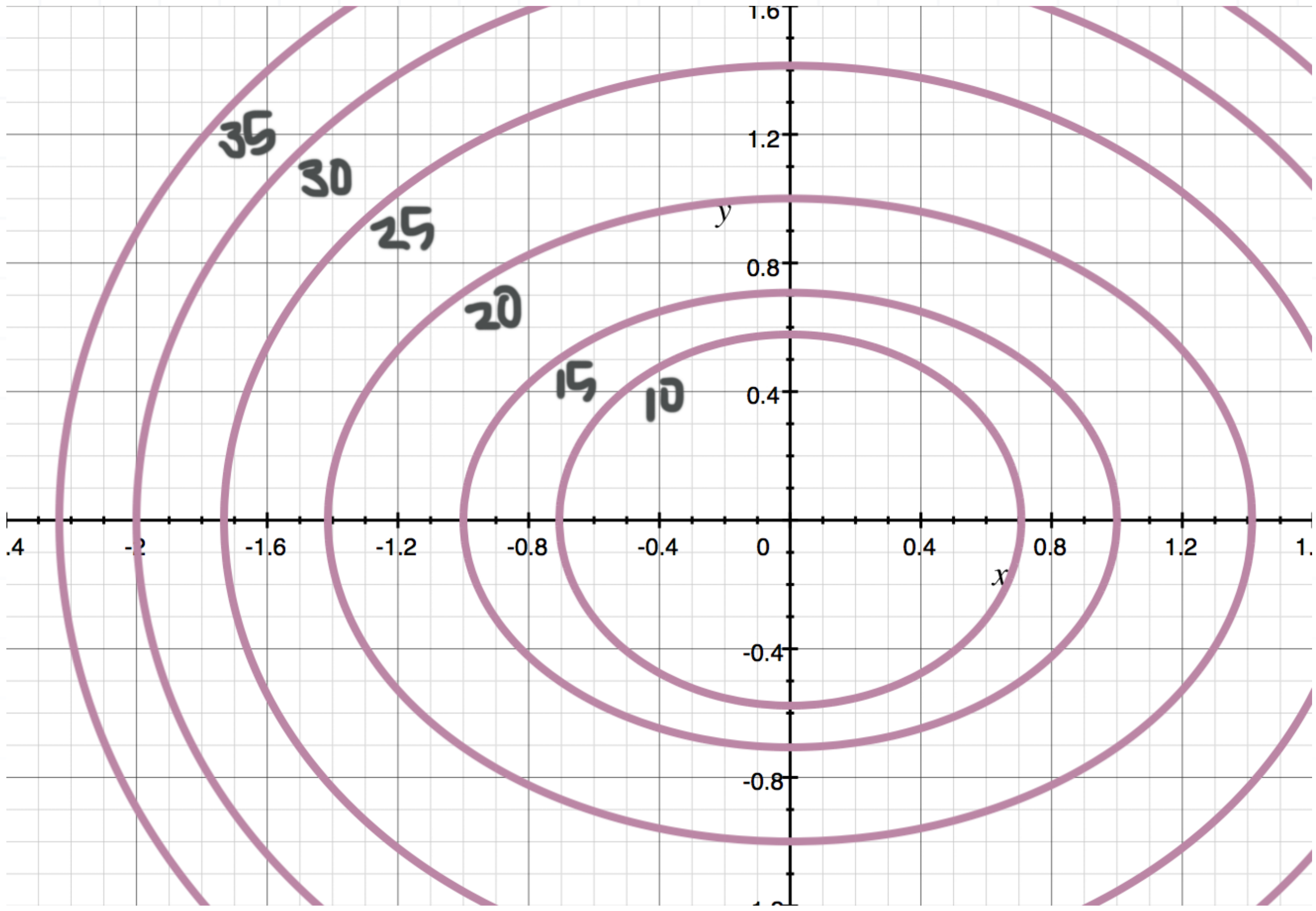
Calculus 3 Workbook Solutions

Double integrals

krista king
MATH

AVERAGE VALUE

■ 1. Use midpoints of squares with side lengths 1 to estimate the average value of the region $R = [-2,1] \times [-2,2]$, given the sketch of level curves.



Solution:

We can find the area of the rectangle, and the area of the smaller squares.

$$A(R) = (3)(4) = 12$$

$$\Delta A = (1)(1) = 1$$

Find estimates of the function's value at each midpoint.

$f(-1.5, -1.5) = 28$ $\quad$ $f(-0.5, -1.5) = 26$ $\quad$ $f(0.5, -1.5) = 26$

$f(-1.5, -0.5) = 18$ $\quad$ $f(-0.5, -0.5) = 12$ $\quad$ $f(0.5, -0.5) = 12$

$f(-1.5,0.5) = 18$ $\quad$ $f(-0.5,0.5) = 12$ $\quad$ $f(0.5,0.5) = 12$

$z(-1.5,1.5) = 28$ $\quad$ $f(-0.5,1.5) = 26$ $\quad$ $f(0.5,1.5) = 26$

So the average value is

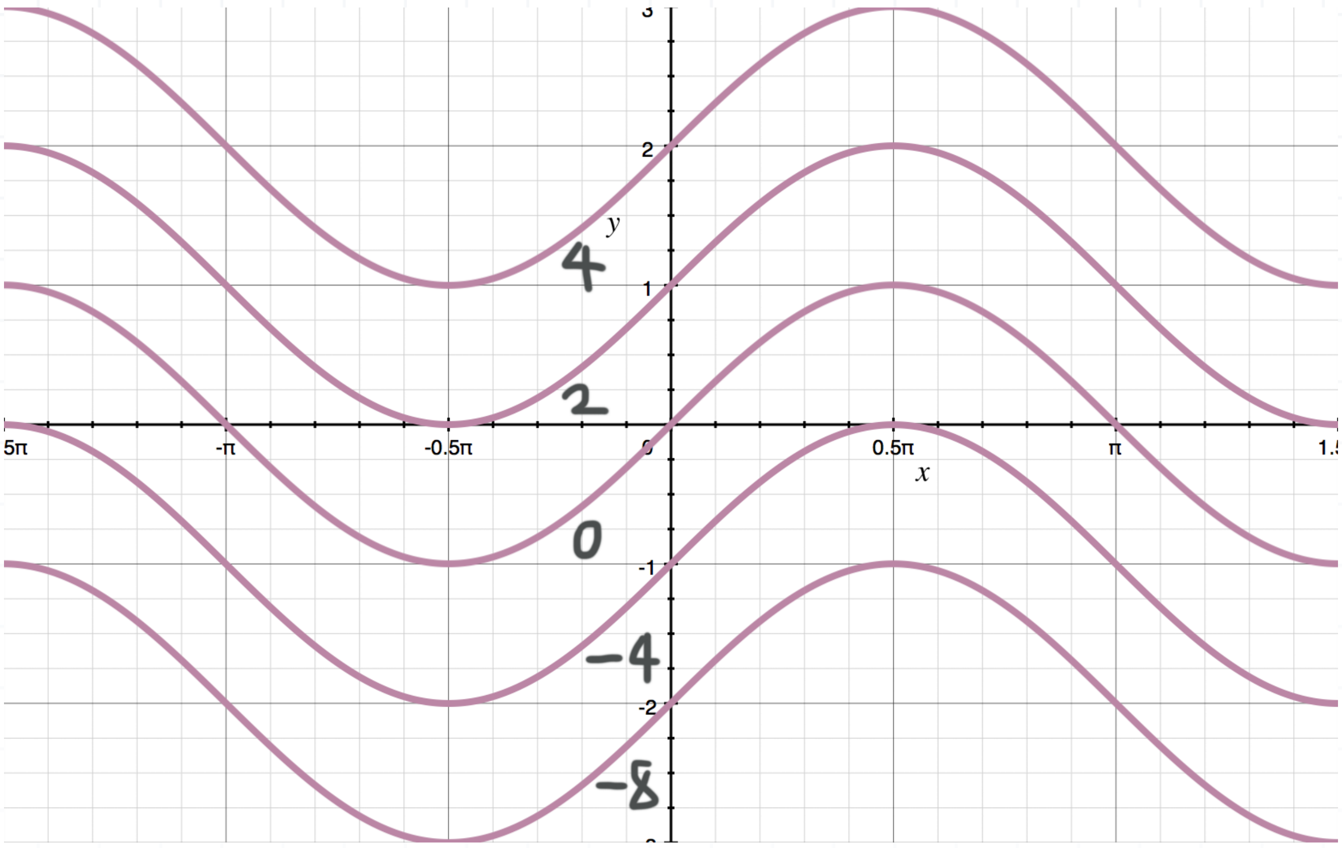
$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)\ \Delta A$$

$$f_{avg} = \frac{1}{12} \cdot 1 \cdot (28 + 26 + 26 + 18 + 12 + 12 + 18 + 12 + 12 + 28 + 26 + 26)$$

$$f_{avg} = \frac{1}{12}(244)$$

$$f_{avg} = \frac{61}{3}$$

■ 2. Use midpoints of rectangles with dimensions $\pi \times 1$ to estimate the average value of the region $R = [-\pi, \pi] \times [-2,2]$, given the sketch of level curves.



Solution:

We can find the area of the rectangle, and the area of the smaller squares.

$$A(R) = (2\pi)(4) = 8\pi$$

$$\Delta A = (\pi)(1) = \pi$$

Find estimates of the function's value at each midpoint.

$$f(-\pi/2, -1.5) = -2 \qquad f(\pi/2, -1.5) = -10$$

$$f(-\pi/2, -0.5) = 1 \qquad f(\pi/2, -0.5) = -6$$

$$f(-\pi/2, 0.5) = 3 \qquad f(\pi/2, 0.5) = -2$$

$$f(-\pi/2, 1.5) = 5 \qquad f(\pi/2, 1.5) = 1$$

So the average value is

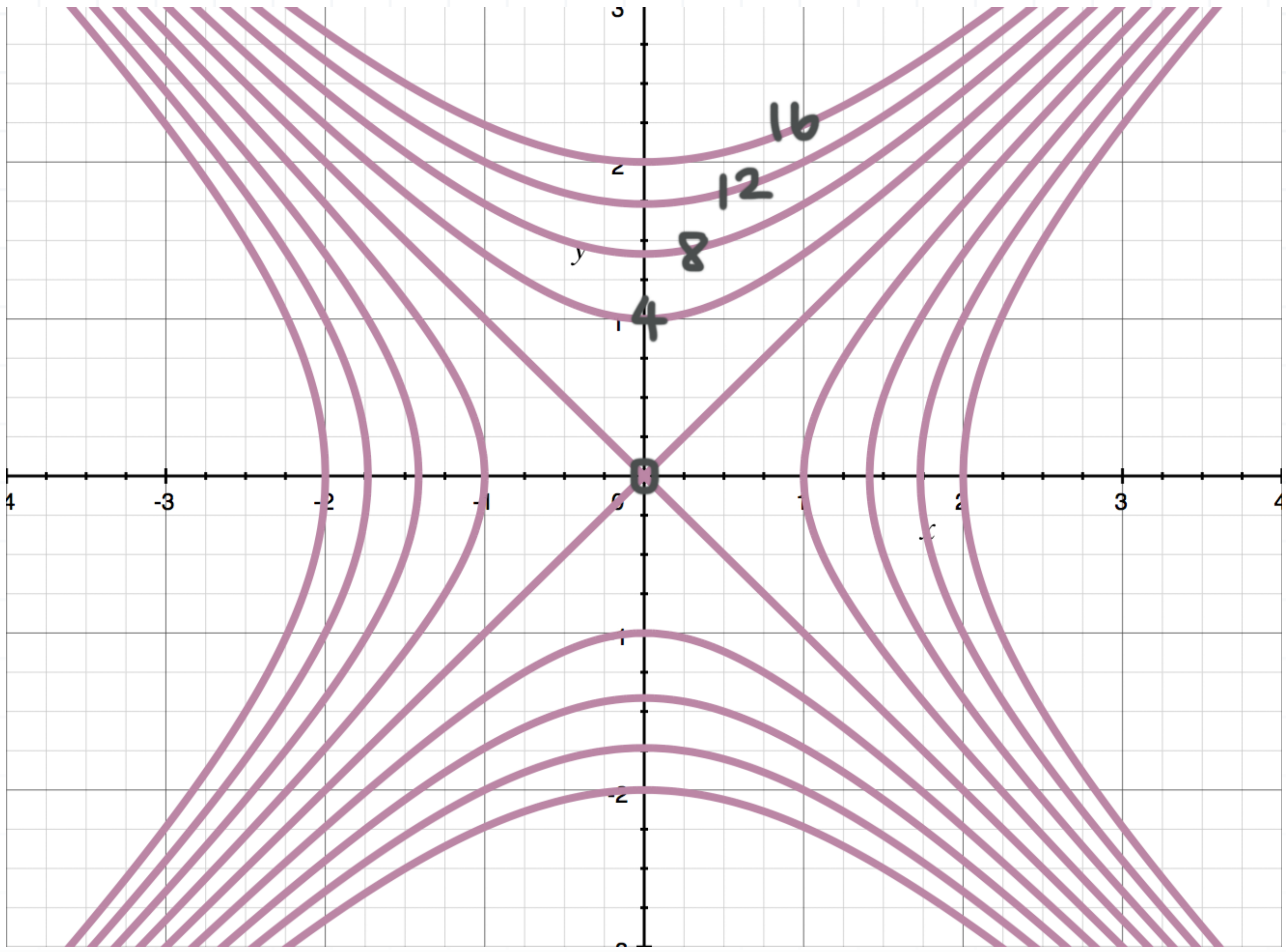
$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)\ \Delta A$$

$$f_{avg} = \frac{1}{8\pi} \cdot \pi \cdot (-2 - 10 + 1 - 6 + 3 - 2 + 5 + 1)$$

$$f_{avg} = \frac{1}{8}(-10)$$

$$f_{avg} = -\frac{5}{4}$$

■ 3. Use midpoints of rectangles with dimensions 2×1 to estimate the average value of the region $R = [-2,2] \times [-2,2]$, given the sketch of level curves.



