Mid - 2

1. (a) Find directional derivative of the function $f = 4xy^2 + 2x^2yz$ at the point $P(1,2,3)$ in the direction of PQ where $Q(5,0,4)$.

Directional derivative $= \bar{e} \cdot \nabla f$

Given $\bar{f} = 4xy^2 + 2x^2yz$ at $P(1,2,3)$

$$\nabla \bar{f} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(4xy^2 + 2x^2yz)$$

$$= (4y^2 + 4xyz)i + (8xy + 2x^2z)j + (2x^2y)k$$

$$(\nabla f)_{\text{at } (1,2,3)} = (4(2)^2 + 4(1)(2)(3))i + (8(1)(2) + 2(1)^2(3))j + 2(1)^2(2)k$$

$$= (16 + 24)i + (16 + 6)j + 4k$$

$$= 40i + 22j + 4k \quad //$$

$$\bar{e} = \frac{\bar{a}}{|a|}$$

$$a = PQ = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

$$= 4i - 2j + k$$

$$\bar{e} = \frac{a}{|a|} = \frac{4i - 2j + k}{\sqrt{4^2 + (-2)^2 + 1^2}} = \frac{4i - 2j + k}{\sqrt{21}} \quad //$$

$$\bar{e} \cdot \nabla f = (40i + 22j + 4k)\left(\frac{4i - 2j + k}{\sqrt{21}}\right)$$

$$= \frac{160 - 44 + 4}{\sqrt{21}} = \frac{120}{\sqrt{21}} \quad //$$

1(b) find the angle b/w the surface $x^2+y^2+z^2=9$ and $z = x^2+y^2-3$ at the point $(2,-1,2)$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

Given $\phi_1 = x^2+y^2+z^2-9$ $\quad \phi_2 = x^2+y^2-z-3$

$$\nabla\phi_1 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-9)$$

$$= 2xi + 2yj + 2zk$$

$$\nabla\phi_1 \text{ at } (2,-1,2) = 2(2)i + 2(-1)j + 2(2)k$$

$$= 4i - 2j + 4k$$

$$\nabla\phi_2 = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2-z-3)$$

$$= 2xi + 2yj - k$$

$$\nabla\phi_2 \text{ at } (2,-1,2) = 2(2)i + 2(-1)j - k$$

$$= 4i - 2j - k$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4i-2j+4k)\cdot(4i-2j-k)}{\sqrt{4^2+(-2)^2+4^2}\cdot\sqrt{4^2+(-2)^2+(-1)^2}}$$

$$= \frac{16+4-4}{\sqrt{36}\sqrt{21}} \Rightarrow \frac{\cancel{16}^{\,8}}{\cancel{6}_{\,3}\sqrt{21}} \Rightarrow \frac{8}{3\sqrt{21}}$$

2. Find constants a, b, c so that the vector

$A = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$

is irrotational.

Given A is a irrotational vector

so $\nabla \times A = 0$

$$\nabla \times A = 0 \Rightarrow \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = 0$$

$$= i(c-(-1)) - j(4-a) + k(b-2) = 0$$

$$= i(c+1) - j(4-a) + k(b-2) = 0$$

$c+1=0$ | $4-a=0$ | $b-2=0$

$\boxed{c=-1}$ | $\boxed{a=4}$ | $\boxed{b=2}$

2b) If $\bar{f} = (x+3y+z)i + (2py-z)j - (xy+3z)k$ is solenoidal then find P.

Given $\bar{f}$ is solenoidal i.e $\nabla \cdot \bar{f} = 0$

$$\nabla \cdot \bar{f} = 0 \Rightarrow \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\bar{f} = 0$$

$$\frac{\partial}{\partial x}(x+3y+z) + \frac{\partial}{\partial y}(2py-z) + \frac{\partial}{\partial z}-(xy+3z) = 0$$

$$1 + 2p - 3 = 0$$

$$2p - 2 = 0$$

$$\boxed{P = 1.}$$

$\therefore$ The value of $P = 1$ //

3. PT $\operatorname{div}(\operatorname{grad} r^m) = m(m+1)r^{m-2}$ (or) $\nabla^2(r^m) = m(m+1)r^{m-2}$

WKT $\bar{r} = \sqrt{x^2+y^2+z^2}$

$$r^2 = x^2+y^2+z^2$$

$$\nabla^2(r^m) = i\frac{\partial^2}{\partial x^2}(r^m) + j\frac{\partial^2}{\partial y^2}(r^m) + k\frac{\partial^2}{\partial z^2}(r^m)$$

