Calculus 3 Workbook Solutions

Lagrange multipliers

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TWO DIMENSIONS, ONE CONSTRAINT

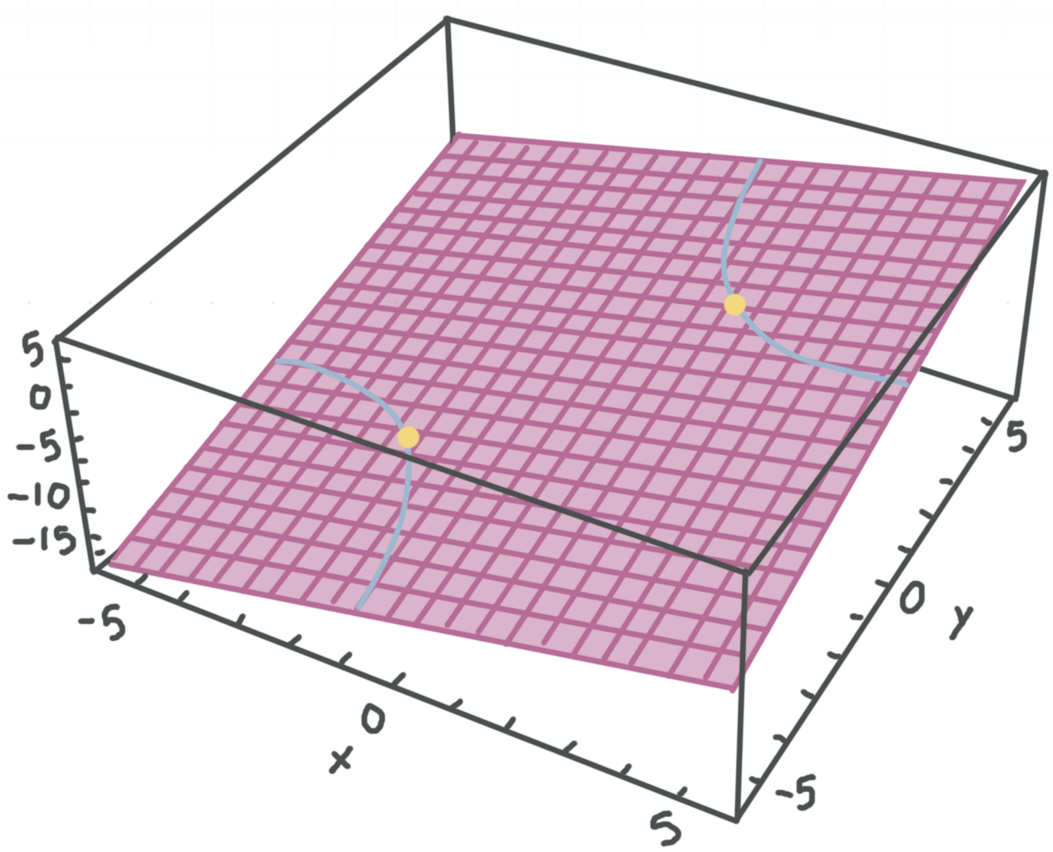
■ 1. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = x + y - 5$$

$$xy = 1$$

Solution:

A sketch of the surface is



Let

$$g(x, y) = xy - 1$$

Create a system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 1 \qquad \frac{\partial g}{\partial x} = y$$

$$\frac{\partial f}{\partial y} = 1 \qquad \frac{\partial g}{\partial y} = x$$

Substitute partial derivatives into the system.

$$1 = y\lambda$$

$$1 = x\lambda$$

Solve the system for λ.

$$x = y = \frac{1}{\lambda} \text{ with } x \neq 0, y \neq 0$$

Plug x into the constraint equation.

$$(y)y = 1$$

$$y^2 = 1$$

$$y = \pm 1$$

Since $x = y$, the solutions are $(1,1,\lambda = 1)$, $(-1,-1,\lambda = -1)$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where $L(x,y) = f(x,y) - \lambda g(x,y)$. The second derivative test for $(1,1)$ with $\lambda = 1$.

$$L(x,y) = f(x,y) - (1)g(x,y)$$

$$L(x,y) = x + y - 5 - (xy - 1)$$

$$L(x,y) = x + y - xy - 4$$

Find second order partial derivatives.

$$L_{xx} = 0$$

$$L_{yy} = 0$$

$$L_{xy} = L_{yx} = -1$$

Substitute $(1,1)$.

$$g_x(1,1) = 1$$

$$g_y(1,1) = 1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix}$$

Then $det(HL) = -2 < 0$. Since $det(HL) < 0$ at $(1,1)$, it's a local minimum, and

$$f(1,1) = (1) + (1) - 5 = -3$$

The second derivative test for $(-1, -1)$ with $\lambda = -1$.

$$L(x, y) = f(x, y) - (-1)g(x, y)$$

$$L(x, y) = x + y - 5 + (xy - 1)$$

$$L(x, y) = x + y + xy - 6$$

Find second order partial derivatives.

$$L_{xx} = 0$$

$$L_{yy} = 0$$

$$L_{xy} = L_{yx} = 1$$

Substitute $(-1, -1)$.

$$g_x(-1, -1) = -1$$

$$g_y(-1, -1) = -1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Then $det(HL) = 2 > 0$. Since $det(HL) > 0$ at $(-1, -1)$, it's a local maximum, and

$$f(-1, -1) = (-1) + (-1) - 5 = -7$$

Then the extrema of the function are

A local maximum of -7 at $(-1, -1)$

A local minimum of -3 at $(1,1)$

■ 2. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = e^{2x+y}$$

$$x^2 + y^2 = 5$$

Solution:

Let

$$g(x, y) = x^2 + y^2 - 5$$

Create a system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 2e^{2x+y} \qquad \frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = e^{2x+y} \qquad \frac{\partial g}{\partial y} = 2y$$

Substitute partial derivatives into the system.

$$2e^{2x+y} = 2x\lambda$$

$$e^{2x+y} = 2y\lambda$$

Solve the system for λ.

$$\frac{e^{2x+y}}{x} = \lambda$$

$$\frac{e^{2x+y}}{2y} = \lambda$$

$$\frac{e^{2x+y}}{x} = \frac{e^{2x+y}}{2y}$$

Since $e^{2x+y} > 0$, we can say $x = 2y$. Plug x into the constraint equation.

$$(2y)^2 + y^2 = 5$$

$$4y^2 + y^2 = 5$$

$$y^2 = 1$$

$$y = \pm 1$$

Substitute to find x.

$$x^2 + (\pm 1)^2 = 5$$

$$x^2 = 4$$

$$x = \pm 2$$

Since $x = 2y$, the solutions to the system are $(2,1)$ and $(-2,-1)$. Calculate λ for each solution pair.

$$\lambda(2,1) = \frac{e^{2(2)+(1)}}{2} = \frac{e^5}{2}$$

$$\lambda(-2,-1) = \frac{e^{2(-2)+(-1)}}{-2} = -\frac{e^{-5}}{2}$$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where $L(x,y) = f(x,y) - \lambda g(x,y)$. The second derivative test for $(2,1)$ with $\lambda = e^5/2$.

$$L(x,y) = f(x,y) - \frac{e^5}{2}g(x,y)$$

$$L(x,y) = e^{2x+y} - \frac{e^5}{2}(x^2 + y^2 - 5)$$

Find second order partial derivatives.

$$L_{xx} = 4e^{2x+y} - e^5$$

$$L_{yy} = e^{2x+y} - e^5$$

$$L_{xy} = L_{yx} = 2e^{2x+y}$$

Then at $(2,1)$,

$$g_x(2,1) = 2(2) = 4$$

$$g_y(2,1) = 2(1) = 2$$

$$L_{xx}(2,1) = 4e^{2(2)+(1)} - e^5 = 3e^5$$

$$L_{yy}(2,1) = e^{2(2)+(1)} - e^5 = 0$$

$$L_{xy}(2,1) = L_{yx}(2,1) = 2e^{2(2)+(1)} = 2e^5$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -4 & -2 \\ -4 & 3e^5 & 2e^5 \\ -2 & 2e^5 & 0 \end{vmatrix}$$

Then $det(HL) = 20e^5 > 0$. Since $det(HL) > 0$ at $(2,1)$, it's a local maximum, and $f(2,1) = e^5$.

The second derivative test for $(-2, -1)$ with $\lambda = -e^{-5}/2$,

$$L(x, y) = f(x, y) + \frac{e^{-5}}{2}g(x, y)$$

$$L(x,y) = e^{2x+y} + \frac{e^{-5}}{2}(x^2 + y^2 - 5)$$

Find second order partial derivatives.

$$L_{xx} = 4e^{2x+y} + e^{-5}$$

$$L_{yy} = e^{2x+y} + e^{-5}$$

$$L_{xy} = L_{yx} = 2e^{2x+y}$$

Substitute $(-2, -1)$.

$$g_x(-2, -1) = 2(2) = 4$$

$$g_y(-2, -1) = 2(1) = 2$$

$$L_{xx}(-2, -1) = 4e^{2(-2)+(-1)} + e^{-5} = 5e^{-5}$$

$$L_{yy}(-2, -1) = e^{2(-2)+(-1)} + e^{-5} = 2e^{-5}$$

$$L_{xy}(-2, -1) = L_{yx}(-2, -1) = 2e^{2(-2)+(-1)} = 2e^{-5}$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -4 & -2 \\ -4 & 5e^{-5} & 2e^{-5} \\ -2 & 2e^{-5} & 2e^{-5} \end{vmatrix}$$

Then $det(HL) = -20e^{-5} < 0$. Since $det(HL) < 0$ at $(-2, -1)$, it's a local minimum, and $f(-2, -1) = e^{-5}$.

Since the restriction curve is a closed circle, the function is continuous, and has one local minimum and one local maximum, and the local minima and maxima are global extrema.

