Lecture 10

Regime Switching Models - Non Linear Models for Returns and Volatility

"A natural approach to modelling economic time series with nonlinear models seems to be to define different states of the world or regimes, and to allow for the possibility that the dynamic behaviour of economic variables depends on the regime that occurs at any given point in time... By 'state-dependent dynamic behaviour' of a time series it is meant that certain properties of the time series, such as its mean, variance and/or autocorrelation, are different in different regimes. As an example, the autocorrelations of stock returns are related to the level of volatility of these returns. In particular, autocorrelations tend to be larger during periods of low volatility and smaller during periods of high volatility. The periods of low and high volatility can be interpreted as distinct regimes - or, put differently, the level of volatility can be regarded as the regime-determining process" (cf. *Nonlinear Time Series Models in Empirical Finance* by Philip Hans Franses and Dick van Dijk (2003), Cambridge University Press - pp. 69 - 70).

For non linear models for returns, we restrict our attention to models that assume that in each of the regimes the dynamic behaviour of the time series can be described adequately by a linear AR model. In other words, the time series is modelled with an AR model, where the autoregressive parameters are allowed to depend on the regime or state.

Two main classes of models can be distinguished. The models in the first class assume that the regimes can be characterised (or determined) by an observable variable. Consequently, the regimes that have occurred in the past and present are known with certainty. The models in the second class assume that the regime cannot actually be observed but is determined by an underlying unobservable stochastic process. This implies that one can never be certain that a particular regime has occurred at a particular point in time, but can only assign probabilities to the occurrence of the different regimes.

Regimes determined by observable variables

The most prominent member of the first class of models, which assume that the regime that occurs at time t can be determined by an observable variable X_t, is the *Threshold Autoregressive (TAR)* model. The simplest TAR model, the one with 2 regimes, is the one which assumes that the regime is determined by the value of X_t relative to a *threshold value*, which we denote as a real constant C; the two regimes being $X_t < C$ and $X_t \geq C$. A special case arises when the threshold variable X_t is taken to be a lagged value of the time series itself - that is, $X_t = Y_{t-d}$ for a certain integer $d > 0$. As in this case the regime (being either $Y_{t-d} < C$, or $Y_{t-d} \geq C$) is determined by the time series itself, the resulting model is called a *Self Exciting TAR (SETAR) model*. The integer d is known as the *delay parameter* of this $SETAR$ model.

Formally, a $2-regime\ SETAR$ model of order p for the time series $\{Y_t\}$, with delay parameter $d = 1$ and threshold value C, is defined by

$$Y_t = \begin{cases} \alpha + \sum_{i=1}^{p} \beta_{1i} Y_{t-i} + \epsilon_t & \text{if } Y_{t-1} < C \\ \\ \alpha + \sum_{i=1}^{p} \beta_{2i} Y_{t-i} + \epsilon_t & \text{if } Y_{t-1} \geq C \end{cases}$$

where *at least* one of the p pairs (β_{1i}, β_{2i}) is such that $\beta_{1i} \neq \beta_{2i}, i = 1, 2, ..., p$.

Let $I[A]$ be the *indicator function* of an event A, where $I[A] = 1$ if event A occurs, and $I[A] = 0$ otherwise. Then the above $SETAR$ model for Y_t becomes

$$Y_t = \alpha + (\sum_{i=1}^{p} \beta_{1i} Y_{t-i}) I[Y_{t-1} < C] + (\sum_{i=1}^{p} \beta_{2i} Y_{t-i}) I[Y_{t-1} \geq C] + \epsilon_t$$

or

$$Y_t = \alpha + (\sum_{i=1}^{p} \beta_{1i} Y_{t-i})(1 - I[Y_{t-1} \geq C] + (\sum_{i=1}^{p} \beta_{2i} Y_{t-i}) I[Y_{t-1} \geq C] + \epsilon_t.$$

The simplest $2-regime\ SETAR$ model of order 1 is given by

$$Y_t = \begin{cases} \alpha + \beta_1 Y_{t-1} + \epsilon_t & \text{if } Y_{t-1} < C \\ \\ \alpha + \beta_2 Y_{t-1} + \epsilon_t & \text{if } Y_{t-1} \geq C \end{cases}$$

which becomes

$$Y_t = \alpha + \beta_1 Y_{t-1}(1 - I[Y_{t-1} \geq C]) + \beta_2 Y_{t-1} I[Y_{t-1} \geq C] + \epsilon_t \qquad (*)$$

Remark: For this simplest $2 - regime\ SETAR$ model of order 1, a necessary and sufficient condition for the stationarity of $\{Y_t\}$ is that

$$\beta_1 < 1, \ \ \beta_2 < 1 \ \text{ and } \ \beta_1\beta_2 < 1,$$

(cf. *Nonlinear Time Series Models in Empirical Finance* by Philip Hans Franses and Dick van Dijk (2003), Cambridge University Press - p. 79).

