Notes on the Repertoire Method for Solving Recurrences

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The repertoire method is presented in *Concrete Mathematics, a Foundation for Computer Science*, by Graham, Knuth, and Patashnik, Addison-Wesley, 1989.

This is the greatest book on discrete math ever written.

Solve:

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

First, get some cases

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= r_{n-1} + 3n + 5
\end{aligned}
$$

$$
\begin{aligned}
r_1 &= r_0 + 3(1) + 5 = 1 + 3 + 5 = 9 \\
r_2 &= r_1 + 3(2) + 5 = 9 + 6 + 5 = 20 \\
r_3 &= r_2 + 3(3) + 5 = 20 + 9 + 5 = 34
\end{aligned}
$$

It's easy enough to do this by hand,
or write a little throw-away program to calculate them for you.

n	0	1	2	3	4	5
r_n	1	9	20	34	51	71

Quick, what's the next number in this sequence?
Hmmm... nothing occurs to me.

Unsimplified cases.

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

Let's try that a little slower:

$$\begin{aligned} r_1 &= r_0 + 3(1) + 5 = 1 + 3 + 5 \\ r_2 &= r_1 + 3(2) + 5 \\ &= 1 + 3 + 5 + 3(2) + 5 \\ &= 1 + 3(3) + 5(2) \\ r_3 &= r_2 + 3(3) + 5 \\ &= 1 + 3(3) + 5(2) + 3(3) + 5 \\ &= 1 + 3(6) + 5(3) \\ r_4 &= r_3 + 3(4) + 5 \\ &= 1 + 3(6) + 5(3) + 3(4) + 5 \\ &= 1 + 3(10) + 5(4) \end{aligned}$$

A pattern in the unsimplified cases.

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= r_{n-1} + 3n + 5
\end{aligned}
$$

$$
\begin{aligned}
r_0 &= 1(1) + 3(0) + 5(0) \\
r_1 &= 1(1) + 3(1) + 5(1) \\
r_2 &= 1(1) + 3(3) + 5(2) \\
r_3 &= 1(1) + 3(6) + 5(3) \\
r_4 &= 1(1) + 3(10) + 5(4)
\end{aligned}
$$

It looks like our solution could be:

$$
r_n = 1A(n) + 3B(n) + 5C(n)
$$

where A, B and C are simple functions of n.

Solve one function with three others.

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= r_{n-1} + 3n + 5
\end{aligned}
$$

$$
\begin{aligned}
r_0 &= 1(1) + 3(0) + 5(0) \\
r_1 &= 1(1) + 3(1) + 5(1) \\
r_2 &= 1(1) + 3(3) + 5(2) \\
r_3 &= 1(1) + 3(6) + 5(3) \\
r_4 &= 1(1) + 3(10) + 5(4) \\
r_n &= 1A(n) + 3B(n) + 5C(n)
\end{aligned}
$$

It's pretty easy to guess that $A(n) = 1$ and $C(n) = n$.

What about $B(n)$?

Use that fantastic brain of yours...

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= r_{n-1} + 3n + 5 \\
r_0 &= 1(1) + 3(0) + 5(0) \\
r_1 &= 1(1) + 3(1) + 5(1) \\
r_2 &= 1(1) + 3(3) + 5(2) \\
r_3 &= 1(1) + 3(6) + 5(3) \\
r_4 &= 1(1) + 3(10) + 5(4) \\
r_n &= 1A(n) + 3B(n) + 5C(n) \\
&= 1 + 3B(n) + 5n
\end{aligned}
$$

n	0	1	2	3	4	5	6	...
$B(n)$	0	1	3	6	10	15	21	...

Where have I seen those numbers before?

Aha!

n	0	1	2	3	4	5	6	...
$B(n)$	0	1	3	6	10	15	21	...

$$\begin{aligned} B(n) &= \sum_{i=0}^{n} i \\ &= \frac{n(n+1)}{2} \end{aligned}$$

We did it!

