Calculus 3
Final Exam Solutions

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MATH

Calculus 3 Final Exam Answer Key

1. (5 pts)	■	B	C	D	E
2. (5 pts)	A	B	■	D	E
3. (5 pts)	A	■	C	D	E
4. (5 pts)	A	■	C	D	E
5. (5 pts)	■	B	C	D	E
6. (5 pts)	A	B	C	D	■
7. (5 pts)	A	B	C	■	E
8. (5 pts)	A	B	C	D	■

9. (15 pts) $\frac{x+1}{-5}=\frac{y+3}{-5}=\frac{z}{-5}$

10. (15 pts) $15,\ 15,\ 15$

11. (15 pts) $V=512\pi$

12. (15 pts) $(\sqrt{2},\ln 2)$

Calculus 3 Final Exam Solutions

1. A. Find the partial derivatives of z with respect to x and y. If $z = 2x^2\cos(3y) - 4\ln x$, then

$$\frac{\partial z}{\partial x} = 4x\cos(3y) - \frac{4}{x}$$

and

$$\frac{\partial z}{\partial y} = 2x^2(-3\sin(3y))$$

$$\frac{\partial z}{\partial y} = -6x^2\sin(3y)$$

Plugging these values into the formula for the differential, we get

$$dz = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy$$

$$dz = \left(4x\cos(3y) - \frac{4}{x}\right)\ dx + (-6x^2\sin(3y))\ dy$$

$$dz = 4x\cos(3y)\ dx - \frac{4}{x}\ dx - 6x^2\sin(3y)\ dy$$

2. C. The problem tells us that $(a, b) = (0, -2)$, so we need to find $f(a, b) = f(0, -2)$.

$$f(x, y) = 4y^3 + x^2y^2$$

$$f(0, -2) = 4(-2)^3 + (0)^2(-2)^2$$

$$f(0, -2) = 4(-8) + 0(4)$$

$$f(0, -2) = -32$$

Then we need to find the partial derivatives of the function with respect to x and y.

$$\frac{\partial f}{\partial x} = 2xy^2$$

$$\frac{\partial f}{\partial x}(0, -2) = 2(0)(-2)^2$$

$$\frac{\partial f}{\partial x}(0, -2) = 0$$

and

$$\frac{\partial f}{\partial y} = 12y^2 + 2x^2y$$

$$\frac{\partial f}{\partial y}(0, -2) = 12(-2)^2 + 2(0)^2(-2)$$

$$\frac{\partial f}{\partial y}(0, -2) = 48$$

Plugging everything we've found into the linear approximation formula, we get

$$L(x, y) = -32 + 0(x - 0) + 48(y - (-2))$$

$$L(x, y) = -32 + 48(y + 2)$$

$$L(x, y) = -32 + 48y + 96$$

$$L(x, y) = 48y + 64$$

3. B. Take partial derivatives of $f(x, y) = 3x^2 - 2y^2 - ax + 2by$.

$$\frac{\partial f}{\partial x} = 6x - a$$

and

$$\frac{\partial f}{\partial y} = -4y + 2b$$

Setting each of these equal to 0 gives a system of equations.

$$6x - a = 0$$

$$-4y + 2b = 0$$

Since we already know the critical point is $(-3,2)$, we can plug $x = -3$ and $y = 2$ into the system to solve for a and b.

$$6(-3) - a = 0$$

$$-18 - a = 0$$

$$a = -18$$

and

$$-4(2) + 2b = 0$$

$$-8 + 2b = 0$$

$$2b = 8$$

$$b = 4$$

4. B. We haven't been given the order of integration as $dy\ dx$ or $dx\ dy$, so we can pick either order. We'll integrate first with respect to y, then with respect to x.

$$\iint_R 2e^{2x+y}\ dA$$

$$\int_0^{\ln 3} \int_1^{\ln 5} 2e^{2x+y}\ dy\ dx$$

Integrate with respect to y, treating x as a constant, then evaluate over the interval.

