Calculus 3 Quizzes

krista king MATH

Topic: Plotting points in three dimensions

Question: Which value does -7 represent in the point $(3,-7,9)$?

Answer choices:

A x

B y

C z

D r

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Solution: B

Three-dimensional coordinate points in rectangular coordinates are given in the form (x, y, z). That means that in the coordinate point $(3, -7, 9)$, $x = 3$, $y = -7$ and $z = 9$.

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Topic: Plotting points in three dimensions

Question: Which value does -5 represent in the point $(-12,8,-5)$?

Answer choices:

A x

B y

C z

D r

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Solution: C

Three-dimensional coordinate points in rectangular coordinates are given in the form (x, y, z). That means that in the coordinate point $(-12,8,-5)$, $x=-12$, $y=8$ and $z=-5$.

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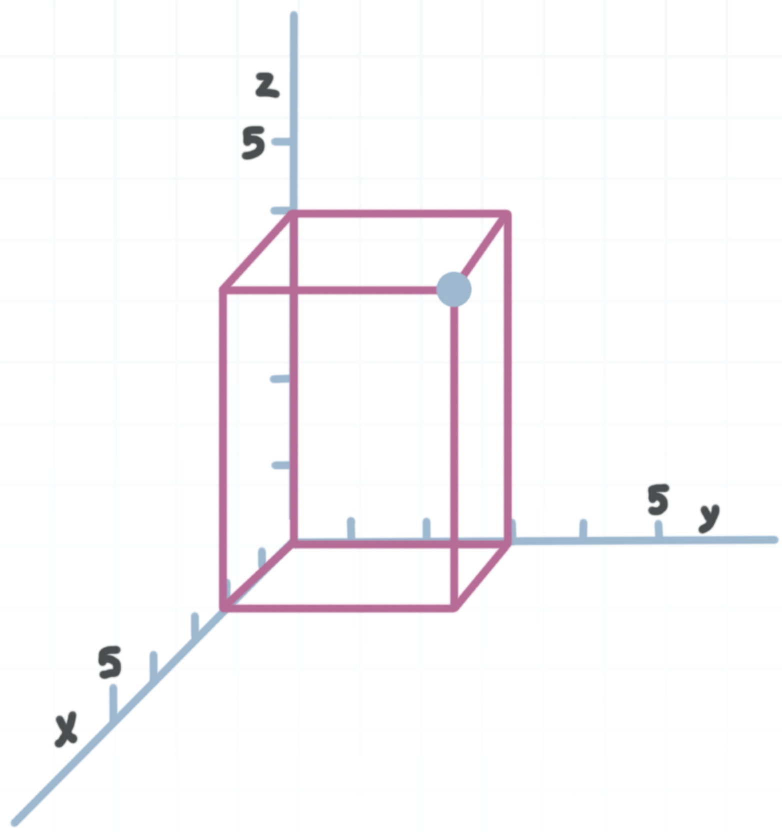
Topic: Plotting points in three dimensions

Question: Which point is shown in light blue?



Answer choices:

A $(3,2,4)$

B $(3,4,2)$

C $(2,4,3)$

D $(2,3,4)$

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Solution: D

To read the point shown on the plot, start by noting the x-coordinate, then the y-coordinate, and finally the z-coordinate. For the given point, the x-value is 2, the y-coordinate is 3, and the z-coordinate is 4. This makes the plotted point $(2,3,4)$.

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Topic: Distance between points in three dimensions

Question: Find the distance between the points.

$$(4, -1, 2)$$

$$(3, 2, -1)$$

Answer choices:

A $\sqrt{19}$

B 7

C $\sqrt{7}$

D 19

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Solution: A

To find the distance between two points in three dimensions, we'll use

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

where (x_1,y_1,z_1) is one point and (x_2,y_2,z_2) is the other point.

$$D=\sqrt{(3-4)^2+\left[2-(-1)\right]^2+(-1-2)^2}$$

$$D=\sqrt{1+9+9}$$

$$D=\sqrt{19}$$

The distance between the points is $\sqrt{19}$.

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Topic: Distance between points in three dimensions

Question: In which plane does the point lie?

$$(6,0,-1)$$

Answer choices:

A yz-plane

B $r\theta$-plane

C xy-plane

D xz-plane

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Solution: D

We know that

- a point with a zero x-value lies in the yz-plane.
- a point with a zero y-value lies in the xz-plane.
- a point with a zero z-value lies in the xy-plane.

Since $(6,0,-1)$ has a zero y-value, it lies in the xz-plane.

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Topic: Distance between points in three dimensions

Question: Which point is closest to the plane?

The xy-plane

Answer choices:

A $(-4,0,9)$

B $(-2,-1,-3)$

C $(0,3,4)$

D $(2,4,-8)$

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Solution: B

To find the point that is closest to the xy-plane, we can take the absolute value of the z-coordinate for each of the answer choices. The value closest to 0, or 0 itself, is the point that's closest to the plane.

If we wanted to find the point closest to the yz-plane, we'd examine the x-coordinate in the same way. If we wanted to find the point closest to the xz-plane, we'd examine the y-coordinate in the same way.

For $(2,4,-8)$, $\quad |z| = |-8| = 8$

For $(0,3,4)$, $\quad |z| = |4| = 4$

For $(-2,-1,-3)$, $\quad |z| = |-3| = 3$

For $(-4,0,9)$, $\quad |z| = |9| = 9$

Since 3 is the value closest to 0, $(-2,-1,-3)$ is the closest point to the xy-plane.

Topic: Center, radius and equation of a sphere

Question: Find the center and radius of the sphere.

$$(x-9)^2+(y-2)^2+(z+1)^2=16$$

Answer choices:

A $(-9,-2,1)$ and $r=4$

B $(-9,-2,1)$ and $r=16$

C $(9,2,-1)$ and $r=4$

D $(9,2,-1)$ and $r=16$

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Solution: C

The standard equation of a sphere is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

where (h,k,l) is the center of the sphere

where r is the radius of the sphere

If we match the equation we've been given to the standard equation

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

$$(x-9)^2+(y-2)^2+(z+1)^2=16$$

we can see that

$$h=9$$

$$k=2$$

$$l=-1$$

So the center of the sphere is $(9,2,-1)$.

If we set the right side of the given equation equal to the right side of the standard equation, we can say that the radius is

$$r^2=16$$

$$r=4$$

In summary, the center of the sphere is $(9,2,-1)$ and its radius is $r=4$.

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Topic: Center, radius and equation of a sphere

Question: Find the center and radius of the sphere.

$$x^2 - 2x + y^2 + 4y + z^2 - 6z = 11$$

Answer choices:

A $(-1,2,-3)$ and $r = 5$

B $(1,-2,3)$ and $r = 5$

C $(-1,2,-3)$ and $r = \sqrt{11}$

D $(1,-2,3)$ and $r = \sqrt{11}$

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Solution: B

The standard equation of a sphere is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

where (h,k,l) is the center of the sphere

where r is the radius of the sphere

In order to get the given equation into standard form, we'll need to complete the square with respect to each variable. Remember that in order to complete the square, we'll take the coefficient on the first-degree term, divide it by 2, square the result, and then add that final value into the quadratic expression (and subtract it back out so that we don't change the value of the equation).

$$x^2-2x+\left(\frac{-2}{2}\right)^2-\left(\frac{-2}{2}\right)^2+y^2+4y+\left(\frac{4}{2}\right)^2-\left(\frac{4}{2}\right)^2$$

$$+z^2-6z+\left(\frac{-6}{2}\right)^2-\left(\frac{-6}{2}\right)^2=11$$

$$x^2-2x+(-1)^2-(-1)^2+y^2+4y+(2)^2-(2)^2+z^2-6z+(-3)^2-(-3)^2=11$$

$$\left[x^2-2x+(-1)^2\right]-(-1)^2+\left[y^2+4y+(2)^2\right]-(2)^2+\left[z^2-6z+(-3)^2\right]-(-3)^2=11$$

$$\left[x^2-2x+(-1)^2\right]+\left[y^2+4y+(2)^2\right]+\left[z^2-6z+(-3)^2\right]=11+(-1)^2+(2)^2+(-3)^2$$

$$\left(x^2-2x+1\right)+\left(y^2+4y+4\right)+\left(z^2-6z+9\right)=11+1+4+9$$

$$(x-1)^2+(y+2)^2+(z-3)^2=25$$

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If we match this transformed equation to the standard equation

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

$$(x-1)^2+(y+2)^2+(z-3)^2=25$$

we can see that

$$h=1$$

$$k=-2$$

$$l=3$$

So the center of the sphere is $(1,-2,3)$.

If we set the right side of the transformed equation equal to the right side of the standard equation, we can say that the radius is

$$r^2=25$$

$$r=5$$

In summary, the center of the sphere is $(1,-2,3)$ and its radius is $r=5$.

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Topic: Center, radius and equation of a sphere

Question: Find the equation of the sphere.

Passing through $(-2,2,0)$

Center at $(-1,1,-1)$

Answer choices:

A $\quad (x-1)^2+(y+1)^2+(z-1)^2=3$

B $\quad (x+1)^2+(y-1)^2+(z+1)^2=\sqrt{3}$

C $\quad (x+1)^2+(y-1)^2+(z+1)^2=3$

D $\quad (x-1)^2+(y+1)^2+(z-1)^2=\sqrt{3}$

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Solution: C

The standard equation of a sphere is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

where (h,k,l) is the center of the sphere

where r is the radius of the sphere

If we don't know the radius, but we have the center and a point on the sphere, then we can calculate the radius as the distance between them, using

$$r=D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

where (x_1,y_1,z_1) is one point on the surface of the sphere

where (x_2,y_2,z_2) is the center point of the sphere

Since we've been given the center of the sphere already as $(-1,1,-1)$, we can plug it into the standard equation of a sphere.

$$\left[x-(-1)\right]^2+(y-1)^2+\left[z-(-1)\right]^2=r^2$$

$$(x+1)^2+(y-1)^2+(z+1)^2=r^2$$

The only value left to find is the radius, so we'll plug the center $(-1,1,-1)$ and the point $(-2,2,0)$ on the surface of the sphere into the distance formula.

$$r=\sqrt{\left[-1-(-2)\right]^2+(1-2)^2+(-1-0)^2}$$

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$$r = \sqrt{(1)^2 + (-1)^2 + (-1)^2}$$

$$r = \sqrt{1 + 1 + 1}$$

$$r = \sqrt{3}$$

Remember that we just found r, but we need r^2 for the equation of the sphere.

$$r = \sqrt{3}$$

$$r^2 = 3$$

Making this substitution into the equation of the sphere gives

$$(x + 1)^2 + (y - 1)^2 + (z + 1)^2 = 3$$

This is the equation of the sphere with center $(-1,1,-1)$ that passes through the $(-2,2,0)$.

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Topic: Describing a region in three dimensional space

Question: Describe the surface in three-dimensional space.

$$x + y = 1$$

Answer choices:

A A horizontal plane that's parallel to the z-axis and intersects the xy-plane in the line $x + y = 1$.

B A horizontal plane that's perpendicular to the z-axis and intersects the xy-plane in the line $x + y = 1$.

C A vertical plane that's perpendicular to the z-axis and intersects the xy-plane in the line $x + y = 1$.

D A vertical plane that's parallel to the z-axis and intersects the xy-plane in the line $x + y = 1$.

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Solution: D

The first thing we can see with our region $x + y = 1$ is that there's no z element. This means that our region has no intersection with the z-axis which means that our region is a plane that's parallel to the z-axis. Next we can solve for the x- and y-intercepts.

The x-intercept occurs where $y = 0$.

$$x + (0) = 1$$

$$x = 1$$

The x-intercept occurs at the point $(1,0,0)$.

The y-intercept occurs where $x = 0$.

$$(0) + y = 1$$

$$y = 1$$

The y-intercept occurs at the point $(0,1,0)$.

These two points show that the xy-plane is intersected in R^3 along the line $x + y = 1$.

Pulling together all of this information we can see that the surface in R^3 of $x + y = 1$ is a vertical plane that's parallel to the z-axis and intersects the xy-plane in the line $x + y = 1$.

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Topic: Describing a region in three dimensional space

Question: Describe the surface in three-dimensional space.

$$x + z = 4$$

Answer choices:

A A plane that's parallel to the y-axis and intersects the xz-plane in the line $x + z = 4$.

B A plane that's parallel to the z-axis and intersects the xy-plane in the line $x + z = 4$.

C A plane that's parallel to the x-axis and intersects the xz-plane in the line $x + z = 4$.

D A plane that's parallel to the x-axis and intersects the yz-plane in the line $x + z = 4$.

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Solution: A

The first thing we can see with our region $x+z=4$ is that there's no y element. This means that our region has no intersection with the y-axis which means that our region is a plane that's parallel to the y-axis. Next we can solve for the x- and z-intercepts.

The x-intercept occurs where $z=0$.

$$x+(0)=4$$

$$x=4$$

The x-intercept occurs at the point $(4,0,0)$.

The z-intercept occurs where $x=0$.

$$(0)+z=4$$

$$z=4$$

The y-intercept occurs at the point $(0,0,4)$.

These two points show that the xz-plane is intersected in R^3 along the line $x+z=4$.

Pulling together all of this information we can see that the surface in R^3 of $x+z=4$ is a plane that is parallel to the y-axis and intersects the xz-plane in the line $x+z=4$.

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Topic: Describing a region in three dimensional space

Question: Describe the surface in three-dimensional space.

$$2y + z = -6$$

Answer choices:

A	A horizontal plane that's parallel to the z-axis and intersects the xy-plane in the line $2y + z = -6$.

B	A plane that's parallel to the x-axis and intersects the yz-plane in the line $2y + z = -6$.

C	A vertical plane that's parallel to the x-axis and intersects the xy-plane in the line $2y + z = -6$.

D	A plane that's parallel to the y-axis and intersects the xz-plane in the line $2y + z = -6$.

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Solution: B

The first thing we can see with our region $2y + z = -6$ is that there's no x element. This means that our region has no intersection with the x-axis which means that our region is a plane that's parallel to the x-axis. Next we can solve for the y- and z-intercepts.

The y-intercept occurs where $z = 0$.

$$2y + (0) = -6$$

$$y = -3$$

The x-intercept occurs at the point $(0, -3, 0)$.

The z-intercept occurs where $y = 0$.

$$2(0) + z = -6$$

$$z = -6$$

The y-intercept occurs at the point $(0, 0, -6)$.

These two points show that the yz-plane is intersected in R^3 along the line $2y + z = -6$.

Pulling together all of this information we can see that the surface in R^3 of $2y + z = -6$ is a plane that's parallel to the x-axis and intersects the yz-plane in the line $2y + z = -6$.

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Topic: Using inequalities to describe the region

Question: What inequality describes the region consisting of all points between (but not on) two spheres of radius r and R, both centered at the origin, where $r < R$?

Answer choices:

A $r^2 \leq x^2 + y^2 + z^2 \leq R^2$

B $r^2 \geq x^2 + y^2 + z^2 \geq R^2$

C $r^2 < x^2 + y^2 + z^2 < R^2$

D $r^2 > x^2 + y^2 + z^2 > R^2$

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Solution: C

The first thing we need to remember is the base equation for a sphere $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ where (h,k,l) is the point at the center of the sphere. Both of our spheres are centered at the origin $(0,0,0)$ which will make the equations for our two spheres

$$x^2+y^2+z^2=r^2$$

$$x^2+y^2+z^2=R^2$$

Now we need to remember that we're looking for the points which exist between the sphere of radius r (which is smaller than the other sphere so exists completely within the larger sphere) and the sphere of radius R but not on either surface. The points that exist within the sphere of radius R but not on either surface are

$$x^2+y^2+z^2<R^2$$

Remember $x^2+y^2+z^2\le R^2$ would be correct only if we also wanted to include the points on the surface as well as the points inside the sphere and $x^2+y^2+z^2=R^2$ would be correct if we only wanted the points on the surface of the sphere.

Next we need to remove the points that are within $x^2+y^2+z^2=r^2$, which are shown by

$$x^2+y^2+z^2>r^2$$

When we put our two inequalities together to describe the region, we get

$$r^2<x^2+y^2+z^2<R^2$$

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Topic: Using inequalities to describe the region

Question: What inequality describes the region consisting of all points between (but not on) two spheres of radius r and R, both centered at $(1,-4,2)$, where $r<R$?

Answer choices:

A $r^2 \le (x+1)^2+(y-4)^2+(z+2)^2 \le R^2$

B $r^2 < (x-1)^2+(y+4)^2+(z-2)^2 < R^2$

C $r^2 < (x+1)^2+(y-4)^2+(z+2)^2 < R^2$

D $r^2 \le (x-1)^2+(y+4)^2+(z-2)^2 \le R^2$

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Solution: B

To describe a region in three dimensional space, you will need to analyze how the region presents itself. The basic equations for the objects used will help define the region. Remember that these equations include the points on the surface of the object as well as inside the object.

The first thing we need to remember is the base equation for a sphere $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ where (h,k,l) is the point at the center of the sphere. Both of our spheres are centered at the point $(1,-4,2)$ which will make the equations for our two spheres

$$(x-1)^2+(y+4)^2+(z-2)^2=r^2$$

$$(x-1)^2+(y+4)^2+(z-2)^2=R^2$$

Now we need to remember that we're looking for the points which exist between the sphere of radius r (which is smaller than the other sphere so exits completely within the larger sphere) and the sphere of radius R but not on either surface. The points that exist within the sphere of radius R but not on either surface are

$$(x-1)^2+(y+4)^2+(z-2)^2<R^2$$

Remember $(x-1)^2+(y+4)^2+(z-2)^2\le R^2$ would be correct only if we also wanted to include the points on the surface as well as the points inside the sphere and $(x-1)^2+(y+4)^2+(z-2)^2=R^2$ would be correct if we only wanted the points on the surface of the sphere.

Next we need to remove the points that are within $(x-1)^2+(y+4)^2+(z-2)^2=r^2$ which are

$$(x-1)^2 + (y+4)^2 + (z-2)^2 > r^2$$

When we put our two inequalities together to describe the region, we get

$$r^2 < (x-1)^2 + (y+4)^2 + (z-2)^2 < R^2$$

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Topic: Using inequalities to describe the region

Question: What inequality describes the region consisting of all points between two spheres of radius r and R, both centered at $(0,2,-1)$, where $r<R$, and the surfaces of both spheres are included in the region?

Answer choices:

A $\quad r^2<x^2+(y-2)^2+(z+1)^2<R^2$

B $\quad r^2<x^2+(y+2)^2+(z-1)^2<R^2$

C $\quad r^2\le x^2+(y+2)^2+(z-1)^2\le R^2$

D $\quad r^2\le x^2+(y-2)^2+(z+1)^2\le R^2$

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Solution: D

The first thing we need to remember is the base equation for a sphere $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ where (h,k,l) is the point at the center of the sphere. Both of our spheres are centered at the point $(0,2,-1)$ which will make the equations for our two spheres

$$(x-0)^2+(y-2)^2+(z+1)^2=r^2$$

$$x^2+(y-2)^2+(z+1)^2=r^2$$

and

$$(x-0)^2+(y-2)^2+(z+1)^2=R^2$$

$$x^2+(y-2)^2+(z+1)^2=R^2$$

Now we need to remember that we're looking for the points which exist between the sphere of radius r (which is smaller than the other sphere so exits completely within the larger sphere) and the sphere of radius R including both surfaces. The points that exist within the sphere of radius R and on its surface are

$$x^2+(y-2)^2+(z+1)^2\le R^2$$

Remember $x^2+(y-2)^2+(z+1)^2<R^2$ would be correct only if we wanted to exclude the points on the surface but use the points inside the sphere and $x^2+(y-2)^2+(z+1)^2=R^2$ would be correct if we only wanted the points on the surface of the sphere.

Next we need to remove the points that are within $x^2+(y-2)^2+(z+1)^2=r^2$ which are

$$x^2 + (y-2)^2 + (z+1)^2 \geq r^2$$

When we put our two inequalities together to describe the region, we get

$$r^2 \leq x^2 + (y-2)^2 + (z+1)^2 \leq R^2$$

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Topic: Sketching graphs of multivariable functions

Question: Which point is part of the function?

$$f(x, y) = 3x - 2\sqrt{y} - 2$$

Answer choices:

A $(1, -1, 1)$

B $(-1, 1, 1)$

C $(1, 1, -1)$

D $(-1, -1, 1)$

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Solution: C

Since we already have the x and y values from the coordinate point in each answer choice, we can plug those values into the function to see if the z value we get back matches the z value from the answer choice.

Plug in $(1,-1,1)$ from answer choice A to see if we get back $z=1$.

$$z = 3x - 2\sqrt{y} - 2$$

$$z = 3(1) - 2\sqrt{(-1)} - 2$$

$$z = 3 - 2\sqrt{-1} - 2$$

We don't get $z=1$, so let's plug in $(-1,1,1)$ from answer choice B to see if we get back $z=1$.

$$z = 3x - 2\sqrt{y} - 2$$

$$z = 3(-1) - 2\sqrt{(1)} - 2$$

$$z = -3 - 2 - 2$$

$$z = -7$$

We don't get $z=1$, so let's plug in $(1,1,-1)$ from answer choice C to see if we get back $z=-1$.

$$z = 3x - 2\sqrt{y} - 2$$

$$z = 3(1) - 2\sqrt{(1)} - 2$$

$$z = 3 - 2 - 2$$

$$z = -1$$

That's the value for z that we were expecting, so we can say that $(1,1,-1)$ is a point on the function.

We could stop here, but if we want to check $(-1,-1,1)$ from answer choice D, we can see that it does not satisfy the equation.

$$z = 3x - 2\sqrt{y} - 2$$

$$z = 3(-1) - 2\sqrt{(-1)} - 2$$

$$z = -3 - 2\sqrt{-1} - 2$$

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Topic: Sketching graphs of multivariable functions

Question: Find the domain of the function.

$$f(x, y) = \sqrt{y - x}$$

Answer choices:

A $y - x \geq 0$

B $y - x > 0$

C $y - x = 0$

D $y - x \leq 0$

Solution: A

The domain of a three-dimensional function is based on where the function exists in the xy-plane.

The function $f(x, y) = \sqrt{y - x}$ has restrictions on its domain because of the square root, since the value underneath a square root cannot be negative, if we want the function to consist only of real numbers.

So we can define the domain as all values that make $y - x$ positive or zero and we can write the domain of this function as

$$y - x \geq 0$$

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Topic: Sketching graphs of multivariable functions

Question: Describe in words the domain of the function.

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2 - 25}}$$

Answer choices:

A All points inside the sphere with radius 5, excluding the sphere itself

B All points outside the sphere with radius 5, excluding the sphere itself

C All points outside the sphere with radius 5, including the sphere itself

D All points inside the sphere with radius 5, including the sphere itself

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Solution: B

The domain of a four dimensional function includes the three dimensional coordinates where the function exists.

The domain of this function is effected by the fact that it's a fraction, and that the function includes a square root. We can't have the denominator be zero, and we can't have the value underneath the square root be negative.

If we start with the fact that the value under the square root can't be negative, we define the domain as

$$x^2+y^2+z^2-25\geq 0$$

But then, since a zero value under the square root would make the denominator zero, which we can't have, we must restrict the domain to

$$x^2+y^2+z^2-25\geq 0$$

$$x^2+y^2+z^2>25$$

We know that the sphere with radius 5 is given by

$$x^2+y^2+z^2=25$$

Which means that $x^2+y^2+z^2>25$ is the area outside the sphere, not including the sphere itself. If the domain were instead $x^2+y^2+z^2\geq 25$, then the sphere itself would be included. But in this case, the domain is only the area outside the sphere.

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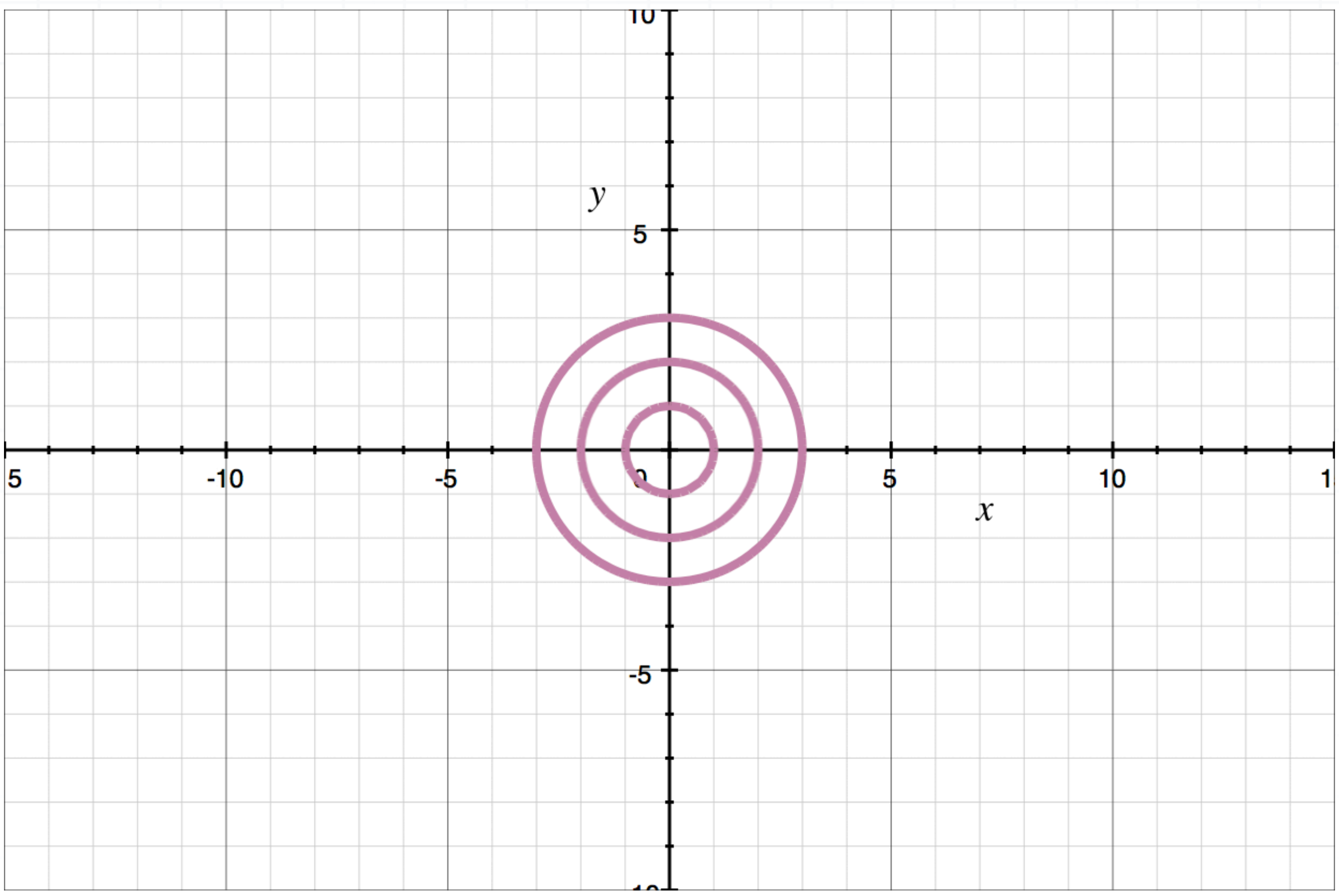
Topic: Sketching level curves of multivariable functions

Question: Which function gives the equation for these level curves?



Answer choices:

A $f(x, y) = \sqrt{x^2 - y^2}$

B $f(x, y) = \sqrt{x + y^2}$

C $f(x, y) = \sqrt{x^2 + y^2}$

D $f(x, y) = \sqrt{x^2 + y}$

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Solution: C

The level curves pictured are circles, which means we need to look in the answer choices for the equation of a circle. We know that the equation of a circle is

$$x^2 + y^2 = r^2$$

where r is the radius of the circle. These level curves each have a different radius, so in effect, we can replace the radius with some arbitrary constant c.

$$x^2 + y^2 = c^2$$

The level curves displayed in this question are likely given by $c = 1, 2, 3$, such that

$$x^2 + y^2 = 1^2$$

$$x^2 + y^2 = 2^2$$

$$x^2 + y^2 = 3^2$$

We want to solve the equation for c, so we get

$$c^2 = x^2 + y^2$$

$$c = \sqrt{x^2 + y^2}$$

If we want to give an equation that represents the level curves in general, regardless of the value of c, then we just replace c with $f(x, y)$, and we get

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$$f(x, y) = \sqrt{x^2 + y^2}$$

This is the equation that represents the level curves.

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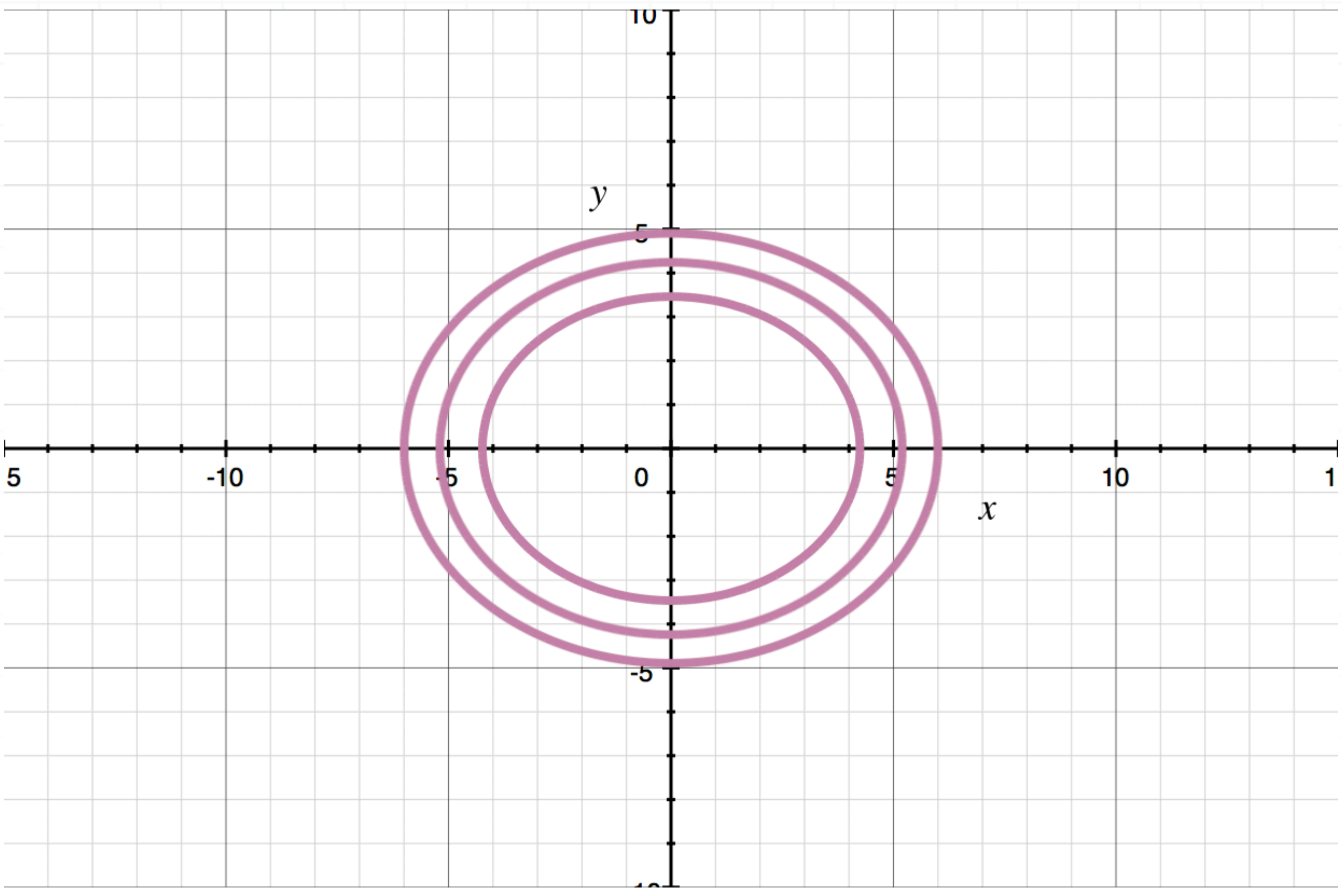
Topic: Sketching level curves of multivariable functions

Question: Which function gives the equation for these level curves?



Answer choices:

A $f(x, y) = \frac{1}{3}x + \frac{1}{2}y^2$

B $f(x, y) = \frac{1}{3}x^2 - \frac{1}{2}y^2$

C $f(x, y) = \frac{1}{3}x^2 + \frac{1}{2}y$

D $f(x, y) = \frac{1}{3}x^2 + \frac{1}{2}y^2$

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Solution: D

The level curves pictured are ellipses, which means we need to look in the answer choices for the equation of an ellipse. We know that the equation of an ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

These level curves each have a different radius, and if we change the constant on the right side of this equation, it changes the radius. So in effect, we can replace 1 with some arbitrary constant c.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=c$$

If we want to give an equation that represents the level curves in general, regardless of the value of c, then we just replace c with $f(x,y)$, and we get

$$f(x,y)=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

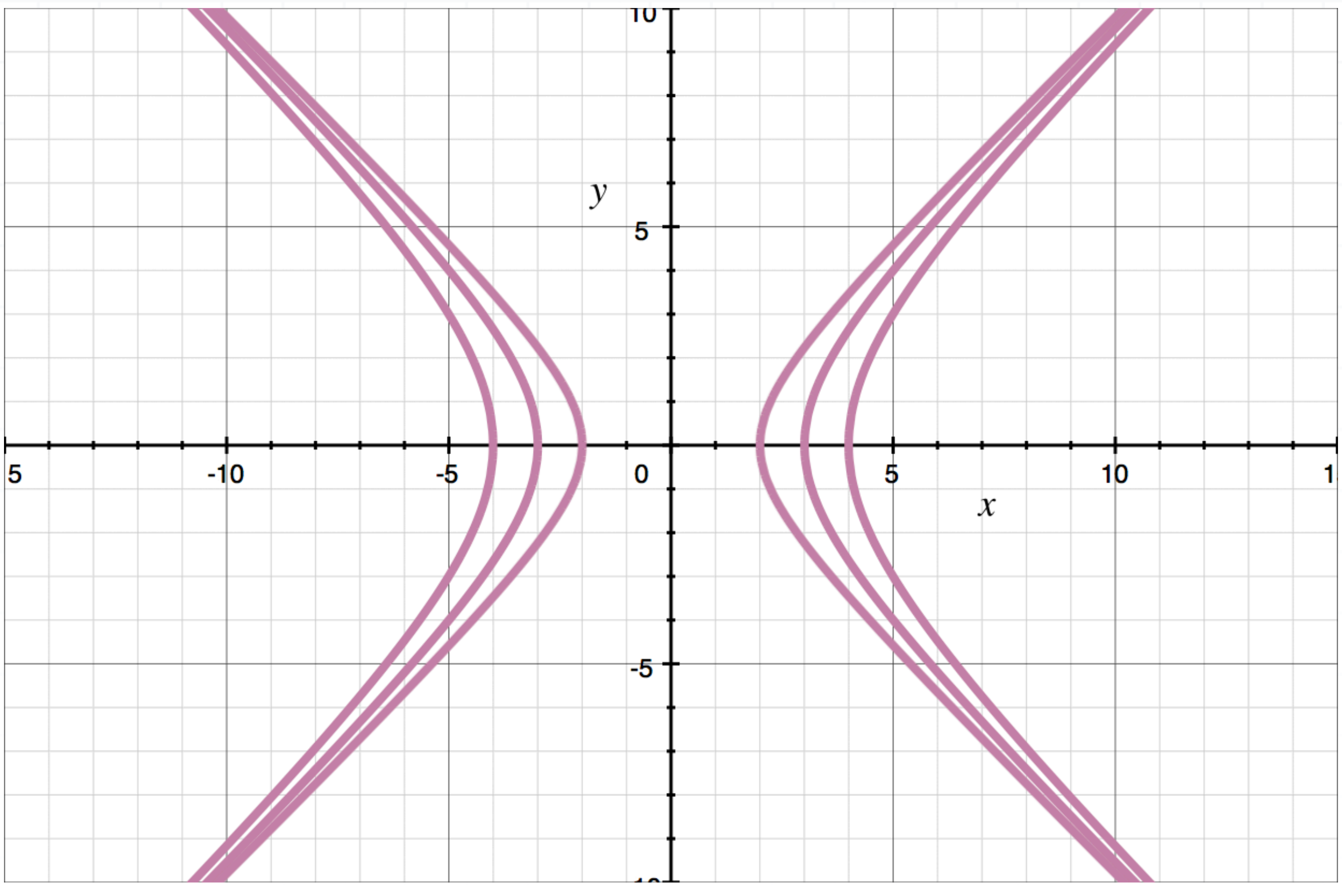
The only equation that matches this format, where x and y are both squared and where we've got an addition sign between the terms instead of a subtraction sign, is

$$f(x,y)=\frac{1}{3}x^2+\frac{1}{2}y^2$$

Therefore, this is the equation that represents the level curves.

Topic: Sketching level curves of multivariable functions

Question: Which function gives the equation for these level curves?



Answer choices:

A $f(x, y) = x^2 - y^2$

B $f(x, y) = x^2 + y^2$

C $f(x, y) = x - y^2$

D $f(x, y) = x^2 - y$

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Solution: A

The level curves pictured are hyperbolas, which means we need to look in the answer choices for the equation of a hyperbola. We know that the equation of a hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{z}{d}$$

These level curves each have a different directrix, so in effect, we can replace z/d with some arbitrary constant c.

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=c$$

The level curves displayed in this question are likely given by $c=1,\ 2,\ 3$, such that

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=3$$

If we want to give an equation that represents the level curves in general, regardless of the value of c, then we just replace c with $f(x,y)$, and we get

$$f(x,y)=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

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The only equation that matches this format, where x and y are both squared and where we've got a subtraction sign between the terms instead of an addition sign, is

$$f(x, y) = x^2 - y^2$$

Therefore, this is the equation that represents the level curves.

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Topic: Matching the function with the graph and level curves

Question: Which of these statements is true?

Answer choices:

A The graph of $x^2 + 3y^2 + z^2 - 12x + 6y - 10z = -27$ is the ellipsoid centered at $(3,-2,5)$.

B The graph of $x^2 + 3y^2 + z^2 + 6x - 12y - 10z = -18$ is the ellipsoid centered at $(3,-2,5)$.

C The graph of $x^2 + 3y^2 + z^2 - 6x + 10y + 12z = -18$ is the ellipsoid centered at $(3,-2,5)$.

D The graph of $x^2 + 3y^2 + z^2 - 6x + 12y - 10z = -19$ is the ellipsoid centered at $(3,-2,5)$.

Solution: D

Answer choice D is the equation that matches the graph. Transform that equation to a standard form, by completing the square with respect to each variable.

$$x^2 + 3y^2 + z^2 - 6x + 12y - 10z = -19$$

$$\left(x^2 - 6x + 9 - 9\right) + \left(3y^2 + 12y + 12 - 12\right) + \left(z^2 - 10z + 25 - 25\right) = -19$$

$$\left(x^2 - 6x + 9\right) + \left(3y^2 + 12y + 12\right) + \left(z^2 - 10z + 25\right) - 9 - 12 - 25 = -19$$

$$(x-3)^2 + 3(y+2)^2 + (z-5)^2 = 27$$

$$\frac{(x-3)^2}{27} + \frac{3(y+2)^2}{27} + \frac{(z-5)^2}{27} = \frac{27}{27}$$

$$\frac{(x-3)^2}{27} + \frac{(y+2)^2}{9} + \frac{(z-5)^2}{27} = 1$$

The center of this quadratic is therefore at $(3,-2,5)$. It's also in the standard form of an ellipsoid.

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Topic: Matching the function with the graph and level curves

Question: Which are level curves of the graph of the function for $k = 0,1,2,3$?

$$f(x, y) = \sqrt{-\left(x^2 + y^2 - 225\right)}$$

Answer choices:

A Circles with radii $15,\ 4\sqrt{14},\ \sqrt{221},\ 6\sqrt{6}$

B Ellipses with semi-axes of $15,\ 4\sqrt{14},\ \sqrt{221},\ 6\sqrt{6}$

C Vertical lines at $x = 15,\ x = 4\sqrt{14},\ x = \sqrt{221},\ x = 6\sqrt{6}$

D Horizontal lines at $y = 15,\ y = 4\sqrt{14},\ y = \sqrt{221},\ y = 6\sqrt{6}$

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Solution: A

The family of level curves of

$$f(x, y) = \sqrt{-\left(x^2 + y^2 - 225\right)}$$

is defined by the family of curves

$$\sqrt{-\left(x^2 + y^2 - 225\right)} = k$$

as transformed below:

$$-\left(x^2 + y^2 - 225\right) = k^2$$

$$225 - x^2 - y^2 = k^2$$

$$x^2 + y^2 = 225 - k^2$$

$$x^2 + y^2 = \left(\sqrt{225 - k^2}\right)^2$$

The equation above defines a family of circles with radii $k = 0,1,2,3$. Therefore, the level curves of the graph of the given function are:

For $k = 0$,

$$x^2 + y^2 = \left(\sqrt{225 - 0^2}\right)^2$$

$$x^2 + y^2 = \left(\sqrt{225}\right)^2$$

$$x^2 + y^2 = 15^2$$

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For $k=1$,

$$x^2+y^2=\left(\sqrt{225-1^2}\right)^2$$

$$x^2+y^2=\left(\sqrt{224}\right)^2$$

$$x^2+y^2=\left(4\sqrt{14}\right)^2$$

For $k=2$,

$$x^2+y^2=\left(\sqrt{225-2^2}\right)^2$$

$$x^2+y^2=\left(\sqrt{221}\right)^2$$

For $k=3$,

$$x^2+y^2=\left(\sqrt{225-3^2}\right)^2$$

$$x^2+y^2=\left(\sqrt{216}\right)^2$$

$$x^2+y^2=\left(6\sqrt{6}\right)^2$$

These level curves are circles with radii 15, $4\sqrt{14}$, $\sqrt{221}$, $6\sqrt{6}$.

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Topic: Matching the function with the graph and level curves

Question: Which answer choice pairs each function with the graph of its own level curve at $k = 2$?

Answer choices:

A $\quad f(x, y) = \dfrac{y - 3}{x + 4}$ level curve for $k = 2$ at $y = 2x - 11$

$\quad f(x, y) = \dfrac{x - 4}{y + 3}$ level curve for $k = 2$ at $y = \dfrac{1}{2}x - 5$

$\quad p(x, y) = \dfrac{x}{2y}$ level curve for $k = 2$ at $y = -\dfrac{x}{4}$

$\quad q(x, y) = \dfrac{y}{2x}$ level curve for $k = 2$ at $y = \dfrac{x}{4}$

B $\quad f(x, y) = \dfrac{y - 3}{x + 4}$ level curve for $k = 2$ at $y = \dfrac{1}{2}x + 5$

$\quad f(x, y) = \dfrac{x - 4}{y + 3}$ level curve for $k = 2$ at $y = 2x + 11$

$\quad p(x, y) = \dfrac{x}{2y}$ level curve for $k = 2$ at $y = \dfrac{x}{4}$

$\quad q(x, y) = \dfrac{y}{2x}$ level curve for $k = 2$ at $y = \dfrac{x}{4}$

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C $\quad f(x,y)=\dfrac{y-3}{x+4}$ level curve for $k=2$ at $y=2x+11$

$f(x,y)=\dfrac{x-4}{y+3}$ level curve for $k=2$ at $y=\dfrac{1}{2}x-5$

$p(x,y)=\dfrac{x}{2y}$ level curve for $k=2$ at $y=\dfrac{x}{4}$

$q(x,y)=\dfrac{y}{2x}$ level curve for $k=2$ at $y=4x$

D $\quad f(x,y)=\dfrac{y-3}{x+4}$ level curve for $k=2$ at $y=2x+11$

$f(x,y)=\dfrac{x-4}{y+3}$ level curve for $k=2$ at $y=\dfrac{1}{2}x-5$

$p(x,y)=\dfrac{x}{2y}$ level curve for $k=2$ at $y=4x$

$q(x,y)=\dfrac{y}{2x}$ level curve for $k=2$ at $y=\dfrac{x}{4}$

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Solution: C

Answer choice C is the only one that correctly matches each function with the graph of its level curve at $k = 2$.

For $\frac{y-3}{x+4} = k$ for $k = 2$:

$$y - 3 = 2(x + 4)$$

$$y - 3 = 2x + 8$$

$$y = 2x + 11$$

For $\frac{x-4}{y+3} = k$ for $k = 2$:

$$\frac{x-4}{y+3} = 2$$

$$x - 4 = 2(y + 3)$$

$$x - 4 = 2y + 6$$

$$x - 10 = 2y$$

$$y = \frac{1}{2}x - 5$$

For $\frac{x}{2y} = k$ for $k = 2$:

$$\frac{x}{2y} = 2$$

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$$4y = x$$

$$y = \frac{x}{4}$$

For $\frac{y}{2x} = k$ for $k = 2$:

$$\frac{y}{2x} = 2$$

$$y = 4x$$

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Topic: Vector and parametric equations of a line

Question: Find the vector equation of the line.

Passing through $(2, -2, -1)$

Perpendicular to $5\mathbf{i} + 6\mathbf{j} - \mathbf{k} = 0$

Answer choices:

A $\quad r = (5 + 2t)\mathbf{i} + (6 - 2t)\mathbf{j} + (-1 - t)\mathbf{k}$

B $\quad r = (2 - 5t)\mathbf{i} + (-2 - 6t)\mathbf{j} + (-1 + t)\mathbf{k}$

C $\quad r = (2 + 5t)\mathbf{i} + (-2 + 6t)\mathbf{j} + (-1 - t)\mathbf{k}$

D $\quad r = (5 - 2t)\mathbf{i} + (6 + 2t)\mathbf{j} + (-1 + t)\mathbf{k}$

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Solution: C

We'll start by converting the given point to its vector equivalent.

$$(2, -2, -1)$$

$$2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$$

We know we're looking for the line perpendicular to $5\mathbf{i} + 6\mathbf{j} - \mathbf{k} = 0$, which means we need the normal line to $5\mathbf{i} + 6\mathbf{j} - \mathbf{k} = 0$, which is $5\mathbf{i} + 6\mathbf{j} - \mathbf{k}$. The line we're looking for will be parallel to $5\mathbf{i} + 6\mathbf{j} - \mathbf{k}$.

Now we're ready to plug into the equation of a line, $r = r_0 + tv$, where r_0 is a point on the line, and where v is a vector parallel to the vector we want.

$$r = r_0 + tv$$

$$r = (2\mathbf{i} - 2\mathbf{j} - \mathbf{k}) + t(5\mathbf{i} + 6\mathbf{j} - \mathbf{k})$$

$$r = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k} + 5t\mathbf{i} + 6t\mathbf{j} - t\mathbf{k}$$

$$r = (2\mathbf{i} + 5t\mathbf{i}) + (-2\mathbf{j} + 6t\mathbf{j}) + (-\mathbf{k} - t\mathbf{k})$$

$$r = (2 + 5t)\mathbf{i} + (-2 + 6t)\mathbf{j} + (-1 - t)\mathbf{k}$$

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Topic: Vector and parametric equations of a line

Question: Find the parametric equations of the line that corresponds to the vector equation.

$$r = (-3 + t)\mathbf{i} + (8t)\mathbf{j} + (1 - 3t)\mathbf{k}$$

Answer choices:

A	$x = 1 + 3t$	$y = -8$	$z = 3 - t$
B	$x = -3 + t$	$y = 8t$	$z = 1 - 3t$
C	$x = 1 - 3t$	$y = 8$	$z = -3 + t$
D	$x = 3 - t$	$y = -8t$	$z = -1 + 3t$

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Solution: B

Given a vector equation

$$r = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

the parametric equations are $x = a$, $y = b$ and $z = c$. So from the vector equation $r = (-3 + t)\mathbf{i} + (8t)\mathbf{j} + (1 - 3t)\mathbf{k}$, we get

$$x = -3 + t$$

$$y = 8t$$

$$z = 1 - 3t$$

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Topic: Vector and parametric equations of a line

Question: Find the parametric equations of the line.

Passing through $(6,0,3)$

Perpendicular to $-\mathbf{i}+3\mathbf{j}+2\mathbf{k}=2$

Answer choices:

A	$x=-6+t$	$y=-3t$	$z=-3-2t$
B	$x=1-6t$	$y=-3$	$z=-2-3t$
C	$x=-1+6t$	$y=3$	$z=2+3t$
D	$x=6-t$	$y=3t$	$z=3+2t$

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Solution: D

We'll start by converting the given point to its vector equivalent.

$$(6,0,3)$$

$$6\mathbf{i}+0\mathbf{j}+3\mathbf{k}$$

$$6\mathbf{i}+3\mathbf{k}$$

We know we're looking for the line perpendicular to $-\mathbf{i}+3\mathbf{j}+2\mathbf{k}=2$, which means we need the normal line to $-\mathbf{i}+3\mathbf{j}+2\mathbf{k}=2$, which is $-\mathbf{i}+3\mathbf{j}+2\mathbf{k}$. The line we're looking for will be parallel to $-\mathbf{i}+3\mathbf{j}+2\mathbf{k}$.

Now we're ready to plug into the equation of a line, $r=r_0+tv$, where r_0 is a point on the line, and where v is a vector parallel to the vector we want.

$$r=r_0+tv$$

$$r=(6\mathbf{i}+3\mathbf{k})+t(-\mathbf{i}+3\mathbf{j}+2\mathbf{k})$$

$$r=6\mathbf{i}+3\mathbf{k}-t\mathbf{i}+3t\mathbf{j}+2t\mathbf{k}$$

$$r=(6\mathbf{i}-t\mathbf{i})+(3t\mathbf{j})+(3\mathbf{k}+2t\mathbf{k})$$

$$r=(6-t)\mathbf{i}+3t\mathbf{j}+(3+2t)\mathbf{k}$$

Now we can turn this into parametric equations. Given a vector equation

$$r=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$$

the parametric equations are $x=a$, $y=b$ and $z=c$. So from the vector equation $r=(6-t)\mathbf{i}+3t\mathbf{j}+(3+2t)\mathbf{k}$, we get

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$$x = 6 - t$$

$$y = 3t$$

$$z = 3 + 2t$$

Topic: Parametric and symmetric equations of the line

Question: Find the symmetric equation of the line.

Passing through $a(1,-3,-1)$

Perpendicular to $\mathbf{i}+4\mathbf{j}-2\mathbf{k}=3$

Answer choices:

A $\quad x-1=\dfrac{y+3}{4}=-\dfrac{z+1}{2}$

B $\quad -x-1=-\dfrac{y-3}{4}=\dfrac{z-1}{2}$

C $\quad -x+1=-\dfrac{y+3}{4}=\dfrac{z+1}{2}$

D $\quad x+1=\dfrac{y-3}{4}=-\dfrac{z-1}{2}$

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Solution: A

The symmetric equation of a line is given by

$$\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}$$

where $a(a_1, a_2, a_3)$ is a point on the line, and where $v\langle v_1, v_2, v_3\rangle$ is a vector parallel to the line. Since we already know the lines passes through $a(1,-3,-1)$, we can plug this into the formula to get

$$\frac{x-1}{v_1}=\frac{y-(-3)}{v_2}=\frac{z-(-1)}{v_3}$$

$$\frac{x-1}{v_1}=\frac{y+3}{v_2}=\frac{z+1}{v_3}$$

Since we want the line that's perpendicular to $\mathbf{i}+4\mathbf{j}-2\mathbf{k}=3$, we could also say we're looking for the line that's parallel to the normal vector of $\mathbf{i}+4\mathbf{j}-2\mathbf{k}=3$. The normal vector is given by the coefficients $\langle 1,4,-2\rangle$, so $\langle 1,4,-2\rangle$ is parallel to the line we need. Therefore we can find the symmetric equation of the line.

$$\frac{x-1}{1}=\frac{y+3}{4}=\frac{z+1}{-2}$$

$$x-1=\frac{y+3}{4}=-\frac{z+1}{2}$$

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Topic: Parametric and symmetric equations of the line

Question: Find the parametric and symmetric equations of the line.

Passes through $a(-1,-1,-1)$

Perpendicular to $-\mathbf{i}-3\mathbf{j}-2\mathbf{k}=5$

Answer choices:

A $\quad x=-1+t \quad y=-1+3t \quad z=-1+2t \quad x-1=\dfrac{y-1}{3}=\dfrac{z-1}{2}$

B $\quad x=-1-t \quad y=-1-3t \quad z=-1-2t \quad x-1=\dfrac{y-1}{3}=\dfrac{z-1}{2}$

C $\quad x=-1-t \quad y=-1-3t \quad z=-1-2t \quad -x-1=-\dfrac{y+1}{3}=-\dfrac{z+1}{2}$

D $\quad x=-1+t \quad y=-1+3t \quad z=-1+2t \quad x+1=\dfrac{y+1}{3}=\dfrac{z+1}{2}$

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Solution: C

The symmetric equation of a line is given by

$$\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}$$

where $a(a_1, a_2, a_3)$ is a point on the line, and where $v\langle v_1, v_2, v_3\rangle$ is a vector parallel to the line. Since we already know the lines passes through $a(-1, -1, -1)$, we can plug this into the formula to get

$$\frac{x-(-1)}{v_1}=\frac{y-(-1)}{v_2}=\frac{z-(-1)}{v_3}$$

$$\frac{x+1}{v_1}=\frac{y+1}{v_2}=\frac{z+1}{v_3}$$

Since we want the line that's perpendicular to $-\mathbf{i}-3\mathbf{j}-2\mathbf{k}=5$, we could also say we're looking for the line that's parallel to the normal vector of $-\mathbf{i}-3\mathbf{j}-2\mathbf{k}=5$. The normal vector is given by the coefficients $\langle -1, -3, -2\rangle$, so $\langle -1, -3, -2\rangle$ is parallel to the line we need. Therefore we can find the symmetric equation of the line.

$$\frac{x+1}{-1}=\frac{y+1}{-3}=\frac{z+1}{-2}$$

$$-x-1=-\frac{y+1}{3}=-\frac{z+1}{2}$$

Now in order to find the parametric equations, we have to first find the vector equation of the line, which is given by

$$r=r_0+tv$$

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where r_0 is a point on the line and where v is a vector that's parallel to the line. The line we want passes through $a(-1,-1,-1)$, which we can rewrite as $-\mathbf{i}-\mathbf{j}-\mathbf{k}$.

Since we want the line that's perpendicular to $-\mathbf{i}-3\mathbf{j}-2\mathbf{k}=5$, we could also say we're looking for the line that's parallel to the normal vector of $-\mathbf{i}-3\mathbf{j}-2\mathbf{k}=5$. The normal vector is given by the coefficients $\langle -1,-3,-2\rangle$, so $\langle -1,-3,-2\rangle$ is parallel to the line we need. We can rewrite this as $-\mathbf{i}-3\mathbf{j}-2\mathbf{k}$.

If we plug both of these into the vector equation of the line, we get

$$r=r_0+tv$$

$$r=(-\mathbf{i}-\mathbf{j}-\mathbf{k})+t(-\mathbf{i}-3\mathbf{j}-2\mathbf{k})$$

$$r=-\mathbf{i}-\mathbf{j}-\mathbf{k}-t\mathbf{i}-3t\mathbf{j}-2t\mathbf{k}$$

$$r=(-\mathbf{i}-t\mathbf{i})+(-\mathbf{j}-3t\mathbf{j})+(-\mathbf{k}-2t\mathbf{k})$$

$$r=(-1-t)\mathbf{i}+(-1-3t)\mathbf{j}+(-1-2t)\mathbf{k}$$

If the vector equation is $r=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, then the parametric equations of the line are given by $x=a$, $y=b$, and $z=c$.

$$x=-1-t$$

$$y=-1-3t$$

$$z=-1-2t$$

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Topic: Parametric and symmetric equations of the line

Question: Find the parametric and symmetric equations of the line.

Passing through $a(4,6,-5)$

Perpendicular to $6\mathbf{i}-2\mathbf{j}+7\mathbf{k}=-2$

Answer choices:

A $x=4-6t$ $y=6+2t$ $z=-5-7t$ $-\frac{x-4}{6}=\frac{y-6}{2}=-\frac{z+5}{7}$

B $x=4+6t$ $y=6-2t$ $z=-5+7t$ $-\frac{x-4}{6}=\frac{y-6}{2}=-\frac{z+5}{7}$

C $x=4-6t$ $y=6+2t$ $z=-5-7t$ $\frac{x-4}{6}=-\frac{y-6}{2}=\frac{z+5}{7}$

D $x=4+6t$ $y=6-2t$ $z=-5+7t$ $\frac{x-4}{6}=-\frac{y-6}{2}=\frac{z+5}{7}$

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Solution: D

The symmetric equation of a line is given by

$$\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}$$

where $a(a_1, a_2, a_3)$ is a point on the line, and where $v\langle v_1, v_2, v_3\rangle$ is a vector parallel to the line. Since we already know the lines passes through $a(4,6,-5)$, we can plug this into the formula to get

$$\frac{x-4}{v_1}=\frac{y-6}{v_2}=\frac{z-(-5)}{v_3}$$

$$\frac{x-4}{v_1}=\frac{y-6}{v_2}=\frac{z+5}{v_3}$$

Since we want the line that's perpendicular to $6\mathbf{i}-2\mathbf{j}+7\mathbf{k}=-2$, we could also say we're looking for the line that's parallel to the normal vector of $6\mathbf{i}-2\mathbf{j}+7\mathbf{k}=-2$. The normal vector is given by the coefficients $\langle 6,-2,7\rangle$, so $\langle 6,-2,7\rangle$ is parallel to the line we need. Therefore we can find the symmetric equation of the line.

$$\frac{x-4}{6}=\frac{y-6}{-2}=\frac{z+5}{7}$$

$$\frac{x-4}{6}=-\frac{y-6}{2}=\frac{z+5}{7}$$

Now in order to find the parametric equations, we have to first find the vector equation of the line, which is given by

$$r=r_0+tv$$

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where r_0 is a point on the line and where v is a vector that's parallel to the line. The line we want passes through $a(4,6,-5)$, which we can rewrite as $4\mathbf{i}+6\mathbf{j}-5\mathbf{k}$.

Since we want the line that's perpendicular to $6\mathbf{i}-2\mathbf{j}+7\mathbf{k}=-2$, we could also say we're looking for the line that's parallel to the normal vector of $6\mathbf{i}-2\mathbf{j}+7\mathbf{k}=-2$. The normal vector is given by the coefficients $\langle 6,-2,7\rangle$, so $\langle 6,-2,7\rangle$ is parallel to the line we need. We can rewrite this as $6\mathbf{i}-2\mathbf{j}+7\mathbf{k}$.

If we plug both of these into the vector equation of the line, we get

$$r=r_0+tv$$

$$r=(4\mathbf{i}+6\mathbf{j}-5\mathbf{k})+t(6\mathbf{i}-2\mathbf{j}+7\mathbf{k})$$

$$r=4\mathbf{i}+6\mathbf{j}-5\mathbf{k}+6t\mathbf{i}-2t\mathbf{j}+7t\mathbf{k}$$

$$r=(4\mathbf{i}+6t\mathbf{i})+(6\mathbf{j}-2t\mathbf{j})+(-5\mathbf{k}+7t\mathbf{k})$$

$$r=(4+6t)\mathbf{i}+(6-2t)\mathbf{j}+(-5+7t)\mathbf{k}$$

If the vector equation is $r=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$, then the parametric equations of the line are given by $x=a$, $y=b$, and $z=c$.

$$x=4+6t$$

$$y=6-2t$$

$$z=-5+7t$$

Topic: Symmetric equations of a line

Question: Find the symmetric equation of the line.

Passing through $a(0,6,-2)$

Perpendicular to $-\mathbf{i}+2\mathbf{j}+4\mathbf{k}=-1$

Answer choices:

A $\quad x=\dfrac{y-6}{2}=\dfrac{z+2}{4}$

B $\quad -x=\dfrac{y-6}{2}=\dfrac{z+2}{4}$

C $\quad x=\dfrac{y+6}{2}=\dfrac{z-2}{4}$

D $\quad -x=\dfrac{y+6}{2}=\dfrac{z-2}{4}$

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Solution: B

The symmetric equation of a line is given by

$$\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}$$

where $a(a_1,a_2,a_3)$ is a point on the line, and where v is a vector parallel to the line.

Since we want the line that's perpendicular to $-\mathbf{i}+2\mathbf{j}+4\mathbf{k}=-1$, we know we want the line that's parallel to the normal vector to $-\mathbf{i}+2\mathbf{j}+4\mathbf{k}=-1$. The normal vector is given by the coefficients, $\langle -1,2,4\rangle$, so the line we want is parallel to $\langle -1,2,4\rangle$.

Plugging all of this and $a(0,6,-2)$ into the formula for the symmetric equation gives

$$\frac{x-0}{-1}=\frac{y-6}{2}=\frac{z-(-2)}{4}$$

$$-x=\frac{y-6}{2}=\frac{z+2}{4}$$

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Topic: Symmetric equations of a line

Question: Find the symmetric equation of the line.

Passing through $a(-3,-2,4)$

Perpendicular to $4\mathbf{i}-10\mathbf{j}-3\mathbf{k}=6$

Answer choices:

A $\quad \dfrac{x+3}{4}=-\dfrac{y+2}{10}=-\dfrac{z-4}{3}$

B $\quad \dfrac{x+3}{4}=\dfrac{y+2}{10}=\dfrac{z-4}{3}$

C $\quad \dfrac{x-3}{4}=-\dfrac{y-2}{10}=-\dfrac{z+4}{3}$

D $\quad \dfrac{x-3}{4}=\dfrac{y-2}{10}=\dfrac{z+4}{3}$

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Solution: A

The symmetric equation of a line is given by

$$\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}$$

where $a(a_1, a_2, a_3)$ is a point on the line, and where v is a vector parallel to the line.

Since we want the line that's perpendicular to $4\mathbf{i}-10\mathbf{j}-3\mathbf{k}=6$, we know we want the line that's parallel to the normal vector to $4\mathbf{i}-10\mathbf{j}-3\mathbf{k}=6$. The normal vector is given by the coefficients, $\langle 4,-10,-3\rangle$, so the line we want is parallel to $\langle 4,-10,-3\rangle$.

Plugging all of this and $a(-3,-2,4)$ into the formula for the symmetric equation gives

$$\frac{x-(-3)}{4}=\frac{y-(-2)}{-10}=\frac{z-4}{-3}$$

$$\frac{x+3}{4}=-\frac{y+2}{10}=-\frac{z-4}{3}$$

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Topic: Symmetric equations of a line

Question: Find the symmetric equation of the line.

Passes through $a(-5,-8,-9)$

Perpendicular to $-5\mathbf{i}+3\mathbf{j}-\mathbf{k}=-8$

Answer choices:

A $\frac{x-5}{5}=\frac{y-8}{3}=z-9$

B $-\frac{x-5}{5}=\frac{y-8}{3}=-z+9$

C $\frac{x+5}{5}=\frac{y+8}{3}=z+9$

D $-\frac{x+5}{5}=\frac{y+8}{3}=-z-9$

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Solution: D

The symmetric equation of a line is given by

$$\frac{x-a_1}{v_1}=\frac{y-a_2}{v_2}=\frac{z-a_3}{v_3}$$

where $a(a_1,a_2,a_3)$ is a point on the line, and where v is a vector parallel to the line.

Since we want the line that's perpendicular to $-5\mathbf{i}+3\mathbf{j}-\mathbf{k}=-8$, we know we want the line that's parallel to the normal vector to $-5\mathbf{i}+3\mathbf{j}-\mathbf{k}=-8$. The normal vector is given by the coefficients, $\langle -5,3,-1\rangle$, so the line we want is parallel to $\langle -5,3,-1\rangle$.

Plugging all of this and $a(-5,-8,-9)$ into the formula for the symmetric equation gives

$$\frac{x-(-5)}{-5}=\frac{y-(-8)}{3}=\frac{z-(-9)}{-1}$$

$$-\frac{x+5}{5}=\frac{y+8}{3}=-z-9$$

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Topic: Parallel, intersecting, skew and perpendicular lines

Question: Are the lines parallel, intersecting, skew, or perpendicular?

L_1: $x_1 = 1 - t$ $y_1 = 1 + 2t$ $z_1 = 1 + t$

L_2: $x_2 = 2 - 2s$ $y_2 = 2 + s$ $z_2 = 3 - s$

Answer choices:

A Parallel

B Intersecting

C Intersecting and perpendicular

D Skew

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Solution: B

If we match the lines we've been given to

$$L_1: \quad x_1 = a_1 + b_1 t \qquad y_1 = c_1 + d_1 t \qquad z_1 = e_1 + f_1 t$$

$$L_2: \quad x_2 = a_2 + b_2 s \qquad y_2 = c_2 + d_2 s \qquad z_2 = e_2 + f_2 s$$

we can plug into the ratio equality to get

$$\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$$

$$\frac{-1}{-2} = \frac{2}{1} = \frac{1}{-1}$$

$$\frac{1}{2} = 2 = -1$$

Because this equation is false, we know that the lines are not parallel. So we'll test to see if the lines are intersecting. In order to do so, we'll need to treat the lines as a system of equations. Because we have two equations for x, we can set them equal to each other. We can do the same for y and z.

[1] $1 - t = 2 - 2s$

[2] $1 + 2t = 2 + s$

[3] $1 + t = 3 - s$

We'll solve [1] for t and then plug it into [2].

[1] $1 - t = 2 - 2s$

$-t = 1 - 2s$

[4] $t = -1 + 2s$

Plugging [4] into [2] gives

$1 + 2(-1 + 2s) = 2 + s$

$1 - 2 + 4s = 2 + s$

$3s = 3$

[5] $s = 1$

Now we'll plug [5] into [4] to find a value for t.

$t = -1 + 2(1)$

$t = -1 + 2$

[6] $t = 1$

Plugging [5] and [6] into [3], the only equation we haven't used yet, gives

[3] $1 + t = 3 - s$

$1 + 1 = 3 - 1$

$2 = 2$

Because this equation is true, we know that the lines are intersecting. If this equation was false, we would have shown that the lines were not parallel, and not intersecting, which would prove that they must be skew.

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To see whether or not these intersecting lines are perpendicular, we'll take their dot product.

$$a \cdot b = (-1)(-2) + (2)(1) + (1)(-1)$$

$$a \cdot b = 2 + 2 - 1$$

$$a \cdot b = 3$$

Because the dot product isn't 0, the lines are intersecting, but not perpendicular.

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Topic: Parallel, intersecting, skew and perpendicular lines

Question: Are the lines parallel, intersecting, skew, or perpendicular?

$$L_1: \quad x_1 = 3 + 4t \qquad y_1 = -2 + 3t \qquad z_1 = 1 - t$$

$$L_2: \quad x_2 = -2 + 8s \qquad y_2 = 4 + 6s \qquad z_2 = 1 - 2s$$

Answer choices:

A Parallel

B Intersecting

C Intersecting and perpendicular

D Skew

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Solution: A

If we match the lines we've been given to

$$L_1: \quad x_1 = a_1 + b_1 t \qquad y_1 = c_1 + d_1 t \qquad z_1 = e_1 + f_1 t$$

$$L_2: \quad x_2 = a_2 + b_2 s \qquad y_2 = c_2 + d_2 s \qquad z_2 = e_2 + f_2 s$$

we can plug into the ratio equality to get

$$\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$$

$$\frac{4}{8} = \frac{3}{6} = \frac{-1}{-2}$$

$$\frac{1}{2} = \frac{1}{2} = \frac{1}{2}$$

Because this equation is true, we know that the lines are parallel.

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Topic: Parallel, intersecting, skew and perpendicular lines

Question: Are the lines parallel, intersecting, skew, or perpendicular?

L_1: $x_1 = 1 - 3t$ $y_1 = -3 + 5t$ $z_1 = 4 + t$

L_2: $x_2 = 2 - 4s$ $y_2 = 6 + 4s$ $z_2 = 3 + 8s$

Answer choices:

A Parallel

B Intersecting

C Intersecting and perpendicular

D Skew

Solution: D

If we match the lines we've been given to

$$L_1: \quad x_1 = a_1 + b_1 t \qquad y_1 = c_1 + d_1 t \qquad z_1 = e_1 + f_1 t$$

$$L_2: \quad x_2 = a_2 + b_2 s \qquad y_2 = c_2 + d_2 s \qquad z_2 = e_2 + f_2 s$$

we can plug into the ratio equality to get

$$\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$$

$$\frac{-3}{-4} = \frac{5}{4} = \frac{1}{8}$$

$$\frac{3}{4} = \frac{5}{4} = \frac{1}{8}$$

Because this equation is false, we know that the lines are not parallel. So we'll test to see if the lines are intersecting. In order to do so, we'll need to treat the lines as a system of equations. Because we have two equations for x, we can set them equal to each other. We can do the same for y and z.

[1] $1 - 3t = 2 - 4s$

[2] $-3 + 5t = 6 + 4s$

[3] $4 + t = 3 + 8s$

We'll solve **[1]** for t and then plug it into **[2]**.

[1] $1 - 3t = 2 - 4s$

$$-3t = 1 - 4s$$

[4] $t = -\dfrac{1}{3} + \dfrac{4}{3}s$

Plugging **[4]** into **[2]** gives

$$-3 + 5\left(-\frac{1}{3} + \frac{4}{3}s\right) = 6 + 4s$$

$$-3 - \frac{5}{3} + \frac{20}{3}s = 6 + 4s$$

$$\frac{20}{3}s - 4s = 6 + 3 + \frac{5}{3}$$

$$20s - 12s = 18 + 9 + 5$$

$$8s = 32$$

[5] $s = 4$

Now we'll plug **[5]** into **[4]** to find a value for t.

$$t = -\frac{1}{3} + \frac{4}{3}(4)$$

$$t = \frac{15}{3}$$

[6] $t = 5$

Plugging **[5]** and **[6]** into **[3]**, the only equation we haven't used yet, gives

[3] $4 + t = 3 + 8s$

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$$4 + 5 = 3 + 8(4)$$

$$9 = 35$$

Because this equation is false, we know that the lines are not intersecting. Since they are not parallel, and not intersecting, they must be skew.

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Topic: Equation of a plane

Question: Find the equation of the plane that includes the points.

$P(1,-1,1)$

$Q(2,-2,0)$

$R(3,3,-1)$

Answer choices:

A $\quad x-y+z=12$

B $\quad x+z=2$

C $\quad x-z=2$

D $\quad -x+y-z=12$

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Solution: B

We'll start by turning the three points we've been given into two vectors.

$$\overrightarrow{PQ} = \left\langle Q_1 - P_1, Q_2 - P_2, Q_3 - P_3 \right\rangle$$

$$\overrightarrow{PQ} = \langle 2 - 1, -2 - (-1), 0 - 1 \rangle$$

$$\overrightarrow{PQ} = \langle 1, -1, -1 \rangle$$

and

$$\overrightarrow{PR} = \left\langle R_1 - P_1, R_2 - P_2, R_3 - P_3 \right\rangle$$

$$\overrightarrow{PR} = \langle 3 - 1, 3 - (-1), -1 - 1 \rangle$$

$$\overrightarrow{PR} = \langle 2, 4, -2 \rangle$$

Now we'll take the cross product of $\overrightarrow{PQ} = \langle 1, -1, -1 \rangle$ and $\overrightarrow{PR} = \langle 2, 4, -2 \rangle$ in order to find the normal vector to the plane.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i}\left(PQ_2PR_3 - PQ_3PR_2\right) - \mathbf{j}\left(PQ_1PR_3 - PQ_3PR_1\right) + \mathbf{k}\left(PQ_1PR_2 - PQ_2PR_1\right)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-1)(-2) - (-1)(4)\right]\mathbf{i} - \left[(1)(-2) - (-1)(2)\right]\mathbf{j} + \left[(1)(4) - (-1)(2)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (2 + 4)\mathbf{i} - (-2 + 2)\mathbf{j} + (4 + 2)\mathbf{k}$$

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$$\overrightarrow{PQ} \times \overrightarrow{PR} = 6\mathbf{i} - 0\mathbf{j} + 6\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 6,0,6 \rangle$$

Now we can plug this normal vector and either of the points we were given into the equation of the plane. We'll get a, b and c from the normal vector, and (x_1, y_1, z_1) from $P(1,-1,1)$, and this will give us the equation of the plane.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$6(x - 1) + 0\left[y - (-1)\right] + 6(z - 1) = 0$$

$$6x - 6 + 6z - 6 = 0$$

$$6x + 6z = 12$$

$$x + z = 2$$

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Topic: Equation of a plane

Question: Find the equation of the plane that includes the points.

$P(3,4,-5)$

$Q(5,-4,11)$

$R(10,-3,-7)$

Answer choices:

A $64x - 58y - 21z = -319$

B $-64x + 58y - 21z = 319$

C $64x - 58y + 21z = -319$

D $64x + 58y + 21z = 319$

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Solution: D

We'll start by turning the three points we've been given into two vectors.

$$\overrightarrow{PQ} = \left\langle Q_1 - P_1, Q_2 - P_2, Q_3 - P_3 \right\rangle$$

$$\overrightarrow{PQ} = \langle 5 - 3, -4 - 4, 11 - (-5) \rangle$$

$$\overrightarrow{PQ} = \langle 2, -8, 16 \rangle$$

and

$$\overrightarrow{PR} = \left\langle R_1 - P_1, R_2 - P_2, R_3 - P_3 \right\rangle$$

$$\overrightarrow{PR} = \langle 10 - 3, -3 - 4, -7 - (-5) \rangle$$

$$\overrightarrow{PR} = \langle 7, -7, -2 \rangle$$

Now we'll take the cross product of $\overrightarrow{PQ} = \langle 2, -8, 16 \rangle$ and $\overrightarrow{PR} = \langle 7, -7, -2 \rangle$ in order to find the normal vector to the plane.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i}\left(PQ_2PR_3 - PQ_3PR_2\right) - \mathbf{j}\left(PQ_1PR_3 - PQ_3PR_1\right) + \mathbf{k}\left(PQ_1PR_2 - PQ_2PR_1\right)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-8)(-2) - (16)(-7)\right]\mathbf{i} - \left[(2)(-2) - (16)(7)\right]\mathbf{j} + \left[(2)(-7) - (-8)(7)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (16 + 112)\mathbf{i} - (-4 - 112)\mathbf{j} + (-14 + 56)\mathbf{k}$$

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$$\overrightarrow{PQ} \times \overrightarrow{PR} = 128\mathbf{i} + 116\mathbf{j} + 42\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 128,116,42 \rangle$$

Now we can plug this normal vector and either of the points we were given into the equation of the plane. We'll get a, b and c from the normal vector, and (x_1, y_1, z_1) from $P(3,4,-5)$, and this will give us the equation of the plane.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$128(x-3) + 116(y-4) + 42\left[z - (-5)\right] = 0$$

$$128(x-3) + 116(y-4) + 42(z+5) = 0$$

$$128x - 384 + 116y - 464 + 42z + 210 = 0$$

$$128x + 116y + 42z - 638 = 0$$

$$128x + 116y + 42z = 638$$

$$64x + 58y + 21z = 319$$

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Topic: Equation of a plane

Question: Find the equation of the plane that includes the points.

$$P(-2,-2,-2)$$

$$Q(4,4,4)$$

$$R(15,-15,-15)$$

Answer choices:

A $x+y-z=0$

B $x-y+z=0$

C $y-z=0$

D $y-z=-8$

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Solution: C

We'll start by turning the three points we've been given into two vectors.

$$\overrightarrow{PQ} = \left\langle Q_1 - P_1, Q_2 - P_2, Q_3 - P_3 \right\rangle$$

$$\overrightarrow{PQ} = \langle 4-(-2), 4-(-2), 4-(-2) \rangle$$

$$\overrightarrow{PQ} = \langle 6,6,6 \rangle$$

and

$$\overrightarrow{PR} = \left\langle R_1 - P_1, R_2 - P_2, R_3 - P_3 \right\rangle$$

$$\overrightarrow{PR} = \langle 15-(-2), -15-(-2), -15-(-2) \rangle$$

$$\overrightarrow{PR} = \langle 17, -13, -13 \rangle$$

Now we'll take the cross product of $\overrightarrow{PQ} = \langle 6,6,6 \rangle$ and $\overrightarrow{PR} = \langle 17, -13, -13 \rangle$ in order to find the normal vector to the plane.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i}\left(PQ_2PR_3 - PQ_3PR_2\right) - \mathbf{j}\left(PQ_1PR_3 - PQ_3PR_1\right) + \mathbf{k}\left(PQ_1PR_2 - PQ_2PR_1\right)$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(6)(-13) - (6)(-13)\right]\mathbf{i} - \left[(6)(-13) - (6)(17)\right]\mathbf{j} + \left[(6)(-13) - (6)(17)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (-78+78)\mathbf{i} - (-78-102)\mathbf{j} + (-78-102)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = 0\mathbf{i} + 180\mathbf{j} - 180\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 0,180,-180 \rangle$$

Now we can plug this normal vector and either of the points we were given into the equation of the plane. We'll get a, b and c from the normal vector, and (x_1, y_1, z_1) from $Q(4,4,4)$, and this will give us the equation of the plane.

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$

$$(0)(x - 4) + (180)(y - 4) + (-180)(z - 4) = 0$$

$$0(x - 4) + 180(y - 4) - 180(z - 4) = 0$$

$$180y - 720 - 180z + 720 = 0$$

$$180y - 180z = 0$$

$$y - z = 0$$

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Topic: Intersection of a line and a plane

Question: Find the point where the line intersects the plane.

Line $x = -t$ $y = 1 + 2t$ $z = 4$

Plane $x + y + z = 4$

Answer choices:

A $(1, -1, 4)$

B $(1, 1, 4)$

C $(-1, -1, -4)$

D $(-1, 1, -4)$

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Solution: A

To find the coordinate point at which the line intersects the plane, we'll take the parametric equations of the line, and plug them into the equation of the plane. With $x = -t$, $y = 1 + 2t$ and $z = 4$, we get

$$x + y + z = 4$$

$$(-t) + (1 + 2t) + (4) = 4$$

$$-t + 1 + 2t + 4 = 4$$

$$t + 5 = 4$$

$$t = -1$$

Now we'll plug $t = -1$ back into the parametric equations.

$$x = -t$$

$$x = -(-1)$$

$$x = 1$$

and

$$y = 1 + 2t$$

$$y = 1 + 2(-1)$$

$$y = -1$$

and

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$$z = 4$$

Putting these values together tells us that the intersection point is $(1, -1, 4)$. Let's make sure that we've got the right point by making sure it still satisfies the equation of the plane.

$$x + y + z = 4$$

$$(1) + (-1) + (4) = 4$$

$$1 - 1 + 4 = 4$$

$$4 = 4$$

The equation is true, so $(1, -1, 4)$ is the point where the line intersects the plane.

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Topic: Intersection of a line and a plane

Question: Find the point where the line intersects the plane.

Line $\quad x = 2 + 2t \quad y = 4 - 5t \quad z = -3 + 2t$

Plane $\quad 4x + y - 3z = 6$

Answer choices:

A $\quad (12,21,7)$

B $\quad (-12,21,-7)$

C $\quad (-12,-21,-7)$

D $\quad (12,-21,7)$

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Solution: D

To find the coordinate point at which the line intersects the plane, we'll take the parametric equations of the line, and plug them into the equation of the plane. With $x = 2 + 2t$, $y = 4 - 5t$ and $z = -3 + 2t$, we get

$$4x + y - 3z = 6$$

$$4(2 + 2t) + (4 - 5t) - 3(-3 + 2t) = 6$$

$$8 + 8t + 4 - 5t + 9 - 6t = 6$$

$$-3t + 21 = 6$$

$$-3t = -15$$

$$t = 5$$

Now we'll plug $t = 5$ back into the parametric equations.

$$x = 2 + 2t$$

$$x = 2 + 2(5)$$

$$x = 12$$

and

$$y = 4 - 5t$$

$$y = 4 - 5(5)$$

$$y = -21$$

and

$$z = -3 + 2t$$

$$z = -3 + 2(5)$$

$$z = 7$$

Putting these values together tells us that the intersection point is $(12, -21, 7)$. Let's make sure that we've got the right point by making sure it still satisfies the equation of the plane.

$$4x + y - 3z = 6$$

$$4(12) + (-21) - 3(7) = 6$$

$$48 - 21 - 21 = 6$$

$$6 = 6$$

The equation is true, so $(12, -21, 7)$ is the point where the line intersects the plane.

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Topic: Intersection of a line and a plane

Question: Find the point where the line intersects the plane.

Line $x = 11 - 5t$ $y = -7 + 9t$ $z = -7 - 6t$

Plane $-5x - 7y + 5z = 12$

Answer choices:

A $\left(-\frac{1,013}{68}, \frac{953}{68}, \frac{158}{68}\right)$

B $\left(-\frac{1,013}{68}, -\frac{953}{68}, -\frac{158}{68}\right)$

C $\left(\frac{1,013}{68}, -\frac{953}{68}, -\frac{158}{68}\right)$

D $\left(\frac{1,013}{68}, \frac{953}{68}, \frac{158}{68}\right)$

Solution: C

To find the coordinate point at which the line intersects the plane, we'll take the parametric equations of the line, and plug them into the equation of the plane. With $x = 11 - 5t$, $y = -7 + 9t$ and $z = -7 - 6t$, we get

$$-5x - 7y + 5z = 12$$

$$-5(11 - 5t) - 7(-7 + 9t) + 5(-7 - 6t) = 12$$

$$-55 + 25t + 49 - 63t - 35 - 30t = 12$$

$$-68t - 41 = 12$$

$$-68t = 53$$

$$t = -\frac{53}{68}$$

Now we'll plug the value we just found for t back into the parametric equations.

$$x = 11 - 5t$$

$$x = 11 - 5\left(-\frac{53}{68}\right)$$

$$x = \frac{748}{68} + \frac{265}{68}$$

$$x = \frac{1{,}013}{68}$$

and

$$y = -7 + 9t$$

$$y = -7 + 9\left(-\frac{53}{68}\right)$$

$$y = -\frac{476}{68} - \frac{477}{68}$$

$$y = -\frac{953}{68}$$

and

$$z = -7 - 6t$$

$$z = -7 - 6\left(-\frac{53}{68}\right)$$

$$z = -\frac{476}{68} + \frac{318}{68}$$

$$z = -\frac{158}{68}$$

Putting these values together tells us that the intersection point is

$$\left(\frac{1{,}013}{68}, -\frac{953}{68}, -\frac{158}{68}\right)$$

Let's make sure that we've got the right point by making sure it still satisfies the equation of the plane.

$$-5x - 7y + 5z = 12$$

$$-5\left(\frac{1,013}{68}\right)-7\left(-\frac{953}{68}\right)+5\left(-\frac{158}{68}\right)=12$$

$$-\frac{5,065}{68}+\frac{6,671}{68}-\frac{790}{68}=12$$

$$\frac{816}{68}=12$$

$$12=12$$

The equation is true, so

$$\left(\frac{1,013}{68},-\frac{953}{68},-\frac{158}{68}\right)$$

is the point where the line intersects the plane.

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Topic: Parallel, perpendicular, and angle between planes

Question: Say whether the planes are parallel or perpendicular, otherwise find the angle between them.

$$2x + 3y - 5z = 3$$

$$4x + 6y - 10z = 17$$

Answer choices:

A Parallel

B Perpendicular

C $\theta = 23.4^\circ$

D $\theta = 66.6^\circ$

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Solution: A

First we'll test to see if planes are parallel by taking the ratio of their components. Since the planes are $2x+3y-5z=3$ and $4x+6y-10z=17$, we get

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$$

$$\frac{2}{4}=\frac{3}{6}=\frac{-5}{-10}$$

$$\frac{1}{2}=\frac{1}{2}=\frac{1}{2}$$

Because this equation is true, we can say that the planes are parallel.

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Topic: Parallel, perpendicular, and angle between planes

Question: Say whether the planes are parallel or perpendicular, otherwise find the angle between them.

$$7x - 5y - 2z = 1$$

$$-2x - 6y + 5z = 7$$

Answer choices:

A Parallel

B Perpendicular

C $\theta = 4.8^\circ$

D $\theta = 85.2^\circ$

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Solution: D

First we'll test to see if planes are parallel by taking the ratio of their components. Since the planes are $7x-5y-2z=1$ and $-2x-6y+5z=7$, we get

$$\frac{a_1}{b_1}=\frac{a_2}{b_2}=\frac{a_3}{b_3}$$

$$\frac{7}{-2}=\frac{-5}{-6}=\frac{-2}{5}$$

$$-\frac{7}{2}=\frac{5}{6}=-\frac{2}{5}$$

Because this equation is not true, the planes are not parallel. So we'll check to see if they're perpendicular by taking the dot product of the components.

$$a\cdot b=(7)(-2)+(-5)(-6)+(-2)(5)$$

$$a\cdot b=-14+30-10$$

$$a\cdot b=6$$

Because the dot product does not equal 0, the planes are not perpendicular. And because we've shown that they're neither parallel nor perpendicular, it means they must be skew, so we'll find the angle between them. To find the angle between them, we'll need the magnitude of the normal vectors of each plane. The normal vectors are given by the components, so the normal vectors are $\langle 7,-5,-2\rangle$ and $\langle -2,-6,5\rangle$. If we use the origin $(0,0,0)$ as (x_1,y_1,z_1), we get

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$$|a| = \sqrt{(a_1 - x_1)^2 + (a_2 - y_1)^2 + (a_3 - z_1)^2}$$

$$|a| = \sqrt{(7 - 0)^2 + (-5 - 0)^2 + (-2 - 0)^2}$$

$$|a| = \sqrt{49 + 25 + 4}$$

$$|a| = \sqrt{78}$$

and

$$|b| = \sqrt{(b_1 - x_1)^2 + (b_2 - y_1)^2 + (b_3 - z_1)^2}$$

$$|b| = \sqrt{(-2 - 0)^2 + (-6 - 0)^2 + (5 - 0)^2}$$

$$|b| = \sqrt{4 + 36 + 25}$$

$$|b| = \sqrt{65}$$

Now we'll plug $a \cdot b = 6$, $|a| = \sqrt{78}$, and $|b| = \sqrt{65}$ into the formula for the angle between planes.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{6}{\sqrt{78}\sqrt{65}}$$

$$\cos\theta = \frac{6}{\sqrt{5{,}070}}$$

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$$\theta = \arccos \frac{6}{\sqrt{5,070}}$$

$$\theta = 85.2^\circ$$

Topic: Parallel, perpendicular, and angle between planes

Question: Say whether the planes are parallel or perpendicular, otherwise find the angle between them.

$$4x + y + 5z = 1$$

$$-2x + 3y + z = 8$$

Answer choices:

A Parallel

B Perpendicular

C $\theta = 11.9^\circ$

D $\theta = 78.1^\circ$

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Solution: B

First we'll test to see if planes are parallel by taking the ratio of their components. Since the planes are $4x + y + 5z = 1$ and $-2x + 3y + z = 8$, we get

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

$$\frac{4}{-2} = \frac{1}{3} = \frac{5}{1}$$

$$-2 = \frac{1}{3} = 5$$

Because this equation is not true, the planes are not parallel. So we'll check to see if they're perpendicular by taking the dot product of the components.

$$a \cdot b = (4)(-2) + (1)(3) + (5)(1)$$

$$a \cdot b = -8 + 3 + 5$$

$$a \cdot b = 0$$

Because the dot product is 0, the planes are perpendicular.

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Topic: Parametric equations for the line of intersection of two planes

Question: Find the parametric equations for the line of intersection of the planes.

$$-x+y-z=2$$

$$x+y+z=4$$

Answer choices:

A $x=2t$ $y=3$ $z=1-2t$

B $x=1-2t$ $y=3$ $z=2t$

C $x=-2t$ $y=3$ $z=1+2t$

D $x=1+2t$ $y=3$ $z=-2t$

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Solution: D

We need to start by finding the vector equation for the line where the planes intersect each other. The formula we'll use is

$$r = r_0 + tv$$

To find v in the formula, we'll take the cross product of the normal vectors of the planes. Since the planes are $-x + y - z = 2$ and $x + y + z = 4$, their normal vectors are $a\langle -1,1,-1\rangle$ and $b\langle 1,1,1\rangle$, respectively. The cross product is given by

$$v = a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$v = a \times b = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v = a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v = \left[(1)(1) - (-1)(1)\right]\mathbf{i} - \left[(-1)(1) - (-1)(1)\right]\mathbf{j} + \left[(-1)(1) - (1)(1)\right]\mathbf{k}$$

$$v = (1 + 1)\mathbf{i} - (-1 + 1)\mathbf{j} + (-1 - 1)\mathbf{k}$$

$$v = 2\mathbf{i} - 0\mathbf{j} - 2\mathbf{k}$$

Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

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$$-x + y - z = 2$$

$$x + y + z = 4$$

then setting $z = 0$ gives

$$-x + y = 2$$

$$x + y = 4$$

If we add these equations together, we get

$$(-x + y) + (x + y) = 2 + 4$$

$$-x + x + y + y = 2 + 4$$

$$0 + 2y = 6$$

$$y = 3$$

Plugging $y = 3$ back into $-x + y = 2$ gives the corresponding value of x.

$$-x + y = 2$$

$$-x + 3 = 2$$

$$-x = -1$$

$$x = 1$$

Putting all of these values together tells us that $(1,3,0)$ is a point on the line of intersection. We'll change this to its vector representation and call it

$$r_0 = \mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$$

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Now we can plug $v = 2\mathbf{i} - 0\mathbf{j} - 2\mathbf{k}$ and $r_0 = \mathbf{i} + 3\mathbf{j} + 0\mathbf{k}$ into the vector equation for the line of intersection.

$$r = r_0 + tv$$

$$r = (\mathbf{i} + 3\mathbf{j} + 0\mathbf{k}) + t(2\mathbf{i} + 0\mathbf{j} - 2\mathbf{k})$$

$$r = \mathbf{i} + 3\mathbf{j} + 0\mathbf{k} + 2t\mathbf{i} + 0t\mathbf{j} - 2t\mathbf{k}$$

$$r = (\mathbf{i} + 2t\mathbf{i}) + (3\mathbf{j} + 0t\mathbf{j}) + (0\mathbf{k} - 2t\mathbf{k})$$

$$r = (\mathbf{i} + 2t\mathbf{i}) + (3\mathbf{j}) + (-2t\mathbf{k})$$

$$r = (1 + 2t)\mathbf{i} + 3\mathbf{j} - 2t\mathbf{k}$$

Now that we have the vector equation for the line of intersection, we can find the parametric equations from the coefficients. The parametric equations are

$$x = 1 + 2t$$

$$y = 3$$

$$z = -2t$$

Topic: Parametric equations for the line of intersection of two planes

Question: Find the parametric equations for the line of intersection of the planes.

$$2x + 4y + z = 1$$

$$x - 3y + 2z = 3$$

Answer choices:

A $\quad x = \frac{3}{2} + 11t \qquad y = -\frac{1}{2} - 3t \qquad z = -10t$

B $\quad x = \frac{3}{2} + 11t \qquad y = \frac{1}{2} - 3t \qquad z = -10t$

C $\quad x = -\frac{15}{8} - 11t \qquad y = -\frac{13}{8} + 2t \qquad z = 10t$

D $\quad x = \frac{15}{8} - 11t \qquad y = \frac{13}{8} + 2t \qquad z = 10t$

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Solution: A

We need to start by finding the vector equation for the line where the planes intersect each other. The formula we'll use is

$$r = r_0 + tv$$

To find v in the formula, we'll take the cross product of the normal vectors of the planes. Since the planes are $2x + 4y + z = 1$ and $x - 3y + 2z = 3$, their normal vectors are $a\langle 2,4,1\rangle$ and $b\langle 1,-3,2\rangle$, respectively. The cross product is given by

$$v = a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$v = a \times b = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v = a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v = \left[(4)(2) - (1)(-3)\right]\mathbf{i} - \left[(2)(2) - (1)(1)\right]\mathbf{j} + \left[(2)(-3) - (4)(1)\right]\mathbf{k}$$

$$v = (8 + 3)\mathbf{i} - (4 - 1)\mathbf{j} + (-6 - 4)\mathbf{k}$$

$$v = 11\mathbf{i} - 3\mathbf{j} - 10\mathbf{k}$$

Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

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$$2x + 4y + z = 1$$

$$x - 3y + 2z = 3$$

then setting $z = 0$ gives

[1] $2x + 4y = 1$

[2] $x - 3y = 3$

If we multiply [2] by 2, we get

[1] $2x + 4y = 1$

[3] $2x - 6y = 6$

Now we can subtract [3] from [1].

$$(2x + 4y) - (2x - 6y) = 1 - 6$$

$$2x - 2x + 4y + 6y = 1 - 6$$

$$10y = -5$$

$$y = -\frac{1}{2}$$

Plugging $y = -1/2$ back into $2x + 4y = 1$ gives the corresponding value of x.

$$2x + 4y = 1$$

$$2x + 4\left(-\frac{1}{2}\right) = 1$$

$$2x - 2 = 1$$

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$$2x=3$$

$$x=\frac{3}{2}$$

Putting all of these values together tells us that

$$\left(\frac{3}{2},-\frac{1}{2},0\right)$$

is a point on the line of intersection. We'll change this to its vector representation and call it

$$r_0=\frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}+0\mathbf{k}$$

Now we can plug $v=11\mathbf{i}-3\mathbf{j}-10\mathbf{k}$ and $r_0=(3/2)\mathbf{i}-(1/2)\mathbf{j}+0\mathbf{k}$ into the vector equation for the line of intersection.

$$r=r_0+tv$$

$$r=\left(\frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}+0\mathbf{k}\right)+t(11\mathbf{i}-3\mathbf{j}-10\mathbf{k})$$

$$r=\frac{3}{2}\mathbf{i}-\frac{1}{2}\mathbf{j}+0\mathbf{k}+11t\mathbf{i}-3t\mathbf{j}-10t\mathbf{k}$$

$$r=\left(\frac{3}{2}\mathbf{i}+11t\mathbf{i}\right)+\left(-\frac{1}{2}\mathbf{j}-3t\mathbf{j}\right)+(0\mathbf{k}-10t\mathbf{k})$$

$$r=\left(\frac{3}{2}+11t\right)\mathbf{i}+\left(-\frac{1}{2}-3t\right)\mathbf{j}-10t\mathbf{k}$$

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Now that we have the vector equation for the line of intersection, we can find the parametric equations from the coefficients. The parametric equations are

$$x = \frac{3}{2} + 11t$$

$$y = -\frac{1}{2} - 3t$$

$$z = -10t$$

Topic: Parametric equations for the line of intersection of two planes

Question: Find the parametric equations for the line of intersection of the planes.

$$-x + 3y + 6z = 3$$

$$6x - 6y + 3z = 9$$

Answer choices:

A $\quad x = -\frac{15}{4} - 45t \qquad y = -\frac{9}{4} - 39t \qquad z = 12t$

B $\quad x = \frac{15}{4} - 45t \qquad y = \frac{9}{4} - 39t \qquad z = 12t$

C $\quad x = \frac{15}{4} + 45t \qquad y = \frac{9}{4} + 39t \qquad z = -12t$

D $\quad x = -\frac{15}{4} + 45t \qquad y = -\frac{9}{4} + 39t \qquad z = -12t$

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Solution: C

We need to start by finding the vector equation for the line where the planes intersect each other. The formula we'll use is

$$r = r_0 + tv$$

To find v in the formula, we'll take the cross product of the normal vectors of the planes. Since the planes are $-x + 3y + 6z = 3$ and $6x - 6y + 3z = 9$, their normal vectors are $a\langle -1,3,6\rangle$ and $b\langle 6,-6,3\rangle$, respectively. The cross product is given by

$$v = a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$v = a \times b = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v = a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v = \left[(3)(3) - (6)(-6)\right]\mathbf{i} - \left[(-1)(3) - (6)(6)\right]\mathbf{j} + \left[(-1)(-6) - (3)(6)\right]\mathbf{k}$$

$$v = (9 + 36)\mathbf{i} - (-3 - 36)\mathbf{j} + (6 - 18)\mathbf{k}$$

$$v = 45\mathbf{i} + 39\mathbf{j} - 12\mathbf{k}$$

Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

$$-x + 3y + 6z = 3$$

$$6x - 6y + 3z = 9$$

then setting $z = 0$ gives

[1] $-x + 3y = 3$

[2] $6x - 6y = 9$

If we multiply [1] by 2, we get

[1] $-2x + 6y = 6$

[3] $6x - 6y = 9$

If we add these equations together, we get

$$(-2x + 6y) + (6x - 6y) = 6 + 9$$

$$-2x + 6x + 6y - 6y = 6 + 9$$

$$4x = 15$$

$$x = \frac{15}{4}$$

Plugging $x = 15/4$ back into $-x + 3y = 3$ gives the corresponding value of y.

$$-x + 3y = 3$$

$$-\frac{15}{4} + 3y = 3$$

$$3y=\frac{12}{4}+\frac{15}{4}$$

$$y=\frac{27}{4}\left(\frac{1}{3}\right)$$

$$y=\frac{27}{12}$$

$$y=\frac{9}{4}$$

Putting all of these values together tells us that

$$\left(\frac{15}{4},\frac{9}{4},0\right)$$

is a point on the line of intersection. We'll change this to its vector representation and call it

$$r_0=\frac{15}{4}\mathbf{i}+\frac{9}{4}\mathbf{j}+0\mathbf{k}$$

Now we can plug $v=45\mathbf{i}+39\mathbf{j}-12\mathbf{k}$ and $r_0=(15/4)\mathbf{i}+(9/4)\mathbf{j}+0\mathbf{k}$ into the vector equation for the line of intersection.

$$r=r_0+tv$$

$$r=\left(\frac{15}{4}\mathbf{i}+\frac{9}{4}\mathbf{j}+0\mathbf{k}\right)+t(45\mathbf{i}+39\mathbf{j}-12\mathbf{k})$$

$$r=\frac{15}{4}\mathbf{i}+\frac{9}{4}\mathbf{j}+0\mathbf{k}+45t\mathbf{i}+39t\mathbf{j}-12t\mathbf{k}$$

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$$r = \left(\frac{15}{4}\mathbf{i} + 45t\mathbf{i}\right) + \left(\frac{9}{4}\mathbf{j} + 39t\mathbf{j}\right) + (0\mathbf{k} - 12t\mathbf{k})$$

$$r = \left(\frac{15}{4} + 45t\right)\mathbf{i} + \left(\frac{9}{4} + 39t\right)\mathbf{j} - 12t\mathbf{k}$$

Now that we have the vector equation for the line of intersection, we can find the parametric equations from the coefficients. The parametric equations are

$$x = \frac{15}{4} + 45t$$

$$y = \frac{9}{4} + 39t$$

$$z = -12t$$

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Topic: Symmetric equations for the line of intersection of two planes

Question: Find the symmetric equations for the line of intersection of the planes.

$$x + y + z = 1$$

$$x - y + z = 3$$

Answer choices:

A $\quad \dfrac{x-2}{2} = \dfrac{z}{2}, \quad y = -1$

B $\quad \dfrac{x-2}{2} = -\dfrac{z}{2}, \quad y = -1$

C $\quad \dfrac{x-2}{2} = \dfrac{z}{2}, \quad y = 1$

D $\quad \dfrac{x-2}{2} = -\dfrac{z}{2}, \quad y = 1$

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Solution: B

The symmetric equations for the line of intersection are given by

$$\frac{x-c_1}{v_1}=\frac{y-c_2}{v_2}=\frac{z-c_3}{v_3}$$

where $c(c_1,c_2,c_3)$ comes from a point on the line of intersection, and where $v(v_1,v_2,v_3)$ is the cross product of the normal vectors of the planes.

The normal vectors of the planes are given by their components, which means that the normal vectors of $x+y+z=1$ and $x-y+z=3$ are $a\langle 1,1,1\rangle$ and $b\langle 1,-1,1\rangle$, respectively. The cross product is given by

$$v=a\times b=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$v=a\times b=\mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$v=a\times b=\left(a_2b_3-a_3b_2\right)\mathbf{i}-\left(a_1b_3-a_3b_1\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v=\left[(1)(1)-(1)(-1)\right]\mathbf{i}-\left[(1)(1)-(1)(1)\right]\mathbf{j}+\left[(1)(-1)-(1)(1)\right]\mathbf{k}$$

$$v=(1+1)\mathbf{i}-(1-1)\mathbf{j}+(-1-1)\mathbf{k}$$

$$v=2\mathbf{i}-0\mathbf{j}-2\mathbf{k}$$

Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

$$x + y + z = 1$$

$$x - y + z = 3$$

then setting $z = 0$ gives

$$x + y = 1$$

$$x - y = 3$$

We can add the equations together to get

$$(x + y) + (x - y) = 1 + 3$$

$$x + x + y - y = 1 + 3$$

$$2x = 4$$

$$x = 2$$

Plugging $x = 2$ back into $x + y = 1$ gives

$$x + y = 1$$

$$2 + y = 1$$

$$y = -1$$

If we put all these values together, we can say that $c(2, -1, 0)$ is a point on the line of intersection.

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Now we'll put $v = 2\mathbf{i} - 0\mathbf{j} - 2\mathbf{k}$ and $c(2, -1, 0)$ into the formula for the symmetric equations for the line of intersection.

$$\frac{x - c_1}{v_1} = \frac{y - c_2}{v_2} = \frac{z - c_3}{v_3}$$

$$\frac{x - 2}{2} = \frac{y - (-1)}{0} = \frac{z - 0}{-2}$$

Since we cannot divide by 0, we pull out the y equation as its own parametric equation, leaving the other two equations as symmetric equations.

$$\frac{x - 2}{2} = \frac{z - 0}{-2}, \quad y - (-1) = 0$$

$$\frac{x - 2}{2} = -\frac{z}{2}, \quad y + 1 = 0$$

$$\frac{x - 2}{2} = -\frac{z}{2}, \quad y = -1$$

Topic: Symmetric equations for the line of intersection of two planes

Question: Find the symmetric equations for the line of intersection of the planes.

$$2x + 2y + 2z = 3$$

$$-2x - y - z = 3$$

Answer choices:

A $\quad x = -\frac{9}{2}, \quad -\frac{y-6}{2} = \frac{z}{2}$

B $\quad x = -\frac{9}{2}, \quad \frac{y-6}{2} = \frac{z}{2}$

C $\quad x = \frac{9}{2}, \quad -\frac{y-6}{2} = \frac{z}{2}$

D $\quad x = \frac{9}{2}, \quad \frac{y-6}{2} = \frac{z}{2}$

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Solution: A

The symmetric equations for the line of intersection are given by

$$\frac{x-c_1}{v_1}=\frac{y-c_2}{v_2}=\frac{z-c_3}{v_3}$$

where $c(c_1, c_2, c_3)$ comes from a point on the line of intersection, and where $v(v_1, v_2, v_3)$ is the cross product of the normal vectors of the planes.

The normal vectors of the planes are given by their components, which means that the normal vectors of $2x+2y+2z=3$ and $-2x-y-z=3$ are $a\langle 2,2,2\rangle$ and $b\langle -2,-1,-1\rangle$, respectively. The cross product is given by

$$v=a\times b=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$v=a\times b=\mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}-\mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}+\mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$v=a\times b=\left(a_2b_3-a_3b_2\right)\mathbf{i}-\left(a_1b_3-a_3b_1\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v=\left[(2)(-1)-(2)(-1)\right]\mathbf{i}-\left[(2)(-1)-(2)(-2)\right]\mathbf{j}+\left[(2)(-1)-(2)(-2)\right]\mathbf{k}$$

$$v=(-2+2)\mathbf{i}-(-2+4)\mathbf{j}+(-2+4)\mathbf{k}$$

$$v=0\mathbf{i}-2\mathbf{j}+2\mathbf{k}$$

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Now we'll need to find a point on the line of intersection, which we can do by setting $z=0$ in both equations, and then solving what remains as a system of equations. If the planes are

$$2x+2y+2z=3$$

$$-2x-y-z=3$$

then setting $z=0$ gives

$$2x+2y=3$$

$$-2x-y=3$$

We can add the equations together to get

$$(2x+2y)+(-2x-y)=3+3$$

$$2x-2x+2y-y=3+3$$

$$y=6$$

Plugging $y=6$ back into $2x+2y=3$ gives

$$2x+2y=3$$

$$2x+2(6)=3$$

$$2x+12=3$$

$$2x=-9$$

$$x=-\frac{9}{2}$$

If we put all these values together, we can say that

$$c\left(-\frac{9}{2},6,0\right)$$

is a point on the line of intersection.

Now we'll put $v = 0\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$ and $c(-9/2,6,0)$ into the formula for the symmetric equations for the line of intersection.

$$\frac{x - c_1}{v_1} = \frac{y - c_2}{v_2} = \frac{z - c_3}{v_3}$$

$$\frac{x - \left(-\frac{9}{2}\right)}{0} = \frac{y - 6}{-2} = \frac{z - 0}{2}$$

Since we cannot divide by 0, we pull out the x equation as its own parametric equation, leaving the other two equations as symmetric equations.

$$x - \left(-\frac{9}{2}\right) = 0, \quad \frac{y - 6}{-2} = \frac{z - 0}{2}$$

$$x + \frac{9}{2} = 0, \quad -\frac{y - 6}{2} = \frac{z}{2}$$

$$x = -\frac{9}{2}, \quad -\frac{y - 6}{2} = \frac{z}{2}$$

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Topic: Symmetric equations for the line of intersection of two planes

Question: Find the symmetric equations for the line of intersection of the planes.

$$2x + 4y + 5z = 7$$

$$3x - 5y + z = 8$$

Answer choices:

A $\quad \dfrac{x + \frac{67}{22}}{29} = \dfrac{y + \frac{5}{22}}{13} = \dfrac{z}{22}$

B $\quad \dfrac{x + \frac{67}{22}}{29} = \dfrac{y + \frac{5}{22}}{13} = -\dfrac{z}{22}$

C $\quad \dfrac{x - \frac{67}{22}}{29} = \dfrac{y - \frac{5}{22}}{13} = -\dfrac{z}{22}$

D $\quad \dfrac{x - \frac{67}{22}}{29} = \dfrac{y - \frac{5}{22}}{13} = \dfrac{z}{22}$

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Solution: C

The symmetric equations for the line of intersection are given by

$$\frac{x-c_1}{v_1}=\frac{y-c_2}{v_2}=\frac{z-c_3}{v_3}$$

where $c(c_1,c_2,c_3)$ comes from a point on the line of intersection, and where $v(v_1,v_2,v_3)$ is the cross product of the normal vectors of the planes.

The normal vectors of the planes are given by their components, which means that the normal vectors of $2x+4y+5z=7$ and $3x-5y+z=8$ are $a\langle 2,4,5\rangle$ and $b\langle 3,-5,1\rangle$, respectively. The cross product is given by

$$v=a\times b=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$$v=a\times b=\mathbf{i}\begin{vmatrix}a_2 & a_3\\ b_2 & b_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}a_1 & a_3\\ b_1 & b_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}a_1 & a_2\\ b_1 & b_2\end{vmatrix}$$

$$v=a\times b=\left(a_2b_3-a_3b_2\right)\mathbf{i}-\left(a_1b_3-a_3b_1\right)\mathbf{j}+\left(a_1b_2-a_2b_1\right)\mathbf{k}$$

which means plugging in the normal vectors gives

$$v=\left[(4)(1)-(5)(-5)\right]\mathbf{i}-\left[(2)(1)-(5)(3)\right]\mathbf{j}+\left[(2)(-5)-(4)(3)\right]\mathbf{k}$$

$$v=(4+25)\mathbf{i}-(2-15)\mathbf{j}+(-10-12)\mathbf{k}$$

$$v=29\mathbf{i}+13\mathbf{j}-22\mathbf{k}$$

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Now we'll need to find a point on the line of intersection, which we can do by setting $z = 0$ in both equations, and then solving what remains as a system of equations. If the planes are

$$2x + 4y + 5z = 7$$

$$3x - 5y + z = 8$$

then setting $z = 0$ gives

[1] $2x + 4y = 7$

[2] $3x - 5y = 8$

We can multiply [1] by 3 and [2] by 2 to get

[3] $6x + 12y = 21$

[4] $6x - 10y = 16$

Now we can subtract [4] from [3] to get

$$(6x + 12y) - (6x - 10y) = 21 - 16$$

$$6x - 6x + 12y + 10y = 21 - 16$$

$$22y = 5$$

$$y = \frac{5}{22}$$

Plugging $y = 5/22$ back into $2x + 4y = 7$ gives

$$2x + 4y = 7$$

$$2x + 4\left(\frac{5}{22}\right) = 7$$

$$2x + \frac{20}{22} = 7$$

$$2x = \frac{154}{22} - \frac{20}{22}$$

$$2x = \frac{134}{22}$$

$$x = \frac{134}{44}$$

$$x = \frac{67}{22}$$

If we put all these values together, we can say that

$$c\left(\frac{67}{22}, \frac{5}{22}, 0\right)$$

is a point on the line of intersection.

Now we'll put $v = 29\mathbf{i} + 13\mathbf{j} - 22\mathbf{k}$ and $c\left(67/22, 5/22, 0\right)$ into the formula for the symmetric equations for the line of intersection.

$$\frac{x - c_1}{v_1} = \frac{y - c_2}{v_2} = \frac{z - c_3}{v_3}$$

$$\frac{x - \frac{67}{22}}{29} = \frac{y - \frac{5}{22}}{13} = \frac{z - 0}{-22}$$

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$$\frac{x-\frac{67}{22}}{29}=\frac{y-\frac{5}{22}}{13}=-\frac{z}{22}$$

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Topic: Distance between a point and a line

Question: Find the distance between the point and the line.

Point $(1,-1,-1)$

Line $x=1-t$ $y=2t$ $z=-1$

Answer choices:

A $\dfrac{1}{\sqrt{5}}$

B $\dfrac{1}{\sqrt{25}}$

C 5

D $\sqrt{5}$

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Solution: A

We have to start by converting the parametric equations to a vector equation. Since we have $x = 1 - t$, $y = 2t$, and $z = -1$, we get

$$r = (1 - t)\mathbf{i} + (2t)\mathbf{j} + (-1)\mathbf{k}$$

$$r = (1 - t)\mathbf{i} + 2t\mathbf{j} - \mathbf{k}$$

Now we'll rearrange the vector equation until it matches the format $r = r_0 + tv$.

$$r = \mathbf{i} - t\mathbf{i} + 2t\mathbf{j} - \mathbf{k}$$

$$r = (\mathbf{i} - \mathbf{k}) + (-t\mathbf{i} + 2t\mathbf{j})$$

$$r = (\mathbf{i} - \mathbf{k}) + t(-\mathbf{i} + 2\mathbf{j})$$

Matching this to $r = r_0 + tv$ gives us $r_0(1,0,-1)$ and $v\langle -1,2,0\rangle$. We'll rename the vector $v\langle -1,2,0\rangle$ to $a\langle -1,2,0\rangle$. We'll set a aside for a moment and work on the vector b, which connects the given point $(1,-1,-1)$ to the point on the line, $r_0(1,0,-1)$.

$$b\langle 1 - 1, -1 - 0, -1 - (-1)\rangle$$

$$b\langle 0,-1,0\rangle$$

Now we'll find the cross product of a and b.

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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$$a \times b = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

$$a \times b = \left[(2)(0) - (0)(-1)\right]\mathbf{i} - \left[(-1)(0) - (0)(0)\right]\mathbf{j} + \left[(-1)(-1) - (2)(0)\right]\mathbf{k}$$

$$a \times b = (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (1 - 0)\mathbf{k}$$

$$a \times b = 0\mathbf{i} - 0\mathbf{j} + 1\mathbf{k}$$

$$a \times b = \langle 0,0,1 \rangle$$

Then we need the magnitude of the cross product of a and b.

$$\left| a \times b \right| = \sqrt{(0)^2 + (0)^2 + (1)^2}$$

$$\left| a \times b \right| = \sqrt{1}$$

$$\left| a \times b \right| = 1$$

We also need the magnitude of $a\langle -1,2,0 \rangle$.

$$|a| = \sqrt{(-1)^2 + (2)^2 + (0)^2}$$

$$|a| = \sqrt{1 + 4}$$

$$|a| = \sqrt{5}$$

Finally, we'll use the distance formula to find the distance from the point to the line.

$$d = \frac{\left|a \times b\right|}{|a|}$$

$$d = \frac{1}{\sqrt{5}}$$

Topic: Distance between a point and a line

Question: Find the distance between the point and the line.

Point $(1,1,1)$

Line $x = 2 + t$ $y = 1 - 2t$ $z = 3t$

Answer choices:

A $\sqrt{\frac{4}{3}}$

B $\sqrt{\frac{7}{12}}$

C $\sqrt{\frac{12}{7}}$

D $\sqrt{\frac{3}{4}}$

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Solution: C

We have to start by converting the parametric equations to a vector equation. Since we have $x = 2 + t$, $y = 1 - 2t$, and $z = 3t$, we get

$$r = (2 + t)\mathbf{i} + (1 - 2t)\mathbf{j} + (3t)\mathbf{k}$$

$$r = (2 + t)\mathbf{i} + (1 - 2t)\mathbf{j} + 3t\mathbf{k}$$

Now we'll rearrange the vector equation until it matches the format $r = r_0 + tv$.

$$r = 2\mathbf{i} + t\mathbf{i} + \mathbf{j} - 2t\mathbf{j} + 3t\mathbf{k}$$

$$r = (2\mathbf{i} + \mathbf{j}) + (t\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k})$$

$$r = (2\mathbf{i} + \mathbf{j}) + t(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

Matching this to $r = r_0 + tv$ gives us $r_0(2,1,0)$ and $v\langle 1, -2, 3\rangle$. We'll rename the vector $v\langle 1, -2, 3\rangle$ to $a\langle 1, -2, 3\rangle$. We'll set a aside for a moment and work on the vector b, which connects the given point $(1,1,1)$ to the point on the line, $r_0(2,1,0)$.

$$b\langle 1 - 2, 1 - 1, 1 - 0\rangle$$

$$b\langle -1, 0, 1\rangle$$

Now we'll find the cross product of a and b.

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

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$$a \times b = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

$$a \times b = \left[(-2)(1) - (3)(0)\right]\mathbf{i} - \left[(1)(1) - (3)(-1)\right]\mathbf{j} + \left[(1)(0) - (-2)(-1)\right]\mathbf{k}$$

$$a \times b = (-2 - 0)\mathbf{i} - (1 + 3)\mathbf{j} + (0 - 2)\mathbf{k}$$

$$a \times b = -2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$$

$$a \times b = \langle -2, -4, -2 \rangle$$

Then we need the magnitude of the cross product of a and b.

$$\left|a \times b\right| = \sqrt{(-2)^2 + (-4)^2 + (-2)^2}$$

$$\left|a \times b\right| = \sqrt{4 + 16 + 4}$$

$$\left|a \times b\right| = \sqrt{24}$$

We also need the magnitude of $a\langle 1, -2, 3\rangle$.

$$|a| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$

$$|a| = \sqrt{1 + 4 + 9}$$

$$|a| = \sqrt{14}$$

Finally, we'll use the distance formula to find the distance from the point to the line.

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$$d = \frac{|a \times b|}{|a|}$$

$$d = \frac{\sqrt{24}}{\sqrt{14}}$$

$$d = \sqrt{\frac{24}{14}}$$

$$d = \sqrt{\frac{12}{7}}$$

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Topic: Distance between a point and a line

Question: Find the distance between the point and the line.

Point $(2,-4,5)$

Line $x=-3+2t$ $y=3+t$ $z=2-5t$

Answer choices:

A $\sqrt{\frac{19}{2}}$

B $\sqrt{\frac{391}{5}}$

C $\sqrt{\frac{2}{19}}$

D $\sqrt{\frac{5}{391}}$

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Solution: B

We have to start by converting the parametric equations to a vector equation. Since we have $x=-3+2t$, $y=3+t$, and $z=2-5t$, we get

$$r=(-3+2t)\mathbf{i}+(3+t)\mathbf{j}+(2-5t)\mathbf{k}$$

Now we'll rearrange the vector equation until it matches the format $r=r_0+tv$.

$$r=-3\mathbf{i}+2t\mathbf{i}+3\mathbf{j}+t\mathbf{j}+2\mathbf{k}-5t\mathbf{k}$$

$$r=(-3\mathbf{i}+3\mathbf{j}+2\mathbf{k})+(2t\mathbf{i}+t\mathbf{j}-5t\mathbf{k})$$

$$r=(-3\mathbf{i}+3\mathbf{j}+2\mathbf{k})+t(2\mathbf{i}+\mathbf{j}-5\mathbf{k})$$

Matching this to $r=r_0+tv$ gives us $r_0(-3,3,2)$ and $v\langle 2,1,-5\rangle$. We'll rename the vector $v\langle 2,1,-5\rangle$ to $a\langle 2,1,-5\rangle$. We'll set a aside for a moment and work on the vector b, which connects the given point $(2,-4,5)$ to the point on the line, $r_0(-3,3,2)$.

$$b\langle 2-(-3),-4-3,5-2\rangle$$

$$b\langle 5,-7,3\rangle$$

Now we'll find the cross product of a and b.

$$a\times b=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

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$$a \times b = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

$$a \times b = \left[(1)(3) - (-5)(-7)\right]\mathbf{i} - \left[(2)(3) - (-5)(5)\right]\mathbf{j} + \left[(2)(-7) - (1)(5)\right]\mathbf{k}$$

$$a \times b = (3 - 35)\mathbf{i} - (6 + 25)\mathbf{j} + (-14 - 5)\mathbf{k}$$

$$a \times b = -32\mathbf{i} - 31\mathbf{j} - 19\mathbf{k}$$

$$a \times b = \langle -32, -31, -19\rangle$$

Then we need the magnitude of the cross product of a and b.

$$\left|a \times b\right| = \sqrt{(-32)^2 + (-31)^2 + (-19)^2}$$

$$\left|a \times b\right| = \sqrt{1{,}024 + 961 + 361}$$

$$\left|a \times b\right| = \sqrt{2{,}346}$$

We also need the magnitude of $a\langle 2,1,-5\rangle$.

$$|a| = \sqrt{(2)^2 + (1)^2 + (-5)^2}$$

$$|a| = \sqrt{4 + 1 + 25}$$

$$|a| = \sqrt{30}$$

Finally, we'll use the distance formula to find the distance from the point to the line.

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$$d = \frac{|a \times b|}{|a|}$$

$$d = \frac{\sqrt{2,346}}{\sqrt{30}}$$

$$d = \sqrt{\frac{2,346}{30}}$$

$$d = \sqrt{\frac{391}{5}}$$

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Topic: Distance between a point and a plane

Question: Find the distance between the point and the plane.

$$(1,1,1)$$

$$-x + 2y - z = 2$$

Answer choices:

A $-\frac{2}{\sqrt{6}}$

B $-\frac{1}{\sqrt{3}}$

C $\frac{2}{\sqrt{6}}$

D $\frac{1}{\sqrt{3}}$

Solution: C

The distance d between a point and a plane is given by the component of b along n, or the scalar projection of b along n. b is the vector connecting a point on the plane to the given point, and n is the normal vector to the plane.

$$d = \left|\text{comp}_n b\right| = \frac{|n \cdot b|}{|n|}$$

We can also write this formula as

$$d = \frac{\left|ax_1 + by_1 + cz_1 + d\right|}{\sqrt{a^2 + b^2 + c^2}}$$

where (x_1, y_1, z_1) is the given point, and where $ax + by + cz = -d$ is the equation of the plane. Since in this case the given point is $(1,1,1)$, we can plug this into the formula to get

$$d = \frac{\left|a(1) + b(1) + c(1) + d\right|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d = \frac{\left|a + b + c + d\right|}{\sqrt{a^2 + b^2 + c^2}}$$

The plane $-x + 2y - z = 2$ tells us that $a = -1$, $b = 2$, $c = -1$, and $d = -2$. Plugging these in and then simplifying gives the distance between the point and the plane.

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$$d = \frac{|-1+2-1-2|}{\sqrt{(-1)^2+(2)^2+(-1)^2}}$$

$$d = \frac{|-2|}{\sqrt{1+4+1}}$$

$$d = \frac{2}{\sqrt{6}}$$

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Topic: Distance between a point and a plane

Question: Find the distance between the point and the plane.

$$(0,3,-2)$$

$$2x+3y+z=-3$$

Answer choices:

A $\frac{5}{\sqrt{7}}$

B $\frac{10}{\sqrt{14}}$

C $-\frac{10}{\sqrt{14}}$

D $-\frac{5}{\sqrt{7}}$

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Solution: B

The distance d between a point and a plane is given by the component of b along n, or the scalar projection of b along n. b is the vector connecting a point on the plane to the given point, and n is the normal vector to the plane.

$$d = \left| \text{comp}_n b \right| = \frac{|n \cdot b|}{|n|}$$

We can also write this formula as

$$d = \frac{\left| ax_1 + by_1 + cz_1 + d \right|}{\sqrt{a^2 + b^2 + c^2}}$$

where (x_1, y_1, z_1) is the given point, and where $ax + by + cz = -d$ is the equation of the plane. Since in this case the given point is $(0,3,-2)$, we can plug this into the formula to get

$$d = \frac{\left| a(0) + b(3) + c(-2) + d \right|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d = \frac{\left| 3b - 2c + d \right|}{\sqrt{a^2 + b^2 + c^2}}$$

The plane $2x + 3y + z = -3$ tells us that $a = 2$, $b = 3$, $c = 1$, and $d = 3$. Plugging these in and then simplifying gives the distance between the point and the plane.

$$d = \frac{|3(3) - 2(1) + 3|}{\sqrt{(2)^2 + (3)^2 + (1)^2}}$$

$$d = \frac{|10|}{\sqrt{4 + 9 + 1}}$$

$$d = \frac{10}{\sqrt{14}}$$

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Topic: Distance between a point and a plane

Question: Find the distance between the point and the plane.

$$(-2,-1,5)$$

$$x-4y-2z=-6$$

Answer choices:

A $-\dfrac{2}{\sqrt{21}}$

B $\dfrac{2}{\sqrt{7}}$

C $-\dfrac{2}{\sqrt{7}}$

D $\dfrac{2}{\sqrt{21}}$

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Solution: D

The distance d between a point and a plane is given by the component of b along n, or the scalar projection of b along n. b is the vector connecting a point on the plane to the given point, and n is the normal vector to the plane.

$$d = \left| \text{comp}_n b \right| = \frac{|n \cdot b|}{|n|}$$

We can also write this formula as

$$d = \frac{\left| ax_1 + by_1 + cz_1 + d \right|}{\sqrt{a^2 + b^2 + c^2}}$$

where (x_1, y_1, z_1) is the given point, and where $ax + by + cz = -d$ is the equation of the plane. Since in this case the given point is $(-2, -1, 5)$, we can plug this into the formula to get

$$d = \frac{\left| a(-2) + b(-1) + c(5) + d \right|}{\sqrt{a^2 + b^2 + c^2}}$$

$$d = \frac{\left| -2a - b + 5c + d \right|}{\sqrt{a^2 + b^2 + c^2}}$$

The plane $x - 4y - 2z = -6$ tells us that $a = 1$, $b = -4$, $c = -2$, and $d = 6$. Plugging these in and then simplifying gives the distance between the point and the plane.

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$$d = \frac{\left|-2(1) - (-4) + 5(-2) + 6\right|}{\sqrt{(1)^2 + (-4)^2 + (-2)^2}}$$

$$d = \frac{\left|-2 + 4 - 10 + 6\right|}{\sqrt{1 + 16 + 4}}$$

$$d = \frac{2}{\sqrt{21}}$$

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Topic: Distance between parallel planes

Question: Find the distance between the parallel planes.

$$3x + 2y - z = 3$$

$$9x + 6y - 3z = 2$$

Answer choices:

A $\dfrac{1}{3\sqrt{2}}$

B $-\dfrac{7}{3\sqrt{14}}$

C $\dfrac{7}{3\sqrt{14}}$

D $-\dfrac{1}{3\sqrt{2}}$

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Solution: C

First we'll confirm that the planes

$$3x + 2y - z = 3$$

$$9x + 6y - 3z = 2$$

are parallel. To test whether the planes are parallel, we'll take the ratio of the components of the normal vectors to each plane.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

where the planes are given in the form

$$a_1x + a_2y + a_3z = c$$

$$b_1x + b_2y + b_3z = d$$

If the ratios are the same, then the planes are parallel.

This means the two normal vectors are $a\langle a_1, a_2, a_3\rangle$ and $b\langle b_1, b_2, b_3\rangle$. First we can determine our normal vectors. For the plane $3x + 2y - z = 3$, we'll get the normal vector $a\langle 3,2,-1\rangle$. For the plane $9x + 6y - 3z = 2$, we'll get the normal vector $b\langle 9,6,-3\rangle$. Now we can set up the ratio

$$\frac{3}{9} = \frac{2}{6} = \frac{-1}{-3}$$

$$\frac{1}{3} = \frac{1}{3} = \frac{1}{3}$$

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We can see that these ratios are all equal, which means that the planes are parallel.

Next we can find a point on one of the planes. We can take the plane $3x + 2y - z = 3$ and set $y = 0$ and $z = 0$.

$$3x + 2(0) - (0) = 3$$

$$3x = 3$$

$$x = 1$$

This means a point on the plane is $(1,0,0)$.

Now we can find the distance from the point to a plane using the distance formula

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

where the point is (x_1, y_1, z_1) and the plane is $ax + by + cz = -d$.

The point $(1,0,0)$ will give us $x_1 = 1$, $y_1 = 0$, and $z_1 = 0$. The plane $9x + 6y - 3z = 2$ will give us $a = 9$, $b = 6$, $c = -3$, and $d = -2$.

$$d = \frac{|(9)(1) + (6)(0) + (-3)(0) + (-2)|}{\sqrt{(9)^2 + (6)^2 + (-3)^2}}$$

$$d = \frac{|9 + 0 + 0 - 2|}{\sqrt{81 + 36 + 9}}$$

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$$d = \frac{|7|}{\sqrt{126}}$$

$$d = \frac{7}{3\sqrt{14}}$$

This is the distance between the planes.

Topic: Distance between parallel planes

Question: Find the distance between the parallel planes.

$$-2x + 1y - 2z = 6$$

$$-8x + 4y - 8z = -3$$

Answer choices:

A $\frac{9}{4}$

B $\frac{7}{4}$

C $\frac{3}{2}$

D $\frac{7}{2}$

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Solution: A

First we'll confirm that the planes

$$-2x + 1y - 2z = 6$$

$$-8x + 4y - 8z = -3$$

are parallel. To test whether the planes are parallel, we'll take the ratio of the components of the normal vectors to each plane.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

where the planes are given in the form

$$a_1x + a_2y + a_3z = c$$

$$b_1x + b_2y + b_3z = d$$

If the ratios are the same, then the planes are parallel.

This means the two normal vectors are $a\langle a_1, a_2, a_3\rangle$ and $b\langle b_1, b_2, b_3\rangle$. First we can determine our normal vectors. For the plane $-2x + 1y - 2z = 6$, we'll get the normal vector $a\langle -2,1,-2\rangle$. For the plane $-8x + 4y - 8z = -3$, we'll get the normal vector $b\langle -8,4,-8\rangle$. Now we can set up the ratio

$$\frac{-2}{-8} = \frac{1}{4} = \frac{-2}{-8}$$

$$\frac{1}{4} = \frac{1}{4} = \frac{1}{4}$$

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We can see that these ratios are all equal, which means that the planes are parallel.

