Topic: Global extrema

Question: Find the difference between the global maximum and minimum, if $f(x, y)$ is defined on a closed isosceles triangle OMN bounded by $x = 0$, $y = 3$, and $y = x$.

$$f(x, y) = x^2 + y^2 - 4xy + 5$$

Answer choices:

A 11

B 27

C 7

D 5

Solution: B

We need to check for critical points in the interior of the triangle, and along each edge of the triangle, including at its vertices.

To find critical points in the triangle's interior, take the first-order partial derivatives of the function.

$$f(x, y) = x^2 + y^2 - 4xy + 5$$

$$f_x(x, y) = 2x - 4y$$

$$f_y(x, y) = 2y - 4x$$

Solve this as a system of equations.

$$4x - 8y + (-4x + 2y) = 0 + (0)$$

$$-8y + 2y = 0$$

$$-6y = 0$$

$$y = 0$$

Then

$$2x - 4y = 0$$

$$2x - 4(0) = 0$$

$$2x = 0$$

$$x = 0$$

This critical point $(0,0)$ lies at one vertex of the triangle, not in the interior of the triangle. We'll set aside that critical point, and check along each side of the triangle.

On the line segment MN, where $y=3$, the x-value varies, but the y-value remains constant at $y=3$.

$$f(x,3)=x^2+3^2-4x(3)+5$$

$$f(x,3)=x^2-12x+14$$

This equation models the function along the boundary $y=3$. Take the partial derivative with respect to x, since x is the value that varies, to find critical points along that boundary.

$$f_x(x,3)=2x-12$$

$$2x-12=0$$

$$x=6$$

This gives the point $(6,3)$, but that lies outside the line segment, since the segment MN is only defined on $0\le x\le 3$, so we can ignore $(6,3)$.

On the line segment OM, where $x=0$, the y-value varies, but the x-value remains constant at $x=0$.

$$f(0,y)=0^2+y^2-4(0)y+5$$

$$f(0,y)=y^2+5$$

This equation models the function along the boundary $x = 0$. Take the partial derivative with respect to y, since y is the value that varies, to find critical points along that boundary.

$$f_y(0,y) = 2y$$

$$2y = 0$$

$$y = 0$$

This gives the point $(0,0)$, which is the point we already found at one vertex of the triangle. So we'll check the last line segment, ON. On the line segment ON, where $y = x$, the x- and y-values both vary, but we can substitute x for y since $y = x$.

$$f(x, x) = x^2 + x^2 - 4x(x) + 5$$

$$f(x, x) = -2x^2 + 5$$

This equation models the function along the boundary $y = x$. Take the partial derivative with respect to x, since that's the variable that remains in the equation, to find critical points along that boundary.

$$f_x(x, x) = -4x$$

$$-4x = 0$$

$$x = 0$$

This gives the point $(0,0)$, which is the point we already found at one vertex of the triangle.

We've looked at the interior of the triangle, and all three of its sides, and only identified the critical point $(0,0)$. The only thing left to check is the value at $(0,0)$, and the other two vertices of the triangle, $(0,3)$ and $(3,3)$.

$$f(0,0) = 0^2 + 0^2 - 4(0)(0) + 5 = 5$$

$$f(0,3) = 0^2 + 3^2 - 4(0)(3) + 5 = 14$$

$$f(3,3) = 3^2 + 3^2 - 4(3)(3) + 5 = -13$$

So the global maximum is at 14, and the global minimum is at -13. Their difference is $14 - (-13) = 14 + 13 = 27$.

Topic: Global extrema

Question: At which boundaries of the square defined on $0 \le x \le 1$ and $0 \le y \le 1$ will the global minimum and global maximum of $f(x, y)$ be -5 and $34/5$?

$$f(x, y) = 6x^2 - 5y^2 + 4xy$$

Answer choices:

A On the unit square $-2 \le x \le 1$ and $-1 \le y \le 1$

B On the unit square $0 \le x \le 1$ and $-1 \le y \le 2$

C On the unit square $0 \le x \le 1$ and $0 \le y \le 1$

D On the unit square $-2 < x < 0$ and $0 < y < 2$

Solution: C

Find $f_x(x,y)$ and $f_y(x,y)$.

$$f_x(x,y) = 12x + 4y$$

$$f_y(x,y) = -10y + 4x$$

Solve the system of equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$.

$$12x + 4y = 0$$

$$-10y + 4x = 0$$

Solving this system of equations, we get $x = 0$ and $y = 0$. When we plug this point into the original function, we get $f(0,0) = 0$.