$$\int_0^{\ln 3} 2e^{2x+y} \Big|_{y=1}^{y=\ln 5}\ dx$$

$$\int_0^{\ln 3} 2e^{2x+\ln 5} - 2e^{2x+1}\ dx$$

$$\int_0^{\ln 3} 2e^{2x}e^{\ln 5} - 2e^{2x}e^{1}\ dx$$

$$\int_0^{\ln 3} 10e^{2x} - 2e \cdot e^{2x}\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$5e^{2x} - e \cdot e^{2x}\Big|_0^{\ln 3}$$

$$5e^{2(\ln 3)} - e \cdot e^{2(\ln 3)} - (5e^{2(0)} - e \cdot e^{2(0)})$$

$$5e^{\ln 3^2} - e \cdot e^{\ln 3^2} - (5 - e)$$

$$5 \cdot 9 - 9e - 5 + e$$

$$40 - 8e$$

5. A. When we convert from rectangular coordinates to polar coordinates, we use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$dA = dy\ dx = r\ dr\ d\theta$$

The region D is bounded by $y = \pm\sqrt{9 - x^2}$, which can be rewritten as

$$y = \pm\sqrt{9 - x^2}$$

$$y^2 = 9 - x^2$$

$$x^2 + y^2 = 9$$

So the region D is the circle centered at the origin with radius 3. We can evaluate it as a type I region, integrating first with respect to y and then with respect to x, where y is defined on $y = \pm\sqrt{9 - x^2}$, and x is defined on $[-3,3]$.

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \cos(3x^2 + 3y^2)\ dy\ dx$$

The limits of integration define the circle with radius 3, which means we can define r over $[0,3]$, and θ over $[0,2\pi]$. And if we use the conversion formulas above to convert the integrand, we get

$$\int_{0}^{2\pi} \int_{0}^{3} r\cos(3r^2)\ dr\ d\theta$$

6. E. Set the curves equal to one another.

$$1 - y = \sqrt{x^2 + (y-2)^2 - 16}$$

$$(1 - y)^2 = x^2 + (y-2)^2 - 16$$

$$1 - 2y + y^2 = x^2 + y^2 - 4y + 4 - 16$$

$$1 + 2y = x^2 - 12$$

$$2y = x^2 - 13$$

$$y = \frac{1}{2}x^2 - \frac{13}{2}$$

Setting $x = t$ gives

$$y = \frac{1}{2}t^2 - \frac{13}{2}$$

Then plugging this value for y into $z = 1 - y$ gives

$$z = 1 - \frac{1}{2}t^2 + \frac{13}{2}$$

$$z = -\frac{1}{2}t^2 + \frac{15}{2}$$

We set $x = t$ and then used that to find values for y and z in terms of t, so our vector function becomes

$$r(t) = t\mathbf{i} + \left(\frac{1}{2}t^2 - \frac{13}{2}\right)\mathbf{j} + \left(-\frac{1}{2}t^2 + \frac{15}{2}\right)\mathbf{k}$$

7. D. Find the derivative of the vector function.

$$r(t) = 6\sin t\mathbf{i} - 6\cos t\mathbf{j} - 8t\mathbf{k}$$

$$r'(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} - 8\mathbf{k}$$

Then find the magnitude of the derivative.

$$|r'(t)| = \sqrt{[r'(t)_1]^2 + [r'(t)_2]^2 + [r'(t)_3]^2}$$

$$|r'(t)| = \sqrt{(6\cos t)^2 + (6\sin t)^2 + (-8)^2}$$

$$|r'(t)| = \sqrt{36\cos^2 t + 36\sin^2 t + 64}$$

$$|r'(t)| = \sqrt{36(\cos^2 t + \sin^2 t) + 64}$$

$$|r'(t)| = \sqrt{36(1) + 64}$$

$$|r'(t)| = \sqrt{100}$$

$$|r'(t)| = 10$$

Solve for the unit tangent vector.

