

EDGE DOMINATING SETS IN GRAPHS*

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Abstract. We prove that the edge dominating set problem for graphs is *NP*-complete even when restricted to planar or bipartite graphs of maximum degree 3. We show as a corollary that the minimum maximal matching and the achromatic number problems are *NP*-complete. A new linear time algorithm for finding minimum independent edge dominating sets in trees is described, based on an observed relationship between edge dominating sets and independent sets in total graphs.

1. Introduction. In this paper we consider only finite, undirected graphs $G(V, E)$ with no parallel edges and no self-loops, where V is the set of the graph vertices and E is the set of its edges. Two vertices u, v of G connected by an edge are called *adjacent vertices* and we denote the edge by (u, v) . Two edges having a vertex in common or a vertex and its incident edge are also called *adjacent*. We say that an edge *dominates* its adjacent edges. A completely connected set of G is a set of vertices whose every two elements are adjacent. An *independent set* is a set of vertices no two of which are adjacent. For a set of edges M we denote by V_M the set of their adjacent vertices. For two sets A, B we denote by $A - B$ the set of elements of A which are not in B . The number of elements of a set A will be denoted by $|A|$.

A *node cover* of a graph G is a set of vertices C such that every edge of G is adjacent to a vertex of C ; $\gamma(G)$ will denote the size of a minimum node cover.

A set of edges of a graph $G(V, E)$ is called a *matching* if no two of its elements are adjacent. A matching is maximal if no other edges can be added to it. A set of edges M of $G(V, E)$ is called an *edge dominating set* if every edge of $E - M$ is adjacent to an element of M . The number of elements of a minimum edge dominating set will be denoted by $\beta(G)$. An *independent edge dominating set* is an edge dominating set in which no two elements are adjacent. An independent edge dominating set is in fact a maximal matching and a minimum independent edge dominating set is a minimum maximal matching (discussed in [4]).

As pointed out in [8], the size of the minimum edge dominating set of a graph G is equal to the size of its minimum independent edge dominating set. In fact, given a minimum edge dominating set F of G we can construct in polynomial time a minimum independent edge dominating set of G as follows: Consider in F two adjacent edges (u, v) and (v, w) , and let S be the set of edges different from (v, w) adjacent to w . It cannot be that $S \cap F \neq \emptyset$ or that the elements of $F - \{(v, w)\}$ dominate all the elements of S since in either case we could drop (v, w) from F , contradicting its minimality. Therefore, there is an edge $(w, z) \in S$, $z \neq v$, such that (w, z) is dominated only by the edge (v, w) of F . Thus, by replacing (v, w) by (w, z) in F we obtain an edge dominating set of the same size having less pairs of adjacent edges. Continuing in this way we obtain a minimum independent edge dominating set having $|F|$ elements.

In the present paper we consider the problem of the existence of a polynomial time algorithm for finding minimum edge dominating sets or equivalently for finding minimum independent edge dominating sets. This problem has some very interesting applications. For example, let A be an $m \times n$ 0-1 matrix, and consider the problem of

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finding a minimum set C of 1's in A such that any other 1 of A is in the same row or column with an element of C . Let us construct a bipartite graph $B(X, Y)$ by corresponding to every row a vertex in X , to every column a vertex in Y and connecting a vertex in X to a vertex in Y by an edge if and only if A has a one at the intersection of the corresponding row and column. It is easy to see that a minimum set C of 1's in A which dominates all the other 1's corresponds to a minimum edge dominating set of $B(X, Y)$ and conversely.

Another application ([10]) is related to a telephone switching network built to route phone calls from incoming lines to outgoing trunks (we assume that a trunk can pass only one phone call at a time). The problem is to find the worst-case behavior of the network, i.e., the minimum number of routed calls when the network is saturated and no calls can be added. For this we construct a bipartite graph B by connecting a line to a trunk if and only if the line can be switched to the trunk. Then the problem is equivalent to finding the size of a minimum independent edge dominating set of B .