Solution:

We can find the area of the rectangle, and the area of the smaller squares.

$$A(R) = (4)(4) = 16$$

$$\Delta A = (2)(1) = 2$$

Find estimates of the function's value at each midpoint. Since the level curves are symmetric about the origin, we can estimate the function values for the midpoints of the small rectangles in the first quadrant only.

$$f(1,0.5) = 3$$

$$f(1,1.5) = 5$$

So the average value is

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) \ \Delta A$$

$$f_{avg} = \frac{1}{16} \cdot 2 \cdot (3 + 5 + 3 + 5 + 3 + 5 + 3 + 5)$$

$$f_{avg} = \frac{1}{8}(32)$$

$$f_{avg} = 4$$

ITERATED INTEGRALS

■ 1. Evaluate the iterated integral.

$$\int_{2}^{4}\int_{1}^{2}\log_2\frac{y^2}{x^4}\ dx\ dy$$

Solution:

Use laws of logs to rewrite the integrand.

$$\int_{2}^{4}\int_{1}^{2}2\log_2 y-4\log_2 x\ dx\ dy$$

Use the integration formula.

$$\int\log_a x\ dx=\frac{x\ln x-x}{\ln a}+C$$

Integrate with respect to x, treating y as a constant.

$$\int_{2}^{4}(2\log_2 y)x-\frac{4x\ln x-4x}{\ln 2}\Big|_{x=1}^{x=2}\ dy$$

$$\int_{2}^{4}(2\log_2 y)(2)-\frac{4(2)\ln 2-4(2)}{\ln 2}-\left((2\log_2 y)(1)-\frac{4(1)\ln 1-4(1)}{\ln 2}\right)\ dy$$

$$\int_{2}^{4}4\log_2 y-\frac{8\ln 2-8}{\ln 2}-\left(2\log_2 y-\frac{4\ln 1-4}{\ln 2}\right)\ dy$$

$$\int_2^4 4\log_2 y - \frac{8\ln 2 - 8}{\ln 2} - \left(2\log_2 y - \frac{4(0) - 4}{\ln 2}\right)\ dy$$

$$\int_2^4 4\log_2 y - \frac{8\ln 2 - 8}{\ln 2} - 2\log_2 y - \frac{4}{\ln 2}\ dy$$

$$\int_2^4 2\log_2 y - \left(\frac{8\ln 2 - 8}{\ln 2} + \frac{4}{\ln 2}\right)\ dy$$

$$\int_2^4 2\log_2 y - \frac{8\ln 2 - 4}{\ln 2}\ dy$$

$$\int_2^4 2\log_2 y - \frac{8\ln 2}{\ln 2} + \frac{4}{\ln 2}\ dy$$

$$\int_2^4 2\log_2 y - 8 + \frac{4}{\ln 2}\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\frac{2y\ln y - 2y}{\ln 2} + \frac{4}{\ln 2}y - 8y\Big|_2^4$$

$$\frac{2(4)\ln(4) - 2(4)}{\ln 2} + \frac{4}{\ln 2}(4) - 8(4) - \left(\frac{2(2)\ln(2) - 2(2)}{\ln 2} + \frac{4}{\ln 2}(2) - 8(2)\right)$$

$$\frac{8\ln 4 - 8}{\ln 2} + \frac{16}{\ln 2} - 32 - \frac{4\ln 2 - 4}{\ln 2} - \frac{8}{\ln 2} + 16$$

$$\frac{8\ln 4 - 8 + 8 - 4\ln 2 + 4}{\ln 2} - 16$$

$$\frac{8\ln 4 - 4\ln 2 + 4}{\ln 2} - \frac{16\ln 2}{\ln 2}$$

$$\frac{8\ln 4 - 4\ln 2 + 4 - 16\ln 2}{\ln 2}$$

$$\frac{8\ln 4 - 20\ln 2 + 4}{\ln 2}$$

$$\frac{8(\ln 2 + \ln 2) - 20\ln 2 + 4}{\ln 2}$$

$$\frac{8\ln 2 + 8\ln 2 - 20\ln 2 + 4}{\ln 2}$$

$$\frac{4 - 4\ln 2}{\ln 2}$$

■ 2. Evaluate the iterated integral.

$$\int_{-5}^{5}\int_{0}^{\pi}(3x^2 - 4x + 10)\sin(y + \pi)\ dy\ dx$$

Solution:

Integrate with respect to y, treating x as a constant.

$$\int_{-5}^{5} -(3x^2 - 4x + 10)\cos(y + \pi)\Big|_{y=0}^{y=\pi}\ dx$$

$$\int_{-5}^{5} -(3x^2 - 4x + 10)\cos(\pi + \pi) + (3x^2 - 4x + 10)\cos(0 + \pi)\ dx$$

$$\int_{-5}^{5} -(3x^2 - 4x + 10)(1) + (3x^2 - 4x + 10)(-1)\ dx$$

$$\int_{-5}^{5} -3x^2 + 4x - 10 - 3x^2 + 4x - 10\ dx$$

$$\int_{-5}^{5} -6x^2 + 8x - 20\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$-2x^3 + 4x^2 - 20x \Big|_{-5}^{5}$$

$$-2(5)^3 + 4(5)^2 - 20(5) - (-2(-5)^3 + 4(-5)^2 - 20(-5))$$

$$-2(125) + 4(25) - 20(5) + 2(-125) - 4(25) + 20(-5)$$

$$-250 + 100 - 100 - 250 - 100 - 100$$

$$-700$$

3. Evaluate the iterated integral.

$$\int_{-1}^{1}\int_{0}^{2} xe^{x^2-3y+1}\ dx\ dy$$

Solution:

Rewrite the integrand using laws of exponents.

$$\int_{-1}^{1}\int_{0}^{2} xe^{x^2-3y+1}\ dx\ dy$$

$$\int_{-1}^{1}\int_{0}^{2} xe^{x^2}\cdot e^{-3y}\cdot e^{1}\ dx\ dy$$

$$e\int_{-1}^{1}\int_{0}^{2} xe^{x^2}\cdot e^{-3y}\ dx\ dy$$

Since e^{-3y} is a constant for the inner integral,

$$e\int_{-1}^{1} e^{-3y}\int_{0}^{2} xe^{x^2}\ dx\ dy$$

$$e\int_{-1}^{1} e^{-3y}\ dy\cdot\int_{0}^{2} xe^{x^2}\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$-\frac{1}{3}e\cdot e^{-3y}\Big|_{-1}^{1}\cdot\int_{0}^{2} xe^{x^2}\ dx$$

$$-\frac{1}{3}e(e^{-3(1)}-e^{-3(-1)})\cdot\int_{0}^{2} xe^{x^2}\ dx$$

$$-\frac{1}{3}e(e^{-3}-e^{3})\cdot\int_{0}^{2} xe^{x^2}\ dx$$

$$\left(-\frac{1}{3}e^{-2}+\frac{1}{3}e^{4}\right)\int_{0}^{2} xe^{x^2}\ dx$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\int_{0}^{2} xe^{x^2}\ dx$$

Use u-substitution,

$$u = x^2$$

$$dx = \frac{1}{2x}du$$

to rewrite the integrand.

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\int_{x=0}^{x=2} xe^{u}\cdot\frac{1}{2x}du$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\cdot\frac{1}{2}\int_{x=0}^{x=2} e^{u}\ du$$

Integrate, then back-substitute, and evaluate over the interval.

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\cdot\frac{1}{2}e^{u}\Big|_{x=0}^{x=2}$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\cdot\frac{1}{2}e^{x^2}\Big|_{x=0}^{x=2}$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\left(\frac{1}{2}e^{2^2}-\frac{1}{2}e^{0^2}\right)$$

$$\left(\frac{1}{3}e^4-\frac{1}{3e^2}\right)\left(\frac{1}{2}e^4-\frac{1}{2}(1)\right)$$

$$\left(\frac{1}{3}e^4-\frac{1}{3e^2}\right)\left(\frac{1}{2}e^4-\frac{1}{2}\right)$$

$$\left(\frac{e^4}{3}-\frac{e^{-2}}{3}\right)\left(\frac{e^4-1}{2}\right)$$

$$\left(\frac{e^4-e^{-2}}{3}\right)\left(\frac{e^4-1}{2}\right)$$

$$\frac{(e^4-e^{-2})(e^4-1)}{6}$$

DOUBLE INTEGRALS

■ 1. Evaluate the double integral, where R is the rectangle $[0,\pi] \times [0,1]$.

$$\iint_R \cos(x - \pi y)\ dx\ dy$$

Solution:

Since we're integrating over R, we'll add bounds to the integrals.

$$\int_0^1 \int_0^\pi \cos(x - \pi y)\ dx\ dy$$

Integrate with respect to x, treating y as a constant.

$$\int_0^1 \sin(x - \pi y)\Big|_{x=0}^{x=\pi}\ dy$$

$$\int_0^1 \sin(\pi - \pi y) - \sin(0 - \pi y)\ dy$$

$$\int_0^1 \sin(\pi - \pi y) - \sin(-\pi y)\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\frac{1}{\pi}\cos(\pi - \pi y) - \frac{1}{\pi}\cos(-\pi y)\Big|_0^1$$

$$\frac{1}{\pi}\cos(\pi-\pi(1))-\frac{1}{\pi}\cos(-\pi(1))-\left(\frac{1}{\pi}\cos(\pi-\pi(0))-\frac{1}{\pi}\cos(-\pi(0))\right)$$

$$\frac{1}{\pi}\cos(\pi-\pi)-\frac{1}{\pi}\cos(-\pi)-\frac{1}{\pi}\cos(\pi)+\frac{1}{\pi}\cos(0)$$

$$\frac{1}{\pi}(1)-\frac{1}{\pi}(-1)-\frac{1}{\pi}(-1)+\frac{1}{\pi}(1)$$

$$\frac{1}{\pi}+\frac{1}{\pi}+\frac{1}{\pi}+\frac{1}{\pi}$$

$$\frac{4}{\pi}$$

■ 2. Evaluate the double integral, where R is the rectangle $[1,3]\times[1,5]$.

$$\int\int_R \frac{1}{(x+y)^2}\ dx\ dy$$

Solution:

Since we're integrating over R, we'll add bounds to the integrals.

$$\int_1^5\int_1^3 \frac{1}{(x+y)^2}\ dx\ dy$$

Integrate with respect to x, treating y as a constant.

$$\int_1^5 -\frac{1}{x+y}\Big|_{x=1}^{x=3}\ dy$$

$$\int_1^5 -\frac{1}{3+y} - \left(-\frac{1}{1+y}\right)\ dy$$

$$\int_1^5 -\frac{1}{3+y} + \frac{1}{1+y}\ dy$$

$$\int_1^5 \frac{1}{1+y} - \frac{1}{3+y}\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\ln|1+y| - \ln|3+y| \Big|_1^5$$

$$\ln|1+5| - \ln|3+5| - \ln|1+1| + \ln|3+1|$$

$$\ln 6 - \ln 8 - \ln 2 + \ln 4$$

$$\ln\frac{6}{8} - \ln 2 + \ln 4$$

$$\ln\frac{\frac{6}{8}}{2} + \ln 4$$

$$\ln\frac{6}{8\cdot 2} + \ln 4$$

$$\ln\frac{6\cdot 4}{8\cdot 2}$$

$$\ln\frac{3}{2}$$

3. Evaluate the double integral, where R is the rectangle $[x, y] = [-\pi/2, \pi/2] \times [0, \pi]$.

$$\int\int_R \cos(x+y) - x\sin(x+y)\ dx\ dy$$

Solution:

