Lecture 7

Conditional Heteroscedastic Time Series Models

In Statistics, a sequence of random variables is called *homoscedastic* if all the random variables of this sequence have the *same* variance - called *homogeneity* or constancy of variance. On the other hand, if at least two random variables have different variances, then the sequence is called *heteroscedastic* - because of the *heterogeneity* in variance.

We recall that, from the structure of a stationary $ARMA\ (p, q)$ model given by

$$Y_t = \alpha + \sum_{i=1}^{p} \beta_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} + \epsilon_t,$$

where $\{\epsilon_t\}$ is white noise and α, β_i's and θ_j's are unknown real constants. In fact, letting $\mathcal{F}_t$ denote all the information up to time t, we see that the above equation can be written as

$$Y_t = E[Y_t \mid \mathcal{F}_{t-1}] + \epsilon_t.$$

From the stationarity of $\{Y_t\}$, we see that this time series has a constant mean and constant variance - that is, constant *unconditional* mean and variance. Now, we see that the conditional mean of Y_t given $\mathcal{F}_{t-1}$ (denoted by $E[Y_t \mid \mathcal{F}_{t-1}]$) is a deterministic (non-random) **linear** function of the elements of $\mathcal{F}_{t-1}$. Additionally, the conditional variance of Y_t given $\mathcal{F}_{t-1}$ is a constant with respect to t, as seen below:

$$\begin{aligned} V[Y_t \mid \mathcal{F}_{t-1}] &= V[\epsilon_t \mid \mathcal{F}_{t-1}] \\ &= V[\epsilon_t] \qquad \text{(as } \epsilon_t \text{ is independent of } \mathcal{F}_{t-1}) \\ &= \sigma^2. \end{aligned}$$

We now define a *nonlinear* time series as follows.

Definition: A given stationary time series $\{Y_t\}$ is called a *nonlinear* time series if

either

(i) $E[Y_t \mid \mathcal{F}_{t-1}]$ is a **nonlinear** function of the elements of $\mathcal{F}_{t-1}$;

or

(ii) $V[Y_t \mid \mathcal{F}_{t-1}]$ is **not** a constant with respect to t.

"Most financial studies involve returns, instead of prices, of financial assets. The two main reasons are

(a) For average investors, return of an asset is a complete and scale-free summary of the investment opportunity.

(b) Return series are easier to handle than the price series because the former have more attractive statistical properties." - cf. *Analysis of Financial Time Series* by Ruey S. Tsay (2010).

Let P_t be the price of an asset at time t. Then, the (one-period) *return* of this asset at t is defined as

$$R_t = P_t/P_{t-1}.$$

Empirically, it has been established that the logarithm of the return series is stable (in our terminology, stationary). Hence, the focus of the analysis is on the *log return* series, denoted by $\{r_t\}$, where

$$r_t = \log R_t = \log P_t - \log P_{t-1}.$$

Towards this, we present the following empirical study which brings out the typical characteristic features of the financial data and the inadequacies of the use of $ARIMA$ models (cf. *Nonlinear Time Series Models in Empirical Finance* by Philip Hans Franses and Dick van Dijk (2003), Cambridge University Press - pp. 6-19.)

The Data

The data that we use to illustrate the typical features of financial time series consist of eight indexes of major stock markets and eight exchange rates vis-a'-vis the US dollar. To be more precise, we employ the indexes of the stock markets in Amsterdam (EOE), Frankfurt (DAX), Hong Kong (Hang Seng), London (FTSE100), New York, (S&P 500), Paris (CAC40), Singapore (Singapore All Shares) and Tokyo (Nikkei). The exchange rates are the Australian dollar, British pound, Canadian dollar, German Deutschmark, Dutch guilder, French franc, Japanese yen and Swiss franc, all expressed as a number of units of the foreign currency per US dollar. The sample period for the stock indexes runs from 6 January 1986 until 31 December 1997, whereas for the exchange rates the sample covers the period from 2 January 1980 until 31 December 1997. The original series are sampled at daily frequency. The sample periods correspond with 3,127 and 4,521 observations for the stock market indexes and exchange rates, respectively. We often analyse the series on a weekly basis, in which case we use observations recorded on Wednesdays. The stock market data have been obtained from *Datastream*, whereas the exchange rate data have been obtained from the *New York Federal Reserve*.

