Topic: Local extrema and saddle points

Question: Where are the local extrema of the function?

$$f(x, y) = 2x^2 + 3y^2 - 4y - 5$$

Answer choices:

A On the plane R^2, where $4x = 0$ and $3y - 2 = 0$.

B On the plane R^2, where $x = 0$ and $6y + 4 = 1$.

C On the plane R^2, where $4x - 6 = 1$ and $y = 0$.

D On the plane R^2, where $4x - 6 = 0$ and $3y - 2 = 0$.

Solution: A

The function is defined everywhere on the real plane R^2. Therefore, its domain is R^2. The partial derivatives of the function are

$$f_x(x, y) = 4x$$

$$f_y(x, y) = 6y - 4$$

Setting these equations equal to 0 gives $4x = 0$ and $6y - 4 = 0$, or $3y - 2 = 0$. The question doesn't require us to go further, but we could solve these equations to say that the extrema of the function occurs at $(0,2/3)$.

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Question: Which equation verifies that $(2,1)$ is the saddle point of the function?

$$f(x, y) = 2x^2 - 6xy - 2x + 12y + 7$$

Answer choices:

A $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) + \left[f_{xy}(2,1)\right]^2 < 0$

B $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) + \left[f_{xy}(2,1)\right]^2 > 0$

C $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) - \left[f_{xy}(2,1)\right]^2 < 0$

D $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) - \left[f_{xy}(2,1)\right]^2 \geq 0$

Solution: C

Find the partial derivatives of $f(x, y)$.

$$f_x(x, y) = 4x - 6y - 2$$

$$f_y(x, y) = -6x + 12$$

Setting these functions equal to 0 gives the following system of equations:

$$4x - 6y - 2 = 0$$

$$-6x + 12 = 0$$

Solving the system, we find that $x = 2$ and $y = 1$. Now we'll take second-order partial derivatives and evaluate them at $(2,1)$.

$$f_{xx}(2, 1) = 4$$

$$f_{yy}(2, 1) = 0$$

$$f_{xy}(2, 1) = -6$$

We'll use the second derivative test to classify $(2,1)$.

$$D(2,1) = f_{xx}(2,1)f_{yy}(2,1) - \left[f_{xy}(2,1)\right]^2$$

$$D(2,1) = (4)(0) - (-6)^2$$

$$D(2,1) < 0$$

Because $D(2,1) < 0$, $(2,1)$ is a saddle point of the function.

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Question: Which statement is true about the local extrema of the function?

$$f(x, y) = 2x^2 + y^3 - 6xy - 12y$$

Answer choices:

A Local minimum at $(6,4)$ Local maximum at $\left(-\frac{3}{2}, -1\right)$

B Local maximum at $(6,4)$ Local minimum at $\left(-\frac{3}{2}, -1\right)$

C Local minimum at $(6,4)$ No local maximum

D Local maximum at $\left(-\frac{3}{2}, -1\right)$ No local minimum

Solution: C

Find the partial derivatives of $f(x, y)$.

$$f_x(x, y) = 4x - 6y$$

$$f_y(x, y) = 3y^2 - 6x - 12$$

Set these equations equal to 0.

$$4x - 6y = 0$$

$$3y^2 - 6x - 12 = 0$$

If we solve the system, we find that the function has critical points

$\left(-\frac{3}{2}, -1\right)$ and $(6,4)$

Find the second-order partial derivatives.

$$f_{xx}(x, y) = 4$$

$$f_{yy}(x, y) = 6y$$

$$f_{xy}(x, y) = -6$$

Use the second derivative test to classify each critical point.

$$D = f_{xx}\left(x, y\right) f_{yy}\left(x, y\right) - \left[f_{xy}\left(x, y\right)\right]^2$$

$$D\left(-\frac{3}{2}, -1\right) = (4)\left(6\left(-1\right)\right) - (-6)^2$$

$$D\left(-\frac{3}{2}, -1\right) = -60 < 0$$

Therefore, this point is a saddle point, not a local extremum.

$$D(6,4) = (4)\left(6(4)\right) - (-6)^2$$

$$D(6,4) = 60 > 0$$

Therefore, this point is an extremum. Since $f_{xx}(6,4) = 4 > 0$, then $(6,4)$ is a local minimum.