$$\frac{\partial^2}{\partial x^2}(r^m) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(r^m)\right)$$

$$= \frac{\partial}{\partial x}\left(m r^{m-1}\frac{\partial r}{\partial x}\right) \qquad \because \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$= \frac{\partial}{\partial x}\left(m \cdot r^{m-1}\cdot\frac{x}{r}\right)$$

$$= \frac{\partial}{\partial x}\left(m \cdot \underbrace{r^{m-2}}_{u} \cdot \underbrace{x}_{v}\right)$$

$\{uv = uv' + u'v\}$

$$= m\left[r^{m-2}(1) - (m-2)r^{m-3}\frac{\partial r}{\partial x}x\right]$$

$$= m\left[r^{m-2} + (m-2)r^{m-4}\cdot x^2\right]$$

$$= m\left(x^2(m-2)r^{m-4} + r^{m-2}\right)$$

lly $\frac{\partial^2}{\partial y^2}(r^m) = m\left[y^2(m-2)r^{m-4} + r^{m-2}\right]$

lly $\frac{\partial^2}{\partial z^2}(r^m) = m\left[z^2(m-2)r^{m-4} + r^{m-2}\right]$

$$\nabla^2(r^m) = m\left[x^2(m-2)r^{m-4} + r^{m-2}\right] + m\left[y^2(m-2)r^{m-4} + r^{m-2}\right]$$
$$+ m\left[z^2(m-2)r^{m-4} + r^{m-2}\right]$$

$$\nabla^2(r^m) = m\left[(x^2+y^2+z^2)(m-2)r^{m-4} + 3r^{m-2}\right]$$

$$= m\left[r^2(m-2)r^{m-4} + 3r^{m-2}\right]$$

$\nabla^2(r^m) = m[(m-2)r^{m-2} + 3r^{m-2})$ [$\because$ taking r^{m-2} common]

$= m[r^{m-2}(m-2+3)]$

$= m[r^{m-2}(m+1)]$

$\therefore$ Hence proved $\nabla^2(r^m) = m[(m+1)r^{m-2}]$ //

4. PT $r^n\bar{r}$ is solenoidal if $n = -3$ //

Given $n = -3$

$\bar{f} = r^n\bar{r}$

$\bar{f} = r^{-3}\bar{r}$ $\quad \because \bar{r} = (xi + yj + zk)$

$\bar{f} = r^{-3}(xi + yj + zk)$

$\bar{f} = r^{-3}xi + r^{-3}yj + r^{-3}zk$.

For solenoidal vector $\nabla \cdot \bar{f} = 0$

$$\nabla \cdot \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(\underset{u}{r^{-3}}\underset{v}{x}) + \frac{\partial}{\partial y}(r^{-3}y) + \frac{\partial}{\partial z}(r^{-3}z)$$

$$= r^{-3}(1) + x(-3)r^{-4}\frac{\partial r}{\partial x} + r^{-3}(1) + y(-3)r^{-4}\frac{\partial r}{\partial y} + r^{-3}(1) + z(-3)r^{-4}\frac{\partial r}{\partial z}$$

$$= r^{-3} - 3r^{-4}x\left(\frac{x}{r}\right) + r^{-3} - 3r^{-4}y\left(\frac{y}{r}\right) + r^{-3} - 3r^{-4}z\left(\frac{z}{r}\right)$$

$$= 3r^{-3} - 3r^{-4}\left(\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right)$$

$$= 3r^{-3} - 3r^{-4}\left(\frac{x^2+y^2+z^2}{r}\right) \quad \{\because x^2+y^2+z^2 = r^2\}$$

$$= 3r^{-3} - 3r^{-4}\left(\frac{r^2}{r}\right) \Rightarrow 3r^{-3} - 3r^{-4}.r \quad -4+1 = -3$$

$$\nabla \cdot \bar{f} = \cancel{3r^{-3}} - \cancel{3r^{-3}} \Rightarrow 0 //$$

$\therefore \nabla \cdot \bar{f} = 0$ when $n = -3$, Hence Proved that $r^n\bar{r}$ is solenoidal when $n = -3$ //

