① 1a)
$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ -2 & -1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$C_2 \to C_2 - 2C_1$

$C_3 \to C_3 - 3C_1$

$C_4 \to C_4 + C_1$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 3 & -3 \\ 0 & -2 & -2 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

$R_4 \leftrightarrow R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & -2 & -2 & 2 \\ 0 & 3 & 3 & -3 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$

$R_4 \to R_4 - 3R_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$C_4 \to C_4 + C_2$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\approx \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$$

$\boxed{\text{Rank} = 2}$

1b)
$$\begin{bmatrix} 2 & 3 & -1 & 1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_1$
$$\begin{bmatrix} 1 & -1 & -2 & -4 \\ 2 & 3 & -1 & 1 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 6R_1$

$$\begin{pmatrix} 1 & -1 & -2 & -4 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$$

$C_2 \to C_2 + C_1$

$C_3 \to C_3 + 2C_1$

$C_4 \to C_4 + 4C_1$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 4 & 9 & 10 \\ 0 & 9 & 12 & 17 \end{pmatrix}$$

$R_3 \to 5R_3 - 4R_2$

$R_4 \to 5R_4 - 9R_2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 3 & 7 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{pmatrix}$$

$C_3 \to 5C_3 - 3C_2$

$C_4 \to 7C_2 - 5C_4$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 33 & 22 \\ 0 & 0 & 33 & 22 \end{pmatrix}$$

$R_2 \to R_2/5$

$R_3 \to R_3/11$

$R_4 \to R_4/11$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 3 & 2 \end{pmatrix}$$

$R_4 \to R_4 - R_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_3 \to C_3/3$

$C_4 \to C_4/2$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$C_4 \to C_4 - C_3$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sim \begin{pmatrix} I_3 & 0 \\ 0 & 0 \end{pmatrix}$$

Rank $= 3$

2a) $4x + 2y + z + 3w = 0$

$6x + 3y + 4z + 7w = 0$

$2x + y + w = 0$

$$\boxed{AX = 0}$$

$$\begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

here $n = 4$

$$\begin{pmatrix} 4 & 2 & 1 & 3 \\ 6 & 3 & 4 & 7 \\ 2 & 1 & 0 & 1 \end{pmatrix}$$

$R_3 \longleftrightarrow R_1$

$$= \begin{pmatrix} 2 & 1 & 0 & 1 \\ 6 & 3 & 4 & 7 \\ 4 & 2 & 1 & 3 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 2R_1$

$$= \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 4 & 4 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

$R_2 \to R_2/4$

$R_3 \to R_3 - R_2$

$$= \begin{pmatrix} 2 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Rank of matrix $= 2 < 4$

infinitely many solutions.

$2x + y + w = 0$

$z + w = 0$

$z = -w$

let $w = t$ (Assume)

$z = -t$

$2x+y+t=0$

$2x+y=-t$

Assuming

$2(-t)+t=-t$

(for satisfying)

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -t \\ t \\ -t \\ t \end{pmatrix} = t\begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$t \Rightarrow$ scalar

It can be anything.

2b) $x+2y+3z=\lambda x$

$3x+y+2z=\lambda y$

$2x+3y+z=\lambda z$

$(1-\lambda)x+2y+3z=0$

$3x+(1-\lambda)y+2z=0$

$2x+3y+(1-\lambda)z=0$

$$\begin{vmatrix} (1-\lambda) & 2 & 3 \\ 3 & (1-\lambda) & 2 \\ 2 & 3 & (1-\lambda) \end{vmatrix} = 0$$

$R_1 \longrightarrow R_1+R_2+R_3$

$$\begin{vmatrix} (6-\lambda) & (6-\lambda) & (6-\lambda) \\ 3 & (1-\lambda) & 2 \\ 2 & 3 & (1-\lambda) \end{vmatrix} = 0$$

$$(6-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ 3 & (1-\lambda) & 2 \\ 2 & 3 & (1-\lambda) \end{vmatrix} = 0$$

$C_2 \longrightarrow C_2 - C_1$

$C_3 \longrightarrow C_3 - C_1$

$$(6-\lambda)\begin{vmatrix} 1 & 0 & 0 \\ 3 & -2-\lambda & -1 \\ 2 & 1 & -1-\lambda \end{vmatrix} = 0$$

$(6-\lambda)\left[(2+\lambda)(1+\lambda)+1\right] = 0$

$(6-\lambda)\left[2+2\lambda+\lambda+\lambda^2+1\right] = 0$

$(6-\lambda)\left[\lambda^2+3\lambda+3\right] = 0$

does not satisfy the given condition

$6-\lambda=0$

$\underline{\underline{\lambda=6}}$

2c) $x+y+z=6$

$x+2y+3z=10$

$x+2y+\lambda z=\mu$

AB = Augmented Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1$

$R_3 \longrightarrow R_3 - R_1$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{bmatrix}$$

i) no solution

$\rho(A) \neq \rho(AB)$

rank of $A \neq$ Rank of AB

$\lambda = 3, \mu \neq 10$

$\rho(A) = 2 \neq 3 = \rho(AB)$

ii) unique solution if $\rho(A) = \rho(AB) = 3$

$\lambda \neq 3$

iii) infinitely many solutions if

$\rho(A) = \rho(AB) < 3$

$\lambda = 3, \mu = 10.$

2d) $3x + y + 2z = 3$

$2x - 3y - z = -3$

$x + 2y + z = 4$

$$\begin{bmatrix} 3 & 1 & 2 & 3 \\ 2 & -3 & -1 & -3 \\ 1 & 2 & 1 & 4 \end{bmatrix} = AB$$

(Augmented Matrix)

$R_3 \longleftrightarrow R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 2 & -3 & -1 & -3 \\ 3 & 1 & 2 & 3 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - 2R_1$

$R_3 \longrightarrow R_3 - 3R_1$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & -7 & -3 & -11 \\ 0 & -5 & -1 & -9 \end{bmatrix}$$

$(-)(-)$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 7 & 3 & 11 \\ 0 & 5 & 1 & 9 \end{bmatrix}$$

$R_3 \longrightarrow 7R_3 - 5R_2$

$$\begin{bmatrix} 1 & 2 & 1 & 4 \\ 0 & 7 & 3 & 11 \\ 0 & 0 & 8 & -8 \end{bmatrix}$$

* $8z = -8$

$\boxed{z = -1}$

* $7y + 3z = 11$

$7y + 3(-1) = 11$

$7y - 3 = 11$

$7y = 14$

$\boxed{y = 2}$

* $x + 2y + z = 4$

$x + 2(2) - 1 = 4$

$x + \cancel{4} - 1 = \cancel{4}$

$\boxed{x = 1}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$$

Proof

$3(1) + \cancel{2} + \cancel{2(-1)} = 3$ ✓

$2(1) - 3(2) + 1 = -3.$

$2 - 6 + 1 = -3$ ✓

$1 + \cancel{2(2)} - \cancel{1} = 4$ ✓

3) $\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

We know that characteristic equation $|A - \lambda I| = 0$

$= \lambda^3 - (\text{trace of Matrix})\lambda^2 + (\text{sum of minor of principal diagonal elements})\lambda - (\det) = 0$

$\Rightarrow \left| \begin{pmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{pmatrix} \right| = 0$

$(1-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$

$(1-\lambda)\left[4 + \lambda^2 - 4\lambda - 1\right] = 0$

$(1-\lambda)\left[\lambda^2 - 4\lambda + 3\right] = 0.$

$\lambda^2 - 4\lambda + 3 - \lambda^3 + 4\lambda^2 - 3\lambda = 0$

$-\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0.$

$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$

By cayley hamilton theorem

$A^3 - 5A^2 + 7A - 3I = 0 \longrightarrow (1)$

Given that.

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 7A^2 + A^2 - 3A + A + I$

$A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$

$A^5(0) + A(0) + A^2 + A + I$ (from (1))

$= A^2 + A + I$

$A^2 = A \cdot A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 4+0+1 & 2+1+1 & 2+0+2 \\ 0+0+0 & 0+1+0 & 0+0+0 \\ 2+0+2 & 1+1+2 & 1+0+2 \end{pmatrix}$$

$$= \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} = A^2+A+I = \begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} + \begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{pmatrix}$$

from ① $\quad A^3 - 5A^2 + 7A - 3I = 0$

$$A^{-1}(A^3 - 5A^2 + 7A - 3I) = 0$$

$$A^2 - 5A + 7I - 3A^{-1} = 0$$

$$A^2 - 5A + 7I = 3A^{-1}$$

$$A^{-1} = \frac{1}{3}(A^2 - 5A + 7I)$$

$$A^{-1} = \frac{1}{3}\left[\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} - 5\begin{pmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} + 7\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$= \frac{1}{3}\left[\begin{pmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{pmatrix} - \begin{pmatrix} 10 & 5 & 5 \\ 0 & 5 & 0 \\ 5 & 5 & 10 \end{pmatrix} + \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}\right]$$

$$= \frac{1}{3}\left[\begin{pmatrix} 12-10 & 4-5 & 4-5 \\ 0-0 & 1+7-5 & 0+0-0 \\ 4-5 & 4-5 & 12-10 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{3}\begin{bmatrix} 2 & -1 & -1 \\ 0 & 3 & 0 \\ -1 & -1 & 2 \end{bmatrix}$$

UNIT-II

1) a) Rolle's theorem (for every Rolle's theorem)

If f is defined is on $[a,b]$ is

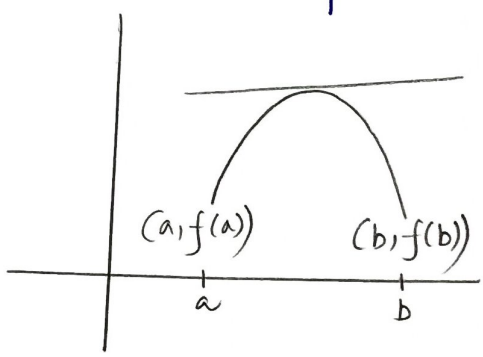
i) continuous on $[a,b]$

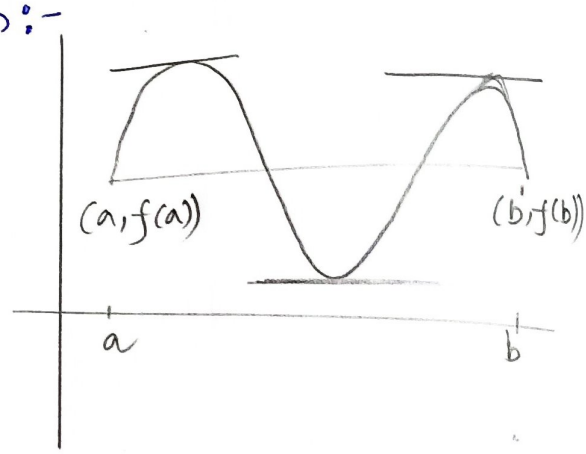
ii) differentiable on (a,b)

iii) $f(a)=f(b)$

then $\exists$ atleast one point $c \in (a,b) \ni f'(c)=0$

$\Rightarrow$ Geometrical interpretation:-





In these graph f is continuous on $[a,b]$ and differentiable on (a,b) and $f(a)=f(b)$ then their exist at least one tangent which is parallel to x-axis

$$f(x)=\log\left[\frac{(x^2+ab)}{x(a+b)}\right] \text{ in } [a,b],\ a>0,\ b>0$$

$$f(x)=\log(x^2+ab)-\log(x(a+b)) \qquad \left[\log\left(\frac{a}{b}\right)=\log a-\log b\right]$$

$$f'(x)=\frac{1}{(x^2+ab)}(2x)-\frac{1}{\cancel{x(a+b)}}\cancel{(a+b)}$$

$$=\frac{2x}{x^2+ab}-\frac{1}{x} \text{ exist in } (a,b)\ a>0,b>0$$

$f(x)=\log\left[\frac{(x^2+ab)}{x(a+b)}\right]$ is defined on $[a,b]$. It is continuous on $[a,b]$, f' exist on $[a,b]$, so it is differentiable on (a,b)

$$f(a) = \log\left[\frac{a^2+ab}{a(a+b)}\right] \qquad f(b) = \log\left[\frac{b^2+ab}{b(a+b)}\right]$$

$$f(a) = \log\left(\frac{a\cancel{(a+b)}}{a\cancel{(a+b)}}\right) \qquad f(b) = \log\left(\frac{b\cancel{(b+a)}}{b\cancel{(a+b)}}\right)$$

$$\log 1 = 0 \qquad \log 1 = 0$$

$$f(a) = f(b)$$

$$f'(c) = \frac{2c}{c^2+ab} - \frac{1}{c} = 0$$

$$\frac{2c}{c^2+ab} = \frac{1}{c}$$

$$2c^2 = c^2+ab$$

$$c^2 = ab$$

$$c = \sqrt{ab} \in (a,b)$$

$\therefore$ Hence verified.

b) $f(x) = x(x+3)e^{-x/2}$ $[-3,0]$

$$= (x^2+3x)e^{-x/2}$$

$$f'(x) = (2x+3)e^{-x/2} + e^{-x/2}\left(\frac{-1}{2}\right)(x^2+3x)$$

$$= e^{-x/2}\left(2x+3-\frac{1}{2}(x^2+3x)\right)$$

$$f(-3) = (-3)(-3+3)e^{-3/2} = 0$$

$$f(0) = 0(0+0)e^{-0/2} = 0$$

b]

$$f(-3) = f(0)$$

$$f'(c) = 0$$

$$e^{-c/2}\left(2c+3-\frac{1}{2}(c^2+3c)\right) = 0 \quad \text{[exponential fun can't be '0']}$$

$$2c+3 = \frac{1}{2}c^2+3c$$

$$4c+6 = c^2+3c$$

$$c^2-c-6 = 0$$

$$c^2-3c+2c-6 = 0$$

$$c(c-3)+2(c-3) = 0$$

$$(c+2)(c-3) = 0$$

$$c = 3 \notin [-3,0]$$

(c) $f(x) = e^{-x}\sin x$ in $[0,\pi]$

$f(0) = e^{-0}\sin 0 = 0$

$f(\pi) = e^{-\pi}\sin\pi = 0$

$f(0) = f(\pi)$

$f'(x) = e^{-x}\cos x + e^{-x}(-1)\sin x$

$= e^{-x}(\cos x - \sin x)$

$f'(c) = e^{-c}(\cos c - \sin c)$

$e^{-c} \neq 0$ (exponential fun)