Since we're integrating over R, we'll add bounds to the integrals.

$$\int_0^{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x+y) - x\sin(x+y)\ dx\ dy$$

Integrate with respect to x, treating y as a constant. Use integration by parts on the second integral.

$$\int_0^2 \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x+y)\ dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(x+y)\ dx\right]\ dy$$

$$\int_0^2 \left[\sin(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(x+y)\ dx\right]\ dy$$

$$u = x$$

$$du = dx$$

$$dv = \sin(x+y)\ dx$$

$$v = -\cos(x+y)$$

$$\int_0^2 \left[\sin(x+y) \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \left(-x\cos(x+y) \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\cos(x+y)\ dx \right) \right]\ dy$$

$$\int_0^2 \left[\sin(x+y) \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \left(-x\cos(x+y) + \sin(x+y) \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \right) \right]\ dy$$

$$\int_0^2 \left[\sin(x+y) + x\cos(x+y) - \sin(x+y) \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \right]\ dy$$

$$\int_0^2 \left[x\cos(x+y) \Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \right]\ dy$$

$$\int_0^2 \left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}+y\right) - \left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}+y\right) \right]\ dy$$

$$\int_0^2 \frac{\pi}{2}\cos\left(\frac{\pi}{2}+y\right) + \frac{\pi}{2}\cos\left(-\frac{\pi}{2}+y\right)\ dy$$

$$\frac{\pi}{2}\int_0^2 \cos\left(\frac{\pi}{2}+y\right) + \cos\left(-\frac{\pi}{2}+y\right)\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+y\right) + \sin\left(-\frac{\pi}{2}+y\right)\right)\Big|_0^2$$

$$\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+2\right) + \sin\left(-\frac{\pi}{2}+2\right)\right) - \frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+0\right) + \sin\left(-\frac{\pi}{2}+0\right)\right)$$

$$\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+2\right)+\sin\left(-\frac{\pi}{2}+2\right)\right)-\frac{\pi}{2}(1+(-1))$$

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}+2\right)+\frac{\pi}{2}\sin\left(-\frac{\pi}{2}+2\right)$$

TYPE I AND II REGIONS

1. Evaluate the double integral if D is the circle centered at the origin with radius 4.

$$\iint_D 4x^2y + 3\ dA$$

Solution:

Since the area of integration is symmetric, the region can be treated either as Type I or Type II. Let's do this as a Type I region.

The equation of the circle centered at the origin with radius 4 is

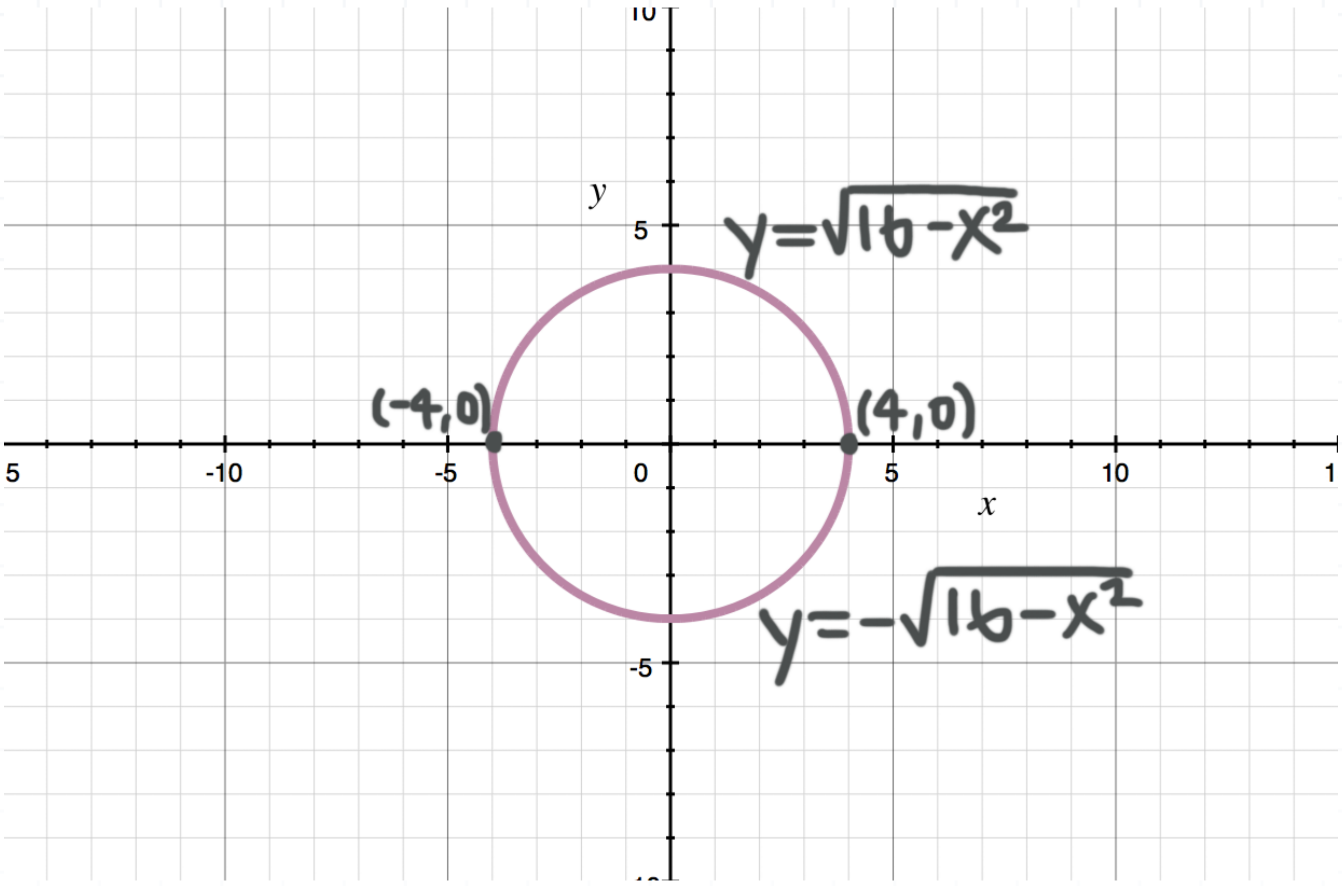
$$x^2 + y^2 = 4^2$$

Solve for y.

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

A sketch of the region is



So for every x from -4 to 4, y changes from $-\sqrt{16-x^2}$ to $\sqrt{16-x^2}$. Therefore, the given integral is equivalent to

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 4x^2y+3\ dy\ dx$$

Integrate with respect to y.

$$\int_{-4}^{4} 2x^2y^2+3y\Big|_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}\ dx$$

$$\int_{-4}^{4} 2x^2(\sqrt{16-x^2})^2+3(\sqrt{16-x^2})-(2x^2(-\sqrt{16-x^2})^2+3(-\sqrt{16-x^2}))\ dx$$

$$\int_{-4}^{4} 2x^2(16-x^2)+3\sqrt{16-x^2}-(2x^2(16-x^2)-3\sqrt{16-x^2})\ dx$$

$$\int_{-4}^{4} 32x^2 - 2x^4 + 3\sqrt{16-x^2} - 32x^2 + 2x^4 + 3\sqrt{16-x^2}\ dx$$

$$\int_{-4}^{4} 6\sqrt{16-x^2}\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$4(16-x^2)^{\frac{3}{2}}\Big|_{-4}^{4}$$

$$4(16-4^2)^{\frac{3}{2}} - 4(16-(-4)^2)^{\frac{3}{2}}$$

$$4(16-16)^{\frac{3}{2}} - 4(16-16)^{\frac{3}{2}}$$

$$0$$

■ 2. Evaluate the double integral if D is the region bounded by the curves $y + x^2 - 4 = 0$ and $y + 2x^2 - 8 = 0$.

$$\iint_D 462y\sqrt{x+2}\ dA$$

Solution:

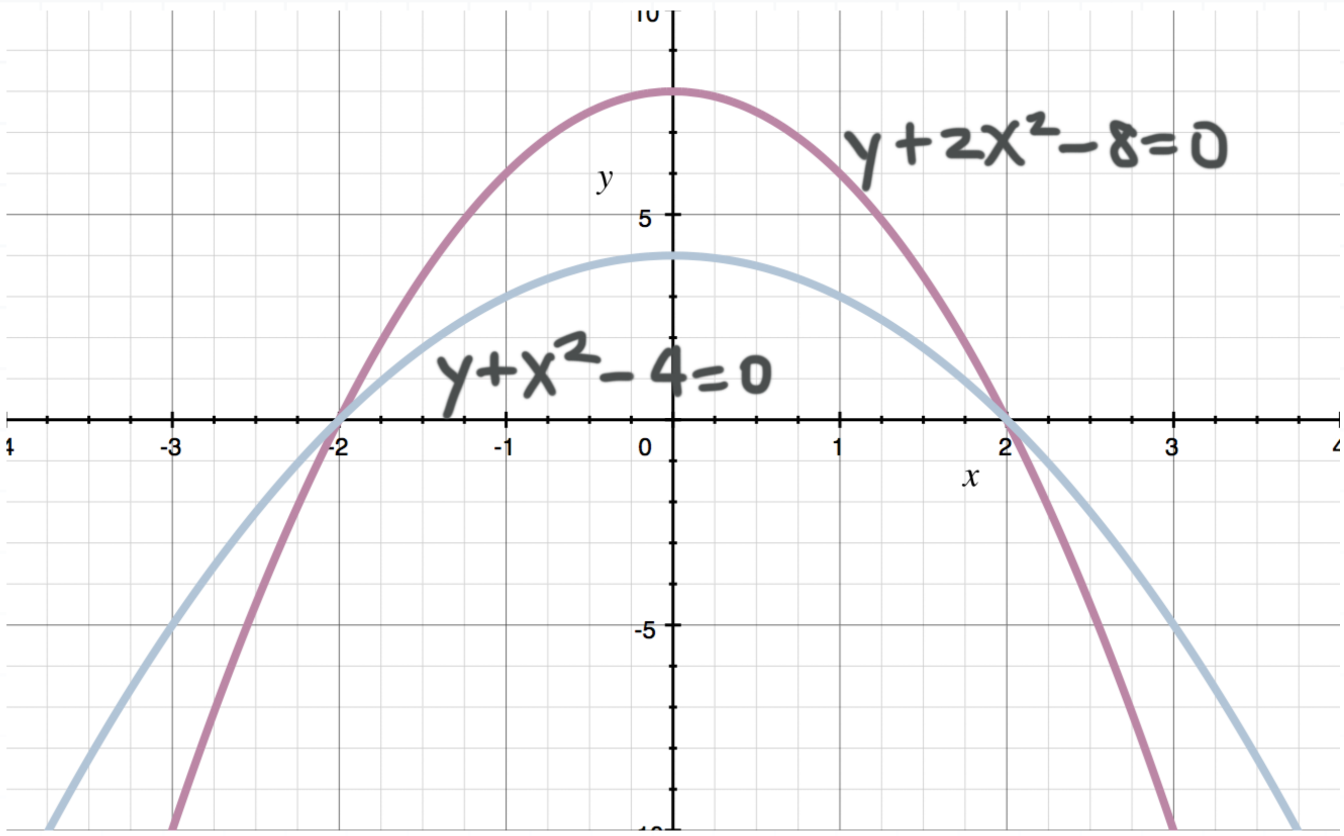
Find the points of intersection of $y = -x^2 + 4$ and $y = -2x^2 + 8$.

$$-x^2 + 4 = -2x^2 + 8$$

$$x^2 = 4$$

$$x = \pm 2$$

A sketch of the curves is



So we have a Type I region, and for every x from -2 to 2, y changes from $-x^2+4$ to $-2x^2+8$. Therefore, the given integral is equivalent to

$$\int_{-2}^{2}\int_{-x^2+4}^{-2x^2+8} 462y\sqrt{x+2}\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$\int_{-2}^{2} 231y^2\sqrt{x+2}\Big|_{y=-x^2+4}^{y=-2x^2+8}\ dx$$

$$\int_{-2}^{2} 231(-2x^2+8)^2\sqrt{x+2} - 231(-x^2+4)^2\sqrt{x+2}\ dx$$

$$231\int_{-2}^{2}(4x^4-32x^2+64)\sqrt{x+2}-(x^4-8x^2+16)\sqrt{x+2}\ dx$$

$$231\int_{-2}^{2}4x^4\sqrt{x+2}-32x^2\sqrt{x+2}+64\sqrt{x+2}$$

$$-x^4\sqrt{x+2}+8x^2\sqrt{x+2}-16\sqrt{x+2}\ dx$$

$$231\int_{-2}^{2}3x^4\sqrt{x+2}-24x^2\sqrt{x+2}+48\sqrt{x+2}\ dx$$

$$693\int_{-2}^{2}x^4\sqrt{x+2}-8x^2\sqrt{x+2}+16\sqrt{x+2}\ dx$$

Substitute $u=x+2$, $x=u-2$, and $du=dx$.

