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EDGECUT POLYNOMIAL OF GRAPHS

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Abstract

In this paper, we introduce the standard edgecut and the edgecut polynomial of a simple finite connected graph. Moreover, we determine the edgecut polynomial of some graphs.

1. Introduction

In social networking systems, in order to improve the efficiency of content delivery and to minimize the total cost, graph partitioning is a vital pre-processing step for many large scale applications that are solved on parallel computing platforms. The majority of multilevel graph partitioning formulations has primarily focused on edgecut based models and has tried to optimize edgecut related objectives. In the edgecut model all the edges split between different partitions account as multiple communication messages and the edgecut metric gives us an approximation of the total communication cost, which are discussed in [1]. Also, the pandemic COVID-19 has revealed

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the necessity of social distancing in public domains so that there arises a need for an efficient machinery to break social gatherings for isolating people. These ideas motivate the authors to introduce the concepts of standard edgecut and edgecut polynomial of finite simple connected graphs. Throughout this paper, G denotes a finite simple connected graph with vertex set and edgeset denoted by $V(G)$ and $E(G)$ respectively.

2. Main Results

In this section, we first introduce the *cycle-number* and *cycle-degree* of a vertex of a finite simple graph and then define the *standard edgecut* and *edgecut polynomial* of a finite simple connected graph G. Moreover, we derive the *edgecut-polynomial* of some special types of graphs.

Definition 1. Let $G = (V, E)$ be a finite simple graph. Let v be a vertex of G. Then the *cycle-number* v_c of v is defined as the length of the largest cycle containing v. If v is not contained in any of the cycles of G, then we define $v_c = 0$.

Definition 2. Let $G = (V, E)$ be a finite simple graph. Then the *cycle-degree* of a vertex $v \in V$ belonging to a cycle is defined as the number of distinct cycles passing through v with cycle-number v_c. If v is not contained in any of the cycles of G, then its cycle-degree to be defined as zero.

Definition 3. Let $G = (V, E)$ be a finite simple graph. Then two vertices of G with same cycle-number are said to be *similar* if they have the same degree and cycle-degree.

Definition 4. Let $G = (V, E)$ be a finite simple connected graph. Then the *standard edgecut* for G is defined inductively as follows.

Step 1. Delete the cutedges of G one by one so that the number of components of G increases exactly by one at each step and let G_1 be the union of all nontrivial components thus obtained.

Step 2. Now one of the similar vertices of G_1 having the minimum degree and maximum cycle-number and cycle-degree is isolated by deleting all the edges incident to it and let G_2 be the union of all nontrivial components thus obtained.

Step 3. If G_2 has cutedges, delete them one by one followed by the isolation of one of the similar vertices as mentioned in step 2.

Step 4. This process is continued by deleting cutedges followed by isolating one vertex at a time till G becomes totally disconnected. The edges removed at each step constitute the standard edgecut for G.

Definition 5. Let $G = (V, E)$ be a simple connected graph of order n. Then the *edgecut polynomial*, $E[G; x]$, of G is defined as:

$$E[G; x] = \sum_{i=1}^{n-1} E(G, i) x^{i+1},$$

where $E(G, i)$ is the number of edges removed at the ith step in the standard edgecut for $1 \leq i \leq n-1$.

We observe the following simple properties of $E[G; x]$:

(i) $E[G; x]$ is a monic polynomial of degree exactly n.

(ii) $E[G; 1] = \sum_{i=1}^{n-1} E(G, i) = |E|$.

(iii) The coefficient of x^2 in $E[G; x]$ is the edge connectivity of the graph G.

(iv) The coefficients of the polynomial $E[G; x]$ always lie between 0 and n.

(v) For any subgraph H of the graph G, $E[G; 1] \geq E[H; 1]$ with equality if and only if $G = H$.

Definition 6. Let $G = (V, E)$ be a simple connected graph of order n. Then the sequence $\{E(G, i)\}_{i=1}^{n-1}$ is defined as the edgecut sequence of G.

Theorem 7. *Let G be a tree of order n. Then, we have*

$$E[G;\ x] = \sum_{i=1}^{n-1} x^{i+1}.$$

Proof. Since every edge of G is a cutedge, deletion of one edge at each step yields $n - (n - i)$ components for $2 \leq i \leq n$.

