SIMATS

School of Engineering

Discrete Mathematics

Science & Humanities

Saveetha Institute of Medical And Technical Sciences,Chennai.

UBA04-DISCRETE MATHEMATICS

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UNIT: I PROPOSITIONAL CALCULUS

Propositional Logic

Logic: Logic is the science of reasoning. It helps us to understand and reason about different mathematical statement.

Proposition: It is a declarative statement that is either true or false but not both. The truthvalue of proposition is T or F

Ex:- The sum of first 'n' positive integer is $\frac{n(n+1)}{2}$

Delhi is the Capital of India

Connectives :-

Logical Operators :-

$\neg P$ – Not P

$P \wedge q$ – P and q

$P \vee q$ – P or q

Conjunction :-

P	q	$P \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction :-

P	q	$P \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation :-

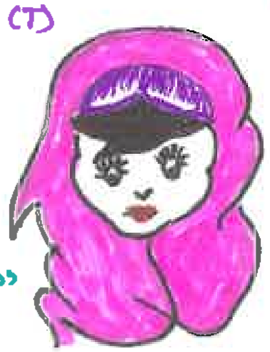
P	$\neg P$
T	F
F	T

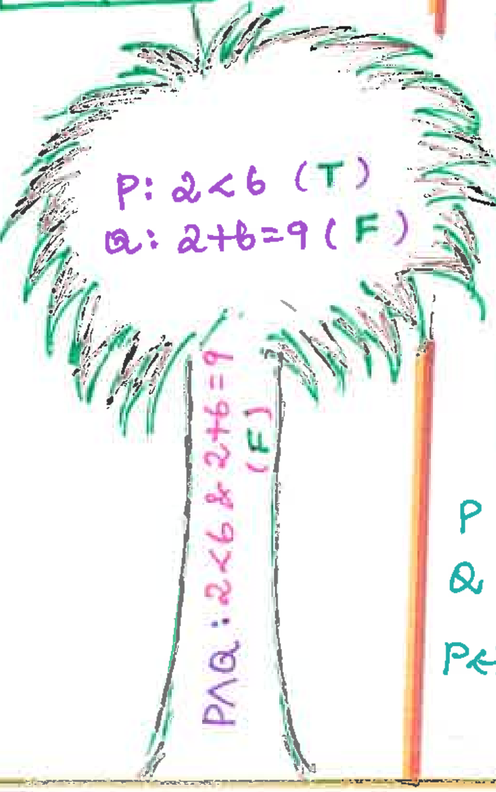
P: It is snowing (T)

Q: I am Cold.

$\boxed{P \wedge Q}$

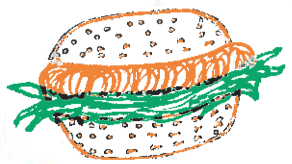
"It is snowing and I am Cold" (T)





* If P and Q are two statement
* Then the stmt. $P \to Q$ which is read as If P Then Q

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T



I am Hungry

P: I am Hungry

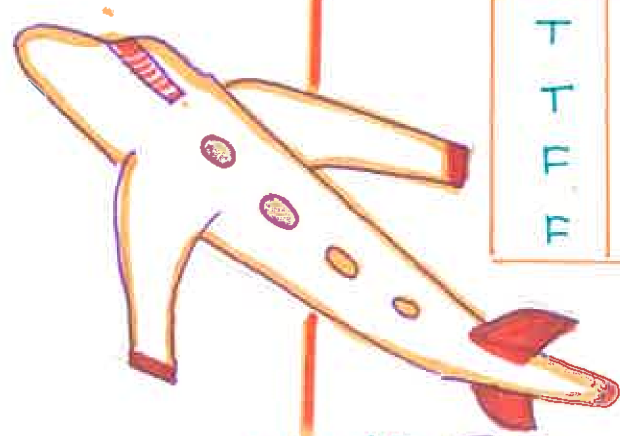
Q: I will Eat

$P \to Q$: If I am Hungry, then I will Eat

Bi-Conditional Statement

* If P and q are 2 stmts.
* Then $P \leftrightarrow Q$ which is read "P&q" is call Bi-Conditional stmt

P	Q	$P \leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T



P: You can take the flight

Q: You buy a ticket

$P \leftrightarrow Q$: You can take the flight if and only if You buy a ticket.

EXAMPLE:

TRUTH TABLE

Construct the truth table for the following proposition

$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$

P	Q	$\neg P$	$\neg Q$	$P \wedge Q$	$\neg(P \wedge Q)$	$\neg P \vee \neg Q$	$\neg(P \wedge Q) \leftrightarrow (\neg P \vee \neg Q)$
T	T	F	F	T	F	F	T
T	F	F	T	F	T	T	T
F	T	T	F	F	T	T	T
F	F	T	T	F	T	T	T

Last Column contains all truth values "T"
From this column we can say that given proposition follows "TAUTOLOGY"

EXAMPLE:

"The crop will be destroyed if there is a flood"

Construct the truth table for

$(P \to Q) \to (Q \to P)$

P	Q	$P \to Q$	$Q \to P$	$(P \to Q) \to (Q \to P)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

Propositions Considered :-

P: Crop will be destroyed

Q: There is a flood.

Negation Example :-

P: Today is Thursday → $\neg P$: Today is not Thursday

ALGEBRA OF PROPOSITION

Contradiction if the truth value of E is false for all combination values of the variable

Tautology if the truth value of E is true for all combination of values of the variable

A negation of contradiction is →

A logical expression E in a contain number of logical variable is said to be :

Examples: Indicate which one are tautology or contradiction

i) $((p \to \neg p)) \to \neg p$

p	$\neg p$	$p \to \neg p$	$((p \to \neg p)) \to \neg p$
T	F	F	T
F	T	T	T

Hence it is a tautology

ii) $(\neg q \wedge p) \wedge q$

p	q	$\neg q$	$\neg q \wedge p$	$(\neg q \wedge p) \wedge q$
T	T	F	F	F
T	F	T	T	F
F	T	F	F	F
F	F	T	F	F

Hence it is a contradiction

Three related conditional statements

Implication : $p \to q$

Contrapositive	Converse	Inverse
$\neg q \to \neg p$	$q \to p$	$\neg p \to \neg q$.
If not q then not p	If q then p	If not p then not q

Compound proposition

Combination of one or more atomic statements using connectives are called compound proposition

Logical equivalence and implications

$p \Leftrightarrow q$

The proposition p & q are called logically equivalent is $p \leftrightarrow q$ is a tautology

$p \Rightarrow q$

Let p & q are two proposition. then we say p implies q is $p \to q$ is a tautology

Logical equivalences

Absorption law

$p \vee (p \wedge q) \Leftrightarrow p$

$p \wedge (p \vee q) \Leftrightarrow p$

Negation law

$p \vee \neg p \Leftrightarrow T$

$p \wedge \neg p \Leftrightarrow F$

Identity laws

$p \wedge T \Leftrightarrow p$

$p \vee F \Leftrightarrow p$

Domination laws

$p \vee T \Leftrightarrow T$

$p \wedge F \Leftrightarrow F$

Distribution laws

$p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$

$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$

De Morgan's law

$\neg (p \wedge q) \Leftrightarrow \neg p \vee \neg q$

$\neg (p \vee q) \Leftrightarrow \neg p \wedge \neg q$

Idempotent laws

$p \vee p \Leftrightarrow p$

$p \wedge p \Leftrightarrow p$

Double negation

$\neg(\neg p) \Leftrightarrow p$

Commutative

$p \vee q \Leftrightarrow q \vee p$

$q \wedge p \Leftrightarrow p \wedge q$

Associative law

$(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$

$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$

Logical equivalences involving conditionals

$p \to q \Leftrightarrow \neg p \vee q$

$p \to q \Leftrightarrow \neg q \to \neg p$

$p \vee q \Leftrightarrow \neg p \to q$

$p \wedge q \Leftrightarrow \neg (p \to \neg q)$

$\neg (p \to q) \Leftrightarrow p \wedge \neg q$

chain rule

Logical equivalences involving Bi-conditional

$p \leftrightarrow q \Leftrightarrow (p \to q) \wedge (q \to p)$

$p \leftrightarrow q \Leftrightarrow (p \wedge q) \vee (\neg p \wedge \neg q)$

$\neg (p \leftrightarrow q) \Leftrightarrow p \leftrightarrow \neg q$.

Examples 2: - S.T $\neg(p \vee (\neg p \wedge q))$ and $\neg p \wedge \neg q$. are logically equivalent 2.

	Reason
$\neg(p \vee (\neg p \wedge q))$	
$\Leftrightarrow \neg p \wedge \neg(\neg p \wedge q)$	De morgan's law
$\Leftrightarrow \neg p \wedge (p \vee \neg q)$	De morgan's law
$\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge \neg q)$	Distribution law
$\Leftrightarrow F \vee (\neg p \wedge \neg q)$	Negation law
$\Leftrightarrow \neg p \wedge \neg q$	identity law.

Examples 3: show the following is an implication $p \to (q \to r) \Rightarrow (p \to q) \to (p \to r)$

Solution :. To prove $(p \to (q \to r)) \to ((p \to q) \to (p \to r))$ is a tautology.

	Reason
i) $p \to (q \to r)$	
$\Leftrightarrow \neg p \vee (q \to r)$	conditional equivalences
$\Leftrightarrow \neg p \vee (\neg q \vee r)$	
$\Leftrightarrow (\neg p \vee \neg q) \vee r$	Associative law
$\Leftrightarrow \neg (p \wedge q) \vee r$	De morgan.

	Reason
ii) $(p \to q) \to (p \to r)$	
$\Leftrightarrow (\neg p \vee q) \to (\neg p \vee r)$	conditional equivalence
$\Leftrightarrow (\neg(\neg p \vee q)) \to \vee (\neg p \vee r)$	
$\Leftrightarrow (p \wedge \neg q) \vee (\neg p \vee r)$	De morgan's & Double negation
$\Leftrightarrow (\neg q \wedge p) \vee (\neg p \vee r)$	commutative
$\Leftrightarrow (\neg q \vee (\neg p \vee r)) \wedge (p (\neg p \vee r))$	Distribution
$\Leftrightarrow ((\neg q \vee \neg p) \vee r) \wedge ((p \vee \neg p) \vee r)$	Associative
$\Leftrightarrow (\neg (q \wedge p) \vee r) \wedge (T \vee r)$	Negation law
$\Leftrightarrow (\neg (q \wedge p) \vee r) \wedge T\ \neg (q \wedge p) \vee r$	Domination law
	Identity law

$\neg (p \wedge q) \vee r \to \neg (q \wedge p) \vee r$

$\Leftrightarrow \neg (\neg (p \wedge q \vee r) \vee (\neg q \wedge p) \vee r$

$\Leftrightarrow T$

Hence the proof.

Elementary Product :-

A Product of the variable & their negation. ex). $P \wedge Q$, $P \wedge \neg P$

Elementary Sum :-

A sum of the variable & their negation. ex) $P \vee Q$, $P \vee \neg P$

Normal form

- **Disjunctive Normal form (DNF)**: statement formula consists of a sum of elementary Product.
- **Conjunctive Normal form (CNF)**: statement formula consist of a sum of elementary sum.

Procedure to obtain DNF & CNF

① Replace $\rightarrow$ & $\leftrightarrow$ by $\wedge, \vee$ & $\neg$

② If negation is present before a given formula apply De' Morgan's law

③ If necessary, Apply other logic equivalences.

Illustration :- Obtain the disjunctive normal form of $P \wedge (P \rightarrow Q)$

let $S \Leftrightarrow P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$

$\Leftrightarrow (P \wedge \neg P) \vee (P \wedge Q)$

which is the required DNF

NORMAL FORMS

Min term $P \wedge Q \wedge \neg R$

The product in which each variable or its negation but not both occurs only once

Max term $P \vee Q \vee \neg R$

The Sum in which each variable or its negation but not both occurs only once.

