Calculus 3 Workbook Solutions

Arc length and curvature

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ARC LENGTH OF A VECTOR FUNCTION

1. Confirm the formula for the arc length $L = 2\pi R$ around the circle by considering the circle's equation as the vector function in polar coordinates, where R is the radius of the circle.

$$\vec{r}(\phi) = \langle R\cos\phi, R\sin\phi\rangle \text{ with } 0 \le \phi \le 2\pi$$

Solution:

Consider the circle centered at the origin with radius R. Rewrite the vector equation in parametric form.

$$x(\phi) = R\cos\phi$$

$$y(\phi) = R\sin\phi$$

Find derivatives.

$$x'(\phi) = -R\sin\phi$$

$$y'(\phi) = R\cos\phi$$

Arc length is given by

$$\int_a^b \sqrt{(x'(\phi))^2 + (y'(\phi))^2}\ d\phi$$

Substitute into the arc length formula.

$$L = \int_0^{2\pi} \sqrt{(-R\sin\phi)^2 + (R\cos\phi)^2}\ d\phi$$

$$L = \int_0^{2\pi} \sqrt{R^2\sin^2\phi + R^2\cos^2\phi}\ d\phi$$

$$L = \int_0^{2\pi} \sqrt{R^2(\sin^2\phi + \cos^2\phi)}\ d\phi$$

$$L = \int_0^{2\pi} \sqrt{R^2}\ d\phi$$

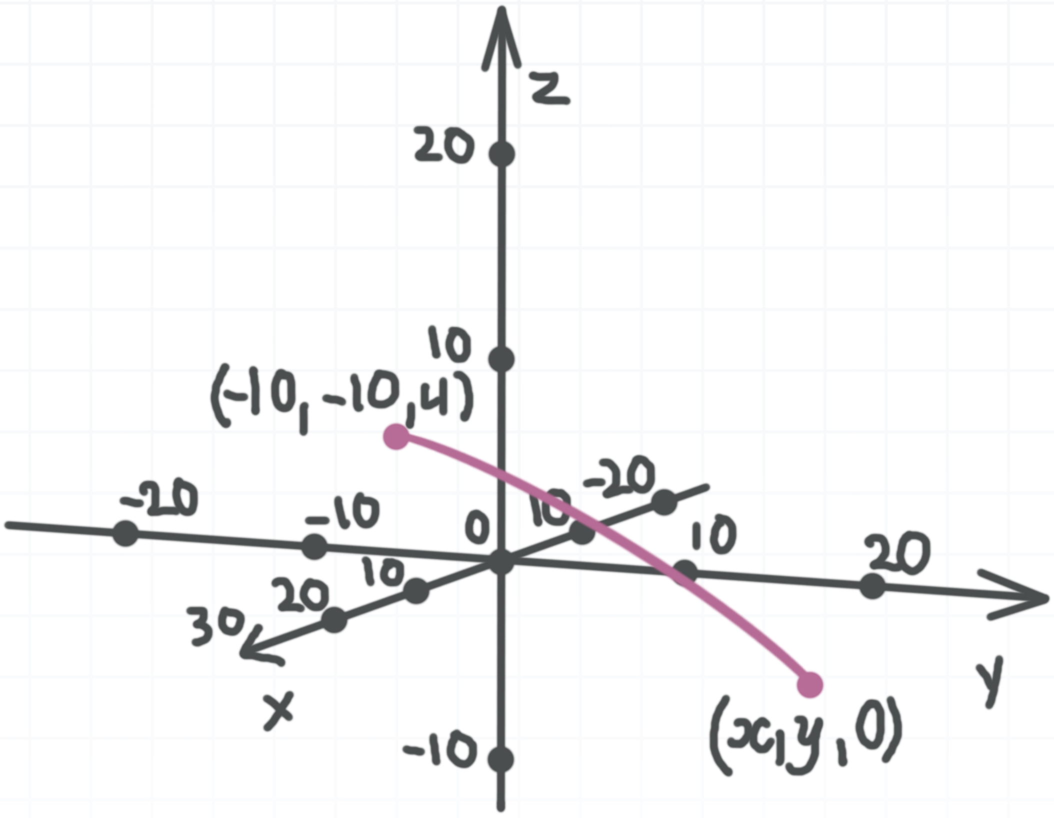
$$L = \int_0^{2\pi} R\ d\phi$$

$$L = R\int_0^{2\pi}\ d\phi$$

$$L = R(2\pi) = 2\pi R$$

■ 2. A cannon ball is shot from the point $A(-10,-10,4)$. Its trajectory can be modeled by the vector function, where $t \geq 0$ is the time. Find the arc length of the ball's trajectory before it hits the ground $z = 0$.

$$\vec{r}(t) = \left\langle t-10, t-10, \frac{-t^2+20t+800}{200} \right\rangle$$



Solution:

Rewrite the vector equation in parametric form.

$$x(t) = t - 10$$

$$y(t) = t - 10$$

$$z(t) = \frac{-t^2 + 20t + 800}{200}$$

Find the value of t when the ball hits the ground by solving the equation $z(t) = 0$ for t.

$$\frac{-t^2 + 20t + 800}{200} = 0$$

$$t^2 - 20t - 800 = 0$$

$$(t-40)(t+20)=0$$

$$t=-20 \text{ or } t=40$$

It’s impossible for $t \geq 0$. So t changes from 0 to 40. Arc length is given by

$$\int_0^{40} \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$$

Find derivatives.

$$x'(t)=1$$

$$y'(t)=1$$

$$z'(t)=\frac{-2t+20}{200}=\frac{10-t}{100}$$

Substitute the derivatives into the arc length formula.

$$L=\int_0^{40} \sqrt{1^2+1^2+\frac{(10-t)^2}{10{,}000}}\ dt$$

$$L=\int_0^{40} \sqrt{2+\frac{(t-10)^2}{10{,}000}}\ dt$$

Make the substitution $x=t-10$, with $dx=dt$, where x changes from -10 to 30.

$$L=\int_{-10}^{30} \sqrt{2+\frac{x^2}{10{,}000}}\ dx$$

$$L = \frac{1}{100} \int_{-10}^{30} \sqrt{20{,}000 + x^2}\ dx$$

Use a trigonometric substitution with the tangent substitution $u = a \tan \theta$, where $a = \sqrt{20{,}000} = 100\sqrt{2}$ and $u = x$.

$$L = \frac{1}{100} \int_{x=-10}^{x=30} \sqrt{20{,}000 + (100\sqrt{2} \tan \theta)^2}\ (100\sqrt{2} \sec^2 \theta\ d\theta)$$

$$L = \int_{x=-10}^{x=30} \sqrt{2} \sec^2 \theta \sqrt{20{,}000 + 20{,}000 \tan^2 \theta}\ d\theta$$

$$L = \int_{x=-10}^{x=30} \sqrt{2} \sec^2 \theta \sqrt{20{,}000(1 + \tan^2 \theta)}\ d\theta$$

$$L = \int_{x=-10}^{x=30} \sqrt{2} \sec^2 \theta \sqrt{20{,}000 \sec^2 \theta}\ d\theta$$

$$L = \int_{x=-10}^{x=30} \sqrt{2} \sec^2 \theta (100\sqrt{2} \sec \theta)\ d\theta$$

$$L = 200 \int_{x=-10}^{x=30} \sec^3 \theta\ d\theta$$

Use integration by parts with $s = \sec \theta$ and $dv = \sec^2 \theta\ d\theta$. Then $ds = \sec \theta \tan \theta\ d\theta$, and $v = \tan \theta$.

$$\int_{x=-10}^{x=30} \sec^3 \theta\ d\theta = \sec \theta \tan \theta \Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} \tan \theta \sec \theta \tan \theta\ d\theta$$

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} \tan^2\theta\sec\theta \ d\theta$$

Use the Pythagorean identity $\tan^2\theta = \sec^2\theta - 1$ to rewrite the integral.

