

73. When a foreign object lodged in the trachea (windpipe) forces a person to cough, the diaphragm thrusts upward causing an increase in pressure in the lungs. This is accompanied by a contraction of the trachea, making a narrower channel for the expelled air to flow through. For a given amount of air to escape in a fixed time, it must move faster through the narrower channel than the wider one. The greater the velocity of the airstream, the greater the force on the foreign object. X rays show that the radius of the circular tracheal tube contracts to about two-thirds of its normal radius during a cough. According to a mathematical model of coughing, the velocity  $v$  of the airstream is related to the radius  $r$  of the trachea by the equation

$$v(r) = k(r_0 - r)r^2 \quad \frac{1}{2}r_0 \leq r \leq r_0$$

where  $k$  is a constant and  $r_0$  is the normal radius of the trachea. The restriction on  $r$  is due to the fact that the tracheal wall stiffens under pressure and a contraction greater than  $\frac{1}{2}r_0$  is prevented (otherwise the person would suffocate).

- (a) Determine the value of  $r$  in the interval  $[\frac{1}{2}r_0, r_0]$  at which  $v$  has an absolute maximum. How does this compare with experimental evidence?

- (b) What is the absolute maximum value of  $v$  on the interval?  
(c) Sketch the graph of  $v$  on the interval  $[0, r_0]$ .

74. Show that 5 is a critical number of the function

$$g(x) = 2 + (x - 5)^3$$

but  $g$  does not have a local extreme value at 5.

75. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

76. If  $f$  has a minimum value at  $c$ , show that the function  $g(x) = -f(x)$  has a maximum value at  $c$ .

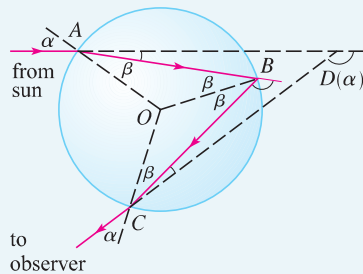
77. Prove Fermat's Theorem for the case in which  $f$  has a local minimum at  $c$ .

78. A cubic function is a polynomial of degree 3; that is, it has the form  $f(x) = ax^3 + bx^2 + cx + d$ , where  $a \neq 0$ .

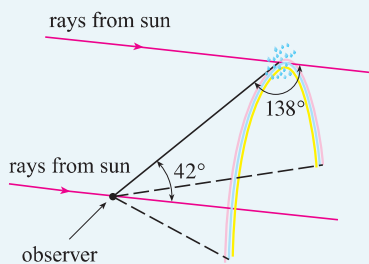
- (a) Show that a cubic function can have two, one, or no critical number(s). Give examples and sketches to illustrate the three possibilities.  
(b) How many local extreme values can a cubic function have?

## APPLIED PROJECT

### THE CALCULUS OF RAINBOWS



Formation of the primary rainbow



Rainbows are created when raindrops scatter sunlight. They have fascinated mankind since ancient times and have inspired attempts at scientific explanation since the time of Aristotle. In this project we use the ideas of Descartes and Newton to explain the shape, location, and colors of rainbows.

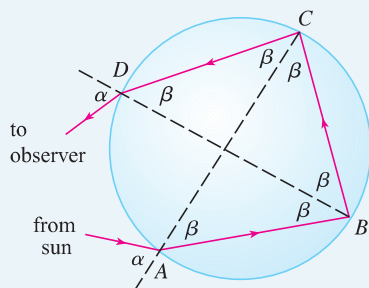
1. The figure shows a ray of sunlight entering a spherical raindrop at  $A$ . Some of the light is reflected, but the line  $AB$  shows the path of the part that enters the drop. Notice that the light is refracted toward the normal line  $AO$  and in fact Snell's Law says that  $\sin \alpha = k \sin \beta$ , where  $\alpha$  is the angle of incidence,  $\beta$  is the angle of refraction, and  $k \approx \frac{4}{3}$  is the index of refraction for water. At  $B$  some of the light passes through the drop and is refracted into the air, but the line  $BC$  shows the part that is reflected. (The angle of incidence equals the angle of reflection.) When the ray reaches  $C$ , part of it is reflected, but for the time being we are more interested in the part that leaves the raindrop at  $C$ . (Notice that it is refracted away from the normal line.) The *angle of deviation*  $D(\alpha)$  is the amount of clockwise rotation that the ray has undergone during this three-stage process. Thus

$$D(\alpha) = (\alpha - \beta) + (\pi - 2\beta) + (\alpha - \beta) = \pi + 2\alpha - 4\beta$$

Show that the minimum value of the deviation is  $D(\alpha) \approx 138^\circ$  and occurs when  $\alpha \approx 59.4^\circ$ .