Principal Disjunctive Normal form

Disjunction of min terms $\rightarrow$ Sum of Product

Illustration :- Obtain PDNF of $P \leftrightarrow Q$. Also find PCNF

let $S \Leftrightarrow P \leftrightarrow Q$

P	Q	S	Min term	Max term
T	T	T	$P \wedge Q$	
T	F	F		$\neg P \vee Q$
F	T	F		$P \vee \neg Q$
F	F	T	$\neg P \wedge \neg Q$	

PDNF

$\therefore S \Leftrightarrow (P \wedge Q) \vee (\neg P \wedge \neg Q)$

PCNF

$S \Leftrightarrow (\neg P \vee Q) \wedge (P \vee \neg Q)$

Principal Conjunctive Normal form

Conjunction of Max terms $\rightarrow$ Product of Sum

PDNF from PCNF :-

① Write the Product of remaining Max term $(\neg S)$

② find the negation of the CNF $\neg S$ by duality principle

$\neg(\neg S) \Leftrightarrow S$

PCNF from PDNF :-

① Write the sum of remaining min term $(\neg S)$

② find the negation of the DNF $\neg S$ by duality Principle

$\neg(\neg S) \Leftrightarrow S$

Illustration :- Obtain the Product of Sums canonical form for $(P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$?

Solution :-

Given $S \Leftrightarrow (P \wedge Q \wedge R) \vee (\neg P \wedge Q \wedge R) \vee (\neg P \wedge \neg Q \wedge \neg R)$

the remaining min term of P, Q & R are $P \wedge \neg Q \wedge R$, $P \wedge Q \wedge \neg R$, $\neg P \wedge \neg Q \wedge R$, $P \wedge \neg Q \wedge \neg R$, $\neg P \wedge Q \wedge \neg R$

$\therefore \neg S \Leftrightarrow (P \wedge \neg Q \wedge R) \vee (P \wedge Q \wedge \neg R) \vee (\neg P \wedge \neg Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R)$

$\neg(\neg S) \Leftrightarrow (\neg P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee R) \wedge (P \vee Q \vee \neg R) \wedge (\neg P \vee \neg Q \vee \neg R) \wedge (P \vee \neg Q \vee R)$

which is the required Product of Sum

Illustration :- Without using truth Table find the PCNF of $(p \vee q) \wedge (r \vee \neg p) \wedge (q \vee \neg r)$

$S \Leftrightarrow ((p \vee q) \vee (r \wedge \neg r)) \wedge ((\neg p \vee r) \vee (q \wedge \neg q)) \wedge ((q \vee \neg r) \vee (p \wedge \neg p))$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee \neg r)$

$\Leftrightarrow (p \vee q \vee r) \wedge (p \vee q \vee \neg r) \wedge (\neg p \vee q \vee r) \wedge (\neg p \vee \neg q \vee r) \wedge (\neg p \vee q \vee \neg r)$ //

Illustration :- Obtain CNF of $P \wedge (P \rightarrow Q)$

let $S \Leftrightarrow P \wedge (P \rightarrow Q)$

$\Leftrightarrow P \wedge (\neg P \vee Q)$ //

Rule P and Rule T

Rule P → A premise may be introduced at any point in the derivation

Rule T → A formula S may be introduced in a derivation is S is tautologically imply the other

Arguments

The set of given statements followed by a conclusion.

Illustration:- "If it rains heavily, then travelling will be difficult". "If students arrive on time, then travelling was not difficult". "They arrive on time therefore, it didnot rain heavily"

Validity of Arguments

Any conclusion which is arrived by following the rules of inference is called a valid conclusion and the argument is called a valid argument

Illustration:- show $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$

Indirect Method

S.N	Statement	Rule
1.	$\neg\neg(P \wedge Q)$	P (assumed)
2.	$P \wedge Q$	T (1)
3.	P	T (2)
4.	$\neg P \wedge \neg Q$	P
5.	$\neg P$	P
6.	$P \wedge \neg P$	Contradiction (F)

Rules of Inference

Direct Proof:- A direct Proof is a Proof in which the truth of the premise directly shows the truth of the conclusion

Indirect Method of Proof:- A Indirect proof proceeds by assuming P is true, but also c is false & then using $P, \neg C$ as well as other premises to deduce a contradiction.

Illustration:- Find the direct Proof of $\neg P \vee Q,\ S \vee P,\ \neg Q \Rightarrow S$

S.N	Statement	Rule
1.	$\neg P \vee Q$	P
2.	$P \to Q$	T
3.	$\neg Q \to \neg P$	T
4	$\neg Q$	P
5.	$\neg P$	(3), (4), T
6.	$S \vee$	P
7.	$\neg S \to P$	T
8	$\neg P \to S$	T
9	S	(5), (8), T

Illustration:- Show that the following Premises are inconsistent.

1. If Jack misses many classes through illness, then he fails high school.
2. If Jack fails high school, then he is uneducated.
3. If Jack read a lot of book, then he is not uneducated.
4. Jack misses many classes through illness and reads a lot of books

Solution:-

A : Jack read a lot of book

E : Jack misses many classes

H : Jack is uneducated

S : Jack fails high school

S.N	Statement	Rule
1.	$E \to S$	Rule P
2.	$S \to H$	P
3.	$E \to H$	T
4.	$A \to \neg H$	P
5.	$H \to \neg A$	T, (4)
6.	$E \to \neg A$	(3), (5), T
7.	$\neg E \vee \neg A$	T
8.	$\neg(E \wedge A)$	T
9	$E \wedge A$	P
10	$(E \wedge A) \wedge \neg(E \wedge A)$	T (8), (9) Contradiction

Illustration:- Find the direct Proof of $P \to (Q \to S),\ \neg R \vee P,\ Q \Rightarrow R \to S$

S.N	Statement	Rule
1.	$\neg R \vee P$	P
2.	$R \to P$	T
3.	R	Assumed Premise
4.	P	T, (2), (3)
5.	$P \to (Q \to S)$	P
6.	$Q \to S$	T, (4), (5)
7.	Q	P
8	$S \quad \therefore R \to S$	T (6), (7)

STATEMENT FUNCTIONS AND QUANTIFIERS

Statement Functions

Simple functions

An expression consist of a predicate symbol & individual variables

Compound stmt. fns.

Combining one or more simple stmt fns by logical connections

Ex:

$T(x)$: "x is a Teacher"

$T(J)$: "John is a Teacher"

Ex:

$M(x) \wedge L(x)$ "x is a number & x is a leader"

$M(x) \rightarrow L(x)$ "If x is a mentor then x is a leader"

2-place Predicates : $\boxed{T(x,y)}$: "x is taller than y"

3-place Predicates : $\boxed{S(V,R,B)}$: "Vinay sits b/w Ralph & Bill"

Predicate :-

Stmt : $x > 4$

Declarate : "x is / greater than 4"

Variable Predicate

Ravi / is a Painter

object Predicate

Quantifiers

"Quantity"

Answer for the Questions "How many"

Universal Quantifiers

All, Some, None, One

All → $\forall x$

Some → $\exists x$ "Existential Quantifiers"

Ex :-

All isosceles triangle are Equiangular.

$(\forall x)[I(x) \rightarrow E(x)]$

$I(x)$ —If & then→ $E(x)$

x is isosceles x is Equiangular

Negation of Quantified stmts :-

Stmt	Negation
$x > 5$	$x \le 5$
"All the integers greater than 5"	"There is an integer less than or equal to 5"
$V(x)$ $G(x)$	$H(x)$ $L(x)$
{G - Greater than 5}	{L = lesser than or equal to 5}
$x \in$ Integer	$x \in$ Integers

Bound Variable

When a Quantifier is used on a predicate Variable x or when a value is assigned to this Variable

Free Variable

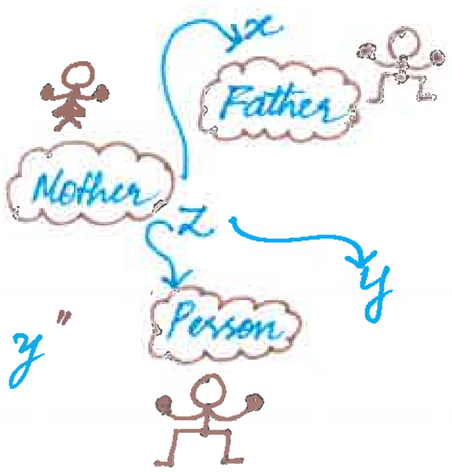
When occurance of Variable x is not bounded by a Quantifier or not set to a particular value

Example :

$P(x)$: x is a person

$F(x,y)$: x is the father of y

$M(x,y)$: x is the Mother of y



Write the Predicate for

"x is the father of the Mother of y"

Solution :

$((\exists z)\ (P(z) \wedge F(x,z) \wedge M(z,y))$

Example : "All the world loves a lover"

$P(x)$: x is a person

$L(x)$: x is a lover

$R(x,y)$: x loves y

$(x)\ (P(x) \rightarrow (y)\ (P(y) \wedge L(y) \rightarrow R(x,y)))$

Nested Quantifiers :-

For any 2-place predicate formula the following possibilities are :

$\forall x \ \forall y \ P(x,y)$

$\exists x \ \exists y \ P(x,y)$

$\forall x \ \exists y \ P(x,y)$

$\exists x \ \forall y \ P(x,y)$

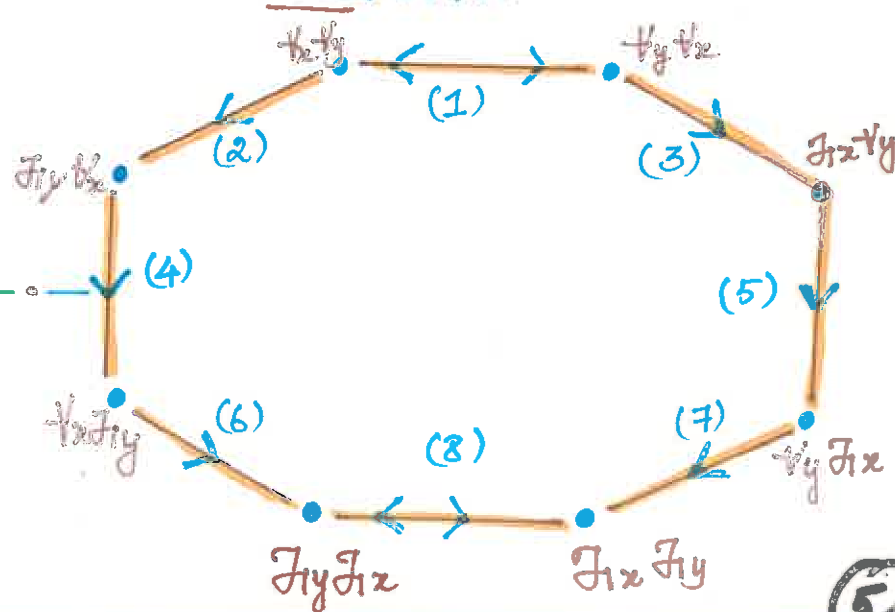
Bound $\forall x\ [P(x) \rightarrow Q(x)]$

Bound $\exists x\ P(x) \wedge Q(x)$ Free

Negate $\forall x \ \exists y\ [P(x) \wedge Q(y)]$

$\neg \{\forall x \ \exists y\ [P(x) \wedge Q(y)]\}$

$\exists x \ \forall y\ [\neg P(x) \vee \neg Q(y)]$



Universal & Existential Rules

Universal specification : Rule US

$\forall x\ P(x) \Rightarrow P(c)$

Existential specification : Rule ES

$\exists x\ P(x) \Rightarrow P(c)$

Universal generalisation : Rule UG

$P(c) \Rightarrow \forall x\ P(x)$

Existential Generalisation : Rule EG

$P(c) \Rightarrow \exists x\ P(x)$

Validity of Stmts:

Example:

Famous Socrates arguement:

'All men are mortal'

'Socrates is a man'

Therefore, 'Socrates is mortal'

Solution :-

$H(x)$: x is a man

$M(x)$: x is mortal

S : socrates

$(\forall x)\,[H(x) \to M(x)] \wedge H(S) \Rightarrow M(S)$

S.no	Stmt	Reason
1	$\forall x\,[H(x) \to M(x)]$	Rule P
2	$H(S) \to M(S)$	Rule US $\{\forall x P(x) \Rightarrow P(c)\}$
3	$H(S)$	Rule P
4	$M(S)$	T, $H(S)$, $H(S) \to M(S)$

Inference Theory of Predicate Calculus

Derivations of formal proof in predicate calculus = Derivations of formal proof in statement calculus.