Next we can find a point on one of the planes. We can take the plane $-2x + 1y - 2z = 6$ and set $y = 0$ and $z = 0$.

$$-2x + 1(0) - 2(0) = 6$$

$$-2x = 6$$

$$x = -3$$

This means a point on the plane is $(-3,0,0)$.

Now we can find the distance from the point to a plane using the distance formula

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

where the point is (x_1, y_1, z_1) and the plane is $ax + by + cz = -d$.

The point $(-3,0,0)$ will give us $x_1 = -3$, $y_1 = 0$, and $z_1 = 0$. The plane $-8x + 4y - 8z = -3$ will give us $a = -8$, $b = 4$, $c = -8$, and $d = 3$.

$$d = \frac{|(-8)(-3) + (4)(0) + (-8)(0) + (3)|}{\sqrt{(-8)^2 + (4)^2 + (-8)^2}}$$

$$d = \frac{|24 + 0 + 0 + 3|}{\sqrt{64 + 16 + 64}}$$

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$$d = \frac{|27|}{\sqrt{144}}$$

$$d = \frac{27}{12}$$

$$d = \frac{9}{4}$$

This is the distance between the planes.

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Topic: Distance between parallel planes

Question: Find the distance between the parallel planes.

$$-6x + 2y + 4z = -12$$

$$-9x + 3y + 6z = 2$$

Answer choices:

A $-\frac{10}{3\sqrt{7}}$

B $-\frac{20}{3\sqrt{14}}$

C $\frac{10}{3\sqrt{7}}$

D $\frac{20}{3\sqrt{14}}$

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Solution: D

First we'll confirm that the planes

$$-6x + 2y + 4z = -12$$

$$-9x + 3y + 6z = 2$$

are parallel. To test whether the planes are parallel, we'll take the ratio of the components of the normal vectors to each plane.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

where the planes are given in the form

$$a_1x + a_2y + a_3z = c$$

$$b_1x + b_2y + b_3z = d$$

If the ratios are the same, then the planes are parallel.

This means the two normal vectors are $a\langle a_1, a_2, a_3\rangle$ and $b\langle b_1, b_2, b_3\rangle$. First we can determine our normal vectors. For the plane $-6x + 2y + 4z = -12$, we'll get the normal vector $a\langle -6,2,4\rangle$. For the plane $-9x + 3y + 6z = 2$, we'll get the normal vector $b\langle -9,3,6\rangle$. Now we can set up the ratio

$$\frac{-6}{-9} = \frac{2}{3} = \frac{4}{6}$$

$$\frac{2}{3} = \frac{2}{3} = \frac{2}{3}$$

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We can see that these ratios are all equal, which means that the planes are parallel.

Next we can find a point on one of the planes. We can take the plane $-6x + 2y + 4z = -12$ and set $y = 0$ and $z = 0$.

$$-6x + 2(0) + 4(0) = -12$$

$$-6x = -12$$

$$x = 2$$

This means a point on the plane is $(2,0,0)$.

Now we can find the distance from the point to a plane using the distance formula

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

where the point is (x_1, y_1, z_1) and the plane is $ax + by + cz = -d$.

The point $(2,0,0)$ will give us $x_1 = 2$, $y_1 = 0$, and $z_1 = 0$. The plane $-9x + 3y + 6z = 2$ will give us $a = -9$, $b = 3$, $c = 6$, and $d = -2$.

$$d = \frac{|(-9)(2) + (3)(0) + (6)(0) + (-2)|}{\sqrt{(-9)^2 + (3)^2 + (6)^2}}$$

$$d = \frac{|-18 + 0 + 0 - 2|}{\sqrt{81 + 9 + 36}}$$

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$$d = \frac{|-20|}{\sqrt{126}}$$

$$d = \frac{20}{3\sqrt{14}}$$

This is the distance between the planes.

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Topic: Reducing equations to standard form

Question: What is the standard form and identity of the figure?

$$y^2 + z^2 = -x^2 + 1$$

Answer choices:

A $\quad x^2 + y^2 + z^2 = 1$ is an elliptic paraboloid

B $\quad x^2 + y^2 + z^2 = 1$ is an elliptic cone

C $\quad x^2 + y^2 + z^2 = 1$ is a cylinder

D $\quad x^2 + y^2 + z^2 = 1$ is a sphere

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Solution: D

The first thing to notice is the presence of the 1 on the right-hand side of the equation. We always like to have that constant alone on one side of the equation by itself, so we'll rearrange the equation.

$$y^2 + z^2 = -x^2 + 1$$

$$x^2 + y^2 + z^2 = 1$$

When all three variables are positive on one side of the equation, and the constant on the other side is 1, the equation $x^2 + y^2 + z^2 = 1$ is a sphere.

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Topic: Reducing equations to standard form

Question: What is the standard form and identity of the figure?

$$\frac{x}{2}+y^2+z^2=0$$

Answer choices:

A $\quad y^2+z^2=-\frac{x}{2}$ is an elliptic cone

B $\quad y^2+z^2=-\frac{x}{2}$ is an elliptic paraboloid

C $\quad y^2+z^2=-\frac{x}{2}$ is a hyperbolic paraboloid

D $\quad y^2+z^2=-\frac{x}{2}$ is an ellipsoid

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Solution: B

The first thing to notice is the presence of the 0 on the right-hand side of the equation. We never want to have a zero value on either side, so we'll rearrange the equation. Since we have two squared variables and one linear variable, we'll move the linear variable to the opposite side of the equation by itself.

$$\frac{x}{2}+y^2+z^2=0$$

$$y^2+z^2=-\frac{x}{2}$$

For a standard form where two of the variables are squared, the third variable is not squared, the equation represents an elliptic paraboloid.

Topic: Reducing equations to standard form

Question: What is the standard form and identity of the figure?

$$x^2 + 4x - y + 4 - z^2 + 2z = 0$$

Answer choices:

A $(x+2)^2 + (z-1)^2 = y - 1$ is a hyperbolic paraboloid centered at $(-2,1,1)$

B $(x+2)^2 - (z-1)^2 = y - 1$ is an elliptic paraboloid centered at $(-2,1,1)$

C $(x+2)^2 - (z-1)^2 = y - 1$ is a hyperbolic paraboloid centered at $(-2,1,1)$

D $(x+2)^2 + (z-1)^2 = y - 1$ is an elliptic paraboloid centered at $(-2,1,1)$

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Solution: C

The first thing to notice is that two of the variables have both squared and non-squared appearances. This indicates that we'll need to complete the square for these variables.

$$\left(x^2+4x\right)-y+4+\left(-z^2+2z\right)=0$$

$$\left(x^2+4x\right)-y+4-\left(z^2-2z\right)=0$$

Complete the squares.

$$\left[x^2+4x+\left(\frac{4}{2}\right)^2\right]-y+4-\left[z^2-2z+\left(\frac{-2}{2}\right)^2\right]-\left(\frac{4}{2}\right)^2+\left(\frac{-2}{2}\right)^2=0$$

$$\left(x^2+4x+4\right)-y+4-\left(z^2-2z+1\right)-4+1=0$$

$$(x+2)^2-y+4-(z-1)^2-4+1=0$$

$$(x+2)^2-(z-1)^2-y+1=0$$

Looking at our reference sheet, we can see that the two squared elements should stay together and the non-squared variable and the number element should be moved to the other side of the equation.

$$(x+2)^2-(z-1)^2=y-1$$

An equation in this form is a hyperbolic paraboloid with center $(-2,1,1)$.

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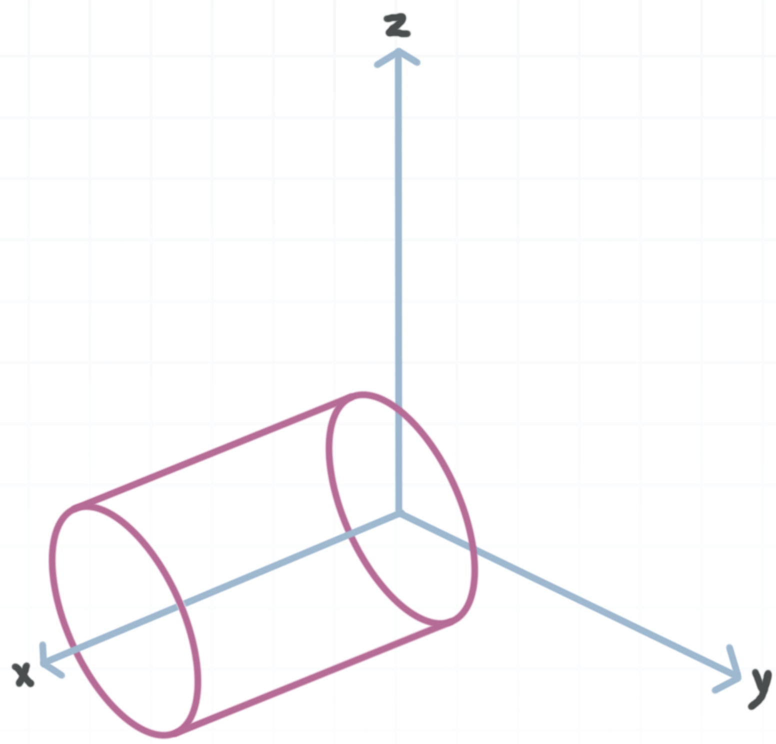
Topic: Sketching the surface

Question: Which equation represents this quadric surface?



Answer choices:

A $x^2 + z^2 = 1$

B $x^2 + y^2 = 1$

C $y^2 + z^2 = 1$

D $x^2 + y^2 + z^2 = 1$

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Solution: C

The first thing we can see in this surface is that it's a cylindrical shape. The standard form of a cylinder is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

for a cylinder with a center at $(0,0,0)$. The cylinder we have does have a center at $(0,0,0)$ but revolves around the x-axis, not the z-axis.

Answer choices A, B and D are all incorrect because the cylinder revolves around the x-axis and never contacts it. Therefore x cannot appear in the correct equation. Answer choice C is correct because the equation represents a cylinder, and there's no x variable in it.

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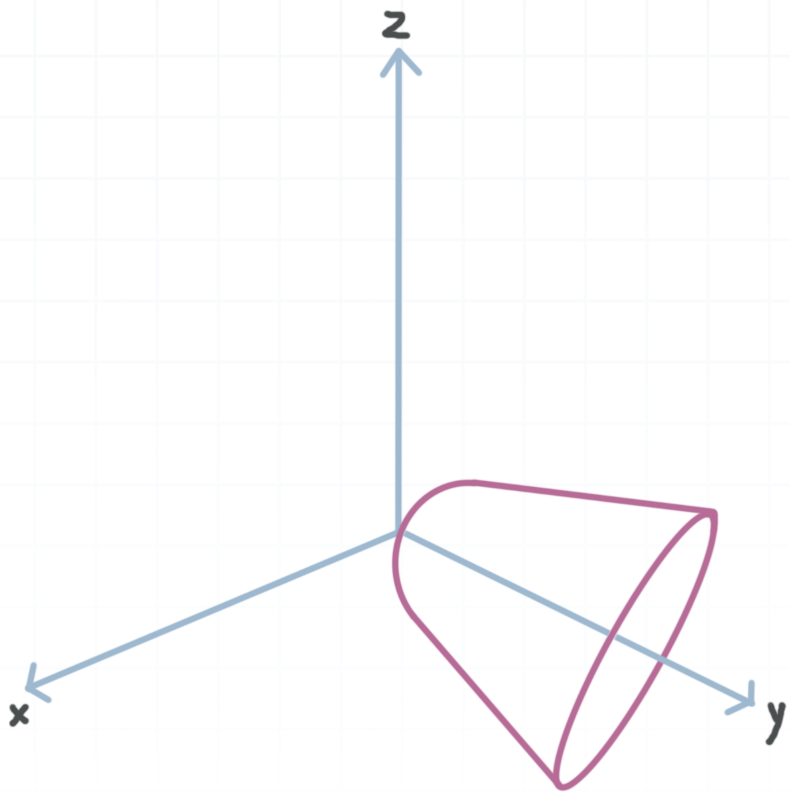
Topic: Sketching the surface

Question: Which equation represents this quadric surface?



Answer choices:

A $x^2 + z^2 = y^2$

B $x^2 + z^2 = y$

C $y^2 + z^2 = x$

D $x^2 + y^2 = z$

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Solution: B

The first thing we can see in this surface is that it's an elliptic paraboloid. The standard form of an elliptic paraboloid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z}{c}$$

for an elliptic paraboloid with a center at $(0,0,0)$. The elliptic paraboloid we have does have a center at $(0,0,0)$ but revolves around the y-axis not the z-axis.

Answer choice A is incorrect because this is not the equation of an elliptic paraboloid. Answer choices C and D are incorrect because these are equation of elliptic paraboloids that revolve around the x-axis and z-axis instead of the y-axis.

Answer choice B is correct because this is the equation of an elliptic paraboloid, and the y variable is not squared, which corresponds to the shape revolving around the y-axis.

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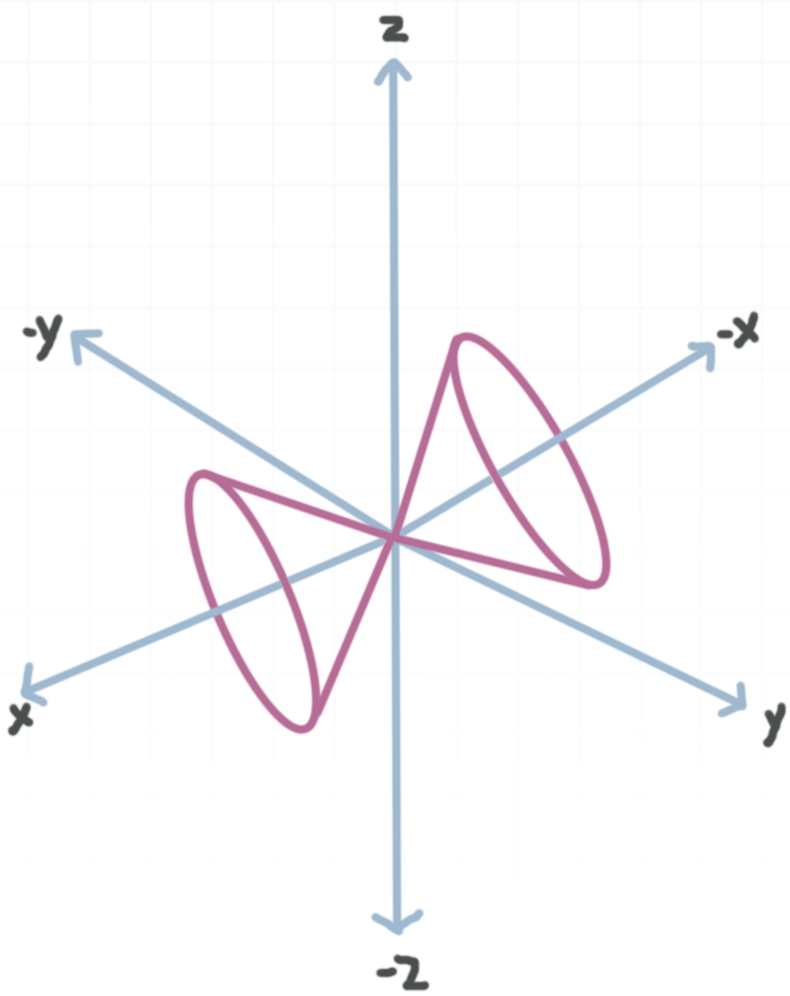
Topic: Sketching the surface

Question: Which equation and description represents this quadric surface?



Answer choices:

A $x^2 + y^2 = z^2$

B $x + y = z$

C $x^2 + z^2 = y^2$

D $y^2 + z^2 = x^2$

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Solution: D

The first thing we can see in this surface is that it's an elliptic cone. The standard form of a cone is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$$

for an elliptic cone with a center at $(0,0,0)$. The elliptic cone we have does have a center at $(0,0,0)$ but revolves around the x-axis not the z-axis.

Answer choices A and C are incorrect because these are the equations of elliptic cones that revolve around the z-axis and y-axis instead of the x-axis. Answer choice B is incorrect because this isn't the equation of an elliptic cone.

Option D is correct because this is the equation of an elliptic cone that revolves around the x-axis.

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Topic: Traces to sketch and identify the surface

Question: What are the traces and the identity of the surface?

$$x^2 + y^2 + z^2 = 1$$

Answer choices:

A The xy, yz, and xz traces are all ellipses, and the surface is an elliptic paraboloid.

B The xy, yz, and xz traces are all parabolas and the surface is an ellipsoid.

C The xy, yz, and xz traces are all ellipses and the surface is an ellipsoid.

D The xy, yz, and xz traces are all parabolas and the surface is an elliptic paraboloid.

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Solution: C

To find the xy trace, substitute $z = 0$ into the equation.

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + y^2 + (0)^2 = 1$$

$$x^2 + y^2 = 1$$

The xy trace is an ellipse with equation $x^2 + y^2 = 1$.

To find the yz trace, substitute $x = 0$ into the equation.

$$x^2 + y^2 + z^2 = 1$$

$$(0)^2 + y^2 + z^2 = 1$$

$$y^2 + z^2 = 1$$

The yz trace is an ellipse with equation $y^2 + z^2 = 1$.

To find the xz trace, substitute $y = 0$ into the equation.

$$x^2 + y^2 + z^2 = 1$$

$$x^2 + (0)^2 + z^2 = 1$$

$$x^2 + z^2 = 1$$

The xz trace is an ellipse with equation $x^2 + z^2 = 1$.

You can sketch the three traces out to generate the shape. When all three traces are ellipses, the surface shape is an ellipsoid.

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Topic: Traces to sketch and identify the surface

Question: What are the traces and the identity of the surface?

$$x^2 + y^2 = z^2$$

Answer choices:

A The yz trace is at the origin, and the xy and xz traces are intersecting lines. The surface shape is a parabolic cone.

B The yz trace is at the origin, and the xy and xz traces are intersecting lines. The surface shape is an elliptic cone.

C The xy trace is at the origin, and the yz and xz traces are intersecting lines. The surface shape is a parabolic cone.

D The xy trace is at the origin, and the yz and xz traces are intersecting lines. The surface shape is an elliptic cone.

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Solution: D

To find the xy trace, substitute $z = 0$ into the equation.

$$x^2 + y^2 = z^2$$

$$x^2 + y^2 = (0)^2$$

$$x^2 + y^2 = 0$$

The xy trace is at the origin.

To find the yz trace, substitute $x = 0$ into the equation.

$$x^2 + y^2 = z^2$$

$$(0)^2 + y^2 = z^2$$

$$y^2 = z^2$$

The yz trace is two intersecting lines.

To find the xz trace, substitute $y = 0$ into the equation.

$$x^2 + y^2 = z^2$$

$$x^2 + (0)^2 = z^2$$

$$x^2 = z^2$$

The xz trace is two intersecting lines.

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You can sketch the three traces out to generate the shape. The xy trace is at the origin, and the yz and xz traces of two intersecting lines each intersect. The surface shape is an elliptic cone.

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Topic: Traces to sketch and identify the surface

Question: What are the traces and the identity of the surface?

$$x^2 + y^2 = z$$

Answer choices:

A The xy trace is at the origin, and the yz and xz traces are parabolas. The surface shape is an elliptic hyperboloid.

B The xy trace is at the origin, and the yz and xz traces are parabolas. The surface shape is an elliptic paraboloid.

C The xy trace is at the origin, and the yz and xz traces are hyperbolas. The surface shape is an elliptic paraboloid.

D The xy trace is at the origin, and the yz and xz traces are hyperbolas. The surface shape is an elliptic hyperboloid.

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Solution: B

To find the xy trace, substitute $z = 0$ into the equation.

$$x^2 + y^2 = z$$

$$x^2 + y^2 = (0)$$

$$x^2 + y^2 = 0$$

The xy trace is at the origin.

To find the yz trace, substitute $x = 0$ into the equation.

$$x^2 + y^2 = z$$

$$(0)^2 + y^2 = z$$

$$y^2 = z$$

The yz trace is a parabola.

To find the xz trace, substitute $y = 0$ into the equation.

$$x^2 + y^2 = z$$

$$x^2 + (0)^2 = z$$

$$x^2 = z$$

The xz trace is a parabola.

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You can sketch the three traces out to generate the shape. The xy trace is at the origin, where the yz and xz traces of parabolas have their vertices. These traces describe an elliptic paraboloid.

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Topic: Domain of a multivariable function

Question: Find the domain of the multivariable function.

$$f(x, y) = \frac{\sqrt{x^2 + y}}{x - 4}$$

Answer choices:

A $y > -x^2$ and $x \neq 4$

B $y < -x^2$ and $x \neq 4$

C $y \geq -x^2$ and $x \neq 4$

D $y \leq -x^2$ and $x \neq 4$

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Solution: C

The domain of a function $f(x, y)$ is the set of all values of x and y that can be plugged into the function and yield valid results.

In the given function,

$$f(x, y) = \frac{\sqrt{x^2 + y}}{x - 4}$$

there are two situations in which we'd be unable to evaluate the function:

a negative result under the square root, or

a 0 result in the denominator.

Therefore, the domain of the function is the set of all values of x and y that avoid either of these scenarios.

First, to avoid taking the square root of a negative number, the quantity under the square root sign must be non-negative (but it can be 0), that is:

$$x^2 + y \geq 0$$

which can be rearranged to $y \geq -x^2$.

Next, to avoid dividing by 0, the denominator of the fraction must not be equal to 0:

$$x - 4 \neq 0$$

Therefore, $x \neq 4$.

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Topic: Domain of a multivariable function

Question: Find the domain of the multivariable function.

$$f(x, y) = x\ln(x + 3y)$$

Answer choices:

A $\quad y > -\frac{x}{3}$

B $\quad y > \frac{x}{3}$

C $\quad y < -\frac{x}{3}$

D $\quad y < \frac{x}{3}$

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Solution: A

The domain of a function $f(x, y)$ is the set of all values of x and y that can be plugged into the function and yield valid results.

In the given function,

$$f(x, y) = x \ln(x + 3y)$$

there is only one situation in which we'd be unable to evaluate the function:

a 0 or negative result inside the natural logarithm.

We would run into problems if the value of $x + 3y$ were 0 or negative, since the natural logarithm ($\ln$) can only accept positive values. As long as we can avoid that situation, we should be able to successfully evaluate $f(x, y)$ for any other values.

Therefore, the domain of our function is the set of values of x and y such that $x + 3y > 0$, which we can rearrange like this:

$$x + 3y > 0$$

$$3y > -x$$

$$y > -\frac{x}{3}$$

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Topic: Domain of a multivariable function

Question: Find the domain of the multivariable function.

$$f(x, y) = \frac{\sqrt{-2x}}{3y^2 - 3}$$

Answer choices:

A $\quad x \geq 0 \quad$ and $\quad y \neq 1$

B $\quad x \leq 0 \quad$ and $\quad y \neq 1$

C $\quad x \geq 0 \quad$ and $\quad y \neq 1 \quad$ and $\quad y \neq -1$

D $\quad x \leq 0 \quad$ and $\quad y \neq 1 \quad$ and $\quad y \neq -1$

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Solution: D

The domain of a function $f(x, y)$ is the set of all values of x and y that can be plugged into the function and yield valid results.

In the given function,

$$f(x, y) = \frac{\sqrt{-2x}}{3y^2 - 3}$$

there are two situations in which we'd be unable to evaluate the function:

a negative result under the square root, or

a 0 result in the denominator.

To avoid taking the square root of a negative number, the expression under the square root sign, $-2x$, must be non-negative:

$$-2x \geq 0$$

$$x \leq 0$$

To avoid dividing by 0, we must make sure the denominator of the fraction $3y^2 - 3$ is non-zero.

$$3y^2 - 3 \neq 0$$

$$3y^2 \neq 3$$

$$y^2 \neq 1$$

$$y \neq \pm 1$$

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The function $f(x, y)$ can be evaluated as long as all of these conditions are true. Therefore, the domain of $f(x, y)$ is

$x \leq 0$ and $y \neq 1$ and $y \neq -1$

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Topic: Limit of a multivariable function

Question: If the limit exists, find its value.

$$\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+2y}$$

Answer choices:

A 1

B $\frac{1}{3}$

C $\frac{1}{2}$

D Does not exist (DNE)

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Solution: B

Before anything else, we always want to try evaluating the limit at the point it approaches.

The limit exists if the answer is a real number or infinite.

The limit does not exist if the function is discontinuous at the point it approaches.

We'll evaluate the limit at the point it approaches.

$$\lim_{(x,y)\to(1,1)} \frac{xy}{x^2+2y}$$

$$\frac{(1)(1)}{(1)^2+2(1)}$$

$$\frac{1}{3}$$

Since we get a real-number answer, the limit exists, and the limit of the function is $1/3$ as we approach $(1,1)$.

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Topic: Limit of a multivariable function

Question: If the limit exists, find its value.

$$\lim_{(x,y,z)\to(1,1,1)} \frac{xy+4z}{2z-x^2+y}$$

Answer choices:

A $\quad \dfrac{5}{2}$

B $\quad 2$

C $\quad 5$

D $\quad$ Does not exist (DNE)

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Solution: A

Before anything else, we always want to try evaluating the limit at the point it approaches.

The limit exists if the answer is a real number or infinite.

The limit does not exist if the function is discontinuous at the point it approaches.

We'll evaluate the limit at the point it approaches.

$$\lim_{(x,y,z)\to(1,1,1)} \frac{xy+4z}{2z-x^2+y}$$

$$\frac{(1)(1)+4(1)}{2(1)-(1)^2+(1)}$$

$$\frac{5}{2}$$

Since we get a real-number answer, the limit exists, and the limit of the function is $5/2$ as we approach $(1,1,1)$.

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Topic: Precise definition of the limit for multivariable functions

Question: Which value of ϵ can be used to apply the definition of limit to $f(x, y)$?

$$f(x, y) = \frac{x + y}{3 + 2\sin x}$$

with $\delta = 0.00007$

Answer choices:

A $\quad 0.00007$

B $\quad 0.00014$

C $\quad 0.00021$

D $\quad 0.00028$

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Solution: B

For all real numbers x,

$$-1 \le \sin x \le 1$$

Multiply all sides of the inequality by 2, and then add 3 to each side.

$$-2 \le 2\sin x \le 2$$

$$-2+3 \le 3+2\sin x \le 2+3$$

$$1 \le 3+2\sin x \le 5$$

Replace each of the three parts with its inverse, while changing the directions of the inequality signs.

$$\frac{1}{1} \ge \frac{1}{3+2\sin x} \ge \frac{1}{5}$$

Multiply all sides by $|x+y|$.

$$|x+y| \ge \frac{|x+y|}{3+2\sin x} \ge \frac{|x+y|}{5}$$

By Triangle Inequality, replace $|x+y|$ by a greater expression $|x|+|y|$:

$$|x|+|y| \ge \frac{|x+y|}{3+2\sin x} \ge \frac{|x+y|}{5}$$

Because $f(0,0)=\dfrac{0+0}{3+2(0)}=0$, then

$$\left|f(x,y)-f(0,0)\right| = \left|\frac{x+y}{3+2\sin x}-0\right|$$

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$$\left|f(x,y) - f(0,0)\right| = \left|\frac{x+y}{3+2\sin x}\right|$$

$$\left|f(x,y) - f(0,0)\right| \le |x| + |y|$$

Applying the given value of $\delta = 0.00007$ to the inequality above results in

$$\left|f(x,y) - f(0,0)\right| = 0.00007 + 0.00007$$

$$\left|f(x,y) - f(0,0)\right| = 0.00014$$

$$\left|f(x,y) - f(0,0)\right| = \epsilon$$

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Topic: Precise definition of the limit for multivariable functions

Question: Using the polar form of the function, which of the following equations or inequalities leads to verification of the limit?

$$f(x, y) = \frac{x^5}{x^2 + y^2}$$

$$\lim_{(x,y)\to(0,0)} f(x, y) = \frac{x^5}{x^2 + y^2} = 0$$

Answer choices:

A $\delta < \epsilon$

B $\delta > \epsilon$

C $\delta = -\epsilon$

D $\delta = \epsilon$

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Solution: D

Use $x = r\cos\theta$ and $y = r\sin\theta$ to convert

$$f(x,y) = \frac{x^5}{x^2+y^2}$$

to

$$f(r,\theta) = \frac{r^5\cos^5\theta}{r^2}$$

$$f(r,\theta) = r^3\cos^5\theta$$

Therefore investigating the limit of $f(x,y)$ is equivalent to investigating

$$\lim_{x\to 0} r^3\cos^5\theta$$

Choosing $\delta = \epsilon$ for an arbitrary $\epsilon > 0$ results in

$$\left|f(r,\theta) - L\right| = \left|r^3\cos^5\theta - 0\right|$$

$$\left|f(r,\theta) - L\right| = \left|r^3\cos^5\theta\right|$$

$$\left|f(r,\theta) - L\right| = |r|^3\left|\cos\theta\right|^5$$

$$\left|f(r,\theta) - L\right| \le |r|$$

$$\left|f(r,\theta) - L\right| < \delta = \epsilon$$

where $0 < |r| < \delta$ holds true for the distance between r and 0.

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Topic: Precise definition of the limit for multivariable functions

Question: Find the condition.

The function

$$f(x, y) = \frac{121xy^2}{x^2 + y^2}$$

is defined on the region $R^2 - (0,0)$. To verify

$$\lim_{(x,y)\to(0,0)} f(x, y) = \frac{121xy^2}{x^2 + y^2} = 0$$

which of the following conditions must be applied to definition of the limit of $f(x, y)$?

Answer choices:

A $\quad \delta < \frac{\epsilon}{121}$

B $\quad \delta \leq \frac{\epsilon}{121}$

C $\quad \epsilon \leq \frac{\delta}{121}$

D $\quad \epsilon > \frac{\delta}{121}$

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Solution: B

By the conceptual definition of the limit of a multivariable function, for any arbitrary number $\epsilon > 0$ we must identify a real number $\delta > 0$, where $|f(x,y) - 0| < \epsilon$ and the inequality

$$0 < \sqrt{x^2 + y^2} < \delta$$

holds true for the distance between $(0,0)$ and (x,y).

That is, for an arbitrary number $\epsilon > 0$, we define the corresponding real number $\delta > 0$ such that the

$$\left| \frac{121xy^2}{x^2 + y^2} - 0 \right| < \epsilon$$

Simplify the left side.

$$\left| \frac{121xy^2}{x^2 + y^2} \right|$$

$$\frac{121\ |x|\ y^2}{x^2 + y^2}$$

$$\frac{121\ |x|\ y^2}{x^2 + y^2} \le 121\,|x|\,(1)$$

$$\frac{121\ |x|\ y^2}{x^2 + y^2} \le (121)\sqrt{x^2}$$

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$$\frac{121\ |x|\ y^2}{x^2+y^2} \le (121)\sqrt{x^2+y^2}$$

$$\frac{121\ |x|\ y^2}{x^2+y^2} < 121\delta$$

Therefore, choosing

$$\left| f(x,y) - 0 \right| < 121\delta$$

and integrating the inequality with

$$0 < \sqrt{x^2+y^2} < \delta$$

implies that

$$\delta \le \frac{\epsilon}{121}$$

leads to

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{121xy^2}{x^2+y^2} = 0$$

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Topic: Discontinuities of multivariable functions

Question: Find any discontinuities of the function.

$$f(x, y) = \sqrt{y - 2x}$$

Answer choices:

A The function is continuous

B The function is discontinuous for $y < 2x$

C The function is discontinuous for $y < -2x$

D The function is discontinuous for $y < \frac{x}{2}$

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Solution: B

A function is discontinuous wherever

it's undefined

it doesn't exist because of a hole in the graph

it doesn't exist because of an asymptote in the graph

its limit along two paths is different at the same point

The given function is a radical function, and radical functions don't exist as real numbers when the value under the square root is negative. Therefore, to find the points where the function is discontinuous, we solve the inequality,

$$y - 2x < 0$$

$$y < 2x$$

The function is discontinuous whenever $y < 2x$.

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Topic: Discontinuities of multivariable functions

Question: Find any discontinuities of the function.

$$f(x, y, z) = \frac{2x + 1 - z}{2y}$$

Answer choices:

A The function is continuous

B The function is discontinuous for $x = 0$

C The function is discontinuous for $y = 0$

D The function is discontinuous for $z = 1$

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Solution: C

A function is discontinuous wherever

it's undefined

it doesn't exist because of a hole in the graph

it doesn't exist because of an asymptote in the graph

its limit along two paths is different at the same point

The given function is a rational function, and rational functions don't exist as real numbers when the value in the denominator is 0. Therefore, to find the points where the function is discontinuous, we solve the equality,

$$2y = 0$$

$$y = 0$$

The function is discontinuous whenever $y = 0$.

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Topic: Discontinuities of multivariable functions

Question: Find any discontinuities of the function.

$$f(x, y, z) = \ln(z - 3) + xy$$

Answer choices:

A The function is continuous

B The function is discontinuous for $z = 3$

C The function is discontinuous for $z < 3$

D The function is discontinuous for $z \leq 3$

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Solution: D

A function is discontinuous wherever

it's undefined

it doesn't exist because of a hole in the graph

it doesn't exist because of an asymptote in the graph

its limit along two paths is different at the same point

The given function is a logarithmic function, and logarithmic functions don't exist as real numbers when the value inside the logarithm is negative or 0. Therefore, to find the points where the function is discontinuous, we solve the inequality,

$$z - 3 \leq 0$$

$$z \leq 3$$

The function is discontinuous whenever $z \leq 3$.

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Topic: Compositions of multivariable functions

Question: Find $f(g(t))$.

$$f(x,y) = -3x^2y^2\cos(x+y)$$

$$g(t) = \langle t^2, t^3 \rangle$$

Answer choices:

A $f(g(t)) = 3t^{10}\cos\left(t+t^3\right)$

B $f(g(t)) = 3t^{10}\cos\left(t^2+t\right)$

C $f(g(t)) = -3t^{10}\cos\left(t^2+t^3\right)$

D $f(g(t)) = -3t^{10}\sin\left(t^2+t^3\right)$

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Solution: C

We're looking for the composition $f(g(t))$, which means we need to plug $g(t)$ into $f(x, y)$.

$$f(g(t)) = f\left(t^2, t^3\right)$$

Because

$$f(x, y) = -3x^2y^2\cos(x + y)$$

we'll be plugging $x = t^2$ and $y = t^3$ into $f(x, y)$.

$$f(t) = -3\left(t^2\right)^2\left(t^3\right)^2\cos\left(t^2 + t^3\right)$$

$$f(t) = -3\left(t^4\right)\left(t^6\right)\cos\left(t^2 + t^3\right)$$

$$f(t) = -3t^{10}\cos\left(t^2 + t^3\right)$$

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Topic: Compositions of multivariable functions

Question: Find $h(f(x,y), g(x,y))$.

$$f(x,y) = x^2 - y^2$$

$$g(x,y) = x^2 + y^2$$

$$h(x,y) = \frac{x-y}{x+y}$$

Answer choices:

A $\quad h(x,y) = -\frac{y^2}{x^2}$

B $\quad h(x,y) = -\frac{x^2}{y^2}$

C $\quad h(x,y) = \frac{y^2}{x^2}$

D $\quad h(x,y) = \frac{x^2}{y^2}$

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Solution: A

We're looking for the composition $h(f(x,y),g(x,y))$, which means we need to plug $f(x,y)$ and $g(x,y)$ into $h(x,y)$.

Because

$$h(x,y)=\frac{x-y}{x+y}$$

we'll be plugging $x=x^2-y^2$ and $y=x^2+y^2$ into $h(x,y)$.

$$h(f(x,y),g(x,y))=\frac{\left(x^2-y^2\right)-\left(x^2+y^2\right)}{\left(x^2-y^2\right)+\left(x^2+y^2\right)}$$

$$h(f(x,y),g(x,y))=\frac{x^2-y^2-x^2-y^2}{x^2-y^2+x^2+y^2}$$

$$h(f(x,y),g(x,y))=\frac{-2y^2}{2x^2}$$

$$h(f(x,y),g(x,y))=-\frac{y^2}{x^2}$$

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Topic: Compositions of multivariable functions

Question: Given the following functions, which compositions are defined?

$$f(x, y) = x - 3y^2$$

$$g(x) = 1 - 3x^2$$

$$h(x, y) = 3x^2 - y$$

$$p(x, y) = x^3 - y^3$$

Answer choices:

A $f(g(x, y))$ and $g(f(x, y))$

B $g(f(x, y))$ and $g(p(x, y))$

C $h(p(x, y))$ and $h(f(x, y))$

D $p(g(x, y))$ and $g(f(x, y))$

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Solution: B

Answer choice B is the only set of compositions where both are defined.

First composition:

$$g(f(x,y)) = g\left(x - 3y^2\right)$$

$$g(f(x,y)) = 1 - 3\left(x - 3y^2\right)^2$$

$$g(f(x,y)) = 1 - 3\left(x^2 - 6xy^2 + 9y^4\right)$$

$$g(f(x,y)) = -3x^2 + 18xy^2 - 27y^4 + 1$$

Second composition:

$$g(p(x,y)) = g\left(x^3 - y^3\right)$$

$$g(p(x,y)) = 1 - 3\left(x^3 - y^3\right)^2$$

$$g(p(x,y)) = 1 - 3\left(x^6 - 2x^3y^3 + y^6\right)$$

$$g(p(x,y)) = -3x^6 + 6x^3y^3 - 3y^6 + 1$$

Topic: Partial derivatives in two variables

Question: Find the partial derivative(s).

Find f_y

for $f(x,y) = 4x^3y^2 + 2x^2y^2 + xy + 3x$

Answer choices:

A $\quad f_y = 24x^2y + 8xy + 1$

B $\quad f_y = 12x^2y^2 + 4xy^2 + y + 3$

C $\quad f_y = 8x^3y + 4x^2y + x$

D $\quad f_y = 12x^2y^2 + 8x^3y + 4x^2y + 4xy^2 + x + y + 3$

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Solution: C

To find the partial derivative f_y, we want to treat x as a constant, while differentiating $f(x, y)$ with respect to y. Remember that if we treat x as a constant, then x^2 and x^3 are also constants.

Therefore,

$$f(x, y) = 4x^3y^2 + 2x^2y^2 + xy + 3x$$

$$f(x, y) = \left(4x^3\right)y^2 + \left(2x^2\right)y^2 + (x)y + (3x)$$

$$f_y = \left(4x^3\right)(2y) + \left(2x^2\right)(2y) + (x)(1) + (3x)(0)$$

$$f_y = 8x^3y + 4x^2y + x$$

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Topic: Partial derivatives in two variables

Question: Find the partial derivative(s).

Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

for $f(x, y) = \sin\left(2x^2y\right)$

Answer choices:

A $\quad \frac{\partial f}{\partial x} = 4xy\cos\left(2x^2y\right)$ and $\frac{\partial f}{\partial y} = 2x^2\cos\left(2x^2y\right)$

B $\quad \frac{\partial f}{\partial x} = \cos(4xy)$ and $\frac{\partial f}{\partial y} = \cos\left(2x^2\right)$

C $\quad \frac{\partial f}{\partial x} = 4xy\cos(4xy)$ and $\frac{\partial f}{\partial y} = 2x^2\cos\left(2x^2\right)$

D $\quad \frac{\partial f}{\partial x} = 2x^2y\cos(4xy)$ and $\frac{\partial f}{\partial y} = 2x^2y\cos\left(2x^2\right)$

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Solution: A

To find $\partial f/\partial x$, we want to treat y as a constant, while differentiating $f(x, y)$ with respect to x. Since our function $f(x, y) = \sin\left(2x^2y\right)$ involves a function within a function, we need to use the chain rule to differentiate it, just like we would for a single variable function.

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\sin\left(2x^2y\right)\right]$$

$$\frac{\partial f}{\partial x} = \cos\left(2x^2y\right) \cdot \frac{\partial}{\partial x}\left(2x^2y\right)$$

$$\frac{\partial f}{\partial x} = \cos\left(2x^2y\right) \cdot (2)(2x)(y)$$

$$\frac{\partial f}{\partial x} = 4xy\cos\left(2x^2y\right)$$

To find $\partial f/\partial y$, we follow a similar process, but this time we hold x as a constant, while differentiating $f(x, y)$ with respect to y.

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\sin\left(2x^2y\right)\right]$$

$$\frac{\partial f}{\partial y} = \cos\left(2x^2y\right) \cdot \frac{\partial}{\partial y}\left(2x^2y\right)$$

$$\frac{\partial f}{\partial y} = \cos\left(2x^2y\right) \cdot \left(2x^2\right)(1)$$

$$\frac{\partial f}{\partial y} = 2x^2\cos\left(2x^2y\right)$$

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Topic: Partial derivatives in two variables

Question: Find f_x and f_y for $f(x, y) = 2x^3 \cos y$.

Answer choices:

A	$f_x = -12x \sin y$	and	$f_y = -6x^2 \cos y$
B	$f_x = 6x^2 \cos y$	and	$f_y = -2x^3 \sin y$
C	$f_x = -6x^2 \sin y + 12x \cos y$	and	$f_y = -2x^3 \cos y - 6x^2 \sin y$
D	$f_x = -6x^2 \sin y$	and	$f_y = -6x^2 \sin y$

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Solution: B

Looking at the function $f(x,y)=2x^3\cos y$, it seems like we might need to use the product rule to differentiate it, since it's the product of two functions ($2x^2$ and $\cos y$). However, since finding partial derivatives involves differentiating with respect to only one variable at a time (while holding the other constant), and since x and y each only appear in one of these functions, there will effectively only be one variable function present in each case, so we can avoid use of the product rule.

For example, to find the partial derivative f_x, we want to treat y as a constant, while differentiating $f(x,y)$ with respect to x. If we treat y as a constant, then $\cos y$ is also effectively a constant, leaving x^3 as the only portion of the original function that needs to be differentiated.

$$f(x,y)=2x^3\cos y$$

$$f(x,y)=(2\cos y)x^3$$

$$f_x=(2\cos y)3x^2$$

$$f_x=6x^2\cos y$$

Similarly, to find f_y, we can treat x (and therefore x^3) as a constant, and differentiate $f(x,y)$ with respect to y.

$$f(x,y)=2x^3\cos y$$

$$f(x,y)=\left(2x^3\right)\cos y$$

$$f_y=\left(2x^3\right)(-\sin y)$$

$$f_y = -2x^3 \sin y$$

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Topic: Partial derivatives in three or more variables

Question: Find the partial derivative(s).

Find f_x, f_y, and f_z

for $f(x, y, z) = 2x^3e^{2y}\ln z$

Answer choices:

A $\quad f_x = 12x^2e^{2y}\ln z \qquad f_y = 4x^3e^{2y}\ln z \qquad f_z = \dfrac{12x^2e^{2y}}{z}$

B $\quad f_x = \dfrac{4x^3e^{2y}}{z} \qquad f_y = \dfrac{6x^2e^{2y}}{z} \qquad f_z = 12x^2e^{2y}\ln z$

C $\quad f_x = 6x^2e^{2y}\ln z \qquad f_y = 4x^3e^{2y}\ln z \qquad f_z = \dfrac{2x^3e^{2y}}{z}$

D $\quad f_x = \dfrac{12x^2e^{2y}}{z} \qquad f_y = \dfrac{12x^2e^{2y}}{z} \qquad f_z = \dfrac{12x^2e^{2y}}{z}$

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Solution: C

To find the partial derivative f_x, we want to treat y and z as constants when we differentiate $f(x, y, z)$ with respect to x. Remember that if we treat y and z as constants, then e^{2y} and $\ln z$ are also constants. Therefore, we don't need to use the product rule here, since we're just differentiating a function of x multiplied by a constant.

$$f(x, y, z) = 2x^3e^{2y}\ln z$$

$$f(x, y, z) = \left(2e^{2y}\ln z\right)x^3$$

$$f_x = \left(2e^{2y}\ln z\right)3x^2$$

$$f_x = 6x^2e^{2y}\ln z$$

Similarly, to find f_y we treat x and z (and therefore x^3 and $\ln z$) as constants, while differentiating $f(x, y, z)$ with respect to y. Note that in this case we need to use the chain rule to differentiate e^{2y}.

$$f(x, y, z) = 2x^3e^{2y}\ln z$$

$$f(x, y, z) = \left(2x^3\ln z\right)e^{2y}$$

$$f_y = \left(2x^3\ln z\right)2e^{2y}$$

$$f_y = 4x^3e^{2y}\ln z$$

Finally, to find f_z, treat x and y as constants while differentiating $f(x, y, z)$ with respect to z.

$$f(x, y, z) = 2x^3e^{2y}\ln z$$

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$$f(x, y, z) = \left(2x^3e^{2y}\right) \ln z$$

$$f_z = \left(2x^3e^{2y}\right) \frac{1}{z}$$

$$f_z = \frac{2x^3e^{2y}}{z}$$

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Topic: Partial derivatives in three or more variables

Question: Find the partial derivative(s).

Find f_z

for $f(x, y, z) = 2x^2yz^3 \ln z$

Answer choices:

A $f_z = 12xz^3 \ln z$

B $f_z = \dfrac{12x}{z}$

C $f_z = 4x^2yz$

D $f_z = 2x^2yz^2 + 6x^2yz^2 \ln z$

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Solution: D

To find the partial derivative f_z, we want to treat x and y as constants, while differentiating $f(x, y, z)$ with respect to z. Remember that if we treat x and y as constants, then x^2y is also a constant, so we can rearrange the function like this:

$$f(x, y, z) = 2x^2yz^3 \ln z$$

$$f(x, y, z) = \left(2x^2y\right) z^3 \ln z$$

This shows us that the function we are differentiating is a product of two expressions containing our variable z, along with a constant multiplier $(2x^2y)$. We must therefore use the product rule to find f_z, and we get

$$f(x, y, z) = \left(2x^2y\right) z^3 \ln z$$

$$f_z = \left(2x^2y\right)\left(z^3 \cdot \frac{1}{z} + 3z^2 \cdot \ln z\right)$$

$$f_z = \left(2x^2y\right)\left(z^2 + 3z^2 \ln z\right)$$

$$f_z = 2x^2yz^2 + 6x^2yz^2 \ln z$$

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Topic: Partial derivatives in three or more variables

Question: Find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ for $f(x, y, z) = x^3y + x^2y^2z + 2yz^2$.

Answer choices:

A $\quad \frac{\partial f}{\partial x} = 2x^2y + 2xy^2z + 2yz^2 \quad \frac{\partial f}{\partial y} = x^3 + 2x^2yz + 2z^2 \quad \frac{\partial f}{\partial z} = x^3y + x^2y^2z + 4yz$

B $\quad \frac{\partial f}{\partial x} = 3x^2y + 2xy^2z \quad \frac{\partial f}{\partial y} = x^3 + 2x^2yz + 2z^2 \quad \frac{\partial f}{\partial z} = x^2y^2 + 4yz$

C $\quad \frac{\partial f}{\partial x} = x^3 + 2x^2y + 4z \quad \frac{\partial f}{\partial y} = 2x^2y + 2xy^2 + 4z \quad \frac{\partial f}{\partial z} = 3x^2 + 4xy + 2z^2$

D $\quad \frac{\partial f}{\partial x} = 2x^2 + 2xy^2 \quad \frac{\partial f}{\partial y} = x^3 + 2x^2y + 2z^2 \quad \frac{\partial f}{\partial z} = x^2y^2 + 4z$

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Solution: B

To find the partial derivative $\partial f/\partial x$, we want to treat y and z as constants, while differentiating $f(x, y, z)$ with respect to x. Remember that if we treat y and z as constants, then y^2 and z^2 are also constants, so we can rearrange and differentiate the function as follows:

$$f(x, y, z) = x^3y + x^2y^2z + 2yz^2$$

$$f(x, y, z) = (y)x^3 + \left(y^2z\right)x^2 + \left(2yz^2\right)$$

$$\frac{\partial f}{\partial x} = (y)3x^2 + \left(y^2z\right)2x + \left(2yz^2\right)0$$

$$\frac{\partial f}{\partial x} = 3x^2y + 2xy^2z$$

Then to find $\partial f/\partial y$, treat x and z as constants while differentiating $f(x, y, z)$ with respect to y.

$$f(x, y, z) = x^3y + x^2y^2z + 2yz^2$$

$$f(x, y, z) = \left(x^3\right)y + \left(x^2z\right)y^2 + \left(2z^2\right)y$$

$$\frac{\partial f}{\partial y} = \left(x^3\right)1 + \left(x^2z\right)2y + \left(2z^2\right)1$$

$$\frac{\partial f}{\partial y} = x^3 + 2x^2yz + 2z^2$$

To find $\partial f/\partial z$, treat x and y as constants while differentiating $f(x, y, z)$ with respect to z.

$$f(x, y, z) = x^3y + x^2y^2z + 2yz^2$$

$$f(x, y, z) = \left(x^3y\right) + \left(x^2y^2\right)z + (2y)z^2$$

$$\frac{\partial f}{\partial z} = \left(x^3y\right)0 + \left(x^2y^2\right)1 + (2y)2z$$

$$\frac{\partial f}{\partial z} = x^2y^2 + 4yz$$

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Topic: Higher order partial derivatives

Question: Find the partial derivative(s).

Find all four second-order partial derivatives.

$$f(x, y) = \sin x \cos 2y$$

Answer choices:

A $\quad f_{xx} = -\sin x \cos 2y \qquad f_{xy} = -2\cos x \sin 2y$

$\quad f_{yy} = -4\sin x \cos 2y \qquad f_{yx} = -2\cos x \sin 2y$

B $\quad f_{xx} = -4\sin x \cos 2y \qquad f_{xy} = -2\cos x \sin 2y$

$\quad f_{yy} = -\sin x \cos 2y \qquad f_{yx} = -2\cos x \sin 2y$

C $\quad f_{xx} = -2\cos x \sin 2y \qquad f_{xy} = -\sin x \cos 2y$

$\quad f_{yy} = -2\cos x \sin 2y \qquad f_{yx} = -4\sin x \cos 2y$

D $\quad f_{xx} = -2\cos x \sin 2y \qquad f_{xy} = -4\sin x \cos 2y$

$\quad f_{yy} = -2\cos x \sin 2y \qquad f_{yx} = -\sin x \cos 2y$

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Solution: A

To find the second-order partial derivatives, first we want to find the first-order partial derivatives f_x (by treating y as a constant and differentiating $f(x, y)$ with respect to x), and f_y (by treating x as a constant and differentiating $f(x, y)$ with respect to y).

$$f(x, y) = \sin x \cos 2y$$

$$f_x = (\cos 2y)(\cos x)$$

$$f_x = \cos x \cos 2y$$

$$f_y = (\sin x)(-2 \sin 2y)$$

$$f_y = -2 \sin x \sin 2y$$

Then, to find the second-order partial derivative f_{xx}, treat y as a constant while differentiating f_x with respect to x.

$$f_x = \cos x \cos 2y$$

$$f_{xx} = (\cos 2y)(-\sin x) = -\sin x \cos 2y$$

And find f_{xy} by treating x as a constant while differentiating f_x with respect to y.

$$f_x = \cos x \cos 2y$$

$$f_{xy} = (\cos x)(-2 \sin 2y) = -2 \cos x \sin 2y$$

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Finally, find f_{yy} and f_{yx} by differentiating f_y with respect to y (while holding x constant), and with respect to x (while holding y constant), respectively:

$$f_y = -2\sin x \sin 2y$$

$$f_{yy} = (-2\sin x)2\cos 2y = -4\sin x \cos 2y$$

$$f_{yx} = (-2\sin 2y)(\cos x) = -2\cos x \sin 2y$$

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Topic: Higher order partial derivatives

Question: Find the partial derivative(s).

Find f_{xzy}

for $f(x, y, z) = \sin\left(xy + z^2\right)$

Answer choices:

A $\quad f_{xzy} = \cos\left(xy + z^2\right) + 2z\cos\left(xy + z^2\right)$

B $\quad f_{xzy} = -2xz\cos\left(xy + z^2\right)$

C $\quad f_{xzy} = -4xz^2\cos\left(xy + z^2\right) + \sin\left(xy + z^2\right)$

D $\quad f_{xzy} = -2z\sin\left(xy + z^2\right) - 2xyz\cos\left(xy + z^2\right)$

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Solution: D

To find the third-order partial derivative f_{xzy}, first we want to find the first-order partial derivative f_x (by treating y and z as constants and differentiating $f(x,y,z)$ with respect to x), using the chain rule.

$$f(x,y,z) = \sin\left(xy + z^2\right)$$

$$f_x = \cos\left(xy + z^2\right) \cdot \frac{\partial}{\partial x}\left(xy + z^2\right)$$

$$f_x = \cos\left(xy + z^2\right) \cdot y = y\cos\left(xy + z^2\right)$$

Next, we can find the second-order partial derivative f_{xz} by treating x and y as constants while differentiating f_x with respect to z (again, using the chain rule).

$$f_x = y\cos\left(xy + z^2\right)$$

$$f_{xz} = y\left[-\sin\left(xy + z^2\right)\right] \cdot \frac{\partial}{\partial z}\left(xy + z^2\right)$$

$$f_{xz} = y\left[-\sin\left(xy + z^2\right)\right] \cdot 2z = -2yz\sin\left(xy + z^2\right)$$

Now we can find the third-order partial derivative f_{xzy} by treating x and z as constants while differentiating f_{xz} with respect to y (we'll use product rule and chain rule since f_{xz} is the product of two expressions containing y):

$$f_{xz} = -2yz\sin\left(xy + z^2\right)$$

$$f_{xzy} = (-2z)\left[\sin\left(xy + z^2\right)\right] + (-2yz)\left[\cos\left(xy + z^2\right)(x)\right]$$

$$f_{xzy} = -2z\sin\left(xy + z^2\right) - 2xyz\cos\left(xy + z^2\right)$$

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Topic: Higher order partial derivatives

Question: Find f_{xyx} and f_{yxy} for $f(x, y) = x^3y^2 + xy^3$.

Answer choices:

A $\quad f_{xyx} = 6x^2 + 3y^2 \quad$ and $\quad f_{yxy} = 6x^2 + 3y^2$

B $\quad f_{xyx} = 12xy \quad$ and $\quad f_{yxy} = 6x^2 + 6y$

C $\quad f_{xyx} = 6y^2 \quad$ and $\quad f_{yxy} = 4x$

D $\quad f_{xyx} = 12x + 6y \quad$ and $\quad f_{yxy} = 12x + 6y$

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Solution: B

To find the third-order partial derivative f_{xyx}, first we want to find the first-order partial derivative f_x by treating y as a constant and differentiating $f(x, y)$ with respect to x.

$$f(x, y) = x^3y^2 + xy^3$$

$$f_x = \left(3x^2\right) y^2 + (1)y^3 = 3x^2y^2 + y^3$$

Next, we can find the second-order partial derivative f_{xy} by treating x as a constant while differentiating f_x with respect to y.

$$f_x = 3x^2y^2 + y^3$$

$$f_{xy} = 3x^2(2y) + \left(3y^2\right) = 6x^2y + 3y^2$$

Now we can find the third-order partial derivative f_{xyx} by again treating y as a constant, this time while differentiating f_{xy} with respect to x.

$$f_{xy} = 6x^2y + 3y^2$$

$$f_{xyx} = 6(2x)y + 3y^2(0) = 12xy$$

Similarly, to find f_{yxy}, we must first find f_y (by treating x as a constant and differentiating $f(x, y)$ with respect to y) and then find f_{yx} (by treating y as a constant while differentiating f_y with respect to x). Then, we can find f_{yxy} by again treating x as a constant, while differentiating f_{yx} with respect to y.

$$f(x, y) = x^3y^2 + xy^3$$

$$f_y = x^3(2y) + x\left(3y^2\right) = 2x^3y + 3xy^2$$

$$f_{yx} = 2\left(3x^2\right)y + 3(1)y^2 = 6x^2y + 3y^2$$

$$f_{yxy} = 6x^2(1) + 3(2y) = 6x^2 + 6y$$

Topic: Differential of a multivariable function

Question: Find the differential of the multivariable function.

$$z = 4x^3 + 2\ln y$$

Answer choices:

A $\quad dz = 12x^2\ dx - \frac{2}{y}\ dy$

B $\quad dz = 6x^2\ dx + \frac{1}{y}\ dy$

C $\quad dz = 12x^2\ dx + \frac{2}{y}\ dy$

D $\quad dz = 6x^2\ dx - \frac{1}{y}\ dy$

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Solution: C

The differential of a multivariable function is given by

$$dz = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy$$

so we'll need to find the partial derivatives of z with respect to x and y. If $z = 4x^3 + 2\ln y$, then

$$\frac{\partial z}{\partial x} = 12x^2$$

and

$$\frac{\partial z}{\partial y} = 2\left(\frac{1}{y}\right)$$

$$\frac{\partial z}{\partial y} = \frac{2}{y}$$

Plugging these values into the formula for the differential, we get

$$dz = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy$$

$$dz = \left(12x^2\right)\ dx + \left(\frac{2}{y}\right)\ dy$$

$$dz = 12x^2\ dx + \frac{2}{y}\ dy$$

This is the differential of the multivariable function.

Topic: Differential of a multivariable function

Question: Find the differential of the multivariable function.

$$z = x\sin(2y) - 13y^2$$

Answer choices:

A $dz = \sin(2y)\ dx + 2x\cos(2y)\ dy - 26y\ dy$

B $dz = \sin y\ dx + 2x\cos y\ dy - 26y\ dy$

C $dz = \sin(2y)\ dx - 2x\cos(2y)\ dy - 26y\ dy$

D $dz = \sin y\ dx - 2x\cos y\ dy - 26y\ dy$

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Solution: A

The differential of a multivariable function is given by

$$dz = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy$$

so we'll need to find the partial derivatives of z with respect to x and y. If $z = x\sin(2y) - 13y^2$, then

$$\frac{\partial z}{\partial x} = \sin(2y)$$

and

$$\frac{\partial z}{\partial y} = x(2)\cos(2y) - 26y$$

$$\frac{\partial z}{\partial y} = 2x\cos(2y) - 26y$$

Plugging these values into the formula for the differential, we get

$$dz = \left[\sin(2y)\right]\ dx + \left[2x\cos(2y) - 26y\right]\ dy$$

$$dz = \sin(2y)\ dx + 2x\cos(2y)\ dy - 26y\ dy$$

This is the differential of the multivariable function.

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Topic: Differential of a multivariable function

Question: Find the differential of the multivariable function.

$$z = 6x^2 \ln(3y) + y^2 \sec(4x)$$

Answer choices:

A $\quad dz = 6x \ln(3y)\ dx + 2y^2 \sec(4x)\tan(4x)\ dx + \frac{3x^2}{y}\ dy + y \sec(4x)\ dy$

B $\quad dz = 12x \ln(3y)\ dx + 4y^2 \sec x \tan x\ \ dx + \frac{6x^2}{y}\ dy + 2y \sec(4x)\ dy$

C $\quad dz = 6x \ln(3y)\ dx + 2y^2 \sec(4x)\csc(4x)\ dx + \frac{3x^2}{y}\ dy + y \sec(4x)\ dy$

D $\quad dz = 12x \ln(3y)\ dx + 4y^2 \sec(4x)\tan(4x)\ dx + \frac{6x^2}{y}\ dy + 2y \sec(4x)\ dy$

Solution: D

The differential of a multivariable function is given by

$$dz = \frac{\partial z}{\partial x}\ dx + \frac{\partial z}{\partial y}\ dy$$

so we'll need to find the partial derivatives of z with respect to x and y. If $z = 6x^2 \ln(3y) + y^2 \sec(4x)$, then

$$\frac{\partial z}{\partial x} = 6(2)x \ln(3y) + (4)\sec(4x)\tan(4x)y^2$$

$$\frac{\partial z}{\partial x} = 12x \ln(3y) + 4y^2 \sec(4x)\tan(4x)$$

and

$$\frac{\partial z}{\partial y} = 6x^2(3)\left(\frac{1}{3y}\right) + 2y \sec(4x)$$

$$\frac{\partial z}{\partial y} = \frac{6x^2}{y} + 2y \sec(4x)$$

Plugging these values into the formula for the differential, we get

$$dz = \left[12x \ln(3y) + 4y^2 \sec(4x)\tan(4x)\right]\ dx + \left[\frac{6x^2}{y} + 2y \sec(4x)\right]\ dy$$

$$dz = 12x \ln(3y)\ dx + 4y^2 \sec(4x)\tan(4x)\ dx + \frac{6x^2}{y}\ dy + 2y \sec(4x)\ dy$$

This is the differential of the multivariable function.

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Topic: Chain rule for multivariable functions

Question: If $x = 1 + t$ and $y = 2 + t^2$, use chain rule to find dz/dt.

$$z = x^2y + x$$

Answer choices:

A $\quad \dfrac{dz}{dt} = t^4 + 2t^3 + 3t^2 + 5t + 3$

B $\quad \dfrac{dz}{dt} = 2t + 1$

C $\quad \dfrac{dz}{dt} = 4t^3 + 6t^2 + 6t + 5$

D $\quad \dfrac{dz}{dt} = 3t^3 + 5t^2 + 4t + 3$

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Solution: C

The chain rule tells us that

$$\frac{dz}{dt}=\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

So in order to find dz/dt, we need to find all the pieces from the right-hand side of the formula above. First, let's find the derivatives dx/dt and dy/dt.

$$x=1+t$$

$$\frac{dx}{dt}=1$$

and

$$y=2+t^2$$

$$\frac{dy}{dt}=2t$$

Now we'll find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$.

$$z=x^2y+x$$

$$\frac{\partial z}{\partial x}=2xy+1$$

$$\frac{\partial z}{\partial y}=x^2$$

Plugging these pieces back into our chain rule formula, we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = (2xy + 1)(1) + \left(x^2\right)(2t)$$

$$\frac{dz}{dt} = 2tx^2 + 2xy + 1$$

We want our answer in terms of t only, so we'll substitute for x and y.

$$\frac{dz}{dt} = 2t(1+t)^2 + 2(1+t)\left(2+t^2\right) + 1$$

$$\frac{dz}{dt} = 2t\left(t^2 + 2t + 1\right) + 2\left(t^3 + t^2 + 2t + 2\right) + 1$$

$$\frac{dz}{dt} = 2t^3 + 4t^2 + 2t + 2t^3 + 2t^2 + 4t + 4 + 1$$

$$\frac{dz}{dt} = 4t^3 + 6t^2 + 6t + 5$$

Topic: Chain rule for multivariable functions

Question: If $x = -\cos t$ and $y = \sin 2t$, use chain rule to find dz/dt.

$$z = x^2 - y^2$$

Answer choices:

A $\quad \dfrac{dz}{dt} = \cos^2 t - \sin^2 2t$

B $\quad \dfrac{dz}{dt} = -2\cos t - 2\sin 2t$

C $\quad \dfrac{dz}{dt} = \sin^2 t - 4\cos^2 2t$

D $\quad \dfrac{dz}{dt} = -2\cos t \sin t - 4\sin 2t \cos 2t$

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Solution: D

The chain rule tells us that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

So in order to find dz/dt, we need to find all the pieces from the right-hand side of the formula above. First, let's find the derivatives dx/dt and dy/dt.

$$x = -\cos t$$

$$\frac{dx}{dt} = \sin t$$

and

$$y = \sin 2t$$

$$\frac{dy}{dt} = 2\cos 2t$$

Now we'll find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$.

$$z = x^2 - y^2$$

$$\frac{\partial z}{\partial x} = 2x$$

$$\frac{\partial z}{\partial y} = -2y$$

Plugging these pieces back into our chain rule formula, we get

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = (2x)(\sin t) + (-2y)(2\cos 2t)$$

$$\frac{dz}{dt} = 2x \sin t - 4y \cos 2t$$

We want our answer in terms of t only, so we'll substitute for x and y.

$$\frac{dz}{dt} = 2(-\cos t)\sin t - 4(\sin 2t)\cos 2t$$

$$\frac{dz}{dt} = -2\cos t \sin t - 4\sin 2t \cos 2t$$

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Topic: Chain rule for multivariable functions

Question: If $x = t^2 + 2$ and $y = -t - 3$, use chain rule to find dz/dt.

$$z = \ln\left(x^2 + y\right)$$

Answer choices:

A $\quad \dfrac{dz}{dt} = \dfrac{4t^3 + 8t - 1}{t^4 + 4t^2 - t + 1}$

B $\quad \dfrac{dz}{dt} = \dfrac{4t^2 + 8t - 1}{t^2 + 3t + 1}$

C $\quad \dfrac{dz}{dt} = \dfrac{1}{t^4 + 4t^2 - t + 1}$

D $\quad \dfrac{dz}{dt} = \dfrac{1}{2t^2 + 4t + 1}$

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Solution: A

The chain rule tells us that

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

So in order to find dz/dt, we need to find all the pieces from the right-hand side of the formula above. First, let's find the derivatives dx/dt and dy/dt.

$$x = t^2 + 2$$

$$\frac{dx}{dt} = 2t$$

and

$$y = -t - 3$$

$$\frac{dy}{dt} = -1$$

Now we'll find the partial derivatives $\partial z/\partial x$ and $\partial z/\partial y$.

$$z = \ln\left(x^2 + y\right)$$

$$\frac{\partial z}{\partial x} = \frac{1}{x^2 + y} \cdot 2x = \frac{2x}{x^2 + y}$$

$$\frac{\partial z}{\partial y} = \frac{1}{x^2 + y} \cdot 1 = \frac{1}{x^2 + y}$$

Plugging these pieces back into our chain rule formula, we get

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$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$\frac{dz}{dt} = \frac{2x}{x^2 + y} \cdot 2t + \frac{1}{x^2 + y} \cdot -1 = \frac{4xt - 1}{x^2 + y}$$

We want our answer in terms of t only, so we'll substitute for x and y.

$$\frac{dz}{dt} = \frac{4\left(t^2 + 2\right)t - 1}{\left(t^2 + 2\right)^2 + (-t - 3)}$$

$$\frac{dz}{dt} = \frac{4t^3 + 8t - 1}{t^4 + 4t^2 - t + 1}$$

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Topic: Implicit differentiation for multivariable functions

Question: Use implicit differentiation to find the derivative of the multivariable function.

$$x^2e^y = 3x^2 + 4y^2$$

Answer choices:

A $\quad \dfrac{dy}{dx} = \dfrac{6x + 2xe^y}{8y + x^2e^y}$

B $\quad \dfrac{dy}{dx} = \dfrac{-6x + 2xe^y}{-8y + x^2e^y}$

C $\quad \dfrac{dy}{dx} = \dfrac{-6x + 2xe^y}{8y - x^2e^y}$

D $\quad \dfrac{dy}{dx} = \dfrac{6x - 2xe^y}{8y - x^2e^y}$

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Solution: C

We can use

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

to find the derivative of a multivariable function, which means we'll need to find the partial derivatives of F with respect to x and y.

First though, we'll collect all of our terms on one side of the equation so that we can set the function equal to $F(x, y)$.

$$x^2e^y = 3x^2 + 4y^2$$

$$0 = 3x^2 + 4y^2 - x^2e^y$$

$$F(x, y) = 3x^2 + 4y^2 - x^2e^y$$

Next we'll find the partial derivatives.

$$\frac{\partial F}{\partial x} = 6x - 2xe^y$$

and

$$\frac{\partial F}{\partial y} = 8y - x^2e^y$$

Plugging these values into the formula for the derivative, we get

$$\frac{dy}{dx} = -\frac{6x - 2xe^y}{8y - x^2e^y}$$

$$\frac{dy}{dx} = \frac{-6x + 2xe^y}{8y - x^2e^y}$$

This is the derivative of the multivariable function.

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Topic: Implicit differentiation for multivariable functions

Question: Use implicit differentiation to find the partial derivatives of the multivariable function.

$$2yz^2 = 4xz + 3y^3$$

Answer choices:

A $\frac{\partial z}{\partial x} = -\frac{4z}{4x - 4yz}$ $\frac{\partial z}{\partial y} = \frac{9y^2 - 2z^2}{4x - 4yz}$

B $\frac{\partial z}{\partial x} = -\frac{4z}{4x - 4yz}$ $\frac{\partial z}{\partial y} = -\frac{9y^2 - 2z^2}{4x - 4yz}$

C $\frac{\partial z}{\partial x} = \frac{4z}{4x - 4yz}$ $\frac{\partial z}{\partial y} = -\frac{9y^2 - 2z^2}{4x - 4yz}$

D $\frac{\partial z}{\partial x} = \frac{4z}{4x - 4yz}$ $\frac{\partial z}{\partial y} = \frac{9y^2 - 2z^2}{4x - 4yz}$

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Solution: B

Normally we'd be able to use

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

to find the derivative of a multivariable function. In this case though, we have a function in terms of three variables, which means we'll have to find partial derivatives, instead of just one derivative. We'll use the formulas

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

So we'll need to find the partial derivatives of F with respect to x, y, and z.

First though, we'll collect all of our terms on one side of the equation so that we can set the function equal to $F(x, y, z)$.

$$2yz^2 = 4xz + 3y^3$$

$$0 = 4xz + 3y^3 - 2yz^2$$

$$F(x, y, z) = 4xz + 3y^3 - 2yz^2$$

Next we'll find the partial derivatives.

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$$\frac{\partial F}{\partial x} = 4z$$

$$\frac{\partial F}{\partial y} = 9y^2 - 2z^2$$

$$\frac{\partial F}{\partial z} = 4x - 4yz$$

Plugging these values into the formulas for the partial derivatives, we get

$$\frac{\partial z}{\partial x} = -\frac{4z}{4x - 4yz}$$

and

$$\frac{\partial z}{\partial y} = -\frac{9y^2 - 2z^2}{4x - 4yz}$$

These are the partial derivatives of the multivariable function.

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Topic: Implicit differentiation for multivariable functions

Question: Use implicit differentiation to find the partial derivatives of the multivariable function.

$$-8y^2\ln(z) = 4e^y + 6x\cos(2z)$$

Answer choices:

A $\quad \dfrac{\partial z}{\partial x} = \dfrac{6\cos(2z)}{12x\sin(2z) - \frac{8y^2}{z}} \qquad \dfrac{\partial z}{\partial y} = \dfrac{4e^y + 16y\ln(z)}{12x\sin(2z) - \frac{8y^2}{z}}$

B $\quad \dfrac{\partial z}{\partial x} = -\dfrac{4e^y + 16y\ln(z)}{12x\sin(2z) + \frac{8y^2}{z}} \qquad \dfrac{\partial z}{\partial y} = -\dfrac{6\cos(2z)}{12x\sin(2z) + \frac{8y^2}{z}}$

C $\quad \dfrac{\partial z}{\partial x} = \dfrac{6\cos(2z)}{12x\sin(2z) + \frac{8y^2}{z}} \qquad \dfrac{\partial z}{\partial y} = \dfrac{4e^y + 16y\ln(z)}{12x\sin(2z) + \frac{8y^2}{z}}$

D $\quad \dfrac{\partial z}{\partial x} = \dfrac{4e^y + 16y\ln(z)}{12x\sin(2z) + \frac{8y^2}{z}} \qquad \dfrac{\partial z}{\partial y} = \dfrac{6\cos(2z)}{12x\sin(2z) + \frac{8y^2}{z}}$

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Solution: A

Normally we'd be able to use

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

to find the derivative of a multivariable function. In this case though, we have a function in terms of three variables, which means we'll have to find partial derivatives, instead of just one derivative. We'll use the formulas

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

So we'll need to find the partial derivatives of F with respect to x, y, and z.

First though, we'll collect all of our terms on one side of the equation so that we can set the function equal to $F(x, y, z)$.

$$-8y^2 \ln(z) = 4e^y + 6x\cos(2z)$$

$$0 = 4e^y + 6x\cos(2z) + 8y^2 \ln(z)$$

$$F(x, y, z) = 4e^y + 6x\cos(2z) + 8y^2 \ln(z)$$

Next we'll find the partial derivatives.

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$$\frac{\partial F}{\partial x} = 6\cos(2z)$$

and

$$\frac{\partial F}{\partial y} = 4e^y + 16y\ln(z)$$

and

$$\frac{\partial F}{\partial z} = 6x(2)\left[-\sin(2z)\right] + 8y^2\left(\frac{1}{z}\right)$$

$$\frac{\partial F}{\partial z} = -12x\sin(2z) + \frac{8y^2}{z}$$

Plugging these values into the formulas for the partial derivatives, we get

$$\frac{\partial z}{\partial x} = -\frac{6\cos(2z)}{-12x\sin(2z) + \frac{8y^2}{z}}$$

$$\frac{\partial z}{\partial x} = \frac{6\cos(2z)}{12x\sin(2z) - \frac{8y^2}{z}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{4e^y + 16y\ln(z)}{-12x\sin(2z) + \frac{8y^2}{z}}$$

$$\frac{\partial z}{\partial y} = \frac{4e^y + 16y\ln(z)}{12x\sin(2z) - \frac{8y^2}{z}}$$

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These are the partial derivatives of the multivariable function.

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Topic: Directional derivatives in the direction of the vector

Question: Find the directional derivative.

$$D_u f(-1,1)$$

$$f(x, y) = 4xy + 7y^2$$

$\vec{u}$ is the unit vector toward $\vec{v} = \langle 1,1 \rangle$

Answer choices:

A $D_u f(-1,1) = 3\sqrt{2}$

B $D_u f(-1,1) = 7\sqrt{2}$

C $D_u f(-1,1) = -3\sqrt{2}$

D $D_u f(-1,1) = -7\sqrt{2}$

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Solution: B

We need to convert the vector $\vec{v} = \langle c, d \rangle$ into a unit vector using the formula

$$\vec{u} = \left\langle \frac{c}{\sqrt{c^2 + d^2}}, \frac{d}{\sqrt{c^2 + d^2}} \right\rangle$$

Since in this problem $\vec{v} = \langle 1,1 \rangle$, we get

$$\vec{u} = \left\langle \frac{1}{\sqrt{1^2 + 1^2}}, \frac{1}{\sqrt{1^2 + 1^2}} \right\rangle$$

$$\vec{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$$

$$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

To get the directional derivative, we'll use

$$D_u f(x, y) = a\left(\frac{\partial F}{\partial x}\right) + b\left(\frac{\partial F}{\partial y}\right)$$

where a and b come from the unit vector $\vec{u} = \langle a, b \rangle$ we found earlier. All we need now are the first order partial derivatives.

$$\frac{\partial F}{\partial x} = 4y$$

and

$$\frac{\partial F}{\partial y} = 4x + 14y$$

Since we were asked to find $D_u f(-1,1)$, we need to evaluate the partial derivatives at $(-1,1)$.

$$\frac{\partial F}{\partial x}(-1,1) = 4(1)$$

$$\frac{\partial F}{\partial x}(-1,1) = 4$$

and

$$\frac{\partial F}{\partial y}(-1,1) = 4(-1) + 14(1)$$

$$\frac{\partial F}{\partial y}(-1,1) = 10$$

Plugging everything into the formula for the directional derivative, we get

$$D_u f(-1,1) = \frac{\sqrt{2}}{2}(4) + \frac{\sqrt{2}}{2}(10)$$

$$D_u f(-1,1) = 2\sqrt{2} + 5\sqrt{2}$$

$$D_u f(-1,1) = 7\sqrt{2}$$

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Topic: Directional derivatives in the direction of the vector

Question: Find the directional derivative.

$$D_u f(2,0)$$

$$f(x,y) = -3xe^{2y} + 2x^3$$

$\vec{u}$ is the unit vector toward $\vec{v} = \langle -1, -1 \rangle$

Answer choices:

A $\quad D_u f(2,0) = -\dfrac{33\sqrt{2}}{2}$

B $\quad D_u f(2,0) = \dfrac{9\sqrt{2}}{2}$

C $\quad D_u f(2,0) = -\dfrac{9\sqrt{2}}{2}$

D $\quad D_u f(2,0) = \dfrac{33\sqrt{2}}{2}$

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Solution: C

We need to convert the vector $\vec{v}=\langle c,d\rangle$ into a unit vector using the formula

$$\vec{u}=\left\langle\frac{c}{\sqrt{c^2+d^2}},\frac{d}{\sqrt{c^2+d^2}}\right\rangle$$

Since in this problem $\vec{v}=\langle -1,-1\rangle$, we get

$$\vec{u}=\left\langle\frac{-1}{\sqrt{(-1)^2+(-1)^2}},\frac{-1}{\sqrt{(-1)^2+(-1)^2}}\right\rangle$$

$$\vec{u}=\left\langle-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle$$

$$\vec{u}=\left\langle-\frac{\sqrt{2}}{2},-\frac{\sqrt{2}}{2}\right\rangle$$

To get the directional derivative, we'll use

$$D_uf(x,y)=a\left(\frac{\partial F}{\partial x}\right)+b\left(\frac{\partial F}{\partial y}\right)$$

where a and b come from the unit vector $\vec{u}=\langle a,b\rangle$ we found earlier. All we need now are the first order partial derivatives.

$$\frac{\partial F}{\partial x}=-3e^{2y}+6x^2$$

and

$$\frac{\partial F}{\partial y} = -6xe^{2y}$$

Since we were asked to find $D_u f(2,0)$, we need to evaluate the partial derivatives at $(2,0)$.

$$\frac{\partial F}{\partial x}(2,0) = -3e^{2(0)} + 6(2)^2$$

$$\frac{\partial F}{\partial x}(2,0) = 21$$

and

$$\frac{\partial F}{\partial y}(2,0) = -6(2)e^{2(0)}$$

$$\frac{\partial F}{\partial y}(2,0) = -12$$

Plugging everything into the formula for the directional derivative, we get

$$D_u f(2,0) = -\frac{\sqrt{2}}{2}(21) - \frac{\sqrt{2}}{2}(-12)$$

$$D_u f(2,0) = -\frac{21\sqrt{2}}{2} + \frac{12\sqrt{2}}{2}$$

$$D_u f(2,0) = -\frac{9\sqrt{2}}{2}$$

Topic: Directional derivatives in the direction of the vector

Question: Find the directional derivative.

$$D_u f(1,0,-1)$$

$$f(x,y,z) = 4x^2e^z - 7\cos y + 16y^2z^3$$

$\vec{u}$ is the unit vector toward $\vec{v} = \langle 1,1,-1 \rangle$

Answer choices:

A $\quad D_u f(1,0,-1) = \dfrac{4\sqrt{3}}{3e}$

B $\quad D_u f(1,0,-1) = -\dfrac{4e\sqrt{3}}{3}$

C $\quad D_u f(1,0,-1) = \dfrac{4e\sqrt{3}}{3}$

D $\quad D_u f(1,0,-1) = -\dfrac{4\sqrt{3}}{3e}$

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Solution: A

We need to convert the vector $\vec{v} = \langle d, e, f \rangle$ into a unit vector using the formula

$$\vec{u} = \left\langle \frac{d}{\sqrt{d^2 + e^2 + f^2}}, \frac{e}{\sqrt{d^2 + e^2 + f^2}}, \frac{f}{\sqrt{d^2 + e^2 + f^2}} \right\rangle$$

Since in this problem $\vec{v} = \langle 1, 1, -1 \rangle$, we get

$$\vec{u} = \left\langle \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}, \frac{1}{\sqrt{1^2 + 1^2 + (-1)^2}}, \frac{-1}{\sqrt{1^2 + 1^2 + (-1)^2}} \right\rangle$$

$$\vec{u} = \left\langle \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \right\rangle$$

$$\vec{u} = \left\langle \frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3} \right\rangle$$

To get the directional derivative, we'll use

$$D_u f(x, y, z) = a\left(\frac{\partial F}{\partial x}\right) + b\left(\frac{\partial F}{\partial y}\right) + c\left(\frac{\partial F}{\partial z}\right)$$

where a, b and c come from the unit vector $\vec{u} = \langle a, b, c \rangle$ we found earlier. All we need now are the first order partial derivatives.

$$\frac{\partial F}{\partial x} = 8xe^z$$

$$\frac{\partial F}{\partial y} = 7\sin y + 32yz^3$$

$$\frac{\partial F}{\partial z} = 4x^2e^z + 48y^2z^2$$

Since we were asked to find $D_uf(1,0,-1)$, we need to evaluate the partial derivatives at $(1,0,-1)$.

$$\frac{\partial F}{\partial x}(1,0,-1) = 8(1)e^{-1} = \frac{8}{e}$$

$$\frac{\partial F}{\partial y}(1,0,-1) = 7\sin 0 + 32(0)(-1)^3 = 0$$

$$\frac{\partial F}{\partial z}(1,0,-1) = 4(1)^2e^{-1} + 48(0)^2(-1)^2 = \frac{4}{e}$$

Plugging everything into the formula for the directional derivative, we get

$$D_uf(1,0,-1) = \frac{\sqrt{3}}{3}\left(\frac{8}{e}\right) + \frac{\sqrt{3}}{3}(0) - \frac{\sqrt{3}}{3}\left(\frac{4}{e}\right)$$

$$D_uf(1,0,-1) = \frac{8\sqrt{3}}{3e} - \frac{4\sqrt{3}}{3e}$$

$$D_uf(1,0,-1) = \frac{4\sqrt{3}}{3e}$$

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Topic: Directional derivatives in the direction of the angle

Question: Find the directional derivative.

$$D_u f(1,1)$$

$$f(x, y) = 2x^2y - 2y^2$$

$\vec{u}$ is the unit vector toward $\theta = \frac{\pi}{4}$

Answer choices:

A $\quad D_u f(1,1) = 3\sqrt{2}$

B $\quad D_u f(1,1) = \frac{3\sqrt{2}}{2}$

C $\quad D_u f(1,1) = \frac{\sqrt{2}}{2}$

D $\quad D_u f(1,1) = \sqrt{2}$

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Solution: D

We need to change the unit vector so that it's in the direction of a specific point, instead of in the direction of an angle. To do that we'll use the formula

$$\vec{u} = \langle \cos\theta, \sin\theta \rangle$$

Plugging in the angle we've been given, we get

$$\vec{u} = \left\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \right\rangle$$

$$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

To get the directional derivative, we'll use

$$D_u f(x,y) = a\left(\frac{\partial F}{\partial x}\right) + b\left(\frac{\partial F}{\partial y}\right)$$

where a and b come from the unit vector $\vec{u} = \langle a, b \rangle$ we found earlier. All we need now are the first order partial derivatives.

$$\frac{\partial F}{\partial x} = 4xy$$

and

$$\frac{\partial F}{\partial y} = 2x^2 - 4y$$

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Since we were asked to find $D_u f(1,1)$, we need to evaluate the partial derivatives at $(1,1)$.

$$\frac{\partial F}{\partial x}(1,1) = 4(1)(1)$$

$$\frac{\partial F}{\partial x}(1,1) = 4$$

and

$$\frac{\partial F}{\partial y}(1,1) = 2(1)^2 - 4(1)$$

$$\frac{\partial F}{\partial y}(1,1) = -2$$

Plugging everything into the formula for the directional derivative, we get

$$D_u f(1,1) = \frac{\sqrt{2}}{2}(4) + \frac{\sqrt{2}}{2}(-2)$$

$$D_u f(1,1) = 2\sqrt{2} - \sqrt{2}$$

$$D_u f(1,1) = \sqrt{2}$$

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Topic: Directional derivatives in the direction of the angle

Question: Find the directional derivative.

$$D_u f(-1,0)$$

$$f(x, y) = 2e^{xy} - 3x^2y^3$$

$\vec{u}$ is the unit vector toward $\theta = \dfrac{2\pi}{3}$

Answer choices:

A $\quad D_u f(1,1) = -\dfrac{\sqrt{3}}{2}$

B $\quad D_u f(-1,0) = -\sqrt{3}$

C $\quad D_u f(1,1) = \dfrac{\sqrt{3}}{2}$

D $\quad D_u f(-1,0) = \sqrt{3}$

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Solution: B

We need to change the unit vector so that it's in the direction of a specific point, instead of in the direction of an angle. To do that we'll use the formula

$$\vec{u} = \langle \cos\theta, \sin\theta \rangle$$

Plugging in the angle we've been given, we get

$$\vec{u} = \left\langle \cos\frac{2\pi}{3}, \sin\frac{2\pi}{3} \right\rangle$$

$$\vec{u} = \left\langle -\frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle$$

To get the directional derivative, we'll use

$$D_u f(x, y) = a\left(\frac{\partial F}{\partial x}\right) + b\left(\frac{\partial F}{\partial y}\right)$$

where a and b come from the unit vector $\vec{u} = \langle a, b \rangle$ we found earlier. All we need now are the first order partial derivatives.

$$\frac{\partial F}{\partial x} = 2ye^{xy} - 6xy^3$$

and

$$\frac{\partial F}{\partial y} = 2xe^{xy} - 9x^2y^2$$

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Since we were asked to find $D_u f(-1,0)$, we need to evaluate the partial derivatives at $(-1,0)$.

$$\frac{\partial F}{\partial x}(-1,0) = 2(0)e^{(-1)(0)} - 6(-1)(0)^3$$

$$\frac{\partial F}{\partial x}(-1,0) = 0$$

and

$$\frac{\partial F}{\partial y}(-1,0) = 2(-1)e^{(-1)(0)} - 9(-1)^2(0)^2$$

$$\frac{\partial F}{\partial y}(-1,0) = -2$$

Plugging everything into the formula for the directional derivative, we get

$$D_u f(-1,0) = -\frac{1}{2}(0) + \frac{\sqrt{3}}{2}(-2)$$

$$D_u f(-1,0) = -\sqrt{3}$$

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Topic: Directional derivatives in the direction of the angle

Question: Find the directional derivative.

$$D_u f(0,2)$$

$$f(x, y) = 4x^2e^{2y} - 2xy^4 + 5e^x$$

$\vec{u}$ is the unit vector toward $\theta = \dfrac{\pi}{4}$

Answer choices:

A $\quad D_u f(0,2) = -\dfrac{27\sqrt{2}}{2}$

B $\quad D_u f(0,2) = -27\sqrt{2}$

C $\quad D_u f(0,2) = 27\sqrt{2}$

D $\quad D_u f(0,2) = \dfrac{27\sqrt{2}}{2}$

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Solution: A

We need to change the unit vector so that it's in the direction of a specific point, instead of in the direction of an angle. To do that we'll use the formula

$$\vec{u} = \langle \cos\theta, \sin\theta \rangle$$

Plugging in the angle we've been given, we get

$$\vec{u} = \left\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \right\rangle$$

$$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

To get the directional derivative, we'll use

$$D_u f(x, y) = a\left(\frac{\partial F}{\partial x}\right) + b\left(\frac{\partial F}{\partial y}\right)$$

where a and b come from the unit vector $\vec{u} = \langle a, b \rangle$ we found earlier. All we need now are the first order partial derivatives.

$$\frac{\partial F}{\partial x} = 8xe^{2y} - 2y^4 + 5e^x$$

and

$$\frac{\partial F}{\partial y} = 8x^2e^{2y} - 8xy^3$$

Since we were asked to find $D_uf(0,2)$, we need to evaluate the partial derivatives at $(0,2)$.

$$\frac{\partial F}{\partial x}(0,2) = 8(0)e^{2(2)} - 2(2)^4 + 5e^0$$

$$\frac{\partial F}{\partial x}(0,2) = -27$$

and

$$\frac{\partial F}{\partial y}(0,2) = 8(0)^2e^{2(2)} - 8(0)(2)^3$$

$$\frac{\partial F}{\partial y}(0,2) = 0$$

Plugging everything into the formula for the directional derivative, we get

$$D_uf(0,2) = \frac{\sqrt{2}}{2}(-27) + \frac{\sqrt{2}}{2}(0)$$

$$D_uf(0,2) = -\frac{27\sqrt{2}}{2}$$

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Topic: Linear approximation in two variables

Question: At the point $(2,1)$, find the linear approximation of $f(x, y) = x^3 - y^3$ and use it to approximate $f(2.05,0.95)$.

Answer choices:

A $L(2.05,0.95) = 7$

B $L(2.05,0.95) = 7.75$

C $L(2.05,0.95) = 7.9$

D $L(2.05,0.95) = 7.75775$

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Solution: B

The linear approximation of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Since we know point $(a, b) = (2,1)$, we can easily find $f(a, b) = f(2,1) = 2^3 - 1^3 = 8 - 1 = 7$. Next, we need to find values for $f_x(a, b) = f_x(2,1)$ and $f_y(a, b) = f_y(2,1)$.

To find the value for $f_x(2,1)$, we have to find the partial derivative $f_x(x, y)$ by treating y (and therefore y^3) as a constant, while differentiating $f(x, y)$ with respect to x, and then plug the given point into the partial derivative.

$$f(x, y) = x^3 - y^3$$

$$f_x(x, y) = 3x^2 - 0 = 3x^2$$

$$f_x(2,1) = 3(2^2) = 12$$

Similarly, we can find the partial derivative $f_y(x, y)$ by treating x (and therefore x^3) as a constant, while differentiating $f(x, y)$ with respect to y. Then we'll plug the given point into the partial derivative.

$$f(x, y) = x^3 - y^3$$

$$f_y(x, y) = 0 - 3y^2 = -3y^2$$

$$f_y(2,1) = -3(1^2) = -3$$

Using these values for $f_x(2,1)$ and $f_y(2,1)$, we can now find the linear approximation of f.

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$L(x, y) = f(2,1) + f_x(2,1)(x - 2) + f_y(2,1)(y - 1)$$

$$L(x, y) = 7 + 12(x - 2) - 3(y - 1)$$

$$L(x, y) = 7 + 12x - 24 - 3y + 3$$

$$L(x, y) = 12x - 3y - 14$$

Since we know that the value of $f(x, y)$ near point (a, b) is approximately equal to the linear approximation of f at (a, b), we can use it to find an approximate value for $f(2.1,0.95)$.

$$f(x, y) \approx L(x, y)$$

$$f(x, y) \approx 12x - 3y - 14$$

$$f(2.05,0.95) \approx 12(2.05) - 3(0.95) - 14$$

$$f(2.05,0.95) \approx 24.6 - 2.85 - 14$$

$$f(2.05,0.95) \approx 7.75$$

Therefore, the linear approximation allows us to approximate the value of $f(2.05,0.95)$ as 7.75. If we compare this to the actual value of $f(2.05,0.95) = 2.05^3 - 0.95^3 = 7.75775$, we see that the linear approximation is actually pretty close to the actual value!

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Topic: Linear approximation in two variables

Question: At the point $(0,0)$, find the linear approximation of $f(x, y) = 2\cos x \sin y + 1$ and use it to approximate $f(0.1,0.1)$.

Answer choices:

A $L(0.1,0.1) = 1.1987$

B $L(0.1,0.1) = 1$

C $L(0.1,0.1) = 1.25$

D $L(0.1,0.1) = 1.2$

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Solution: D

The linearization of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Since we know point $(a, b) = (0,0)$, we can easily find $f(a, b) = f(0,0) = 2\cos 0 \sin 0 + 1 = 1$. Next, we need to find values for $f_x(a, b) = f_x(0,0)$ and $f_y(a, b) = f_y(0,0)$.

To find the value for $f_x(0,0)$, we must first find the partial derivative $f_x(x, y)$ by treating y (and therefore $\sin y$) as a constant, while differentiating $f(x, y)$ with respect to x, and then plug the given point into the partial derivative.

$$f(x, y) = 2\cos x \sin y + 1$$

$$f_x(x, y) = 2(-\sin x)\sin y + 0 = -2\sin x \sin y$$

$$f_x(0,0) = -2\sin 0 \sin 0 = -2(0)(0) = 0$$

Similarly, we can find the partial derivative $f_y(x, y)$ by treating x (and therefore $\cos x$) as a constant, while differentiating $f(x, y)$ with respect to y, and use this to find $f_y(0,0)$:

$$f(x, y) = 2\cos x \sin y + 1$$

$$f_y(x, y) = 2\cos x(\cos y) + 0 = 2\cos x \cos y$$

$$f_y(0,0) = 2\cos 0 \cos 0 = 2(1)(1) = 2$$

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Using these values for $f_x(0,0)$ and $f_y(0,0)$, we can now find the linear approximation of f.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0)$$

$$L(x,y) = 1 + 0(x-0) + 2(y-0)$$

$$L(x,y) = 1 + 2y - 2(0) = 2y + 1$$

Since we know that the value of $f(x,y)$ near point (a,b) is approximately equal to $L(a,b)$, we can find an approximate value for $f(0.1,0.1)$ using the linear approximation.

$$f(x,y) \approx L(x,y)$$

$$f(x,y) \approx 2y + 1$$

$$f(0.1,0.1) \approx 2(0.1) + 1$$

$$f(0.1,0.1) \approx 1.2$$

Therefore, the linear approximation allows us to approximate the value of $f(0.1,0.1)$ as 1.2. If we compare this to the actual value of $f(0.1,0.1) = 2\cos(0.1)\sin(0.1) + 1 \approx 1.1987$, we can see that the linear approximation is pretty close to the actual value!

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Topic: Linear approximation in two variables

Question: Find the linear approximation of the function.

At the point $(1, -3)$, find the linear approximation of $f(x, y) = 2x^2y^2 - 4xy^2 - 3y$ and use it to approximate $f(1.1, -2.9)$.

Answer choices:

A $L(1.1, -2.9) = -8.1$

B $L(1.1, -2.9) = -8$

C $L(1.1, -2.9) = -7.9518$

D $L(1.1, -2.9) = -9$

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Solution: A

The linear approximation of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Since we know point $(a, b) = (1, -3)$, we can easily find $f(a, b) = f(1, -3) = 2(1^2)(-3^2) - 4(1)(-3^2) - 3(-3) = -9$.

Next, we need to find values of the partial derivatives $f_x(a, b) = f_x(1, -3)$ and $f_y(a, b) = f_y(1, -3)$.

$$f(x, y) = 2x^2y^2 - 4xy^2 - 3y$$

$$f_x(x, y) = 2(2x)y^2 - 4(1)y^2 - 0 = 4xy^2 - 4y^2$$

$$f_x(1, -3) = 4(1)(-3)^2 - 4(-3)^2 = 0$$

$$f_y(x, y) = 2x^2(2y) - 4x(2y) - 3(1) = 4x^2y - 8xy - 3$$

$$f_y(1, -3) = 4(1)^2(-3) - 8(1)(-3) - 3 = 9$$

Using these values for $f_x(1, -3)$ and $f_y(1, -3)$, we can now find the linear approximation of f.

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$L(x, y) = f(1, -3) + f_x(1, -3)(x - 1) + f_y(1, -3)(y + 3)$$

$$L(x, y) = -9 + 0(x - 1) + 9(y + 3) = 9y + 18$$

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Since we know that the value of $f(x, y)$ near point (a, b) is approximately equal to $L(a, b)$, we can find an approximate value for $f(1.1, -2.9)$ using the linear approximation.

$$f(x, y) \approx L(x, y) = 9y + 18$$

$$f(1.1, -2.9) \approx 9(-2.9) + 18 = -8.1$$

Which is fairly close to the actual value of f at this point.

$$f(1.1, -2.9) = 2(1.1)^2(-2.9)^2 - 4(1.1)(-2.9)^2 - 3(-2.9) = -7.9518$$

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Topic: Linearization of a multivariable function

Question: At which point is the linearization of $f(x, y) = x^2 - 2xy + y^2 - 3$ defined by $L(x_0, y_0) = -6x + 6y - 12$?

Answer choices:

A $(2,5)$

B $(-2,5)$

C $(3,5)$

D $(5,-3)$

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Solution: A

If we evaluate $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ at the point $(2,5)$, we get

$$f(x, y) = x^2 - 2xy + y^2 - 3$$

$$f(2,5) = (2)^2 - 2(2)(5) + (5)^2 - 3$$

$$f(2,5) = 4 - 20 + 25 - 3$$

$$f(2,5) = 6$$

For the partial derivative with respect to x,

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 - 2xy + y^2 - 3\right)$$

$$f_x(x, y) = 2x - 2y$$

$$f_x(2,5) = \left[2(2) - 2(5)\right]$$

$$f_x(2,5) = -6$$

For the partial derivative with respect to y,

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2 - 2xy + y^2 - 3\right)$$

$$f_y(x, y) = -2x + 2y$$

$$f_y(2,5) = \left[-2(2) + 2(5)\right]$$

$$f_y(2,5) = 6$$

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Plug these values into the linearization formula.

$$L(x_0, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x_0, y_0) = f(2{,}5) + f_x(2{,}5)(x - 2) + f_y(2{,}5)(y - 5)$$

$$L(x_0, y_0) = 6 + (-6)(x - 2) + (6)(y - 5)$$

$$L(x_0, y_0) = 6 - 6x + 12 + 6y - 30$$

$$L(x_0, y_0) = -6x + 6y - 12$$

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Topic: Linearization of a multivariable function

Question: What is the linearization of $f(x, y, z) = x^2 + y^2 - xyz$ at $P_0(1,2,-1)$?

Answer choices:

A $L(x, y, z) = 4x + y - 10z + 8$

B $L(x, y, z) = 8x + 10y - z + 12$

C $L(x, y, z) = 4x + 5y - 2z - 9$

D $L(x, y, z) = 8x - 10y - z + 12$

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Solution: C

If we evaluate $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ at the point $P_0(1,2,-1)$, we get

$$f(x, y, z) = x^2 + y^2 - xyz$$

$$f(1,2,-1) = (1)^2 + (2)^2 - (1)(2)(-1)$$

$$f(1,2,-1) = 1 + 4 + 2$$

$$f(1,2,-1) = 7$$

For the partial derivative with respect to x,

$$f_x(x, y, z) = \frac{\partial}{\partial x}(x^2 + y^2 - xyz)$$

$$f_x(x, y, z) = 2x - yz$$

$$f_x(1,2,-1) = 2(1) - (2)(-1)$$

$$f_x(1,2,-1) = 4$$

For the partial derivative with respect to y,

$$f_y(x, y, z) = \frac{\partial}{\partial y}(x^2 + y^2 - xyz)$$

$$f_y(x, y, z) = 2y - xz$$

$$f_y(1,2,-1) = [2(2) - (1)(-1)]$$

$$f_y(1,2,-1) = 5$$

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For the partial derivative with respect to z,

$$f_z(x,y,z) = \frac{\partial}{\partial z}(x^2 + y^2 - xyz)$$

$$f_z(x,y,z) = (-xy)$$

$$f_z(1,2,-1) = [-(1)(2)]$$

$$f_z(1,2,-1) = -2$$

Plug these values into the linearization formula.

$$L(x_0,y_0,z_0) = f(x_0,y_0,z_0) + f_x(x_0,y_0,z_0)(x - x_0) + f_y(x_0,y_0,z_0)(y - y_0)$$

$$+f_z(x_0,y_0,z_0)(z - z_0)$$

$$L(1,2,-1) = f(1,2,-1) + f_x(1,2,-1)(x-1) + f_y(1,2,-1)(y-2) + f_z(1,2,-1)(z+1)$$

$$L(1,2,-1) = 7 + (4)(x-1) + (5)(y-2) + (-2)(z+1)$$

$$L(1,2,-1) = 7 + 4x - 4 + 5y - 10 - 2z - 2$$

$$L(1,2,-1) = 4x + 5y - 2z - 9$$

Topic: Linearization of a multivariable function

Question: What is the upper bound of the error?

Given the linearization of $f(x, y) = x^2 + xy - y^2 - 1$ at $P_0(3,1)$, find the upper bound of the error $|E|$ over the rectangle defined by $|x - 3| \leq 0.2$ and $|y - 1| \leq 0.2$?

Answer choices:

A 1.06

B 0.06

C 1.16

D 0.16

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Solution: D

If we evaluate $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ at the point $(3,1)$, we get

$$f(x, y) = x^2 + xy - y^2 - 1$$

$$f(3,1) = (3)^2 + (3)(1) - (1)^2 - 1$$

$$f(3,1) = 9 + 3 - 1 - 1$$

$$f(3,1) = 10$$

For the partial derivative with respect to x,

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 + xy - y^2 - 1\right)$$

$$f_x(x, y) = 2x + y$$

$$f_x(3,1) = \left[2(3) + 1\right]$$

$$f_x(3,1) = 7$$

For the partial derivative with respect to y,

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2 + xy - y^2 - 1\right)$$

$$f_y(x, y) = x - 2y$$

$$f_y(3,1) = \left[3 - 2(1)\right]$$

$$f_y(3,1) = 1$$

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Plug these values into the linearization formula.

$$L(x_0, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(3,1) = f(3,1) + f_x(3,1)(x - 3) + f_y(3,1)(y - 1)$$

$$L(3,1) = 10 + (7)(x - 3) + (1)(y - 1)$$

$$L(3,1) = 10 + 7x - 21 + y - 1$$

$$L(3,1) = 7x + y - 12$$

We have $f_{xx} = 2$, $f_{yy} = -2$, and $f_{xy} = 1$. The largest of these three values is 2. Therefore, we can consider a common upper bound 2 on the rectangle R. Then

$$\left| E(x_0, y_0) \right| \le \frac{1}{2}(2)\left(|x - 3| + |y - 1| \right)^2$$

$$\left| E(x_0, y_0) \right| \le \frac{2}{2}(0.2 + 0.2)^2$$

$$\left| E(x_0, y_0) \right| = 0.16$$

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Topic: Gradient vectors

Question: Which function produces the gradient vector at the given point?

$$\nabla f(x, y, z) = 22\mathbf{i} - 20\mathbf{j} + 26\mathbf{k}$$

at $(-1,2,-3)$

Answer choices:

A $f(x, y, z) = x^2 + 2y^2 - 3z^2 + 4xyz$

B $f(x, y, z) = x^2 - 2y^2 - 3z^2 - 4xyz$

C $f(x, y, z) = x^2 - 2y^2 + 3z^2 + 4xyz$

D $f(x, y, z) = x^2 + 2y^2 + 3z^2 - 4xyz$

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Solution: B

Apply the definition of gradient vector to each answer choice. Answer choice B gives

$$\nabla f(x,y,z)=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$$

$$\nabla f(x,y,z)=\nabla\left(x^2-2y^2-3z^2-4xyz\right)$$

$$\nabla f(x,y,z)=\frac{\partial\left(x^2-2y^2-3z^2-4xyz\right)}{\partial x}\mathbf{i}+\frac{\partial\left(x^2-2y^2-3z^2-4xyz\right)}{\partial y}\mathbf{j}$$

$$+\frac{\partial\left(x^2-2y^2-3z^2-4xyz\right)}{\partial z}\mathbf{k}$$

$$\nabla f(x,y,z)=(2x-4yz)\mathbf{i}+(-4y-4xz)\mathbf{j}+(-6z-4xy)\mathbf{k}$$

Evaluate $\nabla f(-1,2,-3)$.

$$\nabla f(-1,2,-3)=\left(2(-1)-4(2)(-3)\right)\mathbf{i}-\left(-4(2)-4(-1)(-3)\right)\mathbf{j}+\left(-6(-3)-4(-1)(2)\right)\mathbf{k}$$

$$\nabla f(-1,2,-3)=(-2+24)\mathbf{i}+(-8-12)\mathbf{j}+(18+8)\mathbf{k}$$

$$\nabla f(-1,2,-3)=22\mathbf{i}-20\mathbf{j}+26\mathbf{k}$$

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Topic: Gradient vectors

Question: Find $\nabla(f/g)$.

$$f(x,y) = x^2y$$

$$g(x,y) = xy^2$$

Answer choices:

A $\nabla\left(\frac{f}{g}\right) = \frac{1}{y}\mathbf{i} + \frac{x}{y^2}\mathbf{j}$

B $\nabla\left(\frac{f}{g}\right) = \frac{1}{y^2}\mathbf{i} + \frac{x}{y}\mathbf{j}$

C $\nabla\left(\frac{f}{g}\right) = \frac{1}{x}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$

D $\nabla\left(\frac{f}{g}\right) = \frac{1}{y}\mathbf{i} - \frac{x}{y^2}\mathbf{j}$

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Solution: D

First we'll find $\nabla f(x,y)$ and $\nabla g(x,y)$.

$$\nabla f(x,y)=\frac{\partial\left(x^2y\right)}{\partial x}\mathbf{i}+\frac{\partial\left(x^2y\right)}{\partial y}\mathbf{j}$$

$$\nabla f(x,y)=2xy\mathbf{i}+x^2\mathbf{j}$$

and

$$\nabla g(x,y)=\frac{\partial\left(xy^2\right)}{\partial x}\mathbf{i}+\frac{\partial\left(xy^2\right)}{\partial y}\mathbf{j}$$

$$\nabla g(x,y)=y^2\mathbf{i}+2xy\mathbf{j}$$

Plug into the formula.

$$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{xy^2\left(2xy\mathbf{i}+x^2\mathbf{j}\right)-x^2y\left(y^2\mathbf{i}+2xy\mathbf{j}\right)}{\left(xy^2\right)^2}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{2x^2y^3\mathbf{i}+x^3y^2\mathbf{j}-x^2y^3\mathbf{i}-2x^3y^2\mathbf{j}}{x^2y^4}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{1}{y}\mathbf{i}-\frac{x}{y^2}\mathbf{j}$$

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Topic: Gradient vectors

Question: Find $\nabla(fg)$ at $(2,-2)$.

$$f(x,y)=x^2-y$$

$$g(x,y)=x+y^2$$

Answer choices:

A $\quad \nabla(fg)=30\mathbf{i}-30\mathbf{j}$

B $\quad \nabla(fg)=36\mathbf{i}-2\mathbf{j}$

C $\quad \nabla(fg)=12\mathbf{i}+36\mathbf{j}$

D $\quad \nabla(fg)=12\mathbf{i}-36\mathbf{j}$

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Solution: A

First find the gradient vector of each function.

$$\nabla f(x,y)=\frac{\partial\left(x^2-y\right)}{\partial x}\mathbf{i}+\frac{\partial\left(x^2-y\right)}{\partial y}\mathbf{j}$$

$$\nabla f(x,y)=2x\mathbf{i}-\mathbf{j}$$

and

$$\nabla g(x,y)=\frac{\partial\left(x+y^2\right)}{\partial x}\mathbf{i}+\frac{\partial\left(x+y^2\right)}{\partial y}\mathbf{j}$$

$$\nabla g(x,y)=\mathbf{i}+2y\mathbf{j}$$

Plug into the formula.

$$\nabla(fg)=f\,\nabla g+g\,\nabla f$$

$$\nabla(fg)=\left(x^2-y\right)\left(\mathbf{i}+2y\mathbf{j}\right)+\left(x+y^2\right)\left(2x\mathbf{i}-\mathbf{j}\right)$$

$$\nabla(fg)=\left(2^2-(-2)\right)\left(\mathbf{i}+2(-2)\mathbf{j}\right)+\left(2+(-2)^2\right)\left(2(2)\mathbf{i}-\mathbf{j}\right)$$

$$\nabla(fg)=(6)\left(\mathbf{i}-4\mathbf{j}\right)+(6)\left(4\mathbf{i}-\mathbf{j}\right)$$

$$\nabla(fg)=30\mathbf{i}-30\mathbf{j}$$

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Topic: Gradient vectors and the tangent plane

Question: Find the gradient vector of the function at $P(1,-1)$.

$$f(x,y)=3x^2y-y^3+2x^2-xy-12$$

Answer choices:

A $\nabla f(1,-1)=\langle 1,-1\rangle$

B $\nabla f(1,-1)=\langle -1,1\rangle$

C $\nabla f(1,-1)=\langle -1,-1\rangle$

D $\nabla f(1,-1)=\langle 1,1\rangle$

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Solution: C

We'll find the partial derivatives of the function so that we can plug them into the formula for the gradient vector.

$$\frac{\partial f}{\partial x} = 6xy + 4x - y$$

$$\frac{\partial f}{\partial y} = 3x^2 - 3y^2 - x$$

Plugging these into the formula for the gradient vector, we get

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x, y) = \langle 6xy + 4x - y,\ 3x^2 - 3y^2 - x \rangle$$

Evaluating the gradient vector at $P(1, -1)$ gives

$$\nabla f(1, -1) = \langle 6(1)(-1) + 4(1) - (-1),\ 3(1)^2 - 3(-1)^2 - (1) \rangle$$

$$\nabla f(1, -1) = \langle -1, -1 \rangle$$

This is the gradient vector of the function at $P(1, -1)$.

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Topic: Gradient vectors and the tangent plane

Question: Use the gradient vector to find the equation of the tangent plane at $P(2,-3)$.

$$f(x,y)=6x^3-4xy^4+3xy+2y^3+9$$

Answer choices:

A $z=261x-924y+2{,}772$

B $z=261x-924y+2{,}631$

C $z=-261x+924y+2{,}772$

D $z=-261x+924y+2{,}631$

Solution: D

We'll find the partial derivatives of the function so that we can plug them into the formula for the gradient vector.

$$\frac{\partial f}{\partial x} = 18x^2 - 4y^4 + 3y$$

$$\frac{\partial f}{\partial y} = -16xy^3 + 3x + 6y^2$$

Plugging these into the formula for the gradient vector, we get

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x, y) = \langle 18x^2 - 4y^4 + 3y, \ -16xy^3 + 3x + 6y^2 \rangle$$

Evaluating the gradient vector at $P(2, -3)$ gives

$$\nabla f(2, -3) = \langle 18(2)^2 - 4(-3)^4 + 3(-3), \ -16(2)(-3)^3 + 3(2) + 6(-3)^2 \rangle$$

$$\nabla f(2, -3) = \langle 72 - 324 - 9, 864 + 6 + 54 \rangle$$

$$\nabla f(2, -3) = \langle -261, 924 \rangle$$

This is the gradient vector of the function at $P(2, -3)$. Now that we've got it, we can find the equation of the tangent plane. If we use the formula

$$a(x - x_0) + b(y - y_0) - (z - z_0) = 0$$

for the tangent plane, then we can take $a = -261$ and $b = 924$ from the gradient vector.

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$$-261(x-2)+924(y-(-3))-(z-z_0)=0$$

$$z-z_0=-261x+522+924y+2{,}772$$

$$z-z_0=-261x+924y+3{,}294$$

We get z_0 by plugging $P(2,-3)$ into $f(x,y)$.

$$f(2,-3)=6(2)^3-4(2)(-3)^4+3(2)(-3)+2(-3)^3+9$$

$$f(2,-3)=6(8)-4(2)(81)+3(2)(-3)+2(-27)+9$$

$$f(2,-3)=48-648-18-54+9$$

$$f(2,-3)=-663$$

Plugging this missing value into the equation we left off with, we get

$$z-(-663)=-261x+924y+3{,}294$$

$$z+663=-261x+924y+3{,}294$$

$$z=-261x+924y+2{,}631$$

This is the equation of the tangent plane at $P(2,-3)$.

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Topic: Gradient vectors and the tangent plane

Question: Use the gradient vector to find the equation of the tangent plane at $P(1,0,-2)$.

$$f(x,y,z)=8z^2-12xy^2+34x^3-12xz^2-3xyz-56$$

Answer choices:

A $27x+3y+8z=11$

B $27x+3y+8z=-11$

C $3x+8y=-13$

D $3x+8y=13$

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Solution: A

The partial derivatives of the function are

$$\frac{\partial f}{\partial x} = -12y^2 + 102x^2 - 12z^2 - 3yz$$

$$\frac{\partial f}{\partial y} = -24xy - 3xz$$

$$\frac{\partial f}{\partial z} = 16z - 24xz - 3xy$$

Plugging these into the formula for the gradient vector, we get

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\nabla f(x, y, z) = \langle -12y^2 + 102x^2 - 12z^2 - 3yz,\ -24xy - 3xz,\ 16z - 24xz - 3xy \rangle$$

Evaluating the gradient vector at $P(1,0,-2)$ gives

$$\nabla f(x, y, z) = \langle -12(0)^2 + 102(1)^2 - 12(-2)^2 - 3(0)(-2),$$

$$-24(1)(0) - 3(1)(-2),\ 16(-2) - 24(1)(-2) - 3(1)(0) \rangle$$

$$\nabla f(x, y, z) = \langle 0 + 102 - 48 - 0,\ 0 + 6,\ -32 + 48 - 0 \rangle$$

$$\nabla f(x, y, z) = \langle 54,\ 6,\ 16 \rangle$$

This is the gradient vector of the function at $P(1,0,-2)$. Now that we've got it, we can find the equation of the tangent plane. If we use the formula

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

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for the tangent plane, then we can take $a=54$ and $b=6$ and $c=16$ from the gradient vector.

$$54(x-1)+6(y-0)+16(z-(-2))=0$$

$$54x-54+6y+16z+32=0$$

$$54x+6y+16z=22$$

$$27x+3y+8z=11$$

This is the equation of the tangent plane at $P(1,0,-2)$.

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Topic: Maximum rate of change and its direction

Question: Find the maximum rate of change and its direction.

$$f(x, y) = 2x^2y - 3y^3$$

at $P(1,1)$

Answer choices:

A $||\nabla f(1,1)|| = \sqrt{65}$ and $\nabla f(1,1) = \langle 4,7 \rangle$

B $||\nabla f(1,1)|| = \sqrt{65}$ and $\nabla f(1,1) = \langle -4,7 \rangle$

C $||\nabla f(1,1)|| = \sqrt{65}$ and $\nabla f(1,1) = \langle 4, -7 \rangle$

D $||\nabla f(1,1)|| = \sqrt{65}$ and $\nabla f(1,1) = \langle -4, -7 \rangle$

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Solution: C

The maximum rate of change of a function at the point $P(x,y)$ is given by

$$||\nabla f(x,y)|| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

so we'll need to start by finding first order partial derivatives.

$$\frac{\partial f}{\partial x} = 4xy$$

and

$$\frac{\partial f}{\partial y} = 2x^2 - 9y^2$$

Evaluating at $P(1,1)$ gives

$$\frac{\partial f}{\partial x}(1,1) = 4(1)(1)$$

$$\frac{\partial f}{\partial x}(1,1) = 4$$

and

$$\frac{\partial f}{\partial y}(1,1) = 2(1)^2 - 9(1)^2$$

$$\frac{\partial f}{\partial y}(1,1) = -7$$

Plugging these into the formula for maximum rate of change, we get

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$$||\nabla f(1,1)|| = \sqrt{(4)^2 + (-7)^2}$$

$$||\nabla f(1,1)|| = \sqrt{16 + 49}$$

$$||\nabla f(1,1)|| = \sqrt{65}$$

The direction of the maximum rate of change is given by the gradient vector.

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Since we already know the value of the partial derivatives once they're evaluated at $P(1,1)$, we can say that the direction of the maximum rate of change is

$$\nabla f(1,1) = \langle 4, -7 \rangle$$

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Topic: Maximum rate of change and its direction

Question: Find the maximum rate of change and its direction.

$$f(x, y) = 5x^2 \cos y - 5x^4$$

at $P(-2,0)$

Answer choices:

A $||\nabla f(-2,0)|| = 140$ and $\nabla f(-2,0) = \langle 140,0 \rangle$

B $||\nabla f(-2,0)|| = \sqrt{140}$ and $\nabla f(-2,0) = \langle 140,0 \rangle$

C $||\nabla f(-2,0)|| = 140$ and $\nabla f(-2,0) = \langle 0,140 \rangle$

D $||\nabla f(-2,0)|| = \sqrt{140}$ and $\nabla f(-2,0) = \langle 0,140 \rangle$

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Solution: A

The maximum rate of change of a function at the point $P(x,y)$ is given by

$$||\nabla f(x,y)|| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

so we'll need to start by finding first order partial derivatives.

$$\frac{\partial f}{\partial x} = 10x\cos y - 20x^3$$

and

$$\frac{\partial f}{\partial y} = -5x^2\sin y$$

Evaluating at $P(-2,0)$ gives

$$\frac{\partial f}{\partial x}(-2,0) = 10(-2)\cos 0 - 20(-2)^3$$

$$\frac{\partial f}{\partial x}(-2,0) = 140$$

and

$$\frac{\partial f}{\partial y}(-2,0) = -5(-2)^2\sin 0$$

$$\frac{\partial f}{\partial y}(-2,0) = 0$$

Plugging these into the formula for maximum rate of change, we get

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$$||\nabla f(-2,0)|| = \sqrt{(140)^2 + (0)^2}$$

$$||\nabla f(-2,0)|| = \sqrt{(140)^2}$$

$$||\nabla f(-2,0)|| = 140$$

The direction of the maximum rate of change is given by the gradient vector.

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Since we already know the value of the partial derivatives once they're evaluated at $P(-2,0)$, we can say that the direction of the maximum rate of change is

$$\nabla f(-2,0) = \langle 140,0 \rangle$$

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Topic: Maximum rate of change and its direction

Question: Find the maximum rate of change and its direction.

$$f(x, y) = 3y^2 - \sin(xy) + 4e^x$$

at $P(0,3)$

Answer choices:

A $\quad ||\nabla f(0,3)|| = 10\sqrt{13}$ and $\nabla f(0,3) = \langle 18,1 \rangle$

B $\quad ||\nabla f(0,3)|| = 5\sqrt{13}$ and $\nabla f(0,3) = \langle 18,1 \rangle$

C $\quad ||\nabla f(0,3)|| = 10\sqrt{13}$ and $\nabla f(0,3) = \langle 1,18 \rangle$

D $\quad ||\nabla f(0,3)|| = 5\sqrt{13}$ and $\nabla f(0,3) = \langle 1,18 \rangle$

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Solution: D

The maximum rate of change of a function at the point $P(x,y)$ is given by

$$||\nabla f(x,y)|| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}$$

so we'll need to start by finding first order partial derivatives.

$$\frac{\partial f}{\partial x} = -y\cos(xy) + 4e^x$$

and

$$\frac{\partial f}{\partial y} = 6y - x\cos(xy)$$

Evaluating at $P(0,3)$ gives

$$\frac{\partial f}{\partial x}(0,3) = -3\cos(0 \cdot 3) + 4e^0$$

$$\frac{\partial f}{\partial x}(0,3) = 1$$

and

$$\frac{\partial f}{\partial y}(0,3) = 6(3) - 0\cos(0 \cdot 3)$$

$$\frac{\partial f}{\partial y}(0,3) = 18$$

Plugging these into the formula for maximum rate of change, we get

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$$||\nabla f(0,3)|| = \sqrt{(1)^2 + (18)^2}$$

$$||\nabla f(0,3)|| = \sqrt{1 + 324}$$

$$||\nabla f(0,3)|| = 5\sqrt{13}$$

The direction of the maximum rate of change is given by the gradient vector.

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Since we already know the value of the partial derivatives once they're evaluated at $P(0,3)$, we can say that the direction of the maximum rate of change is

$$\nabla f(0,3) = \langle 1,18 \rangle$$

Topic: Equation of the tangent plane

Question: Find the equation of the tangent plane.

$$z = 2x^2y^2 - 4xy^2 - 3y$$

at $(1, -3, -9)$

Answer choices:

A $z = 72x + 9y$

B $z = 9y + 18$

C $z = 72x + 9y - 54$

D $z = 9y - 36$

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Solution: B

The formula for the equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

In this problem,

$$z = f(x, y) = 2x^2y^2 - 4xy^2 - 3y$$

and

$$(x_0, y_0, z_0) = (1, -3, -9)$$

To find $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, we'll take first-order partial derivatives of our function,

for f_x:

$$f(x, y) = 2x^2y^2 - 4xy^2 - 3y$$

$$f(x, y) = \left(2y^2\right)x^2 - \left(4y^2\right)x - 3y$$

$$f_x(x, y) = \left(2y^2\right)2x - \left(4y^2\right)1 - 0$$

$$f_x(x, y) = 4xy^2 - 4y^2$$

for f_y:

$$f(x, y) = 2x^2y^2 - 4xy^2 - 3y$$

$$f(x, y) = \left(2x^2\right)y^2 - (4x)y^2 - 3y$$

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$$f_y(x, y) = \left(2x^2\right) 2y - (4x)2y - 3$$

$$f_y(x, y) = 4x^2y - 8xy - 3$$

and then evaluate them at $(1, -3, -9)$, and we'll get

$$f_x(x, y) = 4xy^2 - 4y^2$$

$$f_x(1, -3) = 4(1)(-3)^2 - 4(-3)^2$$

$$f_x(1, -3) = 36 - 36 = 0$$

and

$$f_y(x, y) = 4x^2y - 8xy - 3$$

$$f_y(1, -3) = 4(1)^2(-3) - 8(1)(-3) - 3$$

$$f_y(1, -3) = -12 + 24 - 3 = 9$$

Now we'll plug these values and the given point into the formula for the equation of the tangent plane.

$$z + 9 = f_x(1, -3)(x - 1) + f_y(1, -3)(y + 3)$$

$$z + 9 = 0(x - 1) + 9(y + 3)$$

$$z = 0 + 9y + 27 - 9$$

$$z = 9y + 18$$

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Topic: Equation of the tangent plane

Question: Find the equation of the tangent plane.

$$z = 2\cos x \sin y + 1$$

at $(0,0,1)$

Answer choices:

A $\quad z = 2x + 1$

B $\quad z = -2x + 1$

C $\quad z = 2y + 1$

D $\quad z = -2y + 1$

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Solution: C

The formula for the equation of the tangent plane to the surface $z = f(x, y)$ at (x_0, y_0, z_0) is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

In this problem,

$$z = f(x, y) = 2\cos x \sin y + 1$$

and

$$(x_0, y_0, z_0) = (0,0,1)$$

To find $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$, we'll take first-order partial derivatives of our function,

for f_x:

$$f(x, y) = 2\cos x \sin y + 1$$

$$f(x, y) = (2\sin y)(\cos x) + 1$$

$$f_x(x, y) = (2\sin y)(-\sin x) + 0$$

$$f_x(x, y) = -2\sin x \sin y$$

for f_y:

$$f(x, y) = 2\cos x \sin y + 1$$

$$f(x, y) = (2\cos x)(\sin y) + 1$$

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$$f_y(x, y) = (2\cos x)(\cos y) + 0$$

$$f_y(x, y) = 2\cos x \cos y$$

and then evaluate them at $(0,0,1)$, and we'll get

$$f_x(x, y) = -2\sin x \sin y$$

$$f_x(0,0) = -2\sin 0 \sin 0$$

$$f_x(0,0) = -2(0)(0) = 0$$

and

$$f_y(x, y) = 2\cos x \cos y$$

$$f_y(0,0) = 2\cos 0 \cos 0$$

$$f_y(0,0) = 2(1)(1) = 2$$

Now we'll plug these values and the given point into the formula for the equation of the tangent plane.

$$z - 1 = f_x(0,0)(x - 0) + f_y(0,0)(y - 0)$$

$$z - 1 = 0(x - 0) + 2(y - 0)$$

$$z = 2y - 2(0) + 1$$

$$z = 2y + 1$$

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Topic: Equation of the tangent plane

Question: Find the equation of the tangent plane.

$$z = x^3 - y^3$$

at $(2,1,7)$

Answer choices:

A $z = 11x + 5y - 20$

B $z = 9x + 9y - 20$

C $z = 8x - y - 8$

D $z = 12x - 3y - 14$

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Solution: D

The formula for the equation of the tangent plane to the surface $z=f(x,y)$ at (x_0, y_0, z_0) is

$$z-z_0=f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$$

In this problem,

$$z=f(x,y)=x^3-y^3$$

and

$$(x_0,y_0,z_0)=(2,1,7)$$

To find $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$, we'll take first-order partial derivatives of our function,

for f_x:

$$f(x,y)=x^3-y^3$$

$$f_x(x,y)=3x^2-0$$

$$f_x(x,y)=3x^2$$

for f_y:

$$f(x,y)=x^3-y^3$$

$$f_y(x,y)=0-3y^2$$

$$f_y(x,y)=-3y^2$$

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and then evaluate them at $(2,1,7)$, and we'll get

$$f_x(x, y) = 3x^2$$

$$f_x(2,1) = 3(2)^2 = 12$$

and

$$f_y(x, y) = -3y^2$$

$$f_y(2,1) = -3(1)^2 = -3$$

Now we'll plug these values and the given point into the formula for the equation of the tangent plane.

$$z - 7 = f_x(2,1)(x - 2) + f_y(2,1)(y - 1)$$

$$z - 7 = 12(x - 2) - 3(y - 1)$$

$$z = 12x - 24 - 3y + 3 + 7$$

$$z = 12x - 3y - 14$$

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Topic: Normal line to the surface

Question: Find the equation of the normal line to the tangent plane.

$$-x^2 + 3y^2 + 2z^2 = 12$$

$$\frac{\partial f}{\partial x}(1, -1, 2) = -2$$

$$\frac{\partial f}{\partial y}(1, -1, 2) = -6$$

$$\frac{\partial f}{\partial z}(1, -1, 2) = 8$$

Answer choices:

A $-\dfrac{x-1}{2} = \dfrac{y+1}{6} = \dfrac{z-2}{8}$

B $\dfrac{x-1}{2} = \dfrac{y+1}{6} = \dfrac{z-2}{8}$

C $\dfrac{x-1}{2} = -\dfrac{y+1}{6} = \dfrac{z-2}{8}$

D $-\dfrac{x-1}{2} = -\dfrac{y+1}{6} = \dfrac{z-2}{8}$

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Solution: D

The normal line to the surface of the tangent plane can be represented by the symmetric equations

$$\frac{x-x_0}{\frac{\partial f}{\partial x}}=\frac{y-y_0}{\frac{\partial f}{\partial y}}=\frac{z-z_0}{\frac{\partial f}{\partial z}}$$

where $\partial f/\partial x$, $\partial f/\partial y$, and $\partial f/\partial z$ are the partial derivatives of f with respect to x, y, and z, and (x_0, y_0, z_0) is the point of tangency.

Since we've already been given the partial derivatives at the point of tangency, we can just plug in the partial derivatives and the given point.

$$\frac{x-1}{-2}=\frac{y-(-1)}{-6}=\frac{z-2}{8}$$

$$\frac{x-1}{-2}=\frac{y+1}{-6}=\frac{z-2}{8}$$

$$-\frac{x-1}{2}=-\frac{y+1}{6}=\frac{z-2}{8}$$

These are the symmetric equations that represent the normal line at the point $(1,-1,2)$.

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Topic: Normal line to the surface

Question: Find the equation of the normal line to the tangent plane at the point $(3,5,4)$.

$$2x^2 - y^2 + z^2 = 9$$

Answer choices:

A $\quad \dfrac{x-3}{6} = -\dfrac{y-5}{5} = \dfrac{z-4}{4}$

B $\quad \dfrac{x+3}{6} = \dfrac{y-5}{5} = \dfrac{z-4}{4}$

C $\quad \dfrac{x-3}{6} = -\dfrac{y-2}{2} = z-1$

D $\quad \dfrac{x+3}{6} = \dfrac{y-2}{2} = z-1$

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Solution: A

The partial derivatives of the tangent plane equation are

$$\frac{\partial f}{\partial x} = 4x \qquad \frac{\partial f}{\partial y} = -2y \qquad \frac{\partial f}{\partial z} = 2z$$

and their values at $(3,5,4)$ are

$$\frac{\partial f}{\partial x}(3,5,4) = 4(3) \qquad \frac{\partial f}{\partial y}(3,5,4) = -2(5) \qquad \frac{\partial f}{\partial z}(3,5,4) = 2(4)$$

$$\frac{\partial f}{\partial x}(3,5,4) = 12 \qquad \frac{\partial f}{\partial y}(3,5,4) = -10 \qquad \frac{\partial f}{\partial z}(3,5,4) = 8$$

Plugging these values and $(x_0, y_0, z_0) = (3,5,4)$ into the symmetric equations gives

$$\frac{x - x_0}{\frac{\partial f}{\partial x}} = \frac{y - y_0}{\frac{\partial f}{\partial y}} = \frac{z - z_0}{\frac{\partial f}{\partial z}}$$

$$\frac{x-3}{12} = \frac{y-5}{-10} = \frac{z-4}{8}$$

$$\frac{x-3}{6} = -\frac{y-5}{5} = \frac{z-4}{4}$$

These are the symmetric equations that represent the normal line at the point $(3,5,4)$.

