Proof of the Basis Selection Conjecture

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2019-09-19

1 The Problem Statement

Given a collection of n-vector spaces such that the sum of any k of them has dimention atleast k, prove that we can select a basis for an n-dimentional vector space by selecting one vector from each of the vector spaces.

2 Setting up Vocabulary

2.1 Constructing our Space

Definition 2.1

Minispace M_k refers to k-th one of n vector-spaces we have been given to select the n basis-vectors from.

The sum of all minispaces is a space with dimentionality $\geq n$. Therefore, we can construct this space by starting with a 0 dimentional space and adding in each vector from the minispaces one by one iff the vector is linearly independent of the set of all vectors already included. (We can also do this easily using **Gram Schmidt Orthonormalization**). We shall define some terms on this process.

Note that we have numbered the minispaces in an arbitrary order from 1 to n, and we follow this order when constructing our final space.

Definition 2.2

The **influence space** I_k of a minispace k, is the space spanned by the set of vectors it contributed when the full space was being constructed. The **influence dimentionality** i_k is defined as the dimentionality of the influence space. The set of vectors is contributed is called the **Influence Set** $\mathscr{I}_k$.

Definition 2.3

The **cumulative space** C_k of a minispace k, is the sum of the influence spaces of all minispaces from 1 to k. The **cumulative dimentionality** c_k is defined as the dimentionality of the cumulative space. The set of vectors is contributed is called the **Cumulative Set** $\mathscr{C}_k$.

$$C_k = \sum_{\alpha=1}^{k} I_\alpha \tag{1}$$

$$c_k = \sum_{\alpha=1}^{k} i_\alpha \tag{2}$$

We note that once we have built our full space (which is the same as C_n), if **all our influence dimentionalities are non-zero, we will have a solution to our problem**. Obviously, once we have all non-zero influence dimentionalities, we can throw away any $i_k - 1$ vectors, from our set I_k, then each minispace will contribute exactly one vector, all of them together forming a n-dimentional vector space. So having all non-zero influence dimentionalities solves the problem of each minispace contributing a single vector, our attempt will be to algorithmically construct such a set of influence vectors.

Note 2.1

Since all the vectors in $\mathscr{C}_n$ are linearly independent and span all the minispaces (by definition), we can take a subset of this set to form the basis of the Minispace. So, **Minispace M_k has as it's basis B_k such that $B_k \in \mathscr{C}_k$**.

2.2 Redistributing the influences

We will perform the second part of our algorithm by recursing over each k from n to 1. Here the definitions are crucial, and their purpose will be apparent once we start changing the influence spaces.

For now, for all the following definitions, let's assume we have α minispaces, and we want to convert this problem out to a problem with $\alpha - 1$ minispaces.

Definition 2.4

A **Removable Vector** is a vector that is present in Minispace M_k and in the set C_{k-1}.

Definition 2.5

Presenter Minispaces of a Vector are all the minispaces that contain a certain vector.

Definition 2.6

A **Tight Set of Minispaces** is the minimal set of minispaces such that the cardinality of this set is same as the dimentionality of their sum.

Definition 2.7

In our algorithms, when we reduce to a smaller problem in a smaller space, we will also have to delete some A **Removed Vector** is a vector that may have been removed from the set of all minispaces. **Removed Set** Ω is the set of all Removed vectors.

Whenever a Vector is Removed, all the **presenters of this vector get their dimentionality reduced by 1**. This fact is illustrated by defining the residual spaces as follows:

Definition 2.8

A **Residual MiniSet** $\mathscr{R}_k$ is the difference between the set of vectors in the Influence Set of the minispace $\mathscr{I}_k$ and the Removed Set Ω. So,

$$\mathscr{R}_k = \mathscr{I}_k \setminus \Omega \tag{3}$$

The **Residual MiniSpace** R_k is the span of the Residual Miniset.

This is the entire set of definitions we need. Now let's proceed to the algorithm.

