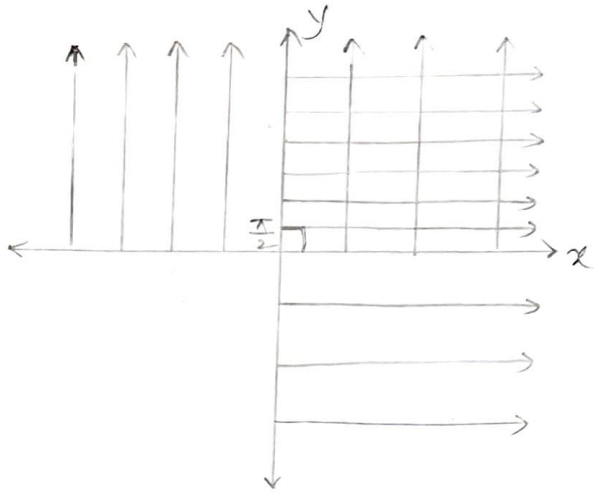
UNIT-01

1.

a) Show that $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, $m>0$, $n>0$



We know that,

$$\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1}\,dx \rightarrow (1)$$

$$\Gamma(m) = 2\int_0^\infty e^{-y^2} y^{2m-1}\,dy \rightarrow (2)$$

multiplying (1) & (2) we get

$$\Gamma(m)\,\Gamma(n) = 4\cdot\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2n-1} y^{2m-1}\,dx\,dy \rightarrow (3)$$

→ Integrand is located in first quadrant as shown in figure

→ In order to evaluate above integral expressions, change the variables from cartesian coordinates to polar coordinates

→ Transformation equations are $x = r\cos\theta$, $y = r\sin\theta$

→ squaring and adding x and y

we get, $x^2 = r^2\cos^2\theta$, $y^2 = r^2\sin^2\theta$

$$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$$
$$= r^2(\sin^2\theta + \cos^2\theta)$$
$$= r^2$$

$$dx\,dy = |J|\,dr\,d\theta \qquad [\because |J| = r]$$
$$= r\,dr\,d\theta$$

For the above region,

'r' is varying from '0' to '∞',

'θ' is varying from '0' to '$\frac{\pi}{2}$'.

$$\therefore \Gamma(m)\Gamma(n) = 4 \cdot \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-(x^2+y^2)} x^{2n-1} y^{2m-1} \, dx \, dy$$

$$= 4 \cdot \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} (r\cos\theta)^{2n-1} (r\sin\theta)^{2m-1} \, r \cdot dr \, d\theta$$

$$= 4 \cdot \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2} r^{2m+2n+1} \cos^{2n-1}\theta \sin^{2m-1}\theta \, dr \, d\theta$$

$$= 2 \cdot \int_{r=0}^{\infty} e^{-r^2} r^{2(m+n)-1} \, dr \cdot 2 \int_{\theta=0}^{\pi/2} \cos^{2n-1}\theta \sin^{2m-1}\theta \, d\theta$$

$$= \Gamma(m+n) \cdot \cancel{2} \cdot \frac{1}{\cancel{2}} \beta \cdot 2 \cdot \int_{\theta=0}^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$= \Gamma(m+n) \cdot \cancel{2} \cdot \frac{1}{\cancel{2}} \beta\left(\frac{2m-\cancel{1}+\cancel{1}}{2}, \frac{2n-\cancel{1}+\cancel{1}}{2}\right)$$

$$= \Gamma(m+n) \cdot \beta(m,n) \qquad \left[\because \int_{\theta=0}^{\pi/2} \sin^m\theta \cos^n\theta = \frac{1}{2}\beta\left(\frac{m+1}{2}, \frac{n+1}{2}\right)\right]$$

$$\therefore \Gamma(m)\Gamma(n) = \Gamma(m+n) \cdot \beta(m,n)$$

$$\boxed{\therefore \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

$\therefore$ Hence proved.

1b. Evaluate $\int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta$

$$\int_0^{\pi/2} (\cot\theta)^{1/2}\, d\theta$$

$$\int_0^{\pi/2} \frac{(\cos\theta)^{1/2}}{(\sin\theta)^{+1/2}}\, d\theta$$

$$\int_0^{\pi/2} \sin^{-1/2}\theta \cos^{1/2}\theta\, d\theta$$

$$= \frac{1}{2} \beta\left(\frac{\frac{-1}{2}+1}{2}, \frac{\frac{1}{2}+1}{2}\right)$$

$$= \frac{1}{2} \beta\left(\frac{+1}{4}, \frac{3}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{1}{4}\right) \cdot \Gamma\left(\frac{3}{4}\right)}{\Gamma\left(\frac{1}{4}+\frac{3}{4}\right)}$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)$$

