Assignment-1

1. Harvard Law School courses often have assigned seating to facilitate the "Socratic method." Suppose that there are 100 first year Harvard Law students, and each takes two courses: Torts and Contracts. Both are held in the same lecture hall (which has 100 seats), and the seating is uniformly random and independent for the two courses.

(a) Find the probability that no one has the same seat for both courses (exactly; you should leave your answer as a sum).
(b) Find a simple but accurate approximation to the probability that no one has the same seat for both courses.
(c) Find a simple but accurate approximation to the probability that at least two students have the same seat for both courses.

Answer:

(a) Define random variable X that marks the number of students that take the same seat in both classes. If we denote with S_j that $j\ th$ student has the same seat, we have following:
$P(X = 0) = 1 - P(X \geq 1) = 1 - P(\cup_j S_J)$
Using the inclusion-exclusion formula and the symmetry, we have

$$P\left(\bigcup_j S_J\right) = \sum_j (-1)^{j-1} \binom{100}{j} P\left(\bigcap_{k=1}^{j} S_k\right)$$

The probability that first j students sit on their seats is $\frac{(100-j)!}{100!}$, Thus, we have

$$P\left(\bigcup_j S_J\right) = \sum_j (-1)^{j-1} \binom{100}{j} \frac{(100-j)!}{100!}$$

$$P(X = 0) = 1 - \sum_j (-1)^{j-1} \binom{100}{j} \frac{(100-j)!}{100!} = 1 - \sum_j (-1)^{j-1} \frac{\cancel{100!}}{j!\,\cancel{(100-j)!}} \frac{\cancel{(100-j)!}}{\cancel{100!}}$$

$$P(X = 0) = 1 - \sum_{j=1}^{100} \frac{(-1)^{j-1}}{j!}$$

$$P(X = 0) = \sum_{j=0}^{100} \frac{(-1)^{j}}{j!}$$

(b) Defining indicator random variables I_j that indicates if S_j has occurred, we have

$$X = \sum_{j=1}^{100} I_j$$

We know that $P(I_j) = \frac{1}{100}$ and that we can approximate

$$P\left((I_j = 1) \cap (I_k = 1)\right) = \frac{1}{100} \times \frac{1}{99} \approx \left(\frac{1}{100}\right)^2 = P(I_j = 1)P(I_k = 1)$$

So, we can consider them as independent random variables. Next, we can approximate X with Poisson distribution with parameter $\lambda = E(X) = 100\,E(I_1) = 1$. So, we have

$$P(X = 0) \approx \frac{1^0}{0!} e^{-1} \approx 0.37$$

(c) Use Poisson approximation to finally obtain that
$P(X \geq 2) = 1 - P(X = 0) - P(X = 1) \approx 1 - e^{-1} - e^{-1} = 1 - 2e^{-1} \approx 0.26$

2. **There are 100 passengers lined up to board an airplane with 100 seats (with each seat assigned to one of the passengers). The first passenger in line crazily decides to sit in a randomly chosen seat (with all seats equally likely). Each subsequent passenger takes his or her assigned seat if available, and otherwise sits in a random available seat. What is the probability that the last passenger in line gets to sit in his or her assigned seat?**

Answer:

The answer is $\frac{1}{2}$.

Case-1: Let consider the first case where there are only 2 people A and B. A pick a seat at random. The only way B gets to pick its designated seat is if A correctly picks its seat.

Therefore $P(B\ gets\ its\ seat)\ =\ P(A\ chooses\ its\ own\ seat)\ = \frac{1}{2}$

Case-2: Let consider the first case where there are only 3 people A, B and C. A picks a seat at random. There are 2 possibilities:

- A pick its own seat
- A picks B's seat and B picks A's seat

Therefore

$$P(C\ gets\ its\ seat) = P(A\ chooses\ its\ own\ seat) + P(A\ choose\ B's) \times P(B\ choose\ A's)$$

$$P(C\ gets\ its\ seat) = \frac{1}{3} + \left(\frac{1}{3}\right) \times \left(\frac{1}{2}\right) = \frac{1}{3} + \frac{1}{6} = 0.5$$