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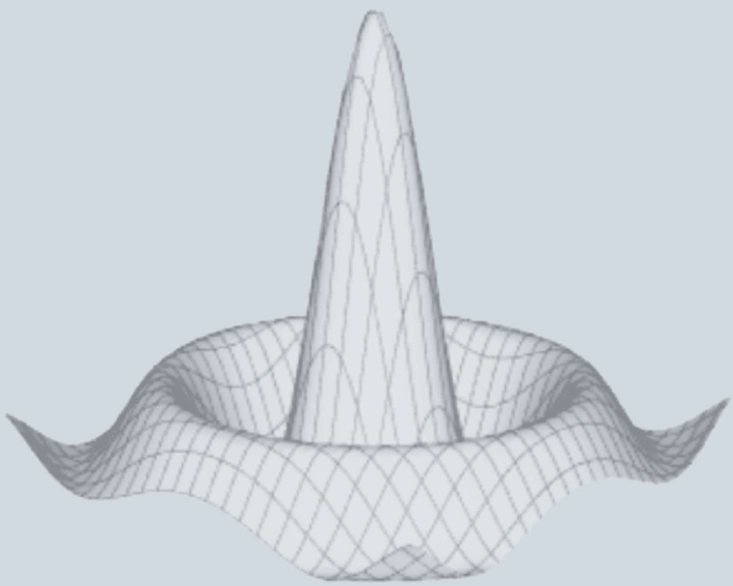

$$\frac{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}} = \left(\frac{1}{1+}\frac{q}{1+}\frac{q^2}{1+}\frac{q^3}{1+}\frac{q^4}{1+\ldots}\right)^3.$$

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Professor Remy Y. Denis started his career as a lecturer in the Department of Mathematics, St. Andrews College, Gorakhpur in July 1963 but soon joined the Department of Mathematics, Gorakhpur University, Gorakhpur in September 1963.

He completed his Ph.D. degree in 1969 and D.Sc. degree in 1994. He has published over 150 research papers on Basic as well as Ordinary Hypergeometric functions, Number Theory, Graph Theory, Ring Theory and Lie Algebra. He has collaborated with many national and foreign experts in various projects and has completed over fifteen research projects sanctioned by UGC, CSIR and DST. He has guided 25 students successfully for Ph.D. degree and two students for D.Sc. degree. He has written two books on Theory of functions of a Complex Variable (1975), Complex Variables & Integral Transforms (1988) and edited two Proceedings, Proceeding of the National Symposium on Special Functions and their applications (1986) and Proceeding of the Second International Conference of SSFA (2001). He has delivered invited talks in the symposiums on special functions and their applications organized during different annual conferences of IMS, Science Congress and others. He has visited several premier research Institutes as IIT Madras, Institute of Mathematical Sciences, Chennai, Ramanujan Mathematics Institute Chennai, AMU, BHU, Chandigarh, Calcutta Mathematical Society and Calcutta University, Tripura Central University, Baroda University, Cochin University of Sciences and Technology, Institute of Mathematical Sciences, Pala, Kerala etc.

He has delivered a series of video undergraduate level class room lectures under a U G C project which have been telecast on the national channel of Door Darshan class room teaching programme. The most remarkable contribution of Prof. Denis is the generalization of most of the results of Ramanujan on continued fractions and q- series. The generalization of the most remarkable result of Ramanujan involving Lambert Series and continued fraction is also credited to Prof. Denis. This fact has been duly acknowledged by Prof .George Andrews and Prof Bruce Berndt in their recently published book "Lost Notebook of Ramanujan ,Part I p.116-117,Springer (2005)".

Prof. Denis is the President of "Indian Society of Mathematics and Mathematical Sciences", Patron in Chief of the "South East Asian Journal of Mathematics and Mathematical Sciences", member of the Editorial Board of several national and international journals. He has published more than 250 Letters, Comments and articles relating to National Integration, Education, Social reforms and Social justice based on Social issues and Human rights in National Newspapers, Bulletins and Journals. Prof. Denis has been the National President of "All India Catholic Union" (2008-2013). He has been member of Mathematical Sciences Panel of UGC (1997-2000), Commission of Education and Culture, Catholic Bishops Conference of India CBCI, New Delhi (1987-1997), Minorities Commission UP (1987-1990), Executive Council and Board of Governors, Allahabad Agricultural Institute Deemed University (2000-2004). Prof. Denis has worked as the Director, U.G.C Coaching Center for Minorities at Gorakhpur University (1992-1999), Organizing Secretary of National Symposium on Special Functions and their Applications held at Gorakhpur University (1986), Organizing Secretary, East Zone All India Inter Varsity Hockey Tournament 1979, Communication In charge, Ramanujan Birth Centenary Annual Conference IMS 1987.

Prof. Denis besides being an excellent academician has been an outstanding football player and represented Gorakhpur University. He is also a very good management and leadership trainer.

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CONTENTS

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NEW GENERALIZED $\alpha - \psi-$GERAGHTY CONTRACTION TYPE MAPS AND FIXED POINTS

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Abstract: In this paper, we introduce the notion of new generalized $\alpha - \psi-$ Geraghty contraction type maps in the context of metric space and establish some fixed point theorems for such maps. This new contraction map is motivated by the different Geraghty contraction type maps introduced by many authors over the years. An example is also given to illustrate our result.

Keywords and Phrases: Metric space, fixed point, generalized $\alpha-$Geraghty contraction type map, generalized $\alpha - \psi-$Geraghty contraction type map, extended generalized $\alpha - \psi-$Geraghty contraction type map, new generalized $\alpha - \psi-$Geraghty contraction type map.

2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

In 1973, Geraghty [5] generalized the Banach contraction principle in the setting of a complete metric space by considering an auxiliary function. This important result of Geraghty was further generalized and improved upon by the works of many authors namely Amini-Harandi & Emami [1], Caballero et al. [3] and Gordji et al. [6] etc. In 2012, Samet et al.[17] defined the notion of $\alpha-\psi-$contractive mappings and obtained remarkable fixed point results. Inspired by this notion of $\alpha-\psi-$contractive mappings, Karapinar & Samet [9] introduced the concept of generalized $\alpha-\psi-$contractive mappings and obtained fixed point results for such mappings. In 2013, Cho et al. [4] defined the concept of generalized $\alpha-$Geraghty contraction type maps in the setting of a metric space and proved the existence and uniqueness of a fixed point of such maps. Further as generalizations of the type of maps defined by Cho et al. [4], Erdal Karapinar [10] introduced the concept of generalized $\alpha-\psi-$Geraghty contraction type maps and proved fixed point results generalizing the results obtained by Cho et al.[4]. Recently, in 2014, Popescu [15] generalized the results of Cho et al. [4] and gave other conditions for the existence and uniqueness of a fixed point of $\alpha-$Geraghty contraction type maps. Then, very recently K. Anthony Singh [7] introduced extended generalized $\alpha-\psi-$ Geraghty contraction type maps and proved some fixed point results generalizing the results of Popescu [15].

In this paper, motivated by the different Geraghty contraction type maps introduced by many authors, we define new generalized $\alpha-\psi-$Geraghty contraction type maps in the setting of metric space and obtain the existence and uniqueness of a fixed point of such maps. We also give an example to illustrate our result.

2. Preliminaries

In this section, we recall some basic definitions and related results on the topic in the literature.

Let $\mathcal{F}$ be the family of all functions $\beta : [0,\infty) \to [0,1)$ which satisfies the condition

$$\lim_{n\to\infty} \beta(t_n) = 1 \quad \text{implies} \quad \lim_{n\to\infty} t_n = 0.$$

By using such a map, Geraghty proved the following interesting result.

Theorem 2.1. [5] *Let (X,d) be a complete metric space and T a mapping on X. Suppose that there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$d(Tx,Ty) \le \beta(d(x,y))d(x,y).$$

Then T has a unique fixed point $x_* \in X$ and $\{T^n x\}$ converges to x_* for each $x \in X$.

Definition 2.2. [17] *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be $\alpha-$admissible if $\alpha(x,y) \ge 1$ implies $\alpha(Tx,Ty) \ge 1$.*

Definition 2.3. [8] *A map $T : X \to X$ is said to be triangular $\alpha-$admissible if*

(T1) T is $\alpha-$admissible,
(T2) $\alpha(x,z)\geq 1$ and $\alpha(z,y)\geq 1$ imply $\alpha(x,y)\geq 1$.

Lemma 2.4. [8] *Let $T:X\to X$ be a triangular $\alpha-$admissible map. Assume that there exists $x_1\in X$ such that $\alpha(x_1,Tx_1)\geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1}=Tx_n$. Then we have $\alpha(x_n,x_m)\geq 1$ for all $m,n\in\mathbb{N}$ with $n<m$.*
Cho et al. [4] introduced the following contraction and proved some interesting fixed point results generalizing many results in the existing literature.

Definition 2.5. [4] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized $\alpha-$Geraghty contraction type map if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)d(Tx,Ty)\leq\beta(M(x,y))M(x,y),$$

where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$.
Erdal Karapinar [10] defined the following class of auxiliary functions.
Let Ψ denote the class of functions $\psi:[0,\infty)\to[0,\infty)$ which satisfy the following conditions:
(a) ψ is nondecreasing;
(b) ψ is subadditive, that is, $\psi(s+t)\leq\psi(s)+\psi(t)$;
(c) ψ is continuous;
(d) $\psi(t)=0\Leftrightarrow t=0$.
Erdal Karapinar [10] also introduced the following contraction and proved some interesting fixed point results generalizing the results of Cho et al.[4].

Definition 2.6. [10] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized $\alpha-\psi-$Geraghty contraction type mapping if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty))\leq\beta(\psi(M(x,y)))\psi(M(x,y)),$$

where $M(x,y)=\max\{d(x,y),d(x,Tx),d(y,Ty)\}$ and $\psi\in\Psi$.
Popescu [15] extended the notion of generalized $\alpha-$Geraghty contraction type map and gave the following definition.

Definition 2.7. [15] *Let (X,d) be a metric space and $\alpha:X\times X\to\mathbb{R}$ be a function. A map $T:X\to X$ is called a generalized $\alpha-$Geraghty contraction type map if there exists $\beta\in\mathcal{F}$ such that for all $x,y\in X$,*

$$\alpha(x,y)d(Tx,Ty)\leq\beta(M_T(x,y))M_T(x,y),$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$.
K. Anthony Singh [7] further introduced the following contraction and proved some fixed point results generalizing the results of Popescu [15].

Definition 2.8. [7] *Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. A map $T : X \to X$ is called an extended generalized $\alpha - \psi-$Geraghty contraction type map if there exists $\beta \in \mathcal{F}$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty)) \le \beta(\psi(M_T(x,y)))\psi(M_T(x,y)),$$

where $M_T(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ *and* $\psi \in \Psi$.

3. Main Results

We now state and prove our main results. Here we introduce the following new definitions. The new contraction map defined below is motivated by the different Geraghty contraction type maps introduced by many authors as in the above section 2.
Let Ω be the family of all functions $\theta : [0,\infty) \to [0,1]$ which satisfy the following conditions
(1) $\theta(t) < 1$ for $t > 0$, and
(2) $\lim_{n\to\infty} \theta(t_n) = 1$ implies $\lim_{n\to\infty} t_n = 0$.

Remark 3.1. *Here instead of the family $\mathcal{F}$ we are introducing a slightly extended family Ω.*

Definition 3.2. *Let (X,d) be a metric space and $\alpha : X \times X \to \mathbb{R}$ be a function. Then the mapping $T : X \to X$ is called a new generalized $\alpha - \psi-$Geraghty contraction type map if there exists $\theta \in \Omega$ such that for all $x, y \in X$,*

$$\alpha(x,y)\psi(d(Tx,Ty)) \le \theta(\psi(N(x,y)))\psi(N(x,y)),$$

where $N(x,y) = \max\left\{d(x,y), \frac{d(x,Tx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Tx)}{2}\right\}$ *and* $\psi \in \Psi$.

Theorem 3.3. *Let (X,d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is a new generalized $\alpha - \psi-$Geraghty contraction type map,
(ii) T is triangular $\alpha-$admissible,
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \ge 1$,
(iv) T is continuous.
Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.
Proof. Let $x_1 \in X$ be such that $\alpha(x_1, Tx_1) \ge 1$. We construct a sequence of

points $\{x_n\}$ in X by $x_{n+1} = Tx_n$ for $n \in \mathbb{N}$. If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is clearly a fixed point of T and the proof is complete. Hence, we suppose that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$.
By hypothesis, $\alpha(x_1, x_2) \geq 1$ and T is triangular $\alpha-$admissible. Therefore by Lemma 2.4., we have

$$\alpha(x_n, x_{n+1}) \geq 1 \quad \text{for all} \quad n \in \mathbb{N}.$$

Then we have

$$\begin{aligned} \psi(d(x_{n+1}, x_{n+2})) &= \psi(d(Tx_n, Tx_{n+1})) \leq \alpha(x_n, x_{n+1})\psi(d(Tx_n, Tx_{n+1})) \\ &\leq \theta(\psi(N(x_n, x_{n+1})))\psi(N(x_n, x_{n+1})) \quad \text{for all} \quad n \in \mathbb{N}. \quad (1) \end{aligned}$$

Here we have

$$\begin{aligned} N(x_n, x_{n+1}) &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})}{2}, \frac{d(x_n, Tx_{n+1}) + d(x_{n+1}, Tx_n)}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n, x_{n+2}) + d(x_{n+1}, x_{n+1})}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}, \frac{d(x_n, x_{n+2})}{2}\right\} \\ &= \max\left\{d(x_n, x_{n+1}), \frac{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})}{2}\right\} \end{aligned}$$

If $d(x_{n+1}, x_{n+2}) \geq d(x_n, x_{n+1})$, then $N(x_n, x_{n+1}) \leq d(x_{n+1}, x_{n+2})$.
Now from (1) and the definition of θ, we have

$$\psi(d(x_{n+1}, x_{n+2})) < \psi(d(x_{n+1}, x_{n+2}),$$

which is a contradiction.
Therefore, we have

$$d(x_{n+1}, x_{n+2}) < d(x_n, x_{n+1}) \quad \text{for all} \quad n \in \mathbb{N}.$$

Thus the sequence $\{d(x_n, x_{n+1})\}$ is nonnegative and nonincreasing and also by then we have $N(x_n, x_{n+1}) = d(x_n, x_{n+1})$.
Now, we prove that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$.
It is clear that $\{d(x_n, x_{n+1})\}$ is a decreasing sequence which is bounded from below. Therefore there exists $r \geq 0$ such that $\lim\limits_{n\to\infty} d(x_n, x_{n+1}) = r$. We show that $r = 0$. We suppose on the contrary that $r > 0$.
Then, we have

$$\frac{\psi(d(x_{n+1}, x_{n+2}))}{\psi(d(x_n, x_{n+1}))} \leq \theta(\psi(d(x_n, x_{n+1}))) < 1.$$

Now by taking limit $n \to \infty$, we have

$$\lim_{n\to\infty} \theta(\psi(d(x_n, x_{n+1}))) = 1.$$

By the property of θ, we have $\lim_{n\to\infty} \psi(d(x_n, x_{n+1})) = 0 \Rightarrow \lim_{n\to\infty} d(x_n, x_{n+1}) = 0$ that is $r = 0$ which is a contradiction. Hence $r = 0$ that is

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2}$$

Now we show that the sequence $\{x_n\}$ is a Cauchy sequence. Let us suppose on the contrary that $\{x_n\}$ is not a Cauchy sequence. Then there exists $\epsilon > 0$ such that, for all positive integers k, there exist $m_k > n_k > k$ with

$$d(x_{m_k}, x_{n_k}) \geq \epsilon \tag{3}$$

Let m_k be the smallest number satisfying the conditions above. Then we have

$$d(x_{m_k-1}, x_{n_k}) < \epsilon \tag{4}$$

By (3) and (4), we have

$$\begin{aligned} \epsilon &\leq d(x_{m_k}, x_{n_k}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{m_k-1}, x_{n_k}) \\ &< d(x_{m_k-1}, x_{m_k}) + \epsilon \end{aligned}$$

that is,

$$\epsilon \leq d(x_{m_k}, x_{n_k}) < \epsilon + d(x_{m_k-1}, x_{m_k}) \quad \text{for all} \quad k \in \mathbb{N}. \tag{5}$$

Then in view of (2) and (5), we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon. \tag{6}$$

Again, we have

$$\begin{aligned} d(x_{m_k}, x_{n_k}) &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{m_k-1}) \\ &\leq d(x_{m_k}, x_{m_k-1}) + d(x_{n_k}, x_{n_k-1}) + d(x_{m_k-1}, x_{n_k-1}) \end{aligned}$$

and

$$d(x_{m_k-1}, x_{n_k-1}) \leq d(x_{m_k-1}, x_{m_k}) + d(x_{n_k-1}, x_{n_k}) + d(x_{m_k}, x_{n_k}).$$

Taking limit as $k \to \infty$ and using (2) and (6), we obtain

$$\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon. \tag{7}$$

Also, we have

$$|d(x_{n_k}, x_{m_k-1}) - d(x_{n_k}, x_{m_k})| \le d(x_{m_k}, x_{m_k-1}).$$

Taking limit as $k \to \infty$, we get

$$\lim_{k\to\infty} d(x_{n_k}, x_{m_k-1}) = \epsilon.$$

Similarly, we have

$$\lim_{k\to\infty} d(x_{m_k}, x_{n_k-1}) = \epsilon.$$

By Lemma 2.4., we get $\alpha(x_{n_k-1}, x_{m_k-1}) \ge 1$. Therefore, we have

$$\begin{aligned}
\psi(d(x_{m_k}, x_{n_k})) &= \psi(d(Tx_{m_k-1}, Tx_{n_k-1})) \\
&\le \alpha(x_{n_k-1}, x_{m_k-1})\psi(d(Tx_{n_k-1}, Tx_{m_k-1})) \\
&\le \theta(\psi(N(x_{n_k-1}, x_{m_k-1})))\psi(N(x_{n_k-1}, x_{m_k-1})).
\end{aligned}$$

Here we have

$$\begin{aligned}
N(x_{n_k-1}, x_{m_k-1}) =& \max\Bigg\{d(x_{n_k-1}, x_{m_k-1}), \frac{d(x_{n_k-1}, Tx_{n_k-1}) + d(x_{m_k-1}, Tx_{m_k-1})}{2}, \\
&\frac{d(x_{n_k-1}, Tx_{m_k-1}) + d(x_{m_k-1}, Tx_{n_k-1})}{2}\Bigg\} \\
=& \max\Bigg\{d(x_{n_k-1}, x_{m_k-1}), \frac{d(x_{n_k-1}, x_{n_k}) + d(x_{m_k-1}, x_{m_k})}{2}, \\
&\frac{d(x_{n_k-1}, x_{m_k}) + d(x_{m_k-1}, x_{n_k})}{2}\Bigg\}
\end{aligned}$$

And we see that

$$\lim_{k\to\infty} N(x_{n_k-1}, x_{m_k-1}) = \epsilon.$$

Now we have

$$\frac{\psi(d(x_{n_k}, x_{m_k}))}{\psi(N(x_{n_k-1}, x_{m_k-1}))} \le \theta(\psi(N(x_{n_k-1}, x_{m_k-1}))) < 1.$$

By using (6) and taking limit as $k \to \infty$ in the above inequality, we obtain

$$\lim_{k\to\infty} \theta(\psi(N(x_{n_k-1}, x_{m_k-1}))) = 1.$$

So, $\lim_{k\to\infty} \psi(N(x_{n_k-1}, x_{m_k-1})) = 0 \Rightarrow \lim_{k\to\infty} N(x_{n_k-1}, x_{m_k-1}) = 0 = \epsilon$, which is a contradiction. Hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that $x_n \to x^*$. As T is continuous, we have $Tx_n \to Tx^*$ that is $\lim_{n\to\infty} x_{n+1} = Tx^*$ and so $x^* = Tx^*$. Hence x^* is a fixed point of T.
Popescu [15] introduced the following two new concepts.

Definition 3.4. [15] *Let $T : X \to X$ be a map and $\alpha : X \times X \to \mathbb{R}$ be a function. Then T is said to be $\alpha-$orbital admissible if $\alpha(x, Tx) \geq 1$ implies $\alpha(Tx, T^2x) \geq 1$.*

Definition 3.5. [15] *A map $T : X \to X$ is said to be triangular $\alpha-$orbital admissible if (T1) T is $\alpha-$orbital admissible, (T2) $\alpha(x, y) \geq 1$ and $\alpha(y, Ty) \geq 1$ imply $\alpha(x, Ty) \geq 1$.*

Lemma 3.6. [15] *Let $T : X \to X$ be a triangular $\alpha-$orbital admissible map. Assume that there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$. Then we have $\alpha(x_n, x_m) \geq 1$ for all $m, n \in \mathbb{N}$ with $n < m$.*
Obviously, every $\alpha-$admissible map is an $\alpha-$orbital admissible map and every triangular $\alpha-$admissible map is a triangular $\alpha-$orbital admissible map. If we replace the condition (ii) of Theorem 3.3. by a weaker condition "T is triangular $\alpha-$orbital admissible", we can still prove the theorem. Thus we have the following theorem:

Theorem 3.7. *Let (X, d) be a complete metric space, $\alpha : X \times X \to \mathbb{R}$ be a function and $T : X \to X$ be a mapping. Suppose that the following conditions hold:*
(i) T is a new generalized $\alpha - \psi-$Geraghty contraction type map,
(ii) T is triangular $\alpha-$orbital admissible,
(iii) there exists $x_1 \in X$ such that $\alpha(x_1, Tx_1) \geq 1$,
(iv) T is continuous.
Then T has a fixed point $x^* \in X$ and $\{T^n x_1\}$ converges to x^*.
For the uniqueness of a fixed point of a new generalized $\alpha - \psi-$Geraghty contraction type map, we consider the following hypothesis:
(G) For any two fixed points x and y of T, there exists $z \in X$ such that $\alpha(x, z) \geq 1$, $\alpha(y, z) \geq 1$ and $\alpha(z, Tz) \geq 1$.

Theorem 3.8. *Adding condition (G) to the hypotheses of Theorem 3.3. (or Theorem 3.7.), we obtain that x^* is the unique fixed point of T.*
Proof. Due to Theorem 3.3. (or Theorem 3.7.), we obtain that $x^* \in X$ is a fixed point of T. Let $y^* \in X$ be another fixed point of T. Then by hypothesis (G), there

exists $z \in X$ such that
$\alpha(x^*, z) \geq 1$, $\alpha(y^*, z) \geq 1$ and $\alpha(z, Tz) \geq 1$.
Since T is triangular $\alpha-$admissible (or triangular $\alpha-$orbital admissible) we get $\alpha(x^*, T^n z) \geq 1$ and $\alpha(y^*, T^n z) \geq 1$ for all $n \in \mathbb{N}$.
Then we have

$$\begin{aligned} \psi(d(x^*, T^{n+1}z)) &\leq \alpha(x^*, T^n z)\psi(d(Tx^*, TT^n z)) \\ &\leq \theta(\psi(N(x^*, T^n z)))\psi(N(x^*, T^n z)), \quad \forall n \in \mathbb{N}. \end{aligned}$$

Here we have

$$\begin{aligned} N(x^*, T^n z) &= \max\left\{d(x^*, T^n z), \frac{d(x^*, Tx^*) + d(T^n z, TT^n z)}{2}, \frac{d(x^*, TT^n z) + d(T^n z, Tx^*)}{2}\right\} \\ &= \max\left\{d(x^*, T^n z), \frac{d(T^n z, T^{n+1}z)}{2}, \frac{d(x^*, T^{n+1}z) + d(T^n z, x^*)}{2}\right\} \end{aligned}$$

By Theorem 3.3. (or Theorem 3.7.) we deduce that the sequence $\{T^n z\}$ converges to a fixed point $z^* \in X$. Then taking limit $n \to \infty$ in the above equality, we get $\lim\limits_{n\to\infty} N(x^*, T^n z) = d(x^*, z^*)$. And let us suppose that $z^* \neq x^*$. Then we have

$$\frac{\psi(d(x^*, T^{n+1}z))}{\psi(N(x^*, T^n z))} \leq \theta(\psi(N(x^*, T^n z))) < 1.$$

And taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \theta(\psi(N(x^*, T^n z))) = 1.$$

Therefore we have $\lim\limits_{n\to\infty} \psi(N(x^*, T^n z)) = 0 \Rightarrow \lim\limits_{n\to\infty} N(x^*, T^n z) = 0$ that is $d(x^*, z^*) = 0$, which is a contradiction. Therefore we must have $z^* = x^*$. Similarly, we get $z^* = y^*$. Thus we have $y^* = x^*$. Hence x^* is the unique fixed point of T.
Here we give an example to illustrate Theorem 3.3.

Example 3.9. Let $X = \{1, 2, 3\}$ with the metric d defined as $d(1, 1) = d(2, 2) = d(3, 3) = 0$, $d(1, 2) = d(2, 1) = 1$, and $d(1, 3) = d(3, 1) = d(2, 3) = d(3, 2) = \frac{1}{2}$. Then (X, d) is a complete metric space. And let $\theta(t) = \frac{1}{1+2t}$ for all $t \geq 0$. Then $\theta \in \Omega$. Also let the function $\psi : [0, \infty) \to [0, \infty)$ be defined as $\psi(t) = \frac{t}{3}$. Then we have $\psi \in \Psi$.
Let a mapping $T : X \to X$ be defined by $T(1) = T(3) = 1, T(2) = 3$. And let a function $\alpha : X \times X \to \mathbb{R}$ be defined by

$$\alpha(x, y) = \begin{cases} 1 & (x = y) \\ \frac{1}{4} & \text{otherwise.} \end{cases}$$

Then, T is triangular $\alpha-$admissible, which is condition (ii) of Theorem 3.3. Condition (iii) of Theorem 3.3. is satisfied with $x_1 = 1$. And condition (iv) of Theorem 3.3. is satisfied because T is continuous. We finally show that condition (i) is also satisfied, that is

$$\alpha(x,y)\psi(d(Tx,Ty)) \leq \theta(\psi(N(x,y)))\psi(N(x,y)).$$

If $(x,y) = (1,1)$ or $(2,2)$ or $(3,3)$ then $d(Tx,Ty) = 0$. Therefore condition (i) is obviously satisfied.
If $(x,y) = (1,3)$ or $(3,1)$ then $d(Tx,Ty) = d(1,1) = 0$. Therefore condition (i) is satisfied.
If $(x,y) = (1,2)$ then we have

$$\alpha(x,y)\psi(d(Tx,Ty)) = \alpha(1,2)\psi(d(T(1),T(2))) = \frac{1}{4}\frac{d(1,3)}{3} = \frac{1}{24}.$$

And

$$\begin{aligned} N(x,y) = N(1,2) &= \max\left\{d(1,2), \frac{d(1,T(1)) + d(2,T(2))}{2}, \frac{d(1,T(2)) + d(2,T(1))}{2}\right\} \\ &= \max\left\{d(1,2), \frac{d(1,1) + d(2,3)}{2}, \frac{d(1,3) + d(2,1)}{2}\right\} \\ &= \max\left\{1, \frac{1}{4}, \frac{3}{4}\right\} = 1. \end{aligned}$$

Therefore, $\theta(\psi(N(x,y)))\psi(N(x,y)) = \dfrac{\psi(N(x,y))}{1 + 2\psi(N(x,y))} = \dfrac{N(1,2)/3}{1 + 2 \times N(1,2)/3} = \dfrac{1/3}{1 + 2 \times 1/3} = \dfrac{1}{5}$. Thus condition (i) is satisfied. Similarly, we see that condition (i) is satisfied for $(x,y) = (2,1)$. If $(x,y) = (2,3)$, then $\alpha(x,y)\psi(d(Tx,Ty)) = \alpha(2,3)\psi(d(T(2),T(3))) = \dfrac{1}{4}\dfrac{d(3,1)}{3} = \dfrac{1}{24}$.
And

$$\begin{aligned} N(x,y) = N(2,3) &= \max\left\{d(2,3), \frac{d(2,T(2)) + d(3,T(3))}{2}, \frac{d(2,T(3)) + d(3,T(2))}{2}\right\} \\ &= \max\left\{d(2,3), \frac{d(2,3) + d(3,1)}{2}, \frac{d(2,1) + d(3,3)}{2}\right\} \\ &= \max\left\{\frac{1}{2}, \frac{\frac{1}{2} + \frac{1}{2}}{2}, \frac{1+0}{2}\right\} = \max\left\{\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right\} = \frac{1}{2}. \end{aligned}$$

Therefore, $\theta(\psi(N(x,y)))\psi(N(x,y)) = \dfrac{\psi(N(x,y))}{1+2\psi(N(x,y))} = \dfrac{N(2,3)/3}{1+2\times N(2,3)/3} = \dfrac{1/6}{1+2\times 1/6} = \dfrac{1}{8}$.
Thus condition (i) is satisfied. Similarly, we see that condition (i) is satisfied for $(x,y)=(3,2)$. Hence all the conditions of Theorem 3.3. are satisfied and T has a unique fixed point $x^*=1$.

4. Conclusion

Recently, fixed-circle problem has been considered and studied by many authors as a geometric generalization of the fixed point theory in metric spaces and its generalizations. In some cases when we do not have uniqueness of the fixed point, such a map sometimes under certain conditions fixes a circle which we call a fixed-circle. Various fixed-circle theorems have been obtained using different approaches (see [11], [12], [13], [14], [18]). Also, in some papers, application of the obtained fixed-circle results was given to discontinuous activation functions on metric spaces. Therefore, it is becoming important and also interesting to investigate new fixed-circle results.

In closing, we want the readers to investigate, under what conditions, we can prove the results in this paper in fixed-circle. In general, we can always seek answer to the question: What is (are) the necessary and sufficient condition(s) for a self-mapping (two or more self-mappings) that make a given circle the fixed-circle (common fixed-circle)?

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ESSENTIAL ASCENT AND ESSENTIAL DESCENT OF LINEAR OPERATORS AND COMPOSITION OPERATORS

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Abstract: In this paper we prove certain results relating to essential ascent and essential descent of linear operator on an arbitrary vector space. Further, we give a complete characterization of essential ascent and essential descent of composition operators on l^p spaces.

Keywords and Phrases: Essential Ascent, Essential Descent, Composition Operator.

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1. Introduction

Let X denote an arbitrary vector space and T be a linear operator on X. Let $D(T)$, $N(T)$ and $R(T)$ denote domain, kernel and range of T respectively. Let $\mathbb{N}$ denote the set of natural numbers.

Definition 1.1. *If there is some integer $n \geq 0$ such that $N(T^n) = N(T^{n+1})$, the smallest such integer is called the ascent of T and is denoted by a(T). If no such integer exists then $a(T) = \infty$; see [13].*

Definition 2.2. *If there is some integer $n \geq 0$ such that $R(T^{n+1}) = R(T^n)$, the smallest such integer is called the descent of T and is denoted by d(T). If no such integer exists then $d(T) = \infty$; see [13].*

Note that if X is a finite dimensional, then a(T) and d(T) are both finite. We state the following well-known result about ascent and descent of linear operators.

Theorem 1.1. *If $D(T) = X$, $a(T)$ and $d(T)$ are both finite, then $a(T) = d(T)$; see [13].*

2. Essential Ascent and Essential Descent of Linear Operators

Definition 2.1. *If there is some integer $n \geq 0$ such that dim $(N(T^{n+1})/N(T^n))$ is finite, the smallest such integer is called the essential ascent of T and is denoted by $a_e(T)$. If no such integer exists then $a_e(T) = \infty$; see [10].*

Definition 2.2. *If there is some integer $n \geq 0$ such that dim $(R(T^n)/R(T^{n+1}))$ is finite, the smallest such integer is called the essential descent of T and is denoted by $d_e(T)$. If no such integer exists then $d_e(T) = \infty$; see [10].*

In [4] Grabiner and Zemanek have proved that if essential ascent and essential descent of a bounded linear operator on Banach space X are both finite, then they must be equal. In this section we give an elementary proof of the same result for a linear operator on an arbitrary vector space X.

Theorem 2.1. *If $a_e(T)$is finite and $d_e(T) = 0$, then $a_e(T) = 0$.*

Proof. Suppose that $a_e(T) \neq 0$, i.e. dim $(N(T)/N(T^0)) = \infty$. This implies that dim $N(T) = \infty$. Hence there exist a sequence of linearly independent vectors $\{y_i\}_{i=1}^{\infty}$ in $N(T)$. Also it is given that $d_e(T) = 0$, i.e. dim $(X/R(T))$ is finite $= n$(say). It follows that $\{y_i + R(T)\}_{i=1}^{n+1}$ are linearly dependent vectors in $X/R(T)$. Hence there exist scalars $\{\alpha_i\}_{i=1}^{n+1}$ not all of them zero such that

$$\sum_{i=1}^{n+1} \alpha_i(y_i + R(T)) = 0 + R(T) = R(T)$$

Then

$$\sum_{i=1}^{n+1} \alpha_i y_i + R(T) = R(T)$$

It follows that

$$\sum_{i=1}^{n+1} \alpha_i y_i \in R(T)$$

Let

$$z_1 = \sum_{i=1}^{n+1} \alpha_i y_i$$

Clearly $z_1 \neq 0$ and $z_1 \in R(T)$. Similarly since $\{y_i + R(T)\}_{i=n+2}^{2n+2}$ are linearly dependent vectors in $X/R(T)$. Then there exist scalars $\{\alpha_i\}_{i=n+2}^{2n+2}$ not all of them zero such that

$$\sum_{i=n+2}^{2n+2} \alpha_i(y_i + R(T)) = 0 + R(T) = R(T)$$

Put

$$z_2 = \sum_{i=n+2}^{2n+2} \alpha_i y_i$$

Clearly $z_2 \neq 0$ and $z_2 \in R(T)$. Continuing we get a sequence of linearly independent vectors $\{z_m\}_{m=1}^{\infty}$ such that $z_m \in R(T)$, since $\{y_i\}_{i=1}^{\infty}$ are linearly independent vectors in $N(T)$. Hence there exist a sequence $\{x_m\}_{m=1}^{\infty}$ of linearly independent vectors in X such that $Tx_m = z_m \neq 0$ but $T^2x_m = 0$ for all $m \geq 1$. Then $\{x_m\}_{m=1}^{\infty} \in N(T^2)$ but $\{x_m\}_{m=1}^{\infty} \notin N(T)$. Therefore $\{x_m + N(T)\}_{m=1}^{\infty}$ are linearly independent vectors in $N(T^2)/N(T)$. Then dim $(N(T^2)/N(T)) = \infty$. By using above argument recursively, we get dim $(N(T^{k+1})/N(T^k)) = \infty$ for each $k \geq 0$. Hence $a_e(T) = \infty$. This is a contradiction. Therefore $a_e(T) = 0$

Theorem 2.2. *If $D(T) = X$, $a_e(T)$ and $d_e(T)$ are both finite, then $a_e(T) = d_e(T)$.*
Proof. Let $p = d_e(T)$; i.e.dim $(R(T^p)/R(T^{p+1}))$ is finite. Let $T_1 (= T\big|_{R(T)})$ be the mapping from $R(T^p)$ to $R(T^{p+1})$. Then $T_1x = Tx$ for each $x \in R(T^p)$. Since $N(T_1^{n+1}) - N(T_1^n) \subseteq N(T^{n+1}) - N(T^n)$, $a_e(T_1) \leq a_e(T)$. Hence $a_e(T_1)$ is finite. Also dim $(R(T_1^0)/R(T_1)) =$ dim $(R(T^p)/R(T^{p+1})) < \infty$. Therefore $d_e(T_1) = 0$, by Theorem (2.1) we get $a_e(T_1) = 0$. Therefore dim $N(T_1)$ is finite. Now we claim that dim $(N(T^{p+1})/N(T^p)) \leq$ dim $N(T_1) < \infty$. Let $\{x_i + N(T^p)\}_{i=1}^n$ are linearly independent vectors in $N(T^{p+1})/N(T^p)$. Clearly $\{x_i\}_{i=1}^n \in N(T^{p+1}) - N(T^p)$. Then $T^{p+1}(x_i) = 0$ but $T^p x_i \neq 0$ for each i, $1 \leq i \leq n$. Let $y_i = T^p x_i \neq 0$ for each i, $1 \leq i \leq n$. Then $\{y_i\}_{i=1}^n \in N(T_1)$. We show that $\{y_i\}_{i=1}^n$ are linearly independent vectors in $N(T_1)$. Suppose $\sum\limits_{i=1}^n \alpha_i y_i = 0$. This implies that $T^p(\sum\limits_{i=1}^n \alpha_i x_i) = 0$, since $y_i = T^p x_i$ for each i, $1 \leq i \leq n$. Hence $\sum\limits_{i=1}^n \alpha_i x_i \in N(T^p)$. Therefore $\sum\limits_{i=1}^n \alpha_i(x_i + N(T^p)) = N(T^p)$ Now, since $\{x_i + N(T^p)\}_{i=1}^n$ are linearly independent vectors in $N(T^{p+1})/N(T^p)$, it follows that $\alpha_i = 0$ for each i, $1 \leq i \leq n$. Hence $\{y_i\}_{i=1}^n$ are linearly independent vectors in $N(T_1)$. Thus dim $(N(T^{p+1})/N(T^p)) \leq$ dim $N(T_1) < \infty$. Hence

$$a_e(T) \leq p = d_e(T) \tag{1}$$

We have only to proof $a_e(T) \geq d_e(T)$. The case when $d_e(T) = 0$ is proved in Theorem (2.1). Therefore we assume that $p = d_e(T) \geq 1$. This implies that dim $(R(T^{p-1})/R(T^p)) = \infty$. Let $\{y_i + R(T^p)\}_{i=1}^{\infty}$ are linearly independent vectors in $R(T^{p-1})/R(T^p)$. Then $\{y_i\}_{i=1}^{\infty}$ are linearly independent vectors in $R(T^{p-1})$. Clearly $y_i \in R(T^{p-1}) - R(T^p)$ for each $i \geq 1$. Then $Ty_i \in R(T^p)$ for each $i \geq 1$. Let $I = \{i \in \mathbb{N} : Ty_i \in R(T^{p+1})\}$.

Case(a): Suppose I is infinite. If $Ty_i \in R(T^{p+1})$ for infinitely many $i \geq 1$. Then $Ty_i = T^{p+1}w_i$ for some $w_i \in X$ and infinitely many $i \geq 1$. It follows that $T(T^{p-1}x_i) = T^{p+1}w_i$ for infinitely many $i \geq 1$, since $y_i \in R(T^{p-1})$ for infinitely many $i \geq 1$. Then $T^p(x_i - Tw_i) = 0$,for infinitely many $i \geq 1$. It follows that $(x_i - Tw_i) \in N(T^p)$ for infinitely many $i \geq 1$. But $T^{p-1}(x_i - Tw_i) = T^{p-1}x_i - T^p w_i = y_i - T^p w_i \neq 0$ for infinitely many $i \geq 1$, since $y_i \in R(T^{p-1}) - R(T^p)$ and $T^p w_i \in R(T^p)$ for infinitely many $i \geq 1$. Then $(x_i - Tw_i) \notin N(T^{p-1})$ for infinitely many $i \geq 1$. We claim that $\{(x_i - Tw_i) + N(T^{p-1}) : i \in I\}$ are linearly independent vectors in $N(T^p)/N(T^{p-1})$. It is sufficient if we prove that every finite subset $\{(x_i - Tw_i) + N(T^{p-1}) : i \in I\}$ are linearly independent vectors in $N(T^p)/N(T^{p-1})$. Suppose $\sum\limits_{i=1}^{k} \alpha_i(x_i - Tw_i) + N(T^{p-1}) = 0 + N(T^{p-1}) = N(T^{p-1})$. Then $\sum\limits_{i=1}^{k} \alpha_i(x_i - Tw_i) \in N(T^{p-1})$. It follows that $T^{p-1}\left\{\sum\limits_{i=1}^{k} \alpha_i(x_i - Tw_i)\right\} = 0$. Then $\sum\limits_{i=1}^{k} \alpha_i(y_i - T^p w_i) = 0$. Therefore $\sum\limits_{i=1}^{k} \alpha_i y_i = \sum\limits_{i=1}^{k} \alpha_i(T^p w_i)$. Since $\{y_i\}_{i=1}^{\infty}$ are linearly independent vectors in $R(T^{p-1}) - R(T^p)$ and $T^p w_i \in R(T^p)$ for infinitely many $i \leq 1$. Hence $\sum\limits_{i=1}^{k} \alpha_i y_i = 0$. It follows that $\alpha_i = 0$ for each $1 \leq i \leq k$. Therefore $\{(x_i - Tw_i) + N(T^{p-1}) : i \in I\}$ are linearly independent vectors in $N(T^p)/N(T^{p-1})$. Since I is infinite set. Thus dim $(N(T^p)/N(T^{p-1})) = \infty$. Hence $a_e(T) \geq p = d_e(T)$.

Case(b) : suppose that I is finite. Leaving the finite members of the of the Sequence $\{y_i\}$ for $i \in I$, the sequence $\{y_i\}_{i \notin I}$ satisfies that $y_i \notin R(T^{p-1}) - R(T^p)$ and $Ty_i \notin R(T^{p+1})$ for each $i \notin I$. So without loss of generality, we may assume that $I = \phi$. Let dim $(R(T^{p-1})/R(T^p))$ is finite $= m$(say). It follows that $\{Ty_i + R(T^p)\}_{i=1}^{m+1}$ are linearly dependent vectors in $R(T^{p-1})/R(T^p)$. Then there exist scalars $\{\alpha_i\}_{i=1}^{m+1}$ not all of them zero such that

$$\sum_{i=1}^{m+1} \alpha_i(Ty_i + R(T^p)) = 0 + R(T^p) = R(T^p)$$

It follows that

$$\sum_{i=1}^{m+1} \alpha_i(Ty_i) + R(T^p) = R(T^p)$$

Hence

$\sum\limits_{i=1}^{m+1} \alpha_i(Ty_i) \in R(T^p)$
Let
$z_1 = \sum_{i=1}^{m+1} \alpha_i y_i$
Clearly $z_1 \neq 0$ and $Tz_1 \in R(T^p)$. Similarly since $\{Ty_i + R(T^p)\}_{i=m+2}^{2m+2}$ are linearly dependent vectors in $R(T^{p-1})/R(T^p)$. Then there exist scalars $\{\alpha_i\}_{i=m+2}^{2m+2}$ not all of them zero such that

$$\sum_{i=m+2}^{2m+2} \alpha_i(Ty_i + R(T^p)) = 0 + R(T^p) = R(T^p)$$

Put

$$z_2 = \sum_{i=m+2}^{2m+2} \alpha_i y_i$$

Clearly $z_2 \neq 0$ and $Tz_2 \in R(T^p)$. Continuing we get a sequence of linearly independent vectors $\{z_i\}_{i=1}^{\infty}$ in $R(T^{p-1})$, such that $\{Tz_i\}_{i=1}^{\infty} \in R(T^p)$. Now we apply case (a)and,it follows that dim $(N(T^{p-1})/N(T^p)) = \infty$. Hence

$$a_e(T) \geq p = d_e(T) \tag{2}$$

Combining equation (1) and (2), we get $a_e(T) = d_e(T)$.

3. Essential Ascent and Essential Descent of Composition Operators

The composition operator C_ϕ on l^p where $1 \leq p < \infty$ induced by a function ϕ on $\mathbb{N}$ into itself, is defined by $C_\phi(f) = f \circ \phi$ for all $f \in l^p$. It is well known that a necessary and sufficient condition that a function ϕ on $\mathbb{N}$ into itself induces a composition operator on l^p is that $\left\{\overline{\overline{\phi^{-1}(n)}} : n \in \mathbb{N}\right\}$ is bounded. Here $\overline{\overline{\phi^{-1}(n)}}$ denotes the number of elements in $\phi^{-1}(n)$; see [12]. We denote by χ_n, the characteristic function of $\{n\}$. Note that a linear operator T belongs to exactly one of the following classes:
1. $a_e(T) = d_e(T) = finite$.
2. $a_e(T) = \infty$ and $d_e(T) = finite$.
3. $a_e(T) = finite$ and $d_e(T) = \infty$.
4. $a_e(T) = d_e(T) = \infty$.

We give examples of composition operators, exactly one for each of the above type, as follows:

Example 3.1. Let ϕ be a self-map on $\mathbb{N}$ defined as:

$$\phi(n) = \begin{cases} \text{n}, & \text{if } n \text{ is odd} \\ \text{n-1}, & \text{if } n \text{ is even .} \end{cases}$$

Then $a_e(C_\phi) = 1$ and $d_e(C_\phi) = 1$.

Example 3.2. Let ϕ be the self-map on $\mathbb{N}$ defined as:

$$\phi(p_k^n) = p_{k+1}^n \text{ for all } k \in \mathbb{N}.$$

Where $\{p_k : k \in \mathbb{N}\}$ denote the enumeration of primes.

$$\text{and } \phi(n) = n \text{ when } n \in \left(\mathbb{N} - \bigcup_{k\in\mathbb{N}} E_k\right)$$
$$\text{where } E_k = \{p_k^n : n \geq 1\} \text{ for each } k \in \mathbb{N}.$$

Then $a_e(C_\phi) = \infty$ and $d_e(C_\phi) = 0$.

Example 3.3. Let ϕ be the self-map on $\mathbb{N}$ defined as:

$$\phi(3n) = 2n \ ;$$

and

$$\phi(3n-2) = \phi(3n-1) = 2n-1.$$

Then $a_e(C_\phi) = 0$ and $d_e(C_\phi) = \infty$.

Example 3.4. Let $P = \bigcup_{k\in\mathbb{N}} \{p_k^n : n \in \mathbb{N}\}$ where p_k denote the k-th prime and $\mathbb{N} - P = \{q_k : k \geq 1\} = \{1, 6, 10, 12, \ldots\ldots\}$. Clearly $\mathbb{N} - P$ is an infinite subset of $\mathbb{N}$ and ϕ be the self-map on $\mathbb{N}$ defined as :

$$\phi(p_k^n) = p_{k+1}^n \text{ for all } k \in \mathbb{N}.$$

$$\phi(q_1) = \phi(q_2) = q_1$$

and

$$\phi(q_{2k-1}) = \phi(q_{2k}) = q_{2k-2} \text{ for each } k \geq 2.$$

Then it is easy to show that $a_e(C_\phi) = \infty$ and $d_e(C_\phi) = \infty$.

In this section we prove results about essential ascent and essential descent of composition operators on l^p spaces where $1 \leq p < \infty$.

Theorem 3.1. *$a_e(C_\phi) = \infty$ if and only if there exists a sequence $\{E_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ such that each E_k is infinite, $E_k \subseteq R(\phi^{k-1})$ and $R(\phi^k) \bigcap E_k = \phi$ for* *each $k \in \mathbb{N}$.*

Proof. Suppose that $a_e(C_\phi) = \infty$. Let $E_k = \{m : m \in R(\phi^{k-1}) - R(\phi^k)\}$. By construction of E_k, it is clear that $E_k \subseteq R(\phi^{k-1})$ and $R(\phi^k) \bigcap E_k = \phi$ for each $k \in \mathbb{N}$. We claim that E_k is infinite set. Suppose on the contrary, there exist a natural number N for which E_N is finite set. Clearly $\{\chi_n + N(C_\phi^{N-1}) : n \in E_N\}$ forms a basis for $N(C_\phi^N)/N(C_\phi^{N-1})$. Let $f = g + N(C_\phi^{N-1})$, where $g = \sum \alpha_n \chi_n$. Then

$N(C_\phi^N)/N(C_\phi^{N-1}) = span\,\{\chi_n + N(C_\phi^{N-1}) : n \in E_N\}$.

Therefore dim $(N(C_\phi^N)/N(C_\phi^{N-1})) \leq \overline{\overline{E_N}} < \infty$. Thus $a_e(C_\phi) \leq (N-1)$. This is a contradiction. Hence E_k is infinite set.

Conversely, assume that there exists a sequence $\{E_k\}_{k=1}^{\infty}$ of subsets of $\mathbb{N}$ such that each E_k is infinite, $E_k \subseteq R(\phi^{k-1})$ and $R(\phi^k) \cap E_k = \phi$ for each $k \in \mathbb{N}$. Hence $E_k \subseteq \{m : m \in R(\phi^{k-1}) - R(\phi^k)\}$. Now we claim that $\{\chi_n + N(C_\phi^{k-1}) : n \in E_k\}$ is linearly independent sequence of $(N(C_\phi^k)/N(C_\phi^{k-1}))$. Claim is achieved if we prove that every finite subset $\{\chi_n + N(C_\phi^{k-1}) : n \in E_k\}$ is linearly independent in $N(C_\phi^k)/N(C_\phi^{k-1})$. Let $\beta_1(\chi_{n_1} + N(C_\phi^{k-1}) + \ldots \cdots + \beta_l(\chi_{n_l} + N(C_\phi^{k-1}) = N(C_\phi^{k-1})$. This implies that $\beta_1\chi_{n_1} + \ldots \cdots + \beta_l\chi_{n_l} \in N(C_\phi^{k-1})$. Therefore $C_\phi^{k-1}(\beta_1\chi_{n_1} + \ldots \cdots + \beta_l\chi_{n_l}) = 0$. $E_k \subseteq R(\phi^{k-1}) - R(\phi^k)\}$ provides that $\{(\phi^{k-1})^{-1}(n_j)\}_{1\leq j\leq l}$ is a collection of non-empty pairwise disjoint sets. Thus $\beta_j\chi_{(\phi^{k-1})^{-1}(n_j)} = 0$ for $1 \leq j \leq l$. Thus $\{\chi_n + N(C_\phi^{k-1}) : n \in E_k\}$ is linearly independent sequence of $N(C_\phi^k)/N(C_\phi^{k-1})$. Since each E_k is infinite set. Therefore dim $(N(C_\phi^k)/N(C_\phi^{k-1})) = \infty$ for each $k \geq 1$. Hence $a_e(C_\phi) = \infty$.

Remark 3.1. *The following example shows that for each $n \in \mathbb{N}$ there exist a composition operator C_ϕ on l^p such that $a_e(C_\phi) = n - 1$.*

Example 3.5. Let n be any fixed natural number and ϕ be a self-map on $\mathbb{N}$ defined as:

$$\phi(m) = \begin{cases} \text{m}, & \text{if } n/(m-1) \\ \text{m-1}, & \text{otherwise.} \end{cases}$$

Then $a_e(C_\phi) = n - 1$ and $d_e(C_\phi) = n - 1$.

Theorem 3.2. $d_e(C_\phi) = \infty$ *if and only if for each $k \geq 0$; $\overline{\overline{\phi^{-1}(n)}} > 1$ for infinitely many $n \in R(\phi^k)$.*

Proof. If possible, suppose $A = \left\{n \in R(\phi^N) : \overline{\overline{\phi^{-1}(n)}} > 1)\right\}$ is finite for some natural number N. We claim that dim $(R(C_\phi^N)/R(C_\phi^{N+1})) < \infty$. Let $f \in R(C_\phi^N)$.

Then $f = C_\phi^N(g)$ for some $g \in l^p$. Let $g = \sum \alpha_n \chi_n$. Then

$$\begin{aligned} C_\phi^N(g) &= \sum_{n\in\ R(\phi^N)} \alpha_n \chi_{(\phi^N)^{-1}(n)} \\ &= \sum_{\substack{n^{'}\in\ R(\phi^N) \\ and\ \overline{\overline{\phi^{-1}(n^{'})}}>1}} \alpha_{n'} \chi_{(\phi^N)^{-1}(n')} + \sum_{\substack{n^{''}\in\ R(\phi^N) \\ and\ \overline{\overline{\phi^{-1}(n^{''})}}=1}} \alpha_{n''} \chi_{(\phi^N)^{-1}(n'')} \end{aligned}$$

i.e.

$$C_\phi^N(g) = h_1 + h_2(say) \tag{3}$$

We claim that $h_2 \in R(C_\phi^{N+1})$. Let $g^{'} = \sum \beta_n \chi_n$, where

$$\beta_n = \begin{cases} \alpha_{\phi^{-1}(n)}, & \text{when } \ n \in R(\phi^{N+1}), \\ \overline{\overline{\phi^{-1}(n)}} = 1 \ \text{ and } \ \overline{\overline{\phi^{-1}(n'')}} = 1 \ whenevern'' \in \phi^{-1}(n) \\ 0, & \text{Otherwise} \end{cases}$$

Then, clearly $g^{'} \in l^p$. Now

$$\begin{aligned} C_\phi^{N+1}(g^{'}) &= \sum_{\substack{n\in\ R(\phi^{N+1}) \\ and\ \overline{\overline{\phi^{-1}(n)}}=1}} \beta_n \chi_{(\phi^{N+1})^{-1}(n)} + \sum_{\substack{n\notin\ R(\phi^{N+1}) \\ or\ \overline{\overline{\phi^{-1}(n)}}>1}} \beta_n \chi_{(\phi^{N+1})^{-1}(n)} \\ &= \sum_{\substack{n\in\ R(\phi^{N+1}) \\ and\ \overline{\overline{\phi^{-1}(n)}}=1}} \alpha_{\phi^{-1}(n)} \chi_{(\phi^{N+1})^{-1}(n)} \end{aligned}$$

Now put $n^{''} = \phi^{-1}(n)$, then $n^{''} \in R(\phi^N)$ and by our assumption we get $\overline{\overline{\phi^{-1}(n)}} = 1 \Leftrightarrow \overline{\overline{\phi^{-1}(n'')}} = 1$(fallows from the definition of β_n). Therefore

$$C_\phi^{N+1}(g^{'}) = \sum_{\substack{n^{''}\in\ R(\phi^N) \\ and\ \overline{\overline{\phi^{-1}(n^{''})}}=1}} \alpha_{n''} \chi_{(\phi^N)^{-1}(n'')} = h_2 \tag{4}$$

Thus $h_2 \in R(C_\phi^{N+1})$.
Combining equation (3) and (4) we get dim $(R(C_\phi^N)/R(C_\phi^{N+1}))$is finite. Thus $d_e(C_\phi) \leq N$.
Conversely, assume that $\overline{\overline{\phi^{-1}(n)}} > 1$ for infinitely many $n \in\ R(\phi^k)$. Let $\{n_m\}_{m=1}^\infty \in R(\phi^k)$ such that $\overline{\overline{\phi^{-1}(n_m)}} > 1$ for each $m \geq 1$. Let $\{\alpha_{n_m}, \beta_{n_m}\} \subseteq \phi^{-1}(n_m)$ Define a sequence $\{f_m\}_{m=1}^\infty$ as follows :

$$f_m(n) = \begin{cases} 1, & \text{if } \phi^{k-1}(n) = \alpha_{n_m} \\ \text{-1}, & \text{if } \phi^{k-1}(n) = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{f_m\}_{m=1}^{\infty} \in l^p$. And also we define a sequence $\{h_m\}_{m=1}^{\infty}$ as follows:

$$h_m(n) = \begin{cases} 1, & \text{if } n = \alpha_{n_m} \\ \text{-1}, & \text{if } n = \beta_{n_m} \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $\{h_m\}_{m=1}^{\infty} \in l^p$ and $f_m = C_{\phi}^{k-1}(h_m) \in R(C_{\phi}^{k-1})$. We claim that $\{f_m\}_{m=1}^{\infty} \notin R(C_{\phi}^{k})$. If possible, assume that $\{f_{m_0}\}_{m_0=1}^{\infty} \in R(C_{\phi}^{k})$ for some $m_0 \geq 1$. This implies that $f_{m_0} = C_{\phi}^{k}(h_0)$ for some $h_0 \in l^p$. Then $f_{m_0} = h_0 \circ \phi^k$. Let $n_m^{(1)}$ and $n_m^{(2)}$ be such that $\phi^{k-1}(n_m^{(1)}) = \alpha_{n_m}$ and $\phi^{k-1}(n_m^{(2)}) = \beta_{n_m}$, where $\phi(\alpha_{n_m}) = \phi(\beta_{n_m}) = n_m$. Then $1 = f_{m_0}(n_m^{(1)}) = h_0(\phi^k(n_m^{(1)})) = h_0(\phi(\alpha_{n_m})) = h_0(\phi(\beta_{n_m})) = h_0(\phi^k(n_m^{(2)})) = f_{m_0}(n_m^{(2)}) = -1$. This is a contradiction. Hence $\{f_m\}_{m=1}^{\infty} \notin R(C_{\phi}^{k})$. Thus sequence $\left\{f_m + R(C_{\phi}^{k})\right\}_{m=1}^{\infty}$ are linearly independent in $R(C_{\phi}^{k-1})/R(C_{\phi}^{k})$. Thus dim $(R(C_{\phi}^{k-1})/R(C_{\phi}^{k}))$ is not finite. Since $k \geq 1$ is arbitrary it follows that $d_e(C_\phi) = \infty$

Remark 3.2. *From example (3.5) it follows that for each $n \in \mathbb{N}$ there exist a composition operator C_ϕ on l^p such that $d_e(C_\phi) = n - 1$.*

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TIME TO REPLACEMENT OF A SYSTEM WITH PERMISSIVE AND OBLIGATORY THRESHOLDS

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Abstract: Shock exerts on the system is a common phenomenon in reliability theory. These shocks will create damage to the system due to its impact. The system receives shocks in two mutually exclusive ways, internally (circuit problem, a heavy supply of voltage, etc.) and externally (shocks by the circumstances). Adequate replacement of the system due to the damages is not realistic since it involves cost. A stochastic model is constructed with three different cases of shocks and the time to replacement of a system is obtained, when the cumulative damages cross its obligatory threshold. The numerical illustration has been made to the mean and variance of time to replacement and the realistic conclusion is presented.

Keywords and Phrases: Time to replacement, cumulative damages, permissive threshold, obligatory threshold, shock model approach.

2010 Mathematics Subject Classification: Primary: 90B25, Secondary: 60K05, 60K20.

1. Introduction

A shocks creating damages to the system placed in the environment is a common phenomenal in reliability theory. The system receives shocks in many different categories and it is classified into two mutual exclusive shocks (i) Internal power supply or voltage problem. (ii) Shocks due to circumstances. These shocks will create damage to the system due to its impact. The time at which the cumulative damages crosses the obligatory threshold, cannot be predicted. Defining a permissive threshold (less than obligatory threshold) fills the gap to expect the replacement of a system. If the cumulative damages cross permissive threshold, it gives alertness about the replacement, where the system may or may not be replaced, whereas the cumulative damages crosses the obligatory threshold, the system is replaced. In this context author [2] has studied many reliability models with various assumptions on damages. Authors in [1], [3] and [4] have studied the concept of manpower planning and the shock model approach. Considering the shock model approach, the replacement of a system is carried out whenever the cumulative damages crosses its obligatory threshold and the system may or may not be replaced if the cumulative damages crosses permissive threshold.

In this paper, the mean and variance of time to replacement of a system is determined for the three different cases of shocks. In case-I: it is assumed that the time between two consecutive shocks forms a sequence of independent and identically distributed random variables. There are some minute shocks that will never create damage to the system. In case-II: it is assumed that the system receives $n + r$ shocks. Of these, n shocks will create the damages with probability $0 < p < 1$ to the system. Some system receives more shocks internally than the shocks due to the external circumstances. In case-III: it is assumed that the shocks received by the system has been classified into two mutually exclusive types.
For these three cases, the mean and variance of time to replacement have been determined when the cumulative damages cross its obligatory threshold and the system may or may not be replaced if the cumulative damages cross the permissive threshold. The results are numerically illustrated and the findings in the illustration coincide with the realistic observation.

2. Model Description:

Consider the system in which its functioning gets affected due to the impact of the shocks. Let $B_i, (i = 1, 2, 3, ...)$ be a stochastic process which represents the damage due to the i^{th} shock with exponential distribution function $G_i(.)$ of parameter $\alpha > 0$. Let A_i be the stochastic process that represents the time between $i - 1^{th}$ and i^{th} shock. Let the probability $0 < q < 1$ represents the system is not replaced after the cumulative damages crosses the permissive threshold. The randomly indexed partial sum S_l represents the cumulative damages to the system by the first l shocks. Let $N(t)$ is the stochastic process that represents the number of shocks exerted to the system up to the time t. Let R is a random variable that represents time to replacement of the system, with distribution function $L(.)$, density function $l(.)$ with Laplace transform $\bar{l}(.)$. Random variable Y represents the permissive threshold for the cumulative damages that follows an exponential distribution with parameter $\gamma_1 > 0$. Let Z be an exponential obligatory threshold for the cumulative damages with parameter $\gamma_2 > 0$. It is assumed that damages created to the system, inter-shock times and the thresholds are stochastically independent.

3. Analytical Results

The analytical results for the mean and variance of time to replacement has been derived for the three different cases of inter-shock times.

Case - I.

In Reliability, there exits some system which receives shocks in a periodic manner. Hence, in this case, it is assumed that time between the two consecutive shocks forms a sequence of independent and identically distributed exponential random variables with parameter $\lambda > 0$. According to the policy, the replacement occurs before the time t is equivalent to the cumulative damages crosses its obligatory threshold and the system is replaced or the cumulative damages cross a permissive threshold and the system is not replaced and the cumulative damages cross the obligatory threshold before the time t. Hence the distribution function of time to replacement is determined as

$$P(R < t) = P(S_{N(t)} > Y)(1-q) + P(S_{N(t)} > Y)(q)P(S_{N(t)} > Z)$$

Using the law of total probability, the distribution function of time to replacement is given by

$$L(t) = 1 - e^{-\lambda t(1-\overline{g}(\gamma_1))} - qe^{-\lambda t(1-\overline{g}(\gamma_2))} + qe^{-\lambda t(2-\overline{g}(\gamma_1)-\overline{g}(\gamma_2))}$$

By differentiating with respect to t, Laplace transform for the probability density function of time to replacement is determined. Now, differentiating the Laplace transform of time to replacement with respect to s, the mean time to replacement is determined at $s = 0$.

$$E(R) = \frac{(\alpha+\gamma_1)}{\lambda\gamma_1} + \frac{q(\alpha+\gamma_2)}{\lambda\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2(\alpha+\gamma_1)(\alpha+\gamma_2) - \alpha(\alpha+\gamma_1) - \alpha(\alpha+\gamma_2))}$$

The second moment of time to replacement is determined by differentiating twice the Laplace transform of time to replacement with respect to s and $s = 0$. From these results, the variance of time to replacement is determined and it is given by

$$V(R) = \frac{2(\alpha+\gamma_1)^2}{(\lambda\gamma_1)^2} + \frac{2q(\alpha+\gamma_2)^2}{(\lambda\gamma_2)^2} - \frac{2q((\alpha+\gamma_1)(\alpha+\gamma_2))^2}{(\lambda(2(\alpha+\gamma_1)(\alpha+\gamma_2) - \alpha(\alpha+\gamma_1) - \alpha(\alpha+\gamma_2)))^2}$$

$$- \left(\frac{(\alpha+\gamma_1)}{\lambda\gamma_1} + \frac{q(\alpha+\gamma_2)}{\lambda\gamma_2} - \frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2(\alpha+\gamma_1)(\alpha+\gamma_2) - \alpha(\alpha+\gamma_1) - \alpha(\alpha+\gamma_2))}\right)^2$$

Case - II.

Now the analytical results for the mean and variance of time to replacement are determined by assuming that the system receives $n+r$ shocks. Of these, n shocks will create the damages with probability $0<p<1$ to the system. By proceeding as in case - I, differentiating the Laplace transform of time to replacement with respect to s, the mean and variance of time to replacement are determined

$$E(R)=\frac{(\alpha+\gamma_1)}{p\lambda\gamma_1}+\frac{q(\alpha+\gamma_2)}{p\lambda\gamma_2}-\frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2p(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}$$

$$V(R)=\frac{2(\alpha+\gamma_1)^2}{(p\lambda\gamma_1)^2}+\frac{2q(\alpha+\gamma_2)^2}{(p\lambda\gamma_2)^2}-\frac{2q((\alpha+\gamma_1)(\alpha+\gamma_2))^2}{(\lambda(2p(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2)))^2}$$

$$-\left(\frac{(\alpha+\gamma_1)}{p\lambda\gamma_1}+\frac{q(\alpha+\gamma_2)}{p\lambda\gamma_2}-\frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{\lambda(2p(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}\right)^2$$

Case - III.

In this case, the Poisson process $N(t)$ that represents the number of shocks exerted to the system is considered as the sum of two (Internal and External) independent Poisson process with parameters $\lambda_1>0$ and $\lambda_2>0$. Now, the probability density function of time to replacement is derived by taking derivative for the distribution function with respect to t. Taking Laplace transform for the probability density function of time to replacement and differentiating the Laplace transform, the moments of time to replacement are determined.

$$E(R)=\frac{(\alpha+\gamma_1)}{(\lambda_1+\lambda_2)\gamma_1}+\frac{q(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)\gamma_2}-\frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)(2(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}$$

The variance of time to replacement is derived by using the first two moments of time to replacement. It is given by

$$V(R)=\frac{2(\alpha+\gamma_1)^2}{((\lambda_1+\lambda_2)\gamma_1)^2}+\frac{2q(\alpha+\gamma_2)^2}{((\lambda_1+\lambda_2)\gamma_2)^2}$$
$$-\frac{2q((\alpha+\gamma_1)(\alpha+\gamma_2))^2}{((\lambda_1+\lambda_2)(2(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2)))^2}$$

$$-\left(\frac{(\alpha+\gamma_1)}{(\lambda_1+\lambda_2)\gamma_1}+\frac{q(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)\gamma_2}-\frac{q(\alpha+\gamma_1)(\alpha+\gamma_2)}{(\lambda_1+\lambda_2)(2(\alpha+\gamma_1)(\alpha+\gamma_2)-\alpha(\alpha+\gamma_1)-\alpha(\alpha+\gamma_2))}\right)^2$$

4. Numerical Illusion

The following tables are the numerical values of the mean and variance of time to replacement for the three cases. By fixing the thresholds (the limits for the system could get from the circumstances with SI units of power) $C_1 = 200$ and $C_2 = 300$ and varying the other parameters λ (in days) and α (damages created to the system due to SI units of power), numerical values of mean and variance of time to replacement (in days) are studied.

Case-I

$1/\lambda$	$1/\alpha$	q	E(R)	V(R)
0.033	0.017	0.4	36.366	1065.4
0.040	0.017	0.4	30.002	725.11
0.050	0.017	0.4	24.002	464.07
0.066	0.017	0.4	18.183	266.34
0.10	0.017	0.4	12.001	116.02
0.028	0.020	0.4	42.861	1479.9
0.028	0.025	0.4	42.862	1479.9
0.028	0.033	0.4	42.865	1480.0
0.028	0.050	0.4	42.867	1480.3
0.028	0.100	0.4	42.877	1480.9
0.028	0.017	0.5	44.646	1514.9
0.028	0.017	0.6	46.432	1543.9
0.028	0.017	0.7	48.218	1565.9
0.028	0.017	0.8	50.004	1581.9
0.028	0.017	0.9	51.789	1591.4

Case-II

p	q	$1/\lambda$	$1/\alpha$	E(R)	V(R)
0.1	0.4	0.033	0.017	363.66	106536
0.1	0.4	0.040	0.017	300.02	72511.1
0.1	0.4	0.050	0.017	240.02	46407.1
0.1	0.4	0.066	0.017	181.88	26634
0.1	0.4	0.10	0.017	120.01	11601
0.1	0.4	0.028	0.020	428.61	147985.8
0.1	0.4	0.028	0.025	428.62	147992.5
0.1	0.4	0.028	0.033	428.64	148003.2
0.1	0.4	0.028	0.050	428.67	148025.9
0.1	0.4	0.028	0.100	428.77	148092.7
0.2	0.4	0.028	0.017	214.32	36995.4
0.3	0.4	0.028	0.017	142.87	16442.4
0.4	0.4	0.028	0.017	107.15	9248.9
0.5	0.4	0.028	0.017	85.721	5919.3
0.6	0.4	0.028	0.017	71.434	4110.6
0.1	0.5	0.028	0.017	446.46	151489.6
0.1	0.6	0.028	0.017	464.32	154359.6
0.1	0.7	0.028	0.017	482.18	156591.7
0.1	0.8	0.028	0.017	500.04	158186.1
0.1	0.9	0.028	0.017	517.89	159142.6

Case-III

$1/\lambda_1$	$1/\lambda_2$	$1/\alpha$	q	E(R)	V(R)
0.033	0.033	0.017	0.4	18.183	266.34
0.040	0.033	0.017	0.4	16.440	217.71
0.050	0.033	0.017	0.4	14.459	168.41
0.066	0.033	0.017	0.4	12.122	118.37
0.10	0.033	0.017	0.4	9.0233	65.588
0.028	0.040	0.017	0.4	17.648	250.90
0.028	0.050	0.017	0.4	15.386	190.69
0.028	0.066	0.017	0.4	12.767	131.30
0.028	0.10	0.017	0.4	9.375	70.812
0.028	0.20	0.017	0.4	5.2636	22.318
0.028	0.033	0.020	0.4	19.673	311.80
0.028	0.033	0.025	0.4	19.674	311.81
0.028	0.033	0.033	0.4	19.675	311.84
0.028	0.033	0.050	0.4	19.677	311.88
0.028	0.033	0.100	0.4	19.681	312.03
0.028	0.033	0.017	0.5	20.493	319.18
0.028	0.033	0.017	0.6	21.313	325.23
0.028	0.033	0.017	0.7	22.133	329.93
0.028	0.033	0.017	0.8	22.952	333.29
0.028	0.033	0.017	0.9	23.772	335.31

5. Numerical Results

In all the three cases, if the probability of not replacing the system ($q > 0$) increases, it elongates the time of replacement. Hence, the mean time to replacement increases. In Case II, if the probability of shocks producing damages to the system ($p > 0$) increases, then the system receives more damages. Hence, the cumulative damages takes less time to cross the breakdown threshold. Thus, the mean time to replacement of a system decreases. For the Cases I and II, if the average inter shocks time increases, then the epoch of occurrences of shocks decreases. Thus, the occurrences of shocks increases, that creates more damages to the system. Hence, the mean time to replacement declines. In Case - III, if the average inter external shocks and inter internal shocks increases, then the epoch of the occurrence of shocks (external and internal shocks) decreases. Thus, the number of shocks exerted to the system increases which creates more damages to the system. This reduces the mean time to replacement.

In Cases I, II and III, if the average damages due to the shocks increases, then the damages exerted to the system decreases. Hence, the cumulative damages takes more time to cross the breakdown threshold. Thus, the mean time to replacement of a system increases. Comparing all the three cases, Case II gives more time to replacement of a system. Perceiving the results for the three cases from the tables, the numerical results coincides with the realistic surveillance.

6. Conclusion:

The general observation of hike in damages leads to the reduction of time to replacement and the time between the shock (inter-shock times) elongates, the replacement time of a system extended. This observation coincides with realistic scrutiny. The concept of considering the independence in the inter-shock times and the damages can be dropped in the future to study the dependence nature of shock and its damages.

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HEMI-SLANT SUBMANIFOLDS OF GENERALIZED D-CONFORMAL DEFORMED β-KENMOTSU MANIFOLD

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Abstract: We study some geometric properties such as integrability, geodesic foliation of hemi-slant submanifolds of generalized D-conformal deformed β-Kenmotsu manifold.

Keywords and Phrases: Hemi-slant submanifolds, generalized D-conformal deformation, integrability, geodesic foliation.

2010 Mathematics Subject Classification: 53C25, 53C40, 53C50.

1. Introduction

Study of slant submanifolds was initiated by Chen [8], as a generalization of both holomorphic and totally real submanifolds of a Kahler manifold. Slant submanifolds have been studied in different kind of structures: almost contact [9], neutral Kahler [2], Lorentzian Sasakian [3], and Sasakian [5] by several geometers. Papaghiuc [12] introduced semi-slant submanifolds of a Kahler manifold as a natural generalization of slant submanifold. Sari and Vanli [13] investigated semi-slant submanifolds of a Lorentz Kenmotsu manifold and obtained some curvature properties for semi-slant submanifold of a Lorentz Kenmotsu space form. Carriazo [6], introduced bi-slant submanifolds of an almost Hermitian manifold as a generalization of semi-slant submanifolds. One of the classes of bi-slant submanifolds is that of anti-slant submanifolds, which are studied by Carriazo [6].

In [1], generalized D-conformal deformations are applied to trans-Sasakian manifolds where the covariant derivatives of the deformed metric is evaluated under the condition that the functions used in deformation depend only on the direction of the characteristic vector field of the trans-Sasakian structure. In this current year 2019, Ozdemir, Aktay and Solgun [10], have derived a relation between the covariant derivatives of the ambient manifold and its generalized D-conformal deformed manifold i.e. ∇ and $\nabla^\star$ for an almost contact metric structure without choosing any condition on the positive functions a and b. The integrability conditions of the distributions of submanifolds have been studied since years. Geometers have proved, if a and b depend only in the direction of ξ then the generalized D-conformal deformed β-Kenmotsu manifold turns into $\beta^\star$-Kenmotsu manifold where $\beta^\star = \frac{1}{a}\beta + \frac{1}{2ab}\xi(b)$.

In the present paper, we study the integrability of hemi-slant submanifolds and totally geodesic foliation of generalized D-conformal deformed β-Kenmotsu manifold.

2. Preliminaries

Let M be an m dimensional almost contact metric manifold with the structure tensors (ϕ, ξ, η, g), where ϕ is a tensor field of type $(1,1)$, ξ a vector field, η a 1-form and g is a Riemannian metric on M [2] satisfying

$$\begin{aligned} &\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \ \ \eta \cdot \phi = 0, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X), \end{aligned} \tag{2.1}$$

for any $X, Y \in \Gamma(TM)$.

Let Φ denote the 2-form in M given by $\Phi(X, Y) = g(X, \phi Y)$.

An almost contact metric manifold (M, ϕ, ξ, η, g) is called β-Kenmotsu manifold, if the relation

$$(\nabla_X \phi)Y = \beta(g(\phi X, Y)\xi - \eta(Y)\phi X), \tag{2.2}$$

is satisfied, where β is a smooth function on M.

For β-Kenmotsu manifold the following relations hold:

$$g(\nabla_X \xi, Y) = g(\nabla_Y \xi, X), \tag{2.3}$$

$$\nabla_X \xi = \beta(X - \eta(X)\xi). \tag{2.4}$$

If we take

$$\phi^\star = \phi, \ \ \xi^\star = \frac{1}{a}\xi, \ \ \eta^\star = a\eta, \ \ g^\star = bg + (a^2 - b)\eta \otimes \eta, \tag{2.5}$$

where a and b are positive functions on M, one can easily check that $(M, \phi^\star, \xi^\star, \eta^\star, g^\star)$ is an almost contact metric manifold too. This deformation is

called a generalized D-conformal deformation.
After this deformation, the derivation of new fundamental 2-form $\tilde{\Phi}$ is

$$d\Phi^{\star} = d\Phi(X,Y,Z) + X(b)\Phi(Y,Z) - Y(b)\Phi(X,Z) + Z(b)\Phi(X,Y). \tag{2.6}$$

Lemma [A]. [10] *Consider a generalized D-conformal deformation of an almost contact metric structure such that* $g(\nabla_X\xi, Y) = g(\nabla_Y\xi, X)$, *where* a *and* b *are positive functions. After a generalized D-conformal deformation, the new Levi-Civita covariant derivative is*

$$\begin{aligned}\nabla^{\star}_X Y = \nabla_X Y &+ \frac{(a^2-b)}{b} g(\nabla_X\xi, Y)\xi + \frac{1}{2b}(X(b)Y + Y(b)X)\\ &+ \frac{1}{a}(X(a)\eta(Y) + Y(a)\eta(X))\xi + \frac{a}{b}\eta(Y)\phi^2(grad\ a)\\ &- \frac{1}{a}\xi(a)\eta(X)\eta(Y)\xi - \frac{1}{2a^2}\xi(b)g(\phi X, \phi Y)\xi\\ &+ \frac{1}{2b}g(\phi X, \phi Y)\phi^2(grad\ a).\end{aligned} \tag{2.7}$$

Theorem [A]. [10] *Let* (M, ϕ, ξ, η, g) *be a* β*-Kenmotsu manifold and consider a generalized D-conformal deformation with* a *and* b *positive functions. If* $grad\ a = g(grad\ a, \xi)\xi$ *and* $grad\ b = g(grad\ b, \xi)\xi$, *then* $(M, \phi^{\star}, \xi^{\star}, \eta^{\star}, g^{\star})$ *is a* $\beta^{\star}$*-Kenmotsu manifold, where*

$$\beta^{\star} = \frac{1}{a}\beta + \frac{1}{2ab}\xi(b). \tag{2.8}$$

For a $\beta^{\star}$*-Kenmotsu manifold, the following relations hold:*

$$grad\ a = \xi(a)\xi \tag{2.9}$$

and

$$(\nabla^{\star}_X\phi^{\star})Y = \beta^{\star}(g^{\star}(\phi^{\star}X, Y)\xi^{\star} - \eta^{\star}Y\phi^{\star}X). \tag{2.10}$$

Let N be n- dimensional immersed submanifold of M. Then the Gauss and Weingarten formulas [3] are respectively, given by

$$\nabla_X Y = \tilde{\nabla}_X Y + \sigma(X,Y) \tag{2.11}$$

and

$$\nabla_X V = -A_V X + \tilde{\nabla}^{\perp}_X V, \tag{2.12}$$

for any $X, Y \in \Gamma(TN)$ and $V \in \Gamma(TN^{\perp})$, where σ, $\tilde{\nabla}$, $\tilde{\nabla}^{\perp}$ and A denote respectively the second fundamental form, Levi-civita connection, the normal connection and the shape operator on the submanifold N. The second fundamental form and shape operator are related by

$$g(A_V X, Y) = g(\sigma(X, Y), V), \tag{2.13}$$

where g denotes the induced metric on N as well as the Riemannian metric g on M.
Let N be a submanifold of an almost contact metric manifold $M(\phi, \xi, \eta, g)$. For $X \in \Gamma(TN)$, we write

$$\phi X = TX + FX, \tag{2.14}$$

where $TX \in \Gamma(TN)$ and $FX \in \Gamma(T^{\perp}N)$.
For $V \in \Gamma(T^{\perp}N)$, we have

$$\phi V = tV + fV, \tag{2.15}$$

where $tV \in \Gamma(TN)$ and $fV \in \Gamma(T^{\perp}N)$.

