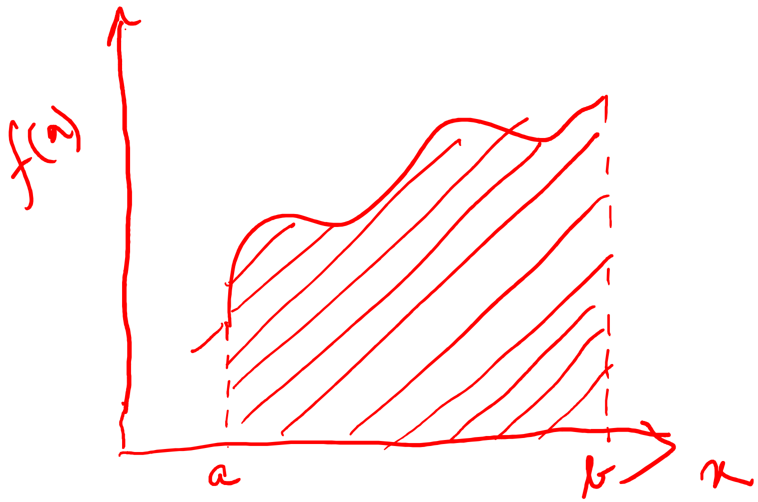
Topics to be covered

- Why we need to do numerical integration?
- Deterministic vs. stochastic methods of integration
- Examples of deterministic methods: Rectangular, trapezoidal, etc. and their limitations
- Random nos., random no generators, $\boxed{\text{testing of random numbers}}$
- Monte Carlo (MC) integration:

    A. Introduction to MC integration
    B. Errors in MC integration
    C. Improvement of MC integration
        1. Hit & miss/ Acceptance & rejection method
        2. Change of variables
        3. Importance sampling
    D. Multi dimensional integration using MC.



$$I = \int_a^b f(x)\,dx$$

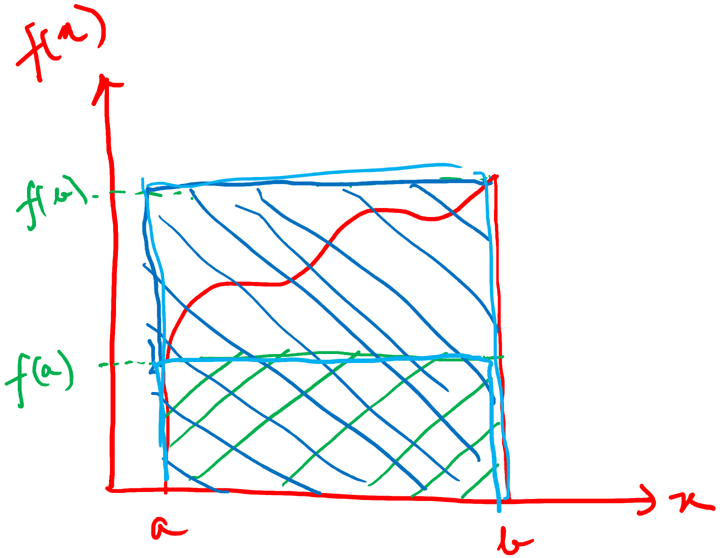
= Area ~~of~~ under the curve.