$=1$

0

$(4)dy$

) prove that $\int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \cdot \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx = \frac{\pi}{4\sqrt{2}}$

consider $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx \cdot \int_0^1 \frac{1}{\sqrt{1+x^4}}\,dx$

consider $I_1 = \int_0^1 \frac{x^2}{\sqrt{1-x^4}}\,dx$

$x^2 = \sin\theta \Rightarrow x = \sqrt{\sin\theta}$

$dx = \frac{1}{2\sqrt{\sin\theta}} \cdot \cos\theta\, d\theta$

$x = 0 \rightarrow 0^2 = \sin\theta$

$\theta = 0$

$x = 1 \Rightarrow 1^2 = \sin\theta$

$\theta = \frac{\pi}{2}$

$I_1 = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \frac{1}{2\sqrt{\sin\theta}} \cdot \cos\theta\, d\theta = \int_0^{\pi/2} \frac{\sin\theta}{\sqrt{\cos^2\theta}} \cdot \frac{1}{2\sqrt{\sin\theta}} \cos\theta\, d\theta$

$= \int_0^{\pi/2} \frac{\sqrt{\sin\theta}}{2} =$

$= \frac{1}{2} \cdot \int_0^{\pi/2} \sin^{1/2}\theta \cdot \cos^0\theta\, d\theta$

$= \frac{1}{2} \times \frac{1}{2} \times 2 \int_0^{\pi/2} \sin^{\left(\frac{2}{2}\cdot\frac{3}{2}-1\right)}\theta \cos^{2\cdot\frac{1}{2}-1}\theta\, d\theta$

$= \frac{1}{4} \times 2 \times \frac{1}{2} \cdot \beta \int_0^{\pi/2} \sin^{\left(2\cdot\left(\frac{3}{4}\right)-1\right)}\theta \cos^{\left(2\cdot\frac{1}{2}-1\right)}\theta\, d\theta$

$= \frac{1}{4} \times 2 \times \frac{1}{2} \cdot \beta\left(\frac{3}{4}, \frac{1}{2}\right) = \frac{1}{4}\beta\left(\frac{3}{4}, \frac{1}{2}\right)$

Consider, $I_2 = \int_0^1 \frac{1}{\sqrt{1+x^4}} dx$

take $x^2 = \tan\theta$

$x = \sqrt{\tan\theta}$

$dx = \frac{1}{2\sqrt{\tan\theta}} \cdot \sec^2\theta d\theta$

limits $\rightarrow$ $x = 0 \Rightarrow \theta^2 = \tan\theta$

$\theta = 0$

$x = 1 \Rightarrow 1^2 = \tan\theta$

$\theta = 1$

$$I_2 = \int_0^1 \frac{1}{\sqrt{1+x^4}} dx$$

$$= \int_0^{\pi/4} \frac{1}{\sqrt{1+\tan^2\theta}} \cdot \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta \, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} \cdot \frac{1}{\sec\theta} \frac{1}{2\sqrt{\tan\theta}} \sec^2\theta d\theta$$

$$= \frac{1}{2} \cdot \int_0^{\pi/4} \frac{1}{\sqrt{\frac{\sin\theta}{\cos\theta} \times \cos^2\theta}} d\theta$$

$$= \frac{1}{2} \cdot \int_0^{\pi/4} \frac{1}{\sqrt{\frac{2\sin\theta\cos\theta}{2}}} = \frac{1}{2} \int_0^{\pi/4} \frac{\sqrt{2}}{\sqrt{\sin 2\theta}} d\theta$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}} d\theta$$

Take,

$2\theta = t$

$\theta = t/2$

$d\theta = dt/2$

$\theta = 0 \rightarrow 2 \cdot 0 = t \Rightarrow t = 0$

$\theta = \pi/4 \rightarrow 2 \cdot \left(\frac{\pi}{4}\right) = t \Rightarrow t = \pi/2$

$$\cancel{I_2 = \frac{1}{\sqrt{2}} \int_0^{\pi/4} \frac{1}{\sqrt{\sin 2\theta}}\, d\theta}$$

$$I_2 = \frac{1}{\sqrt{2}} \frac{2}{2} \int_0^{\pi/2} \frac{1}{\sqrt{\sin t}} \frac{dt}{2}$$

$$= \frac{1}{4\sqrt{2}} \, 2 \cdot \int_0^{\pi/2} \sin^{-1/2} t \cos^0 t \, dt$$

$$= \frac{1}{4\sqrt{2}} \cdot \beta\left(\frac{-1/2+1}{2}, \frac{0+1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \beta\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma\frac{1}{4} \, \Gamma\frac{1}{2}}{\Gamma\frac{1}{4}+\frac{1}{2}} = \frac{1}{4\sqrt{2}} \frac{\Gamma\frac{1}{4} \, \Gamma\frac{1}{2}}{\Gamma\frac{3}{4}}$$