$$693\int_{x=-2}^{x=2}(u-2)^4\sqrt{u}-8(u-2)^2\sqrt{u}+16\sqrt{u}\ du$$

$$693\int_{x=-2}^{x=2}(u^4-8u^3+24u^2-32u+16)\sqrt{u}$$

$$-(8u^2-32u+32)\sqrt{u}+16\sqrt{u}\ du$$

$$693\int_{x=-2}^{x=2}u^{\frac{9}{2}}-8u^{\frac{7}{2}}+24u^{\frac{5}{2}}-8u^{\frac{5}{2}}-32u^{\frac{3}{2}}+32u^{\frac{3}{2}}+16u^{\frac{1}{2}}-32u^{\frac{1}{2}}+16u^{\frac{1}{2}}\ du$$

$$693\int_{x=-2}^{x=2}u^{\frac{9}{2}}-8u^{\frac{7}{2}}+16u^{\frac{5}{2}}\ du$$

Integrate with respect to x, back-substitute, then evaluate over the interval.

$$693\left(\frac{2}{11}u^{\frac{11}{2}}-\frac{16}{9}u^{\frac{9}{2}}+\frac{32}{7}u^{\frac{7}{2}}\right)\Bigg|_{x=-2}^{x=2}$$

$$693\left(\frac{2}{11}(x+2)^{\frac{11}{2}}-\frac{16}{9}(x+2)^{\frac{9}{2}}+\frac{32}{7}(x+2)^{\frac{7}{2}}\right)\Bigg|_{-2}^{2}$$

$$693\left(\frac{2}{11}(2+2)^{\frac{11}{2}}-\frac{16}{9}(2+2)^{\frac{9}{2}}+\frac{32}{7}(2+2)^{\frac{7}{2}}\right)$$

$$-693\left(\frac{2}{11}(-2+2)^{\frac{11}{2}}-\frac{16}{9}(-2+2)^{\frac{9}{2}}+\frac{32}{7}(-2+2)^{\frac{7}{2}}\right)$$

$$693\left(\frac{2}{11}(4)^{\frac{11}{2}}-\frac{16}{9}(4)^{\frac{9}{2}}+\frac{32}{7}(4)^{\frac{7}{2}}\right)$$

$$693\left(\frac{2}{11}2^{11}-\frac{16}{9}2^{9}+\frac{32}{7}2^{7}\right)$$

$$693\left(\frac{2^{12}}{11}-\frac{2^{13}}{9}+\frac{2^{12}}{7}\right)$$

$$693(2^{12})\left(\frac{1}{11}-\frac{2}{9}+\frac{1}{7}\right)$$

Find a common denominator.

$$693(2^{12})\left(\frac{63}{693}-\frac{154}{693}+\frac{99}{693}\right)$$

$$2^{12}(63-154+99)$$

$$2^{12}(8)$$

$$2^{15}$$

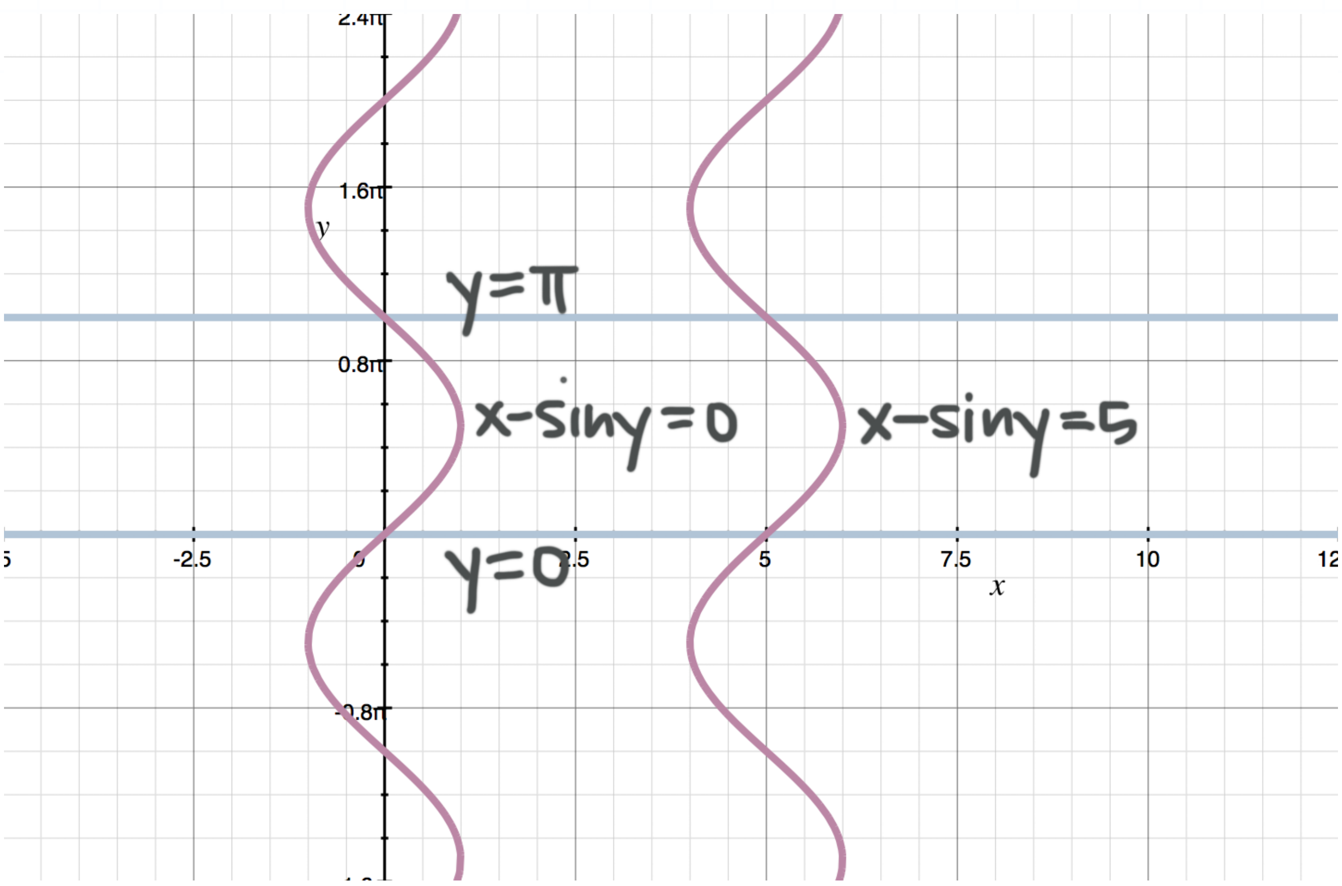
$$32{,}768$$

■ 3. Evaluate the double integral if D is the region bounded by $x - \sin y = 0$, $x - \sin y = 5$, $y = 0$, and $y = \pi$.

$$\iint_D 2x\ dA$$

Solution:

A sketch of the region is



We have a Type II region, and for every y from 0 to π, x changes from $\sin y$ to $\sin y + 5$. Therefore, the given integral is equivalent to

$$\int_0^{\pi} \int_{\sin y}^{\sin y + 5} 2x \ dx \ dy$$

Integrate with respect to x, then evaluate over the interval.

$$\int_0^{\pi} x^2 \Big|_{x=\sin y}^{x=\sin y + 5} \ dy$$

$$\int_0^{\pi} (\sin y + 5)^2 - (\sin y)^2 \ dy$$

$$\int_0^{\pi} \sin^2 y + 10 \sin y + 25 - \sin^2 y \ dy$$

$$\int_0^{\pi} 10 \sin y + 25 \ dy$$

Integrate with respect to y, then evaluate over the interval.

$$25y - 10 \cos y \Big|_0^{\pi}$$

$$25\pi - 10 \cos \pi - (25(0) - 10 \cos 0)$$

$$25\pi - 10 \cos \pi + 10 \cos 0$$

$$25\pi - 10(-1) + 10(1)$$

$$25\pi + 10 + 10$$

$25\pi + 20$

FINDING SURFACE AREA

■ 1. Find area of the surface $z=\sqrt{3}x+y^2+1$ inside the rectangle $-1\le x\le 1$, $0\le y\le 1$.

Solution:

The partial derivatives of z are

$$z_x=\sqrt{3}$$

$$z_y=2y$$

Then the area of the surface is

$$A=\iint_D \sqrt{1+z_x^2+z_y^2}\ dA$$

$$A=\int_{-1}^{1}\int_{0}^{1}\sqrt{1+z_x^2+z_y^2}\ dy\ dx$$

$$A=\int_{-1}^{1}\int_{0}^{1}\sqrt{1+3+4y^2}\ dy\ dx$$

$$A=2\int_{-1}^{1}\int_{0}^{1}\sqrt{1+y^2}\ dy\ dx$$

Use a substitution with $y=\tan u$ and $dy=\sec^2 u\ du$. The bounds for y change from $[0,1]$ to $[0,\pi/4]$.

$$A=2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sqrt{1+\tan^2 u}\sec^2 u\ du\ dx$$

$$A=2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sqrt{\sec^2 u}\sec^2 u\ du\ dx$$

$$A=2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sec u\sec^2 u\ du\ dx$$

$$A=2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sec^3 u\ du\ dx$$

The integral of $\sec^3 x$ can be found with the reduction formula

$$\int \sec^3 x\ dx=\frac{1}{2}\tan x\sec x+\frac{1}{2}\int \sec x\ dx$$

Substitute from the reduction formula into the double integral.

$$A=2\int_{-1}^{1}\left[\frac{1}{2}\tan u\sec u\Big|_0^{\frac{\pi}{4}}+\frac{1}{2}\int_0^{\frac{\pi}{4}}\sec u\ du\right]\ dx$$

$$A=2\int_{-1}^{1}\left[\frac{1}{2}\tan u\sec u\Big|_0^{\frac{\pi}{4}}+\frac{1}{2}\ln|\sec u+\tan u|\Big|_0^{\frac{\pi}{4}}\right]\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{2}\tan u\sec u+\frac{1}{2}\ln|\sec u+\tan u|\Big|_0^{\frac{\pi}{4}}\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{2}\tan\frac{\pi}{4}\sec\frac{\pi}{4}+\frac{1}{2}\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|$$

$$-\left(\frac{1}{2}\tan(0)\sec(0)+\frac{1}{2}\ln|\sec(0)+\tan(0)|\right)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{2}(1)\frac{2}{\sqrt{2}}+\frac{1}{2}\ln\left|\frac{2}{\sqrt{2}}+1\right|-\left(\frac{1}{2}(0)(1)+\frac{1}{2}\ln|1+0|\right)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}+\frac{1}{2}\ln\left(\frac{2}{\sqrt{2}}+1\right)-\frac{1}{2}\ln 1\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}+\frac{1}{2}\ln\left(\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}}+1\right)-\frac{1}{2}(0)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}+\frac{1}{2}\ln(\sqrt{2}+1)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)+\frac{1}{2}\ln(\sqrt{2}+1)\ dx$$

$$A=2\int_{-1}^{1}\frac{\sqrt{2}}{2}+\frac{1}{2}\ln(\sqrt{2}+1)\ dx$$

$$A=\int_{-1}^{1}\sqrt{2}+\ln(\sqrt{2}+1)\ dx$$

The integrand is now a constant (it doesn't include any x variables), so the integral is

$$A=(\sqrt{2}+\ln(\sqrt{2}+1))x\Big|_{-1}^{1}$$

$$A = (\sqrt{2} + \ln(\sqrt{2} + 1))(1) - (\sqrt{2} + \ln(\sqrt{2} + 1))(-1)$$

$$A = (\sqrt{2} + \ln(\sqrt{2} + 1)) + (\sqrt{2} + \ln(\sqrt{2} + 1))$$

$$A = 2(\sqrt{2} + \ln(\sqrt{2} + 1))$$

$$A = 2\sqrt{2} + 2\ln(\sqrt{2} + 1)$$

■ 2. Find area of the surface $z = \ln(\sin(3x)) + 2\sqrt{2}y - 5$ inside the rectangle $\pi/6 \leq x \leq \pi/4$, $0 \leq y \leq 1$.