$$I \approx (b-a)\, f(a)$$

underestimating

$$I \approx (b-a)\, f(b)$$

overestimating

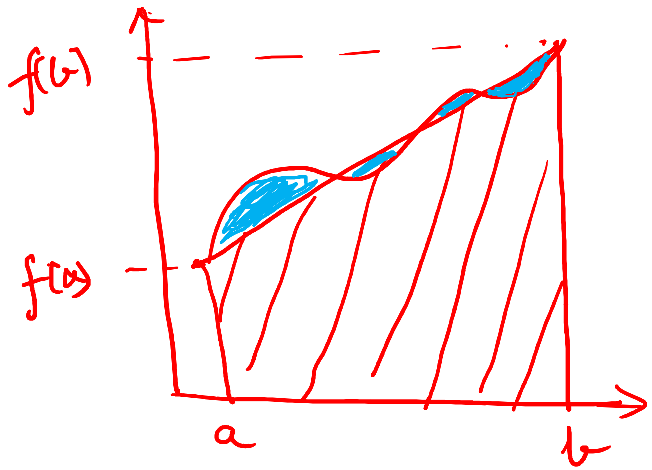


$$\int_a^b \underbrace{f(x)}\, dx$$

$f(x) = x$ or $ax+by$

x	f(x)
-	-
-	-
-	-
-	-

← Don't know f(x) related to x ✗

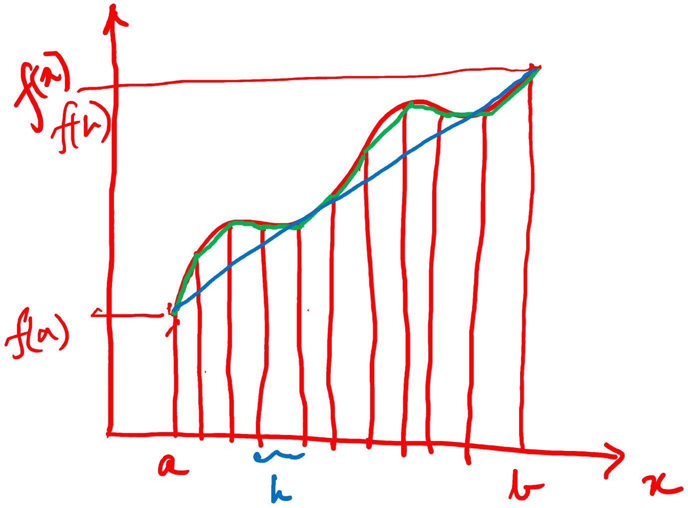


$$I \approx (b-a)\left[\frac{f(b)+f(a)}{2}\right]$$

Trapezoidal rule

Can we improve the estimate?

Composite Trapezoidal Rule



I = Sum over area of all trapezoids

$$h = \frac{b-a}{n} \qquad x_0 = a, \quad x_n = b$$

$$I = \sum_{i=1}^{n} I_i \cdots ① \qquad I_i = h\frac{\left(f(x_{i-1}) + f(x_i)\right)}{2} \cdots ②$$

Plugin ② in ①

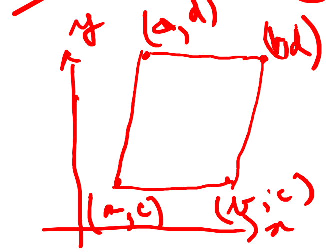
$$I = \frac{h}{2}\left[\{f(x_0) + f(x_1)\} + \{f(x_1) + f(x_2)\} + \cdots\cdots + \{f(x_{n-1}) + f(x_n)\}\right]$$

$$\boxed{I = \frac{h}{2}\left[f(a) + f(b) + 2\sum_{k=1}^{n-1} f(x_k)\right]}$$

$$\boxed{\text{Error} \sim O(h^2) \sim O(n^{-2})}$$

2D

$z = f(x,y)$ $\qquad I = \int_{x=a}^{b}\int_{y=c}^{d} z\,dx\,dy$



$[a,b]$ m subintervals each width

$h = \dfrac{b-a}{m}$, $x_i = x_0 + ih$, $i = 0, 1, \dots, m$

$[c,d]$ n subintervals

$k = \dfrac{d-c}{n}$ $\quad y_j = y_0 + jk$, $j = 0, 1, \dots n$

$$I_{2D} = \frac{hk}{4}\Big[f(a,c) + f(b,c) + f(a,d) + f(b,d) \Big]$$

$$+ 2\sum_{i=1}^{m-1} f(x_i, c) + 2\sum_{i=1}^{m-1} f(x_i, d) + 2\sum_{j=1}^{n-1} f(a, x_j) + 2\sum_{j=1}^{n-2} f(b, x_j)$$

$$+ 4\sum_{j=1}^{n-1}\left(\sum_{i=1}^{m-1} f(x_i, y_i)\right)$$

$$\text{Error} \sim \underline{O(h^2)} + O\underline{(k^2)}$$

$$O(m^{-2}) + O(n^{-2})$$

$O(n^2)$

$k = n$ $\qquad N = n^2$

$n = N^{-2}$

$O(n^2) \sim O((N^{-2})^2)$

$\sim O(\underline{N^{-1}})$

General k dimension, what will error?

$$N = n^k \Rightarrow n = N^{1/k}$$

$$\text{Error} \sim O(n^{-2}) \sim O\left(\left(N^{1/k}\right)^{-2}\right)$$

$$\sim O\left(N^{-2/k}\right)$$

$k \uparrow$ Error grid based methods $\uparrow$

Quality of my integral $\downarrow$

Computational cost??

1D $\sim O(n)$

2D $\sim O(n^2)$

3D $\sim O(n^3)$

⋮

k-D $\sim O(n^k)$

As we increase dimension computational cost increases.

Feel of computational cost

$$E = \frac{\int dR_1 \cdots dR_N \, \Psi^*(R_1,\dots,R_N) \, H(R_1,\dots,R_N) \, \Psi(R_1,\dots,R_N) \, dR_1 \cdots dR_N}{\text{Normalization}}$$

10 particles.

Dimension $- \; 3 \times 10 = 30$

10 grid ~~pts~~ points along each dir.

Total # grid pts. 10^{30}

100 Tflop computer $\Longrightarrow 10^{14}$ floating pt operations / s.

Time taken to compute at 10^{30} grid points

$\boxed{\sim 10^{16}\,s}$

Age of our universe $\sim 4 \cdot 7 \times 10^{17}\,s$

What is the way out?

$$I = \int_{a_1}^{b_1}\int_{a_2}^{b_2}\cdots\int_{a_M}^{M} f(x_1, x_2, \ldots, x_M)\, dx^M$$

