Lecture 3

In **Lecture 2**, we had seen that

(i) White noise is always stationary

(ii) With conditions on the coefficients, $AR(p)$ is staionary

In what follows, we introduce the concept of $ARMA$ (*autoregressive–moving average*) models which obviously combines the AR and MA models.

Autoregressive Moving Average (ARMA) Models

A time series $\{Y_t\}$ is called an $ARMA$ model of order (p, q), denoted by $ARMA(p, q)$, if it has the form

$$Y_t = \alpha + \sum_{i=1}^{p} \beta_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t,$$

or equivalently,

$$Y_t - \sum_{i=1}^{p} \beta_i Y_{t-i} = \alpha + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise and α, β_i's and θ_j's are unknown real constants.

Remark: Note that, while rewriting the first equation in the above definition as the second one, we have just put the autoregressive part on the left side and the moving average part on the right of the second equation. Then we see immediately that the above $ARMA(p, q)$ is stationary only if the autoregressive part (on the left side of the second equation) is stationary. This is because the moving average part on the right side is always stationary.

That is, we require only that the solutions of the p-th degree polynomial equation

$$1 - \beta_1 x - \beta_2 x^2 - ... - \beta_p x^p = 0$$

be strictly greater than 1 in absolute value, for our above $ARMA(p, q)$ to be stationary.

For example, if we build an $ARMA(1,3)$ model for a given data set, we require only that the coefficient β be strictly less than 1 in absolute value, for the stationarity of our model.

A digression: As mentioned in **Lecture 2**, we will now see more details about an interesting $AR(1)$ model which plays a very important role in numerous areas such as Finance, Physics, Biology, Computer Science and so on. Typically, an $AR(1)$ model is described by the equation

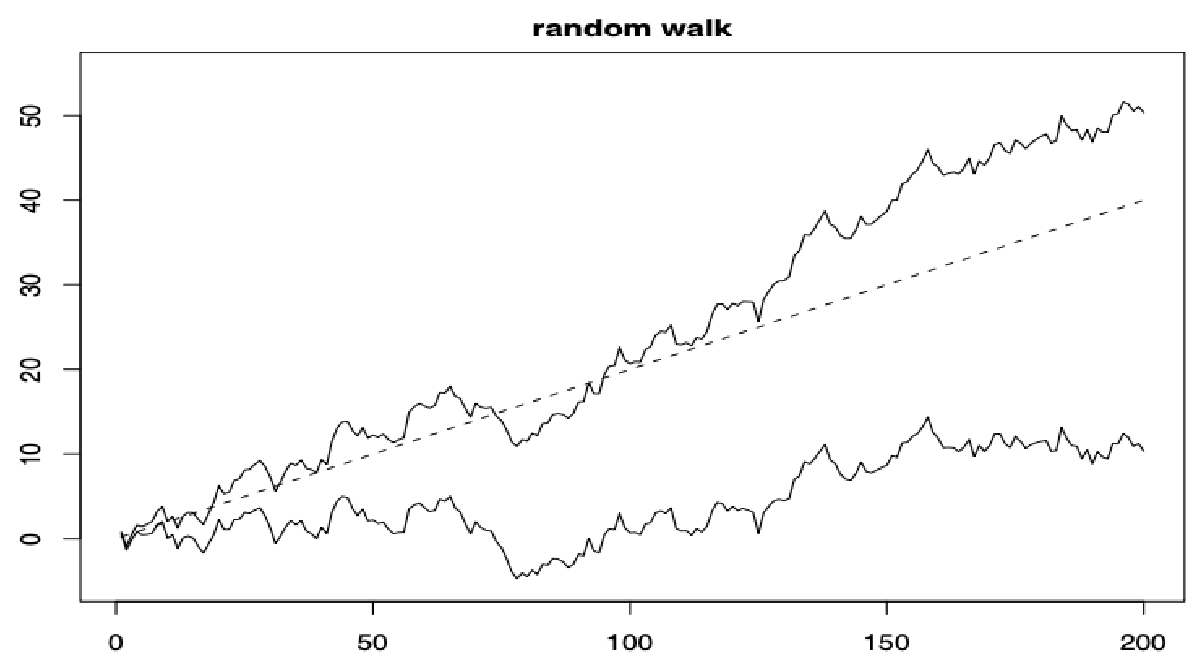
$$Y_t = \alpha + \beta Y_{t-1} + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise, α and β are real constants. Then we know that this model is stationary if $\mid \beta \mid < 1$. Let us not worry about stationarity for the discussion below. Suppose that $\beta = 1$ and that the first observation $Y_1 = 0$. Then the above $AR(1)$ model becomes

$$Y_t = \alpha + Y_{t-1} + \epsilon_t, \quad \text{with } Y_1 = 0. \qquad (*)$$

The model given in the equation (*) above is the well known **Random Walk with Drift** if $\alpha \neq 0$, and just **Random Walk** if $\alpha = 0$.