On the right vertical side of the square, $x = 1$ and $0 \le y \le 1$. Then

$$f(1,y) = -5y^2 + 4y + 6$$

$$f_y(1,y) = -10y + 4$$

This results in a critical number $y = 2/5$.

Now calculate $f(1,y) = -5y^2 + 4y + 6$ for $y = 0$, $y = 2/5$, and $y = 1$.

$$f(1,0) = -5(0)^2 + 4(0) + 6 = 6$$

$$f\left(1,\frac{2}{5}\right) = -5\left(\frac{2}{5}\right)^2 + 4\left(\frac{2}{5}\right) + 6 = \frac{34}{5}$$

$$f(1,1) = -5(1)^2 + 4(1) + 6 = 5$$

On the left vertical boundary side, $x = 0$ and $0 \leq y \leq 1$. Therefore

$$f(0,y) = -5y^2$$

At the point $(0,1)$, the maximum is 0 and the minimum is -5.

On the lower horizontal side of the square, $y = 0$ and $0 \leq x \leq 1$, which implies

$$f(x,0) = 6x^2$$

Thus, the maximum at $x = 1$ and $y = 0$ is 6, and the minimum at $x = 0$ and $y = 0$ is 0.

On the upper horizontal boundary side, $y = 1$ and $0 \leq x \leq 1$. Then

$$f(x,1) = 6x^2 + 4x - 5$$

$$f_x(x,1) = 12x + 4$$

This gives a critical point $x = -1/3$. This point is not within the set $0 \leq x \leq 1$. As a result we must evaluate only $f(x,1)$ at $x = 0$ and $x = 1$:

$$f(0,1) = -5$$

$$f(1,1) = 5$$

Thus the global minimum is -5 at $(0,1)$, and the global maximum is $34/5$ at $(1,2/5)$.

Topic: Global extrema

Question: What is the global maximum of $f(x, y)$ in the region defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$?

$$f(x, y) = 2x^2 + 2y^2 - 4x^2y$$

Answer choices:

A 2

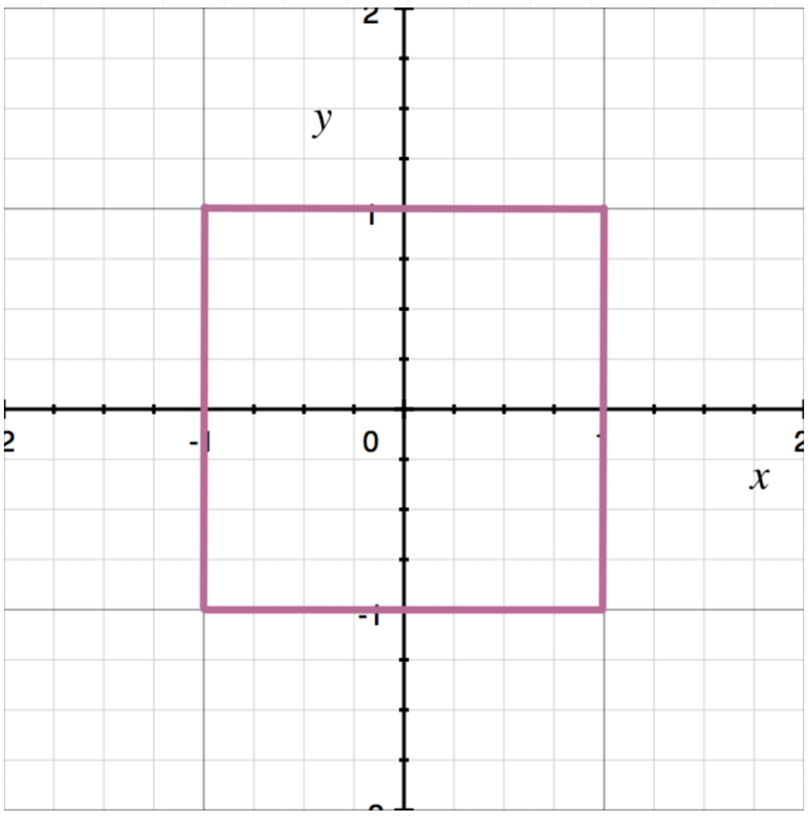
B 6

C 8

D 10

Solution: C

The region is defined over $x = [-1,1]$ and $y = [-1,1]$, which means the region looks like this:



We'll start with the critical points of the function, which we'll find by taking first order partial derivatives of the function, setting them equal to 0, and solving the resulting system of equations. Since $f(x,y) = 2x^2 + 2y^2 - 4x^2y$, we get

$$\frac{\partial f}{\partial x} = 4x - 8xy$$

$$\frac{\partial f}{\partial y} = 4y - 4x^2$$

So we'll solve the system of equations

$$4x - 8xy = 0$$

$$4y - 4x^2 = 0$$

Solving $4y - 4x^2 = 0$ for y gives

$$4y = 4x^2$$

$$y = x^2$$

Plug this value back into $4x - 8xy = 0$.

$$4x - 8xy = 0$$

$$4x - 8x(x^2) = 0$$

$$4x - 8x^3 = 0$$

$$4x(1 - 2x^2) = 0$$

The two solutions are

$$4x = 0$$

$$x = 0$$

and

$$1 - 2x^2 = 0$$

$$1 = 2x^2$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{\sqrt{2}}{2}$$

Given these three x-values, we need to find the corresponding y-values by plugging into $y=x^2$. We get

For $x=0$, $\qquad y=x^2=0^2=0$

For $x=\frac{\sqrt{2}}{2}$, $\qquad y=x^2=\left(\frac{\sqrt{2}}{2}\right)^2=\frac{2}{4}=\frac{1}{2}$

For $x=-\frac{\sqrt{2}}{2}$, $\qquad y=x^2=\left(-\frac{\sqrt{2}}{2}\right)^2=\frac{2}{4}=\frac{1}{2}$

The critical points in the interior of the region are therefore

$(0,0)$, $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$, and $\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)$

Let's go ahead and find the function's value at these three points before we move on to work on the boundaries of the region.

For $x=0$,

$$f(0,0)=2(0)^2+2(0)^2-4(0)^2(0)$$

$$f(0,0)=0$$

For $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$,

$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)=2\left(\frac{\sqrt{2}}{2}\right)^2+2\left(\frac{1}{2}\right)^2-4\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{1}{2}\right)$$

$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)=2\left(\frac{2}{4}\right)+2\left(\frac{1}{4}\right)-4\left(\frac{2}{4}\right)\left(\frac{1}{2}\right)$$

$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)=1+\frac{1}{2}-1$$

$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)=\frac{1}{2}$$

For $\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)$,

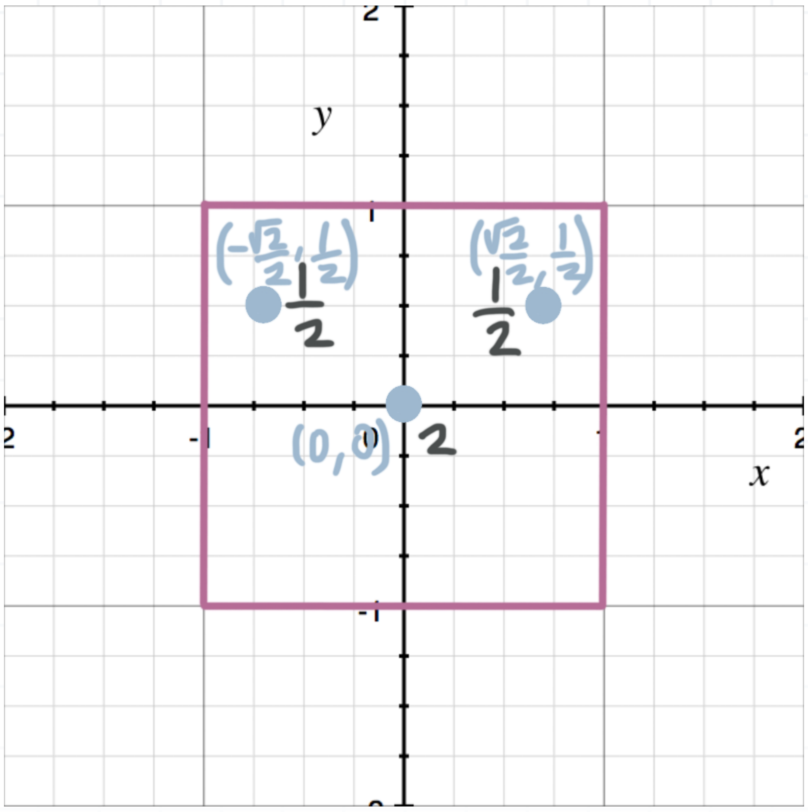
$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=2\left(-\frac{\sqrt{2}}{2}\right)^2+2\left(\frac{1}{2}\right)^2-4\left(-\frac{\sqrt{2}}{2}\right)^2\left(\frac{1}{2}\right)$$