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Topic: Normal line to the surface

Question: Which set of parametric equations defines the normal line to the surface at $(-2,1,\sqrt{5})$?

$$x^2+y^2=z^2$$

Answer choices:

A $x=4t-2$, $y=2t+1$, and $z=-2t+2$

B $x=-4t-2$, $y=2t+1$, and $z=-2\sqrt{5}t+\sqrt{5}$

C $x=2t-2$, $y=-2t+1$, and $z=2\sqrt{5}t-\sqrt{5}$

D $x=-2t-2$, $y=-4t+1$, and $z=t+4$

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Solution: B

First, we'll rewrite the given equation.

$$x^2 + y^2 = z^2$$

$$x^2 + y^2 - z^2 = 0$$

$$f(x, y, x) = x^2 + y^2 - z^2$$

The gradient then is defined as

$$\nabla F(x, y, z) = F_x(x, y, z)\mathbf{i} + F_y(x, y, z)\mathbf{j} + F_z(x, y, z)\mathbf{k}$$

$$\nabla F(x, y, z) = (2x)\mathbf{i} + (2y)\mathbf{j} + (-2z)\mathbf{k}$$

The gradient at the point $(-2,1,\sqrt{5})$ is

$$\nabla F(-2,1,\sqrt{5}) = (2(-2))\mathbf{i} + (2(1))\mathbf{j} + (-2(\sqrt{5}))\mathbf{k}$$

$$\nabla F(-2,1,\sqrt{5}) = -4\mathbf{i} + 2\mathbf{j} - 2\sqrt{5}\mathbf{k}$$

Therefore, the direction numbers of the normal line at $(-2,1,\sqrt{5})$ are -4, 2, and $-2\sqrt{5}$. Then the associated group of symmetric equations is defined by

$$\frac{x+2}{-4} = \frac{y-1}{2} = \frac{z-\sqrt{5}}{-2\sqrt{5}}$$

Setting this compound equation proportional to t and cross multiplying gives us:

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$$\frac{x+2}{-4}=t \qquad \frac{y-1}{2}=t \qquad \frac{z-\sqrt{5}}{-2\sqrt{5}}=t$$

$$x+2=-4t \qquad y-1=2t \qquad z-\sqrt{5}=-2\sqrt{5}t$$

$$x=-4t-2 \qquad y=2t+1 \qquad z=-2\sqrt{5}t+\sqrt{5}$$

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Topic: Critical points

Question: For which values a and b is the critical point of the function $(-8,-5)$?

$$f(x,y)=x^2+y^2-ax+by$$

Answer choices:

A $a=-18$ $b=-10$

B $a=-16$ $b=5$

C $a=-16$ $b=10$

D $a=16$ $b=10$

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Solution: C

Take partial derivatives of $f(x, y)$.

$$\frac{\partial f}{\partial x} = \frac{\partial f(x^2 + y^2 - ax + by)}{\partial x}$$

$$\frac{\partial f}{\partial x} = 2x - a$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f(x^2 + y^2 - ax + by)}{\partial x}$$

$$\frac{\partial f}{\partial y} = 2y + b$$

Setting these equal to 0 gives the system

$$2x - a = 0$$

$$2y + b = 0$$

Since we already know the critical point is $(-8, -5)$, we can plug $x = -8$ and $y = -5$ into the system to solve for a and b.

$$2(-8) - a = 0$$

$$-16 - a = 0$$

$$a = -16$$

and

$$2(-5) + b = 0$$

$$-10 + b = 0$$

$$b = 10$$

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Topic: Critical points

Question: Which function has a critical point that's equidistant from all the three major axes?

Answer choices:

A $f(x, y, z) = x^2 + y^2 + z^2 - x - y - z$

B $f(x, y, z) = x^2 + 2y^2 + 4z^2 + x - y - z$

C $f(x, y, z) = 2x^2 + y^2 + 4z^2 - x - y + z$

D $f(x, y, z) = 2x^2 + 4y^2 + z^2 - x + y - z$

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Solution: A

To test each system, start by taking partial derivatives of $f(x, y, z)$. These are the partial derivatives of $f(x, y, z) = x^2 + y^2 + z^2 - x - y - z$ from answer choice A:

$$\frac{\partial f}{\partial x} = \frac{\partial f(x^2 + y^2 + z^2 - x - y - z)}{\partial x}$$

$$\frac{\partial f}{\partial x} = 2x - 1$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial f(x^2 + y^2 + z^2 - x - y - z)}{\partial y}$$

$$\frac{\partial f}{\partial y} = 2y - 1$$

and

$$\frac{\partial f}{\partial z} = \frac{\partial f(x^2 + y^2 + z^2 - x - y - z)}{\partial z}$$

$$\frac{\partial f}{\partial z} = 2z - 1$$

Setting these equal to 0 gives

$$2x - 1 = 0 \qquad 2y - 1 = 0 \qquad 2z - 1 = 0$$

$$x = \frac{1}{2} \qquad y = \frac{1}{2} \qquad z = \frac{1}{2}$$

Because $x = y = z$, this critical point is equidistant from all three major axes.

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Topic: Critical points

Question: Which two functions have the same critical points?

$$f(x, y) = x^2 + y^2 - 4x - 8y$$

$$g(x, y) = x^2 + 8y^2 - 6x + 15y$$

$$h(x, y) = 2x^2 - 4y^2 - 8x + 32y$$

$$k(x, y) = x^2 - 9y^2 - 5x + 4y$$

Answer choices:

A $f(x, y)$ and $g(x, y)$

B $f(x, y)$ and $h(x, y)$

C $g(x, y)$ and $h(x, y)$

D $g(x, y)$ and $k(x, y)$

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Solution: B

To find the critical point of each function, take its partial derivatives. For $f(x, y)$:

$$\frac{\partial f}{\partial x} = \frac{\partial(x^2 + y^2 - 4x - 8y)}{\partial x}$$

$$\frac{\partial f}{\partial x} = 2x - 4$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial(x^2 + y^2 - 4x - 8y)}{\partial x}$$

$$\frac{\partial f}{\partial y} = 2y - 8$$

Setting these equations equal to 0 gives us the critical point $(2,4)$. For $h(x, y)$:

$$\frac{\partial h}{\partial x} = \frac{\partial(2x^2 - 4y^2 - 8x + 32y)}{\partial x}$$

$$\frac{\partial h}{\partial x} = 4x - 8$$

and

$$\frac{\partial h}{\partial y} = \frac{\partial(2x^2 - 4y^2 - 8x + 32y)}{\partial y}$$

$$\frac{\partial h}{\partial y} = -8y + 32$$

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Setting these equations equal to 0 gives us the critical point $(2,4)$. Therefore, $f(x, y)$ and $h(x, y)$ have the same critical point.

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Topic: Second derivative test

Question: Find the critical points of the function.

$$f(x, y) = 2x^2 + 3xy^2 + y^2 - 2$$

Answer choices:

A $(0,0)$ $\left(-\frac{1}{3}, \frac{2}{3}\right)$ $\left(\frac{1}{3}, -\frac{2}{3}\right)$

B $(0,0)$ $\left(-\frac{1}{3}, \frac{2}{3}\right)$ $\left(-\frac{1}{3}, -\frac{2}{3}\right)$

C $(0,0)$ $\left(-\frac{1}{3}, \frac{2}{3}\right)$ $\left(\frac{1}{3}, \frac{2}{3}\right)$

D $(0,0)$ $\left(-\frac{1}{3}, \frac{2}{3}\right)$ $\left(-\frac{1}{3}, \frac{2}{3}\right)$

Solution: B

To find the critical points of the multivariable function, we'll set the first order partial derivatives equal to 0.

$$\frac{\partial f}{\partial x} = 4x + 3y^2$$

$$4x + 3y^2 = 0$$

$$4x = -3y^2$$

$$x = -\frac{3}{4}y^2$$

and

$$\frac{\partial f}{\partial y} = 6xy + 2y$$

$$6xy + 2y = 0$$

Plugging this value for x into $6xy + 2y = 0$ gives

$$6\left(-\frac{3}{4}y^2\right)y + 2y = 0$$

$$-\frac{9}{2}y^3 + 2y = 0$$

$$y\left(-\frac{9}{2}y^2 + 2\right) = 0$$

$$y = 0$$

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and

$$-\frac{9}{2}y^2+2=0$$

$$-\frac{9}{2}y^2=-2$$

$$y^2=\frac{4}{9}$$

$$y=\pm\frac{2}{3}$$

So we know that $y=0$, $y=2/3$ and $y=-2/3$ are all critical points. Now we just need to solve for their associated x values.

For $y=0$,

$$x=-\frac{3}{4}(0)^2$$

$$x=0$$

The first critical point is $(0,0)$.

For $y=\frac{2}{3}$,

$$x=-\frac{3}{4}\left(\frac{2}{3}\right)^2$$

$$x=-\frac{3}{4}\left(\frac{4}{9}\right)$$

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$$x = -\frac{1}{3}$$

The second critical point is $\left(-\frac{1}{3}, \frac{2}{3}\right)$.

For $y = -\frac{2}{3}$,

$$x = -\frac{3}{4}\left(-\frac{2}{3}\right)^2$$

$$x = -\frac{3}{4}\left(\frac{4}{9}\right)$$

$$x = -\frac{1}{3}$$

The third critical point is $\left(-\frac{1}{3}, -\frac{2}{3}\right)$.

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Topic: Second derivative test

Question: Evaluate the critical points of the function.

$$f(x, y) = 4x^2 - 2xy + 2y^2 - 18$$

Answer choices:

A Local maximum at $(0,0)$

B Saddle point at $(0,0)$

C Local minimum at $(0,0)$

D The test is inconclusive

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Solution: C

Our first step is to find critical points. We'll find the first order partial derivatives of the function, set them equal to 0, and then solve the resulting system of equations for critical points.

$$\frac{\partial f}{\partial x} = 8x - 2y$$

$$8x - 2y = 0$$

$$-2y = -8x$$

$$y = 4x$$

and

$$\frac{\partial f}{\partial y} = -2x + 4y$$

$$-2x + 4y = 0$$

Plugging this value for y into $-2x + 4y = 0$ gives

$$-2x + 4(4x) = 0$$

$$-2x + 16x = 0$$

$$14x = 0$$

$$x = 0$$

So we know that $x = 0$ is a critical point. Now we just need to solve for its associated y value.

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For $x = 0$,

$$y = 4(0)$$

$$y = 0$$

The critical point is $(0,0)$.

Next we'll use the second derivative test to evaluate the critical point, which means we'll need to find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 8$$

$$\frac{\partial^2 f}{\partial y^2} = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2$$

Plugging the second derivatives into the second derivative test gives

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (8)(4) - (-2)^2$$

$$D(x, y) = 32 - 4$$

$$D(x, y) = 28$$

Now we'll evaluate at the critical point $(0,0)$.

$$D(0,0) = 28$$

The second derivative test tells us that,

if $D(x,y) < 0$	then the critical point is a saddle point
if $D(x,y) = 0$	then the second derivative test is inconclusive
if $D(x,y) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$	then the critical point is a local minimum
if $D(x,y) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$	then the critical point is a local maximum

Since

$$D(0,0) = 28 > 0$$

and

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 8 > 0$$

then $(0,0)$ is a local minimum.

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Topic: Second derivative test

Question: Evaluate the critical points of the function.

$$f(x, y) = 8x^3 - xy + 2y^3 - 4$$

Answer choices:

A Saddle point at $(0,0)$ Local minimum at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$

B Saddle point at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$ Local minimum at $(0,0)$

C Saddle point at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$ Local maximum at $(0,0)$

D Saddle point at $(0,0)$ Local maximum at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$

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Solution: A

The first step is to find critical points. We'll find the first order partial derivatives of the function, set them equal to 0, and then solve the resulting system of equations for critical points.

$$\frac{\partial f}{\partial x} = 24x^2 - y$$

$$24x^2 - y = 0$$

$$-y = -24x^2$$

$$y = 24x^2$$

and

$$\frac{\partial f}{\partial y} = -x + 6y^2$$

$$-x + 6y^2 = 0$$

Plugging this value for y into $-x + 6y^2 = 0$ gives

$$-x + 6\left(24x^2\right)^2 = 0$$

$$-x + 3{,}456x^4 = 0$$

$$x\left(-1 + 3{,}456x^3\right) = 0$$

$$x = 0$$

and

$$-1 + 3{,}456x^3 = 0$$

$$3{,}456x^3 = 1$$

$$x^3 = \frac{1}{3{,}456}$$

$$x = \frac{1}{12\sqrt[3]{2}}$$

So we know that $x = 0$ and $x = 1/12\sqrt[3]{2}$ are critical points. Now we just need to solve for their associated y values.

For $x = 0$,

$$y = 24(0)^2$$

$$y = 0$$

The first critical point is $(0,0)$.

For $x = \dfrac{1}{12\sqrt[3]{2}}$,

$$y = 24\left(\frac{1}{12\sqrt[3]{2}}\right)^2$$

$$y = \frac{24}{144\sqrt[3]{4}}$$

$$y = \frac{1}{6\sqrt[3]{4}}$$

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The second critical point is $\left(\frac{1}{12\sqrt[3]{2}},\frac{1}{6\sqrt[3]{4}}\right)$.

Next we'll use the second derivative test to evaluate the critical point, which means we'll need to find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2}=48x$$

$$\frac{\partial^2 f}{\partial y^2}=12y$$

$$\frac{\partial^2 f}{\partial x \partial y}=-1$$

Plugging the second derivatives into the second derivative test gives

$$D(x,y)=\left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right)-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x,y)=(48x)(12y)-(-1)^2$$

$$D(x,y)=576xy-1$$

Now we'll evaluate at both critical points.

For $(0,0)$,

$$D(0,0)=576(0)(0)-1$$

$$D(0,0)=-1$$

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For $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$,

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 576\left(\frac{1}{12\sqrt[3]{2}}\right)\left(\frac{1}{6\sqrt[3]{4}}\right) - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{576}{72\sqrt[3]{2}\sqrt[3]{4}} - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{8}{\sqrt[3]{8}} - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{8}{2} - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 3$$

The second derivative test tells us that,

if $D(x, y) < 0$ then the critical point is a saddle point

if $D(x, y) = 0$ then the second derivative test is inconclusive

if $D(x, y) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ then the critical point is a local minimum

if $D(x, y) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ then the critical point is a local maximum

Since

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$$D(0,0) = -1 < 0$$

then $(0,0)$ is a saddle point.

Since

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 3 > 0$$

and

$$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 48\left(\frac{1}{12\sqrt[3]{2}}\right)$$

$$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{4}{\sqrt[3]{2}} > 0$$

then $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$ is a local minimum.

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Topic: Local extrema and saddle points

Question: Where are the local extrema of the function?

$$f(x, y) = 2x^2 + 3y^2 - 4y - 5$$

Answer choices:

A On the plane R^2, where $4x = 0$ and $3y - 2 = 0$.

B On the plane R^2, where $x = 0$ and $6y + 4 = 1$.

C On the plane R^2, where $4x - 6 = 1$ and $y = 0$.

D On the plane R^2, where $4x - 6 = 0$ and $3y - 2 = 0$.

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Solution: A

The function is defined everywhere on the real plane R^2. Therefore, its domain is R^2. The partial derivatives of the function are

$$f_x(x, y) = 4x$$

$$f_y(x, y) = 6y - 4$$

Setting these equations equal to 0 gives $4x = 0$ and $6y - 4 = 0$, or $3y - 2 = 0$. The question doesn't require us to go further, but we could solve these equations to say that the extrema of the function occurs at $(0,2/3)$.

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Topic: Local extrema and saddle points

Question: Which equation verifies that $(2,1)$ is the saddle point of the function?

$$f(x, y) = 2x^2 - 6xy - 2x + 12y + 7$$

Answer choices:

A $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) + \left[f_{xy}(2,1)\right]^2 < 0$

B $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) + \left[f_{xy}(2,1)\right]^2 > 0$

C $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) - \left[f_{xy}(2,1)\right]^2 < 0$

D $\quad D(2,1) = f_{xx}(2,1)f_{yy}(2,1) - \left[f_{xy}(2,1)\right]^2 \geq 0$

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Solution: C

Find the partial derivatives of $f(x, y)$.

$$f_x(x, y) = 4x - 6y - 2$$

$$f_y(x, y) = -6x + 12$$

Setting these functions equal to 0 gives the following system of equations:

$$4x - 6y - 2 = 0$$

$$-6x + 12 = 0$$

Solving the system, we find that $x = 2$ and $y = 1$. Now we'll take second-order partial derivatives and evaluate them at $(2,1)$.

$$f_{xx}(2, 1) = 4$$

$$f_{yy}(2, 1) = 0$$

$$f_{xy}(2, 1) = -6$$

We'll use the second derivative test to classify $(2,1)$.

$$D(2,1) = f_{xx}(2,1)f_{yy}(2,1) - \left[f_{xy}(2,1)\right]^2$$

$$D(2,1) = (4)(0) - (-6)^2$$

$$D(2,1) < 0$$

Because $D(2,1) < 0$, $(2,1)$ is a saddle point of the function.

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Topic: Local extrema and saddle points

Question: Which statement is true about the local extrema of the function?

$$f(x, y) = 2x^2 + y^3 - 6xy - 12y$$

Answer choices:

A	Local minimum at $(6,4)$	Local maximum at $\left(-\frac{3}{2}, -1\right)$
B	Local maximum at $(6,4)$	Local minimum at $\left(-\frac{3}{2}, -1\right)$
C	Local minimum at $(6,4)$	No local maximum
D	Local maximum at $\left(-\frac{3}{2}, -1\right)$	No local minimum

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Solution: C

Find the partial derivatives of $f(x, y)$.

$$f_x(x, y) = 4x - 6y$$

$$f_y(x, y) = 3y^2 - 6x - 12$$

Set these equations equal to 0.

$$4x - 6y = 0$$

$$3y^2 - 6x - 12 = 0$$

If we solve the system, we find that the function has critical points

$\left(-\frac{3}{2}, -1\right)$ and $(6,4)$

Find the second-order partial derivatives.

$$f_{xx}(x, y) = 4$$

$$f_{yy}(x, y) = 6y$$

$$f_{xy}(x, y) = -6$$

Use the second derivative test to classify each critical point.

$$D = f_{xx}(x, y) f_{yy}(x, y) - \left[f_{xy}(x, y)\right]^2$$

$$D\left(-\frac{3}{2}, -1\right) = (4)\left(6(-1)\right) - (-6)^2$$

$$D\left(-\frac{3}{2}, -1\right) = -60 < 0$$

Therefore, this point is a saddle point, not a local extremum.

$$D(6,4) = (4)\left(6(4)\right) - (-6)^2$$

$$D(6,4) = 60 > 0$$

Therefore, this point is an extremum. Since $f_{xx}(6,4) = 4 > 0$, then $(6,4)$ is a local minimum.

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Topic: Global extrema

Question: Find the difference between the global maximum and minimum, if $f(x, y)$ is defined on a closed isosceles triangle OMN bounded by $x = 0$, $y = 3$, and $y = x$.

$$f(x, y) = x^2 + y^2 - 4xy + 5$$

Answer choices:

A 11

B 27

C 7

D 5

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Solution: B

We need to check for critical points in the interior of the triangle, and along each edge of the triangle, including at its vertices.

To find critical points in the triangle's interior, take the first-order partial derivatives of the function.

$$f(x, y) = x^2 + y^2 - 4xy + 5$$

$$f_x(x, y) = 2x - 4y$$

$$f_y(x, y) = 2y - 4x$$

Solve this as a system of equations.

$$4x - 8y + (-4x + 2y) = 0 + (0)$$

$$-8y + 2y = 0$$

$$-6y = 0$$

$$y = 0$$

Then

$$2x - 4y = 0$$

$$2x - 4(0) = 0$$

$$2x = 0$$

$$x = 0$$

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This critical point $(0,0)$ lies at one vertex of the triangle, not in the interior of the triangle. We'll set aside that critical point, and check along each side of the triangle.

On the line segment MN, where $y=3$, the x-value varies, but the y-value remains constant at $y=3$.

$$f(x,3)=x^2+3^2-4x(3)+5$$

$$f(x,3)=x^2-12x+14$$

This equation models the function along the boundary $y=3$. Take the partial derivative with respect to x, since x is the value that varies, to find critical points along that boundary.

$$f_x(x,3)=2x-12$$

$$2x-12=0$$

$$x=6$$

This gives the point $(6,3)$, but that lies outside the line segment, since the segment MN is only defined on $0\le x\le 3$, so we can ignore $(6,3)$.

On the line segment OM, where $x=0$, the y-value varies, but the x-value remains constant at $x=0$.

$$f(0,y)=0^2+y^2-4(0)y+5$$

$$f(0,y)=y^2+5$$

This equation models the function along the boundary $x = 0$. Take the partial derivative with respect to y, since y is the value that varies, to find critical points along that boundary.

$$f_y(0,y) = 2y$$

$$2y = 0$$

$$y = 0$$

This gives the point $(0,0)$, which is the point we already found at one vertex of the triangle. So we'll check the last line segment, ON. On the line segment ON, where $y = x$, the x- and y-values both vary, but we can substitute x for y since $y = x$.

$$f(x, x) = x^2 + x^2 - 4x\,(x) + 5$$

$$f(x, x) = -2x^2 + 5$$

This equation models the function along the boundary $y = x$. Take the partial derivative with respect to x, since that's the variable that remains in the equation, to find critical points along that boundary.

$$f_x(x, x) = -4x$$

$$-4x = 0$$

$$x = 0$$

This gives the point $(0,0)$, which is the point we already found at one vertex of the triangle.

We've looked at the interior of the triangle, and all three of its sides, and only identified the critical point $(0,0)$. The only thing left to check is the value at $(0,0)$, and the other two vertices of the triangle, $(0,3)$ and $(3,3)$.

$$f(0,0) = 0^2 + 0^2 - 4(0)(0) + 5 = 5$$

$$f(0,3) = 0^2 + 3^2 - 4(0)(3) + 5 = 14$$

$$f(3,3) = 3^2 + 3^2 - 4(3)(3) + 5 = -13$$

So the global maximum is at 14, and the global minimum is at -13. Their difference is $14 - (-13) = 14 + 13 = 27$.

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Topic: Global extrema

Question: At which boundaries of the square defined on $0 \le x \le 1$ and $0 \le y \le 1$ will the global minimum and global maximum of $f(x, y)$ be -5 and $34/5$?

$$f(x, y) = 6x^2 - 5y^2 + 4xy$$

Answer choices:

A On the unit square $-2 \le x \le 1$ and $-1 \le y \le 1$

B On the unit square $0 \le x \le 1$ and $-1 \le y \le 2$

C On the unit square $0 \le x \le 1$ and $0 \le y \le 1$

D On the unit square $-2 < x < 0$ and $0 < y < 2$

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Solution: C

Find $f_x(x,y)$ and $f_y(x,y)$.

$$f_x(x,y) = 12x + 4y$$

$$f_y(x,y) = -10y + 4x$$

Solve the system of equations $f_x(x,y) = 0$ and $f_y(x,y) = 0$.

$$12x + 4y = 0$$

$$-10y + 4x = 0$$

Solving this system of equations, we get $x = 0$ and $y = 0$. When we plug this point into the original function, we get $f(0,0) = 0$.

On the right vertical side of the square, $x = 1$ and $0 \le y \le 1$. Then

$$f(1,y) = -5y^2 + 4y + 6$$

$$f_y(1,y) = -10y + 4$$

This results in a critical number $y = 2/5$.

Now calculate $f(1,y) = -5y^2 + 4y + 6$ for $y = 0$, $y = 2/5$, and $y = 1$.

$$f(1,0) = -5(0)^2 + 4(0) + 6 = 6$$

$$f\left(1,\frac{2}{5}\right) = -5\left(\frac{2}{5}\right)^2 + 4\left(\frac{2}{5}\right) + 6 = \frac{34}{5}$$

$$f(1,1) = -5(1)^2 + 4(1) + 6 = 5$$

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On the left vertical boundary side, $x = 0$ and $0 \le y \le 1$. Therefore

$$f(0,y) = -5y^2$$

At the point $(0,1)$, the maximum is 0 and the minimum is -5.

On the lower horizontal side of the square, $y = 0$ and $0 \le x \le 1$, which implies

$$f(x,0) = 6x^2$$

Thus, the maximum at $x = 1$ and $y = 0$ is 6, and the minimum at $x = 0$ and $y = 0$ is 0.

On the upper horizontal boundary side, $y = 1$ and $0 \le x \le 1$. Then

$$f(x,1) = 6x^2 + 4x - 5$$

$$f_x(x,1) = 12x + 4$$

This gives a critical point $x = -1/3$. This point is not within the set $0 \le x \le 1$. As a result we must evaluate only $f(x,1)$ at $x = 0$ and $x = 1$:

$$f(0,1) = -5$$

$$f(1,1) = 5$$

Thus the global minimum is -5 at $(0,1)$, and the global maximum is $34/5$ at $(1,2/5)$.

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Topic: Global extrema

Question: What is the global maximum of $f(x, y)$ in the region defined by $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$?

$$f(x, y) = 2x^2 + 2y^2 - 4x^2y$$

Answer choices:

A $\quad 2$

B $\quad 6$

C $\quad 8$

D $\quad 10$

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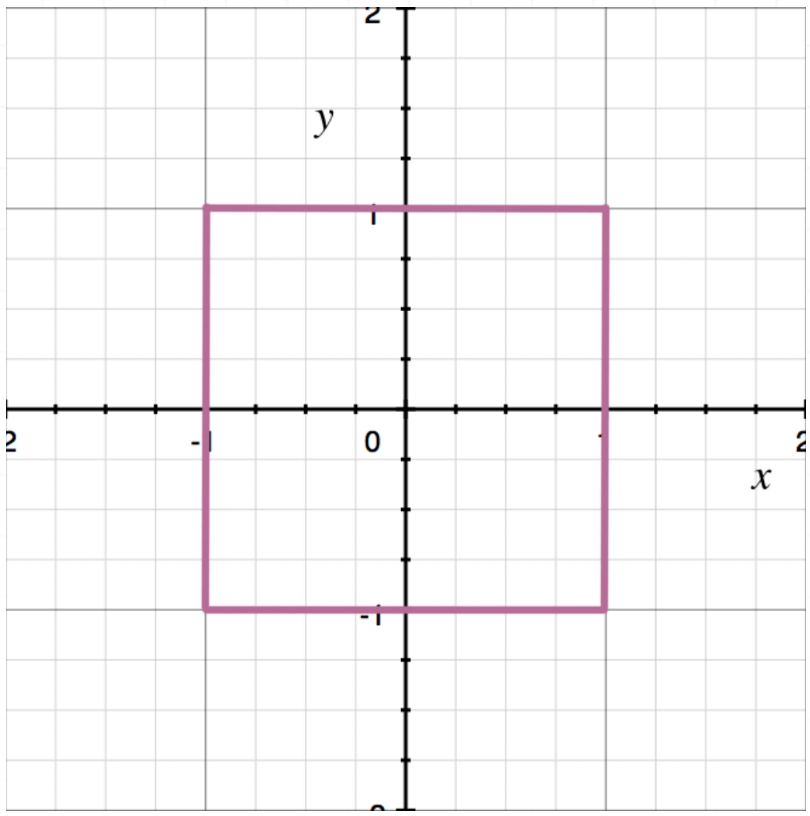
Solution: C

The region is defined over $x = [-1,1]$ and $y = [-1,1]$, which means the region looks like this:



We'll start with the critical points of the function, which we'll find by taking first order partial derivatives of the function, setting them equal to 0, and solving the resulting system of equations. Since $f(x, y) = 2x^2 + 2y^2 - 4x^2y$, we get

$$\frac{\partial f}{\partial x} = 4x - 8xy$$

$$\frac{\partial f}{\partial y} = 4y - 4x^2$$

So we'll solve the system of equations

$$4x - 8xy = 0$$

$$4y - 4x^2 = 0$$

Solving $4y - 4x^2 = 0$ for y gives

$$4y = 4x^2$$

$$y = x^2$$

Plug this value back into $4x - 8xy = 0$.

$$4x - 8xy = 0$$

$$4x - 8x(x^2) = 0$$

$$4x - 8x^3 = 0$$

$$4x(1 - 2x^2) = 0$$

The two solutions are

$$4x = 0$$

$$x = 0$$

and

$$1 - 2x^2 = 0$$

$$1 = 2x^2$$

$$x^2 = \frac{1}{2}$$

$$x = \pm\frac{\sqrt{2}}{2}$$

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Given these three x-values, we need to find the corresponding y-values by plugging into $y = x^2$. We get

For $x = 0$, $\quad y = x^2 = 0^2 = 0$

For $x = \frac{\sqrt{2}}{2}$, $\quad y = x^2 = \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$

For $x = -\frac{\sqrt{2}}{2}$, $\quad y = x^2 = \left(-\frac{\sqrt{2}}{2}\right)^2 = \frac{2}{4} = \frac{1}{2}$

The critical points in the interior of the region are therefore

$(0,0)$, $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$, and $\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)$

Let's go ahead and find the function's value at these three points before we move on to work on the boundaries of the region.

For $x = 0$,

$$f(0,0) = 2(0)^2 + 2(0)^2 - 4(0)^2(0)$$

$$f(0,0) = 0$$

For $\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)$,

$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right) = 2\left(\frac{\sqrt{2}}{2}\right)^2 + 2\left(\frac{1}{2}\right)^2 - 4\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{1}{2}\right)$$

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$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)=2\left(\frac{2}{4}\right)+2\left(\frac{1}{4}\right)-4\left(\frac{2}{4}\right)\left(\frac{1}{2}\right)$$

$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)=1+\frac{1}{2}-1$$

$$f\left(\frac{\sqrt{2}}{2},\frac{1}{2}\right)=\frac{1}{2}$$

For $\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)$,

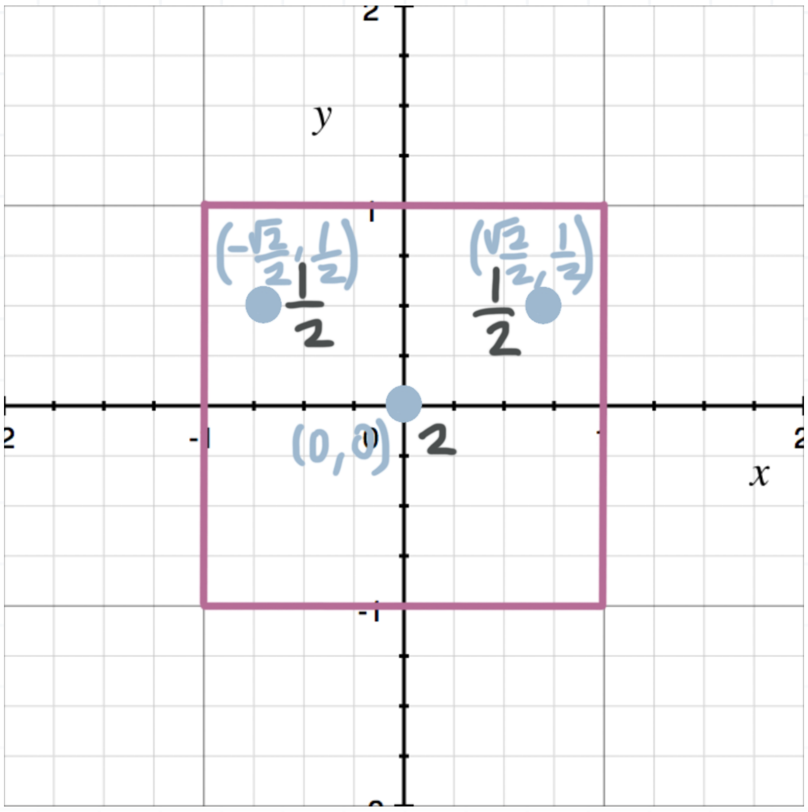
$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=2\left(-\frac{\sqrt{2}}{2}\right)^2+2\left(\frac{1}{2}\right)^2-4\left(-\frac{\sqrt{2}}{2}\right)^2\left(\frac{1}{2}\right)$$

$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=2\left(\frac{2}{4}\right)+2\left(\frac{1}{4}\right)-4\left(\frac{2}{4}\right)\left(\frac{1}{2}\right)$$

$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=1+\frac{1}{2}-1$$

$$f\left(-\frac{\sqrt{2}}{2},\frac{1}{2}\right)=\frac{1}{2}$$

Let’s plot these values into our diagram.



Extrema can only occur at critical points and at the edges of the region. We already now know that the value of the function at each critical point, so our next step is to check the value of the function everywhere along the edges of the region. Comparing the value of the function everywhere along the edges to the value at each critical point, we can say that the largest value we find is the global maximum and that the smallest value we find is the global minimum.

Let's start by looking at the right edge of the region, which corresponds to the line $x=1$, where $-1 \le y \le 1$. Since $x=1$ everywhere along this edge, the function that defines the edge is

$$g(y)=f(1,y)=2(1)^2+2y^2-4(1)^2y$$

$$g(y)=f(1,y)=2+2y^2-4y$$

$$g(y)=f(1,y)=2y^2-4y+2$$

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To figure out the value of the function along the edge, we'll look for critical points of the function that defines the edge, which means we need to take the derivative of the new function.

$$g'(y) = f'(1,y) = 4y - 4$$

Now we'll set the derivative equal to 0 and solve for y.

$$4y - 4 = 0$$

$$4y = 4$$

$$y = 1$$

Putting this together with the value of x along the right edge, the only critical point along the right edge is at $(1,1)$. That's right at the corner of the region, which means that corner is either the lowest point or the highest point along the right edge $x = 1$. So we need to plug that critical point $(1,1)$, and the other corner point along that edge $(1, -1)$ into the original function to compare those values.

$$f(1,1) = 2(1)^2 + 2(1)^2 - 4(1)^2(1)$$

$$f(1,1) = 0$$

and

$$f(1, -1) = 2(1)^2 + 2(-1)^2 - 4(1)^2(-1)$$

$$f(1, -1) = 8$$

Which means the function is decreasing everywhere along that right boundary from the high point at $(1, -1)$ to the low point at $(1,1)$.

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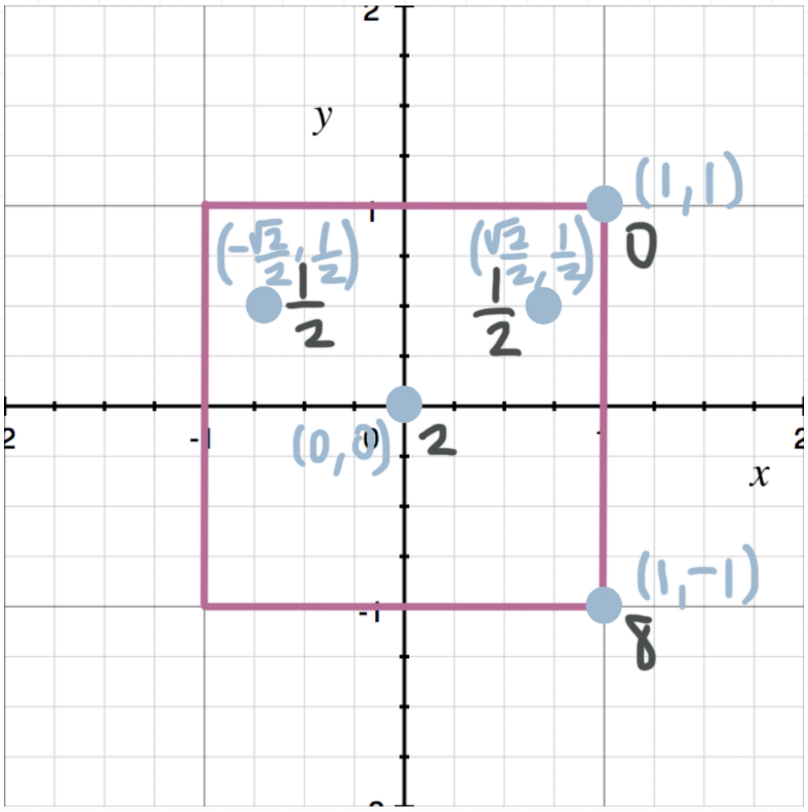

We'll repeat this same process for all four edges of the region. For the left edge of the region, which corresponds to the line $x = -1$, where $-1 \leq y \leq 1$, the function that defines the edge is

$$h(y) = f(-1,y) = 2(-1)^2 + 2y^2 - 4(-1)^2y$$

$$h(y) = f(-1,y) = 2 + 2y^2 - 4y$$

$$h(y) = f(-1,y) = 2y^2 - 4y + 2$$

This is the same function as the right edge, so we already know the derivative is $h'(y) = f'(-1,y) = 4y - 4$, and that the critical point is at $y = 1$. Putting this together with the value of x along the left edge, the only critical point along the left edge is at $(-1,1)$. That's right at the corner of the region, which means that corner is either the lowest point or the highest point along the left edge $x = -1$. So we need to plug that critical point $(-1,1)$, and the other corner point along that edge $(-1,-1)$ into the original function to compare those values.

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$$f(-1,1) = 2(-1)^2 + 2(1)^2 - 4(-1)^2(1)$$

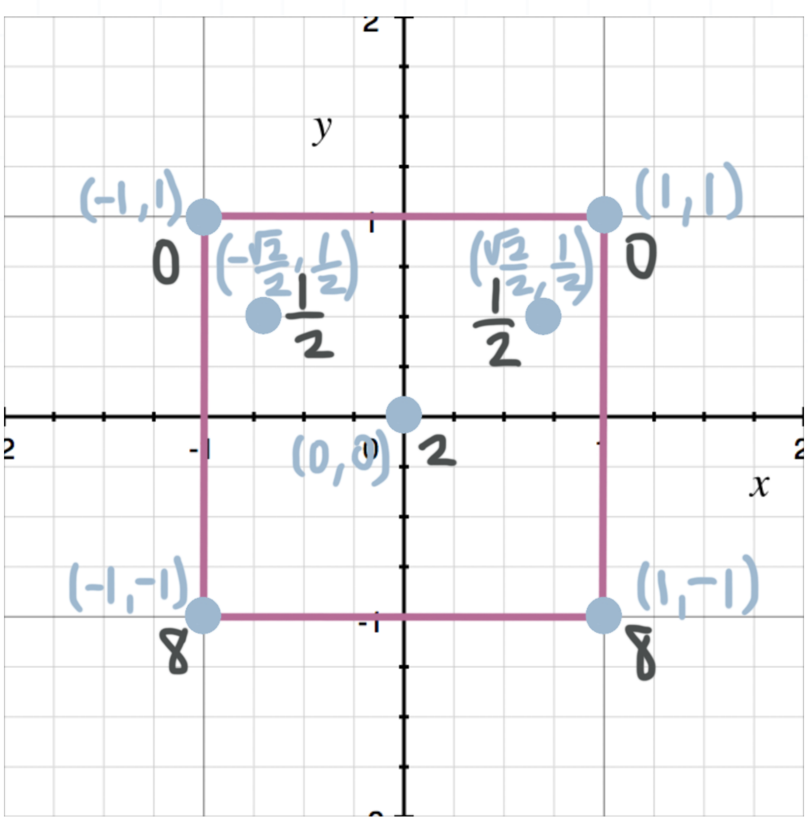
$$f(-1,1) = 0$$

and

$$f(-1,-1) = 2(-1)^2 + 2(-1)^2 - 4(-1)^2(-1)$$

$$f(-1,-1) = 8$$

Which means the function is decreasing everywhere along that left boundary from the high point at $(-1,-1)$ to the low point at $(-1,1)$.



For the top edge of the region, which corresponds to the line $y = 1$, where $-1 \le x \le 1$, the function that defines the edge is

$$m(x) = f(x,1) = 2x^2 + 2(1)^2 - 4x^2(1)$$

$$m(x) = f(x,1) = 2 - 2x^2$$

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Taking the derivative gives

$$m'(x) = f'(x,1) = -4x$$

We'll set the derivative equal to 0 and solve for x.

$$-4x = 0$$

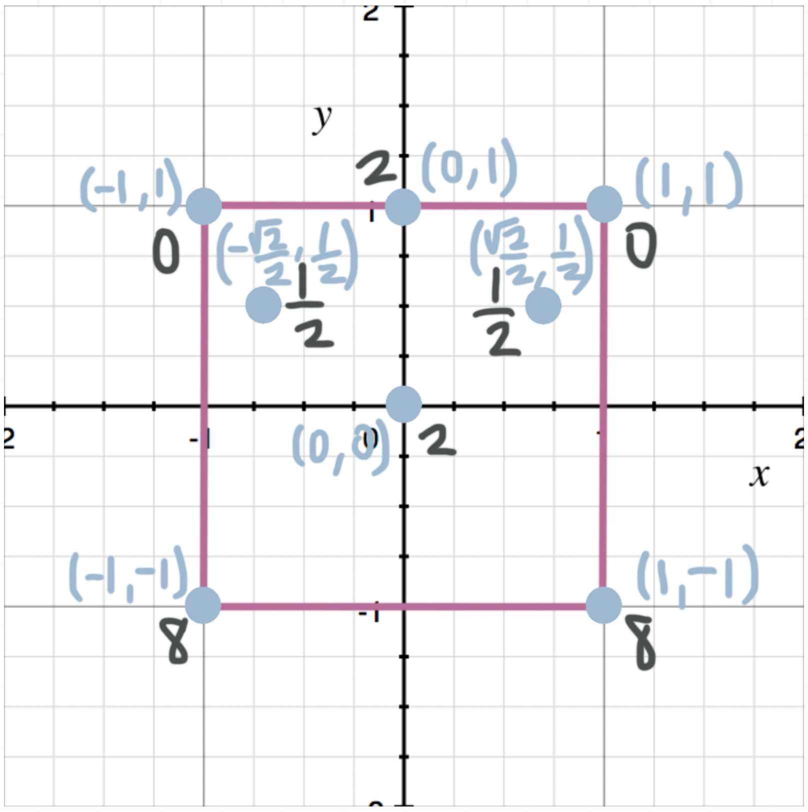
$$x = 0$$

Putting this together with the value of y along the top edge, the only critical point along the top edge is at $(0,1)$. We need to plug that critical point $(0,1)$ into the original function. We already know the values at the two top corners.

$$f(0,1) = 2(0)^2 + 2(1)^2 - 4(0)^2(1)$$

$$f(0,1) = 2$$

Which means the function is increasing from the low point at $(-1,1)$ up to the high point at $(0,1)$, and then decreases from that high point down to the low point at $(1,1)$.



For the bottom edge of the region, which corresponds to the line $y = -1$, where $-1 \leq x \leq 1$, the function that defines the edge is

$$n(x) = f(x, -1) = 2(-1)^2 + 2(-1)^2 - 4x^2(-1)$$

$$n(x) = f(x, -1) = 4 + 4x^2$$

Taking the derivative gives

$$m'(x) = f'(x, -1) = 8x$$

We'll set the derivative equal to 0 and solve for x.

$$8x = 0$$

$$x = 0$$

Putting this together with the value of y along the bottom edge, the only critical point along the bottom edge is at $(0, -1)$. We need to plug that

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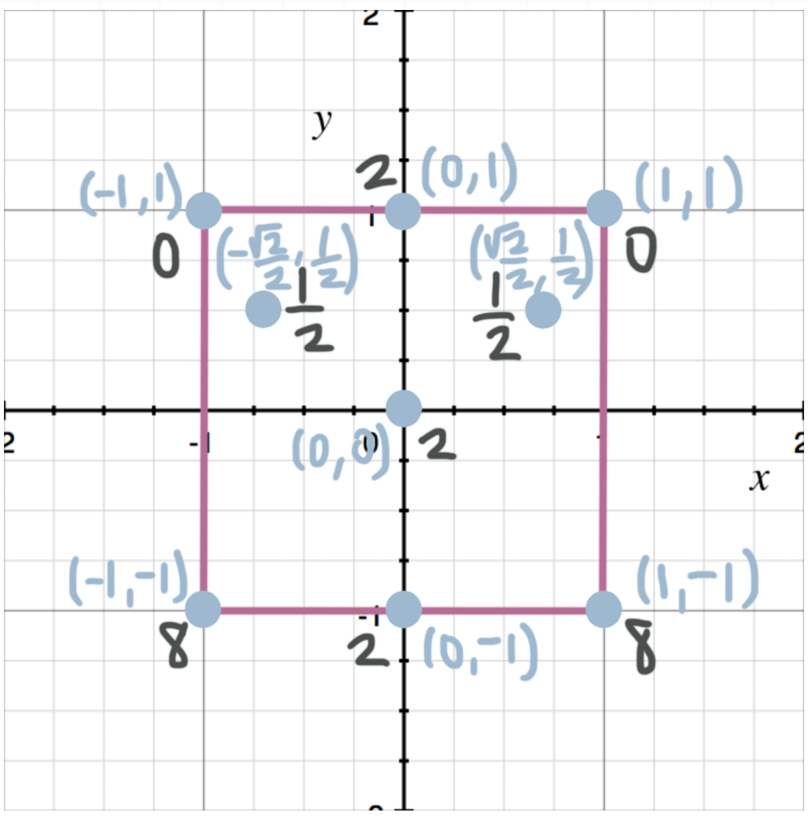
critical point $(0,-1)$ into the original function. We already know the values at the two bottom corners.

$$f(0,-1)=2(0)^2+2(-1)^2-4(0)^2(-1)$$

$$f(0,-1)=2$$

Which means the function is decreasing from the high point at $(-1,-1)$ down to the low point at $(0,-1)$, and then increases from that low point up to the high point at $(1,-1)$.



Looking at all of these values together, we can see that the global maxima of the function over the region exist at $(1,-1)$ and $(-1,-1)$ where the value of the function is 8, which is greater than the values we see everywhere else, which range from 0 to 2.

If we sketch the region in 3D space, we can see these high points.

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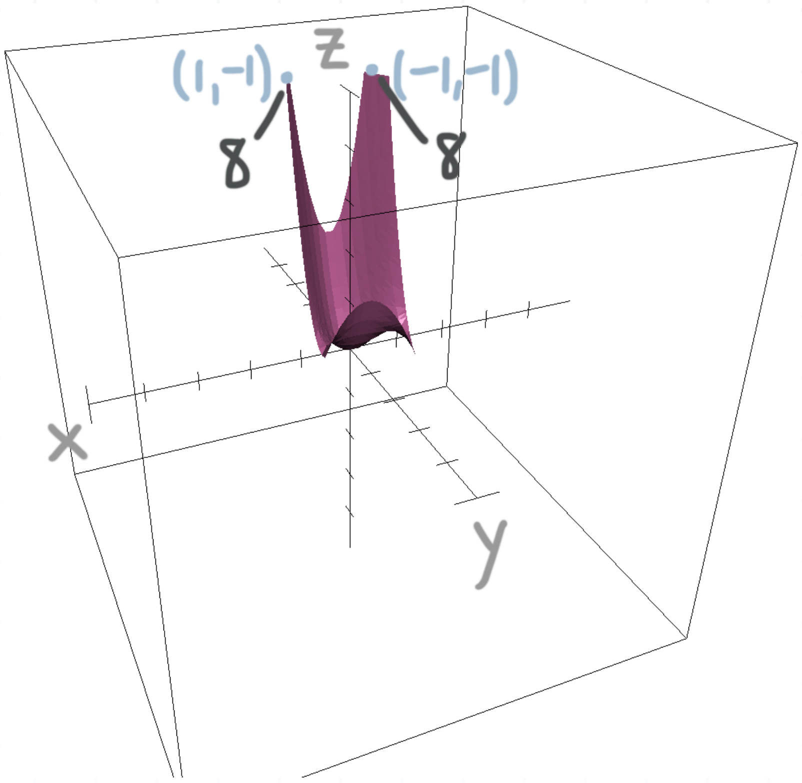
(1,−1)
z
(−1,−1)
8
8
x
y

Topic: Extreme value theorem

Question: Which of these statements is true?

Answer choices:

A The Extreme Value Theorem shows you how to calculate the absolute extrema points in a function $f(x, y)$.

B The Extreme Value Theorem confirms the existence of absolute extrema in a function $f(x, y)$.

C The Extreme Value Theorem states that the absolute extrema can only exist in the interior of the designated bounded set.

D The Extreme Value Theorem states that the absolute extrema can only exist on the boundaries of the designated bounded set.

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Solution: B

The Extreme Value Theorem (EVT) states that if $f(x, y)$ is continuous in some closed, bounded set D, then there are points in D, (x_1, y_1) and (x_2, y_2), such that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum.

This EVT doesn't tell us where the absolute extrema will occur, just that they'll exist.

Answer choice A is incorrect because the theorem does not tell you how to calculate the absolute extrema points in a function $f(x, y)$.

Answer choice B is correct because the theorem confirms the existence of absolute extrema in a function $f(x, y)$.

Option C and D are incorrect because the absolute extrema can occur within the bounded set or on the boundary's edge.

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Topic: Extreme value theorem

Question: Find the absolute minima and absolute maxima of the function.

$$f(x, y) = 3x^2 - 3y^2 + xy + 2$$

for $D = \{(x, y) \mid -1 \leq x \leq 1, -1 \leq y \leq 1\}$

Answer choices:

A Absolute maxima at $(-1,1/6)$ and $(-1,1/6)$

Absolute minima at $(-1,1)$ and $(1, -1)$

B Absolute maxima at $(1,1/6)$ and $(-1,1/6)$

Absolute minima at $(-1,1)$ and $(1, -1)$

C Absolute maxima at $(1,1/6)$ and $(-1, -1/6)$

Absolute minima at $(-1/6,1)$ and $(1/6, -1)$

D Absolute maxima at $(-1,1/6)$ and $(-1, -1/6)$

Absolute minima at $(-1,1)$ and $(1, -1)$

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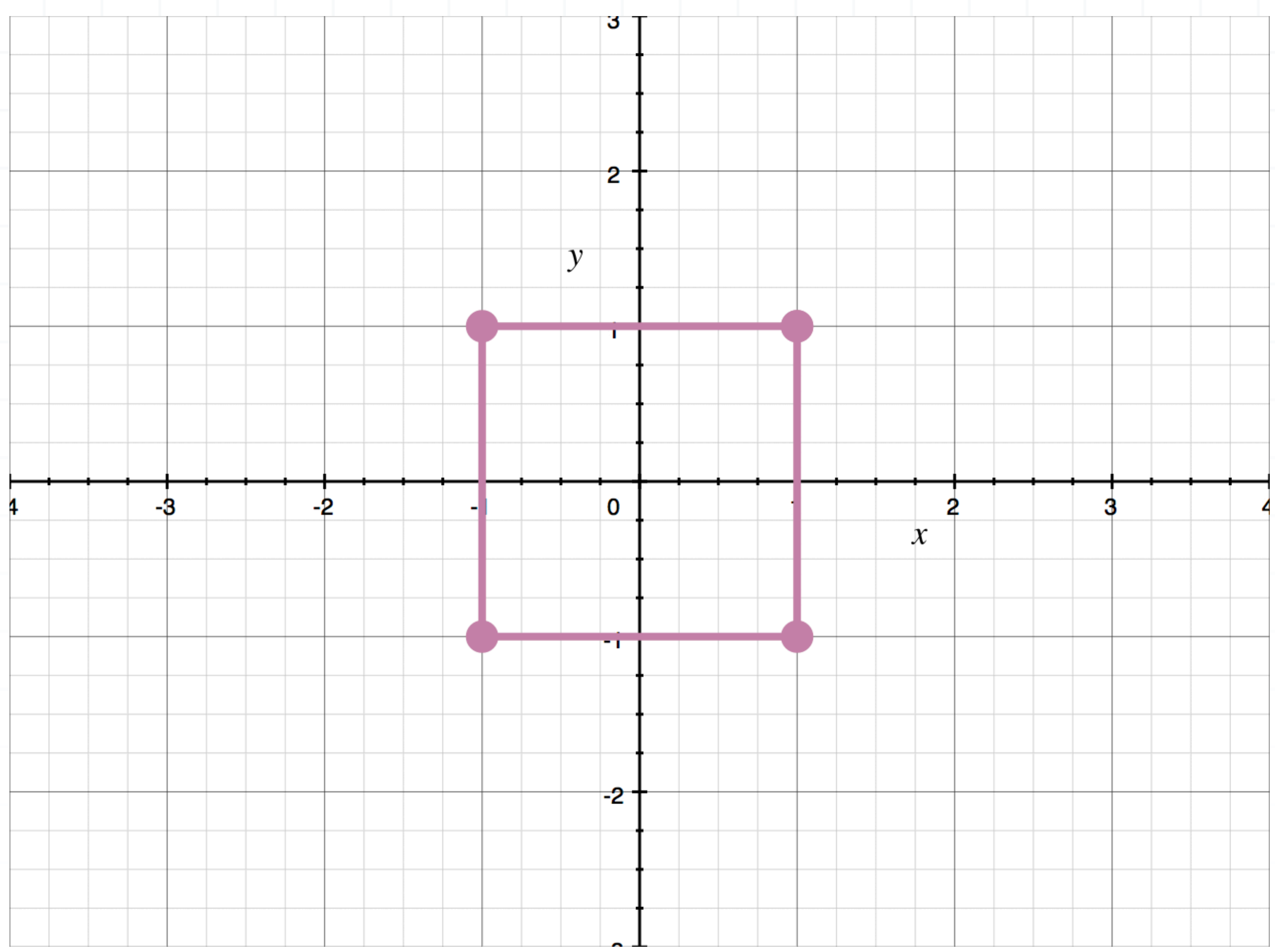
Solution: C

The set D is defined over $x=[-1,1]$ and $y=[-1,1]$, which means the region looks like this:



We'll start with the critical points of the function, which we'll find by taking first order partial derivatives of the function, setting them equal to 0, and solving the resulting system of equations. Since $f(x,y)=3x^2-3y^2+xy+2$, we get

$$\frac{\partial f}{\partial x}=6x+y$$

$$6x+y=0$$

[1] $y=-6x$

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and

$$\frac{\partial f}{\partial y} = -6y + x$$

[2] $-6y + x = 0$

Plugging [1] into [2] gives

$$-6(-6x) + x = 0$$

$$36x + x = 0$$

$$37x = 0$$

$$x = 0$$

The fact that we only found one value tells us that the only critical point is at $x = 0$. Now we just need to find the corresponding value of y, which we'll do by plugging $x = 0$ into $y = -6x$.

$$y = -6(0)$$

$$y = 0$$

Putting $y = 0$ together with $x = 0$, we can say that the only critical point in the region D is $(0,0)$.

To find the value of the function at this critical point, we'll plug $(0,0)$ into $f(x, y)$.

$$f(x, y) = 3x^2 - 3y^2 + xy + 2$$

$$f(0,0) = 3(0)^2 - 3(0)^2 + (0)(0) + 2$$

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$$f(0,0) = 2$$

Extrema can only occur at critical points and at the edges of the region. We now know that the value of the function at the only critical point is $f(0,0) = 2$. Our next step is to check the value of the function everywhere along the edges of the region. Comparing the value of the function along the edges to the value at the critical point, we can say that the largest value we find is the global maximum and that the smallest value we find is the global minimum.

Let's start by looking at the right edge of the region, which corresponds to the line $x = 1$, where $-1 \le y \le 1$. Since $x = 1$ everywhere along this edge, the function that defines the edge is

$$g(y) = f(1,y) = 3(1)^2 - 3y^2 + (1)y + 2$$

$$g(y) = f(1,y) = 3 - 3y^2 + y + 2$$

$$g(y) = f(1,y) = -3y^2 + y + 5$$

To figure out the value of the function along the edge, we'll look for critical points of the function that defines the edge, which means we need to take the derivative of the new function.

$$g'(y) = f'(1,y) = -6y + 1$$

Now we'll set the derivative equal to 0 and solve for y.

$$-6y + 1 = 0$$

$$-6y = -1$$

$$y = \frac{1}{6}$$

With just one critical point along the edge, we know that the highest and lowest values along the edge have to occur at the critical point $y = 1/6$, or at the ends of the interval, $y = [-1,1]$.

At $y = \frac{1}{6}$,

$$g\left(\frac{1}{6}\right) = -3\left(\frac{1}{6}\right)^2 + \frac{1}{6} + 5$$

$$g\left(\frac{1}{6}\right) = -\frac{3}{36} + \frac{6}{36} + \frac{180}{36}$$

$$g\left(\frac{1}{6}\right) \approx 5.08$$

At $y = 1$,

$$g(1) = -3(1)^2 + 1 + 5$$

$$g(1) = -3 + 1 + 5$$

$$g(1) = 3$$

At $y = -1$,

$$g(-1) = -3(-1)^2 + (-1) + 5$$

$$g(-1) = -3 - 1 + 5$$

$$g(-1) = 1$$

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We'll repeat this same process for all four edges of the region. For the left edge of the region, which corresponds to the line $x = -1$, where $-1 \le y \le 1$, the function that defines the edge is

$$h(y) = f(-1,y) = 3(-1)^2 - 3y^2 + (-1)y + 2$$

$$h(y) = f(-1,y) = 3 - 3y^2 - y + 2$$

$$h(y) = f(-1,y) = -3y^2 - y + 5$$

Taking the derivative gives

$$h'(y) = f'(-1,y) = -6y - 1$$

We'll set the derivative equal to 0 and solve for y.

$$-6y - 1 = 0$$

$$-6y = 1$$

$$y = -\frac{1}{6}$$

Evaluating the function at the critical point and at the endpoints of the interval $y = [-1,1]$, we get

At $y = -\frac{1}{6}$,

$$h\left(-\frac{1}{6}\right) = -3\left(-\frac{1}{6}\right)^2 - \left(-\frac{1}{6}\right) + 5$$

$$h\left(-\frac{1}{6}\right) = -\frac{3}{36} + \frac{6}{36} + \frac{180}{36}$$

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$$h\left(-\frac{1}{6}\right) \approx 5.08$$

At $y = 1$,

$$h(1) = -3(1)^2 - (1) + 5$$

$$h(1) = -3 - 1 + 5$$

$$h(1) = 1$$

At $y = -1$,

$$h(-1) = -3(-1)^2 - (-1) + 5$$

$$h(-1) = -3 + 1 + 5$$

$$h(-1) = 3$$

For the top edge of the region, which corresponds to the line $y = 1$, where $-1 \le x \le 1$, the function that defines the edge is

$$m(x) = f(x,1) = 3x^2 - 3(1)^2 + x(1) + 2$$

$$m(x) = f(x,1) = 3x^2 - 3 + x + 2$$

$$m(x) = f(x,1) = 3x^2 + x - 1$$

Taking the derivative gives

$$m'(x) = f'(x,1) = 6x + 1$$

We'll set the derivative equal to 0 and solve for x.

$$6x + 1 = 0$$

$$6x = -1$$

$$x = -\frac{1}{6}$$

Evaluating the function at the critical point and at the endpoints of the interval $x = [-1,1]$, we get

At $x = -\dfrac{1}{6}$,

$$m\left(-\frac{1}{6}\right) = 3\left(-\frac{1}{6}\right)^2 + \left(-\frac{1}{6}\right) - 1$$

$$m\left(-\frac{1}{6}\right) = \frac{3}{36} - \frac{6}{36} - \frac{36}{36}$$

$$m\left(-\frac{1}{6}\right) \approx -1.08$$

At $x = 1$,

$$m(1) = 3(1)^2 + (1) - 1$$

$$m(1) = 3 + 1 - 1$$

$$m(1) = 3$$

At $x = -1$,

$$m(-1) = 3(-1)^2 + (-1) - 1$$

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$$m(-1) = 3 - 1 - 1$$

$$m(-1) = 1$$

For the bottom edge of the region, which corresponds to the line $y = -1$, where $-1 \le x \le 1$, the function that defines the edge is

$$n(x) = f(x, -1) = 3x^2 - 3(-1)^2 + x(-1) + 2$$

$$n(x) = f(x, -1) = 3x^2 - 3 - x + 2$$

$$n(x) = f(x, -1) = 3x^2 - x - 1$$

Taking the derivative gives

$$n'(x) = f'(x, -1) = 6x - 1$$

We'll set the derivative equal to 0 and solve for x.

$$6x - 1 = 0$$

$$6x = 1$$

$$x = \frac{1}{6}$$

Evaluating the function at the critical point and at the endpoints of the interval $x = [-1,1]$, we get

At $x = \frac{1}{6}$,

$$n\left(\frac{1}{6}\right) = 3\left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right) - 1$$

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$$n\left(\frac{1}{6}\right) = \frac{3}{36} - \frac{6}{36} - \frac{36}{36}$$

$$n\left(\frac{1}{6}\right) \approx -1.08$$

At $x = 1$,

$$n(1) = 3(1)^2 - (1) - 1$$

$$n(1) = 3 - 1 - 1$$

$$n(1) = 1$$

At $x = -1$,

$$n(-1) = 3(-1)^2 - (-1) - 1$$

$$n(-1) = 3 + 1 - 1$$

$$n(-1) = 3$$

If we collect all of our values together, we can see the value of the function at various points throughout the region D.

At the critical point	$(0,0)$	$f(x,y) = 2$
Along the right edge at	$(1,1/6)$	$f(x,y) \approx 5.08$
	$(1,1)$	$f(x,y) = 3$
	$(1,-1)$	$f(x,y) = 1$
Along the left edge at	$(-1,-1/6)$	$f(x,y) \approx 5.08$

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	$(-1,1)$	$f(x,y)=1$
	$(-1,-1)$	$f(x,y)=3$
Along the top edge at	$(-1/6,1)$	$f(x,y)\approx-1.08$
	$(1,1)$	$f(x,y)=3$
	$(-1,1)$	$f(x,y)=1$
Along the bottom edge at	$(1/6,-1)$	$f(x,y)\approx-1.08$
	$(1,-1)$	$f(x,y)=1$
	$(-1,-1)$	$f(x,y)=3$

The largest of these values is 5.08 and the smallest is -1.08, which means

the global maxima occur at $(1,1/6)$ and $(-1,-1/6)$

the global minima occur at $(-1/6,1)$ and $(1/6,-1)$

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Topic: Extreme value theorem

Question: Find the absolute minima and absolute maxima of the function.

$$f(x, y) = x^2 - 2y^2 + 4y$$

for $D = \{(x, y) \,|\, x^2 + y^2 \leq 4\}$

Answer choices:

A Absolute maxima at $(4\sqrt{2}/3,2/3)$ and $(-4\sqrt{2}/3,2/3)$

Absolute minimum at $(0, -2)$

B Absolute maxima at $(4\sqrt{2}/3,2/3)$ and $(-4\sqrt{2}/3,2/3)$

Absolute minimum at $(0,2)$

C Absolute maxima at $(4\sqrt{2}/3, -2/3)$ and $(-4\sqrt{2}/3, -2/3)$

Absolute minimum at $(0, -2)$

D Absolute maxima at $(4\sqrt{2}/3, -2/3)$ and $(-4\sqrt{2}/3, -2/3)$

Absolute minimum at $(0,2)$

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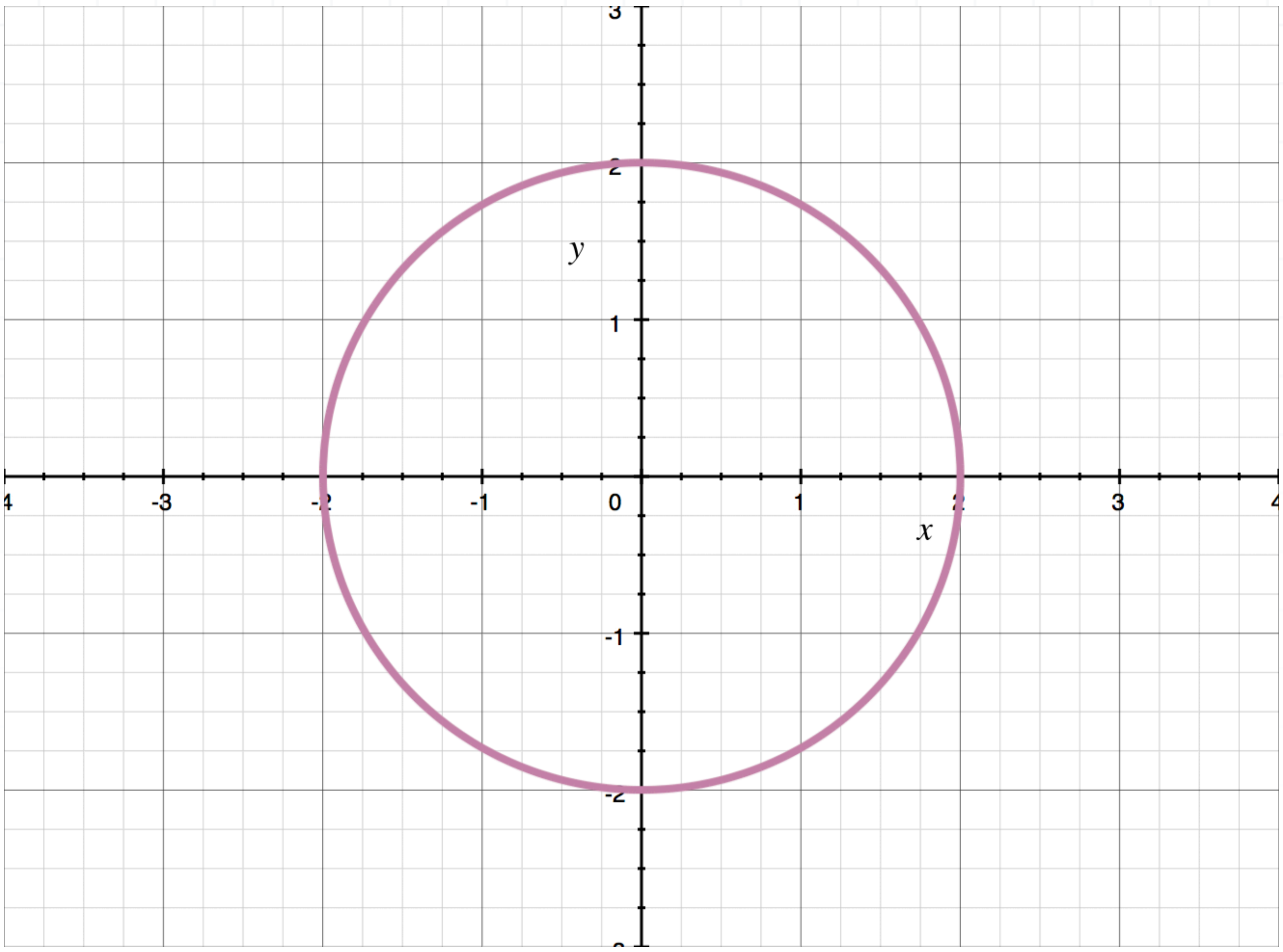
Solution: A

The set D is given by the inequality $x^2 + y^2 \leq 4$, which means that D is the circle centered at the origin with radius 2, and includes all of the area inside the circle, including its boundary at $r = 2$. It looks like this:



We'll start with the critical points of the function, which we'll find by taking first order partial derivatives of the function, setting them equal to 0, and solving the resulting system of equations. Since $f(x, y) = x^2 - 2y^2 + 4y$, we get

$$\frac{\partial f}{\partial x} = 2x$$

$$2x = 0$$

$$x = 0$$

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and

$$\frac{\partial f}{\partial y} = -4y + 4$$

$$-4y + 4 = 0$$

$$-4y = -4$$

$$y = 1$$

The fact that we only found one value for both x and y tells us that we can put them together and say that the only critical point in the region D is at $(0,1)$.

To find the value of the function at this critical point, we'll plug $(0,1)$ into $f(x,y)$.

$$f(x,y) = x^2 - 2y^2 + 4y$$

$$f(0,1) = (0)^2 - 2(1)^2 + 4(1)$$

$$f(0,1) = 2$$

Extrema can only occur at critical points and at the edges of the region. We now know that the value of the function at the only critical point is $f(0,1) = 2$. Our next step is to check the value of the function everywhere along the edges of the region. Comparing the value of the function along the edges to the value at the critical point, we can say that the largest value we find is the global maximum and that the smallest value we find is the global minimum.

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In order to look at the boundary, we need to find an equation that describes the value of the original function along the boundary of D. If we solve the inequality that represents D for x^2 we get

$$x^2 = -y^2 + 4$$

Now we'll make a substitution into the original function, defining this as a new function $g(y)$, that describes the value of $f(x, y)$ along the boundary of the region D.

$$f(x, y) = x^2 - 2y^2 + 4y$$

$$g(y) = \left(-y^2 + 4\right) - 2y^2 + 4y$$

$$g(y) = -3y^2 + 4y + 4$$

Since this new function describes the boundary, and since we're looking for extrema along the boundary, our next step is to find critical points of $g(y)$, which we'll do by taking its derivative.

$$g'(y) = -6y + 4$$

Setting the derivative equal to 0 and solving for y gives

$$-6y + 4 = 0$$

$$-6y = -4$$

$$y = \frac{2}{3}$$

With just one critical point along the edge, we know that the highest and lowest values along the edge have to occur at the critical point $y = 2/3$, or

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at the ends of the interval, $y=[-2,2]$. Remember that $-2 \le y \le 2$ because D is the circle with radius 2.

At $y=2/3$,

$$g\left(\frac{2}{3}\right)=-3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)+4$$

$$g\left(\frac{2}{3}\right)=-\frac{4}{3}+\frac{8}{3}+\frac{12}{3}$$

$$g\left(\frac{2}{3}\right)\approx 5.33$$

At $y=2$,

$$g(2)=-3(2)^2+4(2)+4$$

$$g(2)=-12+8+4$$

$$g(2)=0$$

At $y=-2$,

$$g(-2)=-3(-2)^2+4(-2)+4$$

$$g(-2)=-12-8+4$$

$$g(-2)=-16$$

We need to find the values of x that correspond with $y=2/3$, $y=2$ and $y=-2$ so that we can identify the coordinate points these correspond to. Plugging each of them into $x^2=-y^2+4$ gives

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For $y = 2/3$,

$$x^2 = -\left(\frac{2}{3}\right)^2 + 4$$

$$x^2 = -\frac{4}{9} + \frac{36}{9}$$

$$x^2 = \frac{32}{9}$$

$$x = \pm\frac{4\sqrt{2}}{3}$$

For $y = 2$,

$$x^2 = -2^2 + 4$$

$$x^2 = -4 + 4$$

$$x^2 = 0$$

$$x = 0$$

For $y = -2$,

$$x^2 = -(-2)^2 + 4$$

$$x^2 = -4 + 4$$

$$x^2 = 0$$

$$x = 0$$

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If we collect all of our values together, we can see the value of the function at various points throughout the region D.

At the critical point	$(0,1)$	$f(x,y)=2$
Along the boundary at	$\left(\frac{4\sqrt{2}}{3},\frac{2}{3}\right)$	$f(x,y)\approx 5.33$
	$\left(-\frac{4\sqrt{2}}{3},\frac{2}{3}\right)$	$f(x,y)\approx 5.33$
	$(0,2)$	$f(x,y)=0$
	$(0,-2)$	$f(x,y)=-16$

The largest of these values is 5.33 and the smallest is -16, which means

the global maxima occur at $(4\sqrt{2}/3,2/3)$ and $(-4\sqrt{2}/3,2/3)$

the global minimum occurs at $(0,-2)$

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Topic: Applied optimization

Question: Find the point on the cone closest to the given point.

$$z^2 = x^2 + y^2$$

at $(3,1,0)$

Answer choices:

A $\left(\frac{3}{2}, \frac{1}{2}, \frac{5}{2}\right)$ and $\left(\frac{3}{2}, \frac{1}{2}, -\frac{5}{2}\right)$

B $\left(\frac{3}{2}, \sqrt{\frac{1}{2}}, \sqrt{\frac{5}{2}}\right)$ and $\left(\frac{3}{2}, \sqrt{\frac{1}{2}}, -\sqrt{\frac{5}{2}}\right)$

C $\left(\frac{3}{2}, \frac{1}{2}, \sqrt{\frac{5}{2}}\right)$ and $\left(\frac{3}{2}, \frac{1}{2}, -\sqrt{\frac{5}{2}}\right)$

D $\left(\sqrt{\frac{3}{2}}, \frac{1}{2}, \frac{5}{2}\right)$ and $\left(\sqrt{\frac{3}{2}}, \frac{1}{2}, -\frac{5}{2}\right)$

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Solution: C

We're trying to minimize the distance between $(3,1,0)$ and the surface of the cone. If we're minimizing distance, then we can start with the distance equation, and plug in the point we were given.

$$D=\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

$$D=\sqrt{(x-3)^2+(y-1)^2+(z-0)^2}$$

$$D=\sqrt{(x-3)^2+(y-1)^2+z^2}$$

Because we're optimizing this equation, we need to get it in terms of two variables only. But we were given the equation of the cone as $z^2=x^2+y^2$, so we can substitute into the distance equation for z^2.

$$D=\sqrt{(x-3)^2+(y-1)^2+x^2+y^2}$$

Now we'll square both sides to get rid of the square root.

$$D^2=(x-3)^2+(y-1)^2+x^2+y^2$$

$$D^2=x^2-6x+9+y^2-2y+1+x^2+y^2$$

$$D^2=2x^2-6x+2y^2-2y+10$$

Find first-order partial derivatives of this function.

$$\frac{\partial D^2}{\partial x}=4x-6$$

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$$\frac{\partial D^2}{\partial y} = 4y - 2$$

Set both partial derivatives equal to 0 and solve for x and y.

$$4x - 6 = 0$$

$$4x = 6$$

$$x = \frac{3}{2}$$

and

$$4y - 2 = 0$$

$$4y = 2$$

$$y = \frac{1}{2}$$

So the critical point is given by

$$\left(\frac{3}{2}, \frac{1}{2}\right)$$

Find second-order partial derivatives.

$$\frac{\partial^2 D^2}{\partial x^2} = 4$$

$$\frac{\partial^2 D^2}{\partial y^2} = 4$$

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$$\frac{\partial^2 D^2}{\partial x \partial y} = 0$$

Then we'll plug these into the formula for D.

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (4)(4) - (0)^2$$

$$D(x, y) = 16$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

If $D = 0$, then the second derivative test is inconclusive

If $D > 0$,

and $\frac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\frac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

In this problem, $D > 0$ and $\frac{\partial^2 D^2}{\partial x^2} > 0$, so the critical point is a local minimum.

So now we'll just plug the critical point into the equation $z^2 = x^2 + y^2$ to find the associated z-value.

$$z^2 = x^2 + y^2$$

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$$z^2 = \left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2$$

$$z^2 = \frac{9}{4} + \frac{1}{4}$$

$$z^2 = \frac{10}{4}$$

$$z = \sqrt{\frac{10}{4}}$$

$$z = \pm\sqrt{\frac{5}{2}}$$

Therefore, the points on the cone closest to $(3,1,0)$ are

$$\left(\frac{3}{2}, \frac{1}{2}, \pm\sqrt{\frac{5}{2}}\right)$$

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Topic: Applied optimization

Question: Find three positive numbers with maximum product.

Sum $= 90$

Answer choices:

A $30, 30, 30$

B $30, 40, 20$

C $10, 70, 10$

D $25, 40, 25$

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Solution: A

We need to find three numbers that sum to 90, so we can write one equation that represents the numbers as

$$x + y + z = 90$$

We've been asked to maximize the product of the three numbers, and we can represent this in an equation as

$$P = xyz$$

Since we need to maximize this product equation, we need to get it in terms of just two variables. So we'll solve the sum equation for z and then plug that value into the product equation.

$$x + y + z = 90$$

$$z = 90 - x - y$$

so

$$P = xyz$$

$$P = xy\left(90 - x - y\right)$$

$$P = 90xy - x^2y - xy^2$$

To make things a little easier, we'll change this to

$$f(x, y) = 90xy - x^2y - xy^2$$

Now we'll find the first-order partial derivatives of this function.

$$\frac{\partial f}{\partial x} = 90y - 2xy - y^2$$

$$\frac{\partial f}{\partial y} = 90x - x^2 - 2xy$$

Set both equations equal to 0, solving the first for y and the second for x.

$$90y - 2xy - y^2 = 0$$

$$y(90 - 2x - y) = 0$$

$y = 0$ or $90 - 2x - y = 0$

and

$$90x - x^2 - 2xy = 0$$

$$x(90 - x - 2y) = 0$$

$x = 0$ or $90 - x - 2y = 0$

Because we've been asked for positive numbers, we can't use $x = 0$ or $y = 0$. So we'll solve the other solutions as a system of equations.

$$90 - 2x - y = 0$$

$$90 - x - 2y = 0$$

Change them to

[1] $2x + y = 90$

[2] $x + 2y = 90$

Multiply [1] by 2 to get $2y$ in both equations so that we can cancel it out and solve for x.

[3] $4x + 2y = 180$

[2] $x + 2y = 90$

Subtract [2] from [3].

$$4x + 2y - (x + 2y) = 180 - (90)$$

$$4x + 2y - x - 2y = 90$$

$$3x = 90$$

$$x = 30$$

Plugging $x = 30$ into $90 - 2x - y = 0$ to solve for y gives

$$90 - 2x - y = 0$$

$$90 - 2(30) - y = 0$$

$$90 - 60 - y = 0$$

$$30 - y = 0$$

$$y = 30$$

This gives us the point $(30,30)$ as our critical point. We need to test it to make sure that it gives at maximum, so we'll find the second-order partial derivatives of $f(x, y)$.

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$$\frac{\partial^2 f}{\partial x^2} = -2y$$

$$\frac{\partial^2 f}{\partial y^2} = -2x$$

$$\frac{\partial^2 f}{\partial x \partial y} = 90 - 2x - 2y$$

Then we'll plug these into the formula for D.

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (-2y)(-2x) - (90 - 2x - 2y)^2$$

$$D(x, y) = 4xy - (90 - 2x - 2y)^2$$

Now we'll evaluate D at the critical point $(30,30)$.

$$D(30,30) = 4(30)(30) - \left[90 - 2(30) - 2(30)\right]^2$$

$$D(30,30) = 3{,}600 - (90 - 60 - 60)^2$$

$$D(30,30) = 3{,}600 - (-30)^2$$

$$D(30,30) = 3{,}600 - 900$$

$$D(30,30) = 2{,}700$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

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If $D = 0$, then the second derivative test is inconclusive

If $D > 0$,

and $\dfrac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\dfrac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

In this problem, $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2}(30,30) = -2(30) = -60 < 0$, so the critical point is a local maximum.

So now we'll just plug the critical point into the sum equation $x + y + z = 90$ to find the associated z-value.

$$x + y + z = 90$$

$$30 + 30 + z = 90$$

$$z = 30$$

These are the three positive numbers that sum to 90 and have the maximum possible product.

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Topic: Applied optimization

Question: Find the maximum volume of a rectangular box inscribed in a sphere.

$$r = 4$$

Answer choices:

A $\quad V_B = \dfrac{2{,}048}{9}$

B $\quad V_B = \dfrac{2{,}048}{3\sqrt{3}}$

C $\quad V_B = \dfrac{512}{9}$

D $\quad V_B = \dfrac{512\sqrt{3}}{9}$

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Solution: D

We know the equation for a sphere is

$$x^2+y^2+z^2=r^2$$

Since we've been told that our sphere has radius $r=4$, the equation of the sphere becomes

$$x^2+y^2+z^2=16$$

We also know that the formula for the volume of a box is given by $V=lwh$. Since the sphere is centered at the origin $(0,0,0)$, we'll center the box at the origin also, and give the dimensions of the box in terms of x, y and z instead of l, w and h. If you imagine the box centered at the origin, with half of the box one side of the origin, and half the box on the other, then x describes the width of the right side, and $2x$ describes the full width. In the same way, y describes the length of the back side and $2y$ describes the full length. z describes the height of the top side and $2z$ describes the full height. So the volume of the box is given by

$$V_B=lwh$$

$$V_B=(2x)(2y)(2z)$$

$$V_B=8xyz$$

Because we're trying to maximize the volume of the box, we need to get this equation for volume in terms of just two variables, instead of three. If we solve the equation of the sphere for z, then we can substitute into the volume equation.

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$$x^2 + y^2 + z^2 = 16$$

$$z^2 = 16 - x^2 - y^2$$

$$z = \sqrt{16 - x^2 - y^2}$$

Plug this into the volume equation.

$$V_B = 8xyz$$

$$V_B = 8xy\sqrt{16 - x^2 - y^2}$$

Now we'll start to find critical points by taking first-order partial derivatives of this volume equation. We'll need to use product rule.

$$\frac{\partial V_B}{\partial x} = (8y)\left(\sqrt{16 - x^2 - y^2}\right) + (8xy)\left[\frac{1}{2}\left(16 - x^2 - y^2\right)^{-\frac{1}{2}}(-2x)\right]$$

$$\frac{\partial V_B}{\partial x} = 8y\sqrt{16 - x^2 - y^2} - \frac{8x^2y}{\left(16 - x^2 - y^2\right)^{\frac{1}{2}}}$$

$$\frac{\partial V_B}{\partial x} = 8y\sqrt{16 - x^2 - y^2} - \frac{8x^2y}{\sqrt{16 - x^2 - y^2}}$$

$$\frac{\partial V_B}{\partial x} = 8y\sqrt{16 - x^2 - y^2}\left(\frac{\sqrt{16 - x^2 - y^2}}{\sqrt{16 - x^2 - y^2}}\right) - \frac{8x^2y}{\sqrt{16 - x^2 - y^2}}$$

$$\frac{\partial V_B}{\partial x} = \frac{8y\left(16 - x^2 - y^2\right)}{\sqrt{16 - x^2 - y^2}} - \frac{8x^2y}{\sqrt{16 - x^2 - y^2}}$$

$$\frac{\partial V_B}{\partial x}=\frac{8y\left(16-x^2-y^2\right)-8x^2y}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial x}=\frac{128y-8x^2y-8y^3-8x^2y}{\sqrt{16-x^2-y^2}}$$

[1] $$\frac{\partial V_B}{\partial x}=\frac{128y-16x^2y-8y^3}{\sqrt{16-x^2-y^2}}$$

and

$$\frac{\partial V_B}{\partial y}=(8x)\left(\sqrt{16-x^2-y^2}\right)+(8xy)\left[\frac{1}{2}\left(16-x^2-y^2\right)^{-\frac{1}{2}}(-2y)\right]$$

$$\frac{\partial V_B}{\partial y}=8x\sqrt{16-x^2-y^2}-\frac{8xy^2}{\left(16-x^2-y^2\right)^{\frac{1}{2}}}$$

$$\frac{\partial V_B}{\partial y}=8x\sqrt{16-x^2-y^2}-\frac{8xy^2}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial y}=8x\sqrt{16-x^2-y^2}\left(\frac{\sqrt{16-x^2-y^2}}{\sqrt{16-x^2-y^2}}\right)-\frac{8xy^2}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial y}=\frac{8x\left(16-x^2-y^2\right)}{\sqrt{16-x^2-y^2}}-\frac{8xy^2}{\sqrt{16-x^2-y^2}}$$

$$\frac{\partial V_B}{\partial y}=\frac{8x\left(16-x^2-y^2\right)-8xy^2}{\sqrt{16-x^2-y^2}}$$

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$$\frac{\partial V_B}{\partial y} = \frac{128x - 8x^3 - 8xy^2 - 8xy^2}{\sqrt{16 - x^2 - y^2}}$$

$$[2] \quad \frac{\partial V_B}{\partial y} = \frac{128x - 16xy^2 - 8x^3}{\sqrt{16 - x^2 - y^2}}$$

Now we'll set [1] equal to 0.

$$\frac{128y - 16x^2y - 8y^3}{\sqrt{16 - x^2 - y^2}} = 0$$

This equation is only true when the numerator is equal to 0. Therefore, we can say

$$128y - 16x^2y - 8y^3 = 0$$

$$y\left(128 - 16x^2 - 8y^2\right) = 0$$

This means that either $y = 0$ or $128 - 16x^2 - 8y^2 = 0$. Because y represents the length of the box, and the length can't be 0 otherwise the box wouldn't exist, we have to use $128 - 16x^2 - 8y^2 = 0$.

$$128 - 16x^2 - 8y^2 = 0$$

$$128 = 16x^2 + 8y^2$$

$$[3] \quad 16 = 2x^2 + y^2$$

Now we'll set [2] equal to 0.

$$\frac{128x - 16xy^2 - 8x^3}{\sqrt{16 - x^2 - y^2}} = 0$$

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This equation is only true when the numerator is equal to 0. Therefore, we can say

$$128x - 16xy^2 - 8x^3 = 0$$

$$x\left(128 - 16y^2 - 8x^2\right) = 0$$

This means that either $x = 0$ or $128 - 16y^2 - 8x^2 = 0$. Because x represents the width of the box, and the width can't be 0 otherwise the box wouldn't exist, we have to use $128 - 16y^2 - 8x^2 = 0$.

$$128 - 16y^2 - 8x^2 = 0$$

$$128 = 8x^2 + 16y^2$$

[4] $16 = x^2 + 2y^2$

Now we can put these together as a system of equations.

[3] $16 = 2x^2 + y^2$

[4] $16 = x^2 + 2y^2$

If we multiply [4] by 2, the system becomes

[5] $16 = 2x^2 + y^2$

[6] $32 = 2x^2 + 4y^2$

Subtract [5] from [6] to eliminate x and solve for y.

$$32 - (16) = 2x^2 + 4y^2 - \left(2x^2 + y^2\right)$$

$$32 - 16 = 2x^2 + 4y^2 - 2x^2 - y^2$$

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$$16 = 4y^2 - y^2$$

$$16 = 3y^2$$

$$\frac{16}{3} = y^2$$

$$\sqrt{\frac{16}{3}} = y$$

$$y = \frac{\sqrt{16}}{\sqrt{3}}$$

$$y = \frac{4}{\sqrt{3}}$$

$$y = \frac{4\sqrt{3}}{3}$$

Plug this back into [3] to find the corresponding value of x.

$$16 = 2x^2 + y^2$$

$$16 = 2x^2 + \left(\frac{4\sqrt{3}}{3}\right)^2$$

$$16 = 2x^2 + \frac{16(3)}{9}$$

$$16 = 2x^2 + \frac{48}{9}$$

$$144 = 18x^2 + 48$$

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$$96 = 18x^2$$

$$\frac{16}{3} = x^2$$

$$x = \frac{\sqrt{16}}{\sqrt{3}}$$

$$x = \frac{4}{\sqrt{3}}$$

$$x = \frac{4\sqrt{3}}{3}$$

If we plug this x-value and the y-value we found earlier into the original equation for the sphere, we get the corresponding value for z.

$$x^2 + y^2 + z^2 = 16$$

$$\left(\frac{4\sqrt{3}}{3}\right)^2 + \left(\frac{4\sqrt{3}}{3}\right)^2 + z^2 = 16$$

$$\frac{16(3)}{9} + \frac{16(3)}{9} + z^2 = 16$$

$$\frac{48}{9} + \frac{48}{9} + z^2 = 16$$

$$48 + 48 + 9z^2 = 144$$

$$9z^2 = 48$$

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$$z^2 = \frac{48}{9}$$

$$z^2 = \frac{16}{3}$$

$$z = \frac{\sqrt{16}}{\sqrt{3}}$$

$$z = \frac{4}{\sqrt{3}}$$

$$z = \frac{4\sqrt{3}}{3}$$

So the critical point is

$$\left(\frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}, \frac{4\sqrt{3}}{3}\right)$$

Since this is the only critical point, we can assume that this is the point that gives maximum volume. If we wanted to, we could use the second derivative test to verify this.

So we'll plug this critical point into the volume equation for the box, and we'll get

$$V_B = 8xyz$$

$$V_B = 8\left(\frac{4\sqrt{3}}{3}\right)\left(\frac{4\sqrt{3}}{3}\right)\left(\frac{4\sqrt{3}}{3}\right)$$

$$V_B = 8\left(\frac{64(3)\sqrt{3}}{27}\right)$$

$$V_B = \frac{512\sqrt{3}}{9}$$

This is the maximum volume of the rectangular box inscribed in the sphere.

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Topic: Two dimensions, one constraint

Question: Find the extrema of the function, subject to the given constraint.

$$f(x, y) = x^2 + y^2 + 100$$

when $y + 2x = 6$

Answer choices:

A Local maximum at $\left(\frac{12}{5}, \frac{6}{5}\right)$

B Local minimum at $\left(\frac{6}{5}, \frac{12}{5}\right)$

C Local minimum at $\left(\frac{12}{5}, \frac{6}{5}\right)$

D Local maximum at $\left(\frac{6}{5}, \frac{12}{5}\right)$

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Solution: C

We'll start by moving all terms in the constraint equation to one side, until the equation is equal to 0. Then we'll replace the 0 with $g(x, y)$.

$$y + 2x = 6$$

$$y + 2x - 6 = 0$$

$$g(x, y) = y + 2x - 6$$

Next we'll find the first-order partial derivatives of $f(x, y)$ and $g(x, y)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

and

$$\frac{\partial g}{\partial x} = 2$$

$$\frac{\partial g}{\partial y} = 1$$

We'll multiply the partial derivatives of g by the Lagrange multiplier λ.

$$\frac{\partial g}{\partial x} = 2\lambda$$

$$\frac{\partial g}{\partial y} = \lambda$$

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Then we'll set the partial derivatives of f equal to the corresponding partial derivatives from g, making sure to use the equations that include the Lagrange multiplier.

For the partial derivatives with respect to x we get

$$2x = 2\lambda$$

$$\lambda = x$$

For the partial derivatives with respect to y we get

$$2y = \lambda$$

$$\lambda = 2y$$

Now that we have two equations that are solved for λ, we can set them equal to each other, and then solve this equation for y in terms of x.

$$x = 2y$$

$$y = \frac{x}{2}$$

Plug this value for y back into the constraint equation.

$$y + 2x = 6$$

$$\left(\frac{x}{2}\right) + 2x = 6$$

$$x + 4x = 12$$

$$5x = 12$$

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$$x = \frac{12}{5}$$

Now plug this back into the constraint equation to solve for y.

$$y + 2x = 6$$

$$y + 2\left(\frac{12}{5}\right) = 6$$

$$y + \frac{24}{5} = 6$$

$$y = 6 - \frac{24}{5}$$

$$y = \frac{30 - 24}{5}$$

$$y = \frac{6}{5}$$

Putting these values for x and y together, the critical point is

$$\left(\frac{12}{5}, \frac{6}{5}\right)$$

To say whether this critical point is a maximum or minimum, we'll find second-order partial derivatives of $f(x, y)$.

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

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$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Then we'll plug these into the formula for D.

$$D(x, y, \lambda) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y, \lambda) = (2)(2) - (0)^2$$

$$D(x, y, \lambda) = 4$$

At this point, if we still had variables remaining on the right side of D, we'd evaluate D at the critical point. In this case, since there are no variables, plugging in the critical point won't change the value.

$$D\left(\frac{12}{5}, \frac{6}{5}, \lambda\right) = 4$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

If $D = 0$, then the second derivative test is inconclusive

If $D > 0$,

and $\frac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\frac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

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In this problem, $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so the critical point is a local minimum.

Topic: Two dimensions, one constraint

Question: Find the extrema of the function, subject to the given constraint.

$$f(x, y) = 4x^2 + 6y^2 - 35$$

when $-10y + 5x = 25$

Answer choices:

A Local maximum at $\left(-\frac{20}{11}, \frac{15}{11}\right)$

B Local minimum at $\left(\frac{15}{11}, -\frac{20}{11}\right)$

C Local minimum at $\left(-\frac{20}{11}, \frac{15}{11}\right)$

D Local maximum at $\left(\frac{15}{11}, -\frac{20}{11}\right)$

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Solution: B

We'll start by moving all terms in the constraint equation to one side, until the equation is equal to 0. Then we'll replace the 0 with $g(x, y)$.

$$-10y + 5x = 25$$

$$5x - 10y - 25 = 0$$

$$g(x, y) = 5x - 10y - 25$$

Next we'll find the first-order partial derivatives of $f(x, y)$ and $g(x, y)$.

$$\frac{\partial f}{\partial x} = 8x$$

$$\frac{\partial f}{\partial y} = 12y$$

and

$$\frac{\partial g}{\partial x} = 5$$

$$\frac{\partial g}{\partial y} = -10$$

We'll multiply the partial derivatives of g by the Lagrange multiplier λ.

$$\frac{\partial g}{\partial x} = 5\lambda$$

$$\frac{\partial g}{\partial y} = -10\lambda$$

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Then we'll set the partial derivatives of f equal to the corresponding partial derivatives from g, making sure to use the equations that include the Lagrange multiplier.

For the partial derivatives with respect to x we get

$$8x = 5\lambda$$

$$\lambda = \frac{8}{5}x$$

For the partial derivatives with respect to y we get

$$12y = -10\lambda$$

$$\lambda = -\frac{6}{5}y$$

Now that we have two equations that are solved for λ, we can set them equal to each other, and then solve this equation for y in terms of x.

$$\frac{8}{5}x = -\frac{6}{5}y$$

$$8x = -6y$$

$$y = -\frac{8}{6}x$$

$$y = -\frac{4}{3}x$$

Plug this value for y back into the constraint equation.

$$-10y + 5x = 25$$

$$-10\left(-\frac{4}{3}x\right)+5x=25$$

$$\frac{40}{3}x+5x=25$$

$$40x+15x=75$$

$$55x=75$$

$$x=\frac{75}{55}$$

$$x=\frac{15}{11}$$

Now plug this back into the constraint equation to solve for y.

$$-10y+5x=25$$

$$-10y+5\left(\frac{15}{11}\right)=25$$

$$-110y+75=275$$

$$y=-\frac{200}{110}$$

$$y=-\frac{20}{11}$$

Putting these values for x and y together, the critical point is

$$\left(\frac{15}{11},-\frac{20}{11}\right)$$

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To say whether this critical point is a maximum or minimum, we'll find second-order partial derivatives of $f(x, y)$.

$$\frac{\partial^2 f}{\partial x^2} = 8$$

$$\frac{\partial^2 f}{\partial y^2} = 12$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Then we'll plug these into the formula for D.

$$D(x, y, \lambda) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y, \lambda) = (8)(12) - (0)^2$$

$$D(x, y, \lambda) = 96$$

At this point, if we still had variables remaining on the right side of D, we'd evaluate D at the critical point. In this case, since there are no variables, plugging in the critical point won't change the value.

$$D\left(\frac{15}{11}, -\frac{20}{11}, \lambda\right) = 96$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

If $D = 0$, then the second derivative test is inconclusive

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If $D > 0$,

and $\dfrac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\dfrac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

In this problem, $D > 0$ and $\dfrac{\partial^2 f}{\partial x^2} > 0$, so the critical point is a local minimum.

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Topic: Two dimensions, one constraint

Question: What is the maximum area of the land defined by $f(x, y) = 12xy$, subject to the given constraint?

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

Answer choices:

A 392

B 388

C 380

D 360

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Solution: D

Let

$$g(x, y) = \frac{x^2}{25} + \frac{y^2}{144} - 1$$

Take partial derivatives of the given function $f(x, y) = 12xy$, and set them equal to the corresponding partial derivatives of the constraint equation $g(x, y)$. Don't forget to multiply by λ. Therefore, the equations we're building are

$$f_x(x, y) = \lambda g_x(x, y)$$

$$f_y(x, y) = \lambda g_y(x, y)$$

and we get

$$12y = \frac{2\lambda}{25}x$$

$$12x = \frac{2\lambda}{144}y$$

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

We'll solve the first equation for λ.

$$12y = \frac{2\lambda}{25}x$$

$$\lambda = \frac{150y}{x}$$

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Plug this value into the other equation.

$$12x = \frac{2\left(\frac{150y}{x}\right)}{144}y$$

$$x^2 = \left(\frac{25}{144}\right)y^2$$

Plug this value into the constraint equation for x^2.

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

$$\frac{1}{25}\left[\left(\frac{25}{144}\right)y^2\right] + \frac{1}{144}y^2 = 1$$

$$\frac{1}{144}y^2 + \frac{1}{144}y^2 = 1$$

$$y^2 + y^2 = 144$$

$$2y^2 = 144$$

$$y^2 = 72$$

$$y = \pm 6\sqrt{2}$$

Plug these values into the constraint equation to find corresponding values for x.

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

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$$\frac{x^2}{25} + \frac{\left(\pm 6\sqrt{2}\right)^2}{144} = 1$$

$$\frac{x^2}{25} + \frac{36(2)}{144} = 1$$

$$\frac{x^2}{25} + \frac{1}{2} = 1$$

$$2x^2 + 25 = 50$$

$$x^2 = \frac{25}{2}$$

$$x = \frac{5\sqrt{2}}{2}$$

Plugging this point into the original function gives

$$f(x, y) = 12xy$$

$$f\left(6\sqrt{2}, \frac{5\sqrt{2}}{2}\right) = 12\left(6\sqrt{2}\right)\left(\frac{5\sqrt{2}}{2}\right)$$

$$f\left(6\sqrt{2}, \frac{5\sqrt{2}}{2}\right) = 12(6)(5)$$

$$f\left(6\sqrt{2}, \frac{5\sqrt{2}}{2}\right) = 360$$

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Topic: Three dimensions, one constraint

Question: What is the maximum value of the function subject to the given constraint?

$$f(x, y, z) = 8xyz$$

$$\text{subject to } \frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} = 1$$

Answer choices:

A $\quad 48$

B $\quad 44$

C $\quad 38$

D $\quad 32$

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Solution: A

To maximize

$$f(x, y, z) = 8xyz$$

subject to the constraint

$$g(x, y, z) = \frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} - 1 = 0$$

we'll use the formulas

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$

$$f_y(x, y, z) = \lambda g_y(x, y, z)$$

$$f_z(x, y, z) = \lambda g_z(x, y, z)$$

Applying these formulas to

$$g(x, y, z) = \frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} - 1 = 0$$

gives

$$8yz = \frac{2\lambda x}{3}$$

Multiplying this equation by x gives

$$8xyz = \frac{2\lambda x^2}{3}$$

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$$8xz = \frac{2\lambda y}{12}$$

Multiplying this equation by y gives

$$8xyz = \frac{2\lambda y^2}{12}$$

$$8xy = \frac{2\lambda z}{27}$$

Multiplying this equation by z gives

$$8xyz = \frac{2\lambda z^2}{27}$$

The left sides of all the three conclusions are equal. Hence, their right sides are equal as well:

$$\frac{2\lambda x^2}{3} = \frac{2\lambda y^2}{12} = \frac{2\lambda z^2}{27}$$

$$\frac{x^2}{3} = \frac{y^2}{12} = \frac{z^2}{27}$$

Solve the following system of equations:

$$\frac{x^2}{3} = \frac{y^2}{12} = \frac{z^2}{27}$$

$$\frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} - 1 = 0$$

$$4x^2 = y^2$$

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$$9y^2 = 4z^2$$

$$9x^2 = z^2$$

The solutions of this system of equations are $x = 1$, $y = 2$, and $z = 3$. Then the maximum of the function is

$$f(x, y, z) = 8xyz$$

$$f(x, y, z) = 8(1)(2)(3)$$

$$f(x, y, z) = 48$$

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Topic: Three dimensions, one constraint

Question: What is the minimum value of the function subject to the given constraint?

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to $3x - 4y - 5z = -16$

Answer choices:

A $\frac{256}{25}$

B $\frac{128}{25}$

C $\frac{64}{25}$

D $\frac{32}{25}$

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Solution: B

Let

$$g(x, y, z) = 3x - 4y - 5z + 16$$

Apply the formulas

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$

$$f_y(x, y, z) = \lambda g_y(x, y, z)$$

$$f_z(x, y, z) = \lambda g_z(x, y, z)$$

Then,

$$2x = 3\lambda$$

$$2y = -4\lambda$$

$$2z = -5\lambda$$

$$3x - 4y - 5z = -16$$

We'll solve each of the first three equations for λ.

$$\lambda = \frac{2}{3}x$$

$$\lambda = -\frac{2}{4}y = -\frac{1}{2}y$$

$$\lambda = -\frac{2}{5}z$$

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Therefore, we can say

$$\frac{2}{3}x = -\frac{1}{2}y = -\frac{2}{5}z$$

If we solve this for x, we get

$$x = -\frac{3}{4}y = -\frac{3}{5}z$$

If we solve this for y, we get

$$-\frac{4}{3}x = y = \frac{4}{5}z$$

If we solve this for z, we get

$$-\frac{5}{3}x = \frac{5}{4}y = z$$

Now we can make substitutions into the constraint equation $3x - 4y - 5z = -16$. We'll replace x and y with values in terms of z in order to solve for z.

$$3x - 4y - 5z = -16$$

$$3\left(-\frac{3}{5}z\right) - 4\left(\frac{4}{5}z\right) - 5z = -16$$

$$-\frac{9}{5}z - \frac{16}{5}z - 5z = -16$$

$$-9z - 16z - 25z = -80$$

$$-50z = -80$$

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$$z = \frac{80}{50}$$

$$z = \frac{8}{5}$$

Now we'll replace x and z with values in terms of y in order to solve for y.

$$3x - 4y - 5z = -16$$

$$3\left(-\frac{3}{4}y\right) - 4y - 5\left(\frac{5}{4}y\right) = -16$$

$$-\frac{9}{4}y - 4y - \frac{25}{4}y = -16$$

$$-9y - 16y - 25y = -64$$

$$-50y = -64$$

$$y = \frac{64}{50}$$

$$y = \frac{32}{25}$$

Now we'll replace y and z with values in terms of x in order to solve for x.

$$3x - 4y - 5z = -16$$

$$3x - 4\left(-\frac{4}{3}x\right) - 5\left(-\frac{5}{3}x\right) = -16$$

$$3x + \frac{16}{3}x + \frac{25}{3}x = -16$$

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$$9x + 16x + 25x = -48$$

$$50x = -48$$

$$x = -\frac{48}{50}$$

$$x = -\frac{24}{25}$$

These x, y and z values together represent only one critical point, which means that the minimum of the function is

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$f\left(-\frac{24}{25}, \frac{32}{25}, \frac{8}{5}\right) = \left(-\frac{24}{25}\right)^2 + \left(\frac{32}{25}\right)^2 + \left(\frac{8}{5}\right)^2$$

$$\frac{576}{625} + \frac{1{,}024}{625} + \frac{64}{25}$$

$$\frac{576}{625} + \frac{1{,}024}{625} + \frac{1{,}600}{625}$$

$$\frac{3{,}200}{625}$$

$$\frac{128}{25}$$

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Topic: Three dimensions, one constraint

Question: To the nearest tenth, what is the minimum value of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ on the plane $x + 2y + 3z = 16$?

Answer choices:

A 23.6

B 20.2

C 19.8

D 19.1

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Solution: A

Let

$$g(x, y, z) = x + 2y + 3z - 16 = 0$$

Apply the formulas:

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$

$$f_y(x, y, z) = \lambda g_y(x, y, z)$$

$$f_z(x, y, z) = \lambda g_z(x, y, z)$$

Then,

$$4x = \lambda$$

$$6y = 2\lambda$$

$$2z = 3\lambda$$

$$x + 2y + 3z - 16 = 0$$

We'll solve each of the first three equations for λ.

$$\lambda = 4x$$

$$\lambda = 3y$$

$$\lambda = \frac{2}{3}z$$

Therefore, we can say

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$$4x = 3y = \frac{2}{3}z$$

If we solve this for x, we get

$$x = \frac{3}{4}y = \frac{1}{6}z$$

If we solve this for y, we get

$$\frac{4}{3}x = y = \frac{2}{9}z$$

If we solve this for z, we get

$$6x = \frac{9}{2}y = z$$

Now we can make substitutions into the constraint equation $x + 2y + 3z = 16$. We'll replace x and y with values in terms of z in order to solve for z.

$$x + 2y + 3z = 16$$

$$\frac{1}{6}z + 2\left(\frac{2}{9}z\right) + 3z = 16$$

$$\frac{3}{2}z + 4z + 27z = 144$$

$$3z + 8z + 54z = 288$$

$$65z = 288$$

$$z = \frac{288}{65}$$

Now we'll replace x and z with values in terms of y in order to solve for y.

$$x + 2y + 3z = 16$$

$$\frac{3}{4}y + 2y + 3\left(\frac{9}{2}y\right) = 16$$

$$3y + 8y + 54y = 64$$

$$65y = 64$$

$$y = \frac{64}{65}$$

Now we'll replace y and z with values in terms of x in order to solve for x.

$$x + 2y + 3z = 16$$

$$x + 2\left(\frac{4}{3}x\right) + 3(6x) = 16$$

$$3x + 8x + 54x = 48$$

$$65x = 48$$

$$x = \frac{48}{65}$$

These x, y and z values together represent only one critical point, which means that the minimum of the function is

$$f(x, y, z) = 2x^2 + 3y^2 + z^2$$

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$$f\left(\frac{48}{65},\frac{64}{65},\frac{288}{65}\right)=2\left(\frac{48}{65}\right)^2+3\left(\frac{64}{65}\right)^2+\left(\frac{288}{65}\right)^2$$

$$\frac{2\cdot 48^2}{65^2}+\frac{3\cdot 64^2}{65^2}+\frac{288^2}{65^2}$$

$$\frac{2\cdot 48^2+3\cdot 64^2+288^2}{65^2}$$

$$\frac{4{,}608+12{,}288+82{,}944}{65^2}$$

$$\frac{99{,}840}{65^2}$$

$$\frac{1{,}536}{65}\approx 23.6$$

Topic: Three dimensions, two constraints

Question: Find the extrema of the function, subject to the given constraints.

$$f(x, y, z) = x^2 + y^2 + z^2$$

when $2x + y - z = 4$ and $3x + 5y + 9z = 12$

Answer choices:

A Local maximum at $\left(\frac{76}{49}, -\frac{54}{49}, \frac{10}{49}\right)$

B Local minimum at $\left(\frac{76}{49}, -\frac{54}{49}, \frac{10}{49}\right)$

C Local maximum at $\left(\frac{76}{49}, \frac{54}{49}, \frac{10}{49}\right)$

D Local minimum at $\left(\frac{76}{49}, \frac{54}{49}, \frac{10}{49}\right)$

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Solution: D

We'll start by moving all terms in the constraint equation to one side, until the equation is equal to 0. Then we'll replace the 0 with $g(x, y, z)$ and $h(x, y, z)$.

$$2x + y - z = 4$$

$$2x + y - z - 4 = 0$$

$$g(x, y, z) = 2x + y - z - 4$$

and

$$3x + 5y + 9z = 12$$

$$3x + 5y + 9z - 12 = 0$$

$$h(x, y, z) = 3x + 5y + 9z - 12$$

Next we'll find the first-order partial derivatives of $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial z} = 2z$$

and

$$\frac{\partial g}{\partial x} = 2$$

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$$\frac{\partial g}{\partial y} = 1$$

$$\frac{\partial g}{\partial z} = -1$$

and

$$\frac{\partial h}{\partial x} = 3$$

$$\frac{\partial h}{\partial y} = 5$$

$$\frac{\partial h}{\partial z} = 9$$

We'll multiply the partial derivatives of g by the Lagrange multiplier λ, and the partial derivatives of h by the Lagrange multiplier μ

$$\frac{\partial g}{\partial x} = 2\lambda$$

$$\frac{\partial g}{\partial y} = 1\lambda = \lambda$$

$$\frac{\partial g}{\partial z} = -1\lambda = -\lambda$$

and

$$\frac{\partial h}{\partial x} = 3\mu$$

$$\frac{\partial h}{\partial y} = 5\mu$$

$$\frac{\partial h}{\partial z} = 9\mu$$

Then we'll set the partial derivatives of f equal to the sum of the corresponding partial derivatives from g and h, making sure to use the equations that include the Lagrange multiplier.

For the partial derivatives with respect to x we get

$$2x = 2\lambda + 3\mu$$

$$x = \lambda + \frac{3}{2}\mu$$

For the partial derivatives with respect to y we get

$$2y = \lambda + 5\mu$$

$$y = \frac{1}{2}\lambda + \frac{5}{2}\mu$$

For the partial derivatives with respect to z we get

$$2z = -\lambda + 9\mu$$

$$z = -\frac{1}{2}\lambda + \frac{9}{2}\mu$$

Now we'll take these values for x, y and z and plug them into both constraint equations.

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$$2x + y - z = 4$$

$$2\left(\lambda + \frac{3}{2}\mu\right) + \left(\frac{1}{2}\lambda + \frac{5}{2}\mu\right) - \left(-\frac{1}{2}\lambda + \frac{9}{2}\mu\right) = 4$$

$$2\lambda + 3\mu + \frac{1}{2}\lambda + \frac{5}{2}\mu + \frac{1}{2}\lambda - \frac{9}{2}\mu = 4$$

$$4\lambda + 6\mu + \lambda + 5\mu + \lambda - 9\mu = 8$$

$$6\lambda + 2\mu = 8$$

$$3\lambda + \mu = 4$$

and

$$3x + 5y + 9z = 12$$

$$3\left(\lambda + \frac{3}{2}\mu\right) + 5\left(\frac{1}{2}\lambda + \frac{5}{2}\mu\right) + 9\left(-\frac{1}{2}\lambda + \frac{9}{2}\mu\right) = 12$$

$$3\lambda + \frac{9}{2}\mu + \frac{5}{2}\lambda + \frac{25}{2}\mu - \frac{9}{2}\lambda + \frac{81}{2}\mu = 12$$

$$6\lambda + 9\mu + 5\lambda + 25\mu - 9\lambda + 81\mu = 24$$

$$2\lambda + 115\mu = 24$$

Now we'll solve these remaining equations as a system of equations.

[1] $3\lambda + \mu = 4$

[2] $2\lambda + 115\mu = 24$

We'll multiply [1] by 2 and [2] by 3 to get 6λ in both equations.

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[3] $6\lambda + 2\mu = 8$

[4] $6\lambda + 345\mu = 72$

Now we'll subtract **[3]** from **[4]**, which will cancel λ and allow us to solve for μ.

$$6\lambda + 345\mu - (6\lambda + 2\mu) = 72 - (8)$$

$$343\mu = 64$$

$$\mu = \frac{64}{343}$$

Now plug this back into **[1]** to solve for λ.

$$3\lambda + \mu = 4$$

$$3\lambda + \frac{64}{343} = 4$$

$$1{,}029\lambda + 64 = 1{,}372$$

$$1{,}029\lambda = 1{,}308$$

$$\lambda = \frac{1{,}308}{1{,}029}$$

$$\lambda = \frac{436}{343}$$

Putting these values for λ and μ back into our equations for x, y, and z, we get

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$$x = \lambda + \frac{3}{2}\mu$$

$$x = \frac{436}{343} + \frac{3}{2}\left(\frac{64}{343}\right)$$

$$x = \frac{436}{343} + \frac{192}{686}$$

$$x = \frac{872}{686} + \frac{192}{686}$$

$$x = \frac{1{,}064}{686}$$

$$x = \frac{76}{49}$$

and

$$y = \frac{1}{2}\lambda + \frac{5}{2}\mu$$

$$y = \frac{1}{2}\left(\frac{436}{343}\right) + \frac{5}{2}\left(\frac{64}{343}\right)$$

$$y = \frac{436}{686} + \frac{320}{686}$$

$$y = \frac{756}{686}$$

$$y = \frac{54}{49}$$

and

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$$z = -\frac{1}{2}\lambda + \frac{9}{2}\mu$$

$$z = -\frac{1}{2}\left(\frac{436}{343}\right) + \frac{9}{2}\left(\frac{64}{343}\right)$$

$$z = -\frac{436}{686} + \frac{576}{686}$$

$$z = \frac{140}{686}$$

$$z = \frac{10}{49}$$

The critical point for the function is therefore given by

$$\left(\frac{76}{49}, \frac{54}{49}, \frac{10}{49}\right)$$

$f(x, y, z) = x^2 + y^2 + z^2$ is the standard equation of a paraboloid, and we know that figure will only have one critical point. Since the paraboloid opens up, the critical point must be a minimum.

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Topic: Three dimensions, two constraints

Question: What is the maximum temperature on the intersection of the sphere and the plane?

The function $F(x, y, z) = 5x + 5y + z^2 + 12$ models the temperature where the sphere

$$x^2 + y^2 + z^2 = \frac{21}{4}$$

intersects the plane

$$x + y - z = \frac{3}{2}$$

Answer choices:

A $\quad 27.18$

B $\quad 27.87$

C $\quad 28.68$

D $\quad 29.66$

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Solution: A

The constraints are

$$g(x) = x^2 + y^2 + z^2 - \frac{21}{4}$$

$$h(x) = x + y - z - \frac{3}{2}$$

Then applying ∇F, ∇g, and ∇h gives us

$$F_x(x, y, z) = \lambda g_x(x, y, z) + \rho_x(x, y, z)$$

$$5 = 2\lambda x + \rho$$

and

$$F_y(x, y, z) = \lambda g_y(x, y, z) + \rho g_y(x, y, z)$$

$$5 = 2\lambda y + \rho$$

and

$$F_z(x, y, z) = \lambda g_z(x, y, z) + \rho g_z(x, y, z)$$

$$2z = 2\lambda z - \rho$$

Set up the following system of equations, and then solve.

$$5 = 2\lambda x + \rho$$

$$5 = 2\lambda y + \rho$$

$$2z = 2\lambda z - \rho$$

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$$x^2 + y^2 + z^2 = \frac{21}{4}$$

$$x + y - z = \frac{3}{2}$$

Subtract the first two equations, and simplify.

$$\lambda(x - y) = 0$$

$$2z = 2\lambda z - \rho$$

$$x^2 + y^2 + z^2 = \frac{21}{4}$$

$$x + y - z = \frac{3}{2}$$

The first equation of the system above indicates that either $x = y$ or $\lambda = 0$. Replacing $x = y$ in the fourth and fifth equations results in

$$2x^2 + z^2 = \frac{21}{4}$$

$$2x - z = \frac{3}{2}$$

The solutions of this system are

$$(x, z) = \left(\frac{1 + \sqrt{3}}{2}, \frac{-1 + 2\sqrt{3}}{2}\right)$$

$$(x, z) = \left(\frac{1 - \sqrt{3}}{2}, \frac{-1 - 2\sqrt{3}}{2}\right)$$

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Combining these solutions with their associated y-values gives the critical points

$$(x, z) = \left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\right)$$

$$(x, z) = \left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{-1-2\sqrt{3}}{2}\right)$$

Replacing $\lambda = 0$ in $5 = 2\lambda y + \rho$ results in $\rho = 5$. Replacing these values in $2z = 2\lambda z - \rho$ and solving the system of the last three equations yields the critical points in the imaginary plane which can't be considered.

Therefore we have only two critical points for which the measures of temperature must be calculated:

$$F\left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\right) = 5\left(\frac{1+\sqrt{3}}{2}\right) + 5\left(\frac{1+\sqrt{3}}{2}\right) + \left(\frac{-1+2\sqrt{3}}{2}\right)^2 + 12$$

$$F\left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\right) \approx 27.18$$

$$F\left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{-1-2\sqrt{3}}{2}\right) = 5\left(\frac{1-\sqrt{3}}{2}\right) + 5\left(\frac{1-\sqrt{3}}{2}\right) + \left(\frac{-1-2\sqrt{3}}{2}\right)^2 + 12$$

$$F\left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{-1-2\sqrt{3}}{2}\right) \approx 13.32$$

Comparing the values of the temperature function at two critical points shows that the maximum temperature is 27.18.

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Topic: Three dimensions, two constraints

Question: Find the extrema of the function $F(x, y, z) = x^2 + y^2 + z^2 + 8$ subject to the constraints $x + y - z = 12$ and $x - y - z = 6$.

Answer choices:

A 53.5

B 55.5

C 57.5

D 59.5

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Solution: C

The constraints are

$$g(x) = x + y - z - 12$$

$$h(x) = x - y - z - 6$$

Then applying ∇F, ∇g, and ∇h results in:

$$F_x(x, y, z) = \lambda g_x(x, y, z) + \rho_x(x, y, z)$$

$$2x = \lambda + \rho$$

and

$$F_y(x, y, z) = \lambda g_y(x, y, z) + \rho g_y(x, y, z)$$

$$2y = \lambda - \rho$$

and

$$F_z(x, y, z) = \lambda g_z(x, y, z) + \rho g_z(x, y, z)$$

$$2z = -\lambda - \rho$$

Set up the following system of equations.

$$2x = \lambda + \rho$$

$$2y = \lambda - \rho$$

$$2z = -\lambda - \rho$$

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Solving for x, y, and z, we get

$$x = \frac{\lambda + \rho}{2}$$

$$y = \frac{\lambda - \rho}{2}$$

$$z = -\frac{\lambda + \rho}{2}$$

Now we'll plug these values into $g(x)$ and $h(x)$ to get

$$x + y - z = 12$$

$$\frac{\lambda + \rho}{2} + \frac{\lambda - \rho}{2} - \left(-\frac{\lambda + \rho}{2}\right) = 12$$

$$\frac{\lambda + \rho}{2} + \frac{\lambda - \rho}{2} + \frac{\lambda + \rho}{2} = 12$$

$$\lambda + \rho + \lambda - \rho + \lambda + \rho = 24$$

$$3\lambda + \rho = 24$$

and

$$x - y - z = 6$$

$$\frac{\lambda + \rho}{2} - \frac{\lambda - \rho}{2} - \left(-\frac{\lambda + \rho}{2}\right) = 6$$

$$\frac{\lambda + \rho}{2} - \frac{\lambda - \rho}{2} + \frac{\lambda + \rho}{2} = 6$$

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$$\lambda + \rho - (\lambda - \rho) + \lambda + \rho = 12$$

$$\lambda + \rho - \lambda + \rho + \lambda + \rho = 12$$

$$\lambda + 3\rho = 12$$

If we multiply $\lambda + 3\rho = 12$ by 3, we get $3\lambda + 9\rho = 36$. Now we can subtract $3\lambda + 9\rho = 36$ from $3\lambda + \rho = 24$ in order to solve for ρ.

$$3\lambda + \rho - (3\lambda + 9\rho) = 24 - 36$$

$$3\lambda + \rho - 3\lambda - 9\rho = -12$$

$$\rho - 9\rho = -12$$

$$-8\rho = -12$$

$$\rho = \frac{-12}{-8} = \frac{3}{2}$$

We'll use $\rho = 3/2$ to find the value for λ.

$$3\lambda + \rho = 24$$

$$3\lambda + \frac{3}{2} = 24$$

$$6\lambda + 3 = 48$$

$$6\lambda = 45$$

$$\lambda = \frac{45}{6} = \frac{15}{2}$$

With values of ρ and λ, we can find values for x, y, and z.

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$$2x = \frac{15}{2} + \frac{3}{2}$$

$$2x = \frac{18}{2}$$

$$x = \frac{18}{4} = \frac{9}{2}$$

and

$$2y = \frac{15}{2} - \frac{3}{2}$$

$$2y = \frac{12}{2}$$

$$2y = 6$$

$$y = \frac{6}{2} = 3$$

and

$$2z = -\frac{15}{2} - \frac{3}{2}$$

$$2z = -\frac{18}{2}$$

$$z = -\frac{18}{4} = -\frac{9}{2}$$

The solutions of this system of equations are

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$$\left(\frac{9}{2}, 3, -\frac{9}{2}\right)$$

The value of the given function at the critical point is

$$F(x,y,z) = x^2 + y^2 + z^2 + 8$$

$$F(x,y,z) = \left(\frac{9}{2}\right)^2 + 3^2 + \left(-\frac{9}{2}\right)^2 + 8$$

$$F(x,y,z) = \frac{81}{4} + 9 + \frac{81}{4} + 8$$

$$F(x,y,z) = \frac{162}{4} + 17$$

$$F(x,y,z) = \frac{162}{4} + \frac{68}{4}$$

$$F(x,y,z) = \frac{230}{4}$$

$$F(x,y,z) = 57.5$$

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Topic: Approximating double integrals with rectangles

Question: The rectangle R is defined on $0 \leq x \leq 6$ and $0 \leq y \leq 9$. A solid volume is defined above this rectangle and below $z = 3x + y^2$. Which value most closely approximates the volume if you divide the rectangle into 3×3 squares and use a Riemann sum to approximate the volume?

Answer choices:

A $1{,}200$

B $1{,}150$

C $2{,}800$

D $3{,}000$

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Solution: D

The rectangle below the volume is bounded by the lines $x=0$, $x=6$, $y=0$, and $y=9$. The surface that defines the top of the volume is $z=3x+y^2$. If we divide the area of the rectangle into 3×3 squares, then $\Delta A=9$.

If x is defined from 0 to 6, that means we'll need $(6-0)/3$, or 2 squares across, and if y is defined from 0 to 9, that means we'll need $(9-0)/3$, or 3 squares down.

Therefore, using upper-right-hand corners in a Riemann sum, the estimate for the volume is given by

$$V\approx\sum_{i=1}^{2}\sum_{j=1}^{3}f(x_i,y_j)\Delta A=\Delta A\left[f(3,3)+f(3,6)+f(3,9)+f(6,3)+f(6,6)+f(6,9)\right]$$

If we plug in the values we know, we get

$$V\approx 9\left[\left(3(3)+3^2\right)+\left(3(3)+6^2\right)+\left(3(3)+9^2\right)+\left(3(6)+3^2\right)+\left(3(6)+6^2\right)+\left(3(6)+9^2\right)\right]$$

$$V\approx 9\left[(9+9)+\left(9+36\right)+(9+81)+(18+9)+\left(18+36\right)+(18+81)\right]$$

$$V\approx 9(18+45+90+27+54+99)$$

$$V\approx 2{,}997$$

Topic: Approximating double integrals with rectangles

Question: The rectangle R is defined on the boundary set given in one of the answer choices. A solid volume is defined above this rectangle and below $z = x^2 + xy$. The approximate volume of the region is 33, if you divide the rectangle into 1×1 squares and use a Riemann sum to approximate the volume. Which are the boundaries of the rectangle R?

Answer choices:

A	$0 \le x \le 2$	and	$0 \le y \le 3$
B	$0 \le x \le 4$	and	$0 \le y \le 6$
C	$0 \le x \le 3$	and	$0 \le y \le 5$
D	$0 \le x \le 1$	and	$0 \le y \le 6$

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Solution: A

Since we don't know the bounds yet, let's say that the rectangle below the volume is bounded by the lines $x = a$, $x = b$, $y = c$, and $y = d$. The surface that defines the top of the volume is $z = x^2 + xy$. If we divide the area of the rectangle into 1×1 squares, then $\Delta A = 1$.

If x is defined from a to b, that means we'll need $(b - a)/1$, or $b - a$ squares across, and if y is defined from c to d, that means we'll need $(d - c)/1$, or $d - c$ squares down.

Therefore, using upper-right-hand corners in a Riemann sum, the estimate for the volume is given by

$$V \approx \sum_{i=1}^{b-a} \sum_{j=1}^{d-c} f(x_i, y_j)\Delta A$$

$$V \approx \sum_{i=1}^{b-a} \sum_{j=1}^{d-c} f(x_i, y_j)(1)$$

$$V \approx \sum_{i=1}^{b-a} \sum_{j=1}^{d-c} f(x_i, y_j)$$

Assume that the rectangle is bounded by the interval given in answer choice A, $0 \le x \le 2$ and $0 \le y \le 3$. Then the volume is given by

$$V \approx \sum_{i=1}^{2-0} \sum_{j=1}^{3-0} f(x_i, y_j)$$

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$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(x_i, y_j)$$

Because each square is 1×1, plugging in the upper-right-hand corners gives

$$V \approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(x_i, y_j) = f(1,1) + f(1,2) + f(1,3) + f(2,1) + f(2,2) + f(2,3)$$

Plug in all the other values we know.

$$V \approx \left(1^2 + (1)(1)\right) + \left(1^2 + (1)(2)\right) + \left(1^2 + (1)(3)\right) + \left(2^2 + (2)(1)\right) + \left(2^2 + (2)(2)\right) + \left(2^2 + (2)(3)\right)$$

$$V \approx (1 + 1) + (1 + 2) + (1 + 3) + (4 + 2) + (4 + 4) + (4 + 6)$$

$$V \approx 2 + 3 + 4 + 6 + 8 + 10$$

$$V \approx 33$$

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Topic: Approximating double integrals with rectangles

Question: A solid is defined above the rectangle R and below the surface S. S is defined on the rectangle $0 \le x \le 10$ and $0 \le y \le 15$. Divide the rectangle into 5×5 squares. If using a Riemann sum approximates the volume to be $12{,}750$, then which of the following functions is S?

Answer choices:

A $\quad z = xy + x$

B $\quad z = xy + y$

C $\quad z = xy + 2y$

D $\quad z = xy + 2x$

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Solution: B

The rectangle R below the volume is bounded by the lines $x=0$, $x=10$, $y=0$, and $y=15$. The equation of the surface S that defines the top of the volume is unknown. If we divide the area of the rectangle into 5×5 squares, then $\Delta A=25$.

If x is defined from 0 to 10, that means we'll need $(10-0)/5$, or 2 squares across, and if y is defined from 0 to 15, that means we'll need $(15-0)/5$, or 3 squares down.

Therefore, using upper-right-hand corners in a Riemann sum, the estimate for the volume is given by

$$V\approx\sum_{i=1}^{2}\sum_{j=1}^{3}f(x_i,y_j)\Delta A=\Delta A\left[f(5,5)+f(5,10)+f(5,15)+f(10,5)+f(10,10)+f(10,15)\right]$$

If we plug in the values we know, we get

$$V\approx 25\left[f(5,5)+f(5,10)+f(5,15)+f(10,5)+f(10,10)+f(10,15)\right]$$

If we assume that the function f is given by the surface from answer choice B, $z=xy+y$, then we get

$$V\approx 25\left[((5)(5)+5)+((5)(10)+10)+((5)(15)+15)+((10)(5)+5)+((10)(10)+10)+((10)(15)+15)\right]$$

$$V\approx 25\left[(25+5)+(50+10)+(75+15)+(50+5)+(100+10)+(150+15)\right]$$

$$V\approx 25(30+60+90+55+110+165)$$

$$V\approx 12{,}750$$

Topic: Midpoint rule for double integrals

Question: The value of the integral is to be estimated on the rectangle $R = [0,2] \times [0,8]$, where the rectangle is divided into 2×2 subrectangles. Which expression can be used as Midpoint Rule to estimate the value of the integral?

$$\iint_R x^2 + y^2 \ dA$$

Answer choices:

A $\displaystyle \iint_R x^2 + y^2 \ dA \approx 4\left(\frac{17}{4} + \frac{25}{4} + \frac{145}{4} + \frac{153}{4}\right)$

B $\displaystyle \iint_R x^2 + y^2 \ dA \approx 4\left(\frac{9}{4} + \frac{25}{4} + \frac{136}{4} + \frac{153}{4}\right)$

C $\displaystyle \iint_R x^2 + y^2 \ dA \approx 2\left(\frac{17}{4} + \frac{25}{4} + \frac{145}{4} + \frac{153}{4}\right)$

D $\displaystyle \iint_R x^2 + y^2 \ dA \approx 2\left(\frac{9}{4} + \frac{25}{4} + \frac{136}{4} + \frac{153}{4}\right)$

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Solution: A

The rectangle R is bounded by the lines $x=0$, $x=2$, $y=0$, and $y=8$. Because we want 2 subrectangles across and 2 subrectangles down, that means that we'll have $2 \times 2 = 4$ total subrectangles, each with dimensions $x \times y = 1 \times 4$.