$\int f(x)\,dx$

curve

Area

Integral of 1D fn. $\longrightarrow$ Area under the curve

" 2D " $\longrightarrow$ Volume enclosed

" " M dimensional fn $\longrightarrow$ M+1 dimensional volume

$$I = V^{M+1} = \frac{(b_1-a_1)(b_2-a_2)\cdots(b_M-a_M)}{N_1 \cdot N_2 \cdots N_M} \sum_{i_1=1}^{N_1}\cdots\sum_{i_M=1}^{N_M} f(\underline{x}_i)$$

$N_1 \cdot N_2 \cdots N_M = N$

$$= \underbrace{(b_1-a_1)}\,\underbrace{(b_2-a_2)}\cdots\underbrace{(b_M-a_M)} \cdot \frac{\sum_{i_1=1}^{N_1}\cdots\sum_{i_M=1}^{N_M} f(\underline{x}_i)}{N}$$

$$= V^M \langle f \rangle$$

where $\langle f \rangle = \frac{1}{N}\sum_{i=1}^{N} f(\underline{x}_i)$ where $\underline{x}_i = [x_1, x_2, \ldots x_M]$

$$I \sim V^{M+1} = V^{M} \langle f \rangle$$

$$\langle f \rangle = \frac{1}{N} \sum_{i=1}^{N} f(x_i)$$

Choose x_i's **randomly** such that they lie within the volume V^M

— then compute $\langle f \rangle$ using these random x_i's

$$V^{M+1} \approx V^{M} \langle f \rangle$$

$$\boxed{\text{NEED RANDOM NOS.}}$$

Stochastic Method

Monte Carlo Integration.

Errors in MC integration.

— MC is a statistical average ~~av~~ of the fn.

$$\underbrace{V^{M+1}} = V^{M} \underbrace{\langle f \rangle}$$

what is the error in statistical estimate of $\langle f \rangle$??

Error $\rightarrow$ standard deviation of the probability distribution fn. of $\underbrace{\langle f \rangle}$

For j^{th} measurement $\langle f \rangle_j$ = Do this $N \Rightarrow \langle f \rangle_j = \frac{1}{M}\sum_{i=1}^{M}\left[f(x_i)\right]_j$

$$\underbrace{\langle f \rangle = \frac{1}{N}\sum_{j=1}^{N}\langle f \rangle_j}$$

what is the PDF of $\langle f \rangle_j$

Central Limit Theorem.

Central Limit Theorem

PDF of $\langle f \rangle_i$ is a normal distribution

& $\sigma_N = \frac{\sigma}{\sqrt{N}}$ ← is the standard deviation of the PDF that has been used to compute one measurement of $\langle f \rangle_j$

$N \to \infty$ this is accurate.

Reality N is finite

$$\boxed{\sigma_N = \frac{\sigma}{\sqrt{N-1}}}$$

For $N=1$ $\sigma_N \to \infty$

Compare error of grid based method & MC int.

Grid → $O(N^{-2/k})$

MC Int → $O(N^{-1/2})$

(i) Error in MC is independent of the dimension of integral

(ii) Grid ~~Avode~~ based methods are more accurate for low dimension.

RANDOM Nos :

Natural random # generator (RNG)

www.random.org ← atmos. noises created by lighting discharge in thunder storms

www.randomnumbers.info ← quantum sources

www.coursecire . com ← hardware based random nos.

Random #s

(i) Can't be predicted

(ii) no correlation.

Pseudo-RNG

— linear congruent / ~~mod~~ modulo generator

$$x_i = (a \cdot x_{i-1} + c) \bmod m \quad , \; i > 0$$

$i = 0 \Rightarrow x_0$ is called the seed

$x_i, a, c, m \longrightarrow$ all integers

$m \longrightarrow$ determines the period.

$\underline{a}$ & $\underline{c}$ are crucial to determine the period

$\rightarrow x_i = (6 x_{i-1} + 7) \bmod (\underline{5})$

$x_0 = 2$

$\underbrace{2, 4, 1, 3, 0}, \underbrace{2, 4, 1, 3, 0}, \ldots$

$x_i = (\underline{27}\, x_{i-1} + \underline{11}) \bmod (\underline{54})$

$x_0 = 0$

$\underbrace{11, 38}, \underbrace{11, 38}, \underbrace{11, 38}, \ldots$

Shift register

Properties of random nos. and their tests.

Generate RN uniform distribution, $x_i = [0, 1]$

$$p(x) = 1$$

(i) Average (μ) and Standard deviation (σ)

Suppose we generate N random #s.

what is the average of the k^{th} moment of the set of N random #s.

$$\langle x^k \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i^k \, p(x_i)$$

For uniform distr $p(x_i) = 1 \Rightarrow \langle x^k \rangle = \frac{1}{N} \sum_{i=1}^{N} x_i^k$

Lt $N \to \infty$ Replace $\sum$ by $\int$

$$\langle x^k \rangle = \int_0^1 dx \, x^k = \frac{1}{k+1}$$

$$\mu = \langle x^1 \rangle = 1/2 \qquad \boxed{\mu = 0.5}$$

$$\sigma = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{1}{\sqrt{12}} \simeq 0.2886 \qquad \boxed{\sigma = 0.2886}$$

(b) Degree of correlation (auto-correlation function)