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$T(t) = \frac{6\cos t\mathbf{i} + 6\sin t\mathbf{j} - 8\mathbf{k}}{10}$$

$$T(t) = \frac{6\cos t}{10}\mathbf{i} + \frac{6\sin t}{10}\mathbf{j} - \frac{8}{10}\mathbf{k}$$

$$T(t) = \frac{3}{5}\cos t\mathbf{i} + \frac{3}{5}\sin t\mathbf{j} - \frac{4}{5}\mathbf{k}$$

This is the unit tangent vector. Now we need to find the unit normal vector. We'll take the derivative of the unit tangent vector.

$$T'(t) = -\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j} - 0\mathbf{k}$$

Find the magnitude of this derivative.

$$|T'(t)| = \sqrt{[T'(t)_1]^2 + [T'(t)_2]^2 + [T'(t)_3]^2}$$

$$|T'(t)| = \sqrt{\left(-\frac{3}{5}\sin t\right)^2 + \left(\frac{3}{5}\cos t\right)^2 + (0)^2}$$

$$|T'(t)| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}$$

$$|T'(t)| = \sqrt{\frac{9}{25}(\sin^2 t + \cos^2 t)}$$

$$|T'(t)| = \sqrt{\frac{9}{25}(1)}$$

$$|T'(t)| = \frac{3}{5}$$

Now we can use everything we just found to solve for the unit normal vector.

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$N(t) = \frac{-\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j} - 0\mathbf{k}}{\frac{3}{5}}$$

$$N(t) = \frac{-\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j}}{\frac{3}{5}}$$

$$N(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

Finally, we need to find the unit normal vector at $t = \pi$.

$$N(\pi) = -\sin \pi \mathbf{i} + \cos \pi \mathbf{j}$$

$$N(\pi) = 0\mathbf{i} - 1\mathbf{j}$$

$$N(\pi) = -\mathbf{j}$$

8. E. Find the first derivative of the position vector $r(t) = 2t^2\mathbf{i} - t^2\mathbf{j} + 3t\mathbf{k}$ in order to get velocity,

$$r'(t) = 4t\mathbf{i} - 2t\mathbf{j} + 3\mathbf{k}$$

and the second derivative in order to get acceleration.

$$r''(t) = 4\mathbf{i} - 2\mathbf{j} + 0\mathbf{k}$$

To find the normal component of acceleration, we'll find the cross product $r'(t) \times r''(t)$.

$$r'(t) \times r''(t) = \mathbf{i}[(-2t)(0) - (3)(-2)] - \mathbf{j}[(4t)(0) - (3)(4)]$$

$$+\mathbf{k}[(4t)(-2) - (-2t)(4)]$$

$$r'(t) \times r''(t) = \mathbf{i}(6) - \mathbf{j}(-12) + \mathbf{k}(0)$$

$$r'(t) \times r''(t) = 6\mathbf{i} + 12\mathbf{j} + 0\mathbf{k}$$

Find the magnitude of the cross product.

$$|\, r'(t) \times r''(t) \,| = \sqrt{6^2 + 12^2 + 0^2}$$

$$|r'(t) \times r''(t)| = \sqrt{36 + 144}$$

$$|r'(t) \times r''(t)| = \sqrt{180}$$

$$|r'(t) \times r''(t)| = 6\sqrt{5}$$

Find the magnitude of the first derivative, $r'(t) = 4t\mathbf{i} - 2t\mathbf{j} + 3\mathbf{k}$.

$$|r'(t)| = \sqrt{(r'(t)_1)^2 + (r'(t)_2)^2 + (r'(t)_3)^2}$$

$$|r'(t)| = \sqrt{(4t)^2 + (-2t)^2 + (3)^2}$$

$$|r'(t)| = \sqrt{16t^2 + 4t^2 + 9}$$

$$|r'(t)| = \sqrt{20t^2 + 9}$$

Find the normal component of the acceleration vector.

