Calculus 3 Workbook Solutions

Cylinders and quadric surfaces

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REDUCING EQUATIONS TO STANDARD FORM

■ 1. What is the standard form and identity of the quadratic surface?

$$16x^2 + 49y^2 + 784z^2 + 128x - 294y - 87 = 0$$

Solution:

Isolate the terms with x, y, and z, then complete the squares.

$$(16x^2 + 128x + 256 - 256) + (49y^2 - 294y + 441 - 441) + 784z^2 - 87 = 0$$

$$(16x^2 + 128x + 256) - 256 + (49y^2 - 294y + 441) - 441 + 784z^2 - 87 = 0$$

$$16(x^2 + 8x + 16) + 49(y^2 - 6y + 9) + 784z^2 - 87 - 256 - 441 = 0$$

$$16(x + 4)^2 + 49(y - 3)^2 + 784z^2 - 784 = 0$$

Divide each term by 784 (the least common multiple of 16, 49, and 784).

$$\frac{16(x + 4)^2}{784} + \frac{49(y - 3)^2}{784} + z^2 - 1 = 0$$

$$\frac{(x + 4)^2}{49} + \frac{(y - 3)^2}{16} + z^2 = 1$$

$$\frac{(x + 4)^2}{7^2} + \frac{(y - 3)^2}{4^2} + z^2 = 1$$

The surface is the ellipsoid centered at $(-4,3,0)$.

■ 2. What is the standard form and identity of the quadratic surface?

$$25y^2 + 9z^2 - 50y + -36z - 225x - 839 = 0$$

Solution:

Isolate the terms with x, y, and z, then complete the squares.

$$(25y^2 - 50y + 25 - 25) + (9z^2 - 36z + 36 - 36) - 225x - 839 = 0$$

$$(25y^2 - 50y + 25) - 25 + (9z^2 - 36z + 36) - 36 - 225x - 839 = 0$$

$$25(y^2 - 2y + 1) + 9(z^2 - 4z + 4) - 225x - 839 - 25 - 36 = 0$$

$$25(y - 1)^2 + 9(z - 2)^2 - 225x - 900 = 0$$

$$25(y - 1)^2 + 9(z - 2)^2 - 225(x + 4) = 0$$

Divide each term by 225 (the least common multiple of 25 and 9).

$$\frac{25(y - 1)^2}{225} + \frac{9(z - 2)^2}{225} - (x + 4) = 0$$

$$\frac{(y - 1)^2}{9} + \frac{(z - 2)^2}{25} = x + 4$$

$$x + 4 = \frac{(y - 1)^2}{3^2} + \frac{(z - 2)^2}{5^2}$$

The surface is the elliptic paraboloid with vertex at $(-4,1,2)$.

3. What is the standard form and identity of the quadratic surface?

$$9x^2 - 9y^2 + 4z^2 + 18x - 36y - 8z = 23$$

Solution:

Isolate the terms with x, y, and z, then complete the squares.

$$(9x^2 + 18x + 9 - 9) - (9y^2 + 36y + 36 - 36) + (4z^2 - 8z + 4 - 4) = 23$$

$$(9x^2 + 18x + 9) - 9 - (9y^2 + 36y + 36) + 36 + (4z^2 - 8z + 4) - 4 = 23$$

$$9(x^2 + 2x + 1) - 9(y^2 + 4y + 4) + 4(z^2 - 2z + 1) - 9 + 36 - 4 = 23$$

$$9(x + 1)^2 - 9(y + 2)^2 + 4(z - 1)^2 + 23 = 23$$

Divide each term by 36 (the least common multiple of 4 and 9).