3. Hemi-slant Submanifolds of Generalized D-conformal Deformed β-Kenmotsu Manifold

Definition 3.1. *A submanifold N of M is said to be hemi-slant submanifold of an almost contact metric manifold M if there exists two orthogonal complementary distributions D_1 and D_2 on N such that*

1. $TN = D_1 \oplus D_2 \oplus \langle \xi \rangle$,
2. *the distribution D_2 is slant with slant angle $\theta \neq \frac{\pi}{2}$,*
3. *the distribution D_1 is anti-invariant, i.e. $\phi D_1 \subseteq T^{\perp}N$.*

Let a non-zero $X \in \Gamma(D_2)$, we derive

$$cos\theta_2 = \frac{g^{\star}(\phi^{\star}X, TX)}{\|\phi^{\star}X\|\|TX\|} = \frac{\|TX\|}{\|\phi X\|}. \tag{3.1}$$

Squaring on the both sides of (3.1), we get

$$cos^2\theta_2\|\phi X\|^2 = \|TX\|^2. \tag{3.2}$$

Replacing X by $X + Y$ for $Y \in \Gamma(D_2)$ and a simple calculation gives

$$g^{\star}((T^2 - cos^2\theta_2\phi^2)X, Y) = g^{\star}((T^2 - cos^2\theta_2\phi^2)Y, X). \tag{3.3}$$

But $T^2 - cos^2\theta_2\ \phi^2$ is symmetric and since $X, Y \in \Gamma(D_2)$ then from (3.3) we have

$$T^2X = -cos^2\theta_2 X. \tag{3.4}$$

Therefore we state the following

Proposition 3.1. *Let N be a hemi-slant submanifold of a generalized D-conformal deformed manifold M. Then for any $X \in \Gamma(TN)$,*

1. *if θ_1 is the slant angle with respect to the distribution D_1 then $T^2X = 0$,*
2. *if θ_2 is the slant angle with respect to the distribution D_2 then*
   a. *for $\theta_2 = 0$, we have $T^2X = X$,*
   b. *for $\theta_2 \neq \frac{\pi}{2}$, we have $T^2X = -cos^2\theta_2 X$.*

Let $X, Y \in D_2 \oplus \langle\xi\rangle$ and $Z \in D_1$, then we have

$$g^\star([X,Y], Z) = g^\star(\phi[X,Y], \phi Z) + \eta^\star([X,Y])\eta^\star(Z). \tag{3.5}$$

Since the distribution $Z \in \Gamma(D_1)$, we have $\eta^\star(Z) = 0$ and therefore (3.5) reduces to

$$g^\star([X,Y], Z) = g^\star(-(\nabla^\star_X\phi)Y + (\nabla^\star_Y\phi)X + \nabla^\star_X\phi Y - \nabla^\star_Y\phi X, \phi Z). \tag{3.6}$$

From (2.7), (2.12) and (2.5) and using the definition of hemi-slant submanifold, we obtain

$$g^\star([X,Y], Z) = b\ g(-A_{\phi Y}X + A_{\phi X}Y - \tilde{\nabla}^\perp_Y\phi X + \tilde{\nabla}^\perp_X\phi Y, \phi Z). \tag{3.7}$$

Hence we state the following:

Theorem 3.1. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. Then the distribution $D_2 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(-A_{\phi Y}X + A_{\phi X}Y - \tilde{\nabla}^\perp_Y\phi X + \tilde{\nabla}^\perp_X\phi Y, \phi Z) = 0.$$

In view of the Theorem 3.1 and the Theorem A we state the following:

Corollary 3.1. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If $grad\ a = g(grad\ a, \xi)\xi$ and $grad\ b = g(grad\ b, \xi)\xi$, then the distribution $D_2 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(-A_{\phi Y}X + A_{\phi X}Y - \tilde{\nabla}^\perp_Y\phi X + \tilde{\nabla}^\perp_X\phi Y, \phi Z) = 0.$$

Next let $X, Y \in D_1 \oplus \langle\xi\rangle$ and $Z \in D_2$, then we have

$$g^\star([X,Y],Z) = g^\star(\phi[X,Y],\phi Z) + \eta^\star([X,Y])\eta^\star(Z). \tag{3.8}$$

Since the distribution $Z \in \Gamma(D_2)$, we have $\eta^\star(Z) = 0$ and therefore (3.8) reduces to

$$g^\star([X,Y],Z) = g^\star(-(\tilde{\nabla}_X\phi)Y + (\tilde{\nabla}_Y\phi)X + \tilde{\nabla}_X\phi Y - \tilde{\nabla}_Y\phi X, \phi Z). \tag{3.9}$$

From (2.7), (2.12) and (2.14) and using the definition of hemi-slant submanifold, we obtain

$$g^\star([X,Y],Z) = b\ g(\sigma^\star(X,TY) - \sigma^\star(TX,Y) + \tilde{\nabla}^\perp_X FY - \tilde{\nabla}^\perp_Y FX, \phi Z). \tag{3.10}$$

Hence we state the following:

Theorem 3.2. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. Then the distribution $D_1 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(\sigma^\star(X,TY) - \sigma^\star(TX,Y) + \tilde{\nabla}^\perp_X FY - \tilde{\nabla}^\perp_Y FX, \phi Z) = 0.$$

And similarly we can state the corollary:

Corollary 3.2. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If $grad\ a = g(grad\ a,\xi)\xi$ and $grad\ b = g(grad\ b,\xi)\xi$, then the distribution $D_1 \oplus \langle\xi\rangle$ is integrable if and only if*

$$g(\sigma^\star(X,TY) - \sigma^\star(TX,Y) + \tilde{\nabla}^\perp_X FY - \tilde{\nabla}^\perp_Y FX, \phi Z) = 0.$$

Now let $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, then we may write

$$g^\star(\tilde{\nabla}^\star_Y X, Z) = g^\star(\phi\nabla^\star_Y X, \phi Z). \tag{3.11}$$

Using (2.7) and (2.14) in (3.11), we obtain

$$\begin{aligned} g^\star(\tilde{\nabla}^\star_Y X, Z) = &-g^\star(\nabla_Y X, T^2 Z + FTZ) + g^\star(\nabla_Y\phi X - (\nabla_Y\phi)X, FZ) \\ &+ g^\star(\frac{1}{2}(X(b)\phi Y + Y(b)\phi X) + \frac{a}{b}(\eta(X)\phi^3 grad\ a) \\ &+ \frac{1}{2b}(g(\phi X,\phi Y)\phi^3 grad\ a), \phi Z). \end{aligned} \tag{3.12}$$

Since $Z \in \Gamma(D_2)$ therefore $T^2 Z = 0$, then from (3.12) we get

$$\begin{aligned} g^\star(\tilde{\nabla}^\star_Y X, Z) = &-g^\star(\nabla_Y X, FTZ) + g^\star(\nabla_Y\phi X - (\nabla_Y\phi)X, FZ) \\ &- g^\star(\frac{1}{2}(X(b)\phi^2 Y + Y(b)\phi^2 X) + \frac{a}{b}(\eta(X)\phi^4 grad\ a) \\ &+ \frac{1}{2b}(g(\phi X,\phi Y)\phi^4 grad\ a), Z). \end{aligned} \tag{3.13}$$

From (2.1) and (2.2) and using the property of orthogonal distribution we find

$$g^\star(\tilde{\nabla}^\star_Y X, Z) = b\ g(A_{FZ}\phi X - A_{FTZ}X, Y) + g^\star(\frac{a}{b}\eta(X)\phi^2 grad\ a + \frac{1}{2b}g(\phi X, \phi Y)\phi^2 grad\ a, Z). \tag{3.14}$$

Therefore we state the following

Theorem 3.3. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If grad $a \perp X$ and grad $a \perp \xi$ then the distribution D_1 defines a totally geodesic foliation if and only if $g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0$ or $b = 0$.*

In view of above theorem we have the following corollary:

Corollary 3.3. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If grad $a = g(grad\ a, \xi)\xi$ and grad $b = g(grad\ b, \xi)\xi$, then the distribution D_1 defines a totally geodesic foliation if and only if $g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0$ or $b = 0$.*

Next let $X, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, then we may write

$$g^\star(\tilde{\nabla}^\star_Y X, Z) = g^\star(\phi\nabla^\star_Y X, \phi Z). \tag{3.15}$$

Using (2.7) and (2.14) in (3.15), we obtain

$$\begin{aligned} g^\star(\tilde{\nabla}^\star_Y X, Z) &= -g^\star(\nabla_Y X, T^2 Z + FTZ) + g^\star(\nabla_Y \phi X - (\nabla_Y \phi)X, FZ) \\ &\quad + g^\star(\frac{1}{2}(X(b)\phi Y + Y(b)\phi X) + \frac{a}{b}(\eta(X)\phi^3 grad\ a) \\ &\quad + \frac{1}{2b}(g(\phi X, \phi Y)\phi^3 grad\ a), \phi Z). \end{aligned} \tag{3.16}$$

Since $Z \in \Gamma(D_2)$ therefore $T^2 Z = -cos^2\theta_2$, then from (3.12) we get

$$\begin{aligned} sin^2\theta_2\ g^\star(\tilde{\nabla}^\star_Y X, Z) &= -g^\star(\nabla_Y X, FTZ) + g^\star(\nabla_Y \phi X - (\nabla_Y \phi)X, FZ) \\ &\quad - g^\star(\frac{1}{2}(X(b)\phi^2 Y + Y(b)\phi^2 X) + \frac{a}{b}(\eta(X)\phi^4 grad\ a) \\ &\quad + \frac{1}{2b}(g(\phi X, \phi Y)\phi^4 grad\ a), Z). \end{aligned} \tag{3.17}$$

From (2.1) and (2.2) and using the property of orthogonal distribution we find

$$sin^2\theta_2\ g^\star(\tilde{\nabla}^\star_Y X, Z) = b\ g(A_{FZ}\phi X - A_{FTZ}X, Y) + g^\star(\frac{a}{b}\eta(X)\phi^2 grad\ a + \frac{1}{2b}g(\phi X, \phi Y)\phi^2 grad\ a, Z). \tag{3.18}$$

Hence we state the following theorem:

Theorem 3.4. *Let N be a hemi-slant submanifold of generalized D-conformal deformed β-Kenmotsu manifold. If grad $a \perp X$ and grad $a \perp \xi$ then the distribution D_2 defines a totally geodesic foliation if and only if*
$g(A_{FZ}\phi X - A_{FTZ}X, Y) = 0$ *or* $b = 0$.

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ON EXISTENCE OF $\psi-$HILFER HYBRID FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract: In this paper we derive existence results for the solutions to the first order hybrid fractional differential equations with perturbation of first kind and second kind involving $\psi-$Hilfer fractional derivative using different fixed point theorems. Finally the result is illustrated with an example.

Keywords and Phrases: Fractional differential equation, Hybrid, Fixed point theorem, $\psi-$Hilfer fractional derivative.

2010 Mathematics Subject Classification: 26A33, 34A38, 47H10, 34A12, 34K37.

1. Introduction

For the last few decades, many researchers attracted towards the study of fractional calculus motivated by it's wide application both in pure and applied mathematics [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]. For studying about the dynamical systems described by non linear differential and integral equations, the perturbation techniques are very useful. The perturbed differential equations are categorized into various types. Quadratic perturbations of nonlinear fractional differential equations, which is an important type of these perturbations (Hybrid Differential Equations) have achieved a great deal of interest and attention of several researchers. Dhange and Lakshmikantham [14, 15] and Dhange and Jadhav [16] discussed the existence and uniqueness theorems of the solution to the ordinary first order hybrid differential equations with perturbation of first and second kind respectively. Much work has done in this theory and we refer the readers to the articles [17, 18, 19, 20, 21, 22].

Fractional Calculus is a rich field and we can find several definitions for fractional integrals and fractional derivatives. One who interested in the study of fractional calculus may confuse to select operators. One way to overcome this problem is to consider more general definitions. Recently Sousa and Oliveira [23] proposed a new general fractional derivative, which is named as $\psi-$Hilfer fractional derivative. They derived around 22 types of fractional derivatives and integrals from $\psi-$Hilfer operator. Many works has done on the fractional equations involving $\psi-$Hilfer fractional operator [23, 24, 25, 26, 27, 28].

In this paper we discuss the existences of hybrid fractional differential equations of first and second type involving $\psi-$Hilfer fractional derivative, which are given by

$$\begin{cases} {}^H\mathbb{D}^{\alpha,\beta;\psi}_{0+} \frac{x(t)}{f(t,x(t))} = g(t, x(t)), \quad a.e \quad t \in I = [0, T] \\ I^{1-\gamma,\psi}_{0+} \frac{x(0)}{f(0,x(0))} = x_0 \end{cases} \tag{1}$$

and

$$\begin{cases} {}^H\mathbb{D}^{\alpha,\beta;\psi}_{t_0+}[x(t) - f(t, x(t))] & = g(t, x(t)), \ a.e. \ t \in J = [t_0, t_0 + a] \\ I^{1-\gamma;\psi}_{t_0}[x(t_0) - f(t_0, x(t_0))] & = \sigma \in \mathbb{R} \end{cases} \tag{2}$$

where ${}^H\mathbb{D}^{\alpha,\beta;\psi}(\cdot)$ is the ψ-Hilfer fractional derivative with $0 < \alpha < 1, 0 \leq \beta \leq 1$, $\alpha \leq \gamma = \alpha + \beta - \alpha\beta < 1$ and $f \in C_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R}|\{0\})$ and $g \in \mathcal{C}_{1-\gamma;\psi}(J \times \mathbb{R}, \mathbb{R})$. $J = [t_0, t_0 + a]$ is a bounded interval in $\mathbb{R}$ for some t_0 and $a \in \mathbb{R}$. $I = [t_0, t_0 + a]$,with $t_0 = 0$ and $a = T$.

This paper is organized as follows: Some basic definitions and lemmas are introduced in section 2. It also includes some results required to prove our main result. In section 3 we give the existence result for $\psi-$Hilfer Fractional Hybrid differential equation of first type based on Dhange fixed point theorem. In section 4 we give the existence result for $\psi-$Hilfer Fractional Hybrid differential equation of second type based on fixed point theorem is given. We finished the section with an example and plotted graphs for different functions.

2. Preliminaries

Let $[a, b]$, $(0 < a < b < \infty)$ be finite interval on the half axis $\mathbb{R}^+$ and let $C[a, b]$ be the space of continuous function f on $[a, b]$ with the norm. Define,

$$||f||_{C[a,b]} = \max_{x\in[a,b]} |f(t)|. \tag{3}$$

The weighted space $C_{1-\gamma;\psi}[a,b]$ of continuous f on $(a,b]$ is defined by

$$C_{1-\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; (\psi(t) - \psi(a))^{1-\gamma} f(t) \in C[a,b]\}, \tag{4}$$

$0 \le \gamma < 1$ with the norm

$$||f||_{C_{1-\gamma;\psi}}[a,b] = ||(\psi(t) - \psi(a))^{1-\gamma} f(t)||_{C[a,b]} = \max_{x\in[a,b]} |(\psi(t) - \psi(a))^{1-\gamma} f(t)|. \tag{5}$$

The weighted space $C^n_{\gamma;\psi}[a,b]$ of continuous f on (a,b] is defined by

$$C^n_{\gamma;\psi}[a,b] = \{f : (a,b] \to \mathbb{R}; f(t) \in C^{n-1}[a,b]; f^{(n)}(t) \in C_{\gamma,\psi}[a,b]\}, \tag{6}$$

$0 < \gamma < 1$ with the norm

$$\|f\|_{C^n_{\gamma;\psi}[a,b]} = \sum_{k=0}^{n-1} \|f^{(k)}\|_{C[a,b]} + \|f^{(n)}\|_{C_{\gamma;\psi}[a,b]}. \tag{7}$$

For the weighed function $\mathcal{C}_{1-\gamma;\psi}[a,b]$,

(i) The map $t \to g(t,x)$ is measurable for each $x \in \mathbb{R}$;

(ii) The map $x \to g(t,x)$ is continuous for each $t \in [a,b]$;

(iii) For each $g \in \mathcal{C}_{1-\gamma;\psi}[a,b]$, $g(t,x(t))$ is $\psi-$integrable.

Definition 1. [1] *Let (a,b), $(-\infty \le a < b \le \infty)$ be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha > 0$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(t)$ on (a,b). The left-sided fractional integral of a function f with respect to a function ψ on [a,b] is defined by:*

$$I^{\alpha;\psi}_{a+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} f(s) ds. \tag{8}$$

The right-sided fractional integral is defined in an analogous form.

Definition 2. [23] *Let $n-1 < \alpha < n$ with $n \in \mathbb{N}$, let $I = [a,b]$ be an interval such that $-\infty \le a < b \le \infty$ and let $f, \psi \in C^n[a,b]$ be two functions such that ψ is increasing and $\psi'(t) \neq 0$, for all $t \in I$. The left-sided $\psi-$Hilfer fractional derivative ${}^H\mathbb{D}^{\alpha,\beta;\psi}_{a+}(\cdot)$ of a function f of order α and type $0 \le \beta \le 1$, is defined by*

$$ {}^H\mathbb{D}^{\alpha,\beta;\psi}_{a+} f(t) = I^{\beta(n-\alpha);\psi}_{a+} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I^{(1-\beta)(n-\alpha);\psi}_{a+} f(t). \tag{9}$$

The right-sided $\psi-$Hilfer fractional derivative is defined in an analogous form.

Lemma 1. [23] *If $f \in C^n[a,b], 0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then*

$$I_{a+}^{\alpha;\psi}\ {}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} f(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma-k}}{\Gamma(\gamma-k+1)} f_{\phi}^{[n-k]} I_{a+}^{(1-\beta)(n-\alpha);\psi} f(a). \tag{10}$$

Lemma 2. [23] *If $f \in C^1[a,b], 0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then*

$${}^{H}\mathbb{D}_{a+}^{\alpha,\beta;\psi} I_{a+}^{\alpha;\psi}\ f(t) = f(t). \tag{11}$$

Lemma 3. [6] *Let $\alpha > 0$ and $\delta > 0$. If $f(t) = (\psi(t)-\psi(a))^{\delta-1}$, then*

$$I_{a+}^{\alpha;\psi} f(t) = \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(t)-\psi(a))^{\alpha+\delta-1}. \tag{12}$$

Lemma 4. *Let $\psi \in C^1([a,b],\mathbb{R})$ be a function such that ψ is increasing and $\psi'(t) \neq 0, \forall t \in [a,b]$. If $\gamma = \alpha + \beta(1-\alpha)$, where $0 < \alpha < 1$ and $0 \leq \beta \leq 1$, then $\psi-$Riemann-Liouville fractional integral operator $I_{a+}^{\alpha;\psi}(\cdot)$ is bounded from $C_{1-\gamma;\psi}[a,b]$ to $C_{1-\gamma;\psi}[a,b]$.*

$$\|I_{a+}^{\alpha;\psi} h\|_{C_{1-\gamma;\psi}[a,b]} \leq M \frac{\Gamma(\gamma)}{\Gamma(\gamma+\alpha)} (\psi(t)-\psi(a))^{\alpha},$$

where M is the bound of a bounded function $(\psi(\cdot)-\psi(a))^{1-\gamma} h(\cdot)$.

Theorem 1. [15, 16] *Let S be a non empty, closed, convex and bounded subset of the Banach algebra X, and let $A : X \to X$ and $B : X \to X$ be two operators such that,*

(a) *A is Lipschitzian with a Lipschitz constant α;*

(b) *B is completely continuous;*

(c) *$x = Ax\, Bx \implies x \in S$ for all $y \in S$;*

(d) *$M\zeta(r) < r$, where $M = \|B(S)\| = sup\|B(x)\| : x \in S$;*

then the operator equation $Ax\, Bx = x$ has a solution in S.

Theorem 2. [15, 16] *Let S be a closed convex and bounded subset of the Banach space X and let $A : X \to X$ and $B : S \to X$ be two operators such that,*

(a) *A is a nonlinear contraction;*

(b) B is continuous and compact;

(c) $x = Ax + Bx$ for all $y \in S \implies x \in S$;

Then the operator equation $Ax + By = x$ has a solution in S.

3. $\psi-$Hilfer Fractional Hybrid Differential Equation of the First Type

We take $X = C_{1-\gamma;\psi}([0,T]), T > 0$ through out this section.
we have the following lemma.

Lemma 5. *Any function satisfies IVP(1) will also satisfy the integral equation*

$$\begin{aligned} x(t) &= f(t,x(t))\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} I_{0+}^{1-\gamma;\psi}\left[\frac{x(0)}{f(0,x(0))}\right] \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds, \\ &= f(t,x(t))\left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0 + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds\right], \quad (13) \\ & \qquad t \in [0,T]. \end{aligned}$$

In addition if the function $x \to \frac{x}{f(0,x)}$ is injective, and $I_{0+}^{\alpha;\psi} g(t,x(t))$ is an absolutely continuous function, then the converse is true.

Proof. From lemma (1), the proof is clear [23, 24].

Theorem 3. *Assume the following.*

(H_1) *The function $x \to \frac{x}{f(t,x)}$ is increasing in $\mathbb{R}$, for all $t \in I$.*

(H_2) *There exists a constant $L_f > 0$ such that $|f(t,x) - f(t,y)| \le L_f|x-y|$, for all $t \in I$ and $x, y \in \mathbb{R}$.*

(H_3) *$K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| < 1$, then the $\psi-$Hilfer Hybrid Fractional differential equation has a solution defined on I, where K is the bound of a bounded function $(\psi(\cdot)-\psi(a))^{1-\gamma}g(\cdot)$.*

Then IVP (1) has a mild solution on I.

Proof. We define a subset S of X by $S = \{x \in X : \|x\| \le N\}$,
where,

$$N = \frac{F_0\left[\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)x_0 + \|h\|\frac{1}{\Gamma(\alpha)}\left(\frac{(\psi(T)-\psi(0))^{\alpha}}{\alpha}\right)\right]}{1 - L_f\left(\frac{\psi(T-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right) + \|h\|\frac{1}{\Gamma(\alpha)}\left(\frac{\psi(T)-\psi(0))^{\alpha}}{\alpha}\right)}. \quad (14)$$

and $F_0 = \sup_{t\in I} \|f(t,0)\|_{C_{1-\gamma;\psi}[0,T]}$.

It is clear that S satisfies the hypotheses of Theorem(1).

Also $IVP(1)$ is equivalent to the $\psi-$Hilfer Hybrid volterra Integral equation:

$$x(t) = f(t,x(t)) \left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right], \tag{15}$$

$$t \in [0,T].$$

Define two operators $A : X \to X$ and $B : S \to X$ by:

$$Ax(t) = f(t,x(t)),\ t \in I. \tag{16}$$

$$Bx(t) = \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds. \tag{17}$$

Then equation (15) is transformed into the operator equation as

$$x(t) = Ax(t)Bx(t),\ t \in I. \tag{18}$$

We shall show that the operators A,B satisfy all the conditions of Theorem(1).

Claim I:

Let $x, y \in X$, then by Hypothesis (H_2)

$$|Ax(t) - Ay(t)| = |f(t,x(t)) - f(t,y(t))| \le L_f |x(t) - y(t)| \le L_f \|x - y\|,\ \forall t \in I.$$

Taking supremum over t, we obtain

$$\|Ax - Ay\|_{C_{1-\gamma;\psi}[I\times\mathbb{R},\mathbb{R}|\{0\}]} \le L_f \|x - y\|,$$
$$\text{where } x, y \in X.$$

Claim II:

We show that B is continuous in S. Let x_n be a sequence in S converging to a point $x \in S$, Then by Lebesgue Dominated Convergence Theorem:

$$\lim_{n\to\infty} Bx_n(t)$$
$$= \lim_{n\to\infty} \left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x_n(s))ds \right],$$

$$= \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} \lim_{n\to\infty} g(s,x_n(s))ds.$$

Hence

$$\lim_{n\to\infty} Bx_n(t) = Bx(t),\ \forall t \in I.$$

Claim III:

B is a Compact Operator on S.

First, we show that $B(S)$ is a uniformly bounded set in X. Let $x \in S$, then by hypothesis (H_3), $\forall t \in I$.

$$\begin{aligned}
&|Bx(t)| \\
&\quad \leq \left|\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds, \\
&\quad \leq \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(t)-\psi(0))^{\alpha}\Gamma(1-\alpha)\frac{(\psi(t)-\psi(0))^{\alpha}}{\alpha\Gamma(\alpha)}, \\
&\quad \leq \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}.
\end{aligned}$$

Thus

$$\|Bx\| \leq \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)},\ \forall x \in X.$$

This shows that B is uniformly bounded on S.

Next we show that $B(S)$ is an equicontinuous set on X.
Let $t_1, t_2 \in I$, then for any $x \in S$,

$$\begin{aligned}
&|Bx(t_1) - Bx(t_2)| = \frac{1}{\Gamma(\alpha)}\times \\
&\left|\int_0^{t_1} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds - \int_0^{t_2} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds\right|, \\
&\qquad\qquad \leq \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds, \\
&\qquad\qquad \leq K\frac{1}{\Gamma(\alpha)}\frac{(\psi(t_1)-\psi(t_2))^{\alpha}}{\alpha}.
\end{aligned}$$

Hence for $\epsilon > 0$, there exist a $\delta > 0$ such that, whenever $|t_1 - t_2| < \delta$, then $|B(x(t_1)) - B(x(t_2))| < \epsilon$, $\forall\ t_1, t_2 \in I$ and $\forall\ x \in X$. This shows that $B(S)$ is an equicontinuous set in X. Then by the Arzelá-Ascoli theorem, B is a continuous and compact operator on S.

Claim IV:
The hypothesis (c) of theorem (1) is satisfied.

Let $x \in X$ and $y \in S$ be arbitrary such that $x = AxBy$, then:

$$
\begin{aligned}
|x(t)| &= |Ax(t)||By(t)|,\\
\leq & |f(t,x(t))| \left| \frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right|,\\
\leq & [f(t,x(t)) - f(t,0)] +\\
& \quad |f(t,0)| \left[\frac{(\psi(t)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s,x(s))ds \right],\\
\leq & (L_f|x(t)| + F_0) \left[\frac{(\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)} x_0 + K(\psi(T)-\psi(0))^{2\alpha} \frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} \right].
\end{aligned}
$$

Thus

$$
|x(t)| \leq \frac{F_0\left[\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)x_0 + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}\right]}{1 - L_f\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right) + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}}. \tag{19}
$$

Taking supremum over $t \in I$,

$$
||x|| \leq \frac{F_0\left[\left(\frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}\right)x_0 + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}\right]}{1 - L_f\left(\frac{\psi(T-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}x_0\right) + K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}} = N. \tag{20}
$$

Thus $x \in S$ and hypothesis (c) of Theorem(1) is satisfied. Finally we have,

$$
\begin{aligned}
M = ||B(S)|| = \sup ||Bx|| : x \in S\\
\leq K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0|.
\end{aligned}
$$

and so

$$
\alpha M \leq K(\psi(T)-\psi(0))^{2\alpha}\frac{\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)} + \frac{\psi(T)-\psi(0))^{\gamma-1}}{\Gamma(\gamma)}|x_0| < 1.
$$

Thus all conditions of Theorem(1) are satisfied and hence the operator equation $AxBx = x$ has a solution in S. As a result equation(1) has a solution defined on I. This completes the proof.

4. $\psi-$Hilfer Fractional Hybrid Differential Equation of second type

Consider the $\psi-$Hilfer Fractional Hybrid Differential Equation of the form:

$$\begin{cases} {}^{H}\mathbb{D}_{t_0+}^{\alpha,\beta;\psi}[x(t)-f(t,x(t))] & = g(t,x(t)),\ a.e.\ t\in J=[t_0,t_0+a], \\ I_{t_0}^{1-\gamma;\psi}[x(t_0)-f(t_0,x(t_0))] & = \sigma\in\mathbb{R}. \end{cases} \tag{21}$$

Lemma 6. *Any function satisfies $IVP(2)$ will also satisfy the integral equation*

$$x(t)= f(t,x(t))+\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}I_{t_0+}^{1-\gamma}(x(t_0)-f(t_0,x(t_0)))+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds, \tag{22}$$

$$= f(t,x(t))+\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}g(s,x(s))ds, \tag{23}$$

$$t\in[t_0,t_0+a].$$

In addition if the function $x\to x-f(0,x)$ is injective, and $I_{0+}^{\alpha;\psi}g(t,x(t))$ is an absolutely continuous function, then the converse is true.

Proof. From lemma (1), the proof is clear [23, 24].

Theorem 4. *Assume the following*

($\mathcal{A}1$) *There exists constants $M_f\geq L_f>0$ such that* $|f(t,x)-f(t,y)|\leq\frac{L_f|x-y|}{(M_f+|x-y|)}$ *for all $t\in J$ and $x,y\in\mathbb{R}$;*

($\mathcal{A}2$) *There exists $r>0$ such that* $r>L_f+F_0+\left|\frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right|+\frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}(\psi(t_0+a)-\psi(t_0))^{2\alpha}$, *where $F_0=\sup_{t\in J}|f(t,0)|$;*

($\mathcal{A}3$) *K is the bound of a bounded function $(\psi(\cdot)-\psi(a))^{1-\gamma}g(\cdot)$.*

Then equation (21) has a mild solution on J.

Proof. Let $X=C_{1-\gamma;\psi}([t_0,t_0+a]),T>0$ and define the set $S\subset X$ by $S=\{x\in X:||x||\leq r\}$.

We prove the existence of a mild solution to problem (21) by discussing the solution to the integral equation (23) which is equivalent to the operator equation,

$$Ax(t) + Bx(t) = x(t),\ t \in J. \tag{24}$$

where,

$$\begin{aligned}
&Ax(t) = f(t, x(t)).\\
&Bx(t) = \frac{(\psi(t) - \psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} g(s, x(s))ds.
\end{aligned}$$

Now we prove our Theorem by proving that the conditions of Theorem(2) are satisfied.

Step I:

Using the hypothesis $(\mathcal{A}1)$ we get:

$$\begin{aligned}
|Ax(t) - Ay(t)| =& |f(t, x(t)) - f(t, y(t))|,\\
\leq& \frac{L_f|x(t) - y(t)|}{M_f + |x(t) - y(t)|},\\
\leq& \frac{L_f||x - y||}{M_f + ||x - y||}.
\end{aligned}$$

Thus the operator A is a nonlinear contraction with the function ϕ defined by $\phi(r) = \frac{L_f r}{M_f + r}$.

Step II:

Similarly by Theorem(3), we can prove that B is continuous and compact.

Step III:

Let $x \in X$ be fixed and $y \in S$ be arbitrary such that $x = Ax + By$, then we get:

$$
\begin{aligned}
&|x(t)|\\
&\quad\leq |Ax(t)|+|By(t)|,\\
&\quad\leq |f(t,x(t))|+\left|\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right|\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds,\\
&\quad\leq |f(t,x(t))-f(t,0)|+|f(t,0)|+\left|\frac{(\psi(t)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right|\\
&\qquad+\frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}|g(s,x(s))|ds,\\
&\quad\leq L_f+F_0+\left|\frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right|+\frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}(\psi(t_0+a)-\psi(t_0))^{2\alpha},\\
&\quad\leq r.
\end{aligned}
$$

which proves that $||x||\leq r$. Thus $x\in S$.

Thus the conditions of Theorem (2) are satisfied; then the operator equation $Ax(t)+Bx(t)=x(t)$ has a solution in S which proves the existence of a mild solution to problem (21) in J.

We finish the section with the following example.

Example 1. Consider the $\psi-$Hilfer fractional Hybrid differential equation

$$
\begin{aligned}
&{}^{H}\mathbb{D}_{0+}^{0.5,0;x}\left(x(t)-\frac{sin(t)|x(t)|}{2+|x(t)|}\right)=\frac{tx(t)}{1+|x(t)|},\\
&I_{0+}^{0.5;x}(x(0)-f(0,x(0)))=1,\ t\in[0,\pi].
\end{aligned}
$$

We get that,

$$
\begin{aligned}
|f(t,x(t))-f(t,y(t))| &\leq \frac{|x(t)-y(t)|}{2+|x(t)-y(t)|+|y(t)|},\\
&\leq \frac{|x(t)-y(t)|}{2+|x(t)-y(t)|}.
\end{aligned}
$$

and where $|g(t,x(t))|\leq t$, we get that all hypotheses of theorem (4) are satisfied with

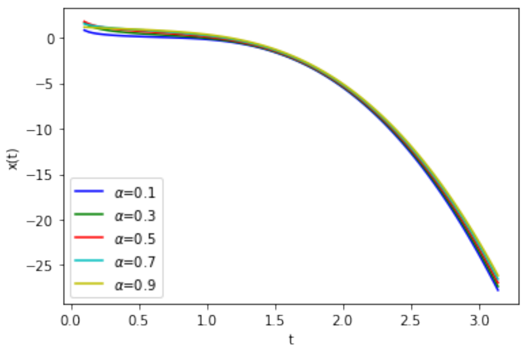
$$L_f=1, M_f=2, T=\pi, F_0=0.$$

We conclude that

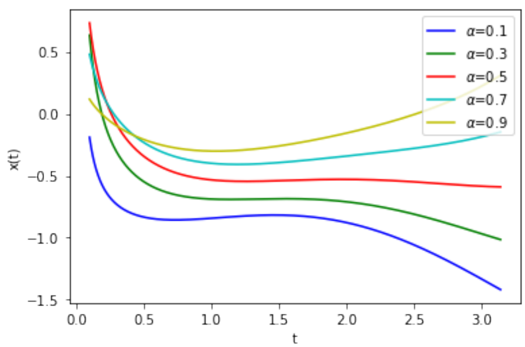
$$L_f + F_0 + \left|\frac{(\psi(t_0+a)-\psi(t_0))^{\gamma-1}}{\Gamma(\gamma)}\sigma\right| + \frac{K\Gamma(1-\alpha)}{\alpha\Gamma(\alpha)}(\psi(t_0+a)-\psi(t_0))^{2\alpha} =$$

$$1+\frac{1}{\pi}+\pi^2\sqrt{\pi} < 19.$$

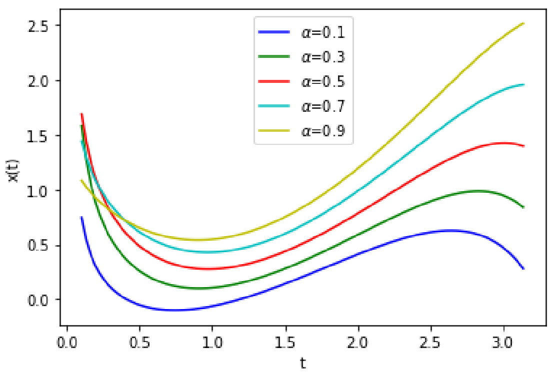
For different $x(t)$ and α the solutions corresponding to the problem are plotted below. In each case red colour indicated the solution of the above problem.



(a) $x(t) = t^3$

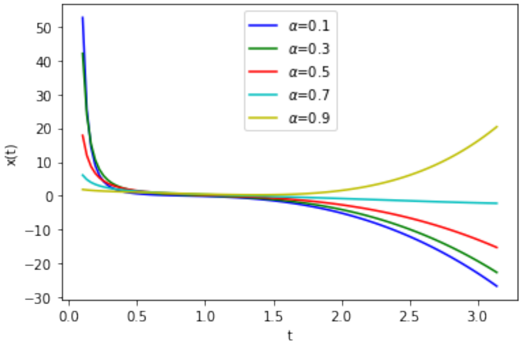


(b) $x(t) = 1 + t$

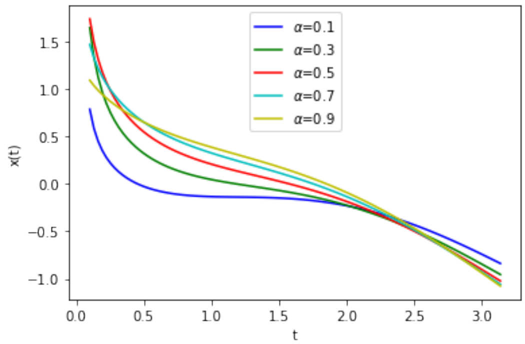


(c) $x(t) = sint$

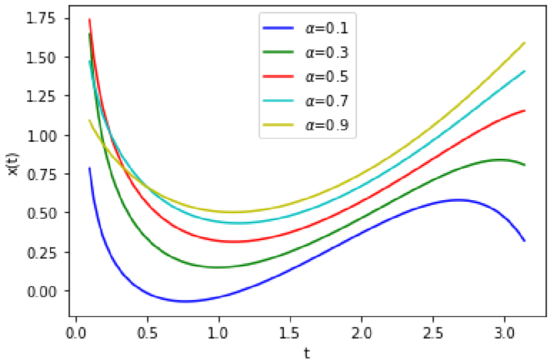
Corresponding to the above example with $f(t) = \frac{sin|x(t)|}{2+|x(t)|}$ and $g(t) = \frac{tx(t)}{1+|x(t)|}$ we plotted some graphs with different $x(t)$ and $\psi(t)$.



(a) $x(t) = t^3, \psi(t) = t^3$



(b) $x(t) = t, \psi(t) = ln(t+1)$



(c) $x(t) = sint, \psi(t) = ln(t+1)$

5. Conclusions

In this paper we proved the existences of two types of Hybrid fractional differential equation involving $\psi-$Hilfer fractional Derivative, which is a generalised fractional derivative using different fixed point theorems and concluded the paper with an example. we plotted some graphs for different values of $\psi(t)$ and $x(t)$ corresponding to the example.

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GENERALIZED *H*- RESOLVENT EQUATION WITH $H - \phi - \eta$ ACCRETIVE OPERATOR

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Abstract: In this paper, we consider extended variational-like inclusion problem (for short EVLIP) which contains many known variational inclusions existing in literature. In connection with EVLIP we consider a generalized resolvent equation problem with H-ϕ-η-accretive operator called generalized H-resolvent equation problem (for short H-REP). To compute the approximate solution of H-REP, we introduce an algorithm. Convergence of sequences procreated by algorithm are also studied.

Keywords and Phrases: Extended Variational-Like Inclusion, Resolvent Operator, Generalized H-Resolvent Equation, Algorithm, Convergence.

2010 Mathematics Subject Classification: 49J40, 47J20, 47H06, 49J53.

1. Introduction, Notations, Definitions and Known Results

In past years, generalized forms of variational inequalities, variational inclusions and variational-like inclusions, have been expansively studied and extended in different directions to study the practical problems arising in optimization, economics, finance, applied science etc. See, for example [1, 3-6, 10, 12-19, 24] and references therein. As we all know that, develop an adept iterative algorithm for approximation solution of variational inclusions is most interesting aspect of variational inclusion theory. It is well known that projection method and Wiener-Hopf equation can not be improved to solve nonlinear variational inequalities and variational inclusions. Then resolvent operator technique is strategic and useful for

approximation solvability of variational inclusions. Lot of studies and research has been done on several techniques for computing the solution of the variational inclusion and variational-like inclusion in the setting of different spaces, see [1, 3-8, 11-19, 21, 22, 24] and references therein.
Fang and Huang [16] have extended the concept of resolvent operators to the new H-accretive operators, which was associated with the m-accretive operators. Ahamd and Ansari [5] have considered generalized variational inclusions (GVI) and they also considered generalized resolvent equation with H-accretive operator called H-resolvent equation (H-RE) and suggested the algorithm for unique solution of GVI and H-RE and studied the convergence of iterative sequences generated by the proposed algorithm.
In this paper, we consider extended variational-like inclusion problem (for short EVLIP) which contains many known variational inclusions existing in literature. In connection with EVLIP we consider a generalized resolvent equation problem with H-ϕ-η-accretive operator called generalized H-resolvent equation problem (for short H-REP). To compute the approximate solution of H-REP, we introduce an algorithm. Convergence of sequences procreated by algorithm are also studied.

Now, we present some basic notations, definitions and known results of functional analysis relevant to our paper. Throughout the paper unless otherwise specified, we assume that X is a real Banach space endowed with a norm $\|.\|$ and topological dual X^*. d is a metric induced by the norm $\|.\|$, $CB(X)$ is the family of all non-empty closed and bounded subsets of X, 2^X is family of all non-empty subsets of X, and $\mathcal{D}(.,.)$ is the Hausdorff metric on $CB(X)$ defined by

$$\mathcal{D}(E, F) = \max\{\sup_{x\in E} d(x, F), \sup_{y\in F} d(E, y)\},$$

where $d(x, F) = \inf_{y\in F} d(x, y)$ and $d(E, y) = \inf_{x\in E} d(x, y)$.

Definition 1.1. [9] *Let X be a real Banach space then generalized duality mapping $J_q : X \to 2^{X^*}$ is defined by*

$$J_q(x) = \{f \in X^* : \langle x, f\rangle = \|x\|^q, \|f\| = \|x\|^{q-1}\}, \forall x \in X,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, $J_q(x) = \|x\|^{q-1} J_2(x)$ for $x \neq 0$ and J_q is a single-valued if X is strictly convex. If X is real Hilbert space, J_2 becomes the identity mapping on X.

Definition 1.2. [2] *A Banach space X is said to be uniformly smooth if for any given $\epsilon > 0$, there exists $\delta > 0$ such that*

$$\frac{\|x + y\| + \|x - y\|}{2} - 1 \le \epsilon\|y\|$$

holds.
The function

$$\rho_X(t) = \sup\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| = 1, \|y\| = t\}.$$

is called the modulus of smoothness of the space X.

Remark 1.3. *The space X is uniformly convex if and only if $\rho_X(\epsilon) > 0$ for all $\epsilon > 0$, and it is called uniformly smooth if and only if $\lim_{t\to 0} \frac{\rho_X(t)}{t} = 0$.*

Definition 1.4. [2] *The space X is called q-uniformly smooth, if there exist a constant $C > 0$ such that*

$$\rho_X(t) \le Ct^q, q > 1.$$

Note that J_q is single valued if X is uniformly smooth. The following inequality in q-uniformly smooth Banach spaces has been proved by Xu [25].

Lemma 1.5. [25] *Let X be a real uniformly smooth Banach space. Then X is q-uniformly smooth if and only if there exists a constant $c_q > 0$ such that for all $x, y \in X$,*

$$\|x+y\|^q \le \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q.$$

Definition 1.6. [3] *Let $A, B : X \to X$ and $\eta, H : X \times X \to X$ be the single valued mappings.*

(i) A is said to be η-accretive, if $\langle Ax - Ay, J_q(\eta(x,y))\rangle \ge 0, \forall x, y \in X$;

(ii) A is said to be strictly η-accretive, if A is η-accretive and equality holds if and only if $x = y$;

(iii) $H(A, .)$ is said to be α-strongly η-accretive with respect to A, if there exist a constant $\alpha > 0$ such that $\langle H(Ax, u) - H(Ay, u), J_q(\eta(x, y))\rangle \ge \alpha\|x - y\|^q, \forall x, y, u \in X$;

(iv) $H(., B)$ is said to be β-relaxed η-accretive with respect to B, if there exist a constant $\beta > 0$ such that $\langle H(u, Bx) - H(u, By), J_q(\eta(x, y))\rangle \ge (-\beta)\|x - y\|^q, \forall x, y, u \in X$;

(v) $H(., .)$ is said to r_1-Lipschitz continuous with respect to A, if there exist a constant $r_1 > 0$ such that $\|H(Ax, u) - H(Ay, u)\| \le r_1\|x - y\|, \forall x, y, u \in X$. In a similar way, we can define the Lipschitz continuity of the mapping $H(., .)$ with respect to B.

(vi) η *is said to be* τ*-Lipschitz continuous, if there exist a constant* $\tau > 0$ *such that* $\|\eta(x,y)\| \leq \tau\|x-y\|, \forall x,y \in X$.

Definition 1.7. [3] *Let* $N, P : X \times X \times X \to X$ *and* $\eta : X \times X \to X$ *be the single valued mappings. Let* $M : X \times X \to 2^X$ *be multi-valued mapping.*

(i) M *is said to be* η*-accretive, if* $\langle u - v, J_q(\eta(x,y))\rangle \geq 0, \forall x,y \in X, u \in M(x,z), v \in M(y,z)$*, for each fixed* $z \in X$*;*

(ii) M *is said to be strictly* η*-accretive, if* M *is* η*-accretive and equality holds if and only if* $x = y$*;*

(iii) N *is said to be* t*-relaxed* η*-accretive in the first argument, if there exist a constant* $t > 0$ *such that* $\langle N(x,u,v) - N(y,u,v), J_q(\eta(x,y))\rangle \geq -t\|x-y\|^q, \forall x,y,u,v \in X$*;*

(iv) N *is said to be* ξ*-Lipschitz continuous in the first argument, if there exists a constant* $\xi > 0$ *such that* $\|N(x,u,v) - N(y,u,v)\| \leq \xi\|x-y\|, \forall x,y,u,v \in X$*. Similarly, we can define the lipschitz continuity of* N *in the second and third argument.*

(v) P *is said to be* ζ*-Lipschitz continuous in the first argument, if there exists a constant* $\zeta > 0$ *such that* $\|P(x,u,v) - P(y,u,v)\| \leq \zeta\|x-y\|, \forall x,y,u,v \in X$*. Similarly, we can define the lipschitz continuity of* P *in the second and third argument.*

Definition 1.8. [16] *The operator* $H : X \to X$ *is said to be*

(i) *accretive if* $\langle H(x) - H(y), J_q(x-y)\rangle \geq 0, \forall x,y \in X$

(ii) *strongly accretive if there exists a constant* $r > 0$ *such that* $\langle H(x) - H(y), J_q(x-y)\rangle \geq r\|x-y\|^q, \forall x,y \in X$

Definition 1.9. [3] *Let* $\phi, A, B : X \to X$ *and* $H, \eta : X \times X \to X$ *be the single-valued mappings. Let* $M : X \times X \to 2^X$ *be a multi-valued mapping.* M *is said to be* $H(.,.) - \phi - \eta$*-accretive operator with respect to mappings* A *and* B*, if for each fixed* $z \in X$*,* $\phi \circ M(.,z)$ *is* η*-accretive in the first argument and* $(H(A,B) + \phi \circ M(.,z))(X) = X$*.*

Theorem 1.10. [3] *Let* $H(A,B)$ *be* α*-strongly* η*-accretive with respect to* A*,* β*-relaxed* η*-accretive with respect to* B*,* $\alpha > \beta$*. Let* M *be an* $H(.,.) - \phi - \eta$*-accretive operator with respect to mappings* A *and* B*. Then the operator* $(H(A,B) + \phi \circ$

$M(.,z))^{-1}$ is single-valued for each fixed $z \in X$.

Definition 1.11 [3] *Let $H(A,B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B, $\alpha > \beta$. Let M be an $H(.,.)-\phi-\eta$-accretive operator with respect to mappings A and B. Then for each fixed $z \in X$, the resolvent operator $R^{H(.,.)-\phi-\eta}_{M(.,z)} : X \to X$ is defined by*

$$R^{H(.,.)-\phi-\eta}_{M(.,z)}(u) = (H(A,B) + \phi \circ M(.,z))^{-1}(u), \forall u \in X.$$

Theorem 1.12. [3] *Let $H(A,B)$ be α-strongly η-accretive with respect to A, β-relaxed η-accretive with respect to B, $\alpha > \beta$ and η is τ-Lipschitz continuous. Let $M : X \times X \to 2^X$ is a $H(.,.)-\phi-\eta$-accretive operator with respect to mappings A and B. Then the resolvent operator $R^{H(.,.)-\phi-\eta}_{M(.,z)} : X \to X$ is $\frac{\tau^{q-1}}{\alpha-\beta}$-Lipschitz continuous i.e.,*

$$\|R^{H(.,.)-\phi-\eta}_{M(.,z)}(u) - R^{H(.,.)-\phi-\eta}_{M(.,z)}(v)\| \leq \frac{\tau^{q-1}}{\alpha-\beta}\|u-v\|, \forall u, v \in X \text{ and each fixed } z \in X.$$

2. Extended Variational-Like Inclusion Problem

Let $G, J, K, L, R, S, T : X \to CB(X)$ be multi-valued mappings. $A, B, \phi : X \to X$, $H, \eta : X \times X \to X$ and $N, P : X \times X \times X \to X$ be single valued mappings. Suppose $M : X \times X \to 2^X$ be a multi-valued mapping such that M is $H(.,.)-\phi-\eta-$ accretive operator.

We consider the following problem of finding $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ and

$$0 \in N(u,v,w) - P(j,k,l) + M(x,z) \tag{2.1}$$

Problem (2.1) is called extended variational-like inclusion problem.

Below are some special cases of our problem:

(i) If $P \equiv R \equiv 0$ and $N(.,.,.) = N(.,.)$ then our problem reduces to the problem considered by Ahmad et al. [3].

(ii) If $P \equiv R \equiv 0$ and $N(.,.,.) = N(.,.)$, X is real Hilbert space and $M(.,z)$ is maximal monotone operator then problem similar to (3.1) was introduced and studied by Huang et al. [20].

(iii) If $P \equiv T \equiv R \equiv G \equiv 0$, S is single-valued and identity mapping and $N(.,.,.) = N(.), M(.,.) = M(.)$ then our problem reduces to the problem considered by Bi et al. [11], that is find $u \in X$ such that $0 \in N(u) + M(u)$.

It is easy to see that (2.1) includes many more known variational inclusions considered and studied in the literature.

3. Generalized H-Resolvent Equation Problem

In this section, we propose the *generalized H-resolvent equation problem* for the case when $H(.,.) = H(.)$ along with some suitable assumption. we consider the following *generalized H-resolvent equation problem* to find $s, x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ such that

$$N(u, v, w) - P(j, k, l) + \phi^{-1} J^{H-\phi-\eta}_{M(.,z)}(s) = 0 \tag{3.1}$$

where $J^{H-\phi-\eta}_{M(.,z)} = I - H(R^{H-\phi-\eta}_{M(.,z)}), I$ is the identity operator, $R^{H-\phi-\eta}_{M(.,z)}$ is the *H-resolvent operator.* The equation (3.1) is called *generalized H-resolvent equation.*

Lemma 3.1. *Let X be a q-uniformly smooth Banach space. $G, J, K, L, R, S, T : X \to CB(X)$ be multi-valued mappings, $H : X \to X$ be single valued mapping and $\phi : X \to X$ be a mapping satisfying $\phi(x + y) = \phi(x) + \phi(y)$ and $ker(\phi) = 0$, where $ker(\phi) = \{x \in X : \phi(x) = 0\}$. Let $\eta : X \times X \to X$ be single valued mappings and $N, P : X \times X \times X \to X$ be also single valued mappings. Let $M : X \times X \to 2^X$ be a multi-valued mapping such that M is $H - \phi - \eta -$ accretive operator. Then (x, u, v, w, z, j, k, l) where $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$, is a solution of problem (3.1) if and only if (x, u, v, w, z, j, k, l) satisfies*

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)]$$

Proof. Let (x, u, v, w, z, j, k, l) where $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ satisfies the above equation, i.e.,

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)]$$

Using the definition of resolvent operator, we have

$$\begin{aligned}
x &= (H(.) + \phi \circ M(., z))^{-1}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)] \\
&\Leftrightarrow H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l) \in H(x) + \phi \circ M(x, z) \\
&\Leftrightarrow 0 \in \phi \circ N(u, v, w) - \phi \circ P(j, k, l) + \phi \circ M(x, z) \\
&\Leftrightarrow 0 \in \phi(N(u, v, w) - P(j, k, l)) + M(x, z)) \\
&\Leftrightarrow \phi^{-1}(0) \in N(u, v, w) - P(j, k, l) + M(x, z) \\
&\Leftrightarrow 0 \in N(u, v, w) - P(j, k, l) + M(x, z).
\end{aligned}$$

This completes the proof.
Now we present an equivalence between (2.1) and (3.1).

Proposition 3.2. [5] *The (2.1) has a solution (x, u, v, w, z, j, k, l) with $x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$ if and only if (3.1) has a solution $(s, x, u, v, w, z, j, k, l)$ with $s, x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$, where*

$$x = R^{H-\phi-\eta}_{M(.,z)}(s) \tag{3.2}$$

and

$$s = H(x) - \phi \circ (N(u, v, w) - P(j, k, l)) \tag{3.3}$$

Proof. Let (x, u, v, w, z, j, k, l) be the solution of (2.1) then by lemma 3.1 it is a solution of following equation

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ (N(u, v, w) - P(j, k, l))] \tag{3.4}$$

Let $s = H(x) - \phi \circ (N(u, v, w) - P(j, k, l))$, then from (3.4), we have

$$x = R^{H-\phi-\eta}_{M(.,z)}(s).$$

By using the fact that $J^{H-\phi-\eta}_{M(.,z)} = I - H(R^{H-\phi-\eta}_{M(.,z)})$, we obtain

$$\begin{aligned}
s &= H(R^{H-\phi-\eta}_{M(.,z)}(s)) - \phi \circ (N(u, v, w) - P(j, k, l)) \\
\Leftrightarrow s - H(R^{H-\phi-\eta}_{M(.,z)}(s)) &= -\phi \circ (N(u, v, w) - P(j, k, l)) \\
\Leftrightarrow [I - H(R^{H-\phi-\eta}_{M(.,z)})](s) &= -\phi \circ (N(u, v, w) - P(j, k, l)) \\
\Leftrightarrow J^{H-\phi-\eta}_{M(.,z)}(s) &= -\phi \circ (N(u, v, w) - P(j, k, l))
\end{aligned}$$

Hence $N(u, v, w) - P(j, k, l) + \phi^{-1} J^{H-\phi-\eta}_{M(.,z)}(s) = 0$
Based on proposition 3.2, we suggest the following iterative method to compute the approximate solution of (3.1).

Algorithm 3.3. *For any given $s_0, x_0 \in X$, $u_0 \in S(x_0)$, $v_0 \in T(x_0)$, $w_0 \in R(x_0)$, $z_0 \in G(x_0)$, $j_0 \in J(x_0)$, $k_0 \in K(x_0)$, and $l_0 \in L(x_0)$, compute the sequences $\{x_n\}$, $\{u_n\}$, $\{v_n\}$, $\{w_n\}$, $\{z_n\}$, $\{j_n\}$, $\{k_n\}$ and $\{l_n\}$ by the following iterative schemes*

$$x_{n+1} = R^{H-\phi-\eta}_{M(.,z)}(s_{n+1}) \tag{3.5}$$

$$u_n \in S(x_n), \|u_n - u_{n+1}\| \leq \mathcal{D}(S(x_n), S(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.6}$$

$$v_n \in T(x_n), \|v_n - v_{n+1}\| \leq \mathcal{D}(T(x_n), T(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.7}$$

$$w_n \in R(x_n), \|w_n - w_{n+1}\| \leq \mathcal{D}(R(x_n), R(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.8}$$

$$z_n \in G(x_n), \|z_n - z_{n+1}\| \leq \mathcal{D}(G(x_n), G(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.9}$$

$$j_n \in J(x_n), \|j_n - j_{n+1}\| \leq \mathcal{D}(J(x_n), J(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.10}$$

$$k_n \in K(x_n), \|k_n - k_{n+1}\| \leq \mathcal{D}(K(x_n), K(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.11}$$

$$l_n \in L(x_n), \|l_n - l_{n+1}\| \leq \mathcal{D}(L(x_n), L(x_{n+1})) + \varepsilon^{n+1}\|x_n - x_{n+1}\| \tag{3.12}$$

$$s_{n+1} = H(x_n) - \phi \circ (N(u_n, v_n, w_n) - P(j_n, k_n, l_n)) \tag{3.13}$$

$n = 0, 1, 2, 3, \ldots$
Now we study the existence of the solution of (3.1) and the convergence of iterative sequences generated by the above algorithm to the exact solution of (3.1).

Theorem 3.4. *Let X be a real q-uniformly smooth Banach space and $H : X \to X$ be a strongly accretive and Lipschitz continuous operator with constant r and γ, respectively. Let $\phi \circ N$ and $\phi \circ P$ be both Lipschitz continuous in all three arguments with constants ξ_1, ξ_2, ξ_3 and $\zeta_1, \zeta_2, \zeta_3$ respectively, also let G, J, K, L, R, S, T be $\mathcal{D}$-Lipschitz continuous with constants $\lambda_G, \lambda_J, \lambda_K, \lambda_L, \lambda_R, \lambda_S$ and λ_T, respectively. Suppose that $M : X \times X \to 2^X$ is $H - \phi - \eta$-accretive multivalued map such that*

$$\begin{aligned} 0 < \frac{1}{r}[\gamma^q - (q - c_q)\{\xi_1(\lambda_S + \varepsilon^n) + \xi_2(\lambda_T + \varepsilon^n) + \xi_3(\lambda_R + \varepsilon^n)\}^q - (q - c_q)\{\zeta_1(\lambda_J + \varepsilon^n) \\ + \zeta_2(\lambda_K + \varepsilon^n) + \zeta_3(\lambda_L + \varepsilon^n)\}^q]^{\frac{1}{q}} < 1 \end{aligned} \tag{3.14}$$

holds. Then there exists a unique solution $(s, x, u, v, w, z, j, k, l)$ with $s, x \in X, u \in S(x), v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$, and the iterative sequences $\{s_n\}, \{x_n\}, \{u_n\}, \{v_n\}, \{w_n\}, \{z_n\}, \{j_n\}, \{k_n\}$ and $\{l_n\}$ generated by Algorithm 3.3 converge to s,x,u,v,w,z,j,k,l strongly in X, respectively.
Proof.

$$\begin{aligned} \|s_{n+1} - s_n\| = \|H(x_n) - H(x_{n-1}) - \phi \circ [(N(u_n, v_n, w_n) - N(u_{n-1}, v_{n-1}, w_{n-1})) \\ - (P(j_n, k_n, l_n) - P(j_{n-1}, k_{n-1}, l_{n-1}))]\| \end{aligned} \tag{3.15}$$

By Lemma 1.5, we have

$$\begin{aligned}
&\|H(x_n)-H(x_{n-1})-\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)\\
&\qquad -P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\le \|H(x_n)-H(x_{n-1})\|^q-q\langle\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)\\
&\qquad -P(j_{n-1},k_{n-1},l_{n-1})),J_q(H(x_n)-H(x_{n-1}))\rangle+c_q\|\phi\circ(N(u_n,v_n,w_n)\\
&\qquad -N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q \qquad (3.16)
\end{aligned}$$

Again by Lemma 1.5,

$$\begin{aligned}
&\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\le \|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|^q-(q-c_q)\|\phi\circ(P(j_n,k_n,l_n)\\
&\qquad -P(j_{n-1},k_{n-1},l_{n-1}))\|^q \qquad (3.17)
\end{aligned}$$

Now,

$$\begin{aligned}
&\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|\\
&\le \|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_n,w_n))+\phi\circ(N(u_{n-1},v_n,w_n)-N(u_{n-1},v_{n-1},w_n))\\
&\qquad +\phi\circ(N(u_{n-1},v_{n-1},w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|\\
&\le \xi_1\|u_n-u_{n-1}\|+\xi_2\|v_n-v_{n-1}\|+\xi_3\|w_n-w_{n-1}\|\le\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)\\
&\qquad +\xi_3(\lambda_R+\varepsilon^n)\}\|x_n-x_{n-1}\|
\end{aligned}$$

So,

$$\begin{aligned}
&\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))\|^q\le\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)\\
&\qquad +\xi_3(\lambda_R+\varepsilon^n)\}^q\|x_n-x_{n-1}\|^q \qquad (3.18)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\|\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q\le\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)\\
&\qquad +\zeta_3(\lambda_L+\varepsilon^n)\}^q\|x_n-x_{n-1}\|^q \qquad (3.19)
\end{aligned}$$

Using (3.18) and (3.19), (3.17) becomes

$$\begin{aligned}
&\|\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)-P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\le[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q-(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)\\
&\qquad +\zeta_3(\lambda_L+\varepsilon^n)\}^q]\|x_n-x_{n-1}\|^q
\end{aligned}$$

Therefore (3.16) becomes

$$\begin{aligned}
&\|H(x_n)-H(x_{n-1})-\phi\circ(N(u_n,v_n,w_n)-N(u_{n-1},v_{n-1},w_{n-1}))-\phi\circ(P(j_n,k_n,l_n)\\
&\qquad -P(j_{n-1},k_{n-1},l_{n-1}))\|^q\\
&\leq \gamma^q\|x_n-x_{n-1}\|^q-(q-c_q)[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q\\
&\qquad -(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)+\zeta_3(\lambda_L+\varepsilon^n)\}^q]^q\|x_n-x_{n-1}\|^q
\end{aligned}$$

So from (3.15), we have

$$\begin{aligned}
\|s_{n+1}-s_n\| &\leq [\gamma^q-(q-c_q)[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q\\
&-(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)+\zeta_2(\lambda_K+\varepsilon^n)+\zeta_3(\lambda_L+\varepsilon^n)\}^q]^q]^{\frac{1}{q}}\|x_n-x_{n-1}\| \quad (3.20)
\end{aligned}$$

By (3.5), we obtain

$$\begin{aligned}
\|x_n-x_{n-1}\| &= \|x_n-x_{n-1}+x_n-x_{n-1}-R^{H-\phi-\eta}_{M(.,z)}(s_n)+R^{H-\phi-\eta}_{M(.,z)}(s_{n-1})\|\\
&\leq 2\|x_n-x_{n-1}\|-\|R^{H-\phi-\eta}_{M(.,z)}(s_n)-R^{H-\phi-\eta}_{M(.,z)}(s_{n-1})\|\\
&\leq 2\|x_n-x_{n-1}\|-\frac{1}{r}\|s_n-s_{n-1}\|
\end{aligned}$$

Where $R^{H-\phi-\eta}_{M(.,z)}$ is $\frac{1}{r}$-Lipschitz continuous.
Therefore,

$$\|x_n-x_{n-1}\|\leq \frac{1}{r}\|s_n-s_{n-1}\| \quad (3.21)$$

By combining (3.20) and (3.21), we get

$$\|s_{n+1}-s_n\|\leq b\|s_n-s_{n-1}\| \quad (3.22)$$

where

$$\begin{aligned}
b=\frac{1}{r}[\gamma^q-(q-c_q)[\{\xi_1(\lambda_S+\varepsilon^n)+\xi_2(\lambda_T+\varepsilon^n)+\xi_3(\lambda_R+\varepsilon^n)\}^q-(q-c_q)\{\zeta_1(\lambda_J+\varepsilon^n)\\
+\zeta_2(\lambda_K+\varepsilon^n)+\zeta_3(\lambda_L+\varepsilon^n)\}^q]^q]^{\frac{1}{q}} \quad (3.23)
\end{aligned}$$

From (3.14), it follows that $0\leq b<1$. Consequently, from (3.22), we see that the sequence $\{s_n\}$ is cauchy sequence in a Banach space X. So there exist $s\in X$ such that $\{s_n\}\to s$ as $n\to\infty$. From (3.21), we know that the sequence $\{x_n\}$ is a cauchy sequence in X, so there exist $x \in X$ such that $\{x_n\} \to x$. Also from Algorithm 3.3, we have

$$\begin{aligned}
\|u_n - u_{n+1}\| &\leq (\lambda_S + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|v_n - v_{n+1}\| &\leq (\lambda_T + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|w_n - w_{n+1}\| &\leq (\lambda_R + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|z_n - z_{n+1}\| &\leq (\lambda_G + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|j_n - j_{n+1}\| &\leq (\lambda_J + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|k_n - k_{n+1}\| &\leq (\lambda_K + \varepsilon^n)\|x_n - x_{n+1}\| \\
\|l_n - l_{n+1}\| &\leq (\lambda_L + \varepsilon^n)\|x_n - x_{n+1}\|.
\end{aligned}$$

and hence $\{u_n\}, \{v_n\}, \{w_n\}, \{z_n\}, \{j_n\}, \{k_n\}$ and $\{l_n\}$ are also cauchy sequences in X, so that there exist u, v, w, z, j, k, l in X such that $\{u_n\} \to u, \{v_n\} \to v, \{w_n\} \to w, \{z_n\} \to z, \{j_n\} \to j, \{k_n\} \to k$ and $\{l_n\} \to l$ as $n \to \infty$. Now using the continuity of operators $R, S, T, G, J, K, L, H, \phi \circ N, \phi \circ P, \eta$ and M and by Algorithm 3.3, we have

$$x = R^{H-\phi-\eta}_{M(.,z)}[H(x) - \phi \circ N(u, v, w) + \phi \circ P(j, k, l)].$$

Now, we shall show that $u \in S(x)$
$d(u, S(x)) \leq \|u - u_n\| + d(u, S(x)) \leq \|u - u_n\| + \mathcal{D}(S(x_n), S(x))$
$\leq \|u - u_n\| + \lambda_S\|x_n - x\| \to 0$ as $n \to \infty$.
$\Rightarrow d(u, S(x)) = 0$, since $S(x) \in CB(X)$ [23], it follows that $u \in S(x)$.
Similarly we can prove that $v \in T(x), w \in R(x), z \in G(x), j \in J(x), k \in K(x)$, and $l \in L(x)$. Let $(s^*, x^*, u^*, v^*, w^*, z^*, j^*, k^*, l^*)$ be another solution of (H-REP). Then by Lemma 3.1, we have

$$x^* = R^{H-\phi-\eta}_{M(.,z)}[H(x^*) - \phi \circ N(u^*, v^*, w^*) + \phi \circ P(j^*, k^*, l^*)]$$

From above two equations and by using the same argument given above we get

$$\|x - x^*\| \leq b\|x - x^*\|,$$

where b is defined by (3.23). Since $0 \leq b < 1$, we get $x = x^*$, then by algorithm 3.3 $(s^*, x^*, u^*, v^*, w^*, z^*, j^*, k^*, l^*)$ is unique solution of (H-REP). This completes the proof.

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PROPERTIES OF FUZZY PERFECT INTRINSIC EDGE-MAGIC GRAPHS

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Abstract: In this paper, we have discussed the idea of fuzzy perfect intrinsic edge-magic labelling and perfect intrinsic edge-magic graphs. We have checked fuzzy path, cycle, paw graph, banner graph, star graph & friendship graph are perfect intrinsic edge-magic graphs with intrinsic super constant. The vital and competent condition also discussed for the fuzzy perfect intrinsic edge-magic graphs. Quasi perfect intrinsic edge-magic graphs are also introduced. Some theorems related to stated graphs have been presented.

Keywords and Phrases: Fuzzy perfect intrinsic edge-magic labelling, quasi perfect intrinsic edge-magic graph, intrinsic super constant, weak constant.

2010 Mathematics Subject Classification: 05C72, 05C78.

1. Introduction

Fuzzy set was firstly introduced by [13]. Then various researches added productive concepts to develop fuzzy sets theory like [10] and [3]. In 1987 Bhattacharya has succeeded to develop the connectivity notions between fuzzy bridge and fuzzy cut nodes [2]. A fuzzy graph contains many properties similar to crisp graph due to generalization of crisp graphs but it diverge at many places.

A crisp graph G is an order pair of vertex-set V and edge set E such that $E \subseteq V \times V$. In addition $v = |V|$ is said to order and $e = |E|$ is called size of the graph G respectively. In a crisp graph, a bijective function $\rho : V \cup E \to N$ that produced a unique positive integer (To each vertex and/or edge) is called a labelling [4]. Introduced the notion of magic graph that the labels vertices and edges are natural numbers from 1 to $|V| + |E|$ such that sum of the labels of vertices and the edge between them must be constant in entire graph [4]. Extended the concept of magic graph with added a property that vertices always get smaller labels than edges and named it super edge magic labelling. Numerous other authors have explored diverse types of different magic graphs [1], [5] & [11]. The subject of edge-magic labelling of graphs had its origin in the work of Kotzig and Rosa on what they called magic valuations of graphs [7]. These labelling are currently referred to as either edge-magic labelling or edge-magic total labelling.

Fuzzy graphs are generalization of graphs. In graphs two vertices are either related or not related to each other. Mathematically, the degree of relationship is either 0 or 1. In fuzzy graphs, the degree of relationship takes values from $[0, 1]$. A fuzzy graph has ability to solve uncertain problems in a wide range of fields. The first definition of a fuzzy graph was introduced by Kaufmann in 1973. Azriel Rosenfield in 1975 [10] developed the structure of fuzzy graphs and obtained analogs of several graph theoretical concepts. In [8], Nagoor Gani et. al. introduced the concepts of fuzzy labelling graphs, fuzzy magic graphs. A fuzzy graph contains many properties similar to crisp graph due to generalization of crisp graphs but it diverge at many places. In this paper we have developed the concept of fuzzy perfect intrinsic edge magic graphs and also we introduced some general form of intrinsic super constant of above graphs. Throughout this paper we only focussed on undirected fuzzy graphs.

2. Preliminaries

Definition 2.1. *A fuzzy graph $G = (\sigma, \mu)$ is a pair of functions $\sigma : V \to [0, 1]$ and $\mu : V \times V \to [0, 1]$ where for all $u, v \in V$, we have $\mu(u, v) \leq \sigma(u)\Lambda\sigma(v)$.*

Definition 2.2. *A path P in a fuzzy graph is a sequence of distinct nodes v_1, v_2, v_3, $\ldots$, v_n such that $\mu(v_i, v_{i+1}) > 0; 1 \leq i \leq n$; here $n \geq 1$ is called the length of the path P. The consecutive pairs (v_i, v_{i+1}) are called the edge of the path.*

Definition 2.3. *A path P is called a cycle if $v_1 = v_n$ and $n \geq 3$ and a cycle is* *called a fuzzy cycle if it contains more than one weakest arc.*

Definition 2.4. *A bijection ω is a function from the set of all nodes and edges of to $[0,1]$ which assign each nodes $\sigma^{\omega}(a)$, $\sigma^{\omega}(b)$ and edge $\mu^{\omega}(a,b)$ a membership value such that $\mu^{\omega}(a,b) \leq \sigma^{\omega}(a)\Lambda\sigma^{\omega}(b)$ for all $a, b \in V$ is called fuzzy labelling. A graph is said to be fuzzy labelling graph if it has a fuzzy labelling and it is denoted by G^{ω}.*

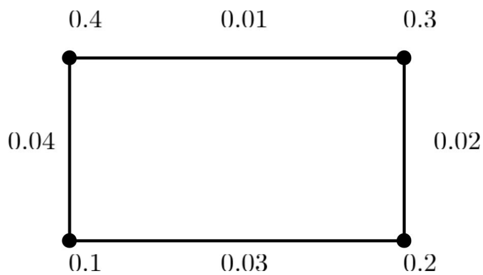

Figure 1: Fuzzy Labelling Graph

Definition 2.5. [6] *A fuzzy labelling graph G is said to be fuzzy intrinsic labelling if $\sigma : V \to [0,1]$ and $\mu : V \times V \to [0,1]$ is bijective such that the membership values of edges and vertices are $\{z, 2z, 3z, \ldots, Nz\}$ without any repetition where N is the total number of vertices and edges and let $z = 0.1$ for $N < 6$ & $z = 0.01$ for $N \geq 6$.*

Definition 2.6. [6] *A fuzzy intrinsic labelling graph is said to be a fuzzy intrinsic edge-magic labelling if it has an intrinsic constant $\lambda_c = \sigma(v_i) + \mu(v_i v_j) + \sigma(v_j)$ for all $v_i, v_j \in V$.*

Definition 2.7. [6] *A fuzzy graph G is said to be intrinsic edge-magic if it satisfies the intrinsic edge-magic labelling with intrinsic constant$'\lambda_c'$.*

Definition 2.8. [6] *An edge-magic constant in a fuzzy intrinsic edge-magic graph is said to be mock constant $'\lambda_m'$ if it is equal to $\sigma(v_i) + \mu(v_i v_j) + \sigma(v_j)$ for some $v_i, v_j \in V$ with $\lambda_c \neq \lambda_m$.*

Definition 2.9. *A fuzzy intrinsic edge-magic labelling graph is said to be fuzzy intrinsic edge-magic graph if it satisfies both vital and competent condition.*

Definition 2.10. *The friendship graph F can be constructed by joining n-copies of the cycle graph C_3 with a common vertex. The graph F_2 is isomorphic to the butterfly graph.*

Definition 2.11. *The Pan graph is the graph obtained by joining a cycle graph to a singleton graph with a bridge. The 3-Pan graph is sometimes known as the Paw graph.*

3. Fuzzy Perfect Intrinsic Edge-magic Graphs

Definition 3.1. *A fuzzy labelling graph G is said to be fuzzy perfect intrinsic labelling if $f : \sigma \to [0,1]$ and $f : \mu \to [0,1]$ is bijective such that the membership values of edges are $\{z, 2z, 3z, \ldots, \varepsilon z\}$ and vertices are $\{(\varepsilon+1)z, (\varepsilon+2)z, ...(\varepsilon+\upsilon)z\}$ where $\varepsilon+\upsilon = N$ is the total number of vertices and edges and let $z = 0.1$ for $N \leq 5$ & $z = 0.01$ for $N > 5$.*

Definition 3.2. *A fuzzy perfect intrinsic labelling graph is said to be a fuzzy perfect intrinsic edge-magic labelling if it has an intrinsic super constant $\lambda_s = \sigma(v_i) + \mu(v_iv_j) + \sigma(v_j)$ for all $v_i, v_j \in V$.*

Definition 3.3. *A fuzzy graph G is said to be perfect intrinsic edge-magic if it satisfies the perfect intrinsic edge-magic labelling with intrinsic super constant $'\lambda_s'$.*

Definition 3.4. *An edge-magic constant in a fuzzy perfect intrinsic edge-magic graph is said to be weak constant $'\lambda_w'$ if it is equal to $\sigma(v_i) + \mu(v_iv_j) + \sigma(v_j)$ for some $v_i, v_j \in V$ with $\lambda_s \neq \lambda_w$.*

Definition 3.5. *A fuzzy graph is said to be a quasi-intrinsic edge-magic graph if it contains atleast one weak constant $'\lambda_w'$which is denoted by $'G_q'$.*
Vital condition: For perfect intrinsic edge-magic, the vital condition is that the above mentioned graph satisfies only the perfect intrinsic edge magic labelling.
Competent condition: A competent condition for perfect intrinsic edge-magic is that if it has the intrinsic super constant for all edges.

Definition 3.6. *A fuzzy perfect intrinsic edge-magic labelling graph is said to be perfect intrinsic edge-magic if it satisfies both vital and competent condition.*

Theorem 3.7. *A path P_n is fuzzy perfect intrinsic edge-magic if $n \geq 2$ where n is length of P_n.*

Theorem 3.8. *A fuzzy cycle $C_n$is fuzzy perfect intrinsic edge-magic iff $n = 3$.*

Theorem 3.9. *A 3-pan graph (paw graph) is fuzzy perfect intrinsic edge-magic.*
Proof. Let G be a 3-Pan graph (Paw graph). Consider the fuzzy perfect intrinsic edge-magic labelling, for 3-pan graph,

$$\begin{aligned}
\sigma(v_{2i}) &= (2n+2-i)z \text{ for } 1 \leq i \leq n (i \text{ is odd }) \\
\sigma(v_n) &= (2n+2)z, \qquad \sigma(v_{n+1}) = 2nz \\
\mu(v_1v_n) &= (n-2)z, \qquad \mu(v_iv_{i+1}) = (n+2-i) \text{ for } 1 \leq i \leq n-1 \\
\mu(v_nv_{n+1}) &= (n-1)z
\end{aligned}$$

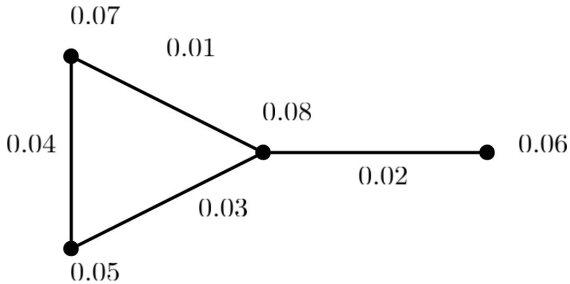

Figure 2:

Now , we consider the above labelling, we get

$$\begin{aligned}\lambda\,(3- \text{ pan graph }) &= \sigma\,(v_n) + \mu\,(v_n v_{n+1}) + \sigma\,(v_{n+1}) \\ &= (2n+2)z + (n-1)z + 2nz \\ &= (5n+1)z \\ \lambda\,(3-pangraph) &= (5n+1)z\end{aligned}$$

Here, intrinsic super constant $\lambda_s = (5n+1)z$ (In general) i.e., $\lambda_s = 0.16$

In the above observation, the 3-pan graph satisfies both vital & competent condition for intrinsic perfect intrinsic edge-magic. We conclude that a 3-pan graph is fuzzy perfect intrinsic edge-magic.

Theorem 3.10. *A 4-pan graph (Banner graph) is a fuzzy quasi perfect intrinsic edge-magic graph with one weak constant.*

Proof. Let G be a 4-Pan graph with $n = 4$. Consider the fuzzy perfect intrinsic edge-magic labelling, we get the following graph.

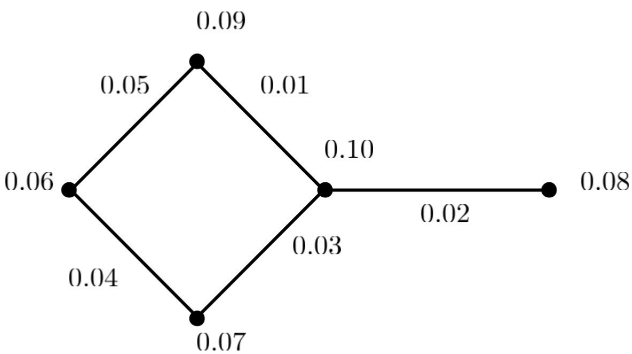

Figure 3: 4-pan graph (Banner Graph)

$$\begin{aligned}
\sigma(v_1) + \mu(v_1v_2) + \sigma(v_2) &= 0.09 + 0.06 + 0.05 = 0.20 = \lambda_s\\
\sigma(v_2) + \mu(v_2v_3) + \sigma(v_3) &= 0.06 + 0.07 + 0.04 = 0.17 = \lambda_w\\
\sigma(v_3) + \mu(v_3v_4) + \sigma(v_4) &= 0.07 + 0.10 + 0.03 = 0.20 = \lambda_s\\
\sigma(v_4) + \mu(v_4v_5) + \sigma(v_5) &= 0.10 + 0.08 + 0.02 = 0.20 = \lambda_s\\
\sigma(v_1) + \mu(v_1v_4) + \sigma(v_4) &= 0.09 + 0.10 + 0.01 = 0.20 = \lambda_s
\end{aligned}$$

Here the 4-pan graph has a weak constant for some edge. But it obviously satisfies perfect intrinsic edge-magic labelling.

We conclude that the Banner graph is a Quasi perfect intrinsic edge-magic.

Theorem 3.11. *The friendship graph F_n is not a perfect intrinsic edge-magic graph for all $n > 1$. (i.e, The cycle C_3 always fuzzy perfect intrinsic edge-magic but the n-copies of C_3 need not be a fuzzy perfect intrinsic edge-magic).*

Theorem 3.12. *The star graph $K_{1,n}$ is a fuzzy perfect intrinsic edge-magic with intrinsic super constant $\lambda_s = (4n + 2)z$ for all $n > 2$ and let $z = 0.1$ for $N < 6$ &$z = 0.01$ for $N \geq 6$.*

Proof. Let G be a star graph $K_{1,n}$ with n-vertices. We put n=3, 4, 5... it exhibits the respective graph is a fuzzy perfect intrinsic edge-magic with intrinsic super constant. Apply the perfect intrinsic edge-magic labelling,

$$\begin{aligned}
\sigma(v) &= (2n + 1)z & \lambda(K_{1,n}) &= \sigma(v) + \mu(vv_i) + \sigma(v_i)\\
\sigma(v_i) &= (n + i)z, \text{ for } 1 \leq i \leq n & &= (2n + 1 + n + n + i)z\\
\mu(vv_i) &= nz & \lambda_s &= (4n + 2)z
\end{aligned}$$

Case (i): Let $n = 3$, we get $\lambda_s = 0.14$

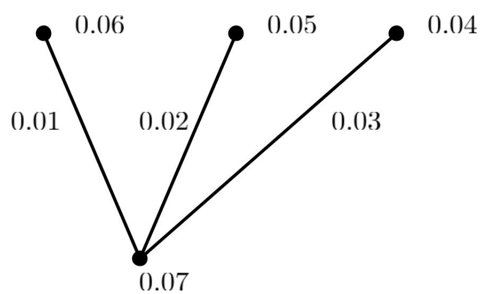

Figure 4:

Intrinsic super constant value for all edges is the following:

$$\sigma(v_1) + \mu(v_1v_2) + \sigma(v_2) = 0.07 + 0.01 + 0.06 = 0.14 = \lambda_s$$
$$\sigma(v_1) + \mu(v_1v_3) + \sigma(v_3) = 0.07 + 0.02 + 0.05 = 0.14 = \lambda_s$$
$$\sigma(v_1) + \mu(v_1v_4) + \sigma(v_4) = 0.07 + 0.03 + 0.04 = 0.14 = \lambda_s$$

Case (ii): Let $n = 4$, we get $\lambda_s = 0.18$

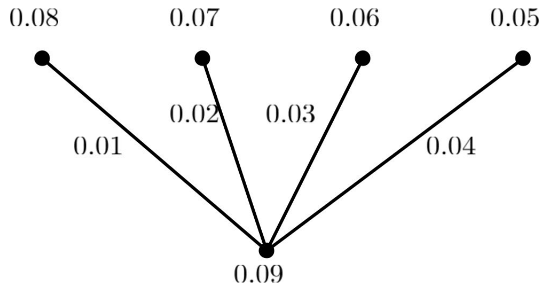

Figure 5:

The intrinsic super constant value for all edges are the following:

$$\sigma(v_1) + \mu(v_1v_2) + \sigma(v_2) = 0.09 + 0.01 + 0.08 = 0.18 = \lambda_s$$
$$\sigma(v_1) + \mu(v_1v_3) + \sigma(v_3) = 0.09 + 0.02 + 0.07 = 0.18 = \lambda_s$$
$$\sigma(v_1) + \mu(v_1v_4) + \sigma(v_4) = 0.09 + 0.03 + 0.06 = 0.18 = \lambda_s$$
$$\sigma(v_1) + \mu(v_1v_5) + \sigma(v_5) = 0.09 + 0.04 + 0.05 = 0.18 = \lambda_s$$

Case(iii): Let $n = 5$, we get $\lambda_s = 0.22$.
Case(iv): Let $n = 6$, we get $\lambda_s = 0.26$.

Continuing this process, we put different values of 'n', it gives intrinsic super constant. Hence the star graph $K_{1,n}$ is fuzzy perfect intrinsic edge-magic for all $n > 2$.

4. Conclusion

In this paper, we discussed the idea of fuzzy perfect intrinsic edge-magic graphs with intrinsic super constant and the fuzzy perfect intrinsic edge magic labelling graphs like fuzzy paths, fuzzy cycles and fuzzy stars are also discussed. We focussed some theorems on fuzzy perfect intrinsic edge-magic graphs. It ought to be note that the necessary and sufficient conditions are given for all the above mentioned graphs.

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HYPERGEOMETRIC FORMS OF SOME FUNCTIONS INVOLVING ARCSINE(x) USING DIFFERENTIAL EQUATION APPROACH

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Abstract: In this paper, by changing the independent and dependent variables in the suitable ordinary differential equations of second and third order and comparing the resulting ordinary differential equations with standard ordinary hypergeometric differential equations of Gauss and Clausen, we obtain the hypergeometric forms of following functions:

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}, \quad [\sin^{-1}(x)]^2 \quad and \quad \sin^{-1}(x).$$

Keywords and Phrases: Hypergeometric functions, Ordinary differential equations.

2010 Mathematics Subject Classification: 33C20, 34-XX.

1. Introduction and Preliminaries

In our investigations, we shall use the following standard notations:
$\mathbb{N} := \{1, 2, 3, \cdots\}; \mathbb{N}_0 := \mathbb{N} \bigcup \{0\}; \mathbb{Z}_0^- := \mathbb{Z}^- \bigcup \{0\} = \{0, -1, -2, -3, \cdots\}$.
The symbols $\mathbb{C}, \mathbb{R}, \mathbb{N}, \mathbb{Z}, \mathbb{R}^+$ and $\mathbb{R}^-$ denote the sets of complex numbers, real numbers, natural numbers, integers, positive and negative real numbers respectively.

Pochhammer symbol:
The Pochhammer symbol (or the *shifted* factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ [9, p. 22 eq(1), p. 32 Q. N.(8) and Q. N.(9)], see also [11, p. 23, eq(22) and eq(23)], is defined by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\backslash\{0\}) \\ \prod\limits_{j=0}^{n-1}(\lambda+j) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ \frac{(-1)^k n!}{(n-k)!} & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ 0 \leqq k \leqq n) \\ 0 & (\lambda = -n;\ \nu = k;\ n, k \in \mathbb{N}_0;\ k > n) \\ \frac{(-1)^k}{(1-\lambda)_k} & (\nu = -k;\ k \in \mathbb{N};\ \lambda \in \mathbb{C}\backslash\mathbb{Z}), \end{cases}$$

it being understood conventionally that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

Generalized hypergeometric function of one variable

A natural generalization of the Gaussian hypergeometric series ${}_2F_1[\alpha, \beta; \gamma; z]$, is accomplished by introducing any arbitrary number of numerator and denominator parameters. Thus, the resulting series

$${}_pF_q\left[\begin{array}{lc} (\alpha_p); & \\ & z \\ (\beta_q); & \end{array}\right] = {}_pF_q\left[\begin{array}{lc} \alpha_1, \alpha_2, \ldots, \alpha_p; & \\ & z \\ \beta_1, \beta_2, \ldots, \beta_q; & \end{array}\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n(\beta_2)_n \ldots (\beta_q)_n} \frac{z^n}{n!}, \tag{1.1}$$

is known as the generalized hypergeometric series, or simply, the generalized hypergeometric function. Here p and q are positive integers or zero and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that

$$\beta_j \neq 0, -1, -2, \ldots;\quad j = 1, 2, \ldots, q.$$

Supposing that none of the numerator and denominator parameters is zero or a negative integer, we note that the ${}_pF_q$ series defined by equation (1.1):

(i) converges for $|z| < \infty$, if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q + 1$,

(iii) diverges for all z, $z \neq 0$, if $p > q + 1$,

(iv) converges absolutely for $|z| = 1$, if $p = q + 1$, and $\Re(\omega) > 0$,

(v) converges conditionally for $|z| = 1 (z \neq 1)$, if $p = q + 1$ and $-1 < \Re(\omega) \leqq 0$,

(vi) diverges for $|z| = 1$, if $p = q + 1$ and $\Re(\omega) \leqq -1$,

where by convention, a product over an empty set is interpreted as 1 and

$$\omega := \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{1.2}$$

$\Re(\omega)$ being the real part of complex number ω.