Solution:

The partial derivatives of z are

$$z_x = \frac{1}{\sin(3x)}(\cos(3x))(3)$$

$$z_x = 3\cot(3x)$$

and

$$z_y = 2\sqrt{2}$$

Then the area of the surface is

$$A = \iint_D \sqrt{1 + z_x^2 + z_y^2}\ dA$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}\sqrt{1+z_x^2+z_y^2}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}\sqrt{1+(3\cot(3x))^2+(2\sqrt{2})^2}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}\sqrt{1+9\cot^2(3x)+8}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}3\sqrt{1+\cot^2(3x)}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}3\sqrt{\csc^2(3x)}\ dx\ dy$$

Since $\csc(3x)\geq 0$ inside the given region,

$$A=\int_0^1\int_{\pi/6}^{\pi/4}3\csc(3x)\ dx\ dy$$

Integrate with respect to x.

$$A=\int_0^1 -3\left(\frac{1}{3}\right)\ln|\csc(3x)+\cot(3x)|\Bigg|_{x=\pi/6}^{x=\pi/4}\ dy$$

$$A=\int_0^1 -\ln|\csc(3x)+\cot(3x)|\Bigg|_{x=\pi/6}^{x=\pi/4}\ dy$$

$$A=\int_0^1 -\ln\left|\csc\left(3\cdot\frac{\pi}{4}\right)+\cot\left(3\cdot\frac{\pi}{4}\right)\right|-\left(-\ln\left|\csc\left(3\cdot\frac{\pi}{6}\right)+\cot\left(3\cdot\frac{\pi}{6}\right)\right|\right)\ dy$$

$$A = \int_0^1 -\ln\left|\csc\frac{3\pi}{4}+\cot\frac{3\pi}{4}\right| + \ln\left|\csc\frac{\pi}{2}+\cot\frac{\pi}{2}\right|\ dy$$

$$A = \int_0^1 -\ln\left|\frac{2}{\sqrt{2}}+(-1)\right| + \ln|1+0|\ dy$$

$$A = \int_0^1 -\ln\left|\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}}-1\right| + 0\ dy$$

$$A = \int_0^1 -\ln(\sqrt{2}-1)\ dy$$

The integrand is now a constant (it doesn't include any y variables), so the integral is

$$A = (-\ln(\sqrt{2}-1))y\Big|_0^1$$

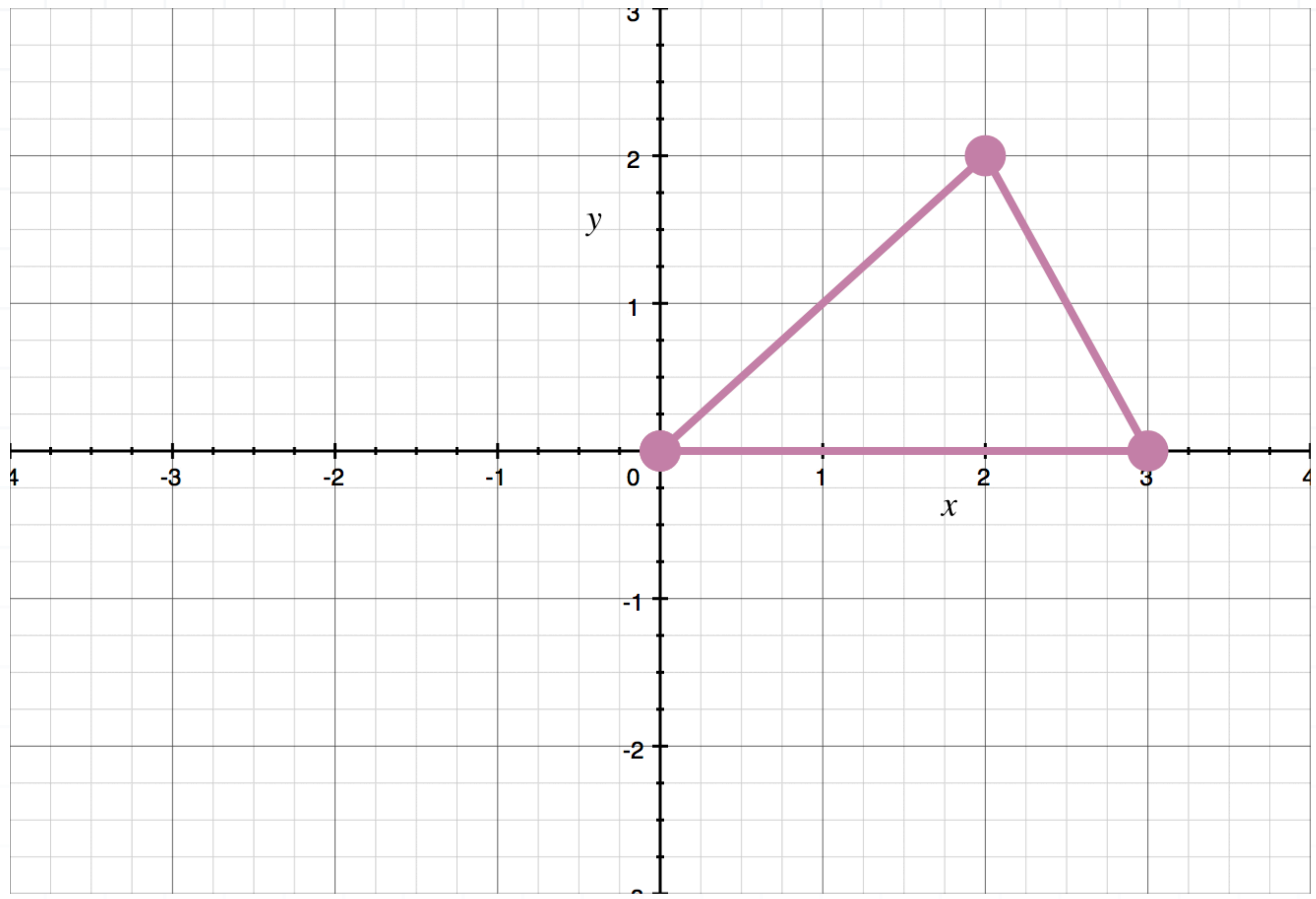
$$A = (-\ln(\sqrt{2}-1))(1) - ((-\ln(\sqrt{2}-1))(0))$$

$$A = -\ln(\sqrt{2}-1)$$

■ 3. Find area of the surface $z = 2(x+3)^{3/2} + 5^{3/2}y - 6$ inside the triangle OAB, if O is the origin and A and B are at $A(3,0)$ and $B(2,2)$.

Solution:

Sketch the triangle.



Based on the sketch, treat the triangle as a Type II region. The equation of line OB is $y = x$, and the equation of line AB is $2x + y = 6$, or

$$x = -\frac{y}{2} + 3$$

Find the partial derivatives of $z = 2(x+3)^{3/2} + 5^{3/2}y - 6$.

$$z_x = 2 \cdot \frac{3}{2}(x+3)^{1/2}$$

$$z_x = 3\sqrt{x+3}$$

and

$$z_y = 5^{3/2}$$

Then the area of the surface is

$$A=\iint_{OAB}\sqrt{1+z_x^2+z_y^2}\ dA$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+z_x^2+z_y^2}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+(3\sqrt{x+3})^2+(5^{3/2})^2}\ dx\ dy$$

since $x+3\geq 0$ inside the triangle OAB,

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+9(x+3)+125}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+9x+27+125}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{9x+153}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}3\sqrt{x+17}\ dx\ dy$$

Integrate with respect to x.

$$A=\int_0^2 3\cdot\frac{2}{3}(x+17)^{\frac{3}{2}}\Big|_{x=y}^{x=-\frac{y}{2}+3}\ dy$$

$$A=\int_0^2 2(x+17)^{\frac{3}{2}}\Big|_{x=y}^{x=-\frac{y}{2}+3}\ dy$$

$$A=\int_0^2 2\left(-\frac{y}{2}+3+17\right)^{\frac{3}{2}}-2(y+17)^{\frac{3}{2}}\ dy$$

$$A=\int_0^2 2\left(-\frac{y}{2}+20\right)^{\frac{3}{2}}-2(y+17)^{\frac{3}{2}}\ dy$$

Integrate with respect to y.

$$A=2\cdot\frac{2}{5}(-2)\left(-\frac{y}{2}+20\right)^{\frac{5}{2}}-2\cdot\frac{2}{5}(y+17)^{\frac{5}{2}}\Big|_0^2$$

$$A=-\frac{8}{5}\left(-\frac{y}{2}+20\right)^{\frac{5}{2}}-\frac{4}{5}(y+17)^{\frac{5}{2}}\Big|_0^2$$

$$A=-\frac{8}{5}\left(-\frac{2}{2}+20\right)^{\frac{5}{2}}-\frac{4}{5}(2+17)^{\frac{5}{2}}-\left[-\frac{8}{5}\left(-\frac{0}{2}+20\right)^{\frac{5}{2}}-\frac{4}{5}(0+17)^{\frac{5}{2}}\right]$$

$$A=-\frac{8}{5}(-1+20)^{\frac{5}{2}}-\frac{4}{5}(19)^{\frac{5}{2}}-\left[-\frac{8}{5}(0+20)^{\frac{5}{2}}-\frac{4}{5}(17)^{\frac{5}{2}}\right]$$

$$A=-\frac{8}{5}(19)^{\frac{5}{2}}-\frac{4}{5}(19)^{\frac{5}{2}}-\left[-\frac{8}{5}(20)^{\frac{5}{2}}-\frac{4}{5}(17)^{\frac{5}{2}}\right]$$

$$A=-\frac{12}{5}(19)^{\frac{5}{2}}+\frac{8}{5}(20)^{\frac{5}{2}}+\frac{4}{5}(17)^{\frac{5}{2}}$$

$$A\approx 38.9$$

FINDING VOLUME

■ 1. Use a double integral to find the volume of the solid that's bounded by the surface and the xy-plane, on $0 \le x \le 2$ and $0 \le y \le \pi/2$.

$$z = \frac{\sin(2y)}{(x+1)^2}$$

Solution:

Treating the region as Type I, the volume is

$$V = \int_0^2 \int_0^{\frac{\pi}{2}} \frac{\sin(2y)}{(x+1)^2}\ dy\ dx$$

Integrate with respect to y by treating x as a constant.

$$V = \int_0^2 \frac{1}{(x+1)^2} \int_0^{\frac{\pi}{2}} \sin(2y)\ dy\ dx$$

$$V = \int_0^2 \frac{1}{(x+1)^2} \left[-\frac{1}{2}\cos(2y) \Big|_{y=0}^{y=\frac{\pi}{2}} \right]\ dx$$

$$V = \int_0^2 \frac{1}{(x+1)^2} \left[-\frac{1}{2}\cos\left(2 \cdot \frac{\pi}{2}\right) - \left(-\frac{1}{2}\cos(2 \cdot 0) \right) \right]\ dx$$

$$V = \int_0^2 \frac{1}{(x+1)^2} \left[-\frac{1}{2}\cos\pi + \frac{1}{2}\cos(0) \right]\ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}\left(-\frac{1}{2}(-1)+\frac{1}{2}(1)\right)\ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}\left(\frac{1}{2}+\frac{1}{2}\right)\ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}(1)\ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}\ dx$$

Integrate with respect to x.

