Graph Theory Exercises

Fan

2023.8.13

Contents

Index

1 Definitions and examples

1.1 Definitions

Problem 1.1.1. If G is a simple graph with at least two vertices, prove that G must contain two or more vertices of the same degree.

Proof. Suppose G has n vertices ($n \geq 2$) and none of its vertices has the same degree. We can also assume that there is no isolated vertice, since there is atmost one isolated vertice and won't contribute to other vertices' degrees. Thus, we have n vertices with at least 1 degree. However, since the graph is simple, each vertices can have n-1 degrees at most. Thus, there must be two vertices with same degree. □

Problem 1.1.2.

1. Show that there are exactly $2^{n(n-1)/2}$ labelled simple graph on v vertices.
2. How many of these have exactly m edge.

Proof. We will prove by induction. When $n = 1$, the only vertice has to be isolated vertice. Thus there is only one possibility, which is agree with $2^{n(n-1)/2}$.

Suppose when $n = k$ then equation is valid. Then, when $n = k + 1$, we can regard this situation as adding one vertice into n vertices. The new vertex can choose whether to adjacent with other vertices, thus we have totally $2^{n(n-1)/2} \cdot 2^n = 2^{n(n+1)/2}$. □

Problem 1.1.3. Let G be a graph with n vertices and m edges, and let v be a vertex of G of degree k and e be an edge of G. How many vertices and edges have $G - e, G - v$ and $G \backslash e$.

$G - e$ has n vertices and $m - 1$ edges. $G - v$ has $n - 1$ vertices and $m - k + l$ edges, where l is the number of the loops incident with v. $G \backslash e$ has $n - 1$ vertices and $m - s$ edges, where s is the number of edges that is in the same edge-family with e.

Problem 1.1.4. If G is a graph without loops, what can you say about the sum of the entries in

1. any row or column of the adjacency matrix of G?
2. any row of the incidence matrix G?
3. any column of the incidence matrix of G?

Nothing. Nothing. It must be 2.

1.2 Challenge problems

Problem 1.2.1. A simple graph that is isomorphic to its complement is self-complementary. Prove that, if G is self-complementary, then G has $4k$ or $4k+1$ vertices, where k is an integer.

Proof. $G \cup G' = K_n$. Since G and G' has same amounts of edges, the number of K_n's edge is even. Thus $n(n-1)/2$ must be an even number. Therefore, $n = 4k$ or $n = 4k+1$. □

Problem 1.2.2. If G is a simple graph with edge-set $E(G)$, the vector space of G is the vector space over the field $Z_2 = \{0, 1\}$ of integers modulo 2, whose elements are subsets of $E(G)$. The sum $E + F$ of two such subsets E and F is the set of edges in E or F but not in both, and scalar multiplication is defined by $1 \cdot E = E$ and $0 \cdot E = \varnothing$. Show that this define a vector space over Z_2, and find a basis for it.

Problem 1.2.3. Prove that, if G is regular of degree k, then $L(G)$ is regular of degree $2k-2$.

Proof. Since G is regular of degree k, for each edge wv, each of its end has other $k-1$ edges that is incident with it. Thus, the vertex isomorphic to the edge vw has $2k-2$ edge that is incident with it. □

Problem 1.2.4. Find an expression for the number of edges of $L(G)$ in terms of the degrees of the vertices of G.

$$\sum_{V(G)} \frac{\deg v(\deg v - 1)}{2}$$

Problem 1.2.5. Show that an infinite graph G can be drawn in Euclidean 3-space if $V(G)$ and $E(G)$ can each be put in one-one correspondence with a subset of the set of real numbers.

Proof. □

2 Paths and cycles

2.1 Connectivity

Problem 2.1.1. Prove that a simple graph and its complement cannot both be disconnected.

Proof. Suppose graph G has at least two component. For vertex v and w, if they are in the different component, then in $\bar{G}$, they are connected. If they are in the same component, then we can choose vertex z in other component, so that in $\bar{G}$, we have $w \to z \to w$. Thus, for every pair of vertices in $\bar{G}$, there exists a path connect them. Therefore, $\bar{G}$ is connected. □

Problem 2.1.2. Prove that a graph is $2-$edge-connected if and only if any two distinct vertices are joined by at least two paths with no edges in common.

Proof. Since G is connected, for any two distinct vertices v and w in G, there exists a path P that joins them. Suppose there is another path P' that also joins them. If there are no two paths that have no edges in common. Then, P and P' □

Problem 2.1.3. Prove that a graph with at least three vertices is $2-$connected if and only if any two distinct vertices are joined by at least two path with no other vertices in common.

Problem 2.1.4. A tournament T is irreducible if it is impossible to split the set of vertices of T into two disjoint set V_1 and V_2 so that each arc joining a vertex of V_1 and a vertex of V_2 is directed from V_1 to V_2. Prove that a tournament is irreducible if and only if it is strongly connected.

2.2 Euler graphs and digraphs

Problem 2.2.1. Let G be a connected graph with $k(> 0)$ vertices of odd degree. Show that the minimum number of trails, that together include every edge of G and that have no edges in common, is $k/2$.

Proof. By Handshaking theorem, k must be an even number.

We then proof by induction. When $k = 2$, there is a semi-Eulerian trail, thus the minimal number is 1. Suppose then $k - 2$ is an even number, the minimal number is $(k - 2)/2$. Then, when there is k vertices with odd degrees. Suppose there are no vertices with odd incidence are adjacent. Then, the original graph can be separated into two graph: one of them consists of all the vertices with even degree and $k-2$ vertices with odd degree and all the edges that is incident with them. By induction hypothesis, this part can have minimal number of $(k - 2)/2$. Another graph consists of all the vertices of even degree and two vertices with odd degree that are not included in another part and the edges incident with them. By corollary 2.3. there is a semi-Eulerian. Thus, the minimal number of second part is 1. Therefore, the minimal number for the original graph is $k/2$.

If there are at least two vertices with odd degree that are adjacent, then, by removing one edge incident with these two vertices, we can get a graph with only $k - 2$ vertices with odd degree. Thus, by induction hypothesis, the total minimal number is $k/2$.

remark. Actually, we only prove that the minimal number is less than $k/2$. We can also prove that the minimal number is greater than $k/2$. When $k = 2$, obviously, the minimal number is greater than 1. Then, on the basis of a given graph, we can add a pair of vertices that is only adjacent with each other. Then, if when there is k vertices with odd degree, the minimal number is greater than $k/2$. Then, when there is $k+2$ vertices with odd degree, the minimal number is greater than $k/2+1=(k+2)/2$. Thus we finish the proof.

