Fundamental Bounds on Learning Performance in Neural Circuits

Raman, Rotondo & O'Leary
Presented by Rylan Schaeffer

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- Synapses in biological networks lack persistence, undergoing significant turnover [3, 9, 8], with magnitude rivaling Hebbian plasticity [5]
- Across species and regions, neurons frequently make multiple synaptic connections to same postsynaptic neuron [1, 2, 4, 6]
- What is the role of these processes? What (dis)advantages do these phenomena confer on biological circuits? [7]

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- For a given task, there is an optimal network size that maximizes rate of error reduction
- Below optimal size, increasing network size causes network to learn faster by minimizing effect of curvature

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- Notation: $c\hat{d}ot$ denotes normalized vector

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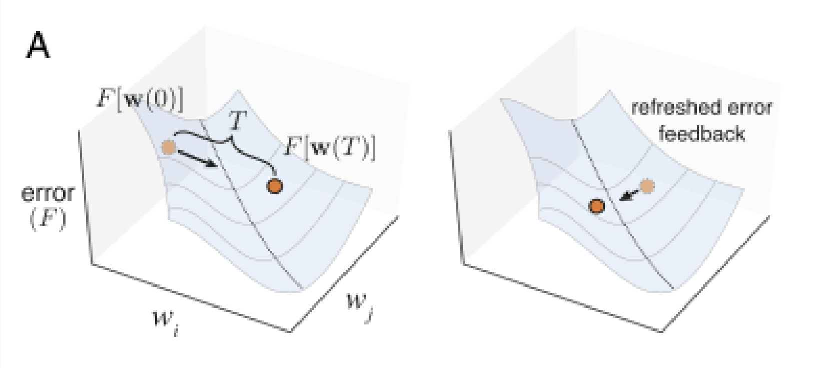
Curvature competes with gradient to accelerate, slow or reverse learning. Fig 3A:



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- Writing the weight change:

$$\dot{w}_T = -\gamma_1 \nabla \hat{F}[w(0)] + \gamma_2 \hat{n}_2 + \gamma_3 \sqrt{\frac{N}{T}} \hat{n}_3$$

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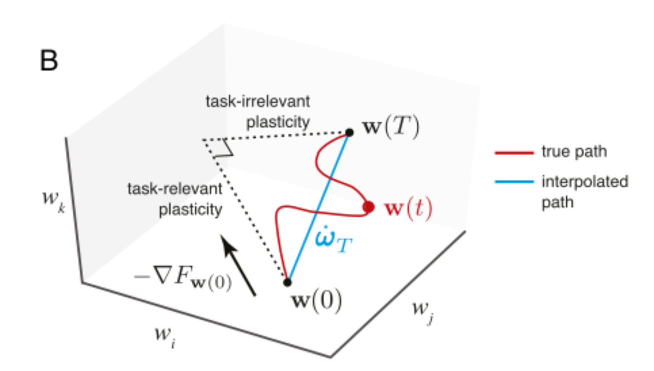

Fig 3. Synaptic noise not pictured!

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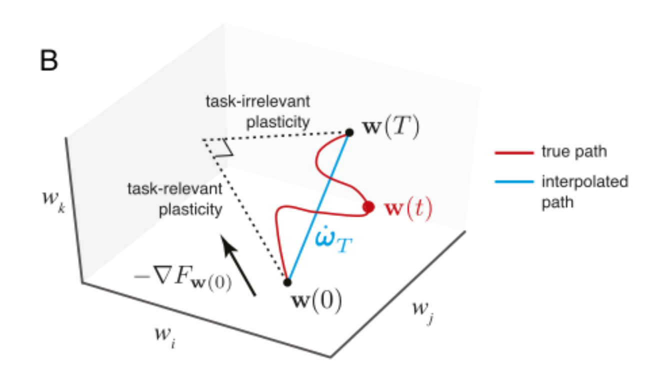

Fig 3. Synaptic noise not pictured! How does each factor affect k?

Weight Change Effect on Learning Rate

$$k \approx -\frac{||\nabla F||_2}{F}\left[\dot{w}_T^T \nabla \hat{F} + T\frac{||\dot{w}_T||_2^2}{2||\nabla F||_2}\dot{\hat{w}}_T^T \nabla^2 F \dot{\hat{w}}_T\right]$$

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Assume (1) $n_2, n_3, \nabla F$ uncorrelated; (2) n_2, n_3 independent from $\nabla^2 F[w]$ i.e. $\langle n_i^T \nabla^2 F n_i \rangle_{\hat{n}_2,\hat{n}_3} = \frac{\mathrm{Tr}(\nabla^2 F)}{N}$.

