{z})$

$$\therefore \quad \frac{\partial u}{\partial \bar{z}} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}} = \frac{1}{2}\frac{\partial u}{\partial x} - \frac{1}{2i}\cdot\frac{\partial u}{\partial y}$$

$$\therefore \quad \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{\partial y}{\partial z}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial y \partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{\partial y}{\partial z}\right]$$

$$= \frac{1}{2}\left[\frac{\partial^2 u}{\partial x^2}\cdot\frac{1}{2} + \frac{\partial^2 u}{\partial x \partial y}\cdot\frac{1}{2i}\right] - \frac{1}{2i}\left[\frac{\partial^2 u}{\partial x \partial y}\cdot\frac{1}{2} + \frac{\partial^2 u}{\partial y^2}\cdot\frac{1}{2i}\right]$$

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$$\therefore \frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{4}\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right] = 0 \qquad \text{[By (i)]}$$

Hence, the required result.

Example 2 : If $u(x, y)$ is a harmonic function then prove that $f(z) = u_x - i\,u_y$ is an analytic function. **(M.U. 2004, 08)**

Sol. : Since u is harmonic $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ ..... (1)

By data $f(z) = u_x - i\,u_y$.

Let $u_x = U$ and $-u_y = V$, so that $f(z) = U + i\,V$.

We have to show that $f(z)$ is analytic.

Now, $U_x = \dfrac{\partial^2 u}{\partial x^2} = -\dfrac{\partial^2 u}{\partial y^2}$ [By (1)] and $U_y = \dfrac{\partial^2 u}{\partial x\,\partial y}$

$V_x = -\dfrac{\partial^2 u}{\partial y\,\partial x}$ and $V_y = -\dfrac{\partial^2 u}{\partial y^2}$ $\therefore$ $U_x = V_y$ and $U_y = -V_x$.

$\therefore$ $f(z) = U + i\,V$ is analytic *i.e.* $f(z) = u_x - i\,u_y$ is analytic.

Example 3 : If u, v are harmonic conjugate functions, show that uv is a harmonic function. **(M.U. 2003)**

Sol. : Let $f(z) = u + iv$ be the analytic function. [See note (2), page 4-26]

$\therefore$ $u_x = v_y$ and $u_y = -v_x$

And u, v are harmonic $\therefore$ $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ and $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$ ..... (1)

Now, $\dfrac{\partial}{\partial x}(uv) = u\dfrac{\partial v}{\partial x} + v\dfrac{\partial u}{\partial x}$

$$\therefore \frac{\partial^2}{\partial x^2}(uv) = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial u}{\partial x} + v\frac{\partial^2 u}{\partial x^2}$$

$$\therefore \frac{\partial^2}{\partial x^2}(uv) = u\frac{\partial^2 v}{\partial x^2} + v\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} \qquad \text{..... (2)}$$

Similarly, we can prove that

$$\frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} + 2\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}$$

But $u_x = v_y$ and $u_y = -v_x$

$$\therefore \frac{\partial^2}{\partial y^2}(uv) = u\frac{\partial^2 v}{\partial y^2} + v\frac{\partial^2 u}{\partial y^2} - 2\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} \qquad \text{..... (3)}$$

Adding (2) and (3), we get

$$\frac{\partial^2}{\partial x^2}(uv) + \frac{\partial^2}{\partial y^2}(uv) = u\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + v\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$= 0 \qquad \text{[By (1)]}$$

$\therefore$ uv is harmonic.

Example 4 : If Φ and Ψ are functions of x and y satisfying Laplace equation and if $u = \Phi_y - \Psi_x$ and $v = \Phi_x + \Psi_y$, prove that $u + iv$ is analytic (holomorphic). **(M.U. 2003)**

Sol. : Since Φ and Ψ satisfy Laplace equation, we have

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \text{..... (1)} \qquad \text{and} \qquad \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \quad \text{..... (2)}$$

Now $u_x = \Phi_{yx} - \Psi_{xx} = \Phi_{xy} + \Psi_{yy}$ [By (2)]

And $u_y = \Phi_{yy} - \Psi_{xy} = -(\Phi_{xx} + \Psi_{xy})$ [By (1)]

Similarly, $v_x = \Phi_{xx} + \Psi_{xy}$ and $v_y = \Phi_{xy} + \Psi_{yy}$

Hence, $u_x = v_y$ and $u_y = -v_x$

Hence, $u + iv$ is analytic.

12. To Find an Analytic Function whose Real or Imaginary Part is given

Method 1 : Let $f(z) = u + iv$ and let u be given. Since, u is given we can find u_x and u_y.

As $f(z)$ is analytic, by C-R equations $u_x = v_y$ and $u_y = -v_x$.

$\therefore$ $f'(z) = u_x + iv_x = u_x - iu_y = \Phi(z)$ say.

Hence, by mere integration $f(z)$ can be obtained. (See Ex. 1, page 4-31)

Note ....

The method can be used only when we are able to express $u_x - iu_y$ as a function of z, say $\Phi(z)$.

Method 2 : Let $f(z) = u + iv$ and let u be given. This means $\dfrac{\partial u}{\partial x}$ and $\dfrac{\partial u}{\partial y}$ are known.

Now, $dv = \dfrac{\partial v}{\partial x}dx + \dfrac{\partial v}{\partial y}dy$

But by C-R equations $u_x = v_y$ and $u_y = -v_x$

$$\therefore dv = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy \qquad \text{..... (1)}$$

Further $\dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right) = -\dfrac{\partial^2 u}{\partial y^2}$ and $\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}$

Since, u is harmonic $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$

i.e. $-\dfrac{\partial^2 u}{\partial y^2} = \dfrac{\partial^2 u}{\partial x^2}$ *i.e.* $\dfrac{\partial}{\partial y}\left(-\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)$

$\therefore$ R.H.S. of (1) is an exact differential.

(Recall that if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ then $M\,dx + N\,dy = du$)

$\therefore$ Integrating (i), $v = \displaystyle\int\left(-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right) + c$

From this we get v then $f(z) = u + iv$.

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Notes ....

1. Recall how we obtain the solution of an exact differential equation $M\,dx + N\,dy = 0$. In this case we have $M = -\frac{\partial u}{\partial y}$ and $N = \frac{\partial u}{\partial x}$. Hence, to obtain v, **integrate** $\left(-\frac{\partial u}{\partial y}\right)$ **w.r.t. x treating y constant and integrate only those terms in** $\frac{\partial u}{\partial x}$ **which are free from x. Their sum plus a constant is equal to v.**

2. If $f(z) = u + iv$ is an analytic function so that u and v both are harmonic functions then u and v are called **harmonic conjugate functions**. Each is called the **harmonic conjugate function** of the other.

3. Instead of u if v is given we can reason exactly as above to find $f(z)$. In this case **integrate** $\frac{\partial v}{\partial y}$ **treating x constant and integrate those terms in** $-\frac{\partial v}{\partial x}$ **which are free from x. Their sum plus a constant is equal to u.**

Method 3 : Milne-Thompson's Method

Since, $z = x + iy,\ \bar{z} = x - iy \qquad \therefore\ x = \frac{z+\bar{z}}{2},\ y = \frac{z-\bar{z}}{2i}$

$$\therefore\ f(z) = u(x,y) + iv(x,y) = u\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right] + iv\left[\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right]$$

This can be regarded as an identity in two independent variables z and $\bar{z}$. We can, therefore, put $\bar{z} = z$ and get

$$f(z) = u(z,0) + iv(z,0)$$

Thus, **$f(z)$ can be obtained in terms of z by putting $x = z$ and $y = 0$ in $f(z) = u(x,y) + iv(x,y)$ when $f(z)$ is analytic.**

Now $f'(z) = u_x + iv_x = u_x - iu_y$ [$\because$ C-R equations]

Let $u_x = \Phi_1(x,y)$ and $u_y = \Phi_2(x,y)$

$$\therefore\ f'(z) = \Phi_1(x,y) - i\,\Phi_2(x,y) = \Phi_1(z,0) - i\,\Phi_2(z,0)$$

Integrating, we get,

$$\boxed{f(z) = \int \Phi_1(z,0)\,dz - i\int \Phi_2(z,0)\,dz + c}$$

Similarly if v is given arguing on the above lines we can show that

$$\boxed{f(z) = \int \psi_1(z,0)\,dz + i\int \psi_2(z,0)\,dz + c}$$

where, $v_y = \psi_1(x,y),\ v_x = \psi_2(x,y)$

Louis Melville Milne-Thompson (1891-1974)

He became Wrangler in 1913 and joined Winchester College as an assistant mathematics master in 1914. He left Winchester College and joined Royal Naval College, Greenwich in 1921. His earlier research was related to tables. He published in 1931 "Standard Four Figure Mathematical Tables" and in the next year he published "Standard Table of Square Root And Jacobian Elliptic Functions". He published his first book on "The Calculus Of Finite Differences" in 1933. In 1938, he published his second book but on different subject "Theoretical Hydrodynamics". He retried from the Royal Naval College in 1956 at the age of 65 and then took various posts as Visiting professor at various institutions throughout the world. He was elected a fellow of the Royal Society of Edinburg, Royal Astronomical Society and Cambridge Philosophical Society. We know him for Milne-Thomson method for finding analytic function.

Method 4 : Let $f(z) = f(z+iy) = u(x,y) + iv(x,y)$

Then $\overline{f(z)} = \overline{f(x+iy)} = u(x,y) - iv(x,y) \qquad \therefore\ f(z) + \overline{f(z)} = 2u(x,y)$

We can consider $\overline{f(z)}$ as a function of $\bar{z}$ and denote it as $\bar{f}(\bar{z})$.

$$u(x,y) = \frac{1}{2}[f(z) + \bar{f}(\bar{z})] = \frac{1}{2}[f(x+iy) - \bar{f}(x-iy)]$$

This can be regarded as an identity and holds even if x and y are complex. Putting $x = \frac{z}{2}$ and $y = \frac{z}{2i}$, we get, from above

$$u(z/2,\ z/2i) = \frac{1}{2}\left[f\left(\frac{z}{2} + i\frac{z}{2i}\right) + \bar{f}\left(\frac{z}{2} - i\frac{z}{2i}\right)\right] = \frac{1}{2}[f(z) + \bar{f}(0)]$$

$$\therefore\ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - \bar{f}(0)$$

We can assume that $\bar{f}(0)$ is purely real and write $\bar{f}(0) = u(0,0)$

$$\therefore\ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0)$$

We may add a purely imaginary constant ci and get

$$\boxed{f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci}$$

Notes ....

1. Note that this method is purely algebraic and does not involve integration.
2. Of these four methods the third method *i.e.* **Milne-Thompson method is commonly used.**

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Type I : To find the analytic function whose real part u is given

Example 1 : Construct an analytic function whose real part is $x^4 - 6x^2y^2 + y^4$.

Sol. : We shall first solve this example by the above four methods and then lay down the procedure to find the analytic function $f(z)$.

Method 1 : Let $u = x^4 - 6x^2y^2 + y^4$ and let $f(z) = u + iv$ be the required function.

$\therefore\ u_x = 4x^3 - 12xy^2\ ;\ u_y = -12x^2y + 4y^3$

As seen above (or see note (2), page 4-8)

$$f'(z) = u_x - iu_y$$
$$= 4x^3 - 12xy^2 + 12ix^2y - 4iy^3$$
$$= 4[x^3 + 3x(iy)^2 + 3x^2(iy) + (iy)^3]$$
$$= 4(x+iy)^3 = 4z^3$$
$$\therefore\ f(z) = \int f'(z)\,dz = \int 4z^3\,dz = z^4 + c.$$

Method 2 : As before $\dfrac{\partial u}{\partial x} = 4x^3 - 12xy^2\ ;\ \dfrac{\partial u}{\partial y} = -12x^2y + 4y^3$.

Since, $f(z)$ is given to be analytic we can use the note 1 given on page 4-29.

$$\int -\frac{\partial u}{\partial y}\,dx = \int -(-12x^2y + 4y^3)\,dx = 4x^3y - 4y^3x$$

And $\int$(terms in $\dfrac{\partial u}{\partial x}$ free from x)$dy = \int 0\,dy = 0$

$$\therefore\ v = 4x^3y - 4xy^3 + 0 = 4x^3y - 4xy^3$$
$$\therefore\ f(z) = u + iv + c = x^4 - 6x^2y^2 + y^4 + 4ix^3y - 4ixy^3 + c$$

Setting $x = z,\ y = 0$, we get $f(z) = z^4 + c$.

Method 3 : *Milne-Thompson Method :* We have as above

$$\Phi_1 = u_x = 4x^3 - 12xy^2\ ;\ \Phi_2 = u_y = -12xy^2 + 4y^3$$
$$\therefore\ f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0)$$
$$= 4z^3 - i(0) \qquad [\text{Putting } x = z,\ y = 0 \text{ in } \Phi_1 \text{ and } \Phi_2]$$
$$\therefore\ f(z) = \int 4z^3\,dz = z^4 + c \text{ as before.}$$

Method 4 : We have

$$u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{z^4}{16} - 6\left(\frac{z^2}{4}\right)\left(\frac{z^2}{4i^2}\right) + \left(\frac{z^4}{16i^4}\right) = \frac{z^4}{16} + \frac{6z^4}{16} + \frac{z^4}{16} = \frac{8z^4}{16}$$

Further $u(0, 0) = 0$

$$\therefore\ f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + c_i = 2\cdot\frac{8z^4}{16} - 0 + c_i = z^4 + c_i$$

Procedure to find $f(z)$ when real part u is given.

1. From u, first find u_x and u_y
2. Then put $\Phi_1 = u_x$ and $\Phi_2 = u_y$

3. Putting $x = z$ and $y = 0$, find $\Phi_1(z, 0)$ and $\Phi_2(z, 0)$.
4. Then $f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$.
5. $f(z) = \int f'(z)\,dz + c$

Example 2 : Construct an analytic function whose real part is $e^x \cos y$. **(M.U. 2019)**

Sol. : Let $u = e^x \cos y$

$$\therefore\ u_x = e^x \cos y \text{ and } u_y = -e^x \sin y$$
$$\therefore\ \Phi_1 = u_x = e^x \cos y,\ \Phi_2 = u_y = -e^x \sin y$$

By Milne-Thompson method

$$\therefore\ f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = e^z - i(0)$$
$$\therefore\ f(z) = \int e^z\,dz = e^z + c \text{ which is the required analytic function.}$$

Example 3 : Find an analytic function whose real part is

$$e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\}$$ **(M.U. 2017, 18, 19)**

Sol. : Let $u = e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\}$

$$\therefore\ u_x = -e^{-x}\left\{(x^2 - y^2)\cos y + 2xy \sin y\right\} + e^{-x}\left\{2x \cos y + 2y \sin y\right\}$$
$$= e^{-x}\left[-(x^2 - y^2)\cos y + 2x \cos y + 2y \sin y - 2xy \sin y\right]$$
$$u_y = e^{-x}\left[-(x^2 - y^2)\sin y - 2y \cos y + 2x \sin y + 2xy \cos y\right]$$
$$\therefore\ \Phi_1 = u_x \text{ and } \Phi_2 = u_y$$

By Milne-Thompson method

$$f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = e^{-z}[-z^2 + 2z]$$
$$\therefore\ f(z) = \int e^{-z}(-z^2 + 2z)\,dz$$

Integrating by parts,

$$f(z) = (-z^2 + 2z)(-e^{-z}) - \int(-e^{-z})(-2z + 2)\,dz$$
$$= e^{-z}(z^2 - 2z) + \int e^{-z}(2 - 2z)\,dz$$

Integrating by parts again,

$$\therefore\ f(z) = e^{-z}(z^2 - 2z) + (2 - 2z)(-e^{-z}) - \int(-e^{-z})(-2)\,dz$$
$$= e^{-z}(z^2 - 2z) - e^{-z}(2 - 2z) + 2e^{-z}$$
$$= z^2 e^{-z} + c.$$

Example 4 : Find the imaginary part of the analytic function whose real part is

$$e^{2x}(x \cos 2y - y \sin 2y).$$ **(M.U. 1993, 2000, 03, 14)** **(M.U. 2004, 05, 10, 19)**

Also verfity that v is harmonic.

Sol. : Let $u = e^{2x}(x \cos 2y - y \sin 2y)$

$$\therefore\ \Phi_1 = u_x = e^{2x}\cdot 2(x \cos 2y - y \sin 2y) + e^{2x}(\cos 2y)$$
$$= e^{2x}(2x \cos 2y - 2y \sin 2y + \cos 2y)$$

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$$\Phi_2 = u_y = e^{2x}(-2x\sin 2y - \sin 2y - 2y\cos 2y)$$

By Milne-Thompson Method

$$\therefore\ f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = e^{2z}(2z+1) - i\,e^{2z}(0) = e^{2z}(2z+1)$$

Integrating by parts,

$$\therefore\ f(z) = \int e^{2z}(2z+1)\,dz = (2z+1)\frac{e^{2z}}{2} - \int \frac{e^{2z}}{2}\cdot 2\cdot dz$$

$$= (2z+1)\frac{e^{2z}}{2} - \int e^{2z}dz = (2z+1)\frac{e^{2z}}{2} - \frac{e^{2z}}{2} = e^{2z}z + c.$$

Now, $f(z) = e^{2(x+iy)}\cdot(x+iy) = e^{2x}\cdot e^{2iy}(x+iy)$

$$= e^{2x}[\cos 2y + i\sin 2y](x+iy)$$

$$\therefore\ v = e^{2x}(y\cos 2y + x\sin 2y)$$

$$\therefore\ \frac{\partial v}{\partial x} = 2e^{2x}(y\cos 2y + x\sin 2y) + e^{2x}(\sin 2y)$$

$$\frac{\partial^2 v}{\partial x^2} = 4e^{2x}(y\cos 2y + x\sin 2y) + 2e^{2x}\sin 2y + 2e^{2x}\sin 2y$$

$$\frac{\partial^2 v}{\partial x^2} = 4e^{2x}(y\cos 2y + x\sin 2y) + 4e^{2x}\sin 2y$$

$$\frac{\partial v}{\partial y} = e^{2x}(\cos 2y - 2y\sin 2y + 2x\cos 2y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^{2x}(-2\sin 2y - 2\sin 2y - 4y\cos 2y - 4x\sin 2y)$$

$$= e^{2x}(-4\sin 2y - 4y\cos 2y - 4x\sin 2y)$$

$$\therefore\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \qquad \therefore\ v \text{ is harmonic.}$$

Example 5 : Find an analytic function $f(z) = u + iv$ where

$$u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\frac{y}{x} + \sin x\cosh y.$$ **(M.U. 2003, 08, 16)**

Sol. : We have $u = \frac{x}{2}\log(x^2+y^2) - y\tan^{-1}\frac{y}{x} + \sin x\cosh y$

$$\therefore\ \Phi_1 = u_x = \frac{1}{2}\log(x^2+y^2) + \frac{x^2}{x^2+y^2} - y\cdot\frac{1}{1+(y^2/x^2)}\left(-\frac{y}{x^2}\right) + \cos x\cosh y$$

$$= \frac{1}{2}\log(x^2+y^2) + \frac{x^2}{x^2+y^2} + \frac{y^2}{x^2+y^2} + \cos x\cosh y$$

And $$\Phi_2 = u_y = \frac{xy}{x^2+y^2} - \tan^{-1}\left(\frac{y}{x}\right) - y\cdot\frac{1}{1+(y^2/x^2)}\cdot\frac{1}{x} + \sin x\sinh y$$

$$= \frac{xy}{x^2+y^2} - \tan^{-1}\frac{y}{x} - \frac{xy}{x^2+y^2} + \sin x\sinh y$$

By Milne-Thomson method,

$$f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0)$$

$$\therefore\ f'(z) = \frac{1}{2}\log z^2 + \frac{z^2}{z^2} + \cos z - i(0)$$

$$= \log z + 1 + \cos z$$

$\therefore$ By integration,

$$f(z) = \int \log z\,dz + \int 1dz + \int \cos z\,dz$$

$$= \log z\cdot z - \int 1\cdot dz + z + \sin z \qquad \text{[By Integrating by parts]}$$

$$= z\log z - z + z + \sin z$$

$$= z\log z + \sin z + c.$$

Example 6 : Find the analytic function whose real part is

$$\frac{\sin 2x}{\cosh 2y + \cos 2x}.$$ **(M.U. 2002, 06, 10, 11, 15)**

Sol. : Let $u = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$

$$\therefore\ \Phi_1 = u_x = \frac{(\cosh 2y + \cos 2x)(2\cos 2x) + \sin 2x\cdot 2\sin 2x}{(\cosh 2y + \cos 2x)^2}$$

$$= \frac{2\cosh 2y\cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$$

$$\Phi_2 = u_y = -\frac{\sin 2x\cdot 2\sinh 2y}{(\cosh 2y + \cos 2x)^2}$$

By Milne-Thompson Method

$$\therefore\ f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0)$$

$$= \frac{2\cos 2z + 2}{(1+\cos 2z)^2} - 0 = \frac{2}{1+\cos 2z} = \sec^2 z$$

(See that $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\sinh 0 = 0$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\cosh 0 = 1$.)

$$\therefore\ f(z) = \int \sec^2 z\,dz = \tan z + c.$$

Type II : To find the analytic function whose imaginary part v is given

Procedure to find $f(z)$ when imaginary part v is given

1. From v first find v_y and v_x.
2. Then put $\Psi_1 = v_y$ and $\Psi_2 = v_x$.
3. Putting $x = z$ and $y = 0$, find $\Psi_1(z, 0)$ and $\Psi_2(z, 0)$.
4. Then $f'(z) = \Psi_1(z, 0) + i\,\Psi_2(z, 0)$.
5. $f(z) = \int f'(z)\,dz + c$

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Example 1 : Find an analytic function whose imaginary part is

$$(x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$$

Sol. : We have $v = (x^4 - 6x^2y^2 + y^4) + (x^2 - y^2) + 2xy$

$$\therefore \quad v_y = \psi_1(x, y) = -12x^2y + 4y^3 - 2y + 2x$$

$$v_x = \psi_2(x, y) = 4x^3 - 12xy^2 + 2x + 2y$$

$$\therefore \quad \psi_1(z, 0) = 2z, \quad \psi_2(z, 0) = 4z^3 + 2z$$

By Milne-Thompson method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore \quad f(z) = \int \psi_1(z, 0)\,dz + i\int \psi_2(z, 0)\,dz$$

$$= \int 2z\,dz + i\int (4z^3 + 2z)\,dz$$

$$= z^2 + i(z^4 + z^2) + c.$$

Example 2 : Find an analytic function whose imaginary part is $e^{-x}(y\cos y - x\sin y)$.

Sol. : We have $v = e^{-x}(y\cos y - x\sin y)$

$$\therefore \quad v_y = \psi_1(x, y) = e^{-x}(\cos y - y\sin y - x\cos y)$$

$$v_x = \psi_2(x, y) = -e^{-x}(y\cos y - x\sin y) + e^{-x}(-\sin y)$$

$$= e^{-x}(-\sin y - y\cos y + x\sin y)$$

$$\therefore \quad \psi_1(z, 0) = e^{-z}(1 - z), \quad \psi_2(z) = 0$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore \quad f(z) = \int \psi_1(z, 0)\,dz + i\int \psi_2(z, 0)\,dz = \int (1 - z)e^{-z}dz$$

$$= (1 - z)(-e^{-z}) - \int (-e^{-z})(-1)\,dz$$

$$= -e^{-z} + ze^{-z} + e^{-z} = ze^{-z} + c.$$

Example 3 : Find an analytic function $f(z)$ whose imaginary part is $e^{-x}(y\sin y + x\cos y)$. **(M.U. 1995, 2007, 09)**

Sol. : We have $v = e^{-x}(y\sin y + x\cos y)$

$$\therefore \quad v_y = \psi_1(x, y) = e^{-x}(\sin y + y\cos y - x\sin y)$$

$$v_x = \psi_2(x, y) = -e^{-x}(y\sin y + x\cos y) + e^{-x}(\cos y)$$

$$\therefore \quad v_x = \psi_2(x, y) = e^{-x}(\cos y - y\sin y - x\cos y)$$

$$\therefore \quad \psi_1(z, 0) = 0, \quad \psi_2(z, 0) = e^{-z}(1 - z)$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore \quad f(z) = \int \psi_1(z, 0)\,dz + i\int \psi_2(z, 0)\,dz = i\int e^{-z}(1 - z)\,dz$$

$$= i\left[(1 - z)(-e^{-z}) - \int -e^{-z}(-1)\,dz\right] = i\left[(1 - z)(-e^{-z}) + e^{-z}\right]$$

$$= ie^{-z}z + c.$$

Example 4 : Find the analytic function whose imaginary part is $\tan^{-1}\dfrac{y}{x}$. **(M.U. 2019)**

Sol : We have $v = \tan^{-1}\dfrac{y}{x}$.

$$\therefore \quad v_y = \psi_1(x, y) = \frac{1}{1 + (y^2/x^2)} \cdot \frac{1}{x} = \frac{x}{x^2 + y^2}$$

$$v_x = \psi_2(x, y) = \frac{1}{1 + (y^2/x^2)}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2 + y^2}$$

$$\therefore \quad \psi_1(z, 0) = \frac{z}{z^2} = \frac{1}{z}; \quad \psi_2(z, 0) = 0$$

By Milne-Thompson method

$$f'(z) = \Psi_1(z, 0) + i\,\Psi_2(z, 0) = \frac{1}{z} \qquad \therefore \quad f(z) = \int \frac{1}{z}dz = \log z + c.$$

Example 5 : If the imaginary part of the analytic function $w = f(z)$ is $v = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$, show that the real part $u = -2xy + \dfrac{y}{x^2 + y^2} + c$. **(M.U. 2007, 08, 13, 14)**

Sol. : We have $v = x^2 - y^2 + \dfrac{x}{x^2 + y^2}$

$$\therefore \quad v_y = \psi_1(x, y) = -2y - \frac{2xy}{(x^2 + y^2)^2}$$

$$v_x = \psi_2(x, y) = 2x - \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\psi_1(z, 0) = 0, \quad \psi_2(z, 0) = 2z - \frac{1}{z^2}$$

By Milne-Thompson Method

$$f'(z) = \psi_1(z, 0) + i\,\psi_2(z, 0)$$

$$\therefore \quad f'(z) = i\left(2z - \frac{1}{z^2}\right) = i\int\left(2z - \frac{1}{z^2}\right)dz = i\left(z^2 + \frac{1}{z}\right)$$

$$= i(x^2 + 2i\,xy - y^2) + i\frac{(x - iy)}{x^2 + y^2}$$

$$= \left(-2xy + \frac{y}{x^2 + y^2}\right) + i\left(x^2 - y^2 + \frac{x}{x^2 + y^2}\right) + c$$

$$\therefore \quad u = -2xy + \frac{y}{x^2 + y^2} + c.$$

Type III : To find the analytic function when $u + v$ or $u - v$ is given

Procedure to find $f(z)$ when $u + v$ is given

1. Let $V = u + v$ (given).
2. Find $\dfrac{\partial V}{\partial x} = \Psi_1(x, y)$ and $\dfrac{\partial V}{\partial y} = \Psi_2(x, y)$.

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3. Find $\Psi'_2(z, 0)$ and $\Psi'_1(z, 0)$.
4. Then $(1 + i) f'(z) = \Psi'_2(z, 0) + i\,\Psi'_1(z, 0)$.
5. By integrating both sides w.r.t. z, we get $f(z)$.

Example 1 : If $f(z) = u + iv$ is analytic and $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$, find $f(z)$.

(M.U. 1998, 2003, 04, 06, 07)

Sol. : $f(z) = u + iv \quad \therefore\ i f(z) = iu - v$

$$\therefore\ (1+i) f(z) = (u - v) + i(u + v) = U + iV \text{ say.}$$

$$\therefore\ (1+i) f'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial y} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} \qquad [\because\ U_x = V_y]$$

(When $u + v$ is given we use $\dfrac{\partial V}{\partial y} + i\dfrac{\partial V}{\partial x}$)

But $\quad V = u + v = \dfrac{\sin 2x}{\{(e^{2y} + e^{-2y}) / 2\} - \cos 2x} = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

$$\therefore\ \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(u + v) = \frac{(\cosh 2y - \cos 2x)(2\cos 2x) - 2\sin 2x \sin 2x}{(\cosh 2y - \cos 2x)^2}$$

$$= [2\cosh 2y \cos 2x - 2] / [\cosh 2y - \cos 2x]^2 = \psi_1(x, y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(u + v) = -\frac{2\sinh y \sin 2x}{[\cosh 2y - \cos 2x]^2} = \psi_2(x, y)$$

By Milne-Thompson method

$$\therefore\ (1+i) f'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \psi_2(z, 0) + i\,\psi_1(z, 0)$$

$$\therefore\ (1+i) f(z) = \int 0 + i\frac{2(\cos 2z - 1)}{(1 - \cos 2z)^2} \cdot dz = -2i\int \frac{dz}{1 - \cos 2z} = -2i\int \frac{dz}{2\sin^2 z}$$

$$\therefore\ (1+i) f(z) = -i\int \operatorname{cosec}^2 z\, dz = i\cot z + c'$$

$$\therefore\ f(z) = \frac{i}{1+i} \cdot \cot z + c.$$

Example 2 : Find an analytic function $f(z) = u + iv$ where $u + v = e^x(\cos y + \sin y)$.

(M.U. 2004, 05, 15)

Sol. : We have $\quad f(z) = u + iv \ \therefore\ i f(z) = iu - v$

$$\therefore\ (1+i) f(z) = (u - v) + i(u + v) = U + iV, \text{ say}$$

$$\therefore\ \frac{\partial V}{\partial x} = \frac{\partial}{\partial x}(u + v) = e^x(\cos y + \sin y) = \psi_1(x, y)$$

$$\frac{\partial V}{\partial y} = \frac{\partial}{\partial y}(u + v) = e^x(-\sin y + \cos y) = \psi_2(x, y)$$

By Milne-Thompson method

$$(1+i) f'(z) = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x} = \psi_2(z, 0) + i\,\psi_1(z, 0)$$

$$= e^z(0 + 1) + i\,e^z(1 + 0) = (1 + i)e^z$$

$$\therefore\ f'(z) = e^z \qquad \therefore\ f(z) = \int e^z dz = e^z + c.$$

Procedure to find $f(z)$ when $u - v$ is given

1. Let $U = u - v$ (given).
2. Find $\dfrac{\partial U}{\partial x} = \Phi_1(x, y)$ and $\dfrac{\partial U}{\partial y} = \Phi_2(x, y)$.
3. Find $\Phi_1(z, 0)$ and $\Phi_2(z, 0)$.
4. Then $(1 + i) f'(z) = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$.
5. Integrating both sides w.r.t. z, we get $f(z)$.

Example 3 : Find the analytic function $f(z) = u + iv$ such that

$$u - v = \frac{\cos x + \sin x - e^{-y}}{2\cos x - e^y - e^{-y}} \text{ when } f\left(\frac{\pi}{2}\right) = 0.$$

(M.U. 2003, 06, 07, 09, 13, 19)

Sol. : We have $\quad f(z) = u + iv \quad \therefore\ i f(z) = iu - v$

$$\therefore\ (1+i) f(z) = (u - v) + i(u + v) = U + iV, \text{ say.} \qquad \text{..... (A)}$$

$$\therefore\ \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(u - v) = \frac{(2\cos x - e^y - e^{-y})(-\sin x + \cos x) - (\cos x + \sin x - e^{-y})(-2\sin x)}{(2\cos x - e^y - e^{-y})^2}$$

$$= \frac{2 + e^y \sin x - e^{-y}\sin x - e^y \cos x - e^{-y}\cos x}{(2\cos x - e^y - e^{-y})^2}$$

$$= \Phi_1(x, y)$$

And $\quad \dfrac{\partial U}{\partial y} = \dfrac{\partial}{\partial y}(u - v) = \dfrac{(2\cos x - e^y - e^{-y})(e^{-y}) - (\cos x + \sin x - e^{-y})(-e^y + e^{-y})}{(2\cos x - e^y - e^{-y})^2}$

$$\frac{\partial U}{\partial y} = \frac{\cos x \cdot e^{-y} + \cos x\, e^y + \sin x\, e^y - \sin x \cdot e^{-y} - 2}{(2\cos x - e^y - e^{-y})^2} = \Phi_2(x, y)$$

(When $u - v$ is given we used $\dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y}$.)

From (A), $\quad (1+i) f'(z) = \dfrac{\partial U}{\partial x} + i\dfrac{\partial V}{\partial x} = \dfrac{\partial U}{\partial x} - i\dfrac{\partial U}{\partial y} \qquad [\because\ U_y = -V_x]$

$$(1+i) f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$

$$= \frac{2 + \sin z - \sin z - \cos z - \cos z}{(2\cos z - 2)^2} - i \cdot \frac{\cos z + \cos z + \sin z - \sin z - 2}{(2\cos z - 2)^2}$$

$$= \frac{2(1 - \cos z)}{4(\cos z - 1)^2} - i\frac{2(\cos z - 1)}{4(\cos z - 1)^2} = -\frac{1}{2(\cos z - 1)} - i \cdot \frac{1}{2(\cos z - 1)}$$

$$= (1 + i) \cdot \frac{1}{2(1 - \cos z)} = (1 + i) \cdot \frac{1}{4\sin^2(z/2)} = \frac{(1+i)}{4} \cdot \operatorname{cosec}^2\frac{z}{2}.$$

$$\therefore\ f'(z) = \frac{1}{4}\operatorname{cosec}^2\frac{z}{2}$$

$$\therefore\ f(z) = \frac{1}{4}\int \operatorname{cosec}^2(z/2)\, dz = -\frac{1}{2}\cot\frac{z}{2} + c$$

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But when $z = \frac{\pi}{2}$, $f(z) = 0$ $\therefore\ 0 = -\frac{1}{2} + c$ $\therefore\ c = \frac{1}{2}$

$$\therefore\ f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right).$$

Example 4 : Find the analytic function $f(z) = u + iv$ in terms of z if $u - v = (x - y)(x^2 + 4xy + y^2)$. **(M.U. 2003, 04, 07, 08, 12, 15)**

Sol. : We have $f(z) = u + iv$

$\therefore\ i f(z) = iu - v$

$\therefore\ (1 + i) f(z) = (u - v) + i(u + v) = U + iV$ say

We have

$$\begin{aligned} U &= u - v \\ &= (x - y)(x^2 + 4xy + y^2) \\ &= x^3 + 4x^2 y + xy^2 - x^2 y - 4xy^2 - y^3 \\ &= x^3 + 3x^2 y - 3xy^2 - y^3 \end{aligned}$$

$$\therefore\ \frac{\partial U}{\partial x} = \frac{\partial (u - v)}{\partial x} = 3x^2 + 6xy - 3y^2 = \Phi_1(x, y)$$

$$\frac{\partial U}{\partial y} = \frac{\partial (u - v)}{\partial y} = 3x^2 - 6xy - 3y^2 = \Phi_2(x, y)$$

$\therefore\ \Phi_1(z, 0) = 3z^2$ and $\Phi_2(z, 0) = 3z^2$.

$$\begin{aligned} \therefore\ (1 + i) f'(z) &= \Phi_1(z, 0) - i\,\Phi_2(z, 0) \\ &= 3z^2 - i \cdot 3z^2 \\ &= -3i^2 z^2 - i \cdot 3z^2 \qquad [\because\ i^2 = -1] \end{aligned}$$

By integration,

$$\begin{aligned} (1 + i) f(z) &= -3i^2 \int z^2\,dz - 3i \int z^2\,dz \\ &= -i^2 z^3 - i z^3 = -i z^3 (1 + i) \end{aligned}$$

$\therefore\ f(z) = -i z^3 + c.$

Example 5 : If $f(z) = u + iv$ is analytic and $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z. **(M.U. 2003, 05, 14, 19)**

Sol. : $\because\ f(z) = u + iv$, $i f(z) = iu - v$.

$\therefore\ (1 + i) f(z) = (u - v) + i(u + v) = U + iV$ say.

Now $\frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(u - v) = e^x (\cos y - \sin y) = \Phi_1(x, y)$

$$\frac{\partial U}{\partial y} = e^x (-\sin y - \cos y) = \Phi_2(x, y)$$

$$\therefore\ (1 + i) f'(z) = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \Phi_1(z, 0) - i\,\Phi_2(z, 0)$$

$$\therefore\ (1 + i) f(z) = \int [e^z + i e^z]\,dz = (1 + i)\int e^z\,dz = (1 + i) e^z + c'$$

$\therefore\ f(z) = e^z + c.$

Example 6 : Find the analytic function $f(z) = u + iv$ if $3u + 2v = y^2 - x^2 + 16xy$. **(M.U. 2002, 08, 13)**

Sol. : Differentiating the given relation w.r.t. x and y

$$3\frac{\partial u}{\partial x} + 2\frac{\partial v}{\partial x} = -2x + 16y \qquad \text{..... (i)}$$

and $$3\frac{\partial u}{\partial y} + 2\frac{\partial v}{\partial y} = 2y + 16x \qquad \text{..... (ii)}$$

But $u_x = v_y$ and $u_y = -v_x$. Hence, from (ii), we get,

$$-3\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial x} = 2y + 16x \qquad \text{..... (iii)}$$

Now, multiply (i) by (3) and (iii) by (2) and add.

$$\therefore\ 13\frac{\partial u}{\partial x} = 26x + 52y \quad i.e.\ \frac{\partial u}{\partial x} = 2x + 4y = \Phi_1(x, y)$$

Again, multiply (i) by (– 2) and (iii) by (3) and add.

$$\therefore\ -13\frac{\partial v}{\partial x} = 52x - 26y \quad i.e.\ \frac{\partial v}{\partial x} = -4x + 2y = \Phi_2(x, y)$$

But $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \Phi_1(x, y) + i\,\Phi_2(x, y) = \Phi_1(z, 0) + i\,\Phi_2(z, 0)$$

$\therefore\ f'(z) = 2z - i \cdot 4z$

$$\therefore\ f(z) = \int 2z\,dz - 2i\int 2z\,dz = z^2 - 2i z^2 + c = (1 - 2i) z^2 + c$$

Example 7 : State true or false with proper justification "There does not exist an analytic function whose real part is $x^3 - 3x^2 y - y^3$". **(M.U. 1995, 2004, 14)**

Sol. : We shall use the theorem (page 4-26) to check whether $u = x^3 - 3x^3 - y^3$ is a real part of some analytic function. By the result referred to above $u = x^3 - 3x^2 y - y^3$ must satisfy Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ if it is a real part of some analytic function.

Now $\frac{\partial u}{\partial x} = 3x^2 - 6xy$, $\frac{\partial^2 u}{\partial x^2} = 6x - 6y$; $\frac{\partial u}{\partial y} = -3x^2 - 3y^2$, $\frac{\partial^2 u}{\partial y^2} = -6y$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 12y \neq 0$$

$\therefore$ There does not exist an analytic function whose real part is $u = x^3 - 3x^2 y - y^3$.

Type IV : To find the analytic function whose real part u is given in polar form

Example 1 : State Laplace's equation in polar form and verify it for $u = r^2 \cos 2\theta$ and also find v and $f(z)$.

Sol. : Laplace's equation in polar form is $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$

$\because\ u = r^2 \cos 2\theta$

$\therefore\ \frac{\partial u}{\partial r} = 2r \cos 2\theta$, $\frac{\partial^2 u}{\partial r^2} = 2\cos 2\theta$; $\frac{\partial u}{\partial \theta} = -2r^2 \sin 2\theta$, $\frac{\partial^2 u}{\partial \theta^2} = -4r^2 \cos 2\theta$

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$$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 2\cos 2\theta + \frac{1}{r}(2r\cos 2\theta) + \frac{1}{r^2}(-4r^2\cos 2\theta)$$
$$= 4\cos 2\theta - 4\cos 2\theta = 0$$

$\therefore$ Laplace's equation is satisfied.

By Cauchy-Riemann equations in polar form $u_r = \frac{1}{r}v_\theta \quad \therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\therefore \frac{\partial v}{\partial \theta} = r(2r\cos 2\theta) = 2r^2\cos 2\theta$$

Integrating w.r.t. θ, $\quad v = r^2 \sin 2\theta + c$

Hence, $f(z) = u + iv$

$$\therefore f(z) = r^2\cos 2\theta + i r^2 \sin 2\theta + c = r^2(\cos 2\theta + i\sin 2\theta) + c$$
$$= r^2 e^{i2\theta} = (re^{i\theta})^2 + c = z^2 + c$$

Example 2 : Verify Laplace's equation for $u = \left(r + \frac{a^2}{r}\right)\cos\theta$. Also find v and $f(z)$.

(M.U. 2004, 14)

Sol. : $\because u = \left(r + \frac{a^2}{r}\right)\cos\theta$

$$\therefore \frac{\partial u}{\partial r} = \left(1 - \frac{a^2}{r^2}\right)\cos\theta, \quad \frac{\partial^2 u}{\partial r^2} = \frac{2a^2}{r^3}\cos\theta$$

$$\frac{\partial u}{\partial \theta} = -\left(r + \frac{a^2}{r}\right)\sin\theta, \quad \frac{\partial^2 u}{\partial \theta^2} = -\left(r + \frac{a^2}{r}\right)\cos\theta$$

$$\therefore \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{2a^2}{r^3}\cos\theta + \frac{1}{r}\cdot\left(1 - \frac{a^2}{r^2}\right)\cos\theta - \frac{1}{r^2}\left(r + \frac{a^2}{r}\right)\cos\theta$$
$$= 0.$$

$\therefore$ Laplace's equation is satisfied.

By Cuachy - Riemann equations in polar form $u_r = \frac{1}{r}v_\theta \quad \therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$.

$$\therefore \left(1 - \frac{a^2}{r^2}\right)\cos\theta = \frac{1}{r}\cdot\frac{\partial v}{\partial \theta} \qquad \therefore \frac{\partial v}{\partial \theta} = \left(r - \frac{a^2}{r}\right)\cos\theta$$

Integrating w.r.t. θ, $\quad v = \left(r - \frac{a^2}{r}\right)\sin\theta + c$.

Hence, $f(z) = u + iv = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta + c$

$$= r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta) + c = z + \frac{a^2}{z} + c.$$

Alternatively we can express u in terms of x and y and use cartesian form of Laplace's equation. However, it may be noted that this method is rather tedius.

Example 3 : If $u = k(1 + \cos\theta)$, find v so that $u + iv$ is analytical. **(M.U. 2007, 17, 18)**

Sol. : Since, $u = k + k\cos\theta$, $\quad \frac{\partial u}{\partial r} = 0$ and $\frac{\partial u}{\partial \theta} = -k\sin\theta$.

But by C-R equations in polar coordinates.

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial u}{\partial \theta} = -r\frac{\partial v}{\partial r} \qquad \therefore \frac{\partial v}{\partial \theta} = 0, \ \frac{\partial v}{\partial r} = -\frac{1}{r}(-k\sin\theta)$$

Integrating the first equation partially w.r.t θ,

$v = f(r)$ where $f(r)$ is an arbitrary function.

(If $v = f(r)$ then $\frac{\partial V}{\partial \theta} = 0$)

$$\therefore \frac{\partial v}{\partial r} = f'(r) = \frac{k\sin\theta}{r} \qquad \therefore v = k\sin\theta\log r + c$$

Hence, the analytic function is

$$f(z) = u + iv = k(1 + \cos\theta) + ik\sin\theta\log r + c$$

Example 4 : If $u = -r^3\sin 3\theta$, find the analytic function $f(z)$ whose real part is u.

(M.U. 2004, 05, 16)

Sol. : We have $\frac{\partial u}{\partial r} = -3r^2\sin\theta$ and $\frac{\partial u}{\partial \theta} = -3r^3\cos 3\theta$

By Cauchy-Riemann equations

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta} \qquad \therefore \frac{\partial v}{\partial \theta} = -3r^3\sin 3\theta.$$

Integrating w.r.t. θ, $\quad v = r^3\cos 3\theta$.

$$\therefore f(z) = u + iv = -r^3\sin 3\theta + i r^3\cos 3\theta$$
$$= i r^3(\cos 3\theta + i\sin 3\theta) = i r^3 e^{i3\theta} = i z^3 + c.$$

Example 5 : Show that $u = r - \frac{a^2}{r}\sin\theta$ cannot be the real part of an analytic function $f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$. **(M.U. 2003, 04)**

Sol. : Left to you. Show that the Cauchy equation in polar form, page 4-23 is not satisfied by u.

EXERCISE - III

1. Prove that $u(x, y) = x^2 - y^2$ and $v(x, y) = -y/(x^2 + y^2)$ are both harmonic functions but $u + iv$ is not analytic. **(M.U. 2003)**
2. Show that there does not exist an analytic function whose real part is
   (i) $3x^2 + \sin x + y^2 + 5y + 4$ **(M.U. 2002)** (ii) $3x^2 - 2x^2y + y^2$ **(M.U. 2003)**
3. Show that the following functions are harmonic
   (i) $e^x\cos y + x^3 - 3xy^2$ **(M.U. 2003)** (ii) $e^{2x}(x\cos 2y - y\sin 2y)$ **(M.U. 2003)**
   (iii) $\log\sqrt{x^2 + y^2}$ **(M.U. 2003)**

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4. Find the analytic function whose real part is

1. $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ **(M.U. 2011, 13)**
2. $u = (x-1)^3 - 3xy^2 + 3y^2$
3. $u = x^2 - y^2 - 5x + y + 2$
4. $u = \log\sqrt{x^2 + y^2}$
5. $u = \sin x \cosh y$
6. $u = e^{2x}(x\cos 2y - y\sin 2y)$
7. $u = e^x(x\cos y - y\sin y)$ **(M.U. 2014)**
8. $u = e^{-x}(x\sin y - y\cos y)$
9. $u = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$ **(M.U. 1995)**
10. $u = \dfrac{x}{2}\log(x^2 + y^2) - y\tan^{-1}\left(\dfrac{y}{x}\right) + \sin x \cosh y$ **(M.U. 2003, 08)**

[Ans. : (1) $z^3 + 3z^2 + c$, (2) $(z-1)^3 + c$, (3) $z^2 - 5z - iz + c$, (4) $\log z + c$, (5) $\sin z + c$, (6) $z e^{2z} + c$, (7) $z e^z + c$, (8) $i z e^{-z} + c$, (9) $\cot z + c$, (10) $z\log z + \sin z + c$. **]**

5. Find the analytic function whose imaginary part is

1. $v = \log(x^2 + y^2) + x - 2y$
2. $v = \dfrac{x-y}{x^2 + y^2}$
3. $v = \cos x \cosh y$
4. $v = \sinh x \cos y$
5. $v = e^x(x\sin y + y\cos y)$ **(M.U. 2005)**
6. $v = e^{-x}(x\sin y - y\cos y)$
7. $v = \dfrac{\sinh 2y}{\cos 2x + \cosh 2y}$
8. $v = \sinh x \sin y$
9. $v = e^{-x}\left[2xy\cos y + (y^2 - x^2)\sin y\right]$ **(M.U. 2003)**
10. $\dfrac{x}{x^2 + y^2} + \cosh x \cos y$ **(M.U. 2002, 09, 14)**
11. $v = \dfrac{y}{x^2 + y^2}$

[Ans. : (1) $(i-2)z + i\log z + c$, (2) $(1+i)\dfrac{1}{z}$, (3) $i\cos z + 1$, (4) $i\sin hz$, (5) $z e^z + c$, (6) $-z e^{-z} + c$, (7) $\tan z + c$, (8) $\sin hz + c$, (9) $e^{-z}\cdot z^2 + c$, (10) $i\left(\dfrac{1}{z} + \cosh hz\right) + c$, (11) $-\dfrac{1}{z} + c$. **]**

6. Find the analytic function $f(z) = u + iv$ in terms of z if

(i) $u - v = (x-y)(x^2 + 4xy + y^2)$ **(M.U. 2003, 04, 07, 08, 12)**

(ii) $u + v = \dfrac{x}{x^2 + y^2}$ **(M.U. 2003, 15)**

(iii) $u - v = \dfrac{\sin x + \sinh y}{\cosh y - \cos x}$

(iv) $u + v = \dfrac{\sin 2x}{\cosh 2y - \cos 2x}$

(v) $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$ and $f(\pi/2) = 0$ **(M.U. 1996)**

(vi) $u - v = x^3 + x^2 - 3xy^2 - y^2 - 3x^2y + y^3 - 2xy$

(vii) $u + v = e^x(\cos y + \sin y) + \dfrac{x-y}{x^2 + y^2}$ **(M.U. 2005)**

[Ans. : (i) $f(z) = -iz^3 + c$, (ii) $\dfrac{1}{1+i}\left(\dfrac{i}{z}\right) + c$, (iii) $\cot\dfrac{z}{2} + c$, (iv) $\dfrac{(1+i)}{2}\cot z + c$, (v) $\cot\left(\dfrac{z}{2}\right) - 1$, (vi) $z^3 + z^2 + c$, (vii) $e^z + \dfrac{1}{z} + c$. **]**

7. Find the analytic function $f(z)$ whose real part is

(i) $r^2\cos 2\theta - r\sin\theta + 2$ **(M.U. 2014)** (ii) $r^n\cos n\theta$ **[Ans. : (i) $z^2 + z + c$, (ii) $z^n + c$.]**

13. To Find an Analytic Function when Harmonic Function is Given

Example 1 : Show that $u = y^3 - 3x^2y$ is a harmonic function. Find its harmonic conjugate and the corresponding analytic function. **(M.U. 2003)**

Sol. : Method 1 : Since, $u = y^3 - 3x^2y$

$$u_x = -6xy,\ u_{xx} = -6y\ ;\ u_y = 3y^2 - 3x^2,\ u_{yy} = 6y$$

$$\therefore\ \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6y + 6y = 0$$

$\therefore$ $u = x^3 - 3x^2y$ is a harmonic function.

Now by the note 1 after § 12, page 4-29.

$$\int -\frac{\partial u}{\partial y}dx = \int -(3y^2 - 3x^2)dx = -3xy^2 + x^3$$

and $\int (\text{terms in } \dfrac{\partial u}{\partial x} \text{ free from } x)dy = \int 0\,dy = 0$

$\therefore$ $v = x^3 - 3xy^2 + c$.

$\therefore$ $f(z) = u + iv = (y^3 - 3x^2y) + i(x^3 - 3xy^2) + c$.

Putting $x = z,\ y = 0$ $\quad\therefore\ f(z) = z^3 i + c$

Method 2 : Since, $u = y^3 - 3x^2y$ by Milne-Thompson method given on page 4-29.

$$u_x = \Phi_1 = -6xy,\quad u_y = \Phi_2 = 3y^2 - 3x^2$$

$\therefore$ $f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = 0 + 3iz^2$

$\therefore$ $f(z) = \int 3iz^2\,dz = iz^3 + c$ as above is the required analytic function.

Now, $f(z) = i(x + iy)^3 = i(x^3 + 3ix^2y - 3xy^2 - iy^3)$

$\therefore$ $u + iv = -3x^2y + y^3 + i(x^3 - 3xy^2)$

$\therefore$ $v = x^3 - 3xy^2$ is the harmonic conjugate.

Note ....

This method is more convenient.

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Method 3 : $u = y^3 - 3x^2y$

$$\therefore\ u\left(\frac{z}{2}, \frac{z}{2i}\right) = \frac{z^3}{8i^3} - 3\left(\frac{z}{2}\right)^2\left(\frac{z}{2i}\right) = -\frac{z^3}{8i} - \frac{3z^3}{8i} = -\frac{z^3}{2i} = \frac{z^3}{2}i$$

$$u(0, 0) = 0$$

$$\therefore\ f(x) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + ci = z^3 i + ci.$$

Example 2 : Show that the function $u = \sin x \cos hy + 2\cos x \sin hy + x^2 - y^2 + 4xy$ satisfies Laplace's equation and find it corresponding analytic function $f(z) = u + iv$.

(M.U. 1998, 2005, 09, 13)

Sol. : We have $\dfrac{\partial u}{\partial x} = \cos x \cos hy - 2\sin x \sin hy + 2x + 4y$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cos hy - 2\cos x \sin hy + 2$$

$$\frac{\partial u}{\partial y} = \sin x \sin hy + 2\cos x \cos hy - 2y + 4x$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cos hy + 2\cos x \sin hy - 2$$

$$\therefore\ \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, u satisfies Laplace's equation.

Now $u_x = \Phi_1(x, y) = \cos x \cos hy - 2\sin x \sin hy + 2x + 4y$

$$\Phi_1(z,0) = \cos z + 2z$$

$$u_y = \Phi_2(x, y) = \sin x \sin hy + 2\cos x \cos hy - 2y + 4x$$

$$\Phi_2(z,0) = 2\cos z + 4z$$

Now we use Milne-Thompson Method given on page 4-29.

(Read the procedure given on page 4-31)

$$\therefore\ f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = (\cos z + 2z) - i(2\cos z + 4z)$$

$$\therefore\ f(z) = \int [(\cos z + 2z) - i(2\cos z + 4z)]\,dz = \sin z + z^2 - i(2\sin z + 2z^2) + c$$

which is the required analytic function.

Example 3 : If $v = e^x \sin y$, prove that v is a harmonic function. Also find the corresponding harmonic conjugate function and analytic function.

(M.U. 2011, 18)

Sol. : We have $\dfrac{\partial v}{\partial x} = e^x \sin y$, $\dfrac{\partial^2 v}{\partial x^2} = e^x \sin y$; $\dfrac{\partial v}{\partial y} = e^x \cos y$, $\dfrac{\partial^2 v}{\partial y^2} = -e^x \sin y$.

$$\therefore\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad \therefore\ v \text{ satisfies Laplace's equation.}$$

Now, we use, Milne-Thompson Method given on page 4-29.

(Read the procedure given on page 4-34)

$v_x = e^x \sin y \quad \therefore\ \psi_2(z,0) = 0$; $\quad v_y = e^x \cos y \quad \therefore\ \psi_1(z,0) = e^z$

$$\therefore\ f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = e^z + 0$$

$$\therefore\ f(z) = e^z + c$$

Now, $f(z) = e^z = e^{x+iy} = e^x \cdot e^{iy} = e^x(\cos y + i \sin y)$

$$\therefore\ u = e^x \cos y$$

Example 4 : Show that the function $v = e^x(x \sin y + y \cos y)$ satisfies Laplace equation and find its corresponding analytic function and its harmonic conjugate.

(M.U. 2013)

Sol. : We have $\dfrac{\partial v}{\partial x} = e^x(x \sin y + y \cos y) + e^x \sin y$

$$\therefore\ \frac{\partial^2 v}{\partial x^2} = e^x(x \sin y + y \cos y) + e^x(\sin y) + e^x \sin y$$

$$\frac{\partial v}{\partial y} = e^x(x \cos y + \cos y - y \sin y)$$

$$\therefore\ \frac{\partial^2 v}{\partial y^2} = e^x(-x \sin y - \sin y - \sin y - y \cos y)$$

$$\therefore\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \qquad \therefore\ v \text{ satisfies Laplace equation.}$$

Now, $v_x = e^x(x \sin y + y \cos y + \sin y)$

$$\therefore\ \Psi_2(z, 0) = 0$$

$$v_y = e^x(x \cos y + \cos y - y \sin y)$$

$$\therefore\ \Psi_1(z, 0) = e^z(z + 1)$$

$$\therefore\ f'(z) = \Psi_1(z, 0) + i\,\Psi_2(z, 0) = e^z(z + 1) + 0$$

$$\therefore\ f(z) = \int e^z(z + 1)\,dz = z e^z \quad \text{which is the required analytic function.}$$

Now, $f(z) = (x + iy)(e^{x+iy}) = (x + iy)(e^x \cdot e^{iy})$

$$= (x + iy)\,e^x(\cos y + i \sin y)$$

$$\therefore\ u = e^x(x \cos y - y \sin y) \quad \text{which is the required harmonic conjugate.}$$

Example 5 : If $v = 3x^2y + 6xy - y^3$, show that v is harmonic and find the corresponding analytic function.

(M.U. 2003, 07, 13, 14, 19)

Sol. : We have

$$\frac{\partial v}{\partial x} = 6xy + 6y, \quad \frac{\partial^2 v}{\partial x^2} = 6y; \quad \frac{\partial v}{\partial y} = 3x^2 + 6x - 3y^2, \quad \frac{\partial^2 v}{\partial y^2} = -6y$$

$$\therefore\ \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 6y - 6y = 0. \qquad \therefore\ v \text{ is satisfies Laplace's equation.}$$

Now we use Milne-Thompson Method given on page 4-29.

$$\therefore\ v_x = 6xy + 6y, \quad \psi_2(z,0) = 0$$

$$v_y = 3x^2 + 6x - 3y^2, \quad \psi_1(z,0) = 3z^2 + 6z$$

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$$\therefore \quad f'(z) = \psi_1(z,0) + i\,\psi_2(z,0) = (3z^2 + 6z) + 0$$

$$\therefore \quad f(z) = \int (3z^2 + 6z)\,dz = (z^3 + 3z^2) + c$$

Example 6 : Show that $u = \cos x \cos hy$ is a harmonic function. Find its harmonic conjugate and corresponding analytic function. **(M.U. 2005, 19)**

Sol. : Since, $u = \cos x \cos hy$

$$\frac{\partial u}{\partial x} = -\sin x \cos hy, \quad \frac{\partial^2 u}{\partial x^2} = -\cos x \cos hy$$

$$\frac{\partial u}{\partial y} = \cos x \sin hy, \quad \frac{\partial^2 u}{\partial y^2} = \cos x \cos hy$$

$$\therefore \quad \nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \qquad \therefore \quad u = \cos x \cos hy \text{ is a harmonic functions.}$$

Now, we use Milne-Thompson Method.

Now, $u_x = \Phi_1(x,y) = -\sin x \cos hy \qquad \therefore \quad \Phi_1(z,0) = -\sin z$

$u_y = \Phi_2(x,y) = \cos x \sin hy \qquad \therefore \quad \Phi_2(z,0) = 0$

$$\therefore \quad f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = -\sin z$$

$\therefore \quad f(z) = \int -\sin z\,dz = \cos z + c$ is the required analytic function.

Now, $f(z) = \cos(x + iy) = \cos x \cos iy - \sin x \sin iy$

$$\therefore \quad u + iv = \cos x \cos hy - i \sin x \sin hy$$

$\therefore \quad v = -\sin x \sin hy$ is the required harmonic conjugate.

Example 7 : Prove that $u = x^2 - y^2,\ v = -\dfrac{y}{x^2 + y^2}$ both u and v satisfy Laplace's equation, but that $u + iv$ is not an analytic function of z. **(M.U. 1996, 2003, 09)**

Sol. : $u_x = 2x,\ u_{xx} = 2\ ;\ u_y = -2y,\ u_{yy} = -2$

$$v_x = \frac{2xy}{(x^2 + y^2)^2}$$

$$\therefore \quad v_{xx} = 2y\left[\frac{(x^2 + y^2)^2 \cdot 1 - x \cdot 2(x^2 + y^2) \cdot 2x}{(x^2 + y^2)^4}\right]$$

$$= \frac{2y(x^2 + y^2)[x^2 + y^2 - 4x^2]}{(x^2 + y^2)^4} = 2y\frac{(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$$v_y = -\left[\frac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2}\right] = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$v_{yy} = \frac{(x^2 + y^2)^2 \cdot 2y - (y^2 - x^2) \cdot 2(x^2 + y^2) \cdot 2y}{(x^2 + y^2)^4}$$

$$= 2y(x^2 + y^2)\frac{[x^2 + y^2 - 2y^2 + 2x^2]}{(x^2 + y^2)^4} = 2y\frac{(3x^2 - y^2)}{(x^2 + y^2)^3} = -2y\frac{(y^2 - 3x^2)}{(x^2 + y^2)^3}$$

$\therefore \quad u_{xx} + u_{yy} = 0$ and $v_{xx} + v_{yy} = 0$

Hence, u, v satisfy Laplace's Equations.

But Cauchy-Riemann equations are not satisfied as $u_x \neq v_y$ and $u_y \neq -v_x$.

Hence, $u + iv$ is not analytic.

Example 8 : Show that the following function satisfies Laplace's equation and find its corresponding analytic function and the harmonic conjugate, $u = \dfrac{1}{2}\log(x^2 + y^2)$. **(M.U. 2002, 05, 06, 08, 10)**

Sol. : We have $\dfrac{\partial u}{\partial x} = \dfrac{x}{x^2 + y^2} \quad \therefore \quad \dfrac{\partial^2 u}{\partial x^2} = \dfrac{(x^2 + y^2) \cdot 1 - x \cdot 2x}{(x^2 + y^2)^2} = \dfrac{-x^2 + y^2}{(x^2 + y^2)^2}$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{y}{x^2 + y^2} \quad \therefore \quad \dfrac{\partial^2 u}{\partial y^2} = \dfrac{(x^2 + y^2) \cdot 1 - y \cdot 2y}{(x^2 + y^2)^2} = \dfrac{x^2 - y^2}{(x^2 + y^2)^2}$

$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{-x^2 + y^2 + x^2 - y^2}{(x^2 + y^2)^2} = 0$$

$\therefore$ u satisfies Laplace's equation.

Now $u_x = \Phi_1(x,y) = \dfrac{x}{x^2 + y^2} \qquad \therefore \quad \Phi_1(z,0) = \dfrac{z}{z^2 + 0} = \dfrac{1}{z}$

$u_y = \Phi_2(x,y) = \dfrac{y}{x^2 + y^2} \qquad \therefore \quad \Phi_2(z,0) = 0$

By Milne-Thompson method,

$$\therefore \quad f'(z) = \Phi_1(z,0) - i\,\Phi_2(z,0) = \frac{1}{z} - i\,0 = \frac{1}{z}$$

$$\therefore \quad f(z) = \int \frac{1}{z}\,dz = \log z + c = \log(x + iy) + c$$

$$\therefore \quad u + iv = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\frac{y}{x} + c \qquad \textbf{[Note this]}$$

(See Applied Mathematics I by the same author.)

$\therefore \quad v = \tan^{-1}\dfrac{y}{x} + c$ is the corresponding harmonic conjugate.

Example 9 : Prove that $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right) = 4\dfrac{\partial^2}{\partial z\,\partial \bar{z}}$.

Sol. : We have $z = x + iy,\ \bar{z} = x - iy$

$$\therefore \quad x = \frac{1}{2}(z + \bar{z}), \quad y = \frac{1}{2i}(z - \bar{z})$$

Treating z and $\bar{z}$ as independent variables

$$\therefore \quad \frac{\partial x}{\partial z} = \frac{1}{2}, \quad \frac{\partial x}{\partial \bar{z}} = \frac{1}{2}, \quad \frac{\partial y}{\partial z} = \frac{1}{2i}, \quad \frac{\partial y}{\partial \bar{z}} = -\frac{1}{2i}.$$

Now, $\dfrac{\partial}{\partial z} = \dfrac{\partial}{\partial x} \cdot \dfrac{\partial x}{\partial z} + \dfrac{\partial}{\partial y} \cdot \dfrac{\partial y}{\partial z} = \dfrac{\partial}{\partial x} \cdot \dfrac{1}{2} + \dfrac{\partial}{\partial y} \cdot \dfrac{1}{2i} = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} - i\dfrac{\partial}{\partial y}\right)$

And $\dfrac{\partial}{\partial \bar{z}} = \dfrac{\partial}{\partial x} \cdot \dfrac{\partial x}{\partial \bar{z}} + \dfrac{\partial}{\partial y} \cdot \dfrac{\partial y}{\partial \bar{z}} = \dfrac{\partial}{\partial x} \cdot \dfrac{1}{2} + \dfrac{\partial}{\partial y} \cdot \left(-\dfrac{1}{2i}\right) = \dfrac{1}{2}\left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$

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$$\therefore \quad \frac{\partial^2}{\partial z\,\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\cdot\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) = \frac{1}{4}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Hence, the result.

Example 10 : If $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 - y^2$, find u.

Sol. : Let $z = x + iy \quad \therefore \bar{z} = x - iy \quad \therefore \quad x = \frac{z+\bar{z}}{2},\ y = \frac{z-\bar{z}}{2i}$

$$\therefore \quad x^2 - y^2 = \frac{1}{4}[(z+\bar{z})^2 + (z-\bar{z})^2] = \frac{1}{2}(z^2 + \bar{z}^2) \qquad \text{..... (1)}$$

By the above result $\quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial z\,\partial \bar{z}} \qquad$ ..... (2)

$$\therefore \quad 4\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = x^2 - y^2 \qquad \text{[By data]}$$

$$\therefore \quad 4\frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{2}(z^2 + \bar{z}^2) \qquad \text{[By (1)]}$$

$$\therefore \quad \frac{\partial^2 u}{\partial z\,\partial \bar{z}} = \frac{1}{8}(z^2 + \bar{z}^2)$$

$\therefore$ By integration w.r.t. $\bar{z}$ $\quad \frac{\partial u}{\partial z} = \frac{1}{8}\cdot z^2\bar{z} + \frac{\bar{z}^3}{24} + f_1(\bar{z})$

where $f_1(\bar{z})$ if a constant of integration and f_1 is an arbitrary function.

Integrating again, w.r.t. z,

$$u = \frac{z^3}{24}\cdot\bar{z} + \frac{\bar{z}^3\cdot z}{24} + f_2(z) + f_1(\bar{z}) \text{ where } f_2(z) \text{ is an arbitrary function of } z.$$

$$\therefore \quad u = \frac{z\bar{z}}{24}(z^2 + \bar{z}^2) + f_1(\bar{z}) + f_2(z)$$

$$= \frac{(x+iy)(x-iy)}{24}[(x+iy)^2 + (x-iy)^2] + f_1(x-iy) + f_2(x+iy)$$

$$= \frac{(x^2-y^2)}{12}\cdot(x^2+y^2) + f_1(x-iy) + f_2(x+iy)$$

$$= \frac{x^4-y^4}{12} + f_1(x-iy) + f_2(x+iy).$$

Example 11 : If $f(z)$ is analytic function, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

(M.U. 1993, 96, 98, 2003, 06, 07)

Sol. : We first note that if $f(z) = u + iv,\ f(\bar{z}) = u - iv$

$$\therefore \quad |f(z)|^2 = u^2 + v^2 = (u+iv)(u-iv) = f(z)\cdot f(\bar{z}) \qquad \text{..... (1)}$$

Also, $\quad f'(z) = u_x + iv_x,\ \ f'(\bar{z}) = u_x - iv_x$

$$\therefore \quad |f'(z)|^2 = u_x^2 + v_x^2 = (u_x + iv_x)(u_x - iv_x) = f'(z)\cdot f'(\bar{z}) \qquad \text{..... (2)}$$

Now, by the result obtained in Ex. 9 above.

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}f(z)\cdot f(\bar{z})$$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}f(z)\cdot f(\bar{z})\right] = 4\frac{\partial}{\partial z}\left[f(z)\cdot\frac{\partial}{\partial \bar{z}}f(\bar{z})\right]$$

$$= 4\frac{\partial}{\partial z}[f(z)\cdot f'(\bar{z})] = 4\cdot f'(\bar{z})\frac{\partial}{\partial z}f(z) = 4\cdot f'(\bar{z})\cdot f'(z)$$

$$= 4|f'(z)|^2 \qquad \text{[By (2)]}$$

Restatement

Since $|f(z)|^2 = u^2 + v^2$ and $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$, the above example can be restated in the following way.

"If $f(z)$ is an anylatic function, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u^2 + v^2) = 4\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]."$$

Example 12 : If $f(z)$ is an analytic function, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2|f'(z)|^{n-2}\cdot|f'(z)|^2.$$

(M.U. 1994, 2002)

Sol. : We first note that

$$|f(z)| = \left\{|f(z)|^2\right\}^{1/2} = [f(z)\cdot f(\bar{z})]^{1/2} \quad \therefore \quad |f(z)|^n = [f(z)\cdot f(\bar{z})]^{n/2} \qquad \text{..... (1)}$$

Now $\quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|f(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}[f(z)\cdot f(\bar{z})]^{n/2}$

$$= 4\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}[f(z)]^{n/2}\cdot[f(\bar{z})]^{n/2}\right] = 4\left[\frac{\partial}{\partial z}\left\{[f(z)]^{n/2}\cdot\frac{\partial}{\partial \bar{z}}[f(\bar{z})]^{n/2}\right\}\right]$$

$$= 4\left[\frac{\partial}{\partial z}[f(z)]^{n/2}\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\right]$$

$$= 4\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\cdot\frac{\partial}{\partial z}\cdot[f(z)]^{n/2}$$

$$= 4\cdot\frac{n}{2}f(\bar{z})^{(n/2)-1}\cdot f'(\bar{z})\cdot\frac{n}{2}\cdot[f(z)]^{(n/2)-1}\cdot f'(z)$$

$$= n^2[f(z)\cdot f(\bar{z})]^{(n/2)-1}\cdot[f'(z)\cdot f'(\bar{z})]$$

But $\quad f(z)\cdot f(\bar{z}) = |f(z)|^2$ and $f'(z)\cdot f'(\bar{z}) = |f'(z)|^2$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^n = n^2\left\{|f(z)|^2\right\}^{(n/2)-1}\cdot|f'(z)|^2$$

$$= n^2|f(z)|^{n-2}\cdot|f'(z)|^2$$

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Example 13 : If $f(z) = u + iv$ is an analytic function, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2|f'(z)|^2$.

(M.U. 2005)

Sol. : We first note that if $f(z) = u + iv$, $f(\bar{z}) = u - iv$

$$\therefore \quad |Rf(z)|^2 = \left|\frac{1}{2}[f(z) + f(\bar{z})]\right|^2 = \frac{1}{4}|f(z) + f(\bar{z})|^2 \qquad \text{..... (1)}$$

And $\quad |f(z)|^2 = u^2 + v^2 = (u + iv)(u - iv) = f(z) \cdot f(\bar{z}) \qquad$ ..... (2)

$$\therefore \quad |Rf(z)|^2 = \frac{1}{4}[f(z) + f(\bar{z})][f(\bar{z}) + f(\bar{z})] = \frac{1}{4}[f(z) + f(\bar{z})] \cdot [f(\bar{z}) + f(z)]$$

$$= \frac{1}{4}[f(z) + f(\bar{z})]^2$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|Rf(z)|^2 = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\frac{1}{4}[f(z) + f(\bar{z})]^2\right\}$$

$$= \frac{\partial^2}{\partial z\,\partial \bar{z}}[f(z) + f(\bar{z})]^2 = \frac{\partial}{\partial z}2[f(z) + f(\bar{z})] \cdot f'(\bar{z})$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^2 = 2f'(\bar{z}) \cdot \frac{\partial}{\partial z}[f(z) + f(\bar{z})] = 2f'(\bar{z}) \cdot f'(z)$$

$$= 2|f'(z)|^2 \qquad [\text{ By (2) }]$$

Example 14 : If $f(z)$ is an analytic function, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n = n(n-1)|Rf(z)|^{n-2} \cdot |f'(z)|^2.$$

Sol. : We know that

$$|Rf(z)|^2 = \left|\frac{1}{2}[f(z) + f(\bar{z})]\right|^2 = \frac{1}{4}|f(z) + f(\bar{z})|^2 \qquad \text{..... (1)}$$

But $\quad |f(z)|^2 = f(z) \cdot f(\bar{z}) \qquad$ ..... (2)

$$\therefore \quad |Rf(z)|^2 = \frac{1}{4}|f(z) + f(\bar{z})|^2 = \frac{1}{4}[f(z) + f(\bar{z})] \cdot [f(\bar{z}) + f(\bar{z})]$$

$$= \frac{1}{4}[f(z) + f(\bar{z})] \cdot [f(\bar{z}) + f(z)] = \frac{1}{4}[f(z) + f(\bar{z})]^2 \qquad \text{..... (3)}$$

$$\therefore \quad |Rf(z)|^n = \left[|Rf(z)|^2\right]^{n/2} = \left\{\frac{1}{4}[f(z) + f(\bar{z})]\right\}^{n/2} = \frac{1}{2^n}[f(z) + f(\bar{z})]^n \qquad \text{..... (4)}$$

Now, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}|Rf(z)|^n = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left\{\frac{1}{2^n}[f(z) + f(\bar{z})]^n\right\}$$

$$= 4 \cdot \frac{1}{2^n}\frac{\partial}{\partial z}\left\{n[f(z) + f(\bar{z})]^{n-1} \cdot f'(\bar{z})\right\} = \frac{n}{2^{n-2}}f'(\bar{z}) \cdot \frac{\partial}{\partial z}[f(z) + f(\bar{z})]^{n-1}$$

$$= \frac{n}{2^{n-2}}f'(z) \cdot (n-1)[f(z) + f(\bar{z})]^{n-2} \cdot f'(z)$$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n = \frac{n(n-1)}{2^{n-2}}[f(z) + f(\bar{z})]^{n-2} \cdot f'(z) \cdot f'(\bar{z})$$

By (2), $\quad f'(z) \cdot f'(\bar{z}) = |f'(z)|^2$

By (4), $\quad \frac{1}{2^{n-2}}[f(z) + f(\bar{z})]^{n-2} = |Rf(z)|^{n-2}$

$$\therefore \quad \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|Rf(z)|^n = n(n-1)|Rf(z)|^{n-2} \cdot |f'(z)|^2$$

Example 15 : If $f(z)$ is an analytic function, prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 0$.

(M.U. 2000, 11)

Sol. : We first note that $\log|f'(z)| = \frac{1}{2}\log|f'(z)|^2$

But $\quad |f'(z)|^2 = f'(z) \cdot f'(\bar{z})$

$$\therefore \quad \log|f'(z)| = \frac{1}{2}\log[f'(z) \cdot f'(\bar{z})] = \frac{1}{2}\log f'(z) + \frac{1}{2}\log f'(\bar{z})$$

Now, $$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log|f'(z)| = 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\log|f'(z)|$$

$$= 4\frac{\partial^2}{\partial z\,\partial \bar{z}}\left[\frac{1}{2}\log f'(z) + \frac{1}{2}\log f'(\bar{z})\right] = 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\log f'(z) + 2\frac{\partial^2}{\partial z\,\partial \bar{z}}\log f'(\bar{z})$$

$$= 2\frac{\partial}{\partial z}\left[\frac{\partial}{\partial \bar{z}}\log f'(z)\right] + 2\frac{\partial}{\partial \bar{z}}\left[\frac{\partial}{\partial z}\log f'(\bar{z})\right] = 2\frac{\partial}{\partial z}(0) + 2\frac{\partial}{\partial \bar{z}}(0) = 0.$$

Example 16 : If Φ and Ψ are functions satisfying Laplace equation, then show that $s + it$ is holomorphic (analytic) where $s = \frac{\partial \Phi}{\partial y} - \frac{\partial \Psi}{\partial x}$ and $t = \frac{\partial \Phi}{\partial x} + \frac{\partial \Psi}{\partial y}$. **(M.U. 1999, 2003, 05)**

Sol.: Since Φ and Ψ satisfy Laplace equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \text{ and } \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} = 0 \qquad \text{..... (1)}$$

Now, $$\frac{\partial s}{\partial x} = \frac{\partial^2 \Phi}{\partial y\,\partial x} - \frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \Phi}{\partial y\,\partial x} + \frac{\partial^2 \Psi}{\partial y^2} \qquad [\text{ By (1) }] \qquad \text{..... (2)}$$

$$\frac{\partial s}{\partial y} = \frac{\partial^2 \Phi}{\partial y^2} - \frac{\partial^2 \Psi}{\partial x\,\partial y} = -\frac{\partial^2 \Phi}{\partial x^2} - \frac{\partial^2 \Psi}{\partial x\,\partial y} \qquad [\text{ By (1) }] \qquad \text{..... (3)}$$

Also, $$\frac{\partial t}{\partial x} = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y\,\partial x} \qquad \text{..... (4)}$$

$$\frac{\partial t}{\partial y} = \frac{\partial^2 \Phi}{\partial x\,\partial y} + \frac{\partial^2 \Psi}{\partial y^2} \qquad \text{..... (5)}$$

From (2) and (5), we have $\quad \frac{\partial s}{\partial x} = \frac{\partial t}{\partial y}$

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From (3) and (4), we have $\frac{\partial s}{\partial y} = -\frac{\partial t}{\partial x}$

Since, $s + it$ satisfies Cauchy-Riemann equations it is analytic.

EXERCISE - IV

1. Prove that $u = e^x \cos y + x^3 - 3xy^2$ is harmonic. **(M.U. 2003)**
2. Check whether $u = x + e^{xy} + y + e^{-xy}$ is harmonic. **(M.U. 2004) [Ans. : No]**
3. Prove that $u = e^x \cos y$ is harmonic. Determine its harmonic conjugate v and the analytic function $f(z)$. **[Ans. : $f(z) = e^z + c$, $v = e^x \sin y$]**
4. Prove that $u = x^2 - y^2 - 2xy - 2x + 3y$ is harmonic. Find the corresponding analytic function $f(z)$. Also find the conjugate function. **[Ans. : $f(z) = (1+i)z^2 - (z+3i)z$, $v = x^2 + 2xy - y^2 - 3x - 2y$]**
5. Show that $f(z) = e^x(\cos y + i \sin y)$ is analytic, it also satisfies Laplace's equation. Find its derivative. **[Ans. : $f(z) = e^z + c$]**
6. Show that the function $u = 3x^2y + 2x^2 - y^3 - 2y^2$ is harmonic. Find the harmonic conjugate function v and express $u + iv$ as an analytic function of z. **(M.U. 2006, 19) [Ans. : $v = 4xy - x^3 + 3xy^2$; $f(z) = -iz^3 + 2z^2 + c$]**
7. Show that $u = e^x(x \cos y - y \sin y)$ is harmonic. Find the harmonic conjugate v and the analytic function $f(z)$. **(M.U. 2009) [Ans. : $v = e^x(x \sin y + y \cos y)$; $f(z) = ze^z + c$]**
8. Show that the following functions are harmonic and find the corres-ponding analytic function $f(z) = u + iv$

   (i) $v = e^{-x}(x \cos y + y \sin y)$, (ii) $v = e^{2x}(y \cos 2y + x \sin 2y)$.

   (iii) $u = e^x \cos y - x^2 + y^2$ (iv) $u = x^4 - 6x^2y^2 + y^4$ **(M.U. 2013)**

   [Ans. : (i) $ize^{-z} + c$, (ii) $ze^{2z} + c$, (iii) $e^z - z^2 + c$, (iv) $z^4 + c$.]
9. Show that the following functions are harmonic and find their corres-ponding analytic functions $f(z) = u + iv$

   (i) $u = x^2 - y^2$, (ii) $u = 2x(1 - y)$ (iii) $u = 3x^2y - y^3$,

   (iv) $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ (v) $u = (x-1)^3 - 3xy^2 + 3y^2$ **(M.U. 2004)**

   [Ans. : (i) $z^2 + c$, (ii) $2z + iz^2 = c$, (iiii) $z^3 + c$, (iv) $z^3 + 3z^2 + c$, (v) $(z-1)^3 + c$.]
10. Find the analytic function $f(z) = u + iv$ where $u = 2axy + b(y^2 - x^2)$. Also verify that u is harmonic. **(M.U. 2004) [Ans. : $f(z) = (a - bi)z^2$]**
11. Show that $u = e^{-2xy} \sin(x^2 - y^2)$ is harmonic. Find the corresponding analytic function $f(z)$ and also find the conjugate function v. **[Ans. : $f(z) = -ie^{iz^2} + c$; $v = -e^{-2xy} \cos(x^2 - y^2) + c'$]**
12. Show that $u = \frac{\sin 2x}{\cosh 2y + \cos 2x}$ is harmonic. Hence, find its harmonic conjugate v and corresponding analytic function $f(z) = u + iv$. **(M.U. 1997, 2006) [Ans. : $f(z) = \tan z + c$]**

13. If u and v are harmonic functions of x and y and $s = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ and $t = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$, prove that $s + it$ is an analytic function of $z = x + iy$.
14. If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function of $z = x + iy$. **(M.U. 2005)**

Remark ....

The solved Example No. 16 above and this example are the same with the difference in wording them.

14. Orthogonal Curves - Orthogonal Trajectories

(a) If $f(z) = u(x, y) + iv(x, y)$ is an analytic function then the curves $u = c_1$ and $v = c_2$ intersect orthogonally. **(M.U. 2003, 06)**

Proof : Let $u = f(x, y) = c_1$ and $v = \Phi(x, y) = c_2$

Then $\left(\frac{dy}{dx}\right)_{u=c_1} = -\frac{\partial f/\partial x}{\partial f/\partial y} = -\frac{\partial u/\partial x}{\partial u/\partial y}$ and $\left(\frac{dy}{dx}\right)_{v=c_2} = -\frac{\partial \Phi/\partial x}{\partial \Phi/\partial y} = -\frac{\partial v/\partial x}{\partial v/\partial y}$

Since, $f(z)$ is analytic C-R equations give $u_x = v_y$ and $u_y = -v_x$

$$\therefore \left(\frac{\partial y}{\partial x}\right)_{u=c_1} \times \left(\frac{dy}{dx}\right)_{v=c_2} = \frac{\partial u/\partial x}{\partial u/\partial y} \times \frac{\partial v/\partial x}{\partial v/\partial y} = \frac{v_y}{-v_x} \cdot \frac{v_x}{v_y} = -1.$$

Hence, $u = c_1$ and $v = c_2$ intersect orthogonally.

(b) Orthogonal Trajectories : By an **orthogonal trajectory** of a family of curves we mean a curve which cuts every member of the given family at right angles. For example, consider a family of straight lines passing through the origin given by $y = mx$, where m is an arbitrary constant.

It is easy to see that these straight lines are cut by a circle with centre at the origin at right angles at every point of intersection. Its equation is of the form $x^2 + y^2 = a^2$ where a is a parameter.

Thus, the family of circles $x^2 + y^2 = a^2$ represents the family of orthogonal trajectories to the family of straight lines given by $y = mx$.

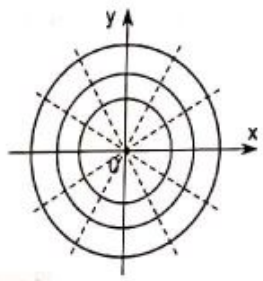

Fig. 4.4

(c) Orthogonal trajectories of the family of curves given by $u = c$.

We have seen that if $f(z) = u + iv$ is an analytic function then the curves $u = c_1$ and $v = c_2$ intersect orthogonally *i.e.* $v = c_2$ is the family of orthogonal trajectories of the family of curves given by $u = c_1$.

Hence, to find the orthogonal trajectory of $u = c_1$ (or $v = c_2$) we find the harmonic conjugate $v = c_2$ (or $u = c_1$) of u (or v).

Example 1 : Find the orthogonal trajectory of the family of curves $x^3y - xy^3 = c$. **(M.U. 2003, 04, 05, 06, 07, 11, 14, 19)**

Sol. : As seen above we have to find the harmonic conjugate of u.

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$\because\ u = x^3y - xy^3 \qquad \therefore\ u_x = 3x^2y - y^3$ and $u_y = x^3 - 3xy^2$

$\therefore\ f'(z) = u_x + iv_x = u_x - iu_y$ [By C-R equations]

$= (3x^2y - y^3) - i(x^3 - xy^2)$

By Milne-Thompson's method, we put $x = z,\ y = 0$

$\therefore\ f'(z) = -iz^3$

$\therefore\ f(z) = -\int iz^3 dz = -i\dfrac{z^4}{4} + c = -\dfrac{i}{4}(x+iy)^4 + c$

$= -\dfrac{i}{4}(x^4 + 4x^3iy - 6x^2y^2 - 4xiy^3 + y^4) + c$

$\therefore$ Imaginary part $v = -\dfrac{1}{4}(x^4 - 6x^2y^2 + y^4)$

Hence, the required orthogonal trajectories are $x^4 - 6x^2y^2 + y^4 = c'$.

Example 2 : Find the orthogonal trajectories of the family of curves $e^{-x}\cos y + xy = \alpha$, where α is the real constant in the xy-plane. **(M.U. 2004, 05, 13, 16, 19)**

Sol. : We have seen above that the orthogonal trajectories of $u = c_1$ are given by $v = c_2$ where v is the harmonic conjugate of u.

$\therefore\ u = e^{-x}\cos y + xy \qquad \therefore\ u_x = -e^{-x}\cos y + y$ and $u_y = -e^{x}\sin y + x$

Also $f'(z) = u_x + iv_x = u_x - iu_y$ [By C-R equations]

$= (-e^{-x}\cos y + y) - i(-e^{-x}\sin y + x)$

By Milne-Thompson's method, we replace x by z and y by zero.

$\therefore\ f'(z) = -e^{-z} - iz$

By integration $f(z) = e^{-z} - i\dfrac{z^2}{2} + c$

$\therefore\ f(z) = e^{-(x+iy)} - i\dfrac{(x+iy)^2}{2} + c$

$= e^{-x}(\cos y - i\sin y) - \dfrac{i}{2}(x^2 + 2ixy - y^2) + c$

$\therefore$ Imaginary part, $\quad v = -e^{-x}\sin y - \dfrac{1}{2}(x^2 - y^2)$

Hence, the required orthogonal trajectories are $e^{-x}\sin y + \dfrac{1}{2}(x^2 - y^2) = c_2$.

Example 3 : Find the orthogonal trajectory of the family of curves given by $2x - x^3 + 3xy^2 = a$. **(M.U. 2002, 04, 16)**

Sol. : Let $u = 2x - x^3 + 3xy^2$

$\therefore\ u_x = 2 - 3x^2 + 3y^2,\ u_y = 6xy$

$\therefore\ f'(z) = u_x + i\,v_x = u_x - i\,u_y$ [By C-R equations]

$= 2 - 3x^2 + 3y^2 - i \cdot 6xy$

By Milne-Thomson method, we put $x = z,\ y = 0$.

$\therefore\ f'(z) = 2 - 3z^2$

Integrating w.r.t. z, we get

$f(z) = 2z - z^3 + c = 2(x+iy) - (x+iy)^3 + c$

$= 2x + 2iy - x^3 - 3ix^2y + 3xy^2 + iy^3 + c$

$\therefore$ Imaginary part is $v = 2y - 3x^2y + y^3$.

$\therefore$ The required orthogonal trajectory is $\quad 2y - 3x^2y + y^3 = c$.

Example 4 : For the function $f(z) = z^3$, verify that the families of curves $u = c_1$ and $v = c_2$ cut orthogonally where c_1 and c_2 are constants and $f(z) = u + iv$. **(M.U. 2004)**

Sol. : $f(z) = (x+iy)^3 = x^3 + 3ix^2y - 3xy^2 - iy^3$

$\therefore\ u = x^3 - 3xy^2,\ \ v = 3x^2y - y^3;\quad u_x = 3x^2 - 3y^2,\ \ u_y = -6xy$

$v_x = 6xy,\ \ v_y = 3x^2 - 3y^2$

$\therefore\ m_1 = \left(\dfrac{dy}{dx}\right)_{u=c_1} = -\dfrac{u_x}{u_y} = -\dfrac{3(x^2 - y^2)}{-6xy}$

$m_2 = \left(\dfrac{dy}{dx}\right)_{u=c_2} = -\dfrac{v_x}{v_y} = -\dfrac{6xy}{3(x^2 - y^2)}$

$\therefore\ m_1 \times m_2 = \dfrac{3(x^2 - y^2)}{6xy} \cdot \left(-\dfrac{6xy}{3(x^2 - y^2)}\right) = -1$

Hence, the families cut orthogonally.

EXERCISE - V

Find the orthogonal trajectories of the family of curves

1. $3x^2y - y^3 = c$ **(M.U. 2002, 04, 07, 15)**
2. $x^2 - y^2 + x = c$ **(M.U. 2019)**
3. $e^{-x}(x\sin y - y\cos y) = c$
4. $x^2 - y^2 - 2xy + 2x - 3y = c$ **(M.U. 2004, 10)**
5. $e^x\cos y - xy = c$ **(M.U. 2007, 08)**
6. $3x^2y + 2x^2 - y^3 - 2y^2 = c$ **(M.U. 2003, 12, 14)**

[**Ans. :** (1) $3xy^2 - x^3 = c$, (2) $2xy + y = c$, (3) $e^{-x}(x\cos y + y\sin y) = c$
(4) $2xy + 2y + x^2 - y^2 + 3y = c$ (5) $e^x\sin y + (x^2 - y^2)/2 = c$,
(6) $4y - 3x^2 + 3xy^2 = c$.]

(d) Angle of Intersection of Two Curves in Polar Coordinates : If the equation of a curve is given in polar coordinates by $r = f(\theta)$ and if Φ is the angle between the radius vector at P and the tangent at P, then $\tan\Phi = r\dfrac{d\theta}{dr}$. (We accept this result without proof.)

If two curves given by polar equations $r = f(\theta)$ and $r = F(\theta)$ intersect at P and if Φ_1 and Φ_2 are the angles between the common radius vector and the tangents to the two curves then it is clear from the figure that the angle between the curves *i.e.* the angle between the tangents is equal to $\Phi_1 \sim \Phi_2$ *i.e.* the difference between Φ_1 and Φ_2.

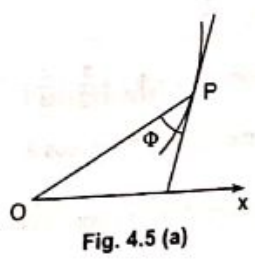

Fig. 4.5 (a)

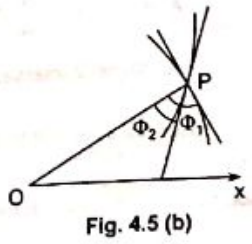

Fig. 4.5 (b)

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The two curves intersect orthogonally if the angle between the tangents to the two curves at the common point is $\pi/2$ *i.e.* if $\Phi_1 - \Phi_2 = \pi/2$.

Example 1 : Prove that the curves $r^n = a \sec n\theta$ and $r^n = b \operatorname{cosec} n\theta$ intersect orthogonally. (M.U. 2006)

Sol. : Let $P(r, \theta)$ be the point of intersection.

For the first curve $\quad n \log r = \log a + \log \sec n\theta$

$$\therefore \frac{n}{r}\cdot\frac{dr}{d\theta} = \frac{1}{\sec n\theta}\cdot \sec n\theta \cdot \tan n\theta \cdot n \quad \therefore \frac{dr}{d\theta} = r \tan n\theta$$

$$\therefore \tan\Phi_1 = r\frac{d\theta}{dr} = \frac{r}{r\tan n\theta} = \cot n\theta \quad \therefore \Phi_1 = \frac{\pi}{2} - n\theta.$$

For the second curve, $\quad n \log r = \log b + \log \operatorname{cosec} n\theta$

$$\therefore \frac{n}{r}\frac{dr}{d\theta} = -\frac{1}{\operatorname{cosec} n\theta}\cdot \operatorname{cosec} n\theta \cdot \cot n\theta \cdot n \quad \therefore \frac{dr}{d\theta} = -r\cot n\theta$$

$$\therefore \tan\Phi_2 = r\frac{d\theta}{dr} = -\frac{r}{r\cot n\theta} = -\tan n\theta \quad \therefore \Phi_2 = -n\theta$$

$$\therefore \Phi_1 - \Phi_2 = \frac{\pi}{2} - n\theta - (-n\theta) = \frac{\pi}{2}.$$

$\therefore$ The curves intersect orthogonally.

Example 2 : Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.

Sol. : If the two curves intersect at $P(r, \theta)$, then by eliminating r, we have $\sin\theta + \cos\theta = 2\sin\theta$

$$\therefore \sin\theta = \cos\theta \quad \therefore \tan\theta = 1 \quad \therefore \theta = \frac{\pi}{4} \text{ at the point of intersection.}$$

Now, for the curve $r = \sin\theta + \cos\theta$

$$\frac{dr}{d\theta} = \cos\theta - \sin\theta \quad \therefore \tan\Phi = r\frac{d\theta}{dr} \quad \therefore \tan\Phi = \frac{\sin\theta + \cos\theta}{\cos\theta - \sin\theta}$$

$$\therefore \text{At } P, \text{ i.e. at } \theta = \frac{\pi}{4}, \ \tan\Phi = \frac{(1/\sqrt{2}) + (1/\sqrt{2})}{(1/\sqrt{2}) - (1/\sqrt{2})} = \infty.$$

$$\therefore \text{At } P, \ \Phi_1 = \frac{\pi}{2}$$

Also for the curve $r = 2\sin\theta$.

$$\frac{dr}{d\theta} = 2\cos\theta \quad \therefore \tan\Phi = r\frac{d\theta}{dr} = \frac{2\sin\theta}{2\cos\theta}$$

$$\therefore \tan\Phi = \tan\theta \quad \therefore \Phi = \theta$$

$$\therefore \text{At } P, \text{ i.e., at } \theta = \frac{\pi}{4}, \ \Phi_2 = \frac{\pi}{4}.$$

$$\therefore \text{Angle of intersection} = \Phi_1 - \Phi_2 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

Example 3 : Prove that the curves $r^2 = a^2 \cos 2\theta$ and $r = a(1 + \cos\theta)$ intersect at an angle $3\sin^{-1}(3/4)^{1/4}$.

Sol. : Let us first find θ at the point of intersection of the two curves.

Eliminating r, $\quad a^2 \cos 2\theta = a^2(1 + \cos\theta)^2 \quad \therefore 2\cos^2\theta - 1 = 1 + 2\cos\theta + \cos^2\theta$

$$\therefore \cos^2\theta - 2\cos\theta - 2 = 0 \quad \therefore \cos\theta = \frac{2 \pm \sqrt{4 - 4(1)(-2)}}{2} = 1 \pm \sqrt{3}.$$

Since, $\cos\theta$ lies between -1 and $+1$.

$$\cos\theta = 1 - \sqrt{3} \quad \therefore 1 - 2\sin^2\frac{\theta}{2} = 1 - \sqrt{3}$$

$$\therefore \sin^2\left(\frac{\theta}{2}\right) = \frac{\sqrt{3}}{2} = \left[\left(\frac{\sqrt{3}}{2}\right)^2\right]^{1/2} = \left(\frac{3}{4}\right)^{1/2} \quad \therefore \sin\left(\frac{\theta}{2}\right) = \left(\frac{3}{4}\right)^{1/4} \quad \therefore \frac{\theta}{2} = \sin^{-1}\left(\frac{3}{4}\right)^{1/4}$$

Now, for the curve $r^2 = a^2 \cos 2\theta$.

$$2r\frac{dr}{d\theta} = a^2 \cdot (-2)\sin 2\theta \quad \therefore \frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}$$

$$\therefore \tan\Phi = r\frac{d\theta}{dr} = r\left(\frac{-r}{a^2 \sin 2\theta}\right) = -\frac{r^2}{a^2 \sin 2\theta}$$

$$= -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta} = -\cot 2\theta = \tan\left(\frac{\pi}{2} + 2\theta\right)$$

$$\therefore \Phi_1 = \frac{\pi}{2} + 2\theta$$

Further for the second curve, $r = a(1 + \cos\theta)$

$$\frac{dr}{d\theta} = -a\sin\theta$$

$$\therefore \tan\Phi = r\frac{d\theta}{dr} = \frac{a(1 + \cos\theta)}{-a\sin\theta} = -\frac{2\cos^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)} = -\cot\frac{\theta}{2} = \tan\left(\frac{\pi}{2} + \frac{\theta}{2}\right)$$

$$\therefore \Phi_2 = \frac{\pi}{2} + \frac{\theta}{2}$$

$$\therefore \text{The angle of intersection} = \Phi_1 - \Phi_2 = \frac{\pi}{2} + 2\theta - \frac{\pi}{2} - \frac{\theta}{2} = 3\left(\frac{\theta}{2}\right) = 3\sin^{-1}\left(\frac{3}{4}\right)^{1/4}.$$

EXERCISE - VI

Prove that the following curves in each case intersect orthogonally.

1. $r = 2\sin\theta$, $r = 2\cos\theta$.
2. $r = a(1 + \cos\theta)$, $r = b(1 - \cos\theta)$
3. $r^2 \sin 2\theta = a^2$, $r^2 \cos 2\theta = b^2$

15. Applications of Analytic Functions

Analytic functions are useful in the study of 'field problems'. For example, in the study of flow of a fluid or of heat or of electricity in a given region analytic functions can be used.

In physical applications of analytic functions, the complex function w is denoted by

$$w = u(x, y) + iv(x, y) = \Phi(x, y) + i\,\psi(x, y)$$

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(a) In the flow of fluids $\Phi(x, y)$ is called velocity potential and $\psi(x, y)$ is called stream functions. The function $w = \Phi(x, y) + i\,\psi(x, y)$ is called potential function $\Phi(x, y) = c_1$ gives family of equipotential curves and $\psi(x, y) = c_2$ gives family of stream lines. $\Phi(x, y) = c_1$ are orthogonal to $\psi(x, y) = c_2$. Stream lines are the paths of fluid particles.

(b) In electrostatic fields $\Phi(x, y)$ is called potential function and $\psi(x, y)$ is known as stream function. $\Phi(x, y) = c_1$ gives family of equipotential curves and $\psi(x, y) = c_2$ gives family of lines of force of electrostatic field. As before $\Phi(x, y) = c_1$ and $\psi(x, y) = c_2$ are orthogonal.

(c) In the study of heat flow, $\Phi(x, y)$ is known as temperature function and $\psi(x, y)$ is known as stream function $\Phi(x, y) = c_1$ gives a family of isothermals and $\psi(x, y) = c_2$ gives a family of heat flow lines.

Example 1 : If $w = \Phi + i\psi$ represents complex potential for an electric field and $\psi = x^2 - y^2 + \frac{x}{x^2+y^2}$, determine Φ. Also show that ψ is harmonic. **(M.U. 2003, 06)**

Sol. : See Ex. 5 on page 4-36.

Example 2 : The potential function of an electrostatic field is given by $\Phi = 3x^2y - y^3$ find the corresponding stream function.

Sol. : See Ex. 1, page 4-44.

EXERCISE - VII

1. In aerodynamics and fluid mechanics the functions Φ and ψ in $f(z) = \Phi + i\psi$ where $f(z)$ is analytic are called potential and stream functions. If potential function $\Phi = x^2 + 4x - y^2 + 2y$, find the stream function. [**Ans. :** $\psi = 2xy + 4y - 2x$]
2. If the potential function is $\log(x^2 + y^2)$, find the flux function and the complex potential function. [**Ans. :** $2\tan^{-1}\frac{y}{x}$, $2\log z + c$]
3. In a two dimensional fluid flow, the stream function is $\psi = \tan^{-1}\left(\frac{y}{x}\right)$ find the velocity potential. [**Ans. :** $\frac{1}{2}\log(x^2 + y^2)$]
4. An electrostatic field in the x-y plane is given by the potential function $\Phi = x^2 - y^2$ find the stream function. [**Ans. :** $\psi = 2xy + c$]

EXERCISE - VIII

Theory

1. Define analytic function and harmonic function. **(M.U. 2007)**
2. State and prove the conditions for a function $w = f(z)$ to be analytic. **(M.U. 1997, 2003, 04)**
3. State and prove Cauchy - Riemann equations. **(M.U. 1999, 2002, 03)**
4. State and prove Cauchy - Riemann equations in polar form. **(M.U. 2002, 04, 08, 11)**
5. If $f(z)$ and $\overline{f(z)}$ are both analytic then show that $f(z)$ is constant. **(M.U. 1993)**

6. If $f(z)$ is analytic and $|f(z)|$ is constant, prove that $f(z)$ is constant. **(M.U. 1994, 1997, 2003)**
7. Prove that real and imaginary parts of an analytic function satisfy Laplace Equation. **(M.U. 1993)**
8. If $f(z)$ is analytic, show that $\frac{\partial f}{\partial \bar{z}} = 0$.
9. If $f(z)$ is analytic, show that $f(z)$ is independent of $\bar{z}$.
10. If $f(z)$ is analytic, show that x and y can occur in $f(z)$ in the combination of $x + iy$ only.
11. Show that real and imaginary parts of an analytic function are harmonic. **(M.U. 1996, 2005, 06, 07)**
12. State true or false with proper justification.
    (i) There does not exist an analytic function whose real part is $x^3 - 3x^2y - y^3$. **(M.U. 2013)**
    (ii) If $f(z)$ and $\overline{f(z)}$ are both analytic then $f(z)$ is a constant function. **(M.U. 1995)**
13. Show that an analytic function with constant modulus is constant. **(M.U. 1994, 99, 2003)**
14. If $f(z) = u + iv$ is analytic in a region R then show that
    (i) u and v are harmonic functions and
    (ii) u = constant and v = constant intersect orthogonally. **(M.U. 1994, 97, 98, 2003)**
15. If $f(z) = u + iv$ is analytic in a region R show that
    (i) $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$
    (ii) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y} = 0$, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ **(M.U. 1995)**
16. If $f(z) = u + iv$ is analytic and $z = re^{i\theta}$, show that
    $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$ **(M.U. 1996, 2002, 03)**
17. If $f(z) = u + iv$ is an analytic function, prove that
    (i) u = constant and v constant are orthogonal trajectories.
    (ii) u and v are harmonic functions.
    (iii) $f(z)$ is constnat if $f'(z) = 0$. **(M.U. 2004)**
    (iv) $u - iv$ is also analytic.
18. If u and v are conjugate harmonic functions, prove that uv is also harmonic. **(M.U. 2003)**

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CHAPTER 5

Correlation

1. Introduction

In Statistics so far we have studied problems involving a single variable. Many a time, we come across problems which involve two or more than two variables. Data relating to two variates are called **Bivariate Data**. If we study bivariate data carefully, we may find some relation between the two variables. For example, if a car-owner maintains the record of petrol consumption and mileage, he will find that there is some relation between the two variables. On the other hand, if we compare the figures of rainfall with the production of cars, we may find that there is no relation between the two variables. If there is any relation between two variables *i.e.* if as one variable changes the other also changes in the same or opposite direction, we say that they are correlated. Thus, correlation means **"the study of existence and the magnitude and direction of variation between two or more variables".**

2. Types of Correlation

Correlation may be classified as :

(1) Positive and negative,

(2) Linear and non-linear.

(1) Positive and Negative Correlation

The distinction between the positive and negative correlation depends upon the direction of change of two variables. If both the variables change in the same direction *i.e.* if, as one variable increases, the other also increases and as one variable decreases, the other also decreases, the correlation is called positive (*e.g.* advertising and sales). If, on the other hand, the variables change in opposite direction *i.e.* if, as one variable increases, the other decreases and vice-versa, then the correlation is called negative (*e.g.* T.V. registrations and cinema attendance).

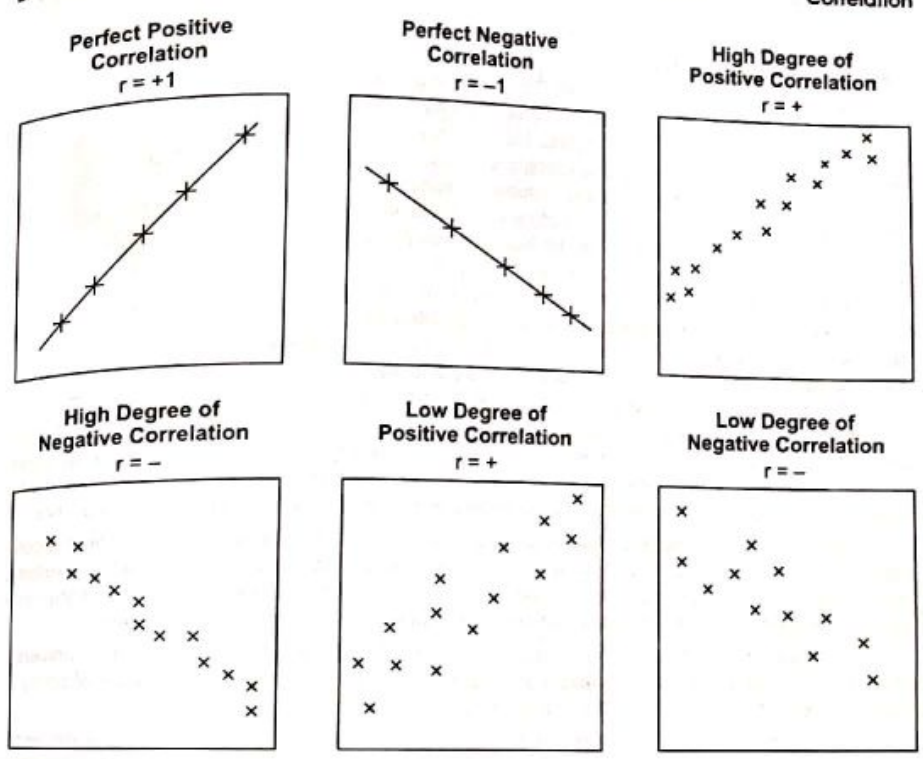
(2) Linear or Non-linear Correlation

This distinction is based upon the nature of the graph of the relation between the variables. If the graph is a straight line the correlation is called linear and if the graph is not a straight line but a curve it is called non-linear or curvi-linear correlation.

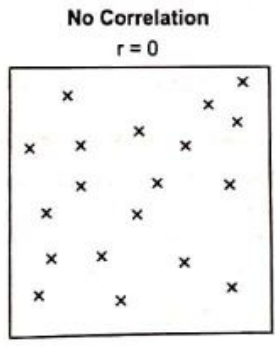
We shall consider the following commonly used methods of studying correlation. (1) Scatter Diagram, (2) Karl Pearson's Coefficient of Correlation, (3) Spearman's Rank-Correlation Coefficient.

3. Scatter Diagram

One of the most simple methods of studying correlation between two variables is to construct a scatter diagram.



To obtain a scatter diagram, one variable is plotted along the *x*-axis and the other along the *y*-axis, on a graph paper. By plotting data in this way, we get points which are generally scattered but which show a pattern. The way in which the points are scattered indicates the degree and direction of correlation. If the points are close to each other we infer that the variables are correlated. If they are spread away from each other, we infer that the variables are not correlated. Moreover, if the points lie in a narrow strip rising from left-hand bottom to the right-hand top, we say that there is positive correlation of high order. If the points lie in a narrow strip, falling from the left-hand top to the right-hand bottom, we say that there is negative correlation of high order. If the points are all spread over, we say that there is zero correlation.



4. Karl Pearson's Coefficient of Correlation

The method of scatter diagram is descriptive in nature and gives only a general idea of correlation. The most commonly used method which gives a mathematical expression for correlation is the one suggested by Karl Pearson (1867-1936) a British Biometrician.

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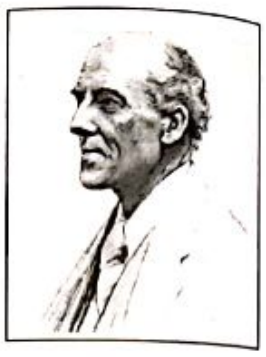
Karl Pearson (1857 - 1936)



Born in London, he went to King's College, Cambridge in 1876 to study mathematics graduating in 1879 as Third Wrangler in the Mathematical Tripos. He then went to Germany to study Physics at the University of Heidelberg. Other subjects he studied in Germany include metaphysics, physiology, Roman Law, German Literature and Socialism. He studied so many subjects because he believed that there was no subject in the universe unworthy of study. Then he returned to London to study Law, although he never practised. In 1881 he returned to mathematics and was appointed as professor of mathematics at University College, London. In 1891 he met Walter Weldon, a zoologist and worked with him in biometry and evolutionary theory. He was introduced to Galton, Darwins cousin and became Galtons statistical heir. He was the first holder of the Galton's Chair of Eugenics. In 1911 he founded the world's first university statistics department at University College, London. He remained with the department until his retirement in 1933 and continued to work until his death in 1936. He thus established the new discipline of mathematical statistics.

His famous book **"The Grammar of Science"** covers several themes that were later to become part of the theories of Einstein and other scientists. He speculated that an observer who travelled at the speed of light would see an eternal now and an observer who travelled faster than light would see time reversal. He also discussed antimatter, fourth dimension and wrinkles in time.

Karl Pearson was awarded many medals including The Darwin Medal, a DSc from university of London. His commitment to socialism and his ideals led him to refuse the honours of being an OBE (Officer of the Order of the British Empire) and knighthood in 1935.

Karl Pearson is known for Karl Pearson's coefficient of correlation, methods of moments, Pearson's system of continuous curves, Chi-distance, Statistical hypothesis testing theory, Statistical decision theory, Pearson's chi-square test, etc.

Just as $\sigma_x^2 = \frac{1}{N}\Sigma(x-\bar{x})^2$ gives us a measure of variation in x and $\sigma_y^2 = \frac{1}{N}\Sigma(y-\bar{y})^2$ gives a measure of variation in y we may expect $\frac{1}{N}\Sigma(x-\bar{x})(y-\bar{y})$ to give the measure of simultaneous variation in x and y. But this will depend upon the units of x and y. To find a ratio which is independent of these units, we divide it by the quantities of the same order that is by $\sigma_x \cdot \sigma_y$. With this view in mind Karl Pearson suggested in 1890 the following coefficient of correlation to measure correlation between x and y. It is denoted by r.

Thus,
$$\boxed{r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{N\sigma_x\sigma_y}} \quad \text{..... (1)}$$

But $\frac{1}{N}\Sigma(x-\bar{x})(y-\bar{y})$ is called the **covariance** between x and y. Hence, from (1), we have

$$\boxed{r = \frac{\text{cov.}(x,y)}{\sigma_x \cdot \sigma_y}} \quad \text{..... (2)}$$

If we put $\sigma_x = \sqrt{\frac{\Sigma(x-\bar{x})^2}{N}}$, $\sigma_y = \sqrt{\frac{\Sigma(y-\bar{y})^2}{N}}$, then

$$\boxed{r = \frac{\Sigma(x-\bar{x})(y-\bar{y})}{\sqrt{\Sigma(x-\bar{x})^2\,\Sigma(y-\bar{y})^2}}} \quad \text{..... (3)}$$

If we write $x-\bar{x} = x'$, $y-\bar{y} = y'$, then

$$\boxed{r = \frac{\Sigma x'y'}{\sqrt{\Sigma x'^2 \cdot \Sigma y'^2}}} \quad \text{..... (4)}$$

The Karl Pearson's coefficient of correlation is also called the **product moment coefficient correlation.**

Further, we can expand (3) and write

$$r = \frac{\Sigma(xy - x\bar{y} - \bar{x}y + \bar{x}\bar{y})}{\sqrt{\Sigma(x^2 - 2x\bar{x} + \bar{x}^2)\cdot\Sigma(y^2 - 2y\bar{y} + \bar{y}^2)}}$$
$$= \frac{\Sigma xy - \bar{y}\Sigma x - \bar{x}\Sigma y + \bar{x}\bar{y}\Sigma 1}{\sqrt{(\Sigma x^2 - 2\bar{x}\Sigma x + \bar{x}^2\Sigma 1)\cdot(\Sigma y^2 - 2\bar{y}\Sigma y + \bar{y}^2\Sigma 1)}}$$

But $\Sigma x = N\bar{x}$, $\Sigma y = N\bar{y}$ and $\Sigma 1 = N$

$$\therefore \quad \boxed{r = \frac{\Sigma xy - N\bar{x}\bar{y}}{\sqrt{(\Sigma x^2 - N\bar{x}^2)\cdot(\Sigma y^2 - N\bar{y}^2)}}} \quad \text{..... (5)}$$

If $\bar{x}$, $\bar{y}$ are integers we take deviations of x and y from them and use the formula (3). If we have to find r from direct values we use the formula (5). This is the most *commonly used formula.*

(i) **Limits for r : $-1 \le r \le 1$** **(M.U. 2004)**

Proof : If we write $E(X) = \mu_X$, $E(Y) = \mu_Y$, then

$$E\left[\left(\frac{X-\mu_X}{\sigma_X}\right) \pm \left(\frac{Y-\mu_Y}{\sigma_Y}\right)\right]^2 \ge 0$$

$$\therefore \quad E\left(\frac{X-\mu_X}{\sigma_X}\right)^2 + E\left(\frac{Y-\mu_Y}{\sigma_Y}\right)^2 \pm 2\frac{E\left[(X-\mu_X)^2(Y-\mu_Y)\right]}{\sigma_X\,\Sigma_Y} \ge 0$$

$\therefore \quad 1 + 1 \pm 2r \ge 0 \qquad \therefore \quad 1 \pm r \ge 0$

$\therefore \quad 1 + r \ge 0 \qquad$ or $\qquad 1 - r \ge 0$

$\therefore \quad 1 \ge -r \qquad$ or $\qquad 1 \ge r$

$\therefore \quad -1 \le r,\ r \le 1 \qquad \therefore \quad -1 \le r \le 1 \qquad [\because E(X-\mu_X)^2 = \sigma_X^2]$

(ii) **Theorems on correlation**

Theorem 1 : If x, y are independent variables they are not correlated.

We accept this theorem without proof.

Theorem 2 : Correlation coefficient is independent of change of origin and change of scale. **(M.U. 2002, 07)**

This means if we write $u_i = \frac{x-a}{h}$, $v_i = \frac{y-b}{k}$, then

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$$\boxed{r_{xy} = r_{uv}} \qquad \text{..... (6)}$$

i.e., the correlation between x and y is equal to the correlation between u and v.

We accept this theorem without proof.

Remark ....

The above theorem can also be stated as :-

"If $x = au + b$, $y = cv + d$ where a, b, c, d are constants then $r_{xy} = r_{uv}$."

Example : Discuss the statement : "If the coefficient of correlation between x and y is negative then the coefficient of correlation between $(-x)$ and $(-y)$ is positive. **(M.U. 1998)**

Sol. : If we write $a = 0$ and $b = 0$, $h = -1$ and $k = -1$ in (A), then by the above theorem since $r_{xy} = r_{uv}$, the coefficient of correlation between $-x$ and $-y$ will be also the same in magnitude and sign as the coefficient of correlation between x and y.

Or since the coefficient of correlation does not change under change of scale and since $-x$ and $-y$ mean the change of scale, the coefficient of correlation between $-x$ and $-y$ will be also negative.

Theorem 3 : *If d_x and d_y denote the deviations of x and y from the assumed means A and B then*

$$\boxed{r = \frac{\Sigma d_x d_y - \frac{(\Sigma d_x)(\Sigma d_y)}{N}}{\sqrt{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{N}}\sqrt{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{N}}}} \qquad \text{..... (7)}$$

We accept this result without proof.

5. Interpretation of the Coefficient of Correlation

1. **$r > 0{\cdot}95$:** If r is greater than 0·95, it indicates high degree of correlation and the value of one variable can be estimated from a known value of the other fairly accurately.

2. **$r > 0{\cdot}75$ but $< 0{\cdot}85$:** If r is greater than 0·75 but less than 0·85, there is probably a definite relationship between the variables and the value of one variable can be roughly estimated from a known value of the other.

3. **$r > 0{\cdot}40$ but $< 0{\cdot}60$:** If r is greater than 0·40 but less than 0·60 there may be some relationship between the two variables. But the value of one variable calculated from a known value of the other cannot be reliable.

4. **$r < 0{\cdot}35$:** If r is less than 0·35 the correlation is poor and one variable cannot be estimated from the other.

5. **r nearly zero :** If r is nearly equal to zero, it indicates that there is probably no relation between the two variables *i.e.* they are independent of each other.

6. Computation of Coefficient of Correlation : *(Ungrouped Data)*

There are three methods of calculating r.

(1) Actual mean method, (2) Step deviation method,

(3) Assumed mean method.

(1) Actual Mean Method

The formula to be used is, $$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}}$$

Steps :

(i) Calculate mean $\overline{X}$ and then take deviation x of X from $\overline{X}$ *i.e.*, calculate $x = X - \overline{X}$.

(ii) Calculate mean $\overline{Y}$ and then take deviation y of Y from $\overline{Y}$ *i.e.*, calculate $y = Y - \overline{Y}$.

(iii) Multiply x by y and prepare the column of xy.

(iv) Take the squares of x and prepare the column of x^2.

(v) Take the squares of y and prepare the column of y^2.

(vi) Apply the above formula.

Example 1 : Find from the following values of the demand and the corresponding price of a commodity, the degree of correlation between the demand and price by computing Karl Pearson's coefficient of correlation.

Demand in quintals : 65, 66, 67, 67, 68, 69, 70, 72.

Price in Paise per k.g. : 67, 68, 65, 68, 72, 72, 69, 71.

Sol. : Let X denote the demand in Quintals and Y denote the price in paise per kg.

Calculation of r between demand and price

Sr. No.	Demand in Qnt. $X - \overline{X}$, $\overline{X} = 68$			Price per Kg. $Y - \overline{Y}$, $\overline{Y} = 69$			Product
	X	x	x^2	Y	y	y^2	xy
1	65	−3	9	67	−2	4	6
2	66	−2	4	68	−1	1	2
3	67	−1	1	65	−4	16	4
4	67	−1	1	68	−1	1	1
5	68	0	0	72	+3	9	0
6	69	+1	1	72	+3	9	3
7	70	+2	4	69	0	0	0
8	72	+4	16	71	+2	4	8
$N = 8$	$\Sigma X = 544$		$\Sigma x^2 = 36$	$\Sigma Y = 552$		$\Sigma y^2 = 44$	$\Sigma xy = +24$

Now, $\overline{X} = \frac{544}{8} = 68$ and $\overline{Y} = \frac{552}{8} = 69$.

$$\therefore \quad r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}}$$

But $\Sigma xy = 24$, $\Sigma x^2 = 36$, $\Sigma y^2 = 44$.

$$\therefore \quad r = \frac{24}{\sqrt{36 \times 44}} = \frac{24}{39{\cdot}80} = 0{\cdot}6030.$$

Example 2 : Calculate Karl Pearson's coefficient of correlation for the following bivariate series.

X : 28, 45, 40, 38, 35, 33, 40, 32, 36, 33

Y : 23, 34, 33, 34, 30, 26, 28, 31, 36, 35 **(M.U. 2015)**

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Sol. :

Calculationof r between X and Y

Sr. No.	$X - \bar{X}$, $\bar{X} = 36$			$Y - \bar{Y}$, $\bar{Y} = 31$			Product
	X	x	x^2	Y	y	y^2	xy
1	28	– 8	64	23	– 8	64	+ 64
2	45	+ 9	81	34	+ 3	9	+ 27
3	40	+ 4	16	33	+ 2	4	+ 8
4	38	+ 2	4	34	+ 3	9	+ 6
5	35	– 1	1	30	– 1	1	+ 1
6	33	– 3	9	26	– 5	25	– 15
7	40	+ 4	16	28	– 3	9	– 12
8	32	– 4	16	31	0	0	0
9	36	0	0	36	+ 5	25	0
10	33	– 3	9	35	+ 4	16	– 12
$N = 10$	$\Sigma X = 360$		$\Sigma x^2 = 216$	$\Sigma Y = 310$		$\Sigma y^2 = 162$	$\Sigma xy = + 97$

Now, $\bar{X} = \frac{360}{10} = 36$ and $\bar{Y} = \frac{310}{10} = 31.$

$$\therefore \quad r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}}$$

But $\Sigma xy = 97,\ \Sigma x^2 = 216,\ \Sigma y^2 = 162.$

$$\therefore \quad r = \frac{97}{\sqrt{216 \times 162}} = \frac{97}{187 \cdot 1} = 0 \cdot 5186.$$

(2) Step-deviation Method

As in the case of mean and standard deviation, to simplify calculations we can use step-deviation method whenever possible for calculating r. But it should be noted that *the result is not to be multiplied by the constant in the final stage*. The reason is the coefficient of correlation is independent of change of origin and change of scale. (See Theorem 2 of r, page 5-4)

Example 1 : Calculate the co-efficient of correlation from the following data.

X: 100, 200, 300, 400, 500.

Y: 30, 40, 50, 60, 70. **(M.U. 2015)**

Sol. :

Calculations of r between X and Y

Sr. No.	$\bar{X} = 300$, $X - \bar{X}$		$\frac{X - \bar{X}}{100}$		$\bar{Y} = 50$, $Y - \bar{Y}$		$\frac{Y - \bar{Y}}{10}$		Product
	X	$X - 300$	x	x^2	Y	$Y - 50$	y	y^2	xy
1	100	– 200	– 2	4	30	– 20	– 2	4	4
2	200	– 100	– 1	1	40	– 10	– 1	1	1
3	300	0	0	0	50	0	0	0	0
4	400	100	1	1	60	10	1	1	1
5	500	200	2	4	70	20	2	4	4
$N = 5$	$\Sigma X = 1500$			$\Sigma x^2 = 10$	$\Sigma Y = 250$			$\Sigma y^2 = 10$	$\Sigma xy = 10$

Now $\bar{X} = \frac{1500}{5} = 300$ and $\bar{Y} = \frac{250}{5} = 50$ $\quad \therefore \quad r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}}$

But $\Sigma xy = 10,\ \Sigma x^2 = 10,\ \Sigma y^2 = 10.$ $\quad \therefore \quad r = \frac{10}{\sqrt{10 \times 10}} = \frac{10}{10} = +1.$

(3) Assumed Mean Method

Since in the calculation of r, deviations are to be squared the calculations will be tedious if the means are not integers but data are in integers. In such cases, we take deviations from an assumed mean conveniently chosen. The corresponding formula is

$$r = \frac{\Sigma d_x d_y - \frac{(\Sigma d_x)(\Sigma d_y)}{N}}{\sqrt{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{N}}\sqrt{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{N}}}$$

where, d_x = deviations of X from an assumed mean, $(X - A)$,

d_y = deviations of Y from an assumed mean, $(Y - B)$,

N = Number of pairs of observations.

Steps :

(i) Assume any mean A for X and calculate deviations d_x of X from A *i.e.*, $d_x = X - A$.

(ii) Assume any mean B for Y and calculate deviations d_y of Y from B *i.e.*, $d_y = Y - B$.

(iii) Take the squares of d_x.

(iv) Take the squares of d_y.

(v) Take the products of d_x and d_y.

(vi) Apply the formula.

Example 4 : Find the co-efficient of correlation for the prices (in Rs.) and sales units.

Price in Rs. : 100, 98, 85, 92, 90, 84, 88, 90, 93, 95.

Sales Units : 500, 610, 700, 630, 670, 800, 800, 750, 700, 690.

Sol. : Let us assume 92 and 670 to be the means of X and Y respectively.

Calculations of r between price and sale

Sr. No.	Price in Rs. $(X - 92)$			Sales Units $(Y - 670)$			Product
	X	d_x	d_x^2	Y	d_y	d_y^2	$d_x d_y$
1	100	+ 8	64	500	– 170	28900	– 1360
2	98	+ 6	36	610	– 60	3600	– 360
3	85	– 7	49	700	+ 30	900	– 210
4	92	0	0	630	– 40	1600	0
5	90	– 2	4	670	0	0	0
6	84	– 8	64	800	+ 130	16900	– 1040
7	88	– 4	16	800	+ 130	16900	– 520
8	90	– 2	4	750	+ 80	6400	– 160
9	93	+ 1	1	700	+ 30	900	+ 30
10	95	+ 3	9	690	+ 20	400	+ 40
$N = 10$	$\Sigma d_x = – 5$		$\Sigma d_x^2 = 247$	$\Sigma d_y = 150$		$\Sigma d_y^2 = 76500$	$\Sigma d_x d_y = – 3560$

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$$\text{Now,}\quad r = \frac{\Sigma d_x d_y - \frac{(\Sigma d_x)(\Sigma d_y)}{N}}{\sqrt{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{N}}\sqrt{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{N}}}$$

But, $\Sigma d_x d_y = -3560,\quad \Sigma d_x = -5,\quad \Sigma d_y = 150$

$N = 10,\quad \Sigma d_x^2 = 247,\quad \Sigma d_y^2 = 76500.$

$$\therefore\quad r = \frac{-3560 - \frac{(-5)\times(150)}{10}}{\sqrt{247 - \frac{25}{10}}\sqrt{76500 - \frac{22500}{100}}}$$

$$= \frac{-3560 + 75}{\sqrt{247 - 2{\cdot}5}\sqrt{76500 - 2250}}$$

$$= \frac{-3485}{\sqrt{244{\cdot}5}\sqrt{74250}}$$

$$= \frac{3485}{4261} = -0{\cdot}8179$$

Example 5 : Calculate the correlation coefficient from the following data.

X : 23, 27, 28, 29, 30, 31, 33, 35, 36, 39.

Y : 18, 22, 23, 24, 25, 26, 28, 29, 30, 32. **(M.U. 2004, 14, 19)**

Sol. : Let us assume 30 and 25 to be the means of x and y respectively.

Calculation of r between X and Y

Sr. No.	$(X-30)$			$(Y-25)$			Product
	X	d_x	d_x^2	Y	d_y	d_y^2	$d_x d_y$
1	23	−7	49	18	−7	49	49
2	27	−3	9	22	−3	9	9
3	28	−2	4	23	−2	4	4
4	29	−1	1	24	−1	1	1
5	30	0	0	25	0	0	0
6	31	+1	1	26	+1	1	1
7	33	+3	9	28	+3	9	9
8	35	+5	25	29	+4	16	20
9	36	+6	36	30	+5	25	30
10	39	+9	81	32	+7	49	63
$N = 10$		$\Sigma d_x = +11$	$\Sigma d_x^2 = 215$		$\Sigma d_y = +7$	$\Sigma d_y^2 = 163$	$\Sigma d_x d_y = 186$

$$\text{Now,}\quad r = \frac{\Sigma d_x d_y - \frac{(\Sigma d_x)(\Sigma d_y)}{N}}{\sqrt{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{N}}\sqrt{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{N}}}$$

But, $\Sigma d_x d_y = 186,\quad \Sigma d_x = 11,\quad \Sigma d_y = 7$

$N = 10,\quad \Sigma d_x^2 = 215,\quad \Sigma d_y^2 = 163.$

$$\therefore\quad r = \frac{186 - \frac{11\times 7}{10}}{\sqrt{215 - \frac{(11)^2}{10}}\sqrt{163 - \frac{(7)^2}{10}}}$$

$$= \frac{186 - 77}{\sqrt{215 - 12{\cdot}1}\sqrt{163 - 4{\cdot}9}}$$

$$= \frac{178{\cdot}3}{\sqrt{202{\cdot}9}\sqrt{158{\cdot}1}} = 0{\cdot}9948$$

7. Direct Method of Calculating Coefficient of Correlation

We can find the coefficient of correlation directly without taking the deviations of x and of y from their respective means. In such cases the following formula is used.

$$\boxed{r = \frac{\Sigma xy - \frac{\Sigma x \Sigma y}{n}}{\sqrt{\left(\Sigma x^2 - \frac{(\Sigma x)^2}{n}\right)\left(\Sigma y^2 - \frac{(\Sigma y)^2}{n}\right)}}} \quad \text{..... (8)}$$

where, x and y are the observed values of the variables and $\bar{x}, \bar{y}$ are their respective means.

The formula can also be written as,

$$r = \frac{\Sigma xy - n\bar{x}\bar{y}}{n\cdot\sqrt{\left(\frac{\Sigma x^2}{n} - \bar{x}^2\right)\left(\frac{\Sigma y^2}{n} - \bar{y}^2\right)}}$$

$$\boxed{r = \frac{\Sigma xy - n\bar{x}\bar{y}}{n\sigma_x\sigma_y}} \quad \text{..... (9)}$$

Example 1 : Calculate the coefficient of correlation between X and Y from the following data.

X : 3, 5, 4, 6, 2.

Y : 3, 4, 5, 2, 6.

Sol. :

Calculations of r

x	x^2	y	y^2	xy
3	9	3	9	9
5	25	4	16	20
4	16	5	25	20
6	36	2	4	12
2	4	6	36	12
$\Sigma x = 20$	$\Sigma x^2 = 90$	$\Sigma y = 20$	$\Sigma y^2 = 90$	$\Sigma xy = 73$

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Since, $\bar{x} = 4$, $\bar{y} = 4$ putting these values in equation (8).

$$\therefore \quad r = \frac{73 - \frac{20^2}{5}}{\sqrt{90 - \frac{20^2}{5}}\sqrt{90 - \frac{20^2}{5}}} = \frac{73 - 80}{10} = -0{\cdot}7.$$

EXERCISE - I

1. Calculate the coefficient of correlation from the following data. Is there any marked correlation between the production and price of tea ?

Production in crores (kgs)	:	34,	27,	31,	38,	38,	36,	39,	40.
Price in Rs. per kg	:	3·75,	4·62,	4·25,	4·12,	4·28,	4·32,	4·21,	4·05.

[**Ans. :** $r = -0{\cdot}48$]

2. Compute the coefficient of correlation between X and Y from their values given blow.

X : 30, 33, 25, 10, 33, 75, 40, 85, 90, 95.

Y : 68, 65, 80, 85, 70, 30, 55, 18, 15, 10. **(M.U. 2015)** [**Ans. :** $r = -0{\cdot}7069$]

3. The following data give the hardness (X) and tensile strength (Y) for some specimens of a material in certain units in a factory. Find the correlation coefficient and interpret your result.

X : 23·3, 17·5, 17·8, 20·7, 18·1, 20·9, 22·9, 20·8.

Y : 4·2, 3·8, 4·6, 3·2, 5·2, 4·7, 4·4, 5·6.

[**Ans. :** $r = -0{\cdot}072$. No correlation]

4. Calculate the product moment coefficient of correlation between the indices of business activity (X) and employment (Y) from the following data.

X : 100, 102, 108, 111, 115, 116, 118.

Y : 110, 100, 104, 108, 112, 116, 120. [**Ans. :** $r = 0{\cdot}75$]

5. Find Karl Pearson's coefficient of correlation between X and Y.

X : 10, 12, 14, 15, 16, 17, 18, 10, 14, 15

Y : 17, 16, 15, 12, 10, 9, 8, 15, 13, 12 [**Ans. :** $r = -0{\cdot}93$]

6. Compute a coefficient of correlation between X and Y.

X : 3, 6, 4, 5, 7

Y : 2, 4, 5, 3, 6 [**Ans. :** $r = 0{\cdot}7$]

7. Calculate the coefficient of correlation between price and demand by direct method.

Price : 2, 3, 4, 7, 4

Demand: 8, 7, 3, 1, 1 [**Ans. :** $-0{\cdot}81$]

8. Calculate the coefficient of correlation between the X and Y by direct method.

X : 8, 8, 7, 5, 6, 2

Y : 3, 4, 10, 13, 22, 8 **(M.U. 2017)** [**Ans. :** $0{\cdot}2646$]

9. Soil temperature (x) and Germination interval (y) for winter wheat in 12 places are as follows.

x (in °F) : 57, 42, 38, 42, 45, 42, 44, 40, 46, 44, 43, 40.

y (days) : 10, 26, 41, 29, 27, 27, 19, 18, 19, 31, 29, 33.