Which means, using midpoints, the Riemann sum estimate is given by

$$\iint_R x^2+y^2\ dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_i,y_j)\Delta A = \Delta A\left[f\left(\frac{1}{2},2\right)+f\left(\frac{3}{2},2\right)+f\left(\frac{1}{2},6\right)+f\left(\frac{3}{2},6\right)\right]$$

Plugging each of the midpoints into the given integrand gives

$$\iint_R x^2+y^2\ dA \approx 4\left[\left(\frac{1}{4}+4\right)+\left(\frac{9}{4}+4\right)+\left(\frac{1}{4}+36\right)+\left(\frac{9}{4}+36\right)\right]$$

$$\iint_R x^2+y^2\ dA \approx 4\left[\left(\frac{1}{4}+\frac{16}{4}\right)+\left(\frac{9}{4}+\frac{16}{4}\right)+\left(\frac{1}{4}+\frac{144}{4}\right)+\left(\frac{9}{4}+\frac{144}{4}\right)\right]$$

$$\iint_R x^2+y^2\ dA \approx 4\left(\frac{17}{4}+\frac{25}{4}+\frac{145}{4}+\frac{153}{4}\right)$$

Topic: Midpoint rule for double integrals

Question: Using midpoint rule to estimate the double integral gave $3{,}584$. Which dimensions describe the rectangle R beneath the volume if $\Delta A = 16$, and the rectangle is divided into $m = 2$ sub-squares across by $n = 4$ sub-squares down?

$$\iint_R xy + x - y \ dA$$

Answer choices:

A $\quad R = [-8,8] \times [0,16]$

B $\quad R = [0,8] \times [-8,16]$

C $\quad R = [0,8] \times [0,16]$

D $\quad R = [0,16] \times [0,8]$

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Solution: C

Because the problem says that we are dividing the underlying rectangle R into sub-squares, we know that ΔA must be given by

$$x^2 = \Delta A$$

$$x^2 = 16$$

$$x = 4$$

So the dimensions of the sub-squares are 4×4. Then, because we know that we have 2 rectangles across by 4 rectangles down, using the Midpoint Rule with the information we've been given, plus the dimensions from answer choice C, we get

$$\iint_R xy + x - y \ dA \approx \sum_{i=1}^{2} \sum_{j=1}^{4} f(x_i, y_j)\Delta A$$

$$= \Delta A \left[f(2,2) + f(2,6) + f(2,10) + f(2,14) + f(6,2) + f(6,6) + f(6,10) + f(6,14)\right]$$

Plugging each of the midpoints into the integrand gives

$$V \approx 16\big[((2)(2) + 2 - 2) + ((2)(6) + 2 - 6) + ((2)(10) + 2 - 10) + ((2)(14) + 2 - 14)$$

$$+((6)(2) + 6 - 2) + ((6)(6) + 6 - 6) + ((6)(10) + 6 - 10) + ((6)(14) + 6 - 14)\big]$$

$$V \approx 16\left[(4) + (12 - 4) + (20 - 8) + (28 - 12) + (12 + 4) + (36) + (60 - 4) + (84 - 8)\right]$$

$$V \approx 16(4 + 8 + 12 + 16 + 16 + 36 + 56 + 76)$$

$$V \approx 3{,}584$$

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Topic: Midpoint rule for double integrals

Question: The double integral is defined on both the rectangle K and the square L. Using midpoint rule, which is the approximation of $V_K - V_L$?

$$\iint x^2 - y \ dA$$

Rectangle K:	$K = [0,6] \times [0,4]$	$m = 3,\ n = 2,\ \Delta A = 4$
Square L:	$L = [0,4] \times [0,4]$	$m = 2,\ n = 1,\ \Delta A = 8$

Answer choices:

A $\quad 166$

B $\quad 184$

C $\quad 128$

D $\quad 126$

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Solution: B

For rectangle K, given $K = [0,6] \times [0,4]$ and that $m = 3$, $n = 2$, and $\Delta A = 4$, we must be dividing K into sub-squares with dimensions 2×2, such that we have $m = 3$ squares across between $x = 0$ and $x = 6$, and $n = 2$ squares down between $y = 0$ and $y = 4$.

Therefore, we can set up the Riemann sum with midpoints as

$$\int\int_K x^2 - y \ dA \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j)\Delta A$$

$$= \Delta A \left[f(1,1) + f(1,3) + f(3,1) + f(3,3) + f(5,1) + f(5,3) \right]$$

If we plug these points into the integrand, and add in $\Delta A = 4$, we get

$$V_K \approx 4 \left[\left(1^2 - 1\right) + \left(1^2 - 3\right) + \left(3^2 - 1\right) + \left(3^2 - 3\right) + \left(5^2 - 1\right) + \left(5^2 - 3\right) \right]$$

$$V_K \approx 4(0 - 2 + 8 + 6 + 24 + 22)$$

$$V_K \approx 232$$

For square L, given $L = [0,4] \times [0,4]$ and that $m = 2$, $n = 1$, and $\Delta A = 8$, we must be dividing L into sub-rectangles with dimensions 2×4, such that we have $m = 2$ rectangles across between $x = 0$ and $x = 4$, and $n = 1$ rectangle down between $y = 0$ and $y = 4$.

Therefore, we can set up the Riemann sum with midpoints as

$$\int\int_L x^2 - y \ dA \approx \sum_{i=1}^{2} \sum_{j=1}^{1} f(x_i, y_j)\Delta A$$

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$$= \Delta A \left[f(1,2) + f(3,2)\right]$$

If we plug these points into the integrand, and add in $\Delta A = 8$, we get

$$V_L \approx 8\left[\left(1^2 - 2\right) + \left(3^2 - 2\right)\right]$$

$$V_L \approx 8(-1 + 7)$$

$$V_L \approx 48$$

Therefore,

$$V_K - V_L = 232 - 48$$

$$V_K - V_L = 184$$

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Topic: Riemann sums for double integrals

Question: Use Riemann sums to approximate the double integral.

$$\iint_R x + y^2 \, dA$$

$$m = n = 2$$

$$R = [0,2] \times [0,2]$$

Answer choices:

A 8

B 16

C 32

D 48

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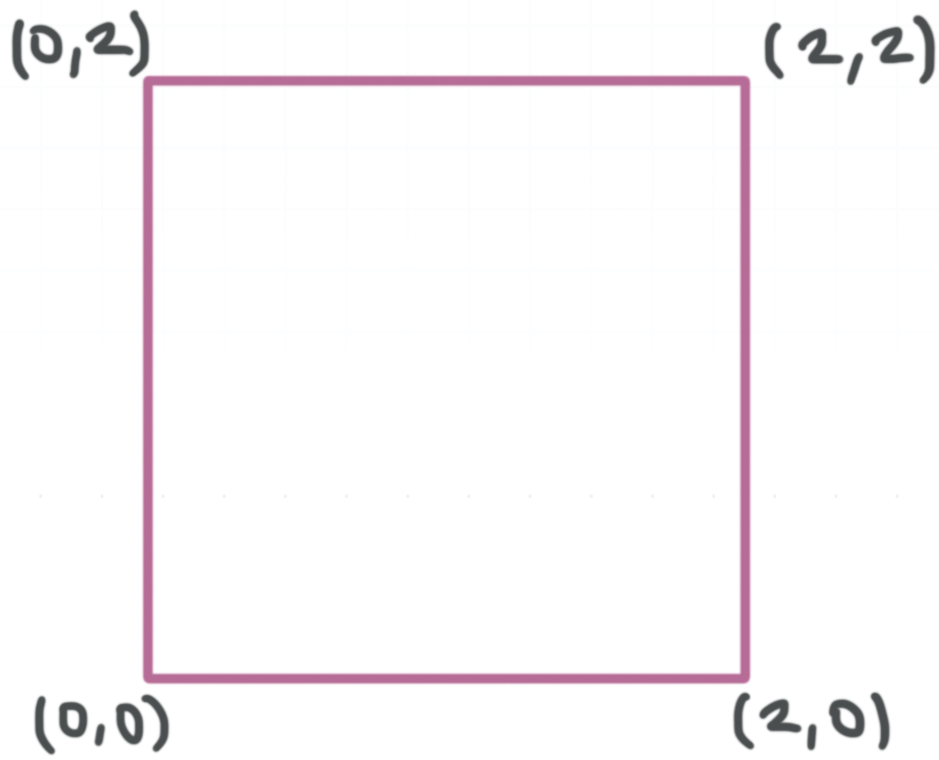
Solution: B

The question is asking us to use Riemann sums to approximate a double integral so we will need to use the formula

$$\iint_R f(x,y)\ dA = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij},y_{ij})\Delta A$$

$$= \Delta A\left[f(x_1,y_1) + f(x_2,y_2) + +f(x_3,y_3) + f(x_4,y_4) + \dots + f(x_n,y_n)\right]$$

To use this formula, we need to find the upper right corner points. The rectangle $R = [0,2] \times [0,2]$ gives us the x interval $[0,2]$ and the y interval $[0,2]$.



Because $m = n = 2$, we'll divide this larger rectangle $R = [0,2] \times [0,2]$ into two parts in the x direction and two parts in the y direction.

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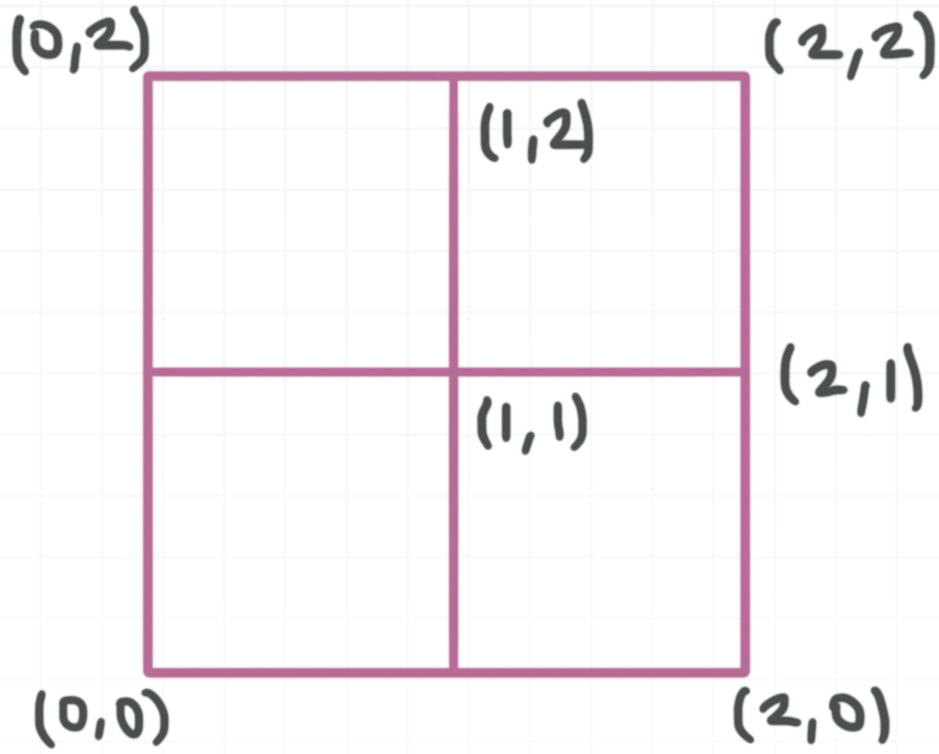

Then the upper right corners of the smaller rectangles are given by

$(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$

Next, we need to solve for ΔA. We'll use the dimensions of one of the smaller rectangles to find ΔA.

$$\Delta A = (\text{length of small rectangle})(\text{width of small rectangle})$$

$$\Delta A = (1)(1)$$

$$\Delta A = 1$$

Now we can plug everything we've found into our Riemann sum formula.

$$\iint_R f(x,y)\ dA = \Delta A\left[f(x_1,y_1) + f(x_2,y_2) + +f(x_3,y_3) + f(x_4,y_4) + \ldots + f(x_n,y_n)\right]$$

$$\iint_R f(x,y)\ dA = \Delta A\left[f(1,1) + f(2,1) + +f(1,2) + f(2,2)\right]$$

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$$\iint_R x+y^2\ dA=(1)\Big[\big(1+(1)^2\big)+\big(2+(1)^2\big)+\big(1+(2)^2\big)+\big(2+(2)^2\big)\Big]$$

$$\iint_R x+y^2\ dA=(1+1)+(2+1)+(1+4)+(2+4)$$

$$\iint_R x+y^2\ dA=2+3+5+6$$

$$\iint_R x+y^2\ dA=16$$

The approximate volume of the double integral is 16.

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Topic: Riemann sums for double integrals

Question: Use Riemann sums to approximate the double integral.

$$\iint_R e^{xy} \, dA$$

$$m = n = 2$$

$$R = [0,2] \times [0,2]$$

Answer choices:

A $\quad 18.024$

B $\quad 27.094$

C $\quad 72.094$

D $\quad 36.047$

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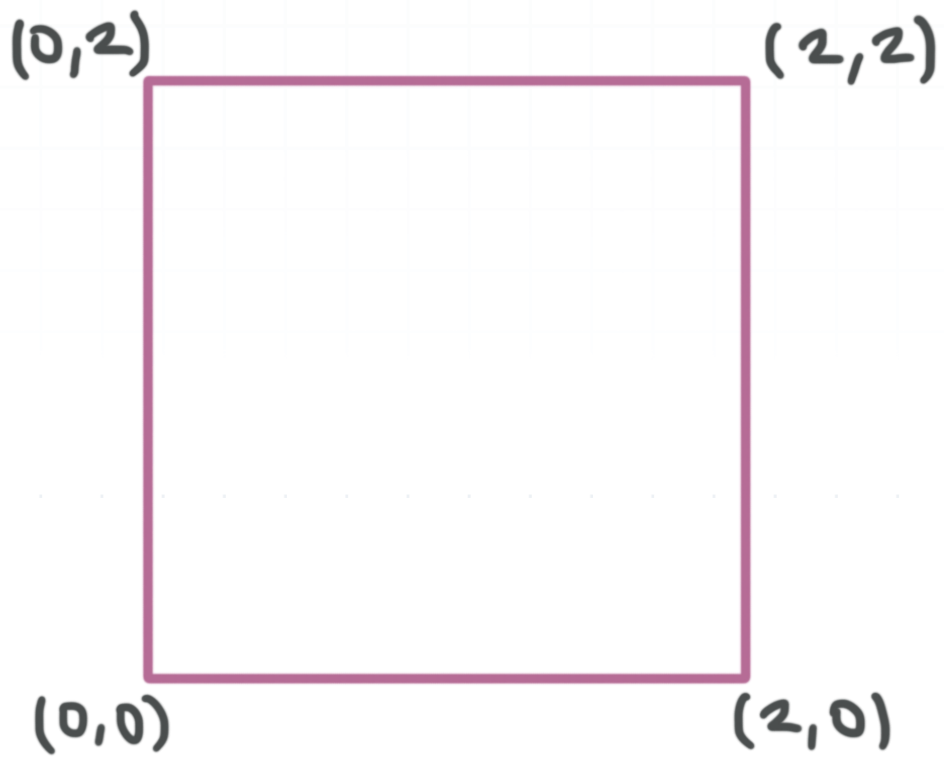
Solution: C

The question is asking us to use Riemann sums to approximate a double integral so we will need to use the formula

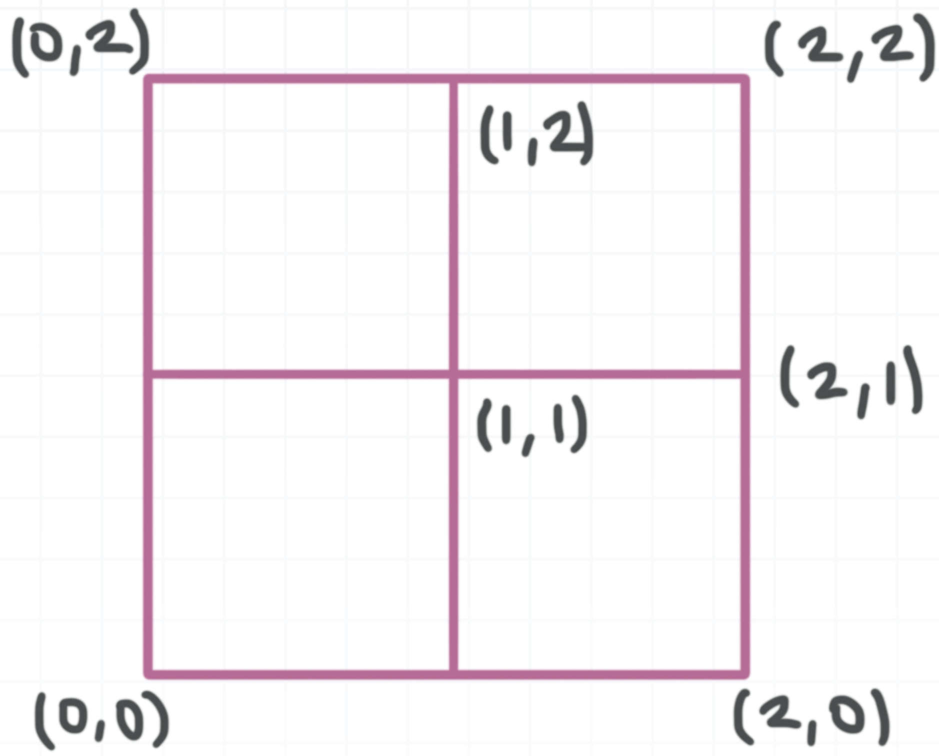
$$\iint_R f(x,y)\ dA = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

$$= \Delta A\left[f(x_1,y_1) + f(x_2,y_2) + +f(x_3,y_3) + f(x_4,y_4) + \ldots + f(x_n,y_n)\right]$$

To use this formula, we need to find the upper right corner points. The rectangle $R = [0,2] \times [0,2]$ gives us the x interval $[0,2]$ and the y interval $[0,2]$.



Because $m = n = 2$, we'll divide this larger rectangle $R = [0,2] \times [0,2]$ into two parts in the x direction and two parts in the y direction.



Then the upper right corners of the smaller rectangles are given by

$(1,1)$, $(2,1)$, $(1,2)$ and $(2,2)$

Next, we need to solve for ΔA. We'll use the dimensions of one of the smaller rectangles to find ΔA.

$$\Delta A = (\text{length of small rectangle})(\text{width of small rectangle})$$

$$\Delta A = (1)(1)$$

$$\Delta A = 1$$

Now we can plug everything we've found into our Riemann sum formula.

$$\iint_R f(x, y)\ dA = \Delta A\left[f(x_1, y_1) + f(x_2, y_2) + +f(x_3, y_3) + f(x_4, y_4) + \dots + f(x_n, y_n)\right]$$

$$\iint_R f(x, y)\ dA = \Delta A\left[f(1,1) + f(2,1) + +f(1,2) + f(2,2)\right]$$

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$$\iint_R e^{xy}\, dA = (1)\left[\left(e^{(1)(1)}\right) + \left(e^{(2)(1)}\right) + \left(e^{(1)(2)}\right) + \left(e^{(2)(2)}\right)\right]$$

$$\iint_R e^{xy}\, dA = e^1 + e^2 + e^2 + e^4$$

$$\iint_R e^{xy}\, dA = e + e^2 + e^2 + e^4$$

We could leave the answer this way, or we could calculate a decimal value for the approximate volume of the double integral. If we do that, then we can say that approximate volume is 72.094.

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Topic: Riemann sums for double integrals

Question: Use Riemann sums to approximate the double integral.

$$\iint_R 3ye^x \, dA$$

$$m = n = 2$$

$$R = [0,3] \times [0,2]$$

Answer choices:

A 110.553

B 663.316

C 221.105

D 331.658

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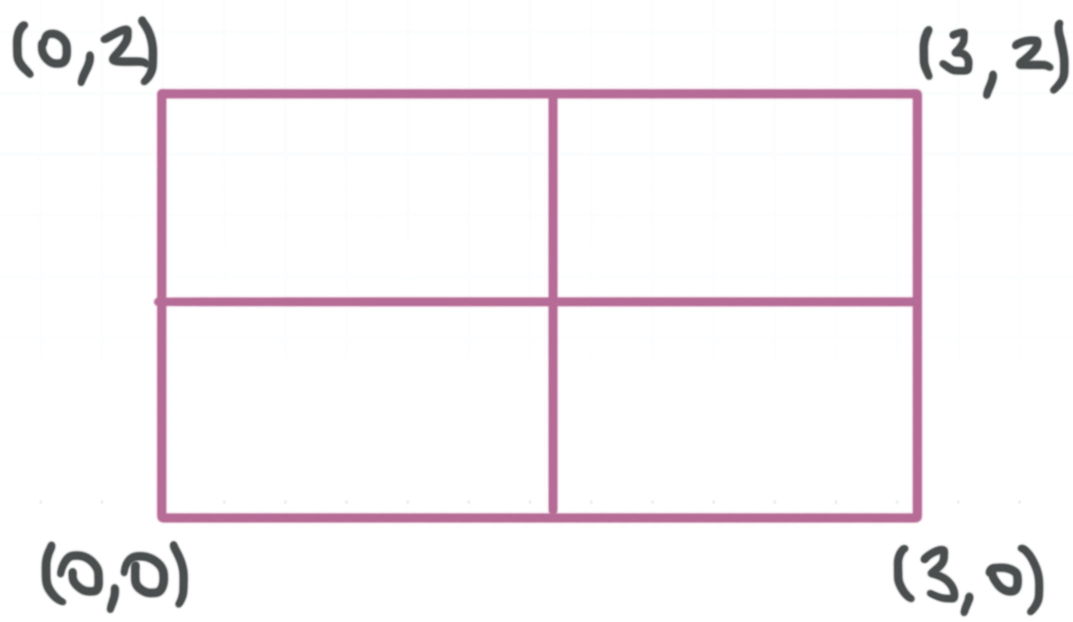
Solution: D

The question is asking us to use Riemann sums to approximate a double integral so we will need to use the formula

$$\iint_R f(x,y)\ dA = \sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}, y_{ij})\Delta A$$

$$= \Delta A\left[f(x_1,y_1) + f(x_2,y_2) + +f(x_3,y_3) + f(x_4,y_4) + \dots + f(x_n,y_n)\right]$$

To use this formula, we need to find the upper right corner points. The rectangle $R = [0,3] \times [0,2]$ gives us the x interval $[0,3]$ and the y interval $[0,2]$.



Because $m = n = 2$, we'll divide this larger rectangle $R = [0,2] \times [0,2]$ into two parts in the x direction and two parts in the y direction.

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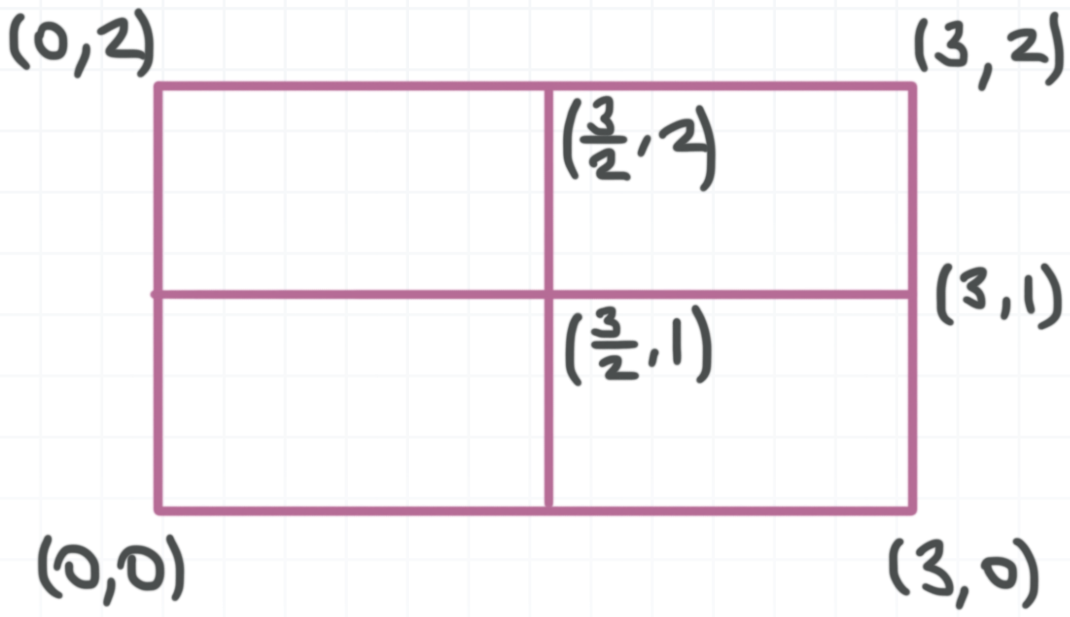

Then the upper right corners of the smaller rectangles are given by

$$\left(\frac{3}{2},1\right),\ (3,1),\ \left(\frac{3}{2},2\right) \text{ and } (3,2)$$

Next, we need to solve for ΔA. We'll use the dimensions of one of the smaller rectangles to find ΔA.

$$\Delta A = (\text{length of small rectangle})(\text{width of small rectangle})$$

$$\Delta A = \left(\frac{3}{2}\right)(1)$$

$$\Delta A = \frac{3}{2}$$

Now we can plug everything we've found into our Riemann sum formula.

$$\iint_R f(x,y)\ dA = \Delta A\left[f(x_1,y_1)+f(x_2,y_2)++f(x_3,y_3)+f(x_4,y_4)+\ldots+f(x_n,y_n)\right]$$

$$\iint_R f(x,y)\ dA = \Delta A\left[f\left(\frac{3}{2},1\right)+f(3,1)++f\left(\frac{3}{2},2\right)+f(3,2)\right]$$

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$$\iint_R 3ye^x\ dA = \frac{3}{2}\left[3(1)e^{\frac{3}{2}} + 3(1)e^3 + 3(2)e^{\frac{3}{2}} + 3(2)e^3\right]$$

$$\iint_R 3ye^x\ dA = \frac{3}{2}\left(3e^{\frac{3}{2}} + 3e^3 + 6e^{\frac{3}{2}} + 6e^3\right)$$

$$\iint_R 3ye^x\ dA = \frac{3}{2}\left(9e^{\frac{3}{2}} + 9e^3\right)$$

$$\iint_R 3ye^x\ dA = \frac{27}{2}e^{\frac{3}{2}}\left(1 + e^{\frac{3}{2}}\right)$$

We could leave the answer this way, or we could calculate a decimal value for the approximate volume of the double integral. If we do that, then we can say that approximate volume is 331.658.

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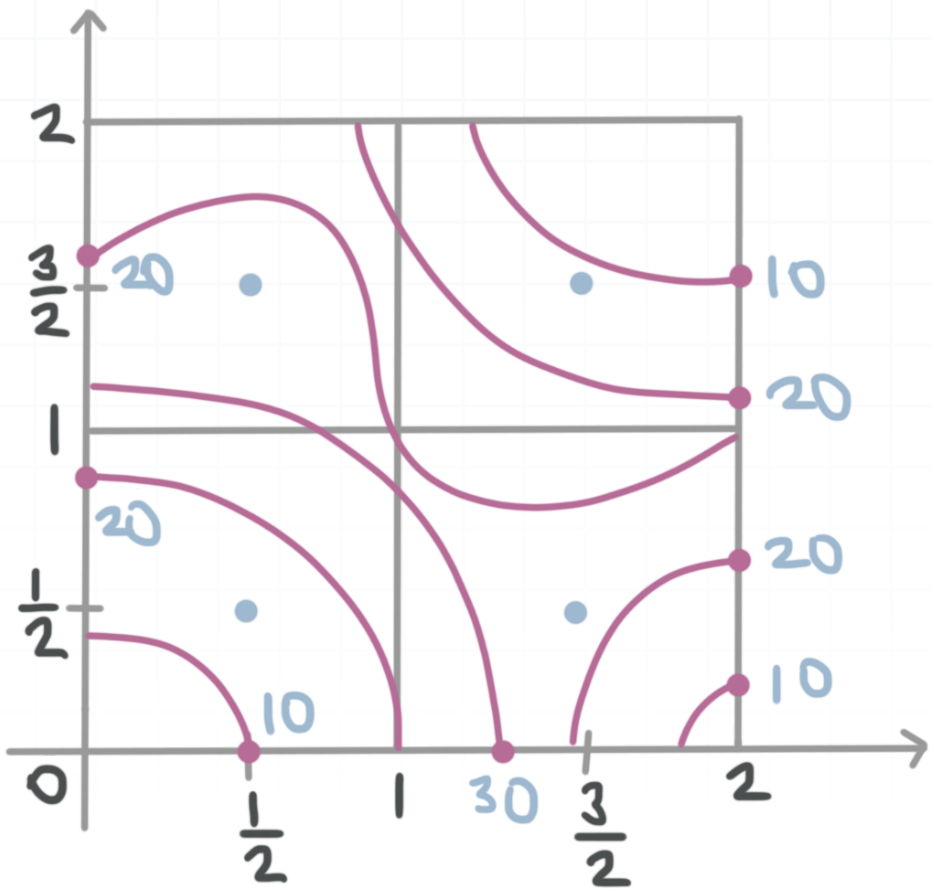
Topic: Average value

Question: Use midpoints to estimate the average value of the region $R = [0,2] \times [0,2]$ with $m = n = 2$ given the sketch of level curves.



Answer choices:

A $\quad 30$

B $\quad 12$

C $\quad 18$

D $\quad 25$

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Solution: C

To approximate the average value over the region, we'll use the average value formula

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y) \Delta A$$

where $A(R)$ is the area of the larger rectangle, and ΔA is the area of one of the smaller rectangles.

Because the larger rectangle has a width of 2 and a height of 2, the area is

$$A(R) = (2)(2)$$

$$A(R) = 4$$

The area of a smaller rectangle ΔA, is given by

$$\Delta A = (1)(1)$$

$$\Delta A = 1$$

The midpoints of the smaller rectangles are

$$\left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{3}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{3}{2}\right) \text{ and } \left(\frac{3}{2}, \frac{3}{2}\right)$$

Since we don't have the equation of the function, we can use the level curves to estimate the value of the function at each midpoint.

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For the point $\left(\frac{1}{2},\frac{1}{2}\right)$, the function is approximately 15, since it's halfway between level curves with values of 10 and 15.

For the point $\left(\frac{3}{2},\frac{1}{2}\right)$, the function is approximately 21, since it's just past a level curve with a value of 20, towards a level curve with a value of 30.

For the point $\left(\frac{1}{2},\frac{3}{2}\right)$, the function is approximately 24, since it's about halfway between level curves with values of 20 and 30.

For the point $\left(\frac{3}{2},\frac{3}{2}\right)$, the function is approximately 12, since it's just past a level curve with a value of 10, towards a level curve with a value of 20.

Now we're ready to estimate the average value.

$$f_{avg}=\frac{1}{A(R)}\iint_R f(x,y)\Delta A$$

$$f_{avg}=\frac{1}{4}\left(15+21+24+12\right)(1)$$

$$f_{avg}=\frac{1}{4}(72)$$

$$f_{avg}=18$$

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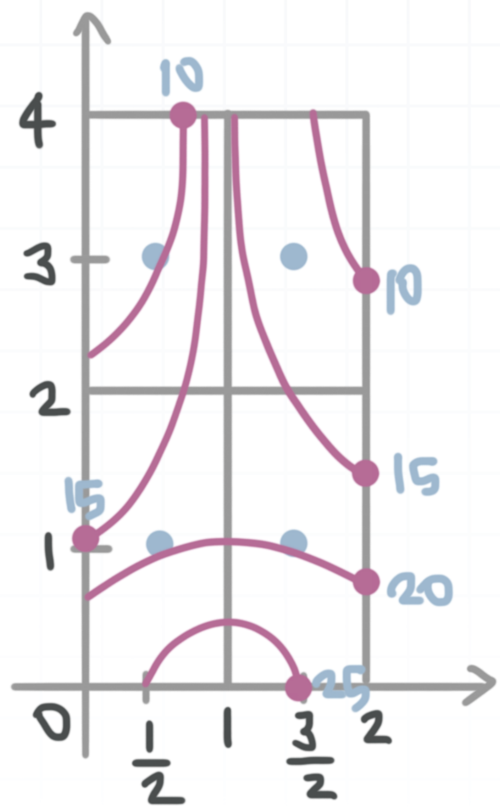
Topic: Average value

Question: Use midpoints to estimate the average value of the region $R = [0,2] \times [0,4]$ with $m = n = 2$ given the sketch of level curves.



Answer choices:

A 16

B 64

C 8

D 32

Solution: A

To approximate the average value over the region, we'll use the average value formula

$$f_{avg} = \frac{1}{A(R)} \int\int_R f(x,y)\Delta A$$

where $A(R)$ is the area of the larger rectangle, and ΔA is the area of one of the smaller rectangles.

Because the larger rectangle has a width of 2 and a height of 4, the area is

$$A(R) = (2)(4)$$

$$A(R) = 8$$

The area of a smaller rectangle ΔA, is given by

$$\Delta A = (1)(2)$$

$$\Delta A = 2$$

The midpoints of the smaller rectangles are

$\left(\frac{1}{2},1\right)$, $\left(\frac{3}{2},1\right)$, $\left(\frac{1}{2},3\right)$ and $\left(\frac{3}{2},3\right)$

Since we don't have the equation of the function, we can use the level curves to estimate the value of the function at each midpoint.

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For the point $\left(\frac{1}{2},1\right)$, the function is approximately 20, since it's right next to a level curve with a value of 20.

For the point $\left(\frac{3}{2},1\right)$, the function is approximately 20, since it's right next to a level curve with a value of 20.

For the point $\left(\frac{1}{2},3\right)$, the function is approximately 10, since it's right next to a level curve with a value of 10.

For the point $\left(\frac{3}{2},3\right)$, the function is approximately 15, since it's halfway between level curves with values of 10 and 15.

Now we're ready to estimate the average value.

$$f_{avg}=\frac{1}{A(R)}\int\int_R f(x,y)\Delta A$$

$$f_{avg}=\frac{1}{8}\left(20+20+10+15\right)(2)$$

$$f_{avg}=\frac{1}{4}\left(65\right)$$

$$f_{avg}=16.25$$

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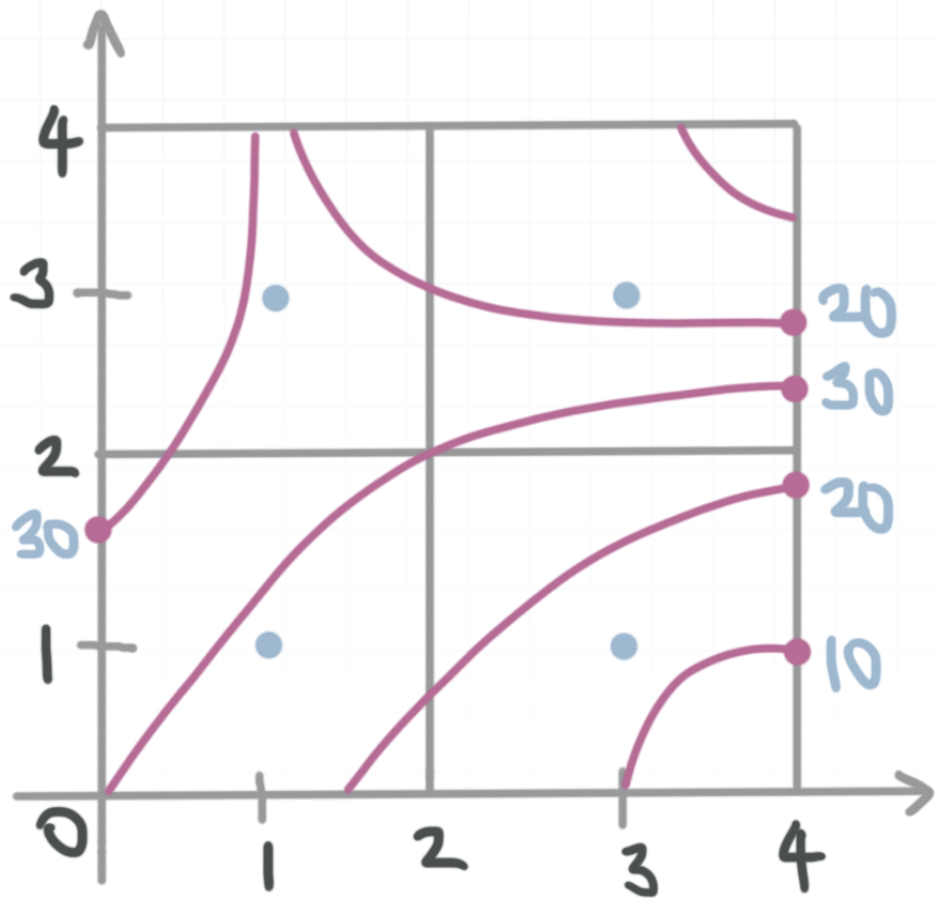
Topic: Average value

Question: Use midpoints to estimate the average value of the region $R = [0,4] \times [0,4]$ with $m = n = 2$ given the sketch of level curves.



Answer choices:

A 88

B 11

C 44

D 21

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Solution: D

To approximate the average value over the region, we'll use the average value formula

$$f_{avg} = \frac{1}{A(R)} \int\int_R f(x, y)\Delta A$$

where $A(R)$ is the area of the larger rectangle, and ΔA is the area of one of the smaller rectangles.

Because the larger rectangle has a width of 2 and a height of 2, the area is

$$A(R) = (4)(4)$$

$$A(R) = 16$$

The area of a smaller rectangle ΔA, is given by

$$\Delta A = (2)(2)$$

$$\Delta A = 4$$

The midpoints of the smaller rectangles are

$(1,1)$, $(3,1)$, $(1,3)$ and $(3,3)$

Since we don't have the equation of the function, we can use the level curves to estimate the value of the function at each midpoint.

For the point $(1,1)$, the function is approximately 29, since it's right next to a level curve with a value of 30, towards a level curve with a value of 20.

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For the point $(3,1)$, the function is approximately 15, since it's halfway between level curves with values of 10 and 20.

For the point $(1,3)$, the function is approximately 29, since it's right next to a level curve with a value of 30, towards a level curve with a value of 20.

For the point $(3,3)$, the function is approximately 10, since it's right next to a level curve with a value of 10.

Now we're ready to estimate the average value.

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x,y) \Delta A$$

$$f_{avg} = \frac{1}{16} \left(29 + 15 + 29 + 10\right)(4)$$

$$f_{avg} = \frac{1}{4}(83)$$

$$f_{avg} = 20.75$$

Topic: Iterated integrals

Question: Evaluate the iterated integral.

$$\int_0^1 \int_0^1 x^3y^2 + e^y \ dy \ dx$$

Answer choices:

A $\quad e - \dfrac{11}{12}$

B $\quad e + \dfrac{11}{12}$

C $\quad e + 1$

D $\quad e - 1$

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Solution: A

When we evaluate an iterated integral, we always start on the inside and work our way out. Since dy is on the inside and dx is on the outside, we'll start by integrating with respect to y. When we integrate with respect to y, we have to treat x like a constant.

$$\int_0^1 \int_0^1 x^3y^2 + e^y \ dy \ dx$$

$$\int_0^1 x^3\left(\frac{y^3}{3}\right) + e^y\Big|_{y=0}^{y=1} \ dx$$

$$\int_0^1 \frac{x^3y^3}{3} + e^y\Big|_{y=0}^{y=1} \ dx$$

Now we can evaluate over the interval $[0,1]$.

$$\int_0^1 \left[\frac{x^3(1)^3}{3} + e^{(1)}\right] - \left[\frac{x^3(0)^3}{3} + e^{(0)}\right] \ dx$$

$$\int_0^1 \left(\frac{x^3}{3} + e\right) - (1) \ dx$$

$$\int_0^1 \frac{x^3}{3} + e - 1 \ dx$$

Now we'll integrate with respect to x and evaluate over the interval $[0,1]$.

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$$\frac{x^4}{12} + xe - x \Big|_0^1$$

$$\left[\frac{(1)^4}{12} + (1)e - (1)\right] - \left[\frac{(0)^4}{12} + (0)e - (0)\right]$$

$$\frac{1}{12} + e - 1$$

$$-\frac{11}{12} + e$$

$$e - \frac{11}{12}$$

This is the volume given by the iterated integral.

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Topic: Iterated integrals

Question: Evaluate the iterated integral.

$$\int_0^3 \int_1^2 2x^3e^{2y} - 3x^2y \ dy \ dx$$

Answer choices:

A -915.484

B $1{,}911.683$

C $-1{,}911.683$

D 915.484

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Solution: D

When we evaluate an iterated integral, we always start on the inside and work our way out. Since dy is on the inside and dx is on the outside, we'll start by integrating with respect to y. When we integrate with respect to y, we have to treat x like a constant.

$$\int_0^3 \int_1^2 2x^3e^{2y} - 3x^2y\ dy\ dx$$

$$\int_0^3 2x^3\left(\frac{e^{2y}}{2}\right) - 3x^2\left(\frac{y^2}{2}\right)\Bigg|_{y=1}^{y=2}\ dx$$

Now we can evaluate over the interval $[1,2]$.

$$\int_0^3 2x^3\left(\frac{e^{2(2)}}{2}\right) - 3x^2\left(\frac{(2)^2}{2}\right) - \left[2x^3\left(\frac{e^{2(1)}}{2}\right) - 3x^2\left(\frac{(1)^2}{2}\right)\right]\ dx$$

$$\int_0^3 x^3e^4 - 6x^2 - \left(x^3e^2 - \frac{3x^2}{2}\right)\ dx$$

$$\int_0^3 x^3e^4 - 6x^2 - x^3e^2 + \frac{3x^2}{2}\ dx$$

$$\int_0^3 x^3e^4 - \frac{12x^2}{2} - x^3e^2 + \frac{3x^2}{2}\ dx$$

$$\int_0^3 \left(e^4 - e^2\right)x^3 - \frac{9x^2}{2}\ dx$$

Now we'll integrate with respect to x and evaluate over the interval $[0,3]$.

$$\left(e^4 - e^2\right)\frac{x^4}{4} - \frac{3x^3}{2}\Bigg|_0^3$$

$$\left(e^4 - e^2\right)\frac{(3)^4}{4} - \frac{3(3)^3}{2} - \left[\left(e^4 - e^2\right)\frac{(0)^4}{4} - \frac{3(0)^3}{2}\right]$$

$$\left(e^4 - e^2\right)\frac{81}{4} - \frac{81}{2}$$

$$\left(e^4 - e^2\right)\frac{81}{4} - (2)\frac{81}{4}$$

$$\left(e^4 - e^2 - 2\right)\frac{81}{4}$$

$$915.48$$

This is the volume given by the iterated integral.

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Topic: Iterated integrals

Question: Find the area bounded by the given curves.

$$3 + 2\sin x$$

$$3 + 2\cos x$$

$$x = \frac{\pi}{3}$$

$$x = \frac{5\pi}{6}$$

Answer choices:

A $\quad 2\left(2 + 2\sqrt{3}\right)$

B $\quad 2\left(2 + 2\sqrt{2}\right)$

C $\quad 2\sqrt{3}$

D $\quad \sqrt{3}$

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Solution: C

Let $dy\ dx$ be the order of the integration. Then the area of the region is given by the iterated integral

$$A=\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}}\int_{3+2\cos x}^{3+2\sin x} dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$A=\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} y\Big|_{y=3+2\cos x}^{y=3+2\sin x} dx$$

$$A=\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} 3+2\sin x-(3+2\cos x)\ dx$$

$$A=\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} 3+2\sin x-3-2\cos x\ dx$$

$$A=\int_{\frac{\pi}{3}}^{\frac{5\pi}{6}} 2\sin x-2\cos x\ dx$$

Now integrate with respect to x, then evaluate over the interval.

$$A=-2\cos x-2\sin x\Big|_{\frac{\pi}{3}}^{\frac{5\pi}{6}}$$

$$A=-2\cos\frac{5\pi}{6}-2\sin\frac{5\pi}{6}-\left(-2\cos\frac{\pi}{3}-2\sin\frac{\pi}{3}\right)$$

$$A = -2\left(-\frac{\sqrt{3}}{2}\right) - 2\left(\frac{1}{2}\right) - \left[-2\left(\frac{1}{2}\right) - 2\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$A = \sqrt{3} - 1 - \left(-1 - \sqrt{3}\right)$$

$$A = \sqrt{3} - 1 + 1 + \sqrt{3}$$

$$A = 2\sqrt{3}$$

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Topic: Double integrals

Question: Evaluate the double integral.

$$\iint_R y^2 \sin x + y^2 \cos(2x)\ dA$$

$$R = \left\{ (x, y) \ \middle|\ 0 \le x \le \frac{\pi}{2}, -2 \le y \le 1 \right\}$$

Answer choices:

A -3

B 1

C 3

D 0

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Solution: C

In this problem, we haven't been given the order of integration inside the integral as $dy\ dx$ or $dx\ dy$, so we can pick either order. We'll integrate first with respect to y, and then with respect to x.

$$\iint_R y^2 \sin x + y^2 \cos(2x)\ dA$$

$$\int_0^{\frac{\pi}{2}} \int_{-2}^{1} y^2 \sin x + y^2 \cos(2x)\ dy\ dx$$

When we integrate with respect to y, we have to treat x like a constant.

$$\int_0^{\frac{\pi}{2}} \frac{1}{3}y^3 \sin x + \frac{1}{3}y^3 \cos(2x) \Bigg|_{y=-2}^{y=1} dx$$

Now we can evaluate over the interval $[-2,1]$.

$$\int_0^{\frac{\pi}{2}} \frac{1}{3}(1)^3 \sin x + \frac{1}{3}(1)^3 \cos(2x) - \left[\frac{1}{3}(-2)^3 \sin x + \frac{1}{3}(-2)^3 \cos(2x)\right]\ dx$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{3}\sin x + \frac{1}{3}\cos(2x) - \left[-\frac{8}{3}\sin x - \frac{8}{3}\cos(2x)\right]\ dx$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{3}\sin x + \frac{1}{3}\cos(2x) + \frac{8}{3}\sin x + \frac{8}{3}\cos(2x)\ dx$$

$$\int_0^{\frac{\pi}{2}} \frac{9}{3}\sin x + \frac{9}{3}\cos(2x)\ dx$$

$$\int_0^{\frac{\pi}{2}} 3\sin x + 3\cos(2x)\ dx$$

Now we'll integrate with respect to x and evaluate over the interval $\left[0,\frac{\pi}{2}\right]$.

$$-3\cos x + \frac{3}{2}\sin(2x)\Big|_0^{\frac{\pi}{2}}$$

$$-3\cos\frac{\pi}{2} + \frac{3}{2}\sin\left(2\frac{\pi}{2}\right) - \left[-3\cos 0 + \frac{3}{2}\sin(2(0))\right]$$

$$-3\cos\frac{\pi}{2} + \frac{3}{2}\sin\pi + 3\cos 0 - \frac{3}{2}\sin 0$$

$$-3(0) + \frac{3}{2}(0) + 3(1) - \frac{3}{2}(0)$$

$$3$$

This is the volume given by the iterated integral.

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Topic: Double integrals

Question: Evaluate the double integral.

$$\int\int_R 3y - 2xy\ dx\ dy$$

where R is the rectangle on the interval

$$2 \le x \le 3$$

$$0 \le y \le 2$$

Answer choices:

A 16

B 11

C -6

D -4

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Solution: D

First, we'll apply the given interval to the double integral, to turn it into an iterated integral.

$$\int\int_R 3y - 2xy\ dx\ dy$$

$$\int_0^2 \int_2^3 3y - 2xy\ dx\ dy$$

Then integrate with respect to x, and evaluate over the interval.

$$\int_0^2 3xy - x^2y\Big|_{x=2}^{x=3}\ dy$$

$$\int_0^2 3(3)y - (3)^2y - \left(3(2)y - (2)^2y\right)\ dy$$

$$\int_0^2 9y - 9y - \left(6y - 4y\right)\ dy$$

$$\int_0^2 -2y\ dy$$

Integrate with respect to y, and evaluate over the interval.

$$-y^2\Big|_0^2$$

$$-(2)^2 - \left(-(0)^2\right)$$

$$-4$$

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Topic: Double integrals

Question: Which double integral is equal to $1/2$?

Answer choices:

A $\quad L = \int_0^1 dy \int_x^1 x - y \, dx$

B $\quad L = \int_0^1 dy \int_x^1 x + y \, dx$

C $\quad L = \int_0^1 dx \int_x^1 x + y \, dy$

D $\quad L = \int_0^1 dx \int_x^1 x - y \, dy$

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Solution: C

Starting with answer choice C,

$$L=\int_0^1 dx\int_x^1 x+y\ dy$$

integrate first with respect to y, then evaluate over the interval.

$$L=\int_0^1 xy+\frac{1}{2}y^2\Big|_{y=x}^{y=1}\ dx$$

$$L=\int_0^1 x(1)+\frac{1}{2}(1)^2-\left(x(x)+\frac{1}{2}(x)^2\right)\ dx$$

$$L=\int_0^1 x+\frac{1}{2}-x^2-\frac{1}{2}x^2\ dx$$

$$L=\int_0^1 -\frac{3}{2}x^2+x+\frac{1}{2}\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$L=-\frac{1}{2}x^3+\frac{1}{2}x^2+\frac{1}{2}x\Big|_0^1$$

$$L=-\frac{1}{2}(1)^3+\frac{1}{2}(1)^2+\frac{1}{2}(1)-\left(-\frac{1}{2}(0)^3+\frac{1}{2}(0)^2+\frac{1}{2}(0)\right)$$

$$L=\frac{1}{2}$$

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Topic: Type I and II regions

Question: Say whether the region is type I or II, then find the volume given by the double integral, if D is the triangle bounded by $x=1$, $y=1$, and $y=-x+4$.

$$\iint_D x^2\ dA$$

Answer choices:

A $\quad 8$

B $\quad 6$

C $\quad 2$

D $\quad 12$

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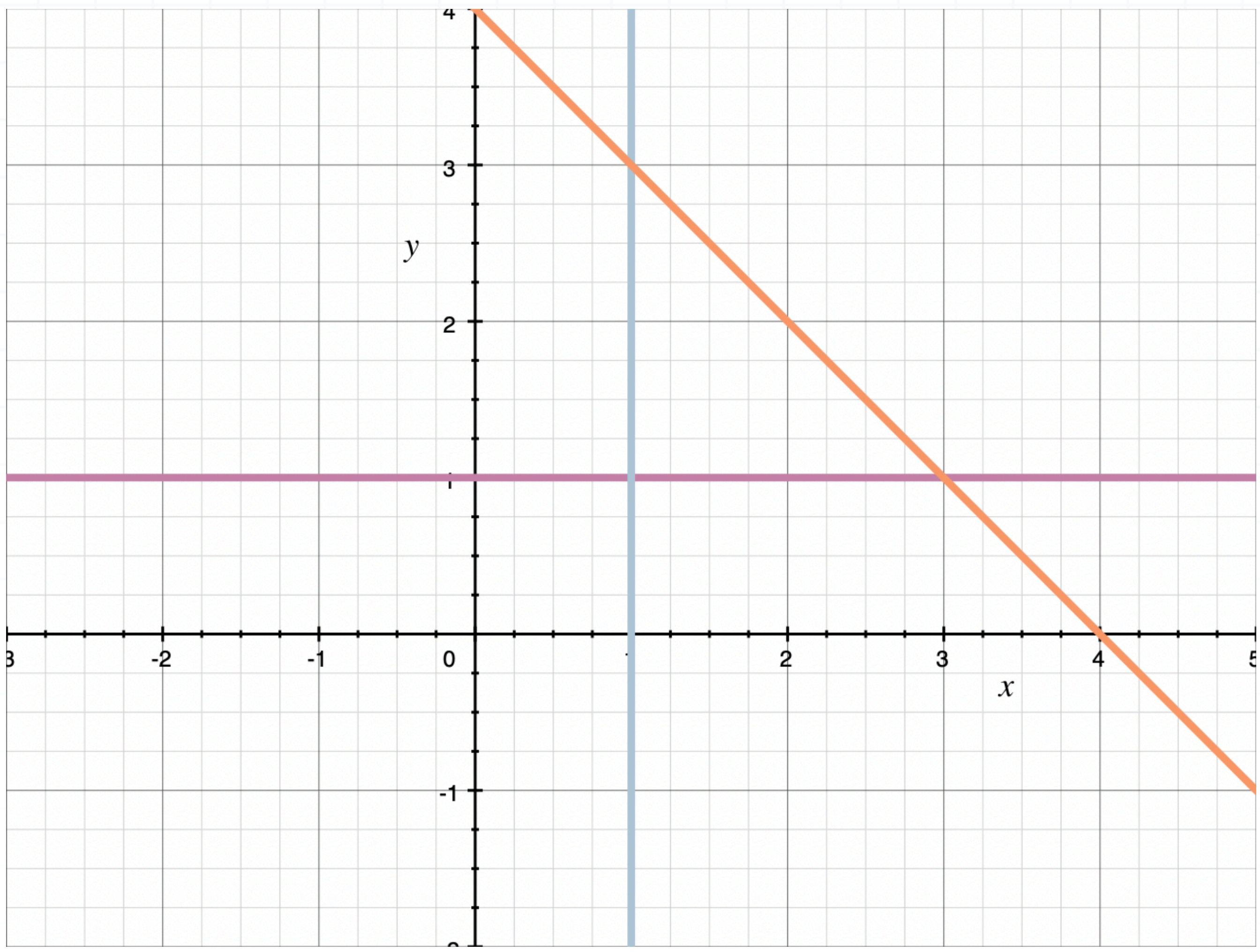
Solution: B

A sketch of the region D is



We can cut the region into uniform slices both horizontally and vertically, which means we can evaluate it as type I or type II. Let's do this as a type II region.

We need to solve $y = -x + 4$ for x, and we find $x = 4 - y$. This equation, along with $x = 1$, define the limits of integration with respect to x. If we look at the sketch of the region, we can see that y is defined on $[1,3]$. Putting all of this into a double integral, we get

$$\iint_D x^2 \, dA$$

$$\int_{1}^{3}\int_{1}^{4-y} x^2\ dx\ dy$$

Integrate with respect to x and evaluate over the interval.

$$\int_{1}^{3} \frac{1}{3}x^3 \Big|_{x=1}^{x=4-y}\ dy$$

$$\int_{1}^{3} \frac{1}{3}(4-y)^3 - \frac{1}{3}(1)^3\ dy$$

$$\int_{1}^{3} \frac{1}{3}(4-y)^3 - \frac{1}{3}\ dy$$

Integrate with respect to y and evaluate over the interval.

$$-\frac{1}{12}(4-y)^4 - \frac{1}{3}y \Big|_{1}^{3}$$

$$\left(-\frac{1}{12}(4-3)^4 - \frac{1}{3}(3)\right) - \left(-\frac{1}{12}(4-1)^4 - \frac{1}{3}(1)\right)$$

$$\left(-\frac{1}{12} - 1\right) - \left(-\frac{81}{12} - \frac{1}{3}\right)$$

$$-\frac{13}{12} + \frac{85}{12}$$

$$\frac{72}{12}$$

$$6$$

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Topic: Type I and II regions

Question: Say whether the region is type I or II, then find the volume given by the double integral, if D is the triangle bounded by $y=1$, $y=x+1$, and $y=-x+4$.

$$\iint_D x^2 \, dA$$

Answer choices:

A $\quad \dfrac{189}{16}$

B $\quad \dfrac{378}{32}$

C $\quad \dfrac{189}{64}$

D $\quad \dfrac{189}{32}$

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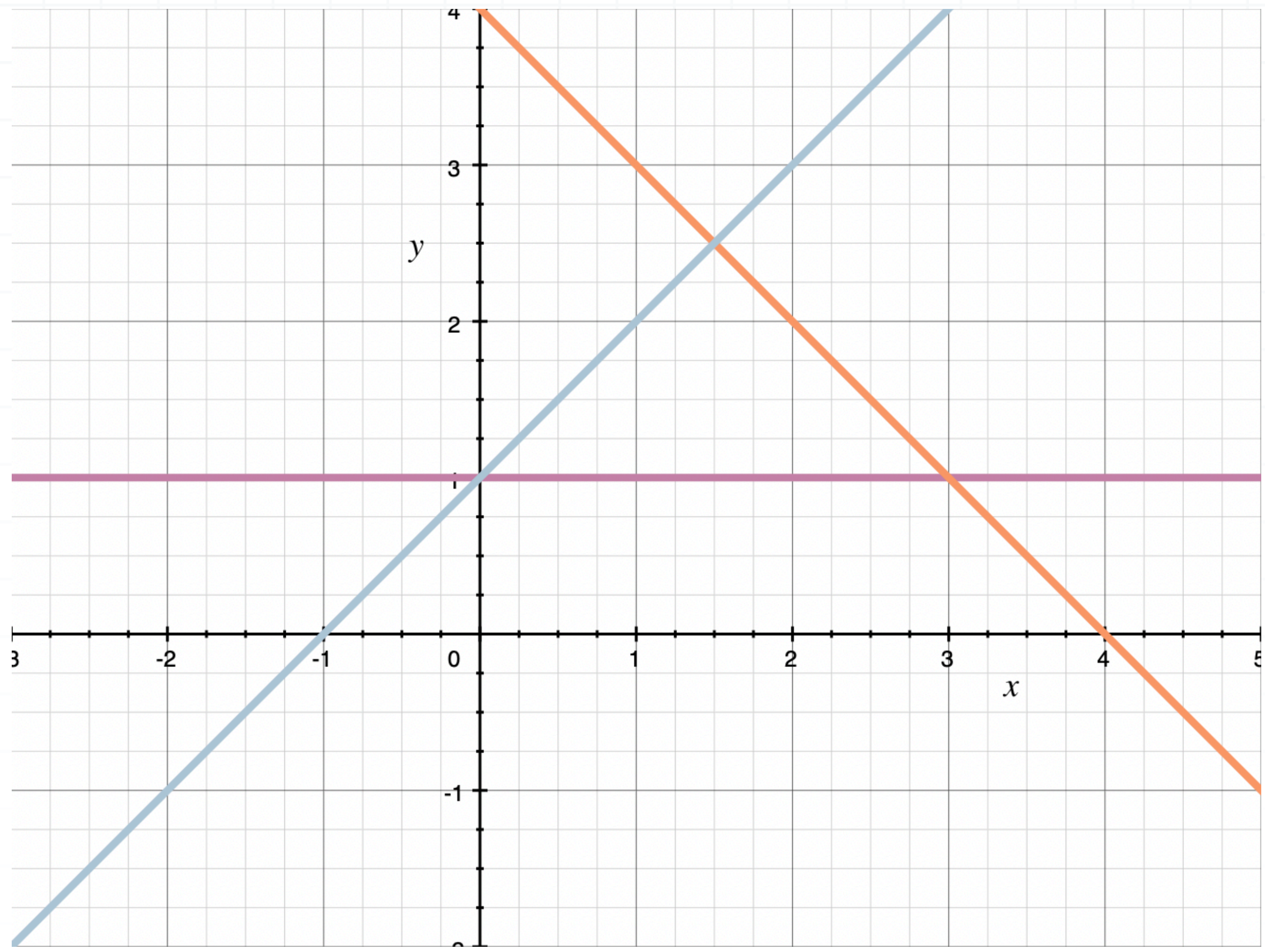
Solution: D

A sketch of the region D is



We can cut the region into uniform slices both horizontally and vertically, which means we can evaluate it as type I or type II. Let's do this as a type II region.

We need to solve $y = x + 1$ and $y = -x + 4$ for x, and we find $x = y - 1$ and $x = 4 - y$. These equations define the limits of integration with respect to x. If we look at the sketch of the region, we can see that y is defined on $[1,2.5]$. Putting all of this into a double integral, we get

$$\iint_D x^2 \, dA$$

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$$\int_{1}^{\frac{5}{2}}\int_{y-1}^{4-y} x^2\ dx\ dy$$

Integrate with respect to x and evaluate over the interval.

$$\int_{1}^{\frac{5}{2}} \frac{1}{3}x^3\Big|_{x=y-1}^{x=4-y}\ dy$$

$$\int_{1}^{\frac{5}{2}} \frac{1}{3}(4-y)^3-\frac{1}{3}(y-1)^3\ dy$$

Integrate with respect to y and evaluate over the interval.

$$-\frac{1}{12}(4-y)^4-\frac{1}{12}(y-1)^4\Big|_{1}^{\frac{5}{2}}$$

$$-\frac{1}{12}\left(4-\frac{5}{2}\right)^4-\frac{1}{12}\left(\frac{5}{2}-1\right)^4-\left(-\frac{1}{12}(4-1)^4-\frac{1}{12}(1-1)^4\right)$$

$$-\frac{1}{12}\left(\frac{3}{2}\right)^4-\frac{1}{12}\left(\frac{3}{2}\right)^4+\frac{81}{12}$$

$$-\frac{27}{64}-\frac{27}{64}+\frac{432}{64}$$

$$\frac{378}{64}$$

$$\frac{189}{32}$$

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Topic: Type I and II regions

Question: Say whether the region is type I or II, then find the volume given by the double integral, if D is the triangle bounded by $y=1$, $y=x+1$, and $y=-x+4$.

$$\iint_D x^2+1\ dA$$

Answer choices:

A $\quad \dfrac{261}{32}$

B $\quad \dfrac{522}{16}$

C $\quad \dfrac{261}{64}$

D $\quad \dfrac{522}{32}$

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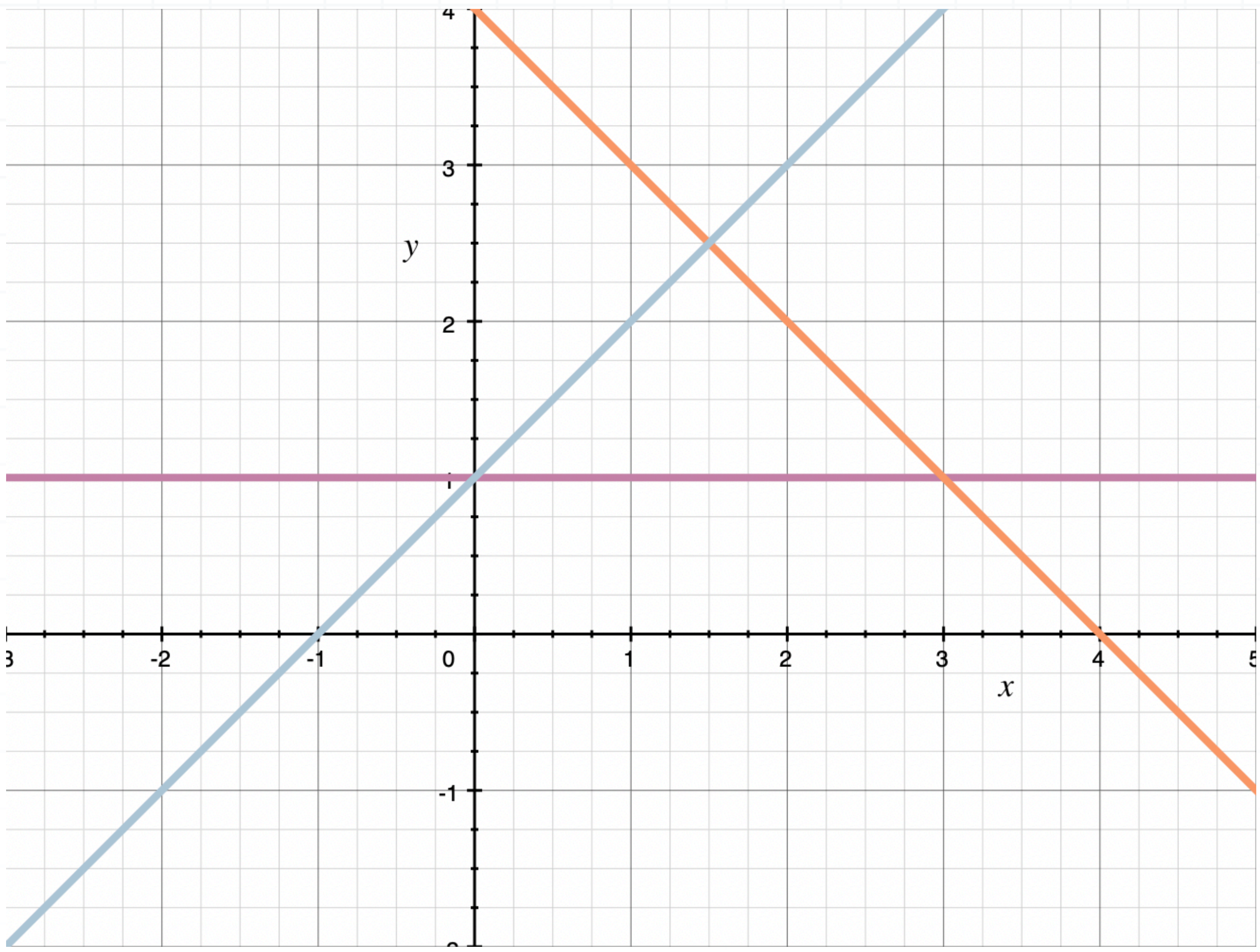
Solution: A

A sketch of the region D is



We can cut the region into uniform slices both horizontally and vertically, which means we can evaluate it as type I or type II. Let's do this as a type II region.

We need to solve $y = x + 1$ and $y = -x + 4$ for x, and we find $x = y - 1$ and $x = 4 - y$. These equations define the limits of integration with respect to x. If we look at the sketch of the region, we can see that y is defined on $[1,2.5]$. Putting all of this into a double integral, we get

$$\iint_D x^2 + 1 \ dA$$

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$$\int_{1}^{\frac{5}{2}}\int_{y-1}^{4-y} x^2+1\ dx\ dy$$

Integrate with respect to x and evaluate over the interval.

$$\int_{1}^{\frac{5}{2}} \frac{1}{3}x^3+x\Big|_{x=y-1}^{x=4-y}\ dy$$

$$\int_{1}^{\frac{5}{2}} \frac{1}{3}(4-y)^3+(4-y)-\left(\frac{1}{3}(y-1)^3+(y-1)\right)\ dy$$

$$\int_{1}^{\frac{5}{2}} \frac{1}{3}(4-y)^3+(4-y)-\frac{1}{3}(y-1)^3-(y-1)\ dy$$

$$\int_{1}^{\frac{5}{2}} \frac{1}{3}(4-y)^3-\frac{1}{3}(y-1)^3-2y+5\ dy$$

Integrate with respect to y and evaluate over the interval.

$$-\frac{1}{12}(4-y)^4-\frac{1}{12}(y-1)^4-y^2+5y\Big|_{1}^{\frac{5}{2}}$$

$$-\frac{1}{12}\left(4-\frac{5}{2}\right)^4-\frac{1}{12}\left(\frac{5}{2}-1\right)^4-\left(\frac{5}{2}\right)^2+5\left(\frac{5}{2}\right)$$

$$-\left(-\frac{1}{12}(4-1)^4-\frac{1}{12}(1-1)^4-(1)^2+5(1)\right)$$

$$-\frac{1}{12}\left(\frac{8}{2}-\frac{5}{2}\right)^4-\frac{1}{12}\left(\frac{5}{2}-\frac{2}{2}\right)^4-\frac{25}{4}+\frac{25}{2}-\left(-\frac{1}{12}(81)+4\right)$$

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$$-\frac{1}{12}\left(\frac{3}{2}\right)^4-\frac{1}{12}\left(\frac{3}{2}\right)^4-\frac{25}{4}+\frac{25}{2}+\frac{81}{12}-4$$

$$-\frac{27}{64}-\frac{27}{64}-\frac{25}{4}+\frac{25}{2}+\frac{27}{4}-4$$

$$\frac{261}{32}$$

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Topic: Finding surface area

Question: Find surface area.

The part of $z = xy$ inside $x^2 + y^2 = 25$

Answer choices:

A $\quad A(S) = \dfrac{50\pi\sqrt{26}}{3}$

B $\quad A(S) = \dfrac{2\pi}{3}\left(\sqrt{26^3} + 1\right)$

C $\quad A(S) = \dfrac{2\pi}{3}\left(\sqrt{26^3} - 1\right)$

D $\quad A(S) = 38\pi\sqrt{26}$

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Solution: C

To find the area of some surface that's bounded by another function, we'll need to take the partial derivatives of the equation of the surface.

$$z = xy$$

$$\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = x$$

Since we're dealing with a double integral, we have to decide whether the region we're looking at is a type 1 or type 2 region. We're looking for the area inside $x^2 + y^2 = 25$, which is a circle. Since we could take uniform slices of the circle horizontally or vertically, we could treat the region as either type 1 or type 2, but we'll go ahead and treat it as a type 1 region. Therefore, we'll integrate first with respect to y, and second with respect to x.

$$A(S) = \int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \ dA$$

$$A(S) = \int\int_D \sqrt{1 + y^2 + x^2} \ dy \ dx$$

To find the limits of integration with respect to y, we'll solve $x^2 + y^2 = 25$ for y.

$$x^2 + y^2 = 25$$

$$y^2 = 25 - x^2$$

$$y = \pm\sqrt{25 - x^2}$$

To find the limits of integration with respect to x, we'll remember that the circle $x^2 + y^2 = 25$ is a circle centered at the origin with radius 5, and therefore is defined for x on $[-5,5]$.

$$A(S) = \int_{-5}^{5} \int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} \sqrt{1 + y^2 + x^2}\ dy\ dx$$

The simplest way to solve this particular integral is to convert it to polar coordinates. Using the conversion formulas

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

the integral becomes

$$A(S) = \int_{x=-5}^{x=5} \int_{y=-\sqrt{25-x^2}}^{y=\sqrt{25-x^2}} r\sqrt{1 + r^2}\ dr\ d\theta$$

Now we need to change the limits of integration from rectangular to polar. Inside the circle $x^2 + y^2 = 25$, r is defined on $[0,5]$, and θ is defined on $[0,2\pi]$.

$$A(S) = \int_{0}^{2\pi} \int_{0}^{5} r\sqrt{1 + r^2}\ dr\ d\theta$$

We'll use u-substitution to take the integral.

$$u = 1 + r^2$$

$$\frac{du}{dr} = 2r$$

$$dr = \frac{du}{2r}$$

Making substitutions into the integral gives

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=5} r\sqrt{u}\left(\frac{du}{2r}\right) d\theta$$

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=5} \frac{1}{2}u^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u.

$$A(S) = \int_0^{2\pi} \frac{1}{3}u^{\frac{3}{2}}\Big|_{r=0}^{r=5} d\theta$$

Back-substitute.

$$A(S) = \int_0^{2\pi} \frac{1}{3}\left(1 + r^2\right)^{\frac{3}{2}}\Big|_0^5 d\theta$$

Evaluate over the interval and simplify.

$$A(S) = \int_0^{2\pi} \frac{1}{3}\left(1 + 5^2\right)^{\frac{3}{2}} - \frac{1}{3}\left(1 + 0^2\right)^{\frac{3}{2}}\ d\theta$$

$$A(S) = \int_0^{2\pi} \frac{1}{3}\left(26^{\frac{3}{2}}\right) - \frac{1}{3}\left(1^{\frac{3}{2}}\right)\ d\theta$$

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$$A(S)=\int_0^{2\pi}\frac{1}{3}\left(26^3\right)^{\frac{1}{2}}-\frac{1}{3}\left(1^3\right)^{\frac{1}{2}}\ d\theta$$

$$A(S)=\int_0^{2\pi}\frac{1}{3}\sqrt{26^3}-\frac{1}{3}\sqrt{1}\ d\theta$$

$$A(S)=\int_0^{2\pi}\frac{1}{3}\sqrt{26^3}-\frac{1}{3}\ d\theta$$

$$A(S)=\int_0^{2\pi}\frac{\sqrt{26^3}-1}{3}\ d\theta$$

Integrate with respect to θ.

$$A(S)=\frac{\sqrt{26^3}-1}{3}\theta\Big|_0^{2\pi}$$

Evaluate over the interval and simplify.

$$A(S)=\frac{\sqrt{26^3}-1}{3}(2\pi)-\frac{\sqrt{26^3}-1}{3}(0)$$

$$A(S)=\frac{2\pi}{3}\left(\sqrt{26^3}-1\right)$$

This is the area of the surface.

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Topic: Finding surface area

Question: Find surface area.

The part of $x^2 + y^2 + z^2 = 4z$ inside $z = x^2 + y^2$

Answer choices:

A $A(S) = 4\pi$

B $A(S) = 2\pi$

C $A(S) = 8\pi$

D $A(S) = 12\pi$

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Solution: A

To find the area of some surface that's bounded by another function, we'll need to take the partial derivatives of the equation of the surface. But first, we'll solve the equation for z by completing the square with respect to z.

$$x^2 + y^2 + z^2 = 4z$$

$$x^2 + y^2 + z^2 - 4z = 0$$

$$x^2 + y^2 + z^2 - 4z + 4 = 0 + 4$$

$$x^2 + y^2 + (z-2)^2 = 4$$

Now we'll solve for z.

$$(z-2)^2 = 4 - x^2 - y^2$$

$$z - 2 = \pm\sqrt{4 - x^2 - y^2}$$

$$z = 2 \pm \sqrt{4 - x^2 - y^2}$$

We're looking for the part of this equation that lies inside $z = x^2 + y^2$. Since $z = x^2 + y^2$ is the paraboloid with vertex at the origin that opens up, we want to take the positive solution of $z = 2 \pm \sqrt{4 - x^2 - y^2}$, since the positive solution is the top part of that curve. We'll take partial derivatives of the curve.

$$z = 2 + \sqrt{4 - x^2 - y^2}$$

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$$\frac{\partial z}{\partial x}=\frac{1}{2}\left(4-x^2-y^2\right)^{-\frac{1}{2}}(-2x)$$

$$\frac{\partial z}{\partial x}=-\frac{x}{\sqrt{4-x^2-y^2}}$$

and

$$\frac{\partial z}{\partial y}=\frac{1}{2}\left(4-x^2-y^2\right)^{-\frac{1}{2}}(-2y)$$

$$\frac{\partial z}{\partial y}=-\frac{y}{\sqrt{4-x^2-y^2}}$$

Since we're dealing with a double integral, we have to decide whether the region we're looking at is a type 1 or type 2 region. We're looking for the area inside $z=x^2+y^2$, which is a paraboloid. Since we could take uniform slices of the circle horizontally or vertically, we could treat the region as either type 1 or type 2, but we'll go ahead and treat it as a type 1 region. Therefore, we'll integrate first with respect to y, and second with respect to x.

$$A(S)=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$A(S)=\iint_D \sqrt{1+\left(-\frac{x}{\sqrt{4-x^2-y^2}}\right)^2+\left(-\frac{y}{\sqrt{4-x^2-y^2}}\right)^2}\ dy\ dx$$

$$A(S)=\iint_D \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\ dy\ dx$$

$$A(S) = \iint_D \sqrt{\frac{4 - x^2 - y^2}{4 - x^2 - y^2} + \frac{x^2 + y^2}{4 - x^2 - y^2}} \ dy \ dx$$

$$A(S) = \iint_D \sqrt{\frac{4 - x^2 - y^2 + x^2 + y^2}{4 - x^2 - y^2}} \ dy \ dx$$

$$A(S) = \iint_D \sqrt{\frac{4}{4 - x^2 - y^2}} \ dy \ dx$$

To find the limits of integration, we'll solve $x^2 + y^2 + z^2 = 4z$ and $z = x^2 + y^2$ as a system of equations to see where they intersect. Since $z = x^2 + y^2$, we'll make a substitution into $x^2 + y^2 + z^2 = 4z$.

$$x^2 + y^2 + z^2 = 4z$$

$$z + z^2 = 4z$$

$$z^2 - 3z = 0$$

$$z(z - 3) = 0$$

$$z = 0, \ 3$$

If we plug these values for z back into $z = x^2 + y^2$, we get two equations:

$$x^2 + y^2 = 0$$

$$x^2 + y^2 = 3$$

The first is the equation of a circle with radius 0, which means it's actually the equation of a single point, and the second is the equation of a circle with radius $\sqrt{3}$. Since we've got equations of circles for the limits of integration, it'll be easier to represent the limits of integration in polar coordinates, so we'll convert the integral. Using the conversion formulas

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

we get

$$A(S) = \iint_D \sqrt{\frac{4}{4 - x^2 - y^2}}\ dy\ dx$$

$$A(S) = \iint_D \sqrt{\frac{4}{4 - \left(x^2 + y^2\right)}}\ dy\ dx$$

$$A(S) = \iint_D r\sqrt{\frac{4}{4 - r^2}}\ dr\ d\theta$$

Adding in the limits of integration gives

$$A(S) = \int_0^{2\pi} \int_0^{\sqrt{3}} r\sqrt{\frac{4}{4 - r^2}}\ dr\ d\theta$$

We'll use u-substitution to take the integral.

$$u = 4 - r^2$$

$$\frac{du}{dr} = -2r$$

$$dr = -\frac{du}{2r}$$

Making substitutions into the integral gives

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=\sqrt{3}} r\sqrt{\frac{4}{u}} \left(-\frac{du}{2r}\right) \ d\theta$$

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=\sqrt{3}} -\frac{1}{2}\sqrt{\frac{4}{u}} \ du \ d\theta$$

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=\sqrt{3}} -\frac{1}{2}\left(2u^{-\frac{1}{2}}\right) \ du \ d\theta$$

$$A(S) = \int_0^{2\pi} \int_{r=0}^{r=\sqrt{3}} -u^{-\frac{1}{2}} \ du \ d\theta$$

Integrate with respect to u.

$$A(S) = \int_0^{2\pi} -2u^{\frac{1}{2}}\Big|_{r=0}^{r=\sqrt{3}} \ d\theta$$

Back-substitute.

$$A(S) = \int_0^{2\pi} -2\left(4-r^2\right)^{\frac{1}{2}}\Big|_0^{\sqrt{3}} \ d\theta$$

Evaluate over the interval and simplify.

$$A(S) = \int_0^{2\pi} -2\left[4-\left(\sqrt{3}\right)^2\right]^{\frac{1}{2}} + 2\left(4-0^2\right)^{\frac{1}{2}} \ d\theta$$

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$$A(S) = \int_0^{2\pi} -2\sqrt{1} + 2\sqrt{4}\ d\theta$$

$$A(S) = \int_0^{2\pi} -2 + 2(2)\ d\theta$$

$$A(S) = \int_0^{2\pi} 2\ d\theta$$

Integrate with respect to θ.

$$A(S) = 2\theta \Big|_0^{2\pi}$$

Evaluate over the interval and simplify.

$$A(S) = 2(2\pi) - 2(0)$$

$$A(S) = 4\pi$$

This is the area of the surface.

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Topic: Finding surface area

Question: Find surface area and give your answer as a decimal.

The surface $z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$ on $0 \le x \le 1$, $0 \le y \le 1$

Answer choices:

A $A(S) \approx 1.4087$

B $A(S) \approx 1.4066$

C $A(S) \approx 1.4213$

D $A(S) \approx 1.4136$

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Solution: B

To find the area of some surface that's bounded by another function, we'll need to take the partial derivatives of the equation of the surface.

$$z = \frac{2}{3}\left(x^{\frac{3}{2}} + y^{\frac{3}{2}}\right)$$

$$\frac{\partial z}{\partial x} = \frac{2}{3}\left(\frac{3}{2}x^{\frac{1}{2}}\right)$$

$$\frac{\partial z}{\partial x} = \sqrt{x}$$

and

$$\frac{\partial z}{\partial y} = \frac{2}{3}\left(\frac{3}{2}y^{\frac{1}{2}}\right)$$

$$\frac{\partial z}{\partial y} = \sqrt{y}$$

Since we're dealing with a double integral, we have to decide whether the region we're looking at is a type 1 or type 2 region. For the area we're looking at, we can take uniform slices horizontally or vertically, so we can treat the region as either type 1 or type 2, but we'll go ahead and treat it as a type 2 region. Therefore, we'll integrate first with respect to x, and second with respect to y.

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \ dA$$

$$A(S)=\iint_D \sqrt{1+\left(\sqrt{x}\right)^2+\left(\sqrt{y}\right)^2}\ dx\ dy$$

$$A(S)=\iint_D \sqrt{1+x+y}\ dx\ dy$$

Applying the interval we were given, we get

$$A(S)=\int_0^1\int_0^1 \sqrt{1+x+y}\ dx\ dy$$

Integrate with respect to x, then evaluate over $[0,1]$.

$$A(S)=\int_0^1 \frac{2}{3}(1+x+y)^{\frac{3}{2}}\Big|_{x=0}^{x=1}\ dy$$

$$A(S)=\int_0^1 \frac{2}{3}(1+1+y)^{\frac{3}{2}}-\frac{2}{3}(1+0+y)^{\frac{3}{2}}\ dy$$

$$A(S)=\frac{2}{3}\int_0^1 (2+y)^{\frac{3}{2}}-(1+y)^{\frac{3}{2}}\ dy$$

Integrate with respect to y, then evaluate over $[0,1]$.

$$A(S)=\frac{2}{3}\left[\frac{2}{5}(2+y)^{\frac{5}{2}}-\frac{2}{5}(1+y)^{\frac{5}{2}}\right]\Bigg|_0^1$$

$$A(S)=\frac{4}{15}\left[(2+y)^{\frac{5}{2}}-(1+y)^{\frac{5}{2}}\right]\Bigg|_0^1$$

$$A(S)=\frac{4}{15}\left[(2+1)^{\frac{5}{2}}-(1+1)^{\frac{5}{2}}\right]-\frac{4}{15}\left[(2+0)^{\frac{5}{2}}-(1+0)^{\frac{5}{2}}\right]$$

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$$A(S)=\frac{4}{15}\left(3^{\frac{5}{2}}-2^{\frac{5}{2}}\right)-\frac{4}{15}\left(2^{\frac{5}{2}}-1^{\frac{5}{2}}\right)$$

$$A(S)=\frac{4}{15}\left(3^{\frac{5}{2}}-2^{\frac{5}{2}}-2^{\frac{5}{2}}+1^{\frac{5}{2}}\right)$$

$$A(S)=\frac{4}{15}\left(\sqrt{3^5}-\sqrt{2^5}-\sqrt{2^5}+\sqrt{1^5}\right)$$

$$A(S)=\frac{4}{15}\left(\sqrt{3\cdot 3\cdot 3\cdot 3\cdot 3}-\sqrt{32}-\sqrt{32}+\sqrt{1}\right)$$

$$A(S)=\frac{4}{15}\left(9\sqrt{3}-4\sqrt{2}-4\sqrt{2}+1\right)$$

$$A(S)=\frac{4}{15}\left(9\sqrt{3}-8\sqrt{2}+1\right)$$

$$A(S)\approx 1.4066$$

This is the area of the surface.

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Topic: Finding volume

Question: Find the volume of the solid in the first quadrant that's bounded by the given lines and planes.

The three coordinate planes

$3x^2 - y^2 + z = 10$

$x = 3$ and $y = 3$

Answer choices:

A 36

B 39

C 42

D 44

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Solution: A

The given solid is positioned in the first quadrant above the square $R = [0,3] \times [0,3]$ and under the surface defined by $3x^2 - y^2 + z = 10$. If we plug all these things into a double integral to find the volume, we get

$$V = \iint_R 10 - 3x^2 + y^2 \ dA$$

$$V = \int_0^3 \int_0^3 10 - 3x^2 + y^2 \ dy \ dx$$

Integrate first with respect to y, and then evaluate over the interval.

$$V = \int_0^3 10y - 3x^2y + \frac{1}{3}y^3 \Big|_{y=0}^{y=3} \ dx$$

$$V = \int_0^3 10(3) - 3x^2(3) + \frac{1}{3}(3)^3 - \left(10(0) - 3x^2(0) + \frac{1}{3}(0)^3\right) \ dx$$

$$V = \int_0^3 30 - 9x^2 + 9 \ dx$$

$$V = \int_0^3 39 - 9x^2 \ dx$$

Integrate with respect to x, and then evaluate over the interval.

$$V = 39x - 3x^3 \Big|_0^3$$

$$V = 39(3) - 3(3)^3 - \left(39(0) - 3(0)^3\right)$$

$$V = 117 - 81$$

$$V = 36$$

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Topic: Finding volume

Question: A solid space is bounded by the paraboloid $z = -4x^2 - y^2 + 16$ and the xy-plane. If the volume of the solid is defined by the single integral, then what are the bounds for y if a double integral is used to calculate the volume of the solid?

$$V = \int_{-2}^{2} \frac{4}{3}\left(\sqrt{16 - 4x^2}\right)^3 \ dx$$

Answer choices:

A $\quad -\sqrt{16 - 4x^2} \le y \le \sqrt{16 - 4x^2}$

B $\quad -\sqrt{1 - 4x^2} \le y \le \sqrt{1 - 4x^2}$

C $\quad -\sqrt{4 - 4x^2} \le y \le \sqrt{16 - 4x^2}$

D $\quad -\sqrt{16 - 4x^2} \le y \le \sqrt{4 - x^2}$

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Solution: A

Because the solid is bounded by the xy-plane (which is where $z=0$), and the paraboloid $z=-4x^2-y^2+16$, we can say that those surfaces meet each other at

$$0=-4x^2-y^2+16$$

$$y^2=16-4x^2$$

$$y=\pm\sqrt{16-4x^2}$$

Therefore

$$-\sqrt{16-4x^2}\le y\le\sqrt{16-4x^2}$$

Let's check ourselves. We already know from the given integral that the bounds for x are given as $x=[-2,2]$. Plugging these bounds and the paraboloid which bounds the volume into a double integral would give

$$V=\int_{-2}^{2}\int_{-\sqrt{16-4x^2}}^{\sqrt{16-4x^2}}-4x^2-y^2+16\ dy\ dx$$

If we integrated with respect to y, we'd get

$$V=\int_{-2}^{2}-4x^2y-\frac{1}{3}y^3+16y\Big|_{y=-\sqrt{16-4x^2}}^{y=\sqrt{16-4x^2}}\ dx$$

$$V=\int_{-2}^{2}-4x^2\sqrt{16-4x^2}-\frac{1}{3}\left(\sqrt{16-4x^2}\right)^3+16\sqrt{16-4x^2}-\left[-4x^2\left(-\sqrt{16-4x^2}\right)-\frac{1}{3}\left(-\sqrt{16-4x^2}\right)^3+16\left(-\sqrt{16-4x^2}\right)\right]\ dx$$

$$V=\int_{-2}^{2}-4x^2\sqrt{16-4x^2}-\frac{1}{3}\left(\sqrt{16-4x^2}\right)^3+16\sqrt{16-4x^2}-4x^2\sqrt{16-4x^2}-\frac{1}{3}\left(\sqrt{16-4x^2}\right)^3+16\sqrt{16-4x^2}\ dx$$

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$$V = \int_{-2}^{2} -8x^2\sqrt{16-4x^2} - \frac{2}{3}\left(\sqrt{16-4x^2}\right)^3 + 32\sqrt{16-4x^2}\ dx$$

$$V = \int_{-2}^{2} (32-8x^2)\sqrt{16-4x^2} - \frac{2}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

$$V = \int_{-2}^{2} 2(16-4x^2)\sqrt{16-4x^2} - \frac{2}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

$$V = \int_{-2}^{2} 2\left(\sqrt{16-4x^2}\right)^3 - \frac{2}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

$$V = \int_{-2}^{2} \frac{4}{3}\left(\sqrt{16-4x^2}\right)^3\ dx$$

Because we got back to the integral we were given, this proves that the bounds we found for y, $-\sqrt{16-4x^2} \le y \le \sqrt{16-4x^2}$, are correct.

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Topic: Finding volume

Question: A solid is positioned above the square $R = [0,3] \times [0,3]$ and below the surface S, and is bounded by the three coordinate planes. If the volume of S is 36, then which function defines the surface S?

Answer choices:

A $\quad 3x^2 - y^2 + z = 10$

B $\quad x^2 - 3y^2 + z = 10$

C $\quad x^2 - y^2 - z = 10$

D $\quad 3x^2 + y^2 + z = 10$

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Solution: A

If we start with answer choice A, we want to solve it for z.

$$3x^2 - y^2 + z = 10$$

$$z = 10 - 3x^2 + y^2$$

Then we can plug this and everything else we were given into a double integral.

$$V = \int\int_R 10 - 3x^2 + y^2 \ dA$$

$$V = \int_0^3 \int_0^3 10 - 3x^2 + y^2 \ dy \ dx$$

Integrate first with respect to y, then evaluate over the interval.

$$V = \int_0^3 10y - 3x^2y + \frac{1}{3}y^3 \Big|_{y=0}^{y=3} \ dx$$

$$V = \int_0^3 10(3) - 3x^2(3) + \frac{1}{3}(3)^3 - \left(10(0) - 3x^2(0) + \frac{1}{3}(0)^3\right) \ dx$$

$$V = \int_0^3 30 - 9x^2 + 9 \ dx$$

$$V = \int_0^3 39 - 9x^2 \ dx$$

Integrate with respect to x, then evaluate over the interval.

$$V = 39x - 3x^3 \Big|_0^3$$

$$V = 39(3) - 3(3)^3 - \left(39(0) - 3(0)^3\right)$$

$$V = 117 - 81$$

$$V = 36$$

Because we were told in the problem that the volume of the solid S was 36, and we got 36 here, we know that the equation given in answer choice A is the correct one.

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Topic: Changing the order of integration

Question: Change the order of integration for the iterated integral.

$$\int_1^2 \int_1^{3-y} x^2 \, dx \, dy$$

Answer choices:

A $\int_1^2 \int_1^{3-x} x^2 \, dy \, dx$

B $\int_1^{3+x} \int_1^2 x^2 \, dy \, dx$

C $\int_1^2 \int_1^{3+x} x^2 \, dy \, dx$

D $\int_1^{3-x} \int_1^2 x^2 \, dy \, dx$

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Solution: A

To change the order of integration of a double integral, we cannot simply reverse the two integrals. The outer integral must have limits that are constants.

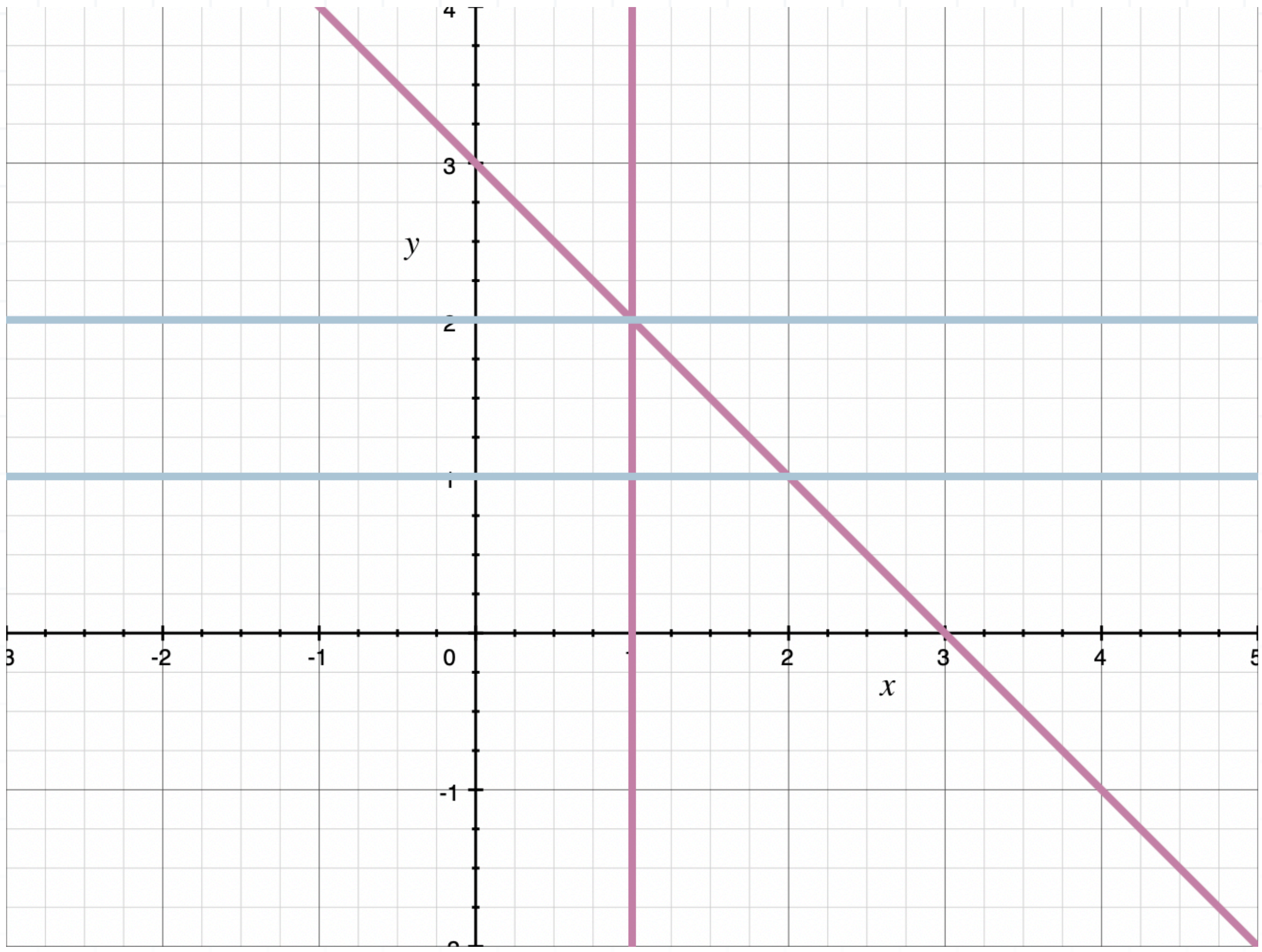
The integral we've been given has $dx\ dy$ on the end of it, so dx is on the inside and dy is on the outside. Which means we've been told to integrate first with respect to x, and then with respect to y. And since we've been asked to switch the order of integration, it means we'll need to change the iterated integral to $dy\ dx$, where we integrate first with respect to y, then x.

Currently, the limits of integration are

$$1 \le x \le 3 - y$$

$$1 \le y \le 2$$

Ideally, we want to sketch the limits. Our sketch will help us confirm what the limits should be when we try to switch them.



To change the order of integration, we want to integrate first with respect to y. That means we need to put our limits of integration for y in terms of x.

The top of the region can be defined by the slanted line $x = 3 - y$. If we solve that for y in terms of x, we get

$$x + y = 3$$

$$y = 3 - x$$

And the lower bound for y can be given by $y = 1$. So the new limits of integration for y are

$$1 \le y \le 3 - x$$

The largest value for which the region is defined for x is $x = 2$, and the lower bound for x is $x = 1$. So the limits of integration for x are

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$$1 \le x \le 2$$

So when we switch the order of integration, we get

$$\int_1^2 \int_1^{3-y} x^2 \ dx \ dy = \int_1^2 \int_1^{3-x} x^2 \ dy \ dx$$

Topic: Changing the order of integration

Question: Change the order of integration for the iterated integral.

$$\int_{-2}^{4}\int_{1}^{x+3} 4x^2y\ dy\ dx$$

Answer choices:

A $\displaystyle\int_{1}^{7}\int_{y+3}^{4} 4x^2y\ dx\ dy$

B $\displaystyle\int_{1}^{7}\int_{4}^{y+3} 4x^2y\ dx\ dy$

C $\displaystyle\int_{1}^{7}\int_{y-3}^{4} 4x^2y\ dx\ dy$

D $\displaystyle\int_{1}^{7}\int_{4}^{y-3} 4x^2y\ dx\ dy$

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Solution: C

To change the order of integration of a double integral, we cannot simply reverse the two integrals. The outer integral must have limits that are constants.

The integral we've been given has $dy\ dx$ on the end of it, so dy is on the inside and dx is on the outside. Which means we've been told to integrate first with respect to y, and then with respect to x. And since we've been asked to switch the order of integration, it means we'll need to change the iterated integral to $dx\ dy$, where we integrate first with respect to x, then y.

Currently, the limits of integration are

$$1 \le y \le x + 3$$

$$-2 \le x \le 4$$

Ideally, we want to sketch the limits. Our sketch will help us confirm what the limits should be when we try to switch them.

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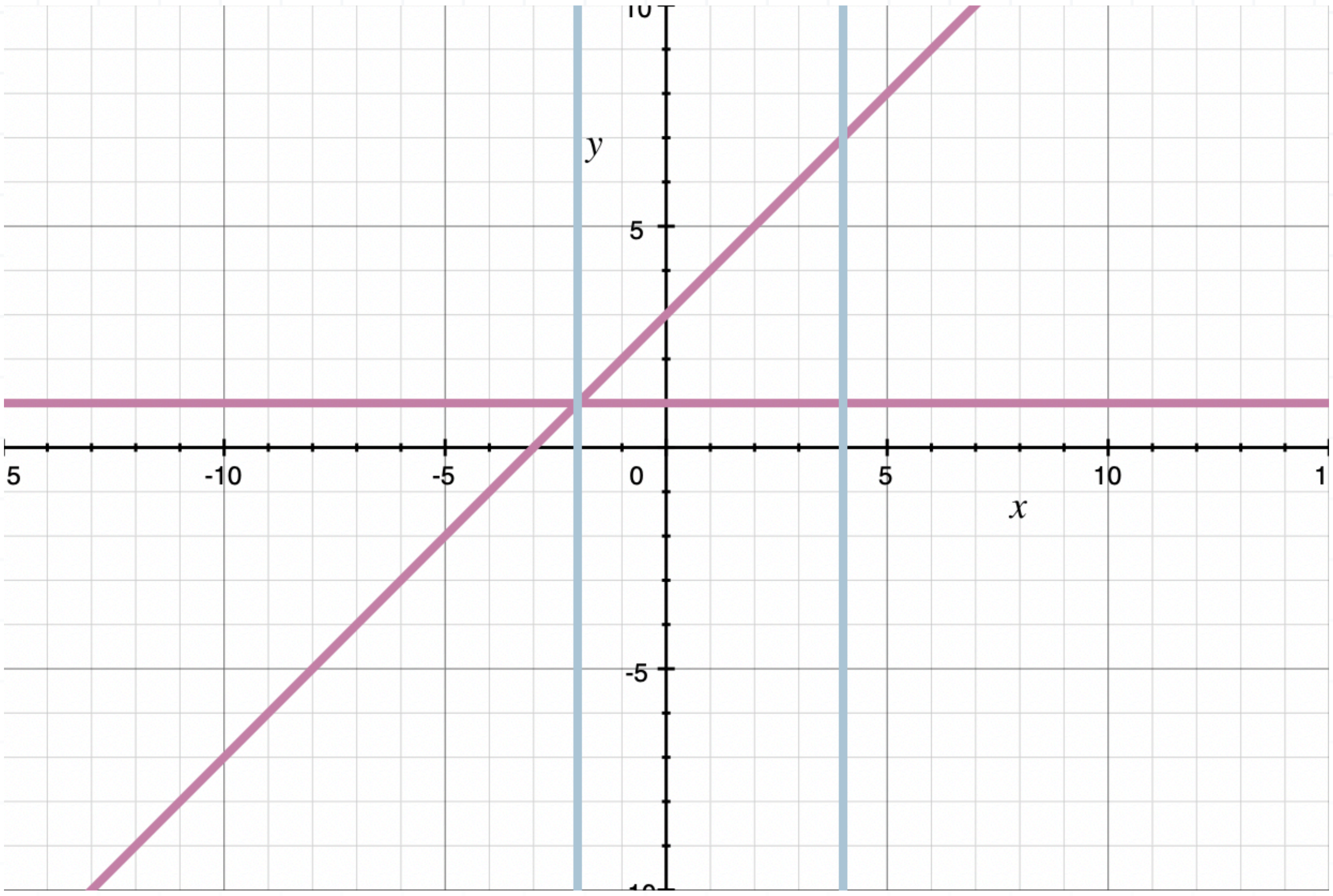

To change the order of integration, we want to integrate first with respect to x. That means we need to put our limits of integration for x in terms of y.

The right side of the region can be defined by the line $x = 4$. And the left side of the region is given by the slanted line $y = x + 3$. If we solve that for x, we get

$$x = y - 3$$

So the new limits of integration for x are

$$y - 3 \leq x \leq 4$$

The largest value for which the region is defined for y is $y = 7$, and the lower bound for y is $y = 1$. So the limits of integration for y are

$$1 \leq y \leq 7$$

So when we switch the order of integration, we get

$$\int_{-2}^{4}\int_{1}^{x+3} 4x^2y\ dy\ dx = \int_{1}^{7}\int_{y-3}^{4} 4x^2y\ dx\ dy$$

Topic: Changing the order of integration

Question: Change the order of integration for the iterated integral.

$$\int_1^3 \int_1^{-y+4} 3e^{xy} \ dx \ dy$$

Answer choices:

A $\quad \displaystyle\int_1^3 \int_1^{-x-4} 3e^{xy} \ dy \ dx$

B $\quad \displaystyle\int_1^3 \int_1^{x+4} 3e^{xy} \ dy \ dx$

C $\quad \displaystyle\int_1^3 \int_1^{x-4} 3e^{xy} \ dy \ dx$

D $\quad \displaystyle\int_1^3 \int_1^{4-x} 3e^{xy} \ dy \ dx$

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Solution: D

To change the order of integration of a double integral, we cannot simply reverse the two integrals. The outer integral must have limits that are constants.

The integral we've been given has $dx\ dy$ on the end of it, so dx is on the inside and dy is on the outside. Which means we've been told to integrate first with respect to x, and then with respect to y. And since we've been asked to switch the order of integration, it means we'll need to change the iterated integral to $dy\ dx$, where we integrate first with respect to y, then x.

Currently, the limits of integration are

$$1 \leq x \leq -y + 4$$

$$1 \leq y \leq 3$$

Ideally, we want to sketch the limits. Our sketch will help us confirm what the limits should be when we try to switch them.

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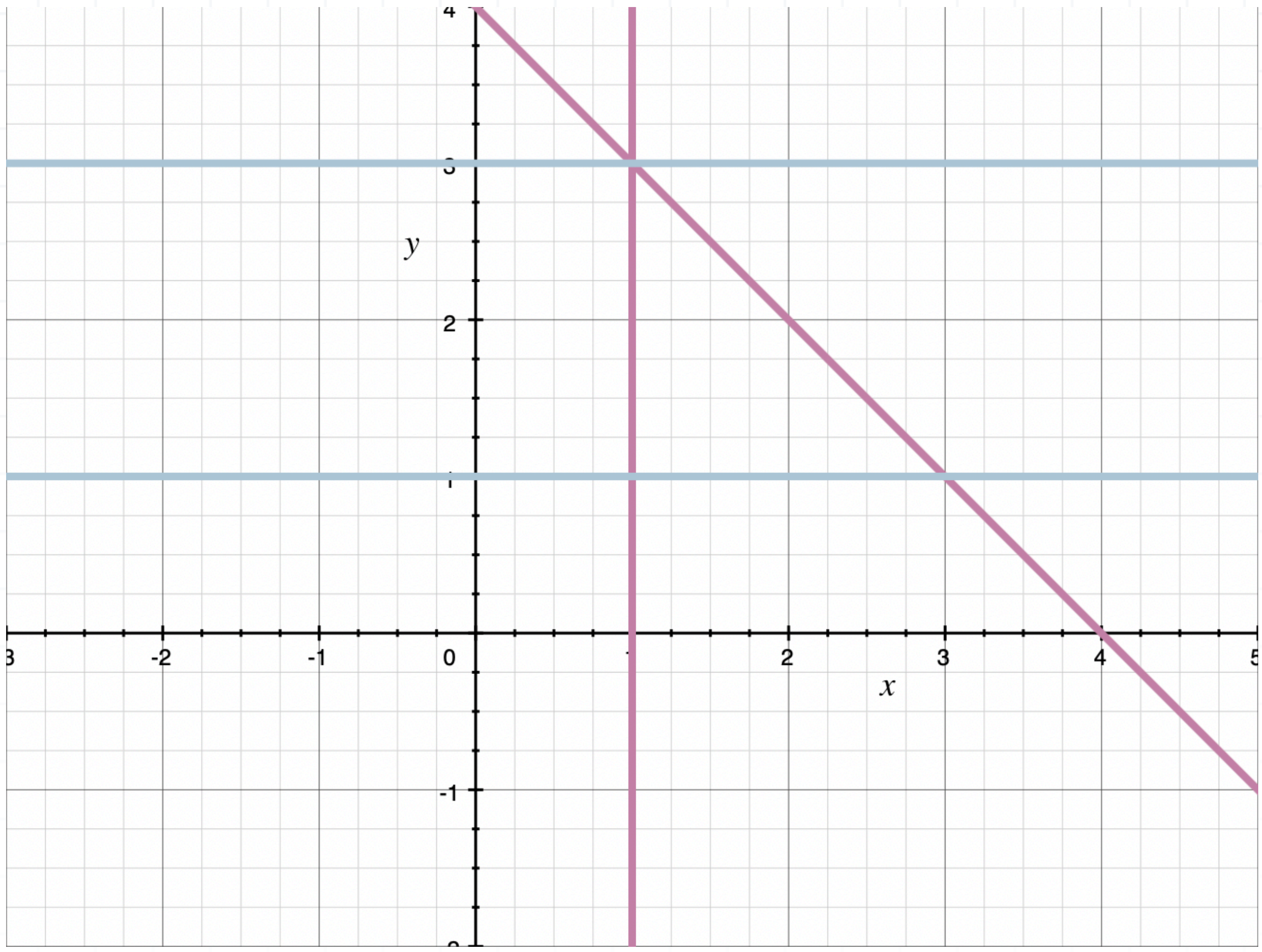

To change the order of integration, we want to integrate first with respect to y. That means we need to put our limits of integration for y in terms of x.

The top of the region can be defined by the slanted line $x = -y + 4$. If we solve that for y in terms of x, we get

$$x + y = 4$$

$$y = 4 - x$$

And the lower bound for y can be given by $y = 1$. So the new limits of integration for y are

$$1 \leq y \leq 4 - x$$

The largest value for which the region is defined for x is $x = 3$, and the lower bound for x is $x = 1$. So the limits of integration for x are

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$$1 \le x \le 3$$

So when we switch the order of integration, we get

$$\int_1^3 \int_1^{-y+4} 3e^{xy}\ dx\ dy = \int_1^3 \int_1^{-x+4} 3e^{xy}\ dy\ dx$$

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Topic: Changing iterated integrals to polar coordinates

Question: Convert the iterated integral to polar coordinates.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2\ dy\ dx$$

Answer choices:

A $\int_{0}^{2\pi}\int_{-1}^{1} r^2\ dr\ d\theta$

B $\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta$

C $\int_{0}^{2\pi}\int_{-1}^{1} r^3\ dr\ d\theta$

D $\int_{0}^{2\pi}\int_{0}^{1} r^2\ dr\ d\theta$

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Solution: B

When we convert from rectangular coordinates to polar coordinates, we use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

We were given the double integral

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2\ dy\ dx$$

We'll convert the integrand first, leaving the limits of integration.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2\ dy\ dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} r^2 r\ dr\ d\theta$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2\ dy\ dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} r^3\ dr\ d\theta$$

Next we need to convert the limits of integration. We know that the limits of integration with respect to y are $y = \pm\sqrt{1-x^2}$. We can rewrite those as

$$y = \pm\sqrt{1-x^2}$$

$$y^2 = 1 - x^2$$

$$x^2 + y^2 = 1$$

Since we can now see that we're talking about the circle with radius 1, and since in polar coordinates r represents radius, the bounds for r have to be $[0,1]$. Similarly, because we're dealing with the entire circle, the limits of integration for the angle θ have to be $[0,2\pi]$. Therefore, the converted integral is

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2 \ dy \ dx = \int_{0}^{2\pi} \int_{0}^{1} r^3 \ dr \ d\theta$$

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Topic: Changing iterated integrals to polar coordinates

Question: Convert the iterated integral to polar coordinates.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)\ dy\ dx$$

Answer choices:

A $\displaystyle\int_0^{2\pi}\int_0^2 \sin r^2\ dr\ d\theta$

B $\displaystyle\int_0^{2\pi}\int_0^2 r\cos r^2\ dr\ d\theta$

C $\displaystyle\int_0^{2\pi}\int_0^2 \cos r^2\ dr\ d\theta$

D $\displaystyle\int_0^{2\pi}\int_0^2 r\sin r^2\ dr\ d\theta$

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Solution: D

When we convert from rectangular coordinates to polar coordinates, we use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

We were given the double integral

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)\ dy\ dx$$

We'll convert the integrand first, leaving the limits of integration.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)\ dy\ dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} r\sin r^2\ dr\ d\theta$$

Next we need to convert the limits of integration. We know that the limits of integration with respect to y are $y = \pm\sqrt{4-x^2}$. We can rewrite those as

$$y = \pm\sqrt{4-x^2}$$

$$y^2 = 4 - x^2$$

$$x^2 + y^2 = 4$$

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Since we can now see that we're talking about the circle with radius 2, and since in polar coordinates r represents radius, the bounds for r have to be $[0,2]$. Similarly, because we're dealing with the entire circle, the limits of integration for the angle θ have to be $[0,2\pi]$. Therefore, the converted integral is

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)\ dy\ dx = \int_{0}^{2\pi}\int_{0}^{2} r\sin r^2\ dr\ d\theta$$

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Topic: Changing iterated integrals to polar coordinates

Question: Convert the iterated integral to polar coordinates.

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \ln\left(x^2+y^2\right)\ dy\ dx$$

Answer choices:

A $\quad \int_{0}^{2\pi}\int_{0}^{4} e^{r^2}\ dr\ d\theta$

B $\quad \int_{0}^{2\pi}\int_{0}^{4} \ln r^2\ dr\ d\theta$

C $\quad \int_{0}^{2\pi}\int_{0}^{4} r\ln r^2\ dr\ d\theta$

D $\quad \int_{0}^{2\pi}\int_{0}^{4} re^{r^2}\ dr\ d\theta$

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Solution: C

When we convert from rectangular coordinates to polar coordinates, we use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

We were given the double integral

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \ln\left(x^2+y^2\right)\ dy\ dx$$

We'll convert the integrand first, leaving the limits of integration.

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \ln\left(x^2+y^2\right)\ dy\ dx = \int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} r\ln r^2\ dr\ d\theta$$

Next we need to convert the limits of integration. We know that the limits of integration with respect to y are $y = \pm\sqrt{16-x^2}$. We can rewrite those as

$$y = \pm\sqrt{16-x^2}$$

$$y^2 = 16 - x^2$$

$$x^2 + y^2 = 16$$

Since we can now see that we're talking about the circle with radius 4, and since in polar coordinates r represents radius, the bounds for r have to be $[0,4]$. Similarly, because we're dealing with the entire circle, the limits of integration for the angle θ have to be $[0,2\pi]$. Therefore, the converted integral is

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \ln\left(x^2+y^2\right)\ dy\ dx = \int_0^{2\pi}\int_0^4 r\ln r^2\ dr\ d\theta$$

Topic: Changing double integrals to polar coordinates

Question: Convert the double integral to polar coordinates.

$$\iint_D x^2 + y^2 \ dA$$

D is bounded by $y = \pm\sqrt{1 - x^2}$

Answer choices:

A $\int_0^{2\pi} \int_0^1 r^3 \ dr \ d\theta$

B $\int_0^{2\pi} \int_0^1 r^2 \ dr \ d\theta$

C $\int_0^{2\pi} \int_{-1}^1 r^2 \ dr \ d\theta$

D $\int_0^{2\pi} \int_{-1}^1 r^3 \ dr \ d\theta$

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Solution: A

When we convert from rectangular coordinates to polar coordinates, we use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

The region D is bounded by $y = \pm\sqrt{1 - x^2}$, which can be rewritten as

$$y = \pm\sqrt{1 - x^2}$$

$$y^2 = 1 - x^2$$

$$x^2 + y^2 = 1$$

So the region D is the circle centered at the origin with radius 1. Which means we can evaluate it as a type I region, and that we'll integrate first with respect to y and then with respect to x. The limits of integration for y will be what we've already been given, and the limits of integration for x will simply be $[-1,1]$.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2\ dy\ dx$$

If we use the conversion formulas above to convert the integrand, but we leave the limits of integration alone, we get

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$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2\ dy\ dx=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} r^2r\ dr\ d\theta$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2\ dy\ dx=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} r^3\ dr\ d\theta$$

Next we need to convert the limits of integration. Since we're talking about the circle with radius 1, and since in polar coordinates r represents radius, the bounds for r have to be $[0,1]$. Similarly, because we're dealing with the entire circle, the limits of integration for the angle θ have to be $[0,2\pi]$. Therefore, the converted integral is

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2dydx=\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta$$

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Topic: Changing double integrals to polar coordinates

Question: Convert the double integral to polar coordinates.

$$\int\int_D e^{x^2+y^2}\ dA$$

where D is bounded by $y = \pm\sqrt{25 - x^2}$

Answer choices:

A $\int_0^{2\pi}\int_0^{25} e^{r^2}\ dr\ d\theta$

B $\int_0^{2\pi}\int_0^{5} e^{r^2}\ dr\ d\theta$

C $\int_0^{2\pi}\int_0^{25} re^{r^2}\ dr\ d\theta$

D $\int_0^{2\pi}\int_0^{5} re^{r^2}\ dr\ d\theta$

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Solution: D

When we convert from rectangular coordinates to polar coordinates, we use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

The region D is bounded by $y = \pm\sqrt{25 - x^2}$, which can be rewritten as

$$y = \pm\sqrt{25 - x^2}$$

$$y^2 = 25 - x^2$$

$$x^2 + y^2 = 25$$

So the region D is the circle centered at the origin with radius 5. Which means we can evaluate it as a type I region, and that we'll integrate first with respect to y and then with respect to x. The limits of integration for y will be what we've already been given, and the limits of integration for x will simply be $[-5,5]$.

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} e^{x^2+y^2}\ dy\ dx$$

If we use the conversion formulas above to convert the integrand, but we leave the limits of integration alone, we get

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$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} e^{x^2+y^2}\ dy\ dx = \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} e^{r^2}r\ dr\ d\theta$$

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} e^{x^2+y^2}\ dy\ dx = \int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} re^{r^2}\ dr\ d\theta$$

Next we need to convert the limits of integration. Since we're talking about the circle with radius 5, and since in polar coordinates r represents radius, the bounds for r have to be $[0,5]$. Similarly, because we're dealing with the entire circle, the limits of integration for the angle θ have to be $[0,2\pi]$. Therefore, the converted integral is

$$\int_{-5}^{5}\int_{-\sqrt{25-x^2}}^{\sqrt{25-x^2}} e^{x^2+y^2}\ dy\ dx = \int_{0}^{2\pi}\int_{0}^{5} re^{r^2}\ dr\ d\theta$$

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Topic: Changing double integrals to polar coordinates

Question: Convert the double integral to polar coordinates.

$$\iint_D \sin\left(x^2+y^2\right)\,dA$$

where D is bounded by $y=\pm\sqrt{4-x^2}$

Answer choices:

A $\int_0^{2\pi}\int_0^4 r\sin r^2\,dr\,d\theta$

B $\int_0^{2\pi}\int_0^2 \sin r^2\,dr\,d\theta$

C $\int_0^{2\pi}\int_0^2 r\sin r^2\,dr\,d\theta$

D $\int_0^{2\pi}\int_0^4 \sin r^2\,dr\,d\theta$

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Solution: C

When we convert from rectangular coordinates to polar coordinates, we use the following conversion formulas.

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

$$dy\ dx = r\ dr\ d\theta$$

The region D is bounded by $y = \pm\sqrt{4 - x^2}$, which can be rewritten as

$$y = \pm\sqrt{4 - x^2}$$

$$y^2 = 4 - x^2$$

$$x^2 + y^2 = 4$$

So the region D is the circle centered at the origin with radius 2. Which means we can evaluate it as a type I region, and that we'll integrate first with respect to y and then with respect to x. The limits of integration for y will be what we've already been given, and the limits of integration for x will simply be $[-2,2]$.

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sin\left(x^2 + y^2\right)\ dy\ dx$$

If we use the conversion formulas above to convert the integrand, but we leave the limits of integration alone, we get

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$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)\ dy\ dx = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} r\sin r^2\ dr\ d\theta$$

Next we need to convert the limits of integration. Since we're talking about the circle with radius 2, and since in polar coordinates r represents radius, the bounds for r have to be $[0,2]$. Similarly, because we're dealing with the entire circle, the limits of integration for the angle θ have to be $[0,2\pi]$. Therefore, the converted integral is

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sin\left(x^2+y^2\right)\ dy\ dx = \int_{0}^{2\pi}\int_{0}^{2} r\sin r^2\ dr\ d\theta$$

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Topic: Sketching area

Question: The area between two polar curves is given by the double integral. Which of the following describes the area given by the double integral?

$$A = \int_{\pi}^{2\pi} \int_{1}^{1-\sin\theta} r\ dr\ d\theta$$

Answer choices:

A The intersection of the unit circle and the cardioid $r = 1 - \sin\theta$

B The union of the unit circle and the cardioid $r = 1 - \sin\theta$

C The region inside the unit circle and outside the cardioid $r = 1 - \sin\theta$

D The region outside the unit circle and inside the cardioid $r = 1 - \sin\theta$

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Solution: D

Because we can see from the double integral that θ is defined on $\pi \leq \theta \leq 2\pi$, then, and the angles between π and 2π all lie below the x-axis where $y = 0$, and since $\sin\theta$ always represents the y-value, we know that

$$\sin\theta < 0$$

$$-\sin\theta > 0$$

$$1 - \sin\theta > 1$$

The graph of $r = 1 - \sin\theta$ is a cardioid and the graph of $r = 1$ is the unit circle (the circle centered at the origin with radius 1). With $1 - \sin\theta > 1$, we know that the cardioid lies outside of the unit circle.

Therefore, since in the given double integral r is given on the interval $r = 1$ to $r = 1 - \sin\theta$, we can say that the area represented by the double integral is the region outside the unit circle but inside the cardioid.

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Topic: Sketching area

Question: What are the boundaries of the region given in the double integral?

$$S = \int_{0}^{\frac{\pi}{3}} \int_{\frac{3}{2\cos\theta}}^{3} 2r \ dr \ d\theta$$

Answer choices:

A The intersection of the area of a circle with radius 3 and the region to the right of $x = 3/2$.

B The intersection of the area of a circle with radius 4 and the region to the left of $x = 2/3$.

C The intersection of the area of a circle with radius 3 and the area above $y = 3/2$.

D The intersection of the area of a circle with radius 4 and the area below $y = 2/3$.

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Solution: A

The boundaries of the integral

$$S = \int_{\frac{3}{2\cos\theta}}^{3} 2r\ dr$$

indicate that the area is part of a circle with radius 3. Therefore, the equation of the circle is $x^2 + y^2 = 9$.

On the other hand, the lower boundary of the given integral indicates that the area is to the right of the vertical line $x = 3/2$, knowing that $r\cos\theta = 3/2$. Thus, the boundaries of the region are given by the intersection of the area of a circle with radius 3 and the region to the right of $x = 3/2$.

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Topic: Sketching area

Question: The area of the region A, which is part of an annulus, is defined by the double integral. What lines are the boundaries of the given region?

$$A = \int_0^{\frac{\pi}{3}} \tan^2\theta \ d\theta \int_a^{3a} r \ dr$$

Answer choices:

A The area between two circles for which one radius is twice the other, and the area is below the line $\theta = 2\pi/3$.

B The area between two circles for which one radius is twice the other, and the area is below the line $\theta = \pi$.

C The area between two circles for which one radius is three times the other, and the area is below the line $\theta = \pi/3$.

D The area between two circles for which one radius is three times the other, and the area is above the line $\theta = \pi/3$.

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Solution: C

The boundaries of the second integral from

$$A = \int_0^{\frac{\pi}{3}} \tan^2\theta \ d\theta \int_a^{3a} r \ dr$$

indicate that the region is part of an annulus formed by two circles for which one radius is three times the other. The boundaries of the first integral verify that the given segment of the annulus is limited to the line passing the origin and forming an angle of $\theta = \pi/3$.

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Topic: Finding area

Question: Find the area given by the double polar integral.

$$\int_{0}^{2\pi}\int_{0}^{2} r\ dr\ d\theta$$

Answer choices:

A $\frac{\pi}{4}$

B 2π

C 4π

D $\frac{\pi}{2}$

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Solution: C

To find area, we just need to evaluate the double integral. We always integrate from the inside out, which means we'll integrate first with respect to r.

$$\int_0^{2\pi}\int_0^2 r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2 r\ dr\ d\theta=\int_0^{2\pi}\frac{1}{2}r^2\Big|_0^2\ d\theta$$

$$\int_0^{2\pi}\int_0^2 r\ dr\ d\theta=\int_0^{2\pi}\frac{1}{2}(2)^2-\frac{1}{2}(0)^2\ d\theta$$

$$\int_0^{2\pi}\int_0^2 r\ dr\ d\theta=\int_0^{2\pi}2\ d\theta$$

Integrate with respect to θ.

$$\int_0^{2\pi}\int_0^2 r\ dr\ d\theta=2\theta\Big|_0^{2\pi}$$

$$\int_0^{2\pi}\int_0^2 r\ dr\ d\theta=2(2\pi)-2(0)$$

$$\int_0^{2\pi}\int_0^2 r\ dr\ d\theta=4\pi$$

This is the area given by the double integral.

Topic: Finding area

Question: Find the area given by the double polar integral.

$$\int_0^{\frac{\pi}{2}} \int_0^4 e^r \ dr \ d\theta$$

Answer choices:

A $\quad \dfrac{\pi}{2}(e^4 - 1)$

B $\quad e^3\pi$

C $\quad e^4\pi$

D $\quad \dfrac{\pi}{2}(e^3 - 1)$

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Solution: A

To find area, we just need to evaluate the double integral. We always integrate from the inside out, which means we'll integrate first with respect to r.

$$\int_0^{\frac{\pi}{2}}\int_0^4 e^r\ dr\ d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^4 e^r\ dr\ d\theta = \int_0^{\frac{\pi}{2}} e^r\Big|_0^4\ d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^4 e^r\ dr\ d\theta = \int_0^{\frac{\pi}{2}} e^4 - e^0\ d\theta$$

$$\int_0^{\frac{\pi}{2}}\int_0^4 e^r\ dr\ d\theta = \int_0^{\frac{\pi}{2}} e^4 - 1\ d\theta$$

Integrate with respect to θ.

$$\int_0^{\frac{\pi}{2}}\int_0^4 e^r\ dr\ d\theta = e^4\theta - \theta\Big|_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}}\int_0^4 e^r\ dr\ d\theta = e^4\left(\frac{\pi}{2}\right) - \frac{\pi}{2} - \left(e^4(0) - (0)\right)$$

$$\int_0^{\frac{\pi}{2}}\int_0^4 e^r\ dr\ d\theta = \frac{\pi}{2}(e^4 - 1)$$

This is the area given by the double integral.

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Topic: Finding area

Question: Find the area of the region.

The region D where D is bounded by $y = \pm\sqrt{1 - x^2}$

$$\iint_D x^2 + y^2 \ dA$$

Answer choices:

A 2π

B $\frac{\pi}{4}$

C π

D $\frac{\pi}{2}$

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Solution: D

We need to turn the given integral

$$\iint_D x^2 + y^2 \ dA$$

into an iterated integral.

We've been told that the region D is bounded by $y = \pm\sqrt{1 - x^2}$. If we rearrange D, we get

$$y = \pm\sqrt{1 - x^2}$$

$$y^2 = 1 - x^2$$

$$x^2 + y^2 = 1$$

This is the circle centered at the origin with radius 1, which means we can define it as a type I region. For a type I region, we'll integrate first with respect to y and then with respect to x. Therefore, the integral will be

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2 \ dy \ dx$$

This integral will be easier to handle in polar form, so we'll convert it, remembering that$r^2 = x^2 + y^2$ and that $dy \ dx = r \ dr \ d\theta$. We'll need to remember to change the limits of integration as well. We know that the region D is everything inside the circle with radius 1, so the bounds for r become $[0,1]$. Since we're dealing with the entire circle, the bounds for θ will be $[0,2\pi]$. The integral becomes

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2\ dy\ dx=\int_{0}^{2\pi}\int_{0}^{1} r^2(r)\ dr\ d\theta$$

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2+y^2\ dy\ dx=\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta$$

Now we'll integrate with respect to r.

$$\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta=\int_{0}^{2\pi} \frac{1}{4}r^4\Big|_0^1\ d\theta$$

$$\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta=\int_{0}^{2\pi} \frac{1}{4}(1)^4-\frac{1}{4}(0)^4\ d\theta$$

$$\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta=\int_{0}^{2\pi} \frac{1}{4}\ d\theta$$

Integrate with respect to θ.

$$\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta=\frac{1}{4}\theta\Big|_0^{2\pi}$$

$$\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta=\frac{1}{4}(2\pi)-\frac{1}{4}(0)$$

$$\int_{0}^{2\pi}\int_{0}^{1} r^3\ dr\ d\theta=\frac{\pi}{2}$$

This is the area given by the double integral.

Topic: Finding volume

Question: Find the volume given by the double integral.

$$\int_0^{2\pi}\int_0^1 r^2\ dr\ d\theta$$

Answer choices:

A $2\pi^3$

B $\frac{2\pi}{3}$

C $\frac{8\pi}{3}$

D $\frac{8\pi^3}{3}$

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Solution: B

We've been given the double integral

$$\int_0^{2\pi}\int_0^1 r^2\ dr\ d\theta$$

in polar coordinates. All we need to do to find the volume it represents is evaluate the integral. We always work from the inside out, which means we'll integrate first with respect to r.

$$V=\int_0^{2\pi}\frac{1}{3}r^3\Big|_0^1\ d\theta$$

$$V=\int_0^{2\pi}\frac{1}{3}(1)^3-\frac{1}{3}(0)^3\ d\theta$$

$$V=\int_0^{2\pi}\frac{1}{3}\ d\theta$$

Integrate with respect to θ.

$$V=\frac{1}{3}\theta\Big|_0^{2\pi}$$

$$V=\frac{1}{3}(2\pi)-\frac{1}{3}(0)$$

$$V=\frac{2\pi}{3}$$

This is the volume given by the double integral.

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Topic: Finding volume

Question: Find the volume of the region.

The region under $-x^2 - y^2 + z = 1$, on $0 \le \theta \le 2\pi$ and $0 \le r \le 2$.

Answer choices:

A $12\pi^2$

B 12π

C 6π

D $6\pi^2$

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Solution: B

We'll rearrange the function we've been given so that it's solved for z.

$$-x^2 - y^2 + z = 1$$

$$z = 1 + x^2 + y^2$$

Since we've been given bounds for the region in terms of polar coordinates, we need to convert this function into polar coordinates, too. We know that $r^2 = x^2 + y^2$, so the function becomes

$$z = 1 + r^2$$

Now we've got everything we need to plug into the volume integral. We know that when we move from rectangular to polar coordinates, that $dy\ dx = r\ dr\ d\theta$. Therefore, the volume integral becomes

$$V = \int_0^{2\pi} \int_0^2 \left(1 + r^2\right) r\ dr\ d\theta$$

$$V = \int_0^{2\pi} \int_0^2 r + r^3\ dr\ d\theta$$

We'll work from the inside out, and integrate first with respect to r.

$$V = \int_0^{2\pi} \frac{1}{2}r^2 + \frac{1}{4}r^4 \Big|_0^2\ d\theta$$

$$V = \int_0^{2\pi} \frac{1}{2}(2)^2 + \frac{1}{4}(2)^4 - \left(\frac{1}{2}(0)^2 + \frac{1}{4}(0)^4\right)\ d\theta$$

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$$V=\int_0^{2\pi}2+4\ d\theta$$

$$V=\int_0^{2\pi}6\ d\theta$$

Integrate with respect to θ.

$$V=6\theta\Big|_0^{2\pi}$$

$$V=6(2\pi)-6(0)$$

$$V=12\pi$$

This is the volume given by the double integral.

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Topic: Finding volume

Question: Find the volume of the region that lies inside $z = x^2 + y^2$ and below the plane $z = 4$.

