Lecture 2

Recall that (from **Lecture 1**), in this course on time series, we will be studying time series models of the form

$$Y_t = g(Y_{t-1}, Y_{t-2}, ...; \epsilon_t, \epsilon_{t-1}, ...),$$

for some function g, where Y_t is the observation made at time t and ϵ_t is the error associated while observing Y_t. Here, it is important to understand the fact that *"current observation is independent of future error(s)"*; that is, $\forall\, t < s, Y_t$ is independent of ϵ_s. This immediately ensures that

$$Cov[Y_t, \epsilon_s] = 0, \;\; \forall\, t < s.$$

Now, we consider the white noise $\{\epsilon_t, t = 1, 2, 3, ...\}$ - (**Definition 2** from **Lecture 1**). We easily see that this is a stationary time series, as it satisfies all the three conditions for stationarity (**Definition 6** from **Lecture 1**), *viz*

(a) $V[\epsilon_t] = \sigma^2$, finite and positive $\quad \forall\, t$.

(b) The mean function $\mu_t = E[\epsilon_t] \equiv 0, \;\; \forall\, t$.

(c) The autocovariance function, as a function of only $\mid t - s \mid$, is given by

$$Cov[\epsilon_t, \epsilon_s] = \lambda(t, s) = \begin{cases} \sigma^2 & \text{if } \mid t - s \mid = 0 \\ 0 & \text{if } \mid t - s \mid > 0 \end{cases}$$

Remark: Note that, in the definition of white noise, we have not assumed any *distribution* of ϵ_t's. We have only assumed uncorrelatedness together with zero mean and finite constant variance. In particular, if each ϵ_t is a normal random variable with mean 0 and variance σ^2, then the white noise $\{\epsilon_t\}$ is a sequence of *i.i.d.* (independent, identically distributed) normal variables, as we know that uncorrelatedness is equivalent to independence for two normal random variables. Such a white noise is known as *Gaussian White Noise*.

In most of our hands-on time series data analysis, it will be assumed that the underlying white noise is indeed Gaussian.

From now on, we will write $\{Y_t\}$ to denote the time series $\{Y_t, t = 1, 2, 3, ...\}$, if no confusion would arise.

Suppose that $\{Y_t\}$ is given to be a stationary time series (that is, $\{Y_t\}$ satisfies all the three conditions given in **Definition 6** from **Lecture 1**), then we easily deduce the following fact(s) about its autocovariance function:

(i) From condition (c) of stationarity, for any given s,

$$\lambda(t-s,t) = \lambda(t,t+s) = \lambda(t+u-s,t+u) = \lambda(t+u,t+u+s) = \lambda(s) \text{ (say)}, \quad \forall\, t, u.$$

The above essentially means, for example, as long as the time difference between two observations is a constant, say $s = 4$, then where on the time axis they are observed does not alter their statistical dependence; that is, $Cov[Y_1, Y_5] = Cov[Y_{13}, Y_{17}] = Cov[Y_{1865}, Y_{1869}] = Cov[Y_{t-4}, Y_t] = Cov[Y_t, Y_{t+4}]$, no matter what t is.

(ii) Constancy of variance - that is,

$$Cov[Y_t, Y_t] = V[Y_t] = Cov[Y_s, Y_s] = V[Y_s] = \delta^2 \text{ (say)}, \ \forall\, t, s.$$

The constancy of variance is due to the fact that the autocovariance function of a stationary time series is a function of the time difference alone; that is $Cov[Y_t, Y_s] = \lambda(\mid t - s \mid)$. In particular,

$$V[Y_t] = Cov[Y_t, Y_t] = \lambda(0) = \delta^2 \text{ (say) \ for any } t.$$

So, a stationary time series always has constant mean and constant variance. In what follows, we introduce the most well known and basic time series models. **We also assume that these models are stationary.**

Autoregressive Models (AR Models)

Let $\{\epsilon_t\}$ be white noise and $\{Y_t\}$ be a stationary time series defined by

$$Y_t = \alpha + \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + ... + \beta_p Y_{t-p} + \epsilon_t,$$

where α and β_i's are unknown, real constants. Then the above model $\{Y_t\}$ is known as an *autoregressive model of order* p (or, simply $AR(p)$).

The simplest AR model, given by$AR(1)$, is of interest, and we will study about it a little later in the context of *random walk*.

Moving Average Models (MA Models)

Let $\{\epsilon_t\}$ be white noise and $\{Y_t\}$ be a time series defined by

$$Y_t = \alpha + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + ... + \theta_q \epsilon_{t-q} + \epsilon_t,$$

where α and θ_j's are unknown, real constants. Then the above model $\{Y_t\}$ is known as a *moving average model of order* q (or, simply $MA(q)$).