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$$a_N = \frac{6\sqrt{5}}{\sqrt{20t^2 + 9}}$$

Rationalize the denominator.

$$a_N = \frac{6\sqrt{5}}{\sqrt{20t^2 + 9}}\left(\frac{\sqrt{20t^2 + 9}}{\sqrt{20t^2 + 9}}\right)$$

$$a_N = \frac{6\sqrt{5}\sqrt{20t^2 + 9}}{20t^2 + 9}$$

$$a_N = \frac{6\sqrt{5(20t^2 + 9)}}{20t^2 + 9}$$

9. The normal vectors of $x - 2y + z = 5$ and $-2x - y + 3z = 5$ are $a\langle 1, -2, 1\rangle$ and $b\langle -2, -1, 3\rangle$, respectively, and the cross product of these normal vectors is

$$v = ((-2)(3) - (1)(-1))\mathbf{i} - ((1)(3) - (1)(-2))\mathbf{j} + ((1)(-1) - (-2)(-2))\mathbf{k}$$

$$v = (-6 + 1)\mathbf{i} - (3 + 2)\mathbf{j} + (-1 - 4)\mathbf{k}$$

$$v = -5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$$

If the planes are $x - 2y + z = 5$ and $-2x - y + 3z = 5$, then setting $z = 0$ gives

$$x - 2y = 5$$

$$-2x - y = 5$$

or

$$2x - 4y = 10$$

$$-2x - y = 5$$

We can add the equations together to get

$$(2x - 4y) + (-2x - y) = 10 + 5$$

$$2x - 2x - 4y - y = 10 + 5$$

$$-5y = 15$$

$$y = -3$$

Plugging $y = -3$ back into $x - 2y = 5$ gives

$$x - 2(-3) = 5$$

$$x + 6 = 5$$

$$x = -1$$

If we put all these values together, we can say that $c(-1,-3,0)$ is a point on the line of intersection. Now we'll put $v = -5\mathbf{i} - 5\mathbf{j} - 5\mathbf{k}$ and $c(-1,-3,0)$ into the formula for the symmetric equations for the line of intersection.

$$\frac{x - c_1}{v_1} = \frac{y - c_2}{v_2} = \frac{z - c_3}{v_3}$$

$$\frac{x - (-1)}{-5} = \frac{y - (-3)}{-5} = \frac{z - 0}{-5}$$

$$\frac{x + 1}{-5} = \frac{y + 3}{-5} = \frac{z}{-5}$$

10. We need to find three numbers that sum to 45, so we can write one equation that represents the numbers as

$$x + y + z = 45$$

We've been asked to maximize the product of the three numbers, and we can represent this in an equation as

$$P = xyz$$

Since we need to maximize this product equation, we need to get it in terms of just two variables. So we'll solve the sum equation for $z = 45 - x - y$ and then plug this value into the product equation.

$$P = xy(45 - x - y)$$

$$P = 45xy - x^2y - xy^2$$

To make things a little easier, we'll change this to

$$f(x, y) = 45xy - x^2y - xy^2$$

Now we'll find the first-order partial derivatives.

$$\frac{\partial f}{\partial x} = 45y - 2xy - y^2$$

$$\frac{\partial f}{\partial y} = 45x - x^2 - 2xy$$

Set both equations equal to 0, solving the first for y and the second for x. We get

$$45y - 2xy - y^2 = 0$$

$$y(45 - 2x - y) = 0$$

$y = 0$ or $45 - 2x - y = 0$

and

$$45x - x^2 - 2xy = 0$$

$$x(45 - x - 2y) = 0$$

$x = 0$ or $45 - x - 2y = 0$

Because we've been asked for positive numbers, we can't use $x = 0$ or $y = 0$. So we'll solve the other solutions as a system of equations.

