DS 503: Advanced Data Analytics

Lecture 6: Compressed Sensing

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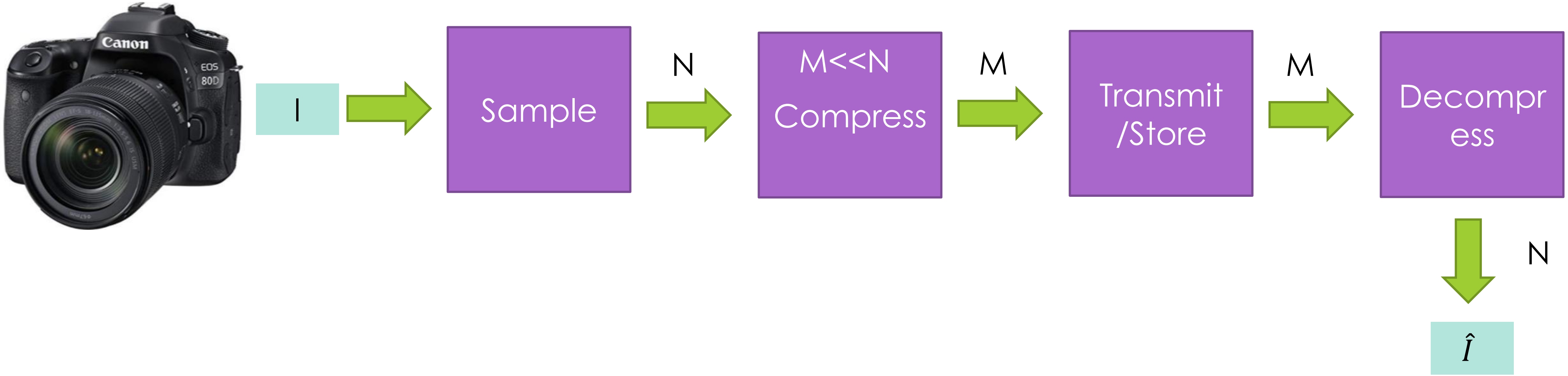
A contemporary paradox

Raw: 15MB

JPEG: 150KB

- Massive data acquisition
- Most of the data is redundant and can be thrown away
- Seems enormously wasteful
- *"One can regard the possibility of digital compression as a failure of sensor design. If it is possible to compress measured data, one might argue that too many measurements were taken." – David Brady (Duke, ECE)*

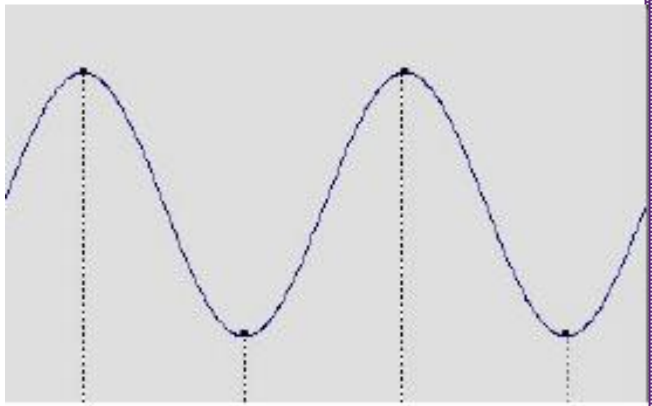
Changing the way design sensors (Image, Voice etc.)



