Finding volume

We can use triple integrals and spherical coordinates to solve for the volume of a solid sphere. The volume formula in rectangular coordinates is

$$V = \iiint_B f(x, y, z)\ dV$$

where B represents the solid sphere and dV can be defined in spherical coordinates as

$$dV = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

To convert in general from rectangular to spherical coordinates, we can use the formulas

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

and

$$\rho^2 = x^2 + y^2 + z^2$$

Remember, rectangular coordinates are given as (x, y, z), and spherical coordinates are given as (ρ, θ, ϕ).

In order to find limits of integration for the triple integral, we'll say that ϕ is defined on the interval $[0,\pi]$ and that θ is defined on the interval $[0,2\pi]$. Then we only have to find an interval for ρ.

Example

Use spherical coordinates to find the volume of the triple integral, where B is a sphere with center $(0,0,0)$ and radius 4.

$$\iiint_B x^2+y^2+z^2\ dV$$

Using the conversion formula $\rho^2=x^2+y^2+z^2$, we can change the given function into spherical notation.

$$\iiint_B x^2+y^2+z^2\ dV=\iiint_B \rho^2\ dV$$

Then we'll use $dV=\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$ to make a substitution for dV.

$$\iiint_B \rho^2\left(\rho^2\sin\phi\ d\rho\ d\theta\ d\phi\right)$$

$$\iiint_B \rho^4\sin\phi\ d\rho\ d\theta\ d\phi$$

Now we'll find limits of integration. We already know the limits of integration for ϕ and θ, since they are always the same if we're dealing with a full sphere, so we get

$$\int_0^{\pi}\int_0^{2\pi}\int \rho^4\sin\phi\ d\rho\ d\theta\ d\phi$$

Since ρ defines the radius of the sphere, and we're told that this sphere has its center at $(0,0,0)$ and radius 4, ρ is defined on $[0,4]$, so

$$\int_0^{\pi}\int_0^{2\pi}\int_0^4 \rho^4 \sin\phi \ d\rho \ d\theta \ d\phi$$

We always integrate inside out, so we'll integrate with respect to ρ first, treating all other variables as constants.

$$V=\int_0^{\pi}\int_0^{2\pi} \frac{1}{5}\rho^5 \sin\phi \Big|_{\rho=0}^{\rho=4} \ d\theta \ d\phi$$

$$V=\int_0^{\pi}\int_0^{2\pi} \frac{1}{5}(4)^5\sin\phi-\frac{1}{5}(0)^5\sin\phi \ d\theta \ d\phi$$

$$V=\int_0^{\pi}\int_0^{2\pi} \frac{1{,}024}{5}\sin\phi \ d\theta \ d\phi$$

Now we'll integrate with respect to θ, treating all other variables as constants.

$$V=\frac{1{,}024}{5}\int_0^{\pi} \theta\sin\phi \Big|_{\theta=0}^{\theta=2\pi} \ d\phi$$

$$V=\frac{1{,}024}{5}\int_0^{\pi} 2\pi\sin\phi-0\sin\phi \ d\phi$$

$$V=\frac{2{,}048\pi}{5}\int_0^{\pi} \sin\phi \ d\phi$$

Finally, we'll integrate with respect to ϕ.

$$V=\frac{2{,}048\pi}{5}\left(-\cos\phi\right)\Big|_0^{\pi}$$

$$V=-\frac{2{,}048\pi\cos\phi}{5}\Bigg|_0^{\pi}$$

$$V=-\frac{2{,}048\pi}{5}\cos\pi-\left[-\frac{2{,}048\pi}{5}\cos 0\right]$$

$$V=-\frac{2{,}048\pi}{5}(-1)+\frac{2{,}048\pi}{5}(1)$$

$$V=\frac{2{,}048\pi}{5}+\frac{2{,}048\pi}{5}$$

$$V=\frac{4{,}096\pi}{5}$$

This is the volume of the sphere.