□

remark. The proof given by the answer is more elegent.

Connect the k vertices in pair by adding $k/2$ edges, then get the Eulerian trail. Finally, remove the added edges to get $k/2$ trail.

However, it seems that it doesn't prove it is the minimal number.

2.3 Hamiltonian graphs and digraphs

Problem 2.3.1. Prove that, for every n, Q_n is Hamiltonian.

Proof. For $n=1,2,3$, it is obvious to give out the Hamiltonian cycle. Suppose when $n=k$, there is a Hamiltonian cycle for Q_n, which is

$$\cdots \to Y \to X \to Z \to \cdots \to Y \to X \to Z \to \cdots$$

where X, Y, Z are all a string consists of n's 1 or 0. Then we can construct the Hamiltonian cycle for Q_{k+1}:

$$\overline{X0} \to \overline{X1} \to \text{(along the direction of the former cycle)} \to \overline{Y1} \to \overline{Y0}$$

$$\to \text{(against the direction of the former cycle)} \to \overline{Z0} \to \overline{X0}$$

□

Problem 2.3.2. Prove that, if G is a bipartite graph with an odd number of vertices, then G is non-Hamiltonian.

Proof. If a graph is non-Hamiltonian, then, any subgraph of it is also non-Hamiltonian. Thus, we only need to prove that a complete bipartite is non-Hamiltonian, which is obvious. □

Problem 2.3.3. Let G be a graph with n vertices and (n-1)(n-2)/2+2 edges. Use Ore's theorem to prove that G is Hamiltonian.

Proof. The graph can be obtained by removing $n-3$'s edges from K_n. Suppose there is a pair of disjoint vertices v, w that doesn't satisfy the condition of theorem 2.13., then, $\deg(v)+\deg(w) \leq n-1$. Thus, we must remove at least $1+(2n-2)-(n-1)=n'$s edges from K_n. Thus, there is no such vertices in the given graph. Therefore, it is a Hamiltonian. □

2.4 Challenge problems

Problem 2.4.1. Let G be a simple garph on $2k$ vertices containing no triangles. Prove, by induction on k, that G has at most k^2 edges.

Proof. When $k = 1$, G has two vertices. Since G is a simple graph, it is obvious that it can have at most 1 edges.

Suppose when $n = k$, in order not to have triangle in G, G can have at most k^2 edges. When $n = k + 1$, there is $2k + 2$ vertices in G. For any $2k$ vertices in the graph G, by induciton hypothesis, there are at most k^2 edges that are incident with any two of them. Then, we can separate the $2k + 2$ vertices into v, w and other $2k$ vertices. Since there is no triangle, there is atmost k^2 edges incident with vertex in the latter group. And there are atmost $2k + 1$ edges left: vw and vz, wz for every vertex z in the latter group. Thus, there is at most, $k^2 + 2k + 1 = (k + 1)^2$ edges.

remark. Such upper bound can be achieved in $K_{k,k}$

□

Problem 2.4.2. Let G be a connected graph with vertex-set $\{v_1, v_2, \cdots, v_n\}$, m edges and t triangles.

1. If A is the adjacency matrix of G, prove that the number of walks of length 2 from v_i to v_j is the ijth entry of the matrix A^2. Deduce that $2m =$ the sum of the diagonal entries of A^2.

2. Obtain a corresponding result for the number of walks of length 3 from v_i to v_j and deduce that $6t =$ the sum of the diagonal entries of A^3.

Proof.

$$A^2(i;j) = \sum_{k=1}^{n} A(i;k)A(k;j)$$

Since

$$\begin{aligned} A(i;k)A(k;j) \neq 0 \quad &\Leftrightarrow A(i;k) \neq 0 \text{ and } A(k;j) \neq 0 \\ &\Leftrightarrow v_i \text{ is adjacent with } v_k \text{ and } v_k \text{ is adjacent with } v_j \end{aligned}$$

If there is n walks, then there is n different k, thus $A^2(i;j) = n$. If $A^2(i;j) = n$, then there should be n different k, thus there is n different walks.

Since every edge provide 2 different walk by traversing along it twice, whose starting point is the two vertices incident with the edges respectively.

We can regard A^3 as A^2A, then we can proof it similarly. □

Problem 2.4.3. 1. Prove that, if two distinct cycles of a graph G each contains an edge e, then G has a cycle that doesn't contain e.

2. Prove a similar result with 'cycle' replaced by 'cutsets'.

Proof. Obviously, the union of two cycle without e is the desitred cycle.

Suppose A, B is the distinct cutset but both have e. Then, $A \cup B$ is no a cutset but has a subset that is cutset. $\square$

Problem 2.4.4. Prove that, if C is a cycle and C^* is a cutset of a connected graph G, then C and C^* have an even number of edges in common.

Proof. Since a cutset will separate a graph into two, say, A and B, and cutset has no unnecessary edges. Then, every edge in the cutset should incident with one vertex in A and one vertex in B. Otherwise, if we remove that edge from cutset, the graph can still be separated into two part.

Thus, if there is one edge that is both in C and C^*. Then, say that edge is ab and a is in A and b is in B. Then, the part of the circle that is connected to the vertex a should be separated into A, to the vertex b separated into B. Thus, there must be even edges that is both in C and C^*. $\square$

Problem 2.4.5. Prove that, if S is any set of edges of G with an even number of edges in common with each cutset of G, then S can be split into edge-disjoint cycles.

Problem 2.4.6. A set E of edges of a graph is independent if E contains no cycle of G. Prove that

1. any subset of an independent set is independent.
2. if I and J are independent sets of edges with $|J| > |I|$, then there is an edge e that lies in J but not in I with the property that $I \cup \{e\}$ is independent.

Prove also that (1) and (2) still hold if we replace the word 'cycle' by 'cutset'.

Proof. The first question is very easy to prove.

For the second question, since $|J| > |I|$, there exists an edge e that lies in J but not in I. Suppose there is no such edges that statify the given conditons, then, for every edge e that lies in J but not in I, $I \cup \{e\}$ is there is a circle. However, I is independent, which means that I has a semi-Eulerian trail and has two vertices with odd degrees. To form a circle, every edge that lies in J but not in I is incident with those two vertices whose degree are odd. Therefore, there can only be one such edge, since J is independent and thus has no multiple edges. We denote that very edge by e. Thus, we can get J by adding e into I. However, this means that there is an Eulerian trail in J, thus there must be a circle in J, which is a contradiction.

When we replace 'cycle' by 'cutset', (1) is still obvious.

remark. Maybe this problem is closely related to the previous two problems.