$\cos c - \sin c = 0$

$\cos c = \sin c$

$\tan c = 1$

$c = \tan^{-1}(1)$

$c = 45^\circ \in [0,\pi]$

$\pi/4$

*2) Lagrange's mean value theorem:-

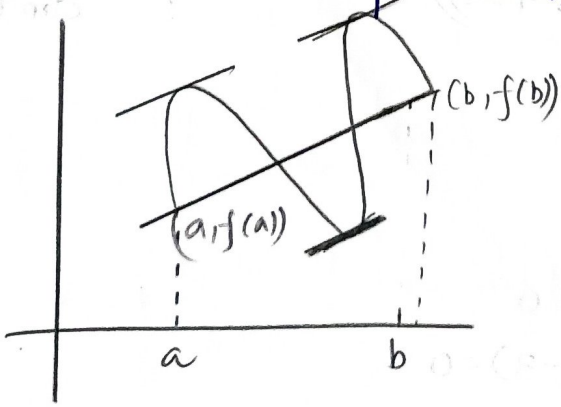
If $f:[a,b] \longrightarrow \mathbb{R}$ be a function

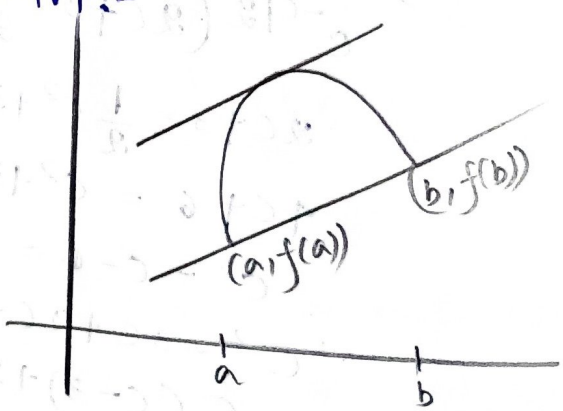
f is continuous on $[a,b]$

f is differentiable on (a,b) then there exists at least one point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

Geometrical interpretation LMVT:-





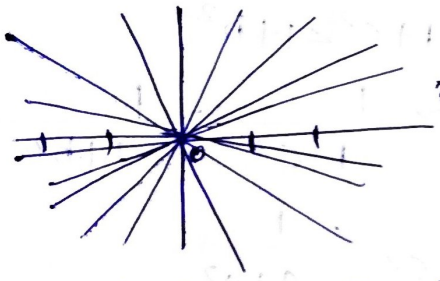
If f defined on $[a,b]$

continuous on $[a,b]$

differentiable on (a,b) then there exist atleast one tangent which is parallel to secant line connecting b/w $f(a)$ and $f(b)$

a) $|x| = f(x)$ in $[-1,1]$

$|x|$ is continuous on $[-1,1]$ and f is non differentiable on $x=0$.



$x=0$ (there are ∞ many tangents).

$\therefore$ It is non differentiable, LMVT not applicable.

b) $f(x) = (x-1)(x-2)(x-3)$ on $[0,4]$

$f(x)$ is a polynomial function so it is continuous on $[0,4]$ and differentiable on $(0,4)$

$f(x) = (x^2-3x+2)(x-3)$

$f(x) = x^2 - 6x + 11x - 6$

$f'(x) = 3x^2 - 12x + 11$

$f'(c) = 3c^2 - 12c + 11$

$f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$f(b) = f(4)$

$= (4-1)(4-2)(4-3)$

$= (3)(2)(1)$

$= 6$

$f(0) = (-1)(-2)(-3) = -6$

$3c^2 - 12c + 11 = \dfrac{(6)-(-6)}{4-0}$

$3c^2 - 12c + 11 = 12/4 = 3$

$3c^2 - 12c + 11 - 3 = 0$

$3c^2 - 12c + 8 = 0$

$c = \dfrac{-(-12) \pm \sqrt{(-12)^2 - 4 \times 3 \times 8}}{2 \times 3}$

$c = \dfrac{12 \pm \sqrt{48}}{6} = \dfrac{12 \pm 4\sqrt{3}}{6}$

$= 2 \pm 2/\sqrt{3} \in [0,4]$

(c) $f(x) = \tan^{-1}x$

$f'(x) = \dfrac{1}{1+x^2}$ $\quad \tan^{-1}(b) = f(b)$

$f(a) = \tan^{-1}x$

LMVT $f'(c) = \dfrac{1}{1+c^2}$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b-a}$$

$c \in (a,b) \Rightarrow a < c < b \Rightarrow a^2 < c^2 < b^2$

$$1+a^2 < 1+c^2 < 1+b^2$$

$$\frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

$$\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b-a} < \frac{1}{1+a^2}$$

$$\frac{b-a}{1+b^2} < \tan^{-1}b < \tan^{-1}a < \frac{b-a}{1+a^2}$$

Take $b = \frac{4}{3}$, $a = 1$

$$\frac{\frac{4}{3} - 1}{1 + \left(\frac{4}{3}\right)^2} < \tan^{-1}\frac{4}{3} - \tan^{-1}(1) < \frac{\frac{4}{3} - 1}{1+1^2}$$

$$\frac{\frac{1}{3}}{1 + \frac{16}{9}} < \tan^{-1}\frac{4}{3} - \pi/4 < \frac{1/3}{2}$$

$$\frac{1}{25/3} + \pi/4 < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \pi/4$$

$$\frac{3}{35} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \pi/4$$

take $b = 2$, $a = 1$

$$\frac{2-1}{1+(2)^2} < \tan^{-1}(2) - \tan^{-1}1 < \frac{2-1}{1+1^2}$$

$$\frac{1}{5} < \tan^{-1}2 - \pi/4 < \frac{1}{2}$$

$$\frac{1}{5} + \frac{\pi}{4} < \tan^{-1}2 < \frac{1}{2} + \pi/4$$

$$\frac{4+5\pi}{20} < \tan^{-1}2 < \frac{\pi+2}{4}$$

3) a)

$f(x) = \cos^{-1}x$ $\qquad$ $f(a) = \cos^{-1}a,\ f(b) = \cos^{-1}(b)$

$f'(x) = \dfrac{-1}{\sqrt{1-x^2}}$ $\qquad$ $f'(c) = \dfrac{-1}{\sqrt{1-c^2}}$

$f'(a) = \dfrac{-1}{\sqrt{1-a^2}}$ $\qquad$ $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$f'(b) = \dfrac{-1}{\sqrt{1-b^2}}$ $\qquad$ $\dfrac{-1}{\sqrt{1-c^2}} = \dfrac{\cos^{-1}b - \cos^{-1}a}{b-a}$

$c \in (a,b)$ $\qquad$ $a < c < b \Rightarrow a^2 < c^2 < b^2$

$-a^2 > -c^2 > -b^2$

$1-a^2 > 1-c^2 > 1-b^2 \Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$$\frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}}$$

$$\frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1}b - \cos^{-1}a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

let $b = 3/5$ $\quad a = 1/2$

$$\frac{-1}{\sqrt{1-(1/2)^2}} > \frac{\cos^{-1}\left(\frac{3}{5}\right) - \cos^{-1}\left(\frac{1}{2}\right)}{3/5 - 1/2} > \frac{-1}{\sqrt{1-(3/5)^2}}$$

$$\frac{-1}{\sqrt{3/4}} > \frac{\cos^{-1}\left(\frac{3}{5}\right) - \pi/3}{1/10} > \frac{-1}{\sqrt{16/25}}$$

$$\frac{-\cancel{2}1}{\cancel{10}_{5}\sqrt{3}} + \pi/3 > \cos^{-1}\left(\frac{3}{5}\right) > \pi/3 - \frac{\cancel{5}1}{\cancel{10}_{2} \times 4}$$

$$\pi/3 - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \pi/3 - \frac{1}{8}$$

b) $f(x) = \sin^{-1} x$

$f(a) = \sin^{-1} a$

$f(b) = \sin^{-1} b$

$f'(x) = \dfrac{1}{\sqrt{1-x^2}}$

$f'(c) = \dfrac{1}{\sqrt{1-c^2}}$

$f'(c) = \dfrac{f(b)-f(a)}{b-a} = \dfrac{\sin^{-1} b - \sin^{-1} a}{b-a}$

$\dfrac{1}{\sqrt{1-c^2}} = \dfrac{\sin^{-1} b - \sin^{-1} a}{b-a}$

$a < c < b$

$a^2 < c^2 < b^2$

$-a^2 > -c^2 > -b^2$

$1-a^2 > 1-c^2 > 1-b^2 \Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\dfrac{1}{\sqrt{1-a^2}} < \dfrac{1}{\sqrt{1-c^2}} < \dfrac{1}{\sqrt{1-b^2}}$

$\dfrac{1}{\sqrt{1-a^2}} < \dfrac{\sin^{-1} b - \sin^{-1} a}{b-a} < \dfrac{1}{\sqrt{1-b^2}}$

$\dfrac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \dfrac{b-a}{\sqrt{1-b^2}}$

let $b = 1/4$, $a = 1/2$

$\dfrac{\frac{1}{4} - \frac{1}{2}}{\sqrt{1-(1/2)^2}} < \sin^{-1}\left(\frac{1}{4}\right) - \sin^{-1}\left(\frac{1}{2}\right) < \dfrac{\frac{1}{4} - \frac{1}{2}}{\sqrt{1-\left(\frac{1}{4}\right)^2}}$

$\dfrac{\frac{1-2}{4}}{\sqrt{3/4}} < \sin^{-1}\left(\frac{1}{4}\right) - \dfrac{\pi}{6} < \dfrac{\frac{1-2}{4}}{\sqrt{\frac{15}{4}}}$

$\dfrac{\pi}{6} + \dfrac{-1/\cancel{4}\,2}{\sqrt{3}/\cancel{2}\,1} < \sin^{-1}\left(\frac{1}{4}\right) < \dfrac{\pi}{6} + \dfrac{\frac{(-1)}{\cancel{2}}}{\frac{\sqrt{15}}{\cancel{2}}}$

$\dfrac{\pi}{6} - \dfrac{1}{2\sqrt{3}} < \sin^{-1}\left(\frac{1}{4}\right) < \dfrac{\pi}{6} - \dfrac{1}{\sqrt{15}}$

4) Cauchy's Mean value theorem :-

If $f, g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$, differentiable on (a,b), $g'(x) \neq 0 \ \forall x \in (a,b)$ then there exist a point $c \in (a,b) \ni \dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

a) $f(x) = \log x$ in $(1, e)$

$f'(x) = 1/x \Rightarrow g(x) = f'(x)$

$f(e) = \log e = 1$

$f(1) = \log 1 = 0$.

$g'(x) = -1/x^2 \neq 0 \ \forall x \in [1, e]$

$g(e) = 1/e$

$g(1) = 1/1 = 1$

By CMVT $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

$$\frac{\frac{1}{c}}{-1/c^2} = \frac{\log e - \log 1}{1/e - 1/1} = \frac{1-0}{1/e - 1} = \frac{1}{\frac{1-e}{e}}$$

$$-c = \frac{e}{1-e}$$

$$c = \frac{e}{e-1}$$

$$c = \frac{e+1-1}{e-1}$$

$$c = 1 + \frac{1}{e-1} \in [1, e]$$

b) $f(x) = \sqrt{x}$ $\quad g(x) = \dfrac{1}{\sqrt{x}}$ $\quad g'(x) = \dfrac{-1}{2x\sqrt{x}}$ $\quad g'(c) = \dfrac{-1}{2c\sqrt{c}}$

$f(a) = \sqrt{a}$ $\quad g(b) = 1/\sqrt{b}$

$f(b) = \sqrt{b}$ $\quad g(a) = 1/\sqrt{a}$

By CMVT $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

$f'(x) = \dfrac{1}{2\sqrt{x}} \Rightarrow f'(c) = \dfrac{1}{2\sqrt{c}}$

$$\frac{\frac{1}{\cancel{2\sqrt{c}}}}{\frac{-1}{\cancel{2c\sqrt{c}}}} = \frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = -c = \frac{\cancel{\sqrt{b}-\sqrt{a}}\ (-1)}{\frac{\cancel{\sqrt{a}-\sqrt{b}}}{\sqrt{ab}}}$$

$c = \sqrt{ab} \in (a,b)$

5)

a) $f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots$

$f(x) = \sin x \Rightarrow f(a) = \sin \pi/4 = 1/\sqrt{2}$

$x - a = x - \pi/4$

$a = \pi/4$

$f'(x) = \cos x \Rightarrow f'(a) \Rightarrow f'(\pi/4) = 1/\sqrt{2}$

$f''(x) = -\sin x \Rightarrow f''(a) \Rightarrow f''(\pi/4) = -1/\sqrt{2}$

$f'''(x) = -\cos x \Rightarrow f'''(a) \Rightarrow f'''(\pi/4) = -1/\sqrt{2}$

$$f(x) = \frac{1}{\sqrt{2}} + \frac{x-\pi/4}{1!}\left(\frac{1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^2}{2!}\left(\frac{-1}{\sqrt{2}}\right) + \frac{(x-\pi/4)^3}{3!}\left(\frac{-1}{\sqrt{2}}\right) + \dots$$

$$= \frac{1}{\sqrt{2}}\left(1 + \frac{x-\pi/4}{1!} - \frac{(x-\pi/4)^2}{2!} - \frac{(x-\pi/4)^3}{3!} + \dots\right)$$

$$= \frac{1}{\sqrt{2}}\left(1 + \frac{x-\pi/4}{1} - \frac{(x-\pi/4)^2}{2} - \frac{(x-\pi/4)^3}{6} + \dots\right)$$

b)