5. Find the work done by the force $\bar{F} = (x^2 - y^2 + x)\hat{i} + (2xy + y)\hat{j}$ which moves a particle in xy plane from $(0,0)$ to $(1,1)$ along the curve $y^2 = x$

Given $\bar{F} = (x^2 - y^2 + x)\hat{i} + (2xy + y)\hat{j}$

$$\int_C \bar{F} \cdot dr = \int_C (x^2 - y^2 + x)\, dx + (2xy + y)\, dy.$$

Converting into single variable

$y^2 = x$

$dx = 2y\, dy \qquad y \to 0 \text{ to } 1$

$$\int_{y=0}^{1} F \cdot dr = \int_{y=0}^{1} ((y^2)^2 - \cancel{y^2} + \cancel{y^2})\, 2y\, dy + (2y^2 \cdot y + y)\, dy$$

$$= \int_{y=0}^{1} y^4 (2y\, dy) + 2y^3\, dy + y\, dy$$

$$= \int_{y=0}^{1} (2y^5 + 2y^3 + y)\, dy.$$

$$= \left.\frac{\cancel{2}y^6}{\cancel{6}_3} + \frac{\cancel{2}^{1} y^4}{\cancel{4}_2} + \frac{y^2}{2}\right]_0^1$$

$$= \left.\frac{y^6}{3} + \frac{y^4}{2} + \frac{y^2}{2}\right]_0^1$$

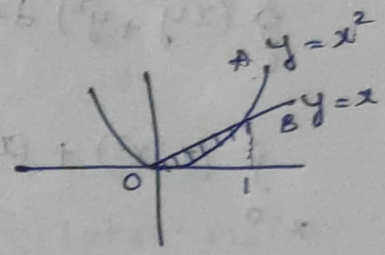
$$= \frac{1}{3} + \frac{1}{2} + \frac{1}{2}$$

$$= \frac{2+3+3}{6} \Rightarrow \frac{8}{6} \Rightarrow \frac{4}{3} \;//$$

6. Verify Green's theorem for $\int_C (xy+y^2)\,dx + x^2\,dy$

where C is bounded by $y=x$ and $y=x^2$

Green's theorem $\Rightarrow \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$



RHS $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$

Given $\int_C \underbrace{(xy+y^2)}_{M}\,dx + \underbrace{x^2}_{N}\,dy$

$y = x^2$ to x, $x \to 0$ to 1

$$\cancel{\int_{y=x^2}^{x} \int_{x=0}^{1} (xy+y^2)\,dx + x^2\,dy.}$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 2x - x - 2y$$

$$= \int_{y=x^2}^{x} \int_{x=0}^{1} (2x - x - 2y)\,dx\,dy.$$

$$= \int_{y=x^2}^{x} \int_{x=0}^{1} x - 2y\,dx\,dy.$$

$$= \int_{x=0}^{1} xy - \cancel{2}\frac{y^2}{\cancel{2}}\Big]_{x^2}^{x} dx$$

$$\int_{x=0}^{1} (x(x-x^2) - (x^2 - x^4))\,dx$$

$$\int_{x=0}^{1} (\cancel{x^2} - x^3 - \cancel{x^2} + x^4)\,dx$$

$$\left(-\frac{x^4}{4} + \frac{x^5}{5}\right]_0^1 \Rightarrow \frac{-1}{4} + \frac{1}{5} \Rightarrow \frac{-1}{20} \;//$$

LHS $\oint M\,dx + N\,dy$.