The extrema of the function are

A global maximum of e^5 at $(2,1)$

A global minimum of e^{-5} at $(-2,-1)$

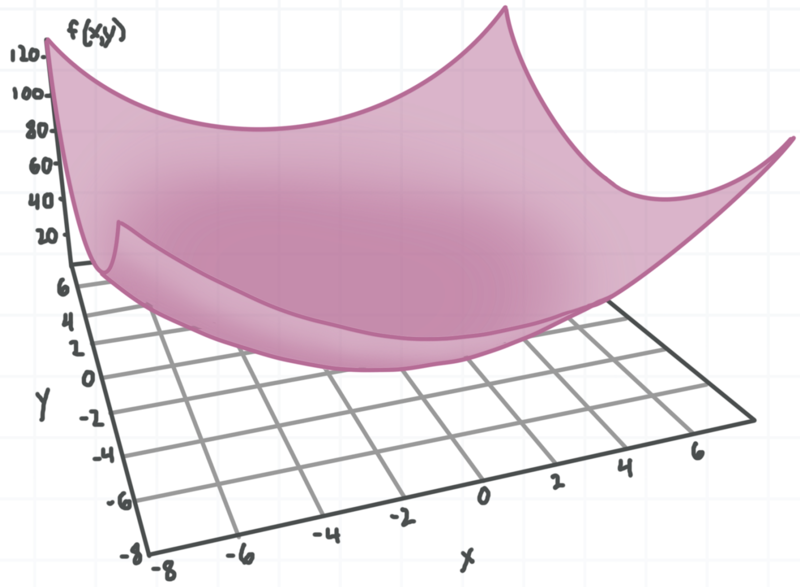
■ 3. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = x^2 + y^2 + 3$$

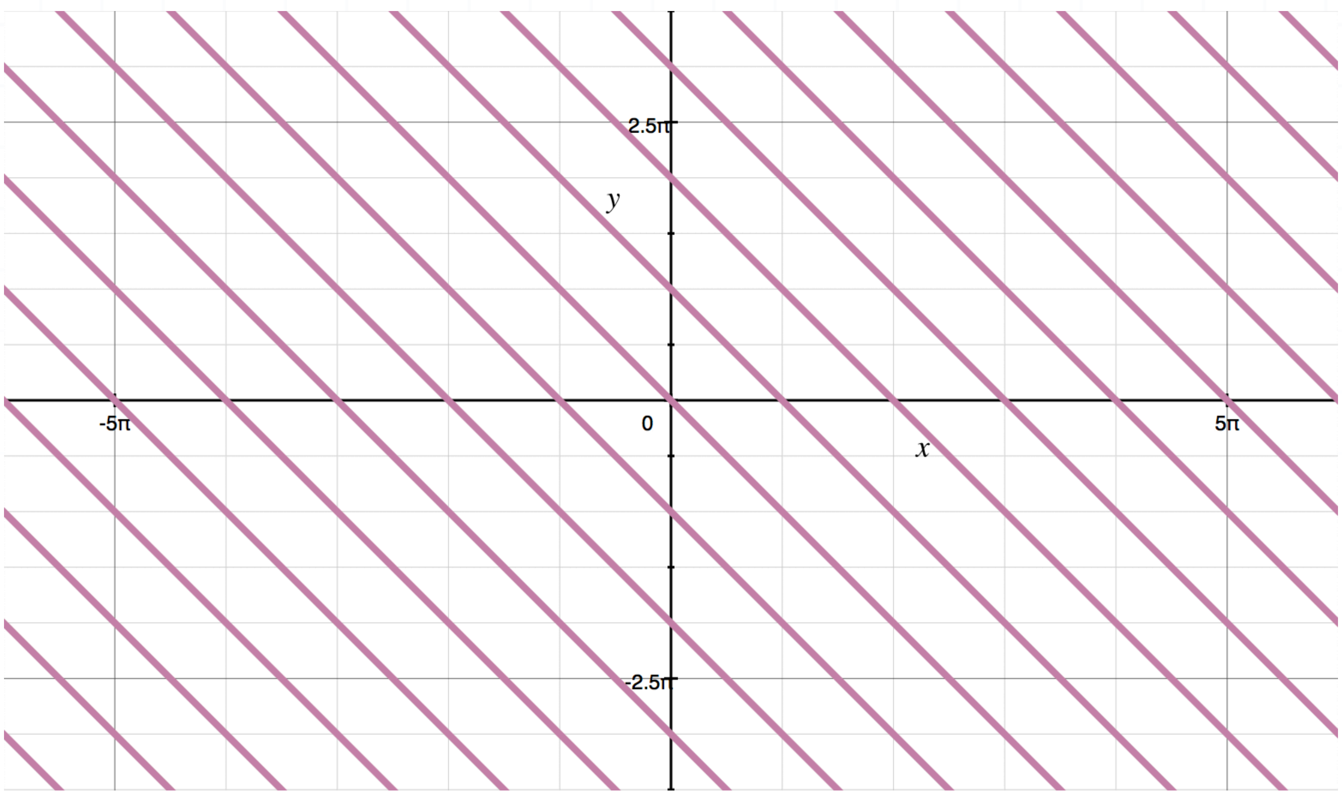
$$\sin(x + y) = 0$$

Solution:

A sketch of the surface is



And it’s subject to the constraint



Let

$$g(x, y) = \sin(x + y)$$

Create a system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial g}{\partial x} = \cos(x+y)$$

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial g}{\partial y} = \cos(x+y)$$

Substitute partial derivatives into the system.

$$2x = \lambda \cos(x+y)$$

$$2y = \lambda \cos(x+y)$$

Solve the system for λ. If $\cos(x+y)=0$, then $x=0$ and $y=0$, but $\cos(x+y)=\cos(0)=1$, so $\cos(x+y)$ can't be 0. Since $\cos(x+y) \neq 0$,

$$\frac{2x}{\cos(x+y)} = \lambda$$

$$\frac{2y}{\cos(x+y)} = \lambda$$

$$\frac{2x}{\cos(x+y)} = \frac{2y}{\cos(x+y)}$$

So $x=y$. Plug x into the constraint equation.

$$\sin(y+y) = 0$$

$$\sin(2y) = 0$$

We get

$$2y = \pi n, \text{ where } n \text{ is any integer}$$

$$y = 0.5\pi n, \text{ where } n \text{ is any integer}$$

Since $x = y$, $x = 0.5\pi n$, where n is any integer, the solution to the system is $(0.5\pi n, 0.5\pi n)$, where n is any integer number.

Calculate λ for the solution.

$$2(0.5\pi n) = \lambda \cos(0.5\pi n + 0.5\pi n)$$

$$\pi n = \lambda \cos(\pi n)$$

Which gives

$$\lambda = \pm \pi n, \text{ where } n \text{ is any integer}$$

$$\lambda = \pi n, \text{ where } n \text{ is an even integer}$$

$$\lambda = -\pi n, \text{ where } n \text{ is an odd integer}$$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where $L(x, y) = f(x, y) - \lambda g(x, y)$.

$$L(x, y) = f(x, y) - (\pm \pi n)g(x, y)$$

$$L(x, y) = x^2 + y^2 + 3 - (\pm \pi n)\sin(x + y)$$

Find second order partial derivatives.

$$L_{xx} = 2 + (\pm \pi n)\sin(x + y)$$

$$L_{yy} = 2 + (\pm \pi n)\sin(x + y)$$

$$L_{xy} = L_{yx} = (\pm \pi n)\sin(x + y)$$

Substitute $(0.5\pi n, 0.5\pi n)$.

$$g_x(0.5\pi n, 0.5\pi n) = \cos(0.5\pi n + 0.5\pi n) = \pm 1(\text{the same sign as } \lambda)$$

$$g_y(0.5\pi n, 0.5\pi n) = \cos(0.5\pi n + 0.5\pi n) = \pm 1(\text{the same sign as } \lambda)$$

$$L_{xx}(0.5\pi n, 0.5\pi n) = 2 + (\pm \pi n)\sin(0.5\pi n + 0.5\pi n) = 2$$

$$L_{yy}(0.5\pi n, 0.5\pi n) = 2 + (\pm \pi n)\sin(0.5\pi n + 0.5\pi n) = 2$$

$$L_{xy}(0.5\pi n, 0.5\pi n) = L_{xy}(0.5\pi n, 0.5\pi n) = (\pm \pi n)\sin(0.5\pi n + 0.5\pi n) = 0$$

Form the bordered Hessian matrix for even n.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

Then $det(HL) = -4 < 0$. Form the bordered Hessian matrix for even n.

$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

Then $det(HL) = -4 < 0$. Since $det(HL) < 0$ at $(0.5\pi n, 0.5\pi n)$, all of them are local minima, and

$$f(0.5\pi n, 0.5\pi n) = (0.5\pi n)^2 + (0.5\pi n)^2 + 3 = \frac{\pi^2 n^2}{2} + 3$$

where n is any integer number.

So the extremum of the function is

A local minima of $\frac{\pi^2 n^2}{2} + 3$ at $(0.5\pi n, 0.5\pi n)$, where n is any integer number.

■ 4. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = \ln \frac{2x - 4}{y^2}$$

$$4x + 8y - 15 = 0$$

Solution:

Expand the logarithm to simplify $f(x, y)$.

$$f(x, y) = \ln(2x - 4) - 2\ln y$$

Let

$$g(x, y) = 4x + 8y - 15$$

Create the system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2}{2x - 4} = \frac{1}{x - 2}$$

$$\frac{\partial g}{\partial x} = 4$$

$$\frac{\partial f}{\partial y} = -\frac{2}{y}$$

$$\frac{\partial g}{\partial y} = 8$$

Substitute partial derivatives into the system of equations.

$$\frac{1}{x - 2} = 4\lambda$$

$$-\frac{2}{y} = 8\lambda$$

Solve the system for λ.

$$\frac{1}{4(x - 2)} = \lambda$$

$$-\frac{1}{4y} = \lambda$$

$$\frac{1}{4(x-2)} = -\frac{1}{4y}$$

$$\frac{1}{x-2} = -\frac{1}{y}$$

$$-y = x - 2$$

$y = -x + 2$ for $x \neq 2$ and $y \neq 0$

Plug y into the constraint equation.

$$4x + 8(-x + 2) - 15 = 0$$

$$-4x + 16 - 15 = 0$$

$$x = \frac{1}{4}$$

$$y = -\frac{1}{4} + 2 = \frac{7}{4}$$

So the solution is $x = 1/4$, $y = 7/4$.

