Jacobian for three variables

In the past we've converted multivariable functions defined in terms of cartesian coordinates x and y into functions defined in terms of polar coordinates r and θ.

Similarly, given a region defined in uvw-space, we can use a Jacobian transformation to redefine it in xyz-space, or vice versa.

Given three equations $x = f(u, v, w)$, $y = g(u, v, w)$, and $z = h(u, v, w)$, the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u} \begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - \frac{\partial x}{\partial v} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} + \frac{\partial x}{\partial w} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial u}\right) + \frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial u}\right)$$

Example

Find the Jacobian of the transformation.

$$x = uw^2$$

$$y = v^3 - 3w$$

$$z = \frac{2uv}{w}$$

Our functions tell us that we have a 3×3 set-up, so we'll use the formula

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial u}\right)$$

$$+\frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial u}\right)$$

We need to start by finding the partial derivatives of x, y and z with respect to u, v and w.

$$\frac{\partial x}{\partial u} = w^2$$

$$\frac{\partial x}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = 2uw$$

and

$$\frac{\partial y}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = 3v^2$$

$$\frac{\partial y}{\partial w} = -3$$

and

$$\frac{\partial z}{\partial u}=\frac{2v}{w}$$

$$\frac{\partial z}{\partial v}=\frac{2u}{w}$$

$$\frac{\partial z}{\partial w}=-\frac{2uv}{w^2}$$

We'll plug the partial derivatives into our formula and get

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=w^2\left[3v^2\left(-\frac{2uv}{w^2}\right)-(-3)\left(\frac{2u}{w}\right)\right]-0\left[0\left(-\frac{2uv}{w^2}\right)-(-3)\left(\frac{2v}{w}\right)\right]$$

$$+2uw\left[0\left(\frac{2u}{w}\right)-3v^2\left(\frac{2v}{w}\right)\right]$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=w^2\left(-\frac{6uv^3}{w^2}+\frac{6u}{w}\right)+2uw\left(-\frac{6v^3}{w}\right)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=-\frac{6uv^3w^2}{w^2}+\frac{6uw^2}{w}-\frac{12uv^3w}{w}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=-6uv^3+6uw-12uv^3$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=-18uv^3+6uw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=6uw-18uv^3$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 6u\left(w - 3v^3\right)$$

This is the Jacobian of the transformation.