Calculus 3 Workbook Solutions

Triple integrals in cylindrical coordinates

krista king MATH

CYLINDRICAL COORDINATES

1. Evaluate the triple integral given in cylindrical coordinates, where $f(r,\theta,z)=(3r-12z^2)\cos\theta$.

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{2}^{3} f(r,\theta,z)\ r\ dr\ d\theta\ dz$$

Solution:

Set up the integral.

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{2}^{3} r(3r-12z^2)\cos\theta\ dr\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{2}^{3} (3r^2-12rz^2)\cos\theta\ dr\ d\theta\ dz$$

Integrate with respect to r.

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (r^3-6r^2z^2)\cos\theta\Big|_{r=2}^{r=3}\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (3^3-6(3)^2z^2)\cos\theta-((2^3-6(2)^2z^2)\cos\theta)\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(27-54z^2)\cos\theta-(8-24z^2)\cos\theta\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}[(27-54z^2)-(8-24z^2)]\cos\theta\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}(19-30z^2)\cos\theta\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_{-1}^{1}(19-30z^2)\sin\theta\Big|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}}\ dz$$

$$\int_{-1}^{1}(19-30z^2)\sin\frac{\pi}{4}-(19-30z^2)\sin\left(-\frac{\pi}{4}\right)\ dz$$

$$\int_{-1}^{1}(19-30z^2)\left(\frac{\sqrt{2}}{2}\right)-(19-30z^2)\left(-\frac{\sqrt{2}}{2}\right)\ dz$$

$$\int_{-1}^{1}(19-30z^2)\left(\frac{\sqrt{2}}{2}\right)+(19-30z^2)\left(\frac{\sqrt{2}}{2}\right)\ dz$$

$$\int_{-1}^{1}2(19-30z^2)\left(\frac{\sqrt{2}}{2}\right)\ dz$$

$$\int_{-1}^{1}19\sqrt{2}-30\sqrt{2}z^2\ dz$$

Integrate with respect to z.

$$19\sqrt{2}z - 10\sqrt{2}z^3\Big|_{-1}^{1}$$

$$19\sqrt{2}(1) - 10\sqrt{2}(1)^3 - (19\sqrt{2}(-1) - 10\sqrt{2}(-1)^3)$$

$$19\sqrt{2} - 10\sqrt{2} - 10\sqrt{2} + 19\sqrt{2}$$

$$38\sqrt{2} - 20\sqrt{2}$$

$$18\sqrt{2}$$

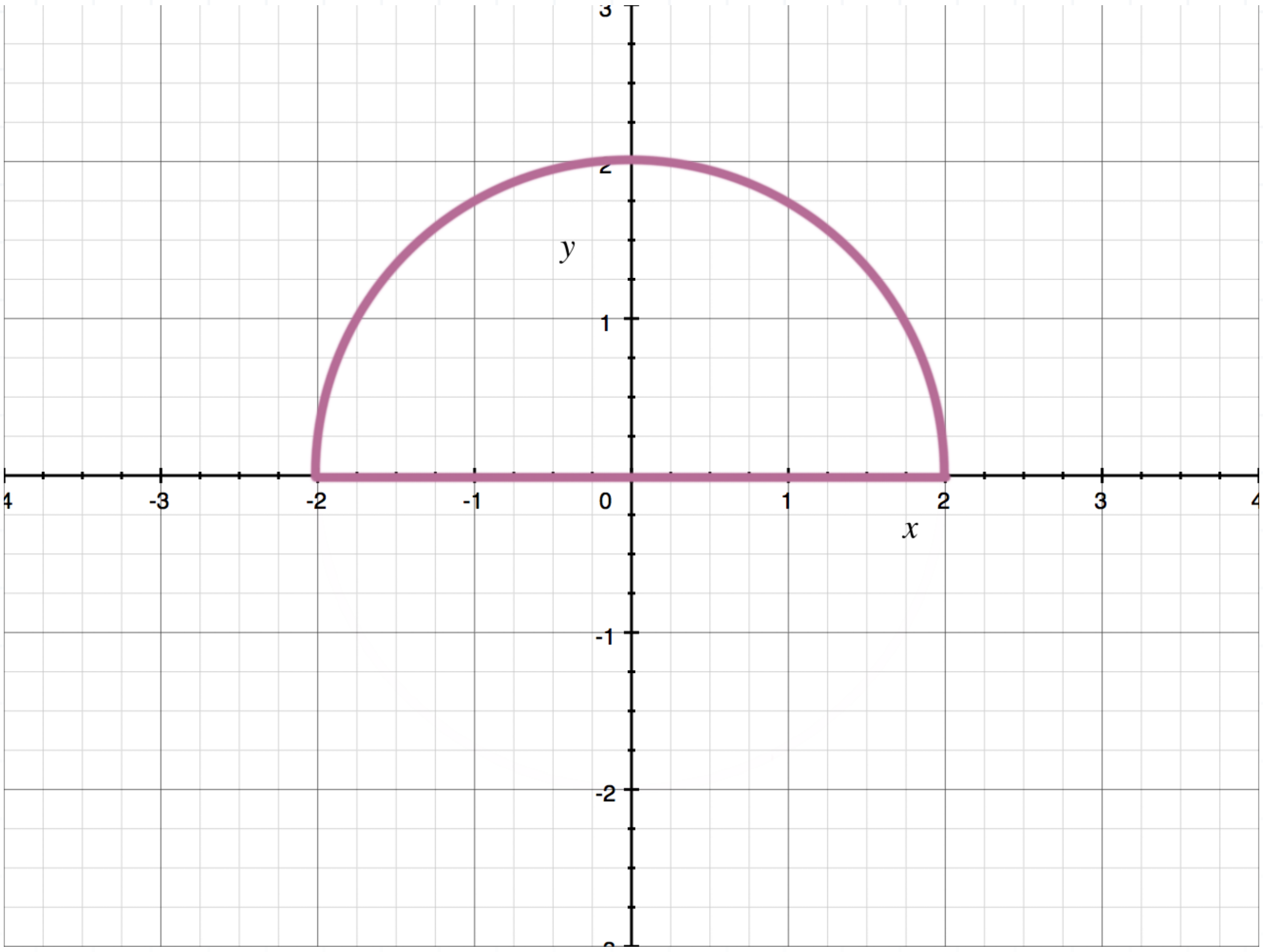
■ 2. Identify the solid given by the following iterated integral in cylindrical coordinates.

$$\int_{-4}^{6}\int_{0}^{\pi}\int_{0}^{2} f(r,\theta,x)\ r\ dr\ d\theta\ dx$$

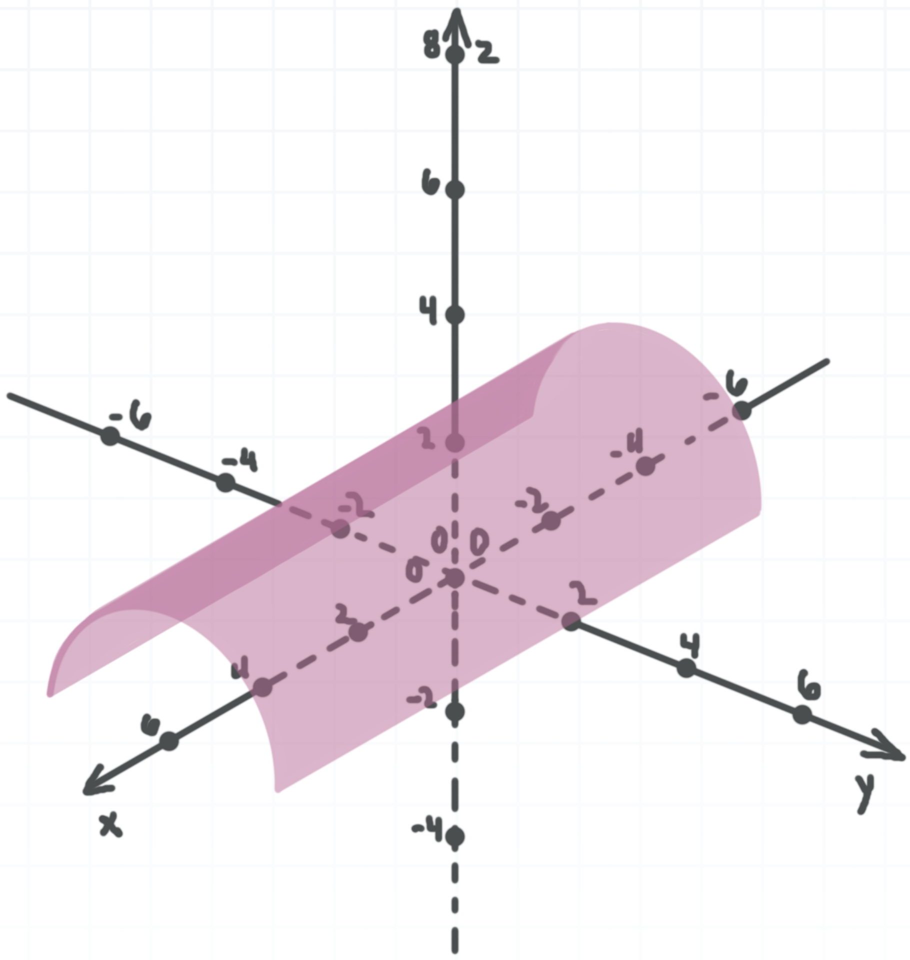
Solution:

The value of x changes from -4 to 6. The polar coordinates r and θ change within the semicircle with center at the x-axis and radius 2, that lies in the plane parallel to yz-plane.

Since the angular coordinate θ changes from 0 to π, the region of integration includes only the points within the semicircle for $z > 0$.



So the region of integration in three dimensions is the half of the cylinder with radius 2 and height 10, with the cylinder's axis parallel to the x-axis, and bases lying in the planes $x = -4$ and $x = 6$, and only the points of the cylinder where $z > 0$.



3. Identify the solid given by the iterated improper integral in cylindrical coordinates.

$$\int_{2}^{\infty}\int_{0}^{2\pi}\int_{0}^{\sqrt{2y-4}} f(r,\theta,y)\ r\ dr\ d\theta\ dy$$

Solution:

The value of y changes from 2 to ∞, and the values of r and θ change within the circle with center at the y-axis that lies in the plane parallel to the xz-plane.

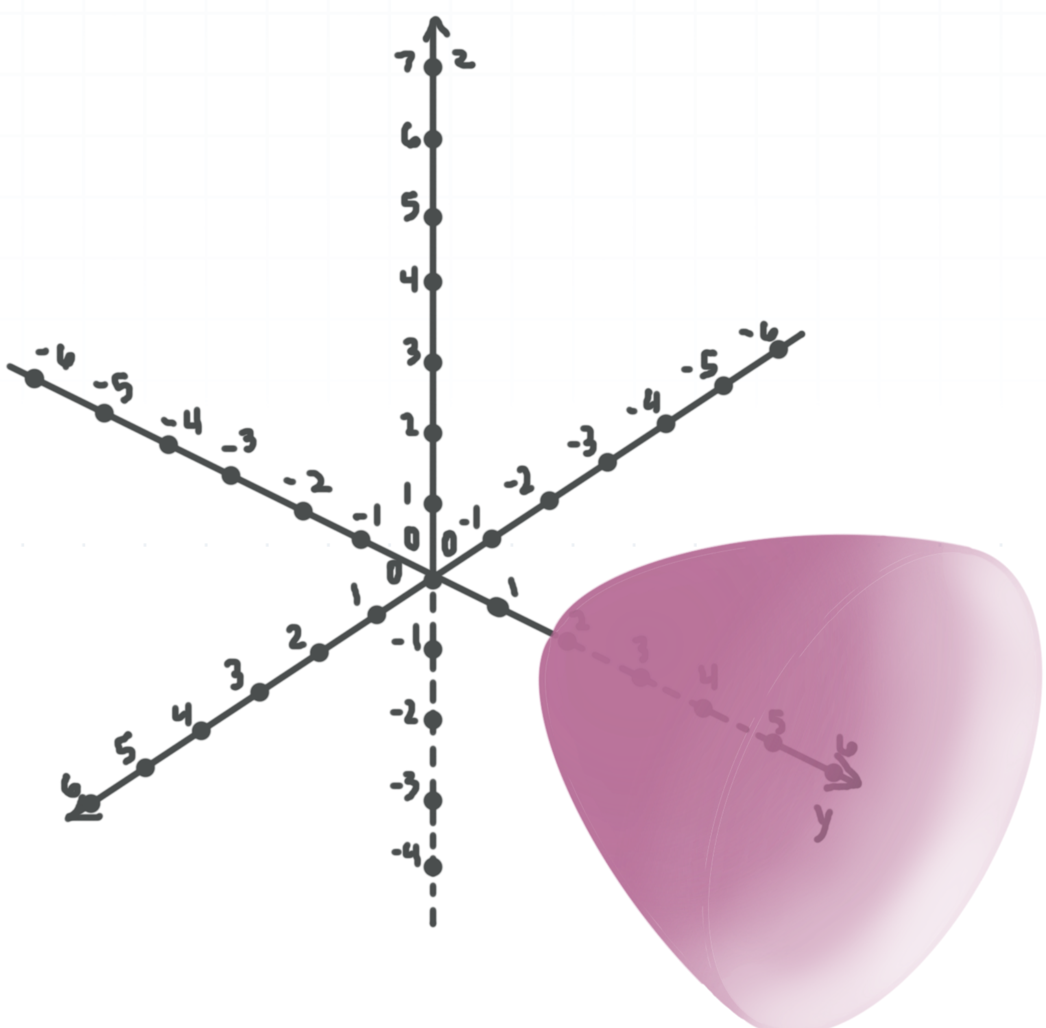
Since r changes from 0 to $\sqrt{2y-4}$, we can find the upper bound for r by substituting $r^2 = x^2 + z^2$ into the equation $r = \sqrt{2y-4}$.

