INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY, HYDERABAD

FIRST YEAR

REAL ANALYSIS, DIGITAL SYSTEMS, DICRETE MATHEMATICS, COMPUTER PROGRAMMING

College Notes

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Chapter 1

Group Theory and Graphs

1.1 Groups and Subgroups

1.1.1 What is a Group?

A group is defined over a Set A and an arbitrary operation $\times$, denoted as: $\langle A|\times\rangle$.

- **Closure:** If a and b are elements in A, then $a \times b$ is also in A.
- **Identity:** There exists e such that $a \times e = a$.
- **Invertability:** There exists a^{-1} such that $a \times a^{-1} = e$.
- **Associativity:** $(a \times b) \times c = a \times (b \times c)$

A Subgroup is a subset of the original group that is itself a group.

One Step Subgroup Test states that if ab^{-1} is in the group H, then H is a subgroup of G.

Two Step Subgroup Test states that if a^{-1} is in H whenever a is in H and ab is in H for all a, b in H, then H is a subgroup of G.

Finite Subgroup Test If H is a non-empty finite subset of a group G, and H is closed under the operation G, then H is a subgroup of G.

Operations that Hold in groups are:

- Uniqueness of Identity (If $x \cdot a = x$ and $x \cdot b = x$, $(\forall x)$, then $a = b = e$)
- Uniqueness of Inverse (If $x \cdot a = e$ and $x \cdot b = e$, $(\exists x)$, then $a = b = x^{-1}$)
- Left and Right Cancellation (If $ab = ac$ then $b = c$. If $ba = ca$ then $b = c$.)
- Socks-Shoes Property ($(ab)^{-1} = b^{-1}a^{-1}$)

1.1.2 Cayley's Table

Cayley's Table is a 2-D matrix of all members of the group a and b on both axis and $a \cdot b$

1.1.3 Subgroups and GCD

Theorem 1.1: Equivalent Cyclic Subpgroups

Let a be an element of order n in a group and let k be a positive integer. Then $\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle$ *and* $|a^k| = n/gcd(n,k)$.

Proof 1.1: Equivalent Cyclic Subpgroups

$(a^{gcd(n,k)})^{\alpha} = a^k$, Since gcd(n,k) divides k. Thereby $\langle a^k \rangle \subseteq \langle a^{gcd(n,k)} \rangle$. Also, $gcd(n,k) = \alpha n + \beta k$, so $a^{gcd(n,k)} = a^{\alpha n + \beta k} = a^{\alpha n} a^{\beta k} = e \cdot a^{\beta k}$, therefore we can state that, $\langle a^{gcdn,k} \rangle \subseteq \langle a^k \rangle$. So we proved that $\langle a^k \rangle = \langle a^{gcd(n,k)} \rangle$.

Next, using the proof in the first part, since the groups are equal their orders are the same, so $|a^k| = |a^{gcd(n,k)}| = \frac{n}{gcd(n,k)}$, since the gcd divides n, it is the least solution to $(a^{gcd(n,k)})^x = a^n$

This has the following crucial corollaries.

- In a finite cyclic group, the order of an element divides the orderof the group.
- Let $|a| = n$. Then $\langle a^i \rangle = \langle a^j \rangle$ if and only if $gcd(n,i) = gcd(n,j)$, and $|a^i| = |a^j|$ if and only if $gcd(n,i) = gcd(n,j)$.
- Let $|a| = n$. Then $\langle a \rangle = \langle a^j \rangle$ if and only if $gcd(n,j) = 1$, and $|a| = |\langle a^j \rangle|$ if and only if $gcd(n,j) = 1$.
- An integer k in Z_n is a generator of Z_n if and only if $gcd(n,k) = 1$.

These facts help us count the number of subgroups in a given set.

1.1.4 Cyclic Groups

Theorem 1.2: Fundamental Theorem of Cyclic Groups

Every subgroup of a cyclic group is cyclic. Moreover, if $|\langle a \rangle|$ = n, then the order of any subgroup of $\langle a \rangle$ is a divisor of n; and, for each positive divisor k of n, the group $\langle a \rangle$ has exactly one subgroup of order k, namely, $\langle a^{n/k} \rangle$.

Theorem 1.3: Number of Elements of Order D

Let G_n be a finite Cyclic Group of order N, if d is a positive Divisor of N, then the number of elements of order D in G of order d is $\phi(d)$.