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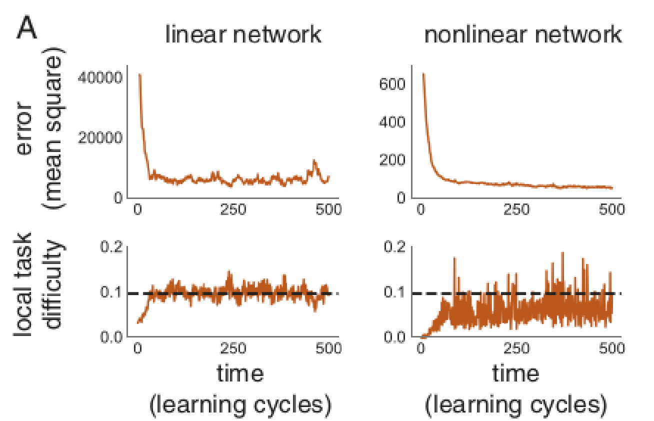
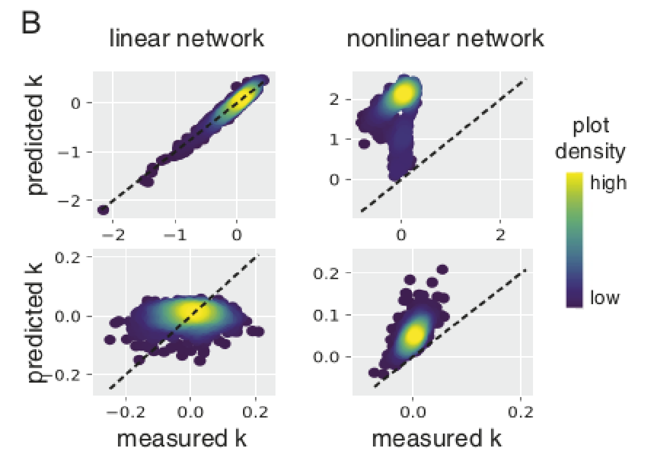
- There exists optimal size N^* that maximizes $\langle k \rangle_{\hat{n}_2,\hat{n}_3}$

Local Task Difficulty



Local Task Difficulty

Top: Low intrinsic noise ($\gamma_3 = 0.05$). Bottom: High intrinsic noise ($\gamma_3 = 0.1$).



Optimal Linear Network Size

- Student-teacher framework with $W \in \mathbb{R}^{o \times i}$:

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- Replace W with $D^T W' B$

$$y = D^T W' B x$$

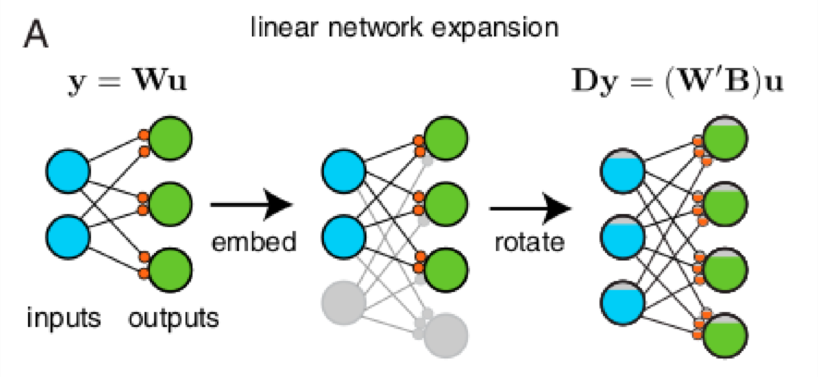
$$F[W'] = F[W]$$

$$||F[W']||_F^2 = ||F[W]||_F^2$$

$$\mathrm{Tr}(\nabla^2 F[W']) = c_2 \,\mathrm{Tr}(\nabla^2 F[W])$$

Optimal Linear Network Size

$$y = D^T W' Bx \iff Dy = W' Bx$$



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- Define $N = io, \tilde{N} = c_1 c_2 io$

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- Thus:

$$\langle k(\tilde{N}) \rangle \approx \frac{-||\nabla F||_2}{F}\left[-\gamma_1 + Tc_2\gamma_1^2 \nabla \hat{F}^T \nabla^2 F \nabla \hat{F} + Tc_2\frac{\mathrm{Tr}(\nabla^2 F)}{2||\nabla F||_2^2}\left[\frac{\gamma_2^2}{\tilde{N}} + \frac{\gamma_3^2}{T}\right]\right]$$

Optimal Linear Network Size

Find N^* that maximizes $k(\tilde{N})$:

$$N^* \approx \frac{T\gamma_2^2}{\gamma_3^2}(1 - \frac{\gamma_1^2}{\gamma_2^2})$$

If no task-irrelevant plasticity, $\gamma_2 = 0 \Rightarrow N^* \approx 0 - \frac{T\gamma_1^2}{\gamma_3^2} < 0 \Rightarrow$ optimal network size is negative?