(I) When $x = \sqrt{t}$, then

$$\frac{dx}{dt} = \frac{1}{2\sqrt{t}}, \tag{1.3}$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx} = 2\sqrt{t}\,\frac{dy}{dt}, \tag{1.4}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(2\sqrt{t}\,\frac{dy}{dt}\right)\frac{dt}{dx}$$

after simplification, we get

$$\frac{d^2y}{dx^2} = 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}, \tag{1.5}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dt}\left(4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt}\right)\frac{dt}{dx}$$

after simplification, we get

$$\frac{d^3y}{dx^3} = 8t^{\frac{3}{2}}\frac{d^3y}{dt^3} + 12\sqrt{t}\frac{d^2y}{dt^2}. \tag{1.6}$$

(II) When $y = z(\sqrt{t})$, where z is the function of t then

$$\frac{dy}{dt} = \sqrt{t}\,\frac{dz}{dt} + \frac{z}{2\sqrt{t}}, \tag{1.7}$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\sqrt{t}\,\frac{dz}{dt} + \frac{z}{2\sqrt{t}}\right)$$

after simplification, we get

$$\frac{d^2y}{dt^2} = \sqrt{t}\,\frac{d^2z}{dt^2} + \frac{1}{\sqrt{t}}\frac{dz}{dt} - \frac{z}{4t^{\frac{3}{2}}}\ . \tag{1.8}$$

(III) When $y = zt$, where z is the function of t then

$$\frac{dy}{dt} = z + t\,\frac{dz}{dt}, \tag{1.9}$$

$$\frac{d^2y}{dt^2} = 2\frac{dz}{dt} + t\,\frac{d^2z}{dt^2}, \tag{1.10}$$

$$\frac{d^3y}{dt^3} = 3\frac{d^2z}{dt^2} + t\,\frac{d^3z}{dt^3}\ . \tag{1.11}$$

(IV) We know that

$$z = {}_2F_1\left[\begin{array}{ccc} a, & b; & \\ & & t \\ & c; & \end{array}\right], \tag{1.12}$$

is one of the series solution of the following Gauss' ordinary hypergeometric homogeneous linear differential equation of second order with variable coefficients

$$t(1-t)\frac{d^2z}{dt^2} + [c-(a+b+1)t]\frac{dz}{dt} - abz = 0\ . \tag{1.13}$$

(V) We know that

$$z = {}_3F_2\left[\begin{array}{ccc} \alpha, & \beta, \ \gamma; & \\ & & t \\ & \lambda, \ \mu; & \end{array}\right], \tag{1.14}$$

is one of the series solution of the following Clausen's ordinary hypergeometric homogeneous linear differential equation of third order with variable coefficients

$$t^2(1-t)\frac{d^3z}{dt^3} + [(1+\lambda+\mu)-(3+\alpha+\beta+\gamma)t]t\frac{d^2z}{dt^2} +$$

$$+[\lambda\mu-(1+\alpha+\beta+\gamma+\alpha\beta+\alpha\gamma+\beta\gamma)t]\frac{dz}{dt} - \alpha\beta\gamma z = 0\ . \tag{1.15}$$

The present article is organized as follows. In section 3, we have derived the hypergeometric forms of some functions involving arcsine function, using differential equation approach. For hypergeometric forms of other mathematical functions and functions of mathematical physics, one can refer the literature [1], [2], [3], [4], [5], [6], [7], [8], [10] and [12], where the proof of hypergeometric forms of related functions are not given. So we are interested to give the proof of hypergeometric forms of some arcsine function using differential equation approach.

2. Some Hypergeometric Forms

When $|x| < 1$, then following hypergeometric forms hold true:

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}} = x\ {}_2F_1\left[\begin{array}{ccc} 1,\ 1; & & \\ & & x^2 \\ \frac{3}{2}; & & \end{array}\right]. \tag{2.1}$$

$$[\sin^{-1}(x)]^2 = x^2\ {}_3F_2\left[\begin{array}{ccc} 1,\ 1,\ 1; & & \\ & & x^2 \\ 2,\ \frac{3}{2}; & & \end{array}\right]. \tag{2.2}$$

$$\sin^{-1}(x) = x\ {}_2F_1\left[\begin{array}{ccc} \frac{1}{2},\ \frac{1}{2}; & & \\ & & x^2 \\ \frac{3}{2}; & & \end{array}\right]. \tag{2.3}$$

3. Proof of Hypergeometric Forms

Proof of hypergeometric form (2.1):

Consider the following function

$$y = \frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}}$$

$$or \quad \sqrt{(1-x^2)}\ y = \sin^{-1}(x). \tag{3.1}$$

Differentiate the equation (3.1) w.r.t. x and use product rule, after simplification we get

$$(1-x^2)\frac{dy}{dx} - xy = 1. \tag{3.2}$$

Again differentiate the equation (3.2) w.r.t. x and apply product rule, after simplification we have

$$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0. \tag{3.3}$$

Put $x = \sqrt{t}$, then use values of equations (1.4) and (1.5) in above differential equation (3.3), after simplification we get

$$t(1-t)\frac{d^2y}{dt^2} + \left\{\frac{1}{2} - 2t\right\}\frac{dy}{dt} - \frac{1}{4}y = 0\ . \tag{3.4}$$

Now substitute $y = z(\sqrt{t})$ and put the values of equations (1.7) and (1.8) in above differential equation (3.4), after simplification we obtain

$$t(1-t)\frac{d^2z}{dt^2} + \left\{\frac{3}{2} - 3t\right\}\frac{dz}{dt} - z = 0 \ . \tag{3.5}$$

Now compare the coefficients of above differential equation (3.5) with Gauss' standard differential equation (1.13), we get

$$c = \frac{3}{2}, \ a + b + 1 = 3 \ and \ ab = 1 \ .$$

Now solve the above algebraic equations simultaneously, we get

$$a = 1, \ b = 1 \ .$$

Therefore the solution of above differential equation (3.5) is given by

$$z = {}_2F_1\left[\begin{array}{ccc} 1, \ 1; & & \\ & & t \\ \frac{3}{2}; & & \end{array}\right],$$

$$y = \sqrt{t} \ {}_2F_1\left[\begin{array}{ccc} 1, \ 1; & & \\ & & t \\ \frac{3}{2}; & & \end{array}\right],$$

$$\frac{\sin^{-1}(x)}{\sqrt{(1-x^2)}} = x \ {}_2F_1\left[\begin{array}{ccc} 1, \ 1; & & \\ & & x^2 \\ \frac{3}{2}; & & \end{array}\right].$$

This completes the proof of hypergeometric form (2.1).

Proof of hypergeometric form (2.2):
Consider the following function

$$y = [\sin^{-1}(x)]^2. \tag{3.6}$$

Differentiate the equation (3.6) w.r.t. x, we get

$$\sqrt{(1-x^2)}\frac{dy}{dx} = 2[\sin^{-1}(x)]. \tag{3.7}$$

Again differentiate the equation (3.7) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2 \ . \tag{3.8}$$

Now again differentiate the equation (3.8) w.r.t. x and apply product rule, after simplification we obtain

$$(1-x^2)\frac{d^3y}{dx^3}-3x\frac{d^2y}{dx^2}-\frac{dy}{dx}=0\ . \tag{3.9}$$

Put $x=\sqrt{t}$, then use the values of equations (1.4), (1.5) and (1.6) in above differential equation (3.9), after simplification we get

$$t(1-t)\frac{d^3y}{dt^3}+\left\{\frac{3}{2}-3t\right\}\frac{d^2y}{dt^2}-\frac{dy}{dt}=0\ . \tag{3.10}$$

Now substitute $y=tz$ and put the values of equations (1.9), (1.10) and (1.11) in above differential equation (3.10), after simplification we have

$$t^2(1-t)\frac{d^3z}{dt^3}+\left\{\frac{9}{2}-6t\right\}t\frac{d^2z}{dt^2}+\{3-7t\}\frac{dz}{dt}-z=0\ . \tag{3.11}$$

Now compare the coefficients of above differential equation (3.11) with Clausen's standard differential equation (1.15), we get

$$1+\lambda+\mu=\frac{9}{2}, 3+\alpha+\beta+\gamma=6, \lambda\mu=3, 1+\alpha+\beta+\gamma+\alpha\beta+\alpha\gamma+\beta\gamma=7 \ and\ \alpha\beta\gamma=1.$$

Now solve the above algebraic equations simultaneously, we get

$$\lambda=2,\ \mu=\frac{3}{2},\ \alpha=1,\ \beta=1,\ \gamma=1\ \ .$$

Therefore the solution of above differential equation (3.11) is given by

$$z={}_3F_2\left[\begin{array}{c}1,1,1;\\ \\ 2,\ \frac{3}{2};\end{array}\ t\right],$$

$$y=t\ {}_3F_2\left[\begin{array}{c}1,1,1;\\ \\ 2,\ \frac{3}{2};\end{array}\ t\right],$$

$$[\sin^{-1}(x)]^2=x^2\ {}_3F_2\left[\begin{array}{c}1,1,1;\\ \\ 2,\ \frac{3}{2};\end{array}\ x^2\right].$$

This completes the proof of hypergeometric form (2.2).

Proof of hypergeometric form (2.3):
Consider the following function

$$y = \sin^{-1}(x)\ . \tag{3.12}$$

Differentiate the equation (3.12) w.r.t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sqrt{(1-x^2)}}$$

$$or \quad \sqrt{(1-x^2)}\,\frac{dy}{dx} = 1 \quad . \tag{3.13}$$

Again differentiate the equation (3.13) w.r.t. x and use product rule, after simplification we have

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0\ . \tag{3.14}$$

Put $x = \sqrt{t}$, then use values of equations (1.4) and (1.5) in above differential equation (3.14), after simplification we obtain

$$t(1-t)\frac{d^2y}{dt^2} + \left\{\frac{1}{2} - t\right\}\frac{dy}{dt} = 0\ . \tag{3.15}$$

Now substitute $y = z(\sqrt{t})$ and put the values of equations (1.7) and (1.8) in above differential equation (3.15), after simplification we have

$$t(1-t)\frac{d^2z}{dt^2} + \left\{\frac{3}{2} - 2t\right\}\frac{dz}{dt} - \frac{1}{4}z = 0\ . \tag{3.16}$$

Now compare the coefficients of above differential equation (3.16) with Gauss' standard differential equation (1.13), we get

$$c = \frac{3}{2},\ a + b + 1 = 2 \ and\ ab = \frac{1}{4} \quad .$$

Now solve the above algebraic equations simultaneously, we get

$$a = \frac{1}{2},\ b = \frac{1}{2} \quad .$$

Therefore the solution of above differential equation (3.16) is given by

$$z = {}_2F_1\left[\begin{array}{cc} \frac{1}{2}, \frac{1}{2}; & \\ & t \\ \frac{3}{2}; & \end{array}\right],$$

$$y = \sqrt{t}\ {}_2F_1\left[\begin{array}{cc} \frac{1}{2}, \frac{1}{2}; & \\ & t \\ \frac{3}{2}; & \end{array}\right],$$

$$\sin^{-1}(x) = x\ {}_2F_1\left[\begin{array}{cc} \frac{1}{2}, \frac{1}{2}; & \\ & x^2 \\ \frac{3}{2}; & \end{array}\right].$$

This completes the proof of hypergeometric form (2.3).

4. Conclusion

In our present investigation, we derived the hypergeometric forms of some functions involving arcsine function by using differential equation approach. Moreover, the results derived in this paper are expected to have useful applications in wide range of problems of Mathematics, Statistics and Physical sciences. Similarly, we can derive the hypergeometric forms of other functions in an analogous manner.

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INDUCED V_4- MAGIC LABELING OF SOME STAR AND PATH RELATED GRAPHS

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Abstract: Let $V_4 = \{0, a, b, c\}$ be the Klein-4-group with identity element 0 and $G = (V(G), E(G))$, be the graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : V(G) \to V_4$ be a vertex labeling and $f^* : E(G) \to V_4$ denote the induced edge labeling of f defined by $f^*(uv) = f(u) + f(v)$ for all $uv \in E(G)$. Then f^* again induces a vertex labeling $f^{**} : V(G) \to V_4$ defined by $f^{**}(u) = \Sigma f^*(uv)$ where the summation is taken over all the vertices v which are adjacent to u. A graph $G = (V(G), E(G))$ is said to be an induced V_4-Magic graph if there exists a non zero vertex labeling $f : V(G) \to V_4$ such that $f \equiv f^{**}$. The function f, so obtained is called an induced V_4-Magic labeling of G. In this paper we discuss Induced V_4 magic labeling of some graphs and the Induced V_4 magic labeling of some star and path related graphs.

Keywords and Phrases: Klein-4-group, Induced V_4-magic graphs.

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1. Introduction

In this paper we consider simple, connected, finite and undirected graphs and the Klein 4-group is denoted by $V_4 = \{0, a, b, c\}$, which is a noncyclic abelian group of order 4 in which every nonidentity element has order 2. We refer to Frank Harary [2] for the standard terminology and notations related to graph theory. Let $G = (V(G), E(G))$, be the graph with vertex set $V(G)$ and edge set $E(G)$. Let $f : V(G) \to V_4$ be a vertex labeling and $f^* : E(G) \to V_4$ denote the induced edge labeling of f defined by $f^*(uv) = f(u) + f(v)$ for all $uv \in E(G)$. Then this f^* induces another vertex labeling denoted by $f^{**} : V(G) \to V_4$ defined by $f^{**}(u) = \Sigma f^*(uv)$ where the summation is taken over all the vertices v which are adjacent to u. A graph $G = (V(G), E(G))$ is said to be an induced V_4-Magic graph denoted by IMV_4G or simply IMG if there exists a non zero labeling $f : V(G) \to V_4$ such that $f \equiv f^{**}$. The function f, so obtained is called a induced V_4-Magic labeling of G or simply induced Magic labeling of G and it is denoted by IMV_4L or simply IML. In this paper we discuss some Induced V_4 magic labeling of the graphs $C_n, K_n, K_{m,n}$, some star related graphs and some path related graphs which belong to the following categories:

(i) $\Gamma(V_4)$:= class of all induced V_4-magic graphs.

(ii) $\Gamma_k(V_4)$:= class of all induced V_4-magic graphs with induced magic labeling f satisfying $f(V(G)) = \{k\}$ for some $k \in V_4$.

(iii) $\Gamma_{k,0}(V_4)$:= class of all induced V_4-magic graphs with induced magic labeling f satisfying $f(V(G)) = \{k, 0\}$ for some $k \in V_4$.

Theorem 1.1. [5] *If f is an induced magic labeling of a graph G and u be a pendent vertex adjacent to a vertex v in G,then $f(v) = 0$.*

Corollary 1.2. [5] *If f is an induced magic labeling of a graph G and $wuvz$ is a path in G with w and z are pendent vertices in G,then $f^*(uv) = 0$.*

Theorem 1.3. [5] *Let f be any vertex labeling of a graph G and u, be a vertex in G with deg$(u) = m$. Then f is an induced magic labeling of G if and only if $(m-1)f(u)+\Sigma f(v) = 0$ where the summation is taken over all the vertices v which are adjacent to u.*

Theorem 1.4. [5] *$C_n \in \Gamma(A)$ if and only if $n \equiv 0$ $(mod\ 3)$, where A is an Abelian group.*

Theorem 1.5. [5] *$P_n \in \Gamma(A)$ if and only if $n \equiv 0$ $(mod\ 3)$, where A is an Abelian group.*

2. Main Results

Theorem 2.1. *For any graph G, $G \notin \Gamma_a(V_4)$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. If possible suppose $f(v_i) = a$ for $i = 1, 2, 3, \cdots, n$, then we have $f^*(v_jv_k) = f(v_j) + f(v_k) = 0$ for all $v_jv_k \in E(G)$. Thus $f^{**}(v_i) = \Sigma f^*(v_iv) = 0$ where the summation is taken over all the vertices v which are adjacent to v_i. Therefore $f^{**} \not\equiv f$, hence f is not an induced V_4 magic labeling and $G \notin \Gamma_a(V_4)$.

Corollary 2.2. *[Degree sum equation of a vertex]*
Let f be any vertex labeling of a graph G and u, be a vertex in G with $deg(u) = m$. Then f is an induced V_4magic labeling if and only if $f(u)+\Sigma f(v) = 0$ or $\Sigma f(v) = 0$ according as $deg(u) = m$ is even or odd, where the summation is taken over all the vertices v which are adjacent to u.
Proof. From Theorem 1.3, we have f is an induced V_4magic labeling of G if and only if $(m-1)f(u) + \Sigma f(v) = 0$, where v is adjacent to u, then the result follows directly from the fact that $f(u) \in V_4$.

3. Induced magic labeling of some graphs

Theorem 3.1. *The complete graph $K_n \in \Gamma(V_4)$ if and only if n is odd.*
Proof. Let $V(K_n) = \{v_1, v_2, v_3, \cdots, v_n\}$. Suppose n is odd. Define $f : V(K_n) \to V_4$ as :

$$f(v_i) = \begin{cases} 0 & \text{if} \quad i = 1 \\ a & \text{if} \quad i = 2, 3, \cdots, n \end{cases}$$

Then, f is an induced magic labeling of K_n. Conversely suppose n is an even number. Then $\deg v_i = n - 1$ is an odd number. Therefore by corollary 2.2 we have, f is an induced magic label if and only if $f$satisfies the following system equations:

$$\begin{aligned} f(v_2) + f(v_3) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ f(v_1) + f(v_3) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ f(v_1) + f(v_2) + f(v_4) + \cdots + f(v_{n-1}) + f(v_n) &= 0 \\ &\vdots \\ f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{n-2}) + f(v_n) &= 0 \\ f(v_1) + f(v_2) + f(v_3) + \cdots + f(v_{n-2}) + f(v_{n-1}) &= 0 \end{aligned}$$

Note that the above system of equations show that $f(v_1) = f(v_2) = f(v_3) = \cdots = f(v_n)$ and which again implies that $(n-1)f(v_1) = 0$, that is $f(v_1) = 0$, thus $f \equiv 0$, which is a contradiction.

The following corollary follows directly from the proof of Theorem 3.1.

Corollary 3.2. $K_n \in \Gamma_{a,0}(V_4)$ *if and only if n is odd.*

Corollary 3.3. *$C_n \in \Gamma_{a,0}(V_4)$ if and only if $n \equiv 0$ $(mod\ 3)$.*
Proof. Consider C_n with vertex set $\{v_1, v_2, \ldots, v_{n-1}, v_n.\}$. Suppose $n \equiv 0$ $(mod\ 3)$. Define $f : V(C_n) \to V_4$ as follows:

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1\ (mod\ 3) \\ 0 & \text{if} \quad i \equiv 2\ (mod\ 3) \end{cases}$$

Then clearly f is Induced V_4 magic labeling of C_n. Converse part follows from Theorem 1.4.

Theorem 3.4. *For $m, n > 1$, the complete bipartite graph $K_{m,n} \in \Gamma(V_4)$ if and only if either m or n is odd.*
Proof. Let $V(K_{m,n}) = \{v_1, v_2, v_3, \cdots, v_m, u_1, u_2, u_3, \cdots, u_n\}$, where $v_i u_j \in E(K_{m,n})$ for $i = 1, 2, 3, \cdots m$, and $j = 1, 2, 3, \cdots, n$.

Case 1: m and n are odd

In this case define $f : V(K_{m,n}) \to V_4$ as follows:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = v_1, u_1 \\ a & \text{if} \quad v = v_2, v_3, v_4, \cdots, v_m \\ b & \text{if} \quad v = u_2, u_3, u_4, \cdots, u_n \end{cases}$$

Case 2: m is odd and n is even

In this case define $g : V(K_{m,n}) \to V_4$ as follows:

$$g(v) = \begin{cases} 0 & \text{if} \quad v = v_1, v_2, v_3, \cdots, v_m \\ a & \text{if} \quad v = u_1, u_2, u_3, \cdots, u_n \end{cases}$$

Case 3: m is even and n is odd

In this case define $h : V(K_{m,n}) \to V_4$ as follows:

$$h(v) = \begin{cases} 0 & \text{if} \quad v = u_1, u_2, u_3, \cdots, u_n \\ a & \text{if} \quad v = v_1, v_2, v_3, \cdots, v_m \end{cases}$$

Then in each case we can easily verify that the vertex labeling f, g and h are induced magic labeling of $K_{m,n}$. Thus $K_{m,n} \in \Gamma(V_4)$ if either m or n is odd. Now consider the following case:

Case 4: m and n are even

If possible suppose $f : V(K_{m,n}) \to V_4$ is an induced magic labeling of $K_{m,n}$. Then by corollary 2.2 f must satisfy the following system of equations:

$$\begin{aligned} f(v_i)+f(u_1)+f(u_2)+f(u_3)+\cdots+f(u_n) &= 0 \text{ for } i=1,2,3,\cdots,m \quad (1)\\ f(u_j)+f(v_1)+f(v_2)+f(v_3)+\cdots+f(v_m) &= 0 \text{ for } j=1,2,3,\cdots,n \quad (2)\end{aligned}$$

Note that the above equations in (1) imply that $f(v_1)=f(v_2)=f(v_3)=\cdots=f(v_n)$ and the equations in (2) imply that $f(u_1)=f(u_2)=f(u_3)=\cdots=f(u_n)$. Thus the above system reduces to:

$$\begin{aligned} f(v_1)+nf(u_1) &= 0\\ mf(v_1)+f(u_1) &= 0\end{aligned}$$

Since both m and n are even the above system implies that $f(v_1)=f(u_1)=0$. Thus $f\equiv 0$ and it is a contradiction to our assumption. Hence in this case $K_{m,n}\notin\Gamma(V_4)$.

4. Induced magic labeling of some star related graphs

Theorem 4.1. *For $n>1$, the star graph $K_{1,n}\in\Gamma(V_4)$.*
Proof. Let $V(K_{1,n})=\{v,v_1,v_2,v_3,\cdots,v_n,\}$, where $vv_i\in E(K_{1,n})$ for $i=1,2,3,\cdots n$.

Case 1: n is even

In this case define $f:V(K_{1,n})\to V_4$ as follows:

$$f(u)=\begin{cases} 0 & \text{if} \quad u=v\\ a & \text{if} \quad u=v_1,v_2,v_3,\cdots,v_n\end{cases}$$

Case 2: n is odd

In this case define $g:V(K_{1,n})\to V_4$ as follows:

$$g(u)=\begin{cases} 0 & \text{if} \quad u=v,v_1\\ a & \text{if} \quad u=v_2,v_3,v_4,\cdots,v_n\end{cases}$$

Then in each case we can easily verify that the vertex labeling f and g are induced magic labeling of $K_{1,n}$. Thus $K_{1,n}\in\Gamma(V_4)$. for all $n>1$.

Corollary 4.2. *For $n>1, K_{1,n}\in\Gamma_{a,0}(V_4)$.*

Definition 4.3. *The Bistar $B_{m,n}$ is the graph obtained by joining the central or apex vertex of $K_{1,m}$ and $K_{1,n}$ by an edge.*

Theorem 4.4. *For the Bistar $B_{m,n}\in\Gamma(V_4)$ for all m and n with $m+n>2$.*
Proof. Let $V(B_{m,n})=\{u,v,v_1,v_2,v_3,\cdots,v_m,u_1,u_2,u_3,\cdots,u_n\}$, where uv, vv_i, $uu_j\in E(B_{m,n})$ for $i=1,2,3,\cdots m$, and $j=1,2,3,\cdots,n$.

Case 1: m and n are even

In this case define $f : V(B_{m,n}) \to V_4$ as follows:

$$f(w) = \begin{cases} 0 & \text{if} \quad w = v, u \\ a & \text{if} \quad w = v_1, v_2, v_3, \cdots, v_m \\ b & \text{if} \quad w = u_1, u_2, u_3, \cdots, u_n \end{cases}$$

Case 2: m is odd and n is even

In this case define $g : V(B_{m,n}) \to V_4$ as follows:

$$g(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1 \\ a & \text{if} \quad w = v_2, v_3, v_4 \cdots, v_m \\ b & \text{if} \quad w = u_1, u_2, u_3, \cdots, u_n \end{cases}$$

Case 3: m is even and n is odd

In this case define $h : V(B_{m,n}) \to V_4$ as follows:

$$h(w) = \begin{cases} 0 & \text{if} \quad w = v, u, u_1 \\ a & \text{if} \quad w = v_1, v_2, v_3, \cdots, v_m \\ b & \text{if} \quad w = u_2, u_3, u_4 \cdots, u_n \end{cases}$$

Case 4: m and n are odd

In this case define $k : V(B_{m,n}) \to V_4$ as follows:

$$k(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1, u_1 \\ a & \text{if} \quad w = v_2, v_3, v_4, \cdots, v_m \\ b & \text{if} \quad w = u_2, u_3, u_4, \cdots, u_n \end{cases}$$

Then in each case we can easily verify that the vertex labeling f, g, h and k are induced magic labeling of $B_{m,n}$. Thus $B_{m,n} \in \Gamma(V_4)$, for all m and n.

Corollary 4.5. *For the Bistar $B_{m,n} \in \Gamma_{a,0}(V_4)$ for all m and n with $m + n > 2$.*
Proof. Let $V(B_{m,n}) = \{u, v, v_1, v_2, v_3, \cdots, v_m, u_1, u_2, u_3, \cdots, u_n\}$, where uv, vv_i, $uu_j \in E(B_{m,n})$ for $i = 1, 2, 3, \cdots m$, and $j = 1, 2, 3, \cdots, n$.

Case 1: m and n are even

In this case define $f : V(B_{m,n}) \to V_4$ as follows:

$$f(w) = \begin{cases} 0 & \text{if} \quad w = v, u \\ a & \text{if} \quad w = v_i, u_j, i = 1, 2, 3, \cdots m, j = 1, 2, 3, \cdots, n. \end{cases}$$

Case 2: m is odd and n is even

In this case define $g : V(B_{m,n}) \to V_4$ as follows:

$$g(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1 \\ a & \text{if} \quad w = v_i, u_j, i = 2, 3, \cdots m, j = 1, 2, 3, \cdots, n. \end{cases}$$

Case 3: m is even and n is odd

In this case define $h : V(B_{m,n}) \to V_4$ as follows:

$$h(w) = \begin{cases} 0 & \text{if} \quad w = v, u, u_1 \\ a & \text{if} \quad w = v_i, u_j, i = 1, 2, 3, \cdots m, j = 2, 3, \cdots, n. \end{cases}$$

Case 4: m and n are odd

In this case define $k : V(B_{m,n}) \to V_4$ as follows:

$$k(w) = \begin{cases} 0 & \text{if} \quad w = v, u, v_1, u_1 \\ a & \text{if} \quad w = v_i, u_j, i = 2, 3, \cdots m, j = 2, 3, \cdots, n. \end{cases}$$

Then in each case we can easily verify that the vertex labeling f, g, h and k are induced magic labeling of $B_{m,n}$. Thus $B_{m,n} \in \Gamma_{a,0}(V_4)$, for all m and n.

Definition 4.6. [3] *Let $< K_{1,n} : m >$ denote the graph obtained by taking m disjoint copies of $K_{1,n}$, and joining a new vertex to the centers of the m copies of $K_{1,n}$.*

Theorem 4.7. *The graph $< K_{1,n} : m > \in \Gamma(V_4)$ for all m, n.*

Proof. Consider the graph $< K_{1,n} : m >$ with $\{v_i, v_{ij} : 1 \leq j \leq n\}$ as the vertex set of i^{th} copy of $K_{1,n}$ with central vertex v_i for $i = 1, 2, 3, \cdots m$ and let v be the unique vertex adjacent to the central vertices v_i in $< K_{1,n} : m >$. Then define $f : V(< K_{1,n} : m >) \to V_4$ as follows:

Case 1: m is odd

Subcase 1: n is odd

Define f as

$$f(u) = \begin{cases} a & \text{if} \quad u = v \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \; j = 1, 2, 3, \cdots, n. \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

Subcase 2: n is even

Define f as

$$f(u) = \begin{cases} a & \text{if} \quad u = v \\ 0 & \text{if} \quad u = v_{11} \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \;\; j = 2, 3, 4, \cdots, n \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

Case 2: m is even

Subcase 1: n is odd

Define f as

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v \\ 0 & \text{if} \quad u = v_{11} \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \;\; j = 2, 3, 4, \cdots, n. \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

Subcase 2: n is even

Define f as

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots m, \;\; j = 1, 2, 3, \cdots, n. \\ 0 & \text{if} \quad u = v_i, i = 1, 2, 3, \cdots m, \end{cases}$$

One can easily verify that the vertex label f defined in all four cases are IML of $< K_{1,n} : m >$.

Definition 4.8. *The (n, k)-Banana tree $Bt(n, k)$ is the graph obtained by starting with n number of $k-$stars and connecting one end vertex from each to a new vertex.*

Theorem 4.9. *The (n, k)-Banana tree $Bt(n, k) \in \Gamma(V_4)$ for all n and k.*
Proof. Consider the graph $Bt(n, k)$. Let $V[Bt(n, k)]$=$\{v, v_i, v_{ij} : 1 \leq i \leq n \;, 1 \leq j \leq k\}$ and $E[Bt(n, k)]$=$\{vv_{i1}, v_iv_{ij}\} : 1 \leq i \leq n, 1 \leq j \leq k\}$.

Case 1 : k is odd

In this case define $f(V(Bt(n, k))) :\to V_4$ by

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v, v_i \text{ for } i = 1, 2, 3, \cdots n \\ 0 & \text{if} \quad u = v_{i1} \text{ for } i = 1, 2, 3, \cdots n \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots n, \;\; j = 2, 3, 4, \cdots, k. \end{cases}$$

Case 2 : k is even

In this case define $f(V(Bt(n,k))) :\to V_4$ by

$$f(u) = \begin{cases} 0 & \text{if} \quad u = v, v_i \text{ for } i = 1, 2, 3, \cdots n \\ 0 & \text{if} \quad u = v_{i1}, v_{i2} \text{ for } i = 1, 2, 3, \cdots n, \\ a & \text{if} \quad u = v_{ij}, i = 1, 2, 3, \cdots n, j = 3, 4, \cdots, k. \end{cases}$$

In both case, we can easily verify that, f is an IML of $Bt(n,k)$. Hence the Proof.

5. Induced magic labeling of some path related graphs

Theorem 5.1. $P_n \in \Gamma_{a,0}(V_4)$ *if and only if* $n \equiv 0\ (mod\ 3)$.
Proof. Consider the path P_n with vertex set $V = \{v_1, v_2, v_3, \ldots, v_{n-1}, v_n\}$ where $n \equiv 0\ (mod\ 3)$. Define $f : V(P_n) \to V_4$ as :

$$f(v_i) = \begin{cases} a & \text{if} \quad i \equiv 0, 1\ (mod\ 3) \\ 0 & \text{if} \quad i \equiv 2\ (mod\ 3) \end{cases}$$

Then, f is an induced magic labeling of P_n. Hence $P_n \in \Gamma_{a,0}(V_4)$.

Conversely if $n \not\equiv 0\ (mod\ 3)$ then by the Theorem 1.5 $P_n \notin \Gamma_{a,0}(V_4)$.

Definition 5.2. *The Corona* $P_n \odot K_1$ *is called the comb graph* CB_n.

Theorem 5.3. *The Comb graph* $CB_n \notin \Gamma(V_4)$ *for all* n.
Proof. Let $\{u_i, v_i : 1 \le i \le n\}$ be the vertex set of CB_n where $v_i(1 \le i \le n)$ are the pendent vertices adjacent to $u_i(1 \le i \le n)$. If possible suppose f is an IML of the graph CB_n. Then from the degree sum equation of the vertices u_i and v_i, we have $f(u_i) = f(v_i) = 0$. That is $f \equiv 0$, which is a contradiction.

Definition 5.4. [3] *A triangular snake graph* TS_n *is obtained from a path* v_1, v_2, $\cdots, v_n$ *by joining* v_i *and* v_{i+1} *to a new vertex* w_i *for* $i = 1, 2, 3, \cdots, n-1$.

Theorem 5.5. *The triangular snake graph* $TS_n \in \Gamma(V_4)$ *for all* n.
Proof. Let $V(TS_n) = \{v_1, v_2, \cdots, v_n, w_1, w_2, w_3, \cdots, w_{n-1}\}$, where v_i's are the vertices of corresponding path P_n and f be an IML of TS_n with $f(v_i) = x_i$ and $f(w_j) = y_j$. Then the vertices v_i and w_j must satisfy the degree sum equation.

Note that the degree sum equation of v_i gives the following system of equations.

$$\begin{aligned} x_1 + x_2 + y_1 &= 0 \\ x_1 + x_2 + x_3 + y_1 + y_2 &= 0 \\ x_2 + x_3 + x_4 + y_2 + y_3 &= 0 \\ &\vdots \\ x_{n-2} + x_{n-1} + x_n + y_{n-2} + y_{n-1} &= 0 \\ x_{n-1} + x_n + y_{n-1} &= 0 \end{aligned}$$

Similarly the degree sum equation of w_j gives the following system of equations.

$$\begin{aligned} x_1 + x_2 + y_1 &= 0 \\ x_2 + x_3 + y_2 &= 0 \\ x_3 + x_4 + y_3 &= 0 \\ &\vdots \\ x_{n-2} + x_{n-1} + y_{n-2} &= 0 \\ x_{n-1} + x_n + y_{n-1} &= 0 \end{aligned}$$

On substituting the second system in the first system of equations we get

$$\begin{aligned} x_1 + x_2 + y_1 &= 0 \\ x_1 + y_1 &= 0 \\ x_2 + y_2 &= 0 \\ &\vdots \\ x_{n-2} + y_{n-2} &= 0 \\ x_{n-1} + x_n + y_{n-1} &= 0 \end{aligned}$$

From the above two system of equations one can easily conclude that

$$\begin{aligned} x_1 &= y_1 \\ x_2 = x_3 = x_4 = \cdots = x_{n-1} &= 0 \\ y_2 = y_3 = y_4 = \cdots = y_{n-2} &= 0 \\ x_n &= y_{n-1} \end{aligned}$$

Thus to get an IML of TS_n we need to define $f : V(TS_n) \to V_4$ as follows:

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, w_1 \\ 0 & \text{if} \quad v = v_i, \text{ for } i = 2, 3, 4, \cdots, n-1 \\ 0 & \text{if} \quad v = w_j, \text{ for } j = 2, 3, 4, \cdots, n-2 \\ b & \text{if} \quad v = v_n, w_{n-1} \end{cases}$$

Hence the proof.

Corollary 5.6. *The triangular snake graph $TS_n \in \Gamma_{a,0}(V_4)$ for all n.*
Proof. Define $f : V(TS_n) \to V_4$ as follows:

$$f(v) = \begin{cases} a & \text{if} \quad v = v_1, w_1 \\ 0 & \text{if} \quad v = v_i, \ \text{for } i = 2, 3, 4, \cdots, n-1 \\ 0 & \text{if} \quad v = w_j, \ \text{for } j = 2, 3, 4, \cdots, n-2 \\ a & \text{if} \quad v = v_n, w_{n-1} \end{cases}$$

Then from Theorem 5.5, f is an IML of TS_n. Hence the corollary follows.

Definition 5.7. [3] *A double triangular snake graph DTS_n consists of two triangular snake graphs that have a common path. That is, a double triangular snake is obtained from a path $v_1, v_2, ..., v_n$ by joining v_i and v_{i+1} to a new vertex w_i for $i = 1, 2, ..., n-1$ and to a new vertex u_i for $i = 1, 2, ..., n-1$.*

Theorem 5.8. *The double triangular snake graph $DTS_n \in \Gamma(V_4)$ if and only if $n \equiv 0\ (mod\ 3)$.*
Proof. Consider a double triangular snake graph DTS_n with vertex set $v_1, v_2, v_3, \cdots, v_n, w_1, w_2, w_3, \cdots, w_{n-1}, u_1, u_2, u_3, \cdots, u_{n-1}$, where $v_1, v_2, ..., v_n$ are the vertices of corresponding path and w_i, u_i are the vertices attached to v_i and v_{i+1} for $i = 1, 2, ..., n-1$. Suppose $n \equiv 0\ (mod\ 3)$. Let $n = 3k$, then define $g : V(DTS_n) \to V_4$ as:

$$g(v) = \begin{cases} 0 & \text{if} \quad v = v_2, v_5, v_8, \cdots, v_{n-4}, v_{n-1} \\ a & \text{if} \quad v = v_1, v_3, v_4, v_6, v_7, \cdots, v_{n-3}, v_{n-2}, v_n \\ 0 & \text{if} \quad v = w_{3j}, \ \text{for } j = 1, 2, 3, \cdots, k-1 \\ a & \text{if} \quad v = w_1, w_2, w_4, w_5, w_7, \cdots, w_{n-2}, w_{n-1} \\ 0 & \text{if} \quad v = u_{3j}, \ \text{for } j = 1, 2, 3, \cdots, k-1 \\ a & \text{if} \quad v = u_1, u_2, u_4, u_5, u_7, \cdots, u_{n-2}, u_{n-1} \end{cases}$$

We can easily prove that g is an IML of DTS_n.
Conversely suppose that $n = 3k+1$ or $n = 3k+2$ for some integer k. If possible suppose f is an IML DTS_n. Then from the degree sum equation of the vertices v_i

we have:

$$\begin{aligned}
f(v_2)+f(u_1)+f(w_1) &= 0\\
f(v_1)+f(v_2)+f(v_3)+f(u_1)+f(u_2)+f(w_1)+f(w_2) &= 0\\
f(v_2)+f(v_3)+f(v_3)+f(u_2)+f(u_3)+f(w_2)+f(w_3) &= 0\\
f(v_3)+f(v_4)+f(v_5)+f(u_3)+f(u_4)+f(w_3)+f(w_4) &= 0\\
&\vdots\\
f(v_{n-2})+f(v_{n-1})+f(v_n)+f(u_{n-2})+f(u_{n-1})+f(w_{n-2})+f(w_{n-1}) &= 0\\
f(v_{n-1})+f(u_{n-1})+f(w_{n-1}) &= 0
\end{aligned}$$

Similarly from the degree sum equation of u_i and w_i we get the following system of equations:

$$\begin{aligned}
f(u_1)+f(v_1)+f(v_2) &= 0\\
f(u_2)+f(v_2)+f(v_3) &= 0\\
f(u_3)+f(v_3)+f(v_4) &= 0\\
&\vdots\\
f(u_{n-1})+f(v_{n-1})+f(v_n) &= 0
\end{aligned}$$

and

$$\begin{aligned}
f(w_1)+f(v_1)+f(v_2) &= 0\\
f(w_2)+f(v_2)+f(v_3) &= 0\\
f(w_3)+f(v_3)+f(v_4) &= 0\\
&\vdots\\
f(w_{n-1})+f(v_{n-1})+f(v_n) &= 0
\end{aligned}$$

On comparing the degree sum equations of u_i and w_i we get $f(u_i) = f(w_i)$, for $i = 1, 2, 3, \cdots, n-1$, and using this in the degree sum equations of v_i we get

$$\begin{aligned}
f(v_2) &= 0\\
f(v_1)+f(v_2)+f(v_3) &= 0\\
f(v_2)+f(v_3)+f(v_3) &= 0\\
f(v_3)+f(v_4)+f(v_5) &= 0\\
&\vdots\\
f(v_{n-2})+f(v_{n-1})+f(v_n) &= 0\\
f(v_{n-1}) &= 0
\end{aligned}$$

On solving this we get $f(v_2) = f(v_5) = f(v_8) = \cdots = 0$, $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots$. On substituting this in the degree sum equation of w_i we get $f(w_3) = f(w_6) = f(w_9) = \cdots = 0, f(w_{n-3}) = f(w_{n-6}) = f(w_{n-9}) = \cdots = 0$ and $f(w_1) = f(w_2) = f(w_4) = f(w_{n-5}) \cdots = f(v_1) = f(w_{n-1}) = f(w_{n-2}) = f(w_{n-4}) = f(w_{n-5}) = \cdots = f(v_1)$

Case 1 : $n = 3k + 1$ for some integer k.

In this case, the above conclusion of the system of equations become,$f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{n-2}) = 0$ and $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = f(v_3) = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots = f(v_{n-1}) = f(v_n)$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots = f(v_4) = f(v_2) = f(v_1)$. Thus $f \equiv 0$.

Case 2 : $n = 3k + 2$ for some integer k.

In this case, the above conclusion of the system of equations become,$f(v_2) = f(v_5) = f(v_8) = \cdots = f(v_{n-3}) = f(v_n) = 0$ and $f(v_{n-1}) = f(v_{n-4}) = f(v_{n-7}) = \cdots = f(v_4) = f(v_1) = 0$, $f(v_1) = f(v_3) = f(v_4) = f(v_6) = f(v_7) = \cdots = f(v_{n-2}) = f(v_{n-1})$ and $f(v_n) = f(v_{n-2}) = f(v_{n-3}) = f(v_{n-5}) = f(v_{n-6}) = \cdots = f(v_3) = f(v_2)$. Thus $f \equiv 0$.

Hence in both cases we get $f \equiv 0$, which is a contradiction. Hence the proof.

Corollary 5.9. *The double triangular snake graph $DTS_n \in \Gamma_{k,0}(V_4)$ if and only if $n \equiv 0$ $(mod\ 3)$.*
Proof. Proof follows from Theorem 5.8.

Definition 5.10. *An open ladder $O(L_n), n \geq 2$ graph is obtained from two paths of length $n-1$ with $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 2 \leq i \leq n-1\}$.*

Theorem 5.11. *For $n \geq 2$, the open ladder, $O(L_n) \in \Gamma(V_4)$ for $n \equiv 0$ $(mod\ 3)$.*
Proof. Consider an open ladder $O(L_n), n \geq 2$, with vertex set $V(G) = \{u_i, v_i : 1 \leq i \leq n\}$ and edge set $E(G) = \{u_iu_{i+1}, v_iv_{i+1} : 1 \leq i \leq n-1\} \cup \{u_iv_i : 2 \leq i \leq n-1\}$. Then for $n \equiv 0$ $(mod\ 3)$, define $f : V(O(L_n)) \to V_4$ as follows:

$$f(v) = \begin{cases} 0 & \text{if} \quad v = u_2, u_5, u_8, \cdots, u_{n-1} \\ 0 & \text{if} \quad v = v_2, v_5, v_8, \cdots, v_{n-1} \\ a & \text{if} \quad \text{otherwise.} \end{cases}$$

Then f is an IML of $O(L_n)$. Hence the proof follows.

Corollary 5.12. *For $n \geq 2$, the open ladder, $O(L_n) \in \Gamma_{k,0}(V_4)$ for $n \equiv 0$ $(mod\ 3)$.*
Proof. Proof follows from Theorem 5.11.

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ON CERTAIN SUMMATION FORMULAE FOR q-HYPERGEOMETRIC SERIES

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Abstract: In this paper, making use of a transformation formula of basic bilateral q series due to Bailey, certain summation formulae of basic bilateral series have been established.

Keywords and Phrases: q-hypergeometric series, q-bilateral hypergeometric series, transformation formula, summation formula.

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1. Introduction, Notations and Definitions

Let q be a fixed complex parameter with $0 < |q| < 1$. The $q-$ shifted factorial is defined for any complex parameter 'a' by

$$(a;q)_\infty = \prod_{r=0}^{\infty}(1-aq^r), \quad (a;q)_k = \frac{(a;q)_\infty}{(aq^k;q)_\infty},$$

where k is any integer.
For brevity, we write

$$(a_1, a_2, ..., a_r; q)_n = (a_1;q)_n (a_2;q)_n ... (a_r;q)_n.$$

Further, recall the definition of basic hypergeometric series

$$ {}_r\Phi_{r-1}\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_{r-1} \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_{r-1}; q)_n}, \tag{1.1}$$

and basic bilateral hypergeometric series

$$ {}_r\Psi_r\left[\begin{array}{l} a_1,a_2,...,a_r;q;z \\ b_1,b_2,...,b_r \end{array}\right] = \sum_{n=-\infty}^{\infty}\frac{(a_1,a_2,...,a_r;q)_n z^n}{(b_1,b_2,...,b_r;q)_n}. \tag{1.2} $$

We shall make use of following results in our analysis. Bailey's transformation formula,

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha,\beta;q;z \\ \gamma,\delta \end{array}\right] = \frac{(\alpha z,\beta z,\gamma q/\alpha\beta z,\delta q/\alpha\beta z;q)_\infty}{(q/\alpha,q/\beta,\gamma,\delta;q)_\infty}\,{}_2\Psi_2\left[\begin{array}{l} \alpha\beta z/\gamma,\alpha\beta z/\delta;q;\frac{\gamma\delta}{\alpha\beta z} \\ \alpha z,\beta z \end{array}\right], \tag{1.3} $$

where $\left|\frac{\gamma\delta}{\alpha\beta}\right| < |z| < 1$.

[Andrews and Berndt 1;(12.4.1) p. 273]

Ramanujan's summation formula

$$ {}_1\Psi_1\left[\begin{array}{l} a;q;z \\ b \end{array}\right] = \frac{(az,q/az,q,b/a;q)_\infty}{(b,q/a,z,b/az;q)_\infty}. \tag{1.4} $$

[Gasper and Rahman 4; App. II (II.29), p. 357]

2. Summation Formulas

In this section we establish summation formulas for basic bilateral hypergeometric series.

(i) Putting $z = q/\alpha$ in (1.3) we find,

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha,\beta;q;q/\alpha \\ \gamma,\delta \end{array}\right] = \frac{(\beta q/\alpha,\gamma/\beta,\delta/\beta;q)_\infty(q;q)_\infty}{(q/\alpha,q/\beta,\gamma,\delta;q)_\infty}\,{}_2\Phi_1\left[\begin{array}{l} \beta q/\alpha,\beta q/\delta;q;\frac{\gamma\delta}{\beta q} \\ \beta q/\alpha \end{array}\right]. \tag{2.1} $$

(ii) Again, taking $\delta = \beta q$ in (2.1) we get the summation formula,

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha,\beta;q;q/\alpha \\ \gamma,\beta q \end{array}\right] = \frac{(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \tag{2.2} $$

which is a known result due to Bhargava and Adiga [2].
Since

$$ {}_2\Psi_2\left[\begin{array}{l} \alpha,\beta;q;1/\alpha \\ \gamma,\beta q \end{array}\right] - \beta\,{}_2\Psi_2\left[\begin{array}{l} \alpha,\beta;q;q/\alpha \\ \gamma,\beta q \end{array}\right] = (1-\beta)\,{}_1\Psi_1\left[\begin{array}{l} \alpha;q;1/\alpha \\ \gamma \end{array}\right]. $$

From (1.4) we find,

$$ {}_1\Psi_1\left[\begin{array}{l}\alpha;q;1/\alpha\\ \gamma\end{array}\right]=0.\quad \text{So,}\quad {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;1/\alpha\\ \gamma,\beta q\end{array}\right]=\beta\,{}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]. \quad (2.3) $$

Now, making use of (2.2), (2.3) yields the summation formula,
(iii)

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;1/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{\beta(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \quad (2.4) $$

Again, let us consider,

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]-\beta\,{}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^2/\alpha\\ \gamma,\beta q\end{array}\right] $$

$$ =\sum_{n=-\infty}^{\infty}\frac{(\alpha;q)_n(1-\beta)}{(\gamma;q)_n}\left(\frac{q}{\alpha}\right)^n=(1-\beta)\,{}_1\Psi_1\left[\begin{array}{l}\alpha;q;q/\alpha\\ \gamma\end{array}\right]. $$

From (1.4) we find,

$$ {}_1\Psi_1\left[\begin{array}{l}\alpha;q;q/\alpha\\ \gamma\end{array}\right]=0\qquad \text{So,}\qquad {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^2/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{1}{\beta}\,{}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]. \quad (2.5) $$

Making use of (2.2) we have,
(iv)

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^2/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{1}{\beta}\frac{(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \quad (2.6) $$

Iterating this process, we have
(v)

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q^{k+1}/\alpha\\ \gamma,\beta q\end{array}\right]=\frac{1}{\beta^k}\frac{(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \quad (2.7) $$

where $|\gamma|<|q^k|<|\alpha|$.
Further, let us consider,

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]=\sum_{n=-\infty}^{\infty}\frac{(\alpha,\beta;q)_n(1/\alpha)^n}{(\gamma,\beta q;q)_n(1-c)}-\sum_{n=-\infty}^{\infty}\frac{(\alpha,\beta;q)_n(q/\alpha)^n c}{(\gamma,\beta q;q)_n(1-c)} $$

$$ =\frac{1}{(1-c)}\,{}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;1/\alpha\\ \gamma,\beta q\end{array}\right]-\frac{c}{(1-c)}\,{}_2\Psi_2\left[\begin{array}{l}\alpha,\beta;q;q/\alpha\\ \gamma,\beta q\end{array}\right]. \quad (2.8) $$

(vi) Making use of (2.2) and (2.4) in (2.8) we get,

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \tag{2.9} $$

which is known summation [Exton 3; App. A (A.25), p. 305].
(vii) Again, if we consider,

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]-\beta\ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]=(1-\beta)\ {}_2\Psi_2\left[\begin{array}{l}\alpha,cq;q;1/\alpha\\ \gamma,c\end{array}\right]. \tag{2.10} $$

Making use of (1.4) it is easy to show that

$$ {}_2\Psi_2\left[\begin{array}{l}\alpha,cq;q;1/\alpha\\ \gamma,c\end{array}\right]=0\quad\text{So,}\quad {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta}\ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;1/\alpha\\ \gamma,\beta q,c\end{array}\right]. \tag{2.11} $$

(viii) Making use of (2.9) in (2.11) we get the summation formula

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta}\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}, \tag{2.12} $$

which is also a known result [Exton 3; App. A (A.24), p. 305].
Proceeding as above

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]-\beta\ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^2/\alpha\\ \gamma,\beta q,c\end{array}\right]=(1-\beta)\ {}_2\Psi_2\left[\begin{array}{l}\alpha,cq;q;q/\alpha\\ \gamma,c\end{array}\right]. \tag{2.13} $$

${}_2\Psi_2$ on the right hand side of (2.13) also vanishes, so;

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^2/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta}\ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q/\alpha\\ \gamma,\beta q,c\end{array}\right]. \tag{2.14} $$

(ix) Making use of (2.12) we have

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^2/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta^2}\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.15} $$

(x) Iterating the above process, we have

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;q^k/\alpha\\ \gamma,\beta q,c\end{array}\right]=\frac{1}{\beta^k}\frac{(1-\beta/c)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.16} $$

Now, let us consider

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{1-d}\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c\end{array}\right]-\frac{d}{1-d}\,{}_3\Psi_3\left[\begin{array}{l}\alpha,\beta,cq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c\end{array}\right]. \tag{2.17}$$

Making use of (2.9) and (2.12) in (2.17) we get,
(xi)

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{(1-\beta/c)(1-\beta/d)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{\beta(1-1/c)(1-1/d)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.18}$$

Again,

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]-\beta\,{}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=(1-\beta)\,{}_3\Psi_3\left[\begin{array}{l}\alpha,cq,dq;q;\frac{1}{\alpha}\\ \gamma,c,d\end{array}\right]. \tag{2.19}$$

Using (1.4) it is easy to show that

$$ {}_3\Psi_3\left[\begin{array}{l}\alpha,cq,dq;q;\frac{1}{\alpha}\\ \gamma,c,d\end{array}\right]=0\ \text{So,}\ \ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{\beta}\,{}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]. \tag{2.20}$$

Making use of (2.18) in (2.20) we get,
(xii)

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{\beta^2}\frac{(1-\beta/c)(1-\beta/d)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(1-1/d)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.21}$$

Iterating this process, we have

$$ {}_4\Psi_4\left[\begin{array}{l}\alpha,\beta,cq,dq;q;\frac{q^k}{\alpha}\\ \gamma,\beta q,c,d\end{array}\right]=\frac{1}{\beta^{k+1}}\frac{(1-\beta/c)(1-\beta/d)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{(1-1/c)(1-1/d)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.22}$$

Also, proceeding as above, it is easy to show that,
(xiv)

$$ {}_5\Psi_5\left[\begin{array}{l}\alpha,\beta,cq,dq,eq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,d,e\end{array}\right]=\frac{1}{\beta^2}\frac{\left(1-\frac{\beta}{c}\right)\left(1-\frac{\beta}{d}\right)\left(1-\frac{\beta}{e}\right)(\beta q/\alpha,\gamma/\beta;q)_\infty(q;q)_\infty^2}{\left(1-\frac{1}{c}\right)\left(1-\frac{1}{d}\right)\left(1-\frac{1}{e}\right)(q/\alpha,q/\beta,\gamma,\beta q;q)_\infty}. \tag{2.23}$$

Iterating the process, we can show that,
(xv)

$$ {}_{r+3}\Psi_{r+3}\left[\begin{array}{l}\alpha,\beta,cq,c_1q,c_2q,...,c_rq;q;\frac{1}{\alpha}\\ \gamma,\beta q,c,c_1,c_2,...,c_r\end{array}\right]$$

$$= \frac{1}{\beta^r} \frac{\left(1-\frac{\beta}{c}\right)\left(1-\frac{\beta}{c_1}\right)\ldots\left(1-\frac{\beta}{c_r}\right)(\beta q/\alpha, \gamma/\beta; q)_\infty (q;q)_\infty^2}{\left(1-\frac{1}{c}\right)\left(1-\frac{1}{c_1}\right)\ldots\left(1-\frac{1}{c_r}\right)(q/\alpha, q/\beta, \gamma, \beta q; q)_\infty}. \tag{2.24}$$

3. Special Cases

In this section we shall deduce certain interesting summation formulae from the results established in previous section.

(i) Taking $\gamma = \alpha q$ in (2.2) we get,

$${}_2\Psi_2 \left[\begin{array}{l} \alpha, \beta; q; q/\alpha \\ \alpha q, \beta q \end{array} \right] = \frac{(\beta q/\alpha, \alpha q/\beta; q)_\infty (q;q)_\infty^2}{(q/\alpha, q/\beta, \alpha q, \beta q; q)_\infty}. \tag{3.1}$$

(ii) As $\alpha \to \infty$, (2.2) yields

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(\gamma;q)_n (1-\beta q^n)} = \frac{(\gamma/\beta; q)_\infty (q;q)_\infty^2}{(q/\beta, \gamma, \beta; q)_\infty}. \tag{3.2}$$

(iii) Taking $\gamma = q$ in (3.2) we find,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(q;q)_n (1-\beta q^n)} = \frac{(q;q)_\infty}{(\beta;q)_\infty}. \tag{3.3}$$

(iv) Replacing q by q^2 and then taking $\beta = q$ in (3.3) we get,

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)}}{(q^2;q^2)_n (1-q^{2n+1})} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \Psi(q), \tag{3.4}$$

where $\Psi(q)$ is a function given in [Andrews, G.E. and Berndt, B.C. 1; (1.1.7), p. 11].

(v) Taking $\gamma = q$ in (2.4) we find,

$${}_2\Phi_1 \left[\begin{array}{l} \alpha, \beta; q; 1/\alpha \\ \beta q \end{array} \right] = \frac{\beta(\beta q/\alpha; q)_\infty (q;q)_\infty}{(q/\alpha, \beta q; q)_\infty}. \tag{3.5}$$

(vi) For $\alpha \to \infty$, (3.5) yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(q;q)_n (1-\beta q^n)} = \frac{\beta(q;q)_\infty}{(\beta;q)_\infty}. \tag{3.6}$$

(vii) Taking $\gamma = \alpha q$ in (2.4) we find,

$$\sum_{n=-\infty}^{\infty} \frac{(1/\alpha)^n}{(1-\alpha q^n)(1-\beta q^n)} = \frac{\beta(\beta q/\alpha, \alpha q/\beta; q)_\infty (q;q)_\infty^2}{(q/\alpha, q/\beta, \alpha, \beta; q)_\infty}. \tag{3.7}$$

(viii) Taking $\alpha \to \infty$, $\gamma = 0$ in (2.2) we get,

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(1-\beta q^n)} = \frac{(q;q)_\infty^2}{(q/\beta,\beta;q)_\infty}. \tag{3.8}$$

(ix) Taking $\alpha \to \infty$ and $\gamma = 0$ in (2.4) we get

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(1-\beta q^n)} = \frac{\beta(q;q)_\infty^2}{(q/\beta,\beta;q)_\infty}. \tag{3.9}$$

(x) Taking $\gamma = q$ in (2.9) we find,

$${}_3\Phi_2\left[\begin{matrix}\alpha,\beta,cq;q;\frac{1}{\alpha}\\ \beta q, c\end{matrix}\right] = \frac{\left(1-\frac{\beta}{c}\right)(\beta q/\alpha;q)_\infty (q;q)_\infty}{\left(1-\frac{1}{c}\right)(q/\alpha,\beta q;q)_\infty}. \tag{3.10}$$

(xi) Taking $\alpha \to \infty$ we have

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2}(1-cq^n)}{(q;q)_n(1-\beta q^n)} = \frac{(1-c)(1-/\beta/c)(q;q)_\infty}{(1-1/c)(\beta;q)_\infty}. \tag{3.11}$$

(xii) As $c \to \infty$, (3.10) yields

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n (q/\alpha)^n}{(q;q)_n(1-\beta q^n)} = \frac{(\beta q/\alpha, q;q)_\infty}{(q/\alpha,\beta;q)_\infty}. \tag{3.12}$$

(xiii) Taking q^8 for q and then $\alpha = \beta = q$ in (3.12) we get,

$$\sum_{n=0}^{\infty} \frac{(q;q^8)_n q^{7n}}{(q^8;q^8)_n(1-q^{8n+1})} = \frac{(q^8;q^8)_\infty^2}{(q,q^7;q^8)_\infty}. \tag{3.13}$$

(xiv) Taking q^8 for q and then $\alpha = \beta = q^3$ in (3.12) we have

$$\sum_{n=0}^{\infty} \frac{(q^3;q^8)_n q^{5n}}{(q^8;q^8)_n(1-q^{8n+3})} = \frac{(q^8;q^8)_\infty^2}{(q^3,q^5;q^8)_\infty}. \tag{3.14}$$

(xv) Taking the ratio of (3.13) and (3.14) and using the result [Andrew and Berndt 1; (6.2.38), p. 154] we get

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(q^3;q^8)_n q^{5n}}{(q^8;q^8)_n(1-q^{8n+3})}}{\displaystyle\sum_{n=0}^{\infty} \frac{(q;q^8)_n q^{7n}}{(q^8;q^8)_n(1-q^{8n+1})}} = \frac{(q,q^7;q^8)_\infty}{(q^3,q^5;q^8)_\infty} = \frac{1}{1+}\,\frac{q+q^2}{1+}\,\frac{q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}. \tag{3.15}$$

(xvi) Replacing q by q^2 and then taking $\alpha = \beta = q$ in (3.12) we find,

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(q^2;q^2)_n(1-q^{2n+1})} = \frac{(q^2;q^2)_\infty^2}{(q;q^2)_\infty^2} = \Psi^2(q), \tag{3.16}$$

(xvii) Replacing q by q^6 and then $\alpha = \beta = q$ in (3.12) we get,

$$\sum_{n=0}^{\infty} \frac{(q;q^6)_n q^{5n}}{(q^6;q^6)_n(1-q^{6n+1})} = \frac{(q^6;q^6)_\infty^2}{(q,q^5;q^6)_\infty}. \tag{3.17}$$

(xviii) Replacing q by q^6 and then $\alpha = \beta = q^3$ in (3.12) we obtain

$$\sum_{n=0}^{\infty} \frac{(q^3;q^6)_n q^{3n}}{(q^6;q^6)_n(1-q^{6n+3})} = \frac{(q^6;q^6)_\infty^2}{(q^3;q^6)_\infty}. \tag{3.18}$$

(xv) Taking the ratio of (3.17) and (3.18) and using the result [Andrew and Berndt 1; (6.2.37), p. 154] we find

$$\frac{\displaystyle\sum_{n=0}^{\infty} \frac{(q^3;q^6)_n q^{3n}}{(q^6;q^6)_n(1-q^{6n+3})}}{\displaystyle\sum_{n=0}^{\infty} \frac{(q;q^6)_n q^{5n}}{(q^6;q^6)_n(1-q^{6n+1})}} = \frac{(q,q^5;q^6)_\infty}{(q^3;q^6)_\infty^2} = \frac{1}{1+}\,\frac{q+q^2}{1+}\,\frac{q^2+q^4}{1+}\,\frac{q^3+q^6}{1+\ldots}. \tag{3.19}$$

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A NEW METHOD FOR SOLVING DODECAGONAL FUZZY ASSIGNMENT PROBLEM

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Abstract: Assignment problem is a special case of linear programming problem. The objective of the optimal assignment is to minimize the total cost or maximize the profit. Fuzzy set theory has been applied in many fields of science, Engineering and Management. In this paper a new ranking method is proposed for solving the dodecagonal fuzzy assignment problem. Fuzzy assignment problem transformed into crisp assignment problem and solved by Hungarian method. A numerical example is presented and the optimal solution is derived by using proposed method.

Keywords and Phrases: Dodecagonal Fuzzy number, Ranking Method, Fuzzy Assignment Problem.

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1. Introduction

Assignment problem is the special case of linear programming problem. Assignment problem can be applied in all fields like Science, Engineering, and Management etc. Assignment problem plays an important role in industry and other applications. In assignment problem " n "jobs are assigned " n "persons depending their on their efficiency to do the job. The objective of the optimal assignment is to minimize the total cost or maximize the profit. Fuzzy assignment problems have received great attention in recent years. Here we investigate a more realistic problem, namely the assignment problem with fuzzy costs or times. The objective is to minimize the cost or to maximize the total profit, subject to some crisp constraints, the objective function is considered also as a fuzzy number.

L. A. Zadeh [20] introduced the concept of fuzzy sets to deal with imprecision, vagueness in real life situations. R. E. Bellmann and L. A. Zadeh [2] proposed the concept of decision making in fuzzy environment. C. J. Lin and U. P. Wen [8] was finding the solution of fuzzy assignment problem using labeling algorithm. H. Basirzadeh [1] approached a new technique for solving fuzzy transportation problem. M. S. Chen [3] proved some theorems and proved a fuzzy assignment model that considers all individuals to have same skills.

Amit Kumar and Anila Gupta [7] were first solved fuzzy assignment problem and travelling salesman problem with different membership function. R. Jahirhussain and P. Jayaraman [5] solved fuzzy assignment problem using robust ranking method. T. S. Pavithra and C. Jenita [14] proposed a new a method for solving a dodecagonal fuzzy assignment problem. S. Manimaran and M. Ananthanarayanan [9] were discussed a comparative study of two fuzzy numbers using average method. Y. L. P. Thorani and N. Ravi Shankar [19] were studied the applications of fuzzy assignment problems. D. Stephen Dinagar and S. Kamalanathan [16] solved a fuzzy assignment problem with two different ranking methods. L. Sujatha and S. Elizabeth [17] solved the fuzzy transportation problem and fuzzy unbalanced assignment problem by using one point method. Anchal Choudhary, R. N. Jat, C. Sharma and Sanjay Jain [4] were applied a branch and bound technique for solving fuzzy assignment problem.

P. Pandian and K. Kavitha [13] solved the assignment problems using parallel moving method. A. Nagoor Gani and V. N. Mohamed [12] proposed a new method for solving assignment problem for trapezoidal fuzzy numbers. Sunil Kumar Mehta, Neha Ishesh Thakur, Parmpreet Kaur [10] had approached numerical methods to find the solution of fuzzy assignment problem. Y. L. P. Thorani and N. Ravi Shankar [18] solved fuzzy assignment problem with generalized fuzzy number. Supriya Kar, KajlaBasu, Sathi Mukherjee [6] finding solution of generalized fuzzy assignment problem with restriction on the cost of both job and person under fuzzy environment. S. Muruganandhan and D. Hema [11] solved fuzzy assignment method using one suffix method. Jatinder Pal Singh and Neha Ishesh Thakur [15] were solved fuzzy transportation problem using dodecagonal fuzzy number.

In this paper a new ranking method is proposed in solving a dodecagonal fuzzy assignment problem. Fuzzy assignment problem can be converted into crisp assignment problem using ranking method and an optimal solution is obtained by using Hungarian method.

2. Preliminaries

Definition 2.1. *A fuzzy set is characterized by a membership function mapping element of a domain space or the universe of discourse X to the unit interval $\{0, 1\}$.*

$$(i.e)A = \{x, \mu_A(x); x \in X\}, \quad here \;\; \mu_A(x) = 1$$

Definition 2.2. *A fuzzy set A of the universe of discourse X is called normal fuzzy set implying that there exist atleast one such that $\mu_A(x) = 1$.*

Definition 2.3. *support of a fuzzy set in the universal set X is the set contains all the elements of X that have a non-zero membership grade in $\widetilde{A}$.*

$$Supp(\widetilde{A}) = \{x \in X \mid \mu_{\widetilde{A}}(x) > 1\}$$

Definition 2.4. *Given a fuzzy set A defined on X and any number $\alpha \in [0, 1]$ the $\alpha-$cut, α_A is the crisp set,*

$$\alpha_A = \{x \in X \mid A_x \geq \alpha, \alpha \in [0, 1]\}$$

Definition 2.5. *A fuzzy set $\widetilde{A}$ defined on the set of real numbers R is said to be fuzzy number if its membership function $\mu_A : R \to [0, 1]$ has the following properties,*
1. A must be a normal and convex fuzzy set.
2. α_A must be a closed interval for every $\alpha \in [0, 1]$.
3. The support of $\widetilde{A}$ must be bounded.

Definition 2.6. *A fuzzy number $\widetilde{A}$ is called triangular function is denoted by $\widetilde{A} = (a_1, a_2, a_3)$ whose membership function is defined as follows,*

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & x < a_1 \\ \frac{x-a_1}{a_1-a_2} & a_1 \leq x \leq a_2 \\ \frac{a_3-x}{a_3-a_2} & a_2 \leq x \leq a_3 \\ 0 & x > a_3 \end{cases}$$

Definition 2.7. *A fuzzy number $\widetilde{A}$ is called trapezoidal function is denoted by*

$\widetilde{A} = (a_1, a_2, a_3, a_4)$ *whose membership function is defined as follows,*

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & x<a_1 \\ \frac{x-a_1}{a_1-a_2} & a_1 \leq x \leq a_2 \\ 1 & a_2 \leq x \leq a_3 \\ \frac{a_4-x}{a_4-a_3} & a_3 \leq x \leq a_4 \\ 0 & x > a_3 \end{cases}$$

3. Dodecagonal Fuzzy Number

Definition 3.1. *A fuzzy number $\widetilde{A}$ is a Dodecagonal fuzzy number defined by $\widetilde{A}$=(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}), where a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12} are real numbers and its membership function is given by,*

$$\mu_{\widetilde{A}}(x) = \begin{cases} 0 & x \leq a_1 \\ k_1\left(\frac{x-a_1}{a_2-a_1}\right) & a_1 \leq x \leq a_2 \\ k_1 & a_2 \leq x \leq a_3 \\ k_1 + (k_2-k_1)\left(\frac{x-a_3}{a_4-a_3}\right) & a_3 \leq x \leq a_4 \\ k_2 & a_4 \leq x \leq a_5 \\ k_2 + (1-k_2)\left(\frac{x-a_5}{a_6-a_5}\right) & a_5 \leq x \leq a_6 \\ 1 & a_6 \leq x \leq a_7 \\ k_2 + (1-k_2)\left(\frac{a_8-x}{a_8-a_7}\right) & a_7 \leq x \leq a_8 \\ k_2 & a_8 \leq x \leq a_9 \\ k_1 + (k_2-k_1)\left(\frac{a_{10}-x}{a_{10}-a_9}\right) & a_9 \leq x \leq a_{10} \\ k_1 & a_{10} \leq x \leq a_{11} \\ k_1\left(\frac{a_{12}-x}{a_{12}-a_{11}}\right) & a_{11} \leq x \leq a_{12} \\ 0 & a_{12} \leq x \end{cases}$$

for $0 < k_1 < k_2 < 1$.

3.1. Arithmetic Operations on Dodecagonal Fuzzy Number

Let $\widetilde{A}_{DDFN}$=(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10}, a_{11}, a_{12}) & $\widetilde{B}_{DDFN}$= (b_1, b_2, b_3, b_4, b_5, b_6, b_7, b_8, b_9, b_{10}, b_{11}, b_{12}), be two dodecagonal fuzzy numbers then the addition, subtraction and scalar multiplications can be performed as follows,
$\widetilde{A}_{DDFN}$+ $\widetilde{B}_{DDFN}$=[a_1+b_1, a_2+b_2, a_3+b_3, a_4+b_4, a_5+b_5, a_6+b_6, a_7+b_7, a_8+b_8, a_9+b_9, $a_{10}+b_{10}$, $a_{11}+b_{11}$, $a_{12}+b_{12}$]

$\widetilde{A}_{DDFN}$ - $\widetilde{B}_{DDFN}$=$[a_1 - b_{12},\ a_2 - b_{11},\ a_3 - b_{10},\ a_4 - b_9,\ a_5 - b_8,\ a_6 - b_7,\ a_7 - b_6,$ $a_8 - b_5,\ a_9 - b_4,\ a_{10} - b_3,\ a_{11} - b_2,\ a_{12} - b_1]$

$\lambda\widetilde{A}_{DDFN}$=$[\lambda a_1,\ \lambda a_2,\ \lambda a_3,\ \lambda a_4,\ \lambda a_5,\ \lambda a_6,\ \lambda a_7,\ \lambda a_8,\ \lambda a_9,\ \lambda a_{10},\ \lambda a_{11},\ \lambda a_{12}]$

$\lambda\widetilde{B}_{DDFN}$= $[\ \lambda b_1,\ \lambda b_2,\ \lambda b_3,\ \lambda b_4,\ \lambda b_5,\ \lambda b_6,\ \lambda b_7,\ \lambda b_8,\ \lambda b_9,\ \lambda b_{10},\ \lambda b_{11},\ \lambda b_{12}]$

3.2. Measure of Fuzzy Number

The measure of $\widetilde{A}_\omega$ is a measure is a function $M_o : R_\omega(I) \to R^+$ which assign a non-negative real numbers $M_0^{DDFN}(\widetilde{A}_\omega)$ that expresses the measure of

$$M_0^{DDFN}(\widetilde{A}_\omega) = \tfrac{1}{2}\int_{\alpha}^{k_1}\left(f_1(r) + \bar{f}_1(r)\right)dr + \tfrac{1}{2}\int_{k_1}^{k_2}\left(g_1(s) + \bar{g}_1(s)\right)ds + \tfrac{1}{2}\int_{k_2}^{\omega}\left(h_1(t) + \bar{h}_1(t)\right)dt \tag{3.1}$$

where $0 \le \alpha < 1$.

4. Proposed Ranking Method

Let $\widetilde{A}$ be a normal dodecagonal fuzzy number. The measure of $\widetilde{A}$ is defined by,

$$M_0^{DDFN}(\widetilde{A}_\omega) = \tfrac{1}{2}\int_{\alpha}^{k_1}\left(f_1(r) + \bar{f}_1(r)\right)dr + \tfrac{1}{2}\int_{k_1}^{k_2}\left(g_1(s) + \bar{g}_1(s)\right)ds + \tfrac{1}{2}\int_{k_2}^{\omega}\left(h_1(t) + \bar{h}_1(t)\right)dt$$

$$M_0^{DDFN}(\widetilde{A}) = \tfrac{1}{4}\{(a_1 + a_2 + a_{11} + a_{12})k_1 + (a_3 + a_4 + a_9 + a_{10})(k_2 - k_1) + (a_5 + a_6 + a_7 + a_8)(1 - k_1)\} \tag{4.1}$$

where $0 < k_1 < k_2 < 1$

5. Mathematical Formulation of Fuzzy Assignment Problem

The assignment problem can be stated in the form of cost matrix of fuzzy numbers as follows:

Persons	Jobs			
	1	**2**	...	**n**
1	$\widetilde{c}_{11}$	$\widetilde{c}_{12}$		$\widetilde{c}_{1n}$
2	$\widetilde{c}_{21}$	$\widetilde{c}_{22}$		$\widetilde{c}_{2n}$
...	...	...	...	...
n	$\widetilde{c}_{11}$	$\widetilde{c}_{12}$		$\widetilde{c}_{nn}$

The mathematical formulation of the fuzzy assignment problem is given by

$$minimize\ \widetilde{Z}^* = \textstyle\sum_{i=1}^{n}\sum_{j=1}^{n}\widetilde{c}_{ij}x_{ij}^*$$

Subject to the constraints $\sum_{i=1}^{n} x_{ij} = 1, i = 1, 2, ..., n$ $\sum_{j=1}^{n} x_{ij} = 1, j = 1, 2, ..., n$ where x_{ij} is the decision variable
and

$$x_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ person assigned to } j^{th} \text{ job} \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

$\widetilde{c}_{ij}$ is the fuzzy assignment cost of i^{th} job to j^{th} person. Hence it cannot be solved directly. For solving the problem, we first defuzzify the fuzzy cost coefficients into crisp ones by the above fuzzy number ranking method (4.1).

5.1. Numerical Example

Consider the following dodecagonal fuzzy assignment problem which consists of four jobs and four machines. The cost matrix $\widetilde{c}_{ij}$ whose costs are dodecagonal fuzzy numbers. Here our objective is to find the optimum assignment so as to minimize the cost (or time).

Table 1: Dodecagonal Fuzzy Assignment Problem

	$\mathbf{W}_1$	$\mathbf{W}_2$	$\mathbf{W}_3$	$\mathbf{W}_4$
$\mathbf{J}_1$	(1,3,5,7,9,11,13, 15,17,19,21,23)	(2,4,6,8,10,12,14, 16,18,20,22,24)	(1,2,3,4,5,6,7, 8,9,10,11,12)	(1,2,3,4,5,7,9, 11,13,17,21,25)
$\mathbf{J}_2$	(3,7,11,13,17,21, 22,25,29,32,40,43)	(2,4,6,8,9,13,15, 16,18,20,21,25)	(2,3,7,8,9,11,13, 15,16,21,33,37)	(1,3,5,7,9,12,15, 18,21,25,29,33)
$\mathbf{J}_3$	(1,2,3,4,7,10,13, 15,16,17,22,26)	(5,8,10,13,16,21, 23,28,31,32,37,39)	(4,6,7,9,10,11,18, 23, 24,26,27,30)	(2,3,5,7,10,13,17, 21,25,29,34,39)
$\mathbf{J}_4$	(2,3,5,7,11,13,17, 19,23,29,31,35)	(1,2,3,4,7,10,13, 16,20,24,28,32)	(1,4,7,10,13,16,19, 22,25,28,31,34)	(3,6,9,12,16,20,24, 29,33,38,40,42)

5.2. Ranking of Dodecagonal Fuzzy Number

In order to find the optimum value of the given dodecagonal fuzzy cost given in table 1, first we convert the fuzzy cost into the crisp cost using the proposed ranking method (4.1). Take the values of $k_1 = 0.3, k_2 = 0.7$. The ranking of fuzzy numbers is done by using proposed ranking method (4.1). The crisp assignment problem of the corresponding dodecagonal fuzzy assignment problem is given in the following table. The given fuzzy assignment problem is a balanced assignment problem. By applying the Hungarian method, we find the optimal assignment schedule and the optimum assignment cost. The optimal assignment schedule is given by $J_1 \to W_3,\ J_2 \to W_4,\ J_3 \to W_1,\ J_4 \to W_2$. The optimal assignment cost =

Table 2: Crisp assignment problem of the corresponding Dodecagonal fuzzy assignment problem

	$\mathbf{W}_1$	$\mathbf{W}_2$	$\mathbf{W}_3$	$\mathbf{W}_4$
$\mathbf{J}_1$	12	13	6.5	9.8
$\mathbf{J}_2$	21.9	13.1	14.4	14.9
$\mathbf{J}_3$	11.2	21.8	61.3	17
$\mathbf{J}_4$	16.2	13.3	17.5	22.7

$6.5+14.9+11.2+13.3 = 45.9$ units

6. Conclusion

Ranking of fuzzy number plays an important role in decision making problems and some other fuzzy application system. Fuzzy numbers must be ranked before an action is taken by a decision maker. Ranking methods which convert a fuzzy number to a crisp number by applying a mapping function. In this paper, a new method is proposed for solving dodecagonal fuzzy assignment problem. The proposed ranking method is simple and easy to calculate rank of fuzzy numbers which also gives perfect solution to the given problem. This ranking method is used to rank the all the Dodecagonal fuzzy numbers. The advantage of the proposed model is illustrated by examples. In future, this method is applied to assigning jobs to suitable persons in a real life problem.

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LIAR'S DOMINATION IN SIERPIŃSKI-LIKE GRAPHS

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Abstract: The vertex set $L \subseteq V(G)$ is a liar's dominating set if and only if it satisfies the following two conditions: (i) L double dominates every $v \in V(G)$ and (ii) for every pair u, v of distinct vertices, $|(N[u] \cup N[v]) \cap L| \geq 3$. The liar's domination number for a graph G is denoted by $\gamma_L(G)$ which is the minimum cardinality of the liar's dominating set L. Liar's domination was introduced by P. J. Slater. In a liar's dominating set it is assumed that any one protective device in its neighborhood of the intruder vertex might misreport the location of an intruder vertex in its closed neighborhood. In this paper, we determine the liar's domination set for Sierpiński-like graphs.

Keywords and Phrases: Domination, Liar's domination, Sierpiński graphs, Sierpiński cycle graphs, Sierpiński complete graphs.

2010 Mathematics Subject Classification: 97K30, 05C85.

1. Introduction

Domination in graphs is a widely researched topic because of its applications in many fields. There are different variations of domination existing in literature that motivates one to explore its applications in any graph or network. In the year 2009 Slater introduced liar's domination. This concept was introduced in such a way that a network is modeled as a graph and all its vertices are the possible locations for the intruder to enter and a dominating set as a set of protection devices placed at a vertex v so that the intruder and its exact location can be detected in its closed neighbourhood even if a protection device is allowed to lie or becomes faulty.

Consider the graph $G = (V, E)$, for $u \in V(G)$ we denote the open and closed neighbourhoods of v as $N(v) = \{u|uv \in E(G)\}$ and $N[v] = N(v) \cup \{v\}$ respectively. A vertex u is said to be dominated by v if $u \in N[v]$. A set $D \subseteq V$ is a dominating set if each vertex $v \in V$ is dominated by a vertex in D or $|N[v] \cap D| \geq 1$ for all $v \in V$. A set $D \subseteq V$ is a double dominating set if each vertex $v \in V$ is dominated by atleast two vertices in D or $|N[v] \cap D| \geq 2$ for all $v \in V$. A set $D \subseteq V$ is a k-tuple dominating set if each vertex $v \in V$ is dominated by atleast k vertices in D or $|N[v] \cap D| \geq k$ for all $v \in V$. P. J. Slater introduced liar's domination with the following characterization: A set $L \in V(G)$ is a liar's dominating set if and only if it satisfies these two conditions (i) $|N[v] \cap L| \geq 2$ and (ii) for any two distinct vertices u, v, $|(N[u] \cup N[v]) \cap L| \geq 3$ [9].

Slater [9] has showed that the liar's dominating set problem is NP-*hard* for general graphs and has specified a lower bound for trees. Roden and Slater [8] proved that the problem is NP-*hard* even for bipartite graphs. Panda and Paul [5,6] have proved that the problem is NP-*hard* for split graphs and chordal graphs and later they suggested a linear time algorithm for proper interval graphs. Liar dominating set for Circulant networks was given by Paul Manuel [4]. B. S. Panda et al. [7] studied the problem for bounded degree graphs and p-claw free graphs. Alimadadi et al. [1] have given the characterization of graphs and trees such that the liar's domination number is $| V |$ and $| V | -1$ respectively. In this paper we determine the liar's domination number for Sierpiński cycle graphs and Sierpiński complete graphs.

2. Sierpiński Graphs

Consider the Sierpiński graph $S(n, G)$ to be a finite undirected graph with the set of vertices $\{1, 2, \ldots, k\}$ where k is an integer, with vertex set $\{1, 2, \ldots, k\}^n$ and edge set $\{u, v\}$ is defined if and only if there exists an $h \in \{1, 2, \ldots, n\}$ such that:

- $u_t = v_t$ for $t = 1, 2, \ldots, h-1$;
- $u_h \neq v_h$;
- $u_t = v_h$ and $v_t = u_h$ for $t = h+1, \ldots, n$.

Here vertex $(u_1, u_2, ..., u_n)$ is represented as $(u_1u_2 \ldots u_n)$ and in Figures as $u_1u_2 \ldots u_n$. The vertices $(1 \ldots 1)$, $(2 \ldots 2)$,$\ldots$,$(k \ldots k)$ are called extreme vertices of $S(n, G)$. Let $n \geq 2$, for $i \in \{1, \ldots, k\}$ $S_i(n-1, G)$ be the subgraph of $S(n, G)$ induced by the vertices of the form $(iv_2 \ldots v_n)$. Note that $S_i(n-1, G)$ is isomorphic to $S(n-1, G)$ [2].

Remark 2.1. *$S(1, G)$ is isomorphic to the graph G and we can construct $S(n +$*

$1, G)$ by copying $|V(G)|$ times $S(n, G)$ and adding an edge between the vertices $ijj...j$ and $jii...i$ which is called as the linking edge in $S(n+1, G)$ [2].

Sierpiński graphs have played an important role in the growing literature of research. The variants of these graphs are numerous and have applications in different fields of mathematics. Klavžar and Milutinović [3] proved that the Sierpiński graphs $S(n, K_3)$ are isomorphic to the Tower of Hanoi graphs on 3 pegs. Many authors have discussed, investigated and given many results regarding the chromatic number, vertex cover number, clique number, domination number and many more.

3. Sierpiński Cycle Graphs

In this section we consider G to be isomorphic to C_4.

Theorem 3.1. [9] *For a cycle C_n we have $\gamma_L(C_n) = \lceil \frac{3n}{4} \rceil$.*

By the above result for $n = 1$, the result is obvious.

