Lecture 11

Regime Switching Models - Non Linear Models for Returns and Volatility (contd...)

In the previous lecture (**Lecture 10**), we introduced the non linear models namely $SETAR$ (self exciting threshold autoregressive) and $STAR$ (smooth transition autoregressive) models, which are models for the return of a financial asset. These essentially capture the change in the behaviour of returns in different regimes. That is, in each different regime, the return has a different AR model where the regimes are characterised by the past return values. In what follows, we will introduce such regime switching models for volatility - the conditional standard deviation of the return.

We also noted that, as a motivation for the study of $GARCH$ models, one of the most prominent empirical facts of returns on financial assets is that their volatility changes over time. In particular, periods of large movements in prices alternate with periods during which prices hardly change. This characteristic feature is commonly referred to as *volatility clustering*. Further, we noted that, under "**Key characteristic features of the financial data**", that *large volatility often follows large negative stock market returns*. That is, *periods of large volatility (large fluctuations in the return values) tend to be triggered by a large negative return*. In other words, the occurrence of a large negative return is often a signal for the beginning of a period of large volatility. This, in turn, emphasises the *assymetric* effect of positive and negative shocks on the volatility of the return, as volatility is modelled as a function of the past shocks in $GARCH$ models.

One of the most fundamental non linear extensions of the $GARCH$ model is the one which is designed to allow for different effects of positive and negative shocks. In 1993, Glosten, Jagannathan and Runkle introduced the following regime switching $GARCH(1,1)$ model defined by

$$\epsilon_t = \sigma_t a_t$$

where

$$\sigma_t^2 = \begin{cases} \phi_0 + \phi_1^{(1)}\epsilon_{t-1}^2 + \gamma\sigma_{t-1}^2 & \text{if } \epsilon_{t-1} < 0 \\ \\ \phi_0 + \phi_1^{(2)}\epsilon_{t-1}^2 + \gamma\sigma_{t-1}^2 & \text{if } \epsilon_{t-1} \geq 0, \end{cases}$$

with $\phi_1^{(1)} \neq \phi_1^{(2)}$.

That is,

$$\sigma_t^2 = \phi_0 + \phi_1^{(1)}\epsilon_{t-1}^2 I[\epsilon_{t-1} < 0] + \phi_1^{(2)}\epsilon_{t-1}^2 I[\epsilon_{t-1} \geq 0] + \gamma\sigma_{t-1}^2.$$

The above model is also called $TGARCH(1,1)$ (Threshold $GARCH(1,1)$). This is the simplest regime switching $GARCH$ model.

Just as we had a smoothly transitioning TAR model, the *Smooth Transition* $GARCH(1,1)$ ($STGARCH(1,1)$ is defined by

$$\sigma_t^2 = \phi_0 + \phi_1^{(1)}\epsilon_{t-1}^2(1 - G(\epsilon_{t-1};\lambda)) + \phi_1^{(2)}\epsilon_{t-1}^2 G(\epsilon_{t-1};\lambda) + \gamma\sigma_{t-1}^2,$$

where

$$G(\epsilon_{t-1};\lambda) = \frac{1}{1 + \exp(-\lambda\epsilon_{t-1})}.$$

Value at Risk (VaR) and RiskMetrics

In Finance, *risk* refers to the degree of uncertainty and/or potential financial loss inherent in an investment decision. Financial risk is classified into four broad categories: market risk, credit risk, liquidity risk, and operational risk. *Value at Risk* (VaR) is mainly concerned with market risk, but the concept is also applicable to other types of risk.