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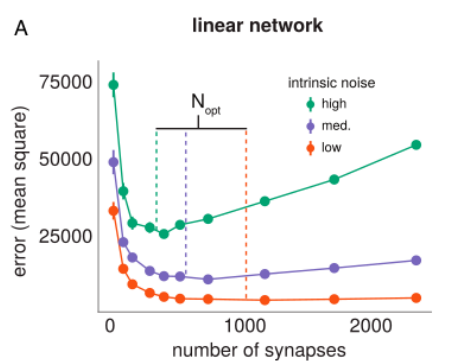

Optimal Non-Linear Network Size

- Student-Teacher framework with logistic sigmoid activation functions :

$$h^{(k)} = \sigma(W^{(k)} h^{k-1})$$

Optimal Non-Linear Network Size

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- Replace W with larger W', with new synaptic weights initialized to zero

Optimal Non-Linear Network Size

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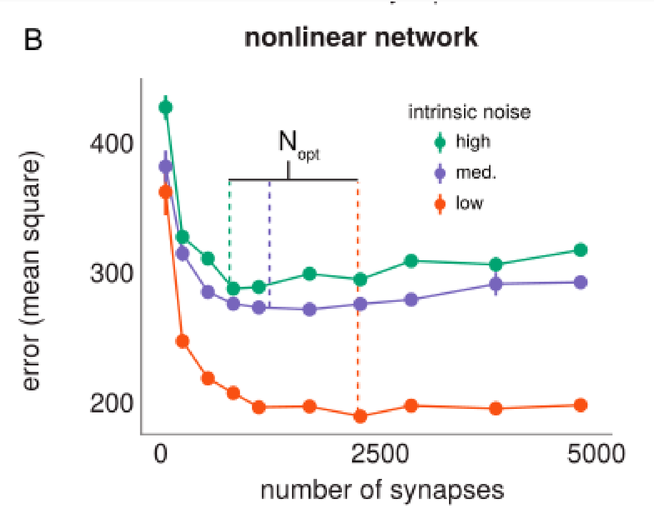
$$h^{(k)} = \sigma(W^{(k)} h^{k-1})$$

- Replace W with larger W', with new synaptic weights initialized to zero
- Through some derivation I didn't have time to read:

$$N^* = \frac{T\gamma_2^2}{\gamma_3^2}\left[\frac{\gamma_1^2 N}{\gamma_2^2 N^*}\right]$$

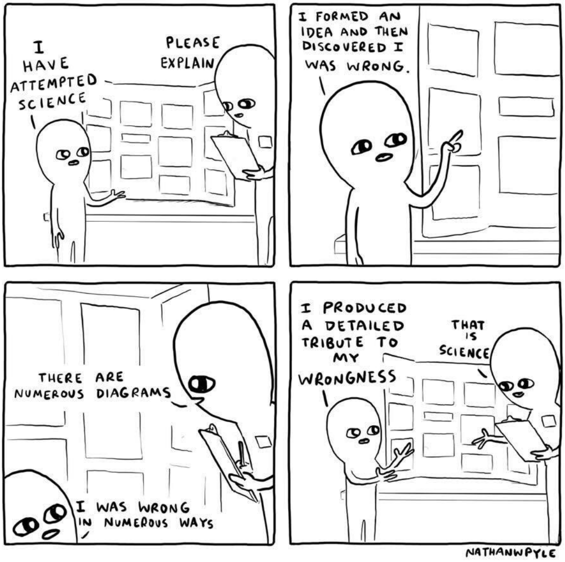
Optimal Non-Linear Network Size



Takeaways

- Larger networks learn better, but
- Intrinsically noisy synapses eventually negate benefits of larger network size
- Experimental Prediction: Circuit size should be inversely proportional to per-synaptic rate of change
- Experimental Prediction: suppression of synaptic noise allows for larger circuit formation

Questions?



I HAVE ATTEMPTED SCIENCE
PLEASE EXPLAIN
I FORMED AN IDEA AND THEN DISCOVERED I WAS WRONG.
THERE ARE NUMEROUS DIAGRAMS
I WAS WRONG IN NUMEROUS WAYS
I PRODUCED A DETAILED TRIBUTE TO MY WRONGNESS
THAT IS SCIENCE
NATHANWPYLE

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