Topic: Applied optimization

Question: Find the point on the cone closest to the given point.

$$z^2 = x^2 + y^2$$

at $(3,1,0)$

Answer choices:

A $\left(\frac{3}{2},\frac{1}{2},\frac{5}{2}\right)$ and $\left(\frac{3}{2},\frac{1}{2},-\frac{5}{2}\right)$

B $\left(\frac{3}{2},\sqrt{\frac{1}{2}},\sqrt{\frac{5}{2}}\right)$ and $\left(\frac{3}{2},\sqrt{\frac{1}{2}},-\sqrt{\frac{5}{2}}\right)$

C $\left(\frac{3}{2},\frac{1}{2},\sqrt{\frac{5}{2}}\right)$ and $\left(\frac{3}{2},\frac{1}{2},-\sqrt{\frac{5}{2}}\right)$

D $\left(\sqrt{\frac{3}{2}},\frac{1}{2},\frac{5}{2}\right)$ and $\left(\sqrt{\frac{3}{2}},\frac{1}{2},-\frac{5}{2}\right)$

Solution: C

We're trying to minimize the distance between $(3,1,0)$ and the surface of the cone. If we're minimizing distance, then we can start with the distance equation, and plug in the point we were given.

$$D = \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2}$$

$$D = \sqrt{(x - 3)^2 + (y - 1)^2 + (z - 0)^2}$$

$$D = \sqrt{(x - 3)^2 + (y - 1)^2 + z^2}$$

Because we're optimizing this equation, we need to get it in terms of two variables only. But we were given the equation of the cone as $z^2 = x^2 + y^2$, so we can substitute into the distance equation for z^2.

$$D = \sqrt{(x - 3)^2 + (y - 1)^2 + x^2 + y^2}$$

Now we'll square both sides to get rid of the square root.

$$D^2 = (x - 3)^2 + (y - 1)^2 + x^2 + y^2$$

$$D^2 = x^2 - 6x + 9 + y^2 - 2y + 1 + x^2 + y^2$$

$$D^2 = 2x^2 - 6x + 2y^2 - 2y + 10$$

Find first-order partial derivatives of this function.

$$\frac{\partial D^2}{\partial x} = 4x - 6$$

$$\frac{\partial D^2}{\partial y} = 4y - 2$$

Set both partial derivatives equal to 0 and solve for x and y.

$$4x - 6 = 0$$

$$4x = 6$$

$$x = \frac{3}{2}$$

and

$$4y - 2 = 0$$

$$4y = 2$$

$$y = \frac{1}{2}$$

So the critical point is given by

$$\left(\frac{3}{2}, \frac{1}{2}\right)$$

Find second-order partial derivatives.

$$\frac{\partial^2 D^2}{\partial x^2} = 4$$

$$\frac{\partial^2 D^2}{\partial y^2} = 4$$

$$\frac{\partial^2 D^2}{\partial x \partial y} = 0$$

Then we'll plug these into the formula for D.

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (4)(4) - (0)^2$$

$$D(x, y) = 16$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

If $D = 0$, then the second derivative test is inconclusive

If $D > 0$,

and $\frac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\frac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

In this problem, $D > 0$ and $\frac{\partial^2 D^2}{\partial x^2} > 0$, so the critical point is a local minimum.

So now we'll just plug the critical point into the equation $z^2 = x^2 + y^2$ to find the associated z-value.

$$z^2 = x^2 + y^2$$

$$z^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$z^2 = \frac{9}{4} + \frac{1}{4}$$

$$z^2 = \frac{10}{4}$$

$$z = \sqrt{\frac{10}{4}}$$

$$z = \pm\sqrt{\frac{5}{2}}$$

Therefore, the points on the cone closest to $(3,1,0)$ are

$$\left(\frac{3}{2}, \frac{1}{2}, \pm\sqrt{\frac{5}{2}}\right)$$

Topic: Applied optimization

Question: Find three positive numbers with maximum product.