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} (\sec^2\theta - 1)\sec\theta \ d\theta$$

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} \sec^3\theta - \sec\theta \ d\theta$$

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} \sec^3\theta \ d\theta + \int_{x=-10}^{x=30} \sec\theta \ d\theta$$

$$2\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} + \int_{x=-10}^{x=30} \sec\theta \ d\theta$$

$$2\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\Big|_{x=-10}^{x=30}$$

$$200\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = 100\sec\theta\tan\theta + 100\ln\left|\sec\theta + \tan\theta\right|\Big|_{x=-10}^{x=30}$$

Now back-substitute into the equation for L.

$$L = 100\sec\theta\tan\theta + 100\ln\left|\sec\theta + \tan\theta\right|\Big|_{x=-10}^{x=30}$$

Back-substitute to put the expression back in terms of x.

$$L = 100\frac{\sqrt{20{,}000 + x^2}}{100\sqrt{2}}\frac{x}{100\sqrt{2}} + 100\ln\left|\frac{\sqrt{20{,}000 + x^2}}{100\sqrt{2}} + \frac{x}{100\sqrt{2}}\right|\Bigg|_{-10}^{30}$$

$$L = \frac{x\sqrt{20{,}000 + x^2}}{200} + 100\ln\left|\frac{x + \sqrt{20{,}000 + x^2}}{100\sqrt{2}}\right|\Bigg|_{-10}^{30}$$

Evaluate over the interval.

$$L = \frac{30\sqrt{20{,}000 + 30^2}}{200} + 100\ln\left|\frac{30 + \sqrt{20{,}000 + 30^2}}{100\sqrt{2}}\right|$$

$$-\left(\frac{-10\sqrt{20{,}000 + (-10)^2}}{200} + 100\ln\left|\frac{-10 + \sqrt{20{,}000 + (-10)^2}}{100\sqrt{2}}\right|\right)$$

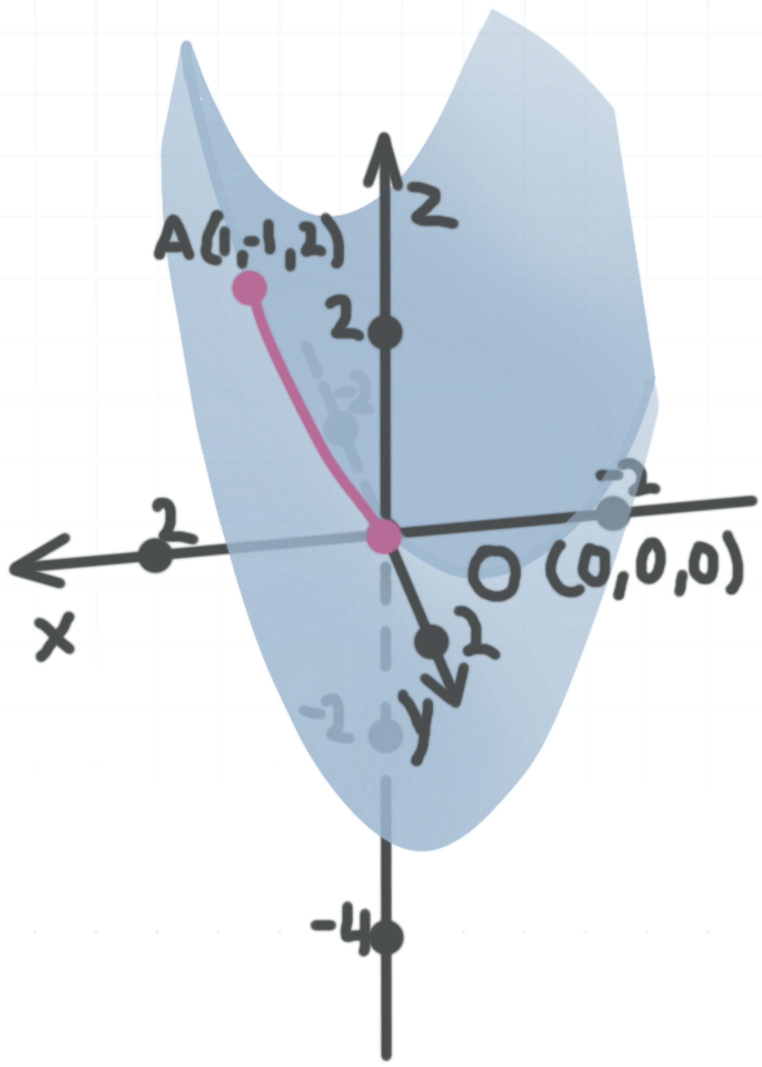
$$L = \frac{3\sqrt{20{,}900}}{20} + 100\ln\left|\frac{30 + \sqrt{20{,}900}}{100\sqrt{2}}\right| + \frac{\sqrt{20{,}100}}{20} - 100\ln\left|\frac{-10 + \sqrt{20{,}100}}{100\sqrt{2}}\right|$$

$$L = \frac{10\sqrt{201} + 30\sqrt{209}}{20} + 100\ln\left|\frac{30 + 10\sqrt{209}}{100\sqrt{2}}\right| - 100\ln\left|\frac{10\sqrt{201} - 10}{100\sqrt{2}}\right|$$

$$L = \frac{\sqrt{201} + 3\sqrt{209}}{2} + 100\ln\left|\frac{3 + \sqrt{209}}{10\sqrt{2}}\right| - 100\ln\left|\frac{\sqrt{201} - 1}{10\sqrt{2}}\right|$$

$$L \approx 56.8964$$

■ 3. Find the arc length of the curve that's the intersection of the cylinder $x^2 - y - z = 0$ and the plane $x + y = 0$, between $O(0,0,0)$ and $A(1,-1,2)$.



Solution:

Let x be t, then

$$t^2 - y - z = 0$$

$$t + y = 0$$

$$y = -t$$

$$z = t^2 - y = t^2 + t$$

So the parametrization of the curve is

$$x(t) = t$$

$$y(t) = -t$$

$$z(t) = t^2 + t$$

Find the limits for t which correspond to O and A.

If $t = 0$, then $x(0) = 0$, $y(0) = 0$, and $z(0) = 0$

If $t = 1$, then $x(1) = 1$, $y(1) = -1$, and $z(1) = 2$

So $0 \le t \le 1$. The arc length is given by

$$\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

Find derivatives.

$$x'(t) = 1$$

$$y'(t) = -1$$

$$z'(t) = 2t + 1$$

Substitute the derivatives into the arc length formula.

$$L=\int_0^1 \sqrt{1^2+(-1)^2+(2t+1)^2}\ dt$$

$$L=\int_0^1 \sqrt{(2t+1)^2+2}\ dt$$

Make the substitution $u=2t+1$, with $du=2\ dt$, and u changing from 1 to 3.

$$L=\frac{1}{2}\int_1^3 \sqrt{u^2+2}\ du$$

$$L=\frac{1}{2}\left[\frac{u\sqrt{u^2+2}}{2}+\ln\left(u+\sqrt{u^2+2}\right)\right]\Bigg|_1^3$$

$$L=\frac{1}{2}\left[\frac{3\sqrt{3^2+2}}{2}+\ln\left(3+\sqrt{3^2+2}\right)\right]-\frac{1}{2}\left[\frac{1\sqrt{1^2+2}}{2}+\ln\left(1+\sqrt{1^2+2}\right)\right]$$

$$L\approx 2.47$$

REPARAMETRIZING THE CURVE

■ 1. Reparametrize $\vec{r}(t)=\langle -3+t,2+2t,6-2t\rangle$ in terms of the arc length measured from $(-3,2,6)$ in the direction of increasing t.

Solution:

To reparametrize a curve $\vec{r}(t)$ in terms of arc length, we need to modify the curve so that the path is the same, but increasing the argument by 1 results in increasing the arc length by 1. This way, inputting a value of s for the curve will result in the curve having arc length s.