$$\frac{9(x + 1)^2}{36} - \frac{9(y + 2)^2}{36} + \frac{4(z - 1)^2}{36} = 0$$

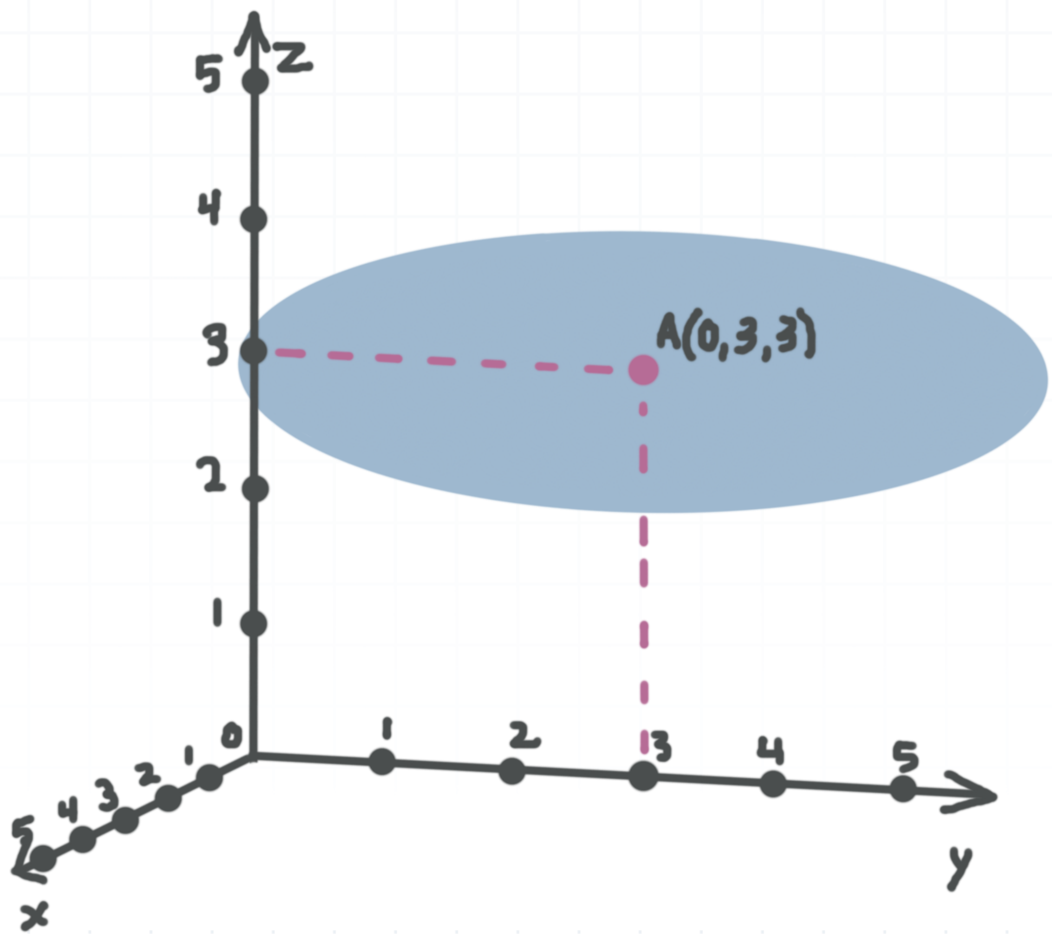
$$\frac{(x + 1)^2}{4} - \frac{(y + 2)^2}{4} + \frac{(z - 1)^2}{9} = 0$$

$$\frac{(x + 1)^2}{2^2} + \frac{(z - 1)^2}{3^2} - \frac{(y + 2)^2}{2^2} = 0$$

The surface is the elliptic cone centered at $(-1, -2, 1)$.

SKETCHING THE SURFACE

■ 1. Find the equation of the surface if its x- and z- principal axes have length 4 and 2 respectively.



Solution:

The surface is the ellipsoid with center $(0,3,3)$. The standard equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1$$

where (h,k,l) is the center, and a, b, and c are the x, y, and z semi-axes.