along OA $\quad y = x^2 \qquad x \to 0$ to 1

$dy = 2x\,dx$

$$\oint_C (xy + y^2)\,dx + x^2\,dy$$

$$\int_{x=0}^{1} (x(x^2) + (x^2)^2)\,dx + x^2(2x\,dx)$$

$$\int_{x=0}^{1} (x^3 + x^4 + 2x^3)\,dx$$

$$\int_{x=0}^{1} (3x^3 + x^4)\,dx$$

$$3\frac{x^4}{4} + \frac{x^5}{5}\Big]_0^1$$

$$\frac{3}{4} + \frac{1}{5} \Rightarrow \frac{19}{20} //$$

along OB $\quad y = x \qquad x \to 1$ to 0

$dy = dx$

$$\int_{x=1}^{0} (x(x) + x^2)\,dx + x^2(dx)$$

$$\int_{x=1}^{0} (x^2 + x^2 + x^2)\,dx \Rightarrow \int_{x=1}^{0} 3x^2\,dx \Rightarrow \frac{3x^3}{3}\Big]_1^0$$

$$\Rightarrow -1$$

$$OA + OB = \frac{19}{20} - 1 \Rightarrow \frac{-1}{20} //$$

LHS = RHS $\therefore$ Hence Green's theorem is verified //

7. Evalute $\iint_S \bar{F}\,\hat{n}\,ds$ where $\bar{F} = z\hat{i} + x\hat{j} - 3y^2 z\,k$ and S is the surface of the cylinder $x^2+y^2=16$ included in the first octant b/w $z=0$ & $z=5$.

Given $\bar{F} = z\hat{i} + x\hat{j} - 3y^2 z\,k$ and $\phi = x^2+y^2-16$

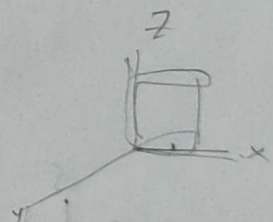
$$n = \text{Unit normal} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\nabla\phi = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x^2+y^2+16)$$

$$= 2x\hat{i} + 2y\hat{j}$$

$$n = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{(2x)^2+(2y)^2}}$$

$$n = \frac{\cancel{2}(x\hat{i}+y\hat{j})}{\cancel{2}\sqrt{x^2+y^2}} \Rightarrow \frac{x\hat{i}+y\hat{j}}{\sqrt{x^2+y^2}} \Rightarrow \frac{x\hat{i}+y\hat{j}}{\sqrt{16}} \Rightarrow \frac{x\hat{i}+y\hat{j}}{4}$$



$$\bar{F}.n = (z\hat{i}+x\hat{j}-3y^2 z k).\left(\frac{x\hat{i}+y\hat{j}}{4}\right)$$

$$= \frac{xz+xy}{4}$$

$$\iint \bar{F}n\,ds = \iint \frac{xz+xy}{4}\,\frac{dx\,dz}{|\bar{n}.\hat{j}|}$$

from surface $x^2+y^2=16$

$$y = \sqrt{16-x^2}$$

put $y=0$ $\boxed{x=4}$

given $z \to 0$ to 5

$x \to 0$ to 4

$$\iint Fn\,ds = \int_{x=0}^{4}\int_{z=0}^{5} \frac{xz+xy}{\cancel{4}} \cdot \frac{dx\,dz}{\left|\frac{x\hat{i}+y\hat{j}}{\cancel{4}}.\hat{j}\right|}$$

$$\iint F \cdot n \, ds = \int_{x=0}^{4} \int_{z=0}^{5} xz + xy \cdot \left(\frac{dx\,dz}{y}\right)$$

$$= \int_{x=0}^{4} \int_{z=0}^{5} \frac{xz + xy}{y} \, (dx\,dz)$$

$$= \int_{x=0}^{4} \int_{z=0}^{5} \left(\frac{xz}{y} + x\right) dx\,dz$$

$\therefore$ We don't have y plane so converting y terms $x^2+y^2=16$

$y = \sqrt{16-x^2}$

$$= \int_{x=0}^{4} \int_{z=0}^{5} \left(\frac{xz}{\sqrt{16-x^2}} + x\right) dx\,dz$$

Multipy & divide by -2.

$$= \int_{x=0}^{4} \int_{z=0}^{5} \left(\frac{-2x}{-2\sqrt{16-x^2}} z + x\right) dx\,dz$$

$$\left\{ \therefore \frac{f'(x)}{\sqrt{f(x)}} dx = 2\sqrt{f(x)} \right\}$$

$$= \int_{z=0}^{5} \frac{1}{-\cancel{2}} \left(\cancel{2}\sqrt{16-x^2}\right)\Big]_0^4 z + \frac{x^2}{2}\Big]_0^4 \, dz$$

$$\cancel{= \left(-\sqrt{16-x^2}\,(z) + \frac{16}{2}\right) dz}$$

$$= \int_{z=0}^{5} \left(-\sqrt{16-4^2} - \left(-\sqrt{16-0^2}\right)\right) z + \frac{4^2}{2} \, dz$$