Rewrite $\vec{r}(t)$ as

$$\vec{r}(t)=\langle -3,2,6\rangle+t\langle 1,2,-2\rangle$$

Since $\vec{r}(t)$ is a linear curve, each unit increase in t corresponds to an increase of arc length by

$$|\langle 1,2,-2\rangle|=\sqrt{1^2+2^2+(-2)^2}=\sqrt{9}=3$$

So

$$3t=s$$

$$t=\frac{s}{3}$$

Substitute $t=s/3$ into $\vec{r}(t)$.

$$\vec{r}(s) = \left\langle -3 + \frac{s}{3}, 2 + 2 \cdot \frac{s}{3}, 6 - 2 \cdot \frac{s}{3} \right\rangle = \left\langle -3 + \frac{s}{3}, 2 + \frac{2}{3}s, 6 - \frac{2}{3}s \right\rangle$$

■ 2. Reparametrize $\vec{r}(t) = \langle 4\cos 3t, -2t, 4\sin 3t \rangle$ in terms of arc length, measured from $(-4,2\pi,0)$.

Solution:

To reparametrize a curve $\vec{r}(t)$ in terms of arc length, we need to modify the curve so that the path is the same, but so that increasing the argument by 1 results in increasing the arc length by 1. This way, inputting a value of s for the curve will result in the curve having arc length s.

Reparametrizing $s(t)$ is given by

$$s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\ du$$

In order to find the initial value of t that corresponds to $(-4,2\pi,0)$, solve the system of equations for t.

$$4\cos 3t = -4$$

$$-2t = 2\pi$$

$$4\sin 3t = 0$$

From the second equation, $t = -\pi$. Check that the other equations hold.

$$4\cos(-3\pi) = 4(-1) = -4$$

$$4\sin(-3\pi) = 0$$

So the initial value of t is $a = -\pi$.

Find the derivatives of each component of the vector function.

$$x'(t) = -12\sin 3t$$

$$y'(t) = -2$$

$$z'(t) = 12\cos 3t$$

Substitute into the formula for arc length.

$$s(t) = \int_{-\pi}^{t} \sqrt{(-12\sin 3u)^2 + (-2)^2 + (12\cos 3u)^2}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{144\ \sin^2 3u + 4 + 144\ \cos^2 3u}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{144(\sin^2 3u + \cos^2 3u) + 4}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{144 + 4}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{148}\ du$$

$$s(t) = 2\sqrt{37}\int_{-\pi}^{t} du$$

$$s(t) = 2\sqrt{37}\,(t+\pi)$$

Solve for t.

$$s = 2\sqrt{37}\,(t+\pi)$$

$$t = \frac{s}{2\sqrt{37}} - \pi$$

Substitute t into $\vec{r}(t)$.

$$\vec{r}(s) = \left\langle 4\cos\left(3\left(\frac{s}{2\sqrt{37}} - \pi\right)\right), \ -2\left(\frac{s}{2\sqrt{37}} - \pi\right), 4\sin\left(3\left(\frac{s}{2\sqrt{37}} - \pi\right)\right)\right\rangle$$

$$\vec{r}(s) = \left\langle -4\cos\left(\frac{3s}{2\sqrt{37}}\right), \ -\frac{s}{\sqrt{37}} + 2\pi, \ -4\sin\left(\frac{3s}{2\sqrt{37}}\right)\right\rangle$$

3. Reparametrize the curve $\vec{r}(t) = \langle 2e^{2t}, e^{2t}\rangle$ in terms of arc length measured from $t=0$. Use the parametrization to find the position after traveling 5 units.

Solution:

To reparametrize a curve $\vec{r}(t)$ in terms of arc length, we need to modify the curve so that the path is the same, but so that increasing the argument by 1 results in increasing the arc length by 1. This way, inputting a value of s for the curve will result in the curve having arc length s.

The reparametrizing function $s(t)$ is given by

$$s(t) = \int_a^t \sqrt{(r_1'(u))^2 + (r_2'(u))^2}\ du$$

Find the derivatives of each component of the vector function.

$$r_1'(t) = 4e^{2t}$$

$$r_2'(t) = 2e^{2t}$$

Substitute the derivatives into the arc length formula.

$$s(t) = \int_0^t \sqrt{(4e^{2u})^2 + (2e^{2u})^2}\ du$$

$$s(t) = \int_0^t \sqrt{16e^{4u} + 4e^{4u}}\ du$$

$$s(t) = \int_0^t \sqrt{20e^{4u}}\ du$$

$$s(t) = \int_0^t 2\sqrt{5}e^{2u}\ du$$

$$s(t) = \sqrt{5}e^{2u}\Big|_0^t$$

$$s(t) = \sqrt{5}e^{2t} - \sqrt{5}e^0$$

$$s(t) = \sqrt{5}e^{2t} - \sqrt{5}$$

Solve for e^{2t} (since we only need e^{2t} for the initial equation).

$$s = \sqrt{5}e^{2t} - \sqrt{5}$$

$$\sqrt{5}e^{2t} = s + \sqrt{5}$$

$$e^{2t} = \frac{s}{\sqrt{5}} + 1$$

Substitute e^{2t} into the vector function.

$$\vec{r}(s) = \left\langle 2\left(\frac{s}{\sqrt{5}} + 1\right), \frac{s}{\sqrt{5}} + 1 \right\rangle$$

$$\vec{r}(s) = \left\langle \frac{2s}{\sqrt{5}} + 2, \frac{s}{\sqrt{5}} + 1 \right\rangle$$

Plug in $s = 5$ to find the position after traveling 5 units.

$$\vec{r}(5) = \left\langle \frac{2 \cdot 5}{\sqrt{5}} + 2, \frac{5}{\sqrt{5}} + 1 \right\rangle$$

$$\vec{r}(5) = \left\langle 2\sqrt{5} + 2, \sqrt{5} + 1 \right\rangle$$

CURVATURE

1. Find the curvature of $f(x) = 2x^2 - 4$ at $x = 1$.

Solution:

The curvature of the vector function is given by

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite $f(x)$ in parametric form for $x = t$ and $y = f(t)$.

$$x(t) = t$$

$$y(t) = 2t^2 - 4$$

Find the derivatives of these functions.

$$x'(t) = 1$$

$$y'(t) = 4t$$

So $\vec{r}'(t) = \langle 1, 4t \rangle$, then find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{(4t)^2 + 1^2} = \sqrt{16t^2 + 1}$$

Therefore,

$$\vec{T}(t) = \frac{\langle 1, 4t \rangle}{\sqrt{16t^2 + 1}}$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{16t^2 + 1}}, \frac{4t}{\sqrt{16t^2 + 1}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{1}{\sqrt{16t^2 + 1}} \right)' = -\frac{16t}{(16t^2 + 1)^{3/2}}$$

$$\vec{T}_2'(t) = \left(\frac{4t}{\sqrt{16t^2 + 1}} \right)' = \frac{4}{(16t^2 + 1)^{3/2}}$$

Plug $t = 1$ into the expressions for $|\vec{r}'(t)|$, and $\vec{T}'(t)$.

$$|\vec{r}'(1)| = \sqrt{16 \cdot 1^2 + 1} = \sqrt{17}$$

$$\vec{T}_1'(1) = -\frac{16 \cdot 1}{(16 \cdot 1^2 + 1)^{3/2}} = -\frac{16}{17\sqrt{17}}$$

$$\vec{T}_2'(1) = \frac{4}{(16 \cdot 1^2 + 1)^{3/2}} = \frac{4}{17\sqrt{17}}$$

Find the magnitude of $\vec{T}'(1)$.

$$|\vec{T}'(1)| = \sqrt{\left(-\frac{16}{17\sqrt{17}}\right)^2 + \left(\frac{4}{17\sqrt{17}}\right)^2} = \frac{4}{17}$$

