Topic: Finding volume

Question: Find the volume of the solid in the first quadrant that's bounded by the given lines and planes.

The three coordinate planes

$3x^2 - y^2 + z = 10$

$x = 3$ and $y = 3$

Answer choices:

A $\quad 36$

B $\quad 39$

C $\quad 42$

D $\quad 44$

Solution: A

The given solid is positioned in the first quadrant above the square $R=[0,3]\times[0,3]$ and under the surface defined by $3x^2-y^2+z=10$. If we plug all these things into a double integral to find the volume, we get

$$V=\int\int_R 10-3x^2+y^2\ dA$$

$$V=\int_0^3\int_0^3 10-3x^2+y^2\ dy\ dx$$

Integrate first with respect to y, and then evaluate over the interval.

$$V=\int_0^3 10y-3x^2y+\frac{1}{3}y^3\Big|_{y=0}^{y=3}\ dx$$

$$V=\int_0^3 10(3)-3x^2(3)+\frac{1}{3}(3)^3-\left(10(0)-3x^2(0)+\frac{1}{3}(0)^3\right)\ dx$$

$$V=\int_0^3 30-9x^2+9\ dx$$

$$V=\int_0^3 39-9x^2\ dx$$

Integrate with respect to x, and then evaluate over the interval.

$$V=39x-3x^3\Big|_0^3$$

$$V=39(3)-3(3)^3-\left(39(0)-3(0)^3\right)$$

$$V = 117 - 81$$

$$V = 36$$

Topic: Finding volume

Question: A solid space is bounded by the paraboloid $z = -4x^2 - y^2 + 16$ and the xy-plane. If the volume of the solid is defined by the single integral, then what are the bounds for y if a double integral is used to calculate the volume of the solid?

$$V = \int_{-2}^{2} \frac{4}{3}\left(\sqrt{16 - 4x^2}\right)^3 \ dx$$

Answer choices:

A $\quad -\sqrt{16 - 4x^2} \le y \le \sqrt{16 - 4x^2}$

B $\quad -\sqrt{1 - 4x^2} \le y \le \sqrt{1 - 4x^2}$

C $\quad -\sqrt{4 - 4x^2} \le y \le \sqrt{16 - 4x^2}$

D $\quad -\sqrt{16 - 4x^2} \le y \le \sqrt{4 - x^2}$

Solution: A

Because the solid is bounded by the xy-plane (which is where $z = 0$), and the paraboloid $z = -4x^2 - y^2 + 16$, we can say that those surfaces meet each other at

$$0 = -4x^2 - y^2 + 16$$

$$y^2 = 16 - 4x^2$$

$$y = \pm\sqrt{16 - 4x^2}$$

Therefore

$$-\sqrt{16 - 4x^2} \le y \le \sqrt{16 - 4x^2}$$

Let's check ourselves. We already know from the given integral that the bounds for x are given as $x = [-2,2]$. Plugging these bounds and the paraboloid which bounds the volume into a double integral would give

$$V = \int_{-2}^{2} \int_{-\sqrt{16-4x^2}}^{\sqrt{16-4x^2}} -4x^2 - y^2 + 16 \ dy \ dx$$

If we integrated with respect to y, we'd get

$$V = \int_{-2}^{2} -4x^2y - \frac{1}{3}y^3 + 16y \Big|_{y=-\sqrt{16-4x^2}}^{y=\sqrt{16-4x^2}} \ dx$$

$$V = \int_{-2}^{2} -4x^2\sqrt{16-4x^2} - \frac{1}{3}\left(\sqrt{16-4x^2}\right)^3 + 16\sqrt{16-4x^2} - \left[-4x^2\left(-\sqrt{16-4x^2}\right) - \frac{1}{3}\left(-\sqrt{16-4x^2}\right)^3 + 16\left(-\sqrt{16-4x^2}\right)\right] \ dx$$

$$V = \int_{-2}^{2} -4x^2\sqrt{16-4x^2} - \frac{1}{3}\left(\sqrt{16-4x^2}\right)^3 + 16\sqrt{16-4x^2} - 4x^2\sqrt{16-4x^2} - \frac{1}{3}\left(\sqrt{16-4x^2}\right)^3 + 16\sqrt{16-4x^2} \ dx$$

$$V=\int_{-2}^{2}-8x^2\sqrt{16-4x^2}-\frac{2}{3}\left(\sqrt{16-4x^2}\right)^3+32\sqrt{16-4x^2}\ dx$$

$$V=\int_{-2}^{2}(32-8x^2)\sqrt{16-4x^2}-\frac{2}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

$$V=\int_{-2}^{2}2(16-4x^2)\sqrt{16-4x^2}-\frac{2}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

$$V=\int_{-2}^{2}2\left(\sqrt{16-4x^2}\right)^3-\frac{2}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

$$V=\int_{-2}^{2}\frac{4}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

Because we got back to the integral we were given, this proves that the bounds we found for y, $-\sqrt{16-4x^2}\le y\le\sqrt{16-4x^2}$, are correct.

Topic: Finding volume

Question: A solid is positioned above the square $R = [0,3] \times [0,3]$ and below the surface S, and is bounded by the three coordinate planes. If the volume of S is 36, then which function defines the surface S?

Answer choices:

A $3x^2 - y^2 + z = 10$

B $x^2 - 3y^2 + z = 10$

C $x^2 - y^2 - z = 10$

D $3x^2 + y^2 + z = 10$

Solution: A

If we start with answer choice A, we want to solve it for z.

$$3x^2 - y^2 + z = 10$$

$$z = 10 - 3x^2 + y^2$$

Then we can plug this and everything else we were given into a double integral.

$$V = \iint_R 10 - 3x^2 + y^2 \ dA$$

$$V = \int_0^3 \int_0^3 10 - 3x^2 + y^2 \ dy \ dx$$

Integrate first with respect to y, then evaluate over the interval.

$$V = \int_0^3 10y - 3x^2y + \frac{1}{3}y^3 \Big|_{y=0}^{y=3} \ dx$$

$$V = \int_0^3 10(3) - 3x^2(3) + \frac{1}{3}(3)^3 - \left(10(0) - 3x^2(0) + \frac{1}{3}(0)^3\right) \ dx$$

$$V = \int_0^3 30 - 9x^2 + 9 \ dx$$

$$V = \int_0^3 39 - 9x^2 \ dx$$

Integrate with respect to x, then evaluate over the interval.

$$V = 39x - 3x^3 \Big|_0^3$$

$$V = 39(3) - 3(3)^3 - \left(39(0) - 3(0)^3\right)$$

$$V = 117 - 81$$

$$V = 36$$

Because we were told in the problem that the volume of the solid S was 36, and we got 36 here, we know that the equation given in answer choice A is the correct one.