Calculus 3 Workbook Solutions

Optimization

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CRITICAL POINTS

■ 1. Find a set of critical points for $f(t, s)$.

$$f(t, s) = \ln \frac{2t^3 - 12t^2 + 1}{s^2 + 6s + 11}$$

Solution:

Rewrite f using laws of logs.

$$f(t, s) = \ln(2t^3 - 12t^2 + 1) - \ln(s^2 + 6s + 11)$$

Use chain rule to find partial derivatives.

$$\frac{\partial f}{\partial t} = \frac{6t^2 - 24t}{2t^3 - 12t^2 + 1}$$

$$\frac{\partial f}{\partial s} = -\frac{2s + 6}{s^2 + 6s + 11}$$

Set both partial derivatives equal to 0 and use those as a system of equations to find critical points.

$$\frac{6t^2 - 24t}{2t^3 - 12t^2 + 1} = 0$$

$$-\frac{2s + 6}{s^2 + 6s + 11} = 0$$

That gives the system

$$6t^2 - 24t = 0$$

$$2s + 6 = 0$$

and then

$$t(t - 4) = 0$$

$$s + 3 = 0$$

The solutions to the system are $(0, -3)$ and $(4, -3)$.

Check if each of these points lie within the domain of f.

$$\frac{2(0)^3 - 12(0)^2 + 1}{(-3)^2 + 6(-3) + 11} = \frac{1}{2} > 0$$

$$\frac{2(4)^3 - 12(4)^2 + 1}{(-3)^2 + 6(-3) + 11} = \frac{-63}{2} < 0$$

So only the point $(0, -3)$ lies within the domain and is a critical point of the function $f(t, s)$.

■ 2. Find and identify a set of critical points for $f(x, y, z)$.

$$f(x, y, z) = x^2 \cos(y + z)$$

Solution:

Use chain rule to find partial derivatives.

$$\frac{\partial f}{\partial x} = 2x\cos(y+z)$$

$$\frac{\partial f}{\partial y} = -x^2\sin(y+z)$$

$$\frac{\partial f}{\partial z} = -x^2\sin(y+z)$$

Set the partial derivatives equal to 0 and use those as a system of equations to find critical points.

$$2x\cos(y+z) = 0$$

$$-x^2\sin(y+z) = 0$$

$$-x^2\sin(y+z) = 0$$

Since the system of equations

$$\sin\theta = 0$$

$$\cos\theta = 0$$

has no real solutions, $\cos(y+z)$ and $\sin(y+z)$ can't be equal to 0 simultaneously. So the solution to the system is $x=0$, or the yz-plane.

■ 3. Find a set of critical points for $f(x_1, x_2, x_3, x_4)$.

$$f(x_1, x_2, x_3, x_4) = x_1^2 - 2x_1x_2 + 2x_2^2 - 4x_3^2 + 4x_1 + 5x_4^2 - 10x_4 + 6$$

Solution:

Use chain rule to find partial derivatives.

$$\frac{\partial f}{\partial x_1} = 2x_1 - 2x_2 + 4$$

$$\frac{\partial f}{\partial x_2} = -2x_1 + 4x_2$$

$$\frac{\partial f}{\partial x_3} = -8x_3$$

$$\frac{\partial f}{\partial x_4} = 10x_4 - 10$$

Setting all the partial derivatives equal to 0 to create a system of equations gives

$$2x_1 - 2x_2 + 4 = 0$$

$$-2x_1 + 4x_2 = 0$$

$$-8x_3 = 0$$

$$10x_4 - 10 = 0$$

The solution to the system is $(x_1, x_2, x_3, x_4) = (-4, -2, 0, 1)$, so this is the critical point of the function.

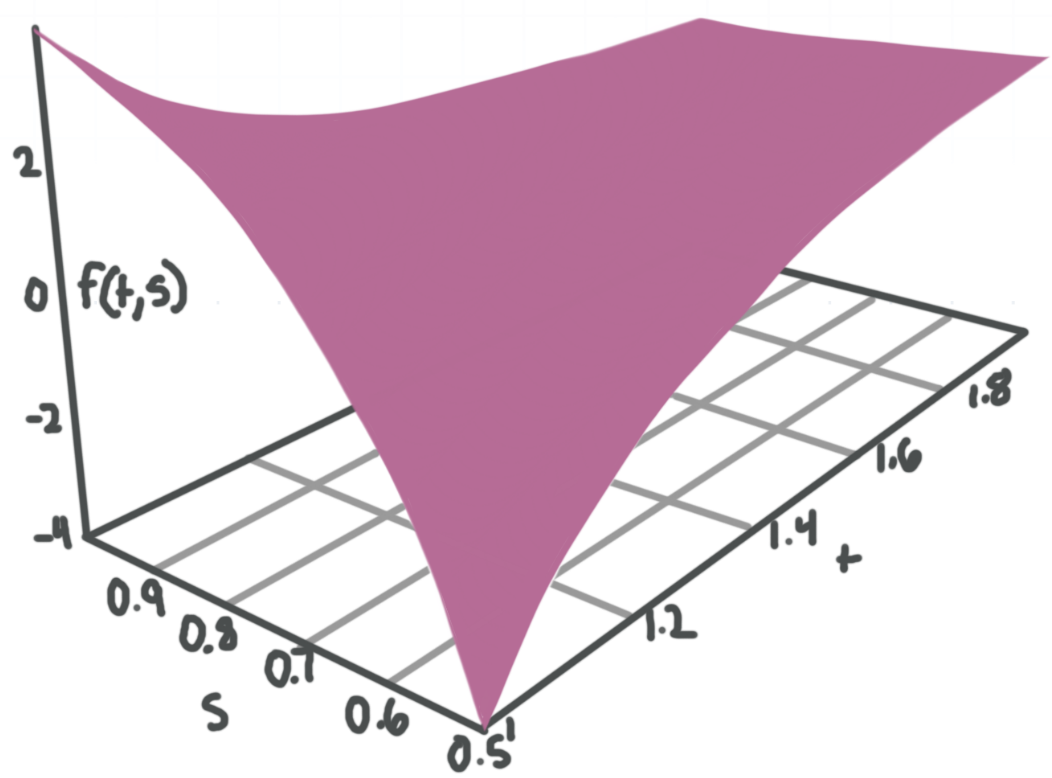
SECOND DERIVATIVE TEST

■ 1. Use the second derivative test to classify the critical points of $f(t, s)$.

$$f(t,s) = \frac{t^5s - 3t + 6s^2}{s^2t^5}$$

Solution:

A sketch of the function is



Rewrite f.

$$f(t,s) = \frac{t^5s}{s^2t^5} - \frac{3t}{s^2t^5} + \frac{6s^2}{s^2t^5}$$

$$f(t,s) = \frac{1}{s} - \frac{3}{s^2t^4} + \frac{6}{t^5}$$

$$f(t,s) = s^{-1} - 3s^{-2}t^{-4} + 6t^{-5}$$

Use chain rule to find first-order partial derivatives.

$$\frac{\partial f}{\partial t} = 12s^{-2}t^{-5} - 30t^{-6}$$

$$\frac{\partial f}{\partial s} = 6s^{-3}t^{-4} - s^{-2}$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$12s^{-2}t^{-5} - 30t^{-6} = 0$$

$$6s^{-3}t^{-4} - s^{-2} = 0$$

Solve the system for $t \neq 0$ and $s \neq 0$.

$$2t - 5s^2 = 0$$

$$st^4 - 6 = 0$$

$$t = 2.5s^2$$

$$s(2.5s^2)^4 - 6 = 0$$

$$s^9 2.5^4 = 6$$

The values that satisfy the system are $s = 2^{\frac{5}{9}}3^{\frac{1}{9}}5^{-\frac{4}{9}} \approx 0.81$ and $t = 3^{\frac{2}{9}}10^{\frac{1}{9}} \approx 1.65$, so the solution is $(1.65, 0.81)$.

Calculate the second-order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial t^2} = -60s^{-2}t^{-6} + 180t^{-7}$$

$$\frac{\partial^2 f}{\partial s^2} = -18s^{-4}t^{-4} + 2s^{-3}$$

$$\frac{\partial^2 f}{\partial t \partial s} = \frac{\partial^2 f}{\partial s \partial t} = -24s^{-3}t^{-5}$$

Substitute $(1.65,0.81)$ for (t, s).

$$\frac{\partial^2 f}{\partial t^2}(1.65,0.81) = -60 \cdot 0.81^{-2}1.65^{-6} + 180 \cdot 1.65^{-7} \approx 0.87$$

$$\frac{\partial^2 f}{\partial s^2}(1.65,0.81) = -18 \cdot 0.81^{-4}1.65^{-4} + 2 \cdot 0.81^{-3} \approx -1.88$$

$$\frac{\partial^2 f}{\partial t \partial s}(1.65,0.81) = -24 \cdot 0.81^{-3}1.65^{-5} \approx -3.69$$

Perform the second derivative test.