Plug the values we've found into the formula for $k(t)$.

$$k(1) = \frac{|\vec{T}'(1)|}{|\vec{r}'(1)|}$$

$$k(1) = \frac{\frac{4}{17}}{\sqrt{17}} = \frac{4\sqrt{17}}{289}$$

■ 2. Find the curvature of the vector function at $t = 0$.

$$\vec{r}(t) = \left\langle 2(2+t)^{3/2}, 6t, 2(2-t)^{3/2} \right\rangle$$

Solution:

The curvature of the vector function is given by the formula

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function $\vec{r}(t)$ in parametric form.

$$x(t) = 2(2+t)^{3/2}$$

$$y(t) = 6t$$

$$z(t) = 2(2-t)^{3/2}$$

Find the derivatives of these functions.

$$x'(t) = 3(2+t)^{1/2} = 3\sqrt{2+t}$$

$$y'(t) = 6$$

$$z'(t) = 3(2-t)^{1/2} = 3\sqrt{2-t}$$

So

$$\vec{r}\,'(t) = \left\langle 3\sqrt{2+t}, 6, 3\sqrt{2-t} \right\rangle$$

Find the magnitude of $\vec{r}\,'(t)$.

$$|\vec{r}\,'(t)| = \sqrt{\left(3\sqrt{2+t}\right)^2 + 6^2 + \left(3\sqrt{2-t}\right)^2}$$

$$|\vec{r}\,'(t)| = \sqrt{9(2+t) + 36 + 9(2-t)} = \sqrt{72} = 6\sqrt{2}$$

Therefore,

$$\vec{T}(t) = \frac{\left\langle 3\sqrt{2+t}, 6, 3\sqrt{2-t} \right\rangle}{6\sqrt{2}}$$

$$\vec{T}(t) = \left\langle \frac{\sqrt{2+t}}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\sqrt{2-t}}{2\sqrt{2}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{\sqrt{2+t}}{2\sqrt{2}} \right)' = \frac{1}{4\sqrt{2}\sqrt{2+t}}$$

$$\vec{T}_2'(t) = \left(\frac{1}{\sqrt{2}} \right)' = 0$$

$$\vec{T}_3'(t) = \left(\frac{\sqrt{2-t}}{2\sqrt{2}} \right)' = -\frac{1}{4\sqrt{2}\sqrt{2-t}}$$

Plug $t = 0$ into the expressions for $\vec{T}'(t)$.

$$\vec{T}_1'(0) = \frac{1}{4\sqrt{2}\sqrt{2+0}} = \frac{1}{8}$$

$$\vec{T}_2'(0) = 0$$

$$\vec{T}_3'(0) = -\frac{1}{4\sqrt{2}\sqrt{2-0}} = -\frac{1}{8}$$

Find the magnitude of $\vec{T}'(0)$.

$$|\vec{T}'(0)| = \sqrt{\left(\frac{1}{8}\right)^2 + 0^2 + \left(-\frac{1}{8}\right)^2} = \frac{1}{4\sqrt{2}}$$

Plug the values we've found into the formula for $k(t)$.

$$k(0) = \frac{|\vec{T}'(0)|}{|\vec{r}'(0)|}$$

$$k(0) = \frac{\frac{1}{4\sqrt{2}}}{6\sqrt{2}} = \frac{1}{48}$$

■ 3. Find the value(s) of t_0 such that the curvature of $\vec{r}(t) = \langle e^t + 5, 2e^t, -2e^t \rangle$ is 0 at $t = t_0$.

Solution:

The curvature of the vector function is given by the formula

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function $\vec{r}(t)$ in parametric form.

$$x(t) = e^t + 5$$

$$y(t) = 2e^t$$

$$z(t) = -2e^t$$

Find the derivatives of these functions.

$$x'(t) = e^t$$

$$y'(t) = 2e^t$$

$$z'(t) = -2e^t$$

So $\vec{r}'(t) = \langle e^t, 2e^t, -2e^t \rangle$. Find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{(e^t)^2 + (2e^t)^2 + (-2e^t)^2}$$

$$|\vec{r}'(t)| = \sqrt{9(e^t)^2} = 3e^t$$

Therefore,

$$\vec{T}(t) = \frac{\langle e^t, 2e^t, -2e^t \rangle}{3e^t}$$

$$\vec{T}(t) = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$$

Since $\vec{T}(t)$ is a constant vector, $\vec{T}'(t) = \langle 0,0,0 \rangle$ and $|\vec{T}'(t)| = 0$. So for any t,

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = 0$$

Since the curvature of the function is 0 at any point, the graph of this function is a line.

MAXIMUM CURVATURE

1. Find the absolute maximum curvature $k(t)$ of $\vec{r}(t) = \langle 2 + \sin t, \cos(t + \pi) \rangle$ on the interval $[0,2\pi]$.

Solution:

The curvature of the vector function is given by

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Use the trigonometric identity $\cos(\phi + \pi) = -\cos\phi$ to simplify the function.

$$\vec{r}(t) = \langle 2 + \sin t, -\cos t \rangle$$

Rewrite the function in parametric form.

$$x(t) = 2 + \sin t$$

$$y(t) = -\cos t$$

Find the derivatives of these equations.

$$x'(t) = \cos t$$

$$y'(t) = \sin t$$

So $\vec{r}\,'(t) = \langle \cos t, \sin t \rangle$, now find the magnitude of $\vec{r}\,'(t)$.

$$|\,\vec{r}\,'(t)\,| = \sqrt{(\cos t)^2 + (\sin t)^2} = 1$$

Therefore,

$$\vec{T}(t) = \frac{\langle \cos t, \sin t \rangle}{1}$$

$$\vec{T}(t) = \langle \cos t, \sin t \rangle$$

Find $\vec{T}\,'(t)$.

$$\vec{T}\,'(t) = \langle -\sin t, \cos t \rangle$$

Find the magnitude of $\vec{T}\,'(t)$.

$$|\,\vec{T}\,'(t)\,| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

So the curvature $k(t)$ is

$$k(t) = \frac{|\,\vec{T}\,'(t)\,|}{|\,\vec{r}\,'(t)\,|}$$

$$k(t) = \frac{1}{1} = 1$$

Since the curvature is a constant function, it reaches the maximum value of 1 at any point on the interval $[0,2\pi]$.

■ 2. Find the absolute minimum and maximum curvature $k(x)$ of the function $f(x) = \ln(6x)$ on the interval $(0,1]$.

Solution:

The curvature of the vector function is given by

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function $f(x)$ in parametric form for $x = t$ and $y = f(t)$.

$$x(t) = t$$

$$y(t) = \ln(6t)$$

Find the derivatives of these equations.

$$x'(t) = 1$$

$$y'(t) = \frac{1}{t}$$

So

$$\vec{r}'(t) = \left\langle 1, \frac{1}{t} \right\rangle$$

Find the magnitude of $\vec{r}\,'(t)$.

$$|\vec{r}\,'(t)| = \sqrt{1^2 + \left(\frac{1}{t}\right)^2} = \frac{\sqrt{t^2+1}}{t}$$

Therefore,

$$\vec{T}(t) = \frac{\left\langle 1, \frac{1}{t} \right\rangle}{\frac{\sqrt{t^2+1}}{t}}$$

$$\vec{T}(t) = \left\langle \frac{t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle$$

Find $\vec{T}\,'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{t}{\sqrt{t^2+1}}\right)' = \frac{1}{(t^2+1)^{3/2}}$$

$$\vec{T}_2'(t) = \left(\frac{1}{\sqrt{t^2+1}}\right)' = -\frac{t}{(t^2+1)^{3/2}}$$

Find the magnitude of $\vec{T}\,'(t)$.