$f(x) = \log_e x$

$x - a = x - 1$

$a = 1$

$f(x) = \log_e x \Rightarrow f'(x) = 1/x = f'(a) = 1/1 = 1$

$f(a) = \log 1 = 0$

$f''(x) = -1/x^2 \Rightarrow f''(1) = -1$

$f'''(x) = +2/x^3 \Rightarrow f'''(1) = +2$

$$f(x) = f(a) + f'(a)\left(\frac{x-a}{1!}\right) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \dots$$

$$= 0 + \frac{(x-1)}{1!}(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(+2) + \dots$$

$$= (x-1)\left[\frac{1}{1!} + \frac{(x-1)}{2!}(-1) + \frac{(x-1)^2}{3!}(+2) + \cdots\right]$$

$$= (x-1)\left[\frac{1}{1!} - \frac{x-1}{2} + \frac{(x-1)^2}{6} \times 2 + \cdots\right]$$

$$= (x-1)\left[\frac{1}{1} - \frac{x-1}{2} + \frac{(x-1)^2}{3} + \cdots\right]$$

$$\log_e 1.1 = (1.1-1)\left[1 - \frac{(1.1-1)}{2} + \frac{(1.1-1)^2}{3} + \cdots\right]$$

$$= (0.1)\left(1 - \frac{0.1}{2} + \frac{(0.1)^2}{3} + \cdots\right]$$

$$= 0.1\left(1 - 0.05 + \frac{0.01}{3}\right]$$

$$= 0.1 - 0.005 + 0.00033$$

$$= 0.10033 - 0.005$$

$$= 0.09533.$$

c) $f(x) = e^{\sin x}$ $\qquad x - a = x - 0$, $a = 0$

$$f'(x) = e^{\sin x}(\sin x)'$$

$$= e^{\sin x}\cos x.$$

$$f'(0) = e^{\sin 0}\cos 0 = e^0(1) = 1$$

$$f''(x) = e^{\sin x}(-\sin x) + \cos x\, e^{\sin x}\cos x.$$

$$= -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x}$$

$$f''(0) = 1$$

$$f'''(x) = -\sin x\, e^{\sin x}\cdot\cos x + e^{\sin x}(-\cos x) + 2\cos x(-\sin x)\, e^{\sin x} + \cos^2 x\, e^{\sin x}\cdot\cos x$$

$$= 0 - 1 + 0 + 1$$

$$= 0$$

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4}f^{iv}(0) + \cdots$$

$$= 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \cdots$$

$$= 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6}(0)$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{6}(0) = 1 + x + \frac{x^2}{2} + \cdots$$

d) $\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} - \frac{x^4}{192} + \cdots$

$$f(x) = \log(1+e^x)$$
$$f(0) = \log(1+e^0)$$
$$= \log(1+1) = \log 2$$

$$f'(x) = \frac{1}{1+e^x} e^x \qquad f'(0) = \frac{e^0}{1+e^0} = \frac{1}{1+1} = \frac{1}{2}$$
$$= \frac{e^x}{1+e^x}$$

$$f''(x) = \frac{e^x(1+e^x) - (e^x)(e^x)}{(1+e^x)^2} = \frac{e^x + \cancel{e^{2x}} - \cancel{e^{2x}}}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2}$$

$$f''(0) = \frac{e^0}{(1+e^0)^2} = \frac{1}{(2)^2} = 1/4$$

$$f'''(x) = \frac{(1+e^x)^2 e^x - e^x(2)(1+e^x)(e^x)}{((1+e^x)^2)^2}$$

$$= \frac{1+e^{2x}+2e^x - 2e^x(e^x+e^{2x})}{(1+e^x)^4} = \frac{1+e^{2x}+2e^x-2e^{2x} - 2e^{3x}}{(1+e^x)^4}$$

$$f'''(0) = \frac{1+1+\cancel{2(1)} - \cancel{2(1)} - 2)}{(2)^4} = \frac{2-2}{2^4} = \underline{\underline{0}}$$

$$f(x) = f(a) + \frac{x-a}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \frac{(x-a)^3}{3!} f'''(a) + \cdots$$

$$f(x) = f(0) + \frac{x-0}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{1!}\left(\frac{1}{2}\right) + \frac{x^2}{2!}\left(\frac{1}{4}\right) + \frac{x^3}{3!}(0) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{2\times 4} + 0 + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{x}{2} + \frac{x^2}{8} + \cdots$$

differentiate above equation w.r.to x

$$\frac{1}{1+e^x}(e^x) = 0 + \frac{1}{2}(x)' + \frac{1}{8}(x^2)' + \cdots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2}(1) + \frac{1}{8}(2x) + \cdots$$

$$\frac{e^x}{1+e^x} = \frac{1}{2} + \frac{1}{4}x + \cdots$$

UNIT-3

2) $u = \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$ $\quad \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0$

$$\frac{du}{dx} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\left(\frac{(x^2-y^2)2y - 2xy(2x)}{(x^2-y^2)^2}\right)$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2}\left(\frac{2x^2y-2y^3-4x^2y}{\cancel{(x^2-y^2)^2}}\right)$$

$$= \frac{1}{(x^2+y^2)^2}\left(-2y^3-2x^2y\right)$$

$$= \frac{1}{(x^2+y^2)^2}\left(\cancel{-2y(x^2+y^2)}\right)$$

$$= \frac{-2y}{(x^2+y^2)^1}$$

$$\frac{d^2u}{dx^2} = \frac{(x^2+y^2)(0) - (-2y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{4xy}{(x^2+y^2)^2}$$

$$\frac{du}{dy} = \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\left(\frac{(x^2-y^2)(2x)-(2xy)(-2y)}{(x^2-y^2)^2}\right)$$

$$= \frac{\cancel{(x^2-y^2)^2}}{(x^2-y^2)^2+4x^2y^2}\left(\frac{2x^3-2xy^2+4xy^2}{\cancel{(x^2-y^2)^2}}\right)$$

$$= \frac{1}{(x^2-y^2)^2+4x^2y^2}\left(2x^3+2xy^2\right)$$

$$= \frac{1}{(x^2+y^2)^{\cancel{2}}}\left(2x\cancel{(x^2+y^2)}\right)$$

$$= \frac{2x}{x^2+y^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{(x^2+y^2)(0) - 2y(2x)}{(x^2+y^2)^2}$$

$$= \frac{0-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-4xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{4xy}{(x^2+y^2)^2} - \frac{4xy}{(x^2+y^2)^2}$$

$$= 0$$

Proved

b) $u = \log(x^3+y^3+z^3-3xyz)$

$$\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}\left(3x^2-3yz\right)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2-3yz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial y} = \frac{3y^2-3xz}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3x^2 - 3yz + 3y^2 - 3xz + 3z^2 - 3xy}{x^3+y^3+z^3-3xyz}$$

$$= \frac{3\cancel{(x^2+y^2+z^2-(xy+yz+zx))}}{(x+y+z)\cancel{(x^2+y^2+z^2-(xy+yz+zx))}}$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{3}{x+y+z}\right) = \frac{-3}{(x+y+z)^2}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial y}\left(\frac{3}{x+y+z}\right) = \frac{-3}{(x+y+z)^2}$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{3}{x+y+z}\right) = \frac{-3}{(x+y+z)^2}$$

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2}$$

Hence proved

2a) $u = f(x-y,\ y-z,\ z-x)$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

$$u = f(a, b, c)$$

$$a = x-y,\ b = y-z,\ c = z-x$$

$$\frac{\partial a}{\partial x} = 1,\ \frac{\partial a}{\partial y} = -1,\ \frac{\partial b}{\partial y} = 1,\ \frac{\partial b}{\partial z} = -1,\ \frac{\partial c}{\partial z} = 1,\ \frac{\partial c}{\partial x}$$

$$\frac{\partial a}{\partial z} = 0,\ \frac{\partial b}{\partial x} = 0,\ \frac{\partial c}{\partial y} = 0$$

$$\left(\frac{\partial a}{\partial x} + \frac{\partial a}{\partial y} + \frac{\partial a}{\partial z}\right) + \left(\frac{\partial b}{\partial x} + \frac{\partial b}{\partial y} + \frac{\partial b}{\partial z}\right) + \left(\frac{\partial c}{\partial x} + \frac{\partial c}{\partial y} + \frac{\partial c}{\partial z}\right)$$

$$(1 + (-1) - 0) + (0 + 1 - 1) + (-1 + 1 + 0)$$

$$= 1 - 1 + 1 - 1 - 1 + 1$$

$$= 3 - 3$$

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

2b) $u = x^2 + y^2 + z^2 \quad x = e^t,\ y = e^t \sin t,\ z = e^t \cos t$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left(\frac{\partial x}{\partial t}\right) + \frac{\partial u}{\partial y}\left(\frac{\partial y}{\partial t}\right) + \frac{\partial u}{\partial z}\left(\frac{\partial z}{\partial t}\right)$$

$$= (2x)(e^t) + 2y(e^t \cos t + e^t \sin t) + 2z(e^t \cos t - e^t \sin t)$$

$$= 2e^t(x + y\cos t + y\sin t + z\cos t - z\sin t)$$

$$x = e^t,\ y = e^t \sin t,\ z = e^t \cos t$$

$$= 2e^t(e^t + e^t \sin t \cos t + e^t \sin t \sin t + e^t \cos t \cos t - e^t \cos t \sin t)$$

$$= 2e^t(e^t + \cancel{e^t \sin t \cos t} + e^t \sin^2 t + e^t \cos^2 t - \cancel{e^t \cos t \sin t})$$

$$= 2e^t(e^t + e^t(\cos^2 t + \sin^2 t))$$

$$= 2e^t(e^t + e^t) = 2e^t(2e^t)$$

$$= 4e^{2t}$$

3) a) $u = x+y+z$, $u = x + uv - \cancel{uvw} + \cancel{uvw}$ $\quad x = u - uv$

$y + z = uv$ $\quad y + uvw = uv \Rightarrow y = uv - uvw$

$z = uvw$

$$\frac{\partial x}{\partial u} = 1 - v, \quad \frac{\partial x}{\partial v} = -u, \quad \frac{\partial x}{\partial w} = 0.$$

$$\frac{\partial y}{\partial u} = v - vw, \quad \frac{\partial y}{\partial v} = u - uw, \quad \frac{\partial y}{\partial w} = -uv$$

$$\frac{\partial z}{\partial u} = vw, \quad \frac{\partial z}{\partial v} = uw, \quad \frac{\partial z}{\partial w} = uv$$

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \partial y/\partial v & \partial y/\partial w \\ \frac{\partial z}{\partial u} & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= (1-v)\big((u-uw)(uv) + (uv)(uw)\big) + u\big((v-vw)(uv) + (uv)(vw)\big)$$

$$= (1-v)(u^2v - \cancel{u^2vw} + \cancel{u^2vw}) + u(uv^2 - \cancel{v^2uw} + \cancel{uv^2w})$$

$$= (1-v)(u^2v) + u(uv^2)$$

$$= u^2v - \cancel{u^2v^2} + \cancel{u^2v^2}$$

$$= u^2v \quad (\therefore \text{Hence proved}).$$

b) $u = 2axy$, $v = a(x^2 - y^2)$ $\quad x = r\cos\theta$, $y = r\sin\theta$.

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$\frac{\partial u}{\partial x} = 2ay \quad \frac{\partial v}{\partial x} = a \cdot 2x, \quad \frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta$$

$$\frac{\partial u}{\partial y} = 2ax, \quad \frac{\partial v}{\partial y} = a(-2y), \quad \frac{\partial x}{\partial \theta} = r(-\sin\theta), \quad \partial y/\partial\theta = r\cos\theta$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\begin{vmatrix} 2ay & 2ax \\ 2ax & -2ay \end{vmatrix} \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4a^2y^2 - 4a^2x^2)(r\cos^2\theta + r\sin^2\theta)$$

$$= -4a^2(x^2+y^2)(r(\cos^2\theta+\sin^2\theta))$$

$$= -4a^2(x^2+y^2)(r(1))$$

$$= -4a^2(r^2\cos^2\theta + r^2\sin^2\theta)(r)$$

$$= -4a^2r(r^2)$$

$$= -4a^2r^3$$

c) $u = \dfrac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{du}{dx} = \frac{1(1-xy) - (-y)(x+y)}{(1-xy)^2}$$

$$= \frac{(1-xy) + y(x+y)}{(1-xy)^2} = \frac{1 - \cancel{xy} + \cancel{xy} + y^2}{(1-xy)^2}$$

$$\frac{du}{dx} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{du}{dy} = \frac{1(1-xy) - (-x)(x+y)}{(1-xy)^2} = \frac{1 - \cancel{xy} + x^2 + \cancel{xy}}{(1-xy)^2}$$

$$\frac{du}{dy} = \frac{1+x^2}{(1-xy)^2}$$

$$\frac{dv}{dx} = \frac{1}{1+x^2}, \quad \frac{dv}{dy} = \frac{1}{1+y^2}$$

$$\begin{vmatrix} \frac{du}{dx} & \frac{dv}{dx} \\ \frac{du}{dy} & \frac{dv}{dy} \end{vmatrix} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1}{1+x^2} \\ \frac{1+x^2}{(1-xy)^2} & \frac{1}{1+y^2} \end{vmatrix}$$

$$= \left(\frac{\cancel{1+y^2}}{(1-xy)^2}\right)\left(\frac{1}{\cancel{1+y^2}}\right) - \left(\frac{1}{\cancel{1+x^2}}\right)\left(\frac{\cancel{1+x^2}}{(1-xy)^2}\right)$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