A third application arises in the approximation of the node cover problem, known to be NP -complete ([9]). Let M be any maximal matching and V_M the set of vertices that are adjacent to the edges of M . Since M is a maximal matching, every edge of the graph is adjacent to a vertex of V_M , i.e., V_M is a node cover. On the other hand any node cover must contain at least one vertex from each edge of M , and therefore must be at least half the size of V_M . Clearly it would be desirable here to first find a minimum maximal matching M .

Unfortunately, as we shall prove in § 2 the edge dominating set problem is NP -complete even when restricted to planar or bipartite graphs with maximum degree 3. Therefore, the above problems are "intractable." In § 3 we show that there is a close relationship between edge dominating sets and independent sets in total graphs. Using this relationship we prove that the achromatic number problem is NP -complete, and describe in § 4 a linear time algorithm for finding minimum independent edge dominating sets in trees. A linear time algorithm for this problem has been described before in [12], but we shall point out why our algorithm is simpler.

2. The NP -completeness of the edge dominating problem. As proved by Cook [2] and Karp [9] there exists a family of problems called NP -complete no member of which is known to have a polynomial time algorithm, but if any of them does have one, then they all have. It seems unlikely that the NP -complete problems have polynomial time algorithms and knowing that a problem is NP -complete may spare some work to researchers. For proving that a problem P is NP -complete it is enough to prove that $P \in NP$ and to show that a known NP -complete problem is reducible to P in polynomial time. The known NP -complete problems used in our reductions are:

The node cover problem on planar cubic graphs [11].

Input: A planar cubic graph $G(V, E)$ and a positive integer k ;

Property: G has a node cover with at most k elements.

The SAT-3 restricted problem [13].

Input: A set of clauses C_1, \dots, C_p containing only variables, with at most three literals per clause, such that every variable occurs two times and its negation once.

Property: There is a truth assignment of zeros and ones to the variables satisfying all the clauses.

Our main task is to prove that the edge dominating set problem is NP -complete even when restricted to planar or bipartite graphs with maximum degree 3. This problem is defined as follows:

The edge dominating set problem.

Input: A graph $G(V, E)$ and a positive integer k ;

Property: G has a set of at most k edges which dominate all the other edges.

THEOREM 1. *The edge dominating set problem for planar graphs with maximum degree 3 is NP-complete.*

Proof. The reduction is from the node cover problem on planar cubic graphs.

Let $G(V, E)$ be a planar cubic graph. We fix an embedding of G on the plane and we replace each node v_i of G by a part H_i shown in Fig. 1.

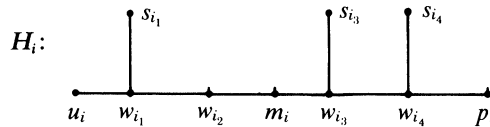


FIG. 1

The three edges incident to v_i are attached to the three nodes u_i, m_i, p_i . (There is some freedom here as to which of the $3!$ possible assignments is chosen. At the end we will see which assignment to choose so as to ensure our result.) The resulting graph G' has obviously maximum degree 3 and is planar (regardless of which assignment is chosen).

Let us prove that $\beta(G') = 2|V| + \gamma(G)$;

(i) Given a node cover C of G we define

$$F = \{(u_i, w_{i1}), (m_i, w_{i3}), (p_i, w_{i4}) | v_i \in C\} \cup \{(w_{i1}, w_{i2}), (w_{i3}, w_{i4}) | v_i \notin C\}.$$

Clearly, F is an edge dominating set (since C is a node-cover) with cardinality $|F| = 2|V| + |C|$, hence $\beta(G') \leq 2|V| + \gamma(G)$.

(ii) Conversely, let F be an edge dominating set of G' . For every H_i , F must contain one edge adjacent to w_{i1}, w_{i3}, w_{i4} in order to dominate the edges $(w_{i1}, s_{i1}), (w_{i3}, s_{i3}), (w_{i4}, s_{i4})$. Therefore F contains at least two edges from each H_i . Moreover if F has exactly two edges from H_i , then one is (w_{i3}, w_{i4}) and the other is (w_{i1}, w_{i2}) or (u_i, w_{i1}) or (s_{i1}, w_{i1}) and in the last two cases the edge of v_i adjacent to m_i must belong to F (to dominate (w_{i2}, m_i)). If F has three or more edges from some H_i we replace them by $(u_i, w_{i1}), (m_i, w_{i3}), (p_i, w_{i4})$. The resulting set is dominating with at least as low cardinality.