Check if the point lies within the domain of $f(x, y)$.

$$\frac{2(0.25) - 4}{(1.75)^2} = -\frac{8}{7} < 0$$

Since there's no critical point inside the domain of $f(x, y)$, it has no extrema.

THREE DIMENSIONS, ONE CONSTRAINT

■ 1. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x, y, z) = x^5 - 160y + 160z$$

$$x + y^2 + z^2 = 0$$

Solution:

Let $g(x, y, z) = x + y^2 + z^2$, and create a system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = 5x^4 \qquad g_x = 1$$

$$f_y = -160 \qquad g_y = 2y$$

$$f_z = 160 \qquad g_z = 2z$$

Substitute partial derivatives into the system.

$$5x^4 = \lambda$$

$$-160 = 2y\lambda$$

$$160 = 2z\lambda$$

Solve the system for λ.

$$5x^4 = \lambda$$

$$-80 = y\lambda$$

$$80 = z\lambda$$

So $x \neq 0$.

$$y = -\frac{80}{\lambda} = -\frac{80}{5x^4} = -\frac{16}{x^4}$$

$$z = \frac{80}{\lambda} = \frac{80}{5x^4} = \frac{16}{x^4}$$

Plug these into the constraint equation.

$$x + \left(-\frac{16}{x^4}\right)^2 + \left(\frac{16}{x^4}\right)^2 = 0$$

$$x + 2 \cdot \frac{256}{x^8} = 0$$

$$x^9 = -512$$

$$x = -2$$

Calculate y, z, and λ.

$$y = -\frac{16}{x^4} = -\frac{16}{(-2)^4} = -1$$

$$z = \frac{16}{x^4} = \frac{16}{(-2)^4} = 1$$

$$\lambda = 5x^4 = 5(-2)^4 = 80$$

So the solution to the system is $(-2, -1, 1)$ with $\lambda = 80$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda\, g(x, y, z)$. The second derivative test for $(-2, -1, 1)$ with $\lambda = 80$.

$$L(x, y, z) = f(x, y, z) - 80g(x, y, z)$$

$$L(x, y, z) = x^5 - 160y + 160z - 80(x + y^2 + z^2)$$

Calculate the second order partial derivatives and substitute $(-2, -1, 1)$.

$$L_{xx} = 20(-2)^3 = -160, \ L_{yy} = -160, \ L_{zz} = -160$$

$$L_{xy} = L_{yx} = 0, \ L_{yz} = L_{zy} = 0, \ L_{xz} = L_{zx} = 0$$

$$g_x = 1, \ g_y = 2(-1) = -2, \ g_z = 2(1) = 2$$

Form the bordered Hessian matrix H_4.

$$\begin{vmatrix} 0 & -1 & 2 & -2 \\ -1 & -160 & 0 & 0 \\ 2 & 0 & -160 & 0 \\ -2 & 0 & 0 & -160 \end{vmatrix}$$

Then $det(H_4) = -230{,}400 < 0$. Since $det(H_4) < 0$ at $(-2, -1, 1)$, it's a local extremum.

$$f(-2, -1, 1) = (-2)^5 - 160(-1) + 160(1) = 288$$

To check if it's local minimum or maximum, form the Hessian submatrix H_3.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

$$\begin{vmatrix} 0 & -1 & 2 \\ -1 & -160 & 0 \\ 2 & 0 & -160 \end{vmatrix}$$

Then $det(H_3) = 800 > 0$. Since $det(H_3) > 0$ at $(-2, -1, 1)$, it's a local maximum.

So the function has a local maximum of 288 at $(-2, -1, 1)$.

■ 2. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y, z) = x + 2y^2 - 3z^2 - 4$$

$$e^x + y - 3z = -\frac{1}{4}$$

Solution:

Let $g(x, y, z) = e^x + y - 3z + 1/4$, and create a system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = 1 \qquad g_x = e^x$$

$$f_y = 4y \qquad g_y = 1$$

$$f_z = -6z \qquad g_z = -3$$

Substitute partial derivatives into the system of equations.

$$1 = e^x \lambda$$

$$4y = \lambda$$

$$-6z = -3\lambda$$

Solve the system.

$$6z = 3\lambda = 3(4y)$$

$$z = 2y$$

$$1 = e^x\lambda = 4ye^x$$

$$\frac{1}{4e^x} = y$$

$$z = \frac{1}{2e^x}$$

Plug these values for y and z into the constraint equation.

$$e^x + \frac{1}{4e^x} - 3\frac{1}{2e^x} + \frac{1}{4} = 0$$

$$e^x - \frac{5}{4e^x} + \frac{1}{4} = 0$$

$$4(e^x)^2 + e^x - 5 = 0$$

$$(e^x - 1)(4e^x + 5) = 0$$

Since $4e^x + 5 > 0$, the only solution is

$$e^x - 1 = 0$$

$$e^x = 1$$

$$x = 0$$

Calculate y, x, λ.

$$y = \frac{1}{4e^0} = 0.25$$

$$z = \frac{1}{2e^0} = 0.5$$

$$\lambda = 4y = 4(0.25) = 1$$

So the solution is $(0,0.25,0.5)$ with $\lambda = 1$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z)$.

$$L(x, y, z) = f(x, y, z) - g(x, y, z)$$

$$L(x, y, z) = x + 2y^2 - 3z^2 - 4 - (e^x + y - 3z + 0.25)$$

$$L(x, y, z) = x + 2y^2 - 3z^2 - e^x - y + 3z - 4.25)$$

Calculate the second order partial derivatives and substitute $(0,0.25,0.5)$.

$$L_{xx} = -e^x = -e^0 = -1,\ L_{yy} = 4,\ L_{zz} = -6$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = e^0 = 1,\ g_y = 1,\ g_z = -3$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -1 & -1 & 3 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 3 & 0 & 0 & -6 \end{vmatrix}$$

Then $det(HL) = 54 > 0$. Since $det(HL) > 0$ at $(0,0.5,0.25)$, it's a saddle point.

So the function has no extrema.

■ 3. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x, y, z) = |x^3y^7z^5|$$

$$3x + 7y + 5z = 60$$

Solution:

Since $f(x, y, z) \geq 0$, and $f(x, y, z) = 0$ when $x = 0$, or $y = 0$, or $z = 0$, all these points are global minima by the definition.

Consider the points where $x \neq 0$, $y \neq 0$, and $z \neq 0$. To simplify the calculations, introduce the helper function

$$F(x, y, z) = \ln(f(x, y, z)) = \ln|x^3y^7z^5|$$

$$F(x, y, z) = 3\ln|x| + 7\ln|y| + 5\ln|z|$$

Since the logarithm is a monotonic increasing function, $\ln f$ has the local extrema at the same points as the initial function f.

Let $g(x, y, z) = 3x + 7y + 5z - 60$, then create the system of simultaneous equations.

$$F_x = \lambda g_x$$

$$F_y = \lambda g_y$$

$$F_z = \lambda g_z$$

Calculate first order partial derivatives.

$$F_x = \frac{3}{x} \qquad g_x = 3$$

$$F_y = \frac{7}{y} \qquad g_y = 7$$

$$F_z = \frac{5}{z} \qquad g_z = 5$$

Substitute partial derivatives into the system of equations.

$$\frac{3}{x} = 3\lambda$$

$$\frac{7}{y} = 7\lambda$$

$$\frac{5}{z} = 5\lambda$$

Solve the system for λ.

$$\frac{1}{x} = \lambda$$

$$\frac{1}{y} = \lambda$$

$$\frac{1}{z} = \lambda$$

$$x = y = z$$

Plug in $y = x$ and $z = x$ into the constraint equation.

$$3x + 7(x) + 5(x) - 60 = 0$$

$$15x - 60 = 0$$

$$x = 4$$

Calculate y, x, and λ.

$$y = z = 4$$

$$\frac{1}{(4)} = \lambda$$

$$\lambda = 0.25$$

The solution to the system is $(4,4,4)$ with $\lambda = 0.25$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = F(x, y, z) - \lambda g(x, y, z)$. The second derivative test for the point $(4,4,4)$ with $\lambda = 0.25$ is

$$L(x, y, z) = F(x, y, z) - 0.25g(x, y, z)$$

$$L(x, y, z) = 3\ln|x| + 7\ln|y| + 5\ln|z| - 0.25(3x + 7y + 5z - 60)$$

Find second order partial derivatives at $(4,4,4)$.

$$L_{xx} = -\frac{3}{(4)^2} = -\frac{3}{16}, \ L_{yy} = -\frac{7}{(4)^2} = -\frac{7}{16}, \ L_{zz} = -\frac{5}{(4)^2} = -\frac{5}{16}$$

$$L_{xy} = L_{yx} = 0, \ L_{yz} = L_{zy} = 0, \ L_{xz} = L_{zx} = 0$$

$$g_x = 3, \ g_y = 7, \ g_z = 5$$

Form the bordered Hessian matrix H_4.

$$\begin{vmatrix} 0 & -3 & -7 & -5 \\ -3 & -\frac{3}{16} & 0 & 0 \\ -7 & 0 & -\frac{7}{16} & 0 \\ -5 & 0 & 0 & -\frac{5}{16} \end{vmatrix}$$

Then $det(H_4) \approx -6.15 < 0$. Since $det(H_4) < 0$ at $(4,4,4)$, it's a local extremum, and

$$f(4,4,4) = |(4)^3(4)^5(4)^7| = 4^{15} = 1,073,741,824$$

To check if it's a local minimum or maximum, form the Hessian submatrix H_3.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

$$\begin{vmatrix} 0 & -3 & -7 \\ -3 & -\frac{3}{16} & 0 \\ -7 & 0 & -\frac{7}{16} \end{vmatrix}$$

Then $det(H_3) = 13.125 > 0$. Since $det(H_3) > 0$ at $(4,4,4)$, it's a local maximum.