$\square$

Problem 2.4.7. Let V be the vector space of a graph G

1. Use corollary 2.4 to show that, if C and D are cycles of G, then their sum $C + D$ can be written as a union of edge-disjoint cycles.
2. Deduce that the set of such unions of cycles of C forms a subspace W of V (the cycle subspace of G), and find its dimension.
3. Show that the set of unions of edge-disjoint cutsets of G forms a subspace W^* of V(the cutset subspace of G), and find its dimension.

Problem 2.4.8. Show that the line graph of a simple Eulerian graph is Eulerian.

Proof. Suppose G is a simple Eulerian graph with Eulerian cycle. Then, every vertices in G has even degree. Thus, for every edges, each vertex that it is incident with is also adjacent to an odd number of edges. Thus, every edge is adjacent with odd numbers of other edges. Therefore, in $L(G)$, every vertex has even degree. Thus, $L(G)$ is also a Eulerian graph. □

Problem 2.4.9. If the line graph of a simple graph G is Eulerian, must G be Eulerian?

Proof. We only need to prove that $L(L(G)) = G$.

Since the number of the vertices in $L(L(G))$ equal the number of the edges in $L(G)$, which is equal to the number of the vertices in G. And each pair of vertices in $L(L(G))$ is adjacent if and only if the corresponding edges is adjacent in $L(G)$, which happens if and only if the corresponding pair of vertices is adjacent in G. Thus, $L(L(G))$ is isomorphic to G. Thus, the answer is yes. □

Problem 2.4.10. Let T be a tournament. The score of a vertex of T is its out-degree, and the score sequence of T is the sequence formed by arming the scores of its vertices in non-decreasing order. Prove that, if $(s_1, s_2, \cdots, s_n)$ is the score-sequence of a tournament T, then

1. $s_1 + s_2 + \cdots + s_n = n(n-1)/2$;
2. for each positive integer $k < n, s_1 + s_2 + \cdots + s_k \geq k(k-1)/2$,with strict inequility for all k if and only if T is strong connected.
3. T is transitive if and only if $s_k = k - 1$ for each k

Proof. The first question is trival, since there are $n(n-1)/2$ edges at all.

For the second question, since for the match in the $k's$ team, no metter who wins, it will contribute to the sum, thus $s_1 + \cdots + s_k \geq k(k-1)/2$. When the equation is valid, these k's team don't win unless they are fighting with the team other than these $k's$ teams. Thus, obviously, the k's team is isolated from the outside, since there is no path from outside from inside. Such path only exists when it is a strict inequility.

□

Problem 2.4.11. Let G be a Hamiltonian graph and let S be any set of k vertices in G. Prove that the graph $G - S$ has at most k component.

Proof. The deletion of k vertices will at most cut the Hamiltonian cycle into $k's$ path, thus at most $k's$ components. □

Problem 2.4.12. What is the maximum number of edge-disjoint Hamiltonian cycles in K_{2k+1}?

3 Trees

3.1 Properties of trees

Problem 3.1.1. Prove that every tree is a bipartite graph

Proof. Choose a vertes v in the tree T, and color it white. Then, since T is connected, there is a shortest path join each othere vertex in T to v. Color the vertex whose corresponding shortest path's leng is even white. If odd, color black. When a vertex is about to color, there suppose to be only one vertex adjacent to it has already been colored, otherwise, there will be a color. Thus, we can get a bipartite. □

Problem 3.1.2. If G is a connected graph, a centre of G is a vertex v with the property that the maximum of the distances betweenn v and the other vertices of G is as small as possible. Prove that every tree has either one centre or two adjacent centres.

Proof. Since we are talking about finite graph, there is at least one such center. Even we remove one end-vertex from the graph, the center stay the same, unless the center is exactly that end-vertex, which will only only happen when there are only two vertices. So there can only be at most 2 center. □

Problem 3.1.3.
1. Let C^* be a set of edges of a graph G. Show that, if C^* has an edge in common with each spanning forest of G, then C^* contains a cutset.
2. Obtain a corresponding result for cycles.

Proof. We only need to prove that if a edge is in every spanning tree of G, then, it is a bridge.

It is obvious that it is in every possible circle or in none of them.

In the first case, then, by removing that edge we can get a tree with $n-1$ edges, which means that there are n edges in the original graph and the circle rank of it is 1. Thus there is only one cicle. Obviously, that edge is not a loop, then, it is in a circle consists of many edges, which is contrary to the hypothesis.

In the second case, suppose only one of the vertices incident with that edge is in a cycle. Then, the part of the graph on the other side of the edge is a tree, and thus it is a bridge. Soppose the two vertices incident with that edge are in the circle C_1 and C_2, then, there is no other edges that is also connected with both cycles, otherwise that edge will be in a cycle. Thus, it is a bridge.

The corresponding result is that 'let C be a set of edges of a graph G, if C has a edge that has no common edge with any spanning forest of G, then C contain a cycle'

We only need to prove that one of the edge of C is a loop. Since that edge have no common edge with any spanning forest, then, that edge must be in a cycle with no more that one edge, which means that it is a loop. $\square$

Problem 3.1.4. Let T_1 and T_2 be spanning trees of a connected graph G.

1. If e is any edge of T_1, show that there exists an edge f of T_2 such that the graph $(T_1 - \{e\}) \cup \{f\}$ (obtained from T_1on replacing e by f) is also a spanning tree.

2. Deduce that T_1 can be 'transformed' into T_2 by replacing the edges of T_1 one at a time by edges of T_2 in such a way that a spanning tree is obtained at each stage.

3.2 Counting trees

Problem 3.2.1. Let $\tau(G)$ be the number of spanning trees in a connected graph G. Prove that, for any edge e, $\tau(G) = \tau(G-e) + \tau(G\backslash e)$.

Proof. First, it is easy to show that, if the deletion of a edge doesn't break the spanning tree, then, it is still the spanning tree of the graph with one edge less.

$\tau(G-e)$ equal the number of the spanning trees that won't be break by the deletion of edge e. For those spanning tree, if we let the original graph change into $G\backslash e$, we will get a circle, because there is already a path included in the spanning tree that join the vertices joined by e. For the spanning trees break by the deletion, by getting $G\backslash e$, we can get a new spanning tree.

Thus we finished the proof.

□

3.3 Challenge problems

Problem 3.3.1. Show that if H and K are subgraphs of a connected graph G, and if $H \cup K$ and $H \cap K$ are defined in the obvious way, then the cutset rank ζ satisfies:

1. $0 \leq \zeta(H) \leq |E(H)|$(the number of edges of H)
2. if H is a subgraph of K, then $\zeta(H) \leq \zeta(K)$
3. $\zeta(H \cup K) + \zeta(H \cap K) \leq \zeta(H) + \zeta(K)$.