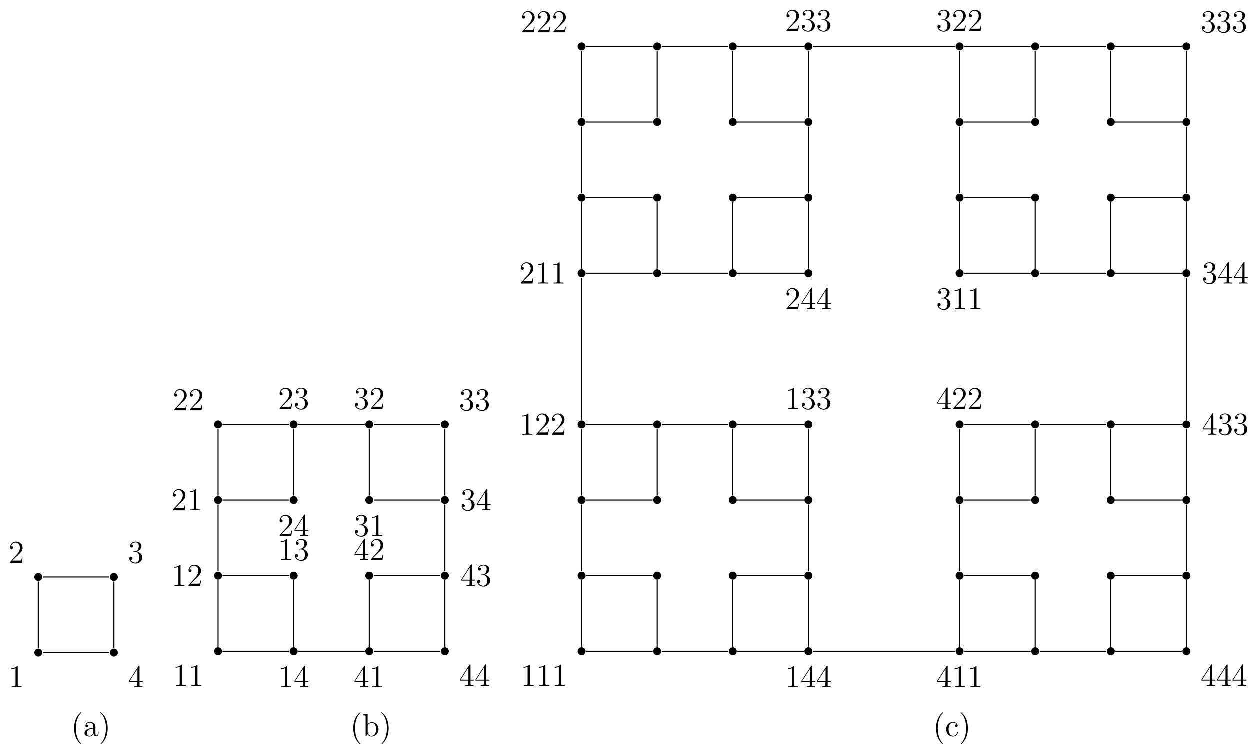

Figure 1: Sierpiński Cycle Graphs $S(1, C_4)$, $S(2, C_4)$, $S(3, C_4)$

Theorem 3.2. *Let G be a Sierpiński cycle graph $S(n, C_4)$, $n \geq 2$ then $\gamma_L S(n, C_4) = 4[\gamma_L S(n-1, C_4)] - 4$.*

Proof. Let us prove the result by the method of induction. For $n = 2$, the vertices are labeled as shown in Figure 1(b). Since the corner vertices ii, $i = 1, 2, 3, 4$ and its diagonally opposite vertices are of degree two, among $(i1, i2, i3, i4)$; $i = 1, 2, 3, 4$ any three of the vertices should be in liar's dominating set. Therefore $L = \{11, 12, 14, 21, 22, 23, 32, 33, 34, 41, 43, 44\}$ is a minimum liar's dominating set. For $n = 3$, we know that $S(3, C_4)$ is constructed from 4 copies of $S(2, C_4)$ namely $S_i(2, C_4)$, $i = 1, 2, 3, 4$. We can obtain the liar's dominating set for $S_i(2, C_4)$ from $S(2, C_4)$. By construction, the extreme vertex $(i \quad i+1 \quad i+1)$ of $S_i(2, C_4)$ is joined by an edge with the extreme vertex $(i+1 \quad i \quad i)$, $i = 1, 2, 3, 4(i \ mod\ 4)$ and in $S(2, C_4)$ all the extreme vertices are in L. In $S(3, C_4)$ we can either have $(i \quad i+1 \quad i+1)$ or $(i+1 \quad i \quad i)$. Without loss of generality suppose $(i \quad i+1 \quad i+1) \in L$ and $(i+1 \quad i \quad i) \notin L$, then also the closed neighbourhood of the vertices $(i+1 \quad i \quad i)$ and its diagonally opposite vertex have atleast 3 vertices in L. Thus $\gamma_L S(3, C_4) = 4[\gamma_L S(2, C_4)] - 4$. Let us assume that the result is true for $S(n, C_4)$, $n < k$. Let $n = k$. Since $S(k, C_4)$ is obtained from 4 copies of $S(k-1, C_4)$ by joining the edge $(i \quad i+1 \quad i+1 \ldots i+1)$ with $(i+1 \quad i \quad i \ldots i)$, $i = 1, 2, 3, 4(i \ mod\ 4)$. Also since minimum liar's dominating set of $S(k-1, C_4)$ includes all the corner vertices, by induction hypothesis in a similar manner $\gamma_L S(k, C_4) = 4[\gamma_L S(k-1, C_4)] - 4$.

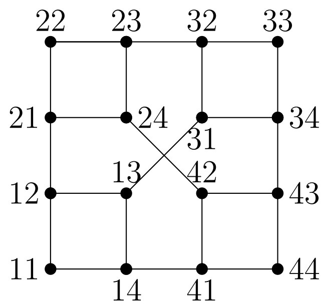

Figure 2: Sierpiński Cycle Graph $S'(2, C_4)$

The graph $S'(2, C_4)$ shown in Figure 2 is obtained by joining the inner vertices 13, 31, 24, 42 of $S(2, C_4)$ with edges called cross edges namely (13, 31), (24, 42). Similarly $S'(3, C_4)$ is obtained by taking four copies of $S'(2, C_4)$ along with linking edges in $S(3, C_4)$ and also the vertices (133, 311) and (244, 422) are joined with edges. In general $S'(n, C_4)$ is obtained in a similar manner from $S'(n-1, C_4)$ which include linking edges and the cross edges (244...4, 422...2) and (133...3, 311...1).

Theorem 3.3. *Let G be a Sierpiński cycle graph $S'(2, C_4)$ then $\gamma_L S'(2, C_4) = 10$.*

Proof. In view of Theorem 3.2, $\gamma_L S(2, C_4) = 12$ where L should have atleast three vertices from each $S_i(1, C_4)$ of $S(2, C_4)$ and since $d(i \quad i+2) = 3$ for $i = 1, 2, 3, 4(i \ mod\ 4)$ let us take $L = \{12, 14, 21, 23, 32, 34, 41, 43, 13, 31, 24, 42\}$ as a

liar's dominating set. In order to get minimum, either (24 and 31) or (13 and 42) can be removed from L. Without loss of generality let (13 and 42) $\in L$. Then also $|(N[31] \cup N[33]) \cap L| = 3$ and $|(N[24] \cup N[22]) \cap L| = 3$. Thus $\gamma_L S'(2, C_4) = 10$.
Note: There are number of minimum liar's dominating sets for $S'(2, C_4)$. In which by symmetry the minimum liar's dominating set that includes all its extreme vertices is $\{11, 12, 13, 22, 23, 32, 33, 42, 43, 44\}$.

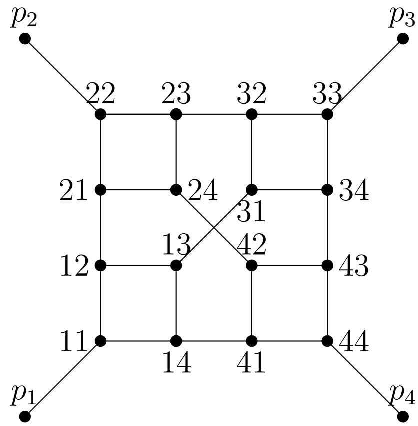

Figure 3: $S'_H(2, C_4)$

α-graph shown in Figure 3 is obtained from $S'(2, C_4)$ by attaching pendant edge at each of the extreme vertices namely ii, $i = 1, 2, 3, 4$ *mod* 4. The end vertex (pendant vertex) in each of the pendant edge are labeled as p_1, p_2, p_3 and p_4.

Remark 3.4. *We prove that the minimum liar's dominating set of $S'(2, C_4)$ in α-graph is same as that of $S'(2, C_4)$. The only case for which we have to verify is for the minimum liar's dominating set which include the corner vertices (i.e) for $L = \{11, 12, 13, 22, 23, 32, 33, 42, 43, 44\}$. Now we prove that instead of $ii \in L$ we cannot include $p_i \in L$. If $p_1 \in L$ and $11 \notin L$ then $|(N[12] \cup N[13]) \cap L| = 2$. And if $p_2 \in L$ and $22 \notin L$ we have $|N[21] \cap L| = 1$. Thus if any of the p_i's are included then minimum liar's dominating set whose cardinality is 10 cannot be obtained.*
Theorem 3.5. *Let G be a Sierpiński cycle graph $S'(n, C_4)$ then $\gamma_L S'(n, C_4) = 4[\gamma_L S'(n-1, C_4)]$.*
Proof. In view of Theorem 3.3, the result is true for $n = 2$. For $n = 3$, consider $S'_1(2, C_4)$ since $S'(3, C_4)$ consists of 4 copies of $S'(2, C_4)$ namely $S'_i(2, C_4)$, $i = 1, 2, 3, 4$. The graph induced by $\langle S'_1(2, C_4), 211, 311, 411\rangle$ is an $\alpha \setminus p_1$-graph. In view of Remark 3.4, the minimum liar's dominating set for this $\alpha \setminus p_1$-graph lies in $S'_1(2, C_4)$. Similar cases are dealt for $S'_i(2, C_4)$, $i = 2, 3, 4$. Therefore $\gamma_L S'(3, C_4) =$

$4[\gamma_L S'(2, C_4)]$. Let us assume that the result is true for $S'(n, C_4)$, $n < k$. Let $n = k$, since $S'(k, C_4)$ is constructed from 4 copies of $S'(k-1, C_4)$ by induction hypothesis we have $\gamma_L S'(k, C_4) = 4[\gamma_L S'(k-1, C_4)]$.

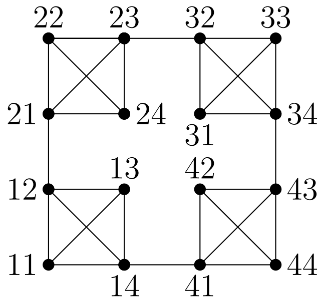

Figure 4: Sierpiński Complete Graph $S(2, K_4)$

4. Sierpiński Complete Graph

The Sierpiński complete graph $S(2, K_4)$ is formed by 4 copies of K_4 by adding linking edges $(12, 21), (23, 32), (34, 43), (14, 41)$ between them. We observe that the minimum liar's dominating set of $S(n, K_4)$ is the same as that of the minimum liar's dominating set of $S(n, C_4)$. $S(1, K_4) \cong K_4$ and by [8] $\gamma_L(K_n) = 3$. Consider $n = 2$, each $S_i(1, C_4)$ should have 3 vertices in L, otherwise $|(N[11] \cup N[13]) \cap L| < 3$. Similarly for the case $S_i(1, K_4)$. Thus in view of Theorem 3.2 we have the following result.

Theorem 4.1. *Let G be a Sierpinski complete graph $S(n, K_4)$ then $\gamma_L S(n, K_4) = 4[\gamma_L S(n-1, K_4)] - 4$.*

Let $S'(n, K_4)$ be obtained from 4 copies of $S'(n-1, K_4)$ together with the linking edges and cross edges.

Theorem 4.2. *Let G be a Sierpiński complete graph $S'(2, K_4)$ then $\gamma_L S'(2, K_4) = 9$.*

Proof. Since $d(ii) = 3$, L should have atleast two and atmost three vertices from each $S'_i(1, K_4)$, $i = 1, 2, 3, 4$. Thus $8 \leq \gamma_L S'(2, K_4) \leq 12$. Now we prove that $\gamma_L S'(2, K_4) > 8$. Suppose $\gamma_L S'(2, K_4) = 8$ then two vertices from $S'_1(1, K_4)$ will be in L. It can be either (11, 12) or (11, 13) or (11, 14) or (12, 13) or (12, 14) or (13, 14). In order to satisfy (ii) condition of minimum liar's dominating set the adjacent vertices of $S'_1(1, K_4)$ should be in L namely 21, 31 and 41. Thus now $|L| = 5$.

Case 1: $11 \notin L$

Without loss of generality let us take 12, 13 $\in L$ along with 21, 31, 41. Now we can take exactly one vertex from each $S'_i(1, K_4)$, $i = 2, 3, 4$. In $S'_2(1, K_4)$ it can be either

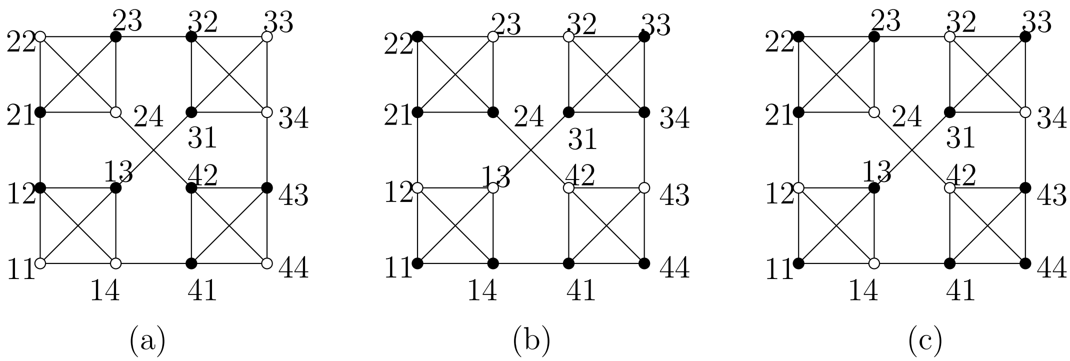

Figure 5: Bold vertices in each case represent the liar's dominating set

22 or 23 or 24. Since 23 and 24 are adjacent vertices of $S'_3(1, K_4)$ and $S'_4(1, K_4)$ respectively, let $23 \in L$. Then 42 should be in L. Also because the adjacent vertex of $S'_3(1, K_4)$ should be in L, $43 \in L$. Thus $|L| > 8$.

Case 2: $11 \in L$

Without loss of generality let us take 11, 12 along with 21, 31, 41 in L. Again either 22 or 23 or $24 \in L$. Suppose $23 \in L$ then 32, 42 should be in L. Now each $S'_i(1, K_4)$ has exactly two vertices. If it is so then the adjacent vertices of $S'_i(1, K_4)$ should be in L. Thus 13 and 43 the adjacent vertices of $S'_3(1, K_4) \in L$. In this case $|L| = 10 > 8$.

Hence $\gamma_L S'(2, K_4) > 8$ in both the cases. See Figure 5(a) in which $\gamma_L S'(2, K_4) = 9$.

Remark 4.3.

1. *In view of case* (2) *in Theorem* 4.2*, we observe that if* $ii \in L$ *then* $|L| \geq 10$*. If* $ii \in L$ *and* $|N(ii) \cap L| = 2$ *then in* $S'_i(1, K_4)$ *all the adjacent vertices of* $S'_j(1, K_4)$ *does not belong to* L*. Also if* $ii \in L$ *and* $|N(ii) \cap L| = 1$ *in* $S'_i(1, K_4)$ *then all the adjacent vertices of* $S'_j(1, K_4)$ *belong to* L*. Moreover in* $S'(2, K_4)$ *exactly two* $S'_i(1, K_4)$ *contain three vertices and other two has two vertices in* L*. Now let us consider the cases in which* $S'(2, K_4)$ *has all corner vertices in* L*, it can be either Figure* 5(*b*) *or* 5(*c*)*.*

2. *Consider* $S'(3, K_4)$ *which can be constructed from Figure* 5(*b*) *or* 5(*c*) *or both.* $S'(3, K_4)$ *obtained so consists of a path* P_4 *in* $G[L]$ *namely* $(111, 114, 141, 144)$ *or* $(111, 113, 131, 133)$ *then we can remove a vertex which will be corner vertex in* $S'_i(2, K_4)$ *of* $S'(3, K_4)$ *other than* iii*,* $i = 1, 2, 3, 4$ *since the adjacent vertex of* $S'_i(2, K_4)$ *namely the corner vertex of* $S'_j(2, K_4)$*,* $j \neq i$ *are in* L*. By doing so, we can remove maximum of 4 vertices in* $S'(3, K_4)$ *which is obtained by*

considering Figure 5(b) and 5(c) which is equal to obtaining minimum liar's dominating set through 5(a). Thus there are 36 *vertices in* $S'(3, K_4)$ *and paths* P_4 *in* $G[L]$ *are reduced to* P_3.

In view of Remark 4.3 we have the following result.

Theorem 4.4. *Let* G *be a Sierpiński complete graph* $S'(n, K_4)$, $n \geq 2$ *then* $\gamma_L S'(n, K_4) = 4[\gamma_L S'(n-1, K_4)]$.

Proof. Let us prove the result by the method of induction. Consider $n = 3$. In view of Theorem 4.2, $\gamma_L S'(2, K_4) = 9$. Since $S'(3, K_4)$ is made up of 4 copies of $S'(2, K_4)$, the minimum liar's dominating set in each of $S'_i(2, K_4)$ is same as that of $S'(2, K_4)$. Thus $\gamma_L S'(3, K_4) = 4\gamma_L S'(2, K_4)$. Suppose in $S'_i(2, K_4)$ the vertices selected in liar's dominating set are by Remark 4.3, (i.e) all the corner vertices are in L then $\gamma_L S'(3, K_4) = 4(10) - 4 = 36$. Since $S'(3, K_4)$ consists of only P_3 in $G[L]$, $\gamma_L S'(4, K_4) = 4[\gamma_L S'(3, K_4)]$. Let us assume that the result is true for $n < k$. Let $n = k$. By induction hypothesis, $\gamma_L S'(n, K_4) = 4[\gamma_L S'(n-1, K_4)]$.

5. Conclusion

Liar's domination is applied to protect a network even when one protection device becomes faulty or is allowed to lie. This paper provides the liar's domination number for Sierpiński-cycle graphs $S(n, C_4)$, and $S'(n, C_4)$ as well as Sierpiński-complete graphs $S(n, K_4)$ and $S'(n, K_4)$. The further work can be extended to families of Sierpiński-graphs $S(n, G)$ namely G isomorphic to other cycle, complete graph, path, star graphs etc.

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UNIFORM BOUNDEDNESS PRINCIPLE AND HAHN-BANACH THEOREM FOR B-LINEAR FUNCTIONAL RELATED TO LINEAR 2-NORMED SPACE

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Abstract: In this paper, we will see that the Cartesian product of two 2-Banach spaces is also 2-Banach space and discuss some properties of closed linear operator in linear 2-normed space. We also describe the concept of different types of continuity of b-linear functional and derive the Uniform Boundedness Principle and Hahn-Banach extension theorem for b-linear functionals in the case of linear 2-normed spaces. We also introduce the notion of weak* convergence for the sequence of bounded b-linear functionals relative to linear 2-normed space.

Keywords and Phrases: Linear 2-normed space, 2-Banach space, Closed operator, Uniform Boundedness Principle, Hahn-Banach extension Theorem.

2010 Mathematics Subject Classification: 46A22, 46B07, 46B25.

1. Introduction

The Uniform boundedness principle is one of the most useful results in functional analysis which was obtained by S. Banach and H. Steinhaus in 1927 and it is also familiar as Banach-Steinhaus Theorem. The Uniform boundedness principle tells us that if a sequence of bounded linear operators $T_n \in B(X, Y)$, where X is a Banach space and Y a normed space, is pointwise bounded, then the sequence $\{T_n\}$ is uniformly bounded.

The Hahn-Banach theorem is another useful and important theorem in functional analysis and it is frequently applied in other branches of mathematics viz., algebra, geometry, optimization, partial differential equation and so on. In fact, in this theorem a bounded linear functional defined on a subspace can be extended into the entire space.

The idea of linear 2-normed space was first introduced by S. Gahler ([3]) and thereafter the geometric structure of linear 2-normed spaces was developed by the great mathematicians like A. White, Y. J. Cho, R. W. Freese, S. C. Gupta and others [4, 5, 6]. In recent times, some important results in classical normed spaces have been proved into 2-norm setting by many researchers.

In this paper, we will see that in the Cartesian product $X \times Y$, we can induce a 2-norm using the 2-norms of X and Y and then we describe the concept of different types of continuity of b-linear functional in the case of linear 2-normed spaces and establish some results related to such types of continuity. In this paper, we are going to construct the uniform boundedness principle and Hahn-Banach extension theorem for a bounded b-linear functional defined on a 2-Banach space. Moreover we introduce a notion of weak * convergence of the sequence of bounded b-linear functionals in linear 2-normed spaces.

2. Preliminaries

Definition 2.1. ([3]) *Let X be a linear space of dimension greater than 1 over the field $\mathbb{K}$, where $\mathbb{K}$ is the real or complex numbers field and $\|\cdot,\cdot\|$ be a $\mathbb{K}$-valued function defined on $X \times X$ satisfying the following conditions:*

(N1) $\|x, y\| = 0$ *if and only if x, y are linearly dependent,*

(N2) $\|x, y\| = \|y, x\|$,

(N3) $\|\alpha x, y\| = |\alpha| \, \|x, y\| \ \forall \ \alpha \in \mathbb{K}$,

(N4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$.

Then $\|\cdot,\cdot\|$ is called a 2-norm on X and the pair $(X, \|\cdot,\cdot\|)$ is called a linear 2-normed space. The non-negativity condition of 2-norm can be obtained by using $(N3)\&(N4)$.

Definition 2.2. ([6]) *Let X be a linear 2-normed space. A sequence $\{x_n\}$ in X is said to be convergent to some $x \in X$ if*

$$\lim_{n \to \infty} \| x_n - x , y \| = 0$$

for every $y \in X$ and it is called Cauchy sequence if

$$\lim_{n , m \to \infty} \| x_n - x_m , z \| = 0$$

for every $z \in X$. X is said to be complete if every Cauchy sequence in this space is convergent in X. A linear 2-normed space is called 2-Banach space if it is complete.

Definition 2.3. ([9]) *Define the following open and closed ball in linear 2-normed space X:*

$$B_e(a , \delta) = \{ x \in X : \| x - a , e \| < \delta \}$$

$$B_e[a , \delta] = \{ x \in X : \| x - a , e \| \leq \delta \} ,$$

where $a , e \in X$ and δ be a positive number.

Definition 2.4. ([9]) *A subset G of a linear 2-normed space X is said to be open in X if for all $a \in G$, there exists $e \in X$ and $\delta > 0$ such that $B_e(a , \delta) \subseteq G$.*

Definition 2.5. ([10]) *Let X be a linear 2-normed space and $A \subseteq X$. Then a point $a \in A$ is said to an interior point of A if there exists $e \in X$ and $\delta > 0$ such that $B_e(a , \delta) \subseteq A$.*

Definition 2.6. ([9]) *Let X be a linear 2-normed space. Then $G \subseteq X$ is said to be dense in X if $V \cap G \neq \phi$ for every open set V in X.*

Definition 2.7. ([10]) *Let X be a linear 2-normed space and $A \subseteq X$. Then the closure of A is denoted by $\overline{A}$ and defined as,*

$$\{ x \in X \mid \exists \{ x_n \} \in A \text{ with } \lim_{n \to \infty} x_n = x \} .$$

The set A is said to be closed if $A = \overline{A}$.

Theorem 2.8. (Baire's theorem for 2-Banach spaces) ([1, 9]) *Let X be a 2-Banach space. Then the intersection of a countable number of dense open subsets of X is dense in X.*

Definition 2.9. ([1]) *Let X and Y be two linear 2-normed spaces over the field*

$\mathbb{K}$. Then a linear operator $T : X \to Y$ is said to be closed if for every $\{x_n\}$ in X with $x_n \to x$ and $T(x_n) \to y$ in Y we have $x \in X$ and $T(x) = y$.

Definition 2.10. ([9]) *Let X and Y be two linear 2-normed spaces over $\mathbb{R}$ and $T : X \to Y$ be a linear operator. The operator T is said to be sequentially continuous at $x \in X$ if and only if for every sequence $\{x_n\}$ in X that converges to x, the sequence $\{T(x_n)\}$ converges to $T(x)$.*

Definition 2.11. ([9]) *Let X and Y be two linear 2-normed spaces over $\mathbb{R}$. If X is finite dimensional, then every linear operator $T : X \to Y$ is sequentially continuous.*

Definition 2.12. ([2]) *Let $(X, \|\cdot,\cdot\|)$ be a linear 2-normed space over the field $\mathbb{K}$ with $b \in X$ be fixed and $\langle b \rangle$ is the subspace of X generated by b. Let W be a subspace of X, then a mapping $T : W \times \langle b \rangle \to \mathbb{K}$ is called a b-linear functional on $W \times \langle b \rangle$, if the following two hold:*

(1) $T(x + y, b) = T(x, b) + T(y, b)\ \forall x, y \in W$.

(2) $T(kx, b) = kT(x, b)\ \forall k \in \mathbb{K}$.

A b-linear functional $T : W \times \langle b \rangle \to \mathbb{K}$ is said to be bounded if there exists a real number $M > 0$ such that

$$|T(x, b)| \leq M\|x, b\|\ \forall x \in W.$$

Now we can define the norm of the b-linear functional $T : W \times \langle b \rangle \to \mathbb{K}$ as

$$\|T\| = \inf\{M > 0 : |T(x, b)| \leq M\|x, b\| \quad \forall x \in W\}.$$

Then one can easily verified that,

$$\|T\| = \sup\{|T(x, b)| : \|x, b\| \leq 1\}$$

$$\|T\| = \sup\{|T(x, b)| : \|x, b\| = 1\}$$

$$\|T\| = \sup\left\{\frac{|T(x, b)|}{\|x, b\|} : \|x, b\| \neq 0\right\}$$

and then

$$|T(x, b)| \leq \|T\|\,\|x, b\|\ \forall x \in W.$$

Let X_b^* denote the Banach space of all bounded b-linear functionals defined on $X \times \langle b \rangle$.

3. Analogous Results in the classical normed spaces to 2-normed spaces

Theorem 3.1. *Let $(X, \|\cdot,\cdot\|_X)$ and $(Y, \|\cdot,\cdot\|_Y)$ be two linear 2-normed linear spaces over the field $\mathbb{K}$, Then in the Cartesian product $X \times Y$, we can induce a 2-norm $\|\cdot,\cdot\|$ using the 2-norms of X and Y. Furthermore, if X and Y are 2-Banach spaces then $X \times Y$ is also 2-Banach space.*

Proof. Define a function $\|\cdot,\cdot\| : (X \times Y) \times (X \times Y) \to \mathbb{R}$ by,

$$\|(x_1, y_1), (x_2, y_2)\| = \|x_1, x_2\|_X + \|y_1, y_2\|_Y$$

for all $(x_1, y_1), (x_2, y_2) \in (X \times Y)$. We now verify that this function is a 2-norm on $X \times Y$.

(N1) Suppose

$$\|(x_1, y_1), (x_2, y_2)\| = 0 \ \forall (x_1, y_1), (x_2, y_2) \in (X \times Y)$$

$\Leftrightarrow \|x_1, x_2\|_X + \|y_1, y_2\|_Y = 0$

$\Leftrightarrow \|x_1, x_2\|_X = 0, \|y_1, y_2\|_Y = 0$ for $x_1, x_2 \in X \&\ y_1, y_2 \in Y$

$\Leftrightarrow \{x_1, x_2\}$ and $\{y_1, y_2\}$ are linearly dependent in X & Y

$\Leftrightarrow (x_1, y_1), (x_2, y_2)$ are linearly dependent in $X \times Y$.

(N2) Now,

$$\begin{aligned}\|(x_1, y_1), (x_2, y_2)\| &= \|x_1, x_2\|_X + \|y_1, y_2\|_Y \\ &= \|x_2, x_1\|_X + \|y_2, y_1\|_Y \\ &= \|(x_2, y_2), (x_1, y_1)\|\end{aligned}$$

for all $(x_1, y_1), (x_2, y_2) \in (X \times Y)$.

(N3) Let $\alpha \in \mathbb{K}$, then

$$\begin{aligned}\|\alpha(x_1, y_1), (x_2, y_2)\| &= \|(\alpha x_1, \alpha y_1), (x_2, y_2)\| \\ &= \|\alpha x_1, x_2\|_X + \|\alpha y_1, y_2\|_Y \\ &= |\alpha|\, \|x_1, x_2\|_X + |\alpha|\, \|y_1, y_2\|_Y \\ &= |\alpha|\, (\|x_1, x_2\|_X + \|y_1, y_2\|_Y) \\ &= |\alpha|\, \|(x_1, y_1), (x_2, y_2)\|\end{aligned}$$

for every $(x_1, y_1), (x_2, y_2) \in (X \times Y)$.

(N4)

$$\|(x_1, y_1) + (x_2, y_2), (x_3, y_3)\| = \|(x_1 + x_2, y_1 + y_2), (x_3, y_3)\|$$

$$= \|x_1 + x_2, x_3\|_X + \|y_1 + y_2, y_3\|_Y$$

$$\leq \|x_1, x_3\|_X + \|x_2, x_3\|_X + \|y_1, y_3\|_Y + \|y_2, y_3\|_Y$$

$$= \|x_1, x_3\|_X + \|y_1, y_3\|_Y + \|x_2, x_3\|_X + \|y_2, y_3\|_Y$$

$$= \|(x_1, y_1), (x_3, y_3)\| + \|(x_2, y_2), (x_3, y_3)\|$$

for every $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in (X \times Y)$.

Thus, $(X \times Y, \|\cdot, \cdot\|)$ becomes a linear 2-normed space.

Second part: Let $\{(x_n, y_n)\}$ be a Cauchy sequence in $X \times Y$. Then

$$\lim_{n, m \to \infty} \|(x_n, y_n) - (x_m, y_m), (z, t)\| = 0 \ \forall (z, t) \in X \times Y$$

$$\Rightarrow \lim_{n, m \to \infty} \|(x_n - x_m, y_n - y_m), (z, t)\| = 0 \ \forall (z, t) \in X \times Y$$

$$\Rightarrow \lim_{n, m \to \infty} (\|x_n - x_m, z\|_X + \|y_n - y_m, t\|_Y) = 0 \qquad .$$

Therefore,

$$\lim_{n, m \to \infty} \|x_n - x_m, z\|_X = 0 \ \forall z \in X$$

and

$$\lim_{n, m \to \infty} \|y_n - y_m, t\|_Y = 0 \ \forall t \in Y.$$

This shows that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X and Y, respectively. Since X and Y are 2-Banach spaces, So there exists points $x \in X$ and $y \in Y$ such that $x_n \to x$ in X and $y_n \to y$ in Y and hence $(x_n, y_n) \to (x, y)$ in $X \times Y$. Therefore, $X \times Y$ is 2-Banach space.

Theorem 3.2. *Let X and Y be two linear 2-normed spaces over the field $\mathbb{K}$ and D be a subspace of X. Then the linear operator $T : D \to Y$ is closed if and only if its Graph is a closed subspace of $X \times Y$.*

Proof. First we suppose that $T : D \to Y$ is closed operator, that is the relation $x_n \in D$, $x_n \to x$ in X, $Tx_n \to y$ in Y implies that $x \in D$ and $Tx = y$. We shall prove that the graph $G_T = \{(x, Tx) : x \in D\}$ is closed in linear 2-normed space $X \times Y$. Let $\{(x_n, Tx_n)\} \subseteq G_T$, $x_n \in D$ and $(x_n, Tx_n) \to (x, y)$ as $n \to \infty$. Therefore,

$$\lim_{n \to \infty} \|(x_n, Tx_n) - (x, y), (z, t)\| = 0 \ \forall (z, t) \in X \times Y$$

$$\Rightarrow \lim_{n \to \infty} \| (x_n - x , T x_n - y) , (z , t) \| = 0 \quad \forall (z , t) \in X \times Y$$

$$\Rightarrow \lim_{n \to \infty} (\|x_n - x, z \|_X + \| T x_n - y , t \|_Y) = 0 \quad \forall (z,t) \in X \times Y.$$

Thus,

$$\lim_{n \to \infty} \| x_n - x , z \|_X = 0 \quad \forall z \in X$$

and

$$\lim_{n \to \infty} \| T x_n - y , t \|_Y = 0 \quad \forall t \in Y .$$

This shows that $x_n \to x$ and $T x_n \to y$ as $n \to \infty$. Since T is closed operator, We have $x \in D$ and $T x = y$ and therefore $(x , y) = (x , T x) \in G_T$. Hence, G_T is closed subspace of linear 2-normed space $X \times Y$.

Conversely, Suppose G_T is closed subspace of linear 2-normed space $X \times Y$. To prove T is closed operator, we consider $x_n \to x$, $x_n \in D$ and $T x_n \to y$. Now,

$$\| (x_n , T x_n) - (x , y) , (z , t) \| = \| (x_n - x , T x_n - y) , (z , t) \|$$

$$= \| x_n - x , z \|_X + \| T x_n - y , t \|_Y . \tag{1}$$

Since $x_n \to x$ and $T x_n \to y$ as $n \to \infty$, then

$$\lim_{n \to \infty} \| x_n - x , z \|_X = 0 \ \forall z \in X$$

and

$$\lim_{n \to \infty} \| T x_n - y , t \|_Y = 0 \ \forall t \in Y .$$

So by (1),

$$\lim_{n \to \infty} \| (x_n , T x_n) - (x , y) , (z , t) \| = 0 \quad \forall (z , t) \in X \times Y .$$

This shows that $(x_n , T x_n) \to (x , y)$ as $n \to \infty$. Since G_T is closed subspace of linear 2-normed space $X \times Y$, it follows that $(x , y) \in G_T$, that is, $x \in D$ and $y = T x$. Hence, T is closed linear operator.

4. Some properties related to b-linear functional

In this section we define different types of continuity of b-linear functional and give some characterizations between them in linear 2-normed spaces.

Theorem 4.1. *Let X be a linear 2-normed space. Then*

$$| \| x , z \| - \| y , z \| | \leq \| x - y , z \| \ \forall x , y , z \in X .$$

Proof. Take $x, y, z \in X$. Then

$$\|x, z\| = \|x - y + y, z\| \leq \|x - y, z\| + \|y, z\|$$

$$\Rightarrow \|x, z\| - \|y, z\| \leq \|x - y, z\|.$$

Also, interchanging x and y, we get that

$$-(\|x, z\| - \|y, z\|) \leq \|y - x, z\| = \|x - y, z\|.$$

Combining the above two inequality the result follows.

Theorem 4.2. *Let T be a bounded b-linear functional on $X \times \langle b \rangle$, where $\langle b \rangle$ is the subspace of the linear 2-normed space X generated by a fixed $b \in X$. Then*

$$|T(x, b) - T(y, b)| \leq \|T\| \|x - y, b\| \quad \forall x, y \in X.$$

Proof. For each $x, y \in X$,

$$\begin{aligned} |T(x, b) - T(y, b)| &= |T(x, b) + T(-y, b)| \\ &= |T(x - y, b)| \\ &\leq \|T\| \|x - y, b\|. \end{aligned}$$

Definition 4.3. *Let T be a b-linear functional defined on $X \times \langle b \rangle$. Then T is said to be b-sequentially continuous at $x \in X$ if for every sequence $\{x_n\}$ converging to x in X, we have $\{T(x_n, b)\}$ converging to $T(x, b)$ in $\mathbb{K}$.*

Theorem 4.4. *Let X be a linear 2-normed space over $\mathbb{K}$ and $b \in X$ be fixed. Then every bounded b-linear functional defined on $X \times \langle b \rangle$ is b-sequentially continuous.*

Proof. Let T be a bounded b-linear functional on $X \times \langle b \rangle$ and $\{x_n\}$ be a sequence converging to x in X. Then,

$$\lim_{n \to \infty} \|x_n - x, z\| = 0 \ \forall z \in X,$$

and for particular $z = b$, we can write, $\lim_{n \to \infty} \|x_n - x, b\| = 0$. Now, using Theorem (4.2), by putting $x = x_n$ and $y = x$, we can write

$$|T(x_n, b) - T(x, b)| \leq \|T\| \|x_n - x, b\|$$

$$\Rightarrow \lim_{n \to \infty} |T(x_n, b) - T(x, b)| \leq \|T\| \lim_{n \to \infty} \|x_n - x, b\|$$

$$\Rightarrow \lim_{n\to\infty} |T(x_n, b) - T(x, b)| = 0 .$$

Therefore, $\{T(x_n, b)\}$ converging to $T(x, b)$ in $\mathbb{K}$. Hence, T is b-sequentially continuous.

Definition 4.5. *Let X be a linear 2-normed space and $b \in X$ be fixed. Then a b-linear functional $T : X \times \langle b\rangle \to \mathbb{K}$ is said to be continuous at $x_0 \in X$ if for any open ball $B(T(x_0, b), \epsilon)$ in $\mathbb{K}$, there exist an open ball $B_e(x_0, \delta)$ in X such that*

$$T(B_e(x_0, \delta), b) \subseteq B(T(x_0, b), \epsilon) .$$

Equivalently, for a given $\epsilon > 0$, there exist some $e \in X$ and $\delta > 0$ such that

$$x \in X,\ \|x - x_0, e\| < \delta \Rightarrow |T(x, b) - T(x_0, b)| < \epsilon .$$

Theorem 4.6. *Let X be a linear 2-normed space and $b \in X$ be fixed. If a b-linear functional T on $X \times \langle b\rangle$ is continuous at 0 then it is continuous on the whole space X.*

Proof. Let $T : X \times \langle b\rangle \to \mathbb{K}$ be a b-linear functional which is continuous at 0 and $x_0 \in X$ be arbitrary. Then for any open ball $B(0, \epsilon)$ in $\mathbb{K}$, we can find an open ball $B_e(0, \delta)$ in X such that

$$T(B_e(0, \delta), b) \subseteq B(T(0, b), \epsilon) = B(0, \epsilon)\ [\because T(0, b) = 0] .$$

Then,

$$T(x, b) - T(x_0, b) = T(x - x_0, b) \in B(0, \epsilon),$$

whenever $x - x_0 \in B_e(0, \delta)$. Thus, if $x \in x_0 + B_e(0, \delta) = B_e(x_0, \delta)$, then

$$T(x, b) \in T(x_0, b) + B(0, \epsilon) = B(T(x_0, b), \epsilon) .$$

Therefore,

$$T(B_e(x_0, \delta), b) \subseteq B(T(x_0, b), \epsilon) .$$

Since x_0 is arbitrary element of X, So T is continuous on X.

Theorem 4.7. *Let X be a linear 2-normed space. Then every continuous b-linear functional defined on $X \times \langle b\rangle$ is b-sequentially continuous.*

Proof. Suppose that $T : X \times \langle b\rangle \to \mathbb{K}$ is continuous at $x \in X$. Then for any open ball $B(T(x, b), \epsilon)$ in $\mathbb{K}$, we can find an open ball $B_e(x, \delta)$ in X such that

$$T(B_e(x, \delta), b) \subseteq B(T(x, b), \epsilon) . \tag{3}$$

Let $\{x_n\}$ be any sequence in X such that $x_n \to x$ as $n \to \infty$. Then, for the open ball $B_e(x, \delta)$, there exist some $K > 0$ such that $x_n \in B_e(x, \delta)$ $\forall\, n \geq K$. Now from (3), it follows that

$$T(x_n, b) \in B\,(T(x, b), \epsilon) \;\; \forall\, n \geq K$$

$$\Rightarrow\; |T(x_n, b) - T(x, b)| < \epsilon \;\; \forall\, n \geq K.$$

Since $B\,(T(x, b), \epsilon)$ is an arbitrary open ball in $\mathbb{K}$, it follows that $T(x_n, b) \to T(x, b)$ as $n \to \infty$. This shows that T is b-sequentially continuous on X.

Theorem 4.8. *Let X be a finite dimensional linear 2-normed space. Then every b-linear functional defined on $X \times \langle b \rangle$ is b-sequentially continuous.*

Proof. Let X be a finite dimensional linear 2-normed space and $T : X \times \langle b \rangle \to \mathbb{K}$ be a b-linear functional. If $X = \{0\}$ then the proof is obvious. Suppose that $X \neq \{0\}$ and let $\{e_1, e_2, \cdots e_m\}$ be a basis for X. We now show that T is b-sequentially continuous. Let $\{x_n\}$ be a sequence in X with $x_n \to x$ as $n \to \infty$. Now, we can write,

$$x_n = a_{n,1} e_1 + a_{n,2} e_2 + \cdots + a_{n,m} e_m$$

and

$$x = a_1 e_1 + a_2 e_2 + \cdots + a_m e_m$$

where $a_{n,j}$, a_1, $a_2 \cdots a_m \in \mathbb{R}$. In Theorem (2.11), it has been shown that $a_{n,j} \to a_j$ as $n \to \infty$ for all j. Then

$$\begin{aligned} T(x_n, b) &= T\,(a_{n,1} e_1 + a_{n,2} e_2 + \cdots + a_{n,m} e_m,\, b) \\ &= a_{n,1} T(e_1, b) + a_{n,2} T(e_2, b) + \cdots + a_{n,m} T(e_m, b) \\ &\to a_1 T(e_1, b) + a_2 T(e_2, b) + \cdots + a_m T(e_m, b), \text{ as } n \to \infty \\ &= T\,(a_1 e_1 + a_2 e_2 + \cdots + a_m e_m, b) = T(x, b). \end{aligned}$$

Thus, we have shown that if $x_n \to x \Rightarrow T(x_n, b) \to T(x, b)$. Therefore, T is b-sequentially continuous.

5. Results in 2-Banach space analogous to Uniform Boundedness Principle and Hahn-Banach Theorem

In this section we give the notion of Pointwise boundedness and Uniformly boundedness of a bounded b-linear functional in linear 2-normed space. We derive an analogue of Uniform Boundedness Principle and Hahn-Banach Extension Theorem for bounded b-linear functional on 2-normed space. We define weak * convergence of sequence of bounded b-linear functionals in linear 2-normed spaces.

Definition 5.1. *A subset A of a linear 2-normed space X is said to be nowhere dense if its closure has empty interior. Thus, A is nowhere dense in X if corresponding to any open ball $B_e(a, \delta)$ in X with $a \in A$, there exists another open ball $B_e(a^{'}, \delta^{'})$ such that $B_e(a^{'}, \delta^{'}) \subset B_e(a, \delta)$ with $A \cap B_e(a^{'}, \delta^{'}) = \phi$.*

Definition 5.2. *Let X be a linear 2-normed space. A set $\mathcal{A}$ of bounded b-linear functionals defined on $X \times \langle b \rangle$ is said to be:*

(a) Pointwise bounded if for each $x \in X$, the set $\{T(x, b) : T \in \mathcal{A}\}$ is a bounded set in $\mathbb{K}$. That is

$$|T(x, b)| \leq K \, \|x, b\| \quad \forall \, x \in X \;\; \& \;\; \forall \, T \in \mathcal{A} \, .$$

(b) Uniformly bounded if there is a constant $K > 0$ such that $\|T\| \leq K \; \forall \, T \in \mathcal{A}$.

Theorem 5.3. *If a set $\mathcal{A}$ of bounded b-linear functionals on $X \times \langle b \rangle$ is uniformly bounded then it is pointwise bounded set.*
Proof. Suppose $\mathcal{A}$ uniformly bounded. Then there is a constant $K > 0$ such that $\| T \| \leq K \quad \forall \, T \in \mathcal{A}$. Let $x \in X$ be given. Then,

$$|\, T \,(\, x \,,\, b \,)\,| \leq \|\, T \,\| \; \|\, x \,,\, b \,\| \leq K \;\; \|\, x \,,\, b \,\| \;\; \forall \, T \in \mathcal{A}$$

and hence $\mathcal{A}$ is pointwise bounded set in $\mathbb{K}$.

Theorem 5.4. *Let X be 2-Banach space over the field $\mathbb{K}$ and $b \in X$ be fixed. If a set $\mathcal{A}$ of bounded b-linear functionals on $X \times \langle b \rangle$ is pointwise bounded, then it is uniformly bounded.*
Proof. For each positive integer n, we consider the set

$$F_n = \{\, x \in X \,:\, |\, T \,(\, x \,,\, b \,)\,| \leq n \;\; \forall \, T \in \mathcal{A} \,\} \; .$$

We now show that F_n is a closed subset of X. Let $x \in \overline{F_n}$ and $\{\, x_k \,\}$ be a sequence in F_n such that $x_k \to x$ as $k \to \infty$. Then

$$|\, T \,(\, x_k \,,\, b \,)\,| \leq n \;\; \forall \, T \in \mathcal{A} \, .$$

Now, by the Theorem (4.4), T is b-sequentially continuous. So

$$\lim_{k \to \infty} |\, T \,(\, x_k \,,\, b \,)\,| = T \,(\, x \,,\, b \,) \, .$$

This shows that $|\, T \,(\, x \,,\, b \,)\,| \leq n \;\; \forall \, T \in \mathcal{A} \Rightarrow x \in F_n$ and hence F_n becomes a closed subset of X for every $n \in \mathbb{N}$. Since $\mathcal{A}$ is pointwise bounded, then the set $\{T(x, b) : T \in \mathcal{A}\}$ is a bounded for each $x \in X$. Thus, we see that for each $x \in X$ is in some F_n and therefore

$$X = \bigcup_{n=1}^{\infty} F_n .$$

Since X is 2-Banach space, by Baire's Category theorem for 2-Banach space, $\exists\, n_0 \in \mathbb{N}$ such that F_{n_0} is not nowhere dense in X i.e F_{n_0} has nonempty interior. Consequently, $\exists$ a non-empty open ball $B_e(x_0, \delta)$ such that $B_e(x_0, \delta) \subset F_{n_0}$. i.e,

$$|T(x, b)| \leq n_0 \quad \forall\, x \in B_e(x_0, \delta) \text{ and } \forall\, T \in \mathcal{A}.$$

The above expression can be written in the form

$$|T\,(B_e(x_0, \delta), b)| \leq n_0 \;\; \forall\, T \in \mathcal{A}.$$

Note that

$$\begin{aligned}
x_0 + \delta B_e(0, 1) &= \{x \in X : x = x_0 + \delta a \;,\; a \in B_e(0, 1)\} \\
&= \{x \in X : x = x_0 + \delta a \;,\; \|a, e\| < 1\} \\
&= \left\{x \in X : \left\| \frac{x - x_0}{\delta}, e \right\| < 1\right\} \\
&= \{x \in X : \|x - x_0, e\| < \delta\} = B_e(x_0, \delta) \\
\Rightarrow B_e(0, 1) &= \frac{B_e(x_0, \delta) - x_0}{\delta}.
\end{aligned}$$

Clearly,

$$|T(x_0, b)| \leq n_0 \quad \forall\, T \in \mathcal{A} \;\; [\because\; x_0 \in B_e(x_0, \delta)].$$

Therefore,

$$\begin{aligned}
|T(B_e(0, 1), b)| &= \left| T\left(\frac{B_e(x_0, \delta) - x_0}{\delta}, b\right)\right| \\
&= \left| \frac{1}{\delta} T\,(B_e(x_0, \delta) - x_0, b)\right| \\
&\leq \frac{1}{\delta}\,\{|T(B_e(x_0, \delta), b)| + |T(x_0, b)|\}
\end{aligned}$$

$$\leq \frac{2n_0}{\delta} \quad \forall T \in \mathcal{A}$$

$$\Rightarrow |T(x, b)| \leq \frac{2n_0}{\delta} \quad \forall x \in B_e(0, 1) \text{ and } \forall T \in \mathcal{A}.$$

Thus,

$$\|T\| = \sup\{|T(x, b)| : x \in B_e(0, 1) \quad \forall T \in \mathcal{A}\} \leq \frac{2n_0}{\delta} \quad \forall T \in \mathcal{A}.$$

This proves that $\mathcal{A}$ is uniformly bounded.

Theorem 5.5. *Let X be a 2-Banach space and X_b^* be the Banach space of all bounded b-linear functionals defined on $X \times \langle b \rangle$. If $\{T_n\} \subseteq X_b^*$ be a sequence such that*

$$\lim_{n \to \infty} T_n(x, b) = T(x, b) \quad \forall x \in X \tag{4}$$

exists, then $T \in X_b^*$.

Proof. Note that (4) defines a mapping $T : X \times \langle b \rangle \to \mathbb{K}$ which is, clearly, a b-linear functional. We need only to show that T is bounded.

Since, for every $x \in X$, $\{T_n(x, b)\}$ convergent sequence in $\mathbb{K}$, then it is bounded in $\mathbb{K}$. By the Theorem (5.4), the set $\{\|T_n\|\}$ is bounded. Then there exists some constant $M > 0$ such that $\|T_n\| \leq M \quad \forall n \in \mathbb{N}$. Therefore,

$$|T_n(x, b)| \leq \|T_n\| \, \|x, b\| \leq M \, \|x, b\| \quad \forall x \in X \;\; \& \;\; \forall n \in \mathbb{N}$$

$$\Rightarrow \lim_{n \to \infty} |T_n(x, b)| \leq M \, \|x, b\| \quad \forall x \in X$$

$$\Rightarrow |T(x, b)| \leq M \, \|x, b\| \quad \forall x \in X.$$

This shows that T is bounded b-linear functional defined on $X \times \langle b \rangle$ and hence $T \in X_b^*$.

Definition 5.6. *A sequence $\{T_n\}$ in X_b^*, where X_b^* is the Banach space of all bounded b-linear functionals defined on $X \times \langle b \rangle$ is said to be b-weak * convergent if there exists an $T \in X_b^*$ such that*

$$\lim_{n \to \infty} T_n(x, b) = T(x, b) \;\; \forall x \in X.$$

*The limits T is called the b-weak * limit of the sequence $\{T_n\}$.*

Definition 5.7. *A subset M of a linear 2-normed space X is said to be a fundamental or total set if the set SpanM is dense in X, that is $\overline{SpanM} = X$.*

Theorem 5.8. *Let X be a 2-Banach space and $\{T_n\} \subseteq X_b^*$ be a sequence. Then $\{T_n\}$ is b-weak * Convergent if and only if the following two conditions hold:*

(a) The sequence $\{\|T_n\|\}$ *is bounded and*

(b) The sequence $\{T_n(x,b)\}$ *is Cauchy sequence for each* $x \in W$*, where* W *is fundamental or total subset of* X*.*

Proof. Let $\{T_n\}$ be b-weak * Convergent in X_b^*. Then

$$\lim_{n \to \infty} T_n(x, b) = T(x, b) \;\forall\, x \in X.$$

This implies that $\{T_n(x, b)\}$ is bounded for each $x \in X$. Since X is 2-Banach space then by Theorem (5.4), we get that $\{\|T_n\|\}$ is bounded and therefore (a) hold. Now from the definition of b-weak * Convergence, $\{T_n(x, b)\}$ is a convergent sequence of numbers for $x \in X$, in particular, for $x \in W$. This proves (b). *Conversely*, suppose that the given two conditions hold. Since the sequence $\{\|T_n\|\}$ is bounded, $\exists$ a constant $K > 0$ such that $\|T_n\| \leq K \;\; \forall\, n \in \mathbb{N}$. Also, $\overline{Span\, W} = X$, then for a given $\epsilon > 0$ and for each $x \in X$, $\exists\, y \in Span\, W$ such that $\|x - y, b\| < \dfrac{\epsilon}{3K}$. Now from the condition (b), it follows that $\{T_n(y, b)\}$ is Cauchy sequence for $y \in Span\, W$ and hence there exists an integer $N > 0$ such that

$$|T_n(y, b) - T_m(y, b)| < \frac{\epsilon}{3} \;\; \forall\, m, n \geq N.$$

Now, for an arbitrary element $x \in X$, we have

$$|T_n(x, b) - T_m(x, b)|$$

$$= |T_n(x, b) - T_n(y, b) + T_n(y, b) - T_m(y, b) + T_m(y, b) - T_m(x, b)|$$
$$\leq |T_n(x, b) - T_n(y, b)| + |T_n(y, b) - T_m(y, b)| + |T_m(y, b) - T_m(x, b)|$$
$$< \|T_n\|\, \|x - y, b\| + |T_n(y, b) - T_m(y, b)| + \|T_m\|\, \|x - y, b\|$$
$$< K \cdot \frac{\epsilon}{3K} + \frac{\epsilon}{3} + K \cdot \frac{\epsilon}{3K} = \epsilon \;\; \forall\, m, n \geq N.$$

Therefore,

$$|T_n(x, b) - T_m(x, b)| < \epsilon \;\; \forall\, m, n \geq N.$$

This shows that $\{T_n(x, b)\}$ is Cauchy sequence in $\mathbb{K}$. But $\mathbb{K}$ is complete, So $\{T_n(x, b)\}$ is converges to $T(x, b)$ in $\mathbb{K}$. Since x is an arbitrary element of X, therefore it follows that

$$\lim_{n\to\infty} T_n(x, b) = T(x, b) \;\forall\, x \in X.$$

Thus, $\{T_n\}$ is b-weak * Convergent sequence in X_b^*.

Definition 5.9. *A linear 2-normed space X is said to be separable if X has a countable dense subset.*

Theorem 5.10. *Let X be a linear 2-normed space over the field $\mathbb{R}$ and W be a subspace of X. Then each bounded b-linear functional T_W defined on $W \times \langle b \rangle$ can be extended onto $X \times \langle b \rangle$ with preservation of the norm. In other words, there exists a bounded b-linear functional T defined on $X \times \langle b \rangle$ such that*

$$T(x, b) = T_W(x, b) \ \forall\, x \in W \quad \& \quad \|T_W\| = \|T\|.$$

Proof. We prove this theorem by assuming X is separable. This theorem also hold for the spaces which are not separable. Let $x_0 \in X - W$ and consider the set $W + x_0 = \{x + t\,x_0 : x \in W$ and t is arbitrary real number $\}$. Clearly, $W + x_0$ is a subspace of X containing W. Let T_W be a bounded b-linear functional defined on $W \times \langle b \rangle$ and further suppose that (x_1, b), $(x_2, b) \in W \times \langle b \rangle$. Now,

$$T_W(x_1, b) - T_W(x_2, b) \leq \|T_W\|\,\|x_1 - x_2, b\|$$

$$\leq \|T_W\|\,(\|x_1 + x_0, b\| + \|x_2 + x_0, b\|)$$

$$\Rightarrow\ T_W(x_1, b) - \|T_W\|\,\|x_1 + x_0, b\| \leq T_W(x_2, b) + \|T_W\|\,\|x_2 + x_0, b\|.$$

Since $(x_1, b), (x_2, b)$ be two arbitrary elements in $W \times \langle b \rangle$, we obtain

$$\sup_{x \in W}\{T_W(x, b) - \|T_W\|\,\|x + x_0, b\|\}$$

$$\leq \inf_{x \in W}\{T_W(x, b) + \|T_W\|\,\|x + x_0, b\|\}$$

and hence we can find a real number α such that

$$\sup_{x \in W}\{T_W(x, b) - \|T_W\|\,\|x + x_0, b\|\} \leq \alpha$$

$$\leq \inf_{x \in W}\{T_W(x, b) + \|T_W\|\,\|x + x_0, b\|\}. \tag{5}$$

Now we define a b-linear functional T_0 on $(W + x_0) \times \langle b \rangle$ by,

$$T_0(y, b) = T_W(x, b) - t\,\alpha \quad \forall\,(y, b) \in (W + x_0) \times \langle b \rangle$$

where $y = x + t\,x_0$, t is a unique real number and α is the real number satisfying (5) and $x \in W$. Clearly, $T_W(y, b) = T_0(y, b)\ \forall\, y \in W$. We now show that T_0 is bounded on $(W + x_0) \times \langle b \rangle$ and $\|T_W\| = \|T_0\|$. For the boundedness part of T_0, we consider the following two cases

(i) First we consider $t > 0$. Since W is a subspace, we get $\dfrac{x}{t} \in W$, whenever $x \in W$ and then the inequality (5) implies that,

$$\begin{aligned} T_0(y, b) &= t \cdot \left\{ \frac{1}{t} T_W(x, b) - \alpha \right\} = t \cdot \left\{ T_W \left(\frac{x}{t}, b \right) - \alpha \right\} \\ &\leq t \cdot \| T_W \| \left\| \frac{x}{t} + x_0, b \right\| \\ &= \| T_W \| \, \| x + t x_0, b \| = \| T_W \| \, \| y, b \| . \end{aligned}$$

(ii) Next we consider $t < 0$ and again using the inequality (5)

$$\begin{aligned} T_W \left(\frac{x}{t}, b \right) - \alpha &\geq - \| T_W \| \left\| \frac{x}{t} + x_0, b \right\| \\ &= - \frac{1}{|t|} \| T_W \| \, \| y, b \| = \frac{1}{t} \| T_W \| \, \| y, b \| . \end{aligned}$$

So,

$$T_0(y, b) = t \cdot \left\{ T_W \left(\frac{x}{t}, b \right) - \alpha \right\} \leq t \cdot \frac{1}{t} \|T_W\| \, \|y, b\| = \|T_W\| \, \|y, b\| .$$

Therefore,

$$T_0(y, b) \leq \| T_W \| \, \| y, b \| \quad \forall (y, b) \in (W + x_0) \times \langle b \rangle . \tag{6}$$

Replacing $-y$ for y in (6), we get

$$T_0(-y, b) \leq \| T_W \| \, \| -y, b \| \Rightarrow -T_0(y, b) \leq \| T_W \| \, \| y, b \| .$$

Combining this with (6), we obtain

$$| T_0(y, b) | \leq \| T_W \| \, \| y, b \| \quad \forall (y, b) \in (W + x_0) \times \langle b \rangle .$$

This shows that T_0 is bounded and $\| T_0 \| \leq \| T_W \|$. Since the domain of T_W is a subset of the domain of T_0, we get $\| T_0 \| \geq \| T_W \|$ and hence $\| T_0 \| = \| T_W \|$. Thus we have seen that $T_0(x, b)$ is the extension of $T_W(x, b)$ onto $(W + x_0) \times \langle b \rangle$ with $\| T_0 \| = \| T_W \|$. Since X is separable, so there exists a countable dense subset D of X. We select elements from this dense subset those belong to $X - W$ and arrange them as a sequence $\{ x_0, x_1, x_2 \cdots \}$. By the previous procedure, we get the extension of $T_W(x, b)$ onto $(W + x_0) \times \langle b \rangle =$

$W_1 \times \langle b \rangle$, $(W_1 + x_1) \times \langle b \rangle = W_2 \times \langle b \rangle$, $(W_2 + x_2) \times \langle b \rangle = W_3 \times \langle b \rangle$ and so on. Then we arrive at a bounded b-linear functional $T_g : W_g \times \langle b \rangle \to \mathbb{K}$, where W_g is everywhere dense in X and that contains W_n for $n = 1, 2, 3 \cdots$ and $\| T_g \| = \| T_W \|$. If $y \in X - W_g$, then there exists a sequence $\{ y_n \}$ in W_g such that $y = \lim\limits_{n \to \infty} y_n$. We now define

$$T(y, b) = \lim_{n \to \infty} T_g(y_n, b).$$

If $y \in W_g$, we can put in particular $y_1 = y_2 = \cdots = y$ and so the b-linear functional $T(y, b)$ is an extension of $T_g(y, b)$ onto $X \times \langle b \rangle$. Now

$$|T(y, b)| = \lim_{n \to \infty} |T_g(y_n, b)| \leq \| T_g \| \lim_{n \to \infty} \| y_n, b \| = \| T_W \| \| y, b \| .$$

This shows that $T(y, b)$ is bounded b-linear functional and $\|T\| \leq \|T_W\|$. Since the domain of T_W is a subset of the domain of T, we get $\|T\| \geq \|T_W\|$ and therefore $\|T\| = \|T_W\|$. Clearly $T(x, b) = T_W(x, b)$ for $x \in W$. This proves the theorem.

Theorem 5.11. *Let X be a linear 2-normed space over the field $\mathbb{R}$ and let x_0 be an arbitrary non-zero element in X. Then there exists a bounded b-linear functional T defined on $X \times \langle b \rangle$ such that*

$$\| T \| = 1 \quad \& \quad T(x_0, b) = \| x_0, b \| .$$

Proof. Consider the set $W = \{ t x_0 |$ where t is a arbitrary real number $\}$. Then it is easy to prove that W is a subspace of X. Define $T_W : W \times \langle b \rangle \to \mathbb{R}$ by,

$$T_W(x, b) = T_W(t x_0, b) = t \| x_0, b \| , \; t \in \mathbb{R} .$$

Note that T_W is a b-linear functional on $W \times \langle b \rangle$ with the property that

$$T_W(x_0, b) = \| x_0, b \| .$$

Further, for any $x \in W$, we have

$$|T_W(x, b)| = |T_W(t x_0, b)| = t \| x_0, b \| = \| t x_0, b \| = \| x, b \| .$$

So, T_W is bounded b-linear functional and $\| T_W \| = 1$. Now, according to the Theorem (5.10), there exists a bounded b-linear functional T defined on $X \times \langle b \rangle$ such that $T(x, b) = T_W(x, b) \;\; \forall \, x \in W$ and $\| T \| = \| T_W \|$. Therefore, $T(x_0, b) = T_W(x_0, b) = \| x_0, b \|$ and $\| T \| = 1$. This completes the proof.

Theorem 5.12. *Let X be a linear 2-normed space over the field $\mathbb{R}$ and let $x \in X$ and X_b^* is the Banach space of all bounded b-linear functionals defined on $X \times \langle b \rangle$. Then*

$$\| x , b \| = \sup \left\{ \frac{| T (x , b) |}{\| T \|} : T \in X_b^* , T \neq 0 \right\} .$$

Proof. If $x = 0$, there is nothing to prove. Let $x \neq 0$ be any element in X. By the Theorem (5.11), there exists a $T_1 \in X_b^*$ such that $T_1(x, b) = \|x, b\|$ and $\|T_1\| = 1$. Therefore,

$$\sup \left\{ \frac{| T (x , b) |}{\| T \|} : T \in X_b^* , T \neq 0 \right\} \geq \frac{| T_1 (x , b) |}{\| T_1 \|} = \| x , b \| . \qquad (7)$$

On the other hand, $| T (x , b) | \leq \| T \| \| x , b \| \quad \forall \, T \in X_b^*$ and we obtain

$$\sup \left\{ \frac{| T (x , b) |}{\| T \|} : T \in X_b^* , T \neq 0 \right\} \leq \| x , b \| . \qquad (8)$$

From (7) and (8), we can write,

$$\| x , b \| = \sup \left\{ \frac{| T (x , b) |}{\| T \|} : T \in X_b^* , T \neq 0 \right\} .$$

This completes the proof.

6. Conclusion

Hahn-Banach theorem, Uniform boundedness principle, also known as Banach-Steinhaus theorem, open mapping theorem, closed graph theorem are most fundamental theorems and determining tools in functional analysis. In this paper, in the settings of linear 2-normed space, we have established necessary and sufficient condition for a linear operator to be closed in terms of its graph, different types of continuity for b-linear functionals and some characterizations of them and finally uniform boundedness principle and Hahn-Banach extension theorem for bounded b-linear functionals. Yet it remains to establish another few important concepts of functional analysis like, reflexivity of linear 2-normed space, Hahn-Banach separation theorem for bounded b-linear functionals etc. in the settings of linear 2-normed space. Also, these results can further be developed in linear n-normed space.

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FUZZY gp*-CLOSED SETS IN FUZZY TOPOLOGICAL SPACE

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Abstract: In this paper fuzzy gp*- closed sets, fuzzy gp* continuous functions, fuzzy gp*-irresolute functions, fuzzy gp*-connectedness and fuzzy T*gp-space are introduced and also their relation with some other fuzzy sets and some of their properties are investigated.

Keywords and Phrases: Fuzzy topological spaces; fuzzy gp*-closed sets; fuzzy gp* continuous functions and fuzzy gp*-irresolute functions; fuzzy gp*-open sets; fuzzy T*gp-space.

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1. Introduction

Fuzzy set theory as introduced by Lotfi A. Zadeh [1] in 1965 is the expansion of the classical set theory and it expanded the basic definition of the classical or crisp sets. So fuzzy mathematics is just a kind of mathematics developed in this framework and fuzzy topology introduced by C.L Chang [2] in 1968 is the generalization of ordinary topology in classical mathematics. Since the introduction of fuzzy sets and fuzzy topological spaces, work started taking place at a good rate in this field of mathematics and various types of fuzzy sets were introduced and studied by various researchers, Like S.S Benchalli and G.P.Siddapur introduced fuzzy g* pre continuous maps[3], Hamid Reza Moradi and Anahid Kamali introduced fuzzy strongly g* -closed sets and g**-closed sets in 2015 [4], And almost all the mathematical, engineering, medicinal etc concepts have been redefined using fuzzy theory and it has further deepened the understanding of basic set theory.

In this paper fuzzy gp*- closed sets is defined and its relation with other sets like fuzzy closed sets, fuzzy g*-closed sets and g*p-closed sets are found and also some other properties of these sets are investigated. Moreover fuzzy gp*-open sets are introduced and their relation with other fuzzy sets are found. Fuzzy gp*- continuous function and fuzzy gp*-irresolute functions are defined and their relation with other fuzzy functions are investigated, also investigated some other properties of these functions. Fuzzy gp*-connectedness and fuzzy T*gp-spaces in fuzzy topological spaces are also introduced and some of their properties are investigated.

2. Preliminaries

Definition 2.1. [1] *Let X be a space of objects, with a generic element of X denoted by x. Then a fuzzy set A in X is a set of ordered pairs $\{(x, f(x))\}$ where $f_A(x)$ is called the membership function which associates each point in X a real number in the interval [0,1].*

Definition 2.2. [2] *A family τ of fuzzy sets of X is called fuzzy topology on X if 0 and 1 belong to τ and τ is closed with respect to arbitrary union and finite intersection. The elements of τ are called fuzzy open sets and there complements are called fuzzy closed sets. The space X with topology τ is called fuzzy topological space denoted by (X,τ).*

Definition 2.3. [2] *For a fuzzy set α of X, the closure cl α and the interior int α of α are defined respectively, as*
$cl\alpha = \wedge\{\mu : \mu \geq \alpha, 1 - \mu \in \tau\}$ *and*
$int\alpha = \vee\{\mu : \mu \leq \alpha, \mu \in \tau\}$

Definition 2.4. [5] *A subset A of X is called fuzzy pre-closed (in short pcl) set if $A \leq cl(int(A))$ and fuzzy pre-open set if $A \leq int(cl(A))$.*

Definition 2.5. [4] *Let (X, τ) be a fuzzy topological space. A fuzzy set A of (X, τ) is called fuzzy strongly g*-closed if $cl(int(A)) \leq H$, whenever $A \leq H$ and H is fuzzy generalized -open in X.*

Definition 2.6. [6] *A fuzzy set A of a fuzzy topological space (X, τ) is called a fuzzy generalized star closed or $g*-$closed if $cl(A) \leq O$ whenever $A \leq O$ and O is fuzzy generalized-open or g-open.*

Definition 2.7. [7] *A fuzzy set A of a fuzzy topological space (X, τ) is called fuzzy generalized closed or g-closed if $cl(A) \leq G$ whenever $A \leq G$ and $G \in \tau$ and is called fuzzy generalized open or g-open if $1 - A$ is fuzzy g-closed.*

Definition 2.8. [8] *A fuzzy set A of a fuzzy topological space (X, τ) is called fuzzy* *generalized pre -closed or gp-closed set if $pcl(A) \leq U$ whenever $A \leq U$ and U is a fuzzy open set in (X, τ). And complement of a Fuzzy gp-closed set is called fuzzy generalized pre-open or gp-open set.*

Definition 2.9. [3] *A fuzzy set A of a fuzzy topological space (X, τ) is called a fuzzy generalized star pre-closed (briefly g*p-closed) set if $pcl(A) \leq U$ whenever $A \leq U$ and U is fuzzy g-open set in (X, τ).*

Definition 2.10. [2] *A function f from a fts (X, τ) to a fts (Y, δ) is fuzzy-continuous iff the inverse of each δ-open fuzzy set in Y is τ-open fuzzy set in X.*

Definition 2.11. [9] *A function f from a fts (X, τ) to a fts (Y, δ) is fuzzy g*-continuous if $f^{-1}(A)$ is fuzzy g*-closed in X for every fuzzy closed set of Y.*

Definition 2.12. [10] *A fuzzy topological space X is said to be fuzzy connected if it has no proper fuzzy clopen set, (A fuzzy set λ in X is proper if $\lambda \neq 0$ and $\lambda \neq 1$, clopen means closed-open).*

Definition 2.13. [9] *A fuzzy topological space (X, τ) is called a fuzzy $T^*_{1/2}$ space if every g*-closed fuzzy set is a closed fuzzy set.*

Definition 2.14. [3] *A fts (X, τ) is called a fuzzy-T^*_p - space if every g*p closed fuzzy set is closed fuzzy set.*

Theorem 1. *Every fuzzy generalized-closed set is fuzzy generalized pre-closed set.*
Proof. Let θ is a fuzzy g-closed set and μ be a fuzzy open set such that $\theta \leq \mu$, then $cl(\theta) \leq \mu$ and hence $pcl(\theta) \leq cl(\theta) \leq \mu$ implies θ is a fuzzy gp-closed set.

Theorem 2. *All fuzzy generalized open sets are fuzzy generalized pre-open sets.*
Proof. Consider θ is a fuzzy generalized open set. Then (1- θ) is a fuzzy generalized closed set. Now by Theorem 1, (1-θ) is a fuzzy generalized pre closed set implying that θ is a fuzzy generalized pre-open set.

3. Fuzzy gp*-closed sets

Definition 3.1. *A fuzzy set λ of a fuzzy topological space (fts) (Y, τ) is called fuzzy generalized pre star closed (briefly fuzzy gp*-closed) if $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy generalized pre-open in Y.*

Example 3.2. Let $Y = \{y\}$ and $\tau = \{0_Y, y_{2/3}, y_{3/4}, 1_Y\}$. Then in this fuzzy topological space (Y, τ), fuzzy sets 0_Y, $A = y_{1/3}$, $B = y_{1/4}$ and 1_Y satisfy the condition $cl(\lambda) \leq \mu$ whenever $\lambda \leq \mu$ and μ is fuzzy generalized pre-open in Y. Implying 0_Y, $A = y_{1/3}$, $B = y_{1/4}$ and 1_Y are fuzzy gp*-closed sets in (Y, τ).

Theorem 3.3. *All fuzzy closed sets are fuzzy gp* closed sets.*

Proof. Consider θ is a fuzzy closed set in fuzzy topological space Y and μ is a fuzzy generalized pre- open set in Y containing $cl(\theta) \leq \theta = \mu$. Implying that $\theta$is a fuzzy gp*-closed set in Y.

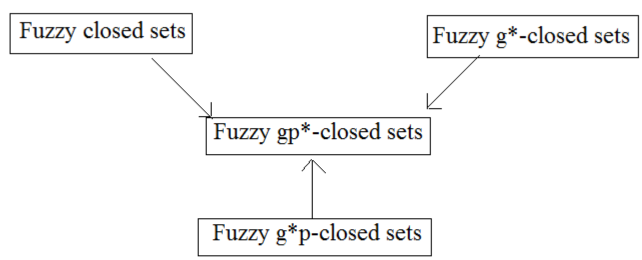
Theorem 3.4. *All fuzzy generalized star pre-closed sets are fuzzy gp* closed.*

Proof. Consider σ is any arbitrary fuzzy generalized star pre closed set in fuzzy topological space (Y, τ). Let σ is contained in fuzzy generalized open μ. Now as every fuzzy generalized open set is fuzzy generalized pre-open set (By Theorem 2) so $pcl(\sigma) \leq cl(\sigma) \leq \mu$. Implying $cl(\sigma) \leq \mu$, Which in turn implies that σ is fuzzy gp*-closed.

Theorem 3.5. *All fuzzy g*-closed sets are fuzzy gp*-closed sets.*

Proof. Consider θ is a fuzzy g*-closed set in fuzzy topological space (Y, τ) and μ is any generalized-open set that contains θ. Now as every fuzzy generalized-open set is fuzzy generalized pre-open set (By theorem 2), so $cl(\theta) \leq \mu$, where μ is a fuzzy generalized pre-open set in Y. Implying that θ is fuzzy gp*-closed set.

Remark 3.6. *The following diagram depicts the relation of fuzzy gp*-closed set and other fuzzy sets discussed above.*



Theorem 3.7. *The Union of two fuzzy gp*-closed sets Δ and ∇ in fuzzy topological space (Y, τ) is also fuzzy gp*-closed set in Y.*

Proof. Suppose that Δ and ∇ are two fuzzy gp*-closed sets in Y. Let μ be a fuzzy generalized pre-open set that contains both Δ and ∇. so $cl(\Delta) \leq \mu$ and

$cl(\nabla) \leq \mu$. Now, as $\Delta \leq \mu$ and $\nabla \leq \mu$ implying that $\Delta \cup \nabla \leq \mu$ which inturn implies $cl(\Delta \cup \nabla) = cl(\Delta) \cup cl(\nabla) \leq \mu$, which gives the required result i.e $\Delta \cup \nabla$ is also a fuzzy gp*-closed set in Y.

Theorem 3.8. *If Δ and ∇ are two fuzzy gp*-closed sets in fuzzy topological space (Y, τ) then $\Delta \cap \nabla$ is also fuzzy gp*-closed in Y.*
Proof. Suppose $\Delta \& \nabla$ are two fuzzy gp*-closed sets in fuzzy topological space Y, such that $\Delta \leq \mu$ and $\nabla \leq \mu$, where μ is a fuzzy generalized pre-open set in Y. Then $cl(\Delta) \leq \mu$, $cl(\nabla) \leq \mu$ therefore $cl(\Delta \cap \nabla) \leq \mu$, where μ is fuzzy generalized pre-open set in Y. Which implies that $\Delta \cap \nabla$ is also fuzzy gp*-closed set in Y.

Theorem 3.9. *Suppose that μ is fuzzy gp*-closed set in fuzzy topological space Δ such that $\mu \leq \nabla \leq \Delta$, then μ is also fuzzy gp*-closed relative to ∇.*
Proof. Given that $\mu \leq \nabla \leq \Delta$, where μ is fuzzy gp*-closed in Δ. Now suppose that $\mu \leq \nabla \cap \theta$, where θ is fuzzy generalized pre-open in Δ. As μ is fuzzy gp*-closed, $\mu \leq \theta$ implies $cl(\mu) \leq \theta$. Which implies that $\nabla \cap cl(\mu) \leq \nabla \cap \theta$ i.e μ is also fuzzy gp*-closed relative to ∇.

4. Fuzzy gp*-open sets

Definition 4.1. *Suppose a fuzzy set λ is fuzzy generalized pre star closed set in fts (Y, τ), Then its complement i,e $1 - \lambda$ is called fuzzy generalized pre star open (briefly fuzzy gp*-open) in (Y, τ).*

Example 4.2. In the fuzzy topological space (Y, τ) defined in example 3.2, the complements of the fuzzy gp*-closed sets 0_Y, $A = y_{1/3}$, $B = y_{1/4}$ and 1_Y are respectively as 1_Y, C=$y_{2/3}$, D=$y_{3/4}$ and 0_Y. Implying 1_Y, C=$y_{2/3}$, D=$y_{3/4}$ and 0_Y are fuzzy gp*-open sets in (Y, τ).

Theorem 4.3. *All fuzzy open sets are fuzzy gp*-open.*
Proof. Consider μ is a fuzzy open set in fuzzy topological space (Y, τ), implies $1 - \mu$ is a fuzzy closed set. Now from the theorem 3.3 all fuzzy closed sets are fuzzy gp*-closed sets. So $1 - \mu$ is also a fuzzy gp*-closed set implying that μ is fuzzy gp*-open in fuzzy topological space (Y, τ).

Theorem 4.4. *The intersection of two fuzzy gp*-open sets Δ and ∇ in fuzzy topological space (Y, τ) is also a fuzzy gp*-open set in (Y, τ).*
Proof. Suppose$\Delta \& \nabla$ are two fuzzy gp*-open sets in fuzzy topological space (Y, τ). Which implies $1 - \Delta$ and $1 - \nabla$ are fuzzy gp*-closed in (Y, τ). Now according to theorem 3.9 $(1 - \Delta) \cup (1 - \nabla)$is also a fuzzy gp*-closed in (Y, τ). So $(1 - \Delta) \cup (1 - \nabla) = (1 - (\Delta \cap \nabla))$ is fuzzy gp*-closed in (Y, τ). Implying that $\Delta \cap \nabla$ is also a fuzzy gp*-open set in (Y, τ).

5. Fuzzy gp*-continuous mappings

Definition 5.1. *If G and H are two fuzzy topological spaces then a mapping $g : G \to H$ is called fuzzy gp*-continuous mapping if $g^{-1}(\phi)$ is fuzzy gp*-open set in G, for every fuzzy open ϕ of H.*

Definition 5.2. *If G and H are two fuzzy topological spaces then a mapping $g : G \to H$ is called fuzzy gp*-irresolute mapping if g^{-1} (ϕ) is fuzzy gp*-closed set in G, for every fuzzy gp*-closed set ϕ of H.*

Theorem 5.3. *A function $g : G \to H$ is fuzzy gp*-continuous if & only if the inverse image of each fuzzy closed set in H is fuzzy gp*-closed set in G.*
Proof. Suppose that G and H are two fuzzy topological spaces and $g : G \to H$ be a fuzzy gp*- continuous function. Let α be a fuzzy closed set in H implies that $1 - \alpha$ is a fuzzy open set in H. Now as g is a fuzzy gp*-continuous function implies $g^{-1}(1 - \alpha) = 1 - g^{-1}(\alpha)$ is a fuzzy gp*-open set in G, implying $g^{-1}(\alpha)$ is a fuzzy gp*-closed set in G. Conversely let's suppose that α is a fuzzy closed set in H and $g^{-1}(\alpha)$ is fuzzy gp*-closed in G. Now $1 - \alpha$ is a fuzzy open set in H and $g^{-1}(1 - \alpha) = 1 - g^{-1}(\alpha)$ is fuzzy gp*-open, which was the required proof.

Theorem 5.4. *All fuzzy continuous functions are fuzzy gp*-continuous.*
Proof. Suppose that G and H are two fuzzy topological spaces and $g : G \to H$ be a fuzzy continuous function. Now , suppose α is a fuzzy open set in H & as g is fuzzy continuous function implies $g^{-1}(\alpha)$ is fuzzy open set in G. So by theorem 4.3 $g^{-1}(\alpha)$ is fuzzy gp*-open set in G, implying that $g : G \to H$ is a fuzzy gp*-continuous function.

Theorem 5.5. *All fuzzy g*-continuous functions are fuzzy gp*-continuous function.*
Proof. Let G and H are two fuzzy topological spaces and $g : G \to H$ be a fuzzy g*-continuous function. Now, suppose α is a fuzzy closed set in H & as g is fuzzy g*-continuous function implies $g^{-1}(\alpha)$ is fuzzy generalized star-closed set in G. Now as by theorem 3.4 All fuzzy generalized star pre-closed sets are fuzzy gp* closed implying that $g^{-1}(\alpha)$ is also a fuzzy gp*-closed set, means g is fuzzy gp*-continuous.

Theorem 5.6. *If G, H and I are fuzzy topological spaces and $j : G \to H$ & $k : H \to I$ are such that k is a fuzzy gp*-continuous function and j is fuzzy gp*-irresolute, then koj is a fuzzy gp* continuous function.*
Proof. Suppose α is a fuzzy closed set in I. Also $(koj)^{-1}(\alpha) = j^{-1}(k^{-1}(\alpha))$. Now as k is fuzzy gp*-continuous, so by its definition $A = k^{-1}(\alpha)$ is a fuzzy gp*-closed set in H. Now as j is a fuzzy gp*-irresolute implies $j^{-1}(A) = j^{-1}(k^{-1}(\alpha))$ is also fuzzy gp*-closed set in G, implying that koj is a fuzzy gp* continuous function.

Theorem 5.7. *Suppose $j : G \to H$ & $k : H \to I$ are such that k is a fuzzy continuous function and j is fuzzy gp*-continuous, then koj is a fuzzy gp* continuous function.*

Proof. Suppose $\alpha \leq I$ be any fuzzy closed set in I. Also $(koj)^{-1}(\alpha) = j^{-1}(k^{-1}(\alpha))$. Now as k is a fuzzy continuous function, implies $A = k^{-1}(\alpha)$ is a fuzzy closed set in H. Now j is a fuzzy gp*-continuous function, implying that $j^{-1}(A) = j^{-1}(k^{-1}(\alpha))$ is a fuzzy gp*-closed set in G. Which shows that koj is a fuzzy gp*-continuous function by theorem 5.3.

Theorem 5.8. *Suppose $j : G \to H$ & $k : H \to I$ are fuzzy gp*-irresolute functions, then koj is a also fuzzy gp*-irresolute function.*

Proof. Let $\alpha \leq I$ be any fuzzy gp*- closed set in I. Also $(koj)^{-1}(\alpha) = j^{-1}(k^{-1}(\alpha))$. Now as k is a fuzzy gp*-irresolute function, implies $A = k^{-1}(\alpha)$ is a fuzzy gp*-closed set in H. Now as j is also fuzzy gp*-irresolute, implying that $j^{-1}(A) = j^{-1}(k^{-1}(\alpha))$ is a fuzzy gp*-closed set in G. So by definition 5.2 koj is also a fuzzy gp*-irresolute function.

6. Fuzzy gp*-connectedness

Definition 6.1. *A fuzzy gp*-connected space is a fuzzy topological space (Y, τ) that cannot be written as the union of two non-empty disjoint fuzzy gp*-open sets in (Y, τ).*

Theorem 6.2. *If (Y, τ) is a fts, then the following are equivalent;*
(a) Y is a fuzzy gp-connected space.*
(b) The only subsets in Y which are both fuzzy gp-open and fuzzy gp*-closed are 0_Y & 1_Y.*

Proof. (a) $\Rightarrow$ (b): Let Y is a fuzzy gp*-connected space. Now, suppose $\alpha < Y$ is both fuzzy gp*-open & fuzzy gp*-closed. Then $1 - \alpha$ is also both fuzzy gp*-closed & fuzzy gp*-open. So $Y = \alpha \vee (1 - \alpha)$ is the union of two disjoint non empty fuzzy gp*-open sets, which contradicts (a). Implying $\alpha = 0_Y$ or $\alpha = 1_y$.
(b) $\Rightarrow$ (a): Suppose α & β are non-empty disjoint fuzzy gp*-open sets such that $Y = \alpha \vee \beta$. Now $\alpha = 1 - \beta$ & $\beta = 1 - \alpha$ are fuzzy gp*-open sets, which in turn implies α & β are also fuzzy gp*-closed sets. Now by (b) $\alpha = 0_Y$ or $\alpha = 1_Y$ implies Y is fuzzy gp*-connected.

Theorem 6.3. *All fuzzy gp*-connected spaces are fuzzy connected spaces.*

Proof. Let Y is a fuzzy gp*-connected space and suppose that Y is not a connected space. Then by Definition 2.12 there exists a non-empty proper fuzzy clopen subset λ in Y. Now as every fuzzy closed set is fuzzy gp*-closed implying that λ is also a non-empty proper subset of Y, which is both fuzzy gp*-closed and fuzzy gp*-open in Y. So by Theorem 6.2 Y is not a fuzzy gp*-connected space, which is a contradiction implying that Y is a connected space.

Theorem 6.4. *Suppose $g : G \to H$ is an onto fuzzy gp*-continuous map and G is a fuzzy gp*-connected space then H is also a fuzzy connected space.*
Proof. Let's suppose that H is not a fuzzy connected space and suppose that $H = M \vee N$, where M & N are disjoint fuzzy non-empty open sets in H. Since g is fuzzy gp*-continuous implies $g^{-}(M)$ & $g^{-}(N)$ are non-empty disjoint fuzzy gp*-open sets in G and as g is onto also implies $G = g^{-1}(M) \vee g^{-1}(N)$, which contradicts fuzzy gp*-connectedness of G. So H is a fuzzy connected space.

Theorem 6.5. *Suppose $g : G \to H$ is an onto fuzzy gp*-irresolute map and G is fuzzy gp*-connected space then H is also a fuzzy gp*- connected space.*
Proof. Let's suppose that H is not a fuzzy gp*-connected space and let's suppose that $H = M \vee N$ where M & N are non-empty fuzzy disjoint gp*-open sets in H. Now, as g is fuzzy gp*-irresolute function implies $g^{-1}(M)$ & $g^{-1}(N)$ are non-empty disjoint fuzzy gp*-open sets in G and as g is onto also implies $G = g^{-1}(M) \vee g^{-1}(N)$, which contradicts fuzzy gp*-connectedness of G. Implies H is a fuzzy connected space.