$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=2\left(\frac{2}{4}\right)+2\left(\frac{1}{4}\right)-4\left(\frac{2}{4}\right)\left(\frac{1}{2}\right)$$

$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=1+\frac{1}{2}-1$$

$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=\frac{1}{2}$$

Let's plot these values into our diagram.



Extrema can only occur at critical points and at the edges of the region. We already now know that the value of the function at each critical point, so our next step is to check the value of the function everywhere along the edges of the region. Comparing the value of the function everywhere along the edges to the value at each critical point, we can say that the largest value we find is the global maximum and that the smallest value we find is the global minimum.

Let's start by looking at the right edge of the region, which corresponds to the line $x=1$, where $-1 \le y \le 1$. Since $x=1$ everywhere along this edge, the function that defines the edge is

$$g(y)=f(1,y)=2(1)^2+2y^2-4(1)^2y$$

$$g(y)=f(1,y)=2+2y^2-4y$$

$$g(y)=f(1,y)=2y^2-4y+2$$

To figure out the value of the function along the edge, we'll look for critical points of the function that defines the edge, which means we need to take the derivative of the new function.

$$g'(y) = f'(1,y) = 4y - 4$$

Now we'll set the derivative equal to 0 and solve for y.

$$4y - 4 = 0$$

$$4y = 4$$

$$y = 1$$

Putting this together with the value of x along the right edge, the only critical point along the right edge is at $(1,1)$. That's right at the corner of the region, which means that corner is either the lowest point or the highest point along the right edge $x = 1$. So we need to plug that critical point $(1,1)$, and the other corner point along that edge $(1, -1)$ into the original function to compare those values.

