Reparametrizing the curve

When we reparametrize a curve, it means that we rewrite it in terms of an independent variable. There's not a specific variable that's always used in reparametrization, but it's common to see s used. Given a vector function

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

the reparametrized curve will be

$$r\left[t(s)\right] = r\left[t(s)\right]_1\mathbf{i} + r\left[t(s)\right]_2\mathbf{j} + r\left[t(s)\right]_3\mathbf{k}$$

To reparametrize the curve with respect to arc length, we'll have to find the arc length of the vector function using

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

where L is the arc length of the vector function, $[a, b]$ is the interval, and dx/dt, dy/dt and dz/dt are the derivatives of the parametric equations for x, y and z, respectively.

Whether the vector function is given as $r(t) = \left\langle r(t)_1, r(t)_2, r(t)_3 \right\rangle$ or $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, the parametric equations of the vector are

$$x = r(t)_1$$

$$y = r(t)_2$$

$$z = r(t)_3$$

Once we have the parametric equations, we can find their derivatives and plug them into the arc length formula. We'll evaluate the integral to find arc length, then take the arc length and set it equal to s, such that $L(t) = s$. We'll solve $L(t) = s$ for t, and then plug the value we found for t back into the original vector function. This will give us the reparametrized curve

$$r\left[t(s)\right] = r\left[t(s)\right]_1 \mathbf{i} + r\left[t(s)\right]_2 \mathbf{j} + r\left[t(s)\right]_3 \mathbf{k}$$

Example

Reparametrize the curve with respect to arc length from $t = 0$ in the direction of increasing t.

$$r(t) = -2t\mathbf{i} + 4t\mathbf{j} + (2 - 4t)\mathbf{k}$$

First, we'll use the coefficients from the vector function to generate parametric equations of the vector function.

$$x = -2t$$

$$y = 4t$$

$$z = 2 - 4t$$

Now we'll take the derivatives of these.

$$\frac{dx}{dt} = -2$$

$$\frac{dy}{dt} = 4$$

$$\frac{dz}{dt} = -4$$

Next, we'll plug the derivatives into the arc length formula. We also know that the limits of integration will be $[0,t]$, since the problem indicated that the arc length should start at 0 and head in the direction of increasing t.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_0^t \sqrt{(-2)^2 + (4)^2 + (-4)^2}\ dt$$

$$L = \int_0^t \sqrt{4 + 16 + 16}\ dt$$

$$L = \int_0^t \sqrt{36}\ dt$$

$$L = \int_0^t 6\ dt$$

$$L = 6t\Big|_0^t$$

$$L = 6(t) - 6(0)$$

$$L = 6t$$

Now we'll set arc length equal to the independent variable s, and we'll get

$$s = 6t$$

Then we'll solve for t.

$$t = \frac{s}{6}$$

We'll substitute the value of t into the original vector function, and we'll get the reparametrized version.

$$r\left[t(s)\right] = -2\left(\frac{s}{6}\right)\mathbf{i} + 4\left(\frac{s}{6}\right)\mathbf{j} + \left[2 - 4\left(\frac{s}{6}\right)\right]\mathbf{k}$$

$$r\left[t(s)\right] = -\frac{1}{3}s\mathbf{i} + \frac{2}{3}s\mathbf{j} + \left(2 - \frac{2}{3}s\right)\mathbf{k}$$

This is the reparametrized curve in terms of arc length.