Figures 1.1 and 1.2 below, offer a first look at the data by showing a selection of the original price series P_t and the corresponding logarithmic returns measured in percentage terms, denoted y_t and computed as

$$Y_t = 100 * (\log P_t - \log P_{t-1}).$$

Strictly speaking, returns should also take into account dividends, but for daily data one often uses Y_t given above. Prices and returns for the Frankfurt, London and Tokyo indexes are shown in Figure 1.1, and prices and returns for the British pound, Japanese yen and Dutch guilder exchange rates are shown in Figure 1.2 (also for the period 1986 - 1997).

Summary statistics for the stock and exchange rate returns are given in Tables 1.1 and 1.2, respectively, for both daily and weekly sampling frequencies. These statistics are used in the discussion of the characteristic features of these series below.

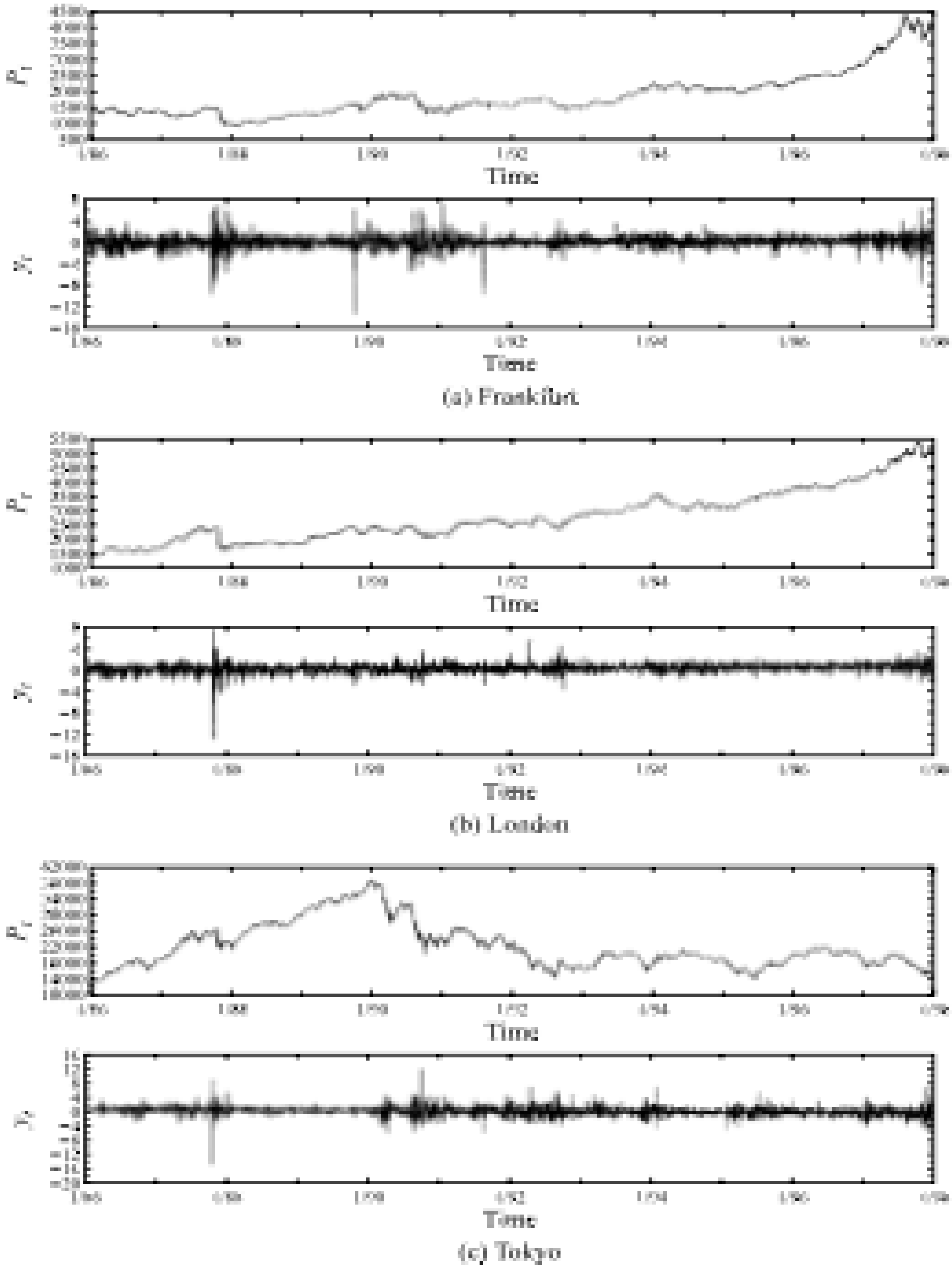
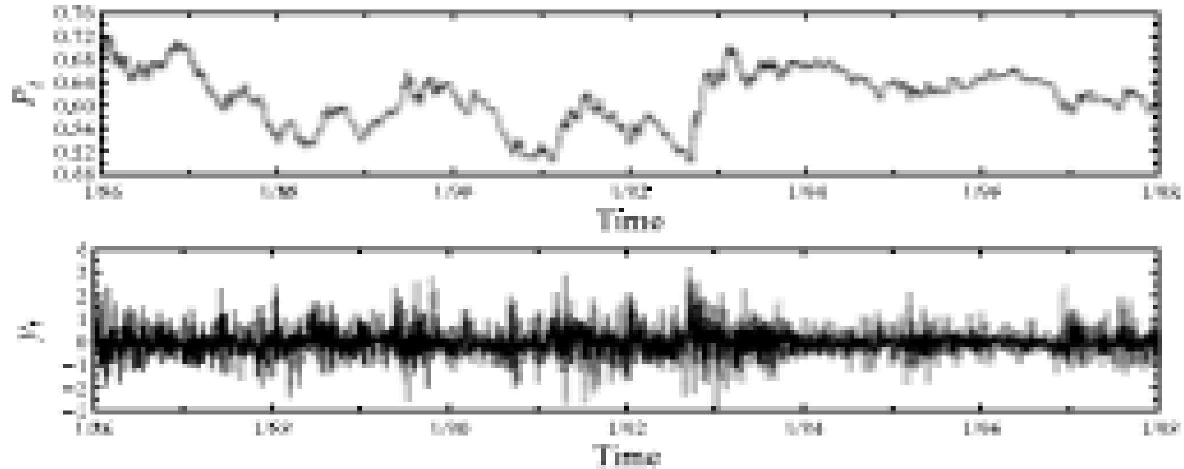
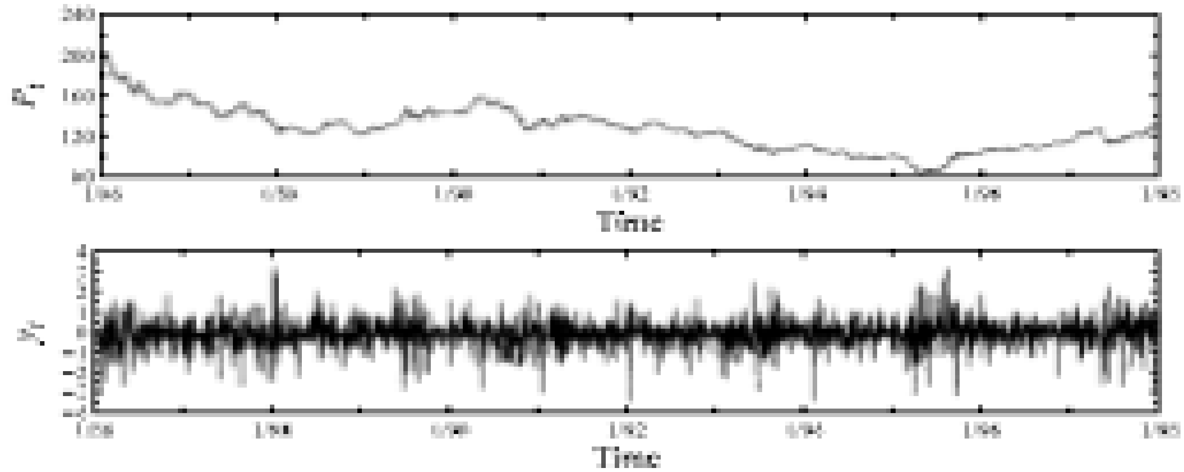