Theorem 8. *If K_n is the complete graph of order n, then we have*

$$E[K_n;\ x] = \sum_{i=1}^{n-1} (n - i) x^{i+1}.$$

Proof. Observe that each vertex of K_n has degree $n - 1$ and cycle-degree $\frac{(n-1)!}{2}$ with cycle-number n. Therefore, any two vertices of K_n are *similar*. So, in order to isolate the vertices one by one $n - i$ edges have to be deleted at each stage yielding K_{n-i} for $1 \leq i \leq n - 1$.

Theorem 9. *If C_n is a cycle graph, then*

$$E[C_n;\ x] = \sum_{i=1}^{n-2} x^{i+2} + 2x^2.$$

Proof. Since every vertex in C_n is of degree 2 and being contained in a unique n-cycle, they are *similar*. Thus in order to isolate a vertex, two edges have to be removed initially resulting in the nontrivial component P_{n-1}.

A lollipop graph $L_{m,n}$ is obtained by joining K_m to a path on n vertices with a bridge.

Theorem 10. *We have the following*

$$E[L_{m,n};\ x] = \sum_{i=1}^{m-1} (m - i) x^{n+i+1} + \sum_{i=1}^{n} x^{i+1}.$$

Proof. In $L_{m,n}$, there are n cutedges and $\binom{m}{2}$ edges as part of K_m.

Thus, the removal of n cutedges of $L_{m,n}$ one by one gives the nontrivial component K_m.

This completes the proof.

A wheel graph W_n for $n > 3$ is obtained by taking the join of the cycle C_{n-1} and K_1.

Theorem 11. *We have the following*

$$E[W_n;\, x] = x^n + \sum_{i=1}^{n-3} 2x^{i+2} + 3x^2.$$

Proof. Here we consider 2 cases:

Case 1. Suppose $n = 4$.

In this case W_4 is the complete graph K_4. Hence the edgecut polynomial of W_4 is $x^4 + 2x^3 + 3x^2$.

Case 2. Suppose $n > 4$.

In this case, every vertex is part of a cycle and there are $n - 1$ vertices (of C_{n-1}) of degree 3 and one vertex of degree $n - 1$. Moreover, each corner vertex has cycle degree $n - 1$ with cycle number n and hence *similar*. After the isolation of any one of the *similar* vertices, there exist two vertices of degree 2, $n - (k + 2)$ vertices of degree 3 and one vertex of degree $n - k$ at each succeeding stage for $2 \leq k \leq n - 2$. Since the vertices of degree 2 are contained in a unique cycle of length $n - k - 1$, for $2 \leq k \leq n - 2$, they are *similar* and hence any one of them can be isolated. The process is continued till C_3 is obtained so that the edgecut sequence of W_n is the sequence $(3, \underbrace{2, 2, \ldots, 2}_{n-3 \text{ times}}, 1)$.

This completes the proof.

A helm H_n is obtained from a wheel graph W_n by adding pendant edges to every vertex on the wheel rim.

Corollary 12. *For* $n \geq 4$, *we have the following*

$$E[H_n; x] = x^{2n-1} + \sum_{i=2}^{n-1} 2x^{2n-i} + \sum_{i=1}^{n} x^{i+1}.$$

Proof. In H_n, the deletion of $n-1$ cutedges on the wheel rim one by one yields the nontrivial component W_n.

Theorem 13. *Let G be a simple connected graph of order n. If G is Hamiltonian, then the vertices of G having same degree are similar.*

Proof. Since G is Hamiltonian, there exists a spanning cycle of G of length n. Also, any other cycle of length n, if exists, traverses through each and every vertex of G. Thus every vertex of G has the same cycle-degree with cycle-number n. Therefore, the vertices of G having the same degree will also have the same cycle number and cycle degree and hence *similar*.

A webgraph WB_n is obtained by joining the pendant vertices of H_n to form a cycle and then adding a single pendant edge to each vertex of the outer cycle.