"Rule P, Rule T and Rule CP are also same"

Logical Equivalences & Implications to be Recalled...

$P \to Q = \neg P \vee Q$

$P,\ P \to Q \Rightarrow Q$

$\neg Q,\ P \to Q \Rightarrow \neg P$

$P \to Q,\ Q \to R = P \to R$

$P \to Q \Leftrightarrow \neg Q \to \neg P$

$P \wedge \neg P = F$ & $P \vee \neg P = T$

Examples:

Show that the premises "One student in the class knows how to write Java programme"; "Everyone who knows how to write in Java, can get a high-paying job" implies the conclusion "Someone in this class can get high-paying job"

Solution:-

$C(x)$: x is in this class

$J(x)$: x knows Java prog

$H(x)$: x can get high-paying job.

Premises:

$\exists x\ [C(x) \wedge J(x)]$

$\forall x\ [J(x) \to H(x)]$

→ are the premises

S.no	Stmt	Reason
1.	$\exists x\,[C(x) \wedge J(x)]$	Rule P
2.	$C(a) \wedge J(a)$	Rule ES
3.	$C(a)$	
4.	$J(a)$	
5.	$\forall x\,[J(x) \to H(x)]$	Rule P
6.	$J(a) \to H(a)$	Rule US
7.	$H(a)$	T, from 4, 6
8.	$C(a) \wedge H(a)$	T, from 3, 7
9.	$\exists x\,[C(x) \wedge H(x)]$	Rule EG

Examples:

Through Indirect method, show that

$\forall x\,[P(x) \vee Q(x)] \Rightarrow [\forall x\,P(x)] \vee [\exists x\,Q(x)]$

Solution:

S.no	Stmt	Reason
1.	$\neg \forall x\,[P(x) \vee Q(x)]$	Rule CP
2.	$\exists x\,\{\neg P(x) \wedge \neg Q(x)\}$	
3.	$\neg P(a) \wedge \neg Q(a)$	Existential Specification
4.	$\neg P(a)$	
5.	$\neg Q(a)$	
6.	$\neg\{P(a) \vee Q(a)\}$	
7.	$\forall x\,[P(x) \vee Q(x)]$	Rule P
8.	$\{P(a) \vee Q(a)\}$	universal specification
9.	F	contradiction by 6 & 8

∴ Our Assumption leads to contradiction

SET THEORY

BASIC CONCEPTS & NOTATIONS

CARDINALITY: Let A be finite set. The number of different elements in A is called cardinality and is denoted as $|A|$, $n(A)$ or $\# A$

Example: $A = \{a, b, c\}$

$n(A) = 3$

Finite & Infinite sets:

* A set which has finite number of elements is a finite set.
* A set which has not finite is called an infinite set.

SINGLETON SET: A set having only one element is called singleton set

SET: A set is a collection of well defined objects called elements

Example:

$N = \{1, 2, 3, \dots\}$ = set of Natural numbers

$X = \{1, 2, a, b, c\}$

$Y = \{0, 1, a, b\}$

SUBSET: Let A and B are two sets. Then A is a subsets of B ($A \subseteq B$) if every element of A is again an element of B

EQUAL SETS: Two sets A & B are equal if and only if $A \leq B$ & $B \leq A$ (have same elements)

Example:

$A = \{a, b, c\}$;

$B = \{b, a, c\}$

UNIVERSAL SET: A set E called a universal set if E is the superset of all sets.

Example: $E = \{1, 2, 3 \dots, 10\}$ $C = \{5, 10\}$

$A = \{1, 3, 5, 7, 9\}$ $B = \{2, 4, 6, 8, 10\}$

Clearly, E is the superset of all A, B, C and hence E is a universal set

POWER SET: The power set of a set A is a collection or family of all subjects of A and is denoted as $P(A)$

NULL SET OR EMPTY SET

A set having no elements is called a Null set or Empty set.

Example: $A = \{x : x \in N, 1 < x < 2\}$

OPERATION ON SETS

DISJOINT SETS: Two sets are said to be disjoint if $A \cap B = \phi$

Example:

$A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$

$A \cap B = \phi$

UNION: Let A and B are two sets. The union of A & B is defined as a set of all elements of A and the set of all elements of B & common elements being taken once is given by $A \cup B = \{x : x \in A \text{ or } x \in B\}$

INTERSECTION: Let A and B are two sets. The intersection of A & B is defined as the set of all elements common to both sets A & B is given by.

Example:

$A = \{1, 2, 3, 4\}$

$B = \{2, 4, 5\}$

$A \cap B = \{2, 4\}$

Illustration:- Let $A = \{1, 2, 3, 4\}$ $B = \{2, 3\}$ find $(A - B) \cup (B - A)$, $(A \cap B) - A$?

Solution:- $A - B = \{1, 4\}$

$B - A = \{\varphi\}$

$(A - B) \cup (B - A) = \{1, 4\}$

$A \cap B = \{2, 3\}$

$(A \cap B) - A = \{\varphi\}$

DIFFERENCE: Let A and B are any two sets. The relative component of B in A written as $A - B$ is the set consisting of all elements of A which are not elements of B That is,

$A - B = \{x : x \in A \text{ \& } x \notin B\}$

Example: $A = \{a, b, c, d, e, f, g\}$

$B = \{a, b, c, d, h, i, j\}$

$A - B = \{e, f, g\}$; $B - A = \{h, i, j\}$

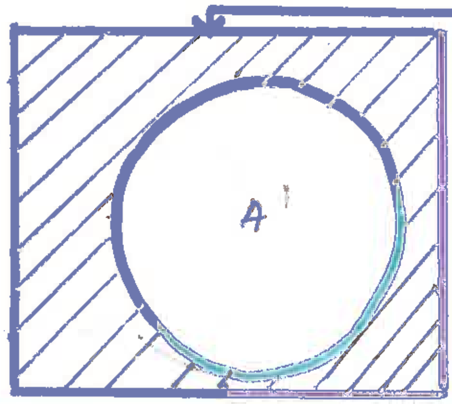
COMPLEMENT SET: Let E be the universal set. for any set A, the relative complement of A with respect to E, that is $E - A$ is called the absolute complement of A. and is denoted by $\sim A$ or A' or A^c

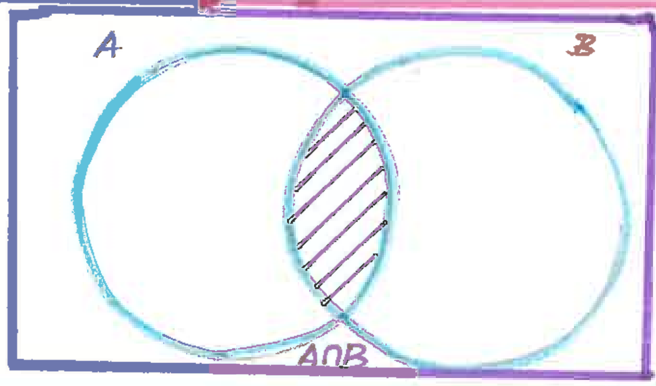
$A^c = E - A = \{x \mid x \in E \text{ and } x \notin A\}$

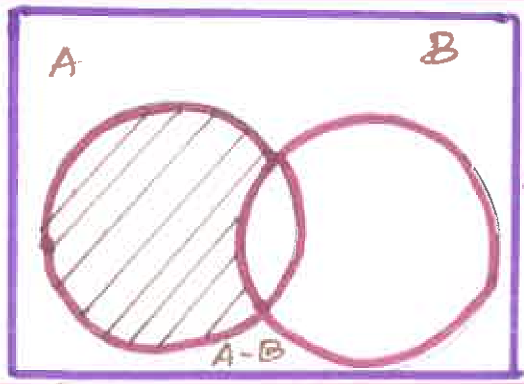
Example:

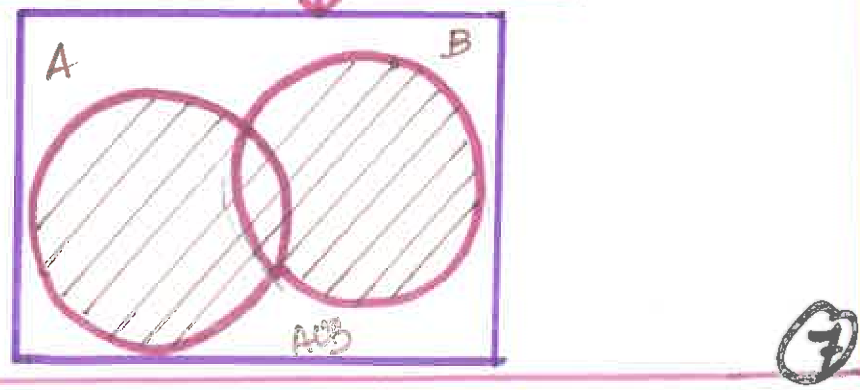
$E = \{1, 2, 3 \dots 10\}$ $A = \{1, 3, 5, 7, 9\}$

$A^c = \{2, 4, 6, 8, 10\}$









Basic laws of set Theory

1) COMMUTATIVE LAWS:-

$A\cup B = B\cup A$; $A\cap B = B\cap A$

2) ASSOCIATIVE LAWS:-

$A\cup(B\cup C) = (A\cup B)\cup C$

$A\cap(B\cap C) = (A\cap B)\cap C$

3) DISTRIBUTIVE LAWS:-

$A\cup(B\cap C) = (A\cup B)\cap(A\cup C)$

$A\cap(B\cup C) = (A\cap B)\cup(A\cap C)$

4). DE MORGAN'S LAWS:-

$(A\cup B)' = A'\cap B'$

$(A\cap B)' = A'\cup B'$

5). IDEMPOTENT LAWS:-

$A\cup A = A$; $A\cap A = A$

6). NEGATION:-

$A\cap A' = \phi$; $A\cup A' = U$

PRINCIPLE OF INCLUSION AND EXCLUSION

1) $n(A\cup B) = n(A)+n(B)-n(A\cap B)$

2) $n(A\cup B\cup C) = n(A)+n(B)+n(C) - n(A\cap B) - n(B\cap C) - n(A\cap C) + n(A\cap B\cap C)$.

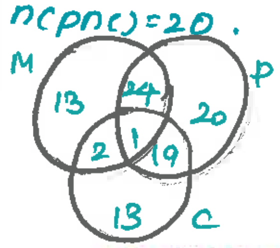
PROBLEM:-

In a survey of 100 students, it was found that 40 studied Mathematics, 64 studied physics, 35 studied chemistry, 1 studied all three subjects 30 studied Mathematics and chemistry and 20 studied physics and chemistry and 25 studied Mathematics and physics find the numbers of students who studied only chemistry and number of students who studied none of these subjects none of these subjects.