$$D(t, s) = \frac{\partial^2 f}{\partial t^2} \cdot \frac{\partial^2 f}{\partial s^2} - \left(\frac{\partial^2 f}{\partial t \partial s}\right)^2$$

$$D(1.65,0.81) = 0.87 \cdot (-1.88) - (-3.69)^2 = -15.2517 < 0$$

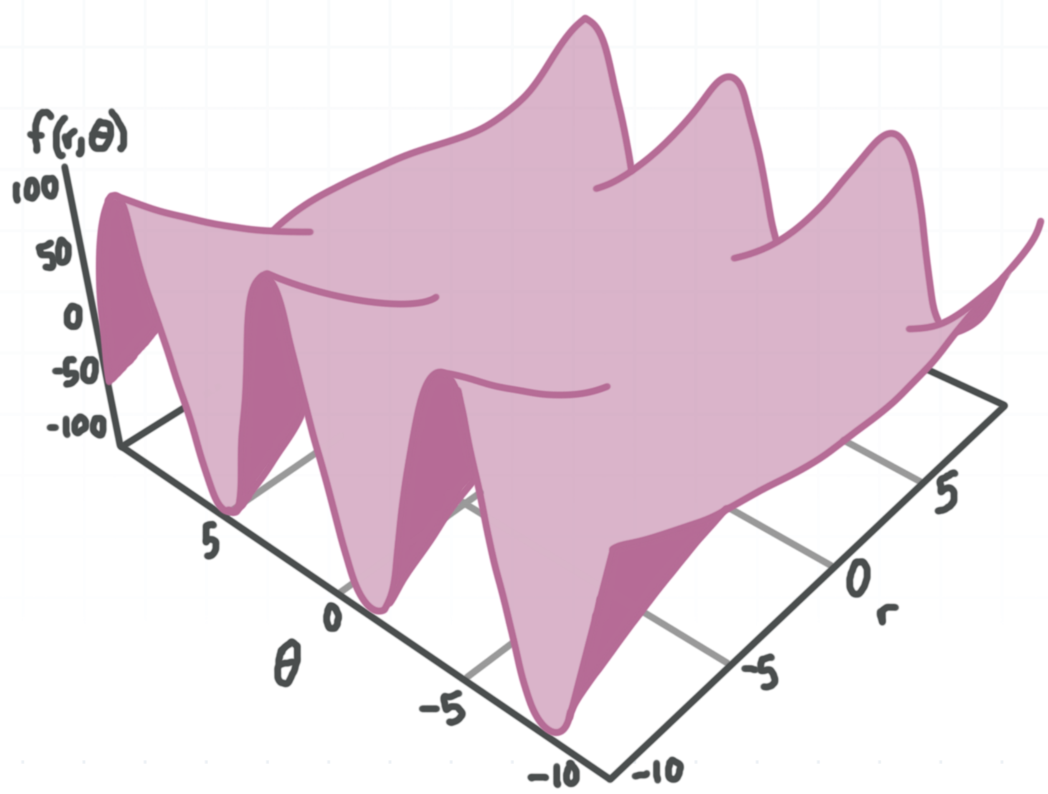
So $(1.65,0.81)$ is a saddle point.

2. Use the second derivative test to classify the critical points of $f(r, \theta)$.

$$f(r, \theta) = (r^2 - 2r - 3)\sin\theta$$

Solution:

A sketch of the function is



Find first-order partial derivatives.

$$\frac{\partial f}{\partial r} = (2r - 2)\sin\theta$$

$$\frac{\partial f}{\partial \theta} = (r^2 - 2r - 3)\cos\theta$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$(2r - 2)\sin\theta = 0$$

$$(r^2 - 2r - 3)\cos\theta = 0$$

Solve the system for r and θ.

$$(r - 1)\sin\theta = 0$$

$$(r + 1)(r - 3)\cos\theta = 0$$

Since $\sin\theta$ and $\cos\theta$ can't be equal to 0 simultaneously, we have the three sets of solutions:

1) $r - 1 = 0$ and $\cos\theta = 0$

$r = 1$ and $\theta = \frac{\pi}{2} + \pi n$ where n is any integer number

2) $r + 1 = 0$ and $\sin\theta = 0$

$r = -1$ and $\theta = \pi m$ where m is any integer number

3) $r - 3 = 0$ and $\sin\theta = 0$

$r = 3$ and $\theta = \pi k$ where k is any integer number

Calculate the second-order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial r^2} = 2\sin\theta$$

$$\frac{\partial^2 f}{\partial \theta^2} = -(r^2 - 2r - 3)\sin\theta$$

$$\frac{\partial^2 f}{\partial r \partial \theta} = \frac{\partial^2 f}{\partial \theta \partial r} = (2r - 2)\cos\theta$$

Perform the second derivative test.

$$D(r, \theta) = \frac{\partial^2 f}{\partial r^2} \cdot \frac{\partial^2 f}{\partial \theta^2} - \left(\frac{\partial^2 f}{\partial r \partial \theta}\right)^2$$

The second derivative test for the first set of solutions is

$$\frac{\partial^2 f}{\partial r^2}\left(1, \frac{\pi}{2} + \pi n\right) = 2\sin\left(\frac{\pi}{2} + \pi n\right) = 2(\text{n is even}) \text{ or } -2(\text{n is odd})$$

$$\frac{\partial^2 f}{\partial \theta^2}\left(1, \frac{\pi}{2} + \pi n\right) = -(1^2 - 2(1) - 3)\sin\left(\frac{\pi}{2} + \pi n\right) = 4(\text{n is even}) \text{ or } -4(\text{n is odd})$$

$$\frac{\partial^2 f}{\partial r \partial \theta}\left(1, \frac{\pi}{2} + \pi n\right) = \frac{\partial^2 f}{\partial \theta \partial r} = (2(1) - 2)\cos\left(\frac{\pi}{2} + \pi n\right) = 0$$

For even n:

$$D\left(1, \frac{\pi}{2} + \pi n\right) = 2 \cdot 4 - 0^2 = 8 > 0$$

For odd n:

$$D\left(1, \frac{\pi}{2} + \pi n\right) = (-2) \cdot (-4) - 0^2 = 8 > 0$$

Since $\partial^2 f/\partial r^2 > 0$ for even n, and < 0 otherwise, the set of critical points $r = 1$, $\theta = \pi/2 + \pi n$ where n is even, are the local minima, and the set of critical points $r = 1$, $\theta = \pi/2 + \pi n$ where n is odd, are the local maxima.

Perform the second derivative test for the second set of solutions is

$$\frac{\partial^2 f}{\partial r^2}(-1,\pi m) = 2\sin(\pi m) = 0$$

$$\frac{\partial^2 f}{\partial \theta^2}(-1,\pi m) = -((-1)^2 - 2(-1) - 3)\sin(\pi m) = 0$$

$$\frac{\partial^2 f}{\partial r \partial \theta}(-1,\pi m) = \frac{\partial^2 f}{\partial \theta \partial r} = (2(-1) - 2)\cos(\pi m) = \pm 4$$

Then

$$D(-1,\pi m) = 0 \cdot 0 - (\pm 4)^2 = -16 < 0$$

Since $D < 0$, the set of critical points $r = -1$, $\theta = \pi m$ where m is an integer, are saddle points.

Perform the second derivative test for the third set of solutions is

$$\frac{\partial^2 f}{\partial r^2}(3,\pi k) = 2\sin\left(\pi k\right) = 0$$

$$\frac{\partial^2 f}{\partial \theta^2}(3,\pi k) = -(3^2 - 2(3) - 3)\sin\left(\pi k\right) = 0$$

$$\frac{\partial^2 f}{\partial r \partial \theta}(3,\pi k) = \frac{\partial^2 f}{\partial \theta \partial r} = (2(3) - 2)\cos\left(\pi k\right) = \pm 4$$

Then

$$D(3,\pi k) = 0 \cdot 0 - (\pm 4)^2 = -16 < 0$$

Since $D < 0$, the set of critical points $r = 3$, $\theta = \pi k$ where k is integer, are saddle points.

Then the extrema of the function are

Local minima: $r = 1$, $\theta = \frac{\pi}{2} + \pi n$ where n is even

Local maxima: $r = 1$, $\theta = \frac{\pi}{2} + \pi n$ where n is odd

Saddle points: $r = -1$, $\theta = \pi m$, and $r = 3$, $\theta = \pi k$ where m and k are integers

■ 3. Find the set of all possible values of a for which $f(x, y)$ has only one local minimum.

$$f(x, y) = x^2 + ay^2 - 4x + 8y - 6$$

Solution:

Calculate the first order partial derivatives:

$$\frac{\partial f}{\partial x} = 2x - 4$$

$$\frac{\partial f}{\partial y} = 2ay + 8$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$2x - 4 = 0$$

$$2ay + 8 = 0$$

The solution to the system is $x = 2$, $y = -4/a$ for $a \neq 0$. If $a = 0$, then the function has no critical points.

Calculate the second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2a$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

The second derivative test gives

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D\left(2, -\frac{4}{a}\right) = 2 \cdot 2a - 0^2 = 4a$$

Since the function must have the local maximum at this critical point, $D > 0$, and

$$4a > 0$$

$$a > 0$$

If $a > 0$, then $D > 0$ and $\partial^2 f/\partial x^2 = 2 > 0$ at the point $(2, -4/a)$. So the function has only one local minimum when $a > 0$.

LOCAL EXTREMA AND SADDLE POINTS

1. Find the local extrema of $f(t, s)$.

$$f(t, s) = \frac{t^3 + s^3 + 1}{ts}$$

Solution:

Rewrite f.

$$f(t, s) = \frac{t^3}{ts} + \frac{s^3}{ts} + \frac{1}{ts}$$

$$f(t, s) = \frac{t^2}{s} + \frac{s^2}{t} + \frac{1}{ts}$$

$$f(t, s) = t^2s^{-1} + s^2t^{-1} + t^{-1}s^{-1}$$

Find first-order partial derivatives.

$$\frac{\partial f}{\partial t} = 2ts^{-1} - s^2t^{-2} - t^{-2}s^{-1}$$

$$\frac{\partial f}{\partial s} = -t^2s^{-2} + 2st^{-1} - t^{-1}s^{-2}$$

Setting the partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$2ts^{-1} - s^2t^{-2} - t^{-2}s^{-1} = 0$$

$$-t^2s^{-2} + 2st^{-1} - t^{-1}s^{-2} = 0$$

Solve the system for $t \neq 0$ and $s \neq 0$.

$$s^3 - 2t^3 + 1 = 0$$

$$t^3 - 2s^3 + 1 = 0$$

Subtract equations to get

$$3t^3 - 3s^3 = 0$$

$$t^3 = s^3$$

$$t = s$$

and then

$$s^3 - 2s^3 + 1 = 0$$

$$-s^3 + 1 = 0$$

$$s = 1$$

So the solution to the system is $(1,1)$.