Answer choices:

A 4π

B 2π

C 8π

D 16π

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Solution: C

Because we're looking for the volume inside $z = x^2 + y^2$ and below $z = 4$, we'll use

$$V = \int\int_D 4 - \left(x^2 + y^2\right)\ dA$$

to represent the volume. We know that $r^2 = x^2 + y^2$, so we'll rewrite the integrand as

$$V = \int\int_D 4 - r^2\ dA$$

In polar coordinates, the bounds will be $0 \le \theta \le 2\pi$ and $0 \le r \le 2$. We also know that $dy\ dx = r\ dr\ d\theta$ whenever we move from rectangular to polar coordinates. Therefore, the volume integral becomes

$$V = \int_0^{2\pi} \int_0^2 \left(4 - r^2\right) r\ dr\ d\theta$$

$$V = \int_0^{2\pi} \int_0^2 4r - r^3\ dr\ d\theta$$

We always work from the inside out, so we'll integrate first with respect to r.

$$V = \int_0^{2\pi} 2r^2 - \frac{1}{4}r^4 \Big|_0^2\ d\theta$$

$$V = \int_0^{2\pi} 2(2)^2 - \frac{1}{4}(2)^4 - \left[2(0)^2 - \frac{1}{4}(0)^4\right]\ d\theta$$

$$V = \int_0^{2\pi} 2(4) - \frac{1}{4}(16)\ d\theta$$

$$V = \int_0^{2\pi} 4\ d\theta$$

Integrate with respect to θ.

$$V = 4\theta \Big|_0^{2\pi}$$

$$V = 4(2\pi) - 4(0)$$

$$V = 8\pi$$

This is the volume given by the double integral.

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Topic: Double integrals to find mass and center of mass

Question: The vertices of a triangle-shaped lamina are $(4,2)$, $(0,2)$, and $(0,0)$. The density of the lamina is defined by $\delta = 4x + 2y$. What is the mass of the lamina?

Answer choices:

A $\quad m = \dfrac{1{,}024}{3}$

B $\quad m = \dfrac{1{,}024}{5}$

C $\quad m = 512$

D $\quad m = 32$

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Solution: D

Of the given vertices $(4,2)$, $(0,2)$, and $(0,0)$, two of them are on the horizontal line $y=2$. And two of the vertices are on the line $x=0$. So $y=2$ and $x=0$ are two boundaries of the triangle.

The third boundary is a line that connects $(4,2)$ to $(0,0)$. The equation of this line is $y=(1/2)x$ or $x=2y$. Now we can plug the bounds and the density equation $\delta=4x+2y$ into the double integral.

$$m=\int_0^2\int_0^{2y}4x+2y\ dx\ dy$$

Integrate with respect to x, then evaluate over the interval.

$$m=\int_0^2 2x^2+2xy\Big|_{x=0}^{x=2y}\ dy$$

$$m=\int_0^2 2(2y)^2+2(2y)y-\left(2(0)^2+2(0)y\right)\ dy$$

$$m=\int_0^2 8y^2+4y^2\ dy$$

$$m=\int_0^2 12y^2\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$m=4y^3\Big|_0^2$$

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$$m = 4(2)^3 - 4(0)^3$$

$$m = 32$$

Topic: Double integrals to find mass and center of mass

Question: A region is bounded by the parabolas $y = 3x - x^2$ and $y = 2x^2 - 6x$. What is the center of the mass of the region?

Answer choices:

A $\left(\frac{3}{2}, -\frac{9}{10}\right)$

B $\left(\frac{3}{2}, \frac{9}{20}\right)$

C $\left(\frac{9}{7}, -\frac{9}{10}\right)$

D $\left(\frac{9}{2}, \frac{9}{2}\right)$

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Solution: A

The points of intersection of the parabolas

$$y = 3x - x^2$$

$$y = 2x^2 - 6x$$

are $(0,0)$, and $(3,0)$. Putting this information into a double integral gives

$$A = \int_R dA$$

$$A = \int_0^3 \int_{2x^2-6x}^{3x-x^2} dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$A = \int_0^3 y\Big|_{y=2x^2-6x}^{y=3x-x^2} dx$$

$$A = \int_0^3 3x - x^2 - \left(2x^2 - 6x\right)\ dx$$

$$A = \int_0^3 3x - x^2 - 2x^2 + 6x\ dx$$

$$A = \int_0^3 9x - 3x^2\ dx$$

Integrate with respect to x, then evaluate over the interval.

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$$A=\frac{9}{2}x^2-x^3\Big|_0^3$$

$$A=\frac{9}{2}(3)^2-(3)^3-\left(\frac{9}{2}(0)^2-(0)^3\right)$$

$$A=\frac{81}{2}-27$$

$$A=\frac{81}{2}-\frac{54}{2}$$

$$A=\frac{27}{2}$$

Now that we have the area, we need to find M_x and M_y.

$$M_x=\int_0^3\int_{2x^2-6x}^{3x-x^2}y\ dy\ dx$$

$$M_x=\int_0^3\frac{1}{2}y^2\Big|_{y=2x^2-6x}^{y=3x-x^2}\ dx$$

$$M_x=\int_0^3\frac{1}{2}\left(3x-x^2\right)^2-\frac{1}{2}\left(2x^2-6x\right)^2\ dx$$

$$M_x=\int_0^3\frac{1}{2}\left(9x^2-6x^3+x^4\right)-\frac{1}{2}\left(4x^4-24x^3+36x^2\right)\ dx$$

$$M_x=\int_0^3\frac{9}{2}x^2-3x^3+\frac{1}{2}x^4-2x^4+12x^3-18x^2\ dx$$

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$$M_x = \int_0^3 -\frac{3}{2}x^4 + 9x^3 - \frac{27}{2}x^2 \ dx$$

$$M_x = -\frac{3}{10}x^5 + \frac{9}{4}x^4 - \frac{9}{2}x^3 \Big|_0^3$$

$$M_x = -\frac{3}{10}(3)^5 + \frac{9}{4}(3)^4 - \frac{9}{2}(3)^3 - \left(-\frac{3}{10}(0)^5 + \frac{9}{4}(0)^4 - \frac{9}{2}(0)^3\right)$$

$$M_x = -\frac{729}{10} + \frac{729}{4} - \frac{243}{2}$$

$$M_x = -\frac{1{,}458}{20} + \frac{3{,}645}{20} - \frac{2{,}430}{20}$$

$$M_x = -\frac{243}{20}$$

And for M_y we get

$$M_y = \int_0^3 \int_{2x^2-6x}^{3x-x^2} x \ dy \ dx$$

$$M_y = \int_0^3 x\left(3x - x^2\right) - x\left(2x^2 - 6x\right) \ dx$$

$$M_y = \int_0^3 3x^2 - x^3 - 2x^3 + 6x^2 \ dx$$

$$M_y = \int_0^3 9x^2 - 3x^3 \ dx$$

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$$M_y = 3x^3 - \frac{3}{4}x^4 \Big|_0^3$$

$$M_y = 3(3)^3 - \frac{3}{4}(3)^4 - \left(3(0)^3 - \frac{3}{4}(0)^4\right)$$

$$M_y = 81 - \frac{243}{4}$$

$$M_y = \frac{81}{4}$$

Now that we have area, plus M_x and M_y, we can find $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{M_y}{A} = \frac{\frac{81}{4}}{\frac{27}{2}} = \frac{3}{2}$$

$$\bar{y} = \frac{M_x}{A} = \frac{-\frac{243}{20}}{\frac{27}{2}} = -\frac{9}{10}$$

Therefore, the center of mass is at

$$\left(\frac{3}{2}, -\frac{9}{10}\right)$$

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Topic: Double integrals to find mass and center of mass

Question: The mass of a circular plate is given, where a is the length of a radius. What is the relationship between the radius of the given circular object and its density?

$$M = \frac{4\pi a^3}{3}$$

Answer choices:

A The density of the circular object is equal to its radius.

B The radius of the circular object is three times its density.

C The density of the circular object is twice its radius.

D The radius of the circular object is twice its density.

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Solution: C

Start with a circle with radius a. Then for the the mass of the plate we get

$$M = \iint_R r\ dA$$

$$M = \int_0^{2\pi} \int_0^a (2r)(r)\ \ dr\ d\theta$$

$$M = \int_0^{2\pi} \frac{2}{3}r^3 \Big|_0^a\ d\theta$$

$$M = \int_0^{2\pi} \frac{2}{3}a^3\ d\theta$$

$$M = \frac{2}{3}a^3\theta \Big|_0^{2\pi}$$

$$M = \frac{4\pi a^3}{3}$$

Because we got the equation of the mass we were given, we know that answer choice C is correct.

Topic: Midpoint rule for triple integrals

Question: Use the midpoint rule for triple integrals to estimate the volume.

$$\iiint_B xyz \ dV$$

where $B = \left\{(x, y, z) \mid 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1\right\}$ and B is divided into 8 sub-boxes of equal size.

Answer choices:

A 0.625

B 0.125

C 0.188

D 0.250

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Solution: B

The midpoint rule formula is

$$V \approx V_a \left[f(x_1, y_1, z_1) + \ldots + f(x_n, y_n, z_n) \right]$$

where V_a is the volume of one of the subdivisions of B, and

where $f(x_1, y_1, z_1) + \ldots + f(x_n, y_n, z_n)$ is the given function at the midpoints of each subdivision of B.

The first step to solve this problem is to figure out the mid-points of the 8 sub-boxes. If the 8 sub-boxes are all equal-sized, it means we divided x in half, then y in half, and then z in half. If we're having trouble, we can always sketch the volume and the sub-boxes to get a better picture.

The midpoints of the sub-boxes are

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \quad \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right) \quad \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}\right) \quad \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right)$$

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right) \quad \left(\frac{3}{4}, \frac{1}{4}, \frac{3}{4}\right) \quad \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right) \quad \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right)$$

We also need to find the volume of an individual sub-box.

$$V_a = lwh$$

$$V_a = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$V_a = \frac{1}{8}$$

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Plugging into the midpoint rule formula gives

$$V \approx \frac{1}{8}\left[f\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{1}{4}\right)\right.$$

$$\left.+f\left(\frac{1}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)\right]$$

Because $f(x,y,z)=xyz$, we'll plug each of these midpoints into that function, and we get

$$V \approx \frac{1}{8}\left[\left(\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\right)+\left(\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\right)+\left(\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right)+\left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right)\right.$$

$$\left.+\left(\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\right)+\left(\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\right)+\left(\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right)+\left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right)\right]$$

$$V \approx \frac{1}{8}\left(\frac{1}{64}+\frac{3}{64}+\frac{3}{64}+\frac{9}{64}+\frac{3}{64}+\frac{9}{64}+\frac{9}{64}+\frac{27}{64}\right)$$

$$V \approx \frac{1}{8}\left(\frac{64}{64}\right)$$

$$V \approx \frac{1}{8}$$

$$V \approx 0.125$$

This is the approximate volume of the triple integral.

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Topic: Midpoint rule for triple integrals

Question: Use the midpoint rule for triple integrals to estimate the volume.

$$\iiint_B \sin(xyz)\ dV$$

where $B = \left\{(x, y, z)\,|\,0 \le x \le 2, 0 \le y \le 2, 0 \le z \le 2\right\}$ and B is divided into 8 sub-boxes.

Answer choices:

A $\quad 2.326$

B $\quad 3.632$

C $\quad 2.632$

D $\quad 3.699$

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Solution: D

The midpoint rule formula is

$$V \approx V_a \left[f(x_1, y_1, z_1) + \dots + f(x_n, y_n, z_n)\right]$$

where V_a is the volume of one of the subdivisions of B, and

where $f(x_1, y_1, z_1) + \dots + f(x_n, y_n, z_n)$ is the given function at the midpoints of each subdivision of B.

The first step to solve this problem is to figure out the mid-points of the 8 sub-boxes. If the 8 sub-boxes are all equal-sized, it means we divided x in half, then y in half, and then z in half. If we're having trouble, we can always sketch the volume and the sub-boxes to get a better picture.

The midpoints of the sub-boxes are

$$\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \qquad \left(\frac{3}{2},\frac{1}{2},\frac{1}{2}\right) \qquad \left(\frac{1}{2},\frac{3}{2},\frac{1}{2}\right) \qquad \left(\frac{3}{2},\frac{3}{2},\frac{1}{2}\right)$$

$$\left(\frac{1}{2},\frac{1}{2},\frac{3}{2}\right) \qquad \left(\frac{3}{2},\frac{1}{2},\frac{3}{2}\right) \qquad \left(\frac{1}{2},\frac{3}{2},\frac{3}{2}\right) \qquad \left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)$$

We also need to find the volume of an individual sub-box.

$$V_a = lwh$$

$$V_a = (1)(1)(1)$$

$$V_a = 1$$

Plugging into the midpoint rule formula gives

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$$V \approx (1)\left[f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)+f\left(\frac{3}{2},\frac{1}{2},\frac{1}{2}\right)+f\left(\frac{1}{2},\frac{3}{2},\frac{1}{2}\right)+f\left(\frac{3}{2},\frac{3}{2},\frac{1}{2}\right)\right.$$

$$\left.+f\left(\frac{1}{2},\frac{1}{2},\frac{3}{2}\right)+f\left(\frac{3}{2},\frac{1}{2},\frac{3}{2}\right)+f\left(\frac{1}{2},\frac{3}{2},\frac{3}{2}\right)+f\left(\frac{3}{2},\frac{3}{2},\frac{3}{2}\right)\right]$$

Because $f(x,y,z)=\sin(xyz)$, we'll plug each of these midpoints into that function, and we get

$$V \approx \left[\sin\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right)+\sin\left(\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\right)+\sin\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\right)+\sin\left(\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\right)\right.$$

$$\left.+\sin\left(\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\right)+\sin\left(\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\right)+\sin\left(\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\right)+\sin\left(\frac{3}{2}\cdot\frac{3}{2}\cdot\frac{3}{2}\right)\right]$$

$$V \approx \left[\sin\left(\frac{1}{8}\right)+\sin\left(\frac{3}{8}\right)+\sin\left(\frac{3}{8}\right)+\sin\left(\frac{9}{8}\right)+\sin\left(\frac{3}{8}\right)+\sin\left(\frac{9}{8}\right)+\sin\left(\frac{9}{8}\right)+\sin\left(\frac{27}{8}\right)\right]$$

$$V \approx \left[\sin\left(\frac{1}{8}\right)+3\sin\left(\frac{3}{8}\right)+3\sin\left(\frac{9}{8}\right)+\sin\left(\frac{27}{8}\right)\right]$$

$$V \approx 3.699$$

This is the approximate volume of the triple integral.

Topic: Midpoint rule for triple integrals

Question: Use the midpoint rule for triple integrals to estimate the volume.

$$\iiint_B 4xyz \ dV$$

where $B = \{(x, y, z) \mid 0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$ and B is divided into 8 sub-boxes.

Answer choices:

A $\quad 0.625$

B $\quad 0.125$

C $\quad 0.500$

D $\quad 0.250$

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Solution: C

The midpoint rule formula is

$$V \approx V_a \left[f(x_1, y_1, z_1) + \ldots + f(x_n, y_n, z_n) \right]$$

where V_a is the volume of one of the subdivisions of B, and

where $f(x_1, y_1, z_1) + \ldots + f(x_n, y_n, z_n)$ is the given function at the midpoints of each subdivision of B.

The first step to solve this problem is to figure out the mid-points of the 8 sub-boxes. If the 8 sub-boxes are all equal-sized, it means we divided x in half, then y in half, and then z in half. If we're having trouble, we can always sketch the volume and the sub-boxes to get a better picture.

The midpoints of the sub-boxes are

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \qquad \left(\frac{3}{4}, \frac{1}{4}, \frac{1}{4}\right) \qquad \left(\frac{1}{4}, \frac{3}{4}, \frac{1}{4}\right) \qquad \left(\frac{3}{4}, \frac{3}{4}, \frac{1}{4}\right)$$

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right) \qquad \left(\frac{3}{4}, \frac{1}{4}, \frac{3}{4}\right) \qquad \left(\frac{1}{4}, \frac{3}{4}, \frac{3}{4}\right) \qquad \left(\frac{3}{4}, \frac{3}{4}, \frac{3}{4}\right)$$

We also need to find the volume of an individual sub-box.

$$V_a = lwh$$

$$V_a = \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$V_a = \frac{1}{8}$$

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Plugging into the midpoint rule formula gives

$$V \approx \frac{1}{8}\left[f\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{1}{4}\right)+f\left(\frac{1}{4},\frac{3}{4},\frac{1}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{1}{4}\right)\right.$$

$$\left.+f\left(\frac{1}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{1}{4},\frac{3}{4}\right)+f\left(\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)+f\left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)\right]$$

Because $f(x, y, z) = 4xyz$, we'll plug each of these midpoints into that function, and we get

$$V \approx \frac{1}{8}\left[4\left(\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\right)+4\left(\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{1}{4}\right)+4\left(\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right)+4\left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{1}{4}\right)\right.$$

$$\left.+4\left(\frac{1}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\right)+4\left(\frac{3}{4}\cdot\frac{1}{4}\cdot\frac{3}{4}\right)+4\left(\frac{1}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right)+4\left(\frac{3}{4}\cdot\frac{3}{4}\cdot\frac{3}{4}\right)\right]$$

$$V \approx \frac{1}{2}\left(\frac{1}{64}+\frac{3}{64}+\frac{3}{64}+\frac{9}{64}+\frac{3}{64}+\frac{9}{64}+\frac{9}{64}+\frac{27}{64}\right)$$

$$V \approx \frac{1}{2}\left(\frac{64}{64}\right)$$

$$V \approx \frac{1}{2}$$

$$V \approx 0.500$$

This is the approximate volume of the triple integral.

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Topic: Iterated integrals

Question: Evaluate the iterated integral.

$$\int_0^1 \int_0^x \int_0^z xyz - 2yz \ dy \ dz \ dx$$

Answer choices:

A 0.008

B -0.029

C 0.029

D -0.008

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Solution: B

When we evaluate an iterated integral, we work from the inside out, integrating first with respect to the innermost variable. In this case, the iterated integral shows that the order of integration is $dy\ dz\ dx$, so we'll integrate first with respect to y.

$$\int_0^1\int_0^x\int_0^z xyz-2yz\ dy\ dz\ dx=\int_0^1\int_0^x \frac{1}{2}xy^2z-y^2z\Big|_0^z\ dz\ dx$$

$$\int_0^1\int_0^x\int_0^z xyz-2yz\ dy\ dz\ dx=\int_0^1\int_0^x \frac{1}{2}x(z)^2z-(z)^2z-\left[\frac{1}{2}x(0)^2z-(0)^2z\right]\ dz\ dx$$

$$\int_0^1\int_0^x\int_0^z xyz-2yz\ dy\ dz\ dx=\int_0^1\int_0^x \frac{1}{2}xz^3-z^3\ dz\ dx$$

Now we'll integrate with respect to z.

$$\int_0^1\int_0^x\int_0^z xyz-2yz\ dy\ dz\ dx=\int_0^1 \frac{1}{8}xz^4-\frac{1}{4}z^4\Big|_0^x\ dx$$

$$\int_0^1\int_0^x\int_0^z xyz-2yz\ dy\ dz\ dx=\int_0^1 \frac{1}{8}x(x)^4-\frac{1}{4}(x)^4-\left[\frac{1}{8}x(0)^4-\frac{1}{4}(0)^4\right]\ dx$$

$$\int_0^1\int_0^x\int_0^z xyz-2yz\ dy\ dz\ dx=\int_0^1 \frac{1}{8}x^5-\frac{1}{4}x^4\ dx$$

Now we'll integrate with respect to x.

$$\int_0^1\int_0^x\int_0^z xyz-2yz\ dy\ dz\ dx=\frac{1}{48}x^6-\frac{1}{20}x^5\Big|_0^1$$

$$\int_0^1 \int_0^x \int_0^z xyz - 2yz \ dy \ dz \ dx = \frac{1}{48}(1)^6 - \frac{1}{20}(1)^5 - \left[\frac{1}{48}(0)^6 - \frac{1}{20}(0)^5\right]$$

$$\int_0^1 \int_0^x \int_0^z xyz - 2yz \ dy \ dz \ dx = \frac{1}{48} - \frac{1}{20}$$

$$\int_0^1 \int_0^x \int_0^z xyz - 2yz \ dy \ dz \ dx = \frac{20}{960} - \frac{48}{960}$$

$$\int_0^1 \int_0^x \int_0^z xyz - 2yz \ dy \ dz \ dx = -\frac{7}{240}$$

$$\int_0^1 \int_0^x \int_0^z xyz - 2yz \ dy \ dz \ dx = -0.029$$

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Topic: Iterated integrals

Question: Evaluate the triple integral over the given region.

$$\iiint_B zy^2 \sin x - z \cos x \ dV$$

$$B = \left\{ (x, y, z) \middle| 0 \le x \le \frac{\pi}{2}, \ 1 \le y \le 2, \ 0 \le z \le y \right\}$$

Answer choices:

A 4.267

B -1.933

C -4.267

D 1.933

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Solution: D

In this problem, we've been given a triple integral, but we don't have the order of integration. To determine the order of integration, we need to look at the bounds for each of the variables. The bounds with the greatest number of variables need to be evaluated first, whereas the bounds with the least number of integrals need to be evaluated last. The bounds that were given for this triple integral were

$$0 \le x \le \frac{\pi}{2}$$

$$1 \le y \le 2$$

$$0 \le z \le y$$

The bounds for x and y are constants; they don't include any variables. But the bounds for z include the variable y. That tells us that we'll need to integrate first with respect to z, and then x and y can come in either order after that.

So we'll set the integration order as $dz\ dy\ dx$, and therefore integrate first with respect to z.

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = \int_0^{\frac{\pi}{2}}\int_1^2 \frac{1}{2}z^2y^2\sin x - \frac{1}{2}z^2\cos x\Big|_0^y\ dy\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = \int_0^{\frac{\pi}{2}}\int_1^2 \frac{1}{2}(y)^2y^2\sin x - \frac{1}{2}(y)^2\cos x - \left[\frac{1}{2}(0)^2y^2\sin x - \frac{1}{2}(0)^2\cos x\right]\ dy\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = \int_0^{\frac{\pi}{2}}\int_1^2 \frac{1}{2}y^4\sin x - \frac{1}{2}y^2\cos x\ dy\ dx$$

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Now we'll integrate with respect to y.

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = \int_0^{\frac{\pi}{2}} \frac{1}{10}y^5\sin x - \frac{1}{6}y^3\cos x\Big|_1^2\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = \int_0^{\frac{\pi}{2}} \frac{1}{10}(2)^5\sin x - \frac{1}{6}(2)^3\cos x - \left[\frac{1}{10}(1)^5\sin x - \frac{1}{6}(1)^3\cos x\right]\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = \int_0^{\frac{\pi}{2}} \frac{32}{10}\sin x - \frac{8}{6}\cos x - \frac{1}{10}\sin x + \frac{1}{6}\cos x\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = \int_0^{\frac{\pi}{2}} \frac{31}{10}\sin x - \frac{7}{6}\cos x\ dx$$

Now we'll integrate with respect to x.

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = -\frac{31}{10}\cos x - \frac{7}{6}\sin x\Big|_0^{\frac{\pi}{2}}$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = -\frac{31}{10}\cos\frac{\pi}{2} - \frac{7}{6}\sin\frac{\pi}{2} - \left[-\frac{31}{10}\cos(0) - \frac{7}{6}\sin(0)\right]$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = -\frac{31}{10}(0) - \frac{7}{6}(1) - \left[-\frac{31}{10}(1) - \frac{7}{6}(0)\right]$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = -\frac{7}{6} + \frac{31}{10}$$

$$\int_0^{\frac{\pi}{2}}\int_1^2\int_0^y zy^2\sin x - z\cos x\ dz\ dy\ dx = -\frac{35}{30} + \frac{93}{30}$$

$$\int_0^{\frac{\pi}{2}} \int_1^2 \int_0^y zy^2 \sin x - z\cos x \ dz \ dy \ dx = \frac{58}{30}$$

$$\int_0^{\frac{\pi}{2}} \int_1^2 \int_0^y zy^2 \sin x - z\cos x \ dz \ dy \ dx = \frac{29}{15}$$

$$\int_0^{\frac{\pi}{2}} \int_1^2 \int_0^y zy^2 \sin x - z\cos x \ dz \ dy \ dx = 1.933$$

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Topic: Iterated integrals

Question: Evaluate the iterated integral.

$$\int_0^2 \int_0^{2z} \int_0^{x^2} 2e^2y + 4yz \ dy \ dx \ dz$$

Answer choices:

A $\quad 738.5$

B $\quad -738.5$

C $\quad 240.9$

D $\quad -240.9$

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Solution: A

When we evaluate an iterated integral, we work from the inside out, integrating first with respect to the innermost variable. In this case, the iterated integral shows that the order of integration is $dy\ dx\ dz$, so we'll integrate first with respect to y.

$$\int_0^2 \int_0^{2z} \int_0^{x^2} 2e^2y + 4yz\ dy\ dx\ dz$$

$$\int_0^2 \int_0^{2z} e^2y^2 + 2y^2z \Big|_0^{x^2}\ dx\ dz$$

$$\int_0^2 \int_0^{2z} e^2(x^2)^2 + 2(x^2)^2z - \left[e^2(0)^2 + 2(0)^2z\right]\ dx\ dz$$

$$\int_0^2 \int_0^{2z} e^2x^4 + 2x^4z\ dx\ dz$$

Now we'll integrate with respect to x.

$$\int_0^2 \frac{1}{5}e^2x^5 + \frac{2}{5}x^5z \Big|_0^{2z}\ dz$$

$$\int_0^2 \frac{1}{5}e^2(2z)^5 + \frac{2}{5}(2z)^5z - \left[\frac{1}{5}e^2(0)^5 + \frac{2}{5}(0)^5z\right]\ dz$$

$$\int_0^2 \frac{32}{5}e^2z^5 + \frac{64}{5}z^6\ dz$$

Now we'll integrate with respect to z.

$$\frac{32}{30}e^2z^6+\frac{64}{35}z^7\Big|_0^2$$

$$\frac{32}{30}e^2(2)^6+\frac{64}{35}(2)^7-\left[\frac{32}{30}e^2(0)^6+\frac{64}{35}(0)^7\right]$$

$$\frac{32}{30}e^2(64)+\frac{64}{35}(128)$$

$$\frac{1{,}024}{15}e^2+\frac{8{,}192}{35}$$

738.5

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Topic: Average value

Question: Find the average value of the function over the region.

Use triple integrals to find the average value of the function over a cube with side length 1, lying in the first octant with one vertex at $(0,0,0)$, and three sides on coordinate axes.

$$f(x, y, z) = xyz$$

Answer choices:

A $\frac{1}{4}$

B $\frac{1}{2}$

C $\frac{1}{8}$

D 1

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Solution: C

To find the average value over the region, we'll use the formula

$$f_{avg} = \frac{1}{V(E)} \iiint_E f(x, y, z)\ dV$$

where $V(E)$ is the volume of the region E.

We can start by calculating $V(E)$. From the question, we know that the side lengths of the cube are all equal to 1. This will make $V(E)$

$$V(E) = (1)(1)(1)$$

$$V(E) = 1$$

Next we need to find the limits of integration for each of the three integrals. All of the sides of the cube start at 0 and extend to 1, so the bounds for each variable will be $[0,1]$. Plugging this into the formula gives us

$$f_{avg} = \frac{1}{1} \int_0^1 \int_0^1 \int_0^1 xyz\ dx\ dy\ dz$$

$$f_{avg} = \int_0^1 \int_0^1 \int_0^1 xyz\ dx\ dy\ dz$$

We always integrate from the inside out, so we'll integrate first with respect to x.

$$f_{avg} = \int_0^1 \int_0^1 \frac{1}{2}x^2yz\Big|_0^1\ dy\ dz$$

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$$f_{avg} = \int_0^1 \int_0^1 \frac{1}{2}(1)^2 yz - \frac{1}{2}(0)^2 yz \ dy \ dz$$

$$f_{avg} = \int_0^1 \int_0^1 \frac{1}{2} yz \ dy \ dz$$

Integrate with respect to y.

$$f_{avg} = \int_0^1 \frac{1}{4} y^2 z \Big|_0^1 \ dz$$

$$f_{avg} = \int_0^1 \frac{1}{4}(1)^2 z - \frac{1}{4}(0)^2 z \ dz$$

$$f_{avg} = \int_0^1 \frac{1}{4} z \ dz$$

Integrate with respect to z.

$$f_{avg} = \frac{1}{8} z^2 \Big|_0^1$$

$$f_{avg} = \frac{1}{8}(1)^2 - \frac{1}{8}(0)^2$$

$$f_{avg} = \frac{1}{8}$$

This is the average value over the region.

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Topic: Average value

Question: Find the average value of the function over the region.

Use triple integrals to find the average value of the function over a cube with side length of 2, lying in the first octant with one vertex at $(0,0,0)$, and three sides on coordinate axes.

$$f(x, y, z) = 2xyz^2$$

Answer choices:

A $\quad 2$

B $\quad \frac{1}{3}$

C $\quad 1$

D $\quad \frac{8}{3}$

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Solution: D

To find the average value over the region, we'll use the formula

$$f_{avg} = \frac{1}{V(E)} \iiint_E f(x, y, z)\ dV$$

where $V(E)$ is the volume of the region E.

We can start by calculating $V(E)$. From the question, we know that the side lengths of the cube are all equal to 2. This will make $V(E)$

$$V(E) = (2)(2)(2)$$

$$V(E) = 8$$

Next we need to find the limits of integration for each of the three integrals. All of the sides of the cube start at 0 and extend to 2, so the bounds for each variable will be $[0,2]$. Plugging this into the formula gives us

$$f_{avg} = \frac{1}{8} \int_0^2 \int_0^2 \int_0^2 2xyz^2\ dx\ dy\ dz$$

We always integrate from the inside out, so we'll integrate first with respect to x.

$$f_{avg} = \frac{1}{8} \int_0^2 \int_0^2 x^2yz^2 \Big|_0^2\ dy\ dz$$

$$f_{avg} = \frac{1}{8} \int_0^2 \int_0^2 (2)^2yz^2 - (0)^2yz^2\ dy\ dz$$

$$f_{avg}=\frac{1}{2}\int_0^2\int_0^2 yz^2\ dy\ dz$$

Integrate with respect to y.

$$f_{avg}=\frac{1}{2}\int_0^2 \frac{1}{2}y^2z^2\Big|_0^2\ dz$$

$$f_{avg}=\frac{1}{4}\int_0^2 y^2z^2\Big|_0^2\ dz$$

$$f_{avg}=\frac{1}{4}\int_0^2 (2)^2z^2-(0)^2z^2\ dz$$

$$f_{avg}=\int_0^2 z^2\ dz$$

Integrate with respect to z.

$$f_{avg}=\frac{1}{3}z^3\Big|_0^2$$

$$f_{avg}=\frac{1}{3}(2)^3-\frac{1}{3}(0)^3$$

$$f_{avg}=\frac{1}{3}(8)$$

$$f_{avg}=\frac{8}{3}$$

This is the average value over the region.

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Topic: Average value

Question: Find the average value of the function over the region.

Use triple integrals to find the average value of the function over a cube with side length 2, lying in the first octant with one vertex at $(0,0,0)$, and three sides on coordinate axes.

$$f(x, y, z) = xyz$$

Answer choices:

A $\frac{1}{2}$

B 1

C $\frac{1}{8}$

D $\frac{1}{4}$

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Solution: B

To find the average value over the region, we'll use the formula

$$f_{avg}=\frac{1}{V(E)}\iiint_E f(x,y,z)\ dV$$

where $V(E)$ is the volume of the region E.

We can start by calculating $V(E)$. From the question, we know that the side lengths of the cube are all equal to 2. This will make $V(E)$

$$V(E)=(2)(2)(2)$$

$$V(E)=8$$

Next we need to find the limits of integration for each of the three integrals. All of the sides of the cube start at 0 and extend to 2, so the bounds for each variable will be $[0,2]$. Plugging this into the formula gives us

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2\int_0^2 xyz\ dx\ dy\ dz$$

We always integrate from the inside out, so we'll integrate first with respect to x.

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 \frac{1}{2}x^2yz\Big|_0^2\ dy\ dz$$

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 \frac{1}{2}(2)^2yz-\frac{1}{2}(0)^2yz\ dy\ dz$$

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$$f_{avg}=\frac{1}{4}\int_0^2\int_0^2 yz\ dy\ dz$$

Integrate with respect to y.

$$f_{avg}=\frac{1}{4}\int_0^2 \frac{1}{2}y^2z\Big|_0^2\ dz$$

$$f_{avg}=\frac{1}{4}\int_0^2 \frac{1}{2}(2)^2z-\frac{1}{2}(0)^2z\ dz$$

$$f_{avg}=\frac{1}{2}\int_0^2 z\ dz$$

Integrate with respect to z.

$$f_{avg}=\frac{1}{2}\left(\frac{1}{2}z^2\right)\Big|_0^2$$

$$f_{avg}=\frac{1}{4}z^2\Big|_0^2$$

$$f_{avg}=\frac{1}{4}(2)^2-\frac{1}{4}(0)^2$$

$$f_{avg}=1$$

This is the average value over the region.

Topic: Finding volume

Question: Find the mass of the solid described by the triple integral.

$$\int_0^2 \int_0^z \int_0^{3y-1} x \, dx \, dy \, dz$$

Answer choices:

A 1

B 3

C $\frac{787}{54}$

D $\frac{787}{108}$

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Solution: B

To find the mass, we'll just need to solve the integral. We can start by solving the innermost integral, which we'll do by integrating with respect to x.

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz = \int_0^2 \int_0^z \frac{1}{2}x^2 \Big|_0^{3y-1} \ dy \ dz$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz = \int_0^2 \int_0^z \frac{1}{2}(3y-1)^2 - \frac{1}{2}(0)^2 \ dy \ dz$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz = \int_0^2 \int_0^z \frac{1}{2}(3y-1)^2 \ dy \ dz$$

Integrate with respect to y.

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz = \int_0^2 \frac{1}{18}(3y-1)^3 \Big|_0^z \ dz$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz = \int_0^2 \frac{1}{18}(3z-1)^3 - \frac{1}{18}(3(0)-1)^3 \ dz$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz = \int_0^2 \frac{1}{18}(3z-1)^3 + \frac{1}{18} \ dz$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x \ dx \ dy \ dz = \frac{1}{18}\int_0^2 (3z-1)^3 + 1 \ dz$$

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Evaluate with respect to z.

$$\int_0^2 \int_0^z \int_0^{3y-1} x\ dx\ dy\ dz = \frac{1}{18}\left[\frac{1}{12}(3z-1)^4 + z\right]\Bigg|_0^2$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x\ dx\ dy\ dz = \frac{1}{18}\left[\frac{1}{12}(3(2)-1)^4 + 2\right] - \frac{1}{18}\left[\frac{1}{12}(3(0)-1)^4 + 0\right]$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x\ dx\ dy\ dz = \frac{1}{18}\left[\frac{1}{12}(5)^4 + 2\right] - \frac{1}{18}\left(\frac{1}{12}\right)$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x\ dx\ dy\ dz = \frac{1}{18}\left(\frac{625}{12} + \frac{24}{12}\right) - \frac{1}{216}$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x\ dx\ dy\ dz = \frac{649}{216} - \frac{1}{216}$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x\ dx\ dy\ dz = \frac{648}{216}$$

$$\int_0^2 \int_0^z \int_0^{3y-1} x\ dx\ dy\ dz = 3$$

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Topic: Finding volume

Question: Use a triple integral to find the volume of the solid tetrahedron enclosed by the given plane and the coordinate planes.

$$x + y + z = 1$$

Answer choices:

A $\frac{7}{6}$

B $\frac{1}{6}$

C $\frac{3}{2}$

D $\frac{1}{2}$

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Solution: B

The first thing we need to do is find the limits of integration for the triple integral, which we can do based on the information we've been given in the problem. Since the tetrahedron is bounded by the coordinate planes and $x+y+z=1$, the limits of integration are

$$0 \le z \le 1-x-y$$

$$x+y+z=1$$

$$z=1-x-y$$

$$0 \le y \le 1-x$$

$$x+y+z=1$$

$$x+y+0=1$$

$$y=1-x$$

$$0 \le x \le 1$$

$$x+y+z=1$$

$$x+0+0=1$$

$$x=1$$

The bounds with the largest number of variables should go on the inside, and the bounds with the fewest number of variables should go on the outside. So we'll order the integrals this way:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx$$

To find volume, we need to solve the integral. We'll integrate from the inside out, starting with z.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 \int_0^{1-x} z\Big|_0^{1-x-y} dy\ dx$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 \int_0^{1-x} 1 - x - y - 0\ dy\ dx$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 \int_0^{1-x} 1 - x - y\ dy\ dx$$

Integrate with respect to y.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 y - xy - \frac{1}{2}y^2\Big|_0^{1-x} dx$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 (1-x) - x(1-x) - \frac{1}{2}(1-x)^2 - \left[0 - x(0) - \frac{1}{2}(0)^2\right]\ dx$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 1 - x - x + x^2 - \frac{1}{2}(1-x)^2\ dx$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \int_0^1 1 - 2x + x^2 - \frac{1}{2}(1-x)^2\ dx$$

Integrate with respect to x.

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = x - x^2 + \frac{1}{3}x^3 + \frac{1}{6}(1-x)^3 \Big|_0^1$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = 1 - (1)^2 + \frac{1}{3}(1)^3 + \frac{1}{6}(1-1)^3 - \left[0 - (0)^2 + \frac{1}{3}(0)^3 + \frac{1}{6}(1-0)^3\right]$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = 1 - 1 + \frac{1}{3} + \frac{1}{6}(0)^3 - \left[\frac{1}{6}(1)^3\right]$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = 1 - 1 + \frac{1}{3} - \frac{1}{6}$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \frac{1}{3} - \frac{1}{6}$$

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} dz\ dy\ dx = \frac{1}{6}$$

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Topic: Finding volume

Question: Use a triple integral to find the volume of the solid tetrahedron enclosed by the given plane and the coordinate planes.

$$x + y + z = 4$$

Answer choices:

A $\frac{32}{2}$

B $\frac{32}{6}$

C $\frac{32}{8}$

D $\frac{32}{3}$

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Solution: D

The first thing we need to do is find the limits of integration for the triple integral, which we can do based on the information we've been given in the problem. Since the tetrahedron is bounded by the coordinate planes and $x+y+z=4$, the limits of integration are

$$0 \le z \le 4 - x - y$$

$$x + y + z = 4$$

$$z = 4 - x - y$$

$$0 \le y \le 4 - x$$

$$x + y + z = 4$$

$$x + y + 0 = 4$$

$$y = 4 - x$$

$$0 \le x \le 4$$

$$x + y + z = 4$$

$$x + 0 + 0 = 4$$

$$x = 4$$

The bounds with the largest number of variables should go on the inside, and the bounds with the fewest number of variables should go on the outside. So we'll order the integrals this way:

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$$\int_0^4 \int_0^{4-x} \int_0^{4-x-y} dz\ dy\ dx$$

To find volume, we need to solve the integral. We'll integrate from the inside out, starting with z.

$$\int_0^4 \int_0^{4-x} z\Big|_0^{4-x-y} dy\ dx$$

$$\int_0^4 \int_0^{4-x} 4 - x - y - 0\ dy\ dx$$

$$\int_0^4 \int_0^{4-x} 4 - x - y\ dy\ dx$$

Integrate with respect to y.

$$\int_0^4 4y - xy - \frac{1}{2}y^2\Big|_0^{4-x} dx$$

$$\int_0^4 4(4-x) - x(4-x) - \frac{1}{2}(4-x)^2 - \left[4(0) - x(0) - \frac{1}{2}(0)^2\right]\ dx$$

$$\int_0^4 16 - 4x - 4x + x^2 - \frac{1}{2}(4-x)^2\ dx$$

$$\int_0^4 16 - 8x + x^2 - \frac{1}{2}(4-x)^2\ dx$$

Integrate with respect to x.

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$$16x - 4x^2 + \frac{1}{3}x^3 + \frac{1}{6}(4-x)^3 \Big|_0^4$$

$$16(4) - 4(4)^2 + \frac{1}{3}(4)^3 + \frac{1}{6}(4-4)^3 - \left[16(0) - 4(0)^2 + \frac{1}{3}(0)^3 + \frac{1}{6}(4-0)^3\right]$$

$$64 - 4(16) + \frac{1}{3}(64) + \frac{1}{6}(0)^3 - \left[\frac{1}{6}(4)^3\right]$$

$$64 - 64 + \frac{64}{3} - \frac{1}{6}(64)$$

$$\frac{64}{3} - \frac{64}{6}$$

$$\frac{64}{6}$$

$$\frac{32}{3}$$

Topic: Expressing the integral six ways

Question: Which of the following represents the triple integral?

$$\iiint_E f(x,y,z)\ dV$$

where $x+y^2+z^2=1$ and $x=0$

Answer choices:

A $\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-1+y^2+z^2}^{1-y^2-z^2} f(x,y,z)\ dx\ dy\ dz$

B $\int_{0}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{0}^{1-y^2-z^2} f(x,y,z)\ dx\ dy\ dz$

C $\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{0}^{1-y^2-z^2} f(x,y,z)\ dx\ dy\ dz$

D $\int_{0}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-1+y^2+z^2}^{1-y^2-z^2} f(x,y,z)\ dx\ dy\ dz$

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Solution: C

Since this is a multiple choice question, the easiest way to figure out which one is correct is to evaluate them directly.

The integral in each answer choice has the integration order

$$dx\ dy\ dz$$

Which means we only need to look at that order of integration to find the correct answer.

We need to focus on the limits of integration. The innermost integral is in terms of x, so the innermost limits of integration will be in terms of x. We know that either the upper or lower limit will be $x = 0$, since it was given in the question. We'll find the other limit by solving $x + y^2 + z^2 = 1$ for x.

$$x + y^2 + z^2 = 1$$

$$x = 1 - y^2 - z^2$$

Now that we know the bounds for x, we can say

$$\int \int \int_0^{1-y^2-z^2} f(x, y, z)\ dx\ dy\ dz$$

Next we'll try to find bounds for the middle integral. We'll set $x = 0$ in $x + y^2 + z^2 = 1$, since by this point in the integral we would have evaluated for x. We'll solve this equation for y.

$$0 + y^2 + z^2 = 1$$

$$y^2 = 1 - z^2$$

$$y = \pm\sqrt{1-z^2}$$

Now that we know the bounds for y, we can say

$$\int\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_0^{1-y^2-z^2} f(x,y,z)\ dx\ dy\ dz$$

To find the limits of integration for the outermost integral, we'll set $x=0$ and $y=0$, since by this point in the integral we would have evaluated for x and y. We'll solve this equation for z.

$$0+(0)^2+z^2=1$$

$$z^2=1$$

$$z=\pm 1$$

Now that we know the bounds for z, we can say

$$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_0^{1-y^2-z^2} f(x,y,z)\ dx\ dy\ dz$$

This is the represents the given volume.

Topic: Expressing the integral six ways

Question: Which of the following represents the triple integral?

$$\iiint_E xyz\ dV$$

where $x^2 + y^2 + z^2 = 16$

Answer choices:

A $\displaystyle \int_{-16}^{16} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-\sqrt{16-x^2-z^2}}^{\sqrt{16-x^2-z^2}} xyz\ dy\ dx\ dz$

B $\displaystyle \int_{-4}^{4} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} xyz\ dz\ dy\ dx$

C $\displaystyle \int_{-4}^{4} \int_{-\sqrt{16+z^2}}^{\sqrt{16+z^2}} \int_{-\sqrt{16-x^2-z^2}}^{\sqrt{16-x^2-z^2}} xyz\ dy\ dx\ dz$

D $\displaystyle \int_{-16}^{16} \int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} \int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} xyz\ dz\ dy\ dx$

Solution: B

Since this is a multiple choice question, the easiest way to figure out which one is correct is to evaluate them directly.

We have a couple different orders of integration in our answer choices.

$$dy\ dx\ dz$$

$$dz\ dy\ dx$$

We'll start by looking at the order of integration $dy\ dx\ dz$.

We need to focus on the limits of integration. The innermost integral is in terms of y, so the innermost limits of integration will be in terms of y. We'll find the limits by solving $x^2 + y^2 + z^2 = 16$ for y.

$$x^2 + y^2 + z^2 = 16$$

$$y^2 = 16 - x^2 - z^2$$

$$y = \pm\sqrt{16 - x^2 - z^2}$$

Now that we know the bounds for y, we can say

$$\int\int\int_{-\sqrt{16-x^2-z^2}}^{\sqrt{16-x^2-z^2}} xyz\ dy\ dx\ dz$$

Next we'll try to find bounds for the middle integral. We'll set $y = 0$ in $x^2 + y^2 + z^2 = 16$, since by this point in the integral we would have evaluated for y. We'll solve this equation for x.

$$x^2 + (0)^2 + z^2 = 16$$

$$x^2 = 16 - z^2$$

$$x = \pm\sqrt{16 - z^2}$$

Now that we know the bounds for x, we can say

$$\int\int_{-\sqrt{16-z^2}}^{\sqrt{16-z^2}}\int_{-\sqrt{16-x^2-z^2}}^{\sqrt{16-x^2-z^2}} xyz\ dy\ dx\ dz$$

To find the limits of integration for the outermost integral, we'll set $y = 0$ and $x = 0$, since by this point in the integral we would have evaluated for y and x. We'll solve this equation for z.

$$(0)^2 + (0)^2 + z^2 = 16$$

$$z^2 = 16$$

$$z = \pm 4$$

Now that we know the bounds for z, we can say

$$\int_{-4}^{4}\int_{-\sqrt{16-z^2}}^{\sqrt{16-z^2}}\int_{-\sqrt{16-x^2-z^2}}^{\sqrt{16-x^2-z^2}} xyz\ dy\ dx\ dz$$

Answer choices A and C were the triple integrals with $dy\ dx\ dz$ as the order of integration. Our answer doesn't match either of those answer choices, which means that we can eliminate answer choices A and C.

So let's look at the other order of integration, $dz\ dy\ dx$.

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The innermost integral is in terms of z, so the innermost limits of integration will be in terms of z. We'll find the limits by solving $x^2+y^2+z^2=16$ for z.

$$x^2+y^2+z^2=16$$

$$z^2=16-x^2-y^2$$

$$z=\pm\sqrt{16-x^2-y^2}$$

Now that we know the bounds for z, we can say

$$\int\int\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-y^2}} xyz\ dz\ dy\ dx$$

Next we'll try to find bounds for the middle integral. We'll set $z=0$ in $x^2+y^2+z^2=16$, since by this point in the integral we would have evaluated for z. We'll solve this equation for y.

$$x^2+y^2+(0)^2=16$$

$$y^2=16-x^2$$

$$y=\pm\sqrt{16-x^2}$$

Now that we know the bounds for y, we can say

$$\int\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-zy^2}} xyz\ dz\ dy\ dx$$

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To find the limits of integration for the outermost integral, we'll set $z=0$ and $y=0$, since by this point in the integral we would have evaluated for z and y. We'll solve this equation for x.

$$x^2+(0)^2+(0)^2=16$$

$$x^2=16$$

$$x=\pm 4$$

Now that we know the bounds for x, we can say

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}}\int_{-\sqrt{16-x^2-y^2}}^{\sqrt{16-x^2-zy^2}} xyz\ dz\ dy\ dx$$

Comparing this to answer choices B and D, we can see that B is the correct answer.

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Topic: Expressing the integral six ways

Question: Which of the following represents the triple integral?

$$\iiint_E xyz\ dV$$

where $x + y^2 + z = 5$, $x = 0$, and $z = 0$

Answer choices:

A $\displaystyle\int_{-5}^{5}\int_{0}^{5-x}\int_{0}^{\sqrt{5-x-z}} xyz\ dy\ dz\ dx$

B $\displaystyle\int_{-5}^{5}\int_{0}^{5-y^2}\int_{0}^{5-x-y^2} xyz\ dz\ dx\ dy$

C $\displaystyle\int_{0}^{\sqrt{5}}\int_{0}^{5-x}\int_{-\sqrt{5-x-z}}^{\sqrt{5-x-z}} xyz\ dy\ dz\ dx$

D $\displaystyle\int_{-\sqrt{5}}^{\sqrt{5}}\int_{0}^{5-y^2}\int_{0}^{5-x-y^2} xyz\ dz\ dx\ dy$

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Solution: D

Since this is a multiple choice question, the easiest way to figure out which one is correct is to evaluate them directly.

We have a couple different orders of integration in our answer choices.

$$dy\ dz\ dx$$

$$dz\ dx\ dy$$

We'll start by looking at the order of integration $dy\ dz\ dx$.

We need to focus on the limits of integration. The innermost integral is in terms of y, so the innermost limits of integration will be in terms of y. We'll find the limits by solving $x + y^2 + z = 5$ for y.

$$x + y^2 + z = 5$$

$$y^2 = 5 - x - z$$

$$y = \pm\sqrt{5 - x - z}$$

Now that we know the bounds for y, we can say

$$\int\int\int_{-\sqrt{5-x-z}}^{\sqrt{5-x-z}} xyz\ dy\ dz\ dx$$

Next we'll try to find bounds for the middle integral. We'll set $y = 0$ in $x + y^2 + z = 5$, since by this point in the integral we would have evaluated for y. We'll solve this equation for z.

$$x + (0)^2 + z = 5$$

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$$z = 5 - x$$

We were told in the original problem that $z = 0$, so putting this together with what we just found, we know the bounds for z, and we can say

$$\int \int_{0}^{5-x} \int_{-\sqrt{5-x-z}}^{\sqrt{5-x-z}} xyz \ dy \ dz \ dx$$

To find the limits of integration for the outermost integral, we'll set $y = 0$ and $z = 0$, since by this point in the integral we would have evaluated for y and z. We'll solve this equation for x.

$$x + (0)^2 + (0) = 5$$

$$x = 5$$

We were told in the original problem that $x = 0$, so putting this together with what we just found, we know the bounds for x, and we can say

$$\int_{0}^{5} \int_{0}^{5-x} \int_{-\sqrt{5-x-z}}^{\sqrt{5-x-z}} xyz \ dy \ dz \ dx$$

Answer choices A and C were the triple integrals with $dy \ dx \ dz$ as the order of integration. Our answer doesn't match either of those answer choices, which means that we can eliminate answer choices A and C.

So let's look at the other order of integration, $dz \ dx \ dy$.

The innermost integral is in terms of z, so the innermost limits of integration will be in terms of z. We'll find the limits by solving $x + y^2 + z = 5$ for z.

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$$x + y^2 + z = 5$$

$$z = 5 - x - y^2$$

We were told in the original problem that $z = 0$, so putting this together with what we just found, we know the bounds for z, and we can say

$$\int\int\int_0^{5-x-y^2} xyz\ dz\ dx\ dy$$

Next we'll try to find bounds for the middle integral. We'll set $z = 0$ in $x + y^2 + z = 5$, since by this point in the integral we would have evaluated for z. We'll solve this equation for x.

$$x + y^2 + (0) = 5$$

$$x = 5 - y^2$$

We were told in the original problem that $x = 0$, so putting this together with what we just found, we know the bounds for x, and we can say

$$\int\int_0^{5-y^2}\int_0^{5-x-y^2} xyz\ dz\ dx\ dy$$

To find the limits of integration for the outermost integral, we'll set $z = 0$ and $x = 0$, since by this point in the integral we would have evaluated for z and x. We'll solve this equation for y.

$$(0) + y^2 + (0) = 5$$

$$y^2 = 5$$

$$y = \pm\sqrt{5}$$

Now that we know the bounds for y, we can say

$$\int_{-\sqrt{5}}^{\sqrt{5}} \int_{0}^{5-y^2} \int_{0}^{5-x-y^2} xyz \ dz \ dx \ dy$$

Comparing this to answer choices B and D, we can see that D is the correct answer.

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Topic: Type I, II and III regions

Question: Evaluate the triple integral.

$$\iiint_E 4x \ dV$$

where E is the region under the plane $2x + 3y + z = 6$ that lies in the first octant.

Answer choices:

A $\quad -18$

B $\quad -180$

C $\quad 18$

D $\quad 180$

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Solution: C

The volume we're trying to find is the region E that lies below

$$2x + 3y + z = 6$$

We can sketch the plane by finding intercepts for each of the major axes. For example, if we set $y = 0$ and $z = 0$,

$$2x + 3(0) + 0 = 6$$

$$2x = 6$$

$$x = 3$$

then we can say that the plane intersects the x-axis at $(3,0,0)$. Using the same process, we know that the y-intercept is $(0,2,0)$ and that the z-intercept is $(0,0,6)$. If we draw a triangle to connect those points, we can see that the volume E is a tetrahedron in the first octant that's sitting on top of a triangle in the xy-plane. That triangle has one vertex at the origin, with two sides extending out along the x and y-axes, and we'll call that triangle the region D.

There's no reason we can't integrate with respect to z first, which means we can treat this as a type I region. So the order of integration either needs to be $dz\ dy\ dx$ or $dz\ dx\ dy$. It doesn't matter which order we choose, as long as z comes first, so we'll choose $dz\ dy\ dx$.

Now we can find the limits of integration for each variable. Since z comes first, we'll need limits of integration for z in terms of x and y. We know we're bounded by the first octant, so the bounds will be

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$$0 \le z \le 6 - 2x - 3y$$

The limits of integration for y need to be in terms of x, so we'll set $z = 0$ into the plane equation since we will have already integrated with respect to z, and we'll get

$$2x + 3y + 0 = 6$$

$$3y = 6 - 2x$$

$$y = 2 - \frac{2}{3}x$$

So the bounds for y will be

$$0 \le y \le 2 - \frac{2}{3}x$$

To find the bounds for x, since we will have already integrated with respect to both z and y, we'll plug $z = 0$ and $y = 0$ into the plane equation.

$$2x + 3(0) + 0 = 6$$

$$2x = 6$$

$$x = 3$$

So the bounds for x will be

$$0 \le x \le 3$$

Plugging everything into the triple integral, we get

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$$\int_0^3 \int_0^{2-\frac{2}{3}x} \int_0^{6-2x-3y} 4x \ dz \ dy \ dx$$

Now we can evaluate. We'll integrate first with respect to z.

$$\int_0^3 \int_0^{2-\frac{2}{3}x} 4xz \Big|_0^{6-2x-3y} \ dy \ dx$$

$$\int_0^3 \int_0^{2-\frac{2}{3}x} 4x(6-2x-3y) - 4x(0) \ dy \ dx$$

$$\int_0^3 \int_0^{2-\frac{2}{3}x} 24x - 8x^2 - 12xy \ dy \ dx$$

Integrate with respect to y.

$$\int_0^3 24xy - 8x^2y - 6xy^2 \Big|_0^{2-\frac{2}{3}x} \ dx$$

$$\int_0^3 24x\left(2-\frac{2}{3}x\right) - 8x^2\left(2-\frac{2}{3}x\right) - 6x\left(2-\frac{2}{3}x\right)^2 - \left[24x(0) - 8x^2(0) - 6x(0)^2\right] \ dx$$

$$\int_0^3 48x - \frac{48}{3}x^2 - 16x^2 + \frac{16}{3}x^3 - 6x\left(4 - \frac{8}{3}x + \frac{4}{9}x^2\right) \ dx$$

$$\int_0^3 48x - \frac{48}{3}x^2 - 16x^2 + \frac{16}{3}x^3 - 24x + \frac{48}{3}x^2 - \frac{24}{9}x^3 \ dx$$

$$\int_0^3 24x - 16x^2 + \frac{24}{9}x^3 \ dx$$

Integrate with respect to x.

$$12x^2 - \frac{16}{3}x^3 + \frac{2}{3}x^4 \Big|_0^3$$

$$12(3)^2 - \frac{16}{3}(3)^3 + \frac{2}{3}(3)^4 - \left[12(0)^2 - \frac{16}{3}(0)^3 + \frac{2}{3}(0)^4\right]$$

$$12(9) - \frac{16}{3}(27) + \frac{2}{3}(81)$$

$$108 - 144 + 54$$

$$18$$

This is the volume given by the triple integral.

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Topic: Type I, II and III regions

Question: Find volume.

Determine the volume of the region that lies behind the plane $x+y+z=8$ and in front of the region in the yz-plane that is bounded by

$$z=\frac{3}{2}\sqrt{y} \text{ and } z=\frac{3}{4}y$$

Answer choices:

A $\frac{49}{5}$

B $\frac{29}{5}$

C $-\frac{29}{5}$

D $-\frac{49}{5}$

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Solution: A

The volume we're trying to find is the region E that lies behind

$$x + y + z = 8$$

We can sketch the plane by finding intercepts for each of the major axes. For example, if we set $y = 0$ and $z = 0$,

$$x + 0 + 0 = 8$$

$$x = 8$$

then we can say that the plane intersects the x-axis at $(8,0,0)$. Using the same process, we know that the y-intercept is $(0,8,0)$ and that the z -intercept is $(0,0,8)$. If we draw a triangle to connect those points, we can see a picture of the plane.

Then, if we sketch

$$z = \frac{3}{2}\sqrt{y} \text{ and } z = \frac{3}{4}y$$

in the yz-plane, we see a small region enclosed by the curves. So the volume we're talking about is the volume sandwiched by that small area, and the plane from earlier.

Since our volume is resting on top of the yz-plane, the easiest way to set up the integral will be as a type II region. That means we'll integrate first with respect to x. So the order of integration either needs to be $dx\ dz\ dy$ or $dx\ dy\ dz$. It doesn't matter which order we choose, as long as x comes first, so we'll choose $dx\ dz\ dy$.

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Now we can find the limits of integration for each variable. Since x comes first, we'll need limits of integration for x in terms of y and z. We know we have to stay in front of the yz-plane, so the bounds will be

$$0 \le x \le 8 - y - z$$

To find the limits of integration for z, we use the curves we were given.

$$\frac{3}{4}y \le z \le \frac{3}{2}\sqrt{y}$$

To find the limits of integration for y, we can set the curves equal to one another.

$$\frac{3}{4}y = \frac{3}{2}\sqrt{y}$$

$$3y = 6\sqrt{y}$$

$$y = 2\sqrt{y}$$

$$y^2 = 4y$$

$$y^2 - 4y = 0$$

$$y(y - 4) = 0$$

$$y = 0, 4$$

So the bounds for y will be

$$0 \le y \le 4$$

Plugging everything into the triple integral, we get

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$$\int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} \int_0^{8-y-z} dx\ dz\ dy$$

Now we can evaluate. We'll integrate first with respect to x.

$$\int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} x\Big|_0^{8-y-z} dz\ dy$$

$$\int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} 8-y-z-0\ dz\ dy$$

$$\int_0^4 \int_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} 8-y-z\ dz\ dy$$

Integrate with respect to z.

$$\int_0^4 8z-yz-\frac{1}{2}z^2\Big|_{\frac{3}{4}y}^{\frac{3}{2}\sqrt{y}} dy$$

$$\int_0^4 8\left(\frac{3}{2}\sqrt{y}\right)-y\left(\frac{3}{2}\sqrt{y}\right)-\frac{1}{2}\left(\frac{3}{2}\sqrt{y}\right)^2-\left[8\left(\frac{3}{4}y\right)-y\left(\frac{3}{4}y\right)-\frac{1}{2}\left(\frac{3}{4}y\right)^2\right] dy$$

$$\int_0^4 \frac{24}{2}\sqrt{y}-\frac{3}{2}y^{\frac{3}{2}}-\frac{9}{8}y-\left(\frac{24}{4}y-\frac{3}{4}y^2-\frac{9}{32}y^2\right) dy$$

$$\int_0^4 \frac{24}{2}\sqrt{y}-\frac{3}{2}y^{\frac{3}{2}}-\frac{9}{8}y-\frac{24}{4}y+\frac{3}{4}y^2+\frac{9}{32}y^2\ dy$$

$$\int_0^4 \frac{24}{2}y^{\frac{1}{2}}-\frac{3}{2}y^{\frac{3}{2}}-\frac{57}{8}y+\frac{33}{32}y^2\ dy$$

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Integrate with respect to y.

$$8y^{\frac{3}{2}} - \frac{3}{5}y^{\frac{5}{2}} - \frac{57}{16}y^2 + \frac{11}{32}y^3 \Big|_0^4$$

$$8(4)^{\frac{3}{2}} - \frac{3}{5}(4)^{\frac{5}{2}} - \frac{57}{16}(4)^2 + \frac{11}{32}(4)^3 - \left[8(0)^{\frac{3}{2}} - \frac{3}{5}(0)^{\frac{5}{2}} - \frac{57}{16}(0)^2 + \frac{11}{32}(0)^3\right]$$

$$8(4)^{\frac{3}{2}} - \frac{3}{5}(4)^{\frac{5}{2}} - \frac{57}{16}(4)^2 + \frac{11}{32}(4)^3$$

$$8(8) - \frac{3}{5}(32) - \frac{57}{16}(16) + \frac{11}{32}(64)$$

$$64 - \frac{96}{5} - 57 + 22$$

$$29 - \frac{96}{5}$$

$$\frac{145}{5} - \frac{96}{5}$$

$$\frac{49}{5}$$

This is the volume given by the triple integral.

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Topic: Type I, II and III regions

Question: Evaluate the triple integral.

$$\iiint_E \sqrt{3x^2+3z^2}\ dV$$

where E is the solid bounded by $y=2x^2+2z^2$ and the plane $y=8$.

Answer choices:

A $\quad -\dfrac{128\pi\sqrt{3}}{5}$

B $\quad -\dfrac{256\pi\sqrt{3}}{15}$

C $\quad \dfrac{128\pi\sqrt{3}}{5}$

D $\quad \dfrac{256\pi\sqrt{3}}{15}$

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Solution: D

We're looking for the volume of the region E that's sitting on the plane $y=8$. But the volume is also contained by $y=2x^2+2z^2$.

Since our volume oriented in terms of y, the easiest way to set up the integral will be to integrate y first. The order of integration either needs to be $dy\ dx\ dz$ or $dy\ dz\ dx$. It doesn't matter which order we choose, as long as y comes first.

Now we can find the limits of integration for each variable. Since y comes first, we'll need limits of integration for y in terms of x and z.

$$2x^2+2z^2 \le y \le 8$$

To find the limits of integration for x and z, it'll be easier to switch to polar coordinates. From a sketch of the region, we can say

$$0 \le r \le 2$$

$$0 \le \theta \le 2\pi$$

Plugging everything into the triple integral, we get

$$\int_0^{2\pi}\int_0^2 \left[\int_{2x^2+2z^2}^{8} \sqrt{3x^2+3z^2}\ dy\right]\ r\ dr\ d\theta$$

Now we can evaluate. We'll integrate first with respect to y.

$$\int_0^{2\pi}\int_0^2 \left[y\sqrt{3x^2+3z^2}\Big|_{2x^2+2z^2}^{8}\right]\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2 \left[8\sqrt{3x^2+3z^2}-\left(2x^2+2z^2\right)\sqrt{3x^2+3z^2}\right]\ r\ dr\ d\theta$$

Now we'll convert to polar coordinates, knowing that $x^2+z^2=r^2$.

$$\int_0^{2\pi}\int_0^2 \left[8\sqrt{3r^2}-\left(2r^2\right)\sqrt{3r^2}\right]\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2 \left[8\sqrt{3}r-2\sqrt{3}r^3\right]\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2 8\sqrt{3}r^2-2\sqrt{3}r^4\ dr\ d\theta$$

Integrate with respect to r.

$$\int_0^{2\pi} \frac{8\sqrt{3}}{3}r^3-\frac{2\sqrt{3}}{5}r^5\Big|_0^2\ d\theta$$

$$\int_0^{2\pi} \frac{8\sqrt{3}}{3}(2)^3-\frac{2\sqrt{3}}{5}(2)^5-\left[\frac{8\sqrt{3}}{3}(0)^3-\frac{2\sqrt{3}}{5}(0)^5\right]\ d\theta$$

$$\int_0^{2\pi} \frac{64\sqrt{3}}{3}-\frac{64\sqrt{3}}{5}\ d\theta$$

Integrate with respect to θ.

$$\frac{64\sqrt{3}}{3}\theta-\frac{64\sqrt{3}}{5}\theta\Big|_0^{2\pi}$$

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$$\frac{64\sqrt{3}}{3}(2\pi) - \frac{64\sqrt{3}}{5}(2\pi) - \left[\frac{64\sqrt{3}}{3}(0) - \frac{64\sqrt{3}}{5}(0)\right]$$

$$\frac{128\pi\sqrt{3}}{3} - \frac{128\pi\sqrt{3}}{5}$$

$$\frac{640\pi\sqrt{3}}{15} - \frac{384\pi\sqrt{3}}{15}$$

$$\frac{256\pi\sqrt{3}}{15}$$

This is the volume given by the triple integral.

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Topic: Cylindrical coordinates

Question: Convert the rectangular point (π, π, π) into cylindrical coordinates.

Answer choices:

A $\left(\sqrt{2}\pi, \frac{5\pi}{4}, \pi\right)$ and $\left(\sqrt{2}\pi, \frac{\pi}{4}, \pi\right)$

B $\left(-\sqrt{2}\pi, \frac{5\pi}{4}, \pi\right)$ and $\left(-\sqrt{2}\pi, \frac{\pi}{4}, \pi\right)$

C $\left(-\sqrt{2}\pi, \frac{\pi}{4}, \pi\right)$ and $\left(\sqrt{2}\pi, \frac{5\pi}{4}, \pi\right)$

D $\left(\sqrt{2}\pi, \frac{\pi}{4}, \pi\right)$ and $\left(-\sqrt{2}\pi, \frac{5\pi}{4}, \pi\right)$

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Solution: D

We'll plug (π, π, π) into the conversion formulas.

$$x = r\cos\theta$$

$$\pi = r\cos\theta$$

$$r = \frac{\pi}{\cos\theta}$$

and

$$y = r\sin\theta$$

$$\pi = r\sin\theta$$

$$r = \frac{\pi}{\sin\theta}$$

and

$$z = z$$

$$z = \pi$$

Since these first two equations are equal to r, we can set them equal to each other.

$$\frac{\pi}{\cos\theta} = \frac{\pi}{\sin\theta}$$

$$\cos\theta = \sin\theta$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

Since we found two values for θ, we'll have two cylindrical coordinate points that can represent the rectangular point (π, π, π). We'll plug these θ values into one of the equations we found for r.

For $\theta = \pi/4$:

$$r = \frac{\pi}{\cos\frac{\pi}{4}}$$

$$r = \frac{\pi}{\frac{\sqrt{2}}{2}}$$

$$r = \frac{2\pi}{\sqrt{2}}$$

$$r = \sqrt{2}\pi$$

For $\theta = 5\pi/4$:

$$r = \frac{\pi}{\cos\frac{5\pi}{4}}$$

$$r = \frac{\pi}{-\frac{\sqrt{2}}{2}}$$

$$r = -\frac{2\pi}{\sqrt{2}}$$

$$r = -\sqrt{2}\pi$$

Putting these values together, we can say that the rectangular point (π, π, π) is the same as the cylindrical points

$$\left(\sqrt{2}\pi, \frac{\pi}{4}, \pi\right) \text{ and } \left(-\sqrt{2}\pi, \frac{5\pi}{4}, \pi\right)$$

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Topic: Cylindrical coordinates

Question: Convert the rectangular point $(0,\pi,-1)$ into cylindrical coordinates.

Answer choices:

A $\left(-\pi,\frac{\pi}{2},-1\right)$ and $\left(\pi,\frac{3\pi}{2},-1\right)$

B $\left(\pi,\frac{\pi}{2},-1\right)$ and $\left(-\pi,\frac{3\pi}{2},-1\right)$

C $\left(-\pi,\frac{3\pi}{2},-1\right)$ and $\left(\pi,\frac{\pi}{2},-1\right)$

D $\left(\pi,\frac{3\pi}{2},-1\right)$ and $\left(-\pi,\frac{\pi}{2},-1\right)$

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Solution: B

We'll plug $(0,\pi,-1)$ into the conversion formulas.

$$x = r\cos\theta$$

$$0 = r\cos\theta$$

$$r = 0 \text{ and } \theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

and

$$y = r\sin\theta$$

$$\pi = r\sin\theta$$

$$r = \frac{\pi}{\sin\theta}$$

and

$$z = z$$

$$z = -1$$

Since these first two equations are equal to r, we can set them equal to each other.

$$0 = \frac{\pi}{\sin\theta}$$

There are no values of θ that make this equation true, which means that $r = 0$ can't be a solution. However, we can plug the values we found for θ into the second equation for r.

Since we found two values for θ, we'll have two cylindrical coordinate points that can represent the rectangular point $(0,\pi,-1)$. We'll plug these θ values into one of the equations we found for r.

For $\theta=\pi/2$:

$$r=\frac{\pi}{\sin\frac{\pi}{2}}$$

$$r=\frac{\pi}{1}$$

$$r=\pi$$

For $\theta=3\pi/2$:

$$r=\frac{\pi}{\sin\frac{3\pi}{2}}$$

$$r=\frac{\pi}{-1}$$

$$r=-\pi$$

Putting these values together, we can say that the rectangular point $(0,\pi,-1)$ is the same as the cylindrical points

$\left(\pi,\frac{\pi}{2},-1\right)$ and $\left(-\pi,\frac{3\pi}{2},-1\right)$

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Topic: Cylindrical coordinates

Question: Convert the rectangular point $(2,-4,3\pi/4)$ into cylindrical coordinates.

Answer choices:

A $\left(-4.471,1.107,\frac{3\pi}{4}\right)$

B $\left(4.471,-1.107,\frac{3\pi}{4}\right)$

C $\left(-\frac{1}{2},2,\frac{3\pi}{4}\right)$

D $\left(\frac{1}{2},-2,\frac{3\pi}{4}\right)$

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Solution: B

We'll plug $(2, -4, 3\pi/4)$ into the conversion formulas.

$$x = r\cos\theta$$

$$2 = r\cos\theta$$

$$r = \frac{2}{\cos\theta}$$

and

$$y = r\sin\theta$$

$$-4 = r\sin\theta$$

$$r = \frac{-4}{\sin\theta}$$

and

$$z = z$$

$$z = \frac{3\pi}{4}$$

Since these first two equations are equal to r, we can set them equal to each other.

$$\frac{2}{\cos\theta} = \frac{-4}{\sin\theta}$$

$$2\sin\theta = -4\cos\theta$$

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$$-\frac{1}{2}\sin\theta = \cos\theta$$

$$-\frac{1}{2} = \frac{\cos\theta}{\sin\theta}$$

$$-\frac{1}{2} = \cot\theta$$

$$\theta \approx -1.107$$

We'll plug this θ value into one of the equations we found for r.

$$r \approx \frac{2}{\cos(-1.107)}$$

$$r \approx 4.471$$

Putting these values together, we can say that the rectangular point $(2,-4,3\pi/4)$ is the same as the cylindrical point

$$\left(4.471,-1.107,\frac{3\pi}{4}\right)$$

Topic: Changing triple integrals to cylindrical coordinates

Question: Which of the following represents this triple integral written in cylindrical coordinates?

$$\int_{1}^{3}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{3} xyz\ dz\ dx\ dy$$

Answer choices:

A $\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{3} r^2\cos\theta\sin\theta\ dz\ dr\ d\theta$

B $\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{3} r^3 z\cos\theta\sin\theta\ dz\ dr\ d\theta$

C $\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{3} r^3\cos\theta\sin\theta\ dz\ dr\ d\theta$

D $\int_{0}^{2\pi}\int_{0}^{1}\int_{r}^{3} r^2 z\cos\theta\sin\theta\ dz\ dr\ d\theta$

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Solution: B

To convert from rectangular to cylindrical coordinates, we use the conversion formulas

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

We'll start by finding new limits of integration for the triple integral we've been given

$$\int_1^3 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{\sqrt{x^2+y^2}}^3 xyz \ dz \ dx \ dy$$

Let's start with the innermost integral, which is in terms of z. The upper limit includes no variables, it's just a constant, so it can be left alone. The lower limit can be converted using the conversion formula $r^2 = x^2 + y^2$.

$$z = \sqrt{x^2 + y^2}$$

$$z = \sqrt{r^2}$$

$$z = r$$

So the new limits for z are $[r,3]$.

Next we can convert the middle integral, which is in terms of x. To convert we will need to remember that $x = r\cos\theta$, and $y = r\sin\theta$. We can start with the upper limit.

$$x = \sqrt{1 - y^2}$$

$$r\cos\theta = \sqrt{1 - (r\sin\theta)^2}$$

$$r\cos\theta = \sqrt{1 - r^2\sin^2(\theta)}$$

$$r^2\cos^2\theta = 1 - r^2\sin^2\theta$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 1$$

$$r^2(\cos^2\theta + \sin^2\theta) = 1$$

We know that $\cos^2\theta + \sin^2\theta = 1$, so

$$r^2(1) = 1$$

$$r = \pm 1$$

But r can never be negative, so we bring up the lower bound to $r = 0$, and then we can say that the bounds for r are $[0,1]$.

Now we can convert the outer integral, which is in terms of θ. The limits for θ will be $[0,2\pi]$, because the limits of integration define the full 360° of a cylinder.

Next we can convert the function $f(x,y,z)=xyz$ into cylindrical coordinates. When we convert to cylindrical coordinates, we always multiply the integrand by r. So the integrand will be

$$f(r,\theta,z)=(r\cos\theta)(r\sin\theta)z(r)$$

$$f(r,\theta,z)=r^3z\sin\theta\cos\theta$$

Finally we can assemble the new triple integral in terms of cylindrical coordinates.

$$\int_1^3\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{\sqrt{x^2+y^2}}^{3} xyz\ dz\ dx\ dy=\int_0^{2\pi}\int_0^1\int_r^3 r^3z\sin\theta\cos\theta\ dz\ dr\ d\theta$$

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Topic: Changing triple integrals to cylindrical coordinates

Question: Which of the following represents this triple integral written in cylindrical coordinates?

$$\int_{1}^{2}\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{4\sqrt{x^2+y^2}}^{10} y^2z\ dz\ dx\ dy$$

Answer choices:

A $\displaystyle\int_{0}^{2\pi}\int_{0}^{2}\int_{4r}^{10} r^3z\sin^2\theta\ dz\ dr\ d\theta$

B $\displaystyle\int_{0}^{2\pi}\int_{0}^{2}\int_{2r}^{10} r^2z\sin^2\theta\ dz\ dr\ d\theta$

C $\displaystyle\int_{0}^{2\pi}\int_{0}^{2}\int_{2r}^{10} r^3z\sin^2\theta\ dz\ dr\ d\theta$

D $\displaystyle\int_{0}^{2\pi}\int_{0}^{2}\int_{4r}^{10} r^2z\sin^2\theta\ dz\ dr\ d\theta$

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Solution: A

To convert from rectangular to cylindrical coordinates, we use the conversion formulas

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

We'll start by finding new limits of integration for the triple integral we've been given

$$\int_1^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{4\sqrt{x^2+y^2}}^{10} y^2z\ dz\ dx\ dy$$

Let's start with the innermost integral, which is in terms of z. The upper limit includes no variables, it's just a constant, so it can be left alone. The lower limit can be converted using the conversion formula $r^2 = x^2 + y^2$.

$$z = 4\sqrt{x^2+y^2}$$

$$z = 4\sqrt{r^2}$$

$$z = 4r$$

So the new limits for z are $[4r, 10]$.

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Next we can convert the middle integral, which is in terms of x. To convert we will need to remember that $x = r\cos\theta$, and $y = r\sin\theta$. We can start with the upper limit.

$$x = \sqrt{4 - y^2}$$

$$r\cos\theta = \sqrt{4 - (r\sin\theta)^2}$$

$$r\cos\theta = \sqrt{4 - r^2\sin^2(\theta)}$$

$$r^2\cos^2\theta = 4 - r^2\sin^2\theta$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 4$$

$$r^2(\cos^2\theta + \sin^2\theta) = 4$$

We know that $\cos^2\theta + \sin^2\theta = 1$, so

$$r^2(1) = 4$$

$$r = \pm 2$$

But r can never be negative, so we bring up the lower bound to $r = 0$, and then we can say that the bounds for r are $[0,2]$.

Now we can convert the outer integral, which is in terms of θ. The limits for θ will be $[0,2\pi]$, because the limits of integration define the full 360° of a cylinder.

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Next we can convert the function $f(x,y,z)=y^2z$ into cylindrical coordinates. When we convert to cylindrical coordinates, we always multiply the integrand by r. So the integrand will be

$$f(r,\theta,z)=(r\sin\theta)^2z(r)$$

$$f(r,\theta,z)=r^3z\sin^2\theta$$

Finally we can assemble the new triple integral in terms of cylindrical coordinates.

$$\int_1^2\int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}}\int_{4\sqrt{x^2+y^2}}^{10}y^2z\ dz\ dx\ dy=\int_0^{2\pi}\int_0^2\int_{4r}^{10}r^3z\sin^2\theta\ dz\ dr\ d\theta$$

Topic: Changing triple integrals to cylindrical coordinates

Question: Which of the following represents this triple integral written in cylindrical coordinates?

$$\int_{2}^{3}\int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}}\int_{9\sqrt{x^2+y^2}}^{12} x^3z\ dz\ dx\ dy$$

Answer choices:

A $\displaystyle\int_{0}^{2\pi}\int_{0}^{4}\int_{3r}^{12} r^3z\cos^3\theta\ dz\ dr\ d\theta$

B $\displaystyle\int_{0}^{2\pi}\int_{0}^{16}\int_{9r}^{12} r^3z\cos^3\theta\ dz\ dr\ d\theta$

C $\displaystyle\int_{0}^{2\pi}\int_{0}^{4}\int_{9r}^{12} r^4z\cos^3\theta\ dz\ dr\ d\theta$

D $\displaystyle\int_{0}^{2\pi}\int_{0}^{16}\int_{3r}^{12} r^4z\cos^3\theta\ dz\ dr\ d\theta$

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Solution: C

To convert from rectangular to cylindrical coordinates, we use the conversion formulas

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

We'll start by finding new limits of integration for the triple integral we've been given

$$\int_2^3 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{9\sqrt{x^2+y^2}}^{12} x^3 z \ dz \ dx \ dy$$

Let's start with the innermost integral, which is in terms of z. The upper limit includes no variables, it's just a constant, so it can be left alone. The lower limit can be converted using the conversion formula $r^2 = x^2 + y^2$.

$$z = 9\sqrt{x^2+y^2}$$

$$z = 9\sqrt{r^2}$$

$$z = 9r$$

So the new limits for z are $[9r,12]$.

Next we can convert the middle integral, which is in terms of x. To convert we will need to remember that $x = r\cos\theta$, and $y = r\sin\theta$. We can start with the upper limit.

$$x = \sqrt{16 - y^2}$$

$$r\cos\theta = \sqrt{16 - (r\sin\theta)^2}$$

$$r\cos\theta = \sqrt{16 - r^2\sin^2\theta}$$

$$r^2\cos^2\theta = 16 - r^2\sin^2\theta$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 16$$

$$r^2(\cos^2\theta + \sin^2\theta) = 16$$

We know that $\cos^2\theta + \sin^2\theta = 1$, so

$$r^2(1) = 16$$

$$r = \pm 4$$

But r can never be negative, so we bring up the lower bound to $r = 0$, and then we can say that the bounds for r are $[0,4]$.

Now we can convert the outer integral, which is in terms of θ. The limits for θ will be $[0,2\pi]$, because the limits of integration define the full 360° of a cylinder.

Next we can convert the function $f(x, y, z) = x^3z$ into cylindrical coordinates. When we convert to cylindrical coordinates, we always multiply the integrand by r. So the integrand will be

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$$f(r,\theta,z) = (r\cos\theta)^3 z(r)$$

$$f(r,\theta,z) = r^4 z\cos^3\theta$$

Finally we can assemble the new triple integral in terms of cylindrical coordinates.

$$\int_2^3 \int_{-\sqrt{16-y^2}}^{\sqrt{16-y^2}} \int_{9\sqrt{x^2+y^2}}^{12} x^3 z\ dz\ dx\ dy = \int_0^{2\pi} \int_0^4 \int_{9r}^{12} r^4 z\cos^3\theta\ dz\ dr\ d\theta$$

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Topic: Finding volume

Question: Use cylindrical coordinates to find the volume of the solid.

Find

$$\iiint_E dV$$

where E is the solid that lies

within the cylinder $x^2 + y^2 = 1$

above the plane $z = 0$

below the cone $z^2 = x^2 + y^2$

Answer choices:

A $\frac{\pi}{3}$

B π

C 2π

D $\frac{2\pi}{3}$

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Solution: D

We've been given equations in rectangular coordinates, so we'll need to convert them to cylindrical coordinates using the conversion formulas

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

We know we need to stay within the cylinder $x^2 + y^2 = 1$, which in cylindrical coordinates will be

$$r^2 = 1$$

$$r = 1$$

In cylindrical coordinates, r represents the distance from the origin, so the limits of integration for r will be $[0,1]$.

Because in cylindrical coordinates $z = z$, that means $z = 0$ remains the same.

We were told we also need to stay below the cone $z^2 = x^2 + y^2$, which in cylindrical coordinates will be

$$z^2 = r^2$$

$$z = r$$

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Putting this together with $z=0$ tells us that the limits of integration for z will be $[0,r]$.

Because we're talking just about staying inside the cylinder, but there's no restriction on where inside the cylinder, that means the limits of integration for θ in cylindrical coordinates will be the full $[0,2\pi]$.

Our next step is to convert the function itself

$$f(x,y,z)=1$$

Since there are no variables in the function, we don't need to apply any conversion formulas. However, whenever we change from rectangular coordinates to cylindrical coordinates, we do need to multiply the integrand by r. Therefore, the integrand will be

$$f(r,\theta,z)=1r$$

$$f(r,\theta,z)=r$$

The triple integral becomes

$$\iiint_E dV=\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta$$

To find volume, we just need to evaluate the integral. We'll work from the inside out, integrating first with respect to z.

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta=\int_0^{2\pi}\int_0^1 rz\Big|_0^r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^1 r(r) - r(0)\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^1 r^2\ dr\ d\theta$$

Integrate with respect to r.

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \int_0^{2\pi} \frac{1}{3}r^3\Big|_0^1\ d\theta$$

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \int_0^{2\pi} \frac{1}{3}(1)^3 - \frac{1}{3}(0)^3\ d\theta$$

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \int_0^{2\pi} \frac{1}{3}\ d\theta$$

Integrate with respect to θ.

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \frac{1}{3}\theta\Big|_0^{2\pi}$$

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \frac{1}{3}(2\pi) - \frac{1}{3}(0)$$

$$\int_0^{2\pi}\int_0^1\int_0^r r\ dz\ dr\ d\theta = \frac{2\pi}{3}$$

This is the volume of the solid E, which we found using cylindrical coordinates.

Topic: Finding volume

Question: Use cylindrical coordinates to find the volume of the solid.

Find

$$\iiint_E y \, dV$$

where E is the solid that lies

within the cylinder $x^2 + y^2 = 4$

above the plane $z = 0$

below the cone $z^2 = 9x^2 + 9y^2$

Answer choices:

A 24

B 0

C π

D 12

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Solution: B

We've been given equations in rectangular coordinates, so we'll need to convert them to cylindrical coordinates using the conversion formulas

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

We know we need to stay within the cylinder $x^2 + y^2 = 4$, which in cylindrical coordinates will be

$$r^2 = 4$$

$$r = 2$$

In cylindrical coordinates, r represents the distance from the origin, so the limits of integration for r will be $[0,2]$.

Because in cylindrical coordinates $z = z$, that means $z = 0$ remains the same.

We were told we also need to stay below the cone $z^2 = 9x^2 + 9y^2$, which in cylindrical coordinates will be

$$z^2 = 9r^2$$

$$z = 3r$$

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Putting this together with $z=0$ tells us that the limits of integration for z will be $[0,3r]$.

Because we're talking just about staying inside the cylinder, but there's no restriction on where inside the cylinder, that means the limits of integration for θ in cylindrical coordinates will be the full $[0,2\pi]$.

Our next step is to convert the function itself

$$f(x,y,z)=y$$

We'll get

$$f(x,y,z)=r\sin\theta$$

And whenever we change from rectangular coordinates to cylindrical coordinates, we do need to multiply the integrand by r. Therefore, the integrand will be

$$f(r,\theta,z)=(r)r\sin\theta$$

$$f(r,\theta,z)=r^2\sin\theta$$

The triple integral becomes

$$\iiint_E dV=\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta$$

To find volume, we just need to evaluate the integral. We'll work from the inside out, integrating first with respect to z.

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta=\int_0^{2\pi}\int_0^2\left(r^2\sin\theta\right)z\Big|_0^{3r}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^2 (3r)\left(r^2\sin\theta\right) - (0)\left(r^2\sin\theta\right)\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = \int_0^{2\pi}\int_0^2 3r^3\sin\theta\ dr\ d\theta$$

Integrate with respect to r.

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = \int_0^{2\pi} \frac{3}{4}r^4\sin\theta\Big|_0^2\ d\theta$$

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = \int_0^{2\pi} \frac{3}{4}(2)^4\sin\theta - \frac{3}{4}(0)^4\sin\theta\ d\theta$$

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = \int_0^{2\pi} 12\sin\theta\ d\theta$$

Integrate with respect to θ.

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = -12\cos\theta\Big|_0^{2\pi}$$

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = -12\cos(2\pi) - (-12\cos(0))$$

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = -12(1) - (-12(1))$$

$$\int_0^{2\pi}\int_0^2\int_0^{3r} r^2\sin\theta\ dz\ dr\ d\theta = 0$$

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This is the volume of the solid E, which we found using cylindrical coordinates.

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Topic: Finding volume

Question: Use cylindrical coordinates to find the volume of the solid.

$$\iiint_E x^2 z \ dV$$

where E is the solid that lies

within the cylinder $x^2 + y^2 = 16$

above the plane $z = 0$

below the cone $z^2 = 4x^2 + 4y^2$

Answer choices:

A $4{,}096\pi$

B $2{,}048\pi$

C $\dfrac{4{,}096\pi}{3}$

D $\dfrac{2{,}048\pi}{3}$

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Solution: C

We've been given equations in rectangular coordinates, so we'll need to convert them to cylindrical coordinates using the conversion formulas

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

We know we need to stay within the cylinder $x^2 + y^2 = 16$, which in cylindrical coordinates will be

$$r^2 = 16$$

$$r = 4$$

In cylindrical coordinates, r represents the distance from the origin, so the limits of integration for r will be $[0,4]$.

Because in cylindrical coordinates $z = z$, that means $z = 0$ remains the same.

We were told we also need to stay below the cone $z^2 = 4x^2 + 4y^2$, which in cylindrical coordinates will be

$$z^2 = 4r^2$$

$$z = 2r$$

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Putting this together with $z=0$ tells us that the limits of integration for z will be $[0,2r]$.

Because we're talking just about staying inside the cylinder, but there's no restriction on where inside the cylinder, that means the limits of integration for θ in cylindrical coordinates will be the full $[0,2\pi]$.

Our next step is to convert the function itself

$$f(x,y,z)=x^2z$$

We'll get

$$f(x,y,z)=r^2z\cos^2\theta$$

And whenever we change from rectangular coordinates to cylindrical coordinates, we do need to multiply the integrand by r. Therefore, the integrand will be

$$f(r,\theta,z)=(r)r^2z\cos^2\theta$$

$$f(r,\theta,z)=r^3z\cos^2\theta$$

The triple integral becomes

$$\iiint_E \ dV=\int_0^{2\pi}\int_0^4\int_0^{2r} r^3z\cos^2\theta\ dz\ dr\ d\theta$$

To find volume, we just need to evaluate the integral. We'll work from the inside out, integrating first with respect to z.

$$\int_0^{2\pi}\int_0^4\int_0^{2r} r^3z\cos^2\theta\ dz\ dr\ d\theta$$

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$$\int_0^{2\pi}\int_0^4 \frac{1}{2}r^3z^2\cos^2\theta\Big|_0^{2r}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^4 \frac{1}{2}r^3(2r)^2\cos^2\theta-\frac{1}{2}r^3(0)^2\cos^2\theta\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^4 2r^5\cos^2\theta\ dr\ d\theta$$

Integrate with respect to r.

$$\int_0^{2\pi}\frac{1}{3}r^6\cos^2\theta\Big|_0^4\ d\theta$$

$$\int_0^{2\pi}\frac{1}{3}(4)^6\cos^2\theta-\frac{1}{3}(0)^6\cos^2\theta\ d\theta$$

$$\int_0^{2\pi}\frac{4{,}096}{3}\cos^2\theta\ d\theta$$

Integrate with respect to θ. We'll need the trig identity

$$\cos^2\theta=\frac{1}{2}(1+\cos(2\theta))$$

in order to simplify the integrand.

$$\int_0^{2\pi}\frac{4{,}096}{3}\left[\frac{1}{2}(1+\cos(2\theta))\right]\ d\theta$$

$$\frac{2{,}048}{3}\int_0^{2\pi}1+\cos(2\theta)\ d\theta$$

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$$\frac{2,048}{3}\left[\theta+\frac{1}{2}\sin(2\theta)\right]\Big|_0^{2\pi}$$

$$\frac{2,048}{3}\left[2\pi+\frac{1}{2}\sin(2(2\pi))\right]-\frac{2,048}{3}\left[0+\frac{1}{2}\sin(2(0))\right]$$

$$\frac{2,048}{3}\left[2\pi+\frac{1}{2}(0)\right]-\frac{2,048}{3}\left[0+\frac{1}{2}(0)\right]$$

$$\frac{4,096\pi}{3}$$

This is the volume of the solid E, which we found using cylindrical coordinates.

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Topic: Spherical coordinates

Question: Convert the spherical coordinates into cartesian coordinates.

$$(0,\pi,1)$$

Answer choices:

A $(1,0,1)$

B $(0,0,0)$

C $(0,1,0)$

D $(1,1,1)$

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Solution: B

Spherical coordinates are given in the form (ρ, θ, ϕ), and to convert them to rectangular coordinates, we'll use these conversion formulas:

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

So we get

$$x = \rho \sin\phi \cos\theta$$

$$x = (0)\sin(1)\cos(\pi)$$

$$x = 0$$

and

$$y = \rho \sin\phi \sin\theta$$

$$y = (0)\sin(1)\sin(\pi)$$

$$y = 0$$

and

$$z = \rho \cos\phi$$

$$z = (0)\cos(1)$$

$$z = 0$$

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The spherical point $(0,\pi,1)$ is the same point in space as the rectangular point $(0,0,0)$.

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Topic: Spherical coordinates

Question: Convert the spherical coordinates into cartesian coordinates.

$$\left(1,\frac{\pi}{2},0\right)$$

Answer choices:

A $(0,0,1)$

B $(1,0,0)$

C $(0,1,0)$

D $(0,0,0)$

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Solution: A

Spherical coordinates are given in the form (ρ, θ, ϕ), and to convert them to rectangular coordinates, we'll use these conversion formulas:

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

So we get

$$x = \rho \sin \phi \cos \theta$$

$$x = (1)\sin(0)\cos\left(\frac{\pi}{2}\right)$$

$$x = 0$$

and

$$y = \rho \sin \phi \sin \theta$$

$$y = (1)\sin(0)\sin\left(\frac{\pi}{2}\right)$$

$$y = 0$$

and

$$z = \rho \cos \phi$$

$$z = (1)\cos(0)$$

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$$z = 1$$

The spherical point $\left(1,\frac{\pi}{2},0\right)$ is the same point in space as the rectangular point $(0,0,1)$.

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Topic: Spherical coordinates

Question: Convert the spherical coordinates into cartesian coordinates.

$$\left(1,2\pi,\frac{\pi}{2}\right)$$

Answer choices:

A $(0,1,0)$

B $(1,1,1)$

C $(0,0,0)$

D $(1,0,0)$

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Solution: D

Spherical coordinates are given in the form (ρ, θ, ϕ), and to convert them to rectangular coordinates, we'll use these conversion formulas:

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

So we get

$$x = \rho \sin \phi \cos \theta$$

$$x = (1)\sin\left(\frac{\pi}{2}\right)\cos(2\pi)$$

$$x = 1$$

and

$$y = \rho \sin \phi \sin \theta$$

$$y = (1)\sin\left(\frac{\pi}{2}\right)\sin(2\pi)$$

$$y = 0$$

and

$$z = \rho \cos \phi$$

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$$z = (1)\cos\left(\frac{\pi}{2}\right)$$

$$z = 0$$

The spherical point $\left(1,2\pi,\frac{\pi}{2}\right)$ is the same point in space as the rectangular point $(1,0,0)$.

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Topic: Changing triple integrals to spherical coordinates

Question: Convert into spherical coordinates.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 2\ dz\ dy\ dx$$

Answer choices:

A $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 2\rho \sin\phi\ d\rho\ d\theta\ d\phi$

B $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 2\rho^2 \cos\phi\ d\rho\ d\theta\ d\phi$

C $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 2\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$

D $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 2\rho \cos\phi\ d\rho\ d\theta\ d\phi$

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Solution: C

Spherical coordinates are given in the form (ρ, θ, ϕ), and to convert them to rectangular coordinates, we'll use these conversion formulas:

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

We also know that

$$\rho^2 = x^2 + y^2 + z^2$$

$$\rho \geq 0$$

$$0 \leq \phi \leq \pi$$

$$dV = \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

The limits of integration have been given to us as

$$0 \leq x \leq 1$$

$$0 \leq y \leq \sqrt{1 - x^2}$$

$$\sqrt{x^2 + y^2} \leq z \leq \sqrt{1 - x^2 - y^2}$$

Because x and y are both greater than 0, we know we're looking at a volume in the first quadrant. Therefore, in spherical coordinates

$$0 \leq \theta \leq \frac{\pi}{2}$$

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For ϕ, we need to look at the lower bound of z,

$$z = \sqrt{x^2 + y^2}$$

If we use $x^2 + y^2 = r^2$, we can say

$$z = \sqrt{r^2}$$

$$z = r$$

We know that $r = \rho \sin \phi$ and that $z = \rho \cos \phi$, so

$$\rho \cos \phi = \rho \sin \phi$$

$$\cos \phi = \sin \phi$$

This equation is only true when $\phi = \pi/4$. Since we found this by looking at the lower bound for z, that means we're looking at the volume above $\phi = \pi/4$. So the bounds for ϕ are

$$0 \leq \phi \leq \frac{\pi}{4}$$

If we look at the upper bound for z and convert it to spherical coordinates, we get

$$z = \sqrt{1 - x^2 - y^2}$$

$$x^2 + y^2 + z^2 = 1$$

$$\rho^2 = 1$$

$$\rho = 1$$

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So we can say that the bounds for ρ are

$$0 \le \rho \le 1$$

We've got the limits of integration done, but now we need to put everything into the integral.

$$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{1-x^2-y^2}} 2 \ dz \ dy \ dx = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^1 2\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

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Topic: Changing triple integrals to spherical coordinates

Question: Convert into spherical coordinates.

$$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z\ dz\ dy\ dx$$

Answer choices:

A $\displaystyle \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^3 \sin\phi\cos\phi\ d\rho\ d\theta\ d\phi$

B $\displaystyle \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^2 \sin\phi\cos\phi\ d\rho\ d\theta\ d\phi$

C $\displaystyle \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^4 \rho^2 \sin\phi\cos\phi\ d\rho\ d\theta\ d\phi$

D $\displaystyle \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^4 \rho^3 \sin\phi\cos\phi\ d\rho\ d\theta\ d\phi$

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Solution: A

Spherical coordinates are given in the form (ρ, θ, ϕ), and to convert them to rectangular coordinates, we'll use these conversion formulas:

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

We also know that

$$\rho^2 = x^2 + y^2 + z^2$$

$$\rho \geq 0$$

$$0 \leq \phi \leq \pi$$

$$dV = \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

The limits of integration have been given to us as

$$0 \leq x \leq 4$$

$$0 \leq y \leq \sqrt{4 - x^2}$$

$$\sqrt{x^2 + y^2} \leq z \leq \sqrt{4 - x^2 - y^2}$$

Because x and y are both greater than 0, we know we're looking at a volume in the first quadrant. Therefore, in spherical coordinates

$$0 \leq \theta \leq \frac{\pi}{2}$$

For ϕ, we need to look at the lower bound of z,

$$z = \sqrt{x^2 + y^2}$$

If we use $x^2 + y^2 = r^2$, we can say

$$z = \sqrt{r^2}$$

$$z = r$$

We know that $r = \rho \sin \phi$ and that $z = \rho \cos \phi$, so

$$\rho \cos \phi = \rho \sin \phi$$

$$\cos \phi = \sin \phi$$

This equation is only true when $\phi = \pi/4$. Since we found this by looking at the lower bound for z, that means we're looking at the volume above $\phi = \pi/4$. So the bounds for ϕ are

$$0 \leq \phi \leq \frac{\pi}{4}$$

If we look at the upper bound for z and convert it to spherical coordinates, we get

$$z = \sqrt{4 - x^2 - y^2}$$

$$x^2 + y^2 + z^2 = 4$$

$$\rho^2 = 4$$

$$\rho = 2$$

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So we can say that the bounds for ρ are

$$0 \leq \rho \leq 2$$

We've got the limits of integration done, but now we need to convert the integrand, z.

$$z = \rho \cos \phi$$

Now we need to put everything into the integral.

$$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z \ dz \ dy \ dx = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^2 \rho \cos \phi \left(\rho^2 \sin \phi \right) \ d\rho \ d\theta \ d\phi$$

$$\int_0^4 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z \ dz \ dy \ dx = \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^2 \rho^3 \sin \phi \cos \phi \ d\rho \ d\theta \ d\phi$$

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Topic: Changing triple integrals to spherical coordinates

Question: Convert into spherical coordinates.

$$\int_0^2 \int_0^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} 3z \ dz \ dy \ dx$$

Answer choices:

A $\quad \displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^4 3\rho^2 \sin\phi \cos\phi \ d\rho \ d\theta \ d\phi$

B $\quad \displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{16} 3\rho^3 \sin\phi \cos\phi \ d\rho \ d\theta \ d\phi$

C $\quad \displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^{16} 3\rho^2 \sin\phi \cos\phi \ d\rho \ d\theta \ d\phi$

D $\quad \displaystyle\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^4 3\rho^3 \sin\phi \cos\phi \ d\rho \ d\theta \ d\phi$

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Solution: D

Spherical coordinates are given in the form (ρ, θ, ϕ), and to convert them to rectangular coordinates, we'll use these conversion formulas:

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

We also know that

$$\rho^2 = x^2 + y^2 + z^2$$

$$\rho \geq 0$$

$$0 \leq \phi \leq \pi$$

$$dV = \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

The limits of integration have been given to us as

$$0 \leq x \leq 2$$

$$0 \leq y \leq \sqrt{9 - x^2}$$

$$\sqrt{x^2 + y^2} \leq z \leq \sqrt{16 - x^2 - y^2}$$

Because x and y are both greater than 0, we know we're looking at a volume in the first quadrant. Therefore, in spherical coordinates

$$0 \leq \theta \leq \frac{\pi}{2}$$

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For ϕ, we need to look at the lower bound of z,

$$z = \sqrt{x^2 + y^2}$$

If we use $x^2 + y^2 = r^2$, we can say

$$z = \sqrt{r^2}$$

$$z = r$$

We know that $r = \rho \sin \phi$ and that $z = \rho \cos \phi$, so

$$\rho \cos \phi = \rho \sin \phi$$

$$\cos \phi = \sin \phi$$

This equation is only true when $\phi = \pi/4$. Since we found this by looking at the lower bound for z, that means we're looking at the volume above $\phi = \pi/4$. So the bounds for ϕ are

$$0 \leq \phi \leq \frac{\pi}{4}$$

If we look at the upper bound for z and convert it to spherical coordinates, we get

$$z = \sqrt{16 - x^2 - y^2}$$

$$x^2 + y^2 + z^2 = 16$$

$$\rho^2 = 16$$

$$\rho = 4$$

So we can say that the bounds for ρ are

$$0 \le \rho \le 4$$

We've got the limits of integration done, but now we need to convert the integrand, z.

$$3z = 3\rho\cos\phi$$

Now we need to put everything into the integral.

$$\int_0^2 \int_0^{\sqrt{9-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{16-x^2-y^2}} 3z\ dz\ dy\ dx$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^4 3\rho\cos\phi\left(\rho^2\sin\phi\right)\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{2}} \int_0^4 3\rho^3\sin\phi\cos\phi\ d\rho\ d\theta\ d\phi$$

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Topic: Finding volume

Question: Use spherical coordinates to find the volume of the triple integral.

$$\iiint_E 4\ dV$$

where E is a sphere with center $(0,0,0)$ and radius 2

Answer choices:

A $\quad \dfrac{128\pi}{3}$

B $\quad -\dfrac{64\pi}{3}$

C $\quad -\dfrac{128\pi}{3}$

D $\quad \dfrac{64\pi}{3}$

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Solution: A

The first thing we can do is find the limits of integration in spherical coordinates. Because ρ represents radius in spherical coordinates, we can say that the bounds for ρ are

$$0 \le \rho \le 2$$

Because we've been told we're finding the volume of a full sphere, the bounds for θ will be

$$0 \le \theta \le 2\pi$$

and the bounds for ϕ will be

$$0 \le \phi \le \pi$$

The given integrand $f(x, y, z) = 4$ doesn't contain any variables, so it can stay just how it is. But when we convert from rectangular to spherical coordinates, $dV = 4\rho^2 \sin \phi$.

Plugging everything into the integral gives

$$\iiint_E 4\ dV = \int_0^{\pi} \int_0^{2\pi} \int_0^2 4\rho^2 \sin \phi\ d\rho\ d\theta\ d\phi$$

We always work from the inside out, which means we'll integrate first with respect to ρ.

$$V = \int_0^{\pi} \int_0^{2\pi} \frac{4}{3}\rho^3 \sin \phi \Big|_0^2\ d\theta\ d\phi$$

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$$V=\int_0^{\pi}\int_0^{2\pi}\frac{4}{3}(2)^3\sin\phi-\frac{4}{3}(0)^3\sin\phi\ d\theta\ d\phi$$

$$V=\int_0^{\pi}\int_0^{2\pi}\frac{32}{3}\sin\phi\ d\theta\ d\phi$$

Integrate with respect to θ.

$$V=\int_0^{\pi}\frac{32}{3}\theta\sin\phi\Big|_0^{2\pi}\ d\phi$$

$$V=\int_0^{\pi}\frac{32}{3}(2\pi)\sin\phi-\frac{32}{3}(0)\sin\phi\ d\phi$$

$$V=\int_0^{\pi}\frac{64\pi}{3}\sin\phi\ d\phi$$

Integrate with respect to ϕ.

$$V=-\frac{64\pi}{3}\cos\phi\Big|_0^{\pi}$$

$$V=-\frac{64\pi}{3}\cos\pi+\frac{64\pi}{3}\cos(0)$$

$$V=-\frac{64\pi}{3}(-1)+\frac{64\pi}{3}(1)$$

$$V=\frac{64\pi}{3}+\frac{64\pi}{3}$$

$$V=\frac{128\pi}{3}$$

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Topic: Finding volume

Question: Use spherical coordinates to find the volume of the triple integral.

$$\iiint_E x^2+y^2+z^2\ dV$$

where B is a sphere with the center $(0,0,0)$ and radius 1

Answer choices:

A $\frac{2\pi}{5}$

B $-\frac{4\pi}{5}$

C $\frac{4\pi}{5}$

D $-\frac{2\pi}{5}$

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Solution: C

The first thing we can do is find the limits of integration in spherical coordinates. Because ρ represents radius in spherical coordinates, we can say that the bounds for ρ are

$$0 \le \rho \le 1$$

Because we've been told we're finding the volume of a full sphere, the bounds for θ will be

$$0 \le \theta \le 2\pi$$

and the bounds for ϕ will be

$$0 \le \phi \le \pi$$

The given integrand $f(x, y, z) = x^2 + y^2 + z^2$ can be converted to spherical coordinates using the conversion formula $\rho^2 = x^2 + y^2 + z^2$, and it becomes $f(\rho, \theta, \phi) = \rho^2$. And when we convert from rectangular to spherical coordinates, $dV = \rho^2 \sin \phi$.

Plugging everything into the integral gives

$$\iiint_E x^2 + y^2 + z^2 \ dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^2 \left(\rho^2 \sin \phi\right) \ d\rho \ d\theta \ d\phi$$

$$\iiint_E x^2 + y^2 + z^2 \ dV = \int_0^{\pi} \int_0^{2\pi} \int_0^1 \rho^4 \sin \phi \ d\rho \ d\theta \ d\phi$$

We always work from the inside out, which means we'll integrate first with respect to ρ.

$$V=\int_0^{\pi}\int_0^{2\pi}\frac{1}{5}\rho^5\sin\phi\Big|_0^1\ d\theta\ d\phi$$

$$V=\int_0^{\pi}\int_0^{2\pi}\frac{1}{5}(1)^5\sin\phi-\frac{1}{5}(0)^5\sin\phi\ d\theta\ d\phi$$

$$V=\int_0^{\pi}\int_0^{2\pi}\frac{1}{5}\sin\phi\ d\theta\ d\phi$$

Integrate with respect to θ.

$$V=\int_0^{\pi}\frac{1}{5}\theta\sin\phi\Big|_0^{2\pi}\ d\phi$$

$$V=\int_0^{\pi}\frac{1}{5}(2\pi)\sin\phi-\frac{1}{5}(0)\sin\phi\ d\phi$$

$$V=\int_0^{\pi}\frac{2\pi}{5}\sin\phi\ d\phi$$

Integrate with respect to ϕ.

$$V=-\frac{2\pi}{5}\cos\phi\Big|_0^{\pi}$$

$$V=-\frac{2\pi}{5}\cos\pi+\frac{2\pi}{5}\cos(0)$$

$$V=-\frac{2\pi}{5}(-1)+\frac{2\pi}{5}(1)$$

$$V=\frac{2\pi}{5}+\frac{2\pi}{5}$$

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$$V = \frac{4\pi}{5}$$

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Topic: Finding volume

Question: Use spherical coordinates to find the volume of the triple integral, where B is the upper half of the sphere $x^2 + y^2 + z^2 = 9$.

$$\iiint_B 2(x^2 + y^2 + z^2)\ dV$$

Answer choices:

A $\quad \dfrac{1{,}458\pi}{5}$

B $\quad \dfrac{972\pi}{5}$

C $\quad \dfrac{486\pi}{5}$

D $\quad \dfrac{1{,}944\pi}{5}$

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Solution: B

The first thing we can do is find the limits of integration in spherical coordinates, and we're looking at the upper half of the sphere $x^2 + y^2 + z^2 = 9$, which is the sphere centered at the origin with radius 3. Because ρ represents radius in spherical coordinates, we can say that the bounds for ρ are

$$0 \leq \rho \leq 3$$

Because we've been told we're finding the volume of a half sphere, but that's still a full 2π rotation around the z-axis, the bounds for θ will be

$$0 \leq \theta \leq 2\pi$$

but the bounds for ϕ will reflect the fact that we just want the half sphere, and they'll be

$$0 \leq \phi \leq \frac{\pi}{2}$$

The given integrand $f(x, y, z) = 2(x^2 + y^2 + z^2)$ can be converted to spherical coordinates using the conversion formula $\rho^2 = x^2 + y^2 + z^2$, and it becomes $f(\rho, \theta, \phi) = 2\rho^2$. And when we convert from rectangular to spherical coordinates, $dV = \rho^2 \sin \phi$.

Plugging everything into the integral gives

$$\iiint_B 2(x^2 + y^2 + z^2) \ dV$$

$$\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^3 2\rho^2(\rho^2 \sin \phi) \ d\rho \ d\theta \ d\phi$$

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$$\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^3 2\rho^4\sin\phi\ d\rho\ d\theta\ d\phi$$

We always work from the inside out, which means we'll integrate first with respect to ρ.

$$V=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\frac{2}{5}\rho^5\sin\phi\Big|_0^3\ d\theta\ d\phi$$

$$V=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\frac{2}{5}(3)^5\sin\phi-\frac{2}{5}(0)^5\sin\phi\ d\theta\ d\phi$$

$$V=\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\frac{486}{5}\sin\phi\ d\theta\ d\phi$$

Integrate with respect to θ.

$$V=\int_0^{\frac{\pi}{2}}\frac{486}{5}\theta\sin\phi\Big|_0^{2\pi}\ d\phi$$

$$V=\int_0^{\frac{\pi}{2}}\frac{486}{5}(2\pi)\sin\phi-\frac{486}{5}(0)\sin\phi\ d\phi$$

$$V=\int_0^{\frac{\pi}{2}}\frac{972\pi}{5}\sin\phi\ d\phi$$

Integrate with respect to ϕ.

$$V=-\frac{972\pi}{5}\cos\phi\Big|_0^{\frac{\pi}{2}}$$

$$V = -\frac{972\pi}{5}\cos\left(\frac{\pi}{2}\right) + \frac{972\pi}{5}\cos(0)$$

$$V = -\frac{972\pi}{5}(0) + \frac{972\pi}{5}(1)$$

$$V = \frac{972\pi}{5}$$

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Topic: Jacobian for two variables

Question: Find the Jacobian of the transformation.

$$x = r^2 \sin\theta$$

$$y = 2r \sin\theta$$

Answer choices:

A $-4r^2 \sin\theta \cos\theta$

B $2r^2 \sin\theta \cos\theta$

C $-2r^2 \sin\theta \cos\theta$

D $4r^2 \sin\theta \cos\theta$

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Solution: B

To find the Jacobian of the transformation, we'll find the partial derivatives of the given functions.

For $x = r^2 \sin \theta$,

$$\frac{\partial x}{\partial r} = 2r \sin \theta$$

$$\frac{\partial x}{\partial \theta} = r^2 \cos \theta$$

For $y = 2r \sin \theta$,

$$\frac{\partial y}{\partial r} = 2 \sin \theta$$

$$\frac{\partial y}{\partial \theta} = 2r \cos \theta$$

Now we'll plug the partial derivatives into the formula for the Jacobian.

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} 2r \sin \theta & r^2 \cos \theta \\ 2 \sin \theta & 2r \cos \theta \end{vmatrix}$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = \left(2r \sin \theta\right)(2r \cos \theta) - \left(2 \sin \theta\right)\left(r^2 \cos \theta\right)$$

$$\frac{\partial(x, y)}{\partial(r, \theta)} = 4r^2 \sin \theta \cos \theta - 2r^2 \sin \theta \cos \theta$$

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$$\frac{\partial(x,y)}{\partial(r,\theta)} = 2r^2 \sin\theta \cos\theta$$

Topic: Jacobian for two variables

Question: Find the Jacobian of the transformation.

$$x = 3r\cos\theta$$

$$y = e^r\sin\theta$$

Answer choices:

A $3e^r\cos^2\theta - 3re^r\sin^2\theta$

B $3e^r\cos^2\theta + 3e^r\sin^2\theta$

C $3e^r\cos^2\theta + 3re^r\sin^2\theta$

D $3e^r\cos^2\theta - 3e^r\sin^2\theta$

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Solution: C

To find the Jacobian of the transformation, we'll find the partial derivatives of the given functions.

For $x = 3r\cos\theta$,

$$\frac{\partial x}{\partial r} = 3\cos\theta$$

$$\frac{\partial x}{\partial \theta} = -3r\sin\theta$$

For $y = e^r\sin\theta$,

$$\frac{\partial y}{\partial r} = e^r\sin\theta$$

$$\frac{\partial y}{\partial \theta} = e^r\cos\theta$$

Now we'll plug the partial derivatives into the formula for the Jacobian.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ e^r\sin\theta & e^r\cos\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = (3\cos\theta)\left(e^r\cos\theta\right) - \left(e^r\sin\theta\right)\left(-3r\sin\theta\right)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = 3e^r\cos^2\theta + 3re^r\sin^2\theta$$

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Topic: Jacobian for two variables

Question: Find the Jacobian of the transformation.

$$x = 4r^2 \sin(2\theta)$$

$$y = 2e^{2r} \cos\theta$$

Answer choices:

A $\quad 16re^{2r} \sin(2\theta)\sin\theta + 32r^2e^{2r} \cos(2\theta)\cos\theta$

B $\quad -16e^{2r} \sin(2\theta)\sin\theta - 32e^{2r} \cos(2\theta)\cos\theta$

C $\quad 16e^{2r} \sin(2\theta)\sin\theta + 32e^{2r} \cos(2\theta)\cos\theta$

D $\quad -16re^{2r} \sin(2\theta)\sin\theta - 32r^2e^{2r} \cos(2\theta)\cos\theta$

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Solution: D

To find the Jacobian of the transformation, we'll find the partial derivatives of the given functions.

For $x = 4r^2\sin(2\theta)$,

$$\frac{\partial x}{\partial r} = 8r\sin(2\theta)$$

$$\frac{\partial x}{\partial \theta} = 8r^2\cos(2\theta)$$

For $y = 2e^{2r}\cos\theta$,

$$\frac{\partial y}{\partial r} = 4e^{2r}\cos\theta$$

$$\frac{\partial y}{\partial \theta} = -2e^{2r}\sin\theta$$

Now we'll plug the partial derivatives into the formula for the Jacobian.

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} 8r\sin(2\theta) & 8r^2\cos(2\theta) \\ 4e^{2r}\cos\theta & -2e^{2r}\sin\theta \end{vmatrix}$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \left(8r\sin(2\theta)\right)\left(-2e^{2r}\sin\theta\right) - \left(4e^{2r}\cos\theta\right)\left(8r^2\cos(2\theta)\right)$$

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$$\frac{\partial(x,y)}{\partial(r,\theta)} = -16re^{2r}\sin(2\theta)\sin\theta - 32r^2e^{2r}\cos(2\theta)\cos\theta$$

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Topic: Jacobian for three variables

Question: Find the Jacobian of the transformation.

$$x = 2uv$$

$$y = vw$$

$$z = 2uw$$

Answer choices:

A $4uvw$

B 0

C $8uvw$

D $2uvw$

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Solution: C

To find the Jacobian of the transformation, we'll find the partial derivatives of the given functions.

For $x = 2uv$,

$$\frac{\partial x}{\partial u} = 2v$$

$$\frac{\partial x}{\partial v} = 2u$$

$$\frac{\partial x}{\partial w} = 0$$

For $y = vw$,

$$\frac{\partial y}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = w$$

$$\frac{\partial y}{\partial w} = v$$

For $z = 2uw$,

$$\frac{\partial z}{\partial u} = 2w$$

$$\frac{\partial z}{\partial v} = 0$$

$$\frac{\partial z}{\partial w} = 2u$$

Now we'll plug the partial derivatives into the formula for the Jacobian.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2v & 2u & 0 \\ 0 & w & v \\ 2w & 0 & 2u \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 2v \begin{vmatrix} w & v \\ 0 & 2u \end{vmatrix} - 2u \begin{vmatrix} 0 & v \\ 2w & 2u \end{vmatrix} + 0 \begin{vmatrix} 0 & w \\ 2w & 0 \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 2v\left[(w)(2u) - (0)(v)\right] - 2u\left[(0)(2u) - (2w)(v)\right]$$

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$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 2v(2uw) - 2u(-2vw)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 4uvw + 4uvw$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 8uvw$$

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Topic: Jacobian for three variables

Question: Find the Jacobian of the transformation.

$$x = 2uvw$$

$$y = v^2w$$

$$z = uw^3$$

Answer choices:

A $18uv^2w^4$

B $14uv^2w^4$

C $6uv^2w^4$

D $10uv^2w^4$

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Solution: D

To find the Jacobian of the transformation, we'll find the partial derivatives of the given functions.

For $x = 2uvw$,

$$\frac{\partial x}{\partial u} = 2vw$$

$$\frac{\partial x}{\partial v} = 2uw$$

$$\frac{\partial x}{\partial w} = 2uv$$

For $y = v^2w$,

$$\frac{\partial y}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = 2vw$$

$$\frac{\partial y}{\partial w} = v^2$$

For $z = uw^3$,

$$\frac{\partial z}{\partial u} = w^3$$

$$\frac{\partial z}{\partial v} = 0$$

$$\frac{\partial z}{\partial w} = 3uw^2$$

Now we'll plug the partial derivatives into the formula for the Jacobian.

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} 2vw & 2uw & 2uv \\ 0 & 2vw & v^2 \\ w^3 & 0 & 3uw^2 \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 2vw \begin{vmatrix} 2vw & v^2 \\ 0 & 3uw^2 \end{vmatrix} - 2uw \begin{vmatrix} 0 & v^2 \\ w^3 & 3uw^2 \end{vmatrix} + 2uv \begin{vmatrix} 0 & 2vw \\ w^3 & 0 \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = 2vw\left[(2vw)(3uw^2) - (0)(v^2)\right] - 2uw\left[(0)(3uw^2) - (w^3)(v^2)\right] + 2uv\left[(0)(0) - (w^3)(2vw)\right]$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 2vw\left(6uvw^3\right) - 2uw\left(-w^3v^2\right) + 2uv\left(-2vw^4\right)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 12uv^2w^4 + 2uv^2w^4 - 4uv^2w^4$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 10uv^2w^4$$

Topic: Jacobian for three variables

Question: Find the Jacobian of the transformation.

$$x = e^{uv}$$

$$y = u$$

$$z = 4vw$$

Answer choices:

A $4uve^{uv}$

B $-4e^{uv}$

C $-4uve^{uv}$

D $4e^{uv}$

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Solution: C

To find the Jacobian of the transformation, we'll find the partial derivatives of the given functions.

For $x = e^{uv}$,

$$\frac{\partial x}{\partial u} = ve^{uv}$$

$$\frac{\partial x}{\partial v} = ue^{uv}$$

$$\frac{\partial x}{\partial w} = 0$$

For $y = u$,

$$\frac{\partial y}{\partial u} = 1$$

$$\frac{\partial y}{\partial v} = 0$$

$$\frac{\partial y}{\partial w} = 0$$

For $z = 4vw$,

$$\frac{\partial z}{\partial u} = 0$$

$$\frac{\partial z}{\partial v} = 4w$$

$$\frac{\partial z}{\partial w} = 4v$$

Now we'll plug the partial derivatives into the formula for the Jacobian.

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} ve^{uv} & ue^{uv} & 0 \\ 1 & 0 & 0 \\ 0 & 4w & 4v \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = ve^{uv} \begin{vmatrix} 0 & 0 \\ 4w & 4v \end{vmatrix} - ue^{uv} \begin{vmatrix} 1 & 0 \\ 0 & 4v \end{vmatrix} + 0 \begin{vmatrix} 1 & 0 \\ 0 & 4w \end{vmatrix}$$

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = ve^{uv} \left[(0)(4v) - (4w)(0)\right] - ue^{uv} \left[(1)(4v) - (0)(0)\right] + 0 \left[(1)(4w) - (0)(0)\right]$$

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$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = ve^{uv}(0) - ue^{uv}(4v)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = -4uve^{uv}$$

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Topic: Evaluating double integrals

Question: Evaluate the integral.

Find $\iint_R x + y \ dA$

where R is a trapezoidal region formed by the lines

$$y = x$$

$$y = -x$$

$$y = -x + 2$$

$$y = x - 2$$

Use the transformation $x = u + v$ and $y = u - v$.

Answer choices:

A 1

B 2

C -1

D -2

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Solution: B

The first step to solve the double integral is transforming the four boundary equations $y = x$, $y = -x$, $y = -x + 2$, and $y = x - 2$ using $x = u + v$ and $y = u - v$.

For $y = x$:

$$u - v = u + v$$

$$-2v = 0$$

$$v = 0$$

For $y = -x$:

$$u - v = -(u + v)$$

$$2u = 0$$

$$u = 0$$

For $y = -x + 2$:

$$u - v = -(u + v) + 2$$

$$2u = 2$$

$$u = 1$$

For $y = x - 2$:

$$u - v = (u + v) - 2$$

$$-2v = -2$$

$$v = 1$$

We've transformed the bounds of the original region R into new bounds for the region S, and they are

$$v = 0$$

$$u = 0$$

$$u = 1$$

$$v = 1$$

We can write these bounds as

$$0 \le u \le 1$$

$$0 \le v \le 1$$

These intervals will be the limits of integration on our transformed double integral.

Now we can solve for the transformed function using the original function

$$f(x, y) = x + y$$

and the two transforms $x = u + v$ and $y = u - v$.

$$f(g(u, v), h(u, v)) = (u + v) + (u - v)$$

$$f(g(u, v), h(u, v)) = 2u$$

This is our transformed function.

Next, we'll find the jacobian, starting with finding the partial derivatives of the transforms.

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = 1$$

$$\frac{\partial y}{\partial v} = -1$$

Then we can find the jacobian

$$\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\frac{\partial (x, y)}{\partial (u, v)} = (1)(-1) - (1)(1)$$

$$\frac{\partial (x, y)}{\partial (u, v)} = -2$$

Now that we've done all of our transforms, we have the new transformed limits of integration, the new transformed function, and the new transformed jacobian. We're ready to plug into the integral.

$$\iint_R f(x,y)\ dA = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$$

$$\iint_R x+ydA = \int_0^1\int_0^1 2u\left|-2\right|\ du\ dv$$

$$\iint_R x+y\ dA = \int_0^1\int_0^1 4u\ du\ dv$$

Integrate from the inside out, so in this case we'll integrate first with respect to u.

$$\iint_R x+y\ dA = \int_0^1 2u^2\Big|_0^1\ dv$$

$$\iint_R x+y\ dA = \int_0^1 2(1)^2-2(0)^2\ dv$$

$$\iint_R x+y\ dA = \int_0^1 2\ dv$$

Integrate with respect to v.

$$\iint_R x+y\ dA = 2v\Big|_0^1$$

$$\iint_R x+y\ dA = 2(1)-2(0)$$

$$\iint_R x+y\ dA = 2$$

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Topic: Evaluating double integrals

Question: Evaluate the integral.

Find $\iint_R x + y\ dA$

where R is a trapezoidal region formed by the lines

$$y = x$$

$$y = -x$$

$$y = -x + 8$$

$$y = x - 8$$

Use the transformation $x = 2u + v$ and $y = 2u - v$.

Answer choices:

A $\quad 128$

B $\quad 32$

C $\quad 64$

D $\quad 16$

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Solution: A

The first step to solve the double integral is transforming the four boundary equations $y = x$, $y = -x$, $y = -x + 8$, and $y = x - 8$ using $x = 2u + v$ and $y = 2u - v$.

For $y = x$:

$$2u - v = 2u + v$$

$$-2v = 0$$

$$v = 0$$

For $y = -x$:

$$2u - v = -(2u + v)$$

$$4u = 0$$

$$u = 0$$

For $y = -x + 8$:

$$2u - v = -(2u + v) + 8$$

$$4u = 8$$

$$u = 2$$

For $y = x - 8$:

$$2u - v = (2u + v) - 8$$

$$-2v = -8$$

$$v = 4$$

We've transformed the bounds of the original region R into new bounds for the region S, and they are

$$v = 0$$

$$u = 0$$

$$u = 2$$

$$v = 4$$

We can write these bounds as

$$0 \leq u \leq 2$$

$$0 \leq v \leq 4$$

These intervals will be the limits of integration on our transformed double integral.

Now we can solve for the transformed function using the original function

$$f(x, y) = x + y$$

and the two transforms $x = 2u + v$ and $y = 2u - v$.

$$f(g(u, v), h(u, v)) = (2u + v) + (2u - v)$$

$$f(g(u, v), h(u, v)) = 4u$$

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This is our transformed function.

Next, we'll find the jacobian, starting with finding the partial derivatives of the transforms.

$$\frac{\partial x}{\partial u} = 2$$

$$\frac{\partial x}{\partial v} = 1$$

$$\frac{\partial y}{\partial u} = 2$$

$$\frac{\partial y}{\partial v} = -1$$

Then we can find the jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = (2)(-1) - (1)(2)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -4$$

Now that we've done all of our transforms, we have the new transformed limits of integration, the new transformed function, and the new transformed jacobian. We're ready to plug into the integral.

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$$\iint_R f(x,y)\ dA = \iint_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$$

$$\iint_R x+y\ dA = \int_0^4\int_0^2 4u|-4|\ du\ dv$$

$$\iint_R x+y\ dA = \int_0^4\int_0^2 16u\ du\ dv$$

Integrate from the inside out, so in this case we'll integrate first with respect to u.

$$\iint_R x+y\ dA = \int_0^4 8u^2\Big|_0^2\ dv$$

$$\iint_R x+y\ dA = \int_0^4 8(2)^2-8(0)^2\ dv$$

$$\iint_R x+y\ dA = \int_0^4 32\ dv$$

Integrate with respect to v.

$$\iint_R x+y\ dA = 32v\Big|_0^4$$

$$\iint_R x+y\ dA = 32(4)-32(0)$$

$$\iint_R x+y\ dA = 128$$

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Topic: Evaluating double integrals

Question: Evaluate the integral. Use the transformation $x = 3u + 2v$ and $y = 3u - 2v$.

$$\iint_R x + y \ dA$$

where R is a trapezoidal region formed by the lines

$y = x$ and $y = -x$

$y = -x + 3$ and $y = x - 3$

Answer choices:

A $\frac{9}{4}$

B $\frac{27}{2}$

C $\frac{9}{2}$

D $\frac{27}{4}$

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Solution: D

The first step to solve the double integral is transforming the four boundary equations $y = x$, $y = -x$, $y = -x + 3$, and $y = x - 3$ using $x = 3u + 2v$ and $y = 3u - 2v$.

For $y = x$:

$$3u - 2v = 3u + 2v$$

$$-4v = 0$$

$$v = 0$$

For $y = -x$:

$$3u - 2v = -(3u + 2v)$$

$$6u = 0$$

$$u = 0$$

For $y = -x + 3$:

$$3u - 2v = -(3u + 2v) + 3$$

$$6u = 3$$

$$u = \frac{1}{2}$$

For $y = x - 3$:

$$3u - 2v = (3u + 2v) - 3$$

$$-4v = -3$$

$$v = \frac{3}{4}$$

We've transformed the bounds of the original region R into new bounds for the region S, and they are

$$v = 0$$

$$u = 0$$

$$u = \frac{1}{2}$$

$$v = \frac{3}{4}$$

We can write these bounds as

$$0 \leq u \leq \frac{1}{2}$$

$$0 \leq v \leq \frac{3}{4}$$

These intervals will be the limits of integration on our transformed double integral.

Now we can solve for the transformed function using the original function

$$f(x, y) = x + y$$

and the two transforms $x = 3u + 2v$ and $y = 3u - 2v$.

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$$f(g(u,v),h(u,v))=(3u+2v)+(3u-2v)$$

$$f(g(u,v),h(u,v))=6u$$

This is our transformed function.

Next, we'll find the jacobian, starting with finding the partial derivatives of the transforms.

$$\frac{\partial x}{\partial u}=3$$

$$\frac{\partial x}{\partial v}=2$$

$$\frac{\partial y}{\partial u}=3$$

$$\frac{\partial y}{\partial v}=-2$$

Then we can find the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=(3)(-2)-(2)(3)$$

$$\frac{\partial(x,y)}{\partial(u,v)}=-12$$

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Now that we've done all of our transforms, we have the new transformed limits of integration, the new transformed function, and the new transformed jacobian. We're ready to plug into the integral.

$$\int\int_S f(g(u,v),h(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\ du\ dv$$

$$\int_0^{\frac{3}{4}}\int_0^{\frac{1}{2}} 6u|-12|\ du\ dv$$

$$\int_0^{\frac{3}{4}}\int_0^{\frac{1}{2}} 72u\ du\ dv$$

Integrate from the inside out, so in this case we'll integrate first with respect to u.

$$\int_0^{\frac{3}{4}} 36u^2\Big|_0^{\frac{1}{2}}\ dv$$

$$\int_0^{\frac{3}{4}} 36\left(\frac{1}{2}\right)^2-36(0)^2\ dv$$

$$\int_0^{\frac{3}{4}} 9\ dv$$

Integrate with respect to v.

$$9v\Big|_0^{\frac{3}{4}}$$

$$9\left(\frac{3}{4}\right)-9(0)$$

$$\frac{27}{4}$$

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Topic: Equations of the transformation

Question: What is the new equation after the transformation?

