Discrete Optimization

The Knapsack Problem: Branch and Bound

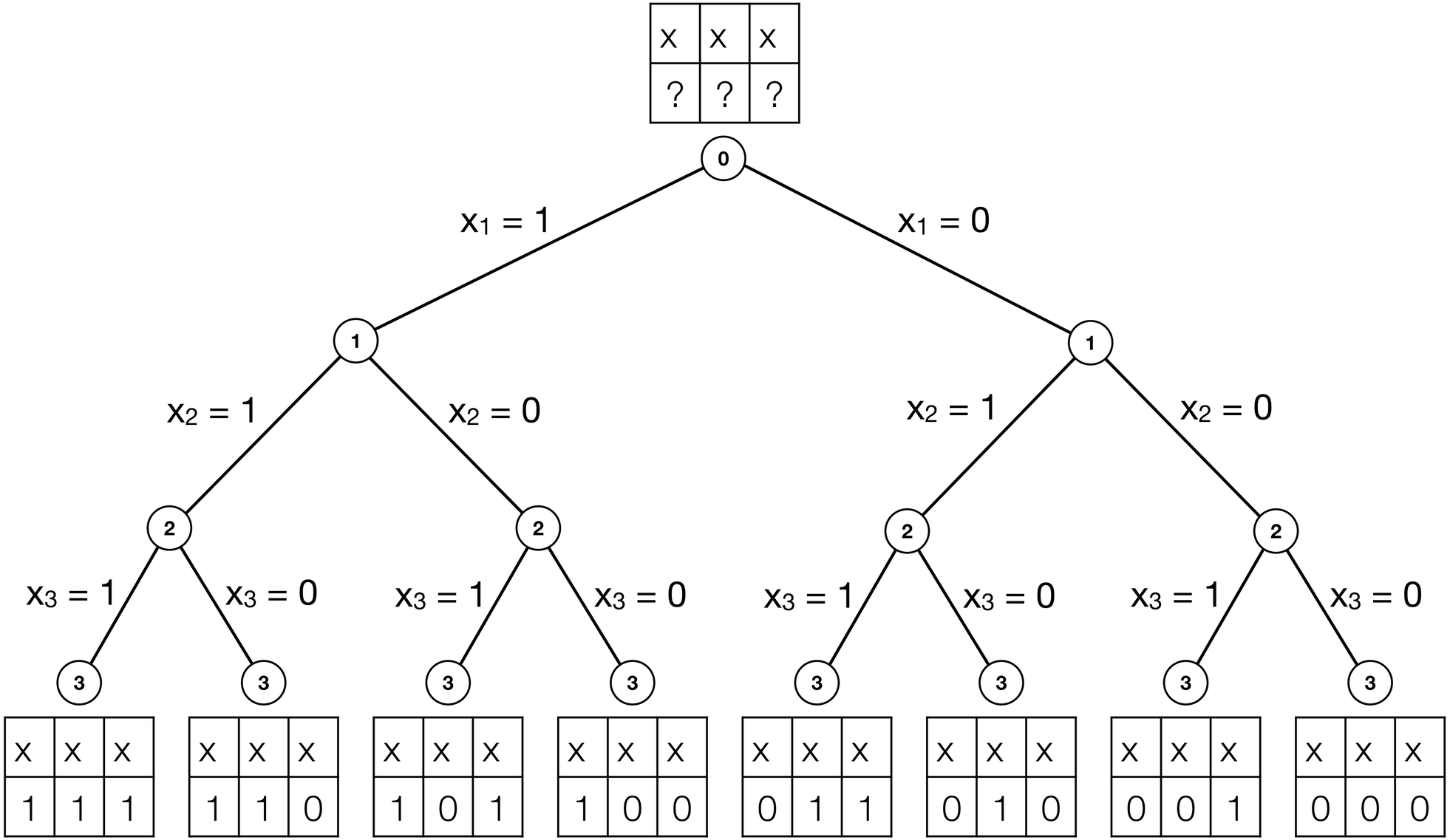
Goals of the Lecture

- Introduce branch and bound
- The value of relaxation

One-Dimensional Knapsack

$$\begin{aligned}
\text{maximize} \quad & 45x_1 + 48x_2 + 35x_3 \\
\text{subject to} \quad & \\
& 5x_1 + 8x_2 + 3x_3 \leq 10 \\
& x_i \in \{0, 1\} \quad (i \in 1..3)
\end{aligned}$$

Exhaustive Search



Branch and Bound

- Iterative two steps
  - branching
  - bounding
- Branching
  - split the problem into a number of subproblems
    - like in exhaustive search
- Bounding
  - find an ***optimistic estimate*** of the best solution to the subproblem
    - maximization: upper bound
    - minimization: lower bound

Branch and Bound

- How to find this optimistic estimate?
  - Relaxation!