- Acquire/Sample (A-to-D converter, digital camera)
- Compress (signal dependent, nonlinear)
- Fundamental Question: Can we directly acquire just the useful part of the signal?
  - Huge Practical Implications: MRI (reduce radiation), Remote Sensing, Networks

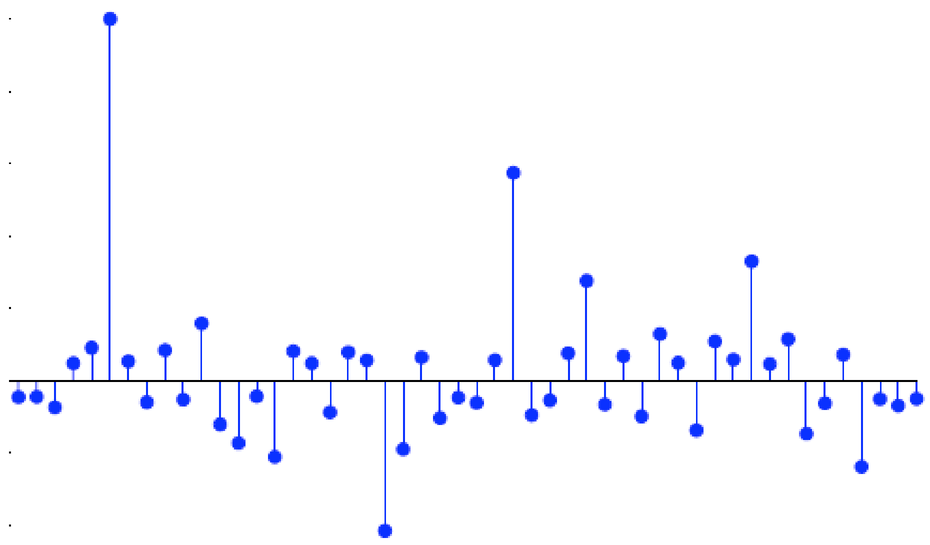
Summary

- Can obtain super-resolved signals from just a few sensors
  - Apparently breaking the Shannon-Nyquist Theorem
    - Requires sampling at twice the highest frequency (f)
    - If f is high, then it is difficult to build circuits to sample at the desired rate
- Sensing is nonadaptive: no effort to understand the signal
  - Sample is a linear functional applied to the signal
- Simple acquisition process followed by numerical optimization
- When will it work? Two conditions:
  - Sparsity: In some domain (choice of basis functions)
  - Incoherence (applied through Isometric property)

