Topic: Finding surface area

Question: Find surface area.

The part of $z = xy$ inside $x^2 + y^2 = 25$

Answer choices:

A $\quad A(S) = \dfrac{50\pi\sqrt{26}}{3}$

B $\quad A(S) = \dfrac{2\pi}{3}\left(\sqrt{26^3} + 1\right)$

C $\quad A(S) = \dfrac{2\pi}{3}\left(\sqrt{26^3} - 1\right)$

D $\quad A(S) = 38\pi\sqrt{26}$

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Solution: C

To find the area of some surface that's bounded by another function, we'll need to take the partial derivatives of the equation of the surface.

$$z = xy$$

$$\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = x$$

Since we're dealing with a double integral, we have to decide whether the region we're looking at is a type 1 or type 2 region. We're looking for the area inside $x^2 + y^2 = 25$, which is a circle. Since we could take uniform slices of the circle horizontally or vertically, we could treat the region as either type 1 or type 2, but we'll go ahead and treat it as a type 1 region. Therefore, we'll integrate first with respect to y, and second with respect to x.

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$A(S) = \iint_D \sqrt{1 + y^2 + x^2}\ dy\ dx$$

To find the limits of integration with respect to y, we'll solve $x^2 + y^2 = 25$ for y.

$$x^2 + y^2 = 25$$

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$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$

To find the limits of integration with respect to x, we'll remember that the circle $x^2 + y^2 = 25$ is a circle centered at the origin with radius 5, and therefore is defined for x on $[-5,5]$.

$$A(S) = \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{1 + y^2 + x^2}\ dy\ dx$$

The simplest way to solve this particular integral is to convert it to polar coordinates. Using the conversion formulas

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

the integral becomes

$$A(S) = \int_{x=-5}^{x=5} \int_{y=-\sqrt{25-x^2}}^{y=\sqrt{25-x^2}} r\sqrt{1 + r^2}\ dr\ d\theta$$

Now we need to change the limits of integration from rectangular to polar. Inside the circle $x^2 + y^2 = 25$, r is defined on $[0,5]$, and θ is defined on $[0,2\pi]$.

$$A(S) = \int_0^{2\pi} \int_0^5 r\sqrt{1 + r^2}\ dr\ d\theta$$

We'll use u-substitution to take the integral.

$$u = 1 + r^2$$

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$$\frac{du}{dr} = 2r$$

$$dr = \frac{du}{2r}$$

Making substitutions into the integral gives

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=5} r\sqrt{u} \left(\frac{du}{2r} \right) d\theta$$

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=5} \frac{1}{2} u^{\frac{1}{2}} \ du \ d\theta$$

Integrate with respect to u.

$$A(S) = \int_0^{2\pi} \frac{1}{3} u^{\frac{3}{2}} \Big|_{r=0}^{r=5} \ d\theta$$

Back-substitute.

$$A(S) = \int_0^{2\pi} \frac{1}{3} \left(1 + r^2 \right)^{\frac{3}{2}} \Big|_0^5 \ d\theta$$

Evaluate over the interval and simplify.

$$A(S) = \int_0^{2\pi} \frac{1}{3} \left(1 + 5^2 \right)^{\frac{3}{2}} - \frac{1}{3} \left(1 + 0^2 \right)^{\frac{3}{2}} \ d\theta$$

$$A(S) = \int_0^{2\pi} \frac{1}{3} \left(26^{\frac{3}{2}} \right) - \frac{1}{3} \left(1^{\frac{3}{2}} \right) \ d\theta$$

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$$A(S)=\int_0^{2\pi}\frac{1}{3}\left(26^3\right)^{\frac{1}{2}}-\frac{1}{3}\left(1^3\right)^{\frac{1}{2}}\ d\theta$$

$$A(S)=\int_0^{2\pi}\frac{1}{3}\sqrt{26^3}-\frac{1}{3}\sqrt{1}\ d\theta$$

$$A(S)=\int_0^{2\pi}\frac{1}{3}\sqrt{26^3}-\frac{1}{3}\ d\theta$$

$$A(S)=\int_0^{2\pi}\frac{\sqrt{26^3}-1}{3}\ d\theta$$

Integrate with respect to θ.