Value at Risk (VaR) calculates the maximum loss expected (or worst case scenario) on an investment, over a given time period and given a specified degree of confidence (or probability). For example, if the 99% one-month VaR is \$1 million, there is 99% confidence ((or) chance (or) probability) that over the next month the portfolio *will not* lose more than \$1 million. Equivalently, the portfolio will lose more than \$1 million over the next month with probability 1%. In this view, one treats

VaR as a measure of loss associated with a rare (or extraordinary) event under normal market conditions. It is used most often by commercial and investment banks to capture the potential loss in value of their traded portfolios from adverse market movements over a specified period; this can then be compared to their available capital and cash reserves to ensure that the losses can be covered without putting the firms at risk. That is, *VaR is used to ensure that the financial institutions can still be in business after a catastrophic event.*

Suppose that at time t, we are interested in the risk of a financial position for the next k periods. Let $\Delta V_t(k)$ be the change in value of the underlying portfolio from time t to $t+k$. (Typically, if P_t is the value (or price) of the portfolio at tme t, then one can define $\Delta V_t(k) = \log(\frac{P_{t+k}}{P_t})$, the k-period log return, or simply as $\Delta V_t(k) = P_{t+k} - P_t$, the k-period simple price change). Note that, at time t, $\Delta V_t(k)$ is a random variable as this change in value involves the future value of the portfolio after k periods. Let $F_k(.)$ be the cumulative distribution function (cdf) of $\Delta V_t(k)$. Now we define VaR.

Definition: Under the above setup, we define the VaR, over the time horizon k with tail probability p, as the quantity which satisfies

$$p = P[\Delta V_t(k) \geq VaR] = 1 - P[\Delta V_t(k) < VaR] = 1 - F_k(VaR). \qquad --- \quad (*)$$

Remarks:

1. From the above equation we see that VaR is such that $F_k(VaR) = 1-p$. Note that VaR is the $100(1-p)$ percentile point of the cdf F_k. Here p denotes the probability of a rare (or extreme) event and hence will be very small. In real applications, for computing the VaR, p is usually taken as 0.01 (1%). The cdf F_k is assumed to be symmetric about 0 and is continuous (so that, for a given p, the computed VaR is the upper $100p$ percentile point of F_k and is unique).

2. Since log returns correspond approximately to percentage changes in value of a financial asset, we use 1-period log returns $Y_t = \log(\frac{P_t}{P_{t-1}})$ in data analysis. The VaR, calculated from the upper percentile of the distribution of Y_{t+1} given the information up to time t, is therefore in percentage. The dollar amount of VaR is then the cash value of the portfolio times the VaR of the log return series. That is,

$$VaR \text{ (dollars)} = \text{Portfolio Value} \times VaR \text{ (of log returns)}.$$

The *RiskMetrics* methodology for calculating VaR assumes that a portfolio or any asset's returns follow a normal distribution. Launched in 1994 by J.P. Morgan, RiskMetrics was upgraded by the company in partnership with Reuters in 1996.

The companies teamed up to make the data used in RiskMetrics widely available to individual investors. RiskMetrics is now owned by MSCI (Morgan Stanley Capital International). In what follows, we give the details of this methodology.

RiskMetrics Methodology for Computing VaR

Let the daily (or 1-period) log return of a portfolio be denoted by

$$Y_t[1] = Y_{t+1} = \log(\frac{P_{t+1}}{P_t}),$$

where P_t is the portfolio price at time t. Then we see that the corresponding k-period return is defined by

$$Y_t[k] = \log(\frac{P_{t+k}}{P_t}) = \sum_{i=1}^{k} \log(\frac{P_{t+i}}{P_{t+i-1}}) = \sum_{i=1}^{k} Y_{t+i}.$$

Let $\mathcal{F}_t$ be the collection of available information up to time t. Then, we assume that $Y_t = \log P_t - \log P_{t-1} = \epsilon_t$ with $\epsilon_t = \sigma_t a_t$, where $\{a_t\}$ is a sequence of iid standard normal variables, a_t is independent of $\mathcal{F}_{t-1}$ and, σ_t^2 is $IGARCH(1,1)$ without intercept; that is,

$$\sigma_t^2 = \theta\sigma_{t-1}^2 + (1-\theta)Y_{t-1}^2, \qquad 0 < \theta < 1. \quad --- (**)$$