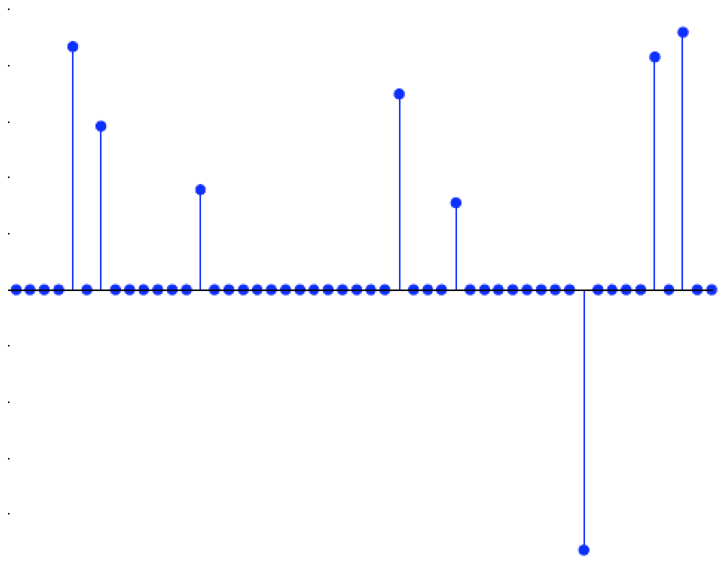


Analog to Digital
RF Receivers
Cameras
Medical Imaging

Sparsity



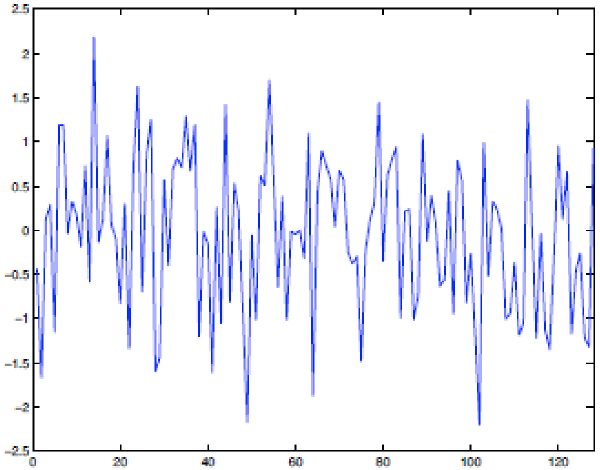
Nearly Sparse Signal



Sparse Signal

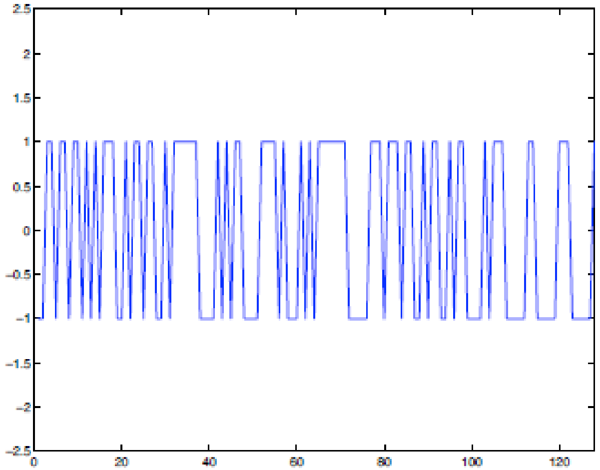
- x: signal coefficients in our convenient representation
- Collect information by measuring largest components of x
- Question: How to choose the basis and measure when the positions are not known in advance?
    - How to reconstruct the signal?

Incoherent (random sensing)



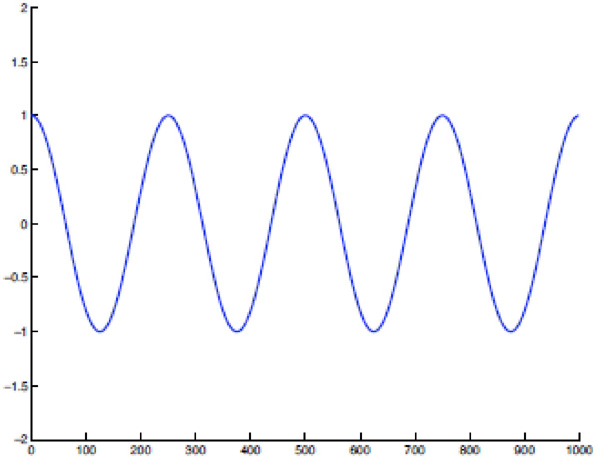
random waveform $N(0,1)$

a_k i.i.d. N(0,1) (white noise)