Theorem 14. *For* $n > 4$, *we have the following*

$$E[WB_n; x] = x^{3n-2} + \sum_{i=1}^{n-3} 2x^{2n+i} + 3x^{2n} + x^{2n-1} + \sum_{i=1}^{n-2} 2x^{n+i} + \sum_{i=1}^{n} x^{i+1}.$$

Proof. Observe that in WB_n, there are $n-1$ cutedges. After deleting the cutedges one by one, the nontrivial component has $n-1$ vertices of degrees 3 and 4 respectively and a single vertex of degree $n-1$. Since all the vertices of degree 3 are *similar*, one such vertex is isolated and the resulting graph consists of two vertices of degree 2, $n-3$ vertices of degree 3, $n-2$ vertices of degree 4 and one vertex of degree $n-1$. Again since the vertices of degree 2 are *similar*, one such vertex is isolated and this process continued till W_n is obtained by isolating all the vertices on the outer cycle. Thus the edgecut sequence of WB_n is given by

$$(\underbrace{1, 1, ..., 1}_{n-1 \text{ times}}, 3, \underbrace{2, 2, ..., 2}_{n-3 \text{ times}}, 1, 3, \underbrace{2, 2, ..., 2}_{n-3 \text{ times}}, 1).$$

This completes the proof.

Theorem 15. *For* $m, n \geq 2,$ *we have the following*

$$E[K_{m,n}; x]$$

$$= \begin{cases} \sum_{i=0}^{m-2} (i+1)[x^{2m-2i} + x^{2m-2i-1}] + mx^2, & \text{if } m = n, \\ \sum_{i=0}^{m-2} (i+1)[x^{n+m-2i} + x^{n+m-2i-1}] + m\sum_{i=2}^{n-m+2} x^i, & \text{if } m \neq n. \end{cases}$$

Proof. Let M and N be the bipartite sets of vertices of $K_{m,n}$ with cardinalities *m* and *n* respectively. We consider two cases:

Case (i). $m = n$

In $K_{2,2}$, all the vertices are of degree 2 and being a Hamiltonian graph, they are *similar*. Thus after isolating any of the vertices of degree 2, the nontrivial component reduces to P_3.

In $K_{m,n}$ for $m, n > 2,$ all the vertices are of degree *n* and since there exists a cycle of length 2*n*, they are *similar*. Thus the isolation of any of the vertices of degree *n*, say from *M* results in the decrease in the degree of the vertices of *N* exactly by 1 and that of *M* remains unchanged. Again the resulting nontrivial component is Hamiltonian and one of the *n* vertices of degree $n-1$ is isolated. Now there are $2(n-1)$ vertices each of degree $n-1$ and the process of isolating the vertices in a similar manner is continued till the nontrivial component becomes $K_{2,2}$. Thus the edgecut sequence is given by $(n, n-1, n-1, ..., n-(n-2), n-(n-2), n-(n-1), n-(n-1)),$ where each $n-i$ for $1 \leq i \leq n-1$ is repeated twice in the sequence.

Case (ii). $m \neq n$

Without loss of generality let us assume that $m < n$. Observe that in $K_{m,n}$, there are *m* vertices of degree *n* and *n* vertices of degree *m*. It can be easily observed that all the vertices of degree *m* are *similar* and hence any one of them can be isolated. Now the degree of vertices in *M* decreases exactly by 1, so that the nontrivial component has *m* vertices of degree $n - 1$ and $n - 1$ vertices of degree *m*. Now further isolation of any of the *similar* vertices of degree *m* is continued till $m = n - k$ for some $k > 0$ and the remaining vertices are isolated one by one as in case (i). Hence the edgecut sequence is $(\underbrace{m, m, ..., m}_{n-m+1 \text{ times}}, m-1, m-1, ..., m-(m-1), m-(m-1))$, where each $m - i$ for $1 \leq i \leq m - 1$ is repeated twice in the sequence.

This completes the proof.

A shell graph S_n is obtained from the cycle graph C_n by adding the edges corresponding to the $n - 3$ concurrent chords of the cycle. The vertex at which all the chords are concurrent is called the *apex* of the shell. Observe that S_n can be considered as the join of P_{n-1} and K_1.

Theorem 16. *For* $n > 2$,

$$E[S_n; x] = x^n + \sum_{i=1}^{n-2} 2x^{i+1}.$$

Proof. For $n = 3$, S_3 is K_3 whose edgecut polynomial is $E[S_3; x] = x^3 + 2x^2$. Since S_n is Hamiltonian, for $n > 3$, the two vertices of degree 2 in S_n are *similar*. After isolating one of the vertices of degree 2, the nontrivial component becomes S_{n-1} and the process of isolating any of the *similar* vertices of degree 2 is continued till C_3 is obtained. Thus the deletion of two edges at each stage followed by the removal of a single edge at the final step makes the graph totally disconnected.

A bow graph is a double shell with same apex in which each shell has any order.