$$A(S)=\frac{\sqrt{26^3}-1}{3}\theta\Big|_0^{2\pi}$$

Evaluate over the interval and simplify.

$$A(S)=\frac{\sqrt{26^3}-1}{3}(2\pi)-\frac{\sqrt{26^3}-1}{3}(0)$$

$$A(S)=\frac{2\pi}{3}\left(\sqrt{26^3}-1\right)$$

This is the area of the surface.

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Topic: Finding surface area

Question: Find surface area.

The part of $x^2 + y^2 + z^2 = 4z$ inside $z = x^2 + y^2$

Answer choices:

A $A(S) = 4\pi$

B $A(S) = 2\pi$

C $A(S) = 8\pi$

D $A(S) = 12\pi$

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Solution: A

To find the area of some surface that's bounded by another function, we'll need to take the partial derivatives of the equation of the surface. But first, we'll solve the equation for z by completing the square with respect to z.

$$x^2 + y^2 + z^2 = 4z$$

$$x^2 + y^2 + z^2 - 4z = 0$$

$$x^2 + y^2 + z^2 - 4z + 4 = 0 + 4$$

$$x^2 + y^2 + (z-2)^2 = 4$$

Now we'll solve for z.

$$(z-2)^2 = 4 - x^2 - y^2$$

$$z - 2 = \pm\sqrt{4 - x^2 - y^2}$$

$$z = 2 \pm \sqrt{4 - x^2 - y^2}$$

We're looking for the part of this equation that lies inside $z = x^2 + y^2$. Since $z = x^2 + y^2$ is the paraboloid with vertex at the origin that opens up, we want to take the positive solution of $z = 2 \pm \sqrt{4 - x^2 - y^2}$, since the positive solution is the top part of that curve. We'll take partial derivatives of the curve.

$$z = 2 + \sqrt{4 - x^2 - y^2}$$

$$\frac{\partial z}{\partial x}=\frac{1}{2}\left(4-x^2-y^2\right)^{-\frac{1}{2}}(-2x)$$

$$\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{4-x^2-y^2}}$$

and

$$\frac{\partial z}{\partial y}=\frac{1}{2}\left(4-x^2-y^2\right)^{-\frac{1}{2}}(-2y)$$

$$\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{4-x^2-y^2}}$$

Since we're dealing with a double integral, we have to decide whether the region we're looking at is a type 1 or type 2 region. We're looking for the area inside $z=x^2+y^2$, which is a paraboloid. Since we could take uniform slices of the circle horizontally or vertically, we could treat the region as either type 1 or type 2, but we'll go ahead and treat it as a type 1 region. Therefore, we'll integrate first with respect to y, and second with respect to x.

$$A(S)=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$A(S)=\iint_D \sqrt{1+\left(-\frac{x}{\sqrt{4-x^2-y^2}}\right)^2+\left(-\frac{y}{\sqrt{4-x^2-y^2}}\right)^2}\ dy\ dx$$

$$A(S)=\iint_D \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\ dy\ dx$$

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$$A(S) = \iint_D \sqrt{\frac{4 - x^2 - y^2}{4 - x^2 - y^2} + \frac{x^2 + y^2}{4 - x^2 - y^2}} \ dy \ dx$$

$$A(S) = \iint_D \sqrt{\frac{4 - x^2 - y^2 + x^2 + y^2}{4 - x^2 - y^2}} \ dy \ dx$$

$$A(S) = \iint_D \sqrt{\frac{4}{4 - x^2 - y^2}} \ dy \ dx$$

To find the limits of integration, we'll solve $x^2 + y^2 + z^2 = 4z$ and $z = x^2 + y^2$ as a system of equations to see where they intersect. Since $z = x^2 + y^2$, we'll make a substitution into $x^2 + y^2 + z^2 = 4z$.

