Topic: Acute angles between the curves

Question: Find the acute angle between the curves.

$$y = x^2$$

$$y = 2x^2 - 4$$

Answer choices:

A 13.8°

B 15.0°

C 3.5°

D 6.9°

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Solution: D

We'll start by setting the curves equal to one another to find the points where they intersect.

$$x^2 = 2x^2 - 4$$

$$-x^2 = -4$$

$$x^2 = 4$$

$$x = \pm 2$$

Since we found two intersection points, we'll have two acute angles. We need to find the y-values for $x = \pm 2$. Since $x = \pm 2$ are the points where the curves intersect, we can plug $x = \pm 2$ into either curve to find the associated y-values.

For $x = 2$,

$$y = x^2$$

$$y = 2^2$$

$$y = 4$$

For $x = -2$,

$$y = x^2$$

$$y = (-2)^2$$

$$y = 4$$

So the intersection points occur at $(2,4)$ and $(-2,4)$.

Now we need the equation of the tangent line for both curves at both intersection points, so we'll find the slope (derivative) of each curve at the intersection points, then plug the slope and the point into the point-slope formula for the equation of a line.

The derivative of $y = x^2$ is $y' = 2x$.

At $(2,4)$, the slope is $y'(2,4) = 2(2) = 4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 4(x - 2)$$

$$y - 4 = 4x - 8$$

$$y = 4x - 4$$

$$-4x + y = -4$$

At $(-2,4)$, the slope is $y'(-2,4) = 2(-2) = -4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -4\left[x - (-2)\right]$$

$$y - 4 = -4x - 8$$

$$y = -4x - 4$$

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$$4x + y = -4$$

The derivative of $y = 2x^2 - 4$ is $y' = 4x$.

At $(2,4)$, the slope is $y'(2,4) = 4(2) = 8$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 8(x - 2)$$

$$y - 4 = 8x - 16$$

$$y = 8x - 12$$

$$-8x + y = -12$$

At $(-2,4)$, the slope is $y'(-2,4) = 4(-2) = -8$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -8\left[x - (-2)\right]$$

$$y - 4 = -8x - 16$$

$$y = -8x - 12$$

$$8x + y = -12$$

So the tangent lines at $(2,4)$ are

$-4x + y = -4$, which is $a = \langle -4,1 \rangle$ in standard vector form

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$-8x + y = -12$, which is $b = \langle -8,1 \rangle$ in standard vector form

And the tangent lines at $(-2,4)$ are

$4x + y = -4$, which is $a = \langle 4,1 \rangle$ in standard vector form

$8x + y = -12$, which is $b = \langle 8,1 \rangle$ in standard vector form

Now that we've found our tangent lines and converted them to standard vector form, we can plug them into the equation for the angle between vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. We'll use the origin $(0,0)$ as the point (x_1, y_1) and then find the length of each vector using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The dot product of the vectors at $(2,4)$ is

$$a \cdot b = (-4)(-8) + (1)(1)$$

$$a \cdot b = 32 + 1$$

$$a \cdot b = 33$$

The length of $a = \langle -4,1 \rangle$ is

$$D_a = \sqrt{(-4 - 0)^2 + (1 - 0)^2}$$

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$$D_a = \sqrt{16+1}$$

$$D_a = |a| = \sqrt{17}$$

The length of $b = \langle -8,1 \rangle$ is

$$D_b = \sqrt{(-8-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{64+1}$$

$$D_b = |b| = \sqrt{65}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(2,4)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{33}{\sqrt{17}\sqrt{65}}$$

$$\cos\theta = \frac{33}{\sqrt{1{,}105}}$$

$$\theta = \arccos\frac{33}{\sqrt{1{,}105}}$$

$$\theta \approx 6.9^\circ$$

Now we'll work on the acute angle between the curves at $(-2,4)$. The dot product of the vectors at $(-2,4)$ is

$$a \cdot b = (4)(8) + (1)(1)$$

$$a \cdot b = 32 + 1$$

$$a \cdot b = 33$$

The length of $a = \langle 4,1 \rangle$ is

$$D_a = \sqrt{(4 - 0)^2 + (1 - 0)^2}$$

$$D_a = \sqrt{16 + 1}$$

$$D_a = |a| = \sqrt{17}$$

The length of $b = \langle 8,1 \rangle$ is

$$D_b = \sqrt{(8 - 0)^2 + (1 - 0)^2}$$

$$D_b = \sqrt{64 + 1}$$

$$D_b = |b| = \sqrt{65}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(-2,4)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{33}{\sqrt{17}\sqrt{65}}$$

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$$\cos\theta = \frac{33}{\sqrt{1{,}105}}$$

$$\theta = \arccos\frac{33}{\sqrt{1{,}105}}$$

$$\theta \approx 6.9^\circ$$

In summary,

the acute angle between the curves at $(2,4)$ is $\theta \approx 6.9^\circ$

the acute angle between the curves at $(-2,4)$ is $\theta \approx 6.9^\circ$

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Topic: Acute angles between the curves