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= r_{n-1} + 3n + 5 \\
r_0 &= 1(1) + 3(0) + 5(0) \\
r_1 &= 1(1) + 3(1) + 5(1) \\
r_2 &= 1(1) + 3(3) + 5(2) \\
r_3 &= 1(1) + 3(6) + 5(3) \\
r_4 &= 1(1) + 3(10) + 5(4) \\
r_n &= 1A(n) + 3B(n) + 5C(n) \\
&= 1 + \frac{3n(n+1)}{2} + 5n \\
&= \frac{3}{2}n^2 + \frac{13}{2}n + 1
\end{aligned}
$$

Summarizing

If our wild assumption is correct, the solution to

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

is

$$r_n = \frac{3}{2}n^2 + \frac{13}{2}n + 1$$

How can we know for sure?

Testing

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

$$r_n = \frac{3}{2}n^2 + \frac{13}{2}n + 1$$

Test it out with another little program:

n	0	1	2	3	4	5
r_n	1	9	20	34	51	71
$(3/2)n^2 + (13/2)n + 1$	1	9	20	34	51	71

Yay! It works! It must be true!

Is there a better way?

Prove it by induction

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= r_{n-1} + 3n + 5
\end{aligned}
$$

$$
r_n = \frac{3}{2}n^2 + \frac{13}{2}n + 1
$$

Now let's start over.

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

Can we use what we've learned in a better way, and without needing the **Aha!** in the middle?

First we generalize:

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

Let's solve this where the constants 1, 3 and 5 have been replaced by variables α, β, and γ:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

But isn't that harder?

Not really.

Cases of our generalized version

Again, let's work up by hand some simple cases.

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

n	0	1	2	3
r_n	α	$\alpha+\beta+\gamma$	$(\alpha+\beta+\gamma)+2\beta+\gamma$ $=\alpha+3\beta+2\gamma$	$(\alpha+3\beta+2\gamma)+3\beta+\gamma$ $=\alpha+6\beta+3\gamma$

Again, it looks like we have some α's, some β's, and some γ's in the solution.

But how many of each?

Wild assumption:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

Let's assume that there are three fixed functions, A, B, and C, such that the solution to the above always has this form:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

We don't know this is true, but the evidence suggests it.

Can we figure out what A, B and C are? Yes!

Is this easier than the original problem? Yes!

Here's How

We assume that *any* recurrence defined by:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

has a solution that looks like:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

no matter what α, β, and γ are.

Different α, β, and γ will define different r_n.

Very Important Point:

Many different recurrences that look like

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

are solved by

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

α, β, and γ might be different for each one, but...

$A(n)$, $B(n)$, and $C(n)$ are the *same* for *all* of them!

What does this buy us?

For any α, β, and γ, the equations:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

are always solved by:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

If we pick really simple functions (with really easy values for α, β and γ) we can *solve* for A, B, and C!

And once we have A, B and C, we have a solution to the general recurrence!

Let's give it a try.

Easy Solutions

Now let's see if some *easy* functions are solutions of these equations, both the recurrence and the closed form, with different α, β, and γ.

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

First Easy Solution

Let's try $r_n = 1$. That's an easy function. Plugging it into:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

gives:

$$\begin{aligned} 1 &= \alpha \\ 1 &= 1 + \beta n + \gamma \\ 1 &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

Solve the result

After plugging, we have:

$$\begin{aligned} 1 &= \alpha \\ 1 &= 1 + \beta n + \gamma \\ 1 &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

Which can be solved for one of the unknown functions:

$$\begin{aligned} \alpha &= 1 \\ \beta &= 0 \\ \gamma &= 0 \\ A(n) &= 1 \end{aligned}$$

We have found $A(n)$!

$r_n = 1$ **has consequences!**

If our wild assumption is correct, $A(n) = 1$ for *all* solutions of our equations, and so this:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

becomes:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha + \beta B(n) + \gamma C(n) \end{aligned}$$

We're making progress!
Our *general* solution is closer to being solved.