The region R is

$$\frac{x^2}{36} + y^2 = 1$$

and is transformed by the following:

$$x = 2u$$

$$y = \frac{v}{3}$$

Answer choices:

A $\quad u^2 + v^2 = 9$

B $\quad u^2 + v^2 = 1$

C $\quad u^2 + v^2 = 36$

D $\quad u^2 + v^2 = 3$

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Solution: A

To determine the new equation(s) when variables are transformed, we can simply substitute the transformation equations into the original function(s).

In this case, we'll substitute

$$x = 2u$$

$$y = \frac{v}{3}$$

into

$$\frac{x^2}{36} + y^2 = 1$$

We get

$$\frac{(2u)^2}{36} + \left(\frac{v}{3}\right)^2 = 1$$

$$\frac{4u^2}{36} + \frac{v^2}{9} = 1$$

$$\frac{u^2}{9} + \frac{v^2}{9} = 1$$

$$u^2 + v^2 = 9$$

The new transformed equation is $u^2 + v^2 = 9$.

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Topic: Equations of the transformation

Question: What are the new equations after the transformation?

The region R is bounded by

$$y = -x + 2$$

$$y = x + 1$$

$$y = \frac{1}{2}x + 2$$

and is transformed by the following:

$$x = u + v$$

$$y = u - v$$

Answer choices:

A $\quad u = 1$, $v = -\frac{1}{2}$, and $v = \frac{1}{3}u + \frac{4}{3}$

B $\quad u = 1$, $v = \frac{1}{2}$, and $v = \frac{1}{3}u - \frac{4}{3}$

C $\quad u = 1$, $v = -\frac{1}{2}$, and $v = \frac{1}{3}u - \frac{4}{3}$

D $\quad u = 1$, $v = \frac{1}{2}$, and $v = \frac{1}{3}u + \frac{4}{3}$

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Solution: C

To determine the new equation(s) when variables are transformed, we can simply substitute the transformation equations into the original function(s).

In this case, we'll substitute

$$x = u + v$$

$$y = u - v$$

into

$$y = -x + 2$$

$$y = x + 1$$

$$y = \frac{1}{2}x + 2$$

Into $y = -x + 2$:

$$(u - v) = -(u + v) + 2$$

$$u - v = -u - v + 2$$

$$2u = 2$$

$$u = 1$$

Into $y = x + 1$:

$$(u - v) = (u + v) + 1$$

$$u - v = u + v + 1$$

$$-2v = 1$$

$$v = -\frac{1}{2}$$

Into $y = \frac{1}{2}x + 2$:

$$(u - v) = \frac{1}{2}(u + v) + 2$$

$$u - v = \frac{1}{2}u + \frac{1}{2}v + 2$$

$$-\frac{3}{2}v = -\frac{1}{2}u + 2$$

$$3v = u - 4$$

$$v = \frac{1}{3}u - \frac{4}{3}$$

The new transformed region is bounded by

$$u = 1$$

$$v = -\frac{1}{2}$$

$$v = \frac{1}{3}u - \frac{4}{3}$$

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Topic: Equations of the transformation

Question: What is the new equation for the region R after it's transformed by $x = 3u$ and $y = 5v$?

$$\frac{x^2}{27} + \frac{y^2}{75} = 1$$

Answer choices:

A $u^2 + v^2 = 9$

B $u^2 + v^2 = 1$

C $u^2 + v^2 = 18$

D $u^2 + v^2 = 3$

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Solution: D

To determine the new equation(s) when variables are transformed, we can simply substitute the transformation equations into the original function(s).

In this case, we'll substitute

$$x = 3u$$

$$y = 5v$$

into

$$\frac{x^2}{27} + \frac{y^2}{75} = 1$$

We get

$$\frac{(3u)^2}{27} + \frac{(5v)^2}{75} = 1$$

$$\frac{9u^2}{27} + \frac{25v^2}{75} = 1$$

$$\frac{u^2}{3} + \frac{v^2}{3} = 1$$

$$u^2 + v^2 = 3$$

The new transformed equation is $u^2 + v^2 = 3$.

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Topic: Image of the set under the transformation

Question: What shape does the new region take?

Region R is the ellipse

$$x^2 + \frac{y^2}{16} = 1$$

and is transformed by the following:

$$x = \frac{u}{2}$$

$$y = 2v$$

Answer choices:

A A disk with radius 1.

B A disk with radius 2.

C A disk with radius 16.

D A disk with radius 4.

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Solution: B

To determine the new region when variables are transformed, we can simply substitute the transformation equations into the original function.

We'll plug the transformations

$$x = \frac{u}{2}$$

$$y = 2v$$

into the ellipse.

$$x^2 + \frac{y^2}{16} = 1$$

$$\left(\frac{u}{2}\right)^2 + \frac{(2v)^2}{16} = 1$$

$$\frac{u^2}{4} + \frac{4v^2}{16} = 1$$

$$\frac{u^2}{4} + \frac{v^2}{4} = 1$$

$$u^2 + v^2 = 4$$

This is the equation of the new region, which means it's a disk with radius 2.

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Topic: Image of the set under the transformation

Question: What shape does the new region take?

Region R is the triangle bounded by

$$y = -x + 6$$

$$y = x + 2$$

$$y = \frac{1}{2}x + 1$$

and is transformed by the following:

$$x = u + v$$

$$y = u - v$$

Answer choices:

A A triangle bounded by $v = 3$, $u = -1$, and $v = \frac{2}{3}u - \frac{1}{3}$.

B A triangle bounded by $v = 3$, $u = -1$, and $v = \frac{1}{3}u - \frac{2}{3}$.

C A triangle bounded by $u = 3$, $v = -1$, and $v = \frac{2}{3}u - \frac{1}{3}$.

D A triangle bounded by $u = 3$, $v = -1$, and $v = \frac{1}{3}u - \frac{2}{3}$.

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Solution: D

To determine the new region when variables are transformed, we can simply substitute the transformation equations into the original function.

We'll plug the transformations

$$x = u + v$$

$$y = u - v$$

into the equations that bound the triangle.

Plugging into $y = -x + 6$:

$$(u - v) = -(u + v) + 6$$

$$u - v = -u - v + 6$$

$$2u = 6$$

$$u = 3$$

Plugging into $y = x + 2$:

$$(u - v) = (u + v) + 2$$

$$u - v = u + v + 2$$

$$-2v = 2$$

$$v = -1$$

Plugging into $y = (1/2)x + 1$:

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$$(u-v)=\frac{1}{2}(u+v)+1$$

$$u-v=\frac{1}{2}u+\frac{1}{2}v+1$$

$$-\frac{3}{2}v=-\frac{1}{2}u+1$$

$$3v=u-2$$

$$v=\frac{1}{3}u-\frac{2}{3}$$

The new triangle is bounded by

$$u=3$$

$$v=-1$$

$$v=\frac{1}{3}u-\frac{2}{3}$$

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Topic: Image of the set under the transformation

Question: The region R is an ellipse transformed by $x = 2u$ and $y = 5v$. What shape does the new region take?

$$\frac{x^2}{36} + \frac{y^2}{225} = 1$$

Answer choices:

A A disk with radius 27.

B A disk with radius 1.

C A disk with radius 3.

D A disk with radius 9.

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Solution: C

To determine the new region when variables are transformed, we can simply substitute the transformation equations into the original function.

We'll plug the transformations

$$x = 2u$$

$$y = 5v$$

into the ellipse.

$$\frac{x^2}{36} + \frac{y^2}{225} = 1$$

$$\frac{(2u)^2}{36} + \frac{(5v)^2}{225} = 1$$

$$\frac{4u^2}{36} + \frac{25v^2}{225} = 1$$

$$\frac{u^2}{9} + \frac{v^2}{9} = 1$$

$$u^2 + v^2 = 9$$

This is the equation of the new region, which means it's a disk with radius 3.

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Topic: Triple integrals to find mass and center of mass

Question: Three edges of a $2 \times 2 \times 2$ cube sit on the three major axes. If the mass of the cube is 32, then which relationship holds true between the density and the dimensions of the cube?

Answer choices:

A The density of the cube is one-half each edge of the cube at any point (x, y, z).

B The density of the cube is equal to the square of each edge of the cube at any point (x, y, z).

C Each edge of the cube is twice the density of the cube at any point (x, y, z).

D Each edge of the cube is equal to the density of the cube at any point (x, y, z).

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Solution: B

Use the triple integral formula for the mass. If we assume, as in answer choice B, that the density of the cube is equal to to square of each edge, then we would write the triple integral as

$$m = \int_0^2 \int_0^2 \int_0^2 4 \ dz \ dy \ dx$$

If this is the correct answer choice, then we should get 32 when we evaluate the integral.

$$m = \int_0^2 \int_0^2 4z \Big|_{z=0}^{z=2} \ dy \ dx$$

$$m = \int_0^2 \int_0^2 4(2) - 4(0) \ dy \ dx$$

$$m = \int_0^2 \int_0^2 8 \ dy \ dx$$

$$m = \int_0^2 8y \Big|_{y=0}^{y=2} \ dx$$

$$m = \int_0^2 8(2) - 8(0) \ dx$$

$$m = \int_0^2 16 \ dx$$

$$m = 16x \Big|_0^2$$

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$$m = 16(2) - 16(0)$$

$$m = 32$$

Topic: Triple integrals to find mass and center of mass

Question: The mass of a unit cube of dimensions $2 \times 2 \times 2$ is 5. Three faces of the cube are in the first octant, and one of its vertices is at the origin. If the density of the cube at any point (x, y, z) is numerically 5 times the square of the distance of the point from the origin, then what is the center of mass of the cube?

Answer choices:

A $\left(\frac{200}{3}, \frac{200}{3}, \frac{400}{3}\right)$

B $\left(\frac{112}{3}, \frac{112}{3}, \frac{112}{3}\right)$

C $(125,125,160)$

D $(160,125,125)$

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Solution: B

We know that the mass is $m = 5$. Now we'll find M_{yz}, the first moment about yz-plane.

$$M_{yz} = \int_0^2 \int_0^2 \int_0^2 5x\left(x^2 + y^2 + z^2\right)\ dz\ dy\ dx$$

$$M_{yz} = \int_0^2 \int_0^2 5x^3z + 5xy^2z + \frac{5}{3}xz^3 \Big|_{z=0}^{z=2}\ dy\ dx$$

$$M_{yz} = \int_0^2 \int_0^2 10x^3 + 10xy^2 + \frac{40}{3}x\ dy\ dx$$

$$M_{yz} = \int_0^2 10x^3y + \frac{10}{3}xy^3 + \frac{40}{3}xy \Big|_{y=0}^{y=2}\ dx$$

$$M_{yz} = \int_0^2 20x^3 + \frac{80}{3}x + \frac{80}{3}x\ dx$$

$$M_{yz} = \int_0^2 20x^3 + \frac{160}{3}x\ dx$$

$$M_{yz} = 5x^4 + \frac{80}{3}x^2 \Big|_0^2$$

$$M_{yz} = 5(2)^4 + \frac{80}{3}(2)^2 - \left(5(0)^4 + \frac{80}{3}(0)^2\right)$$

$$M_{yz} = 80 + \frac{320}{3}$$

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$$M_{yz} = \frac{240}{3} + \frac{320}{3}$$

$$M_{yz} = \frac{560}{3}$$

By symmetry of x, y and z, we conclude that

$$M_{xz} = \frac{560}{3}$$

and

$$M_{xy} = \frac{560}{3}$$

also. Because the mass of the cube is $m = 5$, divide each moment by 5. Then the center of mass is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{560}{15}, \frac{560}{15}, \frac{560}{15}\right) = \left(\frac{112}{3}, \frac{112}{3}, \frac{112}{3}\right)$$

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Topic: Triple integrals to find mass and center of mass

Question: A cube with edge-length 3 units is located in the first octant with three of its faces on the three coordinate planes. The density function is given as

$$f(x, y, z) = 2\left(x^2 + y^2 + z^2\right)$$

Which is the mass of the cube?

Answer choices:

A $\quad m = 27$

B $\quad m = 81$

C $\quad m = 243$

D $\quad m = 486$

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Solution: D

If the density is numerically twice the distance from the origin, then we get the triple integral

$$m=\int_0^3\int_0^3\int_0^3 2\left(x^2+y^2+z^2\right)\ dz\ dy\ dx$$

$$m=\int_0^3\int_0^3\int_0^3 2x^2+2y^2+2z^2\ dz\ dy\ dx$$

Integrate with respect to z, then evaluate over the interval.

$$m=\int_0^3\int_0^3 2x^2z+2y^2z+\frac{2}{3}z^3\Big|_{z=0}^{z=3}\ dy\ dx$$

$$m=\int_0^3\int_0^3 2x^2(3)+2y^2(3)+\frac{2}{3}(3)^3-\left(2x^2(0)+2y^2(0)+\frac{2}{3}(0)^3\right)\ dy\ dx$$

$$m=\int_0^3\int_0^3 6x^2+6y^2+18\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$m=\int_0^3 6x^2y+2y^3+18y\Big|_{y=0}^{y=3}\ dx$$

$$m=\int_0^3 6x^2(3)+2(3)^3+18(3)-\left(6x^2(0)+2(0)^3+18(0)\right)\ dx$$

$$m=\int_0^3 18x^2+54+54\ dx$$

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$$m = \int_0^3 18x^2 + 108 \ dx$$

Integrate with respect to x, then evaluate over the interval.

$$m = 6x^3 + 108x \Big|_0^3$$

$$m = 6(3)^3 + 108(3) - \left(6(0)^3 + 108(0)\right)$$

$$m = 486$$

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Topic: Moments of inertia

Question: A solid object is bounded by the xy-plane and the given surface. The density of the object at any point $M(x, y, z)$ is equal to the distance of M from xy-plane. What is the moment of inertia of the object about the x-axis?

$$z = \sqrt{1 - x^2 - y^2}$$

Answer choices:

A $\quad I_x = \frac{\pi}{8}$

B $\quad I_x = \frac{\pi}{32}$

C $\quad I_x = \frac{5\pi}{16}$

D $\quad I_x = \frac{\pi}{16}$

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Solution: A

By the given relation between density and the distance of M from the xy-plane, we can start with $\delta\left(x,y,z\right)=z$. Then,

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} z\left(y^2+z^2\right)\ dz\ dy\ dx$$

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} y^2z+z^3\ dz\ dy\ dx$$

Integrate with respect to z, then evaluate over the interval.

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}y^2z^2+\frac{1}{4}z^4\Big|_{z=0}^{z=\sqrt{1-x^2-y^2}}\ dy\ dx$$

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}y^2\left(\sqrt{1-x^2-y^2}\right)^2+\frac{1}{4}\left(\sqrt{1-x^2-y^2}\right)^4-\left(\frac{1}{2}y^2(0)^2+\frac{1}{4}(0)^4\right)\ dy\ dx$$

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}y^2\left(1-x^2-y^2\right)+\frac{1}{4}\left(1-x^2-y^2\right)^2\ dy\ dx$$

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}y^2-\frac{1}{2}x^2y^2-\frac{1}{2}y^4$$

$$+\frac{1}{4}\left(1-x^2-y^2-x^2+x^4+x^2y^2-y^2+x^2y^2+y^4\right)\ dy\ dx$$

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{2}y^2-\frac{1}{2}x^2y^2-\frac{1}{2}y^4$$

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$$+\frac{1}{4}-\frac{1}{4}x^2-\frac{1}{4}y^2-\frac{1}{4}x^2+\frac{1}{4}x^4+\frac{1}{4}x^2y^2-\frac{1}{4}y^2+\frac{1}{4}x^2y^2+\frac{1}{4}y^4\ dy\ dx$$

$$I_x=\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\frac{1}{4}x^4-\frac{1}{2}x^2+\frac{1}{4}-\frac{1}{4}y^4\ dy\ dx$$

$$I_x=\frac{1}{4}\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}x^4-2x^2+1-y^4\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$I_x=\frac{1}{4}\int_{-1}^{1}x^4y-2x^2y+y-\frac{1}{5}y^5\Big|_{y=-\sqrt{1-x^2}}^{y=\sqrt{1-x^2}}\ dx$$

$$I_x=\frac{1}{4}\int_{-1}^{1}x^4\sqrt{1-x^2}-2x^2\sqrt{1-x^2}+\sqrt{1-x^2}-\frac{1}{5}\left(\sqrt{1-x^2}\right)^5$$

$$-\left(x^4\left(-\sqrt{1-x^2}\right)-2x^2\left(-\sqrt{1-x^2}\right)+\left(-\sqrt{1-x^2}\right)-\frac{1}{5}\left(-\sqrt{1-x^2}\right)^5\right)\ dx$$

$$I_x=\frac{1}{4}\int_{-1}^{1}x^4\sqrt{1-x^2}-2x^2\sqrt{1-x^2}+\sqrt{1-x^2}-\frac{1}{5}\left(\sqrt{1-x^2}\right)^5$$

$$+x^4\sqrt{1-x^2}-2x^2\sqrt{1-x^2}+\sqrt{1-x^2}-\frac{1}{5}\left(\sqrt{1-x^2}\right)^5\ dx$$

$$I_x=\frac{1}{4}\int_{-1}^{1}2x^4\sqrt{1-x^2}-4x^2\sqrt{1-x^2}+2\sqrt{1-x^2}-\frac{2}{5}\left(\sqrt{1-x^2}\right)^5\ dx$$

$$I_x=\frac{1}{2}\int_{-1}^{1}x^4\sqrt{1-x^2}-2x^2\sqrt{1-x^2}+\sqrt{1-x^2}-\frac{1}{5}\left(\sqrt{1-x^2}\right)^5\ dx$$

$$I_x = \frac{1}{2}\int_{-1}^{1} x^4\sqrt{1-x^2}\ dx - \int_{-1}^{1} x^2\sqrt{1-x^2}\ dx$$

$$+\frac{1}{2}\int_{-1}^{1} \sqrt{1-x^2}\ dx - \frac{1}{10}\int_{-1}^{1} \left(\sqrt{1-x^2}\right)^5\ dx$$

Use trigonometric substitution to integrate with respect to x.

$$a = 1,\ u = x$$

$$x = \sin\theta,\ \sin\theta = \frac{x}{1}$$

$$dx = \cos\theta\ d\theta,\ \theta = \arcsin x$$

Make the substitutions.

$$I_x = \frac{1}{2}\int_{x=-1}^{x=1} \sin^4\theta\sqrt{1-\sin^2\theta}\cos\theta\ d\theta - \int_{x=-1}^{x=1} \sin^2\theta\sqrt{1-\sin^2\theta}\cos\theta\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1} \sqrt{1-\sin^2\theta}\cos\theta\ d\theta - \frac{1}{10}\int_{x=-1}^{x=1} \left(\sqrt{1-\sin^2\theta}\right)^5\cos\theta\ d\theta$$

$$I_x = \frac{1}{2}\int_{x=-1}^{x=1} \sin^4\theta\sqrt{\cos^2\theta}\cos\theta\ d\theta - \int_{x=-1}^{x=1} \sin^2\theta\sqrt{\cos^2\theta}\cos\theta\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1} \sqrt{\cos^2\theta}\cos\theta\ d\theta - \frac{1}{10}\int_{x=-1}^{x=1} \left(\sqrt{\cos^2\theta}\right)^5\cos\theta\ d\theta$$

$$I_x = \frac{1}{2}\int_{x=-1}^{x=1} \sin^4\theta\cos\theta\cos\theta\ d\theta - \int_{x=-1}^{x=1} \sin^2\theta\cos\theta\cos\theta\ d\theta$$

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$$+\frac{1}{2}\int_{x=-1}^{x=1} \cos\theta\cos\theta\ d\theta - \frac{1}{10}\int_{x=-1}^{x=1} (\cos\theta)^5\cos\theta\ d\theta$$

$$I_x = \frac{1}{2}\int_{x=-1}^{x=1} \sin^4\theta\cos^2\theta\ d\theta - \int_{x=-1}^{x=1} \sin^2\theta\cos^2\theta\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1} \cos^2\theta\ d\theta - \frac{1}{10}\int_{x=-1}^{x=1} \cos^6\theta\ d\theta$$

$$I_x = \frac{1}{2}\int_{x=-1}^{x=1} \sin^2\theta(\sin\theta\cos\theta)^2\ d\theta - \int_{x=-1}^{x=1} (\sin\theta\cos\theta)^2\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1} \cos^2\theta\ d\theta - \frac{1}{10}\int_{x=-1}^{x=1} \cos^6\theta\ d\theta$$

Use the trigonometric identity

$$\sin x\cos x = \frac{1}{2}\sin(2x)$$

Substitute.

$$I_x = \frac{1}{2}\int_{x=-1}^{x=1} \sin^2\theta\left(\frac{1}{2}\sin(2\theta)\right)^2\ d\theta - \int_{x=-1}^{x=1} \left(\frac{1}{2}\sin(2\theta)\right)^2\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1} \cos^2\theta\ d\theta - \frac{1}{10}\int_{x=-1}^{x=1} \cos^6\theta\ d\theta$$

$$I_x = \frac{1}{8}\int_{x=-1}^{x=1} \sin^2\theta\sin^2(2\theta)\ d\theta - \frac{1}{4}\int_{x=-1}^{x=1} \sin^2(2\theta)\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1}\cos^2\theta\ d\theta-\frac{1}{10}\int_{x=-1}^{x=1}\cos^6\theta\ d\theta$$

Use the trigonometric identity

$$\sin^2 x=\frac{1}{2}(1-\cos(2x))$$

Substitute.

$$I_x=\frac{1}{8}\int_{x=-1}^{x=1}\frac{1}{2}(1-\cos(2\theta))\frac{1}{2}(1-\cos(2(2\theta)))\ d\theta-\frac{1}{4}\int_{x=-1}^{x=1}\frac{1}{2}(1-\cos(2(2\theta)))\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1}\cos^2\theta\ d\theta-\frac{1}{10}\int_{x=-1}^{x=1}\cos^6\theta\ d\theta$$

$$I_x=\frac{1}{32}\int_{x=-1}^{x=1}(1-\cos(2\theta))(1-\cos(4\theta))\ d\theta-\frac{1}{8}\int_{x=-1}^{x=1}1-\cos(4\theta)\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1}\cos^2\theta\ d\theta-\frac{1}{10}\int_{x=-1}^{x=1}\cos^6\theta\ d\theta$$

$$I_x=\frac{1}{32}\int_{x=-1}^{x=1}1-\cos(4\theta)-\cos(2\theta)+\cos(2\theta)\cos(4\theta)\ d\theta-\frac{1}{8}\int_{x=-1}^{x=1}1-\cos(4\theta)\ d\theta$$

$$+\frac{1}{2}\int_{x=-1}^{x=1}\cos^2\theta\ d\theta-\frac{1}{10}\int_{x=-1}^{x=1}\cos^6\theta\ d\theta$$

Integrate what you can.

$$I_x=\frac{1}{32}\left(\theta-\frac{1}{4}\sin(4\theta)-\frac{1}{2}\sin(2\theta)\right)\Bigg|_{x=-1}^{x=1}$$

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$$+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta-\frac{1}{8}\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Big|_{x=-1}^{x=1}$$

$$+\frac{1}{2}\int_{x=-1}^{x=1}\cos^2\theta\ d\theta-\frac{1}{10}\int_{x=-1}^{x=1}\cos^6\theta\ d\theta$$

Use the trigonometric identity

$$\cos^2 x=\frac{1}{2}(1+\cos(2x))$$

Substitute.

$$I_x=\frac{1}{32}\left(\theta-\frac{1}{4}\sin(4\theta)-\frac{1}{2}\sin(2\theta)\right)\Big|_{x=-1}^{x=1}$$

$$+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta-\frac{1}{8}\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Big|_{x=-1}^{x=1}$$

$$+\frac{1}{2}\int_{x=-1}^{x=1}\frac{1}{2}(1+\cos(2\theta))\ d\theta-\frac{1}{10}\int_{x=-1}^{x=1}\left(\frac{1}{2}(1+\cos(2x))\right)^3\ d\theta$$

$$I_x=\frac{1}{32}\left(\theta-\frac{1}{4}\sin(4\theta)-\frac{1}{2}\sin(2\theta)\right)\Big|_{x=-1}^{x=1}$$

$$+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta-\frac{1}{8}\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Big|_{x=-1}^{x=1}$$

$$+\frac{1}{4}\int_{x=-1}^{x=1}1+\cos(2\theta)\ d\theta-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2x))^3\ d\theta$$

$$I_x=\frac{1}{32}\left(\theta-\frac{1}{4}\sin(4\theta)-\frac{1}{2}\sin(2\theta)\right)\Bigg|_{x=-1}^{x=1}$$

$$+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta-\frac{1}{8}\left(\theta-\frac{1}{4}\sin(4\theta)\right)\Bigg|_{x=-1}^{x=1}$$

$$+\frac{1}{4}\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Bigg|_{x=-1}^{x=1}-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2x))^3\ d\theta$$

$$I_x=\frac{1}{32}\left(\theta-\frac{1}{4}\sin(4\theta)-\frac{1}{2}\sin(2\theta)\right)-\frac{1}{8}\left(\theta-\frac{1}{4}\sin(4\theta)\right)$$

$$+\frac{1}{4}\left(\theta+\frac{1}{2}\sin(2\theta)\right)\Bigg|_{x=-1}^{x=1}+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta$$

$$-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2x))^3\ d\theta$$

$$I_x=\frac{1}{32}\theta-\frac{1}{128}\sin(4\theta)-\frac{1}{64}\sin(2\theta)-\frac{1}{8}\theta+\frac{1}{32}\sin(4\theta)$$

$$+\frac{1}{4}\theta+\frac{1}{8}\sin(2\theta)\Bigg|_{x=-1}^{x=1}+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta$$

$$-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2x))^3\ d\theta$$

$$I_x=\frac{1}{32}\theta+\frac{8}{32}\theta-\frac{4}{32}\theta-\frac{1}{128}\sin(4\theta)+\frac{4}{128}\sin(4\theta)$$

$$-\frac{1}{64}\sin(2\theta)+\frac{8}{64}\sin(2\theta)\Bigg|_{x=-1}^{x=1}+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta$$

$$-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2x))^3\ d\theta$$

$$I_x=\frac{5}{32}\theta+\frac{7}{64}\sin(2\theta)+\frac{3}{128}\sin(4\theta)\Big|_{x=-1}^{x=1}+\frac{1}{32}\int_{x=-1}^{x=1}\cos(2\theta)\cos(4\theta)\ d\theta$$

$$-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2x))^3\ d\theta$$

Use the trigonometric identity

$$\cos A\cos B=\frac{1}{2}\left[\cos(A-B)+\cos(A+B)\right]$$

Substitute.

$$I_x=\frac{5}{32}\theta+\frac{7}{64}\sin(2\theta)+\frac{3}{128}\sin(4\theta)\Big|_{x=-1}^{x=1}$$

$$+\frac{1}{32}\int_{x=-1}^{x=1}\frac{1}{2}\left[\cos(4\theta-2\theta)+\cos(4\theta+2\theta)\right]\ d\theta$$

$$-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2\theta))^3\ d\theta$$

$$I_x=\frac{5}{32}\theta+\frac{7}{64}\sin(2\theta)+\frac{3}{128}\sin(4\theta)\Big|_{x=-1}^{x=1}+\frac{1}{64}\int_{x=-1}^{x=1}\cos(2\theta)+\cos(6\theta)\ d\theta$$

$$-\frac{1}{80}\int_{x=-1}^{x=1}(1+\cos(2\theta))^3\ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{7}{64}\sin(2\theta) + \frac{3}{128}\sin(4\theta)\Big|_{x=-1}^{x=1} + \frac{1}{64}\left(\frac{1}{2}\sin(2\theta) + \frac{1}{6}\sin(6\theta)\right)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} (1+\cos(2\theta))^3 \ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{7}{64}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{128}\sin(2\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} (1+\cos(2\theta))^3 \ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{14}{128}\sin(2\theta) + \frac{1}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} (1+\cos(2\theta))^3 \ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} (1+\cos(2\theta))^3 \ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} (1+2\cos(2\theta)+\cos^2(2\theta))(1+\cos(2\theta)) \ d\theta$$

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$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} 1 + 3\cos(2\theta) + 3\cos^2(2\theta) + \cos^3(2\theta)\ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} 1 + 3\cos(2\theta) + 3\left(\frac{1}{2}(1+\cos(2(2\theta)))\right) + \left(\frac{1}{2}(1+\cos(2(2\theta)))\right)\cos(2\theta)\ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} 1 + 3\cos(2\theta) + \frac{3}{2}(1+\cos(4\theta)) + \frac{1}{2}(1+\cos(4\theta))\cos(2\theta)\ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} 1 + 3\cos(2\theta) + \frac{3}{2} + \frac{3}{2}\cos(4\theta) + \frac{1}{2}(\cos(2\theta) + \cos(2\theta)\cos(4\theta))\ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1} 1 + 3\cos(2\theta) + \frac{3}{2} + \frac{3}{2}\cos(4\theta) + \frac{1}{2}\cos(2\theta) + \frac{1}{2}\cos(2\theta)\cos(4\theta)\ d\theta$$

$$I_x = \frac{5}{32}\theta + \frac{15}{128}\sin(2\theta) + \frac{3}{128}\sin(4\theta) + \frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{80}\int_{x=-1}^{x=1}\frac{5}{2}+\frac{7}{2}\cos(2\theta)+\frac{3}{2}\cos(4\theta)+\frac{1}{2}\cos(2\theta)\cos(4\theta)\ d\theta$$

$$I_x=\frac{5}{32}\theta+\frac{15}{128}\sin(2\theta)+\frac{3}{128}\sin(4\theta)+\frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{160}\int_{x=-1}^{x=1}5+7\cos(2\theta)+3\cos(4\theta)+\cos(2\theta)\cos(4\theta)\ d\theta$$

$$I_x=\frac{5}{32}\theta+\frac{15}{128}\sin(2\theta)+\frac{3}{128}\sin(4\theta)+\frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{160}\int_{x=-1}^{x=1}5+7\cos(2\theta)+3\cos(4\theta)+\frac{1}{2}[\cos(4\theta-2\theta)+\cos(4\theta+2\theta)]\ d\theta$$

$$I_x=\frac{5}{32}\theta+\frac{15}{128}\sin(2\theta)+\frac{3}{128}\sin(4\theta)+\frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{160}\int_{x=-1}^{x=1}5+7\cos(2\theta)+3\cos(4\theta)+\frac{1}{2}\cos(2\theta)+\frac{1}{2}\cos(6\theta)\ d\theta$$

$$I_x=\frac{5}{32}\theta+\frac{15}{128}\sin(2\theta)+\frac{3}{128}\sin(4\theta)+\frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$-\frac{1}{160}\int_{x=-1}^{x=1}5+\frac{15}{2}\cos(2\theta)+3\cos(4\theta)+\frac{1}{2}\cos(6\theta)\ d\theta$$

Integrate.

$$I_x=\frac{5}{32}\theta+\frac{15}{128}\sin(2\theta)+\frac{3}{128}\sin(4\theta)+\frac{1}{384}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

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$$-\frac{5}{160}\theta-\frac{15}{640}\sin(2\theta)-\frac{3}{640}\sin(4\theta)-\frac{1}{1{,}920}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$I_x=\frac{5}{32}\theta+\frac{15}{128}\sin(2\theta)+\frac{3}{128}\sin(4\theta)+\frac{1}{384}\sin(6\theta)$$

$$-\frac{5}{160}\theta-\frac{15}{640}\sin(2\theta)-\frac{3}{640}\sin(4\theta)-\frac{1}{1{,}920}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$I_x=\frac{25}{160}\theta-\frac{5}{160}\theta+\frac{75}{640}\sin(2\theta)-\frac{15}{640}\sin(2\theta)$$

$$+\frac{15}{640}\sin(4\theta)-\frac{3}{640}\sin(4\theta)+\frac{5}{1{,}920}\sin(6\theta)-\frac{1}{1{,}920}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

$$I_x=\frac{1}{8}\theta+\frac{3}{32}\sin(2\theta)+\frac{3}{160}\sin(4\theta)+\frac{1}{480}\sin(6\theta)\Big|_{x=-1}^{x=1}$$

Back-substitute to put the expression in terms of x.

$$I_x=\frac{1}{8}\arcsin x+\frac{3}{32}\sin(2\arcsin x)$$

$$+\frac{3}{160}\sin(4\arcsin x)+\frac{1}{480}\sin(6\arcsin x)\Big|_{x=-1}^{x=1}$$

$$I_x=\frac{1}{8}\arcsin 1+\frac{3}{32}\sin(2\arcsin 1)+\frac{3}{160}\sin(4\arcsin 1)+\frac{1}{480}\sin(6\arcsin 1)$$

$$-\left(\frac{1}{8}\arcsin(-1)+\frac{3}{32}\sin(2\arcsin(-1))+\frac{3}{160}\sin(4\arcsin(-1))+\frac{1}{480}\sin(6\arcsin(-1))\right)$$

$$I_x=\frac{1}{8}\left(\frac{\pi}{2}\right)+\frac{3}{32}\sin\left(2\left(\frac{\pi}{2}\right)\right)+\frac{3}{160}\sin\left(4\left(\frac{\pi}{2}\right)\right)+\frac{1}{480}\sin\left(6\left(\frac{\pi}{2}\right)\right)$$

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$$-\left(\frac{1}{8}\left(-\frac{\pi}{2}\right)+\frac{3}{32}\sin\left(2\left(-\frac{\pi}{2}\right)\right)+\frac{3}{160}\sin\left(4\left(-\frac{\pi}{2}\right)\right)+\frac{1}{480}\sin\left(6\left(-\frac{\pi}{2}\right)\right)\right)$$

$$I_x=\frac{\pi}{16}+\frac{3}{32}\sin\pi+\frac{3}{160}\sin(2\pi)+\frac{1}{480}\sin(3\pi)$$

$$-\left(-\frac{\pi}{16}+\frac{3}{32}\sin(-\pi)+\frac{3}{160}\sin(-2\pi)+\frac{1}{480}\sin(-3\pi)\right)$$

$$I_x=\frac{\pi}{16}+\frac{3}{32}\sin\pi+\frac{3}{160}\sin(2\pi)+\frac{1}{480}\sin(3\pi)$$

$$+\frac{\pi}{16}-\frac{3}{32}\sin(-\pi)-\frac{3}{160}\sin(-2\pi)-\frac{1}{480}\sin(-3\pi)$$

$$I_x=\frac{\pi}{16}+\frac{3}{32}(0)+\frac{3}{160}(0)+\frac{1}{480}(0)+\frac{\pi}{16}-\frac{3}{32}(0)-\frac{3}{160}(0)-\frac{1}{480}(0)$$

$$I_x=\frac{\pi}{16}+\frac{\pi}{16}$$

$$I_x=\frac{\pi}{8}$$

Topic: Moments of inertia

Question: A solid circular cylinder with base radius 4 and height 6 has a density at (x, y, z) that is four times the distance of the point from one of the bases of the cylinder. The moment of inertia of the object about the z-axis is given. What are the boundaries of the solid?

$$I_z = 24 \int_0^4 3x^2\sqrt{16 - x^2} + \left(\sqrt{16 - x^2}\right)^3 \ dx$$

Answer choices:

A $\quad x^2 + y^2 = 4$, $x = 0$, and $z = 6$

B $\quad x + y = 8$, $x = 0$, and $y = 6$

C $\quad x^2 + y^2 = 16$, $z = 0$, and $z = 6$

D $\quad x^2 - y^2 = 16$, $z = 0$, and $z = 6$

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Solution: C

We were told that the density is given by

$$\delta\left(x, y, z\right) = 4z$$

Choosing the boundaries from answer choice C of $x^2 + y^2 = 16$, $z = 0$, and $z = 6$ gives the triple integral as

$$I_z = \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^6 4z\left(x^2 + y^2\right)\ dz\ dy\ dx$$

$$I_z = \int_0^4 \int_0^{\sqrt{16-x^2}} \int_0^6 4x^2z + 4y^2z\ dz\ dy\ dx$$

Integrate first with respect to z, then evaluate over the interval.

$$I_z = \int_0^4 \int_0^{\sqrt{16-x^2}} 2x^2z^2 + 2y^2z^2\Big|_{z=0}^{z=6}\ dy\ dx$$

$$I_z = \int_0^4 \int_0^{\sqrt{16-x^2}} 2x^2(6)^2 + 2y^2(6)^2 - \left(2x^2(0)^2 + 2y^2(0)^2\right)\ dy\ dx$$

$$I_z = \int_0^4 \int_0^{\sqrt{16-x^2}} 72x^2 + 72y^2\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$I_z = \int_0^4 72x^2y + 24y^3\Big|_{y=0}^{y=\sqrt{16-x^2}}\ dx$$

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$$I_z = \int_0^4 72x^2\sqrt{16-x^2} + 24\left(\sqrt{16-x^2}\right)^3 - \left(72x^2(0) + 24(0)^3\right)\ dx$$

$$I_z = 24\int_0^4 3x^2\sqrt{16-x^2} + \left(\sqrt{16-x^2}\right)^3\ dx$$

Because this is the integral we were given, we know that we picked the correct bounds.

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Topic: Moments of inertia

Question: The radius of a solid ball centered at the origin is 1 and its constant density about the diameter is 4. Which is the moment of inertia of the ball?

Answer choices:

A $\quad I = \dfrac{22\pi}{15}$

B $\quad I = \dfrac{16\pi}{15}$

C $\quad I = \dfrac{32\pi}{25}$

D $\quad I = \dfrac{32\pi}{15}$

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Solution: D

Using the known volume of a sphere, we get

$$I = \int_0^{2\pi} \int_0^1 \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} 4r^3 \ dz \ dr \ d\theta$$

$$I = 4 \int_0^{2\pi} \int_0^1 \int_{\sqrt{25-r^2}}^{\sqrt{1-r^2}} r^3 \ dz \ dr \ d\theta$$

Assume that $p = 1 - r^2$. Then,

$$I = 16\pi \int_0^1 (1-p)\sqrt{p} \ dp$$

$$I = \frac{32\pi}{15}$$

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Topic: Vector from two points

Question: Find the vector that connects the points.

$$\overrightarrow{ST}$$

$$S(1,5)$$

$$T(-1,-4)$$

Answer choices:

A $\langle -2,-9 \rangle$

B $\langle -2,9 \rangle$

C $\langle 2,9 \rangle$

D $\langle 2,-9 \rangle$

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Solution: A

To find the vector that connects two points $A(x_A, y_A)$ and $B(x_B, y_B)$, use the formula

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A \rangle$$

or for points in three-dimensional space $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, use

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

The important thing to remember here is that vectors indicate direction, so the vector $\overrightarrow{AB}$ is the vector that starts at A and ends at B. Therefore A is called the "initial point" and B is called the "terminal point". You want to make sure you're subtracting values in the initial point from values in the terminal point. You can think about it as

$$\overrightarrow{AB} = B - A$$

$$\overrightarrow{AB} = \text{terminal} - \text{initial}$$

So the vector $\overrightarrow{ST}$, where S is $S(1,5)$ and T is $T(-1, -4)$ is

$$\overrightarrow{ST} = \langle -1 - 1, -4 - 5 \rangle$$

$$\overrightarrow{ST} = \langle -2, -9 \rangle$$

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Topic: Vector from two points

Question: Find the vector that connects the points.

$$\overrightarrow{RQ}$$

$$Q(0,-1,4)$$

$$R(4,0,-9)$$

Answer choices:

A $\langle 4,1,-13\rangle$

B $\langle -4,-1,-5\rangle$

C $\langle -4,-1,13\rangle$

D $\langle 4,1,5\rangle$

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Solution: C

To find the vector that connects two points $A(x_A, y_A)$ and $B(x_B, y_B)$, use the formula

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A \rangle$$

or for points in three-dimensional space $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, use

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

The important thing to remember here is that vectors indicate direction, so the vector $\overrightarrow{AB}$ is the vector that starts at A and ends at B. Therefore A is called the "initial point" and B is called the "terminal point". You want to make sure you're subtracting values in the initial point from values in the terminal point. You can think about it as

$$\overrightarrow{AB} = B - A$$

$$\overrightarrow{AB} = \text{terminal} - \text{initial}$$

So the vector $\overrightarrow{RQ}$, where R is $R(4,0,-9)$ and Q is $Q(0,-1,4)$ is

$$\overrightarrow{RQ} = \langle 0-4, -1-0, 4-(-9) \rangle$$

$$\overrightarrow{RQ} = \langle -4, -1, 13 \rangle$$

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Topic: Vector from two points

Question: Find the vector that connects the points.

$$\overrightarrow{SR}$$

$$R(11,6,-9)$$

$$S(-8,-9,-14)$$

Answer choices:

A $\langle 3,-3,-25\rangle$

B $\langle 19,15,5\rangle$

C $\langle -3,3,25\rangle$

D $\langle -19,-15,-5\rangle$

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Solution: B

To find the vector that connects two points $A(x_A, y_A)$ and $B(x_B, y_B)$, use the formula

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A \rangle$$

or for points in three-dimensional space $A(x_A, y_A, z_A)$ and $B(x_B, y_B, z_B)$, use

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

The important thing to remember here is that vectors indicate direction, so the vector $\overrightarrow{AB}$ is the vector that starts at A and ends at B. Therefore A is called the “initial point” and B is called the “terminal point”. You want to make sure you're subtracting values in the initial point from values in the terminal point. You can think about it as

$$\overrightarrow{AB} = B - A$$

$$\overrightarrow{AB} = \text{terminal} - \text{initial}$$

So the vector $\overrightarrow{SR}$, where S is $S(-8, -9, -14)$ and R is $R(11,6, -9)$ is

$$\overrightarrow{SR} = \langle 11 - (-8), 6 - (-9), -9 - (-14) \rangle$$

$$\overrightarrow{SR} = \langle 19,15,5 \rangle$$

Topic: Combinations of vectors

Question: Find the combination.

$$\overrightarrow{ED} + \overrightarrow{DF}$$

Answer choices:

A $\overrightarrow{FED}$

B $\overrightarrow{EF}$

C $\overrightarrow{FE}$

D $\overrightarrow{DEF}$

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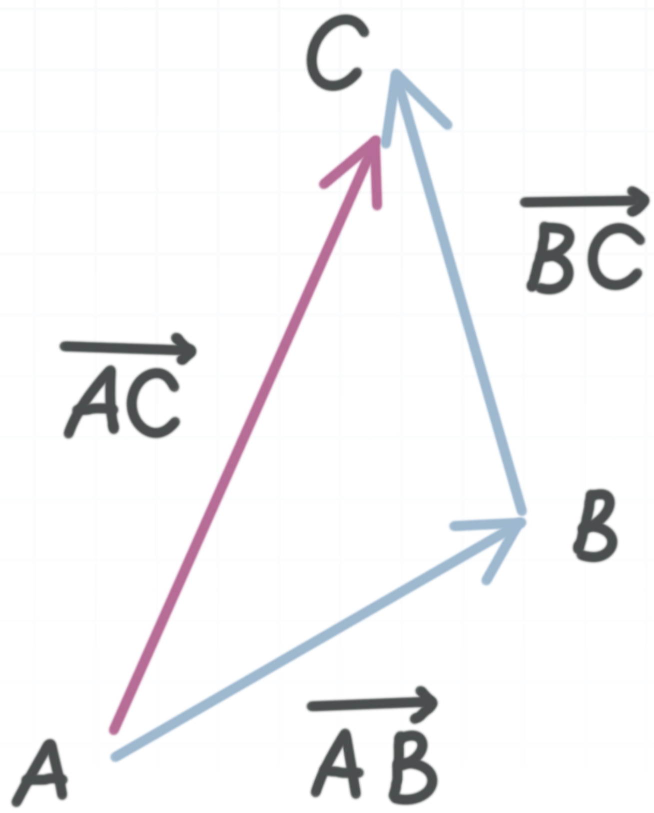
Solution: B

If you start with two vectors and then find their combination, what you're doing is placing the initial point of one of the vectors at the terminal point of the other vector, and then the combination is the vector that connects the initial point of the first with the terminal point of the second.



So $\overrightarrow{AC}$ is the combination of $\overrightarrow{AB}$ and $\overrightarrow{BC}$. In other words

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

And even if we didn't have a picture, we can look at $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and tell that what we're really doing is just bypassing B and connecting A directly to C.

So when we're asked to find $\overrightarrow{ED} + \overrightarrow{DF}$, we can say that we're bypassing D and connecting E directly to F. So the combination is

$$\overrightarrow{ED} + \overrightarrow{DF} = \overrightarrow{EF}$$

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Topic: Combinations of vectors

Question: Find the combination.

$$\overrightarrow{GH} - \overrightarrow{FH}$$

Answer choices:

A $\overrightarrow{HGF}$

B $\overrightarrow{FGH}$

C $\overrightarrow{FG}$

D $\overrightarrow{GF}$

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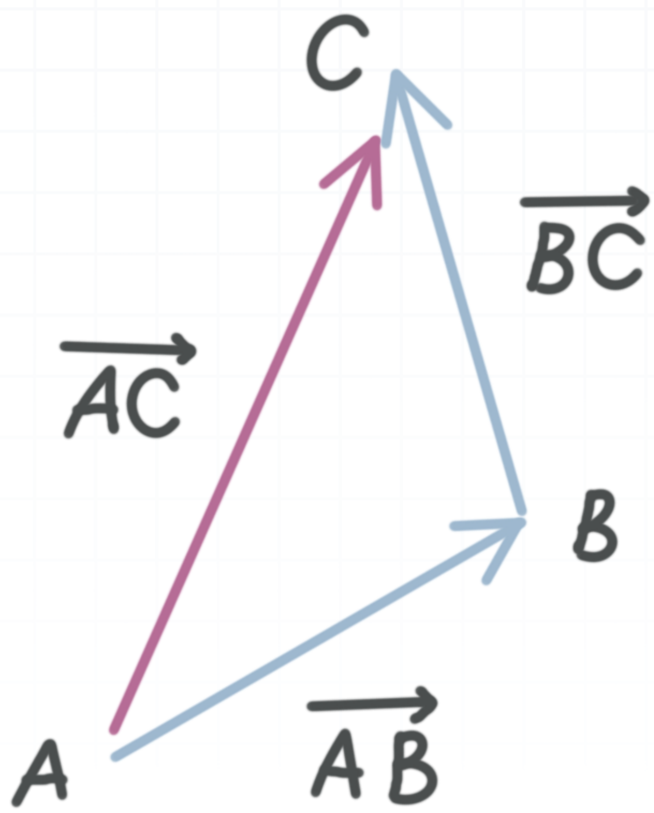
Solution: D

If you start with two vectors and then find their combination, what you're doing is placing the initial point of one of the vectors at the terminal point of the other vector, and then the combination is the vector that connects the initial point of the first with the terminal point of the second.



So $\overrightarrow{AC}$ is the combination of $\overrightarrow{AB}$ and $\overrightarrow{BC}$. In other words

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

And even if we didn't have a picture, we can look at $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and tell that what we're really doing is just bypassing B and connecting A directly to C.

When we've been asked to find the difference between two vectors, we want to turn the subtraction problem into an addition problem, which we can do by flipping the direction of the vector that's being subtracted.

So when we're asked to find $\overrightarrow{GH} - \overrightarrow{FH}$, we can flip around the vector being subtracted and change the subtraction to addition.

$$\overrightarrow{GH} - \overrightarrow{FH}$$

$$\overrightarrow{GH} + \overrightarrow{HF}$$

Then we can say that we're bypassing H and connecting G directly to F. So the combination is

$$\overrightarrow{GH} - \overrightarrow{FH} = \overrightarrow{GF}$$

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Topic: Combinations of vectors

Question: Find the combination.

$$\overrightarrow{BC}-\overrightarrow{DC}+\overrightarrow{DA}$$

Answer choices:

A $\overrightarrow{BA}$

B $\overrightarrow{DB}$

C $\overrightarrow{AC}$

D $\overrightarrow{CD}$

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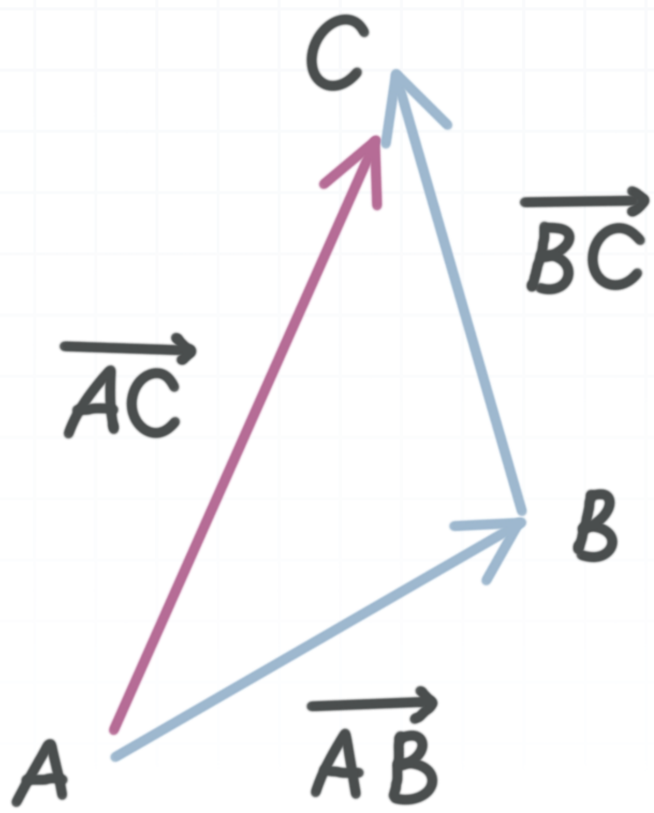
Solution: A

If you start with two vectors and then find their combination, what you're doing is placing the initial point of one of the vectors at the terminal point of the other vector, and then the combination is the vector that connects the initial point of the first with the terminal point of the second.



So $\overrightarrow{AC}$ is the combination of $\overrightarrow{AB}$ and $\overrightarrow{BC}$. In other words

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

And even if we didn't have a picture, we can look at $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and tell that what we're really doing is just bypassing B and connecting A directly to C.

When we've been asked to find the difference between two vectors, we want to turn the subtraction problem into an addition problem, which we can do by flipping the direction of the vector that's being subtracted.

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So when we're asked to find $\overrightarrow{BC} - \overrightarrow{DC} + \overrightarrow{DA}$, we can flip around the vector being subtracted and change the subtraction to addition.

$$\overrightarrow{BC} - \overrightarrow{DC} + \overrightarrow{DA}$$

$$\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$$

Then we can say that we're bypassing C and connecting B directly to D. So the combination becomes

$$\overrightarrow{BD} + \overrightarrow{DA}$$

From here we know that we're bypassing D and connecting B directly to A. So the combination is

$$\overrightarrow{BC} - \overrightarrow{DC} + \overrightarrow{DA} = \overrightarrow{BA}$$

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Topic: Sum of two vectors

Question: Find the sum of the vectors.

$$u = \langle -3,5 \rangle$$

$$v = \langle 6,8 \rangle$$

Answer choices:

A $\quad w = \langle 13,3 \rangle$

B $\quad w = \langle -3, -9 \rangle$

C $\quad w = \langle -9, -3 \rangle$

D $\quad w = \langle 3,13 \rangle$

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Solution: D

To sum the vectors $u = \langle -3,5 \rangle$ and $v = \langle 6,8 \rangle$, we simply add the x-coordinates together to get the new x-coordinate, and we do the same for the y-coordinates. We can call the new vector w.

$$w = \langle -3 + 6, 5 + 8 \rangle$$

$$w = \langle 3, 13 \rangle$$

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Topic: Sum of two vectors

Question: Find the sum of the vectors.

$$u = \langle 3,0,-5\rangle$$

$$v = \langle -1,-4,6\rangle$$

Answer choices:

A $\quad w = \langle 2,-4,1\rangle$

B $\quad w = \langle -11,4,4\rangle$

C $\quad w = \langle 1,-4,2\rangle$

D $\quad w = \langle 4,4,-11\rangle$

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Solution: A

To sum the vectors $u = \langle 3,0, -5 \rangle$ and $v = \langle -1, -4,6 \rangle$, we simply add the x-coordinates together to get the new x-coordinate, and we do the same for the y- and z-coordinates. We can call the new vector w.

$$w = \langle 3 + (-1), 0 + (-4), -5 + 6 \rangle$$

$$w = \langle 2, -4,1 \rangle$$

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Topic: Sum of two vectors

Question: Find the sum of the vectors.

$$u = 6\mathbf{i} - 7\mathbf{j} - \mathbf{k}$$

$$v = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

Answer choices:

A $\quad w = 7\mathbf{i} - 10\mathbf{j} + \mathbf{k}$

B $\quad w = -7\mathbf{i} + 10\mathbf{j} - \mathbf{k}$

C $\quad w = 5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$

D $\quad w = -5\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$

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Solution: C

To sum the vectors we'll take the direction numbers and rewrite the vectors as

$$u = 6\mathbf{i} - 7\mathbf{j} - \mathbf{k}$$

$$u = \langle 6, -7, -1 \rangle$$

and

$$v = -\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

$$v = \langle -1, 3, -2 \rangle$$

Then we simply add the x-coordinates together to get the new x-coordinate, and we do the same for the y- and z-coordinates. We can call the new vector w. Since we want the sum in the same form as the given vectors, we'll insert the sums back into the vector equation as

$$w = \left[6 + (-1)\right]\mathbf{i} + (-7 + 3)\mathbf{j} + \left[-1 + (-2)\right]\mathbf{k}$$

$$w = 5\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$$

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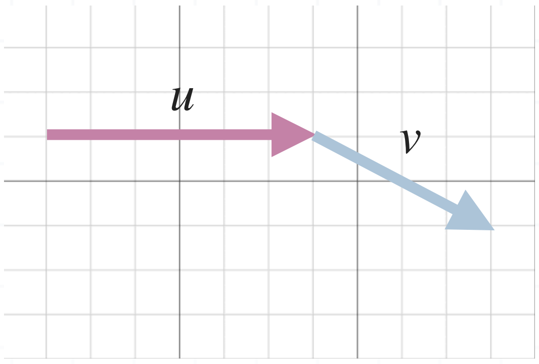
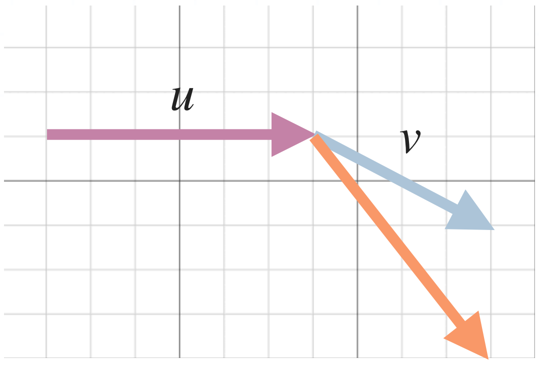
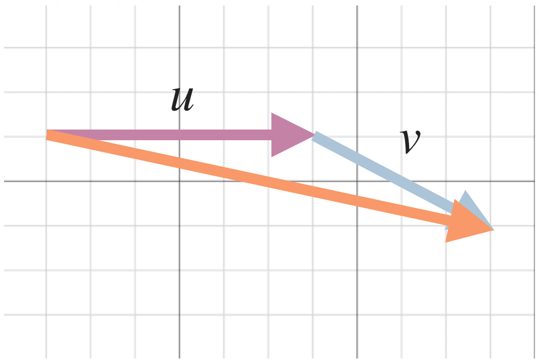
Topic: Copying vectors and using them to draw combinations

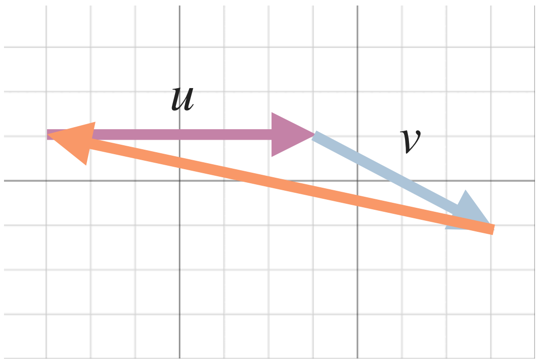
Question: Given the vectors u and v below, which red vector represents $u+v$?

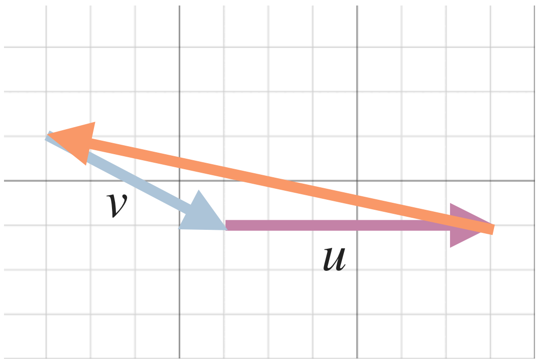


Answer choices:

A


B


C


D


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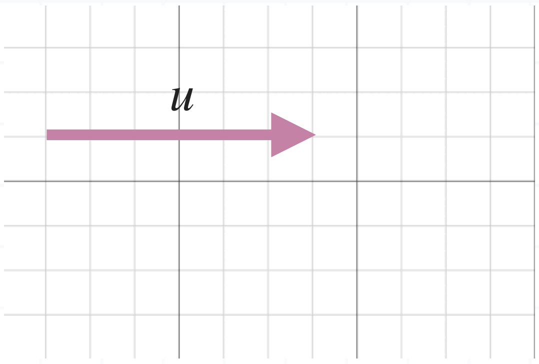
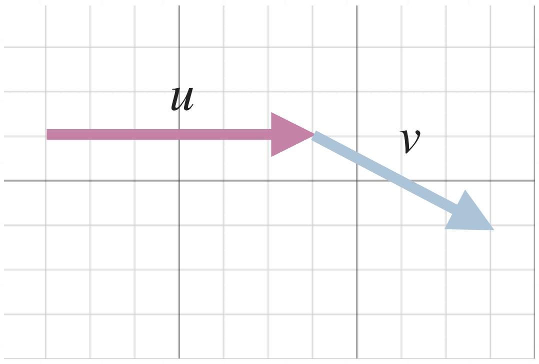
Solution: B

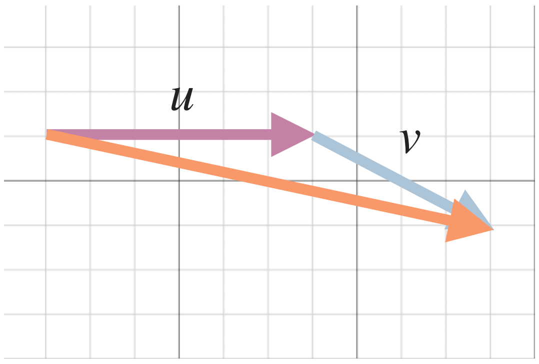
To solve for $u + v$, we start with the vector u.



Then we add the vector v to it by connecting the initial point of v to the terminal point of u.



To find the resultant vector $u + v$, we connect the initial point of u to the terminal point of v.



Remember that the direction of the resultant vector is from the first vector, towards the second vector.

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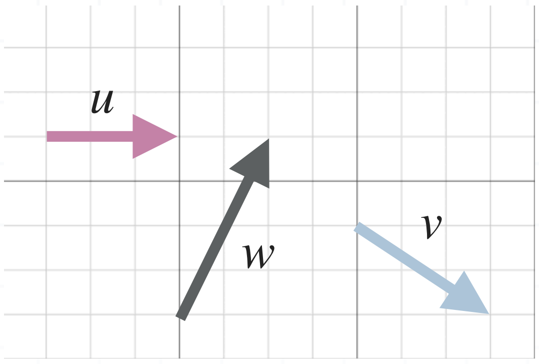
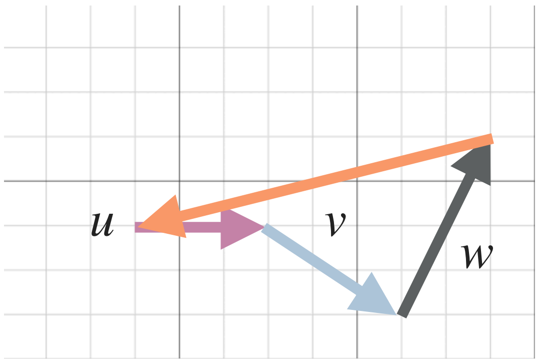
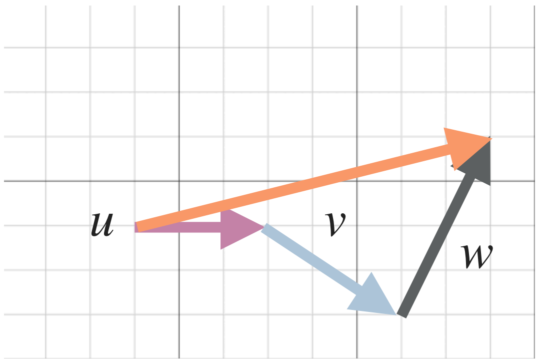
Topic: Copying vectors and using them to draw combinations

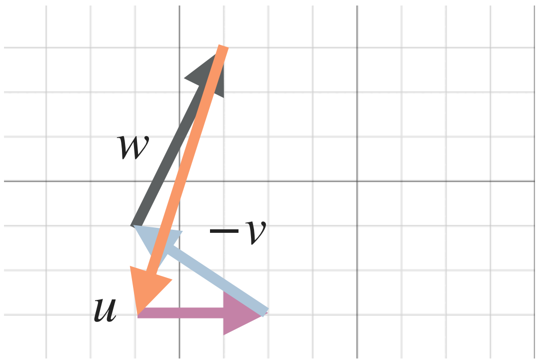
Question: Given the vectors u, v, and w below, which red vector represents $u-v+w$?

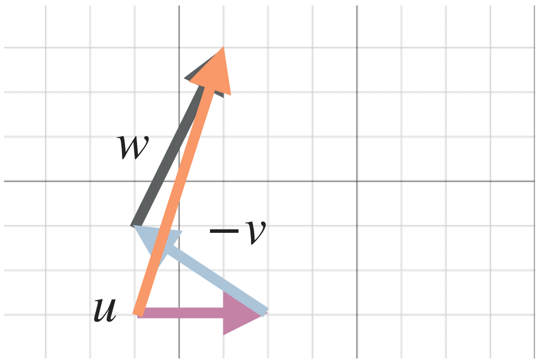


Answer choices:

A


B


C


D


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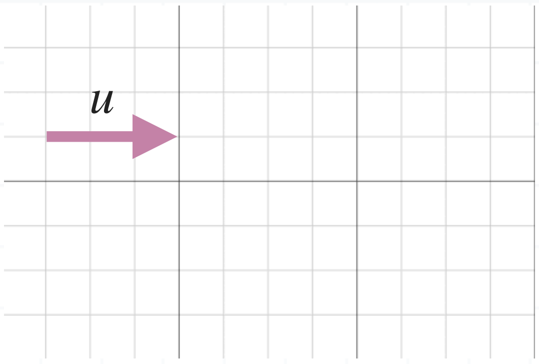
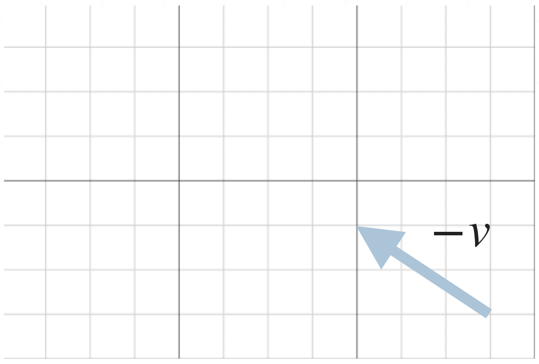
Solution: D

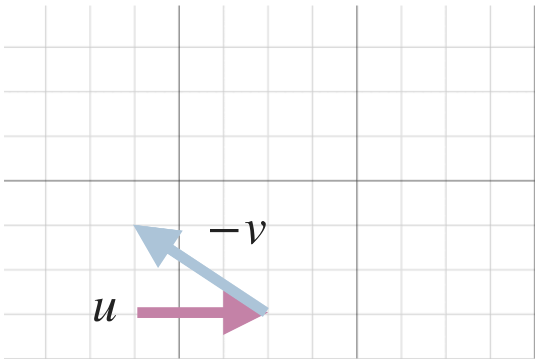
To solve for $u - v + w$, we start with the vector u.



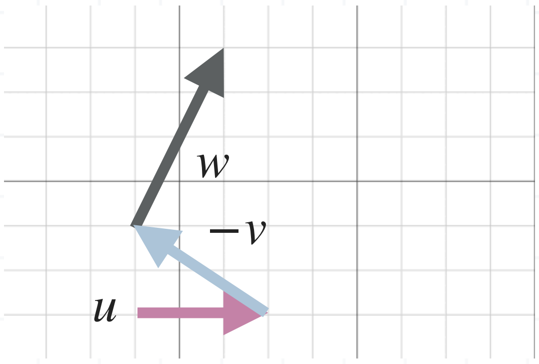
Then we add the vector v to it by connecting the initial point of v to the terminal point of u. However, the question is asking for vector $-v$ so we must first reverse the direction of vector v. This gives us



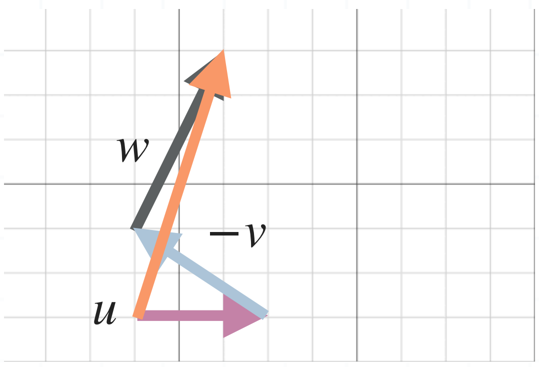
Connecting the initial point of vector $-v$ to the terminal point of vector u looks like



Then we add the vector w to this combination by connecting the initial point of vector w to the terminal point of vector $-v$.



To find the resultant vector $u - v + w$, we connect the initial point of u to the terminal point of w.



Remember that the direction of the resultant vector is from the first vector, towards the last vector.

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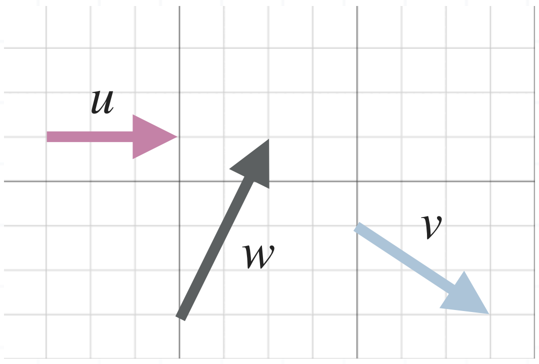
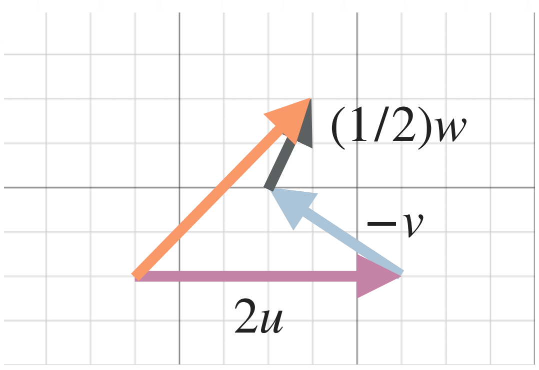
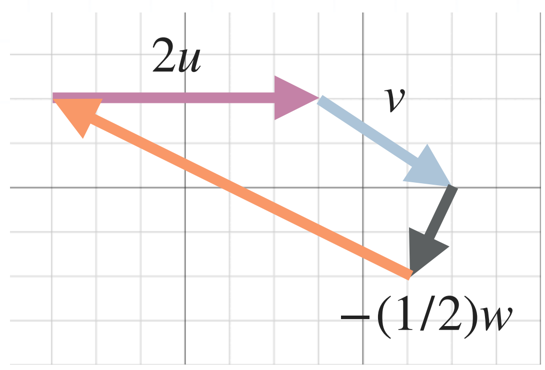
Topic: Copying vectors and using them to draw combinations

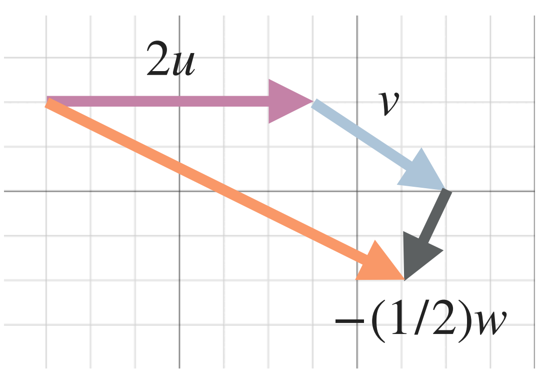
Question: Given the vectors u, v, and w below, which red vector represents $2u + v - (1/2)w$?

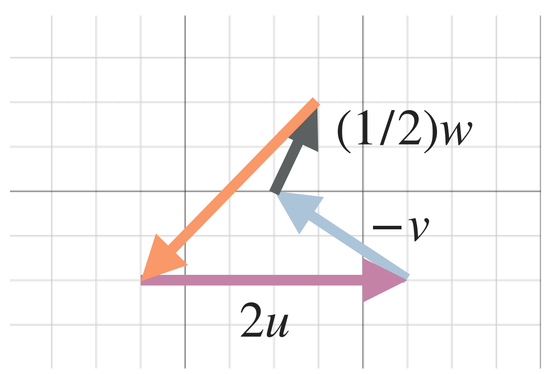


Answer choices:

A


B


C


D


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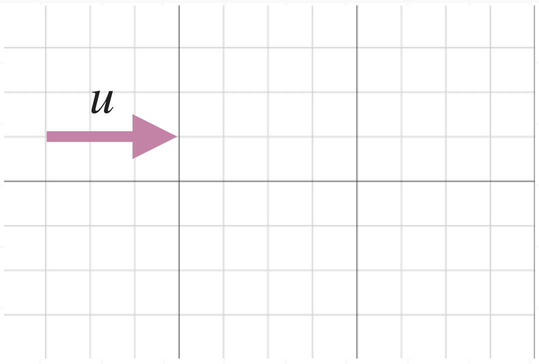
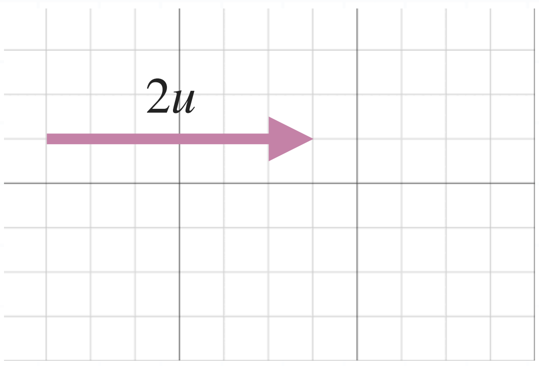
Solution: C

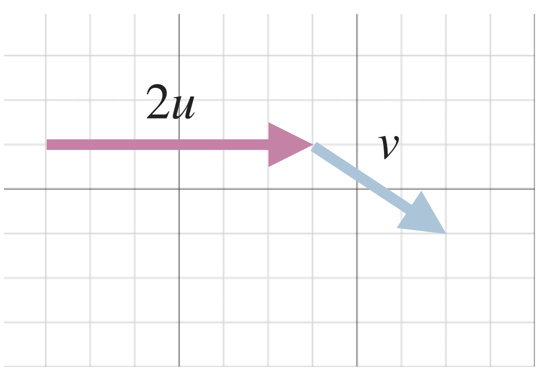
To solve for $2u + v - (1/2)w$, we start with the vector u.



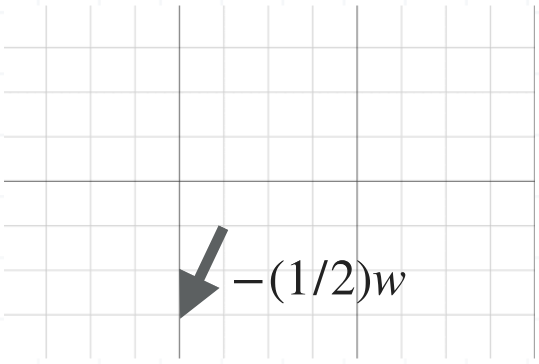
But we need $2u$ (double the magnitude of u).



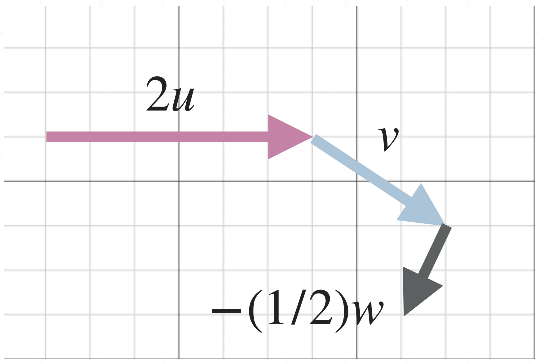
Then we add the vector v to it by connecting the initial point of v to the terminal point of $2u$.



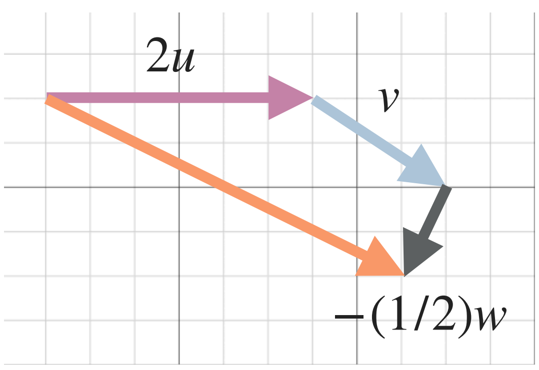
Then we add the vector $-(1/2)w$ to this combination. Remember vector $-(1/2)w$ will go in the opposite direction of vector w because of the negative sign. And vector $-(1/2)w$ will be half the magnitude of vector w because it's multiplied by $1/2$. So vector $-(1/2)w$ is



Now we can add vector $-(1/2)w$ to the rest of our combination by connecting the initial point of vector $-(1/2)w$ to the terminal point of vector v.



To find the resultant vector $2u + v - (1/2)w$, we connect the initial point of $2u$ to the terminal point of w.



Remember that the direction of the resultant vector is from the first vector, towards the last vector.

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Topic: Unit vector in the direction of the given vector

Question: Find the unit vector in the direction of the given vector.

$$\mathbf{i}+2\mathbf{j}-2\mathbf{k}$$

Answer choices:

A $\quad u=\frac{1}{9}\mathbf{i}+\frac{2}{9}\mathbf{j}-\frac{2}{9}\mathbf{k}$

B $\quad u=-\frac{1}{9}\mathbf{i}-\frac{2}{9}\mathbf{j}+\frac{2}{9}\mathbf{k}$

C $\quad u=\frac{1}{3}\mathbf{i}+\frac{2}{3}\mathbf{j}-\frac{2}{3}\mathbf{k}$

D $\quad u=-\frac{1}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{2}{3}\mathbf{k}$

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Solution: C

A unit vector is a vector that has a magnitude of 1. To find the unit vector in the same direction as $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$, we can start by finding the magnitude of $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ using the distance formula.

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

We'll let (x_1,y_1,z_1) be the origin $(0,0,0)$, and we'll take (x_2,y_2,z_2) from the direction numbers of $\mathbf{i}+2\mathbf{j}-2\mathbf{k}$ and say that $(x_2,y_2,z_2)=(1,2,-2)$. We'll plug both of these into the distance formula to find the magnitude.

$$D=\sqrt{(1-0)^2+(2-0)^2+(-2-0)^2}$$

$$D=\sqrt{1+4+4}$$

$$D=\sqrt{9}$$

$$D=3$$

The formula for the unit vector is

$$u=\frac{x}{D}\mathbf{i}+\frac{y}{D}\mathbf{j}+\frac{z}{D}\mathbf{k}$$

where x, y and z are the direction numbers from the given vector

where D is the magnitude we found earlier using the distance formula

Plugging $(1,2,-2)$ and $D=3$ into this formula gives

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$$u = \frac{1}{3}\mathbf{i} + \frac{2}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$$

Topic: Unit vector in the direction of the given vector

Question: Find the unit vector in the direction of the given vector.

$$-4\mathbf{i} + \mathbf{j} - 9\mathbf{k}$$

Answer choices:

A $\quad u = -\frac{4}{7\sqrt{2}}\mathbf{i} + \frac{1}{7\sqrt{2}}\mathbf{j} - \frac{9}{7\sqrt{2}}\mathbf{k}$

B $\quad u = \frac{4}{7\sqrt{2}}\mathbf{i} - \frac{1}{7\sqrt{2}}\mathbf{j} + \frac{9}{7\sqrt{2}}\mathbf{k}$

C $\quad u = -\frac{4}{98}\mathbf{i} + \frac{1}{98}\mathbf{j} - \frac{9}{98}\mathbf{k}$

D $\quad u = \frac{4}{98}\mathbf{i} - \frac{1}{98}\mathbf{j} + \frac{9}{98}\mathbf{k}$

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Solution: A

A unit vector is a vector that has a magnitude of 1. To find the unit vector in the same direction as $-4\mathbf{i}+\mathbf{j}-9\mathbf{k}$, we can start by finding the magnitude of $-4\mathbf{i}+\mathbf{j}-9\mathbf{k}$ using the distance formula.

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

We'll let (x_1,y_1,z_1) be the origin $(0,0,0)$, and we'll take (x_2,y_2,z_2) from the direction numbers of $-4\mathbf{i}+\mathbf{j}-9\mathbf{k}$ and say that $(x_2,y_2,z_2)=(-4,1,-9)$. We'll plug both of these into the distance formula to find the magnitude.

$$D=\sqrt{(-4-0)^2+(1-0)^2+(-9-0)^2}$$

$$D=\sqrt{16+1+81}$$

$$D=\sqrt{98}$$

$$D=\sqrt{49\cdot 2}$$

$$D=7\sqrt{2}$$

The formula for the unit vector is

$$u=\frac{x}{D}\mathbf{i}+\frac{y}{D}\mathbf{j}+\frac{z}{D}\mathbf{k}$$

where x, y and z are the direction numbers from the given vector

where D is the magnitude we found earlier using the distance formula

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Plugging $(-4,1,-9)$ and $D = 7\sqrt{2}$ into this formula gives

$$u = -\frac{4}{7\sqrt{2}}\mathbf{i} + \frac{1}{7\sqrt{2}}\mathbf{j} - \frac{9}{7\sqrt{2}}\mathbf{k}$$

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Topic: Unit vector in the direction of the given vector

Question: Find the unit vector in the direction of the given vector.

$$2\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$$

Answer choices:

A $u = \frac{2}{38}\mathbf{i} - \frac{5}{38}\mathbf{j} + \frac{3}{38}\mathbf{k}$

B $u = -\frac{2}{\sqrt{38}}\mathbf{i} + \frac{5}{\sqrt{38}}\mathbf{j} - \frac{3}{\sqrt{38}}\mathbf{k}$

C $u = -\frac{2}{38}\mathbf{i} + \frac{5}{38}\mathbf{j} - \frac{3}{38}\mathbf{k}$

D $u = \frac{2}{\sqrt{38}}\mathbf{i} - \frac{5}{\sqrt{38}}\mathbf{j} + \frac{3}{\sqrt{38}}\mathbf{k}$

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Solution: D

A unit vector is a vector that has a magnitude of 1. To find the unit vector in the same direction as $2\mathbf{i}-5\mathbf{j}+3\mathbf{k}$, we can start by finding the magnitude of $2\mathbf{i}-5\mathbf{j}+3\mathbf{k}$ using the distance formula.

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

We'll let (x_1,y_1,z_1) be the origin $(0,0,0)$, and we'll take (x_2,y_2,z_2) from the direction numbers of $2\mathbf{i}-5\mathbf{j}+3\mathbf{k}$ and say that $(x_2,y_2,z_2)=(2,-5,3)$. We'll plug both of these into the distance formula to find the magnitude.

$$D=\sqrt{(2-0)^2+(-5-0)^2+(3-0)^2}$$

$$D=\sqrt{4+25+9}$$

$$D=\sqrt{38}$$

The formula for the unit vector is

$$u=\frac{x}{D}\mathbf{i}+\frac{y}{D}\mathbf{j}+\frac{z}{D}\mathbf{k}$$

where x, y and z are the direction numbers from the given vector

where D is the magnitude we found earlier using the distance formula

Plugging $(2,-5,3)$ and $D=\sqrt{38}$ into this formula gives

$$u = \frac{2}{\sqrt{38}}\mathbf{i} - \frac{5}{\sqrt{38}}\mathbf{j} + \frac{3}{\sqrt{38}}\mathbf{k}$$

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Topic: Angle between a vector and the x-axis

Question: What is the angle between the vector $\mathbf{i}+\mathbf{j}$ and the positive direction of the x-axis?

Answer choices:

A 45°

B 135°

C 60°

D 75°

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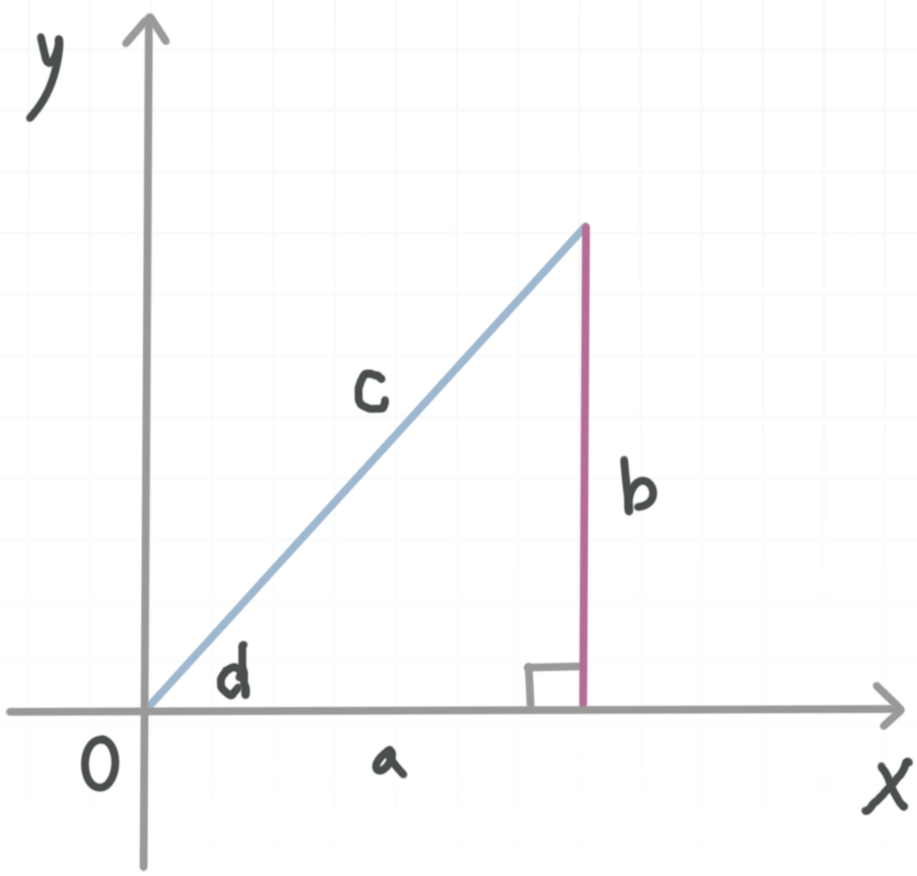
Solution: A

To calculate the angle between a vector $a\mathbf{i} + b\mathbf{j}$ and the positive direction of the x-axis, we start by sketching the vector.



The a represents the coefficient in front of the $\mathbf{i}$ term in the given vector, the b represents the coefficient in front of the $\mathbf{j}$ term in the given vector, the c represents the vector, and the d represents the angle we're trying to solve for. To find the angle, we can use the formula

$$d = \tan^{-1}\left(\frac{b}{a}\right)$$

Remember this formula will give us the answer in radians. You can then use the conversion factor

$$x^\circ = \frac{d \times 180^\circ}{\pi}$$

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To solve for the angle between the vector $\mathbf{i}+\mathbf{j}$ and the positive direction of the x-axis, we can see that $a=1$ and $b=1$. We can then use the formula for the angle.

$$d=\tan^{-1}\left(\frac{b}{a}\right)$$

$$d=\tan^{-1}\left(\frac{1}{1}\right)$$

$$d=\tan^{-1}(1)$$

$$d=0.785$$

Now we can convert from radians to degrees using the conversion factor and $d=0.785$.

$$x^\circ=\frac{d\times 180^\circ}{\pi}$$

$$x^\circ=\frac{(0.785)180^\circ}{\pi}$$

$$x^\circ=45^\circ$$

The angle between the vector $\mathbf{i}+\mathbf{j}$ and the positive direction of the x-axis is 45°.

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Topic: Angle between a vector and the x-axis

Question: What is the angle between the vector $2\mathbf{i}+\mathbf{j}$ and the positive direction of the x-axis?

Answer choices:

A 53.2°

B 116.6°

C 26.6°

D 143.2°

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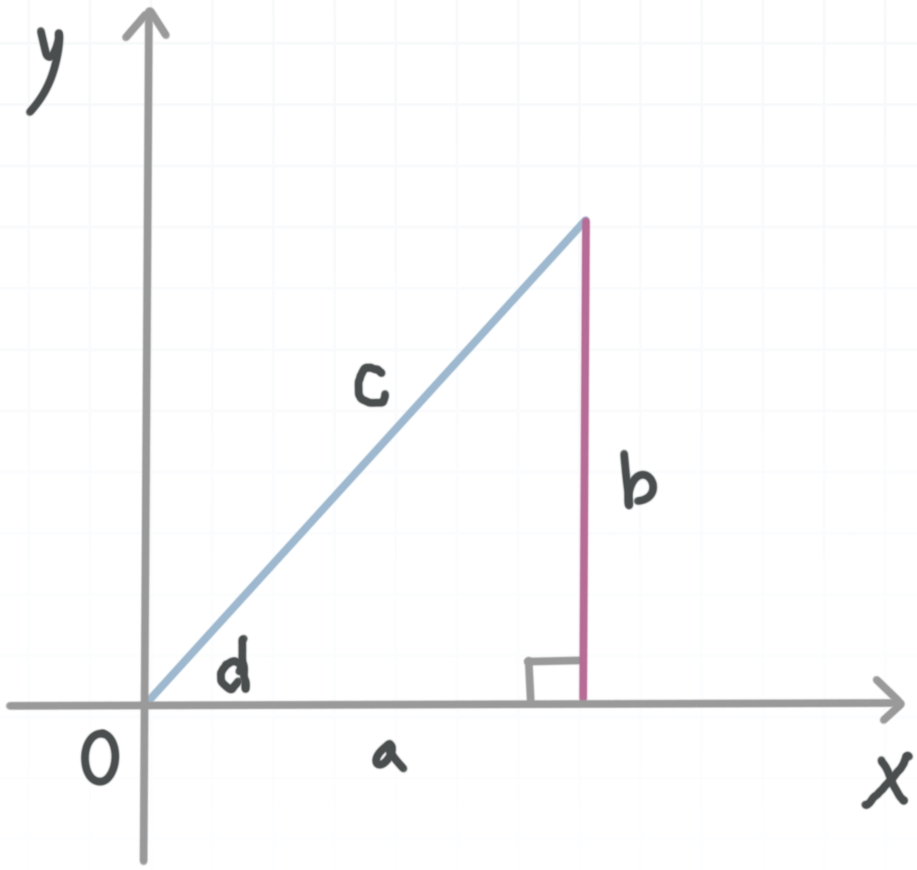
Solution: C

To calculate the angle between a vector $a\mathbf{i} + b\mathbf{j}$ and the positive direction of the x-axis, we start by sketching the vector.



The a represents the coefficient in front of the $\mathbf{i}$ term in the given vector, the b represents the coefficient in front of the $\mathbf{j}$ term in the given vector, the c represents the vector, and the d represents the angle we're trying to solve for. To find the angle, we can use the formula

$$d = \tan^{-1}\left(\frac{b}{a}\right)$$

Remember this formula will give us the answer in radians. You can then use the conversion factor

$$x^\circ = \frac{d \times 180^\circ}{\pi}$$

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To solve for the angle between the vector $2\mathbf{i}+\mathbf{j}$ and the positive direction of the x-axis, we can see that $a=2$ and $b=1$. We can then use the formula for the angle.

$$d=\tan^{-1}\left(\frac{b}{a}\right)$$

$$d=\tan^{-1}\left(\frac{1}{2}\right)$$

$$d=0.464$$

Now we can convert from radians to degrees using the conversion factor and $d=0.464$.

$$x^\circ=\frac{d\times 180^\circ}{\pi}$$

$$x^\circ=\frac{(0.464)180^\circ}{\pi}$$

$$x^\circ=26.6^\circ$$

The angle between the vector $2\mathbf{i}+\mathbf{j}$ and the positive direction of the x-axis is 26.6°.

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Topic: Angle between a vector and the x-axis

Question: What is the angle between the vector $2\mathbf{i}+\sqrt{2}\mathbf{j}$ and the positive direction of the x-axis?

Answer choices:

A 65.2°

B 35.2°

C 125.2°

D 155.2°

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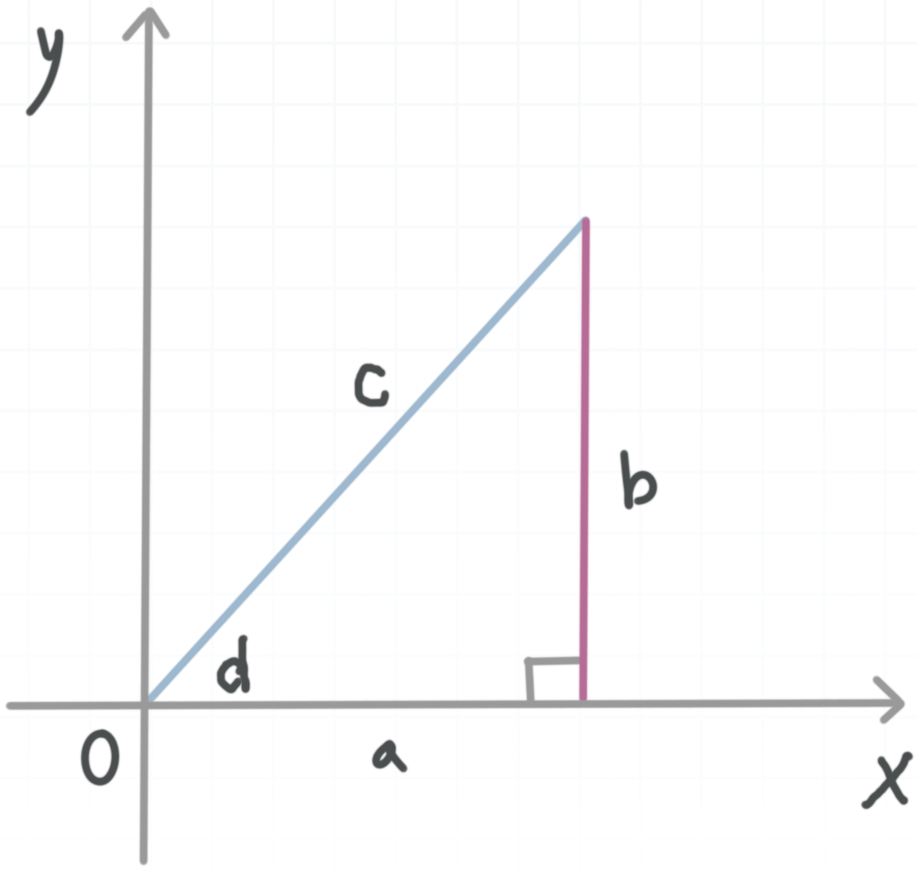
Solution: B

To calculate the angle between a vector $a\mathbf{i}+b\mathbf{j}$ and the positive direction of the x-axis, we start by sketching the vector.



The a represents the coefficient in front of the $\mathbf{i}$ term in the given vector, the b represents the coefficient in front of the $\mathbf{j}$ term in the given vector, the c represents the vector, and the d represents the angle we're trying to solve for. To find the angle, we can use the formula

$$d=\tan^{-1}\left(\frac{b}{a}\right)$$

Remember this formula will give us the answer in radians. You can then use the conversion factor

$$x^\circ=\frac{d\times 180^\circ}{\pi}$$

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To solve for the angle between the vector $2\mathbf{i}+\sqrt{2}\mathbf{j}$ and the positive direction of the x-axis, we can see that $a=2$ and $b=\sqrt{2}$. We can then use the formula for the angle.

$$d=\tan^{-1}\left(\frac{b}{a}\right)$$

$$d=\tan^{-1}\left(\frac{\sqrt{2}}{2}\right)$$

$$d=0.615$$

Now we can convert from radians to degrees using the conversion factor and $d=0.615$.

$$x^\circ=\frac{d\times180^\circ}{\pi}$$

$$x^\circ=\frac{(0.615)180^\circ}{\pi}$$

$$x^\circ=35.2^\circ$$

The angle between the vector $2\mathbf{i}+\sqrt{2}\mathbf{j}$ and the positive direction of the x -axis is 35.2°.

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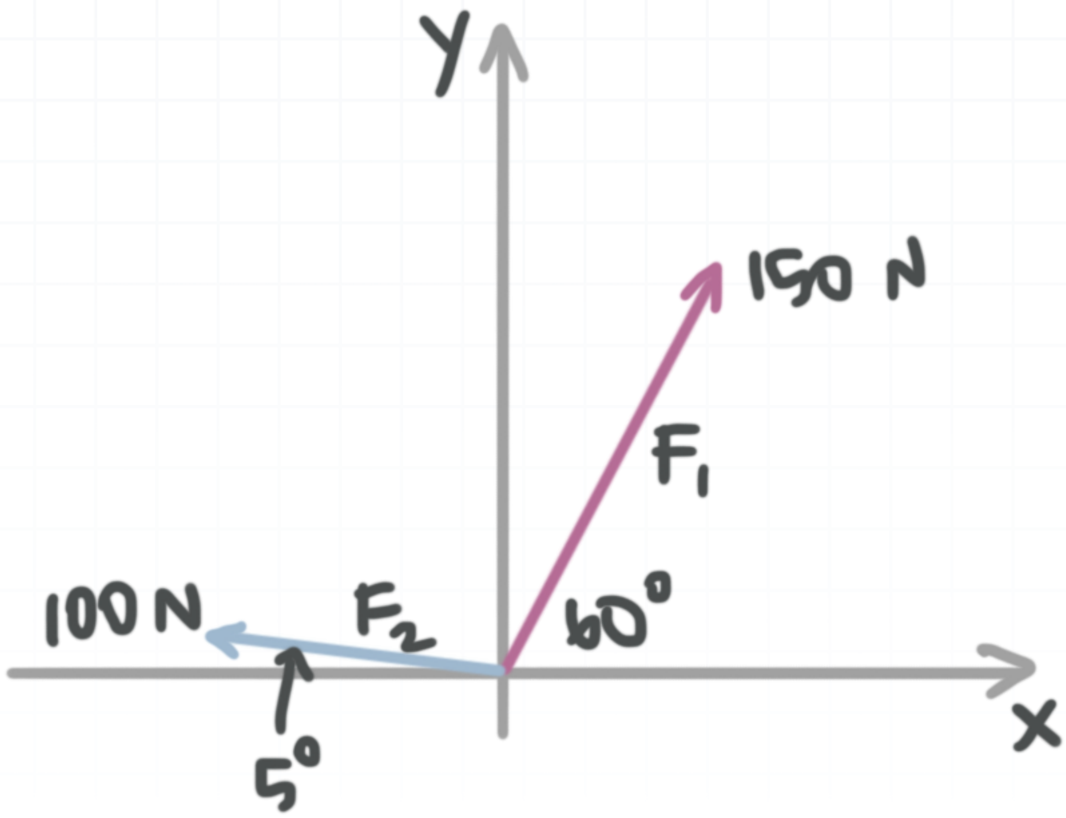
Topic: Magnitude and angle of the resultant force

Question: What is the the magnitude and angle of the resultant force of the vectors?



Answer choices:

A	139.66 N	and	82.99°
B	121.57 N	and	97.01°
C	140.79 N	and	100.07°
D	121.57 N	and	82.99°

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Solution: C

The first vector has a force of 150 N, and a 60° angle from the horizontal axis. Since it's in the first quadrant, we'll use a positive sign on each term.

$$F_1 = 150\cos 60^\circ \mathbf{i} + 150\sin 60^\circ \mathbf{j}$$

$$F_1 = 75\mathbf{i} + 129.90\mathbf{j}$$

$$F_1 = \langle 75,129.90\rangle$$

The second vector has a force of 100 N, and a 5° angle from the horizontal axis. Since it's in the second quadrant, we'll use a negative sign on the x-term and a positive sign on the y-term.

$$F_2 = -100\cos 5^\circ \mathbf{i} + 100\sin 5^\circ \mathbf{j}$$

$$F_2 = -99.62\mathbf{i} + 8.72\mathbf{j}$$

$$F_2 = \langle -99.62,8.72\rangle$$

Add F_1 and F_2 to get the resultant force.

$$F_R = 75\mathbf{i} + 129.90\mathbf{j} - 99.62\mathbf{i} + 8.72\mathbf{j}$$

$$F_R = -24.62\mathbf{i} + 138.62\mathbf{j}$$

$$F_R = \langle -24.62,138.62\rangle$$

Find the magnitude of the resultant force using the distance formula.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

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$$D_R = \sqrt{(-24.62 - 0)^2 + (138.62 - 0)^2}$$

$$D_R = \sqrt{606.14 + 19{,}215.50}$$

$$D_R = 140.79$$

Find the angle of the resultant force.

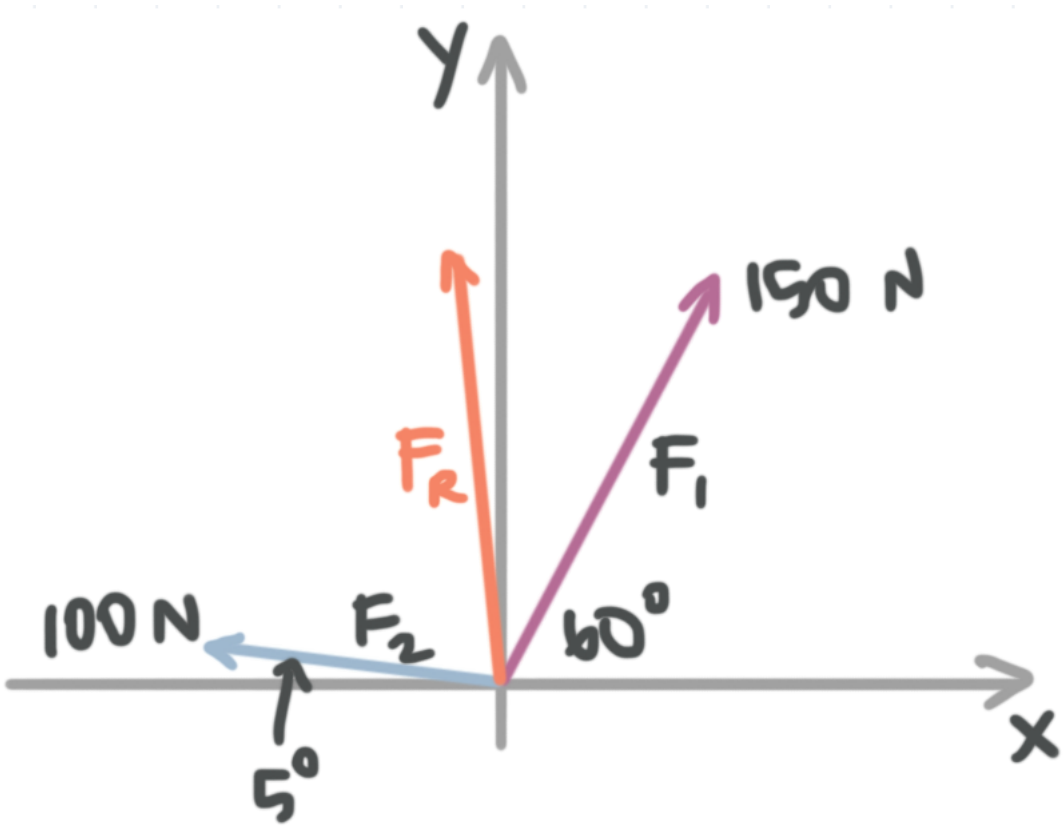
$$\theta_R = 180^\circ - \arctan\frac{|y|}{|x|}$$

$$\theta_R = 180^\circ - \arctan\frac{138.62}{24.62}$$

$$\theta_R = 180^\circ - 79.93^\circ$$

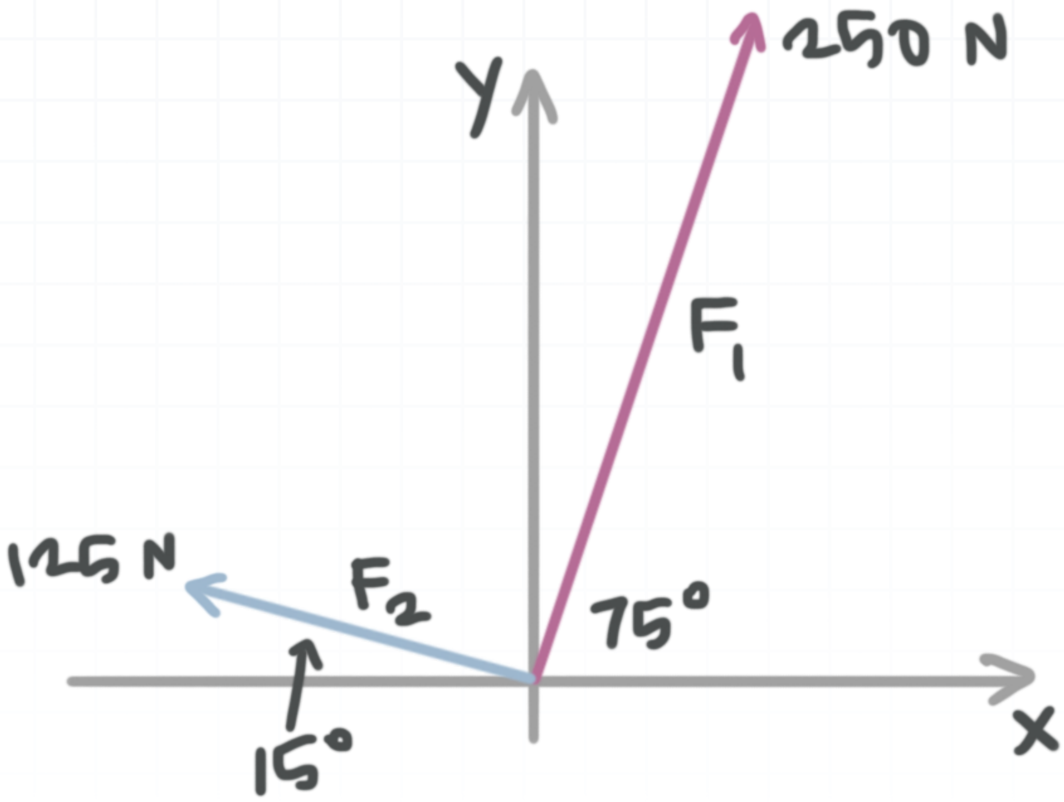
$$\theta_R = 100.07^\circ$$

The magnitude of the resultant force is 140.79 N and the angle of the resultant force is 100.07°.



Topic: Magnitude and angle of the resultant force

Question: What is the the magnitude and angle of the resultant force of the vectors?



Answer choices:

A	279.51 N	and	101.56°
B	279.51 N	and	78.44°
C	217.79 N	and	101.56°
D	217.79 N	and	78.44°

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Solution: A

The first vector has a force of 250 N, and a 75° angle from the horizontal axis. Since it's in the first quadrant, we'll use a positive sign on each term.

$$F_1 = 250\cos 75^\circ \mathbf{i} + 250\sin 75^\circ \mathbf{j}$$

$$F_1 = 64.70\mathbf{i} + 241.48\mathbf{j}$$

$$F_1 = \langle 64.70, 241.48 \rangle$$

The second vector has a force of 125 N, and a 15° angle from the horizontal axis. Since it's in the second quadrant, we'll use a negative sign on the x -term and a positive sign on the y-term.

$$F_2 = -125\cos 15^\circ \mathbf{i} + 125\sin 15^\circ \mathbf{j}$$

$$F_2 = -120.74\mathbf{i} + 32.35\mathbf{j}$$

$$F_2 = \langle -120.74, 32.35 \rangle$$

Add F_1 and F_2 to get the resultant force.

$$F_R = 64.70\mathbf{i} + 241.48\mathbf{j} - 120.74\mathbf{i} + 32.35\mathbf{j}$$

$$F_R = -17.05\mathbf{i} + 138.62\mathbf{j}$$

$$F_R = \langle -56.04, 273.83 \rangle$$

Find the magnitude of the resultant force using the distance formula.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D_R = \sqrt{(-56.04 - 0)^2 + (273.83 - 0)^2}$$

$$D_R = \sqrt{3{,}140.48 + 74{,}982.87}$$

$$D_R = 279.51$$

Find the angle of the resultant force.

$$\theta_R = 180^\circ - \arctan\frac{|y|}{|x|}$$

$$\theta_R = 180^\circ - \arctan\frac{273.83}{56.04}$$

$$\theta_R = 180^\circ - 78.44^\circ$$

$$\theta_R = 101.56^\circ$$

The magnitude of the resultant force is 279.51 N and the angle of the resultant force is 101.56°.

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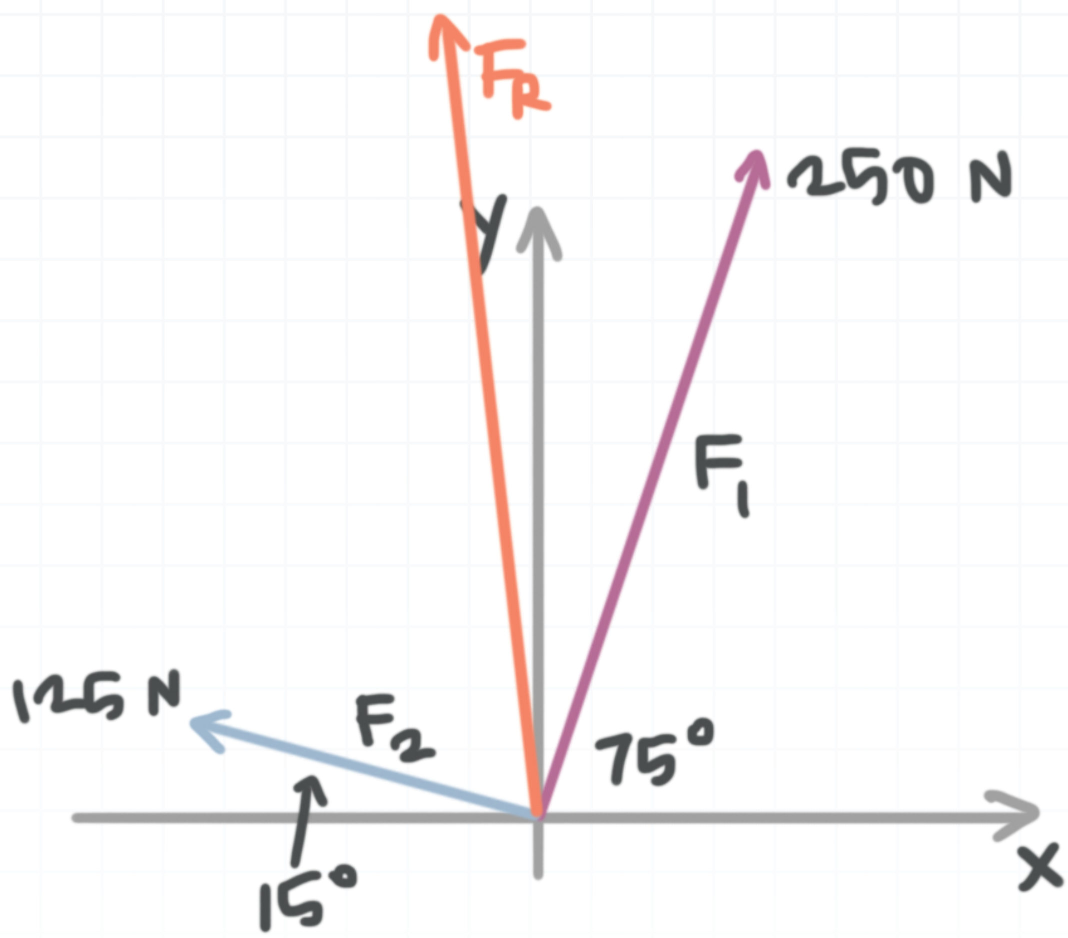
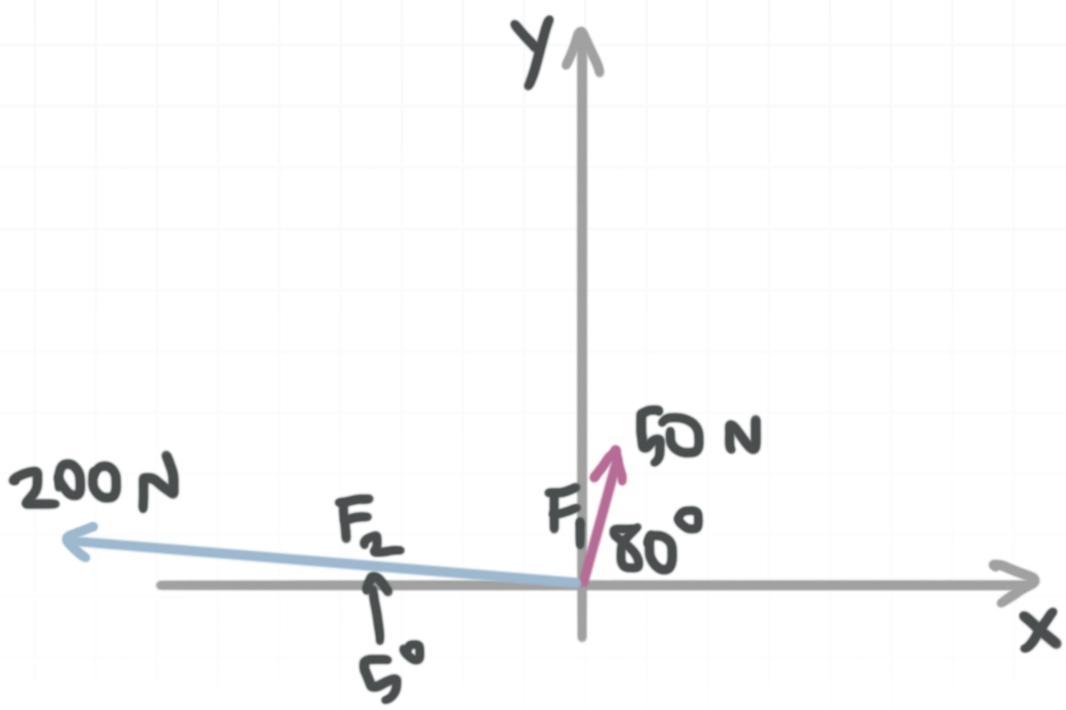


F_R
250 N
y
F_1
125 N
F_2
75°
15°
x

Topic: Magnitude and angle of the resultant force

Question: What is the the magnitude and angle of the resultant force of the vectors?



Answer choices:

A	201.89 N	and	19.28°
B	257.23 N	and	160.72°
C	257.23 N	and	19.28°
D	201.89 N	and	160.72°

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Solution: D

The first vector has a force of 50 N, and an 80° angle from the horizontal axis. Since it's in the first quadrant, we'll use a positive sign on each term.

$$F_1 = 50\cos 80^\circ\mathbf{i} + 50\sin 80^\circ\mathbf{j}$$

$$F_1 = 8.68\mathbf{i} + 49.24\mathbf{j}$$

$$F_1 = \langle 8.68, 49.24\rangle$$

The second vector has a force of 200 N, and a 5° angle from the horizontal axis. Since it's in the second quadrant, we'll use a negative sign on the x-term and a positive sign on the y-term.

$$F_2 = -200\cos 5^\circ\mathbf{i} + 200\sin 5^\circ\mathbf{j}$$

$$F_2 = -199.24\mathbf{i} + 17.43\mathbf{j}$$

$$F_2 = \langle -199.24, 17.43\rangle$$

Add F_1 and F_2 to get the resultant force.

$$F_R = 8.68\mathbf{i} + 49.24\mathbf{j} - 199.24\mathbf{i} + 17.43\mathbf{j}$$

$$F_R = -190.56\mathbf{i} + 66.67\mathbf{j}$$

$$F_R = \langle -190.56, 66.67\rangle$$

Find the magnitude of the resultant force using the distance formula.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

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$$D_R = \sqrt{(-190.56 - 0)^2 + (66.67 - 0)^2}$$

$$D_R = \sqrt{36{,}313.11 + 4{,}444.89}$$

$$D_R = 201.89$$

Find the angle of the resultant force.

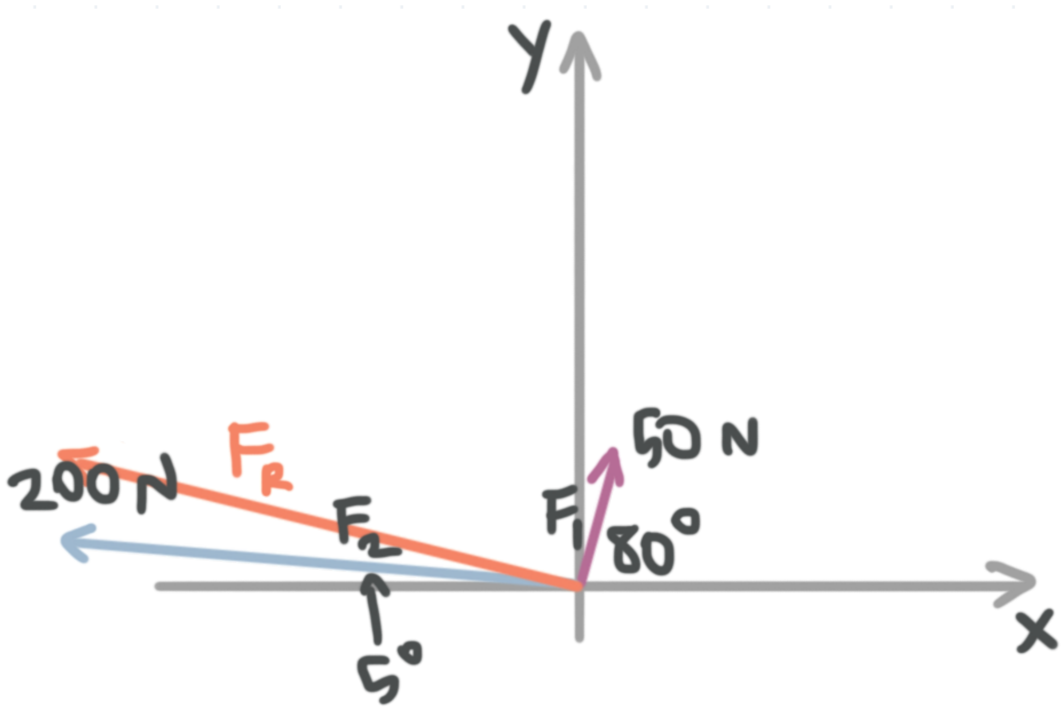
$$\theta_R = 180^\circ - \arctan \frac{|y|}{|x|}$$

$$\theta_R = 180^\circ - \arctan \frac{66.67}{190.56}$$

$$\theta_R = 180^\circ - 19.28^\circ$$

$$\theta_R = 160.72^\circ$$

The magnitude of the resultant force is 201.89 N and the angle of the resultant force is 160.72°.



Topic: Dot product of two vectors

Question: Find the dot product.

$$x = \langle 5, -1 \rangle$$

$$y = \langle 3, 2 \rangle$$

Answer choices:

A 11

B 13

C 9

D 17

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Solution: B

To find the dot product of two vectors, we simply multiply like coordinates together and then add them to each other. So the dot product of $x = \langle 5, -1 \rangle$ and $y = \langle 3,2 \rangle$ can be given by

$$x \cdot y = (5)(3) + (-1)(2)$$

$$x \cdot y = 15 - 2$$

$$x \cdot y = 13$$

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Topic: Dot product of two vectors

Question: Find the dot product.

$$x = \langle -4,0,12 \rangle$$

$$y = \langle 9, -12,8 \rangle$$

Answer choices:

A 48

B 132

C 72

D 60

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Solution: D

To find the dot product of two vectors, we simply multiply like coordinates together and then add them to each other. So the dot product of $x = \langle -4,0,12 \rangle$ and $y = \langle 9,-12,8 \rangle$ can be given by

$$x \cdot y = (-4)(9) + (0)(-12) + (12)(8)$$

$$x \cdot y = -36 + 0 + 96$$

$$x \cdot y = 60$$

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Topic: Dot product of two vectors

Question: Find the dot product.

$$x = -4\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$$

$$y = 6\mathbf{i} - \mathbf{j} - 10\mathbf{k}$$

Answer choices:

A -96

B -44

C -92

D -48

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Solution: C

To find the dot product of two vectors, we simply multiply like coordinates together and then add them to each other.

Taking the direction numbers from the given vectors lets us rewrite them as

$$x = -4\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$$

$$x = \langle -4, -2, 7 \rangle$$

and

$$y = 6\mathbf{i} - \mathbf{j} - 10\mathbf{k}$$

$$y = \langle 6, -1, -10 \rangle$$

So the dot product of $x = -4\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$ and $y = 6\mathbf{i} - \mathbf{j} - 10\mathbf{k}$ can be given by

$$x \cdot y = (-4)(6) + (-2)(-1) + (7)(-10)$$

$$x \cdot y = -24 + 2 - 70$$

$$x \cdot y = -92$$

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Topic: Angle between two vectors

Question: Find the angle between the vectors.

$$a = \langle 2,0,-1 \rangle$$

$$b = \langle -1,4,2 \rangle$$

Answer choices:

A 113°

B 247°

C 293°

D 67°

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Solution: A

The angle between two vectors a and b can be given by

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product

where $|a|$ is the length of the vector a (can also be called D_a)

where $|b|$ is the length of the vector b (can also be called D_b)

We'll start by finding the dot product. To find the dot product of two vectors, we just multiply like coordinates together and then add them to each other.

$$a \cdot b = (2)(-1) + (0)(4) + (-1)(2)$$

$$a \cdot b = -2 + 0 - 2$$

$$a \cdot b = -4$$

Next we'll find the length of each vector using the distance formula. We'll use the origin $(0,0,0)$ as (x_1, y_1, z_1), and we'll take (x_2, y_2, z_2) from the direction numbers of the vector.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The length of $a = \langle 2,0,-1 \rangle$ is

$$D_a = \sqrt{(2-0)^2 + (0-0)^2 + (-1-0)^2}$$

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$$D_a = \sqrt{4+0+1}$$

$$D_a = |a| = \sqrt{5}$$

The length of $b = \langle -1,4,2 \rangle$ is

$$D_b = \sqrt{(-1-0)^2 + (4-0)^2 + (2-0)^2}$$

$$D_b = \sqrt{1+16+4}$$

$$D_b = |b| = \sqrt{21}$$

Plugging everything we've found into the formula gives the angle between the vectors.

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}$$

$$\cos(\theta) = \frac{-4}{\sqrt{5}\sqrt{21}}$$

$$\cos(\theta) = \frac{-4}{\sqrt{105}}$$

$$\theta = \arccos \frac{-4}{\sqrt{105}}$$

$$\theta = 113^\circ$$

Topic: Angle between two vectors

Question: Find the angle between the vectors.

$$a = \langle 3, -2, 1 \rangle$$

$$b = \langle 5, 3, 1 \rangle$$

Answer choices:

A 297°

B 243°

C 63°

D 117°

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Solution: C

The angle between two vectors a and b can be given by

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product

where $|a|$ is the length of the vector a (can also be called D_a)

where $|b|$ is the length of the vector b (can also be called D_b)

We'll start by finding the dot product. To find the dot product of two vectors, we just multiply like coordinates together and then add them to each other.

$$a \cdot b = (3)(5) + (-2)(3) + (1)(1)$$

$$a \cdot b = 15 - 6 + 1$$

$$a \cdot b = 10$$

Next we'll find the length of each vector using the distance formula. We'll use the origin $(0,0,0)$ as (x_1, y_1, z_1), and we'll take (x_2, y_2, z_2) from the direction numbers of the vector.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The length of $a = \langle 3, -2,1 \rangle$ is

$$D_a = \sqrt{(3-0)^2 + (-2-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{9+4+1}$$

$$D_a = |a| = \sqrt{14}$$

The length of $b = \langle 5,3,1 \rangle$ is

$$D_b = \sqrt{(5-0)^2 + (3-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{25+9+1}$$

$$D_b = |b| = \sqrt{35}$$

Plugging everything we've found into the formula gives the angle between the vectors.

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}$$

$$\cos(\theta) = \frac{10}{\sqrt{14}\sqrt{35}}$$

$$\cos(\theta) = \frac{10}{\sqrt{490}}$$

$$\theta = \arccos \frac{10}{\sqrt{490}}$$

$$\theta = 63^\circ$$

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Topic: Angle between two vectors

Question: Find the angle between the vectors.

$$a = \langle 4, -4, -5 \rangle$$

$$b = \langle -2, 4, -3 \rangle$$

Answer choices:

A 283°

B 103°

C 257°

D 77°

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Solution: B

The angle between two vectors a and b can be given by

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product

where $|a|$ is the length of the vector a (can also be called D_a)

where $|b|$ is the length of the vector b (can also be called D_b)

We'll start by finding the dot product. To find the dot product of two vectors, we just multiply like coordinates together and then add them to each other.

$$a \cdot b = (4)(-2) + (-4)(4) + (-5)(-3)$$

$$a \cdot b = -8 - 16 + 15$$

$$a \cdot b = -9$$

Next we'll find the length of each vector using the distance formula. We'll use the origin $(0,0,0)$ as (x_1, y_1, z_1), and we'll take (x_2, y_2, z_2) from the direction numbers of the vector.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

The length of $a = \langle 4, -4, -5 \rangle$ is

$$D_a = \sqrt{(4-0)^2 + (-4-0)^2 + (-5-0)^2}$$

$$D_a = \sqrt{16 + 16 + 25}$$

$$D_a = |a| = \sqrt{57}$$

The length of $b = \langle -2, 4, -3 \rangle$ is

$$D_b = \sqrt{(-2 - 0)^2 + (4 - 0)^2 + (-3 - 0)^2}$$

$$D_b = \sqrt{4 + 16 + 9}$$

$$D_b = |b| = \sqrt{29}$$

Plugging everything we've found into the formula gives the angle between the vectors.

$$\cos(\theta) = \frac{a \cdot b}{|a||b|}$$

$$\cos(\theta) = \frac{-9}{\sqrt{57}\sqrt{29}}$$

$$\cos(\theta) = \frac{-9}{\sqrt{1,653}}$$

$$\theta = \arccos \frac{-9}{\sqrt{1,653}}$$

$$\theta = 103^\circ$$

Topic: Orthogonal, parallel or neither

Question: Say whether the vectors are orthogonal, parallel, or neither.

$$a = \langle 1,2,-1 \rangle$$

$$b = \langle 2,1,4 \rangle$$

Answer choices:

A Orthogonal

B Parallel

C Neither

D Impossible to know

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Solution: A

Two vectors are orthogonal to one another (set at 90° from each other) when their dot product is 0.

Two vectors are parallel they can be in different directions and have different magnitudes, but their direction numbers will be equal to each other, or multiples of one another.

Two vectors that don't meet either of the above criteria will be neither orthogonal nor parallel.

We'll test to see whether or not the vectors are orthogonal by calculating their dot product.

$$a \cdot b = (1)(2) + (2)(1) + (-1)(4)$$

$$a \cdot b = 2 + 2 - 4$$

$$a \cdot b = 0$$

Since the dot product is 0, the vectors are orthogonal.

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Topic: Orthogonal, parallel or neither

Question: Say whether the vectors are orthogonal, parallel, or neither.

$$a = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$b = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Answer choices:

A Orthogonal

B Parallel

C Neither

D Impossible to know

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Solution: B

Two vectors are orthogonal to one another (set at 90° from each other) when their dot product is 0.

Two vectors are parallel they can be in different directions and have different magnitudes, but their direction numbers will be equal to each other, or multiples of one another.

Two vectors that don't meet either of the above criteria will be neither orthogonal nor parallel.

We'll test to see whether or not the vectors are orthogonal by calculating their dot product.

$$a \cdot b = (1)(2) + (1)(2) + (2)(4)$$

$$a \cdot b = 2 + 2 + 8$$

$$a \cdot b = 12$$

Since the dot product is not 0, the vectors are not orthogonal.

Next we'll check to see whether or not the vectors are parallel. If we divide b by a factor of 2, we'll get

$$b = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

$$b = \frac{2}{2}\mathbf{i} + \frac{2}{2}\mathbf{j} + \frac{4}{2}\mathbf{k}$$

$$b = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

If we compare this to a, we can see that both a and b have the direction numbers $a = b = \langle 1,1,2 \rangle$. Therefore the vectors are parallel.

Topic: Orthogonal, parallel or neither

Question: Say whether the vectors are orthogonal, parallel, or neither.

$$a = -2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

$$b = 11\mathbf{i} - 3\mathbf{j} - 7\mathbf{k}$$

Answer choices:

A Orthogonal

B Parallel

C Neither

D Impossible to know

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Solution: C

Two vectors are orthogonal to one another (set at 90° from each other) when their dot product is 0.

Two vectors are parallel they can be in different directions and have different magnitudes, but their direction numbers will be equal to each other, or multiples of one another.

Two vectors that don't meet either of the above criteria will be neither orthogonal nor parallel.

We'll test to see whether or not the vectors are orthogonal by calculating their dot product.

$$a \cdot b = (-2)(11) + (5)(-3) + (7)(-7)$$

$$a \cdot b = -22 - 15 - 49$$

$$a \cdot b = -86$$

Since the dot product is not 0, the vectors are not orthogonal.

Next we'll check to see whether or not the vectors are parallel. Neither vector has any factor that we can remove that would make the two base vectors equal, so we can say that the vectors are not parallel. Therefore the vectors are neither orthogonal nor parallel.

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Topic: Acute angle between the lines

Question: Find the acute angle between the lines.

$$x - 2y = 1$$

$$2x - y = 3$$

Answer choices:

A 12°

B 37°

C 45°

D 53°

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Solution: B

To find the acute angle between two lines, we'll first convert them to standard vector format, so that we can use the formula for the angle between two vectors. In this case, the line $x-2y=1$ will become $a=\langle 1,-2\rangle$ and the line $2x-y=3$ will become $b=\langle 2,-1\rangle$. Now we can use

$$\cos\theta=\frac{a\cdot b}{|a||b|}$$

to find the angle between the vectors. $a\cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. The lengths can also be denoted by D_a and D_b, and are found using the distance formula

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

We'll start by finding the dot product.

$$a\cdot b=(1)(2)+(-2)(-1)$$

$$a\cdot b=2+2$$

$$a\cdot b=4$$

Now we'll find the length of each vector, using the origin $(0,0)$ as (x_1,y_1). The length of a is

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$D_a=\sqrt{(1-0)^2+(-2-0)^2}$$

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$$D_a = \sqrt{1+4}$$

$$D_a = |a| = \sqrt{5}$$

And the length of b is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D_b = \sqrt{(2-0)^2 + (-1-0)^2}$$

$$D_b = \sqrt{4+1}$$

$$D_b = |b| = \sqrt{5}$$

We'll plug everything we've found into the formula for the angle between the vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{4}{\sqrt{5}\sqrt{5}}$$

$$\cos\theta = \frac{4}{5}$$

$$\theta = \arccos\frac{4}{5}$$

$$\theta \approx 37^\circ$$

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Remember, if the angle is greater than 90°, it is not acute. This just means we accidentally found the obtuse angle between the two vectors. If that's the case, we can find the acute by subtracting the obtuse angle from 180°.