Then we see that $Y_t \mid \mathcal{F}_{t-1} \sim N(0, \sigma_t^2)$. Since $Y_t = \epsilon_t \ \forall t$, we observe that $Y_{t+i} \mid \mathcal{F}_t \sim N(0, \sigma_{t+i}^2)$ for each $i = 1, 2, ..., k$. Let $\sigma_t^2[k] = V[Y_t[k] \mid \mathcal{F}_t]$. Then,

$$\begin{aligned}
\sigma_t^2[k] &= V[\sum_{i=1}^{k} Y_{t+i} \mid \mathcal{F}_t] = V[\sum_{i=1}^{k} \epsilon_{t+i} \mid \mathcal{F}_t] \\
&= \sum_{i=1}^{k} V[\epsilon_{t+i} \mid \mathcal{F}_t] + 2\sum_{i<j} Cov[\epsilon_{t+i}, \epsilon_{t+j} \mid \mathcal{F}_t] \\
&= \sum_{i=1}^{k} V[\epsilon_{t+i} \mid \mathcal{F}_t] + 2\sum_{i<j} E[\epsilon_{t+i}\epsilon_{t+j} \mid \mathcal{F}_t] \\
&= \sum_{i=1}^{k} V[\epsilon_{t+i} \mid \mathcal{F}_t] \qquad \text{(Prove this step!)} \\
&= \sum_{i=1}^{k} E[\sigma_{t+i}^2 \mid \mathcal{F}_t]
\end{aligned}$$

Now, using $Y_{t-1} = \epsilon_{t-1} = \sigma_{t-1} a_{t-1}$ in equation (**) above, we get

$$\sigma_t^2 = \sigma_{t-1}^2 + (1-\theta)\sigma_{t-1}^2(a_{t-1}^2 - 1) \qquad \forall\, t.$$

In particular,

$$\sigma_{t+i}^2 = \sigma_{t+i-1}^2 + (1-\theta)\sigma_{t+i-1}^2(a_{t+i-1}^2 - 1) \qquad \text{for each } i = 2, 3, ..., k.$$

Since $E[(a_{t+i-1}^2 - 1) \mid \mathcal{F}_t] = 0,$ for each $i = 2, 3, ..., k$, the above equation shows that

$$E[\sigma_{t+i}^2 \mid \mathcal{F}_t] = E[\sigma_{t+i-1}^2 \mid \mathcal{F}_t], \qquad \text{for each } i = 2, 3, ..., k. \quad ---(***)$$

For the one-step-ahead volatility forecast, equation (**) gives us that $\sigma_{t+1}^2 = \theta\sigma_t^2 + (1-\theta)Y_t^2$. Hence we get that

$$V[Y_{t+i} \mid \mathcal{F}_t] = E[\sigma_{t+i}^2 \mid \mathcal{F}_t] = \sigma_{t+1}^2 \quad \text{for each } i = 1, 2, 3, ..., k.$$

Therefore,

$$V[Y_t[k] \mid \mathcal{F}_t] = \sigma_t^2[k] = \sum_{i=1}^{k} E[\sigma_{t+i}^2 \mid \mathcal{F}_t] = k\sigma_{t+1}^2$$

and hence

$$Y_t[k] \mid \mathcal{F}_t \sim N(0, k\sigma_{t+1}^2).$$

RiskMetrics uses the above result to calculate VaR for a given probability p for the log return as given below.

Recall from Probability that if $X \sim N(0, \sigma^2)$ then the lower and the upper 1% points are given by -2.326 σ and +2.326 σ, respectively. So, given the log return data up to time t and $p = 0.01$ (that is, $100p\% = 1\%$), the VaR for one-day-ahead is given by $VaR = VaR[1] = 2.326\ \sigma_{t+1}$. The VaR for k-days-ahead is given by $VaR[k] = \sqrt{k}\ 2.326\ \sigma_{t+1}$. Consequently, we have that

$$VaR[k] = \sqrt{k} \times VaR,$$

which is the well known *Square Root of Time Rule* in VaR calculation under RiskMetrics.