$$r^2 = 2y - 4$$

$$x^2 + z^2 = 2y - 4$$

$$\frac{x^2}{2} + \frac{z^2}{2} = y - 2$$

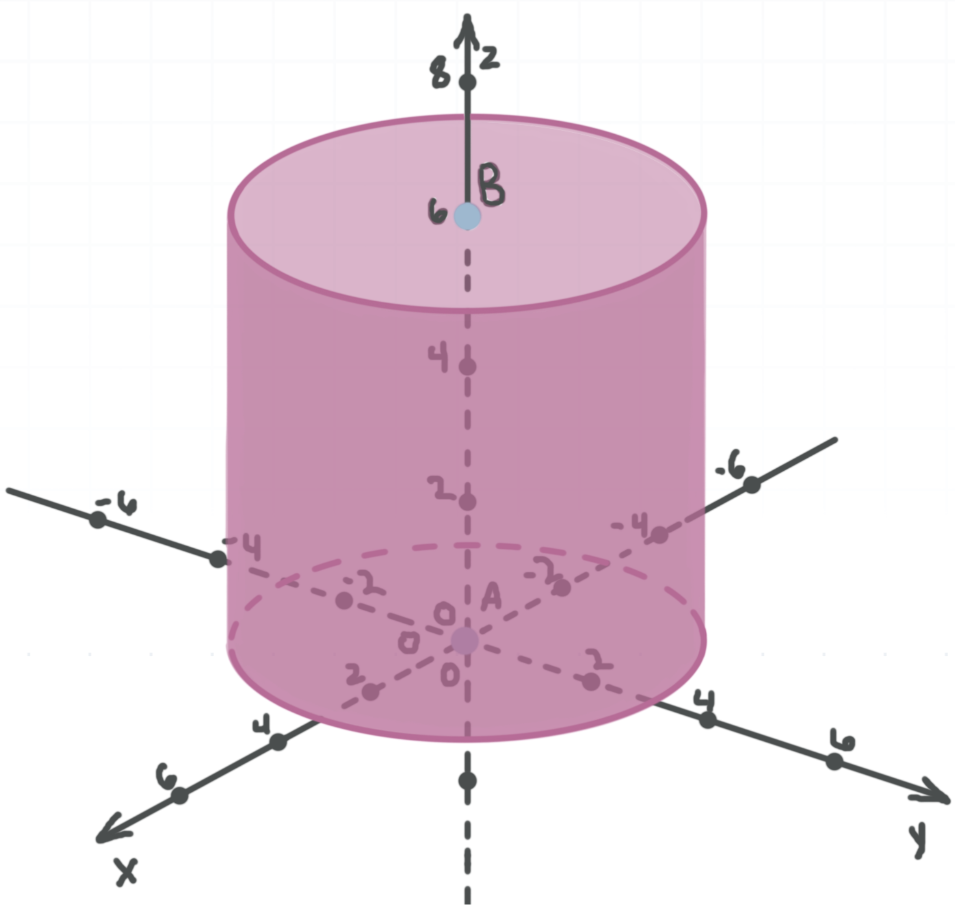
So the upper bound for r is the circular paraboloid with center at the point $(0,2,0)$, whose axis is parallel to the y-axis.



CHANGING TRIPLE INTEGRALS TO CYLINDRICAL COORDINATES

■ 1. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the right circular cylinder with radius 3, height 6, and a base that lies in the xy-plane with center at the origin.

$$\iiint_E (x^2+y^2)2^z \ dV$$



Solution:

The value of z changes from 0 to 6, x and y change within the circle C with radius 3 that lies in the xy-plane with center at the origin. The value of r

changes from 0 to 3, and θ changes from 0 to 2π. Then the integral in cylindrical coordinates is

$$\int_0^6 \int_0^{2\pi} \int_0^3 r^2 2^z \cdot r\ dr\ d\theta\ dz$$

$$\int_0^6 \int_0^{2\pi} \int_0^3 r^3 2^z\ dr\ d\theta\ dz$$

$$\int_0^6 2^z\ dz \cdot \int_0^{2\pi} d\theta \cdot \int_0^3 r^3\ dr$$

Evaluate each integral.

$$\frac{2^z}{\ln 2}\Big|_0^6 \cdot \theta\Big|_0^{2\pi} \cdot \frac{1}{4}r^4\Big|_0^3$$

$$\left(\frac{2^6}{\ln 2} - \frac{2^0}{\ln 2}\right)(2\pi - 0)\left(\frac{1}{4}(3)^4 - \frac{1}{4}(0)^4\right)$$

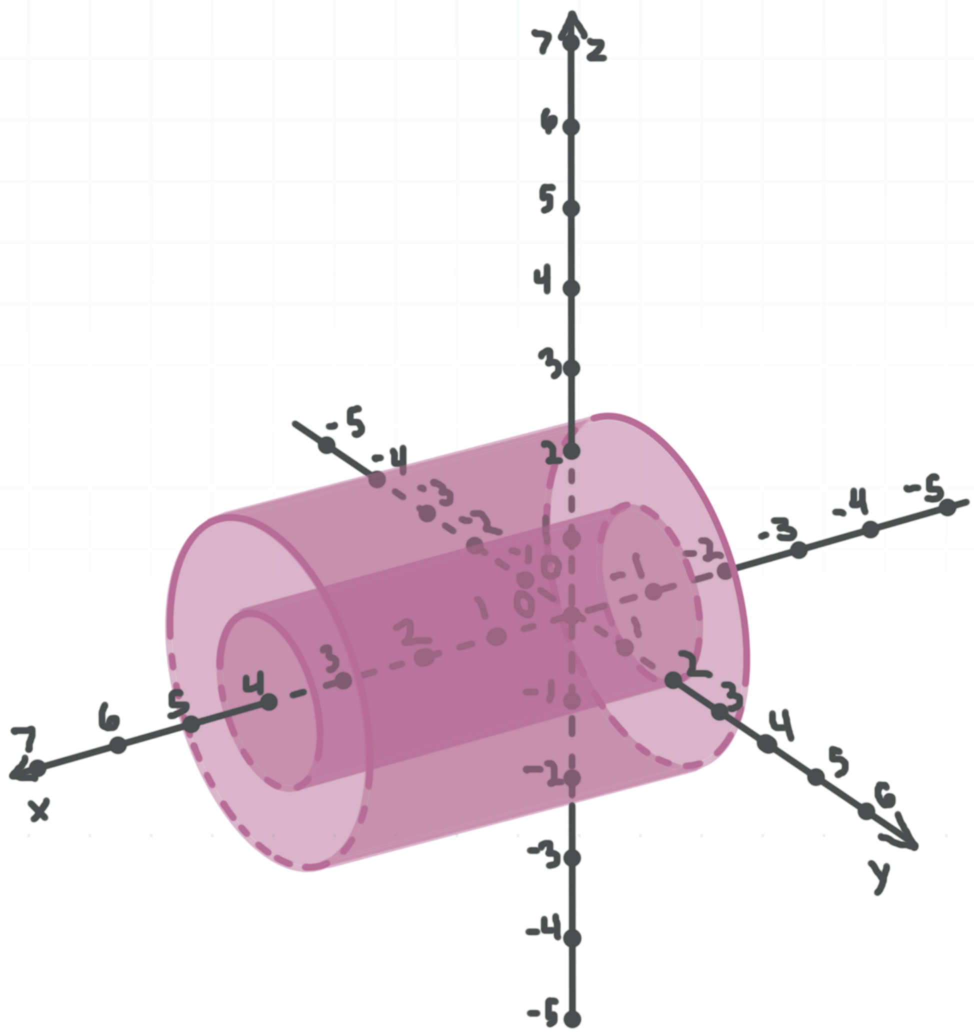
$$\left(\frac{64}{\ln 2} - \frac{1}{\ln 2}\right)(2\pi)\left(\frac{81}{4}\right)$$

$$\frac{63(81)(2\pi)}{4\ln 2}$$

$$\frac{5{,}103\pi}{2\ln 2}$$

■ 2. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the set of points between right circular cylinders with radius 1 and 2, height 5, cylinder axes are x -axis, and bases that lie in the planes $x = -1$ and $x = 4$.

$$\iiint_E \frac{x+y+z}{y^2+z^2}\ dV$$



Solution:

The value of x changes from -1 to 4, and x and y change between the circles C_1 and C_2 with centers at the x-axis, radii 1 and 2 respectively, that lie in the plane parallel to the yz-plane. The value of r changes from 1 to 2, and θ changes from 0 to 2π, so the integral in cylindrical coordinates is

$$\int_{-1}^{4}\int_{0}^{2\pi}\int_{1}^{2}\left(\frac{x}{r^2}+\frac{\cos\theta}{r}+\frac{\sin\theta}{r}\right)\cdot r\ dr\ d\theta\ dx$$

$$\int_{-1}^{4}\int_{0}^{2\pi}\int_{1}^{2}\frac{x}{r}+\cos\theta+\sin\theta\ dr\ d\theta\ dx$$

Integrate with respect to r.

$$\int_{-1}^{4}\int_{0}^{2\pi}x\ln r+r\cos\theta+r\sin\theta\Big|_{r=1}^{r=2}\ d\theta\ dx$$

$$\int_{-1}^{4}\int_{0}^{2\pi}x\ln 2+2\cos\theta+2\sin\theta-(x\ln 1+\cos\theta+\sin\theta)\ d\theta\ dx$$

$$\int_{-1}^{4}\int_{0}^{2\pi}x\ln 2+\cos\theta+\sin\theta\ d\theta\ dx$$

Integrate with respect to θ.

$$\int_{-1}^{4}\theta x\ln 2+\sin\theta-\cos\theta\Big|_{\theta=0}^{\theta=2\pi}\ dx$$

$$\int_{-1}^{4}2\pi x\ln 2+\sin(2\pi)-\cos(2\pi)-(0x\ln 2+\sin(0)-\cos(0))\ dx$$

$$\int_{-1}^{4}2\pi x\ln 2-1+1\ dx$$

$$\int_{-1}^{4}2\pi x\ln 2\ dx$$

Integrate with respect to x.