Calculate the coefficient of correlation between x and y. **(M.U. 2015)** [**Ans. :** $r = -0{\cdot}74$]

10. Find the Karl Pearson's coefficient of correlation between X and Y from the following.

X : 51, 54, 56, 59, 65, 60, 70

Y : 38, 44, 33, 36, 33, 23, 13

[**Ans. :** $r = -0{\cdot}7977$]

8. Spearman's Rank Correlation

(M.U. 2009)

The method developed by Spearman is simpler than Karl Pearson's method since, it depends upon ranks of the items and actual values of the items are not required. Hence, this can be used to study correlation even when actual values are not known. For instance we can study correlation between intelligence and honesty by this method.

Let x_i, y_i be the ranks in the two characteristics of the i-th member where $i = 1, 2, \ldots\ldots n$. We assume that no two members have the same rank either for x or for y. Thus, x and y take all integral values between 1 and n.

$$\therefore \quad \bar{x} = \frac{1}{2}(1 + 2 + 3 + \ldots\ldots + n) = \frac{n+1}{2}$$

Similarly, $\bar{y} = \frac{1}{2}(1 + 2 + 3 + \ldots\ldots + n) = \frac{n+1}{2} \quad \therefore \bar{x} = \bar{y}$

$$\therefore \quad \Sigma(x_i - \bar{x})^2 = \Sigma(x_i^2 - 2x_i\bar{x} + \bar{x}^2) = \Sigma x_i^2 - 2\bar{x}\,\Sigma x_i + \bar{x}^2\Sigma 1$$

$$= \Sigma x_i^2 - 2n\bar{x}^2 + n\bar{x}^2 = \Sigma x_i^2 - n\bar{x}^2$$

$$= (1^2 + 2^2 + \ldots\ldots + n^2) - n\left(\frac{n+1}{2}\right)^2$$

$$\therefore \quad \Sigma(x_i - \bar{x})^2 = \frac{n}{6}(n+1)(2n+1) - \frac{n(n+1)^2}{4} = \frac{1}{12}(n^3 - n)$$

Similarly, $\Sigma(y_i - \bar{y}) = \frac{1}{12}(n^3 - n)$.

If d_i denotes the difference between the ranks of i-th member in the two variables, we have

$$d_i = (x_i - y_i) = (x_i - \bar{x}) - (y_i - \bar{y}) = x_i' - y_i' \text{ (since, } \bar{x}, \bar{y} \text{ are equal)}$$

where, x_i', y_i' denote the deviations of x_i, y_i from their means $\bar{x}$, $\bar{y}$ respectively.

$$\therefore \quad \Sigma d_i^2 = \Sigma x_i'^2 + \Sigma y_i'^2 - 2\Sigma x_i' y_i'$$

$$\therefore \quad \Sigma d_i^2 = \frac{1}{12}(n^3 - n) + \frac{1}{12}(n^3 - n) - 2\Sigma x_i' y_i'$$

$$\therefore \quad \Sigma x_i' y_i' = \frac{1}{2}\left[\frac{n^3 - n}{6} - \Sigma d_i^2\right]$$

But the coefficient of correlation

$$= \frac{\Sigma x_i' y_i'}{\sqrt{\Sigma x_i'^2 \cdot \Sigma y_i'^2}} = \frac{\frac{1}{2}\left[\frac{n^3 - n}{6} - \Sigma d_i^2\right]}{\frac{1}{12}(n^3 - n)} = 1 - \frac{6\Sigma d_i^2}{n^3 - n}$$

This coefficient is denoted by R

$$\therefore \quad \boxed{R = 1 - \frac{6\Sigma d_i^2}{n^3 - n}.} \qquad \ldots\ldots\ldots (10)$$

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The value of R, as of r, lies between +1 and −1. If $R = +1$, there is perfect positive correlation *i.e.* there is complete agreement in the same direction. If $R = -1$, there is perfect negative correlation *i.e.* there is complete agreement but in opposite direction. Generally, the value of R is neither +1 nor −1 but lies somewhere in between. If $R = 0$, there is no correlation between X and Y.

Charles Edward Spearman (1863 - 1945)

Spearman was an English psychologist known for his work in statistics as a pioneer in factor analysis and for Spearman's rank correlation coefficient.

After serving for fifteen years as an officer in the British Army he went to Leipzig, Germany to study experimental psychology and obtained his Ph.D. degree in 1906. He was elected to Royal Society of London in 1924. In 1928 he became Professor of Psychology at University College, London. His many published papers cover a wide field, but he especially did pioneering work in the application of mathematical methods to the analysis of human mind. He discovered the general factor in human intelligence and developed a theory of 'g'.

He was greatly influenced by the work of Galton. He did pioneering work in psychology and developed correlation coefficient known by his name.

(a) Interpretation of $R = +1$ and $R = -1$

Two values of R need special attention. They are + 1 and − 1. $R = +1$, when the scatter diagram is a straight line rising to the right. In this case the ranks of the value of X are the same as the ranks of the values of Y. $R = -1$, when the scatter diagram is a straight line falling to right. In this case, when the ranks of the values of X go on increasing in order, the ranks of the corresponding values of Y go on decreasing in the same order.

For example, consider the following data.

$R = +1$			
X	Y	R_1	R_2
8	115	1	1
11	120	2	2
14	125	3	3
17	130	4	4
20	135	5	5

$R = -1$			
X	Y	R_1	R_2
8	135	1	5
11	130	2	4
14	125	3	3
17	120	4	2
20	115	5	1

Note ....

When the scatter diagram is a straight line $r = +1$ or -1 and also $R = +1$ or -1.

(b) Relation between Spearman's Rank Correlation Coefficient R and Karl Pearson's Correlation Coefficient r.

Generally, for a given distribution the values of Spearman's rank correlation coefficient and Karl Pearson's correlation coefficient are different. Although both of them lie between + 1 and − 1, but actual values of the two coefficients for a given distribution are different. However, if the data are such that if the values of the two variables x and y are arranged in either ascending or descending

order and if they are found to increase or decrease by the equal amount *i.e.* if the difference between two values of x and the difference between the corresponding two values of y is constant, then the two values of R and r are equal. This is illustrated by the following example.

Example 1 : Calculate R and r from the following data.

X : 12 17 22 27 32,

Y : 113 119 117 115 121.

Interpret your result.

Sol. :

Calculation of R and r

Sr. No.	$X - \bar{X}$			$Y - \bar{Y}$			xy	R_1	R_2	$(R_1 - R_2)^2$
	X	x	x^2	Y	y	y^2				
1	12	−10	100	117	−4	16	40	5	5	0
2	17	−5	25	119	2	4	−10	4	2	4
3	22	0	0	117	0	0	0	3	3	0
4	27	5	25	115	−2	4	−10	2	4	4
5	32	10	100	121	4	16	40	1	1	0
$N = 5$	110		250	585		40	60			8

$$R = 1 - \frac{6\Sigma D^2}{N^3 - N} = 1 - \frac{6 \times 8}{125 - 5} = 0{\cdot}6; \qquad \bar{X} = \frac{110}{5} = 22,\ \bar{Y} = \frac{585}{5} = 117$$

$$r = \frac{\Sigma xy}{\sqrt{\Sigma x^2 \cdot \Sigma y^2}} = \frac{60}{\sqrt{250 \times 40}} = 0{\cdot}6.$$

Thus, the values of R and r are equal. It should be noted that the values of X increase by 5 and the values of Y, when arranged in ascending order also increase by the same amount 2 every time.

In general, if the values of x, when arranged in ascending order increase (or decrease) by a fixed amount and if the values of y, when arranged in ascending order increase (or decrease) by another (or the same) fixed amount, then the values of r and R come out to be equal.

(c) Computation of Correlation

There are two types of problems.

(I) When ranks of items are given.

(II) When the actual values of the items are given.

(I) When ranks are given :

Steps : (i) Calculate the difference $D = R_1 - R_2$.

(ii) Calculate : D^2.

(iii) Apply the formula, $R = 1 - \dfrac{6\Sigma D^2}{N^3 - N}$.

(II) When the actual vales are given : We first ascertain the ranks of all items and follow the above procedure.

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Example 1 : Compute Spearman's rank correlation coefficient from the following data.

X : 18 20 34 52 12

Y : 39 23 35 18 46 **(M.U. 2016)**

Sol. : First we give ranks to the data in descending order and then calculate $D^2 = (R_1 - R_2)^2$.

Sol. : **Calculation of R between X and Y**

Serial No.	X	R_1	Y	R_2	D^2 $(R_1 - R_2)^2$
1	18	4	39	2	4
2	20	3	23	4	1
3	34	2	35	3	1
4	52	1	18	5	16
5	12	5	46	1	16
$N = 5$					$\Sigma D^2 = 38$

$$\therefore \quad R = 1 - \frac{6\Sigma D^2}{N^3 - D}. \qquad \text{Here, } \Sigma D^2 = 38,\ N = 5.$$

$$\therefore \quad R = 1 - \frac{6 \times 38}{125 - 5} = 1 - \frac{228}{120} = 1 - 1{\cdot}9 = -0{\cdot}9$$

Example 2 : Calculate the rank correlation coefficient from the following data, relating to the ranks of 10 students in English and Mathematics.

Student No. : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.

Rank in English : 1, 3, 7, 5, 4, 6, 2, 10, 9, 8.

Rank in Mathematics : 3, 1, 4, 5, 6, 9, 7, 8, 10, 2.

Sol. : **Calculation of R between English and Mathematics**

Student No.	Rank in English R_1	Rank in Mathematics R_2	D^2 $(R_1 - R_2)^2$
1	1	3	4
2	3	1	4
3	7	4	9
4	5	5	0
5	4	6	4
6	6	9	9
7	2	7	25
8	10	8	4
9	9	10	1
10	8	2	36
$N = 10$			$\Sigma D^2 = 96$

$$\text{Now,} \quad R = 1 - \frac{6\Sigma D^2}{N^3 - N} \qquad \because \Sigma D^2 = 96,\ N = 10$$

$$\therefore \quad R = 1 - \frac{6 \times 96}{990} = 1 - \frac{96}{165} = 1 - 0{\cdot}5819 = 0{\cdot}4181$$

Example 3 : Calculate Spearman's coefficient of rank correlation from the data on height and weight of eight students.

Height (in inces) : 60 62 64 66 68 70 72 74

Weight (in lbs.) : 92 83 101 110 128 119 137 146 **(M.U. 2016)**

Sol. : **Calculation of R between Height and Weight**

Serial No.	Height	Rank R_1	Weight	Rank R_2	D^2 $(R_1 - R_2)^2$
1	60	1	92	2	1
2	62	2	83	1	1
3	64	3	101	3	0
4	66	4	110	4	0
5	68	5	128	6	1
6	70	6	119	5	1
7	72	7	137	7	0
8	74	8	146	8	0
$N = 8$					$\Sigma D^2 = 4$

$$\text{Now,} \quad R = 1 - \frac{6\Sigma D^2}{N^3 - N} \qquad \because \Sigma D^2 = 4,\ N = 8$$

$$\therefore \quad R = 1 - \frac{6 \times 4}{512 - 8} = 1 - \frac{24}{504} = 1 - 0{\cdot}048 = 0{\cdot}952$$

(d) Equal Ranks

In some cases it may happen that there is a tie between two or more members *i.e.*, they have equal values and hence equal ranks. In such cases we divide the rank among equal members. For instance, if two items have 4th rank we divide the 4th and the next rank 5th between them equally and give $\frac{4+5}{2} = 4{\cdot}5$th rank to each of them. If three items have the same 4th rank, we give each of them $\frac{4+5+6}{3} = 5$th rank.

After assigning ranks in this way an adjustment is necessary. If m is the number of items having equal ranks then the factor $\frac{1}{12}(m^3 - m)$ is added to Σd_i^2. If there are more than one cases of this type this factor is added corresponding to each case. Then,

$$R = 1 - \frac{6\left[\Sigma d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \ldots\right]}{n^3 - n}$$

Example 1 : Obtain the rank correlation coefficient from the following data.

X : 10, 12, 18, 18, 15, 40.

Y : 12, 18, 25, 25, 50, 25. **(M.U. 2004, 05, 10, 14, 19)**

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Sol. : **Calculation of R**

X	Rank R_1	Y	Rank R_2	D^2 $(R_1 - R_2)^2$
10	1	12	1	0·00
12	2	18	2	0·00
18	4·5	25	4	0·25
18	4·5	25	4	0·25
15	3	50	6	9·00
40	6	25	4	4·00
$N = 6$				$\Sigma D^2 = 13·50$

There are two items in X series having equal values at the rank 4. Each is given the rank $= \frac{4+5}{2} = 4·5$. Similarly, there are three items in Y series at the rank 3. Each of them is given the rank $= \frac{3+4+5}{3} = 4$.

$$\therefore \quad R = 1 - \frac{6\left[\Sigma d_i^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2)\right]}{n^3 - n}$$

Since, $\Sigma D^2 = 13·50$, $m_1 = 2$, $m_2 = 3$, $N = 6$.

$$R = 1 - \frac{6\left[13·50 + \frac{1}{12}(8-2) + \frac{1}{12}(27-3)\right]}{216 - 6} = 1 - 0·4571 = 0·5429.$$

Example 2 : Calculate the value of rank correlation coefficient from the following data regarding marks of 6 students in statistics and accountancy in a test :

Marks in Statistics : 40, 42, 45, 35, 36, 39.

Marks in Accountancy : 46, 43, 44, 39, 40, 43. **(M.U. 2014, 19)**

Sol. : **Calculation of R**

X	R_1	Y	R_2	D^2 $(R_1 - R_2)^2$
40	3	46	1	4·00
42	2	43	3·5	2·25
45	1	44	2	1·00
35	6	39	6	0·00
36	5	40	5	0·00
39	4	43	3·5	0·25
$N = 6$				$\Sigma D^2 = 7·50$

$$R = 1 - \frac{6\left\{\Sigma D^2 + \frac{1}{12}(2^3 - 2)\right\}}{N^3 - N} = 1 - \frac{6(7·5 + 0·5)}{216 - 6}$$

$$= 1 - \frac{48}{210} = 1 - 0·229 = 0·771$$

Example 3 : From the following data calculate the coefficient of rank correlation between X and Y.

X : 32, 55, 49, 60, 43, 37, 43, 49, 10, 20.

Y : 40, 30, 70, 20, 30, 50, 72, 60, 45, 25. **(M.U. 2016)**

Sol. : **Calculation of R between X and Y**

X	R_1	Y	R_2	D^2 $(R_1 - R_2)^2$
32	3	40	5	4·00
55	9	30	3·5	30·25
49	7·5	70	9	2·25
60	10	20	1	81·00
43	5·5	30	3·5	4·00
37	4	50	7	9·00
43	5·5	72	10	20·25
49	7·5	60	8	0·25
10	1	45	6	25·00
20	2	25	2	0
$N = 10$				$\Sigma D^2 = 176$

Since there are two items in the X series having equal values at the rank 5 and two at the rank 7 they are given rank 5·5 and 7·5 each respectively. Similarly, in the Y series two items at the rank 3 are given the rank 3·5 each. There are three cases where there is a tie each having 2 times.

$$\therefore \quad R = 1 - \frac{6\left\{\Sigma D^2 + \frac{1}{12}(m_1^3 - m_1) + \frac{1}{12}(m_2^3 - m_2) + \frac{1}{12}(m_3^3 - m_3)\right\}}{N^3 - N}$$

But, $\Sigma D^2 = 176$, $m_1 = m_2 = m_3 = 2$, $N = 10$

$$\therefore \quad R = 1 - \frac{6\left\{176 + \frac{1}{12}(8-2) + \frac{1}{12}(8-2) + \frac{1}{12}(8-2)\right\}}{1000 - 10}$$

$$= 1 - \frac{6(177·5)}{990} = 1 - 1·076 = -0·076.$$

EXERCISE - II

1. Sixteen industries of the State have been ranked as follows according to profits earned is 1980 - 81 and the working capital for the year. Calculate the rank correlation coefficient.

Industry :	A,	B,	C,	D,	E,	F,	G,	H,	I,	J,	K,	L,	M,	N,	O,	P.
Rank (Profit) :	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11	12,	13,	14,	15,	16.
Rank (Capital) :	13,	16,	14,	15,	10,	12,	4,	11,	5,	9,	8,	3,	1,	6,	7,	2.

[Ans. : $R = -0·8176$]

2. Distribution of marks in Economics and Mathematics for ten students in a certain test are given below :

Student No. :	1,	2,	3,	4,	5,	6,	7,	8,	9,	10.
Marks in Eco. :	25,	28,	32,	36,	38,	40,	39,	42,	41,	45.
Marks In Maths. :	70,	80,	85,	75,	59,	65,	48,	50,	54,	66.

Calculate the value of Spearman's Rank correlation coefficient. **[Ans. : – 0·6364]**

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3. Calculate Spearman's coefficient of rank correlation for the following data.

X : 53, 98, 95, 81, 75, 61, 59, 55.

Y : 47, 25, 32, 37, 30, 40, 39, 45. [Ans. : – 0·9048]

4. Calculate Spearman's coefficient of rank correlation from the following data.

X : 35, 38, 43, 30, 54, 68, 70, 92, 44, 56.

Y : 51, 37, 48, 62, 93, 73, 56, 72, 70, 92. [Ans. : 0·59]

5. Calculate Spearman's coefficient of rank correlation for the following data of scores in psychological tests (X) and arithmetical ability (Y) of 10 children.

Child : A, B, C, D, E, F, G, H, I, J.

X : 105, 104, 102, 101, 100, 99, 98, 96, 93, 92.

Y : 101, 103, 100, 98, 95, 96, 104, 92, 97, 94. [Ans. : R = 0·782]

6. Find the rank correlation coefficient between poverty and over crowding of cities from the following data.

Town :	A,	B,	C,	D,	E,	F,	G,	H,	I,	J.
No. of poor families :	17,	13,	15,	16,	6,	11,	14,	9,	7,	12.
Population (over crowding) :	30,	46,	35,	24,	12,	18,	27,	22,	46,	8.

[Ans. : R = 0·73]

7. Calculate the rank coefficient of correlation from the following data.

X : 105, 110, 112, 108, 111, 116, 120, 104, 115, 125.

Y : 39, 41, 45, 38, 48, 58, 60, 35, 54, 69. [Ans. : R = 0·9636]

8. Two judges X, Y ranked 8 candidates as follows. Find the correlation coefficient.

Candidates :	A,	B,	C,	D,	E,	F,	G,	H.
First judge X :	5,	2,	8,	1,	4,	6,	3,	7.
Second judge Y :	4,	5,	7,	3,	2,	8,	1,	6.

[Ans. : R = 0·67]

9. Calculate the rank correlation coefficient from the following data.

Marks in Paper I : 52, 63, 45, 36, 72, 65, 45, 25.

Marks in Paper II : 62, 53, 51, 25, 79, 43, 60, 33. [Ans. : R = 0·648]

Miscellaneous Examples

Example 1 : State true or false with proper justification.

If coefficient of correlation between x and y is negative then the coefficient of correlation between $-x$ and $-y$ is positive. **(M.U. 1998)**

Sol. : Coefficient of correlation between x and y is given by

$$r = \frac{\Sigma xy - \Sigma x \cdot \Sigma y / N}{\sqrt{\Sigma x^2 - (\Sigma x)^2 / N}\sqrt{\Sigma y^2 - (\Sigma y)^2 / N}}$$

If we change the signs of x and y both, since the product and square terms occur, the sign of r will remain the same.

$\therefore$ The statement is false.

Example 2 : The coefficient of rank correlation of the marks obtained by 10 students in Physics and Chemistry was found to be 0·5. It was later discovered that the difference in ranks in the two subjects obtained by one of the students was wrongly taken as 3 instead of 7. Find the correct coefficient of rank correlation. **(M.U. 2005)**

Sol. : Since $R = 1 - \dfrac{6\,\Sigma d_i^2}{n^3 - n}$ and R = 0·5, n = 10.

$$0{\cdot}5 = 1 - \frac{6\,\Sigma d_i^2}{1000 - 10} \quad \therefore\ \Sigma d_i^2 = \frac{495}{6}$$

$$\therefore\ \text{Correct } \Sigma d_i^2 = \text{Incorrect } \Sigma d_i^2 - (\text{Incorrect rank diff.})^2 + (\text{Correct rank diff.})^2$$

$$= \frac{495}{6} - 3^2 + 7^2 = \frac{735}{6}$$

$$\therefore\ \text{Correct } R = 1 - \frac{6 \times (735/6)}{990} = 1 - \frac{735}{990} = 0{\cdot}26.$$

Example 3 : (a) Let r_{xy} = 0·4, Cov. (x, y) = 1·6, σ_y^2 = 25. Find σ_x.

(b) If $R_{x,y}$ = 0·143 and the sum of the squares of the differences between the ranks is 48, find N. **(M.U. 2005)**

Sol. : (a) We have $r = \dfrac{\text{cov}(x, y)}{\sigma_x \cdot \sigma_y}$. But r = 0·4, cov. (x, y) = 1·6, σ_y = 5.

$$\therefore\ 0{\cdot}4 = \frac{1{\cdot}6}{5\,\sigma_x} \quad \therefore\ \sigma_x = \frac{1{\cdot}6}{5 \times 0{\cdot}4} = 0{\cdot}8$$

(b) We have, $R = 1 - \dfrac{6\,\Sigma D^2}{N^3 - N}$. By data, R = 0·143, ΣD^2 = 48.

$$\therefore\ 0{\cdot}143 = 1 - \frac{6 \times 48}{N^3 - N} = 1 - \frac{288}{N^3 - N} \quad \therefore\ \frac{288}{N^3 - N} = 1 - 0{\cdot}143 = 0{\cdot}857$$

$$\therefore\ N^3 - N = \frac{288}{0{\cdot}857} = 336 \quad \therefore\ N^3 - N - 336 = 0$$

$$\therefore\ N^3 - 7N^2 + 7N^2 - 49N + 48N - 336 = 0$$

$\therefore\ (N - 7)(N^2 + 7N + 48) = 0 \quad \therefore\ N = 7$, other roots of N are imaginary.

Example 4 : Calculate the correlation coefficient between x and y from the following data.

N = 10, Σx = 140, Σy = 150, $\Sigma (x - 10)^2$ = 180,

$\Sigma (y - 15)^2$ = 215, $\Sigma (x - 10)(y - 15)$ = 60. **(M.U. 1997, 99)**

Sol. : With usual notation Σd_x^2 = 180, Σd_y^2 = 215, $\Sigma d_x d_y$ = 60.

Now, $\bar{x} = A + \dfrac{\Sigma dx}{N} \quad \therefore\ 14 = 10 + \dfrac{\Sigma d_x}{10} \quad \therefore\ \Sigma d_x = 40$

Similarly, $\bar{y} = B + \dfrac{\Sigma d_y}{N} \quad \therefore\ 15 = 15 + \dfrac{\Sigma d_y}{10} \quad \therefore\ \Sigma d_y = 10$

Now,
$$r = \frac{\Sigma d_x d_y - \dfrac{\Sigma d_x \cdot \Sigma d_y}{N}}{\sqrt{\Sigma d_x^2 - \dfrac{(\Sigma d_x)^2}{N}}\sqrt{\Sigma d_y^2 - \dfrac{(\Sigma d_y)^2}{N}}}$$

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$$\therefore \quad r = \frac{60 - \frac{40 \times 0}{10}}{\sqrt{180 - \frac{(40)^2}{10}}\sqrt{215 - \frac{(0)^2}{10}}} = \frac{60}{\sqrt{20}\sqrt{215}} = 0{\cdot}915.$$

EXERCISE - III

Type I

1. Compute Spearman's rank correlation coefficient from the following data.
   X : 85, 74, 85, 50, 65, 78, 74, 60, 74, 90
   Y : 78, 91, 78, 58, 60, 72, 80, 55, 68, 70. **(M.U. 2007, 18) [Ans. : R = 0·45]**
2. Compute Spearman's rank correlation coefficient from the following data.
   X : 18, 20, 34, 52, 12.
   Y : 39, 23, 35, 18, 46. **(M.U. 2004) [Ans. : – 0·824]**
3. From the following data calculate Spearman's rank correlation between x and y.
   x : 36, 56, 20, 42, 33, 44, 50, 15, 60.
   y : 50, 35, 70, 58, 75, 60, 45, 80, 38. **(M.U. 2010) [Ans. : R = 0·92]**
4. Find the coefficient of correlation (r) between x and y for the following data.
   x : 62 64 65 69 70 71 72 74
   y : 126 125 139 145 165 152 180 208 **(M.U. 2003, 04) [Ans. : 0·9032]**
5. Find Karl Pearson's coefficient of correlation and also, the Spear-man's rank coefficient of correlation for the following data.
   x : 12 17 22 27 32
   y : 113 119 117 115 121
   Also interpret your result. **(M.U. 2019) [Ans. : $r = R$ = 0·6]**
6. The following data gave the growth of employment in lakhs in organised sector in India between 1988 and 1995.
   Year : 1988, 89, 90, 91, 92, 93, 94, 95.
   Public Sector : 98, 101, 104, 107, 113, 120, 125, 128.
   Private Sector : 65, 65, 67, 68, 68, 69, 68, 68.
   Find the correlation coefficient (r) between the employment in public and private sectors and give your comments. **(M.U. 1998) [Ans. : r = 0·98]**
7. Calculate the coefficient of correlation from the following figures. Is there any marked correlation between the production and price of tea ?
   Production in crores of lbs. : 44, 37, 31, 38, 36, 35, 40.
   Price in Rs. per lbs. : 2·75, 3·62, 4·25, 4·12, 4·28, 4·32, 4·05.
8. Draw a scatter diagram to represent the following data.
   X : 2, 4, 5, 6, 8, 11.
   Y : 18, 12, 10, 8, 7, 5.
   Calculate the coefficient of correlation between X and Y for the above data. **(M.U. 1998) [Ans. : r = – 0·92]**

9. Find the coefficient of correlation between height of father and height of son from the following data.
   Height of father : 65, 66, 67, 67, 68, 69, 71, 73.
   Height of son : 67, 68, 64, 68, 72, 70, 69, 70. **[Ans. : r = + 0·55]**
10. Calculate Spearman's coefficient of rank correlation and Pearson's coefficient of correlation from the data on height and weight of eight students. Why the two values are same ?
    Height *(in inches)* : 60, 62, 64, 66, 68, 70, 72, 74.
    Weight *(in lbs.)* : 92, 83, 101, 110, 128, 119, 137, 146.
    [Ans. : $r = R$ = 0·93 ; For both the series, the difference between consecutive terms remains constant if arranged in order. **]**
11. The following table shows the marks obtained by 10 students in Accountancy and Statistics. Find the Spearman's coefficient of rank correlation.
    Student No. : 1, 2, 3, 4, 5, 6, 7, 8, 9, 10.
    Accountancy : 45, 70, 65, 30, 90, 40, 50, 57, 85, 60.
    Statistics : 35, 90, 70, 40, 95, 40, 60, 80, 80, 50.
    Will the result change if the marks in the two subjects of all the students are increased by 5 and 10 respectively ? Will the result change if marks in the two subjects of all the students are halved ? **[Ans. :** r = 0·8606 ; No. : No. **]**

Type II

1. The coefficient of rank correlation between marks in Physics and Chemistry obtained by a group of students is 0·8. If the sum of the squares of differences in ranks in 33, find the number pairs students. **[Ans. :** $N = 10$ **]**
2. Find the number of pairs of observations from the following data.
   $r = 0{\cdot}4,\ \Sigma xy = 108,\ \sigma_y = 3,\ \Sigma x^2 = 900.$
   where x, y are the deviations of x, y from there respective means. **[Ans. :** $N = 10$ **]**
3. Coefficient of correlation between two variables is 0·4. Their covariance is 12. The variance of x is 25. Find the standard deviation of y. **[Ans. :** $\sigma_y = 6$ **]**
4. A computer while calculating the correlation coefficient between two variables x and y, from 25 observations obtained the following results
   $N = 25,\ \Sigma x = 125,\ \Sigma y = 100,\ \Sigma x^2 = 650,\ \Sigma y^2 = 960,\ \Sigma xy = 508$
   where x, y denote the actual values of the variables. Find the value of r. **[Ans. : 0·067]**
5. A sample of 25 pairs of values of x and y lead to the following results.
   $\Sigma x = 127,\ \Sigma y = 100,\ \Sigma x^2 = 760,\ \Sigma y^2 = 449,\ \Sigma xy = 500.$
   Later on it was found that two pairs of values were taken as (8, 14) and (8, 6) instead of correct values (8, 12) and (6, 8).
   Find corrected correlation coefficient between x and y. **(M.U. 2004) [Ans. :** r = – 0·31 **]**
6. Given : Number of pairs of observations = 10
   X series standard deviation = 22·70. Y series standard deviation = 9·592
   Summation of the products of corresponding deviations of X and Y from their respective actual means = – 1439. Find r. **[Ans. :** r = – 0·66 **]**

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EXERCISE - IV

Theory

1. What is meant by correlation ? Describe scatter diagram and interpret.

2. Define - (i) Karl Pearson's coefficient of correlation, (ii) Spearman's rank correlation coefficient, (iii) Coefficient of variation. **(M.U. 2005)**

3. Two variables x and y are connected by the relation $ax + by + c = 0$. Show that the coefficient of correlation is either + 1 or – 1.

4. Define product moment correlation coefficient and show that it is always numerically less than or equal to unity.

5. Define Karl Pearson's Product moment correlation coefficient.

6. Prove that Spearson's rank correlation coefficient R is given by

$$R = 1 - \frac{6\Sigma D^2}{N^3 - N}$$ **(M.U. 1996, 99, 2002, 03, 04, 05, 06, 09)**

7. What is scatter diagram ? How does it help in studying the correlation between two variables. Draw scatter diagrams for $r = +1$, $r = -1$ and $r = 0$.

8. Define the Karl Pearson's coefficient of correlation r between two variables x and y. What is "Spurious Correlation" ?

Interpret the cases $r = +1$, $r = -1$, $r = 0$. Also draw the scatter diagrams corresponding to these cases.

CHAPTER 6

Regression and Fitting of Curves

1. Introduction

We have seen in the previous chapter how to examine and measure in magnitude and direction correlation between two variables. After establishing correlation, it is natural to search for a method which will help us to estimate the value of one variable when that of the other is known. This is achieved by the analysis of regression. Regression can be defined as **'a method of estimating the value of one variable when that of the other is known and when the variables are correlated'.**

The term, regression was first used by Galton. He found that although tall fathers have tall sons, and short fathers have short sons, the average height of sons of tall fathers is less than the average height of their fathers and the average height of sons of short fathers is more than the average height of their fathers. In other words the average height of sons of tall fathers or short fathers will regress or go back to the general average height. This phenomenon was described by him as 'regression.'

2. Lines of Regression

We have seen in the previous chapter that if the variables which are highly correlated are plotted on a graph then the points lie in a narrow strip. If the strip is nearly straight, we may draw a line such that all the points are close to it from both the sides. Such a line can be taken as the representative of the ideal variation. It is called the line of best fit. It is a line such that the sum of the distances of the points from the line is minimum. It is also called **'the line of regression'**. But we do not measure the distance by dropping a perpendicular from a point to the line. We measure, the deviations (i) vertically and (ii) horizontally, and get one line when distances are minimised vertically and second line when distances are minimised horizontally. Thus, we get **two lines of regression.**

(i) Line of regression of Y on X

If we minimise the deviations of the points from the line measured along y-axis we get a line which is called **the line of regression of Y on X**. Its equation is written in the from $Y = a + bX$. **This line is used for estimating the value of Y for a given value of X.** [See Fig. 6.1]

The equation of the line of regression of y on x must be written with y on the left hand side and x and the constant term on the right hand side.

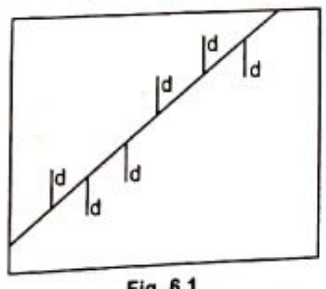

Fig. 6.1
For the Line of Regression of Y on X the distances d are minimised.

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(ii) Line of regression of X on Y

If we minimise the deviations of the points from the line measured along x-axis we get a line which is called **the line of regression of X on Y.** Its equation is written in the form $X = a + bY$. **This line is used for estimating the value of X for a given value of Y.** [See Fig. 6.2]

The equation of the line of regression of x on y must be written with x on the left hand side and y and constant term on the right hand side.

There are two methods of obtaining the lines of regression. The first is graphical, the other is mathematical. They are :

1. The method of Scatter Diagram,
2. The Method of Least Squares.

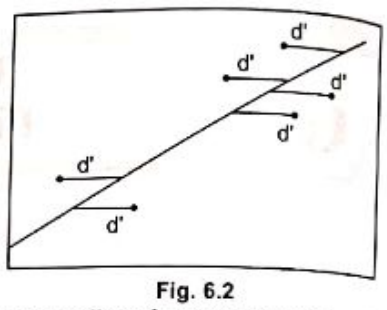

Fig. 6.2
For the line of regression of X on Y the distances d' are minimised.

3. The Method of Scatter Diagram

It is the simplest method of obtaining the lines of regression. The data are plotted on a graph paper by taking the independent variable on x-axis and the dependent variable on y-axis. We thus get points which are generally scattered. If the correlation is perfect *i.e.* if r is equal to one, positive or negative, the points will lie on a line, which is the line of regression. And there is only one line of regression and not two in such cases. However, in practice we rarely come across problems wherein we have perfect correlation. Usually, the points are scattered in a narrow straight strip and we have to find a line which will best represent all the points of the scatter diagram. We draw a line which will be close to all the points as far as possible.

Example : Given the following pairs of values of X and Y.

X :	1,	2,	3,	4,	5,	6,	7,	8,	9,	10,	11,	12.
Y :	5,	6,	5,	6,	6,	8,	7,	9,	8,	9,	10,	11.

Plot the points on a graph and draw a line of regression.

Sol. :

Scatter Diagram and Line of Regression

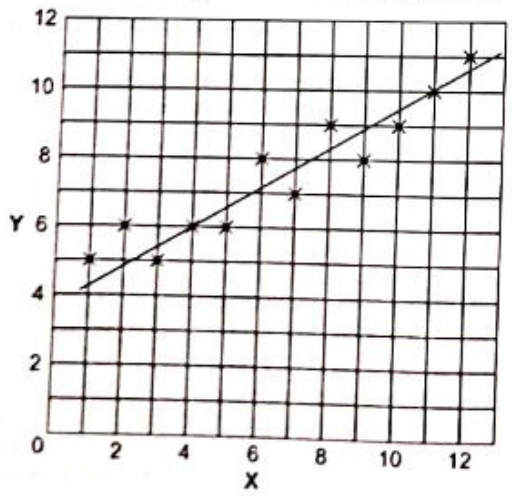

Fig. 6.3

4. The Method of Least Square

This is a mathematical method which gives an objective treatment to find a line of regression.

Let $y = a + bx$ be the equation of the line required. To find the line of regression of y on x we minimise the sum of the absolute distances of the points like $P(x_i, y_i)$ from the line measured along the y-axis. If Q is the point on the line corresponding to $P(x_i, y_i)$ we have to minimise the absolute distance PQ. Since Q lies on $y = a + bx$ its y-coordinate $= a + bx_i$

$$\therefore \quad |PQ| = |y_i - a - bx_i|$$

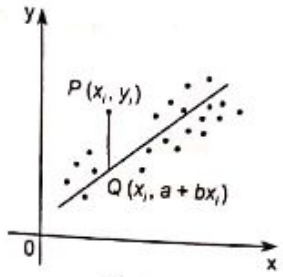

Fig. 6.4

For minimising $|PQ|$ we minimise its squares. Hence, if S denotes the sum of the squares of these distances,

$$S = \Sigma f_i (y_i - a - bx_i)^2 \qquad \text{where } f_i \text{ is the frequency of } (x_i, y_i).$$

We have to find a and b such that S is minimum, the conditions for which are

$$\frac{\partial S}{\partial a} = 2\Sigma f_i(y_i - a - bx_i) = 0 \quad \text{and} \quad \frac{\partial S}{\partial b} = 2\Sigma f_i(y_i - a - bx_i)x_i = 0$$

$$\Sigma f_i (y_i - a - bx_i) = 0 \qquad \text{..... (A)}$$

$$\Sigma x_i f_i (y_i - a - bx_i) = 0 \qquad \text{..... (B)}$$

From (A) we get, $\Sigma f_i y_i - a\Sigma f_i - b\Sigma f_i x_i = 0$

$$\therefore \quad N\bar{y} - aN - bN\bar{x} = 0 \quad \therefore \quad \bar{y} = a + b\bar{x} \qquad \text{..... (C)}$$

which shows that **the line of regression passes through** $(\bar{x}, \bar{y})$.

From (B) we get, $\Sigma f_i x_i y_i - a\Sigma f_i x_i - b\Sigma f_i x_i^2 = 0$ ..... (D)

We now find the values of these expressions in terms of r, σ_x, σ_y.

But since, $r = \dfrac{1}{N}\dfrac{\Sigma f_i(x_i - \bar{x})(y_i - \bar{y})}{\sigma_x \sigma_y}$

$$r = \frac{\Sigma f_i x_i y_i - N\bar{x}\bar{y}}{N\sigma_x \sigma_y} \quad \therefore \quad \Sigma f_i x_i y_i = Nr\sigma_x\sigma_y + N\bar{x}\bar{y}$$

and $\sigma_x^2 = \dfrac{1}{N}\Sigma f_i (x_i - \bar{x})^2 = \dfrac{1}{N}\Sigma f_i x_i^2 - \bar{x}^2 \qquad \therefore \quad \Sigma f_i x_i^2 = N\sigma_x^2 + N\bar{x}^2$

Putting the values of $\Sigma f_i x_i y_i$ and $\Sigma f_i x_i^2$ in D, we get

$$Nr\sigma_x\sigma_y + N\bar{x}\bar{y} = aN\bar{x} + bN\sigma_x^2 + bN\bar{x}^2$$

i.e. $r\sigma_x\sigma_y + \bar{x}\bar{y} = a\bar{x} + b\sigma_x^2 + b\bar{x}^2$ ..... (E)

Multiply (C) by $\bar{x}$ and subtract it form (E)

$$r\sigma_x\sigma_y = b\sigma_x^2 \quad \therefore \quad b = r\frac{\sigma_y}{\sigma_x}$$

Since, the line passes through $(\bar{x}, \bar{y})$ and its slope $b = r\dfrac{\sigma_y}{\sigma_x}$ its equation is

$$\boxed{y - \bar{y} = r\frac{\sigma_y}{\sigma_x}(x - \bar{x})} \qquad \text{..... (F)}$$

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Similarly, the equation of the line of regression of x on y can be shown to be

$$\boxed{x - \bar{x} = r\frac{\sigma_x}{\sigma_y}(y - \bar{y})} \quad \text{..... (G)}$$

Alternative Method : Instead of calculating $\bar{x}, \bar{y}, \sigma_x, \sigma_y$ and r we may use the following method.

(i) The line of regression of y on x

Let the equation of the line of regression of y on x be $y = a + bx$.

Then as before we have to minimise

$S = \Sigma f_i(y_i - a - b x_i)^2$ the conditions of which are

$$\frac{\partial S}{\partial a} = -2\Sigma f_i(y_i - a - bx_i) = 0 \quad \text{and} \quad \frac{\partial S}{\partial b} = -2\Sigma f_i(y_i - a - bx_i)\, x_i = 0$$

i.e. $\Sigma f_i(y_i - a - bx_i) = 0$ and $\Sigma f_i x_i (y_i - a - bx_i) = 0$

i.e. $\Sigma f_i y_i - a\Sigma f_i - b\Sigma f_i x_i = 0$ and $\Sigma f_i x_i y_i - a\Sigma f_i x_i - b\Sigma f_i x_i^2 = 0$.

If the required sums are known, by solving the above two equations simultaneously for a and b we get the required equation.

In particular if all values occur only once *i.e.* if $f_i = 1$ for all i then the above equations take the form

$$\boxed{\Sigma y = aN + b\Sigma x} \quad \text{and} \quad \boxed{\Sigma xy = a\Sigma x + b\Sigma x^2}$$

from which a and b can be calculated.

The above equations are called **normal equations**.

(ii) The line of regression of x on y

Let the equation of the line be $x = a + by$.

Proceeding as above we get the normal equations as

$$\boxed{\Sigma x = aN + b\Sigma y} \quad \text{and} \quad \boxed{\Sigma xy = a\Sigma y + b\Sigma y^2}$$

from which we can find the values of a and b.

Example : Find the equations of the lines of regression from the following data.

x: 5 6 7 8 9

y: 2 4 5 6 8 Also find r.

Sol. : **Calculations of regression**

Sr. No.	x	x^2	y	y^2	xy
1	5	25	2	4	10
2	6	36	4	16	24
3	7	49	5	25	35
4	8	64	6	36	48
5	9	81	8	64	72
$N = 5$	35	255	25	145	189

The equation of the line of regression of y on x is $y = a + bx$ where a, b are given by

$\Sigma y = aN + b\Sigma x$ and $\Sigma xy = a\Sigma x + b\Sigma x^2$

Putting the values of Σx, Σx^2, Σxy, we get

$25 = 5a + 35b$ ..... (i) and $189 = 35a + 255b$ ..... (ii)

Multiply the first by 7 and subtract it from the second.

$$\begin{aligned} 189 &= 35a + 255b \\ 175 &= 35a + 245b \\ \hline 14 &= \quad\quad 10b \end{aligned} \qquad \therefore\ b = 1{\cdot}4$$

Putting this value of b in (i), we get

$25 = 5a + 35\,(1{\cdot}4)$ $\therefore\ 5a = 25 - 49 = -24$ $\therefore\ a = -4{\cdot}8$

$\therefore$ The equation of the line of regression of y on x is

$y = -4{\cdot}8 + 1{\cdot}4x$

The equation of the line of regression of x on y is $x = a + by$ where a, b are given by

$\Sigma x = aN + b\Sigma y$ and $\Sigma xy = a\Sigma y + b\Sigma y^2$

Putting the values of Σx, Σy, Σxy, Σy^2, we get

$35 = 5a + 25b$ ..... (iii) and $189 = 25a + 145b$ ..... (iv)

Multiply (iii) by 5 and subtract it from (iv)

$$\begin{aligned} 189 &= 25a + 145b \\ 175 &= 25a + 120b \\ \hline 14 &= \quad\quad 25b \end{aligned} \qquad \therefore\ b = 0{\cdot}56$$

Putting this value of b in (iii), we get

$35 = 5a + 25\,(0{\cdot}56)$ $\therefore\ 5a = 35 - 14 = 11$ $\therefore\ a = 2{\cdot}2$

$\therefore$ The equation of the line of regression of x on y is

$x = 2{\cdot}2 + 0{\cdot}56y$

Now, $r = \sqrt{b_1 \times b_2} = \sqrt{1{\cdot}4 \times 0{\cdot}56} = 0{\cdot}88$.

5. Calculations of the Equations of the Lines of Regression

There are various methods of calculating the equations of the lines of regression. The choice is yours. We state them below.

(a) By calculating the coefficient of correlation r and standard deviation σ_x and σ_y. The equation of the line of regression of Y on X.

The equation of the line of regression of Y on X is given by,

$$Y - \bar{Y} = b_{yx}(X - \bar{X})$$

where, $\bar{X}$ = the mean of X, $\bar{Y}$ = the mean of Y and $b_{yx} = r\dfrac{\sigma_y}{\sigma_x}$. [See § 6]

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∴ The equation, therefore can be written as

$$\boxed{Y - \overline{Y} = r\frac{\sigma_y}{\sigma_x}(X - \overline{X})} \quad \text{..... (1)}$$

This equation is to be used for calculating the most probable value of Y for a given value of X.

The equation of the line of regression of X on Y.

The equation of the line of regression of X on Y is given by,

$$X - \overline{X} = b_{xy}(Y - \overline{Y})$$

where, $\overline{X}$ = the mean of X, $\overline{Y}$ = the mean of Y and $b_{xy} = r\frac{\sigma_x}{\sigma_y}$.

∴ The equation, therefore can be written as

$$\boxed{X - \overline{X} = r\frac{\sigma_x}{\sigma_y}(Y - \overline{Y})} \quad \text{..... (2)}$$

This equation is to be used for calculating the most probable value of X for a given value of Y.

When the values of X and Y are known, we can as usual calculate r, σ_x and σ_y and then obtain equations of the lines of regression of X and Y.

(b) By calculating a and b directly

Instead of calculating $\overline{X}$, $\overline{Y}$, σ_x, σ_y and r we may calculate a and b directly as explained below.

The line of regression of Y on X : The constants a and b of the equation of the line of regression of Y on X *i.e.* of,

$$Y = a + bX$$

can be obtained by solving the following simultaneous equations.

$$\boxed{\Sigma Y = aN + \Sigma X} \quad \text{and} \quad \boxed{\Sigma YX = a\Sigma X + b\Sigma X^2} \quad \text{..... (3)}$$

The line of regression of X on Y : The constants a and b of the equation of the line of regression of X on Y *i.e.* of,

$$X = a + bY$$

can be obtained by solving the following simultaneous equations,

$$\boxed{\Sigma X = aN + b\Sigma Y} \quad \text{and} \quad \boxed{\Sigma XY = a\Sigma Y + b\Sigma y^2} \quad \text{..... (4)}$$

The equations, are called 'Normal equations'.

The formulae (3) and (4) are convenient when ΣX, ΣY, ΣXY, ΣX^2, ΣY^2 are known.

(c) By taking deviations from the means X and Y

If x and y denote the deviations of X and Y from their means, we know that,

$$b_{yx} = r\frac{\sigma_y}{\sigma_x} = \frac{\Sigma xy}{N\sigma_x\sigma_y}\cdot\frac{\sigma_y}{\sigma_x} = \frac{\Sigma xy}{N\sigma_x^2} = \frac{\Sigma xy}{N\cdot\Sigma x^2/N} = \frac{\Sigma xy}{\Sigma x^2}$$

$$\therefore \boxed{b_{yx} = \frac{\Sigma xy}{\Sigma x^2}} \quad \text{..... (5)}$$

Similarly, we can show that,

$$b_{xy} = r\frac{\sigma_x}{\sigma_y} = \frac{\Sigma xy}{\Sigma y^2} \quad \therefore \quad \boxed{b_{xy} = \frac{\Sigma xy}{\Sigma y^2}} \quad \text{..... (6)}$$

where, $x = X - \overline{X}$, $y = Y - \overline{Y}$.

Hence, the equations of the lines of regression become,

$$\boxed{Y - \overline{Y} = \frac{\Sigma xy}{\Sigma x^2}(X - \overline{X})} \quad \text{..... (7)}$$

and

$$\boxed{X - \overline{X} = \frac{\Sigma xy}{\Sigma y^2}(Y - \overline{Y})} \quad \text{..... (8)}$$

(d) By taking deviations from assumed means

If the deviations of X and Y are taken from assumed means *i.e.* if $d_x = X - A$ and $d_y = Y - B$ then the coefficients b_{yx} and b_{xy} are given by,

$$\boxed{b_{yx} = \frac{\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{N}}{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{N}}} \quad \text{..... (9)}$$

$$\boxed{b_{xy} = \frac{\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{N}}{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{N}}} \quad \text{..... (10)}$$

(e) By using actual values directly

If X, Y are actual values of the two variates then it can be shown that

$$\boxed{b_{yx} = \frac{\Sigma XY - \frac{\Sigma X\cdot\Sigma Y}{N}}{\Sigma X^2 - \frac{(\Sigma X)^2}{N}}} \quad \text{..... (11)}$$

$$\boxed{b_{xy} = \frac{\Sigma XY - \frac{\Sigma X\cdot\Sigma Y}{N}}{\Sigma Y^2 - \frac{(\Sigma Y)^2}{N}}} \quad \text{..... (12)}$$

6. Regression Coefficients

The slope b of the line of regression of y on x *i.e.* b of the equation $y = a + bx$ is called the **coefficient of regression of y on x**. It represents the increment in y for unit change in the value of x. It is denoted by b_{yx}.

∴ b_{yx} = Coefficient of Regression of y on x.

$$b_{yx} = r\frac{\sigma_y}{\sigma_x}$$

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Similarly, the slope b of the line of regression x on y *i.e.* b of the equation $x = a + by$ is called **the coefficient of regression of x on y**. It represents the increment in x for unit change in y. It is denoted by b_{xy}.

$\therefore$ b_{xy} = Coefficient of Regression of x on y.

$$b_{xy} = r\frac{\sigma_x}{\sigma_y}$$

Putting b_{yx} and b_{xy} which are the slopes of the lines of regression in (F) and (G), we can write the equations of lines of regression as

$$\boxed{y - \bar{y} = b_{yx}(x - \bar{x})} \quad \text{..... (H)}$$

and

$$\boxed{x - \bar{x} = b_{xy}(y - \bar{y})} \quad \text{..... (I)}$$

By putting the values of r and σ_y, σ_x in terms of actual values of x and y or by taking deviations from actual means or assumed means, we get the following formulae for b_{yx} and b_{xy}. (We repeat the first two.)

$$\boxed{b_{yx} = r\frac{\sigma_y}{\sigma_x}} \quad \text{..... (1)}$$

$$\boxed{b_{yx} = \frac{\Sigma xy}{\Sigma x^2}} \quad \text{..... (2)}$$

where $x = X - \bar{x}$, $y = Y - \bar{y}$ *i.e.* x, y are deviations of X, Y from actual means.

$$\boxed{b_{yx} = \frac{\Sigma d_x d_y - \dfrac{\Sigma d_x \cdot \Sigma d_y}{N}}{\Sigma d_x^2 - \dfrac{(\Sigma d_x)^2}{N}}} \quad \text{..... (3)}$$

where $d_x = X - A$, $d_y = Y - B$ *i.e.* d_x, d_y are deviations of X, Y from assumed means A and B.

$$\boxed{b_{yx} = \frac{\Sigma XY - \dfrac{\Sigma X \cdot \Sigma Y}{N}}{\Sigma X^2 - \dfrac{(\Sigma X)^2}{N}}} \quad \text{..... (4)}$$

where X, Y are the actual values of the variables.

Also,

$$\boxed{b_{xy} = r\frac{\sigma_x}{\sigma_y}} \quad \text{..... (1')}$$

$$\boxed{b_{xy} = \frac{\Sigma xy}{\Sigma y^2}} \quad \text{..... (2')}$$

where $x = X - \bar{x}$, $y = Y - \bar{y}$ *i.e.* x, y are the deviations of X, Y from actual means

$$\boxed{b_{xy} = \frac{\Sigma d_x d_y - \dfrac{\Sigma d_x \cdot \Sigma d_y}{N}}{\Sigma d_y^2 - \dfrac{(\Sigma d_y)^2}{N}}} \quad \text{..... (3')}$$

where $d_x = X - A$, $d_y = Y - B$ *i.e.* d_x, d_y are the deviations of X, Y from assumed means A and B.

$$\boxed{b_{xy} = \frac{\Sigma XY - \dfrac{\Sigma X \cdot \Sigma Y}{N}}{\Sigma Y^2 - \dfrac{(\Sigma Y)^2}{N}}} \quad \text{..... (4')}$$

where X, Y are the actual values of the variables.

7. Properties of Coefficients of Regression

1. Coefficient of correlation is the geometric mean between the coefficients of regression.

Proof : From the above results we have

$$b_{yx} \cdot b_{xy} = r\frac{\sigma_y}{\sigma_x} \cdot r\frac{\sigma_x}{\sigma_y}$$

$$\therefore \quad \boxed{b_{yx} \cdot b_{xy} = r^2} \quad \text{Hence, the result.}$$

Remark ....

Since the product of b_{yx} and b_{xy} is positive, if one of them is negative, the other also must be negative. In other words both the coefficients of regression are positive or both the coefficients of regression are negative together.

2. If one coefficient of regression is greater than one, the other must be less than one.

Proof : Since $-1 \le r \le 1$, $r^2 \le 1$.

Hence, from the above result,

$$b_{yx} \cdot b_{xy} \le 1 \quad \therefore \; b_{yx} \le \frac{1}{b_{xy}} \qquad \therefore \text{ If } b_{yx} < 1,\; b_{xy} > 1.$$

[See the values of b_{yx} and b_{xy} in Ex. 4, page 6-16. See that b_{yx} is less than one and b_{xy} is greater than one.]

3. Arithmetic mean of the coefficients of regression is greater than or equal to the coefficient of correlation.

Proof : We have to show that $\dfrac{b_{yx} + b_{xy}}{2} \ge r$

i.e. $\dfrac{1}{2}\left(r\dfrac{\sigma_y}{\sigma_x} + r\dfrac{\sigma_x}{\sigma_y}\right) \ge r$ *i.e.* $\dfrac{\sigma_y}{\sigma_x} + \dfrac{\sigma_x}{\sigma_y} \ge 2$

i.e. $\sigma_x^2 + \sigma_y^2 \ge 2\sigma_x\sigma_y$ *i.e.* $\sigma_x^2 - 2\sigma_x\sigma_y + \sigma_y^2 \ge 0$

i.e. $(\sigma_x - \sigma_y)^2 \ge 0$ which is obviously true.

Remark ....

In other words this means **the sum of the two coefficients of regression is greater than or equal to 2r**. We shall verify this in Ex. 2, page 6-19; Ex. 5, page 6-17 and Ex. 6, page 6-22 below. We further note that equality will hold when $\sigma_x = \sigma_y$.

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4. Coefficients of regression are independent of change of origin but not of change of scale.

If $u = ax + h$ and $v = by + k$, then

$$\boxed{b_{uv} = \frac{a}{b} \cdot b_{xy} = \frac{\text{Coefficient of } x}{\text{Coefficient of } y} \cdot b_{xy}}$$

We accept this result without proof.

5. If the correlation is perfect then the two coefficients of regression are reciprocals of each other.

Proof : We have $r = \pm 1$ and $r = \sqrt{b_{yx} \cdot b_{xy}}$ $\quad \therefore \quad \pm 1 = \sqrt{b_{yx} \cdot b_{xy}}$.

Squaring $1 = b_{yx} \cdot b_{xy}$ $\quad \therefore \quad b_{yx} = \dfrac{1}{b_{xy}}$.

e.g., if one coefficient of regression is 0·5 and if the correlation is perfect, then the other coefficient of regression is 2.

Example 1 : State whether the following statement is true or false with reasoning : "The regression coefficients between $2x$ and $2y$ are the same as those between x and y". **(M.U. 1997)**

Sol. : As seen above if $u = ax + h$ and $v = by + k$, $\quad b_{uv} = \dfrac{a}{b} \cdot b_{xy}$.

But by data $u = 2x$ *i.e.* $a = 2$ and $v = 2y$ *i.e.* $b = 2$.

$\therefore \quad b_{vu} = \dfrac{2}{2} \cdot b_{xy} = b_{xy}$. $\qquad$ Hence, the statement is true.

Example 2 : State whether the following statement is true or false : "The lines of regression between x and y are parallel to the lines of regression between $2x$ and $2y$.

Sol. : True. Explanation is left to you.

Example 3 : State whether the following statement is true or false : "The coefficients of regression between x and y are the same as the coefficients of regression between $2x + 5$ and $2y - 7$.

Sol. : True. Explanation is left to you.

Example 4 : If the arithmetic mean of regression coefficients is p and their difference is $2q$. find the correlation coefficient. **(M.U. 1998)**

Sol. : Let the coefficients of regression be b_1 and b_2.

Now by data $\quad \dfrac{b_1 + b_2}{2} = p$ and $b_1 - b_2 = 2q$

$\therefore \quad b_1 + b_2 = 2p$ and $b_1 - b_2 = 2q$

$\therefore \quad b_1 = p + q$ and $b_2 = p - q$

$\therefore$ Coefficient of correlation $= r = \sqrt{b_1 b_2} = \sqrt{p^2 - q^2}$.

Example 5 : State true of false with reasoning : "$2x + y = 3$ and $x = 2y + 3$ cannot be the lines of regression." **(M.U. 2004)**

Sol. : If the first line is the line of regression of y on x it must written as $y = -2x + 3$ and if the second line is the line of regression of x on y, then it must be written as $x = 2y + 3$.

Hence, the coefficients of regression are $b_{yx} = -2$ and $b_{xy} = 2$ which is not possible as one of the them is negative and the other is positive and both are greater 1 numerically.

Now, we consider the lines in other way round. Let the first line be the line of regression of x on y and let the second line be the line of regression of y on x.

$\therefore \quad x = -\dfrac{1}{2}y + \dfrac{3}{2}$ and $y = \dfrac{1}{2}x - \dfrac{3}{2}$.

Hence, the coefficients of regression are

$b_{yx} = -\dfrac{1}{2}$ and $b_{xy} = \dfrac{1}{2}$

which is again not possible because one is positive and the other is negative.

Hence, the statement is true.

Example 6 : State true or false with justification. If two lines of regression are $x + 3y - 5 = 0$ and $4x + 3y - 8 = 0$ then the correlation coefficient is + 0·5. **(M.U. 2003, 14)**

Sol. : Let the line $x + 3y - 5 = 0$ be the line of regression of x on y. Writing it as $x = -3y + 5$, we get

$b_{xy} = -3$.

Let the line $4x + 3y - 8 = 0$ be the line of regression of y on x. Writing it as $3y = -4x + 8$

i.e., as $y = -\dfrac{4}{3}x + 2$, we get $b_{yx} = -\dfrac{4}{3}$.

$\therefore \quad r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{(-3)(-4/3)} = \sqrt{4} = 2$

But r cannot be greater than 1.

Hence, our suppositions are wrong.

Now, let the line $x + 3y - 5 = 0$ be the line of regression of y on x. Writing it as

$3y = -x + 5 \qquad$ *i.e.*, $\quad y = -\dfrac{1}{3}x + \dfrac{6}{3}$, we get $\quad b_{yx} = -\dfrac{1}{3}$.

Let the line $4x + 3y - 8 = 0$ be the line of regression of x on y. Writing it as

$4x = -3y + 8 \quad$ *i.e.*, $\quad x = -\dfrac{3}{4}y + 2$, we get $\quad b_{xy} = -\dfrac{3}{4}$.

Now, $\quad r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{\left(-\dfrac{1}{3}\right)\left(-\dfrac{3}{4}\right)} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2} = 0{\cdot}5$

Hence, the statement is true.

6. Angle between the lines of regression

The equation of the lines of regression of y on x is

$y - \bar{y} = r\dfrac{\sigma_y}{\sigma_x}(x - \bar{x})$. $\qquad$ Hence, its slope $m_1 = r\dfrac{\sigma_y}{\sigma_x}$

The equation of the line of regression of x on y is

$x - \bar{x} = r\dfrac{\sigma_x}{\sigma_y}(y - \bar{y})$

i.e., $\quad y - \bar{y} = \dfrac{\sigma_y}{r \cdot \sigma_x}(x - \bar{x})$ $\qquad$ Hence, its slope $m_2 = \dfrac{\sigma_y}{r \cdot \sigma_x}$.

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If θ is the angle between the lines of regression

$$\therefore \quad \tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{r\dfrac{\sigma_y}{\sigma_x} - \dfrac{\sigma_y}{r\sigma_x}}{1 + \dfrac{r\sigma_y}{\sigma_x} \cdot \dfrac{\sigma_y}{r\sigma_x}} = \frac{\dfrac{(r^2\sigma_y\sigma_x - \sigma_y\sigma_x)}{r\sigma_x^2}}{\dfrac{\sigma_x^2 + \sigma_y^2}{\sigma_x^2}}$$

$$= \frac{(r^2 - 1)}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\right) = \frac{1 - r^2}{r}\left(\frac{\sigma_x\sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$

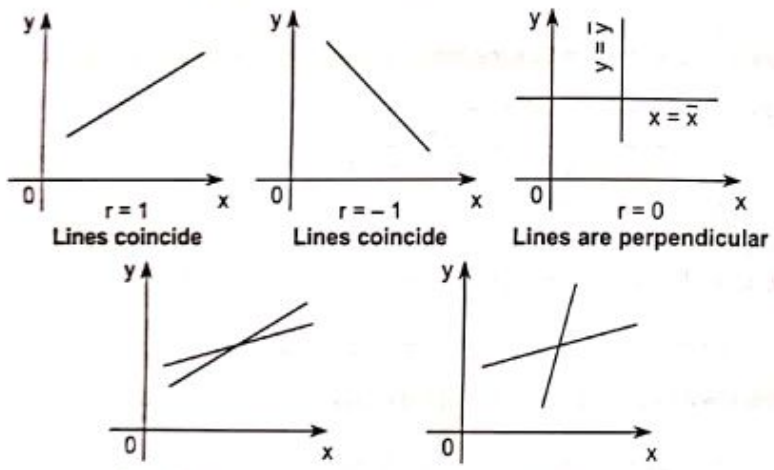
Corollary 1 : If $r = 0$, $\tan\theta = \infty$ $\therefore$ $\theta = \pi/2$.

The lines of regression are perpendicular to each other.

Corollary 2 : If $r = \pm 1$, $\tan\theta = 0$ $\therefore$ $\theta = 0$.

The lines of regression are coincident.

Various Cases Are Shown Below.



Illustrative Examples

Example 1 : A panel of two judges A and B graded dramatic performances by independently awarding marks as follows :

Performance No.	:	1,	2,	3,	4,	5,	6,	7.
Marks by A	:	36,	32,	34,	31,	32,	32,	34.
Marks by B	:	35,	33,	31,	30,	34,	32,	36.

The eighth performance, however, which judge B could not attend, got 38 marks by judge A. If judge B had also been present, how many marks would he be expected to have awarded to the eighth performance ?

Sol. :

Calculations of coefficient of correlation etc.

Sr. No.	$X - \bar{X}$			$Y - \bar{Y}$			Product xy
	X	x	x^2	Y	y	y^2	
1	36	3	9	35	2	4	6
2	32	– 1	1	33	0	0	0
3	34	1	1	31	– 2	4	– 2
4	31	– 2	4	30	– 3	9	6
5	32	– 1	1	34	1	1	– 1
6	32	– 1	1	32	– 1	1	1
7	35	1	1	36	3	9	3
$N = 7$	$\Sigma X = 231$		$\Sigma x^2 = 18$	$\Sigma Y = 231$		$\Sigma y^2 = 28$	$\Sigma xy = 13$

We have to find the marks that would have been awarded by the judge B. Therefore, let the marks given by the judge B be denoted by Y and those given by A by X.

$$\therefore \quad \bar{X} = \frac{\Sigma X}{N} = \frac{231}{7} = 33, \quad \bar{Y} = \frac{\Sigma Y}{N} = \frac{231}{7} = 33$$

$$\therefore \quad b_{yx} = \frac{\Sigma xy}{\Sigma x^2} = \frac{13}{18} = 0{\cdot}72$$

The equation of the line of regression of Y on X is,

$$Y - \bar{Y} = b_{yx}(X - \bar{X}) \quad i.e. \quad Y - 33 = 0{\cdot}72(X - 33)$$

To find the value of Y when $X = 38$, put $X = 38$ in the above equation.

$$\therefore \quad Y - 33 = 0{\cdot}72(38 - 43) = 0{\cdot}72 \times 5 = 3{\cdot}6$$

$$\therefore \quad Y = 33 + 3{\cdot}76 = 36{\cdot}6 = 37 \text{ approximately.}$$

$\therefore$ The judge B would have given 37 marks to the eighth performannce.

Alternatively : **Calculations of regression**

Sr. No.	x	x^2	y	xy
1	36	1296	35	1260
2	32	1024	33	1056
3	34	1156	31	1054
4	31	961	30	930
5	32	1024	34	1088
6	32	1024	32	1024
7	34	1156	36	1224
$N = 7$	231	7641	231	7636

Let the marks given by A be x and those given by B be y.

Then the line of regression of y on x is $y = a + bx$.

And the normal equations are

$$\Sigma y = Na + b\Sigma x \qquad \therefore \quad 231 = 7a + 231b \qquad \text{..... (i)}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \qquad \therefore \quad 7636 = 231a + 7641b \qquad \text{..... (ii)}$$

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Multiply the first by 33 and subtract it from the second.

$$\therefore \quad 7636 = 231a + 7641b$$
$$7623 = 231a + 7626b$$
$$13 = 18b \qquad \therefore \; b = 13/18 = 0{\cdot}72$$

Putting this value in (i), we get,

$$231 = 7a + 231\,(0{\cdot}72) \qquad \therefore \; 7a = 231 - 231\,(0{\cdot}72) = 64{\cdot}68$$

$\therefore \; a = 9{\cdot}24$

$\therefore$ The equation of the line of regression of y on x is

$$y = 9{\cdot}24 + 0{\cdot}72x$$

To estimate y when $x = 38$, we put $x = 38$ in the above equation

$\therefore \; y = 9{\cdot}24 + 0{\cdot}72\,(38) = 36{\cdot}6 = 37$ approximately.

Example 2 : Find the equations of the lines of regression for the following data.

x :	5	6	7	8	9	10	11
y :	11	14	14	15	12	17	16

(M.U. 2003, 05, 16, 17)

Sol. : **Calculations of regression**

Sr. No.	x	x^2	y	y^2	xy
1	5	25	11	121	55
2	6	36	14	196	84
3	7	49	14	196	98
4	8	64	15	225	120
5	9	81	12	144	108
6	10	100	17	289	170
7	11	121	16	256	176
$N = 7$	56	476	99	1427	811

Now, the line of regression of y on x is $y = a + bx$.

The normal equations are

$$\Sigma y = Na + b\,\Sigma x \qquad \therefore \; 99 = 7a + 56b \qquad \text{..... (1)}$$
$$\Sigma xy = a\,\Sigma x + b\,\Sigma x^2 \qquad \therefore \; 811 = 56a + 476b \qquad \text{..... (2)}$$

Multiply the first equation by 56 and the second by 7 and subtract

$$\therefore \quad 5544 = 392a + 3136b$$
$$5677 = 392a + 3332b$$
$$-133 = -196b \qquad \therefore \; b = \frac{133}{196} = 0{\cdot}6789$$

Putting this value of b in (1), we get

$$99 = 7a + 56 \times \frac{133}{196} \qquad \therefore \; 7a = 99 - 38 \qquad \therefore \; 7a = 61 \qquad \therefore \; a = 8{\cdot}7143$$

$\therefore$ The equation of the line of regression of y on x is

$$y = 8{\cdot}7143 + 0{\cdot}6786x$$

Now, the equation of the line of regression of x on y is $x = a + by$.

The normal equations are

$$\Sigma x = Na + b\,\Sigma y \qquad \therefore \; 56 = 7a + 99b \qquad \text{..... (3)}$$
$$\Sigma xy = a\,\Sigma x + b\,\Sigma y^2 \qquad \therefore \; 811 = 99a + 1427b \qquad \text{..... (4)}$$

Multiply the third equation by 99 and the fourth by 7 and subtract

$$\therefore \quad 5544 = 693a + 9801b$$
$$5677 = 693a + 9989b$$
$$133 = 188b \qquad \therefore \; b = \frac{133}{188} = 0{\cdot}7074$$

Putting this value of b in (1), we get

$$56 = 7a + 99 \times \frac{133}{188} \qquad \therefore \; 7a = 56 - 70{\cdot}0372 = -14{\cdot}0372 \qquad \therefore \; a = -2{\cdot}0053$$

$\therefore$ The equation of the line of regression of x on y is

$$x = -2{\cdot}0053 + 0{\cdot}7074y$$

(Further the coefficient of correlation is given by

$$r = \sqrt{b_1 b_2} = \sqrt{0{\cdot}6786 \times 0{\cdot}7074} = 0{\cdot}6928.)$$

Example 3 : From the following table showing age of cars of a certain make and annual maintenance costs, obtain the regression equation for costs related to age.

Age of Cars (years)	:	2,	4,	6,	7,	8,	10,	12.
Annual maintenance Cost (Rs.)	:	1,600,	1,500,	1,800,	1,900,	1,700,	2,100,	2,000.

Find the approximate cost of maintaining a 3 years old car of the same make.

Sol. : **Calculations of regression**

Sr. No.	$X - \overline{X}$			$Y - \overline{Y}$			Product
	X	x	x^2	Y	y	y^2	xy
1	2	−5	25	1,600	−200	40,000	1000
2	4	−3	9	1,500	−300	90,000	900
3	6	−1	1	1,800	0	0	0
4	7	0	0	1,900	100	10,000	0
5	8	1	1	1,700	−100	10,000	−100
6	10	3	9	2,100	300	90,000	900
7	12	5	25	2,000	200	40,000	1000
$N = 7$	$\Sigma X = 49$		$\Sigma x^2 = 70$	$\Sigma Y =$ 12,600		$\Sigma y^2 =$ 2,80,000	$\Sigma xy = 3{,}700$

Let X denote age in years, Y denote cost in Rs.

Now $\overline{X} = \dfrac{\Sigma X}{N} = \dfrac{49}{7} = 7, \quad \overline{Y} = \dfrac{12600}{7} = 1{,}800$

and $b_{yx} = \dfrac{\Sigma xy}{\Sigma x^2} = \dfrac{3700}{40} = 52{\cdot}86$

The equation of the line of regression of Y on X is,

$$Y - \overline{Y} = b_{yx}(X - \overline{X}) \qquad \therefore \; Y - 1800 = 52{\cdot}86\,(X - 7)$$

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To find the value of Y when $X = 3$ put this value in the above equation,

$\therefore$ $Y - 1800 = 52{\cdot}86\,(3 - 7) = -211{\cdot}44$

$\therefore$ $Y = 1800 - 212.64 = 1588{\cdot}56$

$\therefore$ The cost of maintenance of 3 years old car = ₹ 1588·56.

Example 4 : Find the coefficients of regression and hence the equations of the lines of regression for the following data.

X : 78, 36, 98, 25, 75, 82, 90, 62, 65, 39.

Y : 84, 51, 91, 60, 68, 62, 86, 58, 53, 47.

Draw the lines of regression from your equations on the graph. Estimate the value of Y when $X = 50$ and the value of X when $Y = 90$ from the graph.

What is the significance of the point of intersection of the two lines?

Sol. : **Calculations of coefficients of regression**

Sr. No.	$X - \overline{X}$			$Y - \overline{Y}$			Product xy
	X	x	x^2	Y	y	y^2	
1	78	+ 13	169	84	+ 18	324	234
2	36	– 29	841	51	– 15	225	435
3	98	+ 33	1089	91	+ 25	625	825
4	25	– 40	1600	60	– 6	36	240
5	75	+ 10	100	68	+ 2	4	20
6	82	+ 17	289	62	– 4	16	– 68
7	90	+ 25	625	86	+ 20	400	500
8	62	– 3	9	58	– 8	64	24
9	65	0	0	53	– 13	169	0
10	39	– 26	676	47	– 19	361	494
$N = 10$	$\Sigma X = 650$		$\Sigma x^2 = 5398$	$\Sigma Y = 660$		$\Sigma y^2 = 2224$	$\Sigma xy = 2704$

(i) Now $\overline{X} = \dfrac{\Sigma X}{N} = \dfrac{650}{10} = 65$, $\overline{Y} = \dfrac{\Sigma Y}{N} = \dfrac{660}{10} = 66$.

Coefficient of regression of Y on X is,

$$b_{yx} = \frac{\Sigma xy}{\Sigma x^2} = \frac{2704}{5398} = 0{\cdot}5009$$

Coefficient of regression of X on Y is,

$$b_{xy} = \frac{\Sigma xy}{\Sigma y^2} = \frac{2704}{2224} = 1{\cdot}215$$

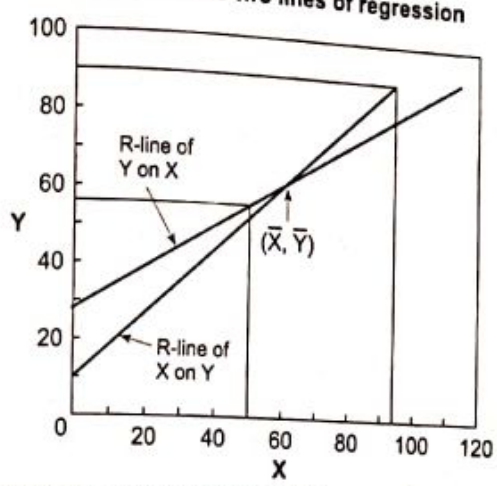
The equation of the line of regression of Y on X is,

$$Y - \overline{Y} = b_{yx}(X - \overline{X}) \qquad \therefore \quad Y - 66 = 0{\cdot}5\,(X - 65)$$

The equation of the line of regression of X on Y is,

$$X - \overline{X} = b_{xy}(Y - \overline{Y}) \qquad \therefore \quad X - 65 = 1{\cdot}2\,(Y - 66)$$

(ii) **Graph of the two lines of regression**



To draw the lines of regression we take two points on each and connect them by a straight line.

From $Y - 66 = 0{\cdot}5\,(X - 65)$

we find that when $X = 65$, $Y = 66$

and when $X = 0$, $Y = 27{\cdot}5$

From $X - 65 = 1{\cdot}2\,(Y - 66)$

we find that when $X = 65$, $Y = 66$

and when $X = 0$, $Y = 12$.

To estimate Y when $X = 50$ we draw a line at $X = 50$, parallel to y-axis to meet the line of regression of Y on X and read the corresponding value of Y. It is 58. To estimate X when $Y = 90$, we draw a line at $Y = 90$, parallel to x-axis, to meet the line of regression of X on Y and read the corresponding value of X. It is 94.

The two lines intersect at the point for which $X = \overline{X} = 65$ and $Y = \overline{Y} = 66$.

Example 5 : A chemical engineer is investigating the effect of process operating temperature X on product yield Y. The study results in the following data.

X : 100, 110, 120, 130, 140, 150, 160, 170, 180, 190.

Y : 45, 51, 54, 61, 66, 70, 74, 78, 85, 89.

Find the equation of the least square line which will enable to predict yield on the basis of temperature. Find also the degree of relationship between the temperature and the yield.

(M.U. 2004, 06, 19)

Also verify that the sum of the coefficients of regression is greater than $2r$.

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Sol. : Calculations of b_{xy}, b_{yx} etc.

Sr. No.	dx			dy			$d_x d_y$
	X	$X-150$	d_x^2	Y	$X-70$	d_y^2	
1	100	– 50	2500	45	– 25	625	1250
2	110	– 40	1600	51	– 19	361	760
3	120	– 30	900	54	– 16	256	480
4	130	– 20	400	61	– 9	81	180
5	140	– 10	100	66	– 4	16	40
6	150	00	000	70	0	0	00
7	160	10	100	74	4	16	40
8	170	20	400	78	8	64	160
9	180	30	900	85	15	225	450
10	190	40	1600	89	19	361	760
$N = 10$		– 50	8500		– 27	2005	4120

$$\overline{X} = A + \Sigma\frac{dx}{N} = 150 - \frac{50}{10} = 145; \qquad \overline{Y} = B + \Sigma\frac{dy}{N} = 70 - \frac{27}{10} = 67{\cdot}3$$

$$b_{yx} = \frac{\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{N}}{\Sigma d_x^2 - \frac{(\Sigma d_x)^2}{N}} = \frac{4120 - \frac{(-50)(-27)}{10}}{8500 - \frac{(-50)^2}{10}} = \frac{4120 - 135}{8500 - 250} = \frac{3985}{8250} = 0{\cdot}483$$

$$b_{xy} = \frac{\Sigma d_x d_y - \frac{\Sigma d_x \Sigma d_y}{N}}{\Sigma d_y^2 - \frac{(\Sigma d_y)^2}{N}} = \frac{4120 - \frac{(-50)(-27)}{10}}{2005 - \frac{(-27)^2}{10}} = \frac{4120 - 135}{2005 - 72{\cdot}9} = \frac{3985}{1932{\cdot}1} = 2{\cdot}06$$

The line of regression of Y on X is

$$Y - \overline{Y} = b_{yx}(X - \overline{X})$$

$\therefore \quad Y - 67{\cdot}3 = 0{\cdot}483\,(X - 145) \qquad \therefore \quad Y = 0{\cdot}483X - 2{\cdot}735$

The coefficient of correlation

$$r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{0{\cdot}483 \times 2{\cdot}06} = 0{\cdot}9975$$

Now, $b_{xy} + b_{yx} = 2{\cdot}060 + 0{\cdot}483 = 2{\cdot}543$ and $2r = 2 \times 0{\cdot}9975 = 1{\cdot}995$

Hence, we see that $b_{yx} + b_{xy} > 2r$.

Miscellaneous Examples

Example 1 : Find the angle between the lines of regression using the following data.

$n = 10,\ \Sigma x = 270,\ \Sigma y = 630,\ \sigma_x = 4,\ \sigma_y = 5,\ r_{xy} = 0{\cdot}6.$ **(M.U. 1998)**

Sol. : The angle between the lines of regression is given by

$$\tan\theta = \left(\frac{1 - r^2}{r}\right)\left(\frac{\sigma_x \cdot \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$

Putting the given values

$$\tan\theta = \left(\frac{1 - 0{\cdot}6^2}{0{\cdot}6}\right)\left(\frac{4 \times 5}{16 + 25}\right) = 0{\cdot}52.$$

Example 2 : Discuss the statement "The sum of the two coefficients of regression is always greater than $2r$ where r is the coefficient of correlation". **(M.U. 1998)**

Sol. : We have proved above (§ 7, 3, page 6-9) that

$$b_{yx} + b_{xy} \geq 2r.$$

Because this statement leads us to

$$(\sigma_x - \sigma_y)^2 \geq 0 \quad \text{which is always true.} \qquad \text{..... (A)}$$

This means the above given statement is partially true. The sum of the two coefficients of regression is greater than $2r$ but not always. The sum can be also equal to $2r$.

The condition (A) shows that if $\sigma_x = \sigma_y$, then the sign of equality will hold and then $b_{yx} + b_{xy}$ will be equal to $2r$.

This is also clear otherwise. From (1) and (1') (page 6-8), we have

$$b_{yx} + b_{xy} = r\frac{\sigma_y}{\sigma_x} + r\frac{\sigma_x}{\sigma_y}$$

If $\sigma_x = \sigma_y$, we will get $b_{yx} + b_{xy} = 2r$.

Example 3 : Given the following results of weights X and heights Y of 1000 men

$\overline{X} = 150$ lbs. $\quad \sigma_x = 20$ lbs.

$\overline{Y} = 68$ inches, $\quad \sigma_y = 2{\cdot}5$ inches, $\quad r = 0{\cdot}6.$

where $\overline{X}$ and $\overline{Y}$ are means of X and Y, σ_x and σ_y are standard deviations of X and Y and r is the correlation coefficient between X and Y.

John weighs 200 lbs., Smith is five feet tall. Estimate the height of John and weight of Smith.

From the value of height of John estimate his weight. Why is it different from 200 ?

Sol. : With the given notation the line of regression of Y on X is

$$Y - \overline{Y} = r\frac{\sigma_y}{\sigma_x}(X - \overline{X})$$

Substituting the given values,

$$Y - 68 = 0{\cdot}6 \times \frac{2{\cdot}5}{20}(X - 150) = \frac{15}{200}(X - 150).$$

Put $X = 200$,

$$\therefore \quad Y - 68 = \frac{15}{200}(200 - 150) = \frac{15}{4} = 3{\cdot}75$$

$$\therefore \quad Y = 68 + 3{\cdot}75 = 71{\cdot}75 \text{ inches.}$$

Now the line of regression of X of Y is

$$X - \overline{X} = r\frac{\sigma_x}{\sigma_y}(Y - \overline{Y})$$

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Substituting the given values,

$$X - 150 = 0{\cdot}6 \times \frac{20}{2{\cdot}5}(Y - 68) = \frac{24}{5}(Y - 68)$$

Put Y = 5 feet = 60 inches.

$$\therefore \quad X - 150 = \frac{24}{5}(60 - 68) = -\frac{192}{5}$$

$$\therefore \quad X = 150 - \frac{192}{5} = 111{\cdot}6 \text{ lbs.}$$

Hence, height of John = 71·75 inches and weight of Smith = 111·6 lbs.

To estimate the weight of John from his height 71·25 we have to use the equation of line of regression of X on Y (and not Y on X).

i.e., $\quad X - 150 = \frac{24}{5}(Y - 68)$

Putting Y = 71·75, we get

$$X - 150 = \frac{24}{5}(3{\cdot}75) = 18 \quad \therefore \; X = 168.$$

The difference is due to the fact that for estimating Y we use one equation and for estimating X we use another equation.

Example 4 : Given $6Y = 5X + 90$, $15X = 8Y + 130$, $\sigma_x^2 = 16$.

Find (i) $\overline{X}$ and $\overline{Y}$, (ii) r and (iii) σ_y^2. **(M.U. 2009, 10, 18, 19)**

Sol. : (i) To find $\overline{X}$ and $\overline{Y}$: We solve the given equations simultaneously. Multiply the first equation by 3.

$\therefore \; -15X + 18Y = 270$ and add $15X - 8Y = 130$

$\therefore \; 10Y = 40 \qquad \therefore \; \overline{Y} = 40$

Putting this value in any of the given equations.

$$6 \times 40 = 5X + 90 \quad \therefore \; X = 30 \quad \therefore \; \overline{X} = 30.$$

(ii) To find r : Suppose the first equation represents, the line of regression of X on Y.

Writing it as $X = \frac{6}{5}Y - 18$, we find $b_{xy} = \frac{6}{5}$.

Suppose the second equation represents the line of regression of Y on X.

Writing it as $Y = \frac{15}{8}X - \frac{130}{8}$, we find $b_{yx} = \frac{15}{8}$.

$$\therefore \; r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\frac{6}{5} \times \frac{15}{8}} = \sqrt{\frac{9}{4}} = \sqrt{2{\cdot}25} = 1{\cdot}5.$$

But the value of r can never be greater than 1 numerically. Hence, our supposition is wrong.

Now treating the first equation as representing the line of regression of Y on X, we write it as.

$$Y = \frac{5}{6}X + 15 \quad \therefore \; b_{yx} = \frac{5}{6}.$$

Treating the second equation as representing the line of regression of X on Y, we write it as.

$$X = \frac{8}{15}Y + \frac{130}{15} \quad \therefore \; b_{xy} = \frac{8}{15}$$

$$\therefore \; r = \sqrt{b_{xy} \times b_{yx}} = \sqrt{\frac{8}{15} \times \frac{5}{6}} = \sqrt{\frac{4}{9}} = \frac{2}{3} = 0{\cdot}667.$$

(iii) To find σ_y. Consider, $\quad b_{yx} = r\frac{\sigma_y}{\sigma_x} \quad \therefore \; \frac{5}{6} = \frac{2}{3} \times \frac{\sigma_y}{4} \quad \therefore \; \sigma_y = 5.$

Example 5 : The equations of the two regression lines are $3x + 2y = 26$ and $6x + y = 31$.

Find : (i) the means of x and y, (ii) coefficient of correlation between x and y,

(iii) σ_y if $\sigma_x = 3$. **(M.U. 2007, 16)**

Sol. : (i) To find $\overline{X}$ and $\overline{Y}$

We solve the equations simultaneously. Multiply the second by 2 and subtract from the first.

$$\therefore \quad \begin{aligned} 3x + 2y &= 26 \\ 12x + 2y &= 62 \\ \hline 9x \quad &= 36 \end{aligned} \qquad \therefore \; x = 4.$$

Putting this value of x in the second equation, we get $24 + y = 31 \qquad \therefore \; y = 7.$

$\therefore \; \overline{X} = 4, \; \overline{Y} = 7.$

(ii) To find r : Suppose the first equation represents the line of regression of X on Y.

Writing it as $3x = -2y + 26. \qquad \therefore \; x = -\frac{2}{3}y + \frac{26}{3}$

$\therefore$ We find that $b_{xy} = -\frac{2}{3}$.

Suppose the second equation represents the line of regression of Y on X.

Writing it as $y = -6x + 31 \qquad \therefore \; b_{yx} = -6.$

$$\therefore \; r = \sqrt{b_{xy} \cdot b_{yx}} = \sqrt{(-2/3)(-6)} = \sqrt{4} = 2$$

But the value of r can never be greater than 1. Hence, our supposition is wrong.

Now, treating the first equation as representing the line of regression on of Y on X, we write it as

$$2y = -3x + 26 \quad \therefore \; y = -\frac{3}{2}x + 13 \quad \therefore \; b_{yx} = -\frac{3}{2}.$$

Treating the second equation as representing the line of regression of X on Y, we write it as

$$6x = -y + 31 \quad \therefore \; x = -\frac{1}{6}y + \frac{31}{6} \quad \therefore \; b_{xy} = -\frac{1}{6}.$$

$$\therefore \; r = \sqrt{b_{yx} \cdot b_{xy}} = \sqrt{\left(-\frac{3}{2}\right)\left(-\frac{1}{6}\right)} = \sqrt{\frac{1}{4}} = \frac{1}{2} = 0{\cdot}5$$

Since, both b_{xy} and b_{yx} are negative r is negative. $\quad \therefore \; r = -0{\cdot}5.$

(iii) To find σ_y : Consider $b_{yx} = r\frac{\sigma_y}{\sigma_x}$.

But $b_{yx} = -\frac{3}{2}$, $r = -\frac{1}{2}$, and $\sigma_x = 3$. $\quad \therefore \; \sigma_y = b_{yx} \cdot \frac{\sigma_x}{r} = \left(-\frac{3}{2}\right) \cdot \frac{3}{(-1/2)} = 9$

Example 6 : The regression lines of a sample are $x + 6y = 6$, and $3x + 2y = 10$. Find (i) sample means $\overline{x}$ and $\overline{y}$, (ii) coefficient of correlation between x and y. Also estimate y when $x = 12$. **(M.U. 2004, 14, 15)**

Also verify that the sum of the coefficients of regressions is greater than $2r$.

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Sol. : (i) Mean $\bar{x}$ and $\bar{y}$ are obtained by solving the two given equations.

$$3x + 18y = 18 \quad \therefore \; y = 1/2$$
$$3x + 2y = 10 \quad \therefore \; x = 3$$
$$16y = 8$$

(ii) If the line $x + 6y = 6$ is the line of regression of y on x, then

$$6y = -x + 6 \;\; i.e. \;\; y = -\frac{1}{6}x + 1 \quad \therefore \; b_{yx} = -\frac{1}{6}$$

If the line $3x + 2y = 10$ is the line of regression of x on y, then

$$3x = -2y + 10 \;\; i.e. \;\; x = -\frac{2}{3}y + \frac{10}{3} \quad \therefore \; b_{xy} = -\frac{2}{3}$$

$$\therefore \; r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{\left(-\frac{1}{6}\right) \times \left(-\frac{2}{3}\right)} = \sqrt{\frac{1}{9}} = \frac{1}{3}$$

Since b_{yx} and b_{xy} are both negative, r is negative $\quad \therefore \; r = -1/3$.

Since $\quad b_{yx} + b_{xy} = \frac{1}{6} + \frac{2}{3} = \frac{5}{6}$ (Numerically)

and $\; 2r = \frac{2}{3}$, we see that $b_{yx} + b_{xy} > 2r$.

(iii) To estimate y when $x = 12$, we use the line of regression of y on x *i.e.* $y = -\frac{1}{6}x + 1$, when $x = 12$, $y = -2 + 1 = -1$.

Example 7 : If the tangent of the angle made by the line of regression of y on x is 0.6 and $\sigma_y = 2\,\sigma_x$, find the correlation coefficient between x and y. **(M.U. 2004, 09, 10, 15)**

Sol. : If the equation of the line of regression of y on x is $y - \bar{y} = b_{yx}(x - \bar{x})$ then we know that b_{yx} is the slope of the line of regression. We are thus, given $b_{yx} = 0{\cdot}6$.

But $\quad b_{yx} = r\frac{\sigma_y}{\sigma_x}$ and $\sigma_y = 2\sigma_x$

Putting these value, $\quad 0{\cdot}6 = r \cdot \frac{2\sigma_x}{\sigma_x} = 2r \quad \therefore \; r = \frac{0{\cdot}6}{2} = 0{\cdot}3.$

Example 8 : If $\sigma_x = \sigma_y = \sigma$ and the angle between the lines of regression is $\tan^{-1} 3$, find the coefficient of correlation. **(M.U. 2004)**

Sol. : We have

$$\tan\theta = \frac{1 - r^2}{r}\left(\frac{\sigma_x \sigma_y}{\sigma_x^2 + \sigma_y^2}\right) \qquad \therefore \; 3 = \frac{1 - r^2}{r} \cdot \left(\frac{\sigma^2}{\sigma^2 + \sigma^2}\right) = \frac{1 - r^2}{2r}$$

$$\therefore \; \frac{1 - r^2}{r} = 6 \quad \therefore \; r^2 + 6r - 1 = 0 \quad \therefore \; r = \frac{-6 \pm \sqrt{36 - 4}}{2} = -3 \pm 2\sqrt{2}$$

Since r cannot be numerically greater than 1, $r = -3 + 2\sqrt{2} = -0{\cdot}17$.

Example 9 : The following data regarding the heights (y) and weights (x) of 100 college students are given

$$\Sigma x = 15000, \; \Sigma x^2 = 2272500, \; \Sigma y = 6800, \; \Sigma y^2 = 463025, \; \Sigma xy = 1022250.$$

Find the coefficient of correlation between height and weight and also the equation of regression of height and weight. **(M.U. 1996)**

Sol. : The coefficients of regression are given by

$$b_{yx} = \frac{\Sigma xy - \frac{\Sigma x \cdot \Sigma y}{N}}{\Sigma x^2 - \frac{(\Sigma x)^2}{N}} = \frac{1022250 - \frac{15000 \times 6800}{100}}{2272500 - \frac{15000^2}{100}} = \frac{2250}{22500} = 0{\cdot}1.$$

$$b_{xy} = \frac{\Sigma xy - \frac{\Sigma x \cdot \Sigma y}{N}}{\Sigma y^2 - \frac{(\Sigma y)^2}{N}} = \frac{1022250 - \frac{15000 \times 6800}{100}}{463025 - \frac{6800^2}{100}} = \frac{2250}{625} = 3{\cdot}6.$$

$$\therefore \; r = \sqrt{b_{yx} \times b_{xy}} = \sqrt{0{\cdot}1 \times 3{\cdot}6} = 0{\cdot}6.$$

The equation of the lines of regression of y on x is

$$y - \bar{y} = b_{yx}(x - \bar{x}) \quad \therefore \; y - 68 = 0{\cdot}1(x - 1500) \quad \therefore \; y = 0{\cdot}1x - 82.$$

Example 10 : It is given that the means of x and y are 5 and 10. If the line of regression of y on x is parallel to the line $20y = 9x + 40$, estimate the value of y for $x = 30$. **(M.U. 1998, 2015)**

Sol. : The line of regression of y on x is $y - \bar{y} = b_{yx}(x - \bar{x})$.

Its slope is b_{yx}. But this line is parallel to $20y = 9x + 40$

i.e. $\quad y = \frac{9}{20}x + 2$ whose slope is $\frac{9}{20}$. $\quad \therefore \; b_{yx} = \frac{9}{20}$.

But by data $\bar{x} = 5$ and $\bar{y} = 10$. Hence, the equation of the line of regression of y on x is

$$y - 10 = \frac{9}{20}(x - 5) \;\; i.e. \;\; y = \frac{9}{20}x + \frac{155}{20}$$

When $x = 30$, $\quad y = \frac{270}{20} + \frac{155}{20} = \frac{425}{20} = 21{\cdot}25.$

Example 11 : If the arithmetic mean of regression coefficients is p and their difference is $2q$, find the correlation coefficient. **(M.U. 1998)**

Sol. : Let the coefficients of regression be b_1 and b_2.

Now by data $\quad \frac{b_1 + b_2}{2} = p$ and $b_1 - b_2 = 2q$

$\therefore \quad b_1 + b_2 = 2p$ and $b_1 - b_2 = 2q$

$\therefore \quad b_1 = p + q$ and $b_2 = p - q$

$\therefore$ Coefficient of correlation $= r = \sqrt{b_1 b_2} = \sqrt{p^2 - q^2}$

EXERCISE - I

Type I

State true or false with proper reasoning.

1. If $r = 0$, the lines of regression are parallel to each other. **[Ans. : False]**
2. The values of r and R can never be equal. **[Ans. : False]**

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3. In a regression analysis it was found that $b_{yx} = 0{\cdot}87$, $b_{xy} = 1{\cdot}55$. These values are not consistent. [**Ans.** : True]

4. The two regression coefficients are both positive or both negative. [**Ans.** : True]

5. $3x + y = 5$ and $2x - 3y = 7$ cannot be lines of regression for any set of values of x and y. [**Ans.** : True]

Type II

1. The following table gives the age of car of a certain make and annual maintenance cost. Obtain the equation of the line of regression of cost on age.

Age of a car :	2	4	6	8
Maintenance :	1	2	2·5	3

(M.U. 1998, 2014) [**Ans.** : $y = 0{\cdot}325x + 0{\cdot}5$]

2. Obtain the equation of the line of regression of y on x from the following data and estimate y for $x = 73$.

x :	70,	72,	74,	76,	78,	80.
y :	163,	170,	179,	188,	196,	220.

(M.U. 1997)

[**Ans.** : $y = 5{\cdot}31x - 212.57$; $y = 175{\cdot}37$]

3. The heights in cms of fathers (x) and of the eldest sons (y) are given below.

x :	165	160	170	163	173	158	178	168	173	170	175	180
y :	173	168	173	165	175	168	173	165	180	170	173	178

Estimate the height of the eldest son if the height of the father is 172 cms. and the height of the father if the height of the eldest son is 173 cm.

Also find the coefficient of correlation between the heights of fathers and sons.

(M.U. 2002, 05)

[**Ans.** : **(i)** $y = 1{\cdot}016x - 5{\cdot}123$, **(ii)** $x = 0{\cdot}476y + 98{\cdot}98$, **(iii)** 169·97, 173·45, **(iv)** $r = 0{\cdot}696$]

4. Find (i) the lines of regression, (ii) coefficient of correlation for the following data.

x :	65	66	67	67	68	69	70	72
y :	67	68	65	66	72	72	69	71

(M.U. 2002, 14, 19)

[**Ans.** : $y = 19{\cdot}79 + 0{\cdot}72x$, **(ii)** $x = 33{\cdot}29 + 0.5y$; $r = 0{\cdot}6$]

5. Find the line of regression for the following data and estimate y corresponding to $x = 15{\cdot}5$.

x :	10	12	13	16	17	20	25
y :	19	22	24	27	29	33	37

(M.U. 2004, 19) [**Ans.** : $y = 0{\cdot}8x + 13{\cdot}23$; 25·63]

Type III

1. Given

	x series	y series
Mean	18	100
S.D.	14	20

$r = 0{\cdot}8$.

Find the most probable value of y when $x = 70$ and most probable value of x when $y = 90$.

[**Ans.** : $y = 159{\cdot}3$, $x = 12{\cdot}4$]

2. Given the following information about marks of 60 students.

	Mathematics	English
Mean	80	50
S.D.	15	10

Coefficient of correlation $r = 0{\cdot}4$. Estimate the marks of the student in mathematics who scored 60 marks in English.

(M.U. 2006) [**Ans.** : 86]

3. You are supplied with the following information. The equation of the lines regression are $2x + 3y + 8 = 0$ and $x + 2y - 5 = 0$.

Find the means of x and y and the coefficient of correlation between them. **(M.U. 1997)**

[**Ans.** : $\bar{x} = -31$, $\bar{y} = 18$, $r = -0{\cdot}87$]

4. From 8 observations the following results were obtained :

$\Sigma x = 59$, $\Sigma y = 40$, $\Sigma x^2 = 524$, $\Sigma y^2 = 256$, $\Sigma xy = 364$.

Find the equation of the line of regression of x on y and the coefficient of correlation.

(M.U. 2018) [**Ans.** : $x = 1{\cdot}5y - 0{\cdot}5$, $r = 0{\cdot}98$]

5. The equations of the two lines of regression are $x = 19{\cdot}13 - 0{\cdot}87y$ and $y = 11{\cdot}64 - 0{\cdot}50x$. Find (i) the means of x and y, (ii) the coefficient of correlation between x and y.

(M.U. 2004) [**Ans.** : **(i)** $\bar{x} = 15{\cdot}79$, $\bar{y} = 3{\cdot}74$; **(ii)** $r = -0{\cdot}66$, $b_{yx} = -0{\cdot}50$, $b_{xy} = -0{\cdot}87$]

6. Out of the two equations given below which can be a line of regression of x on y and why ?

$x + 2y - 6 = 0$ and $2x + 3y - 8 = 0$. **(M.U. 2003)** [**Ans.** : $2x + 3y - 8 = 0$]

7. In a partially destroyed laboratory record of analysis of correlation data the following results are legible. Variance of $x = 9$, equations of the lines of regression

$4x - 5y + 33 = 0$, $20x - 9y - 107 = 0$.

Find (i) the mean values of x and y, (ii) the standard deviation of y, (iii) coefficient of correlation between x and y. **(M.U. 1999, 2003, 19)** [**Ans.** : **(i)** $\bar{x} = 13$, $\bar{y} = 17$, **(ii)** $\sigma_y = 4$, **(iii)** $r = 0{\cdot}6$]

8. Given : var (x) = 25. The equations of the two lines of regression are $5x - y = 22$ and $64x - 45y = 24$.

Find (i) $\bar{x}$ and $\bar{y}$, (ii) r, (iii) σ_y. **(M.U. 1998)**

[**Ans.** : **(i)** $\bar{x} = 6$, $\bar{y} = 8$, **(ii)** $r = 1{\cdot}87$, **(iii)** $\sigma_y = 1/5$]

9. Find the regression coefficients and the coefficient of correlation from the following data where x, y denote the actual values.

$N = 12$, $\Sigma x = 120$, $\Sigma y = 432$, $\Sigma xy = 4992$, $\Sigma x^2 = 1392$, $\Sigma y^2 = 18252$.

[**Ans.** : $b_{yx} = 3{\cdot}5$, $b_{xy} = 0{\cdot}249$, $r = 0{\cdot}93$]

EXERCISE - II

Theory

1. Distinguish between correlation and regression.

2. Explain "the line of regression". Why there are two lines of regression ? **(M.U. 2007)**

3. Explain what you understand by regression. What are lines of regression ? Why are there in general two lines of regression ? When do they coincide, when are they perpendicular ? **(M.U. 2004)**

4. Explain the method of scatter diagram to obtain a line of regression. **(M.U. 2002, 03)**

5. Obtain the equations of lines of regression.

6. Prove that the sum of the coefficients of regression is greater than or equal to $2r$ where r is the coefficient of correlation. [See 3, page 6-9] **(M.U. 1998)**

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7. Find the expression for the acute angle between the lines of regression. **(M.U. 2004, 05)**

8. With usual notation prove that (i) $r = \sqrt{b_{yx} \cdot b_{xy}}$. (ii) $b_{xy} + b_{yx} \geq 2r$.

9. Examine whether the following statement is correct

$b_{xy} = 3{\cdot}2,\ b_{yx} = 0{\cdot}7.$ **[Ans. : No]**

10. If θ is the angle between the two lines of regression, prove that

$$\tan\theta = \left(\frac{1-r^2}{r}\right)\left(\frac{\sigma_x - \sigma_y}{\sigma_x^2 + \sigma_y^2}\right)$$ **(M.U. 1996, 2004, 07)**

8. Fitting of Curves : Introduction

In many social, economical, engineering and physical problems we have a set of values of x and y although we do not know the functional relationship between them. Fitting of a curve to a given set of values means finding a functional relationship between x and y whose curve is the closest possible curve to the given values. The curve so obtained does not pass through all the given points but is close to them to the maximum extent. Finding such a curve for a given set of values is called **Curve Fitting**. The relation in general is assumed to be a linear function $y = a + bx$ or a parabolic function $y = a + bx + cx^2$ or even exponential or logarithmic. The method used for fitting the curve is based on the **principle of least squares**.

9. Fitting a Straight Line by the Method of Least Squares

Suppose we have a set of values (x_i, y_i). Suppose further that we want to fit a straight line $y = a + bx$ to these values. The straight line must be close to the given points to the maximum extent. **The principle of least square states that the straight line should be such that the distances of the given points from the straight line measured along the y-axis must be minimum.** The line obtained in this way is called **the line of best fit**.

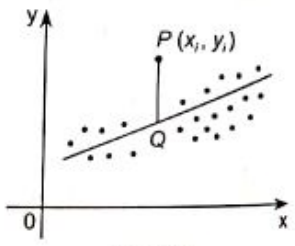

Fig. 6.6

Suppose $P(x_i, y_i)$ is a given point and suppose the equation of the line of best fit be $y = a + bx$. Suppose further that the line parallel to the y-axis through P intersects the line in Q. Now, the coordinates of Q are $(x_i, a + bx_i)$. We want to find a, b such that the distance $|PQ|$ is minimum. But the distance $|PQ|$ is minimum when the square of the distance $(y_i - a - bx_i)^2$ is minimum. This must be true for all points. This means we should have $S \equiv \Sigma (y_i - a - bx_i)^2$ minimum. Since we find a, b such that the sum of the squares S is least, the method is known as the method of least squares.

Now, for S to be minimum the conditions are

$$\frac{\partial S}{\partial a} = 0 \quad \text{and} \quad \frac{\partial s}{\partial b} = 0$$

$\therefore\ \Sigma (y_i - a - bx_i) = 0$ and $\Sigma (y_i - a - bx_i)\, x_i = 0$ $\quad\therefore\ \Sigma y_i = \Sigma a - b \Sigma x_i$

$\therefore\ \Sigma y_i = Na + b \Sigma x_i$ and $\Sigma y_i x_i = a \Sigma x_i + b \Sigma x_i^2$

We shall drop the suffix i and write the equations as

$$\boxed{\begin{aligned} \Sigma y &= Na + b \Sigma x \\ \Sigma xy &= a \Sigma x + b \Sigma x^2 \end{aligned}}$$

These equations are known as **Normal Equations**.

Methods of finding the constants a

1. **Direct Method :** If we find Σx, Σy, etc. and substitute their values in the above equations, we get two equations in two unknowns a and b. Solving these two equations we can find the constants a and b.

2. **Assumed Mean Method :** We may assume a certain mean of x and of y, *i.e.*, we may put $X = x - m$ and $Y = y - n$ and find ΣX, ΣY, etc. put these values in the above two equations.

If necessary, we may divide the deviations by a common factor, if there is any, *i.e.*, we may put $X = \dfrac{x-m}{h}$ and $Y = \dfrac{y-n}{k}$.

Solving these equations, we get a and b and resubstituting the values of X and Y, we get the equation of the line.

3. **Actual Mean Method :** If actual means of x and y are round figures, we put $X = x - \bar{x}$ and $Y = y - \bar{y}$ $\left(\text{or } X = \dfrac{x-\bar{x}}{h},\ Y = \dfrac{y-\bar{y}}{k}\right)$, find ΣX, ΣY, etc. and put these values in the above two equations.

Solving these equations as above and resubstituting the values of X and Y, we get the equation of the line.

Of these three methods we shall see below that the last method is the simplest one.

Example 1 : Fit a straight line to the following data :

(x, y) : (1, 1), (2, 5), (3, 11), (4, 8), (5, 14)

Sol. : We shall obtain the line from the above normal equations.

Calculations of Σx, Σx^2 etc.

x	y	x^2	xy
1	1	1	1
2	5	4	10
3	11	9	33
4	8	16	32
5	14	25	70
$N = 5$, 15	39	55	146

The normal equations are

$\Sigma y = Na + b \Sigma x \quad \therefore\ 39 = 5a + 15b$ ..... (1)

and $\Sigma xy = a \Sigma x + b \Sigma x^2 \quad \therefore\ 146 = 15a + 55b$ ..... (2)

Multiply (1) by 3,

$\therefore\ 117 = 15a + 45b$ and $146 = 15a + 55b$

By subtraction $10b = 29 \quad \therefore\ b = 2{\cdot}9.$

Putting this value of b in (1),

$5a = 39 - 15 \times 2{\cdot}9 = -4{\cdot}5$

$\therefore\ a = -\dfrac{4{\cdot}5}{5} = -0{\cdot}9.$

Hence, the equation of the line is $y = -0{\cdot}9 + 2{\cdot}9x$.

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Example 2 : Fit a straight line $y = a + bx$ to the following data.

x :	0	5	10	15	20	25
y :	12	15	17	22	24	30

Sol. : (1) Direct Method

Calculations of Σx, Σx^2 etc.

x	y	x^2	xy
0	12	0	0
5	15	25	75
10	17	100	170
15	22	225	330
20	24	400	480
25	30	625	750
$N = 6, \Sigma x = 75$	$\Sigma y = 120$	$\Sigma x^2 = 1375$	$\Sigma xy = 1805$

Let the equation be $y = a + bx$.

The normal equations are

$\Sigma y = Na + b\Sigma x$ $\quad\therefore\ 120 = 6a + 75b$ ..... (1)

and $\Sigma xy = a\Sigma x + b\Sigma x^2$ $\quad\therefore\ 1805 = 75a + 1375b$ ..... (2)

Now, to find a, b multiply (1) by 50 and (2) by 4 and subtract.

$$\begin{aligned} 6000 &= 300\,a + 3750\,b \\ 7220 &= 300\,a + 5500\,b \\ \hline 1220 &= \qquad\quad 1750\,b \end{aligned}$$

$\therefore\ b = \dfrac{1220}{1750} = 0{\cdot}6971 = 0{\cdot}7$

Putting this value of b in (1), we get

$120 = 6a + 75\,(0{\cdot}7)$ $\quad\therefore\ 6a = 120 - 52{\cdot}5 = 67{\cdot}5$ $\quad\therefore\ a = 11{\cdot}25.$

$\therefore$ The equation of the line is $y = 11{\cdot}25 + 0{\cdot}7x$.

(2) Assumed Mean Method : We shall assume 10 and 17 as means of x and y, *i.e.*, we shall put $X = x - 10$ and $Y = y - 17$.

Calculations of ΣX, ΣX^2 etc.

x	y	$X = x - 10$	$Y = y - 17$	X^2	XY
0	12	− 10	− 5	100	50
5	15	− 5	− 2	25	10
10	17	0	0	0	0
15	22	5	5	25	25
20	24	10	7	100	70
25	30	15	13	225	195
$N = 6$		15	18	475	350

Let the equation be $Y = a + bX$.

The normal equations are

$\Sigma Y = Na + b\Sigma X$ $\quad\therefore\ 18 = 6a + 15b$ ..... (1)

and $\Sigma XY = a\Sigma X + b\Sigma X^2$ $\quad\therefore\ 350 = 15a + 475$ ..... (2)

Multiply (1) by 5 and (2) by 2 and subtract.

$$\begin{aligned} \therefore\quad 90 &= 30\,a + 75\,b \\ 700 &= 30\,a + 950\,b \\ \hline 610 &= \qquad\quad 875\,b \end{aligned}$$

$\therefore\ b = \dfrac{610}{875} = 0{\cdot}7$

Putting this value of b in (1), we get

$18 = 6a + 15\,(0{\cdot}7)$ $\quad\therefore\ 6a = 18 - 10{\cdot}5$ $\quad\therefore\ a = 1{\cdot}25.$

$\therefore$ The equation is $Y = 1{\cdot}25 + 0{\cdot}7X$.

Resubstituting $Y = y - 17$ and $X = x - 10$, we get

$y - 17 = 1{\cdot}25 + 0{\cdot}7\,(x - 10)$ $\quad\therefore\ y = 0{\cdot}7x + 11{\cdot}25.$

Example 3 : Fit a straight line $y = a + bx$ to the following data.

x :	1	2	3	4	5	6
y :	49	54	60	73	80	86

(M.U. 2014)

Sol. : We shall solve this example by all the three methods.

(a) Direct method :

Calculations of Σx, Σx^2 etc.

x	y	x^2	xy
1	49	1	49
2	54	4	108
3	60	9	180
4	73	16	292
5	80	25	400
6	86	36	516
$N = 6$, 21	402	91	1545

Now, $y = a + bx$

$\Sigma y = Na + b\Sigma x$ $\quad\therefore\ 402 = 6a + 21b$ ..... (1)

and $\Sigma xy = a\Sigma x + b\Sigma x^2$ $\quad\therefore\ 1545 = 21a + 91b$ ..... (2)

Multiply (1) by 7 and (2) by 2 and subtract

$$\begin{aligned} \therefore\quad 2814 &= 42\,a + 147\,b \\ 3090 &= 42\,a + 182\,b \\ \hline 276 &= \qquad\quad 35\,b \end{aligned}$$

$\therefore\ b = 7{\cdot}89$

Putting this value of b in (1), we get

$402 = 6a + 21\,(7{\cdot}89)$

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$\therefore\ 6a = 402 - 165{\cdot}69 = 236{\cdot}31 \qquad \therefore\ a = 39{\cdot}38$

$\therefore\ y = 39{\cdot}38 + 7{\cdot}89x.$

(b) Assumed Mean Method : We shall assume 3 and 60 as means of x and y and find the deviations.

Calculations of $\Sigma X, \Sigma X^2$ etc.

x	y	$X = x - 3$	$Y = y - 60$	X^2	XY
1	49	– 2	– 11	4	22
2	54	– 1	– 6	1	6
3	60	0	0	0	0
4	73	1	13	1	13
5	80	2	20	4	40
6	86	3	26	9	78
$N = 6$		3	42	19	159

Now, $Y = a + bX$

$\Sigma Y = Na + b\Sigma X \qquad \therefore\ 42 = 6a + 3b$ ..... (1)

and $\Sigma XY = a\Sigma X + b\Sigma X^2 \qquad \therefore\ 159 = 3a + 19b$ ..... (2)

Multiply (2) by 2 and subtract (1) from the result.

$$\begin{aligned} \therefore\ 318 &= 6a + 38b \\ 42 &= 6a + 3b \\ \hline 276 &= 35b \end{aligned} \qquad \therefore\ b = 7{\cdot}89$$

Putting this value of b in (1),

$42 = 6a + 3\,(7{\cdot}89) \qquad \therefore\ 6a = 18{\cdot}33 \qquad \therefore\ a = 3{\cdot}05$

$\therefore\ Y = 3.05 + 7.89\,X.$

Putting $X = x - 3,\ Y = y - 60$, we get

$y - 60 = 3{\cdot}05 + 7{\cdot}89\,(x - 3)$

$\therefore\ y = 39{\cdot}38 + 7{\cdot}89\,x.$

(c) Actual Mean method : We shall take deviations from actual means of x and y.

Now, $\bar{x} = \dfrac{21}{6} = 3{\cdot}5,\ \ \bar{y} = \dfrac{402}{6} = 67.$

Calculations of $\Sigma X, \Sigma X^2$ etc.

x	y	$X = x - 3{\cdot}5$	$Y = y - 67$	X^2	XY
1	49	– 2·5	– 18	6·25	45·0
2	54	– 1·5	– 13	2·25	19·5
3	60	– 0·5	– 7	0·25	3·5
4	73	0·5	6	0·25	3·0
5	80	1·5	13	2·25	19·5
6	86	2·5	19	6·25	47·5
$N = 6$		0	0	17·5	138·0

Now, $Y = a + bX$

$\therefore\ \Sigma Y = Na + b\Sigma X$ and $\Sigma XY = a\Sigma X + b\Sigma X^2$

Putting the above values

$0 = 6a + b\,(0) \qquad \therefore\ a = 0$

and $138 = a\,(0) + 17{\cdot}5\,b \qquad \therefore\ b = 7{\cdot}89$

Hence, the equation is $Y = 7{\cdot}89X.$

Putting the values of X, Y, we get

$y - 67 = 7{\cdot}89\,(x - 3{\cdot}5) \qquad \therefore\ y = 39{\cdot}38 + 7{\cdot}89x.$

Note ....

Thus, we see that if we take deviations from $\bar{x}$ and $\bar{y}$, we get $a = 0$ and we get the value of b directly. In this way the last method is more convenient.

Example 4 : Fit a straight line to the following data.

Year x	:	1951	1961	1971	1981	1991
Production y (000 tons)	:	10	12	8	10	15

Also estimate the production in 1987. **(M.U. 2013)**

Sol. : Now the means of x and y are $\bar{x} = 1971$ and $\bar{y} = 11$.

Since $\bar{x}$ and $\bar{y}$ are round figures, we shall take deviations from 1971 and 11.

Calculations of $\Sigma X, \Sigma Y$ etc.

x	y	$X = x - 1971$	$Y = y - 11$	X^2	XY
1951	10	– 20	– 1	400	20
1961	12	– 10	1	100	– 10
1971	8	0	– 3	0	0
1981	10	10	– 1	100	– 10
1991	15	20	4	400	80
$N = 5$		0	0	1000	80

Now, $Y = a + bX$

$\therefore\ \Sigma Y = Na + b\Sigma X$ and $\Sigma XY = a\Sigma X + b\Sigma X^2$

Putting the above values

$0 = 5a + 0 \qquad \therefore\ a = 0$

and $80 = 0 + 1000\,b \qquad \therefore\ b = 0{\cdot}08$

Hence, the equation is $Y = 0{\cdot}08X.$

Putting the values of X, Y, we get

$y - 11 = 0.08\,(x - 1971)$

$\therefore\ y = -\,146{\cdot}68 + 0{\cdot}08x.$

Putting $x = 1987$,

$y = -\,146{\cdot}68 + 0{\cdot}08\,(1987) = 12{\cdot}28.$

$\therefore$ The production in 1987 is 12·28.

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Example 5 : Fit a straight line of the form $y = a + bx$ to the following data and estimate the value of y for $x = 3{\cdot}5$.

x	:	0	1	2	3	4
y	:	1	1·8	3·3	4·5	6·3

(M.U. 2018)

Sol. : We shall solve this problem by both the methods *i.e.*, by direct method and also by deviation method.

(a) Direct method :

Calculations of Σx, Σx^2 etc.

x	y	x^2	xy
0	1·0	0	0·0
1	1·8	1	1·8
2	3·3	4	6·6
3	4·5	9	13·5
4	6·3	16	25·2
$N = 5$, 10	16·9	30	47·1

Let the equation of the line be $y = a + bx$.

Then the normal equations are

$\Sigma y = Na + b\Sigma x \quad \therefore \; 16{\cdot}9 = 5a + 10b$ ..... (1)

and $\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \therefore \; 47{\cdot}1 = 10a + 30b$ ..... (2)

Now, multiply (1) by 2 and subtract the result from (2).

$$\begin{aligned} \therefore \; 47{\cdot}1 &= 10a + 30b \\ 33{\cdot}8 &= 10a + 20b \\ \hline 13{\cdot}3 &= \quad\;\; 10b \end{aligned} \qquad \therefore \; b = 1{\cdot}33$$

Putting this value of b in (1), we get

$16{\cdot}9 = 5a + 13{\cdot}3 \quad \therefore \; 5a = 3{\cdot}6 \quad \therefore \; a = 0{\cdot}72$

Hence, the equation of the line is

$y = 0{\cdot}72 + 1{\cdot}33x$.

(b) Deviation method : We shall take deviations of x from the mid-point of the values of x.

x	$X = x - 2$	$Y = y$	X^2	XY
0	– 2	1.0	4	– 2·0
1	– 1	1.8	1	– 1·8
2	0	3.3	0	0·0
3	1	4.5	1	4·5
4	2	6.3	4	12·6
$N = 5$	0	16.9	10	13·3

Let the equation of the line be $Y = a + bX$.

The normal equations are

$\Sigma Y = Na + b\Sigma X$ and $\Sigma XY = a\Sigma X + b\Sigma X^2$

Putting the values from the table, we get,

$16{\cdot}9 = 5a + b\,(0) \qquad \therefore \; a = \dfrac{16{\cdot}9}{5} = 3{\cdot}38.$

And $13{\cdot}3 = a\,(0) + 10\,b \qquad \therefore \; b = 1{\cdot}33.$

Hence, the equation of the line is $Y = 3{\cdot}38 + 1{\cdot}33X$.

Putting $X = x - 2$ and $Y = y$ we get

$y = 3{\cdot}38 + 1{\cdot}33\,(x - 2) = 3{\cdot}38 - 2{\cdot}66 + 1{\cdot}33\,x$

$\therefore \; y = 0{\cdot}72 + 1{\cdot}33\,x$

Thus, the equation of the line is $y = 0{\cdot}72 + 1{\cdot}33x$.

Putting $x = 3{\cdot}5$, $y = 0{\cdot}72 + 4{\cdot}655 = 5{\cdot}375$.

Hence, when $x = 3{\cdot}5$, $y = 5{\cdot}375$.

Example 6 : Fit a straight line to the following data, with x as independent variable.

x	:	1965	1966	1967	1968	1969
y	:	125	140	165	195	200

(M.U. 2014)

Sol. : The means of x and y are $\bar{x} = 1967$, $\bar{y} = 165$.

Since $\bar{x}$ and $\bar{y}$ are round figures, we shall take deviations from 1967 and 165.

Calculations of ΣX, ΣY etc.

x	y	$X = x - 1967$	$Y = y - 165$	X^2	XY
1965	125	– 2	– 40	4	80
1966	140	– 1	– 25	1	25
1967	165	0	0	0	0
1968	195	1	30	1	1
1969	200	2	35	4	70
$N = 5$		0	0	10	176

Now, $Y = a + bX$

$\therefore \; \Sigma Y = Na + b\Sigma X$ and $\Sigma XY = a\Sigma X + b\Sigma X^2$

Putting the above values

$0 = 5a + 0 \qquad \therefore \; a = 0$

and $176 = 0 + 10\,b \qquad \therefore \; b = 17{\cdot}6$

Hence, the equation of line is $Y = 17{\cdot}6X$.

Putting the values of X and Y, we get

$$\begin{aligned} y - 165 &= 17{\cdot}6\,(x - 1967) \\ &= 165 + 17{\cdot}6\,x - 34619{\cdot}2 \\ \therefore \quad y &= 17.6x - 34454{\cdot}2. \end{aligned}$$

Example 7 : Fit a straight line to the following data.

x	:	1	2	3	4	5	6
y	:	83	92	71	90	160	191

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Sol. : Now, the mean of $x = 3{\cdot}5$. Since the deviations will be in decimals, we multiply them by 2, *i.e.*, we put $2(x - 35) = X$.

The mean of $y = 114{\cdot}5$.

Since, the deviations will be in decimals, we multiply them by 2, *i.e.*, we put $2(y - 114{\cdot}5) = Y$.

Calculations of ΣX, ΣY etc.

x	y	$X = 2(x - 3{\cdot}5)$	$Y = 2(y - 114.5)$	X^2	XY
1	83	– 5	– 63	25	315
2	92	– 3	– 45	9	135
3	71	– 1	– 87	1	87
4	90	1	– 49	1	– 49
5	160	3	91	9	273
6	191	5	153	25	765
$N = 6$		0	0	70	1526

Now, $Y = a + bX$

$\therefore \quad \Sigma Y = Na + b\Sigma X$ and $\Sigma XY = a\Sigma X + b\Sigma X^2$

Putting the above values

$0 = 6a + 0 \qquad \therefore \; a = 0$

and $1526 = 0 + 70\,b \qquad \therefore \; b = 21{\cdot}8$

Hence, the equation of line is $Y = 21{\cdot}8X$.

Putting the values of X and Y, we get

$2(y - 114{\cdot}5) = 21{\cdot}8\,(2)\,(x - 3{\cdot}5)$

$\therefore \; y - 114{\cdot}5 = 21{\cdot}8\,(x - 3{\cdot}5) \qquad \therefore \; y = 38{\cdot}2 + 21{\cdot}8x.$

EXERCISE - III

Class (a) : 4 Marks

1. Fit a straight line to the following data :

x :	0	1	2	3	4	5
y :	1	2	3	4·5	6	7·5

[**Ans. :** $y = 0{\cdot}70 + 1{\cdot}32x$]

2. Fit a straight line to the following data :

x :	100	120	140	160	180	200
y :	0·45	0·55	0·60	0·70	0·80	0·85

[**Ans. :** $y = 0{\cdot}041 + 0{\cdot}0041x$]

3. Fit a first degree curve to the following data and estimate the value of y when $x = 73$.

x :	10	20	30	40	50	60	70	80
y :	1	3	5	10	6	4	2	1

[**Ans. :** $y = 4 - 0{\cdot}071u$ where $u = (x - 45)/5$, $y = 3{\cdot}596$, when $x = 73$]

4. Fit a straight line to the following data and estimate y when $x = 12$

x :	1	2	3	4	5	6	7	8	9	10
y :	52·5	58·7	65·0	70·2	75·4	81·1	87·2	95·5	102·2	108·4

(M.U. 2005) [**Ans. :** $Y = 79{\cdot}62 + 3{\cdot}08X$ where $X = (x - 5{\cdot}5)2$, when $x = 12$, $y = 119{\cdot}66$]

5. Fit a straight line to the following data.
$(x, y) = (-1, -5), (1, 1), (2, 4), (3, 7), (4, 10)$. Estimate y when $x = 7$. **(M.U. 2015)**

[**Ans. :** $y = -2 + 3x$; 19]

10. Fitting a Parabola

Suppose we have a set of values (x_i, y_i). Suppose further that we want to fit a parabola $y = a + bx + cx^2$ to these values. The parabola must be close to the given points as much as possible. The principle of least squares states that the parabola should be such that the distances of the given points from the parabola measured along the y axis must be minimum.

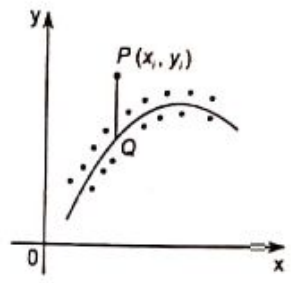

Fig. 6.7

Suppose $P(x_i, y_i)$ is a given point and a line through P parallel to the y axis intersects the curve $y = a + bx + cx^2$ in Q. Then Q is $(x_i, a + bx_i + cx_i^2)$. We find a, b, c such that the distance $|PQ|$ is minimum. But distance $|PQ|$ is minimum when the square of the distance $(y_i - a - bx_i - cx_i^2)^2$ is minimum. This must be true for all points. This means we should have $s = \Sigma(y_i - a - bx_i - cx_i^2)^2$ minimum. Since we find a, b, c such that the sum of the squares S is least the method as you know is known as the **method of least squares.**

Now for s to be minimum, the conditions are

$$\frac{\partial s}{\partial a} = 0, \quad \frac{\partial s}{\partial b} = 0, \quad \frac{\partial s}{\partial c} = 0$$

Differentiating s partially w.r.t. a, b, c, we get

$\therefore \; \Sigma(y_i - a - bx_i - cx_i^2) = 0 ; \qquad \Sigma(y_i - a - bx_i - cx_i^2)\,x_i = 0 ;$

$\Sigma(y_i - a - bx_i - cx_i^2)\,x_i^2 = 0.$

i.e., $\Sigma y_i = Na + b\Sigma x_i + c\Sigma x_i^2 ; \qquad \Sigma x_i y_i = a\Sigma x_i + b\Sigma x_i^2 + c\Sigma x_i^3 ;$

$\Sigma x_i^2 y_i = a\Sigma x_i^2 + b\Sigma x_i^3 + c\Sigma x_i^4.$

where N is the number of observations.

Dropping the suffix, i, we get the equations as

$$\boxed{\begin{aligned} \Sigma y &= Na + b\Sigma x + c\Sigma x^2 \\ \Sigma xy &= a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \\ \Sigma x^2 y &= a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \end{aligned}}$$

These equations are called **normal equations.**

As in the case of straight line for fitting of parabola also we have two methods. (i) Direct Method and (ii) Deviation Method (or Change of Origin Method).

In direct method we use the given values of x and y and find the constants a, b, c.

If the values of x (or y) are evenly spaced we change the origin to the mid-point of x and use the deviations as in Ex. 3 or 4 below.

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Example 1 : By the method of least squares find the best values of a, b, c in the law $y = a + bx + cx^2$ to fit the following data.

x :	-2	-1	0	1	2
y :	$-3{\cdot}150$	$-1{\cdot}390$	$0{\cdot}620$	$2{\cdot}880$	$5{\cdot}378$

(M.U. 2019)

Sol. :

Calculations of Σx, Σy, Σx^2

Sr. No.	x	y	x^2	x^3	x^4	xy	x^2y
1	-2	$-3{\cdot}150$	4	-8	16	$6{\cdot}300$	$-12{\cdot}600$
2	-1	$-1{\cdot}390$	1	-1	1	$1{\cdot}390$	$-1{\cdot}390$
3	0	$0{\cdot}620$	0	0	0	$0{\cdot}000$	$0{\cdot}000$
4	1	$2{\cdot}880$	1	1	1	$2{\cdot}880$	$2{\cdot}880$
5	2	$5{\cdot}378$	4	8	16	$10{\cdot}756$	$21{\cdot}512$
$N = 5$	0	$4{\cdot}338$	10	0	34	$21{\cdot}326$	$10{\cdot}402$

The normal equations are

$$\Sigma y = Na + b\Sigma x + c\Sigma x^2 \qquad \therefore \quad 4{\cdot}338 = 5a + 10c$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \qquad \therefore \quad 21{\cdot}325 = 10b$$

$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \qquad \therefore \quad 10{\cdot}402 = 10a + 34c$$

Solving these three equations, we get

$$a = 0{\cdot}621, \quad b = 2{\cdot}1326, \quad c = 0{\cdot}1233$$

Hence, the law is

$$y = 0{\cdot}621 + 2{\cdot}1326x + 0{\cdot}1233x^2.$$

Example 2 : Fit a second degree curve *i.e.*, parabola to the following data.

x :	0	1	2	3	4
y :	$1{\cdot}0$	$1{\cdot}8$	$1{\cdot}3$	$2{\cdot}5$	$6{\cdot}3$

Sol. : Let the equation of the parabola be $y = a + bx + cx^2$.

Calculations of Σx, Σx^2, etc.

Sr. No.	x	y	x^2	x^3	x^4	xy	x^2y
1	0	$1{\cdot}0$	0	0	0	0	0
2	1	$1{\cdot}8$	1	1	1	$1{\cdot}8$	$1{\cdot}8$
3	2	$1{\cdot}3$	4	8	16	$2{\cdot}6$	$5{\cdot}2$
4	3	$2{\cdot}5$	9	27	81	$7{\cdot}5$	$22{\cdot}5$
5	4	$6{\cdot}3$	16	64	256	$25{\cdot}2$	$100{\cdot}8$
$N = 5$	10	$12{\cdot}9$	30	100	354	$37{\cdot}1$	$130{\cdot}3$

The normal equations are

$$\Sigma y = Na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Putting the values from the table,

$$12{\cdot}9 = 5a + 10b + 30c \qquad \text{..... (1)}$$

$$37{\cdot}1 = 10a + 30b + 100c \qquad \text{..... (2)}$$

$$130{\cdot}3 = 30a + 100b + 354c \qquad \text{..... (3)}$$

We first eliminate a. For this multiply (1) by 2 and subtract the result from the second.

$$\therefore \quad 37{\cdot}1 = 10a + 30b + 100c$$

$$25{\cdot}8 = 10a + 20b + 60c$$

$$11{\cdot}3 = 10b + 40c \qquad \text{..... (4)}$$

Now, multiply (1) by 6 and subtract the result from (3),

$$\therefore \quad 130{\cdot}3 = 30a + 100b + 354c$$

$$77{\cdot}4 = 30a + 60b + 180c$$

$$52{\cdot}9 = 40b + 174c \qquad \text{..... (5)}$$

Now, multiply (4) by 4 and subtract the result from (5).

$$\therefore \quad 52{\cdot}9 = 40b + 174c$$

$$45{\cdot}2 = 40b + 160c$$

$$7{\cdot}7 = 14c \qquad \therefore \quad c = \frac{7{\cdot}7}{14} = 0{\cdot}55.$$

Putting this value of c in (4), we get,

$$11{\cdot}3 = 10b + 40\,(0{\cdot}55)$$

$$\therefore \quad 10b = 11{\cdot}3 - 22 = -10{\cdot}7 \qquad \therefore \quad b = -1{\cdot}07$$

Putting the values of b and c in (1), we get

$$12{\cdot}9 = 5a + 10\,(-1{\cdot}07) + 30\,(0{\cdot}55)$$

$$\therefore \quad 5a = 12{\cdot}9 + 10{\cdot}7 - 16{\cdot}5 = 7{\cdot}1 \qquad \therefore \quad a = 1{\cdot}42$$

Hence, the equation of the parabola is

$$y = 1{\cdot}42 - 1{\cdot}07x + 0{\cdot}55x^2.$$

Example 3 : Fit a non-linear trend of the form $y = a + bx + cx^2$ to the following data.

x :	0	1	2	3	4
y :	$1{\cdot}0$	$1{\cdot}5$	$1{\cdot}5$	$2{\cdot}5$	$3{\cdot}5$

Sol. :

Calculations of ΣX, ΣY, ΣX^2

Sr. No.	x	$Y = y$	Deviations from mid point $X = x - 2$	X^2	X^3	X^4	XY	X^2Y
1	0	$1{\cdot}0$	-2	4	-8	16	$-2{\cdot}0$	$4{\cdot}0$
2	1	$1{\cdot}5$	-1	1	-1	1	$-1{\cdot}5$	$1{\cdot}5$
3	2	$1{\cdot}5$	0	0	0	0	$0{\cdot}0$	$0{\cdot}0$
4	3	$2{\cdot}5$	1	1	1	1	$2{\cdot}5$	$2{\cdot}5$
5	4	$3{\cdot}5$	2	4	8	16	$7{\cdot}0$	$14{\cdot}0$
$N = 5$	10	$10{\cdot}0$	0	10	0	34	$6{\cdot}0$	$22{\cdot}0$

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Thus, we have

$N = 5,\ \Sigma X = 0,\ \Sigma Y = 10,\ \Sigma X^2 = 10,\ \Sigma X^3 = 0,\ \Sigma X^4 = 34,\ \Sigma XY = 6,\ \Sigma X^2 Y = 22.$

Since, we have taken mid-point of X as origin $\Sigma X = 0$ and $\Sigma X^3 = 0$. The normal equations now reduce to

$$\Sigma Y = Na + b\Sigma X^2$$
$$\Sigma XY = a\Sigma X^2$$
$$\Sigma X^2 Y = a\Sigma X^2 + c\Sigma X^4$$

Putting the above values, we get $10 = 5a + 10c$

$\therefore\ 22 = 10a + 34c$ and $6 = 10b$ $\therefore\ b = \frac{6}{10}$

$20 = 10a + 20c$

$2 = 14c$ $\therefore\ c = \frac{2}{14} = \frac{1}{7}$

$\therefore\ 10 = 5a + \frac{10}{7}$ $\therefore\ a = \frac{12}{7}$. $\therefore$ The equation is $Y = \frac{12}{7} + \frac{6}{10}X + \frac{1}{7}X^2$.