Proof. (1) Since the cutset rank equal the $n-1$ for a connected graph with n vertices and the subgraph of a connected graph is also connected, $0 = 1 - 1 \leq \zeta(H) \leq |V(H)| - 1$. What's more, since H is connected, the minimal number of the edge is $|V(H)| - 1$, achieved when H is a tree. Thus we finished the proof of the first inequility.

A subgraph of a connected graph is still connected but may have less vertices, thus we have second inequility.

Obviously, we the sum of the vertices of the graph on each sides are equal. Since H, K are all connected, $H \cap K$ are also connected. Since $H \cup K$ may be disconnected, consist of two component and the formula for cutset tank is $\zeta(G) = n - k$. We got the inequility with $\leq$. □

Problem 3.3.2. Let V be the vector space associated with a simple connected graph G, and let T be a spanning tree of G.

1. Show that the fundamental set of cycles associated with T forms a basis for the cycle subspace W.
2. Obtain a corresponding result for the cutset subspace W^*.
3. Deduce that the dimensions of W and W^* are $\gamma(G)$ and $\zeta(G)$, respectively.

Problem 3.3.3. Use the matrix-tree theorem to prove Cayley's theorem.

Proof. The matrix-tree theorem tells us about how many spanning tree can be construct in a given graph and Cayley's theorem tells us how many tree can be constrcted, given a certain number of vertices. □

Problem 3.3.4. Let $T(n)$ be the number of labelled trees on n vertices.

1. By counting the number of ways of joining a labelled tree on k vertices and one on $n-k$ vertices, prove that

$$2(n-1)T(n) = \sum_{k=1}^{n-1} \binom{n}{k} k(n-k)T(k)T(n-k)$$

2. Deduce the identity

$$\sum_{k=1}^{n-1} \binom{n}{k} k^{k-1}(n-k)^{n-k-1} = 2(n-1)n^{n-2}$$

Proof. If we want to get a tree with n vertices after joining, then, we can get $T(n)$ different trees.

Given n labelled vertices, to run through all the condition, first, we have to classify all the condition according to the number of the vertices in the each components, one is k and the other one is $n-k$, where $k = 1, 2, \cdots, n-1$. Thus we have the $\sum_{k=1}^{n-1}$ on the RHS. Now, given a certain k, we need to run through all the possibility according to the different trees with k vertices an $n-k$ vertices, thus we have $T(k)T(n-k)$ on the RHS. Finally, we have two choose one vertex from each component and join them to form the tree with n vertices. Thus we have $k(n-k)$ on the RHS. Since we have k from 1 to $n-1$, we repeat once. THus we have 2 on the LHS. What's more, since every tree with n vertices can be constrcted by join one vertex from a component with k vertices and another one from a component with $n-k$ vertices, where k keep the same, we actually repeat $(n-1)$ times on the basis of the former repeat, leading to the $(n-1)$ on the LHS. Thus we finished the proof.

By substituting $T(n)$ with n^{n-2}, we can get the second formula. □

4 Planarity

4.1 Planar graph

Problem 4.1.1. Which complete graphs and complete bipartite garphs are planar?

Proof. Since by contracting one vertex to any other vertex of a complete graph, we can get another complete graph with one less vertex, only $K_n(n \leq 4)$ are planar.

In the same manner, we can know that $K_{a,b}$ is planar if and only if at least one of them is less than 3. □

Problem 4.1.2. 1. For which values of k is the $k-$cube Q_k planar?

2. For which values of r, s and t is the complete tripartite graph $K_{r,s,t}$planar?

Proof. For k less than 5, because if $k \geq 5$, then the cube graph can be concracted into a K_5. When $k \geq 5$, there are at least 32 vertices with degree at least 5. So we have enough vertices and degrees to allow it to be contracted into K_5, which is easy to prove.

For the second question, obviously, there can be at most one number that is greater than 2.

Suppose $r \leq s \leq t$. When $r = s = 1$, there is no constrain for t. When $r = 1, s = 2$, if $t \geq 3$, then, one of its subgraph is $K_{3,3}$. Thus $t = 2$. When $r = s = 2$, due to the former discussion, $t \leq 2$. It is easy to construct a planar complete graph $K_{2,2,2}$. Overall, only $K_{1,1,n}, K_{1,2,2}, K_{2,2,2}$ is planar. □

Problem 4.1.3. If two homeomorphic graphs have n_i vertices and m_i edges $(i = 1, 2)$, show that

$$m_1 - n_1 = m_2 - n_2$$

Proof. Suppose the original graph G has n_0 vertices and m_0 edges. When ever we add a vertex on its edge, we got a graph with an extra vertex and an extra edge, thus the difference between the number of the vertices and the number of the edges stay same. □

Problem 4.1.4. A graph is outerplanar if G can be drawn in the plane so that all of its vertices lie on the exterior boundary.

1. Show that K_4 and $K_{2,3}$ are not outerplanar.

2. Deduce that, if G is an outerplanar graph, then G contains no subgraph homeomorphic or contractible to K_4 or $K_{2,3}$.

Proof. The first question can be answered by experienment.

For the second question, suppose G is an outerplanar graph, and contains a subgraph that is homeomorphic or contractible to those two graph. It is obviously that, a subgraph of a outerplanar graph is still an outerplanar graph. What's more, during the process of homeomorphic and contractible, an outerplanar graph won't become a non-outerplanar graph. The rest of the proof is then obvious. □

Problem 4.1.5. By placing the vertices at the point $(1, 1^2, 1^3), (2, 2^2, 2^3), \cdots$ prove that any simple graph can be drawn without crossings in Euclidean three-dimensional space so that each edge is represented by a straight line.

4.2 Euler's formula

Problem 4.2.1.

1. Use Euler's formula to prove that, if G is a connected planar graph of girth 5 with n vertices and m edges, then $m \leq \frac{5}{3}(n - 2)$.Deduce that the Petersen graph is non-planar.

2. Obtain an inequility, generalizing that in part (1), for connected planar graphs of girth r.

Proof. We shall give the genral version directly. Since the girth of the planar graph is r, every face of that graph is at least bounded by r edges. It follows that

$$2m \geq rf,$$

where the factor 2 is because of the fact that each edge is adjacent to 2 faces at the same time.

Since

$$n - m + f = 2,$$

we have

$$2m \geq r(2 + m - n)$$

or

$$m \leq \frac{r}{r-2}(n-2)$$

When $r = 5$, we get the inequility in (1). Petersen graph has 10 vertices and 15 edges, which does not satisfy the inequility when $r = 5$ □

Problem 4.2.2. Let G be a polyhedron (or polyhedral graph), each of whose faces is bounded by a pentagon or hexagon.