The above $SETAR$ model assumes that the border between the two regimes is given by a specific value of the threshold variable Y_{t-1}; that is, the change from one $AR(1)$ model to the other is abrupt (sudden) based on the value of Y_{t-1}. A more gradual transition between the different regimes can be obtained by replacing the indicator function $I[Y_{t-1} \geq C]$ in **(*)** above by a continuous function $G(Y_{t-1}; \gamma, C)$, which changes smoothly from 0 to 1 as Y_{t-1} increases. The resultant model is called a *Smooth Transition AR (STAR)* model and is given by

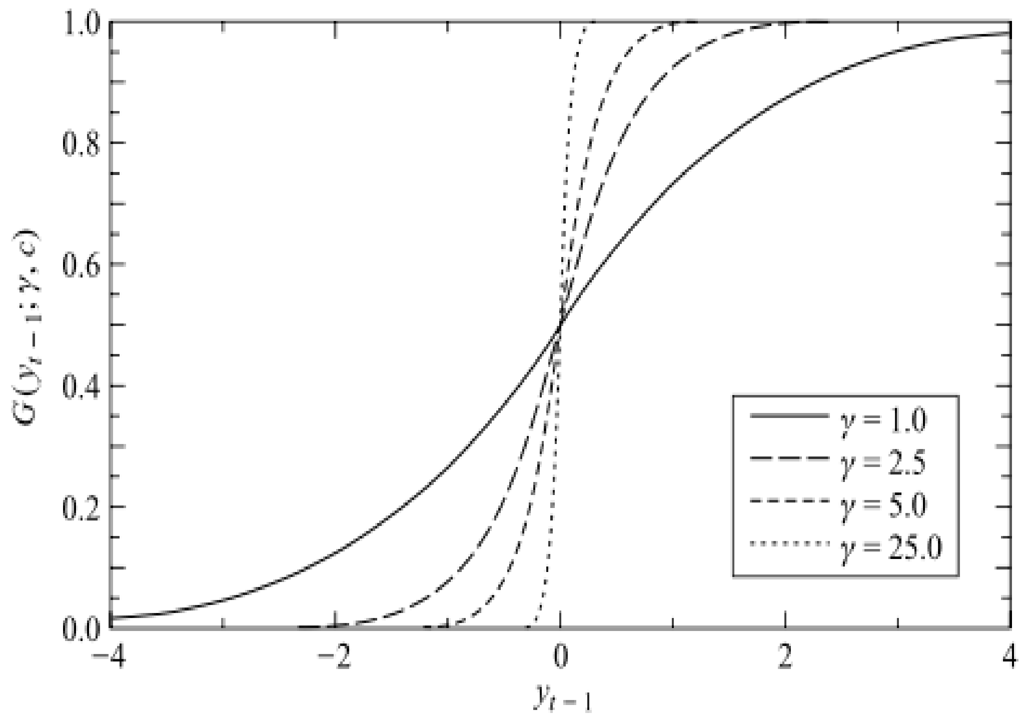
$$Y_t = \alpha + \beta_1 Y_{t-1}(1 - G(Y_{t-1}; \gamma, C)) + \beta_2 Y_{t-1} G(Y_{t-1}; \gamma, C) + \epsilon_t \qquad (**)$$

A popular choice for the so-called *transition function* $G(Y_{t-1}; \gamma, C)$ is the logistic function given by

$$G(Y_{t-1}; \gamma, C) = \frac{1}{1 + \exp(-\gamma(Y_{t-1} - C))} \qquad (***),$$

and the resultant model is then called a *Logistic STAR (LSTAR)* model. The parameter C in **(***)** can be interpreted as the threshold between the two regimes corresponding to $G(Y_{t-1}; \gamma, C) = 0$ and $G(Y_{t-1}; \gamma, C) = 1$, in the sense that the logistic function changes monotonically from 0 to 1 as Y_{t-1} increases from $-\infty$ to $+\infty$, while $G(C; \gamma, C) = 0.5$. The parameter γ determines the smoothness of the change in the value of the logistic function, and thus the transition from one regime to the other.

The picture below shows some examples of the logistic function for various different values of the smoothness parameter γ. From this picture, it is seen that as γ becomes very large, the change of $G(Y_{t-1}; \gamma, C)$ from 0 to 1 becomes almost instantaneous at $Y_{t-1} = C$ and, consequently, the logistic function $G(Y_{t-1}; \gamma, C)$ approaches the indicator function $I[Y_{t-1} \geq C]$. Hence the $SETAR$ model **(*)** can be approximated arbitrarily well by the $LSTAR$ model **(**)** with **(***)**. When $\gamma \to 0$, the logistic function becomes equal to a constant (equal to 0.5) and when $\gamma = 0$, the $STAR$ model reduces to a linear model.



For the sake of completion, below we give the notion of $SETAR$ models with multiple regimes.

SETAR models with multiple regimes

We can also define $SETAR$ model of order p with K regimes, where there are $K-1$ threshold values (with $K \geq 2$) given by $-\infty < C_1 < C_2 < C_3 < ... < C_{K-1} < \infty$ which determine the K regimes given by $Y_{t-d} < C_1$ (Regime 1), $C_1 \leq Y_{t-d} < C_2$ (Regime 2), ... $C_{K-2} \leq Y_{t-d} < C_{K-1}$ (Regime $K-1$), and $Y_{t-d} \geq C_{K-1}$ (Regime K). With $d = 1$, it is given by

$$Y_t = \begin{cases} \alpha + \sum_{i=1}^{p} \beta_{1i} Y_{t-i} + \epsilon_t & \text{if } Y_{t-1} < C_1 \\ \\ \alpha + \sum_{i=1}^{p} \beta_{2i} Y_{t-i} + \epsilon_t & \text{if } C_1 \leq Y_{t-1} < C_2 \\ \vdots \\ \\ \alpha + \sum_{i=1}^{p} \beta_{Ki} Y_{t-i} + \epsilon_t & \text{if } Y_{t-1} \geq C_K \end{cases}$$