For any group (i.e. Including non-cyclic), the number of elements of order d is a multiple of $\phi(d)$.

Proof 1.2: Number of Elements of Order D

If d is a divisor of n, then there is only one subgroup of order D of the group G_n. The elements in that group are those which are of the form $\langle a^x \rangle$, such that D and x are coprime. Therefore there can only be exactly $\phi(d)$ elements that are of order d.

If there are no elements of order D in the group, $\phi(d)|0$. Now let $\langle a_1 \rangle$ be a subgroup of order d, it has $\phi(d)$ elements in the subgroup of order d. So on for each $\langle a_i \rangle$ for all i, therefore a multiple of $\phi(d)$.

1.1.5 Abelian Groups

Theorem 1.4: Fundamental Theorem of Abelian Groups

Every finite Abelian group is a direct product of cyclic groups of prime order power. Moreover, the number of terms in the product and the orders fo the cyclic groups are uniquely determined by the group. This is valid **upto isomorphism**.

Proof 1.3: Fundamental Theorem of Abelian Groups

Lemma 1 Let G be a finite Abelian Group of order $p^n m$, where p is a prime that does not divide m. Then $G = H \times K$, where $H = \{x \in G | x^{p^n} = e\}$ and $K = \{x \in G | x^m = e\}$. Moreover $|H| = p^n$

Lemma 2 Let G be an Abelian group of prime-power order and let a be an element of maximum order in G. Then G can be written in the form $\langle a \rangle = K$

Lemma 3 A finite Abelian group of prime-power order is an internal direct product of cyclic groups.

Lemma 4 G be a finite Abelian group of prime-power order. If $G = H_1 \times H_2 \times ... \times H_m$ and $G = K_1 \times K_2 \times ... \times K_n$, where the H's and K's are nontrivial cyclic subgroups with $|H_1| \geq |H_2| \geq ... \geq |H_m|$ and $|K_1| \geq |K_2| \geq ... \geq |K_n|$, then $m = n$ and $|H_i| = |K_i|$ $(\forall i)$.

Example 1.1: Using Cardinality and Generators

Prove that every abelian group of order that is a product of primes is also cyclic. Given any abelian Group of order $p \cdot q$. If there is no prime such that

1.2 Special Groups and Their Properties

1.2.1 Permutation Groups

A Dihedral Group (D_n) is a group of all permutations of a n-sided Regular Polygon. The number of elements in this group is 2^n.

Definition 2.1: Permutation Groups

Permutation Groups are a group of permutations, where a permutation is a bijective function from a group to itself. Eg.

$$ExamplePermutation = \begin{bmatrix} A_0 & A_1 & A_2 & A_3 & A_4 & ... \\ \alpha(A_0) & \alpha(A_1) & \alpha(A_2) & \alpha(A_3) & \alpha(A_4) & ... \end{bmatrix} \tag{1.1}$$

Disjoint Cycle Notation Any permutation can be written as a product of disjoint cycles. Each cyclic subgroup is expressed as a separate disjoint cycle.

$$(1,3)(2,7)(4,5,6)(8) = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 7 & 1 & 5 & 6 & 4 & 2 & 8 \end{bmatrix} \tag{1.2}$$

These Permutation Cycles can be multiplied together (that is operated by the Group Operator, Function Composition). eg.

$$\begin{aligned} P_1 \times P_2 &= (1,3)(2,7)(4,5,6)(8) \times (1,2,3,7)(6,4,8)(5) \\ &= (1,3)(2,7)(4,5,6)(8)(1,2,3,7)(6,4,8)(5) \\ &= (1,7,3,2)(4,8)(5,6) \end{aligned}$$

We go from Right to Left when multiplying, as function composition applies from Right to left. We see that $1 \to 2 \to 7$, $7 \to 1 \to 3$, $3 \to 7 \to 2$, $2 \to 3 \to 1$ so we get the cycle (1,3,7,2). We continue in this fashion to multiply. By Convention, we can choose not to write down single element cycles.

A Symetric Group (S_n) is a group of all the permutations over the operator Function Composition. There are n! elements in S_n

Every Cycle can be written as a product of Disjoint Cycles Each disjoint cycle is a permutation, and it can always be written as a composition of two or more permutations. Also, any **disjoint cycles can always commute**.

