Lecture 8

Conditional Heteroscedastic Time Series Models - continued...

In *Finance*, **volatility** is defined as the conditional standard deviation of the underlying asset return. Typically, an $ARMA$ model in Finance has the representation given by

$$Y_t = \underbrace{\sum_{i=1}^{p} \beta_i Y_{t-i} + \sum_{j=1}^{q} \theta_j \epsilon_{t-j} +}_{\text{Expected Return}} \underbrace{\epsilon_t.}_{\text{Source of Volatility}}$$

Campbell *et al* (1997, p. 481) argued that "it is both logically inconsistent and statistically inefficient to use volatility measures that are based on the assumption of constant volatility over some period when the resulting series moves through time." In the case of financial data, for example, large and small errors tend to occur in clusters; that is, large returns are followed by more large returns, and small returns by more small returns (cf. *The Econometrics of Financial Markets* by Campbell, J. Y., Lo, A. W., and MacKinlay, A. C. (1997), Princeton, New Jersey: Princeton University Press).

Hence, the most crucial and the fundamental assumption that the errors (or *shocks*, as known in Finance) have constant variance as assumed in the classical (linear) $ARMA$ model analysis becomes clearly untenable. In order to account for the change in variance (called *heteroscedasticity*), Robert F. Engle formulated the concept of *auto-regressive conditional heteroscedastic* ($ARCH$) model in his seminal research paper in 1982. Though the fundamental model for the asset returns retains an $ARMA$ structure, where $Y_t = ExpectedReturn + Error$ (stated earlier above), the errors form a a white noise sequence with the structure given by

$$\epsilon_t = \sigma_t a_t, \text{ where } \sigma_t^2 = \phi_0 + \sum_{i=1}^{m} \phi_i \epsilon_{t-i}^2, \quad \phi_0 > 0, \ \phi_i \geq 0 \ \ \forall\, 1 \leq i \leq m,$$

and $\{a_t\}$ is a sequence of i. i. d. standard normal variables, a_t independent of $\mathcal{F}_{t-1}$.

The above model for the error sequence $\{\epsilon_t\}$ is called an *ARCH* model of *order m*; that is, *ARCH* (m). The *ARCH* are called *volatility models* for the obvious reason that the error is expressed as a product of volatility and a standard normal variable.

Note: As $\{\epsilon_t\}$ is assumed to be a white noise sequence, we have that $E[\epsilon_t] = 0$ and $V[\epsilon_t] =$ finite constant $= \delta^2$ (say). Additionally, we also assume that the sequence $\{\epsilon_t\}$ has finite **constant** fourth moment; that is, $E[\epsilon_t^4] = C < \infty, \ \forall\, t$.

Towards a better understanding of the *ARCH* models, we give the following details.

To put the volatility models in proper perspective, it is informative to consider the conditional mean and variance of Y_t given $\mathcal{F}_{t-1}$; that is, let

$$M_t = E[Y_t \mid \mathcal{F}_{t-1}].$$

Now, consider

$$\begin{aligned} V[Y_t \mid \mathcal{F}_{t-1}] &= E[(Y_t - M_t)^2 \mid \mathcal{F}_{t-1}] \\ &= E[\epsilon_t^2 \mid \mathcal{F}_{t-1}] \\ &= E[\sigma_t^2 a_t^2 \mid \mathcal{F}_{t-1}] \\ &= \sigma_t^2 E[a_t^2], \end{aligned}$$

since σ_t is a function of elements of $\mathcal{F}_{t-1}$ and a_t is independent of $\mathcal{F}_{t-1}$.

Therefore,

$$V[Y_t \mid \mathcal{F}_{t-1}] = \sigma_t^2 \qquad \text{as } E[a_t^2] = 1.$$

Also, note here that

$$E[\epsilon_t \mid \mathcal{F}_{t-1}] = \sigma_t E[a_t \mid \mathcal{F}_{t-1}] = 0$$

and

$$V[\epsilon_t \mid \mathcal{F}_{t-1}] = E[\epsilon_t^2 \mid \mathcal{F}_{t-1}] = \sigma_t^2.$$

From now on, ϵ_t is referred to as the *shock* or *innovation* of an asset return at time t and σ_t is the positive square root of σ_t^2. The model for the conditional mean M_t is referred to as the *mean equation* for Y_t (indicated as the Expected Return in the beginning of this lecture), and the model for σ_t^2 is the volatility equation for Y_t.