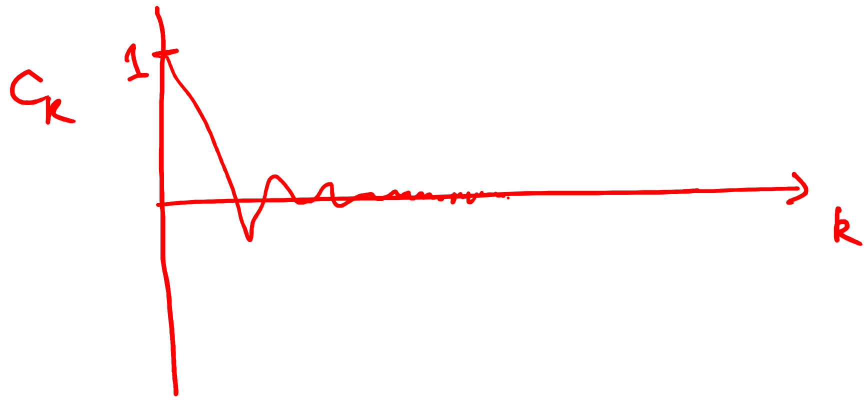
$$C_k = \frac{\langle x_{i+k} x_i \rangle - \langle x_i \rangle^2}{\langle x_i^2 \rangle - \langle x_i \rangle^2} \quad , \quad \langle x_{i+k} x_i \rangle = \frac{1}{N-k} \sum_{i=1}^{N-k} x_i x_{i+k}$$

$k = 0$ $\quad \boxed{C_k = 1}$

For 0 correlation / uncorrelated #s

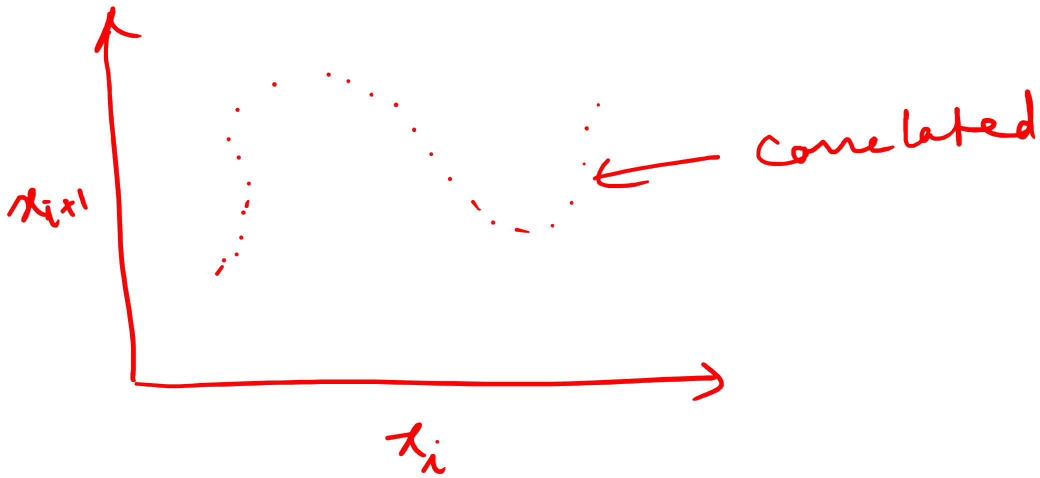
$C_k = 0$

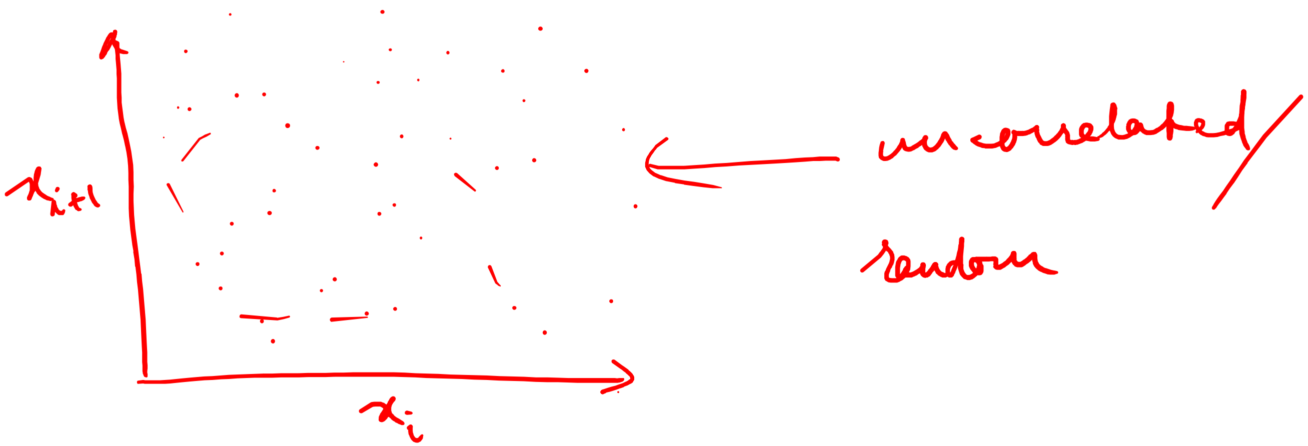
Reality $C_k \to 0$ / very small



(c) Scatter plot

$x_i \quad x_{i+1}$





Random # generators in Fortran

RANDOM_NUMBER (p)

p → array / scalar
REAL

Fortran gives real RN
between [0, 1]
with uniform distribution

RANDOM_SEED ([SIZE, PUT, GET])