Order of a permutation is defined as the Least Common Multiple of the lengths of the disjoint cycles.

Every Cycle can be Written as a Product of 2 cycles Any permutation is a composition of flips. Also by direct computation, we can prove this. Here is an example: $(01234)(56)(789) = (43)(42)(41)(40)(56)(97)(98)$
The representation of any cycle as a composition of 2 cycles can be classified as even or odd, i.e. any cycle C will require either even number of 2-cycles in all possible breakdowns or odd 2-cycles in all possible breakdowns, but it can never be that some 2-cycle decompositions are odd order and some are even order.

1.2.2 Cosets and Lagrange's Theorem

A Coset of a subgroup H in group G is the set $\{ah \mid H \in a\}$ - Left Coset, or $\{ha \mid H \in a\}$ - Right Coset.

Properties of Cosets of subgroup H in group G are:

1. $\mathbf{a \in aH}$, since $e \in H$, so $ae \in aH$.
2. $\mathbf{aH = H \text{ iff } a \in H}$, If $ah = H$ then $a = ae \in aH = H$, Also if $a \in H$ then $aH \subset H$ due to closure, $H \subset aH$ as $h = eh = aa^{-1}h = a(a^{-1}h) \in aH$, as a^{-1} is in H (invertability of a in H and then closure).
3. $(\mathbf{ab})\mathbf{H} = \mathbf{a}(\mathbf{bH})$ **and** $\mathbf{H}(\mathbf{ab}) = (\mathbf{Ha})\mathbf{b}$, as Associativity holds.
4. $\mathbf{aH = bH \text{ iff } a \in bH}$, as if $aH = bH$ then $a = ae \in aH = bH$. The other way, if $a \in bH$ then $aH = (bh)H = b(hH) = bH$ so they are equal sets.
5. $\mathbf{aH = bH \text{ or } aH \cap bH = \Phi}$, Using Property 4, if there exists c in the intersection, then $c \in aH$ and $c \in bH$, so $aH = cH = bH$ thereby $aH = bH$.
6. $\mathbf{aH = bH \text{ iff } a^{-1}b \in H}$, If there is $c \in aH, bH$ so we can say that $a^{-1}c \in H$ and $b^{-1}c \in H \iff c^{-1}b \in H$ due to invertability. Multiplying both elements in the we get $a^{-1}c \cdot c^{-1}b = a^{-1}b \in H$, again by Closure.
7. $|\mathbf{aH}| = |\mathbf{bH}|$, We can show a one-one map $aH \to bH$ using the cancellation property, i.e. $ah = bh$ $(\forall h \in H)$.

8. **aH = Ha iff H = aHa^{-1}**, right-multiply both sides by a^{-1}, so $aHa^{-1} = Haa^{-1} = H$.

9. **aH is a subgroup of G, iff a $\in$ H**, aH must have identity e to be a subgroup, since $aH \cap eH \neq \phi \implies aH = eH = H$. By Property 2, $a \in H$. Conversely if $a \in H$ then $aH = H$ is a subgroup.

Theorem 2.1: Lagrange's Theorem

If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct (loft/right) cosets of H in G are $|G|/|H|$.

Proof 2.1: Lagrange's Theorem

Using the Properties proven above, $|aH| = |bH| = |eH| = |H|$ and $aH = bH$ or $aH \cap bH = \phi$, therefore $|G|$ is divisible by $|H|$.

The fruitfulness of cosets, when applied to Permutation Groups is displayed here:

Stabilizer of a Point: Let G be a group of permutations of Set S. For each i in S, let $stab_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $stab_G(i)$ the stabilizer of i in G.

Orbit of a Point: Let G be a group of permutations of a set S. For each s in S, let $orb_G(s) = \{\phi(s) \mid \phi \in G\}$. The set $orb_G(s)$ is a subset of S called the orbit of s under G.

Eg.: G = {(1),(132)(465)(78),(132)(465),(123)(456),(123)(456)(78),(78)}

Orbits of G	**Stabilizers of G**
$orb_G(1) = \{1,3,2\}$	$stab_G(1) = \{(1),(78)\}$
$orb_G(2) = \{2,1,3\}$	$stab_G(2) = \{(1),(78)\}$
$orb_G(4) = \{4,6,5\}$	$stab_G(4) = \{(1),(78)\}$
$orb_G(7) = \{7,8\}$	$stab_G(7) = \{(1),(132)(465),(123)(456)\}$

Orbit-Stabilizer Theorem states that $|orb_G(i)| \cdot |stab_G(i)| = |G|\ (\forall i)$.

