Finding volume

We can use triple integrals and cylindrical coordinates to solve for the mass of a solid cylinder,

$$\iiint_B f(x, y, z)\ dV$$

where B represents the solid cylinder, f is a function that models density, and dV can be defined in cylindrical coordinates as

$$dV = r\ dz\ dr\ d\theta$$

Remember, rectangular coordinates are given as (x, y, z), and cylindrical coordinates are given as (r, θ, z), and to convert from rectangular to cylindrical coordinates, we can use the formulas, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and $r^2 = x^2 + y^2$.

Let's try an example where we use a triple integral to find mass.

Example

Use cylindrical coordinates to find the mass given by the triple integral, where E is the solid that lies within the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$ and below the cone $z^2 = 9x^2 + 9y^2$.

$$\iiint_E 3x^2\ dV$$

First, we'll convert the function we were given into cylindrical coordinates using the conversion formulas.

$$3x^2$$

$$3(r\cos\theta)^2$$

$$3r^2\cos^2\theta$$

Replacing the original function with this one, and substituting for dV, the integral becomes

$$\iiint_E 3r^2\cos^2\theta(r\ dz\ dr\ d\theta)$$

$$\iiint_E 3r^3\cos^2\theta\ dz\ dr\ d\theta$$

Now we just need to find limits of integration. We've been told that we're interested in the solid that lies inside the cylinder $x^2+y^2=4$. If we convert this to cylindrical coordinates using $r^2=x^2+y^2$, we get

$$r^2=4$$

$$r=2$$

Since r represents radius, and radius can only be positive, we can say that the limits of integration for r are $[0,2]$, and therefore

$$\int\int_0^2\int 3r^3\cos^2\theta\ dz\ dr\ d\theta$$

We've also been told that the solid lies above the plane $z = 0$. Since no conversion is required for z when we're moving from rectangular to cylindrical coordinates, we can leave this as-is. We also know that the solid lies below $z^2 = 9x^2 + 9y^2$. We'll factor out 9 and get $z^2 = 9(x^2 + y^2)$. Using $r^2 = x^2 + y^2$ to convert this to cylindrical coordinates, we get

$$z^2 = 9(x^2 + y^2)$$

$$z^2 = 9(r^2)$$

$$z^2 = 9r^2$$

$$\sqrt{z^2} = \sqrt{9r^2}$$

$$z = 3r$$

Putting these two pieces of information together, we can say that the limits of integration for z are $[0,3r]$, and therefore

$$\int \int_0^2 \int_0^{3r} 3r^3 \cos^2\theta \ dz \ dr \ d\theta$$

For all full cylinders, the limits of integration for θ will be $[0,2\pi]$, therefore

$$\int_0^{2\pi} \int_0^2 \int_0^{3r} 3r^3 \cos^2\theta \ dz \ dr \ d\theta$$

We always integrate from the inside out, so we'll integrate with respect to z first, treating all other variables as constants.

$$\int_0^{2\pi} \int_0^2 3r^3 z \cos^2\theta \Big|_{z=0}^{z=3r} \ dr \ d\theta$$

$$\int_0^{2\pi}\int_0^2 3r^3(3r)\cos^2\theta - 3r^3(0)\cos^2\theta \ dr \ d\theta$$

$$\int_0^{2\pi}\int_0^2 9r^4\cos^2\theta \ dr \ d\theta$$

Now we'll integrate with respect to r, treating all other variables as constants.

$$\int_0^{2\pi} 9\left(\frac{1}{5}\right) r^5\cos^2\theta\Big|_{r=0}^{r=2} \ d\theta$$

$$\int_0^{2\pi} \frac{9}{5}r^5\cos^2\theta\Big|_{r=0}^{r=2} \ d\theta$$

$$\int_0^{2\pi} \frac{9}{5}(2)^5\cos^2\theta - \frac{9}{5}(0)^5\cos^2\theta \ d\theta$$

$$\int_0^{2\pi} \frac{288}{5}\cos^2\theta \ d\theta$$

We'll make a substitution using the trigonometric identity

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

So the integral becomes

$$\int_0^{2\pi} \frac{288}{5}\left\{\frac{1}{2}(1+\cos(2\theta))\right\} \ d\theta$$

$$\int_0^{2\pi} \frac{144}{5}(1+\cos(2\theta))\ d\theta$$

$$\int_0^{2\pi} \frac{144}{5}+\frac{144}{5}\cos(2\theta)\ d\theta$$

$$\frac{144}{5}\theta+\frac{144}{5(2)}\sin(2\theta)\Big|_0^{2\pi}$$

$$\frac{144}{5}\theta+\frac{72}{5}\sin(2\theta)\Big|_0^{2\pi}$$

$$\frac{144}{5}(2\pi)+\frac{72}{5}\sin(2(2\pi))-\left(\frac{144}{5}(0)+\frac{72}{5}\sin(2(0))\right)$$

$$\frac{144}{5}(2\pi)+\frac{72}{5}\sin(4\pi)-\frac{72}{5}\sin 0$$

$$\frac{288\pi}{5}+\frac{72}{5}(0)-\frac{72}{5}(0)$$

$$\frac{288\pi}{5}$$

This is the mass of the cylinder.