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial t^2} = 2s^{-1} + 2s^2t^{-3} + 2s^{-1}t^{-3}$$

$$\frac{\partial^2 f}{\partial s^2} = 2t^2s^{-3} + 2t^{-1} + 2t^{-1}s^{-3}$$

$$\frac{\partial^2 f}{\partial t \partial s} = \frac{\partial^2 f}{\partial s \partial t} = -2ts^{-2} - 2st^{-2} + t^{-2}s^{-2}$$

Evaluate the second order partial derivatives at $(1,1)$.

$$\frac{\partial^2 f}{\partial t^2}(1,1) = 2(1)^{-1} + 2(1)^2(1)^{-3} + 2(1)^{-1}(1)^{-3} = 6$$

$$\frac{\partial^2 f}{\partial s^2}(1,1) = 2(1)^2(1)^{-3} + 2(1)^{-1} + 2(1)^{-1}(1)^{-3} = 6$$

$$\frac{\partial^2 f}{\partial t \partial s}(1,1) = -2(1)(1)^{-2} - 2(1)(1)^{-2} + (1)^{-2}(1)^{-2} = -3$$

Perform the second derivative test.

$$D(t,s) = \frac{\partial^2 f}{\partial t^2} \cdot \frac{\partial^2 f}{\partial s^2} - \left(\frac{\partial^2 f}{\partial t \partial s}\right)^2$$

$$D(1,1) = 6 \cdot 6 - (-3)^2 = 27 > 0$$

Since $D(1,1) > 0$ and $\partial^2 f/\partial t^2(1,1) > 0$, the point $(1,1)$ is a local minima. So find $f(1,1)$.

$$f(1,1) = \frac{(1)^3 + (1)^3 + 1}{(1)(1)} = 3$$

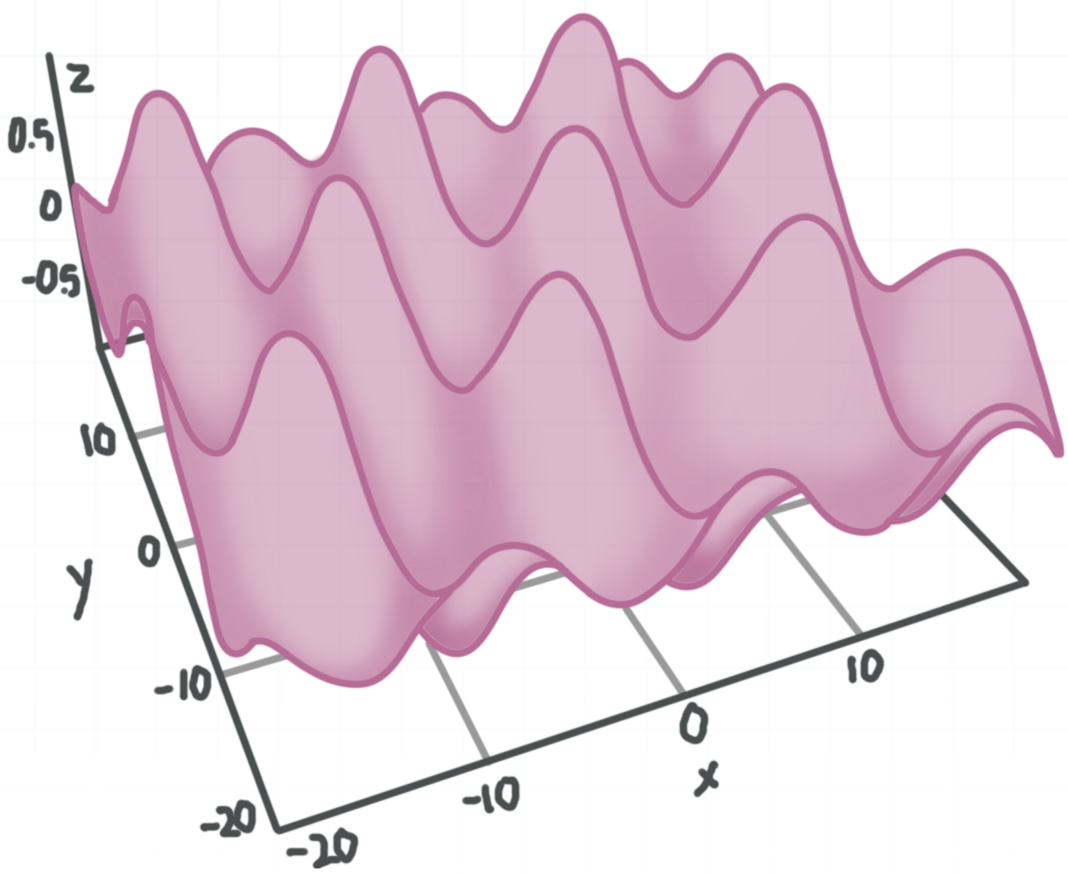
Then $f(1,1) = 3$ is the local minimum.

■ 2. Find the local extrema of $f(x, y)$.

$$f(x,y) = \sin(0.5x)\cos(0.25y)$$

Solution:

A sketch of the surface is



Use chain rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 0.5\cos(0.5x)\cos(0.25y)$$

$$\frac{\partial f}{\partial y} = -0.25\sin(0.5x)\sin(0.25y)$$

Set both partial derivatives equal to 0 to make a system of equations that we can use to find critical points.

$$0.5\cos(0.5x)\cos(0.25y) = 0$$

$$-0.25\sin(0.5x)\sin(0.25y) = 0$$

Solve the system for x and y.

$$\cos(0.5x)\cos(0.25y) = 0$$

$$\sin(0.5x)\sin(0.25y) = 0$$

Since sine and cosine functions of the same angle can't be 0 at the same time, we have two sets of solutions.

1) $\cos(0.5x) = 0$, $\sin(0.25y) = 0$

2) $\cos(0.25y) = 0$, $\sin(0.5x) = 0$

So the solution sets are

1) $0.5x = 0.5\pi + \pi n$, $0.25y = \pi m$

$x = \pi + 2\pi n, y = 4\pi m$ where n and m are integers

2) $0.5x = \pi n$, $0.25y = 0.5\pi + \pi m$

$x = 2\pi n, y = 2\pi + 4\pi m$ where n and m are integers

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2} = -0.5^2 \sin(0.5x)\cos(0.25y)$$

$$\frac{\partial^2 f}{\partial y^2} = -0.5^4 \sin(0.5x)\cos(0.25y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial s \partial t} = -0.5^3 \cos(0.5x)\sin(0.25y)$$

The second derivative test gives

$$D(x,y)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x\partial y}\right)^2$$

The second derivative test for the first solution set, substituting $(\pi+2\pi n, 4\pi m)$ for (x,y) gives

$$\frac{\partial^2 f}{\partial x^2}(\pi+2\pi n,4\pi m)=-0.5^2\sin(0.5(\pi+2\pi n))\cos(0.25(4\pi m))=\pm 0.5^2$$

$$\frac{\partial^2 f}{\partial y^2}(\pi+2\pi n,4\pi m)=-0.5^4\sin(0.5(\pi+2\pi n))\cos(0.25(4\pi m))=\pm 0.5^4$$

$$\frac{\partial^2 f}{\partial x\partial y}(\pi+2\pi n,4\pi m)=-0.5^3\cos(0.5(\pi+2\pi n))\sin(0.25(4\pi m))=0$$

So the second derivative test gives

$$D(\pi+2\pi n,4\pi m)=0.5^6\sin^2(0.5(\pi+2\pi n))\cos^2(0.25(4\pi m))=0.5^6>0$$

When n and m are both even, or when n and m are both odd, $\partial^2 f/\partial x^2(\pi+2\pi n,4\pi m)<0$, so we have a local maximum at the point. Otherwise the point is a local minimum.

Find the value of f.

$$f(\pi+2\pi n,4\pi m)=\sin(0.5(\pi+2\pi n))\cos(0.25(4\pi m))=\sin(0.5\pi+\pi n))\cos(\pi m)=1$$

when n and m are both even, or when n and m are both odd, or equal to -1.

The second derivative test for the second solution set, substituting $(2\pi n, 2\pi + 4\pi m)$ for (x, y) gives

$$\frac{\partial^2 f}{\partial x^2}(2\pi n, 2\pi + 4\pi m) = -0.5^2\sin(0.5(2\pi n))\cos(0.25(2\pi + 4\pi m)) = 0$$

$$\frac{\partial^2 f}{\partial y^2}(2\pi n, 2\pi + 4\pi m) = -0.5^4\sin(0.5(2\pi n))\cos(0.25(2\pi + 4\pi m)) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(2\pi n, 2\pi + 4\pi m) = -0.5^3\cos(0.5(2\pi n))\sin(0.25(2\pi + 4\pi m)) = \pm 0.5^3$$

So the second derivative test gives

$$D(2\pi n, 2\pi + 4\pi m) = 0 \cdot 0 - (\pm 0.5^3)^2 < 0$$

Since $D < 0$ for any n and m, all points from the second solution set are saddle points.