Relation $u = \dfrac{x+y}{1-xy}$

$v = \tan^{-1}x + \tan^{-1}y$

We know that

$$\tan^{-1}\left(\frac{x+y}{1-xy}\right) = \tan^{-1}x + \tan^{-1}y$$

$$\boxed{v = \tan^{-1}u}$$

(d) $u = \dfrac{x^2-y^2}{x^2+y^2}$, $v = \dfrac{2xy}{x^2+y^2}$

$$\frac{du}{dx} = \frac{x^2+y^2(2x) - (x^2-y^2)(2x)}{(x^2+y^2)^2}$$

$$= \frac{\cancel{2x^3} + 2xy^2 - \cancel{2x^3} + 2xy^2}{(x^2+y^2)^2}$$

$$\frac{du}{dx} = \frac{4xy^2}{(x^2+y^2)^2}$$

$$\frac{du}{dy} = \frac{(x^2+y^2)(-2y) - (x^2-y^2)(2y)}{(x^2+y^2)^2}$$

$$= \frac{-2x^2y - \cancel{2y^3} - \cancel{2x^2y} + \cancel{2y^3}}{(x^2+y^2)^2}$$

$$\frac{du}{dy} = \frac{-4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{(x^2+y^2)(2y)-(2x)(2xy)}{(x^2+y^2)^2}$$

$$= \frac{2x^2y+2y^3-4x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2y^3-2x^2y}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{(x^2+y^2)(2x)-(2xy)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^3+2xy^2-4xy^2}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{2x^3-2xy^2}{(x^2+y^2)^2}$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{4xy^2}{(x^2+y^2)^2} & \frac{-4x^2y}{(x^2+y^2)^2} \\ \frac{2y^3-2x^2y}{(x^2+y^2)^2} & \frac{2x^3-2xy^2}{(x^2+y^2)^2} \end{vmatrix}$$

$$= \frac{1}{(x^2+y^2)^2}\left[4xy^2(2x^3-2xy^2)+4x^2y(2y^3-2x^2y)\right]$$

$$= \frac{1}{(x^2+y^2)^2}\left[\cancel{8x^4y^2}-\cancel{8x^2y^4}+\cancel{8x^2y^4}-\cancel{8x^4y^2}\right]$$

$$= \frac{1}{(x^2+y^2)^2}(0)$$

$$= \underline{\underline{0}}$$

u and v are fD

$$u = \frac{x^2-y^2}{x^2+y^2}, \quad v = \frac{2xy}{x^2+y^2}$$

$$u^2 = \frac{(x^2-y^2)^2}{(x^2+y^2)^2} \qquad v^2 = \frac{4x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^4+y^4-2x^2y^2}{(x^2+y^2)^2}$$

$$u^2+v^2 = \frac{x^4+y^4-2x^2y^2+4x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{x^4+y^4+2x^2y^2}{(x^2+y^2)^2}$$

$$= \frac{(x^2+y^2)^2}{(x^2+y^2)^2}$$

$$\boxed{u^2+v^2 = 1}$$

4a) $x^2+y^2+z^2 = r^2$

$$x^2+y^2+z^2-r^2 = 0$$

$$z^2 = r^2-x^2-y^2$$

$$z = \sqrt{r^2-x^2-y^2} \longrightarrow (x)$$

$$\text{volume} = 8xyz$$

$$= 8xy\sqrt{r^2-x^2-y^2}$$

$$\frac{dV}{dx} = 8y\sqrt{r^2-x^2-y^2} + \frac{8xy(-2x)}{2\sqrt{r^2-x^2-y^2}}$$

$$= \frac{8y\left[(r^2-x^2-y^2)-x^2\right]}{\sqrt{r^2-x^2-y^2}}$$

$$\frac{dV}{dx} = 0 \Rightarrow r^2-x^2-y^2-x^2 = 0$$

$$-2x^2-y^2+r^2=0$$

$$2x^2+y^2-r^2=0 \longrightarrow (1)$$

$$\frac{dV}{dy} = 8x\sqrt{r^2-x^2-y^2} + \frac{8xy(-\cancel{2}y)}{\cancel{2}\sqrt{r^2-x^2-y^2}}$$

$$= 8x\left[\sqrt{r^2-x^2-y^2} + \frac{(-y^2)}{\sqrt{r^2-x^2-y^2}}\right]$$

$$= \frac{8x\left((r^2-x^2-y^2)-y^2\right)}{\sqrt{r^2-x^2-y^2}}$$

$$\frac{dV}{dy}=0 \qquad r^2-x^2-y^2-y^2=0$$

$$2y^2+x^2=r^2 \longrightarrow (2)$$

solving (1) and (2)

$$\cancel{2x^2}+y^2-r^2=0$$

$$\cancel{2x^2}+4y^2-2r^2=0$$

$$(-) \quad (-) \quad (+)$$

$$-3y^2+r^2=0$$

$$y=\frac{r}{\sqrt{3}}$$

$$\cancel{2y^2}+x^2-r^2=0$$

$$\cancel{2y^2}+4x^2-2r^2=0$$

$$(-) \quad (-) \quad (+)$$

$$-3x^2+r^2=0$$

$$3x^2=r^2$$

$$x=\frac{r}{\sqrt{3}}$$

substitute x and y values to get z in (2) eq

$$z=\sqrt{r^2-x^2-y^2}$$

$$=\sqrt{r^2-\left(\frac{r}{\sqrt{3}}\right)^2-\frac{r^2}{(\sqrt{3})^2}}$$

$$z = \sqrt{r^2 - \frac{r^2}{3} - \frac{r^2}{3}}$$

$$z = \sqrt{\frac{3r^2 - 2r^2}{3}}$$

$$z = \sqrt{\frac{r^2}{3}}$$

$$z = \frac{r}{\sqrt{3}}$$

We notice that all sides are equal and of length $r/\sqrt{3}$

4b) $S = xy + 2yz + 2zx$ (since open at the top) $\longrightarrow$ ①

Given that its volume $(xyz) = 32$

$$z = 32/xy$$

substitute in ①

$$S = xy + 2\cancel{y}\left(\frac{32}{x\cancel{y}}\right) + 2\cancel{x}\left(\frac{32}{\cancel{x}y}\right)$$

$$S = xy + \frac{64}{x} + \frac{64}{y}$$

$$\frac{dS}{dx} = y - \frac{64}{x^2}$$

$$\frac{dS}{dx} = 0$$

$$y = 64/x^2$$

$$x^2 y = 64$$

$$\frac{dS}{dy} = x - \frac{64}{y^2}$$

$$\frac{dS}{dy} = 0$$

$$xy^2 = 64$$

$$x = 64/y^2$$

$$\left(\frac{64}{y^2}\right)\left(\frac{64}{y^2}\right)y = 64$$

$$\frac{64}{y^3} = 1$$

$$y^3 = 64$$

$y = 64/x^2$

$64/y^2 = x$

$x = 64/16$

$x = 4$

$x = 4,\ y = 4$

Least Material (minimum)

f_{xx} or $f_{yy} > 0$ (Minima Condition)

$$\frac{d^2s}{dx^2} = 0 - \frac{(-128)}{x^3} = \frac{128}{x^3}$$

$$\frac{d^2s}{dy^2} = 0 - \frac{(-128)}{y^3} = \frac{128}{y^3}$$

$$\frac{d^2s}{dx^2} = \frac{128}{x^3} = \frac{128}{4\times4\times4} = 2 > 0$$

(Minimum)

$z = 32/xy$

$z = 32/4\times4 = 2$

The dimensions of the box for least material for its construction are 4, 4, 2.

4c) x, y, z be three parts

$x + z + y = k$ (say)

$x + y + z - k = \phi(x, y, z)$

product $= zy^2x^3$ should be max

$= f(x, y, z) \Rightarrow$ (say)

$F(x, y, z) = f(x, y, z) + \phi(x, y, z)$

$= zy^2x^3 + \lambda(x + y + z - k)$

$$\frac{dF}{dx} = 3x^2zy^2 + \lambda \longrightarrow (2)$$

$$\frac{dF}{dy} = 2yzx^3 \qquad \frac{dF}{dz} = y^2x^3 + \lambda$$

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

$$1 = -3x^2 z y^2$$

$$1 = -y^2 x^3$$

$$1 = -2yzx^3.$$

$$* \; \cancel{-}2\cancel{y}z\cancel{x^3} = \cancel{-}\cancel{x^3}y^{\cancel{2}}$$

$$2z = y$$

$$\cancel{-}3\cancel{x^2}\cancel{z}y^{\cancel{2}} = \cancel{-}2\cancel{y}\cancel{z}x^{\cancel{3}}.$$

$$3y = 2x$$

$$x = \frac{3}{2}y$$

$$* \; \cancel{-}3\cancel{x^2}\cancel{y^2}z = \cancel{-}x^{\cancel{3}\,2}\cancel{y^2}$$

$$3z = x$$

$$\cancel{3}z = \frac{\cancel{3}}{2}y$$

$$z = \frac{1}{2}y$$

$$x + y + z = k.$$

$$\frac{3}{2}y + y + \frac{1}{2}y = k$$

$$y\left(\frac{3}{2} + \frac{1}{2} + 1\right) = k$$

$$y(2+1) = k$$

$$y(3) = k$$

$$k = 3y$$

$$* \; y = k/3$$

$$* \; x = \frac{\cancel{3}}{2}\left(\frac{k}{\cancel{3}}\right)$$

$$x = k/2$$

$$* \; z = \frac{1}{2}\left(\frac{k}{3}\right)$$

$$= \frac{k}{6}$$

$$\therefore (x, y, z) = \left(k/2, \frac{k}{3}, \frac{k}{6}\right)$$

d) $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7 \longrightarrow (1)$

$$f_x = 3x^2 + 3y^2 - 3(2x)$$

$$= 3x^2 + 3y^2 - 6x$$

$$f_x = 0 = 3x^2 + 3y^2 - 6x = 0$$

$$3(x^2 + y^2 - 2x) = 0$$

$$x^2 + y^2 - 2x = 0 \longrightarrow (2)$$

$$f_{xx} = 3(2x) - 6(1)$$

$$= 6x - 6.$$

$f_y = 6xy - 6y$

$f_y = 0$

$6y(x-1) = 0 \longrightarrow (3)$

$y = 0$ (or) $x = 1$

at $y = 0$ in (2)

$x^2 + y^2 - 2x = 0$

$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0, x = 2$

stationary points

$(x,y) = (0,0)$

$(x,y) = (2,0)$

$f_{yy} = 6y - 6$

$f_{xy} = 6(1)y - 0$

$= 6y$

at $x = 1$ in (1)

$1 + y^2 - 2(1) = 0$

$y^2 - 1 = 0$

$y^2 = 1$

$y = \pm 1$

stationary points

$(x,y) = (1,1)$

$(x,y) = (1,-1)$

* $f_{xx} f_{yy} - (f_{xy})^2$ [condition]

$(6x-6)(6y-6) - (6y)^2$

$\Rightarrow$ at $(0,0)$

$(6(0)-6)(6(0)-6) - 6(0)$

$(-6)(-6) > 0$

$f_{xx} < 0$

Maximum at $(0,0)$

$f(x,y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 7$

$f(0,0) = 7$

7 is the Maximum value.

$\Rightarrow$ at $(2,0)$

$(6(2)-6)(6(0)-6) - 6(0)$

$(12-6)(-6) - 0$

$(6)(-6) < 0$

$f_{xx} > 0$

Minimum at $(2,0)$

$f(x,y) = (2)^3 + 3(2)(0)^2 - 3(2)^2 - 0 + 7$

$= 8 + 0 - 3(4) + 7$

$= 8 - 12 + 7 = 3$

Minimum value is 3.

* at $(1,1)$

$(6(1)-6)(6(1)-6)-6(1)$

$(6-6)(6-0)-(6(1))^2$

$-36<0$

$f_{xx}=0$

neither minimum nor maximum

saddle point

* at $(1,-1)$

$(6(1)-6)(6(-1)-6)-(6(-1))^2$

$0-36<0$

$f_{xx}=0$

neither minimum nor

maximum saddle point

5a) $f(x,y) = x^2 + xy + y^2$ in the power of $(x-1)$ and $(y-2)$

$$f(x,y) = x^2 + xy + y^2$$

$$f(a,b) = f(1,2)$$

$$f(1,2) = (1)^2 + (1)(2) + (2)^2 = 1 + 2 + 4 = 7$$

$$f_x = 2x + y \Rightarrow f_x(a,b) \Rightarrow f_x(1,2) = 2(1) + 2 = 4$$

$$f_y = x + 2y \Rightarrow f_y(a,b) \Rightarrow f_y(1,2) = 1 + 2(2) = 5$$

$$f_{xy} = 1 \Rightarrow f_{xy}(a,b) \Rightarrow f_{xy} = 1$$

$$f_{xx} = 2 \Rightarrow f_{yy} = 2 \qquad f_{xxy} = f_x(1) = 0.$$

$$f_{xxx} = 0 \Rightarrow f_{yyy} = 0 \qquad f_{xyy} = f_x(2) = 0.$$

$$f(x,y) = f(a,b) + (x-a) f_x(a,b) + (y-b) f_y(a,b)$$

$$+ \frac{1}{2!}\left((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right)$$

$$+ \frac{1}{3!}\Big[(x-a)^3 f_{xxx}(a,b) + 3(x-a)^2(y-b) f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)$$

$$+ (y-b)^3 f_{yyy}(a,b)\Big] \ldots\ldots$$

$$f(x,y) = 7 + (x-1)4 + (y-2)5 + \frac{1}{2!}\left[(x-1)^2(2) + 2(x-1)(y-2)(1) + (y-2)^2(2)\right] + \frac{1}{3!}\left[(x-1)^3(0) + 3(x-1)^2(y-2)0 + 3(x-1)(y-2)^2(0) + (y-2)^3(0)\right]$$

$$= 7 + (x-1)4 + (y-2)5 + \frac{1}{2!}\left((x^2-2x+1)2 + 2(xy-2x-y+2) + (y^2-4y+4)2\right] + 0$$

$$= 7 + 4x - 4 + 5y - 10 + \frac{1}{\cancel{2}} \times \cancel{2}\left(x^2-2x+1+xy-2x-y+2+y^2-4y+4\right)$$

$$= 7 + \cancel{4x} - 4 + 5y - 10 + x^2 - \cancel{2x} - \cancel{2x} + 1 + xy - y + 2 + y^2 - \cancel{4y}$$

$$= -7 + \cancel{5y} + x^2 + 1 + xy - \cancel{y} + 2 + y^2 - \cancel{4y} + 4$$

$$= x^2 + y^2 + xy - \cancel{7} + \cancel{3} + \cancel{4}$$

$$= x^2 + y^2 + xy$$

5b) $f(x,y) = e^x \log(1+y)$ power of (x and y)