(a) Suppose that F has an edge of the form (u_i, t_j) , $i \neq j$, where t_j is u_j or m_j or p_j . If F has three edges from H_j , we delete (u_i, t_j) from F . If F has two edges from H_j , we replace them and (u_i, t_j) by $(u_j, w_{j1}), (m_j, w_{j3}), (p_j, w_{j4})$.

(b) Suppose that F has an edge of the form (p_i, t_j) , $i \neq j$, where t_j is m_j or p_j . Then we can apply the transformation of (a) above.

Assume now that G' was constructed in such a way that it doesn't have any edges of the form (m_i, m_j) , $i \neq j$. Then F has no edges from the original graph G and therefore whenever F contains exactly two edges from a part H_i these must be (w_{i3}, w_{i4}) and (w_{i1}, w_{i2}) . It follows that for every edge of G' connecting two different parts H_i, H_j , F must contain three edges in at least one of these parts, otherwise the edge is not dominated by F . Therefore the set C of the vertices v_i of G such that F has three edges in H_i is a node cover of G . Hence $\beta(G') \geq 2n + \gamma(G)$.

Thus, the node cover problem for planar cubic graphs is reducible to the edge dominating set problem for planar graphs with maximum degree 3.

It remains to show how to assign the edges adjacent to every v_i such that G' has no edges of the form (m_i, m_j) .

Consider the graph $G'' = (V \cup E, E'')$, where $E'' = \{(v, e) : v \in e\}$. Clearly G'' is a bipartite graph with maximum vertex degree 3. Hence it can be 3-edge colored in

polynomial time (see [5]). Such a coloring yields the desired assignment of edges when the “colors” are u , m , and p . \square

THEOREM 2. *The edge dominating set problem for bipartite graphs with maximum degree 3 is NP-complete.*

Proof. The reduction is from the SAT-3-restricted problem. Consider a set of clauses C_1, C_2, \dots, C_p with variables x_1, x_2, \dots, x_n as input for the SAT-3-restricted problem. We construct a graph $G(V, E)$ as follows:

$$V = \{a_i, b_i, c_i, (a_i, j_1), (a_i, j_1)', (b_i, j_2), (b_i, j_2)', (c_i, j_3), (c_i, j_3)' \mid$$

$$x_i \in C_{j_1}, C_{j_2}; \bar{x}_i \in C_{j_3}; 1 \leq i \leq n\} \cup \{d_i, d_i' \mid 1 \leq i \leq p\};$$

$$E = \{(a_i, c_i), (b_i, c_i), (a_i, (a_i, j_1)), (b_i, (b_i, j_2)), (c_i, (c_i, j_3)), ((a_i, j_1),$$

$$(a_i, j_1)'), ((b_i, j_2), (b_i, j_2)'), ((c_i, j_3), (c_i, j_3)'), ((a_i, j_1), d_{j_1}), ((b_i, j_2), d_{j_2}),$$

$$((c_i, j_3), d_{j_3}) \mid 1 \leq i \leq n\} \cup \{(d_j, d_j') \mid 1 \leq j \leq p\}.$$

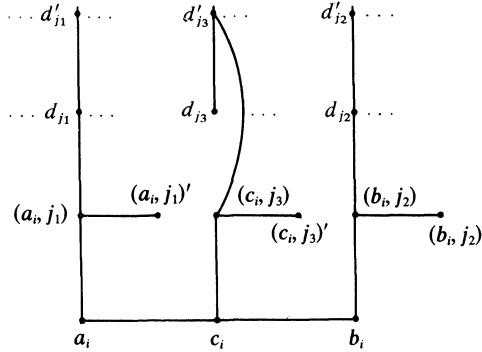


FIG. 2

The pattern corresponding in G to the variable x_i is shown in Fig. 2. A bipartition (S, T) of V is

$$S = \{a_i, b_i, (a_i, j_1)', (b_i, j_2)', (c_i, j_3) \mid 1 \leq i \leq n\} \cup \{d_i \mid 1 \leq i \leq p\},$$

$$T = \{c_i, (a_i, j_1), (b_i, j_2), (c_i, j_3)' \mid 1 \leq i \leq n\} \cup \{d_i' \mid 1 \leq i \leq p\}.$$