$$\pi x^2 \ln 2 \Big|_{-1}^{4}$$

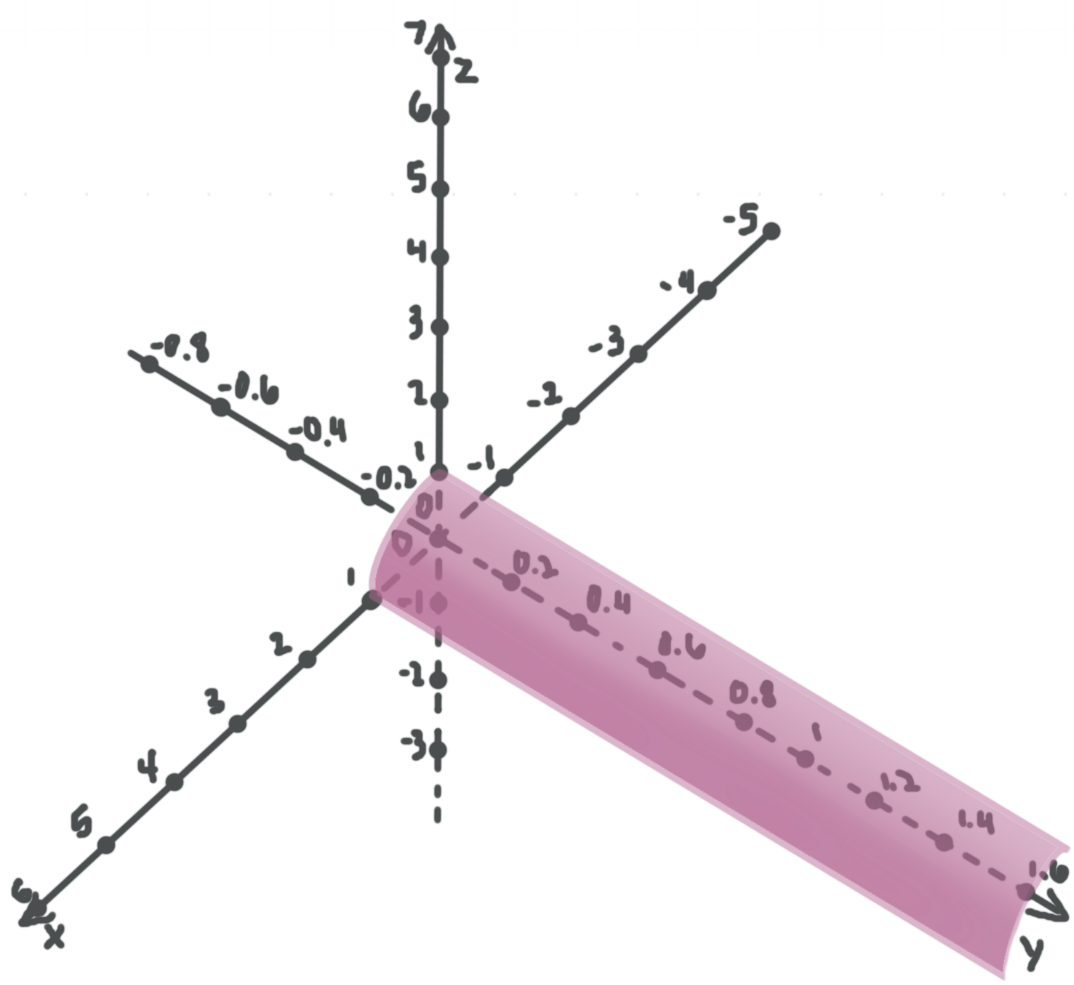
$$\pi(4)^2 \ln 2 - \pi(-1)^2 \ln 2$$

$$16\pi \ln 2 - \pi \ln 2$$

$$15\pi \ln 2$$

■ 3. Evaluate the improper triple integral by changing it to cylindrical coordinates, where E is part of the cylinder $x^2 + z^2 = 1$ that lies in the first octant.

$$\iiint_E 2e^{-x^2-y^2-z^2}\ dV$$



Solution:

The value of y changes from 0 to ∞, x and z change in the first quarter of the circle C with center at the y-axis, radius 1, that lies in the plane parallel to the xz-plane. The values of r change from 0 to 1, and θ changes from 0 to $\pi/2$. So the integral in cylindrical coordinates is

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^1 2e^{-r^2-y^2} \cdot r \ dr \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^1 2re^{-r^2-y^2} \ dr \ d\theta \ dy$$

Integrate with respect to r, using a substitution with $u = r^2$, $du = 2r \ dr$, and where u changes from 0 to 1.

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^1 e^{-u-y^2} \ du \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \frac{1}{-1} e^{-u-y^2} \Big|_{u=0}^{u=1} \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} - e^{-u-y^2} \Big|_{u=0}^{u=1} \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} - e^{-1-y^2} - (-e^{-0-y^2}) \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -e^{-1-y^2} + e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -e^{-1}e^{-y^2} + e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -\frac{1}{e}e^{-y^2} + e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \left(1 - \frac{1}{e}\right) e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \frac{e-1}{e}e^{-y^2}\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_0^\infty \frac{e-1}{e}e^{-y^2}\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\ dy$$

$$\int_0^\infty \frac{e-1}{e}e^{-y^2}\left(\frac{\pi}{2}\right) - \frac{e-1}{e}e^{-y^2}(0)\ dy$$

$$\int_0^\infty \frac{\pi e - \pi}{2e}e^{-y^2}\ dy$$

Integrating with respect to y, remembering that

$$\int_{-\infty}^\infty e^{-t^2}\ dt = \sqrt{\pi}$$

gives us

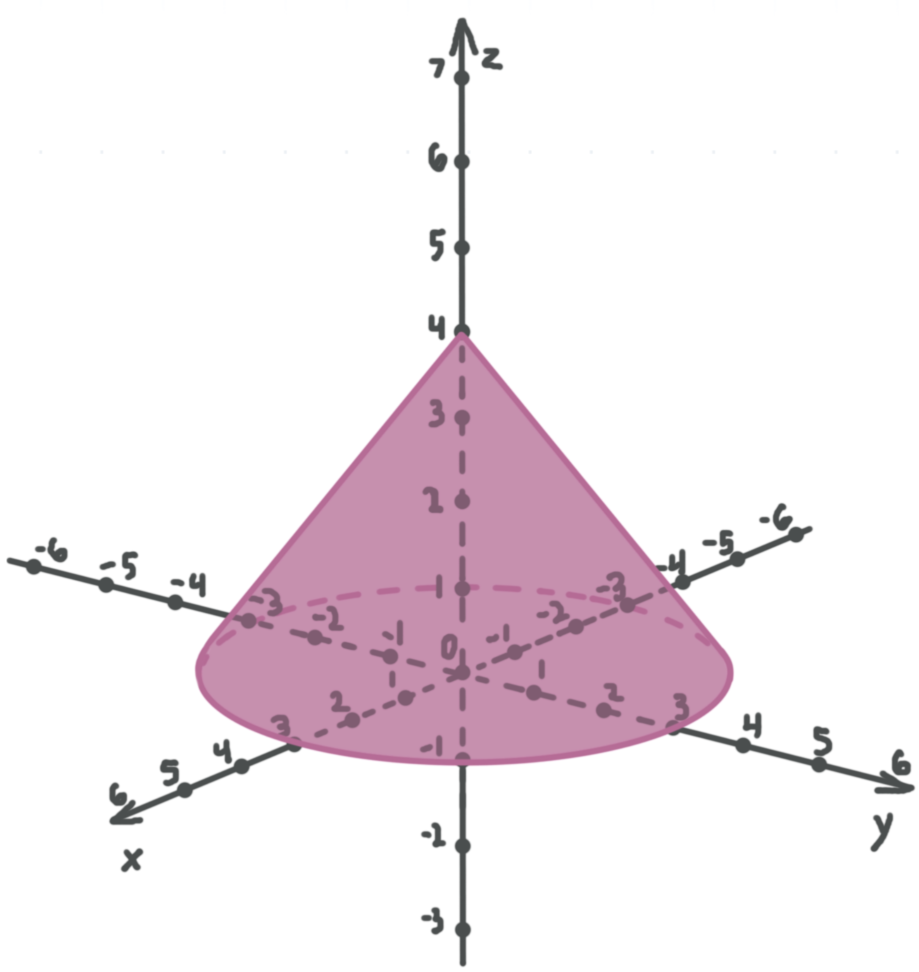
$$\left(\frac{\pi e - \pi}{2e}\right)\left(\frac{\sqrt{\pi}}{2}\right)$$

$$\frac{\pi e\sqrt{\pi} - \pi\sqrt{\pi}}{4e}$$

$$\frac{\pi\sqrt{\pi}(e-1)}{4e}$$

■ 4. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the right circular cone with radius 3, vertex at the point $(0,0,4)$, and a base that lies in the xy-plane with its center at the origin.

$$\iiint_E 48x^2y^2(z+2)\ dV$$



Solution:

The value of z changes from 0 to 4, x and y changes within the circle centered at the origin that lies in the plane parallel to the xy-plane. The value of θ changes from 0 to 2π, and r changes linearly on z, i.e. $r = az + b$. When $z = 0$, $r = 3$, and when $z = 4$, $r = 0$. Substitute these values into the linear equation and solve the resulting system for a and b.

$$3 = a(0) + b$$

$$0 = a(4) + b$$

Then $b = 3$, so $4a + 3 = 0$ and $a = -3/4$. Therefore, $r = -(3/4)z + 3$. The function is

$$48x^2y^2(z+2)$$

$$48r^4 \cdot \cos^2\theta \cdot \sin^2\theta\,(z+2)$$

$$12r^4(2\cos\theta \cdot \sin\theta)^2(z+2)$$

$$12r^4\sin^2(2\theta)(z+2)$$

$$6r^4(1-\cos(4\theta))(z+2)$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^4 \int_0^{2\pi} \int_0^{-\frac{3}{4}z+3} 6r^4(1-\cos(4\theta))(z+2) \cdot r\ dr\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^{-\frac{3}{4}z+3} 6r^5(1-\cos(4\theta))(z+2)\ dr\ d\theta\ dz$$

Integrate with respect to r.

$$\int_0^4 \int_0^{2\pi} r^6(1-\cos(4\theta))(z+2)\Big|_{r=0}^{r=-\frac{3}{4}z+3}\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \left(-\frac{3}{4}z+3\right)^6 (1-\cos(4\theta))(z+2) - 0^6(1-\cos(4\theta))(z+2)\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \left(-\frac{3}{4}z+3\right)^6 (1-\cos(4\theta))(z+2)\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_0^4 \left(-\frac{3}{4}z+3\right)^6 \left(\theta-\frac{1}{4}\sin(4\theta)\right)(z+2)\Big|_{\theta=0}^{\theta=2\pi}\ dz$$

$$\int_0^4 \left(-\frac{3}{4}z+3\right)^6 \left(2\pi-\frac{1}{4}\sin(4(2\pi))\right)(z+2)$$

$$-\left(-\frac{3}{4}z+3\right)^6 \left(0-\frac{1}{4}\sin(4(0))\right)(z+2)\ dz$$

$$\int_0^4 2\pi\left(-\frac{3}{4}z+3\right)^6 (z+2)\ dz$$

Rewrite the function.

$$2\pi\int_0^4\left(\frac{729}{4{,}096}z^6-\frac{2{,}187}{512}z^5+\frac{10{,}935}{256}z^4-\frac{3{,}645}{16}z^3\right.$$

$$\left.+\frac{10{,}935}{16}z^2-\frac{2{,}187}{2}z+729\right)(z+2)\ dz$$

$$2\pi\int_0^4\frac{729}{4{,}096}z^7-\frac{2{,}187}{512}z^6+\frac{10{,}935}{256}z^5-\frac{3{,}645}{16}z^4+\frac{10{,}935}{16}z^3-\frac{2{,}187}{2}z^2+729z$$

$$+\frac{729}{2{,}048}z^6-\frac{2{,}187}{256}z^5+\frac{10{,}935}{128}z^4-\frac{3{,}645}{8}z^3+\frac{10{,}935}{8}z^2-2{,}187z+1{,}458\ dz$$

$$2\pi\int_0^4\frac{729}{4{,}096}z^7-\frac{8{,}019}{2{,}048}z^6+\frac{8{,}748}{256}z^5-\frac{18{,}225}{128}z^4+\frac{3{,}645}{16}z^3$$

$$+\frac{2{,}187}{8}z^2-1{,}458z+1{,}458\ dz$$

Integrate with respect to z.