Theorem 17. *For* $N > 4$, *if* B_N *is a bow graph with N vertices*, *then*

$$E[B_N;\, x] = \sum_{i=N-2}^{N-1} x^{i+1} + \sum_{i=N-3}^{N-2} x^{i+1} + \sum_{i=1}^{N-4} 2x^{i+1}.$$

Proof. We consider two cases:

Case (i). Let the bow graph B_N be the double shell of S_n. Then $N = 2n - 1$. For $N = 5$, B_5 has four *similar* vertices of degree 2 and apex of degree 4. After isolating one of the vertices of degree 2, the nontrivial component becomes a join of C_3 and K_1 so that $E[B_5;\, x] = x^5 + 2x^4 + x^3 + 2x^2$.

For $N > 5$, B_N has four vertices of degree 2, $N - 5$ vertices of degree 3 and one vertex of degree $N - 1$. Since all the vertices of degree 2 are having cycle-degree 1 and cycle-number n, they are *similar*. The deletion of two edges to isolate any such vertex gives a nontrivial component with shells S_n and S_{n-1}. Now we have four vertices of degree 2, $N - 6$ vertices of degree 3 and one vertex of degree $N - 2$ such that the vertices of degree 2 belonging to S_n are having cycle-number n and cycle-degree 1 whereas those belonging to S_{n-1} are having cycle-number $n - 1$ and cycle-degree 1 respectively. Thus we isolate any one of the *similar* vertices of degree 2 as part of S_n and obtain B_{N-1} with both the component shells S_{n-1}. The process of isolation of vertices is continued similarly till both the component shells reduces to S_3 so that the sequence $(\underbrace{2, 2, ..., 2}_{2n-6 \text{ times}}, 2, 1, 2, 1)$ denotes the edgecut sequence of B_N.

Case (ii). Let the bow graph B_N include the shells S_m and S_n, where $m \neq n$.

Then $N = n + m - 1$. Without loss of generality, let us assume that $m < n$. As in case (i), B_N has four vertices of degree 2, $N - 5$ vertices of degree 3 and one vertex of degree $N-1$. Since $m < n$ and the vertices of degree 2 in S_m possesses cycle-degree 1 with cycle-number *m* and those in S_n has cycle degree and cycle-number 1 and *n* respectively, we isolate one of the *similar* vertices of degree 2 in S_n. The process is continued similarly by isolating one of the vertices of degree 2 in S_n with cycle-number $n-k$ for some $k > 0$ till $m = n-k$. After reaching the stage in which $m = n-k$, we may proceed the isolation of the vertices as in case (i) so that the edgecut sequence becomes $(\underbrace{2, 2, 2, \ldots, 2}_{n+m-6 \text{ times}}, 2, 1, 2, 1)$.

This completes the proof.

A butterfly graph is a bow graph along with exactly two pendant edges at the apex.

Corollary 18. *For* $n > 4$, *if* BF_n *is a butterfly graph with n vertices*, *then*

$$E[BF_n;\, x] = \sum_{i=n-2}^{n-1} x^{i+1} + \sum_{i=n-3}^{n-2} x^{i+1} + \sum_{i=3}^{n-4} 2x^{i+1} + \sum_{i=1}^{2} x^{i+1}.$$

A friendship graph F_n is the one-point union of *n* copies of the cycle C_3.

Theorem 19. *For* $n > 1$,

$$E[F_n;\, x] = \sum_{i=1}^{n} x^{2i+1} + \sum_{i=1}^{n} 2x^{2i}.$$

Proof. In F_n, there are $2n$ vertices of degree 2 and a single vertex of degree $2n$. The minimum degree vertices are *similar* because all of them have cycle-degree 1 with cycle-number 3. Thus in order to isolate such a vertex, two edges have to be removed resulting in a unique cutedge which is deleted in the succeeding step and we get F_{n-1}. The process is continued similarly till C_3 is obtained and an ordered pair of edges $(2, 1)$ are removed at each stage in which a cycle vanishes. Thus the edgecut sequence is exactly the ordered pair $(2, 1)$ repeated n times.

A tadpole graph $T_{n,l}$ is a graph obtained by attaching a path P_l to one of the vertices of the cycle C_n by a bridge.