$$V=\int_0^2 (x+1)^{-2}\ dx$$

$$V=-(x+1)^{-1}\Big|_0^2$$

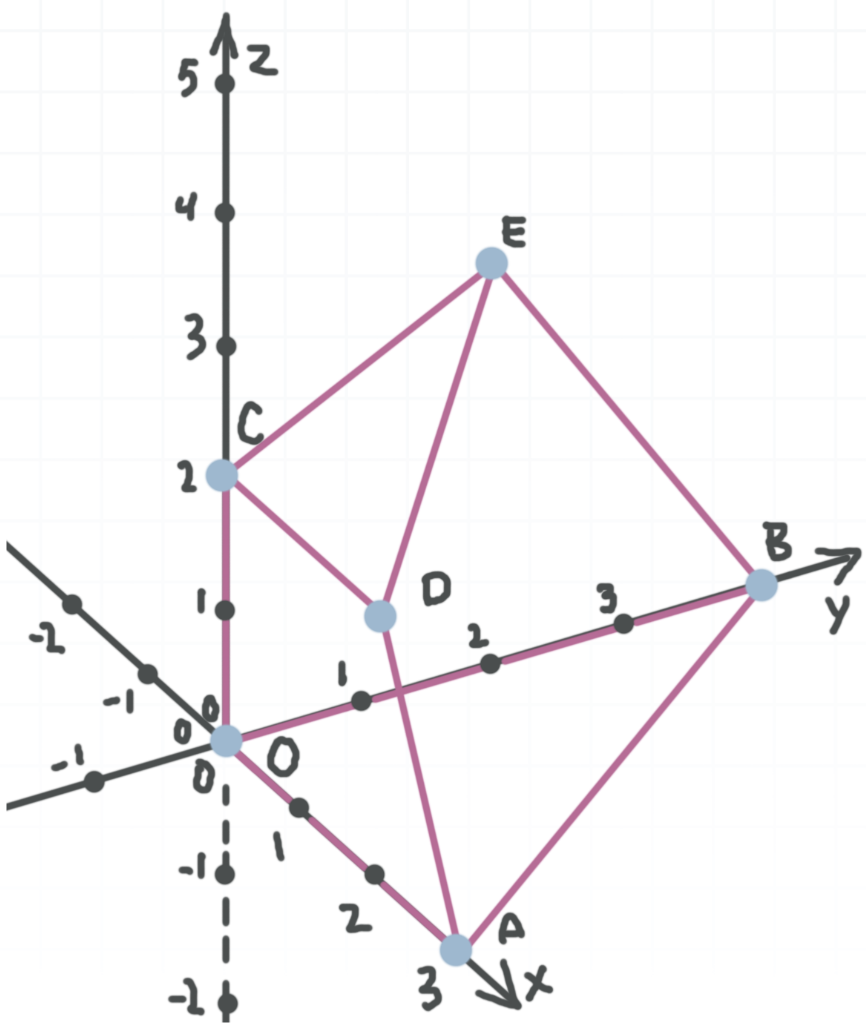
$$V=-\frac{1}{x+1}\Big|_0^2$$

$$V=-\frac{1}{2+1}-\left(-\frac{1}{0+1}\right)$$

$$V=-\frac{1}{3}+\frac{1}{1}$$

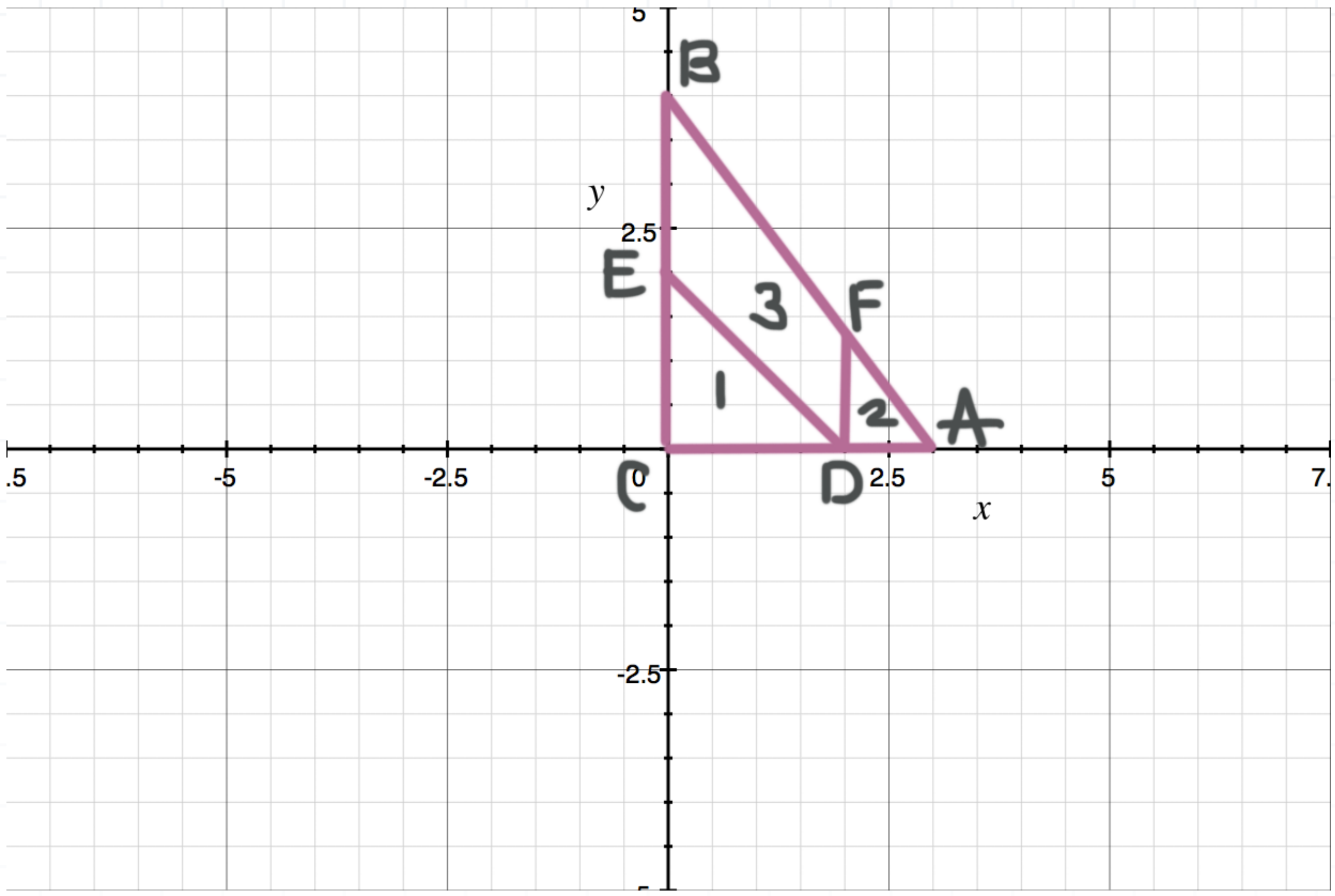
$$V=\frac{2}{3}$$

■ 2. Use a double integral to find the volume of the irregular hexagon $OABCDE$, where O is the origin, and the hexagon's other vertices are $A(3,0,0)$, $B(0,4,0)$, $C(0,0,2)$, $D(2,0,2)$, and $E(0,2,3)$.



Solution:

The hexagon's projection onto the xy-plane is



Let's add a helper point at $F(2,4/3,0)$. Then the hexagon's volume can be calculated as the sum of three volumes: below the triangle $CED(1)$, below the triangle $ADF(2)$, and below the quadrilateral $BEDF(3)$. We can treat all of them as Type I regions.

(1) The equation of the plane CDE that passes through $C(0,0,2)$, $D(2,0,2)$, and $E(0,2,3)$ is

$$-y + 2z - 4 = 0$$

$$2z = 4 + y$$

$$z = 2 + 0.5y$$

The equation of the line that's the projection of ED is $x + y = 2$. So when x is changing from 0 to 2, y is changing from 0 to $2 - x$. Therefore, the volume is

$$V_{CDE} = \int_0^2 \int_0^{2-x} 2 + 0.5x \ dy \ dx$$

Integrate with respect to y, while treating x as a constant.

$$V_{CDE} = \int_0^2 2y + 0.5xy \Big|_{y=0}^{y=2-x} \ dx$$

$$V_{CDE} = \int_0^2 2(2-x) + 0.5x(2-x) - (2(0) + 0.5x(0)) \ dx$$

$$V_{CDE} = \int_0^2 (4-2x) + (x - 0.5x^2) - (0+0) \ dx$$

$$V_{CDE} = \int_0^2 4 - 2x + x - 0.5x^2 \ dx$$

$$V_{CDE} = \int_0^2 4 - x - 0.5x^2 \ dx$$

Integrate with respect to x.

$$V_{CDE} = 4x - 0.5x^2 - \frac{0.5}{3}x^3 \Big|_0^2$$

$$V_{CDE} = 4(2) - 0.5(2)^2 - \frac{0.5}{3}(2)^3 - \left(4(0) - 0.5(0)^2 - \frac{0.5}{3}(0)^3\right)$$

$$V_{CDE} = 4(2) - 0.5(4) - \frac{0.5}{3}(8)$$

$$V_{CDE} = 8 - 2 - \frac{4}{3}$$

$$V_{CDE} = \frac{14}{3}$$

(2) The equation of the plane $ABED$ that passes through $A(3,0,0)$, $B(0,4,0)$, $E(0,2,3)$, and $D(2,0,2)$ is

$$4x + 3y + 2z - 12 = 0$$

$$z = -2x - 1.5y + 6$$

The equation of the line AB is

$$4x + 3y = 12$$

$$3y = -4x + 12$$

$$y = -\frac{4}{3}x + 4$$

So when x is changing from 2 to 3, y is changing from 0 to $(-4/3)x + 4$. Therefore, the volume is

$$V_{AB} = \int_2^3 \int_0^{-\frac{4}{3}x+4} -2x - 1.5y + 6 \ dy \ dx$$

Integrate with respect to y by treating x as a constant.

$$V_{AB} = \int_2^3 -2xy - 0.75y^2 + 6y \Big|_{y=0}^{y=-\frac{4}{3}x+4} \ dx$$

$$V_{AB} = \int_2^3 -2x\left(-\frac{4}{3}x + 4\right) - 0.75\left(-\frac{4}{3}x + 4\right)^2 + 6\left(-\frac{4}{3}x + 4\right)$$

$$-(-2x(0) - 0.75(0)^2 + 6(0))\ dx$$

$$V_{AB} = \int_2^3 \frac{8}{3}x^2 - 8x - \frac{3}{4}\left(\frac{16}{9}x^2 - \frac{32}{3}x + 16\right) - 8x + 24\ dx$$

$$V_{AB} = \int_2^3 \frac{8}{3}x^2 - 8x - \frac{4}{3}x^2 + 8x - 12 - 8x + 24\ dx$$

$$V_{AB} = \int_2^3 \frac{4}{3}x^2 - 8x + 12\ dx$$

Integrate with respect to x.

$$V_{AB} = \frac{4}{9}x^3 - 4x^2 + 12x\Big|_2^3$$

$$V_{AB} = \frac{4}{9}(3)^3 - 4(3)^2 + 12(3) - \left(\frac{4}{9}(2)^3 - 4(2)^2 + 12(2)\right)$$

$$V_{AB} = 12 - 36 + 36 - \frac{32}{9} + 16 - 24$$

$$V_{AB} = \frac{4}{9}$$

(3) The equation of the plane $ABED$ that passes through $A(3,0,0)$, $B(0,4,0)$, $E(0,2,3)$, and $D(2,0,2)$ is

$$z = -2x - 1.5y + 6$$

The equation of the line AB is

$$y = -\frac{4}{3}x + 4$$

The equation of the line that's the projection of ED is

$$y = 2 - x$$

So when x is changing from 0 to 2, y is changing from $2 - x$ to $(-4/3)x + 4$. Therefore, the volume is

$$V_{ED} = \int_0^2 \int_{2-x}^{-\frac{4}{3}x+4} -2x - 1.5y + 6\ dy\ dx$$

Integrate with respect to y by treating x as a constant.

$$V_{ED} = \int_0^2 -2xy - 0.75y^2 + 6y \Big|_{y=2-x}^{y=-\frac{4}{3}x+4}\ dx$$

$$V_{ED} = \int_0^2 -2x\left(-\frac{4}{3}x + 4\right) - 0.75\left(-\frac{4}{3}x + 4\right)^2 + 6\left(-\frac{4}{3}x + 4\right)$$

$$-(-2x(2 - x) - 0.75(2 - x)^2 + 6(2 - x))\ dx$$

$$V_{ED} = \int_0^2 \frac{8}{3}x^2 - 8x - \frac{3}{4}\left(\frac{16}{9}x^2 - \frac{32}{3}x + 16\right) - 8x + 24$$

$$-\left(-4x + 2x^2 - 3 + 3x - \frac{3}{4}x^2 + 12 - 6x\right)\ dx$$

$$V_{ED} = \int_0^2 \frac{8}{3}x^2 - 8x - \frac{4}{3}x^2 + 8x - 12 - 8x + 24$$

$$+4x - 2x^2 + 3 - 3x + \frac{3}{4}x^2 - 12 + 6x\ dx$$

$$V_{ED} = \int_0^2 \frac{8}{3}x^2 - \frac{4}{3}x^2 - 2x^2 + \frac{3}{4}x^2$$

$$-8x + 8x - 8x + 4x - 3x + 6x - 12 + 24 + 3 - 12\ dx$$

$$V_{ED} = \int_0^2 \frac{1}{12}x^2 - x + 3\ dx$$

Integrate with respect to x.

