

Partial derivatives in two variables

By this point we've already learned how to find derivatives of single-variable functions. After learning derivative rules like power rule, product rule, quotient rule, chain rule and others, we're pretty comfortable handling the derivatives of functions like these:

$$f(x) = x^2 + 5$$

$$f(x) = \frac{(x^2 + 4)^3 \sin x}{x^4 + \ln 7x^4}$$

But now it's time to start talking about derivatives of multivariable functions, such as

$$f(x, y) = x^4y^3 + x^3y^2 + \ln xe^y$$

Finding derivatives of a multivariable function like this one may be less challenging than you think, because we're actually only going to take the derivative with respect to one variable at a time. For example, we'll take the derivative with respect to x while we treat y like it's a constant. Then we'll take another derivative of the original function, this one with respect to y , and we'll treat x as a constant.

In that way, we sort of reduce the problem to a single-variable derivative problem, which is a derivative we already know how to handle!

We call these kinds of derivatives “partial derivatives” because we're only taking the derivative of one part (variable) of the function at a time.



Remember the definition of the derivative from single-variable calculus (aka the difference quotient)? Let's adapt that definition so that it works for us for multivariable functions.

We know that, if z is a function defined in terms of x and y , like $z = f(x, y)$, then

The partial derivative of z with respect to x is

$$z_x = f_x(x, y) = \lim_{h \rightarrow 0} \frac{f(x + h, y) - f(x, y)}{h}$$

The partial derivative of z with respect to y is

$$z_y = f_y(x, y) = \lim_{h \rightarrow 0} \frac{f(x, y + h) - f(x, y)}{h}$$

The definition as we've written it here gives two different kinds of notation for the partial derivatives of z : z_x or z_y and $f_x(x, y)$ or $f_y(x, y)$. In fact, there are many ways you might see partial derivatives defined.

The partial derivatives of a function z defined in terms of x and y could be written in all of these ways:

The partial derivative of z with respect to x :

$$f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_x = z_x$$

The partial derivative of z with respect to y :

$$f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_y = z_y$$



Let's use what we've learned so far to work through an example using the difference quotient to find the partial derivatives of a multivariable function.

Example

Using the definition, find the partial derivatives of

$$f(x, y) = 2x^2y$$

For the partial derivative of z with respect to x , we'll substitute $x + h$ into the original function for x .

$$f(x + h, y) = 2(x + h)^2y$$

$$f(x + h, y) = 2(x^2 + 2xh + h^2)y$$

$$f(x + h, y) = 2x^2y + 4xhy + 2h^2y$$

Plugging our values of $f(x, y)$ and $f(x + h, y)$ into the definition, we get

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{2x^2y + 4xhy + 2h^2y - 2x^2y}{h}$$

$$f_x(x, y) = \lim_{h \rightarrow 0} \frac{4xhy + 2h^2y}{h}$$

$$f_x(x, y) = \lim_{h \rightarrow 0} 4xy + 2hy$$

$$f_x(x, y) = \lim_{h \rightarrow 0} 4xy + 2(0)y$$



$$f_x(x, y) = 4xy$$

For the partial derivative of z with respect to y , we'll substitute $y + h$ into the original function for y .

$$f(x, y + h) = 2x^2(y + h)$$

$$f(x, y + h) = 2x^2y + 2x^2h$$

Plugging our values of $f(x, y)$ and $f(x, y + h)$ into the definition, we get

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{2x^2y + 2x^2h - 2x^2y}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} \frac{2x^2h}{h}$$

$$f_y(x, y) = \lim_{h \rightarrow 0} 2x^2$$

$$f_y(x, y) = 2x^2$$

You'll remember from single-variable calculus that using the definition of the derivative was the “long way” that we learned to take the derivative before we learned the derivative rules that made the process faster. The good news is that we can apply all the same derivative rules to multivariable functions to avoid using the difference quotient! We just have to remember to work with only one variable at a time, treating all other variables as constants.



The next example shows how the power rule provides a faster way to find this function's partial derivatives.

Example

Using the power rule, find the partial derivatives of

$$f(x, y) = 2x^2y$$

For the partial derivative of z with respect to x , we treat y as a constant and use power rule to find the derivative.

$$f_x(x, y) = 2 \left(\frac{d}{dx} x^2 \right) y$$

$$f_x(x, y) = 2 (2x) y$$

$$f_x(x, y) = 4xy$$

For the partial derivative of z with respect to y , we treat x as a constant and use power rule to find the derivative.

$$f_y(x, y) = 2x^2 \left(\frac{d}{dy} y \right)$$

$$f_y(x, y) = 2x^2(1)$$

$$f_y(x, y) = 2x^2$$