So the extrema of the function are

Local maxima at $(\pi + 2\pi n, 4\pi m, 1)$ when n and m are either both even or both odd.

Local minima at $(\pi + 2\pi n, 4\pi m, -1)$ when n is even and m is odd, or vice versa.

■ 3. Find the equation(s) of the tangent plane to $f(x, y)$ at the function's local maximum.

$$f(x, y) = -x^2 - 2y^2 + 4x - 12y - 9$$

Solution:

Use power rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = -2x + 4$$

$$\frac{\partial f}{\partial y} = -4y - 12$$

Setting both partial derivatives equal to 0 gives us a system of equations that we can use to find critical points.

$$-2x + 4 = 0$$

$$-4y - 12 = 0$$

The solution to the system is $x = 2$ and $y = -3$.

Calculate second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2}(2, -3) = -2$$

$$\frac{\partial^2 f}{\partial y^2}(2, -3) = -4$$

$$\frac{\partial^2 f}{\partial x \partial y}(2, -3) = \frac{\partial^2 f}{\partial s \partial t} = 0$$

Then the second derivative test gives

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

$$D(2, -3) = (-2) \cdot (-4) - (0)^2 = 8 > 0$$

Since $D > 0$ and $\partial^2 f / \partial x^2 < 0$, the point $(2, -3)$ is a local maximum.

Since $\partial f / \partial x (2, -3) = 0$ and $\partial f / \partial x (2, -3) = 0$, the equation of the tangent plane to the surface $z = f(x, y)$ at $(2, -3)$ is

$$z = f(2, -3) = -2^2 - 2(-3)^2 + 4(2) - 12(-3) - 9 = 13$$

So the equation of the tangent plane is $z = 13$.

4. Find the values of a and b where $f(x, y)$ has a local minimum at $(5, -3)$.

$$f(x, y) = 4x^2 + 2y^4 - ax - by + 5$$

Solution:

Use power rule to calculate first order partial derivatives.

$$\frac{\partial f}{\partial x} = 8x - a$$

$$\frac{\partial f}{\partial y} = 8y^3 - b$$

Setting both partial derivatives equal to 0 gives us a system of equations we can use to find critical points.

$$8x - a = 0$$

$$8y^3 - b = 0$$

The solution to the system is $x = a/8$ and $y = \sqrt[3]{b}/2$.

Setting x and y equal to 5 and -3 respectively and using these equations as a system of simultaneous equations to find the values of a and b gives

$$\frac{a}{8} = 5$$

$$\frac{\sqrt[3]{b}}{2} = -3$$

The solution to this system is $a = 8(5) = 40$ and $b = (2(-3))^3 = -216$.

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2}(5, -3) = 8$$

$$\frac{\partial^2 f}{\partial y^2}(5, -3) = 24(-3)^2 = 216$$

$$\frac{\partial^2 f}{\partial x \partial y}(5, -3) = \frac{\partial^2 f}{\partial s \partial t} = 0$$

The second derivative test gives

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

$$D(5, -3) = 8 \cdot 216 - (0)^2 = 1728 > 0$$

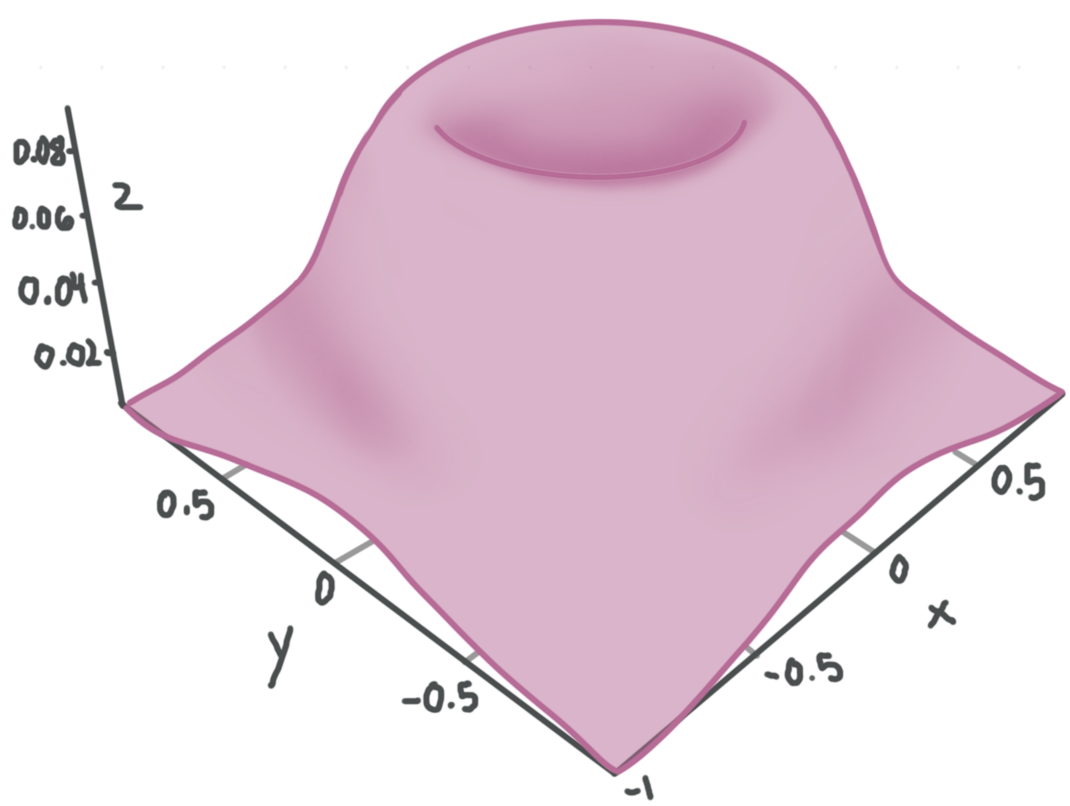
Since $D > 0$ and $\partial^2 f/\partial x^2 > 0$, the point $(5, -3)$ is a local minimum. So the values of a and b are $a = 40$ and $b = -216$.

■ 5. Find and identify the set of local maxima of $f(x, y)$.

$$f(x, y) = e^{-4(x^2+y^2)}(x^2 + y^2)$$

Solution:

A sketch of the surface is



Use product rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = -8xe^{-4(x^2+y^2)}(x^2+y^2) + e^{-4(x^2+y^2)}(2x) = -2xe^{-4(x^2+y^2)}(4x^2+4y^2-1)$$

$$\frac{\partial f}{\partial y} = -8ye^{-4(x^2+y^2)}(x^2+y^2) + e^{-4(x^2+y^2)}(2y) = -2ye^{-4(x^2+y^2)}(4x^2+4y^2-1)$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$-2xe^{-4(x^2+y^2)}(4x^2+4y^2-1) = 0$$

$$-2ye^{-4(x^2+y^2)}(4x^2+4y^2-1) = 0$$

Since $e^a > 0$,

$$x(4x^2+4y^2-1) = 0$$

$$y(4x^2+4y^2-1) = 0$$

Then the solution sets are

1) $4x^2+4y^2-1=0$, or $x^2+y^2=(0.5)^2$, which is a circle with center $(0,0)$ and radius 0.5

2) $x=0$, $y=0$

Since $f(x,y) > 0$ for all (x,y) except $(0,0)$, and $f(0,0)=0$, the point $(0,0)$ is a local (and also global) minimum.

To check if the solution set $(x,y)\,|\,x^2+y^2=(0.5)^2$ are a set of local maxima, we need to perform the second derivative test. Since the points from the set form a continuous curve, all of them may be classified the same way. So we can check any point from the set, for example $(0.5,0)$.

Find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 2e^{-4(x^2+y^2)}(32x^4 + 32x^2y^2 - 20x^2 - 4y^2 + 1)$$

$$\frac{\partial^2 f}{\partial y^2} = 2e^{-4(x^2+y^2)}(32y^4 + 32x^2y^2 - 20y^2 - 4x^2 + 1)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial s \partial t} = 32xye^{-4(x^2+y^2)}(2x^2 + 2y^2 - 1)$$

Evaluate the second order partial derivatives at $(0.5,0)$.

$$\frac{\partial^2 f}{\partial x^2}(0.5,0) = -\frac{4}{e}$$

$$\frac{\partial^2 f}{\partial y^2}(0.5,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0.5,0) = 0$$

The second derivative test gives

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(0.5,0) = -\frac{4}{e} \cdot 0 - (0)^2 = 0$$

So the second derivative test is inconclusive. So consider the arbitrary cross-section through the critical point $(0.5,0)$ along the line $y = t(x - 0.5)$. Substitute $t(x - 0.5)$ for y into the expression for $f(x, y)$.

$$g(x) = f(x, t(x - 0.5)) = e^{-4(x^2+t^2(x-0.5)^2)}(x^2 + t^2(x - 0.5)^2)$$

$$g(x) = e^{(-4(t^2+1)x^2+4t^2x-t^2)}((t^2 + 1)x^2 - t^2x + 0.25t^2)$$

Investigate the sign of $g'(x)$ in the vicinity of the $x = 0.5$.

$$g'(x) = e^{(-4(t^2+1)x^2+4t^2x-t^2)}((-8(t^2 + 1)x + 4t^2)((t^2 + 1)x^2 - t^2x + 0.25t^2) + 2(t^2 + 1)x - t^2)$$

Since $g'(0.5) = 0$ and $g'(0.5^-) > 0$, $g'(0.5+) < 0$, the function $g(x)$ has a local maximum at $x = 0.5$. So since every cross-section through the point $(0.5,0)$ has a local maximum at this point, the function $f(x, y)$ also has a local maximum at $(0.5,0)$.