1. Use Euler's formula to show that G must have at least 12 pentagonal faces.

2. Prove, in addition, that if G is such a polyhedron with exactly three faces meeting at each vertex(such as a football), then G has exactly 12 pentagonal faces.

Proof. □

Problem 4.2.3. Let G be a siple plane graph with fewer than 12 faces, in which each vertex has degree at least 3. Use Euler's formula to show that G has a face bounded by at most four edges.

Proof. If the graph satisfying all the condition have no face bounded by less than 4 edges, then

$$\begin{cases} n - m + f = 2 \\ f < 12 \\ 3n \leq 2m \\ 5f \leq 2m \end{cases}$$

We get

$$15n + 15f = 30 + 15m \leq 10m + 6m$$

or

$$30 \leq m.$$

However

$$f = 2 + m - n \geq 2 + m = \frac{2}{3}m = 2 + \frac{m}{3} = 12$$

which is contrary to the second inequility. $\square$

Problem 4.2.4. Let G be a simple connected cubic graph, and let C_k be the number of $k-$sided faces. By counting the number of vertices and edges of G, prove that

$$3C_3 + 2C_4 + C_5 - C_7 - 2C_8 - 3C_9 - \cdots = 12$$

Proof.

$$\begin{cases} 3n = \sum_{k=3}^{\infty} kC_k \\ n - m + f = 2 \\ 2m = \sum_{k=3}^{\infty} kC_k \\ f = \sum_{k=3}^{\infty} C_k \end{cases}$$

The first equation is about the relationship bewteen the number of vertices and the number of edges in cubic graph. The second equation is the Euler's formula. The third equation is about the number of the edges. The fourth equation is about the number of the faces.

By combining these there equations, we have

$$6\sum_{k=3}^{\infty} C_k - \sum_{k=3}^{\infty} kC_k = 12.$$

$\square$

Problem 4.2.5. Let G be a simple graph with at least 11 vertices, and let $\bar{G}$ be its complement. Prove that G and $\bar{G}$ cannot be both planar.

Proof. If both graph are planar, then

$$\begin{cases} m + \bar{m} = \frac{n(n-1)}{2} \\ n = \bar{n} \geq 11 \\ m \leq 3n - 6 \\ \bar{m} \leq 3\bar{n} - 6 \end{cases}$$

thus we have

$$\frac{2}{n(n-1)} \leq 6n - 12$$

or

$$n^2 - 13n + 24 \leq 0$$

which is contrary to $n \geq 11$. $\square$

4.3 Dual graph

Problem 4.3.1. Use the duality to prove that there exists no plane graph with five faces, each pair of which shares an edge in common.

Proof. If G is a graph satisfying all the conditions, then, its dual graph is K_5, whose dual graph doesn't exists, which is contrary to the fact that the dual graph of a dual graph is its original graph. □

Problem 4.3.2. Prove that if G is a disconnected plane graph, then G^{**} is not isomorphic to G.

Proof. It is obvious that if G is a disconnected plane graph, then, each of its components is a connected plane graph. Since the dual of a connected plane graph is a connected graph, because each pair of faces of the original graph is separated by at least one edge, and the vertex corresponding to the infinite graph of the original graph is obviously adjacent to at least one vertex of each dual graph of the original components. Thus, G^* is connected. So is G^{**}. As a result, G and G^{**} are not isomorphic to each other. □

Problem 4.3.3. Dualize the result of Exercises 4.2.2.

Proof. □

Problem 4.3.4. Prove that, if G is a $3-\textit{connected}$ plane graph, then its geometric dual is a simple graph.

Proof. If its geometric dual is not a simple graph, then there are loops or multiple edges. Since the original graph is also the dual graph of its dual graph, there should be a vertex with 2 degrees or 3 degrees, which is a contradiction. □

Problem 4.3.5. Let G be a connected plane graph, prove that G is bipartite if and only if its dual G^* is Eulerian.

Proof. If G is a bipartite, then, every cycle of G has even length. In other words, every face of G is bounded edges of even number. Thus, every vertex in G^* is of even degrees.

If G^* is Eulerian, then, it is can be split into separate cycles. Then, it is planar and thus have a dual graph. If we color white the vertex, in G, corresponding to the infinite face of G^*. Since the circle form a chain with inclusion relation, we can color the vertex corresponding to the face of the biggest circle, that is not overlapped by other samller cycle ,in each chain black, the second white and so on. Thus we get a bipartite G. □

remark. In the above proof, we regard G^* as geometric dual. It should be considered as abstract dual, because every geometric dual is a abstract dual, but not every abstract dual is a geometric dual.

Problem 4.3.6. Prove that if G is a connected plane graph, then any spanning tree in G correspond to the complement of a spanning tree in G^*.

Proof. Since every cycle in G is corresponding to a cutset in G^*. Then, the number of the edges in the complement of the spanning tree in G equal to the one of the spanning tree in G^*. If we switch G with G^*, the statement is still valid. $\square$

4.4 Graphs on other surfaces

Problem 4.4.1. Show that there is no graph of genus $g \geq 1$ such that is regular of degree 4 and in which each face is triangle.

Proof. If there is such graph that is has genus greater than or equal 1 and satisfies other conditions, then

$$\begin{cases} n - m + f = 2 - 2g \\ g \geq 1 \\ 2m = 4n \\ 3f = 2m \end{cases}$$

It follows that

$$12n - 12m + 12f = 24 - 24g = 6m - 12m + 8m = 2m$$

or

$$12g = 12 - m < 12,$$

which is contrary to the second inequility. $\square$

Problem 4.4.2. Obtain a lower bound, analogous to that of Corollary 4.6. for a graph containing no triangles.

Proof. If in G, there is no triangles, then

$$\begin{cases} n - m + f = 2 - 2g \\ 2m \geq 4f \end{cases}$$

What's more, g is an integer. Thus we have

$$\left\lceil 1 + \frac{1}{4}(m - 2n) \right\rceil \leq g.$$

$\square$

Problem 4.4.3. Deduce that $g(K_{r,s}) \geq \left\lceil \frac{1}{4}(r-2)(s-2) \right\rceil$.

Proof. We can deduce it directly from the result of the last question, since $n = r + s, m = rs$. $\square$

4.5 Challenge problem

Problem 4.5.1. Let G be a planar graph with vertex-set $\{v_1, v_2, \cdots, v_n\}$, and let $p_1, p_2, \cdots, p_n$ be any n distinct points in the plane. Given a heuristic argument to show that G can be drawn in the plane in such a way that the point p_i represents the vertex v_i, for each i.