(i) Given a satisfying assignment of the clauses define the set F of edges as follows:

$$F = \{((a_i, j_1), d_{j_1}), ((b_i, j_2), d_{j_2}), ((c_i, j_3), c_i) \mid x_i = 1\}$$

$$\cup \{((a_i, j_1), a_i), ((b_i, j_2), b_i), ((c_i, j_3), d_{j_3}') \mid x_i = 0\}.$$

F is easily seen to be a dominating edge-set with $|F| = 3n$.

(ii) Let F be a dominating edge-set for G with $|F| = 3n$. Because of the edges $((a_i, j_1), (a_i, j_1)'),$ etc., F must contain at least one edge incident to each of $(a_i, j_1), (b_i, j_2), (c_i, j_3)$ for all i . Since $|F| = 3n$, it does not contain (a_i, c_i) or (b_i, c_i) and therefore it has either $((c_i, j_3), c_i)$ or both $((a_i, j_1), a_i)$ and $((b_i, j_2), b_i)$.

Define a truth assignment τ by setting $x_i = 1$ in the first case and $x_i = 0$ in the second case. Since $|F| = 3n$, for every j we have $(d_j, d_j') \notin F$ and therefore F contains at least one edge incident to either d_j or d_j' . Consequently τ satisfies all clauses.

The graph G constructed above has maximum degree 4. The degree-4 nodes are the d_i 's, where C_i is a 3-literal clause. We can take care of that as follows: Replace the

nodes d_i and d'_i by the graph H of the Fig. 3. The three edges between the nodes, corresponding to variables and d_i are now attached to x, y, z .

It is easy to show by a straightforward case analysis that H has the following properties:

$H - \{x, y, z\}$ consists of three $u-v$ paths of length 4. Therefore it needs at least four edges for its domination, i.e., even if all three edges incident to x, y, z are in F , F must contain four more edges from H .

If at least one of these three edges is in F , then four edges from H still suffice. (In the figure we show them with heavy lines in the case that the edge incident to x is in F .) If however none of these three edges belongs to F , then F must contain at least five edges from H . \square

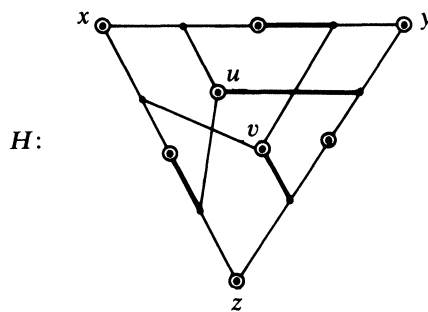


FIG. 3. (Nodes belonging to the one set of the bipartition of H are circled.)

3. Edge dominating sets and total graphs. For a graph $G(V, E)$ we can construct a graph $T(G)$ called the *total graph* of G by corresponding a vertex in $T(G)$ to every element of $V \cup E$ and connecting by an edge two vertices of $T(G)$ if and only if the corresponding elements of $V \cup E$ are adjacent in G . A vertex of $T(G)$ corresponding to a vertex of G is called a *v-vertex* and a vertex corresponding to an edge of G is called an *e-vertex*. These graphs were discussed in [1] and [7].

Consider a graph $G(V, E)$ and its total graph $T(G)$. As we shall prove, there is a direct connection between the minimum independent edge dominating sets of G and the maximum independent sets of $T(G)$. Let M be a set of edges of G .