$$|\vec{T}\,'(t)| = \sqrt{\left(\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(-\frac{t}{(t^2+1)^{3/2}}\right)^2}$$

$$|\vec{T}\,'(t)| = \sqrt{\frac{1}{(t^2+1)^3} + \frac{t^2}{(t^2+1)^3}}$$

$$|\vec{T}'(t)| = \frac{1}{t^2+1}$$

So the curvature is

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$k(t) = \frac{\frac{1}{t^2+1}}{\frac{\sqrt{t^2+1}}{t}} = \frac{t}{(t^2+1)^{3/2}}$$

To find the minimum of the function over $(0,1]$, let's investigate the critical points. Take the derivative.

$$k'(t) = \frac{1-2t^2}{(t^2+1)^{5/2}}$$

Solve the equation $k'(t)=0$ in order to find the critical points.

$$\frac{1-2t^2}{(t^2+1)^{5/2}} = 0$$

$$1-2t^2=0$$

$$t^2 = \frac{1}{2}$$

Since $0 < t \le 1$,

$$t = \frac{1}{\sqrt{2}}$$

Since $k'(t) > 0$ for $0 < t < 1/\sqrt{2}$ and $k'(t) < 0$ for $1/\sqrt{2} < t \le 1$, the point $t = 1/\sqrt{2}$ is the local maximum.

$$k\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(\left(\frac{1}{\sqrt{2}}\right)^2 + 1\right)^{3/2}} = \frac{2\sqrt{3}}{9} \approx 0.38$$

So to find the absolute maximum, we need to compare the values of the function at the endpoints. Since the function isn't defined at $t = 0$, we need to consider the limit of the function when t approaches 0.

$$\lim_{t \to 0} k(t) = \lim_{t \to 0} \frac{t}{(t^2+1)^{3/2}} = \frac{0}{(0^2+1)^{3/2}} = 0$$

$$k(1) = \frac{1}{(1^2+1)^{3/2}} = \frac{\sqrt{2}}{4} \approx 0.35$$

So the absolute maximum is $2\sqrt{3}/9$ at $x = 1/\sqrt{2} = \sqrt{2}/2$, and the absolute minimum does not exist (it exists only as a limit when x approaches 0). So to summarize, the absolute maximum is $2\sqrt{3}/9$ at $x = \sqrt{2}/2$, and the absolute minimum does not exist.

■ 3. Find the absolute maximum curvature $k(t)$ of $\vec{r}(t) = \langle 3t+1, 2.5t^2 - 3, 4 - 4t \rangle$ on the interval $(-\infty, \infty)$.

Solution:

The curvature of the vector function is given by

$$k(t)=\frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}=\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function in parametric form.

$$x(t)=3t+1$$

$$y(t)=2.5t^2-3$$

$$z(t)=4-4t$$

Find the derivatives of these equations.

$$x(t)=3$$

$$y(t)=5t$$

$$z(t)=-4$$

So $\vec{r}'(t)=\langle 3,5t,-4\rangle$, and we can find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)|=\sqrt{3^2+(5t)^2+(-4)^2}=\sqrt{25t^2+25}=5\sqrt{t^2+1}$$

Therefore,

$$\vec{T}(t) = \frac{\langle 3, 5t, -4 \rangle}{5\sqrt{t^2+1}}$$

$$\vec{T}(t) = \left\langle \frac{3}{5\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}}, -\frac{4}{5\sqrt{t^2+1}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{3}{5\sqrt{t^2+1}}\right)' = -\frac{3t}{5(t^2+1)^{3/2}}$$

$$\vec{T}_2'(t) = \left(\frac{t}{\sqrt{t^2+1}}\right)' = \frac{1}{(t^2+1)^{3/2}}$$

$$\vec{T}_3'(t) = \left(-\frac{4}{5\sqrt{t^2+1}}\right)' = \frac{4t}{5(t^2+1)^{3/2}}$$

Find the magnitude of $\vec{T}'(t)$.

$$|\vec{T}'(t)| = \sqrt{\frac{(3t)^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{(4t)^2}{25(t^2+1)^3}}$$

$$|\vec{T}'(t)| = \sqrt{\frac{1}{(t^2+1)^2}}$$

$$|\vec{T}'(t)| = \frac{1}{t^2+1}$$

So the curvature $k(t)$ is

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$k(t) = \frac{\frac{1}{t^2+1}}{5\sqrt{t^2+1}} = \frac{1}{5(t^2+1)^{3/2}}$$

To find the maximum of the function over $(-\infty, \infty)$, let's investigate the critical points. Take the derivative of the curvature function.

$$k'(t) = -\frac{3t}{5(t^2+1)^{5/2}}$$

Solve the equation $k'(t) = 0$ in order to find the critical points.

$$-\frac{3t}{5(t^2+1)^{5/2}} = 0$$

$$t = 0$$

To find the absolute maximum, we need to compare the values of the function at $t = 0$ and when t approaches $-\infty$ and ∞.

$$\lim_{t \to \infty} k(t) = \lim_{t \to \infty} \frac{1}{5(t^2+1)^{3/2}} = 0$$

$$\lim_{t \to -\infty} k(t) = \lim_{t \to -\infty} \frac{1}{5(t^2+1)^{3/2}} = 0$$

$$k(0) = \frac{1}{5(1^2+1)^{3/2}} = \frac{1}{5(2)^{3/2}} = \frac{1}{10\sqrt{2}} = \frac{\sqrt{2}}{20}$$

So the curvature reaches its absolute maximum of $\sqrt{2}/20$ at $t = 0$.

NORMAL AND OSCULATING PLANES

1. Find the point(s) at which the normal plane to the curve $\vec{r}(t)$ is parallel to the y-axis, then find the equation(s) of the normal plane at each point.

$$\vec{r}(t) = \langle 3t^3 - 10t, t^3 - 6t^2 - 15t, 4t + 1 \rangle$$

Solution:

The normal plane is the plane perpendicular to the tangent vector $\vec{r}\,'(t)$ of a space curve. The equation of the normal plane at the point $(x_0, y_0, z_0) = (r_1(t_0), r_2(t_0), r_3(t_0))$ is given by

$$r_1'(t_0)(x - x_0) + r_2'(t_0)(y - y_0) + r_3'(t_0)(z - z_0) = 0$$

Rewrite the function in parametric form.

$$r_1(t) = 3t^3 - 10t$$

$$r_2(t) = t^3 - 6t^2 - 15t$$

$$r_3(t) = 4t + 1$$

Find the derivatives of these equations.

$$r_1'(t) = 9t^2 - 10$$

$$r_2'(t) = 3t^2 - 12t - 15$$

$$r_3'(t) = 4$$

Since the normal plane is parallel to the y-axis at the point t_0, $r_2'(t_0) = 0$. So $t = t_0$ is the solution of the following equation:

$$3t^2 - 12t - 15 = 0$$

$$3(t + 1)(t - 5) = 0$$

So the normal plane is parallel to the y-axis at $t_0 = -1$ or $t_0 = 5$. For $t_0 = -1$,

$$r_1(-1) = 3(-1)^3 - 10(-1) = 7$$

$$r_2(-1) = (-1)^3 - 6(-1)^2 - 15(-1) = 8$$

$$r_3(-1) = 4(-1) + 1 = -3$$

$$r_1'(-1) = 9(-1)^2 - 10 = -1$$

$$r_2'(-1) = 3(-1)^2 - 12(-1) - 15 = 0$$

$$r_3'(-1) = 4$$

So the equation of the normal plane at the point $(7, 8, -3)$ is

$$-1(x - 7) + 4(z + 3) = 0$$

$$-x + 4z + 19 = 0$$

For $t_0 = 5$,

$$r_1(5) = 3(5)^3 - 10(5) = 325$$

$$r_2(5) = (5)^3 - 6(5)^2 - 15(5) = -100$$

$$r_3(5) = 4(5) + 1 = 21$$

$$r_1'(5) = 9(5)^2 - 10 = 215$$

$$r_2'(5) = 3(5)^2 - 12(5) - 15 = 0$$

$$r_3'(5) = 4$$

So the equation of the normal plane at the point $(325, -100, 21)$ is

$$215(x - 325) + 4(z - 21) = 0$$

$$215x + 4z - 69{,}959 = 0$$

■ 2. Find the equation of the osculating plane to $\vec{r}(t) = \langle 12 - 6t, 5t^2 - 10, 7 - 8t \rangle$ at the point $(0,10, -9)$.