$(a,b) = (0,0)$

$f(x,y) = e^x \log(1+y) \rightarrow f(0,0) = e^0 \log(1+0) = 1(0) = 0$

$f_x = e^x \log(1+y) \Rightarrow f_x(0,0) = 0$

$f_y = e^x\left(\frac{1}{1+y}\right) \Rightarrow f_y(0,0) = e^0\left(\frac{1}{1+0}\right) = 1$

$f_{xx} = e^x \log y \Rightarrow f_{xx}(0,0) = 0$

$f_{yy} = e^x\left(\frac{-1}{(1+y)^2}\right) = e^0\left(\frac{-1}{1}\right) = -1$

$f_{xy} = e^x\left(\frac{1}{1+y}\right) \Rightarrow f_{xy}(0,0) = 1$

$f_{xxx} = e^x(\log(1+y)) = 0$

$f_{yyy} = e^x\left(\frac{(1+y)^2(0) - (1)2(1+y)^2(1)}{(1+y)^4}\right) = e^x\left(\frac{+2(1+y)}{(1+y)^4}\right)$

$= e^x\left(\frac{+2}{(1+y)^3}\right)$

$$f_{yyy} \Rightarrow f_{yyy}(a,b) = e^{0}\left(\frac{2}{(1)^2}\right) = 2$$

$$\Rightarrow f_{xxy} = e^{x}\left(\frac{1}{1+y}\right) = 1$$

$$f_{xyy} = e^{x}\left(\frac{-1}{(1+y)^2}\right) = -1$$

$$f(x,y) = f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b)$$

$$+\frac{1}{2!}\left((x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(x,b)\right)$$

$$+\frac{1}{3!}\left((x-a)^3 f_{yy}(a,b) + 3(x-a)^2(y-b)f_{xxy}(a,b) + 3(x-a)(y-b)^2 f_{xyy}(a,b)\right.$$

$$\left. + (y-b)^3 f_{yy}(a,b)\right] + \ldots\ldots$$

$$= 0 + (x)(0) + (y)(1) + \frac{1}{2!}\left((x)^2(0) + 2(x)(y)(1) + (y)^2(-1)\right)$$

$$+\frac{1}{3!}\left[(x)^3(0) + 3x^2y(1) + 3xy^2(-1) + y^3(2)\right]\ldots\ldots$$

$$= 0 + 0 + y + \frac{1}{2!}\left(0 + 2xy - y^2\right) + \frac{1}{3!}\left(0 + 3x^2y - 3xy^2 + 2y^3\right)$$

$$= y + \frac{1}{2!}\left(2xy - y^2\right) + \frac{1}{3!}\left(3x^2y - 3xy^2 + 2y^3\right).$$

UNIT-IV

1) a) $(1+e^{x/y})dx + e^{x/y}(1-x/y)dy = 0$

Homogeneous symstem of equation

$M = 1+e^{x/y}$

$N = e^{x/y}(1-x/y)$

$= e^{x/y} - x/y\,(e^{x/y})$

$$\frac{\partial M}{\partial y} = 0 + e^{x/y}\, x\left(\frac{-1}{y^2}\right)$$

$$= \frac{-x}{y^2}\left(e^{x/y}\right)$$

$$\frac{\partial N}{\partial x} = e^{x/y}\left(\frac{1}{y}\right) - \left(\frac{1}{y}e^{x/y} + \frac{x}{y}e^{x/y}\left(\frac{1}{y}\right)\right)$$

$$= \cancel{\frac{1}{y}e^{x/y}} - \cancel{\frac{1}{y}e^{x/y}} - \frac{x}{y^2}e^{x/y}$$

$$\frac{\partial N}{\partial x} = \frac{-x}{y^2}e^{x/y}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{(exact)}$$

$$\int^{x} M\,dx = \int^{x}(1+e^{x/y})\,dx = \int^{x} 1\,dx + \int^{x} e^{x/y}\,dx.$$

$$= x + \frac{e^{x/y}}{1/y}$$

$$= x + y\,e^{x/y}$$

$\int N\,dy \Rightarrow$ terms not involving x

$\int 0\,dy = 0.$

$$\int^{x} M\,dx + \int N\,dy = c$$

$$x + y\,e^{x/y} = c$$

b) $(x^4+y^4)dx - xy^3 dy = 0$

Homogeneous.

$M = x^4+y^4$

$N = -xy^3$

$\frac{\partial M}{\partial y} = 4y^3$

$\frac{\partial N}{\partial x} = -y^3$

$Mx = (x^4+y^4)x \Rightarrow x^5 + xy^4$

$Ny = (-xy^3)y = -xy^4$

$Mx + Ny = x^5 + \cancel{xy^4} - \cancel{xy^4}$

$= x^5$

$IF = \frac{1}{Mx+Ny} = \frac{1}{x^5}$

$$\left(\frac{x^4+y^4}{x^5}\right)dx - \left(\frac{xy^3}{x^5}\right)dy = 0$$

$$\frac{x^4}{x^5}dx + \frac{y^4}{x^5}dx - \frac{xy^3}{x^5}dy = 0$$

$$\frac{1}{x}dx + \frac{y^4}{x^5}dx - \frac{y^3}{x^4}dy = 0.$$

$$\int^x M dx = \int^x \frac{1}{x}dx + y^4\int x^{-5}dx$$

$$= \log|x| + y^4 \frac{x^{-4}}{-4}$$

$$= \log x + y^4 \frac{1}{4x^4}$$

$$\int N dy = \int 0 = 0.$$

$$\int^x M dx + \int N dy = c$$

$$\log x + y^4 \frac{1}{4x^4} = c.$$

c) $(y^4+2y)dx + (xy^3+2y^4-4x)dy = 0$

$M = y^4+2y \qquad N = xy^3+2y^4-4x$

$\frac{\partial M}{\partial y} = 4y^3+2 \qquad \frac{\partial N}{\partial x} = y^3-4$

$$g(y) = \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)$$

$$= \frac{1}{y^4+2y}\left(y^3-4-4y^3-2\right)$$

$$= \frac{1}{y(y^3+2)}\left(-3y^3-6\right) = \frac{-3\cancel{(y^3+2)}}{y\cancel{(y^3+2)}}$$

$$g(y) = \frac{-3}{y}$$

$$I.F = \int e^{g(y)dy} = \int e^{-3\int y\,dy} = e^{-3\log y} = \frac{1}{y^3}$$

$M_1 dx + N_1 dy = 0$

$$\left(\frac{y^4+2y}{y^3}\right)dx + \left(\frac{xy^3+2y^4-4x}{y^3}\right)dy = 0$$

$$\left(\frac{\cancel{y^4}}{\cancel{y^3}} + \frac{2\cancel{y}}{y^{\cancel{3}2}}\right)dx + \left(\frac{x\cancel{y^3}}{\cancel{y^3}} + \frac{2y\cancel{^4}}{\cancel{y^3}} - \frac{4x}{y^3}\right)dy = 0$$

$$y\,dx + \frac{2}{y^2}dx + x\,dy + 2y\,dy - \frac{4x}{y^3}dy = 0$$

$$\int y\,dx + \int \frac{2}{y^2}dx + \int 2y\,dy = c$$

$$xy + \frac{2x}{y^2} + \frac{\cancel{2}y^2}{\cancel{2}} = c$$

$$xy + \frac{2x}{y^2} + y^2 = c$$

d) $(xy^2\sin xy + y\cos xy)dx + (x^2y\sin xy - x\cos xy)dy = 0$

$$\frac{\partial M}{\partial y} = x(2y)\sin xy + xy^2\cos xy(x) + \cos xy + y(-\sin xy)(x)$$

$$= 2xy\sin xy + x^2y^2\cos xy + \cos xy - xy\sin xy.$$

$$= xy\sin xy + x^2y^2\cos xy + \cos xy$$

$$\frac{\partial N}{\partial x} = 2xy\sin xy + x^2y(\cos xy)(y) - \cos xy - x(-\sin xy)x$$

$$= 2xy\sin xy + x^2y^2\cos xy - \cos xy + x^2\sin xy.$$

$\neq$ not exact

$$Mx = (xy^2\sin xy + y\cos xy)x \Rightarrow x^2y^2\sin xy + xy\cos xy.$$

$$Ny = (x^2y\sin xy - x\cos xy)y \Rightarrow x^2y^2\sin xy - xy\cos xy$$

$$Mx - Ny = \cancel{x^2y^2\sin xy} + xy\cos xy - \cancel{x^2y^2\sin xy} + xy\cos xy.$$

$$Mx - Ny = 2xy\cos xy. \quad \text{Integrating factor} = \frac{1}{Mx - Ny}$$

$$= \frac{1}{2xy\cos xy}.$$

$$\left(\frac{xy^2\sin xy}{2xy\cos xy} + \frac{y\cos xy}{2xy\cos xy}\right)dx + \left(\frac{x^2y\sin xy}{2xy\cos xy} - \frac{x\cos xy}{2xy\cos xy}\right)dy = 0.$$

$$\left(y\tan xy + \frac{1}{x}\right)dx + \left(x\tan xy - \frac{1}{y}\right)dy = 0.$$

$$\int M\,dx + \int N\,dy = 0.$$

$$\int y\tan xy\,dx + \int \frac{1}{x}\,dx - \int \frac{1}{y}\,dy = c$$

$$\cancel{y}\log|\sec xy|\frac{1}{\cancel{y}} + \log x - \log y = c.$$

$$\log|\sec xy| + \log x - \log y = c.$$

2a) $\frac{dy}{dx} + y\sec^2 x = \tan x \sec^2 x$

$P(x) = \sec^2 x$

$Q(x) = \tan x \sec^2 x$

$I.F = e^{\int P(x)dx} = e^{\int \sec^2 x dx}$

$I.F = e^{\tan x}$

General solution $y(I.F) = \int Q(x)(I.F)dx$

$y e^{\tan x} = \int \tan x \sec^2 x e^{\tan x} dx$

$y e^{\tan x} = \int \tan x \, d(e^{\tan x}) dx$

$\int f(x)g(x)dx = f(x)\int g(x) - \int\left[f'(x)\int g(x)dx\right]dx$

$y e^{\tan x} = \tan x \int d(e^{\tan x}) - \int\left[\sec^2 x \int d(e^{\tan x})dx\right]dx$

$y e^{\tan x} = \tan x e^{\tan x} - \int \sec^2 x e^{\tan x} dx$

$y e^{\tan x} = \tan x e^{\tan x} - \int d(e^{\tan x})$

$y e^{\tan x} = \tan x e^{\tan x} - e^{\tan x} + c$

$y \cancel{e^{\tan x}} = \cancel{e^{\tan x}}(\tan x - 1) + c e^{-\tan x}$

$y = \tan x - 1 + e^{-\tan x} c$

2b) $(1+x^2)dy + 2xy dx = \cot x dx$

$(1+x^2)\frac{dy}{dx} + 2xy(1) = \cot x(1)$

$(1+x^2)\frac{dy}{dx} + 2xy = \cot x$

$\frac{dy}{dx} + \frac{2x}{1+x^2}y = \frac{\cot x}{1+x^2}$

$P(x) = \frac{2x}{1+x^2}$

$I.F = e^{\int P(x)dx} = e^{\int \frac{2x}{1+x^2}dx}$

$= e^{\log(1+x^2)}$

$= 1+x^2$

General solution

$$y(I.F) = \int Q(x)(I.F)\,dx$$

$$y\cdot(1+x^2) = \int \frac{(\cot x)}{\cancel{(1+x^2)}}\cancel{(1+x^2)}\,dx$$

$$y\cdot(1+x^2) = \int \cot x\,dx$$

$$y\cdot(1+x^2) = \log(\sin x) + c$$

2c) $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

$$\frac{dx}{dy} = \frac{\tan^{-1}y - x}{1+y^2}$$

$$\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$$

$$\frac{dx}{dx} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$P(y) = \frac{1}{1+y^2}, \quad Q(y) = \frac{\tan^{-1}y}{1+y^2}$$

$$I.F = e^{\int P(y)dy}$$

$$= e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

$$x(I.F) = \int Q(y)\cdot IF\,dy$$

$$x e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y}\,dy$$

$\tan^{-1}y = t$

$\frac{1}{1+y^2}dy = dt$

$$x e^{\tan^{-1}y} = \int t e^t\,dt$$

$$x e^{\tan^{-1}y} = t e^t - e^t + c$$

$$x e^{\tan^{-1}y} = \tan^{-1}y\, e^{\tan^{-1}y} - e^{\tan^{-1}y} + c$$

d) $x\frac{dy}{dx}+y=x^3y^6$

$$\frac{dy}{dx}+\frac{y}{x}=\frac{x^3y^6}{x}$$

$$\frac{dy}{dx}+\frac{y}{x}=x^2y^6$$

$$\frac{1}{y^6}\frac{dy}{dx}+\frac{y}{x}\left(\frac{1}{y^6}\right)=x^2$$

$$y^{-6}\frac{dy}{dx}+\frac{y^{-5}}{x}=x^2$$

$$y^{-5}=v$$

$$-5y^{-6}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\frac{1}{y^6}\frac{dy}{dx}=\frac{-1}{5}\frac{dv}{dx}$$