$$f(1,1) = 2(1)^2 + 2(1)^2 - 4(1)^2(1)$$

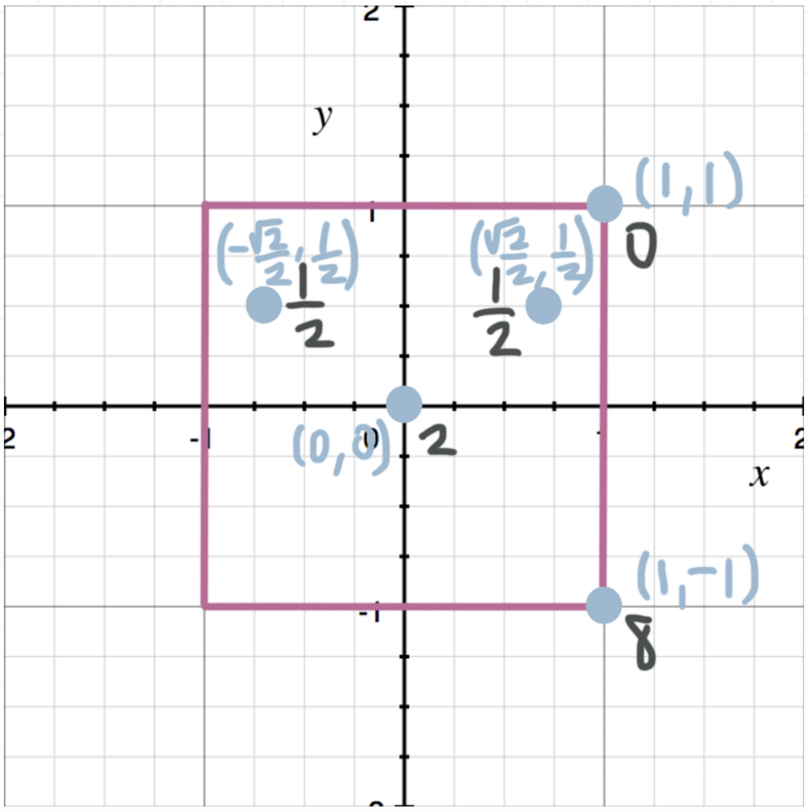
$$f(1,1) = 0$$

and

$$f(1, -1) = 2(1)^2 + 2(-1)^2 - 4(1)^2(-1)$$

$$f(1, -1) = 8$$

Which means the function is decreasing everywhere along that right boundary from the high point at $(1, -1)$ to the low point at $(1,1)$.



We'll repeat this same process for all four edges of the region. For the left edge of the region, which corresponds to the line $x = -1$, where $-1 \le y \le 1$, the function that defines the edge is

$$h(y) = f(-1,y) = 2(-1)^2 + 2y^2 - 4(-1)^2y$$

$$h(y) = f(-1,y) = 2 + 2y^2 - 4y$$

$$h(y) = f(-1,y) = 2y^2 - 4y + 2$$

This is the same function as the right edge, so we already know the derivative is $h'(y) = f'(-1,y) = 4y - 4$, and that the critical point is at $y = 1$. Putting this together with the value of x along the left edge, the only critical point along the left edge is at $(-1,1)$. That's right at the corner of the region, which means that corner is either the lowest point or the highest point along the left edge $x = -1$. So we need to plug that critical point $(-1,1)$, and the other corner point along that edge $(-1,-1)$ into the original function to compare those values.

$$f(-1,1) = 2(-1)^2 + 2(1)^2 - 4(-1)^2(1)$$

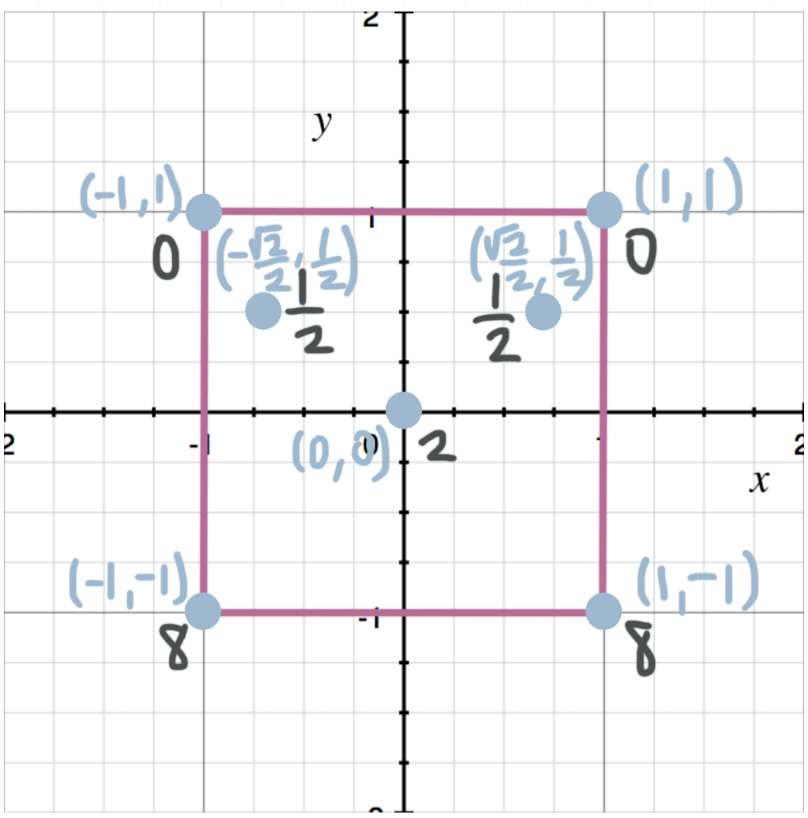
$$f(-1,1) = 0$$

and

$$f(-1,-1) = 2(-1)^2 + 2(-1)^2 - 4(-1)^2(-1)$$

$$f(-1,-1) = 8$$

Which means the function is decreasing everywhere along that left boundary from the high point at $(-1,-1)$ to the low point at $(-1,1)$.