Solution:

The equation of the osculating plane at $(x_0, y_0, z_0) = (r_1(t_0), r_2(t_0), r_3(t_0))$ is given by

$$B_1(t_0)(x - x_0) + B_2(t_0)(y - y_0) + B_3(t_0)(z - z_0) = 0$$

where $\vec{B}(t)$ is the binormal vector such that

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The unit tangent vector $\vec{T}(t)$ is equal to

$$\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

and the unit normal vector $\vec{N}(t)$ is equal to

$$\frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

To find the value of t_0 which corresponds to $(0,10,-9)$, we can solve $r_1(t)=0$, $r_2(t)=10$, or $r_3(t)=-9$ for t. From the first equation,

$$12-6t=0$$

$$t=2$$

We can also check that the other equations hold for $t_0=2$.

$$r_2(2)=5(2)^2-10=10$$

$$r_3(2)=7-8(2)=-9$$

To find the unit tangent vector $\vec{T}(t)$, rewrite the function in parametric form.

$$x(t)=12-6t$$

$$y(t)=5t^2-10$$

$$z(t)=7-8t$$

Find the derivatives of these equations.

$$x'(t)=-6$$

$$y'(t) = 10t$$

$$z'(t) = -8$$

So $\vec{r}'(t) = \langle -6, 10t, -8 \rangle$, and we can find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{(-6)^2 + (10t)^2 + (-8)^2} = \sqrt{100t^2 + 100} = 10\sqrt{t^2 + 1}$$

Therefore,

$$\vec{T}(t) = \frac{\langle -6, 10t, -8 \rangle}{10\sqrt{t^2 + 1}}$$

$$\vec{T}(t) = \left\langle -\frac{3}{5\sqrt{t^2 + 1}}, \frac{t}{\sqrt{t^2 + 1}}, -\frac{4}{5\sqrt{t^2 + 1}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(-\frac{3}{5\sqrt{t^2 + 1}} \right)' = \frac{3t}{5(t^2 + 1)^{3/2}}$$

$$\vec{T}_2'(t) = \left(\frac{t}{\sqrt{t^2 + 1}} \right)' = \frac{1}{(t^2 + 1)^{3/2}}$$

$$\vec{T}_3'(t) = \left(-\frac{4}{5\sqrt{t^2 + 1}} \right)' = \frac{4t}{5(t^2 + 1)^{3/2}}$$

Find the magnitude of $\vec{T}'(t)$.

$$|\vec{T}'(t)|=\sqrt{\frac{(3t)^2}{25(t^2+1)^3}+\frac{1}{(t^2+1)^3}+\frac{(4t)^2}{25(t^2+1)^3}}$$

$$|\vec{T}'(t)|=\sqrt{\frac{1}{(t^2+1)^2}}$$

$$|\vec{T}'(t)|=\frac{1}{t^2+1}$$

So the unit normal vector $\vec{N}(t)$ is

$$\vec{N}(t)=\frac{\left\langle \frac{3t}{5(t^2+1)^{3/2}},\frac{1}{(t^2+1)^{3/2}},\frac{4t}{5(t^2+1)^{3/2}}\right\rangle}{\frac{1}{t^2+1}}$$

$$\vec{N}(t)=\left\langle \frac{3t}{5\sqrt{t^2+1}},\frac{1}{\sqrt{t^2+1}},\frac{4t}{5\sqrt{t^2+1}}\right\rangle$$

Plug $t=2$ into $\vec{T}(t)$ in order to find the unit tangent vector at that point.

$$\vec{T}(2)=\left\langle -\frac{3}{5\sqrt{2^2+1}},\frac{2}{\sqrt{2^2+1}},-\frac{4}{5\sqrt{2^2+1}}\right\rangle$$

$$\vec{T}(2)=\left\langle -\frac{3}{5\sqrt{5}},\frac{2}{\sqrt{5}},-\frac{4}{5\sqrt{5}}\right\rangle$$

Plug $t=2$ into $\vec{N}(t)$ in order to find the unit normal vector at that point.

$$\vec{N}(2)=\left\langle \frac{3\cdot 2}{5\sqrt{2^2+1}},\frac{1}{\sqrt{2^2+1}},\frac{4\cdot 2}{5\sqrt{2^2+1}}\right\rangle$$

$$\vec{N}(2)=\left\langle \frac{6}{5\sqrt{5}},\frac{1}{\sqrt{5}},\frac{8}{5\sqrt{5}} \right\rangle$$

Find the binormal vector using the cross product.

$$\vec{B}(2)=\vec{T}(2)\times\vec{N}(2)$$

$$\vec{B}(2)=\left\langle -\frac{3}{5\sqrt{5}},\frac{2}{\sqrt{5}},-\frac{4}{5\sqrt{5}} \right\rangle \times \left\langle \frac{6}{5\sqrt{5}},\frac{1}{\sqrt{5}},\frac{8}{5\sqrt{5}} \right\rangle$$

The cross product of the two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}\times\vec{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Plug in the vectors $\vec{a}$ and $\vec{b}$,

$$\langle a_1,a_2,a_3\rangle=\left\langle -\frac{3}{5\sqrt{5}},\frac{2}{\sqrt{5}},-\frac{4}{5\sqrt{5}} \right\rangle$$

$$\langle b_1,b_2,b_3\rangle=\left\langle \frac{6}{5\sqrt{5}},\frac{1}{\sqrt{5}},\frac{8}{5\sqrt{5}} \right\rangle$$

to get

$$\vec{B}(2)=\mathbf{i}\left(\frac{2}{\sqrt{5}}\cdot\frac{8}{5\sqrt{5}}+\frac{4}{5\sqrt{5}}\cdot\frac{1}{\sqrt{5}}\right)-\mathbf{j}\left(-\frac{3}{5\sqrt{5}}\cdot\frac{8}{5\sqrt{5}}+\frac{4}{5\sqrt{5}}\cdot\frac{6}{5\sqrt{5}}\right)$$

$$+\mathbf{k}\left(-\frac{3}{5\sqrt{5}}\cdot\frac{1}{\sqrt{5}}-\frac{2}{\sqrt{5}}\cdot\frac{6}{5\sqrt{5}}\right)$$

$$\vec{B}(2) = \left\langle \frac{4}{5}, 0, -\frac{3}{5} \right\rangle$$

The equation of the plane through the point $(0,10,-9)$ and with the normal vector $\vec{B}(2)$ is

$$\frac{4}{5}(x-0) + 0(y-10) - \frac{3}{5}(z+9) = 0$$

$$4x - 3z - 27 = 0$$

■ 3. Use the binormal vector to prove that the graph of the vector function $\vec{r}(t)$ is a planar curve (a curve that lies in a single plane), then find the equation of the plane.

$$\vec{r}(t) = \langle 2\sin t - 2,\ \cos t + 1, 2\cos t + 5 \rangle$$

Solution:

The curve is planar if its binormal vector is constant for any t. In this case the binormal vector is orthogonal to this plane. The binormal vector $\vec{B}(t)$ is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

where the unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

and the unit normal vector $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

To find the unit tangent vector $\vec{T}(t)$, rewrite the function in parametric form.

$$x(t) = 2\sin t - 2$$

$$y(t) = \cos t + 1$$

$$z(t) = 2\cos t + 5$$

Find derivatives of these equations.