1.2.3 Normal and Factor Subgroups

Normal Subgroups are subgroups H of group G, if $aH = Ha\ (\forall A \in G)$, written as $H \triangleleft G$. i.e., $ah = h'a$, the commutations are fudged a bit, when commuting we are allowed to use a different elements from H.
Normal Subgroup Test: A subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H\ (\forall x \in G)$.

Factor Groups (or Quotient Groups) are the groups, such that H is a normal Subgroup of G, $G/H = \langle\{aH \mid a \in G\} \,||\, (aH)(bH) = abH\rangle$. This is because the normal subgroups are such that their left or right cosets are themselves subgroups. The operation is the composition of the operation that generated the cosets.

Internal Direct Product: We say that G is the internal direct product of H and K and write $G = H \times K$ if H and K are normal subgroups of G, $G = HK$, $H \cap K = \{e\}$

1.3 Isomorphism and Homomorphism

1.3.1 Isomorphisms

Definition 3.1: Isomorphism

A Isomorphism is a bijective mapping from one group onto another wherein if $a \times b = c$ then $\Phi(a) \times \Phi(b) = \Phi(c)$. That is the mapping preserve the result of every operation, and every inverse.

Automorphism is an Isomorphism that exists from a group onto itself.

Example 3.1: Disproving Isomorphisms

The Set of Real numbers can never be isomorphic to proper subset of itself under the operation addition.

The homomorphisms of real numbers to its proper subsets under addition are highly constrained: $\phi(a) + \phi(a) = \phi(2a) \implies \phi(m) = qm$. Therefore there can only be a multiplicative map that holds the homomorphism.

Any subgroup of rational numbers can be generated by a subset of the series if prime inverses: $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, \ldots\}$, the set of all prime denominators. Since both of our subgroups are generated by a subset of this set, there must exist a map, just a multiplicative factor, ϕ, from $A \to B$. This never exists, since there is no such factor that maps the set of prime numbers to another set of prime numbers (or the reciprocals thereof).

Example 3.2: Proving Isomorphisms

From $\langle\mathbb{R}|+\rangle$ to $\langle\mathbb{R}|\times\rangle$, show an isomorphism exists.

We have $\phi(x) = 2^x$, the mapping is bijective, and $\phi(x) \times \phi(y) = 2^x \times 2^y = 2^{x+y} = \phi(x+y)$

Cayley's Theorem states that every group is isomorphic to a group of permutations.

1.3.2 Homomorphism

Definition 3.2: Group Homomorphism

Homomorphism ϕ is a mapping from a group G to $\bar{G}$ is a mapping that preserves the group operation, i.e. $\phi(ab) = \phi(a)\phi(b)$ $(\forall\ a, b \in G)$.
(The mapping may not be bijective, as opposed to Isomorphism)

Definition 3.3: Kernel of a Homomorphism

he kernel of a homomorphism ϕ from a group G to a group with identity e is the set $\{x \in G \mid \phi(x) = e\}$. The kernel of ϕ is denoted by $Ker\phi$.
(All elements in G that map to the identity, for isomorphism it's just the identity element - trivial subgroup)

1.4 Rings and Fields

1.4.1 Rings

A Ring is a set with 2 binary operations on it such that the following properties hold true. (Z is a arbitrary set, and + and × are arbitrary operations in $\langle Z|+,\times\rangle$, also + is the first and × is second operation on the ring).

- Associativity over +: (a + b) + c = a + (b + c)
- Commutativity over +: a + b = b + a
- Identity over +: a + 0 = a must exist for some 0
- Invertable over +: a + (-a) = 0 must exist for some -a
- Associativity over ×: $a \times (b \times c) = (a \times b) \times c$.
- Distributivity of × over +: $a \times (b + c) = (b + c) \times a = (a \times b) + (a \times c)$.

In summary, A Ring is a Abelian Group over +, and Associative and Distributive over ×.

A subring S of a ring R is the same operations defined over a subset of elements such that they in themselves form a ring.