Sol:- $n(M)=40$, $n(P)=64$, $n(C)=35$,

$n(M\cap P\cap C)=1$, $n(M\cap P)=25$, $n(M\cap C)=3$,

$n(P\cap C)=20$.



No. of students studied only chemistry $= 13$.

Total no. of students $= n(M)+n(P)+n(C) - n(M\cap P) - n(P\cap C) - n(M\cap C) + n(M\cap P\cap C)$

$= 40+64+35-25-3-20+1 = 92$

No. of students who studied none of the subjects $= n(M\cap P\cap C)' = 100-92 = 8$

PROBLEMS:

1) Show that $(A-B)-C = A-(B\cup C)$

Sol:- $(A-B)-C = (A\cap B')\cap C'$ [By difference set definition]

$= A\cap(B'\cap C')$ [By Associative law]

$= A\cap(B\cup C)'$ [By De morgan's Law]

$= A-(B\cup C)$ [By difference set definition]

SET THEORY

ALGEBRA OF SETS

ORDERED PAIR:- A pair of objects whose components occur in a specific order is called an ordered pair.

FOR EXAMPLE: (a,b), $(1,2)$ are ordered pairs

CARTESIAN PRODUCT:- let A and B are any two sets. The set of all ordered pairs is such that the first member of the ordered pair is an element of A and the second members is an element of B is called the cartesian product of A and B and its written as $A\times B$

That is $A\times B = \{(x,y) \mid x\in A \text{ and } y\in B\}$

EXAMPLES:- $A=\{\alpha,\beta\}$, $B=\{1,2\}$

$A\times B = \{(\alpha,1),(\alpha,2),(\beta,1),(\beta,2)\}$

PARTITION OF A SET:- If S is a non-empty set, then a collection of disjoint non-empty subsets of S whose union is S called a partition of S.

EXAMPLE:-

$A = \{1,2,\ldots,10\}$

$A_1 = \{1,3,5\}$, $A_2 = \{2,4,6,8\}$

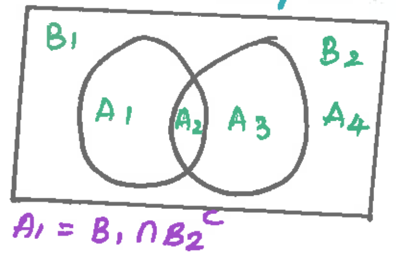
$A_3 = \{7,9\}$, $A_4 = \{10\}$

Then A_1, A_2, A_3, A_4 form a partition of A.

MINSETS:

Let B_1 and B_2 are subsets of a set A

Consider the venn diagram.



Let $A_1 = B_1\cap B_2^c$

$A_2 = B_1\cap B_2$

$A_3 = B_1^c\cap B_2$

$A_4 = B_1^c\cap B_2^c$

Then each of A_1, A_2, A_3, A_4 is called a minset or minterm generated by B_1 and B_2

NOTE:-

For given sets B_1 and B_2 there are 2^2 minsets

If B_1, B_2, B_3 are three sets given, then there are 2^3 minsets. In general for subsets $B_1, B_2, \ldots B_n$ of A, there are 2^n minsets

DEFINITION:- Let A be a set. Let $\{B_1, B_2, \ldots B_n\}$ be two subsets of A. A set of the form $D_1\cap B_2\cap\ldots\cap D_n$, where each D_i may either B_i or B_i^c is called a minset generated by $B_1, B_2, \ldots B_n$.

MAXSET:-

Let A be set. Let $\{B_1, B_2, \ldots B_n\}$ be a set of subsets of A. A set of the form $D_1\cup B_2\cup\ldots\cup D_n$ where each D_i may be either B_i or B_i^c is called a maxset generated by $B_1, B_2, \ldots B_n$

Relation on sets

Relation :-

If A & B are sets, then a relation R from A to B is a subset of $A \times B$

If $x \in A$ is related to an element $y \in B$ under some relation R. then we unite xRy (or) $(x, y) \in R$.

Ex: Let $A = \{1,2,3\}$ & $B = \{6\}$. The relation R is defined such that "Less than"

$R = \{(1,6),(2,6),(3,6)\}$

Types of relation

COMPLEMENTARY RELATION

Let A & B are two finite sets & R be a relation from A to B. Then, the complementary function of R is defined as

$R^c = \{(a,b) \in A \times B / (a,b) \notin R\}$

INVERSE RELATION

Let R be a relation from set A to set B. Then, the inverse relation of R is defined by.

$R^{-1} = \{(b,a) / (a,b) \in R\}$

Ex: Let $A = \{a,b,c\}$ & $B = \{1,2,4\}$ the relation $R = \{(a,1),(a,4),(b,1),(b,4),(c,1),(c,2),(c,4)\}$ find R^c, R^{-1}

Sol: $A \times B = \{(a,1),(a,2),(a,4),(b,1),(b,2),(b,4),(c,1),(c,2),(c,4)\}$

$R^c = \{(a,2),(b,2)\}$

$R^{-1} = \{(1,a),(2,c),(4,a),(1,b),(4,c),(4,b),(1,c)\}$

Operation on relation

UNION & INTERSECTION OF TWO RELATIONS

Let R & S be two relations from a set A to set B. Then $R \cup S$ & $R \cap S$ are defined as

$R \cup S = \{(a,b) / (a,b) \in R \text{ (or) } (a,b) \in S\}$

$R \cap S = \{(a,b) / (a,b) \in R \ \& \ (a,b) \in S\}$

COMPOSITION OF RELATIONS :-

Suppose A, B and C are sets.

R is a relation from A to B.

S is a relation from B to C.

The composition of R and S is written as SoR.

The relation SoR is a relation from A to C and is defined as, if $x \in A$ and $z \in C$, then $x(SoR)z$ if and only if for some $y \in B$, we have xRy and yRz

Example :- Let $A = \{1,2,3\}$, $B = \{2,3,6,8,12\}$, $C = \{13,17,22\}$ and $R = \{(1,2),(1,3),(1,12),(2,3),(2,6),(2,8),(2,12)\}$. $S = \{(2,13),(2,17),(3,13),(3,22),(8,22)\}$. find SoR.

$SoR = \{(1,3),(1,17),(1,22),(2,13),(2,22)\}$.

PROPERTIES OF A RELATION :-

Let A be a non-empty set and R be a binary relation in A.

1) R is said to be reflexive if $aRa \ \forall a \in A$ (or) $(a,a) \in A \ \forall a \in A$.

2) R is said to be symetric if $aRb \Rightarrow bRA \ \forall (a,b) \in A$ (or) $(a,b) \in R \Rightarrow (b,a) \in R$

3) R is said to be transitive if aRb & $bRc \Rightarrow aRc \ \forall a,b,c \in A$ (or) $(a,b) \in R$, $(b,c) \in R \Rightarrow (a,c) \in R$.

4) R is said to be antisymmetric if aRb & $bRa \Rightarrow a=b \ \forall (a,b) \in A$ i.e $(a,b) \in R$ & $(b,a) \in R \Rightarrow a=b \ \forall (a,b) \in A$.

EXAMPLE: i) If $A = \{1,2,3,4\}$ then

(i) The relation $\{(1,3),(2,4)\}$ is not reflexive, not symmetric & not transitive

(ii) The relation $\{(1,1),(1,3),(3,1),(3,4),(4,3)\}$ is symmetric but neither reflexive nor transitive.

(iii) The relation $\{(1,1),(2,2),(3,3),(4,4),(1,3)\}$ is reflexive, transitive but not symmetric.

EXAMPLE: If $A = \{2,4,6,8\}$ $B = \{3,5,7\}$ and if R is defined by $\{(2,3),(2,5),(4,5),(4,7),(6,3),(6,7),(8,7)\}$. Find M_R, $M_{R^{-1}}$, M_{R^c} ?

Sol: $M_R = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

R^{-1} is defined by $\{(3,2),(5,2),(5,4),(7,4),(3,6),(7,6),(7,8)\}$

Now,

$M_{R^{-1}} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} = M_R^T$

$M_{R^c} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$

Representation of a relation

MATRIX OF A RELATION

If $A = \{a_1, a_2, \ldots, a_m\}$ and $B = \{b_1, b_2, \ldots, b_n\}$ are finite elements containing m and n elements respectively and R is a relation from A to B. then R is represented by a m×n matrix is

$M_R = [m_{ij}]$ which is defined as

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

Example: given $A = \{1,2,3,4\}$ and $B = \{x,y,z\}$

let $R = \{(1,y),(1,z),(3,y),(4,x),(4,z)\}$

Then the matrix of R is

$$M_R = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

DIGRAPH OF A RELATION

A relation R on a finite set A can be represented pictorially as:

① A small circle is drawn for each element of A and marked with the corresponding element. These circles are called vertices.

② An arc is drawn from the vertex a_i to the vertex a_j if $a_i \, R \, a_j$. This is called an edge.

This pictorial representation of R is called a directed graph or digraph of R.

In a digraph of R, the indegree of a vertex is the number of edges terminating the vertex. But out degree of a vertex is the number of edges leaving the vertex.

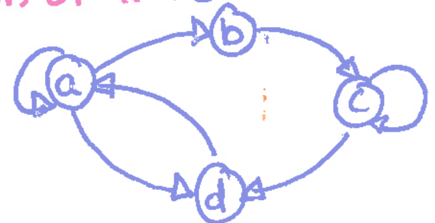
Example: let $A = \{a,b,c,d\}$ and R is a relation on A that has the matrix.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

construct the diagraph of R and list the indegree and out degrees of all vertices.

Solution: $R = \{(a,a),(a,b),(a,d),(b,c),(c,c),(c,d),(d,a)\}$

The digraph of R is



The indegrees and out degrees of all vertices are:

	a	b	c	d
Indegree	2	1	2	2
outdegree	3	1	2	1

EQUIVALENCE RELATION

If R is an equivalence relation on a set A, then the set of all elements of A that are related to an element 'a' of A is called equivalence class of 'a' and denoted by $[a]_R$.

$$[a] = \{x \mid (a,x) \in R\}$$

EQUIVALENCE CLASS

The collection of all equivalence class of elements of A under an equivalence relation R is denoted by A/R and called the quotient set of A by R

$$A/R = \{[a] \mid a \in A\}$$

Example: The relation R on the set $A = \{1,2,3\}$ defined by

$R = \{(1,1),(1,2),(2,1),(2,2),(3,3)\}$ is an equivalence relation since R is reflexive, symmetry and transitive.

$[1] = \{1,2\}$

$[2] = \{1,2\}$

$[3] = \{3\}$

∴ [1], [2], [3] are the equivalence classes of A under R and hence A/R.

Example: examine if the relation R represented by $M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ is an equivalence relation, using the properties of M_R.

Since all the elements in the main diagonal of M_R are equals to 1 each, R is a reflexive relation.

Since M_R is a symmetric matrix, R is a symmetric relation.

$$M_{R^2} = M_R \cdot M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = M_R$$

viz. $R^2 \subseteq R$

∴ R is transitive relation.

Hence R is an equivalence relation.

Example: If $R_1 = \{(0,0),(1,1),(2,2),(3,3)\}$, check for reflexive, symmetric, & transitive.

Sol: R_1 is reflexive, symmetric and transitive

∴ R_1 is an equivalence relation.