LEMMA 1. *If M is a maximal matching of G then $M \cup (V - V_M)$ is a maximal independent set of $T(G)$. Also $|M \cup (V - V_M)| = |V| - |M|$.*

Proof. Since M is a maximal matching of G it follows that $V - V_M$ is an independent set of G , hence $M \cup (V - V_M)$ is independent in $T(G)$. Since every element of $T(G)$ not in $M \cup (V - V_M)$ is incident to M or is in $V - V_M$ it follows that $M \cup (V - V_M)$ is a maximal independent set. Also $|M \cup (V - V_M)| = |M| + |V| - 2|M| = |V| - |M|$. \square

LEMMA 2. *If M is a maximal matching of G and $M \cup (V - V_M)$ is a maximum independent set of $T(G)$ then M is a minimum independent edge dominating set of G .*

Proof. Assume that G has a minimum independent edge dominating set M' such that $|M'| < |M|$. Then M' is a maximal matching of G , and by Lemma 1 $M' \cup (V - V_{M'})$ is a maximal independent set of $T(G)$ such that $|M' \cup (V - V_{M'})| = |V| - |M'|$. Therefore, $M' \cup (V - V_{M'})$ is an independent set greater than $M \cup (V - V_M)$, contradicting our assumption. \square

THEOREM 3. *Let A be a maximum independent set of $T(G)$ and let M be the set of*

e-vertices of A . Then, for every maximal matching M' of G such that $M \subseteq M'$, the set $M' \cup (V - V_{M'})$ is a maximum independent set of $T(G)$ and M' is a minimum independent edge dominating set of G .

Proof. Let M' be any maximal matching of G such that $M \subseteq M'$. Consider an edge $e \in M' - M$. Clearly, e is not adjacent in $T(G)$ to any element of M . Also, e cannot be adjacent to two v -vertices x_1, x_2 of A , otherwise x_1 and x_2 would be adjacent in $T(G)$. Therefore, every element $e \in M' - M$ is adjacent in $T(G)$ to exactly one element x of A and x must be a v -vertex. Hence, by adding $M' - M$ to A and deleting from A the v -vertices adjacent to elements of $M' - M$ we obtain an independent set A' of $T(G)$ such that $|A'| = |A|$ and $A' = M' \cup (V - V_{M'})$. Also, by Lemma 2, M' is a minimum independent edge dominating set of G . \square

THEOREM 4. *If M is a minimum independent edge dominating set of G then $M \cup (V - V_M)$ is a maximum independent set of $T(G)$.*

Proof. By Lemma 1, $M \cup (V - V_M)$ is a maximal independent set of $T(G)$. Let A be a maximum independent set of $T(G)$, and let \bar{M} be the set of e -vertices of A . Let M' be any maximal matching of G such that $\bar{M} \subseteq M'$. By Theorem 3, $M' \cup (V - V_{M'})$ is also a maximum independent set of $T(G)$. Also, $|M' \cup (V - V_{M'})| = |V| - |M'|$, $|M \cup (V - V_M)| = |V| - |M|$ and $|V| - |M'| \geq |V| - |M|$, hence $|M| \geq |M'|$. On the other hand $|M'| \geq |M|$ since M is a minimum independent edge dominating set of G . Therefore $|M| = |M'|$ and $M \cup (V - V_M)$ is a maximum independent set of $T(G)$. \square

From Theorems 3 and 4 we can conclude that for every graph G , we can construct in polynomial time a minimum independent edge dominating set of G from a maximum independent set of $T(G)$ and conversely. Combining this conclusion with Theorem 2 we have

COROLLARY 1. *The independent set problem for the total graphs of bipartite graphs is NP-complete.*

We can describe another related problem. Consider a graph $G(V, E)$. An *achromatic coloring* of G is a coloring of its vertices such that no two adjacent vertices have the same color, and for every two colors i and j there are two adjacent vertices one colored i and one colored j . The achromatic coloring problem is: Given G and a positive integer k , does G have an achromatic coloring with k colors? The maximum k for which G has an achromatic coloring is called the *achromatic number* of G . Given a graph G we can construct the *independence graph* S of G as follows: we represent each independent set of G by a vertex in S , and we connect by an edge two vertices of S if and only if the corresponding independent sets I_1, I_2 of G are disjoint and there are two adjacent vertices v_1, v_2 of G such that $v_1 \in I_1, v_2 \in I_2$. It is not hard to see that every achromatic coloring of G corresponds to a maximal completely connected set of S and conversely. Therefore, it appears that the smaller are the independent sets of G , the easier would be the problem. Then, let us assume that G is the complement of a bipartite graph G' ; i.e., an independent set of G is a vertex or a pair of vertices. It is easy to prove that in this case the independence graph S of G is exactly the complement of the total graph $T(G')$ of G' . Hence the achromatic colorings of G correspond to the maximal independent sets of $T(G')$, and from Corollary 1 we can deduce