Sum $= 90$

Answer choices:

A $30, 30, 30$

B $30, 40, 20$

C $10, 70, 10$

D $25, 40, 25$

Solution: A

We need to find three numbers that sum to 90, so we can write one equation that represents the numbers as

$$x + y + z = 90$$

We've been asked to maximize the product of the three numbers, and we can represent this in an equation as

$$P = xyz$$

Since we need to maximize this product equation, we need to get it in terms of just two variables. So we'll solve the sum equation for z and then plug that value into the product equation.

$$x + y + z = 90$$

$$z = 90 - x - y$$

so

$$P = xyz$$

$$P = xy\left(90 - x - y\right)$$

$$P = 90xy - x^2y - xy^2$$

To make things a little easier, we'll change this to

$$f(x, y) = 90xy - x^2y - xy^2$$

Now we'll find the first-order partial derivatives of this function.

$$\frac{\partial f}{\partial x} = 90y - 2xy - y^2$$

$$\frac{\partial f}{\partial y} = 90x - x^2 - 2xy$$

Set both equations equal to 0, solving the first for y and the second for x.

$$90y - 2xy - y^2 = 0$$

$$y\left(90 - 2x - y\right) = 0$$

$y = 0$ or $90 - 2x - y = 0$

and

$$90x - x^2 - 2xy = 0$$

$$x\left(90 - x - 2y\right) = 0$$

$x = 0$ or $90 - x - 2y = 0$

Because we've been asked for positive numbers, we can't use $x = 0$ or $y = 0$. So we'll solve the other solutions as a system of equations.

$$90 - 2x - y = 0$$

$$90 - x - 2y = 0$$

Change them to

[1] $2x + y = 90$

[2] $x + 2y = 90$

Multiply [1] by 2 to get $2y$ in both equations so that we can cancel it out and solve for x.

[3] $4x + 2y = 180$

[2] $x + 2y = 90$

Subtract [2] from [3].

$$4x + 2y - (x + 2y) = 180 - (90)$$

$$4x + 2y - x - 2y = 90$$

$$3x = 90$$

$$x = 30$$

Plugging $x = 30$ into $90 - 2x - y = 0$ to solve for y gives

$$90 - 2x - y = 0$$

$$90 - 2(30) - y = 0$$

$$90 - 60 - y = 0$$

$$30 - y = 0$$

$$y = 30$$

This gives us the point $(30,30)$ as our critical point. We need to test it to make sure that it gives at maximum, so we'll find the second-order partial derivatives of $f(x, y)$.

$$\frac{\partial^2 f}{\partial x^2} = -2y$$

$$\frac{\partial^2 f}{\partial y^2} = -2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = 90 - 2x - 2y$$

Then we'll plug these into the formula for D.

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (-2y)(-2x) - (90 - 2x - 2y)^2$$

$$D(x, y) = 4xy - (90 - 2x - 2y)^2$$

Now we'll evaluate D at the critical point $(30,30)$.

$$D(30,30) = 4(30)(30) - \left[90 - 2(30) - 2(30)\right]^2$$

$$D(30,30) = 3{,}600 - (90 - 60 - 60)^2$$

$$D(30,30) = 3{,}600 - (-30)^2$$

$$D(30,30) = 3{,}600 - 900$$

$$D(30,30) = 2{,}700$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

If $D = 0$, then the second derivative test is inconclusive

If $D > 0$,

and $\frac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\frac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

In this problem, $D > 0$ and $\frac{\partial^2 f}{\partial x^2}(30,30) = -2(30) = -60 < 0$, so the critical point is a local maximum.

So now we'll just plug the critical point into the sum equation $x + y + z = 90$ to find the associated z-value.

$$x + y + z = 90$$

$$30 + 30 + z = 90$$

$$z = 30$$

These are the three positive numbers that sum to 90 and have the maximum possible product.

