GROVER QUANTUM SEARCH ALGORITHM

5/11/2025

Grover's Algorithm Insights :-

N-items are there & need to search only 1 item

$x_1 \rightarrow$ set of all (N-1) items what we don't want

$x_2 \rightarrow$ set of '1' item what we search

Let x_1 & x_2 are vectors (prepare)

$$|\psi_0\rangle = |x_1\rangle + |x_2\rangle \longrightarrow \text{Full sample space}$$

We can represent these in vector space to visualize [This is one of the way, you can have many other ways]

Let N-items are

$$x_3 = \{y_1, y_2, y_3 \ldots\ldots y_N\}$$

Hadamard state can be represented by :-

$$|x_3\rangle = \frac{1}{\sqrt{N}}\left[|y_1\rangle + |y_2\rangle + |y_3\rangle + \ldots |y_m\rangle + \ldots |y_N\rangle\right]$$

y_m is the one we need to search

$$|x_1\rangle = \frac{1}{\sqrt{N-1}}\left[|y_1\rangle + |y_2\rangle + \ldots |y_n\rangle\right]$$

No "y_m" here

meaning

If, I club all items "we dont want" seperately, they are equiprobable

$$|x_2\rangle = \frac{1}{\sqrt{1}}|y_m\rangle \rightarrow \text{set we "target"}$$

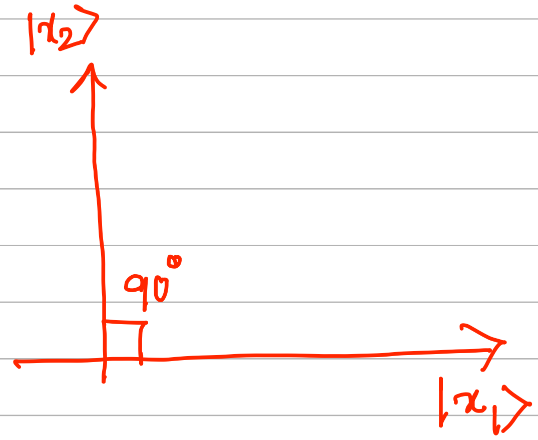
If, I club all items "we want" seperately, they are equiprobable

If we translate the vectors $|x_1\rangle$ & $|x_2\rangle$ into co-ordinate system,

$|x_1\rangle$ & $|x_2\rangle$ are orthogonal vectors

$|x_1\rangle \cdot |x_2\rangle = 0,$

linear combination of all the remaining vectors w.r.t $|x_1\rangle$ & $|x_2\rangle$ lie the plane



Equiprobable vector (or) Hadamard state is also in the plane :- and can be represented by,

$$|x_s\rangle = \frac{1}{\sqrt{N}}\left[|y_1\rangle + |y_2\rangle + |y_3\rangle + \text{------} |y_n\rangle\right]$$

This vector will also lie in $|x_1\rangle, |x_2\rangle$ plane.

A proper linear combination of $|x_2\rangle$ & $|x_1\rangle$ will yield the whole sample space:

$$|\psi_s\rangle = \frac{1}{\sqrt{N}}\left[|y_m\rangle\right] + \frac{1}{\sqrt{N}}\underbrace{\left[|y_1\rangle + |y_2\rangle + \text{---} |y_n\rangle\right]}_{\text{Except } |y_m\rangle}$$