Given below is the graph of the sample paths (that is, two sets of observations/realisations) of two random walks - one with drift $\alpha = 0.2$ and the other with $\alpha = 0$. Here, the white noise is a Gaussian white noise with mean 0 and variance 1 (that is, $\sigma^2 = 1$). The dotted line is the mean function of the random walk with drift $\alpha = 0.2$; a straight line with slope 0.2. This picture is from Shumway & Stoffer's book.



We will now show that the random walk (with or without drift) is *not* stationary. We will also see how to get a stationary model from this random walk.

Noting that $Y_1 = 0$, we have the successive Y_t's as

$$\begin{aligned}
Y_2 &= \alpha + \epsilon_2 \\
Y_3 &= 2\alpha + \epsilon_2 + \epsilon_3 \\
. &= . \\
. &= . \\
. &= . \\
Y_t &= (t-1)\alpha + \sum_{i=2}^{t} \epsilon_i
\end{aligned}$$

That is, in general for $t \geq 2$, with the initial condition that $Y_1 = 0$, we have that

$$Y_t = (t-1)\alpha + \sum_{i=2}^{t} \epsilon_i.$$

Remember that if a time series is stationary, then it should have constant mean and constant variance. So, to prove that it is not stationary, it is enough to prove that the time series does not have either constant mean or constant variance. Now, $\alpha = 0$ or $\neq 0$. Assume that $\alpha \neq 0$. Then, taking expectations on both sides of the last equation above, together with the fact that $E[\epsilon_t] = 0 \; \forall \; t$, we see that the mean function of the random walk with drift is

$$E[Y_t] = (t-1)\alpha, \quad t = 2, 3, ...$$

which is nonconstant; the mean of Y_t varies with t. Hence, when $\alpha \neq 0$, the random walk is not stationary.

Now assume that $\alpha = 0$. Then the random walk becomes

$$Y_t = \sum_{i=2}^{t} \epsilon_i, \quad t = 2, 3, ...$$

Here it is immediate that $E[Y_t] = 0 \;\; \forall \; t \geq 2$. But, by taking variance on both sides and noting that ϵ_t's are uncorrelated, we see that

$$V[Y_t] = (t-1)\sigma^2, \quad t = 2, 3, ...,$$

which again depends on t.

Therefore, a random walk (with or without drift) is not stationary.

Let us now rewrite the equation of the random walk given in (*) earlier as

$$Y_t - Y_{t-1} = \alpha + \epsilon_t.$$

Further, let $C_t = Y_t - Y_{t-1}$ to get a new series given by

$$C_t = \alpha + \epsilon_t, \quad t = 2, 3, ...$$

Then it is very easy to see that the time series $\{C_t\}$, which is called the *first difference series of the original series* $\{Y_t\}$*, or simply the first difference*, is indeed stationary. In fact, if the drift $\alpha = 0$, then $\{C_t\}$ is exactly the white noise. If $\alpha \neq 0$, then $\{C_t\}$ is a sequence of uncorrelated random variables with constant mean α and constant variance σ^2, which has the same autocovariance function as that of the white noise. Hence, $\{C_t\}$ is stationary.

In general, if $\{Y_t\}$ is a given time series which is **not** stationary, then one can possibly look at the first differenced series denoted by $\{Y_t^{(1)}\}$ where

$$Y_t^{(1)} = Y_t - Y_{t-1}, \;\; t = 2, 3, ...$$

and check for the stationarity of this differenced series. If $\{Y_t^{(1)}\}$ is also not stationary we then look at the differenced series of $\{Y_t^{(1)}\}$ series, given by $\{Y_t^{(2)}\}$ where

$$Y_t^{(2)} = Y_t^{(1)} - Y_{t-1}^{(1)}, \;\; t = 3, 4, ...$$

and check for the stationarity of this second differenced series. And, if this $\{Y_t^{(2)}\}$ series is also not stationary, construct its differenced series to check for stationarity. We continue this successive differencing till we get stationarity at a stage d of differencing for the first time. That is, here

$$d = \; \min \{k \;\mid\; \{Y_t^{(k)}\} \text{ is stationary}\}.$$

We will see more on this differencing in the next lecture notes.