Topic: Normal line to the surface

Question: Find the equation of the normal line to the tangent plane.

$$-x^2 + 3y^2 + 2z^2 = 12$$

$$\frac{\partial f}{\partial x}(1,-1,2) = -2$$

$$\frac{\partial f}{\partial y}(1,-1,2) = -6$$

$$\frac{\partial f}{\partial z}(1,-1,2) = 8$$

Answer choices:

A $\quad -\frac{x-1}{2} = \frac{y+1}{6} = \frac{z-2}{8}$

B $\quad \frac{x-1}{2} = \frac{y+1}{6} = \frac{z-2}{8}$

C $\quad \frac{x-1}{2} = -\frac{y+1}{6} = \frac{z-2}{8}$

D $\quad -\frac{x-1}{2} = -\frac{y+1}{6} = \frac{z-2}{8}$

Solution: D

The normal line to the surface of the tangent plane can be represented by the symmetric equations

$$\frac{x-x_0}{\frac{\partial f}{\partial x}}=\frac{y-y_0}{\frac{\partial f}{\partial y}}=\frac{z-z_0}{\frac{\partial f}{\partial z}}$$

where $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ are the partial derivatives of f with respect to x, y, and z, and (x_0, y_0, z_0) is the point of tangency.

Since we've already been given the partial derivatives at the point of tangency, we can just plug in the partial derivatives and the given point.

$$\frac{x-1}{-2}=\frac{y-(-1)}{-6}=\frac{z-2}{8}$$

$$\frac{x-1}{-2}=\frac{y+1}{-6}=\frac{z-2}{8}$$

$$-\frac{x-1}{2}=-\frac{y+1}{6}=\frac{z-2}{8}$$

These are the symmetric equations that represent the normal line at the point $(1,-1,2)$.

Topic: Normal line to the surface

Question: Find the equation of the normal line to the tangent plane at the point $(3,5,4)$.

$$2x^2 - y^2 + z^2 = 9$$

Answer choices:

A $\quad \dfrac{x-3}{6} = -\dfrac{y-5}{5} = \dfrac{z-4}{4}$

B $\quad \dfrac{x+3}{6} = \dfrac{y-5}{5} = \dfrac{z-4}{4}$

C $\quad \dfrac{x-3}{6} = -\dfrac{y-2}{2} = z-1$

D $\quad \dfrac{x+3}{6} = \dfrac{y-2}{2} = z-1$

Solution: A

The partial derivatives of the tangent plane equation are

$$\frac{\partial f}{\partial x}=4x \qquad \frac{\partial f}{\partial y}=-2y \qquad \frac{\partial f}{\partial z}=2z$$

and their values at $(3,5,4)$ are

$$\frac{\partial f}{\partial x}(3,5,4)=4(3) \qquad \frac{\partial f}{\partial y}(3,5,4)=-2(5) \qquad \frac{\partial f}{\partial z}(3,5,4)=2(4)$$

$$\frac{\partial f}{\partial x}(3,5,4)=12 \qquad \frac{\partial f}{\partial y}(3,5,4)=-10 \qquad \frac{\partial f}{\partial z}(3,5,4)=8$$

Plugging these values and $(x_0,y_0,z_0)=(3,5,4)$ into the symmetric equations gives

$$\frac{x-x_0}{\frac{\partial f}{\partial x}}=\frac{y-y_0}{\frac{\partial f}{\partial y}}=\frac{z-z_0}{\frac{\partial f}{\partial z}}$$

$$\frac{x-3}{12}=\frac{y-5}{-10}=\frac{z-4}{8}$$

$$\frac{x-3}{6}=-\frac{y-5}{5}=\frac{z-4}{4}$$

These are the symmetric equations that represent the normal line at the point $(3,5,4)$.

Topic: Normal line to the surface

Question: Which set of parametric equations defines the normal line to the surface at $(-2,1,\sqrt{5})$?

$$x^2+y^2=z^2$$

Answer choices:

A $x=4t-2$, $y=2t+1$, and $z=-2t+2$

B $x=-4t-2$, $y=2t+1$, and $z=-2\sqrt{5}t+\sqrt{5}$

C $x=2t-2$, $y=-2t+1$, and $z=2\sqrt{5}t-\sqrt{5}$

D $x=-2t-2$, $y=-4t+1$, and $z=t+4$

Solution: B

First, we'll rewrite the given equation.

$$x^2 + y^2 = z^2$$

$$x^2 + y^2 - z^2 = 0$$

$$f(x, y, x) = x^2 + y^2 - z^2$$

The gradient then is defined as

$$\nabla F(x, y, z) = F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}$$

$$\nabla F(x, y, z) = (2x)\mathbf{i} + (2y)\mathbf{j} + (-2z)\mathbf{k}$$

The gradient at the point $(-2,1,\sqrt{5})$ is

$$\nabla F(-2,1,\sqrt{5}) = (2(-2))\mathbf{i} + (2(1))\mathbf{j} + (-2(\sqrt{5}))\mathbf{k}$$

$$\nabla F(-2,1,\sqrt{5}) = -4\mathbf{i} + 2\mathbf{j} - 2\sqrt{5}\mathbf{k}$$

Therefore, the direction numbers of the normal line at $(-2,1,\sqrt{5})$ are -4, 2, and $-2\sqrt{5}$. Then the associated group of symmetric equations is defined by

$$\frac{x+2}{-4} = \frac{y-1}{2} = \frac{z-\sqrt{5}}{-2\sqrt{5}}$$

Setting this compound equation proportional to t and cross multiplying gives us:

$$\frac{x+2}{-4}=t \qquad \frac{y-1}{2}=t \qquad \frac{z-\sqrt{5}}{-2\sqrt{5}}=t$$

$$x+2=-4t \qquad y-1=2t \qquad z-\sqrt{5}=-2\sqrt{5}t$$

$$x=-4t-2 \qquad y=2t+1 \qquad z=-2\sqrt{5}t+\sqrt{5}$$