7. Fuzzy T*gp-Space

Definition 7.1. *A fts (Y, τ) is called a fuzzy T*gp-space if every fuzzy gp*-closed set in (Y, τ) is a fuzzy closed set in (Y, τ).*

Theorem 7.2. *Every fuzzy T*gp-space is fuzzy T*p-space.*
Proof. Let Y be a fuzzy T*gp-space. Let A be a fuzzy g*p-closed set in Y. Now by Theorem 3.4 as every fuzzy g*p-closed set is fuzzy gp*-closed set, implies A is fuzzy gp*-closed set in Y. Since Y is a fuzzy T*gp-space, A is a fuzzy closed set in Y. Hence Y is a fuzzy T*p-space.

Theorem 7.3. *Every fuzzy T*gp-space is fuzzy $T^*_{1/2}$ space.*
Proof. Let Y be a fuzzy T*gp-space. Let A be a fuzzy g*-closed set in Y. Now by Theorem 3.6, A is fuzzy gp*-closed set in Y. Since Y is a fuzzy-T*gp-space implies A is fuzzy closed set in Y. Hence Y is a $T^*_{1/2}$ space.

Theorem 7.4. *If G is a fuzzy T*gp-space then G is fuzzy connected iff it is fuzzy gp*-connected.*
Proof. Let G is a fuzzy connected space & suppose that G is not fuzzy gp*-connected. Then there exists two proper fuzzy gp*-open sets M & N of G such that $G = M \vee N$ & $M \wedge N = \phi$, which implies $M = 1 - N$ & $N = 1 - M$ are also fuzzy gp*-closed sets and G is a fuzzy T*gp-space implies M & N are fuzzy

closed sets (by Definition 7.1). So $M = 1 - N$ & $N = 1 - M$ implies M & N are fuzzy open sets & $G = M \vee N$, $M \wedge N = \phi$ contradicts the fuzzy connectedness of G. So G is a fuzzy gp*-connected space. Conversely suppose that G is fuzzy gp*-connected and let G is not fuzzy connected implies there exists two proper fuzzy open subsets M & N of G such that $G = M \vee N$ & $M \wedge N = \phi$. Now, as every fuzzy open set is fuzzy gp*-open, so $G = M \vee N$ contradicts the fuzzy gp*-connectedness of G. Implies G is a fuzzy connected space.

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CONGRUENCES FOR $(4, 5)$-REGULAR BIPARTITIONS INTO DISTINCT PARTS

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Abstract: Let $B_{4,5}(n)$ denote the number of $(4, 5)$-regular bipartitions of a positive integer n into distinct parts. In this paper, we establish many infinite families of congruences modulo powers of 2 for $B_{4,5}(n)$. For example,

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n$$
$$\equiv 2 f_1^3 \pmod 4, \text{ for all } \alpha, \beta, \gamma \geq 0.$$

Keywords and Phrases: Partition identities, Theta–functions, Partition congruences, Regular partition.

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1. Introduction

Throughout this paper, we let $|q| < 1$. We use the standard notation

$$f_k := (q^k; q^k)_\infty.$$

Following Ramanujan, we define

$$f(-q) \ := \ f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty. \tag{1.1}$$

Ramanujan's general theta function $f(a,b)$ [1] is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \ \ |ab| < 1. \tag{1.2}$$

In Ramanujan's notation, Jacobi's famous triple product identity becomes

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{1.3}$$

A partition of a positive integer n is a non-increasing sequence of positive integers whose sum is n. An ℓ-regular partition is a partition in which none of its parts are divisible by ℓ. Let $b_\ell(n)$ denote the number of ℓ-regular partitions of n with $b_\ell(0) = 1$. The generating function for $b_\ell(n)$ is

$$\sum_{n=0}^{\infty} b_\ell(n) q^n = \frac{f_\ell}{f_1}.$$

Recently, arithmetic properties of ℓ-regular partition functions have been studied by a number of mathematicians. Calkin et al. [2] established congruences for 5-regular partitions modulo 2 and 13-regular partitions modulo 2 and 3 using the theory of modular forms. For more details, one can see [3], [5], [6] and [7].
Suppose ℓ, $m > 0$ and $(\ell, m) = 1$. A partition is an (ℓ, m)-regular partitions of the positive integer n if none of the parts are divisible by ℓ or m. Let $a_{\ell,m}(n)$ denote the number of such partitions of n into distinct parts with $a_{\ell,m}(0) = 1$. The generating function is given by

$$\sum_{n=0}^{\infty} a_{\ell,m}(n) q^n = \frac{(-q;q)_\infty(-q^{\ell m};q^{\ell m})_\infty}{(-q^\ell;q^\ell)_\infty(-q^m;q^m)_\infty}. \tag{1.4}$$

For example, there are 3 partitions for $a_{3,5}(11)$, namely

$$11, \ \ 8+2+1, \ \ 7+4.$$

For more details, one can see [9] and [10].
Let $B_{\ell,m}(n)$ denote the number of (ℓ, m)-regular bipartitions of n into distinct parts with $B_{\ell,m}(0) = 1$ and the generating function is given by

$$\sum_{n=0}^{\infty} B_{\ell,m}(n) q^n = \frac{(-q;q)_\infty^2(-q^{\ell m};q^{\ell m})_\infty^2}{(-q^\ell;q^\ell)_\infty^2(-q^m;q^m)_\infty^2} = \frac{f_2^2 f_{2\ell m}^2 f_\ell^2 f_m^2}{f_{2\ell}^2 f_{2m}^2 f_1^2 f_{\ell m}^2}. \tag{1.5}$$

For example, there are 12 bipartitions for $B_{4,5}(6)$, namely

(0, 6), (6, 0), (3, 3), (2+1, 2+1), (3+2+1, 0), (0, 3+2+1)
(1, 3+2), (3+2,1), (2, 3+1), (3+1, 2), (3, 2+1), (2+1, 3).

2. Preliminary Results

Lemma 2.1. *we have*

$$\frac{1}{f_1^4}=\frac{f_4^{14}}{f_2^{14}f_8^4}+4q\frac{f_4^2f_8^4}{f_2^{10}}. \tag{2.1}$$

For proof, see [1, p. 40].

Lemma 2.2. *The following* 3*-dissection holds :*

$$f_1^3=\frac{f_6f_9^6}{f_3f_{18}^3}-3qf_9^3+4q^3\frac{f_3^2f_{18}^6}{f_6^2f_9^2}. \tag{2.2}$$

For proof, see [1, p. 395].

Lemma 2.3. *The following* 2*-dissections hold :*

$$\frac{f_5}{f_1}=\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}} \tag{2.3}$$

and

$$\frac{f_1}{f_5}=\frac{f_2f_8f_{20}^3}{f_4f_{10}^3f_{40}}-q\frac{f_4^2f_{40}}{f_8f_{10}^2}. \tag{2.4}$$

The equation (2.3) was proved by Hirschhorn and Sellers [5], see also [11]. Replacing q by $-q$ in (2.3) and using the fact that

$$(-q;-q)_\infty=\frac{f_2^3}{f_1f_4},$$

we obtain (2.4).

Lemma 2.4. [8] *We have*

$$f_1f_5^3=f_2^3f_{10}-q\frac{f_2^2f_{10}^2f_{20}}{f_4}+2q^2f_4f_{20}^3-2q^3\frac{f_4^4f_{10}f_{40}^2}{f_2f_8^2} \tag{2.5}$$

and

$$\frac{1}{f_1f_5^3}=\frac{f_4f_{20}^3}{f_{10}^8}+q\frac{f_{20}^4}{f_2f_{10}^7}+2q^2\frac{f_4^2f_{20}^6}{f_2^3f_{10}^9}++2q^3\frac{f_4^5f_{20}^3f_{40}^2}{f_2^4f_8^2f_{10}^8}. \tag{2.6}$$

Lemma 2.5. [1, p. 303, Entry 17 (v)] *We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{2.7}$$

where $A(q) = f(-q^3, -q^4)$, $B(q) = f(-q^2, -q^5)$ *and* $C(q) = f(-q, -q^6)$.
We shall prove the following Theorems :

Theorem 2.1. *Let* $r_1 \in \{62, 78\}$, $r_2 \in \{14, 46, 62, 78\}$, $r_3 \in \{14, 62, 158, 206\}$ *and* $r_4 \in \{46, 94, 142, 238\}$. *Then for all* $\alpha, \beta, \gamma \geq 0$, *we have for modulo 16,*

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 5^{2\alpha}n + 6 \cdot 5^{2\alpha} - 1\right)q^n \equiv 8f_4f_5 + 8qf_1^3f_{10}^3, \tag{2.8}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 5^{2\alpha+1}n + 14 \cdot 5^{2\alpha+1} - 1\right)q^n \equiv 8f_1f_{20} + 8f_2^3f_5^3, \tag{2.9}$$

$$B_{4,5}\left(16 \cdot 5^{2\alpha+2}n + r_1 \cdot 5^{2\alpha+1} - 1\right) \equiv 0, \tag{2.10}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 6 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right)q^n \equiv 8f_1^9, \tag{2.11}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1}n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} - 1\right)q^n \equiv 8q^2f_7^9, \tag{2.12}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} - 1\right)q^n \equiv 8qf_5^9, \tag{2.13}$$

$$B_{4,5}\left(16 \cdot 3^{4\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma}n + r_2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.14}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 22 \cdot 3^{4\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right)q^n \equiv 8f_2f_3^3, \tag{2.15}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{4\alpha+1} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}n + 2 \cdot 3^{4\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma+1} - 1\right)q^n$$
$$\equiv 8q^2f_{10}f_{15}^3, \tag{2.16}$$

$$B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_3\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.17}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+38\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n\equiv 8f_1f_6^3, \tag{2.18}$$

$$\sum_{n=0}^{\infty}B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+46\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)q^n\equiv 8q^3f_5f_{30}^3, \tag{2.19}$$

$$B_{4,5}\left(16\cdot 3^{4\alpha+1}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_4\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0. \tag{2.20}$$

Theorem 2.2. *For $\alpha\geq 0$, we have*

$$B_{4,5}(2^{2\alpha+3}n+2^{2\alpha+3}-1)\equiv B_{4,5}(8n+7)\pmod{16}. \tag{2.21}$$

Theorem 2.3. *For $\alpha,\beta\geq 0$, we have for modulo 8,*

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta}n+2^{2\alpha+2}\cdot 5^{2\beta}-1)q^n\equiv 4qf_1f_5^7-2f_1^2f_5^2, \tag{2.22}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta+1}n+2^{2\alpha+2}\cdot 5^{2\beta+1}-1)q^n\equiv 2f_1^2f_5^2+4f_1^7f_5, \tag{2.23}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+4}\cdot 5^{2\beta}n+2^{2\alpha+3}\cdot 5^{2\beta}-1)q^n\equiv 4qf_1f_5^7-2f_1^2f_5^2, \tag{2.24}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+4}\cdot 5^{2\beta+1}n+2^{2\alpha+3}\cdot 5^{2\beta+1}-1)q^n\equiv 2f_1^2f_5^2+4f_1^7f_5, \tag{2.25}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+5}\cdot 5^{2\beta}n+2^{2\alpha+4}\cdot 5^{2\beta}-1)q^n\equiv 4qf_1f_5^7-2f_1^2f_5^2, \tag{2.26}$$

$$\sum_{n=0}^{\infty}B_{4,5}(2^{2\alpha+5}\cdot 5^{2\beta+1}n+2^{2\alpha+4}\cdot 5^{2\beta+1}-1)q^n\equiv 2f_1^2f_5^2+4f_1^7f_5. \tag{2.27}$$

Theorem 2.4. *Let $r_5\in\{22,38\}$, $r_6\in\{34,66\}$, $r_7\in\{26,42,58,74\}$, $r_8\in\{88,152\}$, $r_9\in\{136,264\}$, $r_{10}\in\{104,168,232,296\}$, $r_{11}\in\{176,304\}$, $r_{12}\in$*

$\{272, 528\}$ *and* $r_{13} \in \{208, 336, 464, 592\}$. *Then for all* $\alpha, \beta, \gamma \geq 0$, *we have for modulo 4,*

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_1^3, \tag{2.28}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} - 1\right) q^n \equiv 2f_7^3, \tag{2.29}$$

$$\begin{aligned} &B_{4,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \\ &\equiv \quad \begin{cases} 2 & \textit{if} \ \ n = k(3k+1)/2 \ \ \textit{for some} \ k \in \mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{2.30}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_3^3, \tag{2.31}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 34 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.32}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + r_5 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.33}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_5^3, \tag{2.34}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_6 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.35}$$

$$B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_7 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.36}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_1^3, \tag{2.37}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 2 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} - 1\right) q^n \equiv 2f_7^3, \tag{2.38}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right) \equiv \begin{cases} 2 & \textit{if } \; n=k(3k+1)/2 \; \textit{ for some } k\in\mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \tag{2.39}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_3^3, \tag{2.40}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.41}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_5\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.42}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}-1\right)q^n \equiv 2f_5^3, \tag{2.43}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_6\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.44}$$

$$B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_7\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.45}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)q^n \equiv 2f_1^3, \tag{2.46}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1)q^n \equiv 2f_7^3, \tag{2.47}$$

$$B_{4,5}(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1) \equiv \begin{cases} 2 & \textit{if } \; n=k(3k+1)/2 \; \textit{ for some } k\in\mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \tag{2.48}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+8\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)q^n \equiv 2f_3^3, \tag{2.49}$$

$$B_{4,5}(64\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+136\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)\equiv 0, \tag{2.50}$$

$$B_{4,5}(64\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+r_8\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}-1)\equiv 0, \tag{2.51}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1)q^n \equiv 2f_5^3, \tag{2.52}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + r_9 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.53}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}n + r_{10} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.54}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1)q^n \equiv 2f_1^3, \tag{2.55}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1}n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} - 1)q^n \equiv 2f_7^3, \tag{2.56}$$

$$\begin{aligned} &B_{4,5}(64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \\ \equiv\; &\begin{cases} 2 & \textit{if } \; n = k(3k+1)/2 \;\; \textit{for some } k \in \mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{2.57}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 8 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1)q^n \equiv 2f_3^3, \tag{2.58}$$

$$B_{4,5}(64 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + 136 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.59}$$

$$B_{4,5}(64 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma}n + r_8 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.60}$$

$$\sum_{n=0}^{\infty} B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}n + 8 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma} - 1)q^n \equiv 2f_5^3, \tag{2.61}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma}n + r_9 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.62}$$

$$B_{4,5}(64 \cdot 3^{2\alpha} \cdot 5^{2\beta+3} \cdot 7^{2\gamma}n + r_{10} \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1) \equiv 0, \tag{2.63}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma}n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2f_1^3, \tag{2.64}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+1} n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma+2} - 1\right) q^n \equiv 2 f_7^3, \tag{2.65}$$

$$\begin{aligned} &B_{4,5}\left(128 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \\ &\equiv \begin{cases} 2 & \textit{if} \;\; n = k(3k+1)/2 \;\; \textit{for some } k \in \mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{2.66}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2 f_3^3, \tag{2.67}$$

$$B_{4,5}\left(128 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + 272 \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.68}$$

$$B_{4,5}\left(128 \cdot 3^{2\alpha+2} \cdot 5^{2\beta} \cdot 7^{2\gamma} n + r_{11} \cdot 3^{2\alpha+1} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.69}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2 f_5^3, \tag{2.70}$$

$$B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + r_{12} \cdot 3^{2\alpha} \cdot 5^{2\beta} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.71}$$

$$B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta+2} \cdot 7^{2\gamma} n + r_{13} \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) \equiv 0, \tag{2.72}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2 f_1^3, \tag{2.73}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+1} n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma+2} - 1\right) q^n \equiv 2 f_7^3, \tag{2.74}$$

$$\begin{aligned} &B_{4,5}\left(128 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 16 \cdot 3^{2\alpha} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) \\ &\equiv \begin{cases} 2 & \textit{if} \;\; n = k(3k+1)/2 \;\; \textit{for some } k \in \mathbb{Z}, \\ 0 & \textit{otherwise}, \end{cases} \end{aligned} \tag{2.75}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} n + 16 \cdot 3^{2\alpha+2} \cdot 5^{2\beta+1} \cdot 7^{2\gamma} - 1\right) q^n \equiv 2 f_3^3, \tag{2.76}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+272\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.77}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha+2}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n+r_{11}\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.78}$$

$$\sum_{n=0}^{\infty} B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+16\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}-1\right)q^n\equiv 2f_5^3, \tag{2.79}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r_{12}\cdot 3^{2\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}-1\right)\equiv 0, \tag{2.80}$$

$$B_{4,5}\left(128\cdot 3^{2\alpha}\cdot 5^{2\beta+3}\cdot 7^{2\gamma}n+r_{13}\cdot 3^{2\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}-1\right)\equiv 0. \tag{2.81}$$

3. Proof of the Theorem (2.1).

From (1.5), we find that

$$\sum_{n=0}^{\infty} B_{4,5}(n)q^n=\frac{f_2^2f_4^2f_{40}^2}{f_8^2f_{10}^2f_{20}^2}\times\frac{f_5^2}{f_1^2}. \tag{3.1}$$

Using (2.3) in (3.1) and extracting the terms involving q^{2n+1} from both sides, we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}(2n+1)q^n=2\frac{f_2^5f_{20}^2}{f_4^2f_{10}f_1^3f_5}. \tag{3.2}$$

Using (2.1) and (2.4) in (3.2), we get

$$\sum_{n=0}^{\infty} B_{4,5}(4n+1)q^n=2\frac{f_2^{11}f_{10}^5}{f_1^8f_4^3f_5^4f_{20}}-8q\frac{f_2^2f_4^3f_{10}^2f_{20}}{f_1^5f_5^3} \tag{3.3}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(4n+3)q^n=8\frac{f_4^5f_{10}^5}{f_1^4f_2f_5^4f_{20}}-2\frac{f_2^{14}f_{10}^2f_{20}}{f_1^9f_4^5f_5^3}. \tag{3.4}$$

From the binomial theorem, it is easy to see that for any positive integers k and m,

$$f_k^{2m}\equiv f_{2k}^m \pmod{2}, \tag{3.5}$$
$$f_k^{4m}\equiv f_{2k}^{2m} \pmod{2^2}, \tag{3.6}$$
$$f_k^{8m}\equiv f_{2k}^{4m} \pmod{2^3}. \tag{3.7}$$

From (3.7) along with (2.1) and (2.6) in (3.3), we get, modulo 16,

$$\sum_{n=0}^{\infty} B_{4,5}(8n+1)q^n \equiv 2\frac{f_2 f_{10}}{f_1 f_5} + 8q f_2^2 f_{10} f_1 f_5^3 \tag{3.8}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(8n+5)q^n \equiv 8f_2^4 f_{10} + 8q^2 f_{10}^5 f_1 f_5^3. \tag{3.9}$$

Employing (2.5) in (3.9), we find that

$$\sum_{n=0}^{\infty} B_{4,5}(16n+5)q^n \equiv 8f_4 f_5 + 8q f_1^3 f_{10}^3 \tag{3.10}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(16n+13)q^n \equiv 8q f_5^9. \tag{3.11}$$

The equation (3.10) is $\alpha = 0$ case of (2.8). Suppose the result (2.8) is true for $\alpha \geq 0$. Ramanujan recorded the following identity in his notebooks without proof:

$$f_1 = f_{25}(R(q^5)^{-1} - q - q^2 R(q^5)), \tag{3.12}$$

where $R(q) = \dfrac{f(-q,-q^4)}{f(-q^2,-q^3)}$.
For a proof of (3.12), one can see [4], [12].
Using (3.12) in (2.8) and then extracting the coefficients of q^{5n+4} from both sides, we see that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 5^{2\alpha+1}n + 14\cdot 5^{2\alpha+1} - 1\right) q^n \equiv 8f_1 f_{20} + 8f_2^3 f_5^3, \tag{3.13}$$

which is (2.9). Again using (3.12) in (3.13) and extracting the terms involving q^{5n+1} from both sides, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 5^{2\alpha+2}n + 6\cdot 5^{2\alpha+2} - 1\right) q^n \equiv 8f_4 f_5 + 8q f_1^3 f_{10}^3, \tag{3.14}$$

which implies that the congruence (2.8) is true for $\alpha+1$. Hence, by induction, the congruence (2.8) is true for non-negative integer α.

Extracting the coefficients of q^{5n+3} and q^{5n+4} in (3.13) along with (3.12), we obtain (2.10). Extracting the coefficients of q^{5n+1} from both sides of (3.11), we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(80n+29\right)q^n \equiv 8f_1^9, \tag{3.15}$$

which is $\alpha=\beta=\gamma=0$ case of (2.11). Let us consider the case $\beta=\gamma=0$. Suppose that the congruence (2.11) holds for some integer $\alpha \geq 0$. Employing the equation (2.2) in (2.11) with $\beta=\gamma=0$ and then extracting the coefficients of q^{3n} from both sides, we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+1}n+10\cdot 3^{4\alpha}-1\right)q^n \equiv 8f_1^3+8qf_3^9 \equiv 8f_3+8qf_3^9+8qf_9^3, \tag{3.16}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+2}n+10\cdot 3^{4\alpha+2}-1\right)q^n \equiv 8f_1^9+8f_3^3 \equiv 8qf_3^2f_9^3+8q^2f_3f_9^6+8q^3f_9^9. \tag{3.17}$$

Collecting the coefficients of q^{3n} from both sides, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+3}n+10\cdot 3^{4\alpha+2}-1\right)q^n \equiv 8qf_3^9, \tag{3.18}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(80\cdot 3^{4\alpha+4}n+10\cdot 3^{4\alpha+4}-1\right)q^n \equiv 8f_1^9, \tag{3.19}$$

which implies that the congruence (2.11) is true for $\alpha+1$. By induction, the congruence (2.11) holds for all $\alpha \geq 0$ with $\beta=\gamma=0$.
Now, suppose the congruence (2.11) is true for $\alpha,\beta \geq 0$ with $\gamma=0$. Utilizing (3.12) in (2.11) and then extracting the terms involving q^{5n+4}, we deduce that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+2}n+14\cdot 3^{4\alpha}\cdot 5^{2\beta+2}-1\right)q^n \equiv 8qf_5^9, \tag{3.20}$$

Extracting the coefficient of q^{5n+1} in (3.20), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+3}n+2\cdot 3^{4\alpha+1}\cdot 5^{2\beta+3}-1\right)q^n \equiv 8f_1^9. \tag{3.21}$$

Thus, the congruence (2.11) is true for $\beta+1$. Hence, by mathematical induction, the congruence (2.11) holds for all $\alpha,\beta \geq 0$ with $\gamma=0$. Suppose the congruence (2.11) is true for $\alpha,\beta,\gamma \geq 0$. Employing (2.7) in (2.11) and then extracting the coefficients of q^{7n+4} from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+1}n+2\cdot 3^{4\alpha}\cdot 5^{2\beta+2}\cdot 7^{2\gamma+1}-1\right)q^n \equiv 8q^2 f_7^9, \quad (3.22)$$

which is (2.12). Extracting the coefficients of q^{7n+2} in (3.22), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}n+6\cdot 3^{4\alpha}\cdot 5^{2\beta+1}\cdot 7^{2\gamma+2}-1\right)q^n \equiv 8f_1^9, \quad (3.23)$$

which implies that the congruence (2.11) is true for $\gamma+1$.
Hence, by induction, the congruence (2.11) holds for all integers $\alpha,\beta,\gamma \geq 0$. Employing (3.12) in (2.11) and then collecting the coefficients of q^{5n+4}, we get (2.13). From (2.13), we arrive at (2.14). Utilizing (2.2) in (2.11) and then collecting the coefficients of q^{3n+1} and q^{3n+2}, we obtain (2.15) and (2.18) respectively. From the equations (2.15) and (2.18) along with (3.12), we get (2.16) and (2.19) respectively. From the equations (2.16) and (2.19), we obtain (2.17) and (2.20) respectively.

4. Proof of the Theorem (2.2).

Employing (3.5) and (3.7) in (3.4), we find that, modulo 16,

$$\sum_{n=0}^{\infty} B_{4,5}(4n+3)q^n \equiv 8f_2^7 f_{10} - 2\frac{f_2^2 f_{10}^2 f_{20}}{f_4 f_1 f_5^3}. \quad (4.1)$$

Using (2.6) in (4.1), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}(8n+3)q^n \equiv 8f_2^2 f_1^3 f_5 - 2f_1^2 f_5^2 - 4q\frac{f_2 f_{10}^5}{f_1 f_5^3} \quad (4.2)$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(8n+7)q^n \equiv -2\frac{f_{10} f_1 f_5^3}{f_2} - 4q\frac{f_{20}^2 f_5^2}{f_1^2}. \quad (4.3)$$

Employing (2.3) and (2.5) in (4.3), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}(16n+7)q^n \equiv 8qf_{10}^2 f_1 f_5^3 - 2f_1^2 f_5^2 - 4q\frac{f_{10}^3 f_1^3 f_5}{f_2} \quad (4.4)$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(16n+15)q^n \equiv 2\frac{f_{10}f_1f_5^3}{f_2} - 4f_2^2f_{10}^2. \tag{4.5}$$

Using (2.5) in (4.5), we get

$$\sum_{n=0}^{\infty} B_{4,5}(32n+15)q^n \equiv 4q\frac{f_{10}^3f_1^3f_5}{f_2} - 2f_1^2f_5^2 \tag{4.6}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(32n+31)q^n \equiv -2\frac{f_{10}f_1f_5^3}{f_2} - 4q\frac{f_{20}^2f_5^2}{f_1^2}. \tag{4.7}$$

From the equations (4.3) and (4.7), we have

$$B_{4,5}(32n+31) \equiv B_{4,5}(8n+7). \tag{4.8}$$

Hence, by mathematical induction on α, we obtain (2.21).

5. Proof of the Theorem (2.3)

From the equation (4.2), we get, modulo 8,

$$\sum_{n=0}^{\infty} B_{4,5}(8n+3)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2 \tag{5.1}$$

which is $\alpha \geq 0$ and $\beta = 0$ case of (2.22). Suppose the congruence (2.22) is true for $\alpha, \beta \geq 0$. Using (3.12) in (2.22) and then collecting the coefficients of q^{5n+2}, we get

$$\sum_{n=0}^{\infty} B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta+1}n + 2^{2\alpha+2}\cdot 5^{2\beta+1} - 1)q^n \equiv 2f_1^2f_5^2 + 4f_1^7f_5, \tag{5.2}$$

which proves (2.23). Again collecting the coefficients of q^{5n+2} from (5.2) along with (3.12), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}(2^{2\alpha+3}\cdot 5^{2\beta+2}n + 2^{2\alpha+2}\cdot 5^{2\beta+2} - 1)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2, \tag{5.3}$$

which implies that the congruence (2.22) is true for $\beta+1$. By mathematical induction, the congruence (2.22) is true for all integers $\alpha, \beta \geq 0$. From the equation (4.4), we find that

$$\sum_{n=0}^{\infty} B_{4,5}(2^{2\alpha+4}n + 2^{2\alpha+3} - 1)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2, \tag{5.4}$$

which is $\alpha \geq 0$ and $\beta = 0$ case of (2.24). The rest of the proofs of the identities (2.24) and (2.25) are similar to the proofs of the identities (2.22) and (2.23). So, we omit the details. From the congruence (4.6), we get,

$$\sum_{n=0}^{\infty} B_{4,5}(32n+15)q^n \equiv 4qf_1f_5^7 - 2f_1^2f_5^2, \tag{5.5}$$

which is $\alpha \geq 0$ and $\beta = 0$ case of (2.26). The rest of the proofs of the identities (2.26) and (2.27) are similar to the proofs of the identities (2.22) and (2.23). So, we omit the details.

6. Proof of the Theorem (2.4)

From the equation (3.8), we get, modulo 4,

$$\sum_{n=0}^{\infty} B_{4,5}(8n+1)q^n \equiv 2\frac{f_1f_{10}}{f_5} \tag{6.1}$$

Using (2.4) in (6.1), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}(16n+1)q^n \equiv 2f_1^3 \tag{6.2}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(16n+9)q^n \equiv 2f_5^3. \tag{6.3}$$

The equation (6.2) is $\alpha = \beta = \gamma = 0$ case of (2.28). Suppose that the congruence (2.28) holds for some integer $\alpha \geq 0$ with $\beta = \gamma = 0$. Employing the equation (2.2) in (2.28) with $\beta = \gamma = 0$, we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha}n + 2 \cdot 3^{2\alpha} - 1\right) q^n \equiv 2(f_3 + qf_9^3). \tag{6.4}$$

Extracting the coefficients of q^{3n+1} from (6.4), we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha+1}n + 2 \cdot 3^{2\alpha+2} - 1\right) q^n \equiv 2f_3^3, \tag{6.5}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}\left(16 \cdot 3^{2\alpha+2}n + 2 \cdot 3^{2\alpha+2} - 1\right) q^n \equiv 2f_1^3, \tag{6.6}$$

which implies that the congruence (2.28) is true for $\alpha+1$. Hence, by induction, the congruence (2.28) holds for any non-negative integer α with $\beta=\gamma=0$. Now, suppose that the congruence (2.28) holds for some integers $\alpha,\beta\geq 0$ and $\gamma=0$. Employing the equation (3.12) in the equation (2.28), we find that

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}-1\right)q^n \equiv 2f_{25}^3\left(R(q^5)^{-1}-q-q^2R(q^5)\right)^3. \tag{6.7}$$

Extracting the coefficients of q^{5n+3} in (6.7), we arrive at

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}-1\right)q^n \equiv 2f_5^3. \tag{6.8}$$

Extracting the coefficients of q^{5n} in (6.8), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta+2}-1\right)q^n \equiv 2f_1^3, \tag{6.9}$$

which implies that the congruence (2.28) is true for $\beta+1$. Hence, by induction, the congruence (2.28) holds for any non-negative integers α and β with $\gamma=0$. Suppose that the congruence (2.28) holds for some integers $\alpha,\beta,\gamma\geq 0$. Employing (2.7) in (2.28) and then collecting the coefficients of q^{7n+6} from the resultant equation, we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n \equiv 2f_7^3, \tag{6.10}$$

which proves (2.29) and extracting the coefficients of q^{7n} in (6.10), we obtain

$$\sum_{n=0}^{\infty} B_{4,5}\left(16\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}-1\right)q^n \equiv 2f_1^3, \tag{6.11}$$

which implies that the congruence (2.28) is true for $\gamma+1$. Hence, by induction, the congruence (2.28) holds for any non-negative integers α,β and γ. Employing the equation (2.2) in the equation (2.28) and then extracting the coefficients of q^{3n}, q^{3n+1} and q^{3n+2}, we obtain (2.30), (2.31) and (2.32) respectively. Collecting the coefficients of q^{3n+1} and q^{3n+3} from (2.31), we get (2.33). Using the equation

(2.28) along with the equation (3.12), we obtain (2.34) and (2.35). From the equation (2.34), we get (2.36). Extracting the coefficients of q^{5n} in (6.3), we get

$$\sum_{n=0}^{\infty} B_{4,5}\left(80n+9\right)q^n \equiv 2f_1^3, \tag{6.12}$$

which is $\alpha=\beta=\gamma=0$ case of (2.37). The rest of the proofs of the identities (2.37)- (2.45) are similar to the proofs of the identities (2.28)- (2.36). So, we omit the details. From the equation (4.4), we find that

$$\sum_{n=0}^{\infty} B_{4,5}(32n+7)q^n \equiv \frac{2f_1f_5^3}{f_{10}} \equiv 2f_2^3+2qf_{10}^3, \tag{6.13}$$

which yields

$$\sum_{n=0}^{\infty} B_{4,5}(64n+7)q^n \equiv 2f_1^3 \tag{6.14}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(64n+39)q^n \equiv 2f_5^3. \tag{6.15}$$

The rest of the proofs of the identities (2.46)- (2.63) are similar to the proofs of the identities (2.28)- (2.36). So, we omit the details.
From the congruence (5.5), we have

$$\sum_{n=0}^{\infty} B_{4,5}(64n+15)q^n \equiv \frac{2f_1f_5^3}{f_{10}} \equiv 2f_2^3+2qf_{10}^3, \tag{6.16}$$

which implies

$$\sum_{n=0}^{\infty} B_{4,5}(128n+15)q^n \equiv 2f_1^3 \tag{6.17}$$

and

$$\sum_{n=0}^{\infty} B_{4,5}(128n+79)q^n \equiv 2f_5^3. \tag{6.18}$$

The rest of the proofs of the identities (2.64)-(2.81) are similar to the proofs of the identities (2.28)- (2.36).
So, we omit the details.

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SOME COMMON FIXED POINT RESULTS IN 2-BANACH SPACES

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Abstract: In this paper, we have proved some common fixed point theorems of a family of self maps without continuity in 2-Banach space. We have used functions on ${\mathbb{R}_+}^5$ to $\mathbb{R}_+$ and also generalize many existing results.

Keywords and Phrases: 2-norm, 2-Banach.

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1. Introduction

In 1965, Gahler ([5], [6]) introduced 2-Banach space and Iseki [7] obtained some results on fixed point theorems in 2-Banach spaces. After the introduction of 2-Banach space many research workers have extended fixed point theorems of metric, Banach spaces etc. in the new setup of 2-Banach spaces. Mishra et al. [10], Khan and Khan [8], Saha et al. [12], Mishra et al. [11], Saluja [13], Saluja and Dhakde [14], Das et al. [1], Shrivas [15], Das et al, [2] - [3], Liu et al. [9] and etc. have worked on fixed point and common fixed point theorems in this space. In this paper we also have proved some unique common fixed point theorems in 2-Banach spaces.

2. Definitions and Preliminaries

Gahler [5] has introduced the notion of 2-norm as follows:

2-norm: Let X be a linear space and $\|.,.\|$ is a real valued function defined on X where

i) $\|a,b\| = 0$ if and only if a and b are linearly dependent;

ii) $\|a,b\| = \|b,a\|$;

iii) $\|a,xb\| = |x|\ \|a,b\|$;

iv) $\|a,b+c\| \leq \|a,b\| + \|a,c\|$

for all $a,b,c \in X$ and $x \in \mathbb{R}$. Then $\|.,.\|$ is called a 2-norm and the pair $(X, \|.,.\|)$ is called a 2-norm space.

In this paper, we denote X as a 2-normed space unless otherwise stated.

Convergent: A sequence $\{x_n\}$ in a 2-norm space X is said to be convergent if there is a point $x \in X$ such that $\lim_{n\to\infty} \|x_n - x, a\| = 0$ for all $a \in X$.

Cauchy Sequence: A sequence $\{x_n\}$ in a 2-norm space X is called a Cauchy sequence if $\lim_{n,m\to\infty} \|x_n - x_m, a\| = 0$ for all $a \in X$.

2-Banach Space: A linear 2-norm space is said to be complete if every Cauchy sequence in X is convergent in X. Then we say X is a 2-Banach Space.

Let us consider a function $f : {\mathbb{R}_+}^5 \to \mathbb{R}_+$ given by

$$f(t_1,t_2,t_3,t_4,t_5) = \max\{t_1, \frac{t_2+t_3}{2}, \frac{t_4+t_5}{2}\}; \tag{2.1}$$

$$f(t_1,t_2,t_3,t_4,t_5) = \max\{\frac{t_1+t_2+t_3}{3}, \frac{t_4+t_5}{3}\}. \tag{2.2}$$

3. Main Part

In this part we have proved some unique common fixed point theorems in 2-Banach spaces.

Theorem 3.1. *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*
$\|F_ix - F_jy, p\| \leq \alpha f(\|x-y,p\|, \|x-F_ix,p\|, \|y-F_jy,p\|, \|x-F_jy,p\|, \|y-F_ix,p\|)$, *where $\alpha < 1$ and f satisfies the relation (2.1). Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.*

Proof. Let $\{x_n\}$ be sequence of points of X given by $x_{n+1} = F_ix_n$ with the initial approximation $x_0 \in X$ for a fixed i. If $F_ix_n = x_n$ i.e., $x_{n+1} = x_n$, then x_n is a common fixed point of $\{F_n\}$. So without loss of generality assume $x_{n+1} \neq x_n$.

We now show that $\lim_{n\to\infty} \|x_n - x, p\| = 0$.

Since,

$\|x_{n+1} - x_n, p\| = \|F_ix_n - F_jx_{n-1}, p\|$
$\leq \alpha f(\|x_n - x_{n-1}, p\|, \|x_n - F_ix_n, p\|, \|x_{n-1} - F_jx_{n-1}, p\|, \|x_n - F_jx_{n-1}, p\|, \|x_{n-1} -$

$F_ix_n, p\|)$
$= \alpha f(\|x_n - x_{n-1}, p\|, \|x_n - x_{n+1}, p\|, \|x_{n-1} - x_n, p\|, \|x_n - x_n, p\|, \|x_{n-1} - x_{n+1}, p\|)$
$= \alpha \max\{\|x_n - x_{n-1}, p\|, \frac{\|x_n - x_{n+1}, p\| + \|x_{n-1} - x_n, p\|}{2}, \frac{0 + \|x_{n-1} - x_{n+1}, p\|}{2}\}$
$\leq \alpha \max\{\|x_n - x_{n-1}, p\|, \frac{\|x_n - x_{n+1}, p\| + \|x_{n-1} - x_n, p\|}{2}, \frac{\|x_{n-1} - x_n, p\| + \|x_n - x_{n+1}, p\|}{2}\}$

$$\leq \alpha \max\{\|x_n - x_{n-1}, p\|, \|x_n - x_{n+1}, p\|\}. \tag{3.1}$$

If $\|x_n - x_{n-1}, p\| \leq \|x_n - x_{n+1}, p\|$, then from (3.1), we have
$\|x_{n+1} - x_n, p\| \leq \alpha\|x_{n+1} - x_n, p\|$
implies $1 \leq \alpha$, which is a contradiction.
Therefore $\{\|x_n - x_{n-1}, p\|\}$ is a sequence of real numbers monotone decreasing and bounded below.

Suppose $\lim_{n\to\infty} \|x_n - x_{n-1}, p\| = s$.
Since,
$s = \lim_{n\to\infty} \|x_n - x_{n-1}, p\|$
$= \lim_{n\to\infty} \|F_ix_{n-1} - F_jx_{n-2}, p\|$
$\leq \lim_{n\to\infty} \alpha f(\|x_{n-1} - x_{n-2}, p\|, \|x_{n-1} - F_ix_{n-1}, p\|, \|x_{n-2} - F_jx_{n-2}, p\|,$
$\|x_{n-1} - F_jx_{n-2}, p\|, \|x_{n-2} - F_ix_{n-1}, p\|)$
$\leq \alpha \lim_{n\to\infty} f(\|x_{n-1} - x_{n-2}, p\|, \|x_{n-1} - x_n, p\|, \|x_{n-2} - x_{n-1}, p\|, \|x_{n-1} - x_{n-1}, p\|,$
$\|x_{n-2} - x_n, p\|)$
$= \alpha \lim_{n\to\infty} \max\{\|x_{n-1} - x_{n-2}, p\|, \frac{\|x_{n-1} - x_n, p\| + \|x_{n-2} - x_{n-1}, p\|}{2}, \frac{0 + \|x_{n-2} - x_n, p\|}{2}\}$
$\leq \alpha \lim\limits_{n\to\infty} \max\{\|x_{n-1} - x_{n-2}, p\|, \frac{\|x_{n-1} - x_n, p\| + \|x_{n-2} - x_{n-1}, p\|}{2}, \frac{\|x_{n-2} - x_{n-1}, p\| + \|x_{n-1} - x_n, p\|}{2}\}$
$\leq \alpha\ s$
implies, $s = 0$
i.e., $\lim_{n\to\infty} \|x_n - x, p\| = 0$.

Now, let $n \geq m \in \mathbb{N} \cup \{0\}$. Then
$\|x_{n+1} - x_{m+1}, p\| = \|F_ix_n - F_jx_m, p\|$
$\leq \alpha f(\|x_n - x_m, p\|, \|x_n - F_ix_n, p\|, \|x_m - F_jx_m, p\|, \|x_n - F_jx_m, p\|, \|x_m - F_ix_n, p\|)$
$= \alpha f(\|x_n - x_m, p\|, \|x_n - x_{n+1}, p\|, \|x_m - x_{m+1}, p\|, \|x_n - x_{m+1}, p\|, \|x_m - x_{n+1}, p\|)$
$= \alpha \max\{\|x_n - x_m, p\|, \frac{\|x_n - x_{n+1}, p\| + \|x_m - x_{m+1}, p\|}{2}, \frac{\|x_n - x_{m+1}, p\| + \|x_m - x_{n+1}, p\|}{2}\}$.
Taking limit as $n, m \to \infty$ on the both sides of the above inequality, we get
$\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, p\|$
$\leq \alpha \max\{\lim\limits_{n,m\to\infty} \|x_n - x_m, p\|, 0, \lim\limits_{n,m\to\infty} \frac{\|x_n - x_m, p\| + \|x_m - x_{m+1}, p\| + \|x_m - x_n, p\| + \|x_n - x_{n+1}, p\|}{2}\}$
$= \alpha \max\{\lim_{n,m\to\infty} \|x_n - x_m, p\|, \lim_{n,m\to\infty} \|x_n - x_m, p\|\}$
$= \alpha \lim_{n,m\to\infty} \|x_n - x_m, p\|$,
which implies, $\lim_{n,m\to\infty} \|x_n - x_m, p\| = 0$ [since $\alpha \neq 0$].
Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an $x \in X$ such that $\lim_{n,m\to\infty} \|x_n - x, p\| = 0$.

Now we show that x is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.
Again,

$\|F_i x - x, p\| \leq \|F_i x - x_n, p\| + \|x_n - x, p\|$
$= \|F_i x - F_j x_{n-1}, p\| + \|x_n - x, p\|$
$\leq \alpha f(\|x - x_{n-1}, p\|, \|x - F_i x, p\|, \|x_{n-1} - F_j x_{n-1}, p\|, \|x - F_j x_{n-1}, p\|, \|x_{n-1} - F_i x, p\|) + \|x_n - x, p\|$
$= \alpha f(\|x_n - x_{n-1}, p\|, \|x - F_i x, p\|, \|x_{n-1} - x_n, p\|, \|x - x_n, p\|, \|x_{n-1} - F_i x, p\|) + \|x_n - x, p\|$
$= \alpha \max\{\|x_n - x_{n-1}, p\|, \frac{\|x - F_i x, p\| + \|x_{n-1} - x_n, p\|}{2}, \frac{\|x - x_n, p\| + \|x_{n-1} - F_i x, p\|}{2}\} + \|x_n - x, p\|$.
Taking limit as $n \to \infty$ we get from above

$\lim_{n \to \infty} \|F_i x - x, p\| \leq \alpha \max\{0, \frac{\|F_i x - x, p\|}{2}, \frac{\|F_i x - x, p\|}{2}\} + 0$
i.e., $\|F_i x - x, p\| \leq \alpha \frac{\|F_i x - x, p\|}{2} \leq \alpha \|F_i x - x, p\|$
implies, $\|F_i x - x, p\| = 0$
i.e., $F_i x = x$.
Thus x is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

To show the uniqueness, let x' be another fixed point of $\{F_n\}_{n=1}^{\infty}$.
Since,

$\|x - x', p\| = \|F_i x - F_j x', p\|$
$\leq \alpha f(\|x - x', p\|, \|x - F_i x, p\|, \|x' - F_j x', p\|, \|x - F_j x', p\|, \|x' - F_i x, p\|)$
$= \alpha f(\|x - x', p\|, \|x - x, p\|, \|x' - x', p\|, \|x - x', p\|, \|x' - x, p\|)$
$= \alpha \max\{\|x - x', p\|, 0, \frac{\|x - x', p\| + \|x - x', p\|}{2}\}$
$= \alpha \|x - x', p\|$,
which implies, $\|x - x', p\| = 0$ [since $\alpha \neq 0$]
i.e., $x = x'$.

Hence $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Corollary 3.1. *Let F_1 and F_2 be two self maps on 2-Banach space $(X, \|., .\|)$ satisfying*
$\|F_1 x - F_2 y, p\| \leq \alpha f(\|x - y, p\|, \|x - F_1 x, p\|, \|y - F_2 y, p\|, \|x - F_2 y, p\|, \|y - F_1 x, p\|)$, *where $\alpha < 1$ and f satisfies the relation (2.1). Then F_1 and F_2 have a unique common fixed point in X.*
Proof. Putting $F_i = F_1$ and $F_j = F_2$ in the **Theorem 3.1** we get the result.

Corollary 3.2. *Let F be a self map on 2-Banach space $(X, \|., .\|)$ satisfying*
$\|Fx - Fy, p\| \leq \alpha f(\|x - y, p\|, \|x - Fx, p\|, \|y - Fy, p\|, \|x - Fy, p\|, \|y - Fx, p\|)$, *where $\alpha < 1$ and f satisfies the relation (2.1). Then F have a unique fixed point in X.*
Proof. Putting $F_i = F_j = F$ in the **Theorem 3.1** we get the result.

Theorem 3.2. *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*
$\|F_ix - F_jy, p\| \leq \beta f(\|x - F_ix, p\|, \|y - F_jy, p\|, \|x - F_jy, p\|, \|y - F_ix, p\|, \|x - y, p\|)$, *where $\beta < 1$ and f satisfy the relation (2.2). Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be an initial point. Construct a sequence $\{x_n\}$ in X, for a fixed i, such that $x_{n+1} = F_ix_n$. If $x_{n+1} = x_n$ i.e., $F_ix_n = x_n$, then x_n is a common fixed point of $\{F_n\}_{n=1}^{\infty}$. So without loss of generality, suppose $x_{n+1} \neq x_n \forall n \in \mathbb{N} \cup \{0\}$.
Since,

$\|x_{n+1} - x_n, p\| = \|F_ix_n - F_jx_{n-1}, p\|$
$\leq \beta f(\|x_n - F_ix_n, p\|, \|x_{n-1} - F_jx_{n-1}, p\|, \|x_n - F_jx_{n-1}, p\|, \|x_{n-1} - F_ix_n, p\|, \|x_n - x_{n-1}, p\|)$
$= \beta f(\|x_n - x_{n+1}, p\|, \|x_{n-1} - x_n, p\|, \|x_n - x_n, p\|, \|x_{n-1} - x_{n+1}, p\|, \|x_n - x_{n-1}, p\|)$
$= \beta \max\{\frac{\|x_n - x_{n+1},p\| + \|x_{n-1} - x_n,p\| + \|x_n - x_n,p\|}{3}, \frac{\|x_{n-1} - x_{n+1},p\| + \|x_n - x_{n-1},p\|}{3}\}$
$\leq \beta \max\{\frac{\|x_n - x_{n+1},p\| + \|x_{n-1} - x_n,p\|}{3}, \frac{\|x_{n-1} - x_n,p\| + \|x_n - x_{n+1},p\| + \|x_n - x_{n-1},p\|}{3}\}$

$$\leq \beta \max\{\|x_n - x_{n+1}, p\|, \|x_n - x_{n-1}, p\|\}. \tag{3.2}$$

If $\|x_n - x_{n-1}, p\| \leq \|x_n - x_{n+1}, p\|$, then from (3.2) we get

$\|x_{n+1} - x_n, p\| \leq \beta\|x_{n+1} - x_n, p\|$

which implies, $1 \leq \beta$, a contradiction.
Therefore,

$\|x_{n+1} - x_n, p\| \leq \|x_n - x_{n-1}, p\|$.

Thus $\{\|x_n - x_{n-1}, p\|\}$ is a monotone decreasing sequence of non-negative real numbers. Suppose $\lim_{n\to\infty} \|x_n - x_{n-1}, p\| = r$.
Thus

$r = \lim_{n\to\infty} \|x_n - x_{n-1}, p\| = \lim_{n\to\infty} \|F_ix_{n-1} - F_jx_{n-2}, p\|$
$\leq \beta \lim_{n\to\infty} f(\|x_{n-1} - F_ix_{n-1}, p\|, \|x_{n-2} - F_jx_{n-2}, p\|, \|x_{n-1} - F_jx_{n-2}, p\|,$
$\|x_{n-2} - F_ix_{n-1}, p\|,$
$\|x_{n-1} - x_{n-2}, p\|)$
$= \lim_{n\to\infty} \beta f(\|x_{n-1} - x_n, p\|, \|x_{n-2} - x_{n-1}, p\|, \|x_{n-1} - x_{n-1}, p\|, \|x_{n-2} - x_n, p\|,$
$\|x_{n-1} - x_{n-2}, p\|)$
$= \lim_{n\to\infty} \beta \max\{\frac{(\|x_{n-1} - x_n,p\| + \|x_{n-2} - x_{n-1},p\| + \|x_{n-1} - x_{n-1},p\|}{3}, \frac{\|x_{n-2} - x_n,p\| + \|x_{n-1} - x_{n-2},p\|}{3}\}$
$\leq \beta \lim_{n\to\infty} \max\{\frac{(\|x_{n-1} - x_n,p\| + \|x_{n-2} - x_{n-1},p\| + \|x_{n-1} - x_{n-1},p\|}{3},$
$\frac{\|x_{n-2} - x_{n-1},p\| + \|x_{n-1} - x_n,p\| + \|x_{n-1} - x_{n-2},p\|}{3}\}$
$\leq \beta \lim_{n\to\infty} \max\{\|x_n - x_{n-1}, p\|, \|x_{n-1} - x_{n-2}, p\|\}$
$= \beta \max\{r, r\}$
$= \beta r$

implies, $r = 0$ [as $\beta < 1$]
i.e., $\lim_{n\to\infty} \|x_n - x_{n-1}, p\| = 0$.
Now, for $n \geq m \in \mathbb{N}$,
$\|x_{n+1} - x_{m+1}, p\| = \|F_i x_n - F_j x_m, p\|$
$\leq \beta f(\|x_n - F_i x_n, p\|, \|x_m - F_j x_m, p\|, \|x_n - F_j x_m, p\|, \|x_m - F_i x_n, p\|, \|x_n - x_m, p\|)$
$= \beta f(\|x_n - x_{n+1}, p\|, \|x_m - x_{m+1}, p\|, \|x_n - x_{m+1}, p\|, \|x_m - x_{n+1}, p\|, \|x_n - x_m, p\|)$
$= \beta \max\{\frac{\|x_n - x_{n+1},p\| + \|x_m - x_{m+1},p\| + \|x_n - x_{m+1},p\|}{3}, \frac{\|x_m - x_{n+1},p\| + \|x_n - x_m,p\|}{3}\}$
$\leq \beta \max\{\frac{\|x_n - x_{n+1},p\| + \|x_m - x_{m+1},p\| + \|x_n - x_m,p\| + \|x_m - x_{m+1},p\|}{3}, \frac{\|x_m - x_n,p\| + \|x_n - x_{n+1},p\| + \|x_n - x_m,p\|}{3}\}$
$\leq \beta \max\{\|x_n - x_{n+1}, p\|, \|x_m - x_{m+1}, p\|, \|x_n - x_m, p\|\}$.
Taking limit as $n, m \to \infty$ on the both sides of the above inequality, we get
$\lim_{n,m\to\infty} \|x_{n+1} - x_{m+1}, p\| \leq \beta \lim_{n,m\to\infty} \|x_n - x_m, p\|$
implies, $\lim_{n,m\to\infty} \|x_n - x_m, p\| = 0$.
Thus $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there is an $z \in X$ such that $\lim_{n\to\infty} \|x_n - z, p\| = 0$.
Since $\|F_i z - z, p\| \leq \|F_i z - x_n, p\| + \|x_n - z, p\|$
$= \|F_i z - F_j x_{n-1}, p\| + \|x_n - z, p\|$
$\leq \beta f(\|z - F_i z, p\|, \|x_{n-1} - F_j x_{n-1}, p\|, \|z - F_j x_{n-1}, p\|, \|x_{n-1} - F_i z, p\|, \|z - x_{n-1}, p\|) + \|x_n - z, p\|$
$= \beta f(\|z - F_i z, p\|, \|x_{n-1} - x_n, p\|, \|z - x_n, p\|, \|x_{n-1} - F_i z, p\|, \|z - x_{n-1}, p\|) + \|x_n - z, p\|$
$= \beta \max\{\frac{\|z - F_i z,p\| + \|x_{n-1} - x_n,p\| + \|z - x_n,p\|}{3}, \frac{\|x_{n-1} - F_i z,p\| + \|z - x_{n-1},p\|}{3}\} + \|x_n - z, p\|$.
Taking $\lim_{n\to\infty}$ on the both sides of the above inequality, we have
$\lim\limits_{n\to\infty} \|F_i z - z, p\|$
$\leq \beta \lim\limits_{n\to\infty} \max\{\frac{\|z - F_i z,p\| + \|x_{n-1} - x_n,p\| + \|z - x_n,p\|}{3}, \frac{\|x_{n-1} - F_i z,p\| + \|z - x_{n-1},p\|}{3}\} + \lim\limits_{n\to\infty} \|x_n - z, p\|$
$= \beta \max\{\frac{\|F_i z - z,p\|}{3}, \frac{\|F_i z - z,p\|}{3}\}$
$\leq \beta \|F_i z - z, p\|$
which implies, $(1 - \beta)\|F_i z - z, p\| \leq 0$
i.e., $\|F_i z - z, p\| = 0$
i.e., $F_i z = z$.
So z is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

Let z' be another common fixed point of $\{F_n\}_{n=1}^{\infty}$.
Then ,
$\|z - z', p\| \leq \|F_i z - F_j z', p\|$
$\leq \beta f(\|z - F_i z, p\|, \|z' - F_j z', p\|, \|z - F_j z', p\|, \|z' - F_i z, p\|, \|z - z', p\|)$
$= \beta f(\|z - z, p\|, \|z' - z', p\|, \|z - z', p\|, \|z' - z, p\|, \|z - z', p\|)$
$= \beta \max\{\frac{0 + 0 + \|z - z',p\|}{3}, \frac{\|z - z',p\| + \|z - z',p\|}{3}\}$
$\leq \beta \|z - z', p\|$

implies, $\|z - z', p\| = 0$ i.e., $z = z'$.

Hence $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Corollary 3.3 *Let F_1 and F_2 be two self maps on 2-Banach space $(X, \|., .\|)$ satisfying*
$\|F_1x - F_2y, p\| \leq \beta f(\|x - F_1x, p\|, \|y - F_2y, p\|, \|x - F_2y, p\|, \|y - F_1x, p\|, \|x - y, p\|)$,
where $\beta < 1$ and f satisfy the relation (2.2). Then F_1 and F_2 have a unique common fixed point in X.

Proof. Put $F_i = F_1$ and $F_j = F_2$ in the above **Theorem 3.2** we get the result.

Corollary 3.4. *Let F be a self map on 2-Banach space $(X, \|., .\|)$ satisfying*
$\|Fx - Fy, p\| \leq \beta f(\|x - Fx, p\|, \|y - Fy, p\|, \|x - Fy, p\|, \|y - Fx, p\|, \|x - y, p\|)$,
where $\beta < 1$ and f satisfy the relation (2.2). Then F have a unique fixed point in X.

Proof. Put $F_i = F_j = F$ in the above **Theorem 3.2** we get the result.

Theorem 3.3. *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|., .\|)$ satisfying*
$\|F_ix - F_jy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_jy,p\|+\|y-F_ix,p\|}{1+\|x-F_jy,p\|+\|y-F_ix,p\|} + \beta \max\{\|x - F_jy, p\|, \|y - F_ix, p\|\} + \gamma\|y - F_jy, p\|$,
where α, β, γ are non-negative real numbers and $3\alpha + 2\beta + \gamma < 1$. Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Proof. For an initial approximation $y_0 \in X$ construct a sequence $\{y_n\}$ in X such that $y_{n+1} = F_iy_n$ for a fixed $i = 1, 2, 3, ...$ If $y_n = F_iy_n$ i.e., $y_n = y_{n+1}$, $n = 0, 1, 2, ...$ then y_n is common fixed point of $\{F_n\}_{n=1}^{\infty}$ for all $n = 0, 1, 2, ...$ and the proof is completed.

So we assume that $y_{n+1} \neq y_n \ \ \forall \ \ n \in \mathbb{N} \cup \{0\}$.

Now we show that $\{y_n\}$ is a Cauchy sequence.
Since,

$\|y_{n+1} - y_n, p\| = \|F_iy_n - F_jy_{n-1}, p\|$
$\leq \alpha(\frac{\|y_n-y_{n-1},p\|+\|y_n-F_jy_{n-1},p\|+\|y_{n-1}-F_iy_n,p\|}{1+\|y_n-F_jy_{n-1},p\|+\|y_{n-1}-F_iy_n,p\|}) + \beta \max\{\|y_n - F_jy_{n-1}, p\|, \|y_{n-1} - F_iy_n, p\|\}$
$+ \gamma\|y_{n-1} - F_jy_{n-1}, p\|$
$\leq \alpha(\|y_n - y_{n-1}, p\| + \|y_n - y_n, p\| + \|y_{n-1} - y_{n+1}, p\|) + \beta \max\{\|y_n - y_n, p\|, \|y_{n-1} - y_{n+1}, p\|\} + \gamma\|y_{n-1} - y_n, p\|$
$\leq \alpha(\|y_n - y_{n-1}, p\| + \|y_{n-1} - y_n, p\| + \|y_n - y_{n+1}, p\|) + \beta[\|y_{n-1} - y_n, p\| + \|y_n - y_{n+1}, p\|] + \gamma\|y_{n-1} - y_n, p\|$
implies, $(1 - \alpha - \beta)\|y_{n+1} - y_n, p\| \leq (2\alpha + \beta + \gamma)\|y_n - y_{n-1}, p\|$
i.e., $\|y_{n+1} - y_n, p\| \leq (\frac{2\alpha+\beta+\gamma}{1-\alpha-\beta})\|y_n - y_{n-1}, p\|$
$= k\|y_n - y_{n-1}, p\|$ [where $\frac{2\alpha+\gamma+\beta}{1-\alpha-\beta} = k < 1$]

$\leq k^2\|y_{n-1} - y_{n-2}, p\|$
$\vdots$
$\leq k^n\|y_1 - y_0, p\|$.
Taking limit as $n \to \infty$, we get
$\lim_{n\to\infty} \|y_{n+1} - y_n, p\| = 0$ [as $k < 1$].
Now, let $n \geq m \in \mathbb{N}$. Then
$\|y_n - y_m, p\| = \|F_i y_{n-1} - F_j y_{m-1}, p\|$
$\leq \alpha(\frac{\|y_{n-1}-y_{m-1},p\|+\|y_{n-1}-F_j y_{m-1},p\|+\|y_{m-1}-F_i y_{n-1},p\|}{1+\|y_{n-1}-F_j y_{m-1},p\|+\|y_{m-1}-F_i y_{n-1},p\|})$
$+ \beta \max\{\|y_{n-1} - F_j y_{m-1}, p\|, \|y_{m-1} - F_i y_{n-1}, p\|\} + \gamma\|y_{m-1} - F_j y_{m-1}, p\|$
$\leq \alpha(\|y_{n-1}-y_{m-1}, p\|+\|y_{n-1}-y_m, p\|+\|y_{m-1}-y_n, p\|)+\beta \max\{\|y_{n-1}-y_m, p\|, \|y_{m-1}-y_n, p\|\} + \gamma\|y_{m-1} - y_m, p\|$
$\leq \alpha(\|y_{n-1}-y_{m-1}, p\|+\|y_{n-1}-y_n, p\|+\|y_n-y_m, p\|+\|y_{m-1}-y_m, p\|+\|y_m-y_n, p\|)+\beta \max\{\|y_{n-1}-y_n, p\|+\|y_n-y_m, p\|, \|y_{m-1}-y_m, p\|+\|y_m-y_n, p\|\}+\gamma\|y_{m-1}-y_m, p\|$.
Let $\lim_{n,m\to\infty} \|y_m - y_n, p\| = r$.
Then from above we get
$r \leq \alpha(r + 0 + r + 0 + r) + \beta \max\{0 + r, 0 + r\} + \gamma.0$
implies, $r \leq 3\alpha r + \beta r$
i.e.,$(1 - 3\alpha - \beta)r \leq 0$
i.e., $r = 0$ [since $1 - 3\alpha - \beta \neq 0$]
i.e., $\lim_{n,m\to\infty} \|y_n - y_m, p\| = 0$.
Thus $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists a $y \in X$ such that $\lim_{n\to\infty} \|y_n - y, p\| = 0$.

Since,
$\|F_i y - y, p\| \leq \|F_i y - y_n, p\| + \|y_n - y, p\|$
$= \|F_i y - F_j y_{n-1}, p\| + \|y_n - y, p\|$
$\leq \alpha(\frac{\|y-y_{n-1},p\|+\|y-F_j y_{n-1},p\|+\|y_{n-1}-F_i y,p\|}{1+\|y-F_j y_{n-1},p\|+\|y_{n-1}-F_i y,p\|}) + \beta \max\{\|y - F_j y_{n-1}, p\|, \|y_{n-1} - F_i y, p\|\} + \gamma\|y_{n-1} - F_j y_{n-1}, p\| + \|y_n - y, p\|$
$\leq \alpha(\|y - y_{n-1}, p\| + \|y - y_n, p\| + \|y_{n-1} - F_i y, p\|) + \beta \max\{\|y - y_n, p\|, \|y_{n-1} - F_i y, p\|\} + \gamma\|y_{n-1} - y_n, p\| + \|y_n - y, p\|$.
Taking limit as $n \to \infty$ we get from the above inequality,
$\lim_{n\to\infty} \|F_i y - y, p\| \leq \alpha\|y - F_i y, p\| + \beta\|y - F_i y, p\| + \gamma.0 + 0$
implies, $(1 - \alpha - \beta)\|y - F_i y, p\| \leq 0$
i.e., $\|y - F_i y, p\| = 0$
i.e., $F_i y = y$.
Thus y is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

Let z be another common fixed point of $\{F_n\}_{n=1}^{\infty}$. Then
$\|y - z, p\| = \|F_i y - F_j z, p\|$

$\leq \alpha \frac{\|y-z,p\|+\|y-F_jz,p\|+\|z-F_iy,p\|}{1+\|y-F_jz,p\|+\|z-F_iy,p\|} + \beta \max\{\|y-F_jz,p\|, \|z-F_iy,p\|\} + \gamma\|z-F_jz,p\|$
$\leq \alpha \frac{\|y-z,p\|+\|y-z,p\|+\|z-y,p\|}{1+\|y-z,p\|+\|z-y,p\|} + \beta \max\{\|y-z,p\|, \|z-y,p\|\} + \gamma\|z-z,p\|$
$\leq (3\alpha+\beta+\gamma)\|y-z,p\|$
implies, $(1-3\alpha-\beta-\gamma)\|y-z,p\| \leq 0$
i.e., $\|y-z,p\| = 0$
i.e., $y = z$.

Hence $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Corollary 3.5. *Let F_1 and F_2 be two self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|F_1x - F_2y, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_2y,p\|+\|y-F_1x,p\|}{1+\|x-F_2y,p\|+\|y-F_1x,p\|} + \beta \max\{\|x-F_2y,p\|, \|y-F_1x,p\|\} + \gamma\|y-F_2y,p\|,$
where α, β, γ are non-negative real numbers and $3\alpha+2\beta+\gamma<1$. Then F_1 and F_2 have a unique common fixed point in X.

Proof. Putting $F_i = F_1$ and $F_j = F_2$ in the **Theorem 3.3** we get the desired result.

Corollary 3.6. *Let F be a self map on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|Fx - Fy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-Fy,p\|+\|y-Fx,p\|}{1+\|x-Fy,p\|+\|y-Fx,p\|} + \beta \max\{\|x-Fy,p\|, \|y-Fx,p\|\} + \gamma\|y-Fy,p\|,$
where α, β, γ are non-negative real numbers and $3\alpha+2\beta+\gamma<1$. Then F have a unique fixed point in X.

Proof. Putting $F_i = F_j = F$ in the **Theorem 3.3** we get the desired result.

Theorem 3.4. *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|F_ix - F_jy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_jy,p\|+\|y-F_ix,p\|}{1+\|x-F_jy,p\|+\|y-F_ix,p\|} + \beta \min\{\|x-F_jy,p\|, \|y-F_ix,p\|\} + \gamma\|y-F_jy,p\|,$
where α, β, γ are non-negative real numbers and $3\alpha+2\beta+\gamma<1$. Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Proof. Since $\min\{\|x-F_jy,p\|, \|y-F_ix,p\|\} \leq \max\{\|x-F_jy,p\|, \|y-F_ix,p\|\}$, the result follows from the **Theorem 3.3**.

Theorem 3.5. *Let $\{F_n\}_{n=1}^{\infty}$ be sequence of self maps on 2-Banach space $(X, \|.,.\|)$ satisfying*

$\|F_ix - F_jy, p\|$
$\leq \alpha \frac{\|x-y,p\|+\|x-F_ix,p\|}{1+\|y-F_ix,p\|} + \beta \max\{\|x-F_jy,p\|, \|y-F_jy,p\|\} + \gamma[\|x-F_ix,p\| + \|y-F_jy,p\|],$
where α, β, γ are non-negative real numbers and $2\alpha+\beta+2\gamma<1$. Then $\{F_n\}_{n=1}^{\infty}$ have a unique common fixed point in X.

Proof. With an initial approximation $y_0 \in X$, construct a sequence $\{y_n\}$ such that $y_{n+1} = F_i y_n;\ n = 0, 1, 2, ...$ for a fixed i. Similarly as previous theorems, assume $y_{n+1} \neq y_n, \forall \ \ n \ \ \in \mathbb{N} \cup \{0\}$.

First of all we show that $\{y_n\}$ is a Cauchy sequence.

Since,

$\|y_{n+1} - y_n, p\| = \|F_i y_n - F_j y_{n-1}, p\|$

$\leq \alpha(\frac{\|y_n - y_{n-1}, p\| + \|y_n - F_i y_n, p\|}{1 + \|y_{n-1} - F_i y_n, p\|}) + \beta \max\{\|y_n - F_j y_{n-1}, p\|, \|y_{n-1} - F_j y_{n-1}, p\|\}$

$+\gamma[\|y_n - F_j y_n, p\| + \|y_{n-1} - F_j y_{n-1}, p\|]$

$\leq \alpha(\|y_n - y_{n-1}, p\| + \|y_n - y_{n+1}, p\|) + \beta \max\{\|y_n - y_n, p\|, \|y_{n-1} - y_n, p\|\} + \gamma[\|y_n - y_{n+1}, p\| + \|y_{n-1} - y_n, p\|]$

$= \alpha\|y_n - y_{n-1}, p\| + \alpha\|y_n - y_{n+1}, p\| + \beta\|y_{n-1} - y_n, p\| + \gamma\|y_n - y_{n+1}, p\| + \gamma\|y_{n-1} - y_n, p\|$

implies, $(1 - \alpha - \gamma)\|y_{n+1} - y_n, p\| \leq (\alpha + \beta + \gamma)\|y_n - y_{n-1}, p\|$

$$\text{i.e., } \|y_{n+1} - y_n, p\| \leq (\frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma})\|y_n - y_{n-1}, p\|$$

$$\leq (\frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma})^2\|y_{n-1} - y_{n-2}, p\|$$

$$\vdots$$

$$\leq (\frac{\alpha + \beta + \gamma}{1 - \alpha - \gamma})^n\|y_1 - y_0, p\|.$$

Taking $\lim_{n\to\infty}$ on the both sides of the above inequality, we get

$\lim_{n\to\infty} \|y_{n+1} - y_n, p\| = 0.$

Now, let $n \geq m \in \mathbb{N}$. Then

$\|y_{n+1} - y_{m+1}, p\|$

$= \|F_i y_n - F_j y_m, p\|$

$\leq \alpha \frac{\|y_n - y_m, p\| + \|y_n - F_i y_n, p\|}{1 + \|y_m - F_i y_n, p\|} + \beta \max\{\|y_n - F_j y_m, p\|, \|y_m - F_j y_m, p\|\} + \gamma[\|y_n - F_i y_n, p\| + \|y_m - F_j y_m, p\|]$

$\leq \alpha(\|y_n - y_m, p\| + \|y_n - y_{n+1}, p\|) + \beta \max\{\|y_n - y_{m+1}, p\|, \|y_m - y_{m+1}, p\|\} + \gamma[\|y_n - y_{n+1}, p\| + \|y_m - y_{m+1}, p\|]$.

Taking limit as $n, m \to \infty$ we get from above

$\lim_{n,m\to\infty} \|y_{n+1} - y_{m+1}, p\|$

$\leq \alpha \lim_{n,m\to\infty} \|y_n - y_m, p\| + \beta \lim_{n,m\to\infty} \|y_n - y_{m+1}, p\| + \gamma.0$

$\leq \alpha \lim_{n,m\to\infty} \|y_n - y_m, p\| + \beta \lim_{n,m\to\infty}[\|y_n - y_m, p\| + \|y_m - y_{m+1}, p\|]$

$= (\alpha + \beta) \lim_{n,m\to\infty} \|y_n - y_m, p\|$

implies, $\lim_{n,m\to\infty} \|y_n - y_m, p\| = 0$

i.e., $\{y_n\}$ is a Cauchy sequence. Since X is complete, there exists an $y \in X$ such that $\lim_{n\to\infty} \|y_n - y, p\| = 0$.

Now we show that y is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

Since

$\|F_i y - y, p\| \le \|F_i y - y_n, p\| + \|y_n - y, p\|$
$= \|F_i y - F_j y_{n-1}, p\| + \|y_n - y, p\|$
$\le \alpha \frac{\|y-y_{n-1},p\|+\|y-F_i y,p\|}{1+\|y_{n-1}-F_i y,p\|} + \beta \max\{\|y - F_j y_{n-1}, p\|, \|y_{n-1} - F_j y_{n-1}, p\|\} + \gamma[\|y - F_i y, p\| + \|y_{n-1} - F_j y_{n-1}, p\|] + \|y_n - y, p\|$
$\le \alpha(\|y - y_{n-1}, p\| + \|y - F_i y, p\|) + \beta \max\{\|y - y_n, p\|, \|y_{n-1} - y_n, p\|\} + \gamma[\|y - F_i y, p\| + \|y_{n-1} - y_n, p\|] + \|y_n - y, p\|$.

Taking $\lim_{n\to\infty}$ on the both sides of above inequality, we get

$\lim_{n\to\infty} \|F_i y - y, p\| \le \alpha\|y - F_i y, p\| + \beta.0 + \gamma\|y - F_i y, p\| + 0$

which implies, $(1 - \alpha - \gamma)\|F_i y - y, p\| \le 0$

i.e., $\|F_i y - y, p\| = 0$

i.e., $F_i y = y$.

Thus y is a common fixed point of $\{F_n\}_{n=1}^{\infty}$.

Let y' be another fixed point of $\{F_n\}_{n=1}^{\infty}$. Then

$\|y - y', p\| = \|F_i y - F_j y', p\|$
$\le \alpha(\frac{\|y-y',p\|+\|y-F_i y,p\|}{1+\|y'-F_i y,p\|}) + \beta \max\{\|y - F_j y', p\|, \|y' - F_j y', p\|\} + \gamma[\|y - F_i y, p\| + \|y' - F_j y', p\|]$
$\le \alpha(\|y-y', p\|+\|y-y, p\|)+\beta \max\{\|y-y', p\|, \|y'-y', p\|\}+\gamma[\|y-y, p\|+\|y'-y', p\|]$
$= \alpha\|y - y', p\| + \beta\|y - y', p\|$

that implies, $(1 - \alpha - \beta)\|y - y', p\| \le 0$ i.e., $\|y - y', p\| = 0$

i.e., $y = y'$.

Hence the result.

Corollary 3.7. *Let F_1 and F_2 be two self maps on 2-Banach space $(X, \|., .\|)$ satisfying*

$\|F_1 x - F_2 y, p\|$
$\le \alpha \frac{\|x-y,p\|+\|x-F_1 x,p\|}{1+\|y-F_1 x,p\|} + \beta \max\{\|x - F_2 y, p\|, \|y - F_2 y, p\|\} + \gamma[\|x - F_1 x, p\| + \|y - F_2 y, p\|]$,

where α, β, γ are non-negative real numbers and $2\alpha + \beta + 2\gamma < 1$. Then F_1 and F_2 have a unique common fixed point in X.

Proof. Putting $F_i = F_1$ and $F_j = F_2$ in the **Theorem 3.5** the result holds.

Corollary 3.8. *Let F be a self map on 2-Banach space $(X, \|., .\|)$ satisfying*

$\|Fx - Fy, p\|$
$\le \alpha \frac{\|x-y,p\|+\|x-Fx,p\|}{1+\|y-Fx,p\|} + \beta \max\{\|x - Fy, p\|, \|y - Fy, p\|\} + \gamma[\|x - Fx, p\| + \|y - Fy, p\|]$,

where α, β, γ are non-negative real numbers and $2\alpha + \beta + 2\gamma < 1$. Then F have a unique fixed point in X.

Proof. Putting $F_i = F_j = F$ in the **Theorem 3.5** the result holds.

The next theorem is the generalization of Saluja [13] theorem 3.1. In that theorem T was a continuous self map on X. We have proved it to a family of self maps without continuity as follows:

Theorem 3.6. *Let X be a 2-Banach space(with $dimX \geq 2$) and $\{T_i\}_{i=1}^{\infty}$ be a family of self maps on X satisfying*

$$\|T_i x - T_j y, a\| \leq h \max\{\|x - y, a\|, \tfrac{\|x - T_i x, a\| + \|y - T_j y, a\|}{2}, \tfrac{\|x - T_j y, a\| + \|y - T_i x, a\|}{2}\},$$

where $0 < h < 1$. Then $\{T_i\}_{i=1}^{\infty}$ have a unique common fixed point in X.

Proof. Let $x_0 \in X$ be arbitrary. Then we construct a sequence $\{x_n\}$ such that $x_{n+1} = T_i x_n$ for a fixed i.

We now show that $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = 0$.

Now,

$$\|x_{n+1} - x_n, a\| = \|T_i x_n - T_{n-1}, a\|$$
$$\leq h \max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n - T_i x_n, a\| + \|x_{n-1} - T_j x_{n-1}, a\|}{2}, \tfrac{\|x_n - T_j x_{n-1}, a\| + \|x_{n-1} - T_i x_n, a\|}{2}\}$$
$$= h \max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n - x_{n+1}, a\| + \|x_{n-1} - x_n, a\|}{2}, \tfrac{\|x_n - x_n, a\| + \|x_{n-1} - x_{n+1}, a\|}{2}\}$$
$$\leq h \max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n - x_{n+1}, a\| + \|x_{n-1} - x_n, a\|}{2}, \tfrac{\|x_{n-1} - x_n, a\| + \|x_n - x_{n+1}, a\|}{2}\}$$
$$= h \max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n - x_{n+1}, a\| + \|x_{n-1} - x_n, a\|}{2}\}$$
$$\leq h \max\{\|x_n - x_{n-1}, a\|, \|x_n - x_{n+1}, a\|\}. \tag{3.3}$$

Suppose $\|x_{n-1} - x_n, a\| \leq \|x_n - x_{n+1}, a\|$.

Then from (3.3), $\|x_{n+1} - x_n, a\| \leq h\|x_{n+1} - x_n, a\|$

implies, $1 \leq h$, a contradiction.

Thus $\|x_{n+1} - x_n, a\| \leq \|x_n - x_{n-1}, a\|$.

Therefore, $\{\|x_{n+1} - x_n, a\|\}$ is a sequence of real numbers monotone decreasing and bounded below. Suppose $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = r$.

Suppose $r \neq 0$. Then,

$$r = \lim_{n\to\infty} \|x_{n+1} - x_n, a\| = \lim_{n\to\infty} \|T_i x_n - T_j x_{n-1}, a\|$$
$$\leq \lim_{n\to\infty} h \max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n - T_i x_n, a\| + \|x_{n-1} - T_j x_{n-1}, a\|}{2}, \tfrac{\|x_n - T_j x_{n-1}, a\| + \|x_{n-1} - T_i x_n, a\|}{2}\}$$
$$= h \lim_{n\to\infty} \max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n - x_{n+1}, a\| + \|x_{n-1} - x_n, a\|}{2}, \tfrac{\|x_n - x_n, a\| + \|x_{n-1} - x_{n+1}, a\|}{2}\}$$
$$\leq h \lim_{n\to\infty} \max\{\|x_n - x_{n-1}, a\|, \tfrac{\|x_n - x_{n+1}, a\| + \|x_{n-1} - x_n, a\|}{2}, \tfrac{\|x_{n-1} - x_n, a\| + \|x_n - x_{n+1}, a\|}{2}\}\}$$
$$= h \lim_{n\to\infty} \max\{r, \tfrac{r+r}{2}, \tfrac{r+r}{2}\} = hr$$

implies, $1 \leq h$, a contradiction.

Therefore, $\lim_{n\to\infty} \|x_{n+1} - x_n, a\| = 0$.

Now we show that $\{x_n\}$ is a Cauchy sequence.

Since for $n > m \in \mathbb{N}$,

$\lim_{n,m\to\infty} \|x_n - x_m, a\|$
$\leq \lim_{n,m\to\infty}[\|x_n - x_{n-1}, a\| + \|x_{n-1} - x_m, a\|]$
$= \lim_{n,m\to\infty} \|x_{n-1} - x_m, a\|$
$\vdots$
$\leq \lim_{n,m\to\infty} \|x_m - x_m, a\|$
$= 0.$
Therefore, $\{x_n\}$ is a Cauchy sequence. Since X is a complete space, there exist a $x \in X$ such that $\lim_{n\to\infty} x_n = x$.

Next, we show that x is a fixed point of $\{T_i\}_{i=1}^{\infty}$.
Since
$\lim_{n\to\infty} \|T_i x - x, a\| \leq \lim_{n\to\infty}[\|T_i x - x_n, a\| + \|x_n - x, a\|]$
$= \lim_{n\to\infty} \|T_i x - T_j x_{n-1}, a\| + \lim_{n\to\infty} \|x_n - x, a\|$
$\leq \lim_{n\to\infty} h \max\{\|x - x_{n-1}, a\|, \frac{\|x-T_i x,a\|+\|x_{n-1}-T_j x_{n-1},a\|}{2}, \frac{\|x-T_j x_{n-1},a\|+\|x_{n-1}-T_i x,a\|}{2}\}$
$= h \lim_{n\to\infty} \max\{\|x - x_{n-1}, a\|, \frac{\|x-T_i x,a\|+\|x_{n-1}-x_n,a\|}{2}, \frac{\|x-x_n,a\|+\|x_{n-1}-T_i x,a\|}{2}\}$
$\leq h\|T_i x - x, a\|$
implies, $\|T_i x - x, a\| \neq 0$,
i.e., $T_i x = x$.
Thus x is fixed point of X.

Now we show that x is a unique common fixed point of $\{T_i\}_{i=1}^{\infty}$. Let y be another common fixed point. Then by the given condition, we get
$\|x - y, a\| = \|T_i x - T_j y, a\|$
$\leq h \max\{\|x - y, a\|, \frac{\|x-T_i x,a\|+\|y-T_j y,a\|}{2}, \frac{\|x-T_j y,a\|+\|y-T_i x,a\|}{2}\}$
$= h \max\{\|x - y, a\|, \frac{\|x-x,a\|+\|y-y,a\|}{2}, \frac{\|x-y,a\|+\|y-x,a\|}{2}\}$
$= h\|x - y, a\|$
implies, $\|x - y, a\| = 0$
i.e., $x = y$.
Thus x is a unique common fixed point of $\{T_i\}_{i=1}^{\infty}$.
Hence the theorem.

Corollary 3.9. *Let X be a 2-Banach space(with $dimX \geq 2$) and T_1 and T_2 be two self maps on X satisfying*
$\|T_1 x - T_2 y, a\| \leq h \max\{\|x - y, a\|, \frac{\|x-T_1 x,a\|+\|y-T_2 y,a\|}{2}, \frac{\|x-T_2 y,a\|+\|y-T_1 x,a\|}{2}\}$,
where $0 < h < 1$. Then T_1 and T_2 have a unique common fixed point in X.
Proof. Putting $T_i = T_1$ and $T_j = T_2$ in the above **Theorem 3.6** we have the required result.

This result is same as Saluja ([13]) theorem 3.1 without continuity.

Corollary 3.10. *Let X be a 2-Banach space(with $dimX \geq 2$) and T be a self maps* *on X satisfying*

$$\|Tx - Ty, a\| \leq h \max\{\|x - y, a\|, \tfrac{\|x-Tx,a\|+\|y-Ty,a\|}{2}, \tfrac{\|x-Ty,a\|+\|y-Tx,a\|}{2}\},$$

where $0 < h < 1$. Then T have a unique fixed point in X.

Proof. Putting $T_i = T_j = T$ in the above **Theorem 3.6** we have the desired result.

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HYDRODYNAMIC LUBRICATION OF SYMMETRIC ROLLERS WITH TWO DIMENSIONAL CONSISTENCY VARIATION OF POWER LAW FLUIDS

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Abstract: Roller bearing is one among the varieties of rotating system incorporating the forces between a system and its environment. It also guides the movement of rotation smoothly, and hence, rolling bearings are significant machine elements pertaining to the lifetime of the system and its accuracy. Here, a thick fluid lubrication of rollers along with normal squeezing motion is considered. The consistency variation of power law lubricant on temperature and pressure is taken into account. A specific model for the lubricant consistency is considered to vary along and across the flow directions. Load, traction, temperature and pressure are calculated for various values of the consistency index n and normal velocity q. These are matched well with the previous results.

Keywords and Phrases: Hydrodynamic lubrication, non-Newtonian power law, Consistency variation, Thermal effects, Normal Squeezing, Cylindrical roller bearings.

2010 Mathematics Subject Classification: 76N02.

1. Introduction

The utility of roller bearing to this modern world is bountiful. Its main function is to transmit the load at very low friction. At normal loads, the bearings do experience normal pressure and temperature. At heavy load and high speed, the bearings experience the Himalayan pressure and temperature. In squeezing motion, this pressure and temperature is even more. Here comes the necessity for the application of squeeze-film technology to optimize the pressure and temperature.

Reynolds equation influenced its power to investigate film thickness, viscosity, and density. These properties are temperature and pressure dependent. Cope in 1949, included some changes to classical Reynolds equation like adding viscosity and density variation with fluid flow. All these changes are included to exhibit energy equation responsible for the temperature in the film. He put together continuity equation and energy balance equation in the fluid film to create temperature and pressure distribution. In this connection, researchers attempted to develop an intelligible model. Dowson successfully studied such cases and concluded that the viscosity of Newtonian fluid does not change much in a lightly loaded system. However, in heavily loaded system, such character of the fluid may not be true [1]. Rong-Tsonn and Hamrock [2], made a detailed study on compressibility and squeezing of isothermal Newtonian lubrication. To serve a single non-dimensional squeeze Reynolds number, Usha and Sridharan [3] examined a laminar squeezing flow of an incompressible Newtonian fluid between plane annuli.