random waveform ± 1

a_k i.i.d. 1

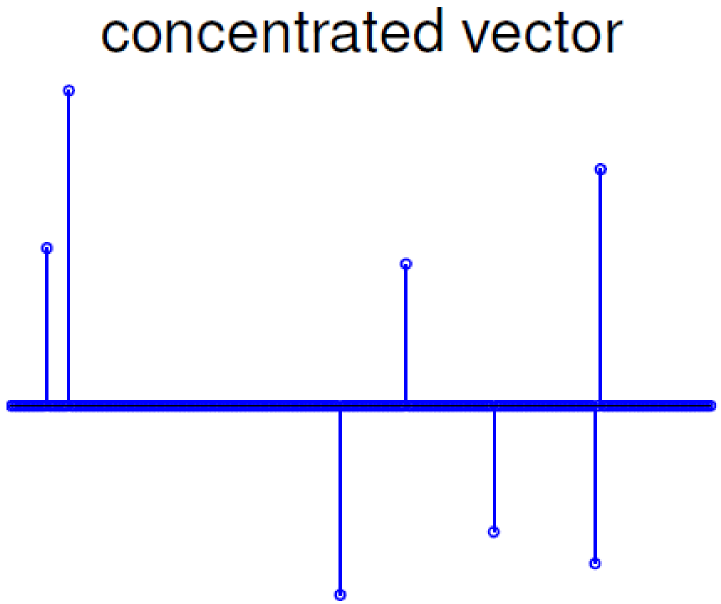


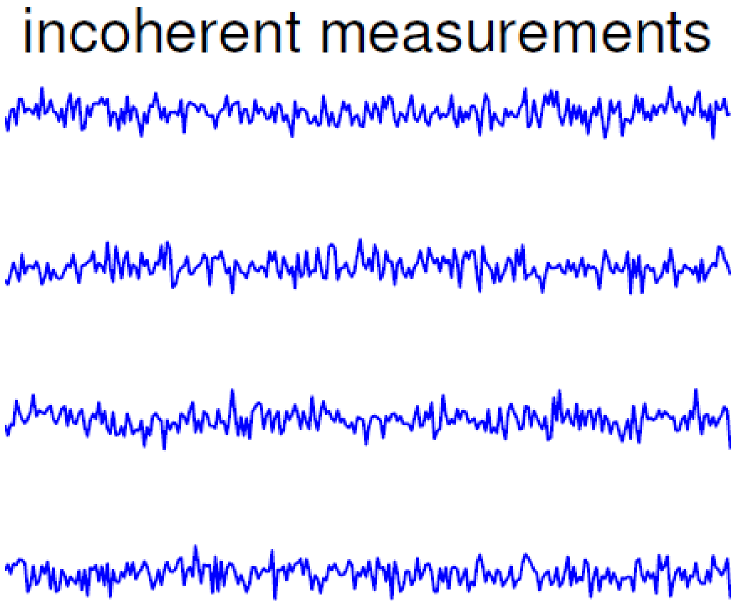
random sinusoid

a_k = exp(i2ω_kt) with i.i.d. frequencies ω_k

- $y =< a_k, x >, k = 1,2,3, \dots, m$: These are m measurements using inner products with x
- Want sensing waveforms as spread out/“incoherent" as possible
- Span of $\{a_k\}$ should be as random as possible (general orientation)
  - $a_k \sim F \ (i.i.d.)$
  - $E[a_k^* a_k] = \mathrm{I}$ and $\mathrm{a_k}$ spread out

Incoherence





- Signal is local, measurements are global
- Each measurement picks up a little information about each component
- Triangulate significant components from measurements
- Formalization: Relies on uncertainty principles between sparsity basis and incoherent measurements

Example of Foundational Result

- Classical viewpoint
  - Measure everything (all the pixels, all the coefficients)
  - Keep d largest coefficients: distortion is $||x - x_d||$
- Compressed sensing viewpoint
  - Take m random measurements: $y_k =< x, a_k >$
  - Reconstruct by linear programming: ($||x||_{l_1} = \sum_i |x_i|$

$$\boxed{x^* = \arg\min ||\check{x}||_{l_1} \ \text{subject to}\ y_k =< \check{x}, a_k > \ , k = 1,2,\ldots,m}$$

- Among all the objects consistent with data, pick *min* l_1

Same performance with about $m = d \log\left(\frac{n}{d}\right)$: $\boxed{||x^* - x||_{l_2} \le ||x - x_d||_{l_2}}$

What is compressive sensing?

- Possibility of compressed data acquisition protocols which directly acquire just the important information
- Incoherent/random measurements → compressed description
- Simultaneous signal acquisition and compression!
- All we need is to decompress...