Progress

The general solution to

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

looks like this:

$$r_n = \alpha + \beta B(n) + \gamma C(n)$$

Now we only need to find $B(n)$ and $C(n)$.

Let's try another easy function!

Let's try $r_n = n$.

Is this a solution? In general, we have:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha + \beta B(n) + \gamma C(n) \end{aligned}$$

Plugging gives:

$$\begin{aligned} 0 &= \alpha \\ n &= n - 1 + \beta n + \gamma \\ n &= \alpha + \beta B(n) + \gamma C(n) \end{aligned}$$

Solving after plugging $r_n = n$

$$
\begin{aligned}
0 &= \alpha \\
n &= n - 1 + \beta n + \gamma \\
1 &= \beta n + \gamma
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 0 \\
\gamma &= 1 \\
n &= \alpha + \beta B(n) + \gamma C(n) \\
&= C(n)
\end{aligned}
$$

Aha! More progress! $C(n) = n$

More Progress

Since $A(n) = 1$ and $C(n) = n$, this:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

becomes:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha + \beta B(n) + \gamma n \end{aligned}$$

Our general solution has become:

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= r_{n-1} + \beta n + \gamma \\
r_n &= \alpha + \beta B(n) + \gamma n
\end{aligned}
$$

Only B left!

Let's find another simple function!

Let's try $r_n = n^2$

Plugging into

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \\ r_n &= \alpha + \beta B(n) + \gamma n \end{aligned}$$

gives

$$\begin{aligned} 0 &= \alpha \\ n^2 &= (n-1)^2 + \beta n + \gamma \\ n^2 &= \alpha + \beta B(n) + \gamma n \end{aligned}$$

Solving after plugging $r_n = n^2$

$$
\begin{aligned}
0 &= \alpha \\
n^2 &= (n-1)^2 + \beta n + \gamma \\
&= n^2 - 2n + 1 + \beta n + \gamma \\
2n - 1 &= \beta n + \gamma
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= 0 \\
\beta &= 2 \\
\gamma &= -1
\end{aligned}
$$

$$
\begin{aligned}
n^2 &= \alpha + \beta B(n) + \gamma n \\
&= 2B(n) - n \\
B(n) &= (n^2 + n)/2
\end{aligned}
$$

Still More Progress!

Since $A(n) = 1$, $B(n) = (n^2 + n)/2$ and $C(n) = n$, we know that the solution to

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

Looks like

$$\begin{aligned} r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \\ &= \alpha + \beta(n^2 + n)/2 + \gamma n \end{aligned}$$

We have solved the general equation!

Assuming our wild assumption was correct...

Note

We found the solution for B without any **Aha!**'s.

$$B(n) = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$$

Summarizing

The solution to any recurrence like this:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

is

$$r_n = \alpha + \beta(n^2 + n)/2 + \gamma n$$

Let's try it out!

The solution to any recurrence like this:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

is

$$r_n = \alpha + \beta(n^2 + n)/2 + \gamma n$$

Here's our original problem:

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

Plug and Chug

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

$$\begin{aligned} r_0 &= 1 \\ r_n &= r_{n-1} + 3n + 5 \end{aligned}$$

$$\alpha = 1 \quad \beta = 3 \quad \gamma = 5$$

$$\begin{aligned} r_n &= \alpha + \beta(n^2 + n)/2 + \gamma n \\ &= 1 + 3(n^2 + n)/2 + 5n \\ &= \frac{3}{2}n^2 + \frac{13}{2}n + 1 \end{aligned}$$

This is the same thing we got the first time.