So the set of local maxima is given by $x^2 + y^2 = (0.5)^2$, which is the circle with center $(0,0)$ and radius 0.5.

■ 6. Find and identify the saddle points and local extrema of $f(x, y)$.

$$f(x, y) = (x - 4)^8 - (y + 7)^{12}$$

Solution:

Use power rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 8(x-4)^7$$

$$\frac{\partial f}{\partial y} = -12(y+7)^{11}$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$8(x-4)^7 = 0$$

$$-12(y+7)^{11} = 0$$

The solution to this system is $x = 4$, $y = -7$.

Find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 8 \cdot 7(x-4)^6$$

$$\frac{\partial^2 f}{\partial y^2} = -12 \cdot 11(y+7)^{10}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Evaluate the second order partial derivatives at $(4, -7)$.

$$\frac{\partial^2 f}{\partial x^2}(4, -7) = 0$$

$$\frac{\partial^2 f}{\partial y^2}(4, -7) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(4,-7)=0$$

Since $D(4,-7)=0$, the second derivative test is inconclusive. Then consider two cross-sections through the critical point $(4,-7)$, along the lines $x=4$ and $y=-7$.

Substitute $x=4$ into the expression for $f(x,y)$.

$$g(y)=f(4,y)=-(y+7)^{12}$$

Since $g(-7)=0$ and $g(y)<0$ when $y\neq-7$, the function has a local maximum at the point.

Substitute $y=-7$ into the expression for $f(x,y)$.

$$h(x)=f(x,-7)=(x-4)^8$$

Since $h(4)=0$ and $h(x)>0$ when $x\neq 4$, the function has a local minimum at the point.

So since two cross sections have minimum and maximum at the critical point $(4,-7)$, it's a saddle point.

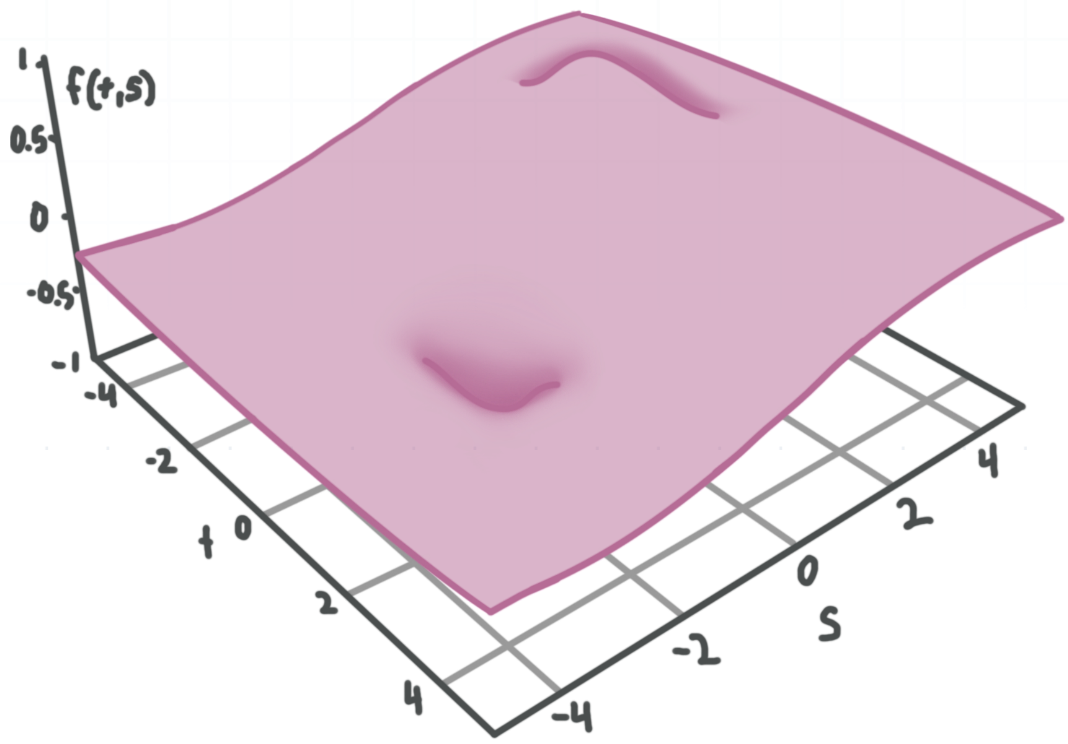
GLOBAL EXTREMA

■ 1. Find the global extrema of $f(t,s)$ over R^2.

$$f(t,s)=\frac{4s}{t^2+2s^2+2}$$

Solution:

A sketch of the surface is



Use quotient rule to find first order partial derivatives.

$$\frac{\partial f}{\partial t}=\frac{-8ts}{(t^2+2s^2+2)^2}$$

$$\frac{\partial f}{\partial s}=\frac{4t^2-8s^2+8}{(t^2+2s^2+2)^2}$$

Setting both of them equal to 0 and using these equations as a system of simultaneous equations to find critical points gives

$$-8ts = 0$$

$$4t^2 - 8s^2 + 8 = 0$$

Then

$$ts = 0$$

$$t^2 - 2s^2 + 2 = 0$$

If $t = 0$, then

$$(0)^2 - 2s^2 + 2 = 0$$

$$s^2 = 1$$

$$s = \pm 1$$

If $s = 0$, then

$$t^2 - 2(0)^2 + 2 = 0$$

$$t^2 = -2$$

and there are no solutions. So the solutions to the system are $(0,1)$ and $(0, -1)$.

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial t^2} = \frac{8s(3t^2 - 2s^2 - 2)}{(t^2 + 2s^2 + 2)^3}$$

$$\frac{\partial^2 f}{\partial s^2} = \frac{16s(-3t^2 + 2s^2 - 6)}{(t^2 + 2s^2 + 2)^3}$$

$$\frac{\partial^2 f}{\partial t \partial s} = \frac{\partial^2 f}{\partial s \partial t} = \frac{8t(-t^2 + 6s^2 - 2)}{(t^2 + 2s^2 + 2)^3}$$

Evaluate the second order partial derivatives at $(0,1)$.

$$\frac{\partial^2 f}{\partial t^2}(0,1) = \frac{8(1)(3(0)^2 - 2(1)^2 - 2)}{((0)^2 + 2(1)^2 + 2)^3} = -0.5$$

$$\frac{\partial^2 f}{\partial s^2}(0,1) = \frac{16(1)(-3(0)^2 + 2(1)^2 - 6)}{((0)^2 + 2(1)^2 + 2)^3} = -1$$

$$\frac{\partial^2 f}{\partial t \partial s}(0,1) = \frac{\partial^2 f}{\partial s \partial t} = \frac{8(0)(-(0)^2 + 6(1)^2 - 2)}{((0)^2 + 2(1)^2 + 2)^3} = 0$$

Perform the second derivative test for $(0,1)$.

$$D(t,s) = \frac{\partial^2 f}{\partial t^2} \cdot \frac{\partial^2 f}{\partial s^2} - \left(\frac{\partial^2 f}{\partial t \partial s}\right)^2$$

$$D(0,1) = (-0.5) \cdot (-1) - (0)^2 = 0.5 > 0$$

So the critical point $(0,1)$ is a local extremum. Since $\partial^2 f/\partial t^2(0,1) < 0$, it's a local maximum.

$$f(0,1) = \frac{4(1)}{(0)^2 + 2(1)^2 + 2} = 1$$

Evaluate the second order partial derivatives at $(0,-1)$.

$$\frac{\partial^2 f}{\partial t^2}(0,-1)=\frac{8(-1)(3(0)^2-2(-1)^2-2)}{((0)^2+2(-1)^2+2)^3}=0.5$$

$$\frac{\partial^2 f}{\partial s^2}(0,-1)=\frac{16(-1)(-3(0)^2+2(-1)^2-6)}{((0)^2+2(-1)^2+2)^3}=1$$

$$\frac{\partial^2 f}{\partial t \partial s}(0,-1)=\frac{\partial^2 f}{\partial s \partial t}=\frac{8(0)(-(0)^2+6(-1)^2-2)}{((0)^2+2(-1)^2+2)^3}=0$$

Perform the second derivative test for $(0,-1)$.

$$D(0,-1)=0.5\cdot 1-(0)^2=0.5>0$$

So the critical point $(0,-1)$ is a local extremum. Since $\partial^2 f/\partial t^2(0,-1)>0$, it's a local minimum.

$$f(0,-1)=\frac{4(-1)}{(0)^2+2(-1)^2+2}=-1$$

To check if these local extrema are global, we need to check the values of $f(t,s)$ on the boundaries of the region, in our case on $t\to\infty$, $s\to\infty$.