For the top edge of the region, which corresponds to the line $y = 1$, where $-1 \leq x \leq 1$, the function that defines the edge is

$$m(x) = f(x,1) = 2x^2 + 2(1)^2 - 4x^2(1)$$

$$m(x) = f(x,1) = 2 - 2x^2$$

Taking the derivative gives

$$m'(x) = f'(x,1) = -4x$$

We'll set the derivative equal to 0 and solve for x.

$$-4x = 0$$

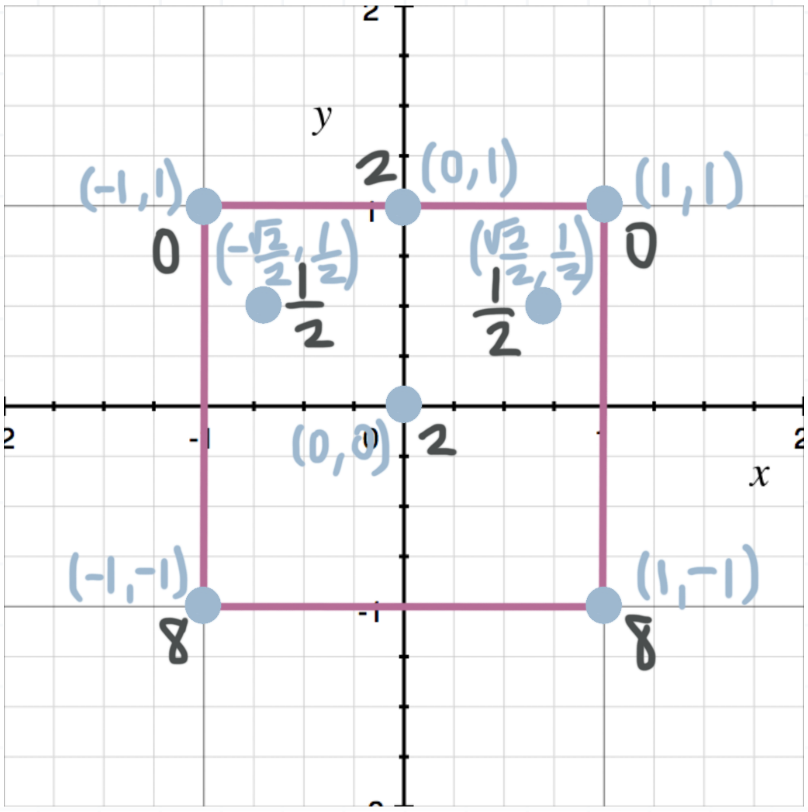
$$x = 0$$

Putting this together with the value of y along the top edge, the only critical point along the top edge is at $(0,1)$. We need to plug that critical point $(0,1)$ into the original function. We already know the values at the two top corners.

$$f(0,1) = 2(0)^2 + 2(1)^2 - 4(0)^2(1)$$

$$f(0,1) = 2$$

Which means the function is increasing from the low point at $(-1,1)$ up to the high point at $(0,1)$, and then decreases from that high point down to the low point at $(1,1)$.



For the bottom edge of the region, which corresponds to the line $y = -1$, where $-1 \le x \le 1$, the function that defines the edge is

$$n(x) = f(x, -1) = 2(-1)^2 + 2(-1)^2 - 4x^2(-1)$$

$$n(x) = f(x, -1) = 4 + 4x^2$$

Taking the derivative gives

$$m'(x) = f'(x, -1) = 8x$$

We'll set the derivative equal to 0 and solve for x.

