Topic: Finding volume

Question: Find the volume given by the double integral.

$$\int_0^{2\pi} \int_0^1 r^2 \, dr \, d\theta$$

Answer choices:

A $2\pi^3$

B $\frac{2\pi}{3}$

C $\frac{8\pi}{3}$

D $\frac{8\pi^3}{3}$

Solution: B

We've been given the double integral

$$\int_0^{2\pi} \int_0^1 r^2 \ dr \ d\theta$$

in polar coordinates. All we need to do to find the volume it represents is evaluate the integral. We always work from the inside out, which means we'll integrate first with respect to r.

$$V = \int_0^{2\pi} \frac{1}{3} r^3 \Big|_0^1 \ d\theta$$

$$V = \int_0^{2\pi} \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3 \ d\theta$$

$$V = \int_0^{2\pi} \frac{1}{3} \ d\theta$$

Integrate with respect to θ.

$$V = \frac{1}{3}\theta \Big|_0^{2\pi}$$

$$V = \frac{1}{3}(2\pi) - \frac{1}{3}(0)$$

$$V = \frac{2\pi}{3}$$

This is the volume given by the double integral.

Topic: Finding volume

Question: Find the volume of the region.

The region under $-x^2 - y^2 + z = 1$, on $0 \le \theta \le 2\pi$ and $0 \le r \le 2$.

Answer choices:

A $12\pi^2$

B 12π

C 6π

D $6\pi^2$

Solution: B

We'll rearrange the function we've been given so that it's solved for z.

$$-x^2 - y^2 + z = 1$$

$$z = 1 + x^2 + y^2$$

Since we've been given bounds for the region in terms of polar coordinates, we need to convert this function into polar coordinates, too. We know that $r^2 = x^2 + y^2$, so the function becomes

$$z = 1 + r^2$$

Now we've got everything we need to plug into the volume integral. We know that when we move from rectangular to polar coordinates, that $dy\ dx = r\ dr\ d\theta$. Therefore, the volume integral becomes

$$V = \int_0^{2\pi} \int_0^2 \left(1 + r^2\right) r\ dr\ d\theta$$

$$V = \int_0^{2\pi} \int_0^2 r + r^3\ dr\ d\theta$$

We'll work from the inside out, and integrate first with respect to r.

$$V = \int_0^{2\pi} \frac{1}{2}r^2 + \frac{1}{4}r^4 \Big|_0^2\ d\theta$$

$$V = \int_0^{2\pi} \frac{1}{2}(2)^2 + \frac{1}{4}(2)^4 - \left(\frac{1}{2}(0)^2 + \frac{1}{4}(0)^4\right)\ d\theta$$

$$V = \int_0^{2\pi} 2 + 4 \ d\theta$$

$$V = \int_0^{2\pi} 6 \ d\theta$$

Integrate with respect to θ.

$$V = 6\theta \Big|_0^{2\pi}$$

$$V = 6(2\pi) - 6(0)$$

$$V = 12\pi$$

This is the volume given by the double integral.

Topic: Finding volume

Question: Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 4$.

Answer choices:

A 4π

B 2π

C 8π

D 16π

Solution: C

Because we're looking for the volume inside $z = x^2 + y^2$ and below $z = 4$, we'll use

$$V = \int\int_D 4 - \left(x^2 + y^2\right)\ dA$$

to represent the volume. We know that $r^2 = x^2 + y^2$, so we'll rewrite the integrand as

$$V = \int\int_D 4 - r^2\ dA$$

In polar coordinates, the bounds will be $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. We also know that $dy\ dx = r\ dr\ d\theta$ whenever we move from rectangular to polar coordinates. Therefore, the volume integral becomes

$$V = \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r\ dr\ d\theta$$

$$V = \int_0^{2\pi} \int_0^2 4r - r^3\ dr\ d\theta$$

We always work from the inside out, so we'll integrate first with respect to r.

$$V = \int_0^{2\pi} 2r^2 - \frac{1}{4}r^4 \Big|_0^2\ d\theta$$

$$V = \int_0^{2\pi} 2(2)^2 - \frac{1}{4}(2)^4 - \left[2(0)^2 - \frac{1}{4}(0)^4\right]\ d\theta$$

$$V=\int_0^{2\pi}2(4)-\frac{1}{4}(16)\ d\theta$$

$$V=\int_0^{2\pi}4\ d\theta$$

Integrate with respect to θ.

$$V=4\theta\Big|_0^{2\pi}$$

$$V=4(2\pi)-4(0)$$

$$V=8\pi$$

This is the volume given by the double integral.