COROLLARY 2. *The achromatic number problem is NP-complete even for complements of bipartite graphs.*

Let us now consider a fixed k . It is NP-complete to determine whether the chromatic number of a graph is at most $k = 3$. But we can answer in polynomial time whether the achromatic number is at least k for any fixed k . This, because the achromatic number of a graph G is at least k if and only if there is a subset W of at most $k(k - 1)$ vertices of G which can be partitioned into k independent sets with at least one

edge connecting two different subsets and the subset W can be chosen in at most $\binom{n}{k(k-1)}$ ways.

4. Minimum independent edge dominating sets in trees. In this section we will use the results of the previous section to describe a linear time algorithm for finding minimum independent edge dominating sets in trees. As we mentioned in the Introduction, a linear time algorithm for this problem has been described before in [12]. Their algorithm finds first a minimum edge dominating set, and then constructs from it a minimum independent edge dominating set by interchanging edges, as described in the Introduction. Our algorithm is somewhat simpler in that it finds directly a minimum independent edge dominating set. We shall first prove a theorem which shows the structure of the total graphs of trees.

A graph G is called *chordal* if every simple circuit with more than three vertices has an edge connecting two nonconsecutive vertices. These graphs were discussed in [3] and [6]. As proved in [3] every chordal graph has a vertex v called *simplicial* such that the set of vertices adjacent to v is a completely connected set.

THEOREM 5. *A connected graph G is a tree if and only if its total graph is chordal.*

Proof. Assume that G is a tree. Let v be a terminal vertex of G , and let $e_v = (u, v)$ be the only edge incident to v in G . In $T(G)$ v is simplicial since it is adjacent only to u and e_v , and u and e_v are also adjacent. Let us delete v from $T(G)$ to obtain T_1 . The vertex e_v of T_1 is also simplicial. Let us delete also e_v from T_1 to obtain T_2 . It is easy to see that T_2 is the total graph of the graph obtained from G by deleting v and e_v . Therefore, by the induction hypothesis T_2 is chordal and so are T_1 and $T(G)$.

Conversely, let us assume that $T(G)$ is chordal. If G has only one edge, it is clearly a tree. Let us assume that G has more than one edge.

Let x be a simplicial vertex of $T(G)$: Let us assume that x is an e -vertex and $x = (u, v)$ in G . Since G has more than one edge and it is connected, there is an edge e incident to u (or v). Then, in $T(G)$ e is not adjacent to v , contradicting the fact that x is simplicial.

Therefore x must be a v -vertex. Let us assume that x is adjacent to two v -vertices x_1, x_2 . Since x is simplicial, x, x_1 and x_2 are mutually adjacent in $T(G)$, and so they are also in G . Denote in G : $e_1 = (x, x_1)$, $e_2 = (x, x_2)$. But then, in $T(G)$, x_2 is not adjacent to e_1 , while e_1 is adjacent to x , contradicting the fact that x is simplicial. Therefore, x can be adjacent in $T(G)$ to only one v -vertex, and to only one e -vertex. (Note that if x is adjacent to two e -vertices, then it has to be adjacent also to two v -vertices.) Therefore x has degree one in G . By deleting x and its edge from G and $T(G)$ we obtain G_1 and T_1 such that T_1 is the total graph of G_1 . Therefore, by the induction hypothesis G_1 is a tree, and by adding to it x and its edge we obtain that G is also a tree. \square

A simple algorithm for finding a minimum independent edge dominating set of a tree G works as follows: Construct $T(G)$. Find a maximum independent set A of the chordal graph $T(G)$, as described in [6]. Let M be the set of e -vertices of A . Then, any maximal matching M_1 containing M is a minimum independent edge dominating set of G . This algorithm requires $O(|V|^2)$ steps.