$$V_{ED} = \frac{1}{36}x^3 - \frac{1}{2}x^2 + 3x\Big|_0^2$$

$$V_{ED} = \frac{1}{36}(2)^3 - \frac{1}{2}(2)^2 + 3(2) - \left(\frac{1}{36}(0)^3 - \frac{1}{2}(0)^2 + 3(0)\right)$$

$$V_{ED} = \frac{2}{9} - 2 + 6$$

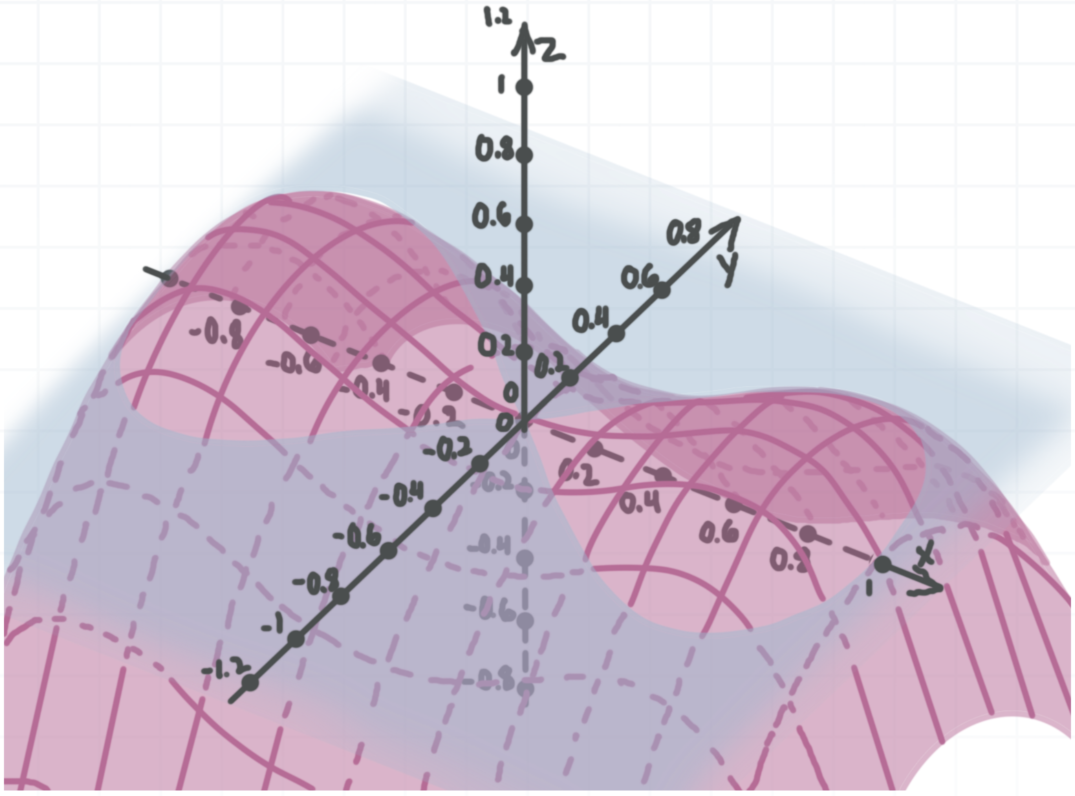
$$V_{ED} = \frac{38}{9}$$

Adding the volumes we found in (1), (2), and (3) gives

$$\frac{14}{3} + \frac{4}{9} + \frac{38}{9}$$

$$\frac{28}{3}$$

■ 3. Use a double integral to find the volume of the solid that's bounded by the surface $z = -x^4 + x^2 - y^2$ and the xy-plane.



Solution:

Since the solid is symmetric, we can calculate its volume within the first quadrant, then multiply the result by 4, to get the total volume from all four quadrants.

We'll consider the intersection of the curve and the xy-plane.

$$-x^4 + x^2 - y^2 = 0$$

$$y^2 = x^2 - x^4$$

The region in the first quadrant is bounded by

$$y = \sqrt{x^2 - x^4} = x\sqrt{1 - x^2}$$

We'll treat the region as Type I, so when x is changing from 0 to 1, y is changing from 0 to $x\sqrt{1 - x^2}$. Therefore, the volume is

$$V=\int_0^1\int_0^{x\sqrt{1-x^2}}-x^4+x^2-y^2\ dy\ dx$$

Integrate with respect to y by treating x as a constant.

$$V=\int_0^1 -x^4y+x^2y-\frac{1}{3}y^3\Big|_{y=0}^{y=x\sqrt{1-x^2}}\ dx$$

$$V=\int_0^1 -x^4(x\sqrt{1-x^2})+x^2(x\sqrt{1-x^2})-\frac{1}{3}(x\sqrt{1-x^2})^3$$

$$-\left(-x^4(0)+x^2(0)-\frac{1}{3}(0)^3\right)\ dx$$

$$V=\int_0^1 -x^5(1-x^2)^{\frac{1}{2}}+x^3(1-x^2)^{\frac{1}{2}}-\frac{1}{3}x^3(1-x^2)^{\frac{3}{2}}\ dx$$

$$V=\int_0^1 (1-x^2)^{\frac{1}{2}}\left(-x^5+x^3-\frac{1}{3}x^3(1-x^2)\right)\ dx$$

$$V=\int_0^1 (1-x^2)^{\frac{1}{2}}\left(-x^5+x^3-\frac{1}{3}x^3+\frac{1}{3}x^5\right)\ dx$$

$$V=\int_0^1 (1-x^2)^{\frac{1}{2}}\left(-\frac{2}{3}x^5+\frac{2}{3}x^3\right)\ dx$$

$$V=\int_0^1 \frac{2}{3}x^3(1-x^2)^{\frac{1}{2}}(1-x^2)\ dx$$

$$V=\int_0^1 \frac{2}{3}x^3(1-x^2)^{\frac{3}{2}}\ dx$$

Use substitution to integrate with respect to x, setting

$$u = 1 - x^2$$

$x^2 = 1 - u$ and $x = (1 - u)^{\frac{1}{2}}$

$\frac{du}{dx} = -2x$, so $du = -2x\ dx$ and $dx = -\frac{du}{2x}$

The bounds on x of $[0,1]$ becomes bounds on u of $[1,0]$.

$$V = \int_1^0 \frac{2}{3}((1-u)^{\frac{1}{2}})^3 u^{\frac{3}{2}} \left(-\frac{du}{2x}\right)$$

$$V = -\frac{1}{3}\int_1^0 \frac{(1-u)^{\frac{3}{2}}}{x} u^{\frac{3}{2}}\ du$$

$$V = -\frac{1}{3}\int_1^0 \frac{(1-u)^{\frac{3}{2}}}{(1-u)^{\frac{1}{2}}} u^{\frac{3}{2}}\ du$$

$$V = -\frac{1}{3}\int_1^0 u^{\frac{3}{2}}(1-u)\ du$$

$$V = -\frac{1}{3}\int_1^0 u^{\frac{3}{2}} - u^{\frac{5}{2}}\ du$$

$$V = -\frac{1}{3}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{7}u^{\frac{7}{2}}\right)\Bigg|_1^0$$

$$V = -\frac{1}{3}\left(\frac{2}{5}(0)^{\frac{5}{2}} - \frac{2}{7}(0)^{\frac{7}{2}}\right) + \frac{1}{3}\left(\frac{2}{5}(1)^{\frac{5}{2}} - \frac{2}{7}(1)^{\frac{7}{2}}\right)$$

$$V = \frac{1}{3}\left(\frac{2}{5} - \frac{2}{7}\right)$$

$$V = \frac{1}{3}\left(\frac{4}{35}\right)$$

$$V = \frac{4}{105}$$

This is the volume in the first quadrant. Multiply by 4 to get the volume in all four quadrants.

$$V = 4 \cdot \frac{4}{105}$$

$$V = \frac{16}{105}$$

CHANGING THE ORDER OF INTEGRATION

■ 1. Change the order of integration of the iterated integral.

$$\int_{-3}^{0}\int_{-\frac{2}{3}\sqrt{9-x^2}}^{\frac{2}{3}\sqrt{9-x^2}} z(x,y)\ dy\ dx$$

Solution:

The double integral represents a Type I region where x is changing from -3 to 0. The bounds on y are

$$y=-\frac{2}{3}\sqrt{9-x^2}$$

$$y=\frac{2}{3}\sqrt{9-x^2}$$

If we square both sides of either equation, we get

$$y^2=\frac{4}{9}(9-x^2)$$

$$9y^2=36-4x^2$$

$$9y^2+4x^2=36$$

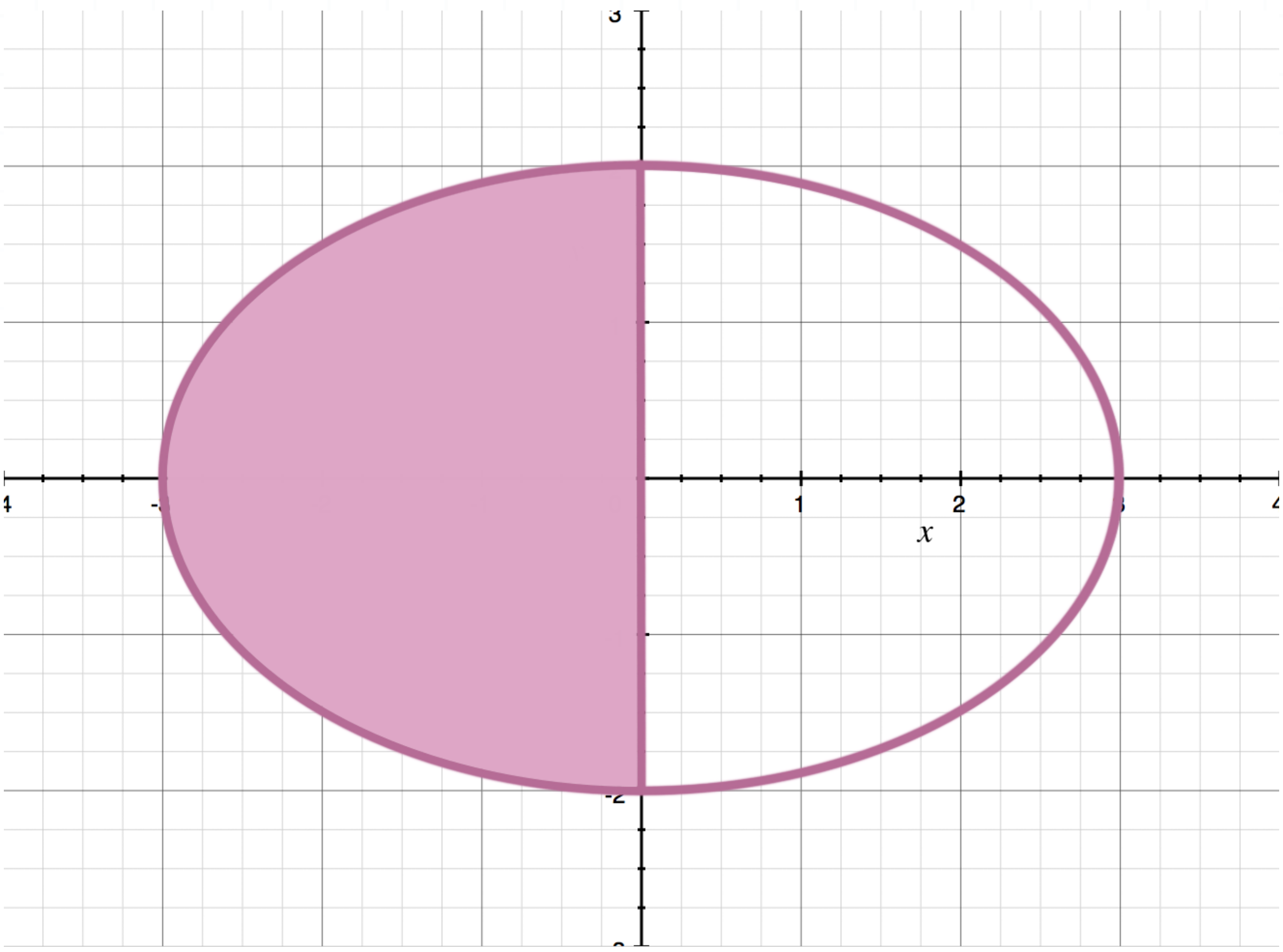
If we divide through by 36, we get an ellipse in standard form.

$$\frac{9}{36}y^2 + \frac{4}{36}x^2 = \frac{36}{36}$$

$$\frac{1}{4}y^2 + \frac{1}{9}x^2 = 1$$

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

Given that the limits given on x are given as $x = [-3,0]$, the region of integration given by the original double integral is the left half of the ellipse with center at the origin, bounded by $x = [-3,3]$ and y bounded by $[-2,2]$.