Proof. □

Problem 4.5.2. If r and s are both even, prove that

$$\mathrm{cr}(K_{r,s}) \leq \frac{1}{16} rs(r-2)(s-2),$$

and obtain corresponding results when r and/or s is odd.

Proof. Place the r vertices along the $x-$axis with $\frac{1}{2}r$ vertices on each side of the origin, and the s vertices along the $y-$axis in a similar way; then join up the vertices by straightline segments and count the crossing. Then we have

$$\begin{aligned} \mathrm{cr}(G) &= 4 \cdot [(\tfrac{r}{2}-1)(\tfrac{s}{2}-2) + (\tfrac{r}{2}-2)(\tfrac{s}{2}-3) + \cdots + 2 \cdot 1] \\ &= \end{aligned}$$

□

Problem 4.5.3. Prove that

$$t(K_{r,s}) \geq \left\lceil \frac{rs}{2r+2s-4} \right\rceil$$

Proof. Since there is no triangles in bipartite, so is the each layer of it. It follows that, for each layer of the graph satisfies

$$m_i \leq 2r + 2s - 4$$

By summing up all $t(K_{r,s})$ inequility, we have

$$\sum_{i=1}^{t(K_{r,s})} m_i = rs \leq (2r+2s-4)t(K_{r,s})$$

or

$$t(K_{r,s}) \geq \left\lceil \frac{rs}{2r+2s-4} \right\rceil$$

□

Problem 4.5.4. By splitting $K_{r,s}$ into a number of copies of $K_{2,s}$, prove that, if r is even, then $t(K_{r,s}) \leq r$, and deduce from the last question that

$$t(K_{r,s}) = \frac{1}{2}r \text{ if } s > \frac{1}{2}(r-2)^2$$

Proof. Obviously,

$$t(K_{r,s}) \leq \frac{r}{2}$$

When $s > \frac{1}{2}(r-2)^2$

$$\left\lceil \frac{rs}{2r+2s-4} \right\rceil > \left\lceil \frac{\frac{1}{2}r(r-2)^2}{2r+(r-2)^2-4} \right\rceil = \frac{r}{2} - 1$$

Thus

$$\frac{r}{2} \geq t(K_{r,s}) \geq \left\lceil \frac{rs}{2r+2s-4} \right\rceil \geq \frac{r}{2}$$

□

Problem 4.5.5. Let G be a polyhedral graph and let W be the cycle subspace of G.

1. Show that the polygons bounding the finite faces of G form a basis for W.
2. Deduce Corollary 4.1.

Problem 4.5.6. A graph G^* is a Whitney dual of G is there is a one-one correspondence between $E(G)$ and $E(G^*)$ such that for each subgraph H or G with $V(H) = V(G)$, the corresponding subgraph H^* of G^* satisfies

$$\gamma(H) + \zeta(\bar{H}^*) - \zeta(G^*)$$

where $\bar{H}^*$ is obtained from G^* by deleting the edges of H^*.

1. Show that this generalizes the idea of a geometric dual.
2. Prove that, if G^* is a Whitney dual, then G is a Whitney dual of G^*

Problem 4.5.7.
1. Let G be a non-planar graph that can be drawn without crossings on a Mobius strip. Prove that, with the usual notation, $n - m + f = 1$
2. Show that K_5 and $K_{3,3}$ can be drawn without crossings on the surface of a Mobius strip.

5 Colouring graphs

5.1 Colouring vertices

Problem 5.1.1. What is the chromatic number of the complete tripartite graph $K_{r,s,t}$?

Proof. Is 3. □

Problem 5.1.2. What is the chromatic number of the $k-$cube Q_k?

Proof. The chromatic number of the cube is 2. When $k = 1, 2, 3$, it is obvious, because there are two vertices in that graph. We are now going to finish the proof by induction. Suppose when $n = k$, Q_k's chromatic number is 2. Since we have already prove that there is a Hamiltonian cycle in Q_k, we can denote Q_k by

$$X_0 \to X_1 \to \cdots \to X_0.$$

Q_{k+1} include two Q_k, and the edge joining $\overline{X_i 0}$ and $\overline{X_i 1}$. Since every vertices in Q_{k+1} is of degree $k+1$, there is no other edges connecting the two Q_k.

By induction hypothesis, we can only use two colours to colour the first Q_k, say white and black. Then, we can colour the second Q_k with the same pattern but in the opposite colour.

It is obvious that we colour the Q_{k+1} with only two colours correctly. □

Problem 5.1.3. Let G be a simple graph with n vertices, which is regular of d. By considering the number of vertices that can be assigned the same colour, prove that $\chi(G) \geq \frac{n}{n-d}$

Proof. Since there are only n vertices in the graph and each vertex is regular of d, the number of vertices that can be assigned the same colour won't be greater than $n - d$, otherwise there must be two adjacent vertices in the same colour. Thus, then number of the colours in that graph is greater than or equal $\frac{n}{n-d}$ □

Problem 5.1.4. Let G be a simple planar graph containing no triangles.

1. Using Euler's formula, show that G contains a vertex of degree at most 3.

2. Use induction to deduce that G is $4-$colourable.

Proof. If G is a simple planar graph containing no triangles and no vertices of degree less than 4, then

$$\begin{cases} n - m + f = 2 \\ 2m \geq 4f \\ 2m \geq 4n \end{cases}$$

It follows that

$$4m \geq 4n + 4f = 8 + 4m,$$

which is absurd.

We will prove that G is $4-$colourable by the induction on the number of the vertices in G. When there are less than 5 vertices in G, the statement is trival. When there are k vertices in G, there is a vertex v whose degree is less than 4. By deleting vertex v, we get a graph with $k-1$ vertices. By the induction hypothesis, that graph can be coloured by 4 vertices. Since v's degree is less than 4, after adding that vertex into the graph, we can always colour it properly. □

5.2 Chromatic polynomials

Problem 5.2.1. 1. Prove that the chromatic polynomial of $K_{2,s}$ is

$$k(k-1)^s + k(k-1)(k-2)^s$$

2. Prove that the chromatic polynomial of C_n is $(k-1)^n + (-1)^n(k-1)$

Proof. Two of the vertices in $K_{2,s}$ can be coloured white, and the other coloured black. If we have other k colours, we can first colour one of the white vertex. Thus we have k choice. If another white vertex is coloured the same, then, each of the black vertex have $k-1$ choices. Otherwise, another white vertex have $k-1$ choice and each of the black vertex have $k-2$ choices. Thus, the total choice we have is

$$k(k-1)^s + k(k-1)(k-2)^s$$

For the second equation, we are going to prove by induction on the edge of the cycle graph. When $n = 1, 2$, the equation is obviously valid. When $n = i$,

$$\begin{aligned} P_{C_i}(k) &= P_{C_i - e}(k) - P_{C_i/e}(k) \\ &= k(k-1)^{i-1} - P_{C_{i-1}} \\ &= k(k-1)^{i-1} - (k-1)^{i-1} - (-1)^{i-1}(k-1) \\ &= (k-1)^i + (-1)^i(k-1) \end{aligned}$$

Thus we finish the proof. □

Problem 5.2.2. Prove that, if G is a disconnected simple graph, then its chromatic polynomial $P_G(k)$ is the product of the chromatic polynomials of its components. What can you say about the degree of the lowest non-vanishing term?