$$x'(t) = 2\cos t$$

$$y'(t) = -\sin t$$

$$z'(t) = -2\sin t$$

So $\vec{r}'(t) = \langle 2\cos t, -\sin t, -2\sin t\rangle$, and we can find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{(2\cos t)^2 + (-\sin t)^2 + (-2\sin t)^2}$$

$$|\vec{r}'(t)| = \sqrt{5\sin^2 t + 4\cos^2 t}$$

$$|\vec{r}'(t)| = \sqrt{4 + \sin^2 t}$$

Therefore,

$$\vec{T}(t) = \frac{\langle 2\cos t, -\sin t, -2\sin t\rangle}{\sqrt{4+\sin^2 t}}$$

$$\vec{T}(t) = \left\langle \frac{2\cos t}{\sqrt{4+\sin^2 t}}, -\frac{\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\sin t}{\sqrt{4+\sin^2 t}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{2\cos t}{\sqrt{4+\sin^2 t}}\right)' = -\frac{10\sin t}{(4+\sin^2 t)^{3/2}}$$

$$\vec{T}_2'(t) = \left(-\frac{\sin t}{\sqrt{4+\sin^2 t}}\right)' = -\frac{4\cos t}{(4+\sin^2 t)^{3/2}}$$

$$\vec{T}_3'(t) = \left(-\frac{2\sin t}{\sqrt{4+\sin^2 t}}\right)' = -\frac{8\cos t}{(4+\sin^2 t)^{3/2}}$$

Find the magnitude of $\vec{T}'(t)$.

$$|\vec{T}'(t)| = \sqrt{\left(-\frac{10\sin t}{(4+\sin^2 t)^{3/2}}\right)^2 + \left(-\frac{4\cos t}{(4+\sin^2 t)^{3/2}}\right)^2 + \left(-\frac{8\cos t}{(4+\sin^2 t)^{3/2}}\right)^2}$$

$$|\vec{T}'(t)| = \sqrt{\frac{20}{(4+\sin^2 t)^2}}$$

$$|\vec{T}'(t)| = \frac{2\sqrt{5}}{4+\sin^2 t}$$

So the unit normal vector $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{\left\langle -\frac{10\sin t}{(4+\sin^2 t)^{3/2}}, -\frac{4\cos t}{(4+\sin^2 t)^{3/2}}, -\frac{8\cos t}{(4+\sin^2 t)^{3/2}} \right\rangle}{\frac{2\sqrt{5}}{4+\sin^2 t}}$$

$$\vec{N}(t) = \left\langle -\frac{10\sin t}{2\sqrt{5}(4+\sin^2 t)^{1/2}}, -\frac{4\cos t}{2\sqrt{5}(4+\sin^2 t)^{1/2}}, -\frac{8\cos t}{2\sqrt{5}(4+\sin^2 t)^{1/2}} \right\rangle$$

$$\vec{N}(t) = \left\langle -\frac{\sqrt{5}\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}}, -\frac{4\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}} \right\rangle$$

Find the unit binormal vector using the cross product.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\vec{B}(t) = \left\langle \frac{2\cos t}{\sqrt{4+\sin^2 t}}, -\frac{\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\sin t}{\sqrt{4+\sin^2 t}} \right\rangle$$

$$\times \left\langle -\frac{\sqrt{5}\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}}, -\frac{4\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}} \right\rangle$$

Factor the denominator out of each vector.

$$\vec{B}(t) = \frac{\langle 2\cos t, -\sin t, -2\sin t \rangle \times \left\langle -\sqrt{5}\sin t, -\frac{2\cos t}{\sqrt{5}}, -\frac{4\cos t}{\sqrt{5}} \right\rangle}{4+\sin^2 t}$$

The cross product of the two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug $\vec{a}$ and $\vec{b}$,

$$\langle a_1, a_2, a_3 \rangle = \langle 2\cos t, -\sin t, -2\sin t \rangle$$

$$\langle b_1, b_2, b_3 \rangle = \left\langle -\sqrt{5}\sin t, -\frac{2\cos t}{\sqrt{5}}, -\frac{4\cos t}{\sqrt{5}} \right\rangle$$

into the cross product formula.

$$\vec{B}(t) = \frac{1}{4+\sin^2 t}\left[\mathbf{i}\left(\sin t \cdot \frac{4\cos t}{\sqrt{5}} - 2\sin t \cdot \frac{2\cos t}{\sqrt{5}}\right)\right.$$

$$-\mathbf{j}\left(-2\cos t \cdot \frac{4\cos t}{\sqrt{5}} - 2\sin t \cdot \sqrt{5}\sin t\right)$$

$$\left.+\vec{k}\left(-2\cos t \cdot \frac{2\cos t}{\sqrt{5}} - \sin t \cdot \sqrt{5}\sin t\right)\right]$$

$$\vec{B}(t) = \frac{\left\langle 0, \frac{8\cos^2 t}{\sqrt{5}} + 2\sqrt{5}\sin^2 t, -\frac{4\cos^2 t}{\sqrt{5}} - \sqrt{5}\sin^2 t \right\rangle}{4+\sin^2 t}$$

$$\vec{B}(t) = \left\langle 0, \frac{8\cos^2 t + 10\sin^2 t}{\sqrt{5}(4+\sin^2 t)}, -\frac{4\cos^2 t + 5\sin^2 t}{\sqrt{5}(4+\sin^2 t)} \right\rangle$$

$$\vec{B}(t) = \left\langle 0, \frac{8+2\sin^2 t}{\sqrt{5}(4+\sin^2 t)}, -\frac{4+\sin^2 t}{\sqrt{5}(4+\sin^2 t)} \right\rangle$$

$$\vec{B}(t) = \left\langle 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

So since $\vec{B}(t)$ is the same for any point on the curve, the curve is planar. And since $\vec{B}(t)$ is orthogonal to the plane which contains the curve, $\vec{B}(t)$ is the normal vector to this plane.

Let's take any point on the curve, for example $t = 0$,

$$x(0) = 2\sin 0 - 2 = -2$$

$$y(0) = \cos 0 + 1 = 2$$

$$z(0) = 2\cos 0 + 5 = 7$$

Then the equation of the plane through $(-2,2,7)$ and with the normal vector

$$\vec{N}(t) = \left\langle 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

is

$$0(x+2) + \frac{2}{\sqrt{5}}(y-2) - \frac{1}{\sqrt{5}}(z-7) = 0$$

$$2y - z + 3 = 0$$

EQUATION OF THE OSCULATING CIRCLE

1. Find the equation of the osculating circle to the curve $\vec{r}(t) = \langle 2 + 5\sin t, 5\cos t - 1 \rangle$ at an arbitrary point.

Solution:

For the parametric curve $\vec{r}(t)$ in two-dimensional space, the signed curvature is given by

$$k(t) = \frac{r_1'(t) \cdot r_2''(t) - r_1''(t) \cdot r_2'(t)}{(r_1'(t)^2 + r_2'(t)^2)^{3/2}}$$

The unit normal vector is

$$\vec{N}(t) = \frac{\left\langle -r_2'(t), r_1'(t) \right\rangle}{|\vec{r}\,'(t)|}$$

The radius of curvature (equal to the radius of the osculating circle) is

$$R(t) = \frac{1}{|k(t)|}$$

and the vector to the center of the osculating circle is

$$\vec{Q}(t) = \vec{r}(t) + \frac{1}{k(t)}\vec{N}(t)$$

Rewrite the function in parametric form.

$$r_1(t) = 2 + 5\sin t$$

$$r_2(t) = 5\cos t - 1$$

The first-order derivatives are

$$r_1'(t) = 5\cos t$$

$$r_2'(t) = -5\sin t$$

The second-order derivatives are

$$r_1''(t) = -5\sin t$$

$$r_2''(t) = -5\cos t$$

Find the magnitude of $\vec{r}\,'(t)$/

$$|\vec{r}\,'(t)| = \sqrt{(5\cos t)^2 + (-5\sin t)^2} = \sqrt{25\cos^2 t + 25\sin^2 t} = \sqrt{25} = 5$$