Topic: Acute angle between the lines

Question: Find the acute angle between the lines.

$$-3x - 1y = 2$$

$$4x + 5y = 5$$

Answer choices:

A 147°

B 123°

C 57°

D 33°

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Solution: D

To find the acute angle between two lines, we'll first convert them to standard vector format, so that we can use the formula for the angle between two vectors. In this case, the line $-3x-1y=2$ will become $a=\langle -3,-1\rangle$ and the line $4x+5y=5$ will become $b=\langle 4,5\rangle$. Now we can use

$$\cos\theta=\frac{a\cdot b}{|a||b|}$$

to find the angle between the vectors. $a\cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. The lengths can also be denoted by D_a and D_b, and are found using the distance formula

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

We'll start by finding the dot product.

$$a\cdot b=(-3)(4)+(-1)(5)$$

$$a\cdot b=-12-5$$

$$a\cdot b=-17$$

Now we'll find the length of each vector, using the origin $(0,0)$ as (x_1,y_1). The length of a is

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$D_a=\sqrt{(-3-0)^2+(-1-0)^2}$$

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$$D_a = \sqrt{9+1}$$

$$D_a = |a| = \sqrt{10}$$

And the length of b is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D_b = \sqrt{(4-0)^2 + (5-0)^2}$$

$$D_b = \sqrt{16+25}$$

$$D_b = |b| = \sqrt{41}$$

We'll plug everything we've found into the formula for the angle between the vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-17}{\sqrt{10}\sqrt{41}}$$

$$\cos\theta = \frac{-17}{\sqrt{410}}$$

$$\theta = \arccos\frac{-17}{\sqrt{410}}$$

$$\theta \approx 147^\circ$$

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Remember, if the angle is greater than 90°, it is not acute. This just means we accidentally found the obtuse angle between the two vectors. If that's the case, we can find the acute by subtracting the obtuse angle from 180°.

$$\theta = 180^\circ - 147^\circ$$

$$\theta \approx 33^\circ$$

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Topic: Acute angle between the lines

Question: Find the acute angle between the lines.

$$-8x - 4y = 7$$

$$11x + 7y = 2$$

Answer choices:

A 15°

B 84°

C 6°

D 174°

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Solution: C

To find the acute angle between two lines, we'll first convert them to standard vector format, so that we can use the formula for the angle between two vectors. In this case, the line $-8x-4y=7$ will become $a=\langle -8,-4\rangle$ and the line $11x+7y=2$ will become $b=\langle 11,7\rangle$. Now we can use

$$\cos\theta=\frac{a\cdot b}{|a||b|}$$

to find the angle between the vectors. $a\cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. The lengths can also be denoted by D_a and D_b, and are found using the distance formula

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

We'll start by finding the dot product.

$$a\cdot b=(-8)(11)+(-4)(7)$$

$$a\cdot b=-88-28$$

$$a\cdot b=-116$$

Now we'll find the length of each vector, using the origin $(0,0)$ as (x_1,y_1). The length of a is

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

$$D_a=\sqrt{(-8-0)^2+(-4-0)^2}$$

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$$D_a = \sqrt{64 + 16}$$

$$D_a = |a| = \sqrt{80}$$

And the length of b is

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

$$D_b = \sqrt{(11 - 0)^2 + (7 - 0)^2}$$

$$D_b = \sqrt{121 + 49}$$

$$D_b = |b| = \sqrt{170}$$

We'll plug everything we've found into the formula for the angle between the vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-116}{\sqrt{80}\sqrt{170}}$$

$$\cos\theta = \frac{-116}{\sqrt{13{,}600}}$$

$$\theta = \arccos\frac{-116}{\sqrt{13{,}600}}$$

$$\theta \approx 174^\circ$$

Remember, if the angle is greater than 90°, it is not acute. This just means we accidentally found the obtuse angle between the two vectors. If that's the case, we can find the acute by subtracting the obtuse angle from 180°.

$$\theta = 180^\circ - 174^\circ$$

$$\theta \approx 6^\circ$$

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Topic: Acute angles between the curves

Question: Find the acute angle between the curves.

$$y = x^2$$

$$y = 2x^2 - 4$$

Answer choices:

A 13.8°

B 15.0°

C 3.5°

D 6.9°

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Solution: D

We'll start by setting the curves equal to one another to find the points where they intersect.

$$x^2 = 2x^2 - 4$$

$$-x^2 = -4$$

$$x^2 = 4$$

$$x = \pm 2$$

Since we found two intersection points, we'll have two acute angles. We need to find the y-values for $x = \pm 2$. Since $x = \pm 2$ are the points where the curves intersect, we can plug $x = \pm 2$ into either curve to find the associated y-values.

For $x = 2$,

$$y = x^2$$

$$y = 2^2$$

$$y = 4$$

For $x = -2$,

$$y = x^2$$

$$y = (-2)^2$$

$$y = 4$$

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So the intersection points occur at $(2,4)$ and $(-2,4)$.

Now we need the equation of the tangent line for both curves at both intersection points, so we'll find the slope (derivative) of each curve at the intersection points, then plug the slope and the point into the point-slope formula for the equation of a line.

The derivative of $y = x^2$ is $y' = 2x$.

At $(2,4)$, the slope is $y'(2,4) = 2(2) = 4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 4(x - 2)$$

$$y - 4 = 4x - 8$$

$$y = 4x - 4$$

$$-4x + y = -4$$

At $(-2,4)$, the slope is $y'(-2,4) = 2(-2) = -4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -4\left[x - (-2)\right]$$

$$y - 4 = -4x - 8$$

$$y = -4x - 4$$

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$$4x + y = -4$$

The derivative of $y = 2x^2 - 4$ is $y' = 4x$.

At $(2,4)$, the slope is $y'(2,4) = 4(2) = 8$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = 8(x - 2)$$

$$y - 4 = 8x - 16$$

$$y = 8x - 12$$

$$-8x + y = -12$$

At $(-2,4)$, the slope is $y'(-2,4) = 4(-2) = -8$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -8\left[x - (-2)\right]$$

$$y - 4 = -8x - 16$$

$$y = -8x - 12$$

$$8x + y = -12$$

So the tangent lines at $(2,4)$ are

$-4x + y = -4$, which is $a = \langle -4,1 \rangle$ in standard vector form

$-8x + y = -12$, which is $b = \langle -8,1 \rangle$ in standard vector form

And the tangent lines at $(-2,4)$ are

$4x + y = -4$, which is $a = \langle 4,1 \rangle$ in standard vector form

$8x + y = -12$, which is $b = \langle 8,1 \rangle$ in standard vector form

Now that we've found our tangent lines and converted them to standard vector form, we can plug them into the equation for the angle between vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. We'll use the origin $(0,0)$ as the point (x_1, y_1) and then find the length of each vector using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The dot product of the vectors at $(2,4)$ is

$$a \cdot b = (-4)(-8) + (1)(1)$$

$$a \cdot b = 32 + 1$$

$$a \cdot b = 33$$

The length of $a = \langle -4,1 \rangle$ is

$$D_a = \sqrt{(-4-0)^2 + (1-0)^2}$$

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$$D_a = \sqrt{16 + 1}$$

$$D_a = |a| = \sqrt{17}$$

The length of $b = \langle -8,1 \rangle$ is

$$D_b = \sqrt{(-8-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{64 + 1}$$

$$D_b = |b| = \sqrt{65}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(2,4)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{33}{\sqrt{17}\sqrt{65}}$$

$$\cos\theta = \frac{33}{\sqrt{1{,}105}}$$

$$\theta = \arccos\frac{33}{\sqrt{1{,}105}}$$

$$\theta \approx 6.9^\circ$$

Now we'll work on the acute angle between the curves at $(-2,4)$. The dot product of the vectors at $(-2,4)$ is

$$a \cdot b = (4)(8) + (1)(1)$$

$$a \cdot b = 32 + 1$$

$$a \cdot b = 33$$

The length of $a = \langle 4,1 \rangle$ is

$$D_a = \sqrt{(4-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{16+1}$$

$$D_a = |a| = \sqrt{17}$$

The length of $b = \langle 8,1 \rangle$ is

$$D_b = \sqrt{(8-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{64+1}$$

$$D_b = |b| = \sqrt{65}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(-2,4)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{33}{\sqrt{17}\sqrt{65}}$$

$$\cos\theta = \frac{33}{\sqrt{1{,}105}}$$

$$\theta = \arccos\frac{33}{\sqrt{1{,}105}}$$

$$\theta \approx 6.9^\circ$$

In summary,

the acute angle between the curves at $(2,4)$ is $\theta \approx 6.9^\circ$

the acute angle between the curves at $(-2,4)$ is $\theta \approx 6.9^\circ$

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Topic: Acute angles between the curves

Question: Find the acute angle between the curves.

$$y = x^2$$

$$y = -x^2 + 18$$

Answer choices:

A 9.5°

B 18.9°

C 161.1°

D 170.5°

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Solution: B

We'll start by setting the curves equal to one another to find the points where they intersect.

$$x^2 = -x^2 + 18$$

$$2x^2 = 18$$

$$x^2 = 9$$

$$x = \pm 3$$

Since we found two intersection points, we'll have two acute angles. We need to find the y-values for $x = \pm 3$. Since $x = \pm 3$ are the points where the curves intersect, we can plug $x = \pm 3$ into either curve to find the associated y-values.

For $x = 3$,

$$y = x^2$$

$$y = 3^2$$

$$y = 9$$

For $x = -3$,

$$y = x^2$$

$$y = (-3)^2$$

$$y = 9$$

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So the intersection points occur at $(3,9)$ and $(-3,9)$.

Now we need the equation of the tangent line for both curves at both intersection points, so we'll find the slope (derivative) of each curve at the intersection points, then plug the slope and the point into the point-slope formula for the equation of a line.

The derivative of $y = x^2$ is $y' = 2x$.

At $(3,9)$, the slope is $y'(3,9) = 2(3) = 6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = 6(x - 3)$$

$$y - 9 = 6x - 18$$

$$y = 6x - 9$$

$$-6x + y = -9$$

At $(-3,9)$, the slope is $y'(-3,9) = 2(-3) = -6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = -6\left[x - (-3)\right]$$

$$y - 9 = -6x - 18$$

$$y = -6x - 9$$

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$$6x + y = -9$$

The derivative of $y = -x^2 + 18$ is $y' = -2x$.

At $(3,9)$, the slope is $y'(3,9) = -2(3) = -6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = -6(x - 3)$$

$$y - 9 = -6x + 18$$

$$y = -6x + 27$$

$$6x + y = 27$$

At $(-3,9)$, the slope is $y'(-3,9) = -2(-3) = 6$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 9 = 6\left[x - (-3)\right]$$

$$y - 9 = 6x + 18$$

$$y = 6x + 27$$

$$-6x + y = 27$$

So the tangent lines at $(3,9)$ are

$-6x + y = -9$, which is $a = \langle -6,1 \rangle$ in standard vector form

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$6x + y = 27$, which is $b = \langle 6,1 \rangle$ in standard vector form

And the tangent lines at $(-3,9)$ are

$6x + y = -9$, which is $a = \langle 6,1 \rangle$ in standard vector form

$-6x + y = 27$, which is $b = \langle -6,1 \rangle$ in standard vector form

Now that we've found our tangent lines and converted them to standard vector form, we can plug them into the equation for the angle between vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. We'll use the origin $(0,0)$ as the point (x_1, y_1) and then find the length of each vector using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The dot product of the vectors at $(3,9)$ is

$$a \cdot b = (-6)(6) + (1)(1)$$

$$a \cdot b = -36 + 1$$

$$a \cdot b = -35$$

The length of $a = \langle -6,1 \rangle$ is

$$D_a = \sqrt{(-6-0)^2 + (1-0)^2}$$

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$$D_a = \sqrt{36 + 1}$$

$$D_a = |a| = \sqrt{37}$$

The length of $b = \langle 6,1 \rangle$ is

$$D_b = \sqrt{(6-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{36 + 1}$$

$$D_b = |b| = \sqrt{37}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(3,9)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-35}{\sqrt{37}\sqrt{37}}$$

$$\cos\theta = -\frac{35}{37}$$

$$\theta = \arccos\left(-\frac{35}{37}\right)$$

$$\theta \approx 18.9^\circ$$

Now we'll work on the acute angle between the curves at $(-3,9)$. The dot product of the vectors at $(-3,9)$ is

$$a \cdot b = (6)(-6) + (1)(1)$$

$$a \cdot b = -36 + 1$$

$$a \cdot b = -35$$

The length of $a = \langle 6,1 \rangle$ is

$$D_a = \sqrt{(6-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{36+1}$$

$$D_a = |a| = \sqrt{37}$$

The length of $b = \langle -6,1 \rangle$ is

$$D_b = \sqrt{(-6-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{36+1}$$

$$D_b = |b| = \sqrt{37}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(-3,9)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-35}{\sqrt{37}\sqrt{37}}$$

$$\cos\theta = -\frac{35}{37}$$

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$$\theta = \arccos\left(-\frac{35}{37}\right)$$

$$\theta \approx 18.9^\circ$$

In summary,

the acute angle between the curves at $(3,9)$ is $\theta \approx 18.9^\circ$

the acute angle between the curves at $(-3,9)$ is $\theta \approx 18.9^\circ$

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Topic: Acute angles between the curves

Question: Find the acute angle between the curves.

$$y = x^2$$

$$y = -2x^2 + 3$$

Answer choices:

A 20.3°

B 139.4°

C 40.6°

D 159.7°

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Solution: C

We'll start by setting the curves equal to one another to find the points where they intersect.

$$x^2 = -2x^2 + 3$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = \pm 1$$

Since we found two intersection points, we'll have two acute angles. We need to find the y-values for $x = \pm 1$. Since $x = \pm 1$ are the points where the curves intersect, we can plug $x = \pm 1$ into either curve to find the associated y-values.

For $x = 1$,

$$y = x^2$$

$$y = 1^2$$

$$y = 1$$

For $x = -1$,

$$y = x^2$$

$$y = (-1)^2$$

$$y = 1$$

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So the intersection points occur at $(1,1)$ and $(-1,1)$.

Now we need the equation of the tangent line for both curves at both intersection points, so we'll find the slope (derivative) of each curve at the intersection points, then plug the slope and the point into the point-slope formula for the equation of a line.

The derivative of $y = x^2$ is $y' = 2x$.

At $(1,1)$, the slope is $y'(1,1) = 2(1) = 2$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 1$$

$$-2x + y = -1$$

At $(-1,1)$, the slope is $y'(-1,1) = 2(-1) = -2$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2\left[x - (-1)\right]$$

$$y - 1 = -2x - 2$$

$$y = -2x - 1$$

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$$2x + y = -1$$

The derivative of $y = -2x^2 + 3$ is $y' = -4x$.

At $(1,1)$, the slope is $y'(1,1) = -4(1) = -4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4(x - 1)$$

$$y - 1 = -4x + 4$$

$$y = -4x + 5$$

$$4x + y = 5$$

At $(-1,1)$, the slope is $y'(-1,1) = -4(-1) = 4$, so the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 4\left[x - (-1)\right]$$

$$y - 1 = 4x + 4$$

$$y = 4x + 5$$

$$-4x + y = 5$$

So the tangent lines at $(1,1)$ are

$-2x + y = -1$, which is $a = \langle -2,1 \rangle$ in standard vector form

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$4x + y = 5$, which is $b = \langle 4,1 \rangle$ in standard vector form

And the tangent lines at $(-1,1)$ are

$2x + y = -1$, which is $a = \langle 2,1 \rangle$ in standard vector form

$-4x + y = 5$, which is $b = \langle -4,1 \rangle$ in standard vector form

Now that we've found our tangent lines and converted them to standard vector form, we can plug them into the equation for the angle between vectors.

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where $a \cdot b$ is the dot product of the vectors, and $|a|$ and $|b|$ are their lengths. We'll use the origin $(0,0)$ as the point (x_1, y_1) and then find the length of each vector using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The dot product of the vectors at $(1,1)$ is

$$a \cdot b = (-2)(4) + (1)(1)$$

$$a \cdot b = -8 + 1$$

$$a \cdot b = -7$$

The length of $a = \langle -2,1 \rangle$ is

$$D_a = \sqrt{(-2-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{4+1}$$

$$D_a = |a| = \sqrt{5}$$

The length of $b = \langle 4,1 \rangle$ is

$$D_b = \sqrt{(4-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{16+1}$$

$$D_b = |b| = \sqrt{17}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(1,1)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-7}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{-7}{\sqrt{85}}$$

$$\theta = \arccos\frac{-7}{\sqrt{85}}$$

$$\theta \approx 139.4^\circ$$

Since we found an obtuse angle instead of an acute angle, we just need to subtract the angle we found from 180° in order to find the associated acute angle.

$$\theta \approx 180^\circ - 139.4^\circ$$

$$\theta \approx 40.6^\circ$$

Now we'll work on the acute angle between the curves at $(-1,1)$. The dot product of the vectors at $(-1,1)$ is

$$a \cdot b = (2)(-4) + (1)(1)$$

$$a \cdot b = -8 + 1$$

$$a \cdot b = -7$$

The length of $a = \langle 2,1 \rangle$ is

$$D_a = \sqrt{(2-0)^2 + (1-0)^2}$$

$$D_a = \sqrt{4+1}$$

$$D_a = |a| = \sqrt{5}$$

The length of $b = \langle -4,1 \rangle$ is

$$D_b = \sqrt{(-4-0)^2 + (1-0)^2}$$

$$D_b = \sqrt{16+1}$$

$$D_b = |b| = \sqrt{17}$$

Plugging everything into the formula for the angle between the vectors, we can say that the acute angle between the curves at $(-1,1)$ is

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

$$\cos\theta = \frac{-7}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{-7}{\sqrt{85}}$$

$$\theta = \arccos\frac{-7}{\sqrt{85}}$$

$$\theta \approx 139.4^\circ$$

Again, we'll subtract the obtuse angle from 180° in order to find the associated acute angle.

$$\theta \approx 180^\circ - 139.4^\circ$$

$$\theta \approx 40.6^\circ$$

In summary,

the acute angle between the curves at $(1,1)$ is $\theta \approx 40.6^\circ$

the acute angle between the curves at $(-1,1)$ is $\theta \approx 40.6^\circ$

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Topic: Direction cosines and direction angles

Question: Find the direction cosines of $a = \langle 5,6 \rangle$.

Answer choices:

A $\cos\alpha = \frac{5}{\sqrt{61}}$ and $\cos\beta = \frac{6}{\sqrt{61}}$

B $\cos\alpha = \frac{5}{6}$ and $\cos\beta = \frac{6}{5}$

C $\cos\alpha = \frac{5}{\sqrt{11}}$ and $\cos\beta = \frac{6}{\sqrt{11}}$

D $\cos\alpha = \frac{5}{\sqrt{121}}$ and $\cos\beta = \frac{6}{\sqrt{121}}$

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Solution: A

We'll start by finding the magnitude of a.

$$|a| = \sqrt{a_1^2 + a_2^2}$$

$$|a| = \sqrt{5^2 + 6^2}$$

$$|a| = \sqrt{25 + 36}$$

$$|a| = \sqrt{61}$$

Then the direction cosines of a are

$$\cos\alpha = \frac{a_1}{|a|} = \frac{5}{\sqrt{61}}$$

$$\cos\beta = \frac{a_2}{|a|} = \frac{6}{\sqrt{61}}$$

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Topic: Direction cosines and direction angles

Question: Find the direction angles, in degrees, of $a = \langle -1, -2, 3 \rangle$.

Answer choices:

A $\alpha \approx 105.5^\circ$, $\beta \approx 122.3^\circ$, and $\gamma \approx 143.3^\circ$

B $\alpha \approx 74.5^\circ$, $\beta \approx 57.7^\circ$, and $\gamma \approx 36.7^\circ$

C $\alpha \approx 105.5^\circ$, $\beta \approx 122.3^\circ$, and $\gamma \approx 36.7^\circ$

D $\alpha \approx 74.5^\circ$, $\beta \approx 57.7^\circ$, and $\gamma \approx 143.3^\circ$

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Solution: C

We find the magnitude of the vector a using the distance formula.

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|a| = \sqrt{(-1)^2 + (-2)^2 + 3^2}$$

$$|a| = \sqrt{1 + 4 + 9}$$

$$|a| = \sqrt{14}$$

Plugging the vector's components and magnitude into the direction cosine formulas, we get

$$\cos\alpha = \frac{-1}{\sqrt{14}} \qquad \cos\beta = \frac{-2}{\sqrt{14}} \qquad \cos\gamma = \frac{3}{\sqrt{14}}$$

Now that we have the direction cosines, we can apply the inverse cosine to both sides of each equation to find the direction angles.

$$\alpha = \arccos\frac{-1}{\sqrt{14}} \qquad \beta = \arccos\frac{-2}{\sqrt{14}} \qquad \gamma = \arccos\frac{3}{\sqrt{14}}$$

$$\alpha \approx 105.5^\circ \qquad \beta \approx 122.3^\circ \qquad \gamma \approx 36.7^\circ$$

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Topic: Direction cosines and direction angles

Question: Find the direction angles, in degrees, of the vector $m = 7\mathbf{i} - \mathbf{j} - 9\mathbf{k}$.

Answer choices:

A $\alpha \approx 127.7^\circ$, $\beta \approx 85.0^\circ$, and $\gamma \approx = 38.2^\circ$

B $\alpha \approx 52.3^\circ$, $\beta \approx 95.0^\circ$, and $\gamma \approx 141.8^\circ$

C $\alpha \approx 127.7^\circ$, $\beta \approx 95.0^\circ$, and $\gamma \approx 141.8^\circ$

D $\alpha \approx 52.3^\circ$, $\beta \approx 85.0^\circ$, and $\gamma \approx 38.2^\circ$

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Solution: B

We find the magnitude of the vector m using the distance formula.

$$|m| = \sqrt{m_1^2 + m_2^2 + m_3^2}$$

$$|m| = \sqrt{7^2 + (-1)^2 + (-9)^2}$$

$$|m| = \sqrt{49 + 1 + 81}$$

$$|m| = \sqrt{131}$$

Plugging the vector's components and magnitude into the direction cosine formulas, we get

$$\cos\alpha = \frac{7}{\sqrt{131}} \qquad \cos\beta = \frac{-1}{\sqrt{131}} \qquad \cos\gamma = \frac{-9}{\sqrt{131}}$$

Now that we have the direction cosines, we can apply the inverse cosine to both sides of each equation to find the direction angles.

$$\alpha = \arccos\frac{7}{\sqrt{131}} \qquad \beta = \arccos\frac{-1}{\sqrt{131}} \qquad \gamma = \arccos\frac{-9}{\sqrt{131}}$$

$$\alpha \approx 52.3^\circ \qquad \beta \approx 95.0^\circ \qquad \gamma \approx 141.8^\circ$$

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Topic: Scalar equation of a line

Question: Find the scalar equations of the line given by the point and the vector.

$$P(-1,3)$$

$$\langle -1, -4 \rangle$$

Answer choices:

A $x = -1 + t$ $y = 3 + 4t$

B $x = -1 + t$ $y = 4 + 3t$

C $x = -1 - t$ $y = 3 - 4t$

D $x = -1 - t$ $y = -4 + 3t$

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Solution: C

To find the scalar equation of a line, we'll use

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

where $P_0(x_0, y_0, z_0)$ is the given point and $v = \langle a, b, c \rangle$ or $v = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is the given vector. These formulas are based on three-dimensional vectors but we can use the same formulas for two dimensional vectors just by ignoring the equation for z.

If we plug the values we've been given into the formulas for x and y, we get

$$x = -1 + (-1)t$$

$$x = -1 - t$$

and

$$y = 3 + (-4)t$$

$$y = 3 - 4t$$

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Topic: Scalar equation of a line

Question: Find the scalar equations of the line given by the point and the vector.

$$P(-4,0,5)$$

$$\langle 7,2,-4 \rangle$$

Answer choices:

A	$x = -4 + 7t$	$y = 2t$	$z = 5 - 4t$
B	$x = -7 - 4t$	$y = -2$	$z = 4 + 5t$
C	$x = 7 - 4t$	$y = 2$	$z = -4 + 5t$
D	$x = -4 - 7t$	$y = -2t$	$z = 5 + 4t$

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Solution: A

To find the scalar equation of a line, we'll use

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

where $P_0(x_0, y_0, z_0)$ is the given point and $v = \langle a, b, c \rangle$ or $v = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is the given vector. These formulas are based on three-dimensional vectors but we can use the same formulas for two dimensional vectors just by ignoring the equation for z.

If we plug the values we've been given into the formulas for x and y, we get

$$x = -4 + 7t$$

and

$$y = 0 + 2t$$

$$y = 2t$$

and

$$z = 5 + (-4)t$$

$$z = 5 - 4t$$

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Topic: Scalar equation of a line

Question: Find the scalar equations of the line given by the point and the vector.

$$P(11, -5, -9)$$

$$\langle -6, -3, 17 \rangle$$

Answer choices:

A $\quad x = 11 + 6t \qquad y = -5 + 3t \qquad z = -9 - 17t$

B $\quad x = -6 - 11t \qquad y = -3 + 5t \qquad z = 17 + 9t$

C $\quad x = -6 + 11t \qquad y = -3 - 5t \qquad z = 17 - 9t$

D $\quad x = 11 - 6t \qquad y = -5 - 3t \qquad z = -9 + 17t$

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Solution: D

To find the scalar equation of a line, we'll use

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

where $P_0(x_0, y_0, z_0)$ is the given point and $v = \langle a, b, c \rangle$ or $v = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ is the given vector. These formulas are based on three-dimensional vectors but we can use the same formulas for two dimensional vectors just by ignoring the equation for z.

If we plug the values we've been given into the formulas for x and y, we get

$$x = 11 + (-6)t$$

$$x = 11 - 6t$$

and

$$y = -5 + (-3)t$$

$$y = -5 - 3t$$

and

$$z = -9 + 17t$$

Topic: Scalar equation of a plane

Question: Find the scalar equation of the plane given by the point and the normal vector.

$$P(1,0,-1)$$

$$\langle 2,1,-2\rangle$$

Answer choices:

A $\quad -2x-y+2z+4=0$

B $\quad 2x+y-2z+4=0$

C $\quad -2x-y+2z-4=0$

D $\quad 2x+y-2z-4=0$

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Solution: D

We'll plug the values from the point $P(1,0,-1)$ into the formula for the tangent plane for $P_0(x_0, y_0, z_0)$.

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$a(x-1)+b(y-0)+c\left[z-(-1)\right]=0$$

$$a(x-1)+by+c(z+1)=0$$

Since $\langle 2,1,-2\rangle$ is the normal vector to the plane, we can plug those values into the equation for $n=\langle a,b,c\rangle$.

$$2(x-1)+1y+(-2)(z+1)=0$$

$$2x-2+y-2z-2=0$$

$$2x+y-2z-4=0$$

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Topic: Scalar equation of a plane

Question: Find the scalar equation of the plane given by the point and the normal vector.

$$P(5,-4,3)$$

$$\langle -3,-3,-3 \rangle$$

Answer choices:

A $-3x-3y+3z+12=0$

B $-3x-3y-3z+12=0$

C $-3x-3y-3z-12=0$

D $-3x-3y+3z-12=0$

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Solution: B

We'll plug the values from the point $P(5,-4,3)$ into the formula for the tangent plane for $P_0(x_0, y_0, z_0)$.

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$a(x-5)+b\left[y-(-4)\right]+c(z-3)=0$$

$$a(x-5)+b(y+4)+c(z-3)=0$$

Since $\langle -3,-3,-3\rangle$ is the normal vector to the plane, we can plug those values into the equation for $n=\langle a,b,c\rangle$.

$$-3(x-5)+(-3)(y+4)+(-3)(z-3)=0$$

$$-3(x-5)-3(y+4)-3(z-3)=0$$

$$-3x+15-3y-12-3z+9=0$$

$$-3x-3y-3z+12=0$$

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Topic: Scalar equation of a plane

Question: Find the scalar equation of the plane given by the point and the normal vector.

$$P(-6,2,-5)$$

$$8\mathbf{i}-5\mathbf{j}+\mathbf{k}$$

Answer choices:

A $8x-5y+z+63=0$

B $8x-5y+z-63=0$

C $-8x+5y-z+63=0$

D $-8x+5y-z-63=0$

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Solution: A

We'll plug the values from the point $P(-6,2,-5)$ into the formula for the tangent plane for $P_0(x_0, y_0, z_0)$.

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$a\left[x-(-6)\right]+b(y-2)+c\left[z-(-5)\right]=0$$

$$a(x+6)+b(y-2)+c(z+5)=0$$

Since $8\mathbf{i}-5\mathbf{j}+\mathbf{k}$ is the normal vector to the plane, we can plug those values into the equation for $n=\langle a,b,c\rangle$.

$$8(x+6)+(-5)(y-2)+1(z+5)=0$$

$$8(x+6)-5(y-2)+z+5=0$$

$$8x+48-5y+10+z+5=0$$

$$8x-5y+z+63=0$$

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Topic: Scalar and vector projections

Question: Find the scalar projection.

b onto a

$$a = \langle 1, -2, 3 \rangle$$

$$b = \langle 4, 0, -1 \rangle$$

Answer choices:

A $\text{comp}_a b = \frac{1}{\sqrt{17}}$

B $\text{comp}_a b = \frac{7}{\sqrt{14}}$

C $\text{comp}_a b = \frac{1}{\sqrt{14}}$

D $\text{comp}_a b = \frac{7}{\sqrt{17}}$

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Solution: C

In order to find the scalar projection of b onto a, we'll first find the dot product of the vectors we've been given. Since $a = \langle 1, -2, 3 \rangle$ and $b = \langle 4, 0, -1 \rangle$, we get

$$a \cdot b = (1)(4) + (-2)(0) + (3)(-1)$$

$$a \cdot b = 4 + 0 - 3$$

$$a \cdot b = 1$$

Since we're looking for the projection of b onto a, we'll find the magnitude of a, using the distance formula and the origin $(0,0,0)$ as (x_1, y_1, z_1).

$$|a| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$|a| = \sqrt{(1 - 0)^2 + (-2 - 0)^2 + (3 - 0)^2}$$

$$|a| = \sqrt{1 + 4 + 9}$$

$$|a| = \sqrt{14}$$

Now we'll plug these pieces into the formula for the scalar projection of b onto a.

$$\text{comp}_a b = \frac{a \cdot b}{|a|}$$

$$\text{comp}_a b = \frac{1}{\sqrt{14}}$$

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Topic: Scalar and vector projections

Question: Find the scalar and vector projections.

b onto a

$$a = \langle 4,3,5 \rangle$$

$$b = \langle -5,5,6 \rangle$$

Answer choices:

A $\quad \text{comp}_a b = \dfrac{5}{\sqrt{2}} \qquad \text{proj}_a b = 2\mathbf{i} + \dfrac{3}{2}\mathbf{j} + \dfrac{5}{2}\mathbf{k}$

B $\quad \text{comp}_a b = \dfrac{5}{\sqrt{2}} \qquad \text{proj}_a b = \dfrac{2}{25}\mathbf{i} + \dfrac{3}{50}\mathbf{j} + \dfrac{1}{10}\mathbf{k}$

C $\quad \text{comp}_a b = \dfrac{25}{\sqrt{86}} \qquad \text{proj}_a b = 2\mathbf{i} + \dfrac{3}{2}\mathbf{j} + \dfrac{5}{2}\mathbf{k}$

D $\quad \text{comp}_a b = \dfrac{25}{\sqrt{86}} \qquad \text{proj}_a b = \dfrac{2}{25}\mathbf{i} + \dfrac{3}{50}\mathbf{j} + \dfrac{1}{10}\mathbf{k}$

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Solution: A

In order to find the scalar projection of b onto a, we'll first find the dot product of the vectors we've been given. Since $a = \langle 4,3,5 \rangle$ and $b = \langle -5,5,6 \rangle$, we get

$$a \cdot b = (4)(-5) + (3)(5) + (5)(6)$$

$$a \cdot b = -20 + 15 + 30$$

$$a \cdot b = 25$$

Since we're looking for the projection of b onto a, we'll find the magnitude of a, using the distance formula and the origin $(0,0,0)$ as (x_1, y_1, z_1).

$$|a| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$|a| = \sqrt{(4 - 0)^2 + (3 - 0)^2 + (5 - 0)^2}$$

$$|a| = \sqrt{16 + 9 + 25}$$

$$|a| = \sqrt{50}$$

$$|a| = \sqrt{25 \cdot 2}$$

$$|a| = 5\sqrt{2}$$

Now we'll plug these pieces into the formula for the scalar projection of b onto a.

$$\text{comp}_a b = \frac{a \cdot b}{|a|}$$

$$\text{comp}_a b = \frac{25}{5\sqrt{2}}$$

$$\text{comp}_a b = \frac{5}{\sqrt{2}}$$

Then to find the vector projection, we'll plug everything we already have into the formula for the vector projection of b onto a. Since $a = \langle 4,3,5 \rangle$, we get

$$\text{proj}_a b = \left(\frac{a \cdot b}{|a|} \right) \frac{a}{|a|}$$

$$\text{proj}_a b = \left(\frac{25}{5\sqrt{2}} \right) \frac{4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}}{5\sqrt{2}}$$

$$\text{proj}_a b = \frac{25(4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k})}{25(2)}$$

$$\text{proj}_a b = \frac{4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}}{2}$$

$$\text{proj}_a b = \frac{4}{2}\mathbf{i} + \frac{3}{2}\mathbf{j} + \frac{5}{2}\mathbf{k}$$

$$\text{proj}_a b = 2\mathbf{i} + \frac{3}{2}\mathbf{j} + \frac{5}{2}\mathbf{k}$$

Topic: Scalar and vector projections

Question: Find the scalar and vector projections.

b onto a

$$a = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$$

$$b = 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$$

Answer choices:

A $\quad \text{comp}_a b = -\frac{1}{\sqrt{29}} \qquad \text{proj}_a b = \frac{4}{29}\mathbf{i} + \frac{3}{29}\mathbf{j} - \frac{1}{29}\mathbf{k}$

B $\quad \text{comp}_a b = \frac{1}{\sqrt{8}} \qquad \text{proj}_a b = \frac{1}{2}\mathbf{i} + \frac{3}{8}\mathbf{j} - \frac{1}{8}\mathbf{k}$

C $\quad \text{comp}_a b = \frac{1}{3} \qquad \text{proj}_a b = \frac{4}{9}\mathbf{i} + \frac{1}{3}\mathbf{j} - \frac{1}{9}\mathbf{k}$

D $\quad \text{comp}_a b = -\frac{1}{\sqrt{26}} \qquad \text{proj}_a b = -\frac{2}{13}\mathbf{i} - \frac{3}{26}\mathbf{j} + \frac{1}{26}\mathbf{k}$

Solution: D

In order to find the scalar projection of b onto a, we'll first find the dot product of the vectors we've been given. Since $a = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$ and $b = 2\mathbf{i} - 4\mathbf{j} - 3\mathbf{k}$, we get

$$a \cdot b = (4)(2) + (3)(-4) + (-1)(-3)$$

$$a \cdot b = 8 - 12 + 3$$

$$a \cdot b = -1$$

Since we're looking for the projection of b onto a, we'll find the magnitude of a, using the distance formula and the origin $(0,0,0)$ as (x_1, y_1, z_1).

$$|a| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$|a| = \sqrt{(4 - 0)^2 + (3 - 0)^2 + (-1 - 0)^2}$$

$$|a| = \sqrt{16 + 9 + 1}$$

$$|a| = \sqrt{26}$$

Now we'll plug these pieces into the formula for the scalar projection of b onto a.

$$\text{comp}_a b = \frac{a \cdot b}{|a|}$$

$$\text{comp}_a b = -\frac{1}{\sqrt{26}}$$

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Then to find the vector projection, we'll plug everything we already have into the formula for the vector projection of b onto a. Since $a = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, we get

$$\text{proj}_a b = \left(\frac{a \cdot b}{|a|}\right)\frac{a}{|a|}$$

$$\text{proj}_a b = \left(-\frac{1}{\sqrt{26}}\right)\frac{4\mathbf{i} + 3\mathbf{j} - \mathbf{k}}{\sqrt{26}}$$

$$\text{proj}_a b = \frac{-4\mathbf{i} - 3\mathbf{j} + \mathbf{k}}{26}$$

$$\text{proj}_a b = -\frac{4}{26}\mathbf{i} - \frac{3}{26}\mathbf{j} + \frac{1}{26}\mathbf{k}$$

$$\text{proj}_a b = -\frac{2}{13}\mathbf{i} - \frac{3}{26}\mathbf{j} + \frac{1}{26}\mathbf{k}$$

Topic: Cross product of two vectors

Question: Find the cross product.

$$a\langle 1,-1,1\rangle$$

$$b\langle -2,1,2\rangle$$

Answer choices:

A $c\langle 3,-4,1\rangle$

B $c\langle -3,-4,-1\rangle$

C $c\langle -3,4,-1\rangle$

D $c\langle 3,4,1\rangle$

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Solution: B

The cross product $a \times b$ of two vectors $a\langle a_1, a_2, a_3\rangle$ and $b\langle b_1, b_2, b_3\rangle$ is given by

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

If we plug our vectors $a\langle 1, -1, 1\rangle$ and $b\langle -2, 1, 2\rangle$ into this last formula, we get

$$a \times b = \mathbf{i}\left[(-1)(2) - (1)(1)\right] - \mathbf{j}\left[(1)(2) - (1)(-2)\right] + \mathbf{k}\left[(1)(1) - (-1)(-2)\right]$$

$$a \times b = \mathbf{i}(-2 - 1) - \mathbf{j}(2 + 2) + \mathbf{k}(1 - 2)$$

$$a \times b = -3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$$

We'll convert this into standard vector form $c\langle c_1, c_2, c_3\rangle$ to get the cross product of the vectors.

$$c\langle -3, -4, -1\rangle$$

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Topic: Cross product of two vectors

Question: Find the cross product.

$$a\langle 4,2,0\rangle$$

$$b\langle -1,-3,1\rangle$$

Answer choices:

A $c\langle -2,4,10\rangle$

B $c\langle -2,-4,10\rangle$

C $c\langle 2,4,-10\rangle$

D $c\langle 2,-4,-10\rangle$

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Solution: D

The cross product $a \times b$ of two vectors $a\langle a_1, a_2, a_3\rangle$ and $b\langle b_1, b_2, b_3\rangle$ is given by

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

If we plug our vectors $a\langle 4,2,0\rangle$ and $b\langle -1,-3,1\rangle$ into this last formula, we get

$$a \times b = \mathbf{i}\left[(2)(1) - (0)(-3)\right] - \mathbf{j}\left[(4)(1) - (0)(-1)\right] + \mathbf{k}\left[(4)(-3) - (2)(-1)\right]$$

$$a \times b = \mathbf{i}(2 - 0) - \mathbf{j}(4 - 0) + \mathbf{k}(-12 + 2)$$

$$a \times b = 2\mathbf{i} - 4\mathbf{j} - 10\mathbf{k}$$

We'll convert this into standard vector form $c\langle c_1, c_2, c_3\rangle$ to get the cross product of the vectors.

$$c\langle 2, -4, -10\rangle$$

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Topic: Cross product of two vectors

Question: Find the cross product.

$$a\langle 6,7,-5\rangle$$

$$b\langle 8,7,-11\rangle$$

Answer choices:

A $c\langle -42,-22,-14\rangle$

B $c\langle -112,106,98\rangle$

C $c\langle -21,13,-7\rangle$

D $c\langle -112,-106,98\rangle$

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Solution: C

The cross product $a \times b$ of two vectors $a\langle a_1, a_2, a_3\rangle$ and $b\langle b_1, b_2, b_3\rangle$ is given by

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

If we plug our vectors $a\langle 6,7,-5\rangle$ and $b\langle 8,7,-11\rangle$ into this last formula, we get

$$a \times b = \mathbf{i}\left[(7)(-11) - (-5)(7)\right] - \mathbf{j}\left[(6)(-11) - (-5)(8)\right] + \mathbf{k}\left[(6)(7) - (7)(8)\right]$$

$$a \times b = \mathbf{i}(-77 + 35) - \mathbf{j}(-66 + 40) + \mathbf{k}(42 - 56)$$

$$a \times b = -42\mathbf{i} + 26\mathbf{j} - 14\mathbf{k}$$

$$a \times b = 2(-21\mathbf{i} + 13\mathbf{j} - 7\mathbf{k})$$

$$a \times b = -21\mathbf{i} + 13\mathbf{j} - 7\mathbf{k}$$

We'll convert this into standard vector form $c\langle c_1, c_2, c_3\rangle$ to get the cross product of the vectors.

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$c\langle -21, 13, -7 \rangle$

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Topic: Vector orthogonal to the plane

Question: Find the vector orthogonal to the plane that includes the given vectors.

$$\overrightarrow{AB} = \langle 2,3,2 \rangle$$

$$\overrightarrow{AC} = \langle -2,1,2 \rangle$$

Answer choices:

A $\langle 1, -2,2 \rangle$

B $\langle 1,2,2 \rangle$

C $\langle 1, -2, -2 \rangle$

D $\langle 1,2, -2 \rangle$

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Solution: A

Given two vectors that lie in the plane, we can find the vector orthogonal to the plane by taking the cross product of the vectors $\overrightarrow{AB} = \langle AB_1, AB_2, AB_3 \rangle$ and $\overrightarrow{AC} = \langle AC_1, AC_2, AC_3 \rangle$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ AB_1 & AB_2 & AB_3 \\ AC_1 & AC_2 & AC_3 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i} \begin{vmatrix} AB_2 & AB_3 \\ AC_2 & AC_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} AB_1 & AB_3 \\ AC_1 & AC_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} AB_1 & AB_2 \\ AC_1 & AC_2 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i} \left(AB_2AC_3 - AB_3AC_2 \right) - \mathbf{j} \left(AB_1AC_3 - AB_3AC_1 \right) + \mathbf{k} \left(AB_1AC_2 - AB_2AC_1 \right)$$

For the given vectors $\overrightarrow{AB} = \langle 2,3,2 \rangle$ and $\overrightarrow{AC} = \langle -2,1,2 \rangle$, we get

$$\overrightarrow{AB} \times \overrightarrow{AC} = \left[(3)(2) - (2)(1) \right] \mathbf{i} - \left[(2)(2) - (2)(-2) \right] \mathbf{j} + \left[(2)(1) - (3)(-2) \right] \mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = (6 - 2)\mathbf{i} - (4 + 4)\mathbf{j} + (2 + 6)\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 4\mathbf{i} - 8\mathbf{j} + 8\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = 4(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

We'll convert this into standard vector form to get the vector orthogonal to the plane.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1, -2,2 \rangle$$

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Topic: Vector orthogonal to the plane

Question: Find the vector orthogonal to the plane that includes the given points.

$$A(1,-1,1)$$

$$B(-2,2,-2)$$

$$C(3,1,-2)$$

Answer choices:

A $\langle -1,5,-4\rangle$

B $\langle 1,-5,-4\rangle$

C $\langle 1,5,4\rangle$

D $\langle -1,-5,4\rangle$

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Solution: C

Since we've been given three points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$, we'll use the points to generate two vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AB} = (AB_1)\mathbf{i} + (AB_2)\mathbf{j} + (AB_3)\mathbf{k}$$

and

$$\overrightarrow{AC} = (c_1 - a_1)\mathbf{i} + (c_2 - a_2)\mathbf{j} + (c_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AC} = (AC_1)\mathbf{i} + (AC_2)\mathbf{j} + (AC_3)\mathbf{k}$$

Plugging in the points we've been given, we get

$$\overrightarrow{AB} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AB} = (-2 - 1)\mathbf{i} + \left[2 - (-1)\right]\mathbf{j} + (-2 - 1)\mathbf{k}$$

$$\overrightarrow{AB} = -3\mathbf{i} + 3\mathbf{j} - 3\mathbf{k}$$

and

$$\overrightarrow{AC} = (c_1 - a_1)\mathbf{i} + (c_2 - a_2)\mathbf{j} + (c_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AC} = (3 - 1)\mathbf{i} + \left[1 - (-1)\right]\mathbf{j} + (-2 - 1)\mathbf{k}$$

$$\overrightarrow{AC} = 2\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$$

If we convert these into standard vector form, we get

$$AB\langle AB_1, AB_2, AB_3\rangle = \langle -3, 3, -3\rangle$$

$$AC\langle AC_1, AC_2, AC_3\rangle = \langle 2,2,-3\rangle$$

Now that we've converted the points into vectors, we'll take the cross product of $\overrightarrow{AB} = \langle AB_1, AB_2, AB_3\rangle$ and $\overrightarrow{AC} = \langle AC_1, AC_2, AC_3\rangle$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ AB_1 & AB_2 & AB_3 \\ AC_1 & AC_2 & AC_3 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i} \begin{vmatrix} AB_2 & AB_3 \\ AC_2 & AC_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} AB_1 & AB_3 \\ AC_1 & AC_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} AB_1 & AB_2 \\ AC_1 & AC_2 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i}\left(AB_2AC_3 - AB_3AC_2\right) - \mathbf{j}\left(AB_1AC_3 - AB_3AC_1\right) + \mathbf{k}\left(AB_1AC_2 - AB_2AC_1\right)$$

For the vectors $\overrightarrow{AB} = \langle -3,3,-3\rangle$ and $\overrightarrow{AC} = \langle 2,2,-3\rangle$, we get

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i}\left[(3)(-3) - (-3)(2)\right] - \mathbf{j}\left[(-3)(-3) - (-3)(2)\right] + \mathbf{k}\left[(-3)(2) - (3)(2)\right]$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i}\left[-9 - (-6)\right] - \mathbf{j}\left[9 - (-6)\right] + \mathbf{k}\left[-6 - 6\right]$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -3\mathbf{i} - 15\mathbf{j} - 12\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -3\left(\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}\right)$$

We'll convert this into standard vector form to get the vector orthogonal to the plane.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle 1,5,4\rangle$$

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Topic: Vector orthogonal to the plane

Question: Find the vector orthogonal to the plane that includes the given points.

$$A(4,-5,-4)$$

$$B(7,-5,0)$$

$$C(11,6,8)$$

Answer choices:

A $\langle -44,8,33\rangle$

B $\langle -44,-8,33\rangle$

C $\langle 7,11,12\rangle$

D $\langle 7,11,-12\rangle$

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Solution: B

Since we've been given three points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$, we'll use the points to generate two vectors, $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AB} = (AB_1)\mathbf{i} + (AB_2)\mathbf{j} + (AB_3)\mathbf{k}$$

and

$$\overrightarrow{AC} = (c_1 - a_1)\mathbf{i} + (c_2 - a_2)\mathbf{j} + (c_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AC} = (AC_1)\mathbf{i} + (AC_2)\mathbf{j} + (AC_3)\mathbf{k}$$

Plugging in the points we've been given, we get

$$\overrightarrow{AB} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AB} = (7 - 4)\mathbf{i} + \left[-5 - (-5)\right]\mathbf{j} + \left[0 - (-4)\right]\mathbf{k}$$

$$\overrightarrow{AB} = 3\mathbf{i} + 0\mathbf{j} + 4\mathbf{k}$$

and

$$\overrightarrow{AC} = (c_1 - a_1)\mathbf{i} + (c_2 - a_2)\mathbf{j} + (c_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AC} = (11 - 4)\mathbf{i} + \left[6 - (-5)\right]\mathbf{j} + \left[8 - (-4)\right]\mathbf{k}$$

$$\overrightarrow{AC} = 7\mathbf{i} + 11\mathbf{j} + 12\mathbf{k}$$

If we convert these into standard vector form, we get

$$AB\langle AB_1, AB_2, AB_3\rangle = \langle 3,0,4\rangle$$

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$$AC\langle AC_1, AC_2, AC_3\rangle = \langle 7,11,12\rangle$$

Now that we've converted the points into vectors, we'll take the cross product of $\overrightarrow{AB} = \langle AB_1, AB_2, AB_3\rangle$ and $\overrightarrow{AC} = \langle AC_1, AC_2, AC_3\rangle$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ AB_1 & AB_2 & AB_3 \\ AC_1 & AC_2 & AC_3 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i} \begin{vmatrix} AB_2 & AB_3 \\ AC_2 & AC_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} AB_1 & AB_3 \\ AC_1 & AC_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} AB_1 & AB_2 \\ AC_1 & AC_2 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i}\left(AB_2AC_3 - AB_3AC_2\right) - \mathbf{j}\left(AB_1AC_3 - AB_3AC_1\right) + \mathbf{k}\left(AB_1AC_2 - AB_2AC_1\right)$$

For the vectors $\overrightarrow{AB} = \langle 3,0,4\rangle$ and $\overrightarrow{AC} = \langle 7,11,12\rangle$, we get

$$\overrightarrow{AB} \times \overrightarrow{AC} = \left[(0)(12) - (4)(11)\right]\mathbf{i} - \left[(3)(12) - (4)(7)\right]\mathbf{j} + \left[(3)(11) - (0)(7)\right]\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = (0 - 44)\mathbf{i} - (36 - 28)\mathbf{j} + (33 - 0)\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -44\mathbf{i} - 8\mathbf{j} + 33\mathbf{k}$$

We'll convert this into standard vector form to get the vector orthogonal to the plane.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle -44, -8, 33\rangle$$

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Topic: Volume of the parallelepiped from vectors

Question: Use the vectors to find the volume of the parallelepiped.

$$a\langle 1,-1,-1\rangle$$

$$b\langle -2,2,-3\rangle$$

$$c\langle 0,2,-1\rangle$$

Answer choices:

A $\quad 10$

B $\quad 4$

C $\quad 44$

D $\quad 1$

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Solution: A

The volume of a parallelepiped is given by the scalar trip product of three vectors that define its edges. To find the scalar triple product, we'll take the cross product of the vectors b and c, and then take the dot product of the result and the vector a. The cross product of $b\langle -2,2,-3\rangle$ and $c\langle 0,2,-1\rangle$ is

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$b \times c = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$b \times c = \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

$$b \times c = \mathbf{i}\left[(2)(-1) - (-3)(2)\right] - \mathbf{j}\left[(-2)(-1) - (-3)(0)\right] + \mathbf{k}\left[(-2)(2) - (2)(0)\right]$$

$$b \times c = \mathbf{i}(-2+6) - \mathbf{j}(2+0) + \mathbf{k}(-4-0)$$

$$b \times c = 4\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}$$

$$b \times c = \langle 4,-2,-4\rangle$$

Now we'll take the dot product of $a\langle 1,-1,-1\rangle$ and $b \times c = \langle 4,-2,-4\rangle$ to find the volume of the parallelepiped.

$$\left|a \cdot (b \times c)\right| = (1)(4) + (-1)(-2) + (-1)(-4)$$

$$\left|a \cdot (b \times c)\right| = 4 + 2 + 4$$

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$$\left| a \cdot (b \times c) \right| = 10$$

Topic: Volume of the parallelepiped from vectors

Question: Use the vectors to find the volume of the parallelepiped.

$$a\langle 4,5,-6\rangle$$

$$b\langle 6,-4,7\rangle$$

$$c\langle -4,-5,4\rangle$$

Answer choices:

A $\quad 60$

B $\quad 76$

C $\quad 92$

D $\quad 108$

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Solution: C

The volume of a parallelepiped is given by the scalar trip product of three vectors that define its edges. To find the scalar triple product, we'll take the cross product of the vectors b and c, and then take the dot product of the result and the vector a. The cross product of $b\langle 6,-4,7\rangle$ and $c\langle -4,-5,4\rangle$ is

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$b \times c = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$b \times c = \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

$$b \times c = \mathbf{i}\left[(-4)(4) - (7)(-5)\right] - \mathbf{j}\left[(6)(4) - (7)(-4)\right] + \mathbf{k}\left[(6)(-5) - (-4)(-4)\right]$$

$$b \times c = \mathbf{i}(-16 + 35) - \mathbf{j}(24 + 28) + \mathbf{k}(-30 - 16)$$

$$b \times c = 19\mathbf{i} - 52\mathbf{j} - 46\mathbf{k}$$

$$b \times c = \langle 19, -52, -46\rangle$$

Now we'll take the dot product of $a\langle 4,5,-6\rangle$ and $b \times c = \langle 19,-52,-46\rangle$ to find the volume of the parallelepiped.

$$\left| a \cdot (b \times c) \right| = (4)(19) + (5)(-52) + (-6)(-46)$$

$$\left| a \cdot (b \times c) \right| = 76 - 260 + 276$$

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$$\left| a \cdot (b \times c) \right| = 92$$

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Topic: Volume of the parallelepiped from vectors

Question: Use the vectors to find the volume of the parallelepiped.

$a\langle 6,2,3\rangle$

$b\langle -4,-3,-4\rangle$

$c\langle 6,3,-5\rangle$

Answer choices:

A 176

B 92

C 38

D 248

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Solution: B

The volume of a parallelepiped is given by the scalar trip product of three vectors that define its edges. To find the scalar triple product, we'll take the cross product of the vectors b and c, and then take the dot product of the result and the vector a. The cross product of $b\langle -4,-3,-4\rangle$ and $c\langle 6,3,-5\rangle$ is

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$b \times c = \mathbf{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$b \times c = \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

$$b \times c = \mathbf{i}\left[(-3)(-5) - (-4)(3)\right] - \mathbf{j}\left[(-4)(-5) - (-4)(6)\right] + \mathbf{k}\left[(-4)(3) - (-3)(6)\right]$$

$$b \times c = \mathbf{i}(15 + 12) - \mathbf{j}(20 + 24) + \mathbf{k}(-12 + 18)$$

$$b \times c = 27\mathbf{i} - 44\mathbf{j} + 6\mathbf{k}$$

$$b \times c = \langle 27, -44, 6\rangle$$

Now we'll take the dot product of $a\langle 6,2,3\rangle$ and $b \times c = \langle 27,-44,6\rangle$ to find the volume of the parallelepiped.

$$\left|a \cdot (b \times c)\right| = (6)(27) + (2)(-44) + (3)(6)$$

$$\left|a \cdot (b \times c)\right| = 162 - 88 + 18$$

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$$\left| a \cdot (b \times c) \right| = 92$$

Topic: Volume of the parallelepiped from adjacent edges

Question: Find the volume of the parallelepiped.

$\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are adjacent

$P(1,1,-1)$

$Q(2,0,-2)$

$R(1,2,-3)$

$S(-1,5,-2)$

Answer choices:

A 11

B 1

C 15

D 27

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Solution: B

First we'll find the vectors PQ, PR and PS. To find a vector from two points, remember that we just subtract the initial point from the terminal point. So, for example, to find PQ, we'll subtract P from Q.

$$\overrightarrow{PQ} = \langle 2-1, 0-1, -2-(-1)\rangle$$

$$\overrightarrow{PQ} = \langle 1, -1, -1\rangle$$

and

$$\overrightarrow{PR} = \langle 1-1, 2-1, -3-(-1)\rangle$$

$$\overrightarrow{PR} = \langle 0, 1, -2\rangle$$

and

$$\overrightarrow{PS} = \langle -1-1, 5-1, -2-(-1)\rangle$$

$$\overrightarrow{PS} = \langle -2, 4, -1\rangle$$

Next we'll find the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i}\begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left(PQ_2PR_3 - PQ_3PR_2\right)\mathbf{i} - \left(PQ_1PR_3 - PQ_3PR_1\right)\mathbf{j} + \left(PQ_1PR_2 - PQ_2PR_1\right)\mathbf{k}$$

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Since $\overrightarrow{PQ} = \langle 1, -1, -1 \rangle$ and $\overrightarrow{PR} = \langle 0,1, -2 \rangle$, we get

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-1)(-2) - (-1)(1)\right]\mathbf{i} - \left[(1)(-2) - (-1)(0)\right]\mathbf{j} + \left[(1)(1) - (-1)(0)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (2 + 1)\mathbf{i} - (-2 + 0)\mathbf{j} + (1 + 0)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 3,2,1 \rangle$$

Finally, we'll take the dot product of $\overrightarrow{PS} = \langle -2,4, -1 \rangle$ and the cross product we just found, which will give us the volume of the parallelepiped.

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|(-2)(3) + (4)(2) + (-1)(1)\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|-6 + 8 - 1\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|1\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = 1$$

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Topic: Volume of the parallelepiped from adjacent edges

Question: Find the volume of the parallelepiped.

$\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are adjacent

$P(2,3,0)$

$Q(1,-3,4)$

$R(-2,6,-2)$

$S(7,5,-2)$

Answer choices:

A $\quad 13$

B $\quad 23$

C $\quad 90$

D $\quad 18$

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Solution: D

First we'll find the vectors PQ, PR and PS. To find a vector from two points, remember that we just subtract the initial point from the terminal point. So, for example, to find PQ, we'll subtract P from Q.

$$\overrightarrow{PQ} = \langle 1-2, -3-3, 4-0 \rangle$$

$$\overrightarrow{PQ} = \langle -1, -6, 4 \rangle$$

and

$$\overrightarrow{PR} = \langle -2-2, 6-3, -2-0 \rangle$$

$$\overrightarrow{PR} = \langle -4, 3, -2 \rangle$$

and

$$\overrightarrow{PS} = \langle 7-2, 5-3, -2-0 \rangle$$

$$\overrightarrow{PS} = \langle 5, 2, -2 \rangle$$

Next we'll find the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left(PQ_2PR_3 - PQ_3PR_2\right)\mathbf{i} - \left(PQ_1PR_3 - PQ_3PR_1\right)\mathbf{j} + \left(PQ_1PR_2 - PQ_2PR_1\right)\mathbf{k}$$

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Since $\overrightarrow{PQ} = \langle -1, -6, 4\rangle$ and $\overrightarrow{PR} = \langle -4, 3, -2\rangle$, we get

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-6)(-2) - (4)(3)\right]\mathbf{i} - \left[(-1)(-2) - (4)(-4)\right]\mathbf{j} + \left[(-1)(3) - (-6)(-4)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (12 - 12)\mathbf{i} - (2 + 16)\mathbf{j} + (-3 - 24)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -18\mathbf{j} - 27\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 0, -18, -27\rangle$$

Finally, we'll take the dot product of $\overrightarrow{PS} = \langle 5, 2, -2\rangle$ and the cross product we just found, which will give us the volume of the parallelepiped.

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|(5)(0) + (2)(-18) + (-2)(-27)\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|0 - 36 + 54\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|18\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = 18$$

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Topic: Volume of the parallelepiped from adjacent edges

Question: Find the volume of the parallelepiped.

$\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$ are adjacent

$P(-2,-4,5)$

$Q(8,-3,7)$

$R(11,3,8)$

$S(3,7,6)$

Answer choices:

A $\quad 143$

B $\quad -29$

C $\quad 42$

D $\quad 123$

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Solution: C

First we'll find the vectors PQ, PR and PS. To find a vector from two points, remember that we just subtract the initial point from the terminal point. So, for example, to find PQ, we'll subtract P from Q.

$$\overrightarrow{PQ} = \langle 8-(-2), -3-(-4), 7-5 \rangle$$

$$\overrightarrow{PQ} = \langle 10,1,2 \rangle$$

and

$$\overrightarrow{PR} = \langle 11-(-2), 3-(-4), 8-5 \rangle$$

$$\overrightarrow{PR} = \langle 13,7,3 \rangle$$

and

$$\overrightarrow{PS} = \langle 3-(-2), 7-(-4), 6-5 \rangle$$

$$\overrightarrow{PS} = \langle 5,11,1 \rangle$$

Next we'll find the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left(PQ_2PR_3 - PQ_3PR_2\right)\mathbf{i} - \left(PQ_1PR_3 - PQ_3PR_1\right)\mathbf{j} + \left(PQ_1PR_2 - PQ_2PR_1\right)\mathbf{k}$$

Since $\overrightarrow{PQ} = \langle 10,1,2 \rangle$ and $\overrightarrow{PR} = \langle 13,7,3 \rangle$, we get

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(1)(3) - (2)(7)\right]\mathbf{i} - \left[(10)(3) - (2)(13)\right]\mathbf{j} + \left[(10)(7) - (1)(13)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (3 - 14)\mathbf{i} - (30 - 26)\mathbf{j} + (70 - 13)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -11\mathbf{i} - 4\mathbf{j} + 57\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -11, -4, 57 \rangle$$

Finally, we'll take the dot product of $\overrightarrow{PS} = \langle 5,11,1 \rangle$ and the cross product we just found, which will give us the volume of the parallelepiped.

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|(5)(-11) + (11)(-4) + (1)(57)\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|-55 - 44 + 57\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = \left|-42\right|$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = 42$$

Topic: Scalar triple product to prove vectors are coplanar

Question: Which of the following statements is true?

Answer choices:

A In order for three vectors to be coplanar, their scalar triple product must be greater than 0.

B In order for three vectors to be coplanar, their scalar triple product must be less than 0.

C In order for three vectors to be coplanar, their scalar triple product must be 0.

D In order for three vectors to be coplanar, their scalar triple product cannot be 0.

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Solution: C

The scalar triple product $\left|a \cdot (b \times c)\right|$ of three vectors $a\langle a_1, a_2, a_3\rangle$, $b\langle b_1, b_2, b_3\rangle$ and $c\langle c_1, c_2, c_3\rangle$ will be 0 when the vectors are coplanar.

This means that answer choices A, B, and D cannot be true, since in each case the scalar triple product $\left|a \cdot (b \times c)\right|$ is not equal to 0.

The answer is C because the scalar triple product $\left|a \cdot (b \times c)\right|$ of three vectors must be 0 in order for the vectors to be coplanar.

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Topic: Scalar triple product to prove vectors are coplanar

Question: Say whether or not the vectors are coplanar.

$$a\langle 1,2,-1\rangle$$

$$b\langle 0,4,-1\rangle$$

$$c\langle 1,1,1\rangle$$

Answer choices:

A Yes, the vectors are coplanar because the scalar triple product isn't 0.

B Yes, the vectors are coplanar because the scalar triple product is 0.

C No, the vectors aren't coplanar because the scalar triple product is 0.

D No, the vectors aren't coplanar because the scalar triple product isn't 0.

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Solution: D

If the scalar triple product of the vectors is 0, it means they're coplanar, which means that they lie in the same plane. To find the value of the scalar triple product, we'll start by taking the cross product $b \times c$ using $b\langle 0,4,-1\rangle$ and $c\langle 1,1,1\rangle$.

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$b \times c = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$b \times c = \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

$$b \times c = \mathbf{i}\left[(4)(1) - (-1)(1)\right] - \mathbf{j}\left[(0)(1) - (-1)(1)\right] + \mathbf{k}\left[(0)(1) - (4)(1)\right]$$

$$b \times c = \mathbf{i}(4+1) - \mathbf{j}(0+1) + \mathbf{k}(0-4)$$

$$b \times c = 5\mathbf{i} - \mathbf{j} - 4\mathbf{k}$$

$$b \times c = \langle 5,-1,-4\rangle$$

Then we'll take the dot product of $a\langle 1,2,-1\rangle$ and $b \times c = \langle 5,-1,-4\rangle$.

$$\left|a \cdot (b \times c)\right| = (1)(5) + (2)(-1) + (-1)(-4)$$

$$\left|a \cdot (b \times c)\right| = 5 - 2 + 4$$

$$\left|a \cdot (b \times c)\right| = 7$$

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Since the scalar triple product isn't 0, it means the vectors aren't coplanar.

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Topic: Scalar triple product to prove vectors are coplanar

Question: Say whether or not the vectors are coplanar.

$a\langle 3, -3, 4\rangle$

$b\langle 2, -2, -2\rangle$

$c\langle 1, -1, 1\rangle$

Answer choices:

A Yes, the vectors are coplanar because the scalar triple product isn't 0.

B Yes, the vectors are coplanar because the scalar triple product is 0.

C No, the vectors aren't coplanar because the scalar triple product is 0.

D No, the vectors aren't coplanar because the scalar triple product isn't 0.

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Solution: B

If the scalar triple product of the vectors is 0, it means they're coplanar, which means that they lie in the same plane. To find the value of the scalar triple product, we'll start by taking the cross product $b \times c$ using $b\langle 2,-2,-2\rangle$ and $c\langle 1,-1,1\rangle$.

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$b \times c = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$b \times c = \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

$$b \times c = \mathbf{i}\left[(-2)(1) - (-2)(-1)\right] - \mathbf{j}\left[(2)(1) - (-2)(1)\right] + \mathbf{k}\left[(2)(-1) - (-2)(1)\right]$$

$$b \times c = \mathbf{i}(-2-2) - \mathbf{j}(2+2) + \mathbf{k}(-2+2)$$

$$b \times c = -4\mathbf{i} - 4\mathbf{j}$$

$$b \times c = \langle -4,-4,0\rangle$$

Then we'll take the dot product of $a\langle 3,-3,4\rangle$ and $b \times c = \langle -4,-4,0\rangle$.

$$\left|a \cdot (b \times c)\right| = (3)(-4) + (-3)(-4) + (4)(0)$$

$$\left|a \cdot (b \times c)\right| = -12 + 12 + 0$$

$$\left|a \cdot (b \times c)\right| = 0$$

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Since the scalar triple product is 0, it means the vectors are coplanar.

Topic: Domain of the vector function

Question: Find the domain of the vector function.

$$r(t) = \left\langle 6t + 1,\ 2t^2,\ \frac{1}{25 - t^2} \right\rangle$$

Answer choices:

A $-5 \le t \le 5$

B $t \neq \pm 5$

C $[\infty, 5]$

D $(-5, 5)$

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Solution: B

For the vector function

$$r(t) = \left\langle 6t + 1,\ 2t^2,\ \frac{1}{25 - t^2} \right\rangle$$

the parameter is t. We'll find the domain for each component of the vector function. The domain for the vector function $r(t)$ will include only the values of t that are included in the domain of all three components.

The domain of $6t + 1$ is the set of all real numbers.

The domain of $2t^2$ is the set of all real numbers.

For the rational function

$$\frac{1}{25 - t^2}$$

we know that the denominator cannot equal 0. So to find the limitations on the domain we can set the denominator equal to 0.

$$25 - t^2 = 0$$

$$t^2 = 25$$

$$t = \pm 5$$

The domain of the third component is $t \neq \pm 5$.

Finally we need to combine these domains. The first and second components have no restrictions on their domains so we can ignore them.

If we look at the third component, we can see that the domain will be

$$t \neq \pm 5$$

The domain of $r(t)$ is $t \neq \pm 5$.

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Topic: Domain of the vector function

Question: Find the domain of the vector function.

$$r(t) = \left\langle t - 2, \frac{1}{t^2 - 16}, \sqrt{t + 3} \right\rangle$$

Answer choices:

A $[-3,4) \cup (4,\infty)$

B $(-3,4) \cup (4,\infty)$

C $(-3,4)$

D $(-4,3)$

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Solution: A

For the vector function

$$r(t) = \left\langle t - 2, \frac{1}{t^2 - 16}, \sqrt{t + 3} \right\rangle$$

the parameter is t. We'll find the domain for each component of the vector function. The domain for the vector function $r(t)$ will include only the values of t that are included in the domain of all three components.

The domain of $t - 2$ is the set of all real numbers.

For the rational function

$$\frac{1}{t^2 - 16}$$

we know that the denominator cannot equal 0. So to find the limitations on the domain we can set the denominator equal to 0.

$$t^2 - 16 = 0$$

$$t^2 = 16$$

$$t = \pm 4$$

The domain of the second component is $t \neq \pm 4$.

For the radical function

$$\sqrt{t + 3}$$

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we know that the value underneath the square root cannot be negative. So we'll restrict that value to only positive values and 0.

$$t + 3 \geq 0$$

$$t \geq -3$$

Finally we need to combine these domains. The first component has no restriction on its domain so we can ignore it. If we look at the second component, we can see that the domain will be $t \neq \pm 4$.

But if we look at the third component, we can see that $t \geq -3$. Which means that we don't have to consider $t = -4$ being excluded from the domain, since it's outside $t \geq -3$ anyway. We only need to exclude $t = 4$ from $t \geq -3$.

The domain of $r(t)$ is $[-3,4) \cup (4,\infty)$.

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Topic: Domain of the vector function

Question: Find the domain of the vector function.

$$r(t) = \left\langle \ln(t-4), \sqrt{t^2+6}, \frac{1}{144-t^2} \right\rangle$$

Answer choices:

A $(4,12)$

B $(4,12]$

C $(4,12) \cup (12,\infty)$

D $-12 \le t \le 12$

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Solution: C

For the vector function

$$r(t) = \left\langle \ln(t-4), \sqrt{t^2+6}, \frac{1}{144-t^2} \right\rangle$$

the parameter is t. We'll find the domain for each component of the vector function. The domain for the vector function $r(t)$ will include only the values of t that are included in the domain of all three components.

For the logarithmic function

$$\ln(t-4)$$

we know that the argument inside the log function must be greater than 0, since the log function is undefined when its argument is 0 or negative.

$$t-4>0$$

$$t>4$$

For the radical function

$$\sqrt{t^2+6}$$

we know that the value underneath the square root cannot be negative. But since t is squared beneath the root, it doesn't matter what value we plug in for t, t^2 will always be positive. And therefore t^2+6 will always be positive. So there's no way to make the value underneath the root positive, which means we can plug in any number we want to for t. So the domain for this component is all real numbers.

For the rational function

$$\frac{1}{144 - t^2}$$

we know that the denominator cannot equal 0. So to find the limitations on the domain we can set the denominator equal to 0.

$$144 - t^2 = 0$$

$$t^2 = 144$$

$$t = \pm 12$$

The domain of the third component is $t \neq \pm 12$.

Finally we need to combine these domains. The first component tells us that $t > 4$. There's no restriction on the domain of the second component, so we can ignore it. If we look at the third component, we can see that the domain can't include $t = \pm 12$. But we've already said that $t > 4$, and the only forbidden value in that interval is $t = 12$.

The domain of $r(t)$ is $(4,12) \cup (12,\infty)$.

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Topic: Limit of a vector function

Question: Find the limit of the vector function.

$$\lim_{t\to 0}\left((t+4)\mathbf{i}+3\mathbf{j}+\frac{2t}{\sin t}\mathbf{k}\right)$$

Answer choices:

A $4\mathbf{i}+3\mathbf{j}+2\mathbf{k}$

B $4\mathbf{i}+3\mathbf{j}$

C $-4\mathbf{i}-3\mathbf{j}$

D $-4\mathbf{i}-3\mathbf{j}-2\mathbf{k}$

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Solution: A

We'll rewrite the limit as three separate limits.

$$\lim_{t\to 0}\left((t+4)\mathbf{i}+3\mathbf{j}+\frac{2t}{\sin t}\mathbf{k}\right)$$

$$\lim_{t\to 0}(t+4)\mathbf{i}+\lim_{t\to 0}3\mathbf{j}+\lim_{t\to 0}\frac{2t}{\sin t}\mathbf{k}$$

Evaluate the first limit.

$$(0+4)\mathbf{i}+\lim_{t\to 0}3\mathbf{j}+\lim_{t\to 0}\frac{2t}{\sin t}\mathbf{k}$$

$$4\mathbf{i}+\lim_{t\to 0}3\mathbf{j}+\lim_{t\to 0}\frac{2t}{\sin t}\mathbf{k}$$

Evaluate the second limit.

$$4\mathbf{i}+3\mathbf{j}+\lim_{t\to 0}\frac{2t}{\sin t}\mathbf{k}$$

If we evaluate the last limit at $t=0$, we'll get a $0/0$ value, which is indeterminate. So we'll use L'Hospital's rule to simplify the function, and then we'll evaluate the limit.

$$4\mathbf{i}+3\mathbf{j}+\lim_{t\to 0}\frac{2}{\cos t}\mathbf{k}$$

$$4\mathbf{i}+3\mathbf{j}+\frac{2}{\cos(0)}\mathbf{k}$$

$$4\mathbf{i}+3\mathbf{j}+\frac{2}{1}\mathbf{k}$$

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$$4\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$$

This is the limit of the vector function.

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Topic: Limit of a vector function

Question: Find the limit of the vector function.

$$\lim_{t \to 0} \left(\ln(2t + e)\mathbf{i} + \frac{6}{\cos t}\mathbf{j} + (t^3 - 1)\mathbf{k} \right)$$

Answer choices:

A $\quad \mathbf{i} - \mathbf{k}$

B $\quad \mathbf{i} + 6\mathbf{j} - \mathbf{k}$

C $\quad -\mathbf{i} - 6\mathbf{j} + \mathbf{k}$

D $\quad -\mathbf{i} + \mathbf{k}$

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Solution: B

We'll rewrite the limit as three separate limits.

$$\lim_{t\to 0}\left(\ln(2t+e)\mathbf{i}+\frac{6}{\cos t}\mathbf{j}+(t^3-1)\mathbf{k}\right)$$

$$\lim_{t\to 0}\ln(2t+e)\mathbf{i}+\lim_{t\to 0}\frac{6}{\cos t}\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

Evaluate the first limit.

$$\ln(2(0)+e)\mathbf{i}+\lim_{t\to 0}\frac{6}{\cos t}\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

$$\ln(e)\mathbf{i}+\lim_{t\to 0}\frac{6}{\cos t}\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

$$1\mathbf{i}+\lim_{t\to 0}\frac{6}{\cos t}\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

$$\mathbf{i}+\lim_{t\to 0}\frac{6}{\cos t}\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

Evaluate the second limit.

$$\mathbf{i}+\frac{6}{\cos(0)}\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

$$\mathbf{i}+\frac{6}{1}\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

$$\mathbf{i}+6\mathbf{j}+\lim_{t\to 0}(t^3-1)\mathbf{k}$$

Evaluate the third limit.

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$$\mathbf{i} + 6\mathbf{j} + (0^3 - 1)\mathbf{k}$$

$$\mathbf{i} + 6\mathbf{j} - 1\mathbf{k}$$

This is the limit of the vector function.

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Topic: Limit of a vector function

Question: Find the limit of the vector function.

$$\lim_{t\to 0}\left(\ln(t^2+e)\mathbf{i}+(4t^2+2)\mathbf{j}+\frac{6t}{\sin(2t)}\mathbf{k}\right)$$

Answer choices:

A $-\mathbf{i}-2\mathbf{j}-6\mathbf{k}$

B $\mathbf{i}+2\mathbf{j}+6\mathbf{k}$

C $-\mathbf{i}-2\mathbf{j}-3\mathbf{k}$

D $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$

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Solution: D

We'll rewrite the limit as three separate limits.

$$\lim_{t\to 0}\left(\ln(t^2+e)\mathbf{i}+(4t^2+2)\mathbf{j}+\frac{6t}{\sin(2t)}\mathbf{k}\right)$$

$$\lim_{t\to 0}\ln(t^2+e)\mathbf{i}+\lim_{t\to 0}(4t^2+2)\mathbf{j}+\lim_{t\to 0}\frac{6t}{\sin(2t)}\mathbf{k}$$

Evaluate the first limit.

$$\ln(0^2+e)\mathbf{i}+\lim_{t\to 0}(4t^2+2)\mathbf{j}+\lim_{t\to 0}\frac{6t}{\sin(2t)}\mathbf{k}$$

$$\ln(e)\mathbf{i}+\lim_{t\to 0}(4t^2+2)\mathbf{j}+\lim_{t\to 0}\frac{6t}{\sin(2t)}\mathbf{k}$$

$$1\mathbf{i}+\lim_{t\to 0}(4t^2+2)\mathbf{j}+\lim_{t\to 0}\frac{6t}{\sin(2t)}\mathbf{k}$$

$$\mathbf{i}+\lim_{t\to 0}(4t^2+2)\mathbf{j}+\lim_{t\to 0}\frac{6t}{\sin(2t)}\mathbf{k}$$

Evaluate the second limit.

$$\mathbf{i}+(4(0)^2+2)\mathbf{j}+\lim_{t\to 0}\frac{6t}{\sin(2t)}\mathbf{k}$$

$$\mathbf{i}+2\mathbf{j}+\lim_{t\to 0}\frac{6t}{\sin(2t)}\mathbf{k}$$

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If we evaluate the last limit at $t=0$, we'll get a $0/0$ value, which is indeterminate. So we'll use L'Hospital's rule to simplify the function, and then we'll evaluate the limit.

$$\mathbf{i}+2\mathbf{j}+\lim_{t\to 0}\frac{6}{2\cos(2t)}\mathbf{k}$$

$$\mathbf{i}+2\mathbf{j}+\frac{6}{2\cos(2(0))}\mathbf{k}$$

$$\mathbf{i}+2\mathbf{j}+\frac{6}{2(1)}\mathbf{k}$$

$$\mathbf{i}+2\mathbf{j}+3\mathbf{k}$$

This is the limit of the vector function.

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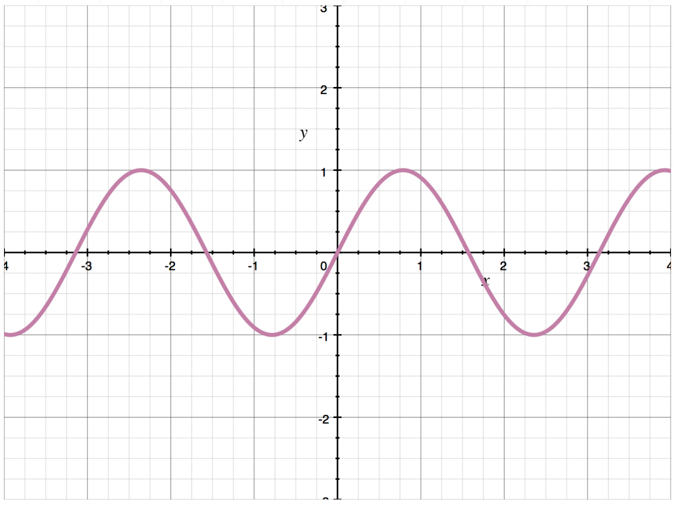
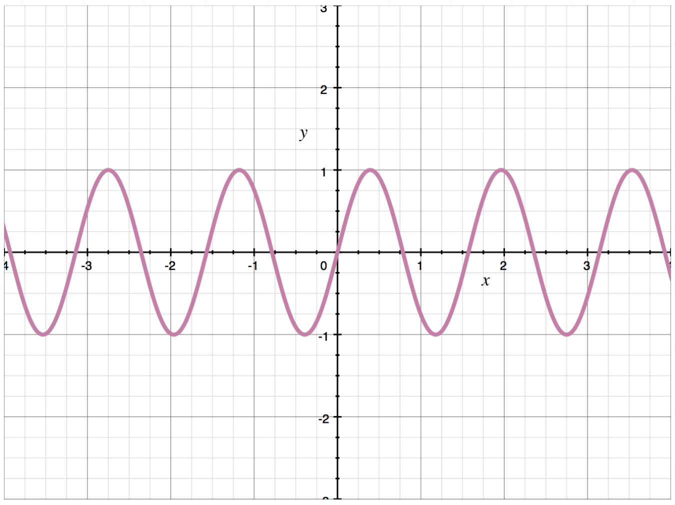
Topic: Sketching the vector equation

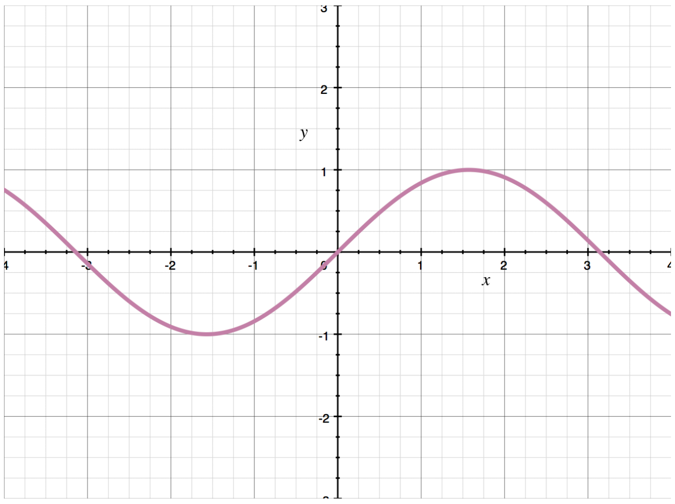
Question: Choose the sketch of the vector equation.

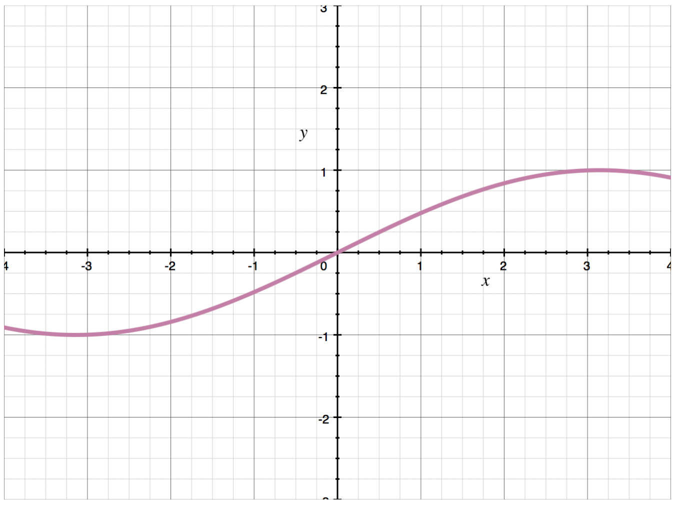
$$r(t) = \langle 2t, \sin t \rangle$$

Answer choices:

A 

B 

C 

D 

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Solution: D

We'll change the vector equation to its parametric form.

$$x = 2t$$

$$y = \sin t$$

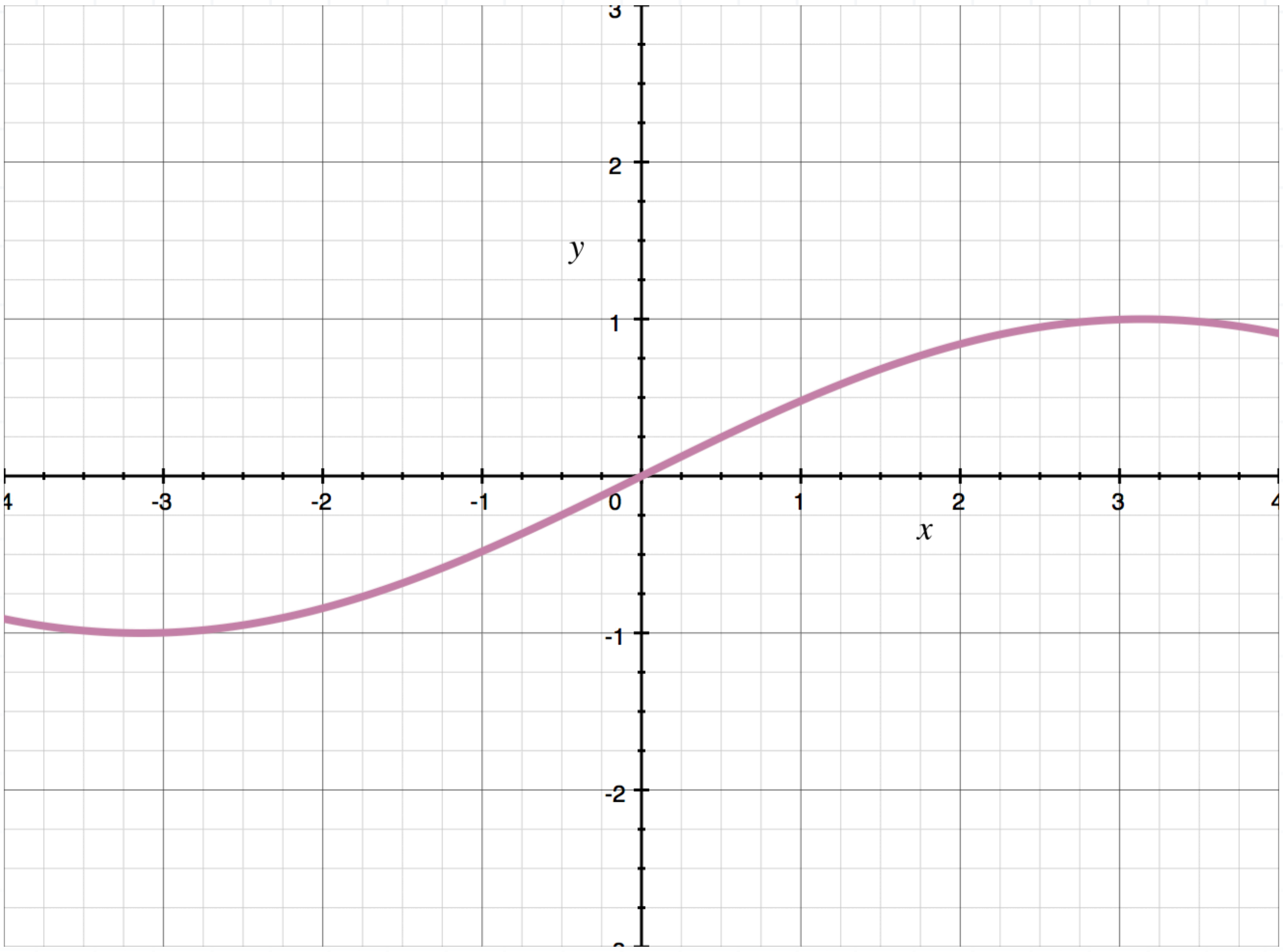
Solve $x = 2t$ for t.

$$t = \frac{1}{2}x$$

Then substitute this into $y = \sin t$ to get an equation for y in terms of x.

$$y = \sin\left(\frac{1}{2}x\right)$$

Sketching this curve gives

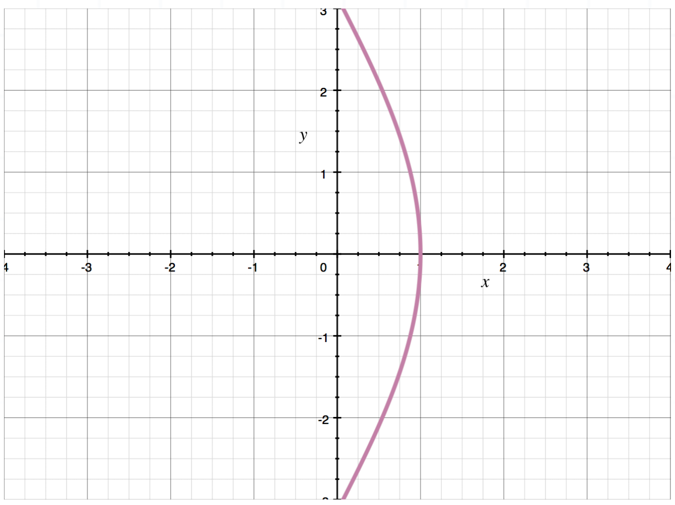

3
2
y
1
-4
-3
-2
-1
0
1
2
3
4
x
-1
-2

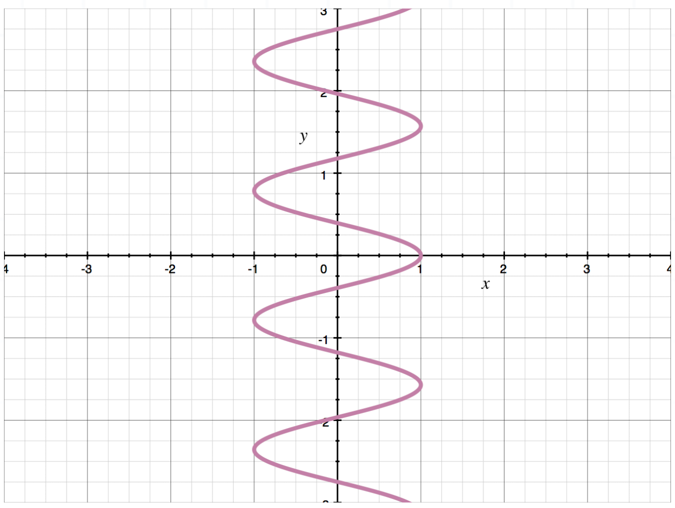
Topic: Sketching the vector equation

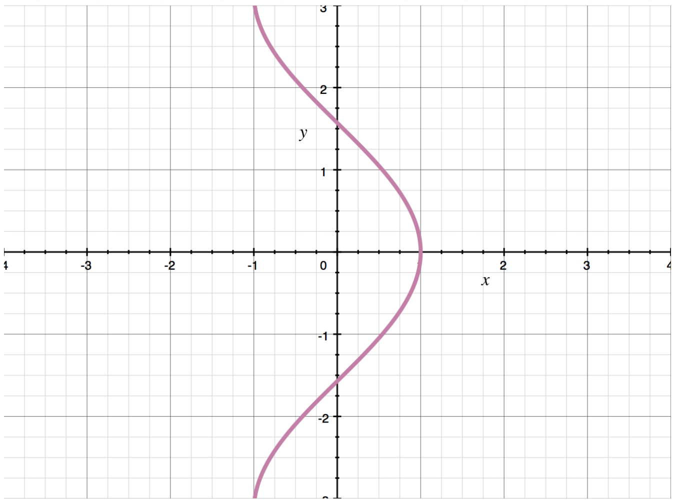
Question: Choose the sketch of the vector equation.

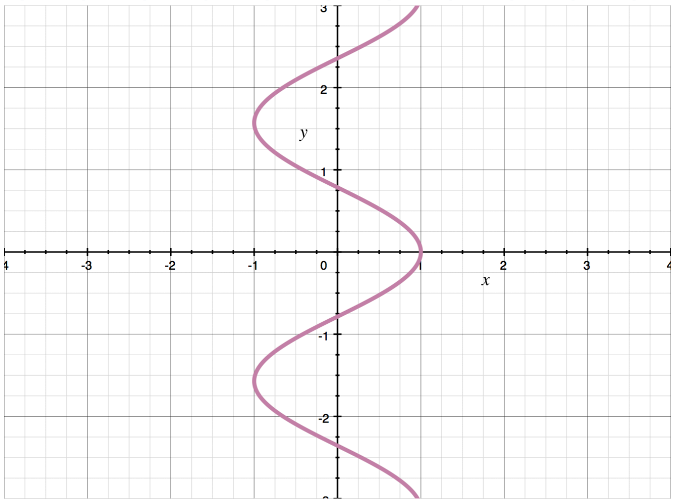
$$r(t) = \left\langle \cos(2t), \frac{1}{2}t \right\rangle$$

Answer choices:

A 

B 

C 

D 

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Solution: B

We'll change the vector equation to its parametric form.

$$x = \cos(2t)$$

$$y = \frac{1}{2}t$$

Solve $y = \frac{1}{2}t$ for t.

$$t = 2y$$

Then substitute this into $x = \cos(2t)$ to get an equation for x in terms of y.

$$x = \cos\left[2(2y)\right]$$

$$x = \cos(4y)$$

Sketching this curve gives

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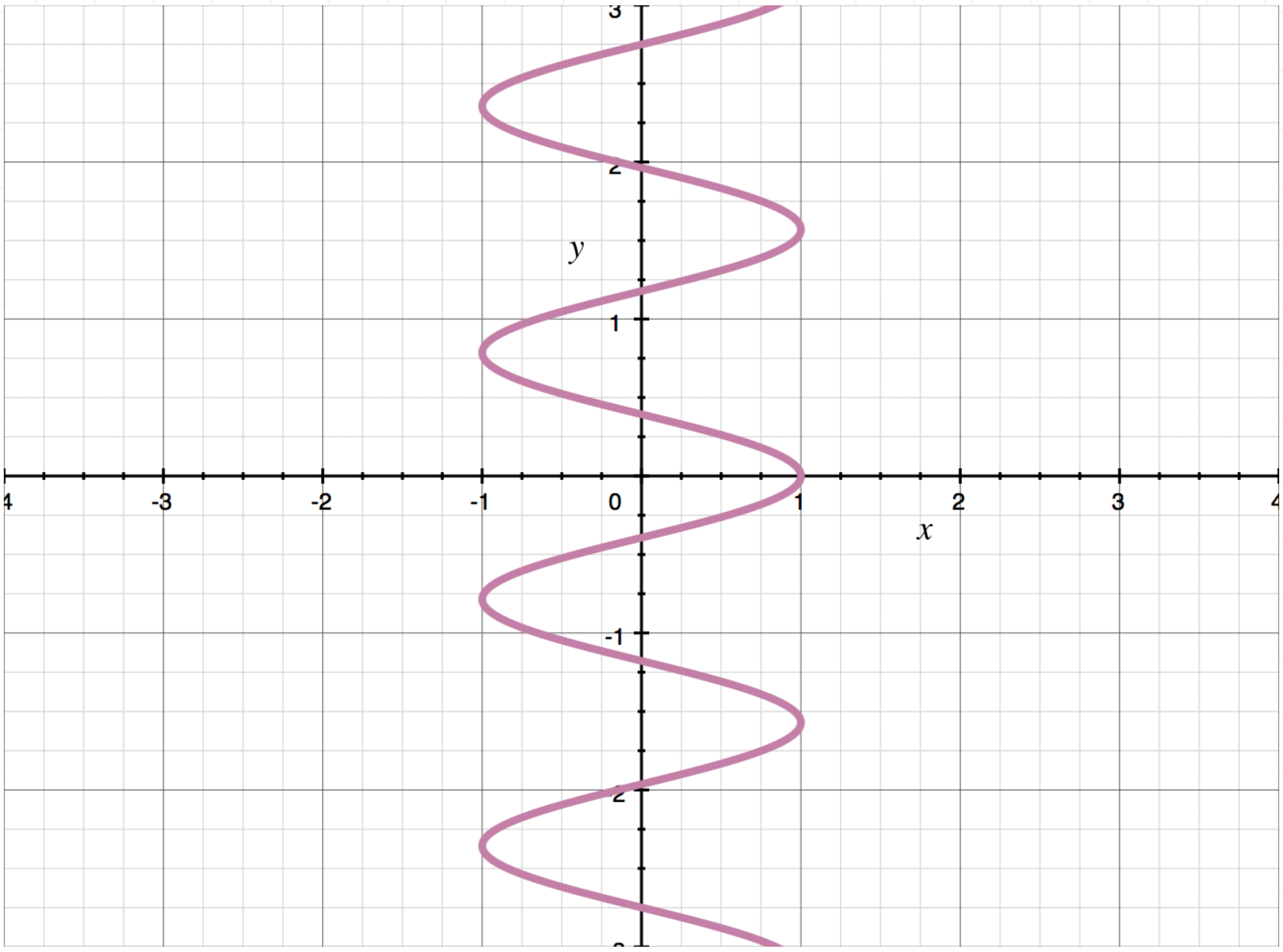

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Topic: Sketching the vector equation

Question: The vector-valued function $\mathbf{s}(t) = 8\cos t\mathbf{i} + 8\sin t\mathbf{j}$ is given on the domain $0 \le t \le 8\pi$. Along which shape does the curve lie?

Answer choices:

A A line with slope 4

B A line with slope 8

C A parabola with vertex $(0,8)$

D A circle with radius 8

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Solution: D

The parametric equations of the functions are

$$x = 8\cos t$$

$$y = 8\sin t$$

or we could write them as

$$x^2 = 64\cos^2 t$$

$$y^2 = 64\sin^2 t$$

If we add the parametric equations together, we get

$$x^2 + y^2 = 64\cos^2 t + 64\sin^2 t$$

$$x^2 + y^2 = 64(\cos^2 t + \sin^2 t)$$

$$x^2 + y^2 = 64$$

The last equation indicates that the function lies along the circle with radius 8.

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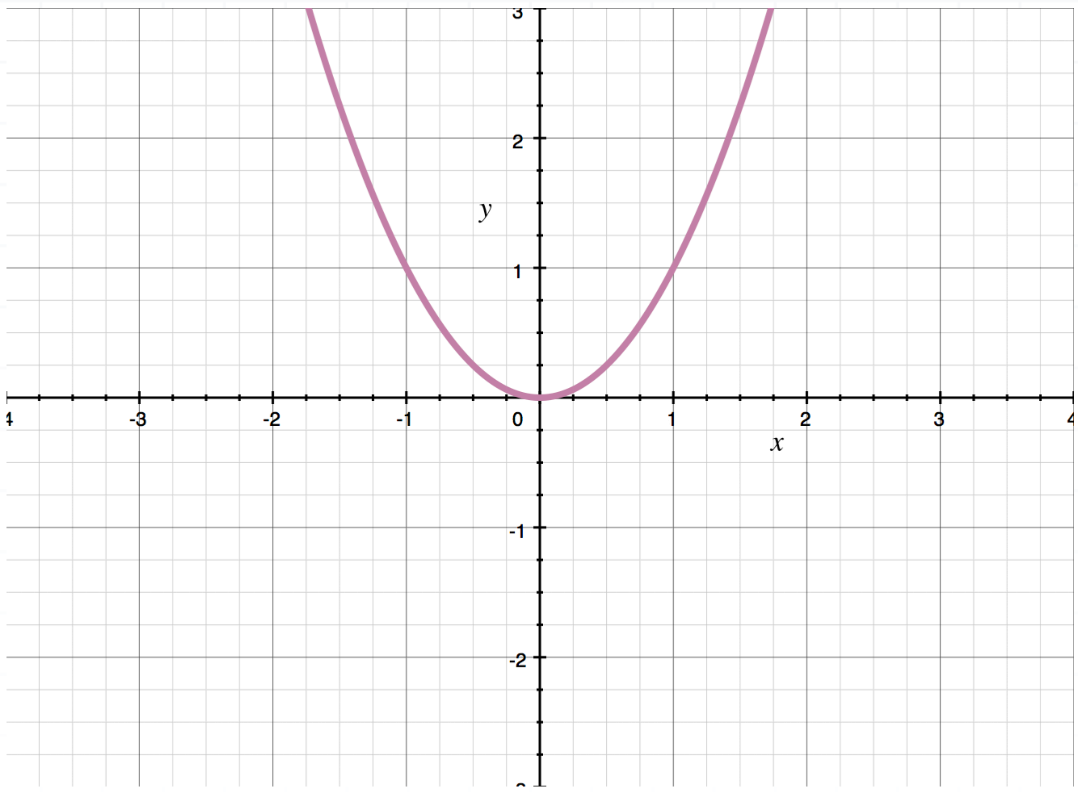
Topic: Projections of the curve

Question: Which vector relates to this sketch of the XY plane projection?



Answer choices:

A $\quad r(t) = \langle t^4, t^3, t^2 \rangle$

B $\quad r(t) = \langle t^2, t, t^3 \rangle$

C $\quad r(t) = \langle t^3, t, t^2 \rangle$

D $\quad r(t) = \langle t, t^2, t^3 \rangle$

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Solution: D

The XY plane projection is the sketch of what the vector looks like from the perspective of the xy-plane. To figure out which would be the correct vector, we can take each answer choice, convert it to its parametric form, and then combine the x and y variables. This equation will give us the XY plane projection.

Changing answer choice A, $r(t) = \langle t^4, t^3, t^2 \rangle$, to its parametric form for the x and y variables gives

$$x = t^4$$

$$y = t^3$$

Solving $x = t^4$ for t gives

$$t = x^{\frac{1}{4}}$$

Plugging $t = x^{\frac{1}{4}}$ into $y = t^3$ gives

$$y = x^{\frac{3}{4}}$$

A sketch of this curve is

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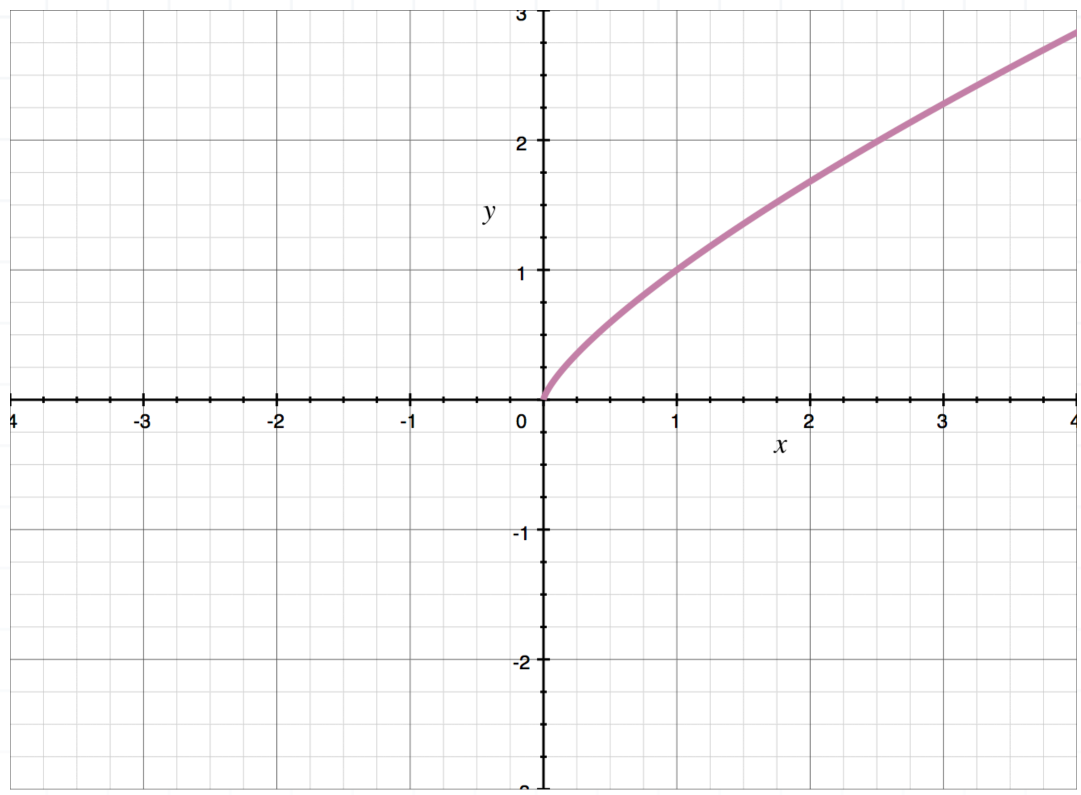

This is not the given sketch for the XY plane projection, so answer choice A is not the right answer.

Changing answer choice B, $r(t) = \langle t^2, t, t^3 \rangle$, to its parametric form for the x and y variables gives

$$x = t^2$$

$$y = t$$

Plugging $y = t$ into $x = t^2$ gives

$$x = y^2$$

A sketch of this curve is

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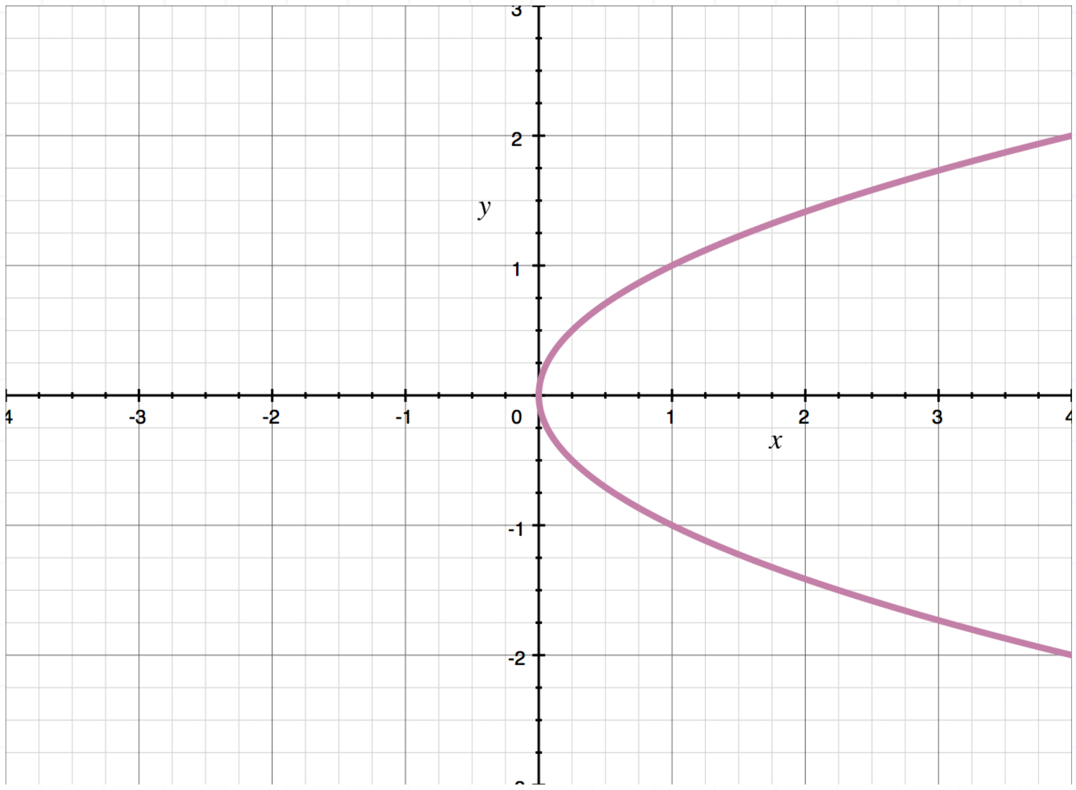


This is not the given sketch for the XY plane projection, so answer choice B is not the right answer.

Changing answer choice C, $r(t) = \langle t^3, t, t^2 \rangle$, to its parametric form for the x and y variables gives

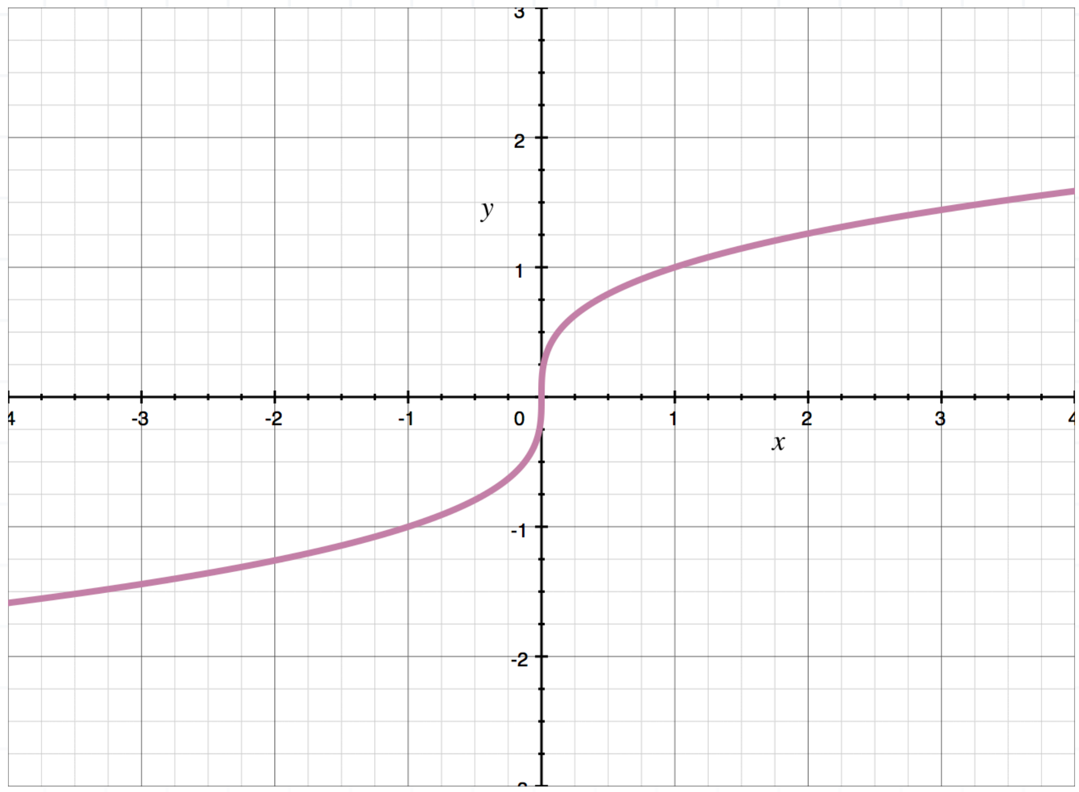
$$x = t^3$$

$$y = t$$

Plugging $y = t$ into $x = t^3$ gives

$$x = y^3$$

A sketch of this curve is



This is not the given sketch for the XY plane projection, so answer choice C is not the right answer.

Changing answer choice D, $r(t) = \langle t, t^2, t^3 \rangle$, to its parametric form for the x and y variables gives

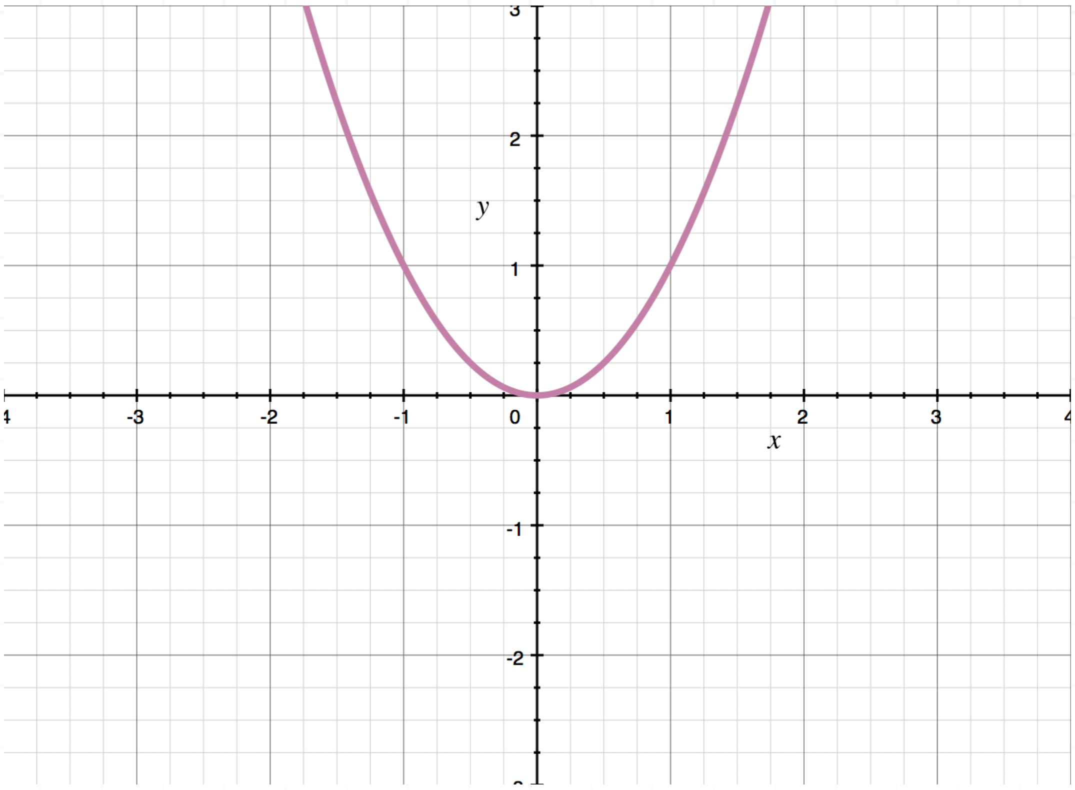
$$x = t$$

$$y = t^2$$

Plugging $x = t$ into $y = t^2$ gives

$$y = x^2$$

A sketch of this curve is



This is the given sketch for the XY plane projection, so answer choice D is the right answer.

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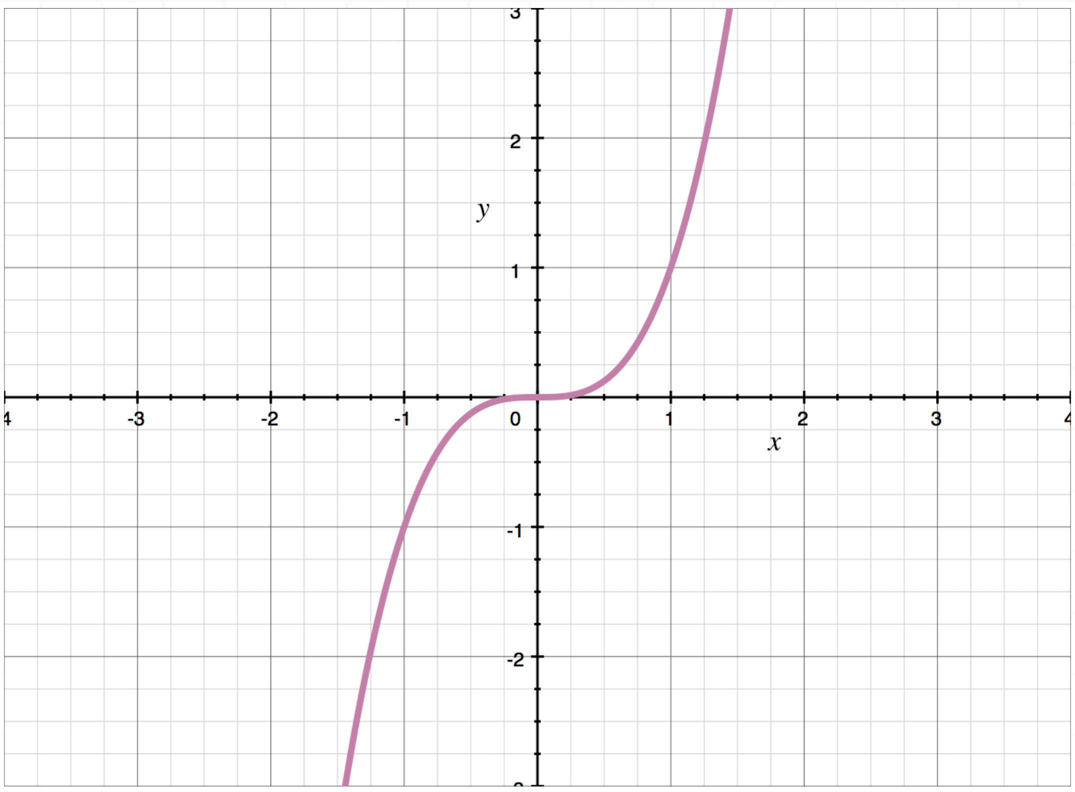
Topic: Projections of the curve

Question: Which vector relates to this sketch of the XY plane projection?



Answer choices:

A $\quad r(t) = \left\langle 2t^3, t^2, \sin t^2 \right\rangle$

B $\quad r(t) = \left\langle t^3, t^2, \cos t^2 \right\rangle$

C $\quad r(t) = \left\langle t, t^3, \cos t^2 \right\rangle$

D $\quad r(t) = \left\langle t, 2t^3, \sin t^2 \right\rangle$

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Solution: C

The XY plane projection is the sketch of what the vector looks like from the perspective of the xy-plane. To figure out which would be the correct vector, we can take each answer choice, convert it to its parametric form, and then combine the x and y variables. This equation will give us the XY plane projection.

Changing answer choice A, $r(t) = \left\langle 2t^3, t^2, \sin t^2 \right\rangle$, to its parametric form for the x and y variables gives

$$x = 2t^3$$

$$y = t^2$$

Solving $y = t^2$ for t gives

$$t = y^{\frac{1}{2}}$$

Plugging $t = y^{\frac{1}{2}}$ into $x = 2t^3$ gives

$$x = 2\left(y^{\frac{1}{2}}\right)^3$$

$$x = 2y^{\frac{3}{2}}$$

A sketch of this curve is

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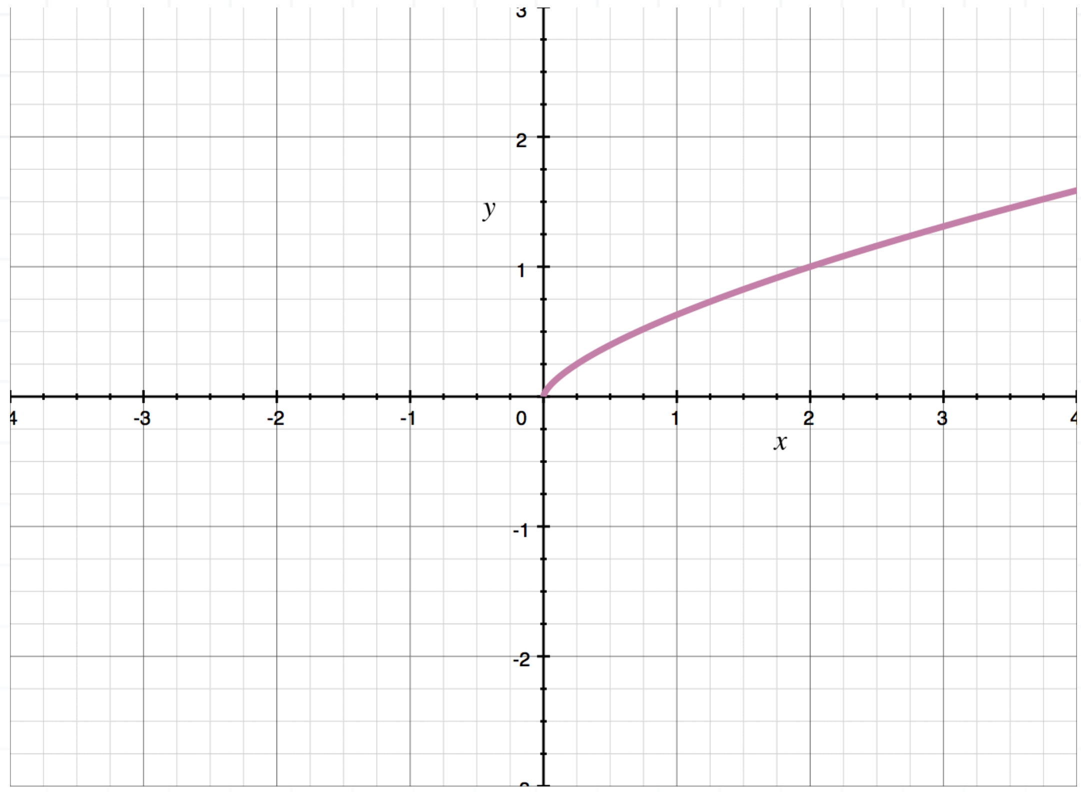

This is not the given sketch for the XY plane projection, so answer choice A is not the right answer.

Changing answer choice B, $r(t) = \langle t^3, t^2, \cos t^2 \rangle$, to its parametric form for the x and y variables gives

$$x = t^3$$

$$y = t^2$$

Solving $y = t^2$ for t gives

$$t = y^{\frac{1}{2}}$$

Plugging $t = y^{\frac{1}{2}}$ into $x = t^3$ gives

$$x = \left(y^{\frac{1}{2}}\right)^3$$

$$x = y^{\frac{3}{2}}$$

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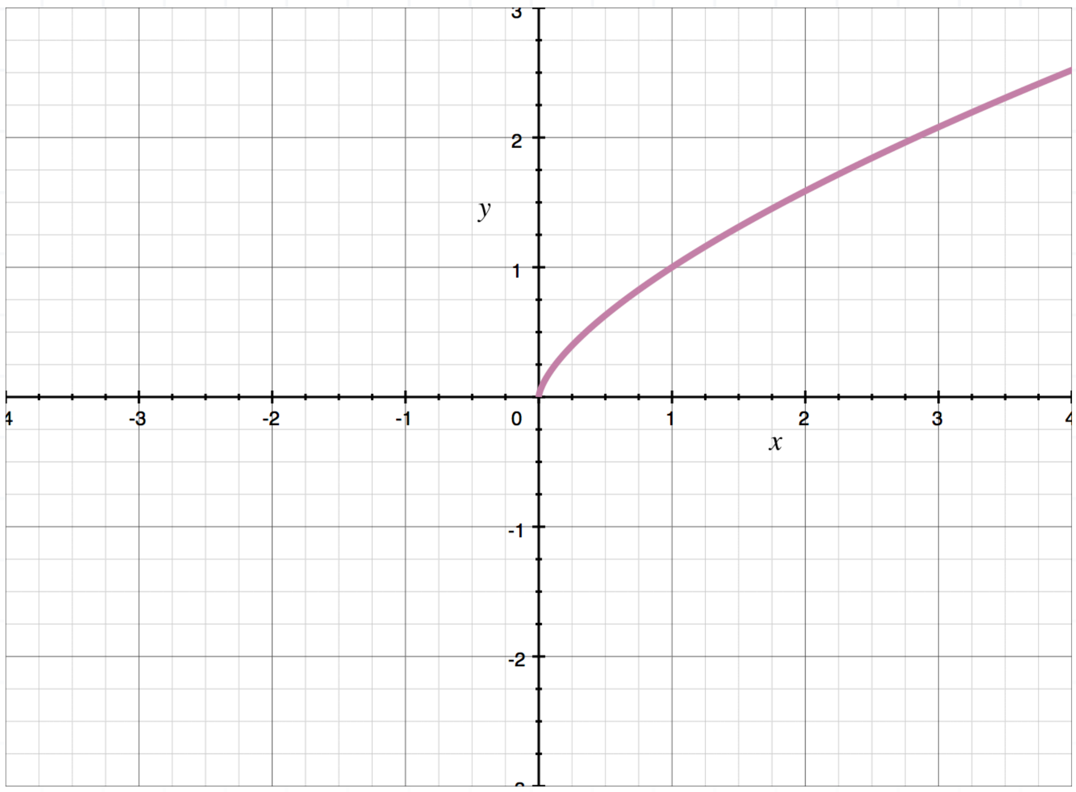
A sketch of this curve is similar to the last curve, like



This is not the given sketch for the XY plane projection, so answer choice B is not the right answer.

Changing answer choice C, $r(t)=\langle t,t^3,\cos t^2\rangle$, to its parametric form for the x and y variables gives

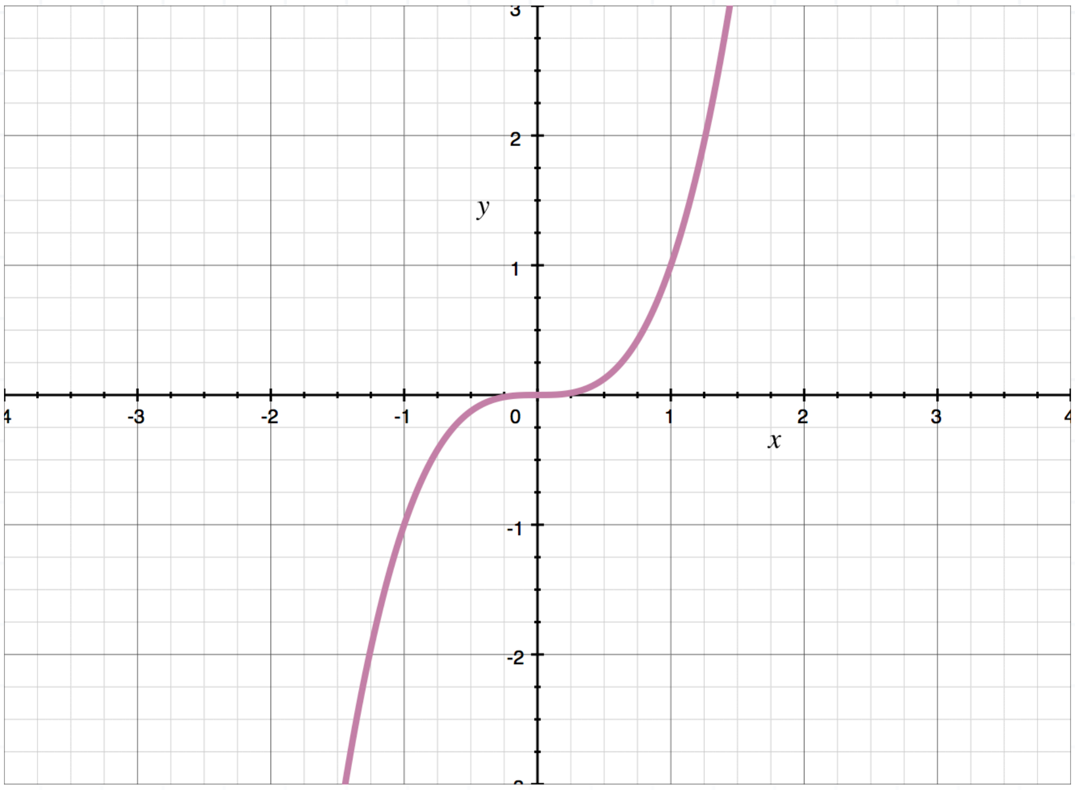
$$x=t$$

$$y=t^3$$

Plugging $x=t$ into $y=t^3$ gives

$$y=x^3$$

A sketch of this curve is



This is the given sketch for the XY plane projection, so answer choice C is the right answer.

Changing answer choice D, $r(t) = \langle t, 2t^3, \sin t^2 \rangle$, to its parametric form for the x and y variables gives

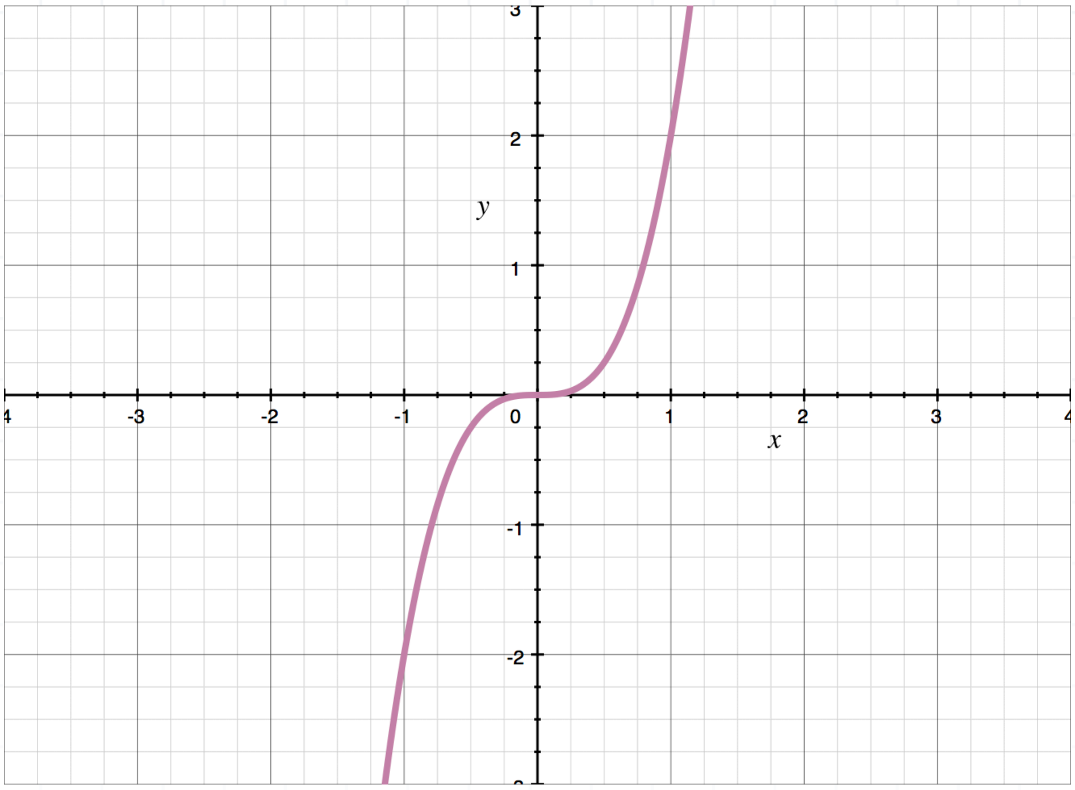
$$x = t$$

$$y = 2t^3$$

Plugging $x = t$ into $y = 2t^3$ gives

$$y = 2x^3$$

A sketch of this curve is



This is not the given sketch for the XY plane projection, so answer choice D is not the right answer.