We can construct a more efficient algorithm by finding a maximum independent set of $T(G)$ and at the same time a minimum independent edge dominating set of G directly from the tree G . We do this in the following way.

We assume that the tree is rooted at any vertex r of it. Let k be the height of G and let A_i , $0 \leq i \leq k$, be the vertices of G which are at distance i from the root. We arrange the vertices and edges of G on levels by putting on level $2i$ the elements of A_i , and on

level $2i + 1$ the edges connecting the vertices of A_i to their sons. Now, we construct a set of edges M (at the beginning $M = \emptyset$) and we label the vertices and edges of G as follows: We go on every level starting with level $2k$ and down to level 0. On level $2j$, we traverse its vertices in some order, let us say from left to right, and for every unlabeled vertex v_j encountered we label by " $-v_j$ " the unlabeled elements on levels $2j$, $2j - 1$, $2j - 2$ which are adjacent to v_j . On level $2j - 1$, we traverse similarly its edges, and for every unlabeled edge e we label by "+" all the elements on levels $2j - 1$, $2j - 2$, $2j - 3$ which are adjacent to e (overwriting any previous labels), and we add e to M . Now, suppose that some edge e' from the same level $2j - 1$ was previously unlabeled, i.e., e' was labeled for the first time from e . Let v be the vertex of e' at level $2j$. We claim that v has label " $-w$ " for some son w of it. For, if v is unlabeled then e' would have been labeled " $-v$ " from vertex v ; if v is labeled "+" from some edge at level $2j + 1$, then e' would have been labeled also "+" from the same edge. For every such edge e' that was previously unlabeled we add the edge (v, w) to M . Finally on level 0 if the root r has label " $-u$," for some son u of it, we add the edge (r, u) to M .

It is easy to see that the set M obtained in the above algorithm is a matching. Suppose that there is an edge $e = (v_i, v_{i-1})$ which is not adjacent to any edge of M , with $v_i \in A_i$, $v_{i-1} \in A_{i-1}$. Then e must be labeled " $-v_i$," v_{i-1} must be labeled " $-w$ " for some son w of it (possibly v_i), and thus, if v_{i-1} is the root then $(v_{i-1}, w) \in M$. If v_{i-1} is not the root then the edge e' that is adjacent to v_{i-1} at level $2i - 3$ is still unlabeled when we start traversing the elements of its level. Thus either it remains unlabeled, in which case $e' \in M$, or it is labeled "+" from some edge on the same level, in which case $(v_{i-1}, w) \in M$. Therefore, M is a maximal matching of G . Now, consider the set S of unlabeled vertices and edges. It is easy to see that S is an independent set of $T(G)$. For every $x \in S$ let B_x be the set containing x and the elements labeled from x . Clearly B_x is a completely connected set of $T(G)$. Thus, the family $\{B_x\}_{x \in S}$ is a covering by completely connected sets of $T(G)$. Since no independent set can contain more than one element from each B_x , S must be a maximum independent set. Since M is a maximal matching that contains all the edges of S , it must be by Theorem 3 a minimum independent edge dominating set of G . With an adequate data structure the algorithm works in $O(|V|)$ steps.