$a = 4/2 = 2$

$b = 3$

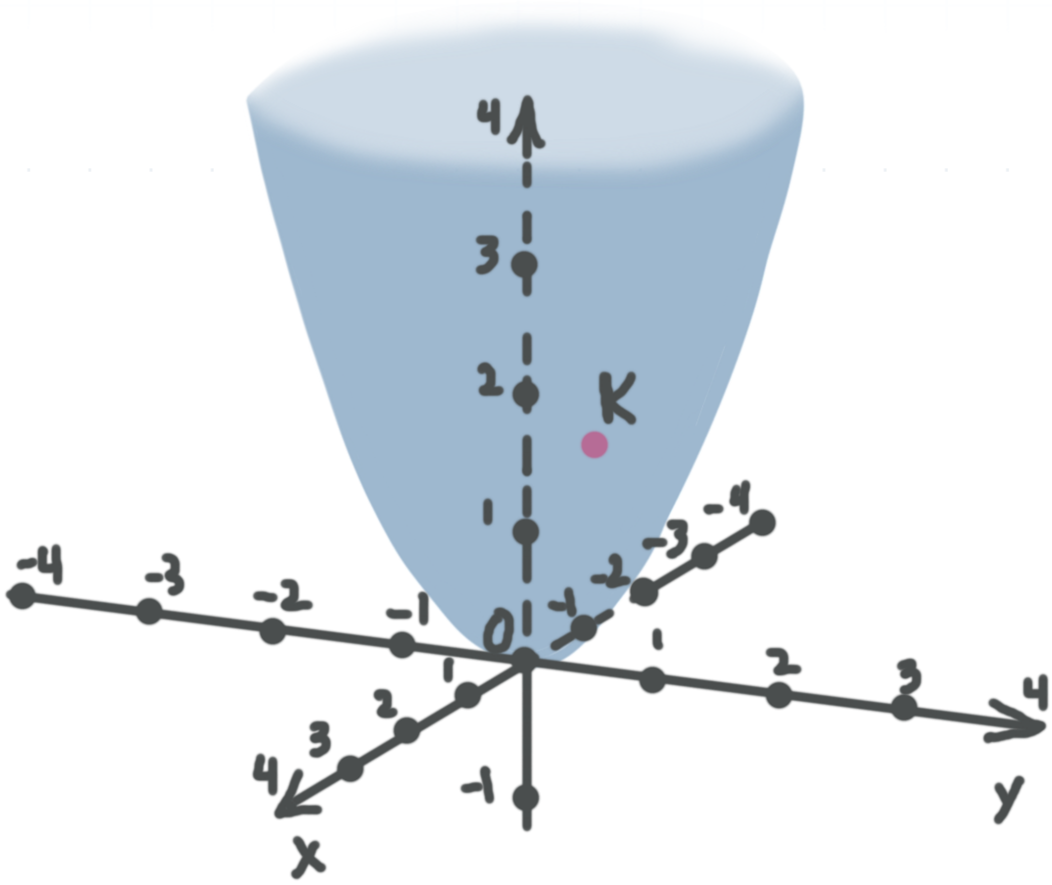
$c = 2/2 = 1$

And the center is $h = 0$, $k = 3$, and $l = 3$. Substitute these into the equation.

$$\frac{(x-0)^2}{2^2} + \frac{(y-3)^2}{3^2} + \frac{(z-3)^2}{1^2} = 1$$

$$\frac{x^2}{2^2} + \frac{(y-3)^2}{3^2} + (z-3)^2 = 1$$

■ 2. Find the equation of the circular paraboloid that passes through $K(1,1,2)$.



Solution:

The standard equation of circular paraboloid with a z-axis of symmetry is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{a^2}=z-l$$

where (h,k,l) is the center. Since from the picture $h=k=l=0$, the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=z$$

To find the value of the constant a, substitute $K(1,1,2)$ into the equation.