$$8x = 0$$

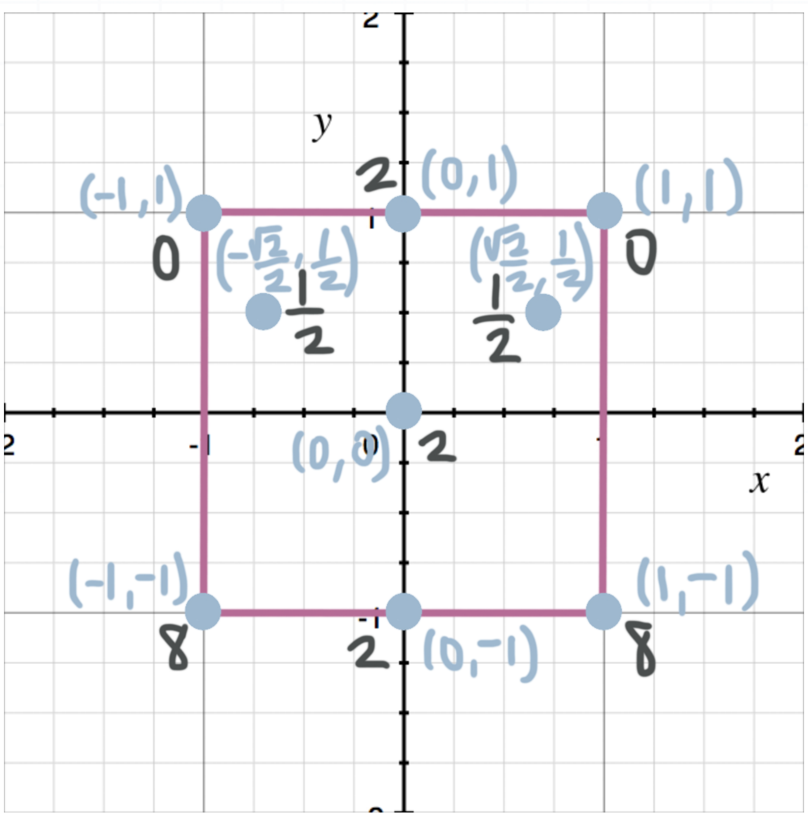
$$x = 0$$

Putting this together with the value of y along the bottom edge, the only critical point along the bottom edge is at $(0, -1)$. We need to plug that critical point $(0,-1)$ into the original function. We already know the values at the two bottom corners.

$$f(0,-1)=2(0)^2+2(-1)^2-4(0)^2(-1)$$

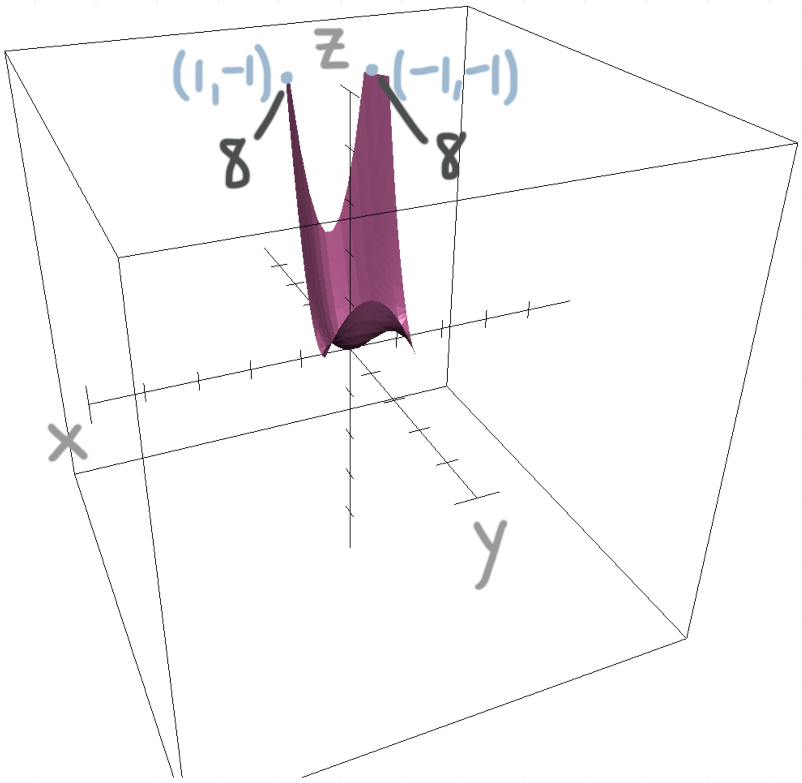
$$f(0,-1)=2$$

Which means the function is decreasing from the high point at $(-1,-1)$ down to the low point at $(0,-1)$, and then increases from that low point up to the high point at $(1,-1)$.



Looking at all of these values together, we can see that the global maxima of the function over the region exist at $(1,-1)$ and $(-1,-1)$ where the value of the function is 8, which is greater than the values we see everywhere else, which range from 0 to 2.

If we sketch the region in 3D space, we can see these high points.


(1,−1)
z
(−1,−1)
8
8
x
y