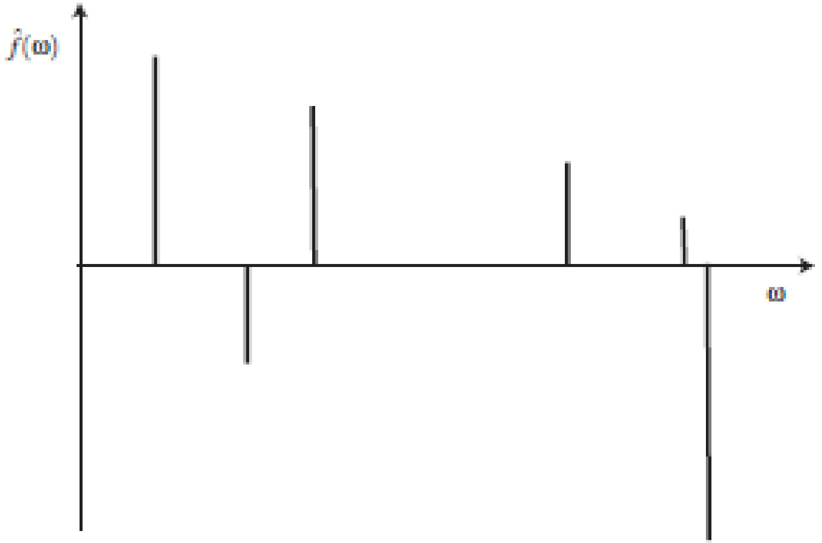
Three surprises:

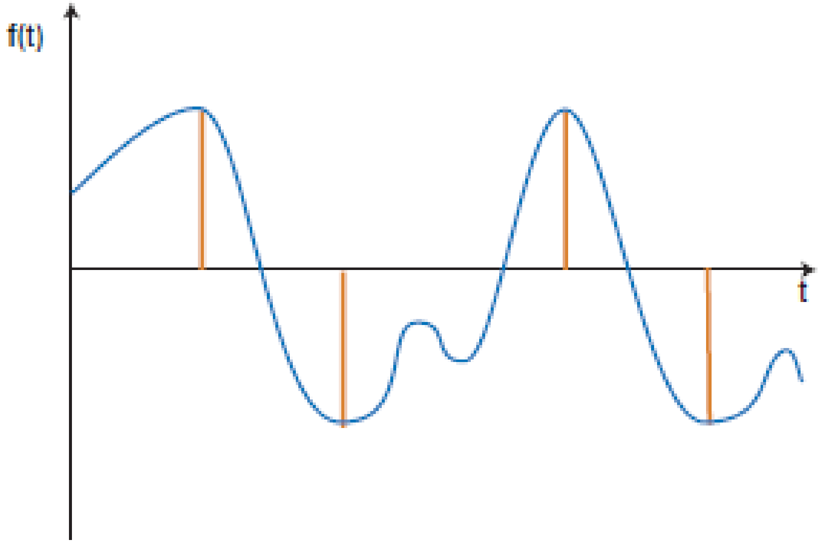
- Sensing is ultra efficient and nonadaptive
- Recovery is possible by tractable optimization
- Sensing/recovery is robust to noise (and other imperfections)

Analog to Digital Conversion

If `information bandwidth' less than total bandwidth, then should be able to

- sample below Nyquist without information loss
- recover missing samples by convex optimization