$\frac{16}{2}$.

$$= \int_{z=0}^{5} 0 + 4z + 8 \, dz$$

$$= \frac{\cancel{4}^{2} z^2}{\cancel{2}} + 8z\Big]_0^5$$

$$= 2(5^2) + 8(5)$$

$$= 50 + 40$$

$$= 90 \;//$$

8. Verify Gauss divergence theorem for $F = 4xz\hat{i} - y^2\hat{j} + yz\hat{k}$ taken over the surface of the cube bounded by $x=0, x=a, y=0, y=a, z=0, z=a$.

Gauss divergence is $\iint_S F \cdot n\, ds = \iiint_S \text{div} \cdot \bar{F}\, dv$

$$\text{div}\, \bar{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$= 4z - 2y + y \Rightarrow 4z - y$$

$$\iiint \text{div}\, \bar{F} = \iiint (4z - y)\, dx\, dy\, dz$$

Given $x \to 0$ to a
$y \to 0$ to a
$z \to 0$ to a

$$= \int_{x=0}^{a} \int_{y=0}^{a} \int_{z=0}^{a} (4z - y)\, dx\, dy\, dz$$

$$= \int_{y=0}^{a} \int_{z=0}^{a} 4zx - yx \Big]_0^a dy\, dz$$

$$\int_{y=0}^{a} \int_{z=0}^{a} (4za - ya)\, dy\, dz$$

$$\int_{z=0}^{a} 4azy - \frac{y^2}{2} a \Big]_0^a dz \rightarrow \int_{z=0}^{a} \left(4a^2 z - \frac{a^3}{2}\right) dz$$

$$= \cancel{4}^{2} a^2 \frac{z^2}{\cancel{2}} - \frac{a^3}{2} z \Big]_0^a$$

$$= 2a^4 - \frac{a^4}{2}$$

$$= \frac{4a^4 - a^4}{2}$$

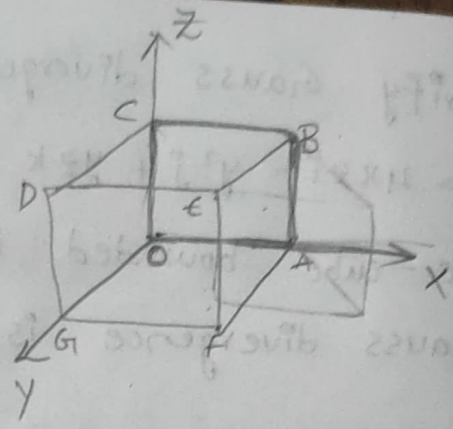
$$= \frac{3a^4}{2}$$

along OABC $y=0$

$x \to 0$ to a

$z \to 0$ to a

$n = -j$



$$\iint f\,n\,ds = \int_{x=0}^{a}\int_{y=0}^{a} -y^2 \frac{dx\,dz}{|-j \cdot j|}$$

$$= 0$$

along DEFG $y = a$ $x \to 0$ to a $n = j$

$z \to 0$ to a

$$\iint f\,n\,ds = \int_{x=0}^{a}\int_{z=0}^{a} -y^2 \frac{dx\,dz}{|j \cdot j|}$$

$$= \int_{x=0}^{a}\int_{z=0}^{a} -a^2\,dx\,dz$$

$$= -a^2\, x\Big]_0^a \cdot z\Big]_0^a$$

$$= -a^2(a)(a)$$

$$= -a^4$$

along ABEF $x = a$, $y \to 0$ to a, $z \to 0$ to a $n = i$

$$\iint f\,n\,ds = \int_{y=0}^{a}\int_{z=0}^{a} 4xz\,dy\,dz$$

$$= \int_{y=0}^{a}\int_{z=0}^{a} 4az\,dy\,dz$$

$$= \cancel{4}^{2} a \frac{z^2}{\cancel{2}}\Big]_0^a\, y\Big]_0^a$$

$$= 2a^4$$

along OCDG $x=0$ $y \to 0$ to a, $z \to 0$ to a, $n = -i$

$$\iint F \cdot n \, ds = \int_{y=0}^{a}\int_{z=0}^{a} \cdot 4\overset{\to 0}{x}z \, dy\, dz$$