$$45 - 2x - y = 0$$

$$45 - x - 2y = 0$$

Change them to

$$2x + y = 45$$

$$x + 2y = 45$$

and then to

$$4x + 2y = 90$$

$$x + 2y = 45$$

Subtract $x + 2y = 45$ from $4x + 2y = 90$.

$$4x + 2y - (x + 2y) = 90 - (45)$$

$$4x + 2y - x - 2y = 45$$

$$3x = 45$$

$$x = 15$$

Plugging $x = 15$ back into $45 - 2x - y = 0$ to solve for y gives

$$45 - 2(15) - y = 0$$

$$45 - 30 - y = 0$$

$$15 - y = 0$$

$$y = 15$$

This gives us the point $(15,15)$ as our critical point. We need to test it to make sure that it gives a maximum, so we'll find the second-order partial derivatives of $f(x, y)$.

$$\frac{\partial^2 f}{\partial x^2} = -2y$$

$$\frac{\partial^2 f}{\partial y^2} = -2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = 45 - 2x - 2y$$

Then we'll plug these into the formula for D.

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (-2y)(-2x) - (45 - 2x - 2y)^2$$

$$D(x, y) = 4xy - (45 - 2x - 2y)^2$$

Now we'll evaluate D at the critical point $(15,15)$.

$$D(15,15) = 4(15)(15) - (45 - 2(15) - 2(15))^2$$

$$D(15,15) = 900 - (45 - 30 - 30)^2$$

$$D(15,15) = 900 - (-15)^2$$

$$D(15,15) = 900 - 225$$

$$D(15,15) = 675$$

Because $D > 0$ and

$$\frac{\partial^2 f}{\partial x^2}(15,15) = -2(15) = -30 < 0$$

the critical point is a local maximum. Now we'll just plug the critical point into the sum equation $x + y + z = 45$ to find the associated z -value.

$$15 + 15 + z = 45$$

$$z = 15$$

Therefore, 15, 15, and 15 are the three positive numbers that sum to 45 and have the maximum possible product.

11. We need to determine the bounds for ρ, θ, and ϕ, given that we're integrating over the upper half of $x^2 + y^2 + z^2 = 16$. Because ρ

represents radius in spherical coordinates, we can say that the bounds for ρ are

$$0 \le \rho \le 4$$

The half sphere is also defined for $0 \le \theta \le 2\pi$, but ϕ will reflect the fact that we just want the half sphere,

$$0 \le \phi \le \frac{\pi}{2}$$

The given integrand $f(x, y, z) = 8z$ can be converted to spherical coordinates using the conversion formula $z = \rho \cos\phi$, and it becomes $f(\rho, \theta, \phi) = 8\rho\cos\phi$. We also know $dV = \rho^2 \sin\phi$.

Plugging everything into the integral gives

$$V = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^4 8\rho^3 \sin\phi \cos\phi \ d\rho \ d\theta \ d\phi$$

$$V = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^4 4\rho^3 \sin(2\phi) \ d\rho \ d\theta \ d\phi$$

Integrate with respect to ρ.

$$V = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \rho^4 \sin(2\phi)\Big|_{\rho=0}^{\rho=4} \ d\theta \ d\phi$$

$$V = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} (4)^4\sin(2\phi) - (0)^4\sin(2\phi) \ d\theta \ d\phi$$

$$V = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 256 \sin(2\phi) \ d\theta \ d\phi$$

Integrate with respect to θ.

$$V = \int_0^{\frac{\pi}{2}} 256\theta \sin(2\phi) \Big|_{\theta=0}^{\theta=2\pi} \ d\phi$$

$$V = \int_0^{\frac{\pi}{2}} 256(2\pi)\sin(2\phi) - 256(0)\sin(2\phi) \ d\phi$$

$$V = \int_0^{\frac{\pi}{2}} 512\pi \sin(2\phi) \ d\phi$$

Integrate with respect to ϕ.