From the structure of the volatility equation, it is seen that large past squared shocks $\epsilon_{t-i}^2, i = 1, 2, ..., m$ imply a large conditional variance σ_t^2 for the innovation

ϵ_t. Consequently, ϵ_t **tends** to assume a large value (in modulus). This means that, under the *ARCH* framework, large shocks tend to be followed by another large shock. Here we use the word 'tend' because a large variance does not necessarily produce a large realisation. It only says that the probability of obtaining a large value of a variable with larger variance is greater than that of the one with smaller variance. This feature is similar to the volatility clusterings observed in asset returns.

Note: Given $\mathcal{F}_{t-1}$, note that the volatility equation is a deterministic one, and **not** random; that is, given $\mathcal{F}_{t-1}$, σ_t^2 is non-random.

Now, we will explain the presence of "auto-regressive" in the name *ARCH*. Consider the volatility equation given by

$$\sigma_t^2 = \phi_0 \ + \ \sum_{i=1}^{m} \phi_i \epsilon_{t-i}^2.$$

From this we see that

$$\begin{aligned}
\epsilon_t^2 &= \sigma_t^2 + (\epsilon_t^2 - \sigma_t^2) \\
\epsilon_t^2 &= (\phi_0 \ + \ \sum_{i=1}^{m} \phi_i \epsilon_{t-i}^2) + \sigma_t^2 (a_t^2 - 1) \qquad \text{as } \epsilon_t = \sigma_t a_t \\
\epsilon_t^2 &= \phi_0 \ + \ \sum_{i=1}^{m} \phi_i \epsilon_{t-i}^2 + \eta_t \quad \text{(say)}
\end{aligned}$$

From the last equation above we see that we are able to represent the sequence $\{\epsilon_t^2\}$ as an auto-regressive model of order m. Hence the term "auto-regressive" in the name "auto-regressive conditional heteroscedastic model". Now, the term "conditional heteroscedastic" in the above name corresponds to the fact that the conditional variance of ϵ_t changes with t; that is, given $\mathcal{F}_{t-1}$,

$$V[\epsilon_t \mid \mathcal{F}_{t-1}] = \sigma_t^2.$$

Important Remark: In the *ARCH* model, even though the shock/error sequence $\{\epsilon_t\}$ is uncorrelated (that is, the ϵ_t's do not have a linear relationship), there *exists* a **non-linear** relationship among the shocks, as demonstrated in the last of the three equations above.

In what follows, we shall prove that, for the simplest model *ARCH* (1), the common *kurtosis* of the shock sequence $\{\epsilon_t\}$ is **strictly greater than** 3 (the kurtosis of the normal distribution). That is, we will show that extreme values occur more often in the case of $ARCH$ models, than in the case of the classical $ARMA$ models - which will be a justification for using the $ARCH$ models over $ARMA$ models for the analysis of financial data.

Consider the simplest $ARCH$ model given by $ARCH$ (1). So, we have

$$\epsilon_t = \sigma_t a_t, \text{ where } \quad \sigma_t^2 = \phi_0 \ + \ \phi_1 \epsilon_{t-1}^2, \quad \phi_0 > 0, \ \phi_1 \geq 0,$$

and $\{a_t\}$ is a sequence of i. i. d. standard normal variables, a_t independent of $\mathcal{F}_{t-1}$. Recall that we have also assumed that the sequence $\{\epsilon_t\}$ has finite **constant** fourth moment; that is, $E[\epsilon_t^4] = C < \infty, \ \forall \ t$. Since $\{\epsilon_t\}$ is white noise, the stationary mean is given by

$$\begin{aligned} E[\epsilon_t] &= E[E[\epsilon_t \mid \mathcal{F}_{t-1}]] \\ &= E[\sigma_t E[a_t \mid \mathcal{F}_{t-1}]] \\ &= E[\sigma_t E[a_t]] = 0. \end{aligned}$$