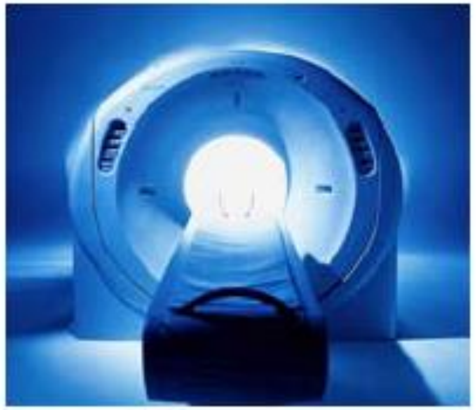




Optical Systems

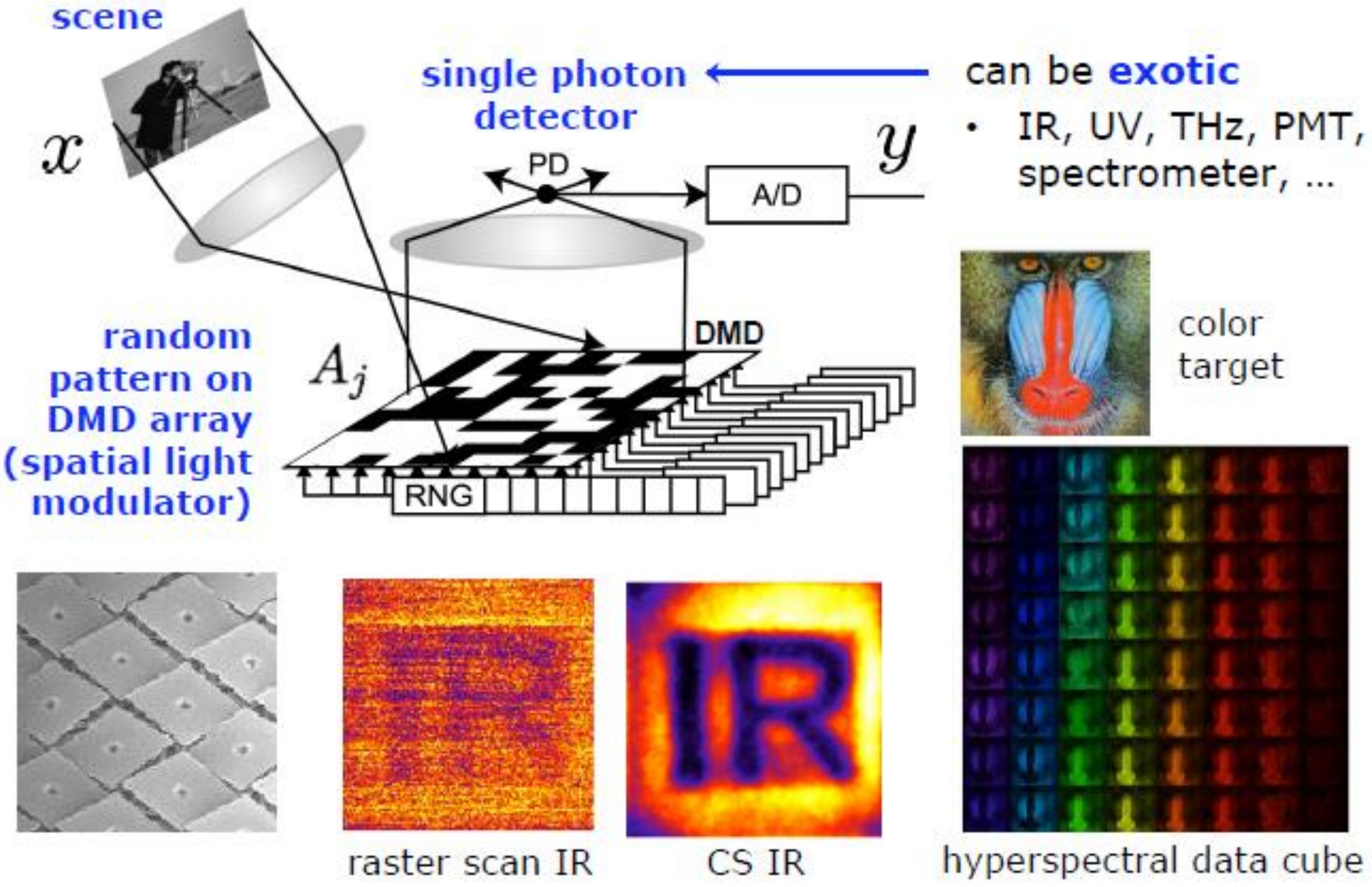
- Direct sampling: analog/digital photography, mid 19th century
- Indirect sampling: acquisition in a transformed domain, second half of 20th century; e.g. CT, MRI
- Compressive sampling: acquisition in an incoherent domain
  - Design incoherent analog sensors rather than usual pixels
  - Pay-off: need far fewer sensors

The first photograph?