Subring Test A non-empty subset of a ring is a subring if it is closed unber subtraction $(a-b)$ and multiplication $(a \times b)$.

1.4.2 The Integer Domain

Rings were invented to abstract the algebraic properties of Integers. However they lose essential features in this abstraction, those of **Existance of Unity, Commutativity, and Cancellation**

Zero Divisors are elements of a Commutative Ring such that there is a non-zero element $b \in R$ with $ab = 0$.

Integral Domain is a commutative ring with unity and no zero-divisors. (It may also be defined as a Commutative Ring with Cancellation, Equivalent)

1.4.3 Fields

A Field is a commutative ring with unity in which every nonzero element is a unit (i.e. Invertable).
Every finite integral domain is a field. (Finite Order: So every element is invertible)

Characteristic of a Ring is the least positive integer n such that nx = 0 for all x in the Ring. If no such integer exists then it is 0.
If R is a ring with unity (1), then the characteristic of n is the order of 1, unless its ∞ in which case characteristic is 0.
Characteristic of an integral domain can only be 0 or prime. Its because if the field is finite, then $0 = n \cdot 1 = (p \cdot 1)(q \cdot 1)$. So, either $p \cdot 1 = 0$, or $q \cdot 1 = 0$. Since p or q are smaller than n, then p or q is the characteristic, not n.

Chapter 2

Classical Mechanics

2.1 Vector Calculus

2.1.1 Vector Algebra

Scalar Triple Product between any three vectors is defined as

$$\vec{A} \cdot \vec{B} \times \vec{C} = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix}$$

Vector Triple Product between 3 vectors can be simplified down to

$$\vec{A} \times \vec{B} \times \vec{C} = B(A \cdot C) - C(A \cdot B)$$

2.1.2 Gradient, Divergence, and Curl

The Gradient of a scalar field in Cartesian Coordinates is defined as:

$$\vec{\nabla}\psi = \frac{\partial\psi}{\partial x}\hat{x} + \frac{\partial\psi}{\partial y}\hat{y} + \frac{\partial\psi}{\partial z}\hat{z} \tag{2.1}$$

The Divergence of a vector field in Cartesian Coordinates is defined as:

$$\vec{\nabla} \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \tag{2.2}$$

The Curl of a vector field in Cartesian Coordinates is defined as:

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} \tag{2.3}$$

2.1.3 Understanding the Fields

Divergence is the net flux coming out of an infinitesimally small volume per unit area, i.e. .

Curl is the amount of the circulation of a vector field, that is the integral of the vector at each point along the boundary.

2.1.4 Differential Operators in Curvilinear Coordinates

The Gradient in generalized Curvilinear Coordinates is expressed as:

$$\vec{\nabla}\psi = \frac{1}{h_1}\frac{\partial\psi}{\partial q_1}\hat{q}_1 + \frac{1}{h_2}\frac{\partial\psi}{\partial q_2}\hat{q}_2 + \frac{1}{h_3}\frac{\partial\psi}{\partial q_3}\hat{q}_3 \tag{2.4}$$

The Divergence in generalized Curvilinear Coordinates is expressed as:

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial A_1 h_2 h_3}{\partial q_1} + \frac{\partial A_2 h_3 h_1}{\partial q_2} + \frac{\partial A_3 h_2 h_1}{\partial q_3}\right) \tag{2.5}$$

The Curl in generalized Curvilinear Coordinates is expressed as:

$$\vec{\nabla}\times\vec{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{q}_1 h_1 & \hat{q}_2 h_2 & \hat{q}_3 h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix} \tag{2.6}$$

2.1.5 Fundamental Theorems

The Fundamental Theorem of Gradient states that the gradient of a scalar field is analytic, so the integral along any path in the field is the same, and equal to difference of value of the scalar field between those two points.

$$\int_a^b (\vec{\nabla}T)\cdot d\vec{l} = T(b) - T(a) \tag{2.7}$$

The Fundamental Theorem of Divergence states the the integral of the Divergence over a volume is the same as the closed integral of the vector field over the surface that bounds the said volume. This is the **Gauss Divergence Theorem**.

$$\iiint (\vec{\nabla}\cdot\vec{V})d\tau = \oiint_\tau \vec{V}\cdot d\vec{s} \tag{2.8}$$

The Fundamental Theorem of Curl states that the integral of the Curl over a surface is the same as the closed integral of the vector field over the boundary of said surface. This is also called the **Stokes Theorem**

$$\iint (\vec{\nabla}\times\vec{V})\cdot d\vec{s} = \oint_l \vec{V}\cdot d\vec{l} \tag{2.9}$$

2.2 From Newton to Lagrange

2.2.1 Center of mass frame

We are expressing values in the stationary coordinate frame (of L, T) in terms of the the Center of mass frame (r', v', p'), and those of the system / center of mass itself (R, V, P).