$$I = I_1 \cdot I_2 = \left(\frac{\frac{1}{4} \cancel{\Gamma\frac{3}{4}} \sqrt{\pi}}{\Gamma\frac{5}{4}}\right)\left(\frac{\frac{1}{4\sqrt{2}} \Gamma\frac{1}{4} \, \Gamma\frac{1}{2}}{\cancel{\Gamma\frac{3}{4}}}\right)$$

$$= \cancel{\frac{1}{4}} \cancel{\sqrt{\pi}} = \frac{1}{4} \cdot \frac{1}{4\sqrt{2}} \frac{\sqrt{\pi}\sqrt{\pi} \, \Gamma\frac{1}{4}}{\left(\frac{5}{4}-1\right) \Gamma\frac{5}{4}-1}$$

$$= \frac{1}{\cancel{4}} \cdot \frac{1}{4\sqrt{2}} \frac{\pi \, \cancel{\Gamma\frac{1}{4}}}{\frac{1}{\cancel{4}} \cdot \cancel{\Gamma\frac{1}{4}}}$$

$$\boxed{\therefore \; I = \frac{\pi}{4\sqrt{2}}}$$

) Change the Order of integration in $\int_0^1 \int_x^{2-x} xy\,dy\,dx$

Region is bounded by curves $x=0,\ x=1$

$y=x^2,\ y=2-x$

$x+y=2$

After change of order of integration dx should be 1st differential. i.e, limits of x should be interms of y.

lets pass horizontal strip and divide the region at $y=1$,

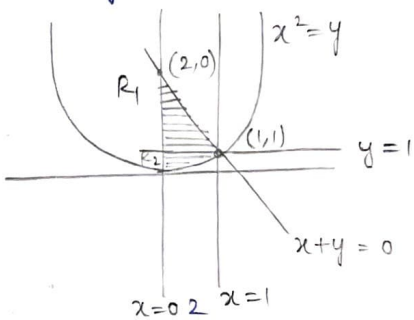
The regions are R_1 & R_2

For R_1 – strip is entering at $x=0$ and leaving at $x+y=2$,

$x=2-y$ & y varying from $y=1$ to $y=2$

For R_2 – strip is entering at $x=0$ and leaving at $x^2=y$

$x=\sqrt{y}$ & y varies from $y=0$ to $y=1$



$$I=\int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy+\int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$=\int_{y=0}^{1} y\cdot\left[\frac{x^2}{2}\right]_0^{\sqrt{y}}dy+\int_{y=1}^{2} y\cdot\left[\frac{x^2}{2}\right]_0^{2-y}dy=\int_{y=0}^{1}\frac{y}{2}\cdot(y)\,dy+\int_{y=1}^{2}\frac{y}{2}\left(4+y^2-4y\right)dy$$

$$=\frac{1}{2}\cdot\int_0^1 y^2\,dy+\frac{1}{2}\int_{y=1}^{2} 4y+y^3-4y^2$$

$$=\frac{1}{2}\left[\frac{y^3}{3}\right]_0^1+\frac{1}{2}\left[\frac{4y^2}{2}+\frac{y^4}{4}-\frac{4\cdot y^3}{3}\right]_1^2$$

$$=\frac{1}{6}+\frac{1}{2}\left[2(4-1)+\frac{1}{4}(2^4-1)-\frac{4}{3}(8-1)\right]$$

$$=\frac{1}{6}+\frac{1}{2}\left[6+\frac{15}{4}-\frac{4}{3}(7)\right]=\frac{1}{6}+\frac{5}{24}$$

$$=\frac{3}{8}$$

3) Show that $\displaystyle\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2-y^2}{x^2+y^2}\,dx\,dy = 8a^2\left(\frac{\pi}{2}-\frac{5}{3}\right)$

Given that region is bounded by curves,

$x = \frac{y^2}{4a} \Rightarrow y^2 = 4ax,\ x = y,\ y = 0,\ y = 4a$

let $x = r\cos\theta$, $y = r\sin\theta$

$dx\,dy = |J|\,dr\,d\theta$

$= r\,dr\,d\theta$

From region,

θ lies between $\frac{\pi}{4}$ to $\frac{\pi}{2}$

r lies between 0 to boundary curve

$y^2 = 4ax$

$r^2\sin^2\theta = 4ar\cos\theta$

$r = \frac{4a\cos\theta}{\sin^2\theta}$

r lies between 0 to $\frac{4a\cos\theta}{\sin^2\theta}$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\, r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \frac{\cancel{r^2}(\cos^2\theta - \sin^2\theta)}{\cancel{r^2}(\cos^2\theta + \sin^2\theta)}\, r\,dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/2}\int_{r=0}^{\frac{4a\cos\theta}{\sin^2\theta}} \cos 2\theta \cdot r \cdot dr\,d\theta$$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/2} \cos 2\theta \left[\frac{r^2}{2}\right]_0^{\frac{4a\cos\theta}{\sin^2\theta}} = \int_{\theta=\frac{\pi}{4}}^{\pi/2} \frac{\cos 2\theta}{2}\left[\frac{16a^2\cos^2\theta}{\sin^4\theta}\right]$$