$$x^2 + y^2 + z^2 = 4z$$

$$z + z^2 = 4z$$

$$z^2 - 3z = 0$$

$$z(z - 3) = 0$$

$$z = 0, \ 3$$

If we plug these values for z back into $z = x^2 + y^2$, we get two equations:

$$x^2 + y^2 = 0$$

$$x^2 + y^2 = 3$$

The first is the equation of a circle with radius 0, which means it's actually the equation of a single point, and the second is the equation of a circle with radius $\sqrt{3}$. Since we've got equations of circles for the limits of integration, it'll be easier to represent the limits of integration in polar coordinates, so we'll convert the integral. Using the conversion formulas

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

we get

$$A(S) = \iint_D \sqrt{\frac{4}{4 - x^2 - y^2}}\ dy\ dx$$

$$A(S) = \iint_D \sqrt{\frac{4}{4 - \left(x^2 + y^2\right)}}\ dy\ dx$$

$$A(S) = \iint_D r\sqrt{\frac{4}{4 - r^2}}\ dr\ d\theta$$

Adding in the limits of integration gives

$$A(S) = \int_0^{2\pi} \int_0^{\sqrt{3}} r\sqrt{\frac{4}{4 - r^2}}\ dr\ d\theta$$

We'll use u-substitution to take the integral.

$$u = 4 - r^2$$

$$\frac{du}{dr} = -2r$$

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$$dr = -\frac{du}{2r}$$

Making substitutions into the integral gives

$$A(S) = \int_{0}^{2\pi} \int_{r=0}^{r=\sqrt{3}} r\sqrt{\frac{4}{u}}\left(-\frac{du}{2r}\right)\ d\theta$$

$$A(S) = \int_{0}^{2\pi} \int_{r=0}^{r=\sqrt{3}} -\frac{1}{2}\sqrt{\frac{4}{u}}\ du\ d\theta$$

$$A(S) = \int_{0}^{2\pi} \int_{r=0}^{r=\sqrt{3}} -\frac{1}{2}\left(2u^{-\frac{1}{2}}\right)\ du\ d\theta$$

$$A(S) = \int_{0}^{2\pi} \int_{r=0}^{r=\sqrt{3}} -u^{-\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u.

$$A(S) = \int_{0}^{2\pi} -2u^{\frac{1}{2}}\Big|_{r=0}^{r=\sqrt{3}}\ d\theta$$

Back-substitute.

$$A(S) = \int_{0}^{2\pi} -2\left(4-r^2\right)^{\frac{1}{2}}\Big|_{0}^{\sqrt{3}}\ d\theta$$

Evaluate over the interval and simplify.

$$A(S) = \int_{0}^{2\pi} -2\left[4-\left(\sqrt{3}\right)^2\right]^{\frac{1}{2}} + 2\left(4-0^2\right)^{\frac{1}{2}}\ d\theta$$

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$$A(S) = \int_0^{2\pi} -2\sqrt{1} + 2\sqrt{4}\ d\theta$$

$$A(S) = \int_0^{2\pi} -2 + 2(2)\ d\theta$$

$$A(S) = \int_0^{2\pi} 2\ d\theta$$

Integrate with respect to θ.

$$A(S) = 2\theta \Big|_0^{2\pi}$$

Evaluate over the interval and simplify.

$$A(S) = 2(2\pi) - 2(0)$$

$$A(S) = 4\pi$$

This is the area of the surface.

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Topic: Finding surface area

Question: Find surface area and give your answer as a decimal.