Now putting $Y = y$ and $X = x - 2$, we get,

$$y = \frac{12}{7} + \frac{6}{10}(x-2) + \frac{1}{7}(x-2)^2 = \frac{12}{7} + \frac{6}{10}x - \frac{12}{10} + \frac{1}{7}x^2 - \frac{4}{7}x + \frac{4}{7}$$

$$\therefore\ y = \frac{38}{35} + \frac{1}{35}x + \frac{1}{7}x^2.$$

Example 4 : Fit a second degree parabolic curve to the following data.

x :	1,	2,	3,	4,	5,	6,	7,	8,	9.
y :	2,	6,	7,	8,	10,	11,	11,	10,	9.

(M.U. 2004, 07, 12)

Sol. : Since the values of x are odd and are equally spaced we change x to X by the relation $X = x - 5$ and put $y = Y$.

Let the equation of the parabola be $Y = a + bX + cX^2$. Then the normal equations are

$$\Sigma Y = Na + b\Sigma X + c\Sigma X^2$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$
$$\Sigma X^2 Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

Calculations of $\Sigma X, \Sigma X^2$ etc.

Sr. No.	x	$X = x - 5$	$Y = y$	X^2	X^3	X^4	XY	X^2Y
1	1	−4	2	16	−64	256	−8	32
2	2	−3	6	9	−27	81	−18	54
3	3	−2	7	4	−8	16	−14	28
4	4	−1	8	1	−1	1	−8	8
5	5	0	10	0	0	0	0	0
6	6	1	11	1	1	1	11	11
7	7	2	11	4	8	16	22	44
8	8	3	10	9	27	81	30	90
9	9	4	9	16	64	256	36	144
$N = 9$		0	74	60	0	708	51	411

Now the normal equations become

$74 = 9a + 60c$ ..... (1)

$51 = 60b$ $\therefore\ b = 0{\cdot}85$ ..... (2)

$411 = 60a + 708c$ ..... (3)

Now to find a and c multiply (1) by 60 and (3) by 9.

$\therefore\ 4440 = 540a + 3600c$

$3699 = 540a + 6372c$

$741 = -2772c$ $\therefore\ c = -0{\cdot}2673$

Now, $9a = 74 - 60c = 74 + 16{\cdot}038 = 90{\cdot}038$

$\therefore\ a = 10{\cdot}0042 = 10$

$\therefore$ The equation of the parabola is $Y = 10 + 0{\cdot}85X - 0{\cdot}27X^2$.

i.e. $y = 10 + 0{\cdot}85(x-5) - 0{\cdot}27(x-5)^2$ $\therefore\ y = -1 + 3{\cdot}55x - 0{\cdot}27x^2$.

Example 5 : Fit a second degree parabolic curve to the following data and estimate the production in 1982.

Year (x)	:	1974	75	76	77	78	79	80	91
Production (y) (in tons)	:	12	14	26	42	40	50	52	53

Sol. : Since the values of X though not odd are equispaced we change x to X by the relation $X = (x - 1977{\cdot}5)2$ and we put $y = Y$.

Let the equation of the parabola be $Y = a + bX + cX^2$. Then the normal equations are

$$\Sigma Y = Na + b\Sigma X + c\Sigma X^2$$
$$\Sigma XY = a\Sigma X + b\Sigma X^2 + c\Sigma X^3$$
$$\Sigma X^2 Y = a\Sigma X^2 + b\Sigma X^3 + c\Sigma X^4$$

Calculations of $\Sigma X, \Sigma X^2$ etc.

Sr. No.	x	X	Y	X^2	X^3	X^4	XY	X^2Y
1	1974	−7	12	49	−343	2401	−84	588
2	1975	−5	14	25	−125	625	−70	350
3	1976	−3	26	9	−27	81	−78	234
4	1977	−1	42	1	−1	1	−42	42
5	1978	1	40	1	1	1	40	40
6	1979	3	50	9	27	81	150	450
7	1980	5	52	25	125	625	260	1300
8	1981	7	53	49	343	2401	371	2597
$N = 9$		0	289	168	0	6216	547	5601

Now, the normal equations become

$289 = 8a + 168c$ ..... (1)

$547 = 168b$ $\therefore\ b = 3{\cdot}2559$ ..... (2)

$5601 = 168a + 6216c$ ..... (3)

Multiply (1) by 21 and subtract (3) from the result,

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$$6069 = 168a + 3528c$$
$$5601 = 168a + 6216c$$
$$468 = \quad -2688c \qquad \therefore\ c = -0{\cdot}1741$$

Now $289 = 8a + 168c$

$\therefore\ 289 = 8a + (168)(-0{\cdot}1741) \qquad \therefore\ a = 39{\cdot}7811$

The equation to the curve is $Y = 39{\cdot}78 + 3{\cdot}26X - 0{\cdot}17X^2$.

Putting $Y = y$ and $X = (x - 1977{\cdot}5)2$, the equation of the parabola is

$$y = 39{\cdot}78 + 3{\cdot}26\,(x - 1977{\cdot}5)2 - 0{\cdot}17\,(x - 1977{\cdot}5)^2 \cdot 4$$
$$y = -2671997{\cdot}77 + 2695{\cdot}92x - 0{\cdot}68x^2.$$

Putting $x = 1982$, we get $y = 55{\cdot}35$.

EXERCISE - IV

1. Fit a parabola to the following data and estimate the value of y for $x = 6$.

x :	1	2	3	4	5
y :	25	28	33	39	46

(M.U. 2006)

[**Ans.** : $Y = -0{\cdot}086 + 5{\cdot}3X + 0{\cdot}643X^2$ where $X = x - 3$, $Y = y - 33$, $y = 54{\cdot}6$]

2. Fit a second parabola to the following data taking x as the independent variable and shifting the origin to 2 for x.

x :	0	1	2	3	4
y :	1	1·8	1·3	2·5	6·3

Find the difference between the actual value and estimated value of y when $x = 2$.

[**Ans.** : $Y = 1{\cdot}48 + 1{\cdot}13X + 0{\cdot}55X^2$ where $X = x - 2$. Difference $= -0{\cdot}18$]

3. Fit a parabola to the following data :

x :	−2	−1	0	1	2
y :	1·0	1·8	1·3	2·5	6·3

(M.U. 2006) [**Ans.** : $y = 1{\cdot}48 + 1{\cdot}13x + 0{\cdot}55x^2$]

4. Fit a parabola to the following data :

x :	−2	−1	0	1	2
y :	−3·150	−1·390	0·620	2·880	5·378

[**Ans.** : $y = 0{\cdot}621 + 2{\cdot}1326x + 0{\cdot}1233x^2$]

5. Fit a second degree curve to the following data and estimate the value of y when $x = 80$

x :	10	20	30	40	50	60	70
y :	20	60	70	80	90	100	100

[**Ans.** : Put $X = (x - 40) / 10$, $Y = y / 10$, $Y = 8{\cdot}381 + 1{\cdot}2143X - 0{\cdot}2381X^2$ when $x = 80$, $y = 94{\cdot}286$]

6. Fit a parabola to the following data and estimate y when $x = 10$.

x :	1	2	3	4	5	6	7	8	9
y :	2	6	7	8	10	11	11	10	9

(M.U. 2004)

[**Ans.** : $Y = 3 + 0{\cdot}85X - 0{\cdot}27X^2$ where $X = x - 5$, $Y = y - 7$, $y = 7{\cdot}5$]

7. Fit a second degree curve to the following data and estimate the production in 1975.

Year :	1921,	31,	41,	51,	61,	71,	81.
Production (in tons) :	3,	5,	9,	10,	12,	14,	15.

[**Ans.** : $Y = 10{\cdot}33 + 2{\cdot}0357X - 0{\cdot}1547X^2$ where $X = (x - 1951) / 10$. Production $= 14{\cdot}32$]

8. Fit a second degree curve to the following data

Year :	1965,	66,	67,	68,	69,	70,	71,	72.
Profit (in Crores Rs.) :	125,	140,	165,	195,	200,	215,	220,	230.

Also estimate the profit in 1973.

[**Ans.** : $Y = 194{\cdot}68 + 7{\cdot}68X - 0{\cdot}40X^2$ where $X = (x - 1968{\cdot}5) / 2$. And in 1973 profit = 231·4 crores of Rs.]

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CHAPTER 7

Probability

1. Introduction

You were introduced probability in the X-th standard and you have studied laws of probability, Bayes' theorem probability distributions, particularly Binomial distribution in XII-th standard. In this chapter we shall briefly review probability and then study once again Bays' Theorem in detail. Then we shall study probability distributions in general.

2. Terms used in Axiomatic Theory

The limitations of classical theory of probability are removed by putting the theory in axiomatic form. This approach was suggested by **A. N. Kolmogorov**, a Russian mathematician in 1933.

It should be noted that as in any other axiomatic theory (*e.g.* Geometry) we start with certain undefined terms, state certain axioms about them and then deduce theorems about these terms strictly from the axioms by rules of logic. Before developing the axiomatic probability theory we explain below certain peculiar terms used therein.

Andrey Nikolaevich Kolmogorov (1903 - 1987)

A well-known Russian mathema-tician known for his great contributions to the fields of probability theory, topology, intuitionistic logic, classical mechanics and others. In 1922 he constructed a Fourier Series that diverges almost everywhere. His pioneering work "About The Analytical Methods Of Probability Theory" was published in 1931, in which year he become professor at Moscow University. In 1933 he published "Foundations of The Theory of Probability" laying the foundation of modern axiomatic theory. In 1939 he become a member of U.S.S.R. Academy of Sciences. He along with British mathematician Chapman developed "The Chapman-Kolmogorov Equations" in Random processes. He was a founder of algorithm complexity theory known as "Kolmogorov Complexity Theory". A quotation of Kolmogorov. "Every mathematician believes that he is ahead of all others. The reason why they do not say this in public is they are intelligent people".

(1) Sample space

The aggregate or the set of all possible outcomes of an experiment is called *sample space*, the outcomes *i.e.* the members of the set themselves being called *sample points*. Sample space is denoted by S or Ω and sample points by $w_1, w_2, w_3, \ldots$. Thus, $\Omega = \{w_1, w_2, w_3, \ldots\}$.

Example 1 : In a random experiment of tossing a coin the sample points are its outcomes a head (H) and a tail (T) and the sample space is

$$\Omega = \{H, T\}.$$

Example 2 : In a random experiment of tossing a dice the sample points are 1, 2, 3, 4, 5 and 6 the sample space is

$$\Omega = \{1, 2, 3, 4, 5, 6\}.$$

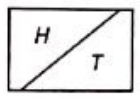
Example 3 : In a throw of two coins the sample points are (H, H), (H, T), (T, H), (T, T), and the sample space is

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}.$$

(2) Event

A subset of a sample space is called an *event*.

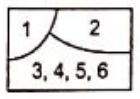
(i) In a throw of a coin $\Omega = \{H, T\}$. We have shown two events



$A_1 = \{H\}$,

$A_2 = \{T\}$.

(ii) In a throw of a die $\Omega = \{1, 2, 3, 4, 5, 6\}$, there are a number of events.



$A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3, 4, 5, 6\}$, ...

$B_1 = \{1, 2\}$, $B_2 = \{1, 3\}$ .....

$C_1 = \{1, 2, 3\}$, $C_2 = \{1, 2, 4\}$, ... etc.

We have shown the first three.

(iii) In a throw of a coin and a die there are again a number of events

$A_1 = \{(H, 1), (H, 2)\}$

$A_2 = \{(H, 1), (H, 3), (H, 5),\}$ .....

$B_1 = \{(T, 1), (T, 2)\}$

$B_2 = \{(T, 1), (T, 3), (T, 5)\}$ ..... etc.

3. Axiomatic Definition of Probability

Let E be a random experiment and S be the sample space. We define the probability $P(A)$ for every element A of S (*i.e.* for every subset A of the sample space S) satisfying the following axioms :

1. $P(A) \geq 0$ (Axiom 1)
2. $P(S) = 1$ (Axiom 2)
3. $P(A \cup B) = P(A) + P(B)$ (Axiom 3)

if A and B are any two exclusive events (*i.e.* they are disjoint sets).

Explanation : The first axiom states that the probability of any event is greater than zero, which means the probability of any event cannot be negative.

The second axiom stales that the **sum** of all the probabilities is equal to 1. This together with the first axiom states that the probability of an event must be less than or equal to 1 *i.e.* $P(A) \leq 1$.

The third axiom states that if two events are mutually exclusive, then the probability that either of them will occur is equal to the some of their probabilities.

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For our study the classical definition is sufficient. We give that definition in terms of sets again for ready reference.

Definition : If a sample space S has n points which are equally likely and mutually exclusive and an event A has m points then the ratio m / n is called probability of A and is denoted by $P(A)$. [Fig. 7.1 (a)]

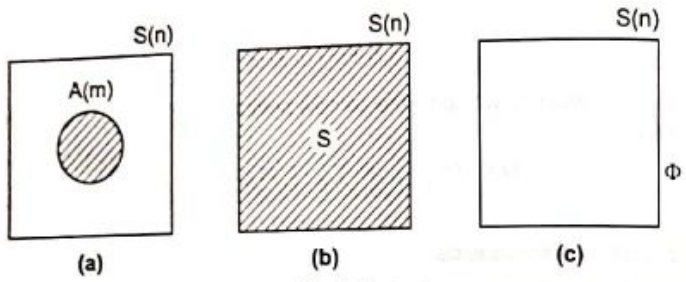

Fig. 7.1

Thus,
$$\boxed{P(A) = \frac{m}{n} = \frac{\text{number of points in } A}{\text{number of point in } S}}$$

Note ....

We shall be concerned with only such random experiments in which all sample points are equally likely. And to find the probability of an event in such a random experiment we need to know the number of points in sample space S and the number of points in the event A.

4. Laws of Probability

There are two laws. The probability of **A or B** *i.e.* $P(A \cup B)$ is given by the *law of addition* and the probability of **A and B** *i.e.* $P(A \cap B)$ is given by the *law of multiplication*.

Theorem 1 : For any two events A and B the probability that exactly B will occur is given by

$$\boxed{P(B \cap \overline{A}) = P(B) - P(A \cap B)}$$

and that exactly A will occur is given by

$$\boxed{P(A \cap \overline{B}) = P(A) - P(A \cap B)}$$

where $\overline{A}$ denotes the complement of A and $\overline{B}$ denotes the complement of B.

Proof : To prove this result we first express the event B as the union of two exclusive events $A \cap B$ and $\overline{A} \cap B$.

$\therefore \quad B = (A \cap B) \cup (\overline{A} \cap B)$

Since the events on the r.h.s. are exclusive

$P(B) = P(A \cap B) + P(\overline{A} \cap B)$ [By Axiom 3]

$\therefore \quad P(B \cap \overline{A}) = P(B) - P(A \cap B)$

Similarly, we can prove that

$P(A \cap \overline{B}) = P(A) - P(A \cap B)$

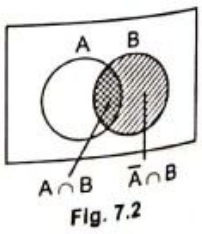

Fig. 7.2

Theorem 2 : (Additive Theorem) (Two Events)

Probability that atleast one of the events A and B will occur is given by

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Proof : We first express the event $A \cup B$ is the union of two exclusive events A and $\overline{A} \cap B$.

$A \cup B = A \cup (\overline{A} \cap B)$

$\therefore \quad P(A \cup B) = P[A \cup (\overline{A} \cap B)]$

But the events on the r.h.s. are mutually exclusive.

$\therefore \quad P(A \cup B) = P(A) + P(\overline{A} \cap B)$ [By Axiom 3]

$= P(A) + P(B) - P(A \cap B)$ [By Theorem 2]

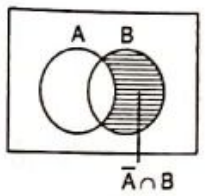

Fig. 7.3

Corollary 1 : If A and B are two mutually exclusive events then the probability that either A or B will happen is the sum of the probabilities of A and B *i.e.*

$$\boxed{P(A \cup B) = P(A) + P(B)}$$

Proof : Since events are exclusive $A \cap B = \Phi$.

$\therefore \quad P(A \cap B) = P(\Phi) = 0$ [By Theorem 1]

Hence, from the above theorem,

$P(A \cup B) = P(A) + P(B)$

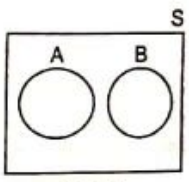

Fig. 7.4

Corollary 2 : If $A, B, C, \ldots, K$ are mutually exclusive events such that their union is the whole of sample space then

$$\boxed{P(A) + P(B) + P(C) + \ldots + P(K) = 1}$$

Proof : Since $A \cup B \cup C \cup \ldots \cup K = S$

$\therefore \quad P(A \cup B \cup C \cup \ldots \cup K) = P(S)$

Since, events are mutually exclusive, we get, from above,

$P(A) + P(B) + P(C) + \ldots + P(K) = P(S)$

But $P(S) = 1$ [By Axiom 2]

$\therefore \quad P(A) + P(B) + P(C) + \ldots + P(K) = 1$

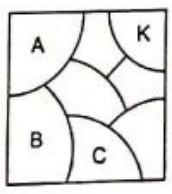

Fig. 7.5

Note ....

The group of all possible events $A, B, C, \ldots, K$ of the sample space S such that

$$P(A) + P(B) + P(C) + \ldots + P(K) = 1$$

is called **exhaustive.** *e.g.* in the toss of a coin H, T; in the toss of a dice 1, 2, 3, 4, 5, 6 are exhaustive events.

Theorem 3 : *Addition Theorem (Three Events)* : If A, B, C are any three events then the probability that *at least one* of them will occur is given by

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C).$$

Proof : We prove this theorem by considering the union of A and $(B \cup C)$ and applying the above theorem.

$\therefore \quad P(A \cup (B \cup C)) = P(A) + P(B \cup C) - P(A \cap (B \cup C))$

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But $P(B \cup C) = P(B) + P(C) - P(B \cap C)$

And $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

$$\therefore \quad P(A \cap (B \cup C)) = P[(A \cap B) \cup (A \cap C)]$$
$$= P(A \cap B) + P(A \cap C) - P[(A \cap B) \cap (A \cap C)]$$
$$\therefore \quad P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(B \cap C)$$
$$- P(B \cap C) - P(A \cap B) - P(A \cap C) + P(A \cap B \cap C)$$

Corollary : If A, B, C are three events which are pairwise exclusive then

$$\boxed{P(A \cup B \cup C) = P(A) + P(B) + P(C)}$$

Proof : Since, A, B are exclusive $A \cap B = \Phi$. Since, $P(\Phi) = 0$, $P(A \cap B) = 0$. Similarly, $P(B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$, $P(C \cap A) = 0$, $P(A \cap B \cap C) = 0$.

Hence, from the above result,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C).$$

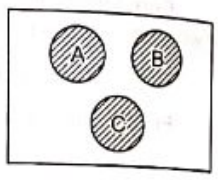

Fig. 7.6

Example : Three horses A, B and C are in a race. A is twice as likely to win as B and B is twice as likely to win as C. What are the probabilities of their winning ?

Sol. : Let $P(C) = x$ then $P(B) = 2x$ and $P(A) = 4x$.

But the sum of all probabilities is 1.

$$\therefore \quad P(A) + P(B) + P(C) = 1 \qquad \therefore \quad 4x + 2x + x = 1 \qquad \therefore \quad x = \frac{1}{7}$$

$$\therefore \quad P(A) = \frac{4}{7}, \quad P(B) = \frac{2}{7}, \quad P(C) = \frac{1}{7}.$$

Complementary Events

It is clear that $A \cup \overline{A} = S$ *i.e.* the union of the event and its complement is the whole of sample space S and $\overline{A} \cap A = \Phi$ *i.e.* the event A and its complement are exclusive.

Theorem 4 : $\boxed{P(\overline{A}) = 1 - P(A)}$

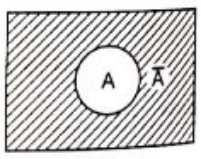

Fig. 7.7

Proof : Since $A \cup \overline{A} = S$, $P(A \cup \overline{A}) = P(S)$. But $P(S) = 1$ and A, $\overline{A}$ are exclusive.

By Axiom 2, $\quad P(A \cup \overline{A}) = P(A) + P(\overline{A})$

$\therefore \quad P(A) + P(\overline{A}) = 1 \qquad \therefore \quad P(\overline{A}) = 1 - P(A)$

Note ....

Since A and $\overline{A}$ are exclusive, we have $A \cup \overline{A} = S$ and $A \cap \overline{A} = \Phi$.

Corollary : Probability of an event is always less than or equal to one.

i.e. $\boxed{P(A) \le 1}$

Proof : $P(A) = 1 - P(\overline{A})$. But $P(\overline{A}) \ge 0$ by axiom 1.

$\therefore \quad P(A) \le 1.$

De Morgan's Laws

Since an event is a subset of sample space, De Morgan's are laws are applicable to events. Thus, we get

$$\boxed{\begin{aligned} P(\overline{A \cup B}) &= P(\overline{A} \cap \overline{B}) \\ P(\overline{A \cap B}) &= P(\overline{A} \cup \overline{B}) \end{aligned}}$$

Augustus De Morgan (1806 - 1871)

Augustus De Morgan was born in Madurai, Tamil Nadu. His father was a colonel in the Indian army. His family returned to England when he was 7 months old. When in schools he mastered Latin, Greek and Hebrew and developed strong interest in mathematics.

He was a fellow of the Astronomical Society and a founder of London Mathematical Society. De Morgan greatly influenced the development of mathematics in the 19th century. He was a prolific writer and wrote over 1000 articles in more than 15 journals, in addition to a number of books known for clarity, logical presentation and minute details. He made original contributions to analysis and logic. He coined the term mathematical induction and gave first precise definition of limit in his book "The Differential And Integral Calculus".

Example 1 : Find the probability that a card drawn will be black or picture ?

Sol. : Let A = the card is picture and B = the card is black.

Now A and B are not mutually exclusive events. There are 6 black pictures

$$P(A) = \frac{12}{52}, \quad P(B) = \frac{26}{52}, \quad P(A \cap B) = \frac{6}{52}$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$= \frac{12}{52} + \frac{26}{52} - \frac{6}{52} = \frac{32}{32} = \frac{8}{13}.$$

Note ....

Many examples of this type can be more easily solved by counting the number of points in the desired event. In the above example $A \cup B$ = {Black or Picture Cards} has 26 + 12 – 6 = 32 points.

$$\therefore \quad P(A \cup B) = \frac{32}{52} = \frac{8}{13}.$$

Example 2 : Two cards are drawn from a pack of cards. Find the probability that they will be both red or both pictures.

Sol. : Let A = { both red } and B = { both pictures }

$$P(A) = \frac{{}^{26}C_2}{{}^{52}C_2}, \quad P(B) = \frac{{}^{12}C_2}{{}^{52}C_2}$$

But A and B are not exclusive. There are six red picture cards.

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$$\therefore \quad P(A \cap B) = \frac{{}^{6}C_2}{{}^{52}C_2},$$

$$\therefore \quad P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= \frac{{}^{26}C_2}{{}^{52}C_2} + \frac{{}^{12}C_2}{{}^{52}C_2} - \frac{{}^{6}C_2}{{}^{52}C_2} = \frac{203}{663}.$$

5. Conditional Probability

Consider general case of conditional probability (See Fig. 7.8 on page 7-8). Let n = the number of points in S, m_1 = number of points in A, m_2 = number of points in B, m_{12} = number of points in A and B both.

Then, $P(A) = \frac{m_1}{n}$, $P(B) = \frac{m_2}{n}$ and $P(A \cap B) = \frac{m_{12}}{n}$.

Now, suppose in a trial we know the result partially *i.e.* we know that A has occurred. What is the probability **now** that B has occurred along with A ?

Since we know that A has occurred the outcome of the trial is one of those points in A *i.e.* one of m_1 (and not n) And of these, m_{12} are in B. Hence, the probability that B will occur, when A has already occurred is (m_{12} / m_1). This is called conditional probability of B under the condition that A has occurred. It is denoted by $P(B / A)$.

Thus, $\quad P(B / A) = \frac{m_{12}}{m_1}$.

Definition : Let A and B be any two events in a sample space S. The probability that B will occur, given that A has already occurred is called the **conditional probability of B** and is denoted by $P(B / A)$.

Similarly the probability that A will occur given that B has already occurred is called the **conditional probability of A** and is denoted by $P(A / B)$.

Example : Suppose there are 100 students in a class and the results of an examination of the class are given in the following table.

	Passed	Failed	Total
Boys	28	32	60
Girls	26	14	40
Total	54	46	100

Let us define two events A, B as follows :

A = a student has passed, $\quad B$ = a student is a male student.

Suppose a student is selected at random and is known to be a male student. What is the probability that this student has passed ? In symbols, we want to find $P(A / B)$.

Since the student is a male student, the sample space of B has 60 points. Of these 28 have passed i.e. A has now 28 points. Hence,

$$\therefore \quad P(A / B) = \frac{28}{60} = \frac{7}{15}$$

But, from the table we see that 28 is the number of points in $A \cap B$ and 60 is the number of points in B

$$\boxed{P(A / B) = \frac{\text{No. of points in } A \cap B}{\text{No. of points in } B}} \quad \text{..... (1)}$$

or

$$\boxed{P(A / B) = \frac{\text{No. of points in } A \text{ out of } B}{\text{No. of points in } B}} \quad \text{..... (2)}$$

Similarly, we can find that

$$\boxed{P(B / A) = \frac{\text{No. of points in } B \cap A}{\text{No. of points in } A}} \quad \text{..... (3)}$$

or

$$\boxed{P(B / A) = \frac{\text{No. of points in } B \text{ out of } A}{\text{No. of points in } A}} \quad \text{..... (4)}$$

Further, we see that, (1) can be written as

$$P(A / B) = \frac{(\text{No. of points in } A \cap B) / \text{No. of points in } S}{(\text{No. of points in } B) / \text{No. of points in } S}$$

$$\therefore \quad P(A / B) = \frac{P(A \cap B)}{P(B)}$$

Similarly, from (3) we can get $\quad P(B / A) = \frac{P(A \cap B)}{P(A)}$.

From these we get $\quad P(A \cap B) = P(A / B) \times P(B)$ and $P(A \cap B) = P(B / A) \times P(A)$

which is called the law of multiplication of probability.

Theorem 5 : *Multiplication Theorem*

If A and B are two events and neither is null then the probability that both of them will occur is given by

$$\boxed{P(A \cap B) = P(A) \times P(B / A)} \quad \text{or} \quad \boxed{P(A \cap B) = P(B) \times P(A / B)}$$

where $P(A / B)$ and $P(B / A)$ denote the conditional probabilities which are greater than zero.

Proof : Let the number of points in A be m_1 and those in B be m_2. Let the number of points in $(A \cap B)$ be m_{12} and let n be the total number of points in S.

$$P(A) = \frac{m_1}{n}, \quad P(B) = \frac{m_2}{n}$$

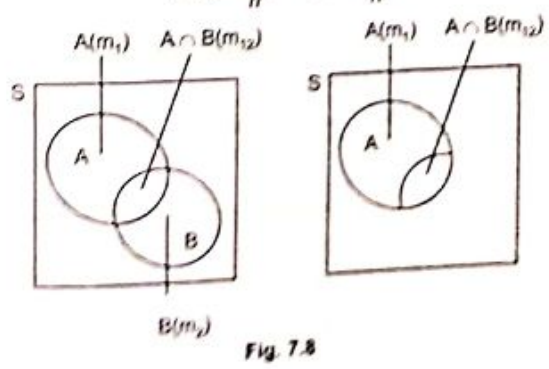

Fig. 7.8

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To find the probability of B when A has happened, we have to consider the sample space of A which has m_1 points. In this sample space only B can occur (along with A) which has m_{12} points.

$$\therefore \quad P(B/A) = \frac{\text{No. of points in } A \cap B}{\text{No. of points in } A} = \frac{m_{12}}{m_1}$$

$$\text{Now,} \quad P(A \cap B) = \frac{m_{12}}{n} = \frac{m_1}{n} \times \frac{m_{12}}{m_1}$$

$$\therefore \quad P(A \cap B) = P(A) \times P(B/A)$$

Similarly, we can prove that $P(A \cap B) = P(B) \times P(A/B)$.

Example 1 : There are 11 tickets in a box bearing numbers 1 to 11. Three tickets are drawn one after the other without replacement. Find the probability that they are drawn in the order bearing (i) even, odd, even number, (ii) odd, odd, even number. **(M.U. 1997)**

Sol. : This is an example on conditional probability. Out of 11 tickets, 5 are even and 6 are odd.

$$P(\text{even, odd, even}) = P(\text{even}) \cdot P(\text{odd}) \cdot P(\text{even})$$

$$= \frac{5}{11} \cdot \frac{6}{10} \cdot \frac{4}{9} = \frac{4}{33}$$

$$P(\text{odd, odd, even}) = P(\text{odd}) \cdot P(\text{odd}) \cdot P(\text{even})$$

$$= \frac{6}{11} \cdot \frac{5}{10} \cdot \frac{5}{9} = \frac{5}{33}$$

Example 2 : In a bag there are 4 white and 3 black balls. If four balls are drawn one by one at random without replacement, what is probability that the balls so drawn are alternately of different colours ?

Sol. : Try.

$$P(A) = \frac{4}{7} \cdot \frac{3}{6} \cdot \frac{3}{5} \cdot \frac{2}{4} = \frac{3}{35}; \qquad P(B) = \frac{3}{7} \cdot \frac{4}{6} \cdot \frac{2}{5} \cdot \frac{3}{4} = \frac{3}{35}$$

$$P(\text{Alternate colour}) = \frac{3}{35} + \frac{3}{35} = \frac{6}{35}.$$

Example 3 : In a certain college 4% of the boys and 1% of the girls are taller than 1·8 mts. Furthermore 60% of the students are girls. If a student selected at random is taller than 1·8 mts, what is the probability that the student was a boy ? **(M.U. 2002, 03)**

Sol. : For convenience suppose there are 1000 students in the college.

Then we can easily prepare the following table.

	Taller than 1·8	Less than 1·8	Total
Boys	16	384	400
Girsl	6	594	600
Total	22	978	1000

Since the student selected at random is found to be taller than 1·8, the student is one of 22. But out of these 22 students 16 are boys.

$$\therefore \quad \text{Required Probability} = \frac{16}{22} = \frac{8}{11}.$$

6. Partition of Sample Space

Definition : The events $A_1, A_2, \ldots, A_n$ are said to represent a partition of sample space S if

(i) $A_i \cap A_j = 0$ **for all** $i \neq j$

and (ii) $A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n = S$.

In words : The events $A_1, A_2, \ldots, A_n$ are mutually exclusive and exhaustive *i.e.* when the experiment is performed *one and only one* of the events A_i occurs and must occur.

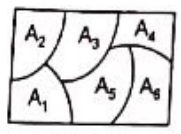

Fig. 7.9

For example, in tossing of a die the events $A_1 = \{1, 2\}$, $A_2 = \{3, 4, 5\}$, $A_3 = \{6\}$ represent a partition of the sample space S. However, $B_1 = \{1, 2, 3\}$, $B_2 = \{3, 4\}$, $B_3 = \{4, 5, 6\}$ do not represent a partition (why ?). Also, $C_1 = \{1\}$, $C_2 = \{3, 4\}$, $C_3 = \{6\}$ do not represent a partition. (why ?)

Example 1 : A, B and C are bidding for a contract. It is believed that A has exactly half the chance than B has. B in turn is 4 / 5th as likely as C to get the contract. What is the probability for each to get the contract ? **(M.U. 2004)**

Sol. : Let $P(A)$, $P(B)$, $P(C)$ denote the probabilities that A, B, C will get the contract respectively. Further, let $P(C) = x$. Then,

$$P(B) = \frac{4}{5}x \text{ and } P(A) = \frac{1}{2} \cdot \frac{4}{5}x = \frac{2}{5}x$$

Since one of A, B, C must get the contract, the three events partition the sample space

$$x + \frac{4}{5}x + \frac{2}{5}x = 1 \quad \therefore \quad 11x = 5 \quad \therefore \quad x = \frac{5}{11}.$$

$$\therefore \quad P(A) = \frac{2}{11}, \; P(B) = \frac{4}{11}, \; P(C) = \frac{5}{11}$$

Example 2 : If the events A, B and C form a partition of the sample space and $P(A) = 2P(B) = 3P(C)$, find (i) $P(A \cup B)$, (ii) $P(B \cup C)$.

Sol. : Let $P(A) = x$, then $P(B) = \frac{x}{2}$ and $P(C) = \frac{x}{3}$.

Since A, B and C form partition S. These are exhaustive events.

$$\therefore \quad P(A) + P(B) + P(C) = 1$$

$$\therefore \quad x + \frac{x}{2} + \frac{x}{3} = 1 \quad \therefore \quad \frac{11}{6}x = 1 \quad \therefore \quad x = \frac{6}{11}$$

$$\therefore \quad P(A) = \frac{6}{11}, \; P(B) = \frac{3}{11}, \; P(C) = \frac{2}{11}.$$

Since, the events are mutually exclusive.

$$P(A \cup B) = P(A) + P(B) = \frac{6}{11} + \frac{3}{11} = \frac{9}{11}$$

$$\text{and} \quad P(B \cup C) = P(B) + P(C) = \frac{3}{11} + \frac{2}{11} = \frac{5}{11}.$$

Example 4 : If the events A, B and C form a partition of the sample space S and $3P(A) = 2P(B) = 6P(C)$, find $P(A \cup B)$. **[Ans. : 5 / 6]**

Please do it.

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Example 5 : If the events A, B, C and D form a partition of the sample space S and the probabilities of A and C are equal, the event B is twice as likely as A and the event D is twice as likely as B, find $P(A \cup B \cup C)$. **[Ans. : 1/2]**

Please do it.

Theorem on Total Probability

Let $A_1, A_2, A_3, \ldots, A_n$ be a partition of S and B be some event defined on S. Then,

$$P(B) = P(B/A_1) \times P(A_1) + P(B/A_2) \times P(A_2) + \ldots + P(B/A_n) \times P(A_n)$$

We accept this theorem without proof.

Fig. 7.10

If we denote the probabilities $P(A_i)$ by p_i and the conditional probabilities $P(B/A_i)$ by p'_i then, the above theorem can be stated as

$$\boxed{P(B) = p_1 p'_1 + p_2 p'_2 + p_3 p'_3 + \ldots p_n p'_n}$$

Remark ....

The above theorem is known as the theorem on **total probability**. In some problems it may be difficult to evaluate $P(B)$ directly. But when additional information that A_i has occurred is known, we may evaluate $P(B/A)$ and use the above theorem to find $P(B)$.

Example 1 : Four roads lead away from a jail. A prisoner trying to escape from the jail selects a road at random. If road A is selected, the probability of escaping is 1/8, for road B, it is 1/6, for road C it is 1/4 and for road D it is 9/10.

What is the probability that a prisoner will succeed in escaping from the jail ? **(M.U. 2005)**

Sol. : Let E = Success in Escaping.

p_1 = selecting road A, p_2 = selecting road B,

p_3 = selecting road C, p_4 = selecting road D.

$p_1 = P(A) = 1/4$, $p_2 = P(B) = 1/4$,

$p_3 = P(C) = 1/4$, $p_4 = P(D) = 1/4$.

$p_1' = P(E/A) = 1/8$, $p_2' = P(E/B) = 1/6$,

$p_3' = P(E/C) = 1/4$, $p_4' = P(E/D) = 9/10$.

$$\therefore \quad P(E) = p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'$$

$$\therefore \quad P(E) = \frac{1}{4}\cdot\frac{1}{8} + \frac{1}{4}\cdot\frac{1}{6} + \frac{1}{4}\cdot\frac{1}{4} + \frac{1}{4}\cdot\frac{9}{10}$$

$$= \frac{1}{4}\left(\frac{1}{8} + \frac{1}{6} + \frac{1}{4} + \frac{9}{10}\right) = \frac{1}{4}\cdot\frac{163}{120} = \frac{163}{480}.$$

Example 2 : In a box there are four tags numbered 1 and 6 tags numbered 2. There are two urns U_1 and U_2 containing 3 red and 7 black balls and 8 red and 2 black balls respectively. One tag is drawn from the box and one ball is drawn from the urn whose number is found on the tag drawn. Find the probability that a red ball is drawn.

Sol. : Try. **[Ans. : 0·6]**

EXERCISE - I

1. The probabilities that three students A, B and C will pass the common entrance test for engineering are 4/9, 2/9 and 1/3 respectively. The probabilities that they will get admission in the same engineering college are 3/10, 1/2 and 4/5 respectively.

Find the probability that they will get admission in the same engineering college.

[Ans. : 23/45]

2. The chances that A, B and C will be the Education Minister of Government of India are in the ratio 4 : 1 : 2. The probabilities that they will introduce reservations in professional colleges for backward classes are 0·3, 0·8 and 0·5 respectively.

Find the probability that the bill for reservation will be introduced. **[Ans. : 3/7]**

3. In a factory an article is produced on three machines. Their respective productions are 300 units by A, 250 units by B and 450 units by C. It is found that the percentages of defective articles for A, B, C are 1, 1·2 and 2 selected at random from a days production (which are mixed).

Find the probability that the selected article is defective. **[Ans. : 0·015]**

We now state an important theorem known as Bayes' Theorem. It enables us to evaluate what may be called **reverse probabilities**. Suppose there are two boxes (I and II) which contain 2 white and 3 black balls; and 3 white and 4 black balls. If a box is chosen at random and a ball is drawn from it, what is the probability that the ball drawn is white ? We know how to calculate this probability. Now, consider the question : If the ball drawn is known to be white, what is the probability that it was drawn from the 1st box ? The question can be answered by Bayes' Theorem. If the 'result' is known, Bayes' Theorem enables us to find the probability of the 'cause'. For this reason it is also sometimes known as the formula for the **"probability of causes"**.

7. Bayes' Theorem

Let the events $A_1, A_2, \ldots, A_n$ represent a partition of the sample space S. Let B be any other event defined on S. If $P(A_i) \neq 0$, $i = 1, 2, \ldots, n$ and $P(B) \neq 0$ then

$$P(A_i/B) = \frac{P(A_i) \times P(B/A_i)}{\Sigma P(A_i) \times P(B/A_i)}$$

If we write $p_1 = P(A_1)$, $p_2 = P(A_2)$, $p_3 = P(A_3)$ ..... etc.

and $p'_1 = P(B/A_1)$, $p'_2 = P(B/A_2)$, $p'_3 = P(B/A_3)$ ..... etc.

then Bayes' theorem can be stated as

$$\boxed{P(A_i/B) = \frac{p_i p'_i}{p_1 p'_1 + p_2 p'_2 + \ldots + p_n p'_n}}$$

Proof : We have by conditional probability

$$P(A_i/B) = \frac{P(A_i \cap B)}{P(B)} \qquad \ldots\ldots\ldots\ldots \text{(i)}$$

But $$P(B/A_i) = \frac{P(B \cap A_i)}{P(A_i)} \quad \therefore \quad P(A_i \cap B) = P(A_i) \cdot P(B/A_i) \qquad \ldots\ldots\ldots\ldots \text{(ii)}$$

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From (i) and (ii), we get

$$\therefore \quad P(A_i / B) = \frac{P(A_i) \cdot P(B / A_i)}{P(B)} = \frac{p_i p'_i}{P(B)} \qquad \text{..... (iii)}$$

But $\quad B = (B \cap A_1) \cup (B \cap A_2) \ldots\ldots (B \cap A_n)$

$$\therefore \quad P(B) = P(B \cap A_1) + P(B \cap A_2) + \ldots\ldots + P(B \cap A_n)$$

But $\quad P(B \cap A_i) = P(A_i) \cdot P(B / A_i)$ etc. $= p_1 p'_1$ etc.

Hence, from (iii) $\quad P(A_j / B) = \dfrac{p_i p'_i}{p_1 p'_1 + p_2 p'_2 + \ldots\ldots + p_n p'_n}$

Thomas Bayes (1701 - 1761)

Thomas Bayes was an English mathematician known for the theorem that bears his name. This theorem was published after his death by Richar Price. He published two works in his lifetime, one theological and one mathematical. He became a Fellow of the Royal Society in 1742. It is said that he learned mathematics and probability from a book by De Moivre. In his later years he took deep interest in probability.

Example 1 : There are in a bag three true coins and one false coin with head on both sides. A coin is chosen at random and tossed four times. If head occurs all the four times, what is the probability that the false coin was chosen and used ? **(M.U. 2003)**

Sol. : $\quad P$ (selecting true coin) $= p_1 = \dfrac{3}{4}$. $\quad P$ (selecting false coin) $= p_2 = \dfrac{1}{4}$.

$p'_1 = P$ (getting all four heads with true coin) $= \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2} = \dfrac{1}{16}$.

$p'_2 = P$ (getting all four heads with false coin) $= 1 \cdot 1 \cdot 1 \cdot 1 = 1$.

$$\therefore \quad \text{Required Probability} = \frac{p_2 p'_2}{p_1 p'_1 + p_2 p'_2} = \frac{(1/4) \cdot 1}{(3/4) \cdot (1/16) + (1/4) \cdot 1} = \frac{16}{19}.$$

Example 2 : A coin is tossed. If it turns up heads two balls are drawn from urn A otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. What is the probability that urn A was used, given that both balls drawn are black ? **(M.U. 2001, 04)**

Sol. : We have $p_1 = P(H) = \dfrac{1}{2}$, $p_2 = P(T) = \dfrac{1}{2}$

$p'_1 = P$ (two black balls from A) $= \dfrac{{}^3C_2}{{}^8C_2}$

$p'_2 = P$ (two black balls from B) $= \dfrac{{}^7C_2}{{}^8C_2}$

$$\therefore \quad \text{Required Probability} = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{\frac{1}{2} \cdot \frac{{}^3C_2}{{}^8C_2}}{\frac{1}{2} \cdot \frac{{}^3C_2}{{}^8C_2} + \frac{1}{2} \cdot \frac{{}^7C_2}{{}^8C_2}}$$

$$= \frac{{}^3C_2}{{}^3C_2 + {}^7C_2} = \frac{3 \cdot 2/2 \cdot 1}{(3 \cdot 2/2 \cdot 1) + (7 \cdot 6/2 \cdot 1)}$$

$$= \frac{6}{6 + 42} = \frac{6}{48} = \frac{1}{8}.$$

Example 3 : In a certain test there are multiple choice questions. There are four possible answers to each question and one of them is correct. An intelligent student can solve 90% questions correctly by reasoning and for the remaining 10% questions he gives answers by guessing. A weak student can solve 20% questions correctly by reasoning and for the remaining 80% questions he gives answers by guessing. An intelligent students gets the correct answer, what is the probability that he was guessing ? **(M.U. 2004)**

Sol. : Consider the intelligent student.

Let $\quad p_1 =$ answering by reasoning $= \dfrac{90}{100} = \dfrac{9}{10}$.

$p_2 =$ answering by guessing $= \dfrac{10}{100} = \dfrac{1}{10}$.

$p_1' =$ answer is correct (by reasoning) $= 1$

$p_2' =$ answering is correct (by guessing) $= \dfrac{1}{4}$.

$$\text{Required Probability} = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2'} = \frac{(1/10) \cdot (1/4)}{(9/10) \cdot 1 + (1/10) \cdot (1/4)}$$

$$= \frac{1/40}{37/40} = \frac{1}{37}.$$

Example 4 : A certain test for a particular cancer is known to be 95% accurate. A person submits to the test and the result is positive. Suppose that a person comes from a population of 100,000 where 2000 people suffer from that disease. What can we conclude about the probability that the person under test has that particular cancer ? **(M.U. 2006)**

Sol. : We have

$p_1 =$ probability a person has the cancer $= \dfrac{2000}{100{,}000} = \dfrac{2}{100} = 0{\cdot}02$

$p_2 =$ probability that a person does not have the cancer $= 1 - 0{\cdot}02 = 0{\cdot}98$

$p_1' =$ test is positive when a person has cancer $= \dfrac{95}{100} = 0{\cdot}95$

$p_2' =$ test is positive when person does not have a cancer $= \dfrac{5}{100} = 0{\cdot}05$

$$\therefore \quad \text{Required probability} = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2'}$$

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$$\therefore \quad \text{Required probability} = \frac{(2/100)\cdot(95/100)}{(2/100)\cdot(95/100)+(98/100)\cdot(5/100)}$$

$$= \frac{190}{680} = 0{\cdot}279$$

Example 5 : A bag contains 7 red and 3 black balls and another bag contains 4 red and 5 black balls. One ball is transferred from the first bag to the second bag and then a ball is drawn from the second bag. If this ball happens to be red, find the probability that a black ball was transferred. **(M.U. 2002)**

Sol. : We have

p_1 = Probability of transferring black ball $= \frac{3}{10}$

p'_1 = Probability of now drawing a red ball $= \frac{4}{10}$

p_2 = Probability of transferring red ball $= \frac{7}{10}$

p'_2 = Probability of now drawing red ball $= \frac{5}{10}$

$$\therefore \quad \text{Required Probability} = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{(3/10)\cdot(4/10)}{(3/10)(4/10)+(7/10)(5/10)}$$

$$= \frac{12}{47}.$$

Example 6 : A man speaks truth 3 times out of 5. When a die is thrown, he states that it gave an ace. What is the probability that this event has actually happened ? **(M.U. 2019)**

Sol. : We have

p_1 = Probability he speaks truth $= \frac{3}{5}$; p'_1 = Probability of an ace $= \frac{1}{6}$

p_2 = Probability he speaks a lie $= \frac{2}{5}$; p'_2 = Probability of not ace $= \frac{5}{6}$

$$P(\text{he speaks truth when ace has occured}) = \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2}$$

$$= \frac{(3/5)(1/6)}{(3/5)(1/6)+(2/5)(5/6)} = \frac{3}{3+10} = \frac{3}{13}.$$

Example 7 : A lot of IC chips is known to contain 3% defective chips. Each chip is tested before delivery but the tester is not completely reliable. It is known that :

P (Tester says the chip is good / The chip is actually good) = 0·95 and P (Tester says the chip is defective / The chip is actually defective) = 0·96.

If a tested chip is declared defective by the tester. What is the probability that it is actually defective ?

Sol. : Let p_1 = Chip is defective = 0·03

p'_1 = Tester says the chip is defective / The chip is defective = 0·96

p_2 = Chip is good = 0·97

p'_2 = Tester says the chip is defective / The chip is good = 0·05.

By Bayes' Theorem,

P (Chip is defective / Tester says it is defective)

$$= \frac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2} = \frac{0{\cdot}03 \times 0{\cdot}96}{0{\cdot}03 \times 0{\cdot}96 + 0{\cdot}97 \times 0{\cdot}05} = 0{\cdot}37.$$

Example 8 : A binary communication transmitter sends data as one of two types of signals denoted by 0 or 1. Due to noise, sometimes a transmitted 1 is received as 0 and vice versa.

If the probability that a transmitted 0 is correctly received as 0 is 0·9 and the probability that a transmitted 1 is correctly received as 1 is 0·8 and if the probability of transmitting 0 is 0·45, find the probability that (i) a 1 is received, (ii) a 0 is received, (iii) a 1 was transmitted given that 1 was received, (iv) a 0 was transmitted given that a 0 was received, (v) the error has occurred.

Sol. : We are given that

$P(T_0)$ = a 0 is transmitted = 0·45

$P(T_1)$ = a 1 is transmitted = $1 - P(T_0) = 1 - p_1 = 0{\cdot}55$

$P(R_0 / T_0)$ = a 0 is received when a 0 was transmitted = 0·9

$P(R_1 / T_0)$ = a 1 received when a 0 was transmitted

$= 1 - 0{\cdot}9 = 0{\cdot}1$

$P(R_1 / T_1)$ = a 1 is received when a 1 was transmitted = 0·8

$P(R_0 / T_1)$ = a 0 is received when a 1 was transmitted

$= 1 - 0{\cdot}8 = 0{\cdot}2.$

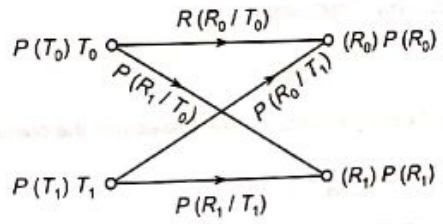

Fig. 7.11

Now, we calculate the required probabilities as follows :

(i) P (1 is received) = P (1 is received when 1 is transmitted) + P (1 is received when 0 is transmitted)

$$\therefore \quad P(R_1) = P(R_1 / T_1)\cdot P(T_1) + P(R_1 / T_0)\cdot P(T_0)$$

$$= 0{\cdot}8 \times 0{\cdot}55 + 0{\cdot}1 \times 0{\cdot}45$$

$$= 0{\cdot}485$$

(ii) P (0 is received) = P (0 is received when 0 is transmitted) + P (0 is received when 1 is transmitted)

$$\therefore \quad P(R_0) = P(R_0 / T_0)\cdot P(T_0) + P(R_0 / T_1)\cdot P(T_1)$$

$$= 0{\cdot}9 \times 0{\cdot}45 + 0{\cdot}2 \times 0{\cdot}55$$

$$= 0{\cdot}515$$

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Now, by Bayes' Theorem

(iii) P (1 was transmitted given that 1 was received) *i.e.*

$$P(T_1/R_1) = \frac{P(R_1/T_1)\,P(T_1)}{P(R_1)} = \frac{0.8 \times 0.55}{0.485} = 0.907$$

(iv) P (0 was transmitted given that 0 was received) *i.e.*

$$P(T_0/R_0) = \frac{P(R_0/T_0)\cdot P(T_0)}{P(R_0)} = \frac{0.9 \times 0.45}{0.515} = 0.786$$

(v) P (Error) = P (0 was received when 1 is transmitted given that 1 was transmitted) + P (1 was received when 0 was transmitted given that 0 was transmitted)

$$= P(R_0/T_1)\cdot P(T_1) + P(R_1/T_0)\cdot P(T_0)$$

$$= 0.2 \times 0.55 + 0.1 \times 0.45 = 0.155.$$

Example 9 : A box contains three biased coins A, B and C. The probability that a head will result when A is tossed is 1/3, when B is tossed, it is 2/3 and when C is tossed, it is 3/4.

(a) If one of the coins is chosen at random and is tossed 3 times, head resulted twice and tail once. What is the probability that the coin chosen was A ?

(b) What is the probability of getting head when a coin selected at random is tossed once ?

(c) What is the probability that we would get two heads in the first three tosses and a head again in the fourth toss with the same coin ?

Sol. : We have p_1 = Probability of choosing $A = \frac{1}{3}$.

p_2 = Probability of choosing $B = \frac{1}{3}$.

p_3 = Probability of choosing $C = \frac{1}{3}$.

p_1' = Probability of getting 2 heads in three tosses with the coin A

$$= {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = 3\cdot\frac{1}{9}\cdot\frac{2}{3} = \frac{2}{9}.$$

p_2' = Probability of getting 2 heads in three tosses with the coin B

$$= {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = 3\cdot\frac{4}{9}\cdot\frac{1}{3} = \frac{4}{9}.$$

p_3' = Probability of getting 2 heads in three tosses with the coin C

$$= {}^3C_2\left(\frac{3}{4}\right)^2\cdot\left(\frac{1}{4}\right) = 3\cdot\frac{9}{16}\cdot\frac{1}{4} = \frac{27}{64}.$$

$\therefore$ Required Probability $= \dfrac{p_1 p'_1}{p_1 p'_1 + p_2 p'_2 + p_3 p_3'}$

$$= \frac{(1/3)(2/9)}{(1/3)\cdot(2/9) + (1/3)\cdot(4/9) + (1/3)\cdot(27/64)}$$

$$= \frac{(2/9)}{(2/9) + (4/9) + (27/64)} = \frac{128}{627}.$$

(b) We do not know which coin was tossed.

If the coin was A, the probability that it will give head

= (Prob. of choosing A) × (Prob. of giving head)

$$= \frac{1}{3}\cdot\frac{1}{3} = \frac{1}{9}.$$

If the coin was B, probability of getting head.

= (Prob. of choosing B) × (Prob. of giving head)

$$= \frac{1}{3}\cdot\frac{2}{3} = \frac{2}{9}.$$

If the coin was C, probability of getting head

= (Prob. of choosing C) × (Prob. of giving head)

$$= \frac{1}{3}\cdot\frac{3}{4} = \frac{1}{4}$$

$\therefore$ The required probability $= \frac{1}{9} + \frac{2}{9} + \frac{1}{4} = \frac{7}{12}$.

(c) Now, getting two heads in the first three tosses and a head in the fourth toss with A

= P (Choosing A) × P (Getting two heads in three tosses with A)

× P (Getting a head in the fourth toss with A)

$$= p_1\cdot p_1'\cdot\frac{1}{3} = \frac{1}{3}\cdot\frac{2}{9}\cdot\frac{1}{3} = \frac{2}{81}.$$

Similarly, getting two heads in the first three tosses and a head in the fourth toss with B

$$= p_2\cdot p_2'\cdot\frac{2}{3} = \frac{1}{3}\cdot\frac{4}{9}\cdot\frac{2}{3} = \frac{8}{81}$$

And getting two heads in the first three tosses and a head in the fourth toss with C

$$= p_3\, p_3'\cdot\frac{3}{4} = \frac{1}{3}\cdot\frac{27}{64}\cdot\frac{3}{4} = \frac{27}{256}$$

$\therefore$ Required Probability $= \frac{2}{81} + \frac{8}{81} + \frac{27}{256} = 0.23.$

Example 10 : A bag contains five balls, the colours of which are not known. Two balls were drawn from the bag and they were found to be white. What is the probability that all balls are white ? **(M.U. 2005)**

Sol. : Since two balls drawn are white, the bag may contain 2 white or 3 white or 4 white or 5 white balls.

Let these events be denoted by A_1, A_2, A_3, A_4 respectively. We can assume that the probabilities of these events are equal.

Let $p_1 = P(A_1),\ p_2 = P(A_2),\ p_3 = P(A_3),\ p_4 = P(A_4)$.

$\therefore\ p_1 = p_2 = p_3 = p_4 = \frac{1}{4}$

Now, two balls out of 5 can be drawn in ${}^5C_2$ ways.

$\therefore\ p_1'$ = P (drawing two balls when two balls are white) $= \dfrac{{}^2C_2}{{}^5C_2} = \dfrac{2\cdot 1}{5\cdot 4} = \dfrac{2}{20}$.

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$$p_2' = P \text{ (drawing two white balls when 3 balls are white)} = \frac{{}^3C_2}{{}^5C_2} = \frac{3 \cdot 2}{5 \cdot 4} = \frac{6}{20}.$$

$$p_3' = P \text{ (drawing 2 white balls when 4 balls are white)} = \frac{{}^4C_2}{{}^5C_2} = \frac{4 \times 3}{5 \times 4} = \frac{12}{20}.$$

$$p_4' = P \text{ (drawing 2 white balls when 5 balls are white)} = \frac{{}^5C_2}{{}^5C_2} = \frac{5 \cdot 4}{5 \cdot 4} = \frac{20}{20}.$$

$\therefore$ By Baye's theorem Required Probability

$$= \frac{p_4 p_4'}{p_1 p_1' + p_2 p_2' + p_3 p_3' + p_4 p_4'}$$

$$= \frac{(1/4) \cdot (20/20)}{(1/4) \cdot (2/20) + (1/4) \cdot (6/20) + (1/4) \cdot (12/20) + (1/4) \cdot (20/20)}$$

$$= \frac{20}{2 + 6 + 12 + 20} = \frac{20}{40} = \frac{1}{2}.$$

Example 11 : There are three boxes containing respectively 1 white 2 red, 3 black balls; 2 white, 3 red and 1 black ball; 3 white, 1 red and 2 black balls. A box is chosen at random and two balls are drawn from it. The two balls are found to be one red and one white. Find the probability that those have come from box 1, box 2 and box 3.

Sol. : Since there are three boxes, say, A_1, A_2, A_3, then

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3} \qquad \therefore \quad p_1 = p_2 = p_3 = \frac{1}{3}$$

Let B be the event that one ball is white and the other is red. Then,

$$p_1' = P(B/A_1) = \frac{{}^1C_1 \cdot {}^2C_1}{{}^6C_2} = \frac{1 \cdot 2 \cdot 2}{6 \cdot 5} = \frac{2}{15}$$

$$p_2' = P(B/A_2) = \frac{{}^2C_1 \cdot {}^3C_1}{{}^6C_2} = \frac{2 \cdot 3 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

$$p_3' = P(B/A_3) = \frac{{}^3C_1 \cdot {}^1C_1}{{}^6C_2} = \frac{3 \cdot 1 \cdot 2}{6 \cdot 5} = \frac{6}{15}$$

By Baye's Theorem,

$$P(A_1/B) = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2' + p_3 p_3'}$$

$$\therefore \quad P(A_1/B) = \frac{(1/3)(2/15)}{(1/3)(2/15) + (1/3)(6/15) + (1/3)(3/15)} = \frac{2/45}{11/45} = \frac{2}{11}$$

$$P(A_2/B) = \frac{p_2 p_2'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{(1/3)(6/15)}{11/45} = \frac{6}{11}$$

$$P(A_3/B) = \frac{p_3 p_3'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{(1/3)(3/15)}{11/45} = \frac{3}{11}.$$

Example 12 : Three factories A, B, C produce 30 %, 50 % and 20 % of the total production of an item. Out of their production 80 %, 50 % and 10 % are defective. An item is chosen at random and found to be defective. Find the probability that it was produced by the factory A. **(M.U. 2019)**

Sol. : $p_1 = P$ (item is produced by A) = 0·3

$p_2 = P$ (item is produced by B) = 0·5

$p_3 = P$ (item is produced by C) = 0·2

Let D be the event that the item is defective then

$$p_1' = P(D/A) = 0{\cdot}8, \quad p_2' = P(D/B) = 0{\cdot}5, \quad p_3' = P(D/C) = 0{\cdot}1$$

Now, the required event is A/D.

$$\therefore \quad P(A/D) = \frac{p_1 p_1'}{p_1 p_1' + p_2 p_2' + p_3 p_3'} = \frac{0{\cdot}3 \times 0{\cdot}8}{0{\cdot}3 \times 0{\cdot}8 + 0{\cdot}5 \times 0{\cdot}5 + 0{\cdot}2 \times 0{\cdot}1}$$

$$= \frac{0{\cdot}24}{0{\cdot}24 + 0{\cdot}25 + 0{\cdot}02} = \frac{0{\cdot}24}{0{\cdot}51} = 0{\cdot}47.$$

EXERCISE - II

1. If $P(A) = 0{\cdot}7$, $P(B) = 0{\cdot}6$, $P(A/B) = 0{\cdot}4$, find $P(B/A)$. **[Ans. : 12 / 35]**

2. If A_1 and A_2 are two mutually exclusive and exhaustive events of sample space S and if B is another event defined on S, such that $P(A_1) = 0{\cdot}6$, $P(A_2) = 0{\cdot}4$ and $P(B/A_1) = 0{\cdot}4$, $P(B/A_2) = 0{\cdot}5$, find $P(B)$. **[Ans. : 0·44]**

3. A bag contains two coins one of which is a false coin with head on both sides and the other is a true coin. When a coin, taken at random from the bag, is tossed, it gave a head. What is the probability that the true coin was taken and tossed ? **[Ans. : 1 / 3]**

4. A bag contains two dice, one of which is regular and fair and the other is false with number 6 on all its faces. A dice was drawn from the bag and tossed. It gave 6. What is the probability that the dice obtained was the false one ? **[Ans. : 6 / 7]**

5. In a bolt factory, machines A, B, C produce respectively 25 %, 35 % and 40 %. Of their output 5 %, 4 % and 2 % are defective. A bolt is drawn at random from a days production and is found defective. What is the probability that it was produced by machines A, B, C ? **(M.U. 1997)**

[Ans. : 25 / 69, 28 / 69, 16 / 69]

6. A manufacturing firm produces steel pipes in three plants with daily production of 500, 1000 and 2000 units. According to past experience it is known that the fraction of defective output produced by the three plants are respectively 0·005, 0·008 and 0·010.

If a pipe is selected at random from a day's output and was found to be defective, what is the probability that it came from the first plant ? **[Ans. : 0·08]**

7. A given lot of I.C. chips contains 3 % defective chips. Each chip is tested before delivery. The tester itself is not totally reliable, such that P (tester shows the chip is good / chip is actually good) = 0·96 and P (tester shows chip is defective / chip is actually defective) = 0·94. If a tested chip is shown to be defective, what is the probability that it is actually defective ? **[Ans. : 0·42]**

8. A binary communication system transmits and also receives data as '0' and '1'. Due to noise a transmitted '0' is sometimes received as '1' and a transmitted '1' is some times received as '0'. The probability that a transmitted '0' is correctly received as '0' is 0·95 and the probability that a transmitted '1' is correctly received as '1' is 0·90, probability of transmitting '0' is 0·45. If a signal is sent find :

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(i) Probability that a '1' is received

(ii) Probability that a '0' is received.

(iii) Probability that a '1' is transmitted, given that a '1' is received.

(iv) Probability that a '0' is transmitted, given that a '0' is received.

[**Ans. : (i)** 0·5175, **(ii)** 0·4825, **(iii)** 0·9565, **(iv)** 0·8860, **(v)** 0·0775]

9. A newly constructed flyover is likely to collapse. The chance that the design is faulty is 0·5. The chance that the flyover will collapse if the design is faulty is 0·95 otherwise it is 0·30. The flyover collapsed. What is the probability that it collapsed because of faulty design ? **(M.U. 2004)**

[**Ans. :** 0·76]

10. Three identical urns have the following composition of black and white balls :

First urn : 2 black, 1 white

Second urn : 1 black, 2 white

Third urn : 2 black, 2 white

One of the urns is selected at random and one ball is drawn. What is the probability of drawing a white ball ? [**Ans. :** 1/ 2]

11. There are three urns having the following compositions of black and white balls :

Urn 1 : 7 white, 3 black balls,

Urn 2 : 4 white, 6 black balls,

Urn 3 : 2 white, 8 black balls.

One of the urns is chosen at random with probabilities 0·2, 0·6 and 0·2 respectively. From the chosen urn two balls are drawn at random without replacement. Calculate the probabilities that both these balls are white. [**Ans. :** 8 / 45]

12. Three boxes A, B, C contain 30, 50, 20 bulbs respectively. The percentages of defective bulbs in these boxes are 4, 3 and 5 respectively. A box is selected at random and a bulb is drawn. What is the probability that the bulb will be defective ? [**Ans. :** 0·037]

13. A man is equally likely to choose any one of three routes C_1, C_2, C_3 from his house to the railway station. The probabilities of missing the train by the routes C_1, C_2 and C_3 are 2 / 5, 3 / 10, 1 / 20. He sets out on a day and misses the train. What is the probability that the route C_3 was selected ? [**Ans. :** 1/ 15]

14. A factory manufacturing television sets gets its supply of components from 4 units A, B, C, D in the following proportion 15%, 20%, 30% and 35% respectively. It was found that of the supply received 1%, 2%, 2% and 3% articles are defective. A television set was chosen at random from the total output and was found to be defective. What is the probability that it came from unit D ?

(M.U. 2002) [**Ans. :** 21 / 44]

15. A man has three coins A, B, C. A is unbiased. The probability that head will result when B is tossed is 2 / 3. The probability that head will result when C is tossed is 1/ 3. If one of the coins is chosen at random and is tossed three times it gave two heads and one tail. Find the probability that the coin A was chosen. **(M.U. 1997, 2002)** [**Ans. :** 9 / 25]

16. Bag I contains 6 blue and 4 red balls.

Bag II contains 2 blue and 6 red balls.

Bag III contains 1 blue and 8 red balls.

A bag is chosen at random and two balls are drawn from it. If both the balls were found to be blue, find the probability that bag II was chosen. **(M.U. 2004)** [**Ans. :** 3 / 31]

17. A coin is tossed. If it turns up head two balls are drawn from urn A , otherwise two balls are drawn from urn B. Urn A contains 3 black and 5 white balls. Urn B contains 7 black and one white ball. In both cases selection is made with replacement. What is the probability that A was chosen given that both the balls drawn are black. **(M.U. 2002)** [**Ans. :** 9 / 58]

18. An urn A contains ten red and three black balls. Another urn B contains three red and five black balls. Two balls are transferred from urn A to the urn B without noticing their colour. One ball now is drawn from the urn B and it is found to be red. What is the probability that one red and one black ball have been transferred ? [**Ans. :** 20 / 59]

19. There are three boxes A, B and C. The probability of getting a white ball from the box A is 1/ 3, from the box B is 2 / 3 and from the box C is 3 / 4.

A box is chosen at random and three balls are drawn from it (without replacement) and it was found that two of them were white.

(a) What is the probability that the box A was chosen ?

(b) If a ball is drawn from a box selected at random, what is the probability that it will be white ?

(c) What is the probability that we would get two white balls in the first three draws and a white ball again in the fourth draw ?

(**Hint :** See Solved Ex. 9) [**Ans. : (a)** 128 / 627, **(b)** 7 / 12, **(c)** 0·23]

20. For a certain binary communication channel, the probability that a transmitted '0' is received as a '0' is 0·95 while the probability that a transmitted '1' is received as '1' is 0.90. If the probability of transmitting a '0' is 0·4, find the probability that (i) a '1' is received, (ii) a '1' was transmitted given that '1' was received. [**Ans. : (i)** 0·56, **(ii)** 27 / 28]

8. Random Variable

In the previous chapter we learnt how to find the probability of an outcome and the laws of probability. We saw that the outcome of an experiment can be anything : it may be colour (black-white-red .....) of a ball, a gender (male-female) of a child, a suit (club-diamond-spade-heart) of a card or a number (1 – 2 – 3 – 4 – 5 – 6) of a die or a logical answer (yes - no) of a question or result of a toss (head - tail) of a coin. In most of the problems the outcome of an experiment is a number *e.g.* the salary of a person, the height of a student, the temperature at a place, the rainfall on a particular day. However, when the outcome is not a number we can express these outcomes in numbers by agreeing to denote,

(a) head by 1 and tail by 0 (b) boy by 1 and girl by 0

(c) yes by 1 and no by 0 (d) club by 1, diamond by 2, spade by 3 and heart by 4

(e) red by 1, white by 2 and black by 3, etc.

In probability problems it is found convenient to think of a variable and consider the values of the variable which describe the outcomes of the experiment. In the toss of a coin the variable takes values 1 and 0; in the selection of a child it takes values 1 and 0, in the answers of the question it takes the values 1 and 0, in drawing a card it takes values 1, 2, 3, 4, in drawing a ball it takes values 1, 2, 3 etc. This variable may take discrete values or may take any value in a range continuously *e.g.* the arrival time of a bus at a stop, say, between 9 a.m. and 9.10 a.m. may be any number between the range. Such a variable is called a **random variable**. Actually random variable is a misnomer. It is a function which assigns a real number to the outcome of an experiment. A random variable is denoted by X and a particular value of X is denoted by x. In the toss of a coin X assigns the value 1 to H and 0 to T, in the selection of a ball X assigns value 1 to red, 2 to white and 3 to black.

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In these cases X takes discrete values and is called a discrete random variable. But in the case of arrival time of a bus X takes continuously any value between 9 am. and 9.10 am. and hence X is a continuous random variable. In this way, we consider X as a function from sample space S to the set of real numbers R. Thus we get the following definition.

(a) Definition

Let E be an experiment and S be the sample space associated with it. A function X assigning to every element s of S one and only one real number $x = X(s)$ of R is called a *random variable*.

Since X is a function whose domain is the set of outcomes of an experiment and whose range is a part or the whole of real line ($-\infty < x < \infty$), it can be shown pictorially as a mapping from the sample space to the real line.

The random variable X can be discrete or continuous depending upon the nature of its domain.

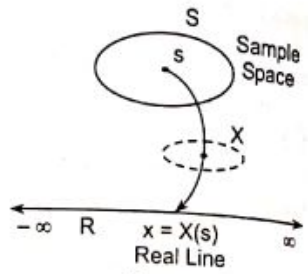

Fig. 7.12

Remarks ....

1. In simple words a variable used to denote the numerical value of the outcome of an experiment is called the random variable, abbreviated as r.v.
2. X is a *function* and still we call it a variable.
3. We are not interested in the *functional nature* of X but in the *values of X*.
4. X must be single valued *i.e.* for every s of S there corresponds *exactly one value* of X. Different elements of S may lead to the same value of X (See Example 2). But two values of X cannot be assigned to the same sample point.
5. We shall denote *random variables* by capital letters $X, Y, Z, \ldots$ and shall denote the *unknown values* of these random variables by small letters $x, y, z, \ldots, x_1, x_2, \ldots y_1, y_2, \ldots z_1, z_2, \ldots$ etc. This is an important distinction and students should note it carefully. With this notation it is meaningless to write $P(x \geq 10)$ say since x being a value of X either is or is not ≥ 10. Instead we should write $P(X \geq 10)$.

Example 1 : Suppose the experiment E is to toss a fair coin.

Then $S = \{H, T\}$. If X is the random variable denoting the number of heads then we have $X(H) = 1$ and $X(T) = 0$. [See Fig. 7.13]

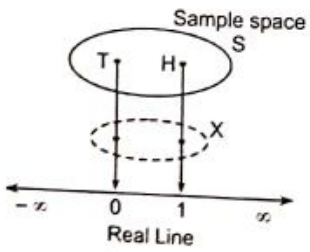

Fig. 7.13

Example 2 : Suppose the experiment E is to toss two fair coins.

Then $S = \{(H, H), (H, T), (T, H), (T, T)\}$. If X is the random variable denoting the number of heads then $X(H, H) = 2$, $X(H, T) = 1$, $X(T, H) = 1$, $X(T, T) = 0$.

[See Fig. 7.14]

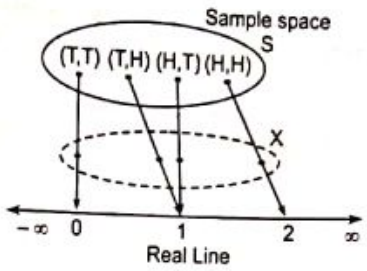

Fig. 7.14

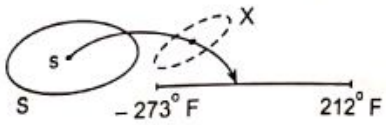

Fig. 7.15

Example 3 : Suppose the experiment is to record the temperature at a place.

If X denoted the temperature then X can take any value from -273° F to 212° F say. [See Fig. 7.15]

Example 4 : Suppose the experiment is to record the time required to complete a software project.

If X denotes the time then X can take any value (theoretically) from 0 to ∞. [See Fig. 7.16]

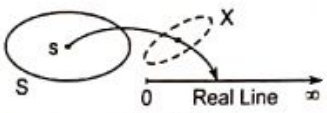

Fig. 7.16

In examples 1 and 2, X takes discrete values and in examples 3 and 4, X takes continuously all values between a specified interval. In the first case X is called a ***discrete random variable*** in the second case it is called ***continuous random variable***.

(b) Definition

Let X be a random variable. If X takes finite or countably infinite values $x_0, x_1, x_2, \ldots$ then X is called a ***discrete random variable***.

(c) Definition

Let X be a random variable. If X takes uncountably infinite values in a given interval then X is called a ***continuous random variable***.

9. Probability Distribution of a Discrete Random Variable

As we know already, with every possible outcome of an experiment there will be associated its probability. We shall be interested in the values of the random variable X along with their probabilities. If x_i is the value of X and $P(x_i)$ is the probability of x_i then set of pairs $\{x_i, P(x_i)\}$ is called the **probability distribution**.

Definition : Let X be a **discrete random variable**. Let $x_1, x_2, \ldots x_n \ldots$ be the possible values of X. With each possible outcome x_i we associate a number $p(x_i) = P(X = x_i)$ called the probability of x_i. The numbers $p(x_i)$, $i = 1, 2, \ldots n \ldots$ must satisfy the following conditions :

1. $p(x_i) \geq 0$ for all i
2. $\sum_{i=1} p(x_i) = 1$

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The function p is called the **probability function** or **probability mass function (p.m.f)** or **probability density function (p.d.f.)** of the random variable X and the set of pairs (x_i, p_i) is called the probability distribution of X.

The probability distribution of a discrete random variable X taking values $x_1, x_2, x_3 \ldots\ldots x_n \ldots\ldots$ with probabilities $p_1, p_2, p_3 \ldots\ldots p_n \ldots\ldots$ where $p_i \geq 0$ and $\Sigma\, p_i = 1$ can be given in tabular form as

X	x_1	x_2	x_3	.....	x_n	.....
$P(x_i)$	p_1	p_2	p_3	.....	p_n	.....

Example 1 : State true or false with justification :

(a) A random variable X takes values 0, 1, 2 and 3 then $p(X = x) = \dfrac{x-1}{2}$ can be its probability distribution.

(b) A random variable takes values 0, 1, 2 and $p(x) = \dfrac{x+1}{3}$ is its probability distribution.

Sol. : As seen above for a probability distribution, two conditions must be satisfied.

(i) Each probability must be equal to or greater than zero but less than one.

(ii) The sum of all probabilities must be equal to unity.

Putting $x = 0, 1, 2, 3$ in (a), we get

$$P(0) = -\frac{1}{2},\ P(1) = 0,\ P(2) = \frac{1}{2},\ P(3) = 1$$

Since, the probability cannot be negative, $P(X = x) = \dfrac{x-1}{2}$ cannot be a probability distribution.

Putting $x = 0, 1, 2, 3$, in (b), we get

$$P(0) = \frac{1}{3},\ P(1) = \frac{2}{3},\ P(2) = 1.$$

Although all probabilities are positive, the sum of all the probabilities is 2, (greater than 1). Hence, $P(X = x) = \dfrac{x+1}{3}$ also cannot be a probability distribution.

Example 2 : From the past experience it was found that the daily demand at an autogarage was as under.

Daily Demand	:	5	6	7
Probability	:	0·25	0·65	0·10

Check if this is a probability distribution. Find also the probability that over a period of two days the number of demands would be 11 or 12.

Sol. : Since the sum of all probabilities = 0·25 + 0·65 + 0·10 = 1, it is a probability distribution.

P (11 requests over two days)

= P (5 requests on the first day and 6 on the second)

+ P (6 request on the first day and 5 on the second)

= (0·25 × 0·65) + (0·65 × 0·25)

= 0·1625 + 0·1625 = 0·325

P (12 requests over two days)

= P (5 requests on the first day and 7 on the second)

+ P (6 requests on the first day and 6 on the second)

+ P (7 requests on the first day and 5 on the second)

= (0·25 × 0·10) + (0·65 × 0·65) + (0·10 × 0·25)

= 0·025 + 0·4224 + 0·025 = 0·7975.

Example 3 : Find the probability distribution of number of heads (X) obtained when a fair coin is tossed 4 times.

Sol. : When a coin is tossed 4 times, there are $2^4 = 16$ outcomes which are listed below.

HHHH, HHHT, HHTH, HHTT, HTHH, HTHT, HTTH, HTTT,

THHH, THHT, THTH, THTT, TTHH, TTHT, TTTH, TTTT.

(Write first H, T, H, T alternately for 16 times. Then 2 H's and 2 T's alternately, then 4 H's and 4 T's alternatively and lastly 8 H's and 8 T's alternately). Hence, the probability distribution of X is

X	:	0	1	2	3	4
$P(X = x)$	:	1 / 16	4 / 16	6 / 16	4 / 16	1 / 16

Example 4 : Write down the probability distribution of the sum of numbers appearing on the toss of two unbiased dice. **(M.U. 2004, 06)**

Sol. : When two dice are thrown, we get the sum of numbers as shown below by slant dotted lines.

(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

It is easy to see that the sum 2 appears once, 3 twice, 4 thrice etc. The probability of each single event is 1/ 36.

The probability distribution obtained is shown below.

X	2	3	4	5	6	7	8	9	10	11	12
$P(X = x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Example 5 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9. **(M.U. 1997)**

Sol. : $P(X \text{ is odd}) = P(X = 3, 5, 7, 9 \text{ or } 11)$

$= P(X = 3) + P(X = 5) + P(X = 7) + P(X = 9) + P(X = 11)$

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$\therefore\ P(X \text{ is odd}) = \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36} = \frac{18}{36} = \frac{1}{2}$

$P(3 \le X \le 9) = P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9)$

$= P(X = 3) + P(X = 4) + \ldots + P(X = 9)$

$= \frac{2}{36} + \frac{3}{36} + \ldots + \frac{4}{36} = \frac{29}{36}.$

Example 6 : Two unbiased dice are thrown. Let x_1 and x_2 denote the scores obtained and y denote the maximum of them *i.e.* $y = \max.(x_1, x_2)$. Find the probability distribution of Y.

Sol. : Refer to the chart of Ex. 4 above. It is easy to see that y takes values, 2, 3, 4, 5, 6 with probabilities 1/36, 3/36, 5/36, 7/36, 9/36, 11/36 respectively as shown by thick arrows.

Y	1	2	3	4	5	6
$P(Y = y)$	1/36	3/36	5/36	7/36	9/36	11/36

Example 7 : For the above distribution, (i) find the probability that X is an odd number, (ii) find the probability that X lies between 3 and 9.

Sol. : $P(X \text{ is odd}) = P(X = 3, 5, 7, 9 \text{ or } 11)$

$= P(X = 3) + P(X = 5) + P(X = 7) + P(X = 9) + P(X = 11)$

$= \frac{2}{36} + \frac{4}{36} + \frac{6}{36} + \frac{4}{36} + \frac{2}{36} = \frac{18}{36} = \frac{1}{2}$

$P(3 \le X \le 9) = P(X = 3, 4, 5, 6, 7, 8 \text{ or } 9)$

$= P(X = 3) + P(X = 4) + \ldots + P(X = 9)$

$= \frac{2}{36} + \frac{3}{36} + \ldots + \frac{4}{36} = \frac{29}{36}.$

Example 8 : The probability mass function of a random variable X is zero except at the points $X = 0, 1, 2$. At these points it has the values $P(0) = 3c^3$, $P(1) = 4c - 10c^2$, $P(2) = 5c - 1$.

(i) Determine c. (ii) Find $P(X < 1)$, $P(1 < X \le 2)$, $P(0 < X \le 2)$. **(M.U. 2001)**

Sol. : Since $\Sigma p_i = 1$, we have, $P(0) + P(1) + P(2) = 1$.

$\therefore\ 3c^3 - 10c^2 + 4c + 5c - 1 = 1 \qquad \therefore\ 3c^3 - 10c^2 + 9c - 2 = 0$

$(3c - 1)(c - 2)(c - 1) \qquad \therefore\ c = 1/3$

(The other values are not admissible. Why?)

$\therefore$ The probability distribution is

X	0	1	2
$P(X = x)$	1/9	2/9	2/3

$\therefore\ P(X < 1) = P(X = 0) = \frac{1}{9}; \quad P(1 < X \le 2) = P(X = 2) = \frac{2}{3};$

$P(0 < X \le 2) = P(X = 1) + P(X = 2) = \frac{2}{9} + \frac{2}{3} = \frac{8}{9}.$

Example 9 : A random variable X has the following probability distribution

X	:	0	1	2	3	4	5	6	7
$P(X = x)$	:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k. (ii) $P\left(\frac{1.5 < X < 4.5}{X > 2}\right)$. (iii) The smallest value of λ for which $P(X \le \lambda) > \frac{1}{2}$.

Sol. : (i) Since $\Sigma p(x_i) = 1$, we get

$10k^2 + 9k = 1 \qquad \therefore\ 10k^2 + 9k - 1 = 0$

$\therefore\ 10k^2 + 10k - k - 1 = 0 \qquad \therefore\ 10k(k + 1) - 1(k + 1) = 0$

$\therefore\ (10k - 1)(k + 1) = 0 \qquad \therefore\ k = 1/10.$ k cannot be -1.

$\therefore$ The probability distribution of X is

X	:	0	1	2	3	4	5	6	7
$P(X = x)$	:	0	1/10	2/10	2/10	3/10	1/100	2/100	17/100

(ii) Now, $P(A/B) = \frac{P(A \cap B)}{P(B)}$

$\therefore\ P\left(\frac{1.5 < X < 4.5}{X > 2}\right) = \frac{P[(1.5 < X < 4.5) \cap (X > 2)]}{P(X > 2)} = \frac{P(2 < X < 4.5)}{P(X > 2)}$

$= \frac{P(X = 3, 4)}{P(X = 3, 4, 5, 6, 7)} = \frac{5/10}{70/100} = \frac{5}{7}.$

Now from the table we find that

$P(X \le 3) = P(X = 0) + P(X = 1) + P(X = 2) + P(X = 3)$

$= 0 + \frac{1}{10} + \frac{2}{10} + \frac{2}{10} = \frac{5}{10} = \frac{1}{2}.$

Hence, $P(X \le 4) = \frac{8}{10} > \frac{1}{2}$. Hence, $\lambda = 4$.

Example 10 : An urn contains 4 white and 3 red balls. Find the probability distribution of the number of red balls in three draws made successively with replacement from the urn. **(M.U. 2007)**

Sol. : We get the following probabilities.

No. of red balls	Probability
0	$\frac{4}{7} \cdot \frac{4}{7} \cdot \frac{4}{7}$
1	$\frac{3}{7} \cdot \frac{4}{7} \cdot \frac{4}{7}$, $\frac{4}{7} \cdot \frac{3}{7} \cdot \frac{4}{7}$, $\frac{4}{7} \cdot \frac{4}{7} \cdot \frac{3}{7}$
2	$\frac{3}{7} \cdot \frac{3}{7} \cdot \frac{4}{7}$, $\frac{3}{7} \cdot \frac{4}{7} \cdot \frac{3}{7}$, $\frac{4}{7} \cdot \frac{3}{7} \cdot \frac{3}{7}$
3	$\frac{3}{7} \cdot \frac{3}{7} \cdot \frac{3}{7}$

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∴ Probability distribution is

x	0	1	2	3	Total
$p(x)$	$\frac{64}{343}$	$\frac{144}{343}$	$\frac{108}{343}$	$\frac{27}{343}$	1

Example 11 : A random variable X has the following probability function :

X	:	1	2	3	4	5	6	7
$P(X = x)$	:	k	$2k$	$3k$	k^2	$k^2 + k$	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X < 5)$, (iii) $P(X > 5)$, (iv) $P\left(\frac{X < 5}{2 < X \le 6}\right)$, (v) $P\left(\frac{X = 4}{3 \le X \le 5}\right)$.

(M.U. 2019)

Sol. : Since $\Sigma\, p(x_i) = 1$,

$$k + 2k + 3k + k^2 + k^2 + k + 2k^2 + 4k^2 = 1$$

$\therefore\ 8k^2 + 7k - 1 = 0 \quad \therefore\ (8k - 1)(k + 1) = 0$

$\therefore\ k = 1/8$ or $k = -1$ which is impossible (why ?)

Thus, we have the following probability distribution.

X	:	1	2	3	4	5	6	7
$P(X = x)$	:	1/8	2/8	3/8	1/64	9/64	2/64	4/64

(i) $P(X < 5) = P(X = 1, 2, 3, 4,)$

$= P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$

$= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{1}{64} = \frac{49}{64}$.

(ii) $P(X > 5) = P(X = 6, 7) = P(X = 6) + P(X = 7)$

$= \frac{2}{64} + \frac{4}{64} = \frac{6}{64} = \frac{3}{32}$.

(iii) $P\left(\frac{X < 5}{2 < X \le 6}\right) = \frac{P(X < 5 \cap 2 < X \le 6)}{P(2 < X \le 6)} = \frac{P(2 < X < 5)}{P(2 < X \le 6)}$

$= \frac{P(X = 3, 4)}{P(X = 3, 4, 5, 6)} = \frac{25/64}{36/64} = \frac{25}{36}$.

(iv) $P\left(\frac{X = 4}{3 \le X \le 5}\right) = \frac{P(X = 4 \cap 3 \le X \le 5)}{P(3 \le X \le 5)}$

$= \frac{P(X = 4)}{P(X = 3, 4, 5)} = \frac{1/64}{34/64} = \frac{1}{64}$.

Example 12 : The probability of a man hitting the target is 1/4. How many times must he fire so that the probability of his hitting the target at least once is greater than 2/3? **(M.U. 2016)**

Sol. : We are given that the probability of hitting the target $p = 1/4$.

Probability of not hitting the target, $q = 1 - \frac{1}{4} = \frac{3}{4}$.

∴ Probability of not hitting the target in n trials

$$= \left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\ldots\ldots(n \text{ times}) = \left(\frac{3}{4}\right)^n$$

∴ Probability of hitting the target at least once in n trials

$$= 1 - \left(\frac{3}{4}\right)^n$$

We want this probability to be greater 2/3.

$\therefore\ 1 - \left(\frac{3}{4}\right)^n > \frac{2}{3} \qquad \therefore\ 1 - \left(\frac{2}{3}\right) > \left(\frac{3}{4}\right)^n$

$\therefore\ \frac{1}{3} > \left(\frac{3}{4}\right)^n \qquad \therefore\ \left(\frac{3}{4}\right)^n < \frac{1}{3}$

Taking logarithms of both sides

$n\,(\log 3 - \log 4) < -\log 3$

$\therefore\ n\,(0{\cdot}4771 - 0{\cdot}6021) < -\log 3 \qquad \therefore\ -n\,(0{\cdot}1250) < -0{\cdot}4771$

$\therefore\ n > \frac{0{\cdot}4771}{0{\cdot}1250} = 3{\cdot}81 \qquad \therefore\ n > 4$

∴ He must fire at least 4 times, so that he will hit the target at least once, with probability of success greater 2/3.

EXERCISE - II

1. If $P(X = x) = x/25$, $x = 1, 3, 5, 7, 9$, find $P(X = 1 \text{ or } 3)$ and $P(4 < X < 8)$.

[Ans. : (i) 4 / 25, (ii) 12 / 25]

2. Verify whether the following functions can be considered as p.m.f. and if so find $P(X = 1 \text{ or } 3)$. Give reasons.

(i) $P(X = x) = \frac{1}{5}$, $x = 0, 1, 2, 3, 4$. (ii) $P(X = x) = \frac{x^2 + 1}{18}$, $x = 0, 1, 2, 3$.

(iii) $P(X = x) = \frac{x^2 - 2}{8}$, $x = 1, 2, 3$. (iv) $P(X = x) = \frac{2x + 1}{18}$, $x = 0, 1, 2, 3$.

[Ans. : (i) Yes, 2 / 5 ; (ii) Yes, 2 / 3 ; (iii) No ; (iv) No]

3. If the p.m.f. $P(X = x)$ of a discrete random variate which assumes values x_1, x_2, x_3 such that $P(x_1) = 2P(x_2) = 3P(x_3)$, obtain the probability distribution of X.

[Ans. : (i) $P(x_1) = 2/11$, (ii) $P(x_2) = 3/11$, (iii) $P(x_3) = 6/11$]

4. Presuming the daily demand to be independent, find the probability that over a two -days period the number of requests at a service station will be (i) 9, (ii) 10, if the past record show that the demand was 4, 5 or 6 with probabilities 0·50, 0·40 or 0·10 respectively.

(Hint : The event can occur as (i) (4, 5), (5, 4); (ii) (4, 6), (6, 4), (5, 5).)

[Ans. : (i) 0.40, (ii) 0·26]

5. Find the probability distribution and the probability mass function of the number of points obtained when a fair die is tossed. **[Ans. : $P(X = x_i) = 1/6$, $i = 1, 2, 3, 4, 5, 6$]**

6. Find the probability distribution and the p.m.f. of the number of heads obtained when an unbiased coin is tossed three times.

$\left[\textbf{Ans. : } P(X = x) = \frac{{}^3C_x}{2^3},\ x = 0, 1, 2, 3\right]$

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7. The probability density function of a random variable X is

X	0	1	2	3	4	5	6
$P(X = x)$	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X < 4)$, $P(3 < X \le 6)$. **(M.U. 2001, 05, 10, 15, 19)**

[Ans. : $k = 1/49$, $16/49$, $33/49$]

8. A random variable X has the following probability function

X	1	2	3	4	5	6	7
$P(X = x)$	k	$2k$	$3k$	k^2	$k^2 + k$	$2k^2$	$4k^2$

Find (i) k, (ii) $P(X < 5)$, (iii) $P(X > 5)$, (iv) $P(0 \le X \le 5)$.

[Ans. : (i) $k = 1/8$, (ii) $49/64$, (iii) $3/32$, (iv) $29/32$]

9. A discrete random variable X has the following probability distribution

X	-2	-1	0	1	2	3
$P(X = x)$	0·1	k	0·2	$2k$	0·3	$3k$

Find (i) k, (ii) $P(X \ge 2)$, (iii) $P(-2 < X < 2)$. **(M.U. 2009)**

[Ans. : (i) $k = 1/15$, (ii) $1/2$, (iii) $2/5$]

10. Given the following probability function of a discrete random variable X

X	0	1	2	3	4	5	6	7
$P(X = x)$	0	c	$2c$	$2c$	$3c$	c^2	$2c^2$	$7c^2 + c$

(i) Find c, (ii) Find $P(X \ge 6)$, (iii) $P(X < 6)$, (iv) Find k if, $P(X \le k) > 1/2$, where k is a positive integer, (v) $P(1{\cdot}5 < X < 4{\cdot}5 / X > 2)$. **(M.U. 1996, 2003, 05)**

[Ans. : (i) $c = 0{\cdot}1$, (ii) 0·19, (iii) 0·81, (iv) $k = 4$, (v) $5/7$]

11. In the example 9 above, find

(i) $P(-1 \le X \le 1 / -2 \le X \le 3)$, (ii) $P(X \le 2 / 0 \le X \le 4)$, (iii) $P(X \le 1 / X \le 2)$.

[Ans. : (i) $\frac{1}{2}$, (ii) $\frac{19}{26}$, (iii) $\frac{15}{24}$.]

12. A random variable X takes values $-2, -1, 0, 1, 2$ such that $P(X > 0) = P(X = 0) = P(X < 0)$; $P(X = -2) = P(X = -1)$, $P(X = 1) = P(X = 2)$. Obtain the probability distribution of X.

Also find (i) $P\left(\frac{-1 \le X \le 1}{-2 \le X \le 0}\right)$, (ii) $P\left(\frac{X = 1}{0 \le X \le 2}\right)$.

[Ans. :

X :	-2	-1	0	1	2
$P(X = x)$:	1/6	1/6	1/3	1/6	1/6

(i) 4/5, (ii) 1/5]

13. A random variable X takes values 0, 1, 2, ... n with probabilities proportional to ${}^nC_0$, ${}^nC_1$, ${}^nC_2$, ... ${}^nC_n$. Find the proportionality constant.

(**Hint :** $k\,[{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_n] = 1 \quad \therefore \; k \cdot 2^n = 1$) **[Ans. :** $k = 2^{-n}$]

14. A random variable X assumes four values with probabilities $(1 + 3x)/4$, $(1 - x)/4$, $(1 + 2x)/4$ and $(1 - 4x)/4$. For what value of x do these values represent the probability distribution of X? **(M.U. 2004)**

[Ans. : $\Sigma p_i = 1$. But $\frac{1+3x}{4} \ge 0$ if $x \ge -\frac{1}{3}$ and also $\frac{1-4x}{4} \ge 0$ if $x \le \frac{1}{4}$ $\therefore -\frac{1}{3} \le x \le \frac{1}{4}$]

15. The amount of bread X (in hundred kgs) that a certain bakery is able to sell in a day is a random variable with probability density function is given by

$$f_X(x) = \begin{cases} Ax, & 0 < x < 5 \\ A(10 - x), & 5 < x < 10 \\ 0, & \text{elsewhere} \end{cases}$$

Find (i) A, (ii) the probabilities of the events : B, the amount of bread sold in a day is more than 500 kgs, C : the amount of bread sold in a day is less than 500 kgs, D : the amount of bread sold in a day is between 250 kgs and 750 kgs. (iii) Are the events B and C exclusive ? (iv) Are the events B and D exclusive. **(M.U. 2005) [Ans. : (i)** $A = 1/25$, **(ii)** $1/2$, $1/2$, 0.75, **(iii)** Yes, **(iv)** No]

10. Distribution Function of a Discrete Random Variable X

Probability distribution of X gives us the probability $p(x)$ that X will take a particular value x. Sometimes we need to know the probability that X will take a value less than or equal to a given value x. This probability is obtained by adding the probabilities of all values less than or equal to x.

Suppose, X is a discrete random variable taking values $x_1, x_2 \ldots\ldots x_n$ with probabilities $p(x_i)$, $i = 1, 2, \ldots\ldots n \ldots\ldots$ such that

(i) $p(x_i) \ge 0$ for all i,

(ii) $\Sigma\, p(x_i) = 1$ and consider the following table.

X	x_1	x_2	x_3	.....	x_n	.....
$F(x_i) = F(X = x_i)$	$p(x_i)$	$\sum_1^2 p(x_i)$	$\sum_1^3 p(x_i)$	.....	$\sum_1^n p(x_i)$	.....

The table states that

$$F(x_1) = P(a \le X < x) = p(x_1)$$
$$F(x_2) = P(X \le x_2) = p(a \le X \le x_1) + p(x_1 \le X \le x_2)$$
$$= p(x_1) + p(x_2)$$
$$F(x_3) = P(X \le x_3) = p(a \le X \le x_1) + p(x_1 \le X \le x_2) + p(x_2 \le X \le x_3)$$
$$= p(x_1) + p(x_2) + p(x_3).$$

And so on.

The graph shown in the Fig. 7.17 shows this diagrammatically.

The function F is called the **distribution function**. We have more precise definition as follows.

(a) Definition

Let X be a discrete random variable taking values $x_1, x_2, \ldots\ldots$ such that $x_1 < x_2 < x_3 \ldots\ldots$ with probabilities $p(x_1), p(x_2), \ldots\ldots$ such that $p(x_i) \ge 0$ for all i and $\Sigma\, p(x_i) = 1$.

Consider F defined by $\quad F(x_i) = p(X \le x_i)$, $i = 1, 2, 3, \ldots\ldots$

i.e. $\quad F(x_i) = p(x_1) + p(x_2) + \ldots\ldots + p(x_i)$

then the function F is called the **cumulative distribution function** or simply **distribution function** and the set of pairs $\{x_i, F(x_i)\}$ is called the **cumulative probability distribution.**

Note ....

The distribution function is to the probability mass function as cumulative frequency distribution is to frequency distribution.

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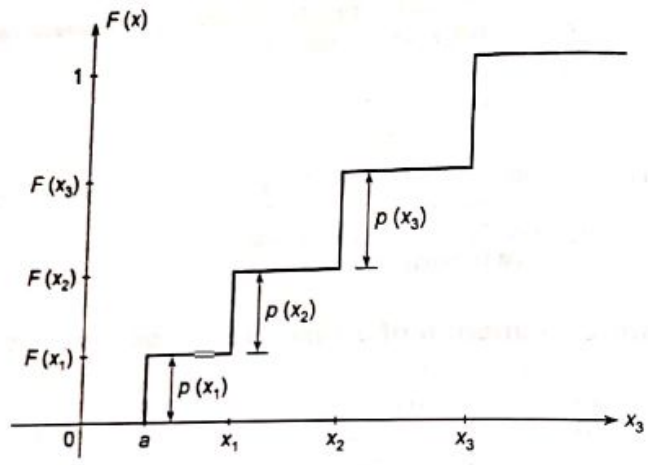

Fig. 7.17 : Graph of Distribution Function.

(b) Important Properties of Distribution function

The distribution function F of a random variable X has the following important properties.

1. $0 \le F(x) \le 1$

Proof : Since, $F(x_1) = p(x_1)$, $F(x_2) = p(x_1) + p(x_2)$

..... $F(x_n) = p(x_1) + p(x_2) + ... + p(x_n)$

and $0 \le p(x_i) \le 1$ for every i and $\Sigma\, p(x_i) = 1$, it is clear that $0 \le F(x) \le 1$.

2. $F(x) = 0$ for $x < a$ and $F(x) = 1$ for $x > b$ where $a < x_1 < x_2 < ..... x_n < b$.

Proof : If $x < a$ where $a < x_1$, $p(x) = 0$ and hence, $F(x) = 0$ for $x < a$.

If $x > b$ then $F(x) = p(x_1) + p(x_2) + ... + p(x_n) = 1$.

3. $F(x)$ is a step function

Proof : By definition, $F(x_1) = F(X < x_1) = p(x_1)$

$F(x_2) = F(X < x_2) = p(x_1) + p(x_2)$

i.e. $F(X)$ has the same value $p(x_1)$ for $x_1 \le x \le x_2$,

and the same value $p(x_1) + p(x_2)$ for $x_2 \le x \le x_3$ .....

Hence, the graph of $F(x)$ is made up of horizontal line segments taking "jumps" at the possible values x_i of X. The jump is of magnitude $p(x_i) = P(X = x_i)$.

Hence, $F(x)$ is a step functions.

Example 1 : A random variable X has the probability function given below :

$f(x) = k$ if $x = 0$; $f(x) = 2k$ if $x = 1$;

$f(x) = 3k$ if $x = 2$; $f(x) = 0$ otherwise.

(i) Determine the value of k. (ii) Evaluate $P(X < 2)$, $P(X \le 2)$, $P(0 < X < 2)$, (iii) Obtain the distribution function.

Sol. : The probability distribution can be tabulated as

X	0	1	2
$p(x)$	k	$2k$	$3k$

(i) Since $\Sigma\, p_i = 1$, $k + 2k + 3k = 1$ $\therefore$ $k = 1/6$.

(ii) $P(X < 2) = P(X = 0) + P(X = 1) = k + 2k = 3/6 = 1/2$

$P(X \le 2) = P(X = 0) + P(X = 1) + P(X = 2) = 6k = 1$.

$P(0 < X < 2) = P(X = 1) = 2k = 1/3$.

(iii) Distribution function of X is

X	0	1	2
$F(x)$	1/6	1/2	1

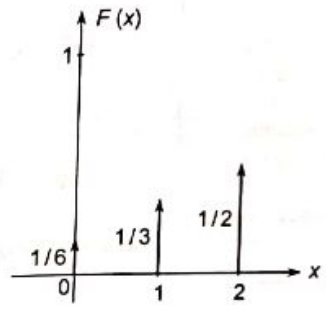

Fig. 7.18 (a)
Probability Density Function.

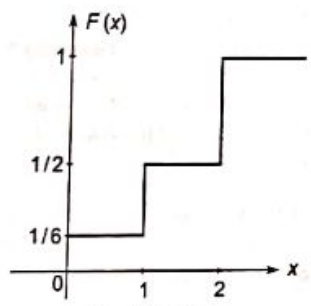

Fig. 7.18 (b)
Distribution Function.

EXERCISE - III

1. A random variable takes values 1, 2, 3, 4 such that $2P(X = 1) = 3P(X = 2) = P(X = 3) = 5P(X = 4)$. Find the probability distribution and the cumulative distribution function. **(M.U. 2004)**

[Ans. :

x	1	2	3	4
$P(x)$	15/61	10/61	30/61	6/61
$F(x)$	15/61	25/61	55/61	1

2. A shipment of 8 computers contains 3 that are defective. If a college makes a random purchase of 2 of these computers, find the probability distribution of the defective computers. Find also distribution function. **(M.U. 2005)**

[Ans. :

x	0	1	2
$P(x)$	10/28	15/28	3/28
$F(x)$	10/28	25/28	1

3. The probability density function of a random variable X is

X	:	0	1	2	3	4	5	6
$P(X = x)$	:	k	$3k$	$5k$	$7k$	$9k$	$11k$	$13k$

Find $P(X < 4)$, $P(3 < X \le 6)$. **[Ans. : k = 1/49, 16/49, 33/49]**

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11. Continuous Random Variable

Definition : A random variable is called a **continuous random variable** if it takes all values between an interval (a, b).

For example, age, height, weight are continuous random variables.

12. Probability Density Function of A Continuous Random Variable

Let $y = f(x)$ be a continuous function of x such that the area $f(x)\,\delta x$ represents the probability that X will lie in the interval $(x, x + \delta x)$. Symbolically,

$$P(x \le X \le x + \delta x) = f_x(x)\,\delta x$$

where, $f_x(x)$ denotes the value of $f(x)$ at x.

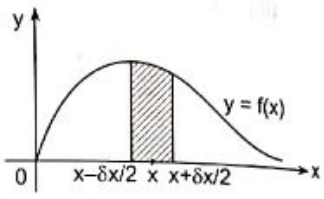

Fig. 7.19

The adjoining figure denotes the curve $y = f(x)$ and the area under the curve in the interval $(x, x + \delta x)$. The function satisfying certain conditions giving the probability that x will lie between certain limits is called **probability density function**, or simply **density function** of a continuous random variable X and is abbreviated as p.d.f. The curve given by $y = f(x)$ is called the **probability density curve** or simply **probability curve**. The expression $f(x)\,dx$ is usually denoted by $df(x)$ and is known as **probability differential**.

Definition : A continuous function $y = f(x)$ such that

(i) $f(x)$ is integrable. (ii) $f(x) \ge 0$

(iii) $\int_a^b f(x)\,dx = 1$ if X lies in $[a, b]$ and

(iv) $\int_\alpha^\beta f(x)\,dx = P(\alpha \le X \le \beta)$ where $a < \alpha < \beta < b$

is called **probability desnity function** of a continuous random variable X.

Thus, for a continuous random variale X,

$$\boxed{P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx}$$

Clearly $\int_\alpha^\beta f(x)\,dx$ represents the area under the curve $y = f(x)$, the x-axis and the ordinates at $x = \alpha$ and $x = \beta$. Further since, the total probability is one, if X lies in the interval $[a, b]$ then $\int_a^b f(x)\,dx = 1$. For a continuous random variable X the range may be finite $[a, b]$ or infinite $[-\infty, \infty]$.

Properties of Probability Density Function

The probability density function $f(x)$ has the following properties.

(i) $f(x) \ge 0$, $-\infty < x < \infty$ (*i.e.* the curve $y = f(x)$ lies above the x-axis in the first and second quadrants only)

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$ (*i.e.* the total area under the curve and the x-axis is one.)

(iii) The probability that $\alpha \le X \le \beta$ is given by $P(\alpha \le X \le \beta) = \int_\alpha^\beta f(x)\,dx$.

Notes ....

1. The property (1) and the property (2) can be used to verify whether a given function $f(x)$ can be a probability density function.

2. You know that for A discrete random variable the probability at $X = c$ may not be zero. But, in a continuous random variable $P(X = c)$ is always zero because $P(X = c) = \int_c^c f_X(x)\,dx$ and this definite integral is zero. Hence, for a continuous random variable X,

$$P(\alpha \le X \le \beta) = P(\alpha < X < \beta) = P(\alpha < X \le \beta) = P(\alpha < X < B)$$

In other words we may include or may not include the end-points in the interval.

3. Any function $f(x)$ of a real variable x can be a probability density function if it satisfies the first two properties given above *viz.* $f(x)$ is non-negative for all value of x and $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

Sometimes $\int_{-\infty}^{\infty} f(x)\,dx$ is not equal to 1 but $\int_{-\infty}^{\infty} k f(x) = 1$ for some value of k.

In such cases k is called the ***normalising factor*** or ***normalisation constant***.

Example 1 : Find the normalising factor, k if the following function is a probability density function

$$f(x) = \begin{cases} k(1 - x^2) & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Also find $P(0{\cdot}1 < X < 0{\cdot}2)$ and $P(X > 0{\cdot}5)$.

Sol. : Since $0 < x < 1$, $f(x) \ge 0$ for all x.

Now, $\int_0^1 f(x)\,dx = k\int_0^1 (1 - x^2)\,dx = k\left[x - \frac{x^3}{3}\right]_0^1 = k\frac{2}{3}$

But this must be equal to 1.

$$\therefore \frac{2k}{3} = 1 \quad \therefore k = \frac{3}{2}$$

(i) $P(0{\cdot}1 < X < 0{\cdot}2) = \int_{0{\cdot}1}^{0{\cdot}2} \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0{\cdot}1}^{0{\cdot}2}$

$$= \frac{3}{2}\left[\left\{0{\cdot}2 - \frac{(0{\cdot}2)^3}{3}\right\} - \left\{0{\cdot}1 - \frac{(0{\cdot}1)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0{\cdot}0977] = 0{\cdot}146$$

(ii) $P(X > 0{\cdot}5) = \int_{0{\cdot}5}^{1} \frac{3}{2}(1 - x^2)\,dx = \frac{3}{2}\left[x - \frac{x^3}{3}\right]_{0{\cdot}5}^{1}$

$$= \frac{3}{2}\left[\left\{1 - \frac{1}{3}\right\} - \left\{0{\cdot}5 - \frac{(0{\cdot}5)^3}{3}\right\}\right]$$

$$= \frac{3}{2}[0{\cdot}667 - 0{\cdot}458] = 0{\cdot}313$$

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Example 2 : A continuous random variable X has the following probability law

$$f(x) = kx^2, \quad 0 \le x \le 2$$

Determine k and find the probabilities that (i) $0.2 \le X \le 0.5$, (ii) $X \ge 3/4$ given that $X \ge 1/2$. **(M.U. 2005)**

Sol. : Since the total probability *i.e.* the total area is unity

$$\int_a^b f(x)\,dx = \int_0^2 kx^2 dx = 1$$

$$k\left[\frac{x^3}{3}\right]_0^2 = 1 \quad \therefore\ k \cdot \frac{8}{3} = 1 \quad \therefore\ k = \frac{3}{8}$$

(i) $$P(0{\cdot}2 \le X \le 0{\cdot}5) = \frac{3}{8}\int_{0\cdot2}^{0\cdot5} x^2 dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot2}^{0\cdot5} = \frac{1}{8}\left[0{\cdot}5^3 - 0{\cdot}3^3\right] = 0{\cdot}0123$$

(ii) Let $A = (X \ge 1/2)$, $B = (X \ge 3/4)$

$$\therefore\ P(A) = P(X \ge 1/2) = \frac{3}{8}\int_{0\cdot5}^{2} x^2 dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot5}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0{\cdot}5^3\right] = 0{\cdot}984$$

$$P(B) = P(X \ge 3/4) = \frac{3}{8}\int_{0\cdot75}^{2} x^2 dx = \frac{3}{8}\left[\frac{x^3}{3}\right]_{0\cdot75}^{2}$$

$$= \frac{1}{8}\left[2^3 - 0{\cdot}75^3\right] = 0{\cdot}947$$

$$P(A \cap B) = P(B) = 0{\cdot}947$$

$$\therefore\ P(B/A) = \frac{P(A \cap B)}{P(A)} = \frac{0{\cdot}947}{0{\cdot}984} = 0{\cdot}96$$

Example 3 : Let X be a continuous random variable with probability distribution

$$p(x) = \begin{cases} \dfrac{x}{6} + k & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and find $P(1 \le x \le 2)$. **(M.U. 1998, 2004)**

Sol. : Since the total probability is one

$$\int_{-\infty}^{\infty} p(x)\,dx = \int_0^3 \left(\frac{x}{6} + x\right)dx = \left[\frac{x^2}{12} + kx\right]_0^3 = \frac{3}{4} + 3k = 1$$

$$\therefore\ 3\left(\frac{1}{4} + k\right) = 1 \quad \therefore\ \frac{1}{4} + k = \frac{1}{3} \quad \therefore\ k = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$$

$$\therefore\ p(x) = \begin{cases} \dfrac{x}{6} + \dfrac{1}{12} & \text{if } 0 \le x \le 3 \\ 0 & \text{elsewhere} \end{cases}$$

$$\therefore\ P(1 \le x \le 2) = \int_1^2 \left(\frac{x}{6} + \frac{1}{12}\right)dx = \left[\frac{x^2}{12} + \frac{x}{12}\right]_1^2$$

$$= \frac{1}{12}\left[(4+2) - (1+1)\right] = \frac{1}{12}(4) = \frac{1}{3}.$$

Example 4 : Let X be a continuous random variable with p.d.f. $f(x) = kx(1-x)$, $0 \le x \le 1$. Find k and determine a number b such that $P(X \le b) = P(X \ge b)$. **(M.U. 2003, 11, 15, 19)**

Sol. : Since $\int_{-\infty}^{\infty} f(x)\,dx = 1$, we have

$$k\int_0^1 (x - x^2)\,dx = 1 \quad \therefore\ k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore\ k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore\ k = 6.$$

Since, the total probability is 1 and $P(x \le b) = P(x \ge b)$, $P(x \le b) = 1/2$.

$$\therefore\ \int_0^b f(x)\,dx = \frac{1}{2}$$

$$\therefore\ 6\int_0^b (x - x^2)\,dx = \frac{1}{2} \quad \therefore\ \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^b = \frac{1}{12}$$

$$\therefore\ \frac{b^2}{2} - \frac{b^3}{3} = \frac{1}{12} \quad \therefore\ 6b^2 - 4b^3 = 1 \quad \therefore\ 4b^3 - 6b^2 - 1 = 0$$

$$\therefore\ 4b^3 - 2b^2 - 4b^2 - 2b + 2b - 1 = 0$$

$$\therefore\ (2b - 1)(2b^2 - 2b + 1) = 0 \quad \therefore\ b = 1/2.$$

Example 5 : The probability that a person will die in the time interval (t_1, t_2) is given by

$$P(t_1 \le t \le t_2) = \int_{t_1}^{t_2} f(t)\,dt$$

where, $$f(t) = \begin{cases} 3 \times 10^{-9}(100t - t^2)^2, & 0 < t < 100 \\ 0, & \text{elsewhere.} \end{cases}$$

Find (i) the probability that Mr. X will die between the ages 60 and 70. (ii) the probability that he will die between the ages 60 and 70, given that he has survived upto age 60. **(M.U. 2005)**

Sol. : (i) $$P(60 \le t \le 70) = \int_{60}^{70} 3 \times 10^{-9}(100t - t^2)^2\,dt$$

$$= 3 \times 10^{-9}\int_{60}^{70}(100^2 t^2 - 200t^3 + t^4)\,dt$$

$$= 3 \times 10^{-9}\left[100^2\frac{t^3}{3} - 200\frac{t^4}{4} + \frac{t^5}{5}\right]_{60}^{70}$$

$$= 3 \times 10^{-9}\left[27{\cdot}89 \times 10^8 - 22{\cdot}75 \times 10^8\right]$$

$$= 0{\cdot}1542.$$

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(ii) $$P\left(\frac{60 \le T \le 70}{T \ge 60}\right) = \frac{P(60 \le t \le 70 \cap t \ge 60)}{P(t \ge 60)} = \frac{P(60 \le t \le 70)}{P(60 \le t \le 100)}$$

$$= \frac{\int_{60}^{70} f(t)\,dt}{\int_{60}^{100} f(t)\,dt} = \frac{0.1542}{0.3174} = 0.4858.$$

EXERCISE - IV

1. A function is defined as

$$f(x) = \begin{cases} 0 & \text{for } x < 2 \\ \dfrac{2x+3}{18} & \text{for } 2 \le x \le 4 \\ 0 & \text{for } x > 4 \end{cases}$$

Show that $f(x)$ is a probability density function and find the probability that $2 < x < 3$.

(M.U. 2005) [Ans. : 5 / 18]

2. A random variable X has the probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x \ge 0 \\ 0 & \text{for } x < 0 \end{cases}$$

Find $P(1 \le X \le 3)$, $P(X \ge 0.5)$. [Ans. : (i) 0.133, (ii) 0.368]

3. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \end{cases}$$

Find (i) the value of k, (ii) $P(x \le 1.5)$. [Ans. : (i) $k = 2/3$, (ii) 2 / 3]

4. A continuous random variable has the following probability density function.

$$f(x) = \begin{cases} (x/4) + k & 0 \le x \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

Evaluate k and $P(1 \le X \le 2)$. [Ans. : $k = 1/4$, 5 / 8]

5. Find the value of k such that the following will be the probability density function. Find also $P(x \le 1.5)$.

$$f(x) = \begin{cases} kx & 0 \le x \le 1 \\ k & 1 \le x \le 2 \\ k(3-x) & 2 \le k \le 3 \end{cases}$$

(M.U. 2003) [Ans. : $k = \frac{1}{2}, \frac{1}{2}$]

13. Continuous Distribution Function

Probability distribution of X or the probability density function of X helps us to find the probability that X will be within a given interval $[a, b]$ *i.e.* $P(a \le X \le b) = \int_a^b f(x)\,dx$, other conditions being satisfied.

However, sometimes we need to know that probability that X will be less than a given value x. For a continuous random variable X, this probability is obtained by integrating $f(x)$ from $-\infty$ (or the lower limit of the interval) to x. The function so obtained is called **distribution function.**

Definition : If X is a continuous random variable X, having the probability density function $f(x)$ then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt, \quad -\infty < x < \infty$$

is called **distribution function** or **cumulative distribution function** of the random variable X.

Some Important Properties of Distribution Function $F(x)$ of a Continuous Random Variable

1. The function $F(x)$ is defined for every real number x.
2. Since $F(x)$ denotes probability and probability of X lies between 0 and 1,
   $0 \le F(x) \le 1$.
3. $F(x)$ is a non-decreasing function which means if $x_1 \le x_2$, then $F(x_1) \le F(x_2)$.
4. **The derivative of $F(x)$** *i.e.* $F'(x)$ exists at all points (except perhaps at a finite number of points) and **is equal to the probability density function $f(x)$.**

$$F'(x) = \frac{d}{dx}F(x) = f(x) \ge 0 \text{ provided the derivative exists.}$$

5. If $F(x)$ is a distribution function of a continuous random variable then
   $P(a \le X \le b) = F(b) - F(a)$.

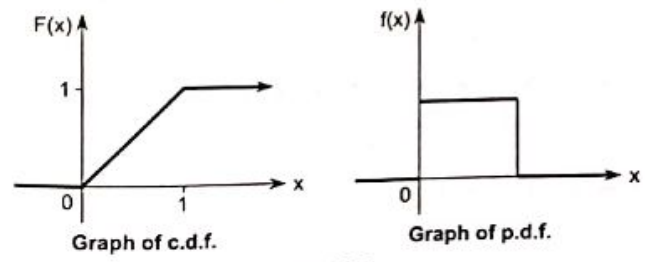
Example 1 : A continuous variable X has the following distribution function

$$F(x) = \begin{cases} 0, & x \le 0 \\ x, & 0 \le x \le 1 \\ 1, & 1 \le x \end{cases} \qquad \text{..... (1)}$$

Find the probability density function and draw the graphs of both p.d.f. and c.d.f.

Sol. : The p.d.f. is

$$f(x) = F'(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & 1 < x \end{cases}$$



Graph of c.d.f. Graph of p.d.f.

Fig. 7.20

From the graph of $F(x)$ we see that $F(x)$ is continuous at all points including $x = 0$ and $x = 1$. $f(x)$ is obtained by differentiating $F(x)$.

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Example 2 : For the distribution function given below, find p.d.f.

$$F(x) = \begin{cases} 0 \ , & x < 0 \\ 1 - e^{-x/4}, & x \geq 0 \end{cases}$$

Also find the probabilities : $P(X \leq 4)$, $P(X \geq 8)$, $P(4 \leq X \leq 8)$.

Sol. : $F(x)$ satisfies all the conditions of a distribution function. If $f(x)$ is the corresponding probability density function

$$f(x) = F'(x) = \begin{cases} \frac{1}{4} e^{-x/4}, & x \geq 0 \\ 0 \ , & x < 0 \end{cases}$$

Now, we have to verify that $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\therefore \int_{-\infty}^{0} f(x)\,dx + \int_0^{\infty} \frac{1}{4} e^{-x/4}\,dx = 0 + \int_0^{\infty} \frac{1}{4} e^{-x/4} dx = \frac{1}{4}\left[\frac{e^{-x/4}}{-1/4}\right]_0^{\infty}$$

$$= -\left[e^{-x/4}\right]_0^{\infty} = -[0-1] = 1$$

Hence, $F(x)$ is a distribution function.

Now, $P(X \leq 4) = F(4) = 1 - e^{-1} = 1 - \frac{1}{e} = \frac{e-1}{e}$

$$P(X \geq 8) = 1 - P(x \leq 8) = 1 - F(8)$$

$$= 1 - \left[1 - e^{-2}\right] = e^{-2} = 1/e^2$$

$$P(4 \leq X \leq 8) = F(8) - F(4) = (1 - e^{-2}) - (1 - e^{-1})$$

$$= e^{-1} - e^{-2} = \frac{1}{e} - \frac{1}{e^2} = \frac{e-1}{e^2}$$

Example 3 : If $f(x) = \begin{cases} x e^{-x^2/2} & x \geq 0 \\ 0 & x < 0 \end{cases}$

(i) Show that $f(x)$ is a probability density function. (ii) Find its distribution function.

Sol. : If $f(x)$ is a p.d.f., we must have $f(x) \geq 0$ where $x \geq 0$ and $\int_0^{\infty} f(x)\,dx = 1$.

Clearly $f(x) = xe^{-x^2/2} \geq 0$ for $x \geq 0$.

Now, $\int_0^{\infty} x e^{-x^2/2} dx = \int_0^{\infty} e^{-t}\,dt \quad \left[t = x^2/2\right]$

$$= \left[-e^{-t}\right]_0^{\infty} = -[0-1] = 1$$

$\therefore$ $f(x)$ is a probability density function.

Now, its distribution function is given by

$$F(x) = \int_0^x f(x)\,dx = \int_0^x x e^{-x^2/2} dx = -\left[e^{-x^2/2}\right]_0^x \qquad \text{[As above]}$$

$$= -\left[e^{-x^2/2} - 1\right] = 1 - e^{-x^2/2}, \quad x \geq 0.$$

EXERCISE - V

1. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 0 \ , & x < -1 \\ \frac{x+1}{4}, & -1 \leq x \leq 3 \\ 1 \ , & 3 < x \end{cases}$$

Find the probability density function. Draw the graphs of both p.d.f. and c.d.f.

2. The c.d.f. of a continuous random variable X is given by

$$F(x) = \begin{cases} 0, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Find the p.d.f. Draw the graphs of both p.d.f. and c.d.f.

Also find $P\left(\frac{1}{2} \leq X \leq \frac{4}{5}\right)$. **[Ans. : 0·195]**

3. Find the distribution functions corresponding to the following probability density functions.

(i) $f(x) = \begin{cases} \frac{1}{2} x^2 e^{-x}, & 0 \leq x < \infty \\ 0 \ , & \text{otherwise} \end{cases}$ (ii) $f(x) = \begin{cases} x \ , & 0 \leq x \leq 1 \\ 2 - x, & 1 \leq x \leq 2 \\ 0 \ , & \text{otherwise} \end{cases}$

(iii) $f(x) = \begin{cases} \lambda (x-1)^4, & 1 \leq x \leq 3, \lambda > 0 \\ 0, & \text{otherwise} \end{cases}$

[Ans. : (i) $F(x) = \begin{cases} 1 - e^{-x}\left(1 + x + \frac{x^2}{2}\right), & x \geq 0 \\ 0 \ , & \text{otherwise} \end{cases}$

(ii) $F(x) = \begin{cases} 0 \ , & x < 0 \\ x^2/2 \ , & 0 \leq x \leq 1 \\ 2x - 0{\cdot}5x^2 - 1, & 1 \leq x \leq 2 \\ 1 \ , & x > 2 \end{cases}$

(iii) $\lambda = \frac{5}{32}$; $F(x) = \begin{cases} 0 \ , & x \leq 1 \\ \frac{(x-1)^5}{32}, & 1 \leq x \leq 3 \\ 1 \ , & x \geq 3 \end{cases}$]

4. A continuous random variable X has the following probability density function

$$f(x) = \frac{a}{x^5}, \quad 2 \leq x \leq 10$$

Determine the constant a, distribution function of X and find the probability of the event $4 \leq x \leq 7$.

[Ans. : (i) $a = \frac{2500}{39}$, (ii) $\frac{625}{39}\left[\frac{1}{16} - \frac{1}{x^4}\right]$, (iii) 0·056]

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5. The distribution function of a random variable X is given by

$$F(x) = \begin{cases} 1-(1+x)e^{-x}, & x \geq 0 \\ 0 \quad , & x < 0 \end{cases}$$

Find the probability density function and find $P(0 \leq X \leq 1)$.

[**Ans. : (i)** $f(x) = x e^{-x}$, **(ii)** $P(0 \leq X \leq 1) = F(1) - F(0) = \frac{e-2}{e}$]

EXERCISE - VI

Theory

1. State and prove Bayes' Theorem.
2. Define the following terms giving suitable examples
   (a) Random variable
   (b) Discrete random variable
   (c) Continuous random variable
   (d) Probability density function
   (e) Distribution function
3. State the properties of probability density function.
4. State the properties of distribution function.

CHAPTER 8

Mathematical Expectation

1. Introduction

Suppose two coins are tossed twenty times. Let X be the number of heads obtained in a toss. X then, takes values 0, 1 and 2. Suppose further that no heads, one head and two heads were obtained 4, 10, 6 times respectively. Then, the average number of heads per toss

$$= \frac{4(0) + 10(1) + 6(2)}{6 + 10 + 4} = 1{\cdot}1$$

This is the average value and is not necessarily a possible outcome of the toss.

The ratios 4 / 20, 10 / 20, 6 / 20 of 0, 1, 2 heads to the total number of tosses are the relative frequencies of $X = 0, 1, 2$. If the experiment is repeated very large number of times, we know that, these relative frequencies tend to the probabilities 1 / 4, 1 / 2, 1 / 4 of 0, 1, 2 heads because in the toss of two coins we have the following.

Sample space	:	HH	$\underbrace{HT, TH}$	TT
Probability	:	1/4	1/2	1/4

The average calculated with probabilities in place of relative frequencies above is called **expected value** or **mathematical expectation** and is denoted by $E(X)$. Thus,

$$E(X) = \frac{1}{4}(0) + \frac{1}{2}(1) + \frac{1}{4}(2) = 1$$

$E(X)$ = sum of the products of the values and their probabilities

$= p_1 x_1 + p_2 x_2 + p_3 x_3 + \ldots\ldots$

This means, a person who throws two coins over and over again will get one head per toss on the average. This suggests us that the expected value of X can be obtained by multiplying the values of X by their respective probabilities and taking the sum. This leads us to the following definition of expectation of a discrete random variable X.

2. Expectation of a Random Variable

(a) Definition : If a **discrete random variable** X assumes values $x_1, x_2, \ldots\ldots x_n \ldots\ldots$ with probabilities $p_1, p_2, \ldots\ldots p_n \ldots\ldots$ respectively then the **mathematical expectation of** X denoted by $E(X)$ (if it exists) is defined by

$$\boxed{E(X) = p_1 x_1 + p_2 x_2 + \ldots\ldots + p_n x_n + \ldots\ldots}$$

i.e. $\boxed{E(X) = \Sigma\, p_i x_i}$ where $\Sigma\, p_i = 1$.

if $\Sigma\, p_i x_i$ is absolutely convergent.

This value is also referred to as *mean value* of X. It is also denoted by μ_1'. $\therefore\ \mu_1' = E(X)$.

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Notation : In this chapter we shall slightly deviate from our previous notation. Instead of denoting $P(X = x)$ by $p(x_i)$ we shall denote it simply by p_i. This will be found more convenient while dealing with expectations.

(b) Definition : Let X be a **continuous random variable** with probability density function $f(x)$. Then the **mathematical expectation of X**, denoted by $E(X)$ (if it exists), is defined by

$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx} \qquad \text{where, } \int_{-\infty}^{\infty} f(x)\,dx = 1$$

if the integral is absolutely convergent.

Notes ....

1. If X assumes only a finite number of values then $E(X) = \Sigma\, p_i x_i$ and can be considered as "weighted average" of the values $x_1, x_2, \ldots\ldots x_n$ with weights $p_1, p_2 \ldots\ldots p_n$.
2. If all values $x_1, x_2, \ldots\ldots x_n$ are equiprobable *i.e.* $p_1 = p_2 = \ldots\ldots = p_n = 1/n$ then $E(X) = (1/n)\Sigma x_i$ and can be seen to be simple arithmetic mean of the n values $x_1, x_2, \ldots\ldots x_n$.
3. One should guard oneself from being mislead by the term 'expectation'. $E(X)$ does *not* give us the value of X, we can expect in a single trial. In the Example 4 below $E(X) = 7/2$ is not even a possible value of X when a die is tossed.

   $E(X)$ denotes *mathematical* expectation of X in the sense that if we toss a die for a fairly large number of times, observe the frequencies of the outcomes 1, 2, 3, 4, 5, 6 then the average of these values will be closer to $7/2$ the more often the die were tossed.
4. $E(X)$ is expressed in the *same units* as X.
5. **Expectation of a constant is constant**

   (i) $E(c) = \Sigma\, p_i c = c\,\Sigma\, p_i = c \qquad [\because \Sigma\, p_i = 1]$

   (ii) $E(c) = \int_{-\infty}^{\infty} c\, f(x)\,dx = c\int_{-\infty}^{\infty} f(x)\,dx = c \qquad \left[\because \int_{-\infty}^{\infty} f(x)\,dx = 1\right]$

Example 1 : A fair coin is tossed 3 times. A person received ₹ X^2 if he gets X heads. Find his expectation. **(M.U. 2004)**

Sol. : When a coin is tossed three times, the sample space is

HHH, HHT, HTH, HTT, THH, THT, TTH, TTT

The probability distribution of X is

X	0	1	2	3
$P(X = x)$	1/8	3/8	3/8	1/8

Now, X^2 takes the following values

X^2	0	1	2	3
$P(X^2)$	1/8	3/8	3/8	1/8

$$\therefore \quad E(X) = \Sigma\, p_i x_i = \frac{1}{8} \times 0 + \frac{3}{8} \times 1 + \frac{3}{8} \times 4 + \frac{1}{8} \times 9$$

$$= \frac{3 + 12 + 9}{8} = \frac{24}{8} = 3 \text{ ₹}$$

Example 2 : There are 10 counters in a bag, 6 of which are worth 5 rupees each while the remaining 4 are of equal but unknown value. If the expectation of drawing a single counter at random is 4 rupees, find the unknown value. **(M.U. 2015)**

Sol. : Let x be the value of the remaining 4 counters.

$$P(\text{of counter worth of ₹ } 5) = \frac{6}{10}$$

$$P(\text{of counter of unknown value}) = \frac{4}{10}$$

$$E(X) = \Sigma\, p_i x_i \quad \therefore \quad 4 = \frac{6}{10} \cdot 5 + \frac{4}{10} \cdot x$$

$$\therefore \quad 40 = 30 + 4x \qquad \therefore \quad 4x = 10 \qquad \therefore \quad x = \text{₹ } 2{\cdot}5.$$

Example 3 : A fair coin is tossed till a head appears. What is the expectation of the number of tosses required ? **(M.U. 1996, 2010)**

Sol. : Let X denote the order of the toss at which we get the first head. We have

Event	*H*	*TH*	*TTH*	*TTTH*	.....
X	1	2	3	4	.....
$P(X = x)$	$\frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$	$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{16}$	.....

$$\therefore \quad E(X) = \Sigma\, p_i x_i = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{8}\right) + 4\left(\frac{1}{16}\right) + \ldots\ldots$$

Let
$$S = \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \ldots\ldots$$

$$\frac{1}{2} S = \qquad \frac{1}{4} + 2 \cdot \frac{1}{8} + 3 \cdot \frac{1}{16} + \ldots\ldots$$

$$\therefore \quad S - \frac{1}{2} S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\ldots$$

$$\therefore \quad \frac{1}{2} S = \frac{1}{2} \cdot \frac{1}{1 - (1/2)} = 1 \qquad \left[\text{G.P. } S_\infty = \frac{a}{1 - r}\right]$$

$$\therefore \quad E(X) = 2.$$

Example 4 : Find the expectation of (i) the sum, (ii) the product of the number of points on the throw of n dice. **(M.U. 2004, 06)**

Sol. : Let X_i denote the number of points on the i th dice.

Then if S denotes the sum of the points of n dice then $S = \sum_{i=1}^{n} X_i$.

$$\text{Now, } \quad E(X_i) = \Sigma\, p_i x_i = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \ldots\ldots + \frac{1}{6} \cdot 6$$

$$\therefore \quad E(X_i) = \frac{1}{6}(1 + 2 + 3 + 3 + 4 + 5 + 6) = \frac{21}{6} = \frac{7}{2}$$

$$\therefore \quad S = \sum_{i=1}^{n} X_i = n \cdot \frac{7}{2} = \frac{7n}{2}$$

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If π denotes the product of the points

$$E(\pi) = E(X_1) \cdot E(X_2) \ldots\ldots E(X_n) = \left(\frac{7}{2}\right) \cdot \left(\frac{7}{2}\right) \ldots\ldots \cdot \left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^n.$$

Example 5 : A box contains n tickets numbered 1, 2, ..., n. If m tickets are drawn at random from the box. What is the expectation of the sum of the numbers on the tickets drawn ? **(M.U. 2001)**

Sol. : Let X_i denote the number on the i th ticket drawn.

