Lecture 9

Conditional Heteroscedastic Time Series Models - continued...

Ljung - Box Test

We recall the definition of the *autocorrelation function* (ACF) of a given time series (**Definition 5, Lecture 1**).

The *autocorrelation function* $\rho(t,s)$ of a time series $\{Y_t . t = 1, 2, 3, ...\}$ is defined as

$$\rho(t,s) = \frac{Cov[Y_t, Y_s]}{\sqrt{V[Y_t]\ V[Y_s]}} = \frac{\lambda(t,s)}{\sqrt{V[Y_t]\ V[Y_s]}}, \quad t, s = 1, 2, 3, ...,$$

where $\lambda(.,.)$ is the associated *autocovariance function*. When the time series $\{Y_t\}$ is stationary, then the form of the ACF becomes

$$\rho(t,s) = \frac{\lambda(t,s)}{\lambda(0)} \quad t, s = 1, 2, 3, ...,$$

where $\lambda(0) = \lambda(t,t)\ \ \forall\, t$ is the common variance of $\{Y_t\}$. Equivalently,

$$\rho(k) = \frac{\lambda(t, t-k)}{\lambda(0)} = \frac{\lambda(t+k, t)}{\lambda(0)} = \frac{\lambda(k)}{\lambda(0)}, \quad k = 1, 2, 3, ...,$$

with $\rho(0) \equiv 1$.

Let $\{Y_1, Y_2, ..., Y_T\}$ be a sample of size T from the stationary $\{Y_t\}$. Then the sample analogue of the ACF, denoted by $\hat{\rho}(k)$, is given by

$$\hat{\rho}(k) = \frac{\sum_{t=k+1}^{T}(Y_t - \bar{Y})(Y_{t-k} - \bar{Y})}{\sum_{t=1}^{T}(Y_t - \bar{Y})^2}, \qquad k = 0, 1, 2, ..., T-1,$$

where $\bar{Y}$ is the *sample mean* of the observations Y_t's, given by $\bar{Y} = \frac{1}{T}\sum_{t=1}^{T} Y_t$.

In time series analysis, it is often required to test jointly that several autocorrelations are zero. Let $L \geq 1$ be a given number and let $\{Y_1, Y_2, ..., Y_T\}$ be a sample of size T from the stationary time series $\{Y_t\}$. To test H_0 : the first L autocorrelations of a given time series are zero (that is, $H_0 : \rho(1) = \rho(2) = ... = \rho(L) = 0$), Ljung and Box (1978) came up with the test statistic $Q(L)$ given by

$$Q(L) = T(T+2) \sum_{k=1}^{L} \frac{\hat{\rho}^2(k)}{T-k}.$$

Decision Rule: Reject H_0 if $Q(L) > \chi^2_\alpha(L)$, where $\chi^2_\alpha(L)$ is the $100(1-\alpha)$th percentile of the chi-square distribution with L degrees of freedom.

Most software packages will provide the p value of $Q(L)$. Then, the decision rule is to reject H_0 if the p value is less than or equal to α, the given level of significance (which is 0.05 by default in most of the packages).

In practice, the choice of L may affect the performance of the Ljung - Box test statistic $Q(L)$. Several values of L are often used. Simulation studies suggest that the choice of $L \approx \log T$ provides better power performance.

Generalised ARCH (GARCH) Models

Let $\{Y_t\}$ be a stationary asset return time series with $Y_t = \textit{Expected Return} + \textit{Error}$, where the errors $\{\epsilon_t\}$ is a white noise sequence defined by

$$\epsilon_t = \sigma_t a_t,$$

where

$$\sigma_t^2 = \phi_0 \ + \sum_{i=1}^{m} \phi_i \epsilon_{t-i}^2 + \sum_{j=1}^{n} \gamma_j \sigma_{t-j}^2,$$

$$\phi_0 > 0,\ \phi_i \geq 0,\ \gamma_j \geq 0 \ \ \forall\ 1 \leq i \leq m, 1 \leq j \leq n,$$

and $\{a_t\}$ is a sequence of i.i.d. standard normal variables, a_t independent of $\mathcal{F}_{t-1}$. Then the above model is called a $GARCH(m, n)$ model.

Remarks:

(1) Note the a $GARCH(m, n)$ model reduces to $ARCH(m)$ when $n = 0$.

(2) A sufficient condition for the unconditional variance of ϵ_t to be finite (in both $ARCH(m)$ and $GARCH(m, n)$ models) is that the sum of all the coefficients in the structure of σ_t^2 is strictly less than 1.