Topic: Applied optimization

Question: Find the maximum volume of a rectangular box inscribed in a sphere.

$$r = 4$$

Answer choices:

A $\quad V_B = \dfrac{2{,}048}{9}$

B $\quad V_B = \dfrac{2{,}048}{3\sqrt{3}}$

C $\quad V_B = \dfrac{512}{9}$

D $\quad V_B = \dfrac{512\sqrt{3}}{9}$

Solution: D

We know the equation for a sphere is

$$x^2 + y^2 + z^2 = r^2$$

Since we've been told that our sphere has radius $r = 4$, the equation of the sphere becomes

$$x^2 + y^2 + z^2 = 16$$

We also know that the formula for the volume of a box is given by $V = lwh$. Since the sphere is centered at the origin $(0,0,0)$, we'll center the box at the origin also, and give the dimensions of the box in terms of x, y and z instead of l, w and h. If you imagine the box centered at the origin, with half of the box one side of the origin, and half the box on the other, then x describes the width of the right side, and $2x$ describes the full width. In the same way, y describes the length of the back side and $2y$ describes the full length. z describes the height of the top side and $2z$ describes the full height. So the volume of the box is given by

$$V_B = lwh$$

$$V_B = (2x)(2y)(2z)$$

$$V_B = 8xyz$$

Because we're trying to maximize the volume of the box, we need to get this equation for volume in terms of just two variables, instead of three. If we solve the equation of the sphere for z, then we can substitute into the volume equation.

$$x^2+y^2+z^2=16$$

$$z^2=16-x^2-y^2$$

$$z=\sqrt{16-x^2-y^2}$$

Plug this into the volume equation.

$$V_B=8xyz$$

$$V_B=8xy\sqrt{16-x^2-y^2}$$

Now we'll start to find critical points by taking first-order partial derivatives of this volume equation. We'll need to use product rule.

$$\frac{\partial V_B}{\partial x}=(8y)\left(\sqrt{16-x^2-y^2}\right)+(8xy)\left[\frac{1}{2}\left(16-x^2-y^2\right)^{-\frac{1}{2}}(-2x)\right]$$

$$\frac{\partial V_B}{\partial x}=8y\sqrt{16-x^2-y^2}-\frac{8x^2y}{\left(16-x^2-y^2\right)^{\frac{1}{2}}}$$

$$\frac{\partial V_B}{\partial x}=8y\sqrt{16-x^2-y^2}-\frac{8x^2y}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial x}=8y\sqrt{16-x^2-y^2}\left(\frac{\sqrt{16-x^2-y^2}}{\sqrt{16-x^2-y^2}}\right)-\frac{8x^2y}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial x}=\frac{8y\left(16-x^2-y^2\right)}{\sqrt{16-x^2-y^2}}-\frac{8x^2y}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial x}=\frac{8y\left(16-x^2-y^2\right)-8x^2y}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial x}=\frac{128y-8x^2y-8y^3-8x^2y}{\sqrt{16-x^2-y^2}}$$

[1] $$\frac{\partial V_B}{\partial x}=\frac{128y-16x^2y-8y^3}{\sqrt{16-x^2-y^2}}$$

and

$$\frac{\partial V_B}{\partial y}=(8x)\left(\sqrt{16-x^2-y^2}\right)+(8xy)\left[\frac{1}{2}\left(16-x^2-y^2\right)^{-\frac{1}{2}}(-2y)\right]$$

$$\frac{\partial V_B}{\partial y}=8x\sqrt{16-x^2-y^2}-\frac{8xy^2}{\left(16-x^2-y^2\right)^{\frac{1}{2}}}$$

$$\frac{\partial V_B}{\partial y}=8x\sqrt{16-x^2-y^2}-\frac{8xy^2}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial y}=8x\sqrt{16-x^2-y^2}\left(\frac{\sqrt{16-x^2-y^2}}{\sqrt{16-x^2-y^2}}\right)-\frac{8xy^2}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial y}=\frac{8x\left(16-x^2-y^2\right)}{\sqrt{16-x^2-y^2}}-\frac{8xy^2}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial y}=\frac{8x\left(16-x^2-y^2\right)-8xy^2}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial y}=\frac{128x-8x^3-8xy^2-8xy^2}{\sqrt{16-x^2-y^2}}$$

[2] $$\frac{\partial V_B}{\partial y}=\frac{128x-16xy^2-8x^3}{\sqrt{16-x^2-y^2}}$$

Now we'll set [1] equal to 0.

$$\frac{128y-16x^2y-8y^3}{\sqrt{16-x^2-y^2}}=0$$

This equation is only true when the numerator is equal to 0. Therefore, we can say

$$128y-16x^2y-8y^3=0$$

$$y\left(128-16x^2-8y^2\right)=0$$

This means that either $y=0$ or $128-16x^2-8y^2=0$. Because y represents the length of the box, and the length can't be 0 otherwise the box wouldn't exist, we have to use $128-16x^2-8y^2=0$.