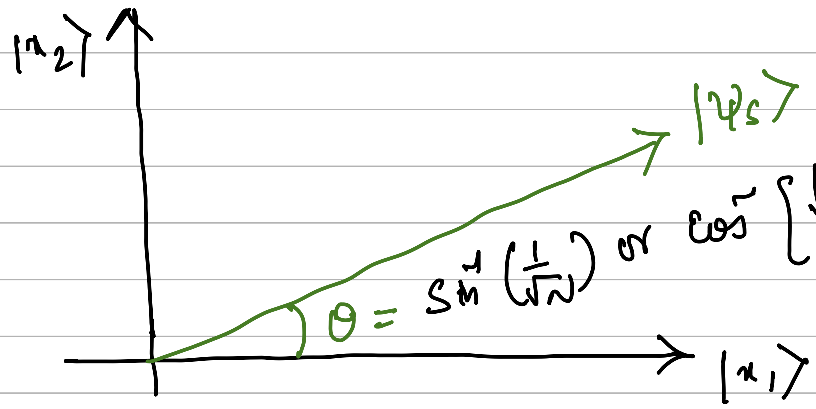
$$|\psi_s\rangle = \frac{1}{\sqrt{N}}|x_2\rangle + \frac{1}{\sqrt{N}} \cdot \frac{\sqrt{N-1}}{\sqrt{N-1}} \cdot \left[|y_1\rangle + |y_2\rangle + \cdots |y_n\rangle\right]$$

Except $|y_m\rangle$

Linear Combination of $|x_1\rangle, |x_2\rangle$

$$|\psi_s\rangle = \frac{1}{\sqrt{N}}|x_2\rangle + \frac{\sqrt{N-1}}{\sqrt{N}}|x_1\rangle$$

Representation of $|\psi_s\rangle$, in $|x_1\rangle$, $|x_2\rangle$ plane



$$|\psi_s\rangle = \underbrace{\frac{\sqrt{N-1}}{\sqrt{N}}|x_1\rangle}_{\text{unit vector}} + \underbrace{\frac{1}{\sqrt{N}}|x_2\rangle}_{\text{unit vector.}}$$