3 The Steps of the Algorithm

3.1 Construction of the Space C_n

Of course, this is trivial, as stated earlier. We can keep constructing C_k from C_{k-1} by adding in linearly independent vectors one by one from M_k that are not in the space C_{k-1}. This gives us all the influence variables, cumulative variables, basis of choice for each minispace, etc. that we need.

3.2 Constraints when Reducing the Problem

Following is the key property we want to hold every time we reduce our problem. This property is initially assigned by the problem statement itself, when we have $k = n$, $\Omega = \Phi$, so all residual spaces are identical to minispaces.

Property 3.1

Given a set of k Residual MiniSpaces, the dimentionality of sum of any α of these Residual MiniSpaces is atleast α. We shall call this Property $\mathscr{P}$.

We know that if this property **is held for the subproblem of size k with removed set Ω, then the property $\mathscr{P}$ is held trivially for the subproblem of size** $k-1$, as long as no new vector is added to Ω. This is true because the Residual Spaces stay the same, and if the Property holds for a set of Residual Spaces, it also holds for all the subsets of those Residual Spaces, which is trivial from the definition of the property.

3.3 Reducing the Problem

Let's say we have a problem of size k. This problem contains the residual spaces R_1 to R_k (represented implicitly as the Minispaces and the Removed Set Ω). These residual spaces must hold the Property $\mathscr{P}$.

3.3.1 Non-Zero Influence

If $i_k > 0$, then we can just drop the k-th residual space. This is because each vector in I_k is not in the span of the MiniSpaces M_1 to M_{k-1} from our greedy construction technique, and therefore is not in the span of the Residual Spaces R_1 to R_{k-1}. So removing any of these vectors will not affect our residual spaces at all, therefore we won't bother to add this to Removed set. The property is held trivially, as the residual spaces are left unchanged.

3.3.2 Zero Influence

Here we need to perform an **Influence Transfer**. This means that we will take a vector that exists in the k-th residual space, but not in it's influence set as it's in the influence set of some other vector space that came before it. Such a vector is termed a Removable vector. We will delete the removable vector from the influence set to which it currently belongs, and add it to the influence set $\mathscr{I}_k$. We obviously recompute all of I_α, i_α, C_α, c_α, $\mathscr{C}_\alpha$ to be consistent with our modified influence sets.

The removed vector needs to be added to the set Ω, so that it's deleted from all the basis sets, so that none of the residual spaces in the next stage will contain this vector.

Once we have performed influence transfer, we know that $i_k = 1$, we can just drop the k-th residual space, and reduce our problem to one of size k-1. **This is only true if the Property $\mathscr{P}$ is held even by the new residual spaces since we added an element to Ω**. We need to prove that amongst all the Removable Vectors, there is atleast one, which when removed, allows the property to be held. This will be done in the next section.

3.4 Termination

If we keep recursing, we continue to reduce the size of our problem down to 0. Now we have $i_k \geq 1 \ \forall \ k \in (1..n)$. We have the corresponding influence sets $\mathscr{I}_k$, each having cardinality atleast 1. Also, all these vectors in the influence sets are linearly independent, since we started by having all linearly independent vectors and we have only transfered vectors from one space to another.

Now we can pick one element from each $\mathscr{I}_k$, and span the Target Space we desired, here our algorithm is complete.

4 Proof of Holding the Property

In the previous section, we presented the fact that remove one of the removable without destroying our property $\surd$, we would have a guarentee of our algorithm continuing till a problem of size 0, there proving that a way to take n basis vectors one from each minispace always exists. Now, let's try to prove that. This is the crux of the entire proof, shows the power of the property $\mathscr{P}$ in constructing said basis.

4.1 Properties of Tight Sets

Theorem 4.1

The set of all tight sets is disjoint. That is, if vector $\vec{v} \in \mathscr{T}_1$ and $\vec{v} \in \mathscr{T}_2$, then $\mathscr{T}_1 = \mathscr{T}_2$.