$$= 0$$

along BCDE $z=a$, $x \to 0$ to a, $y \to 0$ to a, $n = k$

$$\iint F \cdot n \, ds = \int_{x=0}^{a}\int_{y=0}^{a} y\overset{\to a}{z} \, dx\, dy.$$

$$= \int_{x=0}^{a}\int_{y=0}^{a} az\, dx\, dy.$$

$$= a\left.\frac{y^2}{2}\right]_0^a \left.x\right]_0^a$$

$$= \frac{a^4}{2} \,//$$

along OAFG $z=0$, $x \to 0$ to a, $y \to 0$ to a; $n = -k$.

$$\iint F \cdot n \, ds = \int_{x=0}^{a}\int_{y=0}^{a} y\overset{\to 0}{z} \, dx\, dy$$

$$= 0 \,//$$

OABC + DEFG + ABEF + OCDG + BCDF + OAFG =

$$0 + (-a^4) + 2a^4 + 0 + \frac{a^4}{2} + 0$$

$$= a^4 + \frac{a^4}{2}$$

$$= \frac{2a^4 + a^4}{2}$$

$$= \frac{3a^4}{2}$$

$\therefore$ Hence verified that $\iint_S F \cdot n \, ds = \iiint_S \operatorname{div} F \, dv$

9. find (a) $L\{te^{-t}\sin 3t)$

$$L(tf(t)) = (-1)^1 \frac{d}{ds} f(s)$$

$$= (-1)^1 \frac{d}{ds} L(f(t)).$$

$$L(f(t)) = L(e^{-t}\sin 3t)$$

$$= \frac{3}{(s+1)^2+3^2}$$

$$L(tf(t)) = L(t.e^{-t}\sin 3t)$$

$$= (-1)\frac{d}{ds} L(ft))$$

$$\left\{\frac{u}{v}dv = \frac{u'v-uv'}{v^2}\right\}$$

$$= (-1)\frac{d}{ds}\left[\frac{3}{(s+1)^2+3^2}\right]$$

$$\frac{3(2(s+1))(1)-0}{s(s+1)}$$

$$= -1\left(\frac{0-3(2(s+1))}{((s+1)^2+3^2)^2}\right)$$

$$uv - uv'$$

$$0-3(2$$

$$L(t.e^{-t}\sin 3t) = \frac{6(s+1)}{((s+1)^2+3^2)^2}$$

9b] $\int_0^\infty \left(\frac{\cos 5t - \cos 3t}{t}\right) dt$

$$L\left(\frac{\cos 5t - \cos 3t}{t}\right) = \int_s^\infty L(\cos 5t) - L(\cos 3t)\, ds$$

$$\left\{\therefore L\left\{\frac{L(f(t))}{t}\right\} = \int_s^\infty \bar{f}(s)\, ds\right\}$$

$$= \int_s^\infty \left(\frac{s}{s^2+25} - \frac{s}{s^2+9}\right) ds$$

$$\left\{\therefore \int \frac{f'(x)}{f(x)}\, dx = \log f(x)\right\}$$

Multiply and divide by 2

$$= \frac{1}{2}\int_s^\infty \left(\frac{2s}{s^2+25} - \frac{2s}{s^2+9}\right) ds$$

$$= \frac{1}{2}\left[\log(s^2+25) - \log(s^2+9)\right]_s^\infty$$

$$= \frac{1}{2}\log\left(\frac{s^2+25}{s^2+9}\right)_s^\infty$$

$$= \frac{1}{2}\left[\log \frac{\cancel{s^2}\left(1+\frac{25}{s^2}\right)}{\cancel{s^2}\left(1+\frac{9}{s^2}\right)}\right]_s^\infty \Rightarrow \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+25}{s^2+9}\right)\right]$$

$$= \frac{1}{2}\left[0 - \frac{s^2+25}{s^2+9}\right]$$

$$= \frac{1}{2}\left(\frac{s^2+9}{s^2+25}\right)$$

$$L\left(\frac{\cos 5t - \cos 3t}{t}\right) = \underbrace{\frac{1}{2}\log\left(\frac{s^2+9}{s^2+25}\right)}_{f(s)}$$