$$-\frac{1}{5}\frac{dv}{dx}+\frac{v}{x}=x^2$$

$$\frac{dv}{dx}-\frac{5v}{x}=-5x^2$$

$$P(x)=-5/x,$$

$$Q(x)=-5x^2$$

$$I.F=e^{\int P(x)dx}=e^{\int -5/x\,dx}$$

$$=e^{-5\log x}=\frac{1}{x^5}$$

General solution

(v) $y.I.F=\int Q(x)\,I.F\,dx$

(v) $y\left(\frac{1}{x^5}\right)=\int -5x^2\left(\frac{1}{x^5}\right)dx$

$$y^{-5}\left(\frac{1}{x^5}\right)=-5\int x^{-3}dx$$

$$\frac{1}{(xy)^5}=\frac{5}{2x^2}+C$$

$$\frac{1}{(xy)^5}=\frac{5}{2x^2}+C$$

e) $\sec^2y\frac{dy}{dx}+2x\tan y=x^3$

$$\tan y=t$$

$$\sec^2y\frac{dy}{dx}=\frac{dt}{dx}$$

$$\frac{dt}{dx}+2xt=x^3$$

$$P(x)=2x$$

$$Q(x)=x^3$$

$$I.F=\int e^{P(x)dx}=\int e^{2x\,dx}$$

$$=e^{x^2}$$

$$t\,I.F=\int Q(x)\,I.F\,dx$$

$$\tan y\,e^{x^2}=\int x^3e^{x^2}dx$$

$$x^2=v$$

$$2x\,dx=dv$$

$$\tan y\,e^{x^2}=\frac{1}{2}\int e^v v\,dv$$

$$\tan y\,e^{x^2}=\frac{1}{2}\int ve^v dv$$

$$\tan y\,e^{x^2}=\frac{1}{2}\left[ve^v-e^v\right]+c$$

e)

$$\tan y\, e^{x^2} = \frac{1}{2} e^{x^2}(x^2-1) + c$$

$$\tan y\, \cancel{e^{x^2}} = \frac{1}{2} \cancel{e^{x^2}}(x^2-1) + ce^{-x^2}$$

$$\tan y = \frac{1}{2}(x^2-1) + ce^{-x^2}$$

3A)

$$ay^2 = x^3$$

$$a2y\left(\frac{dy}{dx}\right) = 3x^2$$

$$ay^2 = x^3$$

$$a = \frac{x^3}{y^2}$$

$$\frac{x^{\cancel{3}}}{y^{\cancel{2}}} \times 2\cancel{y}\left(\frac{dy}{dx}\right) = 3\cancel{x^2}$$

$$\frac{2x}{y}\left(\frac{dy}{dx}\right) = 3$$

$$\frac{2x}{y} = \frac{3dx}{dy}$$

Replacing $\frac{dy}{dx}$ with $\frac{-dx}{dy}$

$$\frac{2x}{y}\left(\frac{-dx}{dy}\right) = 3$$

$$-2x\,dx = 3y\,dy$$

Take integral.

$$-\int 2x\,dx = \int 3y\,dy$$

$$-\frac{\cancel{2}x^2}{\cancel{2}} = \frac{3y^2}{2} + c$$

$$-x^2 = y^2 \times \frac{3}{2} + c$$

$$-2x^2 = 3y^2 + c$$

$$3y^2 + 2x^2 + c = 0$$

$$3y^2 + 2x^2 = -c$$

$$\frac{x^2}{3c} + \frac{y^2}{2c} = -1$$

3)b)

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1 \longrightarrow (1)$$

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0$$

$$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0$$

Let $\frac{dy}{dx} = p$.

$$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}(p) = 0$$

$$x(b^2+\lambda) + y(a^2+\lambda)p = 0$$

$$xb^2 + x\lambda + (ya^2 + y\lambda)p = 0$$

$$xb^2 + x\lambda + ya^2p + y\lambda p = 0$$

$$\lambda(x+py) = -(b^2x + pya^2)$$

$$\lambda = \frac{-(b^2x+pya^2)}{x+py}$$

$$a^2+\lambda = a^2 - \frac{(b^2x+pya^2)}{x+py} = \frac{a^2x + \cancel{pa^2y} - b^2x - \cancel{pya^2}}{x+py}$$

$$a^2+\lambda = \frac{a^2x - b^2x}{x+py}$$

$$b^2+\lambda = b^2 - \frac{(b^2x+pya^2)}{x+py} = \frac{\cancel{b^2x} + b^2py - \cancel{b^2x} - pya^2}{x+py}$$

$$b^2+\lambda = \frac{py(b^2-a^2)}{x+py}$$

put $a^2+\lambda$ and $b^2+\lambda$ values in (1)

$$\frac{x^{\cancel{2}}}{\frac{\cancel{x}(a^2-b^2)}{x+py}} + \frac{y^{\cancel{2}}}{\frac{p\cancel{y}(b^2-a^2)}{x+py}} = 1 \Rightarrow \frac{x(x+py)}{a^2-b^2} - \frac{y(x+py)}{p(a^2-b^2)} = 1$$

$$\frac{x+py}{a^2-b^2}\left(x - \frac{y}{p}\right) = 1 \longrightarrow (2)$$

WKT $\frac{dy}{dx} = p$

$$\frac{-dx}{dy} = \frac{-1}{p}$$

Replace p with $\frac{-1}{p}$ in above (2) equ

$$\frac{x + \left(\frac{-1}{p}\right)y}{a^2 - b^2}\left(x - \frac{y}{\frac{-1}{p}}\right) = 1$$

$$\left(\frac{x - \frac{y}{p}}{a^2 - b^2}\right)(x + py) = 1 \longrightarrow (3)$$

equ (2) = equ (3)

self orthogonal.

3c) $r^n = a^n \cos n\theta$

$$a^n = \frac{r^n}{\cos n\theta}$$

$r^n = a^n \cos n\theta$ differentiate w.r.to θ.

$$\cancel{n} r^{n-1}\frac{dr}{d\theta} = a^n(-\sin n\theta)\cancel{n}$$

$$\frac{\cancel{r^n}}{r}\frac{dr}{d\theta} = \frac{\cancel{r^n}}{\cos n\theta}(-\sin n\theta)$$

$$\frac{1}{r}\frac{dr}{d\theta} = -\tan n\theta$$

Replace $\frac{dr}{d\theta}$ with $-r^2\frac{d\theta}{dr}$

$$\left(\frac{1}{\cancel{r}}\right)\left(\cancel{-}r^{\cancel{2}}\frac{d\theta}{dr}\right) = \cancel{-}\tan n\theta$$

$$r\frac{d\theta}{dr} = \tan n\theta$$

$$\frac{d\theta}{\tan n\theta} = \frac{dr}{r}$$

$$\cot n\theta\, d\theta = \frac{dr}{r}$$ Take Integral.

$$\int \cot n\theta\, d\theta = \int \frac{dr}{r}$$

$$\log r = \frac{1}{n}\log|\sin n\theta| + \log c.$$

$$\log r^n = \log|\sin n\theta| + \log c^n$$

$$r^n = \sin n\theta \cdot c^n$$

4a) $\theta = 300°F,\ \theta_0 = 20°F,\ t = 0$

$t = 10\text{min},\ \theta = 250°F$

$t = 20$ min

i) $\theta - \theta_0 = Ce^{-kt}$

$300 - 20 = Ce^{0}$

$280 = C$

ii) $\theta - \theta_0 = Ce^{-kt}$

$250 - 20 = 280\, e^{-10k}$

$23\cancel{0} = 28\cancel{0}\, e^{-10k}$

$\frac{23}{28} = e^{-10k}$

$-10k = \log\left(\frac{23}{28}\right)$

$k = \frac{-1}{10}\log\left(\frac{23}{28}\right)$

$k = \frac{1}{10}\log\left(\frac{28}{23}\right)$

iii) $\theta - \theta_0 = Ce^{-kt}$

$\theta - 20 = 280\, e^{\frac{1}{10}\log\left(\frac{23}{28}\right)\times 20}$

$\theta = 20 + 280\, e^{\frac{1}{10}\log\left(\frac{23}{28}\right)\times 2\cancel{0}}$

$\theta = 20 + 280\, e^{\log\left(\frac{23}{28}\right)^2}$

$\theta = 20 + 280\left(\frac{23}{28}\right)^2$

$\theta = 20 + 188.92$

$\theta = 208.92$

4b) $\theta = 94.4°F$ $t = 0$, at $= 9:40$

$\theta = 89.2°F$ $t = 80$ mins at $11:00$

$\theta_0 = 68°F$

$\theta - 68 = Ce^{-kt}$

$\theta - 68 = Ce^{0}$

$C = 94.4 - 68°$

$C = 26.4$

finding k

$89.2 - 68 = (26.4)e^{-k80}$

$e^{-k80} = \frac{26.4}{21.2}$

$k = \frac{-1}{80}\log\left(\frac{26.4}{21.2}\right)$

$\theta = 98.6°F$ (Normal human body temperature)

$98.6 - 68 = 26.4\, e^{-\frac{1}{80}\log\left(\frac{26.4}{21.2}\right)t}$

$\log\left(\frac{30.6}{26.4}\right) = \frac{1}{80}\log\left(\frac{21.2}{26.4}\right)t$ (take ln)

$0.1476 = \frac{1}{80}(-0.2193)t$

$t = -53.84$ min ago.

Time of death $= 9:40 - 53.86$ minutes

$\approx 8:46$ pm.

5a) $\frac{dN}{dt} = kN \Rightarrow \frac{dN}{N} = k\,dt \Rightarrow \int \frac{dN}{N} = \int k\,dt$

$\log N = kt + \log c \Rightarrow N = ce^{-kt}$

$t = 0$ sec, $N = 100$

$100 = ce^{0} \Rightarrow c = 100$

When $t = 3600$ sec, $N = 332$

$332 = 100\, e^{3600k}$

$e^{3600k} = \frac{332}{100}$

$t = \frac{3}{2}$ hours $\Rightarrow 5400$ sec then $N = 100\, e^{5400k}$

$N = 100 \left[e^{3600k}\right]^{3/2}$

$N = 100 \left[\frac{332}{100}\right]^{3/2}$

$N = \frac{1}{10}(6049.3) \Rightarrow 604.9$

$N = \underline{605}$

5b) $m = ce^{-kt}$

When $t = 0$, let $m = m_1$

$m_1 = c$

when $t = 10$, $m = \frac{70m}{100}$

$\frac{70m_1}{100} = ce^{-10k}$

$\frac{70m_1}{100} = m_1 e^{-10k} \Rightarrow e^{-10k} = \frac{7}{10}$

$-10k = \log\frac{7}{10}$

$k = \frac{-1}{10}\log\left(\frac{7}{10}\right) \Rightarrow k = \frac{1}{10}\log\left(\frac{10}{7}\right)$

Required time at 1 ps

$\frac{10m_1}{100} = ce^{-kt} = \frac{10m_1}{100} = m_1 e^{-kt} \Rightarrow \frac{1}{10} = e^{-kt}$

$-kt = \log\left(\frac{1}{10}\right) \Rightarrow t = \frac{1}{k}\log(10)$

$t = \frac{10\log(10)}{\log 10 - \log 7} = \frac{10(1)}{1 - 0.845} = \frac{10}{0.155}$

$= 64.5$ days.

UNIT-V

1a) $(D^2-2D+1)y = xe^x \sin x$

A.E $f(m)=0$

$m^2-2m+1=0$

$m^2-m-m+1=0$

$m(m-1)-1(m-1)=0$

$(m-1)(m-1)=0$

$m=1,1$

$y_c = (c_1+c_2x)e^x$

$$y_p = \frac{xe^x \sin x}{D^2-2D+1}$$

$$= \frac{1}{D^2-2D+1} xe^x \sin x$$

$$= e^x \frac{1}{(D+1)^2-2(D+1)+1} x\sin x$$

$$= e^x \frac{1}{D^2+\cancel{1}+\cancel{2D}-\cancel{2D}-\cancel{2}+\cancel{1}} x\sin x$$

$$= e^x \left[\frac{1}{D^2}\right] x\sin x$$

$$= e^x \left[x\frac{1}{D^2}\sin x\right] - e^x \frac{2D}{(D^2)^2}\sin x$$

$$= -xe^x \sin x - 2e^x \cos x$$

$$= -e^x(x\sin x + 2\cos x)$$

$y = y_c + y_p$

$y = (c_1+c_2x)e^x - e^x(x\sin x + 2\cos x)$

1b) $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x$

A.E $f(m)=0$

$m^2-4m+4=0$

$m^2-2m-2m+4=0$

$m(m-2)-2(m-2)=0$

$(m-2)(m-2)=0$

$m=2,2$

$$y_c = (c_1 + c_2 x)e^{x}$$

$$y_p = \frac{1}{D^2 - 4D + 4} 8x^2 e^{2x} \sin 2x$$

$$= 8e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= 8e^{2x} \frac{1}{D^2 + \cancel{4} + \cancel{4D} - \cancel{4D} - \cancel{8} + \cancel{4}} x^2 \sin 2x$$

$$= 8e^{2x} \left[\frac{1}{D^2}\right] x^2 \sin 2x$$

$$= 8e^{2x} \frac{1}{D} \frac{1}{D} x^2 \sin 2x$$

$$= 8e^{2x} \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= 8e^{2x} \frac{1}{D} \left[-\frac{x^2}{2} \cos 2x + \frac{x}{2} \sin 2x + \frac{1}{4} \cos 2x\right]$$

$$= 8e^{2x} \left[-\frac{1}{2} \int x^2 \cos 2x \, dx + \frac{1}{2} \int x \sin 2x \, dx + \frac{1}{4} \int \cos 2x \, dx\right.$$

$$= 8e^{2x} \left[\left[-\frac{1}{2} \left[x^2 \frac{\sin 2x}{2} - \int 2x \frac{\sin 2x}{2} dx\right] + \frac{1}{2} \int x \sin 2x \, dx + \frac{1}{4} \int \cos 2x \, dx\right]\right.$$

$$= 8e^{2x} \left[-\frac{x^2 \sin 2x}{4} + \int x \sin 2x \, dx + \frac{1}{4} \int \cos 2x \, dx\right]$$

$$= 8e^{2x} \left[-\frac{x^2 \sin 2x}{4} + x \frac{(-\cos 2x)}{2} - \int \frac{1(-\cos 2x)}{2} dx + \frac{1}{4} \int \cos 2x \, dx\right]$$

$$= 8e^{2x} \left[-\frac{x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{4} \int \cos 2x \, dx\right]$$

$$= 8e^{2x} \left[-\frac{x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x\right]$$

$$y = y_c + y_p$$

$$y = (c_1 + c_2 x)e^{x} + 8e^{2x} \left[-\frac{x^2}{4} \sin 2x - \frac{x}{2} \cos 2x + \frac{3}{8} \sin 2x\right]$$

$$y = (c_1 + c_2 x)e^{x} + e^{2x} \left[\frac{-8x^2}{4} \sin 2x - 8 \frac{x}{2} \cos 2x + \frac{3 \times \cancel{8}}{\cancel{8}} \sin 2x\right]$$

$$y = (c_1 + c_2 x)e^{x} + e^{2x} \left[-2x^2 \sin 2x - 4x \cos 2x + 3 \sin 2x\right]$$

