Topic: Precise definition of the limit for multivariable functions

Question: Which value of ϵ can be used to apply the definition of limit to $f(x,y)$?

$$f(x,y)=\frac{x+y}{3+2\sin x}$$

with $\delta = 0.00007$

Answer choices:

A $\quad 0.00007$

B $\quad 0.00014$

C $\quad 0.00021$

D $\quad 0.00028$

Solution: B

For all real numbers x,

$$-1 \le \sin x \le 1$$

Multiply all sides of the inequality by 2, and then add 3 to each side.

$$-2 \le 2\sin x \le 2$$

$$-2+3 \le 3+2\sin x \le 2+3$$

$$1 \le 3+2\sin x \le 5$$

Replace each of the three parts with its inverse, while changing the directions of the inequality signs.

$$\frac{1}{1} \ge \frac{1}{3+2\sin x} \ge \frac{1}{5}$$

Multiply all sides by $|x+y|$.

$$|x+y| \ge \frac{|x+y|}{3+2\sin x} \ge \frac{|x+y|}{5}$$

By Triangle Inequality, replace $|x+y|$ by a greater expression $|x|+|y|$:

$$|x|+|y| \ge \frac{|x+y|}{3+2\sin x} \ge \frac{|x+y|}{5}$$

Because $f(0,0)=\dfrac{0+0}{3+2(0)}=0$, then

$$\left|f(x,y)-f(0,0)\right| = \left|\frac{x+y}{3+2\sin x}-0\right|$$

$$\left|f(x,y) - f(0,0)\right| = \left|\frac{x+y}{3+2\sin x}\right|$$

$$\left|f(x,y) - f(0,0)\right| \le |x| + |y|$$

Applying the given value of $\delta = 0.00007$ to the inequality above results in

$$\left|f(x,y) - f(0,0)\right| = 0.00007 + 0.00007$$

$$\left|f(x,y) - f(0,0)\right| = 0.00014$$

$$\left|f(x,y) - f(0,0)\right| = \epsilon$$

Topic: Precise definition of the limit for multivariable functions

Question: Using the polar form of the function, which of the following equations or inequalities leads to verification of the limit?

$$f(x, y) = \frac{x^5}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \frac{x^5}{x^2 + y^2} = 0$$

Answer choices:

A $\delta < \epsilon$

B $\delta > \epsilon$

C $\delta = -\epsilon$

D $\delta = \epsilon$

Solution: D

Use $x = r\cos\theta$ and $y = r\sin\theta$ to convert

$$f(x,y) = \frac{x^5}{x^2+y^2}$$

to

$$f(r,\theta) = \frac{r^5\cos^5\theta}{r^2}$$

$$f(r,\theta) = r^3\cos^5\theta$$

Therefore investigating the limit of $f(x,y)$ is equivalent to investigating

$$\lim_{x\to 0} r^3\cos^5\theta$$

Choosing $\delta = \epsilon$ for an arbitrary $\epsilon > 0$ results in

$$\left|f(r,\theta) - L\right| = \left|r^3\cos^5\theta - 0\right|$$

$$\left|f(r,\theta) - L\right| = \left|r^3\cos^5\theta\right|$$

$$\left|f(r,\theta) - L\right| = |r|^3\left|\cos\theta\right|^5$$

$$\left|f(r,\theta) - L\right| \le |r|$$

$$\left|f(r,\theta) - L\right| < \delta = \epsilon$$

where $0 < |r| < \delta$ holds true for the distance between r and 0.

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Question: Find the condition.

The function

$$f(x, y) = \frac{121xy^2}{x^2 + y^2}$$

is defined on the region $R^2 - (0,0)$. To verify

$$\lim_{(x,y)\to(0,0)} f(x, y) = \frac{121xy^2}{x^2 + y^2} = 0$$

which of the following conditions must be applied to definition of the limit of $f(x, y)$?

Answer choices:

A $\delta < \frac{\epsilon}{121}$

B $\delta \le \frac{\epsilon}{121}$

C $\epsilon \le \frac{\delta}{121}$

D $\epsilon > \frac{\delta}{121}$

Solution: B

By the conceptual definition of the limit of a multivariable function, for any arbitrary number $\epsilon > 0$ we must identify a real number $\delta > 0$, where $|f(x, y) - 0| < \epsilon$ and the inequality

$$0 < \sqrt{x^2 + y^2} < \delta$$

holds true for the distance between $(0,0)$ and (x, y).

That is, for an arbitrary number $\epsilon > 0$, we define the corresponding real number $\delta > 0$ such that the

$$\left| \frac{121xy^2}{x^2 + y^2} - 0 \right| < \epsilon$$

Simplify the left side.

$$\left| \frac{121xy^2}{x^2 + y^2} \right|$$

$$\frac{121\ |x|\ y^2}{x^2 + y^2}$$

$$\frac{121\ |x|\ y^2}{x^2 + y^2} \le 121|x|(1)$$

$$\frac{121\ |x|\ y^2}{x^2 + y^2} \le (121)\sqrt{x^2}$$

$$\frac{121\ |x|\ y^2}{x^2+y^2} \leq (121)\sqrt{x^2+y^2}$$

$$\frac{121\ |x|\ y^2}{x^2+y^2} < 121\delta$$

Therefore, choosing

$$\left|f(x,y) - 0\right| < 121\delta$$

and integrating the inequality with

$$0 < \sqrt{x^2+y^2} < \delta$$

implies that

$$\delta \leq \frac{\epsilon}{121}$$

leads to

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{121xy^2}{x^2+y^2} = 0$$