$$= \int_{\theta=\frac{\pi}{4}}^{\pi/2} (1-2\sin^2\theta)\,\frac{16a^2}{2}\cdot\frac{\cos^2\theta}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\theta=\frac{\pi}{4}}^{\pi/2} \frac{(1-2\sin^2\theta)(1-\sin^2\theta)}{\sin^4\theta}\,d\theta$$

$$= 8a^2\int_{\pi/4}^{\pi/2} \frac{1-3\sin^2\theta+2\sin^4\theta}{\sin^4\theta}\,d\theta$$

$$= 8a^2\cdot\int_{\pi/4}^{\pi/2} \operatorname{cosec}^4\theta - 3\operatorname{cosec}^2\theta + 2 \quad d\theta$$

$$= 8a^2\cdot\int_{\pi/4}^{\pi/2} \operatorname{cosec}^2\theta\,(1+\cos^2\theta) - 3\operatorname{cosec}^2\theta + 2\,d\theta$$

$$= 8a^2\cdot\int_{\pi/4}^{\pi/2} \left[\cot^2\theta\operatorname{cosec}^2\theta - 2\operatorname{cosec}^2\theta + 2\right]d\theta$$

take $\cot\theta = t$

$-\operatorname{cosec}^2\theta\,d\theta = dt$

limits $\theta = \frac{\pi}{4}$, $t = 1$

$\theta = \frac{\pi}{2}$, $t = 0$

$$= 8a^2\int_{\pi/4}^{\pi/2} \cot^2\theta\operatorname{cosec}^2\theta + 2(\cot\theta)_{\pi/4}^{\pi/2} + 2(\theta)_{\pi/4}^{\pi/2}$$

$$= 8a^2 \int_{1}^{0} t^2(-dt) + 2(0-(-1)) + 2\left[\frac{\pi}{2} - \frac{\pi}{4}\right]$$

$$= 8a^2 . \left[\int_0^1 t^2 dt\right] - 2 + \frac{\pi}{2}$$

$$= 8a^2 . \left[\left[\frac{t^3}{3}\right]_0^1 - 2 + \frac{\pi}{2}\right]$$

$$= 8a^2 \left[\frac{1}{3} - 2 + \frac{\pi}{2}\right]$$

$$= 8a^2 \left[\frac{\pi}{2} - \frac{5}{3}\right]$$

$\therefore$ Hence proved.

4) Evaluate $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dy dx$ by changing to polar coordinates

let $x = r\cos\theta$

$y = r\sin\theta$

$dxdy = |J| dr d\theta$

$= r dr d\theta$

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$= r^2$

The region of integration is

$\therefore$ r limits are 0 to a

θ limits are 0 to $\frac{\pi}{2}$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 . r\, dr . d\theta$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^3 \, dr \cdot d\theta$$

$$= \int_{\theta=0}^{\pi/2} \left. \frac{r^4}{4} \right]_0^a d\theta$$

$$= \frac{a^4}{4} \cdot \int_{\theta=0}^{\pi/2} d\theta$$

$$= \frac{a^4}{4} \cdot \left[\theta \right]_0^{\pi/2}$$

$$= \frac{a^4}{4} \left[\frac{\pi}{2} - 0 \right]$$

$$= \frac{a^4}{4} \cdot \frac{\pi}{2}$$

$$= \frac{\pi a^4}{8}$$

UNIT-2

Integral calculus - 2-11

5. Evaluate

(a) $\int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dx\,dy\,dz$

$$I = \int_{-1}^{1}\int_{0}^{z}\int_{x-z}^{x+z} (x+y+z)\, dx\,dy\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z}\int_{y=x-z}^{x+z} (x+y+z)\, dy\,dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} \left[xy + \frac{y^2}{2} + zy\right]_{x-z}^{x+z} dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} \left[x(x+z) + \frac{(x+z)^2}{2} + z(x+z)\right] - \left[x(x-z) + \frac{(x-z)^2}{2} + z(x-z)\right] dx\,dz$$

$$= \int_{z=-1}^{1}\int_{x=0}^{z} x^2 + zx + \frac{(x+z)^2}{2} + zx + z^2 - x^2 + \cancel{xz} - \frac{(x-z)^2}{2} - \cancel{zx} + z^2$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} 2zx + \frac{x^2}{2} + \frac{z^2}{2} + \frac{2zx}{2} + 2z^2 - \frac{x^2}{2} - \frac{z^2}{2} + \frac{2zx}{2}$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} (2zx + 2z^2 + 2xz)\, dx\, dz$$