Proof. Since the colouring of each components won't interfere each other, given certain kinds of colour, the number of the colouring plan is exactly the product of each component's number of colouring plan. Thus we finished the proof. Since each component's chromatic polynomial's constant term is 0, and it is not a zero polynomial, thus, the degree of the lowest non-vanishing term is greater than the number of the components. □

Problem 5.2.3. Let G be a simple graph with n vertices and m edges. Use induction on m, together with Theorem 5.6. to prove that

1. the coefficient of k^{n-1} is $-m$
2. the coefficients of $P_G(k)$ alternate in sign.

Proof. When $m = 0$, the chromatic polynomial is k^n, when $m = 1$, the chromatic polynomial is $k^{n-1}(k-1)$, which all satisfies the two statements. Suppose when there are i edges the statements are valid. Then, when we have a simple graph G with $n+1$ vertices, since $P_G = P_{G-e} - P_{G/e}$, where e is an edge in G and thus the two chromatic polynomial are belong to simple graph with i edges. What's more, the highest degree of each chromatic polynomial is n and $n-1$ respectively. Since the k^{n-1}'s coefficients is $m-1$ and 1 respectively, we have proved the first statement. Since the nth, $n-2$th,$\cdots$ terms of the first chromatic polynomial have positive coefficients, by induction hypothesis and the fact that the highest term's coefficient is 1, and the other coefficient is negative. Meanwhile, in the second chromatic polynomial, the sign is inversed, which means that the first chromatic polynomial has the same sign as $-P_{G/e}$. Thus, the second statement is true.
remark. It seems that when proving the second statement, we still need the inducion hypothesis on the number of vertices.

□

Problem 5.2.4. Prove that if

$$P_G(k) = k(k-1)^n,$$

then G is a tree on n vertices.

Proof. If there are several components, when the degree of the lowest term is greater than 1, which is contrary to the condition. Thus, G is a connected graph.

The second coefficient is $n-1$, thus G has $n-1$ edges and is a tree. □

5.3 Colouring graph

Problem 5.3.1. The plane is divided into a finite number of regions by drawing infinite straight lines in an arbitrary manner. Show that these regions can be 2-coloured.

Proof. We are going to prove this by induction on the number of line.

Since we place the lines in an arbitrary way, we can regard every vertices in the plane as the crossing of two line. Therefore, whenever we place a new line in the plane, it will lie on a sequence of face the first and the last of which is infinite region. Thus, the original graph is separate into to part by the added line. If we inverse the colour of the regions in the one side, then we can get a new 2-coloured plane. □

5.4 Colouring edges

Problem 5.4.1. Prove that if G is a cubic Hamiltonian graph, then $\chi'(G) = 3$.

Proof. By Handshaking theorem, there must be even number of vertices in G. Thus we can 2-colour the Hamiltonian cycle in it and colour the other edges with the third colour, which is valid since the vertices is of regular 3. □

5.5 Challenge problems

Problem 5.5.1. A graph is $k-$critical if $\chi(G) = k$ and if the deletion of any vertex yields a graph with smaller chromatic number. Prove that if G is $k-$critical, then

1. every vertex of G has degree at least $k - 1$
2. G has no cut-vertices.

Proof. If there is a vertex, say v, with degree less than $k-1$. Then after the deletion of v, we could $k - 1-$colour the remaining graph. However, if we reinstate v, since v is with degree less than $k - 1$, among the $k - 1$ colour, there is at least one colour left for v. Thus the original graph can actully $k - 1-$coloured, which is contrary to $\chi(G) = k$.

□

Problem 5.5.2. Give a upperbound for the least degree of vertices in G and the number of colours needed to colour it, if

1. G has girth r
2. G has thickness t

Proof. Let Δ be the least degree of vertices in G, then

$$\begin{cases} \Delta n \leq 2m \\ rf \leq 2m \\ n - m + f = 2 \end{cases}$$

It follows that

$$\Delta \leq \frac{n-2}{n} \cdot \frac{r}{r-2} \cdot 2 < \frac{2r}{r-2}.$$

We are now going to prove that G is $\Delta+1-$colourable by induction. Say v is the vertex in G that is of degree less than or equal Δ. After the deletion of v, we can $\Delta + 1-$colour the remainging graph by induction hypothesis. However, since v is less then $\Delta+1$ degree, when we reinstate v in the graph, we can always colour it properly.

For the second condition,

$$\begin{cases} t \geq \lceil m/3n-6 \rceil \geq m/3n-6 \\ \Delta n \leq 2m \\ n-m+f=2 \end{cases}$$

It follows that

$$2t(3-\frac{6}{n}) \geq \Delta$$

□

Problem 5.5.3. Let G be a countable graph, each finite subgraph of which is $k-$colourable.

1. Prove that G is $k-$colourable.

2. Deduce that every countable planar graph is $4-$colourable.

Proof. We begin our proof by choosing a vertex from G, say v, and fix the colour of it in every coloured graph containing it. (We can alway achieve this by switch colours.) Then, we are going to use tree to represent the relation between each graph whose center is v. One of the end-vertex of the tree is v, regarded as the zero level. In the nth level, the vertices present the graph consists of all the vertices that can be join with v by a path whose length is less than or equal n, and the edges incident with them. What's more, each graph is corresponding to a unique $k-$colourable colouring, which is assure by the condition. There is no edges join the two vertices in the same level of tree, and the vertex is adjacent to the vertex in the next level if it is the subgraph of it. It is obviously that the tree is a countable graph and there is a path whose end-vertex is v and extend to infinite level. The existence of the path show that we can the whole graph G is $k-$colourable.

The second follows immediately from the fact that every finite graph is $4-$colourable. □

Problem 5.5.4. 1. Let G be a simple graph which is not a null graph. Prove that $\chi'(G) = \chi(L(G))$.