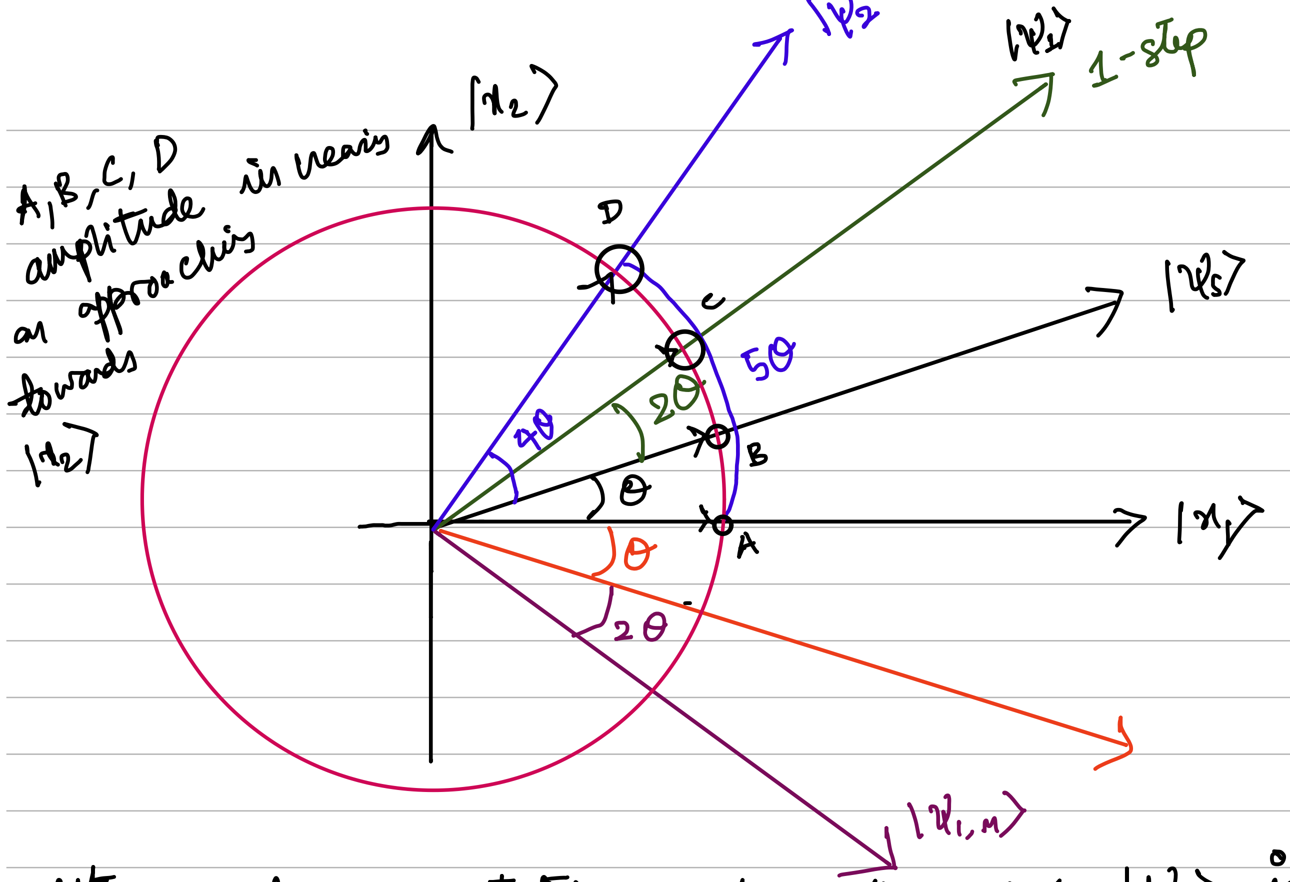
= (or) Can be represented in phasor

$$\cos\theta\,|x_1\rangle + \sin\theta\,|x_2\rangle$$

$$\cos\theta = \frac{\sqrt{N-1}}{\sqrt{N}}, \quad \sin\theta = \frac{1}{\sqrt{N}}.$$

$$\boxed{\theta = \tan^{-1}\left\{\frac{\frac{1}{\sqrt{N}}}{\frac{\sqrt{N-1}}{\sqrt{N}}}\right\} \qquad \tan\theta = \left[\frac{1}{\sqrt{N-1}}\right]}$$

Now, $|\psi_s\rangle$ need to be assigned in other vector and rotate continuously around $|x_1\rangle$ & Hadamad vector. Hadaramad vector (or) a state is the one we can easily generate using Hadamad gate. so, we will have that vector easily.



After one step the total angle, the vector $|\psi_s\rangle$ is done with $|x_1\rangle$ is $= \theta + 2\theta = 3\theta$

Second step:

Rotate $|\psi_1\rangle$ w.r.t $|x_1\rangle$ with 3θ & rotate the $|\psi_{1,M}\rangle$ along $|\psi_s\rangle$ with 4θ, this will make $|\psi_2\rangle$ create an angle with $|x_1\rangle$ as:

$$\theta + 4\theta = 5\theta$$

Step 1: $\longrightarrow$ Rotation $= 2[1]\theta + \theta = 3\theta$

Step 2: $\longrightarrow$ Rotation $= 2[2]\theta + \theta = 5\theta$

generalizing, we will get,

Step K: $\longrightarrow$ Rotation $= (2K)\theta + \theta = (2K+1)\theta$ -

The Vector $|\psi_s\rangle$ Eventually reaching $|x_2\rangle$, which is what we want to search.

The moment $|\psi_s\rangle$ is equal to $|x_2\rangle$, that means the item in set sample space is said to found.

The condition, $|\psi_s\rangle$ becomes $|x_2\rangle$ means, the angle between $|\psi_s|$ & $|x_p\rangle$ is $90°$.

So,
$$(2K+1)\theta = \pi/2$$
$$(2K+1)\sin^{-1}\left(\frac{1}{\sqrt{N}}\right) = \pi/2$$
$$\sin\left[\frac{\pi}{2[2K+1]}\right] = \frac{1}{\sqrt{N}}$$

where K is number of steps, one cycle of rotation is done,

If $N=4$,
$$\sin\left[\frac{\pi}{2[2K+1]}\right] = \frac{1}{2}$$
$$\frac{\cancel{\pi}}{\cancel{2}[2K+1]} = \frac{\cancel{\pi}}{\cancel{6}3} \Rightarrow 2K+1 = 3$$
$$2K = 2$$
$$\boxed{K=1}$$