Then $S = X_1 + X_2 + \ldots\ldots + X_m$

Now, $E(X_i) = \Sigma p_i x_i = \frac{1}{n} \cdot 1 + \frac{1}{n} \cdot 2 + \ldots\ldots + \frac{1}{n} \cdot n$

$$= \frac{1}{n}(1 + 2 + \ldots\ldots + n) = \frac{1}{n} \cdot \frac{n}{2} \cdot (n+1) = \frac{n+1}{2}$$

$$\therefore \quad E(S) = \sum_{i=1}^{m} E(X_i) = \frac{m(n+1)}{2}.$$

Example 6 : Three urns contain respectively 3 green and 2 white balls, 5 green and 6 white balls, 2 green and 4 white balls. One ball is drawn from each urn. Find the expected number of white ball drawn. **(M.U. 2007, 09)**

Sol. : If X denotes the number of white balls drawn from an urn then the expectation of X is as follows.

$$E(X) = p_i x_i = p_1 x_1 + p_2 x_2$$

x takes two values. $X = 1$ if white ball is drawn and $X = 0$ if greenball is drawn.

$\therefore$ From first urn : $E(X_1) = 1 \times \frac{2}{5} + 0 \times \frac{3}{5} = \frac{2}{5}$

From second urn : $E(X_2) = 1 \cdot \frac{6}{11} + 0 \cdot \frac{5}{11} = \frac{6}{11}$

$E(X_3) = 1 \cdot \frac{4}{6} + 0 \cdot \frac{2}{6} = \frac{2}{3}$

$\therefore$ The required expectation $= E(X_1) + E(X_2) + E(X_3)$

$$= \frac{2}{5} + \frac{6}{11} + \frac{2}{3} = \frac{266}{165} = 1.61$$

Example 7 : A box contains 2^n tickets of which ${}^nC_r$ tickets bear the number r ($r = 0, 1, 2, \ldots\ldots n$). A group of m tickets is drawn. What is the expectation of the sum of their numbers ?

Sol. : Let $X_1, X_2, \ldots\ldots, X_m$ be the variables denoting the number on the first, second, ..., m th ticket.

If S is the sum of the numbers on the tickets drawn then

$$S = \Sigma X_i \text{ and } E(S) = \Sigma E(X_i)$$

Now X_i is a random variable which can take any one of the value 0, 1, 2, ..., n with probabilities ${}^nC_0 / 2^n, {}^nC_1 / 2^n, \ldots\ldots, {}^nC_n / 2^n$.

$$\therefore \quad E(X_i) = \Sigma p_i x_i$$

$$= \frac{1}{2^n}\left[0 \cdot {}^nC_0 + 1 \cdot {}^nC_1 + 2 \cdot {}^nC_2 + 3 \cdot {}^nC_3 + \ldots\ldots + n \cdot {}^nC_n\right]$$

$$= \frac{1}{2^n}\left[1 \cdot n + 2 \cdot \frac{n(n-1)}{2!} + 3 \cdot \frac{n(n-1)(n-2)}{3!} + \ldots\ldots + n \cdot 1\right]$$

$$= \frac{n}{2^n}\left[1 + (n-1) + \frac{(n-1)(n-2)}{2!} + \ldots\ldots + 1\right]$$

$$\therefore \quad E(X_i) = \frac{n}{2^n}[1+1]^{n-1} = \frac{n}{2^n} \cdot 2^{n-1} = \frac{n}{2}.$$

$$\therefore \quad E(S) = \Sigma E(X_i) = m \cdot \frac{n}{2} = \frac{mn}{2}.$$

Cor. 1 : If two tickets are drawn then putting $m = 2$, we get

$$E(\text{Sum}) = 2 \cdot \frac{n}{2} = n.$$

(M.U. 2003)

Example 8 : A box contains 'a' white balls and 'b' black balls. 'c' balls are drawn from the box at random. Find the expected value of the number of white balls. **(M.U. 2005)**

Sol. : Let X_i be the variable denoting the result of the i th draw.

Let $X_i = 1$ if i th ball drawn is white and $X_i = 0$ if i th ball drawn is black.

Since, 'c' balls are drawn the sum of the white ball will be

$$S = X_1 + X_2 + \ldots\ldots + X_c = \sum_{i=1}^{c} X_i$$

Now, $P(X_i = 1) = P(\text{drawing a white ball}) = \frac{a}{a+b}$

$P(X_i = 0) = P(\text{drawing a black ball}) = \frac{b}{a+b}$

$E(X_i) = 1 \cdot P(X_i = 1) + 0 \cdot P(X_i = 0)$

$$= 1 \cdot \frac{a}{a+b} + 0 \cdot \frac{b}{a+b} = \frac{a}{a+b}$$

$$\therefore \quad E(S) = E(X_1) + E(X_2) + \ldots\ldots + E(X_c)$$

$$= c \cdot \frac{a}{a+b} = \frac{ac}{a+b}.$$

Example 9 : A die is thrown until a five is obtained, find the expectation of the number of throws.

Sol. : Probability of getting 5 in the first toss = 1 / 6.

Probability of getting 5 in the second = (5 / 6) · (1 / 6).

Probability of getting 5 in third = (5 / 6) (5 / 6) (1 / 6) and so on.

and X takes values 1, 2, 3, ...

$$E(X) = 1(1/6) + 2(5/6)(1/6) + 3(5/6)^2(1/6) + \ldots\ldots$$

$$= (1/6)[1 + 2x + 3x^2 + \ldots\ldots] \qquad \text{where, } x = 5/6$$

$$= (1/6)(1-x)^{-2} = (1/6)[1 - (5/6)]^{-2}$$

$$= 6.$$

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Example 10 : A and B throw a fair die for a stake of ₹ 44, which is won by the player who throws 6 first. If A starts first, find their expectations.

Sol. : A can win the game, in the first throw or in the third throw or in the fifth throw and so on.

$$P(\text{A winning}) = \frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\cdot\frac{1}{6} + \left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\cdot\frac{1}{6} + \ldots\ldots$$

$$\therefore\ P(\text{A winning}) = \frac{1}{6}\left[1 + \left(\frac{25}{36}\right) + \left(\frac{25}{36}\right)^2 + \ldots\ldots\right] = \frac{1}{6}\cdot\frac{1}{1-(25/36)} = \frac{6}{11}.$$

$$P(\text{B winning}) = 1 - P(\text{A winning}) = \frac{5}{11}$$

$$\therefore\ \text{Expectation of A} = p\cdot x = \frac{6}{11}\cdot 44 = ₹\ 24$$

$$\text{Expectation of B} = p\cdot x = \frac{5}{11}\cdot 44 = ₹\ 20.$$

Example 11 : *A, B, C, D* cut a pack of cards successfully in the order mentioned. The person who cuts a spade first wins ₹ 175. Find their expectations.

Sol. : Probability of cutting a spade $= \frac{13}{52} = \frac{1}{4}$.

Let *A* denote the success of *A* and $\overline{A}$ denote failure of *A* and so on.

Probability of *A*'s success

$$= P(A) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\,A) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\cdot\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\,A) + \ldots\ldots$$

$$= \frac{1}{4} + \left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right) + \left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right) + \ldots\ldots$$

$$= \frac{1}{4} + \frac{81}{256}\cdot\frac{1}{4} + \left(\frac{81}{256}\right)^2\cdot\frac{1}{4} + \ldots\ldots = \frac{1}{4}\cdot\frac{1}{1-(81/256)}$$

$$= \frac{1}{4}\cdot\frac{256}{256-81} = \frac{1}{4}\cdot\frac{256}{175} = \frac{64}{175}.$$

Probability of *B*'s success

$$= P(\overline{A}\,B) + P(\overline{A}\,\overline{B}\,\overline{C}\,\overline{D}\cdot\overline{A}\,B) + \ldots\ldots$$

$$= \frac{3}{4}\cdot\frac{1}{4} + \left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right)\left(\frac{3}{4}\cdot\frac{1}{4}\right) + \ldots\ldots = \frac{3}{16}\left[1 + \frac{81}{256} + \ldots\ldots\right]$$

$$= \frac{3}{16}\left[\frac{1}{1-(81/256)}\right] = \frac{3}{16}\left[\frac{256}{175}\right] = \frac{48}{175}.$$

Probability of *C*'s success

$$= \frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4} + \ldots\ldots = \frac{9}{64}\left[\frac{1}{1-(81/256)}\right] = \frac{9}{64}\cdot\frac{256}{256-81} = \frac{36}{175}.$$

Probability of *D*'s success

$$= 1 - [P(A) + P(B) + P(C)] = 1 - \left[\frac{64}{175} + \frac{48}{175} + \frac{36}{175}\right] = \frac{27}{175}.$$

Now, the probabilities of *A, B, C, D*

$$\therefore\ E(A) = p\,x = \frac{64}{175}\times 175 = ₹\ 64, \qquad E(B) = p\,x = \frac{48}{175}\times 175 = ₹\ 48,$$

$$E(C) = p\,x = \frac{36}{175}\times 175 = ₹\ 36, \qquad E(D) = p\,x = \frac{27}{175}\times 175 = ₹\ 27.$$

Example 12 : Find the expectation of number of failures preceding the first success in an infinite series of independent trials with constant probabilities *p* and *q* of success and failure respectively. **(M.U. 1999, 2003, 17, 19)**

Sol. : We have the following probability distribution

X	:	0	1	2	3	....
$P(X = x)$	:	p	qp	q^2p	q^3p	

Since, we may get success in the first trial where the number of failures $X = 0$ and the probability is *p* ; we may get success in the second trial when the number of failures $X = 1$ and the probability is *q p* and so on.

$$\therefore\ E(X) = \Sigma\, p_i x_i = p\,(0) + qp\,(1) + q^2p\,(2) + q^3p\,(3) + \ldots$$

$$= qp\left[1 + 2q + 3q^2 + \ldots\ldots\right] = qp\,(1-q)^{-2} = \frac{qp}{p^2} = \frac{q}{p}.$$

Example 13 : The daily consumption of electric power (in million kwh) is a random variable *X* with probability distribution function

$$f(x) = \begin{cases} kx\,e^{-x/3} & \text{for } x > 0 \\ 0 & \text{for } x \le 0 \end{cases}$$

Find the value of *k*, the expectation of *x* and the probability that on a given day the electric consumption is more than expected value. **(M.U. 2003, 04, 16)**

Sol. : We must have

$$\int_{-\infty}^{\infty} f(x)\,dx = 1 \ \ i.e.\ \ k\int_0^{\infty} x\,e^{-x/3}\,dx = 1$$

$$\therefore\ k\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right) - (1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_0^{\infty} = 1$$

$$\therefore\ k\left[0 + 9\right] = 1 \quad \therefore\ 9k = 1 \quad \therefore\ k = 1/9$$

$$E(X) = \int_{-\infty}^{x} x\,f(x)\,dx = \frac{1}{9}\int_0^{\infty} x^2\cdot e^{-x/3}\cdot dx$$

$$= \frac{1}{9}\left[x^2\left(\frac{e^{-x/3}}{-1/3}\right) - (2x)\left(\frac{e^{-x/3}}{1/9}\right) + 2\left(\frac{e^{-x/3}}{-1/27}\right)\right]_0^{\infty}$$

$$= \frac{1}{9}\left[0 - 0 + 0 + 54\right] = 6$$

$$\therefore\ P(X > 6) = \frac{1}{9}\int_6^{\infty} x\cdot e^{-x/3}\cdot dx$$

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$$\therefore\ P(X>6)=\frac{1}{9}\left[(x)\left(\frac{e^{-x/3}}{-1/3}\right)-(1)\left(\frac{e^{-x/3}}{1/9}\right)\right]_6^{\infty}$$

$$=\frac{1}{9}\left[(0-0)-(-18e^{-2}-9e^{-2})\right]$$

$$=3e^{-2}=0.406$$

Example 14 : Find k and then $E(X)$ if X has the p.d.f.

$$f(x)=\begin{cases}k\,x(2-x), & 0\le x\le 2,\ k>0\\ 0, & \text{elsewhere}\end{cases}$$

Sol. : Now $\int_0^2 kx(2-x)\,dx=k\int_0^2 2x-x^2\,dx=k\left[x^2-\frac{1}{3}x^3\right]_0^2=k\cdot\frac{4}{3}$

$$\therefore\ k\cdot\frac{4}{3}=1\quad\therefore\ k=\frac{3}{4}$$

By definition

$$E(X)=\int_0^2 x\,f(x)\,dx=\int_0^2 x\cdot\frac{3}{4}x(2-x)\,dx=\frac{3}{4}\int_0^2(2x^2-x^3)\,dx$$

$$\therefore\ E(X)=\frac{3}{4}\left[\frac{2x^3}{3}-\frac{x^4}{4}\right]_0^2=\frac{3}{4}\left[\frac{16}{3}-\frac{16}{4}\right]=\frac{3}{4}\cdot\frac{16}{12}=1.$$

Example 15 : Find k and then $E(X)$ for the p.d.f.

$$f(x)=\begin{cases}k(x-x^2), & 0\le x\le 1,\ k>0\\ 0, & \text{elsewhere}\end{cases}$$

Sol. : Now $k\int_0^1(x-x^2)\,dx=k\cdot\left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1=k\left[\frac{1}{2}-\frac{1}{3}\right]=k\cdot\frac{1}{6}$

But $k\cdot\frac{1}{6}=1\quad\therefore\ k=6$

By definition

$$E(X)=\int_0^1 x\,f(x)\,dx=\int_0^1 x\cdot 6(x-x^2)\,dx=6\left[\frac{x^3}{3}-\frac{x^4}{4}\right]_0^1=\frac{6}{12}=\frac{1}{2}.$$

EXERCISE - I

1. From an urn containing 3 red balls and 2 white balls, a man is to draw 2 balls at random without replacement. He gets ₹ 20 for each red ball and ₹ 10 for each white ball he draws. Find his expectation. **[Ans. : ₹ 32]**
2. Two urns contain respectively 5 white and 3 black balls; 2 white and 3 black balls. One ball is drawn from each urn. Find the expected number of white balls drawn. **[Ans. : 41 / 40]**
3. A and B toss a fair coin alternately. One who gets a head first wins ₹ 12. A starts. Find their mathematical expectations. **[Ans. : ₹ 8, 4]**

4. A, B, and C toss a fair coin. The first one to throw a head wins the game and gets ₹ 28. If A starts, find their mathematical expectations. **[Ans. : ₹ 16, 8, 4]**
5. A, B and C draw a card in that order from a well shuffled pack of 52 cards. The first to draw a diamond wins ₹ 740. If A starts, find their expectations. **[Ans. : ₹ 320, 240, 180]**
6. Two unbiased dice are thrown. Find the expectation of the sum. **(M.U. 2007) [Ans. : 7]**
7. A man with n keys in his pocket wants to open the door of his case by trying the keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door if unsuccessful keys are kept aside. **(M.U. 1998)**
   [Ans. : (i) $(n+1)/2$, (ii) $(n^2-1)/12$]
8. A player throwing an ordinary die is to receive ₹ $1/2^n$ where n denotes the number of throws required to get first 3. Find his expectation. **(M.U. 2001, 04) [ans. : ₹ 1/ 7]**
9. Three fair coins are tossed. Find the expectation and the variance of number of heads.
   (M.U. 2004) [Ans. : $\bar{x}=3/2$, Var $(X)=3/4$]
10. From a box containing n tickets bearing numbers 1, 2, 3, ..., n a ticket is drawn. If X denotes the number on the ticket drawn, find the mean and variance of X.
    [Ans. : $\bar{x}=(n+1)/2$, Var $(X)=(n^2-1)/2$]
11. In a game of chance a man is allowed to throw a fair coin indefinitely. He receives rupees 1, 2, 3, ... if he throws a head at the 1st, 2nd, 3rd, ... trial respectively. If the entry fee to participate in the game is ₹ 2, find the expected value of his net gain. **[Ans. : Zero]**
12. A person draws 3 balls from a bag containing 3 white, 4 red and 5 black balls. He is offered ₹ 10, ₹ 5 and ₹ 2 if he draws 3 balls of the same colour, 2 balls of the same colour and 1 ball of each colour respectively. Find his expectation. **(M.U. 2004)**
    [Ans. : $p_1=\frac{3}{44}$, $p_2=\frac{29}{44}$, $p_3=\frac{12}{44}$; ₹ 4·52]
13. A continuous random variable X has the density function $f(x)=k(1+x)$ where $2\le x\le 5$. Find k, $P(x\le 4)$ and $E(X)$. **(M.U. 2005) [Ans. : $k=\frac{2}{27}$; $\frac{16}{27}$; $\frac{11}{3}$]**

3. Expectation of a Function of a Random Variable X

We can now extend the concept of expectation of a random variable to the function of a random variable.

1. Definition : Let X be a discrete random variable taking values $x_1, x_2, \ldots, x_n$ with probabilities $p_1, p_2, \ldots, p_n$ and $g(X)$ be a function of X then **mathematical expectation of $g(X)$** (if it exists) is defined by

$$\boxed{E[g(X)]=\sum p_i\,g(x_i)}$$

2. Definition : Let X be a continuous random variable with p.d.f. $f(x)$, let $g(X)$ be a function such that $g(X)$ is a random variable then $E[g(X)]$ (if it exists) is defined by,

$$\boxed{E[g(X)]=\int_{-\infty}^{\infty} g(x)\cdot f(x)\,dx}$$

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Notes ....

1. If $g(X) = aX^n$, then $E[g(X)] = E[aX^n] = \Sigma p_i a x_i^n = a \Sigma p_i x_i^n = a E(X^n)$

   And $E[g(X)] = E[aX^n] = \int_{-\infty}^{\infty} ax^n f(x)\,dx = a\int_{-\infty}^{\infty} x^n f(x)\,dx = a E(X^n)$

   e.g. $E(aX) = a E(X)$

   e.g. $E(aX^2) = a E(X^2)$, $E(aX^3) = a E(X^3)$

2. If $g(X) = aX + b$, then $E[g(X)] = E[aX + b] = a E(X) + b$.
3. If should be noted that, $E(X^2) \neq [E(X)]^2$ and $E(1/X) \neq 1/E(X)$
4. Putting $a = 1$, $E(X^n) = \Sigma p_i x_i^n$ and $E(X^n) = \int_{-\infty}^{\infty} x^n f(x)\,dx$

   In particular $E(X^2) = \Sigma p_i x_i^2$ and $E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

   $E(X^2)$ is denoted by μ_2'

   $\therefore \mu_2' = \Sigma p_i x_i^2$ or $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

4. Mean and Variance

If we know the probability density function, discrete or continuous, we can find the mean and variance of the random variable as follows.

$$\mu_1' = \text{Mean} = E(X) = \Sigma p_i x_i \quad \text{or} \quad \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx$$

We then find $\mu_2' = E(X^2) = \Sigma p_i x_i^2$ or $\mu_2' = \int_{-\infty}^{\infty} x^2 f(x)\,dx$

Now,
$$\text{Var}(X) = E(X - \bar{X})^2 = E[X - E(X)]^2$$
$$= E\left[X^2 - 2X E(X) + \{E(X)\}^2\right]$$
$$= E(X^2) - 2E(X)\cdot E(X) + [E(X)]^2$$
$$\boxed{\text{Var.}(X) = E(X^2) - [E(X)]^2}$$

But $E(X^2) = \mu_2'$ and $E(X) = \mu_1'$

$\therefore$ $\boxed{\text{Var.}(X) = \mu_2' - \mu_1'^2}$

Type I : Examples on Mean and Variance of a Discrete Probability Distribution

Example 1 : If X denotes the smaller of the two numbers that appear when a pair of dice is thrown, find the probability distribution of X, and also the mean and variance of X. **(M.U. 2004)**

Sol. : Refer to the table of Ex. 4 page 7-26.

We see that the number 1 appears as smaller (including equality) of the two numbers in 11 cases out of 36, the number 2 appears in 9 cases, 3 in 7 cases and so on.

The probability distribution of X is as given below.

X	:	1	2	3	4	5	6
$P(X = x)$	:	1/36	9/36	7/36	5/36	3/36	1/36

$$E(X) = \Sigma p_i x_i = \frac{11}{36}(1) + \frac{9}{36}(2) + \frac{7}{36}(3) + \frac{5}{36}(4) + \frac{3}{36}(5) + \frac{1}{36}(6) = 2.5278$$

$$E(X^2) = \frac{11}{36}(1^2) + \frac{9}{36}(2^2) + \frac{7}{36}(3^2) + \frac{5}{36}(4^2) + \frac{3}{36}(5^2) + \frac{1}{36}(6^2)$$
$$= \frac{301}{36} = 8.3611$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 8.3611 - (2.5278)^2 = 1.97$$

Example 2 : A discrete random variable has the probability density function given below.

X	:	-2	-1	0	1	2	3
$P(X = x)$	:	0.2	k	0.1	$2k$	0.1	$2k$

Find k, the mean and variance. **(M.U. 1997, 2001, 18, 19)**

Solution : We must have $\Sigma p_i = 1$.

$$\therefore \quad 5k + 0.4 = 1 \quad \therefore \quad 5k = 0.6 \quad \therefore \quad k = \frac{0.6}{5} = \frac{3}{25}$$

Hence, the probability distribution is

X	:	-2	-1	0	1	2	3
$P(X = x)$	:	2/10	3/25	1/10	6/25	1/10	6/25

Now, $\text{Mean} = E(X) = \Sigma p_i x_i = -\frac{4}{10} - \frac{3}{25} + 0 + \frac{6}{25} + \frac{2}{10} + \frac{18}{25} = \frac{60}{250} = \frac{6}{25}$

$$E(X^2) = \Sigma p_i x_i^2 = \frac{2}{10}(4) + \frac{3}{25}(1) + 0 + \frac{6}{25}(1) + \frac{1}{10}(4) + \frac{6}{25}(9) = \frac{73}{250}$$

$$\therefore \quad \text{Variance} = \sigma^2 = E(X^2) - [E(X)]^2 = \frac{73}{250} - \frac{36}{625} = \frac{293}{625}.$$

Example 3 : Find the value of k from the following data.

X	:	0	10	15
$P(x)$	:	$\frac{k-6}{5}$	$\frac{2}{k}$	$\frac{14}{5k}$

Also find the distribution function and the expectation of the distribution.

Sol. : Since $\Sigma p_i = 1$.

$$\frac{k-6}{5} + \frac{2}{k} + \frac{14}{5k} = 1 \quad \therefore \quad k^2 - 11k + 14 = 0$$

$$\therefore \quad (k-8)(k-3) = 0 \quad \therefore \quad k = 8 \text{ or } 3$$

But when $k = 3$ $P(x = 0) = \frac{3-6}{5} = -\frac{3}{5}$ which is impossible. $\therefore$ $k = 8$.

$\therefore$ The p.d.f. and distribution function are

X	:	0	10	15
$P(x)$	:	2/5	1/4	7/20
$F(x)$	:	2/5	13/20	1

$$\therefore \quad E(X) = \Sigma p_i x_i = \frac{2}{5}(0) + \frac{1}{4}(10) + \frac{7}{20}(15) = \frac{5}{2} + \frac{21}{4} = \frac{31}{4}.$$

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Example 4 : If the mean of the following distribution is 16 find m, n and variance

X	:	8	12	16	20	24
$P(X = x)$	:	1/8	m	n	1/4	1/12

(M.U. 2006, 16, 19)

Sol. : Since $\Sigma p_i = 1$,

$$\frac{1}{8} + m + n + \frac{1}{4} + \frac{1}{12} = 1 \quad \therefore \quad m + n = \frac{13}{24} \qquad \text{..... (1)}$$

Since mean = 16, $\Sigma p_i x_i = 16$

$$\therefore \quad 1 + 12m + 16n + 5 + 2 = 16$$

$$\therefore \quad 12m + 16n = 8 \qquad \therefore \quad 3m + 4n = 2 \qquad \text{..... (2)}$$

Multiply (1) by 3 and subtract from (2),

$$\therefore \quad 3m + 4n = 2 \;;\; 3m + 3n = \frac{13}{8} \qquad \therefore \quad n = \frac{3}{8}$$

$$\therefore \quad m + n = \frac{13}{24} \text{ gives } \quad m + \frac{3}{8} = \frac{13}{24} \quad \therefore \quad m = \frac{4}{24} = \frac{1}{6}$$

To find variance, consider

$$E(X^2) = \Sigma p_i x_i^2 = \frac{1}{8}(64) + \frac{1}{6}(144) + \frac{3}{8}(256) + \frac{1}{4}(400) + \frac{1}{12}(576) = 276$$

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = 276 - 16^2 = 20.$$

Example 5 : A woman with n keys with her, wants to open the door of her house by trying keys independently and randomly one by one. Find the mean and the variance of the number of trials required to open the door, if unsuccessful keys are kept aside. **(M.U. 2016)**

Sol. : If unsuccessful keys are kept aside, she will get success in the first trial, or second trial or third trial and so on, the random variable X of the successful trial will take values 1, 2, 3, ..., n.

$$\therefore \quad P(\text{Success in the first trial}) = \frac{1}{n}$$

$$P(\text{Failure in the first trial}) = 1 - \frac{1}{n}$$

If there is failure in the first trial, the key is eliminated. There are now $(n - 1)$ keys.

$$\therefore \quad P(\text{Success in the second trial}) = \frac{1}{n-1}$$

$\therefore$ P (Failure in the first trial and success in the second trial)

$$= \left(1 - \frac{1}{n}\right)\left(\frac{1}{n-1}\right) = \frac{n-1}{n} \cdot \frac{1}{n-1} = \frac{1}{n}$$

$\therefore$ P (Failure in the first trial, failure in the second trial and success in the third trial)

$$= \left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n-1}\right)\left(\frac{1}{n-2}\right)$$

$$= \frac{(n-1)}{n} \cdot \frac{(n-2)}{(n-1)} \cdot \frac{1}{(n-2)} = \frac{1}{n}$$

Thus, the probability of success at any trial remains constant $= \frac{1}{n}$.

Thus, the probability distribution of X is

X	:	1	2	3	4	....	n
$P(X = x)$	:	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	$\frac{1}{n}$	....	$\frac{1}{n}$

$$\therefore \quad E(X) = \Sigma x\, p(x) \cdot x = \frac{1}{n} \cdot 1 + \frac{1}{n} \cdot 2 + \frac{1}{n} \cdot 3 + ... + \frac{1}{n} \cdot n$$

$$= \frac{1}{n}(1 + 2 + 3 + ... + n) = \frac{1}{n} \cdot \frac{n}{2} \cdot (n + 1)$$

$$\therefore \quad E(X) = \frac{n+1}{2} \qquad \left[\because 1 + 2 + ... + n = \frac{n+1}{2}\right]$$

$$E(X^2) = \Sigma p(x) x^2 = \frac{1}{n} \cdot 1^2 + \frac{1}{n} \cdot 2^2 + \frac{1}{n} \cdot 3^2 + ... + \frac{1}{n} \cdot n^2$$

$$= \frac{1}{n}(1^2 + 2^2 + 3^2 + ... + n^2) = \frac{1}{n} \cdot \frac{n}{6} \cdot (n + 1)(2n + 1)$$

$$= \frac{(n+1)(2n+1)}{6} \qquad \left[\because 1^2 + 2^2 + ... + n^2 = \frac{n}{6}(n + 1)(2n + 1)\right]$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4}$$

$$= \frac{2(2n^2 + 3n + 1) - 3(n^2 + 2n + 1)}{12} = \frac{n^2 - 1}{12}.$$

Type II : Examples on Mean and Variance of a Continuous Probability Distribution

Example 1 : A continuous random variable X has the p.d.f. defined by $f(x) = A + Bx$, $0 \le x \le 1$. If the mean of the distribution is 1/ 3, find A and B. **(M.U. 2004, 14)**

Sol. : Since $f(x)$ is a probability distribution

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

By data $\int_0^1 (A + Bx)\,dx = 1$

$$\therefore \quad \left[AX + \frac{Bx^2}{2}\right]_0^1 = 1 \quad \therefore \quad A + \frac{B}{2} = 1 \qquad \text{..... (i)}$$

Since the mean is $\frac{1}{3}$. $\int_0^1 x\, f(x)\,dx = \frac{1}{3}$ $\quad \therefore \quad \int_0^1 (A + Bx)\, x\,dx = \frac{1}{3}$

$$\therefore \quad \int_0^1 (Ax + Bx^2)\,dx = \frac{1}{3} \quad \therefore \quad \left[A\frac{x^2}{2} + \frac{Bx^3}{3}\right]_0^1 = \frac{1}{3}$$

$$\therefore \quad \frac{A}{2} + \frac{B}{3} = \frac{1}{3} \quad \therefore \quad 3A + 2B = 2 \qquad \text{..... (ii)}$$

Solving the equations (i) and (ii), we get $A = 2$, $B = -2$.

$\therefore$ The p.d.f. is $f(x) = 2 - 2x$, $0 \le x \le 1$

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Example 2 : The distribution function of a r.v. X is given by $F_X(x) = 1 - (1 + x)\, e^{-x},\ x \geq 0$. Find the mean and variance. **(M.U. 2011)**

Sol. : We have

$$f_X(x) = \frac{d\,F_X(x)}{dx} = (1+x)e^{-x} - e^{-x} = x\,e^{-x}, \quad x \geq 0$$

$$\therefore \quad \text{Mean } \overline{X} = \int_0^\infty x \cdot f_X(x)\,dx = \int_0^\infty x^2\, e^{-x}\,dx$$

$$= \left[x^2(-e^{-x}) - 2x(e^{-x}) + 2(1)\cdot(-e^{-x})\right]_0^\infty = 2$$

$$E(X^2) = \int_0^\infty x^2 f_X(x)\,dx = \int_0^\infty x^3 \cdot e^{-x}\,dx$$

$$E(X^2) = \left[x^3(-e^{x}) - 3x^2(e^{-x}) + 6x(-e^{-x}) - 6(e^{-x})\right]_0^\infty = 6$$

$$\therefore \quad V(X) = E(X^2) - [E(X)]^2 = 6 - 4 = 2.$$

Example 3 : A continuous random variable X has the p.d.f. $f(x) = kx^2 e^{-x},\ x \geq 0$. Find k, mean and variance. **(M.U. 2004)**

Solution : We must have $\int_0^\infty k\,x^2 e^{-x} \cdot dx = 1$

$$\therefore \quad k\left[x^2(-e^{-x}) - \int -e^{-x} 2x\,dx\right]_0^\infty = 1$$

$$\therefore \quad k\left[-x^2e^{-x} + 2x(-e^{-x}) - \int -2e^{-x}dx\right]_0^\infty = 1$$

[Integrating by parts]

$$k\left[-x^2e^{-x} - 2x\,e^{-x} - 2e^{-x}\right]_0^\infty = 1$$

$$k[0 - (-0 - 0 - 2)] = 1 \quad \therefore\ 2k = 1 \quad \therefore\ k = \frac{1}{2}.$$

Now, mean $\overline{X} = \int_0^\infty x\,f(x)\,dx = \int_0^\infty \frac{1}{2}x^3e^{-x}dx$

$$= \frac{1}{2}\left[x^3(-e^{-x}) - (3x^2)(e^{-x}) + (6x)(-e^{-x}) - (6)(e^{-x})\right]_0^\infty$$

(By the generalised rule of integration by parts.

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots..$$

where dashes denote the derivatives and suffixes denote the integrals.)

$$\therefore \quad \overline{X} = \frac{1}{2}[0 - (-6)] = \frac{1}{2}\cdot 6 = 3$$

Now, $\mu_2' = \frac{1}{2}\int_0^\infty x^2 f(x)\,dx = \frac{1}{2}\int_0^\infty x^2 \cdot x^2 e^{-x}\,dx$

$$= \frac{1}{4}\int_0^\infty x^4 \cdot e^{-x}dx$$

$$= \frac{1}{2}\left[x^4(-e^{-x}) - (4x^3)(e^{-x}) + (12x^2)(-e^{-x}) - (24x)(e^{-x}) + 24(-e^{-x})\right]_0^\infty$$

$$\therefore \quad \mu_2' = \frac{1}{2}[0 - (-24)] = \frac{24}{2} = 12$$

$$\therefore \quad \text{Variance} = \mu_2' - \mu_1'^2 = 12 - 9 = 3.$$

Example 4 : If X is a continuous random variable with probability density function given by

$$f(x) = \begin{cases} k(x - x^3), & 0 \leq x \leq 1 \\ 0, & \text{otherwise} \end{cases}$$

Find (i) k, (ii) mean, (iii) variance. **(M.U. 2016)**

Sol. : (i) We have $\int_0^1 k(x - x^3)\,dx = 1$

$$\therefore \quad k\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1 = 1 \quad \therefore \quad k\left[\frac{1}{2} - \frac{1}{4}\right] = 1 \quad \therefore \quad k \cdot \frac{1}{4} = 1 \quad \therefore \quad k = 4$$

(ii) Mean, $\overline{X} = \mu_1' = \int_0^1 x\,f(x)\,dx = \int_0^1 x \cdot 4(x - x^3)\,dx$

$$= 4\int_0^1 (x^2 - x^4)\,dx = 4\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = 4\left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8}{15}$$

$$\mu_2' = \int_0^1 x^2 f(x)\,dx = \int_0^1 x^2 \cdot 4(x - x^3)\,dx = 4\int_0^1 (x^3 - x^5)\,dx$$

$$= 4 \cdot \left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1 = 4 \cdot \left[\frac{1}{4} - \frac{1}{6}\right] = 4 \cdot \frac{1}{12} = \frac{1}{3}$$

(iii) Variance $(X) = \mu_2' - \mu_1'^2 = \frac{1}{3} - \frac{64}{225} = \frac{33}{225} = 0{\cdot}1467.$

Example 5 : Find the value of k, if the function

$$f(x) = kx^2(1 - x^3), \qquad 0 \leq x \leq 1$$
$$= 0 \qquad \text{otherwise}$$

is a probability density function. Also find $P(0 \leq x \leq 1/2)$ and the mean and variance. **(M.U. 2017, 19)**

Sol. : We have $\int_0^1 kx^2(1 - x^3)\,dx = 1 \quad \therefore \quad \int_0^1 k(x^2 - x^5)\,dx = 1$

$$\therefore \quad k\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^1 = 1 \quad \therefore \quad k\left[\frac{1}{3} - \frac{1}{6}\right] = 1 \quad \therefore \quad k \cdot \frac{1}{6} = 1 \quad \therefore \quad k = 6.$$

Now, $P(0 \leq x \leq 2) = 6\int_0^{1/2} (x^2 - x^5)\,dx = 6\left[\frac{x^3}{3} - \frac{x^6}{6}\right]_0^{1/2} = 6\left[\frac{1}{8\cdot 3} - \frac{1}{64\cdot 6}\right] = \frac{15}{64}.$

Mean $\overline{x} = \mu_1' = \int_0^1 x\,f(x)\,dx = 6\int_0^1 x \cdot x^2(1 - x^3)\,dx = 6\int_0^1 (x^3 - x^6)\,dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^7}{7}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{7}\right] = \frac{18}{28} = \frac{9}{14}$$

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$$\mu_2' = \int_0^1 x^2 f(x)\,dx = 6\int_0^1 x^2\left[x^2(1-x^3)\right]dx = 6\int_0^1 (x^4 - x^7)\,dx$$

$$= 6\left[\frac{x^5}{5} - \frac{x^8}{8}\right]_0^1 = 6\left[\frac{1}{5} - \frac{1}{8}\right] = \frac{18}{40} = \frac{9}{20}$$

$$\text{Variance} = \mu_2' - \mu_1' = \frac{9}{20} - \frac{91}{196} = \frac{441-405}{980} = \frac{36}{980} = \frac{9}{245}.$$

Example 6 : A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} kx & 0 \le x \le 2 \\ 2k & 2 \le x \le 4 \\ 6k - kx & 4 \le x \le 6 \end{cases}$$

Find k, $P(1 \le x \le 3)$ and the mean. **(M.U. 2019)**

Sol. : For p.d.f. we must have $\int_{-\infty}^{\infty} f(x)\,dx = 1$.

$$\therefore \int_0^2 kx\,dx + \int_2^4 2k\,dx + \int_4^6 (6k - kx)\,dx = 1$$

$$\therefore k\left[\frac{x^2}{2}\right]_0^2 + 2k[x]_2^4 + k\left[6x - \frac{x^2}{2}\right]_4^6 = 1$$

$$\frac{k}{2}[4-0] + 2k[4-2] + k[(36-18)-(24-8)] = 1$$

$$2k + 4k + 6k = 1 \quad \therefore 12k = 1 \quad \therefore k = \frac{1}{2}$$

$$\therefore P(1 \le x \le 3) = \int_1^2 \frac{x}{12}\,dx + \int_2^3 \frac{1}{6}\,dx = \frac{1}{12}\left[\frac{x^2}{2}\right]_1^2 + \frac{1}{6}[x]_2^3 = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}.$$

$$\text{Mean } \bar{x} = \mu_1' = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \frac{1}{12}\int_0^2 x \cdot x\,dx + \frac{1}{6}\int_2^4 x\,dx + \frac{1}{12}\int_4^6 x(6-x)\,dx$$

$$\therefore \text{Mean } \bar{x} = \frac{1}{12}\left[\frac{x^3}{3}\right]_0^2 + \frac{1}{6}\left[\frac{x^2}{2}\right]_2^4 + \frac{1}{12}\left[3x^2 - \frac{x^3}{3}\right]_4^6$$

$$= \frac{8}{36} + \frac{1}{12}[16-4] + \frac{1}{12}\left[\left(108 - \frac{216}{3}\right) - \left(48 - \frac{64}{3}\right)\right]$$

$$= \frac{2}{9} + 1 + \frac{1}{12} + \frac{28}{3} = \frac{383}{36}.$$

Example 7 : If the distribution function of a random variable is given by

$$F(x) = \begin{cases} 1 - 4/x^2, & x > 2 \\ 0, & x \le 2 \end{cases}$$

find (i) $P(x < 3)$, (ii) $P(4 < x < 5)$, (iii) mean and the variance.

Sol. : The probability density function $f(x)$ is given by

$$f(x) = F'(x) = \begin{cases} 8/x^3, & x > 2 \\ 0, & x \le 2 \end{cases}$$

(i) $P(x < 3) = \int_2^3 \frac{8}{x^3}\,dx = \left[-\frac{8}{2x^2}\right]_2^3 = \left[-\frac{4}{x^2}\right]_2^3 = -4\left[\frac{1}{9} - \frac{1}{4}\right] = \frac{5}{9}.$

(ii) $P(4 < x < 5) = \int_4^5 \frac{8}{x^3}\,dx = \left[-\frac{4}{x^2}\right]_4^5 = -4\left[\frac{1}{25} - \frac{1}{16}\right] = 0{\cdot}09$

(iii) $\mu_1' = \int_2^{\infty} x \cdot \frac{8}{x^3}\,dx = \int_2^{\infty} \frac{8}{x^2}\,dx = 8\left[-\frac{1}{x}\right]_0^{\infty} = -8\left[0 - \frac{1}{2}\right] = 4$

$$\mu_2' = \int_2^{\infty} x^2 \cdot \frac{8}{x^3}\,dx = 8\int_2^{\infty} \frac{1}{x}\,dx = 8[\log x]_2^{\infty} = \infty$$

$\therefore$ Var. $(x) = \mu_2' - \mu_1'^2$ $\quad\therefore$ Var. (x) does not exist.

Example 8 : A continuous random variable has probability density function

$$f(x) = k(x - x^2), \quad 0 \le x \le 1$$

Find (i) k, (ii) mean, (iii) variance. **(M.U. 1997, 2001, 03, 15, 16, 19)**

Sol. : (i) We have

$$\int_0^1 k(x - x^2)\,dx = 1 \quad \therefore k\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 1$$

$$\therefore k\left[\frac{1}{2} - \frac{1}{3}\right] = 1 \quad \therefore k \cdot \frac{1}{6} = 1 \quad \therefore k = 6.$$

(ii) $\mu_1' = \int_0^1 x \cdot 6(x - x^2)\,dx = 6\int_0^1 (x^2 - x^3)\,dx$

$$= 6\left[\frac{x^3}{3} - \frac{x^4}{4}\right]_0^1 = 6\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{1}{2}$$

(iii) $\mu_2' = \int_0^1 x^2 \cdot 6(x - x^2)\,dx = 6\int_0^1 (x^3 - x^4)\,dx$

$$= 6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]_0^1 = 6\left[\frac{1}{4} - \frac{1}{5}\right] = \frac{6}{20} = \frac{3}{10}$$

$$\therefore \text{Var. } (x) = \mu_2' - \mu_1'^2 = \frac{3}{10} - \frac{1}{4} = \frac{1}{20}$$

EXERCISE - II

Type - I

1. The probability distribution of a random variable X is given by

X :	-2	-1	0	1	2	3
$P(X = x)$:	$0{\cdot}1$	k	$0{\cdot}2$	$2k$	$0{\cdot}3$	k

Find k, the mean and variance. **(M.U. 2018) [Ans. : (i)** $k = 0{\cdot}1$, **(ii)** $\bar{X} = 0{\cdot}8$, **(iii)** Var. $2{\cdot}16$ **]**

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2. If X denotes the larger of the two numbers that appear when a pair of dice is thrown, find the probability distribution of X, and also the mean and variance of X.

[**Ans. :**

X	:	1	2	3	4	5	6
$P(X = x)$	:	1/36	3/36	5/36	7/36	9/36	11/36

Mean = 4·47, Var. = 1·97]

3. Given the following distribution

X	:	−3	−2	−1	0	1	2
$P(X = x)$	:	0·01	0·1	0·2	0·3	0·2	0·15

Find (i) $P(X \geq 1)$, (ii) $P(X < 0)$, (iii) $E(X)$, (iv) $V(X)$. **(M.U. 1998)**

[**Ans. : (i)** 0·35, **(ii)** 0·35, **(iii)** 0·05, **(iv)** 1·8475]

4. Find the value of k from the following data.

X	:	0	10	15
$P(X = x)$	:	$(k-6)/5$	$2/k$	$14/5k$

Also find the distribution function and expectation of X. **(M.U. 2003)**

[**Ans. :** $k = 8$ or 3, 3 is impossible since $P(X = 0) = -\frac{3}{5}$ for $k = 3$

X	:	0	10	15
$F(x)$	:	2/5	13/20	1

$E(X) = 31/4$]

5. A random variable X has the probability law $P(X = x) = 1/n$, $x = 1, 2, \ldots\ldots n$. Find $E(X)$ and $V(X)$. [**Ans. : (i)** $(n+1)/2$, **(ii)** $(n^2-1)/12$]

6. A random variable X has the probability distribution

X	:	−2	−1	0	1	2	3
$p(x)$	:	0·1	k	0·2	$2k$	0·3	k

Find k and then mean and variance of X. [**Ans. :** $k = 0{\cdot}1$, 0·8, 2·16]

7. Find the mean and the variance of the following distribution

X	:	1	3	4	5
$P(X = x)$	:	0·4	0·1	0·2	0·3

[**Ans. : (i)** $\overline{X} = 3$, **(ii)** Var. = 3]

8. A random variable X has the probability distribution $P(X = 0) = P(X = 2) = p$, $P(X = 1) = 1 - 2p$ and $0 \leq p \leq 2/3$.

For what value of p is the Var. (X) maximum ? [**Ans. :** $p = 2/3$]

Type - II

1. Find the mean and the variance of the following distribution

$$f(x) = \begin{cases} 1-x, & 0 < x < 1 \\ x-1, & 1 < x < 2 \end{cases}$$

(M.U. 2006) [**Ans. : (i)** $\overline{X} = 1$, **(ii)** Var. = 1/2]

2. If the probability density function is given by

$$f(x) = \begin{cases} kx^2(1-x^3), & 0 \leq x \leq 1 \\ 0, & \text{elsewhere} \end{cases}$$

(i) Find k, (ii) $P(0 < X < 1/2)$, (iii) $\overline{X}$, (iv) σ^2. **(M.U. 1998, 2003, 04)**

[**Ans. : (i)** $k = 6$, **(ii)** 15/64, **(iii)** 9/14, **(iv)** 9/245]

3. If the probability density of a random variable is given by

$$f(x) = \begin{cases} kx, & 0 \leq x \leq 2 \\ 2k, & 2 \leq x \leq 4 \\ 6k - kx, & 4 \leq x \leq 6 \end{cases}$$

Find (i) k, (ii) $P(1 \leq X \leq 3)$, (iii) $\overline{X}$. **(M.U. 2002, 03, 05)** [**Ans. : (i)** 1/8, **(ii)** 7/16, **(iii)** 7/4]

4. Find the mean and the variance of

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2-x, & 1 \leq x \leq 2 \end{cases}$$

(M.U. 2009) [**Ans. :** 1, 5/2]

5. If the probability density of a random variable is given by

(a) $f(x) = \begin{cases} kxe^{-x/3}, & x > 0 \\ 0, & x \leq 0 \end{cases}$ **(M.U. 2005)** (b) $f(x) = \begin{cases} kx^2e^{-x}, & x > 0 \\ 0, & x < 0 \end{cases}$ **(M.U. 2007)**

(i) Find k, (ii) $\overline{X}$, (iii) σ^2. [**Ans. : (a) (i)** 1/9, **(ii)** 6, **(iii)** 18 ; **(b) (i)** 1/2, **(ii)** 3, **(iii)** 3]

6. A continuous random variable X has p.d.f. $f(x) = kx^2e^{-x}$, $x \geq 0$. Find k, mean and variance. **(M.U. 2004)** [**Ans. : (i)** 1/2, **(ii)** 3, **(iii)** 3]

7. If the probability density of a random variable is given by

$$f(x) = \begin{cases} kxe^{-x/3}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

(i) Find k, (ii) $\overline{X}$, (iii) σ^2. **(M.U. 2005)** [**Ans. : (i)** 1/9, **(ii)** 6, **(iii)** 18]

8. A continuous random variable has probability density function

$$f(x) = \begin{cases} 2e^{-2x} & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

Find (i) $E(X)$, (ii) $E(X^2)$, (iii) Var (X), (iv) S.D. of X. **(M.U. 2005)**

[**Ans. : (i)** $\frac{1}{2}$, **(ii)** $\frac{1}{2}$, **(iii)** $\frac{1}{4}$, **(iv)** $\frac{1}{2}$.]

9. The length of time (in minutes) a lady speaks on telephone is found to be a random variable with probability density function

$$f(x) = \begin{cases} Ae^{-x/5} & \text{for } x \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

Find A and the probability that she will speak for (i) more than 10 minutes, (ii) less than 5 minutes, (iii) between 5 and 10 minutes. **(M.U. 2004)**

[**Ans. : (i)** $A = \frac{1}{5}$, **(ii)** $\frac{1}{e^2}$, **(iii)** $\frac{(e-1)}{e}$, **(iv)** $\frac{(e-1)}{e^2}$.]

10. The probability density function of a random variable is given by

$$f(x) = ke^{-x/\sigma}, \quad 0 < x < \infty$$

Find the mean and standard deviation of X. **(M.U. 2002)** [**Ans. : (i)** σ, **(ii)** σ]

11. A random variable X has the p.d.f. $f(x) = \frac{k}{1+x^2}$, $-\infty < x < \infty$. **(M.U. 2004)**

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Determine k and the distribution function. Evaluate (i) $P(x \geq 0)$, (ii) Mean, (iii) Variance.

[**Ans. :** $k = \frac{1}{\pi}$, $F(x) = \frac{1}{\pi}\left[\tan^{-1} x + \frac{\pi}{2}\right]$, $P(x \geq 0) = \frac{1}{2}$, $\overline{X} = 0$, variance does not exist.]

12. If $f(x) = \begin{cases} x e^{-x^2/2}, & x \geq 0 \\ 0, & x < 0 \end{cases}$

prove that (i) $f(x)$ is a probability density function and (ii) obtain distribution function $F(x)$.

(M.U. 1996) [**Ans. :** (i) $\int_0^\infty x e^{-x^2/2} dx = \int_0^\infty e^{-t} dt = 1$ where $t = \frac{x^2}{2}$; (ii) $F(x) = 1 - e^{-x^2/2}$]

13. A continuous random variable X takes values between 2 and 5. Its density function is $f(x) = k(1 + x)$. Find k and $P(x < 4)$. **(M.U. 1996)** [**Ans. :** $k = 2/27$, $16/27$]

14. The distribution function of a continuous random variable X is given by

$$F(x) = 1 - (1 + x) e^{-x}, x \geq 0.$$

Find the density function, mean and variance. [**Ans. :** $f(x) = xe^{-x}$, $x \geq 0$; 2 ; 2]

15. A continuous random variable X has a probability density function $f(x) = 3x^2$, $0 \leq x \leq 1$. Find a and b such that (i) $P(X \leq a) = P(X \geq a)$, (ii) $P(X > b) = 0.005$.

[**Ans. :** (i) $a = \sqrt[3]{1/2}$, (ii) $b = \sqrt[3]{95}$]

16. If $f(x)$ is probability density function of a continuous random variate, find k, mean and variance.

$$f(x) = \begin{cases} kx^2 & 0 \leq x \leq 1 \\ (2-x)^2 & 1 \leq x \leq 2 \end{cases}$$

(M.U. 2002) [**Ans. :** $k = 2$, $\overline{x} = \frac{11}{12}$, Var. = 0·626]

17. A continuous random variable X has the probability density function given by

$$f(x) = \begin{cases} 2ax + b & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

If the mean of the distribution is 3, find the constants a and b. **(M.U. 1996)**

[**Ans. :** $a = \frac{3}{2}$, $b = -\frac{5}{2}$]

18. The probability density function of a random variable is given by

$$f(x) = \begin{cases} 0 & x < 2 \\ \frac{2x+3}{18} & 2 \leq x \leq 4 \\ 0 & x > 4 \end{cases}$$

Find the mean and variance. **(M.U. 2001, 02)** [**Ans. :** $\overline{X} = \frac{83}{27}$, Var. = 0·33]

19. Prove that $f(x) = \begin{cases} 1 - |1 - x| & 0 \leq x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$

is a probability density function. Find its mean and variance. **(M.U. 1998)**

[**Ans. :** $\overline{X} = 12$, Var. = $\frac{1}{2}$]

20. A continuous random variable X has the following probability density function

$$f(x) = \begin{cases} x^3 & 0 \leq x \leq 1 \\ (2-x)^3 & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Find $P(0{\cdot}5 \leq x \leq 1{\cdot}5)$ and the mean of the distribution. **(M.U. 1998)** [**Ans. :** $\frac{15}{32}$, $\overline{X} = \frac{1}{2}$]

21. The probability density function of a continuous random variable X is given by

$$f(x) = kx(2 - x), \ 0 \leq x \leq 2.$$

Find k, mean and variance. **(M.U. 1997, 99)** [**Ans. :** $k = \frac{3}{4}$, $\overline{X} = 1$, $V(X) = \frac{1}{5}$]

5. Laws of Expectation

We shall now develop some laws or theorems that will simplify the calculations of mathematical expectation of functions of random variables. These rules help us to calculate expectations in terms of known or easily computable expectations.

Theorem 1 : If X is a discrete random variable such that $x_i \geq 0$ for all i, then

$$\boxed{E(X) \geq 0}$$

Proof : Since $x_1, x_2, \ldots\ldots x_n$ are all non-negative and since, $p_1, p_2, \ldots\ldots p_n$ are also non-negative

$\therefore \ \Sigma p_i x_i \geq 0 \qquad \therefore \ E(X) \geq 0$

Theorem 2 : If X is a discrete (or continuous) random variate, a and b are constants then

$$\boxed{E(aX + b) = aE(X) + b}$$

Proof : We shall prove the result for discrete random variable and leave the continuous case to you. Let X take values $x_1, x_2, \ldots\ldots x_n$ with probabilities $p_1, p_2, \ldots\ldots p_n$. Then by definition,

$$\begin{aligned} E(aX + b) &= \Sigma p_i (a x_i + b) \\ &= p_1 (ax_1 + b) + p_2 (ax_2 + b) + \ldots\ldots + p_n (ax_n + b) \\ &= a (p_1x_1 + p_2x_2 + \ldots\ldots + p_nx_n) + b (p_1 + p_2 + \ldots\ldots + p_n) \\ &= a E(X) + b \qquad [\because \Sigma p_i = 1] \end{aligned}$$

Cor. 1 : Putting $a = 0$.

$E(b) = b$ *i.e.* **expectation of a constant is the constant itself.**

Cor. 2 : Putting $b = 0$.

$E(ax) = a E(X)$ *i.e.* for calculations **the constant can be taken out.**

Cor. 3 : Putting $a = 1$, $b = -\overline{X}$, $E(X - \overline{X}) = 0$.

Theorem 3 : Theorem of Addition (a) Discrete Variates

The expectation of the sum (or difference) of two (discrete) variates is equal to the sum (or difference) of their expectations.

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In symbols, $\boxed{E(X \pm Y) = E(X) \pm (Y)}$

We shall accept the result without proof.

(a) Theorem of Addition (Continuous Variates)

The expectation of the sum (or difference) of two continuous variates is equal to the sum (or difference) of their expectations.

In symbols, $\boxed{E(X \pm Y) = E(X) \pm (Y)}$

We shall accept the result without proof.

Generalisation : The mathematical expectation of the sum (or difference) of n random variables is equal to sum (or difference) of their expectations provided all the expectations exist.

$$E(X_1 \pm X_2 \pm \ldots\ldots \pm X_n) = E(X_1) \pm E(X_2) \pm \ldots\ldots \pm E(X_n).$$

Theorem 4 : Theorem of Multiplication (a) Discrete variables : The expectation of the product of two independent variates is equal to the product of their expectations if the expectations exist.

In symbols, $\boxed{E(XY) = E(X) \cdot E(Y)}$

We shall accept the theorem without proof.

(b) Theorem of Multiplication (continuous variable)

The expectation of the product of two **independent variates** (discrete or continuous) is equal to the product of their expectations if the expectations exist.

In symbols, $\boxed{E(XY) = E(X) \cdot E(Y)}$

We shall accept the theorem without proof.

Generalisation : The mathematical expectation of the product of a number of independent continuous random variates is equal to the product of their expectations, provided all the expectations exist.

$$E(X, Y, Z \ldots\ldots W) = E(X) \cdot E(Y) \cdot E(Z) \ldots\ldots E(W).$$

6. Properties of Variance

We now prove some properties of variance which are useful in calculating variance.

1. Variance of a constant is zero, $V(C) = 0$.

Proof : $V(C) = E(C^2) - [E(C)]^2 = C^2 - C^2 = 0.$

2. If X is a random variate and a, b are constants then

$$\boxed{V(aX + b) = a^2 V(X)}$$ **(M.U. 2006)**

Proof : By definition,

$$\begin{aligned} V(aX+b) &= E[(aX+b) - E(aX+b)]^2 = E[(aX+b) - a E(X) - b)]^2 \\ &= E[a\{X - E(X)\}]^2 = E[a^2\{X - E(X)\}^2] \\ &= a^2 E[X - E(X)]^2 = a^2 V(X). \end{aligned}$$

Cor. 1 : $\boxed{V(aX) = a^2 V(X)}$

Putting $b = 0$, we get the result.

Cor. 2 : $\boxed{V(X + b) = V(X)}$

Putting $a = 1$, we get the result.

Note ....

Note that although $E(aX + b) = aE(X) + b$ we do **not** have $V(aX + b) = aV(X) + b$. Instead, we have $V(aX + b) = a^2 V(X)$.

3. $$\boxed{V(a_1X_1 + a_2X_2) = a_1^2 V(X_1) + a_2^2 V(X_2)}$$

where X_1 and X_2 are independent random variates.

Proof : Let $Y = a_1X_1 + a_2X_2$ where X_1 and X_2 are independent.

$$\begin{aligned} \therefore \quad \overline{Y} &= E(a_1X_1 + a_2X_2) \\ &= E(a_1X_1) + E(a_2X_2) \qquad \text{[By Theorem 3, page 8-21]} \\ &= a_1 E(X_1) + a_2 E(X_2) \end{aligned}$$

(This result is true even if X_1, X_2 are not independent.)

$$\begin{aligned} \therefore \quad V(Y) &= E[(a_1X_1 + a_2X_2) - \{a_1 E(X_1) + a_2 E(X_2)\}]^2 \\ &= E[a_1\{X_1 - E(X_1)\} + a_2\{X_2 - E(X_2)\}]^2 \\ &= E\Big[a_1^2\{X_1 - E(X_1)\}^2 + a_2^2\{X_2 - E(X_2)\}^2 \\ &\qquad + 2a_1a_2\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}\Big] \\ &= a_1^2 E\{X_1 - E(X_1)\}^2 + a_2^2 E\{X_2 - E(X_2)\}^2 \\ &\qquad + 2a_1a_2 E[\{X_1 - E(X_1)\}\{X_2 - E(X_2)\}] \\ &= a_1^2 E[X_1 - E(X_1)]^2 + a_2 E[X_2 - E(X_2)]^2 \\ &\qquad + 2a_1a_2[E(X_1)E(X_2) - E(X_1)E(X_2) - E(X_1)E(X_2) + E(X_1)E(X_2)] \end{aligned}$$

[$\because$ $E(X_1X_2) = E(X_1) \cdot E(X_2)$ as X_1, X_2 are independent.]

$\therefore$ $V(a_1X_1 + a_2X_2) = a_1^2 V(X_1) + a_2^2 V(X_2)$

Cor. 1 : If $a_1 = 1$, $a_2 = 1$, we get

$$\boxed{V(X_1 + X_2) = V(X_1) + V(X_2)}$$

If $a_1 = 1$, $a_2 = -1$, we get

$$\boxed{V(X_1 - X_2) = V(X_1) + V(X_2)}$$

Example 1 : If $E(X) = 2$ and $V(X) = 5$, find $E(3X + 2)$ and $V(3X + 2)$.

Sol. :

$E(3X + 2) = 3E(X) + 2 = 8$

$V(3X + 2) = 9V(X) = 45.$

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TIME TO GET STARTED

Example 2 : If $E(X) = 1$ and $E(X^2) = 4$, find the mean and variance of $Y = 2X - 3$.

Sol. :
$$E(Y) = E(2X-3) = E(2X) - E(3)$$
$$= 2E(X) - 3 = 2 \cdot 1 - 3 = -1.$$
$$E(Y^2) = E[(2X-3)^2] = E[4X^2 - 12X + 9]$$
$$= E(4X^2) - E(12X) + E(9)$$
$$= 4E(X^2) - 12E(X) + 9$$
$$= 4 \cdot 4 - 12 \cdot 1 + 9 = 13$$
$$V(Y) = E(Y^2) - [E(Y)]^2 = 13 - (-1)^2 = 12.$$

Alternatively : $V(X) = E(X^2) - [E(X)]^2 = 4 - 1 = 3$

$\therefore V(2X-3) = 4V(X) = 4 \cdot 3 = 12$ [By Cor. 1, 2, page 8-22, 8-23]

Example 3 : If $E(X) = 10$, $\sigma_x^2 = 1$, and $Y = 2X(X+20)$, find $E(Y)$.

Sol. :
$$E(Y) = E[2X^2 + 20X] = E(2X^2) + E(20X)$$
$$= 2E(X^2) + 20E(X)$$

Now, $E(X) = 10$

and $V(X) = E(X^2) - [E(X)]^2 = 1$

$\therefore E(X^2) - 100 = 1 \quad \therefore E(X^2) = 101$

$\therefore E(Y) = 2(101) + 20(10) = 202 + 200 = 402.$

Example 4 : Suppose that X is a r.v. with $E(X) = 10$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1.

Sol. : We have, $E(Y) = E[aX - b] = aE(X) - b$

$\therefore 0 = a10 - b \quad \therefore b = 10a$

And $V(Y) = V(aX - b) = a^2 V(X)$

$\therefore 1 = a^2 \cdot 25 \quad \therefore a = 1/5.$

Thus, $a = 1/5$ and $b = 2$.

Example 5 : If X_1 has mean 4 and variance 9 and X_2 has mean -2 variance 4, and the two are independent, find $E(2X_1 + X_2 - 3)$ and $V(2X_1 + X_2 - 3)$. **(M.U. 2019)**

Sol. : We have $E(X_1) = 4$, $V(X_1) = 9$, $E(X_2) = -2$ and $V(X_2) = 4$.

$$\therefore E(2X_1 + X_2 - 3) = E(2X_1 + X_2) - 3 = 2E(X_1) + E(X_2) - 3$$
$$= 2(4) + (-2) - 3 = 3.$$
$$V(2X_1 + X_2 - 3) = V(2X_1 + X_2) \quad \text{[By above Cor. 2]}$$
$$= 2^2 V(X_1) + V(X_2)$$
$$= 4(9) + 5 = 41.$$

EXERCISE - III

1. If $E(X) = 3$, $V(X) = 5$ and $Y = 3X + 4$, find $E(Y)$ and $V(Y)$. **[Ans. : 13 ; 45]**
2. Find $V(X+2)$ and $V(3X+2)$. **[Ans. : $V(X)$, $9V(X)$]**
3. Let X be a random variate with $E(X) = 12$ and $V(X) = 16$. Find the positive values of a and b such that $Y = aX - b$ has expectations 0 and variance 1. **[Ans. : $a = 1/4$, $b = 3$]**

4. If X and Y are independent random variate, prove that
$$V(X+Y) = V(X) + V(Y)$$
(M.U. 2003)
5. Let X be a random variate with $E(X) = 15$ and $V(X) = 25$. Find the positive values of a and b such that $Y = aX - b$ has expectation 0 and variance 1. **[Ans. : 1 / 5, 3]**
6. If X_1 has mean 5 and variance 5, X_2 has mean -2 and variance 3, and if X_1, X_2 are independent, find : (i) $E(X_1 + X_2)$, $V(X_1 + X_2)$
   (ii) $E(X_1 - X_2)$, $V(X_1 - X_2)$
   (iii) $E(2X_1 + 3X_2 - 5)$, $V(2X_1 + 3X_2 - 5)$ **[Ans. : (i) 3, 8 ; (ii) 7, 8 ; (iii) –1, 52]**

7. Covariance and Correlation

Definition : If X and Y are discrete random variates then the covariance between them denoted by cov. (X, Y) is defined by

$$\boxed{\text{cov.}(X, Y) = E[\{X - E(X)\} \cdot \{Y - E(Y)\}]} \quad \text{..... (1)}$$

or $$\text{cov.}(X,Y) = \frac{1}{n}\Sigma(x_i - \bar{X})(y_i - \bar{Y})$$

On simplification, we get another expression for cov. (X, Y)

Now, $$\text{cov.}(X, Y) = E[XY - XE(Y) - YE(X) + E(X)E(Y)]$$
$$= E(XY) - E(X)E(Y) - E(Y)E(X) + E(X) \cdot E(Y)$$
$$\boxed{\text{cov.}(X, Y) = E(XY) - E(X)E(Y)} \quad \text{..... (2)}$$

Example 1 : Prove that cov. $(aX, bY) = ab$ cov. (X, Y)

Sol. : $$\text{Cov.}(aX, bY) = E(ab\,XY) - E(aX)E(bY) = ab\,E(XY) - ab\,E(X)E(Y)$$
$$= ab[E(XY) - E(X)E(Y)] = ab \text{ cov.}(X, Y)$$

Definition of Correlation Coefficient r

Definition : If X, Y are two discrete random variates, then the correlation coefficient between them denoted by r is defined by

$$\boxed{r = \frac{\text{cov.}(X,Y)}{\sigma_x \sigma_y}}$$

where σ_x, σ_y are the standard deviations of x and y.

Theorem : If X, Y independent random variates, then they are not correlated.

Proof : By definition, coefficient of correlation $r = \dfrac{\text{cov.}(X,Y)}{\sigma_x \sigma_y}$.

But cov. $(x, y) = E(X, Y) - E(X) \cdot E(Y)$. [By (2)]

Since x, y are independent $E(X, Y) = E(X) \cdot E(Y)$

$\therefore$ cov. $(X, Y) = E(X) \cdot E(Y) - E(X) \cdot E(Y) = 0$

$\therefore r = 0$

Note ....

However, the converse is not true *i.e.* if $r = 0$, it does not necessarily mean that the variables are not independent.

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Consider the following series.

X	Y	$x_i - \overline{X}$	$y_i - \overline{Y}$	$(x_i - \overline{X})(y_i - \overline{Y})$
4	8	4	3	12
2	2	2	−3	−6
−2	2	−2	−3	6
−4	8	−4	3	−12
$\overline{X} = 0$	$\overline{Y} = 5$			Total 0

Since $\Sigma(x_i - \overline{X})(y_i - \overline{y}) = 0$, $\text{cov.}(X, Y) = 0$ $\therefore r = 0$

But it is clear that the above values of x and y satisfy the relation $y = \frac{1}{2}x^2$. This means X and Y are not independent.

Example : Let X and Y be two variates taking values 0 and 1. If $r(X, Y) = 0$, show that $X = 1$ and $Y = 1$ are independent events. **(M.U. 2004)**

Sol. : Let X take values 0, 1 with probability p and q.

Let Y take values 0, 1 with probabilities p' and q'.

Now, $r(X,Y) = \dfrac{\text{cov}(X,Y)}{\sigma_x \sigma_y}$ $\because r = 0$, $\text{cov}(X,Y) = 0$

But $\text{cov}(X, Y) = E(X, Y) - E(X) \cdot E(Y)$

$\therefore E(X, Y) - E(X)E(Y) = 0$

$\therefore E(X, Y) = E(X) \cdot E(Y)$ ..... (A)

Now, $E(X) = \Sigma p_i x_i = p \times 0 + q \times 1 = q$

$E(Y) = \Sigma p_i y_i = p' \times 0 + q' \times 1 = q'$

Now, $E(XY)\, \Sigma p_{ij} \cdot x_i \cdot y_j = p_{00}(0 \times 0) + p_{01}(0 \times 1) + p_{10}(1 \times 0) + p_{11}(1 \times 1) = p_{11}$

$\therefore$ By (A) $p_{11} = qq'$

$\therefore P(X = 1, Y = 1) = p(X = 1) \cdot p(Y = 1)$

Hence, the events $X = 1$ and $Y = 1$ are independent.

8. Raw and Central Moments

The following mathematical expectations have special significance in the study of probability distributions and hence, they are known by special names. They are denoted by special symbols.

(1) r-th moment about the origin (μ_r')

If we put $g(X) = X^r$, in § 3 (page 8-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r' = E(X^r) = \Sigma p_i x_i^r} \quad \text{or} \quad \boxed{\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x)\,dx}$$

This is called the r-th moment of the probability distribution of X about the origin, denoted by μ_r'.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0' = 1}$

because $\mu_0' = E(X^0) = \Sigma p_i (x_i)^0 = \Sigma p_i = 1$

or $\mu_0' = E(X^0) = \int_{-\infty}^{\infty} x^0 f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1' = \overline{X}}$

because $\mu_1' = E(X) = \Sigma p_i x_i = \overline{X}$

or $\mu_1' = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx = \overline{X}$

(2) r-th moment about the value a (μ_r')

If we put $g(X) = (X - a)^r$, in § 3 (page 8-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r' = E(X - a)^r = \Sigma p_i (x_i - a)^r}$$

or $$\boxed{\mu_r' = E(X - a)^r = \int_{-\infty}^{\infty} (x - a)^r f(x)\,dx}$$

This is called the r-th raw moment of the probability distribution of X about a denoted by μ_r'.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0' = 0}$

because $\mu_0' = E(X - a)^0 = \Sigma p_i (x_i - a)^0 = \Sigma p_i = 1.$

or $\mu_0' = E(X - a)^0 = \int_{-\infty}^{\infty} (x - a)^0 f(x)\,dx$

$= \int_{-\infty}^{\infty} f(x)\,dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1' = \overline{X} - a}$

because $\mu_1' = E(X - a) = \Sigma p_i (x_i - a) = \Sigma p_i x_i - a \Sigma p_i = \overline{X} - a$

or $\mu_1' = E(X - a) = \int_{-\infty}^{\infty} (x - a) f(x)\,dx$

$= \int_{-\infty}^{\infty} x f(x)\,dx - a \int_{-\infty}^{\infty} f(x)\,dx = \overline{X} - a$

(3) r-th moment about the mean (μ_r)

If we put $g(X) = (X - \overline{X})^r$, in § 3 (page 8-9), then the expectation is denoted by μ_r'. Thus,

$$\boxed{\mu_r = E[(X - \overline{X})^r] = \Sigma p_i (x_i - \overline{X})^r}$$

or $$\boxed{\mu_r = E(X - \overline{X})^r = \int_{-\infty}^{\infty} (x - \overline{X}) f(x)\,dx}$$

This is called the r-th moment of the probability distribution of X about the mean $\overline{X}$, denoted by μ_r.

Particular cases : (i) If $r = 0$, we get $\boxed{\mu_0 = 1}$

because $\mu_0 = \Sigma p_i (x_i - \overline{X})^0 = \Sigma p_i = 1$

or $\mu_0 = \int_{-\infty}^{\infty} (x - \overline{X})^0 f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

(ii) If $r = 1$, we get $\boxed{\mu_1 = 0}$

because $\mu_1 = \Sigma p_i (x_i - \overline{X}) = \Sigma p_i x_i - \overline{X} \Sigma p_i = \overline{X} - \overline{X} = 0.$

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or $\mu_1 = \int_{-\infty}^{\infty} (x - \overline{X}) f(x)\,dx$

$$= \int_{-\infty}^{\infty} x f(x)\,dx - \overline{X} \int_{-\infty}^{\infty} f(x)\,dx$$

$$= \overline{X} - \overline{X} = 0$$

9. Moment Generating Function

(a) Definition (Discrete Random Variable) : The moment generating function (m.g.f.) of a discrete random variate X about a denoted by $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \qquad \therefore \quad \boxed{M_a(t) = \Sigma p_i e^{t(x_i - a)}} \qquad \text{..... (1)}$$

The m.g.f. is a function of the real parameter t. The subscript a shows the point about which the m.g.f. is taken.

Expanding the exponential in (1), we get

$$M_a(t) = \Sigma p_i \left[1 + \frac{t}{1!}(x_i - a) + \frac{t^2}{2!}(x_i - a)^2 + \frac{t^3}{3!}(x_i - a)^3 + \ldots\ldots\right]$$

$$= \Sigma p_i + \frac{t}{1!}\Sigma p_i (x_i - a) + \frac{t^2}{2!}\Sigma p_i (x_i - a)^2 + \frac{t^3}{3!}\Sigma p_i (x_i - a)^3 + \ldots\ldots \qquad \text{..... (1a)}$$

But $\Sigma p_i (x_i - a)^r$ is the r-th moment of X about a *i.e.* μ'_r. Hence, we have

$$M_a(t) = 1 + \mu'_1 \cdot \frac{t}{1!} + \mu'_2 \cdot \frac{t^2}{2!} + \mu'_3 \cdot \frac{t^3}{3!} + \ldots\ldots + \mu'_r \cdot \frac{t^r}{r!} + \ldots\ldots \qquad \text{..... (2)}$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a *i.e.* μ'_r. In this way $M_a(t)$ generates moments. This is the reason why the function $M_a(t)$ is called the **moment generating function**.

Thus, $\boxed{\mu'_r = \text{coefficient of } \frac{t^r}{r!}}$

(b) Definition (Continuous Random Variable) : The moment generating function (m.g.f.) of a continuous random variate X about a denoted $M_a(t)$ is defined by

$$M_a(t) = E[e^{t(x-a)}] \qquad \therefore \quad \boxed{M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} \cdot f(x)\,dx} \qquad \text{..... (3)}$$

where $f(x)$ is the p.d.f. of X.

Expanding the exponential in (3), we get,

$$M_a(t) = \int_{-\infty}^{\infty} f(x) \cdot \left[1 + \frac{t}{1}(x - a) + \frac{t^2}{2!}(x - a)^2 + \frac{t^3}{3!}(x - a)^3 + \ldots\ldots\right] dx$$

$$M_a(t) = \int_{-\infty}^{\infty} f(x)\,dx + \frac{t}{1!}\int_{-\infty}^{\infty} (x - a) f(x)\,dx$$

$$+ \frac{t^2}{2!}\int_{-\infty}^{\infty} (x - a)^2 f(x)\,dx + \frac{t^3}{3!}\int_{-\infty}^{\infty} (x - a)^3 f(x)\,dx + \ldots\ldots \qquad \text{..... (3a)}$$

But $\int_{-\infty}^{\infty} (x - a)^r f(x)\,dx$ is the r-th moment μ'_r of X about a. Hence,

$$M_a(t) = 1 + \mu_1' \cdot \frac{t}{1!} + \mu_2' \cdot \frac{t^2}{2!} + \mu_3' \cdot \frac{t^3}{3!} + \ldots\ldots + \mu_r' \cdot \frac{t^r}{r!} + \ldots\ldots \qquad \text{..... (4)}$$

Thus, the coefficient of $(t^r / r!)$ is the r-th moment of X about a.

(c) To find moments of various orders from m.g.f. : It is clear from (2) and (4) that the moment of order r is the coefficient of $(t^r / r!)$ in the expansion of m.g.f. Hence, one way of obtaining various moments is to obtain expansion of the m.g.f. of X and find the coefficient of $(t^r / r!)$ in the expansion.

However, in practice many a time obtaining the expansion of m.g.f. is not convenient. In such cases we differentiate m.g.f. w.r.t. t for r times and equate it to zero to get μ_r.

Thus, differentiating $M_a(t)$ from (2) (or from 4), successively, we get,

$$\frac{d}{dt}[M_a(t)] = \mu_1' + \mu_2' t + \mu_3' \frac{t^2}{2!} + \ldots\ldots$$

Putting $t = 0$, $\frac{d}{dt}[M_a(t)]_{t=0} = \mu_1'$

$$\frac{d^2}{dt^2}[M_a(t)] = \mu_2' + \mu_3' t + \mu_4' \frac{t^2}{2!} + \ldots\ldots$$

Putting $t = 0$, $\frac{d^2}{dt^2}[M_a(t)]_{t=0} = \mu_2'$

In general, $\boxed{\frac{d^r}{dt^r}[M_a(t)]_{t=0} = \mu_r'}$

(d) Moment generating function about origin : Putting $a = 0$ in (1), we get,

$$\boxed{M_0 = \Sigma p_i e^{t x_i}} \qquad \text{..... (5)}$$

Putting $a = 0$ in (3), we get,

$$\boxed{M_0(t) = \int_{-\infty}^{\infty} e^{t x} f(x)\,dx} \qquad \text{..... (6)}$$

Putting $a = 0$ in (1a), we get since $\Sigma p_i x_i^r = \mu_r'$ about the origin.

$$M_0(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\ldots + \mu_r' \frac{t^2}{r!} + \ldots\ldots \qquad \therefore \quad M_0(t) = \Sigma \mu_r' \frac{t^r}{r!}$$

Note ....

It is assumed that the r.h.s. of (5) and (6) is absolutely convergent.

(e) If $L(t) = \log M(t)$ where $M(t)$ is the moment generating function of a random variable, Prove that the mean = $L'(0)$ and variance = $L''(0)$. **(M.U. 2006)**

Proof : We have $M(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2} + \ldots\ldots$

$$\therefore \quad L(t) = \log M(t) = \log\left[1 + \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\ldots\right)\right]$$

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$$= \left(\mu_1' t + \mu_2' \frac{t^2}{2} + \ldots\right) - \frac{1}{2}\left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right)^2 + \ldots$$

$$= \left(\mu_1' t + \mu_2' \frac{t^2}{2!} + \ldots\right) - \frac{1}{2}\left(\mu_1'^2 t^2 + \mu_1'\mu_2' t^3 + \ldots\right) + \ldots$$

$$\therefore \quad L'(t) = \mu_1' + \mu_2' t - \mu_1'^2 \cdot t - \frac{3}{2}\mu_1'\mu_2' t^2 + \text{terms in higher powers of } t \qquad \ldots\ldots\ldots\ldots\ldots\ldots \text{(A)}$$

Putting $t = 0$, $\therefore$ $L'(0) = \mu_1'$

Now, $L''(t) = \mu_2' - \mu_1'^2 - 3\mu_1'\mu_1'^2 t$ + terms in higher powers of t

Putting $t = 0$, $L''(0) = \mu_2' - \mu_1'^2$ $\qquad \ldots\ldots\ldots\ldots\ldots\ldots$ (B)

Hence, Mean, $\mu = L'(0)$, Variance, $\mu_2 = L''(0)$.

(f) Moment generating function of the sum of two independent random variates : ***"The moment generating function of the sum of two independent random variates is equal to the product of the m.g.f.s of the two variates."***

Proof : Let X, Y be two independent random variates then the m.g.f. of their sum $X + Y$ about the origin is given by

$$M_{(X+Y)}(t) = E[e^{t(X+Y)}] = E(e^{tX} \cdot e^{tY})$$

$$= E(e^{tX}) \cdot E(e^{tY}) \qquad [\because X \text{ and } Y \text{ are independent}]$$

$$\therefore \quad \boxed{M_{X+Y}(t) = M_X(t) \cdot M_Y(t)}$$

Generalisation : If $X_1, X_2, \ldots X_n$ are n independent random variates, then the m.g.f. of their sum is equal to the product of their m.g.f.s

$$M_{X_1+X_2+\ldots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \ldots M_{X_n}(t)$$

(g) Uniqueness of Moment Generating Function : This is a very important property of m.g.f. It states that the m.g.f. of a distribution, if it exists, uniquely determines the distribution. In other words it means that for a given probability distribution there is one and only one m.g.f. and corresponding a given m.g.f. there is one and only one probability distribution. Thus, if m.g.f. of X and m.g.f. of Y are equal then X and Y must be identical.

Example 1 : Find the M.G.F. of the following distribution

X	:	-2	3	1
$P(X = x)$	:	1/3	1/2	1/6

Hence, find first four central moments. **(M.U. 2010, 15)**

Sol. : Since we want central moments we shall find m.g.f. about the mean.

Now, $\bar{X} = \Sigma p_i x_i = -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{-4+9+1}{6} = \frac{6}{6} = 1$

$\therefore$ M.G.F. about the mean

$$M_{\bar{x}}(t) = \Sigma p_i e^{t(x_i - \bar{x})} = \frac{1}{3} \cdot e^{t(-2-1)} + \frac{1}{2} e^{t(3-1)} + \frac{1}{6} e^{t(1-1)}$$

$$= \frac{1}{3} e^{-3t} + \frac{1}{2} e^{2t} + \frac{1}{6}$$

$$M_{\bar{x}}(t) = \frac{1}{3}\left[1 - 3t + \frac{9t^2}{2!} - \frac{27t^3}{3!} + \frac{81t^4}{4!} - \ldots\right] + \frac{1}{2}\left[1 + 2t + \frac{4t^2}{2!} + \frac{8t^3}{3!} + \frac{16t^4}{4!} + \ldots\right] + \frac{1}{6}$$

Now, μ_r = Coeff. of $\frac{t^r}{r!}$

$\therefore$ μ_0 = Coeff. of t^0 = constant term $= \frac{1}{3} + \frac{1}{2} + \frac{1}{6} = 1$

$\therefore$ μ_1 = Coeff. of $\frac{t}{1!} = \frac{1}{3}(-3) + \frac{1}{2}(2) = 0$

μ_2 = Coeff. of $\frac{t^2}{2!} = \frac{9}{3} + \frac{4}{2} = 5$

μ_3 = Coeff. of $\frac{t^3}{3!} = \frac{1}{3}(-27) + \frac{1}{2}(8) = -5$

μ_4 = Coeff. of $\frac{t^4}{4!} = \frac{1}{3}(81) + \frac{1}{2}(16) = 35.$

Example 2 : Find the m.g.f. of the random variable X about the origin whose p.m.f. is given above in Ex. 1. Also find the first two moments about the ogirion. **(M.U. 2004)**

Sol. : By definition

$$\mu_0(t) = E(e^{tx}) = \Sigma p_i e^{tx_i} = \frac{1}{3} \cdot e^{-2t} + \frac{1}{2} e^{3t} + \frac{1}{6} e^t$$

Now, $\mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[-\frac{2}{3} e^{-2t} + \frac{3}{2} e^{3t} + \frac{1}{6} e^t\right]_{t=0}$

$$= -\frac{2}{3} + \frac{3}{2} + \frac{1}{6} = \frac{6}{6} = 1$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \left[\frac{4}{3} e^{-2t} + \frac{9}{2} e^{3t} + \frac{1}{6} e^t\right]_{t=0}$$

$$= \frac{4}{3} + \frac{9}{2} + \frac{1}{6} = \frac{36}{6} = 6.$$

Example 3 : A random variable X has probability density function $1/(2^x)$, $x = 1, 2, 3 \ldots$ Find the m.g.f. and hence, find the mean and variance.

Sol. : Since, $P(X = x) = \frac{1}{2^x}$, $x = 1, 2, 3 \ldots$

$$M_0(t) = E(e^{tx}) = \Sigma p e^{tx}$$

$$= \Sigma \frac{1}{2^x} e^{tx} = \sum_{x=1} \left(\frac{e^t}{2}\right)^x = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots$$

$$= \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\right] = \frac{e^t}{2}\left[1 - \frac{e^t}{2}\right]^{-1}$$

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$$\therefore \quad M_0(t) = \frac{e^t}{2} \cdot \frac{1}{1-(e^t/2)} = \frac{e^t}{2} \cdot \frac{2}{2-e^t} = \frac{e^t}{2-e^t}$$

$$\therefore \quad \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[\frac{(2-e^t)e^t - e^t(-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= 2\left[\frac{e^t}{(2-e^t)^2}\right]_{t=0} = 2$$

$$\therefore \quad \mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = 2 \cdot \left[\frac{(2-e^t)^2 \cdot e^t - e^t \cdot 2(2-e^t) \cdot (-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= 2 \cdot \left[\frac{(2-e^t) \cdot e^t + 2e^{2t}}{(2-e^t)^3}\right]_{t=0} = \frac{2(1+2)}{1} = 6$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = 6 - 4 = 2.$$

$\therefore$ Mean = Variance = 2.

Example 4 : If X denotes the outcome when a fair die is tossed, find M.G.F. of X and hence, find the mean and variance of X. **(M.U. 2005, 09, 18, 19)**

Sol. : We have, here X taking values 1, 2, 3, 4, 5, 6 each with probability $1/6$.

$$M_0(t) = E(e^{tx_i}) = \Sigma p_i e^{tx_i}$$

$$= \frac{1}{6}e^t + \frac{1}{6}e^{2t} + \frac{1}{6}e^{3t} + \ldots + \frac{1}{6}e^{6t}$$

$$= \frac{1}{6}(e^t + e^{2t} + \ldots + e^{6t})$$

$$\therefore \quad \mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 2e^{2t} + \ldots + 6e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1 + 2 + \ldots + 6] = \frac{21}{6} = \frac{7}{2}$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \frac{1}{6}\left[e^t + 4e^{2t} + 9e^{3t} + \ldots + 36e^{6t}\right]_{t=0}$$

$$= \frac{1}{6}[1 + 4 + 9 + \ldots + 36] = \frac{91}{6}$$

$$\therefore \quad \mu_2 = \mu_2' - \mu_1'^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}.$$

$\therefore$ Mean $= \frac{7}{2}$ and Variance $= \frac{35}{12}$.

Example 5 : Find the m.g.f. of a random variable X if the r-th moment about the origin is given by $\mu_r' = r!$.

Sol. : Be definition, $\mu_r' = E(x^r) = r!$

$$\therefore \quad E(x) = 1, \quad E(x^2) = 2! \qquad E(x^3) = 3! \ldots \qquad \ldots\ldots\ldots\ldots (1)$$

Now, $M_0(t) = E(e^{tx})$

$$= E\left[1 + tx + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \ldots\right]$$

$$= E(1) + tE(x) + \frac{t^2}{2!}E(x^2) + \frac{t^3}{3!}E(x^3) + \ldots$$

Putting the values of $E(X)$, $E(X)^2$, ... from (1)

$$M_0(t) = 1 + t + \frac{t^2}{2!} \cdot 2! + \frac{t^3}{3!} \cdot 3! + \ldots$$

$$= 1 + t + t^2 + t^3 + \ldots = (1-t)^{-1} = \frac{1}{1-t}.$$

Example 6 : Find the m.g.f. of a random variable whose p.m.f. is

$$P(X = x) = \left(\frac{1}{2}\right)^x, \quad x = 1, 2, 3 \ldots$$
$$= 0 \qquad \text{else where}$$

Hence, find the mean and variance of X.

Sol. : By definition

$$M_0(t) = E(e^{tx}) = \Sigma p_i e^{tx_i} = \Sigma \frac{1}{2^x} e^{tx} = \Sigma \left(\frac{e^t}{2}\right)^x$$

$$M_0(t) = \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \ldots = \frac{e^t}{2}\left[1 + \left(\frac{e^t}{2}\right) + \left(\frac{e^t}{2}\right)^2 + \ldots\right]$$

$$= \frac{e^t}{2} \frac{1}{1-(e^t/2)} = \frac{e^t}{2-e^t}.$$

To find the mean and variance

$$\mu_1' = \left[\frac{d}{dt} M_0(t)\right]_{t=0} = \left[\frac{(2-e^t) \cdot e^t - e^t \cdot (-e^t)}{(2-e^t)^2}\right]_{t=0}$$

$$= \left[\frac{2e^t}{(2-e^t)^2}\right]_{t=0} = 2$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_0(t)\right]_{t=0} = \left[\frac{(2-e^t)^2 \cdot e^t - e^t \cdot 2(2-e^t)(-e^t)}{(2-e^t)^4}\right]_{t=0}$$

$$= \frac{2[1+2]}{1} = 6$$

$\therefore$ Variance $= \mu_2' - \mu_1'^2 = 6 - 4 = 2.$

Remark ....

Mean and variance of the above distribution can be obtained using $E(X)$ and $E(X)^2$. Find them.

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Example 7 : A random variable X has the following probability density function

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0. & \text{otherwise} \end{cases}$$

Find m.g.f., μ_r', mean, and variance.

Sol. : We have

$$M_0(t) = E(e^{tx}) = \int_0^1 e^{tx}(1)\,dx = \left[\frac{e^{tx}}{t}\right]_0^1 = \frac{1}{t}\left[e^t - 1\right]$$

$$= \frac{1}{t}\left[1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots - 1\right] = \frac{1}{t}\left[t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots\right]$$

$$= 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots\ldots + \frac{t^r}{(r+1)!}$$

$$\therefore \quad \mu_r' = \text{Coefficient of } \frac{t^r}{r!} = \frac{1}{r+1}, \quad r = 1, 2, \ldots\ldots$$

Putting $r = 1, 2$

$$\therefore \quad \text{Mean} = \mu_1' = \frac{1}{2}\,;\ \mu_2' = \frac{1}{3} \qquad \therefore \quad \text{Var. } (X) = \mu_2' - \mu_1'^2 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Example 8 : A random variable X has the following probability density function

$$f(x) = \begin{cases} ke^{-kx}, & x > 0,\ k > 0 \\ 0, & \text{elsewhere} \end{cases}$$

Find the m.g.f. and hence, the mean and variance.

Sol. : We have

$$M_0(t) = E(e^{tx}) = \int_0^\infty e^{tx} \cdot k e^{-kx}\,dx = k\int_0^\infty e^{(t-k)x}dx$$

$$= \frac{k}{t-k}\left[e^{(t-k)x}\right]_0^\infty = \frac{k}{t-k}[0-1] = \frac{k}{k-t} \qquad [t \neq k]$$

$$\text{Now, } \quad M_0(t) = \frac{k}{k[1-(t/k)]} = \left[1 - \frac{t}{k}\right]^{-1} = 1 + \frac{t}{k} + \frac{t^2}{k^2} + \frac{t^3}{k^3} + \ldots\ldots$$

$$\therefore \quad \mu_1' = \text{Coefficient of } t = \frac{1}{k}; \qquad \mu_2' = \text{Coefficient of } \frac{t^2}{2!} = \frac{2}{k^2}$$

$$\therefore \quad \text{Mean} = \mu_1' = \frac{1}{k}; \quad \text{Var. } (X) = \mu_2' - \mu_1'^2 = \frac{2}{k^2} - \frac{1}{k^2} = \frac{1}{k^2}.$$

Example 9 : If a random variable has the moment generating function $M_t = \dfrac{3}{3-t}$, obtain the mean and the standard deviation. **(M.U. 2010)**

Sol. : We have

$$M_0(t) = \frac{3}{3-t} = \frac{3}{3[1-(t/3)]} = \left(1 - \frac{t}{3}\right)^{-1} = 1 + \frac{t}{3} + \frac{t^2}{9} + \frac{t^3}{27} + \ldots\ldots\infty.$$

$$\text{Mean} = E(X) = \text{Coefficient of } \frac{t}{1!} = \frac{1}{3}$$

$$\mu_2' = E(X^2) = \text{Coefficient of } \frac{t^2}{2!} = \frac{2}{9}.$$

$$\therefore \quad \text{Var}(X) = \mu_2' - \mu_1^2 = \frac{2}{9} - \frac{1}{9} = \frac{1}{9} \qquad \therefore \quad \text{S.D.} = \frac{1}{3}.$$

EXERCISE - IV

1. A random variable takes values $X = 0, 1$ with probabilities q and p respectively, such that $q + p = 1$. Find the moment generating function of X and show that all moments about the origin are equal to p.

   [Ans. : $M_0(t) = qe^0 + pe^t = 1 + (e^t - 1)p = 1 + \left(t + \dfrac{t^2}{2!} + \dfrac{t^3}{3!} + \ldots\ldots\right) \quad \therefore \ \mu_r' = p$ **]**

2. A random variable X-has the m.g.f. given by $M_0(t) = \dfrac{2}{2-t}$. Find the standard deviation of X.

   [Ans. : $\sigma = 1/2$ **]**

3. A random variable X has the probability distribution

   $$P(X = x) = \frac{1}{8}\,{}^3C_x, \quad x = 0, 1, 2, 3.$$

   Find the moment generating function of X and then find mean and variance. **(M.U. 2003)**

   [Ans. : (i) $\dfrac{1}{8}(1 + e^t)$, (ii) $\dfrac{3}{2}, \dfrac{3}{4}$. **]**

4. A random variable X has the following probability distribution.

X	:	0	1	2	3
$P(X = x)$	:	1/6	1/3	1/3	1/6

   Compute (i) Moment generating function about the origin, (ii) first two raw moments.

   [Ans. : (i) $\dfrac{1}{6} + \dfrac{1}{3}e^t + \dfrac{1}{3}e^{2t} + \dfrac{1}{6}e^{3t}$, (ii) $\dfrac{3}{2}, \dfrac{19}{6}$. **]**

5. A random variable X has the following probability distribution.

X	:	0	1	2
$P(X = x)$	:	1/3	1/3	1/3

   Find (i) the moment generating function, (ii) find mean and variance.

   [Ans. : (i) $\dfrac{1}{3}(1 + e^t + e^{2t})$, (ii) $1, \dfrac{2}{3}$. **]**

6. Find the m.g.f. of the random variable having the following probability density function. Also find the mean and variance.

   (i) $f(x) = \begin{cases} 1/2, & -1 \le x < 1 \\ 0, & \text{elsewhere} \end{cases}$ (ii) $f(x) = \begin{cases} e^{-(x-5)}, & x \ge 5 \\ 0, & \text{elsewhere} \end{cases}$

   $\left[\textbf{Ans. : (i)} \ \dfrac{(e^t - e^{-t})}{2t}, 0, \dfrac{1}{3}\ ;\ \textbf{(ii)} \ \dfrac{e^{5t}}{1-t}, 6, 1\right]$

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7. A random variable X has the probability density function

$$f(x) = \begin{cases} 1/3, & -1 < x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

Find the m.g.f. of X. $\left[\text{Ans. : } M_0(t) = \dfrac{e^{2t} - e^{-t}}{3t},\ t \neq 0\right]$

8. A random variable X has the following density function

$$f(x) = \begin{cases} 2e^{-2x}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Find the m.g.f. and hence, its mean and variance.

$\left[\text{Ans. : } M_0(t) = \dfrac{2}{2-t}\ ;\ t \neq 2,\ \ \mu_1 = \dfrac{1}{2},\ \ \mu_2 = \dfrac{1}{4}\right]$

9. Find the m.g.f. of the random variable having probability density function

$$f(x) = \begin{cases} x, & 0 \leq x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & \text{elsewhere} \end{cases}$$

$\left[\text{Ans. : } M_0(t) = \left(\dfrac{e^t - 1}{t}\right)^2 ;\ t \neq 0\right]$

EXERCISE - V

Theory

1. Define mathematical expectation.
2. Define expectation of a function of a random variable X.
   (a) when X is a discrete r.v., (b) when X is a continuous r.v.
3. Explain the following terms :
   (i) Expectation of a random variable.
   (ii) Variance of a random variable. **(M.U. 1998)**
4. Explain the following terms
   (i) Moments about origin. (ii) Moment Generating Function. **(M.U. 2002)**
5. Prove that $V(X) = E(X^2) - \{E(X)\}^2$.
6. If X is a random variable and $Y = aX + b$ show that
   (i) $\mu_Y = a\mu_X + b$, (ii) $\sigma_Y^2 = a^2\sigma_X^2$
   where μ denotes the mean and σ denotes the standard deviation.
7. Prove that $V(aX + b) = a^2 V(x)$.
8. Define Moment Generating Function of a random variable X about a and explain how you will get moments from it.
9. Derive the formulae for r th moment about any point and r th moment about the mean. **(M.U. 1999)**
10. If $L(t) = \log M(t)$ where $M(t)$ is the m.g.f. of a discrete random variable, then prove that mean $= L'(0)$ and variance $= L''(0)$ **(M.U. 2006)**

SELF LEARNING TOPICS

Module 1

LAPLACE TRANSFORM OF PERIODIC FUNCTIONS, HEAVISIDE'S UNIT STEP FUNCTION, DIRAC DELTA FUNCTION, SPECIAL FUNCTIONS (ERROR AND BESSEL)

1. Laplace Transform of Periodic Functions

If $f(t)$ is a periodic function of period a, show that

$$\boxed{L f(t) = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t)\, dt} \quad \text{(M.U. 1995, 96, 2006, 07)} \qquad \text{..... (1)}$$

Proof. : Since $f(t)$ is periodic with period a, $f(t) = f(t + a) = f(t + 2a) = ...$

$$\therefore\ L f(t) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^a e^{-st} f(t)\, dt + \int_a^{2a} e^{-st} f(t)\, dt + ...\infty$$

Now $\displaystyle\int_a^{2a} e^{-st} f(t)\, dt = \int_0^a e^{-s(u+a)} f(u + a)\, du$ [where $t = u + a$]

$$= e^{-as} \int_0^a e^{-su} f(u + a)\, du$$

$$= e^{-as} \int_0^a e^{-st} f(t + a)\, dt \qquad \text{(Changing } u \text{ to } t\text{)}$$

$$= e^{-as} \int_0^a e^{-st} f(t)\, dt \qquad [f(t + a) = f(t)]$$

Similarly, we can show that the other integrals are also equal to $e^{-2as}\int_0^a e^{-st} f(t)\, dt$ and so on.

$$\therefore\ L f(t) = \left(1 + e^{-as} + e^{-2as} + ... \infty\right)\int_0^a e^{-st} f(t)\, dt.$$

$$\therefore\ \boxed{L f(t) = \frac{1}{1 - e^{-as}} \int_0^a e^{-st} f(t)\, dt} \qquad \left[\text{For a G.P. } s_\infty = \frac{a}{1-r}\right] \qquad \text{..... (2)}$$

Example 1 : Find Laplace transform of

(i) $f(t) = K\dfrac{t}{T}$ for $0 < t < T$ and $f(t) = f(t + T)$.

(ii) $f(t) = 1$, for $0 \leq t < a$ and $f(t) = -1$, $a < t < 2a$ and $f(t)$ is periodic with period $2a$. **(M.U. 1999, 2003, 06, 11)**

(iii) $f(t) = e^t$ for $0 < t < \pi$ and $f(t) = f(t + 2\pi)$

Sol. : (i) Since $f(t)$ is periodic with period T

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$$L f(t) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} f(t)\,dt = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}\cdot\frac{Kt}{T}\,dt$$

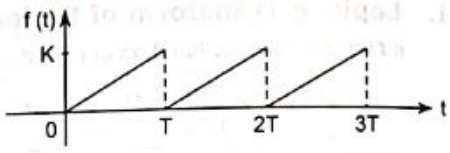
$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\int_a^T e^{-st}\cdot t\cdot dt$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[-t\cdot\frac{e^{-st}}{s}-\frac{e^{-st}}{s^2}\right]_0^T$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[-\frac{Te^{-sT}}{s}-\frac{e^{-st}}{s^2}+\frac{1}{s^2}\right]$$

$$= \frac{1}{1-e^{-sT}}\cdot\frac{K}{T}\left[\frac{1}{s^2}\left(1-e^{-sT}\right)-\frac{Te^{-sT}}{s}\right]$$

$$= K\left[\frac{1}{Ts^2}-\frac{e^{-sT}}{s(1-e^{-st})}\right]$$



The function $f(t) = k\cdot\frac{t}{T}$ is known as "saw-tooth wave" function and its graph is as shown in adjoing figure.

(ii) Since $f(t)$ is periodic with period $2a$,

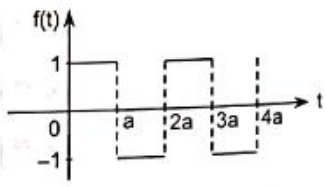
$$L f(t) = \frac{1}{1-e^{-2as}}\int_0^{2a} e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2as}}\left[\int_0^a e^{-st}(1)\,dt + \int_a^{2a} e^{-st}(-1)\,dt\right]$$

$$= \frac{1}{1-e^{-2as}}\left[\left\{-\frac{e^{-st}}{s}\right\}_0^a + \left\{\frac{e^{-st}}{s}\right\}_a^{2a}\right]$$

$$= \frac{1}{s}\cdot\frac{1}{1-e^{-2as}}\cdot(1-e^{-as})^2 = \frac{1}{s}\cdot\frac{1-e^{-as}}{1+e^{-as}}$$

$$= \frac{1}{s}\cdot\left[\frac{e^{as/2}-e^{-as/2}}{e^{as/2}+{}^{-as/2}}\right] = \frac{1}{s}\tanh\left\{\frac{as}{2}\right\}.$$



The function is known as "square wave" function and its graph is shown in figure.

(iii) Since $f(t)$ is periodic with period 2π.

$$L f(t) = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}\cdot f(t)\,dt = \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{-st}\cdot e^t\,dt$$

$$= \frac{1}{1-e^{-2\pi s}}\int_0^{2\pi} e^{(1-s)t}\,dt = \frac{1}{1-e^{-2\pi s}}\left[\frac{e^{(1-s)t}}{1-s}\right]_0^{2\pi}$$

$$= \frac{1}{(1-e^{-2\pi s})(1-s)}\left[e^{(1-s)2\pi}-1\right]$$

Example 2 : Find Laplace transform of $f(t) = a\sin pt,\ 0 < t < \pi/p,\ f(t) = 0,\ \pi/p < t < 2\pi/p$ and $f(t) = f(t + 2\pi/p)$. **(M.U. 1995, 2002, 08, 13)**

Sol. : $$L f(t) = \frac{1}{1-e^{-(2\pi/p)s}}\int_0^{\pi/p} e^{-st} a\sin pt\,dt$$

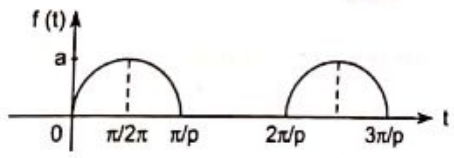
$$= \frac{a}{1-e^{-(2\pi/p)s}}\left[\frac{1}{s^2+p^2}\cdot e^{-st}(-s\sin pt - p\cos pt)\right]_0^{\pi/p}$$

$$= \frac{a}{1-e^{-(2\pi/p)s}}\cdot\frac{1}{(s^2+p^2)}\cdot\left[e^{-s\pi/p}(-s\sin\pi - p\cos\pi) - e^0(-s\sin 0 - p\cos 0)\right]$$

$$= \frac{a}{1-e^{-(2\pi/p)s}}\cdot\frac{1}{s^2+p^2}\left[p\,e^{-s\pi/p}+p\right]$$

$$= \frac{ap}{1-e^{-(s\pi/p)}}\cdot\frac{1}{s^2+p^2}$$

The function $f(t) = a\sin pt$ is known as "half-sine wave rectifier" function and its graph is shown below.



Example 3 : Find the Laplace transform of $f(t) = |\sin pt|,\ t \ge 0$. **(M.U. 2003)**

Sol. : We first note that

$$f\left(t+\frac{\pi}{p}\right) = \left|\sin p\left(t+\frac{\pi}{p}\right)\right| = |\sin(pt+\pi)| = |\sin pt|$$

$\therefore$ $f(t)$ is a periodic function with period π/p.

Now, $$L f(t) = \frac{1}{1-e^{-\pi s/p}}\int_0^{\pi/p} e^{-st}|\sin pt|\,dt$$

$$\therefore\quad L f(t) = \frac{1}{1-e^{-\pi s/p}}\int_0^{\infty} e^{-st}\sin pt\,dt \qquad \left[\because \sin pt > 0,\ \text{for } 0 \le t \le \frac{\pi}{p}\right]$$

$$= \frac{1}{1-e^{-\pi s/p}}\left[\frac{e^{-st}}{s^2+p^2}(-s\sin pt - p\cos pt)\right]_0^{\pi/p}$$

$$\therefore\quad L[f(t)] = \frac{1}{1-e^{-\pi s/p}}\cdot\frac{1}{(s^2+p^2)}\left[e^{-s\pi/p}(0+p)-(0-p)\right]$$

$$= \frac{1}{s^2+p^2}\cdot\frac{1}{1-e^{-\pi s/p}}\cdot p\cdot(1+e^{-\pi s/p})$$

$$= \frac{p}{s^2+p^2}\cdot\left(\frac{e^{\pi s/2p}+e^{-\pi s/2p}}{e^{\pi s/2p}-e^{-\pi s/2p}}\right)$$

$$= \frac{p}{s^2+p^2}\cdot\coth\left(\frac{\pi s}{2p}\right)$$

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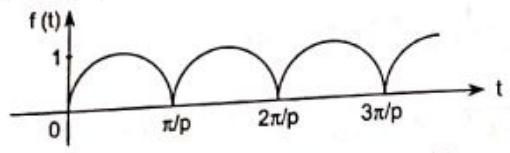
The function $f(t) = |\sin pt|$ is known as "full-sine wave rectifier" and its graph is shown below.



Example 4 : Find Laplace transform of $f(t) = \sin 2t,\ 0 < t < \pi/2,$ $f(t) = 0,\ \pi/2 < t < \pi$ and $f(t) = f(t+\pi)$. **(M.U. 1995, 2002, 05, 12, 16)**

Sol. : In the above Ex. 2, put $p = 2,\ a = 1$.

$$\therefore \quad L f(t) = \frac{1}{1-e^{-\pi/s}} \cdot \frac{1}{s^2+2^2} \cdot \left[2e^{-\pi s/2} + 2\right]$$

$$= \frac{2}{1-e^{-\pi s/2}} \cdot \frac{1}{s^2+2^2}.$$

Example 5 : Find $L f(t)$ where, $f(t) = t,\ 0 < t < 1$; $f(t) = 0,\ 1 < t < 2$ and $f(t+2) = f(t)$ for $t > 0$. **(M.U. 1996, 2018, 19)**

Sol. : Since $f(t)$ is periodic with period $a = 2$, we have

$$L f(t) = \frac{1}{1-e^{-as}} \int_0^a e^{-st} f(t)\,dt = \frac{1}{1-e^{-2s}} \int_0^a e^{-st} f(t)\,dt$$

$$= \frac{1}{1-e^{-2s}} \left[\int_0^1 e^{-st} \cdot t\,dt + \int_1^2 e^{-st} \cdot 0 \cdot dt\right]$$

$$= \frac{1}{1-e^{-2s}} \left[t\left(-\frac{e^{-st}}{s}\right) - \left(\frac{e^{-st}}{s^2}\right)(1)\right]_0^1$$

$$= \frac{1}{1-e^{-2s}} \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2}\right]$$

$$\therefore \quad L f(t) = \frac{1}{s^2(1-e^{-2s})}(1 - e^{-s} - s e^{-s}).$$

EXERCISE - I

Find the Laplace transform of

1. $f(t) = \frac{t}{a},\ 0 < t \le a$; $f(t) = \frac{1}{a}(2a - t),\ a < t < 2a$ and $f(t) = f(t+2a)$. **(M.U. 2002, 04, 11)**
2. $f(t) = t,\ 0 < t < \pi$; $f(t) = \pi - t,\ \pi < t < 2\pi$ and $f(t) = f(t+2\pi)$.
3. $f(t) = t^2,\ 0 < t < 2$, where $f(t)$ is a periodic function with period 2.
4. $f(t) = t,\ 0 < t < 1$ and $f(t)$ is of period 1.
5. $f(t) = \begin{cases} E, & 0 \le t \le (p/2) \\ -E, & (p/2) \le t \le p \end{cases}$, $f(t+p) = f(t)$. **(M.U. 2002, 03, 14)**

[Ans. : (1) $\frac{1}{as^2} \tanh\left(\frac{as}{2}\right)$. (2) $\frac{1-(1+\pi s)e^{-\pi s}}{s^2(1+e^{-\pi s})}$,

(3) $\frac{2}{1-e^{-2s}}\left[\frac{1}{s^3} - \frac{1}{s^3} \cdot e^{-2s} - \frac{2}{s} \cdot e^{-2s} - \frac{2}{s^2} \cdot e^{-2s}\right]$,

(4) $\frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$, (5) $\frac{E}{s} \tanh\left(\frac{sp}{4}\right)$.]

2. Laplace Transforms of Two Special Functions

We shall now obtain Laplace transforms of some special functions which we come across in discussions of engineering problems. These are

(i) Heaviside's Unit Step Function, (ii) Dirac-Delta Function

3. Heaviside's Unit Step Function

The function takes only two values 0 and 1. When x is negative the value of the function is zero and when x is positive its value is 1. It is denoted by $H(t)$ [or $U(t)$], H for Heaviside (U for unit). Thus, the value of $H(t)$ is one to the right of the origin and is zero to the left of the origin. Obviously, it is a discontinuous function.

We define it as $H(t) = \begin{cases} 0, & t < 1 \\ 1, & t \ge 1 \end{cases}$

The function takes a jump of unit magnitude and remains there thereafter. The graph of the function is shown in the Fig. A.

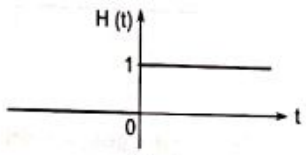


Fig. A

Displaced Unit Step Function

If the origin is shifted to a point $t = a$ *i.e.* if the function is zero before $t = a$ and takes a jump of unit magnitude at $t = a$ and remains there thereafter, the function is called **displaced unit step function.** It is defined by

$$H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \ge a \end{cases}$$

The graph of the function is shown in the Fig. B.

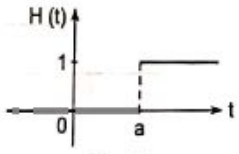
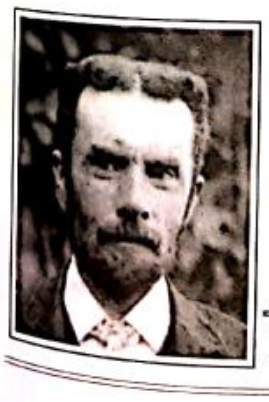


Fig. B

Oliver Heaviside (1850 - 1925)



Although he was a bright student while in school, his parents could not keep him at school after he was 16, so he continued studying by himself and had no further formal education. He adapted complex numbers to the study of electrical circuits, invented mathematical techniques for the solution of differential equations, reformulated Maxwell's field equations and independently coformulated vector analysis. He was, thus, an electrical engineer, mathematician and physicist. He invented the Heaviside step function and employed it to model the current in an electric circuit. He developed the transmission line theory and advanced the idea that the Earth's uppermost atmosphere contained an ionised layer known as ionosphere.

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Functions Represented by Unit Step Functions

In many engineering discussions, it is convenient to represent a function with the help of unit step function. We shall denote the given function by $f(t)$ and consider the following five cases.

Case I : $f(t)\,H(t)$: Since $H(t)$ is zero for $t < 0$ and $H(t)$ is unity for $t \geq 0$, it is easy to see that the product $f(t)\,H(t)$ will be zero for $t < 0$ and $f(t) \cdot H(t)$ will remain $f(t)$ for $t \geq 0$. This means by taking the product $f(t) \cdot H(t)$ **the part of $f(t)$ to the left of the origin is cut off.** Thus

$$f(t) \cdot H(t) = \begin{cases} 0, & t < 0 \\ f(t), & t \geq 0 \end{cases}$$

Suppose $f(t) = t^2$, a parabola then $f(t)\,H(t)$ will give only the branch of the parabola on the right. In the following diagram we shall show both the curves $f(t) = t^2$ and $f(t) \cdot H(t)$.

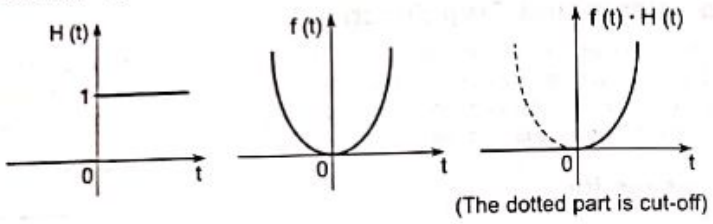

Fig. C

If $f(t)$ is some **general function**, we get the following diagrams.

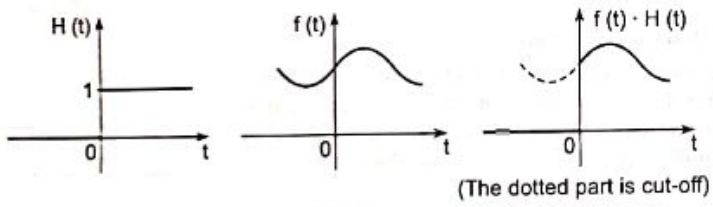

Fig. D

Case II : $f(t)\,H(t-a)$: Since $H(t-a)$ is zero for $t < a$ and $H(t-a)$ is unity for $t \geq a$, it is easy to see that the product $f(t)\,H(t-a)$ will be zero for $t < a$ and $f(t) \cdot H(t-a)$ will remain $f(t)$ for $t \geq a$. This means by taking the product $f(t) \cdot H(t-a)$, **the part of $f(t)$ to the left of $t = a$ is cut-off.** Thus,

$$f(t)H(t-a) = \begin{cases} 0, & t < a \\ f(t), & t \leq a \end{cases}$$

Suppose as before $f(t) = t^2$ is a parabola and $a = 2$ then $f(t)\,H(t-2)$ will give the branch of the parabola to the right of $t = 2$. This is shown in the following diagram.

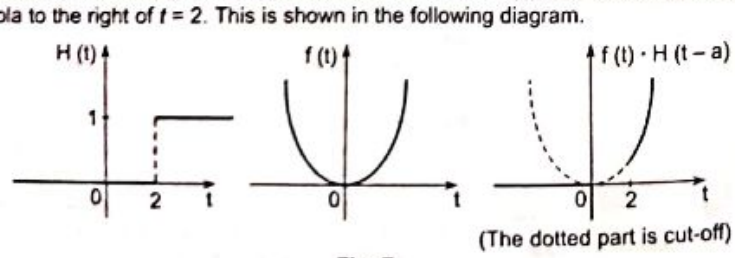

Fig. E

If $f(t)$ is a some **general function** then we get the following diagrams.

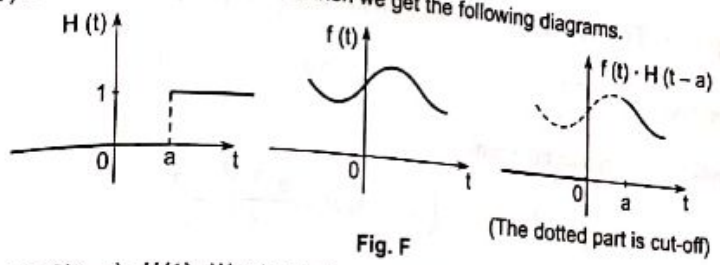

Fig. F

Case III : $f(t-a) \cdot H(t)$: We already know that the curve $y = f(x-a)$ is the same as the curve $y = f(x)$ with the only difference that the origin is shifted to a. The shape of the curve remains unchanged. Combining this with case I, it is easy to see that $f(t-a)\,H(t)$ will represent the part of the curve $f(t-a)$ to the right of the origin.

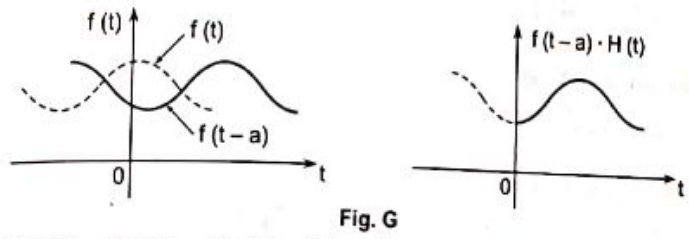

Fig. G

Case IV : $f(t-a)\,H(t-b)$: Since $f(t-a)$ is a curve $f(t)$ with the origin shifted to a and $H(t-b)$ is zero before $t = b$ and unity after $t = b$, it is easy to see that $f(t-a)\,H(t-b)$ will give the part of the shifted curve $f(t-a)$ to the right of $t = b$ cutting off the part before $t = b$.

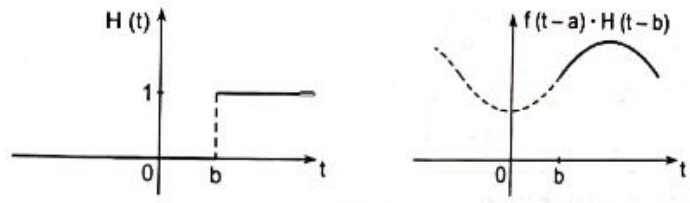

Fig. H

Case V : Representation of the part of the curve $f(t)$ from $t = a$ to $t = b$

We first see that $H(t-a)$ is a unit function to the right of $t = a$ and $H(t-b)$ is a unit function to the right of $t = b$. Hence $H(t-a) - H(t-b)$ is the part of the unit function between $t = a$ and $t = b$. Hence, $f(t)\,[H(t-a) - H(t-b)]$ will be zero before $t = a$, will be $f(t)$ between $t = a$ and $t = b$, and zero after $t = b$. Since the part of the function $f(t)$ is cut-off before $x = a$ and also that part of $f(t)$ is cut-off after $t = b$, only retaining the part between $t = a$ and $t = b$ thus, function $f(t)\,[H(t-a) - H(t-b)]$ is called **filter function.**

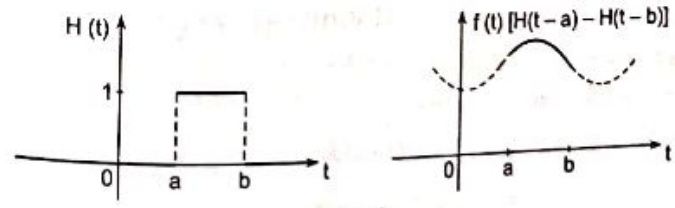

Fig. I

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4. Laplace Transform of Heaviside's Unit Step Function $H(t)$

Since, $H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$

By definition of Laplace transform

$$L[H(t)] = \int_0^\infty e^{-st} H(t)\,dt = \int_0^\infty e^{-st}(1)\,dt = \left[\frac{e^{-st}}{-s}\right]_0^\infty = \frac{1}{s}$$

$$\therefore \quad \boxed{L[H(t)] = \frac{1}{s}} \qquad \text{..... (3)}$$

Cor. : $\boxed{L^{-1}\left[\frac{1}{s}\right] = H(t)}$ ..... (4)

Laplace Transform of $H(t-a)$

Since, $H(t-a) = \begin{cases} 0, & t < a \\ 1, & t \geq a \end{cases}$

By definition of Laplace transform

$$L[H(t-a)] = \int_0^\infty e^{-st} H(t-a)\,dt = \int_0^a 0\,dt + \int_a^\infty e^{-st}(1)\,dt$$

$$= 0 + \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{1}{s}e^{-as}$$

$$\therefore \quad \boxed{L[H(t-a)] = \frac{1}{s}\cdot e^{-as}} \qquad \text{..... (5)}$$

Cor. : $\boxed{L^{-1}\left[\frac{1}{s}\cdot e^{-as}\right] = H(t-a)}$ ..... (6)

Laplace Transform of $f(t)\,H(t-a)$

Since, $f(t)H(t-a) = \begin{cases} 0, & t < a \\ f(t), & t \geq a \end{cases}$

By definition of Laplace transform, we get

$$L[f(t)H(t-a)] = \int_0^\infty e^{-st}[f(t)H(t-a)]\,dt$$

$$= \int_0^a e^{-st}[f(t)H(t-a)]\,dt + \int_a^\infty e^{-st}[f(t)H(t-a)]\,dt$$

But $f(t)H(t-a) = \begin{cases} 0, & t < a \\ f(t), & t \geq a \end{cases}$ $\quad \therefore \; L[f(t)H(t-a)] = 0 + \int_a^\infty e^{-st} f(t)\,dt$

Now, put $t - a = u$ $\quad \therefore \; dt = du$

When $t = a$, $u = 0$; when $t = \infty$, $u = \infty$

$$\therefore \; L[f(t)\cdot H(t-a)] = \int_0^\infty e^{-s(a+u)} f(a+u)\,du$$

$$= e^{-as}\int_0^\infty e^{-su} f(u+a)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t+a)\,dt$$

$$\therefore \quad \boxed{L[f(t)\cdot H(t-a)] = e^{-as} L[f(t+a)]} \qquad \text{..... (7)}$$

Putting $a = 0$, we get

$$\boxed{L[f(t)\,H(t)] = L[f(t)]} \qquad \text{..... (8)}$$

Procedure to find $L[f(t)\,H(t-a)]$

(i) First find $f(t+a)$.

(ii) Then find $L[f(t+a)]$.

(iii) Multiply the result by e^{-as}.

This is the required Laplace transform.

Example 1 : Find $L[t^2 H(t-3)]$.

Sol. : (i) Here $f(t) = t^2$ and $a = 3$.

$$\therefore \quad f(t+3) = (t+3)^2 = t^3 + 6t + 9$$

(ii) $$L[f(t+3)] = L(t^2 + 6t + 9) = L(t^2) + 6\,L(t) + 9\,L(1)$$

$$= \frac{2}{s^3} + 6\cdot\frac{1}{s^2} + 9\cdot\frac{1}{s}$$

(iii) $$L[t^2 H(t-3)] = e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$$

Example 2 : Find $L[(1 + 2t - t^2 + t^3)\,H(t-1)]$.

Sol. : (i) Here $f(t) = 1 + 2t - t^2 + t^3$ and $a = 1$.

$$\therefore \quad f(t+1) = 1 + 2(t+1) - (t+1)^2 + (t+1)^3$$

$$= 1 + 2t + 2 - t^2 - 2t - 1 + t^3 + 3t^2 + 3t + 1$$

$$= t^3 + 2t^2 + 3t + 3$$

(ii) $$L[f(t+1)] = L(t^3 + 2t^2 + 3t + 3)$$

$$= L(t^3) + 2L(t^2) + 3L(t) + 3L(1)$$

$$= \frac{3!}{s^4} + 2\cdot\frac{2!}{s^3} + 3\cdot\frac{1}{s^2} + 3\cdot\frac{1}{s}$$

(iii) $$L[(1 + 2t - t^2 + t^3)H(t-1)] = e^{-s}\left[\frac{6}{s^4} + \frac{4}{s^3} + \frac{3}{s^2} + \frac{3}{s}\right]$$

Example 3 : Find Laplace transform of $(1 + 2t - 3t^2 + 4t^3)\,H(t-2)$. **(M.U. 1998, 2004, 07)**

Sol. : (i) Here $f(t) = 1 + 2t - 3t^2 + 4t^3$ and $a = 2$.

$$\therefore \quad f(t+2) = 1 + 2(t+2) - 3(t+2)^2 + 4(t+2)^3$$

$$= 1 + 2t + 4 - 3(t^2 + 4t + 4) + 4(t^3 + 6t^2 + 12t + 8)$$

$$= 4t^3 + 21t^2 + 38t + 25$$

(ii) $$L[f(t+2)] = L(4t^3 + 21t^2 + 38t + 25)$$

$$= 4L(t^3) + 21\,L(t^2) + 38\,L(t) + 25\,L(1)$$

$$= 4\cdot\frac{3!}{s^4} + 21\cdot\frac{2!}{s^3} + 38\cdot\frac{1}{s^2} + 25\cdot\frac{1}{s}$$

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(iii) $L[f(t)H(t-2)] = e^{-2s}\left[\frac{24}{s^4}+\frac{42}{s^3}+\frac{38}{s^2}+\frac{25}{s}\right]$.

Example 4 : Express the following function as Heaviside unit step function and find its Laplace transform.

$$f(t) = \begin{cases} 0, & 0 < t < 2 \\ (t-2)^2, & t > 2 \end{cases}$$

Sol. : Since $H(t-2) = \begin{cases} 0, & t < 2 \\ 1, & t > 2 \end{cases}$

$\therefore \quad f(t) = f(t)\cdot H(t-2) = (t-2)^2 H(t-2)$

(I) Here $f(t) = (t-2)^2$ and $a = 2$.

$\therefore \quad f(t+2) = (t+2-2)^2 = t^2$

(II) $L[f(t+2)] = L(t^2) = \frac{2!}{s^3}$

(III) $L[f(t)\cdot H(t-2)] = e^{-2s}\cdot\frac{2}{s^3}$.

Example 5 : Find Laplace transform of $\sin t\left[H\left(t-\frac{\pi}{2}\right)\right] - H\left(t-\frac{3\pi}{2}\right)$. **(M.U. 1996)**

Sol. : (a) (I) Here $f(t) = \sin t$ and $a = \frac{\pi}{2}$.

$\therefore \quad f\left(t+\frac{\pi}{2}\right) = \sin\left(t+\frac{\pi}{2}\right) = \cos t$

(II) $L\left[f\left(t+\frac{\pi}{2}\right)\right] = L\cos t = \frac{s}{s^2+1}$ (III) $L\left[f(t)H\left(t-\frac{\pi}{2}\right)\right] = e^{-\pi s/2}\cdot\frac{s}{s^2+1}$

(b) We know that [By (5), page S-8]

$LH(t-a) = \frac{1}{s}e^{-as} \qquad \therefore \quad LH\left(t-\frac{3\pi}{2}\right) = \frac{1}{s}\cdot e^{-3\pi s/2}$

Hence, $L\left[\sin t\, H\left(t-\frac{\pi}{2}\right) - H\left(t-\frac{3\pi}{2}\right)\right] = e^{-\pi s/2}\cdot\frac{s}{s^2+1} - e^{-3\pi s/2}\cdot\frac{1}{s}$.

Example 6 : Find Laplace transform of $e^{-t}\cos t\cdot H(t-\pi)$.

Sol. : (I) Here $f(t) = e^{-t}\cos t$ and $a = \pi$.

$\therefore \quad f(t+\pi) = e^{-(t+\pi)}\cos(t+\pi)$

$= -e^{-(t+\pi)}\cos t = -e^{-\pi}\cdot e^{-t}\cos t$

(II) $L[f(t+\pi)] = L[-e^{-\pi}\cdot e^{-t}\cos t]$

$= -e^{-\pi}\cdot L(e^{-t}\cos t) = -e^{-\pi}\cdot\frac{s+1}{(s+1)^2+1}$

(III) $L[f(t+\pi)H(t-\pi)] = -e^{-\pi s}\cdot e^{-\pi}\cdot\frac{s+1}{s^2+2s+2}$

$= -e^{-\pi(s+1)}\cdot\frac{s+1}{s^2+2s+2}$.

Example 7 : Find Laplace transform of $(t-1)^2\cdot H(t-1)$. **(M.U. 2011)**

Sol. : (I) Here $f(t) = (t-1)^2$ and $a = 1$.

$\therefore \quad f(t+1) = (t+1-1)^2 = t^2$

(II) $L[f(t+1)] = L(t^2) = \frac{2}{s^3}$

(III) $L[f(t+1)H(t-1)] = e^{-s}\cdot\frac{2}{s^3}$.

Example 8 : Find Laplace transform of $e^{-3t}H(t-2)$. **(M.U. 2011)**

Sol. : (I) Here $f(t) = e^{-3t}$ and $a = 2$.

$\therefore \quad f(t+2) = e^{-3(t+2)} = e^{-6}\cdot e^{-3t}$

(II) $L[f(t+2)] = L(e^{-6}\cdot e^{-3t}) = e^{-6}\cdot L(e^{-3t}) = e^{-6}\cdot\frac{1}{s+3}$.

(III) $L[f(t)\cdot H(t-2)] = e^{-2s}\cdot e^{-6}\cdot\frac{1}{s+3} = e^{-2(s+3)}\cdot\frac{1}{s+3}$.

Example 9 : Find Laplace transform of $e^{4t}\cdot H(t-2)$.

Sol. : (I) Here $f(t) = e^{4t}$ and $a = 2$.

$\therefore \quad f(t+2) = e^{4(t+2)} = e^{4t+8} = e^8\cdot e^{4t}$

(II) $L[f(t+2)] = L(e^8\cdot e^{4t}) = e^8\cdot L(e^{4t}) = e^8\cdot\frac{1}{s-4}$

(III) $L[f(t)\cdot H(t-2)] = e^{-2s}\cdot e^8\cdot\frac{1}{s-4} = e^{-2(s-4)}\cdot\frac{1}{s-4}$.

Example 10 : Find Laplace transform of $te^{-2t}\cdot H(t-1)$.

Sol. : (I) Here $f(t) = te^{-2t}$ and $a = 1$.

$\therefore \quad f(t+1) = (t+1)e^{-2(t+1)} = e^{-2}\cdot e^{-2t}(t+1)$

(I) $L[f(t+1)] = e^{-2}\cdot L[e^{-2t}(t+1)] = e^{-2}\left[L(e^{-2t}\cdot t) + L(e^{-2t})\right]$

$= e^{-2}\left[\frac{1}{(s+2)^2}+\frac{1}{s+2}\right]$ [See Ex. 1(I), page 1-29]

(III) $L[f(t)\cdot H(t-1)] = e^{-s}\cdot e^{-2}\left[\frac{1+s+2}{(s+2)^2}\right] = e^{-(s+2)}\cdot\frac{s+3}{(s+2)^2}$.

5. Evaluation of the Integral $\int_0^\infty e^{-at} f(t)H(t-a)\,dt$

We have see on pages 1-13, 1-24, 1-35, 1-43 and 1-65, how Laplace Transforms can be used to find certain integrals. We shall use the same technique to find definite integrals of Heaviside step functions.

Example 1 : Using Laplace transform evaluate

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)H(t-2)\,dt.$$ **(M.U. 2003)**

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Sol. : We have proved above in Ex. 3, page S-9 that

$$L(1+2t-3t^2+4t^3)H(t-2) = e^{-2s}\left[\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right]$$

By definition of Laplace transform, this means,

$$\int_0^\infty e^{-st}(1+2t-3t^2+4t^3)H(t-2)\,dt = e^{-2s}\left[\frac{25}{s}+\frac{38}{s^2}+\frac{42}{s^3}+\frac{24}{s^4}\right]$$

Putting $s = 1$, we get

$$\int_0^\infty e^{-t}(1+2t-3t^2+4t^3)H(t-2)\,dt = e^{-2}\left[\frac{25}{1}+\frac{38}{1^2}+\frac{42}{1^3}+\frac{24}{1^4}\right] = \frac{129}{e^2}.$$

Example 2 : Using Laplace Transform evaluate

$$\int_0^\infty e^{-t}(1+2t-t^2+t^3)H(t-1)\,dt.$$ **(M.U. 2003, 07, 10)**

Sol. : We have proved above in Ex. 2, page S-9 that

$$L[(1+2t-t^2+t^3)H(t-1)] = e^{-s}\left[\frac{3}{s}+\frac{3}{s^2}+\frac{4}{s^3}+\frac{6}{s^4}\right]$$

By definition of Laplace transform this means,

$$\int_0^\infty e^{-st}(1+2t-t^2+t^3)H(t-1)\,dt = e^{-s}\left[\frac{3}{s}+\frac{3}{s^2}+\frac{4}{s^3}+\frac{6}{s^4}\right]$$

Putting $s = 1$,

$$\int_0^\infty e^{-t}(1+2t-t^2+t^3)H(t-1)\,dt = e^{-1}\left[\frac{3}{1}+\frac{3}{1^2}+\frac{4}{1^3}+\frac{6}{1^4}\right] = \frac{16}{e}.$$

EXERCISE - II

1. Express the following functions in terms of Heaviside unit step function and hence find the Laplace transform.

(i) $f(t) = \begin{cases}(t-3)^4, & t>3\\ 0, & 0<t<3\end{cases}$ (ii) $f(t) = \begin{cases}0, & 0<t<4\\ (t-4)^3, & t>4\end{cases}$ **(M.U. 2011)**

(iii) $f(x) = \begin{cases}0, & 0<t<2\\ 1-2t-t^2, & t>2\end{cases}$

[Ans. : (i) $(t-3)^4 H(t-3)$; $e^{-3s}\cdot\frac{4!}{s^5}$, (ii) $(t-4)^3 H(t-4)$; $e^{-4s}\cdot\frac{3!}{s^4}$,

(iii) $(1-2t-t^2)H(t-2)$; $e^{-2s}\left[\frac{1}{s}-\frac{2}{s^2}-\frac{2}{s^3}\right]$

2. Find the Laplace transform of the following functions.

(i) $t\,H(t-2)$ (ii) $\sin t\,H(t-\pi)$ **(M.U. 2005)**

(iii) $t^4 H(t-2)$ (iv) $(1+3t-4t^2+2t^3)\,H(t-3)$

(v) $t^2 H(t-2)$ **(M.U. 1997)** (vi) $e^{-t}\sin t\,H(t-\pi)$ **(M.U. 2005)**

[Ans. : (i) $e^{-2s}\left[\frac{2}{s}+\frac{1}{s^2}\right]$, (ii) $e^{-\pi s}\cdot\frac{-1}{s^2+1}$,

(iii) $e^{-4s}\left[\frac{16}{s}+\frac{32}{s^2}+\frac{48}{s^3}+\frac{48}{s^4}+\frac{24}{s^5}\right]$, (iv) $e^{-3s}\left[\frac{28}{s}+\frac{33}{s^2}+\frac{28}{s^3}+\frac{12}{s^4}\right]$,

(v) $e^{-2s}\left[\frac{4}{s}+\frac{4}{s^2}+\frac{2}{s^3}\right]$, (vi) $e^{-\pi(s+1)}\cdot\frac{1}{s^2+2s+2}$.]