$$V = -256\pi \cos(2\phi) \Big|_0^{\frac{\pi}{2}}$$

$$V = -256\pi \cos \pi + 256\pi \cos(0)$$

$$V = -256\pi(-1) + 256\pi(1)$$

$$V = 256\pi + 256\pi$$

$$V = 512\pi$$

12. Start by finding the function's first and second derivatives.

$$f(x) = 2 \ln x$$

$$f'(x) = \frac{2}{x}$$

$$f''(x) = -\frac{2}{x^2}$$

Take the absolute value of the second derivative.

$$|f''(x)| = \frac{2}{x^2}$$

Find curvature.

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

$$\kappa(x) = \frac{\frac{2}{x^2}}{\left[1 + \left(\frac{2}{x}\right)^2\right]^{\frac{3}{2}}}$$

$$\kappa(x) = \frac{\frac{2}{x^2}}{\left(1 + \frac{4}{x^2}\right)^{\frac{3}{2}}}$$

Use quotient rule to find the derivative of curvature.

$$\kappa'(x) = \frac{-\frac{4}{x^3}\left(1 + \frac{4}{x^2}\right)^{\frac{3}{2}} - \frac{2}{x^2}\left[\frac{3}{2}\left(1 + \frac{4}{x^2}\right)^{\frac{1}{2}}\left(-\frac{8}{x^3}\right)\right]}{\left[\left(1 + \frac{4}{x^2}\right)^{\frac{3}{2}}\right]^2}$$

$$\kappa'(x)=\frac{-\frac{4}{x^3}\left(1+\frac{4}{x^2}\right)^{\frac{3}{2}}+\frac{24}{x^5}\left(1+\frac{4}{x^2}\right)^{\frac{1}{2}}}{\left(1+\frac{4}{x^2}\right)^{3}}$$

Cancel a common factor of $(1+(4/x^2))^{1/2}$.

$$\kappa'(x)=\frac{-\frac{4}{x^3}\left(1+\frac{4}{x^2}\right)+\frac{24}{x^5}}{\left(1+\frac{4}{x^2}\right)^{\frac{5}{2}}}$$

$$\kappa'(x)=\frac{-\frac{4}{x^3}-\frac{16}{x^5}+\frac{24}{x^5}}{\left(1+\frac{4}{x^2}\right)^{\frac{5}{2}}}$$

$$\kappa'(x)=\frac{\frac{8}{x^5}-\frac{4}{x^3}}{\left(1+\frac{4}{x^2}\right)^{\frac{5}{2}}}$$

$$\kappa'(x)=\frac{\frac{4}{x^3}\left(\frac{2}{x^2}-1\right)}{\left(1+\frac{4}{x^2}\right)^{\frac{5}{2}}}$$

To find critical points, we'll set the derivative equal to 0 and solve for x. The only way the derivative will be 0 is if the numerator is 0.

$$\frac{\frac{4}{x^3}\left(\frac{2}{x^2}-1\right)}{\left(1+\frac{4}{x^2}\right)^{\frac{5}{2}}}=0$$

$$\frac{4}{x^3}\left(\frac{2}{x^2}-1\right)=0$$

$$\frac{4}{x^3}=0 \text{ or } \frac{2}{x^2}-1=0$$

There's no solution to the first equation, so we'll just solve the second one.

$$\frac{2}{x^2}=1$$

$$x^2=2$$

$$x=\pm\sqrt{2}$$

We discard the negative option, since it's not in the function's domain, then we'll plug the positive option back into the original function $f(x)=2\ln x$ to find the associated y-value.

$$y=2\ln x$$

$$y=2\ln\sqrt{2}$$

$$y=2\ln 2^{\frac{1}{2}}$$

$$y=2\left(\frac{1}{2}\right)\ln 2$$

$$y=\ln 2$$

Therefore, the function has maximum curvature at

$$(\sqrt{2},\ln 2)$$

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