SIZE → Scalar, integer, INTENT (OUT)
No. of integers used to hold the seed

PUT → Arrays, integer, INTENT [IN]
Size array ≥ # returned by SIZE

value to be used as initial seed

GET → Arrays, integers, INTENT [OUT]
Current value of the seed

Usage of RANDOM NUMBER generator in Fortran

ubuntu [Running] - Oracle VM VirtualBox

random_number_var_seed.f90 + (~/mc_int/computational_physics_2014/programs) - VIM

```
PROGRAM ranseed
! how to change the seed for generating random nos.
  IMPLICIT NONE

  ! ----- variables for portable seed setting -----
  INTEGER :: i_seed, i, n=6
  INTEGER, DIMENSION(:), ALLOCATABLE :: a_seed
  INTEGER, DIMENSION(1:8) :: dt_seed
  ! ----- end of variables for seed setting -----
  REAL :: r

  ! ----- Set up random seed portably -----
  CALL RANDOM_SEED(size=i_seed)
  write(*,*) i_seed
  ALLOCATE(a_seed(1:i_seed))
  CALL RANDOM_SEED(get=a_seed)
  CALL DATE_AND_TIME(values=dt_seed)
  DO i=1,8
     a_seed(i)=dt_seed(i)
  END DO
!  a_seed(i_seed)=dt_seed(8); a_seed(1)=dt_seed(8)*dt_seed(7)*dt_seed(6)
  CALL RANDOM_SEED(put=a_seed)
  DEALLOCATE(a_seed)
  ! ----- Done setting up random seed -----
  DO i=1,n
     CALL RANDOM_NUMBER(r)
     WRITE(6,*) 'random number is ',r
  END DO
END PROGRAM ranseed
```

RN

SIZE → #, INTENT(OUT)

PUT → array, " (IN)

GET → array, " (OUT)

DATE_AND_TIME
- → YEAR
- → MONTH
- → DAY
- → Diff.
- → HR
- → MIN
- → SEC.
- → mill. sec.

Usage of RANDOM NUMBER generator in Fortran

```
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$ f95 ran
ran2.f                      random_number.f90             random_number_var_seed.f90  random_seed.f90
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$ f95 random_number_var_seed.f90
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$ ./a.out
 random number is   0.997559547
 random number is   0.566824675
 random number is   0.965915322
 random number is   0.747927666
 random number is   0.367390871
 random number is   0.480636895
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$ ./a.out
 random number is   0.997559547
 random number is   0.566824675
 random number is   0.965915322
 random number is   0.747927666
 random number is   0.367390871
 random number is   0.480636895
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$
```

Usage of RANDOM NUMBER generator in Fortran

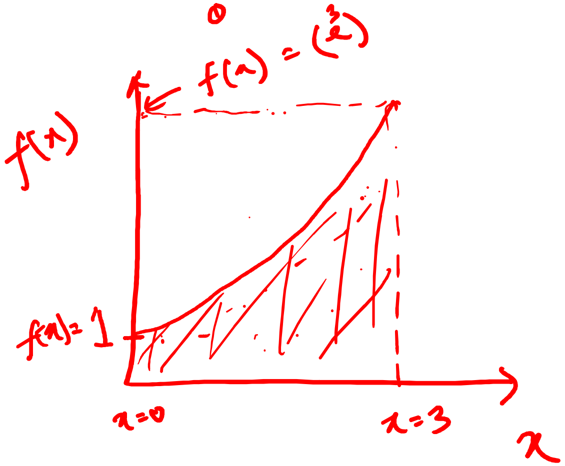
```
ubuntu [Running] - Oracle VM VirtualBox
pghosh@pghosh-VirtualBox: ~/mc_int/computational_physics_2014/programs
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$ f95 random_number_var_seed.f90
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$ ./a.out
 random number is   0.505306959
 random number is   0.962101996
 random number is   0.666360795
 random number is   0.141213417
 random number is   0.273788571
 random number is   0.954724312
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$ ./a.out
 random number is   0.441632032
 random number is   0.813283026
 random number is   0.907419145
 random number is   0.150628090
 random number is   0.804107785
 random number is   0.703152657
pghosh@pghosh-VirtualBox:~/mc_int/computational_physics_2014/programs$
```

ran0, ran1, ...