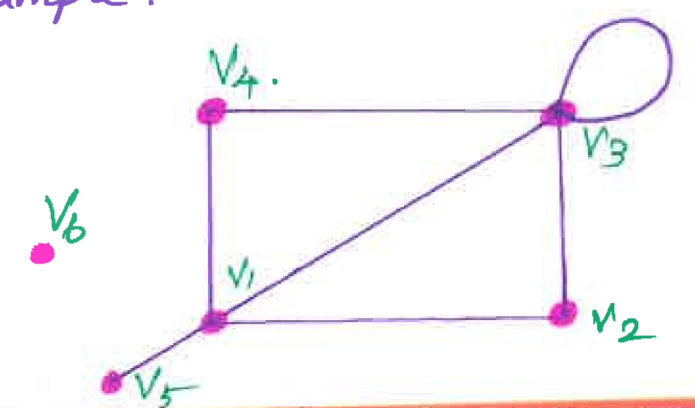
GRAPH TERMINOLOGY

TYPES OF GRAPHS

Special Types of Graphs

Degree of a Vertex:-
The Number of Edges incident at the Vertex V_i

Example :-



$\deg(V_1) = 4$	$\deg(V_4) = 3$
$\deg(V_2) = 2$	$\deg(V_5) = 1$
$\deg(V_3) = 5$	$\deg(V_6) = 0$

Indegree of Vertex :-
Number of edges ends with "v"

Out Degree of Vertex:-
Number of Edges starts with "v"

NOTE : A loop of a Vertex Contributes "1" both in Indegree and Out degree of the Vertex.

Hand Shaking Theorem:-
Let $G = (V,E)$ - undirected graph with "e" edges Then

$$\sum_{v \in V} \deg(v) = 2e$$

Theorem:
If an undirected graph, the Number of odd degree Vertices are even.

$$\sum \deg(v_i) + \sum \deg(v_j) = 2e$$
$$\sum \deg(v_i) = 2e - \sum \deg(v_j)$$

Theorem:
The Maximum number of Edges in a Simple graph with n Vertices $= \frac{n(n-1)}{2}$.

Ex: Simple, graph with 15 Vertices.

$$2e = \sum d(v)$$
$$2e = 15 \times 5$$
$$e = 75/2$$

Regular:
If Every Vertex of a Simple has Same degree then graph is Regular.

Complete:
If there exist an Edge b/w Every point of Vertices then Such graph is Complete

Complete Bipartiele:

A bipartiele graph with partitions V_1, V_2 is Complete if Every vertex in V_1 is adjecent to Every Vertex V_2 But not adjacent within V_1 & V_2 itself.

Subgraph:
A graph $H = (V', E')$ - Subgph of $G = (V,E)$ if $V' \leq V$, $E' \leq E$.

INTRODUCTION TO GRAPHS.

GRAPHS :-

If set of nodes (points) and edges of the graph and ϕ is a mapping from the set of ordered or unordered pair of elements of V.

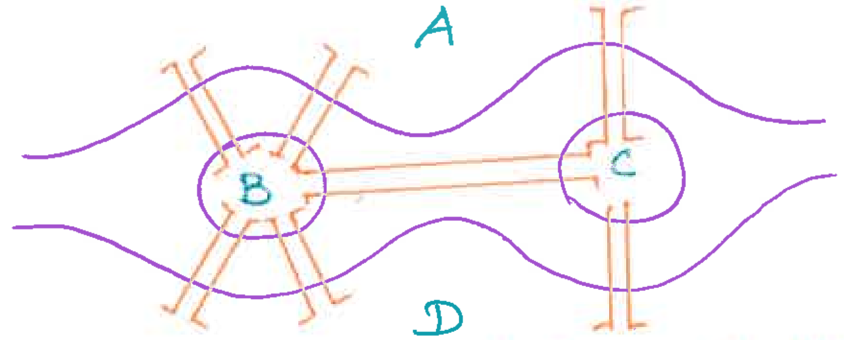
Isolated Vertex :

If the degree of the vertex of a graph is "zero"

Pendent Vertex :

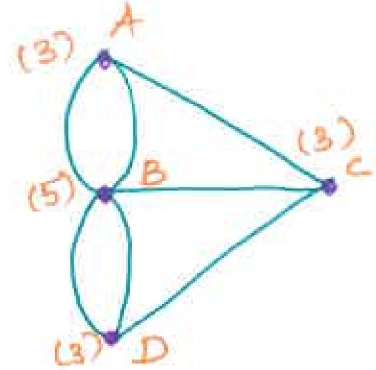
If the degree of a vertex of a graph is "One"

Konigsberg Bridge Problem :-

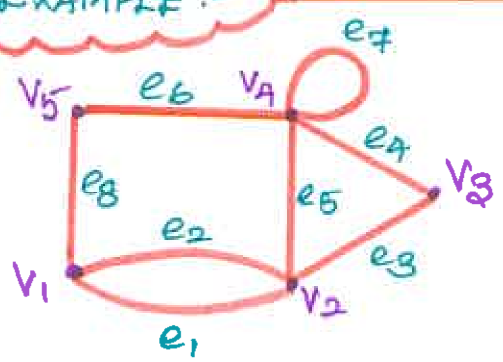


Pragel river contains 7 bridges = 7 edges
4 land areas = 4 Vertices

Diagramatical Representation of Graph



EXAMPLE :-



Vertices : V_1, V_2, V_3, V_4, V_5

Edges :- $e_1, e_2, \cdots e_8$

Self Loop

An edge from v_i to v_i

Parallel Edges

Two edges have same end points

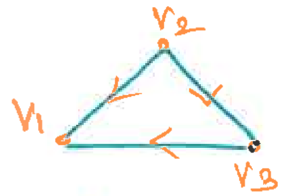
Adjacency & Incidency

If 2 vertices have common edge.

TYPES OF GRAPH.

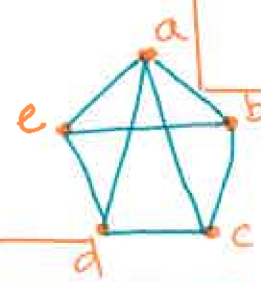
Directed Graph :-

If every edges are directed edges then the Graph is Directed Graph or Digraph



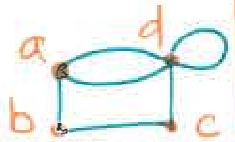
Simple Graph :-

In which graph neither Self loop nor parallel edges are allowed



Pseudo Graph :-

Loops, Parallel edges are allowed

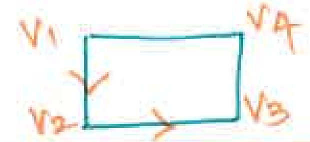


Undirected Graph :-

If Every edge is undirected then the graph is Undirected graph.

Mixed Graph

If a graph contains directed and undirected edges

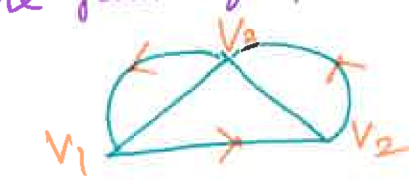


Finite Graph

If a graph contains finite number of Edge set and Vertex set

Multi Graph

If a graph contains some parallel edges, then the given graph is Multigraph

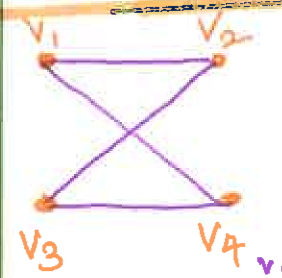


REPRESENTATION & CHARACTERISTIC OF A GRAPH

MATRIX REPRESENTATION of a Graph.

Adjacency Matrix

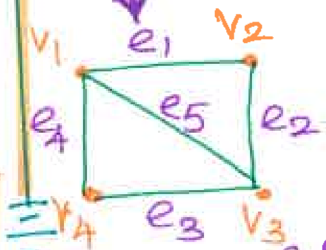
$A=[a_{ij}]=\begin{cases}1; & v_iv_j\in G\\ 0; & \text{otherwise}\end{cases}$



$$A=[a_{ij}]_{n\times n}=\begin{array}{c|c} & \begin{matrix}v_1 & v_2 & v_3 & v_4\end{matrix}\\ \begin{matrix}v_1\\v_2\\v_3\\v_4\end{matrix} & \begin{bmatrix}0&1&1&0\\1&0&0&1\\1&0&0&1\\0&1&1&0\end{bmatrix}\end{array}$$

Incidence Matrix

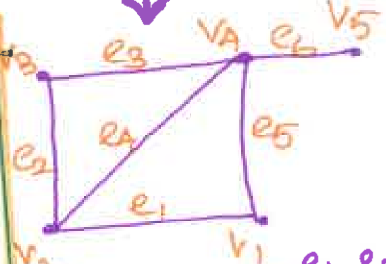
$B=[b_{ij}]=\begin{cases}1, & e_j \text{ incident on } v_i\\ 0, & \text{otherwise}\end{cases}$



$$B=[b_{ij}]_{n\times e}=\begin{array}{c|c} & \begin{matrix}e_1 & e_2 & e_3 & e_4 & e_5\end{matrix}\\ \begin{matrix}v_1\\v_2\\v_3\\v_4\end{matrix} & \begin{bmatrix}1&0&0&1&1\\1&1&0&0&0\\0&1&1&0&1\\0&0&1&1&0\end{bmatrix}\end{array}$$

Path Matrix.

$P=[P_{ij}]=\begin{cases}1; & \exists \text{ a path } v_i \text{ to } v_j\\ 0; & \text{otherwise}\end{cases}$



$$P(V_2,V_4)=\begin{array}{c|c} & \begin{matrix}e_1 & e_2 & e_3 & e_4 & e_5 & e_6\end{matrix}\\ \begin{matrix}P_1\\P_2\\P_3\end{matrix} & \begin{bmatrix}0&0&0&1&0&0\\1&0&0&0&1&0\\0&1&1&0&0&0\end{bmatrix}\end{array}$$

* Sum of entries in i^{th} row $= d(v_i)$
* Entries along leading diagonal are "0" ⇔ Graph has no self loop.

* Each edge incident on exactly 2 vertices
* Each column has exactly 2
* Rank of B = $\boxed{3}$

* Column of 0's correspond to edge does not lie on any path b/w v_i & v_j
* A Column of all 1's correspond to edge lie in every path b/w v_i & v_j

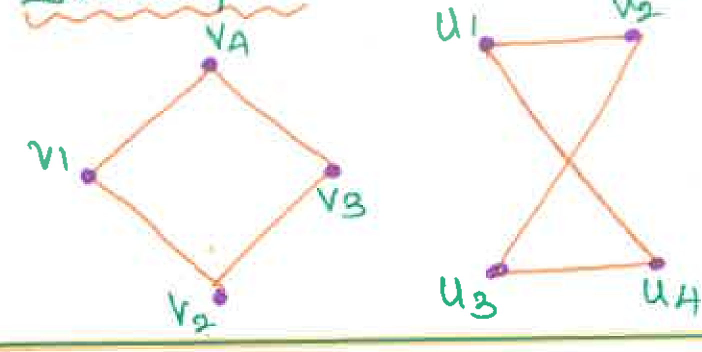
ISOMORPHISM :-

Two Graphs G_1 & G_2 are said to be Isomorphic to each other if "∃ One-One Correspondence b/w Vertex Sets preserves adjacency"

* $|V(G_1)| = |V(G_2)|$
* $|E(G_1)| = |E(G_2)|$
* Equal no. of vertices with given degree

Examples for Isomorphic & Non-Isomorphic Graphs.

Isomorphic :-



Non-Isomorphic :-

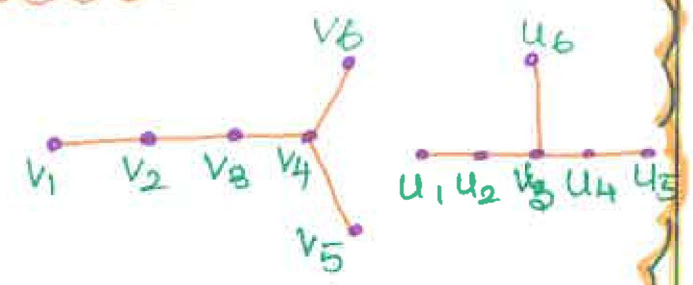
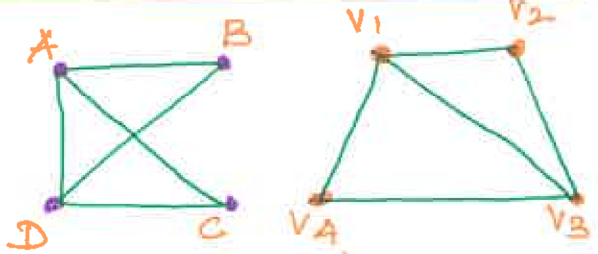


Illustration :-1.