3. Using Laplace transform evaluate the following integrals

(i) $\int_0^\infty e^{-2t}(1+t+t^2)H(t-3)\,dt$ **[Ans. :** $\frac{17}{2e^6}$]

(ii) $\int_0^\infty e^{-t}(1+3t+t^2)H(t-2)$ **[Ans. :** $\frac{20}{e^2}$]

6. Laplace Transform of f (t – a) · H (t – a) **(M.U. 1995, 2004, 06)**

Since $f(t-a)\cdot H(t-a) = \begin{cases}0, & t<a\\ f(t-a), & t\ge a\end{cases}$ by definition of Laplace transform,

$$L[f(t-a)H(t-a)] = \int_0^\infty e^{-st}[f(t-a)\cdot H(t-a)]\,dt$$

$$= \int_0^a e^{-st}[f(t-a)\cdot H(t-a)]\,dt + \int_a^\infty e^{-st}[f(t-a)\cdot H(t-a)]\,dt$$

$$= 0 + \int_a^\infty e^{-st} f(t-a)\,dt$$ [Case (iv), page S-7]

Now, put $t-a = u$, $dt = du$.

When $t = a$, $u = 0$; when $t = \infty$, $u = \infty$

$$L[f(t-a)H(t-a)] = \int_0^\infty e^{-s(a+u)}\cdot f(u)\,du = e^{-as}\int_0^\infty e^{-su} f(u)\,du$$

$$= e^{-as}\int_0^\infty e^{-st} f(t)\,dt$$

[Since a definite integral does not depend upon the variable u or t but on the function f.]

$\therefore$ $\boxed{L[f(t-a)H(t-a)] = e^{-as}L f(t)}$ ..... (9)

$\therefore$ $\boxed{L[f(t-a)H(t-a)] = e^{-as}\Phi(s)}$ where $\Phi(s) = L[f(t)]$

Cor : Taking the inverse Laplace transform, we get

$\boxed{L^{-1}[e^{-as}\Phi(s)] = f(t-a)H(t-a)}$ where $f(t) = L^{-1}\Phi(s)$ ..... (10)

In particular putting $a = 0$, in (27), we get $L[f(t)H(t)] = L f(t)$.

A useful Result

If a given function is a step function having many steps then it is convenient to express it as a unit-step function as explained below and then write down it Laplace transform.

Suppose $f(t) = \begin{cases}f_1(t), & 0<t<a\\ f_2(t), & t>a\end{cases}$

Since, $H(t) = \begin{cases}0, & t<0\\ 1, & t>0\end{cases}$ and $H(t-a) = \begin{cases}0, & t<a\\ 1, & t>a\end{cases}$

$H(t) - H(t-a) = 1$ for $0<t<a$

[See case (v), page S-7 with $a = 0$, $b = a$]

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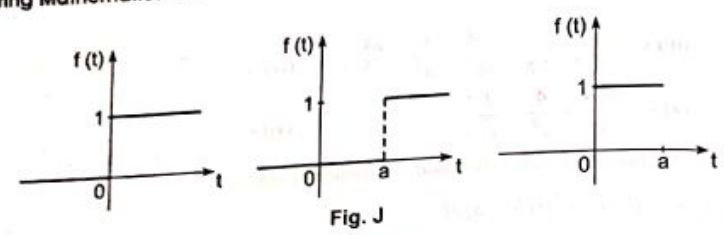

Fig. J

$$\therefore \quad \boxed{f(t) = f_1(t)\,[H(t) - H(t-a)] + f_2(t)\,H(t-a)} \qquad \text{..... (11)}$$

The above result can be extended further. Thus, if

$$f(t) = \begin{cases} f_1(t), & 0 < t < a \\ f_2(t), & a < t < b \\ f_3(t), & b < t < c \\ f_4(t), & t > c \end{cases}$$

then following the above discussion we can write,

$$f(t) = f_1(t)\,[H(t) - H(t-a)] + f_2(t)\,[H(t-a) - H(t-b)] + f_3(t)\,[H(t-b) - H(t-c)] + f_4(t)\,H(t-c)$$

After simplification, we get

$$\boxed{\begin{aligned} f(t) &= f_1(t)\,H(t) + [f_2(t) - f_1(t)]\,H(t-a) \\ &\quad + [f_3(t) - f_2(t)]\,H(t-b) + [f_4(t) - f_3(t)]\,H(t-c) \end{aligned}} \qquad \text{..... (12)}$$

Example 1 : Express the following functions as Heaviside's unit step functions and find their Laplace transforms.

1. $f(t) = \begin{cases} 0, & 0 < t < \pi/2 \\ \cos t, & \pi/2 < t < 3\pi/2 \\ 0, & t > 3\pi/2 \end{cases}$

2. $f(t) = \begin{cases} 0, & 0 < t < 1 \\ t^2, & 1 < t < 3 \\ 0, & t > 3 \end{cases}$

3. $f(t) = \begin{cases} 2t, & 0 < t < 1 \\ 3t^2, & t > 1 \end{cases}$

4. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \cos t, & t < \pi \end{cases}$

5. $f(t) = \begin{cases} e^t \sin t, & 0 < t < \pi \\ e^t \cos t, & t > \pi \end{cases}$

6. $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \cos 2t, & \pi < t < 2\pi \\ \cos 3t, & t > 2\pi \end{cases}$

(M.U. 2008)

Sol. : Following the procedure given on page S-7, we have

(1) $f(t) = \cos t\left[H\left(t - \frac{\pi}{2}\right) - H\left(t - \frac{3\pi}{2}\right)\right] = \cos t\, H\left(t - \frac{\pi}{2}\right) - \cos t\, H\left(t - \frac{3\pi}{2}\right)$

(i) $f_1(t) = \cos t,\ a = \frac{\pi}{2} \qquad \therefore\ f_1\left(t + \frac{\pi}{2}\right) = \cos\left(t + \frac{\pi}{2}\right) = -\sin t$

$L\left[f_1\left(t + \frac{\pi}{2}\right)\right] = L(-\sin t) = -\frac{1}{s^2+1}$ and $f_2(t) = \cos t,\ a = \frac{3\pi}{2}$

$\therefore\ f_2\left(t + \frac{3\pi}{2}\right) = \cos\left(t + \frac{3\pi}{2}\right) = \sin t; \qquad L\left[f_2\left(t + \frac{3\pi}{2}\right)\right] = L(\sin t) = \frac{1}{s^2+1}$

(ii) $L[f(t)] = e^{-\pi s/2}\left(-\frac{1}{s^2+1}\right) - e^{-3\pi s/2} \cdot \frac{1}{s^2+1}$

$= -(e^{-\pi s/2} + e^{-3\pi s/2}) \cdot \frac{1}{s^2+1}$

(2) $f(t) = t^2\,[H(t-1) - H(t-3)] = t^2 H(t-1) - t^2 H(t-3)$

(i) $f_1(t+1) = (t+1)^2 = t^2 + 2t + 1,\ a = 1.$

$\therefore\ L[f_1(t+1)] = L(t^2 + 2t + 1) = \frac{2}{s^3} + 2 \cdot \frac{1}{s^2} + \frac{1}{s}$

$f_2(t+3) = (t+3)^2 = t^2 + 6t + 9,\ a = 3.$

$\therefore\ L[f(t+3)] = \frac{2}{s^3} + 6 \cdot \frac{1}{s^2} + 9 \cdot \frac{1}{s}$

(ii) $L[f(t)] = e^{-s}\left[\frac{2}{s^3} + \frac{2}{s^2} + \frac{1}{s}\right] + e^{-3s}\left[\frac{2}{s^3} + \frac{6}{s^2} + \frac{9}{s}\right]$

(3) $f(t) = 2t\,H(t) + (3t^2 - 2t)\,H(t-1)$

(i) $f_1(t) = 2t,\ a = 0 \qquad \therefore\ f_1(t+0) = 2(t+0) = 2t$

$L[f_1(t)] = L(2t) = 2 \cdot \frac{1}{s^2}$

(ii) $f_2(t) = 3t^2 - 2t,\ a = 1$

$\therefore\ f_2(t+1) = 3(t+1)^2 - 2(t+1) = 3(t^2 + 2t + 1) - 2(t+1)$

$= 3t^2 + 4t + 1$

$\therefore\ L[f_2(t+1)] = L(3t^2 + 4t + 1) = 3 \cdot L(t^2) + 4\,L(t) + L(1)$

$= 3 \cdot \frac{2}{s^3} + 4 \cdot \frac{1}{s^2} + \frac{1}{s}$

(ii) $L[f(t)] = \frac{2}{s^2} + e^{-s}\left[\frac{6}{s^3} + \frac{4}{s^3} + \frac{1}{s}\right]$

(4) $f(t) = \sin t \cdot H(t) + (\cos t - \sin t)\,H(t-\pi)$

(i) $f_1(t) = \sin t,\ a = 0 \qquad \therefore\ L[f_1(t)] = L(\sin t) = \frac{1}{s^2+1}$

(ii) $f_2(t) = \cos t - \sin t, \quad a = \pi$

$\therefore\ L[f_2(t+\pi)] = \cos(t+\pi) - \sin(t+\pi) = -\cos t + \sin t$

$= -\frac{s}{s^2+1} + \frac{1}{s^2+1}$

$\therefore\ L[f_2(t)] = \frac{1-s}{s^2+1}$

(iii) $L[f(t)] = \frac{1}{s^2+1} + e^{-\pi s} \cdot \left(\frac{1-s}{s^2+1}\right).$

(5) $f(t) = e^t \sin t \cdot H(t) + e^t(\cos t - \sin t)\,H(t-\pi)$

(i) $f_1(t) = e^t \sin t, \quad a = 0$

$\therefore\ L[f_1(t)] = L(e^t \sin t) = \frac{s-1}{(s-1)^2+1} = \frac{s-1}{s^2 - 2s + 2}$

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(ii) $f_2(t) = e^t \cos t - e^t \sin t, \quad a = \pi$

$$\therefore \; L[f_2(t+\pi)] = e^{t-\pi}\cos(t+\pi) - e^{t+\pi}\sin(t+\pi)$$
$$= L(-e^{\pi+t}\sin t) + L(e^{\pi+t}\cos t)$$
$$= -e^{\pi}\cdot L(e^t \sin t) + e^{\pi}\cdot L(e^t \cos t)$$
$$= -e^{-\pi}\cdot\frac{s-1}{(s-1)^2+1} + e^{\pi}\cdot\frac{1}{(s-1)^2+1}$$
$$= -e^{-\pi}\cdot\frac{s-2}{s^2-2s+2}$$

(iii) $L[f(t)] = \dfrac{s-1}{s^2-2s+2} - e^{-\pi-\pi s}\cdot\dfrac{s-2}{s^2-2s+2}$

$$\therefore \; L[f(t)] = \frac{s-1}{s^2-2s+2} - e^{-\pi(s+1)}\cdot\frac{s-2}{s^2-2s+2}$$

(6) $f(t) = \cos t\cdot H(t) + (\cos 2t - \cos t)\,H(t-\pi) + (\cos 3t - \cos 2t)\,H(t-2\pi)$

(i) $f_1(t) = \cos t, \quad a = 0 \qquad \therefore \; L[f_1(t)] = L(\cos t) = \dfrac{s}{s^2+1}$

(ii) $f_2(t) = \cos 2t - \cos t, \quad a = \pi$

$$\therefore \; f_2(t+\pi) = \cos 2(t+\pi) - \cos(t+\pi) = \cos 2t + \cos t$$
$$= L\cos 2t + L\cos t = \frac{s}{s^2+4} + \frac{s}{s^2+1}$$

(iii) $f_3(t) = \cos 3t - \cos 2t, \quad a = 2\pi.$

$$\therefore \; f_3(t+2\pi) = \cos 3(t+2\pi) - \cos 2(t+2\pi)$$
$$= \cos(3t+6\pi) - \cos(2t+4\pi)$$
$$= \cos 3t - \cos 2t$$
$$\therefore \; L[f_3(t+2\pi)] = \frac{s}{s^2+9} - \frac{s}{s^2+4}$$
$$\therefore \; L[f(t)] = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s}{s^2+4} + \frac{s}{s^2+1}\right) + e^{-2\pi s}\left(\frac{s}{s^2+9} - \frac{s}{s^2+4}\right)$$

Example 2 : Express the following function in terms of unit step function and hence obtain it Laplace transform

$$f(t) = \begin{cases} 2t, & 0 < t < \pi \\ 1, & t \ge \pi \end{cases}$$

Sol. : $f(t) = 2t\cdot H(t) + (1-2t)\,H(t-\pi)$

(i) $f_1(t) = 2t, \quad a = 0 \qquad \therefore \; L[f_1(t)] = L(2t) = 2\cdot\dfrac{1}{s^2}$

(ii) $f_2(t) = 1 - 2t, \quad a = \pi \qquad \therefore \; f_2(t+\pi) = 1 - 2(t+\pi) = 1 - 2t - 2\pi$

$$\therefore \; L[f_2(t+\pi)] = L(1) - L(2t) - L(2\pi) = \frac{1}{s} - 2\cdot\frac{1}{s^2} - \frac{2\pi}{s}$$

(iii) $L[f(t)] = \dfrac{2}{s^2} + e^{-\pi s}\left(\dfrac{1}{s} - \dfrac{2}{s^2} - \dfrac{2\pi}{s}\right).$

Example 3 : Express the following staircase function as Heaviside's unit step function and find its Laplace transform.

$$f(t) = \begin{cases} 1, & 0 < t < 1 \\ 2, & 1 < t < 2 \\ 3, & 2 < t < 3 \\ \ldots & \ldots\ldots \\ \ldots & \ldots\ldots \end{cases}$$

Sol. : As discussed on page S-14 [See (12)], Heaviside representation of $f(t)$ is

$$f(t) = 1\cdot H(t) + (2-1)\,H(t-1) + (3-2)\,H(t-2) + \ldots$$
$$= H(t) + H(t-1) + H(t-2) + \ldots$$
$$\therefore \; L[f(t)] = L[H(t)] + L[H(t-1)] + L[H(t-2)] + \ldots$$

But by (3) and (5), page S-8

$$L[H(t)] = \frac{1}{s} \quad \text{and} \quad L[H(t-a)] = \frac{1}{s}e^{-as}$$

Hence, $L[f(t)] = \dfrac{1}{s} + \dfrac{1}{s}e^{-s} + \dfrac{1}{s}\cdot e^{-2s} + \dfrac{1}{s}\cdot e^{-3s} + \ldots$

$$= \frac{1}{s}[1 + e^{-s} + e^{-2s} + e^{-3s} + \ldots]$$
$$= \frac{1}{s}\cdot\frac{1}{(1-e^{-s})} \quad [\text{G.P. with } r = e^{-s}]$$
$$= \frac{1}{s(1-e^{-s})}$$

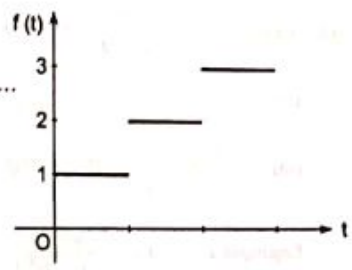

Fig. K

This stair-case function is shown in Fig. K.

EXERCISE - III

Express the following functions in terms of Heaviside's unit step function and hence find their Laplace transform.

1. $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin t, & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

2. $f(t) = \begin{cases} 0, & 0 < t < \pi \\ \sin 2t, & \pi < t < 2\pi \\ 0, & t > 2\pi \end{cases}$

3. $f(t) = \begin{cases} t, & 0 < t < 2 \\ t^2, & t > 2 \end{cases}$

4. $f(t) = \begin{cases} \cos t, & 0 < t < \pi/2 \\ \sin t, & t > \pi/2 \end{cases}$

5. $f(t) = \begin{cases} e^t \cos t, & 0 < t < \pi \\ e^t \sin t, & t > \pi \end{cases}$

6. $f(t) = \begin{cases} \sin t, & 0 < t < \pi \\ \sin 2t, & \pi < t < \pi \\ \sin 3t, & t > 2\pi \end{cases}$

[Ans. : (1) $-(e^{-\pi s} + e^{-2\pi s})\cdot\dfrac{1}{s^2+1}$, (2) $(e^{-\pi s} - e^{-2\pi s})\cdot\dfrac{2}{s^2+4}$,

(3) $\dfrac{1}{s^2} + e^{-2s}\left[\dfrac{2}{s^3} + \dfrac{4}{s^2} + \dfrac{4}{s}\right]$, (4) $\dfrac{s}{s^2+1} + e^{-\pi s/2}\left[\dfrac{s}{s^2+1} + \dfrac{1}{s^2+1}\right]$,

(5) $\dfrac{s-1}{s^2-2s+2} + e^{-\pi(s-1)}\cdot\dfrac{s-2}{s^2-2s+2}$,

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(6) $\dfrac{1}{s^2+1}+e^{-\pi s}\left[\dfrac{2}{s^2+4}+\dfrac{1}{s^2+1}\right]+e^{-2\pi s}\left[\dfrac{3}{s^2+9}+\dfrac{2}{s^2+4}\right]$

7. Inverse Laplace Transforms of $e^{-as}\,\Phi(s)$

Procedure to find $L^{-1}[e^{-as}\Phi(s)]$

1. First find $\Phi(s)$ and a.
2. Then find $L^{-1}[\Phi(s)] = f(t)$, say.
3. Then $L^{-1}[e^{-as}\Phi(s)] = f(t-a)\,H(t-a)$.

Example 1 : Find $L^{-1}\left[\dfrac{8e^{-3s}}{s^2+4}\right]$. **(M.U. 1998)**

Sol. : (i) Here $\Phi(s) = \dfrac{8}{s^2+4}$ and $a = 3$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{8}{s^2+4}\right] = 4L^{-1}\left[\dfrac{2}{s^2+2^2}\right]$ $\quad\therefore\ f(t) = 4\sin 2t$

(iii) $L^{-1}\left[\dfrac{8e^{-3s}}{s^2+4}\right] = f(t-a)\,H(t-a) = 4\sin 2(t-3)\,H(t-3)$

Example 2 : Find $L^{-1}\left[\dfrac{e^{-5s}}{(s-2)^4}\right]$. **(M.U. 1999)**

Sol. : (i) Here $\Phi(s) = \dfrac{1}{(s-2)^4}$ and $a = 5$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{1}{(s-2)^4}\right] = e^{2t}\cdot L^{-1}\left[\dfrac{1}{s^4}\right]$ $\quad\therefore\ f(t) = e^{2t}\cdot\dfrac{t^3}{3!}$

(iii) $L^{-1}\left[\dfrac{e^{-5s}}{(s-2)^4}\right] = f(t-a)\,H(t-a) = e^{2(t-5)}\cdot\dfrac{(t-5)^3}{3!}H(t-3)$.

Example 3 : Find $L^{-1}\left[\dfrac{e^{-3s}}{(s+4)^3}\right]$. **(M.U. 1996, 2012)**

Sol. : (i) Here $\Phi(s) = \dfrac{1}{(s+4)^3}$ and $a = 3$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{1}{(s+4)^3}\right] = e^{-4t}\cdot L^{-1}\left[\dfrac{1}{s^3}\right] = e^{-4t}\cdot\dfrac{t^2}{2}$ $\quad\therefore\ f(t) = e^{-4t}\cdot\dfrac{t^2}{2}$

(iii) $L^{-1}\left[\dfrac{e^{-3s}}{(s+4)^3}\right] = f(t-a)H(t-a) = e^{-4(t-3)}\cdot\dfrac{(t-3)^2}{2!}H(t-3)$

Example 4 : Find $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2-2s+2}\right]$. **(M.U. 1995, 15, 19)**

Sol. : (i) Here $\Phi(s) = \dfrac{1}{s^2-2s+2}$ and $a = \pi$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{1}{(s-1)^2+1}\right] = e^{t}\cdot L^{-1}\left[\dfrac{1}{s^2+1}\right] = e^{t}\sin t$ $\quad\therefore\ f(t) = e^{t}\sin t$

(iii) $L^{-1}\left[\dfrac{e^{-\pi s}}{s^2-2s+2}\right] = f(t-a)\,H(t-a) = e^{(t-\pi)}\sin(t-\pi)\,H(t-\pi)$

Example 5 : Find inverse Laplace transform of $\dfrac{e^{-as}}{(s+b)^{5/2}}$. **(M.U. 2002, 04)**

Sol. : (i) Here $\Phi(s) = \dfrac{1}{(s+b)^{5/2}}$ and $a = a$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{1}{(s+b)^{5/2}}\right] = e^{-bt}\cdot L^{-1}\left[\dfrac{1}{s^{5/2}}\right]$

$$= e^{-bt}\cdot\frac{t^{3/2}}{\Gamma 5/2} = e^{-bt}\cdot\frac{t^{3/2}}{(3/2)(1/2)\Gamma 1/2} = \frac{4}{3\sqrt{\pi}}\cdot e^{-bt}\,t^{3/2}$$

$\therefore\ f(t) = \dfrac{4}{3\sqrt{\pi}}\cdot e^{-bt}\,t^{3/2}$

(iii) $L^{-1}\left[\dfrac{e^{-as}}{(s+b)^{5/2}}\right] = f(t-a)\,H(t-a) = \dfrac{4}{3\sqrt{\pi}}\cdot e^{-b(t-a)}\cdot(t-a)^{3/2}\,H(t-a)$.

Example 6 : Find inverse Laplace transform of $e^{-s}\left\{\dfrac{1-\sqrt{s}}{s^2}\right\}^2$. **(M.U. 2009)**

Sol. : (i) Here $\Phi(s) = \left(\dfrac{1-\sqrt{s}}{s^2}\right)^2$ and $a = 1$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left(\dfrac{1-\sqrt{s}}{s^2}\right)^2 = L^{-1}\left(\dfrac{1-2\sqrt{s}+s}{s^4}\right) = L^{-1}\left[\dfrac{1}{s^4}-\dfrac{2}{s^{7/2}}+\dfrac{1}{s^3}\right]$

$$= \frac{t^3}{3!} - 2\cdot\frac{t^{5/2}}{\Gamma 7/2} + \frac{t^2}{2} \qquad \left[\Gamma\frac{7}{2} = \left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\frac{1}{2} = \frac{15}{8}\sqrt{\pi}\right]$$

$\therefore\ L^{-1}[\Phi(s)] = \dfrac{t^3}{6} - \dfrac{16}{15\sqrt{\pi}}\cdot t^{5/2} + \dfrac{t^2}{2}$

$\therefore\ f(t) = \dfrac{t^3}{6} - \dfrac{16}{15\sqrt{\pi}}\cdot t^{5/2} + \dfrac{t^2}{2}$

(iii) $L^{-1}\left[e^{-s}\left\{\dfrac{1-\sqrt{s}}{s^2}\right\}^2\right] = f(t-a)\,H(t-a)$

$$= \left[\frac{(t-1)^3}{6} - \frac{16}{15\sqrt{\pi}}(t-1)^{5/2} + \frac{(t-1)^2}{2}\right]H(t-1)$$

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Example 7 : Find $L^{-1}\left[\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right]$. **(M.U. 1997, 2000, 05, 13)**

Sol. : (i) We have $L^{-1}\left[\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right] = e^4 \cdot L^{-1}\left[\dfrac{e^{-3s}}{(s+4)^{5/2}}\right]$

Here $\Phi(s) = \dfrac{1}{(s+4)^{5/2}}$ and $a = 3$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{1}{(s+4)^{5/2}}\right] = \dfrac{4}{3\sqrt{\pi}} \cdot e^{-4t}\, t^{3/2}$ [By Ex. 5 above]

$\therefore\ f(t) = \dfrac{4}{3\sqrt{\pi}} \cdot e^{-4t}\, t^{3/2}$

(iii) $L^{-1}\left[\dfrac{e^{4-3s}}{(s+4)^{5/2}}\right] = f(t-a)\, H(t-a)$

$$= e^4 \cdot \frac{4}{3\sqrt{\pi}} \cdot e^{-4(t-3)} (t-3)^{3/2}\, H(t-3)$$

$$= \frac{4}{3\sqrt{\pi}} \cdot e^{-4(t-4)} \cdot (t-3)^{3/2}\, H(t-3) \qquad [\because e^4 = e^{-4(-1)}]$$

Example 8 : Find $L^{-1}\left[\dfrac{e^{-2s}}{s^2+8s+25}\right]$. **(M.U. 1999, 2006, 12)**

Sol. : (i) Here $\Phi(s) = \dfrac{1}{s^2+8s+25}$ and $a = 2$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{1}{s^2+8s+25}\right] = L^{-1}\left[\dfrac{1}{(s+4)^2+3^2}\right]$

$$= e^{-4t} \cdot L^{-1}\left[\frac{1}{s^2+3^2}\right] = e^{-4t} \cdot \frac{1}{3}\sin 3t$$

$\therefore\ f(t) = \dfrac{1}{3} e^{-4t} \sin 3t$

(iii) $L^{-1}\left[\dfrac{e^{-2s}}{s^2+8s+25}\right] = f(t-2)\, H(t-2) = \dfrac{1}{3} e^{-4(t-2)} \sin 3(t-2) \cdot H(t-2)$.

Example 9 : Find $L^{-1}\left[\dfrac{s e^{-as}}{s^2+3s+2}\right]$.

Sol. : (i) Here $\Phi(s) = \dfrac{s}{s^2+3s+2}$ and $a = a$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{s}{s^2+3s+2}\right] = L^{-1}\left[\dfrac{2}{s+2} - \dfrac{1}{s+1}\right]$

$$= L^{-1}\left[\frac{2}{s+2}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

$\therefore\ f(t) = 2e^{-2t} - e^{-t}$.

(iii) $L^{-1}\left[\dfrac{s e^{-as}}{s^2+3s+2}\right] = f(t-a)\, H(t-a) = [2e^{-2(t-a)} - e^{-(t-a)}]\, H(t-a)$.

Example 10 : Find $L^{-1}\left[\dfrac{s e^{-\pi s}}{s^2+2s+2}\right]$. **(M.U. 2012, 14)**

Sol. : (i) Here $\Phi(s) = \dfrac{s}{s^2+2s+2}$ and $a = \pi$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{s}{(s^2+2s+2)}\right] = L^{-1}\left[\dfrac{(s+1)-1}{(s+1)^2+1}\right]$

$$= L^{-1}\left[\frac{(s+1)}{(s+1)^2+1}\right] - L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$$

$$= e^{-t} \cdot L^{-1}\left[\frac{s}{s^2+1}\right] - e^{-t} \cdot L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= e^{-t}\cos t - e^{-t}\sin t = e^{-t}[\cos t - \sin t]$$

$\therefore\ f(t) = e^{-t}[\cos t - \sin t]$

(iii) $L^{-1}\left[\dfrac{s e^{-\pi s}}{s^2+2s+2}\right] = f(t-a)\, H(t-a)$

$$= e^{-(t-\pi)}[\cos(t-\pi) - \sin(t-\pi)] \cdot H(t-\pi)$$

Example 11 : Find $L^{-1}\left[\dfrac{(s+1)e^{-s}}{s^2+s+1}\right]$. **(M.U. 1997, 2006, 10, 11, 12)**

Sol. : (i) Here $\Phi(s) = \dfrac{s+1}{s^2+s+1}$ and $a = 1$.

(ii) $L^{-1}[\Phi(s)] = L^{-1}\left[\dfrac{s+1}{s^2+s+1}\right] = L^{-1}\left[\dfrac{[s+(1/2)]+(1/2)}{[s+(1/2)]^2+(3/4)}\right]$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}\left[\frac{s+(1/2)}{[s+(1/2)]^2+(\sqrt{3}/2)^2}\right] + L^{-1}\left[\frac{1/2}{[s+(1/2)]^2+(\sqrt{3}/2)^2}\right]$$

$$= e^{-t/2} \cdot L^{-1}\left[\frac{s}{s^2+(\sqrt{3}/2)^2}\right] + \frac{1}{2} e^{-t/2} \cdot L^{-1}\left[\frac{1}{s^2+(\sqrt{3}/2)^2}\right]$$

$$\therefore\ f(t) = e^{-t/2}\cos\left(\frac{\sqrt{3}\,t}{2}\right) + \frac{1}{2} \cdot \frac{2}{\sqrt{3}} \cdot e^{-t/2}\sin\left(\frac{\sqrt{3}\,t}{2}\right)$$

(iii) $L^{-1}\left[\dfrac{(s+1)e^{-s}}{s^2+s+1}\right] = f(t-a)\, H(t-a)$

$$= e^{-(t-1)/2}\left[\cos\left(\frac{\sqrt{3}(t-1)}{2}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}(t-1)}{2}\right)\right] H(t-1).$$

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Example 12 : Find $L^{-1}\left[e^{-4s}\cdot\frac{s}{(s+4)^3}\right]$. **(M.U. 2013)**

Sol. : (i) Here $\Phi(s)=\frac{s}{(s+4)^3}$ and $a=4$.

(ii) $$L^{-1}[\Phi(s)]=L^{-1}\left[\frac{s}{(s+4)^3}\right]=L^{-1}\left[\frac{(s+4)-4}{(s+4)^3}\right]$$

$$=L^{-1}\left[\frac{1}{(s+4)^2}\right]-4L^{-1}\left[\frac{1}{(s+4)^3}\right]$$

$$\therefore\ L^{-1}[\Phi(s)]=e^{-4t}\cdot L^{-1}\left[\frac{1}{s^2}\right]-4\cdot e^{-4t}\cdot L^{-1}\left[\frac{1}{s^3}\right]$$

$$\therefore\ f(t)=e^{-4t}\cdot t-4e^{-4t}\cdot\frac{t^2}{2}$$

(iii) $$L^{-1}\left[e^{-4s}\cdot\frac{s}{(s+4)^3}\right]=f(t-a)\cdot H(t-a)$$

$$=e^{-4(t-4)}\cdot(t-4)\cdot H(t-4)-4\cdot e^{-4(t-4)}\cdot\frac{(t-4)^2}{2}\cdot H(t-4)$$

EXERCISE - IV

Find the inverse transform of

1. $\frac{e^{-s}}{(s+1)^2}$, 2. $\frac{e^{-\pi s}}{(s^2+9)}$, 3. $e^{-s}\frac{(1+\sqrt{s})}{s^3}$, 4. $\frac{e^{-as}}{s-b}$, 5. $\frac{se^{-as}}{s^2+b^2}$,

(M.U. 2009)

6. $\frac{se^{-s/2}+\pi e^{-s}}{s^2+\pi^2}$, 7. $\frac{se^{-3s}}{s^2-1}$, 8. $\frac{e^{-bs}}{s^2(s+a)}$, 9. $\frac{e^{-\pi s}}{s^2(s^2+1)}$, 10. $\frac{e^{-4s}}{\sqrt{2s+7}}$

(M.U. 2003)

11. $\frac{se^{-as}}{s^2+3s+2}$

[Ans. : (1) $e^{-(t-1)}(t-1)H(t-1)$, (2) $\frac{1}{3}\sin 3(t-\pi)H(t-\pi)$,

(3) $\left[\frac{(t-1)^2}{2}+\frac{4(t-1)^{3/2}}{3\sqrt{\pi}}\right]H(t-1)$, (4) $e^{b(t-a)}\cdot H(t-a)$,

(5) $\cos b(t-a)\cdot H(t-a)$, (6) $\sin\pi t\left[H\left(t-\frac{1}{2}\right)+H(t-1)\right]$,

(7) $\cos h(t-3)\cdot H(t-3)$, (8) $\frac{1}{a^2}\left[a(t-b)-1+e^{-a(t-b)}\right]\cdot H(t-b)$,

(9) $[(t-\pi)+\sin(t-\pi)\cdot H(t-\pi)]$, (10) $\frac{e^{-7(t-4)/2}}{\sqrt{2\pi(t-4)}}\cdot H(t-4)$]

(11) $[2e^{-2(t-a)}-e^{-(t-a)}]\cdot H(t-a)$]

8. Dirac-delta Function (Unit-impulse Function)

Consider the function $F(t)$ defined by,

$$F(t)=0,\ t<a$$
$$=\frac{1}{\varepsilon},\ a\le t\le a+\varepsilon$$
$$=0,\ t>a+\varepsilon$$

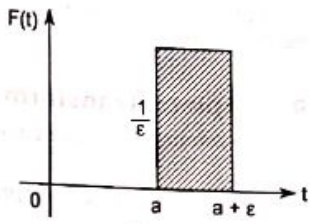

Fig. L

The function is represented by the adjoining figure.

Integrating $F(t)$, we get,

$$\int_0^\infty F(t)\,dt=\int_a^{a+\varepsilon}\frac{1}{\varepsilon}dt=\frac{1}{\varepsilon}[t]_a^{a+\varepsilon}$$
$$=\frac{1}{\varepsilon}[a+\varepsilon-a]=\frac{\varepsilon}{\varepsilon}=1 \text{ for all } \varepsilon.$$

As $\varepsilon\to 0$, the function $F(t)$ tends to infinity at a and is zero everywhere else. But the integral of $F(t)$ is unity.

If $F(t)$ represents a force acting for a short time ε at time $t=a$ then the integral $\lim_{\varepsilon\to 0}\int_0^\infty F(t)\,dt\ (=1)$ represents **unit-impulse** at $t=a$. Hence, the limiting form of $F(t)$ (as $\varepsilon\to 0$) is known as **unit impulse function or Dirac-delta function** and is denoted by $\delta(t-a)$.

$$\therefore\ \delta(t-a)=\lim_{\varepsilon\to 0}F(t)$$

When $a=0$, the unit function is $\boxed{\delta(t)=\lim_{\varepsilon\to 0}F(t)}$

Definition : The function $F(t)$ defined by

$$F(t)=\begin{cases}0, & t<a\\ 1/\varepsilon, & a<t<a+\varepsilon\\ 0, & t>a+\varepsilon\end{cases}$$

where $\int_0^\infty F(t)\,dt=1$ for all ε is called **Dirac-delta function.**

Paul Dirac (1902 - 1984)

Dirac first studied electrical engineering and graduated with first class honours Bachelor of Science degree in engineering. Although he was offered a scholarship for further studies in engineering and as it was not sufficient to live and study, he hook up an offer to study for a Bachelor of Arts degree in mathematics at the University of Bristol free of charge. He completed his Ph.D. in 1926 with the thesis on quantum mechanics from University of Cambridge. His doctoral advisers were Homi Bhabha and Hansh Chandra Mehta among others and later on Fred Hoyle was one of his doctoral students. Dirac shared the 1933 Nobel Prize for Physics with Erwin Schrodinger. After his death in 1984 the Institute of Physics, the United Kingdom's professional body for physicists established the Pual Dirac Medal in his memory for "outstanding contribution to theoretical physics". Stephen Hawking (1987) was the first recipient of this medal.

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Geometrical Interpretation of Dirac's δ function

As we can see from the above figure that as the height of the rectangle increases and the width of the rectangle decreases in such a way that the area of the rectangle remains unity, we get the Dirac's delta function.

9. Laplace Transform of Dirac-delta Function $L[\delta(t-a)]$

By definition of Laplace transform,

$$L[F(t)] = \int_0^\infty e^{-st} F(t)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st}dt = \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_a^{a+\varepsilon}$$

$$= -\frac{1}{\varepsilon s}\left[e^{-s(a+\varepsilon)} - e^{-as}\right] = \frac{1}{s}\cdot e^{-as}\left[\frac{1-e^{-\varepsilon s}}{\varepsilon}\right]$$

$$\therefore\ L[\delta(t-a)] = \lim_{\varepsilon\to 0} L[F(t)] = \frac{1}{s}e^{-as}\lim_{\varepsilon\to 0}\left(\frac{1-e^{-\varepsilon s}}{\varepsilon}\right)$$

$$= \frac{1}{s}e^{-as}\lim_{\varepsilon\to 0}\frac{-e^{-\varepsilon s}(-s)}{1} \qquad \text{[By L'Hospital's Rule]}$$

$$= \frac{1}{s}e^{-as}\cdot s = e^{-as}$$

$$\therefore\ \boxed{L[\delta(t-a)] = e^{-as}} \qquad \text{..... (13)}$$

Cor. : Putting $a = 0$, $\boxed{L[\delta(t)] = 1}$ ..... (14)

10. Inverse Laplace Transform

From the above results, we get,

$$\boxed{L^{-1}(e^{-as}) = \delta(t-a) \text{ and } L^{-1}(1) = \delta(t)} \qquad \text{..... (15)}$$

Relation between Heaviside's Unit Step Function and Dirac's Delta Function

The function $F(t)$ can be expressed in terms of unit step function as follows.

$$F(t) = \frac{1}{\varepsilon}[H(t-a) - H(t-a-\varepsilon)]$$

$$\therefore\ \lim_{\varepsilon\to 0} F(t) = \lim_{\varepsilon\to 0}\frac{H(t-a) - H(t-a-\varepsilon)}{\varepsilon}$$

Now, the left hand side is Dirac's delta function $\delta(t-a)$ and the right hand side is the derivative of Heaviside's unit step function $H(t-a)$

$$\therefore\ \boxed{\delta(t-a) = H'(t-a)} \qquad \text{..... (16)}$$

11. Laplace Transform of $f(t)\,\delta(t-a)$ (M.U. 2011)

By definition, $L\,f(t)\delta(t-a) = \lim_{\varepsilon\to 0}\int_0^\infty e^{-st} f(t)\cdot F(t)\,dt$

Now, $\int_0^\infty e^{-st} f(t)\cdot F(t)\,dt = \frac{1}{\varepsilon}\int_a^{a+\varepsilon} e^{-st} f(t)\cdot 1\cdot dt$

$$= \frac{1}{\varepsilon}\cdot\varepsilon\cdot e^{-s(a+\theta\varepsilon)}\cdot f(a+\theta\varepsilon) \qquad \text{..... (A)}$$

[By mean value theorem for integrals,

$$\int_0^{a+h} f(x)\,dx = h\,f(a+\theta h),\ 0<\theta<1.]$$

Taking the limit as $\varepsilon \to 0$, from (A), we get,

$$\boxed{L\,f(t)\,\delta(t-a) = e^{-as} f(a)} \qquad \text{..... (17)}$$

Taking inverse Laplace transform,

$$\boxed{L^{-1}[e^{-as} f(a)] = f(t)\cdot\delta(t-a)} \qquad \text{..... (18)}$$

Example 1 : Find $L^{-1}\left[\frac{s}{s+1}\right]$. (M.U. 2005)

Sol. : We have

$$L^{-1}\left[\frac{s}{s+1}\right] = L^{-1}\left[\frac{(s+1)-1}{s+1}\right] = L^{-1}(1) - L^{-1}\left[\frac{1}{s+1}\right]$$

$$= \delta t - e^{-t}.$$

Example 2 : Find the inverse Laplace transform of

(i) $\frac{s^2+6s+6}{s^2+5s+6}$ (ii) $\frac{s^2+4s+2}{s^2+3s+2}$

Sol. : (i) Since the degree of the numerator is equal to the degree of the denominator, we first divide the numerator by the denominator.

$$\therefore\ s^2+5s+6 \overline{\big)\ s^2+6s+6\ \big(}\ 1$$
$$\underline{-\,s^2 \pm 5s \pm 6}$$
$$s$$

$$\therefore\ \Phi(s) = 1 + \frac{s}{s^2+5s+6} = 1 + \frac{s}{(s+2)(s+3)}$$

$$= 1 - \frac{2}{s+2} + \frac{3}{s+3} \qquad \text{[By partial fraction]}$$

$$\therefore\ L^{-1}[\Phi(s)] = L^{-1}(1) - 2L^{-1}\left(\frac{1}{s+2}\right) + 3L^{-1}\left(\frac{1}{s+3}\right)$$

$$= \delta(t) - 2e^{-2t} + 3e^{-3t}$$

(ii) Since the degree of the numerator is equal to the degree of the denominator, we first divide the numerator by the denominator.

$$\therefore\ s^2+3s+2 \overline{\big)\ s^2+4s+2\ \big(}\ 1$$
$$\underline{-\,s^2 \pm 3s \pm 2}$$
$$s$$

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$$\therefore \quad \Phi(s) = 1 + \frac{s}{s^2+3s+2} = 1 + \frac{1}{s+1} - \frac{1}{s+2} \qquad \text{[By partial fraction]}$$

$$\therefore \quad L^{-1}[\Phi(s)] = L^{-1}(1) + L^{-1}\left(\frac{1}{s+1}\right) - L^{-1}\left(\frac{1}{s+2}\right)$$

$$= \delta(t) + e^{-t} - e^{-2t}$$

Example 3 : Find $L(\sin 2t) \cdot \delta(t-2)$.

Sol. : (i) Here $f(t) = \sin 2t$ and $a = 2$.

(ii) $L[f(t)\,\delta(t-a)] = e^{-as} f(a)$

$$L[\sin 2t \cdot \delta(t-2)] = e^{-2s} \sin 2(2) = e^{-2s} \sin 4.$$

Example 4 : Find $L^{-1}[e^{-2s} \sin 2]$.

Sol. : We know that $L[e^{-as} f(a)] = f(t)\,\delta(t-a)$ [By (18), page S-25]

Here, $a = 2$, $f(a) = \sin 2$.

$$\therefore \quad L^{-1}[e^{-2a} \sin 2] = \sin t \cdot \delta(t-2).$$

Example 5 : Find Laplace transform of $t\,H(t-4) + t^2\,\delta(t-4)$. **(M.U. 2012)**

Sol. : (a) See procedure given on page S-9.

(i) Here $f(t) = t$ and $a = 4$. $\therefore$ $f(t+4) = t+4$

(ii) $L[f(t+4)] = L(t+4) = L(t) + 4\,L(1) = \frac{1}{s^2} + 4 \cdot \frac{1}{s}$

(iii) $L[t\,H(t-4)] = e^{-4s}\left[\frac{1}{s^2} + \frac{4}{s}\right]$

(b) (i) Here $f(t) = t^2$ and $a = 4$.

(ii) $L[f(t)\,\delta(t-a)] = e^{-as} f(a) = e^{-4s}(4)^2 = 16\,e^{-as}$ [By (17), page S-25]

Hence, $L[t\,H(t-4) + t^2\,d(t-4)] = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right) + 16\,e^{-4s}$

$$= \frac{e^{-4s}}{s^2}[1 + 4s + 16s^2]$$

Example 6 : Find $L[t^2 H(t-2) - \cos h\,t\,\delta(t-4)]$. **(M.U. 2011)**

Sol. : (a) (i) Here $f(t) = t^2$ and $a = 2$. $\therefore$ $f(t+2) = (t+2)^2 = t^2 + 4t + 4$

(ii) $L[f(t+2)] = L[t^2 + 4t + 4] = \frac{2}{s^3} + 4 \cdot \frac{1}{s^2} + 4 \cdot \frac{1}{s}$

(iii) $L[t^2 H(t-2)] = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right]$

(b) Here, $f(t) = \cos h\,t$ and $a = 4$.

$$\therefore \quad L[f(t)\,\delta(t-a)] = e^{-as} f(a) = e^{-4s} \cos h\,4$$

Hence, $L[t^2 H(t-2) - \cos h\,t\,\delta(t-4)] = e^{-2s}\left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s}\right] - e^{-4s} \cos h\,4.$

12. Evaluation of the Integral $\int_0^\infty e^{-at} f(t)\,\delta(t-a)\,dt$

Example 1 : Evaluate $\int_0^\infty e^{-5t}\,\delta(t-3)\,dt$.

Sol. : We know that by (17), page S-25

$$L[f(t) \cdot \delta(t-a)] = e^{-as} f(a)$$

This means, $\int_0^\infty e^{-st} \cdot f(t)\,\delta(t-a)\,dt = e^{-as} f(a)$

Comparing this with the given integral, we find that $s = 5$, $f(t) = 1$, $a = 3$ and $f(a) = 1$.

$$\therefore \quad \int_0^\infty e^{-5t}\,\delta(t-3)\,dt = e^{-15} \cdot 1 = e^{-15}.$$

Example 2 : Evaluate $\int_0^\infty \sin 3t\,\delta\left(t - \frac{\pi}{6}\right)dt$.

Sol. : We know that by (17), page S-25

$$L[f(t) \cdot \delta(t-a)] = e^{-as} f(a)$$

This means, $\int_0^\infty e^{-st} f(t) \cdot \delta(t-a)\,dt = e^{-as} f(a)$

Comparing this with the given integral, we find that

$$s = 0,\ f(t) = \sin 3t,\ a = \frac{\pi}{6},\ f(a) = \sin 3\left(\frac{\pi}{6}\right) = \sin\left(\frac{\pi}{2}\right) = 1.$$

$$\therefore \quad \int_0^\infty \sin 3t \cdot \delta\left(t - \frac{\pi}{6}\right)dt = e^0 \cdot 1 = 1.$$

EXERCISE - V

(1) Find Laplace transform of

1. $\sin t \cdot \delta\left(t - \frac{\pi}{2}\right) - t^2\,\delta(t-2)$
2. $t^4\,H(t-2) + t^2\,\delta(t-2)$.

[**Ans. : (1)** $e^{-\pi s/2} - 4e^{-2s}$, **(2)** $e^{-2s}\left[4 + \frac{16}{s} + \frac{32}{s^2} + \frac{48}{s^3} + \frac{48}{s^4} + \frac{42}{s^5}\right]$.]

(2) Evaluate the following integrals.

1. $\int_0^\infty e^{-3t}\,\delta(t-4)\,dt$
2. $\int_0^\infty \sin 4t \cdot \delta\left(t - \frac{\pi}{8}\right)dt$
3. $\int_0^\infty e^{-2t} \cdot t^2\,\delta(t-1)\,dt$

[**Ans. : (1)** e^{-12}, **(2)** 1, **(3)** $\frac{1}{e^2}$]

(3) Find inverse Laplace transform of

1. $\frac{s}{s+a}$,
2. $e^{-as} \sin a$,
3. $e^{-3s} \cos 3$.

[**Ans. : (1)** $\delta(t) - e^{-at}$, **(2)** $\sin t \cdot \delta(t-a)$, **(3)** $\cos t \cdot \delta(t-3)$]

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EXERCISE - VI

Theory

1. If $f(t)$ is a periodic function of period a, prove that

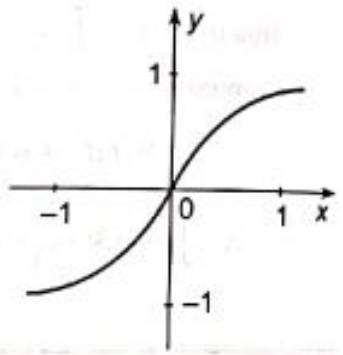
$$L[f(t)] = \frac{1}{1-e^{-as}}\int_0^a e^{-st} f(t)\,dt.$$ **(M.U. 1996, 97, 2003, 07, 08)**

2. Define Heaviside's unit step function $H(t-a)$ and obtain its Laplace transform. **(M.U. 2005)**
3. Define Heaviside unit step function and obtain Laplace transform of $f(t-a)\cdot H(t-a)$.
4. Define Dirac-delta function and obtain its Laplace transform. **(M.U. 2004)**

ERROR FUNCTION

1. Introduction

We shall now study a special type of function known as **"Error Function"**. It is also known as **"Error Integral"**. It has applications in Probability Theory, Thermodynamics and Heat Conduction problems.



2. Definition

The function of x defined by the integral

$$\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$$

is called **error function** of x and is denoted by $erf(x)$. Thus,

$$\boxed{erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt} \quad \text{..... (1)}$$

The function of x defined by the integral

$$\frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

is called **complementary error function** and is denoted by $erf_c(x)$.

Thus,

$$\boxed{erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt} \quad \text{..... (2)}$$

3. Properties of Error Functions

(a) The value of the error function at $x = 0$ is 0

i.e. $\boxed{erf(0) = 0}$ ..... (3)

Proof : Putting $x = 0$ in (1) $\quad erf(0) = \frac{2}{\sqrt{\pi}}\int_0^0 e^{-t^2}dt = 0.$

(b) The value of the error function at infinity is unity

i.e. $\boxed{erf(\infty) = 1}$

Proof : From (1) again $\quad erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1.$ ..... (4)

[The result $\int_0^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}$ is proved in Engg. Mathematics - II in Gama Function.]

(c) $erf(x) + erf_c(x) = 1$

Proof : From the definitions (1) and (2) ..... (5)

$$erf(x) + erf_c(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt + \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

$$= \frac{2}{\sqrt{\pi}}\left[\int_0^x e^{-t^2}dt + \int_x^\infty e^{-t^2}dt\right] = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt$$

$$= \frac{2}{\sqrt{\pi}}\cdot\frac{\sqrt{\pi}}{2} = 1$$

(d) $erf(x)$ is an odd function *i.e.* $erf(-x) = -erf(x)$ ..... (6)

Proof : Changing x to $-x$ in (1), $\quad erf(-x) = \frac{2}{\sqrt{\pi}}\int_0^{-x} e^{-t^2}dt$

Now, put $t = -u \quad \therefore \quad dt = -du$

When $t = 0$, $u = 0$ and when $t = -x$, $u = x$

$$\therefore \quad erf(-x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}(-du) = -\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du = -erf(x)$$

(e) $erf_c(-x) = 1 + erf(x)$ ..... (7)

Proof : From (5), $\quad erf_c(x) = 1 - erf(x)$

Changing x to $-x$, $\quad erf_c(-x) = 1 - erf(-x)$

$= 1 + erf(x)$ [By (6)]

(f) $erf_c(-x) + erf_c(x) = 2$ ..... (8)

Proof : From (5), $\quad erf_c(x) = 1 - erf(x)$

Changing x to $-x$, $\quad erf_c(-x) = 1 - erf(-x)$

$= 1 + erf(x)$ [By (6)]

$= 1 + [1 - erf_c(x)]$ [By (5) again]

$\therefore \quad erf_c(-x) + erf_c(x) = 2$

4. Expression of erf (x) in Series

We have $\quad erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ **(S.U. 2015, 16)**

$$= \frac{2}{\sqrt{\pi}}\int_0^x\left[1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \ldots\right]dt$$

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$$= \frac{2}{\sqrt{\pi}}\left[t - \frac{t^3}{3 \cdot 1!} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \ldots\right]_0^x$$

$$erf\,(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + \ldots\right] \qquad \ldots\ldots\ldots\ldots (1)$$

Further changing x to $\sqrt{x}$, we get,

$$erf\,(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}} e^{-t^2}\,dt$$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{x}}\left[1 - \frac{t^2}{1!} + \frac{t^4}{2!} - \frac{t^6}{3!} + \ldots\right]dt$$

$$= \frac{2}{\sqrt{\pi}}\left[t - \frac{t^3}{3 \cdot 1!} + \frac{t^5}{5 \cdot 2!} - \frac{t^7}{7 \cdot 3!} + \ldots\right]_0^{\sqrt{x}}$$

$$erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\left[x^{1/2} - \frac{x^{3/2}}{3 \cdot 1!} + \frac{x^{5/2}}{5 \cdot 2!} - \frac{x^{7/2}}{7 \cdot 3!} + \ldots\right].$$

Example 1 : Compute $erf\,(0{\cdot}3)$ correct to three decimal places.

Sol. : We have $\quad erf(x) = \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3 \cdot 1!} + \frac{x^5}{5 \cdot 2!} \ldots\ldots\right]$

Putting $x = 0.3$

$$erf\,(0{\cdot}3) = \frac{2}{\sqrt{\pi}}\left[0{\cdot}3 - \frac{(0{\cdot}3)^3}{3 \cdot 1!} + \frac{(0{\cdot}3)^5}{5 \cdot 2!} \ldots\ldots\right] = 0{\cdot}3248.$$

Example 2 : Compute $erf\,(0{\cdot}5)$ correct to four decimal places.

Sol. : As above

$$erf\,(0{\cdot}5) = \frac{2}{\sqrt{\pi}}\left[0{\cdot}5 - \frac{(0{\cdot}5)^3}{3 \cdot 1!} + \frac{(0{\cdot}5)^5}{5 \cdot 2!} \ldots\ldots\right] = 0{\cdot}5204.$$

Example 3 : Prove that $erf(x) = \alpha\,(x\sqrt{2})$ where $\alpha\,(x)$ is a function of x given by

$$\alpha\,(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-t^2/2}\,dt.$$

Sol. : Changing x to $x\sqrt{2}$ in the given function $\alpha\,(x)$, we get

$$\alpha\,(x\sqrt{2}) = \sqrt{\frac{2}{\pi}}\int_0^{x\sqrt{2}} e^{-t^2/2}\,dt$$

Put $\frac{t^2}{2} = u^2,\ \ t = \sqrt{2} \cdot u \quad \therefore\ dt = \sqrt{2}\,du$

When $t = 0$, $u = 0$; when $t = x\sqrt{2}$, $u = x$.

$$\therefore\ \alpha\,(x\sqrt{2}) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-u^2}\sqrt{2}\,du = \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du = erf(x).$$

Example 4 : Show that $\int_a^b e^{-x^2}dt = \frac{\sqrt{\pi}}{2}\left[erf\,(b) - erf\,(a)\right]$.

Sol. : By definition, $\quad erf(b) = \frac{2}{\sqrt{\pi}}\int_0^b e^{-t^2}dt\ ; \quad erf(a) = \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}\,dt$

We can write, assuming $a < b$,

$$erf(b) = \frac{2}{\sqrt{\pi}}\left[\int_0^a e^{-t^2}dt + \int_a^b e^{-t^2}\,dt\right]$$

$$\therefore\ erf(b) - erf(a) = \frac{2}{\sqrt{\pi}}\left[\int_0^a e^{-t^2}\,dt + \int_a^b e^{-t^2}\,dt - \int_0^a e^{-t^2}dt\right]$$

$$= \frac{2}{\sqrt{\pi}}\int_a^b e^{-t^2}dt$$

$$\therefore\ \int_a^b e^{-t^2}dt = \frac{\sqrt{\pi}}{2}\left[erf\,(b) - erf\,(a)\right]$$

EXERCISE - I

1. Define $erf\,(x)$ and prove that $erf\,(\infty) = 1$.
2. Prove that $erf\,(-x) = -\,erf\,(x)$.
3. Prove that $erf_c\,(-x) + erf_c\,(x) = 2$.
4. Show that $erf\,(0) = 0$ and $erf\,(\infty) = 1$.
5. Show that $erf_c\,(-x) = 1 + erf\,(x)$.
6. Show that $erf(\sqrt{x}) = \frac{2}{\sqrt{\pi}}\left[x^{1/2} - \frac{x^{3/2}}{3 \cdot 1!} + \frac{x^{5/2}}{5 \cdot 2!} - \frac{x^{7/2}}{7 \cdot 3!} + \ldots\right]$
7. Express $erf\,(x)$ as an infinite series.
8. Express $erf\,(x)$ as an infinite series and hence find the value of $erf\,(0{\cdot}3)$.

5. Differentiation of Error Function

Example 1 : Prove that $\frac{d}{dx}[erf\,(x)] = \frac{2}{\sqrt{\pi}}e^{-x^2}$ and hence show that

$$\frac{d}{dx}[erf\,(ax^n)] = \frac{2an}{\sqrt{\pi}}x^{n-1} \cdot e^{-a^2x^{2n}}.$$

Sol. : We have, by definition, $\quad erf\,(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$

Applying the Leibnitz's rule of differentiation under integral sign with variable limits, we get,

$$\frac{d}{dx}[erf\,(x)] = \frac{2}{\sqrt{\pi}}\left[\int_0^x \frac{\partial}{\partial x}(e^{-t^2})\,dt + \frac{d}{dx}(x) \cdot e^{-x^2} - \frac{d}{dx}(0) \cdot e^{-0}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + e^{-x^2} - 0\right] = \frac{2}{\sqrt{\pi}}e^{-x^2}.$$

Now, $erf\,(ax^n) = \frac{2}{\sqrt{\pi}}\int_0^{ax^n} e^{-t^2}dt$. Applying the Leibnitz's rule,

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$$\frac{d}{dx}[erf(ax^n)] = \frac{2}{\sqrt{\pi}}\left[\int_0^{ax^n}\frac{\partial}{\partial x}(e^{-t^2})dt + \frac{d}{dx}(ax^n)\cdot e^{-a^2x^{2n}} - \frac{d}{dx}(0)\cdot e^{-0}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + an\,x^{n-1}e^{-a^2x^{2n}} - 0\right]$$

$$= \frac{2an}{\sqrt{\pi}}\cdot x^{n-1}e^{-a^2x^{2n}}.$$

Example 2 : Prove that $\frac{d}{dx}[erf_c(ax)] = -\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$.

Sol. : We have, by definition,

$$erf_c(x) = \frac{2}{\sqrt{\pi}}\int_x^{\infty} e^{-t^2}dt$$

$$\therefore\quad erf_c(ax) = \frac{2}{\sqrt{\pi}}\int_{ax}^{\infty} e^{-t^2}dt$$

Applying the Leibnitz's rule of differentiation under the integral sign with variable limits, we get,

$$\frac{d}{dx}[erf_c(ax)] = \frac{2}{\sqrt{\pi}}\left[\int_{ax}^{\infty}\frac{\partial}{\partial x}(e^{-t^2})dt + \frac{d}{dx}(\infty)\cdot e^{-\infty} - \frac{d}{dx}(ax)\cdot e^{-a^2x^2}\right]$$

$$= \frac{2}{\sqrt{\pi}}\left[0 + 0 - a\cdot e^{-a^2x^2}\right] = -\frac{2a}{\sqrt{\pi}}e^{-a^2x^2}.$$

Example 3 : Prove that $\frac{1}{x}\frac{d}{da}erf_c(ax) = -\frac{1}{a}\frac{d}{dx}erf(ax)$.

Sol. : We have, by definition $\quad erf_c(ax) = \frac{2}{\sqrt{\pi}}\int_{ax}^{\infty} e^{-t^2}dt$ ..... (1)

Applying Leibnitz's rule, we get,

$$\frac{d}{da}[erf_c(ax)] = \frac{2}{\sqrt{\pi}}\left[\int_{ax}^{\infty}\frac{\partial}{\partial a}(e^{-t^2})dt + \frac{d}{da}(\infty)e^{-\infty} - \frac{d}{da}(ax)\cdot e^{-a^2x^2}\right]$$

$$= \frac{2}{\sqrt{\pi}}[0 + 0 - x\cdot e^{-a^2x^2}] = -\frac{2x}{\sqrt{\pi}}\cdot e^{-a^2x^2} \quad \text{..... (2)}$$

Again by definition, $\quad erf(ax) = \frac{2}{\sqrt{\pi}}\int_0^{ax} e^{-t^2}dt$

Applying Leibnitz's rule,

$$\frac{d}{dx}erf(ax) = \frac{2}{\sqrt{\pi}}\left[\int_0^{ax}\frac{\partial}{\partial x}(e^{-t^2})dt + \frac{d}{dx}(ax)e^{-a^2x^2} - \frac{d}{dx}(0)e^{-0}\right]$$

$$= \frac{2}{\sqrt{\pi}}[0 + a\,e^{-a^2x^2} - 0] = \frac{2a}{\sqrt{\pi}}e^{-a^2x^2} \quad \text{..... (3)}$$

From (2) and (3), we get,

$$\frac{1}{x}\frac{d}{da}erf_c(ax) = -\frac{1}{a}erf(ax).$$

6. Integration of Error Function

Example 1 : Prove that $\int_0^x erf(at)dt = x\,erf(ax) + \frac{1}{a\sqrt{\pi}}e^{-a^2x^2} - \frac{1}{a\sqrt{\pi}}$. **(S.U. 2014)**

Sol. : We integrate the l.h.s. by parts,

$$\int_0^x erf(at)\cdot 1\cdot dt = \left[erf(at)\cdot t\right]_0^x - \int_0^x t\cdot\frac{d}{dt}erf(at)\cdot dt$$

$$= x\,erf(ax) - \int_0^x t\cdot\frac{2a}{\sqrt{\pi}}e^{-a^2t^2}\cdot dt$$

$$\left[\because \frac{d}{dt}erf(at) = \frac{2a\,e^{-a^2t^2}}{\sqrt{\pi}}\right.$$ by Ex. 1, page S-31, putting $n = 1$ and $x = t$]

$$= x\cdot erf(ax) + \frac{1}{a\sqrt{\pi}}\int_0^x e^{-a^2t^2}(-2a^2t)dt$$

$$= x\cdot erf(ax) + \frac{1}{a\sqrt{\pi}}\left[e^{-a^2t^2}\right]_0^x$$

$$= x\cdot erf(ax) + \frac{1}{a\sqrt{\pi}}e^{-a^2x^2} - \frac{1}{a\sqrt{\pi}}.$$

Example 2 : Prove that $\int_0^x erf_c(at)dt = x\cdot erf_c(ax) - \frac{1}{a\sqrt{\pi}}e^{-a^2x^2} + \frac{1}{a\sqrt{\pi}}$.

Sol. : As in the above example we integrate l.h.s. by parts.

$$\int_0^x erf_c(at)\cdot 1\cdot dt = \left[erf_c(at)\cdot t\right]_0^x - \int_0^x t\cdot\frac{d}{dt}erf_c(at)\cdot dt$$

[But $\frac{d}{dt}erf_c(at) = -\frac{2a}{\sqrt{\pi}}e^{-a^2t^2}$ by Ex. 2, page S-32]

$$= x\cdot erf_c(ax) - \int_0^x t\left(-\frac{2a}{\sqrt{\pi}}\right)e^{-a^2t^2}dt$$

$$= x\cdot erf_c(ax) - \frac{1}{a\sqrt{\pi}}\int_0^x e^{-a^2t^2}(-2a^2t)dt$$

$$= x\cdot erf_c(ax) - \frac{1}{a\sqrt{\pi}}\left[e^{-a^2t^2}\right]_0^x$$

$$= x\cdot erf_c(ax) - \frac{1}{a\sqrt{\pi}}e^{-a^2x^2} + \frac{1}{a\sqrt{\pi}}.$$

Example 3 : Show that $\int_a^b e^{-t^2}dt = \frac{\sqrt{\pi}}{2}[erf(b) - erf(a)]$.

Sol. : By definition, $\quad erf(b) = \frac{2}{\sqrt{\pi}}\int_0^b e^{-t^2}dt\ ;\quad erf(a) = \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}dt$

We can write, assuming $a < b$,

$$erf(b) = \frac{2}{\sqrt{\pi}}\left[\int_0^a e^{-t^2}dt + \int_a^b e^{-t^2}dt\right]$$

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$$\therefore \quad erf(b) - erf(a) = \frac{2}{\sqrt{\pi}}\left[\int_0^a e^{-t^2}dt + \int_a^b e^{-t^2}dt - \int_0^a e^{-t^2}dt\right] = \frac{2}{\sqrt{\pi}}\int_a^b e^{-t^2}dt$$

$$\therefore \quad \int_a^b e^{-t^2}dt = \frac{\sqrt{\pi}}{2}[erf(b) - erf(a)].$$

Example 4 : Prove that $\int_0^\infty e^{-(x+a)^2}dx = \frac{\sqrt{\pi}}{2}[1 - erf(a)]$.

Sol. : Putting $t = x + a$, we have when $x = 0$, $t = a$ and when $x = \infty$, $t = \infty$

$$\therefore \int_0^\infty e^{-(x+a)^2}dx = \int_a^\infty e^{-t^2}dt = \frac{\sqrt{\pi}}{2}erf_c(a) = \frac{\sqrt{\pi}}{2}[1 - erf(a)].$$

Example 5 : Prove that $\int_0^\infty e^{-x^2-2ax} \cdot dx = \frac{\sqrt{\pi}}{2} \cdot e^{x^2}[1 - erf(a)]$.

Sol. : We have

$$\int_0^\infty e^{-x^2-2ax} \cdot dx = \int_0^\infty e^{-x^2-2ax-a^2} \cdot e^{a^2}\, dx$$

$$= e^{a^2}\int_0^\infty e^{-(x+a)^2}dx$$

Putting $x + a = t$ as before

$$\int_0^\infty e^{-x^2-2ax}dx = e^{a^2} \cdot \int_a^\infty e^{-t^2}dt = e^{a^2} \cdot \frac{\sqrt{\pi}}{2}erf_c(a)$$

$$= e^{a^2}\frac{\sqrt{\pi}}{2}[1 - erf(a)].$$

Example 6 : Prove that $\int_0^t erf(ax)dx + \int_0^t erf_c(ax)dx = t$.

Sol. : We have

$$\int_0^t erf(ax)dx + \int_0^t erf_c(ax)dx = \int_0^t [erf(ax) + erf_c(ax)]dx$$

$$= \int_0^t 1 \cdot dt \qquad \text{[By (5) § 3, page S-29]}$$

$$= [t]_0^t = t$$

Example 7 : Prove that $erf_c\left(\frac{s}{2}\right) = \sum_0^\infty \frac{(-1)^r s^r}{r!\,\Gamma(1-(r/2))}$.

Sol. : By the result obtained (1) in § 4, page S-30,

$$erf\left(\frac{s}{2}\right) = \frac{2}{\sqrt{\pi}}\left[\frac{s}{2} - \frac{s^3}{3 \cdot 1!\,2^3} + \frac{s^5}{5 \cdot 2!\,2^5} \cdots\right]$$

$$\therefore \quad erf_c\left(\frac{s}{2}\right) = 1 - erf\left(\frac{s}{2}\right) = 1 - \frac{2}{\sqrt{\pi}}\left[\frac{s}{2} - \frac{s^3}{3 \cdot 1!\,2^3} + \frac{s^5}{5 \cdot 2!\,2^5} \cdots\right]$$

But $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$\therefore \quad erf_c\left(\frac{s}{2}\right) = 1 - \frac{s}{\Gamma(1/2)} + \frac{s^3}{3 \cdot 1!\,2^2\,\Gamma(1/2)} - \frac{s^5}{5 \cdot 2!\,2^4\,\Gamma(1/2)} + \cdots \quad \text{..... (1)}$$

We know that $\Gamma(n+1) = n\Gamma(n)$ *i.e.* $\Gamma(n) = (n-1)\Gamma(n-1)$

$$\therefore \quad \Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\Gamma\left(-\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\Gamma\left(1-\frac{3}{2}\right)$$

Similarly, $\Gamma\left(-\frac{1}{2}\right) = \left(-\frac{3}{2}\right)\Gamma\left(-\frac{3}{2}\right) = \left(-\frac{3}{2}\right)\Gamma\left(1-\frac{5}{2}\right)$

$$\therefore \quad \Gamma\left(\frac{1}{2}\right) = \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\Gamma\left(1-\frac{5}{2}\right)$$

Putting these values successively on the r.h.s. of (1)

$$erf_c\left(\frac{s}{2}\right) = 1 - \frac{s}{\Gamma\left(1-\frac{1}{2}\right)} + \frac{s^3}{3 \cdot 1 \cdot 2^2\left(-\frac{1}{2}\right)\Gamma\left(1-\frac{3}{2}\right)} - \frac{s^5}{5 \cdot 2!\,2^4\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\Gamma\left(1-\frac{5}{2}\right)} + \cdots$$

$$= 1 - \frac{s}{\Gamma\left(1-\frac{1}{2}\right)} - \frac{s^3}{3!\,\Gamma\left(1-\frac{3}{2}\right)} - \frac{s^5}{5!\,\Gamma\left(1-\frac{5}{2}\right)} \cdots$$

$$= \sum \frac{(1)^r}{r!\,\Gamma(1-(r/2))}s^r$$

(When r is even $\Gamma\left(1-\frac{r}{2}\right)$ is infinite and hence the coefficient of s^r is zero).

EXERCISE - II

1. Prove that $\frac{d}{dx}erf(ax) = \frac{2q}{\sqrt{\pi}}e^{-a^2x^2}$.
2. Prove that $\frac{d}{dx}(erf\sqrt{x}) = \frac{e^{-x}}{\sqrt{\pi x}}$.
3. Prove that $\int_0^x erf(at)dt + \int_0^x erf_c(at)dt = x$.
   (**Hint :** See solved Ex. 1 and 2 of § 6, page S-33.)
4. Show that $\frac{d}{dx}(erf\,ax) = \frac{2a}{\sqrt{\pi}}e^{-a^2x^2}$.

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BESSEL'S FUNCTIONS

1. Introduction

We have already studied two special functions *viz.* Gamma Functions and Beta Functions last year in Applied Mathematics II. You might remember that they were defined as definite integrals. In this chapter we shall study one more special function viz. Bessel function which is defined as a solution of a particular differential equation. You will come across Bessel's equations in many engineering problems such as heat conduction, vibrations of circular membrane etc.

Definition : The second order differential equation

$$\boxed{x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0} \qquad \text{..... (1)}$$

where n is a real number is called **Bessel's equation of order n** and its particular solutions are called **Bessel's functions of order n**.

Friedrich Wilhelm Bessel (1784-1846)

Friedrich Wilhelm Bessel was a German Mathematician. At the age of 14, he left school to become an apprentice in a commercial import-export firm; he had no formal education after this. Here he developed interest in astronomy and mathematics. In 1804, he published his first research paper on Halley's Comet. In 1809, at the age of 26 Bessel was appointed director of Frederick William III Observatory and Professor of Astronomy. He was elected to Berlin academy in 1912. He was the first astronomer to predict the existence of 'dark stars'. The Royal Astronomical Society awarded him the gold medal. We know him for Bessel Functions.

2. Solution of Bessel's Equation

The above Bessel's equation cannot be solved by any of the methods that we have studied before. The solution is not in the usual form also. But it is given in the form of an infinite convergent series. We shall assume the following results.

The solution of the Bessel's equation of order n

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \qquad \text{..... (1)}$$

is the infinite series

$$\boxed{y=\sum_{m=0}^{\infty}a_m x^{m+r}} \qquad \text{..... (2)}$$

where a_m are constants to be determined.

(Since n appears in the equation (1), we use the index m instead of n.)

Now, differentiating (2), w.r.t. x,

$$\frac{dy}{dx}=\sum_{m=0}^{\infty}a_m(m+r)x^{m+r-1}$$

and $$\frac{d^2y}{dx^2}=\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r-2}$$

Since (2) is a solution of (1), putting these values of y', y'' and y from (2) in (1), we get

$$\sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r}+\sum_{m=0}^{\infty}a_m(m+r)x^{m+r}$$
$$+\sum_{m=0}^{\infty}a_m x^{m+r+2}-n^2\sum_{m=0}^{\infty}a_m x^{m+r}=0 \qquad \text{..... (3)}$$

Since, (3) is an identity, the **coefficients of all powers of x** must be equal to zero.

To make the powers of x equal, we replace m by $m-2$ in the third term of (3) and let m vary from 2 to ∞ instead of 0 to ∞.

$$\therefore \sum_{m=0}^{\infty}a_m(m+r)(m+r-1)x^{m+r}+\sum_{m=0}^{\infty}a_m(m+r)x^{m+r}$$
$$+\sum_{m=2}^{\infty}a_{m-2}x^{m+r}-n^2\sum_{m=0}^{\infty}a_m x^{m+r}=0 \qquad \text{..... (4)}$$

$$\therefore \sum_{m=0}^{\infty}a_m[(m+r)^2-(m+r)+(m+r)-n^2]x^{m+r}+\sum_{m=2}^{\infty}a_{m-2}x^{m+r}=0$$

$$\therefore \sum_{m=0}^{\infty}a_m[(m+r)^2-n^2]x^{m+r}+\sum_{m=2}^{\infty}a_{m-2}x^{m+r}=0$$

Separating the terms $m=0$, $m=1$ from the first term of the above equation, we get,

$$a_0(r^2-n^2)x^r+a_1[(r+1)^2-n^2]x^{r+1}$$
$$+\sum_{m=2}^{\infty}a_m[(m+r)^2-n^2]x^{m+r}+\sum_{m=2}^{\infty}a_{m-2}x^{m+r}=0$$

$$\therefore a_0(r^2-n^2)x^r+a_1[(r+1)^2-n^2]x^{r+1}$$
$$+\sum_{m=2}^{\infty}[a_m\{(m+r)^2-n^2\}+a_{m-2}]x^{m+r}=0$$

Equating to zero the coefficients of x^r, x^{r+1}, x^{m+r}, we get

$$a_0(r^2-n^2)=0,\quad a_1[(r+1)^2-n^2]=0$$
$$a_m[(m+r)^2-n^2]+a_{m-2}=0,\quad m\geq 2 \qquad \text{..... (5)}$$

$\therefore$ $a_0=0$ or $r^2-n^2=0$

Since a_0 is arbitrary, we assume $a_0\neq 0$, $\therefore$ $r^2-n^2=0$.

This is called **indicial equation.**

The roots of this equation are $r=n$ or $r=-n$.

Further $a_1[(r+1)^2-n^2]=0$ gives $a_1=0$ or $(r+1)^2-n^2=0$.

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But since $r = \pm n$, $(r+1)^2 - n^2 \neq 0$. Hence, $a_1 = 0$.

Now, we consider two cases $r = n$ and $r = -n$ separately.

Case I : $r = n$

We have from (5)

$$a_m[(m+r)^2 - n^2] + a_{m-2} = 0 \quad \text{for } m \geq 2$$

$$\therefore \quad a_m = -\frac{a_{m-2}}{(m+r)^2 - n^2}, \quad m \geq 2$$

Putting $r = n$,

$$a_m = -\frac{a_{m-2}}{(m+n)^2 - n^2}, \quad m \geq 2$$

$$= -\frac{a_{m-2}}{(m+n-n)(m+n+n)} = -\frac{a_{m-2}}{m(m+2n)}, \quad m \geq 2 \qquad \text{..... (A)}$$

For $m = 2$, $a_2 = -\dfrac{a_0}{2(2+2n)} = -\dfrac{a_0}{2 \cdot 2(1+n)}$ ..... (B)

For $m = 3$, $a_3 = -\dfrac{a_1}{3(3+2n)} = 0 \quad [\because a_1 = 0]$

For $m = 4$, $a_4 = -\dfrac{a_2}{4(4+2n)} = -\dfrac{a_2}{4 \cdot 2(2+n)}$

$$= -\frac{-a_0}{2 \cdot 2(1+n) \times 4 \cdot 2(2+n)} \qquad [\text{By (B)}]$$

$$= \frac{(-1)^2 a_0}{2^2 \cdot 1 \cdot 2 \cdot 2^2 (1+n)(2+n)}$$

For $m = 5$, $a_5 = -\dfrac{a_3}{5(5+2n)} = 0 \quad [\because a_3 = 0]$

In general, $a_{2m} = \dfrac{(-1)^m a_0}{2^m m! \cdot 2^m (1+n)(2+n)\ldots(m+n)}$

$$= \frac{(-1)^m a_0}{2^{2m} m!\,(n+1)(n+2)\ldots(n+m)} \qquad \text{..... (6)}$$

and $a_1 = a_3 = a_5 = a_7 = \ldots = 0$

i.e. $a_{2m+1} = 0 \quad$ for $m \geq 0$ ..... (7)

$\therefore$ The solution of the equation from (1) is given by

$$y = \sum_{m=0}^{\infty} a_m x^{m+r} = \sum_{m=0}^{\infty} a_m x^{m+n} \quad [\because r = n] \qquad \text{..... (8)}$$

Using (6) and (7) (*i.e.* changing m to $2m$ and m to $2m+1$), we get, from (8)

$$y = \sum_{m=0}^{\infty} a_{2m} x^{2m+n} + \sum_{m=0}^{\infty} a_{2m+1} x^{2m+n+1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m a_0}{2^{2m} m!(n+1)(n+2)\ldots(n+m)} x^{2m+n} \quad [\because a_{2m+1} = 0]$$

Since a_0 is arbitrary, we take $\quad a_0 = \dfrac{1}{2^n \lfloor n+1}$

$\therefore$ The solution is ..... (9)

$$y = \sum \frac{(-1)^m x^{2m+n}}{2^{2m} m!\, 2^n \lfloor n+1 (n+1)(n+2)\ldots(n+m)}$$

But $\lfloor n+1 = n \lfloor n = \lfloor n \cdot n$.

$$\therefore \quad \lfloor n+1 (n+1)(n+2)\ldots(n+m) = \lfloor n+2 (n+2)\ldots(n+m)$$

$$= \lfloor n+3 (n+3)\ldots(n+m)$$

$$= \lfloor n+m (n+m) = \lfloor n+m+1$$

$$\therefore \quad y = \sum \frac{(-1)^m x^{2m+n}}{2^{2m+n} \cdot m! \lfloor n+m+1} = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m! \lfloor n+m+1} \qquad \text{..... (10)}$$

This solution (and hence the series) is denoted by $J_n(x)$ and is called **Bessel's Function of the first kind of order n**. Thus,

$$\boxed{J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m! \lfloor n+m+1}} \qquad \text{..... (11)}$$

Case II : $r = -n$

Proceeding as above or by replacing n by $-n$, which is another root of the indicial equation, we get another solution of the equation denoted by $J_{-n}(x)$.

Replacing n by $-n$ in (11), we get

$$\boxed{J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m-n}}{m! \lfloor -n+m+1}} \qquad \text{..... (12)}$$

This is called Bessel's function of the **first kind of order $-n$**.

When n is neither zero, nor an integer positive or negative, **the general solution of the Bessel's differential equation** of order n is

$$\boxed{y = A\,J_n(x) + B\,J_{-n}(x)} \qquad \text{..... (C)}$$

where A and B are arbitrary constants.

Remark ....

The general solution of a differential equation of second order must contain two arbitrary constants.

Particular case when $n = 0$

When $n = 0$, $J_n(x)$ and $J_{-n}(x)$ both become the same function denoted by $J_0(x)$. In this case $y = A\,J_n(x) + B\,J_{-n}(x)$ becomes $y = (A+B)\,J_0(x)$ and is not the general solution of Bessel's equation (1) as it contains only one arbitrary constant.

Now, putting $n = 0$ in (11) [or (12)], we get

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m}}{m! \lfloor m+1} = 1 - \frac{x^2}{2!} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots$$

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In this case, the general solution of the equation (1) can be shown to be

$$y = (A + B\log x)\cdot J_0(x) + B\left[\frac{x^2}{2^2} - \frac{x^4}{2^2\cdot 4^2}\cdot\left(1+\frac{1}{2}\right) + \frac{x^6}{2^2\cdot 4^2\cdot 6^2}\cdot\left(1+\frac{1}{2}+\frac{1}{3}\right)+\ldots\right]$$

where A and B are arbitrary constants and $J_0(x)$ is given as above. (We assume this result.)

Particular case when n is an integer

When n is an integer, the two functions $J_n(x)$ and $J_{-n}(x)$ are not independent but they are related by the relation

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \text{[See Ex. 4, page S-51]}$$

Hence, in this case again

$$y = A\,J_n(x) + B\,J_{-n}(x) = A\,J_n(x) + B\,(-1)^n J_n(x)$$
$$= [\,A + B\,(-1)^n\,]\,J_n(x)$$

and is not the general solution of Bessel's equation (1) as it contains only one arbitrary constant.

In this case the general solution of the equation (1) can be shown to be of the form

$$y = A\,J_n(x) + B\,Y_n(x) \qquad \text{where } Y_n(x) = J_n(x)\int\frac{dx}{x\,J_n^2(2)}.$$

Y_n is called **Bessel's function of the second kind of order n or Neumann function** (Again we shall assume this result).

In what follows we shall be concerned mainly with (C), page S-39. For the sake of brevity we repeat it.

Bessel Function : The solution of the equation $x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2 - n^2)y = 0$ is known as **Bessel's equation** and its solution is given by

$$y = A\,J_n(x) + B\,J_{-n}(x) \qquad \text{where, } A \text{ and } B \text{ are constants}$$

and $$J_n(x) = \sum\frac{(-1)^m\,(x/2)^{2m+n}}{m!\,\lfloor n+m+1}; \qquad J_{-n}(x) = \sum\frac{(-1)^m\,(x/2)^{2m-n}}{m!\,\lfloor -n+m+1}.$$

$J_n(x)$ and $J_{-n}(x)$ are called Bessel's functions.

3. Recurrence Formulae for $J_n(x)$

In general a relation which connects a term a_n of a sequence with some (previous) terms of the sequence is called a **recurrence formula**. In particular, expressions from which $J_n(x)$ can be obtained successively are called **recurrence formulae** for $J_n(x)$. We shall discuss some important recurrence, formulae for $J_n(x)$.

Formula I : $\boxed{x\,J_n'(x) = n\,J_n(x) - x\,J_{n+1}(x)}$ ..... (1)

Proof : We have,

$$J_n(x) = \sum\frac{(-1)^m\,(x/2)^{n+2m}}{m!\,\lfloor n+m+1}$$

Differentiating w.r.t. x,

$$J_n'(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\,(n+2m)}{m!\,\lfloor n+m+1}\cdot\left(\frac{x}{2}\right)^{m+2m-1}\cdot\left(\frac{1}{2}\right)$$

Separating the two terms, we get

$$J_n'(x) = n\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\lfloor n+m+1}\cdot\left(\frac{x}{2}\right)^{n+2m}\cdot\left(\frac{2}{x}\right)\cdot\left(\frac{1}{2}\right)$$
$$+\sum_{m=0}^{\infty}\frac{(-1)^m\cdot 2m}{m!\,\lfloor n+m+1}\cdot\left(\frac{x}{2}\right)^{n+2m}\cdot\left(\frac{2}{x}\right)\cdot\left(\frac{1}{2}\right)$$

Multiplying both sides by x, we get

$$x\,J_n'(x) = n\sum\frac{(-1)^m}{m!\,\lfloor n+m+1}\cdot\left(\frac{x}{2}\right)^{n+2m} + \sum_{m=0}^{\infty}\frac{(-1)^m\cdot 2m}{m!\,\lfloor n+m+1}\cdot\left(\frac{x}{2}\right)^{n+2m}$$

Now, the first sum on the r.h.s. is $J_n(x)$ and the first term of the second sum on the r.h.s. is zero. ($\because m = 0$)

Hence, we start the second sum from $m = 1$ and also write $m! = m\,(m-1)!$.

$$\therefore\quad x\,J_n'(x) = n\,J_n(x) + \sum_{m=1}^{\infty}\frac{(-1)^m\cdot 2m}{m(m-1)!\,\lfloor n+m+1}\cdot\left(\frac{x}{2}\right)^{n+2m}$$
$$= n\,J_n(x) + \sum_{m=1}^{\infty}\frac{(-1)^m\cdot 2}{(m-1)!\,\lfloor n+m+1}\cdot\left(\frac{x}{2}\right)^{n+2m}$$

Now, we put $m - 1 = k$. When $m = 1$, $k = 0$ and when $m = \infty$, $k = \infty$.

$$\therefore\quad x\cdot J_n'(x) = n\,J_n(x) + \sum_{k=0}^{\infty}\frac{(-1)^{k+1}\cdot 2}{k!\,\lfloor n+k+2}\cdot\left(\frac{x}{2}\right)^{n+2(k+1)}$$
$$= n\,J_n(x) - \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\lfloor (n+1)+k+1}\cdot 2\cdot\left(\frac{x}{2}\right)^{(n+1)+2k}\cdot\left(\frac{x}{2}\right)$$
$$= n\,J_n(x) - \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\,\lfloor (n+1)+k+1}\cdot\left(\frac{x}{2}\right)^{(n+1)+2k}\cdot x$$
$$= n\,J_n(x) - x\,J_{n+1}(x)$$

Formula II : $\boxed{x\,J_n'(x) = -n\,J_n(x) + x\,J_{n-1}(x)}$ ..... (2)

Proof : We have $$J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m\,(x/2)^{n+2m}}{m!\,\lfloor n+m+1}$$

Differentiating w.r.t. x, we get

$$J'_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m(n+2m)(x/2)^{n+2m-1}\cdot(1/2)}{m!\,\lfloor (n+m)+1}$$

$$\therefore\quad x\,J'_n(x) = x\sum_{m=0}^{\infty}\frac{(-1)^m[2n+2m-n](x/2)^{n+2m-1}\cdot(1/2)}{m!\,\lfloor (n+m)+1}$$
$$= x\sum_{m=0}^{\infty}\frac{(-1)^m[2(n+m)-n](x/2)^{n+2m}}{m!\,\lfloor (n+m)+1}\cdot\frac{2}{x}\cdot\frac{1}{2}$$

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$$\therefore \quad x J'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m [2(n+m)-n]}{m!\,\Gamma(n+m)+1} \cdot \left(\frac{x}{2}\right)^{n+2m}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m 2(n+m)(x/2)^{n+2m}}{m!\,\Gamma(n+m)+1} - \sum_{m=0}^{\infty} \frac{(-1)^m n\,(x/2)^{n+2m}}{m!\,\Gamma(n+m)+1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m 2(x/2)^{n+2m-1} \cdot (x/2)}{m!\,\Gamma n+m} - n \sum_{m=0}^{\infty} \frac{(-1)^m\,(x/2)^{n+2m}}{m!\,\Gamma(n+m)+1}$$

$$\left[\because \Gamma n+m+1 = (n+m)\Gamma n+m\right]$$

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{(n-1)+2m}}{m!\,\Gamma(n-1)+m+1} - n J_n(x)$$

$$= x J_{n-1}(x) - n J_n(x)$$

Hence, $x J'_n(x) = -n J_n(x) + x J_{n-1}(x)$.

Formula III : $\boxed{2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x)}$

Proof : We have $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!\,\Gamma(n+m)+1}$

Differentiating w.r.t. x, we get

$$J_n'(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+2m)(x/2)^{n+2m-1}(1/2)}{m!\,\Gamma(n+m)+1}$$

We split the term $(n + 2m)$ as $(n + m) + m$,

$$\therefore \quad 2J'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (n+m+m)(x/2)^{n+2m-1}}{m!\,\Gamma(n+m)+1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)(x/2)^{n+2m-1}}{m!\,\Gamma(n+m)+1} + \sum_{m=0}^{\infty} \frac{(-1)^m \cdot m \cdot (x/2)^{n+2m-1}}{m!\,\Gamma(n+m)+1}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)(x/2)^{(n-1)+2m}}{m!\,(n+m)\,\Gamma n+m} + \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{(n+1)-2m}}{(m-1)!\,\Gamma n+m+1}$$

$$\left[\because \Gamma n+1 = n\Gamma n\right]$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{(n-1)+2m}}{m!\,\Gamma(n+m-1)+1} + \sum_{m=0}^{\infty} \frac{(-1)^m \cdot m \cdot (x/2)^{(n-1)+2m}}{m!\,\Gamma(n+m)+1}$$

Since the non-zero term in the second term starts from $m = 1$, we have

$$\therefore \quad 2J'_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{(n-1)+2m}}{m!\,\Gamma(n-1)+m+1} + \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{n-1+2m}}{(m-1)!\,\Gamma(n+m)+1}$$

(We put $m - 1 = k$ in the second term so that the series starts from $k = 0$.)

$$\therefore \quad 2J_n'(x) = J_{n-1}(x) + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (x/2)^{(n-1)+2(k+1)}}{k!\,\Gamma(n+1)+k+1}$$

$$\therefore \quad 2J_n'(x) = J_{n-1}(x) + (-1) \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{(n+1)+2k}}{k!\,\Gamma(n+1)+k+1}$$

$$\therefore \quad 2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

Hence, we get $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$.

Note ....

The above relation can also be obtained by adding formulae (1) and (2).

Formula IV : $\boxed{2n\, J_n(x) = x\, J_{n-1}(x) + x\, J_{n+1}(x)}$

Proof : We have $J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{n+2m}}{m!\,\Gamma(n+m)+1}$

$$\therefore \quad 2n J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m 2n (x/2)^{n+2m}}{m!\,\Gamma(n+m)+1} = \sum_{m=0}^{\infty} \frac{(-1)^m 2n (x/2)^{n+2m-1} \cdot (x/2)}{m!\,\Gamma(n+m)+1}$$

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m \cdot n \cdot (x/2)^{n-1+2m}}{m!\,\Gamma(n+m)+1} = x \sum_{m=0}^{\infty} \frac{(-1)^m [(n+m)-m] \cdot (x/2)^{n-1+2m}}{m!\,\Gamma(n+m)+1}$$

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m (n+m)(x/2)^{n-1+2m}}{m!\,\Gamma(n+m)+1} - x \sum_{m=0}^{\infty} \frac{(-1)^m \cdot m \cdot (x/2)^{n-1+2m}}{m!\,\Gamma(n+m)+1}$$

Since non-zero term starts from $m = 1$ in the second term, we have

$$2n J_n(x) = x \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (x/2)^{n-1+2m}}{m!\,\Gamma n+m} - x \sum_{m=1}^{\infty} \frac{(-1)^m \cdot (x/2)^{n-1+2m}}{(m-1)!\,\Gamma(n+m)+1}$$

(We put $m - 1 = k$ in the second term, so that the series starts from $k = 0$.)

$$= x \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (x/2)^{n-1+2m}}{m!\,\Gamma(n-1)+m+1} - x \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (x/2)^{n-1+2(k+1)}}{(k+1-1)!\,\Gamma(n+1)+k+1}$$

$$= x J_{n-1}(x) - x(-1) \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,\Gamma(n+1)+k+1} \cdot \left(\frac{x}{2}\right)^{(n+1)+2k}$$

$$\therefore \quad 2n J_n(x) = x J_{n-1}(x) + x J_{n+1}(x).$$

Note ....

The above relation can also be obtained from formulae (1) and (2) as follows :

From (1), $n J_n(x) = x J_n'(x) + x J_{n+1}(x)$

and from (2), $n J_n(x) = x J_{n-1}(x) - x J_n'(x)$

Adding these relations, we get

$$2n J_n(x) = x J_{n-1}(x) + x J_{n+1}(x).$$

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Example : Prove that $J_{n+3}(x) + J_{n+5}(x) = (n+4)\cdot\frac{2}{x}J_{n+4}$.

Sol. : By recurrence relation (IV), we have

$$J_{n-1}(x) + J_{n+1}(x) = \frac{2n}{x}J_n(x)$$

Changing n to $n+4$, we get

$$J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4)J_{n+4}(x).$$

Formula V : $\boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}$

Proof : We have $J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}}{m!\,\Gamma(n+m)+1}$

Multiplying both sides by x^n, we get

$$\therefore\ x^n J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}\cdot x^n}{m!\,\Gamma(n+m)+1} = \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{2n+2m}\cdot 2^n}{m!\,\Gamma(n+m)+1}$$

$$\therefore\ \frac{d}{dx}\left[x^n J_n(x)\right] = \sum_{m=0}^{\infty}\frac{(-1)^m (2n+2m)(x/2)^{2n+2m-1}\cdot(1/2)\cdot 2^n}{m!\,\Gamma(n+m)+1}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m 2(n+m)(x/2)^{n+2m-1}\cdot(1/2)\cdot(x/2)^n\cdot 2^n}{m!\,\Gamma(n+m)+1}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{(n-1)+2m}\cdot x^n}{m!\,\Gamma n+m} \qquad \left[\because\ \Gamma n+m+1 = (n+m)\Gamma n+m\right]$$

$$= x^n \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{(n-1)+2m}}{m!\,\Gamma(n-1)+m+1}$$

$$= x^n\cdot J_{n-1}(x)$$

Aliter : By recurrence formula II, we have

$$x J_n'(x) = -n J_n(x) + x J_{n-1}(x)$$

Multiplying by x^{n-1} both sides, we have

$$x^n J_n'(x) = -n x^{n-1} J_n(x) + x^n J_{n-1}(x)$$

$$\therefore\ x^n J_n'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

$$\therefore\ \frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x).$$

Corollary : By integrating both sides, the above formula can also be written in integral form as

Formula V (A) : $\boxed{\int x^n J_{n-1}(x)\,dx = x^n J_n(x)}$

the constant of integration is assumed.

In particular, putting $n = 1$, $\int x J_0(x)\,dx = x J_1(x)$.

Formula VI : $\boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)}$

Proof : We have $J_n(x) = \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}}{m!\,\Gamma(n+m)+1}$

Multiplying both sides by x^{-n}, we get,

$$x^{-n} J_n(x) = x^{-n}\sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}}{m!\,\Gamma(n+m)+1} = \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{2m}}{m!\,\Gamma n+m+1\cdot 2^n}$$

$$\left[\because\ (x/2)^{n+2m} = (x/2)^n\cdot(x/2)^{2m} = (x^n)(1/2)^n\cdot(x/2)^{2m}\right]$$

Differentiating both sides w.r.t. x, we get

$$\therefore\ \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{m=0}^{\infty}\frac{(-1)^m 2m(x/2)^{2m-1}\cdot(1/2)}{m!\,\Gamma n+m+1\cdot 2^n}$$

Non-zero terms on the r.h.s. start from $m = 1$.

$$\therefore\ \frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{m=1}^{\infty}\frac{(-1)^m\cdot(x/2)^{2m-1}\cdot(1/2^n)}{(m-1)!\,\Gamma(n+m)+1}$$

$$= x^{-n}\sum_{m=1}^{\infty}\frac{(-1)^m\cdot(x/2)^{2m-1}\cdot(x^n/2^n)}{(m-1)!\,\Gamma(n+m)+1} \qquad \text{[Note this]}$$

$$= x^{-n}\sum_{m=1}^{\infty}\frac{(-1)^m\cdot(x/2)^{n+2m-1}}{(m-1)!\,\Gamma(n+m)+1}$$

We now put $m - 1 = k$ on the r.h.s., so that the sum starts from $k = 0$.

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\cdot(x/2)^{n+2(k+1)-1}}{k!\,\Gamma(n+k+1)+1}$$

$$= -x^{-n}\sum_{k=0}^{\infty}\frac{(-1)^k\cdot(x/2)^{(n+1)+2k}}{k!\,\Gamma(n+1)+k+1}$$

$$= -x^{-n} J_{n+1}(x)$$

Hence, $\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$.

Putting $n = 0$, $\boxed{J_0'(x) = -J_1(x)}$ ..... (A)

Corollary : The above formula can also be stated in integral form as

Formula VI (A) : $\boxed{\int x^{-n} J_{n+1}(x)\,dx = -x^{-n} J_n(x)}$

the constant of integration is assumed.

In particular, putting $n = 0$, $\boxed{\int J_1(x)\,dx = -J_0(x)}$ ..... (B)

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4. Bessel's Functions of Order ± 1/2, ± 3/2, ± 5/2, ... etc.

Bessel's functions of order $n = \pm 1/2, \pm 3/2, \pm 5/2, \ldots$ can be obtained from the recurrence relation (IV) and using $J_{1/2}(x)$, $J_{-1/2}(x)$. We can thus express these Bessel's functions in terms of $\sin x$ and $\cos x$.

Example 1 : Prove that $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}} \cdot \sin x$. **(M.U. 2008, 14, 15, 18, 19)**

Sol. : Putting $n = 1/2$ in (11), page S-39, *i.e.*, in

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\,\Gamma(n+m+1)} \qquad \therefore \quad J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+(1/2)}}{m!\,\Gamma(m+(3/2))}$$

But $$\Gamma\left(m+\frac{3}{2}\right) = \left(m+\frac{1}{2}\right)\left(m-\frac{1}{2}\right)\cdot\left(m-\frac{3}{2}\right)\cdots\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

$$\therefore \quad \Gamma\left(m+\frac{3}{2}\right) = \frac{(2m+1)(2m-1)(2m-3)\ldots\ldots 3\cdot 1\cdot\sqrt{\pi}}{2^{m+1}}$$

Multiplying (and dividing) by $2 \cdot 4 \cdot 6 \ldots\ldots (2m-2)(2m)$ in the numerator and denominator on r.h.s.

$$\Gamma\left(m+\frac{3}{2}\right) = \frac{(2m+1)(2m)(2m-1)\ldots\ldots 3\cdot 2\cdot 1}{2^{m+1}\cdot 2\cdot 4\cdot 6\ldots\ldots(2m-2)(2m)}\cdot\sqrt{\pi}$$

$$= \frac{1\cdot 2\cdot 3\ldots\ldots(2m-1)(2m)(2m+1)}{2^{m+1}\cdot 2^m\cdot 1\cdot 2\cdot 3\ldots\ldots(m-1)m}\cdot\sqrt{\pi} = \frac{(2m+1)!}{2^{2m+1}\cdot m!}\cdot\sqrt{\pi}$$

$$J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\, 2^{2m+1}\cdot m!}{m!\,(2m+1)!}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{x^{2m+(1/2)}}{2^{2m+(1/2)}}$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m\cdot x^{2m+1}}{(2m+1)!}\cdot\frac{x^{-1/2}}{\sqrt{\pi}}\cdot 2^{1/2} = \sqrt{\frac{2}{\pi x}}\sum_{m=0}^{\infty}\frac{(-1)^m\cdot x^{2m+1}}{(2m+1)!}$$

$$= \sqrt{\frac{2}{\pi x}}\cdot\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right) = \sqrt{\frac{2}{\pi x}}\cdot\sin x$$

Alternatively, we have, by putting $n = 1/2$ in $J_n(x)$, as above

$$J_{1/2}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m\cdot(x/2)^{2m+(1/2)}}{m!\,\Gamma(m+(3/2))}$$

Instead of considering the general term as above, we may consider, each term on the right hand side separately *i.e.* by putting $m = 0, 1, 2, \ldots\ldots$

$$J_{1/2}(x) = \frac{1}{0!\,\Gamma(3/2)}\cdot\left(\frac{x}{2}\right)^{1/2} + \frac{(-1)^1}{1!\,\Gamma(5/2)}\cdot\left(\frac{x}{2}\right)^{3/2} + \frac{(-1)^2}{2!\,\Gamma(7/2)}\cdot\left(\frac{x}{2}\right)^{9/2} + \ldots\ldots$$

$$= \sqrt{\frac{x}{2}}\cdot\left[\frac{1}{\Gamma(3/2)} - \frac{1}{\Gamma(5/2)}\cdot\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma(7/2)}\cdot\left(\frac{x}{2}\right)^4 + \ldots\ldots\right]$$

Now, $$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(1+\frac{1}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{1}{2}\cdot\sqrt{\pi}$$

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(1+\frac{3}{2}\right) = \frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{3}{2^2}\sqrt{\pi}$$

$$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(1+\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\left(\frac{3}{2}\right) = \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi} = \frac{5\cdot 3}{2^3}\sqrt{\pi}$$

$$\therefore \quad J_{1/2}(x) = \sqrt{\frac{x}{2}}\cdot\left[\frac{1}{(1/2)\cdot\sqrt{\pi}} - \frac{1}{3/2^2}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{x^2}{2^2} + \frac{1}{2!(5\cdot 3/2^3)}\cdot\frac{1}{\sqrt{\pi}}\cdot\frac{x^4}{2^4} - \ldots\ldots\right]$$

$$= \sqrt{\frac{x}{2}}\cdot\frac{1}{\sqrt{\pi}}\cdot 2\cdot\frac{1}{x}\left[x - \frac{x^3}{2\cdot 3} + \frac{x^5}{2\cdot 3\cdot 4\cdot 5} - \ldots\ldots\right]$$

$$\therefore \quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\right] = \sqrt{\frac{2}{\pi x}}\cdot\sin x.$$

Example 2 : Evaluate $\int_0^{\infty} \sqrt{x}\cdot e^{-x} J_{1/2}(x)\,dx$.

Sol. : Putting the value of $J_{1/2}(x)$ from Ex. 1, we have

$$I = \int_0^{\infty} \sqrt{x}\cdot e^{-x}\cdot\sqrt{\frac{2}{\pi}}\cdot\frac{1}{\sqrt{x}}\sin x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\sin x\,dx = \sqrt{\frac{2}{\pi}}\cdot\left[\frac{e^{-x}}{2}(-\sin x - \cos x)\right]_0^{\infty}$$

$$= \frac{1}{\sqrt{2\pi}}[0+1] = \frac{1}{\sqrt{2\pi}}.$$

Example 3 : Prove that $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cdot\cos x$.

Sol. : We have the recurrence formula (V)

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n\cdot J_{n-1}(x)$$

Putting $n = \dfrac{1}{2}$, we get $$\frac{d}{dx}\left[x^{1/2} J_{1/2}(x)\right] = x^{1/2} J_{-1/2}(x)$$

Putting the value of $J_{1/2}(x)$ [By Ex. 1],

$$\frac{d}{dx}\left[x^{1/2}\cdot\sqrt{\frac{2}{\pi x}}\cdot\sin x\right] = x^{1/2}\cdot J_{-1/2}(x)$$

$$\therefore \quad \sqrt{\frac{2}{\pi}}\cdot\frac{d}{dx}[\sin x] = x^{1/2} J_{-1/2}(x) \qquad \therefore \quad \sqrt{\frac{2}{\pi}}\cdot\cos x = x^{1/2} J_{-1/2}(x)$$

$$\therefore \quad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cdot\cos x$$

Example 4 : Prove that $J_{1/2}(x) = \tan x\cdot J_{-1/2}(x)$.

Sol. : We have proved in Ex. 1, page S-46 that $J_{1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cdot\sin x$

and in previous Ex. 3, that $J_{-1/2}(x) = \sqrt{\dfrac{2}{\pi x}}\cdot\cos x$

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By division, we get

$$\frac{J_{1/2}(x)}{J_{-1/2}(x)} = \tan x \qquad \therefore \quad J_{1/2}(x) = \tan x \cdot J_{-1/2}(x)$$

Example 5 : Prove that $J_{1/2}^2(x) + J_{-1/2}^2(x) = \frac{2}{\pi x}$.

Sol. : By squaring and adding the above results of Ex. 1 and Ex. 3, we get the required result.

Example 6 : Prove that $J_{3/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \left(\frac{\sin x}{x} - \cos x\right)$. **(M.U. 2013, 16)**

Sol. : We have from the recurrence formula IV,

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Putting $n = \frac{1}{2}$, we get $\quad J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$

Putting the values obtained in Ex. 1 and Ex. 3,

$$J_{3/2}(x) = \frac{1}{x} \cdot \sqrt{\frac{2}{\pi x}} \cdot \sin x - \sqrt{\frac{2}{\pi x}} \cdot \cos x = \sqrt{\frac{2}{\pi x}} \cdot \left(\frac{\sin x}{x} - \cos x\right).$$

Example 7 : Prove that $J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \cdot \left(\frac{\cos x}{x} + \sin x\right)$.

Sol. : We have from the recurrence formula IV,

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x)$$

Putting $n = -\frac{1}{2}$, we get $\quad J_{-3/2}(x) = -\frac{1}{x} J_{-1/2}(x) - J_{1/2}(x)$

Putting the values obtained in Ex. 1 and Ex. 3,

$$J_{-3/2}(x) = -\sqrt{\frac{2}{\pi x}} \cdot \cos x \cdot \frac{1}{x} - \sqrt{\frac{2}{\pi x}} \cdot \sin x = -\sqrt{\frac{2}{\pi x}} \cdot \left(\frac{\cos x}{x} + \sin x\right).$$

Cor. : $$J_{3/2}^2(x) + J_{-3/2}^2(x) = \frac{2}{\pi x}\left[\frac{\sin^2 x}{x^2} + \cos^2 x - \frac{2 \sin x \cos x}{x} + \frac{\cos^2 x}{x^2} + \sin^2 x + \frac{2 \sin x \cos x}{x}\right]$$

$$\therefore \quad J_{3/2}^2(x) + J_{-3/2}^2(x) = \frac{2}{\pi x}\left[\frac{1}{x^2} + 1\right] = \frac{2(1 + x^2)}{\pi x^3}.$$

Example 8 : Prove that $J_{5/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \left\{\frac{3 - x^2}{x^2} \sin x - \frac{3}{x} \cos x\right\}$. **(M.U. 2008, 14)**

Sol. : Putting $n = \frac{1}{2}$, in (IV) *i.e.*, in $\quad J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$ ..... (1)

we get $\quad J_{3/2}(x) = \frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)$ ..... (2)

Putting $n = \frac{3}{2}$ in (1), we get, $\quad J_{5/2}(x) = \frac{3}{x} J_{3/2}(x) - J_{1/2}(x)$ ..... (3)

Putting $J_{3/2}(x)$ from (2) in (3),

$$J_{5/2}(x) = \frac{3}{x}\left[\frac{1}{x} J_{1/2}(x) - J_{-1/2}(x)\right] - J_{1/2}(x) = \left(\frac{3 - x^2}{x^2}\right) J_{1/2}(x) - \frac{3}{x} J_{-1/2}(x)$$

But, $\quad J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \sin x$ and $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cdot \cos x$

$$\therefore \quad J_{5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3 - x^2}{x^2} \sin x - \frac{3}{x} \cos x\right]$$

Example 9 : Prove that $J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3}{x} \sin x + \frac{(3 - x^2)}{x^2} \cos x\right]$. **(M.U. 2018)**

Sol. : Putting $n = -3/2$, in formula IV *i.e.*, in

$$J_{n-1}(x) = \frac{2n}{x} J_n(x) - J_{n+1}(x), \text{ we get}$$

$$J_{-5/2}(x) = -\frac{3}{x} J_{-3/2}(x) - J_{-1/2}(x).$$

Putting $J_{-3/2}(x)$ from Ex. 7 and $J_{-1/2}(x)$ from Ex. 3, we get

$$J_{-5/2}(x) = -\frac{3}{x}\left[-\sqrt{\frac{2}{\pi x}} \cdot \left(\frac{\cos x}{x} + \sin x\right)\right] - \sqrt{\frac{2}{\pi x}} \cdot \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3 \cos x}{x^2} + \frac{3 \sin x}{x} - \cos x\right]$$

$$\therefore \quad J_{-5/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\frac{3 \sin x}{x} + \frac{(3 - x^2)}{x^2} \cos x\right].$$

Note ....

The expressions of $J_{1/2}(x)$, $J_{-1/2}(x)$, $J_{-3/2}(x)$ etc. in terms of $\sin x$ and $\cos x$ are called expressions in **closed forms**. In general $J_{n/2}(x)$ can be expressed in closed form if n is an integer positive or negative.

5. Bessel's Functions of Integral Values

Bessel's function of order $n = 0, 1$ can be obtained from the result (11), page S-39 and then using the recurrence formula IV we can find Bessel's function for $n = 2, 3, 4, 5, \ldots$ Further using $J_{-n}(x) = (-1)^n J_n(x)$, (See Ex. 4, page S-51) we can find $J_{-2}, J_{-3}, J_{-4} \ldots$ etc.

Example 1 : Find $J_0(x)$ and $J_1(x)$.

Sol. : We have $\quad J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m! \, \overline{|n+m+1}}$

But $\overline{|n+m+1} = (n+m)!$

$$\therefore \quad J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m! \, (n+m)!} \quad \text{where } n \text{ is an integer.}$$

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Putting $n = 0$, we get,

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{(m!)^2} \cdot \left(\frac{x}{2}\right)^{2m}$$

$$= 1 - \frac{x^2}{(1!)^2 \cdot 2^2} + \frac{x^4}{(2!)^2 \cdot 2^4} - \frac{x^6}{(3!)^2 \cdot 2^6} + \ldots..$$

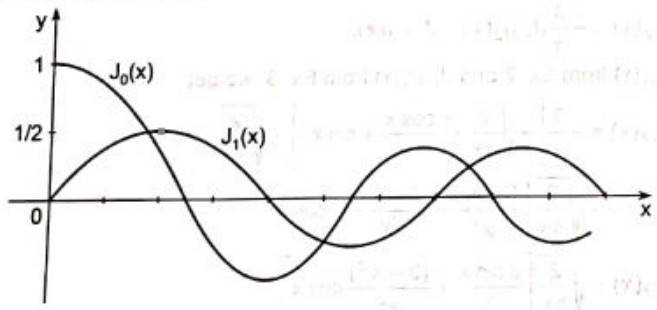
$$= 1 - \frac{x^2}{1^2 \cdot 2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots..$$

Putting $n = 1$, we get,

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+1)!} \cdot \left(\frac{x}{2}\right)^{2m+1} = \frac{x}{2} - \frac{1}{1!\,2!} \cdot \frac{x^3}{2^3} + \frac{1}{2!\,3!} \cdot \frac{x^5}{2^5} - \ldots..$$

$$= \frac{x}{2}\left[1 - \frac{1}{1!\,2!} \cdot \frac{x^2}{2^2} + \frac{1}{2!\,3!} \cdot \frac{x^4}{2^4} - \frac{1}{3!\,4!} \cdot \frac{x^6}{2^6} + \ldots..\right]$$

The graphs of these functions are shown below. Both these functions are oscillatory with varying periods and decreasing amplitudes.



Example 2 : Prove that $J_0'''(x) = \frac{1}{x} J_0(x) + \left(\frac{2}{x^2} - 1\right) J_0'(x)$. **(M.U. 2010)**

Sol. : We have proved in Ex. 1 above that

$$J_0(x) = 1 - \frac{x^2}{1^2 \cdot 2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \frac{x^8}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8^2} - \ldots..$$

$$\therefore \quad J_0'(x) = -\frac{x}{2} + \frac{x^3}{2^2 \cdot 4} - \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \frac{x^7}{2^2 \cdot 4^2 \cdot 6^2 \cdot 8} - \ldots.$$

$$J_0''(x) = -\frac{1}{2} + \frac{3x^2}{2^2 \cdot 4} - \frac{5x^4}{2^2 \cdot 4^2 \cdot 6} + \ldots.$$

$$J_0'''(x) = \frac{3 \cdot 2 \cdot x}{2^2 \cdot 4} - \frac{5 \cdot 4 \cdot x^3}{2^2 \cdot 4^2 \cdot 6} + \ldots. = \frac{3x}{8} - \frac{5x^3}{96} + \ldots.$$

Now, $$\text{r.h.s.} = \frac{1}{x}\left[1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \ldots.\right] + \frac{2}{x^2}\left[-\frac{x}{2} + \frac{x^3}{2^2 \cdot 4} - \frac{x^5}{2^2 \cdot 4^2 \cdot 6} + \ldots.\right]$$

$$- \left[-\frac{x}{2} + \frac{x^3}{2^2 \cdot 4} - \frac{x^5}{2^2 \cdot 4^2 \cdot 6}\right]$$

$$= \frac{1}{x} - \frac{x}{2^2} + \frac{x^3}{2^2 \cdot 4^2} - \ldots. - \frac{1}{x} + \frac{x}{2^2 \cdot 2} - \frac{x^3}{2^2 \cdot 4^2 \cdot 3} + \ldots. + \frac{x}{2} - \frac{x^3}{2^2 \cdot 4} + \ldots.$$

$$= (1-1)\frac{1}{x} + \left(-\frac{1}{4} + \frac{1}{8} + \frac{1}{2}\right)x + \left(\frac{1}{64} - \frac{1}{192} - \frac{1}{16}\right)x^3 + \ldots.$$

$$\therefore \quad \text{r.h.s.} = \frac{3}{8}x - \frac{5x^3}{96} + \ldots.$$

$$\therefore \quad \text{L.H.S.} = \text{R.H.S.}$$

Example 3 : Obtain $J_2(x)$, $J_3(x)$, $J_4(x)$, $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Sol. : We have obtained above $J_0(x)$ and $J_1(x)$.

We have from the recurrence formula IV, page S-43

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad \ldots\ldots\ldots\ldots (1)$$

Putting $n = 1, 2, 3, 4$, successively and suppressing the argument x (whose existence is assumed)

$$J_2 = \frac{2}{x} J_1 - J_0 \quad \ldots\ldots\ldots\ldots (2)$$

$$J_3 = \frac{4}{x} J_2 - J_1 \quad \ldots\ldots\ldots\ldots (3)$$

$$J_4 = \frac{6}{x} J_3 - J_2 \quad \ldots\ldots\ldots\ldots (4)$$

$$J_5 = \frac{8}{x} J_4 - J_3 \quad \ldots\ldots\ldots\ldots (5)$$

Putting J_2 from (2) in (3), we get,

$$J_3 = \frac{4}{x}\left[\frac{2}{x} J_1 - J_0\right] - J_1 = \left[\frac{8}{x^2} - 1\right] J_1 - \frac{4}{x} J_0 \quad \ldots\ldots\ldots\ldots (6)$$

Putting J_3 from (6) and J_2 from (2) in (4),

$$J_4 = \frac{6}{x}\left\{\left[\frac{8}{x^2} - 1\right] J_1 - \frac{4}{x} J_0\right\} - \left[\frac{2}{x} J_1 - J_0\right] \quad \ldots\ldots\ldots\ldots (7)$$

$$J_4 = \left[\frac{48}{x^3} - \frac{8}{x}\right] J_1 + \left[1 - \frac{24}{x^2}\right] J_0 \quad \textbf{(M.U. 2015, 16)} \quad \ldots\ldots\ldots\ldots (8)$$

Putting J_4 from (8) and J_3 from (6) in (5),

$$J_5 = \frac{8}{x}\left\{\left[\frac{48}{x^3} - \frac{8}{x}\right] J_1 + \left[1 - \frac{24}{x^2}\right] J_0\right\} - \left[\frac{8}{x^2} - 1\right] J_1 + \frac{4}{x} J_0$$

$$= \left[\frac{384}{x^4} - \frac{72}{x^2} + 1\right] J_1 + \left[\frac{12}{x} - \frac{192}{x^2}\right] J_0.$$

Example 4 : Prove that $J_{-n}(x) = (-1)^n J_n(x)$ where n is a positive integer. **(M.U. 2010)**

Sol. : Change n to $-n$ in $$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\,\lfloor n+m+1}.$$

we get $$J_{-n}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m-n}}{m!\,\lfloor m-n+1}$$

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We split the sum on the r.h.s. into two sums.

$$J_{-n}(x) = \sum_{m=0}^{m=n-1} \frac{(-1)^m (x/2)^{2m-n}}{m!\,\overline{|m-n+1}} + \sum_{m=n}^{\infty} \frac{(-1)^m (x/2)^{2m-n}}{m!\,\overline{|m-n+1}}$$

But the first term on the right hand side is zero because

$$\overline{|-\text{ve integer}} = \infty$$

$$\therefore \quad J_{-n}(x) = \sum_{m=n}^{\infty} \frac{(-1)^m (x/2)^{2m-n}}{m!\,\overline{|m-n+1}}$$

Now, putting $m = n + k$, when $m = n$, $k = 0$ and when $m = \infty$, $k = \infty$,

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^{n+k} (x/2)^{n+2k}}{(n+k)!\,\overline{|k+1}} = (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{(n+k)!\,\overline{|k+1}}$$

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{2k+n}}{k!\,\overline{|n+k+1}} = (-1)^n J_n(x) \qquad \text{[See (A) below]}$$

Putting $n = 1, 2, 3, \ldots$ we get

$$J_{-1}(x) = (-1)^1 J_1(x) = -J_1(x)$$
$$J_{-2}(x) = (-1)^2 J_2(x) = J_2(x)$$
$$J_{-3}(x) = (-1)^3 J_3(x) = -J_3(x)$$
$$J_{-4}(x) = (-1)^4 J_4(x) = J_4(x)$$
$$\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots$$

Thus, if n is odd $\quad J_{-n}(x) = -J_n(x)$

and if n is even $\quad J_{-n}(x) = J_n(x)$.

Note ....

The above results should be committed to memory and can be used as standard formulae.

Example 5 : Prove that $J_n(x)$ is an even function if n is even integer and is an odd function if n is odd integer. **(M.U. 2009)**

Sol. : If n is an integer then since $\overline{|n+m+1} = (m+n)!$, we have ..... (A)

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\,(m+n)!} \qquad \therefore \quad J_n(-x) = \sum_{m=0}^{\infty} \frac{(-1)^m (-x)^{2m+n}}{m!\,(m+n)!\,2^{2m+n}}$$

If n is even $(2m + n)$ is even and $(-x)^{2m+n} = x^{2m+n}$.

$$\therefore \quad J_n(-x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\,(m+n)!} = J_n(x)$$

Hence, $J_n(x)$ is even.

Again if n is odd then $(2m + n)$ is odd and hence,

$$(-x)^{2m+n} = -(x^{2m+n})$$

$$\therefore \quad J_n(-x) = \sum_{m=0}^{\infty} \frac{(-1)^m \cdot (-1)(x/2)^{2m+n}}{m!\,(m+n)!} = -\sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\,(m+n)!} = -J_n(x)$$

Hence, $J_n(x)$ is odd.

6. Examples Based on Recurrence Formulae

Example 1 : Prove that $6J_3(x) = x\,J_2(x) + x\,J_4(x)$.

Hence, deduce that $J_2(6) - J_3(6) + J_4(6) = 0$.

Sol. : We have by formula IV,

$$2n\,J_n(x) = x\,J_{n-1}(x) + x\,J_{n+1}(x)$$

Putting $n = 3$, we get,

$$2 \cdot 3\,J_3(x) = x\,J_2(x) + x\,J_4(x)$$

Now put $x = 6$,

$$\therefore \quad 6\,J_3(6) = 6\,J_2(6) + 6\,J_4(6)$$
$$\therefore \quad J_2(6) - J_3(6) + J_4(6) = 0.$$

Example 2 : Obtain $J_6(x)$ in terms of $J_0(x)$ and $J_1(x)$.

Sol. : We have, by Formula IV, page S-43

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Putting $n = 5$, $\quad J_6(x) = \frac{10}{x} J_5(x) - J_4(x)$

Putting the values of $J_5(x)$ and $J_4(x)$ obtained on page S-51, we get

$$J_6(x) = \frac{10}{x}\left[\left\{\frac{384}{x^4} - \frac{72}{x^2} + 1\right\} J_1(x) + \left\{\frac{12}{x} - \frac{192}{x^2}\right\} J_0(x)\right]$$
$$- \left[\frac{48}{x^3} - \frac{8}{x}\right] J_1(x) - \left[1 - \frac{24}{x^2}\right] J_0(x)$$
$$= \left[\frac{3840}{x^5} - \frac{768}{x^3} + \frac{18}{x}\right] J_1(x) - \left[1 - \frac{12}{x} + \frac{168}{x^2}\right] J_0(x)$$

Example 3 : Prove that $\quad x\,[J_{n+3}(x) + J_{n+5}(x)] = 2\,(n+4)\,J_{n+4}(x)$.

Sol. : By the recurrence formula IV, we have

$$2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$$

Changing n to $(n + 4)$, we get,

$$2\,(n+4)\,J_{n+4}(x) = x\,[J_{n+3}(x) + J_{n+5}(x)] \quad \text{which is the required result.}$$

Example 4 : Prove that

(i) $\frac{1}{2} \cdot xJ_n(x) = (n+1)J_{n+1}(x) - (n+3)J_{n+3}(x) + (n+5)J_{n+5}(x) - \frac{1}{2}xJ_{n+6}(x)$

(ii) $\frac{1}{2}xJ_{n-1}(x) = n\,J_n(x) - (n+2)J_{n+2}(x) + (n+4)J_{n+4}(x) - \frac{1}{2}xJ_{n+5}(x)$

Sol. : By the recurrence formula IV, we have

$$2n\,J_n(x) = x\,[J_{n-1}(x) + J_{n+1}(x)]$$

Changing n to $n + 1$,

$$2(n+1)J_{n+1}(x) = x\,[J_n(x) + J_{n+2}(x)] \qquad \text{..... (1)}$$

$$\therefore \quad \frac{1}{2} \cdot x\,J_n(x) = (n+1)J_{n+1}(x) - \frac{1}{2}x\,J_{n+2}(x)$$

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Changing n to $n + 2$ in (1), we get

$$\frac{1}{2} \cdot x \cdot J_{n+2}(x) = (n+3) J_{n+3}(x) - \frac{1}{2} x J_{n+4}(x)$$

Hence (1), becomes

$$\frac{1}{2} x J_n(x) = (n+1) J_{n+1}(x) - (n+3) J_{n+3}(x) + \frac{1}{2} \cdot x J_{n+4}(x) \quad \text{..... (2)}$$

Changing n to $n + 4$ in (1), we get,

$$\frac{1}{2} \cdot x \cdot J_{n+4}(x) = (n+5) J_{n+5}(x) - \frac{1}{2} x J_{n+6}(x)$$

Hence, from (2), we get

$$\frac{1}{2} \cdot x \cdot J_n(x) = (n+1) J_{n+1}(x) - (n+3) J_{n+3}(x) + (n+5) J_{n+5}(x) - \frac{1}{2} x J_{n+6}(x) \quad \text{..... (3)}$$

Changing n to $(n - 1)$ in (3), we get the second result.

Example 5 : Prove that $4 J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)$. **(M.U. 2009, 17, 18)**

Sol. : By the recurrence formula III, we have

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad \text{..... (1)}$$

Differentiating both sides w.r.t. x,

$$2 J_n''(x) = J'_{n-1}(x) - J'_{n+1}(x) \quad \text{..... (2)}$$

Changing n to $n - 1$ and $n + 1$ in (1), we get

$$2 J'_{n-1}(x) = J_{n-2}(x) - J_n(x)$$

and $\quad 2J'_{n+1}(x) = J_n(x) - J_{n+2}(x)$

Hence, from (2), we get,

$$2 J_n''(x) = \frac{1}{2} J_{n-2}(x) - \frac{1}{2} J_n(x) - \frac{1}{2} J_n(x) + \frac{1}{2} J_{n+2}(x)$$

$$\therefore \quad 4 J_n''(x) = J_{n-2}(x) - 2 J_n(x) + J_{n+2}(x).$$

Example 6 : Prove that $8 J_n'''(x) = J_{n-3}(x) - 3 J_{n-1}(x) + 3 J_{n+1}(x) - J_{n+3}(x)$.

Hence, deduce that $J_3(x) - 3 J_1(x) + 4 J_0'''(x) = 0$.

Sol. : From the result proved in Ex. 5 above

$$J_n''(x) = \frac{1}{4}\left[J_{n-2}(x) - 2 J_n(x) + J_{n+2}(x)\right]$$

Differentiating w.r.t. x,

$$J_n'''(x) = \frac{1}{4}\left[J_{n-2}'(x) - 2 J_n'(x) + J_{n+2}'(x)\right]$$

Multiplying throughout by 8,

$$8 J_n'''(x) = 2[J_{n-2}'(x) - 2 J_n'(x) + J_{n+2}'(x)] \quad \text{..... (1)}$$

But by formula III, page S-42,

$$2 J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \quad \text{..... (2)}$$

Changing n to $n - 2$, in (2),

$$2 J_{n-2}'(x) = J_{n-3}(x) - J_{n-1}(x) \quad \text{..... (3)}$$

Changing n to $n + 2$, in (2),

$$2J_{n+2}(x) = J_{n+1}(x) - J_{n+3}(x) \quad \text{..... (4)}$$

Putting the values from (2), (3), (4) in r.h.s. of (1), we get

$$8J_n'''(x) = [\, J_{n-3}(x) - J_{n-1}(x) \,] - 2\,[\, J_{n-1}(x) - J_{n+1}(x) \,] + [\, J_{n+1}(x) - J_{n+3}(x) \,]$$

$$\therefore \quad 8J_n'''(x) = J_{n-3}(x) - 3\, J_{n-1}(x) + 3\, J_{n+1}(x) - J_{n+3}(x)$$

Now, for deduction, put $n = 0$

$$\therefore \quad 8J_0'''(x) = J_{-3}(x) - 3\, J_{-1}(x) + 3\, J_1(x) - J_3(x)$$

But by Ex. 4, page S-51,

$$J_{-n}(x) = (-1)^n J_n(x) \qquad \therefore \quad J_{-1}(x) = -J_1(x) \text{ and } J_{-3}(x) = -J_3(x)$$

Hence, $\quad 8J_0'''(x) = -J_3(x) + 3\, J_1(x) + 3\, J_1(x) - J_3(x) = 6\, J_1(x) - 2\, J_3(x)$

$$\therefore \quad J_3(x) - 3\, J_1(x) + 4\, J_0'''(x) = 0.$$

Example 7 : Prove that $\quad x\, J_n'(x) = n\, J_n(x) - 2\,(n+2)\, J_{n+2}(x) + 2\,(n+4)\, J_n(x) - x\, J_{n+5}(x)$

Sol. : By the recurrence formula II, we have,

$$x\, J_n'(x) = -n\, J_n(x) + x\, J_{n-1}(x)$$

But by the result (ii) of the above Example 4,

$$x\, J_{n-1}(x) = 2n\, J_n(x) - 2\,(n+2)\, J_{n+2}(x) + 2\,(n+4)\, J_{n+4}(x) - x\, J_{n+5}(x)$$

Hence, $\quad x\, J_n'(x) = n\, J_n(x) - 2\,(n+2)\, J_{n+2}(x) + 2\,(n+4)\, J_{n+4}(x) - x\, J_{n+5}(x)$

Example 8 : Prove that $x^2 J_n''(x) = (n^2 - n - x^2)\, J_n + x\, J_{n+1}$; $n = 0, 1, 2, \ldots$ **(M.U. 2011)**

Sol. : By the recurrence formula I, we have,

$$x\, J_n'(x) = n\, J_n(x) - x\, J_{n+1}(x) \quad \text{..... (1)}$$

Differentiating (1), we get,

$$x\, J_n''(x) + J_n'(x) = n\, J_n'(x) - x\, J'_{n+1}(x) - J_{n+1}(x)$$

Multiplying by x,

$$x^2 J_n''(x) + x\, J_n'(x) = nx\, J_n'(x) - x^2 J'_{n+1}(x) - x\, J_{n+1}(x) \quad \text{..... (2)}$$

By the recurrence formula II, we have,

$$x\, J_n'(x) = -n\, J_n(x) + x\, J_{n-1}(x)$$

Replacing n by $(n + 1)$, we get,

$$x\, J'_{n+1}(x) = -(n+1)\, J_{n+1}(x) + x\, J_n(x) \quad \text{..... (3)}$$

Putting the values of $x\, J_n'(x)$ and $x\, J'_{n+1}(x)$ from (1) and (3) in (2), we have,

$$x^2 J_n''(x) + n\, J_n(x) - x\, J_{n+1}(x) = n^2 J_n(x) - nx\, J_{n+1}(x) + (n+1)\, x\, J_{n+1}(x) - x^2 J_n(x) - x\, J_{n+1}(x)$$

$$\therefore \quad x^2 J_n''(x) = (n^2 - n - x^2)\, J_n(x) + x\, J_{n+1}(x).$$

Aliter : We know that Bessel's equation is

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2)\, y = 0$$

Since $y = J_n(x)$ is the solution of this equation, $J_n(x)$ satisfies it.

$$\therefore \quad x^2 J_n''(x) + x\, J_n'(x) + (x^2 - n^2)\, J_n(x) = 0$$

But by recurrence formula (I), $\quad x\, J_n'(x) = n\, J_n(x) - x\, J_{n+1}(x)$

Putting this value of $x\, J_n'(x)$ above, we get,

$$x^2 J_n''(x) + n\, J_n(x) - x\, J_{n+1}(x) + (x^2 - n^2)\, J_n(x) = 0$$

$$\therefore \quad x^2 J_n''(x) = (n^2 - n - x^2)\, J_n(x) + x\, J_{n+1}(x).$$

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Example 9 : Prove that $\frac{d}{dx}[x\,J_n(x)J_{n+1}(x)] = x\,[J_n^2(x) - J_{n+1}^2(x)]$. **(M.U. 2008, 10)**

Sol. : By the rule of differentiation of a product

$$\frac{d}{dx}[x\,J_n(x)J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x\,J_n'(x)J_{n+1}(x) + x\,J_n(x)J'_{n+1}(x) \quad \text{..... (1)}$$

But by recurrence formula (I),

$$J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \quad \text{..... (2)}$$

Changing n to $n+1$, we can write

$$J_{n+1}'(x) = \frac{n+1}{x}J_{n+1}(x) - J_{n+2}(x) \quad \text{..... (3)}$$

Putting (2) and (3) in (1), we have,

$$\frac{d}{dx}[x\,J_n(x)\cdot J_{n+1}(x)] = J_n(x)J_{n+1}(x) + x\left[\frac{n}{x}J_n(x) - J_{n+1}(x)\right]J_{n+1}(x) + x\,J_n(x)\left[\frac{n+1}{x}J_{n+1}(x) - J_{n+2}(x)\right]$$

$$= J_n(x)J_{n+1}(x) + n\,J_n(x)J_{n+1}(x) - x\,J_{n+1}^2(x) + (n+1)J_n(x)J_{n+1}(x) - x\,J_n(x)J_{n+2}(x)$$

$$= 2(n+1)J_n(x)\cdot J_{n+1}(x) - x\,J_{n+1}^2(x) - x\,J_n(x)J_{n+2}(x) \quad \text{..... (4)}$$

By recurrence relation (IV), we have

$$J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x)$$

$$\therefore\ J_{n+2}(x) = \frac{2(n+1)}{x}J_{n+1}(x) - J_n(x)$$

Putting this value in (4), we get,

$$\frac{d}{dx}[x\,J_n(x)\cdot J_{n+1}(x)] = 2(n+1)J_n(x)\cdot J_{n+1}(x) - x\,J_{n+1}^2(x) - x\,J_n(x)\left[\frac{2(n+1)}{x}J_{n+1}(x) - J_n(x)\right]$$

$$= 2(n+1)J_n(x)J_{n+1}(x) - x\,J_{n+1}^2(x) - 2(n+1)J_n(x)J_{n+1}(x) + x\,J_n^2(x)$$

$$= x\,J_n^2(x) - x\,J_{n+1}^2(x)$$

$$= x\,[J_n^2(x) - J_{n+1}^2(x)]$$

Example 10 : Prove that $J_5''(x) = \left[\frac{30}{x^2} - 1\right]J_5(x) - \frac{1}{x}J_4(x)$.