We want $\int_0^\infty$

$$\int_0^\infty e^{-st} f(t)\, dt = f(s)$$

let $s=0$ $\quad \int_0^\infty e^{-0.t}\left(\frac{\cos 5t - \cos 3t}{t}\right) dt = \frac{1}{2}\log\left(\frac{0^2+9}{0^2+25}\right)$

$$\int_0^\infty \left(\frac{\cos 5t - \cos 3t}{t}\right) dt = \frac{1}{2}\log\left(\frac{3}{5}\right)^2$$

$$= \log\left(\frac{3}{5}\right)^{2/2}$$

$$= \log\frac{3}{5}$$

10. Using convolution theorem to evaluate

a] $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}$

$$L^{-1}(f(s).G(s)) = L^{-1}(f(s)) * L^{-1}(G(s))$$

$$= L^{-1}\left(\frac{1}{s+1}\right) * L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \underbrace{e^{-t}}_{f(t)} * \underbrace{\sin t}_{g(t)}$$

$$f(t) * g(t) = \int_{u=0}^{t} f(t-u)\, g(u)\, du$$

$$= \int_{u=0}^{t} \underbrace{e^{-(t-u)}}_{u} . \underbrace{\sin u\, du}_{v} \Rightarrow \int_{u=0}^{t} e^{(u-t)} . \sin u\, du = A$$

$$\{uv\, dx = u\int v\, dx - \int(u'\int v\, dx))\, dx\}$$

$$A = e^{(u-t)}\int_0^t \sin u\, du - \int_0^t e^{(u-t)}\int \sin u\, du\,]\, du$$

$$A = e^{(u-t)}(-\cos u)_0^t - \int_0^t \underbrace{e^{(u-t)}}_{u}\underbrace{(-\cos u)}_{v}\, du$$

$$A = e^{(u-t)}(-\cos t - (-\cos 0) + e^{(u-t)}\int_0^t \cos u\, du - \int_0^t e^{(u-t)}\int \cos u\, du$$

$$A = e^{(u-t)}(1-\cos t) + e^{(u-t)}\sin u)_0^t - \underbrace{\int_0^t e^{(u-t)}\sin u\, du}_{A}$$

$$A = e^{(u-t)}(1-\cos t) + e^{(u-t)}(\sin t) - A$$

$$2A = e^{(u-t)}(1-\cos t + e^{(u-t)}(\sin t)$$

$$A = \frac{1}{2} e^{(u-t)}(1+\sin t - \cos t) \quad //$$

b) $L^{-1}\left\{\dfrac{s}{(s^2+a^2)(s^2+b^2)}\right\}$

$$L^{-1}(F(s)\cdot G(s)) = L^{-1}(F(s)) * L^{-1}(G(s))$$

$$= L^{-1}\left(\frac{s}{s^2+a^2}\right) * L^{-1}\left(\frac{s}{s^2+b^2}\right)$$

$$= \cos at * \cos bt$$

$$f(t) * g(t) = \int_{u=0}^{t} \cos a(t-u) * \cos b(u)\, du$$

$$= \int_{u=0}^{t} \cos(at-au)\cos bu\, du$$

$$= \frac{1}{2}\int_{u=0}^{t} 2\cos(at-au)\cos bu\, du$$

{∵ multiply & divide by 2}

$\{2\cos A\cos B = \cos(A+B)+\cos(A-B)\}$

$$= \frac{1}{2}\int_0^t (\cos(at-au+bu)+\cos(at-au-bu))\, du$$

$$= \frac{1}{2}\int_0^t (\cos(at-u(a-b))+\cos(at-u(a+b)))\, du$$

$$= \frac{1}{2}\left[\frac{\sin(at-u(a-b))}{-(a-b)} + \frac{\sin(at-u(a+b))}{-(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(\cancel{at}-\cancel{at}+bt)-\sin(at)}{-(a-b)} + \frac{\sin(\cancel{at}-\cancel{at}-bt)+\sin(at)}{-(a+b)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin bt-\sin at}{b-a} + \frac{\sin(-bt)+\sin at}{-(a+b)}\right]$$