$$\frac{1^2}{a^2}+\frac{1^2}{a^2}=2$$

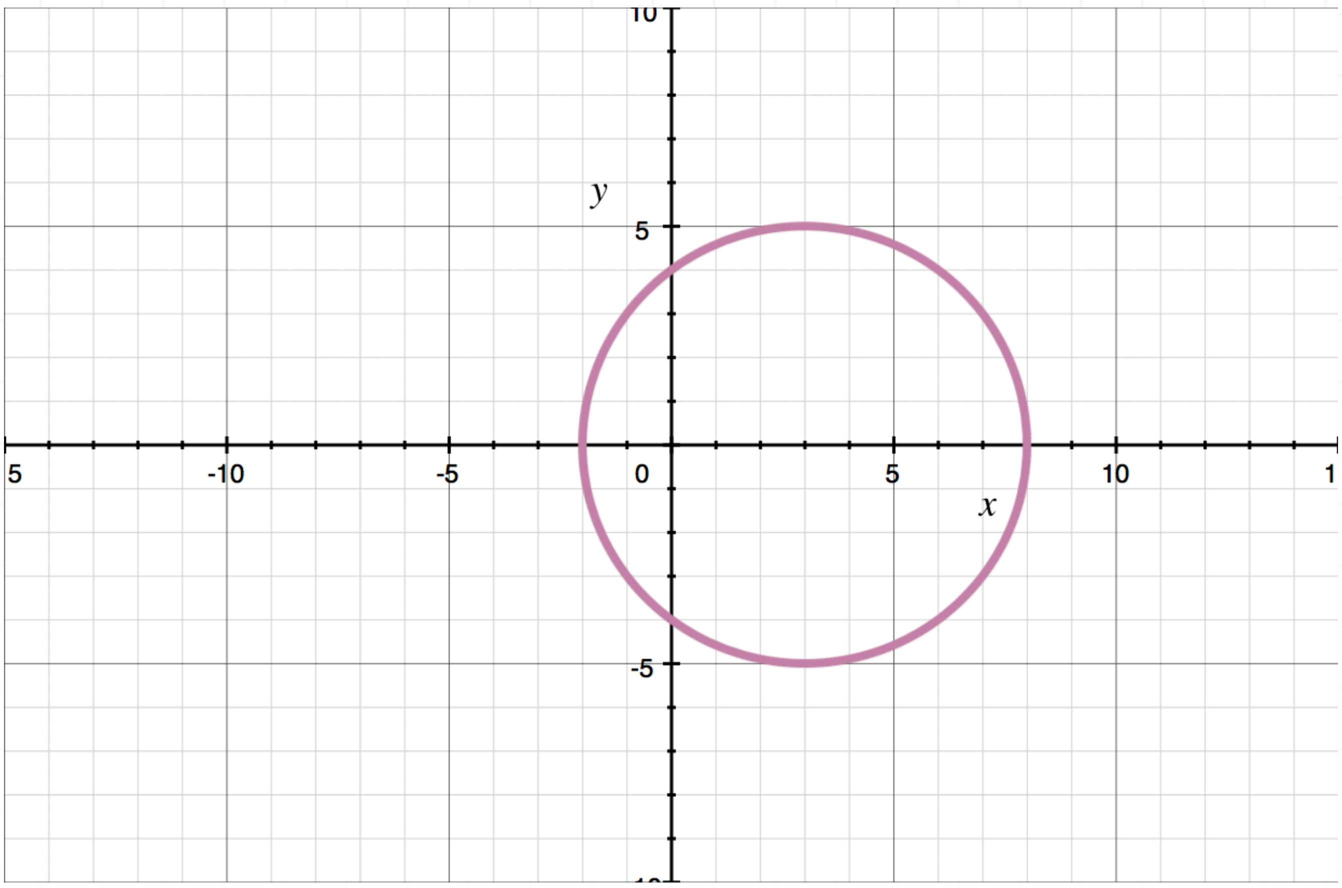
$$\frac{2}{a^2}=2$$

$$a^2=1$$

Since a is positive, $a=1$. So the equation is

$$x^2+y^2=z$$

■ 3. Find the equation of the surface obtained by rotating the circle about the x-axis.



Solution:

The standard equation of a sphere is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

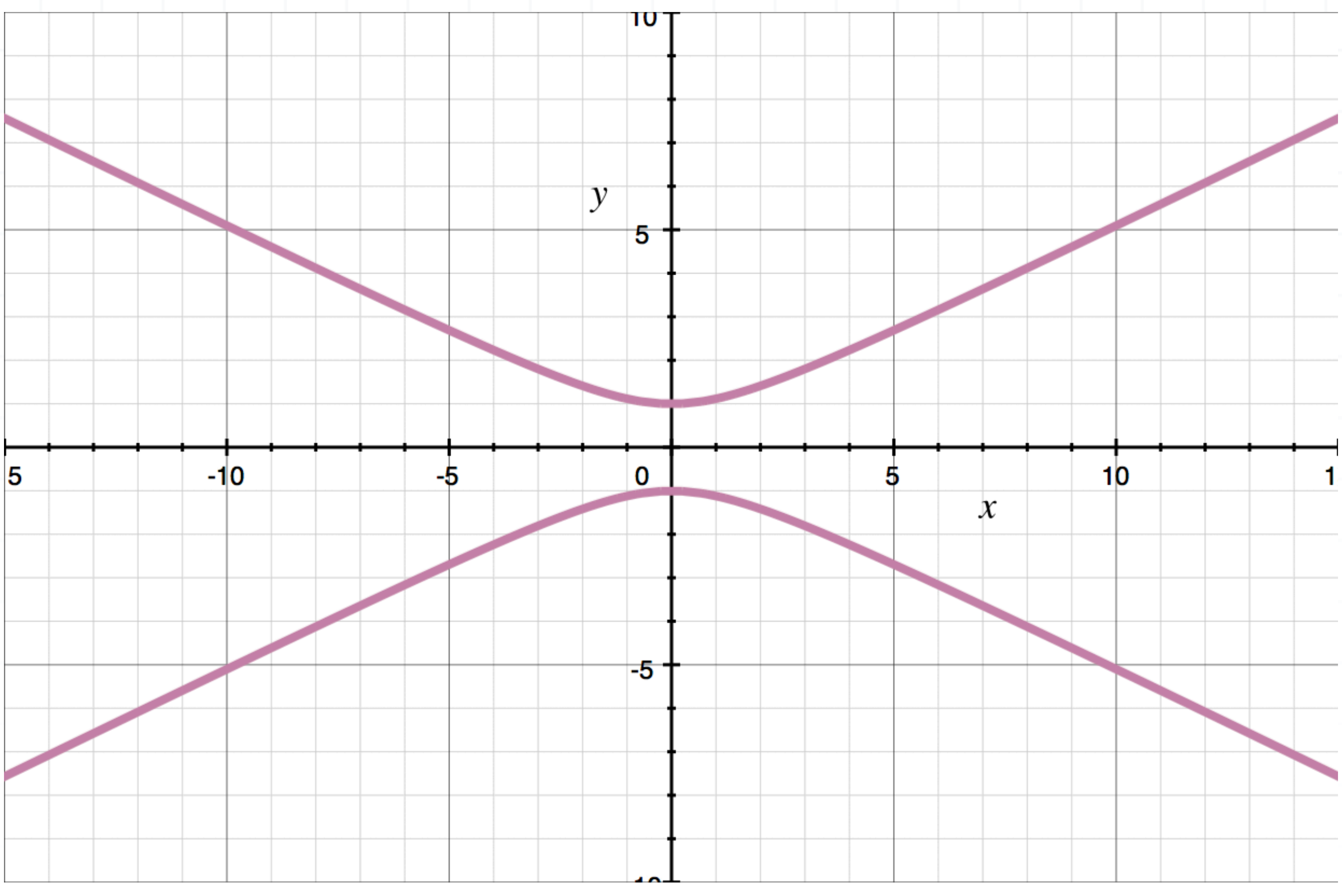
where (h,k,l) is the center and r is the radius.