If N is very large :- since N is large.
$$\sin\left[\frac{\pi}{2[2K+1]}\right] = \frac{1}{\sqrt{N}}, \quad \frac{\pi}{2[2K+1]} = \frac{1}{\sqrt{N}}.$$

$$(2K+1) = \frac{\pi}{2\sqrt{N}}$$

$$K \approx \left[\frac{\pi}{2\sqrt{N}} - 1\right] \frac{1}{2}$$

$$\boxed{K \approx \frac{1}{2}\left[\frac{\pi}{2\sqrt{N}} - 1\right]} \rightarrow \text{only if } \sqrt{N} \text{ is very large}$$

Case (ii) :- If M - items of N - items are needed to search where $M \leq N$

Let N - items are

$$x_s = \{y_1, y_2, y_3 \,.\,.\,.\,.\,.\, y_N\}$$

Hadamard state can be represented by :-

$$|x_s\rangle = \frac{1}{\sqrt{N}}\left[|y_1\rangle + |y_2\rangle + |y_3\rangle + \ldots |y_m\rangle + \ldots |y_n\rangle\right]$$

$x_1 \rightarrow$ set of all (N-M) items what we don't want

$x_2 \rightarrow$ set of M item what we search

$$|x_s\rangle = \frac{1}{\sqrt{N}}\left[|y_1\rangle + |y_2\rangle + \ldots |y_p\rangle + |y_{p+1}\rangle + \ldots |y_{p+(M-1)}\rangle + \ldots |y_N\rangle\right]$$

$$|x_s\rangle = \frac{1}{\sqrt{N}}\left[|y_p\rangle + |y_{p+1}\rangle + \ldots |y_{p+(M-1)}\rangle\right] + \frac{1}{\sqrt{N}}\left[|y_1\rangle + |y_2\rangle + \ldots |y_N\rangle\right]$$

M items removed from list

$$|x_2\rangle = \frac{1}{\sqrt{M}}\left[|y_p\rangle + |y_{p+1}\rangle + \cdots \cdots |y_{p+(M-1)}\rangle\right]$$

$\Downarrow$

This is the state we are looking for

$$|x_1\rangle = \frac{1}{\sqrt{N-M}}\left[|y_1\rangle + |y_2\rangle + \cdots |y_{p-1}\rangle + \cdots |y_{p+M}\rangle + \cdots |y_N\rangle\right]$$

$\Downarrow$

This is the state we do not want.

If we represent $|x_1\rangle$ and $|x_2\rangle$ in the 3-D space, it can be represented as two orthogonal vectors.

$$|x_1\rangle \cdot |x_2\rangle = 0 \Rightarrow |x_1\rangle \;\&\; |x_2\rangle \text{ are orthogonal.}$$

$|x_s\rangle$ can be represented as linear combination of $|x_1\rangle$ & $|x_2\rangle$.