To change the order of integration, solve the equation of the ellipse for x.

$$9y^2 + 4x^2 = 36$$

$$4x^2 = 36 - 9y^2$$

$$2x = \pm\sqrt{36 - 9y^2}$$

$$2x = \pm\sqrt{9(4 - y^2)}$$

$$2x = \pm 3\sqrt{4 - y^2}$$

$$x = \pm\frac{3}{2}\sqrt{4 - y^2}$$

So while y changes from $y = -2$ to $y = 2$, x changes from

$$x = -\frac{3}{2}\sqrt{4 - y^2}$$

to $x = 0$. Therefore, we can rewrite the original integral as

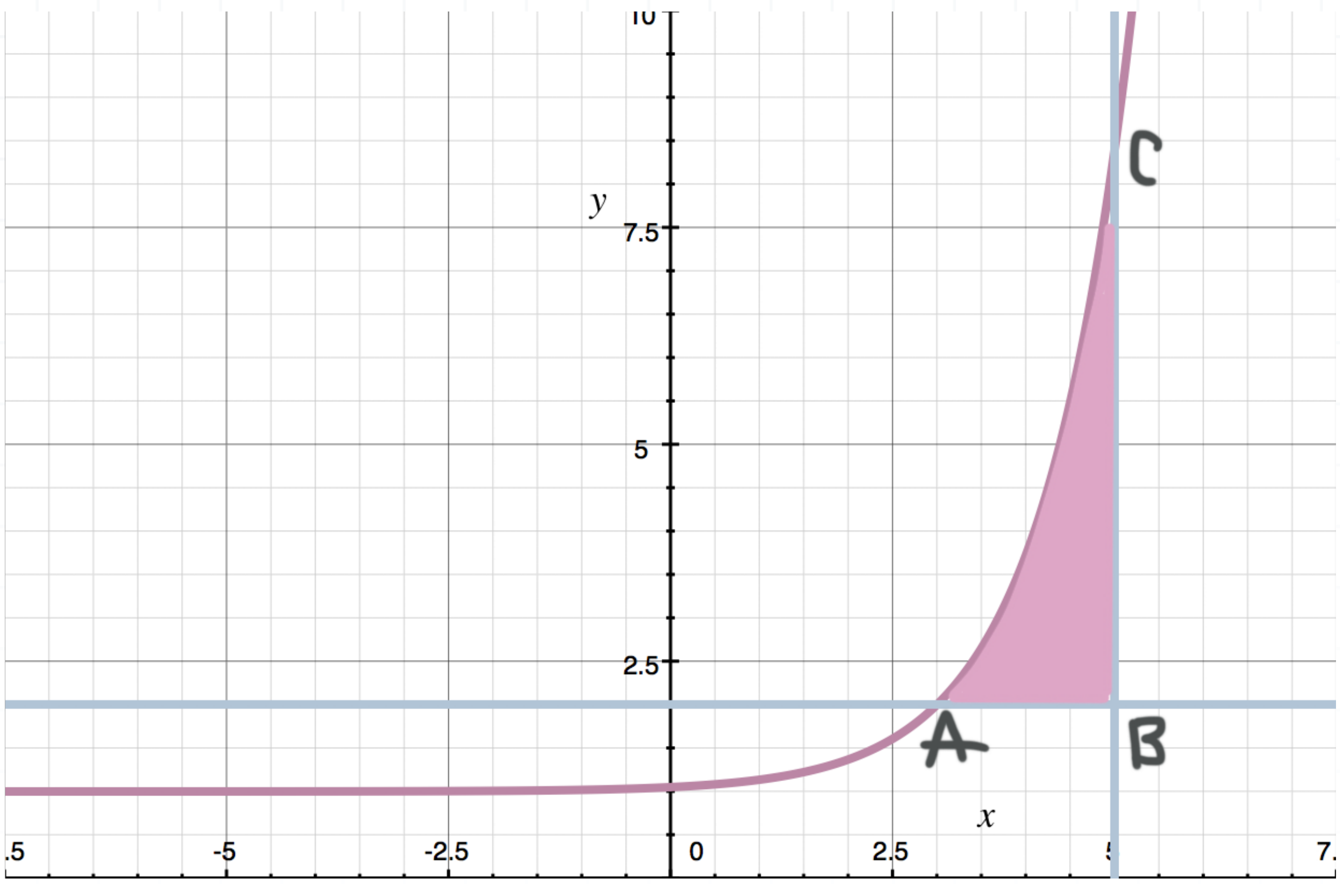
$$\int_{-2}^{2}\int_{-\frac{3}{2}\sqrt{4-y^2}}^{0} z(x, y)\ dx\ dy$$

■ 2. Change the order of integration of the iterated integral.

$$\int_{3}^{5}\int_{2}^{e^{x-3}+1} z(x, y)\ dy\ dx$$

Solution:

The double integral represents a Type I region where x is changing from 3 to 5, and y is changing from 2 to $e^{x-3}+1$. So the region is ABC, where AB and BA are lines, and AC is the curve $y=e^{x-3}+1$.



To find the coordinates of C, plug $x=5$ into the equation of the curve.

$$y(5)=e^{5-3}+1=e^2+1\approx 8.4$$

Solve $y=e^{x-3}+1$ for x.

$$y=e^{x-3}+1$$

$$y-1=e^{x-3}$$

$$x-3=\ln(y-1)$$

$$x=3+\ln(y-1)$$

So when y is changing from $y = 2$ to $y = e^2 + 1$, x is changing from $x = 3 + \ln(y - 1)$ to $x = 5$. Therefore, we can rewrite the original integral as

$$\int_{2}^{e^2+1} \int_{3+\ln(y-1)}^{5} z(x, y)\ dx\ dy$$

■ 3. Change the order of integration of the iterated integral.

$$\int_{-2}^{2} \int_{\frac{1}{4}x^4-x^2}^{0} z(x, y)\ dy\ dx$$

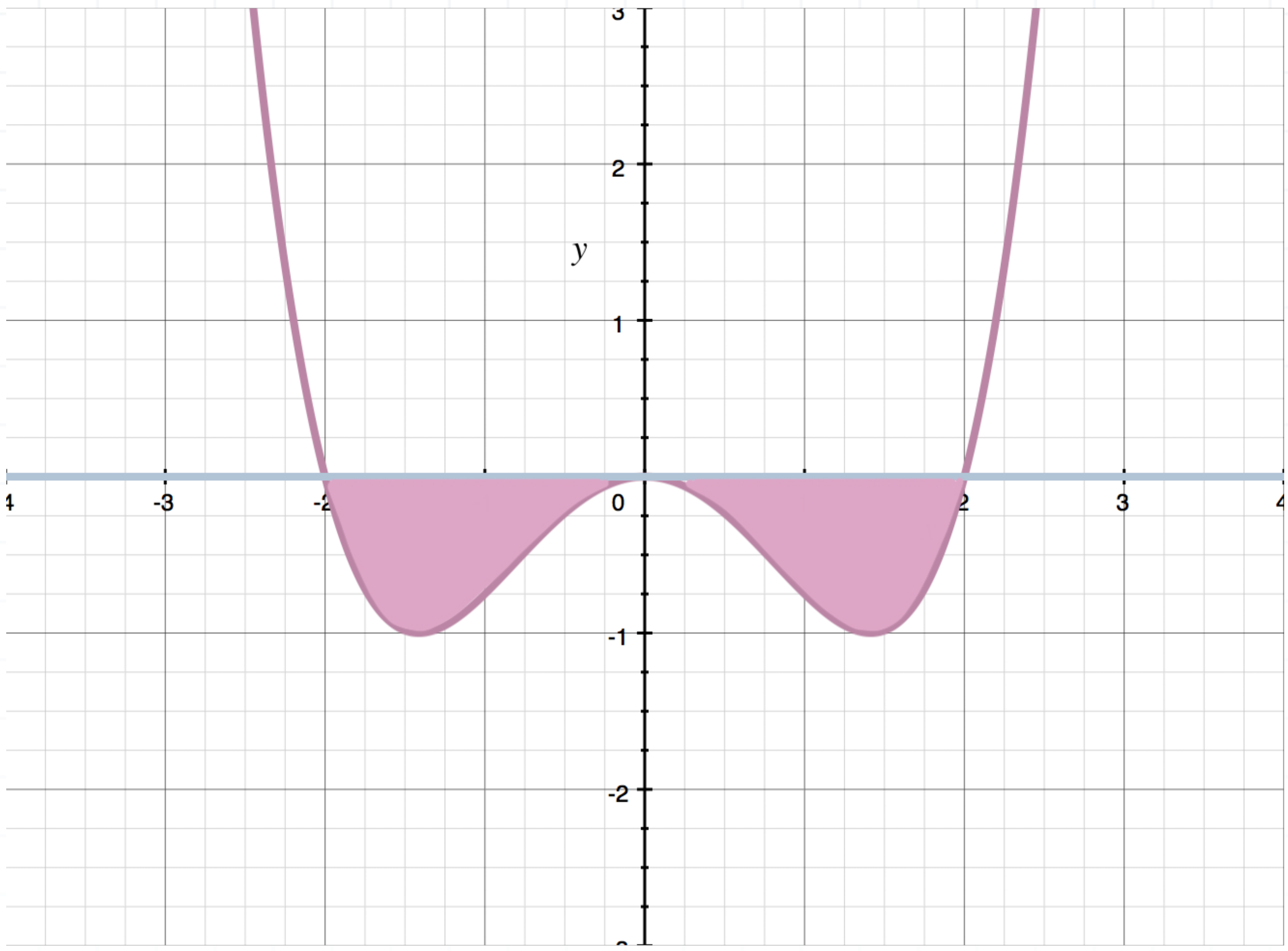
Solution:

The double integral represents a Type I region where x is changing from -2 to 2, and y is changing from $(1/4)x^4 - x^2$ to 0. So the region consists of two parts:

(1) the region on $x = [-2,0]$, and

(2) the region on $x = [0,2]$.

A sketch of both parts is



The area of region (1) is given by

$$\int_{-2}^{0}\int_{\frac{1}{4}x^4-x^2}^{0} z(x,y)\ dy\ dx$$

To find the bounds on y, we need to find the local minimum of $(1/4)x^4-x^2$.

$$y' = x^3 - 2x$$

$$y' = x(x^2-2)$$

$$x(x^2-2) = 0$$

$$x = 0,\ \ \pm\sqrt{2}$$

These are the critical points of the curve, which we can see in the sketch of the curve. The local minimum exists in the first region at $x = -\sqrt{2}$, and exists in the second region at $x = \sqrt{2}$.

$$y(-\sqrt{2}) = \frac{(-\sqrt{2})^4}{4} - (-\sqrt{2})^2$$

$$y(-\sqrt{2}) = \frac{4}{4} - 2$$

$$y(-\sqrt{2}) = 1 - 2$$

$$y(-\sqrt{2}) = -1$$

So y is changing from -1 to 0. To find the bounds on x, solve the equation $y = (1/4)x^4 - x^2$ for x.

$$y = \frac{1}{4}x^4 - x^2$$

$$4y = x^4 - 4x^2$$

$$x^4 - 4x^2 - 4y = 0$$

Use the quadratic equation to find the values of x that satisfy the equation.

$$x^2 = 2 \pm 2\sqrt{y+1}$$

$$x = \pm\sqrt{2 \pm 2\sqrt{y+1}}$$

This gives four solutions in total. For the first region (the region on the left), the value of x changes from

$$x = -\sqrt{2 + 2\sqrt{y+1}} \text{ to } x = -\sqrt{2 - 2\sqrt{y+1}}$$

Therefore, the first region (the region on the left), can be represented by

$$\int_{-1}^{0} \int_{-\sqrt{2+2\sqrt{y+1}}}^{-\sqrt{2-2\sqrt{y+1}}} z(x,y) \ dx \ dy$$

For the second region (the region on the right), the value of x changes from

$$x = \sqrt{2 - 2\sqrt{y+1}} \text{ to } x = \sqrt{2 + 2\sqrt{y+1}}$$

Therefore, the second region (the region on the right), can be represented by

$$\int_{-1}^{0} \int_{\sqrt{2-2\sqrt{y+1}}}^{\sqrt{2+2\sqrt{y+1}}} z(x,y) \ dx \ dy$$

Then both regions together can be represented as

$$\int_{-1}^{0} \int_{-\sqrt{2-2\sqrt{y+1}}}^{-\sqrt{2+2\sqrt{y+1}}} z(x,y) \ dx \ dy + \int_{-1}^{0} \int_{\sqrt{2-2\sqrt{y+1}}}^{\sqrt{2+2\sqrt{y+1}}} z(x,y) \ dx \ dy$$