$$2\pi\left(\frac{729}{32{,}768}z^8-\frac{8{,}019}{14{,}336}z^7+\frac{729}{128}z^6-\frac{3{,}645}{128}z^5+\frac{3{,}645}{64}z^4\right.$$

$$\left.\left.+\frac{729}{8}z^3-729z^2+1{,}458z\right)\right|_0^4$$

$$2\pi\left(\frac{729}{32{,}768}(4)^8-\frac{8{,}019}{14{,}336}(4)^7+\frac{729}{128}(4)^6-\frac{3{,}645}{128}(4)^5+\frac{3{,}645}{64}(4)^4\right.$$

$$+\frac{729}{8}(4)^3-729(4)^2+1{,}458(4)\Bigg)$$

$$2\pi\Bigg(729(2)-\frac{8{,}019}{7}(8)+729(32)-3{,}645(8)+3{,}645(4)$$

$$+729(8)-729(16)+1{,}458(4)\Bigg)$$

$$2\pi\left(1{,}458-\frac{64{,}152}{7}+23{,}328-29{,}160+14{,}580+5{,}832-11{,}664+5{,}832\right)$$

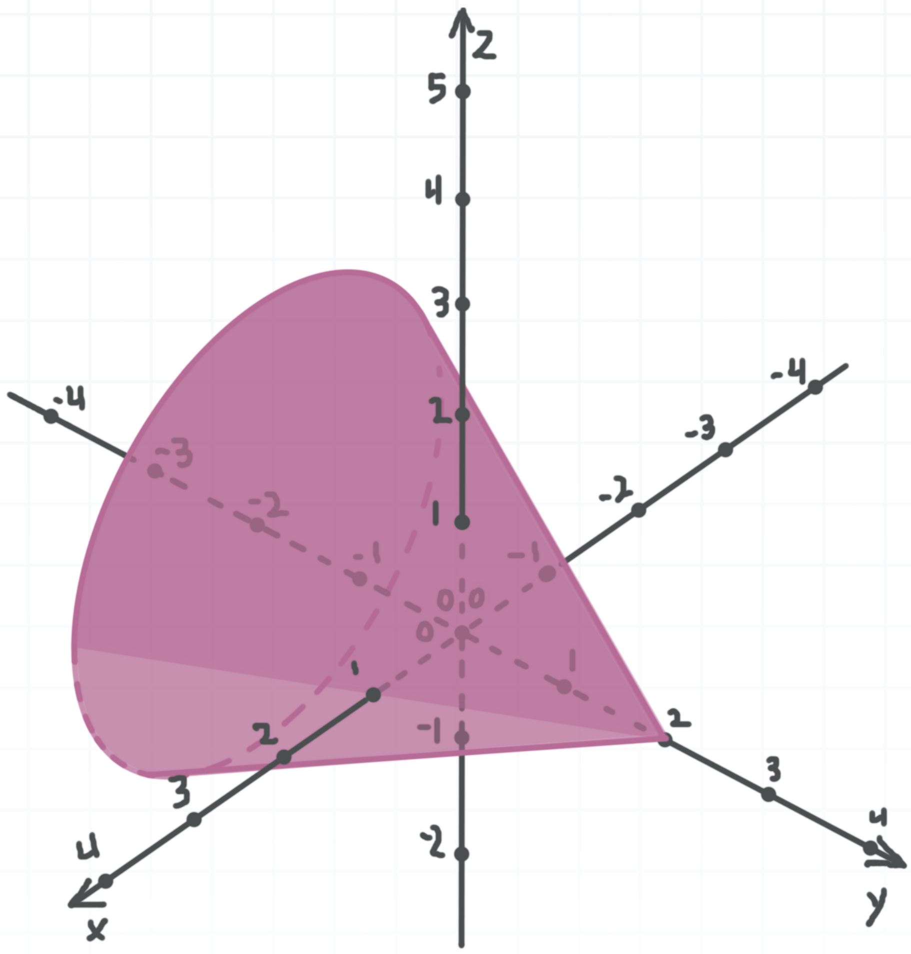
$$2\pi\left(10{,}206-\frac{64{,}152}{7}\right)$$

$$2\pi\left(\frac{71{,}442}{7}-\frac{64{,}152}{7}\right)$$

$$\frac{14{,}580\pi}{7}$$

■ 5. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the right circular cone with radius 2, vertex at the point $(0,2,0)$, and a base that lies in the plane $y=-2$ with its center at the point $(0,-2,0)$.

$$\iiint_E \frac{(x+z)^2}{(y+4)}\ dV$$



Solution:

The value of y changes from -2 to 2, x and z change within the circle C with center at the y-axis, that lies in the plane parallel to the xz-plane. The value of θ changes from 0 to 2π, and r changes linearly on y, i.e. $r = ay + b$. When $y = -2$, $r = 2$, and when $y = 2$, $r = 0$. Substitute the values into the linear equation and solve the resulting system for a and b.

$$2 = a(-2) + b$$

$$0 = a(2) + b$$

$b = 1$ and $a = -0.5$

So $r = -0.5y + 1$. The function is

$$\frac{(x+z)^2}{(y+4)}$$

$$\frac{x^2+z^2+2xz}{(y+4)}$$

$$\frac{r^2+2r^2\cos\theta\sin\theta}{y+4}$$

$$\frac{r^2+r^2\sin 2\theta}{y+4}$$

Therefore, the integral in cylindrical coordinates is

$$\int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{-0.5y+1}\frac{r^2+r^2\sin 2\theta}{y+4}\cdot r\ dr\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{-0.5y+1}\frac{r^3(1+\sin 2\theta)}{y+4}\ dr\ d\theta\ dy$$

Integrate with respect to r.

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{r^4(1+\sin 2\theta)}{4(y+4)}\Bigg|_{r=0}^{r=-0.5y+1}\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{(-0.5y+1)^4(1+\sin 2\theta)}{4(y+4)}-\frac{0^4(1+\sin 2\theta)}{4(y+4)}\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{(-0.5y+1)^4(1+\sin 2\theta)}{4(y+4)}\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{(-0.5y+1)^4}{4(y+4)}(1+\sin 2\theta)\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(\theta-\frac{1}{2}\cos(2\theta)\right)\Bigg|_{\theta=0}^{\theta=2\pi}\ dy$$

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(2\pi-\frac{1}{2}\cos(2(2\pi))\right)-\frac{(-0.5y+1)^4}{4(y+4)}\left(0-\frac{1}{2}\cos(2(0))\right)\ dy$$

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(2\pi-\frac{1}{2}\right)+\frac{(-0.5y+1)^4}{8(y+4)}\ dy$$

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(2\pi-\frac{1}{2}+\frac{1}{2}\right)\ dy$$

$$\int_{-2}^{2}\frac{\pi(-0.5y+1)^4}{2(y+4)}\ dy$$

Simplify the function.

$$\int_{-2}^{2}\frac{\pi\left(\frac{1}{16}y^4-\frac{2}{4}y^3+\frac{3}{2}y^2-2y+1\right)}{2y+8}\ dy$$

$$\int_{-2}^{2}\frac{\frac{\pi}{16}(y^4-8y^3+48y^2-32y+16)}{2(y+4)}\ dy$$

$$\frac{\pi}{32}\int_{-2}^{2}\frac{y^4-8y^3+48y^2-32y+16}{y+4}\ dy$$

Use long division of polynomials.

$$\frac{\pi}{32}\int_{-2}^{2} y^3 - 12y^2 + 96y - 416 + \frac{1{,}680}{y+4}\ dy$$

Integrate with respect to y.

$$\frac{\pi}{32}\left(\frac{1}{4}y^4 - 4y^3 + 48y^2 - 416y + 1{,}680\ln(y+4)\right)\Bigg|_{-2}^{2}$$

$$\frac{\pi}{32}\left(\frac{1}{4}(2)^4 - 4(2)^3 + 48(2)^2 - 416(2) + 1{,}680\ln(2+4)\right)$$

$$-\frac{\pi}{32}\left(\frac{1}{4}(-2)^4 - 4(-2)^3 + 48(-2)^2 - 416(-2) + 1{,}680\ln(-2+4)\right)$$

$$\frac{\pi}{32}(4 - 32 + 192 - 832 + 1{,}680\ln 6) - \frac{\pi}{32}(4 + 32 + 192 + 832 + 1{,}680\ln 2)$$

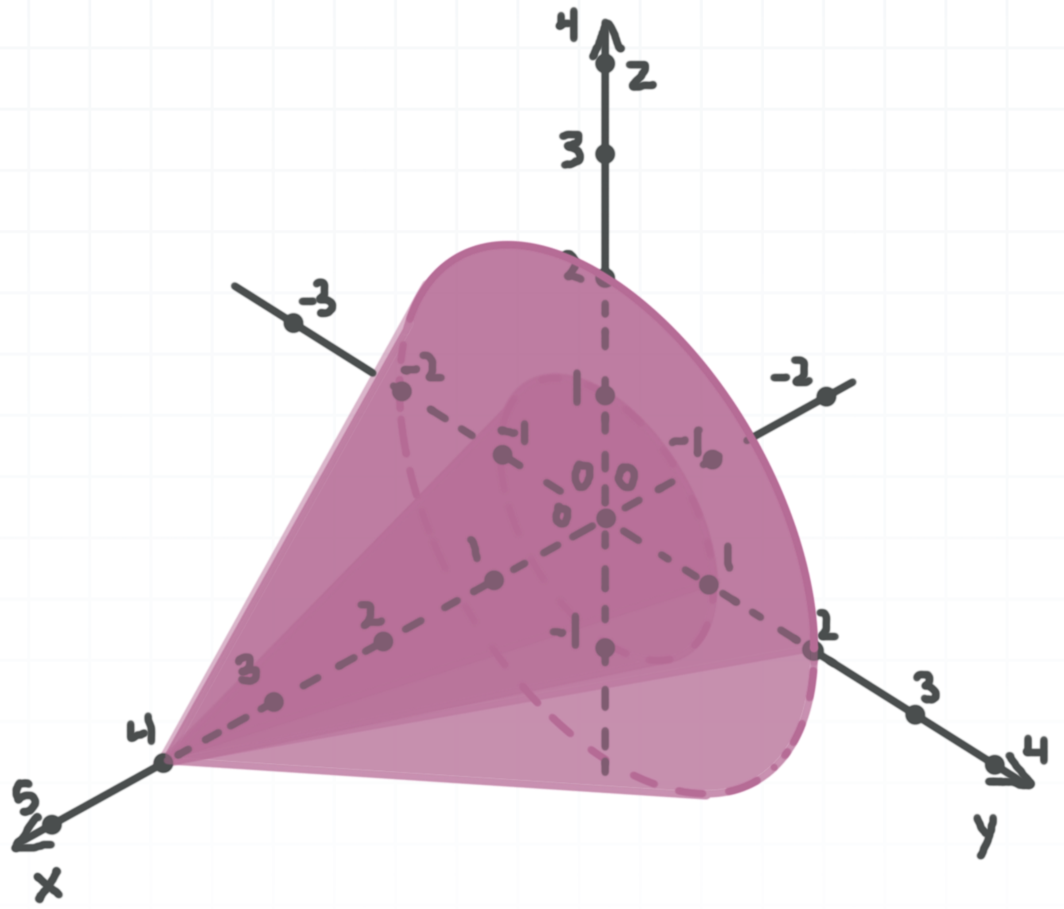
$$\frac{\pi}{32}(4 - 32 + 192 - 832 + 1{,}680\ln 6 - 4 - 32 - 192 - 832 - 1{,}680\ln 2)$$

$$\frac{\pi}{32}(1{,}680\ln 6 - 1{,}680\ln 2 - 1{,}728)$$

$$\frac{3\pi}{2}(35\ln 6 - 35\ln 2 - 36)$$

■ 6. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the set of points between two right circular cones with radii 1 and 2, vertexes at the point $(4,0,0)$, and bases that lie in the yz-plane with center at the origin.

$$\iiint_E x^2 - y^2 - z^2 \, dV$$



Solution:

The value of x changes from 0 to 4, and y and z change between the circles with centers at the x-axis that lie in the plane parallel to the yz-plane. The value of θ changes from 0 to 2π, and r changes between $r_1(x)$ and $r_2(x)$.