The extrema of the function are

A local maximum of $1,073,741,824$ at $(4,4,4)$

Local (and also global) minima of all points where $x = 0$, or $y = 0$, or $z = 0$.

■ 4. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x, y, z) = x + yz + 2y$$

$$y^2 + xyz = 2$$

Solution:

Let $g(x, y, z) = y^2 + xyz - 2$, then create a system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = 1 \qquad g_x = yz$$

$$f_y = z + 2 \qquad g_y = 2y + xz$$

$$f_z = y \qquad g_z = xy$$

Substitute partial derivatives into the system.

$$1 = yz\lambda$$

$$z + 2 = (2y + xz)\lambda$$

$$y = xy\lambda$$

Solve the system for λ. Since $y \neq 0$ (or else $1 = (0)z\lambda$, which is impossible),

$$1 = x\lambda$$

$$\frac{1}{x} = \lambda$$

Plug this into the second equation.

$$z + 2 = (2y + xz)\frac{1}{x}$$

$$x(z + 2) = 2y + xz$$

$$2x = 2y$$

$$x = y$$

Substitute into the first equation.

$$1 = yz\frac{1}{x}$$

$$x = yz$$

Since $x = y$ and $x = yz$, then $1 = z$. Plug in $z = 1$ and $y = x$ into the constraint equation.

$$(x)^2 + x(x)(1) = 2$$

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = \pm 1$$

Calculate y and λ.

$$y = x = \pm 1$$

$$\lambda = \frac{1}{\pm 1} = \pm 1$$

The solutions are

$(1,1,1)$ with $\lambda = 1$

$(-1,-1,1)$ with $\lambda = -1$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z)$. The second derivative test for $(1,1,1)$ with $\lambda = 1$:

$$L(x, y, z) = f(x, y, z) - 1 \cdot g(x, y, z)$$

$$L(x, y, z) = x + yz + 2y - (y^2 + xyz - 2)$$

$$L(x, y, z) = x + yz + 2y - y^2 - xyz + 2$$

Find second order partial derivatives at $(1,1,1)$.

$$L_{xx} = 0,\ L_{yy} = -2,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = -z = -1,\ L_{yz} = L_{zy} = 1 - x = 0,\ L_{xz} = L_{zx} = -y = -1$$

$$g_x = yz = (1)(1) = 1,\ g_y = 2y + xz = 2(1) + (1)(1) = 3,\ g_z = xy = (1)(1) = 1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -1 & -3 & -1 \\ -1 & 0 & -1 & -1 \\ -3 & -1 & -2 & 0 \\ -1 & -1 & 0 & 0 \end{vmatrix}$$

Then $det(HL) = 8 > 0$. Since $det(HL) > 0$ at $(1,1,1)$, it's a saddle point.

The second derivative test for $(-1,-1,1)$ with $\lambda = -1$.

$$L(x, y, z) = f(x, y, z) + 1 \cdot g(x, y, z)$$

$$L(x, y, z) = x + yz + 2y + (y^2 + xyz - 2)$$

$$L(x, y, z) = x + yz + 2y + y^2 + xyz - 2$$

Find second order partial derivatives at $(-1,-1,1)$.

$$L_{xx} = 0,\ L_{yy} = 2,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = z = 1,\ L_{yz} = L_{zy} = x + 1 = 0,\ L_{xz} = L_{zx} = y = -1$$

$$g_x = yz = (-1)(1) = -1$$

$$g_y = 2y + xz = 2(-1) + (-1)(1) = -3$$

$$g_z = xy = (-1)(-1) = 1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 1 & 3 & -1 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{vmatrix}$$

Then $det(HL) = 8 > 0$. Since $det(HL) > 0$ at $(-1, -1, 1)$, it's a saddle point. So the function has no extrema.

■ 5. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y, z) = \ln \frac{xy + 3y}{z - 2}$$

$$x + y + 3z = 6$$

Solution:

Rewrite $f(x, y, z)$ as

$$f(x, y, z) = \ln \frac{(x + 3)y}{z - 2}$$

$$f(x, y, z) = \ln(x + 3) + \ln y - \ln(z - 2)$$

And let

$$g(x, y, z) = x + y + 3z - 6 = 0$$

Create the system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = \frac{1}{x+3} \qquad g_x = 1$$

$$f_y = \frac{1}{y} \qquad g_y = 1$$

$$f_z = -\frac{1}{z-2} \qquad g_z = 3$$

Substitute partial derivatives into the system of equations.

$$\frac{1}{x+3} = \lambda$$

$$\frac{1}{y} = \lambda$$

$$-\frac{1}{z-2} = 3\lambda$$

Solve the system for λ.

$$y = \frac{1}{\lambda}$$

$$x + 3 = \frac{1}{\lambda}, \ x = \frac{1}{\lambda} - 3 = y - 3$$

$$z - 2 = -\frac{1}{3\lambda}, \ z = -\frac{1}{3\lambda} + 2 = -\frac{1}{3}y + 2$$

Plug these values for x and z into the constraint equation.

$$y - 3 + y + 3\left(-\frac{1}{3}y + 2\right) = 6$$

$$2y - 3 - y + 6 = 6$$

$$y - 3 = 0$$

$$y = 3$$

Then

$$x = 3 - 3 = 0$$

$$z = -\frac{1}{3} \cdot 3 + 2 = 1$$

The solution to the system is $x = 0$, $y = 3$, and $z = 1$. Check to see if the point lies within the domain of $f(x, y, z)$.

$$\frac{0 \cdot 3 + 3 \cdot 3}{1 - 2} = -9 < 0$$

Since there is no critical point inside the domain of $f(x, y, z)$, it has no extrema.

■ 6. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x, y, z) = \sin^2 x \cdot \sin 2y \cdot \sin z$$

$$2x + 2y + z = \frac{2}{3}\pi \text{ with } x > 0,\ y > 0,\ z > 0$$

Solution:

To simplify the calculations, introduce the helper function.

$$F(x, y, z) = \ln(f(x, y, z))$$

$$F(x, y, z) = \ln(\sin^2 x \cdot \sin 2y \cdot \sin z)$$

$$F(x, y, z) = 2\ln(\sin x) + \ln(\sin 2y) + \ln(\sin z)$$

Since the logarithm is a monotonic increasing function, $\ln f$ has the local extrema at the same points as the initial function f.

And let

$$g(x, y, z) = 2x + 2y + z - \frac{2}{3}\pi$$

Build a system of equations.

$$F_x = \lambda g_x$$

$$F_y = \lambda g_y$$

$$F_z = \lambda g_z$$

Find first order partial derivatives.

$$F_x = 2\frac{\cos x}{\sin x} = 2\cot x \qquad g_x = 2$$

$$F_y = 2\frac{\cos 2y}{\sin 2y} = 2\cot 2y \qquad g_y = 2$$

$$F_z = \frac{\cos zx}{\sin z} = \cot z \qquad g_z = 1$$

Substitute partial derivatives into the system.

$$2\cot x = 2\lambda$$

$$2\cot 2y = 2\lambda$$

$$\cot z = \lambda$$

Solve the system for λ.

$$\cot x = \lambda$$

$$\cot 2y = \lambda$$

$$\cot z = \lambda$$

So $\cot x = \cot 2y = \cot z$. Since $x > 0$, $y > 0$, $z > 0$, and $2x + 2y + z = (2/3)\pi$.

$$0 < x < \frac{2}{3}\pi$$

$$0 < 2y < \frac{2}{3}\pi$$

$$0 < z < \frac{2}{3}\pi$$

Since $\cot x$ is a one to one function on $(0,(2/3)\pi)$ if $\cot x = \cot 2y = \cot z$, then $x = 2y = z$. Plug $x = 2y$ and $z = 2y$ into the constraint equation.

$$2(2y) + 2y + 2y = \frac{2}{3}\pi$$

$$8y = \frac{2}{3}\pi$$

$$y = \frac{\pi}{12}$$

Calculate x, z, λ.

$$x = z = 2 \cdot \frac{\pi}{12} = \frac{\pi}{6}$$

$$\lambda = \cot\frac{\pi}{6} = \sqrt{3}$$

The solution to the system is

$$\left(\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{6}\right) \text{ with } \lambda = \sqrt{3}$$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = F(x, y, z) - \lambda g(x, y, z)$. The second derivative test for $(\pi/6,\pi/12,\pi/6)$ with $\lambda = \sqrt{3}$.

$$L(x, y, z) = F(x, y, z) - \sqrt{3}g(x, y, z)$$

$$L(x, y, z) = 2\ln(\sin x) + \ln(\sin 2y) + \ln(\sin z) - \sqrt{3}\left(2x + 2y + z - \frac{2}{3}\pi\right)$$

Find second order partial derivatives at $(\pi/6,\pi/12,\pi/6)$.

$$L_{xx} = -2\csc^2\frac{\pi}{6} = -8$$

$$L_{yy} = -4\csc^2\left(2\cdot\frac{\pi}{12}\right) = -16$$

$$L_{zz} = -\csc^2\frac{\pi}{6} = -4$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 2,\ g_y = 2,\ g_z = 1$$

Form the bordered Hessian matrix H_4.

$$\begin{vmatrix} 0 & -2 & -2 & -1 \\ -2 & -8 & 0 & 0 \\ -2 & 0 & -16 & 0 \\ -1 & 0 & 0 & -4 \end{vmatrix}$$

Then $det(H_4) = -512 < 0$. Since $det(H_4) < 0$ at $(\pi/6,\pi/12,\pi/6)$, it's a local extremum,

$$f\left(\frac{\pi}{6},\frac{\pi}{12},\frac{\pi}{6}\right) = \sin^2\frac{\pi}{6}\cdot\sin\left(2\cdot\frac{\pi}{12}\right)\cdot\sin\frac{\pi}{6}$$

$$f\left(\frac{\pi}{6},\frac{\pi}{12},\frac{\pi}{6}\right) = \sin^4\frac{\pi}{6}$$

$$f\left(\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{6}\right) = \frac{1}{16}$$

To check if it's a local minimum or maximum, form the Hessian submatrix H_3.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

$$\begin{vmatrix} 0 & -2 & -2 \\ -2 & -8 & 0 \\ -2 & 0 & -16 \end{vmatrix}$$

Then $det(H_3) = 96 > 0$. Since $det(H_3) > 0$ at $(\pi/6, \pi/12, \pi/6)$, it's a local maximum. So the function has a local maximum of $1/16$ at $(\pi/6, \pi/12, \pi/6)$.