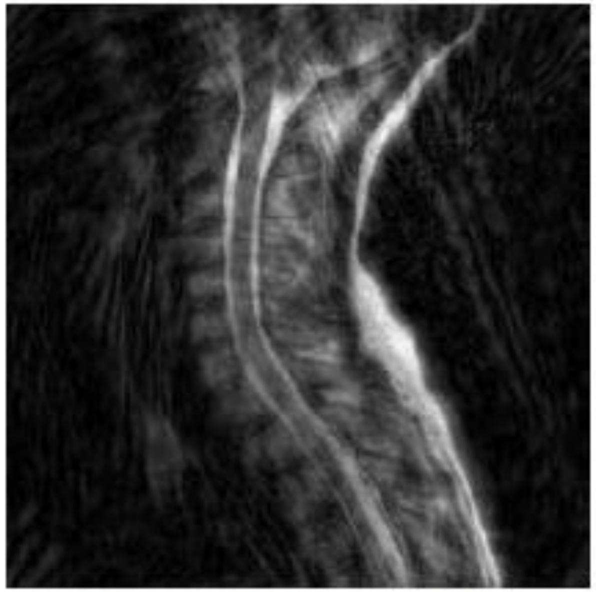
CT scanner

Single Pixel Camera (compressed sensing)



Fast Magnetic Resonance Imaging (MRI)