The radius of the sphere is equal to the radius of the given circle, $r=5$, and the center of the sphere has the same coordinates as the center of the circle in space, $(3,0,0)$, so the equation of the sphere is

$$(x-3)^2+y^2+z^2=5^2$$

4. Determine the identity and the equation of the surface obtained by rotating the hyperbola about the x-axis.

$$\frac{x^2}{2^2} - z^2 = -1$$



Solution:

The surface obtained by rotating this hyperbola about the x-axis is a circular hyperboloid of two sheets (special case of elliptic hyperboloid of two sheets). The standard equation of a circular hyperboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} + \frac{(z-l)^2}{c^2} = -1$$

where (h, k, l) is the center, c is the z semi axis (distance from the vertex of one sheet to the center), and a is a constant.

The semi axis of the hyperboloid is equal to the semi axis of hyperbola, $c = 1$, and the center of the hyperboloid has the same coordinates as the center of the hyperbola in space, $(0,0,0)$.

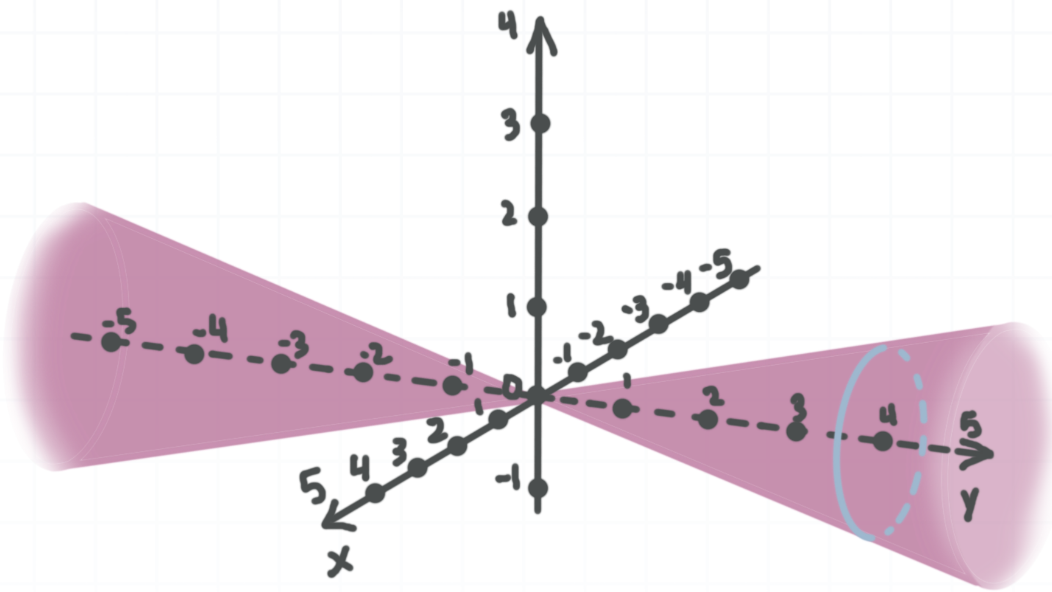
The value of the constant a can be determined from the hyperbola equation. Since the x-term is $x^2/2^2$, then $a = 2$.