The surface $z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$ on $0 \leq x \leq 1$, $0 \leq y \leq 1$

Answer choices:

A $A(S) \approx 1.4087$

B $A(S) \approx 1.4066$

C $A(S) \approx 1.4213$

D $A(S) \approx 1.4136$

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Solution: B

To find the area of some surface that's bounded by another function, we'll need to take the partial derivatives of the equation of the surface.

$$z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$$

$$\frac{\partial z}{\partial x} = \frac{2}{3}\left(\frac{3}{2}x^{\frac{1}{2}}\right)$$

$$\frac{\partial z}{\partial x} = \sqrt{x}$$

and

$$\frac{\partial z}{\partial y} = \frac{2}{3}\left(\frac{3}{2}y^{\frac{1}{2}}\right)$$

$$\frac{\partial z}{\partial y} = \sqrt{y}$$

Since we're dealing with a double integral, we have to decide whether the region we're looking at is a type 1 or type 2 region. For the area we're looking at, we can take uniform slices horizontally or vertically, so we can treat the region as either type 1 or type 2, but we'll go ahead and treat it as a type 2 region. Therefore, we'll integrate first with respect to x, and second with respect to y.

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

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$$A(S)=\iint_D \sqrt{1+\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2}\ dx\ dy$$

$$A(S)=\iint_D \sqrt{1+x+y}\ dx\ dy$$

Applying the interval we were given, we get

$$A(S)=\int_0^1\int_0^1 \sqrt{1+x+y}\ dx\ dy$$

Integrate with respect to x, then evaluate over $[0,1]$.

$$A(S)=\int_0^1 \frac{2}{3}(1+x+y)^{\frac{3}{2}}\Big|_{x=0}^{x=1}\ dy$$

$$A(S)=\int_0^1 \frac{2}{3}(1+1+y)^{\frac{3}{2}}-\frac{2}{3}(1+0+y)^{\frac{3}{2}}\ dy$$

$$A(S)=\frac{2}{3}\int_0^1 (2+y)^{\frac{3}{2}}-(1+y)^{\frac{3}{2}}\ dy$$

Integrate with respect to y, then evaluate over $[0,1]$.

$$A(S)=\frac{2}{3}\left[\frac{2}{5}(2+y)^{\frac{5}{2}}-\frac{2}{5}(1+y)^{\frac{5}{2}}\right]\Bigg|_0^1$$

$$A(S)=\frac{4}{15}\left[(2+y)^{\frac{5}{2}}-(1+y)^{\frac{5}{2}}\right]\Bigg|_0^1$$

$$A(S)=\frac{4}{15}\left[(2+1)^{\frac{5}{2}}-(1+1)^{\frac{5}{2}}\right]-\frac{4}{15}\left[(2+0)^{\frac{5}{2}}-(1+0)^{\frac{5}{2}}\right]$$

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$$A(S) = \frac{4}{15}\left(3^{\frac{5}{2}} - 2^{\frac{5}{2}}\right) - \frac{4}{15}\left(2^{\frac{5}{2}} - 1^{\frac{5}{2}}\right)$$

$$A(S) = \frac{4}{15}\left(3^{\frac{5}{2}} - 2^{\frac{5}{2}} - 2^{\frac{5}{2}} + 1^{\frac{5}{2}}\right)$$

$$A(S) = \frac{4}{15}\left(\sqrt{3^5} - \sqrt{2^5} - \sqrt{2^5} + \sqrt{1^5}\right)$$

$$A(S) = \frac{4}{15}\left(\sqrt{3 \cdot 3 \cdot 3 \cdot 3 \cdot 3} - \sqrt{32} - \sqrt{32} + \sqrt{1}\right)$$

$$A(S) = \frac{4}{15}\left(9\sqrt{3} - 4\sqrt{2} - 4\sqrt{2} + 1\right)$$

$$A(S) = \frac{4}{15}\left(9\sqrt{3} - 8\sqrt{2} + 1\right)$$

$$A(S) \approx 1.4066$$

This is the area of the surface.

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