Volume of the parallelepiped from adjacent edges

If we need to find the volume of a parallelepiped and we're given three adjacent edges of it, all we have to do is find the scalar triple product of the three vectors that define the edges:

$$\left|\overrightarrow{PS}\cdot\left(\overrightarrow{PQ}\times\overrightarrow{PR}\right)\right|$$

where $\overrightarrow{PS}$, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are the three adjacent edges.

First we'll find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$, then we'll find the cross product $\overrightarrow{PQ}\times\overrightarrow{PR}$ using the 3×3 matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$= \left(PQ_2PR_3 - PQ_3PR_2\right)\mathbf{i} - \left(PQ_1PR_3 - PQ_3PR_1\right)\mathbf{j} + \left(PQ_1PR_2 - PQ_2PR_1\right)\mathbf{k}$$

We'll convert the result of the cross product into standard vector form, and then take the dot product of $\overrightarrow{PS}\langle PS_1, PS_2, PS_3\rangle$ and the vector result of $\overrightarrow{PQ}\times\overrightarrow{PR}$. The final answer is the value of the scalar triple product, which is the volume of the parallelepiped.

Example

Find the volume of the parallelepiped given by the adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$.

$$P(5,1,-2)$$

$$Q(0,-1,3)$$

$$R(3,2,-4)$$

$$S(1,-2,0)$$

We need to start by using the four points to find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$, since these are the three adjacent edges of the parallelepiped.

$$\overrightarrow{PQ}=\langle 0-5,-1-1,3-(-2)\rangle$$

$$\overrightarrow{PQ}=\langle -5,-2,5\rangle$$

and

$$\overrightarrow{PR}=\langle 3-5,2-1,-4-(-2)\rangle$$

$$\overrightarrow{PR}=\langle -2,1,-2\rangle$$

and

$$\overrightarrow{PS}=\langle 1-5,-2-1,0-(-2)\rangle$$

$$\overrightarrow{PS}=\langle -4,-3,2\rangle$$

Now we need to take the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -5 & -2 & 5\\ -2 & 1 & -2\end{vmatrix}$$

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\mathbf{i}\begin{vmatrix}-2 & 5\\ 1 & -2\end{vmatrix}-\mathbf{j}\begin{vmatrix}-5 & 5\\ -2 & -2\end{vmatrix}+\mathbf{k}\begin{vmatrix}-5 & -2\\ -2 & 1\end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-2)(-2) - (5)(1)\right] \mathbf{i} - \left[(-5)(-2) - (5)(-2)\right] \mathbf{j} + \left[(-5)(1) - (-2)(-2)\right] \mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4-5)\mathbf{i} - (10+10)\mathbf{j} + (-5-4)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -\mathbf{i} - 20\mathbf{j} - 9\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, -20, -9 \rangle$$

Taking the dot product of $\overrightarrow{PS} = \langle -4, -3, 2 \rangle$ and $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, -20, -9 \rangle$, we get

$$\left| \overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR} \right) \right| = (-4)(-1) + (-3)(-20) + (2)(-9)$$

$$\left| \overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR} \right) \right| = 4 + 60 - 18$$

$$\left| \overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR} \right) \right| = 46$$

The volume of the parallelepiped is 46.