Just as we justified the AR in the term $ARCH$, we shall show that $GARCH$ involves $ARMA$. Towards this,

$$\begin{aligned}
\epsilon_t^2 &= \sigma_t^2 + (\epsilon_t^2 - \sigma_t^2) \\
\epsilon_t^2 &= \phi_0 \; + \; \sum_{i=1}^{m} \phi_i \epsilon_{t-i}^2 + \; \sum_{j=1}^{n} \gamma_j \sigma_{t-j}^2 + (\epsilon_t^2 - \sigma_t^2) \\
\epsilon_t^2 &= \underbrace{\phi_0 \; + \; \sum_{i=1}^{\max(m,n)} (\phi_i + \gamma_i)\epsilon_{t-i}^2}_{AR} \; + \; \underbrace{\eta_t - \sum_{j=1}^{n} \gamma_j \eta_{t-j}}_{MA}, \qquad (1)
\end{aligned}$$

where $\eta_t = \epsilon_t^2 - \sigma_t^2$, $\phi_i = 0$ for $i > m$ and $\gamma_j = 0$ for $j > n$.

Note: Just as we showed that. if $\phi_1 < \sqrt{1/3}$, then the kurtosis of ϵ_t in $ARCH(1)$ is strictly larger than the normal kurtosis which is 3, for $GARCH(1,1)$ we can show that, if $(1 - 2\phi_1^2 - (\phi_1 + \gamma_1)^2) > 0$, then the kurtosis of ϵ_t in $GARCH(1,1)$ is strictly larger than 3.

Model Building:

Let $\{Y_t\}$ be an asset return series and let $\{Y_1, Y_2, ..., Y_T\}$ be a sample of size T from $\{Y_t\}$. To build an $ARCH$ ($GARCH$) model, execute the following steps:

Step 1: Make sure that the data is clean and check for the stationarity (as you did while building the $ARMA$ models. If there is stationarity then go to **Step 2** below; else, make it stationary and then proceed to **Step 2** below.

Step 2: Fit the appropriate $ARMA$ model to the stationary time series gotten from **Step 1** above.

Step 3: Let $\{e_1, e_2, ..., e_T\}$ be the set of *residuals* derived from the fitted $ARMA$ model in **Step 2** above. Run the Ljung - Box test on the following data sets:

(a) $\{e_1, e_2, ..., e_T\}$ - the set of residuals

(b) $\{e_1^2, e_2^2, ..., e_T^2\}$ - the set of squared residuals

Step 4: If the Ljung - Box test of the null hypothesis (that the first L autocorrelations are zero) is **accepted** for **3(a)** *and* is **rejected** for **3(b)** above, go to **Step 5** below.

Step 5: Using the library *rugarch* in *R* (or, the proper syntax in *Python*), build the appropriate $(G)ARCH$ model for the data, and get the forecasts of $Y_t, t \geq T+1$ (using the built-in forecast commands).

Even though the general $GARCH(m,n)$ model might be of theoretical interest, the $GARCH(1,1)$ model often appears adequate in practice (cf. "$ARCH$ modelling in finance: a review of the theory and empirical evidence" by Bollerslev, Chou and Kroner, *Journal of Econometrics* **52**, p. 5 - 59, 1992). Hence, as a variation of the $GARCH$ model, in what follows, we will just introduce the notion of an $IGARCH$ model (without getting into the details).

Integrated GARCH (IGARCH) Model

Consider a $GARCH(m,n)$ model. If the sum of the coefficients in the AR part of the structure of ϵ_t^2 (equation (1)) is 1, then we are said to have an integrated $GARCH(m,n)$ model. The simplest $IGARCH$ model is given by $IGARCH(1,1)$ which is defined by

$$\epsilon_t = \sigma_t a_t, \text{ where } \quad \sigma_t^2 = \phi_0 \ + \ \phi_1 \epsilon_{t-1}^2 \ + \ (1-\phi_1)\sigma_{t-1}^2, \quad \phi_0 > 0, \ 0 < \phi_1 < 1$$

and $\{a_t\}$ is a sequence of i.i.d. standard normal variables, a_t independent of $\mathcal{F}_{t-1}$.

Just to illustrate the importance of this basic $IGARCH(1,1)$ model...

Value at Risk (VaR) is one of the most important and fundamental notions in Finance. *RiskMetrics* is a widely used methodology (in the financial sector) that an investor can use to calculate the VaR of a portfolio of investments. Launched in 1994 by J.P. Morgan, RiskMetrics was upgraded by the company in partnership with Reuters in 1996. The above methodology has been built on the assumption that the daily return of a portfolio has an $IGARCH(1,1)$ model.

If time permits, we will see some discussion on VaR and RiskMetrics methodology later in this course.