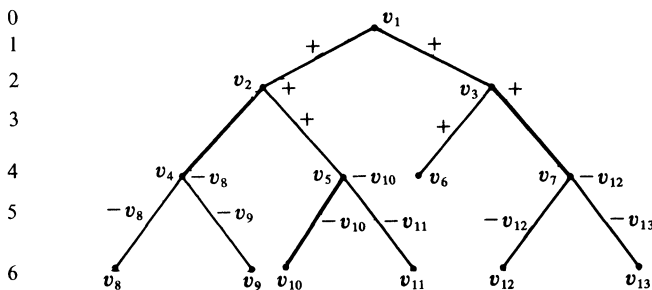


FIG. 4

Consider for example the tree G of Fig. 4. The levels $0, \dots, 6$ are noted on the side. Starting on level 6, we label (v_8, v_4) and v_4 by " $-v_8$," (v_9, v_4) by " $-v_9$ " and we proceed similarly with the rest of the vertices of level 6. Continuing on level 5, we encounter no unlabeled elements. On level 4 we label (v_5, v_3) and v_3 by " $-v_5$." On level 3, the edge (v_4, v_2) is unlabeled, hence we add it to M and we label "+" the elements (v_5, v_2) , v_2 and (v_2, v_1) . Since (v_5, v_2) was previously unlabeled, we add (v_5, v_{10}) to M . Similarly we add (v_7, v_3) to M , and label "+" v_3 , (v_6, v_3) (overwriting their previous labels) and the edge (v_3, v_1) . On levels 2 and 1 there are no unlabeled

elements, and on level 0 the root is unlabeled. Therefore, a minimum independent edge dominating set of G is $M = \{(v_4, v_2), (v_{10}, v_5), (v_7, v_3)\}$, and a maximum independent set of $T(G)$ is $S = \{v_8, v_9, v_{10}, v_{11}, v_{12}, v_{13}, v_6, (v_4, v_2), (v_7, v_3), v_1\}$.

5. Conclusions. We considered the problem of finding a minimum (independent) edge dominating set in graphs. We proved that this problem is NP -complete even when restricted to planar or bipartite graphs of maximum degree 3. Also, we described a linear time algorithm which solves the problem for trees.

Three other related problems were proven to be NP -complete:

- (a) the independent set problem for total graphs of bipartite graphs;
- (b) the achromatic number problem, even when restricted to complements of bipartite graphs;
- (c) finding in an 0–1 matrix a set of k 1's which dominate all the other 1's.

REFERENCES

- [1] M. BEHZAD, *The total chromatic number of a graph: A survey*, Proc. Conf. Combinatorial Mathematics (Oxford, England, 1969), Academic Press, New York, 1970.
- [2] S. A. COOK, *The complexity of theorem-proving procedures*, Proc. 4th Annual ACM Symposium on Theory of Computing (1971), pp. 151–158.
- [3] G. A. DIRAC, *On rigid circuit graphs*, Abh. Math. Sem. Univ. Hamburg, 25 (1961), pp. 71–76.
- [4] R. FORCADE, *Smallest maximal matching in the graph of the d -dimensional cube*, J. Combinatorial Theory Ser. B, 14 (1973), pp. 153–156.
- [5] H. N. GABOW AND O. KARIV, *Algorithms for edge coloring bipartite graphs*, Proc. 10th Annual ACM Symposium on Theory of Computing (1978), pp. 184–192.
- [6] F. GAVRIL, *Algorithms for minimum coloring, maximum clique, minimum covering by cliques and maximum independent set of a chordal graph*, SIAM J. Comput., 1 (1972), pp. 180–187.
- [7] ———, *A recognition algorithm for the total graphs*, Networks, 8 (1978), pp. 121–133.
- [8] F. HARARY, *Graph theory*, Addison-Wesley, Reading, MA, 1969.
- [9] R. M. KARP, *Reducibility among combinatorial problems*, Complexity of Computer Computations, R. E. Miller and J. W. Thatcher, eds., Plenum Press, New York, 1972, pp. 85–104.
- [10] C. L. LIU, *Introduction to Combinatorial Mathematics*, McGraw-Hill, New York, 1968.
- [11] D. MAIER AND J. A. STORER, *A note on the complexity of the superstring problem*, Tech. Rep. 233, Computer Science Laboratory, Princeton Univ., Princeton, NJ, 1977.
- [12] S. MITCHELL AND S. HEDETNIEMI, *Edge domination in trees*, Proc. Eighth Southeastern Conf. on Combinatorics, Graph Theory and Computing, Utilitas Mathematica, Winnipeg, 1977.
- [13] M. YANNAKAKIS, *Edge-deletion problems*, Tech. Rep. 249, Computer Science Laboratory, Princeton, Univ., Princeton, NJ, 1978.