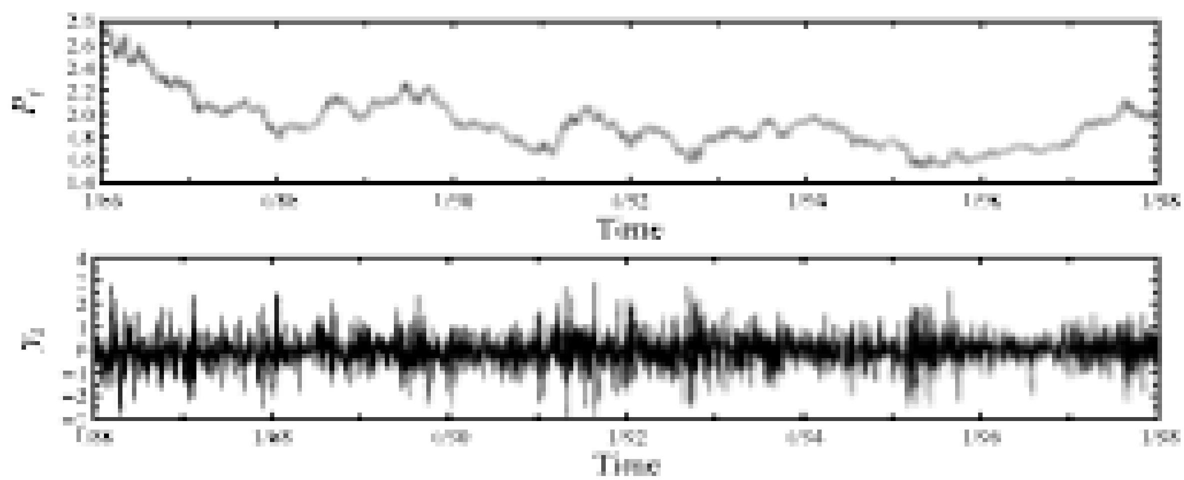
Figure 1.1 Daily observations on the level (upper panel) and returns (lower panel) of (a) the Frankfurt, (b) the London and (c) the Tokyo stock indexes, from 6 January 1986 until 31 December 1997



(a) British pound



(b) Japanese yen



(c) Dutch guilder

Figure 1.2 Daily observations on the level (upper panel) and returns (lower panel) of (a) the British pound, (b) the Japanese yen and (c) the Dutch guilder exchange rates *vis-à-vis* the US dollar, from 6 January 1986 until 31 December 1997

Table 1.1 *Summary statistics for stock returns*

Stock market	Mean	Med	Min	Max	Var	Skew	Kurt
Daily returns							
Amsterdam	0.038	0.029	−12.788	11.179	1.279	−0.693	19.795
Frankfurt	0.035	0.026	−13.710	7.288	1.520	−0.946	15.066
Hong Kong	0.057	0.022	−40.542	17.247	2.867	−5.003	119.241
London	0.041	0.027	−13.029	7.597	0.845	−1.590	27.408
New York	0.049	0.038	−22.833	8.709	0.987	−4.299	99.680
Paris	0.026	0.000	−10.138	8.225	1.437	−0.529	10.560
Singapore	0.019	0.000	−9.403	14.313	1.021	−0.247	28.146
Tokyo	0.005	0.000	−16.135	12.430	1.842	−0.213	14.798
Weekly returns							
Amsterdam	0.190	0.339	−19.962	7.953	5.853	−1.389	11.929
Frankfurt	0.169	0.354	−18.881	8.250	6.989	−1.060	8.093
Hong Kong	0.283	0.556	−34.969	11.046	13.681	−2.190	18.258
London	0.207	0.305	−17.817	9.822	4.617	−1.478	15.548
New York	0.246	0.400	−16.663	6.505	4.251	−1.370	11.257
Paris	0.128	0.272	−20.941	11.594	8.092	−0.995	9.167
Singapore	0.091	0.110	−27.335	10.510	6.986	−2.168	23.509
Tokyo	0.025	0.261	−10.892	12.139	8.305	−0.398	4.897

Notes: Summary statistics for returns on stock market indexes.
The sample period is 6 January 1986 until 31 December 1997, which equals 3,127 (625) daily (weekly) observations.