$$\lim_{t\to\infty, s\to\infty} f(t,s)$$

$$\lim_{t\to\infty, s\to\infty} \frac{4s}{t^2+2s^2+2}$$

$$\lim_{t\to\infty, s\to\infty} \frac{4}{t^2/s+2s+2/s}$$

$$\frac{4}{\infty}=0$$

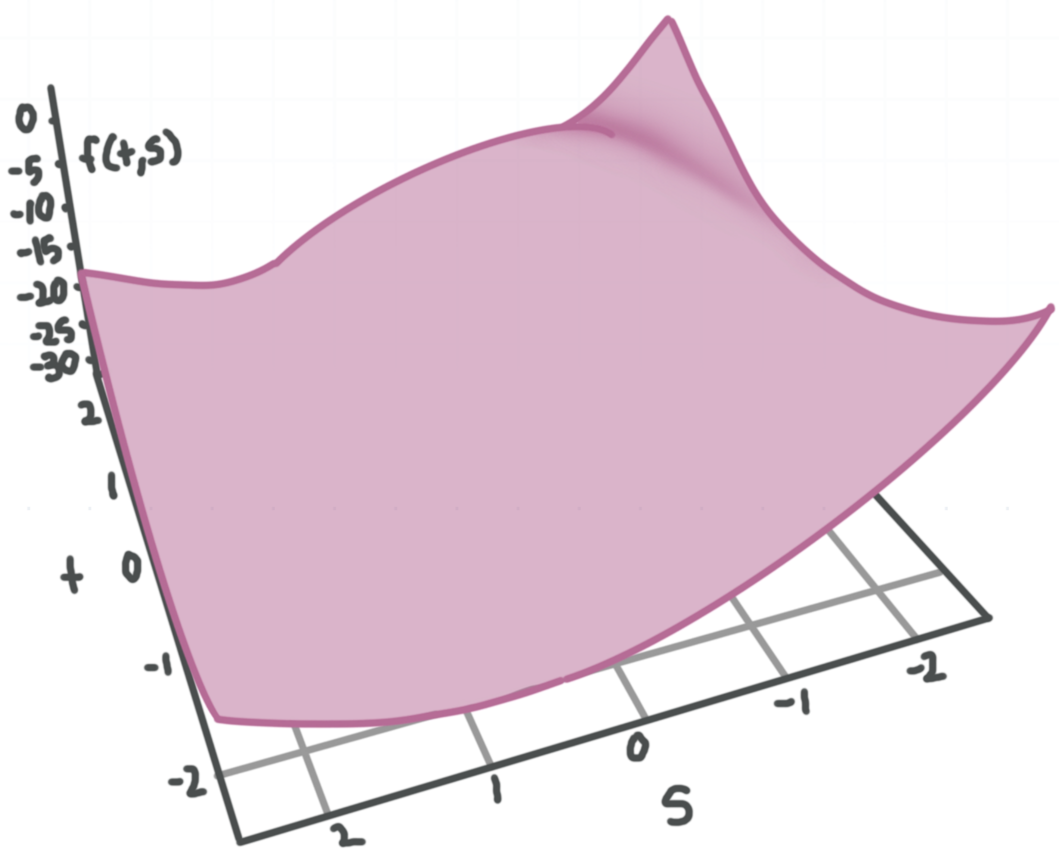
So 1 is the global maximum at $(0,1)$ and -1 is the global minimum at $(0,-1)$.

■ 2. Find the global extrema of $f(t,s)$ over R^2.

$$f(t,s)=t^2s^2-4t^2-4s^2-4s+1$$

Solution:

A sketch of the surface is



Use power rule to find first order partial derivatives.

$$\frac{\partial f}{\partial t}=2ts^2-8t$$

$$\frac{\partial f}{\partial s}=2t^2s-8s-4$$

Setting both partial derivatives equal to 0 gives us a system of equations we can use to find critical points.

$$2ts^2 - 8t = 0$$

$$2t^2s - 8s - 4 = 0$$

and then

$$t(s^2 - 4) = 0$$

$$t^2s - 4s - 2 = 0$$

Consider the first equation. If $t = 0$, then

$$-4s - 2 = 0$$

$$s = -0.5$$

If $s = 2$, then

$$t^2(2) - 4(2) - 2 = 0$$

$$2t^2 - 10 = 0$$

$$t = \pm\sqrt{5}$$

If $s = -2$, then

$$t^2(-2) - 4(-2) - 2 = 0$$

$$-2t^2 + 6 = 0$$

$$t = \pm\sqrt{3}$$

So the solutions to the system are $(0,-0.5)$, $(\pm\sqrt{5},2)$, and $(\pm\sqrt{3},-2)$.

So we have five critical points, and we can perform the second derivative test and get the result that the point $(0,-0.5)$ is a local maximum, and four other points are saddle points. However, it's not necessary here, because we can see that the function has no global minima/maxima because it tends to infinity for big t and s.

For example, consider the path $t=0$, $s\to\infty$.

$$\lim_{t=0,s\to\infty} f(t,s)$$

$$\lim_{t=0,s\to\infty} (t^2s^2-4t^2-4s^2-4s+1)$$

$$\lim_{s\to\infty} (-4s^2-4s+1)=-\infty$$

So the function has no global minimum because it tends to $-\infty$ for some t and s.

As another example, consider the path $t=s$, $s\to\infty$.

$$\lim_{t=s,s\to\infty} f(t,s)$$

$$\lim_{t=s,s\to\infty} (t^2s^2-4t^2-4s^2-4s+1)$$

$$\lim_{s\to\infty} (s^2s^2-4s^2-4s^2-4s+1)$$

$$\lim_{s\to\infty} (s^4-8s^2-4s+1)$$

$$\lim_{s\to\infty} s^4\left(1-\frac{8}{s^2}-\frac{4}{s^3}+\frac{1}{s^4}\right)$$

$$\infty \cdot 1 = \infty$$

So the function has no global maximum because it tends to ∞ for some values of t and s. So the global extrema do not exist.

■ 3. Find the global extrema of $f(x,y)$ over R^2.

$$f(x,y)=\frac{\sin(3x)}{x^2+3y^2}$$

Solution:

Let's investigate the behavior of the function when either x or y approaches infinity.

$$|f(x,y)|=\left|\frac{\sin(3x)}{x^2+3y^2}\right|=\frac{|\sin(3x)|}{|x^2+3y^2|}\le\frac{1}{|x^2+3y^2|}=\frac{1}{x^2+3y^2}$$

So $f(x,y)\to 0$ as x or y approach infinity. Which means the function can have global extrema if it has no infinite discontinuity. But it has one at the point $(0,0)$.

So consider the path $y=0$, $x\to 0^+$.

$$\lim_{x\to 0^+,y=0} f(x,y)$$

$$\lim_{x \to 0^+, y=0} \frac{\sin(3x)}{x^2 + 3y^2}$$

$$\lim_{x \to 0^+} \frac{\sin(3x)}{x^2}$$

$$\lim_{x \to 0^+} \frac{\sin(3x)}{x} \lim_{x \to 0^+} \frac{1}{x}$$

$$3 \cdot \infty = \infty$$

So the function has no global maximum because it tends to ∞ for some x and y.

Then consider the path $y = 0$, $x \to 0^-$.

$$\lim_{x \to 0^-, y=0} f(x, y)$$

$$\lim_{x \to 0^-, y=0} \frac{\sin(3x)}{x^2 + 3y^2}$$

$$\lim_{x \to 0^-} \frac{\sin(3x)}{x^2}$$

$$\lim_{x \to 0^-} \frac{\sin(3x)}{x} \lim_{x \to 0^-} \frac{1}{x}$$

$$3 \cdot -\infty = -\infty$$

So the function has no global minimum because it tends to $-\infty$ for some x and y. So the global extrema do not exist.

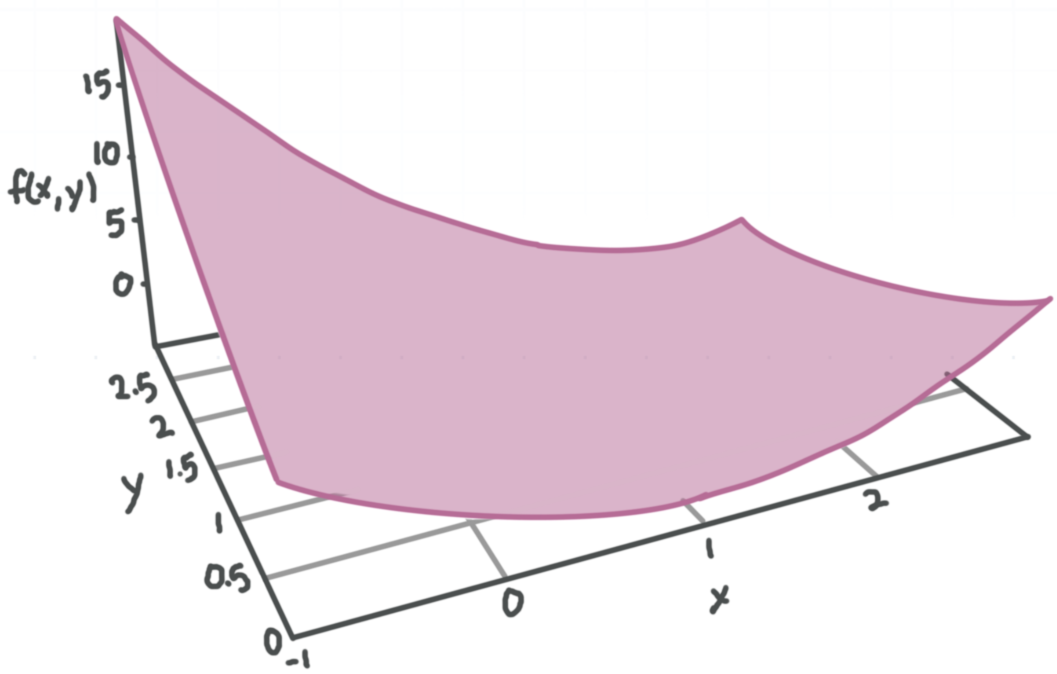
EXTREME VALUE THEOREM

■ 1. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of $f(x, y)$ on the closed rectangle $-1 \le x \le 3$, $0 \le y \le 3$.

$$f(x, y) = 2x^2 - 2xy + y^2 - 4x - 1$$

Solution:

A sketch of the surface is



Calculate the first order partial derivatives:

$$\frac{\partial f}{\partial x} = 4x - 2y - 4$$

$$\frac{\partial f}{\partial y} = -2x + 2y$$

Setting both partial derivatives equal to 0 and using these as a system of equations to find critical points gives

$$4x - 2y - 4 = 0$$

$$-2x + 2y = 0$$

The solution to the system is $(2,2)$.

