Gradient vectors and the tangent plane

We previously learned how to find the gradient vector at a specific point. We just use the formula

$$\nabla f(x,y,z)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

where (x,y,z) is the point we're interested in. If the result of evaluating the gradient vector at the point (x,y,z) gives us

$$\nabla f(x,y,z)=\langle a,b,c\rangle$$

then a, b, and c represent the slope of the original function in the x, y, and z directions, respectively. Therefore, if we're interested in finding the equation of the tangent plane at $P(x_0,y_0,z_0)$, then we can plug the values of a, b, and c, along with the point $P(x_0,y_0,z_0)$, into the formula for the equation of the tangent plane

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

where a, b, and c come from $\nabla f(x,y,z)=\langle a,b,c\rangle$, and x_0, y_0, and z_0 come from the given point $P(x_0,y_0,z_0)$.

Example

Find the gradient vector of the function and use it to find the equation of the tangent plane at $P(3,4,-287)$.

$$x^4-5x^3y-y^2+3y^4+z=6$$

Rearrange the function.

$$f(x, y, z) = x^4 - 5x^3y - y^2 + 3y^4 + z - 6$$

Now we'll start with the partial derivatives of the given function f.

$$\frac{\partial f}{\partial x} = 4x^3 - 15x^2y$$

$$\frac{\partial f}{\partial y} = -5x^3 - 2y + 12y^3$$

$$\frac{\partial f}{\partial z} = 1$$

To find the gradient vector at the point we're interested in, we'll plug the partial derivatives in to the formula for the gradient vector, and then evaluate at the point of interest.

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\nabla f(x, y) = \langle 4x^3 - 15x^2y, -5x^3 - 2y + 12y^3, 1 \rangle$$

Evaluating at $P(3,4,-287)$, we get

$$\nabla f(3,4) = \langle 4(3)^3 - 15(3)^2(4), -5(3)^3 - 2(4) + 12(4)^3, 0 \rangle$$

$$\nabla f(3,4) = \langle -432, 625, 0 \rangle$$

Now we have everything we need to find the equation of the tangent plane.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-432(x - 3) + 625(y - 4) + 0(z + 287) = 0$$

$$-432x + 1{,}296 + 625y - 2{,}500 = 0$$

$$-432x + 625y - 1{,}204 = 0$$