$$= 2\int_{z=-1}^{1} \int_{x=0}^{z} \left[(2zx + z^2)\, dx\right] dz$$

$$= 2\int_{z=-1}^{1} \left[2z \cdot \frac{x^2}{2} + z^2 x\right]_0^z dz$$

$$= 2\int_{z=-1}^{1} \frac{2z}{2}[z^2 - 0] + z^2[z-0]\, dz$$

$$= 2\int_{z=-1}^{1} z^3 + z^3\, dz = 2\int_{z=-1}^{1} 2z^3\, dz$$

$$= 4 \cdot \left[\frac{z^4}{4}\right]_{-1}^{1}$$

$$= [-1+1]$$

$$= 0 \quad //$$

(6.) $\int_0^1\int_1^2\int_2^3 (x+y+z)\,dx\,dy\,dz$

$$I = \int_0^1\int_1^2\int_2^3 (x+y+z)\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=1}^{2}\int_{x=2}^{3} (x+y+z)\,dx\,dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=1}^{2} \left[\frac{x^2}{2} + xy + zx\right]_2^3 dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=1}^{2} \left[\frac{3^2}{2} + 3y + 3z\right] - \left[\frac{2^2}{2} + 2y + 2z\right] dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=1}^{2} \left[\frac{9}{2} + 3y + 3z - \frac{4}{2} - 2y - 2z\right] dy\,dz$$

$$= \int_{z=0}^{1}\int_{y=1}^{2} \left[\frac{5}{2} + y + z\right] dy\,dz$$

$$= \int_{z=0}^{1} \left[\frac{5}{2}y + \frac{y^2}{2} + zy\right]_1^2 dz$$

$$= \int_{z=0}^{1} \left[\frac{5}{2}y + \frac{y^2}{2} + zy \right]_1^2 dz$$

$$= \int_{z=0}^{1} \frac{5}{2}(2-1) + \frac{1}{2}(4-1) + z(2-1) \, dz$$

$$= \int_{z=0}^{1} \frac{5}{2} + \frac{3}{2} + z \, dz$$

$$= \int_{z=0}^{1} 4 + z \, dz$$

$$= \left[4z + \frac{z^2}{2} \right]_0^1$$

$$= 4(1-0) + \frac{1}{2}(1-0)$$

$$= 4 + \frac{1}{2}$$

$$= \frac{9}{2}$$

6. Find the volume of the surface in first octant and bounded by $2x+3y+4z=12$

Given that,

The volume of the surface is located in first octant and bounded by the line $2x+3y+4z=12$.

$\therefore$ x limits are 0 to $\frac{(12-3y-4z)}{2}$

y limits are 0 to $\frac{12-4z}{3}$

z limits are 0 to $\frac{12}{4} = 3$

$$\therefore \iiint_V dx\,dy\,dz = \int_{z=0}^{3}\int_{y=0}^{\frac{12-4z}{3}}\int_{x=0}^{\frac{12-3y-4z}{2}} dx\,dy\,dz$$

$$= \int_{z=0}^{3}\int_{y=0}^{\frac{12-4z}{3}} [x]_0^{\frac{12-3y-4z}{2}}\,dy\,dz$$

$$= \int_{z=0}^{3} \int_{y=0}^{\frac{12-4z}{3}} \left[\frac{1}{2} \left[12 - 3y - 4z - 0 \right] dy \right] dz$$

$$= \int_{z=0}^{3} \int_{y=0}^{\frac{12-4z}{3}} \left[6 - \frac{3}{2} y - 2z \right] dy \cdot dz$$

$$= \int_{z=0}^{3} \left[6y - \frac{3}{2} \cdot \frac{y^2}{2} - 2zy \right]_0^{\frac{12-4z}{3}} dz$$

$$= \int_{z=0}^{3} 6 \left[\frac{12-4z}{3} - 0 \right] - \frac{3}{4} \left[\frac{12-4z}{3} \right]^2 - 2z \left[\frac{12-4z}{3} - 0 \right] dz$$

$$= \int_{z=0}^{3} 24 - 8z - \frac{\cancel{3}}{4} \left[\frac{144 + 16z^2 - 96z}{9_3} \right] - \left[\frac{24z - 8z^2}{3} \right] dz$$

$$= \int_{z=0}^{3} 24 - 8z - \frac{144}{12} - \frac{\overset{4}{\cancel{16}} z^2}{\cancel{12}_3} + \frac{\overset{8}{\cancel{96}} z}{\cancel{12}} - \frac{\overset{8}{\cancel{24}} z}{\cancel{3}} + \frac{8z^2}{3} dz$$

$$= \int_{z=0}^{3} 24 - 8z - 12 - \frac{4z^2}{3} + 8z - 8z + \frac{8z^2}{3} dz$$

$$= \int_{z=0}^{3} 12 - 8z + \frac{4z^2}{3} \, dz$$

$$= \left[12z - \frac{8z^2}{2} + \frac{4}{3} \cdot \frac{z^3}{3} \right]_0^3$$

$$= 12(3) - 4(9-0) + \frac{4}{9}(3^3 - 0)$$

$$= 36 - 36 + \frac{4}{\cancel{9}} \cdot \overset{3}{\cancel{27}}$$

$$= 4 \times 3$$

$$= 12$$

7. Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing into spherical co-ordinates.