Another Method.

$$y_p = \text{Imaginary part } 8e^{2x}\frac{1}{D^2}x^2e^{2ix}$$

$$= \text{IP. } 8e^{2x}e^{2ix}\frac{1}{(D+2i)^2}x^2$$

$$= \text{Imag } 8e^{2x}e^{2ix}\frac{1}{\left(2i\left(1+\frac{D}{2i}\right)\right)^2}x^2$$

$$= \text{Imag } 8e^{2x}e^{2ix}\frac{1}{-4}\left(1-\frac{iD}{2}\right)^{-2}x^2$$

$$= \text{Im}\frac{8}{-4}e^{2x}e^{2ix}\left[1+\frac{2iD}{2}+\frac{3(iD)^2}{4}+\cdots\right]x^2$$

$$= \text{Imag } -2e^{2x}e^{2ix}\left[x^2+2ix-\frac{3}{2}\right]$$

$$= \text{Imag } e^{2x}(\cos 2x+i\sin 2x)\left(-2x^2-4ix+\frac{3}{2}\times 2\right)$$

$$= \text{Imag } e^{2x}(\cos 2x+i\sin 2x)\left((3-2x^2)-4ix\right)$$

$$= 8e^{2x}\left[\frac{-x^2}{4}\sin 2x-\frac{x}{2}\cos 2x+\frac{3}{8}\sin 2x\right]$$

10) $(D^2+1)y = x^2e^{3x}$

$$f(m) = 0$$

$$m^2+1 = 0$$

$$m^2 = -1$$

$$m^2 = i^2$$

$$m = \pm i$$

$$y_c = c_1\cos x + c_2\sin x$$

$$y_p = \frac{1}{(D^2+1)}x^2e^{3x}$$

$$y_p = e^{3x}\frac{1}{(D+3)^2+1}x^2$$

$$y_p = e^{3x}\frac{1}{D^2+9+6D+1}x^2$$

$$y_p = e^{3x} \frac{1}{D^2+6D+10} x^2$$

$$y_p = e^{3x} \frac{1}{10\left(1+\frac{D^2+6D}{10}\right)} x^2$$

$$y_p = e^{3x} \frac{1}{10}\left(1+\left(\frac{D^2+6D}{10}\right)\right)^{-1} x^2$$

$$y_p = e^{3x} \frac{1}{10}\left(1-\left(\frac{D^2+6D}{10}\right)+\left(\frac{D^2+6D}{10}\right)^2+\cdots\right) x^2$$

$$= e^{3x} \frac{1}{10}\left(1-\frac{D^2}{10}-\frac{6D}{10}+\frac{D^4}{100}+\frac{36D^2}{100}+\frac{12D^3}{100}\right) x^2$$

$$= e^{3x} \frac{1}{10}\left(x^2-\frac{2}{10}-\frac{6x}{100}+0+\frac{36(2)}{100}+\frac{12(0)}{100}\right)$$

$$= e^{3x} \frac{1}{10}\left(x^2-\frac{1}{5}-\frac{3x}{50}+\frac{36}{50}\right)$$

$$= e^{3x} \frac{1}{10}\left(x^2-\frac{1}{5}-\frac{3x}{50}+\frac{18}{25}\right)$$

$$= e^{3x} \frac{1}{10}\left(x^2-\frac{3x}{50}+\frac{18}{25}-\frac{1}{5}\right)$$

$$= e^{3x} \frac{1}{10}\left(x^2-\frac{3x}{50}+\frac{18-5}{25}\right)$$

$$= e^{3x} \frac{1}{10}\left(x^2-\frac{3x}{50}+\frac{13}{25}\right)$$

$$= e^{3x}\left(\frac{x^2}{10}-\frac{3x}{500}+\frac{13}{250}\right)$$

2a) $(D^2-3D+2)y = \sin(e^{-x})$

$a_2 = 1,\ a_1 = -3,\ a_0 = 2$

(D^2-3D+2)

$f(m) = 0$

$m^2-3m+2=0$

$m^2-m-2m+2=0$

$m(m-1)-2(m-2)=0$

$(m-1)(m-2)=0$

$m = 1, 2$

$y_c = c_1 e^{x} + c_2 e^{2x}$

$$y_c = c_1 e^{x} + c_2 e^{2x}$$

$$y_1(x) = e^{x}$$

$$y_2(x) = e^{2x}$$

$$Q(x) = \sin(e^{-x})$$

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{Q(x)}{a_2} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & e^{2x} \\ \frac{\sin(e^{-x})}{1} & 2e^{2x} \end{vmatrix}}{\begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix}}$$

$$= \frac{e^{2x}(\sin(e^{-x}))}{2e^{x}e^{2x} - e^{x}e^{2x}} = \frac{e^{2x}(\sin(e^{-x}))}{2e^{3x} - e^{3x}}$$

$$= \frac{\cancel{e^{2x}}(\sin(e^{-x}))}{e^{x}\cancel{e^{2x}}}$$

$$u' = \frac{\sin(e^{-x})}{e^{x}} \qquad u' = e^{-x}\sin(e^{-x})$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1^{2} & \frac{Q(x)}{a_2} \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} e^{x} & 0 \\ e^{x} & \frac{\sin(e^{-x})}{1} \end{vmatrix}}{\begin{vmatrix} e^{x} & e^{2x} \\ e^{x} & 2e^{2x} \end{vmatrix}}$$

$$= \frac{e^{x}\sin(e^{-x})}{2e^{2x}e^{x} - e^{2x}e^{x}} = \frac{\cancel{e^{x}}\sin(e^{-x})}{e^{2x}\cancel{e^{x}}}$$

$$v' = \frac{\sin(e^{-x})}{e^{2x}} = e^{-2x}\sin(e^{-x})$$

$$u = \int u' = \int e^{-x}\sin(e^{-x})\,dx$$

$$= \qquad e^{-x} = t$$

$$-e^{-x}dx = dt$$

$$= \int -\sin t\,dt$$

$$= -(-\cos t)$$

$$= +\cos t \rightarrow \cos(e^{-x})$$

$$v = \int v' = \int e^{-2x} \sin(e^{-x})\,dx$$

$$= \int e^{-x} e^{-x} \sin(e^{-x})\,dx$$

$$e^{-x} = t$$

$$-e^{-x}dx = dt$$

$$= \int t \sin t\,(-dt)$$

$$= -\int t \sin t\,dt$$

$$\int f(x)g(x)\,dx = f(x)\int g(x)\,dx - \int f'(x)\int g(x)\,dx\Big]dx.$$

$$= -\left[t\int \sin t\,dt - \int (t)'\int \sin t\,dt\Big]dt\right]$$

$$= -\left[t(-\cos t) + \int 1 \cos t\,dt\right]$$

$$= t\cos t - \sin t$$

$$= e^{-x}\cos(e^{-x}) - \sin e^{-x}$$

$$y_p = u\,y_1(x) + v\,(y_2(x))$$

$$= \cos(e^{-x})e^{x} + \left(e^{-x}\cos(e^{-x}) - \sin e^{-x}\right)(e^{2x})$$

$$= e^{x}\cos(e^{-x}) + e^{x}\cos(e^{-x}) - e^{2x}\sin e^{-x})$$

$$= 2e^{x}\cos(e^{-x}) - e^{2x}\sin(e^{-x})$$

$$y = y_c + y_p$$

$$y = c_1 e^{x} + c_2 e^{2x} + 2e^{x}\cos(e^{-x}) - e^{2x}\sin(e^{-x})$$

2b) $(D^3 - D^2 - 6D)y = 1 + x^2$

$f(m) = 0$

$m^3 - m^2 - 6m = 0$

$m(m^2 - m - 6) = 0$

$m(m^2 + 2m - 3m - 6) = 0$

$m(m(m+2) - 3(m+2)) = 0$

$m(m-3)(m+2) = 0$

$m = 0, +3, -2$

$y_c = c_1 e^{+0x} + c_2 e^{3x} + c_3 e^{-2x}$

$y_c = c_1 + c_2 e^{3x} + c_3 e^{-2x}$

$$y_p = \frac{1}{(D^3 - D^2 - 6D)}(1 + x^2)$$

$$= \frac{1}{-6D\left(1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right)}(1 + x^2)$$

$$= \frac{-1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(1 + x^2)$$

$$= \frac{-1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 + \cdots\right](1 + x^2)$$

$$= \frac{-1}{6D}\left(1 - \frac{D}{6} + \frac{D^2}{6} + \frac{1}{36}(D^2 + D^4 - 2D^3)\right)(1 + x^2)$$

$$= \frac{-1}{6D}\left[(1 + x^2) - \frac{(0 + 2x)}{6} + \frac{(0+2)}{6} + \frac{1}{36}(0+2) + 0 + 0\right]$$

$$= \frac{-1}{6D}\left[1 + x^2 - \frac{x}{3} + \frac{1}{3} + \frac{1}{18}\right] \Rightarrow \frac{-1}{6D}\left[x^2 - \frac{x}{3} + \frac{7}{18} + 1\right]$$

$$= \frac{-1}{6D}\left[x^2 - \frac{x}{3} + \frac{7+18}{18}\right] = \frac{-1}{6D}\left[x^2 - \frac{x}{3} + \frac{25}{18}\right]$$

$$= \frac{-1}{6}\left[\frac{x^3}{3} - \frac{1}{3}\frac{x^2}{2} + \frac{25}{18}(x)\right]$$

$$= -\left[\frac{x^3}{18} - \frac{x^2}{36} + \frac{25}{108}x\right]$$

$y = y_c + y_p$

2) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = -2\cosh x.$

$(D^2+4D+5)y = -2\cosh x \Rightarrow (D^2+4D+5)y = -(e^x+e^{-x})$

W.K.T $\cosh x = \frac{e^x+e^{-x}}{2}$

$f(m) = 0$

$m^2+4m+5=0$

$m = \frac{-4 \pm \sqrt{16-20}}{2}$

$= -2 \pm i$

$y_c = e^{-2x}(c_1\cos x + c_2 \sin x)$

$y_p = \frac{-e^x - e^{-x}}{D^2+4D+5}$

$y_p = \frac{-e^x}{D^2+4D+5} - \frac{e^{-x}}{D^2+4D+5}$

$= \frac{-e^x}{1+4+5} - \frac{e^{-x}}{(-1)^2+4(-1)+5}$

$y_p = \frac{-e^x}{10} - \frac{e^{-x}}{2}$

$y = y_c + y_p$

$y = e^{-2x}(c_1\cos x + c_2 \sin x) + \left(\frac{-e^x}{10} - \frac{e^{-x}}{2}\right) \longrightarrow (1).$

$y=0, \frac{dy}{dx} = 1$ at $x=0$.

differentiate (1) w.r.to x

$\frac{dy}{dx} = e^{-2x}(c_1(-\sin x) + c_2\cos x) + e^{-2x}(-2)(c_1\cos x + c_2 \sin x)$
$- \frac{e^x}{10} + \frac{e^{-x}}{2}$

$\frac{dy}{dx} = e^{-2x}\left[-c_1\sin x + c_2\cos x - 2c_1\cos x + (-2)c_2\sin x\right]$
$+ \frac{e^{-x}}{2} - \frac{e^x}{10} \longrightarrow (2).$

$y=0$ at $x=0$ (1)

$0 = e^{-0}(c_1(1) + c_2(0)) + \left(\frac{-e^0}{10} - \frac{e^{-0}}{2}\right)$

$0 = c_1 - \left(\frac{1}{10} + \frac{1}{2}\right) \Rightarrow c_1 = \frac{3}{5}$

$\frac{1+5}{10} \Rightarrow 6/10$

$\frac{dy}{dx} = 1$ at $x=0$ in ②

$$1 = e^{-0}\left(-c_1(0) + c_2(1) - 2c_1(1) + 0\right) + \frac{1}{2}(1) - \frac{1}{10}(1).$$

$$1 = c_2 - 2c_1 + \frac{1}{2} - \frac{1}{10}.$$

$$1 = c_2 - 2c_1 + \frac{5-1}{10}$$

$$1 = c_2 - 2c_1 + \frac{4}{10}.$$

$$1 - \frac{4}{10} = c_2 - 2c_1$$

$$\frac{6}{10} = c_2 - 2c_1 \qquad c_1 = 3/5$$

$$\frac{3}{5} = c_2 - 2c_1$$

$$\frac{3}{5} + \frac{6}{5} = c_2$$

$$c_2 = 9/5.$$

c_1 and c_2 values in ①.

$$y = \frac{3}{5}e^{-2x}\cos x + \frac{9}{5}e^{-2x}\sin x - \left(\frac{e^{x}}{10} + \frac{e^{-x}}{2}\right)$$

$$y = \frac{3}{5}e^{-2x}\left(\cos x + 3\sin x\right) - \left(\frac{e^{x}}{10} + \frac{e^{-x}}{2}\right)$$