Angular Momentum of a group of particles is:

$$L = \vec{R} \times M\vec{V} + \Sigma_i(\vec{r}_i' \times \vec{p}_i') \tag{2.10}$$

Kinetic Energy of a group of particles is:

$$T = \frac{1}{2}Mv^2 + \frac{1}{2}\Sigma m_i v_i'^2 \tag{2.11}$$

2.2.2 Constraints

Based on Strictness Constraints can be:

- **Holonomic Constraints**: Contriants of the form $f(r_1, r_2, r_3 ... r_n) = 0$ are called Holonomic. Examples: Motion on a fixed path, rigid body contriants ($(r_i - r_j)^2 - c_i j^2 = 0$).
- **Non-Holonomic Contriants**: Constraints which can be written in the form $f(r_1, r_2, r_3 ... r_n) = 0$. Examples: Falling off the surface of a sphere ($r^2 - c^2 \geq 0$).

Given 3N independent coordinates, and K constraints, we can eliminate the dependent coordinates to get N-K remaining generalized coordinates

Dependence on Time for classification of contriants

- Rheonomous contriants: Explicitly dependent on time.
- Scleronomous contriants: Has no explicit dependence on time.

2.3 Central Force Problem

2.3.1 Solving Periodic Motion in 1-Dimension

Definition 3.1

Eleptic integrals arose from the problem of solving for the arc length of an ellipse. Today they are defined as integrals of the form $\int_c^x R(t, \sqrt{P(t)})dt$, *where R is a rational function, P(t) is a polynomial of degree 3 or 4 with no repeated roots, and c is a constant of integration*

Eleptic Integral of the First Kind are:

$$F(\phi;k) = \int_0^{\phi} \frac{d\theta}{\sqrt{1-k^2 sin^2(\theta)}} \tag{2.12}$$

and they are called **Complete if** $\phi = \pi/2$, Incomplete otherwise. They can be equivalently represented as:

$$F(x;k) = \int_0^{x} \frac{dt}{\sqrt{1-k^2 sin^2(\theta)}} \tag{2.13}$$

By putting in $t = sin(\theta)$ and $x = sin(\phi)$.

The Solution to the Complete Integrals of the First Kind is a Power Series

$$F(k) = \frac{\pi}{2}(\sum_{n=0}^{\infty}(\frac{(2n-1)!!}{(2n)!!}))k^{2n} \tag{2.14}$$

$$= \frac{\pi}{2}(\sum_{n=0}^{\infty}(\frac{(2n)!}{2^{2n}(n)!^2})^2 k^{2n}) \tag{2.15}$$

$$= \frac{\pi}{2}(1+(\frac{1}{2})^2 k + (\frac{1\cdot 3}{2\cdot 4})^2 k^2 + ...) \tag{2.16}$$

2.4 Collisions and Scattering

In the computations here, it is enough to use Conservation of Energy and Momentum to derive the Mechanics of the system.

2.4.1 Disintegration of Particles

We are interested in the statistical disrtibution of the particles as the system disintegrates. Assuming isotropy over space, we can say from the center of mass frame that the probability of a particle going in the solid angle range ω to $\omega + d\omega$ is:

$$dp(n) = \frac{d\omega}{4\pi} = \frac{2\pi sin(\theta_0)d\theta_0}{4\pi} = \frac{1}{2} sin(\theta_0)d\theta_0 \tag{2.17}$$

2.5 Small Oscillations

2.5.1 Forced Oscillations

We write the potential expeanded out as a taylor series and see that the first term is a total time derivative and can be ignored from the Lagrangian. The second term is the force.

$$U = \frac{1}{2}kx^2 + U(x_0,t) + x[\frac{\partial U(x,t)}{\partial t}]_{x=0} \tag{2.18}$$

$$\ddot{x} - \omega^2 x = F(x,t) \tag{2.19}$$

2.6 Hamiltons Equations of Motion

2.6.1 The Formulation

We opt for a different picture from the Lagrangian formulation, seeking to describe the system by first order equitaions of motion, using 2n variable, the position coordinates (q_i) and the conjugate momenta (p_i) which constitute the phase space.

$$p_i = \frac{\partial L(q, \dot{q}, t)}{\partial \dot{q}} \tag{2.20}$$

2.7 Relativistic Mechanics

2.7.1 Michaelson Morely interferometry

The Ether contradiction

The velocity of light should be same in a ether frame, take earth moving parallel and perpendicular to the said frame.