$$= \frac{1}{2}\left[\frac{\sin bt-\sin at+\sin bt-\sin at}{(b-a)(a+b)}\right]$$

$$= \frac{1}{2}\left[\frac{2\sin bt-2\sin at}{(b-a)(a+b)}\right] \Rightarrow \frac{1}{\cancel{2}}\left[\frac{\cancel{2}(\sin bt-\sin at)}{(b-a)(a+b)}\right]$$

$$= \frac{\sin bt-\sin at}{(b-a)(a+b)}$$

11. By using Laplace transform, find the solution of the following differential equations.

(a) $\frac{d^2y}{dt} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, $y(0)=0$, $y(0)=1$

Sol:- $y''(t) + 2y'(t) + 5y(t) = e^{-t}\sin t$

Apply laplace transform on both sides,

$L(y''(t) + 2y'(t) + 5y(t)) = L(e^{-t}\sin t)$

$$\left\{ \begin{array}{l} L(f''(t) = s^2F(s) - sf(0) - f'(0) \\ L(f'(t) = sF(s) - f(0) \end{array} \right\}$$

$\Rightarrow s^2Y(s) - s\overset{\to 0}{y(0)} - \overset{\to 1}{y'(0)} + 2sY(s) + 5Y(s) = \frac{1}{(s+1)^2+1}$

$\Rightarrow s^2Y(s) - 1 + 2sY(s) + 5Y(s) = \frac{1}{s^2+2s+1+1}$

$\Rightarrow s^2Y(s) + 2sY(s) + 5Y(s) - 1 = \frac{1}{s^2+2s+2}$

$Y(s)[s^2 + 2s + 5) = \frac{1}{s^2+2s+2} + 1$

$Y(s)[s^2+2s+5] = \frac{1+s^2+2s+2}{(s^2+2s+2)}$

$Y(s) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$

By reducing it into partial fractions

$\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)} = \frac{As+B}{(s^2+2s+2)} + \frac{Cs+D}{(s^2+2s+5)}$

$$\frac{s^2+2s+3}{\cancel{(s^2+2s+2)(s^2+2s+5)}} = \frac{(As+B)(s^2+2s+5)+(Cs+D)(s^2+2s+2)}{(s^2+2s+2)\cancel{(s^2+2s+5)}}$$

$$s^2+2s+3 = As^3+2As^2+5As+s^2B+2sB+5B+Cs^3+2Cs^2+2Cs + Ds^2+2Ds+2D$$

$$s^2+2s+3 = s^3(A+C)+s^2(2A+B+2C+D)+s(5A+2B+2C+2D) + 5B+2D$$

On comparing coefficients

$A+C=0$

$\boxed{A=-C}$

$\therefore A=0$

$C=0$

$2A+B+2C+D=1$

$2(A+C)+B+D=1$

$0+B+D=1$

$B=1-D$

$5A+2B+2C+2D=2$

$5A+2(B+D)+2C=2$

$5A+2(1)+2(-A)=2$

$3A+2=2$

$3A=0$

$\boxed{A=0}$

$5B+2D$

$5(1-D)+2D=3$

$5-5D+2D=3$

$-3D=-2$

$\boxed{D=\frac{2}{3}}$

$B=1-D$

$B=1-\frac{2}{3}$

$B=\frac{1}{3}$

$$Y(s) = \frac{0(s)+\frac{1}{3}}{(s^2+2s+2)} + \frac{0(s)+\frac{2}{3}}{(s^2+2s+5)}$$

Apply L^{-1} on both sides.

$$L^{-1}Y(s) = L^{-1}\left(\frac{\frac{1}{3}}{(s+1)^2+1} + \frac{2/3}{(s+1)^2+2^2}\right)$$

$$= \frac{1}{3}L^{-1}\left(\frac{1}{(s+1)^2+1}\right) + \frac{1}{3}\left(\frac{2}{(s+1)^2+2^2}\right)$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{1}{3}e^{-t}\sin 2t \Rightarrow \frac{1}{3}e^{-t}[\sin t - \sin 2t] \;//$$