THREE DIMENSIONS, TWO CONSTRAINTS

■ 1. Use Lagrange multipliers to find the shortest distance from the vertex of the elliptic paraboloid $(x-2)^2+2(y+1)^2=3z+6$ to the line that's the intersection of the planes $x+3y+5z=18$ and $3x+5y+z=28$.

Solution:

The vertex of the elliptic paraboloid $(x-2)^2+2(y+1)^2=3(z+2)$ is the point $(2,-1,-2)$.

The square of the distance has a minimum at the same point as the distance itself. So let's minimize the square of the distance from $(2,-1,-2)$. The function to be minimized is $f(x,y,z)=(x-2)^2+(y+1)^2+(z+2)^2$.

Let $g(x,y,z)=x+3y+5z-18$ and $h(x,y,z)=3x+5y+z-28$, then create the system of equations.

$$f_x=\lambda g_x+\mu h_x$$

$$f_y=\lambda g_y+\mu h_y$$

$$f_z=\lambda g_z+\mu h_z$$

To build the system, we'll need first order partial derivatives of f, g, and h.

$f_x=2(x-2)$ $\qquad g_x=1$ $\qquad h_x=3$

$$f_y = 2(y+1) \qquad g_y = 3 \qquad h_y = 5$$

$$f_z = 2(z+2) \qquad g_z = 5 \qquad h_z = 1$$

Plug into the system.

$$2(x-2) = \lambda + 3\mu$$

$$2(y+1) = 3\lambda + 5\mu$$

$$2(z+2) = 5\lambda + \mu$$

Solve the system for x, y, and z.

$$x = \frac{\lambda + 3\mu}{2} + 2$$

$$y = \frac{3\lambda + 5\mu}{2} - 1$$

$$z = \frac{5\lambda + \mu}{2} - 2$$

Plug these into the constraint equations.

$$\frac{\lambda + 3\mu}{2} + 2 + 3\left(\frac{3\lambda + 5\mu}{2} - 1\right) + 5\left(\frac{5\lambda + \mu}{2} - 2\right) = 18$$

$$3\left(\frac{\lambda + 3\mu}{2} + 2\right) + 5\left(\frac{3\lambda + 5\mu}{2} - 1\right) + \frac{5\lambda + \mu}{2} - 2 = 28$$

Simplify both equations to get

$$35\lambda + 23\mu = 58$$

$$23\lambda + 35\mu = 58$$

The solution of the system is $\lambda = 1$, $\mu = 1$, so plug these back into the equations for x, y, and z,

$$x = \frac{1 + 3 \cdot 1}{2} + 2 = 4$$

$$y = \frac{3 \cdot 1 + 5 \cdot 1}{2} - 1 = 3$$

$$z = \frac{5 \cdot 1 + 1}{2} - 2 = 1$$

and the plug these into the equation for $f(x, y, z)$.

$$f(4,3,1) = (4 - 2)^2 + (3 + 1)^2 + (1 + 2)^2 = 29$$

Since the distance function from the point to the line cloud have only one minimum, the critical point $(4,3,1)$ is the global minimum, and the distance is

$$d = \sqrt{29}$$

■ 2. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = x^2 + 2y - 3z^2$$

$4x - y = 0$ and $y + 8z = 0$

Solution:

Let $g(x, y, z) = 4x - y$ and $h(x, y, z) = y + 8z$, then create the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

To build the system, we'll need first order partial derivatives of f, g, and h.

$$f_x = 2x \qquad g_x = 4 \qquad h_x = 0$$

$$f_y = 2 \qquad g_y = -1 \qquad h_y = 1$$

$$f_z = -6z \qquad g_z = 0 \qquad h_z = 8$$

Plug into the system.

$$2x = 4\lambda$$

$$2 = -\lambda + \mu$$

$$-6z = 8\mu$$

So

$$x = 2\lambda$$

$$\lambda = \mu - 2$$

$$z = -\frac{4}{3}\mu$$

Plug these into the constraint equations.

$$4(2\mu - 4) - y = 0$$

$$y + 8\left(-\frac{4}{3}\mu\right) = 0$$

Simplify both equations to get

$$8\mu - 16 = y$$

$$y = \frac{32}{3}\mu$$

So

$$8\mu - 16 = \frac{32}{3}\mu$$

$$\mu = -6$$

And then $\lambda = -6 - 2 = -8$. So plug these back into the equations for x, y, and z.

$$x = 2(-8) = -16$$

$$y = 8 \cdot (-6) - 16 = -64$$

$$z = -\frac{4}{3}(-6) = -8$$

The the solution to the system is $(-16, -64, -8)$, $\lambda = -8$, and $\mu = -6$.

Perform the second derivative test for constrained extrema. The bordered Hessian matrix is

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$. The second derivative test for the solution to the system is

$$L(x, y, z) = f(x, y, z) + 8 \cdot g(x, y, z) + 6 \cdot h(x, y, z)$$

$$L(x, y, z) = x^2 + 2y - 3z^2 + 8(4x - y) + 6(y + 8z)$$

$$L(x, y, z) = x^2 + 32x - 3z^2 + 48z$$

Find second order partial derivatives and evaluate them at $(-16, -64, -8)$.

$$L_{xx} = 2$$

$$L_{yy} = 0$$

$$L_{zz} = -6$$

$$L_{xy} = L_{yx} = 0$$

$$L_{yz} = L_{zy} = 0$$

$$L_{xz} = L_{zx} = 0$$

And we already have $g_x = 4$, $g_y = -1$, $g_z = 0$ and $h_x = 0$, $h_y = 1$, and $h_z = 8$. So the Hessian matrix becomes

$$\begin{vmatrix} 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 & -8 \\ -4 & 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & -6 \end{vmatrix}$$

The determinant is $det(HL) = 32 > 0$. Since $det(HL) > 0$ at $(-16, -64, -8)$, this point is a local minimum.

So the function has a local minimum of -8 at $(-16, -64, -8)$.

3. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = z$$

$4x + 2y + 3z = -2$ and $3x^2 + y^2 - z^2 = 5$

Solution:

Let $g(x, y, z) = 4x + 2y + 3z + 2$ and $h(x, y, z) = 3x^2 + y^2 - z^2 - 5$, then create the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

To build the system, we'll need first order partial derivatives of f, g, and h.

$$f_x = 0 \qquad g_x = 4 \qquad h_x = 6x$$

$$f_y = 0 \qquad g_y = 2 \qquad h_y = 2y$$

$$f_z = 1 \qquad g_z = 3 \qquad h_z = -2z$$

Plug into the system.

$$0 = 4\lambda + 6x\mu$$

$$0 = 2\lambda + 2y\mu$$

$$1 = 3\lambda - 2z\mu$$

Solve the system for x, y, and z.

$$3x\mu = -2\lambda$$

$$y\mu = -\lambda$$

$$2z\mu = 3\lambda - 1$$

Since $\mu \neq 0$ (or else $\lambda = 0$ and $1 = 0 - 0$, which is impossible), we can say

$$x = -\frac{2\lambda}{3\mu}$$

$$y = -\frac{\lambda}{\mu}$$

$$z = \frac{3\lambda - 1}{2\mu}$$

Plug these into the constraint equations.

$$4\left(-\frac{2\lambda}{3\mu}\right) + 2\left(-\frac{\lambda}{\mu}\right) + 3\left(\frac{3\lambda - 1}{2\mu}\right) = -2$$

$$3\left(-\frac{2\lambda}{3\mu}\right)^2 + \left(-\frac{\lambda}{\mu}\right)^2 - \left(\frac{3\lambda - 1}{2\mu}\right)^2 = 5$$

Simplify both equations to get

$$\lambda = 12\mu - 9$$

$$\lambda^2 + 18\lambda - 3 = 60\mu^2$$

Plug $\lambda = 12\mu - 9$ into the second constraint equation.

$$(12\mu - 9)^2 + 18(12\mu - 9) - 3 = 60\mu^2$$

$$84\mu^2 - 84 = 0$$

So $\mu = 1$ or $\mu = -1$. Calculate λ, x, y, z for $\mu = 1$:

$$\lambda = 12 \cdot 1 - 9 = 3$$

$$x = -\frac{2 \cdot 3}{3 \cdot 1} = -2$$

$$y = -\frac{3}{1} = -3$$

$$z = \frac{3 \cdot 3 - 1}{2 \cdot 1} = 4$$

Calculate λ, x, y, z for $\mu = -1$:

$$\lambda = 12 \cdot (-1) - 9 = -21$$

$$x = -\frac{2 \cdot (-21)}{3 \cdot (-1)} = -14$$

$$y = -\frac{(-21)}{(-1)} = -21$$

$$z = \frac{3 \cdot (-21) - 1}{2 \cdot (-1)} = 32$$

The solutions to the system are

$(-2, -3,4)$ with $\lambda = 3$ and $\mu = 1$

$(-14, -21,32)$ with $\lambda = -21$ and $\mu = -1$

Perform the second derivative test for constrained extrema using the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$.