2.7.2 Lorrentz Transforms

Derivations

Let's try the transform $\boldsymbol{x' = k(x - vt)}$, since it's linear, one-one, simple, and easily reduces to classical. From the other frame to the first, this is $\boldsymbol{x = k(x' - vt')}$. Here we have expolited the first postulate of special relativity.

Substituting the first equation into the second, we get $x = k^2(x - vt) + kvt'$

$$t' = kt + (\frac{1-k^2}{kv})x \tag{2.21}$$

Now the calculate k, we shall exploit the second postulate. Replacing the t' by x' and then x, we get:

$$k(x - vt) = ckt + (\frac{1-k^2}{kv})cx$$

Solving for x, we get

$$ct = x = ct[\frac{1 + \frac{v}{c}}{1 - (\frac{1}{k^2} - 1)\frac{c}{v}}]$$

And from here we get the value of k, and so the Lorrentz Transforms.

Results

$$\begin{aligned} x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} && (2.22) \\ t' &= \frac{t - \frac{vx}{c^2}}{\sqrt{1 - v^2/c^2}} && (2.23) \\ y' &= y && (2.24) \\ z' &= z && (2.25) \end{aligned}$$

Velocity Addition

We take the derivative of the x-coordinate with respect to time, x and t in the respective frames. We get the following results.

$$\begin{aligned} V_x &= \frac{V'_x + v}{1 + \frac{vV'_x}{c^2}} && (2.26) \\ V_y &= \frac{V'_y\sqrt{1 - v^2/c^2}}{1 + \frac{vV'_x}{c^2}} && (2.27) \\ V_z &= \frac{V'_z\sqrt{1 - v^2/c^2}}{1 + \frac{vV'_x}{c^2}} && (2.28) \end{aligned}$$

Simulteneity

Two simulatenous events in one frame are separated by some time in another, which is given by

$$t'_2 - t'_1 = \frac{t_0 - vx_2/c^2}{\sqrt{1 - v^2/c^2}} - \frac{t_0 - vx_1/c^2}{\sqrt{1 - v^2/c^2}} = \frac{v(x_2 - x_1)/c^2}{\sqrt{1 - v^2/c^2}} \qquad (2.29)$$

2.7.3 Momentum and Energy

Relativistic Momentum

Particle A be static in frame S_A and Particle B be static in S_B. They are spearated by distance Y and thrown towards each other with velocity V_A and V_B. Now we have as momenta (Measuring purely in frame S_A):

$$\begin{aligned} p_A &= m_A V_A = m_A(\frac{Y}{T_A}) \\ p_B &= m_B V_B = m_B\sqrt{1 - v^2/c^2}(\frac{Y}{T_B}) \end{aligned}$$

To resolve this, conservation of momentum that is, in both frames, we must have:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \qquad (2.30)$$

However, since relativistic mass does not make sense, we call the rest mass m and regard this only as an increase in momentum (and not actually in mass).

Relativistic Energy

Kinetic Energy (T) = $\int (v) d(\frac{mv}{\sqrt{1-v^2/c^2}})$. So, we get:

$$KE = (\gamma - 1)mc^2 \tag{2.31}$$

Where we interpret the rest energy as mc^2 and the total energy as $E = KE + mc^2 = \gamma mc^2$

Note 7.1

The following approxiamtion make calcutation convienient at lower velocities (as compared to light).

$$\frac{1}{\sqrt{1-v^2/c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \quad for\ v << c \tag{2.32}$$

Energy and Momentum

By feeding in the value of E^2 and p^2c^2, we can obtain the relation

$$E^2 = (mc^2)^2 + p^2c^2 \tag{2.33}$$

even if the particle has 0 rest mass.