Summations

Recurrences like these:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= r_{n-1} + \beta n + \gamma \end{aligned}$$

with solution:

$$r_n = \alpha + \beta(n^2 + n)/2 + \gamma n$$

Can be used to solve summations like these:

$$\sum_{i=0}^{n} (3i + 2)$$

Summations

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= r_{n-1} + \beta n + \gamma
\end{aligned}
$$

$$
r_n = \alpha + \beta(n^2 + n)/2 + \gamma n
$$

Let's define S_n like this:

$$
S_n = \sum_{i=0}^{n} (3i + 2)
$$

Then we can write a recurrence for S_n:

$$
\begin{aligned}
S_0 &= 2 \\
S_n &= S_{n-1} + 3n + 2
\end{aligned}
$$

Summations

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= r_{n-1} + \beta n + \gamma
\end{aligned}
$$

$$
r_n = \alpha + \beta(n^2 + n)/2 + \gamma n
$$

$$
\begin{aligned}
S_0 &= 2 \\
S_n &= S_{n-1} + 3n + 2 \\
\alpha &= 2 \\
\beta &= 3 \\
\gamma &= 2 \\
S_n &= 2 + (3/2)(n^2 + n) + 2n \\
&= (3/2)n^2 + (7/2)n + 2 \\
&= \sum_{i=0}^{n} (3i + 2)
\end{aligned}
$$

Summarizing Summations

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= r_{n-1} + \beta n + \gamma
\end{aligned}
$$

$$
r_n = \alpha + \beta(n^2+n)/2 + \gamma n
$$

$$
\sum_{i=0}^{n}(3i+2) = (3/2)n^2 + (7/2)n + 2
$$

You can also solve this the old fashioned way:

$$
\begin{aligned}
\sum_{i=0}^{n}(3i+2) &= \sum_{i=0}^{n} 3i + \sum_{i=0}^{n} 2 \\
&= 3\sum_{i=0}^{n} i + \sum_{i=0}^{n} 2 \\
&= 3(n(n+1)/2) + 2(n+1)
\end{aligned}
$$

Did we get the same answer?

Let's try something harder.

Solve this one:

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= 2r_{n-1} + n
\end{aligned}
$$

(It's the one from your textbook.)

Cases

$$\begin{aligned} r_0 &= 1 \\ r_n &= 2r_{n-1} + n \end{aligned}$$

n	0	1	2	3	4	5
r_n	1	3	8	19	42	89

Recognize those numbers?

Me neither.

First generalize

$$\begin{aligned} r_0 &= 1 \\ r_n &= 2r_{n-1} + n \end{aligned}$$

$$\begin{aligned} r_0 &= \alpha \\ r_n &= \beta r_{n-1} + \gamma n \end{aligned}$$

Cases first

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= 2r_{n-1} + n
\end{aligned}
$$

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= \beta r_{n-1} + \gamma n
\end{aligned}
$$

n	0	1	2	3
r_n	α	$\beta\alpha+\gamma$	$(\beta\alpha+\gamma)\beta+2\gamma$	$((\beta\alpha+\gamma)\beta+2\gamma)\beta+3\gamma$

This doesn't look promising, since the α's β's and γ's are getting mixed up. Perhaps it's *too* general?

Let's try again

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= 2r_{n-1} + n
\end{aligned}
$$

Don't generalize on the 2:

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma
\end{aligned}
$$

We also threw in an extra constant, γ.
In our example, it is 0. Did we need that?

Try leaving it out and seeing how far you get.
The trick is to generalize *just enough*, but not too much.

Cases again

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma
\end{aligned}
$$

n	0	1	2	3
r_n	α	$2\alpha + \beta + \gamma$	$2(2\alpha + \beta + \gamma) + 2\beta + \gamma$ $= 4\alpha + 4\beta + 3\gamma$	$2(4\alpha + 4\beta + 3\gamma) + 3\beta + \gamma$ $= 8\alpha + 11\beta + 7\gamma$

OK! Still a mess, but at least the α's, β's and γ's are not getting mixed up. This gives us hope that there is some solution of the form

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

Somewhat justified wild assumption:

All solutions to:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \end{aligned}$$

Have the form:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(n)$$

for some fixed functions A, B and C.

Let's go!