The significance of the minimum deviation is that when  $\alpha \approx 59.4^\circ$  we have  $D'(\alpha) \approx 0$ , so  $\Delta D / \Delta \alpha \approx 0$ . This means that many rays with  $\alpha \approx 59.4^\circ$  become deviated by approximately the same amount. It is the *concentration* of rays coming from near the direction of minimum deviation that creates the brightness of the primary rainbow. The figure at the left shows that the angle of elevation from the observer up to the highest point on the rainbow is  $180^\circ - 138^\circ = 42^\circ$ . (This angle is called the *rainbow angle*.)

2. Problem 1 explains the location of the primary rainbow, but how do we explain the colors? Sunlight comprises a range of wavelengths, from the red range through orange, yellow,



Formation of the secondary rainbow



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green, blue, indigo, and violet. As Newton discovered in his prism experiments of 1666, the index of refraction is different for each color. (The effect is called *dispersion*.) For red light the refractive index is  $k \approx 1.3318$  whereas for violet light it is  $k \approx 1.3435$ . By repeating the calculation of Problem 1 for these values of  $k$ , show that the rainbow angle is about  $42.3^\circ$  for the red bow and  $40.6^\circ$  for the violet bow. So the rainbow really consists of seven individual bows corresponding to the seven colors.

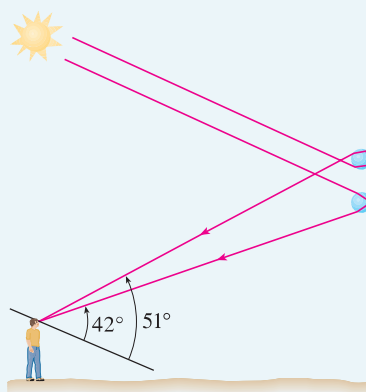
3. Perhaps you have seen a fainter secondary rainbow above the primary bow. That results from the part of a ray that enters a raindrop and is refracted at  $A$ , reflected twice (at  $B$  and  $C$ ), and refracted as it leaves the drop at  $D$  (see the figure). This time the deviation angle  $D(\alpha)$  is the total amount of counterclockwise rotation that the ray undergoes in this four-stage process. Show that

$$D(\alpha) = 2\alpha - 6\beta + 2\pi$$

and  $D(\alpha)$  has a minimum value when

$$\cos \alpha = \sqrt{\frac{k^2 - 1}{8}}$$

Taking  $k = \frac{4}{3}$ , show that the minimum deviation is about  $129^\circ$  and so the rainbow angle for the secondary rainbow is about  $51^\circ$ , as shown in the figure.



4. Show that the colors in the secondary rainbow appear in the opposite order from those in the primary rainbow.

## 4.2 THE MEAN VALUE THEOREM

We will see that many of the results of this chapter depend on one central fact, which is called the Mean Value Theorem. But to arrive at the Mean Value Theorem we first need the following result.

■ Rolle's Theorem was first published in 1691 by the French mathematician Michel Rolle (1652–1719) in a book entitled *Méthode pour résoudre les égalités*. He was a vocal critic of the methods of his day and attacked calculus as being a "collection of ingenious fallacies." Later, however, he became convinced of the essential correctness of the methods of calculus.

**ROLLE'S THEOREM** Let  $f$  be a function that satisfies the following three hypotheses:

1.  $f$  is continuous on the closed interval  $[a, b]$ .
2.  $f$  is differentiable on the open interval  $(a, b)$ .
3.  $f(a) = f(b)$

Then there is a number  $c$  in  $(a, b)$  such that  $f'(c) = 0$ .