Let's find $r_2 = ax + b$. When $x = 0$, $r = 2$, and when $x = 4$, $r = 0$. Substitute the values into the linear equation and solve the resulting system for a and b.

$$2 = a(0) + b$$

$$0 = a(4) + b$$

$$b = 2 \text{ and } a = -0.5$$

So the upper bound is $r_2 = -0.5x + 2$. Let's find $r_1 = ax + b$. When $x = 0$, $r = 1$, and when $x = 4$, $r = 0$. Substitute the values into the linear equation and solve the resulting system for a and b.

$$1 = a(0) + b$$

$$0 = a(4) + b$$

$$b = 1 \text{ and } a = -0.25$$

So lower bound is $r_1 = -0.25x + 1$, and the integral in cylindrical coordinates is

$$\int_0^4 \int_0^{2\pi} \int_{-0.25x+1}^{-0.5x+2} (x^2 - r^2) \cdot r \ dr \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \int_{-0.25x+1}^{-0.5x+2} x^2 r - r^3 \ dr \ d\theta \ dx$$

Integrate with respect to r.

$$\int_0^4 \int_0^{2\pi} \frac{1}{2}x^2 r^2 - \frac{1}{4}r^4 \Big|_{r=-0.25x+1}^{r=-0.5x+2} \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \frac{(-0.5x+2)^2 x^2}{2} - \frac{(-0.5x+2)^4}{4}$$

$$-\frac{x^2(-0.25x+1)^2}{2} + \frac{(-0.25x+1)^4}{4} \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \frac{(-0.5x+2)^2x^2 - x^2(-0.25x+1)^2}{2} + \frac{(-0.25x+1)^4 - (-0.5x+2)^4}{4} \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \frac{81}{1{,}024}x^4 - \frac{33}{64}x^3 + \frac{3}{32}x^2 + \frac{15}{4}x - \frac{15}{4} \ d\theta \ dx$$

Integrate with respect to θ.

$$\int_0^4 \frac{81}{1{,}024}x^4\theta - \frac{33}{64}x^3\theta + \frac{3}{32}x^2\theta + \frac{15}{4}x\theta - \frac{15}{4}\theta \Big|_{\theta=0}^{\theta=2\pi} \ dx$$

$$\int_0^4 \frac{81}{1{,}024}x^4(2\pi) - \frac{33}{64}x^3(2\pi) + \frac{3}{32}x^2(2\pi) + \frac{15}{4}x(2\pi) - \frac{15}{4}(2\pi) \ dx$$

$$\int_0^4 \frac{81\pi}{512}x^4 - \frac{33\pi}{32}x^3 + \frac{3\pi}{16}x^2 + \frac{15\pi}{2}x - \frac{15\pi}{2} \ dx$$

Integrate with respect to x.

$$\frac{81\pi}{2{,}560}x^5 - \frac{33\pi}{128}x^4 + \frac{3\pi}{48}x^3 + \frac{15\pi}{4}x^2 - \frac{15\pi}{2}x \Big|_0^4$$

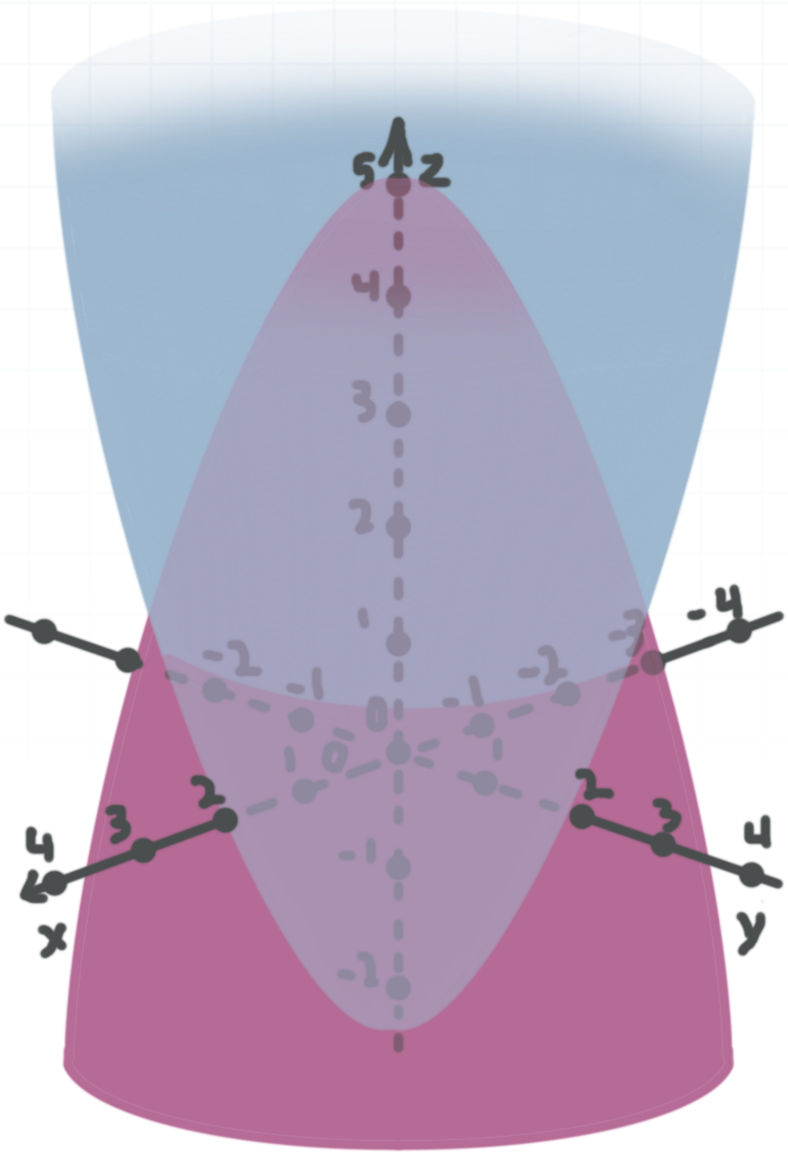
$$\frac{81\pi}{2{,}560}(4)^5 - \frac{33\pi}{128}(4)^4 + \frac{3\pi}{48}(4)^3 + \frac{15\pi}{4}(4)^2 - \frac{15\pi}{2}(4)$$

$$\frac{162\pi}{5} - 66\pi + 4\pi + 60\pi - 30\pi$$

$$\frac{2\pi}{5}$$

■ 7. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the solid bounded by the surfaces $x^2+y^2+z-5=0$ and $x^2+y^2-z-3=0$.

$$\iiint_E 3\sqrt{x^2+y^2}\ dV$$



Solution:

The given surfaces are circular paraboloids, concave up and down, with vertexes at the points $(0,0,5)$ and $(0,0,-3)$, respectively.

To find the set of points where the paraboloids intersect, solve their equations as a system.

$$x^2 + y^2 + z - 5 = 0$$

$$x^2 + y^2 - z - 3 = 0$$

The curve of intersection of these two paraboloids are the circle with center $(0,0,1)$ and radius 2, that lies in the plane $z = 1$.

The value of z changes from -3 to 5, and θ changes from 0 to 2π. Since r has different upper bounds for $z \le 1$ and $z \ge 1$, we need to split the triple integral into the two iterated integrals.

For z from -3 to 1, we get

$$x^2 + y^2 - z - 3 = 0$$

$$r^2 - z - 3 = 0$$

$$r = \sqrt{3 + z}$$

And for z from 1 to 5, we get

$$x^2 + y^2 + z - 5 = 0$$

$$r^2 + z - 5 = 0$$

$$r = \sqrt{5 - z}$$

Therefore, the integral in cylindrical coordinates is the sum of the two iterated integrals.

$$\int_{-3}^{1}\int_{0}^{2\pi}\int_{0}^{\sqrt{3+z}} 3r \cdot r \ dr \ d\theta \ dz + \int_{1}^{5}\int_{0}^{2\pi}\int_{0}^{\sqrt{5-z}} 3r \cdot r \ dr \ d\theta \ dz$$

$$\int_{-3}^{1}\int_{0}^{2\pi}\int_{0}^{\sqrt{3+z}} 3r^2 \ dr \ d\theta \ dz + \int_{1}^{5}\int_{0}^{2\pi}\int_{0}^{\sqrt{5-z}} 3r^2 \ dr \ d\theta \ dz$$

Integrate with respect to r.

$$\int_{-3}^{1}\int_{0}^{2\pi} (\sqrt{3+z})^3 \ d\theta \ dz + \int_{1}^{5}\int_{0}^{2\pi} (\sqrt{5-z})^3 \ d\theta \ dz$$

$$\int_{-3}^{1}\int_{0}^{2\pi} (3+z)^{\frac{3}{2}} \ d\theta \ dz + \int_{1}^{5}\int_{0}^{2\pi} (5-z)^{\frac{3}{2}} \ d\theta \ dz$$

Integrate with respect to θ.

$$\int_{-3}^{1} (3+z)^{\frac{3}{2}}\theta\Big|_{\theta=0}^{\theta=2\pi} \ dz + \int_{1}^{5} (5-z)^{\frac{3}{2}}\theta\Big|_{\theta=0}^{\theta=2\pi} \ dz$$

$$\int_{-3}^{1} (3+z)^{\frac{3}{2}}(2\pi) \ dz + \int_{1}^{5} (5-z)^{\frac{3}{2}}(2\pi) \ dz$$

$$\int_{-3}^{1} 2\pi(3+z)^{\frac{3}{2}} \ dz + \int_{1}^{5} 2\pi(5-z)^{\frac{3}{2}} \ dz$$

Integrate with respect to z.

$$\frac{4\pi}{5}(3+z)^{\frac{5}{2}}\Big|_{-3}^{1} - \frac{4\pi}{5}(5-z)^{\frac{5}{2}}\Big|_{1}^{5}$$

$$\frac{4\pi}{5}(3+1)^{\frac{5}{2}} - \frac{4\pi}{5}(3-3)^{\frac{5}{2}} - \left(\frac{4\pi}{5}(5-5)^{\frac{5}{2}} - \frac{4\pi}{5}(5-1)^{\frac{5}{2}}\right)$$

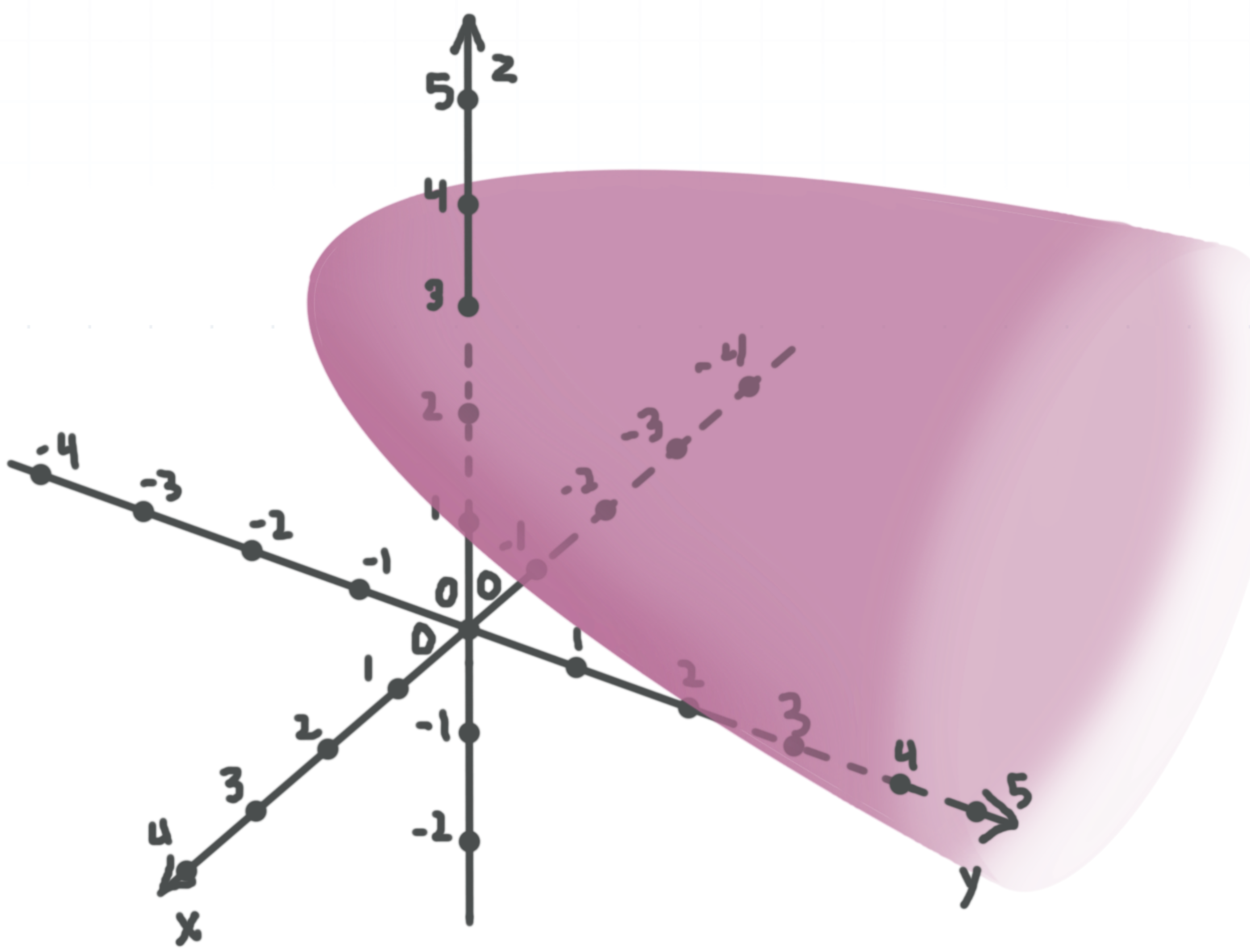
$$\frac{4\pi}{5}(4)^{\frac{5}{2}}+\frac{4\pi}{5}(4)^{\frac{5}{2}}$$