Sol. : By Formula II, page S-41, we have

$$x\,J_n'(x) = -n\,J_n(x) + x\,J_{n-1}(x)$$

$$\therefore\ J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x) \quad \text{..... (1)}$$

Differentiating w.r.t. x,

$$J_n''(x) = \frac{n}{x^2}J_n(x) - \frac{n}{x}J_n'(x) + J_{n-1}'(x)$$

Putting $n = 5$, $\quad J_5''(x) = \frac{5}{x^2}J_5(x) - \frac{5}{x}J_5'(x) + J_4'(x)$ ..... (2)

Putting $n = 5$ in (1), we get $\quad J_5'(x) = -\frac{5}{x}J_5(x) + J_4(x)$ ..... (3)

Now, by formula I, page S-40 $\quad J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

Putting $n = 4$, $\quad J_4'(x) = \frac{4}{x}J_4(x) - J_5(x)$ ..... (4)

Putting the values of $J_5'(x)$ and $J_4'(x)$ from (3) and (4) in (2), we get

$$J_5''(x) = \frac{5}{x^2}J_5(x) - \frac{5}{x}\left[-\frac{5}{x}J_5(x) + J_4(x)\right] + \left[\frac{4}{x}J_4(x) - J_5(x)\right]$$

$$= \left[\frac{5}{x^2} + \frac{25}{x^2} - 1\right]J_5(x) + \left[\frac{4}{x} - \frac{5}{x}\right]J_4(x)$$

$$= \left[\frac{30}{x^2} - 1\right]J_5(x) - \frac{1}{x}J_4(x).$$

Example 11 : Prove that $\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2\left[\frac{n}{x}J_n^2(x) - \frac{(n+1)}{x}J_{n+1}^2(x)\right]$.

Sol. : By differentiation,

$$\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2J_n(x)J_n'(x) + 2J_{n+1}(x)J_{n+1}'(x) \quad \text{..... (1)}$$

But by recurrence formula I,

$$x\,J_n'(x) = n\,J_n(x) - x\,J_{n+1}(x)$$

$$\therefore\ J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x) \quad \text{..... (2)}$$

By recurrence formula II,

$$x\,J_n'(x) = -n\,J_n(x) + x\,J_{n-1}(x)$$

$$\therefore\ J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x)$$

Changing n to $(n+1)$, we have

$$J_{n+1}'(x) = -\frac{(n+1)}{x}J_{n+1}(x) + J_n(x) \quad \text{..... (3)}$$

Putting the values of $J_n'(x)$ and $J_{n+1}'(x)$ from (2) and (3) in (1), we get,

$$\frac{d}{dx}\left[J_n^2(x) + J_{n+1}^2(x)\right] = 2J_n(x)\cdot\left[\frac{n}{x}\cdot J_n(x) - J_{n+1}(x)\right] + 2J_{n+1}(x)\left[-\frac{(n+1)}{x}J_{n+1}(x) + J_n(x)\right]$$

$$= 2\left[\frac{n}{x}\cdot J_n^2(x) - \frac{(n+1)}{x}J_{n+1}^2(x)\right] \text{ as required.}$$

Example 12 : Prove that

(i) $J_0'(x) = -J_1(x)$, (ii) $2J_0''(x) = J_2(x) - J_0(x)$

(iii) $J_2(x) = J_0''(x) - J_0'(x)/x$ (iv) $J_3(x) + 3J_0(x) + 4J_0''(x) = 0$

(M.U. 2013) **(M.U. 2009, 10, 11, 19)**

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Sol. : (i) From the recurrence formula I, we have,

$$x\,J_n'(x) = n\,J_n(x) - x\,J_{n+1}(x)$$

Replacing n by zero, we have

$$x\,J_0'(x) = 0 - x\,J_1(x) \qquad \therefore\ J_0'(x) = -J_1(x)$$

Alternatively :

Putting $n = 0$ in Formula (VI), page S-45,

$$\frac{d}{dx}[J_0(x)] = -J_1(x) \qquad \therefore\ J_0'(x) = -J_1(x)$$

(ii) From recurrence formula III, we have

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \qquad \text{..... (1)}$$

Differentiating, we get,

$$2J_n''(x) = J'_{n-1}(x) - J'_{n+1}(x) \qquad \text{..... (2)}$$

Changing n to $n-1$ in (1),

$$2\,J'_{n-1}(x) = J_{n-2}(x) - J_n(x) \qquad \text{..... (3)}$$

Changing n to $n+1$ in (1),

$$2\,J'_{n+1}(x) = J_n(x) - J_{n+2}(x) \qquad \text{..... (4)}$$

Putting the values of $J'_{n-1}(x)$ and $J'_{n+1}(x)$ from (3) and (4) in (2), we have

$$2J_n''(x) = \frac{1}{2}\left[J_{n-2}(x) - J_n(x)\right] - \frac{1}{2}\left[J_n(x) - J_{n+2}(x)\right]$$

$$\therefore\ 4J_n''(x) = J_{n-2}(x) - 2J_n(x) + J_{n+2}(x) \qquad \text{..... (5)}$$

Putting $n = 0$, we get,

$$4J_0''(x) = J_{-2}(x) - 2J_0(x) + J_2(x)$$

But by Ex. 4, page S-51, $J_{-n}(x) = (-1)^n J_n \quad \therefore\ J_{-2}(x) = J_2(x)$

$$\therefore\ 4J_0''(x) = 2J_2(x) - 2J_0(x)$$

$$\therefore\ 2J_0''(x) = J_2(x) - J_0(x)$$

(iii) From the recurrence formula I, we have

$$x\,J_n'(x) = n\,J_n(x) - x\,J_{n+1}(x) \qquad \text{..... (6)}$$

Putting $n = 1$, $x\,J_1'(x) = J_1(x) - x\,J_2(x)$

Putting $n = 0$, $x\,J_0'(x) = -x\,J_1(x)$

$\therefore\ J_0'(x) = -J_1(x)$. Differentiating it, $\therefore\ J_0''(x) = -J_1'(x)$

Putting the values of $J_1(x)$ and $J_1'(x)$ in (6),

$$-x \cdot J_0''(x) = -J_0'(x) - x\,J_2(x)$$

$$\therefore\ J_2(x) = J_0''(x) - J_0'(x)/x$$

(iv) Differentiating (5) obtained above, we have

$$4J_n'''(x) = J_{n-2}'(x) - 2J_n'(x) + J'_{n+2}(x) \qquad \text{..... (7)}$$

Changing n to $n-2$ in (1),

$$2\,J'_{n-2}(x) = J_{n-3}(x) - J_{n-1}(x) \qquad \text{..... (8)}$$

Changing n to $n+2$ in (1),

$$2\,J'_{n+2}(x) = J_{n+1}(x) - J_{n+3}(x) \qquad \text{..... (9)}$$

Putting the values of $J'_{n-2}(x)$, $J'_{n+2}(x)$ and $J_n'(x)$ from (8), (9) and (1) in (7), we get,

$$4J_n'''(x) = \frac{1}{2}\left[J_{n-3}(x) - J_{n-1}(x)\right] - \left[J_{n-1}(x) - J_{n+1}(x)\right] + \frac{1}{2}\left[J_{n+1}(x) - J_{n+3}(x)\right]$$

$$\therefore\ 8J_n'''(x) = J_{n-3}(x) - 3J_{n-1}(x) + 3J_{n+1}(x) - J_{n+3}(x)$$

Changing n to 0, $8J_0'''(x) = J_{-3}(x) - 3J_{-1}(x) + 3J_1(x) - J_3(x)$

But $J_{-n}(x) = (-1)^n J_n(x) \quad \therefore\ J_{-3}(x) = -J_3(x)$ and $J_{-1} = -J_1(x)$

$$\therefore\ 8J_0'''(x) = -2J_3(x) - 6J_1(x) \qquad \therefore\ J_3(x) = -3J_1(x) - 4J_0'''(n)$$

But $J_1(x) = -J_0'(x) \quad \therefore\ J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0.$

Aliter : We may also obtain the above result by differentiating $J_0'(x)$ twice as follows.

We have by result (1), $J_0'(x) = -J_1(x)$

Differentiating it, we get $J_0''(x) = -J_1'(x)$

But by recurrence formula III,

$$2\,J_n'(x) = J_{n-1}(x) - J_{n+1}(x) \qquad \text{..... (A)}$$

Putting $n = 1$, $2J_1'(x) = J_0(x) - J_2(x)$

$$\therefore\ J_0''(x) = -\frac{1}{2}[J_0(x) - J_2(x)]$$

Differentiating this again,

$$J_0'''(x) = -\frac{1}{2}[J_0'(x) - J_2'(x)] \qquad \text{..... (B)}$$

Putting $n = 2$ in (A), $2\,J_2'(x) = J_1(x) - J_3(x)$.

Hence, from (B), we get

$$J_0'''(x) = -\frac{1}{2}\left[J_0'(x) - \frac{1}{2}\{J_1(x) - J_3(x)\}\right] = -\frac{1}{2}J_0'(x) + \frac{1}{4}J_1(x) - \frac{1}{4}J_3(x)$$

$$= -\frac{1}{2}J_0'(x) - \frac{1}{4}J_0'(x) - \frac{1}{4}J_3(x) \qquad [\because\ J_1(x) = -J_0'(x)]$$

$$\therefore\ 4J_0'''(x) + 3J_0'(x) + J_3(x) = 0.$$

Example 13 : Prove that $J_2'(x) = \left(1 - \frac{4}{x^2}\right)J_1(x) + \frac{2}{x}J_0(x)$. **(M.U. 2009, 16, 19)**

Sol. : We have by recurrence formula II,

$$x\,J_n'(x) = -n\,J_n(x) + x\,J_{n-1}(x)$$

Putting $n = 2$, we get $x\,J_2'(x) = -2J_2(x) + x\,J_1(x)$

$$\therefore\ J_2'(x) = -\frac{2}{x}J_2(x) + J_1(x) \qquad \text{..... (A)}$$

By comparing this with the required result, it is clear that we have to express J_2 in terms of J_1 and J_0.

Now, by recurrence formula I, $\quad J_n'(x) = \frac{n}{x}J_n(x) - J_{n+1}(x)$

and by the recurrence formula II as above $\quad J_n'(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x)$

Hence, equating the two, we get $\quad \frac{n}{x}J_n(x) - J_{n+1}(x) = -\frac{n}{x}J_n(x) + J_{n-1}(x)$

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$$\therefore \quad J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x)$$

Putting $n = 1$, we get $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$

Putting this value of $J_2(x)$ in (A), we get

$$J_2'(x) = -\frac{2}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] + J_1(x) = \left(1 - \frac{4}{x^2}\right) J_1(x) + \frac{2}{x} J_0(x).$$

Example 14 : Prove that $\frac{d}{dx}\left[x^{n/2} J_n\left(\sqrt{x}\right)\right] = \frac{1}{2} \cdot x^{(n-1)/2} J_{n-1}\left(\sqrt{x}\right)$. **(M.U. 2013)**

Sol. : Consider the integral $\int \frac{1}{2} \cdot x^{(n-1)/2} J_{n-1}\left(\sqrt{x}\right) dx$ and put $\sqrt{x} = t$

$$\therefore \quad \frac{1}{2\sqrt{x}} dx = dt \quad \text{and} \quad dx = 2\sqrt{x}\, dt = 2t\, dt$$

$$\therefore \quad \int \frac{1}{2} \cdot x^{(n-1)/2} J_{n-1}\left(\sqrt{x}\right) dx = \int \frac{1}{2} \cdot t^{n-1} \cdot J_{n-1}(t) \cdot (2t)\, dt$$

$$= \int t^n J_{n-1}(t)\, dt = t^n J_n(t) \qquad \text{[By formula V(A), page S-44]}$$

$$= x^{n/2} J_n\left(\sqrt{x}\right)$$

Differentiating both sides, we get

$$\frac{1}{2} \cdot x^{(n-1)/2} J_{n-1}\left(\sqrt{x}\right) = \frac{d}{dx}\left[x^{n/2} J_n\left(\sqrt{x}\right)\right].$$

Example 15 : Prove that $\frac{d}{dx}\left[x^{an} J_n(bx^a)\right] = ab \cdot x^{(n+1)a-1} J_{n-1}(bx^a)$.

Sol. : Consider the integral $\int ab \cdot x^{(n+1)a-1} \cdot J_{n-1}(bx^a)\, dx$ and put $bx^a = t$.

i.e. $\quad x = \left(\frac{t}{b}\right)^{1/a} \quad \therefore \quad dx = \frac{1}{b^{1/a}} \cdot \frac{1}{a} (t)^{(1/a)-1} dt$

$$\therefore \int ab \cdot x^{(n+1)a-1} \cdot J_{n-1}(bx^a)\, dx$$

$$= ab \int \left(\frac{t}{b}\right)^{[(n+1)a-1]/a} \cdot J_{n-1}(t) \cdot \frac{1}{a} \cdot \frac{1}{b^{1/a}} \cdot (t)^{(1-a)/a}\, dt$$

$$= \frac{b}{b^{(n+1)} \cdot b^{-1/a}} \cdot \frac{1}{b^{1/a}} \int t^n \cdot J_{n-1}(t)\, dt$$

$$= \frac{1}{b^n} \int t^n \cdot J_{n-1}(t)\, dt = \frac{1}{b^n} t^n J_n(t) \qquad \text{[By formula V(A), page S-44]}$$

$$= \frac{1}{b^n} (bx^a)^n \cdot J_n(bx^a) = x^{an} J_n(bx^a)$$

Differentiating both sides, we get

$$ab \cdot x^{(n+1)a-1} \cdot J_{n-1}(bx^a) = \frac{d}{dx}\left[x^{an} J_n(bx^a)\right].$$

Example 16 : Prove that $\frac{d}{dx}\left[x^n J_n(ax)\right] = ax^n J_{n-1}(ax)$.

Sol. : Consider the integral $\int ax^n \cdot J_{n-1}(ax)\, dx$ and put $ax = t$.

$$\therefore \quad \int ax^n \cdot J_{n-1}(ax)\, dx = \int a \cdot \frac{t^n}{a^n} J_{n-1}(t) \cdot \frac{dt}{a}$$

$$= \frac{1}{a^n} \int t^n J_{n-1}(t)\, dt = \frac{1}{a^n} \cdot t^n J_n(t) \qquad \text{[By formula V(A), page S-44]}$$

$$= \frac{1}{a^n} (ax)^n J_n(ax) = x^n J_n(ax)$$

Differentiating both sides, we get

$$ax^n J_{n-1}(ax) = \frac{d}{dx}\left[x^n J_n(ax)\right].$$

Example 17 : Prove that $\frac{d}{dx}[x^2 J_3(2x)] = 2x^2 J_2(2x) - x J_3(2x)$.

Sol. : We have

$$\frac{d}{dx}[x^2 J_3(2x)] = \frac{d}{dx}\left[\frac{1}{x} \cdot x^3 J_3(2x)\right] \qquad \text{[Note this step]}$$

$$= \frac{1}{x} \cdot \frac{d}{dx}[x^3 J_3(2x)] + x^3 J_3(2x) \frac{d}{dx}\left(\frac{1}{x}\right)$$

But $\frac{d}{dx}[x^n J_n(ax)] = ax^n J_{n-1}(ax)$ by the above example.

Putting $n = 3$ and $a = 2$,

$$\frac{d}{dx}[x^3 J_3(2x)] = 2x^3 J_2(2x)$$

$$\therefore \quad \frac{d}{dx}[x^2 J_3(2x)] = \frac{1}{x} \cdot [2x^3 J_2(2x)] - \frac{1}{x^2} \cdot x^3 J_3(2x)$$

$$= 2x^2 J_2(2x) - x J_3(2x).$$

Example 18 : Prove that $\frac{d}{dx}[J_n^2(x)] = \frac{x}{2n}[J_{n-1}^2(x) - J_{n+1}^2(x)]$.

Sol. : We have $\frac{d}{dx}[J_n^2(x)] = 2 J_n(x) \cdot J_n'(x)$

But by formula (III), $\quad J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

and by formula (IV), $\quad J_n(x) = \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

Hence, $\quad \frac{d}{dx}[J_n^2(x)] = 2 \cdot \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \cdot \frac{x}{2n}[J_{n-1}(x) + J_{n+1}(x)]$

$$= \frac{x}{2n}[J_{n-1}^2(x) - J_{n+1}^2(x)].$$

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7. Examples Based on Integration

Example 1 : Prove that $\int J_0(\sqrt{x})\,dx = 2\sqrt{x}\,J_1(\sqrt{x}) + c$.

Sol. : Putting $\sqrt{x} = t$ and $\frac{1}{2\sqrt{x}}dx = dt$ *i.e.*, $dx = 2\sqrt{x}\,dt = 2t\,dt$, we get

$$\int J_0(\sqrt{x})\,dx = \int J_0(t)\cdot 2(t)\,dt = 2\int t\,J_0(t)\,dt$$

But by Formula V (A), page S-44

$$\int t\,J_0(t)\,dt = t\,J_1(t) + c$$

$$\therefore \int J_0(\sqrt{x})\,dx = 2t\cdot J_1(t) + c = 2\sqrt{x}\cdot J_1(\sqrt{x}) + c.$$

Example 2 : Prove that $\int J_0(x)\cdot J_1(x)\,dx = -\frac{1}{2}[J_0(x)]^2$.

Sol. : We know that

$$\int J_1(x)\,dx = -J_0(x) \qquad \therefore \quad J_1(x) = -J_0'(x)$$

Hence, $\int J_0(x)\cdot J_1(x)\,dx = -\int J_0(x)\cdot J_0'(x)\,dx$

$$= -\frac{1}{2}[J_0(x)]^2 \qquad [\text{ Put } J_0(x) = t\,]$$

Example 3 : Prove that $\int x^3\cdot J_0(x)\,dx = x^3\cdot J_1(x) - 2x^2 J_2(x)$.

Sol. : We write $x^3\cdot J_0(x)$ as $x^2\cdot x\,J_0(x)$ and integrate it by parts,

$$\therefore \int x^3\cdot J_0(x)\,dx = \int x^2\cdot[x\,J_0(x)]\,dx$$

$$= x^2\int x\,J_0(x)\,dx - \int\left[\int x\cdot J_0(x)\,dx\right]\cdot 2x\,dx$$

But $\int x^n\cdot J_{n-1}(x)\,dx = x^n J_n(x)$ [By Formula V(A)]

$$\therefore \int x^3\cdot J_0(x)\,dx = x^2[x\cdot J_1(x)] - \int[x\cdot J_1(x)]\cdot 2x\,dx$$

$$= x^3\cdot J_1(x) - 2\int x^2\cdot J_1(x)\,dx$$

$$= x^3\cdot J_1(x) - 2x^2\cdot J_2(x) \qquad [\text{ By Formula V(A) }]$$

Example 4 : Prove that $\int_0^1 x^{5/2}\cdot J_{3/2}(ax)\,dx = \frac{1}{a}J_{5/2}(a)$. **(M.U. 2008, 10)**

Sol. : By recurrence formula V, we have

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

Integrating both sides, we get,

$$\int x^n J_{n-1}(x)\,dx = x^n J_n(x) \qquad \text{.....(1)}$$

Now putting $ax = t$ in the given integral, we have

$$\int_0^1 x^{5/2} J_{3/2}(ax)\,dx = \int_0^a \left(\frac{t}{a}\right)^{5/2}\cdot J_{3/2}(t)\frac{dt}{a} = \frac{1}{a^{7/2}}\int_0^a t^{5/2}\cdot J_{3/2}(t)\,dt$$

Putting $n = 5/2$ in (1), we get,

$$\int x^{5/2} J_{3/2}(x)\,dx = x^{5/2}\cdot J_{5/2}(x)$$

$$\therefore \int t^{5/2} J_{3/2}(t)\,dt = t^{5/2}\cdot J_{5/2}(t)$$

Hence, we have

$$\int_0^1 x^{5/2}\cdot J_{3/2}(ax)\,dx = \frac{1}{a^{7/2}}\left[t^{5/2}\cdot J_{5/2}(t)\right]$$

$$= \frac{1}{a^{7/2}}[a^{5/2}\cdot J_{5/2}(a)] = \frac{1}{a}\cdot J_{5/2}(a)$$

Example 5 : Prove that $\int J_3(x)\,dx = -\frac{2J_1(x)}{x} - J_2(x)$. **(M.U. 2010, 11, 13, 16)**

Sol. : By recurrence formula (VI), we have,

$$\frac{d}{dx}\left[-x^{-n}J_n(x)\right] = x^{-n}J_{n+1}(x)$$

Integrating both sides, we get,

$$\int x^{-n}J_{n+1}(x)\,dx = -x^{-n}J_n(x) \qquad \text{.....(1)}$$

Now, we write the l.h.s. of the required result as

$$\int J_3(x)\,dx = \int x^2\,[x^{-2}J_3(x)]\,dx \qquad [\text{ Note this }]$$

Integrating the r.h.s. by parts,

$$\int J_3(x)\,dx = x^2\int x^{-2}J_3(x)\,dx - \int\left[\int x^{-2}J_3(x)\,dx\,(2x)\right]dx$$

Putting $n = 2$ in (1) above,

$$\int x^{-2}J_3(x)\,dx = -x^{-2}J_2(x)$$

$$\therefore \int J_3(x)\,dx = x^2(-x^{-2})J_2(x) - \int -x^{-2}J_2(x)(2x)\,dx$$

$$= -J_2(x) + 2\int x^{-1}J_2(x)\,dx$$

Putting $n = 1$ in (1) above,

$$\int x^{-1}J_2(x)\,dx = -x^{-1}J_1(x) \qquad \therefore \int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x)$$

Example 6 : Evaluate $\int J_3(x)\,dx$ and express the result in terms of J_0 and J_1. **(M.U. 2014)**

Sol. : We have proved in the above example

$$\int J_3(x)\,dx = -J_2(x) - \frac{2}{x}J_1(x)$$

But by formula (IV), page S-43

$$J_n(x) = \frac{x}{2n}[J_{n+1}(x) + J_{n-1}(x)]$$

Putting $n = 1$,

$$J_1(x) = \frac{x}{2}[J_2(x) + J_0(x)] \qquad \therefore \quad J_2(x) = \frac{2}{x}J_1(x) - J_0(x)$$

Putting this value above

$$\int J_3(x)\,dx = J_0(x) - \frac{4}{x}J_1(x).$$

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Example 7 : Prove that $\int J_0(x)\cos x\,dx = x\,J_0(x)\cos x + x\,J_1(x)\sin x$.

Sol. :
$$I = \int J_0(x)\cos x\,dx = \int [J_0(x)\cos x]\cdot 1\cdot dx$$
$$= J_0(x)\cos x\int 1dx - \int\left[\int 1\cdot dx\cdot\frac{d}{dx}J_0(x)\cos x\right]dx$$
$$= J_0(x)\cos x\cdot x - \int x\left[-J_0(x)\sin x + J_0'(x)\cos x\right]dx$$
$$= x\cdot J_0(x)\cos x - \int x\,J_0'(x)\cos x\,dx + \int x\,J_0(x)\sin x\,dx$$

Putting $n = 1$, in formula V, we get $x\,J_0(x) = \frac{d}{dx}[x\,J_1(x)]$

$$\therefore\quad I = x\cdot J_0(x)\cos x - \int x\,J_0'(x)\cos x\,dx + \int \sin x\cdot\frac{d}{dx}[x\,J_1(x)]dx$$

Now, integrating by parts the last term, we get

$$\therefore\quad I = x\cdot J_0(x)\cos x - \int x\,J_0'(x)\cos x\,dx + \sin x\cdot[x\,J_1(x)] - \int x\,J_1(x)\cdot(\cos x)\,dx$$

But $J_0'(x) = -J_1(x)$ [By Ex. 12 (i), page S-57]

Hence, the second and the fourth terms get cancelled.

$$\therefore\quad I = x\,J_0(x)\cos x + x\,J_1(x)\sin x.$$

Example 8 : Prove that $\int J_5(x)\,dx = -J_4(x) - \frac{4}{x}J_3(x) - \frac{8}{x^2}J_2(x)$.

Sol. : We write $J_5(x)$ as follows and integrate it by parts

$$J_5(x) = \int x^4\cdot[x^{-4}J_5(x)]\,dx \qquad \text{[Note this]}$$
$$\therefore\quad J_5(x) = x^4\left[\int x^{-4}J_5(x)\,dx\right] - \int\left[\int x^{-4}J_5(x)\,dx\right](4x^3)\,dx$$

But by formula VI(A),

$$\int x^{-n}\cdot J_{n+1}(x)\,dx = -x^{-n}J_n(x)$$
$$\therefore\quad J_5(x) = x^4[-x^{-4}J_4(x)] - 4\int x^3\cdot[-x^{-4}J_4(x)]\,dx$$
$$= -J_4(x) + 4\int x^{-1}\cdot J_4(x)\,dx$$

[We write $x^{-1} = x^2\cdot x^{-3}$]

$$\therefore\quad J_5(x) = -J_4(x) + 4\int x^2\cdot[x^{-3}J_4(x)]\,dx$$
$$= -J_4(x) + 4\left[x^2\cdot\int x^{-3}J_4(x)\,dx - \int\left\{\int x^{-3}J_4(x)\,dx\cdot 2x\right\}dx\right]$$

[By VI (A) as above]

$$= -J_4(x) + 4\left[x^2[-x^{-3}J_3(x)] - \int[-x^{-3}J_3(x)]\cdot 2x\,dx\right]$$
$$= -J_4(x) + 4\left[-x^{-1}\cdot J_3(x) + 2\int x^{-2}J_3(x)\,dx\right]$$

[By VI (A) as above]

$$= -J_4(x) + 4\left[-x^{-1}\cdot J_3(x) + 2[-x^{-2}J(x)]\right]$$
$$\therefore\quad J_5(x) = -J_4(x) - \frac{4}{x}J_3(x) - \frac{8}{x^2}J_2(x).$$

Example 9 : Prove that $\int x\cdot J_{2/3}(x^{3/2})\,dx = -\frac{2}{3}x^{-1/2}J_{-1/3}(x^{3/2})$. **(M.U. 2009)**

Sol. : Putting $x^{3/2} = t$ *i.e.* $x = t^{2/3}$, we have $dx = \frac{2}{3}t^{-1/3}\,dt$.

$$\therefore\quad \int x\cdot J_{2/3}(x^{3/2})\,dx = \int t^{2/3}\cdot J_{2/3}(t)\cdot\frac{2}{3}t^{-1/3}\,dt$$
$$\therefore\quad \int x\cdot J_{2/3}(x^{3/2})\,dx = \frac{2}{3}\int t^{1/3}J_{2/3}(t)\,dt$$

But $\int x^{-n}\cdot J_{n+1}(x)\,dx = -x^{-n}J_n(x)$ [By Formula VI(A)]

Putting $n + 1 = \frac{2}{3}$ *i.e.* $n = -\frac{1}{3}$, we get the r.h.s. as

$$\int x\cdot J_{2/3}(x^{3/2})\,dx = \frac{2}{3}\left[-t^{-1/3}J_{-1/3}(t)\right]$$

Resubstituting $t = x^{3/2}$,

$$\therefore\quad \int x\cdot J_{2/3}(x^{3/2})\,dx = -\frac{2}{3}\cdot(x^{3/2})^{-1/3}\cdot J_{-1/3}(x^{3/2})$$
$$= -\frac{2}{3}\cdot x^{-1/2}\cdot J_{-1/3}(x^{3/2}).$$

Example 10 : Prove that $\int x^4\,J_1(x)\,dx = x^4\,J_2(x) - 2x^3J_3(x)$. **(M.U. 2015)**

Sol. : We have $\int x^4J_1(x)\,dx = \int x^2\cdot x^2\,J_1(x)\,dx$

Integrating by parts,

$$\int x^4\cdot J_1(x)\,dx = x^2\int x^2\cdot J_1(x)\,dx - \int\left[\int x^2J_1(x)\,dx\right]\cdot 2x\,dx$$

But by formula V(A), page S-44,

$$\int x^2\cdot J_1(x)\,dx = x^2\,J_2(x) \qquad [\,n = 2\,]$$

and $$\int x^3\cdot J_1(x)\,dx = x^3\,J_2(x) \qquad [\,n = 2\,]$$

$$\therefore\quad \int x^4\cdot J_1(x)\,dx = x^2\cdot x^2\cdot J_2(x) - 2\int x^3\cdot J_2(x)\,dx$$
$$= x^4\cdot J_2(x) - 2x^3\cdot J_3(x) \qquad \text{[By V(A), } n = 3\text{]}$$

Example 11 : Prove that $\int x\log x\cdot J_0(x)\,dx = x\log x\cdot J_1(x) + J_0(x)$.

Sol. : We integrate the l.h.s. by parts by writing

$$\int x\log x\cdot J_0(x)\,dx = \int \log x\cdot x\,J_0(x)\,dx$$
$$\therefore\quad \int x\log x\cdot J_0(x)\,dx = \log x\cdot\int x\,J_0(x)\,dx - \int\left[\int x\,J_0(x)\,dx\right]\cdot\frac{1}{x}dx$$

But $\int x\cdot J_0(x)\,dx = x\,J_1(x)$ [By formula V(A)]

$$\therefore\quad \int x\log x\cdot J_0(x)\,dx = \log x\cdot x\,J_1(x) - \int x\cdot J_1(x)\cdot\frac{1}{x}dx$$
$$= x\log x\cdot J_1(x) - \int J_1(x)\,dx$$

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But $\int J_1(x)\,dx = -J_0(x)$ [By (B), page S-45]

$\therefore \int x \log x \cdot J_0(x)\,dx = x \log x \cdot J_1(x) + J_0(x)$

Example 12 : Prove that $J_{n+1}(x) = x\int_0^1 J_n(xy)\cdot y^{n+1}\,dy$. **(M.U. 2010)**

Sol. : Putting $xy = t$ *i.e.*, $y = \frac{t}{x}$ in r.h.s. we get, $dy = \frac{dt}{x}$, when $y = 0$, $t = 0$ and when $y = 1$, $t = x$.

$$\therefore \quad \text{R.H.S.} = x\int_0^x J_n(t)\cdot\frac{t^{n+1}}{x^{n+1}}\cdot\frac{dt}{x} = \frac{1}{x^{n+1}}\int_0^x t^{n+1}J_n(t)\,dt$$

$$= \frac{1}{x^{n+1}}\left[t^{n+1}J_{n+1}(t)\right]_0^x \qquad \text{[By Formula V(A), page S-44]}$$

$$= \frac{1}{x^{n+1}}\left[x^{n+1}J_{n+1}(x)\right] = J_{n+1}(x).$$

Example 13 : Prove that $\int_0^b x\,J_0(ax)\,dx = \frac{b}{a}J_1(ab)$. **(M.U. 2016)**

Sol. : Putting $ax = t$ and $a\,dx = dt$, when $x = 0$, $t = 0$; when $x = b$, $t = ab$, we get

$$\int_0^b x\,J_0(ax)\,dx = \int_0^{ab}\frac{t}{a}J_0(t)\frac{dt}{a} = \frac{1}{a^2}\int_0^{ab} t\,J_0(t)\,dt \qquad \text{..... (A)}$$

But by recurrence relation (V), page S-44

$$x^n J_{n-1}(x) = \frac{d}{dx}[x^n J_n(x)]$$

Putting $n = 1$, we get

$$x\,J_0(x) = \frac{d}{dx}[x\,J_1(x)] \quad \textit{i.e.,} \quad t\,J_0(t) = \frac{d}{dt}[t\,J_1(t)]$$

Putting this value in (A),

$$\int_0^b x\,J_0(ax)\,dx = \frac{1}{a^2}\int_0^{ab}\frac{d}{dt}[t\,J_1(t)]\,dt = \frac{1}{a^2}\left[t\,J_1(t)\right]_0^{ab}$$

$$= \frac{1}{a^2}[ab\,J_1(ab) - 0] = \frac{b}{a}J_1(ab)$$

Hence, the result.

EXERCISE - I

1. Show that $J_1''(x) = -J_1(x) + \frac{1}{x}\cdot J_2(x)$.
2. Prove that $\frac{d}{dx}[J_n^2(x)] = \frac{x}{2n}[J_{n-1}^2(x) - J_{n+1}^2(n)]$.
3. Show that $\frac{d}{dx}[x^2 J_{n-1}(x)\cdot J_{n+1}(x)] = 2x^2 J_n(x) J_n'(x)$.
4. Prove that $\int x^2 J_1(x)\,dx = 2x\,J_1(x) - x^2 J_0(x)$.
5. Prove that $\int x^4 J_1(x)\,dx = x^4 J_2(x) - 2x^3 J_3(x)$. **(M.U. 2015)**

6. Prove that $\int x^3 J_3(x)\,dx = -x^3 J_2(x) - 5x^2 J_1(x) - 15\,x\,J_0(x) + 15\int J_0(x)\,dx$
7. Prove that $\int x\,J_0^2(x)\,dx = \frac{x^2}{2}[J_0^2(x) + J_1^2(x)]$.
8. Prove that $J_{7/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\left(\frac{15 - 6x^2}{x^3}\right)\sin x - \left(\frac{15}{x^2} - 1\right)\cos x\right]$.

8. Generating Function for $J_n(x)$

Bessel's functions of integral order *i.e.* $J_0(x)$, $J_1(x)$, $J_{-1}(x)$, $J_2(x)$, $J_{-2}(x)$ ..... are also known as **Bessel's Coefficients**. These can be obtained as the coefficients of t^n in the expansion of $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$, $(t \neq 0)$ as shown below. Hence, the function $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$ is called the **generating function** of Bessel's coefficients.

Theorem : For a positive integer n, a function $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)}$ generates the Bessel's function J_n.

i.e. $e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \sum_{-\infty}^{\infty} J_n(x)t^n$, $x \in R$ and $|t| > 0$

Proof : We know that $e^x = 1 + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

$$\therefore \quad e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = e^{\frac{xt}{2} - \frac{x}{2t}} = e^{\frac{xt}{2}}\cdot e^{\frac{-x}{2t}}$$

$$e^{\frac{x}{2}\left(t - \frac{1}{t}\right)} = \left[1 + \left(\frac{xt}{2}\right) + \frac{1}{2!}\left(\frac{xt}{2}\right)^2 + \ldots + \frac{1}{n!}\left(\frac{xt}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{xt}{2}\right)^{n+1} + \frac{1}{(n+2)!}\left(\frac{xt}{2}\right)^{n+2} + \ldots\right]\times\left[1 + \left(\frac{-x}{2t}\right) + \frac{1}{2!}\left(\frac{-x}{2t}\right)^2 + \ldots + \frac{1}{n!}\left(\frac{-x}{2t}\right)^n + \frac{1}{(n+1)!}\left(\frac{-x}{2t}\right)^{n+1} + \frac{1}{(n+2)!}\left(\frac{-x}{2t}\right)^{n+2} + \ldots\right] \qquad \text{..... (I)}$$

We collect the coefficients of various powers of t.

Case I : Coefficient of $t^n = \frac{1}{n!}\left(\frac{x}{2}\right)^n(1) + \frac{1}{(n+1)!}\left(\frac{x}{2}\right)^{n+1}\left(\frac{-x}{2}\right) + \frac{1}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+2}\left(\frac{-x}{2}\right)^2 + \ldots$

$$= \frac{(-1)^0}{0!\cdot n!}\left(\frac{x}{2}\right)^{n+0} + \frac{(-1)^1}{1!\cdot(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!\cdot(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \ldots$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}}{m!(n+m)!} = \sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{2m+n}}{m!\,\lfloor n+m+1}$$

$$= J_n(x)$$

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Case II : The coefficient of t^{-n}

$$= \frac{1}{n!}\left(\frac{-x}{2}\right)^n + \frac{1}{(n+1)!}\left(\frac{-x}{2}\right)^{n+1}\left(\frac{x}{2}\right) + \frac{1}{2!(n+2)!}\left(\frac{-x}{2}\right)^{n+2}\left(\frac{x}{2}\right)^2 + \ldots\ldots$$

$$= \frac{(-1)^n}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^{n+1}}{(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^{n+2}}{2!(n+2)}\left(\frac{x}{2}\right)^{n+4} + \ldots\ldots$$

$$= (-1)^n\left[\frac{1}{n!}\left(\frac{x}{2}\right)^n + \frac{(-1)^1}{1!(n+1)!}\left(\frac{x}{2}\right)^{n+2} + \frac{(-1)^2}{2!(n+2)!}\left(\frac{x}{2}\right)^{n+4} + \ldots\ldots\right]$$

$$= (-1)^n J_n(x) = J_{-n}(x) \qquad \text{[By Ex. 4, page S-51]}$$

Case III : The term independent of t

$$= 1 - \frac{x^2}{2^2} + \frac{1}{(2!)^2}\cdot\frac{x^4}{4^2} + \ldots\ldots = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \ldots\ldots$$

$$= J_0(x)$$

On the r.h.s. of (I), the index of t varies from $-\infty$ to ∞.

Hence, $$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \ldots\ldots + t^{-2}J_{-2}(x) + t^{-1}J_{-1}(x) + t^0 J_0(x) + t^1 J_1(x) + t^2 J_2(x) + \ldots\ldots \qquad \ldots\ldots\ldots\ldots \text{(A)}$$

$$\therefore\ e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

Note ....

We know that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = \sum_{n=-\infty}^{\infty} t^n J_n(x)$$

$$= J_0(x) + t\cdot J_1(x) + t^2 J_2(x) + \ldots\ldots + t^{-1}J_{-1}(x) + t^{-2}J_{-2}(x) + \ldots\ldots$$

Since $J_{-n}(x) = (-1)^n J_n(x)$

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + tJ_1(x) + t^2 J_2(x) + \ldots\ldots + (-1)^1 t^{-1}J_1(x) + (-1)^2 t^{-2}J_2(x) + \ldots\ldots$$

$$= J_0(x) + \left(t - \frac{1}{t}\right)J_1(x) + \left(t^2 + \frac{1}{t^2}\right)J_2(x) + \left(t^3 - \frac{1}{t^3}\right)J_3(x) + \ldots\ldots$$

From (A) we see that for integral values of n, $J_n(x)$ = coefficient of t^n in the expansion of $e^{(x/2)[t-(1/t)]}$.

Jacobi's Series

Example 1 : Show that

$$\cos(x\sin\theta) = J_0(x) + 2\cos 2\theta\, J_2(x) + 2\cos 4\theta\, J_4(x) + \ldots\ldots$$

and $$\sin(x\sin\theta) = 2\sin\theta\, J_1(x) + 2\sin 3\theta\, J_3(x) + 2\sin 5\theta\, J_5(x) + \ldots\ldots$$

Hence, prove that

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - 2J_6(x) + \ldots\ldots$$

and $$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \ldots\ldots$$

Sol. : We know that

$$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = J_0(x) + \left(t - \frac{1}{t}\right)J_1(x) + \left(t^2 + \frac{1}{t^2}\right)J_2(x) + \left(t^3 - \frac{1}{t^3}\right)J_3(x) + \ldots\ldots \qquad \ldots\ldots\ldots\ldots (1)$$

Put $t = e^{i\theta}$ $\therefore\ t^{-1} = e^{-i\theta}$

$$\therefore\ t^n - \frac{1}{t^n} = e^{in\theta} - e^{-in\theta} = (\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)$$

$$= 2i\sin n\theta,\quad n = 1, 2, \ldots\ldots$$

$$\therefore\ \left(t - \frac{1}{t}\right) = 2i\sin\theta,\ \left(t^2 - \frac{1}{t^2}\right) = 2i\sin 2\theta, \ldots\ldots \qquad \ldots\ldots\ldots\ldots (2)$$

And $$t^n + \frac{1}{t^n} = e^{in\theta} + e^{in\theta} = (\cos n\theta + i\sin n\theta) + (\cos n\theta - i\sin n\theta)$$

$$= 2\cos n\theta,\quad n = 1, 2, 3, \ldots\ldots$$

$$\therefore\ \left(t + \frac{1}{t}\right) = 2\cos\theta,\ \left(t^2 + \frac{1}{t^2}\right) = 2\cos 2\theta, \ldots\ldots \qquad \ldots\ldots\ldots\ldots (3)$$

Also $$e^{\frac{x}{2}\left(t-\frac{1}{t}\right)} = e^{\frac{x}{2}\left[e^{i\theta} - e^{-i\theta}\right]} = e^{\frac{x}{2}[2i\sin\theta]} = e^{ix\sin\theta}$$

$$= \cos(x\sin\theta) + i\sin(x\sin\theta) \qquad \ldots\ldots\ldots\ldots (4)$$

Therefore, putting the values from (2), (3) and (4) in (1) we get

$$\cos(x\sin\theta) + i\sin(x\sin\theta) = J_0(x) + 2i\sin\theta\, J_1(x) + 2\cos 2\theta\, J_2(x) + 2i\sin 3\theta\, J_3(x) + \ldots\ldots$$

$$= [\, J_0(x) + 2\cos 2\theta\, J_2(x) + 2\cos 4\theta\, J_4(x) + \ldots\ldots]$$

$$+ i\,[2\sin\theta\, J_1(x) + 2\sin 3\theta\, J_3(x) + \ldots\ldots] \qquad \ldots\ldots\ldots\ldots (5)$$

Equating the real and imaginary parts, we get,

$$\cos(x\sin\theta) = J_0(x) + 2\cos 2\theta\, J_2(x) + 2\cos 4\theta\, J_4(x) + \ldots\ldots \qquad \ldots\ldots\ldots\ldots (6)$$

and $$\sin(x\sin\theta) = 2\sin\theta\, J_1(x) + 2\sin 3\theta\, J_3(x) + \ldots\ldots \qquad \ldots\ldots\ldots\ldots (7)$$

When $\theta = \pi/2$, we have $\sin\theta = 1$ and

$$\cos 2n\theta = \cos\left(2n\frac{\pi}{2}\right) = \cos(n\pi) = +1 \quad \text{if } n \text{ is even}$$

$$= -1 \quad \text{if } n \text{ is odd}$$

and $$\sin(2n+1)\theta = \sin\left[(2n+1)\frac{\pi}{2}\right] = \sin\left[n\pi + \frac{\pi}{2}\right]$$

$$= \cos n\pi = +1 \quad \text{if } n \text{ is even}$$

$$= -1 \quad \text{if } n \text{ is odd.}$$

Therefore, putting $\theta = \frac{\pi}{2}$, from (6) and (7), we get

$$\cos x = J_0(x) - 2J_2(x) + 2J_4(x) - \ldots\ldots$$

$$\sin x = 2J_1(x) - 2J_3(x) + 2J_5(x) - \ldots\ldots$$

Remark ....

The series (6) and (7) are known as Jacobi's series.

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Example 2 : Prove that $J_0^2 + 2J_1^2 + 2J_2^2 + \ldots = 1$.

Sol. : We have from the above example,

$$\cos(x\sin\theta) = J_0 + 2J_2 \cdot \cos 2\theta + 2J_4 \cos 4\theta + \ldots \quad \ldots\ldots (1)$$

$$\sin(x\sin\theta) = 2J_1 \sin\theta + 2J_3 \sin 3\theta + 2J_5 \sin 5\theta + \ldots \quad \ldots\ldots (2)$$

Squaring (1) and integrating from 0 to π w.r.t. θ, we have

$$\int_0^\pi \cos^2(x\sin\theta)\,d\theta = \int_0^\pi (J_0 + 2J_2\cos 2\theta + \ldots)^2\,d\theta$$

$$= \int_0^\pi J_0^2 d\theta + 4\int_0^\pi J_2^2\cos^2 2\theta\,d\theta + 4\int_0^\pi J_4^2\cos^4\theta\,d\theta + \ldots$$

$$\ldots + 4\int_0^\pi J_0J_2\cos 2\theta\,d\theta + 4\int_0^\pi J_0J_4\cos 4\theta\,d\theta + \ldots$$

$$\ldots + 8J_2J_4\int_0^\pi \cos 2\theta\cos 4\theta\,d\theta + 8J_2J_6\int_0^\pi \cos 2\theta\cos 6\theta + \ldots$$

Now, $\int_0^\pi 2\cos^2 n\theta\,d\theta = \int_0^\pi (1+\cos 2n\theta)\,d\theta = \left[\theta + \frac{\sin 2n\theta}{2n}\right]_0^\pi = \pi$

Also, $\int_0^\pi \cos n\theta\,d\theta = \left[\frac{\sin n\theta}{n}\right]_0^\pi = 0$

$$\int_0^\pi \cos n\theta\cos m\theta\,d\theta = \frac{1}{2}\int_0^\pi [\cos(n+m)\theta + \cos(n-m)\theta]\,d\theta$$

$$= \frac{1}{2}\left[\frac{\sin(n+m)\theta}{n+m} + \frac{\sin(n-m)\theta}{n-m}\right]_0^\pi = 0$$

$$\therefore \int_0^\pi \cos^2(x\sin\theta)\,d\theta = J_0^2\pi + 2J_2^2\pi + 2J_4^2\pi + \ldots \quad \ldots\ldots (3)$$

Similarly, squaring (2) and integrating from 0 to π w.r.t. θ, we can show that

$$\int_0^\pi \sin^2(x\sin\theta)\,d\theta = 2J_1^2\pi + 2J_3^2\pi + 2J_5^2\pi + \ldots \quad \ldots\ldots (4)$$

Adding (3) and (4), we get,

$$\pi\left[J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + 2J_4^2 + \ldots\right] = \int_0^\pi \cos^2(x\sin\theta)\,d\theta + \int_0^\pi \sin^2(x\sin\theta)\,d\theta$$

$$= \int_0^\pi d\theta = \pi$$

$$\therefore J_0^2 + 2J_1^2 + 2J_2^2 + 2J_3^2 + 2J_4^2 + \ldots = 1.$$

9. Bessel's Integral

Prove that (i) $\int_0^\pi \cos x\sin\theta\,d\theta = \pi J_0$

(ii) $\int_0^\pi \cos(x\theta - x\sin\theta)\,d\theta = \pi J_n(x)$.

Proof : In Ex. 1 above we have proved that[See (6) and (7)]

$$\cos(x\sin\theta) = J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots \quad \ldots\ldots (1)$$

$$\sin(x\sin\theta) = 2[J_1(x)\sin\theta + J_3(x)\sin 3\theta + J_5(x)\sin 5\theta + \ldots] \quad \ldots\ldots (2)$$

Integrating (1) w.r.t. q between 0 and p, we get

$$\int_0^\pi \cos(x\sin\theta)\,d\theta = \int_0^\pi [J_0(x) + 2J_2(x)\cos 2\theta + 2J_4(x)\cos 4\theta + \ldots]\,d\theta$$

$$\therefore \int_0^\pi \cos(x\sin\theta)\,d\theta = J_0(x)\int_0^\pi d\theta + 2J_2(x)\int_0^\pi \cos 2\theta\,d\theta + 2J_4(x)\int_0^\pi \cos 4\theta\,d\theta + \ldots$$

$$= J_0(x)[\theta]_0^\pi + 2J_2(x)\left[\frac{\sin 2\theta}{2}\right]_0^\pi + 2J_4(x)\left[\frac{\sin 4\theta}{4}\right]_0^\pi + \ldots$$

$$= \pi J_0(x) \quad \text{[Other integrals being zero]}$$

Again multiply (1) by $\cos n\theta$ and integrate between 0 and π.

$$\therefore \int_0^\pi \cos(x\sin\theta)\cos n\theta\,d\theta = \int_0^\pi J_0(x)\cos n\theta\,d\theta + \int_0^\pi 2J_2(x)\cos 2\theta\cos n\theta\,d\theta$$

$$+ \int_0^\pi 2J_4(x)\cos 4\theta\cos n\theta\,d\theta + \ldots$$

Now, each integral is of the form,

$$I = \int_0^\pi 2\cos m\theta\cos n\theta\,d\theta = \int_0^\pi [\cos(m+n)\theta + \cos(m-n)\theta]\,d\theta$$

$$\therefore I = \left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right]_0^\pi \quad \ldots\ldots (A)$$

$= 0$ if $m \neq n$ (m, n are both positive integers)

If $m = n$, then $I = \int_0^\pi 2\cos^2 n\theta\,d\theta = \int_0^\pi (1+\cos 2n\theta)\,d\theta = \left[\theta + \frac{\sin 2n\theta}{2n}\right]_0^\pi = \pi$

(m takes values 0, 2, 4, ..., n is any integer)

If $m = n$, n is even and the integral is zero. If $m \neq n$, n is odd by (A) and the integral is π.

$$\therefore \int_0^\pi \cos(x\sin\theta)\cos n\theta\,d\theta = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \pi J_n(x) & \text{if } n \text{ is even} \end{cases} \quad \ldots\ldots (3)$$

Again multiply (2) by $\sin n\theta$ and integrate between 0 and π.

$$\therefore \int_0^\pi \sin(x\sin\theta)\sin n\theta\,d\theta = 2\int_0^\pi J_1\sin\theta\sin n\theta\,d\theta + 2\int_0^\pi J_3\sin 3\theta\sin n\theta\,d\theta + \ldots$$

Now, each integral is of the form,

$$I = \int_0^\pi 2\sin m\theta\sin n\theta\,d\theta = -\int_0^\pi [\cos(m+n)\theta - \cos(m-n)\theta]\,d\theta$$

$$= -\left[\frac{\sin(m+n)\theta}{m+n} - \frac{\sin(m-n)\theta}{m-n}\right]_0^\pi \quad \ldots\ldots (B)$$

$= 0$ if $m \neq n$ (m and n are both positive integers)

If $m = n$, then $I = \int_0^\pi 2\sin^2 n\theta\,d\theta = \int_0^\pi (1-\cos 2n\theta)\,d\theta = \left[\theta - \frac{\sin 2n\theta}{2n}\right]_0^\pi = \pi$

Here, m takes values 1, 3, 5, ... and n is any integer. If $m = n$, n is odd and the integral is zero by (B). If $m \neq n$, n is even and the integral is π.

$$\therefore \int_0^\pi \sin(x\sin\theta)\sin n\theta\,d\theta = \begin{cases} 0 & \text{if } n \text{ is even} \\ \pi J_n(x) & \text{if } n \text{ is odd} \end{cases} \quad \ldots\ldots (4)$$

Combining (3) and (4), we get,

$$\int_0^\pi [\cos(x\sin\theta)\cos n\theta + \sin(x\sin\theta)\cos\theta]\,d\theta = \pi J_n(x)$$

$$\therefore \int_0^\pi \cos(x\sin\theta - n\theta)\,d\theta = \pi J_n(x)$$

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$$\therefore \quad J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta$$

This is known as **Bessel's integral** form of $J_n(x)$.

10. Orthogonal Property of Bessel Functions

The following property of Bessel functions is known as **orthogonal property** of Bessel Functions.

If α, β are the roots of $J_n(ax) = 0$, then

$$\int_0^a x\,J_n(\alpha x)\,J_n(\beta x)\,dx = 0 \qquad \text{if } \alpha \neq \beta$$

$$= \frac{a^2}{2} J_{n+1}{}^2(a\alpha) \qquad \text{if } \alpha = \beta$$

Proof : Consider the following two Bessel's equations

$$x^2\frac{d^2u}{dx^2} + x\frac{du}{dx} + (\alpha^2x^2 - n^2)u = 0$$

i.e., $\quad x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0 \qquad$ ..... (1)

and $\quad x^2\dfrac{d^2v}{dx^2} + x\dfrac{dv}{dx} + (\beta^2x^2 - n^2)v = 0$

i.e., $\quad x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0 \qquad$ ..... (2)

Let the solutions of these two equations be $u = J_n(\alpha x)$ and $v = J_n(\beta x)$.

Now, multiply (1) by $\dfrac{v}{x}$ and (2) by $\dfrac{u}{x}$ and subtract.

$$\therefore \quad xu''v + u'v + (\alpha^2x^2 - n^2)\frac{uv}{x} - xuv'' - uv' - (\beta^2x^2 - n^2)\frac{uv}{x} = 0$$

$$\therefore \quad x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)\,xuv = 0$$

$$\therefore \quad x(u''v - uv'') + (u'v - uv') = (\beta - \alpha^2)\,xuv \qquad \text{..... (3)}$$

But $\quad \dfrac{d}{dx}[x(u'v - uv')] = x(u''v + u'v' - u'v' - uv'') + (u'v - uv')$

$$= x(u''v - uv'') + (u'v - uv')$$

Hence, the equation (3) can be written as

$$\frac{d}{dx}[x(u'v - uv')] = (\beta^2 - \alpha^2)\,xuv \qquad \text{..... (4)}$$

Integrating the equation (4) from 0 to a, we get

$$\int_0^a \frac{d}{dx}[x(u'v - uv')]\,dx = (\beta^2 - \alpha^2)\int_0^a xuv\,dx$$

$$\therefore \quad [x(u'v - uv')]_0^a = (\beta^2 - \alpha^2)\int_0^a xuv\,dx$$

$$\therefore \quad [a(u'v - uv')]_{\text{at } x=a} = (\beta^2 - \alpha^2)\int_0^a xuv\,dx \qquad \text{..... (5)}$$

But $\quad u = J_n(\alpha x) \qquad \therefore \quad u' = \dfrac{d}{dx}[J_n(\alpha x)] = \alpha J_n'(\alpha x)$

and $\quad v = J_n(\beta x) \qquad \therefore \quad v' = \dfrac{d}{dx}[J_n(\beta x)] = \beta J_n(\beta x)$

$$\therefore \quad (\beta^2 - \alpha^2)\int_0^a x\,uv\,dx = a[\alpha J_n'(\alpha a)J_n(\beta a) - \beta J_n(\alpha a)J_n'(\beta a)]$$

$$\therefore \quad \int_0^a x\,uv\,dx = \frac{a}{\beta^2 - \alpha^2}[\alpha J_n'(\alpha a)J_n(\beta a) - \beta J_n(\alpha a)J_n'(\beta a)] \qquad \text{..... (6)}$$

Case I : If $\alpha \neq \beta$ *i.e.*, if α, β are two distinct roots of $J_n(ax) = 0$, then $J_n(a\alpha) = 0$ and $J_n(a\beta) = 0$ and hence r.h.s. of (6) is zero.

$$\therefore \quad \int_0^a x\,uv\,dx = 0 \qquad \therefore \quad \int_0^a x\,J_n(\alpha x)\cdot J_n(\beta x)\,dx = 0$$

Case II : If $\alpha = \beta$ *i.e.*, the root of $J_n(ax) = 0$ are equal then the r.h.s. of (6) is of the form $\dfrac{0}{0}$. (Put $\beta = \alpha$ in r.h.s. of (6) and verify it.)

Hence, assuming α is a root of $J_n(ax) = 0$ we evaluate the r.h.s. of (6) as $\beta \to \alpha$.

i.e., $$\lim_{\beta\to\alpha}\int_0^a x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\left(\frac{a}{\beta^2 - \alpha^2}\right)[\alpha J_n'(\alpha a)J_n(\beta a) - 0]$$

because $J_n(a\alpha) = 0$.

Now, applying L'Hospital's rule (*i.e.*, differentiating w.r.t. β)

$$\lim_{\beta\to\alpha}\int_0^a x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \lim_{\beta\to\alpha}\frac{a}{2\beta}[\alpha J_n'(\alpha a)\cdot a J_n'(\beta a)]$$

$$= \frac{a^2}{2\alpha}\cdot\alpha\cdot[J_n'(\alpha a)]^2$$

$$= \frac{a^2}{2}[J_n'(\alpha a)]^2 \qquad \text{..... (7)}$$

But by formula (I), page S-40

$$x\,J_n'(x) = n\,J_n(x) - x\,J_{n+1}(x)$$

Putting $x = a\alpha$, we have

$$a\alpha\,J_n'(a\alpha) = n\,J_n(a\alpha) - a\alpha\,J_{n+1}(a\alpha)$$

But $\quad J_n(a\alpha) = 0$

$$\therefore \quad a\alpha\,J_n'(a\alpha) = -a\alpha\,J_{n+1}(a\alpha)$$

i.e., $\quad J_n'(a\alpha) = -J_{n+1}(a\alpha)$

Hence, from (7), we get if $\alpha \neq \beta$,

$$\int_0^a x\,J_n(\alpha x)\,J_n(\beta x)\,dx = \frac{a^2}{2}\cdot J_{n+1}{}^2(a\alpha) \qquad \text{..... (8)}$$

11. Bessel-Fourier Series of $f(x)$

We can expand a function $f(x)$ in Bessel-Fourier series by using orthogonality property of Bessel function in the range 0 to a.

Let $\quad f(x) = c_1J_n(\lambda_1x) + c_2J_n(\lambda_2x) + \ldots + c_nJ_n(\lambda_nx) + \ldots^{\infty}$

$$= \sum_{i=1}^{\infty} c_i\,J_n(\lambda_ix) \qquad \text{..... (9)}$$

where $\lambda_1, \lambda_2 \ldots \lambda_n, \ldots$ are the roots of the equation $J_n(\lambda a) = 0$.

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To find c_i we multiply both sides of (1) by $x\,J_n(\lambda_i x)$ and integrate w.r.t. x between the limits 0 and a. By orthogonality property all integrals on the r.h.s. will be zero except the one containing λ_i *i.e.* c_i. Thus, we have

$$\int_0^a x\,J_n(\lambda_i x)f(x)dx = c_1\int_0^a x\,J_n(\lambda_1 x)J_n(\lambda_i x)dx$$
$$+\,c_2\int_0^a x\,J_n(\lambda_2 x)J_n(\lambda_i x)dx + \ldots\ldots$$
$$+\,c_i\int_0^a x\,J_n(\lambda_i x)\cdot J_n(\lambda_i x)dx + \ldots\ldots$$
$$= c_i\int_0^a x\,J_n^2(\lambda_i x)dx = c_i\frac{a^2}{2}J_{n+1}^2(\lambda_i a) \qquad \text{[By (8) above]}$$
$$\therefore\quad c_i = \frac{2}{a^2 J_{n+1}^2(\lambda_i a)}\int_0^a x\,J_n(\lambda_i x)\cdot f(x)dx \qquad \ldots\ldots\ldots\ldots (10)$$

Example 1 : Expand $f(x) = 1$ in $(0 < x < 1)$ in a series as

$$1 = \Sigma\frac{2}{\lambda_n J_1(\lambda_n)}\cdot J_0(\lambda_n x) \qquad \text{where } \lambda_1, \lambda_2, \ldots\ldots, \lambda_n, \ldots\ldots \text{ are positive roots of } J_0(x) = 0.$$

Sol. : We have seen above that if $f(x)$ can be expanded as

$$f(x) = \sum_{i=1}^{\infty} c_i\,J_n(\lambda_i x) \qquad \ldots\ldots\ldots\ldots (1)$$

then $\quad c_i = \dfrac{2}{a^2 J_{n+1}^2(\lambda_i a)}\cdot\int_0^a x\cdot J_n(\lambda_i x)f(x)dx$.

Here, we have $f(x) = 1$, $a = 1$ and $n = 0$ by data

$$c_i = \frac{2}{J_1^2(\lambda_i)}\int_0^1 x\,J_0(\lambda_i x)\cdot 1\cdot dx$$

Putting $\lambda_i x = t$, $\lambda_i dx = dt$. When $x = 0$, $t = 0$; when $x = 1$, $t = \lambda_i$

$$\therefore\quad c_i = \frac{2}{J_1^2(\lambda_i)}\int_0^{\lambda_i}\frac{t}{\lambda_i}\cdot J_0(t)\cdot\frac{dt}{\lambda_i} = \frac{2}{\lambda_i^2 J_1^2(\lambda_i)}\int_0^{\lambda_i} t\cdot J_0(t)dt \qquad \ldots\ldots\ldots\ldots (2)$$

But by formula (V) page S-44,

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x) \qquad \therefore\quad x^n J_n(x) = \int x^n J_{n-1}(x)dx$$

Putting $n = 1$, we have

$$x\,J_1(x) = \int x\,J_0(x)dx \qquad \therefore\quad \int_0^{\lambda_i} x\,J_0(x)dx = \left[x\,J_1(x)\right]_0^{\lambda_i}$$

Hence, from (2), we get

$$c_i = \frac{2}{\lambda_i J_1^2(\lambda_i)}\cdot\left[t\,J_1(t)\right]_0^{\lambda_i} = \frac{2}{\lambda_i^2 J_1^2(\lambda_i)}\cdot\lambda_i J_1(\lambda_i) = \frac{2}{\lambda_i J_1(\lambda_i)}$$

Hence, from (1), we get

$$1 = f(x) = \sum_{i=1}^{\infty}\frac{2}{\lambda_i J_1(\lambda_i)}\cdot J_0(\lambda_i x) \qquad [\because\ n = 0 \text{ by data}]$$
$$\therefore\quad \frac{1}{2} = \sum_{i=1}^{\infty}\frac{1}{\lambda_i J_1(\lambda_i)}J_0(\lambda_i x)$$

Varying i from 1 to ∞,

$$\frac{1}{2} = \frac{J_0(\lambda_1 x)}{\lambda_1 J_1(\lambda_1)} + \frac{J_0(\lambda_2 x)}{\lambda_2 J_1(\lambda_2)} + \frac{J_0(\lambda_3 x)}{\lambda_3 J_1(\lambda_3)} + \ldots\ldots$$

Example 2 : Show that Bessel-Fourier series in $J_2(\lambda_n x)$ for $f(x) = x^2$, $(0 < x < a)$ where $(\lambda_n x)$ are positive roots of $J_2(x) = 0$ is

$$x^2 = 2a\,\Sigma\frac{J_2(\lambda_n x)}{\lambda_n J_3(\lambda_n a)}$$

Sol. : Let the Bessel-Fourier series be

$$x^2 = f(x) = \Sigma\,c_i\,J_2(\lambda_i x) \qquad \ldots\ldots\ldots\ldots (1)$$

To evaluate c_i we multiply both sides of (1) by $x\,J_2(\lambda_n x)$ and integrate between the limits 0 and a.

$$\therefore\quad \int_0^a x^2\cdot x\,J_2(\lambda_n x)dx = \sum_{i=1}^{\infty} c_i\int_0^a x\,J_2(\lambda_n x)\cdot J_2(\lambda_i x)dx$$
$$= c_n\int_0^a x\,J_2^2(\lambda_n x)dx \quad [\because \text{ By § 10, page S-72, the integral is zero where } \lambda_n \neq \lambda_i]$$
$$= c_n\cdot\frac{a^2}{2}J_3^2(\lambda_n a) \quad [\because \text{ From (8), page S-73, putting } n = 2]$$
$$\therefore\quad c_n = \frac{2}{a^2 J_3^2(\lambda_n a)}\int_0^a x^3 J_2(\lambda_n x)dx$$

But by formula (V), page S-44

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x) \qquad \therefore\quad x^n\cdot J_n(x) = \int x^n J_{n-1}(x)dx$$
$$\therefore\quad \int_0^a x^3 J_2(\lambda_n x)dx = \left[\frac{x^3 J_3(\lambda_n x)}{\lambda_n}\right]_0^a = \frac{a^3\cdot J_3(\lambda_n a)}{\lambda_n} \qquad [\text{Put } \lambda_n x = t]$$

Hence, $\quad c_n = \dfrac{2}{a^2 J_3^2(\lambda_n a)}\cdot\dfrac{a^3\cdot J_3(\lambda_n a)}{\lambda_n} = \dfrac{2a}{\lambda_n J_3(\lambda_n a)}$

Hence, from (1), we get,

$$x^2 = 2a\sum_{n=1}^{\infty}\frac{1}{\lambda_n J_3(\lambda_n a)}\cdot J_2(\lambda_n x)$$

Example 3 : Expand $f(x) = 4x - x^3$ in $(0 < x < 2)$ in a series as

$$4x - x^3 = 8\,\Sigma\frac{J_2(2\lambda_n)}{\lambda_n^2 J_2^2(2\lambda_n)}\cdot J_1(\lambda_n x)$$

where $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n, \ldots\ldots$ are the positive roots of $J_1(2\lambda) = 0$.

Sol. : Let the Bessel-Fourier series be

$$4x - x^3 = f(x) = \Sigma\,c_i\,J_1(\lambda_i x) \qquad \ldots\ldots\ldots\ldots (1)$$

To evaluate c_i we multiply both sides of (1) by $x\,J_1(\lambda_n x)$ and integrate between the limits 0 and 2

$$\therefore\quad \int_0^2(4x - x^3)x\,J_1(\lambda_n x)dx = \sum_{i=1}^{\infty} c_i\int_0^2 x\,J_1(\lambda_n x)\cdot J_1(\lambda_i x)dx$$
$$= c_n\int_0^2 x\,J_1^2(\lambda_n x)dx \quad [\because \text{ By § 10, page S-72, the integral is zero where } \lambda_n \neq \lambda_i]$$

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$$= c_n \cdot \frac{4}{2} \cdot J_2^2(2\lambda_n) \quad [\because \text{ From (8), page S-73, putting } n = 2]$$

$$\therefore \quad c_n = \frac{1}{2 \cdot J_2^2(2\lambda_n)} \cdot \int_0^2 (4x - x^3) \cdot x \cdot J_1(\lambda_n x)\,dx \qquad \text{..... (2)}$$

Integrating by parts, we get,

$$\int_0^2 (4x - x^3) \cdot x \cdot J_1(\lambda_n x)\,dx = \int_0^2 (4 - x^2)\,x^2 \cdot J_1(\lambda_n x)\,dx$$

$$= \left[(4 - x^2) \cdot x^2 \cdot \frac{J_2(\lambda_n x)}{\lambda_n}\right]_0^2 - \int_0^2 (-2x) \cdot x^2 \cdot \frac{J_2(\lambda_n x)}{\lambda_n}\,dx$$

[$\because$ By formula (V), page S-44

$$\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)$$

$\therefore \quad x^n J_n(x) = \int x^n J_{n-1}(x)\,dx$. Put $\lambda_n x = t$]

$$\therefore \quad I = 0 + \int_0^2 \frac{2}{\lambda_n} \cdot x^3 \cdot J_2(\lambda_n x)\,dx \quad \therefore \quad I = \frac{2}{\lambda_n}\left[x^3 \cdot \frac{1}{\lambda_n} J_3(\lambda_n x)\right]_0^2 = \frac{2}{\lambda_n^2} \cdot 8 \cdot J_3(2\lambda_n)$$

Hence, from (2),

$$\therefore \quad c_n = \frac{1}{2 \cdot J_2^2(2\lambda_n)} \cdot \frac{16}{\lambda_n^2} \cdot J_3(2\lambda_n) = \frac{8\,J_3(2\lambda_n)}{\lambda_n^2\,J_2^2(2\lambda_n)}$$

Hence, from (1), we have

$$4x - x^3 = 8 \sum_{n=1}^{\infty} \frac{J_3(2\lambda_n)}{\lambda_n^2 \cdot J_2(2\lambda_n)} \cdot J_1(\lambda_n x).$$

Miscellaneous Examples

Example 1 : Prove that $y = x^{-n} J_n(x)$ is a solution of

$$x\frac{d^2y}{dx^2} + (1 + 2n)\frac{dy}{dx} + xy = 0. \qquad \textbf{(M.U. 2008)}$$

Sol. : By data $y = x^{-n} J_n(x)$

$$\therefore \quad \frac{dy}{dx} = -n\,x^{-n-1} J_n(x) + x^{-n} J_n'(x)$$

And $$\frac{d^2y}{dx^2} = (-n)(-n-1)\,x^{-n-2} J_n(x) - n\,x^{-n-1} J_n'(x) - n\,x^{-n-1} J_n'(x) + x^{-n} J_n''(x)$$

$$= n(n+1)\,x^{-n-2} J_n(x) - 2n\,x^{-n-1} J_n'(x) + x^{-n} J_n''(x)$$

Putting these values in the l.h.s. of the given equation, we have

$$\text{l.h.s.} = n(n+1)\,x^{-n-1} J_n(x) - 2n\,x^{-n} J_n'(x) + x^{-n+1} J_n''(x) - n\,x^{-n-1} J_n(x) + x^{-n} J_n'(x)$$
$$- 2n^2 x^{-n-1} J_n(x) + 2n\,x^{-n} J_n'(x) + x^{-n+1} J_n(x)$$

$$= x^{-n-1}\Big[n(n+1) J_n(x) - 2nx\,J_n'(x) + x^2 J_n''(x)$$
$$- n\,J_n(x) + x\,J_n'(x) - 2n^2 J_n(x) + 2n\,x J_n'(x) + x^2 J_n(x)\Big]$$

$$\text{l.h.s.} = x^{-n-1}\left[x^2 J_n''(x) + x\,J_n'(x) + (x^2 - n^2) J_n(x)\right] \qquad \text{..... (2)}$$

But $J_n(x)$ is a solution of Bessel's equation.

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0$$

$$\therefore \quad x^2 J_n''(x) + x\,J_n'(x) + (x^2 - n^2) J_n(x) = 0$$

Hence, l.h.s. = 0 = r.h.s.

Example 2 : Prove that $y = \sqrt{x} \cdot J_n(x)$ is a solution of the equation

$$x^2\frac{d^2y}{dx^2} + \left(x^2 - n^2 + \frac{1}{4}\right)y = 0. \qquad \textbf{(M.U. 2010)}$$

Sol. : $\because \quad y = x^{1/2} J_n(x), \quad \dfrac{dy}{dx} = \dfrac{1}{2}x^{-1/2} J_n(x) + x^{1/2} J_n'(x)$

$$\frac{d^2y}{dx^2} = -\frac{1}{4}x^{-3/2} J_n(x) + \frac{1}{2}x^{-1/2} J_n'(x) + \frac{1}{2}x^{-1/2} J_n'(x) + x^{1/2} J_n''(x)$$

$$= -\frac{1}{4}x^{-3/2} J_n(x) + x^{-1/2} J_n'(x) + x^{1/2} J_n''(x)$$

Putting these values in l.h.s. of the given equation,

$$\text{l.h.s.} = -\frac{1}{4}x^{1/2} J_n(x) + x^{3/2} J_n'(x) + x^{5/2} J_n''(x) + x^{5/2} J_n(x) - n^2 \cdot x^{1/2} J_n(x) + \frac{1}{4}x^{1/2} J_n(x)$$

$$= x^{1/2}\left[x^2 J_n''(x) + x\,J_n'(x) + (x^2 - n^2) J_n(x)\right]$$

Since $J_n(x)$ satisfies Bessel's equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)\,y = 0, \text{ the above bracket is zero.}$$

$\therefore$ l.h.s. = 0 = r.h.s.

Hence, the result.

Example 3 : Prove that $\displaystyle\int_0^{\pi/2} J_1(x\cos\theta)\,d\theta = \frac{1 - \cos x}{x}$.

Sol. : By definition

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\,\overline{|n+m+1}}$$

Putting $n = 1$, we have

$$J_1(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\overline{|m+2}} \cdot \left(\frac{x}{2}\right)^{2m+1}$$

$$\therefore \quad J_1(x\cos\theta) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+1)!} \cdot \left(\frac{x}{2}\right)^{2m+1} (\cos\theta)^{2m+1}$$

$$\therefore \quad \int_0^{\pi/2} J_1(x\cos\theta)\,d\theta = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+1)!} \cdot \left(\frac{x}{2}\right)^{2m+1} \cdot \int_0^{\pi/2} \cos^{2m+1}\theta\,d\theta$$

$$= \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,(m+1)!} \cdot \left(\frac{x}{2}\right)^{2m+1} \cdot \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdots\cdots \frac{2}{3} \cdot 1 \qquad \text{..... (1)}$$

[By reduction formula]

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Now, $\dfrac{2m}{2m+1}\cdot\dfrac{2m-2}{2m-1}\cdots\cdots\dfrac{2}{3}\cdot 1$

$$= \frac{(2m)(2m-2)\ldots\ldots 2\cdot 1}{(2m+1)(2m)(2m-1)(2m-2)\ldots\ldots}\cdot\frac{(2m)(2m-2)\ldots\ldots 2\cdot 1}{3\cdot 2\cdot 1}$$

$$= \frac{[2m\cdot 2(m-1)\ldots\ldots 2\cdot(1)\cdot 1][2\cdot m\cdot 2(m-1)\ldots\ldots 2(1)\cdot 1]}{(2m+1)!}$$

$$= \frac{(2^m m!)(2^m m!)}{(2m+1)!}$$

Putting this value in (1),

$$\int_0^{\pi/2} J_1(x\cos\theta)\,d\theta = \sum \frac{(-1)^m}{m!\,(m+1)!}\cdot\left(\frac{x}{2}\right)^{2m+1}\cdot\frac{2^{2m}\cdot(m!)^2}{(2m+1)!}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+1}}{2\cdot(m+1)\cdot(2m+1)!} = \frac{1}{x}\sum_{m=0}^{\infty}\frac{(-1)^m x^{2m+2}}{(2m+2)!}$$

$$= \frac{1}{x}\left[\frac{x^2}{2!}-\frac{x^4}{4!}+\frac{x^6}{6!}-\ldots\ldots\right] = \frac{1}{x}\left[1-\left\{1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots\ldots\right\}\right]$$

$$= \frac{1}{x}[1-\cos x]$$

Example 4 : Prove that $y=\sqrt{x}\,J_{1/3}\left(\frac{2}{3}x^{3/2}\right)$ is a solution of the equation $\dfrac{d^2y}{dx^2}+xy=0$. Also find the general solution of the given equation.

Sol. : We shall reduce the given differential equation to Bessel's equation by proper substitution.

By data $y=\sqrt{x}\,J_{1/3}\left(\frac{2}{3}x^{3/2}\right)$ *i.e.*, $\dfrac{y}{\sqrt{x}} = J_{1/3}\left(\frac{2}{3}x^{3/2}\right)$

Let us put $\dfrac{y}{\sqrt{x}} = z$ and $\frac{2}{3}x^{3/2} = t$.

Now, we shall transform the given differential equation into the equation in z and t where z is a function of t.

Now, $\because\ z = \dfrac{y}{\sqrt{x}},\ y = \sqrt{x}\,z,$

$$\therefore\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}z+\sqrt{x}\cdot\frac{dz}{dx} = \frac{1}{2\sqrt{x}}z+\sqrt{x}\left[\frac{dz}{dt}\cdot\frac{dt}{dx}\right] \qquad \text{[By Chain Rule]}$$

But $t = \frac{2}{3}x^{3/2}$ $\quad\therefore\ \dfrac{dt}{dx} = \frac{2}{3}\cdot\frac{3}{2}\cdot x^{1/2} = \sqrt{x}$

$$\therefore\ \frac{dy}{dx} = \frac{1}{2\sqrt{x}}\cdot z+\sqrt{x}\cdot\frac{dz}{dt}\cdot\sqrt{x} = \frac{z}{2\sqrt{x}}+x\frac{dz}{dt}$$

Differentiating again w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2}z+\frac{1}{2\sqrt{x}}\cdot\frac{dz}{dt}\frac{dt}{dx}+x\frac{d}{dt}\left(\frac{dz}{dt}\right)\cdot\frac{dt}{dx}+\frac{dz}{dt}$$

$$\therefore\ \frac{d^2y}{dx^2} = -\frac{1}{4}\cdot\frac{z}{x^{3/2}}+\frac{1}{2\sqrt{x}}\cdot\frac{dz}{dt}\cdot\frac{dt}{dx}+x\frac{d^2z}{dt^2}\cdot\frac{dt}{dx}+\frac{dz}{dt}$$

Putting $\dfrac{dt}{dx} = \sqrt{x}$, we get

$$\frac{d^2y}{dx^2} = -\frac{1}{4}\cdot\frac{z}{x^{3/2}}+\frac{1}{2\sqrt{x}}\cdot\frac{dz}{dt}\cdot\sqrt{x}+x\frac{d^2z}{dt^2}\cdot\sqrt{x}+\frac{dz}{dt}$$

$$= -\frac{1}{4}\cdot\frac{z}{x^{3/2}}+\frac{1}{2}\cdot\frac{dz}{dt}+x^{3/2}\frac{d^2z}{dt^2}+\frac{dz}{dt}$$

Rearranging the terms, we get,

$$\frac{d^2y}{dx^2} = x^{3/2}\frac{d^2z}{dt^2}+\frac{3}{2}\cdot\frac{dz}{dt}-\frac{z}{4\cdot x^{3/2}}$$

Putting this value in the given equation, $\dfrac{d^2y}{dx^2}+xy=0$, we get,

$$x^{3/2}\frac{d^2z}{dt^2}+\frac{3}{2}\cdot\frac{dz}{dt}-\frac{z}{4\cdot x^{3/2}}+x\cdot\sqrt{x}\,z=0$$

But $\frac{2}{3}x^{3/2} = t \quad\therefore\ x^{3/2} = \frac{3}{2}t.$

$\therefore$ The above equation becomes,

$$\frac{3t}{2}\cdot\frac{d^2z}{dt^2}+\frac{3}{2}\cdot\frac{dz}{dt}-\frac{z}{6t}+\frac{3t}{2}z=0$$

Multiplying the above equation by $\frac{2}{3}t$, we get,

$$t^2\frac{d^2z}{dt^2}+t\frac{dz}{dt}-\frac{z}{9}+t^2z=0 \qquad \therefore\ t^2\frac{d^2z}{dt^2}+t\frac{dz}{dt}+\left(t^2-\frac{1}{9}\right)z=0 \qquad \text{..... (A)}$$

Comparing this with Bessel's equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-n^2)y=0 \qquad \text{..... (B)}$$

(given in (1), page S-36), we see that (A) is a Bessel equation with $n^2 = \frac{1}{9}$ *i.e.*, $n = \pm\frac{1}{3}$.

Hence, one of the solution of (A) is $z = J_n(t) = J_{1/3}(t)$ *i.e.*, one of the solution of the given equation is

$$\frac{y}{\sqrt{x}} = J_{1/3}(t) \quad i.e., \quad y = \sqrt{x}\,J_{1/3}\left(\frac{2}{3}x^{3/2}\right).$$

But we know that the general solution of (B) is

$y = A\,J_n(x)+B\,J_{-n}(x)$ [See (C), page S-39]

$\therefore$ We see that the general solution of (A) is

$z = A\,J_{1/3}(t)+B\,J_{-1/3}(t)$ ..... (C)

Resubstituting $z = \dfrac{y}{\sqrt{x}}$ and $t = \frac{2}{3}x^{3/2}$, we see that the general solution of (A) is

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$$\frac{y}{\sqrt{x}} = A J_{1/3}(t) + B J_{-1/3}(t)$$

$$\therefore \quad y = \sqrt{x} \cdot A \cdot J_{1/3}\left(\frac{2}{3}x^{3/2}\right) + \sqrt{x} \cdot B \cdot J_{-1/3}\left(\frac{2}{3}x^{3/2}\right)$$

Example 5 : Show that $y = \sqrt{x} \cdot J_{1/4}\left(\frac{1}{2}x^2\right)$ is a solution of $\frac{d^2y}{dx^2} + x^2 y = 0$. Also find the general solution of the given equation.

Sol. : As above we shall reduce the given differential equation to Bessel's equation by proper substitution.

By data $y = \sqrt{x} \cdot J_{1/4}\left(\frac{1}{2}x^2\right)$ *i.e.*, $\frac{y}{\sqrt{x}} = J_{1/4}\left(\frac{1}{2}x^2\right)$

Let us put $\frac{y}{\sqrt{x}} = z$ and $\frac{1}{2}x^2 = t$.

Now, we shall transform the given differential into the equation in z and t where z is a function of t.

$$\because \quad z = \frac{y}{\sqrt{x}}, \quad y = \sqrt{x}\, z$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} z + \sqrt{x} \cdot \frac{dz}{dx} = \frac{1}{2\sqrt{x}} z + \sqrt{x}\left[\frac{dz}{dt} \cdot \frac{dt}{dx}\right] \qquad \text{[By Chain Rule]}$$

But $\quad t = \frac{1}{2}x^2 \quad \therefore \quad \frac{dt}{dx} = x$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot z + \sqrt{x} \cdot \frac{dz}{dt} \cdot x = \frac{1}{2\sqrt{x}} z + x^{3/2} \frac{dz}{dt}.$$

Differentiating this again w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} \cdot z + \frac{1}{2\sqrt{x}} \cdot \frac{dz}{dt}\left(\frac{dt}{dx}\right) + x^{3/2} \frac{d}{dt}\left(\frac{dz}{dt}\right) \cdot \frac{dt}{dx} + \frac{3}{2} \cdot x^{1/2} \frac{dz}{dt}$$

But $\quad \frac{dt}{dx} = x.$

$$\therefore \quad \frac{d^2y}{dx^2} = -\frac{1}{4} \cdot \frac{z}{x^{3/2}} + \frac{1}{2\sqrt{x}} \cdot x \frac{dz}{dt} + x^{3/2} \cdot \frac{d^2z}{dt^2} x + \frac{3}{2} x^{1/2} \frac{dz}{dt}$$

$$= -\frac{1}{4} \cdot \frac{z}{x^{3/2}} + \frac{1}{2}\sqrt{x} \cdot \frac{dz}{dt} + x^{5/2} \frac{d^2z}{dt^2} + \frac{3}{2}\sqrt{x} \cdot \frac{dz}{dt}$$

$$= -\frac{1}{4} \cdot \frac{z}{x^{3/2}} + 2\sqrt{x} \frac{dz}{dt} + x^{5/2} \frac{d^2z}{dt^2}$$

Rearranging the terms, we get

$$\frac{d^2y}{dx^2} = x^{5/2} \frac{d^2z}{dt^2} + 2\sqrt{x} \cdot \frac{dz}{dt} - \frac{1}{4} \cdot \frac{z}{x^{3/2}}$$

Putting this value in the given equation, we get

$$x^{5/2} \frac{d^2z}{dt^2} + 2\sqrt{x} \cdot \frac{dz}{dt} - \frac{1}{4} \cdot \frac{z}{x^{3/2}} + x^2\sqrt{x} \cdot z = 0$$

Multiplying this by $x^{3/2}$, we get,

$$x^4 \frac{d^2z}{dt^2} + 2x^2 \frac{dz}{dt} - \frac{z}{4} + x^4 z = 0$$

But $x^2 = 2t$. $\quad \therefore$ The above equation then becomes,

$$4t^2 \frac{dz}{dt^2} + 4t \frac{dz}{dt} - \frac{z}{4} + 4t^2 = 0$$

Dividing by 4 throughout,

$$t^2 \frac{d^2z}{dt^2} + t \frac{dz}{dt} + \left(t^2 - \frac{1}{16}\right) z = 0 \qquad \text{..... (A)}$$

Comparing this with the Bessel's equation,

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0 \qquad \text{..... (B)}$$

(given in (1), page S-36), we see that (A) is a Bessel's equation with $n^2 = \frac{1}{16}$ *i.e.*, $n = \pm\frac{1}{4}$.

Hence, one of the solution of (A) is $z = J_n(t) = J_{1/4}(t)$ *i.e.*, one of the solution of the given equation is

$$\frac{y}{\sqrt{x}} = J_{1/4}\left(\frac{1}{2}x^2\right) \quad \textit{i.e.}, \quad y = \sqrt{x} \cdot J_{1/4}\left(\frac{1}{2}x^2\right)$$

But we know that the general solution of (B) is

$$y = A J_n(x) + B J_{-n}(x) \qquad \text{[See (C), page S-39]}$$

$\therefore$ We see that the general solution of (A) is $z = A J_{1/4}(t) + B J_{-1/4}(t)$.

Resubstituting $z = \frac{y}{\sqrt{x}}$ and $t = \frac{1}{2}x^2$, we see that the general solution of (A) is

$$\frac{y}{\sqrt{x}} = A J_{1/4}(t) + B J_{-1/4}(t)$$

$$\therefore \quad y = \sqrt{x} \cdot A \cdot J_{1/4}\left(\frac{1}{2}x^2\right) + \sqrt{x} \cdot B \cdot J_{-1/4}\left(\frac{1}{2}x^2\right).$$

Example 6 : Show that the general solution of $\frac{d^2y}{dx^2} + 4x^2 y = 0$ is

$$y = \sqrt{x}\left[A J_{1/4}(x^2) + B J_{-1/4}(x^2)\right]. \qquad \textbf{(M.U. 2014)}$$

Sol. : As above we shall reduce the given differential equation to Bessel's equation by proper substitution.

By data $\quad y = \sqrt{x} \cdot A \cdot J_{1/4}(x^2) + \sqrt{x} \cdot B \cdot J_{-1/4}(x^2)$

Let us put $\frac{y}{\sqrt{x}} = z$ and $t = x^2$.

Now, we shall transform the given differential equation into the equation in z and t where z is a function of t.

$$\because \quad z = \frac{y}{\sqrt{x}}, \quad y = \sqrt{x} \cdot z$$

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$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot z + \sqrt{x}\,\frac{dz}{dx} = \frac{1}{2\sqrt{x}} \cdot z + \sqrt{x}\left[\frac{dz}{dt} \cdot \frac{dt}{dx}\right]$$

But $t = x^2 \quad \therefore \quad \frac{dt}{dx} = 2x$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2\sqrt{x}} \cdot z + \sqrt{x} \cdot \frac{dz}{dt} \cdot 2x = \frac{1}{2\sqrt{x}} \cdot z + 2x^{3/2}\frac{dz}{dt}$$

Differentiating this again, w.r.t. x,

$$\frac{d^2y}{dx^2} = \frac{1}{2}\left(-\frac{1}{2}\right)x^{-3/2} \cdot z + \frac{1}{2\sqrt{x}} \cdot \frac{dz}{dt} \cdot \frac{dt}{dx} + 2x^{3/2}\frac{d}{dt}\left(\frac{dz}{dt}\right) \cdot \frac{dt}{dx} + 2 \cdot \frac{3}{2}x^{1/2}\frac{dz}{dt}$$

$$= -\frac{1}{4} \cdot \frac{z}{x^{3/2}} + \frac{1}{2\sqrt{x}} \cdot 2x\frac{dz}{dt} + 2x^{3/2} \cdot \frac{d^2z}{dt^2} \cdot 2x + 3x^{1/2} \cdot \frac{dz}{dt}$$

$$= -\frac{1}{4} \cdot \frac{z}{x^{3/2}} + 4x^{1/2}\frac{dz}{dt} + 4x^{5/2}\frac{d^2z}{dt^2}$$

Rearranging the terms, we get

$$\frac{d^2y}{dx^2} = 4x^{5/2}\frac{d^2z}{dt^2} + 4x^{1/2}\frac{dz}{dt} - \frac{1}{4} \cdot \frac{z}{x^{3/2}}$$

Putting this value in the given equation, we get

$$4x^{5/2}\frac{d^2z}{dt^2} + 4x^{1/2}\frac{dz}{dt} - \frac{1}{4} \cdot \frac{z}{x^{3/2}} + 4x^2 \cdot \sqrt{x} \cdot z = 0$$

Multiplying this equation by $x^{3/2}$, we get

$$4x^4 \cdot \frac{d^2z}{dt^2} + 4x^2\frac{dz}{dt} - \frac{z}{4} + 4x^4 z = 0$$

But $x^2 = t$. Therefore, the above equation becomes,

$$4t^2\frac{d^2z}{dt^2} + 4t\frac{dz}{dt} + \left(4t^2 - \frac{1}{4}\right)z = 0$$

Dividing by 4 throughout,

$$t^2\frac{d^2z}{dt^2} + t\frac{dz}{dt} + \left(t^2 - \frac{1}{16}\right)z = 0 \qquad \text{..... (A)}$$

Comparing this with the Bessel's equation,

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \qquad \text{..... (B)}$$

(given in (1), page S-39), we see that (A) is Bessel's equation with $n^2 = \frac{1}{16}$ *i.e.*, $n = \pm\frac{1}{4}$.

But we know that the general solution of (B) is $y = A\,J_n(x) + B\,J_{-n}(x)$.

$\therefore$ The general solution of (A) is $z = A\,J_{1/4}(t) + B\,J_{-1/4}(t)$.

Resubstituting $z = \frac{y}{\sqrt{x}}$ and $t = x^2$, we see that the general solution of (A) is

$$\frac{y}{\sqrt{x}} = A\,J_{1/4}(x^2) + B\,J_{-1/4}(x^2) \quad \textit{i.e.,} \quad y = \sqrt{x}\left[A\,J_{1/4}(x^2) + B\,J_{-1/4}(x^2)\right].$$

MISCELLANEOUS EXERCISE

1. Prove that $y = x^n J_n(x)$ is a solution of the equation $x\frac{d^2y}{dx^2} + (1-2n)\frac{dy}{dx} + xy = 0$.

2. Using Jacobi's series prove that

(i) $\sin(x\cos\theta) = 2\,[J_1(x)\cos\theta - J_3(x)\cos 3\theta + J_5(x)\cos 5\theta - \ldots]$

(ii) $\frac{x}{2} = J_1(x) + 3\,J_3(x) + 5\,J_5(x) + \ldots$

(iii) $x\cos x = 2\left[1^2 \cdot J_1(x) - 3^2 \cdot J_3(x) + 5^2 J_5(x) + \ldots\right]$

[**Hint :** Differentiate the series for $\sin(x\sin\theta)$ twice and put $\theta = \pi/2$]

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Module 2

APPLICATIONS TO SOLVE INITIAL AND BOUNDARY VALUE PROBLEMS INVOLVING ORDINARY DIFFERENTIAL EQUATIONS

Applications of Laplace Transforms

In this article we shall see how Laplace transforms can be profitably used to solve differential equations. For the use of Laplace transforms to solve differential equations we need the following results. If Laplace transform of y *i.e.* $L(y)$ is denoted by $\bar{y}$ then [See (1), (2), (3) of § 14, page 1-52] we have

$$\boxed{\begin{aligned} L(y') &= s\bar{y} - y(0) \\ L(y'') &= s^2\bar{y} - sy(0) - y'(0) \\ L(y''') &= s^3\bar{y} - s^2y - sy'(0) - y''(0) \end{aligned}} \qquad \text{..... (1)}$$

Example 1 : Solve using Laplace transforms $3\dfrac{dy}{dt} + 2y = e^{3t}$, $y = 1$ at $t = 0$. **(M.U. 2002)**

Sol. : Taking Laplace transforms of both sides,

$$3L(y') + 2L(y) = L(e^{3t})$$

$$3[s\bar{y} - y(0)] + 2\bar{y} = \frac{1}{s-3}. \quad \text{But } y(0) = 1$$

$$\therefore \quad 3[s\bar{y} - 1] + 2\bar{y} = \frac{1}{s-3}$$

$$\therefore \quad (3s+2)\bar{y} = \frac{1}{s-3} + 3 = \frac{3s-8}{s-3}$$

$$\bar{y} = \frac{3s-8}{(s-3)(3s+2)}$$

$$\therefore \quad \bar{y} = \frac{30}{11}\cdot\frac{1}{3s+2} + \frac{1}{11}\cdot\frac{1}{(s-3)} \qquad \text{[By partial fractions]}$$

$$\therefore \quad \bar{y} = \frac{10}{11}\cdot\frac{1}{s+(2/3)} + \frac{1}{11}\cdot\frac{1}{s-3}$$

Taking inverse Laplace transforms

$$y = \frac{10}{11}L^{-1}\left[\frac{1}{s+(2/3)}\right] + \frac{1}{11}L^{-1}\left[\frac{1}{s-3}\right] = \frac{10}{11}e^{-(2/3)t} + \frac{1}{11}e^{3t}.$$

Example 2 : Solve using Laplace transforms $\dfrac{dy}{dx} + 3y = 2 + e^{-t}$, if $y = 1$ at $t = 0$.

Sol. : Taking Laplace transforms of both sides,

$$L(y') + 3L(y) = L(2) + L(e^{-t})$$

$$s\bar{y} - y(0) + 3\bar{y} = 2\frac{1}{s} + \frac{1}{s+1}. \quad \text{But } y(0) = 1$$

$$\therefore \quad (s+3)\bar{y} = \frac{2}{s} + \frac{1}{s+1} + 1 = \frac{s^2+4s+2}{s(s+1)}$$

$$\therefore \quad \bar{y} = \frac{s^2+4s+2}{s(s+1)(s+3)}$$

$\therefore$ By partial fractions $\quad \bar{y} = \dfrac{2}{3}\cdot\dfrac{1}{s} + \dfrac{1}{2}\cdot\dfrac{1}{s+1} - \dfrac{1}{6}\cdot\dfrac{1}{s+3}$

Taking inverse Laplace transforms, $\quad y = \dfrac{2}{3} + \dfrac{1}{2}\cdot e^{-t} - \dfrac{1}{6}e^{-3t}$

Example 3 : Solve using Laplace transforms $R\dfrac{dQ}{dt} + \dfrac{Q}{C} = V$, $Q = 0$ when $t = 0$. **(M.U. 2016)**

Sol. : Taking Laplace transforms of both sides,

$$RL(Q') + \frac{1}{C}L(Q) = VL(1)$$

$$R[s\bar{Q} - Q(0)] + \frac{1}{C}\bar{Q} = \frac{V}{s}. \quad \text{But } Q(0) = 0$$

$$\therefore \quad \left(Rs + \frac{1}{C}\right)\bar{Q} = \frac{V}{s} \qquad \therefore \quad \bar{Q} = \frac{V}{s}\cdot\frac{1}{[Rs + (1/C)]} = \frac{VC}{s(RCs+1)}$$

$$\therefore \quad \bar{Q} = VCL^{-1}\frac{1}{s} - VRC^2L^{-1}\frac{1}{RCs+1} = VCL^{-1}\frac{1}{s} - VCL^{-1}\left[\frac{1}{s+1/(RC)}\right]$$

Taking inverse Laplace transforms,

$$Q = VC - VCe^{-t/RC} = VC(1 - e^{-t/RC})$$

Example 4 : Solve using Laplace transforms $L\dfrac{dI}{dt} + RI = Ee^{-at}$, where $I(0) = 0$.

Sol. : Taking Laplace transforms of both sides,

$$LL[I'] + RL(I) = EL(e^{-at})$$

$$L[s\bar{I} - I(0)] + R\bar{I} = \frac{E}{s+a}. \quad \text{But } I(0) = 0$$

$$\therefore \quad (Ls + R)\bar{I} = \frac{E}{s+a}$$

$$\therefore \quad \bar{I} = \frac{E}{(s+a)(Ls+R)} \qquad \text{[By partial fractions]}$$

$$= \frac{E}{R-La}\cdot\frac{1}{s+a} - \frac{EL}{R-La}\cdot\frac{1}{Ls+R}$$

Taking inverse Laplace transforms

$$I = \frac{E}{R-La}e^{-at} - \frac{E}{R-La}e^{-(R/L)t} = \frac{E}{R-La}\left[e^{-at} - e^{-(R/L)t}\right]$$

Example 5 : Using Laplace transform solve the following differential equation. **(M.U. 1993)**

$$\frac{dx}{dt} + x = \sin\omega t, \quad x(0) = 2.$$

Sol. : Let $\bar{x}$ be the Laplace transform of x *i.e.* let $L(x) = \bar{x}$.

Taking Laplace transform of both sides of the given equation, ... (1)

$$L(x') + L(x) = L\sin(\omega t)$$

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But $L(x') = s(\bar{x}) - x(0) = s\bar{x} - 2$

Hence, the equation (1) becomes

$$s\bar{x} - 2 + \bar{x} = \frac{\omega}{s^2+\omega^2} \quad \therefore (s+1)\bar{x} = 2 + \frac{\omega}{s^2+\omega^2}$$

$$\therefore (s+1)\bar{x} = \frac{2s^2+2\omega^2+\omega}{s^2+\omega^2}$$

$$\therefore \bar{x} = \frac{2s^2+2\omega^2+\omega}{(s^2+\omega^2)(s+1)} \quad \text{[By partial fractions]}$$

$$\frac{2\omega^2+\omega+2}{(s^2+\omega^2)(s+1)}$$

$$= \frac{2\omega^2+\omega+2}{1+\omega^2}\cdot\frac{1}{s+1} + \frac{-\omega s+\omega}{(1+\omega^2)(s^2+\omega^2)}$$

$$= \frac{2\omega^2+\omega+2}{1+\omega^2}\cdot\frac{1}{s+1} - \frac{\omega}{1+\omega^2}\cdot\frac{s}{s^2+\omega^2} + \frac{\omega}{1+\omega^2}\cdot\frac{1}{s^2+\omega^2}$$

Taking inverse Laplace transforms,

$$x = \frac{2\omega^2+\omega+2}{1+\omega^2}L^{-1}\left(\frac{1}{s+1}\right) - \frac{\omega}{1+\omega^2}\cdot L^{-1}\left(\frac{s}{s^2+\omega^2}\right) + \frac{\omega}{1+\omega^2}L^{-1}\left(\frac{1}{s^2+\omega^2}\right)$$

$$= \frac{2\omega^2+\omega+2}{1+\omega^2}\cdot e^{-t} - \frac{\omega}{1+\omega^2}\cdot\cos\omega t + \frac{\omega}{1+\omega^2}\cdot\frac{1}{\omega}\sin\omega t$$

$$= \frac{1}{1+\omega^2}\left[(2\omega^2+\omega+2)e^{-t} - \omega\cos\omega t + \sin\omega t\right]$$

Example 6 : Solve $(D^2 - 3D + 2)y = 4e^{2t}$, with $y(0) = -3$ and $y'(0) = 5$.

(M.U. 2004, 07, 08, 14, 17, 18, 19)

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - 3L(y') + 2L(y) = 4L(e^{2t})$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} + 3$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} + 3s - 5$

$\therefore$ The equation becomes,

$$(s^2\bar{y} + 3s - 5) - 3(s\bar{y} + 3) + 2\bar{y} = 4\frac{1}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} = \frac{4}{s-2} + 14 - 3s = \frac{-3s^2+20s-24}{s-2}$$

$$\therefore \bar{y} = \frac{-3s^2+20s-24}{(s-2)(s^2-3s+2)} = \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$

By partial fractions, $\bar{y} = -\frac{7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

Taking inverse Laplace transform,

$$y = -7L^{-1}\left(\frac{1}{s-1}\right) + 4L^{-1}\frac{1}{s-2} + 4L^{-1}\frac{1}{(s-2)^2}$$

$$\therefore y = -7e^t L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s} + 4e^{2t}L^{-1}\frac{1}{s^2} = -7e^t + 4e^{2t} + 4te^{2t}$$

$\therefore$ The solution is $y = -7e^t + 4e^{2t} + 4te^{2t}$.

Example 7 : Solve $(D^2 - D - 2)y = 20\sin 2t$, with $y(0) = 1$ and $y'(0) = 2$. **(M.U. 2005, 13)**

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') - L(y') - 2L(y) = 20L(\sin 2t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 1$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s - 2$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - s - 2) - (s\bar{y} - 1) - 2\bar{y} = 20\frac{2}{s^2+4}$$

$$\therefore (s^2 - s - 2)\bar{y} = \frac{40}{s^2+4} + s + 1 = \frac{s^3+s^2+4s+44}{s^2+4}$$

$$\therefore \bar{y} = \frac{s^3+s^2+4s+44}{(s^2+4)(s^2-s-2)} \quad \therefore \bar{y} = -\frac{8}{3}\cdot\frac{1}{s+1} + \frac{8}{3}\cdot\frac{1}{s-2} + \frac{s-6}{s^2+4}$$

Taking inverse Laplace transform,

$$y = \frac{-8}{3}L^{-1}\left(\frac{1}{s+1}\right) + \frac{8}{3}L^{-1}\left(\frac{1}{s-2}\right) + L^{-1}\frac{s}{s^2+4} - 6L^{-1}\frac{1}{s^2+4}$$

$$= -\frac{8}{3}e^{-t} + \frac{8}{3}e^{-2t} + \cos 2t - 3\sin 2t.$$

Example 8 : Using Laplace Transform solve $(D^2 + 3D + 2)y = e^{-2t}\sin t$, $y(0) = 0$, $y'(0) = 0$.

Sol. : Let $L(y) = \bar{y}$. Then taking the Laplace transform of both sides

$$L(y'') + 3L(y') + 2L(y) = L(e^{-2t}\sin t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y}$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y}$

And $L(e^{-2t}\sin t) = \frac{1}{(s+2)^2+1}$

$\therefore$ The equation becomes,

$$s^2\bar{y} + 3s\bar{y} + 2\bar{y} = \frac{1}{s^2+4s+5}$$

$$\therefore (s^2+3s+2)\bar{y} = \frac{1}{s^2+4s+5} \quad \therefore \bar{y} = \frac{1}{(s^2+3s+2)(s^2+4s+5)}$$

Let $\frac{1}{(s^2+3s+2)(s^2+4s+5)} = \frac{a}{s+1} + \frac{b}{s+2} + \frac{cs+d}{s^2+4s+5}$

Solving this, we get $a = \frac{1}{2}$, $b = -1$, $c = \frac{1}{2}$, $d = \frac{1}{2}$

$$\therefore \bar{y} = \frac{1}{2}\cdot\frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2}\left(\frac{s+1}{s^2+4s+5}\right)$$

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Taking inverse Laplace transform,

$$y = \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{(s+2)-1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}L^{-1}\left[\frac{s+2}{(s^2+2)^2+1}\right] - \frac{1}{2}L^{-1}\left[\frac{1}{(s+2)^2+1}\right]$$

$$= \frac{1}{2}L^{-1}\left[\frac{1}{s+1}\right] - L^{-1}\left[\frac{1}{s+2}\right] + \frac{1}{2}\cdot e^{-2t}L^{-1}\left[\frac{s}{s^2+1}\right] - \frac{1}{2}e^{-2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= \frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}\cdot e^{-2t}\cos t - \frac{1}{2}e^{-2t}\sin t$$

Example 9 : Solve $(D^2 + 3D + 2)\,y = 2\,(t^2 + t + 1)$ with $y(0) = 2$ and $y'(0) = 0$.

Sol. : Let $L(y) = \bar{y}$. Then, taking Laplace transform,

$$L(y'') + 3L(y') + 2L(y) = 2L(t^2 + t + 1)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 2$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 2s$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - 2s) + 3(s\bar{y} - 2) + 2\bar{y} = 2\left(\frac{2}{s^3} + \frac{1}{s^2} + \frac{1}{s}\right)$$

$$\therefore\ (s^2 + 3s + 2)\,\bar{y} = 2\frac{(2 + s + s^2)}{s^3} + 2s + 6 = \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3}$$

By partial fractions,

$$\therefore\ \bar{y} = \frac{2(s^4 + 3s^3 + s^2 + s + 2)}{s^3(s^2 + 3s + 2)} = \frac{3}{s} - \frac{2}{s^2} + \frac{2}{s^3} - \frac{1}{s+2}$$

Taking inverse Laplace transform,

$$y = 3L^{-1}\frac{1}{s} - 2L^{-1}\frac{1}{s^2} + 2L^{-1}\frac{1}{s^3} - L^{-1}\frac{1}{s+2}$$

$$y = 3 - 2t + t^2 - e^{-2t}$$

Example 10 : Use Laplace transform to solve,

$$\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 8y = 1 \text{ where, } y(0) = 0,\ y'(0) = 1.$$ **(M.U. 1994, 2009, 10, 18)**

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of the both sides,

$$L(y'') + 4L(y') + 8L(y) = L(1) \qquad \text{..... (1)}$$

Now, $L(y') = s\bar{y} - y(0) = s\bar{y}$

$$L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1 \text{ and } L(1) = \frac{1}{s}$$

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - 1 + 4s\bar{y} + 8\bar{y} = \frac{1}{s} \qquad \therefore\ \bar{y}(s^2 + 4s + 8) = \frac{1}{s} + 1 = \frac{s+1}{s}$$

$$\therefore\ \bar{y} = \frac{s+1}{s(s^2 + 4s + 8)} \qquad \therefore\ y = L^{-1}(\bar{y}) = L^{-1}\frac{s+1}{s(s^2 + 4s + 8)}$$

We obtain the L^{-1} by partial fractions

$$\therefore\ y = L^{-1}\left[\frac{1}{8}\cdot\frac{1}{s} - \frac{1}{8}\cdot\frac{s}{s^2 + 4s + 8} + \frac{1}{2}\cdot\frac{1}{s^2 + 4s + 8}\right]$$

$$= \frac{1}{8}L^{-1}\left(\frac{1}{s}\right) - \frac{1}{8}L^{-1}\frac{(s+2)-2}{(s+2)^2 + 2^2} + \frac{1}{2}L^{-1}\frac{1}{(s+2)^2 + 2^2}$$

$$= \frac{1}{8}\cdot 1 - \frac{1}{8}e^{-2t}L^{-1}\frac{s}{s^2 + 2^2} + \frac{6}{8}e^{-2t}L^{-1}\frac{1}{s^2 + 2^2}$$

$$\therefore\ y = \frac{1}{8} - \frac{1}{8}e^{-2t}\cos 2t + \frac{3}{8}e^{-2t}\sin 2t.$$

Example 11 : Using Laplace transform solve $\frac{d^2y}{dt^2} + y = t,\ y(0) = 1,\ y'(0) = 0.$

(M.U. 1995, 2009, 13, 14, 15, 16)

Sol. : Let $\bar{y}$ be the Laplace transform of y *i.e.* let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y'') + L(y) = L(t)$$

Now, $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - s$ and $L(t) = \frac{1}{s^2}$.

$\therefore$ The equation (1) becomes

$$s^2\bar{y} - s + \bar{y} = \frac{1}{s^2} \qquad \therefore\ s^2\bar{y} + \bar{y} = s + \frac{1}{s^2} = \frac{s^3 + 1}{s^2}$$

$$\therefore\ (s^2 + 1)\,\bar{y} = \frac{s^3 + 1}{s^2} \qquad \therefore\ \bar{y} = \frac{s^3 + 1}{s^2(s^2 + 1)}$$

Let $\frac{s^3 + 1}{s^2(s^2 + 1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{cs + d}{s^2 + 1}$

$$\therefore\ s^3 + 1 = as(s^2 + 1) + b(s^2 + 1) + (cs + d)s^2$$

$$= (a + c)s^3 + (b + d)s^2 + as + b$$

Equating like powers of s,

$$a + c = 1,\ b + d = 0,\ a = 0,\ b = 1 \qquad \therefore\ a = 0,\ b = 1,\ c = 1,\ d = -1$$

$$\therefore\ \bar{y} = \frac{1}{s^2} + \frac{s-1}{s^2 + 1} = \frac{1}{s^2} + \frac{s}{s^2 + 1} - \frac{1}{s^2 + 1}$$

Taking inverse Laplace transform

$$y = L^{-1}\left(\frac{1}{s^2}\right) + L^{-1}\left(\frac{s}{s^2 + 1}\right) - L^{2}\left(\frac{1}{s^2 + 1}\right) = t + \cos t - \sin t.$$

Example 12 : Solve by using Laplace transform $(D^2 + 2D + 5)\,y = e^{-t}\sin t$, when $y(0) = 0$, $y'(0) = 1$.

(M.U. 1995, 2003, 05, 06, 07, 11, 14, 15, 19)

Sol. : Let $L(y) = \bar{y}$. Then taking Laplace transform of both sides,

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$$L(y'') + 2L(y') + 5L(y) = L(e^{-t}\sin t)$$

But $L(y') = s\bar{y} - y(0) = s\bar{y}$. And $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 1$.

And $Le^{-t}\sin t = \dfrac{1}{(s+1)^2+1}$

$\therefore$ The equation becomes

$$(s^2\bar{y} - 1) + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$\therefore\ (s^2+2s+5)\bar{y} = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$

$$\therefore\ \bar{y} = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $\dfrac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \dfrac{as+b}{(s^2+2s+5)} + \dfrac{cs+d}{(s^2+2s+2)}$

After simplification, we get

$$s^2+2s+s = (a+c)s^3 + (2a+b+2c+d)s^2 + (2a+2b+5c+2d)s + (2b+5d)$$

Equating the coefficients of like powers of s, we get,

$$a+c=0,\ 2a+b+2c+d=1,\ 2a+2b+5c+2d=2,\ 2b+5d=3$$

$$\therefore\ a=0,\ b=\frac{2}{3},\ c=0,\ d=\frac{1}{3}$$

$$\therefore\ \bar{y} = \frac{2}{3}\cdot\frac{1}{s^2+2s+5} + \frac{1}{3}\cdot\frac{1}{s^2+2s+2}$$

$$= \frac{2}{3}\cdot\frac{1}{(s+1)^2+2^2} + \frac{1}{3}\cdot\frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplace transform

$$y = \frac{2}{3}\cdot e^{-t}\cdot L^{-1}\left[\frac{1}{s^2+2^2}\right] + \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= \frac{2}{3}e^{-t}\cdot\frac{1}{2}\sin 2t + \frac{1}{3}e^{-t}\sin t = \frac{e^{-t}}{3}(\sin 2t + \sin t).$$

Example 13 : Solve using Laplace transform $\dfrac{d^2y}{dt^2} + 2\dfrac{dy}{dt} + y = 3te^{-t}$ given $y(0) = 4$ and $y'(0) = 2$. **(M.U. 2002, 15, 19)**

Sol. : Let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y'') + 2L(y') + L(y) = L(3te^{-t})$$

But $L(y') = s\bar{y} - y(0) = s\bar{y} - 4$

and $L(y'') = s^2\bar{y} - sy(0) - y'(0) = s^2\bar{y} - 4s - 2$

$$L(e^{-t}) = \frac{1}{s+1} \qquad \therefore\ L[te^{-t}] = -\frac{d}{ds}\left(\frac{1}{s+1}\right) = \frac{1}{(s+1)^2}$$

$\therefore$ The equation becomes,

$$(s^2\bar{y} - 4s - 2) + 2(s\bar{y} - 4) + \bar{y} = 3\cdot\frac{1}{(s+1)^2}$$

$$\therefore\ (s^2+2s+1)\bar{y} - 4s - 10 = \frac{3}{(s+1)^2}$$

$$\therefore\ (s+1)^2\bar{y} = \frac{3}{(s+1)^2} + 4s + 10$$

$$\therefore\ \bar{y} = \frac{3}{(s+1)^4} + \frac{4s+10}{(s+1)^2} = \frac{3}{(s+1)^4} + \frac{4s+4}{(s+1)^2} + \frac{6}{(s+1)^2}$$

$$= \frac{3}{(s+1)^4} + \frac{4}{s+1} + \frac{6}{(s+1)^2}$$

Taking the inverse Laplace transform of both sides,

$$y = L^{-1}\left[\frac{3}{(s+1)^4}\right] + 4L^{-1}\left[\frac{1}{s+1}\right] + 6L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$= 3e^{-t}L^{-1}\left[\frac{1}{s^4}\right] + 4e^{-t}L^{-1}\left[\frac{1}{s}\right] + 6e^{-t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= 3e^{-t}\cdot\frac{t^3}{3!} + 4e^{-t}\cdot 1 + 6e^{-t}\cdot t = 3e^{-t}\left[\frac{t^3}{2} + 6t + 4\right]$$

Example 14 : Solve using Laplace transform $\dfrac{d^2y}{df^2} + 9y = 18t$, given that $y(0) = 0$ and $y(\pi/2) = 0$. **(M.U. 1996, 98, 2004, 05, 07, 13, 19)**

Sol. : Let $L(y) = \bar{y}$. Then taking Laplace transforms of both sides

$$L(y'') + 9L(y) = 18L(t) \qquad \text{..... (1)}$$

But $L(y'') = s^2\bar{y} - sy(0) - y'(0)$; $L(t) = \dfrac{1}{s^2}$

Since, $y'(0)$ is not given let us assume $y'(0) = A$. **[Note this]**

Hence, (1) becomes $s^2\bar{y} - A + 9\bar{y} = \dfrac{18}{s^2}$

$$\therefore\ (s^2+9)\bar{y} = \frac{18}{s^2} + A \qquad \therefore\ \bar{y} = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$

$$\therefore\ \bar{y} = \frac{18}{9}\left[\frac{1}{s^2} - \frac{1}{s^2+9}\right] + \frac{A}{s^2+9} = \frac{2}{s^2} + \left(\frac{A-2}{s^2+9}\right)$$

$$\therefore\ y = 2L^{-1}\left(\frac{1}{s^2}\right) + (A-2)L^{-1}\frac{1}{s^2+9} \qquad \therefore\ y = 2t + \frac{A-2}{3}\sin 3t$$

To find A we put $t = \dfrac{\pi}{2}$ and use that $y\left(\dfrac{\pi}{2}\right) = 0$.

$$\therefore\ 0 = 2\cdot\frac{\pi}{2} + \left(\frac{A-2}{3}\right)\sin\left(\frac{3\pi}{2}\right) \qquad \therefore\ 0 = \pi - \frac{(A-2)}{3}$$

$$\therefore\ 0 = 3\pi - A + 2 \qquad \therefore\ A = 3\pi + 2$$

$$\therefore\ y = 2t + \pi\sin 3t.$$

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Example 15 : Solve $(D^3 - 2D^2 + 5D)y = 0$, with $y(0) = 0$, $y'(0) = 0$, $y''(0) = 1$. **(M.U. 2004, 05)**

Sol. : Let $L(y) = \bar{y}$.

Taking Laplace transform of both sides,

$$L(y''') - 2L(y'') + 5L(y') = 0$$

$$L(y') = s(\bar{y}) - y(0),\ L(y'') = s^2\bar{y} - s\,y(0) - y'(0)$$

$$L(y''') = s^3\bar{y} - s^2 y(0) - sy'(0) - y''(0)$$

From given conditions,

$$L(y') = s\bar{y},\ L(y'') = s^2\bar{y},\ L(y''') = s^3\bar{y} - 1.$$

$\therefore$ The equation becomes

$$s^3\bar{y} - 1 - 2s^2\bar{y} + 5s\,\bar{y} = 0 \qquad \therefore\ \bar{y} = \frac{1}{s^3 - 2s^2 + 5s}$$

Taking inverse Laplace transform

$$y = L^{-1}\left[\frac{1}{s^3 - 2s^2 + 5s}\right] = L^{-1}\left[\frac{1}{s(s^2 - 2s + 5)}\right] = L^{-1}\left[\frac{1}{s[(s-1)^2 + 2^2]}\right]$$

We obtain the inverse by convolution theorem.

Let $\Phi_1(s) = \dfrac{1}{(s-1)^2 + 2^2}$ and $\Phi_2(s) = \dfrac{1}{s}$ $\quad\therefore\ \Phi(s) = \Phi_1(s)\cdot\Phi_2(s)$

$$\therefore\ f_1(t) = L^{-1}\Phi_1(s) = L^{-1}\left[\frac{1}{(s-1)^2 + 2^2}\right] = e^t \cdot L^{-1}\frac{1}{s^2 + 2^2} = \frac{1}{2}\cdot e^t \cdot \sin 2t$$

$$f_2(t) = L^{-1}\Phi_2(s) = L^{-1}\left(\frac{1}{s}\right) = 1$$

$$\therefore\ f_1(u) = \frac{1}{2}e^u \sin 2u.$$

Then by Cor. (16A) page 2-15,

$$\therefore\ L^{-1}\Phi(s) = \int_0^t \frac{1}{2}e^u \sin 2u\,du = \frac{1}{2}\cdot\frac{1}{5}\cdot\left[e^u(\sin 2u - 2\cos 2u)\right]_0^t$$

$$\therefore\ y = \frac{1}{10}\left[e^t(\sin 2t - 2\cos 2t) + 2\right]$$

$\therefore$ The solution is $y = \dfrac{1}{5} - \dfrac{1}{5}e^t \cos 2t + \dfrac{1}{10}e^t \sin 2t.$

***Example 16 :** Solve the equation $y + \int_0^t y\,dt = 1 - e^{-t}$. **(M.U. 2015)**

Sol. : Let $L(y) = \bar{y}$. Taking the Laplace transform of both sides, we get,

$$L(y) + L\left[\int_0^t y\,dt\right] = L(1) + L(e^{-t})$$

Since, $L\left[\int_0^t y\,dt\right] = \int_0^\infty e^{-st}\int_0^\infty y\,dt = \left[\int_0^t y\,dt \cdot \dfrac{e^{-st}}{s}\right]_0^\infty - \int_0^\infty -\dfrac{e^{-st}}{s}\cdot y\,dt$

$$= 0 + \frac{1}{s}\int_0^\infty e^{-st} y\,dt = \frac{1}{s}L(y) = \frac{1}{s}\bar{y}$$

and $Le^{-t} = \dfrac{1}{s+1}$, the equation becomes $\bar{y} + \dfrac{\bar{y}}{s} = \dfrac{1}{s} - \dfrac{1}{s+1} = \dfrac{1}{s(s+1)}$.

$$\therefore\ \bar{y}\frac{(s+1)}{s} = \frac{1}{s(s+1)} \qquad \therefore\ \bar{y} = \frac{1}{(s+1)^2}$$

$$\therefore\ y = L^{-1}\frac{1}{(s+1)^2} = e^{-t}L^{-1}\frac{1}{s^2} = e^{-t}\cdot t \qquad \therefore\ y = te^{-t}.$$

***Example 17 :** Solve the following equation by using Laplace transform

$$\frac{dy}{dt} + 2y + \int_0^t y\,dt = \sin t, \text{ given that } y(0) = 1.$$

(M.U. 1999, 2006, 09, 14)

Sol. : Let $L(y) = \bar{y}$. Taking Laplace transform of both sides, we get

$$L(y') + 2L(y) + L\left[\int_0^t y\,dt\right] = L(\sin t)$$

But $L(y') = sL(y) - y(0) = s\bar{y} - 1$

$$L\left[\int_0^t y\,dt\right] = \frac{1}{s}L(y) = \frac{1}{s}\bar{y},\quad L(\sin t) = \frac{1}{s^2+1}$$

$\therefore$ The equation becomes

$$s\bar{y} - 1 + 2\bar{y} + \frac{1}{s}\bar{y} = \frac{1}{s^2+1} \qquad \therefore\ \left(s + 2 + \frac{1}{s}\right)\bar{y} = \frac{1}{s^2+1} + 1 = \frac{s^2+1+1}{s^2+1}$$

$$\therefore\ \frac{(s^2+2s+1)}{s}\bar{y} = \frac{(s^2+2)}{s^2+1} \qquad \therefore\ \bar{y} = \frac{s(s^2+2)}{(s+1)^2(s^2+1)}$$

Let $\dfrac{s(s^2+2)}{(s+1)^2(s^2+1)} = \dfrac{a}{s+1} + \dfrac{b}{(s+1)^2} + \dfrac{cs+d}{s^2+1}$

$$\therefore\ s(s^2+2) = a(s+1)(s^2+1) + b(s^2+1) + (cs+d)(s+1)^2$$

Putting $s = -1$, $-3 = 2b$ $\quad\therefore\ b = -3/2$. Putting $s = 0$, $0 = a + b + d$.

Equating the coefficients of s^2 and s^3.

$$0 = a + b + 2c + d \text{ and } 1 = a + c$$

$\because\ b = -3/2,\ a + d = 3/2$ and $a + 2c + d = 3/2$.

But $a + d = 3/2$ $\quad\therefore\ 2c = 0$ $\quad\therefore\ c = 0$

$\because\ 1 = a + c$ and $c = 0$ $\quad\therefore\ a = 1$

$\because\ a + d = 3/2$ and $a = 1$ $\quad\therefore\ d = 1/2$

$\therefore\ a = 1,\ b = -3/2,\ c = 0,\ d = 1/2$

$$\therefore\ \bar{y} = \frac{1}{s+1} - \frac{3}{2}\cdot\frac{1}{(s+1)^2} + \frac{1}{2}\cdot\frac{1}{s^2+1}$$

$$\therefore\ y = L^{-1}\left(\frac{1}{s+1}\right) - \frac{3}{2}e^{-t}L^{-1}\frac{1}{s^2} + \frac{1}{2}L^{-1}\frac{1}{s^2+1}$$

$$\therefore\ y = e^{-t} - \frac{3}{2}e^{-t}\cdot t + \frac{1}{2}\sin t.$$

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EXERCISE

Using Laplace transform solve the following differential equations with the given conditions.

1. $(3D+2)y = e^{3t}$, $y(0) = 1$. **(M.U. 2002)**
2. $\frac{dy}{dt} + 2y = e^{-3t}$, $y(0) = 1$
3. $\frac{dy}{dt} + y = e^{-2t}$, $y(0) = 0$
4. $\frac{dy}{dt} + 2y = 5$, $y(0) = 1$
5. $\frac{dy}{dt} + 2y = \sin t$, $y(0) = 0$
6. $\frac{dy}{dt} + y = \cos 2t$, $y(0) = 1$
7. $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - 3y = 0$; at $x = 0$, $y = 0$, $\frac{dy}{dx} = 4$.
8. $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} = -8t$; $x(0) = 0 = x'(0)$.
9. $(D^2+9)y = 18t$; $y(0) = 0$, $y'(0) = 0$.
10. $f''(t) + f'(t) = t$; $f(0) = 1$, $f'(0) = -1$.
11. $(D^2 - 3D + 2)y = 2e^{3t}$; $y = 2$, $y' = 3$ at $t = 0$.
*12. $(D^2+1)y = \sin t$; $y(0) = 1$, $y'(0) = -\frac{1}{2}$.
13. $(D^2 - 2D + 2)x = 0$; $x(0) = x'(0) = 1$.
14. $y'' - 2y' + y = e^t$; $y(0) = 2$, $y'(0) = -1$. **(M.U. 2019)**
15. $(D^2 + D - 2)x = 2(1 + t - t^2)$; $x = 0$, $Dx = 3$ for $t = 0$. **(M.U. 2006)**
16. $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = 2$; $y(0) = 1$, $y'(0) = 0$.
*17. $(D^2 - 3D + 2)y = 4t + e^{3t}$ if $y = 1$, $Dy = -1$ at $t = 0$.
*18. $(D^3 + 2D^2 - D - 2)y = 0$ if $y = 1$, $Dy = 2$, $D^2y = 0$ at $y = 0$.
19. $(D^2 + 4D + 3)y = e^{-t}$; $y(0) = y'(0) = 1$.
20. $(D^2 + D)y = t^2 + 2t$ at $t = 0$, $y = 4$ and $Dy = 2$. **(M.U. 1997)**
21. $(D^2 - 2D + 1)x = e^t$ with the conditions $x = 2$, $Dx = -1$ at $t = 0$. **(M.U. 1997, 2000, 19)**
22. $(D+1)^2y = 6te^{-t}$ with $y(0) = 2$, $y'(0) = 5$. **(M.U. 1994)**
23. $\frac{d^2y}{dx^2} + 16y = \delta(t)$ given that $y = 0$, $\frac{dy}{dt} = 0$ at $t = 0$.
24. $(D^2 - 3D + 2)y = 4e^{2t}$ at $t = 0$, $y = -3$ and $Dy = 5$. **(M.U. 2004)**
25. $(D^2 - 2D - 8)y = 4$, $y(0) = 0$ and $y'(0) = 1$. **(M.U. 2003, 04, 13)**
26. $(D^2 + 2D + 1)y = 3te^{-t}$, $y(0) = 4$, $y'(0) = 2$. **(M.U. 2002)**
27. $(D^2 + D)y = t^2 + 2t$, $y(0) = 4$, $y'(0) = -2$. **(M.U. 2004)**
28. $(D^2 - 4)y = 3e^t$, $y(0) = 0$, $y'(0) = 3$. **(M.U. 2004, 19)**
29. $\frac{d^2y}{dt^2} + y = t$, $y(0) = 1$, $y'(0) = 0$. **(M.U. 2006)**

30. $2y'' + 5y' + 2y = e^{-2t}$, $y(0) = y'(0) = 1$. **(M.U. 2005)**
31. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 2y = e^{3t}$, given $y(0) = 0$ and $y'(0) = 1$. **(M.U. 2003)**
32. $\frac{d^2y}{dt} + 9y = \cos 2t$, $y(0) = 1$ and $y(\pi/2) = -1$. **(M.U. 2002)**
33. $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} - 3y = \sin t$, $y(0) = 0$ and $y'(0) = 0$. **(M.U. 2014)**

Ans.: (1) $y = \frac{1}{11}e^{3t} + \frac{10}{11}e^{-2t/3}$, (2) $y = 2e^{-2t} - e^{-3t}$, (3) $y = e^{-t} - e^{-2t}$, (4) $y = \frac{5}{2} - \frac{3}{2}e^{-2t}$,

(5) $y = -\frac{1}{5}\cos t + \frac{2}{5}\sin t + \frac{1}{5}e^{-2t}$, (6) $y = \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t + \frac{4}{5}e^{-t}$,

(7) $y = e^x - e^{-3x}$, (8) $x = -\frac{1}{8} + \frac{1}{2}t - t^2 + \frac{e^{-4t}}{8}$, (9) $y = 2\left[t - \frac{1}{3}\sin 3t\right]$,

(10) $f(t) = 1 - t + \frac{t^2}{2}$, (11) $y = 2e^t - e^{2t} + e^{3t}$, (12) $y = \left(1 - \frac{t}{2}\right)\cos t$,

(13) $x = e^t \cos t$, (14) $y = e^t\left(2 - 3t + \frac{t^2}{2}\right)$, (15) $x = t^2 + e^t - e^{-2t}$,

(16) $y = -\frac{1}{3} + \frac{8}{15}e^{3t} + \frac{4}{5}e^{-2t}$, (17) $y = -\frac{1}{2}e^t - 2e^{2t} + \frac{1}{2}e^{3t} + 2t + 3$,

(18) $y = \frac{1}{3}(5e^t + e^{-2t}) - e^{-t}$, (19) $y = \frac{7}{4}e^{-t} - \frac{3}{4}e^{-3t} + \frac{1}{2}te^{-t}$,

(20) $y = \frac{t^3}{3} + 2e^{-t} + 2$, (21) $x = 2e^t - 3e^t \cdot t + \frac{t^2}{2}e^t$,

(22) $y = e^{-t}(t^3 + 7t + 2)$, (23) $y = \frac{1}{4}\sin 4t$,

(24) $y = -7e^t + 4e^{2t} + 4te^{2t}$, (25) $y = \frac{1}{6}[e^{4t} + 2e^{-2t} - 3]$,

(26) $y = e^{-t}\left(4 + 6t + \frac{t^3}{2}\right)$, (27) $y = 2 + 2e^{-t} + \frac{t^3}{3}$,

(28) $y = -e^t + \frac{3}{2}e^{2t} - \frac{1}{2}e^{-2t}$, (29) $y = t - \sin t + \cos t$,

(30) $y = \frac{20}{9}e^{-t/2} - \frac{11}{2}e^{-2t} - \frac{t}{3}e^{-2t}$, (31) $y = \frac{1}{6}e^t - \frac{4}{15}e^{-2t} + \frac{e^{3t}}{10}$,

(32) $y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$. (33) $y = -\frac{1}{40}e^{-3t} + \frac{1}{8}e^t - \frac{1}{10}\cos t - \frac{1}{5}\sin t$

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Module 3

COMPLEX FORM OF FOURIER SERIES, ORTHOGONAL AND ORTHONORMAL SET OF FUNCTIONS, FOURIER TRANSFORM

1. Complex Form of Fourier Series

Let $f(x)$ be defined in the interval $(C, C + 2l)$. The complex form of Fourier Series for $f(x)$ in this interval is given by

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}}$$

where, $$\boxed{C_n = \frac{1}{2l}\int_C^{C+2l} f(x) e^{-in\pi x/l} dx} \qquad n = 0, \pm 1, \pm 2, \ldots$$

Proof : Consider

$$f(x) = a_0 + \sum_1^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_1^{\infty} b_n \sin\frac{n\pi x}{l}$$

where, a_0, a_n and b_n are as given in (5) on page 3-6.

$$\therefore\ f(x) = a_0 + \sum_1^{\infty} a_n \left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) + \sum_1^{\infty} b_n \left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2i}\right)$$

$$= a_0 + \sum_1^{\infty} a_n \left(\frac{e^{in\pi x/l} + e^{-in\pi x/l}}{2}\right) - \sum_1^{\infty} i\, b_n \left(\frac{e^{in\pi x/l} - e^{-in\pi x/l}}{2}\right)$$

$$= a_0 + \sum_1^{\infty} \left(\frac{a_n - ib_n}{2}\right) e^{in\pi x/l} + \sum_1^{\infty} \left(\frac{a_n + ib_n}{2}\right) e^{-in\pi x/l}$$

$$= C_0 + \sum_1^{\infty} C_n e^{in\pi x/l} + \sum_1^{\infty} C_{-n} e^{-in\pi x/l} \quad \text{where, } C_n = \frac{a_n - ib_n}{2},\ C_{-n} = \frac{a_n + ib_n}{2}$$

$$\therefore\ f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}$$

where, $$C_0 = a_0 = \frac{1}{2l}\int_C^{C+2l} f(x)\,dx$$

$$C_n = \frac{a_n - ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}dx - i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} - i\sin\frac{n\pi x}{l}\right)dx \qquad \left[\because e^{-i\theta} = \cos\theta - i\sin\theta\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x) e^{-in\pi x/l} dx \qquad \ldots\ldots\ldots\ldots (1)$$

$$C_{-n} = \frac{a_n + ib_n}{2} = \frac{1}{2l}\left[\int_C^{C+2l} f(x)\cos\frac{n\pi x}{l}dx + i\int_C^{C+2l} f(x)\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{1}{2l}\int_C^{C+2l} f(x)\left(\cos\frac{n\pi x}{l} + i\sin\frac{n\pi x}{l}\right)dx \qquad \left[\because e^{i\theta} = \cos\theta + i\sin\theta\right]$$

$$C_{-n} = \frac{1}{2l}\int_C^{C+2l} f(x) e^{in\pi x/l} dx \qquad \ldots\ldots\ldots\ldots (2)$$

Combining these results (1) and (2), we get

$$C_n = \frac{1}{2l}\int_C^{C+2l} f(x) e^{-in\pi x/l} dx, \quad n = 0, \pm 1, \pm 2, \ldots$$

Cor. 1 : If the interval is C to C + 2π replacing l by π in the above result

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2\pi}\int_C^{C+2\pi} f(x) e^{-inx} dx}$$

Cor. 2 : If the interval is $(0, 2l)$, putting C = 0 in the above result

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-in\pi x/l} dx}$$

Cor. 3 : If the interval is $(0, 2\pi)$, putting $l = \pi$ in the above corollary 2,

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx} dx}$$

Cor. 4 : If the interval is $(-l, l)$, putting $C = -l$ in the above result

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x/l} dx}$$

Cor. 5 : If the interval is $(-\pi, \pi)$, putting $l = \pi$ in corollary 4,

$$\boxed{f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}} \quad \text{where,} \quad \boxed{C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx}$$

Example 1 : Obtain complex form of Fourier Series for $f(x) = e^{ax}$ in $(-\pi, \pi)$ where a is not an integer. **(M.U. 2001, 02, 04, 05, 11, 12, 13, 19)**

Hence deduce that when α is a constant other than an integer

(i) $\cos\alpha x = \dfrac{\sin\pi\alpha}{\pi}\sum \dfrac{(-1)^n \alpha}{(\alpha^2 - n^2)} \cdot e^{inx}$ **(M.U. 2009)**

(ii) $\sin\alpha x = \dfrac{\sin\pi\alpha}{i\pi}\sum \dfrac{(-1)^n \cdot n}{(\alpha^2 - n^2)} \cdot e^{inx}$ **(M.U. 2001, 02)**

Sol. : By corollary 5 above, the complex form of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$$

where, $$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ax} \cdot e^{-inx} dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{(a-in)x}}{a - in}\right]_{-\pi}^{\pi} = \frac{1}{2\pi(a - in)}\left[e^{(a-in)\pi} - e^{-(a-in)\pi}\right]$$

$$= \frac{1}{2\pi(a - in)}\left[e^{a\pi} \cdot e^{-in\pi} - e^{-a\pi} \cdot e^{in\pi}\right]$$

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But $\quad e^{\pm in\pi} = e^{i(\pm n\pi)} = \cos(\pm n\pi) + i\sin(\pm n\pi) = (-1)^n + i(0) = (-1)^n$

$$\therefore \quad C_n = \frac{1}{2\pi(a-in)}\left[(-1)^n e^{a\pi} - (-1)^n e^{-a\pi}\right] = \frac{(-1)^n}{\pi(a-in)}\left(\frac{e^{a\pi}-e^{-a\pi}}{2}\right)$$

$$= \frac{(-1)^n}{\pi(a-in)}\sin h\, a\pi = \frac{(-1)^n \sin h\, a\pi}{\pi(a-in)}\cdot\frac{(a+in)}{(a+in)}$$

$$= \frac{(-1)^n \sin h\, a\pi\,(a+in)}{\pi(a^2+n^2)}$$

Hence, $$e^{ax} = \sum_{-\infty}^{\infty}\frac{(-1)^n \sin h\, a\pi\cdot(a+in)}{\pi(a^2+n^2)}e^{inx} \quad \text{..... (1)}$$

For deductions, replace a by $i\alpha$ in (1)

$$\therefore \quad e^{i\alpha x} = \sum\frac{(-1)^n \sin h\, i\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(i\alpha+in)\cdot e^{inx}$$

$$= \sum\frac{(-1)^n (i)\sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(i\alpha+in)\cdot e^{inx}$$

$$\therefore \quad e^{i\alpha x} = \sum\frac{(-1)^n \sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-\alpha-n)\cdot e^{inx} \quad \text{..... (2)}$$

Now, replace a by $-i\alpha$ in (1)

$$\therefore \quad e^{-i\alpha x} = \sum\frac{(-1)^n \sin h(-i\alpha\pi)}{\pi(-\alpha^2+n^2)}\cdot(-i\alpha+in)\cdot e^{inx}$$

$$= \sum\frac{(-1)^n (-i)\sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-i\alpha+in)\cdot e^{inx}$$

$$\therefore \quad e^{-i\alpha x} = \sum\frac{(-1)^n \sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-\alpha+n)\cdot e^{inx} \quad \text{..... (3)}$$

Adding (2) and (3) and dividing by 2; and subtracting (3) from (2) and dividing by $2i$,

$$\therefore \quad \cos\alpha x = \frac{e^{i\alpha x}+e^{-i\alpha x}}{2} = \sum\frac{(-1)^n \sin\alpha\pi}{\pi(-\alpha^2+n^2)}\cdot(-\alpha)\cdot e^{inx} = \frac{\sin\alpha\pi}{\pi}\sum\frac{(-1)^n\cdot\alpha}{(\alpha^2-n^2)}\cdot e^{inx}$$

$$\therefore \quad \sin\alpha x = \frac{e^{i\alpha x}-e^{-i\alpha x}}{2i} = \sum\frac{(-1)^n \sin\alpha\pi}{\pi i(-\alpha^2+n^2)}\cdot(-n)\cdot e^{inx} = \frac{\sin\alpha\pi}{\pi i}\sum\frac{(-1)^n\cdot n}{(\alpha^2-n^2)}\cdot e^{inx}.$$

Example 2 : Obtain the complex form of Fourier Series for $f(x) = e^{-ax}$ in $(-\pi, \pi)$. **(M.U. 2014)**

Sol. : Changing the sign of a, we get from the above result (1),

$$f(x) = \sum_{-\infty}^{\infty}\frac{(-1)^n \sin h(-a\pi)\cdot(-a+in)}{\pi(a^2+n^2)}e^{inx}$$

But $\quad \sin h(-a\pi) = \dfrac{e^{-a\pi}-e^{a\pi}}{2} = -\dfrac{e^{a\pi}-e^{-a\pi}}{2} = -\sin h\, a\pi$

$$\therefore \quad e^{-ax} = \sum\frac{(-1)^n \sin h\, a\pi\cdot(a-in)}{\pi(a^2+n^2)}e^{inx}$$

Or proceeding as above obtain the result independently.