Find L at $(-2, -3,4)$ with $\lambda = 3$ and $\mu = 1$.

$$L(x, y, z) = f(x, y, z) - 3 \cdot g(x, y, z) - 1 \cdot h(x, y, z)$$

$$L(x, y, z) = z - 3(4x + 2y + 3z + 2) - (3x^2 + y^2 - z^2 - 5)$$

$$L(x, y, z) = -3x^2 - y^2 + z^2 - 12x - 6y - 8z - 1$$

Find second order partial derivatives and evaluate at $(-2, -3, 4)$.

$$L_{xx} = -6$$

$$L_{yy} = -2$$

$$L_{zz} = 2$$

$$L_{xy} = L_{yx} = 0$$

$$L_{yz} = L_{zy} = 0$$

$$L_{xz} = L_{zx} = 0$$

And we already have $g_x = 4$, $g_y = 2$, $g_z = 3$ and $h_x = -12$, $h_y = -6$, and $h_z = -8$. So the Hessian matrix becomes

$$\begin{vmatrix} 0 & 0 & -4 & -2 & -3 \\ 0 & 0 & 12 & 6 & 8 \\ -4 & 12 & -6 & 0 & 0 \\ -2 & 6 & 0 & -2 & 0 \\ -3 & 8 & 0 & 0 & 2 \end{vmatrix}$$

The determinant is $det(HL) = -56 < 0$. Since $det(HL) < 0$ at $(-2, -3, 4)$, this point is a local maximum. So the function has a local maximum of 4 at $(-2, -3, 4)$.

Find L at $(-14,-21,32)$ with $\lambda=-21$ and $\mu=-1$.

$$L(x,y,z)=f(x,y,z)+21\cdot g(x,y,z)+1\cdot h(x,y,z)$$

$$L(x,y,z)=z+21(4x+2y+3z+2)+(3x^2+y^2-z^2-5)$$

$$L(x,y,z)=3x^2+y^2-z^2+84x+42y+64z+37$$

Find second order partial derivatives and evaluate at $(-14,-21,32)$.

$$L_{xx}=6$$

$$L_{yy}=2$$

$$L_{zz}=-2$$

$$L_{xy}=L_{yx}=0$$

$$L_{yz}=L_{zy}=0$$

$$L_{xz}=L_{zx}=0$$

And we already have $g_x=4$, $g_y=2$, $g_z=3$ and $h_x=-84$, $h_y=126$, and $h_z=-256$. So the Hessian matrix becomes

$$\begin{vmatrix} 0 & 0 & -4 & -2 & -3 \\ 0 & 0 & 84 & -126 & 256 \\ -4 & 84 & 6 & 0 & 0 \\ -2 & -126 & 0 & 2 & 0 \\ -3 & 256 & 0 & 0 & -2 \end{vmatrix}$$

The determinant is $det(HL) = 5{,}041{,}400 > 0$. Since $det(HL) > 0$ at $(-14, -21{,}32)$, this point is a local minimum. So the function has a local maximum of 32 at $(-14, -21{,}32)$.

So the extrema of the function are

A local maximum of 4 at $(-2, -3{,}4)$

A local minimum of 32 at $(-14, -21{,}32)$

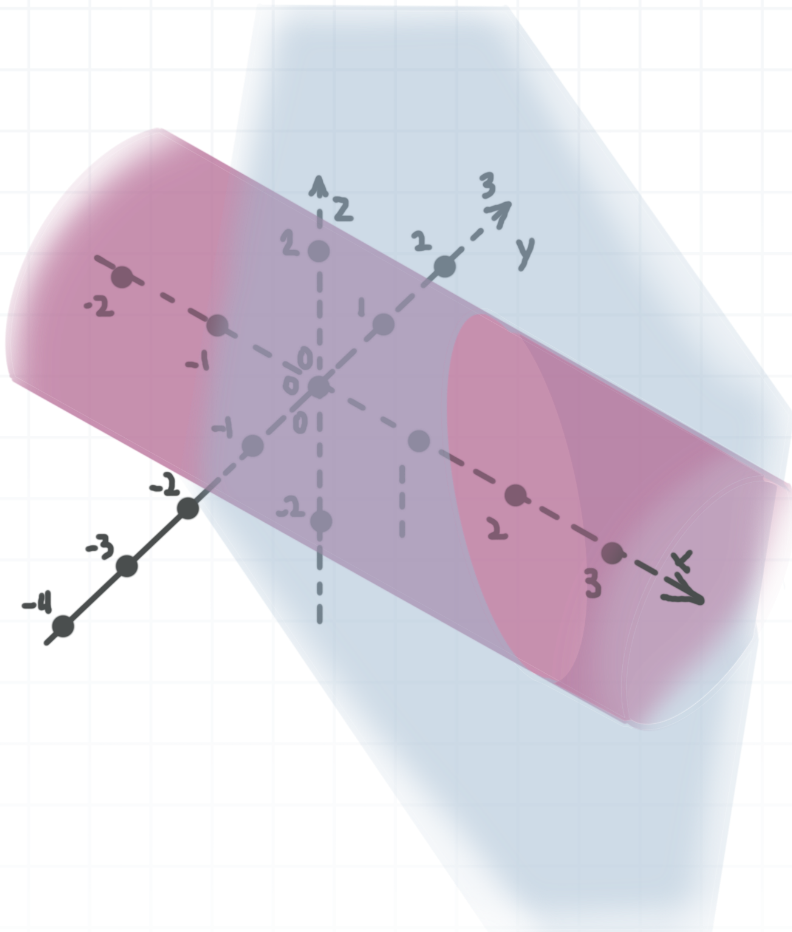
■ 4. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = 2\ln x + z$$

$2x + y + z = 4$ and $y^2 + z^2 = 2$

Solution:

A sketch of the surface is



Let $g(x, y, z) = 2x + y + z - 4$ and $h(x, y, z) = y^2 + z^2 - 2$. Build the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

Calculate first order partial derivatives.

$$f_x = \frac{2}{x} \qquad g_x = 2 \qquad h_x = 0$$

$$f_y = 0 \qquad g_y = 1 \qquad h_y = 2y$$

$$f_z = 1 \qquad g_z = 1 \qquad h_z = 2z$$

Substitute the partial derivatives into the system of equations.

$$\frac{2}{x} = 2\lambda$$

$$0 = \lambda + 2y\mu$$

$$1 = \lambda + 2z\mu$$

Solve the system for x, y, and z.

Since $\lambda \neq 0$ (or else $1/x = 0$, which is impossible) and $\mu \neq 0$ (or else $\lambda = 0$ and $\lambda = 1$, which is impossible),

$$x = \frac{1}{\lambda}$$

$$y = \frac{-\lambda}{2\mu}$$

$$z = \frac{1-\lambda}{2\mu}$$

Plug these into the constraint equations.

$$2\left(\frac{1}{\lambda}\right) + \left(\frac{-\lambda}{2\mu}\right) + \left(\frac{1-\lambda}{2\mu}\right) = 4$$

$$\left(\frac{-\lambda}{2\mu}\right)^2 + \left(\frac{1-\lambda}{2\mu}\right)^2 = 2$$

Simplify the system.

$$-2\lambda^2 + \lambda + 4\mu - 8\lambda\mu = 0$$

$$2\lambda^2 - 2\lambda + 1 - 8\mu^2 = 0$$

Factor the first equation to get

$$(\lambda + 4\mu)(2\lambda - 1) = 0$$

$$\lambda = -4\mu \text{ or } \lambda = \frac{1}{2}$$

First plug $\mu = -\frac{\lambda}{4}$ into the second equation.

$$2\lambda^2 - 2\lambda + 1 - 8\left(-\frac{\lambda}{4}\right)^2 = 0$$

$$3\lambda^2 - 4\lambda + 2 = 0$$

This equation has no solutions.

Then plug $\lambda = 1/2$ into the second equation.

$$2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1 - 8\mu^2 = 0$$

$$\mu = \frac{1}{4} \text{ or } \mu = -\frac{1}{4}$$

For $\lambda = 1/2$ and $\mu = 1/4$, find x, y, z, and $f(x, y, z)$.

$$x = \frac{1}{\frac{1}{2}} = 2$$

$$y = -\frac{\frac{1}{2}}{2 \cdot \frac{1}{4}} = -1$$

$$z = \frac{1 - \frac{1}{2}}{2 \cdot \frac{1}{4}} = 1$$

$$f(2, -1, 1) = 2 \ln 2 + 1$$

Then for $\lambda = 1/2$ and $\mu = -1/4$, find x, y, z, and $f(x, y, z)$.

$$x = \frac{1}{\frac{1}{2}} = 2$$

$$y = -\frac{\frac{1}{2}}{2 \cdot \left(-\frac{1}{4}\right)} = 1$$

$$z = \frac{1 - \frac{1}{2}}{2 \cdot \left(-\frac{1}{4}\right)} = -1$$

$$f(2, 1, -1) = 2 \ln 2 - 1$$

Since the curve of intersection of the two constraints equations, a cylinder and a plane, is a closed continuous curve, then by the extreme value theorem the function has one global minimum and one global maximum. Since we have only two critical points $(2, -1, 1)$ and $(2, 1, -1)$, one of them is the maximum, and the other is the minimum. Because $f(2, -1, 1) > f(2, 1, -1)$, $(2, -1, 1)$ is the maximum.