$$\frac{128\pi}{5}+\frac{128\pi}{5}$$

$$\frac{256\pi}{5}$$

■ 8. Evaluate the triple improper integral by changing it to cylindrical coordinates, where E is the interior of the surface $(x+1)^2+(z-2)^2=y+2$.

$$\iiint_E xz10^{-y}\ dV$$



Solution:

The surface is a circular paraboloid with vertex $(-1,-2,2)$, and an axis parallel to the y-axis. Use conversion formulas to move the vertex to the origin.

$$x_1 = x + 1 \text{ so } x = x_1 - 1$$

$$y_1 = y + 2 \text{ so } y = y_1 - 2$$

$$z_1 = z - 2 \text{ so } z = z_1 + 2$$

The function transforms to

$$xz10^{-y} = (x_1 - 1)(z_1 + 2)10^{-y_1+2}$$

$$xz10^{-y} = 100(x_1 - 1)(z_1 + 2)10^{-y_1}$$

The surface transforms to

$$x_1^2 + z_1^2 = y_1$$

Therefore, the given triple integral is equal to

$$\iiint_{E_1} 100(x-1)(z+2)10^{-y}\ dV$$

Inside the surface $x^2 + z^2 = y$, the value of y changes from 0 to ∞, x and z change within the circle C with center at the y-axis, that lies in the plane parallel to the xz-plane. The value of θ changes from 0 to 2π, and since the upper bound is $r^2 = y$, r changes from 0 to $\sqrt{y}$. The function is

$$100(x-1)(z+2)10^{-y}$$

$$100(r\sin\theta-1)(r\cos\theta+2)10^{-y}$$

$$100(r^2\sin\theta\cos\theta+2r\sin\theta-r\cos\theta-2)10^{-y}$$

$$(50r^2\sin 2\theta+200r\sin\theta-100r\cos\theta-200)10^{-y}$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^\infty\int_0^{2\pi}\int_0^{\sqrt{y}}(50r^2\sin 2\theta+200r\sin\theta-100r\cos\theta-200)10^{-y}\cdot r\ dr\ d\theta\ dy$$

$$\int_0^\infty\int_0^{2\pi}\int_0^{\sqrt{y}}(50r^3\sin 2\theta+200r^2\sin\theta-100r^2\cos\theta-200r)10^{-y}\ dr\ d\theta\ dy$$

Since the integrals of $\cos\theta$, $\sin\theta$, and $\sin 2\theta$ over $[0,2\pi]$ are 0, the integral simplifies to

$$\int_0^\infty\int_0^{2\pi}\int_0^{\sqrt{y}}-200(10^{-y})r\ dr\ d\theta\ dy$$

Integrate with respect to r.

$$\int_0^\infty\int_0^{2\pi}-100(10^{-y})r^2\Big|_{r=0}^{r=\sqrt{y}}\ d\theta\ dy$$

$$\int_0^\infty\int_0^{2\pi}-100(10^{-y})(\sqrt{y})^2+100(10^{-y})(0)^2\ d\theta\ dy$$

$$\int_0^\infty\int_0^{2\pi}-100y(10^{-y})\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_0^\infty -100y(10^{-y})\theta\Big|_{\theta=0}^{\theta=2\pi}\ dy$$

$$\int_0^\infty -100y(10^{-y})(2\pi)+100y(10^{-y})(0)\ dy$$

$$\int_0^\infty -200\pi y(10^{-y})\ dy$$

Integrate with respect to y using integration by parts with $u=y$, $du=dy$, $dv=10^{-y}\ dy$, and $v=-(10^{-y})/(\ln 10)$.

$$-200\pi\left[-\frac{y10^{-y}}{\ln 10}\Big|_0^\infty+\frac{1}{\ln 10}\int_0^\infty 10^{-y}\ dy\right]$$

$$-200\pi\left[-\frac{y10^{-y}}{\ln 10}-\frac{10^{-y}}{\ln^2 10}\Big|_0^\infty\right]$$

$$200\pi\left[\frac{y10^{-y}}{\ln 10}+\frac{10^{-y}}{\ln^2 10}\Big|_0^\infty\right]$$

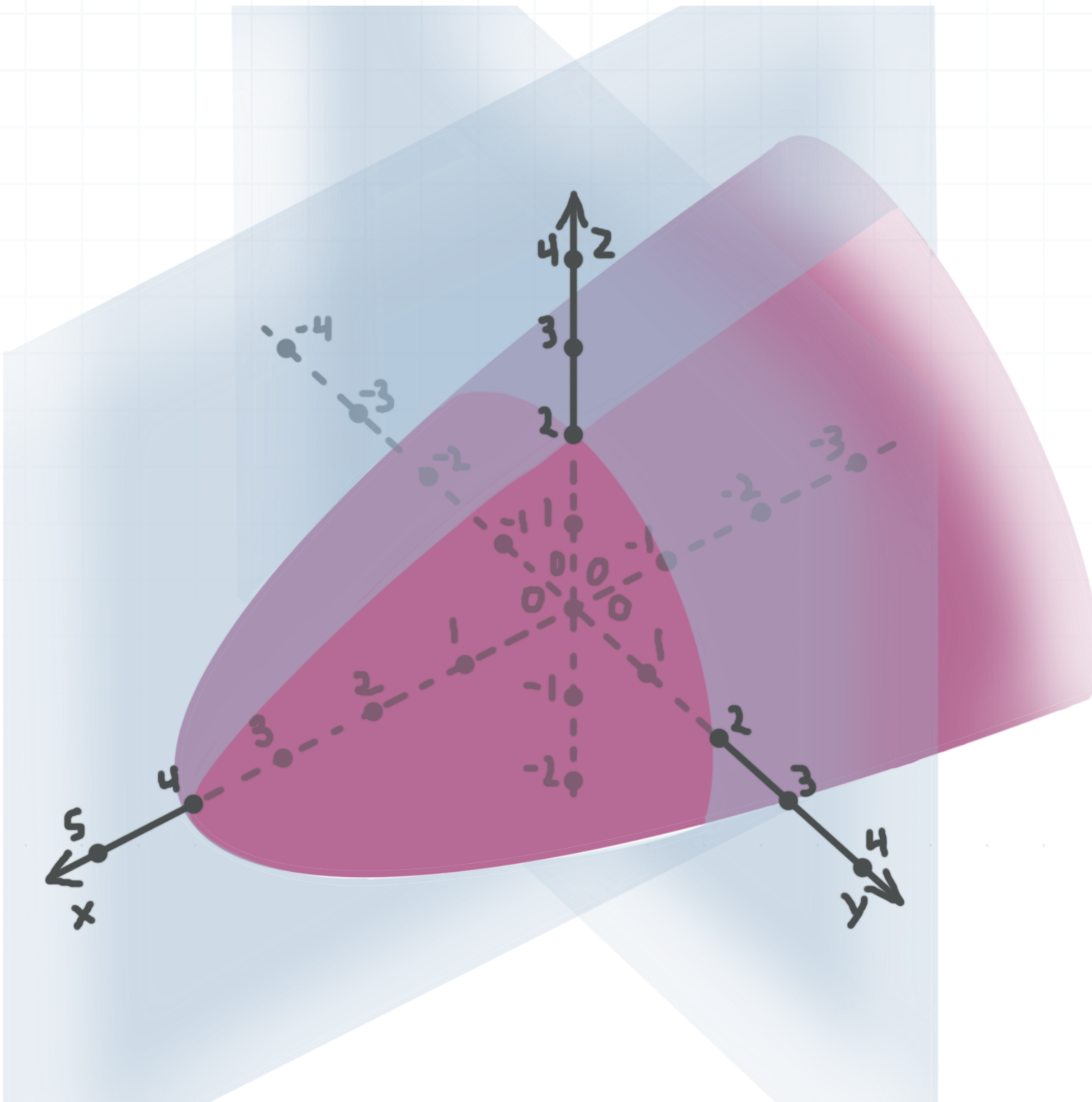
$$200\pi\lim_{t\to\infty}\left[\frac{t10^{-t}}{\ln 10}+\frac{10^{-t}}{\ln^2 10}\right]-200\pi\left[\frac{(0)10^{-0}}{\ln 10}+\frac{10^{-0}}{\ln^2 10}\right]$$

$$0-200\pi\cdot\frac{10^0}{\ln^2 10}$$

$$-\frac{200\pi}{\ln^2 10}$$

9. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the interior of the surface $y^2 + z^2 + x - 4 = 0$ that lies within the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$).

$$\iiint_E x + 8yz \ dV$$



Solution:

The surface is the circular paraboloid with vertex $(4,0,0)$ whose axis is the x-axis. The value of x changes from 0 to 4, y and z change within the circle C with center at the x-axis that lies in the plane parallel to the yz-plane. The value of θ changes from 0 to $\pi/2$, and since the upper bound is $r^2+x-4=0$, r changes from 0 to $\sqrt{4-x}$. The function is

$$x+8yz$$

$$x+8(r\cos\theta)(r\sin\theta)$$

$$x+8r^2\cos\theta\sin\theta$$

$$x+4r^2\sin 2\theta$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^4\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{4-x}}(x+4r^2\sin 2\theta)\cdot r\ dr\ d\theta\ dx$$

$$\int_0^4\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{4-x}}xr+4r^3\sin 2\theta\ dr\ d\theta\ dx$$

Integrate with respect to r.

$$\int_0^4\int_0^{\frac{\pi}{2}}\frac{1}{2}xr^2+r^4\sin 2\theta\Big|_{r=0}^{r=\sqrt{4-x}}\ d\theta\ dx$$

$$\int_0^4\int_0^{\frac{\pi}{2}}\frac{1}{2}x(\sqrt{4-x})^2+(\sqrt{4-x})^4\sin 2\theta-\left(\frac{1}{2}x(0)^2+0^4\sin 2\theta\right)\ d\theta\ dx$$

$$\int_0^4\int_0^{\frac{\pi}{2}}\frac{1}{2}x(4-x)+(4-x)^2\sin 2\theta\ d\theta\ dx$$

Integrate with respect to θ.

$$\int_0^4 \frac{1}{2}x(4-x)\theta - \frac{1}{2}(4-x)^2\cos 2\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \ dx$$

$$\int_0^4 \frac{\pi x(4-x)}{4} + (4-x)^2 \ dx$$

$$\int_0^4 (4-x)\left(\frac{\pi x}{4} + 4 - x\right) \ dx$$

$$\int_0^4 \pi x + 16 - 4x - \frac{\pi x^2}{4} - 4x + x^2 \ dx$$

$$\int_0^4 \pi x + 16 - 8x - \frac{\pi x^2}{4} + x^2 \ dx$$

Integrate with respect to x.