Example 3 : Obtain the complex form of Fourier Series for $f(x) = \cos h\, ax$ in $(-\pi, \pi)$ where a is not an integer.

Sol. : As proved in (1) in example 1,

$$e^{ax} = \sum\frac{(-1)^n \sin h\, a\pi\cdot(a+in)}{\pi(a^2+n^2)}e^{inx}$$

Changing the sign of a, we get,

$$e^{-ax} = \sum\frac{(-1)^n \sin h(-a\pi)\cdot(-a+in)}{\pi(a^2+n^2)}e^{inx}$$

But $\sin h(-a\pi) = -\sin h\, a\pi$.

$$e^{-ax} = \sum\frac{(-1)^n \sin h\, a\pi\cdot(a-in)}{\pi(a^2+n^2)}e^{inx}$$

$$\text{Now, } \cos h\, ax = \frac{e^{ax}+e^{-ax}}{2} = \frac{\sin h\, a\pi}{2}\left[\sum\frac{(-1)^n(a+in)}{(a^2+n^2)}e^{inx} + \sum\frac{(-1)^n(a-in)}{(a^2+n^2)}e^{inx}\right]$$

$$= a\cdot\sin h\, a\pi\sum\frac{(-1)^n}{(a^2+n^2)}e^{in\pi x}$$

Example 4 : Obtain the complex form of Fourier Series for $f(x) = e^{ax}$ in $(-l, l)$.

(M.U. 1993, 2003, 12, 13, 15, 19)

Sol. : By corollary 4 above, the complex form of Fourier series of $f(x) = e^{ax}$ is given by

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{..... (1)}$$

where, $$C_n = \frac{1}{2l}\int_{-l}^{l} f(x)e^{-in\pi x/l} = \frac{1}{2l}\int_{-l}^{l} e^{ax}\cdot e^{-in\pi x/l}\cdot dx$$

$$C_n = \frac{1}{2l}\int_{-l}^{l} e^{(a-in\pi/l)x}\,dx = \frac{1}{2l}\left[\frac{e^{(a-in\pi/l)x}}{(a-in\pi/l)}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{(a-in\pi/l)l} - e^{-(a-in\pi/l)l}}{(a-in\pi/l)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{al}\cdot e^{-in\pi} - e^{-al}\cdot e^{in\pi}}{(al-in\pi)}\right]$$

Now, $e^{\pm in\pi} = e^{i(\pm n\pi)} = \cos(\pm n\pi) + i\sin(\pm n\pi) = (-1)^n + i\cdot 0 = (-1)^n$

$$\therefore \quad C_n = \frac{e^{al}(-1)^n - e^{-al}(-1)^n}{2(al-in\pi)} = \frac{(-1)^n(e^{al}-e^{-al})}{2(al-in\pi)}$$

$$= \frac{(-1)^n\cdot\sin h\, al}{al-in\pi}\cdot\frac{al+in\pi}{al+in\pi} = \frac{(-1)^n \sin h\, al\,(al+in\pi)}{a^2l^2+n^2\pi^2}$$

$$\therefore \quad e^{ax} = \sum_{-\infty}^{\infty}\frac{(-1)^n \sin h\, al\cdot(al+in\pi)}{a^2l^2+n^2\pi^2}e^{in\pi x/l}$$

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Example 5 : Obtain the complex form of Fourier Series for $f(x) = \cos h\, ax$ in $(-l, l)$. **(M.U. 2002, 05, 09, 11)**

Sol. : By example (4) above

$$e^{ax} = \sum \frac{(-1)^n \sin h\, al \cdot (al + in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

Changing the sign of a, we get

$$e^{-ax} = \sum \frac{(-1)^n \sin h(-al)(-al + in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

But $\sin h(-x) = \dfrac{e^{-x} - e^{x}}{2} = -\left(\dfrac{e^{x} - e^{-x}}{2}\right) = -\sin h\, x$

$$\therefore \quad e^{-ax} = \sum \frac{(-1)^n \sin h\, al\,(al - in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

$$\therefore \cos h\, ax = \frac{e^{ax} + e^{-ax}}{2}$$

$$= \frac{\sin h\, al}{2}\left[\sum \frac{(-1)^n (al + in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l} + \sum \frac{(-1)^n (al - in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}\right]$$

$$= al \sin h\, al \sum \frac{(-1)^n}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

Example 6 : Obtain complex form of Fourier Series for $f(x) = \sin h\, ax$ in $(-l, l)$. **(M.U. 2014, 19)**

Sol. : Using the results obtained in example 5, we get

$$\sin h\, ax = \frac{e^{ax} - e^{-ax}}{2}$$

$$= \frac{\sin h\, al}{2}\left[\sum \frac{(-1)^n (al + in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l} - \sum \frac{(-1)^n (al - in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}\right]$$

$$= \sin h\, al \sum \frac{(-1)^n\, in\pi}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

Example 7 : Obtain complex form of Fourier Series for $f(x) = \cos h\, ax + \sin h\, ax$ in $(-l, l)$. **(M.U. 2002, 07, 08, 11)**

Sol. : By using the results obtained in Examples 5 and 6,

$$f(x) = \cos h\, ax + \sin h\, ax = \sin h\, al \sum \frac{(-1)^n (al + in\pi)}{a^2l^2 + n^2\pi^2} e^{in\pi x/l}$$

Example 8 : Obtain complex form of Fourier series for $f(x) = \cos h\, 2x + \sin h\, 2x$ in $(-2, 2)$. **(M.U. 2002, 15)**

Sol. : We have by corollary (4),

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \qquad \text{where } C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x/l}\, dx$$

Here, $l = 2$.

$$\therefore \quad f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/2} \qquad \text{where } C_n = \frac{1}{4}\int_{-2}^{2} f(x) e^{-in\pi x/2}\, dx$$

But $\quad f(x) = \cos h\, 2x + \sin h\, 2x$

$$\therefore \quad f(x) = \frac{e^{2x} + e^{-2x}}{2} + \frac{e^{2x} - e^{-2x}}{2} = e^{2x}$$

$$\therefore \quad C_n = \frac{1}{4}\int_{-2}^{2} e^{2x} \cdot e^{-in\pi x/2}\, dx = \frac{1}{4}\int_{-2}^{2} e^{[2-(in\pi/2)]x}\, dx$$

$$= \frac{1}{4}\left[\frac{e^{[2-(in\pi/2)]x}}{2-(in\pi/2)}\right]_{-2}^{2} = \frac{1}{4}\cdot\frac{2}{4-in\pi}\left[e^{4-in\pi} - e^{-4+in\pi}\right]$$

But $\quad e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n \pm 0 = (-1)^n$ **[Note this]**

$$\therefore \quad C_n = \frac{1}{2(4-in\pi)}\left[(-1)^n e^4 - (-1)^n e^{-4}\right]$$

$$= \frac{1}{2}\cdot\frac{(4+in\pi)}{(16+n^2\pi^2)}\cdot(-1)^n\cdot[e^4 - e^{-4}]$$

$$= \frac{(4+in\pi)\cdot(-1)^n}{16+n^2\pi^2}\cdot\frac{e^4 - e^{-4}}{2}$$

$$= \sin h\, 4 \cdot \frac{(-1)^n \cdot (4+in\pi)}{16+n^2\pi^2}$$

Hence, complex form of Fourier series for $f(x)$ is

$$f(x) = \sin h\, 4 \sum_{-\infty}^{\infty} \frac{(-1)^n (4+in\pi)}{16+n^2\pi^2} \cdot e^{in\pi x/2}.$$

(or, put $a = 2$, $l = 2$ in Ex. 7.)

Example 9 : Obtain complex form of Fourier Series for

$f(x) = \cos h\, 3x + \sin h\, 3x$ in $(-3, 3)$. **(M.U. 2008, 14)**

Sol. : We have by corollary 4

$$f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/l} \qquad \text{where } C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-in\pi x/l}\, dx.$$

Here, $l = 3$.

$$\therefore \quad f(x) = \sum_{-\infty}^{\infty} C_n e^{in\pi x/3} \qquad \text{where } C_n = \frac{1}{6}\int_{-3}^{3} f(x)\cdot e^{-in\pi x/3}\, dx$$

But $\quad f(x) = \cos h\, 3x + \sin h\, 3x = \dfrac{e^{3x} + e^{-3x}}{2} + \dfrac{e^{3x} - e^{-3x}}{2} = e^{3x}$

$$\therefore \quad C_n = \frac{1}{6}\int_{-3}^{3} e^{3x} \cdot e^{-in\pi x/3}\, dx = \frac{1}{6}\int_{-3}^{3} e^{(3-in\pi/3)x}\, dx$$

$$= \frac{1}{6}\left[\frac{e^{(3-in\pi/3)x}}{3-in\pi/3}\right]_{-3}^{3} = \frac{1}{6}\cdot\frac{3}{(9-in\pi)}\left[e^{9-in\pi} - e^{-9+in\pi}\right]$$

But $\quad e^{\pm in\pi} = \cos n\pi \pm i \sin n\pi = (-1)^n \pm 0 = (-1)^n$

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$$\therefore \quad C_n = \frac{1}{2(9-in\pi)}[(-1)^n \cdot e^9 - (-1)^n e^{-9}]$$

$$= \frac{1}{2} \cdot \frac{(9+in\pi)(-1)^n}{81+n^2\pi^2} \cdot [e^9 - e^{-9}] = \frac{(-1)^n (9+in\pi)}{81+n^2\pi^2} \sinh 9$$

$\therefore$ Complex form of Fourier Series for

$$f(x) = \sinh 9 \cdot \sum_{-\infty}^{\infty} \frac{(-1)^n (9+in\pi)}{81+n^2\pi^2} \cdot e^{in\pi x/3}.$$

Example 10 : Find complex form of Fourier Series for $\cos ax$, where a is not an integer in $(-\pi, \pi)$. **(M.U. 2011, 19)**

Sol. : We have $\cos ax = \dfrac{e^{aix} + e^{-aix}}{2}$

Replacing a by ai in (1) in example 1, we get

$$e^{aix} = \sum_{-\infty}^{\infty} \frac{(-1)^n \sinh ai\pi \cdot (ai+in)}{\pi(-a^2+n^2)} e^{inx} \qquad [\because -a^2+n^2 \neq 0]$$

$$= \frac{-i \sinh ai\pi}{\pi} \sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx}$$

Since, $\sin h\, ix = i \sin x$, we get

$$e^{aix} = \frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx}$$

Changing the sign of a,

$$e^{-aix} = \frac{\sin(-a\pi)}{\pi} \sum \frac{(-1)^n (-a+n)}{(a^2-n^2)} e^{inx} = -\frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (-a+n)}{(a^2-n^2)} e^{inx}$$

$$= \frac{\sin a\pi}{\pi} \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx}$$

$$\therefore \quad \cos ax = \frac{e^{aix} + e^{-aix}}{2}$$

$$= \frac{\sin a\pi}{2\pi} \left[\sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx} + \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx} \right]$$

$$= \frac{\sin a\pi}{2\pi} \sum \frac{(-1)^n \cdot 2a}{(a^2-n^2)} e^{inx} = \frac{a \sin a\pi}{\pi} \sum \frac{(-1)^n}{(a^2-n^2)} e^{inx}.$$

Example 11 : Find complex from of Fourier Series for $\sin ax$, where a is not an integer in $(-\pi, \pi)$. **(M.U. 2004, 14)**

Sol. : Using the results obtained in the above example, we get,

$$\sin ax = \frac{e^{aix} - e^{-aix}}{2i} = \frac{\sin a\pi}{2\pi i} \left[\sum \frac{(-1)^n (a+n)}{(a^2-n^2)} e^{inx} - \sum \frac{(-1)^n (a-n)}{(a^2-n^2)} e^{inx} \right]$$

$$\therefore \quad \sin ax = \frac{\sin a\pi}{\pi i} \sum (-1)^n \cdot \frac{n}{(a^2-n^2)} \cdot e^{inx}$$

Example 12 : Obtain the complex form of Fourier series for $f(x) = e^{ax}$ in $(0, a)$. **(M.U. 2003, 10, 13, 14, 16, 19)**

Sol. : By corollary (2), page S-97, putting $2l = a$ i.e. $l = a/2$, we get

$$f(x) = \sum_{-\infty}^{\infty} C_n \cdot e^{2in\pi x/a}$$

where, $$C_n = \frac{1}{a} \int_0^a e^{ax} \cdot e^{-2in\pi x/a} dx$$

$$\therefore \quad C_n = \frac{1}{a} \int_0^a e^{(a-2in\pi/a)x} dx = \frac{1}{a} \left[\frac{e^{(a-2in\pi/a)x}}{(a-2in\pi/a)} \right]_0^a$$

$$= \frac{1}{a} \cdot \frac{a}{(a^2-2in\pi)} \cdot \left[e^{(a^2-2in\pi)} - 1 \right]$$

$$= \frac{1}{(a^2-2in\pi)} \left[e^{a^2} \cdot e^{-2in\pi} - 1 \right]$$

$$= \frac{1}{(a^2-2in\pi)} (e^{a^2} - 1) \qquad [\because e^{-2in\pi} = \cos 2n\pi - i \sin 2n\pi = 1]$$

$$\therefore \quad e^{ax} = (e^{a^2} - 1) \sum_{-\infty}^{\infty} \frac{e^{2in\pi x/a}}{(a^2-2in\pi)}.$$

Example 13 : Obtain the complex form of Fourier Series for $f(x) = e^{2x}$ in $(0, 2)$. **(M.U. 2003, 16)**

Sol. : In the result obtained in Ex. 12 above replace a *by* 2 and get

$$e^{2x} = (e^4 - 1) \sum_{-\infty}^{\infty} \frac{e^{in\pi x}}{(4-2in\pi)}$$

Example 14 : Find the complex form of Fourier Series for $f(x) = \begin{cases} 0, & 0 < x < l \\ a, & l < x < 2l \end{cases}$. **(M.U. 2008)**

Sol. : By corollary 2, the complex form of Fourier Series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l} \qquad \text{where, } C_n = \frac{1}{2l} \int_0^{2l} f(x) e^{-in\pi x/l} dx$$

$$\therefore \quad C_n = \frac{1}{2l} \left[\int_0^l 0 \cdot dx + \int_l^{2l} a e^{-in\pi x/l} dx \right] \qquad \text{..... (1)}$$

$$= \frac{a}{2l} \int_l^{2l} e^{-in\pi x/l} dx = \frac{a}{2l} \left[\frac{e^{-in\pi x/l}}{-in\pi/l} \right]_l^{2l} \qquad \text{..... (2)}$$

$$= \frac{-a}{2in\pi} \left[e^{-2in\pi} - e^{-in\pi} \right] \quad \text{except when } n = 0.$$

Case 1 : Where $n = 0$ from (1)

$$C_0 = \frac{1}{2l} \left[\int_0^l 0\, dx + \int_l^{2l} a e^0 dx \right] = \frac{1}{2l} \int_l^{2l} a\, dx = \frac{a}{2l}[2l - l] = \frac{al}{2l} = \frac{a}{2}$$

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Case 2 : When $n = \pm 1, \pm 3, \ldots\ldots$ from (2)

$$C_1 = \frac{-a}{2i\pi}\left[e^{-2i\pi} - e^{-i\pi}\right]$$

$$= \frac{-a}{2i\pi}\left[\cos(-2\pi) + i\sin(-2\pi) - \cos(-\pi) - i\sin(-\pi)\right]$$

$$= \frac{-a}{2i\pi}\left[1 + i(0) - (-1) + i(0)\right] = \frac{-a}{2i\pi}\cdot 2 = \frac{ai}{\pi}$$

$$C_{-1} = \frac{a}{2i\pi}\left[\cos 2\pi + i\sin 2\pi - \cos\pi - i\sin\pi\right]$$

$$= \frac{a}{2i\pi}\left[1 + i(0) - (-1) + i0\right] = \frac{a}{2i\pi}\cdot 2 = -\frac{ai}{\pi}$$

Similarly, $C_3 = \frac{ia}{3\pi}$, $C_{-3} = -\frac{ai}{3\pi}$; $C_5 = \frac{ia}{5\pi}$, $C_{-5} = -\frac{ai}{5\pi}$.

Case 3 : When $n = \pm 2, \pm 4, \ldots\ldots$

$$C_2 = \frac{-a}{4i\pi}\left[e^{-4i\pi} - e^{-2i\pi}\right]$$

$$= \frac{-a}{4i\pi}\left[\cos(-4\pi) + i\sin(-4\pi) - \cos(-2\pi) + i\sin(-2\pi)\right]$$

$$\therefore \quad C_2 = \frac{-a}{4i\pi}\left[1 + i(0) - (1) - i(0)\right] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots\ldots = 0$

$$\therefore \quad f(x) = \frac{a}{2} + \frac{ai}{\pi}\left[(e^u - e^{-u}) + \frac{1}{3}(e^{3u} - e^{-3u}) + \ldots\right] \text{ where } u = \frac{i\pi x}{l}$$

Example 15 : If $f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 < x < 2 \end{cases}$

$f(x+2) = f(x)$, find complex form of Fourier Series. **(M.U. 1996)**

Sol. : By corollary 2, the complex form of Fourier Series in $(0, 2l)$ is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l} \quad \text{where, } C_n = \frac{1}{2l}\int_0^{2l} f(x) e^{-in\pi x/l}\,dx$$

Since, in this case $l = 1$, $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x}$

and $$C_n = \frac{1}{2}\int_0^2 f(x) e^{-in\pi x}\,dx \qquad \ldots\ldots\ldots\ldots (1)$$

$$= \frac{1}{2}\left[\int_0^1 1\cdot e^{-in\pi x}dx + \int_1^2 0\cdot e^{-in\pi x}dx\right]$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi x}}{-in\pi}\right]_0^1 = \frac{1}{-2in\pi}\left[e^{-in\pi} - 1\right]$$

$$= \frac{1}{2in\pi}\left[1 - e^{-in\pi}\right] \text{ except when } n = 0. \qquad \ldots\ldots\ldots\ldots (2)$$

Case 1 : When $n = 0$, from (1)

$$C_0 = \frac{1}{2}\left[\int_0^1 1\cdot dx + \int_1^2 0\cdot dx\right] = \frac{1}{2}\left[x\right]_0^1 = \frac{1}{2}$$

Case 2 : When $n = \pm 1, \pm 3, \pm 5, \ldots\ldots$ from (2)

$$C_1 = \frac{1}{2i\pi}\left[1 - e^{-i\pi}\right] = \frac{1}{2i\pi}\left[1 - \{\cos\pi - i\sin\pi\}\right]$$

$$= \frac{1}{2i\pi}\left[1 - (-1)\right] = \frac{2}{2i\pi} = \frac{1}{i\pi}$$

$$C_{-1} = \frac{1}{-2i\pi}\left[1 - e^{i\pi}\right] = -\frac{1}{2i\pi}\left[1 - \{\cos\pi + i\sin\pi\}\right]$$

$$= -\frac{1}{2i\pi}\left[1 - (-1)\right] = -\frac{1}{i\pi}$$

Similarly, $C_3 = \frac{1}{3i\pi}$, $C_{-3} = -\frac{1}{3i\pi}$, $C_5 = \frac{1}{5i\pi}$, $C_{-5} = -\frac{1}{5i\pi}, \ldots\ldots$

Case 3 : When $n = \pm 2, \pm 4, \ldots\ldots$ from (2)

$$C_2 = \frac{1}{4i\pi}\left[1 - e^{-2i\pi}\right] = \frac{1}{4i\pi}\left[1 - \{\cos 2\pi + i\sin 2\pi\}\right] = \frac{1}{4i\pi}\left[1 - 1\right] = 0$$

Similarly, $C_{-2} = 0$ and $C_4 = C_{-4} = C_6 = C_{-6} = \ldots\ldots = 0$

Hence, $$f(x) = \frac{1}{2} + \frac{1}{i\pi}(e^{i\pi x} - e^{-i\pi x}) + \frac{1}{3i\pi}(e^{3i\pi x} - e^{-3i\pi x}) + \ldots\ldots$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\left(\frac{e^{i\pi x} - e^{-i\pi x}}{2i}\right) + \frac{1}{3}\left(\frac{e^{3i\pi x} - e^{-3i\pi x}}{2i}\right) + \ldots\right]$$

$$= \frac{1}{2} + \frac{2}{\pi}\left[\frac{\sin\pi x}{1} + \frac{\sin 3\pi x}{3} + \ldots\right]$$

Example 16 : Find the complex form of the Fourier series for $f(x) = 2x$ in $(0, 2\pi)$. **(M.U. 2009)**

Sol. : By corollary (3), page S-97, $f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$ where, $C_n = \frac{1}{2\pi}\int_0^{2\pi} f(x) e^{-inx}\,dx$

Hence, $$C_n = \frac{1}{2\pi}\int_0^{2\pi} 2x\cdot e^{-inx}\,dx = \frac{1}{\pi}\int_0^{2\pi} x e^{-inx}\,dx \qquad \ldots\ldots\ldots\ldots (1)$$

$$= \frac{1}{\pi}\left[x\cdot\frac{e^{-inx}}{-in} - \int\frac{e^{-inx}}{-in}\cdot 1\cdot dx\right]_0^{2\pi} = \frac{1}{\pi}\left[-\frac{x}{in}e^{-inx} - \frac{e^{-inx}}{(in)^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2\pi e^{-i2n\pi}}{in} + \frac{e^{-in2\pi}}{n^2} - 0 - \frac{1}{n^2}\right] \quad \text{for } n \neq 0, (\because i^2 = -1)$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2}\right] \quad \text{for } n \neq 0 \quad [\because e^{-i2n\pi} = \cos(2n\pi) - i\sin(2n\pi) = 1]$$

$$= \frac{1}{\pi}\left[-\frac{2\pi}{in} + \frac{1}{n^2} - \frac{1}{n^2}\right] \quad \text{for } n \neq 0$$

$$\therefore \quad C_n = -\frac{2}{in} = -\frac{2i}{i^2 n} = \frac{2i}{n} \quad \text{for } n \neq 0$$

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For $n = 0$, we get from (1),

$$C_0 = \frac{1}{2\pi}\int_0^{2\pi} 2x \cdot e^0\, dx = \frac{1}{2\pi}\left[x^2\right]_0^{2\pi} = \frac{1}{2\pi}\cdot 4\pi^2 = 2\pi.$$

Hence, $f(x) = 2\pi + 2i \sum_{n=-\infty}^{\infty} \frac{1}{n}\cdot e^{inx}$, for $n \neq 0$

Putting $n = \pm 1, \pm 2, \pm 3, \ldots\ldots$, we get

$$f(x) = 2\pi + 2i\left[\frac{e^{ix}}{1} + \frac{e^{2ix}}{2} + \frac{e^{3ix}}{3} + \ldots - \frac{e^{-ix}}{1} - \frac{e^{-2ix}}{2} - \frac{e^{-3ix}}{3} - \ldots\right]$$

$$= 2\pi + 2i\left[\frac{e^{ix} - e^{-ix}}{1} + \frac{1}{2}\left(\frac{e^{2ix} - e^{-2ix}}{1}\right) + \frac{1}{3}\left(\frac{e^{3ix} - e^{-3ix}}{1}\right) + \ldots\right]$$

$$= 2\pi + 4i^2\left[\frac{(e^{ix} - e^{-ix})}{2i} + \frac{1}{2}\cdot\frac{(e^{2ix} - e^{-2ix})}{2i} + \frac{1}{3}\cdot\frac{(e^{3ix} - e^{-3ix})}{2i} + \ldots\right]$$

$$= 2\pi - 4\left[\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \ldots\right]$$

EXERCISE - I

1. Find complex form of Fourier Series for $f(x) = e^{-x}$ in $(-1, 1)$. **(M.U. 2009, 14)**

   $\left[\textbf{Ans. :}\ f(x) = \sum_{-\infty}^{\infty} \frac{(-1)^n (1 - in\pi)\sinh 1}{1 + n^2\pi^2} e^{in\pi x}\right]$

2. Find the complex form of $f(x) = e^x$ in $(-\pi, \pi)$. **(M.U. 1999, 2000, 04)**

   $\left[\textbf{Ans. :}\ f(x) = \sum \frac{(-1)^n \sinh \pi (1 + in)}{\pi(1^2 + n^2)} e^{inx}\right]$

3. Obtain complex form of Fourier series for $f(x) = e^{ax}$, $-1 < x < 1$. **(M.U. 1996, 2003)**

   $\left[\textbf{Ans. :}\ f(x) = \sum \frac{(-1)^n \sinh a(a + in\pi)}{a^2 + n^2\pi^2} e^{in\pi}\right]$

4. Find the complex form of Fourier Series for $f(x) = \sin hx$ in $(-l, l)$. **(M.U. 1997)**

   $\left[\textbf{Ans. :}\ f(x) = \sinh l \cdot (i\pi) \sum \frac{(-1)^n n}{l^2 + n^2\pi^2} e^{in\pi x/l}\right]$

5. Find the complex form of Fourier Series for $f(x) = \cosh 2x + \sinh 2x$ in $(-5, 5)$.
   (**Hint :** In solved example 7, put $a = 2$ and $l = 5$.) **(M.U. 1999, 2003, 11)**

6. Find the complex form of Fourier Series of $\cosh 3x + \sinh 3x$ in $(-\pi, \pi)$. **(M.U. 1999, 2003, 11)**

7. Find complex form of Fourier Seires of $f(x) = \cos hx + \sin hx$ in $(-\pi, \pi)$. **(M.U. 2003, 09)**

8. Find the complex form of Fourier series for $f(x) = e^x$ in $(0, 2\pi)$.

   $\left[\textbf{Ans. :}\ e^x = \frac{e^{2\pi} - 1}{2\pi} \sum_{-\infty}^{\infty} \frac{1 + in}{n^2 + 1} \cdot e^{inx}\right]$

2. Orthogonality, Orthonormality

We first define these terms and then solve some problems based on this definition.

Definition 1 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots\ldots f_n(x) \ldots\ldots$ is said to be **orthogonal** on (a, b) if

$$\int_a^b f_m(x) f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), \ldots\ldots f_n(x) \ldots\ldots$ is orthogonal on (a, b) if

$$\boxed{\int_a^b f_m(x) f_n(x)\, dx = 0 \quad \text{if } m \neq n} \quad \text{and} \quad \boxed{\int_a^b [f_m(x)]^2 dx \neq 0}$$

Definition 2 : A set of functions $f_1(x), f_2(x), f_3(x), \ldots\ldots f_n(x) \ldots\ldots$ is said to be **orthonormal** on (a, b) if

$$\int_a^b f_m(x) f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ = 1, & \text{if } m = n \end{cases}$$

In other words a set of functions $f_1(x), f_2(x), f_3(x), \ldots\ldots f_n(x) \ldots\ldots$ is said to be orthonormal on (a, b) if

$$\boxed{\int_a^b f_m(x) f_n(x)\, dx = 0 \quad \text{if } m \neq n} \quad \text{and} \quad \boxed{\int_a^b [f_m(x)]^2 dx = 1}$$

Note ....

Every orthonormal set of functions is orthogonal but the converse may not be true.

Example 1 : Show that the set of functions $\cos nx$, $n = 1, 2, 3, \ldots\ldots$ is orthogonal on $(0, 2\pi)$. **(M.U. 1994, 2010, 14, 18, 19)**

Sol. : We have $f_n(x) = \cos nx$.

$$\therefore \int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos mx \cdot \cos nx\, dx$$

$$= \frac{1}{2}\int_0^{2\pi} [\cos(m + n)x + \cos(m - n)x]\, dx$$

$$= \frac{1}{2}\left[\frac{\sin(m + n)x}{m + n} + \frac{\sin(m - n)x}{m - n}\right]_0^{2\pi}$$

Now, two cases arise.

Case 1 : When $m \neq n$, then $\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx = 0$

Case 2 : When $m = n$, then $\int_0^{2\pi} f_n(x) \cdot f_n(x)\, dx = \int_0^{2\pi} \cos^2 nx\, dx$

$$\therefore \int_0^{2\pi} [f_n(x)]^2 dx = \int_0^{2\pi}\left(\frac{1 + \cos 2nx}{2}\right) dx = \frac{1}{2}\left[x + \frac{\sin 2nx}{2n}\right]_0^{2\pi} = \pi \neq 0 \quad \ldots\ldots\ldots\ldots (1)$$

Since, $\int_0^{2\pi} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

The given set of functions is orthogonal on $[0, 2\pi]$.

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Example 2 : How can you construct orthonormal set of functions from the set given in the above example number 1?

Sol. : If the set of functions is to be orthonormal, we should have

$$\int_0^{2\pi} [f_n(x)]^2\, dx = 1$$

For this we divide (1) by π and write it as

$$\int_0^{2\pi} \frac{1}{\pi}[f_n(x)]^2\, dx = \pi \cdot \frac{1}{\pi} = 1 \qquad i.e. \qquad \int_0^{2\pi} \frac{1}{\sqrt{\pi}} f_n(x) \cdot \frac{1}{\sqrt{\pi}} f_n(x)\, dx = 1$$

This is obviously an orthonormal set, where $\phi_n(x) = \frac{1}{\sqrt{\pi}} \cos n\, x$.

Hence, the required orthonormal set is $\frac{1}{\sqrt{\pi}} \cos x,\ \frac{1}{\sqrt{\pi}} \cos 2x,\ \frac{1}{\sqrt{\pi}} \cos 3x, \ldots$

Example 3 : Show that the set of functions { $\sin x$, $\sin 3x$, $\sin 5x$, ... } *i.e.*, { $\sin (2n + 1) x$}, $n = 0, 1, 2, \ldots$ is orthogonal over $[0, \pi / 2]$. Hence, construct orthonormal set of functions.

(M.U. 2002, 04, 06, 08, 09, 10, 13)

Sol. : We have $f(x) = \sin (2n + 1) x$.

$$\therefore \int_0^{\pi/2} f_m(n) \cdot f_n(x)\, dx = \int_0^{\pi/2} \sin(2m+1)x \cdot \sin(2n+1)x\, dx$$

$$= -\frac{1}{2}\int_0^{\pi/2} [\cos(2m+2n+2)x - \cos(2m-2n)x]\, dx$$

$$= -\frac{1}{2}\left[\frac{\sin(2m+2n+2)x}{(2m+2n+2)} - \frac{\sin(2m-2n)x}{(2m-2n)}\right]_0^{\pi/2}$$

Now, two cases arise.

Case 1 : If $m \neq n$, then $\int_0^{\pi/2} f_m(x) \cdot f_n(x) = 0$

Case 2 : If $m = n$, then $\int_0^{\pi/2} f_n(x) \cdot f_n(x)\, dx = \int_0^{\pi/2} \sin^2(2n+1)x\, dx$

$$\therefore \int_0^{\pi/2} [f_n(x)]^2\, dx = \int_0^{\pi/2} \left(\frac{1 - \cos 2(2n+1)x}{2}\right) dx = \frac{1}{2}\left[x - \frac{\sin 2(2n+1)x}{2(2n+1)}\right]_0^{\pi/2}$$

$$\therefore \int_0^{\pi/2} [f_n(x)]^2\, dx = \frac{\pi}{4} \neq 0 \qquad \ldots\ldots\ldots\ldots (1)$$

Since, $\int_0^{\pi/2} f_m(x) \cdot f_n(x)\, dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

the given set of functions is orthogonal on $[0, \pi / 2]$.

Now, if the set is to be orthonormal, then we should have,

$$\int_0^{\pi/2} [f_n(x)]^2\, dx = 1$$

For this, we divide (1) by $\pi / 4$ and write it as

$$\int_0^{\pi/2} \frac{4}{\pi}[f_n(x)]^2\, dx = \frac{4}{\pi} \cdot \frac{\pi}{4} = 1 \qquad i.e. \qquad \int_0^{\pi/2} \frac{2}{\sqrt{\pi}} f_n(x) \cdot \frac{2}{\sqrt{\pi}} f_n(x)\, dx = 1$$

This is obviously an orthonormal set, where $\phi_n(x) = \frac{2}{\sqrt{\pi}} \sin(2n+1)x$.

Hence, the required orthonormal set of functions is $\frac{2}{\sqrt{\pi}} \sin x,\ \frac{2}{\sqrt{\pi}} \sin 3x,\ \frac{2}{\sqrt{\pi}} \sin 5x, \ldots$

Example 4 : Is $S = \left\{\sin\left(\frac{\pi x}{4}\right), \sin\left(\frac{3\pi x}{4}\right), \sin\left(\frac{5\pi x}{4}\right), \ldots\right\}$ orthogonal in (0, 1) ?

(M.U. 2004)

Sol. : We have

$$\int_0^1 f_m(x) \cdot f_n(x)\, dx = \int_0^1 \sin\frac{(2m+1)\pi x}{4} \sin\frac{(2n+1)\pi x}{4} \cdot dx$$

$$= -\frac{1}{2}\left[\int_0^1 \left(\frac{\cos(2m+2n+2)\pi x}{4} - \frac{\cos(2m-2n)\pi x}{4}\right) dx\right]$$

$$= -\frac{1}{2}\left[\int_0^1 \left[\cos\frac{(m+n+1)\pi x}{2} - \cos\frac{(m-n)\pi x}{2}\right] dx\right]$$

$$= -\frac{1}{2}\left[\sin\frac{(m+n+1)\pi x}{2} \cdot \frac{2}{(m+n+1)\pi} - \sin\frac{(m-n)\pi x}{2} \cdot \frac{2}{(m-n)\pi}\right]_0^1$$

$$\neq 0$$

$\therefore$ S is not orthogonal.

Example 5 : Show that the set of functions

$$\sin\left(\frac{\pi x}{2L}\right),\ \sin\left(\frac{3\pi x}{2L}\right),\ \sin\left(\frac{5\pi x}{2L}\right), \ldots \text{ is orthogonal over } (0, L).$$

(M.U. 2002, 05, 06, 09, 12, 13)

Construct corresponding orthonormal set. **(M.U. 2016)**

Sol. : We have $f_n(x) = \sin\frac{(2n+1)\pi x}{2L}$, $n = 0, 1, 2, \ldots$

$$\therefore \int_0^L f_m(x) \cdot f_n(x)\, dx = \int_0^L \sin\frac{(2m+1)\pi x}{2L} \cdot \sin\frac{(2n+1)\pi x}{2L}\, dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{2m+2n+2}{2L}\right)\pi x - \cos\left(\frac{2m-2n}{2L}\right)\pi x\right] dx$$

$$= -\frac{1}{2}\int_0^L \left[\cos\left(\frac{m+n+1}{L}\right)\pi x - \cos\left(\frac{m-n}{L}\right)x\right] dx$$

$$= -\frac{1}{2}\left[\frac{\sin\{(m+n+1)/L\}\pi x}{(m+n+1)\pi/L} - \frac{\sin\{(m-n)/L\}\pi x}{(m-n)\pi/L}\right]_0^L$$

$$= -\frac{1}{2}\left[\frac{\sin(m+n+1)\pi}{(m+n+1)\pi/L} - \frac{\sin(m-n)\pi}{(m-n)\pi/L}\right] \qquad \ldots\ldots\ldots\ldots (1)$$

Now, two cases arise.

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Case 1 : If $m \neq n$, then since m, n are integers from (1)

$$\int_0^L f_m(x) \cdot f_n(x)\,dx = 0$$

Case 2 : If $m = n$, then

$$\int_0^L f_n(x) \cdot f_n(x)\,dx = \int_0^L \sin^2 \frac{(2n+1)\pi x}{2L}\,dx$$

$$\therefore \quad \int_0^L [f_n(x)]^2\,dx = \int_0^L \left[\frac{1-\cos 2\{(2n+1)/2L\}\pi x}{2}\right]dx = \frac{1}{2}\left[x - \frac{\sin 2\{(2n+1)/2L\}\pi x}{(2n+1)\pi/L}\right]_0^L$$

$$\therefore \quad \int_0^L [f_n(x)]^2\,dx = \frac{1}{2}\left[x - \frac{\sin\{(2n+1)/L\}\pi x}{(2n+1)\pi/L}\right]_0^L = \frac{L}{2} \neq 0 \qquad \text{..... (2)}$$

Since, $\int_0^L f_m(x) \cdot f_n(x)\,dx \begin{cases} = 0, & \text{if } m \neq n \\ \neq 0, & \text{if } m = n \end{cases}$

the given set of functions is orthogonal on $[0, L]$.

Now, if the set is orthonormal then we should have

$$\int_0^L [f_n(x)]^2 dx = 1$$

For this, we divide (2) by $\frac{2}{L}$ and write it as

$$\int_0^L \frac{2}{L} \cdot [f_n(x)]^2 dx = \frac{2}{L} \cdot \frac{L}{2} = 1$$

i.e., $\int_0^{\pi/2} \sqrt{\frac{2}{L}}\, f_n(x) \cdot \sqrt{\frac{2}{L}}\, f_n(x)\,dx = 1$

This is obviously orthonormal set where

$$\Phi_n(x) = \sqrt{\frac{2}{L}} \cdot f_u(x) = \sqrt{\frac{2}{L}} \cdot \sin\left[\frac{(2n+1)\pi x}{2L}\right]$$

Hence, the required orthonormal set of function is

$$\sqrt{\frac{2}{L}} \sin\left(\frac{\pi x}{2L}\right),\ \sqrt{\frac{2}{L}} \sin\left(\frac{3\pi x}{2L}\right),\ \sqrt{\frac{2}{L}} \sin\left(\frac{5\pi x}{2L}\right), \ldots$$

Example 6 : Show that the set of functions

$$1,\ \sin\frac{\pi x}{L},\ \cos\frac{\pi x}{L},\ \sin\frac{2\pi x}{L},\ \cos\frac{2\pi x}{L},\ \ldots$$

form an orthogonal set in $(-L, L)$ and construct an orthonormal set. **(M.U. 1998, 2002, 16)**

Sol. : Let $f_n(x) = \sin\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots$

$$g_n(x) = \cos\frac{n\pi x}{L}, \quad n = 0, 1, 2, \ldots$$

(a) $$\int_{-L}^L f_m(x) \cdot f_n(x)\,dx = \int_{-L}^L \sin\frac{m\pi x}{L} \cdot \sin\frac{n\pi x}{L}\,dx \qquad \text{..... (1)}$$

$$= -\frac{1}{2}\int_{-L}^L \left[\cos\left(\frac{(m+n)\pi x}{L}\right) - \cos\left(\frac{(m-n)\pi x}{L}\right)\right]dx$$

$$\therefore \quad \int_{-L}^L f_m(x) \cdot f_n(x)\,dx = -\frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\pi x}{(m+n)\pi/L} - \frac{\sin\{(m-n)/L\}\pi x}{(m-n)\pi/L}\right]_{-L}^L$$

Now, two cases arise.

Case 1 : If $m \neq n$, then $\int_{-L}^L f_m(x) \cdot f_n(x)\,dx = 0$.

Case 2 : If $m = n$, then from (1)

$$\int_{-L}^L f_n(x)\,dx \cdot f_n(x)\,dx = \int_{-L}^L \sin^2 \frac{n\pi x}{L}\,dx$$

$$\therefore \quad \int_{-L}^L [f_n(x)]^2\,dx = \int_{-L}^L \left(\frac{1-\cos(2n\pi x/L)}{2}\right)dx$$

$$= \frac{1}{2}\left[x - \frac{\sin(2n\pi x)/L}{2n\pi/L}\right]_{-L}^L = L \neq 0$$

(b) $$\int_{-L}^L g_m(x) \cdot g_n(x)\,dx = \int_{-L}^L \cos\frac{m\pi x}{L} \cdot \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_{-L}^L \left[\cos\left(\frac{(m+n)\pi x}{L}\right) + \cos\left(\frac{(m-n)\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[\frac{\sin\{(m+n)/L\}\pi x}{(m+n)\pi/L} + \frac{\sin\{(m-n)/L\}\pi x)}{(m-n)\pi/L}\right]_{-L}^L$$

Again two cases arise.

Case 1 : When $m \neq n$, then $\int_{-L}^L g_m(x) \cdot g_n(x)\,dx = 0$.

Case 2 : If $m = n$, then $\int_{-L}^L g_n(x) \cdot g_n(x)\,dx = \int_{-L}^L \cos^2 \frac{m\pi x}{L}\,dx$

$$\therefore \quad \int_{-L}^L [g_n(x)]^2\,dx = \int_{-L}^L \left(\frac{1+\cos(2n\pi x/L)}{2}\right)dx$$

$$\therefore \quad \int_{-L}^L [g_n(x)]^2\,dx = \frac{1}{2}\left[x + \frac{\sin(2n\pi x/L)}{2n\pi/L}\right]_{-L}^L = L \neq 0$$

(c) $$\int_{-L}^L f_m(x) \cdot g_n(x)\,dx = \int_{-L}^L \sin\frac{m\pi x}{L} \cdot \cos\frac{n\pi x}{L}\,dx$$

$$= \frac{1}{2}\int_{-L}^L \left[\sin\left(\frac{(m+n)\pi x}{L}\right) + \sin\left(\frac{(m-n)\pi x}{L}\right)\right]dx$$

$$= \frac{1}{2}\left[-\frac{\cos\{(m+n)/L\}\pi x}{(m+n)/L} - \frac{\cos\{(m-n)/L\}\pi x}{(m-n)/L}\right]_{-L}^L$$

Again two cases arise

Case 1 : If $m \neq n$, then $\int_{-L}^L f_m(x) \cdot g_n(x)\,dx = 0$

Case 2 : If $m = n$, then $f_n(x) \cdot g_n(x) = [f_n(x)]^2$ or $[g_n(x)]^2$

and we have already proved above that

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$$\int_{-L}^{L}[f_n(x)]^2\,dx = L \text{ and } \int_{-L}^{L}[g_n(x)]^2\,dx = L$$

and for the first term *i.e.*, for 1, $\int_{-L}^{L} 1dx = 2L$.

Hence, the given sequence is orthogonal.

For orthonormality the value of the integral L must be 1. Hence, if each term is divided by $\sqrt{L}$ then the set will be orthonormal. Hence, the set $\frac{1}{\sqrt{2L}}, \frac{1}{\sqrt{L}}\sin x, \frac{1}{\sqrt{L}}\cos x, \frac{1}{\sqrt{L}}\sin 2x, \ldots$ is an orthonormal set.

Example 7 : Prove that $f_1(x) = 1$, $f_2(x) = x$, $f_3(x) = (3x^2 - 1)/2$ are orthogonal over $(-1, 1)$.

(M.U. 1997, 2002, 03, 04, 08, 14, 15, 19)

Sol. : We have

$$\int_{-1}^{1} f_1(x)\cdot f_2(x)\,dx = \int_{-1}^{1} x\,dx = \left[\frac{x^2}{2}\right]_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_1(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{1}{2}(3x^2-1)\,dx = \frac{1}{2}\left[x^3 - x\right]_{-1}^{1} = 0$$

$$\int_{-1}^{1} f_2(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{x}{2}(3x^2-1)\,dx = \frac{1}{2}\int_{-1}^{1}(3x^3 - x)\,dx = \frac{1}{2}\left[\frac{3x^4}{4} - \frac{x^2}{2}\right]_{-1}^{1} = 0$$

Further $\int_{-1}^{1} f_1(x)\cdot f_1(x)\,dx = \int_{-1}^{1} 1\cdot 1\cdot dx = [x]_{-1}^{1} = 2 \neq 0$

$$\int_{-1}^{1} f_2(x)\cdot f_2(x)\,dx = \int_{-1}^{1} x\cdot x\,dx = \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \neq 0$$

$$\int_{-1}^{1} f_3(x)\cdot f_3(x)\,dx = \int_{-1}^{1}\frac{1}{4}(9x^4 - 6x^2 + 1)\,dx = \frac{1}{4}\left[\frac{9x^5}{5} - \frac{6x^3}{3} + x\right]_{-1}^{1}$$

$$\therefore \int_{-1}^{1} f_3(x)\cdot f_3(x)\,dx = \frac{1}{4}\left[\left(\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} + 2 - 1\right)\right] = \frac{1}{4}\left(\frac{18}{5} - 4 + 2\right)$$

$$= \frac{1}{4}\cdot\frac{8}{5} = \frac{2}{5} \neq 0$$

Hence, the given set is orthogonal on $[-1, 1]$.

Example 8 : Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on $(-1, 1)$. Determine the constants a and b such that the function $f_3(x) = -1 + ax + bx^2$ is orthogonal to both f_1 and f_2 on that interval.

(M.U. 2003, 05, 06, 07, 09, 12, 19)

Sol. : We have already proved the first part above.

Now, if $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ we should have,

(i) $\int_{-1}^{1} f_1(x)\cdot f_3(x)\,dx = 0 \quad \therefore \int_{-1}^{1} 1\cdot(-1 + ax + bx^2)\,dx = 0$

$$\therefore \left[-x + \frac{ax^2}{2} + \frac{bx^3}{3}\right]_{-1}^{1} = 0 \quad \therefore \left(-1 + \frac{a}{2} + \frac{b}{3}\right) - \left(+1 + \frac{a}{2} - \frac{b}{3}\right) = 0$$

$$\therefore -2 + \frac{2b}{3} = 0 \quad \therefore b = 3.$$

And (ii) $\int_{-1}^{1} f_2(x)\cdot f_3(x)\,dx = 0 \quad \therefore \int_{-1}^{1} x\cdot(-1 + ax + bx^2)\,dx = 0$

$$\therefore \left[\frac{-x^2}{2} + \frac{ax^3}{3} + \frac{bx^4}{4}\right]_{-1}^{1} = 0 \quad \therefore \left(-\frac{1}{2} + \frac{a}{3} + \frac{b}{4}\right) - \left(-\frac{1}{2} - \frac{a}{3} + \frac{b}{4}\right) = 0$$

$$\therefore \frac{2a}{3} = 0 \quad \therefore a = 0$$

$$\therefore f_3(x) = 3x^2 - 1$$

Now, $\int_{-1}^{1}[f_3(x)]^2\,dx = \int_{-1}^{1}(3x^2 - 1)^2\,dx = \int_{-1}^{1}(9x^4 - 6x^2 + 1)\,dx$

$$= \left[\frac{9x^5}{5} - \frac{6x^3}{3} + x\right]_{-1}^{1} = \left(\frac{9}{5} - 2 + 1\right) - \left(-\frac{9}{5} - 12 - 1\right)$$

$$= \frac{18}{5} - 4 + 2 = \frac{18}{5} - 2 = \frac{9}{5}$$

The required function $f_3(x) = 3x^2 - 1$.

Example 9 : If $f_i(x)$, $i = 1, 2, 3, \ldots$ is a set of orthogonal functions in $[a, b]$ and $g(x) = \sum_{i=1}^{\infty} a_i f_i(x)$, then find a_i.

(M.U. 2003)

Sol. : We have $g(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) + \ldots\infty$

Multiply both sides by $f_1(x)$,

$$\therefore f_1(x)g(x) = a_1[f_1(x)]^2 + a_2 f_1(x)\cdot f_2(x) + a_3 f_1(x) f_3(x) + \ldots\infty \quad \ldots\ldots (1)$$

Now, integrate both sides w.r.t. x from a to b.

$$\therefore \int_a^b f_1(x)g(x)\,dx = a_1\int_a^b [f_1(x)]^2 dx + a_2\int_a^b f_1(x)\cdot f_2(x)\,dx + a_3\int_a^b f_1(x)\cdot f_3(x)\,dx + \ldots\infty \quad \ldots\ldots (2)$$

But by definition of orthogonal functions

$$\int_a^b f_m(x) f_n(x) = 0 \text{ if } m \neq n \quad \text{and} \quad \int_a^b [f_m(x)]^2 dx \neq 0$$

Hence, on the r.h.s. of (2) all the integrals except the first are zero and the first not zero.

$$\therefore \int_a^b f_1(x)g(x)\,dx = a_1\int_a^b [f_1(x)]^2 dx \quad \therefore a_1 = \frac{\int_a^b f_1(x)\cdot g(x)\,dx}{\int_a^b [f_1(x)]^2 dx}$$

Similarly, by multiplying (1) successivaly by $f_2(x)$, $f_3(x)$ ..... and integrating both sides w.r.t. x from a to b, we can obtain the values of a_2, a_3, ...

Thus, in general, we have $a_i = \dfrac{\int_a^b f_i(x)g(x)\,dx}{\int_a^b [f_i(x)]^2 dx}$.

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Example 10 : If $f(x) = C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)$, where C_1, C_2, C_3 constants and Φ_1, Φ_2, Φ_3 are orthonormal sets on (a, b), show that

$$\int_a^b [f(x)]^2\, dx = C_1^2 + C_2^2 + C_3^2$$ **(M.U. 2002, 07, 08, 10)**

Sol. : Since, Φ_1, Φ_2, Φ_3 are orthonormal

$$\int_a^b [\Phi_1(x)]^2\, dx = \int_a^b [\Phi_2(x)]^2\, dx = \int_a^b [\Phi_3(x)]^2\, dx = 1 \quad \text{..... (1)}$$

and $$\int_a^b \Phi_m(x)\Phi_n(x)\, dx = 0 \text{ when } m \neq n \quad \text{..... (2)}$$

Now $$\int_a^b [f(x)]^2 dx = \int_a^b [C_1\Phi_1(x) + C_2\Phi_2(x) + C_3\Phi_3(x)]^2 dx$$

$$\therefore \int_a^b [f(x)]^2 dx = \int_a^b \Big[C_1^2\{\Phi_1(x)\}^2 + C_2^2\{\Phi_2(x)\}^2 + C_3^2\{\Phi_3(x)\}^2 + 2C_1C_2\,\Phi_1(x)\Phi_2(x) + 2C_1C_3\,\Phi_1(x)\Phi_3(x) + 2C_2C_3\,\Phi_2(x)\Phi_3(x)\Big] dx$$

$$\therefore \int_a^b [f(x)]^2 dx = C_1^2\int_a^b [\Phi_1(x)]^2 dx + C_2^2\int_a^b [\Phi_2(x)]^2 dx + C_3^2\int_a^b [\Phi_3(x)]^2 dx + 2C_1C_2\int_a^b \Phi_1(x)\Phi_2(x)\, dx + 2C_1C_3\int_a^b \Phi_1(x)\Phi_3(x)\, dx + 2C_2C_2\int_a^b \Phi_1(x)\Phi_2(x)\, dx$$

$$= C_1^2 + C_2^2 + C_3^2 \quad \text{by (1) and (2).}$$

EXERCISE - II

1. Show that the set of functions $\frac{\cos x}{\sqrt{\pi}}, \frac{\cos 2x}{\sqrt{\pi}}, \frac{\cos 3x}{\sqrt{\pi}}, \ldots$ form a normal set in the interval $[-\pi, \pi]$. **(M.U. 2003)**
2. Show that the set of functions $\frac{\sin x}{\sqrt{\pi}}, \frac{\sin 2x}{\sqrt{\pi}}, \frac{\sin 3x}{\sqrt{\pi}}, \ldots$ form a normal set in the interval $[-\pi, \pi]$. **(M.U. 2011)**
3. Show that the set of functions $\cos x, \cos 2x, \cos 3x, \ldots$ is a set of orthogonal functions over $[-\pi, \pi]$. Hence, construct a set of orthonormal functions. **(M.U. 1995, 98, 2005, 12)**
4. Prove that $\sin x, \sin 2x, \sin 3x, \ldots$ is orthogonal on $[0, 2\pi]$ and construct orthonormal set of functions. **(M.U. 1994, 97, 99, 2003, 05, 19)**
5. Is the set of functions $\sin\left(\frac{\pi x}{l}\right), \sin\left(\frac{3\pi x}{l}\right), \sin\left(\frac{5\pi x}{l}\right), \ldots$ orthogonal over $(0, l)$. **(M.U. 2003, 11)** **[Ans. : Yes]**
6. Is the set of functions $\cos x, \cos 3x, \cos 5x, \ldots$ orthogonal over $(0, \pi/2)$. **(M.U. 2003)** **[Ans. : No]**
7. Show that the set of functions $\sin x, \sin 2x, \sin 3x, \ldots$ is orthogonal on the interval $[0, \pi]$. **(M.U. 1999, 2003)**
8. Show that the set of functions $\Phi_n(x) = \sin\left(\frac{n\pi x}{l}\right)$, $n = 1, 2, 3, \ldots$ is orthogonal in $(0, l)$. **(M.U. 2010)**

9. Show that the set of functions $\cos x, \cos 2x, \cos 3x, \ldots$ is orthogonal on $[-\pi, \pi]$. **(M.U. 2003, 06)**
10. Show that the set of functions $1, \sin x, \cos x, \sin 2x, \cos 2x, \ldots$ is orthogonal on $(0, 2\pi)$ but not on $(0, \pi)$. How can you convert the set orthonormal on $(0, 2\pi)$? Write down the orthonormal set. **(M.U. 2003)** **[Ans. :** $\frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}}\sin x, \frac{1}{\sqrt{\pi}}\cos x, \frac{1}{\sqrt{\pi}}\sin 2x, \frac{1}{\sqrt{\pi}}\cos 2x, \ldots$ **]**
11. Show that the following set of functions is orthonormal on $(0, \infty)$
$$\left\{e^{-x/2},\ e^{-x/2}(1-x),\ \frac{1}{2}e^{-x/2}(x^2 - 4x + 2)\right\}$$ **(M.U. 2003, 04, 07)**
12. Show that the functions $\{\sin(2n-1)x\}$, $n = 1, 2, 3, \ldots, \infty$ are orthogonal on $[0, \pi/2]$. Hence, construct an orthonormal set of functions from this. **(M.U. 2011)**

EXERCISE - III

Theory

1. State complex form of Fourier Series. **(M.U. 2004)**

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FOURIER TRANSFORMS

1. Introduction

In Fourier Transforms, we shall first state the Fourier Integral Theorem and then consider Fourier Transform. Fourier transforms transform a non-periodic function $f(t)$ in time-domain into a function $F(\lambda)$ in frequency domain. Fourier transform are highly useful in the study of conduction of heat, wave propagation, communication, etc.

2. Fourier Integral Theorem

If $f(x)$ satisfies Dirichlet's conditions (stated under Fourier Series) in each finite interval $-l \le x \le l$ and if $f(x)$ is integrable in $-\infty$ to ∞ then Fourier Integral Theorem states that

$$\boxed{f(x) = \frac{1}{\pi}\int_{\omega=0}^{\infty}\int_{s=-\infty}^{\infty} f(s)\cos\omega(s-x)\,d\omega\,ds} \quad \text{..... (1)}$$

We assume this result without proof.

3. Fourier Sine and Cosine Integrals

The above integral can be written as

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(s)\{\cos\omega s\cos\omega x\}\,d\omega\,ds + \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(s)\{\sin\omega s\sin\omega x\}\,d\omega\,ds$$

i.e. $$f(x) = \frac{1}{\pi}\int_0^{\infty}\cos\omega x\int_{-\infty}^{\infty} f(s)\cos\omega s\,d\omega\,ds + \frac{1}{\pi}\int_0^{\infty}\sin\omega x\int_{-\infty}^{\infty} f(s)\sin\omega s\,d\omega\,ds$$

(a) Fourier Cosine Integral

When $f(x)$ is an even function, $f(s)$ will be even but $f(s)\sin\omega s$ will be **odd** function and $f(s)\cos\omega s$ will be **even** function. Hence, the second integral will be zero and we will get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\cos\omega x\,d\omega\int_0^{\infty} f(s)\cos\omega s\,ds} \quad \text{..... (2)}$$

This is called **Fourier Cosine Integral.**

(b) Fourier Sine Integral

When $f(x)$ is an odd function, $f(s)$ will be odd but $f(s)\sin\omega s$ will be **even** and $f(s)\cos\omega s$ will be **odd**. Hence, the first integral will be zero and we get

$$\boxed{f(x) = \frac{2}{\pi}\int_0^{\infty}\sin\omega x\,d\omega\int_0^{\infty} f(s)\sin\omega s\,ds} \quad \text{..... (3)}$$

This is called **Fourier Sine Integral.**

Note ....

If the given function $f(x)$ is even then we can use (2) and obtain the Fourier Cosine Integral representation of $f(x)$ (See Ex. 6, page S-124). If the given function $f(x)$ is odd then we can use (3) and obtain the Fourier Sine Integral representation of $f(x)$ (See Ex. 1, page S-120).

Type I : Fourier Integral Representation

Example 1 : Express the function $f(x) = \begin{cases} 1 & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$ as Fourier Integral.

Hence, evaluate $\int_0^{\infty}\frac{\sin\omega\cdot\sin\omega x}{\omega}\,d\omega$. (M.U. 1997, 99, 2002, 03)

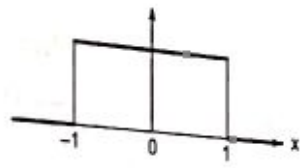
(M.U. 1994, 95, 2003, 11, 12)

Sol. : The Fourier Integral for $f(x)$ is

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} f(s)\cos\omega(s-x)\,d\omega\,ds$$

[By data $f(s) = 0$ from $-\infty$ to -1, $f(s) = 1$ from -1 to 1 and $f(s) = 0$ from 1 to ∞.]

The graph of $f(x)$ is shown in adjoining figure.



Hence, $$f(x) = \frac{1}{\pi}\int_0^{\infty}\int_{-1}^{1} 1\cdot\cos\omega(s-x)\,d\omega\,ds$$

$$f(x) = \frac{1}{\pi}\int_0^{\infty}\left[\frac{\sin\omega(s-x)}{\omega}\right]_{-1}^{1} d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(1-x)-\sin\omega(-1-x)}{\omega}\,d\omega$$

$$= \frac{1}{\pi}\int_0^{\infty}\frac{\sin\omega(1-x)+\sin\omega(1+x)}{\omega}\,d\omega$$

$$= \frac{2}{\pi}\int_0^{\infty}\frac{\sin\omega\cos\omega x}{\omega}\,d\omega$$

This is the Fourier Integral representation of $f(x)$ given in the example.

$$\therefore \int_0^{\infty}\frac{\sin\omega\cdot\cos\omega x}{\omega}\,d\omega = \frac{\pi}{2}\cdot f(x)$$

$$= \begin{cases} \frac{\pi}{2} & \text{for } f(x)=1 \text{ when } |x|<1 \\ 0 & \text{for } f(x)=0 \text{ when } |x|>1 \end{cases} \quad \text{..... (1)}$$

At $|x| = 1$ *i.e.* $x = \pm 1$, $f(x)$ is discontinuous and the integral

$$= \frac{\pi}{2}\cdot\frac{1}{2}\left\{\lim_{x\to 1^-} f(x) + \lim_{x\to 1^+} f(x)\right\} = \frac{\pi}{4}[1+0] = \frac{\pi}{4}. \quad \text{..... (2)}$$

From (1) and (2), we get

$$\int_0^{\infty}\frac{\sin\omega\cdot\cos\omega x}{\omega}\,d\omega = \begin{cases} \pi/2 & \text{when } |x|<1 \\ 0 & \text{when } |x|>1 \\ \pi/4 & \text{when } |x|=1 \end{cases}$$

Cor. : Putting $x = 0$ in the above result, we get

$$\int_0^{\infty}\frac{\sin\omega}{\omega}\,d\omega = \frac{\pi}{2}.$$

Note ....

Unfortunately there is no uniformity in the notation of Fourier Integral and Fourier transforms. Some authors use λ or α in place of ω and t in place of s.

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Example 2 : Find the Fourier Integral representation of

$$f(x)=\begin{cases}0, & x<0\\ 1/2, & x=0\\ e^{-x}, & x>0\end{cases}$$ **(M.U. 2010, 11)**

Sol. : The Fourier Integral of $f(x)$ is

$$f(x)=\frac{1}{\pi}\left[\int_0^\infty\int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds\right]$$

$$=\frac{1}{\pi}\left[\int_0^\infty\int_{-\infty}^0 0\,d\omega\,ds+\int_0^\infty\int_0^\infty e^{-s}\cos\omega(s-x)\,d\omega\,ds\right]$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\infty e^{-s}(\cos\omega s\cos\omega x+\sin\omega s\sin\omega x)\,d\omega\,ds$$

$$=\frac{1}{\pi}\int_0^\infty\left\{\cos\omega x\int_0^\infty e^{-s}\cos\omega s\,ds+\sin\omega x\int_0^\infty e^{-s}\sin\omega s\,ds\right\}d\omega$$

But $\int e^{ax}\cos bx\,dx=\frac{1}{a^2+b^2}\left[e^{ax}(a\cos bx+b\sin bx)\right]$

and $\int e^{ax}\sin bx\,dx=\frac{1}{a^2+b^2}\left[e^{ax}(a\sin bx-b\cos bx)\right]$

$$\therefore\quad f(x)=\frac{1}{\pi}\int_0^\infty\left\{\cos\omega x\left[\frac{1}{1+\omega^2}e^{-s}(-\cos\omega s+\omega\sin\omega s)\right]_0^\infty\right.$$

$$\left.+\sin\omega x\left[\frac{1}{1+\omega^2}e^{-s}(-\sin\omega s-\omega\cos\omega s)\right]_0^\infty\right\}d\omega$$

$$\therefore\quad f(x)=\frac{1}{\pi}\int_0^\infty\left(\frac{\cos\omega x}{1+\omega^2}+\frac{\omega\sin\omega x}{1+\omega^2}\right)d\omega=\frac{1}{\pi}\int_0^\infty\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}\,d\omega$$

And when $x=0$,

$$f(0)=\frac{1}{\pi}\int_0^\infty\frac{1}{1+\omega^2}\,d\omega=\frac{1}{\pi}\left[\tan^{-1}\omega\right]_0^\infty=\frac{1}{\pi}\cdot\frac{\pi}{2}=\frac{1}{2}.$$

Hence, the Fourier Integral representation of $f(x)$ is

$$f(x)=\begin{cases}0, & x<0\\ 1/2, & x=0\\ \frac{1}{\pi}\int_0^\infty\frac{\cos\omega x+\omega\sin\omega x}{1+\omega^2}\,d\omega, & x>0\end{cases}$$

Example 3 : Express the function $f(x)=\begin{cases}-e^{kx} & \text{for } x<0\\ e^{-kx} & \text{for } x>0\end{cases}$

as Fourier Integral and hence, prove that $\int_0^\infty\frac{\omega\sin\omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}e^{-kx}$ if $x>0,\ k>0$. **(M.U. 2010)**

Sol. : The Fourier Integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds$$

But since the given function $f(x)$ is an odd function we use (3) of § 3, page S-116.

$$f(x)=\frac{2}{\pi}\int_0^\infty\sin\omega x\int_0^\infty e^{-ks}\sin\omega s\,d\omega\,ds$$

$$=\frac{2}{\pi}\int_0^\infty\sin\omega x\left[\frac{1}{k^2+\omega^2}e^{-ks}(-k\sin\omega s-\omega\cos\omega s)\right]_0^\infty d\omega$$

$$=\frac{2}{\pi}\int_0^\infty\sin\omega x\cdot\frac{\omega}{k^2+\omega^2}\,d\omega$$

This is the Fourier integral representation of $f(x)$.

$$\therefore\quad\int_0^\infty\frac{\omega\sin\omega x}{\omega^2+k^2}\,d\omega=\frac{\pi}{2}f(x)=\frac{\pi}{2}e^{-kx}\ \text{ if } x>0.$$

Example 4 : Express the function $f(x)=\begin{cases}\sin x, & 0<x\le\pi\\ 0, & x<0,\ x>\pi\end{cases}$

as Fourier Integral and prove that $f(x)=\frac{1}{\pi}\int_0^\infty\frac{\sin\omega x+\cos[\omega(\pi-x)]}{1-\omega^2}\,d\omega$. **(M.U. 2001, 06)**

Hence, deduce that $\int_0^\infty\frac{\cos(\omega\pi/2)}{1-\omega^2}\,d\omega=\frac{\pi}{2}$.

Sol. : The Fourier integral for $f(x)$ is

$$f(x)=\frac{1}{\pi}\int_0^\infty\int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi\sin s\cos\omega(s-x)\,d\omega\,ds$$

$$=\frac{1}{2\pi}\int_0^\infty\int_0^\pi 2\sin s\cos\omega(s-x)\,d\omega\,ds$$

$$=\frac{1}{2\pi}\int_0^\infty\int_0^\pi\left[\sin(s+\omega s-\omega x)+\sin(s-\omega s+\omega x)\right]d\omega\,ds$$

$$=\frac{1}{2\pi}\int_0^\infty\left[-\frac{\cos(s+\omega s-\omega x)}{1+\omega}-\frac{\cos(s-\omega s+\omega x)}{1-\omega}\right]_0^\pi d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\left[-\frac{\cos(\pi+\pi\omega-\omega x)}{1+\omega}-\frac{\cos(\pi-\pi\omega+\omega x)}{1-\omega}+\frac{\cos\omega x}{1+\omega}+\frac{\cos\omega x}{1-\omega}\right]d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\left[\frac{\cos(\pi\omega-\omega x)}{1+\omega}+\frac{\cos(-\pi\omega+\omega x)}{1-\omega}+\frac{\cos\omega x}{1+\omega}+\frac{\cos\omega x}{1-\omega}\right]d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\left[\left(\frac{1}{1+\omega}\right)\{\cos\omega x+\cos\omega(\pi-x)\}+\left(\frac{1}{1-\omega}\right)\{\cos\omega x+\cos\omega(\pi-x)\}\right]d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\left(\frac{1}{1+\omega}+\frac{1}{1-\omega}\right)\{\cos\omega x+\cos\omega(\pi-x)\}\,d\omega$$

$$=\frac{1}{2\pi}\int_0^\infty\frac{2}{(1-\omega^2)}\{\cos\omega x+\cos\omega(\pi-x)\}\,d\omega$$

$$\therefore\quad f(x)=\frac{1}{\pi}\int_0^\infty\left[\frac{\cos\omega x+\cos\omega(\pi-x)}{(1-\omega^2)}\right]d\omega$$

This is the Fourier integral representation of $f(x)$.

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Putting $x = \pi/2$, we get,

$$\sin\frac{\pi}{2} = \frac{1}{\pi}\int_0^\infty \frac{\left[\cos\frac{\pi\omega}{2} + \cos\frac{\pi\omega}{2}\right]}{(1-\omega^2)}d\omega = \frac{2}{\pi}\int_0^\infty \frac{\cos\pi\omega/2}{1-\omega^2}d\omega$$

$$\therefore \quad 1 = \frac{2}{\pi}\int_0^\infty \frac{\cos(\pi\omega/2)}{1-\omega^2}d\omega \qquad \therefore \quad \frac{\pi}{2} = \int_0^\infty \frac{\cos(\pi\omega/2)}{1-\omega^2}d\omega$$

Example 5 : Find the Fourier integral representation of $f(x) = e^{-|x|}$, $-\infty < x < \infty$. **(M.U. 2010, 17, 18)**

Sol. : The Fourier integral of $f(x)$ is

$$f(x) = \frac{1}{\pi}\left[\int_0^\infty \int_{-\infty}^\infty f(s)\cos\omega(s-x)\,d\omega\,ds\right]$$

Since $f(x)$ is an even function, we have by (2) of § 2, page S-116

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s\,d\omega\,ds = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-s}\cos\omega s\,ds\,d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{e^{-s}}{1+\omega^2}(-\cos\omega s + \omega\sin\omega s)\right]_0^\infty d\omega$$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\left[0 + \frac{1}{1+\omega^2}\right]d\omega = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{1+\omega^2}d\omega.$$

Type II : Fourier Sine Integral Representation

Example 1 : Express the function $f(x) = \begin{cases}\sin x, & |x| < \pi \\ 0, & |x| > \pi\end{cases}$

as Fourier sine integral and evaluate $\int_0^\infty \frac{\sin\omega x \cdot \sin\pi\omega}{1-\omega^2}d\omega$.

Sol. : The Fourier Sine Integral of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s)\sin\omega s\,d\omega\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\pi \sin s\sin\omega s\,d\omega\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \left(-\frac{1}{2}\right)\int_0^\pi [\cos s(1+\omega) - \cos s(1-\omega)]\,d\omega\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x\left(-\frac{1}{2}\right)\left[\frac{\sin s(1+\omega)}{1+\omega} - \frac{\sin s(1-\omega)}{1-\omega}\right]_0^\pi d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \left(-\frac{1}{2}\right)\left[-\frac{2\sin\pi\omega}{1-\omega^2}\right]d\omega$$

$[\because \sin(\pi+\theta) = -\sin\theta$ and $\sin(\pi-\theta) = \sin\theta]$

$$\therefore \quad f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin\omega x\sin\pi\omega}{1-\omega^2}dx$$

$$\therefore \quad \int_0^\infty \frac{\sin\omega x \cdot \sin\pi\omega}{1-\omega^2}dx = \frac{\pi}{2}f(x) = \frac{\pi}{2}\begin{cases}\sin x, & |x| < \pi \\ 0, & |x| > \pi\end{cases}$$

Example 2 : Express the function $f(x) = \begin{cases}\pi/2 & \text{for } 0 < x < \pi \\ 0 & \text{for } x > \pi\end{cases}$

as Fourier sine Integral and show that $\int_0^\infty \frac{1-\cos\pi\omega}{\omega}\sin\omega x\,d\omega = \frac{\pi}{2}$ when $0 < x < \pi$. **(M.U. 1998)**

Sol. : Fourier sine integral is,

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\pi \frac{\pi}{2}\cdot\sin\omega s\,d\omega\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \frac{\pi}{2}\left[-\frac{\cos\omega s}{\omega}\right]_0^\pi d\omega$$

$$= \int_0^\infty \sin\omega x\left[\frac{-\cos\pi\omega + 1}{\omega}\right]d\omega$$

$$\therefore \quad f(x) = \int_0^\infty \frac{1-\cos\pi\omega}{\omega}\cdot\sin\omega x\,d\omega$$

$$\therefore \quad \int_0^\infty \frac{1-\cos\pi\omega}{\omega}\cdot\sin\omega x\cdot d\omega = f(x) = \frac{\pi}{2} \text{ when } 0 < x < \pi.$$

Example 3 : Find Fourier Sine integral representation for $f(x) = \frac{e^{-ax}}{x}$. **(M.U. 2004, 09, 16)**

Sol. : By (3) Fourier Sine integral is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s)\sin\omega s\,d\omega\,ds = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty \frac{e^{-as}}{s}\sin\omega s\,ds\,d\omega$$

To evaluate $\int_0^\infty \frac{e^{-as}}{s}\sin\omega s\,ds$ we use the rule of differentiation under the integral sign.

Let $\quad I = \int_0^\infty \frac{e^{-as}}{s}\cdot\sin\omega s\,ds$

$$\therefore \quad \frac{dI}{d\omega} = \int_0^\infty \frac{\partial}{\partial\omega}\left(\frac{e^{-as}}{s}\cdot\sin\omega s\right)ds = \int_0^\infty \frac{e^{-as}}{s}\cdot(\cos\omega s)\cdot s\,ds$$

$$= \int_0^\infty e^{-as}\cos\omega s\,ds = \frac{1}{a^2+\omega^2}\left[e^{-as}(-a\cos\omega s + \omega\sin\omega s)\right]_0^\infty$$

$$\therefore \quad \frac{dI}{d\omega} = \frac{1}{a^2+\omega^2}(a) = \frac{a}{a^2+\omega^2}$$

Integrating w.r.t. ω, $\quad I = a\cdot\frac{1}{a}\tan^{-1}\frac{\omega}{a} + C$

$$\therefore \quad \int_0^\infty \frac{e^{-as}}{s}\cdot\sin\omega s\,ds = \tan^{-1}\frac{\omega}{a} + C$$

To find C, we put $\omega = 0$. $\quad \therefore \quad 0 = 0 + C \quad \therefore \quad C = 0.$

$$\therefore \quad \int_0^\infty \frac{e^{-as}}{s}\cdot\sin\omega s\,ds = \tan^{-1}\frac{\omega}{a}$$

Hence, $f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x\cdot\tan^{-1}\frac{\omega}{a}d\omega$.

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Example 4 : Find Fourier Sine integral of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2, \\ 0, & x > 2 \end{cases}$ **(M.U. 1999)**

Sol. : Fourier Sine integral of $f(x)$ is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \int_0^\infty f(s)\sin\omega s \cdot d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \left[\int_0^1 s\sin\omega s\, ds + \int_1^2 (2-s)\sin\omega s\, ds + \int_2^\infty 0 \cdot \sin\omega s\, ds\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \left\{\left[s\left(-\frac{\cos\omega s}{\omega}\right) - \left(-\frac{\sin\omega s}{\omega^2}\right)(1)\right]_0^1 + \left[(2-s)\left(-\frac{\cos\omega s}{\omega}\right) - \left(-\frac{\sin\omega s}{\omega^2}\right)(-1)\right]_1^2\right\} d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \sin\omega x \left\{\left[-\frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right] + \left[0 - \frac{\sin 2\omega}{\omega^2} + \frac{\cos\omega}{\omega} + \frac{\sin\omega}{\omega^2}\right]\right\} d\omega$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \sin\omega x \cdot \frac{(2\sin\omega - \sin 2\omega)}{\omega^2}\, d\omega$$

Type III : Fourier Cosine Integral Representation

Example 1 : Find the Fourier Cosine integral representation of the function

$$f(x) = \begin{cases} x^2, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. : Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^a s^2 \cos\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left[s^2\left(\frac{\sin\omega s}{\omega}\right) - \left(-\frac{\cos\omega s}{\omega^2}\right)(2s) + \left(-\frac{\sin\omega s}{\omega^n}\right)(2)\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left[\frac{a^2\sin a\omega}{\omega} + \frac{2a\cos a\omega}{\omega^2} - \frac{2\sin a\omega}{\omega^3}\right] d\omega$$

Example 2 : Find the Fourier cosine integral representation of the function $f(x) = e^{-ax}$, $x > 0$ and hence, show that

$$\int_0^\infty \frac{\cos\omega s}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}, \quad x \geq 0.$$

Sol. : Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-as}\cos\omega x\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left[\frac{e^{-ax}}{a^2+\omega^2}(-a\cos\omega x + s\sin\omega s)\right]_0^\infty d\omega$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \left[\frac{a}{a^2+\omega^2}\right] d\omega = \frac{2a}{\pi}\int_0^\infty \frac{\cos\omega x}{a^2+\omega^2}\, d\omega$$

For deduction put $a = 1$,

$$\therefore\ e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega \qquad \therefore\ \int_0^\infty \frac{\cos\omega x}{1+\omega^2}\, d\omega = \frac{\pi}{2}e^{-x}.$$

Example 3 : Find Fourier cosine integral representation of the function

$$f(x) = \begin{cases} \cos x, & |x| < (\pi/2) \\ 0, & |x| > (\pi/2) \end{cases}$$

Sol. : Fourier cosine integral representation of $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^{\pi/2} \cos s\cos\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right)\int_0^{\pi/2} [\cos(1+\omega)s + \cos(1-\omega)s]\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right)\left[\frac{\sin(1+\omega)s}{1+\omega} + \frac{\sin(1-\omega)s}{1-\omega}\right]_0^{\pi/2} ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right)\left[\frac{\sin\pi(1+\omega)/2}{1+\omega} + \frac{\sin\pi(1-\omega)/2}{1-\omega}\right] d\omega$$

But $\sin\left(\frac{\pi}{2} + \frac{\pi\omega}{2}\right) = \cos\frac{\pi\omega}{2}$ and $\sin\left(\frac{\pi}{2} - \frac{\pi\omega}{2}\right) = \cos\frac{\pi\omega}{2}$.

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \cdot \left(\frac{1}{2}\right) \cdot \left[\frac{\cos(\pi\omega/2)}{1+\omega} + \frac{\cos(\pi\omega/2)}{1-\omega}\right] d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \left(\frac{1}{2}\right) \cdot \frac{2\cdot\cos(\pi\omega/2)}{1-\omega^2}\, d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\cos\omega x \cdot \cos(\pi\omega/2)}{1-\omega^2}\, d\omega.$$

Example 4 : Using Fourier Cosine Integral prove that

$$e^{-x}\cos x = \frac{2}{\pi}\int_0^\infty \frac{(\omega^2+2)}{(\omega^4+4)} \cdot \cos\omega x\, d\omega$$ **(M.U. 2002, 05, 07)**

Sol. : By Fourier cosine integral formula

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty f(s)\cos\omega s\, d\omega\, ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-s}\cos s \cdot \cos\omega s\, d\omega\, ds$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega x \int_0^\infty e^{-s}[\cos(\omega+1)s + \cos(\omega-1)s]\, d\omega\, ds$$

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$$\therefore\ f(x) = \frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{1}{1+(\omega+1)^2}\cdot e^{-s}\{-\cos(\omega+1)s+(\omega+1)\sin(\omega+1)s\}_0^\infty\right.$$

$$\left.+\frac{1}{1+(\omega-1)^2}\cdot e^{-s}\{-\cos(\omega-1)s+(\omega-1)\sin(\omega-1)s\}_0^\infty\right]$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{1}{1+(\omega+1)^2}+\frac{1}{1+(\omega-1)^2}\right]d\omega$$

$$= \frac{1}{\pi}\int_0^\infty \cos\omega x\left[\frac{\omega^2-2\omega+2+\omega^2+2\omega+2}{\{(\omega^2+2)+2\omega\}\{(\omega^2+2)+2\omega\}}\right]d\omega$$

$$\therefore\ f(x) = \frac{1}{\pi}\int_0^\infty \cos\omega x\cdot\frac{2(\omega^2+2)}{(\omega^2+2)^2-4\omega^2}\cdot d\omega = \frac{1}{\pi}\int_0^\infty \cos\omega x\cdot\frac{(\omega^2+2)}{(\omega^4+4)}d\omega.$$

Example 5 : Find Fourier cosine integral for $f(x) = \begin{cases}1-x^2, & 0\le x\le 1\\ 0, & x>1\end{cases}$.

Hence, evaluate $\int_0^\infty\left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\cdot dx$. **(M.U. 2003, 19)**

Sol. : Fourier cosine integral for $f(x)$ is

$$f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\int_0^\infty f(t)\cos\omega t\,dt\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\int_0^\infty (1-t^2)\cos\omega t\,dt\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{\left[(1-t^2)\frac{\sin\omega t}{\omega}-(-2t)\left(-\frac{\cos\omega t}{\omega^2}\right)+(-2)\left(-\frac{\sin\omega t}{\omega^3}\right)_0^1\right]\right\}d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left\{-2\cdot\frac{\cos\omega}{\omega^2}+\frac{2\sin\omega}{\omega^3}\right\}d\omega$$

$$1-x^2 = \frac{4}{\pi}\int_0^\infty \cos\omega x\left(\frac{\sin\omega-\omega\cos\omega}{\omega^3}\right)d\omega$$

Now put $x=\frac{1}{2}$. $\therefore\ \frac{3\pi}{16} = \int_0^\infty \frac{\sin\omega-\omega\cos\omega}{\omega^3}\cos\frac{\omega}{2}\cdot d\omega$

Example 6 : Find Fourier integral representation for

$$f(x) = \begin{cases}1-x^2 & \text{for } |x|\le 1\\ 0 & \text{for } |x|>1\end{cases}$$ **(M.U. 1998, 2003, 08, 12)**

Sol. : By data $f(s)=0$ from $-\infty$ to -1, $f(s)=1-s^2$ from -1 to 1 and $f(s)=0$ from 1 to ∞.

Also $f(-s) = 1-(-s)^2 = 1-s^2$

$= f(s)$ from -1 to 1

Hence, $f(s)$ is an even function and we use (2) and obtain Fourier cosine integral representation of $f(x)$.

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\,d\omega\,ds = \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\int_0^1 (1-s^2)\cos\omega s\,ds\right]d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[(1-s^2)\left(\frac{\sin\omega s}{\omega}\right)-\left(-\frac{\cos\omega s}{\omega^2}\right)(-2s)+\left(-\frac{\sin\omega s}{\omega^3}\right)(-2)\right]_0^1 d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[0-\frac{2\cos\omega}{\omega^2}+\frac{2\sin\omega}{\omega^3}\right]d\omega$$

$$\therefore\ f(x) = \frac{4}{\pi}\int_0^\infty \frac{\sin\omega-\omega\cos\omega}{\omega^3}\cdot\cos\omega x\,d\omega.$$

Example 7 : Find Fourier integral representation of

$$f(x) = \begin{cases}e^{ax} & x\le 0,\ a>0\\ e^{-ax} & x\ge 0,\ a>0\end{cases}$$ **(M.U. 1996, 97, 2002, 09, 19)**

Hence, show that $\int_0^\infty \frac{\cos\omega x}{\omega^2+a^2}d\omega = \frac{\pi}{2a}e^{-ax},\ x>0,\ a>0.$

Sol. : Since, $f(x)$ is an even function we use (2).

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\,d\omega\,ds = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty e^{-as}\cos\omega s\,d\omega\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{1}{a^2+\omega^2}\cdot e^{-as}(-a\cos\omega s+\omega\sin\omega s)\right]_0^\infty d\omega$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\cdot\frac{a}{a^2+\omega^2}d\omega$$

$$\therefore\ \int_0^\infty \frac{\cos\omega x}{a^2+\omega^2}d\omega = \frac{\pi}{2a}f(x) = \frac{\pi}{2a}e^{-ax},\ x>0,\ a>0$$

Example 8 : Find Fourier Integral representation of

$$f(x) = x,\quad 0<x<a$$
$$= 0,\quad x>a$$
$$f(-x) = f(x)$$ **(M.U. 1995)**

Sol. : Since, $f(x)$ is even function we use (2).

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^\infty f(s)\cos\omega s\,d\omega\,ds = \frac{2}{\pi}\int_0^\infty \cos\omega x\int_0^a s\cos\omega s\,d\omega\,ds$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{s(\sin\omega s)}{\omega}-\int\frac{\sin\omega s}{\omega}(1)\cdot ds\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{s(\sin\omega s)}{\omega}+\frac{\cos\omega s}{\omega^2}\right]_0^a d\omega$$

$$= \frac{2}{\pi}\int_0^\infty \cos\omega x\left[\frac{a\sin a\omega}{\omega}+\frac{\cos a\omega}{\omega^2}-\frac{1}{\omega^2}\right]d\omega$$

$$\therefore\ f(x) = \frac{2}{\pi}\int_0^\infty \cos\omega x\left(\frac{a\omega\sin a\omega+\cos a\omega-1}{\omega^2}\right)d\omega$$

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$$\therefore \quad F(s) = \frac{1}{\sqrt{2\pi}\, i(\omega + s)}\left(e^{i(\omega+s)b} - e^{i(\omega+s)a}\right).$$

Example 2 : Find the Fourier Transform of $f(x) = e^{-x^2/2}$.

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\cdot e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-1/2(x-is)^2}\cdot e^{-s^2/2}\,dx = \frac{e^{-s^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-1/2(x-is)^2}\,dx$$

Now, put $\frac{1}{\sqrt{2}}(x - is) = y \quad \therefore\ dx = \sqrt{2}\cdot dy$

$$\therefore \quad F(s) = \frac{e^{-s^2/2}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-y^2}\cdot\sqrt{2}\cdot dy$$

$$\therefore \quad F(s) = \frac{e^{-s^2/2}}{\sqrt{\pi}}\cdot\sqrt{\pi} = e^{-s^2/2}. \qquad \left[\because \int_{-\infty}^{\infty} e^{-y^2}dy = \sqrt{\pi}\right]$$

Example 3 : Find the Fourier Transform of $f(x) = \begin{cases} 1/2\epsilon, & |x| \le \epsilon \\ 0, & |x| > \epsilon \end{cases}$

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\epsilon}^{\epsilon}\frac{1}{2\epsilon}\cdot e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}\cdot 2\epsilon}\cdot\left[\frac{e^{isx}}{is}\right]_{-\epsilon}^{\epsilon} = \frac{1}{\sqrt{2\pi}\cdot s\epsilon}\left[\frac{e^{is\epsilon} - e^{-is\epsilon}}{2}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{\sin s\epsilon}{s\epsilon}.$$

Example 4 : Find the Fourier Transform of $f(x) = e^{-|x|}$.

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot(\cos sx + i\sin sx)\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot\cos s\,x\,dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|x|}\cdot\sin sx\,dx$$

Since the first integral is even and the second is odd (and hence zero.)

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}}\int_0^{\infty} e^{-|x|}\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x}\cos sx\,dx \qquad [\because |x| = x \text{ when } x > 0]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+s^2}(-\cos sx + s\sin sx)\right]_0^{\infty}$$

$$\therefore \quad F(s) = \sqrt{\frac{2}{\pi}}\left[0 + \frac{1}{1+s^2}\right] = \sqrt{\frac{2}{\pi}}\cdot\frac{1}{1+s^2}.$$

Example 5 : Find the Fourier Transform of $f(x) = \begin{cases} k, & |x| < a \\ 0, & |x| > a \end{cases}$

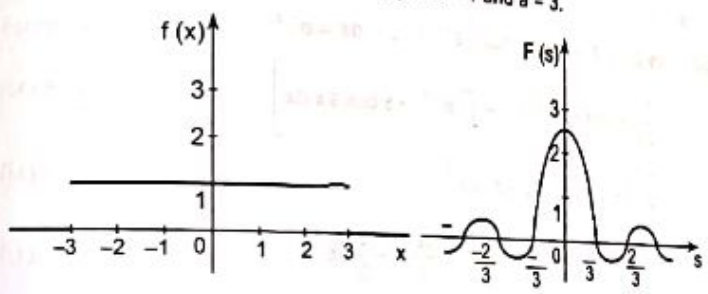
Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-a}^{a} k\cdot e^{isx}\,dx$$

$$= \frac{k}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-a}^{a} = \frac{k}{\sqrt{2\pi}\cdot s}\left[\frac{e^{isa} - e^{-isa}}{i}\right]$$

$$= \frac{k\cdot 2}{\sqrt{2\pi}\cdot s}\left[\frac{e^{isa} - e^{-isa}}{2i}\right] = \frac{k}{s}\cdot\sqrt{\frac{2}{\pi}}\cdot\sin sa.$$

We show below the graph of $f(x)$ and $F(s)$ at $k = 1$ and $a = 3$.



Example 6 : Find the Fourier transform of $f(x) = \begin{cases} 1-|x|, & \text{if } |x| < 1 \\ 0, & \text{if } |x| > 1 \end{cases}$

Sol. : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx = \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)e^{isx}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)[\cos sx + i\sin sx]\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)\cos sx\,dx + \frac{i}{\sqrt{2}}\int_{-1}^{1}(1-|x|)\sin sx\,dx$$

But the first integral is even and the second is odd hence zero.

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}}\int_0^1(1-|x|)\cos sx\,dx = \sqrt{\frac{2}{\pi}}\int_0^1(1-x)\cos sx\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(1-x)\frac{\sin sx}{s} - \left(-\frac{\cos sx}{s^2}\right)(-1)\right]_0^1$$

$$= \sqrt{\frac{2}{\pi}}\left[\left(-\frac{\cos s}{s^2}\right) + \frac{1}{s^2}\right] = \sqrt{\frac{2}{\pi}}\cdot\left(\frac{1-\cos s}{s^2}\right)$$

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Example 7 : Find the Fourier transform of $f(x) = e^{-x^2}$.

Sol. : By definition

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} \cdot e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-x^2} (\cos sx + i \sin sx)\, dx$$

Now, the first integral is even and the second is odd and hence zero.

$$\therefore \quad F(s) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2} \cos sx\, ds \qquad \text{..... (1)}$$

Let $\quad I = \int_0^{\infty} e^{-x^2} \cos sx\, dx \qquad$ ..... (2)

Differentiating under the integral sign with respect to s,

$$\frac{dI}{ds} = \int_0^{\infty} e^{-x^2}(-x) \sin sx\, dx = \frac{1}{2} \int_0^{\infty} \{e^{-x^2}(-2x)\} \cdot \sin sx\, dx$$

Integrating by parts and noting that $\int e^{-x^2}(-2x)\, dx = e^{-x^2}$

$$= \frac{1}{2}\left[\left\{\sin sx \cdot (e^{-x^2})\right\}_0^{\infty} - \int_0^{\infty} e^{-x^2} \cdot s \cos sx\, dx\right]$$

$$= \frac{1}{2}\left[0 - s\int_0^{\infty} e^{-x^2} \cos sx\, dx\right]$$

$$\therefore \quad \frac{dI}{ds} = -s\frac{I}{2} \quad [\text{By (2)}] \qquad \therefore \quad \frac{dI}{I} = -\frac{s}{2} ds$$

Integrating, we get

$$\log I = -\frac{s^2}{4} + \log c \qquad \therefore \quad \log \frac{I}{c} = -\frac{s^2}{4} \qquad \therefore \quad \frac{I}{c} = e^{-s^2/4}$$

Putting $s = 0$, $I(0) = c$.

But from (2), $\quad I(0) = \int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2} \qquad$ [By Gamma function]

$$\therefore \quad c = \frac{\sqrt{\pi}}{2} \qquad \therefore \quad I = \frac{\sqrt{\pi}}{2} \cdot e^{-s^2/4}$$

Hence, from (1),

$$F(s) = \sqrt{\frac{2}{\pi}} \cdot \frac{\sqrt{\pi}}{2} \cdot e^{-s^2/4} = \frac{1}{\sqrt{2}} \cdot e^{-s^2/4} = \frac{e^{-s^2/4}}{\sqrt{2}}.$$

Example 8 : Find the Fourier transform of

$$f(x) = \begin{cases} 1 + \frac{x}{a}, & -a < x < 0 \\ 1 - \frac{x}{a}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

Sol. : By definition,

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \cdot e^{isx}\, dx$$

$$\therefore \quad F(s) = \frac{1}{\sqrt{2\pi}}\left[\int_{-a}^{0}\left(1 + \frac{x}{a}\right)e^{isx} dx + \int_0^a \left(1 - \frac{x}{a}\right)e^{isx}\, dx\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left\{\left(1 + \frac{x}{a}\right)\frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2} \cdot \left(\frac{1}{a}\right)\right\}_{-a}^{0} + \left\{\left(1 - \frac{x}{a}\right) \cdot \frac{e^{isx}}{is} - \frac{e^{isx}}{i^2 s^2}\left(-\frac{1}{a}\right)\right\}_0^a\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\left\{\frac{1}{is} + \frac{1}{as^2} - 0 - \frac{e^{-isa}}{as^2}\right\} + \left\{0 - \frac{e^{isa}}{as^2} - \frac{1}{is} + \frac{1}{as^2}\right\}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} - \frac{1}{as^2}(e^{isa} + e^{-isa})\right] = \frac{1}{\sqrt{2\pi}}\left[\frac{2}{as^2} - \frac{2\cos as}{as^2}\right]$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{as^2}(1 - \cos as)$$

EXERCISE - II

Find the Fourier Transform of the following :

1. $f(x) = \begin{cases} 1, & a < x < b \\ 0, & x < a,\ x > b \end{cases}$ $\quad \left[\text{Ans. : } \frac{1}{\sqrt{2\pi} \cdot is}(e^{isb} - e^{isa})\right]$

2. $f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$ $\quad \left[\text{Ans. : } \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s} \cdot \sin sa\right]$

3. $f(x) = e^{-a|x|}$ $\quad \left[\text{Ans. : } \sqrt{\frac{2}{\pi}} \cdot \frac{a}{s^2 + a^2}\right]$

   (Hint : $f(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{(a+is)x} dx + \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-(a-is)x}$

   $= \frac{1}{\sqrt{2\pi}} \cdot \left[\frac{1}{a + is} + \frac{1}{a - is}\right] = \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2 + s^2}$)

4. $f(x) = \begin{cases} x^2, & |x| < a \\ 0, & |x| > a \end{cases}$ $\quad \left[\text{Ans. : } \sqrt{\frac{2}{\pi}} \cdot \frac{1}{s^3}\left[(a^2 s^2 - 2)\sin as + 2as \cos as\right]\right]$

6. $f(x) = \begin{cases} x, & |x| \le a \\ 0, & |x| > a \end{cases}$ $\quad \left[\text{Ans. : } \frac{2i}{s^2} \cdot \frac{1}{\sqrt{2\pi}}\left[\sin sa - as \cos sa\right]\right]$

7. $f(x) = \begin{cases} \frac{\pi}{2} \cos x, & |x| \le \pi \\ 0, & |x| > \pi \end{cases}$ $\quad \left[\text{Ans. : } x \cdot \frac{s \sin s\pi}{1 - s^2}\right]$

5. Inverse Fourier Transform

If $F(s)$ is the Fourier transform of $f(x)$ and if $f(x)$ satisfies certain conditions (i.e. Dirichlets conditions) in every finite interval $(-l, l)$ and if $\int_{-\infty}^{\infty} |f(x)| dx$ is convergent then at every point of continuity of $f(x)$

$$\boxed{f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s) \cdot e^{-isx}\, ds}$$

$f(x)$ is called the **Inverse Fourier Transform** of $F(s)$.

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Note ....

Some authors define Fourier Transforms in different ways

1. $F(s)=\int_{-\infty}^{\infty} f(x)e^{-isx}dx$ and $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{isx}\,ds$
2. $F(s)=\int_{-\infty}^{\infty} f(x)e^{isx}dx$ and $f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}\,ds$
3. $F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-isx}dx$ and $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)e^{isx}\,ds$

Even some authors use p in place of s, and $\tilde{f}(p)$ in place of $F(s)$, some use λ in place of s. We advise students to use the notation used in this book as it is more common and more convenient.

Example 1 : Find the Fourier transform of $f(x)=\begin{cases}1, & |x|<k\\ 0, & |x|>k\end{cases}$ and hence, evaluate

(i) $\int_{-\infty}^{\infty}\frac{\sin sk\cos sx}{s}ds$ (ii) $\int_{-\infty}^{\infty}\frac{\sin ks}{s}ds$ **(M.U. 2009)** (iii) $\int_{-\infty}^{\infty}\frac{\sin s}{s}ds$

Sol. : By definition

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}dx=\frac{1}{\sqrt{2\pi}}\int_{-k}^{k}1\cdot e^{isx}\,dx$$

$$=\frac{1}{\sqrt{2\pi}}\left[\frac{e^{isx}}{is}\right]_{-k}^{k}=\frac{1}{\sqrt{2\pi}}\cdot\frac{2}{s}\left[\frac{e^{isk}-e^{-isk}}{2i}\right]$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{1}{s}\cdot\sin sk \text{ for } s\neq 0$$

For $s=0$, $F(s)=\frac{1}{\sqrt{2\pi}}\int_{-k}^{k}dx=\frac{1}{\sqrt{2\pi}}[k+k]=\frac{2k}{\sqrt{2\pi}}$

Now, we use inverse Fourier Transform. We know that if

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx \quad \text{then,} \quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\,ds$$

(i) $$\therefore\ f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\cdot\frac{1}{s}\cdot\sin sk\cdot e^{-isx}\cdot ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{(\cos sx-i\sin sx)}{s}\sin sk\cdot ds$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx\cdot\sin sk}{s}ds-\frac{i}{\pi}\int_{-\infty}^{\infty}\frac{\sin sx\sin sk}{s}ds$$

The second integral being odd is zero.

$$\therefore\ f(x)=\begin{cases}1, & |x|<k\\ 0, & |x|>k\end{cases}=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx\cdot\sin sk}{s}\cdot ds$$

$$\therefore\ \int_{-\infty}^{\infty}\frac{\cos sx\cdot\sin sk}{s}ds=\begin{cases}\pi, & |x|<k\\ 0, & |x|>k\end{cases}$$

(ii) In the above result, if we put $x=0$, we put

$$\int_{-\infty}^{\infty}\frac{\sin ks}{s}ds=\pi \quad \therefore\ 2\int_{0}^{\infty}\frac{\sin ks}{s}ds=\pi \quad \therefore\ \int_{0}^{\infty}\frac{\sin ks}{s}ds=\frac{\pi}{2}.$$

Note ....

From the result (ii) above, we get

$$\boxed{\int_{0}^{\infty}\frac{\sin kx}{x}dx=\frac{\pi}{2}}$$

This is an important integral and can be used as a standard result when required. You are advised to memories it and also the following result.

(iii) In the above result put $k=1$,

$$\therefore\ \boxed{\int_{0}^{\infty}\frac{\sin s}{s}ds=\frac{\pi}{2}}$$

Example 2 : Find the Fourier transform of $f(x)=\begin{cases}(1-x^2), & |x|\le 1\\ 0, & |x|>1\end{cases}$

and hence evaluate $\int_{0}^{\infty}\left(\frac{x\cos x-\sin x}{x^3}\right)\cdot\cos\frac{x}{2}dx$. **(M.U. 2008, 09)**

Sol. : By definition,

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{isx}\,dx=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{isx}\,dx$$

Integrating by parts, the integral I is given by

$$I=(1-x^2)\cdot\frac{e^{isx}}{is}-\int\frac{e^{isx}}{is}(-2x)dx$$

$$=(1-x^2)\cdot\frac{e^{isx}}{is}+\frac{2}{is}\left[x\cdot\frac{e^{isx}}{is}-\int\frac{e^{isx}}{is}\cdot 1\cdot dx\right]$$

$$\therefore\ I=(1-x^2)\cdot\frac{e^{isx}}{is}+\frac{2}{is}\cdot\left[x\cdot\frac{e^{isx}}{is}-\frac{e^{isx}}{i^2s^2}\right]$$

$$\therefore\ F(s)=\frac{1}{\sqrt{2\pi}}\left[\left(\frac{(1-x^2)e^{isx}}{is}\right)_{-1}^{+1}+\frac{2}{is}\left(\frac{xe^{isx}}{is}\right)_{-1}^{+1}+\frac{2}{is}\left(\frac{e^{isx}}{s^2}\right)_{-1}^{+1}\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[0-\frac{4}{s^2}\left(\frac{e^{is}+e^{-is}}{2}\right)+\frac{4}{s^3}\left(\frac{e^{is}-e^{-is}}{2i}\right)\right]$$

$$=\frac{1}{\sqrt{2\pi}}\left[-\frac{4}{s^2}\cos s+\frac{4}{s^3}\sin s\right]=-2\cdot\sqrt{\frac{2}{\pi}}\cdot\left(\frac{s\cos s-\sin s}{s^3}\right)$$

Now, we use inverse Fourier Transform. We know that if

$$F(s)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{isx}\,dx \quad \text{then,} \quad f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(s)\cdot e^{-isx}\,ds$$

$$\therefore\ f(x)=\frac{1}{\sqrt{2\pi}}\left(-2\cdot\sqrt{\frac{2}{\pi}}\right)\int_{-\infty}^{\infty}\left(\frac{s\cos s-\sin s}{s^3}\right)e^{-isx}\,ds$$

$$=-\frac{2}{\pi}\int_{-\infty}^{\infty}\cos sx\left(\frac{s\cos s-\sin s}{s^3}\right)dx+i\frac{2}{\pi}\int_{-\infty}^{\infty}\sin sx\left(\frac{s\cos-\sin s}{s^3}\right)ds$$

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Now, the second integral being odd is zero.

$$\therefore \quad f(x) = \begin{cases} (1-x^2), & |x| \le 1 \\ 0, & |x| > 1 \end{cases} = -\frac{2}{\pi}\int_{-\infty}^{\infty} \cos sx \left(\frac{s\cos s - \sin s}{s^3}\right) ds$$

Now, we put $x = \frac{1}{2}$,

$$\therefore \quad \frac{3}{4} = -\frac{2}{\pi}\int_{-\infty}^{\infty} \cos\left(\frac{s}{2}\right)\left(\frac{s\cos s - \sin s}{s^3}\right) ds$$

$$\therefore \quad \int_{-\infty}^{\infty} \cos\left(\frac{s}{2}\right)\cdot\left(\frac{s\cos s - \sin s}{s^3}\right) ds = -\frac{3\pi}{8}$$

$$\therefore \quad \int_{0}^{\infty} \cos\left(\frac{x}{2}\right)\cdot\left(\frac{x\cos x - \sin x}{x^3}\right) dx = -\frac{3\pi}{16}.$$

EXERCISE - III

Find the inverse Fourier Transform of $F(s) = e^{-|s|a}$. $\left[\text{Ans. : } \sqrt{\frac{2}{\pi}}\cdot\frac{a}{a^2+x^2}\right]$

(**Hint :** $f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(a-ix)s}\, ds + \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-(a+ix)s}\, ds$

For $|s| = -s$ if $s \le 0$ and $|s| = s$ if $s \ge 0$.)

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Module 4

CONFORMAL MAPPING, LINEAR, BILINEAR MAPPING, CROSS RATIO, FIXED POINTS AND STANDARD TRANSFORMATIONS

1. Mapping of a Complex Function

We know that a real continuous function $y = f(x)$ can be represented by a smooth curve on xy plane. We also know that a real continuous function $z = f(x, y)$ can be represented by a surface in space. But, how can we represent the complex function $w = f(z)$ i.e. $u + iv = f(x + iy)$ graphically ? Here, we have two dependent variables u, v and two independent variable x, y i.e. in all four variables and to represent such a function graphically we will need four diamensional space. Since this is impractical we take two complex planes - one z-plane and another w-plane. In the z-plane we plot the point $z = x + iy$ and in w-plane we plot the corresponding point $w = u + iv$. When the point z describes a curve C in z-plane the corresponding point w will describe a curve C' in w-plane, if the correspondence is one-one. Thus, **the complex function $w = f(z)$ defines a mapping or transformation of the z-plane into the w-plane.** The corresponding points, curves or regions of the two planes are called the **images** of each other.

Example 1 : Determine the region D' in w-plane corresponding to the region D in the z-plane given by $x = 0$, $y = 0$, $x = 1$, $y = 1$ under the transformation $w = z + (2 - i)$.

Sol. : Since $w = z + (2 - i)$, we have

$$u + iv = x + iy + 2 - i = (x + 2) + i(y - 1)$$

$$\therefore \quad u = x + 2,\ v = y - 1.$$

When $x = 0$, $u = 2$, when $x = 1$, $u = 3$ and when $y = 0$, $v = -1$, when $y = 1$, $v = 0$.

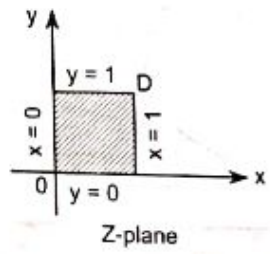

Fig. 4.1 (a)

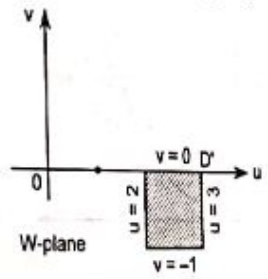

Fig. 4.1 (b)

The region D in the z-plane, is transformed into the region D' in the w-plane.

Example 2 : Determine the region in the w-plane into which the region bounded by the lines $x = 0$, $y = 0$, $x = 1$, $y = 1$ in the z-plane is transformed under the transformation

$$w = (1 + i)\, z - (1 - i).$$

Sol. : Since $w = (1 + i)\, z - (1 - i)$

We have $u + iv = (1 + i)(x + iy) - (1 - i)$

$\therefore \quad u + iv = (x - y - 1) + i(x + y + 1) \qquad \therefore \quad u = x - y - 1,\ v = x + y + 1$

When $x = 0$, $u = -y - 1$ and $v = y + 1$. Adding $u + v = 0$ which is a straight line.

When $y = 0$, $u = x - 1$, $v = x + 1$. Subtracting $u - v = -2$, which is a straight line.

When $x = 1$, $u = 1 - y - 1$, $v = 1 + y + 1$. Adding $u + v = 2$ which is a straight line

When $y = 1$, $u = x - 1 - 1$, $v = x + 1 + 1$. Subtracting $u - v = -4$ which is a straight line.

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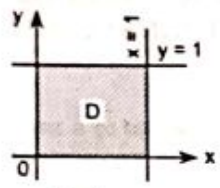
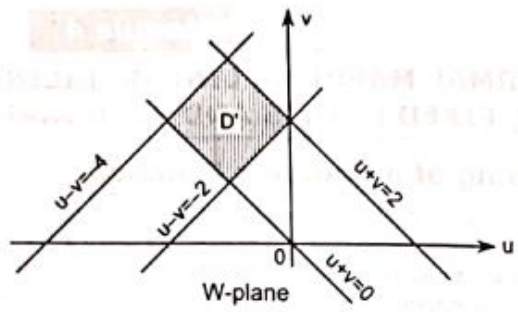


Z-plane

Fig. 4.2 (a)



W-plane

Fig. 4.2 (b)

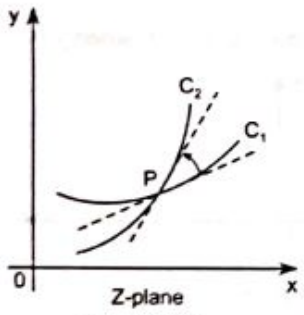
The region D in the z-plane is transformed into the region D' in the w-plane.

2. Conformal Mapping

Let two curves C_1 and C_2 in the z-plane intersect at the point P and let the corresponding curves C'_1 and C'_2 in the w-plane intersect at the point P' under the mapping $w = f(z)$.

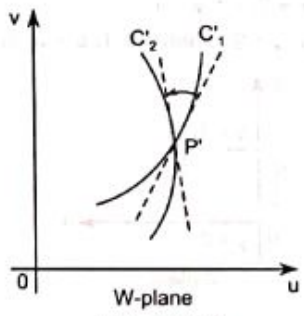
If the angle of intersection of the curves C_1 and C_2 at P is equal to the angle of intersection of the curves C'_1 and C'_2 at P' both in magnitude and sense then the mapping is said to be CONFORMAL at P.

If the angle between C_1, C_2 at P is equal to the angle between C'_1, C'_2 at P' in magnitude only then the mapping is said to be **isogonal.**



Z-plane

Fig. 4.3 (a)



W-plane

Fig. 4.3 (b)

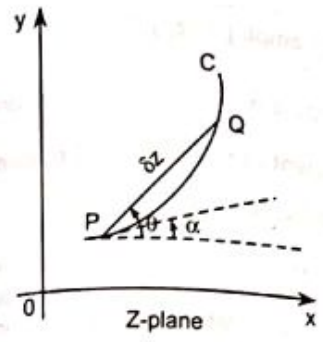
3. Conformal Property of Analytic Functions

Theorem : If $w = f(z)$ is an analytic function and $f'(z) \neq 0$ in a region R of z-plane then the mapping $w = f(z)$ is conformal at all points of R.

Proof : Let $P(z)$ be a point in the region R of z-plane and $P'(w)$ be the corresponding point in the region R' of w-plane. Let $Q(z + \delta z)$ be a neighbouring point on a curve C in R through P and $Q'(w + \delta w)$ be the corresponding neighbouring point on the curve C' in R.

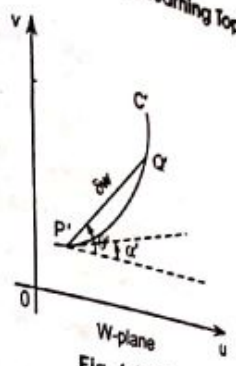
Let the complex numbers δz and δw be expressed in exponential form as

$$\delta z = re^{i\theta} \text{ and } \delta w = r'e^{i\theta'}. \quad \therefore \quad \frac{\delta w}{\delta z} = \frac{r'}{r} e^{i(\theta' - \theta)}$$



Z-plane

Fig. 4.4 (a)



W-plane

Fig. 4.4 (b)

Let the tangents at P and P' make angles α and α' with the x and u axis respectively. Hence, as $\delta z \to 0$, $\theta \to \alpha$, $\delta w \to 0$ and $\theta' \to \alpha'$.

$$\therefore \quad f'(z) = \frac{dw}{dz} = \lim_{\delta z \to 0} \frac{\delta w}{\delta z} = \lim_{\delta z \to 0} \frac{r'}{r} e^{i(\theta' - \theta)}$$

Since $f(z)$ is analytic $f'(z) \neq 0$. Let $f'(z) = Re^{i\Phi}$. Then $R = |f'(z)|$ and Φ = amplitude of $f'(z)$.

$$\therefore \quad Re^{i\Phi} = \lim_{\delta z \to 0} \frac{r'}{r} e^{i(\theta' - \theta)} \qquad \therefore \quad R = \lim_{\delta z \to 0} \frac{r'}{r} \text{ and } \Phi = \lim_{\delta z \to 0} (\theta' - \theta) = \alpha' - \alpha$$

Let us consider another curve C_1 through P and let C'_1 be the corresponding curve in the w plane through P'. If the tangent to the curve C_1 makes an angle β with the x-axis and the tangent to the curve C'_1 makes an angle β' with the u-axis then as above.

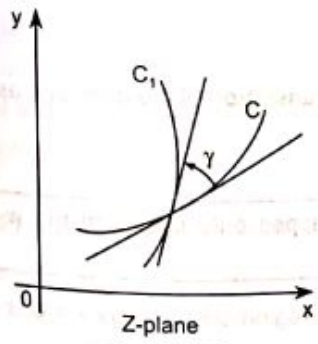
$$\Phi = \beta' - \beta \qquad \therefore \quad \alpha' - \alpha = \beta' - \beta \quad \therefore \beta - \alpha = \beta' - \alpha' = \gamma \text{ say.}$$

Thus, the angle between the curves C, C_1, in z-plane is equal to the angle between the curves C', C'_1 in the w-plane in magnitude and sense.

Hence, the mapping by an analytic function $w = f(z)$ is conformal at each points where $f'(z) \neq 0$.

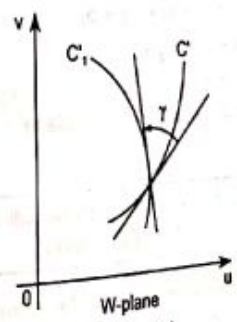
Notes ....

1. A point at which $f'(z) = 0$ is called a **critical point** of the transformation $w = f(z)$.
2. Since $\alpha' = \alpha + \Phi$ the curve C' at P' is rotated through an angle Φ = amp. $[f'(z)]$ under the transformation. This angle is called **the angle of rotation.**



Z-plane

Fig. 4.5 (a)



W-plane

Fig. 4.5 (b)

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$\therefore$ **angle of rotation = ampl. [$f'(z)$]**

3. Since $R = \lim_{\delta z \to 0} \frac{r'}{r}$ a small segment in w-plane is $R = |f'(z)|$ times the corresponding small segment in z-plane. The ratio R is called the **coefficient of linear magnification.**

$\therefore$ **Coeff. of Linear Magnification = $|f'(z)|$**

4. Since in conformal mapping angle is preserved, if two curves in z-plane are orthogonal then their images in w-plane will be also orthogonal and vice versa. In particular since the lines $x = \lambda$, $y = \mu$ are orthogonal their images under conformal transformation will be orthogonal.

Example 1 : Find the angle of rotation and the coefficient of magnification at $z = 2 + i$ under the transformation $w = z^2$.

Sol. : $\because$ $w = z^2$, $\frac{dw}{dz} = 2z$

The transformation is conformal except at $z = 0$.

Hence, the transformation is conformed at $z = 2 + i$

$\therefore$ $f'(z_0) = 2z_0 = 2(2 + i) = 4 + 2i$

Angle of rotation = ampl. [$f'(z_0)$]

$$= \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{2}{4} = \tan^{-1} 0.5.$$

Coefficient of magnification = $|f'(z_0)| = \sqrt{16 + 4} = 2\sqrt{5}$.

Example 2 : Find the angle of rotation and the coefficient of magnification at $z = 1 + i$ under the transformation $w = z^2$. **[Ans. :** $\pi / 4$; $2\sqrt{2}$]

4. Some Standard Transformations

(i) Translation $\boxed{w = z + c}$ where c is a complex constant.

Let $z = x + iy$, $c = a + ib$ and $w = u + iv$

Now, $u + iv = (x + iy) + (a + ib)$

$= (x + a) + i(y + b)$

$\therefore$ $u = x + a$, $v = y + b$.

Thus, the transformation $w = z + c$ is simply a **translation** of the axes and as such **preserves the shape and size** of the region in z-plane.

Remark ....

In particular, circles in the z-plane will be mapped onto circles in the w-plane under the transformation $w = z + c$ (See Ex. 2 below).

Example 1 : Find the image of the rectangular region bounded by $x = 0$, $x = 3$, $y = 0$, $y = 2$ under the transformation $w = z + (1 + i)$. **(M.U. 2015)**

Sol. : Since $u + iv = (x + iy) + (1 + i) = (x + 1) + i(y + 1)$

$\therefore$ $u = x + 1$, $v = y + 1$

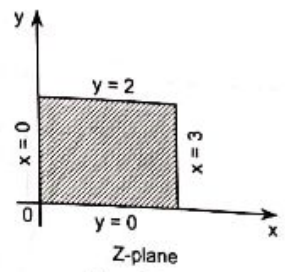

Fig. 4.6 (a)

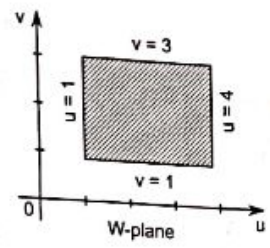

Fig. 4.6 (b)

When $x = 0$, $u = 1$ and when $x = 3$, $u = 4$.

When $y = 0$, $v = 1$ and when $y = 2$, $v = 3$.

Thus, the rectangular region of sides 3 and 2 in the z-plane is transformed into a rectangular region of sides 3 and 2 in the w-plane. Only the rectangle is shifted in the position in the w-plane preserving its size and shape.

Example 2 : Find the image of the circle $|z| = 2$ under the transformation $w = z + 3 + 2i$. **(M.U. 2002, 19)**

Sol. : We have $w = z + 3 + 2i$.

$\therefore$ $u + iv = (x + iy) + (3 + 2i)$

$= (x + 3) + i(y + 2)$

$\therefore$ $u = x + 3$, $v = y + 2$.

But $|z| = 2$ means $\sqrt{x^2 + y^2} = 2$ $\therefore$ $x^2 + y^2 = 4$.

Hence, $(u - 3)^2 + (v - 2)^2 = 4$.

Thus, the circle $x^2 + y^2 = 4$ will be transformed to the circle $(u - 3)^2 + (v - 2)^2 = 4$.

Note ....

Note that the circle is transformed into a circle with the same radius but with translation of the centre from (0, 0) to (3, 2).

Example 3 : Find the image of the triangular region whose vertices are i, $1 + i$, $1 - i$ under the transformation $w = z + 4 - 2i$. Draw the sketch. **(M.U. 2015)**

Sol. : We have $w = z + - 2i$

$\therefore$ $u + iv = x + iy + 4 - 2i = (x + 4) + i(y - 2)$

Equating real and imaginary parts $u = x + 4$, $v = y - 2$.

Now, the point i is $x = 0$, $y = 1$.

$\therefore$ $u = 4$, $v = -1$

The point $1 + i$ is $x = 1$, $y = 1$.

$\therefore$ $u = 5$, $v = -1$

The point $1 - i$ is $x = 1$, $y = -1$.

$\therefore$ $u = 5$, $v = -3$

Thus, the triangular region whose vertices are i, $1 + i$, $1 - i$ in the x-y plane is transformed into a triangular region whose vertices are (4, – 1), (5, – 1) and (5, – 3) in the u-v plane.

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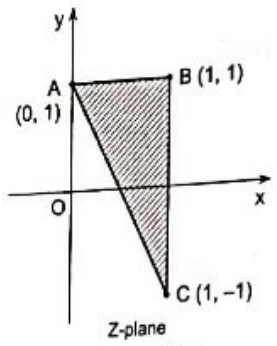

Fig. 4.7 (a)

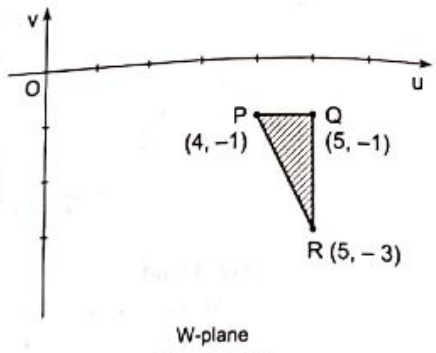

Fig. 4.7 (b)

EXERCISE - I

1. Find the image of the rectangular region bounded by $x = 0$, $y = 0$, $x = 1$, $y = 2$ under the transformation $w = z + (2 - i)$. Draw the sketch.

[**Ans. :** A rectangular region bounded by $u = 2$, $u = 3$, $v = -1$, $v = 1$]

2. Find the image of the triangular region whose vertices are i, $1 + i$, $1 - i$ under the transformation $w = z + 4 - 2i$. **(M.U. 2015)**

[**Ans. :** The triangular region whose vertices are $4 - i$, $5 - i$, $5 - 3i$]

3. Find the map of the circle $|z| = k$ under the transformation $w = z + 4 + 3i$.

[**Ans. :** $u + iv = (x + 4) + i(y + 3)$ $\therefore$ $x = u - 4$, $y = v - 3$

$\therefore$ $x^2 + y^2 = k^2$ becomes $(u - 4)^2 + (v - 3)^2 = k^2$]

(ii) Rotation And Magnification $\boxed{w = cz}$ where c is a complex number.

Let $w = Re^{i\Phi}$, $z = re^{i\theta}$, $c = \rho e^{i\alpha}$

Now $Re^{i\Phi} = re^{i\theta} \cdot \rho e^{i\alpha} = r\rho e^{i(\theta+\alpha)}$ $\therefore$ $R = r\rho$, $\Phi = \theta + \alpha$

Thus, the transformation maps a point $P(r, \theta)$ in the z-plane into a point $P'(r\rho, \theta + \alpha)$ in the w-plane. The radius vector r of the point P is **magnified by $\rho = |c|$ and is rotated through an angle α = amp. (c).** Hence, it maps figures in the z-plane into **geometrically similar figures** in the w-plane. But the figures are magnified and rotated.

Remark ....

In particular circles in the z-plane will be mapped onto circles in the w-plane under the transformation $w = cz$.

Example 1 : Find the image of the region bounded by $x = 0$, $x = 2$, $y = 0$, $y = 2$ in the z-plane under transformation $w = (1 + i)z$. **(M.U. 2005, 13, 14)**

Sol. : Since $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2} \cdot e^{i\pi/4}$

$\therefore$ $w = cz$ becomes $Re^{i\Phi} = \sqrt{2} \cdot e^{i\pi/4} \cdot re^{i\theta} = \sqrt{2} \cdot re^{i(\theta+\pi/4)}$

The square in z-plane is transformed into a square in the w-plane but its sides are magnified by $\sqrt{2}$ and it is rotated through an angle of $\pi / 4$.

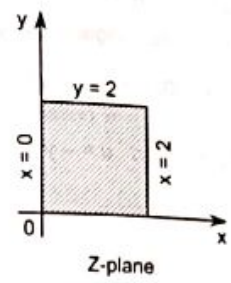

Fig. 4.8 (a)

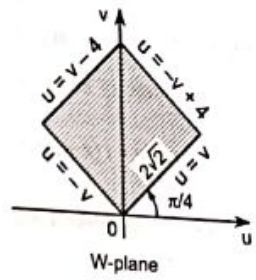

Fig. 4.8 (b)

Alternatively We may write $z = x + iy$.

Then $u + iv = (1 + i)(x + iy) = (x - y) + i(x + y)$ $\therefore$ $u = x - y$, $v = x + y$

When $x = 0$, $u = -y$, $v = y$ $\therefore$ $u = -v$

When $x = 2$, $u = 2 - y$, $v = 2 + y$ $\therefore$ $u + v = 4$

When $y = 0$, $u = x$, $v = x$ $\therefore$ $u = v$

When $y = 2$, $u = x - 2$, $v = x + 2$ $\therefore$ $u - v = -4$

Thus, we get the same square as above.

Example 2 : Consider the transformation $w = (1 + i)z + (2 - i)$ and determine the region in the w-plane into which the rectangular region bounded by $x = 0$, $y = 0$, $x = 1$, $y = 2$ in the z-plane is mapped under this transformation. Sketch the regions. **(M.U. 2000, 17)**

Sol. : We have $u + iv = (1 + i)(x + iy) + 2 - i$

$= x + ix + iy - y + 2 - i$

$= (x - y + 2) + i(x + y - 1)$

$\therefore$ $u = x - y + 2$, $v = x + y - 1$

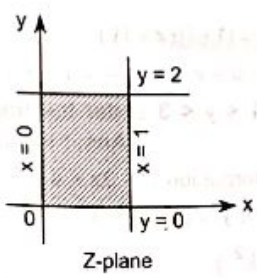

Fig. 4.9 (a)

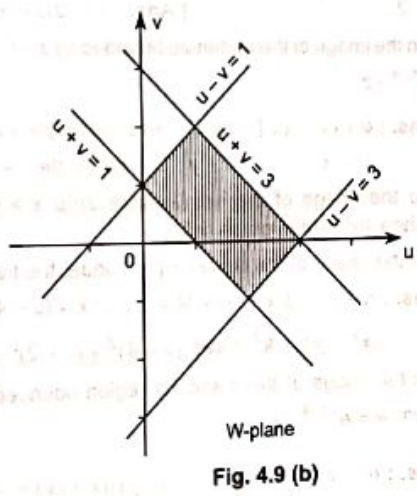

Fig. 4.9 (b)

When $x = 0$, $u = -y + 2$, $v = y - 1$, by addition $u + v = 1$.

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When $y = 0$ $u = x + 2,\ v = x - 1,$ by subtraction, $u - v = 3$

When $x = 1,$ $u = -y + 3,\ v = y,$ by addition, $u + v = 3$

When $y = 2,$ $u = x,\ v = x + 1,$ by subtraction, $u - v = -1$

Thus, the rectangle in z-plane is mapped onto another rectangle in the w-plane but rotated.

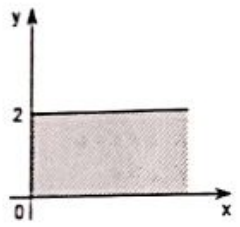
Example 3 : Find the image of semi-infinite strip $x > 0,\ 0 < y < 2$ under the transformation $w = iz + 1$.

Sol. : $\because\ w = iz + 1\ \ \therefore\ u + iv = i(x + iy) + 1 = ix + y + 1\ \ \therefore\ u = -y + 1$ and $v = x$.

When $x > 0,$ $v > 0$

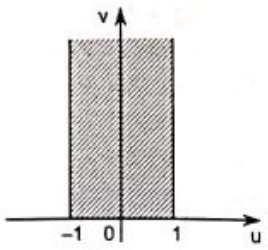
When $y = 0,$ $u = 1$ and when $y = 2,\ u = -1.$

When $0 < y < 2,$ $-1 < u < 1.$



Z-plane

Fig. 4.10 (a)



W-plane

Fig. 4.10 (b)

Thus, the strip $x > 0,\ 0 < y < 2$ in the z-plane is mapped on to the strip $v > 0,\ -1 < u < 1$.

EXERCISE - II

1. Find the image of the triangle whose vertices are (0, 1), (2, 2) (2, 0) under the transformation $w = 4z$. [**Ans. :** Use $u + iv = 4x + 4iy$. The triangle whose vertices are (0, 4), (8, 8), (8, 0)]

2. Find the image of the triangle whose vertices $i,\ 1 + i,\ 1 - i$ under the transformation $w = 3z + 4 - 2i$. [**Ans. :** The triangle whose vertices are (4, 1), (7, 1), (7, −5)]

3. Find the image of the rectangle bounded by $x = 0, y = 0, x = 2, y = 3$ under the transformation $w = \sqrt{2} \cdot e^{i\pi/4} \cdot z$. **(M.U. 1999)**

[**Ans. :** $u + iv = \sqrt{2}[\cos(\pi/4) + i\sin(\pi/4)](x + iy) = (1 + i)(x + iy)$

$\therefore u = x - y,\ v = x + y$. The rectangle $u = -v,\ u = v,\ u + v = 4,\ u - v = -6$]

4. Find the image of the semi-infinite strip $x > 0,\ 1 < y < 3$ under the transformation. $w = iz + 2$. Show the region graphically. [**Ans. :** $v > 0,\ -1 < u < 1$]

5. Find the image of the circle $|z| = k$ under the transformation $w = 3z + 4 + 2i$.

[**Ans. :** $u + iv = 3(x + iy) + (4 + 2i)\ \therefore\ x = (u - 4)/3,\ y = (v - 2)/3$

$\therefore x^2 + y^2 = k^2$ gives $(u - 4)^2 + (v - 2)^2 = 9k^2$]

6. Find the image of the triangular region bounded by $x = 0,\ y = 0,\ x + y = \sqrt{2}$ under the transformation $w = e^{i\pi/4} \cdot z$.

[**Ans. :** $u + iv = \left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right](x + iy) = \frac{1}{\sqrt{2}}(x - y) + \frac{i}{\sqrt{2}}(x + y)$

The triangular region bounded by $u = v,\ u = -v,\ v = 1$.]

(iii) Inversion and Reflection $\boxed{w = \frac{1}{z}}$

Let $z = re^{i\theta}$ and $w = Re^{i\Phi}$

$\therefore\ Re^{i\Phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}\qquad \therefore\ R = \frac{1}{r},\ \Phi = -\theta$

Thus, the transformation $w = \frac{1}{z}$ maps the point $P(r, \theta)$ in the z-plane onto the point $P'\left(\frac{1}{r}, -\theta\right)$ of the w-plane.