Let's try $r_n = 1$

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \\ r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

$$\begin{aligned} 1 &= \alpha \\ 1 &= 2 + \beta n + \gamma \\ 1 &= \alpha A(n) + \beta B(n) + \gamma C(n) \end{aligned}$$

$$\begin{aligned} \alpha &= 1 \\ \beta &= 0 \\ \gamma &= -1 \\ 1 &= A(n) - C(n) \end{aligned}$$

Plugging $r_n = 1$

We concluded that

$$C(n) = A(n) - 1$$

so

$$\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma \\
r_n &= \alpha A(n) + \beta B(n) + \gamma C(n) \\
&= \alpha A(n) + \beta B(n) + \gamma (A(n) - 1) \\
&= (\alpha + \gamma) A(n) + \beta B(n) - \gamma
\end{aligned}$$

Let's try $r_n = n$

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \\ r_n &= (\alpha+\gamma)A(n) + \beta B(n) - \gamma \end{aligned}$$

$$\begin{aligned} 0 &= \alpha \\ n &= 2(n-1) + \beta n + \gamma \\ &= 2n - 2 + \beta n + \gamma \\ -n+2 &= \beta n + \gamma \\ \beta &= -1 \\ \gamma &= 2 \\ n &= (\alpha+\gamma)A(n) + \beta B(n) - \gamma \\ &= 2A(n) - B(n) - 2 \\ B(n) &= 2A(n) - n - 2 \end{aligned}$$

More progress

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma \\
B(n) &= 2A(n) - n - 2 \\
r_n &= (\alpha + \gamma)A(n) + \beta B(n) - \gamma \\
&= (\alpha + \gamma)A(n) + \beta(2A(n) - n - 2) - \gamma \\
&= (\alpha + \gamma)A(n) + 2\beta A(n) - \beta n - 2\beta - \gamma \\
&= (\alpha + 2\beta + \gamma)A(n) - (n+2)\beta - \gamma
\end{aligned}
$$

Let's try $r_n = n^2$

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma \\
r_n &= (\alpha + 2\beta + \gamma)A(n) - (n+2)\beta - \gamma \\
0 &= \alpha \\
n^2 &= 2(n-1)^2 + \beta n + \gamma \\
&= 2n^2 - 4n + 2 + \beta n + \gamma \\
-n^2 + 4n - 2 &= \beta n + \gamma
\end{aligned}
$$

This will go nowhere. Since we don't have something for the n^2 term, we can't solve.

It seems that our recurrence is *not* true for $r_n = n^2$, no matter what constants we choose.

We need a different function to try.

Let's try $r_n = 2^n$

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \\ r_n &= (\alpha + 2\beta + \gamma)A(n) - (n+2)\beta - \gamma \end{aligned}$$

$$\begin{aligned} 1 &= \alpha \\ 2^n &= 2(2^{n-1}) + \beta n + \gamma \end{aligned}$$

This one's easy! $\beta = \gamma = 0$

Why?

Plugging $r_n = 2^n$

$$\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma
\end{aligned}$$

$$\begin{aligned}
1 &= \alpha \\
2^n &= 2(2^{n-1}) + \beta n + \gamma \\
&= 2^n + \beta n + \gamma \\
\beta &= 0 \\
\gamma &= 0 \\
2^n &= (\alpha + 2\beta + \gamma)A(n) - (n+2)\beta - \gamma \\
&= A(n)
\end{aligned}$$

And we have A

Summarizing

If our wild assumption is correct, then

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma
\end{aligned}
$$

has the general solution:

$$
r_n = (\alpha + 2\beta + \gamma)2^n - (n+2)\beta - \gamma
$$

Let's try our specific problem again

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma \\
r_n &= (\alpha + 2\beta + \gamma)2^n - (n+2)\beta - \gamma
\end{aligned}
$$

$$
\begin{aligned}
r_0 &= 1 \\
r_n &= 2r_{n-1} + n
\end{aligned}
$$

$$
\begin{aligned}
\alpha &= 1 \\
\beta &= 1 \\
\gamma &= 0 \\
r_n &= 3(2^n) - n - 2
\end{aligned}
$$

Is this solution correct?