Question: Find the acute angle between the curves.

$$y = x^2$$

$$y = -x^2 + 18$$

Answer choices:

A 9.5°

B 18.9°

C 161.1°

D 170.5°

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Solution: B

We'll start by setting the curves equal to one another to find the points where they intersect.

$$x^2 = -x^2 + 18$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Since we found two intersection points, we'll have two acute angles. We need to find the y-values for $x = \pm 3$. Since $x = \pm 3$ are the points where the curves intersect, we can plug $x = \pm 3$ into either curve to find the associated y-values.

For $x = 3$,

$$y = x^2$$

$$y = 3^2$$

$$y = 9$$

For $x = -3$,

$$y = x^2$$

$$y = (-3)^2$$

$$y = 9$$

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So the intersection points occur at $(3,9)$ and $(-3,9)$.

Now we need the equation of the tangent line for both curves at both intersection points, so we'll find the slope (derivative) of each curve at the intersection points, then plug the slope and the point into the point-slope formula for the equation of a line.

The derivative of $y = x^2$ is $y' = 2x$.

At $(3,9)$, the slope is $y'(3,9) = 2(3) = 6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = 6(x - 3)$$

$$y - 9 = 6x - 18$$

$$y = 6x - 9$$

$$-6x + y = -9$$

At $(-3,9)$, the slope is $y'(-3,9) = 2(-3) = -6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = -6\left[x - (-3)\right]$$

$$y - 9 = -6x - 18$$

$$y = -6x - 9$$

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$$6x + y = -9$$

The derivative of $y = -x^2 + 18$ is $y' = -2x$.

At $(3,9)$, the slope is $y'(3,9) = -2(3) = -6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = -6(x - 3)$$

$$y - 9 = -6x + 18$$

$$y = -6x + 27$$

$$6x + y = 27$$

At $(-3,9)$, the slope is $y'(-3,9) = -2(-3) = 6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = 6\left[x - (-3)\right]$$

$$y - 9 = 6x + 18$$

$$y = 6x + 27$$

$$-6x + y = 27$$

So the tangent lines at $(3,9)$ are

$-6x + y = -9$, which is $a = \langle -6,1 \rangle$ in standard vector form

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$6x+y=27$, which is $b=\langle 6,1\rangle$ in standard vector form

And the tangent lines at $(-3,9)$ are

$6x+y=-9$, which is $a=\langle 6,1\rangle$ in standard vector form

$-6x+y=27$, which is $b=\langle -6,1\rangle$ in standard vector form

Now that we've found our tangent lines and converted them to standard vector form, we can plug them into the equation for the angle between vectors.

$$\cos\theta=\frac{a\cdot b}{|a||b|}$$

where $a\cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. We'll use the origin $(0,0)$ as the point (x_1,y_1) and then find the length of each vector using the distance formula

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

The dot product of the vectors at $(3,9)$ is

$$a\cdot b=(-6)(6)+(1)(1)$$

$$a\cdot b=-36+1$$

$$a\cdot b=-35$$

The length of $a=\langle -6,1\rangle$ is

$$D_a=\sqrt{(-6-0)^2+(1-0)^2}$$

$$D_a = \sqrt{36+1}$$

$$D_a = |a| = \sqrt{37}$$

The length of $b = \langle 6,1 \rangle$ is

$$D_b = \sqrt{(6-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{36+1}$$

$$D_b = |b| = \sqrt{37}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(3,9)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-35}{\sqrt{37}\sqrt{37}}$$

$$\cos\theta = -\frac{35}{37}$$

$$\theta = \arccos\left(-\frac{35}{37}\right)$$

$$\theta \approx 18.9^\circ$$

Now we'll work on the acute angle between the curves at $(-3,9)$. The dot product of the vectors at $(-3,9)$ is

$$a \cdot b = (6)(-6) + (1)(1)$$

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$$a \cdot b = -36 + 1$$

$$a \cdot b = -35$$

The length of $a = \langle 6,1 \rangle$ is

$$D_a = \sqrt{(6-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{36+1}$$

$$D_a = |a| = \sqrt{37}$$

The length of $b = \langle -6,1 \rangle$ is

$$D_b = \sqrt{(-6-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{36+1}$$

$$D_b = |b| = \sqrt{37}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(-3,9)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-35}{\sqrt{37}\sqrt{37}}$$

$$\cos\theta = -\frac{35}{37}$$

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$$\theta = \arccos\left(-\frac{35}{37}\right)$$

$$\theta \approx 18.9^\circ$$

In summary,

the acute angle between the curves at $(3,9)$ is $\theta \approx 18.9^\circ$

the acute angle between the curves at $(-3,9)$ is $\theta \approx 18.9^\circ$

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Topic: Acute angles between the curves

Question: Find the acute angle between the curves.

$$y = x^2$$

$$y = -2x^2 + 3$$

Answer choices:

A 20.3°

B 139.4°

C 40.6°

D 159.7°

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Solution: C

We'll start by setting the curves equal to one another to find the points where they intersect.

$$x^2 = -2x^2 + 3$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

Since we found two intersection points, we'll have two acute angles. We need to find the y-values for $x = \pm 1$. Since $x = \pm 1$ are the points where the curves intersect, we can plug $x = \pm 1$ into either curve to find the associated y-values.