Given that,

$z=0$ to $z=\sqrt{1-x^2-y^2} \Rightarrow x^2+y^2+z^2=1$

The volume of integration is located in the first octant.

Transformation equations from cartesian co-ordinates to special co-ordinates are given by

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$dx\,dy\,dz = r^2\sin\theta\,dr\,d\theta\,d\phi$

let us consider that

$$x^2+y^2+z^2 = r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta$$

$$= r^2(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta)$$

$$= r^2(\sin^2\theta(\cos^2\phi + \sin^2\phi) + \cos^2\theta)$$

$$= r^2(\sin^2\theta(1) + \cos^2\theta)$$

$$= r^2(1)$$

$$= r^2$$

1
0

for the given volume

r limits are 0 to 1

θ limits are 0 to $\pi/2$

ϕ limits are 0 to $\pi/2$

Then,

$$I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{r=0}^{1} \frac{r^2 \sin\theta \, dr \, d\theta \, d\phi}{\sqrt{1-r^2}}$$

when, let $r = \sin t$

$\Rightarrow dr = \cos t \cdot dt$

when $r = 0$, $t = 0$

$r = 1$, $t = \pi/2$

$$\therefore I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{t=0}^{\pi/2} \frac{\sin^2 t}{\sqrt{1-\sin^2 t}} \sin\theta \cos t \, dt \, d\theta \, d\phi$$

$$I = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{t=0}^{\pi/2} \frac{\sin^2 t}{\cos t} \cdot \cos t \sin\theta \, dt \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{t=0}^{\pi/2} \frac{1-\cos 2t}{2} \sin\theta \, dt \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{0=t}^{\pi/2} \frac{1}{2}\left[t - \frac{\cos 2t}{2}\right] dt \sin\theta \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{\pi/2} \sin\theta \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{1}{2}\left[\frac{\pi}{2} - 0\right] - \left[0 - 0\right] \sin\theta \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \frac{\pi}{4} \sin\theta \, d\theta \, d\phi$$

$$= \int_{\phi=0}^{\pi/2} \frac{\pi}{4} \left[-\cos\theta\right]_0^{\pi/2} d\phi$$

$$= \int_{\phi=0}^{\pi/2} \frac{\pi}{4} \left[-\cos \pi/2 + \cos 0\right] d\phi$$

$$= \int_{\phi=0}^{\pi/2} \frac{\pi}{4} \, d\phi$$

$$= \left[\frac{\pi}{4}\phi\right]_0^{\pi/2}$$

$$= \frac{\pi}{4} \cdot \frac{\pi}{2} = \frac{\pi^2}{8} \quad //$$

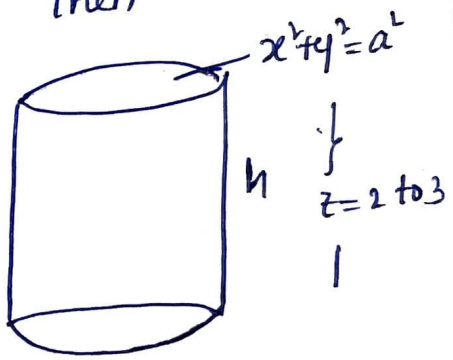
8. Changing into cylindrical co-ordinates, evaluate circular cylinder $\iiint_D z(x^2+y^2)\,dx\,dy\,dz$, D: $x^2+y^2 \le 1$, $2 \le z \le 3$.

Given that,

The volume of the integration is bounded by a cylinder $x^2+y^2=a^2$, $z=2$ and $z=3$.

Transformation equations from cartesian co-ordinates to cylindrical co-ordinate system are given by,

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$ then



$dx\,dy\,dz = |J|\,dr\,d\theta\,dz$

$= r\,dr\,d\theta\,dz$

For the given volume of cylinder,

The 'z' limits are 2 to 3.

'θ' limits are 0 to 2π

'r' limits are 0 to 1

$x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2$

$$\therefore I = \int_{z=2}^{3} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} z(x^2+y^2)\, r\, dr\, d\theta\, dz$$

$$I = \int_{z=2}^{3} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} z \cdot (r^2)\, r\, dr\, d\theta\, dz$$

$$= \int_{z=2}^{3} \int_{\theta=0}^{2\pi} z \left[\frac{r^4}{4}\right]_0^1 d\theta\, dz$$