Further Cheng and Sternlicht [4] analysed EHD rolling/sliding line contact problem and obtained solutions for equation of motion and heat transfer equation considering viscosity to be a function of x and y both. Ghosh and Hamrock [5] also solved such problem using finite difference method while analyzing the film shape by subjecting each rectangular area to a uniform pressure. Unlike Hertzian pressure, the temperature there was not found to be continuous. However, in both cases, squeezing effects were ignored.

Primarily lubricants staged a different character at shear stresses and high pressure. Secondly, Newtonian behavior is seldom exhibited in fluids such as molten plastic, slurries and pulps. Besides, the presence of high molecular weight polymers earns them the label of non-Newtonian. Within the general framework of non-Newtonian fluids, the power law model was more attractive. Prasad et al. [6] exposed the consistency variation of power law fluids with temperature and pressure and displayed how the pressure peak is dragged away from the center line of contact by temperature influence. The trend was found exactly opposite to that of cavitated points.

A semi solid substance like grease is one of the most used non Newtonian lubricants. Neurouth et al. [7] proposed a thermal model for grease-lubricated thrust ball bearings. Here, thermal network method was utilized to study the temperature distribution. In addition, the bearings' power loss was calculated numerically, and concluded that the bearing's inner temperature distribution has minimum effect because of heat convection. Ai et al. [8] developed a feasible thermal model for a grease lubricated double-row tapered bearing by capitalizing Ohm's law to describe the behavior of Herschel – Bulkley model. Various heat sources like: frictional heat dissipation between roller's large end, frictional heat of roller race-way contact, roller's viscous drag loss, raceway flange contact, were incorporated within the model. The increase in speed and grease filling ratio increases the bearing's temperature. This is clearly evident from their numerical results. Yan et al.[9] model aimed to measure the roller inner ring flange contact of tapered roller bearing and roller raceway contact used in super speed train. It concluded that inner ring flange receives highest temperature [10]. Viscosity variation because of temperature and pressure was presented by Osterle and Saibel [11]. Cameron [12] analysed the variation of viscosity across the fluid film thickness. Recently, Prasad et al. [13, 14] proposed a solution to asymmetric rollers using power law fluids. However, consistency variation along y direction was not considered.

Further, the deep interaction between the rolling elements and fluid makes the flow complex inside the bearing cavity. This was presented by Gao et al. [15]. Later, Gao et al. [16] tried to demonstrate two phase flow behavior in roller bearings with under-race lubrication.

In continuation, the study of thermo-hydrodynamic lubrication for a heavily loaded rigid cylindrical line contact with incompressible power law fluids is the goal of this study including cavitations. It is assumed that the lubricant consistency varies with pressure and film temperature under thermal and isothermal boundaries. Under the said condition, the heat effect of the fluid is observed for the rolling with and without squeezing motion including lubricant consistency variation across the fluid flow direction. The modified energy and Reynolds equations are obtained and solved yielding temperature and pressure.

2. Nomenclature

α : pressure coefficient
β : temperature coefficient
h : film thickness
h_0 : minimum film thickness
k : thermal conductivity of the lubricant
m : lubricant consistency
m_0 : consistency at ambient pressure and temperature
p : hydrodynamic pressure
q : squeezing parameter
R : radius of cylinder
T : lubricant temperature
T_0 : ambient temperature
T_{Fh} : traction force
u : velocity of the lubricant in x-direction
U : velocity of the lubricant at y=h/2
v : velocity of the lubricant in y-direction
V : normal velocity (V/2) of either cylinder
x_1 : point of maximum pressure
x_2 : cavitation point

Subscripts: The subscript 1 and 2 denote respectively the corresponding quantities in the inlet and the outlet regions.

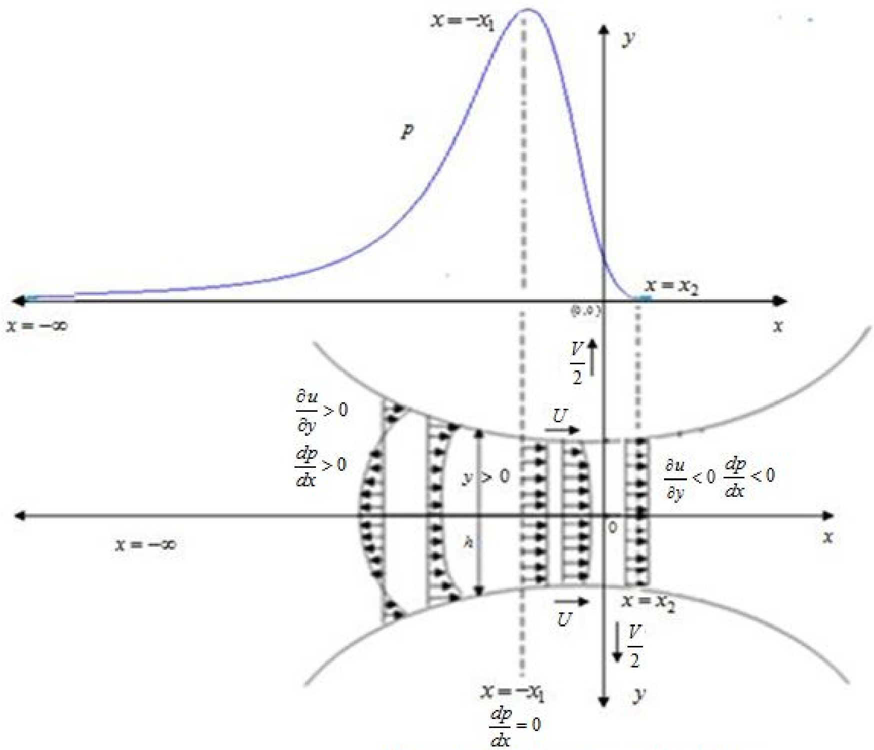

Fig.1: Lubrication of symmetric rollers

3. Mathematical Analysis

The governing equations for the one dimensional fluid flow are [17, 18]

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{1}$$

$$\frac{dp}{dx}=\frac{\partial \tau}{\partial y} \tag{2}$$

$$k\frac{\partial^2 T}{\partial y^2}+\tau\frac{\partial\, u}{\partial\, y}=0 \tag{3}$$

Where k is the thermal conductivity of the lubricant and is assumed to be constant,p is the hydrodynamic pressure, T being temperature due to heat dissipation; u and v are the velocities of the fluids in x and y directions respectively. τ, the shear stress relation for a non-Newtonian power law fluid is:

$$\tau=m\left|\frac{\partial u}{\partial y}\right|^{n-1}\frac{\partial u}{\partial y} \tag{4}$$

Where, the lubricant consistency m of the powerlaw fluid is assumed to vary as

$$m=m_{\,0}\; e^{\alpha\, p\;-\beta\;(T-T_0)} \tag{5}$$

The boundary conditions for the system under consideration are:

$$\frac{\partial u}{\partial y}=0\;at\,y\;=0;u=U at\;y=\frac{h}{2}; \tag{6}$$

The boundary conditions for the heat equation (3) are

$$\frac{\partial T}{\partial y}=0\;at\;y=0;T=T_h\;at\;y=\frac{h}{2};\quad where\;\;h=h_{\,0}+\frac{x^2}{2R} \tag{7}$$

As the consistency m and temperature T being two dimensional, the equations (1) (2) and (3) cannot be solved analytically. Hence, an intuitive mathematical relationship for the consistency m is assumed to be [17]:

$$m=m_0/\left(A_0+A_1y+A_2y^2\right)^{\;n} \tag{8}$$

Where A_0, A_1andA_2 are functions of x only and are unknowns right now.
Integrating equation (1) with respect to ‘y’ with the assumed boundary conditions

(6), one can get for the region $-\infty < x \leq -x_1$

$$u_1 = U + [\frac{1}{m_0}(\frac{dp_1}{dx})]^{\frac{1}{n}}[\frac{nA_0}{n+1}(y^{\frac{n+1}{n}} - (\frac{h}{2})^{\frac{n+1}{n}}) + \frac{nA_1}{2n+1} (y^{\frac{2n+1}{n}} - (\frac{h}{2})^{\frac{2n+1}{n}}) + \frac{nA_2}{3n+1}(y^{\frac{3n+1}{n}} - (\frac{h}{2})^{\frac{3n+1}{n}})] \quad (9)$$

Similarly, for the region $-x_1 \leq x \leq x_2$, one can have

$$u_2 = U - [\frac{1}{m_0}(\frac{-dp_2}{dx})]^{\frac{1}{n}}[\frac{nA_0}{n+1}(y^{\frac{n+1}{n}} - (\frac{h}{2})^{\frac{n+1}{n}}) + \frac{nA_1}{2n+1} (y^{\frac{2n+1}{n}} - (\frac{h}{2})^{\frac{2n+1}{n}}) + \frac{nA_2}{3n+1}(y^{\frac{3n+1}{n}} - (\frac{h}{2})^{\frac{3n+1}{n}})] \quad (10)$$

Integration of the continuity equation (2) with the boundary conditions (6) together with the specified conditions: $v_{h/2} = \frac{U}{2}\frac{dh}{dx} + \frac{V}{2}$; $v_0 = 0$ at y =0 gives

$$\frac{\partial}{\partial x}\int_0^{h/2} u\, dy = -\frac{V}{2} \quad (11)$$

Further integration of the above equation with the assumed conditions: $\frac{dp_1}{dx} = 0$ at $x = -x_1$ and $h = h_1$, one can get

$$\frac{dp_1}{dx} = m_0\left(\frac{2}{h}\right)\left[\left(\frac{2}{h^2}\right)\frac{(U(h-h_1) + V(x+x_1))}{[\frac{nA_0}{2n+1} + \frac{nA_1}{3n+1}(\frac{h}{2}) + \frac{nA_2}{4n+1}(\frac{h}{2})^2)]}\right]^n ; -\infty < x \leq -x_1 \quad (12)$$

Similarly for the other region

$$\frac{dp_2}{dx} = -m_0\left(\frac{2}{h}\right)\left[\left(\frac{2}{h^2}\right)\frac{-(U(h-h_1) + V(x+x_1))}{[\frac{nA_0}{2n+1} + \frac{nA_1}{3n+1}(\frac{h}{2}) + \frac{nA_2}{4n+1}(\frac{h}{2})^2)]}\right]^n ; -x_1 \leq x \leq x_2 \quad (13)$$

The constants, A_0, $A_1$and A_2, as mentioned in equation (8) , are calculated using the heat energy equation (3) with the boundary conditions mentioned in (7); and are obtained as:
$A_0 = 0$, $A_1 = 0$ and $A_2 = (4/h^2)\left(1/\overline{E}\right)^{\frac{1}{n}}$;
Making the above two equations dimensionless, these equations (12) and (13) are reduced to

$$\frac{d\overline{p_1}}{d\overline{x}} = \frac{\overline{m_0}\ \ \overline{E}\ \ \overline{f}^{\,n}}{\overline{h}^{\,2n+1}}; \quad -\infty < \overline{x} \leq -\overline{x_1} \quad (14)$$

$$\frac{d\overline{p_2}}{d\overline{x}} = -\frac{\overline{m_0}\ \overline{E}\ \overline{g}^{\,n}}{\overline{h}^{\,2n+1}}; \quad -\overline{x_1} \le \overline{x} \le \overline{x_2} \tag{15}$$

Where

$$\overline{x} = x/\sqrt{2Rh_0}; \overline{h} = h/h_0;\ \overline{g} = -\overline{f};\ \overline{E} = e^{\overline{p}-\overline{T_h}+\overline{T_0}};$$

$$\overline{p} = \alpha p;\ \overline{m} = 2mc_n\alpha; \overline{f} = \overline{x}^{\,2} - \overline{x_1}^{\,2} + 2q\left(\overline{x} + \overline{x_1}\right);$$

$$q = (V/2U)\sqrt{2R/h_0};\ \ c_n = \left(2\left(4n+1\right)/n\right)^n \sqrt{2R/h_0}\left(U/h_0\right)^n;$$

Solving the energy equation (3) with the boundary condition given in (7), one can get in dimensionless

$$\overline{T_1} = \overline{T_h} - \frac{\overline{m_0}\ \overline{E}\,\overline{f}^{\,n+1}\,\overline{\gamma}\ \bar{l}}{\overline{h}^{\,\frac{2n^2+5n+1}{n}}}; \quad -\infty < \overline{x} \le -\overline{x_1} \tag{16}$$

$$\overline{T_2} = \overline{T_h} - \frac{\overline{m_0}\ \overline{E}\ \overline{g}^{\,n+1}\,\overline{\gamma}\ \bar{l}}{\overline{h}^{\,\frac{2n^2+5n+1}{n}}}; \quad -\overline{x_1} \le \overline{x} \le \overline{x_2} \tag{17}$$

where $\overline{\gamma} = \frac{\beta U h_0}{k\alpha}\sqrt{\frac{h_0}{2R}};\ \overline{T} = \beta T;\ \overline{y} = y/h_0;\ \bar{l} = 2^{\frac{3n+1}{n}}\left(\frac{n}{5n+1}\right)\left(\overline{y}^{\frac{5n+1}{n}} - \left(\frac{\overline{h}}{2}\right)^{\frac{5n+1}{n}}\right);$

Mean Temperature

The mean temperature T_m , defined as $T_m = \frac{2}{h}\int_0^{h/2} T\ dy$ is also calculated using the dimensionless scheme as

$$\overline{T_{m_1}} = \overline{T_h} + \frac{\overline{m_0}\ \overline{E}\,\overline{f}^{\,n+1}\,\overline{\gamma}\ d_n}{\overline{h}^{\,2n}}; \quad -\infty < \overline{x} \le -\overline{x_1} \tag{18}$$

$$\overline{T_{m_2}} = \overline{T_h} + \frac{\overline{m_0}\ \overline{E}\,\overline{g}^{\,n+1}\,\overline{\gamma}\ d_n}{\overline{h}^{\,2n}}; \quad -\overline{x_1} \le \overline{x} \le \overline{x_2} \tag{19}$$

Where $d_n = \frac{n}{4(6n+1)}$;

Load and Traction

The load component in x-direction is given by

$$w_x = -\int_{h_1}^{h_2} p\,dh = -2\int_0^{h} p\,dh = \frac{1}{R}\int_{-\infty}^{x_2} x^2\,\frac{dp}{dx}\,dx \tag{20}$$

The dimensionless load $\overline{w_x} = \frac{W_x\alpha}{2h_0}$ is given by

$$\overline{w_x} = \int_{-\infty}^{\overline{x_2}} \overline{x}^{\,2}\,\frac{d\overline{p}}{d\overline{x}}\,d\overline{x} \tag{21}$$

The load component in y-direction is given by

$$w_y = \int_{-\infty}^{x_2} p\, dx \tag{22}$$

The dimensionless load $\overline{w_y} = \frac{W\alpha}{\sqrt{2Rh_0}}$ is calculated as

$$\overline{w_y} = \int_{-\infty}^{\overline{x_2}} \overline{x}\, \frac{d\overline{p}}{d\overline{x}}\, d\overline{x} \tag{23}$$

The load $\overline{W}$ is calculated by

$$\overline{W} = \sqrt{\overline{w}_x^{\,2} + \overline{w}_y^{\,2}} \tag{24}$$

Traction can be defined as the friction between drive wheel and the surface it moves upon. It is the amount of force a wheel can apply to a surface before it slips.
Hence, the surface traction force $T_F h$, obtained from the integration of shear stress τ over the entire length , may be written as

$$T_{Fh} \;=\; \int_{-\infty}^{x_2} \left(\frac{h}{2}\left(\frac{dp}{dx}\right)\right)\; dx \tag{25}$$

Then, the dimensionless traction may be written as

$$\overline{T_{Fh}} \;=\; \int_{-\infty}^{\overline{x_2}} \overline{h}\; \left(\frac{d\overline{p}}{d\overline{x}}\right)\; d\overline{x}\,; \tag{26}$$

Finally, the consistency expression comes out to be

$$\overline{m} = \frac{\overline{m_0}\;\overline{E}\;\overline{h}^{\;2n}}{4^n\,\overline{y}^{\;2n}} \tag{27}$$

4. Results and Discussion

Thermal effect of fluid film lubrication of identical rollers by power law fluids under rolling and squeezing motions is studied. The lubricant consistency is encouraged to vary exponentially with pressure and the film temperature. The altered Reynolds and energy equations are derived and solved for pressure and temperature. For the numerical calculations the following set of values are used:
R = 0.03m; U = 4 m sec^{-1}; α= 1.6 x$10^{-8}$$pa^{-1}m^2$; h_0= 6 x 10^{-5}m; $\overline{\gamma}$ = 4;
-0.1 < q < 0.1; 0.4 $\leq$ n $\leq$ 1.15 and $0 \leq \overline{T_h} - \overline{T_0} \leq 5$;
The present paper attempts to show the consistency variation $\overline{m}$ of the lubricant

in a steady state thermal lubrication. Pressure $\overline{p}$, temperature $\overline{T}$, mean temperature $\overline{T_m}$ and velocity $\overline{u}$ are marked as functions of the power – law flow behavior index n, and squeezing parameter q. From the graph shown below, it can be interpreted that the variations in $\overline{p}$, $\overline{T_m}$ and $\overline{m}$ with $\overline{x}$ for various values of n and q do not change the general shape of the profile.

4.1. Pressure Distribution

The distribution of pressure $\overline{p}$ against $\overline{x}$ for various values of n for a fixed q, and for various values of q for a fixed n, are shown respectively in Fig. 2 & Fig. 3 and Fig. 4 & Fig. 5 for incompressible fluids in both thermal and iso-thermal cases. From the graph, it is possibly learnt that $\overline{p}$ increases continuously in the input region and decreases in the outlet region [20]. Once the pressure reaches the zenith, $\overline{p}$ falls down, making a steep slope in graph and reduces to the ambient pressure $\overline{p_2}= 0$ at the point of cavitation $\overline{x} = \overline{-x_2}$. The behavior of $\overline{p}$ against $\overline{x}$ for n (fixed) and varying q is similar to that of Dowson et al. [1].

For a fixed value of q, $\overline{p}$ increases significantly with n, especially when n is greater than or equal to 1. It is observed that fixed q value brings both the points of cavitation and maximum pressure nearer to the centre line of contact, (the origin 0) as n increases. Fixed n value increases $\overline{p}$ considerably as q decreases and the cavitation points move slowly towards the centre line of contact as q increases [21]. The change in pressure with respect to q accounts for the observation that as the surfaces approach each other, comparatively more pressure is generated. Maximum pressure with velocity is presented in Fig. 6 and Fig. 7 which are similar to the work done by Yu Chen et al [22].

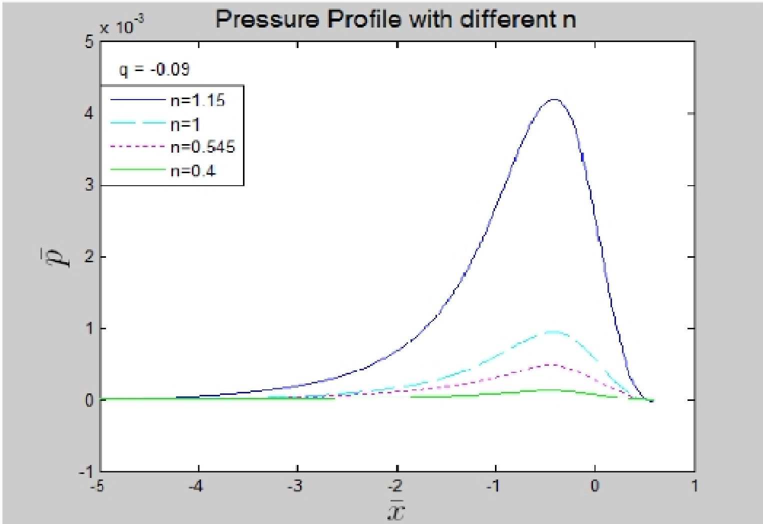

Fig.2: $\bar{p}$ Vs $\bar{x}$

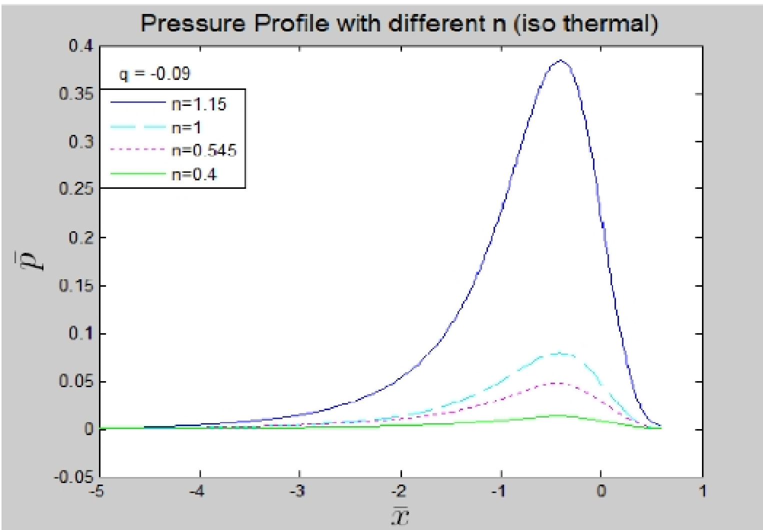

Fig.3: $\bar{p}$ Vs $\bar{x}$

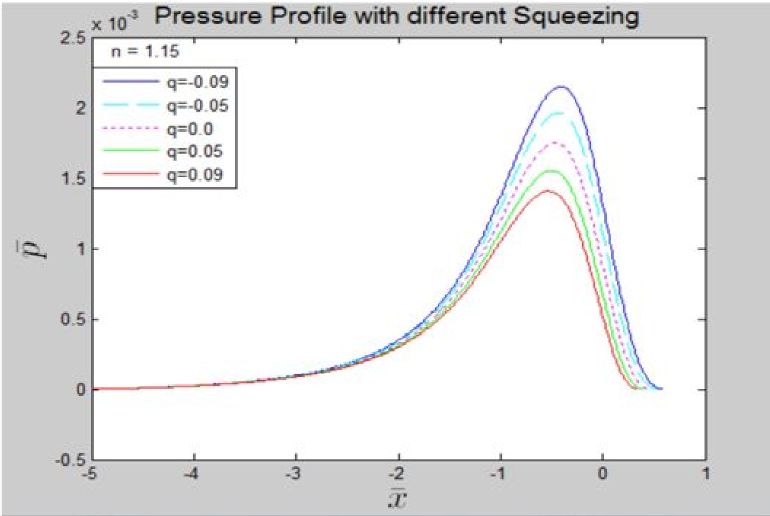

Fig.4: $\bar{p}$ Vs $\bar{x}$

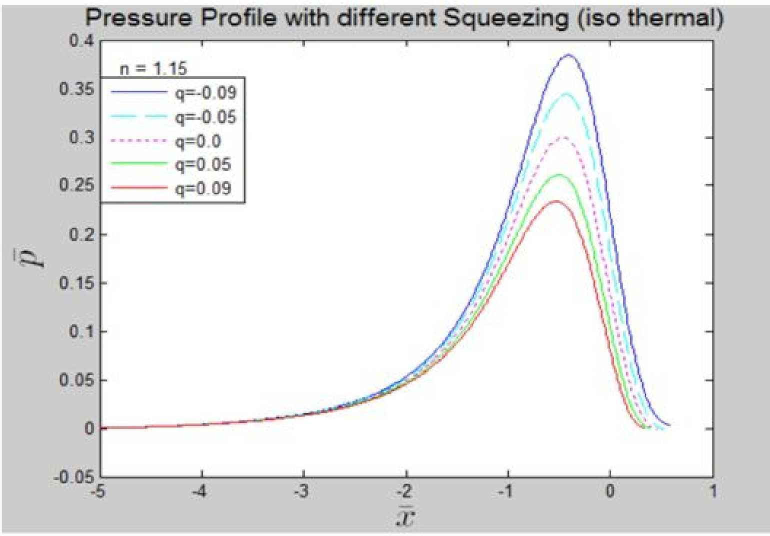

Fig.5: $\bar{p}$ Vs $\bar{x}$

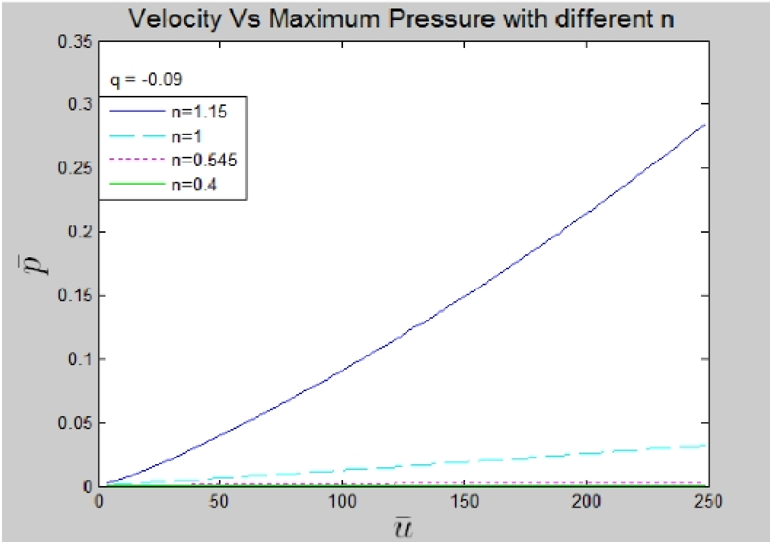

Fig.6: Max $\bar{p}$ Vs $\bar{u}$

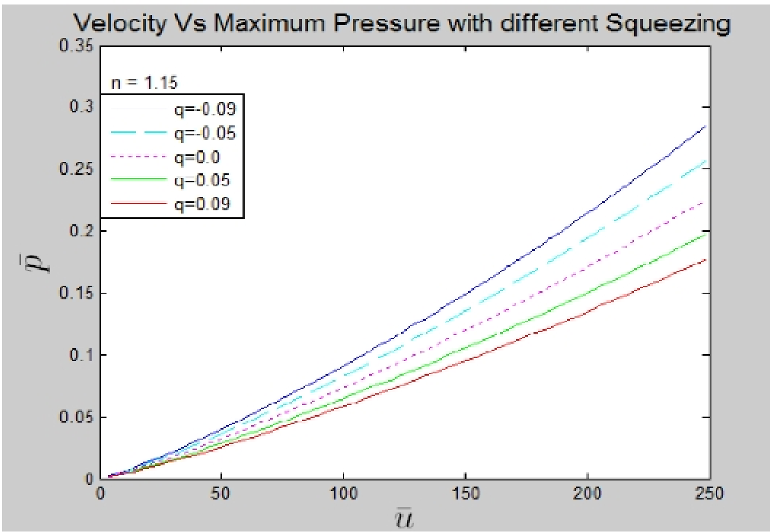

Fig.7: Max $\bar{p}$ Vs $\bar{u}$

4.2. Temperature Distribution

The mean temperature distribution $\overline{T_m}$ according to the n values is shown in Fig. 8 and Fig. 9. It is noteworthy to understand that mean temperature $\overline{T_m}$ increases in inner area. And the increase is ceased at the maximum pressure point, $\overline{x} = \overline{-x_1}$ and then $\overline{T_m}$ comes down slowly to the outer region. The increase $\overline{T_m}$ in the inner area is because of the dragging action of the faster layers in the high-pressure region generates more viscous dissipation in the convergence zone and results in more temperature [23]. From Fig. 8, it is learnt that $\overline{T_m}$ increases with n [21, 24, 25]. An increase in n denotes an enhanced effective viscosity. This increases the resistance to the motion, leading to a higher viscous dissipation.
Similarly, $\overline{T_m}$ increases as q decreases with fixed n. The similar trend was also reported earlier [21]. Mean temperature- velocity graph is given in Fig.10 which is similar to that of Yu Chen et al [22]. A two dimensional temperature $\overline{T}$ profiles for a fixed n and q is given in Fig. 11.

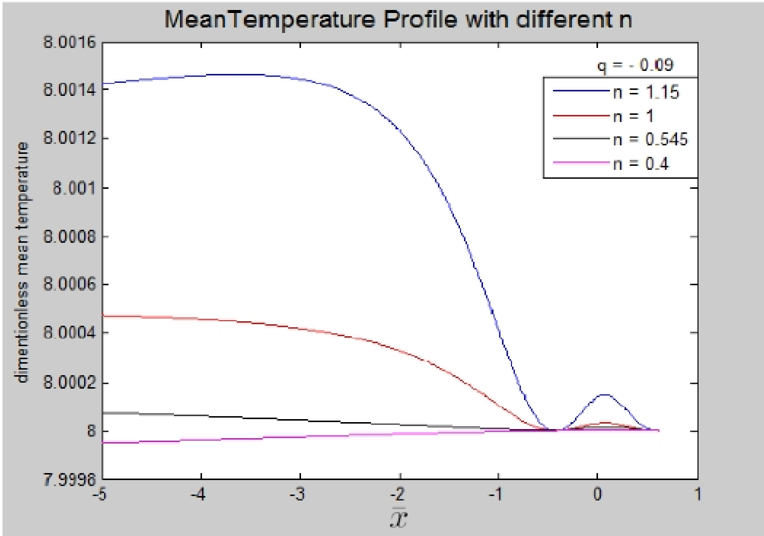

Fig.8: $\bar{T}_m$ Vs $\bar{x}$

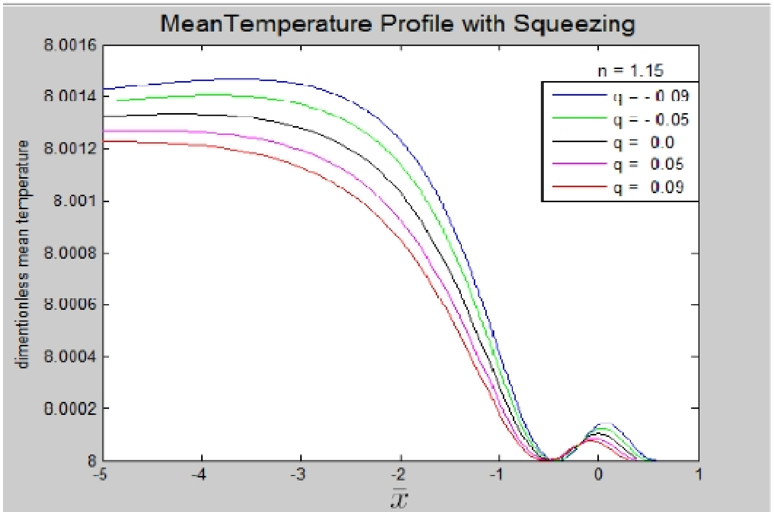

Fig.9: $\bar{T}_m$ Vs $\bar{x}$

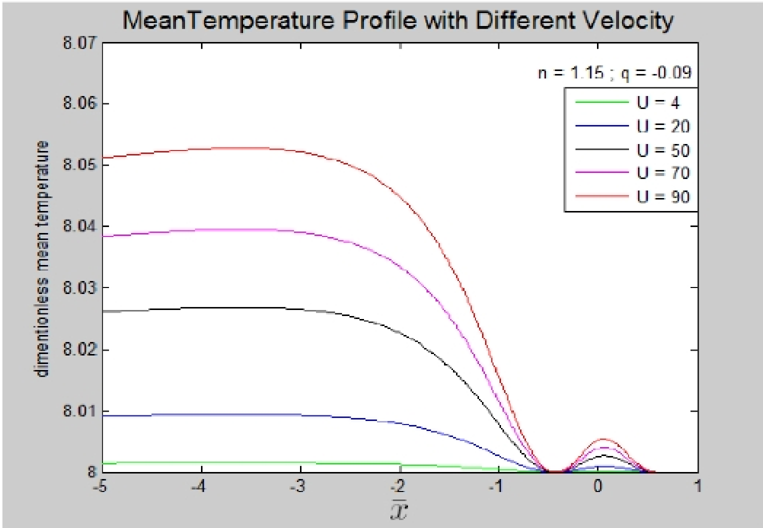

Fig.10: $\overline{T}_m$ Vs $\overline{x}$

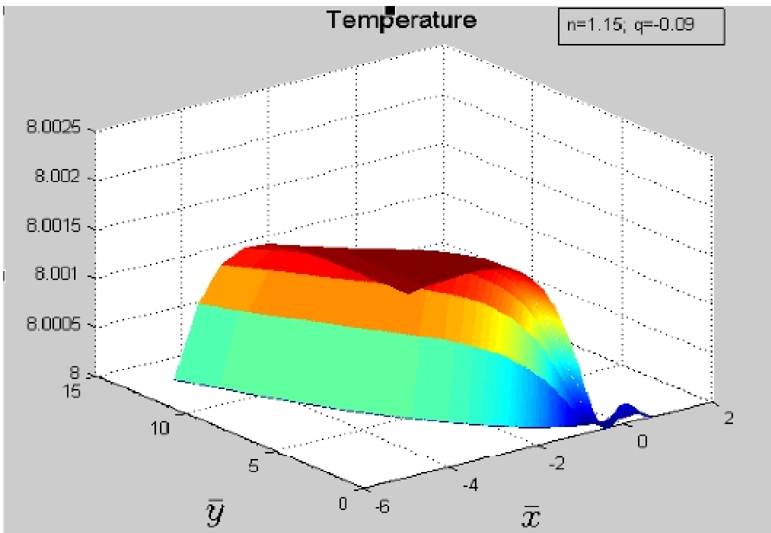

Fig.11: $\overline{T}$ Vs ($\overline{x}$ and $\overline{y}$)

4.3. Load and Traction

Load capacity and traction force are the salient features of bearings. It is presented in Table-1 with different n and q values for thermal conditions only. From the table it can be clearly stated that both the load $\overline{W}$ and the traction force $\overline{T}_{F_h}$, increases with n, which is in accordance with the previous findings [6, 21] for symmetric rollers and [13, 14] for asymmetric rollers, and the coefficient of traction decreases with n. Load versus speed graphs, given in Fig. 12 & Fig. 13, and are similar to the result of Yu Chen et al [22].

Table:1

n / m_0	q=-0.09	q=-0.05	q=0.00	q=0.05	q=0.09
x_1 values					
1.15/0.56	0.418464	0.440518	0.471939	0.503947	0.531958
1.00/0.75	0.420000	0.441891	0.474049	0.505410	0.534398
0.545/86.0	0.424184	0.445567	0.480227	0.508814	0.541901
0.40/128.0	0.427138	0.445961	0.480836	0.508454	0.542826
x_2 values					
1.15/0.56	0.598464	0.540518	0.471939	0.403947	0.351958
1.00/0.75	0.600000	0.541891	0.474049	0.405410	0.354398
0.545/86.0	0.604184	0.545567	0.480227	0.408814	0.361901
0.40/128.0	0.607138	0.545961	0.480836	0.408454	0.362826

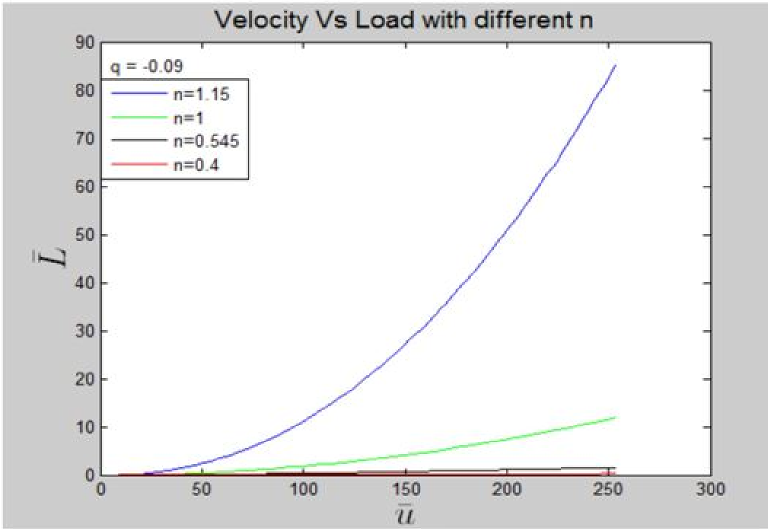

Fig.12: $\bar{L}$ Vs $\bar{u}$

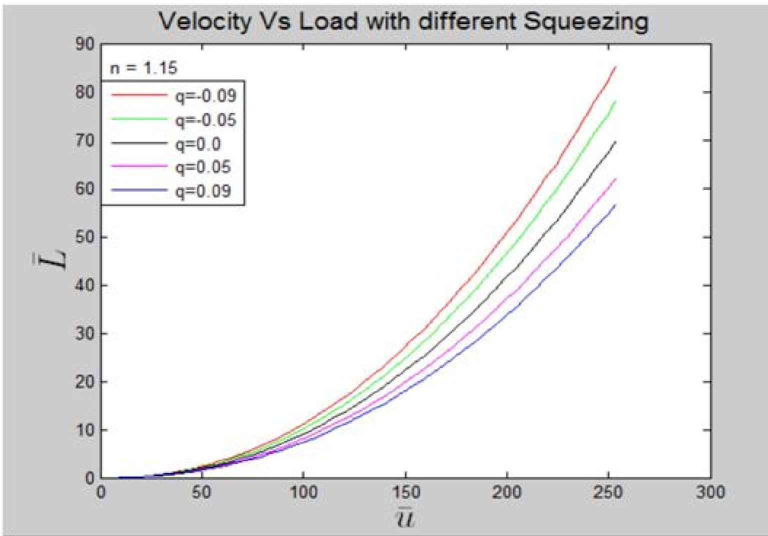

Fig.13: $\bar{L}$ Vs $\bar{u}$

Table: 2

n / m_0	q=-0.09	q=-0.05	q=0.00	q=0.05	q=0.09
Tangential load					
1.15/0.56	0.005246	0.005010	0.004713	0.004425	0.004195
1.00/0.75	0.001315	0.001266	0.001202	0.001140	0.001088
0.545/86.0	0.000970	0.000953	0.000929	0.000904	0.000882
0.40/128.0	0.000291	0.000288	0.000283	0.000277	0.000272
Normal load					
1.15/0.56	0.003119	0.002860	0.002554	0.002277	0.002076
1.00/0.75	0.000758	0.000702	0.000633	0.000571	0.000525
0.545/86.0	0.000503	0.000479	0.000447	0.000416	0.000393
0.40/128.0	0.000146	0.000140	0.000132	0.000124	0.000118
Load					
1.15/0.56	0.006103	0.005769	0.005360	0.004977	0.004681
1.00/0.75	0.001518	0.001447	0.001358	0.001275	0.001208
0.545/86.0	0.001092	0.001066	0.001031	0.000995	0.000965
0.40/128.0	0.000326	0.000320	0.000312	0.000304	0.000297
Traction					
1.15/0.56	0.005221	0.005001	0.004694	0.004427	0.004188
1.00/0.75	0.001310	0.001264	0.001196	0.001141	0.001084
0.545/86.0	0.000969	0.000955	0.000932	0.000908	0.000880
0.40/128.0	0.000290	0.000289	0.000286	0.000279	0.000273
Co-efficient of Traction					
1.15/0.56	0.855425	0.866792	0.875809	0.889533	0.894670
1.00/0.75	0.863184	0.873468	0.880952	0.895066	0.897378
0.545/86.0	0.887202	0.896060	0.903870	0.911924	0.911129
0.40/128.0	0.890755	0.903815	0.914467	0.916847	0.918300

4.4. Consistency

The main feature of this article is to study two-dimensional change in the consistency $(\overline{m})$ of the power law fluids with pressure and the two-dimensional temperature as shown in the below figures. The overall consistency changes with $\overline{x}$ at different positions of the lubricant heights above the x-axis is shown in Fig. 14: This indicates basically the dominance of temperature over the pressure (refer the equation number (27)). Further, $(\overline{m})$ changes enormously with $^{-}$y. This shows that changes much near the symmetric line of the two cylinders and least near the moving surfaces. The consistency versus speed curve is given in Fig.15, and is similar to the result of Espejel [23]. A one dimensional consistency variation in $(\overline{m})$ with $(\overline{x})$ for different flow index n is given in Fig. 16, and a two dimensional variation is shown in Fig. 17. Hence, the consideration of the consistency variation with pressure and temperature is well justified.

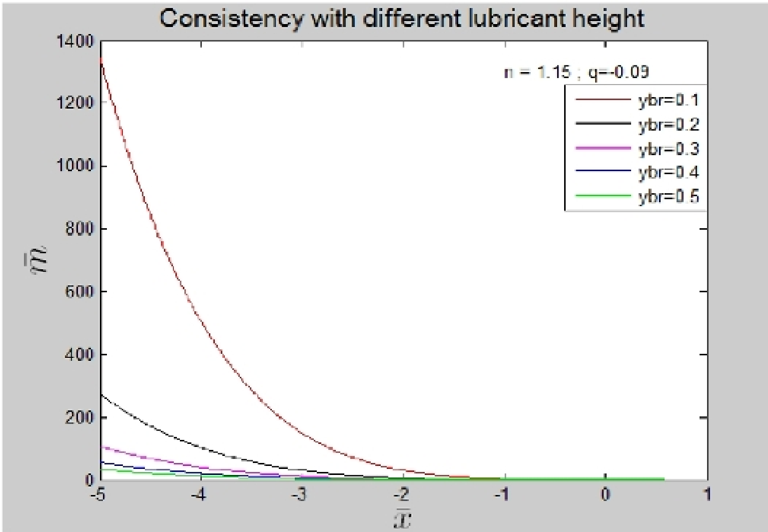

Fig.14: $\overline{m}$ Vs $\overline{x}$

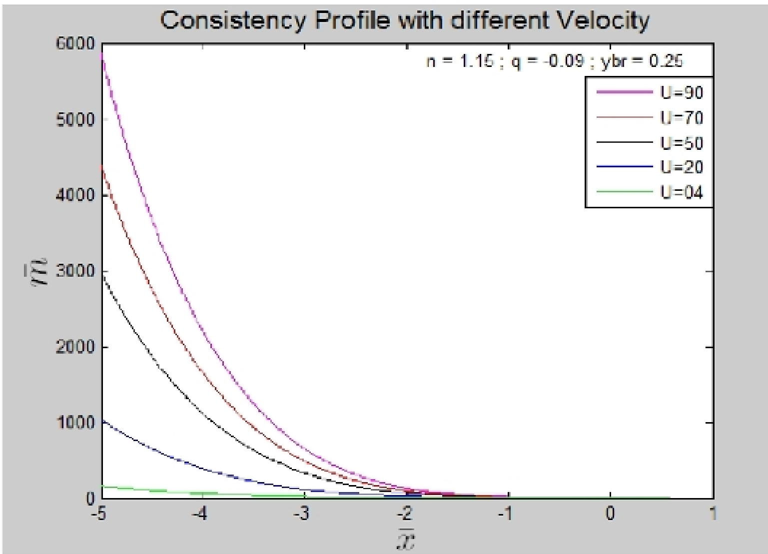

Fig.15: $\overline{m}$ Vs $\overline{x}$

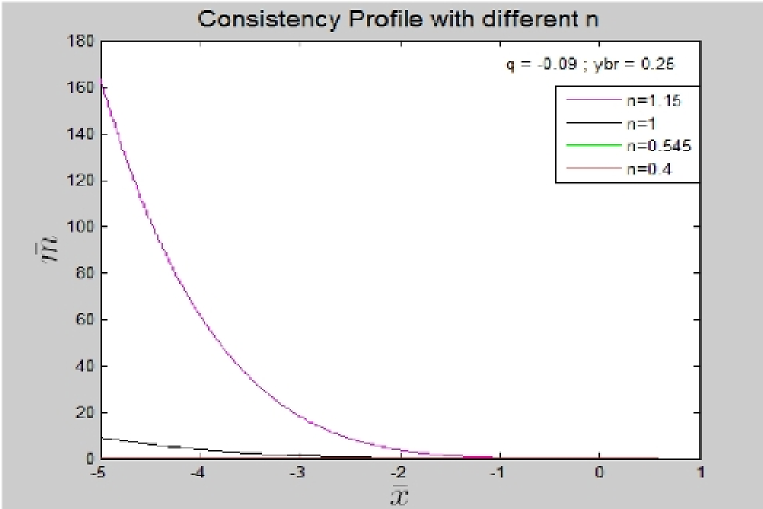

Fig.16: $\overline{m}$ Vs $\overline{x}$

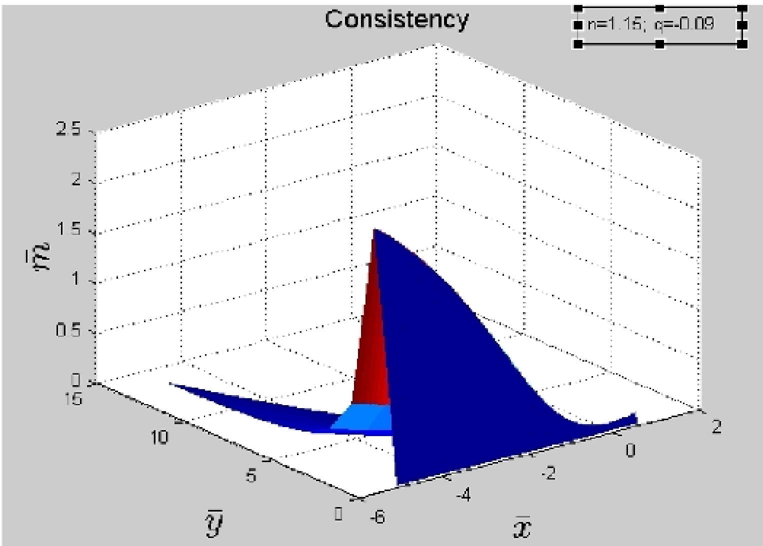

Fig.17: $\overline{m}$ Vs ($\overline{x}$ and $\overline{y}$)

5. Conclusion

To study the consistency variation $\overline{m}$ of the power law fluids, a two dimensional temperature $\overline{T}$ is incorporated to the flow index n and the squeezing parameters q. To analyze the effects of pressure and temperature on the consistency $\overline{m}$ of the fluids, a descriptive method is adopted. Loads and tractions are also studied along with consistency variation. It is found that the pressure and the corresponding temperature increase with n and decrease as q increases numerically. The point of pressure peak moves away from the centre line of contact as q increases. The same trend of pressure with q and n is seen with load and traction. A two-dimensional temperature profile and consistency variation curves are also drawn for some fixed values of n and q which are expectedly nowhere found in the literature. These are the following important observations:

* $\overline{T}, \overline{p}$, n and q are incorporated to observe the consistency variation $\overline{m}$ of the power law fluids.
* Descriptive/ semi-analytical method is adopted for the study of the effects of pressure and temperature on consistency.
* Besides consistency variation, load and traction are also reviewed.
* It is observed that the temperature and pressure upsurge with n and downfalls as q rises numerically. As q increases, the pressure point sidelines from the center line of contact.
* Load and traction exhibit a similar pattern of pressure with q and n.
* For fixed values of n and q, two dimensional consistency variation along with temperature curves are drawn.
* The results are in natural agreement with the previous findings.

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EXISTENCE AND UNIQUENESS SOLUTIONS OF FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH INFINITE POINT CONDITIONS

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Abstract: In this article, we prove the existence of solutions of fractional integro-differential equations with infinite point conditions by using fractional calculus and fixed point theorems. Further continuous dependence on initial point, on nonlocal data, on the functional is also studied. Finally, the obtained results are verified with the help of some examples.

Keywords and Phrases: Functional-differential equations with fractional derivatives, Nonlinear differential equations in abstract spaces, Initial value problems, Fixed point theorems.

2010 Mathematics Subject Classification: 34K37, 34G60, 34A12, 47H10.

1. Introduction

The subject of fractional calculus and fractional differential equations is a rapidly growing area of mathematics. There are many applications of this subject in many field such as engineering, viscoelasticity, economics and biological sciences. There are many remarkable research articles in which theory regarding the existence and uniqueness of solutions established. One can see the research articles [3, 5, 11, 13, 15] for more details. The basic theory of fractional calculus and fractional differential can be found in many books like [4, 6, 9, 16, 19, 30, 32]. In the literature, it has been seen that functional integral and fractional differential equations are closely related. For detailed work one can see the references [8, 16, 26, 27]. Fixed point theory is a great tool to study the existence and uniqueness of solutions of fractional differential equations. Theory and applications of fixed point theory can be found in [1, 7, 14, 29] and the references therein. For some interesting recent work one can see the research articles [18, 20, 21, 22, 23, 26, 27, 28].

Very recently Al-Syed and Ahmad [2] discussed the existence of solutions for the following initial value problems of the functional integro-differential equation

$$\frac{du}{dt} = h(t, u(t), \int_0^t g(s, u(s))ds),$$

with nonlocal condition

$$u(0) + \sum_{i=1}^{n} q_i u(\sigma_i) = u_0, \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T].$$

Motivated by this work we study fractional case of the above work and consider the following functional fractional integro-differential equations

$$^C D^p u(t) = h(t, u(t), \int_0^t g(s, u(s))ds) \tag{1.1}$$

with nonlocal condition

$$u(0) + \sum_{i=1}^{n} q_i u(\sigma_i) = u_0, \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T]. \tag{1.2}$$

Where $^C D^p$ denotes the Caputo fractional derivative of order $p \in (0, 1]$, $t \in J = [0, T]$, $u : J \to X$, $C[J, X]$ denote the Banach space of all continuous functions from J to X with the norm $\|u\| = \sup_{t \in J} |u(t)|, h : J \times X \times X \to X; g : J \times X \to X$ are given functions. We will prove the existence and uniqueness of solution $u \in C[J, X]$, under certain conditions. Where X is the Banach space with the norm $\| \cdot \|$. Also we will study the continuous dependence of the solution on u_0, on the nonlocal-data q_j and on the functional g.

For application point of view, we also study the initial value problem (1.1)-(1.2) if $\sum_{i=1}^{n} q_i$ is convergent.

2. Preliminaries

Definition 2.1. *The fractional integral operator (in Riemann-Liouville sense) of order $p > 0$ of the function u is defined as*

$$I^p u(t) = \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} u(s) ds,$$

where $\Gamma(.)$ denotes the Euler gamma function.

Definition 2.2. *We define the fractional derivative of u of order $p > 0$ in Caputo sense as*

$$^C D^p u(t) = \frac{1}{\Gamma(1-p)} \int_0^t (t-s)^{-p} u'(s) ds, \tag{2.1}$$

where $0 < p \leq 1$ and $u'(s) = \frac{du(s)}{ds}$.
Consider the initial value problem (1.1)-(1.2) with the following assumptions

H_1. Let $h : J \times X \times X \to X$ satisfies the Carathedory condition. There exist a function $\phi \in L^1[0,T]$ and a positive constant $k_1 > 0$, such that

$$|h(t,x,y)| \leq \phi(t) + k_1|x| + k_1|y|.$$

H_2. Let $g : J \times X \to X$ satisfies the Carathedory condition. There exist a function $\psi \in L^1[0,T]$ and a positive constant $k_2 > 0$, such that

$$|g(t,y)| \leq \psi(t) + k_2|y|.$$

H_3. $\sup\limits_{\sigma_i \in [0,1]} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \phi(s) ds \leq M_1$, $\sup\limits_{\sigma_i \in [0,1]} \int_0^{\sigma_i} \int_0^{s} (\sigma_i - s)^{p-1} \psi(\theta) d\theta ds \leq M_2$.

H_4. $\frac{1}{\Gamma(p+1)} \left(1 + E\sum_{i=1}^{n} q_i\right)\left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right) < 1$, where $E = (1 + \sum_{i=1}^{n} q_i)^{-1}$.

Definition 2.3. *A function $u \in C[J,X]$ is said to be the solution of the initial value problem (1.1)-(1.2) if it satisfies the equations (1.1)-(1.2).*

Lemma 2.4. *The solution of initial value problem (1.1)-(1.2) can be represented* *by the following integral equation*

$$\begin{aligned} u(t) = \quad & E\big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\big] \\ & + \frac{1}{\Gamma p} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds, \quad (2.2) \end{aligned}$$

where $E = (1 + \sum_{i=1}^{n} q_i)^{-1}$.

Proof. Let u be a solution of the fractional initial value problem (1.1)-(1.2). Applying Riemann-Liouville operator on both sides of (1.1). We get

$$u(t) = u(0) + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds. \qquad (2.3)$$

Using the nonlocal condition (1.2), we get

$$\sum_{i=1}^{n} q_i u(\sigma_i) = u_0 \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds,$$

since, $\sum_{i=1}^{n} q_i u(\sigma_i) = u_0 - u(0)$, we get

$$u_0 - u(0) = u_0 \sum_{i=1}^{n} q_i + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds,$$

which gives

$$u(0) = \frac{1}{1 + \sum_{i=1}^{n} q_i} \big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\big]. \qquad (2.4)$$

Using (2.3) and (2.4), we obtain

$$\begin{aligned} u(t) \quad = \quad & \frac{1}{1 + \sum_{i=1}^{n} q_i} \big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\big] \\ & + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds. \end{aligned}$$

3. Existence of Solution

Theorem 3.1. *Let the assumptions* $H_1 - H_4$ *are satisfied. Then initial value* *problem (1.1)-(1.2) has at least one solution $u \in C[J, X]$.*

Proof. Define the operator associated with the integral equation (2.2)

$$
\begin{aligned}
Fu(t) &= E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Big] \\
&\quad + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds.
\end{aligned}
$$

Let $Q_r = \{u \in \mathbb{R} : \|u\| \le r\}$, where $r = \frac{E|u_0| + \frac{1}{\Gamma(p+1)}\left(1+E\sum_{i=1}^n q_i\right)\left(M_1 + k_1 M_2\right)}{1 - \left[\frac{1}{\Gamma(p+1)}\left(1+E\sum_{i=1}^n q_i\right)\left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right)\right]}$, it is clear that Q_r is nonempty, closed, bounded and convex subset of $C[0, T]$. Then we have, for $u \in Q_r$

$$
\begin{aligned}
|Fu(t)| &\le E\Big[|u_0| + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} |h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)|ds\Big] \\
&\quad + \frac{1}{\Gamma p} \int_0^t (t-s)^{p-1} |h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)|ds \\
&\le E\Big[|u_0| + \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Big(\phi(s) + k_1|u(s)| + k_1 \int_0^s |g(\theta, u(\theta))|d\theta\Big)ds\Big] \\
&\quad + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big(\phi(s) + k_1|u(s)| + \int_0^s |g(\theta, u(\theta))|d\theta\Big)ds \\
&\le E\Big[|u_0| \\
&\quad + \sum_{i=1}^{n} q_i \Big(\frac{M_1}{\Gamma(p+1)} + \frac{k_1 T^p r}{\Gamma(p+1)} + \frac{k_1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Big(\int_0^s (\psi(\theta) + k_2|u(\theta)|)d\theta\Big)ds\Big)\Big] \\
&\quad + \frac{M_1}{\Gamma(p+1)} + \frac{k_1 T^p r}{\Gamma(p+1)} + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Big(\int_0^s (\psi(\theta) + k_2|u(\theta)|)d\theta\Big)ds \\
&\le E|u_0| + \frac{E}{\Gamma(p+1)} \sum_{i=1}^{n} q_i \Big(M_1 + k_1 T^p r + k_1 M_2 + \frac{k_1 k_2 T^{p+1} r}{p+1}\Big) \\
&\quad + \frac{1}{\Gamma(p+1)} \Big(M_1 + k_1 T^p r + k_1 M_2 + \frac{k_1 k_2 T^{p+1} r}{p+1}\Big) \\
&\le E|u_0| + \frac{1}{\Gamma(p+1)} \Big(1 + E\sum_{i=1}^{n} q_i\Big) \Big(M_1 + k_1 T^p r + k_1 M_2 + \frac{k_1 k_2 T^{p+1} r}{p+1}\Big) = r.
\end{aligned}
$$

Then $F : Q_r \to Q_r$ and the class of functions $\{Fu\}$ is uniformly bounded in Q_r.

Now, let $t_1, t_2 \in (0,1]$ such that $|t_2 - t_1| < \delta$, then

$$\begin{aligned}
|Fu(t_2) - Fu(t_1)| &= \frac{1}{\Gamma(p)}\bigg| \int_0^{t_2} (t_2 - s)^{p-1} h\bigg(s, u(s), \int_0^s g(\theta, u(\theta))d\theta\bigg)ds \\
&\quad - \int_0^{t_1} (t_1 - s)^{p-1} h\bigg(s, u(s), \int_0^s g(\theta, u(\theta))d\theta\bigg)ds\bigg| \\
&\le \frac{1}{\Gamma(p)} \int_0^{t_1} \bigg|\big[(t_2 - s)^{p-1} - (t_1 - s)^{p-1}\big] h\bigg(s, u(s), \int_0^s g(\theta, u(\theta))d\theta\bigg)\bigg| ds \\
&\quad + \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} \bigg|(t_2 - s)^{p-1} h\bigg(s, u(s), \int_0^s g(\theta, u(\theta))d\theta\bigg)\bigg| ds \\
&\le \frac{1}{\Gamma(p)} \int_0^{t_1} \big|(t_2 - s)^{p-1} - (t_1 - s)^{p-1}\big| \\
&\quad \times \bigg(\phi(s) + k_1|u(s)| + k_1 \int_0^s |g(\theta, u(\theta))|d\theta\bigg) ds \\
&\quad + \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} \big|(t_2 - s)^{p-1}\big| \bigg(\phi(s) + k_1|u(s)| + k_1 \int_0^s |g(\theta, u(\theta))|d\theta\bigg) ds \\
&\le \frac{1}{\Gamma(p)} \int_0^{t_1} \big[(t_2 - s)^{p-1} - (t_1 - s)^{p-1}\big] \phi(s) ds \\
&\quad + \frac{k_1 r}{\Gamma(p)} \int_0^{t_1} \big[(t_2 - s)^{p-1} - (t_1 - s)^{p-1}\big] ds \\
&\quad + \frac{k_1}{\Gamma(p)} \int_0^{t_1} \big[(t_2 - s)^{p-1} - (t_1 - s)^{p-1}\big] \int_0^s \big(\psi(\theta) + k_2|u(\theta)|\big) d\theta ds \\
&\quad + \frac{1}{\Gamma(p)} \int_{t_1}^{t_2} \big[(t_2 - s)^{p-1}\big] \phi(s) ds \\
&\quad + \frac{k_1 r}{\Gamma(p)} \int_{t_1}^{t_2} \big[(t_2 - s)^{p-1}\big] ds \\
&\quad + \frac{k_1}{\Gamma(p)} \int_{t_1}^{t_2} \big[(t_2 - s)^{p-1} \int_0^s \big(\psi(\theta) + k_2|u(\theta)|\big) d\theta ds.
\end{aligned}$$

We see that $(t-s)^{p-1} \in L^{\frac{1}{1-p_1}}[0,t]$ for $t \in [0,T]$ and $p_1 \in [0,p)$. Let $d = \frac{p-1}{1-p_1}$, $N_1 = \|\phi(s)\|_{L^{\frac{1}{1-p_1}}[0,T]}$ and $N_2 = \| \int_0^s \big(\psi(\theta) + k_2|u(\theta)|\big)d\theta\|_{L^{\frac{1}{1-p_1}}[0,T]}$.
Now we apply the Hölder inequality [32]

$$|Fu(t_2) - Fu(t_1)| \le \frac{1}{\Gamma(p)}\bigg(\int_0^{t_1} \big((t_2 - s)^{p-1} - (t_1 - s)^{p-1}\big)^{\frac{1}{1-p_1}} ds\bigg)^{1-p_1} \|\phi(s)\|_{L^{\frac{1}{1-p_1}}[0,t]}$$

$$
\begin{aligned}
&+\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p-t_2^p+t_1^p\big)\\
&+\frac{k_1}{\Gamma(p)}\left(\int_0^{t_1}\big((t_2-s)^{p-1}-(t_1-s)^{p-1}\big)^{\frac{1}{1-p_1}}ds\right)^{1-p_1}\\
&\times\|\int_0^s\big(\psi(\theta)+k_2|u(\theta)|\big)d\theta\|_{L^{\frac{1}{1-p_1}}[0,t]}\\
&+\frac{1}{\Gamma(p)}\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{p-1}{1-p_1}}ds\right)^{1-p_1}\|\phi(s)\|_{L^{\frac{1}{1-p_1}}[0,T]}-\frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p\\
&+\frac{k_1}{\Gamma(p)}\left(\int_{t_1}^{t_2}(t_2-s)^{\frac{p-1}{1-p_1}}ds\right)^{1-p_1}\|\int_0^s\big(\psi(\theta)+k_2|u(\theta)|\big)d\theta\|_{L^{\frac{1}{1-p_1}}[0,t]}\\
\leq\;&\frac{N_1}{\Gamma(p)}\left(\int_0^{t_1}\big((t_2-s)^d-(t_1-s)^d\big)ds\right)^{1-p_1}\\
&+\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p-t_2^p+t_1^p\big)\\
&+\frac{k_1 N_2}{\Gamma(p)}\left(\int_0^{t_1}\big((t_2-s)^d-(t_1-s)^d\big)ds\right)^{1-p_1}\\
&-\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}-\frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p\\
&-\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}\\
\leq\;&\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}\Big((t_1)^{1+d}-(t_2)^{1+d}+(t_2-t_1)^{1+d}\Big)^{1-p_1}\\
&+\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p-t_2^p+t_1^p\big)\\
&+\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}\Big((t_1)^{1+b}-(t_2)^{1+b}+(t_2-t_1)^{1+b}\Big)^{1-p_1}\\
&-\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}-\frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p\\
&-\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}\\
\leq\;&\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{k_1 r}{\Gamma(p+1)}\big((t_2-t_1)^p - t_2^p + t_1^p\big)\\
&+\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}\\
&-\frac{N_1}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)} - \frac{k_1 r}{\Gamma(p+1)}(t_2-t_1)^p\\
&-\frac{k_1 N_2}{\Gamma(p)(1+d)^{1-p_1}}(t_2-t_1)^{(1+d)(1-p_1)}.
\end{aligned}
$$

Which shows that the class of functions $\{Fu\}$ is equi-continuous in Q_r.
Let $u_n \in Q_r$ such that $u_n \to u$ as $n \to \infty$. Then by the assumption $H_1 - H_2$, it is clear that $h(t, u_n(t), v_n(t)) \to h(t, u(t), v(t))$ and $g(t, u_n(t)) \to g(t, u(t))$. Also

$$
\begin{aligned}
\lim_{n\to\infty} Fu_n(t) = \quad &\lim_{n\to\infty}\Big[E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Big]\\
&+\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} g(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Big].
\end{aligned}
\tag{3.1}
$$

By using assumption $H_1 - H_2$ and Lebesgue Dominated convergence Theorem [12], from (3.1) we obtain

$$
\begin{aligned}
\lim_{n\to\infty} Fu_n(t) \;=\; &\lim_{n\to\infty}\Big[E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Big]\\
&+\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\Big] = Fu(t).
\end{aligned}
$$

Which shows that $Fu_n \to Fu$ as $n \to \infty$. Therefore F is continuous.

$$
\lim_{t\to 0} u(t) = E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Big] \in C[0, T].
$$

Then by Schauder fixed point Theorem [1] there exist at least one solution $u \in C[J, X]$ of the integral equation (2.2).

4. Infinite-Point Boundary Condition

Theorem 4.1. *Let assumption $H_1 - H_4$ are satisfied and*

$$
M = M_1 + \frac{k_1\|u\|}{p} + k_2 M_2 + \frac{k_1 k_2 \|u\|}{p(p+1)}.
$$

Then the initial value problem (1.1)-(1.2) has at least one solution $u \in C[J, X]$.
Proof. Let the assumptions of Theorem 3.1 be satisfied. Let $t_n, t_n = \sum_{i=1}^{n} q_i$ be convergent sequence, then

$$u_n(t) = \frac{1}{1+\sum_{i=1}^{n} q_i}\left[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\right] + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds. \tag{4.1}$$

Taking the limit to (4.1), as $n \to \infty$, we have

$$\begin{aligned}\lim_{n\to\infty} u_n(t) &= \lim_{n\to\infty}\left[\frac{1}{1+\sum_{i=1}^{n} q_i}\left[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s),\right.\right.\\ &\left.\left.\int_0^s g(\theta, u(\theta))d\theta)ds\right] + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds\right]\\ &= \lim_{n\to\infty} \frac{1}{1+\sum_{i=1}^{n} q_i}\left[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s),\right.\\ &\left.\int_0^s g(\theta, u(\theta))d\theta)ds\right] + \lim_{n\to\infty} \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u_n(s), \int_0^s g(\theta, u_n(\theta))d\theta)ds.\end{aligned} \tag{4.2}$$

Now $|q_i u(\sigma_i)| \leq |q_i| \|u\|$, therefore by the comparison test $\sum_{i=1}^{\infty} q_i u(\sigma_i)$ is convergent. Also

$$\begin{aligned}\left|\int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\right| &\leq \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \big(\phi(s) + k_1|u(s)|\\ &\quad + k_1 \int_0^s g(\theta, u(\theta)d\theta)\big)ds\\ &\leq \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \big(\phi(s) + k_1|u(s)|\\ &\quad + k_2 \int_0^s (\psi(s) + k_2|u(s)|)d\theta\big)ds\\ &\leq M_1 + \frac{k_1\|u\|}{p} + k_2 M_2 + \frac{k_1 k_2 \|u\|}{p(p+1)}\\ &\leq M,\end{aligned}$$

then $\left|q_i \int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\right| \leq |q_i|.M$ and by the comparison test $\sum_{i=1}^{\infty} q_i \int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds$ is convergent.
Now, using assumption $H_1 - H_2$ and Lebesgue Dominated convergence Theorem [12], from (4.2) we obtain

$$u(t) = \frac{1}{1+\sum_{i=1}^{\infty} q_i}\left[u_0 - \sum_{i=1}^{\infty} q_i \frac{1}{\Gamma(p)}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds\right] + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}h(s,u(s),\int_0^s g(\theta,u(\theta))d\theta)ds. \tag{4.3}$$

Hence, the theorem is proved.

5. Uniqueness of Solution

Consider the following assumptions
H_5. Let $h : J \times X \times X \to X$ is measurable in t for any $x, y \in X$ and satisfies the Lipschitz condition

$$|h(t,x,y) - h(t,u,v)| \leq k_1|x-u| + k_1|y-v|, \tag{5.1}$$

H_6. Let $g : J \times X \to X$ is measurable in t for any $x \in X$ and satisfies the Lipschitz condition

$$|g(t,x) - g(t,u)| \leq k_2|x-u|, \tag{5.2}$$

H_7. Let there exists constants L_1 and L_2 such that

$$\sup_{\sigma_i\in[0,T]}\int_0^{\sigma_i}(\sigma_i - s)^{p-1}|g(s,0,0)|ds \leq L_1, \ \sup_{\sigma_i\in[0,T]}\int_0^{\sigma_i}\int_0^s(\sigma_i - s)^{p-1}|h(s,0)|d\theta ds \leq L_2,$$

Theorem 5.1. *Let the assumptions H_5-H_7 are satisfied. Then the initial value problem (1.1)-(1.2) has a unique solution.*
Proof. From assumption H_5 we have h is measurable in t for any $u, v \in \mathbb{R}$ and satisfies the lipschitz condition, then it is continuous for $x, y \in \mathbb{R}, \forall t \in [0,T]$, and

$$|h(t,x,y)| \leq k_1|x| + k_1|y| + |g(t,0,0)|.$$

Which shows that assumption H_1 is satisfied. In a similar way, we can show that assumption H_2 is also satisfied with the help of assumption H_6. Therefore Theorem

3.1 ensures the existence of solution of initial value problem (1.1)-(1.2). Let u, v be two solutions of (1.1)-(1.2), then

$$
\begin{aligned}
|u(t)-v(t)| = &\Bigg| E\left[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\right] \\
&+\frac{1}{\Gamma p}\int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\
&-E\left[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, v(s), \int_0^s g(\theta, v(\theta))d\theta)ds\right] \\
&+\frac{1}{\Gamma p}\int_0^t (t-s)^{p-1} h(s, v(s), \int_0^s g(\theta, v(\theta))d\theta)ds \Bigg| \\
\leq\ & E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta) \\
&-h(s, v(s), \int_0^s g(\theta, v(\theta))d\theta)\Bigg| ds \\
&+\frac{1}{\Gamma p}\int_0^t (t-s)^{p-1} \Bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta) - h(s, v(s), \int_0^s g(\theta, v(\theta))d\theta)\Bigg| ds \\
\leq\ & E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \left(k_1\|u-v\| + k_1 \int_0^s |g(\theta, u(\theta)) - g(\theta, v(\theta))| d\theta \right) ds \\
&+\frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1} \left(k_1\|u-v\| + k_1 \int_0^s |g(\theta, u(\theta)) - g(\theta, v(\theta))| d\theta \right) ds \\
\leq\ & \frac{k_1 E T^p \|u-v\| \sum_{i=1}^{n} q_i}{\Gamma(p+1)} + \frac{k_1 k_2 E T^{p+1} \|u-v\| \sum_{i=1}^{n} q_i}{\Gamma(p+2)} \\
&+\frac{k_1 T^p \|u-v\|}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1} \|u-v\|}{\Gamma(p+2)} \\
&= \frac{1}{\Gamma(p+1)} \left(1 + E\sum_{i=1}^{n} q_i\right) \left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right) \|u-v\|.
\end{aligned}
$$

Which gives

$$
\left(1 - \frac{1}{\Gamma(p+1)} \left(1 + E\sum_{i=1}^{n} q_i\right) \left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right)\right) \|u-v\| \leq 0.
$$

Since $\frac{1}{\Gamma(p+1)} \left(1 + E\sum_{i=1}^{n} q_i\right) \left(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\right) < 1$, therefore $u(t) = v(t)$ and the solution of the initial value problem (1.1)-(1.2) is unique.

6. Continuous Dependence

6.1. Continuous Dependence on u_0

Definition 6.1. *Let u^* is the solution of the initial value problem*

$$^{C}D^{p}u^*(t) = h(t, u^*(t), \int_0^t f(s, u^*(s))ds) \tag{6.1}$$

with nonlocal condition

$$u(0) + \sum_{i=1}^{n} q_i u^*(\sigma_i) = u_0^*, \quad \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T]. \tag{6.2}$$

Then, the solution $u \in C[J, X]$ of initial value problem (1.1)-(1.2) is said to be continuously depends on u_0, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) > 0 \quad s.t. \quad |u_0 - u_0^*| < \delta \implies \|u - u^*\| < \epsilon.$$

Theorem 6.2. *Let the assumptions H_5-H_7 are satisfied. Then the solution of initial value problem (1.1)-(1.2) continuously depends on u_0.*

Proof. Let u, u^* be two solutions of the initial value problem (1.1)-(1.2) and (6.1)-(6.2) respectively. Then

$$\begin{aligned} |u(t) - u^*(t)| = & \Bigg| E\Bigg[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Bigg] \\ & + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\ & - E\Bigg[u_0^* - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds\Bigg] \\ & + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds \Bigg| \\ \leq & \; E|u_0 - u_0^*| + E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \Bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\ & - h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds \Bigg| + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \Bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta \\ & - h(s, u^*(s), \int_0^s f(\theta, u^*(\theta))d\theta \Bigg| ds \end{aligned}$$

$$
\begin{aligned}
\leq\ & E|u_0 - u_0^*| + E\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1}\Big(k_1\|u - u^*\| \\
& + k_2 \int_0^s |g(\theta, u(\theta)) - g(\theta, u^*(\theta))|d\theta\Big)ds + \frac{1}{\Gamma(p)}\int_0^t (t-s)^{p-1}\Big(k_1\|u-u^*\| \\
& + k_2 \int_0^s |g(\theta, u(\theta)) - g(\theta, u^*(\theta))|d\theta\Big)ds \\
\leq\ & E|u_0 - u_0^*| + \frac{k_1 T^p \|u-u^*\| E\sum_{i=1}^n q_i}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1}\|u-u^*\| E\sum_{i=1}^n q_i}{\Gamma(p+2)} \\
& + \frac{k_1 T^p\|u-u^*\|}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1}\|u-u^*\|}{\Gamma(p+2)} \\
\leq\ & E\delta + \frac{1}{\Gamma(p+1)}\Big(1 + E\sum_{i=1}^n q_i\Big)\Big(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1}\Big)\|u-u^*\|.
\end{aligned}
$$

Which gives

$$
\|u - u^*\| \leq \frac{E\delta}{\left[1 - \frac{1}{\Gamma(p+1)}\Big(1 + E\sum_{i=1}^n q_i\Big)\Big(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{\alpha+1}\Big)\right]} = \epsilon
$$

Thus, the solution of initial value problem (1.1)-(1.2) continuously depends on u_0^*.

6.2. Continuous Dependence on the Nonlocal Data q_i

Definition 6.3. *Let u^* is the solution of the initial value problem*

$$
{}^C D^p u^*(t) = h(t, u^*(t), \int_0^t g(s, u^*(s))ds) \tag{6.3}
$$

with nonlocal condition

$$
u(0) + \sum_{i=1}^n q_i^* u^*(\sigma_i) = u_0, \quad \sum_{i=1}^n q_i^* > 0, \quad \sigma_i \in (0, T]. \tag{6.4}
$$

Then, the solution $u \in C[J, X]$ of initial value problem (1.1)-(1.2) is said to be continuously depends on nonlocal data q_i, if

$$
\forall \epsilon > 0, \quad \exists \ \ \delta(\epsilon) > 0 \ \ s.t. \ \ |u_0 - u_0^*| < \delta \implies \|u - u^*\| < \epsilon.
$$

Theorem 6.4. *Let the assumptions H_5-H_7 are satisfied. Then the solution of initial value problem (1.1)-(1.2) continuously depends on the nonlocal data q_i.*

Proof. Let u, u^* be two solutions of the initial value problem (1.1)-(1.2) and (6.3)-(6.4) respectively. Then

$$\begin{aligned}
|u(t)-u^*(t)| = &\Big|E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Big] \\
&+\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\
&-E^*\Big[u_0^* - \sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds\Big] \\
&+\frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds\Big| \\
\le\ & EE^* n\delta|u_0| + \Big|E^* \sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\
&-E \sum_{j=1}^{n} q_i^* \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds \\
&+E\Big[\sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds \\
&-\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Big]\Big| \\
&+\frac{k_1 T^p \|u-u^*\|}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1} \|u-u^*\|}{\Gamma(p+2)} \\
\le\ & EE^* n\delta|u_0| + n\delta\Big[\Big(\frac{k_1 T^p}{\Gamma(p+1)} + \frac{k_1 k_2 T^{\alpha+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^p L_1}{\Gamma(p+1)} + \frac{k_2 T^{p+1} L_2}{\Gamma(p+2)}\Big] \sum_{i=1}^{n} q_i \\
&+E\Big|\Big[\sum_{i=1}^{n} q_i^* \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds \\
&-\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds \\
&+\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g(\theta, u^*(\theta))d\theta)ds \\
&-\sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Big]\Big| \\
&+\frac{k_1 T^p \|u-u^*\|}{\Gamma(p+1)} + \frac{k_1 k_2 T^{p+1} \|u-u^*\|}{\Gamma(p+2)}
\end{aligned}$$

$$\begin{aligned}
&\leq \quad EE^*n\delta|u_0| + n\delta\Big[\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{\alpha+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big]\sum_{i=1}^{n} q_i \\
&\quad +E\Big[n\delta\Big[\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big] \\
&\quad +\sum_{i=1}^{n} q_i\Big(\frac{k_1T^p\|u-u^*\|}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}\|u-u^*\|}{\Gamma(p+2)}\Big)\Big] \\
&\quad +\frac{k_1T^p\|u-u^*\|}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}\|u-u^*\|}{\Gamma(p+2)} \\
&\leq \quad EE^*n\delta|u_0| + n\delta\Big[\Big(\frac{a_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big] \\
&\quad \Big(E + \sum_{i=1}^{n} q_i\Big) + \Big(1 + E\sum_{i=1}^{n} q_i\Big)\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u-u^*\|.
\end{aligned}$$

Hence

$$\|u-u^*\| \leq \frac{EE^*n\|u_0\| + n\Big[\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)\|u^*\| + \frac{T^pL_1}{\Gamma(p+1)} + \frac{k_2T^{p+1}L_2}{\Gamma(p+2)}\Big]\Big(E + \sum_{i=1}^n q_i\Big) + \Big(1 + E\sum_{i=1}^n q_i\Big)}{1 - \Big(1 + E\sum_{i=1}^n q_i\Big)\Big(\frac{k_1T^p}{\Gamma(p+1)} + \frac{k_1k_2T^{p+1}}{\Gamma(p+2)}\Big)}\delta = \epsilon,$$

where $E^* = (1 + \sum_{i=1}^n q_i^*)^{-1}$. Then the solution of the initial value problem (1.1)-(1.2) continuously depends on the nonlocal data q_i.

6.3. Continuous Dependence on the Functional g

Definition 6.5. *Let u^* is the solution of the initial value problem*

$$^CD^pu^*(t) = h(t, u^*(t), \int_0^t g^*(s, u^*(s))ds) \tag{6.5}$$

with nonlocal condition

$$u(0) + \sum_{i=1}^{n} q_iu^*(\sigma_i) = u_0, \quad \sum_{i=1}^{n} q_i > 0, \quad \sigma_i \in (0, T]. \tag{6.6}$$

Then, the solution $u \in C[J, X]$ of initial value problem (1.1)-(1.2) is said to be continuously depends on the functional g, if

$$\forall \epsilon > 0, \quad \exists \quad \delta(\epsilon) > 0 \quad s.t. \quad |g - g^*| < \delta \implies \|u - u^*\| < \epsilon.$$

Theorem 6.6. *Let the assumptions H_5-H_7 are satisfied. Then the solution of initial value problem (1.1)-(1.2) continuously depends on the functional g.*

Proof. Let u, u^* be two solutions of the initial value problem (1.1)-(1.2) and (6.5)-(6.6) respectively. Then

$$
\begin{aligned}
|u(t)-u^*(t)| &= \bigg| E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds\Big] \\
&+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta)ds \\
&- E\Big[u_0 - \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta)ds\Big] \\
&+ \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta)ds \bigg| \\
&\le E \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta) \\
&- h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta) \bigg| ds + \frac{1}{\Gamma(p)} \int_0^t (t-s)^{p-1} \bigg| h(s, u(s), \int_0^s g(\theta, u(\theta))d\theta \\
&- h(s, u^*(s), \int_0^s g^*(\theta, u^*(\theta))d\theta \bigg| ds \\
&\le E \sum_{i=1}^{n} q_i \frac{1}{\Gamma(p)} \int_0^{\sigma_i} (\sigma_i - s)^{p-1} \bigg(k_1 \|u - u^*\| + k_1 \int_0^s |g(\theta, u(\theta)) - g^*(\theta, u^*(\theta))| d\theta \bigg) ds \\
&+ \int_0^t (t-s)^{p-1} \bigg(k_1 \|u - u^*\| + k_1 \int_0^s |g(\theta, u(\theta)) - g^*(\theta, u^*(\theta))| d\theta \bigg) ds \\
&\le \frac{1}{\Gamma(p+1)} \bigg(1 + E \sum_{i=1}^{n} q_i \bigg) \frac{k_1 T^{p+1} \delta}{p+1} \\
&+ \frac{1}{\Gamma(p+1)} \bigg(1 + E \sum_{i=1}^{n} q_i \bigg) \bigg(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1} \bigg) \|u - u^*\|.
\end{aligned}
$$

Hence

$$
\|u - u^*\| \le \frac{\frac{1}{\Gamma(p+1)} \Big(1 + E \sum_{i=1}^{n} q_i \Big) \frac{k_1 T^{p+1} \delta}{p+1}}{1 - \frac{1}{\Gamma(p+1)} \Big(1 + E \sum_{i=1}^{n} q_i \Big) \Big(k_1 T^p + \frac{k_1 k_2 T^{p+1}}{p+1} \Big)} = \epsilon.
$$

Then the solution of the initial value problem (1.1)-(1.2) continuously depends upon the functional g.

7. Examples

Example 7.1. Consider the following nonlinear fractional order integro-differential equation for $p \in (0,1]$

$$ {}^{C}D^{p}u(t) = (1+t)^2 + \frac{u(t)}{3+t^2} + \int_0^t \frac{1}{4}\bigg(\sin(2s+2) + \frac{s^2u(s)}{3(1+u(s))}\bigg)ds, \quad a.e\ t \in (0,1] \tag{7.1}$$

with infinite point boundary condition

$$ u(0) + \sum_{i=1}^{\infty} \frac{1}{3^i} u\bigg(\frac{i}{i+1}\bigg) = u_0. \tag{7.2}$$

Set

$$ h(t,u(t),\int_0^t g(s,u(s))ds) = (1+t)^2 + \frac{u(t)}{3+t^2} + \int_0^t \frac{1}{4}\bigg(\sin(2s+2) + \frac{s^2u(s)}{3(1+u(s))}\bigg)ds. $$

Then

$$ \begin{aligned} |h(t,u(t),\int_0^t g(s,u(s))ds)| \quad &= (1+t)^2 \\ &+\frac{1}{3}\bigg(|u(t)| + \int_0^t \frac{3}{4}\bigg|\bigg(\sin(2s+2) + \frac{s^2u(s)}{3(1+u(s))}\bigg)\bigg|ds\bigg), \end{aligned} $$

and also

$$ |g(s,u(s))| = \frac{3}{4}|\sin(2s+2)| + \frac{3}{12}|u(s)|. $$

With $\phi(t) = (1+t)^2 \in L^1[0,1]$, $\varphi(t) = \frac{3}{4}|\sin(2s+2)| \in L^1[0,1]$, $k_1 = \frac{1}{3}, k_2 = \frac{3}{12}$, $\frac{1}{\Gamma(\alpha+1)}\bigg(1 + E\sum_{i=1}^{n} q_i\bigg)\bigg(k_1T^p + \frac{k_1k_2T^{p+1}}{p+1}\bigg) = \frac{1}{\Gamma(p+1)}\bigg(1 + \frac{\frac{1}{2}}{1+\frac{1}{2}}\bigg)\bigg(\frac{1}{3} + \frac{\frac{1}{3}\cdot\frac{3}{12}}{p+1}\bigg) < 1$, $\forall p \in (0,1]$, all the assumption $H_1 - H_4$ of Theorem 3.1 are satisfied. Therefore by applying the Theorem 3.1 with convergent series $\sum_{i=1}^{\infty}\frac{1}{3^i}$, IVP (7.1)-(7.2) has a solution u.