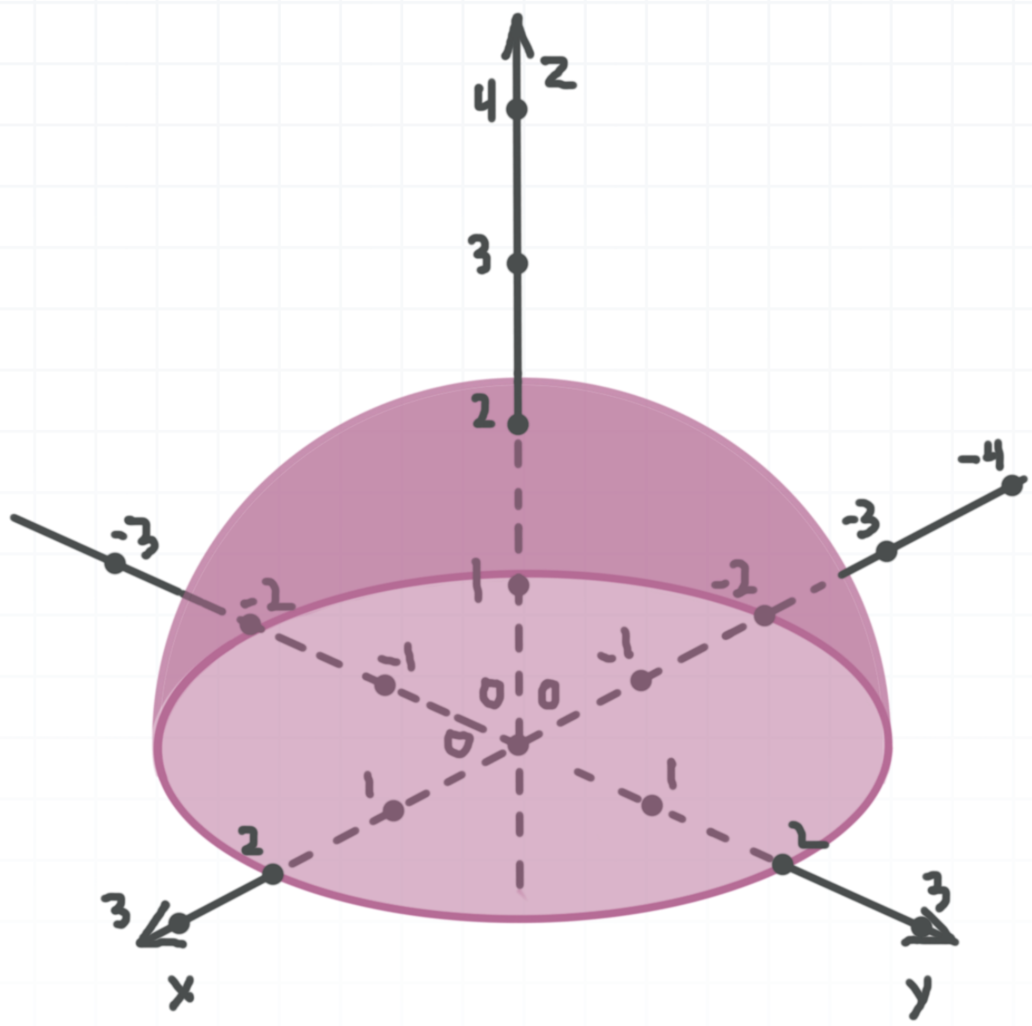
$$\frac{\pi x^2}{2} + 16x - 4x^2 - \frac{\pi x^3}{12} + \frac{1}{3}x^3 \Big|_0^4$$

$$\frac{\pi(4)^2}{2} + 16(4) - 4(4)^2 - \frac{\pi(4)^3}{12} + \frac{1}{3}(4)^3$$

$$\frac{8\pi + 64}{3}$$

■ 10. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the hemisphere with center at the origin, radius 2, and $z \geq 0$.

$$\iiint_E 4x^2 + 4y^2 - 12z^2 \ dV$$



Solution:

The equation of the sphere with center at the origin and radius 2 is

$$x^2 + y^2 + z^2 = 4$$

The value of z changes from 0 to 2, x and y change within the circle C with center at the z-axis, that lies in the plane parallel to the xy-plane. The value of θ changes from 0 to $\pi/2$, and since the upper bound is $r^2 + z^2 = 4$, r changes from 0 to $\sqrt{4 - z^2}$. The function is

$$4x^2 + 4y^2 - 12z^2 = 4r^2 - 12z^2$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^2 \int_0^{2\pi} \int_0^{\sqrt{4-z^2}} (4r^2 - 12z^2) \cdot r \ dr \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} \int_0^{\sqrt{4-z^2}} 4r^3 - 12z^2 r \ dr \ d\theta \ dz$$

Integrate with respect to r.

$$\int_0^2 \int_0^{2\pi} r^4 - 6z^2r^2 \Big|_{r=0}^{r=\sqrt{4-z^2}} \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} 16 - 8z^2 + z^4 - 6z^2(4 - z^2) \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} 16 - 8z^2 + z^4 - 24z^2 + 6z^4 \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} 7z^4 - 32z^2 + 16 \ d\theta \ dz$$

Integrate with respect to θ.

$$\int_0^2 7z^4\theta - 32z^2\theta + 16\theta \Big|_{\theta=0}^{\theta=2\pi} \ dz$$

$$\int_0^2 7z^4(2\pi) - 32z^2(2\pi) + 16(2\pi) - (7z^4(0) - 32z^2(0) + 16(0)) \ dz$$

$$\int_0^2 14\pi z^4 - 64\pi z^2 + 32\pi \ dz$$

Integrate with respect to θ.

$$\frac{14\pi}{5}z^5 - \frac{64\pi}{3}z^3 + 32\pi z\Big|_0^2$$

$$\frac{14\pi}{5}(2)^5 - \frac{64\pi}{3}(2)^3 + 32\pi(2) - \left(\frac{14\pi}{5}(0)^5 - \frac{64\pi}{3}(0)^3 + 32\pi(0)\right)$$

$$\frac{448\pi}{5} - \frac{512\pi}{3} + 64\pi$$

$$\frac{1{,}344\pi}{15} - \frac{2{,}560\pi}{15} + \frac{960\pi}{15}$$

$$-\frac{256\pi}{15}$$

FINDING VOLUME

■ 1. Evaluate the integral.

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} 4rz\ dz\ dr\ d\theta$$

Solution:

Integrate first with respect to z, then evaluate over the interval.

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} 4rz\ dz\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 2rz^2\Big|_{z=0}^{z=\sqrt{4-r^2}}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 2r(\sqrt{4-r^2})^2 - 2r(0)^2\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 2r(4-r^2)\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 8r-2r^3\ dr\ d\theta$$

Then integrate with respect to r, and evaluate over the interval.

$$\int_0^{2\pi} 4r^2 - \frac{1}{2}r^4 \Big|_{r=0}^{r=1} \ d\theta$$

$$\int_0^{2\pi} 4(1)^2 - \frac{1}{2}(1)^4 - \left(4(0)^2 - \frac{1}{2}(0)^4\right) \ d\theta$$

$$\int_0^{2\pi} \frac{7}{2} \ d\theta$$

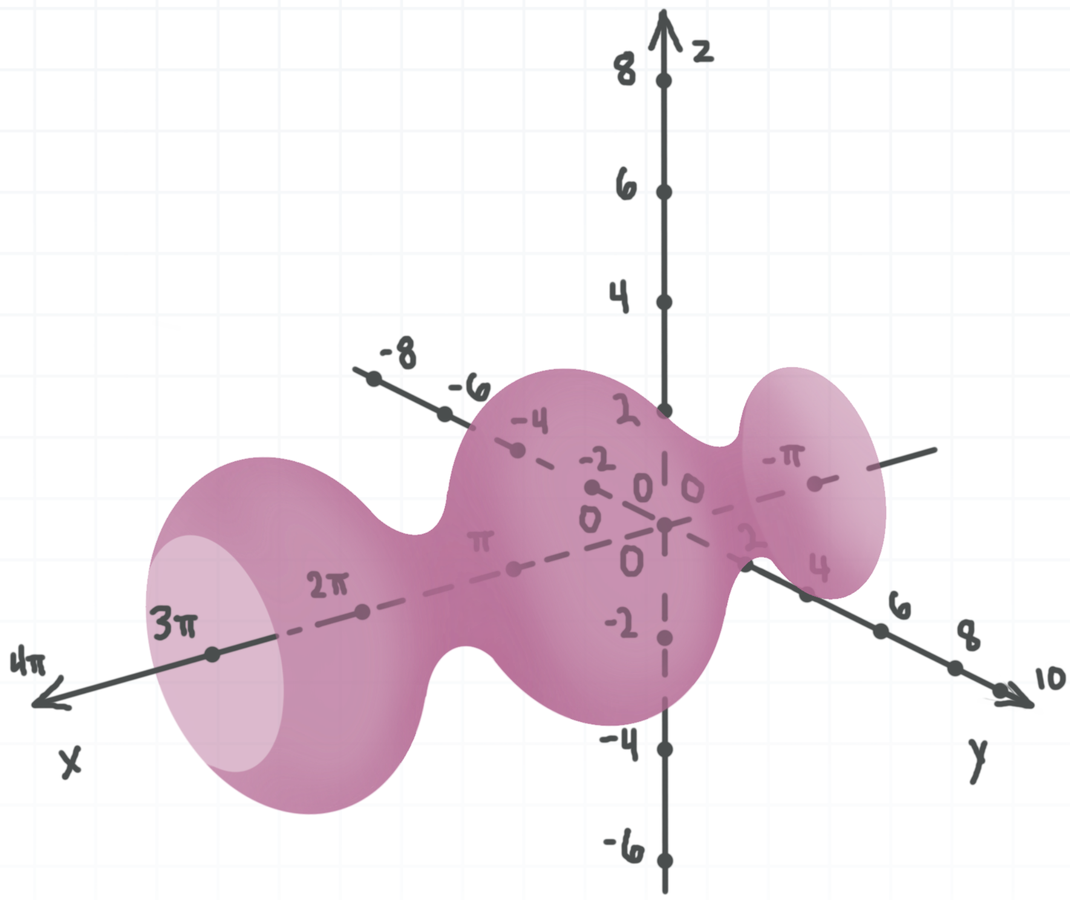
Then integrate with respect to θ, and evaluate over the interval.

$$\frac{7}{2}\theta \Big|_0^{2\pi}$$

$$\frac{7}{2}(2\pi) - \frac{7}{2}(0)$$

$$7\pi$$

■ 2. Use a triple integral in cylindrical coordinates to find the volume of the solid E, where E is the set of points within the surface of revolution created by rotating the curve $z = 2 + \sin x$ around the x-axis, and bounded by the planes $x = -\pi$ and $x = 3\pi$.



Solution:

The value of x changes from $-\pi$ to 3π, y and z change within the circle C with center at the x-axis, that lies in the plane parallel to the yz-plane. The value of θ changes from 0 to 2π, and since the upper bound is $z = 2 + \sin x$, r changes from 0 to $2 + \sin x$.