Theorem 20. *For* $n > 2$ *and for any* l,

$$E[T_{n,l};\, x] = \sum_{i=l+1}^{n+l-1} x^{i+1} + \sum_{i=1}^{l+1} x^{i+1}.$$

Proof. Observe that in $T_{n,l}$, there are l cutedges and n edges as part of C_n. Thus the $n + l - 1$ tuple $(\underbrace{1, 1, 1, \ldots, 1}_{l \text{ times}}, 2, \underbrace{1, 1, \ldots, 1}_{n-2 \text{ times}})$ represents the edgecut cut sequence of $T_{n,l}$.

The n-barbell graph $B_{n,1}$ is a graph obtained by connecting two copies of complete graph K_n by a bridge.

Theorem 21.

$$E[B_{n,1};\, x] = \sum\nolimits_{i=n+1}^{2n-1} (2n - i) x^{i+1} + \sum\nolimits_{i=2}^{n} (n - (i-1)) x^{i+1} + x^2.$$

Proof. In $B_{n,1}$, the bridge connecting K_n's, which is a cutedge, is removed so that two copies of K_n are obtained with all vertices similar. After isolating any of the vertices of degree $n - 1$ from one of the components, the degree of all other vertices of that component decreases by 1. Thus, further isolation of vertices are performed in the same component followed by a similar isolation in the other.

This completes the proof.

The windmill graph $W_n^{(m)}$ is obtained by taking m copies of K_n with a vertex in common.

Theorem 22.

$$E[W_n^{(m)};\ x] = \sum_{i=1}^{n-1}(n-i)\sum_{j=1}^{m} x^{(j-1)n-(j-(i+2))}.$$

Proof. Since all the vertices except the common vertex are *similar* with minimum degree $n-1,$ one such vertex is isolated so that the degree of all the vertices belonging to that particular K_n decreases by 1 so that further isolation of $n-2$ vertices (except the common vertex) is carried out in it. Similarly, the process of isolation of vertices is repeated in all other K_n's so that the $n-1$ tuple $(n-1, n-2, ..., 2, 1)$ repeated m times gives the edgecut sequence of $W_n^{(m)}$.

Dutch windmill graph $D_n^{(m)}$ is a windmill graph $W_n^{(m)}$ with $n = 3.$

Corollary 23.

$$E[D_n^{(m)};\ x] = \sum_{i=1}^{m} x^{2i+1} + \sum_{i=1}^{m} 2x^{2i}.$$

An armed crown $C_n \odot P_m$ is a graph obtained by attaching a path P_m to every vertex of the cycle C_n.

Theorem 24.

$$E[C_n \odot P_m;\ x] = \sum_{i=nm+1}^{nm+n-1} x^{i+1} + \sum_{i=1}^{nm+1} x^{i+1}.$$

Proof. Observe that in $C_n \odot P_m,$ there are *nm* cutedges. The deletion of all those cutedges one by one gives C_n so that the edgecut sequence is given by $(\underbrace{1, 1, 1, ..., 1}_{nm \text{ times}}, 2, \underbrace{1, 1, ..., 1}_{n-2 \text{ times}}).$

The *splitting graph* $S(G)$ of a graph G is obtained by adding new vertices v' to G corresponding to the vertices v of G and then joining the vertex v' to all the vertices of G adjacent to v in G. The vertex v' corresponding to v is called the *tag vertex* of v [2].

Theorem 25. *For* $n \geq 2$,

$$E[S(P_n);\, x] = x^{2n} + \sum_{i=1}^{n-1} x^{2i+1} + \sum_{i=2}^{n-1} 2x^{2i} + x^2.$$

Proof. For $n = 2$, $S(P_2)$ is P_4 so that its edgecut polynomial is $x^4 + x^3 + x^2$. In $S(P_n)$ for $n > 2$, the new vertices v' adjoined has the same degree as the corresponding vertex v in P_n, whereas the degree of the vertices belonging to P_n is doubled. Thus there are two pendant vertices, n vertices of degree 2 and $n - 2$ vertices of degree 4. Since both the pendant vertices are not a part of any cycle, their cycle-degree is zero and hence *similar* so that both of them are isolated one after the other. Now we have n vertices of degree 2, 2 vertices of degree 3 and $n - 4$ vertices of degree 4. Since the nontrivial component is Hamiltonian, all the vertices of degree 2 are *similar* so that any one of them can be isolated. The isolation of such a vertex results in the formation of a pendant edge, which is deleted in the next step to isolate the vertex associated with it. At this stage, the nontrivial component has $n - 1$ vertices of degree 2, 2 vertices of degree 3 and $n - 5$ vertices of degree 4, and being Hamiltonian, all the vertices of minimum degree are *similar*. Thus the process of isolation of vertices is proceeded similarly till C_4 is obtained. Thus the edgecut sequence of $S(P_n)$ is given by

$$(1, 1, \underbrace{2, 1, 2, 1, \ldots, 2, 1}_{2n-6 \text{ times}}, 2, 1, 1).$$

This completes the proof.