A couple of remarks in order:

(i) In the definition of AR model, we assume that $\{Y_t\}$ is a *stationary* time series; *but*, we do not assume this in the definition of MA model. We will see the reason for this a little later, through the computation of the stationary mean and variance of these models.

(ii) The nomenclature in the above two types of models is quite clear. In the AR model, Y_t is *regressed* on its own past variables, and hence the name *autoregressive* model. Whereas, in the MA model above, Y_t is a *weighted moving (or sliding) average* of the present and the past errors, with weights θ_j's. Hence the name MA model.

Let us now get the explicit forms of the stationary mean and stationary variance of the simplest AR model, namely $AR(1)$.

Before we start, let us recall the following from Probability:

Let $X_1, X_2, ..., X_n$ and $Y_1, Y_2, ..., Y_m$ be $m+n$ random variables with finite second moments. Further, let $a_1, a_2, ..., a_n$ and $b_1, b_2, ..., b_m$ be real constants. Then,

$$E[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i E[X_i],$$

$$V[\sum_{i=1}^{n} a_i X_i] = \sum_{i=1}^{n} a_i^2 V[X_i] + \sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} a_i a_j Cov[X_i, X_j],$$

and

$$Cov[\sum_{i=1}^{n} a_i X_i, \sum_{j=1}^{m} b_j Y_j] = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j Cov[X_i, Y_j].$$

AR(1)

Let $\{Y_t\}$ be a stationary $AR(1)$ model given by $Y_t = \alpha + \beta Y_{t-1} + \epsilon_t, \;\; t = 2, 3, ...,$ where $\{\epsilon_t\}$ is white noise. Since $\{Y_t\}$ is stationary, we know that it has constant mean μ and constant variance δ^2; that is, $E[Y_t] = \mu$ and $V[Y_t] = \delta^2, \; \forall \; t$. First, taking expectations on both sides of the $AR(1)$ model equation above, we have for each t that,

$$\begin{aligned} E[Y_t] &= \alpha + \beta E[Y_{t-1}] + E[\epsilon_t] \\ \mu &= \alpha + \beta\mu + 0 \quad (\text{since } \; E[\epsilon_t] = 0) \end{aligned}$$

Hence, the stationary mean of $\{Y_t\}$ is given by

$$\mu = \alpha/(1-\beta)$$

. This μ is finite, as the variance δ^2 is finite - recall the fact that, if a higher order moment of a random variable exists, then *all* its lower order moments exist. Therefore, we get that $\beta \neq 1$ as a condition on our $AR(1)$ structure. Turning to the variance, we have for each t that,

$$\begin{aligned} V[Y_t] &= V[\alpha + \beta Y_{t-1} + \epsilon_t] \\ &= \beta^2 V[Y_{t-1}] + V[\epsilon_t] + 2\beta Cov[Y_{t-1}, \epsilon_t] \end{aligned}$$

But, since $Cov[Y_{t-1}, \epsilon_t] = 0$ and $V[\epsilon_t] = \sigma^2$, we see that the stationary variance of $\{Y_t\}$ is given by

$$\delta^2 = V[Y_t] = \sigma^2/(1-\beta^2), \;\; \forall\, t.$$

Again, by the positivity and finiteness of the variance of a stationary time series, we get the condition on the coefficient β as $\beta^2 < 1$, or equivalently as $\mid \beta \mid < 1$, which implies the earlier condition that $\beta \neq 1$ for the existence finite mean.

Therefore, we have that $\mid \beta \mid < 1$ is a sufficient condition for the stationarity of the $AR(1)$ model. No condition on the intercept α is needed.

Let us now compute the autocovariance function of $AR(1)$. It is given by $Cov[Y_t, Y_s] = \lambda(t, s)$. Consider

$$\begin{aligned} \lambda(t, t+1) &= Cov[Y_t, Y_{t+1}] \\ &= Cov[Y_t, \alpha + \beta Y_t + \epsilon_{t+1}] \\ &= Cov[Y_t, \alpha] + \beta Cov[Y_t, Y_t] + Cov[Y_t, \epsilon_{t+1}] \\ &= 0 + \beta V[Y_t] + 0 \\ &= \beta\, \sigma^2/(1-\beta^2) \end{aligned}$$

So, $\lambda(t, t+1) = \beta\, \sigma^2/(1-\beta^2), \;\; \forall\, t$. One can see, using induction, that

$$\lambda(t, t+s) = \beta^s\, \sigma^2/(1-\beta^2).$$

Or equivalently, that

$$\lambda(t, s) = \beta^{|t-s|}\, \sigma^2/(1-\beta^2), \;\; \forall\, t, s.$$

We now state the following without proof.

For a general AR(p) model (where p is any natural number), a sufficient condition for the stationarity for AR(p) is that the solutions of the p-th degree polynomial equation

$$1 - \beta_1 x - \beta_2 x^2 - ... - \beta_p x^p = 0$$

are strictly greater than 1 *in absolute value.*