Table 1.2 *Summary statistics for exchange rate returns*

Currency	Mean	Med	Min	Max	Var	Skew	Kurt
Daily returns							
Australian dollar	0.012	−0.012	−5.074	10.554	0.377	1.893	35.076
British pound	0.006	0.000	−4.589	3.843	0.442	0.058	5.932
Canadian dollar	0.006	0.000	−1.864	1.728	0.076	0.101	6.578
Dutch guilder	−0.000	0.012	−3.985	3.188	0.464	−0.143	4.971
French franc	0.008	0.016	−3.876	5.875	0.457	0.054	6.638
German Dmark	−0.001	0.017	−4.141	3.227	0.475	−0.136	4.921
Japanese yen	−0.016	0.006	−5.630	3.366	0.478	−0.541	6.898
Swiss franc	−0.003	0.020	−4.408	3.300	0.582	−0.188	4.557
Weekly returns							
Australian dollar	0.057	−0.022	−5.526	10.815	1.731	1.454	11.906
British pound	0.033	−0.027	−7.397	8.669	2.385	0.218	6.069
Canadian dollar	0.022	0.016	−2.551	2.300	0.343	0.040	4.093
Dutch guilder	0.007	0.051	−7.673	7.212	2.416	−0.155	4.518
French franc	0.043	0.074	−7.741	6.858	2.383	−0.014	5.006
German Dmark	0.005	0.052	−8.113	7.274	2.483	−0.168	4.545
Japanese yen	−0.064	0.059	−6.546	6.582	2.192	−0.419	4.595
Swiss franc	−0.008	0.105	−7.969	6.636	2.929	−0.314	3.930

Notes: Summary statistics for exchange rate returns.
The sample period is 2 January 1980 until 31 December 1997, which equals 4,521 (939) daily (weekly) observations.

Key characteristic features of the financial data

1. Large returns occur more often than expected

One of the usual assumptions in the (theoretical) finance literature is that the logarithmic returns Y_t are normally distributed random variables, with mean μ and variance σ^2; that is,

$$Y_t \sim N(\mu, \sigma^2).$$

Kurtosis is a measure of thickness of the tail of a probability distribution. The kurtosis of the normal distribution is 3. A distribution with kurtosis bigger than 3 has "fatter tails" than that of the normal distribution; that is, the chances of occurrence of extreme observations (or, events) from this distribution are greater than for those from the normal distribution.

The kurtosis of a random variable X (with mean μ and variance σ^2) is defined as

$$K_X = E[\frac{(X-\mu)^4}{\sigma^4}]$$

and its sample analogue is defined as

$$\hat{K}_X = \frac{1}{n}\sum_{i=1}^{n}\frac{(X_i-\hat{\mu})^4}{\hat{\sigma}^4},$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are respectively the sample mean and the sample variance based on the sample $\{X_1, X_2, ..., X_n\}$.

One of the features which stands out most prominently from the last columns of Tables 1.1 and 1.2 is that the kurtosis of all series is much larger than the normal kurtosis value of 3, especially for the daily series. This reflects the fact that the tails of the distributions of these series are fatter than the tails of the normal distribution. Put differently, large observations occur (much) more often than one might expect for a normally distributed variable.

2. Large stock market returns are often negative

Skewness is a measure of *asymmetry* of the distribution of a random variable X and is defined by

$$S_X = E[\frac{(X-\mu)^3}{\sigma^3}].$$

Its sample analogue is defined as

$$\hat{S}_X = \frac{1}{n}\sum_{i=1}^{n}\frac{(X_i-\hat{\mu})^3}{\hat{\sigma}^3},$$

where $\hat{\mu}$ and $\hat{\sigma}^2$ are respectively the sample mean and the sample variance.

All symmetric distributions, including the normal distribution, have skewness equal to zero. From Table 1.1, it is seen that the stock return series all have negative skewness, which implies that the left tail of the distribution is fatter than the right tail, or that large negative returns tend to occur more often than large positive ones.

3. Large returns tend to occur in clusters

From Figures 1.1 and 1.2 it appears that relatively volatile periods, characterised by large price changes - and, hence, large returns - alternate with more tranquil periods in which prices remain more or less stable and returns are, consequently, small. In other words, large returns seem to occur in clusters.

4. Large volatility often follows large negative stock market returns

Though it is difficult to see this clearly from Figure 1.1, it can be examined to verify that periods of large volatility (large fluctuations in the return values) tend to be triggered by a large negative return. In other words, the occurrence of a large negative return is often a signal for the beginning of a period of large volatility.

To summarise, the typical features of financial time series documented above (points **1** to **4**) seem to require nonlinear models, simply because the classical $ARIMA$ models would not be able to generate data that have these features.