Problem Solved

$$r_0 = 1$$
$$r_n = 2r_{n-1} + n$$

$$r_n = 3(2^n) - n - 2$$

Cases give us confidence:

n	0	1	2	3	4	5
r_n	1	3	8	19	42	89
$3(2^n) - n - 2$	1	3	8	19	42	89

But to be *really* sure, prove it by induction!

Reprise

Remember, to solve a specific problem:

$$\begin{aligned} r_0 &= 1 \\ r_n &= 2r_{n-1} + n \end{aligned}$$

We solved a general problem:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \end{aligned}$$

Reprise

To solve the general problem:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \end{aligned}$$

We investigated some small cases, and guessed that all solutions looked like:

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(N)$$

for some functions A, B, and C.

Reprise

To solve for A, B, and C, we tried to find simple functions $r_n = f(n)$ and constants α, β and γ that satisfied these conditions:

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \end{aligned}$$

and then plugged them into

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(N)$$

to learn something about A, B, and C.

Reprise

We tried $r_n = 1$ and found that this worked with $\alpha = 1$, $\beta = 0$ and $\gamma = -1$.

$$
\begin{aligned}
r_0 &= \alpha \\
r_n &= 2r_{n-1} + \beta n + \gamma
\end{aligned}
$$

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(N)$$

Which got us the equation

$$1 = A(n) - C(n)$$

Reprise

We tried $r_n = n$ and found that this worked with $\alpha = 0$, $\beta = -1$ and $\gamma = 2$.

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \end{aligned}$$

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(N)$$

Which got us the equation

$$n = -B(n) + 2C(n)$$

Reprise

We tried $r_n = 2^n$ and found that this worked with $\alpha = 1$, $\beta = 0$ and $\gamma = 0$.

$$\begin{aligned} r_0 &= \alpha \\ r_n &= 2r_{n-1} + \beta n + \gamma \end{aligned}$$

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(N)$$

Which got us the equation

$$2^n = A(n)$$

Reprise

The first time through, we plugged each of our equations in to simplify the expression

$$r_n \;=\; \alpha A(n) + \beta B(n) + \gamma C(N)$$

resulting (eventually) in

$$r_n \;=\; (\alpha + 2\beta + \gamma)2^n - (n+2)\beta - \gamma$$

Reprise

This time, let's just collect our three equations:

$$\begin{array}{rcllll} 1 & = & A(n) & & -C(n) \\ n & = & & -B(n) & +2C(n) \\ 2^n & = & A(n) & & \end{array}$$

Clearly we have a system of three equations in three unknowns.

We know how to solve these!

Reprise

$$\begin{array}{rcllll} 1 & = & A(n) & & -C(n) \\ n & = & & -B(n) & +2C(n) \\ 2^n & = & A(n) & & \end{array}$$

Subtracting the first from the third gives:

$$2^n - 1 \; = \; C(n)$$

Adding twice the first to the second gives:

$$\begin{aligned} n+2 &= 2A(n) - B(n) \\ &= 2(2^n) - B(n) \\ B(n) &= 2^{n+1} - n - 2 \end{aligned}$$

Reprise

Solving

$$\begin{array}{rcrrr} 1 & = & A(n) & & -C(n) \\ n & = & & -B(n) & +2C(n) \\ 2^n & = & A(n) & & \end{array}$$

gives

$$\begin{aligned} A(n) &= 2^n \\ B(n) &= 2^{n+1} - n - 2 \\ C(n) &= 2^n - 1 \end{aligned}$$

Reprise

Plugging

$$\begin{aligned} A(n) &= 2^n \\ B(n) &= 2^{n+1} - n - 2 \\ C(n) &= 2^n - 1 \end{aligned}$$

into

$$r_n = \alpha A(n) + \beta B(n) + \gamma C(N)$$

gives

$$r_n = \alpha(2^n) + \beta(2^{n+1} - n - 2) + \gamma(2^n - 1)$$

which, on rearranging, is equivalent to our original

$$r_n = (\alpha + 2\beta + \gamma)2^n - (n+2)\beta - \gamma$$