Polynomial functions are always continuous on R^2. So we consider the continuous function over a closed region, and by the extreme value theorem it has a global maximum and a global minimum.

So we don't need to perform a secondary derivative test in order to find global extrema. We just need to determine the values of the function in every critical point (including boundaries), and find the greatest/the least of them.

Find critical points on the boundary $x = -1$.

$$g(y) = f(-1,y)$$

$$g(y) = 2(-1)^2 - 2(-1)y + y^2 - 4(-1) - 1$$

$$g(y) = y^2 + 2y + 5$$

$$g'(y) = 2y + 2 = 0$$

$$y = -1$$

The point is out of the region, so we should check only the endpoints $(-1,0)$ and $(-1,3)$.

Find critical points on the boundary $x = 3$.

$$g(y) = f(3,y)$$

$$g(y) = 2(3)^2 - 2(3)y + y^2 - 4(3) - 1$$

$$g(y) = y^2 - 6y + 5$$

$$g'(y) = 2y - 6 = 0$$

$$y = 3$$

The point is the endpoint of the region, so we should check the endpoints $(3,0)$ and $(3,3)$.

Find critical points on the boundary $y = 0$.

$$h(x) = f(x,0)$$

$$h(x) = 2x^2 - 2x(0) + (0)^2 - 4x - 1$$

$$h(x) = 2x^2 - 4x - 1$$

$$h'(x) = 4x - 4 = 0$$

$$x = 1$$

So we should check the critical point $(1,0)$ and the endpoints $(-1,0)$ and $(3,0)$.

Find critical points on the boundary $y = 3$.

$$h(x) = f(x,3)$$

$$h(x) = 2x^2 - 2x(3) + (3)^2 - 4x - 1$$

$$h(x) = 2x^2 - 10x + 8$$

$$h'(x) = 4x - 10 = 0$$

$$x = 2.5$$

So we should check the critical point $(2.5,3)$ and the endpoints $(-1,3)$ and $(3,3)$.

Collect all of the points together.

$$f(2,2) = -5$$

$$f(-1,0) = 5$$

$$f(-1,3) = 20$$

$$f(3,0) = 5$$

$$f(3,3) = -4$$

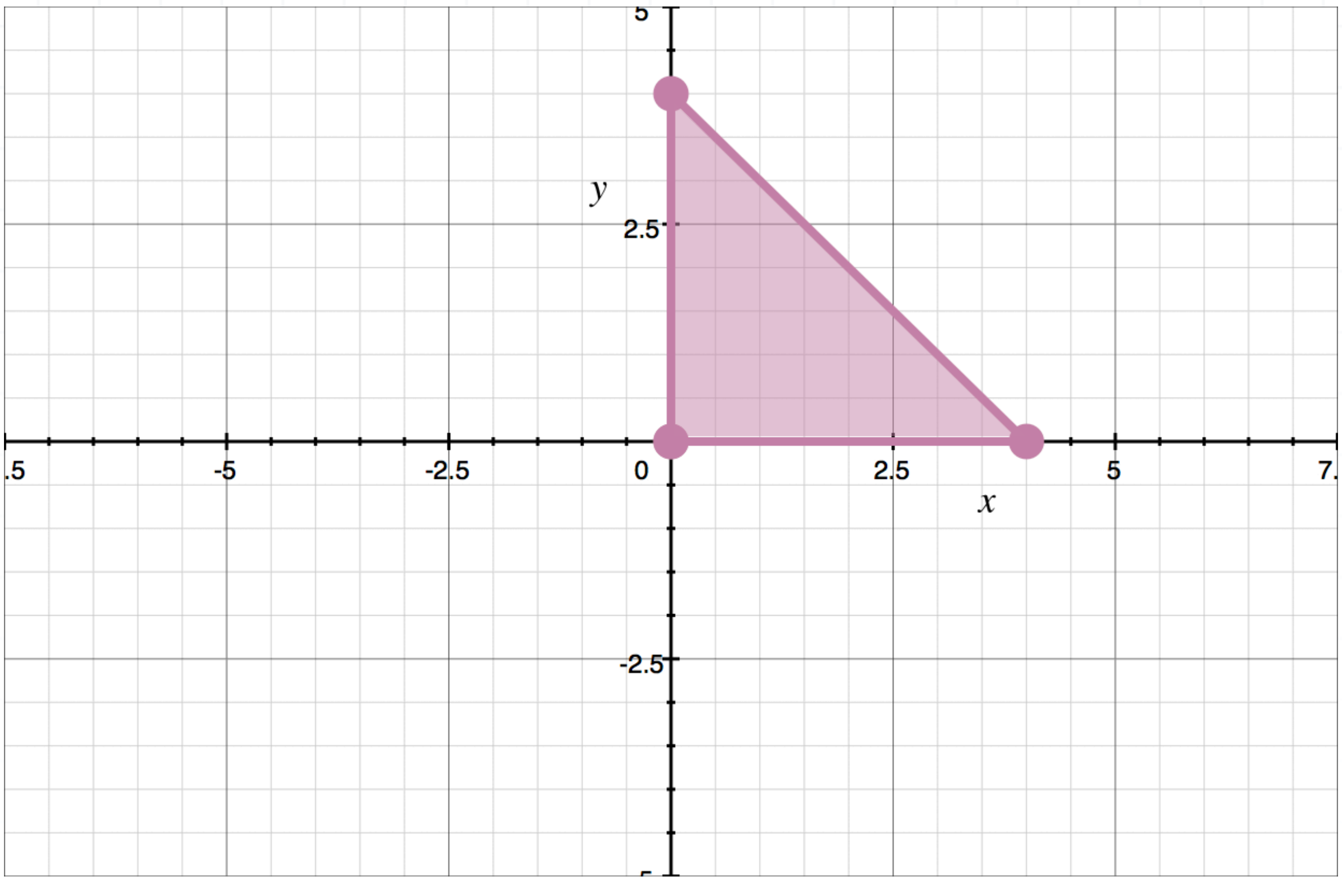
$$f(1,0) = -3$$

$$f(2.5,3) = -4.5$$

Comparing all of these, we can say that 20 is the global maximum at $(-1,3)$, and -5 is the global minimum at $(2,2)$.

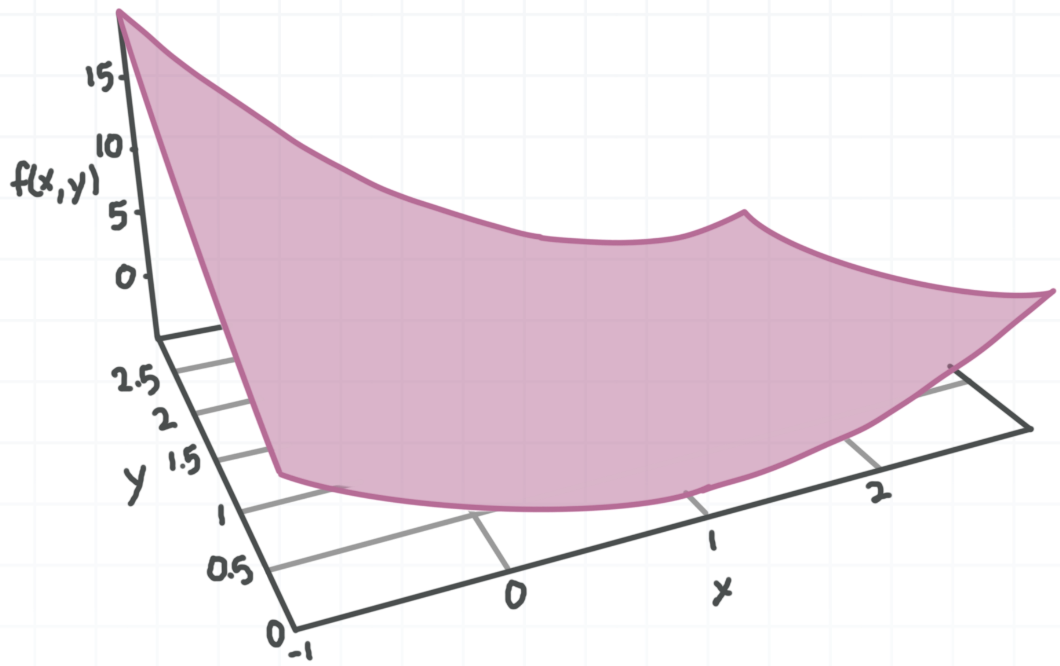
2. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of $f(x,y)$ on a closed triangle bounded by $x=0$, $y=0$, and $x+y-4=0$.

$$f(x,y)=\ln(x^2+y^2-2y-x+2)$$



Solution:

A sketch of the surface is



Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2x - 1}{x^2 + y^2 - 2y - x + 2}$$

$$\frac{\partial f}{\partial y} = \frac{2y - 2}{x^2 + y^2 - 2y - x + 2}$$

Since $x^2 + y^2 - 2y - x + 2$ is always positive, setting both numerators equal to 0 and using these equations as a system of equations to find critical points gives

$$2x - 1 = 0$$

$$2y - 2 = 0$$

The solution to the system is $(0.5,1)$.

The logarithmic function $\ln a$ is continuous on its domain $a > 0$, and $x^2 + y^2 - 2y - x + 2$ is always positive. So we consider the continuous function over a closed region, and by the extreme value theorem it has a global maximum and a global minimum.