Example 7.2. Consider the following nonlinear fractional order integro-differential equation for $\alpha \in (0,1]$

$$ {}^{C}D^{p}u(t) = t^7 + t^3e^{-2t} + 1 + \frac{u(t)}{t+2} $$

$$+\int_0^t \frac{1}{5}\bigg(\cos^2(2s+2)+\frac{s^3u(s)}{4e^{|u(s)|}}\bigg)ds, \quad a.e\ t\in(0,1], \tag{7.3}$$

with infinite point boundary condition

$$u(0)+\sum_{i=1}^{\infty}\frac{1}{4^i}u\bigg(\frac{i^3+i^2-1}{i^3+i^2}\bigg)=u_0. \tag{7.4}$$

Set

$$\begin{aligned} h(t,u(t),\int_0^t g(s,u(s))ds) &= t^7+t^3e^{-2t}+1+\frac{u(t)}{t+2} \\ &\quad +\int_0^t \frac{1}{5}\bigg(\cos^2(2s+2)+\frac{s^3u(s)}{4e^{|u(s)|}}\bigg)ds. \end{aligned}$$

Then

$$\begin{aligned} |h(t,u(t),\int_0^t g(s,u(s))ds)| &\le t^7+t^3e^{-2t}+1 \\ &\quad +\frac{1}{2}\bigg(|u|+\frac{2}{5}\int_0^t \bigg|\bigg(\cos^2(2s+2)+\frac{s^3u(s)}{4e^{|u(s)|}}\bigg)\bigg|ds\bigg) \end{aligned}$$

and also

$$|g(s,u(s))|=\frac{2}{5}|\cos^2(2s+2)|+\frac{1}{10}|u|.$$

All the assumption H_1-H_4 of Theorem 3.1 are satisfied with $\phi(t)=t^7+t^3e^{-2t}+1\in L^1[0,1]$, $\varphi(t)=\frac{2}{5}|\cos^2(2s+2)|\in L^1[0,1]$, $k_1=\frac{1}{2}, k_2=\frac{1}{10}$, $\frac{1}{\Gamma(\alpha+1)}\bigg(1+E\sum_{i=1}^n q_i\bigg)\bigg(k_1T^p+\frac{k_1k_2T^{p+1}}{p+1}\bigg)=\frac{1}{\Gamma(p+1)}\bigg(1+\frac{\frac{1}{3}}{1+\frac{1}{3}}\bigg)\bigg(\frac{1}{3}+\frac{\frac{1}{3}\cdot\frac{1}{10}}{p+1}\bigg)<1$, $\forall p\in(0,1]$, all the assumption H_1-H_4 of Theorem 3.1 are satisfied. Therefore by applying the Theorem 3.1 with convergent series $\sum_{i=1}^{\infty}\frac{1}{4^i}$, IVP (7.1)-(7.2) has a solution u.

8. Conclusion

In this paper Caputo fractional differential equations are studied with infinite point boundary conditions. The statement of the initial value problem is set up and an interpretation of the solutions is given. Further continuous dependence on initial point, on nonlocal data, on the functional is also studied. The fixed point theorems are used to prove main results. The obtained results are verified by some examples.

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CLASSES OF BI-UNIVALENT FUNCTIONS DEFINED BY CONVOLUTION

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Abstract: In this paper, we introduce two new subclasses of the function class Σ of bi-univalent functions defined in the open unit disc. Furthermore, we find estimates on the coefficient $|a_2|$ and we obtain the Fekete-Szegö inequalities for certain classes $Q_{\Sigma}(\alpha; \Psi(z))$ and $\mathcal{P}_{\Sigma}(\beta; \Psi(z))$ of bi-univalent functions.

Keywords and Phrases: Bi-univalent, Fekete-Szegö, Hadamard product.

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1. Introduction

Let $\mathcal{A}$ denote the class of analytic functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (z \in \mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}) \tag{1.1}$$

and $\mathcal{S}$ be the subclass of $\mathcal{A}$, which are univalent functions.

Let $\Psi \in \mathcal{S}$ be given by

$$\Psi(z) = z + \sum_{k=2}^{\infty} b_k z^k, \qquad b_k \neq 0, \tag{1.2}$$

the Hadamard product (or convolution) of f and Ψ is given by

$$(f * \Psi)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k = (\Psi * f)(z). \tag{1.3}$$

For the parameter λ $(\lambda > 0)$ and an analytic function $f \in \mathcal{A}$, we obtain the following integral operator (see Irmak [3, with $p = 1$])

$$\mathcal{Q}^{\lambda} [f] (z) = \frac{2^{\lambda}}{z\Gamma(\lambda)} \int_0^z \left[\log\left(\frac{z}{t}\right)\right]^{\lambda-1} f(t)dt, \tag{1.4}$$

where the function Γ is well-known gamma function.$f(z) \in \mathcal{A}$, we can easy determine that

$$\mathcal{Q}^{\lambda} [f] (z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1}\right)^{\lambda} a_k z^k \qquad (\lambda > 0). \tag{1.5}$$

It is readily verified from (1.5) that

$$z \left(\mathcal{Q}^{\lambda} [f] (z)\right)' = 2\mathcal{Q}^{\lambda-1} [f] (z) - \mathcal{Q}^{\lambda} [f] (z) \qquad (\lambda > 0). \tag{1.6}$$

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \qquad (z \in \mathbb{U})$$

and

$$f(f^{-1}(w)) = w \qquad \left(|w| < r_0(f);\ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots .$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both f and f^{-1} are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1). For a brief history and interesting examples in the class Σ (see [4]).

Brannan and Taha [5] (see also [31]) introduced certain subclasses of the bi-univalent functions class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order α $(0 \le \alpha < 1)$, respectively (see [4]). Thus, following Brannan and Taha [5], a function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*_\Sigma(\alpha)$ of strongly bi-starlike functions of order α $(0 < \alpha \le 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \le 1;\ z \in \mathbb{U}) \tag{1.7}$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \le 1;\ w \in \mathbb{U}), \tag{1.8}$$

where the function g is given by

$$g(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots \tag{1.9}$$

and g is the extension of f^{-1} to $\mathbb{U}$. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α $(0 < \alpha \le 1)$, corresponding to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [5] and [31]).

Recently, the subject of bi-univalent functions actually, revived by Srivastava et al. [29] by their pioneering work on the study of coefficient problems. Various subclasses of the bi-univalent function class Σ were introduced and non-sharp estimates on the first two coefficients $|a_2|$ and $|a_3|$ in the Taylor-Maclaurin series expansion (1.1) were found in several recent investigations (see, for example, [1-3, 6-20, 24-38, 30, 32])and including the references therein. The afore-cited all these papers on the subject were actually motivated by the pioneering work of Srivastava et al. [29]. However, the problem to find the coefficient bounds on $|a_n|$ $(n = 3, 4, \dots)$ for functions $f \in \Sigma$ is still an open problem.

Motivated in this line we consider the following comprehensive classes of bi-univalent function:

Definition 1.1. *Let f given by (1.1) and Ψ given by (1.2), respectively, then f is said to be in the class $Q_{\Sigma}(\alpha;\Psi(z))$ if the following conditions are satisfied:*

$$f \in \Sigma \quad \textit{and} \quad \left|\arg\left(\frac{z\,(f*\Psi)'(z)}{(f*\Psi)(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0<\alpha\le 1;\ z\in\mathbb{U}) \tag{1.10}$$

and

$$\left|\arg\left(\frac{w\,(g*\Psi)'(w)}{(g*\Psi)(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0<\alpha\le 1;\ w\in\mathbb{U}), \tag{1.11}$$

where the function g is given by (1.9).

Definition 2.2. *Let f given by (1.1) and Ψ given by (1.2), respectively, then f is said to be in the class $\mathcal{P}_{\Sigma}(\beta;\Psi(z))$ if the following conditions are satisfied:*

$$f \in \Sigma \quad \textit{and} \quad \Re\left(\frac{z\,(f*\Psi)'(z)}{(f*\Psi)(z)}\right) > \beta \qquad (0\le\beta<1;\ z\in\mathbb{U}) \tag{1.12}$$

and

$$\Re\left(\frac{w\,(g*\Psi)'(w)}{(g*\Psi)(w)}\right) > \beta \qquad (0\le\beta<1;\ w\in\mathbb{U}), \tag{1.13}$$

where the function g is given by (1.9).

We note that for suitable choices of Ψ we obtain the following subclasses:

(i) Putting $\Psi(z) = \dfrac{z}{1-z}$ in Definition 1.1, we obtain the following subclass $Q_{\Sigma}(\alpha)$, defined as follows:

$$\left\{ f\in\Sigma \ \text{ and } \ \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0<\alpha\le 1;\ z\in\mathbb{U}) \right\}$$

and

$$\left|\arg\left(\frac{wf'(w)}{f(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0<\alpha\le 1;\ w\in\mathbb{U}).$$

(ii) Putting $\Psi(z) = z + \sum\limits_{k=2}^{\infty}\left(\dfrac{2}{k+1}\right)^{\lambda} z^k$ in Definition (1.1), we obtain the following subclass $Q_{\Sigma}(\lambda;\alpha)$, defined as follows:

$$\left\{ f\in\Sigma \ \text{ and } \ \left|\arg\left(\frac{z\left(\mathcal{Q}^{\lambda}[f](z)\right)'}{\mathcal{Q}^{\lambda}[f](z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0<\alpha\le 1;\ \lambda>0;\ z\in\mathbb{U}) \right\}$$

and

$$\left|\arg\left(\frac{w\left(\mathcal{Q}^{\lambda}\left[f\right](w)\right)'}{\mathcal{Q}^{\lambda}\left[f\right](w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1;\ \lambda > 0;\ w \in \mathbb{U}),$$

where $\mathcal{Q}^{\lambda}$ is defined by (1.5).

(iii) Putting $\Psi(z) = z + \sum\limits_{k=2}^{\infty} \dfrac{m^{k-1}}{(k-1)!} e^{-m} z^k$ $(m > 0)$ in Definition 1.1, we obtain the following subclass $\mathcal{D}_{\Sigma}^{m}(\alpha)$, defined as follows:

$$\left\{ f \in \Sigma \ \text{ and } \ \left|\arg\left(\frac{z\left(\mathcal{I}^{m} f(z)\right)'}{\mathcal{I}^{m} f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1;\ m > 0;\ z \in \mathbb{U}) \right\}$$

and

$$\left|\arg\left(\frac{w\left(\mathcal{I}^{m} f(w)\right)'}{\mathcal{I}^{m} f(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1;\ m > 0;\ w \in \mathbb{U}),$$

where $\mathcal{I}^{m}$ is a Poisson distribution series [22].

(iv) Putting

$$\Psi(z) = z + \sum_{k=2}^{\infty} \left(\frac{\ell + \gamma(k-1) + 1}{1 + \ell}\right)^{n} z^k \quad (\gamma \geq 0;\ \ell > -1;\ n \in \mathbb{Z} = \{0, \pm 1, \cdots\})$$

in Definition 1.1, we obtain the following subclass $\mathcal{P}_{\Sigma}^{n}(\gamma, \ell; \alpha)$, defined as follows:

$$\left\{ f \in \Sigma \ \text{ and } \ \left|\arg\left(\frac{z\left(\mathcal{J}^{n}(\gamma,\ell) f(z)\right)'}{\mathcal{J}^{n}(\gamma,\ell) f(z)}\right)\right| < \frac{\alpha\pi}{2} \, (0 < \alpha \leq 1; \gamma \geq 0; \ell > -1; n \in \mathbb{Z}; z \in \mathbb{U}) \right\},$$

and

$$\left|\arg\left(\frac{w\left(\mathcal{J}^{n}(\gamma,\ell) f(w)\right)'}{\mathcal{J}^{n}(\gamma,\ell) f(w)}\right)\right| < \frac{\alpha\pi}{2} \ \ (0 < \alpha \leq 1;\ \gamma \geq 0;\ \ell > -1;\ n \in \mathbb{Z};\ w \in \mathbb{U}),$$

where $\mathcal{J}^{n}(\gamma, \ell)$ is the Prajapat operator [23].

To prove our results, we need the following lemmas.

Lemma 1.1. [21] *If $h \in P$ then $|d_k| \leq 2$ for each k, where P is the family of all functions h analytic in $\mathbb{U}$ for which $\Re(h(z)) > 0$, $h(z) = 1 + d_1 z + d_2 z^2 + \cdots$ for $z \in \mathbb{U}$.*

Lemma 1.2. [32] *Let $k, l \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{C}$. If $|z_1| < \mathcal{R}$ and $|z_2| < \mathcal{R}$, then*

$$|(k+l) z_1 + (k-l) z_2| \leq \begin{cases} 2|k|\mathcal{R} & ; |k| \geq |l|, \\ \\ 2|l|\mathcal{R} & ; |k| \leq |l|. \end{cases} \tag{1.14}$$

2. Fekete-Szegö inequalities for the function class $Q_{\Sigma}(\alpha;\ \Psi(z))$

Unless otherwise mentioned, we assume throughout this paper that $0 < \alpha \leq 1,\ 0 \leq \beta < 1,\ z \in \mathbb{U},\ \lambda,\ m > 0$ and Ψ is given by (1.2).

Theorem 2.1. *Let f given by (1.1) belongs to the class $Q_{\Sigma}(\alpha;\Psi(z))$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{b_2^2 + \alpha\left[4b_3 - 3b_2^2\right]}}, \tag{2.1}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{4\alpha^2}{(\alpha+1)\,b_2^2}\left(\dfrac{b_2^2}{b_3} - \mu\right) & ; \dfrac{\alpha^2}{(\alpha+1)\,b_2^2}\left(\dfrac{b_2^2}{b_3} - \mu\right) > \dfrac{\alpha}{4b_3} \\ \dfrac{\alpha}{b_3} & ; \dfrac{\alpha^2}{(\alpha+1)\,b_2^2}\left(\dfrac{b_2^2}{b_3} - \mu\right) < \dfrac{\alpha}{4b_3} \end{cases}. \tag{2.2}$$

Proof. It follows from (1.10) and (1.11) that

$$\frac{z\,(f*\Psi)'\,(z)}{(f*\Psi)\,(z)} = [p(z)]^{\alpha} \tag{2.3}$$

and

$$\frac{w\,(g*\Psi)'\,(w)}{(g*\Psi)\,(w)} = [q(w)]^{\alpha}\,, \tag{2.4}$$

where p and q in P and have the forms

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \tag{2.5}$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + q_3 w^3 + \cdots\,. \tag{2.6}$$

Now, equating the coefficients in (2.3) and (2.4), we get

$$a_2 b_2 = p_1 \alpha, \tag{2.7}$$

$$2a_3 b_3 - a_2^2 b_2^2 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2, \tag{2.8}$$

$$-a_2 b_2 = q_1 \alpha, \tag{2.9}$$

and

$$3a_2^2b_2^2 - 2a_3b_3 = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.10}$$

From (2.7) and (2.9), we get

$$p_1 = -q_1 \tag{2.11}$$

and

$$2a_2^2b_2^2 = \alpha^2\left(p_1^2 + q_1^2\right). \tag{2.12}$$

From (2.8), (2.10) and (2.12), we obtain

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{b_2^2 + \alpha\left[4b_3 - 3b_2^2\right]}. \tag{2.13}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \frac{2\alpha}{\sqrt{b_2^2 + \alpha\left[4b_3 - 3b_2^2\right]}}.$$

This gives the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3 - \mu a_2^2|$, by summing and subtracting (2.8) and (2.10), we get

$$4a_3b_3 - 4a_2^2b_2^2 = \alpha(p_2 - q_2) + \frac{\alpha(\alpha-1)}{2}\left(p_1^2 - q_1^2\right). \tag{2.14}$$

It follows from (2.11), (2.12) and (2.14) that

$$a_3 = \frac{a_2^2b_2^2}{b_3} + \frac{\alpha(p_2-q_2)}{4b_3}. \tag{2.15}$$

Applying (2.7) and (2.9), we dispose of p_1 and q_1, we have

$$a_3 - \mu a_2^2 = p_2\left[\frac{\alpha^2}{(\alpha+1)b_2^2}\left(\frac{b_2^2}{b_3} - \mu\right) + \frac{\alpha}{4b_3}\right] + q_2\left[\frac{\alpha^2}{(\alpha+1)b_2^2}\left(\frac{b_2^2}{b_3} - \mu\right) - \frac{\alpha}{4b_3}\right].$$

From Lemma 1 and Lemma 1.2, we get the desired result (2.2). This completes the proof of Theorem 2.1.

Putting $\Psi(z) = \dfrac{z}{1-z}$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.1. *Let f given by (1.1) belongs to the class $Q_\Sigma(\alpha,\gamma)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{1+\alpha}}, \tag{2.16}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{4\alpha^2}{\alpha+1}(1-\mu) & ;\dfrac{\alpha^2}{(\alpha+1)}(1-\mu) > \dfrac{\alpha}{4} \\ \alpha & ;\dfrac{\alpha^2}{(\alpha+1)}(1-\mu) < \dfrac{\alpha}{4} \end{cases}. \tag{2.17}$$

Putting $\Psi(z) = z + \sum\limits_{k=2}^{\infty}\left(\dfrac{2}{k+1}\right)^{\lambda} z^k$ $(\lambda > 0)$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.2. *Let f given by (1.1) belongs to the class $Q_{\Sigma}(\alpha;\lambda)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left(\frac{2}{3}\right)^{2\lambda} + \alpha\left[4\left(\frac{2}{4}\right)^{\lambda} - 3\left(\frac{2}{3}\right)^{2\lambda}\right]}}, \tag{2.18}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{4\alpha^2}{(\alpha+1)\left(\frac{2}{3}\right)^{2\lambda}}\left(\dfrac{\left(\frac{2}{3}\right)^{2\lambda}}{\left(\frac{2}{4}\right)^{\lambda}} - \mu\right) & ;\dfrac{\alpha^2}{(\alpha+1)\left(\frac{2}{3}\right)^{2\lambda}}\left(\dfrac{\left(\frac{2}{3}\right)^{2\lambda}}{\left(\frac{2}{4}\right)^{\lambda}} - \mu\right) > \dfrac{\alpha}{4\left(\frac{2}{4}\right)^{\lambda}} \\ \dfrac{\alpha}{\left(\frac{2}{4}\right)^{\lambda}} & ;\dfrac{\alpha^2}{(\alpha+1)\left(\frac{2}{3}\right)^{2\lambda}}\left(\dfrac{\left(\frac{2}{3}\right)^{2\lambda}}{\left(\frac{2}{4}\right)^{\lambda}} - \mu\right) < \dfrac{\alpha}{4\left(\frac{2}{4}\right)^{\lambda}} \end{cases}. \tag{2.19}$$

Putting $\Psi(z) = z + \sum\limits_{k=2}^{\infty}\dfrac{m^{k-1}}{(k-1)!}e^{-m}z^k$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.3. *Let f given by (1.1) belongs to the class $\mathcal{D}_{\Sigma}^{m}(\alpha)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{m^2e^{-2m} + \alpha\left[2m^2e^{-m} - 3m^2e^{-2m}\right]}}, \tag{2.20}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{4\alpha^2}{(\alpha+1)\, m^2 e^{-2m}}\left(2e^{-m}-\mu\right) & ; \dfrac{\alpha^2}{(\alpha+1)\, m^2 e^{-2m}}\left(2e^{-m}-\mu\right) > \dfrac{\alpha}{2m^2 e^{-m}} \\ \dfrac{2\alpha}{m^2 e^{-m}} & ; \dfrac{\alpha^2}{(\alpha+1)\, m^2 e^{-2m}}\left(2e^{-m}-\mu\right) < \dfrac{\alpha}{2m^2 e^{-m}} \end{cases}. \tag{2.21}$$

Putting $\Psi(z) = z + \sum_{k=2}^{\infty}\left(\frac{\ell+1+\gamma(k-1)}{1+\ell}\right)^n z^k$ in Theorem 2.1, we obtain the following corollary:

Corollary 2.4. *Let f given by (1.1) belongs to the class $\mathcal{P}_{\Sigma}^{n}(\gamma,\ell;\alpha)$, then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{\left(\frac{\gamma+\ell+1}{1+\ell}\right)^n + \alpha\left[4\left(\frac{2\gamma+\ell+1}{1+\ell}\right)^n - 3\left(\frac{\gamma+\ell+1}{1+\ell}\right)^{2n}\right]}}, \tag{2.22}$$

and

$$\left|a_3-\mu a_2^2\right| \leq \begin{cases} \dfrac{4\alpha^2}{(\alpha+1)}\left(\left(\dfrac{1+\ell}{2\gamma+\ell+1}\right)^n - \mu\left(\dfrac{1+\ell}{\gamma+\ell+1}\right)^{2n}\right); \\ \qquad \dfrac{\alpha^2}{(\alpha+1)}\left(\left(\dfrac{1+\ell}{2\gamma+\ell+1}\right)^n - \mu\left(\dfrac{1+\ell}{\gamma+\ell+1}\right)^{2n}\right) > \dfrac{\alpha}{4}\left(\dfrac{1+\ell}{2\gamma+\ell+1}\right)^n \\ \alpha\left(\dfrac{1+\ell}{2\gamma+\ell+1}\right)^n; \\ \qquad \dfrac{4\alpha^2}{(\alpha+1)}\left(\left(\dfrac{1+\ell}{2\gamma+\ell+1}\right)^n - \mu\left(\dfrac{1+\ell}{\gamma+\ell+1}\right)^{2n}\right) < \dfrac{\alpha}{4}\left(\dfrac{1+\ell}{2\gamma+\ell+1}\right)^n \end{cases}. \tag{2.23}$$

3. Fekete-Szegö inequalities for the function class $P_{\Sigma}(\beta;\Psi(z))$

This section begins by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $P_{\Sigma}(\beta;\Psi(z))$.

Theorem 3.1. *Let f given by (1.1) belongs to the class $P_{\Sigma}(\beta;\Psi(z))$, then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2b_3-b_2^2}}, \tag{3.1}$$

$$|a_3| \leq \frac{4(1-\beta)^2}{(1+\gamma)^2 b_2^2} + \frac{(1-\beta)}{(1+2\gamma)\, b_3}. \tag{3.2}$$

and for $\mu \in \mathbb{R}$

$$\left|a_3 - \mu a_2^2\right| \leq \begin{cases} \dfrac{2(1-\beta)}{2b_3 - b_2^2}\left(\dfrac{b_2^2}{b_3} - \mu\right) & ; \dfrac{1}{4b_3 - 2b_2^2}\left(\dfrac{b_2^2}{b_3} - \mu\right) \geq \dfrac{1}{4b_3} \\ \dfrac{1-\beta}{b_3} & ; \dfrac{1}{4b_3 - 2b_2^2}\left(\dfrac{b_2^2}{b_3} - \mu\right) \leq \dfrac{1}{4b_3} \end{cases}. \tag{3.3}$$

Proof. It follows from (1.12) and (1.13) that

$$\frac{z\,(f*\Psi)'(z)}{(f*\Psi)(z)} = \beta + (1-\beta)\,p(z) \tag{3.4}$$

and

$$\frac{w\,(g*\Psi)'(w)}{(g*\Psi)(w)} = \beta + (1-\beta)\,q(w), \tag{3.5}$$

where p and q have the forms (2.5) and (2.6), respectively. Equating the coefficients in (3.4) and (3.5), we get

$$a_2 b_2 = (1-\beta)\,p_1, \tag{3.6}$$

$$2a_3 b_3 - a_2^2 b_2^2 = (1-\beta)\,p_2, \tag{3.7}$$

$$-a_2 b_2 = (1-\beta)\,q_1, \tag{3.8}$$

and

$$3a_2^2 b_2^2 - 2a_3 b_3 = (1-\beta)\,q_2. \tag{3.9}$$

From (3.6) and (3.8), we get

$$p_1 = -q_1 \tag{3.10}$$

and

$$2a_2^2 b_2^2 = (1-\beta)^2\left(p_1^2 + q_1^2\right). \tag{3.11}$$

From (3.7) and (3.9), we obtain

$$a_2^2 = \frac{(1-\beta)(p_2 + q_2)}{4b_3 - 2b_2^2}. \tag{3.12}$$

Applying Lemma 1.1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{2b_3 - b_2^2}}.$$

This gives the bound on $|a_2|$ as asserted in (3.1).

Next, in order to find the bound on $|a_3 - \mu a_2^2|$, by summing and subtracting (3.7) and (3.9), we get

$$4a_3b_3 - 4a_2^2b_2^2 = (1-\beta)(p_2 - q_2). \tag{3.13}$$

It follows from (3.10), (3.11) and (3.13) that

$$a_3 = \frac{a_2^2b_2^2}{b_3} + \frac{(1-\beta)(p_2-q_2)}{4b_3}. \tag{3.14}$$

We have

$$a_3 - \mu a_2^2 = p_2\left[\frac{(1-\beta)}{4b_3 - 2b_2^2}\left(\frac{b_2^2}{b_3} - \mu\right) + \frac{1-\beta}{4b_3}\right] + q_2\left[\frac{(1-\beta)}{4b_3 - 2b_2^2}\left(\frac{b_2^2}{b_3} - \mu\right) - \frac{1-\beta}{4b_3}\right].$$

From Lemma1 and Lemma 1.2, we get the desired result (3.3). This completes the proof of Theorem 3.1.

Remark 3.1. *By specializing the parameter b_k, we obtain various results for different operators defined in the introduction.*

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PARAMETER ESTIMATION OF NAKAGAMI DISTRIBUTION UNDER PRECAUTIONARY LOSS FUNCTION

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Abstract: In this paper Bayes estimators of the scale parameter of Nakagami distribution have been obtained by taking quasi, inverted gamma and uniform prior distribution using the precautionary loss function. These are compared with the corresponding estimators with squared loss function.

Keywords and Phrases: Nakagami Distribution, Bayesian method,Inverted Gamma, Precautionary Loss Function.

2010 Mathematics Subject Classification: 60E05, 62E15, 62H10, 62H12.

1. Introduction

Nakagami distribution can be considered as a flexible lifetime distribution [1]. It is also widely considered in biomedical fields. Shanker et al. [2] and Tsui et al. [3] use the Nakagami distribution to model ultrasound data in medical imaging studies. This distribution is extensively used in reliability theory and reliability engineering and to model the constant hazard rate portion because of its memory less property.

The probability density function of the Nakagami distribution [4] is given by

$$f(x;\theta,k)=\frac{2k^k}{\Gamma(k)\theta^k}x^{2k-1}e^{-\frac{k}{\theta}x^2} \quad ; \quad x>0,\ k>0,\ \theta>0. \tag{1}$$

where θ and k are called scale and shape parameter respectively.
The joint density function of (1) is given by

$$f(\underline{x};\theta,k)=\frac{(2k^k)^n}{(\Gamma(k))^n\theta^{nk}}\prod_{i=1}^{n}x_i^{2k-1}e^{-\frac{k}{\theta}\sum_{i=1}^{n}x_i^2} \tag{2}$$

The maximum likelihood estimator of θ when k is known is given by

$$\hat{\theta}=\frac{1}{n}\sum_{i=1}^{n}x_i^2 \tag{3}$$

In Bayesian analysis the fundamental problem are that of the choice of prior distribution $g(\theta)$ and a loss function $L(\hat{\theta},\theta)$. The squared error loss function for the scale parameter θ are defined as

$$L(\hat{\theta},\theta)=(\hat{\theta}-\theta)^2 \tag{4}$$

The Bayes estimator under the above loss function, say, $\hat{\theta}_s$ is the posterior mean, i.e,

$$\hat{\theta}_s=E(\theta) \tag{5}$$

This loss function is often used because it does not lead to extensive numerical computations but several authors (Ferguson [5], Berger [6], Zellner [7], Basu and Ebrahimi [8]) have recognized that the inappropriateness of using symmetric loss function. J. G. Norstrom [9] introduced an alternative asymmetric precautionary loss function and also presented a general class of precautionary loss functions with quadratic loss function as a special case. A very useful and simple asymmetric precautionary loss function is given as

$$L(\hat{\theta},\theta)=\frac{(\hat{\theta}-\theta)^2}{\hat{\theta}}. \tag{6}$$

The Bayes estimator under precautionary loss function is denoted by $\hat{\theta}_p$ and is obtained by solving the following equation.

$$\hat{\theta}_p=\left[E(\theta^2)\right]^{\frac{1}{2}}. \tag{7}$$

Let us consider three prior distributions of θ to obtain the Bayes estimators which are given by

(i) Quasi-prior: For the situation where the experimenter has no prior information about the parameter θ, one may use the quasi density as given by

$$g_1(\theta) = \frac{1}{\theta^d} \; ; \quad \theta > 0, \; d \geq 0, \tag{8}$$

where $d = 0$ leads to a diffuse prior and $d = 1$, a non-informative prior.

(ii) Inverted gamma prior: The most widely used prior distribution of θ is the inverted gamma distribution with parameters α and $\beta(> 0)$ with probability density function given by

$$g_2(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{-(\alpha+1)} e^{-\frac{\beta}{\theta}} \quad ; \quad \theta > 0, \tag{9}$$

The main reason for general acceptability is the mathematical tractability resulting from the fact that the inverted gamma distribution is conjugate prior for θ.

(iii) Uniform prior: It frequently happens that the life tester known in advance that the probable values of θ lie over a finite range $[\alpha, \beta]$ but he does not have any strong opinion about any subset of values over this range. In such a case a uniform distribution over $[\alpha, \beta]$ may be a good approximation.

$$g_3(\theta) = \frac{1}{\beta - \alpha}; \quad 0 < \alpha \leq \theta \leq \beta. \tag{10}$$

The object of the present paper is to obtain the Bayes estimators of θ using above three prior distributions under precautionary loss function and to study their performance.

2. Bayes Estimators under $g_1(\theta)$

The posterior density of θ under $g_1(\theta)$, on using (2), is given by

$$f(\theta/\underline{x}) = \frac{\left(k \sum_{i=1}^{n} x_i^2\right)^{(nk+d-1)}}{\Gamma(nk+d-1)} \theta^{-(nk+d)} e^{-\frac{1}{\theta} k \sum_{i=1}^{n} x_i^2} \quad ; \quad \theta > 0. \tag{11}$$

The Bayes estimator under squared error loss function comes out to be

$$\hat{\theta}_s = \frac{k \sum_{i=1}^{n} x_i^2}{nk + d - 2}. \tag{12}$$

From equation (7), on using (11), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_p[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}k\sum_{i=1}^{n}x_i^2. \tag{13}$$

The risk function of the estimators $\hat{\theta}_s$ and $\hat{\theta}_p$ relative to precautionary loss function, denoted by $R_p(\hat{\theta}_s)$ and $R_p(\hat{\theta}_p)$, respectively, are as follows.

$$R_p(\hat{\theta}_s) = \theta\left[\left(\frac{nk+d-2}{nk-1}\right) + \left(\frac{nk}{nk+d-2}\right) - 2\right] \tag{14}$$

$$R_p(\hat{\theta}_p) = \theta\left[\frac{[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}}{nk-1} + \frac{nk}{[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}} - 2\right] \tag{15}$$

The risk function of the estimators $\hat{\theta}_s$ and $\hat{\theta}_p$ relative to squared error loss function, denoted by $R_s(\hat{\theta}_s)$ and $R_s(\hat{\theta}_p)$, respectively, and are given by

$$R_s(\hat{\theta}_s) = \theta^2\left[\frac{nk(nk+1)}{(nk+d-2)^2} - \frac{2nk}{(nk+d-2)} + 1\right] \tag{16}$$

$$R_s(\hat{\theta}_p) = \theta^2\left[\frac{nk(nk+1)}{[(nk+d-2)(nk+d-3)]} - \frac{2nk}{[(nk+d-2)(nk+d-3)]^{\frac{1}{2}}} + 1\right] \tag{17}$$

In general neither of the estimators uniformly dominates the other. For example, if $k = 1, n = 5,\ d = 1$, then

$$\frac{R_s(\hat{\theta}_s)}{\theta^2} = 0.375 < 0.613 = \frac{R_s(\hat{\theta}_p)}{\theta^2}$$
$$\frac{R_p(\hat{\theta}_s)}{\theta} = 0.25 < 0.31 = \frac{R_p(\hat{\theta}_p)}{\theta}$$

If $k = 1,\ n = 5,\ d = 5$ then

$$\frac{R_s(\hat{\theta}_s)}{\theta^2} = 0.219 < 0.199 = \frac{R_s(\hat{\theta}_p)}{\theta^2}$$
$$\frac{R_p(\hat{\theta}_s)}{\theta} = 0.625 < 0.539 = \frac{R_p(\hat{\theta}_p)}{\theta}$$

3. Bayes Estimators under $g_2(\theta)$

Under $g_2(\theta)$, the posterior density of θ, using equation (2), is obtained as

$$f(\theta/\underline{x}) = \frac{\left(\beta + k\sum_{i=1}^{n} x_i^2\right)^{(nk+\alpha)}}{\Gamma(nk+\alpha)} \theta^{-(nk+\alpha-1)} e^{-\frac{1}{\theta}\left(\beta + k\sum_{i=1}^{n} x_i^2\right)}; \quad \theta > 0. \tag{18}$$

The Bayes estimator under squared error loss function on using (18) comes out to be

$$\hat{\theta}_s^* = \frac{\beta + k\sum_{i=1}^{n} x_i^2}{nk+\alpha-1}. \tag{19}$$

From equation (7), on using (18), the Bayes estimator of θ under precautionary loss function is obtained as

$$\hat{\theta}_p^*[(nk+\alpha-1)(nk+\alpha-2)]^{\frac{1}{2}} \left(\beta + k\sum_{i=1}^{n} x_i^2\right). \tag{20}$$

The risk function of the estimators $\hat{\theta}_s^*$ and $\hat{\theta}_p^*$ relative to squared error loss function are given by

$$R_s\left(\hat{\theta}_s^*\right) = \theta^2 \left[\frac{nk(nk+1) + 2nk\left(\frac{\beta}{\theta}\right) + \left(\frac{\beta}{\theta}\right)^2}{(nk+\alpha-1)^2} - \frac{2\left(nk + \frac{\beta}{\theta}\right)}{(nk+\alpha-1)} + 1\right] \tag{21}$$

$$R_s\left(\hat{\theta}_p^*\right) = \theta^2 \left[C^2 \left\{nk(nk+1) + 2nk\left(\frac{\beta}{\theta}\right) + \left(\frac{\beta}{\theta}\right)^2\right\} - 2C\left(nk + \frac{\beta}{\theta}\right) + 1\right] \tag{22}$$

where $C = [(nk+\alpha-1)(nk+\alpha-2)]^{-\frac{1}{2}}$.
The Bayes risk associated with estimators $\hat{\theta}_s^*$ and $\hat{\theta}_p^*$ relative to squared error loss function are given by

$$r_s\left(\hat{\theta}_s^*\right) = \frac{\beta^2}{(\alpha-1)(\alpha-2)(nk+\alpha-1)} \tag{23}$$

$$r_s\left(\hat{\theta}_p^*\right) = \beta^2 \left[\frac{nk(nk+1)C^2 - 2nkC + 1}{(\alpha-1)(\alpha-2)} + \frac{2C(nkC-1)}{(\alpha-1)} + C^2\right] \tag{24}$$

In this case the risk functions relative to precautionary loss function and the corresponding Bayes risks can not be obtained in closed forms.

4. Bayes Estimators under $g_3(\theta)$

Under $g_3(\theta)$, using (2), the posterior density of θ, is given by

$$f(\theta/\underline{x}) = \frac{\left(k\sum_{i=1}^{n} x_i^2\right)^{(nk-1)} \theta^{-nk} e^{-\frac{1}{\theta}\left(k\sum_{i=1}^{n} x_i^2\right)}}{Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\alpha}, nk-1\right) - Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\beta}, nk-1\right)}; 0 < \alpha \le \theta \le \beta. \quad (25)$$

where, $Ig(x,n) = \int_0^x e^{-t} t^{n-1} dt.$

The Bayes estimator under squared error loss function is given by

$$\hat{\theta_s^{**}} = \left[\frac{Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\alpha}, nk-2\right) - Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\beta}, nk-2\right)}{Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\alpha}, nk-1\right) - Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\beta}, nk-1\right)}\right] k\sum_{i=1}^{n} x_i^2 \quad (26)$$

From equation (7), using (26), the Bayes estimator of θ under precautionary loss function is given by

$$\hat{\theta_p^{**}} = \left[\frac{Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\alpha}, nk-3\right) - Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\beta}, nk-3\right)}{Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\alpha}, nk-1\right) - Ig\left(\frac{k\sum_{i=1}^{n} x_i^2}{\beta}, nk-1\right)}\right] k\sum_{i=1}^{n} x_i^2 \quad (27)$$

The equations (26) and (27), can be solved numerically. In this case the risk function and the corresponding Bayes risks can not be obtained in a closed form.

5. Conclusions

From the given example in section (2), it is clear that neither of the estimators uniformly dominates the other. We therefore recommend that the estimator's choice lies according to the value of 'd' in the quasi density used as the prior distribution which in turn depends on the situation at hand.

The risk function and Bayes risks under the natural conjugate are dependent on the population parameter θ and θ is not reparable, therefore, comparison could only be done by using numerical techniques.

Also, it is clear that from the equations (26) and (27) that only numerical solutions exist for the estimators $\hat{\theta}_s^{**}$ and $\hat{\theta}_p^{**}$. In this case the risk functions and Bayes risk cannot be obtained in closed forms. Thus, the comparison could only be done after obtaining the results numerically, which depends on the value of the parameter itself.

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γ_e - GRAPHS OF GRAPHS

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Abstract: A set $S \subseteq V$ is an equitable dominating set of a graph $G = (V, E)$ if every vertex in $V - S$ is equitably adjacent to at least one vertex in S. The equitable domination number $\gamma_e(G)$ of G equals the minimum cardinality of an equitable dominating set S in G; we say that such a set S is a γ_e-set. In this paper we consider the family of all $\gamma_e - sets$ in a graph G and we define the $\gamma_e - graph$ $G(\gamma_e) = (V(\gamma_e), E(\gamma_e))$ of G to be the graph whose vertices $V(\gamma_e)$ correspond 1-to-1 with the γ_e-sets of G, and two γ_e-sets, say D_1 and D_2, are adjacent in $E(\gamma_e)$ if there exists a vertex $v \in D_1$ and a vertex $w \in D_2$ such that v is adjacent to w and $D_1 = D_2 - \{w\} \cup \{v\}$, or equivalently, $D_2 = D_1 - \{v\} \cup \{w\}$. In this paper we initiate the study of γ_e- graph of graphs.

Keywords and Phrases: Equitable dominating set.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

A set $S \subseteq V$ is an *dominating set* of a graph $G = (V, E)$ if every vertex in $V - S$ is adjacent to at least one vertex in S. The *domination number* $\gamma(G)$ of G equals the minimum cardinality of a dominating set S. For a comprehensive survey of domination in graphs, refer [4, 5]. For graph theoretic terminologies we refer [3].

In [8], the concept of γ - graphs was introduced by N. Sridharan, et al. With a slight modification G. H. Fricke, S. T. Hedetniemi, et al. defined γ - graphs and derived interesting results [2]. The concept of equitability was first introduced by W. Meyer in his paper titled *Equitable Coloring* ([6]), where two subsets are equitable if there cardinalities differ by atmost one. E. Sampath Kumar introduced a new idea of degree equitability among vertices of a graph. Two vertices of a graph are degree equitable if their degree difference is atmost 1. In [1] degree equitable domination was defined and studied. In this paper we make a study of γ_e - graph, where γ_e is the equitable domination number of a graph.

Let $G = (V, E)$ be a graph of order n. A vertex $u \in V$ is said to be degree equitable with a vertex $v \in V$ if $|d(u) - d(v)| \leq 1$, where $d(u)$ is the degree of the vertex u in G. The open equitable neighborhood of a vertex v is the set $N_e(v) = \{u \in V | uv \in E, |d(u) - d(v)| \leq 1\}$. The closed equitable neighborhood of v is the set $N_e[v] = N_e(v) \cup \{v\}$. If $S \subseteq V$ then $N_e(S) = \cup_{v \in S} N_e(v)$ and $N_e[S] = N_e(S) \cup S$. The private equitable neighbor of a vertex v with respect to S is defined by $pne[u, S] = N_e[u] - N_e[S - \{u\}]$. The maximum and minimum equitable degree of a vertex in G are respectively denoted by $\Delta_e(G)$ and $\delta_e(G)$ and is defined as $\Delta_e(G) = max_{u \in V} |N_e(u)|$ and $\delta_e(G) = min_{u \in V} |N_e(u)|$.

A set $S \subseteq V$ is an equitable dominating set of G if for every vertex $v \in V - S$ there exists a vertex $u \in S$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leq 1$. The equitable domination number $\gamma_e(G)$ of G equals the minimum cardinality of an equitable dominating set S in G. Define γ_e - graph, $G(\gamma_e) = (V(\gamma_e), E(\gamma_e))$ of G to be the graph whose vertex set $V(\gamma_e)$ correspond 1 to 1 with the γ_e - sets, D_1 and D_2, form an edge in $E(\gamma_e)$ if there exists a vertex $v \in D_1$ and $w \in D_2$ such that v is adjacent to w and $D_1 = D_2 - \{w\} \cup \{v\}$ and $D_2 = D_1 - \{v\} \cup \{w\}$.

For any two graphs G_1 and G_2 the Cartesian product denoted by $G_1 \square G_2$ is defined as, $G_1 \square G_2 = (V(G_1) \times V(G_2), E(G_1) \square E(G_2))$, where two vertices $(u_1, v_1), (u_2, v_2)$ are adjacent in $G_1 \square G_2$ if and only if either $u_1 = u_2$ and v_1 is adjacent to v_2 in G_2 or u_1 is adjacent to u_2 in G_1 and $v_1 = v_2$. The Cartesian product of two paths is called the *grid graph*.

2. γ_e- graphs of some standard graphs

Proposition 2.1. *A graph $G = (V, E)$ has a unique γ_e- set, if and only if $G(\gamma_e) \simeq K_1$.*

Corollary 2.1. $K_{1,n}(\gamma_e) \simeq K_1$.

Proposition 2.2. $\overline{K_n}(\gamma_e) \simeq K_1$, *and* $K_n(\gamma_e) = K_n$.

Proposition 2.3. *For $m, n \geq 3$,* $K_{m,n}(\gamma_e) \simeq \begin{cases} \overline{K_{mn}}, & \text{if } |m-n| \leq 1 \\ K_1, & \text{if } |m-n| \geq 2 \end{cases}$

Remark 2.1. $K_{2,3}(\gamma_e) \simeq K_{1,6}$

Proposition 2.4. $P_n(\gamma_e) \simeq \begin{cases} K_1, & \text{if } n \equiv 0(mod\ 3) \\ P_{\lceil \frac{n}{3} \rceil + 1}, & \text{if } n \equiv 2(mod\ 3) \end{cases}$

Proposition 2.5. $C_n(\gamma_e) \simeq \begin{cases} \overline{K_3}, & \text{if } n \equiv 0(mod\ 3) \\ C_n, & \text{if } n \equiv 2(mod\ 3) \end{cases}$

Proposition 2.6. *For $k \geq 2$, $(P_2 \square P_{2k+1})(\gamma_e) \simeq \overline{K_2}$.*

Proof. Since there are exactly two γ_e - sets for $P_2 \square P_{2k+1}$ which are not adjacent, we have $(P_2 \square P_{2k+1})(\gamma_e) \simeq \overline{K_2}$.

Proposition 2.7. *$G(\gamma_e)$ is a connected graph, for $G \simeq P_n$ or $C_n, n \equiv 1(mod\ 3)$.*

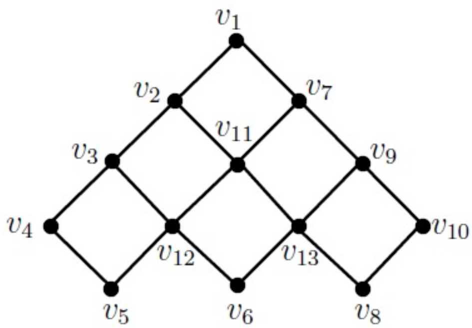
Proof. We claim that every γ_e - set of $G = P_{3k+1}$ is some number of swaps from the γ_e-set $S = \{1, 4, 7, \cdots, 3k+1\}$, thus showing that $G(\gamma_e)$ is connected for these graphs. Let $X = \{x_0, x_1, \cdots, x_k\}$, where $x_0 < x_1 < \cdots < x_k$, be a γ_e - set for G. Consider the vector $D = [D_0, D_1, \cdots, D_k] = [x_0 - 1, x_1 - 4, \cdots, x_{k+1} - (3k+1)]$. If $G = P_{3k+1}$, then $-1 \leq D[i] \leq 1$, for $0 \leq i \leq k$. To see that $D[i] \leq 1$, $0 \leq i \leq k$, suppose to the contrary that j is the first position where $D[j] > 1$. Note that $j > 0$ since otherwise $x_0 \geq 3$, and no vertex in X dominates vertex 1. Thus, for $j > 0$, $x_j - (3j+1) > 1$ but $x_{j-1} - (3(j-1)+1) \leq 1$. However, this implies that no vertex in X dominates vertex $3j+1$, a contradiction. A similar argument shows $D[i] \geq -1$, $0 \leq i \leq k$. Further, if j is the first position where $D[j] < 0$ then for all $\ell > j$, $D[\ell] = -1$. To see this, suppose to the contrary that $D[\ell] \geq 0$ but $D[\ell - 1] = -1$ for some $\ell > j$. Thus $x_\ell \geq 3\ell + 1$ and $x_{\ell-1} < 3(\ell-1)+1$, and this leaves vertex $3(\ell-1)+2 = 3\ell - 1$ undominated in G, a contradiction. A similar argument shows that if j is the last occurrence such that $D[j] = 1$, then for all $\ell < j$, $D[\ell] = 1$. This implies that the vector D consists of a run of $1's$ followed by a run of $0's$ and then a run of $-1's$, where each of these runs is of possibly length 0. To find a path from the vertex corresponding to X in $G(\gamma_e)$ to the vertex corresponding to S, find the last occurrence of a 1 in D, call this position j. Note that $j < k$, since $x_k \leq 3k+1$. Since $D[j+1] \leq 0$, x_j has one external private neighbor, namely $x_j - 1$. The set $X' = X - \{x_j\} \cup \{x_j - 1\}$ is then a γ_e - set of G. Since $D[j-1] = 1$, this swap decreases the number of external private neighbors of x_{j-1} to one. Hence we can perform the swap $X' - \{x_{j-1}\} \cup \{x_{j-1} - 1\}$ and produce a γ_e - set. This process continues until the swap of x_0 for 1 occurs. Then starting with the earliest occurrence of -1 in D, say at vertex x_ℓ, we perform the swap of x_ℓ for $x_\ell + 1$. Note that $\ell > 0$ since $x_0 \geq 1$. Thus this swap can occur since $D[\ell - 1] \geq 0$ leaving x_ℓ with only one external private neighbor, namely $x_\ell + 1$. We continue this second swapping process until x_k swaps with $x_{k+1} = 3k + 1$. Thus each dominating set X is some number of swaps away from S, and each swap under the above process produces a γ_e-set. By the cyclic nature, a similar argument proves C_{3k+1} is connected.

Definition 2.1. [2] *A stepgrid* $SG(k)$ *is an induced subgraph of the grid graph* $P_k \square P_k$*, defined as* $SG(k) = (V(k), E(k))$ *where* $V(k) = \{(i,j) : 1 \leq i, j \leq k, i+j \leq k+2\}$*, and* $E(k) = \{((i,j),(i',j')) : i = i', j' = j+1; i' = i+1, j = j'\}$*.*

Proposition 2.8. $P_n(\gamma_e) \simeq SG\left(\left\lceil \frac{n}{3} \right\rceil\right)$*, for* $n \equiv\ 1(mod\ 3)$*.*

Proof. Let S be a γ_e - set of P_{3k+1}. Every γ_e - set S' of P_{3k+1} is some number of swaps away from S. Let the swaps $(S' - \{i\} \cup \{i+1\})$ and $(S' - \{i\} \cup \{i-1\})$ be called as swaps of type-1 and type-2 respectively. By these swaps each vertex in $P_{3k+1}(\gamma_e)$ can be associated with an ordered pair (i, j), where i denotes the number of swaps of type-1 and j is the number of swaps of type-2 required to convert S to S'. The vertices 1 and $3k + 1$ can be swapped only once with an external private equitable neighbour freeing the other vertices to swap with newly created external private equitable neighbours. The other vertices $2 \leq i \leq 3k$ can be swapped once on either side. Hence, for each ordered pair (i, j) we have the relation $1 \leq i \leq k$, $1 \leq j \leq k$ and $i + j \leq k + 2$. If $l = i + 1$ and $m = j + 1$, we have $1 \leq l \leq k + 1$, $1 \leq m \leq k + 1$ and $l + m \leq (k + 1) + 2$.

Example 2.1. The following is the γ_e graph of P_{10}. Let $v_1 = \{1, 4, 7, 10\}$, $v_2 = \{2, 4, 7, 10\}$, $v_3 = \{2, 5, 7, 10\}$, $v_4 = \{2, 5, 8, 10\}$, $v_5 = \{2, 5, 8, 9\}$, $v_6 = \{2, 5, 6, 9\}$, $v_7 = \{1, 4, 7, 9\}$, $v_8 = \{2, 3, 6, 9\}$, $v_9 = \{1, 4, 6, 9\}$, $v_{10} = \{1, 3, 6, 9\}$, $v_{11} = \{2, 4, 7, 9\}$, $v_{12} = \{2, 5, 7, 9\}$, $v_{13} = \{2, 4, 6, 9\}$.



Proposition 2.9. *If $G \cup H$ denotes the disjoint union of two graphs G and H, then $(G \cup H)(\gamma_e) \simeq G(\gamma_e) \Box H(\gamma_e)$.*

3. γ_e - graphs of trees

Theorem 3.1. *The γ_e - graph $T(\gamma_e)$ of every tree is connected.*

Proof. We prove this result by induction on the order of the tree T. For $n \leq 3$, $T \simeq K_1, K_2$ or P_3 and $K_1(\gamma_e) \simeq K_1$, $K_2(\gamma_e) \simeq K_2$ and $P_3(\gamma_e) \simeq K_1$. Hence the result holds. Assume that the result holds for all trees T of order $n \leq m$. We prove the result is true for $n = m + 1$. If $T \simeq K_{1,m}$, then $T(\gamma_e) \simeq K_1$. Hence assume that $T \not\simeq K_{1,m}$.

Let T be the tree of order $m+1$ obtained from a tree T' by attaching a pendant v to the vertex $u \in T'$. Clearly, $T'(\gamma_e)$ is connected by induction.

Case 1: $u \in D$ for some γ_e - set D of T'.

Subcase 1.1: If $deg_{T'}(u) = 1$, then $deg_T(u) = 2$. Hence, there exist at least two γ_e - sets for T, D_1, D_2 containing u and v which are adjacent. Thus $T(\gamma_e)$ is connected.

Subcase 1.2: If $deg_{T'}(u) \geq 2$, then v is an equitable isolate of T. Hence v belongs to every γ_e - set of T. Since $T'(\gamma_e)$ is connected and v belongs to every γ_e - set of T, $T(\gamma_e)$ is connected.

Case 2: $u \notin D$ for any γ_e - set D of T'.

Subcase 2.1: If $deg_{T'}(u) = 1$, then $deg_T(u) = 2$. Clearly, $u \in D$ for some γ_e - set of T. Therefore, there exist at least two γ_e - sets for T, D_1, D_2 containing u and v which are adjacent. Thus $T(\gamma_e)$ is connected.

Subcase 2.2: If $deg_{T'}(u) \geq 2$, then v is an equitable isolate of T. Hence v belongs to every γ_e - set of T. Since $T'(\gamma_e)$ is connected and v belongs to every γ_e - set of T, $T(\gamma_e)$ is connected.

Hence, $T(\gamma_e)$ is connected for every tree T.

Theorem 3.2. *For any triangle-free graph G, $G(\gamma_e)$ is triangle free.*

Proof. Suppose $G(\gamma_e)$ contains a triangle of 3 vertices corresponding to γ_e-sets S_1, S_2, and S_3. Since (S_1, S_2) corresponds to an edge in $G(\gamma_e)$, $S_2 = S_1 - \{x\} \cup \{y\}$ for some $x, y \in V(G)$ such that $xy \in E(G)$. Further, since (S_2, S_3) corresponds to an edge in $G(\gamma_e)$, $S_3 = S_2 - \{c\} \cup \{d\}$ for some $c, d \in V(G)$ such that $cd \in E(G)$. However, $S_3 = S_2 - \{c\} \cup \{d\} = S_1 - \{x, c\} \cup \{y, d\}$. But since (S_1, S_3) corresponds to an edge in $G(\gamma_e)$, $S_3 = S_1 - \{a\} \cup \{b\}$ for some $a, b \in V(G)$ such that $ab \in E(G)$. Since S_3 cannot be obtained in two swaps from S_1, it must be the case that $x = a, c = y$, and $b = d$. But this implies that $xy, xb, yb \in E(G)$, a contradiction since G is triangle-free. Thus for any triangle - free graph G, there is no K_3 induced subgraph in $G(\gamma_e)$.

Corollary 3.1. *For any tree T, $T(\gamma_e)$ is triangle free.*

Theorem 3.3. *For any tree T, $T(\gamma_e)$ is C_n-free, for any odd $n \geq 3$.*
Proof. Suppose $T(\gamma_e)$ contains a cycle C_k, of $k \geq 3$, k odd. Let x be a vertex in C_k, and let S be the γ_e - set corresponding to vertex x. Let y and z be the two vertices on C_k of distance $m = \frac{k-1}{2}$ swaps away from x with corresponding γ_e-sets S_1 and S_2. That is, there is a path P_1 corresponding to a series of vertex swaps, say x_1 for y_1, x_2 for y_2, $\cdots$, x_m for y_m, so that $S_1 = (S - X) \cup Y$, where $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_m\}$. Likewise, there is a path P_2 corresponding to a series of vertex swaps, say w_1 for z_1, w_2 for z_2, $\cdots$, w_m for z_m, so that $S_2 = (S - W) \cup Z$, where $W = \{w_1, w_2, \cdots, w_m\}$ and $Z = \{z_1, z_2, \cdots, z_m\}$. However, since $yz \in E(T(\gamma_e))$, $S_2 = S_1 - \{a\} \cup \{b\}$ for some $a, b \in V(T)$. Thus, it must be the case that the set $X = W - \{w_j\} \cup \{x_i\}$ and $Y = Z - \{z_j\} \cup \{y_i\}$. This implies that $S_2 = S_1 - \{y_j\} \cup \{x_i\}$ and $x_i y_j \in E(T)$. Since x_i was swapped for y_i and x_j was swapped for y_j in P_1, we also know that $x_i y_i \in E(T)$ and $x_j y_j \in E(T)$. Now both x_i and y_i are in S_2, so there exists a swap x_l for y_i in P_2 which implies $x_l y_i \in E(T)$. However, in path P_1, x_l was swapped for y_l, and thus $x_l y_l \in E(T)$. Similarly, $y_l \in S_2$, so there exists some x_s so that in path P_2, x_s was swapped for y_l. We can continue to find these alternating P_1 and P_2 swaps, but, since m is finite, we must reach a vertex y_q which swapped with x_j in P_2, thus creating a cycle in T and contradicting the fact that T is cycle-free. Hence, $T(\gamma_e)$ is free of odd cycles.

Proposition 3.1. *For $n \geq 4$, $SK_{1,n}(\gamma_e) \simeq Q_n$ where $SK_{1,n}$ is subdivision of the star $K_{1,n}$*
Proof. Let $u, u_1, u_2, \cdots, u_n$ be the vertices of $K_{1,n}$ centered at u. Let u_i' be the vertex adjacent with corresponding u_i for $1 \leq i \leq n$ in $SK_{1,n}$. The central vertex of $SK_{1,n}$ is the only equitable isolate and hence belongs to every γ_e - set of $SK_{1,n}$. Thus $\gamma_e(SK_{1,n}) = n+1$. Every γ_e - set containing u_i can be replaced by u_i' for each $1 \leq i \leq n$ and vice versa. Therefore, there are 2^n γ_e - sets for $SK_{1,n}$ and hence 2^n vertices in $SK_{1,n}(\gamma_e)$ in which two vertices are adjacent if they differ in exactly one ordinate and each vertex has degree $n-1$. Thus, $SK_{1,n}(\gamma_e) \simeq Q_n$

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ON FOUR TUPLE OF DISTINCT INTEGERS SUCH THAT THE SUM OF ANY TWO OF THEM IS CUBE OF A POSITIVE INTEGER

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Abstract: In this article we have discussed determination of distinct positive integers a, b, c, d such that $a + b, a + c, b + c, a + d, b + d, c + d$ are cubes of positive integers with
(i) at least three numbers, say a, b, c are positive.
(ii) all four numbers a, b, c, d are positive.
We can obtain infinitely many four tuples from a single four-tuple.

Keywords and Phrases: Perfect squares, cubes, cubefree numbers, taxicab numbers, cubefree taxicab numbers, primes.

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1. Introduction

Number theory holds a distinguished position in mathematics for its many results which are profound and yet easy to state. Many of the problems in Number Theory arise from the role of addition and multiplication. One important class of such problems in which numbers can be expressed as sum of some numbers defined multiplicatively. This gives rise to the Pythagorean numbers, triangular numbers,

taxicab numbers, the four-square theorem of Lagrange, Goldbach conjecture and many such numbers and theorems, conjectures.
A remarkable aspect of Number Theory is that there is something in it for every one from puzzles as entertainment for layman to many open problems for scholars and mathematicians. For such problems, one may refer [3] or any standard book on Number Theory.
Perfect cubic numbers are $1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64, \cdots$ and for any integer k, with $m^3 < k < (m+1)^3$; is not a perfect cube number, for any integer m.
$k-$tuple of positive integers ($k \geq 3$), for any $m \in \mathbb{N}, (4m^3, 4m^3, \cdots, 4m^3)$ is such that the sum of its any two coordinates is $(2m)^3$, cube of the positive integer $2m$. Taking $m = 1, 2, 3, \cdots$, we get such infinitely many $k-$tuples. For any $p, q \in \mathbb{N}$ with $p^3 > 4q^3$, the $k-$tuple of positive integers $(p^3 - 4q^3, 4q^3, 4q^3, \cdots, 4q^3)$ is such that the sum of its any two coordinates is cube of a positive integer, that is p^3 or $(2q)^3$. Such $k-$tuples are infinitely many and first coordinate is different from the other coordinates. Above are trivial examples of $k-$tuples in which sum of any two coordinates is cube of a positive integer.

1.1. Four tuples of distinct positive integers such that sum of any two of its coordinates is a perfect square [1], [2], [5]

Examples of (a, b, c, d) as a four tuple of distinct positive integers such that $a+b, a+c, b+c, a+d, b+d, c+d$ are perfect squares are:
$(18, 882, 2482, 4743)$, $(4190, 10290, 39074, 83426)$, $(7070, 29794, 71330, 172706)$, $(55967,\ 78722,\ 27554, 10082)$, $(15710, 86690, 157346, 27554)$.
Following are four-tuples of integers where one coordinate is negative and sum of any two coordinates is a perfect square:
$(-286, 386, 770, 1730), (-126, 130, 270, 1026)$.

1.2. For determination of distinct $a, b, c, d \in \mathbb{Z}$, such that $a+b$, $a+c$, $b+c$, $a+d$, $b+d$, $c+d$ are cube of positive integers

We consider $a, b, c, d, p, q, r, s, t, u \in \mathbb{Z}$ with $a < b < c, p < q < r$ and $a+b = p^3, a+c = q^3, b+c = r^3, a+d = s^3, b+d = t^3, c+d = u^3$.

Above equations in matrix form is $\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p^3 \\ q^3 \\ r^3 \\ s^3 \\ t^3 \\ u^3 \end{bmatrix}$

Premultiplying by $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix}$,

and noting $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

we get,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p^3 \\ q^3 \\ r^3 \\ s^3 \\ t^3 \\ u^3 \end{bmatrix}$$

which gives,

$$\begin{aligned} a &= \frac{1}{2}(p^3 + q^3 - r^3) \\ b &= \frac{1}{2}(p^3 - q^3 + r^3) \\ c &= \frac{1}{2}(-p^3 + q^3 + r^3) \\ d &= \frac{1}{2}(-p^3 - q^3 + r^3 + 2s^3) \end{aligned} \tag{1}$$

It is easy to prove that, a, b, c are relatively prime if and only if p, q, r are relatively prime

$[\gcd(a, b, c) = v > 1 \Rightarrow \gcd(a + b, a + c, b + c) \geq v \Rightarrow \gcd(p, q, r) > 1$ etc].

Note that $p \in \mathbb{N} \Rightarrow p, q, r \in \mathbb{N}$.

By (1), for $\gcd(a, b, c) = 1$; exactly one from p, q, r is even and remaining two are odd numbers.

Again by (1); a, b, c, d are integers if and only if all integers p, q, r are even or exactly one of them is even.

In above [in equations (1) etc] the role of t^3, u^3 is invisible.

Here clearly $a + b = p^3, a + c = q^3, b + c = r^3$ and $a + d = s^3$.

Now $t^3 = b + d = -q^3 + r^3 + s^3, u^3 = c + d = -p^3 + r^3 + s^3$
$\Rightarrow s^3 + r^3 = t^3 + q^3 = u^3 + p^3$
This gives a taxicab number , expressed as a sum of two positive algebraic cubes in three distinct ways.
By Fermat's last theorem, $a + b + c + d$ never be a cube of any positive integer.

2. Taxicab Number [4], [6], [7]

There is a famous story in the mathematical folklore concerning the brilliant Indian mathematician Ramanujan. When Ramanujan was at Cambridge working with Hardy (1913), he felled ill and had to be admitted to a hospital at Putney. Hardy came to visit him, and remarked that he came in taxicab numbered 1729, which he found to be a dull number. Ramanujan noticed that this was actually a quite interesting number. It is the smallest number which can be expressed as the sum of two positive cubes in two different ways. As a result, the taxicab numbers are defined as those m for which there are solutions in positive integers to the equation, $m = x^3 + y^3 = u^3 + v^3$ for which $\{x, y\} \neq \{u, v\}$.
The nth taxicab number is a positive integer that can be expressed as a sum of two cubes of positive integers in n different ways. The smallest nth taxicab number is denoted by $T_a(n)$.
The concept of second taxicab number was first mentioned in 1657 by Bernard Frenicle de Bersy and was made famous in the early 20th century by a story involving Srinivasa Ramanujan and G. H. Hardy. In 1938, G. H. Hardy and E. M. Wright proved that such numbers exist for all positive integers n, and their proof is easily converted into a program to generate such numbers. However, the proof makes no claim at all about whether these generated numbers are the smallest positive and thus it cannot be used to find the actual value of $T_a(n)$.
Following six taxicab (smallest in size) are known.
$T_a(1) = 2 = 1^3 + 1^3$
$T_a(2) = 1729 = 1^3 + 12^3 = 9^3 + 10^3$
$T_a(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$
$T_a(4) = 6963472309248 = 2421^3 + 19083^3 = 5436^3 + 18948^3 = 10200^3 + 18072^3$
$= 13322^3 + 16630^3$
$T_a(5) = 48988659276962496 = 38787^3 + 365757^3 = 107839^3 + 362753^3 = 205292^3 + 342952^3 = 221424^3 + 336588^3 = 231518^3 + 331954^3$
$T_a(6) = 24153319581254312065344 = 582162^3 + 28906206^3 = 3064173^3 + 28894803^3$
$= 8519281^3 + 28657487^3 = 16218068^3 + 27093208^3 = 17492496^3 + 26590452^3$
$= 18289922^3 + 26224366^3$
$T_a(2)$ is also known as the Hardy-Ramanujan number. The subsequent taxicab numbers were found with the help of supercomputers. John Leech obtained $T_a(3)$

in 1957. E. Resenstiel, J. A. Dardis and C. R. Rosenstiel found $T_a(4)$ in 1991. J. A. Dardis found $T_a(5)$ in 1994 and it was confirmed by David W. Wilson in 1999. $T_a(6)$ was announced by Uwe Hollerbach on March 9, 2008.
Cubefree number means a positive integer that is not divisible by any p^3 where p is a prime. If a cubefree taxicab number T is written as $T = x^3 + y^3$, then x and y are relatively prime. Among the taxicab numbers $T_a(n), 1 \leq n \leq 6$, only $T_a(1)$ and $T_a(2)$ are cubefree taxicab numbers. The smallest cubefree taxicab number with three representations was discovered by Paul Vojta in 1981 while he was a graduate student. It is
$15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3 = 3^2 \times 5 \times 7 \times 31 \times 37 \times 199 \times 211$.
The smallest cubefree taxicab number with four representations was discovered by Staurt GAscoigne and independently by Duncon Moore in 2003. It is
$1801049058342701083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3 = 600259^3 + 1165884^3$.
Positive integers with three representations, not cube free are:
$87539319 = 436^3 + 167^3 = 423^3 + 228^3 = 44^3 + 255^3 = 3^3 \times 7 \times 31 \times 67 \times 223$
$1148834232 = 1044^3 + 222^3 = 920^3 + 718^3 = 846^3 + 816^3 = 2^3 \times 3^3 \times 7 \times 13 \times 211 \times 277$.

3. Main Results

Consider a (taxicab) number which is expressed as a sum of two cubes of positive integers in three different ways. Let it be $s^3 + r^3 = t^3 + q^3 = u^3 + p^3 \qquad \cdots (*)$
where $p, q, r \in \mathbb{N}$ are all even or exactly one of them is even.
Let $p < q < r$ and $p^3 + q^3 > r^3$.
Take

$$\begin{aligned} a &= \frac{1}{2}(p^3 + q^3 - r^3) \\ b &= \frac{1}{2}(p^3 - q^3 + r^3) \\ c &= \frac{1}{2}(-p^3 + q^3 + r^3) \\ d &= \frac{1}{2}(-p^3 - q^3 + r^3 + 2s^3) \end{aligned}$$

Then $a, b, c \in \mathbb{N}$ and $a < b < c, d = s^3 - a$ and $d \in \mathbb{N}$ iff $s^3 > a$.
Clearly $a + b = p^3, a + c = q^3, b + c = r^3, a + d = s^3$ and $b + d = s^3 + r^3 - q^3 = t^3$, $c + d = s^3 + r^3 - p^3 = u^3$ by (*).
Hence (a, b, c, d) is a four tuple of integers where a, b, c are positive and sum of any two of its coordinates is a cube of a positive integer.
Example 3.1. $T_a(3) = 87539319 = 167^3 + 436^3 = 228^3 + 423^3 = 255^3 + 414^3$

Taking $p = 255, q = 423, r = 436, s = 167$; (1) gives

$$a = \frac{1}{2}(255^3 + 423^3 - 436^3) = \frac{1}{2}(16581375 + 75686967 - 82881856) = \frac{9386486}{2} = 4693243,$$
$$b = \frac{1}{2}(255^3 - 423^3 + 436^3) = \frac{1}{2}(16581375 - 75686967 + 82881856) = \frac{23776264}{2} = 118881132,$$
$$c = \frac{1}{2}(-16581375 + 75686967 + 82881856) = \frac{141987448}{2} = 70993724,$$
$$d = \frac{1}{2}(-16581375 - 75686967 + 82881856 + 2 \times 167^3) = \frac{1}{2}(-938648 + 9314926)$$
$$= \frac{-71560}{2} = -35780$$

Clearly, $a + b = 4693243 + 1188813 = 16581375 = 255^3$,
$a + c = 75686967 = 423^3, a + d = 4657463 = 167^3, b + c = 82881856 = 436^3$
$b + d = 11888132 - 35780 = 11852352 = 228^3$,
$c + d = 70993724 - 35780 = 70957944 = 414^3$.
Thus $(4693243, 11888132, 70993724, -35780)$ is a four-tuple of integers such that sum of any two coordinates is cube of a positive integer.

Example 3.2. $15170835645 = 517^3 + 2468^3 = 709^3 + 2456^3 = 1733^3 + 2152^3$ is the smallest cubefree taxicab number with three representations.
Taking $p = 2152, q = 2456, r = 2468$ and $s = 517$, then by (1),

$$a = \frac{1}{2}(p^3 + q^3 - r^3) = 4873961696,$$
$$b = \frac{1}{2}(p^3 - q^3 + r^3) = 5092174112,$$
$$c = \frac{1}{2}(-p^3 + q^3 + r^3) = 9940473120$$
$$d = s^3 - a = -4735773283$$

where $a+b = 2152^3, a+c = 2456^3, b+c = 2468^3, a+d = 517^3, b+d = 709^3, c+d = 1733^3$.
$\Rightarrow$ Four tuple of integers $(4873961696, 5092174112, 9940473120, -4735773283)$ is such that sum of any two of its coordinates is cube of a positive integer.

Example 3.3. Positive integer, five representations, not cubefree:
$26059452841000 = 29620^3 + 4170^3 = 28810^3 + 12900^3 = 28423^3 + 14577^3$
$= 28423^3 + 14577^3 = 24940^3 + 21930^3$
Dividing by 8, from above we get
$3257431605125 = 14810^3 + 2085^3 = 14405^3 + 6450^3 = 12470^3 + 10965^3$

Taking $p = 10965, q = 14405, r = 14810$ and $s = 2085$, we get

$$\begin{aligned}
a &= \frac{1}{2}(p^3 + q^3 - r^3) = 529531610625,\\
b &= \frac{1}{2}(p^3 - q^3 + r^3) = 788803771500,\\
c &= \frac{1}{2}(-p^3 + q^3 + r^3) = 2459563869500\\
d &= s^3 - a = -520467646500
\end{aligned}$$

$\Rightarrow a+b = 10965^3, a+c = 14405^3, b+c = 14810^3, a+d = 2085^3, b+d = 6450^3, c+d = 12470^3$

$\Rightarrow (529531610625, 788803771500, 2459563869500, -520467646500)$ is a four-tuple where sum of its any two coordinates is a cube of positive integer.

Example 3.4. We have $143604279 = 522^3 + 111^3 = 460^3 + 359^3 = 423^3 + 408^3$

Taking $p = 408, q = 460, r = 522, s = 111$ we get by (1)

$a = 11508332, b = 56408980, c = 85827668, d = -10140701.$

$\Rightarrow (11508332, 56408980, 85827668, -10140701)$ is a four-tuple where sum of its any two coordinates is a cube of positive integer.

Example 3.5. $T_a(5)$ gives,

$48988659276962496 = 205292^3 + 342952^3 = 221424^3 + 336588^3 = 231518^3 + 331954^3$

Taking $p = 331954, q = 336588, r = 342952, s = 205292$, we get by (1),

$a = 17118752226891364, b = 19391636926444 7300, c = 20945030988258108,$

$d = -8535530906734276$

$\Rightarrow a + b = 331954^3, a + c = 336588^3, b + c = 342952^3,$

$a + d = 205292^3, b + d = 221424^3, c + d = 231518^3$

$\Rightarrow (17118752226891364, 193916369264447300, 20945030988258108,$

$-\ 8535530906734276)$ is a four-tuple where sum of its any two coordinates is cube of a positive integer.

Example 3.6. We have

$1801049058342701083 = 92227^3 + 1216500^3 = 136635^3 + 1216102^3 = 341995^3 + 1207602^3$

Taking $p = 1207602, q = 1216102, r = 1216500, s = 92227$ and using (1), we get

$a = 879641367671452208, b = 881407757105599000, c = 918856835019401000,$

$d = -878856901453751125$

$\Rightarrow a + b = 1207602^3, a + c = 1216102^3, b + c = 1216500^3,$

$a + d = 922273^3, b + d = 136635^3, c + d = 341995^3$

$\Rightarrow (879641367671452208, 881407757105599000, 918856835019401000,$

$-\ 878856901453751125)$ is a four-tuple where sum of its any two coordinates is cube of a positive integer.

Theorem 3.7. *There exist four-tuple of distinct positive integers such that sum of any two of its coordinates is cube of a positive integer.*

Proof. $T_a(6)$ gives
$24153319581254312065344 = 16218068^3 + 27093208^3 = 17492496^3 + 26590452^3$
$= 18289922^3 + 26224366^3$
Taking $p = 16218068, q = 17492496, r = 18289922, s = 26224366$, we get by (1),

$$a = \frac{1}{2}(16218068^3 + 17492496^3 - 18289922^3) = 1749942657207366722460,$$
$$b = \frac{1}{2}(16218068^3 - 17492496^3 + 18289922^3) = 2515826512048505687972,$$
$$c = \frac{1}{2}(-16218068^3 + 17492496^3 + 18289922^3) = 3602540998645922917476,$$
$$d = 26224366^3 - a = 16285009413352516737436$$

Here $a + b = 16218068^3, a + c = 17492496^3, b + c = 18289922^3$
$a + d = 26224366^3, b + d = 26590452^3, c + d = 27093208^3$
$\Rightarrow$ $(1749942657207366722460, 2515826512048505687972, 3602540998645922917476, 16285009413352516737436)$ is a four-tuple of distinct positive integers where sum of any two of its coordinates is cube of a positive integer.

4. Conclusion

From above theorem, there is a four tuple of distinct positive integers such that sum of its any two coordinates is cube of a positive integer. If (a, b, c, d) be such a four tuple of distinct positive integers, then for any $n \in \mathbb{N}, (4n^3a, 4n^3b, 4n^3c, 4n^3d)$ is a four tuple of distinct positive integers where sum of any two of its coordinates is cube of a positive integer. Thus there are infinitely many such four tuples of distinct positive integers.

For any integer $n \geq 4$, if we have $p, q, r, s, t, u \in \mathbb{N}$ such that $s^n + r^n = t^n + q^n = u^n + p^n$ with $p < q < r$ and $s^n > p^n + q^n - r^n > 0$, then taking

$$a = \frac{1}{2}(p^n + q^n - r^n)$$
$$b = \frac{1}{2}(p^n - q^n + r^n)$$
$$c = \frac{1}{2}(-p^n + q^n + r^n)$$
$$d = s^n - a$$

(where all p, q, r are even or exactly one of them is even) we get a four-tuple of positive integers (a, b, c, d) such that sum of its any two coordinates is nth power of a positive integer, i.e. $a+b = p^n, a+c = q^n, b+c = r^n, a+d = s^n, b+d = t^n, c+d = u^n$.

5. Future Scope

For budding researchers, there are open problems to determine four-tuple (a, b, c, d) of positive integers, such that sum of any two of them is fourth, fifth, ... power of a positive integer.

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EQUIVALENCIES OF CORDIAL LABELING AND SUM DIVISOR CORDIAL LABELING

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Abstract: In this paper, it has been proved that every tree T is SDC (Sum divisor cordial), every graceful graph with certain condition is SDC and the cordial labeling, sum divisor cordial labeling for a graph G are equivalent.

Keywords and Phrases: Sum divisor cordial graph, graceful graph, cordial graph, tree.

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1. Introduction

We begin with a simple, undirected finite graph G with $p = |V(G)|$, the number of vertices in G and $q = |E(G)|$, the number of edges in G. For all basic terminology, definitions and standard notations, we follows Harary [3]. Gallian [2] survey provide vast amount of literature on different type graph labeling.

Rosa [6] defined α-labeling (α-Graceful labeling) as a graceful labeling f with an additional property that there is an integer $k(0 \leq k < q)$ such that for any

$uv \in E(G)$, $\min\{f(u), f(v)\} \leq k < \max\{f(u), f(v)\}$. An α-graceful graph is necessarily a bipartite graph with partition $V_1 = \{w \in V(G)/f(w) \leq k\}$ and $V_2 = V(G) - V_1$. The cordial labeling introduced by Cahit [1] is a weaker version of graceful labeling.

Lourdusamy and Patrick [5] defined the concept of Sum Divisor Cordial (SDC) labeling. For a graph G, $f : V(G) \longrightarrow \{1, 2, \ldots, p\}$ is a bijection and its edge labeling function $f^\star : E(G) \longrightarrow \{0, 1\}$ defined as

$$f^\star(uv) = \begin{cases} 1; & \text{if } f(u) + f(v) \text{ is even} \\ 0; & \text{Otherwise .} \end{cases}$$

Then the function f is called a SDC labeling, if $|e_f(0) - e_f(1)| \in \{0, 1\}$, where $e_f(0)$= the number of edge in G with 0 label under $f^\star$ and $e_f(1)$= the number of edge in G with 1 label under $f^\star$. A graph G is called SDC, if it admits an SDC labeling.

Let $f : V(G) \longrightarrow \{0, 1\}$ be a function and its edge labeling function $f^\star : E(G) \longrightarrow \{0, 1\}$ define as

$$f^\star(uv) = \begin{cases} 1; & \text{if } f(u) + f(v) \text{ is odd} \\ 0; & \text{Otherwise .} \end{cases}$$

Then the function f is called a cordial labeling, if $|e_f(0) - e_f(1)| \in \{0, 1\}$ and $|v_f(0) - v_f(1)| \in \{0, 1\}$, where $e_f(0)$ and $e_f(1)$ are earlier discussed, while $v_f(0)$= the number of vertices in G with 0 label under f, $v_f(1)$= the number of vertices in G with 1 label under f. A graph G is called cordial, if it admits a cordial labeling.

Let $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ be an injective function and its edge induce labeling function $f^\star : E(G) \longrightarrow \{1, 2, \ldots, q\}$ defined as $f^\star(uv) = |f(u) - f(v)|$, $\forall$ $uv \in E(G)$. f is called a graceful labeling, if $f^\star$ is a bijective and G is called a graceful graph.

The Ringel-Kotzig tree conjecture has been the focus of many papers. Cahit [1] prove that all tree are cordial. Shridevi et al [7] defined the concept of odd even graceful labeling. Kaneria et al [4] have discussed some graceful related labeling and their applications. They also prove that graceful labeling and odd even graceful labeling are equivalent. In a graph G, $v \in V(G)$ is called a pendent vertex, if its degree is one and $e \in E(G)$ is called a pendant edge, if one end vertex of e is a pendant vertex.

2. Main Results

Theorem 2.1. *Every tree T is SDC.*

Proof. To prove this result, induction on $n = |V(T)|$ will be taken. If $n \in \{1, 2\}$, then T is either $K_1 = P_1$ or $K_2 = P_2$. Since, every path P_n is SDC proved by Lourdusamy and Patrick [**?**], T is SDC in both the cases. Since, every tree T has at least two pendant vertices, let $v_1 \in V(T)$ and degree of v_1 is one.

$T_1 = T - \{v_1\} = (V(T) - \{v_1\}, E(T) - \{e_1\})$, where e_1 is the pendent edge of T, which is incident with v_1 in T and T_1 is also a tree with $|V(T_1)| = n - 1$.
By induction hypothesis T_1 admits a SDC labeling say $f : V(T_1) \longrightarrow \{1, 2, \ldots, n-1\}$. Let $e_1 = v_0v_1$, for some $v_0 \in V(T_1)$. Following cases will be considered to define SDC labeling $g : V(T) \longrightarrow \{1, 2, \ldots, n\}$.

Case-**(A) :**

$$(i) e_f(0) = e_f(1) \text{ in } T_1$$
$$(ii) e_f(0) = e_f(1) + 1 \text{ in } T_1 \text{ and } f(v_0) = \text{ odd}$$
$$(ii) e_f(0) + 1 = e_f(1) \text{ in } T_1 \text{ and } f(v_0) = \text{ even}$$

Take $g = f$ on $V(T_1)$ and $g(v_1) = n$.

Case-**(B) :**

$$(iv) e_f(0) = e_f(1) + 1 \text{ in } T_1 \text{ and } f(v_0) = \text{ even}$$
$$(v) e_f(0) + 1 = e_f(1) \text{ in } T_1 \text{ and } f(v_0) = \text{ odd}$$

Take $g = n - f$ on $V(T_1)$ and $g(v_1) = n$.

By definition of $g = V(T) \longrightarrow \{1, 2, \ldots, n\}$, g is a bijection. By following subcases, it will be proved that $|e_g(0) - e_g(1)| \in \{0, 1\}$ in T.

Case-**(i) :** $e_f(0) = e_f(1)$ in T_1.

Since, $g^\star(v_0v_1) = \{0, 1\}$ and $e_f(0) = e_g(0) = e_g(1) = e_f(1)$ in T_1, $|e_g(0) - e_g(1)| \in \{0, 1\}$ hold, in this case. In rest cases (ii) to (v), the number of vertices in T_1 is even, as $|E(T_1)|=$ odd. Hence, n is odd.

Case-**(ii) :** $e_f(0) = e_f(1) + 1$ in T_1 and $f(v_0)=$ odd.

Since, $e_f(0) = e_g(0)$, $e_f(1) = e_g(1)$ in T_1 and $g^\star(v_0v_1) = 1$, we obtained $e_g(1) = e_g(0)$ in T. Hence, $|e_g(0) - e_g(1)| = 0 \in \{0, 1\}$, in this case.

Case-**(iii) :** $e_f(0) + 1 = e_f(1)$ in T_1 and $f(v_0)=$ even.

Since, $e_f(0) = e_g(0)$, $e_f(1) = e_g(1)$ in T_1 and $g^\star(v_0v_1) = 0$, we obtained $e_g(1) = e_g(0)$, in T. Hence, $|e_g(0) - e_g(1)| \in \{0.1\}$, in this case.

Case-**(iv) :** $e_f(0) = e_f(1) + 1$ in T_1 and $f(v_0)=$ even.

Since, $g(v_0) = n - f(v_0)=$ odd, $e_f(0) = e_g(0)$, $e_f(1) = e_g(1)$, we obtained $g^\star(v_0v_1) = 1$. Hence, $e_g(1) = e_g(0)$ in T and so, $|e_g(0) - e_g(1)| \in \{0.1\}$, in this case.

Case-**(v) :** $e_f(0) + 1 = e_f(1)$ in T_1 and $f(v_0)=$ odd.

Since, $g(v_0) = n - f(v_0)$= even, $e_f(0) = e_g(0)$, $e_f(1) = e_g(1)$, in T_1, we obtained $g^\star(v_0v_1) = 0$. Hence, $e_g(0) = e_g(1)$ in T and so, $|e_g(0) - e_g(1)| \in \{0.1\}$, in this case.

Thus, it has been proved that the condition $|e_g(0) - e_g(1)| \in \{0, 1\}$ hold in any case. Hence, g is a SDC labeling for T. Therefore, T is SDC.

Theorem 2.2. *A graceful graph G with $|v_f(odd) - v_f(even)| \leq 1$ is SDC.*

Proof. Let $f : V(G) \longrightarrow \{0, 1, 2, \ldots, q\}$ be a graceful labeling for G. So f is a injective map and its edge induced map $f^\star : E(G) \longrightarrow \{1, 2, \ldots, q\}$ is bijective. Here v_f(odd) is the number of vertices in G with odd label and v_f(even) is the number of vertices in G with even label under f. Let $V_fG(\text{Odd}) = \{v \in V(G)/f(v) = \text{odd}\}$ and $V_fG(\text{Even}) = V(G) - V_fG(\text{Odd})$.

Following three cases will be considered to define SDC labeling $g : V(G) \longrightarrow \{1, 2, \ldots, p\}$.

Case-**(I) :** p is even. Let $t = \frac{p}{2}$ and take $V_fG(\text{Odd}) = \{v_1, v_3, \ldots, v_{2t-1}\}$, V_fG (Even) $= \{v_2, v_4, \ldots, v_{2t}\}$. Define $g(v_i) = i$, $\forall\ i = 1, 2, \ldots, p$.

It can be observe that for any $uv \in E(G)$, $f^\star(uv)$= even $= |f(u) - f(v)|$, gives $g^\star(uv) = 1$ as $f(u)$, $f(v)$ both lies in same sets from $V_fG(\text{Odd})$, $V_fG(\text{Even})$ and $f^\star(uv)$= odd $= |f(u) - f(v)|$, gives $g^\star(uv) = 0$ as $f(u)$, $f(v)$ both lies in different sets from $V_fG(\text{Odd})$, $V_fG(\text{Even})$.