So the extrema of the function are

The local and global maximum is $2\ln 2 + 1$ at $(2, -1,1)$

The local and global minimum is $2\ln 2 - 1$ at $(2,1, -1)$

■ 5. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = 2\ln x - \ln y^4 + z^2 + 5$$

$2x + 3z^2 = 12$ and $4y + z^2 = 4$

Solution:

Rewrite the function as

$$f(x, y, z) = 2\ln x - 4\ln|y| + z^2 + 5$$

The two constraint functions are $g(x, y, z) = 2x + 3z^2 - 12$ and $h(x, y, z) = 4y + z^2 - 4$. Create the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

Calculate first order partial derivatives.

$$f_x = \frac{2}{x} \qquad g_x = 2 \qquad h_x = 0$$

$$f_y = -\frac{4}{y} \qquad g_y = 0 \qquad h_y = 4$$

$$f_z = 2z \qquad g_z = 6z \qquad h_z = 2z$$

Substitute partial derivatives into the system.

$$\frac{2}{x} = 2\lambda$$

$$-\frac{4}{y} = 4\mu$$

$$2z = 6z\lambda + 2z\mu$$

Simplify the system.

$$\frac{1}{x} = \lambda$$

$$\frac{1}{y} = -\mu$$

$$z(3\lambda + \mu - 1) = 0$$

From the third equation, $z = 0$ or $3\lambda + \mu - 1 = 0$. First, plug $z = 0$ into the constraint equations.

$$2x = 12$$

$$4y = 4$$

So

$$x = 6$$

$$y = 1$$

$$\lambda = \frac{1}{6}$$

$$\mu = -1$$

So the solution to the system is $(6,1,0)$, with $\lambda = 1/6$ and $\mu = -1$.

Second, if $3\lambda + \mu - 1 = 0$, since $\lambda \neq 0$ and $\mu \neq 0$, then

$$\mu = 1 - 3\lambda$$

$$x = \frac{1}{\lambda}$$

$$y = -\frac{1}{1 - 3\lambda}$$

Plug in these values for x and y into the constraint equations.

$$2\left(\frac{1}{\lambda}\right) + 3z^2 = 12$$

$$4\left(-\frac{1}{1 - 3\lambda}\right) + z^2 = 4$$

Simplify the system.

$$\lambda(3z^2 - 12) = -2$$

$$\lambda(3z^2 - 12) = z^2 - 8$$

So $z^2 - 8 = -2$, $z = \sqrt{6}$ or $z = -\sqrt{6}$.

From the first equation,

$$\lambda(3 \cdot 6 - 12) = -2$$

$$\lambda = -\frac{1}{3}$$

Calculate μ, x, and y for $z^2 = 6$ and $\lambda = -1/3$.

$$\mu = 1 - 3\left(-\frac{1}{3}\right) = 2$$

$$x = \frac{1}{-\frac{1}{3}} = -3$$

$$y = -\frac{1}{2}$$

So the solutions are

$$\left(-3, -\frac{1}{2}, \sqrt{6}\right) \text{ with } \lambda = -\frac{1}{3} \text{ and } \mu = 2$$

$$\left(-3, -\frac{1}{2}, -\sqrt{6}\right) \text{ with } \lambda = -\frac{1}{3} \text{ and } \mu = 2$$

Check to see if the points lie within the domain of $f(x, y, z)$. $x = -3 < 0$, so neither of these points are critical points.

Perform the second derivative test for constrained extrema at the point $(6,1,0)$. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$. The second derivative test for the point $(6,1,0)$ with $\lambda = 1/6$ and $\mu = -1$ is

$$L(x, y, z) = f(x, y, z) - \frac{1}{6} \cdot g(x, y, z) + h(x, y, z)$$

$$L(x, y, z) = 2 \ln x - \ln y^4 + z^2 + 5 - \frac{1}{6}(2x + 3z^2 - 12) + 4y + z^2 - 4$$

$$L(x, y, z) = 2 \ln x - \ln y^4 - \frac{x}{3} + 4y + \frac{3z^2}{2} + 3$$

Calculate the second order partial derivatives and substitute $(6,1,0)$.

$$L_{xx} = -\frac{1}{18}$$

$$L_{yy} = 4$$

$$L_{zz} = 3$$

$$L_{xy} = L_{yx} = 0$$

$$L_{yz} = L_{zy} = 0$$

$$L_{xz} = L_{zx} = 0$$

And we already know $g_x = 2$, $g_y = 0$, $g_z = 0$, and $h_x = 0$, $h_y = 4$, and $h_z = 0$. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ -2 & 0 & -\frac{1}{18} & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$$

The determinant is $det(HL) = 192 > 0$. Since $det(HL) > 0$ at $(6,1,0)$, this point is a local minimum. So the function has a local minimum of $2 \ln 6 + 5$ at $(6,1,0)$.

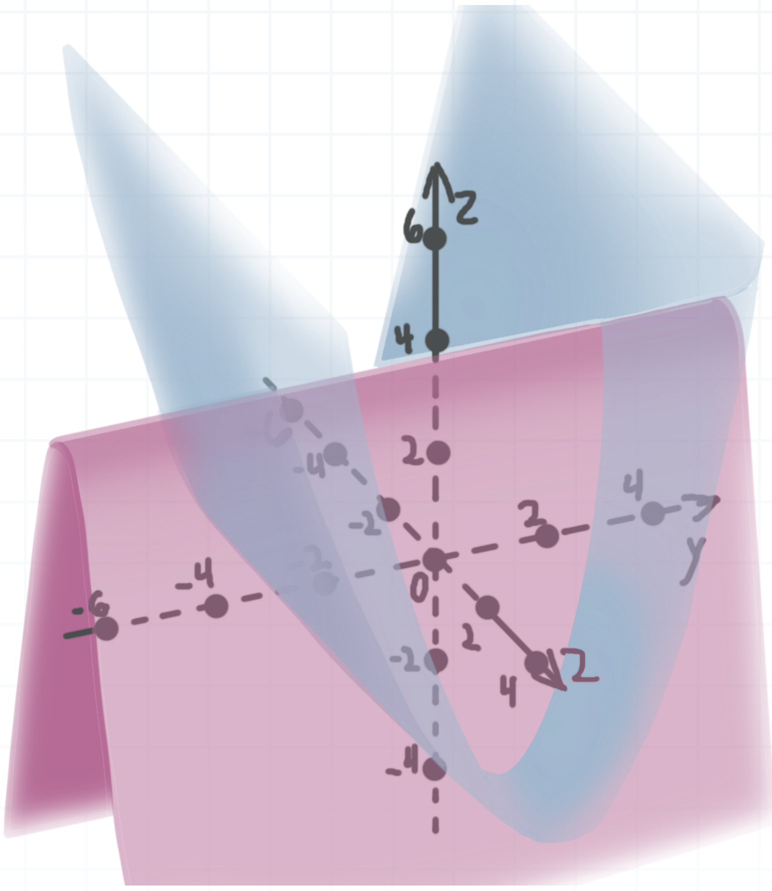
6. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = 4x^3 + 2y^3 - 4z + 1$$

$3y^2 - 4z = 12$ and $6x^2 + 4z = 15$

Solution:

A sketch of the surface is



The two constraint functions are $g(x, y, z) = 3y^2 - 4z - 12$ and $h(x, y, z) = 6x^2 + 4z - 15$.

Create the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

Calculate the first order partial derivatives.

$f_x = 12x^2$ $\qquad g_x = 0$ $\qquad h_x = 12x$

$f_y = 6y^2$ $\qquad g_y = 6y$ $\qquad h_y = 0$

$f_z = -4$ $\qquad g_z = -4$ $\qquad h_z = 4$

Substitute partial derivatives into the system.

$$12x^2 = 12x\mu$$

$$6y^2 = 6y\lambda$$

$$-4 = -4\lambda + 4\mu$$

Simplify the system.

$$x(x - \mu) = 0$$

$$y(y - \lambda) = 0$$

$$\lambda = \mu + 1$$

So from the first equation, $x = 0$ or $x = \mu$. First, plug $x = 0$ into the constraint equations.

$$3y^2 - 4z = 12$$

$$4z = 15$$

$$z = \frac{15}{4}$$

and then

$$y^2 = \frac{12 + 4 \cdot \frac{15}{4}}{3} = 9$$

$y = 3$ or $y = -3$

Calculate λ and μ for $(0,3,15/4)$.

$$\lambda = 3$$

$$\mu = 3 - 1 = 2$$

Calculate λ and μ for the point $(0, -3, 15/4)$.

$$\lambda = -3$$

$$\mu = -3 - 1 = -4$$

So the solutions are

$\left(0, 3, \frac{15}{4}\right)$ with $\lambda = 3$ and $\mu = 2$

$\left(0, -3, \frac{15}{4}\right)$ with $\lambda = -3$ and $\mu = -4$

Second, if $x = \mu$, then from the second equation, $y = 0$ or $y = \lambda$. Plug in $y = 0$ into the constraint equations.

$$-4z = 12$$

$$6x^2 + 4z = 15$$

$$z = -3$$

Then

$$x^2 = \frac{15 - 4(-3)}{6} = \frac{9}{2}$$

$x = \frac{3}{\sqrt{2}}$ or $x = -\frac{3}{\sqrt{2}}$

Calculate λ and μ for the point $(3/\sqrt{2},0,-3)$.

$$\mu = \frac{3}{\sqrt{2}}$$

$$\lambda = \frac{3}{\sqrt{2}} + 1$$

Calculate λ and μ for the point $(-3/\sqrt{2},0,-3)$.

$$\mu = -\frac{3}{\sqrt{2}}$$

$$\lambda = -\frac{3}{\sqrt{2}} + 1$$

So the solutions are

$$\left(-\frac{3}{\sqrt{2}},0,-3\right) \text{ with } \lambda = -\frac{3}{\sqrt{2}} + 1 \text{ and } \mu = -\frac{3}{\sqrt{2}}$$

$$\left(\frac{3}{\sqrt{2}},0,-3\right) \text{ with } \lambda = \frac{3}{\sqrt{2}} + 1 \text{ and } \mu = \frac{3}{\sqrt{2}}$$