Remark ....

If $r = k$, then $R = \frac{1}{k} = k'$, hence, the circles in the z-plane will be mapped onto circles in the w-plane under the transformation $w = \frac{1}{z}$.

Example 1 : Find the images of the following under the transformation $w = \frac{1}{z}$.

(i) $z = \frac{\sqrt{5}}{2} + i$ (ii) $z = \frac{2\sqrt{5}}{9} + \frac{4}{9}i$ (iii) $|z| = 1$

(iv) $|z| < 1$ (v) $|z| > 1$ (vi) $|z| = k$.

Sol. : Since $w = \frac{1}{z}$ as above $Re^{i\Phi} = \frac{1}{r} \cdot e^{-i\theta}\ \therefore R = \frac{1}{r},\ \Phi = -\theta$

(i) Now, $w = \frac{\sqrt{5}}{2} + i = \frac{3}{2}e^{i\theta}$ where $\theta = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$\left[\because\ x + iy = \sqrt{x^2 + y^2}\, e^{i\theta}\ \text{where}\ \theta = \tan^{-1}(y/x)\right]$

$\therefore\ R = \frac{1}{r} = \frac{2}{3}$ and $\Phi = -\theta = -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

Thus, the point $P_1\left(\frac{3}{2}, \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ is mapped onto the point $P'_1\left(\frac{2}{3}, -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$.

(ii) $z = \frac{2\sqrt{5}}{9} + \frac{4}{9}i = \frac{2}{3}e^{i\theta}$ where $\theta = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$\left[\because\ x + iy = \sqrt{x^2 + y^2}\, e^{i\theta}\ \text{where}\ \theta = \tan^{-1}(y/x)\right]$

$\therefore\ R = \frac{1}{r} = \frac{3}{2}$ and $\Phi = -\theta = -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

Thus, the point $P_2\left(\frac{2}{3}, \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$ is mapped onto the point $P'_2\left(\frac{3}{2}, -\tan^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)$.

(iii) $|z| = 1$ is a circle $r = 1\ \ \therefore\ R = \frac{1}{r} = 1\ \ \therefore\ |w| = 1$

(iv) $|z| < 1$ is the interior of the circle $r = 1$ *i.e.* the region for which $r < 1$.

$\therefore\ R = \frac{1}{r} > 1$ *i.e.* the exterior of the circle $|w| = 1$.

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(v) $|z| > 1$ is the exterior of the circle $r = 1$ *i.e.* the region for which $r > 1$.

$\therefore\ R = \frac{1}{r} < 1$ *i.e.* the interior of the circle $|w| = 1$.

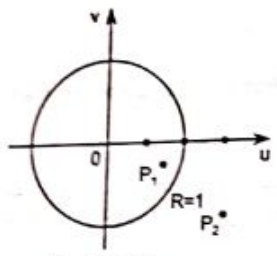

Fig. 4.11 (a)

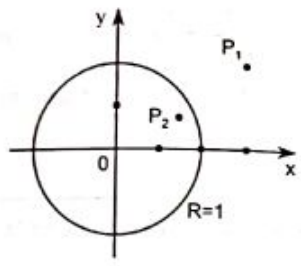

Fig. 4.11 (b)

Thus, the unit circle $|z| = 1$ of the z-plane is mapped onto the unit circle $|w| = 1$ of the w-plane, the interior of the unit circle of z-plane is mapped onto the exterior of the unit circle of the w-plane and the exterior of the unit circle of the z-plane is mapped onto the interior of the unit circle of the w-plane.

(vi) $|z| = k$ is a circle $r = k$ $\therefore\ R = \frac{1}{r} = \frac{1}{k}$ $\therefore\ |w| = \frac{1}{k}$.

Thus, a circle with centre at the origin in the z-plane is mapped onto the circle with centre at the origin in the w-plane.

If the circle with centre at the origin in the z-plane is inside (outside) the unit circle $|z| = 1$ then its image in the w-plane is a circle with centre at the origin outside (inside) the unit circle $|w| = 1$.

Remark ....

Let us superimpose w-plane on z-plane. Let P be (r, θ) and P_1 be $\left(\frac{1}{r}, \theta\right)$. Then as $OP \cdot OP_1 = r \cdot \frac{1}{r} = 1$, the point P_1 is the **inverse** of P w.r.t. the unit circle. If P' is $\left(\frac{1}{r}, -\theta\right)$, then P' is the **reflection** of P_1 in the x-axis.

Thus, the transformation $w = \frac{1}{z}$ is an inversion of z w.r.t. the unit circle and reflection of the inverse in the real axis thereafter.

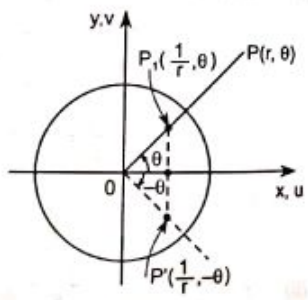

Fig. 4.12

EXERCISE - III

1. Find the images of the following under the transformation $w = \frac{1}{z}$.

(i) The points (1, 0), (0, 1) (−1, 0), (0, −1)

(ii) The points (2, 2), $\left(\frac{1}{2}, \frac{1}{2}\right)$ (iii) $|z| = 1$ (iv) $|z| = 3$ (v) $|z| = \frac{1}{3}$

[**Ans.** : (i) (1, 0), (0, − 1), (− 1, 0), (0, 1) (ii) $\left(\frac{1}{4}, -\frac{1}{4}\right)$, (1, − 1),

(iii) $|w| = 1$, (iv) $|w| = \frac{1}{3}$ (v) $|w| = 3$]

2. Show that under the transformation $w = \frac{1}{z}$ the interior (exterior) of the circle $|z| = 1$ is mapped onto the exterior (interior) of the circle $|w| = 1$.

(iv) Mapping under $w = \frac{k}{z}$ where, k is real

Example 1 : Find the image of $|z - ai| = a$ under the transformation $w = 1/z$. **(M.U. 1999, 2003, 04, 07, 08, 19)**

Sol. : The given curve is $|z - ai| = a$

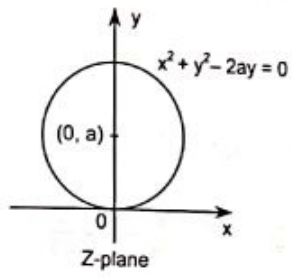

Fig. 4.13 (a)

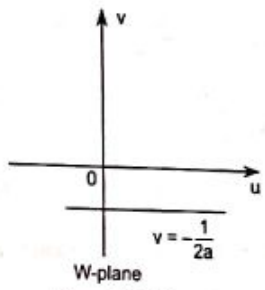

Fig. 4.13 (b)

i.e. $|(x + iy) - ai| = a$ *i.e.* $|x + i(y - a)| = a$

$\therefore\ x^2 + (y - a)^2 = a^2$ *i.e.* $x^2 + y^2 - 2ay = 0$ ..... (i)

This is a circle with centre at (0, a) and radius a.

Now, $w = \frac{1}{z}$ *i.e.* $z = \frac{1}{w}$ $\therefore\ x + iy = \frac{1}{u + iv} = \frac{u - iv}{u^2 + v^2}$

$\therefore\ x = \frac{u}{u^2 + v^2}$ and $y = -\frac{v}{u^2 + v^2}$ ..... (ii)

Putting the values of x and y from (i) in (ii)

$$\frac{u^2}{(u^2 + v^2)^2} + \frac{v^2}{(u^2 + v^2)^2} + \frac{2av}{(u^2 + v^2)} = 0 \quad \text{..... (iii)}$$

$$\therefore\ \frac{u^2 + v^2}{(u^2 + v^2)^2} + \frac{2av}{u^2 + v^2} = 0 \text{ i.e. } 1 + 2av = 0 \text{ i.e. } v = -\frac{1}{2a} \quad \text{..... (iii)}$$

which is a straight line.

Thus, the given circle in z-plane is mapped onto a straight line in w-plane.

Remark ....

$w = 1/z$ transforms a circle passing through the origin in the z-plane to a straight line **not** passing through the origin in the w-plane. **(M.U. 2003)**

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Example 2 : Find the image of the circle $(x-3)^2 + y^2 = 2$ under the transfor-mation $w = 1/z$. **(M.U. 1996, 2006, 07)**

Sol. : The given circle has centre at (3, 0) and radius $\sqrt{2}$.

Now $w = \frac{1}{z}$ $\therefore z = \frac{1}{w}$ $\therefore x + iy = \frac{1}{u+iv} = \frac{u-iv}{u^2+v^2}$

$\therefore x = \frac{u}{u^2+v^2}$ and $y = -\frac{v}{u^2+v^2}$ ..... (i)

The equation of the circle is $x^2 + y^2 - 6x + 7 = 0$ ..... (ii)

Eliminating x and y from (i) and (ii) we get

$$\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2} - \frac{6u}{(u^2+v^2)} + 7 = 0$$

$$\therefore \frac{u^2+v^2}{(u^2+v^2)^2} - \frac{6u}{(u^2+v^2)} + 7 = 0$$

$$\therefore \frac{1}{u^2+v^2} - \frac{6u}{u^2+v^2} + 7 = 0 \qquad 1 - 6u + 7(u^2+v^2) = 0$$

$$\therefore u^2 - \frac{6}{7}u + \frac{1}{7} + v^2 = 0 \qquad \therefore \left(u - \frac{3}{7}\right)^2 + v^2 = \frac{9}{49} - \frac{1}{7} = \frac{2}{49}$$

which is a circle with centre (3/7, 0) and radius $\sqrt{2}/7$.

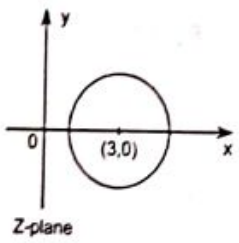

Fig. 4.14 (a)

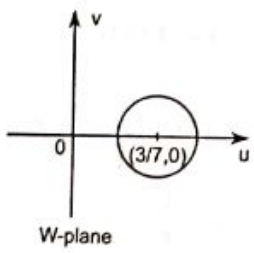

Fig. 4.14 (b)

Example 3 : Show that the function $w = 4/z$ transforms the straight lines $x = c$ in the z-plane into circles in the w-plane. **(M.U. 1993, 2016)**

Sol. : We have $w = \frac{4}{z} = \frac{4}{x+iy}$

$$\therefore w = u + iv = \frac{4}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{4(x-iy)}{x^2+y^2}$$

$$u = 4 \cdot \frac{x}{x^2+y^2}, \quad v = -4 \cdot \frac{y}{x^2+y^2}$$

When $x = c$, $u = 4\frac{c}{c^2+y^2}$ and $v = -\frac{4y}{c^2+y^2}$

We have to eliminate y between these two. Now, squaring and adding, we get,

$$u^2 + v^2 = 16\frac{(c^2+y^2)}{(c^2+y^2)^2} = \frac{16}{c^2+y^2} = 16 \cdot \frac{u}{4c} = \frac{4u}{c}$$

$$\therefore u^2 + v^2 - \frac{4u}{c} = 0 \quad \therefore \left(u - \frac{2}{c}\right)^2 + v^2 = \frac{4}{c^2} = \left(\frac{2}{c}\right)^2$$

This is a family of circles with centre at (2 / c, 0) and radius 2 / c.

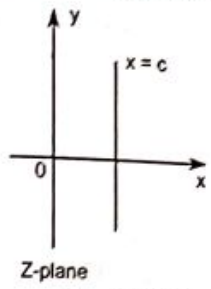

Fig. 4.15 (a)

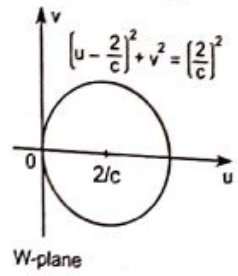

Fig. 4.15 (b)

Example 4 : Find the images of the infinite strips (i) $\frac{1}{2b} < y < \frac{1}{2a}$, (ii) $0 < y < \frac{1}{2a}$ under the transformation $w = 1/z$. Show the regions graphically ($a > 0$, $b > 0$, $a < b$).

Sol. : We have $u + iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$ $\therefore u = \frac{x}{x^2+y^2}$ and $v = -\frac{y}{x^2+y^2}$

To obtain the image of the line y equal to a constant we have to eliminate x from above.

Now $\frac{u}{v} = -\frac{x}{y}$ *i.e.* $x = -\frac{uy}{v}$

$$\therefore v = -\frac{y}{\frac{u^2y^2}{v^2} + y^2} = -\frac{v^2}{u^2+v^2} \cdot \frac{1}{y} \quad \therefore y = -\frac{v}{u^2+v^2}$$

(i) If $y = \frac{1}{2b}$, $-\frac{v}{u^2+v^2} = \frac{1}{2b}$ $\therefore u^2 + v^2 + 2bv = 0$ $\therefore u^2 + (v+b)^2 = b^2$

The line $y = 1/2b$ is transformed into a circle with centre $(0, -b)$ and radius b

(ii) If $y = \frac{1}{2a}$, $-\frac{v}{u^2+v^2} = \frac{1}{2a}$ $\therefore u^2 + v^2 + 2av = 0$ $\therefore u^2 + (u+a)^2 = a^2$

The line $y = 1/2a$ is transformed into a circle with centre $(0, -a)$ and radius a.

Hence, the region $\frac{1}{2b} < y < \frac{1}{2a}$ is mapped into the region between the above two circles as shown in the figure below.

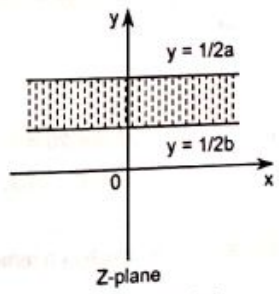

Fig. 4.16 (a)

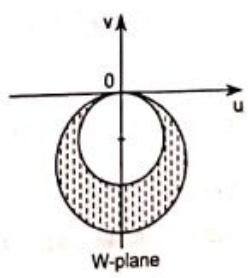

Fig. 4.16 (b)

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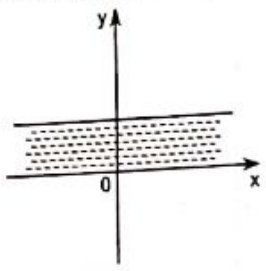
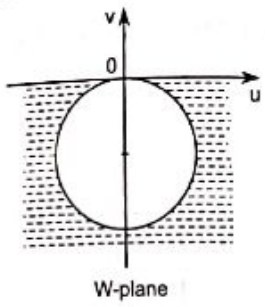
(iii) As b becomes larger and larger the line $y = 1/2b$ comes closer and closer to the x-axis and the circle with centre b becomes larger and larger.



Z-plane

Fig. 4.17 (a)



W-plane

Fig. 4.17 (b)

Hence, the strip $0 < y < \frac{1}{2a}$ is mapped onto the region outside the circle $u^2 + (v + a)^2 = a^2$ below the real line.

Example 5 : Prove that the circle $|z - 2| = 3$ is mapped into the circle $\left|w + \frac{2}{5}\right| = \frac{9}{25}$ under the transformation $w = 1/z$.

Sol. : We have $|z - 2| = 3$

$$\therefore \left|\frac{1}{w} - 2\right| = 3 \quad \therefore \left|\frac{1-2w}{w}\right| = 3$$

$$\therefore |1 - 2w| = 3|w| \quad \therefore |1 - 2(u + iv)| = 3|u + iv|$$

$$\therefore (1 - 2u)^2 + 4v^2 = 9(u^2 + v^2) \quad \therefore 5u^2 + 5v^2 + 4u - 1 = 0$$

$$\therefore u^2 + v^2 + \frac{4u}{5} - \frac{1}{5} = 0$$

$$\left(u + \frac{2}{5}\right)^2 + v^2 = \frac{1}{5} + \frac{4}{25} = \frac{9}{25} \qquad \therefore \left|\left(u + \frac{2}{5}\right) + iv\right| = \frac{9}{25}$$

$$\therefore \left|(u + iv) + \frac{2}{5}\right| = \frac{9}{25} \qquad \therefore \left|w + \frac{2}{5}\right| = \frac{9}{25}$$

(Alternatively you can use the method of example 1.)

Example 6 : Show that the image of the rectangular hyperbola $x^2 - y^2 = 1$ under the transformation $w = 1/z$ is the lemniscate $\rho^2 = \cos 2\Phi$. Also draw the sketches.

(M.U. 1997, 2005, 04, 06)

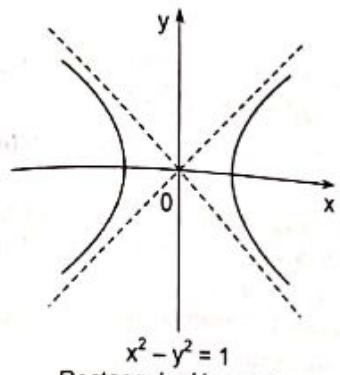
Sol. : Putting $z = re^{i\theta}$ and $w = \rho e^{i\Phi}$ in $w = 1/z$, we get

$$\rho e^{i\Phi} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta} \qquad \therefore \rho = \frac{1}{r}, \ \Phi = -\theta.$$

Since $x^2 - y^2 = 1$, we get $r^2(\cos^2\theta - \sin^2\theta) = 1 \qquad \therefore r^2 \cos 2\theta = 1.$

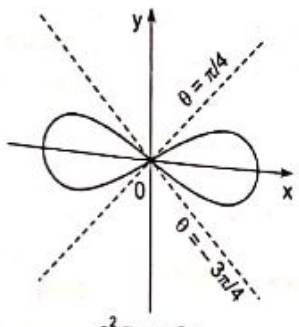
$$\therefore r^2 = \frac{1}{\cos 2\theta} \qquad \therefore \frac{1}{r^2} = \cos 2\theta$$

Putting $\rho = \frac{1}{r}$ and $\Phi = -\theta$, we get $\rho^2 = \cos(-2\Phi) = \cos 2\Phi$, which is a lemniscate.



$x^2 - y^2 = 1$
Rectangular Hyperbola

Fig. 4.18 (a)



$\rho^2 = \cos 2\Phi$
Lemniscate

Fig. 4.18 (b)

EXERCISE - VI

1. Find the image of the circle with centre at (0, 3) and radius 3 *i.e.* of $|z - 3i| = 3$ in the z-plane into the w-plane under the transformation $w = 1/z$. **(M.U. 2003, 05, 07)**

[**Ans. :** Circle is $x^2 + y^2 - 6y = 0$; $v = -1/6$. Put $a = 3$ in Ex. 1, page S-145.]

2. Prove that under the transformation $w = 1/z$ in general circles in the z-plane are transformed into the circles in the w-plane. **(M.U. 2003)**

3. Find the image of the strip $2 \le x \le 4$ in the z-plane under the trans-formation $w = 2/z$.

[**Ans. :** Region bounded by the circles $u^2 + v^2 = u$, and $u^2 + v^2 = u/2$.]

4. Find the image of the infinite strip (i) $\frac{1}{6} \le y < \frac{1}{4}$, (ii) $0 \le y < \frac{1}{4}$ under the transformation $w = 1/z$. Show the regions graphically.

[**Ans. :** (i) Region between the two circles $u^2 + v^2 + 6v = 0$, $u^2 + v^2 + 4v = 0$,

(ii) Region outside the circle $u^2 + v^2 + 4v = 0$.]

5. Prove that the circle $|z - 3| = 5$ is mapped onto the circle $\left|w + \frac{3}{16}\right| = \frac{5}{16}$ under the transformation $w = 1/z$. **(M.U. 1996)**

6. Find the image of the line $y - x + 1 = 0$ under the transformation $w = 1/z$. Also find the image of $y - x = 0$. Draw rough sketches. **(M.U. 1995, 2003, 06, 19)**

[**Ans. :** The circle with centre (1/2, 1/2) and radius $1/\sqrt{2}$.]

7. Find the images of the following curves under the transformation $w = 1/z$. (i) The line $2y - 2x = 1$, (ii) The circle $|z - 4| = 6$.

[**Ans. :** (i) The circle $u^2 + v^2 - 2u - 2v = 0$, (ii) The circle $20u^2 + 20v^2 + 8u - 1 = 0$.]

8. Find the image of the strip $\frac{1}{4} \le y \le \frac{1}{2}$ under the transformation $w = 1/z$. Draw the rough sketches.

[**Ans. :** Region bounded by the circles $u^2 + v^2 + 4v = 0$ and $u^2 + v^2 + 2v = 0$.]

9. Find the images of the following under the transformation $w = 1/z$.

(i) $|z| = 2$ (ii) $|z - 1| = 1$, $|z - i| = 1$ **(M.U. 2003)**

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(iii) $x + y = 1$ (iv) $x = y$

[**Ans. :** See solved Ex. 1 above. **(i)** The circle $x^2 + y^2 = 4$ maps onto the circle $u^2 + v^2 = 1/4$, **(ii)** The lines $u = 1/2$, $v = -1/2$. **(iii)** The circle $(u - 1/2)^2 + (v + 1/2)^2 = 1/2$, **(iv)** $u + v = 0$, a line.]

10. Show that the transformation $w = 1/z$ maps the circle **(i)** $|z - a| = a$, $a > 0$ **(ii)** $|z - ai| = a$ into a straight line. **(M.U. 1995)**

11. Find the image of **(i)** $x^2 + y^2 = 2x$, **(ii)** $x^2 + y^2 = 2y$ and **(ii)** $x = 2y$ under the transformation $w = 1/z$. **(M.U. 2004)** [**Ans. :** **(i)** $2u = -1$, **(ii)** $2v = -1$, **(iii)** $u = -2v$]

12. Show that under the transformation $w = 1/z$, the circle $|z - 3i| = 3$ is mapped on to the line $6u + 1 = 0$. **(M.U. 2005)** (Compare this example with the solved Ex. No. 1.)

13. Find the image of $y = x + (1/2)$ under the transformation $w = 1/z$.

[**Ans. :** $(u + 1)^2 + (v + 1)^2 = 2$]

5. Bilinear Transformation $w = \dfrac{az + b}{cz + d}$

The transformation $w = \dfrac{az + b}{cz + d}$ ..... (1)

where a, b, c, d are complex constants and $ad - bc \neq 0$ is called **bilinear** or **Mobius transformation.**

Since $\dfrac{dw}{dz} = \dfrac{(cz + d)a - (az + b)c}{(cz + d)^2} = \dfrac{ad - bc}{(cz + d)^2} \neq 0$ the transformation is conformal.

From (1) we get $wcz + wd = az + b$

i.e. $z(cw - a) = b - wd$ *i.e.* $z = \dfrac{-wd + b}{cw - a}$ ..... (2)

This inverse mapping is also bilinear.

Notes ....

1. The transformation (1) can be written as $cwz + wd - az - b = 0$

   It is linear in both w and d and hence, the name bilinear transformation.

2. From (1) we see that except $z = -d/c$ to every point of z-plane there is a unique point on the w-plane. From (2) We see that except $w = a/c$ to every point of w-plane there is a unique point of z-plane. Considering the two exceptional points at points at infinity we can say that **there is one to one correspondence between the points of the two planes under a bilinear transformation.**

3. **"A bilinear transformation is a combination of three basic trans-formations viz. (i) translation, (ii) rotation and magnification and (iii) inversion and reflection."**

For, we can write $w = \dfrac{az + b}{cz + d}$ as $w = \dfrac{a}{c} + \dfrac{bc - ad}{c^2} \cdot \dfrac{1}{z + d/c}$ [By actual division]

Now consider the transformations

$w_1 = z + \dfrac{d}{c}$ which is a translation

$w_2 = \dfrac{1}{w_1}$ which is inversion and reflection

$w_3 = \dfrac{bc - ad}{c^2} \cdot w_2$ which is rotation and magnification

$w = \dfrac{a}{c} + w_3$ which is again translation.

Thus, by these transformations we successively go from z-plane to w_1-plane from w_1-plane to w_2-plane, from w_2-plane to w_3-plane and finally from w_3-plane to w-plane. Hence, the above result.

Remark ....

(i) $ad - bc$ is called the **determinant** of the transformation.

(ii) If $ad - bc = 1$ then the transformation is said to be **normalised.**

(iii) A bilinear transformation is also known as **Mobius Transformation** who first studied it.

Theorem : A bilinear transformation, in general, maps circles into circles.

Proof : Let the bilinear transformation be **(M.U. 1996, 2003, 04)**

$$w = \frac{az + b}{cz + d}$$

As above, by actual division, we can write it as

$$w = \frac{a}{c} + \frac{bc - ad}{c^2} \cdot \frac{1}{z + d/c}.$$

Now, consider the transformation $w_1 = z + \dfrac{d}{c}$ which a translation.

Since in the transformation $w = z + k$ size and shape is preserved, circles in z-plane will be transformed into circles in w_1-plane.

Now, consider the transformation $w_2 = 1/w_1$, which is inversion and reflection.

Since in the transformation $w = 1/z$ circles are mapped onto circles, circles in w_1-plane will be mapped onto circles in w_2-plane.

Now, consider the transformation $w_3 = \dfrac{bc - ad}{c^2} \cdot w_2$, which is a rotation and magnification.

Since in the transformation $w = kz$ figures are only rotated and magnified, circles in w_2-plane will be transformed into circles in w_3-plane.

Now, consider the transformation $w = \dfrac{a}{c} + w_3$, which again is a translation.

Hence, circles in w_3-plane will be transformed into circles in the w-plane.

Hence, circles in z-plane will be transformed into circles in w-plane.

Note ....

The above theorem is true **in general.** But there are exceptions.

6. Cross-Ratio

If z_1, z_2, z_3, z_4 are any four points then their cross ratio **in this order is** defined as $\dfrac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$ and is denoted by (z_1, z_2, z_3, z_4).

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Notes ....

1. The cross-ratio can be easily remembered as follows. We start with z_1, and write the elements in the given order $z_1 - z_2$, $z_2 - z_3$, $z_3 - z_4$ $z_4 - z_1$ in cyclic order. We then put the first element in the numerator, second in the denominator, third in the numerator and fourth in the denominator.

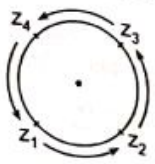

Fig. 4.19

2. If z_1, z_2, z_3, z_4 are the given four points they can be ordered 4 at a time amongst themselves in 4! = 24 ways. This means from 4 points we can have 24 cross-ratios. But, it can be seen that there are only 6 different cross ratios. For, we have

$$(z_1, z_2, z_3, z_4) = (z_2, z_1, z_4, z_3) = (z_4, z_3, z_2, z_1) = (z_3, z_4, z_1, z_2) \quad \text{..... (1)}$$
$$(z_1, z_2, z_4, z_3) = (z_2, z_1, z_3, z_4) = (z_3, z_4, z_2, z_1) = (z_4, z_3, z_1, z_2) \quad \text{..... (2)}$$
$$(z_1, z_3, z_2, z_4) = (z_3, z_1, z_4, z_2) = (z_4, z_2, z_3, z_1) = (z_2, z_4, z_1, z_3) \quad \text{..... (3)}$$
$$(z_1, z_3, z_4, z_2) = (z_3, z_1, z_2, z_4) = (z_2, z_4, z_3, z_1) = (z_4, z_2, z_1, z_3) \quad \text{..... (4)}$$
$$(z_1, z_4, z_2, z_3) = (z_4, z_1, z_3, z_2) = (z_3, z_2, z_4, z_1) = (z_2, z_3, z_1, z_4) \quad \text{..... (5)}$$
$$(z_1, z_4, z_3, z_2) = (z_4, z_1, z_2, z_3) = (z_2, z_3, z_4, z_1) = (z_3, z_2, z_1, z_4)$$

To obtain the second, third and fourth cross-ratio in any row, first inter-change first and second elements and third and forth elements (z_2, z_1, z_4, z_3) then we interchange the first pair with the second pair (z_4, z_3, z_2, z_1) ; and again interchange the first and second elements and third and fourth elements in the previous ratio. The cross-ratios of the first row can be seen to be,

$$\frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)} = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_1 - z_4)(z_3 - z_2)}$$
$$= \frac{(z_4 - z_3)(z_2 - z_1)}{(z_3 - z_2)(z_1 - z_4)} = \frac{(z_3 - z_4)(z_1 - z_2)}{(z_4 - z_1)(z_2 - z_3)}$$

(a) Cross-ratio preservation property

If w_1, w_2, w_3, w_4 are the images in the w-plane of the four distinct points z_1, z_2, z_3, z_4 in the z-plane under a bilinear transformation

$$w = \frac{az + b}{cz + d} \quad (ad - bc \neq 0)$$

then the cross-ratio is preserved *i.e.* the cross-ratio of w_1, w_2, w_3, w_4 is equal to the cross-ratio of z_1, z_2, z_3, z_4.

(M.U. 1994, 97, 2002, 03, 04)

i.e. $$(w_1, w_2, w_3, w_4) = (z_1, z_2, z_3, z_4)$$

Proof : We have $w_i = \dfrac{az_i + b}{cz_i + d}$, $i = 1, 2, 3, 4$

$$\therefore \quad w_1 - w_2 = \frac{a z_1 + b}{c z_1 + d} - \frac{a z_2 + b}{c z_2 + d}$$
$$= \frac{ac\, z_1 z_2 + ad\, z_1 + bc\, z_2 + bd - ac\, z_1 z_2 - ad\, z_2 - bc\, z_1 - bd}{(cz_1 + d)(cz_2 + d)}$$
$$\therefore \quad w_1 - w_2 = \frac{(ad - bc)(z_1 - z_2)}{(cz_1 + d)(cz_2 + d)}$$

Similarly, $$w_2 - w_3 = \frac{a z_2 + b}{c z_2 + d} - \frac{a z_3 + b}{c z_3 + d} = \frac{(ad - bc)(z_2 - z_3)}{(c z_2 + d)(c z_3 + d)}$$

$$w_3 - w_4 = \frac{a z_3 + b}{c z_3 + d} - \frac{a z_4 + b}{c z_4 + d} = \frac{(ad - bc)(z_3 - z_4)}{(c z_3 + d)(c z_4 + d)}$$
$$w_4 - w_1 = \frac{a z_4 + b}{c z_4 + d} - \frac{a z_1 + b}{c z_1 + d} = \frac{(ad - bc)(z_4 - z_1)}{(c z_4 + d)(c z_1 + d)}$$
$$\therefore \quad \frac{w_1 - w_2}{w_2 - w_3} \cdot \frac{w_3 - w_4}{w_4 - w_1} = \frac{(ab - bc)(z_1 - z_2)}{(c z_1 + d)(c z_2 + d)} \cdot \frac{(c z_2 + d)(c z_3 + d)}{(ad - bc)(z_2 - z_3)}$$
$$\times \frac{(ad - bc)(z_3 - z_4)}{(c z_3 + d)(c z_4 + d)} \cdot \frac{(c z_4 + d)(c z_1 + d)}{(ad - bc)(z_4 - z_1)}$$
$$= \frac{(z_1 - z_2)}{(z_2 - z_3)} \cdot \frac{(z_3 - z_4)}{(z_4 - z_1)}$$
$$\therefore \quad \frac{w_1 - w_2}{w_2 - w_3} \cdot \frac{w_3 - w_4}{w_4 - w_1} = \frac{z_1 - z_2}{z_2 - z_3} \cdot \frac{z_3 - z_4}{z_4 - z_1}.$$

Hence, the result.

Remark ....

By taking the reciprocal of the above equality and by replacing w_4 by general point w and also z_4 by z, we get

$$\boxed{\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}}$$

The above equality is easy to remember and can be conveniently used to obtain bilinear transformations.

Example 1 : Find the bilinear transformation which maps the points 2, i, –2 onto the points 1, i, –1 by using cross-ratio property. **(M.U. 1993, 2000, 02, 03, 05, 07)**

Sol. : We have $$\frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)} = \frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)}$$

Putting $z_1 = 2$, $z_2 = i$, $z_3 = -2$ and $w_1 = 1$, $w_2 = i$, $w_3 = -1$, we get

$$\frac{(w - 1)(i + 1)}{(1 - i)(-1 - w)} = \frac{(z - 2)(i + 2)}{(2 - i)(-2 - z)}$$
$$\therefore \quad \frac{(w - 1)(1 + i)(1 + i)}{(-1 - w)(1 - i)(1 + i)} = \frac{(z - 2)(2 + i)(2 + i)}{(-z - 2)(2 - i)(2 + i)}$$
$$\frac{(w - 1)(1 + i)^2}{(w + 1)(1 - i^2)} = \frac{(z - 2)(2 + i)^2}{(z + 2)(4 - i^2)}$$
$$\frac{(w - 1) \cdot (2i)}{(w + 1) \cdot 2} = \frac{(z - 2)(3 + 4i)}{(z + 2) \cdot 5}$$
$$\frac{w - 1}{w + 1} = \frac{(z - 2)(3 + 4i)}{(z + 2) \cdot 5i} = \frac{(z - 2)(4 - 3i)}{(z + 2) \cdot 5}$$

By componendo and dividendo,

$$\frac{2w}{2} = \frac{(z - 2)(4 - 3i) + (z + 2) \cdot 5}{(z + 2) \cdot 5 - (z - 2)(4 - 3i)}$$
$$\therefore \quad w = \frac{4z - 8 - 3iz + 6i + 5z + 10}{5z + 10 - 4z + 8 + 3iz - 6i} = \frac{9z - 3iz + 2 + 6i}{z + 3iz + 18 - 6i}$$
$$= \frac{3z(3 - i) + 2(1 + 3i)}{z(1 + 3i) + 6(3 - i)} = \frac{3z + 2\{(1 + 3i)/(3 - i)\}}{z\{(1 + 3i)/(3 - i)\} + 6}$$

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Now $\frac{1+3i}{3-i} = \frac{1+3i}{3-i} \cdot \frac{3+i}{3+i} = \frac{3+9i+i-3}{9-i^2} = \frac{10i}{10} = i$

$\therefore \quad w = \frac{3z+2i}{zi+6}$ which is the required bilinear transformation.

Note : We shall learn another method of finding bilinear transformation on page S-159.

Example 2 : Find the bilinear transformation which maps the points $z = \infty, i, 0$ onto the points $0, i, \infty$ by using cross-ratio property. **(M.U. 2003, 04, 05, 09)**

Sol. : We have the transformation

$$\frac{(w-w_1)(w_2-w_3)}{(w_1-w_2)(w_3-w)} = \frac{(z-z_1)(z_2-z_3)}{(z_1-z_2)(z_3-z)}$$

Since, $z_1 = \infty$ and $w_3 = \infty$, we divide the N and D of l.h.s. by w_3 and N and D of r.h.s. by z_1.

$$\therefore \quad \frac{(w-w_1)\left(\frac{w_2}{w_3}-1\right)}{(w_1-w_2)\left(1-\frac{w}{w_3}\right)} = \frac{\left(\frac{z}{z_1}-1\right)(z_2-z_3)}{\left(1-\frac{z_2}{z_1}\right)(z_3-z)}$$

Since, $w_3 = \infty$, $w_2 / w_3 = 0$, $w / w_3 = 0$.

Also since $z_1 = \infty$, $z / z_1 = 0$, $z_2 / z_1 = 0$

$$\therefore \quad \frac{(w-w_1)(-1)}{(w_1-w_2)} = \frac{(-1)(z_2-z_3)}{(z_3-z)}$$

Now put $z_2 = i$, $z_3 = 0$ and $w_1 = 0$ and $w_2 = i$.

$$\therefore \quad \frac{w-0}{0-i} = \frac{i-0}{0-z} \qquad \therefore \quad \frac{w}{-i} = \frac{i}{-z} \qquad \therefore \quad w = \frac{-i^2}{-z} = -\frac{1}{z}$$

7. Fixed points of a Bilinear Transformation

Definition : The points which coincide with their transforms under a bilinear transformation are called **fixed points** of the bilinear transform.

Consider a bilinear transform

$$w = \frac{az+b}{cz+d} \qquad (ad - bc \neq 0)$$

The fixed points of this transformation are obtained by putting $w = z$ *i.e.* by

$$z = \frac{az+b}{cz+d} \quad \textit{i.e.} \text{ by the quadratic equation.}$$

$$cz^2 + (d-a)z - b = 0 \qquad \text{..... (1)}$$

In other words the roots of this equation are the fixed points.

Case I : If $c \neq 0$, the roots of the equation (1) are given by

$$z = \frac{-(d-a) \pm \sqrt{(d-a)^2 + 4bc}}{2c}$$

The roots are distinct *i.e.* the points are distinct if $(d-a)^2 + 4bc \neq 0$.

The roots are identical *i.e.* the points are coincident (one point) if $(d-a)^2 + 4bc = 0$.

Case II : If $c = 0$ but $d \neq 0$ then from (1) again, we get, $w = \frac{a}{d}z + \frac{b}{d}$.

If z is infinity then obviously w is infinity. So one fixed point is infinity. The other fixed point is given by

$$z = \frac{a}{d}z + \frac{b}{d} \quad \textit{i.e.} \ (a-d)z + b = 0 \qquad \textit{i.e.,} \qquad z = -\frac{b}{a-d} \ \text{ if } a \neq d$$

If $a = d$ there is only one fixed point and it is infinity.

Thus, by the bilinear transformation, we have

(i) two finite fixed points if $c \neq 0$ and $(d-a)^2 + 4bc \neq 0$.

(ii) only one finite fixed point if $c \neq 0$ and $(d-a)^2 + 4bc = 0$. This bilinear transformation is called **parabolic.**

(iii) one finite fixed point and other infinite fixed point if $c = 0$ and $a - d \neq 0$. ..... (A)

(iv) only one infinite fixed point if $c = 0$ and $a - d = 0$. ..... (B)

Theorem 1 : Every bilinear transformation with two finite fixed points α, β can be put in the form

$$\frac{w-\alpha}{w-\beta} = \lambda \cdot \frac{z-\alpha}{z-\beta}$$ **(M.U. 2002, 06)**

Proof : Let $w = \frac{az+b}{cz+d}$ be the given bilinear transformation with α, β as fixed points.

For fixed points, we put, $w = z \quad \therefore \quad z = \frac{az+b}{cz+d} \quad \therefore cz^2 + (d-a)z - b = 0$

Since α, β are fixed points

$$c\alpha^2 + (d-a)\alpha - b = 0 \ \textit{ i.e. } \ b - d\alpha = c\alpha^2 - a\alpha$$

and $\quad c\beta^2 + (d-a)\beta - b = 0 \ \textit{ i.e. } \ b - d\beta = c\beta^2 - a\beta \qquad \text{..... (1)}$

Now, $$\frac{w-\alpha}{w-\beta} = \frac{\frac{az+b}{cz+d} - \alpha}{\frac{az+b}{cz+d} - \beta} = \frac{(a-c\alpha)z + (b-d\alpha)}{(a-c\beta)z + (b-d\beta)}$$

By using (1), we get,

$$\frac{w-\alpha}{w-\beta} = \frac{(a-c\alpha)z + (c\alpha^2 - a\alpha)}{(a-c\beta)z + (c\beta^2 - a\beta)} = \frac{(a-c\alpha)(z-\alpha)}{(a-c\beta)(z-\beta)}$$

$$= \lambda \cdot \frac{z-\alpha}{z-\beta} \quad \text{where } \lambda = \frac{a-c\alpha}{a-c\beta}$$

Theorem 2 : Every bilinear transformation which has only one fixed point α can be put in the form, called **normal form.**

$$\frac{1}{w-\alpha} = \frac{1}{z-\alpha} + k$$ **(M.U. 1996, 2005, 08)**

Proof : Let $w = \frac{az+b}{cz+d}$ be the given bilinear transformation.

For fixed point we put $w = z \quad \therefore \quad z = \frac{az+b}{cz+d} \quad \therefore cz^2 + (d-a)z - b = 0 \qquad \text{..... (A)}$

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Since α is the only one fixed point

$$cz^2 + (d-a)z - b \equiv c(z-\alpha)^2 = c(z^2 - 2z\alpha + \alpha^2)$$

Equating the coefficients of like powers of z

$$d - a = -2\alpha c \quad \text{and} \quad -b = c\alpha^2$$

i.e. $d = a - 2\alpha c$ and $b = -c\alpha^2$ ..... (1)

Now, $w - \alpha = \dfrac{az+b}{cz+d} - \alpha = \dfrac{(a-c\alpha)z + (b - d\alpha)}{cz+d}$

$$\therefore \frac{1}{w-\alpha} = \frac{cz+d}{(a-c\alpha)z + (b-d\alpha)}$$

$$= \frac{cz + a - 2c\alpha}{(a-c\alpha)z - c\alpha^2 - a\alpha + 2c\alpha^2} \qquad [\text{By (1)}]$$

$$= \frac{cz + a - 2c\alpha}{(a-c\alpha)z - \alpha(a-c\alpha)} = \frac{(a-c\alpha) + c(z-\alpha)}{(a-c\alpha)(z-\alpha)} = \frac{1}{z-\alpha} + \frac{c}{a-c\alpha}$$

$$= \frac{1}{z-\alpha} + k \text{ where } k = \frac{c}{a-c\alpha}.$$

Example : Find the fixed points of $w = \dfrac{3z-4}{z-1}$. Also express it in the normal form $\dfrac{1}{w-\alpha} = \dfrac{1}{z-\alpha} + \lambda$, where λ is a constant and α is the fixed point. Is this transformation parabolic ?

(M.U. 2003, 15, 19)

Sol. : Fixed points are given by

$$z = \frac{3z-4}{z-1} \quad \text{i.e., } z^2 - z = 3z - 4$$

$$\therefore z^2 - 4z + 4 = 0 \quad \therefore (z-2)^2 = 0 \quad \therefore z = 2, 2.$$

Hence, $z = 2$ is the only fixed point.

Now subtracting the fixed point 2 from both sides of the equation $w = \dfrac{3z-4}{z-1}$, we get

$$w - 2 = \frac{3z-4}{z-1} - 2 = \frac{3z-4-2z+2}{z-1} = \frac{z-2}{z-1}$$

$$\therefore \frac{1}{w-2} = \frac{z-1}{z-2} = \frac{(z-2)+1}{z-2} = \frac{1}{z-2} + 1 \text{ which is the required normal form.}$$

Since $z = 2$ is the only one fixed point, the transformation is parabolic.

Theorem 3 : Every bilinear transformation which has one finite fixed point α and the other fixed point ∞ can be put in the form $w - \alpha = \lambda(z - \alpha)$.

Proof : Let $w = \dfrac{az+b}{cz+d}$ be the given bilinear transformation.

Since it has one finite fixed point and the other infinite fixed point from (A), in case (ii), page S-155, it follows that $c = 0$ and $a - d \neq 0$ and it takes the form

$$w = \frac{a}{d}z + \frac{b}{d}.$$

Since α is a fixed point $\alpha = \dfrac{a}{d}\alpha + \dfrac{b}{d}$.

Subtracting we get,

$$w - \alpha = \frac{a}{d}(z-\alpha) = \lambda(z-\alpha) \text{ where } \lambda = \frac{a}{d}.$$

Remark ....

If ∞ is the only fixed point of the transformation then $c = 0$, $a = d$ and $w = \dfrac{az+b}{cz+d}$ [from (B), page S-155] takes the form $w = z + \dfrac{b}{d}$ which is a translation.

Example : Find the fixed points of the bilinear transformation

(i) $w = \dfrac{1+3iz}{i+z}$ **(M.U. 1993, 2002)**

(ii) $w = \dfrac{z-1}{z+1}$. Is this transformation parabolic ?

(iii) $w = \dfrac{z-4}{2z-5}$ **(M.U. 1995, 2002)**

(iv) $w = \dfrac{2z+1}{z+2}$. Is the transformation conformal ? **(M.U. 2002)**

Sol. : (i) The fixed points are given by $z = \dfrac{1+3iz}{i+z}$

$$\therefore iz + z^2 = 1 + 3iz \quad \therefore z^2 - 2iz - 1 = 0$$

$$\therefore z^2 - 2iz + i^2 = 0 \quad \therefore (z-i)^2 = 0$$

$\therefore z = i$ is the only one fixed point of the transformation.

(ii) The fixed points are given by

$$z = \frac{z-1}{z+1} \quad \therefore z^2 + z = z - 1$$

$$\therefore z^2 + 1 = 0 \quad \therefore z^2 - i^2 = 0$$

$\therefore (z+i)(z-i) = 0 \quad \therefore z = i, -i$ are the finite fixed points.

Since there are two distinct fixed points, the mapping is not parabolic.

(iii) The fixed points are given by

$$z = \frac{z-4}{2z-5} \quad \therefore 2z^2 - 5z = z - 4$$

$$\therefore 2z^2 - 6z + 4 = 0 \quad \therefore 2(z^2 - 3z + 2) = 0$$

$$\therefore z^2 - 3z + 2 = 0 \quad \therefore (z-2)(z-1) = 0$$

$\therefore z = 1, 2$ are the fixed points.

(iv) The fixed points are given by

$$z = \frac{2z+1}{z+2} \quad \therefore z^2 = 1 \quad \therefore z = \pm 1$$

The mapping is conformal (See theorem 4 below).

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Theorem 4 : Every bilinear transformation represents a one to one conformal mapping. **(M.U. 1995)**

Proof : We know that the transformation $w = \dfrac{az+b}{cz+d}$

where a, b, c, d are complex constants and $ad - bc \neq 0$ is called a bilinear transformation.

Now, differentiating this equation w.r.t. z, we get

$$\frac{dw}{dz} = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{ad-bc}{(cz+d)^2}$$

If $z = -\dfrac{d}{c}$, $\dfrac{dw}{dz} = \infty$ and if $z = \infty$, $\dfrac{dw}{dz} = 0$.

Thus the bilinear transformation $w = \dfrac{az+b}{cz+d}$ is conformal except at $z = -\dfrac{d}{c}$ and $z = \infty$ which are critical points of the transformation. However by suitably defining the conformal mapping at these points we can overcome this difficulty (unfortunately we do not go into details of this aspect in this book).

Hence, **every bilinear transformation is conformal.**

Example : Find the invariant points of the bilinear transformation

$$w = -\frac{(2z+4i)}{(iz+1)}$$

and prove that these two points together with any point z and its image form a set of points whose cross-ratio is constant. **(M.U. 2003, 05)**

Sol. : The fixed points are given by $z = -\dfrac{2z+4i}{iz+1}$

$\therefore\ iz^2 + z + 2z + 4i = 0 \qquad \therefore\ iz^2 + 3z + 4i = 0$

$\therefore\ i^2z^2 + 3iz + 4i^2 = 0 \qquad \therefore\ -z^2 + 3iz - 4 = 0$

$\therefore\ z^2 - 3iz + 4 = 0 \qquad \therefore\ (z+i)(z-4i) = 0 \qquad \therefore\ z = -i,\ 4i.$

Now, consider the cross-ratio of $z_1 = z$, $z_2 = -i$, $z_3 = 4i$,

$$z_4 = -\frac{(2z+4i)}{(iz+1)} \quad \text{(image of } z\text{)}$$

$$\text{Cross-ratio} = \frac{(z+i)\left(4i + \dfrac{2z+4i}{iz+1}\right)}{(-i-4i)\left(-\dfrac{2z+4i}{iz+1} - z\right)} = \frac{(z+i)(-2z+8i)}{5i(2z+4i+iz^2+z)}$$

$$= \frac{-2(z+i)(z-4i)}{-5(z^2-3iz+4)} = \frac{2}{5}\cdot\frac{(z+i)(z-4i)}{(z+i)(z-4i)} = \frac{2}{5}, \text{ a constant.}$$

EXERCISE - V

1. Find the fixed points of bilinear transformation $w = \dfrac{-1+(2+i)z}{i+2+z}$. **[Ans. : $z = \pm i$]**

2. Find the fixed points of bilinear transformation

(i) $\dfrac{1+3iz}{3i+z}$ (ii) $\dfrac{2z-2+iz}{i+z}$ **(M.U. 1998)** (iii) $\dfrac{2z-5}{z+4}$ **(M.U. 1997)** (iv) $-\dfrac{2z+4i}{iz+1}$ (v) $-\dfrac{z+2}{2z+3}$ **(M.U. 2002)**

[Ans. : (i) $z = \pm 1$, (ii) $z = 1 \pm i$, (iii) $z = -1 \pm 2i$, (iv) $z = -i, 4/i$, (v) $z = -1, -1$ **]**

3. Find the fixed points of (i) $w = \dfrac{4z-9}{z-2}$ (ii) $w = \dfrac{3z-2}{2z-1}$.

Also express it in the normal form.

[Ans. : (i) $z = 3, 3$; $\dfrac{1}{w-3} = \dfrac{1}{z-3} + 1$, (ii) $z = 1, 1$; $\dfrac{1}{w-1} = \dfrac{1}{z-1} + 2$ **]**

4. Find the fixed point of

(i) $w = \dfrac{3z-5i}{iz-1}$ **(M.U. 2003)** (ii) $w = \dfrac{2z+6}{z+7}$ **(M.U. 2003)**

(iii) $w = \dfrac{3z-5i}{iz-1}$. Is this mapping conformal ? **(M.U. 2003)**

[Ans. : (i) $5/i$, i, (ii) $-6, 1$, (iii) $-2i \pm 1$; Yes. **]**

8. Mapping Under Bilinear Transformation $w = \dfrac{az+b}{cz+d}$

Example 1 : Find the image of the circle $|z| = k$ where k is real under the bilinear transformation

$$w = \frac{5-4z}{4z-3}. \qquad \textbf{(M.U. 2002, 05, 12, 13)}$$

Sol. : Since $w = \dfrac{5-4z}{4z-3}$, we have $4wz - 3w = 5 - 4z$.

$\therefore\ z(4w+4) = 5 + 3w \qquad \therefore\ z = \dfrac{5+3w}{4(w+1)}$

When $|z| = k$, $\left|\dfrac{5+3w}{4(w+1)}\right| = k \qquad \therefore\ \left|\dfrac{5+3(u+iv)}{(u+iv)+1}\right| = 4k \qquad \therefore\ \left|\dfrac{(5+3u)+i3v}{(u+1)+iv}\right| = 4k$

$\therefore\ (5+3u)^2 + 9v^2 = 16k^2[(u+1)^2 + v^2]$

$\therefore\ 25 + 30u + 9u^2 + 9v^2 = 16k^2(u^2 + 2u + 1 + v^2)$

$\therefore\ (16k^2 - 9)(u^2 + v^2) + (32k^2 - 30)u + (16k^2 - 25) = 0$ which is a circle.

Remark ....

We have already proved that under a bilinear transformation, in general, circles in the z-plane are mapped onto the circles in the w-plane.

Example 2 : Under the transformation $w = \dfrac{z-1}{z+1}$, show that the map of the straight line $y = x$ is a circle and find its centre and radius. **(M.U. 1994, 2002)**

Sol. : We have $w = \dfrac{z-1}{z+1}$ $\therefore$ We express z in terms of w as follows **(Note this)**.

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By componendo and dividendo

$$\frac{w+1}{1-w} = z \qquad \therefore \frac{(u+iv)+1}{1-(u+iv)} = x+iy$$

$$\therefore \frac{(u+1)+iv}{(1-u)-iv} \cdot \frac{(1-u)+iv}{(1-u)+iv} = x+iy$$

$$\therefore \frac{(1-u^2)+iv(1-u)+iv(1+u)-v^2}{(1-u)^2+v^2} = x+iy$$

$$\therefore \frac{(1-u^2-v^2)+2iv}{1+u^2+v^2-2u} = x+iy$$

$\therefore$ Equating real and imaginary parts

$$x = \frac{1-u^2-v^2}{1+u^2+v^2-2u}, \quad y = \frac{2v}{1+u^2+v^2-2u}$$

Since $y = x$, we get, $1-u^2-v^2 = 2v$

$$\therefore u^2+v^2+2v = 1 \qquad \therefore u^2+(v+1)^2 = 2$$

Hence, the map of $y = x$ is a circle with centre at $(0, -1)$ and radius $\sqrt{2}$.

Example 3 : Show that under the transformation $w = \frac{z-i}{z+i}$, real axis in the z-plane is mapped onto the circle $|w| = 1$. **(M.U. 1997, 2019)**

Sol. : We have $w = \frac{z-i}{z+i}$ $\therefore wz + wi = z - i$

$$\therefore wi + i = z - wz \qquad \therefore i(w+1) = z(1-w)$$

$$\therefore z = i\frac{(1+w)}{(1-w)} \qquad \therefore x+iy = i \cdot \frac{[1+(u+iv)]}{1-(u+iv)}$$

$$\therefore x+iy = i\left[\frac{(1+u)+iv}{(1-u)-iv} \cdot \frac{(1-u)+iv}{(1-u)+iv}\right] = i\left[\frac{(1-u^2)+iv(1+u)+iv(1-u)-v^2}{(1-u)^2+v^2}\right]$$

$$\therefore x = -\frac{2v}{(1-u)^2+v^2}, \quad y = \frac{1-u^2-v^2}{(1-u)^2+v^2}$$

For the real axis $y = 0$.

$$\therefore 1-u^2-v^2 = 0 \qquad \therefore u^2+v^2 = 1 \qquad \therefore |w| = 1$$

$\therefore$ The real axis $y = 0$ in the z-plane is mapped onto the unit circle $|w| = 1$ in the w-plane.

Example 4 : Show that the map of the real axis of the z-plane is a circle under the transformation $w = \frac{2}{z+i}$. Find its centre and the radius. **(M.U. 2014, 19)**

Sol. : We have $w = \frac{2}{z+i}$ $\therefore z+i = \frac{2}{w}$

$$\therefore (x+iy)+i = \frac{2}{u+iv} = \frac{2}{(u+iv)} \cdot \frac{(u-iv)}{(u-iv)} \qquad \therefore x+i(y+1) = \frac{2(u-iv)}{u^2+v^2}$$

$$\therefore x = \frac{2u}{u^2+v^2} \text{ and } y+1 = -\frac{2v}{u^2+v^2}$$

For the real axis *i.e.* the x-axis, $y = 0$.

$$\therefore 1 = -\frac{2v}{u^2+v^2} \qquad \therefore u^2+v^2+2v = 0 \quad \therefore u^2+(v+1)^2 = 1$$

$\therefore$ The map is a circle with centre at $(0, -1)$ and radius 1.

Example 5 : Prove that the transformation $w = \frac{1}{z+i}$ transforms the real axis of the z-plane into a circle in the w-plane. **(M.U. 2005, 13, 15)**

Sol. : We have $w = \frac{1}{z+i}$

$$\therefore z+i = \frac{1}{W} = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv} = \frac{u-iv}{u^2+v^2}$$

$$\therefore x+iy+i = \frac{u-iv}{u^2+v^2}$$

$$\therefore x+i(y+1) = \frac{u}{u+v^2} - i \cdot \frac{v}{u^2+v^2}$$

Equating real and imaginary parts, $x = \frac{u}{u^2+v^2}$ and $y+1 = -\frac{v}{u^2+v^2}$.

Now, the real axis in the z-plane is given by $y = 0$.

Putting $y = 0$ in the second part,

$$1 = -\frac{v}{u^2+v^2} \qquad \therefore u^2+v^2+v = 0 \qquad \therefore u^2+\left(v+\frac{1}{2}\right)^2 = \left(\frac{1}{2}\right)^2$$

This is a circle with centre $\left(0, -\frac{1}{2}\right)$ and radius $\frac{1}{2}$ in the w-plane.

Hence, the result.

Example 6 : Find the mapping of the y-axis under the transformation $w = \frac{1}{kz+1}$, where k is real. **(M.U. 2006)**

Sol. : We have $w = \frac{1}{kz+1}$ $\therefore kz+1 = \frac{1}{w}$

$$\therefore k(x+iy)+1 = \frac{1}{u+iv} = \frac{1}{u+iv} \cdot \frac{u-iv}{u-iv}$$

$$\therefore (kx+1)+iky = \frac{u}{u^2+v^2} - \frac{iv}{u^2+v^2}$$

$$\therefore kx+1 = \frac{u}{u^2+v^2} \text{ and } ky = -\frac{v}{u^2+v^2}$$

For the y-axis, $x = 0$

$$\therefore 1 = \frac{u}{u^2+v^2} \quad \therefore u^2+v^2-u = 0$$

$$\therefore \left(u^2-u+\frac{1}{4}\right)+v^2 = \frac{1}{4} \qquad \therefore \left(u-\frac{1}{2}\right)^2+v^2 = \left(\frac{1}{2}\right)^2$$

$\therefore$ The map of the y-axis is a circle with centre at $(1/2, 0)$ and radius $= 1/2$.

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Example 7 : Show that $w = i\left(\frac{1-z}{1+z}\right)$ transforms the circle $|z| = 1$ onto the real axis of the w-plane and the interior of the circle $|z| < 1$ onto the upper half of the w-plane. **(M.U. 2003, 05)**

Sol. : We have $w = i\left(\frac{1-z}{1+z}\right)$. $\therefore\ w + wz = i - iz \quad \therefore\ z(w+i) = i - w$

$$\therefore\ z = \frac{i-w}{i+w} = \frac{i-(u+iv)}{i+(u+iv)} = \frac{i(-v+1)-u}{i(v+1)+u}$$

Since $|z| = 1$, we get $\left|\frac{i(-v+1)-u}{i(v+1)+u}\right| = 1$

$\therefore\ |i(-v+1)-u| = |i(v+1)+u| \qquad \therefore\ (1-v)^2 + u^2 = (1+v)^2 + u^2$

$\therefore\ 1 - 2v + v^2 + u^2 = 1 + 2v + v^2 + u^2 \qquad \therefore\ -4v = 0$ ..... (A)

$\therefore\ v = 0$

$\therefore$ The circle $|z| = 1$ is mapped onto the real axis in w-plane.

Now when $|z| < 1$ we have from the above relation (A).

$-4v < 0$ *i.e.* $-v < 0$ *i.e.* $v > 0$ *i.e.* the upper half of the w-plane.

Example 8 : Prove that $w = i\left(\frac{z-i}{z+i}\right)$ maps upper half of the z-plane into the interior of the unit circle in the w-plane.

Sol. : Since $w = i\left(\frac{z-i}{z+i}\right)$, we get $zw + iw = iz + 1 \quad \therefore\ z(w-i) = 1 - iw$

$$\therefore\ z = \frac{1-iw}{w-i} \quad \therefore\ x + iy = \frac{1-i(u+iv)}{u+iv-i}$$

$$\therefore\ x + iy = \frac{(1+v)-iu}{u+i(v-1)} \cdot \frac{u-i(v-1)}{u-i(v-1)} = \frac{(1+v)u - u(v-1)}{u^2+(v-1)^2} - i \cdot \frac{u^2+v^2-1}{u^2+(v-1)^2}$$

$$\therefore\ y = -\frac{u^2+v^2-1}{u^2+(v-1)^2}$$

For the upper half of the z-plane $y > 0$,

$$\therefore\ -\frac{(u^2+v^2-1)}{u^2+(v-1)^2} > 0 \quad \therefore\ -(u^2+v^2-1) > 0$$

$$\therefore\ u^2+v^2-1 < 0 \quad \therefore\ u^2+v^2 < 1 \quad \therefore\ |w| < 1$$

$\therefore$ The upper of the z-plane is mapped onto the interior of the unit circle with centre at the origin.

EXERCISE - VI

1. Find the image of the real axis in the z-plane onto the w-plane under the transformation $w = \frac{1}{z+i}$. **(M.U. 2005, 13)**

[**Ans. :** $z + i = \frac{1}{w} = \frac{u-iv}{u^2+v^2}$, $y + 1 = -\frac{v}{u^2+v^2}$ when $y = 0$, $u^2 + v^2 + v = 0$.
The circle withe centre $(0, -1/2)$ and radius $1/2$.]

2. Show that under the transformation $w = \frac{i-z}{i+z}$, the circle $|z| = 1$ is mapped onto the imaginary axis of the w-plane.

[**Ans. :** $u + iv = \frac{-(x^2+y^2)+1+2ix}{x^2+y^2+2y+1}$ when $x^2 + y^2 = 1$, $u = 0$
i.e. the imaginary axis.]

3. Show that under the transformation $w = \frac{2z+3}{z-4}$, the circle $x^2 + y^2 = 4x$ in the z-plane is transformed into the straight line $4u + 3 = 0$ in the w-plane. **(M.U. 2003, 07)**

[**Ans. :** $u + iv = \frac{2(x^2+y^2)-5x-12-11iy}{x^2+y^2-8x+16}$

when $x^2 + y^2 = 4x$, $u + iv = \frac{(3x-12)-11iy}{16-4x} \quad \therefore\ 4u + 3 = 0$.]

4. Show that under the transformation $w = \frac{5-4z}{4z-2}$ the circle $|z| = 1$ in the z-plane is transformed into a circle of unity in the w-plane. Also find the centre of the circle. **(M.U. 1998)**

[**Ans. :** $z = \frac{2w+5}{4(w+1)} \quad \therefore |z| = 1$ gives $\left|\frac{2w+5}{w+1}\right| = 4$

$\therefore\ (2u+5)^2 + 4v^2 = 16\left[(u+1)^2 + v^2\right]$

$\therefore\ u^2 + v^2 + u - 3/4 = 0$. Centre $\left(-\frac{1}{2}, 0\right)$, $r = 1$.]

5. Show that the transformation $w = \frac{iz+2}{4z+i}$ maps the real axis in the z-plane into a circle in the w-plane. Find the centre and radius of the circle. Also find the points on the z-plane which is mapped on to the centre of the circle. **(M.U. 2000, 02, 08)**

[**Ans. :** Centre $(0, 7/8)$, radius $= 9/8$; $(0, -1/4)$]

6. Find the image of the circle $x^2 + y^2 = 1$, under the transformation $w = \frac{5-4z}{4z-2}$.

(M.U. 2002, 03, 12) [**Ans. :** Circle with centre $(-1/2, 0)$ and radius 1.]

7. Find the image of $|z| < 1$ under the transformation $w = \frac{1+iz}{1-iz}$. **(M.U. 2004)**

[**Ans. :** $u > 0$]

9. To Find Bilinear Transformation $w = \frac{az+b}{cz+d}$

Example 1 : Find the bilinear transformation which maps $z = 2, 1, 0$ onto $w = 1, 0, i$. **(M.U. 1998, 2002)**

Sol. : Let the transformation be $w = \frac{az+b}{cz+d}$ ..... (1)

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Putting the given values of z and w we get,

$$1 = \frac{2a+b}{2c+d}, \quad 0 = \frac{a+b}{c+d}, \quad i = \frac{b}{d}$$

From the third we get $b = di$.

From the second we get $a = -b = -di$

Putting these values in the first, $2c + d = 2a + b$, we get $2c + d = -2di + di$

$\therefore\ 2c = -di - d \quad \therefore\ c = -(i+1)\,d/2$

Hence, from (1), we get,

$$w = \frac{-diz + di}{\{-(i+1)dz/2\} + d} = \frac{2(-iz+i)}{-(i+1)z+2} = \frac{2(z-1)}{(1-i)z+2i}.$$

Note ....

You can verify the transformation by putting the values of z.

Example 2 : Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. **(M.U. 1996, 2002, 03, 13, 15)**

Hence, find the fixed points of the transformation and the image of $|z| < 1$. **(M.U. 2004, 06, 07, 08, 10, 19)**

Sol. : Let the transformation be $\quad w = \dfrac{az+b}{cz+d}$ ..... (1)

Putting the given values of z and w, we get,

$$i = \frac{a+b}{c+d}, \quad 0 = \frac{ai+b}{ci+d}, \quad -i = \frac{-a+b}{-c+d}$$

From these equalities, we get,

$(a+b) - i(c+d) = 0$ ..... (2) ; $\quad b + ia = 0$ ..... (3)

$(-a+b) + i(-c+d) = 0$ ..... (4)

Adding (2) and (4), we get, $2b - 2ic = 0 \quad \therefore\ c = \dfrac{b}{i}$.

But from (3), $\quad b = -ia \quad \therefore\ c = -a$.

Substracting (4) from (2), we get, $2a - 2id = 0 \quad \therefore\ d = \dfrac{a}{i} = -ia$.

Putting the values $b = -ia$, $c = -a$ and $d = -ia$ in (1) we get

$$w = \frac{az - ia}{-az - ia} = \frac{z-i}{-z-i}$$

$\therefore\ w = \dfrac{i-z}{i+z}$ is the required bilinear transformation.

Now, the fixed points of the transformation are given by

$$\frac{i-z}{i+z} = z \quad \therefore\ i - z = iz + z^2 \quad \therefore\ z^2 + (1+i)z - i = 0$$

$$\therefore\ z = \frac{-(1+i) \pm \sqrt{(1+i)^2 + 4i}}{2} = \frac{-(1+i) \pm \sqrt{6i}}{2}$$

For the second part refer to the next Ex. 3.

Example 3 : Find the bilinear transformation which maps the points $z = 1, i, -1$ onto the points $w = i, 0, -i$. Hence, find the image of $|z| < 1$ onto the w-plane. **(M.U. 1999, 2003, 04, 09, 18, 19)**

Sol. : As proved in the above example,

$$w = \frac{az - ai}{-az - ai} = \frac{i-z}{i+z} \quad \text{..... (1)}$$

Now, from (1) we get $wi + wz = i - z$.

$\therefore\ wi - i = -z(1+w) \quad \therefore\ z = i \cdot \dfrac{1-w}{1+w}$

Further, $|z| < 1$ is mapped onto the region

$$\left| i\frac{(1-w)}{(1+w)} \right| < 1 \quad \therefore\ |i|\frac{|1-w|}{|1+w|} < 1$$

$\therefore\ |1-w| < |1+w| \qquad [\because |i| = 1]$

$\therefore\ |(1-u) - iv| < |(1+u) + iv|$

$\therefore\ (1-u)^2 + v^2 < (1+u)^2 + v^2 \quad \therefore\ -2u < 2u$

$\therefore\ -4u < 0 \quad \therefore\ -u < 0 \quad \therefore\ u > 0$

Hence, the interior of the circle $x^2 + y^2 = 1$ in the z-plane is mapped onto entire half of the w-plane to the right of imaginary axis.

The regions in the z-plane and w-plane are shown below.

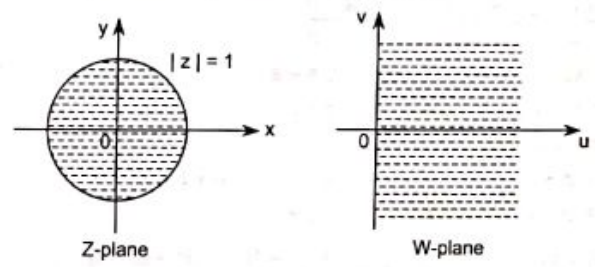

Fig. 4.20 (a) **Fig. 4.20 (b)**

Example 4 : Find the bilinear transformation which maps the points $1, -i, 2$ on z-plane onto $0, 2, -i$ respectively of w-plane. **(M.U. 1993)**

Sol. : Let the transformation be $\quad w = \dfrac{az+b}{cz+d}$ ..... (1)

Putting the given values, $\quad 0 = \dfrac{a+b}{c+d}, \quad 2 = \dfrac{-ai+b}{-ci+d}, \quad -i = \dfrac{2a+b}{2c+d}$

From these equations, we get, $\quad a + b = 0$ ..... (2)

$(a - 2c)\,i + (2d - b) = 0$ ..... (3)

$(2c + d)\,i + (2a + b) = 0$ ..... (4)

From (2) we get $b = -a$.

Putting this value of b in (3) and (4), we get

$(a - 2c)\,i + (2d + a) = 0$ ..... (5)

$(2c + d)\,i + a = 0$ ..... (6)

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Adding (5) and (6), we get,

$(a+d)i + 2(a+d) = 0 \quad \therefore (a+d)(i+2) = 0$

$\therefore d = -a. \quad [\because i \neq -2]$

Putting the values of d and b in $2 = \dfrac{-ai+b}{-ci+d}$, we get $2 = \dfrac{-ai-a}{-ci-a} = \dfrac{a(1+i)}{ci+a}$.

$\therefore 2ci + 2a = a + ai \quad \therefore 2ci = -a + ai \quad \therefore 2ci = ai^2 + ai = ai(1+i)$

$\therefore 2c = (1+i)a \quad \therefore c = \left(\dfrac{1+i}{2}\right)a$

Putting the values of b, c, d in (1),

$$w = \frac{az-a}{\left(\frac{1+i}{2}\right)az - a} = \frac{z-1}{\left(\frac{1+i}{2}\right)z - 1} \quad \therefore w = \frac{2(z-1)}{(1+i)z-2}$$

Example 5 : Find the bilinear transformation under which $1, i, -1$ from the z-plane are mapped onto $0, 1, \infty$ of w-plane.

Further show that under this transformation the unit circle in w-plane is mapped onto a straight line in the z-plane. Write the name of this line. **(M.U. 1997, 2007, 14)**

Sol. : Let the transformation be $w = \dfrac{az+b}{cz+d}$

Putting the given values of z and w, we get,

$$0 = \frac{a+b}{c+d}, \quad 1 = \frac{ai+b}{ci+d}, \quad \infty = \frac{-a+b}{-c+d}$$

From the first we get, $a + b = 0 \quad \therefore b = -a$

From the last we get, $-c + d = 0 \quad \therefore d = c$

From the second we get,

$ai + b = ci + d \quad \therefore ai - a = ci + c \quad \therefore ai + b = ci + d$

$\therefore ai - a = ci + c \quad \therefore a(i-1) = c(i+1)$

$\therefore c = a\dfrac{(i-1)}{(i+1)} \quad \therefore c = a\dfrac{(i-1)}{(i+1)} \cdot \dfrac{(i-1)}{(i-1)} = a\dfrac{(i^2-2i+1)}{i^2-1} = ai$

$\therefore d = c = ai \quad \therefore w = \dfrac{az-a}{aiz+ai} = \dfrac{z-1}{i(z+1)} = -i\dfrac{(z-1)}{(z+1)}$

Now, when $|w| = 1$, $\left|-i \cdot \dfrac{(z-1)}{(z+1)}\right| = 1$

$\therefore |z-1| = |z+1| \quad [\because |i| = 1]$

$\therefore |(x-1) - iy| = |(x+1) + iy|$

$\therefore (x-1)^2 + y^2 = (x+1)^2 + y^2$

$-2x = 2x \quad \therefore 4x = 0 \quad \therefore x = 0$

Hence, the map is the y-axis.

Example 6 : Find bilinear transformation that maps $0, 1, \infty$ of the z-plane onto $-5, -1, 3$ of w-plane. **(M.U. 2004, 13, 15)**

Sol. : Let the transformation be $w = \dfrac{az+b}{cz+d}$.

Putting the given values of z and w.

(i) When $z = 0$, $w = -5$, we get $-5 = \dfrac{b}{d} \quad \therefore b = -5d$ ..... (1)

(ii) When $z = 1$, $w = -1$, we get $-1 = \dfrac{a+b}{c+d} \quad \therefore a + b = -c - d$ ..... (2)

(iii) When $z = \infty$, $w = 3$.

Now, $w = \dfrac{a+(b/z)}{c+(d/z)} \quad \therefore 3 = \dfrac{a}{c} \quad \therefore a = 3c$ ..... (3)

From (2), we have

$a = -b - c - d = 5d - c - d = -c + 4d$

From (3), we have

$3c = a = -c + 4d \quad \therefore 4c = 4d \quad \therefore c = d.$

But, $a = 3c = 3d. \quad \therefore w = \dfrac{az+b}{cz+d} = \dfrac{3dz-5d}{dz+d} = \dfrac{3z-5}{z+1}$.

Example 7 : Find the bilinear transformation which maps the points $2, i, -2$ onto the points $1, i, -1$ **(M.U. 2014)**

Sol. : Let the required transformation be $w = \dfrac{az+b}{cz+d}$

By data, when $z = 2, i, -2$, $w = 1, i, -1$.

Putting these values, we get

$1 = \dfrac{2a+b}{2c+d} \quad \therefore 2a + b = 2c + d \quad \therefore 2a + b - 2c - d = 0$ ..... (1)

$i = \dfrac{ai+b}{ci+d} \quad \therefore ai + b = -c + di \quad \therefore ai + b + c - di = 0$ ..... (2)

$-1 = \dfrac{-2a+b}{-2c+d} \quad \therefore -2a + b = 2c - d \quad \therefore -2a + b - 2c + d = 0$ ..... (3)

From (1) and (3), we get $2b - 4c = 0 \quad \therefore b = 2c$

Multiply (1) by $i \quad \therefore 2ai + bi - 2ci - di = 0$ ..... (4)

Subtract (2) from (4) $ai + b(i-1) - c(2i+1) = 0$

But $b = 2c$, $\quad \therefore ai + 2c(i-1) - c(2i+1) = 0$

$\therefore ai - 3c = 0 \quad \therefore a = 3c/i = -3ci$

Putting the values of a and b in (1), we get

$-6ci + 2c - 2c - d = 0 \quad \therefore d = -6ci$

Putting the values of a, b, d

$$w = \frac{-3ciz + 2c}{cz - 6ci} = \frac{-3zi + 2}{z - 6i}$$

Multiply by i in the numerator and denominator of r.h.s.

$$w = \frac{-3z(i)^2 + 2i}{zi - 6(i)^2} = \frac{3z + 2i}{zi + 6}.$$

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EXERCISE - VII

1. Find the bilinear transformation which maps the points $z = 1, -i, -1$ onto the points $w = i, 0, -i$.
(M.U. 2019) [Ans. : $w = \dfrac{z+i}{z-i}$]

2. Find the bilinear transformation which maps the points
(i) $z = 0, -i, -1$ onto the points $w = i, 1, 0$. Is the transformation parabolic ? (M.U. 2008)
(ii) $z = 1, -1, \infty$ onto the points $w = 1+i, 1-i, 1$.
[Ans. : (i) $w = \dfrac{i(1+z)}{1-z}$; No, (ii) $w = \dfrac{z+i}{z}$]

3. Find the bilinear transformation which maps the points
(i) $z = -1, 1, \infty$ onto the points $w = -i, -1, i$. (M.U. 2003, 08, 14) [Ans. : $w = \dfrac{iz-2+i}{z+1-2i}$]
(ii) $z = 2, 1, 0$ onto the points $w = 1, 0, i$. (M.U. 2002) [Ans. : $w = \dfrac{2i(z-1)}{z(1+i)-2}$]
(iii) $z = -1, 0, 1$ onto the points $w = -1, -i, 1$. (M.U. 2003) [Ans. : $w = \dfrac{z-i}{1-iz}$]
(iv) $z = -i, 0, i$ onto the points $w = -1, i, 1$. (M.U. 2005) [Ans. : $w = \left(\dfrac{1-z}{1+z}\right)\cdot i$]
(v) $z = i, -1, 1$ onto the points $w = 0, 1, \infty$. (M.U. 2003, 19) [Ans. : $w = \dfrac{2(z-i)}{(1+i)(z-1)}$]
(vi) $z = 0, 1, \infty$ onto the points $w = -5, -1, 3$. (M.U. 2004, 13, 15) [Ans. : $w = \dfrac{3z-5}{z+1}$]
(vii) $z = -2, i, 2$ onto the points $w = 0, i, -i$. (M.U. 2004) [Ans. : $w = \dfrac{z+2}{i(3z-2)}$]

4. Find the bilinear transformation which maps $z = \infty, i, 0$ onto the points $w = 0, i, \infty$.
(M.U. 1998, 2006, 09) [Ans. : $w = -1/z$]

5. Find the bilinear transformation which maps the points $0, i, -2i$ of z-plane onto the points $-4i, \infty, 0$ respectively of w-plane. Also obtain fixed points of the transformation.
(M.U. 1995, 2006) [Ans. : $w = -\dfrac{2z+4i}{iz+1}$, $z = -\dfrac{4}{i}, -i$]

6. Find the bilinear transformation which maps the points $z = 0, i, -1$ onto $w = i, 1, 0$.
(M.U. 1999) [Ans. : $w = \dfrac{z+1}{(2-i)z-i}$]

EXERCISE - VIII

Theory

1. Define conformal mapping and show that the mapping defined by an analytic function is conformal at all points where $f'(z) \neq 0$. (M.U. 1993)

2. State and prove that cross-ratio preservation property of a bilinear transformation. Hence, find the bilinear transformation which maps the points $z = 2, i, -2$ onto the points $w = 1, i, -1$.
(M.U. 1993, 2002) [Ans. : $w = \dfrac{-3iz+2}{z-6i}$]

3. Define the cross-ratio of the numbers z_1, z_2, z_3 and z_4. Prove that the cross ratio remains invariant under a bilinear transformation. (M.U. 1994, 97, 2002)

4. If w_1, w_2, w_3, w_4 are distinct images of z_1, z_2, z_3, z_4 (all different) under the transformation $w = \dfrac{az+b}{cz+d}$, $(ad - bc \neq 0)$, then show that $\dfrac{(z_1-z_2)}{(z_2-z_3)}\cdot\dfrac{(z_3-z_4)}{(z_4-z_1)} = \dfrac{(w_1-w_2)}{(w_2-w_3)}\cdot\dfrac{(w_3-w_4)}{(w_4-w_1)}$.
(M.U. 1995)

5. Prove that, in general, a bilinear transformation maps a circle into a circle. (M.U. 1996)

6. Prove that a Bilinear transformation with single fixed point α can be put in the form $\dfrac{1}{w-\alpha} = \dfrac{1}{z-\alpha} + k$ where k is a constant. (M.U. 1996)

7. Define bilinear transformation and prove that under this transformation the cross-ratio is invariant. (M.U. 1998, 2002, 03)

8. Out of the following two statements one statement is true. Prove that statement.
(i) Every bilinear transformation is conformal.
(ii) If u and v are harmonic functions then $f(z) = u + iv$ is analytic function. (M.U. 1995)

9. If $f(z)$ is analytic and $f'(z) \neq 0$ in R, prove that $w = f(z)$ is conformal in R. (M.U. 1997)

10. Define bilinear transformation. Show that every bilinear trans-formation is a combination of (i) translation, (ii) Rotation and Magnification, (iii) Inversion. (M.U. 1999)

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Module 5

COVARIANCE, FITTING OF EXPONENTIAL CURVE

1. Covariance

Refer § 7, page 8-25.

2. Fitting Exponential Curve

If we have to fit an exponential curve of the form $y = ax^b$ or $y = ab^x$ or $y = ae^x$, we take logarithm of both sides. Then the given exponential equation transforms into a linear equation. We then use the above technique.

Suppose the law as $y = ab^x$.

Taking logarithm of both sides to the base 10, we get

$$\log_{10} y = \log_{10} a + x \log_{10} b$$

Putting $\log_{10} y = Y$, $\log_{10} a = A$, $x = X$, $\log_{10} b = B$, we get

$$Y = A + BX$$

This is, a linear equation in X and Y and we can obtain the constants A and B by the method explained in § 2 and resubstituting for A and B, we can obtain the equation $y = ab^x$.

If the law is $y = ax^b$ taking logarithm of both sides, we get

$$\log y = \log a + b \log x.$$

Putting $\log y = Y$, $\log a = A$, $b = B$ and $\log x = X$, we get

$$Y = A + BX.$$

Again using the technique of § 2, we can find A and B and then a and b.

Example 1 : Fit a curve $y = ab^x$ to the following data, using the method of least squares.

x :	2	3	4	5	6
y :	144	172·8	207·4	248·8	298·5

Sol. : Taking logarithms of $y = ab^x$, we get $\log y = \log a + (\log b)\, x$. Let us put $\log y = Y$, $\log a = A$, $\log b = B$ and $x = X$. Then the law becomes $Y = A + BX$.

Calculations of ΣX, ΣY, etc.

Sr. No.	x	y	X	Y	X^2	XY
1	2	144·0	2	2·1584	4	4·3168
2	3	172·8	3	2·2375	9	6·7125
3	4	207·4	4	2·3168	16	9·2672
4	5	248·8	5	2·3959	25	11·9795
5	6	298·5	6	2·4749	36	14·8494
$N = 5$			20	11·5835	90	47·1254

Now, we have

$$\Sigma Y = NA + B\Sigma X \qquad \therefore\ 11{\cdot}5835 = 5A + 20B$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \qquad \therefore\ 47{\cdot}1254 = 20A + 90B$$

Solving these equations, we get,

$$B = 0{\cdot}07914 \quad \text{and} \quad A = 2{\cdot}0001$$

Hence, b = antilog 0·07914 = 1·199, a = antilog 2·0001 = 100.

$\therefore$ The law is $y = 100\,(1{\cdot}199)^x$.

Example 2 : Find the law of the form $y = ab^x$ to the following data. **(M.U. 2005)**

x :	1	2	3	4	5	6	7	8
y :	1	1·2	1·8	2·5	3·6	4·7	6·6	9·1

Sol. : Taking logarithm of both sides of $y = ab^x$, we get

$$\log y = \log a + x \log b.$$

Putting $\log y = Y$, $\log a = A$, $x = X$, $\log b = B$, we get $Y = A + BX$.

We, now, form the following table.

Calculations of ΣX, ΣY, etc.

Sr. No.	$X = x$	y	$Y = \log y$	XY	X^2
1	1	1·0	0·0000	0·0000	1
2	2	1·2	0·0792	0·1584	4
3	3	1·8	0·2553	0·7689	9
4	4	2·5	0·3979	1·5916	16
5	5	3·6	0·5563	2·7815	25
6	6	4·7	0·6721	4·0326	36
7	7	6·6	0·8195	5·7365	49
8	8	9·1	0·9590	7·6720	64
$N = 8$	36		3·7393	22·7385	204

Now, $\quad Y = A + BX$

$$\therefore\ \Sigma Y = NA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Putting $N = 8$, $\Sigma X = 36$, $\Sigma Y = 3{\cdot}7393$, $\Sigma XY = 22{\cdot}7385$, $\Sigma X^2 = 204$, we get

$$3{\cdot}7393 = 8A + 36B$$

$$22{\cdot}7385 = 36A + 204B$$

Solving the two equations, we get,

$$A = -0{\cdot}1657, \quad B = 0{\cdot}1407$$

$\therefore\ a$ = anti-log $(-0{\cdot}1657) = 0{\cdot}6828$

b = anti-log $(0{\cdot}1407) = 1{\cdot}3828$

Hence, the required law correct to two places of decimals is

$$y = 0{\cdot}68\,(1.38)^x.$$

Example 3 : Fit a curve of the form $y = ax^b$ to the following data.

x :	1	2	3	4	5	6
y :	120	90	60	20	11	5

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Sol. : Taking logarithms of both sides of $y = ax^b$, we get

$$\log y = \log a + b \log x.$$

Putting $\log y = Y$, $\log a = A$, $b = B$, $\log x = X$, we get

$$Y = A + BX.$$

Calculations of ΣX, ΣY, etc.

Sr. No.	x	y	$X = \log x$	$Y = \log y$	XY	X^2
1	1	120	0·0000	2·0792	0·0000	0·0000
2	2	90	0·3010	1·9542	0·5882	0·0906
3	3	60	0·4771	1·7782	0·8484	0·2276
4	4	20	0·6021	1·3010	0·7833	0·3625
5	5	11	0·6990	1·0414	0·7279	0·4886
6	6	5	0·7781	0·6990	0·5439	0·6054
$N = 6$			2·8573	8·8530	3·4917	1·7747

Now, the normal equations are

$$\Sigma Y = NA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Putting the values from the table

$$8{\cdot}8530 = 6A + 2{\cdot}8573B \quad \text{..... (1)}$$

$$3{\cdot}4917 = 2{\cdot}8573A + 1{\cdot}7747B \quad \text{..... (2)}$$

Multiply (1) by 2·8573 and (2) by 6 and subtract.

$$25{\cdot}2957 = 17{\cdot}1438A + 8{\cdot}1642B$$

$$20{\cdot}9502 = 17{\cdot}1438A + 10{\cdot}6482B$$

$$4.3455 = -2.4840B$$

$\therefore$ $B = -1{\cdot}7494$

Putting this value of B in (1), we get

$6A = 8{\cdot}8530 + 4{\cdot}9991$ $\quad\therefore$ $A = 2{\cdot}3087$

$\therefore$ $a = \text{antilog } A = \text{antilog } 2{\cdot}3087 = 203{\cdot}56$

and $b = B = -1{\cdot}7494$

Hence, we get $y = 203{\cdot}86x^{-1{\cdot}494}$.

Example 4 : Fit a curve of the form $y = ab^x$ to the following data.

x :	1	2	3	4	5	6
y :	151	100	61	50	20	8

(M.U. 2017)

Sol. : Let the equation of the curve be $y = ab^x$.

Taking logarithms of both sides, we get

$$\log y = \log a + x \log b$$

Let $\log y = Y$, $\log a = A$, $\log b = B$, $X = x$.

$\therefore$ $Y = A + BX$.

Calculations of ΣX, ΣY, etc.

Sr. No.	$X = x$	y	$Y = \log y$	X^2	XY
1	1	151	2·1790	1	2·1790
2	2	100	2·0000	4	4·0000
3	3	61	1·7853	9	5·3559
4	4	50	1·6990	16	6·7960
5	5	20	1·3010	25	6·5050
6	6	8	0·9031	36	5·4186
$N = 6$	21		9·8674	91	30·2545

Now, the normal equations are

$$\Sigma Y = NA + B\Sigma X$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2$$

Putting the above values, we get,

$$9{\cdot}8674 = 6A + 21B \quad \text{..... (1)}$$

$$30{\cdot}2545 = 21A + 91B \quad \text{..... (2)}$$

Multiply (1) by 7 and (2) by 2 and subtract.

$$69{\cdot}0718 = 42A + 147B$$

$$60{\cdot}5090 = 42A + 182B$$

$$8{\cdot}5628 = -35B$$

$\therefore$ $B = -0{\cdot}2447$

Putting the value of B in (1), we get

$$9{\cdot}8674 = 6A + 21(-0{\cdot}2447)$$

$\therefore$ $6A = 9{\cdot}8674 + 5{\cdot}1387$

$6A = 15{\cdot}0061$ $\quad\therefore$ $A = 2{\cdot}5010$

$\therefore$ $a = \text{antilog } A = 316{\cdot}96$

$b = \text{antilog } B = 0{\cdot}5692$

$\therefore$ $y = 316{\cdot}96\,(0{\cdot}5692)^x$.

EXERCISE

Class (c) : 8 Marks

1. Fit a curve of the form $y = ab^x$ to the following data.

x :	0	2	4
y :	5·012	10·000	31·620

[Ans. : $y = 4{\cdot}68\,(1{\cdot}58)^x$]

2. Fit a curve of the form $y = ax^b$ to the following data by the method of least square.

x :	1	2	3	4
y :	2·50	8·00	19·00	50·00

(M.U. 2019) [Ans. : $y = 2{\cdot}22\,x^{2{\cdot}09}$]

3. Fit a curve of the form $y = ax^b$ to the following data by the method of least squares.

x :	1	2	3	4
y :	76	210	390	605

[Ans. : $y = 75{\cdot}2\,x^{1{\cdot}51}$]

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4. Fit a curve of the form $y = ab^x$ to the following data using the method of least squares.

x :	2	3	4	5	6
y :	144	172·8	207·4	248·8	298·5

[**Ans.** : $y = 100\,(1{\cdot}999)^x$]

5. Fit a curve of the form $y = ab^x$ to the following data using the method of least squares.

x :	2	3	4	5	6
y :	34·385	79·0855	181·90	418·36	962·23

(M.U. 2007)

[**Ans.** : $y = 6{\cdot}5\,(2{\cdot}3)^x$]

Module 6

SKEWNESS AND KURTOSIS OF DISTRIBUTION (DATA)

1. Introduction

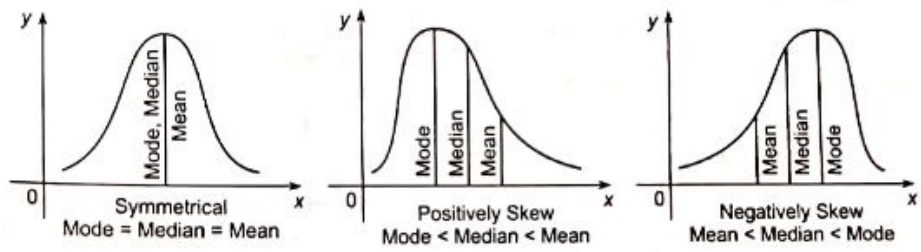
We know that raw data are voluminous and not easy to grasp. We condense them by calculating a measure of central tendency such as mean, median, mode, etc. and also by calculating a measure of dispersion such as range, quartile deviation, standard deviation, etc. But these two figures are not enough to enable us to draw sufficient inferences about data. A frequency distribution has two more characteristics symmetry and flatness. A graph of a frequency distribution may be symmetrical or skew. It can also be flat or peaked. These characteristics are measured from the coefficient of skewness and kurtosis respectively.

2. Skewness

It may happen that two distributions have the same mean and the same standard deviation; but one is symmetrical and the other is not. A distribution which is not symmetrical about the mode, is called skew. By skewness we mean the lack of symmetry of the distribution.

For a symmetrical distribution, the values, at equal distances on either side of the mode, have equal frequencies. As a result of this, the mean, mode and median-all coincide for a symmetrical distribution. The frequency curve of a symmetrical distribution rises slowly, reaches a maximum and falls *equally* slowly.

On the other hand, for a skew distribution, the mean, mode and median do not coincide. Skewness is called positive or negative according as mean and median are to the right or to the left of mode. A positively skew distribution curve rises rapidly, reaches a maximum and falls *slowly*. A negatively skew distribution curve rises slowly, reaches a maximum and falls *rapidly*. These curves are shown in following figures.



3. Measures of Skewness

Like measures of dispersion, measures of skewness are of two types (i) Absolute measures of skewness and (ii) Relative measures of skewness.

Since for a skew distribution mean and mode do not coincide, the difference between them can be used as a measure of skewness. Similarly, since for a skew distribution the two quartiles are not equidistant from the median, the difference between the distances of the two quartiles from the median can be used as a measure of skewness. The first method was suggested by Karl Pearson and the second by Bowley. The magnitude of the measure of skewness shows the extent of skewness and the sign shows the nature-positive or negative. The greater the measure, the greater is the

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skewness. If the measure is positive, skewness is positive, if the measure is negative, the skewness is negative.

(a) Measures of Skewness Based on Mean

Since in a skew distribution mean and mode do not coincide, the distance between them is used to measure skewness.

$$\boxed{\textbf{Absolute Skewness = Mean – Mode}}$$

If mode is ill-defined but the distribution is moderately skew, we use the formula : Mean – Mode = 3 (Mean – Median). In such a case,

$$\boxed{\textbf{Absolute Skewness = 3 (Mean – Median)}}$$

However, since these measures are expressed in the units of distribution, they are not very useful for comparing skewness of distributions which are measured in different units. Hence, for comparing two or more distributions we use another measure of skewness. It is called the relative measure of skewness and is obtained by dividing the above measures by standard deviation (σ). This is Karl Pearson's coefficient of skewness. It is defined as

$$\boxed{\textbf{Karl Pearson's coefficient of skewness} = \frac{\textbf{Mean – Mode}}{\sigma}}$$

In case the mode is ill-defined, the coefficient of skewness is obtained from the following formula.

$$\boxed{\textbf{Karl Pearson's coefficient of skewness} = \frac{\textbf{3 (Mean – Median)}}{\sigma}}$$

The value of this coefficient usually lies between + 1 and – 1. If the distribution is symmetrical, the mean and mode coincide and consequently the coefficient is zero. If the coefficient is positive the distribution is positively skew and if it is negative the distribution is negatively skew.

(d) Measures of Skewness Based on Quartiles

In a skew distribution, the two quartiles are not equidistant from the median. This property can be used to measure skewness of a distribution. The difference between the distances of the two quartiles from the median is defined as skewness.

Absolute skewness = (Q_1 – Median) – (Median – Q_3)

$$\boxed{\textbf{Absolute skewness} = Q_3 + Q_1 - 2\ \textbf{Median}}$$

This is an absolute measure and hence, cannot be used to compare skewness of two or more distributions. The corresponding relative measure of skewness called Bowley's coefficient of skewness is obtained by dividing the above measure by interquartile range.

$$\boxed{\textbf{Bowley's coefficient of skewness} = \frac{Q_3 + Q_1 - 2\ \textbf{Median}}{Q_3 - Q_1}}$$

As before, if this coefficient is zero, the distribution is symmetrical. If it is positive, the distribution is positively skew. If it is negative, the distribution is negatively skew. This coefficient also lies between + 1 and – 1.

If the coefficient of skewness is + 1, then

$$1 = \frac{Q_3 - Q_1 - 2\ \text{Median}}{Q_3 - Q_1} \qquad \therefore\ Q_3 - Q_1 = Q_3 + Q_1 - 2\ \text{Median} \qquad \therefore\ \text{Median} = Q_1.$$

If the coefficient of skewness is – 1, then

$$-1 = \frac{Q_3 + Q_1 - 2\ \text{Median}}{Q_3 - Q_1} \qquad \therefore\ -Q_3 + Q_1 = Q_3 + Q_1 - 2\ \text{Median} \qquad \therefore\ \text{Median} = Q_3.$$

Thus, when the coefficient of skewness is +1, the median coincides with the first quartile and when the coefficient of skewness is – 1, the median coincides with the third quartile.

Example 1 : Given that A.M. = 160, Mode = 157, S.D. = 50. Find (i) Karl Pearson's coefficient of skewness, (ii) Coefficient of variation.

Sol. : We have Karl Pearson's coefficient of

$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

But A.M. = 160, Mode = 157 and σ = 50.

$$\therefore \quad \text{Skewness} = \frac{160 - 157}{50} = \frac{3}{50} = 0.6$$

$$\text{Coefficient of variation} = \frac{\sigma}{\bar{x}} \times 100 = \frac{50}{160} \times 100 = 31{\cdot}25.$$

Example 2 : For a moderately skew distribution the mean, median and Karl Pearson's coefficient of skewness are 86, 80 and 0·42, respectively. Find the mode and coefficient of variation.

Sol. : We have, $\bar{x}$ = 86, Median = 80, Coefficient of skewness = 0.42

$$\text{Now, the coefficient of skewness} = \frac{3(\text{Mean} - \text{Median})}{\sigma}$$

Putting the given values

$$0.42 = \frac{3(86 - 80)}{\sigma} \qquad \therefore\ \sigma = \frac{18}{0{\cdot}42} = 42{\cdot}85$$

Further, the coefficient of

$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

$$\therefore \quad 0.42 = \frac{86 - \text{Mode}}{18/0{\cdot}42} \qquad \therefore\ 86 - \text{Mode} = 18$$

$$\therefore \quad \text{Mode} = 86 - 18 = 68.$$

Example 3 : For a given frequency distribution mean, mode and Karl Pearson's coefficient of skewness are 120, 123, – 3 respectively. Find C.V.

Sol. : We have Karl Pearson's coefficient of

$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\sigma}$$

But Mean = 120, mode = 123 and skewness = – 3

$$\therefore \quad -3 = \frac{120 - 123}{\sigma} \qquad \therefore\ -3 = -\frac{3}{\sigma} \qquad \therefore\ \sigma = 1.$$

$$\text{Coefficient of variation} = \frac{\sigma}{\bar{x}} \times 100 = \frac{1}{120} \times 100 = \frac{5}{6}.$$

Example 4 : From the data given below calculate Karl Pearson's coefficient of skewness and interpret your result.

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Wages (₹) : 70-80, 80-90, 90-100, 100-110, 110-120, 120-130, 130-140, 140-150

No. of persons : 12, 18, 35, 42, 50, 45, 20, 8.

Sol. : **Calculation of Karl Pearson's Coefficient of Skewness**

Wages	No. of persons	Mid-points	$m - 105$	$d / 10$		
x_i	f_i	m_i	d_i	d_i'	$f_i d_i'$	$f_i d_i'^2$
70-80	12	75	– 30	– 3	– 36	108
80-90	18	85	– 20	– 2	– 36	72
90-100	35	95	– 10	– 1	– 35	35
100-110	42	105	0	0	0	0
110-120	50	115	10	1	50	50
120-130	45	125	20	2	90	180
130-140	20	135	30	3	60	180
140-150	8	145	40	4	32	128
Total	230				125	753

$$\text{Mean, } \bar{x} = A + \frac{\Sigma f_i d_i}{N} \times h$$

Here, $A = 105$, $\Sigma f_i d_i = 125$, $N = 230$, $h = 10$

$$\therefore \quad \text{Mean, } \bar{x} = 105 + \frac{125}{230} \times 10 = 105 + 5{\cdot}43 = 110.43$$

$$\text{Mode} = L_1 + \frac{f_1 - f_0}{2f_1 - f_0 - f_2} \times i$$

Modal class is 110-120. Here, $L_1 = 110$, $f_1 = 50$, $f_0 = 42$, $f_2 = 45$, $i = 10$.

$$\therefore \quad \text{Mode} = 110 + \frac{50 - 42}{100 - 42 - 45} \times 10$$

$$= 110 + \frac{8}{13} \times 10 = 116{\cdot}15$$

$$\text{Standard deviation, } \sigma = \sqrt{\frac{\Sigma f_i d_i'^2}{N} - \left(\frac{\Sigma f_i d_i'}{N}\right)^2} \times h = \sqrt{\frac{753}{230} - \left(\frac{125}{230}\right)^2} \times 10$$

$$= \sqrt{3{\cdot}274 - 0{\cdot}294} \times 10 = \sqrt{2{\cdot}98} \times 10$$

$$= 1.755 \times 10 = 17.55$$

$$\text{Coefficient of skewness} = \frac{\text{Mean} - \text{Mode}}{\sigma} = \frac{110{\cdot}13 - 116{\cdot}5}{17{\cdot}55} = -\frac{5{\cdot}72}{17{\cdot}55} = -0{\cdot}326.$$

Since, the coefficient is negative, the distribution is negatively skewed.

Example 5 : From the following data, calculate Bowley's coefficient of skewness :

Weekly earnings in ₹	10-12,	12-14,	14-16,	16-18,	18-20,	20-22,	22-24,	24-26,
No. of employees	3,	6,	10,	15,	24,	42,	75,	90,
	26-28,	28-30,	30-32,	32-34,	34-36,	36-38,	38-40	
	79,	55,	36,	26,	16,	16,	7.	

Sol. : We have to calculate median and the two quartiles first.

$$\text{Median} = \text{size of } (N/2)^{th} \text{ item}$$

$$= \text{size of } 250^{th} \text{ item.}$$

$\therefore$ Median class is 24-26.

Calculation of Median

Weekly earnings in ₹ (x_i)	No. of employees (f_i)	c.f.
10-12	3	3
12-14	6	9
14-16	10	19
16-18	15	34
18-20	24	58
20-22	42	100
22-24	75	175
24-26	90	265
26-28	79	344
28-30	55	399
30-32	36	435
32-34	26	461
34-36	16	477
36-38	16	493
38-40	7	500
	$N = 500$	

$$\text{Median} = L_1 + \frac{(N/2) - \text{c.f.}}{f} \times i$$

$$= 24 + \frac{250 - 175}{90} \times 2$$

$$= 24 + \frac{75}{90} \times 2$$

$$= 24 + 1{\cdot}66$$

$\therefore$ Median = ₹ 25·66

The first quartile,

$$Q_1 = \text{Size of } (N/4)^{th} \text{ item}$$

$$= \text{Size of } 125^{th} \text{ item}$$

$$Q_1 = L_1 + \frac{(N/4) - \text{c.f.}}{f} \times i$$

$$= 22 + \frac{125 - 100}{75} \times 2$$

$$= 22 + \frac{25}{75} \times 2$$

$\therefore \quad Q_1$ = ₹ 22·66

The third quartile,

$$Q_3 = \text{Size of } (3N/4)^{th} \text{ item} = \text{Size of } 375^{th} \text{ item}$$

$$Q_3 = L_1 + \frac{(3N/4) - \text{c.f.}}{f} \times i = 28 + \frac{375 - 344}{55} \times 2$$

$$\therefore \quad Q_3 = 28 + \frac{31}{55} \times 2 = 28 + 1{\cdot}13 = ₹\ 29{\cdot}13.$$

Now, Bowley's Coefficient of Skewness

$$= \frac{Q_3 + Q_1 - 2\text{Median}}{Q_3 - Q_1} = \frac{29{\cdot}13 + 22{\cdot}66 - 2(25{\cdot}66)}{29{\cdot}13 - 22{\cdot}66}$$

$$= \frac{51{\cdot}79 - 51{\cdot}32}{6{\cdot}47} = \frac{0{\cdot}47}{6{\cdot}47} = 0{\cdot}073.$$

4. Kurtosis

The term kurtosis is derived from a Greek word meaning bulging or convexity. In addition to the characteristics considered so far *viz.* measures of central tendency, measures of dispersion and measures of skewness, we may need to know whether the frequency distribution curve is peaked or flat around the central value, usually the mode. This character of the curve is indicated by the measure of **Kurtosis**.

Two distributions may be perfectly symmetrical about the mode but one may be flat around the mode and the other maybe peaked around it. To understand this nature of a given frequency distribution curve, we compare it with a perfectly symmetrical and in a sense ideal curve which is

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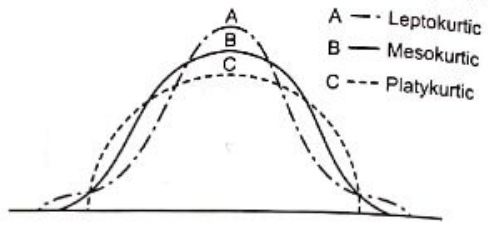
neither flat nor peaked, called the normal curve. If a curve is more peaked than the normal curve it is called **leptokurtic**. If one the other hand it is more flat around the mode than the normal, it is called, **platykurtic**. The normal curve itself is called, **mesokurtic**. In a platykurtic curve the values of the variable are clustered around the mode and in a mesokurtic curve the values are spread evenly around the mode. The name kurtosis was suggested by Karl Pearson.

The adjoining diagram shows the nature of the three types of curves. The curve A is more peaked than the normal curve and is leptokurtic. The curve B is the normal one and is mesokurtic. C is more flat and is platykurtic.



5. Measures of Kurtosis

Kurtosis is measured by the coefficient $\beta_2 = \frac{\mu_4}{\mu_2}$.

For a normal or mesokurtic curve $\beta_2 = 3$ and hence, $\beta_2 > 3$ the curve is leptokurtic and if $\beta_2 < 3$ the curve is platykurtic. For this reason, kurtosis can also be measured by the difference $\beta_2 - 3$ which is denoted by γ_2. Hence, $\gamma_2 = \beta_2 - 3$. For a normal or mesokurtic curve $\gamma_2 = 0$, for a leptokurtic curve γ_2 is positive and for platykurtic curve γ_2 is negative.

6. Person's β and γ Coefficients

Karl Pearson defined the following four coefficients, based upon the first four moments about mean :

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} \text{ and } \gamma_1 = \sqrt{\beta_1}; \qquad \beta_2 = \frac{\mu_4}{\mu_2^2} \text{ and } \gamma_2 = \beta_2 - 3$$

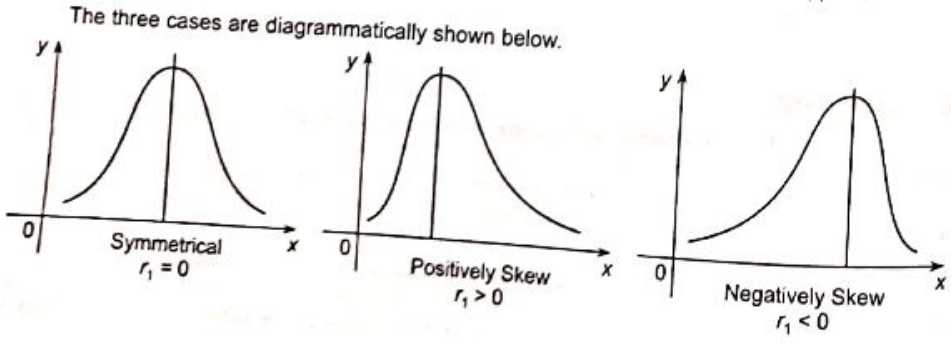
As seen above β_2 and γ_2 are used to measure kurtosis. β_1 and γ_1 are used to measure skewness.

Coefficient of skewness = $\gamma_1 = \sqrt{\beta_1}$ (with the sign of μ_3)

Three special cases

Three cases of r_1 deserve special attention. If $r_1 = 0$, *i.e.*, if $\mu_3 = 0$, it means skewness is zero. Hence, if $\gamma_1 = 0$, the curve is perfectly symmetrical. If $r_1 > 0$, skewness is greater than zero. Hence, if $\gamma_1 > 0$, the curve is positively skew. If $\gamma_1 < 0$, skewness is less than zero. Hence, if $\gamma_1 < 0$, the curve is negatively skew.

The three cases are diagrammatically shown below.



Further, since for symmetric distribution $\gamma_1 = 0$, which means $\beta_1 = 0$, *i.e.*, $\mu_3 = 0$. Now, from the relation for μ_3', we have

$$\mu_3' = \mu_3 + 3\,\mu_2\,\mu_1' + \mu_1'^3$$

Since, for symmetrical distribution $\mu_3 = 0$, we get

$$\mu_3' = 3\,\mu_2\,\mu_1' + \mu_1'^3, \text{ i.e., } \frac{\mu_3}{\mu_1'} = 3\,\mu_2 + \mu_1'^2$$

Note ....

We first see that μ_r has the dimension of (variate)^r. Hence, μ_2^3 has 6th dimension and μ_2^3 has also 6th dimension. Hence, $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ is of zeroth dimension and hence, is a pure number. Similarly, $\beta_2 = \frac{\mu_4}{\mu_2^2}$ is also of zeroth dimension. Hence, β_1, β_2 (and also γ_1 and γ_2) are pure numbers and as such are independent of scale and origin.

Example 1 : Second, third and fourth central moments of a variable are 19, 97, 29, 26, 866 respectively. Calculate the beta coefficients correct to three decimal places.

Sol. : We are given that $\mu_2 = 19{\cdot}67$, $\mu_3 = 29{\cdot}26$, $\mu_4 = 866$.

By definition,
$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(29{\cdot}26)^2}{(19{\cdot}67)^2} = 0{\cdot}113$$
$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{866}{(19{\cdot}67)^2} = 2{\cdot}24$$

Example 2 : For any frequency distribution, show that $\beta_2 \geq 1$.

Sol. : Let us consider a frequency distribution $\frac{x_i}{f_i}$, $i = 1, 2, 3, \ldots, n$.

We have to show that $\beta_2 = \frac{\mu_4}{\mu_2^2} \geq 1$, *i.e.*, $\mu_4 \geq \mu_2^2$.

i.e.,
$$\frac{1}{N}\Sigma f_i\,(x_i - \bar{x})^4 \geq \left[\frac{1}{N}\Sigma f_i (x_i - \bar{x})^2\right]^2$$

If we put $(x_i - \bar{x})^2 = z_i$, then we have to prove that

$$\frac{1}{N}\Sigma f_i\, z_i^2 \geq \left[\frac{1}{N}\Sigma f_i\, z_i\right]^2$$

i.e.,
$$\frac{1}{N}\Sigma f_i\, z_i^2 - \left[\frac{1}{N}\Sigma f_i\, z_i\right]^3 \geq 0, \text{ i.e., } \sigma_z^2 > 0$$

which is true because variance is always positive. Hence, the result.

Example 3 : The first four moments of a distribution are 1, 4, 10 and 46 respectively. Compute the first four central moments and Beta constants. Comment upon the nature of the distribution.

Sol. : We are not given the value of *A* about which the moments are calculate, however, this value is not required since, we do not need the actual mean.

We are given, $\mu_1' = 1$, $\mu_2' = 4$, $\mu_3' = 10$, $\mu_4' = 46$.

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Now, $\mu_2 = \mu_2' - \mu_1'^2 = 4 - 1 = 3$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= 10 - 3(4)(1) + 2(1)^2 = 10 - 12 + 2 = 0$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^3$

$= 46 - 4(10)(1) + 6(4)(1)^2 - 3(1)^4$

$= 46 - 40 + 24 - 3 = 27$

$\therefore \quad \beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{0}{27} = 0; \qquad \beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{27}{9} = 3.$

Since, $\beta_1 = 0$, the distribution has no skewness.

Since, $\beta_2 = 3$, the distribution is mesokurtic.

Example 4 : Find the mean, variance and β_1 and β_2 for the following distribution.

Class interval	:	0-10,	10-20,	20-30,	30-40
Frequency	:	1,	3,	4,	2

Sol. : **Calculation of mean etc.**

Class	Frequency	Mid Point	$\frac{m-25}{10}$				
x_i	f_i	m_i	d_i'	$f_i d_i'$	$f_i d_i'^2$	$f_i d_i'^3$	$f_i d_i'^4$
0-10	1	5	– 2	– 2	4	– 8	16
10-20	3	15	– 1	– 3	3	– 3	3
20-30	4	25	0	0	0	0	0
30-40	2	35	1	2	2	2	2
Total	10			– 3	9	– 9	21

Moments about the assumed mean 25 are given by

$$\mu_1' = \frac{\Sigma f_i d_i'}{N} \cdot h = -\frac{3}{10} \cdot 10 = -3; \qquad \mu_2' = \frac{\Sigma f_i d_i'^2}{N} \cdot h^2 = \frac{9}{10} \cdot 10^2 = 90;$$

$$\mu_3' = \frac{\Sigma f_i d_i'^3}{N} \cdot h^3 = -\frac{9}{10} \cdot 10^3 = -900; \qquad \mu_4' = \frac{\Sigma f_i d_i'^4}{N} \cdot h^4 = \frac{21}{10} \cdot 10^4 = 21000$$

Arithmetic mean $= A + \dfrac{\Sigma f_i d_i'}{N} \cdot h = 25 - \dfrac{3}{10} \cdot 10 = 22$

Variance, $\sigma^2 = \mu_2 = \mu_2' - \mu_1'^2 = 90 - 9 = 81$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$= -900 - 3(90)(3) + 2(27) = -1556$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 21000 - 4(-900)(-3) + 6(90)(-3)^2 - 3(-3)^4$

$= 21000 - 10800 + 4860 - 243 = 14817$

Now, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^2} = \dfrac{(-1556)^2}{(81)^2} = 4{\cdot}555; \qquad \beta_2 = \dfrac{\mu_4}{\mu_2^2} = \dfrac{14817}{81^2} = 2{\cdot}258.$

Miscellaneous Examples

Example 1 : Find the coefficient of variation of a frequency distribution. Given that, its mean is 120. Mode is 123 and Karl Pearson's coefficients of skewness is – 3.

Sol. : We have, Karl Pearson's coefficient of skewness $= \dfrac{\text{Mean} - \text{Mode}}{\sigma}$

By data, $-3 = \dfrac{120 - 123}{\sigma} \qquad \therefore -3 = -\dfrac{3}{\sigma} \qquad \therefore \sigma = 1$

Coefficient of variation $= \dfrac{\sigma}{x} \times 100 = \dfrac{1}{120} \times 100 = \dfrac{5}{6}.$

Example 2 : In a frequency distribution, the coefficient of skewness based upon the quartiles is 0·6. If the sum of the upper and lower quartiles is 100 and the median is 38, find the value of the upper quartile.

Sol. : $S_k = \dfrac{Q_3 + Q_1 - 2\,\text{Median}}{Q_3 - Q_1}$

But $Q_3 + Q_1 = 100$, Median = 38, $S_k = 0{\cdot}6$.

$0.6 = \dfrac{100 - 76}{Q_3 - Q_1} = \dfrac{24}{Q_3 - Q_1} \qquad \therefore Q_3 - Q_1 = 40$

But $Q_3 + Q_1 = 100$.

Adding $2Q_3 = 140 \qquad \therefore Q_3 = 70$ and $Q_1 = 100 - Q_3 = 30$.

Example 3 : From a moderately skew frequency distribution of retail prices for shoes it is found that the mean price is ₹ 20 and the median price is ₹ 17. If the coefficient of variation is 20%, find the Pearson's coefficient of skewness of the distribution.

Sol. : C.V. $= \dfrac{\sigma}{\bar{x}} \times 100$, C.V. = 20, $\bar{x} = 20$

$\therefore \quad 20 = \dfrac{\sigma}{20} \times 100, \quad \sigma = 4$

$S_k = \dfrac{3\,(\text{Mean} - \text{Median})}{\sigma} = \dfrac{3(20 - 17)}{4} = 2{\cdot}25$

Example 4 : The sum of 20 observations is 300 and the sum of their squares is 5000. If the median is 15, find the coefficient of variation and coefficient of skewness.

Sol. : We have, $\sigma = \sqrt{\dfrac{\Sigma x^2}{N} - \left(\dfrac{\Sigma x}{N}\right)^2}$

But $\Sigma x^2 = 5000$, $\Sigma x = 300$, $N = 20$

$\therefore \quad \sigma = \sqrt{\dfrac{5000}{200} - \left(\dfrac{300}{20}\right)^2} = \sqrt{250 - (15)^2} = \sqrt{25} = 5$

But $\bar{x} = \dfrac{\Sigma x}{N} = \dfrac{300}{20} = 15$

$\therefore$ Mode = 3 Median – 2 Mean = 3 (15) – 2 (15) = 15

C.V. $= \dfrac{\sigma}{\bar{x}} \times 100 = \dfrac{5}{15} \times 100 = 33{\cdot}33; \quad S_k = \dfrac{\text{Mean} - \text{Mode}}{\sigma} = \dfrac{15 - 15}{5} = 0.$

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Example 5 : A frequency distribution gave the following results : (i) Coefficient of variation = 5, (ii) Standard deviation = 2, (iii) Karl Pearson's Coefficient of skewness - 0·5.

Find the mean and mode of the distribution.

Sol. : C.V. $= \frac{\sigma}{\bar{x}} \times 100$. But C.V. = 5, $\sigma = 2$.

$\therefore \quad 5 = \frac{2}{\bar{x}} \times 100 \qquad \therefore \quad \bar{x} = 40 \qquad \therefore$ Mean = 40

$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma}$. But $S_k = 0{\cdot}5$, Mean = 40, $\sigma = 2$.

$\therefore \quad 0{\cdot}5 = \frac{40 - \text{Mode}}{2} \qquad \therefore \quad 1 = 40 - \text{Mode} \qquad \therefore$ Mode = 39.

Example 6 : In a certain distribution the following results were obtained —

$\bar{x} = 45$, Median = 48, Coefficient of skewness = – 4.

Estimate the value of the standard deviation.

Sol. : Mode = 3 Median – 2 Mean = 3 (48) – 2 (45) = 54.

$S_k = \frac{\text{Mean} - \text{Mode}}{\sigma} \qquad \therefore \quad -4 = \frac{45 - 54}{\sigma} \qquad \therefore \quad \sigma = \frac{9}{4} = 2{\cdot}25.$

Example 7 : For a moderately skew distribution mean is 200, coefficient of variation is 8 and Pearson's coefficient of skewness is 0·3. Find the mode and the median.

Sol. : We have, $\bar{x} = 200$, C.V. = 8 and $S_k = 0{\cdot}3$.

But C.V. $= \frac{\sigma}{\bar{x}} \times 100 \qquad \therefore \quad 8 = \frac{\sigma}{200} \times 100 \qquad \therefore \quad \sigma = 16.$

$\therefore \quad S_k = \frac{\text{Mean} - \text{Mode}}{\sigma} \qquad \therefore \quad 0.3 = \frac{200 - \text{Mode}}{16} \qquad \therefore$ Mode = 200 – 4.8 = 195·2

Further, for a moderately skew distribution.

Mode = 3 Median – 2 Mean

$\therefore$ Median = (1/3) (Mode + 2 Mean)

= (1/3) (195.2 + 400) = 198·4.

EXERCISE - I

1. For symmetric distribution ($\mu_3 = 0$), show that $\frac{\mu_3'}{\mu_1'} = 3\,\mu_2 + \mu_1'^2$.
2. What do you understand by Kurtosis? Define γ_2 and explain how Kurtosis is measured by γ_2.
3. What are moments? Define raw and central moments of r-th order. Show that $\mu_2 = \mu_2' - \mu_1'^2$.
4. Define r-th central moment in γ_2. Explain how γ_2 is used for measuring Kurtosis.

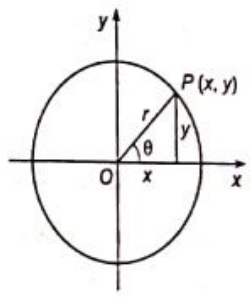
List of Formulae

1. Trigonometic Formulae

1. Definitions of Trigonometric Ratios

$\sin\theta = \frac{y}{r}; \qquad \cos\theta = \frac{x}{r}; \qquad \tan\theta = \frac{y}{x}$

$\text{cosec}\,\theta = \frac{1}{\sin\theta}; \qquad \sec\theta = \frac{1}{\cos\theta}; \qquad \cot\theta = \frac{1}{\tan\theta}$



2. Fundamental Trigonometic Identities

(A) $\sin(-\theta) = -\sin\theta, \qquad \cos(-\theta) = \cos\theta$

$\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta, \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta,$

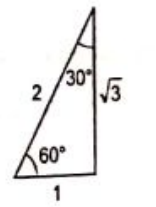
$\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta, \qquad \cos\left(\frac{\pi}{2} + \theta\right) = -\sin\theta,$

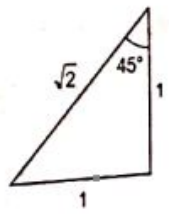
$\sin(\pi - \theta) = \sin\theta, \qquad \cos(\pi - \theta) = -\cos\theta,$

$\sin(\pi + \theta) = -\sin\theta, \qquad \cos(\pi + \theta) = -\cos\theta,$

$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta, \qquad \cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta,$

$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta, \qquad \cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta.$

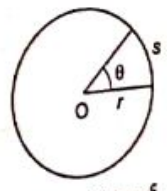




(B)

Quadrant	sin θ	cos θ	tan θ
I	+	+	+
II	+	–	–
III	–	–	+
IV	–	+	–

θ	0°/360°	30°	45°	60°	90°	180°	270°/–90°
sin θ	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	0	– 1
cos θ	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	– 1	0



$s = r\theta, \; 180° = \pi^c$

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(C) $\sin^2\theta + \cos^2\theta = 1$, $\sec^2\theta = 1 + \tan^2\theta$, $\operatorname{cosec}^2\theta = 1 + \cot^2\theta$.

(D) $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

(E) $\sin 2\theta = 2\sin\theta\cos\theta$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta \qquad \cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

$$\sin 2\theta = \frac{2\tan\theta}{1+\tan^2\theta} \qquad \cos 2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} \qquad \tan 3\theta = \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}$$

(F) $\sin C + \sin D = 2\sin\frac{C+D}{2}\cos\frac{C-D}{2}$; $\sin C - \sin D = 2\cos\frac{C+D}{2}\sin\frac{C-D}{2}$;

$\cos C + \cos D = 2\cos\frac{C+D}{2}\cos\frac{C-D}{2}$; $\cos C - \cos D = -2\sin\frac{C+D}{2}\sin\frac{C-D}{2}$;

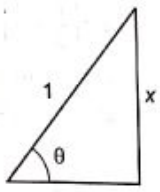
$2\sin A\cos B = \sin(A+B) + \sin(A-B)$; $2\cos A\sin B = \sin(A+B) - \sin(A-B)$;

$2\cos A\cos B = \cos(A+B) + \cos(A-B)$; $-2\sin A\sin B = \cos(A+B) - \cos(A-B)$.

(G) $\sin\theta = x$, $\operatorname{cosec}\theta = \frac{1}{x}$

$\therefore \sin^{-1}x = \operatorname{cosec}^{-1}\frac{1}{x}$, $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $\tan^{-1}x = \cot^{-1}\frac{1}{x}$,

$\sin(\sin^{-1}x) = x$, $\cos(\cos^{-1}x) = x$, $\tan(\tan^{-1}x) = x$.



(H) $\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$, $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$,

$\operatorname{sech} x = \frac{2}{e^x + e^{-x}}$, $\operatorname{cosech} x = \frac{2}{e^x - e^{-x}}$, $\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}}$.

(I) $\cosh^2 x - \sinh^2 x = 1$, $\operatorname{sech}^2 x = 1 - \tanh^2 x$

$\operatorname{cosech}^2 x = \coth^2 x - 1$, $\sinh 2x = 2\sinh x\cosh x$

$\cosh 2x = 2\cosh^2 x - 1 = 2\sinh^2 x + 1$.

(J) $\sinh^{-1}x = \operatorname{cosech}^{-1}\frac{1}{x}$, $\cosh^{-1}x = \operatorname{sech}^{-1}\frac{1}{x}$, $\tanh^{-1}x = \coth^{-1}\frac{1}{x}$.

2. Algebraic Formulae

(A) $a^m \times a^n = a^{m+n}$ $\frac{a^m}{a^n} = a^{m-n}$ $a^m b^m = (ab)^m$

$(a^m)^n = a^{mn}$ $a^0 = 1$, $a^{-r} = \frac{1}{a^r}$ $a^{p/q} = \sqrt[q]{a^p}$

(B)
$$a^2 - b^2 = (a-b)(a+b)$$
$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$
$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$
$$a^4 - b^4 = (a^2 + b^2)(a+b)(a-b)$$
$$(a+b)^2 = a^2 + 2ab + b^2$$
$$(a-b)^2 = a^2 - 2ab + b^2$$
$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = (a^3 + b^3) + 3ab(a+b)$$
$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3 = (a^3 - b^3) - 3ab(a-b)$$
$$(a+b+c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$$
$$(a+b+c)^3 = a^3 + b^3 + c^3 + 3a^2b + 3ab^2 + 3b^2c + 3bc^2 + 3c^2a + 3a^2c + 6abc$$
$$= (a^3 + b^3 + c^3) + 3(a+b+c)\cdot(ab+bc+ca) - 3abc.$$
$$a^2 + b^2 = (a+b)^2 - 2ab = (a-b)^2 + 2ab$$
$$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$$
$$a^3 - b^3 = (a-b)^3 + 3ab(a-b)$$
$$a^3 + b^3 + c^3 = (a+b+c)^3 - 3(a+b+c)(ab+bc+ca) + 3abc$$
$$= 3abc \text{ if } a+b+c = 0$$

(C) $\log mn = \log m + \log n$, $\log\frac{m}{n} = \log m - \log n$

$\log m^n = n\log m$ $\log_b m = \frac{\log a^m}{\log a^b}$

(D) If $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Sum of the roots $= -\frac{b}{a}$, Product of the roots $= \frac{c}{a}$

(E)
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

(a) If two rows or columns are identical then the determinant is zero.

(b) If rows and columns are interchanged the determinant is not changed.

(c) If two rows or columns are interchanged the determinant changes its sign.

(d) If each element of a row or column is multiplied by a constant then the determinant is multiplied by that constant.

(e) The value of a determinant is unchanged if equimultiples of a row or a column are added to the corresponding elements of any other row or column.

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Cramer's Rule

If $a_1 x + b_1 y + c_1 z = d_1$

$a_2 x + b_2 y + c_2 z = d_2$

$a_3 x + b_3 y + c_3 z = d_3$,

then $x = \frac{D_x}{D}$, $y = \frac{D_y}{D}$, $z = \frac{D_z}{D}$ where, $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

and D_x, D_y, D_z are obtained by replacing the coefficients of x, y, z respectively by d_1, d_2, d_3.

3. Differentiation Formulae

(A) $\lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta} \right) = 1$, $\lim_{n \to \infty} \left(1 + \frac{1}{n} \right)^n = e$,

$\lim_{y \to 0} (1 + y)^{1/y} = e$, $\lim_{x \to 0} \frac{a^x - 1}{x} = \log_e a$.

(B)

1. If $y = x^n$, $\frac{dy}{dx} = n x^{n-1}$
2. If $y = \sin x$, $\frac{dy}{dx} = \cos x$
3. If $y = \cos x$, $\frac{dy}{dx} = -\sin x$
4. If $y = \tan x$, $\frac{dy}{dx} = \sec^2 x$
5. If $y = \operatorname{cosec} x$, $\frac{dy}{dx} = -\operatorname{cosec} x \cot x$
6. If $y = \sec x$, $\frac{dy}{dx} = \sec x \tan x$
7. If $y = \cot x$, $\frac{dy}{dx} = -\operatorname{cosec}^2 x$
8. If $y = e^x$, $\frac{dy}{dx} = e^x$
9. If $y = a^x$, $\frac{dy}{dx} = a^x \log a$
10. If $y = \log_e x$, $\frac{dy}{dx} = \frac{1}{x}$
11. If $y = \log_a x$, $\frac{dy}{dx} = \frac{1}{x \log a}$
12. If $y = \sin^{-1} x$, $\frac{dy}{dx} = \frac{1}{\sqrt{1 - x^2}}$
13. If $y = \cos^{-1} x$, $\frac{dy}{dx} = -\frac{1}{\sqrt{1 - x^2}}$

14. If $y = \tan^{-1} x$, $\frac{dy}{dx} = \frac{1}{1 + x^2}$
15. If $y = \sec^{-1} x$, $\frac{dy}{dx} = \frac{1}{x\sqrt{x^2 - 1}}$
16. If $y = \operatorname{cosec}^{-1} x$, $\frac{dy}{dx} = -\frac{1}{x\sqrt{x^2 - 1}}$
17. If $y = \cot^{-1} x$, $\frac{dy}{dx} = -\frac{1}{1 + x^2}$
18. If $y = \sin hx$, $\frac{dy}{dx} = \cos h\, x$
19. If $y = \cos hx$, $\frac{dy}{dx} = \sin h\, x$
20. If $y = \tan hx$, $\frac{dy}{dx} = \sec h^2 x$
21. If $y = \operatorname{cosec} hx$, $\frac{dy}{dx} = -\operatorname{cosec} h\, x \cot x$
22. If $y = \sec hx$, $\frac{dy}{dx} = \sec h\, x \tan h\, x$
23. If $y = \cot hx$, $\frac{dy}{dx} = -\operatorname{cosec} h^2 x$
24. If $y = \sin h^{-1} x$, $\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}$
25. If $y = \cos h^{-1} x$, $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 - 1}}$
26. If $y = \tan h^{-1} x$, $\frac{dy}{dx} = \frac{1}{1 - x^2}$
27. If $y = \sec h^{-1} x$, $\frac{dy}{dx} = -\frac{1}{x\sqrt{1 - x^2}}$
28. If $y = \operatorname{cosec} h^{-1} x$, $\frac{dy}{dx} = -\frac{1}{|x|\sqrt{1 + x^2}}$
29. If $y = \cot h^{-1} x$, $\frac{dy}{dx} = \frac{1}{1 - x^2}$

(C)

1. If $y = u \pm v$, $\frac{dy}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$
2. If $y = uv$, $\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$
3. If $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$
4. If $y = x^x$, $\frac{dy}{dx} = x^x (1 + \log x)$
5. If $x = f(t)$, $y = \Phi(t)$, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

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4. Integration Formulae

$\int \mathrm{I} \cdot \mathrm{II} \cdot dx = \mathrm{I} \cdot \int \mathrm{II}\, dx - \int\left[\int \mathrm{II}\, dx\right] \cdot \frac{d\,\mathrm{I}}{dx} \cdot dx$; $\int e^x [f(x) + f'(x)]\, dx = e^x f(x)$

1. $\int x^n\, dx = \frac{x^{n+1}}{n+1}$ if $n \neq -1$
2. $\int \frac{dx}{x} = \log x$
3. $\int \sin x\, dx = -\cos x$
4. $\int \cos x\, dx = \sin x$
5. $\int \sec^2 x\, dx = \tan x$
6. $\int \operatorname{cosec}^2 x\, dx = -\cot x$
7. $\int \sec x \tan x\, dx = \sec x$
8. $\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$
9. $\int \tan x\, dx = \log \sec x$
10. $\int \cot x\, dx = -\log \operatorname{cosec} x = \log \sin x$
11. $\int \sec x\, dx = \log\left\{\tan\left(\frac{x}{2} + \frac{\pi}{4}\right)\right\} = \log(\sec x + \tan x)$
12. $\int \operatorname{cosec} x\, dx = \log\left(\tan \frac{x}{2}\right) = \log(\operatorname{cosec} x - \cot x)$
13. $\int e^x\, dx = e^x$
14. $\int a^x dx = \frac{a^x}{\log a}$
15. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a}$
16. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \log\left(x + \sqrt{x^2 - a^2}\right)$
17. $\int \frac{dx}{\sqrt{x^2 + a^2}} = \log\left(x + \sqrt{x^2 + a^2}\right)$
18. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$
19. $\int \frac{dx}{x^2 - a^2} = \frac{1}{2a} \log\left(\frac{x - a}{x + a}\right)$
20. $\int \frac{dx}{a^2 - x^2} = \frac{1}{2a} \log\left(\frac{a + x}{a - x}\right)$
21. $\int \frac{dx}{x\sqrt{x^2 - 1}} = \sec^{-1} x$
22. $\int e^{ax} \sin bx\, dx = \frac{1}{a^2 + b^2} \cdot e^{ax}(a \sin bx - b \cos bx)$
23. $\int e^{ax} \cos bx\, dx = \frac{1}{a^2 + b^2} \cdot e^{ax}(a \cos bx + b \sin bx)$
24. $\int \sqrt{a^2 - x^2}\, dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right)$
25. $\int \sqrt{x^2 + a^2}\, dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2} \log\left(x + \sqrt{x^2 + a^2}\right)$
26. $\int \sqrt{x^2 - a^2}\, dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2} \log\left(x + \sqrt{x^2 - a^2}\right)$
27. $\int \sinh x\, dx = \cosh x$
28. $\int \cosh x\, dx = \sinh x$
29. $\int \tanh x\, dx = \log(\cosh x)$
30. $\int \operatorname{sech} x\, dx = \sin^{-1}(\tanh x)$
31. $\int \operatorname{cosech} x\, dx = \tan\left|\tanh \frac{x}{2}\right|$
32. $\int \coth x\, dx = \log|\sinh x|$

Definite Integrals

1. $\int_0^a f(x)\, dx = \int_0^a f(a - x)\, dx$
2. $\int_a^b f(x)\, dx = \int_a^b f(a + b - x)\, dx$

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