Establish the Isomorphism of two graphs given by considering their adjacency matrices



G_1, G_2

$$A_1 \text{ of } G_1=\begin{bmatrix}0&1&1&1\\1&0&0&1\\1&0&0&1\\1&1&1&0\end{bmatrix}\qquad A_2 \text{ of } G_2=\begin{bmatrix}0&1&1&1\\1&0&1&0\\1&1&0&1\\1&0&1&0\end{bmatrix}$$

Conversion of A_2 into A_1 :-

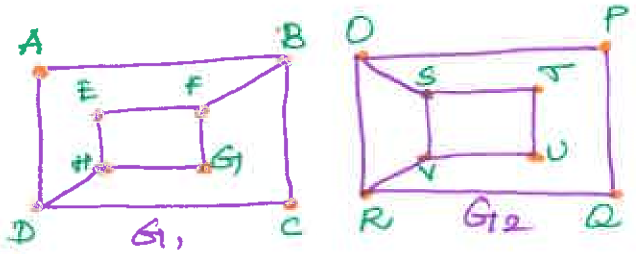
$$A_2=\begin{array}{c|c} & \begin{matrix}C_1 & C_2 & C_3 & C_4\end{matrix}\\ & \begin{bmatrix}0&1&1&1\\1&0&1&0\\1&1&0&1\\1&0&1&0\end{bmatrix}\end{array}\begin{matrix}\to R_1\\ \to R_2\\ \to R_3\\ \to R_4.\end{matrix}$$

$$\cong\begin{bmatrix}0&1&1&1\\1&0&0&1\\1&1&1&0\\1&0&0&1\end{bmatrix}\quad C_3 \Leftrightarrow C_4 \text{ Swapped.}$$

$$\cong\begin{bmatrix}0&1&1&1\\1&0&0&1\\1&0&0&1\\1&1&1&0\end{bmatrix}\quad R_3 \Leftrightarrow R_4 \text{ Swapped.}$$

$= A_1$ ∴ Given G_1, G_2 are Isomorphic.

Some more Illustrations :-...



* $|V(G_1)| = |V(G_2)| = 8$
* $|E(G_1)| = |E(G_2)| = 10$
* Degree sequence are same

$G_1 = 2, 3, 2, 3, 2, 3, 2, 3$

$G_2 = 3, 2, 2, 3, 3, 2, 2, 3$

But, Correspondence between degree sequence does not match

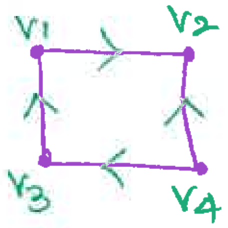
∴ Both are "Not Isomorphic"

Connectivity - Euler & Hamilton Path

Connectivity.

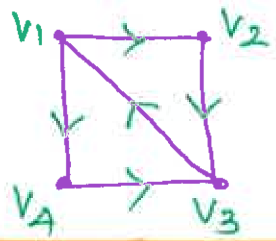
Unilaterally Connected.

If any pair of nodes of G atleast one of the nodes of the pair is reachable.



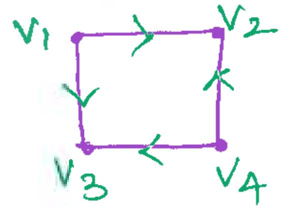
Strongly Connected

In a simple digraph, if for any pair of nodes of G are reachable from one another



Weakly Connected

In a simple digraph, if it is connected in undirected graph when edge direction is neglected.

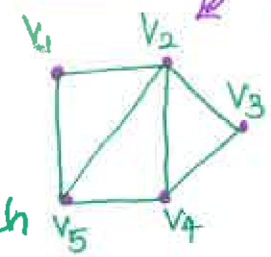


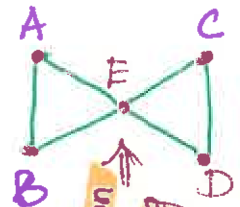
Eulerian

Eulerian Path:-

If path of a graph G is Euler if it contains each edge of the graph exactly once.

Vertex Repetition not allowed





Edge repetition - not happening

Eulerian Circuit:

Circuit having all edges exactly once

Eulerian Graph:-

If Graph containing Eulerian Circuit or Cycle.

Hamiltonian

Hamiltonian Path:-

If path is Hamiltonian if it includes each vertex of G exactly once.

(Vertex Repetition - Not allowed)

Hamiltonian Circuit:

If circuit is Hamiltonian if it includes each vertex of G exactly once except starting & Ending vertex.

Hamiltonian Graph:-

If Graph Containing Hamiltonian Circuit

Path of a Graph:-

Finite alternating sequence of vertices and edges begining and ending with vertices such that each edge is incident on vertex preceding and following it.

Path:-

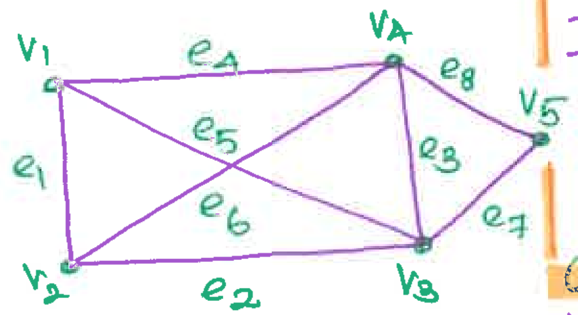
$V_1 e_1 V_2 e_2 V_3 e_5 V_1 e_1 V_2$

Circuit:-

$V_1 e_1 V_2 e_2 V_3 e_3 V_4 e_6 V_2 e_1 V_1$

Circuit of a graph:-

If starting and ending vertex are same in a path, then the path is said to be a circuit



Theorem:-

If Connected graph G is Eulerian iff every vertex of G is of even degree.

G-Eulerian ⇒ Euler Circuit exists

In euler circuit, each vertex has preceding and following edge ⇒ Each vertex has 2 counts of edges ⇒ All Vertex is of even degree.

Converse:- If all vertex has even degree then $C = V, V_1, V_2, \dots V_{n-1}, V$

Subgraph H can be obtained by deleting all edges of C from G.

∵ G-Connected, H & C must have common vertex u

Beginning with u construct C_1 of H.

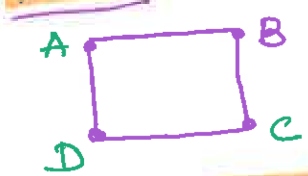
Combining C, C_1 ⇒ Larger Circuit C_2

If it is Eulerian ⇒ G - Eulerian

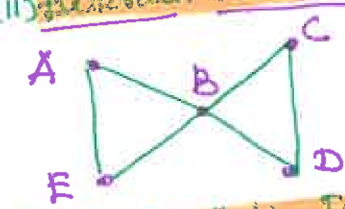
Otherwise extend until make G - Eulerian.

Provide Examples for

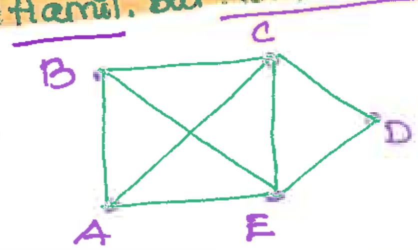
(i) Eulerian & Hamiltonian



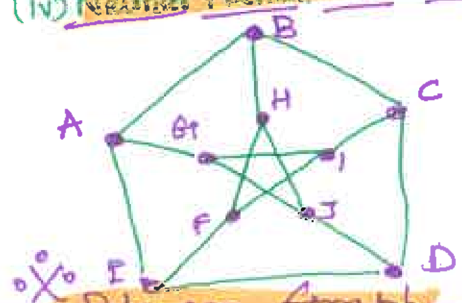
(ii) Eulerian but not Hamil.



(iii) Hamil. But Not Eulerian



(iv) Neither Hamil Nor Eulerian



Peterson Graph

Partialy Ordering Set (POSET)

A set together with a partial Ordering relation R is called POSET

Recall: Partialy Ordring Relation
↓
Reflexive
↓
Anti symmetric
↓
Transitive

Hasse Diagram
↓
The pictorial Representation of a POSET

Illustration :-

$A = \{a\}$ $B = \{a,b\}$ $C = \{a,b,c\}$

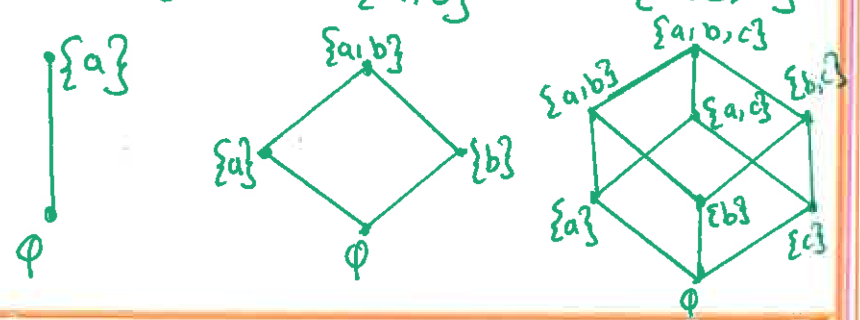
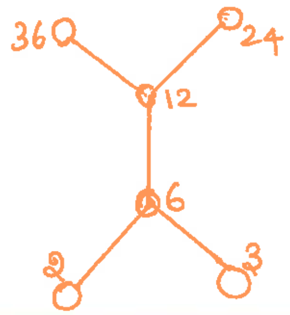


Illustration :- $X = \{2,3,6,12,24,36\}$

$R = \{<a,b> / a/b\}$



Least Upper bound

Let $(P,\le)$ be a Poset and $A \subseteq P$. An element $a \in P$ is said to be LUB or Supremum of A is

Partial Ordering Set

(i) a is upperbound of A

(ii) $a < c$, where c is any other upper bound of A

Greatest Lower Bound

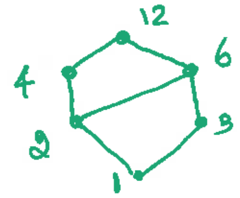
Let $(P,\le)$ be a Poset and $A \subseteq P$. An element $b \in P$ is said to be GLB of A is

(i) b is a lower bound of A

(ii) $b \ge d$, where d is any other lower bound of A

Illustration :- Consider $X = \{1,2,3,4,6,12\}$

$R = \{<a,b> / a/b\}$ Find LUB and GLB for the Poset (X,R).



① $UB\{1,3\} = \{3,6,12\}$

$LUB\{1,3\} = 3$

② $UB\{1,2\} = \{2,4,12\}$

$LUB\{1,2\} = 2$

③ $LB\{1,3\} = \{1\}$

$GLB\{1,3\} = 1$

④ $UB\{2,3,6\} = \{6,12\}$

$LUB\{2,3,6\} = \{6\}$

⑤ $LB\{2,3,6\} = \{1\}$

$GLB\{2,3,6\} = \{1\}$

LATTICE
↓

A Lattice is a POSET $(L,\le)$ in which for every pair of elements $a,b \in L$, both GLB & LUB exists.