Last, if $x = \mu$ and $y = \lambda$ then, from the third equation, $y = x + 1$. Plug $y = x + 1$ into the constraint equations.

$$3(x+1)^2 - 4z = 12$$

$$6x^2 + 4z = 15$$

$$4z = 3(x+1)^2 - 12 = 15 - 6x^2$$

Solve the equation $3(x+1)^2 - 12 = 15 - 6x^2$ for x.

$$9x^2 + 6x - 24 = 0$$

$$x = -2 \text{ or } x = \frac{4}{3}$$

Calculate y, z, λ, and μ for $x = -2$.

$$y = -2 + 1 = -1$$

$$z = \frac{15 - 6(-2)^2}{4} = -\frac{9}{4}$$

$$\mu = -2$$

$$\lambda = -1$$

Calculate y, z, λ, and μ for $x = 4/3$.

$$y = \frac{4}{3} + 1 = \frac{7}{3}$$

$$z = \frac{15 - 6(\frac{4}{3})^2}{4} = \frac{13}{12}$$

$$\mu = \frac{4}{3}$$

$$\lambda = \frac{7}{3}$$

So the solutions are

$$\left(-2,-1,-\frac{9}{4}\right) \text{ with } \lambda=-1 \text{ and } \mu=-2$$

$$\left(\frac{4}{3},\frac{7}{3},\frac{13}{12}\right) \text{ with } \lambda=\frac{7}{3} \text{ and } \mu=\frac{4}{3}$$

So we have six critical points.

1) $\left(0,3,\frac{15}{4}\right)$ with $\lambda=3$ and $\mu=2$

2) $\left(0,-3,\frac{15}{4}\right)$ with $\lambda=-3$ and $\mu=-4$

3) $\left(-\frac{3}{\sqrt{2}},0,-3\right)$ with $\lambda=-\frac{3}{\sqrt{2}}+1$ and $\mu=-\frac{3}{\sqrt{2}}$

4) $\left(\frac{3}{\sqrt{2}},0,-3\right)$ with $\lambda=\frac{3}{\sqrt{2}}+1$ and $\mu=\frac{3}{\sqrt{2}}$

5) $\left(-2,-1,-\frac{9}{4}\right)$ with $\lambda=-1$ and $\mu=-2$

6) $\left(\frac{4}{3},\frac{7}{3},\frac{13}{12}\right)$ with $\lambda=\frac{7}{3}$ and $\mu=\frac{4}{3}$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$.

1) The second derivative test for $(0,3,15/4)$ with $\lambda = 3$ and $\mu = 2$:

$$L(x, y, z) = f(x, y, z) - 3 \cdot g(x, y, z) - 2 \cdot h(x, y, z)$$

$$L(x, y, z) = 4x^3 + 2y^3 - 4z + 1 - 3(3y^2 - 4z - 12) - 2(6x^2 + 4z - 15)$$

$$L(x, y, z) = 4x^3 - 12x^2 + 2y^3 - 9y^2 + 67$$

Find second order partial derivatives at $(0,3,15/4)$.

$$L_{xx} = -24, \ L_{yy} = 18, \ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0, \ L_{yz} = L_{zy} = 0, \ L_{xz} = L_{zx} = 0$$

$$g_x = 0, \ g_y = 18, \ g_z = -4$$

$$h_x = 0, \ h_y = 0, \ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & -18 & 4 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & -24 & 0 & 0 \\ -18 & 0 & 0 & 18 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) = -124{,}416 < 0$. Since $det(HL) < 0$ at $(0,3,15/4)$, it's a local maximum, and $f(0,3,15/4) = 40$.

2) The second derivative test for $(0,-3,15/4)$ with $\lambda = -3$ and $\mu = -4$:

$$L(x,y,z) = f(x,y,z) + 3 \cdot g(x,y,z) + 4 \cdot h(x,y,z)$$

$$L(x,y,z) = 4x^3 + 2y^3 - 4z + 1 + 3(3y^2 - 4z - 12) + 4(6x^2 + 4z - 15)$$

$$L(x,y,z) = 4x^3 + 24x^2 + 2y^3 + 9y^2 - 95$$

Find second order partial derivatives at $(0,-3,15/4)$.

$$L_{xx} = 48,\ L_{yy} = -18,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = -18,\ g_z = -4$$

$$h_x = 0,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 18 & 4 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 48 & 0 & 0 \\ 18 & 0 & 0 & -18 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) = 248{,}832 > 0$. Since $det(HL) > 0$ at $(0, -3, 15/4)$, it's a local maximum, and $f(0, -3, 15/4) = -68$.

3) The second derivative test for $(-3/\sqrt{2}, 0, -3)$ with $\lambda = -3/\sqrt{2} + 1$ and $\mu = -3/\sqrt{2}$:

We don't need high accuracy here because we only need to estimate the determinant value. So to simplify calculations, let's consider the values rounded to one decimal place.

$$L(x, y, z) = f(x, y, z) + 1.1 \cdot g(x, y, z) + 2.1 \cdot h(x, y, z)$$

$$L(x, y, z) = 4x^3 + 2y^3 - 4z + 1 + 1.1(3y^2 - 4z - 12) + 2.1(6x^2 + 4z - 15)$$

$$L(x, y, z) = 4x^3 + 12.6x^2 + 2y^3 + 3.3y^2 - 43.7$$

Find second order partial derivatives at $(-2.1, 0, -3)$.

$$L_{xx} = -25.2,\ L_{yy} = 6.6,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = 0,\ g_z = -4$$

$$h_x = -25.2,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 25.2 & 0 & -4 \\ 0 & 25.2 & -25.2 & 0 & 0 \\ 0 & 0 & 0 & 6.6 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) \approx 67{,}060 > 0$. Since $det(HL) > 0$ at $(-3/\sqrt{2},0,-3)$, it's a local minimum, and $f(-3/\sqrt{2},0,-3) = 13 - 27\sqrt{2}$.

4) The second derivative test for $(3/\sqrt{2},0,-3)$ with $\lambda = 3/\sqrt{2} + 1$ and $\mu = 3/\sqrt{2}$:

$$L(x,y,z) = f(x,y,z) - 3.1 \cdot g(x,y,z) - 2.1 \cdot h(x,y,z)$$

$$L(x,y,z) = 4x^3 + 2y^3 - 4z + 1 - 3.1(3y^2 - 4z - 12) - 2.1(6x^2 + 4z - 15)$$

$$L(x,y,z) = 4x^3 - 12.6x^2 + 2y^3 - 9.3y^2 + 69.7$$

Find second order partial derivatives at $(3/\sqrt{2},0,-3)$.

$$L_{xx} = 25.2,\ L_{yy} = -18.6,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = 0,\ g_z = -4$$

$$h_x = 25.2,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -25.2 & 0 & -4 \\ 0 & -25.2 & 25.2 & 0 & 0 \\ 0 & 0 & 0 & -18.6 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) \approx -188{,}988 < 0$. Since $det(HL) < 0$ at $(3/\sqrt{2},0,-3)$, it's a local maximum, and $f(3/\sqrt{2},0,-3) = 13 + 27\sqrt{2}$.

5) The second derivative test for $(-2,-1,-2.25)$ with $\lambda = -1$ and $\mu = -2$.

$$L(x,y,z) = f(x,y,z) + 1 \cdot g(x,y,z) + 2 \cdot h(x,y,z)$$

$$L(x,y,z) = 4x^3 + 2y^3 - 4z + 1 + (3y^2 - 4z - 12) + 2(6x^2 + 4z - 15)$$

$$L(x,y,z) = 4x^3 + 12x^2 + 2y^3 + 3y^2 - 41$$

Find second order partial derivatives at $(-2,-1,-2.25)$.

$$L_{xx} = -24,\ L_{yy} = -6,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = -6,\ g_z = -4$$

$$h_x = -24,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 24 & 0 & -4 \\ 0 & 24 & -24 & 0 & 0 \\ 6 & 0 & 0 & -6 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) = -69{,}120 < 0$. Since $det(HL) < 0$ at $(-2,-1,-2.25)$, it's a local maximum, and $f(-2,-1,-2.25) = -24$.

6) The second derivative test for $(4/3, 7/3, 13/12)$ with $\lambda = 7/3$ and $\mu = 4/3$.

$$L(x,y,z) = f(x,y,z) - 2.3 \cdot g(x,y,z) - 1.3 \cdot h(x,y,z)$$

$$L(x, y, z) = 4x^3 + 2y^3 - 4z + 1 - 2.3(3y^2 - 4z - 12) - 1.3(6x^2 + 4z - 15)$$

$$L(x, y, z) = 4x^3 - 7.8x^2 + 2y^3 - 6.9y^2 + 48.1$$

Find second order partial derivatives at $(4/3,7/3,13/12)$.

$$L_{xx} = 15.6,\ L_{yy} = 13.8,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = 13.8,\ g_z = -4$$

$$h_x = 15.6,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & -13.8 & 4 \\ 0 & 0 & -15.6 & 0 & -4 \\ 0 & -15.6 & 15.6 & 0 & 0 \\ -13.8 & 0 & 0 & 13.8 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) \approx 101{,}268 > 0$. Since $det(HL) > 0$ at $(4/3,7/3,13/12)$, it's a local minimum, and $f(4/3,7/3,13/12) = 284/9$.

So the extrema of the function are

A local maximum of 40 at $(0,3,15/4)$

A local minimum of -68 at $(0,-3,15/4)$

A local minimum of $13 - 27\sqrt{2}$ at $(-3/\sqrt{2},0,-3)$

A local maximum of $13 + 27\sqrt{2}$ at $(3/\sqrt{2}, 0, -3)$

A local maximum of -24 at $(-2, -1, -2.25)$

A local minimum of $284/9$ at $(4/3, 7/3, 13/12)$

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