Acceptance / Rejection Method   Hit or Miss Method

After von Neumann

$$I = \int_0^3 \exp(x)\,dx$$



$x \in [0, 3]$

$y \in [0, e^3]$

Uniform distr. [0, 1]

for x [0, 3]

for y [0, e^3]

$x = 3 * \text{ran}$

$y = \exp(3) * \text{ran}$

$s = 0$

If $y < \exp(x)$ $s = s+1$

Repeat N no. of times

Fraction of #s in shaded region $= \frac{s}{N}$

$$\boxed{I \simeq 3 \times \exp(3) \times \frac{s}{N}}$$

Improvement of MC

(i) Error ⇒ smaller variance

(ii) Fewer MC sampling

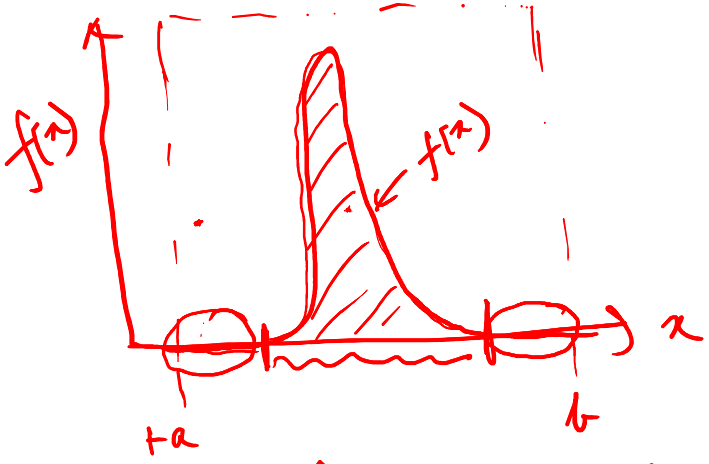
(i) <u>Change of variables</u>

RNG (Fortran) $x \in [0,1]$

$$I = \int_0^1 (\ ) \, dx \qquad \int_{-\infty}^{\infty} \qquad \int_0^{\infty}$$

Need map $x \in [0,1] \longrightarrow y \in [a, b]$??

(ii) Sampling $\Longrightarrow$ Shape of the integrand.



$$I = \int_a^b f(x)\,dx$$

Quantum Mechanics : $p(x) = \psi^*(x)\,\psi(x)$

$\psi \to$ normalized

$$H(x)\psi(x) = E\psi(x)$$

$$\langle H\rangle = \int dx\, \psi^*(x)\, H(x)\, \psi(x) = \int dx\, \psi^*(x)\,\psi(x) \frac{H(x)\,\psi(x)}{\psi(x)}\,dx$$

$$= \int dx\, p(x)\, \tilde{H}(x)\, dx$$

$$\tilde{H}(x) = \frac{H(x)\,\psi(x)}{\psi(x)} = E$$

$$= \int dx\, p(x)\, E$$

$$= E$$

$\because \int dx\, p(x) = 1$

– Ensures smooth behaviour in the neighbourhood of the exact solution.

Change variables :   Starting : PDF $p(x) = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$

$$\int_{-\infty}^{\infty} p(x)\, dx = 1$$

$x \to y$ Conservation of probability

y PDF $\to$ $p(y)$

$$\underbrace{p(y)}\, dy = \underline{p(x)}\, dx = dx$$

$$\int_0^{y'} p(y')\, dy = x$$

$$x(y) = \underbrace{\int_0^{y'} p(y')\, dy}$$

cumulative distribution of $p(y)$

$y(x)$.

Ex (1)

$$p(y)\,dy = \frac{dy}{b-a} \qquad a \leq y \leq b$$

$$= 0 \qquad \text{Elsewhere.}$$

$$\frac{dy}{b-a} = f(x)\,dx = 1\,dx$$

$$x(y) = \int_a^{y'} \frac{dy}{(b-a)} = \frac{y-a}{b-a}.$$

$$\boxed{y = a + (b-a)x}$$

$$[0,1] \to [a,b]$$

Ex 2

Exponential distr :

How to apply this to perform the integration ??

$$I = \int_0^{\infty} F(y)\,dy = \int_0^{\infty} e^{-y} \frac{F(y)}{e^{-y}}\,dy \qquad dx$$

$$G(y) = \frac{F(y)}{e^{-y}}$$

$$= \int_0^1 G(y(x))\,dx$$

$$\simeq \frac{1}{N} \sum_{i=1}^{N} G(y(x_i))$$

where x_i is a random # $[0,1]$

$$p(y)\,dy = e^{-y}\,dy$$

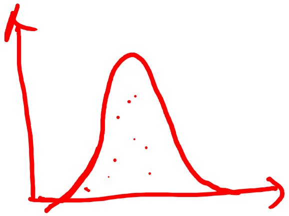
$$e^{-y}\,dy = dx$$

$$x = \int_0^{y} e^{-y'}\,dy' = 1 - e^{-y}$$

$$y(x) = -\ln(1-x)$$

$x = [0,1]$ $\quad y \in [0,\infty)$

Importance Sampling:



PDF $p(y) \rightarrow$ matches a fn F defined in the interval $[a, b]$

Norm. $\int_a^b p(y)\,dy = 1$

$$I = \int_a^b f(y)\,dy = \int_a^b p(y)\,\frac{F(y)}{p(y)}\,dy$$

$$= \int_{\tilde{a}}^{\tilde{b}} \frac{F(y(x))}{p(y(x))}\,dx.$$

$$x(y) = \int_a^y p(y')\,dy' = \cancel{z(\ )}\ z$$

Invert

$$\cancel{y(x)} = z$$

$$I \simeq \frac{1}{N} \sum_{i=1}^{N} \frac{F(y(x_i))}{p(y(x_i))} \leftarrow \tilde{F}$$

