Topic: Parametric equations for the line of intersection of two planes

Question: Find the parametric equations for the line of intersection of the planes.

$$-x+y-z=2$$

$$x+y+z=4$$

Answer choices:

A	$x=2t$	$y=3$	$z=1-2t$
B	$x=1-2t$	$y=3$	$z=2t$
C	$x=-2t$	$y=3$	$z=1+2t$
D	$x=1+2t$	$y=3$	$z=-2t$

Solution: D

We need to start by finding the vector equation for the line where the planes intersect each other. The formula we'll use is

$$r = r_0 + tv$$

To find v in the formula, we'll take the cross product of the normal vectors of the planes. Since the planes are $-x + y - z = 2$ and $x + y + z = 4$, their normal vectors are $a\langle -1,1,-1\rangle$ and $b\langle 1,1,1\rangle$, respectively. The cross product is given by

$$v = a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$v = a \times b = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v = a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v = \left[(1)(1) - (-1)(1)\right]\mathbf{i} - \left[(-1)(1) - (-1)(1)\right]\mathbf{j} + \left[(-1)(1) - (1)(1)\right]\mathbf{k}$$

$$v = (1+1)\mathbf{i} - (-1+1)\mathbf{j} + (-1-1)\mathbf{k}$$

$$v = 2\mathbf{i} - 0\mathbf{j} - 2\mathbf{k}$$

Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

$$-x + y - z = 2$$

$$x + y + z = 4$$

then setting $z = 0$ gives

$$-x + y = 2$$

$$x + y = 4$$

If we add these equations together, we get

$$(-x + y) + (x + y) = 2 + 4$$

$$-x + x + y + y = 2 + 4$$

$$0 + 2y = 6$$

$$y = 3$$

Plugging $y = 3$ back into $-x + y = 2$ gives the corresponding value of x.

$$-x + y = 2$$

$$-x + 3 = 2$$

$$-x = -1$$

$$x = 1$$

Putting all of these values together tells us that $(1,3,0)$ is a point on the line of intersection. We'll change this to its vector representation and call it

$$r_0 = \mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$$

Now we can plug $v = 2\mathbf{i} - 0\mathbf{j} - 2\mathbf{k}$ and $r_0 = \mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$ into the vector equation for the line of intersection.

$$r = r_0 + tv$$

$$r = (\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) + t(2\mathbf{i} + 0\mathbf{j} - 2\mathbf{k})$$

$$r = \mathbf{i} + 3\mathbf{j} + 0\mathbf{k} + 2t\mathbf{i} + 0t\mathbf{j} - 2t\mathbf{k}$$

$$r = (\mathbf{i} + 2t\mathbf{i}) + (3\mathbf{j} + 0t\mathbf{j}) + (0\mathbf{k} - 2t\mathbf{k})$$

$$r = (\mathbf{i} + 2t\mathbf{i}) + (3\mathbf{j}) + (-2t\mathbf{k})$$

$$r = (1 + 2t)\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k}$$

Now that we have the vector equation for the line of intersection, we can find the parametric equations from the coefficients. The parametric equations are

$$x = 1 + 2t$$

$$y = 3$$

$$z = -2t$$

Topic: Parametric equations for the line of intersection of two planes

Question: Find the parametric equations for the line of intersection of the planes.

$$2x + 4y + z = 1$$

$$x - 3y + 2z = 3$$

Answer choices:

A $\quad x = \frac{3}{2} + 11t \qquad y = -\frac{1}{2} - 3t \qquad z = -10t$

B $\quad x = \frac{3}{2} + 11t \qquad y = \frac{1}{2} - 3t \qquad z = -10t$

C $\quad x = -\frac{15}{8} - 11t \qquad y = -\frac{13}{8} + 2t \qquad z = 10t$

D $\quad x = \frac{15}{8} - 11t \qquad y = \frac{13}{8} + 2t \qquad z = 10t$

Solution: A

We need to start by finding the vector equation for the line where the planes intersect each other. The formula we'll use is

$$r = r_0 + tv$$

To find v in the formula, we'll take the cross product of the normal vectors of the planes. Since the planes are $2x + 4y + z = 1$ and $x - 3y + 2z = 3$, their normal vectors are $a\langle 2,4,1\rangle$ and $b\langle 1,-3,2\rangle$, respectively. The cross product is given by

$$v = a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$v = a \times b = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v = a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v = \left[(4)(2) - (1)(-3)\right]\mathbf{i} - \left[(2)(2) - (1)(1)\right]\mathbf{j} + \left[(2)(-3) - (4)(1)\right]\mathbf{k}$$

$$v = (8 + 3)\mathbf{i} - (4 - 1)\mathbf{j} + (-6 - 4)\mathbf{k}$$

$$v = 11\mathbf{i} - 3\mathbf{j} - 10\mathbf{k}$$

Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

$$2x + 4y + z = 1$$

$$x - 3y + 2z = 3$$

then setting $z = 0$ gives

[1] $2x + 4y = 1$

[2] $x - 3y = 3$

If we multiply **[2]** by 2, we get

[1] $2x + 4y = 1$

[3] $2x - 6y = 6$

Now we can subtract **[3]** from **[1]**.

$$(2x + 4y) - (2x - 6y) = 1 - 6$$

$$2x - 2x + 4y + 6y = 1 - 6$$

$$10y = -5$$

$$y = -\frac{1}{2}$$

Plugging $y = -1/2$ back into $2x + 4y = 1$ gives the corresponding value of x.

$$2x + 4y = 1$$

$$2x + 4\left(-\frac{1}{2}\right) = 1$$

$$2x - 2 = 1$$

$$2x = 3$$

$$x = \frac{3}{2}$$

Putting all of these values together tells us that

$$\left(\frac{3}{2}, -\frac{1}{2}, 0\right)$$

is a point on the line of intersection. We'll change this to its vector representation and call it

$$r_0 = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 0\mathbf{k}$$

Now we can plug $v = 11\mathbf{i} - 3\mathbf{j} - 10\mathbf{k}$ and $r_0 = (3/2)\mathbf{i} - (1/2)\mathbf{j} + 0\mathbf{k}$ into the vector equation for the line of intersection.

$$r = r_0 + tv$$

$$r = \left(\frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 0\mathbf{k}\right) + t(11\mathbf{i} - 3\mathbf{j} - 10\mathbf{k})$$

$$r = \frac{3}{2}\mathbf{i} - \frac{1}{2}\mathbf{j} + 0\mathbf{k} + 11t\mathbf{i} - 3t\mathbf{j} - 10t\mathbf{k}$$

$$r = \left(\frac{3}{2}\mathbf{i} + 11t\mathbf{i}\right) + \left(-\frac{1}{2}\mathbf{j} - 3t\mathbf{j}\right) + (0\mathbf{k} - 10t\mathbf{k})$$

$$r = \left(\frac{3}{2} + 11t\right)\mathbf{i} + \left(-\frac{1}{2} - 3t\right)\mathbf{j} - 10t\mathbf{k}$$

Now that we have the vector equation for the line of intersection, we can find the parametric equations from the coefficients. The parametric equations are

$$x = \frac{3}{2} + 11t$$

$$y = -\frac{1}{2} - 3t$$

$$z = -10t$$

Topic: Parametric equations for the line of intersection of two planes

Question: Find the parametric equations for the line of intersection of the planes.

$$-x + 3y + 6z = 3$$

$$6x - 6y + 3z = 9$$

Answer choices:

A $\quad x = -\frac{15}{4} - 45t \qquad y = -\frac{9}{4} - 39t \qquad z = 12t$

B $\quad x = \frac{15}{4} - 45t \qquad y = \frac{9}{4} - 39t \qquad z = 12t$

C $\quad x = \frac{15}{4} + 45t \qquad y = \frac{9}{4} + 39t \qquad z = -12t$

D $\quad x = -\frac{15}{4} + 45t \qquad y = -\frac{9}{4} + 39t \qquad z = -12t$

Solution: C

We need to start by finding the vector equation for the line where the planes intersect each other. The formula we'll use is

$$r = r_0 + tv$$

To find v in the formula, we'll take the cross product of the normal vectors of the planes. Since the planes are $-x + 3y + 6z = 3$ and $6x - 6y + 3z = 9$, their normal vectors are $a\langle -1,3,6\rangle$ and $b\langle 6,-6,3\rangle$, respectively. The cross product is given by

$$v = a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$v = a \times b = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v = a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v = \left[(3)(3) - (6)(-6)\right]\mathbf{i} - \left[(-1)(3) - (6)(6)\right]\mathbf{j} + \left[(-1)(-6) - (3)(6)\right]\mathbf{k}$$

$$v = (9 + 36)\mathbf{i} - (-3 - 36)\mathbf{j} + (6 - 18)\mathbf{k}$$

$$v = 45\mathbf{i} + 39\mathbf{j} - 12\mathbf{k}$$

Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

$$-x + 3y + 6z = 3$$

$$6x - 6y + 3z = 9$$

then setting $z = 0$ gives

[1] $-x + 3y = 3$

[2] $6x - 6y = 9$

If we multiply [1] by 2, we get

[1] $-2x + 6y = 6$

[3] $6x - 6y = 9$

If we add these equations together, we get

$$(-2x + 6y) + (6x - 6y) = 6 + 9$$

$$-2x + 6x + 6y - 6y = 6 + 9$$

$$4x = 15$$

$$x = \frac{15}{4}$$

Plugging $x = 15/4$ back into $-x + 3y = 3$ gives the corresponding value of y.

$$-x + 3y = 3$$

$$-\frac{15}{4} + 3y = 3$$

$$3y = \frac{12}{4} + \frac{15}{4}$$

$$y = \frac{27}{4}\left(\frac{1}{3}\right)$$

$$y = \frac{27}{12}$$

$$y = \frac{9}{4}$$

Putting all of these values together tells us that

$$\left(\frac{15}{4}, \frac{9}{4}, 0\right)$$

is a point on the line of intersection. We'll change this to its vector representation and call it

$$r_0 = \frac{15}{4}\mathbf{i} + \frac{9}{4}\mathbf{j} + 0\mathbf{k}$$

Now we can plug $v = 45\mathbf{i} + 39\mathbf{j} - 12\mathbf{k}$ and $r_0 = (15/4)\mathbf{i} + (9/4)\mathbf{j} + 0\mathbf{k}$ into the vector equation for the line of intersection.

$$r = r_0 + tv$$

$$r = \left(\frac{15}{4}\mathbf{i} + \frac{9}{4}\mathbf{j} + 0\mathbf{k}\right) + t(45\mathbf{i} + 39\mathbf{j} - 12\mathbf{k})$$

$$r = \frac{15}{4}\mathbf{i} + \frac{9}{4}\mathbf{j} + 0\mathbf{k} + 45t\mathbf{i} + 39t\mathbf{j} - 12t\mathbf{k}$$

$$r = \left(\frac{15}{4}\mathbf{i} + 45t\mathbf{i}\right) + \left(\frac{9}{4}\mathbf{j} + 39t\mathbf{j}\right) + (0\mathbf{k} - 12t\mathbf{k})$$

$$r = \left(\frac{15}{4} + 45t\right)\mathbf{i} + \left(\frac{9}{4} + 39t\right)\mathbf{j} - 12t\mathbf{k}$$

Now that we have the vector equation for the line of intersection, we can find the parametric equations from the coefficients. The parametric equations are

$$x = \frac{15}{4} + 45t$$

$$y = \frac{9}{4} + 39t$$

$$z = -12t$$