The stationary variance is

$$\begin{aligned} V[\epsilon_t] &= \delta^2 = E[\epsilon_t^2] \qquad \text{as } E[\epsilon_t] = 0 \\ &= E[E[\epsilon_t^2 \mid \mathcal{F}_{t-1}]] = E[\sigma_t^2 E[a_t^2 \mid \mathcal{F}_{t-1}]] \\ &= E[\phi_0 + \phi_1 \epsilon_{t-1}^2] \\ &= \phi_0 + \phi_1 \delta^2 \end{aligned}$$

That is, we have that the stationary variance is the finite positive constant given by $\delta^2 = \frac{\phi_0}{1-\phi_1}$, which implies that $0 \leq \phi_1 < 1$.

As $E[\epsilon_t] = 0, and V[\epsilon_t] = \delta^2$, and we have assumed that $E[\epsilon_t^4] = C < \infty, \ \forall \ t$, the common kurtosis κ of the sequence $\{\epsilon_t\}$ is given by

$$\kappa = \frac{E[\epsilon_t^4]}{V^2[\epsilon_t]} = \frac{E[\epsilon_t^4]}{\delta^4},$$

where $\delta^2 = \frac{\phi_0}{1-\phi_1}$.

Now, the numerator of κ is

$$\begin{aligned} E[\epsilon_t^4] &= C = E[E[\epsilon_t^4 \mid \mathcal{F}_{t-1}]] \\ &= E[\sigma_t^4 E[a_t^4 \mid \mathcal{F}_{t-1}]] = E[\sigma_t^4 E[a_t^4]] \\ &= E[3(\phi_0 \ + \ \phi_1 \epsilon_{t-1}^2)^2] \quad (\text{as } a_t \sim N(0,1) \text{ and are i.i.d. with kurtosis} = 3) \\ &= 3E[\phi_0^2 + \phi_1^2 \epsilon_{t-1}^4 + 2\phi_0 \phi_1 \epsilon_{t-1}^2] \\ &= 3\phi_0^2 + 3\phi_1^2 C + 6\phi_0 \phi_1 \delta^2 \end{aligned}$$

Plugging the value of $\delta^2 = \frac{\phi_0}{1-\phi_1}$ on the right side of the last equation above and solving for C, we get that

$$C = E[\epsilon_t^4] = \frac{3\phi_0^2(1+\phi_1)}{(1-\phi_1)(1-3\phi_1^2)}.$$

Since we have assumed that C is a finite positive constant (apart from the earlier condition on ϕ_1 that $0 \leq \phi_1 < 1$), we now additionally have another restriction on ϕ_1 that $0 \leq \phi_1^2 < 1/3$. That is, the stationarity and the constancy of the fourth moment of ϵ_t's force that

$$0 \leq \phi_1 < 1/\sqrt{3}.$$

Now, we see that the kurtosis κ of the ϵ_t's is given by

$$\begin{aligned} \kappa &= \frac{E[\epsilon_t^4]}{\delta^4} \\ &= \frac{\frac{3\phi_0^2(1+\phi_1)}{(1-\phi_1)(1-3\phi_1^2)}}{\frac{\phi_0^2}{(1-\phi_1)^2}} \\ &= 3\frac{(1-\phi_1^2)}{(1-3\phi_1^2)} \\ &> 3 \qquad\quad (\text{since } 0 \leq \phi_1 < 1/\sqrt{3}) \end{aligned}$$

Hence, the $ARCH$ (1) shock sequence for the asset return model is more likely (than a Gaussian white noise) to result in extreme values (or, outliers). This conforms with the empirical study finding we saw in **Lecture 7** that the extreme values of the asset return appear more often than that implied by an i.i.d. sequence of Gaussian shocks.

The above phenomenon (the more frequent occurrence of extreme values) continues to be true for $ARCH$ (m) models for $m \geq 2$ also (with more complex expressions).