To prove this theorem, let's assume the contrary - Let there exist two tight sets such that their intersection is not null. We shall label these sets.

Let the two sets be $\mathscr{T}_1$ and $\mathscr{T}_2$, such that $\mathscr{T}_1 \cap \mathscr{T}_2 \neq \Phi$. Lets label the intersection and differences as follows:

- $\alpha = \mathscr{T}_1 \cap \mathscr{T}_2^C$
- $\beta = \mathscr{T}_1^C \cap \mathscr{T}_2$
- $\gamma = \mathscr{T}_1 \cup \mathscr{T}_2$

Where γ is not null. α, β, γ are obviously disjoint. We shall represent the cardinality of these sets as $|\alpha|$, $|\beta|$, $|\gamma|$ and their dimentionalities as $dim(\Sigma\alpha), dim(\Sigma\beta), dim(\Sigma\gamma)$ (Note that α, β, γ are not vector spaces, they are sets of vector spaces, so their cardinality is the normal cardinality of a set and their dimentionality is the dimentionality of their sum). Since these are tight sets in a valid subproblem, Property $\mathscr{P}$ holds. So dimentionality of each set is greater than or equal to its cardinality.

Since any **Tight-Set is minimal**, any subsets of a tight-set is non-tight. Therefore $dim(\Sigma\gamma) > |\gamma|$ (strict). So $dim(\Sigma\mathscr{T}_1 \backslash \Sigma\gamma) < |\alpha|$, as $dim(\alpha+\gamma) = |\alpha+\gamma|$. $((\Sigma\mathscr{T}_1 \backslash \Sigma\gamma)$ is not the same as α, it's a subspace in the span of alpha, so it does not violate Property $\mathscr{P}$).

$$
\begin{aligned}
dim(\mathscr{T}_1 + \mathscr{T}_2) &= dim(\mathscr{T}_2) + dim(\mathscr{T}_1 \backslash \mathscr{T}_2) \\
&= |\mathscr{T}_2| + dim(\mathscr{T}_1 \backslash \gamma) \\
&< |\mathscr{T}_2| + |\alpha| \\
&< |\mathscr{T}_1 + \mathscr{T}_2|
\end{aligned}
$$

Which violates our property $\mathscr{P}$. Therefore, we conclude that as since tight sets are minimal by definition, they are disjoint.

4.2 Selection of a Removable Vector

In our algorithm, we must chose a removable vector that allows the residual spaces to retain the property $\mathscr{P}$. This is a problem iff the removable vector belongs to a Tight Set in the problem of size k-1.

We want to show that all the removable vectors at any subproblem k cannot simultaneously belong to a tight-set, obviously if $i_k = 0$ (otherwise we don't have the problem of removable vectors).

Property 4.1

The union of Tight Sets has dimentionality equal to cardinality. (This is easy to see since the tight sets are disjoint, so the cardinality is just the sum of the cardinalities, and dimentionality can be atmost the sum of cardinalities, but due to $\mathscr{P}$ exactly equal.)

All the basis vectors of space M_k are removable vectors, since they are not in it's influence set, so they must have been in atleast one of the minispaces that came before it. Since each of these vectors is in a tight space in the k-1 subproblem, the union of these tight spaces has dimention equal to the cardinality of the union of the tight sets. But M_k is fully contained in the span of all the tight sets, and M_k cannot be in the k-1 problem tight-sets. So, for the problem of size k, the Property $\mathscr{P}$ is broken, since the cardinality is now one greater than the dimentionality.

Therefore we conclude that all removable vectors cannot be in the tight-sets, so we will always have a way to add one of the removable vectors to Ω and still hold our Property $\mathscr{P}$.

5 Conclusion

We have shown that our algorithm can construct our target space, and proved that the algorithm always has a way to continue, therefore we have proved that for any set of n vector-spaces that hold the property $\mathscr{P}$, we can always select one vector from each space such that they span an n-dimentional vector space.