Equation of the tangent plane

The tangent plane is the 3D version of the tangent line. Remember how we learned that the tangent line was a line that just barely skims against the graph of a function, intersecting it at only one point? Well, put that function in 3D space, and now the tangent line becomes a tangent plane.

Think about it as a flat surface, like a sheet of paper, that comes to balance on top of the graph, intersecting the graph at only one point. The equation of the tangent plane then gives us the slope of the function at the point where they intersect each other, and we can use the equation of the tangent plane to figure out how fast the function is increasing or decreasing at that intersection point.

To find the equation of the tangent plane to a three-dimensional function at a specific point (x_1, y_1, z_1), we'll use the formula

$$z - z_1 = \frac{\partial z}{\partial x}(x_1, y_1, z_1)(x - x_1) + \frac{\partial z}{\partial y}(x_1, y_1, z_1)(y - y_1)$$

where

$\frac{\partial z}{\partial x}(x_1, y_1, z_1)$ is the partial derivative of z with respect to x at the point (x_1, y_1, z_1), and

$\frac{\partial z}{\partial y}(x_1, y_1, z_1)$ is the partial derivative of z with respect to y at the point (x_1, y_1, z_1)

We want to tackle tangent plane problems in three steps:

1. Find the partial derivatives of the function with respect to each variable

2. Evaluate the partial derivatives at the given point to find the slope in each direction

3. Plug the slopes and the given point into the formula for the equation of the tangent plane

$$z - z_1 = \frac{\partial z}{\partial x}(x_1, y_1, z_1)(x - x_1) + \frac{\partial z}{\partial y}(x_1, y_1, z_1)(y - y_1)$$

Example

Find the equation of the tangent plane of the function at the given point.

$$z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$$

at $P(-4,3,5)$

We'll start by finding partial derivatives of the function z with respect to x and y.

$$\frac{\partial z}{\partial x} = \frac{2x}{16}$$

$$\frac{\partial z}{\partial x} = \frac{x}{8}$$

and

$$\frac{\partial z}{\partial y} = \frac{2y}{9}$$

Plugging the given point $P(-4,3,5)$ into the partial derivatives, we'll get values for the slope in each direction.

$$\frac{\partial z}{\partial x} P(-4,3,5) = \frac{-4}{8}$$

$$\frac{\partial z}{\partial x} P(-4,3,5) = -\frac{1}{2}$$

and

$$\frac{\partial z}{\partial y}(-4,3,5) = \frac{2(3)}{9}$$

$$\frac{\partial z}{\partial y}(-4,3,5) = \frac{2}{3}$$

Now we'll plug the slope in each direction and the point $P(-4,3,5)$ into the formula for the equation of the tangent plane.

$$z - z_1 = \frac{\partial z}{\partial x}(x_1, y_1, z_1)(x - x_1) + \frac{\partial z}{\partial y}(x_1, y_1, z_1)(y - y_1)$$

$$z - 5 = -\frac{1}{2}\left[x - (-4)\right] + \frac{2}{3}(y - 3)$$

$$z - 5 = -\frac{1}{2}(x + 4) + \frac{2}{3}y - 2$$

$$z - 5 = -\frac{1}{2}x - 2 + \frac{2}{3}y - 2$$

$$z = -\frac{1}{2}x + \frac{2}{3}y + 1$$