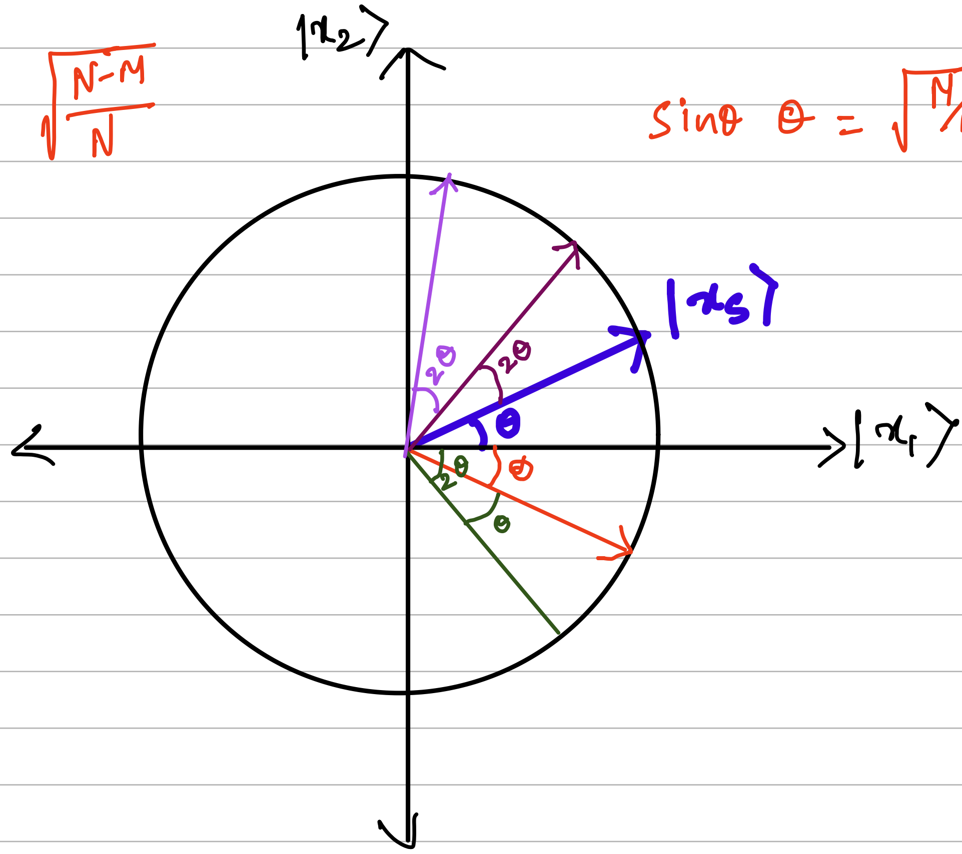
$$|x_s\rangle = \frac{1}{\sqrt{N}}\cdot\frac{\sqrt{M}}{\sqrt{M}}\left[|y_p\rangle + |y_{p+1}\rangle + \cdots |y_{p+(M-1)}\rangle\right] +$$

$$\frac{1}{\sqrt{N}}\cdot\frac{\sqrt{N-M}}{\sqrt{N-M}}\left[|y_1\rangle + |y_2\rangle + \cdots |y_{p-1}\rangle + \cdots |y_{p+M}\rangle + \cdots |y_N\rangle\right]$$

$$= \sqrt{\frac{M}{N}}\cdot|x_2\rangle + \frac{\sqrt{N-M}}{\sqrt{N}}\cdot|x_1\rangle$$

$$\boxed{|x_s\rangle = \sqrt{\frac{N-M}{N}}\,|x_1\rangle + \sqrt{\frac{M}{N}}\cdot|x_2\rangle}$$

$\cos\theta = \sqrt{\frac{N-M}{N}}$ $\sin\theta = \sqrt{M/N}$



After K-steps, the angle it is translated to is:

$$(2K+1)\,\theta$$

This angle should be equal to $\pi/2$

$$(2K+1)\,\theta = \pi/2$$

$$\theta = \frac{\pi}{2[2K+1]}$$

$$\sin\left[\frac{\pi}{2[2K+1]}\right] = \sqrt{\frac{M}{N}}$$

If $\sqrt{\frac{M}{N}}$ is very small, meaning, "N" is very large & M is few elements say, it is fraction of "N", then $\sin\theta = \theta$

$$\frac{\pi}{2[2K+1]} = \sqrt{\frac{M}{N}}$$

$$\Rightarrow 2[2K+1] = \frac{\pi\sqrt{N}}{\sqrt{M}}$$

$$(2K+1) = \frac{\pi\sqrt{N}}{2\sqrt{M}}$$

$$K = \left[\frac{\pi\sqrt{N}}{2\sqrt{M}} - 1\right]\frac{1}{2}$$

$$K = \frac{\pi\sqrt{N} - 2\sqrt{M}}{4\sqrt{M}} \qquad \delta = \frac{M}{N}$$

$$K = \left[\frac{\pi}{4}\sqrt{\frac{1}{\delta}} - \frac{1}{2}\right] \Rightarrow O\left(\sqrt{1/\delta}\right)$$

(OR)

$$K = \frac{1}{2}\left[\frac{\pi}{2}\sqrt{\frac{N}{M}} - 1\right] = O\left(\sqrt{\frac{N}{M}}\right)$$

If more number of similar items to be searched, from a set of "N" items, the less are the number of iterations.