2. Prove theorem 5.12. in the case $\Delta = 3$.

Proof. If $\chi'(G) = k$, then, the edge of G can be $k-$coloured, but cannot $(k-1)-$coloured. Thus, we can colour the vertices in $L(G)$ the same colour as their corresponding edges in G, and get a $k-$coloured $L(G)$, since the vertices in $L(G)$ are adjacent if and only if their corresponding edges in G are also adjacent, which follows that they don't have the same colour. The converse is also true, which means that $L(G)$ cannot be $(k-1)-$coloured. Thus we finish the proof.

We are now going to prove that if G is a simple graph with largest vertex-degree 3, then

$$3 \leq \chi'(G) \leq 4$$

By the last statement we have proved, we only need to prove that

$$3 \leq \chi(L(G)) \leq 4.$$

Since G is a simple graph with largest vertex-degree 3, by theorem 5.2.

$$\chi(L(G)) \leq \Delta L(G) \leq 2 + 2 = 4$$

What's more, suppose v is a vertex in G that is of degree 3. Then, v and the edges incident with it correspond to a triangle in $L(G)$, thus

$$\chi(L(G)) \geq 3.$$

□

Problem 5.5.5. Prove that, if a toroidal graph is embeded on the surface of a torus, then its faces can be coloured with seven colours.

Problem 5.5.6. Let G be a simple graph with an odd number of vertices. Prove that if G is regular of degree Δ, then $\chi'(G) = \Delta + 1$.

6 Matching, marriage and Menger's theorem

6.1 Hall's 'marriage' theorem

Let B be a set of boys, and suppose that each boy in B wishes to marry more than one of his girl friends. Find a necessary and sufficient condition for the harem problem to have a solution.

Proof. If a boy want to marry k gir friends, then, we can replace him with k person who have the same set of girl friends. □

Problem 6.1.1. Let E be the set $\{1, 2, \cdots, 50\}$. How many different transverals has the family $\{\{1, 2\}, \{2, 3\}, \{3, 4\}, \cdots, \{50, 1\}\}$?

Proof. If we pick 1 from $\{1, 2\}$, then we can only pick 50 from $\{50, 1\}$, and then 49 from $\{49, 50\}$ and so on. If we pick 2 from $\{1, 2\}$, then we can only pick 3 from $\{2, 3\}$, 4 from $\{3, 4\}$ and so on. Therefore, we only have 2 possible transveral. □

Problem 6.1.2. Write the statement of Corollary 6.1. in marriage terminology.

Proof. If there are m girls, each of whom knows several boys, then a necessary and sufficient condition for only t girls can find proper boys who are different from each other is that any k girls collectively know at least $k + t - m$ boys. □

6.2 Network flows

Problem 6.2.1. Consider a network with several sources and sinks. Show how the analysis of the flows in this network can be reduced to the standard case by the addition of a new'source vertex' and 'sink vertex.'

Proof. By adding a new source vertex as the source of all the original source and a new sink vertex as the sink of all the original sink. □

6.3 Challenge problems

Problem 6.3.1.

1. Use the marriage condition to show that if each girl has $r(\geq 1)$ boy friends and each boy has r girl friends, then the marriage problem has a solution.
2. Use the result of (1) to prove that, if G is bipartite graph which is regular of degree r, then G has a complete matching. Deduce that the chromatic index of G is r.

Proof. For $k \leq m$, if any k girls only collectively know $s(\leq k-1)$ boys. It follows that

$$r \cdot s = k \cdot r$$

or

$$s = k$$

which is absurd. Therefore, for any k girls, they must collectively know at least k boys. By theorem 6.1., we have a solution.

We can colour boys white and girls black, then the solution is the desired complete mathching.

We only need to provide a $r-$colours drawing. We can index the white vertices with $a_1, a_2, \cdots, a_n$ and black vertices with $b_1, b_2, \cdots, b_n$. Then, we join a_i with $b_i, b_{i+1}, b_{i+2}, b_{i+3}, \cdots, b_{i+r}$ and colour the edges with colour $1, 2, \cdot, r$ respectively. □

Problem 6.3.2. Suppose that the marriage condition is satisfied, and that each of the m girls knows at least t boys. Show that the marriages can be arranged in at least $t!$ ways if $t \leq m$, and in at least $\frac{t!}{(t-m)!}$ ways if $t > m$.

Proof. When $m = 1$, there is only 1 girl and when she know t boys, there is exactly $t = \frac{t!}{(t-1)!}$ choice. Suppose the statement is true when m is less than k. Then, when $m = k$, □

Problem 6.3.3. Let E and $\mathcal{F}$ have their usual meanings, let T_1 and T_2 be transverals of $\mathcal{F}$, and let x be an element of T_1. Show that there exists an element y of T_2 such that $(T_1 - \{x\}) \cup \{y\}$ is also a transveral of $\mathcal{F}$.

Proof. □

Problem 6.3.4. Let $\mathcal{F}$ be a family consisting of m non-empty subsets of E, and let A be a subset of E. By applying theorem 6.1. to the family consisting of $\mathcal{F}$, together with $|E| - m$ copies of $E - A$, prove that there is a transveral of $\mathcal{F}$ containing A if and only if

1. $\mathcal{F}$ has a transveral
2. A is a partial transveral of $\mathcal{F}$

Problem 6.3.5. The rank $r(A)$ of a subset A of E is the number of elements in the largest partial transveral of $\mathcal{F}$ contained in A. Show that

1. $0 \leq r(A) \leq |A|$
2. if $A \subseteq B \subseteq E$, then $r(A) \leq r(B)$
3. if $A, B \subseteq E$, then $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$.

Proof. Since a partial transveral is a non-empty set and it is contained in A, the first inequility is obvious.

Whenever a partial transveral is in A, it is also in B, thus the second inequility is also obvious.

□

Problem 6.3.6. Let E be a countable set, and let $\mathcal{F} = (S_1, S_2, \cdots)$ be a countable family of non-empty finite subsets of E Defining a transveral of $\mathcal{F}$ in the natural way, show, by theorem 2.7. that $\mathcal{F}$ has a transveral if and only if the union of any k subset S_i contains at least k elements, for all finite k.

Proof. $(\rightarrow)$ is obvious.

$(\leftarrow)$ we can construct a tree, in which every vertex stand for a transveral of finite family. According to theorem 2.7., such tree reaches to infinity. □

Problem 6.3.7. Prove theorem 6.4.

Problem 6.3.8. Show how the theorem 6.7.can be used to prove theorem 6.1.