Definition 26. Let G be a graph with edgecut sequence $\{a_i\}_{i=2}^{n}$ satisfying $a_2 = a_j$ for $j = 2, 3, \ldots, n_1$ and $a_2 \neq a_{n_1+1}$ for some $n_1 \leq n$. Then the character edgecut sequence of G is defined to be the sequence $\{a_i\}_{i=n_1}^{n}$. Note that the character edgecut sequence of a tree is defined to be the sequence $\{1\}$.

The derivative G' of a graph G is a graph obtained from G by deleting all the pendant vertices of G.

Theorem 27. *Let G be a graph on n vertices, among which n_1 are pendant vertices. If $\{a_i\}_{i=2}^{n}$ is the edgecut sequence of G, then*

$$E[G';\, x] = \sum_{i=2}^{n-n_1} a_{n_1+i} x^i.$$

Also G and G' will have the same character edgecut sequence iff G' has at least 1 *cutedge.*

Proof. The removal of the n_1 pendant vertices is accompanied by the deletion of the same number of cutedges associated with those vertices and since we are deleting all the possible cutedges initially in the standard edgecut of a graph, the edgecut sequence of G' is in fact a left shift of length n_1 of the edgecut sequence of G.

A caterpillar is a tree graph whose derivative is a path graph.

Corollary 28. *Let G be a caterpillar on n vertices, among which n_1 are pendant vertices. Then G and G' will have the same character edgecut sequence. Also,*

$$E[G';\, x] = \sum_{i=2}^{n-n_1+1} x^i.$$

A snake graph $S_{n,m}$ is obtained from a path P_n by replacing each edge of P_n by the cycle graph C_m [4].

Theorem 29. *For $n, m > 2,$ the edgecut sequence of $S_{n,m}$ is given by*

$$(2, \underbrace{1, 1, 1, \ldots, 1}_{m-2 \text{ times}}, \ldots, 2, \underbrace{1, 1, \ldots, 1}_{m-2 \text{ times}}),$$

where the finite subsequence $(2, \underbrace{1, 1, \ldots, 1}_{m-2 \text{ times}})$ is repeated $n-1$ times.

Proof. In $S_{n,m}$ there are $n-2$ vertices of degree 4 and the remaining vertices of degree 2 are *similar*. Since the graph has no cutedges, without loss generality, one of the *similar* vertices of degree 2 is isolated from the cycle on the left most end of $S_{n,m}$ and this results in the formation of $m-2$ cutedges. After deleting all the cutedges one by one, the resulting nontrivial component is $S_{n-1,m}$ and the process is continued as earlier till all the vertices are isolated. Thus the edgecut sequence of $S_{n,m}$ is the sequence $(2, \underbrace{1, 1, ..., 1}_{m-2 \text{ times}})$ repeated $n-1$ times.

A flower graph $f_{n\times m}$ is a graph with $n(m-1)$ vertices and *nm* edges, in which *n* vertices form an *n*-cycle and *n* sets of $m-2$ vertices form *m*-cycles around the *n* cycle so that each *m*-cycle uniquely intersects with the *n*-cycle on a single edge. The *n*-cycle is called the center and *m*-cycles are called petals of $f_{n\times m}$ [3].

Theorem 30. *For* $n, m > 3,$ *the edgecut sequence of* $f_{n\times m}$ *is given by*

$$(2, \underbrace{1, 1, ..., 1}_{m-3 \text{ times}}, ..., 2, \underbrace{1, 1, ..., 1}_{m-3 \text{ times}}, 2, \underbrace{1, 1, ..., 1}_{n-2 \text{ times}}),$$

where the finite subsequence $(2, \underbrace{1, 1, ..., 1}_{m-3 \text{ times}})$ *is repeated n times*.