For $x = 1$,

$$y = x^2$$

$$y = 1^2$$

$$y = 1$$

For $x = -1$,

$$y = x^2$$

$$y = (-1)^2$$

$$y = 1$$

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So the intersection points occur at $(1,1)$ and $(-1,1)$.

Now we need the equation of the tangent line for both curves at both intersection points, so we'll find the slope (derivative) of each curve at the intersection points, then plug the slope and the point into the point-slope formula for the equation of a line.

The derivative of $y = x^2$ is $y' = 2x$.

At $(1,1)$, the slope is $y'(1,1) = 2(1) = 2$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 1$$

$$-2x + y = -1$$

At $(-1,1)$, the slope is $y'(-1,1) = 2(-1) = -2$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2\left[x - (-1)\right]$$

$$y - 1 = -2x - 2$$

$$y = -2x - 1$$

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$$2x + y = -1$$

The derivative of $y = -2x^2 + 3$ is $y' = -4x$.

At $(1,1)$, the slope is $y'(1,1) = -4(1) = -4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4(x - 1)$$

$$y - 1 = -4x + 4$$

$$y = -4x + 5$$

$$4x + y = 5$$

At $(-1,1)$, the slope is $y'(-1,1) = -4(-1) = 4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 4\left[x - (-1)\right]$$

$$y - 1 = 4x + 4$$

$$y = 4x + 5$$

$$-4x + y = 5$$

So the tangent lines at $(1,1)$ are

$-2x + y = -1$, which is $a = \langle -2,1 \rangle$ in standard vector form

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$4x + y = 5$, which is $b = \langle 4,1 \rangle$ in standard vector form

And the tangent lines at $(-1,1)$ are

$2x + y = -1$, which is $a = \langle 2,1 \rangle$ in standard vector form

$-4x + y = 5$, which is $b = \langle -4,1 \rangle$ in standard vector form

Now that we've found our tangent lines and converted them to standard vector form, we can plug them into the equation for the angle between vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. We'll use the origin $(0,0)$ as the point (x_1, y_1) and then find the length of each vector using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The dot product of the vectors at $(1,1)$ is

$$a \cdot b = (-2)(4) + (1)(1)$$

$$a \cdot b = -8 + 1$$

$$a \cdot b = -7$$

The length of $a = \langle -2,1 \rangle$ is

$$D_a = \sqrt{(-2-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{4+1}$$

$$D_a = |a| = \sqrt{5}$$

The length of $b = \langle 4,1 \rangle$ is

$$D_b = \sqrt{(4-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{16+1}$$

$$D_b = |b| = \sqrt{17}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(1,1)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-7}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{-7}{\sqrt{85}}$$

$$\theta = \arccos\frac{-7}{\sqrt{85}}$$

$$\theta \approx 139.4^\circ$$

Since we found an obtuse angle instead of an acute angle, we just need to subtract the angle we found from 180° in order to find the associated acute angle.

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$$\theta \approx 180^\circ - 139.4^\circ$$

$$\theta \approx 40.6^\circ$$

Now we'll work on the acute angle between the curves at $(-1,1)$. The dot product of the vectors at $(-1,1)$ is

$$a \cdot b = (2)(-4) + (1)(1)$$

$$a \cdot b = -8 + 1$$

$$a \cdot b = -7$$

The length of $a = \langle 2,1 \rangle$ is

$$D_a = \sqrt{(2-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{4+1}$$

$$D_a = |a| = \sqrt{5}$$

The length of $b = \langle -4,1 \rangle$ is

$$D_b = \sqrt{(-4-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{16+1}$$

$$D_b = |b| = \sqrt{17}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(-1,1)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-7}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{-7}{\sqrt{85}}$$

$$\theta = \arccos\frac{-7}{\sqrt{85}}$$

$$\theta \approx 139.4^\circ$$

Again, we'll subtract the obtuse angle from 180° in order to find the associated acute angle.

$$\theta \approx 180^\circ - 139.4^\circ$$

$$\theta \approx 40.6^\circ$$

In summary,

the acute angle between the curves at $(1,1)$ is $\theta \approx 40.6^\circ$

the acute angle between the curves at $(-1,1)$ is $\theta \approx 40.6^\circ$

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