Cond. on $\underline{p(y)}$

(i) Normalizable, +ve definite

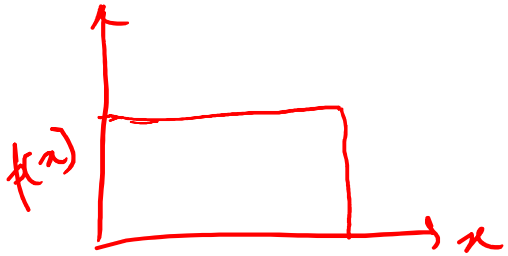
(ii) Integrable analytically

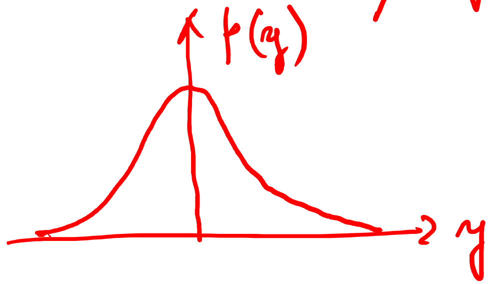
(iii) Invertible.

Error estimate. $\sigma^2 = \frac{1}{N}\sum_{i=1}^{N}(\tilde{F})^2 - \left(\frac{1}{N}\sum_{i=1}^{N}\tilde{F}\right)^2$

$$\tilde{F} = \frac{F(y(x_i))}{p(y(x_i))}$$

Uniform dist. $\longrightarrow$ normal distribution / Gaussian





$$\underbrace{p(x)}_{=1}\, dx = \underbrace{p(y)\, dy}$$

$$x(y) = \int_0^y e^{-y'^2/2}\, dy' \longleftarrow$$

Not possible to integrate analytically

Error fn.

Box - Muller Transform:

$$\underbrace{p(y_1, y_2, \ldots)}_{\text{Joint probability}} \, dy_1 \, dy_2 \ldots = \underbrace{p(x_1, x_2, \ldots)} \underbrace{\left| \frac{\partial(x_1, x_2, \ldots)}{\partial(y_1, y_2, \ldots)} \right|}_{\text{Jacobian det.}} dx_1 \, dx_2 \ldots \quad \text{—} \; (1)$$

Jacobian det. $\Downarrow$ Derivative of all x's w.r.t. y

$$\frac{\partial x_i}{\partial y_j}$$

$$p(x_1, x_2) = e^{-(x_1^2 + x_2^2)/2} \quad \text{—} \; (2)$$

Switch to polar coord.

Let $x_1 = y_1 \cos y_2$

$x_2 = y_1 \sin y_2$

$$\left. \begin{aligned} y_1^2 &= x_1^2 + x_2^2 \\ y_2 &= \tan^{-1}\left(\frac{x_2}{x_1}\right) \end{aligned} \right\} \quad \text{—} \; (3)$$

$$p(y_1, y_2)\, dy_1\, dy_2 = e^{-(x_1^2+x_2^2)/2} \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{vmatrix} dy_1\, dy_2$$

$x_1 = y_1 \cos y_2$

$x_2 = y_1 \sin y_2$

$$= e^{-\underline{(x_1^2+x_2^2)}/2} \begin{vmatrix} \cos y_2 & -y_1 \sin y_2 \\ \sin y_2 & y_1 \cos y_2 \end{vmatrix} dy_1\, dy_2$$

$$\boxed{p(y_1, y_2)\, dy_1\, dy_2 = e^{-y_1^2/2}\, y_1\, dy_1\, dy_2}$$

$y_2 \in [0, 2\pi]$ Multiply RN from uniform disk. $\leftarrow [0,1]$
by 2π

$y_1 = (x_1^2 + x_2^2)^{1/2} \Rightarrow y_1 \in [0, \infty)$

Let $u = \frac{1}{2} y_1^2 \Rightarrow y_1 = \sqrt{2u}$

$du = y_1 \, dy_1$

$e^{-y_1^2/2} y_1 \, dy_1 = \underbrace{e^{-u} du}$ New PDF

$u = -\ln(1-x')$ $\quad x' \in [0,1]$

$$\boxed{\begin{aligned} x_1 &= y_1 \cos y_2 = \sqrt{-2\ln(1-x')} \cos y_2 \\ x_2 &= y_1 \sin y_2 = \sqrt{-2\ln(1-x')} \sin y_2 \end{aligned}}$$

$y_2 \in [0, 2\pi]$, $\quad x' \in (0,1)$

Some more simplification.