Proof. Observe that in $f_{n\times m}$, the *n* vertices which form the center are of degree 4 and all the other vertices have degree 2. Since $f_{n\times m}$ is Hamiltonian, all the vertices of degree 2, which are on the petals, are *similar*. Thus one of the vertices of degree 2 is isolated and as a result $m-3$ cutedges are generated in the respective petal. After deleting all the cutedges, the nontrivial component is a flower graph with $n-1$ petals and the process is continued similarly till all the petals are removed resulting in an *n*-cycle as the nontrivial component.

This completes the proof.

A chaplet graph $C_p \odot C_q^t$, where $p, q, t \geq 3$ is obtained by taking one-point union of t-copies of the cycle C_q and attaching the same to each vertex of the cycle C_p [8].

Theorem 31. *The edgecut sequence of* $C_p \odot C_q^t$ *is given by*

$$(2, \underbrace{1, \ldots, 1}_{q-2 \text{ times}}, 2, \underbrace{1, \ldots, 1}_{q-2 \text{ times}}, \ldots, 2, \underbrace{1, \ldots, 1}_{q-2 \text{ times}}, 2, \underbrace{1, \ldots, 1}_{p-2 \text{ times}}),$$

where the finite subsequence $(2, \underbrace{1, \ldots, 1}_{q-2 \text{ times}})$ *is repeated pt times.*

Proof. Observe that in $C_p \odot C_q^t$, there are p vertices of degree $2(t+1)$ and all other remaining vertices are of degree 2. The vertices of degree 2 are *similar* with cycle-number q and cycle-degree 1. Thus each of the pt cycles of length q are removed one by one by deleting $(2, \underbrace{1, \ldots, 1}_{q-2\ times})$ edges respectively at each stage till we obtain C_p as the nontrivial component, from which $(2, \underbrace{1, \ldots, 1}_{p-2\ times})$ edges are deleted respectively in the above order.

Let v_0 be a specified vertex of a graph G. Let $G_{v_0}(m)$ be a graph obtained from G by identifying the vertex v_0 of G with an end vertex of the path P_{m+1} with $m+1$ vertices.

Theorem 32. *Let G be a graph with n vertices and let* $v_0 \in V(G)$. *Then*

$$E[G_{v_0}(m); x] = \sum_{i=2}^{m+1} x^i + \sum_{i=m+2}^{m+n} a_i x^i,$$

where $\{a_{i-m}\}_{i=m+2}^{m+n}$ *is the edgecut sequence of G.*

Proof. The deletion of the m cutedges of $G_{v_0}(m)$ as a part of P_{m+1} gives G as the nontrivial component. Now, the result follows from the fact that $\{a_i\}_{i=2}^{n}$ is the edgecut sequence of G.

Let G_1 and G_2 be two disjoint graphs. Let $(G_1, G_2)_{u,v}(m)$ be a graph obtained by identifying the vertices u of G_1 and v of G_2 with the end vertices of a path P_m.

Corollary 33. *Let G_1 and G_2 be two disjoint graphs with n_1 and n_2 vertices respectively and let $u \in V(G_1)$ and $v \in V(G_2)$. Then,*

$$E[(G_1, G_2)_{u,v}(m); x] = \sum_{i=m+1}^{m+n_1+n_2-2} a_i x^i + \sum_{i=2}^{m} x^i,$$

where $\{a_{i-(m-2)}\}_{i=m+1}^{m+n_1+n_2-2}$ is the edgecut sequence of the graph obtained by linking G_1 and G_2 by a bridge, except for the first term.

Let G be a graph. The duplication of a vertex v of G is the graph $G^{\bullet}$ obtained by adding a vertex v' in G with $N(v') = N(v)$.

Theorem 34. *For $n > 4$,*

$$E[C_n^{\bullet}; x] = \sum_{i=n}^{n+1} x^i + \sum_{i=3}^{n-2} x^i + \sum_{i\in\{2,\, n-1\}} 2x^i.$$

Proof. Observe that in $C_n^{\bullet}$, the two vertices which are adjacent to v' are of degree 3 and all other vertices are of degree 2. Since the duplicated vertex v' has cycle-number n and cycle-degree 1 and all the vertices of degree 2 in C_n are *similar* with cycle-number n and cycle-degree 2, one of the vertices of degree 2 in C_n is isolated. This results in the formation of $n-4$ cutedges in the nontrivial component whose deletion produces C_4. Thus the edgecut sequence of $C_n^{\bullet}$ is given by $(2, \underbrace{1, \ldots, 1}_{n-4 \text{ times}}, 2, 1, 1)$.