3a) $(D^2 - 2D)y = e^{x}\sin x$

$a_2 = 1,\ a_2 = -2,\ a_0 = 0.$

$f(m) = 0$

$m^2 - 2m = 0.$

$m(m-2) = 0$

$m = 0,\ m = 2$

$y_c = (c_1 e^{0x} + c_2 e^{2x})$

$y_c = c_1 + c_2 e^{2x}$

$y_1 = 1,\ y_2 = e^{2x}.$

$Q(x) = e^{x}\sin x.$

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{Q(x)}{a_2} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & e^{2x} \\ \frac{e^x \sin x}{1} & 2e^{2x} \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ e^{2x} & 2e^{2x} \end{vmatrix}} \Rightarrow \frac{-\cancel{e^{2x}} e^x \sin x}{2\cancel{e^{2x}}}$$

$$\Rightarrow -\frac{e^x \sin x}{2}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{Q(x)}{a_2} \end{vmatrix}}{\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 1 & 0 \\ 0 & \frac{e^x \sin x}{1} \end{vmatrix}}{\begin{vmatrix} 1 & 0 \\ e^{2x} & 2e^{2x} \end{vmatrix}} = \frac{e^x \sin x}{2e^{2x}}$$

$$v' = \frac{e^x \sin x \times e^{-2x}}{2} \qquad v' = \frac{e^{-x} \sin x}{2}$$

$$u = \int u' = -\frac{1}{2} \int e^x \sin x \, dx$$

$$= -\frac{1}{2}\left[\frac{1}{2}\left(\sin x \, e^x - \cos x \, e^x\right)\right]$$

$$= -\frac{1}{4} e^x (\sin x - \cos x)$$

$$v = \int v' = \frac{1}{2} \int e^{-x} \sin x$$

$$= \frac{1}{2}\left[\frac{-1}{2} e^{-x} (\sin x + \cos x)\right]$$

$$= -\frac{1}{4} e^{-x} (\sin x + \cos x)$$

$$y_p = u y_1(x) + v y_2(x)$$

$$= -\frac{1}{4} e^x (\sin x - \cos x)(1) - \frac{1}{4} e^{-x} (\sin x + \cos x) e^{2x}$$

$$= -\frac{1}{4} e^x (\sin x - \cos x) - \frac{1}{4} e^x (\sin x + \cos x)$$

$$= -\frac{1}{4} e^x \sin x + \cancel{\frac{1}{4} e^x \cos x} - \frac{1}{4} e^x \sin x - \cancel{\frac{1}{4} e^x \cos x}$$

$$= -\frac{2}{4} e^x \sin x \qquad y = y_c + y_p$$

$$= -\frac{1}{2} e^x \sin x$$

3b) $(D^2+a^2)y = \sec ax$

$a_2 = 1, a_1 = 0, a_0 = a^2$

$Q(x) = \sec ax$

$(D^2+a^2)y = \sec ax$

$f(m) = m^2 + a^2 = 0$

$m^2 = -a^2$ $\qquad y_1(x) = \cos ax$

$m^2 = a^2 i^2$ $\qquad y_2(x) = \sin ax$

$m = \pm ai$

$y_c = C_1 \cos ax + C_2 \sin ax$

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{Q(x)}{a_2} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & \sin ax \\ \sec ax & a\cos ax \end{vmatrix}}{\begin{vmatrix} \cos ax & -a\sin ax \\ \sin ax & a\cos ax \end{vmatrix}}$$

$$= \frac{-\sin ax\left(\frac{1}{\cos ax}\right)}{a\cos^2 ax + a\sin^2 ax} = \frac{-\tan ax}{a(1)} = \frac{-\tan ax}{a}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{Q(x)}{a_2} \end{vmatrix}}{\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} \cos ax & 0 \\ -a\sin ax & \sec ax \end{vmatrix}}{\begin{vmatrix} \cos ax & -a\sin ax \\ \sin ax & a\cos ax \end{vmatrix}}$$

$$= \frac{\cos ax(\sec ax)}{a\cos^2 ax + a\sin^2 ax} = \frac{1}{a(1)} = \frac{1}{a}$$

$$u = \int u' = -\frac{1}{a}\int \tan ax\,dx = \frac{-1}{a}\left(-\log(\cos ax)\right) \times \frac{1}{a}$$

$$= \frac{1}{a^2}\log|\cos ax|$$

$$v = \int v' = \frac{1}{a}\int 1\,dx = \frac{x}{a}$$

$$y_p = u y_1(x) + v y_2(x)$$

$$= \frac{1}{a^2}\log(\cos ax).\cos ax + \frac{x}{a}\sin ax$$

$y = y_c + y_p$

3c) $(D^2+3D+2)y = e^{e^x}$

$f(m) = 0$

$m^2+3m+2=0$

$m^2+m+2m+2=0$

$m(m+1)+2(m+1)=0$

$(m+2)(m+1)=0$

$m=-1,-2$

$y_c = c_1e^{-x}+c_2e^{-2x}$. $\quad Q(x) = e^{e^x}$

$y_1(x) = e^{-x}$

$y_2(x) = e^{-2x}$.

$$u' = \frac{\begin{vmatrix} 0 & y_2 \\ \frac{Q(x)}{a_2} & y_2' \end{vmatrix}}{\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} 0 & e^{-2x} \\ e^{e^x} & -2e^{-2x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & -e^{-x} \\ e^{-2x} & -2e^{-2x} \end{vmatrix}}$$

$$u' = \frac{-e^{e^x}e^{-2x}}{-2e^{-2x}e^{-x}+e^{-x}e^{-2x}} = \frac{-e^{e^x}}{-2e^{-x}+e^{-x}} = \frac{+e^{e^x}}{+e^{-x}} = e^{e^x}e^{x}$$

$$v' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & \frac{Q(x)}{a_2} \end{vmatrix}}{\begin{vmatrix} y_1 & y_1' \\ y_2 & y_2' \end{vmatrix}} = \frac{\begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & e^{e^x} \end{vmatrix}}{\begin{vmatrix} e^{-x} & -e^{-2x} \\ e^{-2x} & -2e^{-2x} \end{vmatrix}} = \frac{e^{-x}e^{e^x}}{-e^{-x}2e^{-2x}+e^{-2x}e^{-x}}$$

$$v' = \frac{\cancel{e^{-x}}e^{e^x}}{-\cancel{e^{-x}}e^{-2x}} = \frac{e^{e^x}}{-e^{-2x}} = -e^{e^x}.e^{2x}$$

$e^x = t$

$e^x dx = dt$

$$u = \int u' = \int e^{e^x}e^x dx$$

$$= \int e^t dt$$

$$u = e^t$$

$$u = e^{e^x}$$

$$v = \int v' = -\int e^{e^x} e^{2x}\, dx$$

$$= -\int e^{e^x} e^x e^x\, dx \qquad e^x = t$$

$$e^x dx = dt$$

$$= -\int e^t t\, dt$$

$$= -\left[te^t - e^t\right]$$

$$= e^t - te^t$$

$$= e^{e^x} - e^x e^{e^x}.$$

$$y_p = y_1(x)u + v y_2(x)$$

$$= e^{-x}\left(e^{e^x}\right) + \left(e^{e^x} - e^x e^{e^x}\right)\left(e^{-2x}\right)$$

$$= \cancel{e^{-x} e^{e^x}} + e^{e^x} e^{-2x} - \cancel{e^{-x} e^{e^x}}$$

$$= e^{e^x} e^{-2x}.$$

$$y_p = e^{e^x - 2x}.$$

$$y = y_c + y_p$$

$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{e^x - 2x}.$$

4) a) $\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + \frac{y}{x^2} = \frac{2\log x}{x^2}$

$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 2\log x$. $\quad x = e^z$, $z = \log x$.

$(x^2D - xD + 1)y = 2\log x$

$x\frac{dy}{dx} = Dy$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$D(D-1)y - Dy + y = 2z$

$(D^2 - D - D + 1)y = 2z$

$(D^2 - 2D + 1)y = 2z$

~~A.E~~ $f(m) = 0$

$m^2 - 2m + 1 = 0$.

$m^2 - m - m + 1 = 0$

$m(m-1) - 1(m-1) = 0$

$(m-1)(m-1) = 0$

$m = 1, 1$

$y_c = (c_1 + c_2 z)e^z$

$y_p = \frac{1}{D^2 - 2D + 1} 2z$

$= \frac{2z}{(D-1)^2} \Rightarrow \frac{2z}{(1-D)^2}$

$= (1-D)^{-2} 2z$

$= (1 + 2D + 3D^2 + \cdots) 2z$

$= 2(z + 2(1) + 3(0) + \cdots)$

$= 2(z+2)$

$y_p = 2z + 4$

$y_p = 2\log x + 4$

$y = y_c + y_p$

$y = (c_1 + c_2 \log x)x + 2\log x + 4$.

(b) $(2x-1)^2\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}-2y=8x^2-2x+3$

$(2x-1)=e^z \Rightarrow 2x=e^z+1$

$x=\frac{e^z+1}{2}$

$\log(2x-1)=z$

$(2x-1)^2\frac{d^2y}{dx^2} \Rightarrow \theta(\theta-1)\times(2)^2 \Rightarrow 4\theta(\theta-1)$

$(2x-1)\frac{dy}{dx} \Rightarrow 2\theta$

$(4\theta(\theta-1)+2\theta-2)y=8\left(\frac{e^z+1}{2}\right)^2-\cancel{2}\left(\frac{e^z+1}{\cancel{2}}\right)+3$

$(4\theta^2-4\theta+2\theta-2)y=8\left(\frac{e^{2z}+2e^z+1}{4}\right)-e^z-2+3$

$(4\theta^2-2\theta-2)y=2e^{2z}+4e^z+\cancel{2}-e^z-\cancel{2}+3$

$(4\theta^2-2\theta-2)y=2e^{2z}+4e^z-e^z+3$

$(4\theta^2-2\theta-2)y=2e^{2z}+3e^z+3.$

$f(m)=0$

$4m^2-2m-2=0.$

$2m^2-m-1=0.$

$2m^2-2m+m-1=0$

$2m(m-1)+1(m-1)=0.$

$(m-1)(2m+1)=0.$

$m=1,\ -1/2$

$y_c=c_1e^z+c_2e^{-x/2}$

$y_p=\frac{2e^{2z}}{4\theta^2-2\theta-2}+\frac{3e^z}{4\theta^2-2\theta-2}+\frac{3e^0}{4\theta^2-2\theta-2}$

$y_p=\frac{2e^{2z}}{4(4)-4(2)-2}+\frac{3ze^z}{4(2)(1)-2(1)}+\frac{3e^0}{4(0)-2(0)-2}$

$y_p=\frac{2e^{2z}}{16-10}+\frac{3ze^z}{8-2}+\frac{3}{-2}$

$y_p=\frac{2e^{2z}}{6}+\frac{3ze^z}{6}-\frac{3}{2}$

$$y_p = \frac{1}{3} e^{2z} + \frac{1}{2} z e^{z} - \frac{3}{2}$$

$$y_p = \frac{1}{3} e^{2(\log(2x-1))} + \frac{1}{2} \log(2x-1) e^{\log(2x-1)} - \frac{3}{2}$$

$$= \frac{1}{3} e^{\log(2x-1)^2} + \frac{1}{2} \log(2x-1)(2x-1) - \frac{3}{2}$$

$$= \frac{1}{3}(2x-1)^2 + \frac{1}{2} \log(2x-1)(2x-1) - \frac{3}{2}$$

c) $$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$$

$$x = e^{z}$$

$$z = \log x$$

$$xD = \theta$$

$$x^2 D^2 = \theta(\theta-1)$$

$$x^2 D^3 = \theta(\theta-1)(\theta-2)$$

$$\left(\theta(\theta-1)(\theta-2)\right) + 2\left(\theta(\theta-1)+2\right)y = 10\left(e^{z} + \frac{1}{e^{z}}\right)$$

$$\left((\theta^2-\theta)(\theta-2) + 2(\theta^2-\theta)+2\right)y = 10(e^{z}+e^{-z})$$

$$\left(\theta^3 - \cancel{2\theta^2} - \theta^2 + \cancel{2\theta} + \cancel{2\theta^2} - \cancel{2\theta} + 2\right)y = 10(e^{z}+e^{-z})$$

$$(\theta^3 - \theta^2 + 2)y = 10(e^{z}+e^{-z})$$

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & 0 & -1 & 2 & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$m = -1,\ 1 \pm i$

$$\theta^2 - 2\theta + 2 = 0$$

$$\frac{+2 \pm \sqrt{4 - 4(1)(2)}}{2} \Rightarrow \frac{2 \pm \sqrt{4-8}}{2}$$

$$\frac{2 \pm \sqrt{4i^2}}{2}$$

$$\frac{2 \pm 2i}{2} \Rightarrow 1 \pm i$$

$$y_c = c_1 e^{-z} + e^{z}(c_2 \cos z + c_3 \sin z)$$

$$y_p = \frac{1}{(\theta^3 - \theta^2 + 2)} 10\left(e^{z} + e^{-z}\right)$$

$$= 10\left[\frac{e^{z}}{\theta^3 - \theta^2 + 2} + \frac{e^{-z}}{\theta^3 - \theta^2 + 2}\right]$$

$$= 10\left[\frac{e^{z}}{(1) - 1 + 2} + \frac{e^{-z}(z)}{3(-1)^2 - 2(-1) + 0}\right]$$

$$= 10\left[\frac{e^{z}}{2} + \frac{z e^{-z}}{3 + 2}\right]$$

$$= 10\left[\frac{e^{z}}{2} + \frac{z e^{-z}}{5}\right]$$

$$y = y_c + y_p$$

$$y = c_1 e^{-z} + e^{z}(c_2 \cos z + c_3 \sin z) + 10\left(\frac{e^{z}}{2} + \frac{z e^{-z}}{5}\right)$$

$=$ $\quad e^{z} = x$

$\quad z = \log x.$

$$= \frac{c_1}{x} + x\left(c_2 \cos(\log x) + c_3 \sin(\log x)\right) + 10\left(\frac{x}{2} + \frac{\log x}{5}\left(\frac{1}{x}\right)\right)$$

$$= \frac{c_1}{x} + x\left[\left(c_2 \cos(\log x)\right) + c_3 \sin(\log x)\right) + \frac{x}{5} + 2\log x\right] x$$