So we don't need to perform a secondary derivative test in order to find the global extrema. We just need to determine the values of the function at every critical point (including boundaries), and find the greatest/the least of them.

The vertices of the triangle are $(0,0)$, $(0,4)$, $(4,0)$.

Find critical points on the boundary $x = 0$.

$$g(y) = f(0,y)$$

$$g(y) = \ln((0)^2 + y^2 - 2y - (0) + 2)$$

$$g(y) = \ln(y^2 - 2y + 2)$$

$$g'(y) = \frac{2y - 2}{y^2 - 2y + 2} = 0$$

$$y = 1$$

So we should check the critical point $(0,1)$.

Find critical points on the boundary $y = 0$.

$$h(x) = f(x,0)$$

$$h(x) = \ln(x^2 + (0)^2 - 2(0) - x + 2)$$

$$h(x) = \ln(x^2 - x + 2)$$

$$h'(x) = \frac{2x - 1}{x^2 - x + 2} = 0$$

$$x = 0.5$$

So we should check the critical point $(0.5,0)$.

Find critical points on the boundary $x+y-4=0$, or $y=4-x$.

$$h(x)=f(x,4-x)$$

$$h(x)=\ln(x^2+(4-x)^2-2(4-x)-x+2)$$

$$h(x)=\ln(2x^2-7x+10)$$

$$h'(x)=4x-7=0$$

$x=1.75$ and $y=4-1.75=2.25$

So we should check the critical point $(1.75,2.25)$.

Collect all of the points together.

$$f(0.5,1)=\ln(0.75)$$

$$f(0,0)=\ln(2)$$

$$f(0,4)=\ln(10)$$

$$f(4,0)=\ln(14)$$

$$f(0,1)=\ln(1)$$

$$f(0.5,0)=\ln(1.75)$$

$$f(1.75,2.25)=\ln(3.875)$$

Since the logarithm is an increasing function, then $\ln(14)$ is a maximum, and $\ln(0.75)$ is a minimum. So $\ln(14)$ is the global maximum at $(4,0)$, and $\ln(0.75)$ is the global minimum at $(0.5,1)$.

■ 3. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of $f(x, y)$ on the closed rectangle $-\pi \le x \le \pi$, $-1 \le y \le 3$.

$$f(x, y) = y^2 \tan(2x)$$

Solution:

The tangent function is continuous at any point within its domain. The points excluded from domain are

$$\frac{\pi}{2} + \pi n \text{ for any integer } n$$

So we have the inequality

$$2x \neq \frac{\pi}{2} + \pi n \text{ for any integer } n$$

$$x \neq \frac{\pi}{4} + \frac{\pi}{2} n \text{ for any integer } n$$

Consider the point

$$y = 1,\ n = 0,\ x = \frac{\pi}{4} + \frac{\pi}{2} n = \frac{\pi}{4}$$

The point $(\pi/4,1)$ lies within the given closed rectangle, and the function $f(x,y)$ is not continuous in it. So the Extreme Value Theorem does not apply in this case.

And because of this, there's no global extrema on the given rectangle since $f(x,y)$ tends to ∞ or $-\infty$ as x approaches $\pi/4$ for any y.

■ 4. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of the function $f(x,y)$ on the closed rectangle $-\pi \le x \le \pi$, $-2 \le y \le 2$.

$$f(x,y) = (y^2 + 2y + 3)\tan\left(\frac{x}{4}\right)$$

Solution:

The tangent function is continuous at any point within its domain. The points excluded from domain are

$$\frac{\pi}{2} + \pi n \text{ for any integer } n$$

So we have the inequality

$$\frac{x}{4} \neq \frac{\pi}{2} + \pi n \text{ for any integer } n$$

$$x \neq 2\pi + 4\pi n \text{ for any integer } n$$

Since none of these points lie within the given interval $-\pi \leq x \leq \pi$, the function $\tan(x/4)$ is continuous on it. Also, because the polynomial function $y^2 + 2y + 3$ is continuous for any values of x and y, the function $f(x, y)$ is continuous on the given rectangle.

So Extreme Value Theorem applies in this case, and the function has a global maximum and a global minimum. To find the global extrema, we need to determine the values of function in every critical point (including boundaries), and find the greatest/the least of them.

Find the first order partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{1}{4}(y^2 + 2y + 3)\sec^2\left(\frac{x}{4}\right)$$

$$\frac{\partial f}{\partial y} = (2y + 2)\tan\left(\frac{x}{4}\right)$$

Setting both of them equal to 0 and using these equations as a system of equations to find critical points gives

$$\frac{1}{4}(y^2 + 2y + 3)\sec^2\left(\frac{x}{4}\right) = 0$$

$$(2y + 2)\tan\left(\frac{x}{4}\right) = 0$$

Since $(y^2 + 2y + 3)\sec^2(x/4) > 0$ for any (x, y), the system has no solutions. So there are no critical points inside the rectangle.

Find critical points on the boundary $x = -\pi$.

$$g(y) = f(-\pi, y) = (y^2 + 2y + 3)$$

$$\tan\left(\frac{-\pi}{4}\right) = -(y^2 + 2y + 3)$$

$$g'(y) = -2y - 2 = 0$$

$$y = -1$$

So we should check the critical point $(-\pi, -1)$ and the endpoints $(-\pi, -2)$ and $(-\pi, 2)$.

Find critical points on the boundary $x = \pi$.

$$g(y) = f(\pi, y)$$

$$g(y) = (y^2 + 2y + 3)\tan\left(\frac{\pi}{4}\right)$$

$$g(y) = (y^2 + 2y + 3)$$

$$g'(y) = 2y + 2 = 0$$

$$y = -1$$

So we should check the critical point $(\pi, 1)$ and the endpoints $(\pi, -2)$ and $(\pi, 2)$.

Find critical points on the boundary $y = -2$.

$$h(x) = f(x, -2)$$

$$h(x) = ((-2)^2 + 2(-2) + 3)\tan\left(\frac{x}{4}\right)$$

$$h(x) = 3\tan\left(\frac{x}{4}\right)$$

$$h'(x) = \frac{3}{4}\sec^2\left(\frac{x}{4}\right) > 0$$

There are no critical points on the boundary, so we should check the endpoints $(-\pi, -2)$ and $(\pi, -2)$.

Find critical points on the boundary $y = 2$.

$$h(x) = f(x,2)$$

$$h(x) = ((2)^2 + 2(2) + 3)\tan\left(\frac{x}{4}\right)$$

$$h(x) = 11\tan\left(\frac{x}{4}\right)$$

$$h'(x) = \frac{11}{4}\sec^2\left(\frac{x}{4}\right) > 0$$

There are no critical points on the boundary, so we should check the endpoints $(-\pi,2)$ and $(\pi,2)$.

Collect all of the points together.

$$f(-\pi,2) = -11$$

$$f(\pi,2) = 11$$

$$f(-\pi, -2) = -3$$

$$f(\pi, -2) = 3$$

$$f(-\pi, -1) = -2$$

$$f(\pi, 1) = 6$$

Comparing all of these, we can say that 11 is the global maximum at $(\pi, 2)$, and -11 is the global minimum at $(-\pi, 2)$.

5. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of the function $f(x, y)$ on the closed circle with center at the origin and radius 1.

$$f(x, y) = x^2 + y^2 - 2x + 2\sqrt{3}y - 3$$

Solution:

The polynomial function is continuous at any point. So the Extreme Value Theorem applies in this case, and the function has a global maximum and a global minimum.

To find the global extrema, we need to determine the values of the function at every critical point (including boundaries), and find the greatest/the least of them.

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 2x - 2$$

$$\frac{\partial f}{\partial y} = 2y + 2\sqrt{3}$$

Setting both of them equal to 0 and using these equations as a system of equations to find critical points gives

$$2x - 2 = 0$$

$$2y + 2\sqrt{3} = 0$$

The solution to the system is $(1, -\sqrt{3})$. The point lies outside the given circle. To find the critical points on the boundary, consider the function using polar coordinates.

Convert into polar coordinates (r, θ) assuming $r = 1$ on the circle.

$$r \geq 0,\ 0 \leq \theta < 2\pi$$

$$x = r\cos\theta = \cos\theta$$

$$y = r\sin\theta = \sin\theta$$

$r^2 = x^2 + y^2 = 1$ on the circle

Substitute into the function equation.

$$g(\theta) = f(\cos\theta, \sin\theta)$$

$$g(\theta) = 1 - 2\cos\theta + 2\sqrt{3}\sin\theta - 3$$

$$g(\theta) = -2\cos\theta + 2\sqrt{3}\sin\theta - 2$$

$$g'(\theta) = 2\sin\theta + 2\sqrt{3}\cos\theta = 0$$

$$\sin\theta = -\sqrt{3}\cos\theta$$

$$\tan\theta = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3} \text{ or } \theta = \frac{5\pi}{3}$$

So there are two critical points on the circle, $(1,2\pi/3)$ and $(1,5\pi/3)$.

$$g\left(\frac{2\pi}{3}\right) = -2\cos\left(\frac{2\pi}{3}\right) + 2\sqrt{3}\sin\left(\frac{2\pi}{3}\right) - 2 = 2$$

$$g\left(\frac{5\pi}{3}\right) = -2\cos\left(\frac{5\pi}{3}\right) + 2\sqrt{3}\sin\left(\frac{5\pi}{3}\right) - 2 = -6$$

So we have a global maximum of 2 and global minimum of -6. Return back to (x,y) coordinates.

$$x_1 = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$y_1 = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$x_2 = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$$

$$y_2 = \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

So the extrema of the function are

A global maximum of 2 at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$

A global minimum of -6 at $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

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