Note :- Lattice $(L,\le)$ has a binary operation * $(\wedge)$ and ⊕ $(\vee)$, a Lattice can be denoted by triplet $(L,*,\oplus)$ or $(L,\wedge,\vee)$ or $(L,\cdot,+)$

Complement of an element 'a' in a Lattice $(L,\wedge,\vee,0,1)$

$a \in L$, we say 'b' is complement of A if $a \wedge b = 0$ & $a \vee b = 1$ & $a' = b$

Properties of Lattice

Property L1 :- Idempotent $a \vee a = a$ & $a \wedge a = a$

Proof :- $LUB(a,a) = GLB(a,a) = a$

Property L2 :- Commutative $a \vee b = b \vee a$

$a \wedge b = b \wedge a$

$a \vee b = LUB(a,b) = LUB(b,a) = b \vee a$

$a \wedge b = GLB(a,b) = GLB(b,a) = b \wedge a$

Property L3 :- Absorption $a \vee (a \wedge b) = a \wedge (a \vee b) = a$

Property L4 :- Associative

$a \vee (b \vee c) = (a \vee b) \vee c$

$a \wedge (b \wedge c) = (a \wedge b) \wedge c$

Property 2 :- $a \le b \Longleftrightarrow a \wedge b = a \Longleftrightarrow a \vee b = b$

$a \le b$ $\quad$ $LUB(a,b)$ $\quad$ let $a \vee b = b$

$b \le b$ $\quad$ $= a \vee b$ $\quad$ only possible $a \le b$

$a \vee b \le b$ $\quad$ $b \le a \vee b$ $\quad$ $\Rightarrow a \le b \Longleftrightarrow a \vee b = b$

Property 3 :- Isotonic $b \le c$ $\Rightarrow a \vee b \le a \vee c$, $a \wedge b \le a \wedge c$

Proof :- $a \vee c = (a \vee b) \vee (a \vee c)$

$a \wedge b = (a \wedge b) \wedge (a \wedge c)$

Property 4 :- Distributive Inequality

(i) $a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$

Proof :- $a \le (a \vee b) \wedge (a \vee c)$ $\quad$ $b \wedge c \le (a \vee b) \wedge (a \vee c)$

$\Rightarrow a \vee (b \wedge c) \le (a \vee b) \wedge (a \vee c)$

$a \ge (a \wedge b) \vee (a \wedge c)$ $\quad$ $b \vee c \ge (a \wedge b) \vee (a \wedge c)$

$\Rightarrow a \wedge (b \vee c) \ge (a \wedge b) \vee (a \wedge c)$

Property 5 :- Modular Inequality

$a,b,c \in L$, $a \le c \Rightarrow a \vee (b \wedge c) \le (a \vee b) \wedge c$

LATTICES AS ALGEBRAIC SYSTEMS

SUBLATTICE :-

Let $(L, *, \oplus)$ be a lattice. A non empty subset M of L is called Sublattice of L iff M is closed under the same operations $*$ and $\oplus$ of L.

(ie) $a * b \in M$, $a \oplus b \in M$ $\forall a, b \in M$.

Note :- Every Singleton of a lattice L is a Sublattice of L.

Illustration :- I

Let $S = \{a, b, c, d\}$ then the powerset $P(S)$ of S consisting all subsets of S. Then $(P(S), \cap, \cup)$ is a lattice where $*$ is $\cap$

Then P.T. $\{\phi, \{a\}, \{a,c\}, \{c\}, \{a,b,c\}\}$ is a sublattice

Let $A = \{\phi, \{a\}, \{a,c\}, \{c\}, \{a,b,c\}\} \subseteq P(S)$

Using Cayley table :-

$\cup$	ϕ	$\{a\}$	$\{c\}$	$\{a,c\}$	$\{a,b,c\}$
ϕ	ϕ	$\{a\}$	$\{c\}$	$\{a,c\}$	$\{a,b,c\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a,c\}$	$\{a,c\}$	$\{a,b,c\}$
$\{c\}$	$\{c\}$	$\{a,c\}$	$\{c\}$	$\{a,c\}$	$\{a,b,c\}$
$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,b,c\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$

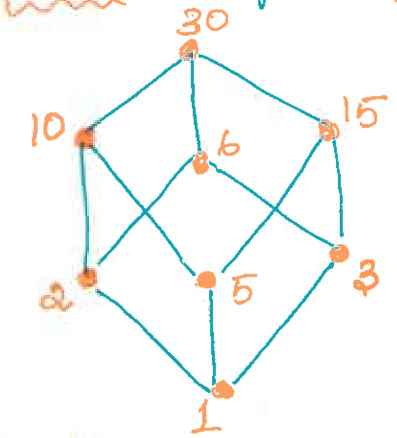
$\cap$	ϕ	$\{a\}$	$\{c\}$	$\{a,c\}$	$\{a,b,c\}$
ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$\{a\}$	ϕ	$\{a\}$	ϕ	$\{a\}$	$\{a\}$
$\{c\}$	ϕ	ϕ	$\{c\}$	$\{c\}$	$\{c\}$
$\{a,c\}$	ϕ	$\{a\}$	$\{c\}$	$\{a,c\}$	$\{a,c\}$
$\{a,b,c\}$	ϕ	$\{a\}$	$\{c\}$	$\{a,c\}$	$\{a,b,c\}$

$\therefore$ We can easily verify that A is closed under $\cap$, $\cup$

Illustration :- II

Find some sublattices of the Lattice $S_{30} = \{1,2,3,5,6,10,15,30\}$

Hasse Diagram of S_{30}



We know that S_{30}-lattice

$a * b = GLB\{a,b\} = GCD(a,b)$

$a \oplus b = LUB(a,b) = LCM(a,b)$

$S_6 = \{1,2,3,6\}$ - Sublattice as it is closed under $*, \oplus$

$S_{10} = \{1,2,5,10\}$ - Sublattice

$S_{15} = \{1,3,5,15\}$ - Sublattice

$S_5 = \{1,5\}$ - Sublattice with 2 elts.

$M_5 = \{5,10,15,30\}$ - Sublattice with 4 elts

$\{1,5,15,30\}$; $\{2,6,10,30\}$

$\{1,5,10,30\}$; $\{1,2,6,30\}$

$\{1,3,6,30\}$; $\{3,6,15,30\}$ etc...

Direct Product of lattices

Let $(L, *, \oplus)$ and $(M, \wedge, \vee)$ be 2 lattices and $L \times M$ be the cartesian product of L & M.

Let $+$ and $\cdot$ be binary operations on $L \times M$ defined as follows.

For any ordered pairs,

$(a_1, b_1), (a_2, b_2) \in L \times M$

$(a_1, b_1) \cdot (a_2, b_2) = (a_1 * a_2, b_1 \wedge b_2)$

$(a_1, b_1) + (a_2, b_2) = (a_1 \oplus a_2, b_1 \vee b_2)$

Then the algebraic system $(L \times M, \cdot, +)$ is called as direct product of $(L, *, \oplus)$ and $(M, \wedge, \vee)$.

HOMOMORPHISM :-

Let $(L, *, \oplus)$ and $(M, \wedge, \vee)$ be two lattices. A mapping $f: L \to M$ is called 'Lattice Homomorphism from $(L, *, \oplus)$ to $(M, \wedge, \vee)$.

if $f(a * b) = f(a) \wedge f(b)$

and

$f(a \oplus b) = f(a) \vee f(b)$

$\forall a, b \in L$:

Isomorphism :- A homomorphism $f: L \to M$ is Isomorphism if f is One-One and Onto.

Endomorphism :- Homomorphism $f: L \to L$ - Endomorphism

Automorphism :- Isomorphism $f: L \to L$ - Automorphism

Some Special Lattices

Complete Lattice :-

A lattice $(L, *, \oplus)$ is complete if every nonempty subset has a least upper bound and greatest lower bound.

Bounded Lattice :-

A lattice $(L, *, \oplus)$ is said to be bounded if it has a greatest element 1 and a least element 0.

(ie) $0 \leq a \leq 1$ $\forall a \in L$.

A Bounded lattice is denoted by $(L, *, \oplus, 0, 1)$.

Some Important Properties

* Every finite Lattice L is Bounded.
* $(P(S), \cup, \cap)$ - Bounded Lattice with $0 = \phi$ and $1 = S$.
* $(N, \leq), (Z, \leq), (R, \leq)$ $\hookrightarrow$ Not Bounded.
* Lattice of divisors of n S_n - has lower bound 1.
* A Bounded lattices satisfies.

$$\boxed{\begin{matrix} a \oplus 0 = a \ ; \ a * 1 = a \\ a \oplus 1 = 1 \ ; \ a * 0 = 0 \end{matrix}}$$

BOOLEAN ALGEBRA

DEFINITION

A complemented distributive Lattice

(Notation :- $(B, \wedge, \vee, 0, 1)$)

Two element Boolean algebra

lower bound 0

upper bound 1

ALGEBRA LAWS

A Boolean algebra is a non-empty set with 2 binary operations $\wedge$ and $\vee$ and is satisfied by the following conditions.

① $a+a=a$ → $a \cdot a = a$

② $a+b=b+a$ → $a \cdot b = b \cdot a$

③ $a+0=a$ → $a \cdot 1 = a$

④ $a+(b+c)=(a+b)+c$ → $a \cdot (b \cdot c) = (a \cdot b) \cdot c$

⑤ $a+1=1$ → $a \cdot 0 = 0$

⑥ $a+(b \cdot c) = (a+b) \cdot (a+c)$ → $a \cdot (b+c) = (a \cdot b) + a \cdot c$

⑦ $a+a'=1$ → $a \cdot a' = 1$

⑧ $a+(a \cdot b)=a$ → $a \cdot (a+b) = a$

⑨ $(a+b)' = a' \cdot b'$ → $(a \cdot b)' = a'+b'$

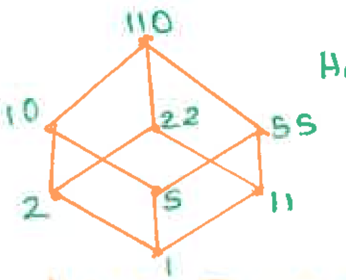
⑩ $(a')' = a$

Illustration :- Prove that D_{110} is a Boolean Algebra and $D_{110} = \{1, 2, 5, 11, 10, 22, 55, 110\}$

Proof :-

'0' element $= 1$

'1' element $= 110$



Hasse Diagram

Here $a \wedge b = GCD(a,b)$

$a \vee b = LCM(a,b)$

$1' = 110$ $5' = 22$ $2' = 55$ $11' = 10$ Each element has a complement Hence D_{110} is a Complement Lattice

Sub Boolean Algebra

If C is a nonempty subset of a Boolean (B) algebra such that C itself is a Boolean algebra w.r.t operations of B.

Illustration :- Consider the Boolean Algebra S_{30}. Determine which of the following are Sub algebra of S_{30}.