Case-**(II) :** $|V_fG(\text{Odd})| = v_f(\text{odd}) > v_f(\text{even})$ and p is odd. Let $t = \frac{p-1}{2}$ and take $V_fG(\text{Odd}) = \{v_1, v_3, \ldots, v_p\}$, $V_fG(\text{Even}) = \{v_2, v_4, \ldots, v_{2t}\}$. Define $g(v_i) = i$, $\forall$ $i = 1, 2, \ldots, p$.

It can be observe that for any edge $uv \in E(G)$, $f^\star(uv) = |f(u) - f(v)|$= even and so, $f(u), f(v)$ both lies in same set from $V_fG(\text{Odd}), V_fG(\text{Even})$. Implies $g^\star(uv) = 1$. Similarly $f^\star(uv)$= odd gives $g^\star(uv) = 0$, as $f(u), f(v)$ both lies in different sets from $V_fG(\text{Odd}), V_fG(\text{Even})$.

Case-**(III) :** $v_f(\text{odd}) < v_f(\text{even})$ and p is odd. Take $V_fG(\text{Odd}) = \{v_1, v_3, \ldots, v_{2t-1}\}$, $V_fG(\text{Even}) = \{v_0, v_2, v_4, \ldots, v_{2t}\}$. Define $g(v_i) = i + 1$, $\forall\ i = 1, 2, \ldots, p$.

It can be observe that for any edge $uv \in E(G)$, whenever $f^\star(uv)$= even $\Rightarrow$ $g^\star(uv) = 1$ and similar way $f^\star(uv)$= odd $\Rightarrow g^\star(uv) = 0$.

Thus, it has been proved that

$$f^\star(e) = \text{even} \Leftrightarrow g^\star(e) = 1 \text{ and}$$
$$f^\star(e) = \text{odd} \Leftrightarrow g^\star(e) = 0, \forall e \in E(G).$$

Since, G is graceful, $|e_f(\text{even}) - e_f(\text{odd})| \in \{0, 1\}$, where e_f(even)= number of edges in G with label even and e_f(odd)= number of edges in G with label odd under $f^\star$. Hence, $|e_g(0) - e_g(1)| \in \{0, 1\}$, g is a SDC labeling. Hence, G is SDC.

Theorem 2.3. *Every cordial graph G is SDC.*

Proof. Let $f : V(G) \longrightarrow \{0,1\}$ and its edge induced function $f^{\star} : E(G) \longrightarrow \{0,1\}$ be a cordial labeling for G. So, it satisfies the condition $|e_f(0) - e_f(1)|$, $|v_f(0) - v_f(1)| \in \{0,1\}$. Following two cases will be considered to define SDC labeling $g : V(G) \longrightarrow \{1,2,\ldots,p\}$.

Case-**(I) :** $v_f(0) = v_f(1)$ in G under f.

In this case p should be even. Take $t = \frac{p}{2}$, $V_fG(1) = \{v_1, v_3, \ldots, v_{2t-1}\}$ and $V_fG(0) = \{v_2, v_4, \ldots, v_{2t}\}$. Define $g(v_i) = i, \forall i = 1,,2,\ldots,p$.

It can be observe that for any $uv \in E(G)$, whenever $f^{\star}(uv) = 0 \Rightarrow f(u) + f(v)$ is even.

$\Rightarrow f(u), f(v)$ both lies in same set from $V_fG(0), V_fG(1)$.

$\Rightarrow g^{\star}(uv) = 1$. Similarly $f^{\star}(uv) = 1 \Rightarrow g^{\star}(uv) = 0$.

Thus, $f^{\star}(uv) = 1 - g^{\star}(uv)$, $\forall uv \in E(G)$.

Case-**(II) :** $|v_f(0) - v_f(1)| = 1$

In this case p should be odd. Since, $1 - f$ is also a cordial labeling for G with $v_f(0) = v_{1-f}(1)$, $v_f(1) = v_{1-f}(0)$ and $e_f(i) = e_{1-f}(i)$ $\forall i = 1,2$, without loss of generality we may assume that $v_f(1) = v_f(0) + 1$. Take $t = \frac{p-1}{2}$, $V_fG(0) = \{v_2, v_4, \ldots, v_{2t}\}$ and $V_fG(1) = \{v_1, v_3, \ldots, v_p\}$. Define $g(v_i) = i, \forall i = 1,2,\ldots,p$.

It can be observe that for any edge $uv \in E(G)$, whenever $f^{\star}(uv) = 1 \Rightarrow f(u) + f(v) = 1$.

$\Rightarrow f(u), f(v)$ both lies in different sets from $V_fG(0), V_fG(1)$.

$\Rightarrow g^{\star}(uv) = 0$. Similarly $f^{\star}(uv) = 1 \Rightarrow g^{\star}(uv) = 0$.

Thus, $g^{\star}(uv) = 1 - f^{\star}(uv)$, $\forall uv \in E(G)$.

Thus, we assert that for any edge $e \in E(G)$, $f^{\star}(e) = 1 - g^{\star}(e)$. Moreover g is a bijective map. Since, G is cordial under f, we obtained $|e_g(o) - e_g(1)| = |e_f(o) - e_f(1)| \in \{0,1\}$. Hence G is SDC.

Theorem 2.4. *Every SDC graph G is cordial.*

Proof. Let $f : V(G) \longrightarrow \{1,2,\ldots,p\}$ is a SDC labeling for G and $f^{\star} : E(G) \longrightarrow \{0,1\}$ define by

$$f^{\star}(uv) = \begin{cases} 1; & \text{if } f(u) + f(v) \text{ is even} \\ 0; & \text{if } f(u) + f(v) \text{ is odd.} \end{cases}$$

Define $g : V(G) \longrightarrow \{0,1\}$ follows by

$$g(v) = \begin{cases} 0; & \text{if } f(v) \text{ is even} \\ 1; & \text{if } f(v) \text{ is odd}, \forall v \in V(G). \end{cases}$$

It can be observe that $v_g(0) = |V_fG(\text{Even})|$ and $v_f(1) = |V_fG(\text{Odd})|$, where $V_fG(\text{Odd}) = \{v \in V(G)/f(v) \text{is odd}\}$, $V_fG(\text{Even}) = \{v \in V(G)/f(v) \text{is even}\}$. Since, g is bijective $|V_fG(\text{Odd})| - |V_fG(\text{Even})|$ is either 0 or 1.

$\Rightarrow v_g(1) - v_g(0) \in \{0, 1\}$.
Moreover for any edge $uv \in E(G)$,

$$\begin{aligned} f^{\star}(uv) = 0 &\Leftrightarrow f(u) + f(v) \text{ is odd} \\ &\Leftrightarrow f(u), f(v) \text{ lies in different sets from } V_fG(\text{Odd}), V_fG(\text{Even}) \\ &\Leftrightarrow g^{\star}(uv) = 1. \end{aligned}$$

and

$$\begin{aligned} f^{\star}(uv) = 1 &\Leftrightarrow f(u) + f(v) \text{ is even} \\ &\Leftrightarrow f(u), f(v) \text{ lies in same set from } V_fG(\text{Odd}), V_fG(\text{Even}) \\ &\Leftrightarrow g^{\star}(uv) = 0. \end{aligned}$$

Therefor, $e_f(0) = e_g(1)$, $e_g(0) = e_f(1)$ in G under f and g. Hence, g satisfied the conditions $|e_g(0) - e| = |e_g(0) - e_g(1)|$, $|v_g(1) - v_g(0)| \in \{0, 1\}$ and so, g is a cordial labeling for G. Therefore, G is cordial.

3. Concluding Remarks

In Theorem 2.1, we prove that every tree is SDC by induction hypothesis. In Theorem 2.2, a graceful graph G with $|v_f(\text{odd}) - v_f(\text{even})| \le 1$ is SDC. We generalized result of Lourdusamy and Patrick [5] in Theorem 2.3, that every cordial graph G is SDC also every SDC graph G is cordial, in Theorem 2.4.

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RESTRICTED MINUS DOMINATION NUMBER OF A GRAPH

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Abstract: A restricted minus dominating function on a graph $G = (V, E)$ is a function $f : V \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 0$ for every vertex $v \in V$ and a vertex assigned 0 is adjacent to at least one vertex assigned 1. The restricted minus domination number $\gamma_r^-(G) = min\{w(f) : f$ is restricted minus dominating function$\}$. In this paper, we initiate the study of $\gamma_r^-(G)$ and its relationship with sign and minus domination are investigated. Many of the known bounds of $\gamma_r^-(G)$ are immediate consequence of our results.

Keywords and Phrases: Graph, domination number, minus domination number, restricted minus domination.

2010 Mathematics Subject Classification: 05C69, 05C70.

1. Introduction

All graphs considered in this paper are finite, simple, and undirected. For a general reference on graph theory, the reader is directed to [8]. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Let $n = |V|$ and $m = |E|$ denote the number of vertices and edges of a graph G, respectively. For any vertex v

of G, let $N(v)$ and $N[v]$ denote its open and closed neighborhoods respectively. $\alpha_0(G)(\alpha_1(G))$, is the minimum number of vertices (edges) in a vertex (edge) cover of G. $\beta_0(G)(\beta_1(G))$, is the minimum number of vertices (edges) in a maximal independent set of vertex (edge) of G. Let $deg(v)$ be the degree of a vertex v in G, $\triangle(G)$ and $\delta(G)$ be maximum and minimum degree of vertices of G, respectively. The complement G^c of a graph G is the graph having the same set of vertices as G denoted by V^c and in which two vertices are adjacent, if and only if they are not adjacent in G. A tree T is an acyclic connected graph.

A dominating set $D \subseteq V$ for a graph G is such that each $v \in V$ is either in D or adjacent to a vertex of D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. For complete review on domination and its related parameters, refer [1], [9] and [10].

For any real valued function $f : V \to R$ the weight of f is denoted and defined as $w(f) = \sum_{v \in V} f(v)$.

A sign dominating function (SDF) of a graph G is a function $f : V \to \{-1, 1\}$ such that $f(N[v]) \geq 1$ for all $v \in V$. The sign domination number of a graph G is $\gamma_s(G) = min\{w(f) : f$ is sign dominating function$\}$. For more details on sign domination, we refer [3] and [14].

A minus dominating function (MDF) of a graph G is a function $g : V \to \{-1, 0, 1\}$ such that $g(N[v]) \geq 1$ for all $v \in V$. The minus domination number of a graph G is $\gamma^-(G) = min\{w(g) : g$ is minus dominating function$\}$. For more details on minus domination, we refer [2], [5], [7], [11], [12] and [13].

A restricted minus dominating function (RMDF) on a graph G is a function $f : V \to \{-1, 0, 1\}$ such that $f(N[v]) \geq 0$ for every vertex $v \in V$ and a vertex assigned 0 is adjacent to at least one vertex assigned 1. The restricted minus domination number $\gamma_r^-(G) = min\{w(f) : f$ is restricted minus dominating function$\}$. Let $|V_{-1}|$, $|V_0|$ and $|V_1|$ denote number of vertices assigned -1, 0 and 1 respectively.

2. Existing Result

Theorem 2.1. [4] *For any tree T, $\gamma^-(T) \geq 1$ with equality if and only if $T \cong K_{1,n-1}$.*

Theorem 2.2. [6] *Let G be a graph with n vertices. If $\gamma_s(G) = 0$, then $n \geq 6$.*

Theorem 2.3. [6] *For any graph G, $\gamma_s(G) = n$ if any only if every non isolated vertex is either an endvertex or adjacent to an endvertex.*

3. Results

We start with the couple of observations, which we use in sequel.

Observation 3.1. *A vertex which is assigned -1 is always adjacent to at least one* *vertex assigned 1.*

Proof. Since weight of every vertex of a graph G should not be negative it implies every vertex $v \in V$ which is assigned -1 should be adjacent to at least one vertex assigned 1 such that $f(N[v]) \geq 0$.

Observation 3.2. *By the definitions of $\alpha_o(G), \alpha_1(G), \beta_o(G)$ and $\beta_1(G)$, Clearly, $\gamma_r^-(G) < min\{\alpha_o(G), \alpha_1(G), \beta_o(G), \beta_1(G)\}$.*

Theorem 3.1. *For any path P_n with $n \geq 1$ and Cycle C_n with $n \geq 3$ vertices,*

$$\gamma_r^-(P_n) = \gamma_r^-(C_n) = \begin{cases} 1 & \textit{if } \; n{=}3k{+}1, \\ 0 & \textit{otherwise}, \end{cases}$$

where k is a positive integer.

Proof. The result can be easily checked for $n = 1$ and 2. We shall prove the result for $n \geq 3$ vertices. For any positive integer k, if there are $3k$-vertices, then -1, 1, 0 is assigned k-times. Hence $\gamma_r^-(G) = 0$. If there are $(3k+1)$-vertices, then as usual $3k$-vertices are assigned -1, 1, 0 in order. Since the last vertex among $3k$-vertices is assigned 0 and $(3k+1)^{th}$ vertex say v can neither be assigned 0 as it will not be adjacent to 1 nor -1 as $f(N[v]) = -1$. Hence it should be assigned 1. For such assignment $\gamma_r^-(G) = 1$. If there are $(3k+2)$-vertices, then -1, 1, 0 are assigned to $3k$-vertices in order. $(3k+1)^{th}$ vertex is assigned 1 and $(3k+2)^{nd}$ vertex can be assigned either 0 or -1. Since the restricted minus domination number of G is minimum of such assignments, we assign -1 to the last vertex. Hence $\gamma_r^-(G) = |V_1| - |V_{-1}| = 0$.

Theorem 3.2. *For any complete bipartite graph $K_{p,q}$ with bipartitions $|P_1| = p$ and $|P_2| = q$,*

$$\gamma_r^-(K_{p,q}) = 1.$$

Proof. Let $f : V \to \{-1, 0, 1\}$ be a restricted minus dominating function.

Case 1. If the number of vertices assigned 1 is equal to number of vertices assigned -1, then for any vertex $v \in V_{-1}$, $f(N[v]) < 0$.

Case 2. If the number of vertices assigned 1 is less than the number of vertices assigned -1 then there is at least one vertex $v \in V$ such that $f(N[v]) < 0$.

From the above two cases $\gamma_r^-(K_{p,q}) > 0$ and $|V_1| > |V_{-1}|$.

If $|V_1| = |V_{-1}| + 1$ then $f(N[v]) > 0$ for all $v \in V$. Hence $\gamma_r^-(K_{p,q}) = 1$. Thus $\gamma_r^-(G) = 1$.

To prove our next result, we make use of the following definition:

A graph G is outerplanar if it has a crossing-free embedding in the plane such that all vertices are on the same face.

Theorem 3.3. *For any positive integer k, there exist an outerplanar graph G with $\gamma_r^-(G_k) \leq -k$.*
Proof. Consider the outerplanar graph G_k which can be constructed as in Figure 1. Then there are $(3k+3)$-vertices out of which $(2k+2)$ vertices are of degree 1. By assigning -1 to $2k$ vertices of degree 1, 1 to k vertices of degree 5 and 0 to remaining vertices produces RMDF f of G_k of weight $k - 2k = -k$ as illustrated. This implies that the restricted minus domination number $\gamma_r^-(G) \leq -k$.

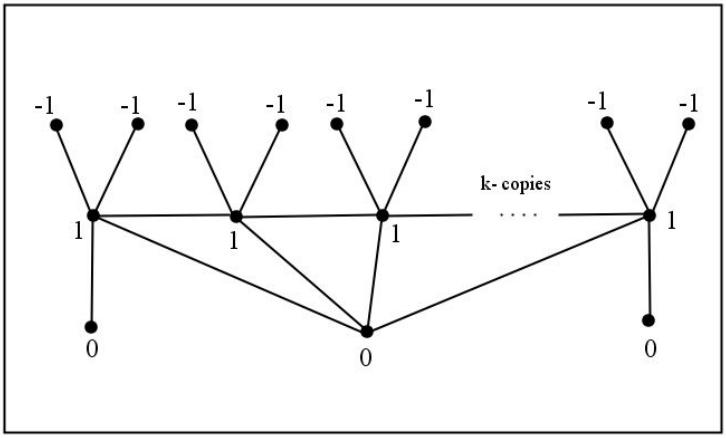

Figure-1: An outerplanar graph G_k with $\gamma_r^-(G_k) \leq -k$

Theorem 3.4. *For any connected graph G, $\gamma_r^-(G) = 0$ if and only if $|V_1| = |V_{-1}|$.*
Proof. As vertices assigned 0 is adjacent to at least one vertex assigned 1, implies that V_1 dominates vertices of V_0. Hence $\gamma_r^-(G) = |V_1| - |V_{-1}|$. Suppose $|V_1| = |V_{-1}|$. This implies that $\gamma_r^-(G) = 0$. On the other hand, if $\gamma_r^-(G) = 0$, then $|V_1| - |V_{-1}| = 0$.

Theorem 3.5. *Let G be a nontrivial graph with $\triangle(G) = n - 1$. Then*

(i) $\gamma_r^-(G) = 0$.

(ii) $\gamma_r^-(G) \leq \gamma_r^-(G^c)$.

Proof. Let G be a nontrivial graph with n-vertices.
(i) Let v be a vertex of degree $n - 1$. If we assign 1 to vertex v, assign -1 to a vertex adjacent to v and remaining $(n-2)$-vertices are assigned 0, then such an assignment satisfies both the conditions RMDF. Hence $\gamma_r^-(G) = 0$. (ii) If G is a graph with $\triangle(G) = n - 1$, then by (i), $\gamma_r^-(G) = 0$. Also, the graph G^c is a disconnected graph, this implies that $\gamma_r^-(G) \leq \gamma_r^-(G^c)$.

Theorem 3.6. *For any connected graph G,*

$$\gamma_r^-(G) \leq \gamma(G).$$

Proof. Let $f : V \to \{0, 1\}$ be a dominating function and $g : V \to \{-1, 0, 1\}$ be RMDF on a graph G. Then $f(N[v]) \geq 1$ and $g(N[v]) \geq 0$ for every $v \in V$. As $\gamma(G) \geq 1$ and due to the fact of the Theorem 3.3, the result follows.

Theorem 3.7. *For any nontrivial graph G, $\gamma_r^-(G) \leq n - \Delta(G)$. Further, the bound is attained if the graph G is totally disconnected.*

Proof. Let G be a graph with n-vertices. Then, we consider the following cases:

Case 1. If $\Delta(G) = 0$, then $G \cong K_n^c$ and $n - \Delta(G) = n$. We have $\gamma_r^-(G) = n$.

Case 2. If $\Delta(G) = 1$, then $G \cong K_2$ and $n - \Delta(G) = 1$. Here, one vertex is assigned 1 and other vertex is assigned -1. Then $\gamma_r^-(G) = 0$.

Case 3. If $\Delta(G) = n - 1$, then $n - \Delta(G) = 1$. Then by Theorem 3.5, $\gamma_r^-(G) = 0$.

Case 4. If $\Delta(G) = k$ other than above considerations, then $n - \Delta(G) = n - k > 1$. We have $\gamma_r^-(G) < n - k$.

Hence, from all the above cases the result is proven.

Theorem 3.8. *For any tree T,*

$$\gamma_r^-(T) \leq \gamma^-(T).$$

Proof. Let T be a tree. Then by Theorem 2.1, $\gamma^-(T) \geq 1$ and by Theorem 3.3, we have $\gamma_r^-(T) \leq \gamma^-(T)$.

There is no good relation between $\gamma_r^-(G)$ and $\gamma^-(G)$ except for tree. For illustration, consider the graphs G_1, G_2 and G_3.

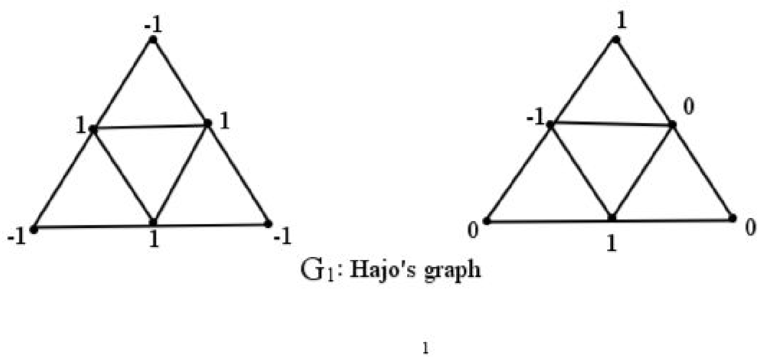

Figure-2 Graphs with $\gamma^-(G_1)$ and $\gamma_r^-(G_1)$.

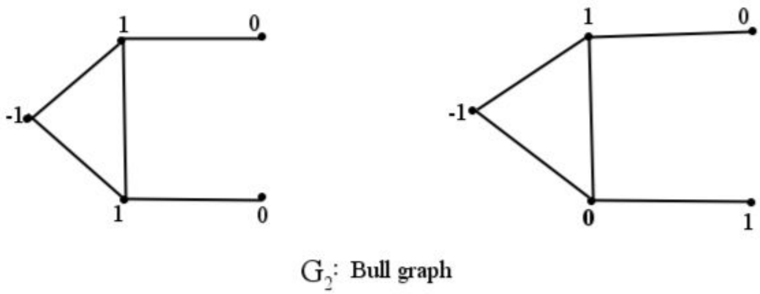

Figure-3 Graphs with $\gamma^-(G_2)$ and $\gamma_r^-(G_2)$.

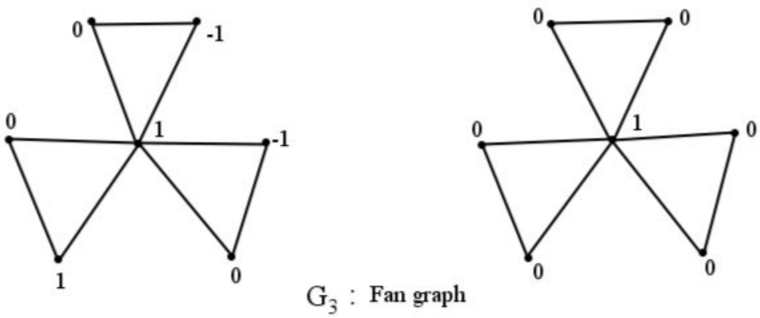

Figure-4 Graphs with $\gamma_r^-(G_3)$ and $\gamma^-(G_3)$.

In Figure 2, we have $\gamma^-(G_1) < \gamma_r^-(G_1)$.
In Figure 3, we have $\gamma_r^-(G_2) = \gamma^-(G_2)$.
In Figure.4, we have $\gamma^-(G_3) > \gamma_r^-(G_3)$.

Theorem 3.9. *Let G be a graph with $n \geq 1$ vertices. Then $\gamma_s(G) = \gamma_r^-(G) = n$ if and only if $G \cong K_n^c$.*

Proof. Let $G \cong K_n^c$. Then, under RMDF and SDF every vertex of G is assigned 1. Hence $\gamma_s(G) = \gamma_r^-(G) = n$. On the other hand, let $\gamma_s(G) = \gamma_r^-(G) = n$. By Theorem 2.3, $\gamma_s(G) = n$ if and only if every vertex of G is either endvertex or support vertex. For any graph G other than K_n^c, where every vertex of G is either endvertex or support vertex, we get a contradiction. Hence the result.

Theorem 3.10. *Let G be a graph with $n \geq 6$ vertices. If $|V_1| = |V_{-1}|$, then*

$$\gamma_r^-(G) = \gamma_s(G).$$

Proof. By Theorem 3.4 and Theorem 2.2, the desired result follows.

Open Problem: For which class of graphs G is

1. $\gamma_r^-(G) = \gamma(G)$.
2. $\gamma_r^-(G) = \gamma^-(G)$.
3. $\gamma_r^-(G) = \gamma_s(G)$.

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HYPOTHESIS OF VALUE DISTRIBUTION AND ITS ASSOCIATED PROBLEMS OF INFINITE DIMENSION

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Abstract: In this current paper, we introduced the overture of the subsequent field given by the span of a finite number of vectors as follows: **(1)** The complete normed inner product space of Nevanlinna theory. **(2)** A complete normed vector space of Nevanlinna theory over the real or complex field.

Keywords and Phrases: Nevanlinna theory, infinite-dimensional space, E-valued function, Hilbert space, Banach space.

2010 Mathematics Subject Classification: Primary 30D35.

1. Introduction and Results

Ever since first and second fundamental theorem of Nevanlinna appeared in 1925 [2]. Many authors have researched on this theorem taking the range of the function in infinite-dimensional Banach space and Hilbert space. In this paper, we are examined initial and subsequent theorem of Nevanlinna E-value in Hilbert space as well as Banach space via representation sections by the concept and properties of meromorphic maps.

2. Theory of Value Distribution of Infinite-dimension

2.1. Theory of Value distribution in Hilbert space

Before establishing Nevanlinna formula of $Poisson - Jensen$, the vector-valued function of finite dimension was extended from the classical Nevanlinna theory of meromorphic function by Ziggler [13] in 1982. Followed by this in 1997, with complete orthogonal basis $\{e_j\}_i^\infty$ from the theory of Nevanlinna of infinite-dimension in Hilbert space E was extended by Hu and Yang [7]. Using the results of [4], the hypothesis that a function of a complex variable is analytic in a sphere and along its boundary attains its maximum absolute numerical quantity and a method for assigning values to certain improper integrals in the sense of Bochner and the Laurent's extension of a vector valued holo-morphic function are along the span in E were established the first fundamental theorem of E-valued Nevanlinna (see [7] for details).

Let $H(z) = (h_1(z), ...h_j(z), ...)$ be a meromorphic E-valued function on $\mathbb{C}$, if $\Sigma_{j=1}^{\infty}|h_j(z)|$ is uniform convergence in any dense subset D in $\mathbb{C}$ then $H(z)$ is said to be of dense coordinate convergence.

Let E_1 be a space that is wholly contained in another space of E, which comprise of all the components of $H(z)$ in C. Without careful proof (cf. [7]), E_1 satisfies the second main theorem of Nevanlinna E-value if $H(z)$ is compact coordinate convergence on $\mathbb{C}$.

On the disk $|z| < R$, $H(z)$ is meromorphic E-Valued function to $E - \{0\}$, the volume component of the curve $\sim H$ defined as $\frac{1}{2} \triangle \;\; log||H(z)|| \; dx \wedge dy$, using correspondent results of [13], alongside the meromorphic vector operator $H(z)$, the volume function can be represented by

$$V(r, b) = \int_0^v \frac{v(t, b)}{t} dt,$$

hence

$$V(t, b) = \frac{1}{2\pi} \int_{S(0,t)} \triangle \; log||F(z) - b|| \;\; dx \wedge dy, \quad S(0, t) = Z : |Z| < t.$$

In 1998, by using the corresponding outcome of [4], Liu and Hu [12] extracted the Hilbert space of first main theorem of Nevanlinna and gave a attentive verification as follows:

2.2. E-valued Nevanlinna's first fundamental theorem

A meromorphic E-valued mapping denoted by $h(z)$ in C_R, where $0 < r < R$, $b \in E$ and b,

$$\mathtt{T}(\mathtt{r}, h) = \mathtt{V}(\mathtt{r}, b) + \mathtt{N}(\mathtt{r}, b) + \mathtt{m}(\mathtt{r}, b) + \mathtt{log}||c_q(b)|| + \epsilon(\mathtt{r}, b).$$

a function $\epsilon(\mathtt{r}, b)$ in such a way that

$$|\epsilon(\mathtt{r}, b)| \leq \mathtt{log}^+||b|| + \mathtt{log}2, \quad \epsilon(\mathtt{r}, 0) \equiv \mathtt{0}.$$

$$\mathtt{m}(\mathtt{r}, b) = \frac{1}{2\pi}\int_0^{2\pi} log^+ \frac{1}{||\mathtt{f}(re^{i\theta}) - b||} d\theta,$$

$$\mathtt{N}(\mathtt{r}, b) = \mathtt{n}(0, b)\ \mathtt{log}\ \mathtt{r} + \int_0^r \frac{n(\mathtt{t}, b) - \mathtt{n}(0, b)}{\mathtt{t}} dt$$

$$T(r, h) = m(r, h) + N(r, h)$$

On the closed disk $|z| < R$, we denote h a unstable meromorphic E-valued function, then

1. For $\mathtt{R} > \mathtt{r} > 0$, $\mathtt{T}(\mathtt{r}, h)$ is an increasing function of $\mathtt{r}$.
2. For $\mathtt{R} > \mathtt{r} > 0$, $\mathsf{T}(\mathsf{r}, h)$ is a convex function of $log\, r$.
3. $\mathtt{n}(\mathtt{r}, h_j) \leq \mathtt{n}(\mathtt{r}, h)$.
4. If $\mathtt{N}(\mathtt{r}, h_j) \leq \mathtt{N}(\mathtt{r}, h)$ and $\mathtt{T}(\mathtt{r}, h_j) \leq \mathtt{T}(\mathtt{r}, h)$, then $\mathtt{z} = 0$ is not a pole of h.
5. If $\mathtt{N}(\mathtt{r}, h_j) + O(1) \leq \mathtt{N}(\mathtt{r}, h)$ and $\mathtt{T}(\mathtt{r}, h_j) \leq \mathtt{T}(\mathtt{r}, h) + \mathtt{O}(1)$, then $z = 0$ is a pole of h.

Specifically, Hilbert space of the Nevanlinna second main theorem is obtained by Hu and Yang [8] in 1992, and is given as follows:

2.3. E-valued Nevanlinna's second fundamental theorem

A non-constant meromorphic E-valued mappings of dense projection in C_R is given by $h(z)$ and $b^{[\mathtt{k}]} \in \mathtt{E}\bigcup\{\infty\}(\mathtt{k} = 1, 2, ..., \mathtt{q})$ be $\mathtt{q} \geq 3$ well defined finite points or infinite points. Subsequently

$$\sum_{k=1}^{q} m(\mathtt{r}, b^{[\mathtt{k}]}) + J(\mathtt{r}, h) \leq \mathrm{T}(\mathsf{r}, h) - \mathtt{N}_1(\mathtt{r}) + \mathtt{S}(\mathtt{r}),$$

given

$$\mathtt{N}_1(\mathtt{r}) = \mathtt{N}(\mathtt{r}, 0, h') + 2\mathsf{N}(\mathsf{r}h) - \mathtt{N}(\mathtt{r}, h')$$

in addition

$$\mathtt{J}(r, h) = \int_0^{\mathtt{r}} \frac{dt}{2\pi} \int_{C_r} \triangle log||h'(\chi)||\ \ d\sigma \wedge dr.$$

If $\mathtt{R}$ is plus infinity, then $\mathtt{S}(\mathtt{r})$ persuade $S(\mathtt{r}) = \mathtt{O}\{logT(\mathtt{r}, \mathtt{f})\} + \mathtt{O}(logr)$ as $\mathtt{r} \longrightarrow +\infty$ outward a set L of atypical intervals of specific quantity

$$\int_L dr < +\infty.$$

Further fascinating structure of the E-valued Nevanlinna second main theorem is expressed as,

$$(\mathtt{q}-1)\mathtt{T}(\mathtt{r},h)+J(\mathtt{r},h)+N_1(\mathtt{r})\leq\sum_{\mathtt{k}=1}^{\mathtt{q}+1}[\mathtt{V}(\mathtt{r},b^{[\mathtt{k}]})+\mathtt{N}(\mathtt{r},b^{[\mathtt{k}]})]+\mathtt{S}(\mathtt{r}).$$

Or

$$(q-2)\mathtt{T}(\mathtt{r},h)+J(\mathtt{r},h)\leq\sum_{\mathtt{k}=1}^{\mathtt{q}}[\mathtt{V}(\mathtt{r},b^{[\mathtt{k}]})+\mathtt{N}(\mathtt{r},b^{[\mathtt{k}]})]+\mathtt{S}(\mathtt{r}),$$

accompanied by

$$\overline{\mathsf{N}}(\mathtt{r},b)=\overline{\mathtt{n}}(0,b)\mathtt{log}\ \mathtt{r}+\int_0^{\mathtt{r}}\frac{\overline{\mathtt{n}}(\mathtt{t},b)-\overline{\mathtt{n}}(\mathtt{t},b)}{t}dt,$$

where the solution $\overline{n}(t,b)$ is counted only once, which indicates the solutions of $h(\mathsf{z})-b$ in $|\mathsf{Z}|\leq\mathtt{t}$.

3. Nevanlinna Theory in Banach Spaces

3.1. Nevanlinna Theory on one complex variable in Banach spaces

A holo-morphic $\mathtt{E}$-valued function $h(\mathtt{z})$ on Ω is forenamed to be of dense projection if and only if $||P_nh(z)-h(z)||<\epsilon$, as adequately huge n in any stable dense subset $\Omega_1\subset\Omega$.
Following we define the meromorphic E-valued functions of Borel exceptional values. Suppose $\mathtt{E}$ be a composite Banach space and $\mathbb{C}$ is the Argand plane. Enable $\Omega=\mathtt{C}_{\mathtt{r}}=\{\mathsf{Z}:|\mathsf{Z}|<\mathtt{r}\}$.

Definition 3.1. *An $\mathtt{E}$-valued Picard exceptional value (evp) in a non-constant meromorphic $\mathtt{E}$-valued function $\boldsymbol{h}$ of extremity $\boldsymbol{b}\in E\bigcup\{\infty\}$ is defined by $V(\boldsymbol{r},b)+N(\boldsymbol{r},b)=O\{log\ r\}$. We sound b an $\mathtt{E}$-valued evp of h for zeros of order $\leq\boldsymbol{k}$ provided*

$$V(\boldsymbol{r},b)+\overline{N}_k(r,b,f)=O\{log\ r\}.$$

3.2. Deficiency relation of Nevanlinna E-valued function

Acceptable with the features of dense projection we define an E-valued function which is meromorphic of $h(z)$. Summing over all points for the countable set $\{b\in E\bigcup\{\infty\}:\Theta(b)>0\}$, we have

$$\sum_b[\delta(b)+\theta(b)]+\delta_J\leq\sum_b\Theta(b)+\delta_J\leq 2,$$

where

$$\delta(b) = \underline{lim}_{r\to R}\frac{m(\mathtt{r},\mathtt{b})}{\mathtt{T}(\mathtt{r},\mathtt{b})}$$

is the *deficiency of the point* b,

$$\theta(b) = \underline{lim}_{r\to R}\frac{\mathtt{N}(\mathtt{r},b) - \overline{N}(\mathtt{r},b)}{T(\mathtt{r},h)}$$

is the *index of multiplicity of* b.

$$\Theta(b) = \underline{lim}_{r\to R}\frac{\mathtt{m}(\mathtt{r},b) + \mathtt{N}(\mathtt{r},b) - \overline{\mathtt{N}}(\mathtt{r},b)}{\mathtt{T}(\mathtt{r},h)},$$

and

$$\delta_J = \underline{lim}_{r\to R}\frac{J(r,h)}{T(r,h)}$$

is the *Ricci index of* h.

On the contrary, by utilizing the *Green's residue theorem* (see [10]) for the instance of holo-morphic mapping on complex manifolds in $\mathbb{C}^m$ to infinite dimension(cf. [9]) the Nevanlinna first main theorem can established as follows:

Theorem (Holomorphic Hermitian line bundles in the first main theorem of Nevanlinna): *Enable a parabolic manifold* (M,Υ) *of size* m, *and a Riemann sphere* $\hat{N}$ *on* H, *as well meromorphic map* $h : M \longrightarrow \hat{N}$ *independent of* $\boldsymbol{u}$ *(i.e.,* $h(\mathtt{M}{-}I_J)$*) is not hold in the zero set* $Z(u)$ *with features of dense projection. Presume a holo-morphic line bundle* $\overline{W} = L$. *Subsequently*

$$T_h(\boldsymbol{r},\boldsymbol{s},\boldsymbol{L},\boldsymbol{k}) = N_{h,u}(r,s,L) + M_{h,u}(r,L,k) - Mh,u(s,L,k), \quad 0 < s < r.$$

Where $T_h(\boldsymbol{r},\boldsymbol{s},\boldsymbol{L},\boldsymbol{k})$ *is the characteristic function,* $m_{f,\boldsymbol{u}}(\boldsymbol{r},\boldsymbol{L},\boldsymbol{k})$ *is the compensation function and* $N_{f,\boldsymbol{u}}(\boldsymbol{r},\boldsymbol{s},\boldsymbol{L})$ *is the valence function.*

Theorem (The first main theorem of Nevanlinna for an operation): *Suppose that* $h = (h_1, h_2, ..., h_k)$ *is independent of* θ. *After that the compensation function*

$$m_{\dot{\theta}f}(\boldsymbol{r}) = m_{h_1\dot{\theta}\ldots\dot{\theta}h_k}(a) = \int_{M_0(r)} log\frac{||\theta||}{||\dot{\theta}f||}, \quad \sigma \geq 0.$$

subsist $\forall\ \ \mathrm{r} \in R_T$ *and enlarge to a continuous function on* $[\mathit{0},+\infty)$ *alike that for* $0 > \boldsymbol{s} > \boldsymbol{r}$,

$$\sum_{s=1}^{k} \mathit{T}_{fj}(\boldsymbol{r},\boldsymbol{s}) = N_{\dot{\theta}f}(\boldsymbol{r},\boldsymbol{s}) + M_{\dot{\theta}f}(\boldsymbol{r}) - M_{\dot{\theta}f}(s) + T_{\theta f}(r,s)$$

where

$$T_{fj}(r,s) = \int_s^r A_{fj}(t)(\frac{dt}{t})$$

also

$$A_{fj}(t) = (\frac{1}{t^{2m-2}}) \int_{M_0[t]} h_j^* \wedge \nu^{m-1}$$

3.3. Second main theorem of Nevanlinna in Banach spaces for small function

Here, we undertake E_0 is a Banach algebra, that is to say $h(\not\equiv 0)$ is a meromorphic function of E_0-value with its derivative $h^j (j = 1, ..., q)$, and that one of its differential polynomials can be represented as

$$P(z) = P(h, h', ..., h^q) = \sum_{\mathtt{k}=1}^{\mathtt{P}} a_k(z) \prod_{\mathtt{j}=0}^{q} \{h^{(j)}(z)\}^{s_{kj}}$$

given $a_{\mathtt{k}}(\mathtt{z})(\mathtt{k} = 1, ..., \mathtt{p})$ are meromorphic E_0-valued functions.

Let us take non-linear, linearly independent, E_0-valued meromorphic function $\psi_k(z)$ for $(\mathtt{k} = 1, 2, ..., t)$, and assume EMF, a set is made up of all meromorphic E_0-valued functions on $\mathbb{C}$. For non-linear meromorphic E_0-valued functions $\psi_{\mathtt{k}}(\mathtt{z})(\mathtt{k} = 1, 2, ..., t; t \geq \mathtt{1})$ and a meromorphic E_0-valued function $h(\mathtt{z})$, define $A_0 = \mathtt{W}(\psi_1, ...\psi_t)$ mean the Wronskian of $\psi_k(\mathtt{z})(\mathtt{k} = 1, 2, ..., t)$ also $\mathtt{W}(\psi_1, ..., \psi_t, \mathtt{f})$ mean the Wronskian of $\psi_k(\mathtt{z})(\mathtt{k} = 1, 2, .., t)$ as well h, where

$$A_0 = \begin{bmatrix} \psi_1 & \psi_2 & \dots \psi_t \\ \psi_1' & \psi_2' & \dots \psi_t' \\ \vdots & \vdots & \vdots \\ \psi_1^{t-1} & \psi_2^{t-1} & \dots \psi_t^{t-1} \end{bmatrix}$$

Assume that $I = J\{E_r(\psi_1, ..., \psi_t)\}$ with dimension EMF_ν of basis $\{\alpha_{\mathtt{j}}(\mathtt{z})\}_{j=1}^{\mathtt{l}}$ and $I' = \mathtt{J}\{\mathtt{E}_{\nu+1}(\psi_1, ..., \psi_t)\}$ with dimension $EMF_{(\nu+1)}$ of basis $\{\beta_{\mathtt{j}}(\mathtt{z})\}_{j=1}^{I'}$.

Suppose ψ_k^{-1} exists. Describe a "linear operator" $L(h) = W\ (\psi_1, ..., \psi_t, h)\ W^{-1}$ $(\psi_1, ..., \psi_t)$. Conclude that the inverse function of $\mathtt{F}(\mathtt{z}) = \mathtt{L}(h)$ is proportionate as $\mathtt{h}(\mathtt{z}) = \sum_{k=1}^{t} c_k(\mathtt{z})\psi_k(\mathtt{z})$, where $\mathtt{c_k}(\mathtt{k} = 1, 2, ..., t)$ are meromorphic E_0-valued function persuading the succeeding relation

$$c_k' = (-1)^{t-k} \phi_k A_0^{-1} H,$$

for $\mathtt{k} = 1, 2, ..., p$ and after the cancellation of horizontal p and the vertical line k in t_0, we define a vector-valued determinant ϕ_k.

Assume that

$$\phi_j(z) = \sum_{k=1}^{t} c_{jk}\psi_k(z), \ \ j = 1, ..., s(s \geq 2)$$

where c_{jk} are constants.

By using the results of chapter 1 in [11] we state an E_0-valued small function [8] of second fundamental theorem of Nevanlinna is as follows:

Theorem: *If* $h - \phi_j^{-1}(j = 1, ...s)$ *with* $P(h)^{-1}$ *exists. then*

$$\sum_{j=1}^{s} m(r, (h - \phi_j)^{-1}) \leq T(r, h) + \frac{I'}{I} N(r, h) - \frac{1}{I} N(r, (P(h)^{-1})) + S(r, h),$$

and

$$\sum_{j=1}^{s} m(r, (h - \phi_j)^{-1}) \leq T(r, h) + (I + \frac{I - 1}{2})\overline{N}(r, h) - \frac{1}{I} N(r, (P(h)^{-1})) + S(r, h),$$

where $S(r, h)$ *satisfies* $lim_{r\to\infty} \frac{S(r,h)}{T(r,h)} = 0$, *outward a set* σ *of specific quantity of the atypical intervals ,* I *also* I' *are positive integer as well* $P(h) = W(\beta_1, ..., \beta_r, h_{\alpha_1}, ...h_{\alpha_I})$ *is the vector-valued Wronskian of* β_i, *also* $h_{\alpha k}$, *being* $i = 1, ..., I'; k = 1, ..., I$.

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SPACE-TIME ADMITTING GENERALIZED CONFORMAL CURVATURE TENSOR

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Abstract: The object of the present paper is to study space-time admitting generalized conformal curvature tensor.

Keywords and Phrases: Conformal curvature tensor; $\mathcal{Z}$-tensor; Generalized conformal curvature tensor; Einstein field equations; Perfect fluid space-time.

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1. Introduction

The aim of the present work is to study certain investigations in general theory of relativity and cosmology by the coordinate free method of differential geometry. The basic difference between Riemannian and semi-Riemannian geometry is (i) the existence of null vector (i.e. $g(v,v) = 0$, for $v \neq 0$, where g is the metric tensor) in semi-Riemannian manifold but not Riemannian manifold, (ii) the signature of metric tensor g in semi-Riemannian manifold is $(-,-,...-,+,+,...,+)$ but in a Riemannian manifold the signature of g is $(+,+,...,+)$. Lorentzian manifold is a spacial case of semi-Riemannian manifold. The signature of metric tensor g in Lorentzian manifold is $(-,+,+,...,+)$. A Lorentzian manifold consists of three types of vectors such as timelike (i.e. $g(v,v) < 0$), spacelike (i.e. $g(v,v) > 0$) and null vector (i.e. $g(v,v) = 0$, for $v \neq 0$). In general, a Lorentzian manifold (M,g) may not have a globally timelike vector field. If (M,g) admits a globally timelike vector field, it is called time orientable Lorentzian manifold, physically known as space-time. The foundation of general relativity is based on a 4−dimensional space-time manifold which is the stage of present modeling of the physical world a torsionless, time-oriented Lorentzian manifold (M, g).

An $n-$dimensoinal generalized Robertson-Walker (GRW) space-time with $n \geq 3$ is a Lorentzian manifold which is a warped product of an open interval I of $\Re$ and an $(n-1)-$dimensional Riemannian manifold ([10], [11], [12]). These Lorentzian manifold broadly extends the classical Robertson-Walker (RW) space-time. RW space-time is regarded as cosmological models since it is spatially homogenous and spatially isotropic whereas GRW space-time serve as inhomogeneous extension of RW space-times that admit an isotropic radiation [18]. A Lorentzian manifold named "Perfect fluid space-time" if its Ricci tensor S has the form

$$S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y), \tag{1.1}$$

where α and β are scalars, A is a non-zero one-form such that $g(X, v) = A(X)$ for all v and v is the velocity vector field such that $g(v, v) = -1$. Perfect- fluid space-times in a language of differential geometry are called quasi-Einstein spaces where A is metrically equivalent to a unit space-like vector field. Einstein's field equation without cosmological constant is given by [16]

$$S(X, Y) - \frac{r}{2} g(X, Y) = kT(X, Y). \tag{1.2}$$

The paper is organized as follows: After Preliminaries in Section 3, we deduce the basic algebraic properties of generalized conformal curvature tensor. Next in Section 4, it is proven that a 4−dimensional Ricci simple generalized conformally flat space-time is a perfect fluid space-time. Moreover the space-time is RW space-time. Finally, it is shown that a 4−dimensional Ricci simple conservative generalized conformal curvature tensor with constant ψ is a GRW space-time.

2. Preliminaries

The Weyl conformal curvature tensor is the traceless part of Riemann tensor given as [13]

$$\begin{aligned} \mathcal{C}(U, V, X, Y) =& \mathcal{R}(U, V, X, Y) - \frac{1}{n-2}[S(V, X)g(U, Y) \\ &- S(U, X)g(V, Y) + S(U, Y)g(V, X) - S(V, Y)g(U, X)] \\ &+ \frac{r}{(n-1)(n-2)}[g(V, X)g(U, Y) - g(U, X)g(V, Y)], \end{aligned} \tag{2.1}$$

where $\mathcal{R}$ Riemann curvature tensor and r denotes the scalar curvature. The number of algebraically independent components of the Ricci and the Weyl tensors equals that of the Riemann tensor. Since in general relativity only the Ricci tensor is coupled to matter by the Einstein's fields equations, the conformal curvature tensor describes the pure gravity degrees of freedom. Divergence of conformal curvature tensor is given by

$$\begin{aligned}(div\mathcal{C})(U,V)X =&\frac{n-3}{n-2}[(\nabla_U S)(V,X)-(\nabla_V S)(U,X)\\&-\frac{1}{2(n-1)}\{g(V,X)dr(U)-g(U,X)dr(V)\}].\end{aligned} \tag{2.2}$$

A symmetric $(0,2)$ type tensor field E on a semi-Riemannian manifold (M^n, g) is said to be a Codazzi tensor if it satisfies the Codazzi equation

$$(\nabla_U E)(V,X) = (\nabla_V E)(U,X), \tag{2.3}$$

for arbitrary vector fields U, V and X. The geometrical and topological consequences of the existence of a non-trivial Codazzi tensor on a Riemannian manifold have been studied by Derdzinski and Shen [6].

In 2012, Mantica and Suh [9] introduced a new generalized $(0,2)$ symmetric tensor $\mathcal{Z}$ and studied various geometric properties of it an Riemannian manifold. A new tensor $\mathcal{Z}$ is defined as:

$$\mathcal{Z}(X,Y) = S(X,Y) + \psi g(X,Y), \tag{2.4}$$

where ψ is an arbitrary scalar function and name is generalized $\mathcal{Z}-$tensor.

Definition 2.1. *A Riemannian manifold (M^n, g) of dimension n $(n > 3)$ is said to be Ricci simple tensor [5] if its Ricci tensor $S(X,Y)$ satisfies the condition*

$$S(X,Y) = -rA(X)A(Y), \tag{2.5}$$

where r and A is scalar curvature and unit time-like vector field respectively. This condition has a geometric meaning that a unit time like vector A becomes a principle vector of Ricci operator.

3. Generalized Conformal Curvature Tensor

In view of equation (2.4), equation (2.1) takes the form

$$\begin{aligned}\mathcal{C}(U,V,X,Y) =&\mathcal{R}(U,V,X,Y) - \frac{1}{n-2}[\mathcal{Z}(V,X)g(U,Y)\\&-\mathcal{Z}(U,X)g(V,Y)+\mathcal{Z}(U,Y)g(V,X)-\mathcal{Z}(V,Y)g(U,X)]\\&+\frac{r}{(n-1)(n-2)}[g(V,X)g(U,Y)-g(U,X)g(V,Y)]\\&+\frac{2\psi}{(n-2)}[g(V,X)g(U,Y)-g(U,X)g(V,Y)].\end{aligned} \tag{3.1}$$

Define

$$\mathcal{C}^*(U,V,X,Y) = \mathcal{R}(U,V,X,Y) - \frac{1}{n-2}[\mathcal{Z}(V,X)g(U,Y) - \mathcal{Z}(U,X)g(V,Y) + \mathcal{Z}(U,Y)g(V,X) - \mathcal{Z}(V,Y)g(U,X)] + \frac{r}{(n-1)(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]. \tag{3.2}$$

Thus from above equation, equation (3.1) reduces to

$$\mathcal{C}(U,V,X,Y) = \mathcal{C}^*(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)],$$

which gives

$$\mathcal{C}^*(U,V,X,Y) = \mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)], \tag{3.3}$$

where $\mathcal{C}^*(U,V,X,Y)$ is called generalized conformal curvature tensor.

If $\psi = 0$, then from equation (3.3), we obtain

$$\mathcal{C}^*(U,V,X,Y) = \mathcal{C}(U,V,X,Y). \tag{3.4}$$

Thus we can state as follows-

Theorem 3.1. *A generalized conformal curvature tensor reduces to conformal curvature tensor provided that the scalar function ψ vanishes.*

Now, interchanging the places of U and V in equation (3.3), we obtain

$$\mathcal{C}^*(V,U,X,Y) = \mathcal{C}(V,U,X,Y) - \frac{2\psi}{(n-2)}[g(U,X)g(V,Y) - g(V,X)g(U,Y)]$$

i.e.

$$\mathcal{C}^*(V,U,X,Y) = -\mathcal{C}(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] = -\mathcal{C}^*(V,U,X,Y)$$

i.e.

$$\mathcal{C}^*(U,V,X,Y) = -\mathcal{C}^*(V,U,X,Y),$$

which gives

$$\mathcal{C}^*(U,V,X,Y) + \mathcal{C}^*(V,U,X,Y) = 0,$$

which shows that generalized conformal curvature tensor is skew-symmetric with respect to first two slots.

Interchanging the places of X and Y in equation (3.3), we obtain

$$\mathcal{C}^*(U,V,Y,X) = \mathcal{C}(U,V,Y,X) - \frac{2\psi}{(n-2)}[g(V,Y)g(U,X) - g(U,Y)g(V,X)]$$

i.e.

$$\mathcal{C}^*(U,V,Y,X) = -\mathcal{C}(U,V,X,Y) + \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] = -\mathcal{C}^*(U,V,Y,X)$$

i.e.

$$\mathcal{C}^*(U,V,X,Y) = -\mathcal{C}^*(U,V,Y,X),$$

which gives

$$\mathcal{C}^*(U,V,X,Y) + \mathcal{C}^*(U,V,Y,X) = 0,$$

which shows that generalized conformal curvature tensor is skew-symmetric with respect to last two slots.

Again interchanging pair of slots in equation (3.3), we obtain

$$\mathcal{C}^*(X,Y,U,V) = \mathcal{C}(X,Y,U,V) - \frac{2\psi}{(n-2)}[g(Y,U)g(X,V) - g(X,U)g(Y,V)]$$

i.e.

$$\mathcal{C}^*(X,Y,U,V) = \mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]$$

i.e.

$$\mathcal{C}^*(U,V,X,Y) = \mathcal{C}^*(X,Y,U,V),$$

which gives

$$\mathcal{C}^*(U,V,X,Y) - \mathcal{C}^*(X,Y,U,V) = 0,$$

which shows that generalized conformal curvature tensor is symmetric on pair slots. Thus we can state as follows-

Theorem 3.2. *A generalized conformal curvature tensor on (M^n, g) is*
(1) *skew-symmetric in first two slots,*
(2) *skew-symmetric in lost two slots,*
(3) *symmetric in pair of slots.*

Now, writing two more equations by the cyclic permutations of U, V and X of equation (3.3), we obtain

$$\mathcal{C}^*(V, X, U, Y) = \mathcal{C}(V, X, U, Y) - \frac{2\psi}{(n-2)}[g(X, U)g(V, Y) - g(V, U)g(X, Y)], \quad (3.5)$$

and

$$\mathcal{C}^*(X, U, V, Y) = \mathcal{C}(X, U, V, Y) - \frac{2\psi}{(n-2)}[g(U, V)g(X, Y) - g(X, V)g(U, Y)], \quad (3.6)$$

Adding equations (3.3), (3.5) and (3.6), we obtain

$$\mathcal{C}^*(U, V, X, Y) + \mathcal{C}^*(V, X, U, Y) + \mathcal{C}^*(X, U, V, Y) = 0, \quad (3.7)$$

which shows that generalized conformal curvature tensor satisfied Bianchi's first identity. Thus we can state as follows-

Theorem 3.3. *A generalized conformal curvature tensor on (M^n, g) satisfies Bianchi's first identity.*

Now, taking the covariant derivative of equation (3.2), with respect to U, we obtain

$$\begin{aligned}(\nabla_U \mathcal{C}^*)(V, X, Y, W) &= (\nabla_U \mathcal{R})(V, X, Y, W) - \frac{1}{n-2}[g(V, W)(\nabla_U \mathcal{Z})(X, Y) \\ &- g(X, W)(\nabla_U \mathcal{Z})(V, Y) + g(X, Y)(\nabla_U \mathcal{Z})(V, W) - g(V, Y)(\nabla_U \mathcal{Z})(X, W)] \\ &+ \frac{dr(U)}{(n-1)(n-2)}[g(X, Y)g(V, W) - g(V, Y)g(X, W)].\end{aligned} \quad (3.8)$$

Writing two more equations by the cyclic permutations of U, V and X from equation (3.8), we obtain

$$\begin{aligned}(\nabla_V \mathcal{C}^*)(X, U, Y, W) &= (\nabla_V \mathcal{R})(X, U, Y, W) - \frac{1}{n-2}[g(X, W)(\nabla_V \mathcal{Z})(U, Y) \\ &- g(U, W)(\nabla_V \mathcal{Z})(X, Y) + g(U, Y)(\nabla_V \mathcal{Z})(X, W) - g(X, Y)(\nabla_V \mathcal{Z})(U, W)] \\ &+ \frac{dr(V)}{(n-1)(n-2)}[g(U, Y)g(X, W) - g(X, Y)g(U, W)],\end{aligned} \quad (3.9)$$

and

$$(\nabla_X \mathcal{C}^*)(U,V,Y,W) = (\nabla_X \mathcal{R})(U,V,Y,W) - \frac{1}{n-2}[g(U,W)(\nabla_X \mathcal{Z})(V,Y) - g(V,W)(\nabla_X \mathcal{Z})(U,Y) + g(V,Y)(\nabla_X \mathcal{Z})(U,W) - g(U,Y)(\nabla_X \mathcal{Z})(V,W)] + \frac{dr(X)}{(n-1)(n-2)}[g(V,Y)g(U,W) - g(U,Y)g(V,W)]. \tag{3.10}$$

Adding equations (3.8), (3.9) and (3.10) with the fact that $(\nabla_U \mathcal{R})(V,X,Y,W) + (\nabla_V \mathcal{R})(X,U,Y,W) + (\nabla_X \mathcal{R})(U,V,Y,W) = 0$, we get

$$\begin{aligned}(\nabla_U \mathcal{C}^*)(V,X,Y,W) &+ (\nabla_V \mathcal{C}^*)(X,U,Y,W) + (\nabla_X \mathcal{C}^*)(U,V,Y,W)\\
&= -\frac{1}{n-2}[g(V,W)\{(\nabla_U \mathcal{Z})(X,Y) - (\nabla_X \mathcal{Z})(U,Y)\}\\
&- g(X,W)\{(\nabla_U \mathcal{Z})(V,Y) - (\nabla_V \mathcal{Z})(U,Y)\}\\
&+ g(X,Y)\{(\nabla_U \mathcal{Z})(V,W) - (\nabla_V \mathcal{Z})(U,W)\}\\
&- g(V,Y)\{(\nabla_U \mathcal{Z})(X,W) - (\nabla_X \mathcal{Z})(U,W)\}\\
&- g(U,W)\{(\nabla_V \mathcal{Z})(X,Y) - (\nabla_X \mathcal{Z})(V,Y)\}\\
&+ g(U,Y)\{(\nabla_V \mathcal{Z})(X,W) - (\nabla_X \mathcal{Z})(V,W)\}]\\
&+ \frac{dr(U)}{(n-1)(n-2)}[g(X,Y)g(V,W) - g(V,Y)g(X,W)]\\
&+ \frac{dr(V)}{(n-1)(n-2)}[g(U,Y)g(X,W) - g(X,Y)g(U,W)]\\
&+ \frac{dr(X)}{(n-1)(n-2)}[g(V,Y)g(U,W) - g(U,Y)g(V,W)].\end{aligned} \tag{3.11}$$

Assuming that $\mathcal{Z}-$tensor is Codazzi tensor, then in view of equation (2.4), Ricci tensor $S(X,Y)$ is also Codazzi tensor, i.e. $(\nabla_U S)(X,Y) - (\nabla_X S)(U,Y) = 0$, which gives that the scalar curvature tensor r is constant. Thus above equation (3.11), reduces to

$$(\nabla_U \mathcal{C}^*)(V,X,Y,W) + (\nabla_V \mathcal{C}^*)(X,U,Y,W) + (\nabla_X \mathcal{C}^*)(U,V,Y,W) = 0. \tag{3.12}$$

Thus we can state as follows-

Theorem 3.4. *A generalized conformal curvature tensor on (M^n, g) satisfies Bianchi's second identity, if the $\mathcal{Z}-$tensor is Codazzi tensor.*

4. Generalized Conformally Flat Space-time

We consider a 4−dimensinal generalized conformally flat manifold (M, g) with Lorentzian metric g. From equation (3.3), we have

$$\mathcal{C}(U,V,X,Y) - \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] = 0.$$

i.e.

$$\mathcal{C}(U,V,X,Y) = \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)]. \tag{4.1}$$

Using equation (2.1) in equation (4.1), we have

$$\begin{aligned}\mathcal{R}(U,V,X,Y) &- \frac{1}{n-2}[S(V,X)g(U,Y) - S(U,X)g(V,Y) + S(U,Y)g(V,X) \\ &- S(V,Y)g(U,X)] + \frac{r}{(n-1)(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] \\ &= \frac{2\psi}{(n-2)}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{4.2}$$

For $n = 4$, above equation (4.2) takes the form

$$\begin{aligned}\mathcal{R}(U,V,X,Y) =& \frac{1}{2}[S(V,X)g(U,Y) - S(U,X)g(V,Y) \\ &+ S(U,Y)g(V,X) - S(V,Y)g(U,X)] \\ &+ \frac{6\psi - r}{6}[g(V,X)g(U,Y) - g(U,X)g(V,Y)].\end{aligned} \tag{4.3}$$

Assuming space-time is Ricci simple, then in view of equation (2.5) above equation reduces to

$$\begin{aligned}\mathcal{R}(U,V,X,Y) =& \frac{1}{2}[-rA(V)A(X)g(U,Y) + rA(U)A(X)g(V,Y) \\ &- rA(U)A(Y)g(V,X) + rA(V)A(Y)g(U,X)] \\ &+ \frac{6\psi - r}{6}[g(V,X)g(U,Y) - g(U,X)g(V,Y)],\end{aligned} \tag{4.4}$$

which gives

$$\begin{aligned}\mathcal{R}(U,V,X,Y) =& \frac{6\psi - r}{6}[g(V,X)g(U,Y) - g(U,X)g(V,Y)] \\ &- \frac{r}{2}[g(U,Y)A(V)A(X) + g(V,X)A(U)A(Y) \\ &- g(V,Y)A(U)A(X) - g(U,X)A(V)A(Y)],\end{aligned} \tag{4.5}$$

which is of the form of quasi-constant curvature tensor.
Contracting equation (4.5), we obtain

$$\begin{aligned} S(V,X) &= \frac{6\psi - r}{6}[4g(V,X) - g(V,X)] \\ &\quad - \frac{r}{2}[4A(V)A(X) - g(V,X) - A(V)A(X) - A(V)A(X)], \end{aligned} \tag{4.6}$$

i.e.

$$S(V,X) = \frac{6\psi - r}{2} g(V,X) + \frac{r}{2} g(V,X) - rA(V)A(X),$$

which in view of equation (2.5) becomes

$$S(V,X) = 3\psi g(V,X) + S(V,X),$$

which yields to

$$\psi = 0. \tag{4.7}$$

In view of equation (4.7), equation (4.1) gives

$$\mathcal{C}(U,V,X,Y) = 0.$$

This shows that generalized conformally Ricci simple flat space-time is conformally flat. Thus we can state as follows-

Theorem 4.1. *A 4−dimensional Ricci simple generalized conformally flat space-time M is conformally flat perfect fluid space-time.*

In [11], the authors have shown that an $n-$dimensional ($n \geq 4$) Ricci simple conformally flat perfect fluid space-time is RW space-time. Thus we can state as follows-

Corollary 4.2. *A 4−dimensional Ricci simple generalized conformally flat space-time M is RW space-time.*

5. Conservative Generalized Conformal Space-time

From equation (3.3), generalized conformal curvature tensor is given by

$$\mathcal{C}^*(U,V)X = \mathcal{C}(U,V)X - \frac{2\psi}{(n-2)}[g(V,X)U - g(U,X)V]. \tag{5.1}$$

The divergence of $C^*(U,V)X$ is defined as

$$(div\mathcal{C}^*)(U,V)X = g((\nabla_{e_i}\mathcal{C}^*)(U,V)X, e_i)$$

i.e.

$$(div\mathcal{C}^*)(U,V)X = g((\nabla_{e_i}\mathcal{C})(U,V)X, e_i) - \frac{2}{n-2}[g((\nabla_{e_i}\psi)\{g(V,X)U - g(U,X)V\}, e_i)],$$

which gives

$$(div\mathcal{C}^*)(U,V)X = (div\mathcal{C})(U,V)X - \frac{2}{(n-2)}[(U\psi)g(V,X) - (V\psi)g(U,X)]. \quad (5.2)$$

From equations (2.2) and (5.2), we obtain

$$\begin{aligned}(div\mathcal{C}^*)(U,V)X =& \frac{n-3}{n-2}[(\nabla_U S)(V,X) - (\nabla_V S)(U,X) \\ & - \frac{1}{2(n-1)}\{g(V,X)dr(U) - g(U,X)dr(V)\}] \\ & - \frac{2}{(n-2)}[(U\psi)g(V,X) - (V\psi)g(U,X)].\end{aligned} \quad (5.3)$$

If scalar function ψ is constant then from equation (5.2), we obtain

$$(div\mathcal{C}^*)(U,V)X = (div\mathcal{C})(U,V)X. \quad (5.4)$$

If $(div\mathcal{C}^*)(U,V)X = 0$ then from equation (5.4), we obtain

$$(div\mathcal{C})(U,V)X = 0.$$

Thus we can state as follows-

Theorem 5.1. *A $4-$dimensional relativistic conservative generalized conformal curvature space-time M admitting constant scalar function ψ is conservative conformal curvature tensor.*

In [11], the authors have shown that an $n-$dimensional $(n \geq 4)$ Ricci simple conservative conformal curvature space-time is a GRW space-time with Einstein fibers. Thus we can state as follows-

Corollary 5.2. *A $4-$dimensional Ricci simple conservative generalized conformal curvature space-time M admitting constant scalar function ψ a GRW space-time with Einstein fibers.*

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SEVERAL GENERATING FUNCTIONS USING GENERALIZED LUCAS SEQUENCES

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Abstract: In this paper I have obtained the generating functions up to third order of generalized sequences defined by Goksal Bilgici. Also I have presented several generating functions of several sequences as particular cases.

Keywords and Phrases: Generating functions, generalized sequences.

2010 Mathematics Subject Classification: 11B39.

1. Introduction

Many authors [1, 3] generalized sequences differently. In [2] Goksal Bilgici defined generalized sequences $\{f_n\}_{n=0}^{\infty}$ and $\{l_n\}_{n=0}^{\infty}$. We can write l_n after some modification as follows:

$$l_n = 2al_{n-1} - (a^2 - b)l_{n-2} \qquad n \geq 2 \tag{1.1}$$

where $l_0 = 2$, $l_1 = 2a$.

Clearly, for $(a, b) = \left(\frac{1}{2}, \frac{5}{4}\right), \left(\frac{1}{2}, \frac{9}{4}\right), (1, 2)$ the sequence $\{l_n\}_{n=0}^{\infty}$ reduces the Classical Lucas, Jacobsthal-Lucas and Pell-Lucas sequences, respectively. In this note I have obtained the generating functions up to third order of generalized sequence and hence find

1. Generating functions up to third order of Lucas sequence.
2. Generating functions up to third order of Jacobsthal-Lucas sequence.

3. Generating functions up to third order of Pell-Lucas sequence.
The $\{l_n\}$ can also be expressed by the closed form formula.

$$l_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.2}$$

where α and β are the roots of equation $x^2 - 2ax + (a^2 - b) = 0$.
So that

$$\alpha = a + \sqrt{b} \ \text{ and } \quad \beta = a - \sqrt{b} \tag{1.3}$$

This gives

$$\alpha + \beta = 2a, \quad \alpha\beta = a^2 - b, \quad \alpha - \beta = 2\sqrt{b} \tag{1.4}$$

2. Generating Functions of $\{l_n\}$

Let us solve second order linear recurrence by method of generating function. Let sequence of integer $\{l_n\}$ defined as follows:

$$l_{n+2} - 2al_{n+1} + (a^2 - b)l_n = 0 \quad n \geq 0, \tag{2.1}$$

where $l_0 = 2$, and $l_1 = 2a$.

Theorem 2.1. Generating function of sequence of integer $\{l_n\}$ is given by

$$\sum_{n=0}^{\infty} l_n x^n = \frac{A_1}{B_1}, \tag{2.2}$$

where $A_1 = 2(1 - ax)$ and $B_1 = 1 - 2ax + (a^2 - b)x^2$.
Proof. Multiplying x^n on both the sides of (2.1) and taking sum from 0 to ∞.

$$\sum_{n=0}^{\infty} l_{n+2}x^n - 2a\sum_{n=0}^{\infty} l_{n+1}x^n + (a^2 - b)\sum_{n=0}^{\infty} l_n x^n = 0$$

$$\frac{1}{x^2}\left[\sum_{n=0}^{\infty} l_n x^n - l_0 - l_1 x\right] - \frac{2a}{x}\left[\sum_{n=0}^{\infty} l_n x^n - l_0\right] + (a^2 - b)\sum_{n=0}^{\infty} l_n x^n = 0$$

$$\sum_{n=0}^{\infty} l_n x^n = \frac{g(x)}{[1 - 2ax + (a^2 - b)x^2]}, \tag{2.3}$$

where $g(x) = l_0 + (l_1 - 2al_0)x$.

Now since $[1-2ax+(a^2-b)x^2]\sum_{n=0}^{\infty} l_n x^n = g(x)$ solving and neglecting terms contains second and higher power of x. Putting $g(x)$ or alternatively putting initial values in (2.3)

$$\sum_{n=0}^{\infty} l_n x^n = \frac{2(1-ax)}{1-2ax+(a^2-b)x^2} \tag{2.4}$$

Now we proceed to find some more generating functions of $\{l_n\}$.
Let $F(x) = \sum_{n=0}^{\infty} l_n x^n = \frac{A_1}{B_1}$ where $A_1 = 2(1-ax)$ and $B_1 = 1-2ax+(a^2-b)x^2$. Then

$$\sum_{n=0}^{\infty} l_{n+1} x^n = \frac{F(x)-l_0}{x} \Rightarrow \sum_{n=0}^{\infty} l_{n+1} x^n = \frac{1}{x}\left[\frac{A_1}{B_1} - 2\right] \quad \text{Since} \quad l_0 = 2$$

$$\sum_{n=0}^{\infty} l_{n+1} x^n = \frac{P_1}{B_1} \quad \text{where} \ P_1 = 2a - 2(a^2-b)x \quad \text{and} \ B_1 = 1-2ax+(a^2-b)x^2. \tag{2.5}$$

$$\text{Again} \quad \sum_{n=0}^{\infty} l_{n+2} x^n = \frac{1}{x}\left[\sum_{n=0}^{\infty} l_{n+1} x^n - l_1\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2} x^n = \frac{1}{x}\left[\frac{P_1}{B_1} - l_1\right]$$

$$\sum_{n=0}^{\infty} l_{n+2} x^n = \frac{1}{x}\left[\frac{P_1}{B_1} - 2a\right] \quad \text{Since } l_1 = 2a$$

$$\sum_{n=0}^{\infty} l_{n+2} x^n = \frac{P_2}{B_1} \quad \text{where} \quad P_2 = 2(a^2+b) - 2a(a^2-b)x. \tag{2.6}$$

$$\text{So in general} \quad \sum_{n=0}^{\infty} l_{n+k} x^n = \frac{P_k}{B_1} \quad \text{where} \quad P_k = l_k - (a^2-b)l_{k-1}x. \tag{2.7}$$

Particular Cases. Now on setting value of a and b in (2.4) to (2.6)

Generating Function of Lucas Sequence On setting $a=\frac{1}{2},\ b=\frac{5}{4}$	Generating Function of Jacobsthal - Lucas Sequence On setting $a=\frac{1}{2},\ b=\frac{9}{4}$	Generating Function of Pell-Lucas Sequence On setting $a=1,\ b=2$
$\sum_{n=0}^{\infty} L_n x^n = \frac{2-x}{(1-x-x^2)}$	$\sum_{n=0}^{\infty} j_n x^n = \frac{2-x}{(1-x-2x^2)}$	$\sum_{n=0}^{\infty} Q_n x^n = \frac{2(1-x)}{(1-2x-x^2)}$
$\sum_{n=0}^{\infty} L_{n+1} x^n = \frac{1+2x}{(1-x-x^2)}$	$\sum_{n=0}^{\infty} j_{n+1} x^n = \frac{1+4x}{(1-x-2x^2)}$	$\sum_{n=0}^{\infty} Q_{n+1} x^n = \frac{2(1+x)}{(1-2x-x^2)}$
$\sum_{n=0}^{\infty} L_{n+2} x^n = \frac{3+x}{(1-x-x^2)}$	$\sum_{n=0}^{\infty} j_{n+2} x^n = \frac{5+2x}{(1-x-2x^2)}$	$\sum_{n=0}^{\infty} Q_{n+2} x^n = \frac{6+2x}{(1-2x-x^2)}$

3. Generating Functions of $\{l_n^2\}$

In this section, again using same method we will find generating functions of $\{l_n^2\}$.

Theorem 3.1. *Generating functions of sequence of integer $\{l_n^2\}$ is given by*

$$\sum_{n=0}^{\infty} l_n^2 x^n = \frac{A_2}{B_2}, \tag{3.1}$$

where $A_2 = 4-4(2a^2+b)x+4a^2(a^2-b)x^2$ and $B_2 = 1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3$.

Proof. To find p^{th} order generating function for $\{l_n\}$ we have to expand $\{l_n^p\}$ by the Binomial theorem for which we will use (1.2). This gives $\{l_n^p\}$ as a linear combination of α^{np}, $\alpha^{n(p-1)}\beta^n, ..., \alpha^n\beta^{n(p-1)}, \beta^{np}$. So this generating function has denominator as $(1-\alpha^p x)$ $(1-\alpha^{p-1}\beta x)...(1-\alpha\beta^{p-1}x)$ $(1-\beta^p x)$. Hence to find second order generating function for $\{l_n\}$ we have to expand $\{l_n^2\}$ by the Binomial theorem for which we will use (1.2). So that we can express as linear combination of $(\alpha-\beta)^2(1-\alpha^2x)(1-\beta^2x)(1-\alpha\beta x)$ and using (1.4) we get denominator of generating functions for $\{l_n\}$ as $B_2 = 1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3$.
Consider

$$\sum_{n=0}^{\infty} l_n^2 x^n = \frac{g(x)}{1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3} \tag{3.2}$$

$$g(x) = \left[1-(3a^2+b)x+(a^2-b)(3a^2+b)x^2-(a^2-b)^3x^3\right]\sum_{n=0}^{\infty} l_n^2 x^n$$

Considering power of x up to two and neglecting higher powers

$$g(x) = 4 - 4(2a^2 + b)x + 4a^2(a^2 - b)x^2$$

Substituting value of $g(x)$ in (3.2) we get required result. Now we proceed to find some more generating functions of $\{l_n^2\}$.

Let $F_1(x) = \sum_{n=0}^{\infty} l_n^2 x^n = \frac{A_2}{B_2}$ where $A_2 = 4 - 4(2a^2 + b)x + 4a^2(a^2 - b)x^2$ and $B_2 = 1 - (3a^2 + b)x + (a^2 - b)(3a^2 + b)x^2 - (a^2 - b)^3 x^3$. Then

$$\sum_{n=0}^{\infty} l_{n+1}^2 x^n = \frac{F_1(x) - l_0^2}{x} \Rightarrow \sum_{n=0}^{\infty} l_{n+1}^2 x^n = \frac{1}{x}\left[\frac{A_2}{B_2} - 4\right] \quad \text{Since} \quad l_0 = 2$$

$$\sum_{n=0}^{\infty} l_{n+1}^2 x^n = \frac{Q_2}{B_2} \quad \text{where} \quad Q_2 = 4a^2 - 4(2a^4 - a^2 b - b^2)x + 4(a^2 - b)^3 x^2. \tag{3.3}$$

Again

$$\sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{1}{x}\left[\sum_{n=0}^{\infty} l_{n+1}^2 x^n - l_1^2\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{1}{x}\left[\frac{Q_2}{B_2} - l_1^2\right]$$

$$\sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{1}{x}\left[\frac{Q_2}{B_2} - 4a^2\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^2 x^n = \frac{Q_3}{B_2} \tag{3.4}$$

where $Q_3 = 4a^2 - (a^2 - b)(3a^2 + b)x + (a^2 - b)^3 x^2$.

Particular Cases. On setting value of a, b in (3.1), (3.3) and (3.4).

Generating Function of Lucas Sequence On setting $a = \frac{1}{2},\ b = \frac{5}{4}$	**Generating Function of Jacobsthal-Lucas Sequence On setting** $a = \frac{1}{2},\ b = \frac{9}{4}$	**Generating Function of Pell-Lucas Sequence On setting** $a = 1,\ b = 2$
$\sum_{n=0}^{\infty} L_n^2 x^n = \frac{4 - 7x - x^2}{(1 - 2x - 2x^2 + x^3)}$	$\sum_{n=0}^{\infty} j_n^2 x^n = \frac{4 - 11x - 2x^2}{(1 - 3x - 6x^2 + 8x^3)}$	$\sum_{n=0}^{\infty} Q_n^2 x^n = \frac{4 - 16x - 4x^2}{(1 - 5x - 5x^2 + x^3)}$
$\sum_{n=0}^{\infty} L_{n+1}^2 x^n = \frac{1 + 7x - 4x^2}{(1 - 2x - 2x^2 + x^3)}$	$\sum_{n=0}^{\infty} j_{n+1}^2 x^n = \frac{1 + 22x - 32x^2}{(1 - 3x - 6x^2 + 8x^3)}$	$\sum_{n=0}^{\infty} Q_{n+1}^2 x^n = \frac{4 + 16x - 4x^2}{(1 - 5x - 5x^2 + x^3)}$
$\sum_{n=0}^{\infty} L_{n+2}^2 x^n = \frac{9 - 2x - x^2}{(1 - 2x - 2x^2 + x^3)}$	$\sum_{n=0}^{\infty} j_{n+2}^2 x^n = \frac{25 - 26x - 8x^2}{(1 - 3x - 6x^2 + 8x^3)}$	$\sum_{n=0}^{\infty} Q_{n+2}^2 x^n = \frac{36 + 16x - 4x^2}{(1 - 5x - 5x^2 + x^3)}$

4. Generating Functions of $\{l_n^3\}$

In this section, again using same method generating functions of $\{l_n^3\}$ is obtained.

Theorem 4.1. *Generating function of sequence of integer $\{l_n^3\}$ is given by*

$$\sum_{n=0}^{\infty} l_n^3 x^n = \frac{A_3}{B_3}, \tag{4.1}$$

where $A_3 = x + 4a(a^2 - b)x^2 + (a^2 - b)^3 x^3$ and
$B_3 = 1 - 4a(a^2+b)x + (6a^6 + 2a^4 b - 6a^2b^2 - 2b^3)x^2 - (4a^9 + 8a^3b^3 - 8a^7 b - 4ab^4)x^3 + (a^{12} + b^6 + 15a^8b^2 + 15a^4b^4 - 20a^6b^3 - 6a^{10}b - 6a^2b^5)x^4$.

Proof. To find third order generating functions for $\{l_n^3\}$ we have to expand $\{l_n^3\}$ by the Binomial theorem for which we will use (1.2). Consider

$$\sum_{n=0}^{\infty} l_n^3 x^n = \frac{g(x)}{1 - 4a(a^2+b)x + (6a^6 + 2a^4b - 6a^2b^2 - 2b^3)x^2 - (4a^9 + 8a^3b^3 - 8a^7b - 4ab^4)x^3 + (a^{12} + b^6 + 15a^8b^2 + 15a^4b^4 - 20a^6b^3 - 6a^{10}b - 6a^2b^5)x^4} \tag{4.2}$$

$g(x) = [1 - 4a(a^2 + b)x + (6a^6 + 2a^4b - 6a^2b^2 - 2b^3)x^2 - (4a^9 + 8a^3b^3 - 8a^7b - 4ab^4)x^3 + (a^{12} + b^6 + 15a^8b^2 + 15a^4b^4 - 20a^6b^3 - 6a^{10}b - 6a^2b^5)x^4] \sum_{n=0}^{\infty} l_n^3 x^n$
Considering power of x up to three and neglecting higher powers

$$g(x) = 8 - 8a(3a^2 + 4b)x + 8(3a^6 - b^3 - 3a^2b^2 + a^4b)x^2 - 8(a^9 - a^3b^3 - 3a^7b + 3a^5b^2)x^3$$

Substituting value of $g(x)$ in (4.2) we get required result. Now we proceed to find some more generating functions of $\{l_n^3\}$.
Let

$$F_2(x) = \sum_{n=0}^{\infty} l_n^3 x^n = \frac{A_3}{B_3}$$

where $A_3 = 8 - 8a(3a^2+4b)x + 8(3a^6 - b^3 - 3a^2b^2 + a^4b)x^2 - 8(a^9 - a^3b^3 - 3a^7b + 3a^5b^2)x^3$ and
$B_3 = 1 - 4a(a^2+b)x + (6a^6 + 2a^4b - 6a^2b^2 - 2b^3)x^2 - (4a^9 + 8a^3b^3 - 8a^7b - 4ab^4)x^3 + (a^{12} + b^{16} + 15a^8b^2 + 15a^4b^4 - 20a^6b^3 - 6a^{10}b - 6a^2b^5)x^4$
Then

$$\sum_{n=0}^{\infty} l_{n+1}^3 x^n = \frac{F_2(x) - l_0^3}{x} \Rightarrow \sum_{n=0}^{\infty} l_{n+1}^3 x^n = \frac{1}{x}\left[\frac{A_3}{B_3} - 8\right] \quad \text{Since} \quad l_0 = 2$$

$$\sum_{n=0}^{\infty} l_{n+1}^3 x^n = \frac{R_3}{B_3} \tag{4.3}$$

where $R_3 = 8a^3 - 8(3a^6 + a^4 b - 3a^2 b^2 - b^3)x + 8(3a^9 + 9a^3 b^3 - 5a^7 b - 3a^5 b^2 - 4ab^4)x^2 - 8(a^{12} + b^6 + 15a^8 b^2 + 15a^4 b^4 - 20a^6 b^3 - 6a^{10} b - 6a^2 b^5)x^3$.
Again

$$\sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{1}{x}\left[\sum_{n=0}^{\infty} l_{n+1}^3 x^n - l_1^3\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{1}{x}\left[\frac{R_3}{B_3} - l_1^3\right]$$

$$\sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{1}{x}\left[\frac{R_3}{B_3} - 8a^3\right] \Rightarrow \sum_{n=0}^{\infty} l_{n+2}^3 x^n = \frac{R_4}{B_3} \quad \text{Since} \quad l_1 = 2a \tag{4.4}$$

where $R_4 = 8(a^6 + b^3 + 3a^2 b^2 + 3a^4 b) - 8(3a^9 - 11a^3 b^3 + 7a^7 b - 3a^5 b^2 + 4ab^4)x + 8(3a^{12} - b^6 - 15a^8 b^2 - 19a^4 b^4 + 28a^6 b^3 - 2a^{10} b + 6a^2 b^5)x^2 - 8(a^{15} + a^3 b^6 + 15a^{11} b^2 + 15a^7 b^4 - 20a^9 b^3 - 6a^{13} b - 6a^5 b^5)x^3$.
Particular Cases: Now setting value of a, b in (4.1), (4.3) and (4.4).

Generating Function of Lucas Sequence **On setting** $a = \frac{1}{2},\ b = \frac{5}{4}$	**Generating Function of Jacobsthal-Lucas Sequence** **On setting** $a = \frac{1}{2},\ b = \frac{9}{4}$	**Generating Function of Pell-Lucas Sequence** **On setting** $a=1,\ b=2$
$\sum_{n=0}^{\infty} L_n^3 x^n = \frac{x-2x^2-x^3}{1-3x-6x^2+3x^3+x^4}$	$\sum_{n=0}^{\infty} j_n^3 x^n = \frac{8-39x-120x^2+8x^3}{1-5x-30x^2+40x^3+64x^4}$	$\sum_{n=0}^{\infty} Q_n^3 x^n = \frac{8-88x-120x^2+8x^3}{1-12x-30x^2+12x^3+x^4}$
$\sum_{n=0}^{\infty} L_{n+1}^3 x^n = \frac{1-2x-x^2}{1-3x-6x^2+3x^3+x^4}$	$\sum_{n=0}^{\infty} j_{n+1}^3 x^n = \frac{1-120x-312x^2-512x^3}{1-5x-30x^2+40x^3+64x^4}$	$\sum_{n=0}^{\infty} Q_{n+1}^3 x^n = \frac{8+120x-88x^2-8x^3}{1-12x-30x^2+12x^3+x^4}$
$\sum_{n=0}^{\infty} L_{n+2}^3 x^n = \frac{1+5x-3x^2-x^3}{1-3x-6x^2+3x^3+x^4}$	$\sum_{n=0}^{\infty} j_{n+2}^3 x^n = \frac{125-282x-552x^2-64x^3}{1-5x-30x^2+40x^3+64x^4}$	$\sum_{n=0}^{\infty} Q_{n+2}^3 x^n = \frac{216+152x-104x^2-8x^3}{1-12x-30x^2+12x^3+x^4}$

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CONTENTS

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