Calculus 3 Workbook Solutions

Approximating triple integrals

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MIDPOINT RULE FOR TRIPLE INTEGRALS

■ 1. Use the midpoint rule to approximate the value of the triple integral, using boxes with sides $2 \times 2 \times \pi$.

$$\int_{-2}^{2}\int_{0}^{4}\int_{-2\pi}^{2\pi} x^2 y \cos z \ dz \ dy \ dx$$

Solution:

The volume of integration consists of eight boxes, so the Riemann sum estimate is given by

$$\int_{-2}^{2}\int_{0}^{4}\int_{-2\pi}^{2\pi} x^2 y \cos(z) \ dz \ dy \ dx \approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V$$

where $\Delta V = 2 \times 2 \times \pi = 4\pi$ is the volume of each box. The midpoints of these boxes are

$$(-1,1,-\pi) \qquad (-1,1,\pi) \qquad (-1,3,-\pi) \qquad (-1,3,\pi)$$

$$(1,1,-\pi) \qquad (1,1,\pi) \qquad (1,3,-\pi) \qquad (1,3,\pi)$$

Find $f(x, y, z)$ for each point.

$$f(-1,1,-\pi) = (-1)^2 \cdot (1) \cdot \cos(-\pi) = -1$$

$$f(-1,1,\pi) = (-1)^2 \cdot (1) \cdot \cos(\pi) = -1$$

$$f(-1,3,-\pi) = (-1)^2 \cdot (3) \cdot \cos(-\pi) = -3$$

$$f(-1,3,\pi) = (-1)^2 \cdot (3) \cdot \cos(\pi) = -3$$

$$f(1,1,-\pi) = (1)^2 \cdot (1) \cdot \cos(-\pi) = -1$$

$$f(1,1,\pi) = (1)^2 \cdot (1) \cdot \cos(\pi) = -1$$

$$f(1,3,-\pi) = (1)^2 \cdot (3) \cdot \cos(-\pi) = -3$$

$$f(1,3,\pi) = (1)^2 \cdot (3) \cdot \cos(\pi) = -3$$

Then the Riemann sum is

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V = 4\pi(-1-1-3-3-1-1-3-3)$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V = 4\pi(-16)$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V = -64\pi$$

■ 2. Use the midpoint rule to approximate the value of the triple integral, where D is the cube with opposite corners $(0,1,-1)$ and $(4,5,3)$. Use cubes with side length 2.

$$\iiint_D \log_2((x+1)^5 y^2 (z+2))\ dV$$

Solution:

Use laws of logs to simplify the integrand.

$$\iiint_D 5\log_2(x+1) + 2\log_2 y + \log_2(z+2)\ dV$$

The volume of integration consists of eight cubes. The Riemann sum estimate is given by

$$\int_0^4 \int_1^5 \int_{-1}^3 (5\log_2(x+1) + 2\log_2 y + \log_2(z+2))\ dz\ dy\ dx$$

$$\approx \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V$$

where $\Delta V = 2 \times 2 \times 2 = 8$ is the volume of each cube. The midpoints of these boxes are

$(1,2,0)$ $\quad$ $(1,2,2)$ $\quad$ $(1,4,0)$ $\quad$ $(1,4,2)$

$(3,2,0)$ $\quad$ $(3,2,2)$ $\quad$ $(3,4,0)$ $\quad$ $(3,4,2)$

Find $f(x, y, z)$ for each point.

$$f(1,2,0) = 5\log_2(1+1) + 2\log_2 2 + \log_2(0+2) = 5 \cdot 1 + 2 \cdot 1 + 1 = 8$$

$$f(1,2,2) = 5\log_2(1+1) + 2\log_2 2 + \log_2(2+2) = 5 \cdot 1 + 2 \cdot 1 + 2 = 9$$

$$f(1,4,0) = 5\log_2(1+1) + 2\log_2 4 + \log_2(0+2) = 5 \cdot 1 + 2 \cdot 2 + 1 = 10$$

$$f(1,4,2) = 5\log_2(1+1) + 2\log_2 4 + \log_2(2+2) = 5 \cdot 1 + 2 \cdot 2 + 2 = 11$$

$$f(3,2,0) = 5\log_2(3+1) + 2\log_2 2 + \log_2(0+2) = 5 \cdot 2 + 2 \cdot 1 + 1 = 13$$

$$f(3,2,2) = 5\log_2(3+1) + 2\log_2 2 + \log_2(2+2) = 5 \cdot 2 + 2 \cdot 1 + 2 = 14$$

$$f(3,4,0) = 5\log_2(3+1) + 2\log_2 4 + \log_2(0+2) = 5 \cdot 2 + 2 \cdot 2 + 1 = 15$$

$$f(3,4,2) = 5\log_2(3+1) + 2\log_2 4 + \log_2(2+2) = 5 \cdot 2 + 2 \cdot 2 + 2 = 16$$

Then the Riemann sum is

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V = (8)(8 + 9 + 10 + 11 + 13 + 14 + 15 + 16)$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V = 768$$

■ 3. Use the midpoint rule to approximate the value of the improper triple integral. Use cubes with side length 1.

$$\int_0^1 \int_0^1 \int_0^\infty \log_4(x)\frac{(y-1)^3}{z^2}\ dz\ dy\ dx$$

Solution:

The volume of integration consists of an infinite number of boxes. The Riemann sum estimate is therefore given by

$$\int_0^1 \int_0^1 \int_0^\infty \log_4(x)\frac{(y-1)^3}{z^2}\ dz\ dy\ dx \approx \sum_{k=1}^{\infty} f\left(\frac{1}{2},\frac{1}{2},z_k\right)\Delta V$$

where $\Delta V = 1 \times 1 \times 1 = 1$ is the volume of each cube, and $z_k = (2k-1)/2$ for k from 1 to ∞. Find $f(1/2,1/2,z_k)$.

$$\log_4\left(\frac{1}{2}\right) \cdot \frac{\left(\frac{1}{2}-1\right)^3}{\left(\frac{2k-1}{2}\right)^2}$$

$$-\frac{1}{2} \cdot \left(-\frac{1}{8}\right) \cdot 4 \cdot \frac{1}{(2k-1)^2}$$

$$\frac{1}{4} \cdot \frac{1}{(2k-1)^2}$$

So the Riemann sum is

$$\sum_{k=1}^{\infty} f\left(\frac{1}{2},\frac{1}{2},z_k\right)\Delta V$$

$$\sum_{k=1}^{\infty} \frac{1}{4} \cdot \frac{1}{(2k-1)^2}(1)$$

$$\frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\frac{1}{4}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}\right)$$

$$\frac{1}{4}\left(\frac{\pi^2}{8}\right)$$

$$\frac{\pi^2}{32}$$

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