The unit normal vector is

$$\vec{N}(t) = \frac{\left\langle -r_2'(t), r_1'(t)\right\rangle}{|\vec{r}\,'(t)|}$$

$$\vec{N}(t) = \frac{\langle 5\sin t, 5\cos t\rangle}{5}$$

$$\vec{N}(t) = \langle \sin t,\ \cos t\rangle$$

The signed curvature is

$$k(t)=\frac{r_1'(t)\cdot r_2''(t)-r_1''(t)\cdot r_2'(t)}{(r_1'(t)^2+r_2'(t)^2)^{3/2}}$$

$$k(t)=\frac{5\cos t\cdot(-5\cos t)-(-5\sin t)\cdot(-5\sin t)}{((5\cos t)^2+(-5\sin t)^2)^{3/2}}$$

$$k(t)=\frac{-25}{25^{3/2}}=-\frac{1}{5}$$

The radius of curvature is

$$R(t)=\frac{1}{|k(t)|}$$

$$R(t)=\frac{1}{\left|-\frac{1}{5}\right|}=5$$

The vector to the center of the osculating circle is

$$\vec{Q}(t)=\vec{r}(t)+\frac{1}{k(t)}\vec{N}(t)$$

$$\vec{Q}(t)=\langle 2+5\sin t,5\cos t-1\rangle-5\langle\sin t,\ \cos t\rangle$$

$$\vec{Q}(t)=\langle 2,-1\rangle$$

So the osculating circle is the circle with center $(2,-1)$ and radius 5. The osculating curve is this circle itself, and so the equation of the osculating circle is

$$(x-2)^2+(y+1)^2=25$$

■ 2. Find the center and radius of the osculating circle to the curve $\overrightarrow{r}(t)$ at the point $(7,6)$.

$$\overrightarrow{r}(t) = \langle 4(5-t)^{5/2} + 3, 24t - 90 \rangle$$

Solution:

In order to find the value of t that corresponds to $(7,6)$, solve the system of equations for t.

$$4(5-t)^{5/2} + 3 = 7$$

$$24t - 90 = 6$$

From the first equation, we get

$$4(5-t)^{5/2} = 4$$

$$(5-t)^{5/2} = 1$$

$$t = 4$$

Check if the second equation holds for $t = 4$.

$$24(4) - 90 = 6$$

For the parametric curve $\overrightarrow{r}(t)$ in two-dimensional space, the signed curvature is given by

$$k(t) = \frac{r_1'(t) \cdot r_2''(t) - r_1''(t) \cdot r_2'(t)}{(r_1'(t)^2 + r_2'(t)^2)^{3/2}}$$

The unit normal vector is

$$\vec{N}(t) = \frac{\langle -r_2'(t), r_1'(t) \rangle}{|\vec{r}'(t)|}$$

The radius of curvature (equal to the radius of the osculating circle) is

$$R(t) = \frac{1}{|k(t)|}$$

and the vector to the center of the osculating circle is

$$\vec{Q}(t) = \vec{r}(t) + \frac{1}{k(t)}\vec{N}(t)$$

Rewrite the function in parametric form.

$$r_1(t) = 4(5-t)^{5/2} + 3$$

$$r_2(t) = 24t - 90$$

The first-order derivatives are

$$r_1'(t) = -10(5-t)^{3/2}$$

$$r_2'(t) = 24$$

The second-order derivatives are

$$r_1''(t) = 15(5-t)^{1/2} = 15\sqrt{5-t}$$

$$r_2''(t) = 0$$

Since we don't need the curvature and other parameters in general form, we'll use their values at $t = 4$.

$$r_1'(4) = -10,\ r_2'(4) = 24$$

$$r_1''(4) = 15,\ r_2''(4) = 0$$

Find the magnitude of $\vec{r}\,'(4)$.

$$|\vec{r}\,'(4)| = \sqrt{(-10)^2 + 24^2} = 26$$

The unit normal vector is

$$\vec{N}(4) = \frac{\langle -r_2'(4), r_1'(4) \rangle}{|\vec{r}\,'(4)|}$$

$$\vec{N}(4) = \frac{\langle -24, -10 \rangle}{26}$$

$$\vec{N}(4) = \left\langle -\frac{12}{13}, -\frac{5}{13} \right\rangle$$

The signed curvature is

$$k(4) = \frac{r_1'(4) \cdot r_2''(4) - r_1''(4) \cdot r_2'(4)}{(r_1'(4)^2 + r_2'(4)^2)^{3/2}}$$

$$k(4) = \frac{-10 \cdot 0 - 15 \cdot 24}{((-10)^2 + 24^2)^{3/2}}$$

$$k(4) = -\frac{360}{26^3}$$

$$k(4) = -\frac{45}{2{,}197}$$

The radius of curvature is

$$R(t) = \frac{1}{|k(t)|}$$

$$R(4) = \frac{1}{\left|-\frac{45}{2{,}197}\right|} = \frac{2{,}197}{45} \approx 48.8$$

The vector to the center of the osculating circle is

$$\vec{Q}(4) = \vec{r}(4) + \frac{1}{k(4)}\vec{N}(4)$$

$$\vec{Q}(4) = \langle 7,6\rangle - \frac{2{,}197}{45}\left\langle -\frac{12}{13}, -\frac{5}{13}\right\rangle$$

$$\vec{Q}(4) = \left\langle \frac{781}{15}, \frac{232}{9}\right\rangle \approx \langle 52.1, 25.8\rangle$$

So the osculating circle has its center at $(52.1, 25.8)$ and a radius of 48.8.

■ 3. Find the point(s) on the curve $\vec{r}(t) = \langle t^2 + 3, 2t - 5\rangle$ where the osculating circle has a radius of 2.

Solution:

For the parametric curve $\vec{r}(t)$ in two-dimensional space, the signed curvature is given by

$$k(t)=\frac{r_1'(t)\cdot r_2''(t)-r_1''(t)\cdot r_2'(t)}{(r_1'(t)^2+r_2'(t)^2)^{3/2}}$$

The radius of curvature (equal to the radius of the osculating circle) is

$$R(t)=\frac{1}{|k(t)|}$$

Since the radius is 2, we need to solve the equation for t.

$$|k(t)|=\frac{1}{2}$$

Rewrite the function in parametric form.

$$r_1(t)=t^2+3$$

$$r_2(t)=2t-5$$

The first-order derivatives are

$$r_1'(t)=2t$$

$$r_2'(t)=2$$

The second-order derivatives are

$$r_1''(t)=2$$

$$r_2''(t)=0$$

The signed curvature is

$$k(t) = \frac{r_1'(t) \cdot r_2''(t) - r_1''(t) \cdot r_2'(t)}{(r_1'(t)^2 + r_2'(t)^2)^{3/2}}$$

$$k(t) = \frac{2t \cdot 0 - 2 \cdot 2}{((2t)^2 + 2^2)^{3/2}}$$

$$k(t) = \frac{-4}{(4t^2 + 4)^{3/2}}$$

$$k(t) = \frac{-4}{8(t^2 + 1)^{3/2}}$$

$$k(t) = -\frac{1}{2(t^2 + 1)^{3/2}}$$

Solve the equation $|k(t)| = 1/2$. Since $k(t)$ is always negative,

$$-\frac{1}{2(t^2 + 1)^{3/2}} = -\frac{1}{2}$$

$$(t^2 + 1)^{3/2} = 1$$

$$(t^2 + 1)^3 = 1$$

$$t^2 + 1 = 1$$

$$t = 0$$

The coordinates of the points on the curve for $t = 0$ are

$r_1(0) = 0^2 + 3 = 3$ and $r_2(0) = 2(0) - 5 = -5$

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