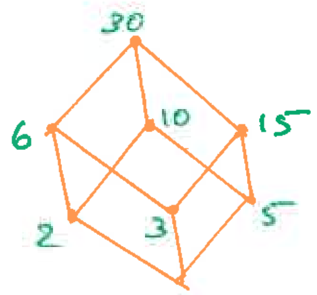
(i) $\{1, 2, 15, 30\}$ (iii) $\{1, 5, 6, 30\}$

(ii) $\{1, 3, 10, 30\}$ (iv) $\{1, 2, 3, 6\}$

Solution :- $S_{30} = \{1, 2, 3, 5, 6, 15, 30\}$

$a * b = \gcd\{a, b\}$, $a + b = \text{lcm}\{a, b\}$

'0' element $= 1$, '1' element $= 30$



Hasse Diagram

$1' = 30$ $30' = 1$

$2' = 15$ $15' = 2$

(i) $B_1 = \{1, 2, 15, 30\}$

$\cdot$	1	2	15	30
1	1	1	1	1
2	1	2	1	2
15	1	1	15	15
30	1	2	15	30

$\therefore B_1$ is Boolean Subalgebra

Similarly $B_2 = \{1, 3, 10, 30\}$, $B_3 = \{1, 5, 6, 30\}$ are Boolean sub algebra

$L_1 = \{1, 2, 3, 6\}$ is a Lattice and it is not a sub algebra as $30 \in L_1$ is it is not closed under

Characteristic of Boolean Algebra

De Morgan's Law :-

$(a+b)' = a' \cdot b'$ and $(a \cdot b)' = a' + b'$ $\forall a, b \in B$

Proof :- "If y is complement of x, then $x+y=1$ & $x \cdot y = 0$"

Now $(a+b)+a'b' = \{(a+b)+a'\} \cdot \{(a+b)+b'\}$

$= \{(b+a)+a'\} \cdot \{(a+b)+b'\}$

$= \{b+(a+a')\} \cdot \{a+(b+b')\}$

$= (b+1) \cdot (a+1)$

$= 1 \cdot 1 = 1$ —①

Similarly $(a+b) \cdot a'b' = 0$ —②

from ① & ② we get $(a+b)' = a'b'$

Expression of Boolean function in Canonical form

Truth Table Method

consider the function $f(x,y,z)$ whose truth table is

x	y	z	f	Min	Max
1	1	1	0		$x'+y'+z'$
1	1	0	1	xyz'	
1	0	1	1	$xy'z$	
1	0	0	1	$xy'z'$	
0	1	1	0		$x+y'+z'$
0	1	0	0		$x+y'+z$
0	0	1	0		$x+y+z'$
0	0	0	0		$x+y+z$

The DNF of f is $xyz'+xy'z+xy'z' = f$

The CNF of f is:

$f = (x'+y'+z')(x+y'+z')(x+y'+z)(x+y+z')(x+y+z)$

Algebraic Method

consider the Boolean function $f(x,y,z) = x(y'+z')$ Express it in the sum of product (DNF)

$f = xy' + xz'$

$= xy' \cdot (z+z') + xz'(y+y')$

$= xy'z + xy'z' + xyz' + xy'z'$

$= xy'z + xy'z' + xyz'$

Product of Sum (CNF)

$f = x(y'+z')$

$= (x+yy') \cdot (y'+z'+xx')$

$= (x+y) \cdot (x+y') \cdot (y'+z'+x)(y'+z'+x')$

$= (x+y+zz')(x+y'zz')(x+y'+z')(x'+y'+z')$

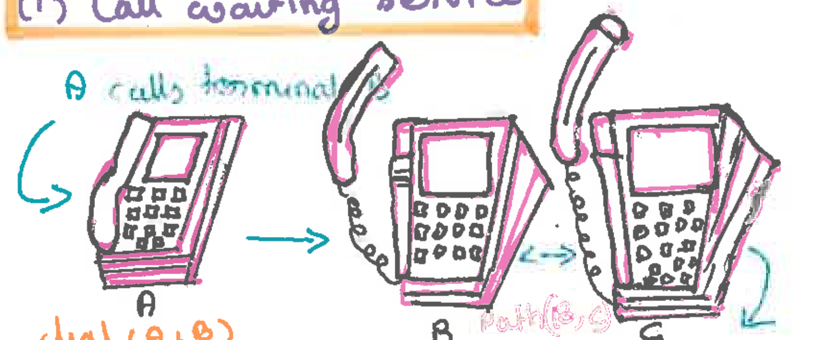
$= (x+y+z) \cdot (x+y+z') \cdot (x+y+z) \cdot (x+y'+z')(x'+y'+z')$

APPLICATIONS OF DISCRETE MATHEMATICS

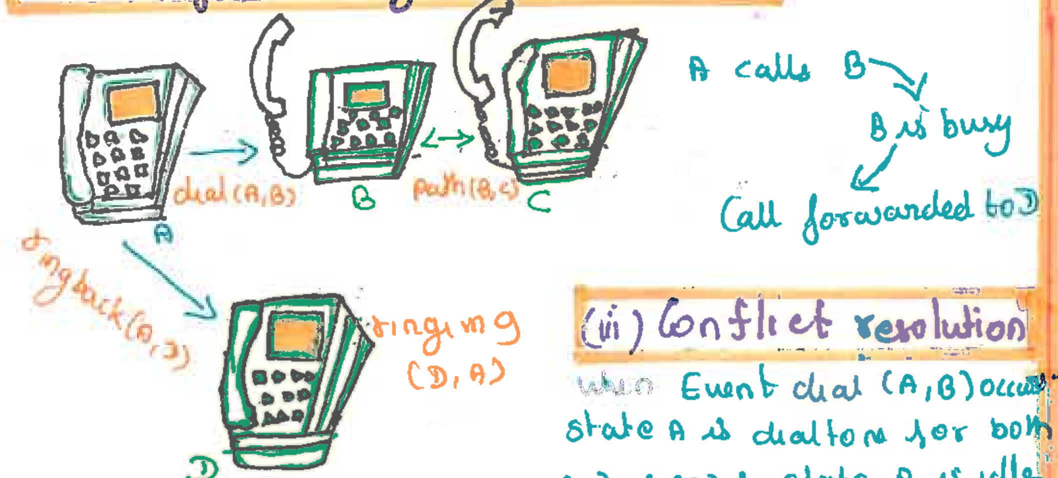
TELECOMMUNICATION SERVICES

When telecommunication services are described in predicate logic, there are possibilities for conflict with other telecommunication services

(i) Call waiting service



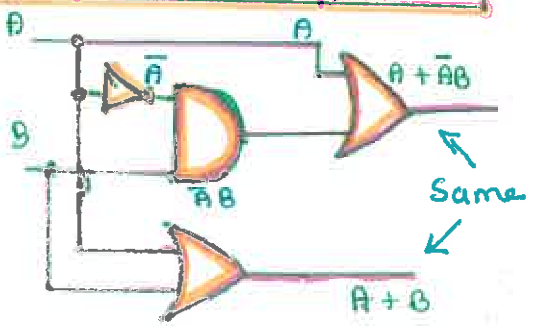
(ii) Call forwarding variable service



(iii) Conflict resolution

When Event dial (A,B) occurs state A is dialton for both (1) & (2) & state B is idle for both (1) & (2). Initial state (1) & (2) are the same for the event dial (A,B) so system cannot choose one rule in a consistent manner. This required conflict resolution.

Logical Operations in Digital Computer



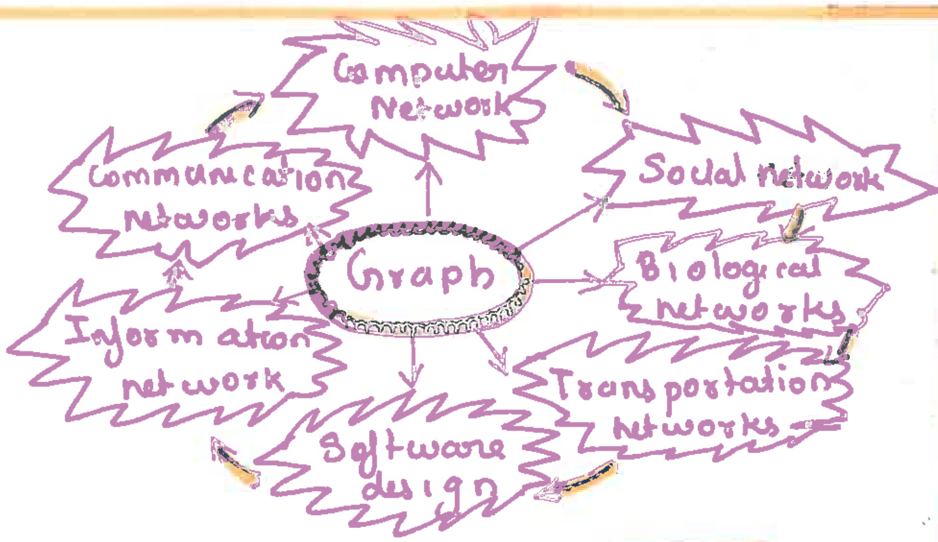
$A+\bar{A}B = A+B$

$A+\bar{A}B \rightarrow A+AB+\bar{A}B$

$\rightarrow A+B(A+\bar{A})$

$\rightarrow A+B(1)$

$\rightarrow A+B$



Purpose of studying Graph Theory

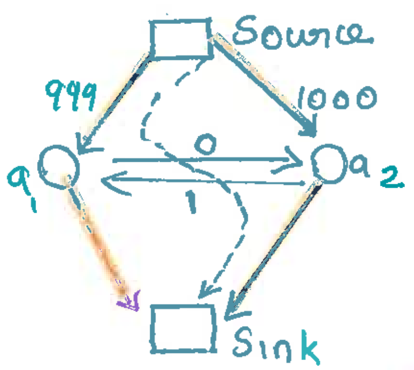
* Set of techniques for solving real-world problems - particularly for different kinds of optimization

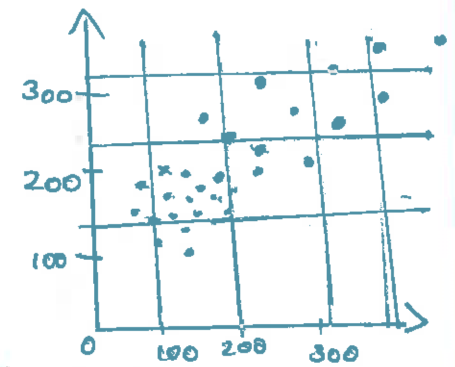
* Graph theory is useful for analyzing "Things are connected to other things" which applies almost everywhere.

* Some difficult problems become easy when representing through Graph theory.

Rules of Inferences

Argumenting path based Algorithms

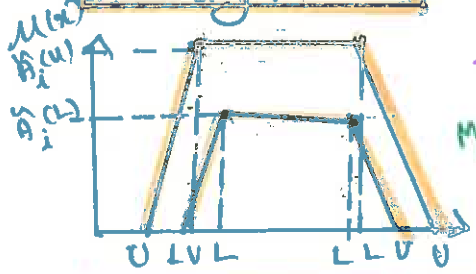




Inferential Modeling fuzzy set

Advances in Set Theory

Fuzzy Set



Trapezoidal Membership function $(x, \mu_{\tilde{A}}(x))$

FSE in big Data

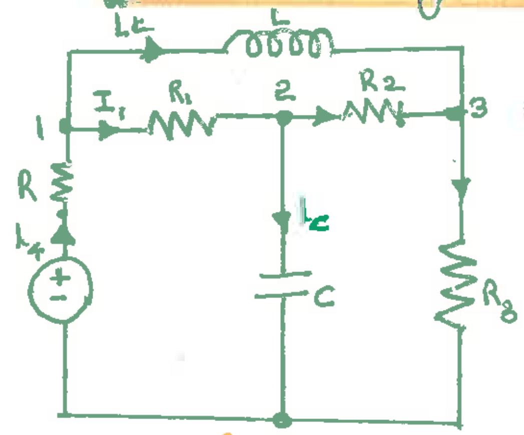
Validity → Big Data ← Volerability

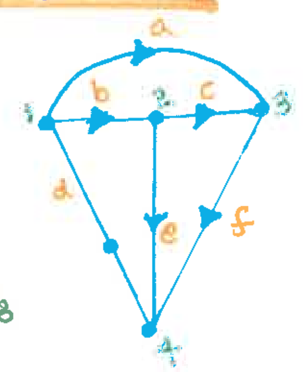
Big Data → Volatility, Visualisation

Logic and Inferencing

Obtaing Implication of Given facts and rules – Hallmark of Intelligence

Graph Theory In ECE





Electric Circuit and its equivalent Graph

SIMATS

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