$$128-16x^2-8y^2=0$$

$$128=16x^2+8y^2$$

[3] $$16=2x^2+y^2$$

Now we'll set [2] equal to 0.

$$\frac{128x-16xy^2-8x^3}{\sqrt{16-x^2-y^2}}=0$$

This equation is only true when the numerator is equal to 0. Therefore, we can say

$$128x - 16xy^2 - 8x^3 = 0$$

$$x\left(128 - 16y^2 - 8x^2\right) = 0$$

This means that either $x = 0$ or $128 - 16y^2 - 8x^2 = 0$. Because x represents the width of the box, and the width can't be 0 otherwise the box wouldn't exist, we have to use $128 - 16y^2 - 8x^2 = 0$.

$$128 - 16y^2 - 8x^2 = 0$$

$$128 = 8x^2 + 16y^2$$

[4] $16 = x^2 + 2y^2$

Now we can put these together as a system of equations.

[3] $16 = 2x^2 + y^2$

[4] $16 = x^2 + 2y^2$

If we multiply [4] by 2, the system becomes

[5] $16 = 2x^2 + y^2$

[6] $32 = 2x^2 + 4y^2$

Subtract [5] from [6] to eliminate x and solve for y.

$$32 - (16) = 2x^2 + 4y^2 - \left(2x^2 + y^2\right)$$

$$32 - 16 = 2x^2 + 4y^2 - 2x^2 - y^2$$

$$16 = 4y^2 - y^2$$

$$16 = 3y^2$$

$$\frac{16}{3} = y^2$$

$$\sqrt{\frac{16}{3}} = y$$

$$y = \frac{\sqrt{16}}{\sqrt{3}}$$

$$y = \frac{4}{\sqrt{3}}$$

$$y = \frac{4\sqrt{3}}{3}$$

Plug this back into [3] to find the corresponding value of x.

$$16 = 2x^2 + y^2$$

$$16 = 2x^2 + \left(\frac{4\sqrt{3}}{3}\right)^2$$

$$16 = 2x^2 + \frac{16(3)}{9}$$

$$16 = 2x^2 + \frac{48}{9}$$

$$144 = 18x^2 + 48$$

$$96 = 18x^2$$

$$\frac{16}{3} = x^2$$

$$x = \frac{\sqrt{16}}{\sqrt{3}}$$

$$x = \frac{4}{\sqrt{3}}$$

$$x = \frac{4\sqrt{3}}{3}$$

If we plug this x-value and the y-value we found earlier into the original equation for the sphere, we get the corresponding value for z.

$$x^2 + y^2 + z^2 = 16$$

$$\left(\frac{4\sqrt{3}}{3}\right)^2 + \left(\frac{4\sqrt{3}}{3}\right)^2 + z^2 = 16$$

$$\frac{16(3)}{9} + \frac{16(3)}{9} + z^2 = 16$$

$$\frac{48}{9} + \frac{48}{9} + z^2 = 16$$

$$48 + 48 + 9z^2 = 144$$

$$9z^2 = 48$$

$$z^2 = \frac{48}{9}$$

$$z^2 = \frac{16}{3}$$

$$z = \frac{\sqrt{16}}{\sqrt{3}}$$

$$z = \frac{4}{\sqrt{3}}$$

$$z = \frac{4\sqrt{3}}{3}$$

So the critical point is

$$\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right)$$

Since this is the only critical point, we can assume that this is the point that gives maximum volume. If we wanted to, we could use the second derivative test to verify this.

So we'll plug this critical point into the volume equation for the box, and we'll get

$$V_B = 8xyz$$

$$V_B = 8\left(\frac{4\sqrt{3}}{3}\right)\left(\frac{4\sqrt{3}}{3}\right)\left(\frac{4\sqrt{3}}{3}\right)$$

$$V_B = 8\left(\frac{64(3)\sqrt{3}}{27}\right)$$

$$V_B = \frac{512\sqrt{3}}{9}$$

This is the maximum volume of the rectangular box inscribed in the sphere.