So the equation of the circular hyperboloid of two sheets is

$$\frac{x^2}{2^2} + \frac{y^2}{2^2} - z^2 = -1$$

TRACES TO SKETCH AND IDENTIFY THE SURFACE

■ 1. Find the identity and the equation of the surface that has a trace $x^2 + z^2 = 1$ for $y = 4$.



Solution:

The surface in the picture is the cone with center at $(0,0,0)$. Since it has a circular trace, it is a circular cone.

The standard equation of a circular cone is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{a^2} = 0$$

where (h,k,l) is the center, and a and b are the constants. Since $h = k = l = 0$, we get

$$\frac{(x-0)^2}{a^2} - \frac{(y-0)^2}{b^2} + \frac{(z-0)^2}{a^2} = 0$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{a^2} = 0$$

Substitute $y = 4$ to get a trace.

$$\frac{x^2}{a^2} - \frac{4^2}{b^2} + \frac{z^2}{a^2} = 0$$

$$x^2 + z^2 = \frac{4^2 a^2}{b^2}$$

Since the trace is $x^2 + z^2 = 1$,

$$\frac{4^2 a^2}{b^2} = 1$$

$$b^2 = 4^2 a^2$$

Substitute $4^2 a^2$ for b^2 into the equation of the cone.

$$\frac{x^2}{a^2} - \frac{y^2}{4^2 a^2} + \frac{z^2}{a^2} = 0$$

$$x^2 - \frac{y^2}{4^2} + z^2 = 0$$

■ 2. Find the trace of the surface in the plane $y = 7$ and identify it.

$$\frac{(x+5)^2}{81} + \frac{(y-3)^2}{4} - \frac{(z+8)^2}{49} = 1$$

Solution:

Substitute the value of 7 for y into the surface equation to get a trace.

$$\frac{(x+5)^2}{81}+\frac{(7-3)^2}{4}-\frac{(z+8)^2}{49}=1$$

$$\frac{(x+5)^2}{9^2}+\frac{(4)^2}{4}-\frac{(z+8)^2}{7^2}=1$$

$$\frac{(x+5)^2}{9^2}+4-\frac{(z+8)^2}{7^2}=1$$

$$\frac{(x+5)^2}{9^2}-\frac{(z+8)^2}{7^2}=-3$$

This is the hyperbola with center $(-5,7,-8)$ and equation

$$\frac{(x+5)^2}{9^2}-\frac{(z+8)^2}{7^2}=-3$$

■ 3. Find the traces of the surface in the planes $x=-2$, $y=8$, and $z=-4$ and use them to identify the surface.

$$\frac{(x+2)^2}{49}+\frac{(y-8)^2}{16}=z+5$$

Solution:

For $x = -2$,

$$\frac{(-2+2)^2}{49} + \frac{(y-8)^2}{16} = z + 5$$

$$\frac{(y-8)^2}{16} = z + 5$$

This is a parabola in the plane $x = -2$ with vertex $(-2,8,-5)$.

For $y = 8$,

$$\frac{(x+2)^2}{49} + \frac{(8-8)^2}{16} = z + 5$$

$$\frac{(y-8)^2}{16} = z + 5$$

This is a parabola in the plane $x = -2$ with vertex $(-2,8,-5)$.

For $z = -4$,

$$\frac{(x+2)^2}{49} + \frac{(y-8)^2}{16} = -4 + 5$$

$$\frac{(x+2)^2}{7^2} + \frac{(y-8)^2}{4^2} = 1$$

This is an ellipse in the plane $z = -4$ with center $(-2,8,-4)$ and semi axes 7 and 4.

Since the traces of this surface are an ellipse and two parabolas, the surface is elliptic paraboloid.

W W W . K R I S T A K I N G M A T H . C O M