Fully sampled

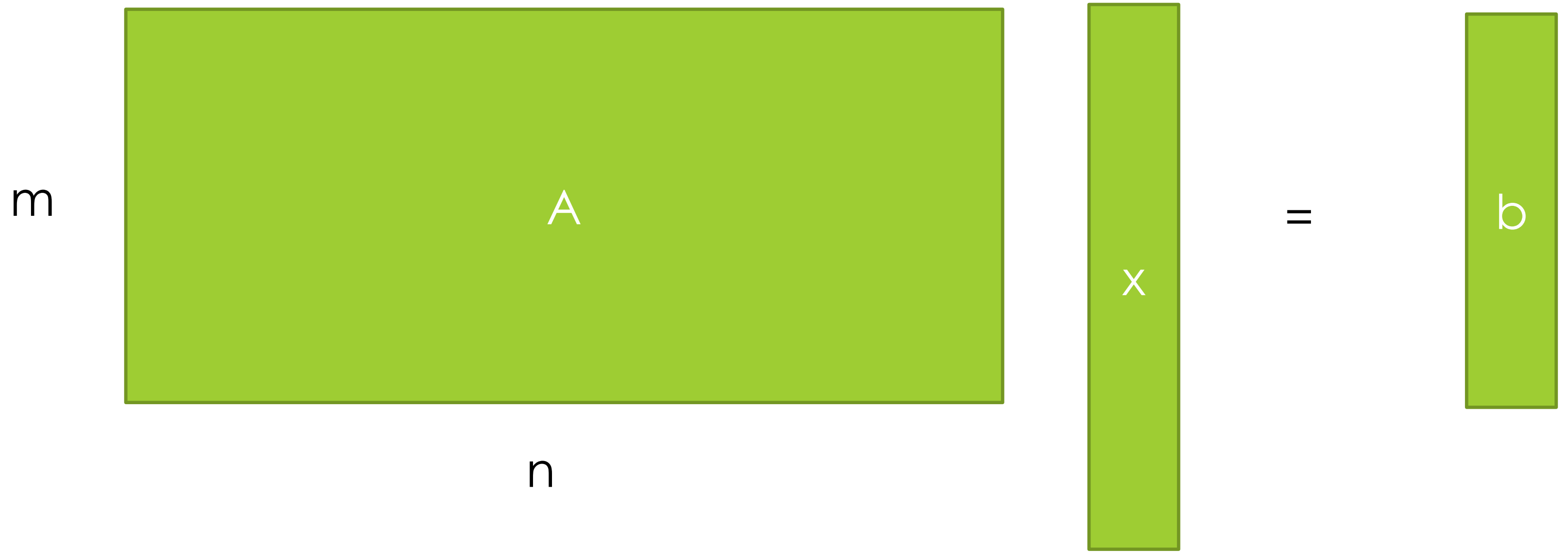


6 × undersampled
classical

6 × undersampled
CS

- Goal: Sample less to speed up MR imaging process

Sparsest solutions of Linear equations



- Find a sparsest solution of linear system:

$$(P_0) \qquad \boxed{\min\{||x||_0 : Ax = b, x \in R^n\}}$$

Where $||x||_0$ = number of non zeros of x and $A \in R^{mxn}$ with m<n

The solution of P_0 is in general not unique and this problem is NP Hard.

Basis Pursuit

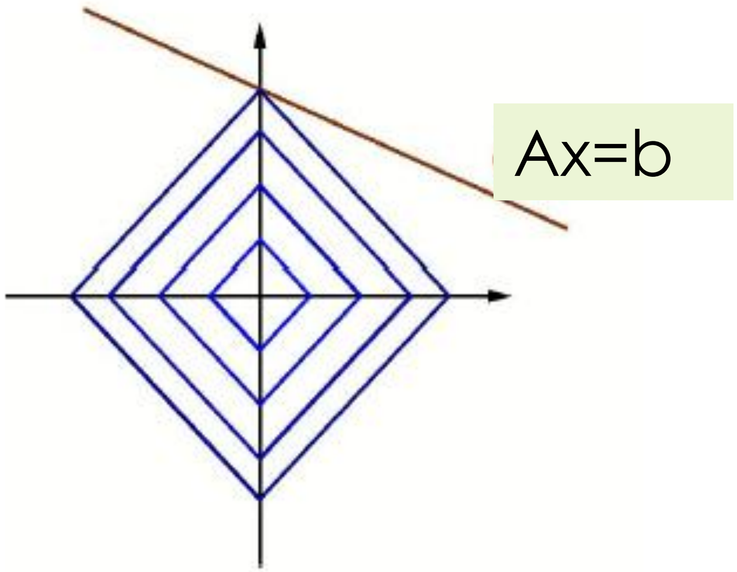
- Let's try to solve this in L_1 norm

$$(P_1) \qquad \boxed{\min\{||x||_1 : Ax = b, x \in R^n\}}$$

Where $||x||_1 = \sum_i |x_i|$ and $A \in R^{mxn}$ with m<n

The solution of P_1 is possible in polynomial time by linear programming

$$\min 1^T y$$
$$\text{s.t. } Ax=b \text{ and } -y \leq x \leq y$$



Sparse Recovery and Mutual Incoherence

<u>**Mutual incoherence (column vectors of A):**</u>

$$M(A) = \max_{i \neq j} |a_i^T a_j|$$

Where $A = [a_1 \ \ldots a_n] \in R^{mxn}$ and $||a_i||_2 = 1$

Suppose that for the sparsest solution x^* we have

$$||x^*||_0 < \frac{\sqrt{2} - \frac{1}{2}}{M(A)}$$

Then the solution of P_1 is equal to the solution of P_0

Sparse Recovery and RIP

Restricted Isometry Property of Order k

Let δ_k be the smallest number such that

$$(1-\delta_k)\left||x|\right|_2^2 \leq |Ax|_2^2 \leq (1+\delta_k)\left||x|\right|_2^2$$

for all k-sparse vectors $x \in R^n$ where $A = [a_1 \;\dots a_n\,] \in R^{mxn}$

Theorem

If $\delta_{2k} < \sqrt{2}-1$, then for all k-sparse vectors x such that Ax=b, the solution of P_1 is equal to the solution of P_0.

Approximate Recovery and RIP