$$= \int_{z=2}^{3} \int_{\theta=0}^{2\pi} \frac{z}{4}[1-0]\, d\theta\, dz$$

$$= \int_{z=2}^{3} \int_{\theta=0}^{2\pi} \frac{z}{4}\, d\theta\, dz$$

$$= \int_{z=2}^{3} \frac{z}{4}[\theta]_0^{2\pi}\, dz$$

$$= \int_{z=2}^{3} \frac{z}{4}[2\pi-0]\, dz = \int_{z=2}^{3} \frac{z}{4}(2\pi)$$

$$= \int_{z=2}^{3} \frac{z}{2}\pi\, dz$$

$=$

$$= \frac{\pi}{2} \int_{z=2}^{3} z \, dz$$

$$= \frac{\pi}{2} \left[\frac{z^2}{2} \right]_2^3$$

$$= \frac{\pi}{2 \cdot 2} \left[3^2 - 2^2 \right]$$

$$= \frac{\pi}{4} [9 - 4]$$

$$= \frac{5\pi}{4}$$

UNIT - III

Vector Differentiation

9a. find the directional derivative of the function $f = 4xy^2 + 2x^2yz$ at the point $P(1,2,3)$ in the direction of PQ where $Q(5,0,4)$

Let $\phi(x,y,z) = 4xy^2 + 2x^2yz$

Consider Grad $\phi = \nabla\phi$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\phi$$

$$= i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$= i(4y^2 + 4xyz) + j(8xy + 2x^2z) + k(2x^2y)$$

Grad $\phi = \nabla\phi$ at point $P(1,2,3)$

$$\nabla\phi\Big|_{(1,2,3)} = i(4(4) + 4(1)(2)(3)) + j(8(1)(2) + 2(1)^2(3)) + k(2(1)(2))$$

$$= i(40) + j(22) + k(4)$$

$$= 40i + 22j + 4k$$

Consider vector $\overline{PQ} = \overline{OQ} - \overline{OP}$

$$= 5i + 0j + 4k - (i + 2j + 3k)$$

$$= 4i - 2j + k$$

$$= \bar{a} \text{ (Say)}$$

Directional derivative of ϕ in the direction of vector $\bar{a}$ is

$$\nabla\phi \cdot \frac{\bar{a}}{|\bar{a}|} = (40i+22j+4k) \cdot \frac{(4i-2j+k)}{\sqrt{16+4+1}}$$

$$= \frac{160-44+4}{\sqrt{21}}$$

$$= \frac{120}{\sqrt{21}}$$

9b. find the angle between the surfaces $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the point $(2,-1,2)$.

Let $\phi_1 = x^2+y^2+z^2-9$, $\phi_2 = x^2+y^2-z-3$

Normal to ϕ_1 is given by

$$\nabla\phi_1 = i\frac{\partial\phi_1}{\partial x} + j\frac{\partial\phi_1}{\partial y} + k\frac{\partial\phi_1}{\partial z}$$

$$= i(2x) + j(2y) + k(2z)$$

Normal to ϕ_1 at $P(2,-1,2)$ is

$$\nabla\phi_1\Big|_{(2,-1,2)} = 2(2)i + j(2(-1)) + k(2)(2)$$

$$= 4i-2j+4k$$

$$= \bar{a} \text{ (Say)}$$

Normal to ϕ_2 is given by

$$\nabla \phi_2 = i\frac{\partial \phi_2}{\partial x} + j\frac{\partial \phi_2}{\partial y} + k\frac{\partial \phi_2}{\partial z}$$

$$= i(2x) + j(2y) + k(-1)$$

$$= 2xi + 2yj - k$$

Normal at $(2,-1,2)$

$$\nabla \phi_2 \Big|_{(2,-1,2)} = i(2)(2) + j(2)(-1) - k$$

$$= 4i - 2j - k$$

$$= \bar{b} \text{ (Say)}$$

Let θ be the angle between two surfaces ($\bar{a}$ & $\bar{b}$)

$$\cos\theta = \frac{\bar{a}\cdot\bar{b}}{|\bar{a}|\cdot|\bar{b}|}$$

$$= \frac{(4i-2j+4k)\cdot(4i-2j-k)}{\sqrt{16+4+16}\,\sqrt{16+4+1}}$$

$$= \frac{16+4-4}{\sqrt{36}\,\sqrt{21}}$$

$$= \frac{16}{6\sqrt{21}}$$

$$= \frac{8}{3\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

10. Prove that $r^n\bar{r}$ is Solenoidal if $n=-3$.

Let $\bar{r} = xi + yj + zk$

$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$. ——— (1)

Differentiating with x partially

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Consider $\operatorname{div} r^n\bar{r} = \nabla \cdot r^n\bar{r}$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(r^n xi + r^n yj + r^n zk)$$

$$= \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$

$$= r^n.1 + xnr^{n-1}\frac{\partial r}{\partial x} + r^n.1 + ynr^{n-1}\frac{\partial r}{\partial y} + r^n.1 + znr^{n-1}\frac{\partial r}{\partial z}$$

$$= 3r^n + nr^{n-1}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= 3r^n + nr^{n-1}\left[x\cdot\frac{x}{r} + y\cdot\frac{y}{r} + z\cdot\frac{z}{r}\right]$$

$$= 3r^n + n\frac{r^{n-1}}{r}\left[x^2+y^2+z^2\right]$$

$$= 3r^n + nr^{n-2}(r^2)$$

$= 3r^n + nr^n$

$= (n+3)r^n$

If $n = -3$

$\text{div } r^n \cdot \bar{r} = (-3+3)r^{-3}$

$= 0$

$\therefore r^n \bar{r}$ is Solenoidal if $n = -3$

11a. find the constants a, b, c so that the vector $A = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$ is irrotational.