This completes the proof.

Theorem 35. *For $n > 4$,*

$$E[P_n^{\bullet}; x] = \sum_{i=n}^{n+1} x^i + 2x^{n-1} + \sum_{i=2}^{n-2} x^i.$$

Proof. In $P_n^{\bullet}$, there are $n-3$ cutedges, each of which is deleted one by one to obtain C_4. Thus the edgecut sequence of $P_n^{\bullet}$ is given by $(\underbrace{1, 1, \ldots, 1}_{n-3 \text{ times}}, 2, 1, 1)$.

This completes the proof.

Theorem 36. *Let G be a graph and let v be the vertex of G of minimum degree such that v and* v' *are similar in* $G^{\bullet}$. *If* $E[G; x] = \sum_{i=2}^{n} a_i x^i$, *then* $E[G^{\bullet}; x] = \sum_{i=3}^{n+1} a_i x^i + 2x^2$, *where* $G^{\bullet}$ *is the duplication of the vertex v of G.*

Proof. Since v is the vertex of minimum degree, the vertex v' of $G^{\bullet}$ will also have the same degree, say d, which is strictly less than the degree of all other vertices of $G^{\bullet}$, except v. Since v and v' are *similar*, v' can be isolated from $G^{\bullet}$ in the first step itself. Then the nontrivial component becomes G, which completes the proof.

Corollary 37. *Let G be a graph and let v be a vertex of G having minimum degree d. Let* v' *be similar to all the vertices of* $G^{\bullet}$ *having degree d in G. If* $E[G; x] = \sum_{i=2}^{n} a_i x^i$, *then* $E[G^{\bullet}; x] = \sum_{i=3}^{n+1} a_i x^i + 2x^2$.

Let G be a graph. Let G'' be the graph obtained from $G^{\bullet}$ such that v and v' are adjacent in G''.

Theorem 38. *For* $n > 4$,

$$E[C_n''; x] = x^{n+1} + \sum_{i=n-1}^{n} 2x^i + \sum_{i=2}^{n-2} x^i + 2x^2.$$

Proof. In C_n'', there are $n-3$ vertices, as part of C_n, of degree 2 and all the remaining vertices are of degree 3. Since C_n'' is Hamiltonian, one of the vertices of degree 2 is isolated which results in the formation of $n-4$

cutedges. The cutedges are deleted one by one and the nontrivial component is a union of two triangles on four vertices whose edgecut sequence is $(2, 2, 1)$.

This completes the proof.

Theorem 39. *For* $n > 3$,

$$E[P_n''; x] = x^{n+1} + \sum_{i=n-1}^{n} 2x^i + \sum_{i=2}^{n-2} x^i,$$

where the duplicated vertex is a non-pendant vertex of P_n. *If the duplicated vertex is a pendant vertex of* P_n, *then*

$$E[P_n''; x] = x^{n+1} + 2x^n + \sum_{i=2}^{n-1} x^i.$$

Proof. Suppose that the duplicated vertex is a non-pendant vertex of P_n. Then, P_n'', has $n - 3$ cutedges, which are deleted one by one so that the resulting nontrivial component is a union of two triangles on four vertices whose edgecut sequence is $(2, 2, 1)$.

On the other hand, if the duplicated vertex is a pendant vertex of P_n, then P_n'' has $n - 2$ cutedges, whose deletion results in the formation of C_3 as the nontrivial component.

This completes the proof.

Theorem 40. *Let G be a tree. If the duplicated vertex of G is a pendant vertex, then G and* $G^{\bullet}$ *will have the same character edgecut sequence.*

Proof. Let v be a pendant vertex of G. Then the vertex v' of $G^{\bullet}$ is also a pendant vertex so that all the edges of $G^{\bullet}$ are also cutedges. Thus $G^{\bullet}$ is also a tree which implies that G and $G^{\bullet}$ will have the same character edgecut sequence $\{1\}$.

Theorem 41. *If G and H are isomorphic graphs, then* $E[G; x] = E[H; x]$.

Proof. Since degree and cycle-degree of each vertex of a graph is preserved under an isomorphism, it follows that G and H will have the same edgecut sequence. That is, $E[G; x] = E[H; x]$.

The converse of Theorem 41 is not true. To prove this, consider the graphs P_{n+1} and $K_{1,n}$. Observe that both graphs have the same edgecut polynomial. But they are not isomorphic.

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