Then the integral in cylindrical coordinates is

$$\int_{-\pi}^{3\pi} \int_{0}^{2\pi} \int_{0}^{2+\sin x} r \ dr \ d\theta \ dx$$

Integrate with respect to r.

$$\int_{-\pi}^{3\pi} \int_{0}^{2\pi} \frac{1}{2} r^2 \Big|_{0}^{2+\sin x} \ d\theta \ dx$$

$$\int_{-\pi}^{3\pi}\int_{0}^{2\pi}\frac{1}{2}(2+\sin x)^2\ d\theta\ dx$$

$$\int_{-\pi}^{3\pi}\int_{0}^{2\pi}2+2\sin x+\frac{1}{2}\sin^2 x\ d\theta\ dx$$

Integrate with respect to θ.

$$\int_{-\pi}^{3\pi}2\theta+2\theta\sin x+\frac{1}{2}\theta\sin^2 x\Big|_{\theta=0}^{\theta=2\pi}\ dx$$

$$\int_{-\pi}^{3\pi}2(2\pi)+2(2\pi)\sin x+\frac{1}{2}(2\pi)\sin^2 x\ dx$$

$$\int_{-\pi}^{3\pi}4\pi+4\pi\sin x+\pi\sin^2 x\ dx$$

Integrate with respect to x.

$$4\pi x-4\pi\cos x+\frac{\pi\sin^3 x}{3\cos x}\Big|_{-\pi}^{3\pi}$$

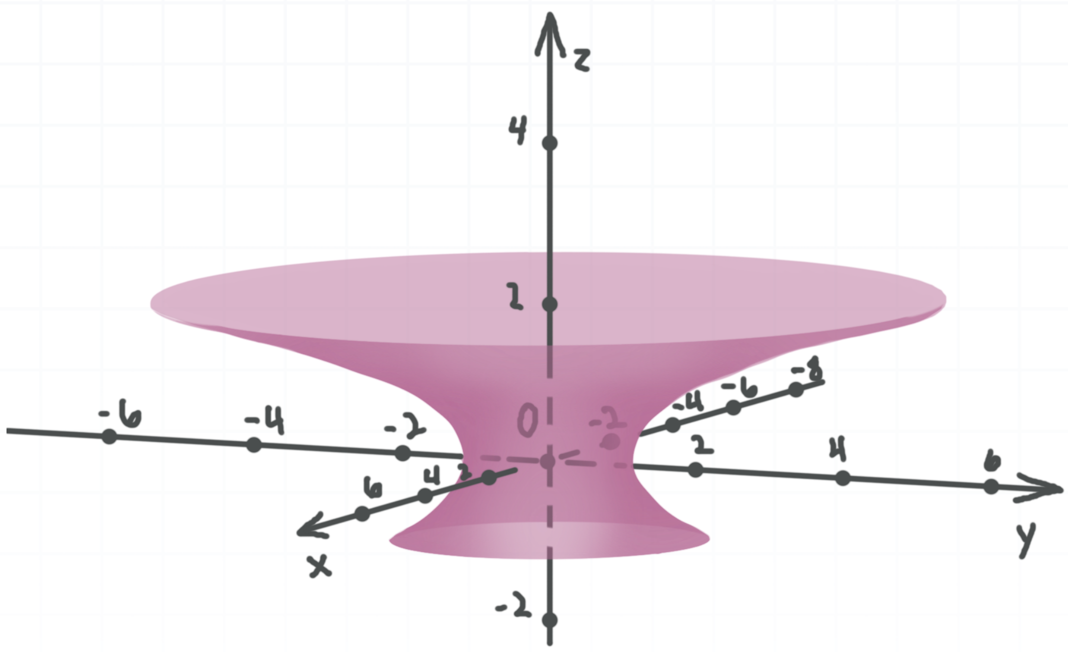
$$4\pi(3\pi)-4\pi\cos(3\pi)+\frac{\pi\sin^3(3\pi)}{3\cos(3\pi)}-\left(4\pi(-\pi)-4\pi\cos(-\pi)+\frac{\pi\sin^3(-\pi)}{3\cos(-\pi)}\right)$$

$$12\pi^2-4\pi(-1)+\frac{\pi(0)}{3(-1)}-\left(-4\pi^2-4\pi(-1)+\frac{\pi(0)}{3(-1)}\right)$$

$$12\pi^2+4\pi+4\pi^2-4\pi$$

$$16\pi^2$$

■ 3. Use a triple integral in cylindrical coordinates to find the volume of the solid E, where E is the set of points within the surface of revolution created by rotating the curve $x = z^2 + 1$ around the z-axis, and bounded by the planes $z = -1$ and $z = 2$.



Solution:

The value of z changes from -1 to 2, x and y change within the circle C with center at the z-axis, that lies in the plane parallel to the xy-plane. The value of θ changes from 0 to 2π, and since the upper bound is $x = z^2 + 1$, r changes from 0 to $z^2 + 1$. Therefore, the integral in cylindrical coordinates is

$$\int_{-1}^{2} \int_{0}^{2\pi} \int_{0}^{z^2+1} r \ dr \ d\theta \ dz$$

Integrate with respect to r.

$$\int_{-1}^{2}\int_{0}^{2\pi}\frac{1}{2}r^2\Big|_{0}^{z^2+1}\ d\theta\ dz$$

$$\int_{-1}^{2}\int_{0}^{2\pi}\frac{1}{2}(z^2+1)^2\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_{-1}^{2}\frac{1}{2}(z^2+1)^2\theta\Big|_{\theta=0}^{\theta=2\pi}\ dz$$

$$\int_{-1}^{2}\frac{1}{2}(z^2+1)^2(2\pi)\ dz$$

$$\int_{-1}^{2}\pi(z^2+1)^2\ dz$$

$$\int_{-1}^{2}\pi z^4+2\pi z^2+\pi\ dz$$

Integrate with respect to z.

$$\frac{1}{5}\pi z^5+\frac{2}{3}\pi z^3+\pi z\Big|_{-1}^{2}$$

$$\frac{1}{5}\pi(2)^5+\frac{2}{3}\pi(2)^3+\pi(2)-\left(\frac{1}{5}\pi(-1)^5+\frac{2}{3}\pi(-1)^3+\pi(-1)\right)$$

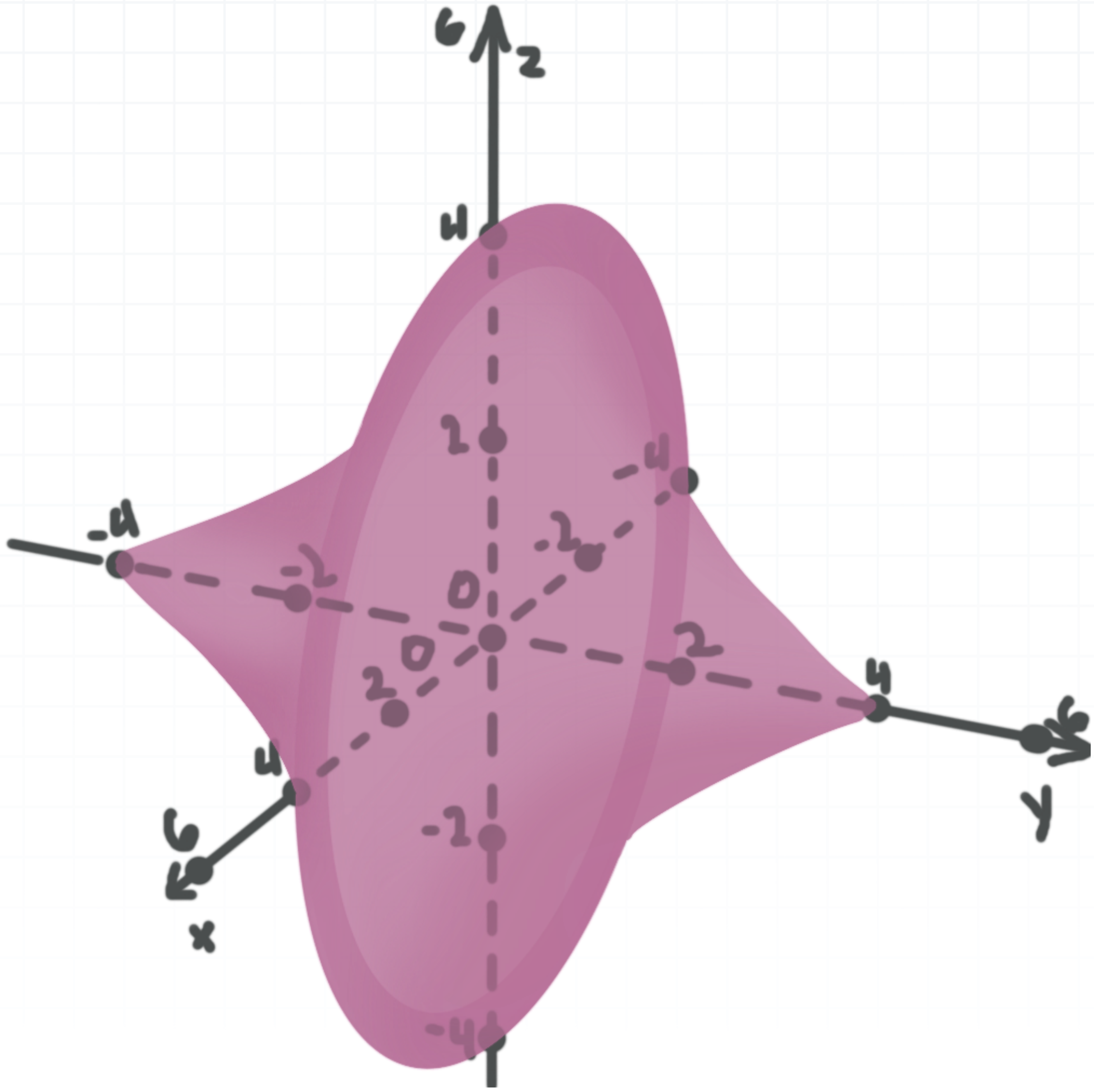
$$\frac{32}{5}\pi+\frac{16}{3}\pi+2\pi-\left(-\frac{1}{5}\pi-\frac{2}{3}\pi-\pi\right)$$

$$\frac{32}{5}\pi + \frac{16}{3}\pi + 2\pi + \frac{1}{5}\pi + \frac{2}{3}\pi + \pi$$

$$\frac{99}{15}\pi + \frac{90}{15}\pi + \frac{45}{15}\pi$$

$$\frac{78}{5}\pi$$

■ 4. Use a triple integral in cylindrical coordinates to find the volume of the solid E, where E is the set of the points within the surface of revolution created by rotating the curve $x = 4 - 2\sqrt{|y|}$ around the y-axis, and bounded by the planes $y = -4$ and $y = 4$.



Solution:

The value of y changes from -4 to 4, x and z change within the circle C with center at the y-axis, that lies in the plane parallel to the xz-plane. The value of θ changes from 0 to 2π, and since the upper bound is $x = 4 - 2\sqrt{|y|}$, r changes from 0 to $4 - 2\sqrt{|y|}$. Therefore, the integral in cylindrical coordinates is

$$\int_{-4}^{4}\int_{0}^{2\pi}\int_{0}^{4-2\sqrt{|y|}} r\ dr\ d\theta\ dy$$

Since the volumes are equal for y from -4 to 0, and from 0 to 4, the volume integral can be simplified to

$$\int_{0}^{4}\int_{0}^{2\pi}\int_{0}^{4-2\sqrt{y}} 2r\ dr\ d\theta\ dy$$

Integrate with respect to r.

$$\int_{0}^{4}\int_{0}^{2\pi} r^2\Big|_{0}^{4-2\sqrt{y}}\ d\theta\ dy$$

$$\int_{0}^{4}\int_{0}^{2\pi} (4-2\sqrt{y})^2\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_{0}^{4} (4-2\sqrt{y})^2\theta\Big|_{\theta=0}^{\theta=2\pi}\ dy$$

$$\int_{0}^{4} 2\pi(4-2\sqrt{y})^2\ dy$$

$$\int_{0}^{4} 2\pi(16-16\sqrt{y}+4y)\ dy$$

$$8\pi\int_{0}^{4} 4-4\sqrt{y}+y\ dy$$

Integrate with respect to y.

$$8\pi\left(4y-\frac{8}{3}y^{\frac{3}{2}}+\frac{1}{2}y^2\right)\Big|_0^4$$

$$8\pi\left(4(4)-\frac{8}{3}(4)^{\frac{3}{2}}+\frac{1}{2}(4)^2\right)$$

$$8\pi\left(16-\frac{64}{3}+8\right)$$

$$64\pi\left(\frac{9}{3}-\frac{8}{3}\right)$$

$$\frac{64\pi}{3}$$

W W W . K R I S T A K I N G M A T H . C O M