Given that,

$\bar{f} = (x+2y+az)i + (bx-3y-z)j + (4x+cy+2z)k$

Also vector $\bar{f}$ is irrotational

$\therefore \text{Curl } \bar{f} = \bar{0}$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix} = \bar{0}$$

$= i(c-(-1)) - j(4-a) + k(b-2) = 0i + 0j + 0k$

$= (c+1)i + j(a-4) + k(b-2) = 0i + 0j + 0k$

On Comparing i, j, k coefficients.

$a-4=0$, $b-2=0$, $c+1=0$

$\boxed{a=4}$ $\boxed{b=2}$ $\boxed{c=-1}$

b. If $\bar{F} = (x+3y+z)i + (2py-z)j - (xy+3z)k$ is solenoidal, then find p

Given that,

$$\bar{F} = (x+3y+z)i + (2py-z)j - (xy+3z)k$$

is Solenoidal

$\therefore$ div $\bar{F} = 0$

$$= \nabla \cdot \bar{F}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)(x+3y+z)i + (2py-z)j - (xy+3z)k$$

$$= \frac{\partial}{\partial x}(x+3y+z) + \frac{\partial}{\partial y}(2py-z) + \frac{\partial}{\partial z}(-xy-3z) = 0$$

$$= 1 + 2p - 3 = 0$$

$$2p - 2 = 0$$

$$\boxed{p = 1}$$

12. Prove that div. (grad r^m) $= m(m+1)r^{m-2}$

(or)

$$\nabla^2(r^m) = m(m+1)r^{m-2}$$

Let $\bar{r} = xi + yj + zk$

$$r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^2 = x^2+y^2+z^2 \quad \text{—— (1)}$$

Differentiate eqn (1) w.r.t x partially

$$2r\frac{\partial r}{\partial x} = 2x$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}$$

Illy; $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

Consider (grad r^m) $= i\frac{\partial}{\partial x}r^m + j\frac{\partial}{\partial y}r^m + k\frac{\partial}{\partial z}r^m$

$$= i\, mr^{m-1}\frac{\partial r}{\partial x} + j \cdot mr^{m-1}\frac{\partial r}{\partial y} + k\, mr^{m-1}\frac{\partial r}{\partial z}$$

$$= mr^{m-1}\left[i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}\right]$$

$$= mr^{m-1}\left[i \cdot \frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right]$$

$$= mr^{m-1-1}\left[xi + yj + zk\right]$$

$$= mr^{m-2}\,\bar{r}$$

Consider div (grad r^m)

$$= \text{div} \cdot m r^{m-2} \bar{r}$$

$$= \nabla \cdot m r^{m-2} \bar{r}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot m r^{m-2} (xi + yj + zk)$$

$$= \frac{\partial}{\partial x}(m r^{m-2} x) + \frac{\partial}{\partial y}(m r^{m-2} y) + \frac{\partial}{\partial z}(m r^{m-2} z)$$

$$= m\left[\left(x \cdot (m-2) r^{m-3} \frac{\partial r}{\partial x} + r^{m-2} \right) + \left(y (m-2) r^{m-3} \frac{\partial r}{\partial y} + r^{m-2} \right) \right.$$

$$\left. + z (m-2) r^{m-3} \frac{\partial r}{\partial z} + r^{m-2} \right]$$

$$= m\left[3r^{m-2} + (m-2) r^{m-3} \left[x \cdot \frac{\partial r}{\partial x} + y \cdot \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z} \right] \right]$$

$$= m\left[3r^{m-2} + (m-2) r^{m-3} \left[x \cdot \frac{x}{r} + y \cdot \frac{y}{r} + z \cdot \frac{z}{r} \right] \right]$$

$$= m\left[3r^{m-2} + (m-2) \frac{r^{m-3}}{r} (x^2 + y^2 + z^2) \right]$$

$$= m\left[3r^{m-2} + (m-2) r^{m-4} (r^2) \right]$$

$$= m\left[3r^{m-2} + (m-2) r^{m-2} \right]$$

$$= m \cdot r^{m-2} (3 + m - 2)$$

$$= m(m+1) r^{m-2}$$

$$= RHS$$

Hence Proved.