Let $1-x' = s$ $\quad$ $x'=0 \quad s=1$

$x'=1 \quad s=0$

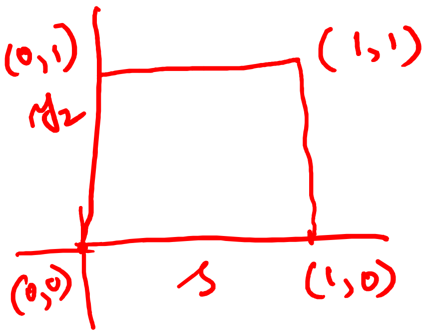
$$x_1 = \sqrt{-2\ln s}\,\cos y_2$$
$$x_2 = \sqrt{-2\ln s}\,\sin y_2$$

$$x_1 = \sqrt{-2\ln s}\,\cos \overbrace{2\pi y_2}^{\theta}$$
$$x_2 = \sqrt{-2\ln s}\,\sin 2\pi y_2$$

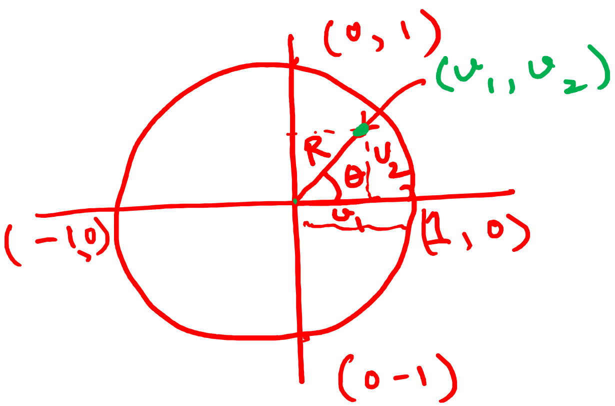
$s \in (0,1]$

$y_2 \in [0,1]$

Box Muller



Pick up uniform deviates in an unit sq.



Let v_1, v_2

ordinate $\quad$ abscissa

$$R^2 = v_1^2 + v_2^2$$

$$s = R^2 = v_1^2 + v_2^2$$

$\cos\theta = \frac{v_1}{R}$ $\quad$ $\sin\theta = v_2/R$.

(1) 2 random #s v_1 & v_2 from uniform dist [0,1]

(2) Compute $s \Longrightarrow$ IF $s=0$ OR $s \geq 1$ throw away v_1 & v_2 → Try the step again

else x_1, x_2

$U_1, U_2 \longrightarrow$ uniform distr. $[0, 1]$

$$\boxed{\begin{aligned} x_1 &= \sqrt{-2\ln R^2}\;\frac{U_1}{R} \\[2ex] x_2 &= \sqrt{-2\ln R^2}\;\frac{U_2}{R} \end{aligned}}$$

Normal / ~~gau~~ Gaussian distr.

Advantage of Box. Mueller Tr.

conv.

$$x_1 = \sqrt{-2\ln s}\ \cos(2\pi y_2)$$

$$x_2 = \sqrt{-2\ln s}\ \sin(2\pi y_2)$$

4 Multiplication

$2 * \ln s$ (1)

$2\pi * y_2$ (1)

$\sqrt{()} * \cos/\sin$ (2)

1 Sq root

1 ln

2 Trigonometric operation

BM

$$x_1 = \sqrt{-2\ln R^2}\ \frac{v_1}{R}$$

$$x_2 = \sqrt{-2\ln R^2}\ \frac{v_2}{R}$$

$R^2 = v_1^2 + v_2^2$

(i) No trigonometric fn. evaluation

expensive

5 Multiplication. $R^2 \to (2) \to v_1 * v_1$; $v_2 * v_2$

$2 * \ln \to (1)$

$\sqrt{\ } * \frac{v_1}{R}$; $\sqrt{\ } * \frac{v_2}{R}$ (2)

✓ 1 square root

2 division

✓ 1 ln

Multi-dimensional Integration.

$\underline{x} = [x_1, x_2, x_3]$

$\underline{y} = [x_4, x_5, x_6]$

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} d\underline{x}\, d\underline{y}\, g(\underline{x}, \underline{y})$$

6-dimensional Integral.

$e^{-x} \frac{f(x)}{e^{-x}}$

$$g(\underline{x}, \underline{y}) = \exp\left(-\underline{x}^2 - \underline{y}^2 - (\underline{x} - \underline{y})^2/2\right)$$

* Importance sampling

Gaussian PDF : $\frac{1}{\sqrt{\pi}} e^{-x^2}$

$\mu = 0$

$\sigma = \frac{1}{\sqrt{2}}$

$$I = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{i=1}^{6} \frac{e^{-x_i^2}}{\sqrt{\pi}} \frac{\exp(-\underline{x}^2 - \underline{y}^2 - (\underline{x} - \underline{y})^2/2)}{\prod_{i=1}^{6} \frac{1}{\sqrt{\pi}} e^{-x_i^2}} dx_1 dx_2 \cdots dx_6$$

$$I = \pi^3 \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} e^{-\left[(x_1 - x_4)^2 + (x_2 - x_5)^2 + (x_3 - x_6)^2\right]/2}\, dy_1\, dy_2 \cdots dy_6$$

Avg of this fn.

(1) Brute force MC ←

(2) Importance sampling ←

Remember to change σ_{BM} to $\sigma = 1/\sqrt{2}$