- Optimization is the art of relaxation

A Knapsack Model

$$
\begin{aligned}
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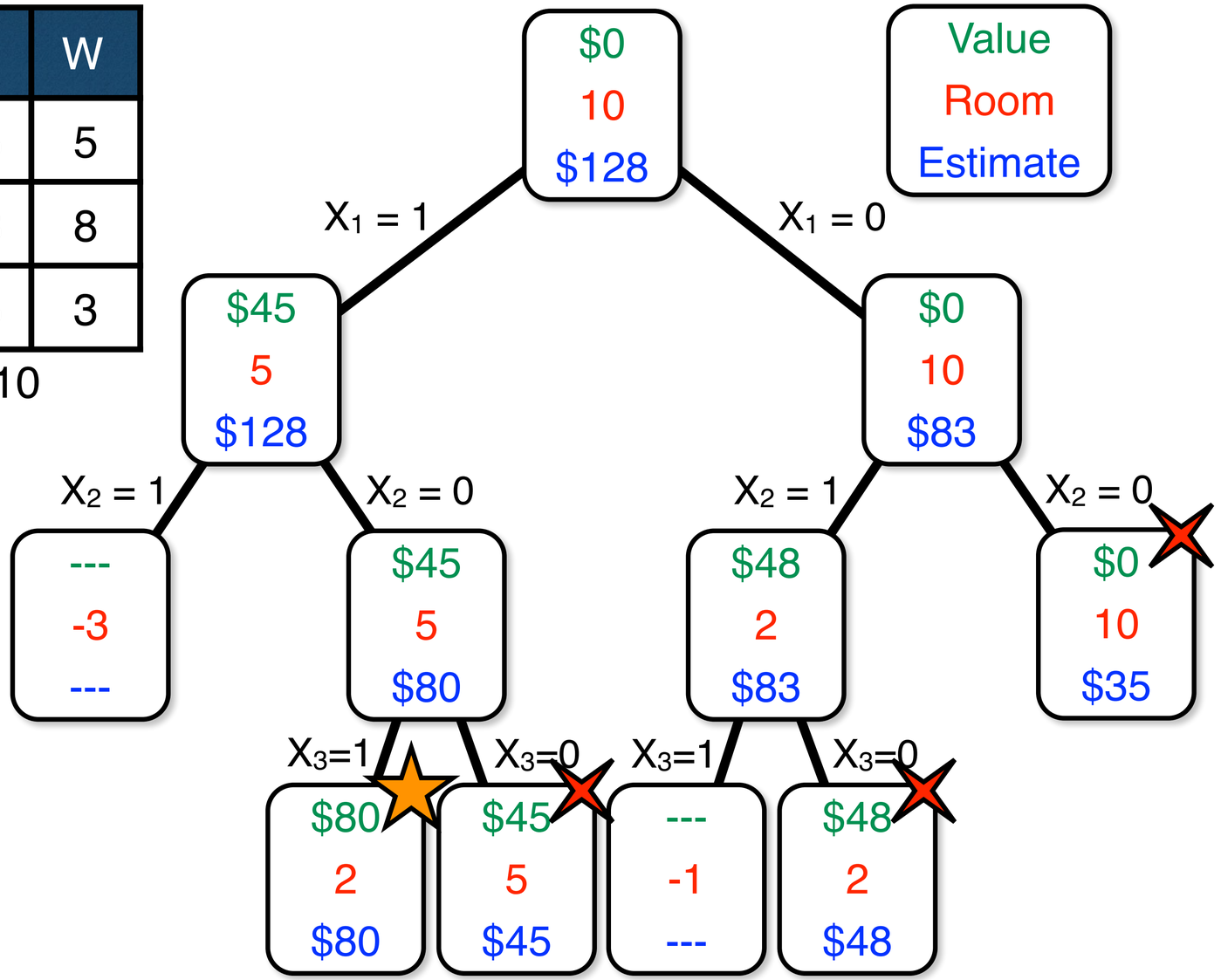
& 5x_1 + 8x_2 + 3x_3 \leq 10 \\
& x_i \in \{0, 1\} \quad (i \in 1..3)
\end{aligned}
$$

- What can we relax?
  - we can relax the capacity constraint

Depth-First Branch and Bound

i	V	W
1	45	5
2	48	8
3	35	3

K = 10



A Knapsack Model

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\begin{aligned}
\text{maximize} \quad & 45x_1 + 48x_2 + 35x_3 \\
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& x_i \in \{0, 1\} \quad (i \in 1..3)
\end{aligned}
$$

- Can we relax something else?

A Knapsack Model

- What if the items are bars of Belgian chocolate?
  - In that case, we could actually take a fraction of the bar!

subject

- This is
  - we will
  - we rela

A Knapsack Model

- Can we solve a knapsack when we can take parts of the items?
  - order the items by decreasing value of V_i/W_i
  - "most value per kilo"

$$\begin{aligned} \text{maximize} \quad & 45x_1 + 48x_2 + 35x_3 \\ \text{subject to} \quad & \\ & 5x_1 + 8x_2 + 3x_3 \leq 10 \\ & 0 \leq x_i \leq 1 \quad (i \in 1..3) \end{aligned}$$

A Knapsack Model

- How to solve the relaxation now?
  - select the items while the capacity is not exhausted
  - select a fraction of the last item

$$\begin{aligned} \text{maximize} \quad & 45x_1 + 48x_2 + 35x_3 \\ \text{subject to} \quad & \\ & 5x_1 + 8x_2 + 3x_3 \leq 10 \\ & 0 \leq x_i \leq 1 \quad (i \in 1..3) \end{aligned}$$

- In this example,
  - $V_1/W_1 = 9$, $V_2/W_2 = 6$, $V_3/W_3 = 11.7$
  - select items 3 and 1
  - select 1/4 of item 2
  - estimation: 92

A Knapsack Model

- Why is correct?

$$\text{let } x_i = \frac{y_i}{v_i}$$

$$\begin{aligned} &\text{maximize} && \textstyle\sum_{i \in 1..j} y_i \\ &\text{subject to} && \\ & && \textstyle\sum_{i \in 1..j} \frac{w_i}{v_i} y_i \leq K \\ & && 0 \leq y_i \leq 1 \quad (i \in 1..j) \end{aligned}$$

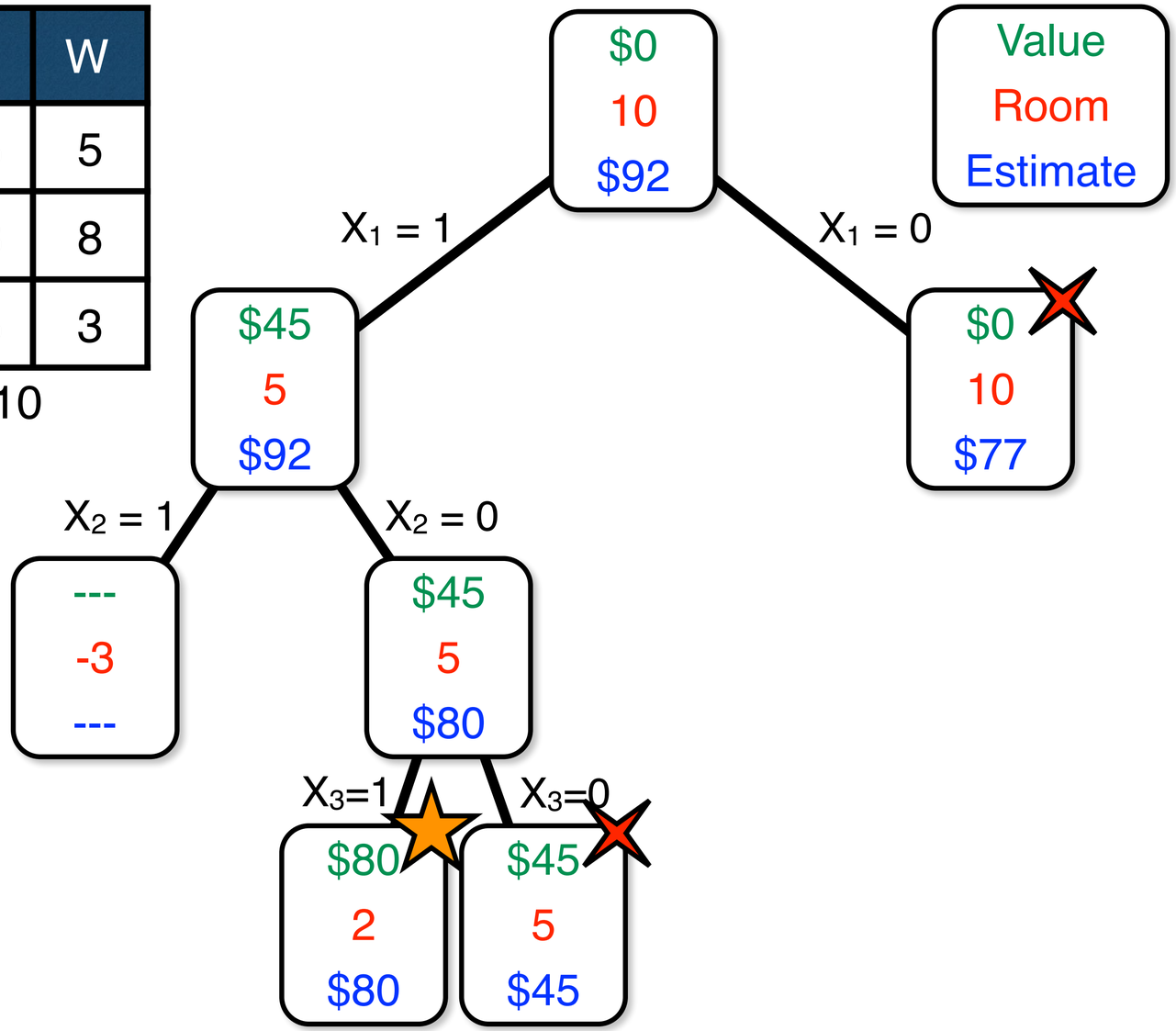
A Knapsack Model

- Of course, if the items are archeological artifacts...

Depth-First Branch and Bound

i	V	W
1	45	5
2	48	8
3	35	3

K = 10



Until Next Time

Citations

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