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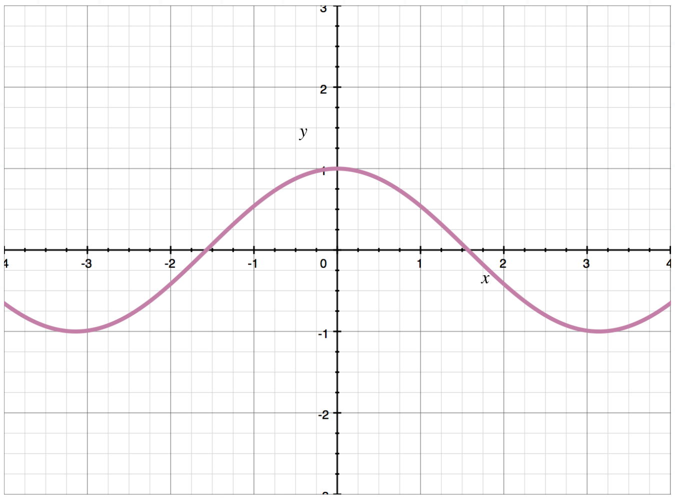
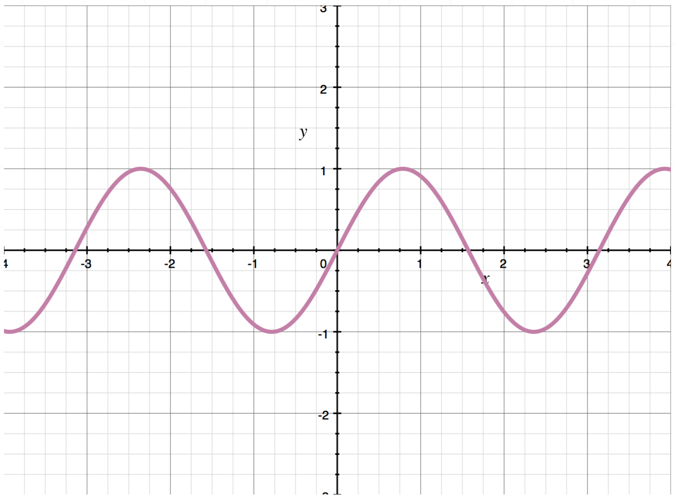
Topic: Projections of the curve

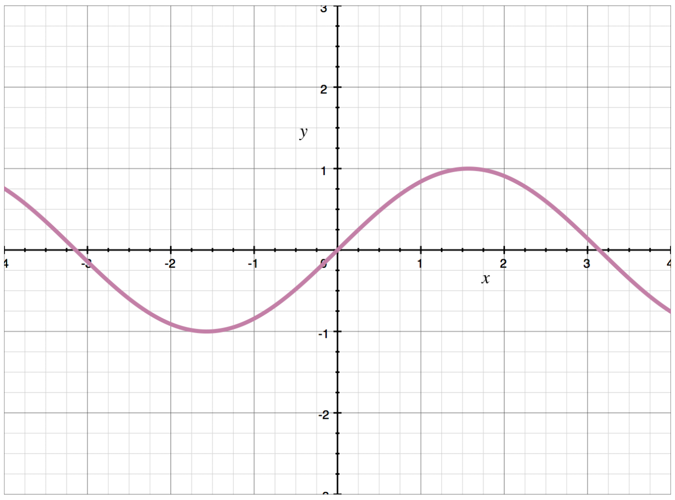
Question: Which is the YZ plane projection of the vector?

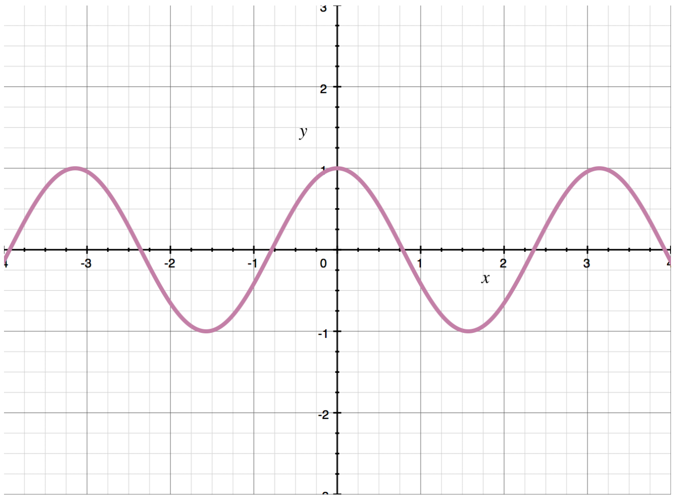
$$r(t) = \langle 3 \ln t, t, \sin t \rangle$$

Answer choices:

A 

B 

C 

D 

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Solution: C

The YZ plane projection is the sketch of what the vector looks like from the perspective of the yz-plane.

We'll change $r(t) = \langle 3\ln t, t, \sin t \rangle$ to its parametric form for the y and z variables.

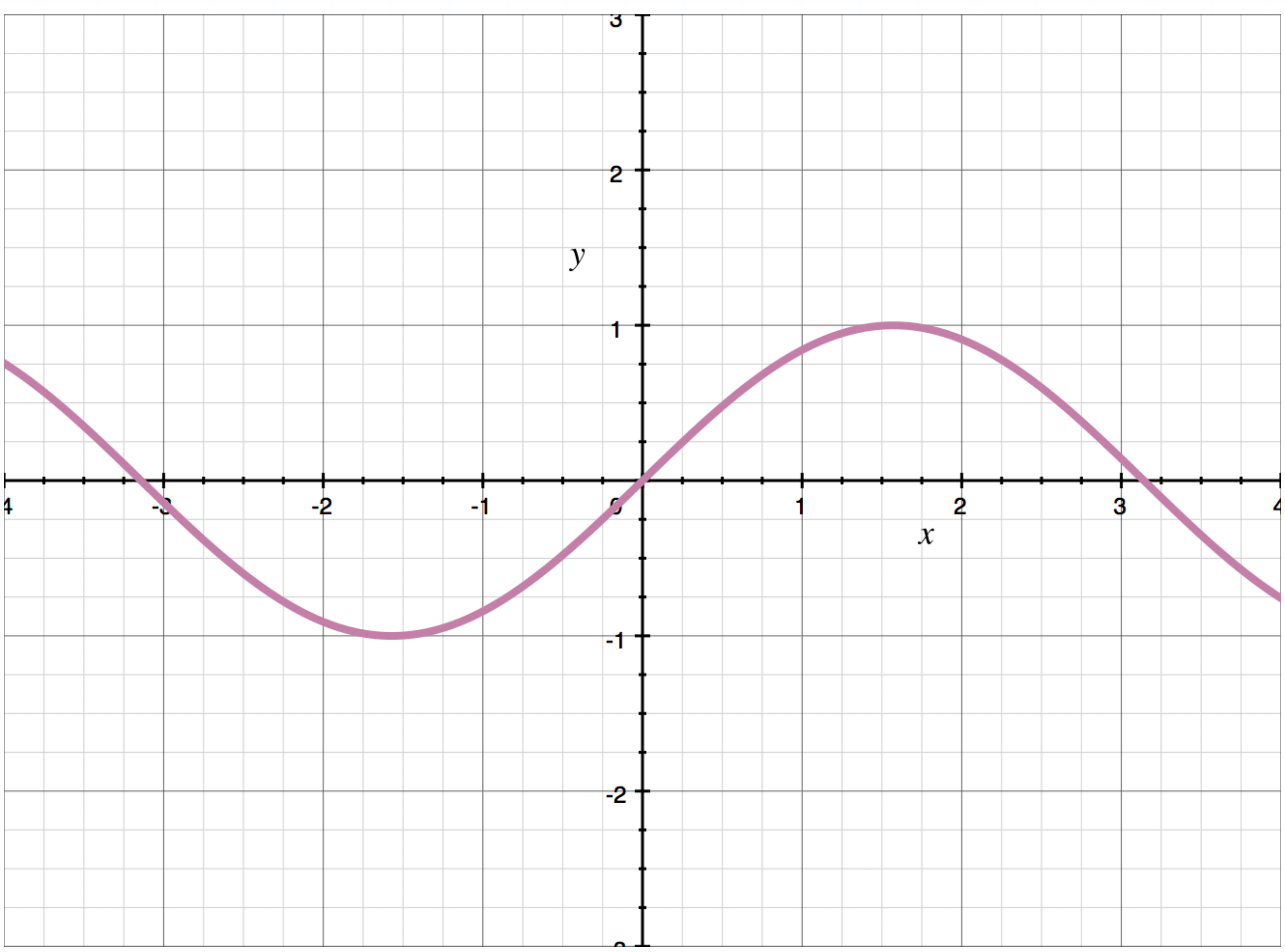
$$y = t$$

$$z = \sin t$$

Then we'll substitute $y = t$ into $z = \sin t$.

$$z = \sin y$$

A sketch of $z = \sin y$ looks like



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Topic: Vector and parametric equations of a line segment

Question: Find the vector equation of the line segment.

$P(1,0,1)$

$Q(1,1,1)$

Answer choices:

A $\quad r(t) = \langle 1,t,1 \rangle$ where $0 \leq t \leq 1$

B $\quad r(t) = \langle 0,t,0 \rangle$ where $0 \leq t \leq 1$

C $\quad r(t) = \langle t,1,t \rangle$ where $0 \leq t \leq 1$

D $\quad r(t) = \langle t,0,t \rangle$ where $0 \leq t \leq 1$

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Solution: A

First we'll change the given points $P(1,0,1)$ and $Q(1,1,1)$ to their vector equivalents. Using the origin $(0,0,0)$ as the initial point, and the given point as the terminal point of the vector, then they become

$$r_0\langle 1,0,1\rangle$$

$$r_1\langle 1,1,1\rangle$$

Now we can use the vector equation of a line segment

$$r(t) = (1-t)r_0 + tr_1$$

where $0 \le t \le 1$

In this equation, r_0 is the vector from the first point $r_0\langle 1,0,1\rangle$ and r_1 is the vector from the second point $r_1\langle 1,1,1\rangle$.

$$r(t) = (1-t)\langle 1,0,1\rangle + t\langle 1,1,1\rangle$$

$$r(t) = \langle 1-t,0,1-t\rangle + \langle t,t,t\rangle$$

$$r(t) = \langle 1,t,1\rangle$$

This is the vector equation of the line segment.

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Topic: Vector and parametric equations of a line segment

Question: Find the parametric equations of the line segment.

$P(3,2,4)$

$Q(1,0,-3)$

Answer choices:

A $\quad x_{r(t)} = 3 - 4t \quad y_{r(t)} = 2 - 2t \quad z_{r(t)} = 4 - t$

B $\quad x_{r(t)} = 3 - 2t \quad y_{r(t)} = 2 - 2t \quad z_{r(t)} = 4 - 7t$

C $\quad x_{r(t)} = 3 + 4t \quad y_{r(t)} = 2 + 2t \quad z_{r(t)} = 4 + t$

D $\quad x_{r(t)} = 3 + 2t \quad y_{r(t)} = 2 + 2t \quad z_{r(t)} = 4 + 7t$

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Solution: B

First we'll change the given points $P(3,2,4)$ and $Q(1,0,-3)$ to their vector equivalents. Using the origin $(0,0,0)$ as the initial point, and the given point as the terminal point of the vector, then they become

$$r_0\langle 3,2,4\rangle$$

$$r_1\langle 1,0,-3\rangle$$

Now we can use the vector equation of a line segment

$$r(t)=(1-t)r_0+tr_1$$

where $0\le t\le 1$

In this equation, r_0 is the vector from the first point $r_0\langle 3,2,4\rangle$ and r_1 is the vector from the second point $r_1\langle 1,0,-3\rangle$.

$$r(t)=(1-t)\langle 3,2,4\rangle+t\langle 1,0,-3\rangle$$

$$r(t)=\langle 3-3t,2-2t,4-4t\rangle+\langle t,0,-3t\rangle$$

$$r(t)=\langle 3-2t,2-2t,4-7t\rangle$$

This is the vector equation of the line segment. Next we can find the parametric equations of the line segment $r(t)=\langle 3-2t,2-2t,4-7t\rangle$ remembering that

$$x=r_{t_1}$$

$$y=r_{t_2}$$

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$$z = r_{t_3}$$

This will give us

$$x = 3 - 2t$$

$$y = 2 - 2t$$

$$z = 4 - 7t$$

These are the parametric equations of the line segment.

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Topic: Vector and parametric equations of a line segment

Question: Find the parametric equations of the line segment.

$P(-2,-4,0)$

$Q(2,3,5)$

Answer choices:

A	$x_{r(t)}=2$	$y_{r(t)}=4-t$	$z_{r(t)}=5t$
B	$x_{r(t)}=-2$	$y_{r(t)}=-4+t$	$z_{r(t)}=-5t$
C	$x_{r(t)}=2-4t$	$y_{r(t)}=4-7t$	$z_{r(t)}=-5t$
D	$x_{r(t)}=-2+4t$	$y_{r(t)}=-4+7t$	$z_{r(t)}=5t$

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Solution: D

First we'll change the given points $P(-2, -4,0)$ and $Q(2,3,5)$ to their vector equivalents. Using the origin $(0,0,0)$ as the initial point, and the given point as the terminal point of the vector, then they become

$$r_0\langle -2, -4,0\rangle$$

$$r_1\langle 2,3,5\rangle$$

Now we can use the vector equation of a line segment

$$r(t) = (1-t)r_0 + tr_1$$

where $0 \le t \le 1$

In this equation, r_0 is the vector from the first point $r_0\langle -2, -4,0\rangle$ and r_1 is the vector from the second point $r_1\langle 2,3,5\rangle$.

$$r(t) = (1-t)\langle -2, -4,0\rangle + t\langle 2,3,5\rangle$$

$$r(t) = \langle -2+2t, -4+4t,0\rangle + \langle 2t,3t,5t\rangle$$

$$r(t) = \langle -2+4t, -4+7t,5t\rangle$$

This is the vector equation of the line segment. Next we can find the parametric equations of the line segment $r(t) = \langle -2+4t, -4+7t,5t\rangle$ remembering that

$$x = r_{t_1}$$

$$y = r_{t_2}$$

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$$z = r_{t_3}$$

This will give us

$$x = -2 + 4t$$

$$y = -4 + 7t$$

$$z = 5t$$

These are the parametric equations of the line segment.

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Topic: Vector function for the curve of intersection of two surfaces

Question: Find the vector function for the curve of intersection of the surfaces.

Sphere: $z=\sqrt{x^2+y^2-25}$

Plane: $z=1+x$

Answer choices:

A $\quad r(t)=\left(\frac{1}{2}t^2+13\right)\mathbf{i}+t\mathbf{j}+\left(\frac{1}{2}t^2+12\right)\mathbf{k}$

B $\quad r(t)=\left(\frac{1}{2}t^2+13\right)\mathbf{i}+t\mathbf{j}+\left(\frac{1}{2}t^2-12\right)\mathbf{k}$

C $\quad r(t)=\left(\frac{1}{2}t^2-13\right)\mathbf{i}+t\mathbf{j}+\left(\frac{1}{2}t^2-12\right)\mathbf{k}$

D $\quad r(t)=\left(\frac{1}{2}t^2-13\right)\mathbf{i}+t\mathbf{j}+\left(\frac{1}{2}t^2+12\right)\mathbf{k}$

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Solution: C

First, we'll solve both equations for the same variable. The equations we've been given in this problem are both already solved for z, so we can go ahead and set them equal to each other, and then solve for x.

$$1 + x = \sqrt{x^2 + y^2 - 25}$$

$$(1 + x)^2 = x^2 + y^2 - 25$$

$$x^2 + 2x + 1 = x^2 + y^2 - 25$$

$$2x + 1 = y^2 - 25$$

$$2x = y^2 - 26$$

$$x = \frac{1}{2}y^2 - 13$$

Set $y = t$ in this equation.

$$x = \frac{1}{2}t^2 - 13$$

Plug this value of x into $z = 1 + x$.

$$z = 1 + \frac{1}{2}t^2 - 13$$

$$z = \frac{1}{2}t^2 - 12$$

We set $y = t$ originally and used that to find values for x and z in terms of t, and so our vector function is

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$$r(t)=\left(\frac{1}{2}t^2-13\right)\mathbf{i}+t\mathbf{j}+\left(\frac{1}{2}t^2-12\right)\mathbf{k}$$

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Topic: Vector function for the curve of intersection of two surfaces

Question: Find the vector function for the curve of intersection of the surfaces.

Cone: $z=\sqrt{x^2+y^2}$

Plane: $z=y-1$

Answer choices:

A $\quad r(t)=t\mathbf{i}+\left(-\frac{1}{2}t^2+\frac{1}{2}\right)\mathbf{j}+\left(-\frac{1}{2}t^2-\frac{1}{2}\right)\mathbf{k}$

B $\quad r(t)=t\mathbf{i}+\left(-\frac{1}{2}t^2+\frac{1}{2}\right)\mathbf{j}+\left(-\frac{1}{2}t^2+\frac{1}{2}\right)\mathbf{k}$

C $\quad r(t)=t\mathbf{i}+\left(\frac{1}{2}t^2+\frac{1}{2}\right)\mathbf{j}+\left(\frac{1}{2}t^2-\frac{1}{2}\right)\mathbf{k}$

D $\quad r(t)=t\mathbf{i}+\left(\frac{1}{2}t^2+\frac{1}{2}\right)\mathbf{j}+\left(\frac{1}{2}t^2+\frac{1}{2}\right)\mathbf{k}$

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Solution: A

First, we'll solve both equations for the same variable. The equations we've been given in this problem are both already solved for z, so we can go ahead and set them equal to each other, and then solve for y.

$$y-1=\sqrt{x^2+y^2}$$

$$(y-1)^2=x^2+y^2$$

$$y^2-2y+1=x^2+y^2$$

$$-2y+1=x^2$$

$$-2y=x^2-1$$

$$y=-\frac{1}{2}x^2+\frac{1}{2}$$

Set $x=t$ in this equation.

$$y=-\frac{1}{2}t^2+\frac{1}{2}$$

Plug this value of y into $z=y-1$.

$$z=-\frac{1}{2}t^2+\frac{1}{2}-1$$

$$z=-\frac{1}{2}t^2-\frac{1}{2}$$

We set $x=t$ originally and used that to find values for y and z in terms of t, and so our vector function is

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$$r(t) = t\mathbf{i} + \left(-\frac{1}{2}t^2 + \frac{1}{2}\right)\mathbf{j} + \left(-\frac{1}{2}t^2 - \frac{1}{2}\right)\mathbf{k}$$

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Topic: Vector function for the curve of intersection of two surfaces

Question: Find the vector function for the curve of intersection of the surfaces.

Ellipsoid: $z=\sqrt{x^2+\frac{1}{3}y^2-4}$

Plane: $z=x+2$

Answer choices:

A $\quad r(t)=\left(\frac{1}{12}t^2+2\right)\mathbf{i}+t\mathbf{j}+\left(\frac{1}{12}t^2+4\right)\mathbf{k}$

B $\quad r(t)=\left(\frac{1}{12}t^2-2\right)\mathbf{i}+t\mathbf{j}+\frac{1}{12}t^2\mathbf{k}$

C $\quad r(t)=\left(-\frac{1}{12}t^2-2\right)\mathbf{i}+t\mathbf{j}+\frac{1}{12}t^2\mathbf{k}$

D $\quad r(t)=\left(\frac{1}{12}t^2+2\right)\mathbf{i}+t\mathbf{j}+\left(\frac{1}{12}t^2-4\right)\mathbf{k}$

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Solution: B

First, we'll solve both equations for the same variable. The equations we've been given in this problem are both already solved for z, so we can go ahead and set them equal to each other, and then solve for x.

$$x+2=\sqrt{x^2+\frac{1}{3}y^2-4}$$

$$(x+2)^2=x^2+\frac{1}{3}y^2-4$$

$$x^2+4x+4=x^2+\frac{1}{3}y^2-4$$

$$4x+4=\frac{1}{3}y^2-4$$

$$4x=\frac{1}{3}y^2-8$$

$$x=\frac{1}{12}y^2-2$$

Set $y=t$ in this equation.

$$x=\frac{1}{12}t^2-2$$

Plug this value of x into $z=x+2$.

$$z=\frac{1}{12}t^2-2+2$$

$$z = \frac{1}{12}t^2$$

We set $y = t$ originally and used that to find values for x and z in terms of t, and so our vector function is

$$r(t) = \left(\frac{1}{12}t^2 - 2\right)\mathbf{i} + t\mathbf{j} + \frac{1}{12}t^2\mathbf{k}$$

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Topic: Derivative of a vector function

Question: Find the derivative of the vector function.

$$r(t) = 4t\mathbf{a} + 2t^3\mathbf{b} - \mathbf{c}$$

Answer choices:

A $\quad r'(t) = 4\mathbf{a} + 6t^3\mathbf{b}$

B $\quad r'(t) = 4\mathbf{a} + 3t^2\mathbf{b}$

C $\quad r'(t) = 4\mathbf{a} + 2t^2\mathbf{b}$

D $\quad r'(t) = 4\mathbf{a} + 6t^2\mathbf{b}$

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Solution: D

To find the derivative, we'll differentiation with respect to t. We can differentiate each term individually.

$$r'(t) = 4\mathbf{a} + 2(3)t^2\mathbf{b} + (0)\mathbf{c}$$

$$r'(t) = 4\mathbf{a} + 6t^2\mathbf{b}$$

This is the derivative of the vector function, given in the same form as the original function.

Topic: Derivative of a vector function

Question: Find the derivative of the vector function.

$$r(t) = \ln\left(t^3\right)\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}$$

Answer choices:

A $\quad r'(t) = \dfrac{3}{t^3}\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$

B $\quad r'(t) = \dfrac{3}{t}\mathbf{i} - 4\cos t\mathbf{j} + \mathbf{k}$

C $\quad r'(t) = \dfrac{3}{t}\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$

D $\quad r'(t) = \dfrac{3}{t^3}\mathbf{i} - 4\cos t\mathbf{j} + \mathbf{k}$

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Solution: C

To find the derivative, we'll differentiation with respect to t. We can differentiate each term individually.

$$r'(t) = \frac{1}{t^3}\left(3t^2\right)\mathbf{i} + 4\cos t\mathbf{j} + (1)\mathbf{k}$$

$$r'(t) = \frac{3t^2}{t^3}\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$$

$$r'(t) = \frac{3}{t}\mathbf{i} + 4\cos t\mathbf{j} + \mathbf{k}$$

This is the derivative of the vector function, given in the same form as the original function.

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Topic: Derivative of a vector function

Question: Find the derivative of the vector function.

$$r(t) = \left\langle te^{4t}, \cos(3t), 5t^4 \right\rangle$$

Answer choices:

A $\quad r'(t) = \left\langle e^{4t} + 4e^{4t}, 3\sin(3t), 20t^3 \right\rangle$

B $\quad r'(t) = \left\langle e^{4t} + 4te^{4t}, -3\sin(3t), 20t^3 \right\rangle$

C $\quad r'(t) = \left\langle e^{4t} + 4e^{4t}, -3\sin(3t), 20t^3 \right\rangle$

D $\quad r'(t) = \left\langle e^{4t} + 4te^{4t}, 3\sin(3t), 20t^3 \right\rangle$

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Solution: B

To find the derivative, we'll differentiation with respect to t. We can differentiate each term individually. We'll need to use product rule to take the derivative of te^{4t}.

$$r'(t) = \left\langle (1)\left(e^{4t}\right) + (t)\left(4e^{4t}\right), -(3)\sin(3t), 5(4)t^3 \right\rangle$$

$$r'(t) = \langle e^{4t} + 4te^{4t}, -3\sin(3t), 20t^3 \rangle$$

This is the derivative of the vector function, given in the same form as the original function.

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Topic: Unit tangent vector

Question: Find the unit tangent vector.

$$r(t) = 3t^2\mathbf{i} - 4\mathbf{j} - t^3\mathbf{k}$$

at $t = 2$

Answer choices:

A $\quad T(2) = \frac{6}{\sqrt{6}}\mathbf{i} - \frac{6}{\sqrt{6}}\mathbf{k}$

B $\quad T(2) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}$

C $\quad T(2) = \frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{k}$

D $\quad T(2) = \frac{6}{\sqrt{6}}\mathbf{i} + \frac{6}{\sqrt{6}}\mathbf{k}$

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Solution: B

First find the derivative of the vector function with respect to t.

$$r'(t) = 6t\mathbf{i} - 0\mathbf{j} - 3t^2\mathbf{k}$$

$$r'(t) = 6t\mathbf{i} - 3t^2\mathbf{k}$$

Now we'll plug $t = 2$ into the derivative.

$$r'(2) = 6(2)\mathbf{i} - 3(2)^2\mathbf{k}$$

$$r'(2) = 12\mathbf{i} - 12\mathbf{k}$$

Next we'll find the magnitude of the derivate at $t = 2$.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left| r'(2) \right| = \sqrt{\left[r'(2)_1\right]^2 + \left[r'(2)_2\right]^2 + \left[r'(2)_3\right]^2}$$

$$\left| r'(2) \right| = \sqrt{(12)^2 + (0)^2 + (-12)^2}$$

$$\left| r'(2) \right| = \sqrt{144 + 144}$$

$$\left| r'(2) \right| = \sqrt{288}$$

$$\left| r'(2) \right| = 12\sqrt{2}$$

Now we can use everything we just found to find the unit tangent vector at $t = 2$.

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$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(2) = \frac{r'(2)}{\left| r'(2) \right|}$$

$$T(2) = \frac{12\mathbf{i} - 12\mathbf{k}}{12\sqrt{2}}$$

$$T(2) = \frac{\mathbf{i} - \mathbf{k}}{\sqrt{2}}$$

$$T(2) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{k}$$

This is the unit tangent vector at $t = 2$.

Topic: Unit tangent vector

Question: Find the unit tangent vector.

$$r(t) = -t^2\mathbf{i} + t\mathbf{j} + 2\ln(3t)\mathbf{k}$$

at $t = 1$

Answer choices:

A $\quad T(1) = \frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$

B $\quad T(1) = \frac{2}{3}\mathbf{i} - \frac{1}{3}\mathbf{j} - \frac{2}{3}\mathbf{k}$

C $\quad T(1) = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$

D $\quad T(1) = -\frac{2}{3}\mathbf{i} + \frac{1}{3}\mathbf{j} + \frac{2}{3}\mathbf{k}$

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Solution: D

First find the derivative of the vector function with respect to t.

$$r'(t) = -2t\mathbf{i} + (1)\mathbf{j} + 2\left(\frac{1}{3t}\right)(3)\mathbf{k}$$

$$r'(t) = -2t\mathbf{i} + \mathbf{j} + \frac{2}{t}\mathbf{k}$$

Now we'll plug $t = 1$ into the derivative.

$$r'(1) = -2(1)\mathbf{i} + \mathbf{j} + \frac{2}{(1)}\mathbf{k}$$

$$r'(1) = -2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Next we'll find the magnitude of the derivate at $t = 1$.

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left|r'(1)\right| = \sqrt{\left[r'(1)_1\right]^2 + \left[r'(1)_2\right]^2 + \left[r'(1)_3\right]^2}$$

$$\left|r'(1)\right| = \sqrt{(-2)^2 + (1)^2 + (2)^2}$$

$$\left|r'(1)\right| = \sqrt{4 + 1 + 4}$$

$$\left|r'(1)\right| = \sqrt{9}$$

$$\left|r'(1)\right| = \sqrt{3}$$

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Now we can use everything we just found to find the unit tangent vector at $t = 1$.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(1) = \frac{r'(1)}{\left| r'(1) \right|}$$

$$T(1) = \frac{-2\mathbf{i} + \mathbf{j} + 2\mathbf{k}}{\sqrt{3}}$$

$$T(1) = -\frac{2}{\sqrt{3}}\mathbf{i} + \frac{1}{\sqrt{3}}\mathbf{j} + \frac{2}{\sqrt{3}}\mathbf{k}$$

This is the unit tangent vector at $t = 1$.

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Topic: Unit tangent vector

Question: Find the unit tangent vector.

$$r(t) = 2\sin(3t)\mathbf{i} - \cos(4t)\mathbf{j} + 4t\mathbf{k}$$

at $t = 0$

Answer choices:

A $\quad T(0) = -\frac{6}{\sqrt{10}}\mathbf{i} - \frac{4}{\sqrt{10}}\mathbf{k}$

B $\quad T(0) = \frac{6}{\sqrt{10}}\mathbf{i} + \frac{4}{\sqrt{10}}\mathbf{k}$

C $\quad T(0) = \frac{3}{\sqrt{13}}\mathbf{i} + \frac{2}{\sqrt{13}}\mathbf{k}$

D $\quad T(0) = -\frac{3}{\sqrt{13}}\mathbf{i} - \frac{2}{\sqrt{13}}\mathbf{k}$

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Solution: C

First find the derivative of the vector function with respect to t.

$$r'(t) = 2\cos(3t)(3)\mathbf{i} + \sin(4t)(4)\mathbf{j} + 4\mathbf{k}$$

$$r'(t) = 6\cos(3t)\mathbf{i} + 4\sin(4t)\mathbf{j} + 4\mathbf{k}$$

Now we'll plug $t = 0$ into the derivative.

$$r'(0) = 6\cos(3(0))\mathbf{i} + 4\sin(4(0))\mathbf{j} + 4\mathbf{k}$$

$$r'(0) = 6(1)\mathbf{i} + 4(0)\mathbf{j} + 4\mathbf{k}$$

$$r'(0) = 6\mathbf{i} + 4\mathbf{k}$$

Next we'll find the magnitude of the derivate at $t = 0$.

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left|r'(0)\right| = \sqrt{\left[r'(0)_1\right]^2 + \left[r'(0)_2\right]^2 + \left[r'(0)_3\right]^2}$$

$$\left|r'(0)\right| = \sqrt{(6)^2 + (0)^2 + (4)^2}$$

$$\left|r'(0)\right| = \sqrt{36 + 0 + 16}$$

$$\left|r'(0)\right| = \sqrt{52}$$

$$\left|r'(0)\right| = 2\sqrt{13}$$

Now we can use everything we just found to find the unit tangent vector at $t = 0$.

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$$T(t)=\frac{r'(t)}{\left|r'(t)\right|}$$

$$T(0)=\frac{r'(0)}{\left|r'(0)\right|}$$

$$T(0)=\frac{6\mathbf{i}+4\mathbf{k}}{2\sqrt{13}}$$

$$T(0)=\frac{6}{2\sqrt{13}}\mathbf{i}+\frac{4}{2\sqrt{13}}\mathbf{k}$$

$$T(0)=\frac{3}{\sqrt{13}}\mathbf{i}+\frac{2}{\sqrt{13}}\mathbf{k}$$

This is the unit tangent vector at $t=0$.

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Topic: Parametric equations of the tangent line

Question: Find the parametric equations of the tangent line.

$$x = 3t^3, \ y = t^2, \ z = 2t$$

at $P(3,1,2)$

Answer choices:

A	$x = 3 + 9t$	$y = 1 + 2t$	$z = 2 + 2t$
B	$x = 3 - 9t$	$y = 1 - 2t$	$z = 2 - 2t$
C	$x = 9 + 3t$	$y = 2 + t$	$z = 2 + 2t$
D	$x = 9 - 3t$	$y = 2 - t$	$z = 2 - 2t$

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Solution: A

The first thing we need to do is find the value of the parameter t that corresponds with the given point $P(3,1,2)$. We'll plug the value of x from the point into the parametric equation for x.

$$x = 3t^3$$

$$3 = 3t^3$$

$$1 = t^3$$

$$t = 1$$

If we try $t = 1$ in $y = t^2$ and $z = 2t$, along with $y = 1$ and $z = 2$ from the given point $P(3,1,2)$, we get

$$y = t^2$$

$$1 = (1)^2$$

$$1 = 1$$

and

$$z = 2t$$

$$2 = 2(1)$$

$$2 = 2$$

Because all of these equations are true, we can conclude that the value of the parameter is $t = 1$.

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Now we'll convert the given parametric equations into vector form. So $x = 3t^3$, $y = t^2$, and $z = 2t$ become

$$r(t) = \left\langle 3t^3, t^2, 2t \right\rangle$$

Take the derivative of this vector.

$$r'(t) = \left\langle 9t^2, 2t, 2 \right\rangle$$

Plug the parameter $t = 1$ into this derivative vector.

$$r'(1) = \left\langle 9(1)^2, 2(1), 2 \right\rangle$$

$$r'(1) = \langle 9,2,2 \rangle$$

Now we can use everything we just found to get the parametric equations of the tangent line. x_1, y_1 and z_1 will come from $P(3,1,2)$, and $r'(t)_1$, $r'(t)_2$ and $r'(t)_3$ will come from $r'(1) = \langle 9,2,2 \rangle$. We get

$$x = x_1 + r'(t)_1 t$$

$$x = 3 + 9t$$

and

$$y = y_1 + r'(t)_2 t$$

$$y = 1 + 2t$$

and

$$z = z_1 + r'(t)_3 t$$

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$$z = 2 + 2t$$

These are the parametric equations of the tangent line.

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Topic: Parametric equations of the tangent line

Question: Find the parametric equations of the tangent line.

$$x = e^t,\ y = 4t^2,\ z = 2e^t \sin(2t)$$

at $P(1,0,0)$

Answer choices:

A	$x = 1 - t$	$y = 0$	$z = 4$
B	$x = 1 + t$	$y = 0$	$z = 4$
C	$x = 1 + t$	$y = 0$	$z = 4t$
D	$x = 1 - t$	$y = 0$	$z = -4t$

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Solution: C

The first thing we need to do is find the value of the parameter t that corresponds with the given point $P(1,0,0)$. We'll plug the value of x from the point into the parametric equation for x.

$$x = e^t$$

$$1 = e^t$$

$$\ln 1 = \ln(e^t)$$

$$t = 0$$

If we try $t = 0$ in $y = 4t^2$ and $z = 2e^t \sin(2t)$, along with $y = 0$ and $z = 0$ from the given point $P(1,0,0)$, we get

$$y = 4t^2$$

$$0 = 4(0)^2$$

$$0 = 0$$

and

$$z = 2e^t \sin(2t)$$

$$0 = 2e^{(0)} \sin(2(0))$$

$$0 = 2(1)(0)$$

$$0 = 0$$

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Because all of these equations are true, we can conclude that the value of the parameter is $t = 0$.

Now we'll convert the given parametric equations into vector form. So $x = e^t$, $y = 4t^2$, and $z = 2e^t \sin(2t)$ become

$$r(t) = \left\langle e^t, 4t^2, 2e^t \sin(2t) \right\rangle$$

Take the derivative of this vector, using product rule to find the derivative of $2e^t \sin(2t)$.

$$r'(t) = \left\langle e^t, 8t, 2e^t \sin(2t) + 2e^t \cos(2t)(2) \right\rangle$$

$$r'(t) = \left\langle e^t, 8t, 2e^t \sin(2t) + 4e^t \cos(2t) \right\rangle$$

Plug the parameter $t = 0$ into this derivative vector.

$$r'(0) = \left\langle e^{(0)}, 8(0), 2e^{(0)} \sin(2(0)) + 4e^{(0)} \cos(2(0)) \right\rangle$$

$$r'(0) = \langle 1, 0, 2(1)(0) + 4(1)(1) \rangle$$

$$r'(0) = \langle 1, 0, 4 \rangle$$

Now we can use everything we just found to get the parametric equations of the tangent line. x_1, y_1 and z_1 will come from $P(1,0,0)$, and $r'(t)_1$, $r'(t)_2$ and $r'(t)_3$ will come from $r'(0) = \langle 1,0,4 \rangle$. We get

$$x = x_1 + r'(t)_1 t$$

$$x = 1 + 1t$$

$$x = 1 + t$$

and

$$y = y_1 + r'(t)_2 t$$

$$y = 0 + 0t$$

$$y = 0$$

and

$$z = z_1 + r'(t)_3 t$$

$$z = 0 + 4t$$

$$z = 4t$$

These are the parametric equations of the tangent line.

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Topic: Parametric equations of the tangent line

Question: Find the parametric equations of the tangent line.

$$x = e^{-t}, \ y = e^{-t}\cos t, \ z = 4e^{-t}$$

at $P(1,1,4)$

Answer choices:

A	$x = 1 + t$	$y = 1 + t$	$z = 1 + t$
B	$x = 1 - t$	$y = 1 - t$	$z = 4 - 4t$
C	$x = 1 + t$	$y = 1 + t$	$z = 4 + 4t$
D	$x = 1 - t$	$y = 1 - t$	$z = 1 - t$

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Solution: B

The first thing we need to do is find the value of the parameter t that corresponds with the given point $P(1,1,4)$. We'll plug the value of x from the point into the parametric equation for x.

$$x = e^{-t}$$

$$1 = e^{-t}$$

$$\ln 1 = \ln\left(e^{-t}\right)$$

$$0 = -t$$

$$t = 0$$

If we try $t = 0$ in $y = e^{-t}\cos t$ and $z = 4e^{-t}$, along with $y = 1$ and $z = 4$ from the given point $P(1,1,4)$, we get

$$y = e^{-t}\cos t$$

$$1 = e^{-(0)}\cos(0)$$

$$1 = (1)(1)$$

$$1 = 1$$

and

$$z = 4e^{-t}$$

$$4 = 4e^{-(0)}$$

$$4 = 4(1)$$

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$$4 = 4$$

Because all of these equations are true, we can conclude that the value of the parameter is $t = 0$.

Now we'll convert the given parametric equations into vector form. So $x = e^{-t}$, $y = e^{-t}\cos t$, and $z = 4e^{-t}$ become

$$r(t) = \left\langle e^{-t}, e^{-t}\cos t, 4e^{-t} \right\rangle$$

Take the derivative of this vector, using product rule to find the derivative of $e^{-t}\cos t$.

$$r'(t) = \langle -e^{-t}, -e^{-t}\cos t + e^{-t}(-\sin t), -4e^{-t} \rangle$$

$$r'(t) = \langle -e^{-t}, -e^{-t}\cos t - e^{-t}\sin t, -4e^{-t} \rangle$$

Plug the parameter $t = 0$ into this derivative vector.

$$r'(0) = \left\langle -e^{-(0)}, -e^{-(0)}\cos(0) - e^{-(0)}\sin(0), -4e^{-(0)} \right\rangle$$

$$r'(0) = \langle -1, -(1)(1) - (1)(0), -4(1) \rangle$$

$$r'(0) = \langle -1, -1, -4 \rangle$$

Now we can use everything we just found to get the parametric equations of the tangent line. x_1, y_1 and z_1 will come from $P(1,1,4)$, and $r'(t)_1$, $r'(t)_2$ and $r'(t)_3$ will come from $r'(0) = \langle -1, -1, -4 \rangle$. We get

$$x = x_1 + r'(t)_1 t$$

$$x = 1 - 1t$$

$$x = 1 - t$$

and

$$y = y_1 + r'(t)_2 t$$

$$y = 1 - 1t$$

$$y = 1 - t$$

and

$$z = z_1 + r'(t)_3 t$$

$$z = 4 - 4t$$

These are the parametric equations of the tangent line.

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Topic: Integral of a vector function

Question: Find the integral of the vector function.

$$\int_0^{\pi} 7t^2\mathbf{i} - e^{2t}\mathbf{j} + \sin(3t)\mathbf{k}\ dt$$

Answer choices:

A $\quad \dfrac{7\pi^3}{3}\mathbf{i} - \left(\dfrac{1}{2}e^{2\pi} - \dfrac{1}{2}\right)\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

B $\quad \dfrac{7\pi^3}{3}\mathbf{i} - \left(e^{2\pi} - 1\right)\mathbf{j} + \dfrac{2}{3}\mathbf{k}$

C $\quad \dfrac{7\pi^3}{3}\mathbf{i} - \left(\dfrac{1}{2}e^{2\pi} - \dfrac{1}{2}\right)\mathbf{j}$

D $\quad \dfrac{7\pi^3}{3}\mathbf{i} - \left(e^{2\pi} - 1\right)\mathbf{j}$

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Solution: A

First we'll rewrite the integral by splitting apart the terms.

$$\int_0^{\pi} 7t^2\mathbf{i} - e^{2t}\mathbf{j} + \sin(3t)\mathbf{k}\ dt$$

$$\int_0^{\pi} 7t^2\ dt\mathbf{i} - \int_0^{\pi} e^{2t}\ dt\mathbf{j} + \int_0^{\pi} \sin(3t)\ dt\mathbf{k}$$

Integrate and then evaluate over the interval.

$$\frac{7}{3}t^3\Big|_0^{\pi}\mathbf{i} - \frac{1}{2}e^{2t}\Big|_0^{\pi}\mathbf{j} - \frac{1}{3}\cos(3t)\Big|_0^{\pi}\mathbf{k}$$

$$\left[\frac{7}{3}(\pi)^3 - \frac{7}{3}(0)^3\right]\mathbf{i} - \left[\frac{1}{2}e^{2(\pi)} - \frac{1}{2}e^{2(0)}\right]\mathbf{j} - \left[\frac{1}{3}\cos(3(\pi)) - \frac{1}{3}\cos(3(0))\right]\mathbf{k}$$

$$\frac{7\pi^3}{3}\mathbf{i} - \left[\frac{1}{2}e^{2\pi} - \frac{1}{2}(1)\right]\mathbf{j} - \left[\frac{1}{3}(-1) - \frac{1}{3}(1)\right]\mathbf{k}$$

$$\frac{7\pi^3}{3}\mathbf{i} - \left(\frac{1}{2}e^{2\pi} - \frac{1}{2}\right)\mathbf{j} + \frac{2}{3}\mathbf{k}$$

This is the integral of the vector function.

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Topic: Integral of a vector function

Question: Find the integral of the vector function.

$$\int_0^{\frac{\pi}{2}} \frac{1}{t+3}\mathbf{i} + \sin t \cos t\mathbf{j} + t^3\mathbf{k}\ dt$$

Answer choices:

A $\ln\left(\frac{6+\pi}{2}\right)\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi^4}{16}\mathbf{k}$

B $\ln\left(\frac{6+\pi}{6}\right)\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi^4}{16}\mathbf{k}$

C $\ln\left(\frac{6+\pi}{2}\right)\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi^4}{64}\mathbf{k}$

D $\ln\left(\frac{6+\pi}{6}\right)\mathbf{i} + \frac{1}{2}\mathbf{j} + \frac{\pi^4}{64}\mathbf{k}$

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Solution: D

First we'll rewrite the integral by splitting apart the terms.

$$\int_0^{\frac{\pi}{2}} \frac{1}{t+3}\mathbf{i} + \sin t \cos t\mathbf{j} + t^3\mathbf{k}\ dt$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{t+3}\ dt\mathbf{i} + \int_0^{\frac{\pi}{2}} \sin t \cos t\ dt\mathbf{j} + \int_0^{\frac{\pi}{2}} t^3\ dt\mathbf{k}$$

Integrate, using u-substitution to find the integral of $\sin t \cos t$.

$$\ln|t+3|\Big|_0^{\frac{\pi}{2}}\mathbf{i} + \int_0^{\frac{\pi}{2}} \sin t \cos t\ dt\mathbf{j} + \frac{1}{4}t^4\Big|_0^{\frac{\pi}{2}}\mathbf{k}$$

$u = \sin t$ and $\frac{du}{dt} = \cos t$, so $du = \cos t\ dt$, or $dt = \frac{du}{\cos t}$

$$\ln|t+3|\Big|_0^{\frac{\pi}{2}}\mathbf{i} + \int_{t=0}^{t=\frac{\pi}{2}} u\cos t\left(\frac{du}{\cos t}\right)\mathbf{j} + \frac{1}{4}t^4\Big|_0^{\frac{\pi}{2}}\mathbf{k}$$

$$\ln|t+3|\Big|_0^{\frac{\pi}{2}}\mathbf{i} + \int_{t=0}^{t=\frac{\pi}{2}} u\ du\mathbf{j} + \frac{1}{4}t^4\Big|_0^{\frac{\pi}{2}}\mathbf{k}$$

$$\ln|t+3|\Big|_0^{\frac{\pi}{2}}\mathbf{i} + \frac{1}{2}u^2\Big|_{t=0}^{t=\frac{\pi}{2}}\mathbf{j} + \frac{1}{4}t^4\Big|_0^{\frac{\pi}{2}}\mathbf{k}$$

Back-substitute and evaluate over the interval.

$$\ln|t+3|\Big|_0^{\frac{\pi}{2}}\mathbf{i} + \frac{1}{2}\sin^2 t\Big|_0^{\frac{\pi}{2}}\mathbf{j} + \frac{1}{4}t^4\Big|_0^{\frac{\pi}{2}}\mathbf{k}$$

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$$\left[\ln\left|\frac{\pi}{2}+3\right|-\ln|0+3|\right]\mathbf{i}+\left[\frac{1}{2}\sin^2\left(\frac{\pi}{2}\right)-\frac{1}{2}\sin^2(0)\right]\mathbf{j}+\left[\frac{1}{4}\left(\frac{\pi}{2}\right)^4-\frac{1}{4}(0)^4\right]\mathbf{k}$$

$$\left[\ln\left(\frac{\pi}{2}+\frac{6}{2}\right)-\ln 3\right]\mathbf{i}+\left[\frac{1}{2}(1)-\frac{1}{2}(0)\right]\mathbf{j}+\frac{\pi^4}{64}\mathbf{k}$$

$$\left[\ln\left(\frac{6+\pi}{2}\cdot\frac{1}{3}\right)\right]\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{\pi^4}{64}\mathbf{k}$$

$$\ln\left(\frac{6+\pi}{6}\right)\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{\pi^4}{64}\mathbf{k}$$

This is the integral of the vector function.

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Topic: Integral of a vector function

Question: Find the integral of the vector function.

$$\int_0^{\frac{\pi}{4}} \sec^2 t\,\mathbf{i} + \frac{1}{t+5}\mathbf{j} + \sin t\cos^2 t\mathbf{k}\ dt$$

Answer choices:

A $\quad \mathbf{i} + \ln\left(\frac{20+\pi}{4}\right)\mathbf{j} + \frac{\sqrt{2}-4}{12}\mathbf{k}$

B $\quad \mathbf{i} + \ln\left(\frac{20+\pi}{20}\right)\mathbf{j} + \frac{\sqrt{2}}{12}\mathbf{k}$

C $\quad \mathbf{i} + \ln\left(\frac{20+\pi}{20}\right)\mathbf{j} - \frac{\sqrt{2}-4}{12}\mathbf{k}$

D $\quad \mathbf{i} + \ln\left(\frac{20+\pi}{4}\right)\mathbf{j} + \frac{\sqrt{2}}{12}\mathbf{k}$

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Solution: C

First we'll rewrite the integral by splitting apart the terms.

$$\int_0^{\frac{\pi}{4}} \sec^2 t\mathbf{i} + \frac{1}{t+5}\mathbf{j} + \sin t \cos^2 t\mathbf{k}\ dt$$

$$\int_0^{\frac{\pi}{4}} \sec^2 t\ dt\mathbf{i} + \int_0^{\frac{\pi}{4}} \frac{1}{t+5}\ dt\mathbf{j} + \int_0^{\frac{\pi}{4}} \sin t \cos^2 t\ dt\mathbf{k}$$

Integrate, using u-substitution to find the integral of $\sin t \cos^2 t$.

$$\tan t \Big|_0^{\frac{\pi}{4}} \mathbf{i} + \ln|t+5| \Big|_0^{\frac{\pi}{4}} \mathbf{j} + \int_0^{\frac{\pi}{4}} \sin t \cos^2 t\ dt\mathbf{k}$$

$u = \cos t$ and $\dfrac{du}{dt} = -\sin t$, so $du = -\sin t\ dt$, or $dt = -\dfrac{du}{\sin t}$

$$\tan t \Big|_0^{\frac{\pi}{4}} \mathbf{i} + \ln|t+5| \Big|_0^{\frac{\pi}{4}} \mathbf{j} + \int_{t=0}^{t=\frac{\pi}{4}} \sin t u^2 \left(-\frac{du}{\sin t}\right) \mathbf{k}$$

$$\tan t \Big|_0^{\frac{\pi}{4}} \mathbf{i} + \ln|t+5| \Big|_0^{\frac{\pi}{4}} \mathbf{j} - \int_{t=0}^{t=\frac{\pi}{4}} u^2\ du\mathbf{k}$$

$$\tan t \Big|_0^{\frac{\pi}{4}} \mathbf{i} + \ln|t+5| \Big|_0^{\frac{\pi}{4}} \mathbf{j} - \frac{1}{3}u^3 \Big|_{t=0}^{t=\frac{\pi}{4}} \mathbf{k}$$

Back-substitute and evaluate over the interval.

$$\tan t \Big|_0^{\frac{\pi}{4}} \mathbf{i} + \ln|t+5| \Big|_0^{\frac{\pi}{4}} \mathbf{j} - \frac{1}{3}\cos^3 t \Big|_0^{\frac{\pi}{4}} \mathbf{k}$$

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$$\left(\tan\frac{\pi}{4}-\tan 0\right)\mathbf{i}+\left(\ln\left|\frac{\pi}{4}+5\right|-\ln|0+5|\right)\mathbf{j}-\left(\frac{1}{3}\cos^3\frac{\pi}{4}-\frac{1}{3}\cos^3 0\right)\mathbf{k}$$

$$(1-0)\mathbf{i}+\left[\ln\left(\frac{\pi}{4}+\frac{20}{4}\right)-\ln 5\right]\mathbf{j}-\left[\frac{1}{3}\left(\frac{\sqrt{2}}{2}\right)^3-\frac{1}{3}(1)\right]\mathbf{k}$$

$$\mathbf{i}+\left[\ln\left(\frac{20+\pi}{4}\right)-\ln 5\right]\mathbf{j}-\left[\frac{1}{3}\left(\frac{2\sqrt{2}}{8}\right)-\frac{1}{3}\right]\mathbf{k}$$

$$\mathbf{i}+\left[\ln\left(\frac{20+\pi}{4}\cdot\frac{1}{5}\right)\right]\mathbf{j}-\left(\frac{\sqrt{2}}{12}-\frac{4}{12}\right)\mathbf{k}$$

$$\mathbf{i}+\ln\left(\frac{20+\pi}{20}\right)\mathbf{j}-\frac{\sqrt{2}-4}{12}\mathbf{k}$$

This is the integral of the vector function.

Topic: Arc length of a vector function

Question: Find the arc length of the vector function.

$$r(t) = \frac{1}{3}t^3\mathbf{i} + \frac{4}{5}t^{\frac{5}{2}}\mathbf{j} + t^2\mathbf{k}$$

when $0 \le t \le 1$

Answer choices:

A $\frac{1}{4}$

B $\frac{4}{3}$

C $\frac{1}{3}$

D $\frac{3}{4}$

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Solution: B

First we'll turn the vector equation into parametric equations.

$r(t) = \frac{1}{3}t^3\mathbf{i} + \frac{4}{5}t^{\frac{5}{2}}\mathbf{j} + t^2\mathbf{k}$ becomes

$$x = \frac{1}{3}t^3$$

$$y = \frac{4}{5}t^{\frac{5}{2}}$$

$$z = t^2$$

Then we'll take the derivative of these.

$$\frac{dx}{dt} = t^2$$

$$\frac{dy}{dt} = 2t^{\frac{3}{2}}$$

$$\frac{dz}{dt} = 2t$$

Our limits of integration are given by $0 \le t \le 1$, so we can plug all of this into the arc length formula and integrate.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_0^1 \sqrt{(t^2)^2 + \left(2t^{\frac{3}{2}}\right)^2 + (2t)^2}\ dt$$

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$$L=\int_0^1 \sqrt{t^4+4t^3+4t^2}\ dt$$

$$L=\int_0^1 \sqrt{t^2\left(t^2+4t+4\right)}\ dt$$

$$L=\int_0^1 t\sqrt{(t+2)^2}\ dt$$

$$L=\int_0^1 t(t+2)\ dt$$

$$L=\int_0^1 t^2+2t\ dt$$

$$L=\frac{1}{3}t^3+t^2\Big|_0^1$$

Evaluate over the interval.

$$L=\left[\frac{1}{3}(1)^3+(1)^2\right]-\left[\frac{1}{3}(0)^3+(0)^2\right]$$

$$L=\frac{1}{3}+1$$

$$L=\frac{4}{3}$$

This is the arc length of the vector function.

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Topic: Arc length of a vector function

Question: Find the arc length of the vector function.

$$r(t) = \sin t\mathbf{i} + 5t\mathbf{j} + \cos t\mathbf{k}$$

when $0 \leq t \leq 1$

Answer choices:

A $\sqrt{26}$

B $5\sqrt{13}$

C 5

D $2\sqrt{13}$

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Solution: A

First we'll turn the vector equation into parametric equations.

$r(t) = \sin t\mathbf{i} + 5t\mathbf{j} + \cos t\mathbf{k}$ becomes

$$x = \sin t$$

$$y = 5t$$

$$z = \cos t$$

Then we'll take the derivative of these.

$$\frac{dx}{dt} = \cos t$$

$$\frac{dy}{dt} = 5$$

$$\frac{dz}{dt} = -\sin t$$

Our limits of integration are given by $0 \le t \le 1$, so we can plug all of this into the arc length formula and integrate.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_0^1 \sqrt{(\cos t)^2 + (5)^2 + (-\sin t)^2}\ dt$$

$$L = \int_0^1 \sqrt{\cos^2 t + 25 + \sin^2 t}\ dt$$

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$$L = \int_0^1 \sqrt{25 + \left(\sin^2 t + \cos^2 t\right)} \ dt$$

$$L = \int_0^1 \sqrt{25 + 1} \ dt$$

$$L = \int_0^1 \sqrt{26} \ dt$$

$$L = \sqrt{26}t \Big|_0^1$$

Evaluate over the interval.

$$L = \sqrt{26}(1) - \sqrt{26}(0)$$

$$L = \sqrt{26}$$

This is the arc length of the vector function.

Topic: Arc length of a vector function

Question: Find the arc length of the vector function.

$$r(t) = 3\cos t\mathbf{i} + 4t\mathbf{j} + 3\sin t\mathbf{k}$$

when $0 \le t \le 1$

Answer choices:

A $4\sqrt{3}$

B $\sqrt{15}$

C $\sqrt{17}$

D 5

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Solution: D

First we'll turn the vector equation into parametric equations.

$r(t) = 3\cos t\mathbf{i} + 4t\mathbf{j} + 3\sin t\mathbf{k}$ becomes

$$x = 3\cos t$$

$$y = 4t$$

$$z = 3\sin t$$

Then we'll take the derivative of these.

$$\frac{dx}{dt} = -3\sin t$$

$$\frac{dy}{dt} = 4$$

$$\frac{dz}{dt} = 3\cos t$$

Our limits of integration are given by $0 \le t \le 1$, so we can plug all of this into the arc length formula and integrate.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_0^1 \sqrt{\left(-3\sin t\right)^2 + (4)^2 + (3\cos t)^2}\ dt$$

$$L = \int_0^1 \sqrt{9\sin^2 t + 16 + 9\cos^2 t}\ dt$$

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$$L=\int_0^1 \sqrt{16+9\left(\sin^2 t+\cos^2 t\right)}\ dt$$

$$L=\int_0^1 \sqrt{16+9(1)}\ dt$$

$$L=\int_0^1 \sqrt{25}\ dt$$

$$L=\int_0^1 5\ dt$$

$$L=5t\Big|_0^1$$

Evaluate over the interval.

$$L=5(1)-5(0)$$

$$L=5$$

This is the arc length of the vector function.

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Topic: Reparametrizing the curve

Question: Reparametrize the curve of the vector function from $t = 0$ in the direction of increasing t.

$$r(t) = 3t\mathbf{i} - t\mathbf{j} + (3 + t)\mathbf{k}$$

Answer choices:

A $\quad r(t(s)) = \frac{3}{\sqrt{5}}s\mathbf{i} - \frac{1}{\sqrt{5}}s\mathbf{j} + \left(3 + \frac{1}{\sqrt{5}}s\right)\mathbf{k}$

B $\quad r(t(s)) = -\frac{3}{\sqrt{5}}s\mathbf{i} + \frac{1}{\sqrt{5}}s\mathbf{j} - \left(3 + \frac{1}{\sqrt{5}}s\right)\mathbf{k}$

C $\quad r(t(s)) = \frac{3}{\sqrt{11}}s\mathbf{i} - \frac{1}{\sqrt{11}}s\mathbf{j} + \left(3 + \frac{1}{\sqrt{11}}s\right)\mathbf{k}$

D $\quad r(t(s)) = -\frac{3}{\sqrt{11}}s\mathbf{i} + \frac{1}{\sqrt{11}}s\mathbf{j} - \left(3 + \frac{1}{\sqrt{11}}s\right)\mathbf{k}$

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Solution: C

First we'll turn the vector equation into parametric equations.

$r(t) = 3t\mathbf{i} - t\mathbf{j} + (3 + t)\mathbf{k}$ becomes

$$x = 3t$$

$$y = -t$$

$$z = 3 + t$$

Then we'll take the derivative of these.

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = -1$$

$$\frac{dz}{dt} = 1$$

Since we're told that we'll start at $t = 0$ and move in the direction of increasing t, the limits of integration are given by $[0,t]$. Now we can plug everything we have into the arc length formula and integrate.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_0^t \sqrt{(3)^2 + (-1)^2 + (1)^2}\ dt$$

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$$L=\int_0^t \sqrt{9+1+1}\ dt$$

$$L=\int_0^t \sqrt{11}\ dt$$

$$L=\sqrt{11}t\Big|_0^t$$

Evaluate over the interval.

$$L=\sqrt{11}(t)-\sqrt{11}(0)$$

$$L=\sqrt{11}t$$

Now we can set $L=s$ and then solve for t.

$$s=\sqrt{11}t$$

$$t=\frac{1}{\sqrt{11}}s$$

Now we can finally reparametrize the curve by substituting this value for t into the vector function.

$$r(t(s))=3\left(\frac{1}{\sqrt{11}}s\right)\mathbf{i}-\left(\frac{1}{\sqrt{11}}s\right)\mathbf{j}+\left[3+\left(\frac{1}{\sqrt{11}}s\right)\right]\mathbf{k}$$

$$r(t(s))=\frac{3}{\sqrt{11}}s\mathbf{i}-\frac{1}{\sqrt{11}}s\mathbf{j}+\left(3+\frac{1}{\sqrt{11}}s\right)\mathbf{k}$$

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Topic: Reparametrizing the curve

Question: Reparametrize the curve of the vector function from $t=0$ in the direction of increasing t.

$$r(t)=(1-3t)\mathbf{i}+6t\mathbf{j}+(4-5t)\mathbf{k}$$

Answer choices:

A $$r(t(s))=\left(1+\frac{3}{\sqrt{70}}s\right)\mathbf{i}+\frac{6}{\sqrt{70}}s\mathbf{j}+\left(4+\frac{5}{\sqrt{70}}s\right)\mathbf{k}$$

B $$r(t(s))=\left(1-\frac{3}{\sqrt{70}}s\right)\mathbf{i}+\frac{6}{\sqrt{70}}s\mathbf{j}+\left(4-\frac{5}{\sqrt{70}}s\right)\mathbf{k}$$

C $$r(t(s))=\left(1+\frac{3}{\sqrt{14}}s\right)\mathbf{i}+\frac{6}{\sqrt{14}}s\mathbf{j}+\left(4+\frac{5}{\sqrt{14}}s\right)\mathbf{k}$$

D $$r(t(s))=\left(1-\frac{3}{\sqrt{14}}s\right)\mathbf{i}+\frac{6}{\sqrt{14}}s\mathbf{j}+\left(4-\frac{5}{\sqrt{14}}s\right)\mathbf{k}$$

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Solution: B

First we'll turn the vector equation into parametric equations.

$r(t) = (1 - 3t)\mathbf{i} + 6t\mathbf{j} + (4 - 5t)\mathbf{k}$ becomes

$$x = 1 - 3t$$

$$y = 6t$$

$$z = 4 - 5t$$

Then we'll take the derivative of these.

$$\frac{dx}{dt} = -3$$

$$\frac{dy}{dt} = 6$$

$$\frac{dz}{dt} = -5$$

Since we're told that we'll start at $t = 0$ and move in the direction of increasing t, the limits of integration are given by $[0,t]$. Now we can plug everything we have into the arc length formula and integrate.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \ dt$$

$$L = \int_0^t \sqrt{(-3)^2 + (6)^2 + (-5)^2} \ dt$$

$$L=\int_0^t \sqrt{9+36+25}\ dt$$

$$L=\int_0^t \sqrt{70}\ dt$$

$$L=\sqrt{70}t\Big|_0^t$$

Evaluate over the interval.

$$L=\sqrt{70}(t)-\sqrt{70}(0)$$

$$L=\sqrt{70}t$$

Now we can set $L=s$ and then solve for t.

$$s=\sqrt{70}t$$

$$t=\frac{1}{\sqrt{70}}s$$

Now we can finally reparametrize the curve by substituting this value for t into the vector function.

$$r(t(s))=\left[1-3\left(\frac{1}{\sqrt{70}}s\right)\right]\mathbf{i}+6\left(\frac{1}{\sqrt{70}}s\right)\mathbf{j}+\left[4-5\left(\frac{1}{\sqrt{70}}s\right)\right]\mathbf{k}$$

$$r(t(s))=\left(1-\frac{3}{\sqrt{70}}s\right)\mathbf{i}+\frac{6}{\sqrt{70}}s\mathbf{j}+\left(4-\frac{5}{\sqrt{70}}s\right)\mathbf{k}$$

Topic: Reparametrizing the curve

Question: Reparametrize the curve of the vector function from $t=0$ in the direction of decreasing t.

$$r(t)=t\mathbf{i}+(t-4)\mathbf{j}+7t\mathbf{k}$$

Answer choices:

A $\quad r(t(s))=-\frac{1}{\sqrt{9}}s\mathbf{i}-\left(\frac{1}{\sqrt{9}}s-4\right)\mathbf{j}-\frac{7}{\sqrt{9}}s\mathbf{k}$

B $\quad r(t(s))=\frac{1}{\sqrt{9}}s\mathbf{i}+\left(\frac{1}{\sqrt{9}}s-4\right)\mathbf{j}+\frac{7}{\sqrt{9}}s\mathbf{k}$

C $\quad r(t(s))=\frac{1}{\sqrt{51}}s\mathbf{i}+\left(\frac{1}{\sqrt{51}}s-4\right)\mathbf{j}+\frac{7}{\sqrt{51}}s\mathbf{k}$

D $\quad r(t(s))=-\frac{1}{\sqrt{51}}s\mathbf{i}-\left(\frac{1}{\sqrt{51}}s+4\right)\mathbf{j}-\frac{7}{\sqrt{51}}s\mathbf{k}$

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Solution: D

First we'll turn the vector equation into parametric equations.

$r(t) = t\mathbf{i} + (t-4)\mathbf{j} + 7t\mathbf{k}$ becomes

$$x = t$$

$$y = t - 4$$

$$z = 7t$$

Then we'll take the derivative of these.

$$\frac{dx}{dt} = 1$$

$$\frac{dy}{dt} = 1$$

$$\frac{dz}{dt} = 7$$

Since we're told that we'll start at $t = 0$ and move in the direction of decreasing t, the limits of integration are given by $[t,0]$. Now we can plug everything we have into the arc length formula and integrate.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_t^0 \sqrt{(1)^2 + (1)^2 + (7)^2}\ dt$$

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$$L=\int_t^0 \sqrt{1+1+49}\ dt$$

$$L=\int_t^0 \sqrt{51}\ dt$$

$$L=\sqrt{51}t\Big|_t^0$$

Evaluate over the interval.

$$L=\sqrt{51}(0)-\sqrt{51}(t)$$

$$L=-\sqrt{51}t$$

Now we can set $L=s$ and then solve for t.

$$s=-\sqrt{51}t$$

$$t=-\frac{1}{\sqrt{51}}s$$

Now we can finally reparametrize the curve by substituting this value for t into the vector function.

$$r(t(s))=\left(-\frac{1}{\sqrt{51}}s\right)\mathbf{i}+\left[\left(-\frac{1}{\sqrt{51}}s\right)-4\right]\mathbf{j}+7\left(-\frac{1}{\sqrt{51}}s\right)\mathbf{k}$$

$$r(t(s))=-\frac{1}{\sqrt{51}}s\mathbf{i}-\left(\frac{1}{\sqrt{51}}s+4\right)\mathbf{j}-\frac{7}{\sqrt{51}}s\mathbf{k}$$

Topic: Unit tangent and unit normal vectors

Question: Find the unit tangent vector.

$$r(t) = 2t\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$$

Answer choices:

A $\quad T(t) = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{\sin t}{\sqrt{5}}\mathbf{j} + \frac{\cos t}{\sqrt{5}}\mathbf{k}$

B $\quad T(t) = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{\cos t}{\sqrt{5}}\mathbf{j} - \frac{\sin t}{\sqrt{5}}\mathbf{k}$

C $\quad T(t) = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{\sin t}{\sqrt{5}}\mathbf{j} - \frac{\cos t}{\sqrt{5}}\mathbf{k}$

D $\quad T(t) = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{\sin t}{\sqrt{5}}\mathbf{k}$

Solution: B

We'll first find the derivative of the vector function.

$$r(t) = 2t\mathbf{i} - \sin t\mathbf{j} + \cos t\mathbf{k}$$

$$r'(t) = 2\mathbf{i} - \cos t\mathbf{j} - \sin t\mathbf{k}$$

Then we'll find the magnitude of the derivative.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left| r'(t) \right| = \sqrt{(2)^2 + (-\cos t)^2 + (-\sin t)^2}$$

$$\left| r'(t) \right| = \sqrt{4 + \cos^2 t + \sin^2 t}$$

$$\left| r'(t) \right| = \sqrt{4 + 1}$$

$$\left| r'(t) \right| = \sqrt{5}$$

Now we'll use everything we just found to solve for the unit tangent vector.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(t) = \frac{2\mathbf{i} - \cos t\mathbf{j} - \sin t\mathbf{k}}{\sqrt{5}}$$

$$T(t) = \frac{2}{\sqrt{5}}\mathbf{i} - \frac{\cos t}{\sqrt{5}}\mathbf{j} - \frac{\sin t}{\sqrt{5}}\mathbf{k}$$

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This is the unit tangent vector.

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Topic: Unit tangent and unit normal vectors

Question: Find the unit tangent and unit normal vectors.

$$r(t) = 5\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k}$$

Answer choices:

A $\quad T(t) = -\cos t\mathbf{j} + \sin t\mathbf{k} \qquad N(t) = \sin t\mathbf{j} + \cos t\mathbf{k}$

B $\quad T(t) = \sin t\mathbf{j} - \cos t\mathbf{k} \qquad N(t) = -\cos t\mathbf{j} - \sin t\mathbf{k}$

C $\quad T(t) = \cos t\mathbf{j} - \sin t\mathbf{k} \qquad N(t) = -\sin t\mathbf{j} - \cos t\mathbf{k}$

D $\quad T(t) = -\sin t\mathbf{j} + \cos t\mathbf{k} \qquad N(t) = \cos t\mathbf{j} + \sin t\mathbf{k}$

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Solution: C

We'll first find the derivative of the vector function.

$$r(t) = 5\mathbf{i} + 4\sin t\mathbf{j} + 4\cos t\mathbf{k}$$

$$r'(t) = 0\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}$$

Then we'll find the magnitude of the derivative.

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left|r'(t)\right| = \sqrt{(0)^2 + (4\cos t)^2 + (-4\sin t)^2}$$

$$\left|r'(t)\right| = \sqrt{16\cos^2 t + 16\sin^2 t}$$

$$\left|r'(t)\right| = \sqrt{16\left(\cos^2 t + \sin^2 t\right)}$$

$$\left|r'(t)\right| = \sqrt{16(1)}$$

$$\left|r'(t)\right| = 4$$

Now we'll use everything we just found to solve for the unit tangent vector.

$$T(t) = \frac{r'(t)}{\left|r'(t)\right|}$$

$$T(t) = \frac{0\mathbf{i} + 4\cos t\mathbf{j} - 4\sin t\mathbf{k}}{4}$$

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$$T(t) = \frac{0}{4}\mathbf{i} + \frac{4\cos t}{4}\mathbf{j} - \frac{4\sin t}{4}\mathbf{k}$$

$$T(t) = \cos t\mathbf{j} - \sin t\mathbf{k}$$

This is the unit tangent vector, and now we need to find the unit normal vector. We'll take the derivative of the unit tangent vector.

$$T'(t) = -\sin t\mathbf{j} - \cos t\mathbf{k}$$

Then we have to find the magnitude of this derivative.

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right| = \sqrt{(0)^2 + (-\sin t)^2 + (-\cos t)^2}$$

$$\left|T'(t)\right| = \sqrt{\sin^2 t + \cos^2 t}$$

$$\left|T'(t)\right| = \sqrt{1}$$

$$\left|T'(t)\right| = 1$$

Now we can use everything we just found to solve for the unit normal vector.

$$N(t) = \frac{T'(t)}{\left|T'(t)\right|}$$

$$N(t) = \frac{-\sin t\mathbf{j} - \cos t\mathbf{k}}{1}$$

$$N(t) = -\sin t\mathbf{j} - \cos t\mathbf{k}$$

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This is the unit normal vector.

Topic: Unit tangent and unit normal vectors

Question: Find the unit tangent and unit normal vectors.

$$r(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 4t\mathbf{k}$$

Answer choices:

A $\quad T(t) = \frac{3}{5}\cos t\mathbf{i} - \frac{3}{5}\sin t\mathbf{j} - \frac{4}{5}\mathbf{k} \qquad N(t) = \sin t\mathbf{i} + \cos t\mathbf{j}$

B $\quad T(t) = -\frac{3}{5}\cos t\mathbf{i} + \frac{3}{5}\sin t\mathbf{j} + \frac{4}{5}\mathbf{k} \qquad N(t) = -\sin t\mathbf{i} - \cos t\mathbf{j}$

C $\quad T(t) = \frac{3}{5}\sin t\mathbf{i} - \frac{3}{5}\cos t\mathbf{j} - \frac{4}{5}\mathbf{k} \qquad N(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$

D $\quad T(t) = -\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j} + \frac{4}{5}\mathbf{k} \qquad N(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$

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Solution: D

We'll first find the derivative of the vector function.

$$r(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + 4t\mathbf{k}$$

$$r'(t) = -3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 4\mathbf{k}$$

Then we'll find the magnitude of the derivative.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left| r'(t) \right| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + (4)^2}$$

$$\left| r'(t) \right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16}$$

$$\left| r'(t) \right| = \sqrt{9\left(\sin^2 t + \cos^2 t\right) + 16}$$

$$\left| r'(t) \right| = \sqrt{9(1) + 16}$$

$$\left| r'(t) \right| = \sqrt{25}$$

$$\left| r'(t) \right| = 5$$

Now we'll use everything we just found to solve for the unit tangent vector.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(t) = \frac{-3\sin t\mathbf{i} + 3\cos t\mathbf{j} + 4\mathbf{k}}{5}$$

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$$T(t) = -\frac{3\sin t}{5}\mathbf{i} + \frac{3\cos t}{5}\mathbf{j} + \frac{4}{5}\mathbf{k}$$

$$T(t) = -\frac{3}{5}\sin t\mathbf{i} + \frac{3}{5}\cos t\mathbf{j} + \frac{4}{5}\mathbf{k}$$

This is the unit tangent vector, and now we need to find the unit normal vector. We'll take the derivative of the unit tangent vector.

$$T'(t) = -\frac{3}{5}\cos t\mathbf{i} - \frac{3}{5}\sin t\mathbf{j} + 0\mathbf{k}$$

Then we have to find the magnitude of this derivative.

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right| = \sqrt{\left(-\frac{3}{5}\cos t\right)^2 + \left(-\frac{3}{5}\sin t\right)^2 + (0)^2}$$

$$\left|T'(t)\right| = \sqrt{\frac{9}{25}\cos^2 t + \frac{9}{25}\sin^2 t}$$

$$\left|T'(t)\right| = \sqrt{\frac{9}{25}\left(\cos^2 t + \sin^2 t\right)}$$

$$\left|T'(t)\right| = \sqrt{\frac{9}{25}(1)}$$

$$\left|T'(t)\right| = \frac{3}{5}$$

Now we can use everything we just found to solve for the unit normal vector.

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$$N(t) = \frac{T'(t)}{\left|T'(t)\right|}$$

$$N(t) = \frac{-\frac{3}{5}\cos t\mathbf{i} - \frac{3}{5}\sin t\mathbf{j} + 0\mathbf{k}}{\frac{3}{5}}$$

$$N(t) = \frac{-\frac{3}{5}\cos t\mathbf{i} - \frac{3}{5}\sin t\mathbf{j}}{\frac{3}{5}}$$

$$N(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

This is the unit normal vector.

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Topic: Curvature

Question: Find the curvature of the vector function.

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

Answer choices:

A $\kappa(t) = \frac{1}{\sqrt{5}}$

B $\kappa(t) = \sqrt{5}$

C $\kappa(t) = \frac{1}{5}$

D $\kappa(t) = 5$

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Solution: C

First we'll find the derivative of the vector function.

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

$$r'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$$

Then we'll find the magnitude of this derivative.

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left|r'(t)\right| = \sqrt{(-\sin t)^2 + (\cos t)^2 + (2)^2}$$

$$\left|r'(t)\right| = \sqrt{\left(\sin^2 t + \cos^2 t\right) + 4}$$

$$\left|r'(t)\right| = \sqrt{(1) + 4}$$

$$\left|r'(t)\right| = \sqrt{5}$$

And now we can use everything we just found to find the unit tangent vector.

$$T(t) = \frac{r'(t)}{\left|r'(t)\right|}$$

$$T(t) = \frac{-\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}}{\sqrt{5}}$$

$$T(t) = -\frac{\sin t}{\sqrt{5}}\mathbf{i} + \frac{\cos t}{\sqrt{5}}\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

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$$T(t) = -\frac{1}{\sqrt{5}}\sin t\mathbf{i} + \frac{1}{\sqrt{5}}\cos t\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

To find the curvature of the vector function, we'll first need to find the derivative of this unit tangent vector.

$$T(t) = -\frac{1}{\sqrt{5}}\sin t\mathbf{i} + \frac{1}{\sqrt{5}}\cos t\mathbf{j} + \frac{2}{\sqrt{5}}\mathbf{k}$$

$$T'(t) = -\frac{1}{\sqrt{5}}\cos t\mathbf{i} - \frac{1}{\sqrt{5}}\sin t\mathbf{j} + (0)\mathbf{k}$$

Then we'll find the magnitude of this derivative.

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right| = \sqrt{\left(-\frac{1}{\sqrt{5}}\cos t\right)^2 + \left(-\frac{1}{\sqrt{5}}\sin t\right)^2 + (0)^2}$$

$$\left|T'(t)\right| = \sqrt{\frac{1}{5}\cos^2 t + \frac{1}{5}\sin^2 t}$$

$$\left|T'(t)\right| = \sqrt{\frac{1}{5}\left(\cos^2 t + \sin^2 t\right)}$$

$$\left|T'(t)\right| = \sqrt{\frac{1}{5}(1)}$$

$$\left|T'(t)\right| = \frac{1}{\sqrt{5}}$$

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And now we can use everything we just found to solve for the curvature of the vector function.

$$\kappa(t) = \frac{\left| T'(t) \right|}{\left| r'(t) \right|}$$

$$\kappa(t) = \frac{\frac{1}{\sqrt{5}}}{\sqrt{5}}$$

$$\kappa(t) = \frac{1}{\sqrt{5}\sqrt{5}}$$

$$\kappa(t) = \frac{1}{5}$$

This is the curvature of the vector function.

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Topic: Curvature

Question: Find the curvature of the vector function.

$$r(t) = 4t\mathbf{i} - 3\cos t\mathbf{j} - 3\sin t\mathbf{k}$$

Answer choices:

A $\kappa(t) = \dfrac{1}{25}$

B $\kappa(t) = \dfrac{3}{5}$

C $\kappa(t) = \dfrac{1}{5}$

D $\kappa(t) = \dfrac{3}{25}$

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Solution: D

First we'll find the derivative of the vector function.

$$r(t) = 4t\mathbf{i} - 3\cos t\mathbf{j} - 3\sin t\mathbf{k}$$

$$r'(t) = 4\mathbf{i} + 3\sin t\mathbf{j} - 3\cos t\mathbf{k}$$

Then we'll find the magnitude of this derivative.

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left|r'(t)\right| = \sqrt{(4)^2 + (3\sin t)^2 + (-3\cos t)^2}$$

$$\left|r'(t)\right| = \sqrt{16 + 9\sin^2 t + 9\cos^2 t}$$

$$\left|r'(t)\right| = \sqrt{16 + 9\left(\sin^2 t + \cos^2 t\right)}$$

$$\left|r'(t)\right| = \sqrt{16 + 9(1)}$$

$$\left|r'(t)\right| = \sqrt{25}$$

$$\left|r'(t)\right| = 5$$

And now we can use everything we just found to find the unit tangent vector.

$$T(t) = \frac{r'(t)}{\left|r'(t)\right|}$$

$$T(t) = \frac{4\mathbf{i} + 3\sin t\mathbf{j} - 3\cos t\mathbf{k}}{5}$$

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$$T(t)=\frac{4}{5}\mathbf{i}+\frac{3}{5}\sin t\mathbf{j}-\frac{3}{5}\cos t\mathbf{k}$$

To find the curvature of the vector function, we'll first need to find the derivative of this unit tangent vector.

$$T(t)=\frac{4}{5}\mathbf{i}+\frac{3}{5}\sin t\mathbf{j}-\frac{3}{5}\cos t\mathbf{k}$$

$$T'(t)=(0)\mathbf{i}+\frac{3}{5}\cos t\mathbf{j}+\frac{3}{5}\sin t\mathbf{k}$$

Then we'll find the magnitude of this derivative.

$$\left|T'(t)\right|=\sqrt{\left[T'(t)_1\right]^2+\left[T'(t)_2\right]^2+\left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right|=\sqrt{(0)^2+\left(\frac{3}{5}\cos t\right)^2+\left(\frac{3}{5}\sin t\right)^2}$$

$$\left|T'(t)\right|=\sqrt{\frac{9}{25}\cos^2 t+\frac{9}{25}\sin^2 t}$$

$$\left|T'(t)\right|=\sqrt{\frac{9}{25}\left(\cos^2 t+\sin^2 t\right)}$$

$$\left|T'(t)\right|=\sqrt{\frac{9}{25}(1)}$$

$$\left|T'(t)\right|=\frac{3}{5}$$

And now we can use everything we just found to solve for the curvature of the vector function.

$$\kappa(t) = \frac{\left|T'(t)\right|}{\left|r'(t)\right|}$$

$$\kappa(t) = \frac{\frac{3}{5}}{5}$$

$$\kappa(t) = \frac{3}{5 \cdot 5}$$

$$\kappa(t) = \frac{3}{25}$$

This is the curvature of the vector function.

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Topic: Curvature

Question: Find the curvature of the vector function.

$$r(t) = 6\sin t\mathbf{i} - 8t\mathbf{j} + 6\cos t\mathbf{k}$$

Answer choices:

A $\quad \kappa(t) = \dfrac{3}{5}$

B $\quad \kappa(t) = \dfrac{3}{50}$

C $\quad \kappa(t) = \dfrac{3}{25}$

D $\quad \kappa(t) = \dfrac{3}{10}$

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Solution: B

First we'll find the derivative of the vector function.

$$r(t) = 6\sin t\mathbf{i} - 8t\mathbf{j} + 6\cos t\mathbf{k}$$

$$r(t) = 6\cos t\mathbf{i} - 8\mathbf{j} - 6\sin t\mathbf{k}$$

Then we'll find the magnitude of this derivative.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left| r'(t) \right| = \sqrt{(6\cos t)^2 + (-8)^2 + (-6\sin t)^2}$$

$$\left| r'(t) \right| = \sqrt{36\cos^2 t + 64 + 36\sin^2 t}$$

$$\left| r'(t) \right| = \sqrt{64 + 36\left(\cos^2 t + \sin^2 t\right)}$$

$$\left| r'(t) \right| = \sqrt{64 + 36(1)}$$

$$\left| r'(t) \right| = \sqrt{100}$$

$$\left| r'(t) \right| = 10$$

And now we can use everything we just found to find the unit tangent vector.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(t) = \frac{6\cos t\mathbf{i} - 8\mathbf{j} - 6\sin t\mathbf{k}}{10}$$

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$$T(t) = \frac{6}{10}\cos t\mathbf{i} - \frac{8}{10}\mathbf{j} - \frac{6}{10}\sin t\mathbf{k}$$

$$T(t) = \frac{3}{5}\cos t\mathbf{i} - \frac{4}{5}\mathbf{j} - \frac{3}{5}\sin t\mathbf{k}$$

To find the curvature of the vector function, we'll first need to find the derivative of this unit tangent vector.

$$T(t) = \frac{3}{5}\cos t\mathbf{i} - \frac{4}{5}\mathbf{j} - \frac{3}{5}\sin t\mathbf{k}$$

$$T'(t) = -\frac{3}{5}\sin t\mathbf{i} - (0)\mathbf{j} - \frac{3}{5}\cos t\mathbf{k}$$

Then we'll find the magnitude of this derivative.

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right| = \sqrt{\left(-\frac{3}{5}\sin t\right)^2 + (0)^2 + \left(-\frac{3}{5}\cos t\right)^2}$$

$$\left|T'(t)\right| = \sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t}$$

$$\left|T'(t)\right| = \sqrt{\frac{9}{25}\left(\sin^2 t + \cos^2 t\right)}$$

$$\left|T'(t)\right| = \sqrt{\frac{9}{25}(1)}$$

$$\left|T'(t)\right| = \frac{3}{5}$$

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And now we can use everything we just found to solve for the curvature of the vector function.

$$\kappa(t) = \frac{\left|T'(t)\right|}{\left|r'(t)\right|}$$

$$\kappa(t) = \frac{\frac{3}{5}}{10}$$

$$\kappa(t) = \frac{3}{5 \cdot 10}$$

$$\kappa(t) = \frac{3}{50}$$

This is the curvature of the vector function.

Topic: Maximum curvature

Question: Find the point where the function has maximum curvature.

$$f(x) = -e^x$$

Answer choices:

A $\left(-\frac{1}{2}\ln 2, -\frac{1}{\sqrt{2}}\right)$

B $\left(\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$

C $\left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$

D $\left(\frac{1}{2}\ln 2, -\frac{1}{\sqrt{2}}\right)$

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Solution: A

To find the curvature of a function, we can start by finding the first derivative of the function.

$$f(x) = -e^x$$

$$f'(x) = -e^x$$

Then we'll find the second derivative.

$$f''(x) = -e^x$$

Then we'll take its absolute value.

$$|f''(x)| = e^x$$

With all this information, we're able to find curvature.

$$K(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{\frac{3}{2}}}$$

$$K(x) = \frac{e^x}{\left[1 + (-e^x)^2\right]^{\frac{3}{2}}}$$

$$K(x) = \frac{e^x}{\left(1 + e^{2x}\right)^{\frac{3}{2}}}$$

We'll find the derivative of curvature.

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$$K'(x)=\frac{e^x\left(1+e^{2x}\right)^{\frac{3}{2}}-e^x\left[\frac{3}{2}\left(1+e^{2x}\right)^{\frac{1}{2}}\left(2e^{2x}\right)\right]}{\left[\left(1+e^{2x}\right)^{\frac{3}{2}}\right]^2}$$

$$K'(x)=\frac{e^x\left(1+e^{2x}\right)^{\frac{3}{2}}-3e^{3x}\left(1+e^{2x}\right)^{\frac{1}{2}}}{\left(1+e^{2x}\right)^3}$$

Cancel out a common factor of $(1+e^{2x})^{1/2}$.

$$K'(x)=\frac{e^x\left(1+e^{2x}\right)-3e^{3x}}{\left(1+e^{2x}\right)^{\frac{5}{2}}}$$

$$K'(x)=\frac{e^x+e^{3x}-3e^{3x}}{\left(1+e^{2x}\right)^{\frac{5}{2}}}$$

$$K'(x)=\frac{e^x-2e^{3x}}{\left(1+e^{2x}\right)^{\frac{5}{2}}}$$

$$K'(x)=\frac{e^x\left(1-2e^{2x}\right)}{\left(1+e^{2x}\right)^{\frac{5}{2}}}$$

To find critical points, we'll set the derivative equal to 0 and solve for x. The only way that the above derivative function will be equal to 0 is if its numerator is equal to 0.

$$0=\frac{e^x\left(1-2e^{2x}\right)}{\left(1+e^{2x}\right)^{\frac{5}{2}}}$$

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$$0 = e^x\left(1 - 2e^{2x}\right)$$

$e^x = 0$ or $1 - 2e^{2x} = 0$

There's no point at which $e^x = 0$, so we'll solve $1 - 2e^{2x} = 0$ to find critical points.

$$2e^{2x} = 1$$

$$e^{2x} = \frac{1}{2}$$

$$\ln e^{2x} = \ln\frac{1}{2}$$

$$2x = \ln\frac{1}{2}$$

$$2x = \ln 2^{-1}$$

$$x = \frac{1}{2}\ln 2^{-1}$$

$$x = -\frac{1}{2}\ln 2$$

We'll plug this back into the original function $f(x) = -e^x$ to find the associated y-value.

$$y = -e^x$$

$$y = -e^{-\frac{1}{2}\ln 2}$$

$$y = -e^{\ln 2^{-\frac{1}{2}}}$$

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$$y = -2^{-\frac{1}{2}}$$

$$y = -\frac{1}{\sqrt{2}}$$

Putting these together, we can say that the function has maximum curvature at

$$\left(-\frac{1}{2}\ln 2, -\frac{1}{\sqrt{2}}\right)$$

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Topic: Maximum curvature

Question: Find the point where the function $f(x) = (1/3)x^3$ has maximum curvature. To find the curvature of a function $f(x)$, use the formula below.

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

Answer choices:

A $\left(\frac{1}{\sqrt[4]{5}}, \frac{1}{3\sqrt[4]{5}}\right)$ only

B $\left(\frac{1}{\sqrt[4]{5}}, \frac{1}{3\sqrt[4]{5}}\right)$ and $\left(-\frac{1}{\sqrt[4]{5}}, -\frac{1}{3\sqrt[4]{5}}\right)$

C $\left(\frac{1}{\sqrt[4]{5}}, \frac{1}{3\sqrt[4]{125}}\right)$ only

D $\left(\frac{1}{\sqrt[4]{5}}, \frac{1}{3\sqrt[4]{125}}\right)$ and $\left(-\frac{1}{\sqrt[4]{5}}, -\frac{1}{3\sqrt[4]{125}}\right)$

Solution: D

Based on the formula for the curvature of $f(x)$, we'll need to find the function's first and second derivatives.

$$f(x) = \frac{1}{3}x^3$$

$$f'(x) = x^2$$

$$f''(x) = 2x$$

Find curvature.

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{\frac{3}{2}}}$$

$$\kappa(x) = \frac{|2x|}{[1 + (x^2)^2]^{\frac{3}{2}}}$$

$$\kappa(x) = \frac{2x}{(1 + x^4)^{\frac{3}{2}}}$$

Now find the derivative of curvature.

$$\kappa'(x) = \frac{2(1 + x^4)^{\frac{3}{2}} - 2x\left(\frac{3}{2}(1 + x^4)^{\frac{1}{2}}\right)(4x^3)}{[(1 + x^4)^{\frac{3}{2}}]^2}$$

$$\kappa'(x) = \frac{2(1 + x^4)^{\frac{3}{2}} - 12x^4(1 + x^4)^{\frac{1}{2}}}{(1 + x^4)^3}$$

Cancel a common factor of $(1 + x^4)^{1/2}$.

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$$\kappa'(x) = \frac{2(1 + x^4) - 12x^4}{(1 + x^4)^{\frac{5}{2}}}$$

$$\kappa'(x) = \frac{2 + 2x^4 - 12x^4}{(1 + x^4)^{\frac{5}{2}}}$$

$$\kappa'(x) = \frac{2 - 10x^4}{(1 + x^4)^{\frac{5}{2}}}$$

$$\kappa'(x) = \frac{2(1 - 5x^4)}{(1 + x^4)^{\frac{5}{2}}}$$

To find critical points, set the derivative equal to 0 and solve for x. The only way that the derivative function will be 0 is if its numerator is 0.

$$\frac{2(1 - 5x^4)}{(1 + x^4)^{\frac{5}{2}}} = 0$$

$$2(1 - 5x^4) = 0$$

$$1 - 5x^4 = 0$$

$$5x^4 = 1$$

$$x^4 = \frac{1}{5}$$

$$x = \pm \frac{1}{\sqrt[4]{5}}$$

We'll plug these values back into the original function $f(x) = (1/3)x^3$ to find the associated y-values.

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$$y = \frac{1}{3}\left(\frac{1}{\sqrt[4]{5}}\right)^3$$

$$y = \frac{1}{3}\left(\frac{1}{\sqrt[4]{125}}\right)$$

$$y = \frac{1}{3\sqrt[4]{125}}$$

and

$$y = \frac{1}{3}\left(-\frac{1}{\sqrt[4]{5}}\right)^3$$

$$y = \frac{1}{3}\left(-\frac{1}{\sqrt[4]{125}}\right)$$

$$y = -\frac{1}{3\sqrt[4]{125}}$$

Putting these together, we can say that the function reaches its maximum curvature at two points.

$$\left(\frac{1}{\sqrt[4]{5}}, \frac{1}{3\sqrt[4]{125}}\right)$$

$$\left(-\frac{1}{\sqrt[4]{5}}, -\frac{1}{3\sqrt[4]{125}}\right)$$

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Topic: Maximum curvature

Question: Find the point where the function has maximum curvature.

$$f(x) = e^{-x}$$

Answer choices:

A $\left(-\frac{1}{2}\ln 2, -\frac{1}{\sqrt{2}}\right)$

B $\left(\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$

C $\left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$

D $\left(\frac{1}{2}\ln 2, -\frac{1}{\sqrt{2}}\right)$

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Solution: B

To find the curvature of a function, we can start by finding the first derivative of the function.

$$f(x) = e^{-x}$$

$$f'(x) = -e^{-x}$$

Then we'll find the second derivative.

$$f''(x) = e^{-x}$$

Then we'll take its absolute value.

$$|f''(x)| = e^{-x}$$

With all this information, we're able to find curvature.

$$K(x) = \frac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{\frac{3}{2}}}$$

$$K(x) = \frac{e^{-x}}{\left[1 + (-e^{-x})^2\right]^{\frac{3}{2}}}$$

$$K(x) = \frac{e^{-x}}{\left(1 + e^{-2x}\right)^{\frac{3}{2}}}$$

We'll find the derivative of curvature.

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$$K'(x)=\frac{-e^{-x}\left(1+e^{-2x}\right)^{\frac{3}{2}}-e^{-x}\left[\frac{3}{2}\left(1+e^{-2x}\right)^{\frac{1}{2}}\right]\left(-2e^{-2x}\right)}{\left[\left(1+e^{-2x}\right)^{\frac{3}{2}}\right]^2}$$

$$K'(x)=\frac{-e^{-x}\left(1+e^{-2x}\right)^{\frac{3}{2}}+3e^{-3x}\left(1+e^{-2x}\right)^{\frac{1}{2}}}{\left(1+e^{-2x}\right)^3}$$

Cancel out a common factor of $(1+e^{-2x})^{1/2}$.

$$K'(x)=\frac{-e^{-x}\left(1+e^{-2x}\right)+3e^{-3x}}{\left(1+e^{-2x}\right)^{\frac{5}{2}}}$$

$$K'(x)=\frac{-e^{-x}-e^{-3x}+3e^{-3x}}{\left(1+e^{-2x}\right)^{\frac{5}{2}}}$$

$$K'(x)=\frac{2e^{-3x}-e^{-x}}{\left(1+e^{-2x}\right)^{\frac{5}{2}}}$$

$$K'(x)=\frac{e^{-x}\left(2e^{-2x}-1\right)}{\left(1+e^{-2x}\right)^{\frac{5}{2}}}$$

To find critical points, we'll set the derivative equal to 0 and solve for x. The only way that the above derivative function will be equal to 0 is if its numerator is equal to 0.

$$0=\frac{e^{-x}\left(2e^{-2x}-1\right)}{\left(1+e^{-2x}\right)^{\frac{5}{2}}}$$

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$$0 = e^{-x}\left(2e^{-2x} - 1\right)$$

$$e^{-x} = 0 \text{ or } 2e^{-2x} - 1 = 0$$

There's no point at which $e^{-x} = 0$, so we'll solve $2e^{-2x} - 1 = 0$ to find critical points.

$$2e^{-2x} - 1 = 0$$

$$2e^{-2x} = 1$$

$$e^{-2x} = \frac{1}{2}$$

$$\ln e^{-2x} = \ln \frac{1}{2}$$

$$-2x = \ln \frac{1}{2}$$

$$-2x = \ln 2^{-1}$$

$$x = -\frac{1}{2} \ln 2^{-1}$$

$$x = \frac{1}{2} \ln 2$$

We'll plug this back into the original function $f(x) = e^{-x}$ to find the associated y-value.

$$y = e^{-x}$$

$$y = e^{-\frac{1}{2} \ln 2}$$

$$y = e^{\ln 2^{-\frac{1}{2}}}$$

$$y = 2^{-\frac{1}{2}}$$

$$y = \frac{1}{\sqrt{2}}$$

Putting these together, we can say that the function has maximum curvature at

$$\left(\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$$

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Topic: Normal and osculating planes

Question: Find the equation of the normal plane at the point.

$$x = 2t^2,\ y = t,\ z = 6$$

$$(2,1,6)$$

Answer choices:

A $\quad x + y = 0$

B $\quad 4x + y = 0$

C $\quad 4x + y = 9$

D $\quad 4x + y = -9$

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Solution: C

Combine the parametric equations

$$x = 2t^2$$

$$y = t$$

$$z = 6$$

into one vector function.

$$r(t) = 2t^2\mathbf{i} + t\mathbf{j} + 6\mathbf{k}$$

Find the derivative of the vector function.

$$r'(t) = 4t\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

Now we need to find the value of the parameter t. Using the parametric equation $y = t$ and the given point $(2,1,6)$, we can see that $y = 1$, so it must also be true that $t = 1$.

Evaluate the derivative at $t = 1$.

$$r'(1) = 4(1)\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

$$r'(1) = 4\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

The coefficients in this equation give the direction numbers $\langle 4,1,0 \rangle$. Plugging these, and $(2,1,6)$ into the equation of a plane will give the equation of the normal plane.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$4(x-2)+1(y-1)+0(z-6)=0$$

$$4(x-2)+(y-1)=0$$

$$4x-8+y-1=0$$

$$4x+y=9$$

This is the equation of the normal plane.

Topic: Normal and osculating planes

Question: Find the equations of the normal and osculating planes at the point.

$$x = t, \ y = \sin t, \ z = -\cos t$$

$$(0,0,-1)$$

Answer choices:

A Normal plane: $x + y = 0$ Osculating plane: $x - y = 0$

B Normal plane: $x - y = 0$ Osculating plane: $x + y = 0$

C Normal plane: $x + y = \pi$ Osculating plane: $x - y = \pi$

D Normal plane: $x - y = \pi$ Osculating plane: $x + y = \pi$

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Solution: A

Combine the parametric equations

$$x = t$$

$$y = \sin t$$

$$z = -\cos t$$

into one vector function.

$$r(t) = t\mathbf{i} + \sin t\mathbf{j} - \cos t\mathbf{k}$$

Find the derivative of the vector function.

$$r'(t) = \mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$$

Now we need to find the value of the parameter t. Using the parametric equation $x = t$ and the given point $(0,0,-1)$, we can see that $x = 0$, so it must also be true that $t = 0$.

Evaluate the derivative at $t = 0$.

$$r'(0) = \mathbf{i} + \cos 0\mathbf{j} + \sin 0\mathbf{k}$$

$$r'(0) = \mathbf{i} + 1\mathbf{j} + 0\mathbf{k}$$

$$r'(0) = \mathbf{i} + \mathbf{j}$$

The coefficients in this equation give the direction numbers $\langle 1,1,0 \rangle$. Plugging these, and $(0,0,-1)$ into the equation of a plane will give the equation of the normal plane.

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$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$1(x - 0) + 1(y - 0) + 0(z - (-1)) = 0$$

$$(x - 0) + (y - 0) = 0$$

$$x + y = 0$$

This is the equation of the normal plane.

To find the osculating plane, we'll need to first find the binormal vector. This means we'll need to find the unit tangent vector and the unit normal vector.

To find the unit tangent vector, start by taking the magnitude of the derivative, $r'(t) = \mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k}$.

$$|r'(t)| = \sqrt{(r'(t)_1)^2 + (r'(t)_2)^2 + (r'(t)_3)^2}$$

$$|r'(t)| = \sqrt{(1)^2 + (\cos t)^2 + (\sin t)^2}$$

$$|r'(t)| = \sqrt{1 + (\cos^2 t + \sin^2 t)}$$

Using the identity $\sin^2 t + \cos^2 t = 1$,

$$|r'(t)| = \sqrt{1 + 1}$$

$$|r'(t)| = \sqrt{2}$$

We'll find the unit tangent vector.

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$$T(t)=\frac{r'(t)}{|r'(t)|}$$

$$T(t)=\frac{\mathbf{i}+\cos t\mathbf{j}+\sin t\mathbf{k}}{\sqrt{2}}$$

$$T(t)=\frac{1}{\sqrt{2}}\mathbf{i}+\frac{1}{\sqrt{2}}\cos t\mathbf{j}+\frac{1}{\sqrt{2}}\sin t\mathbf{k}$$

To find the unit normal vector, start by finding the derivative of this unit tangent vector.

$$T'(t)=0\mathbf{i}-\frac{1}{\sqrt{2}}\sin t\mathbf{j}+\frac{1}{\sqrt{2}}\cos t\mathbf{k}$$

Find the magnitude of this derivative.

$$|T'(t)|=\sqrt{(T'(t)_1)^2+(T'(t)_2)^2+(T'(t)_3)^2}$$

$$|T'(t)|=\sqrt{(0)^2+\left(-\frac{1}{\sqrt{2}}\sin t\right)^2+\left(\frac{1}{\sqrt{2}}\cos t\right)^2}$$

$$|T'(t)|=\sqrt{\frac{1}{2}\sin^2 t+\frac{1}{2}\cos^2 t}$$

$$|T'(t)|=\sqrt{\frac{1}{2}\left(\sin^2 t+\cos^2 t\right)}$$

Using the identity $\sin^2 t+\cos^2 t=1$,

$$|T'(t)| = \sqrt{\frac{1}{2}(1)}$$

$$|T'(t)| = \frac{1}{\sqrt{2}}$$

Find the unit normal vector.

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$N(t) = \frac{-\frac{1}{\sqrt{2}}\sin t\mathbf{j} + \frac{1}{\sqrt{2}}\cos t\mathbf{k}}{\frac{1}{\sqrt{2}}}$$

$$N(t) = \left(\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right)\frac{-\sin t\mathbf{j} + \cos t\mathbf{k}}{1}$$

$$N(t) = -\sin t\mathbf{j} + \cos t\mathbf{k}$$

Now we can find the binormal.

$$B(t) = T(t) \times N(t)$$

To find the cross product $T(t) \times N(t)$, we'll use the cross product formula.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

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$$\vec{a} \times \vec{b} = \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

So the binormal vector is

$$B(t) = \mathbf{i}\left[\left(\frac{1}{\sqrt{2}}\cos t\right)(\cos t) - \left(\frac{1}{\sqrt{2}}\sin t\right)(-\sin t)\right] - \mathbf{j}\left[\left(\frac{1}{\sqrt{2}}\right)(\cos t) - \left(\frac{1}{\sqrt{2}}\sin t\right)(0)\right]$$

$$+\mathbf{k}\left[\left(\frac{1}{\sqrt{2}}\right)(-\sin t) - \left(\frac{1}{\sqrt{2}}\cos t\right)(0)\right]$$

$$B(t) = \mathbf{i}\left[\frac{1}{\sqrt{2}}\cos^2 t + \frac{1}{\sqrt{2}}\sin^2 t\right] - \mathbf{j}\left[\frac{1}{\sqrt{2}}\cos t\right] + \mathbf{k}\left[-\frac{1}{\sqrt{2}}\sin t\right]$$

$$B(t) = \left[\frac{1}{\sqrt{2}}\left(\cos^2 t + \sin^2 t\right)\right]\mathbf{i} - \frac{1}{\sqrt{2}}\cos t\mathbf{j} - \frac{1}{\sqrt{2}}\sin t\mathbf{k}$$

$$B(t) = \left[\frac{1}{\sqrt{2}}(1)\right]\mathbf{i} - \frac{1}{\sqrt{2}}\cos t\mathbf{j} - \frac{1}{\sqrt{2}}\sin t\mathbf{k}$$

$$B(t) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\cos t\mathbf{j} - \frac{1}{\sqrt{2}}\sin t\mathbf{k}$$

Evaluate the binormal vector at $t = 0$.

$$B(0) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\cos 0\mathbf{j} - \frac{1}{\sqrt{2}}\sin 0\mathbf{k}$$

$$B(0) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}(1)\mathbf{j} - \frac{1}{\sqrt{2}}(0)\mathbf{k}$$

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$$B(0) = \frac{1}{\sqrt{2}}\mathbf{i} - \frac{1}{\sqrt{2}}\mathbf{j} + 0\mathbf{k}$$

From this equation for the binormal vector, we get the direction numbers

$$\left\langle \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right\rangle$$

Find the equation of the osculating plane using these direction numbers and the point $(0,0,-1)$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\frac{1}{\sqrt{2}}(x - 0) - \frac{1}{\sqrt{2}}(y - 0) + 0(z - (-1)) = 0$$

$$\frac{1}{\sqrt{2}}(x - 0) - \frac{1}{\sqrt{2}}(y - 0) = 0$$

$$\frac{1}{\sqrt{2}}x - \frac{1}{\sqrt{2}}y = 0$$

$$x - y = 0$$

This is the equation of the osculating plane.

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Topic: Normal and osculating planes

Question: Find the equations of the normal and osculating planes at $(0,0,-4)$.

$$x = t$$

$$y = 4\sin t$$

$$z = -4\cos t$$

Answer choices:

A Normal plane: $4x - y = 0$ Osculating plane: $x - 4y = 0$

B Normal plane: $4x - y = 0$ Osculating plane: $x + 4y = 0$

C Normal plane: $x - 4y = 0$ Osculating plane: $4x + y = 0$

D Normal plane: $x + 4y = 0$ Osculating plane: $4x - y = 0$

Solution: D

Combine the parametric equations

$$x = t$$

$$y = 4\sin t$$

$$z = -4\cos t$$

into one vector function.

$$r(t) = t\mathbf{i} + 4\sin t\mathbf{j} - 4\cos t\mathbf{k}$$

Find the derivative of the vector function.

$$r'(t) = \mathbf{i} + 4\cos t\mathbf{j} + 4\sin t\mathbf{k}$$

Now we need to find the value of the parameter t. Using the parametric equation $x = t$ and the given point $(0,0,-4)$, we can see that $x = 0$, so it must also be true that $t = 0$.

Evaluate the derivative at $t = 0$.

$$r'(0) = \mathbf{i} + 4\cos(0)\mathbf{j} + 4\sin(0)\mathbf{k}$$

$$r'(0) = \mathbf{i} + 4(1)\mathbf{j} + 4(0)\mathbf{k}$$

$$r'(0) = \mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$$

The coefficients in this equation give the direction numbers $\langle 1,4,0 \rangle$. Plugging these, and $(0,0,-4)$ into the equation of a plane will give the equation of the normal plane.

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$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$1(x - 0) + 4(y - 0) + 0(z - (-4)) = 0$$

$$x + 4y = 0$$

This is the equation of the normal plane.

To find the osculating plane, we'll need to first find the binormal vector. This means we'll need to find the unit tangent vector and the unit normal vector.

To find the unit tangent vector, start by taking the magnitude of the derivative, $r'(t) = \mathbf{i} + 4\cos t\mathbf{j} + 4\sin t\mathbf{k}$.

$$|r'(t)| = \sqrt{(r'(t)_1)^2 + (r'(t)_2)^2 + (r'(t)_3)^2}$$

$$|r'(t)| = \sqrt{(1)^2 + (4\cos t)^2 + (4\sin t)^2}$$

$$|r'(t)| = \sqrt{1 + 16(\cos^2 t + \sin^2 t)}$$

Using the identity $\sin^2 t + \cos^2 t = 1$,

$$|r'(t)| = \sqrt{1 + 16(1)}$$

$$|r'(t)| = \sqrt{17}$$

We'll find the unit tangent vector.

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

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$$T(t)=\frac{\mathbf{i}+4\cos t\mathbf{j}+4\sin t\mathbf{k}}{\sqrt{17}}$$

$$T(t)=\frac{1}{\sqrt{17}}\mathbf{i}+\frac{4}{\sqrt{17}}\cos t\mathbf{j}+\frac{4}{\sqrt{17}}\sin t\mathbf{k}$$

To find the unit normal vector, start by finding the derivative of this unit tangent vector.

$$T'(t)=0\mathbf{i}-\frac{4}{\sqrt{17}}\sin t\mathbf{j}+\frac{4}{\sqrt{17}}\cos t\mathbf{k}$$

Find the magnitude of this derivative.

$$|T'(t)|=\sqrt{(T'(t)_1)^2+(T'(t)_2)^2+(T'(t)_3)^2}$$

$$|T'(t)|=\sqrt{(0)^2+\left(-\frac{4}{\sqrt{17}}\sin t\right)^2+\left(\frac{4}{\sqrt{17}}\cos t\right)^2}$$

$$|T'(t)|=\sqrt{\frac{16}{17}\sin^2 t+\frac{16}{17}\cos^2 t}$$

$$|T'(t)|=\sqrt{\frac{16}{17}(\sin^2 t+\cos^2 t)}$$

Using the identity $\sin^2 t+\cos^2 t=1$,

$$|T'(t)|=\sqrt{\frac{16}{17}(1)}$$

$$|T'(t)| = \frac{4}{\sqrt{17}}$$

Find the unit normal vector.

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$N(t) = \frac{-\frac{4}{\sqrt{17}} \sin t\mathbf{j} + \frac{4}{\sqrt{17}} \cos t\mathbf{k}}{\frac{4}{\sqrt{17}}}$$

$$N(t) = \left(\frac{\frac{4}{\sqrt{17}}}{\frac{4}{\sqrt{17}}}\right) \frac{-\sin t\mathbf{j} + \cos t\mathbf{k}}{1}$$

$$N(t) = -\sin t\mathbf{j} + \cos t\mathbf{k}$$

Now we can find the binormal vector.

$$B(t) = T(t) \times N(t)$$

To find $T(t) \times N(t)$, we'll use the cross product formula.

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2 b_3 - a_3 b_2) - \mathbf{j}(a_1 b_3 - a_3 b_1) + \mathbf{k}(a_1 b_2 - a_2 b_1)$$

So the binormal vector is

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$$B(t)=\mathbf{i}\left[\left(\frac{4}{\sqrt{17}}\cos t\right)(\cos t)-\left(\frac{4}{\sqrt{17}}\sin t\right)(-\sin t)\right]$$

$$-\mathbf{j}\left[\left(\frac{1}{\sqrt{17}}\right)(\cos t)-\left(\frac{4}{\sqrt{17}}\sin t\right)(0)\right]$$

$$+\mathbf{k}\left[\left(\frac{1}{\sqrt{17}}\right)(-\sin t)-\left(\frac{4}{\sqrt{17}}\cos t\right)(0)\right]$$

$$B(t)=\left(\frac{4}{\sqrt{17}}\cos^2 t+\frac{4}{\sqrt{17}}\sin^2 t\right)\mathbf{i}-\left(\frac{1}{\sqrt{17}}\cos t\right)\mathbf{j}+\left(-\frac{1}{\sqrt{17}}\sin t\right)\mathbf{k}$$

$$B(t)=\frac{4}{\sqrt{17}}(\cos^2 t+\sin^2 t)\mathbf{i}-\frac{1}{\sqrt{17}}\cos t\mathbf{j}-\frac{1}{\sqrt{17}}\sin t\mathbf{k}$$

$$B(t)=\frac{4}{\sqrt{17}}(1)\mathbf{i}-\frac{1}{\sqrt{17}}\cos t\mathbf{j}-\frac{1}{\sqrt{17}}\sin t\mathbf{k}$$

$$B(t)=\frac{4}{\sqrt{17}}\mathbf{i}-\frac{1}{\sqrt{17}}\cos t\mathbf{j}-\frac{1}{\sqrt{17}}\sin t\mathbf{k}$$

Evaluate the binormal vector at $t=0$.

$$B(0)=\frac{4}{\sqrt{17}}\mathbf{i}-\frac{1}{\sqrt{17}}\cos(0)\mathbf{j}-\frac{1}{\sqrt{17}}\sin(0)\mathbf{k}$$

$$B(0)=\frac{4}{\sqrt{17}}\mathbf{i}-\frac{1}{\sqrt{17}}(1)\mathbf{j}-\frac{1}{\sqrt{17}}(0)\mathbf{k}$$

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$$B(0) = \frac{4}{\sqrt{17}}\mathbf{i} - \frac{1}{\sqrt{17}}\mathbf{j} + 0\mathbf{k}$$

From this equation for the binormal vector, we get the direction numbers

$$\left\langle \frac{4}{\sqrt{17}}, -\frac{1}{\sqrt{17}}, 0 \right\rangle$$

Find the equation of the osculating plane using these direction numbers and the point $(0,0,-4)$.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$\frac{4}{\sqrt{17}}(x - 0) - \frac{1}{\sqrt{17}}(y - 0) + 0(z - (-4)) = 0$$

$$\frac{4}{\sqrt{17}}x - \frac{1}{\sqrt{17}}y = 0$$

$$4x - y = 0$$

This is the equation of the osculating plane.

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Topic: Equation of the osculating circle

Question: Find the radius of the osculating circle of the parabola at $x = 4$.

$$f(x) = -\frac{1}{8}x^2 + 2x$$

Answer choices:

A $\quad r = \dfrac{1}{8\sqrt{2}}$

B $\quad r = \dfrac{8}{\sqrt{2}}$

C $\quad r = 8\sqrt{2}$

D $\quad r = 16\sqrt{2}$

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Solution: C

Find the value of the first derivative at $x = 4$.

$$f'(x) = -\frac{1}{4}x + 2$$

$$f'(4) = -\frac{1}{4}(4) + 2 = 1$$

Find the value of the second derivative at $x = 4$.

$$f''(x) = -\frac{1}{4}$$

$$f''(4) = -\frac{1}{4}$$

Plugging these values into the formula for curvature, we get

$$K = \frac{|f''(x)|}{|1 + (f'(x))^2|^{\frac{3}{2}}}$$

$$K = \frac{\left|-\frac{1}{4}\right|}{|1 + 1^2|^{\frac{3}{2}}} = \frac{\frac{1}{4}}{2\sqrt{2}} = \frac{\sqrt{2}}{16}$$

The radius is then

$$r = \frac{1}{K} = \frac{1}{\frac{\sqrt{2}}{16}} = 8\sqrt{2}$$

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Topic: Equation of the osculating circle

Question: The radius of the osculating circle of which curve is approximately 66.287 units at the point $(2,1)$?

Answer choices:

A $\quad f(x) = x^2 + x + 5$

B $\quad f(x) = x^2 + 2x - 5$

C $\quad f(x) = x^2 + 5x + 1$

D $\quad f(x) = x^2 - 5x + 5$

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Solution: A

Given the function

$$f(x) = x^2 + x + 5$$

we'll find the value of the first derivative at $x = 2$.

$$f'(x) = 2x + 1$$

$$f'(2) = 2(2) + 1 = 5$$

Find the value of the second derivative at $x = 2$.

$$f''(x) = 2$$

$$f''(2) = 2$$

Plugging these values into the formula for curvature, we get

$$K = \frac{\left|f''(x)\right|}{\left|1 + \left(f'(x)\right)^2\right|^{\frac{3}{2}}}$$

$$K = \frac{|2|}{|1 + 5^2|^{\frac{3}{2}}} = \frac{2}{26^{\frac{3}{2}}} \approx 0.015$$

The radius is then

$$r = \frac{1}{K} = \frac{1}{0.015} \approx 66.287$$

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Topic: Equation of the osculating circle

Question: At which point is the radius of the osculating circle of the curve $\sqrt{17^3}$ units long?

$$f(x) = -\frac{1}{2}x^2 + x - 4$$

Answer choices:

A $(1,3)$

B $(-2,2)$

C $(3,-1)$

D $(-3,1)$

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Solution: D

The first and second derivatives of the given function are

$$f'(x) = -x + 1$$

$$f''(x) = -1$$

Find $f'(-3)$ and $f''(-3)$.

$$f'(-3) = -(-3) + 1 = 4$$

$$f''(-3) = -1$$

Plugging these values into the formula for curvature, we get

$$K = \frac{\left|f''(x)\right|}{\left|1 + \left(f'(x)\right)^2\right|^{\frac{3}{2}}}$$

$$K = \frac{\left|-1\right|}{\left|1 + (4)^2\right|^{\frac{3}{2}}} = \frac{1}{\left|17\right|^{\frac{3}{2}}} = \frac{1}{\sqrt{17^3}}$$

Then the radius is

$$r = \frac{1}{K} = \frac{1}{\frac{1}{\sqrt{17^3}}} = \sqrt{17^3}$$

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Topic: Velocity and acceleration vectors

Question: Find the velocity of the vector function.

$$r(t) = 2t\mathbf{i} - t^3\mathbf{j} - 4\mathbf{k}$$

Answer choices:

A $\quad v(t) = \langle -2t, t^3, 4 \rangle$

B $\quad v(t) = \langle 2, -3t^2, 0 \rangle$

C $\quad v(t) = \langle 2t, -t^3, -4 \rangle$

D $\quad v(t) = \langle -2, 3t^2, 0 \rangle$

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Solution: B

To find the velocity of the vector function, we'll take the derivative of the vector function.

$$r(t) = 2t\mathbf{i} - t^3\mathbf{j} - 4\mathbf{k}$$

$$r'(t) = v(t) = 2\mathbf{i} - 3t^2\mathbf{j} + 0\mathbf{k}$$

Rewriting the velocity equation in vector form gives

$$v(t) = \left\langle 2, -3t^2, 0 \right\rangle$$

This is the velocity of the vector function.

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Topic: Velocity and acceleration vectors

Question: Find the acceleration of the vector function.

$$r(t) = \sin t\mathbf{i} + 4t\mathbf{j} - 5t^4\mathbf{k}$$

Answer choices:

A $\quad a(t) = \langle -\cos t, 4, -20t^3 \rangle$

B $\quad a(t) = \langle \sin t, 0, -60t^2 \rangle$

C $\quad a(t) = \langle \cos t, 4, -20t^3 \rangle$

D $\quad a(t) = \langle -\sin t, 0, -60t^2 \rangle$

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Solution: D

To find the velocity of the vector function, we'll take the derivative of the vector function.

$$r(t) = \sin t\mathbf{i} + 4t\mathbf{j} - 5t^4\mathbf{k}$$

$$r'(t) = v(t) = \cos t\mathbf{i} + 4\mathbf{j} - 20t^3\mathbf{k}$$

To find the acceleration of the vector function, we'll take the second derivative of the vector function.

$$r''(t) = v'(t) = a(t) = -\sin t\mathbf{i} + 0\mathbf{j} - 60t^2\mathbf{k}$$

Rewriting the acceleration equation in vector form gives

$$a(t) = \left\langle -\sin t, 0, -60t^2 \right\rangle$$

This is the acceleration of the vector function.

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Topic: Velocity and acceleration vectors

Question: Find the speed of the vector function.

$$r(t) = e^{-2t}\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

when $t = 0$

Answer choices:

A $|v(0)| = \sqrt{3}$

B $|v(0)| = \sqrt{5}$

C $|v(0)| = 3$

D $|v(0)| = 5$

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Solution: C

To find the velocity of the vector function, we'll take the derivative of the vector function.

$$r(t) = e^{-2t}\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

$$r'(t) = v(t) = -2e^{-2t}\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$$

Now we'll use this velocity equation and the formula for speed.

$$|v(t)| = \sqrt{(v(t)_1)^2 + (v(t)_2)^2 + (v(t)_3)^2}$$

$$|v(t)| = \sqrt{\left(-2e^{-2t}\right)^2 + (\cos t)^2 + (2)^2}$$

$$|v(t)| = \sqrt{4e^{-4t} + \cos^2 t + 4}$$

To find speed when $t = 0$, we'll plug $t = 0$ into this speed equation.

$$|v(0)| = \sqrt{4e^{-4(0)} + \cos^2(0) + 4}$$

$$|v(0)| = \sqrt{4(1) + (1) + 4}$$

$$|v(0)| = \sqrt{9}$$

$$|v(0)| = 3$$

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Topic: Velocity, acceleration and speed, given position

Question: Find the velocity of the position function.

$$r(t) = -\mathbf{i} + e^t\mathbf{j} + 3t^2\mathbf{k}$$

Answer choices:

A $\quad v(t) = e^t\mathbf{j} + 6t\mathbf{k}$

B $\quad v(t) = -e^t\mathbf{j} + 6t\mathbf{k}$

C $\quad v(t) = -e^t\mathbf{j} - 6t\mathbf{k}$

D $\quad v(t) = e^t\mathbf{j} - 6t\mathbf{k}$

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Solution: A

To find the velocity of the position function, we'll take the derivative of the position function.

$$r(t) = -\mathbf{i} + e^t\mathbf{j} + 3t^2\mathbf{k}$$

$$r(t) = v(t) = 0\mathbf{i} + e^t\mathbf{j} + 6t\mathbf{k}$$

The velocity of the position function is

$$v(t) = e^t\mathbf{j} + 6t\mathbf{k}$$

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Topic: Velocity, acceleration and speed, given position

Question: Find the acceleration of the position function.

$$r(t) = -5t^2\mathbf{i} + \cos t\mathbf{j} + e^{3t}\mathbf{k}$$

Answer choices:

A $\quad a(t) = -10t\mathbf{i} + \sin t\mathbf{j} + 3e^{3t}\mathbf{k}$

B $\quad a(t) = -10t\mathbf{i} - \sin t\mathbf{j} + 3e^{3t}\mathbf{k}$

C $\quad a(t) = -10\mathbf{i} - \cos t\mathbf{j} + 9e^{3t}\mathbf{k}$

D $\quad a(t) = -10\mathbf{i} + \cos t\mathbf{j} + 9e^{3t}\mathbf{k}$

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Solution: C

To find the velocity of the position function, we'll take the derivative of the position function.

$$r(t) = -5t^2\mathbf{i} + \cos t\mathbf{j} + e^{3t}\mathbf{k}$$

$$r'(t) = v(t) = -10t\mathbf{i} - \sin t\mathbf{j} + 3e^{3t}\mathbf{k}$$

To find the acceleration of the position function, we'll take the second derivative of the position function.

$$r''(t) = v'(t) = a(t) = -10\mathbf{i} - \cos t\mathbf{j} + 9e^{3t}\mathbf{k}$$

The acceleration of the position function is

$$a(t) = -10\mathbf{i} - \cos t\mathbf{j} + 9e^{3t}\mathbf{k}$$

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Topic: Velocity, acceleration and speed, given position

Question: Find the speed of the position function.

$$r(t) = 4t\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$$

Answer choices:

A $|v(t)| = 25$

B $|v(t)| = 5$

C $|v(t)| = \sqrt{7}$

D $|v(t)| = 1$

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Solution: B

To find the velocity of the position function, we'll take the derivative of the position function.

$$r(t) = 4t\mathbf{i} - 3\sin t\mathbf{j} + 3\cos t\mathbf{k}$$

$$r'(t) = v(t) = 4\mathbf{i} - 3\cos t\mathbf{j} - 3\sin t\mathbf{k}$$

Now we'll use this velocity equation and the formula for speed.

$$|v(t)| = \sqrt{(v(t)_1)^2 + (v(t)_2)^2 + (v(t)_3)^2}$$

$$|v(t)| = \sqrt{(4)^2 + (-3\cos t)^2 + (-3\sin t)^2}$$

$$|v(t)| = \sqrt{16 + 9\left(\cos^2 t + \sin^2 t\right)}$$

Use the identity $\sin^2 x + \cos^2 x = 1$.

$$|v(t)| = \sqrt{16 + 9(1)}$$

$$|v(t)| = \sqrt{25}$$

$$|v(t)| = 5$$

This is the speed of the position function.

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Topic: Velocity and position given acceleration and initial conditions

Question: Find the velocity and position of the acceleration function.

$$a(t) = 6t\mathbf{i} + 2\mathbf{j}$$

when $v(0) = \mathbf{k}$ and $r(0) = \mathbf{j}$

Answer choices:

A $\quad v(t) = t^2\mathbf{i} + t\mathbf{j}$ and $r(t) = t^3\mathbf{i} + (t^2 + 1)\mathbf{j}$

B $\quad v(t) = 3t^2\mathbf{i} + 2t\mathbf{j}$ and $r(t) = t^3\mathbf{i} + (t^2 + 1)\mathbf{j}$

C $\quad v(t) = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$ and $r(t) = t^3\mathbf{i} + (t^2 + 1)\mathbf{j} + t\mathbf{k}$

D $\quad v(t) = t^2\mathbf{i} + t\mathbf{j} + \mathbf{k}$ and $r(t) = t^3\mathbf{i} + (t^2 + 1)\mathbf{j} + t\mathbf{k}$

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Solution: C

To find velocity, we integrate the acceleration function. Remember that $\mathbf{k}$ is missing, but we should include it as $0\mathbf{k}$.

$$v(t) = \int a(t)\ dt$$

$$v(t) = \int 6t\mathbf{i} + 2\mathbf{j}\ dt$$

$$v(t) = \int 6t\mathbf{i} + 2\mathbf{j} + 0\mathbf{k}\ dt$$

Integrate the acceleration function to get velocity. We need to include a constant of integration in each coefficient.

$$v(t) = (3t^2 + b)\mathbf{i} + (2t + c)\mathbf{j} + (d)\mathbf{k}$$

We were told that $v(0) = \mathbf{k}$, which tells us that

$$v(0) = 0\mathbf{i} + 0\mathbf{j} + \mathbf{k}$$

Now we can substitute.

$$0\mathbf{i} + 0\mathbf{j} + \mathbf{k} = (3(0)^2 + b)\mathbf{i} + (2(0) + c)\mathbf{j} + (d)\mathbf{k}$$

$$0\mathbf{i} + 0\mathbf{j} + \mathbf{k} = (b)\mathbf{i} + (c)\mathbf{j} + (d)\mathbf{k}$$

Equating coefficients gives us $b = 0$, $c = 0$ and $d = 1$. We can substitute these values into $v(t) = (3t^2 + b)\mathbf{i} + (2t + c)\mathbf{j} + (d)\mathbf{k}$.

$$v(t) = (3t^2 + 0)\mathbf{i} + (2t + 0)\mathbf{j} + (1)\mathbf{k}$$

$$v(t) = 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}$$

To find position, we integrate the velocity function.

$$r(t) = \int v(t)\ dt$$

$$r(t) = \int 3t^2\mathbf{i} + 2t\mathbf{j} + \mathbf{k}\ dt$$

Integrate the velocity function to get position. We need to include a constant of integration in each coefficient.

$$r(t) = (t^3 + m)\mathbf{i} + (t^2 + n)\mathbf{j} + (t + p)\mathbf{k}$$

We were told that $r(0) = \mathbf{j}$, which tells us that

$$r(0) = 0\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

Now we can substitute.

$$0\mathbf{i} + \mathbf{j} + 0\mathbf{k} = ((0)^3 + m)\mathbf{i} + ((0)^2 + n)\mathbf{j} + ((0) + p)\mathbf{k}$$

$$0\mathbf{i} + \mathbf{j} + 0\mathbf{k} = (m)\mathbf{i} + (n)\mathbf{j} + (p)\mathbf{k}$$

Equating coefficients gives us $m = 0$, $n = 1$ and $p = 0$. We can substitute these values into $r(t) = (t^3 + m)\mathbf{i} + (t^2 + n)\mathbf{j} + (t + p)\mathbf{k}$.

$$r(t) = (t^3 + 0)\mathbf{i} + (t^2 + 1)\mathbf{j} + (t + 0)\mathbf{k}$$

$$r(t) = t^3\mathbf{i} + (t^2 + 1)\mathbf{j} + t\mathbf{k}$$

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Topic: Velocity and position given acceleration and initial conditions

Question: Find the velocity and position of the acceleration function.

$$a(t) = 4\mathbf{i} + 6t\mathbf{k}$$

when $v(0) = 4\mathbf{j}$ and $r(0) = \mathbf{k}$

Answer choices:

A $\quad v(t) = 4t\mathbf{i} + 4\mathbf{j} + 3t^2\mathbf{k}$ and $r(t) = 2t^2\mathbf{i} + 2t\mathbf{j} + t^3\mathbf{k}$

B $\quad v(t) = 4t\mathbf{i} + 4\mathbf{j} + 3t^2\mathbf{k}$ and $r(t) = 2t^2\mathbf{i} + 4t\mathbf{j} + (t^3 + 1)\mathbf{k}$

C $\quad v(t) = 4t\mathbf{i} + 3t^2\mathbf{k}$ and $r(t) = 2t^2\mathbf{i} + t^3\mathbf{k}$

D $\quad v(t) = 4t\mathbf{i} + 3t^2\mathbf{k}$ and $r(t) = 2t^2\mathbf{i} + (t^3 + 1)\mathbf{k}$

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Solution: B

To find velocity, we integrate the acceleration function. Remember that $\mathbf{j}$ is missing, but we should include it as $0\mathbf{j}$.

$$v(t) = \int a(t)\ dt$$

$$v(t) = \int 4\mathbf{i} + 6t\mathbf{k}\ dt$$

$$v(t) = \int 4\mathbf{i} + 0\mathbf{j} + 6t\mathbf{k}\ dt$$

Integrate the acceleration function to get velocity. We need to include a constant of integration in each coefficient.

$$v(t) = (4t + b)\mathbf{i} + (c)\mathbf{j} + (3t^2 + d)\mathbf{k}$$

We were told that $v(0) = 4\mathbf{j}$, which tells us that

$$v(0) = 0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k}$$

Now we can substitute.

$$0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} = (4t + b)\mathbf{i} + (c)\mathbf{j} + (3t^2 + d)\mathbf{k}$$

$$0\mathbf{i} + 4\mathbf{j} + 0\mathbf{k} = (b)\mathbf{i} + (c)\mathbf{j} + (d)\mathbf{k}$$

Equating coefficients gives us $b = 0$, $c = 4$ and $d = 0$. We can substitute these values into $v(t) = (4t + b)\mathbf{i} + (c)\mathbf{j} + (3t^2 + d)\mathbf{k}$.

$$v(t) = (4t + 0)\mathbf{i} + (4)\mathbf{j} + (3t^2 + 0)\mathbf{k}$$

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$$v(t) = 4t\mathbf{i} + 4\mathbf{j} + 3t^2\mathbf{k}$$

To find position, we integrate the velocity function.

$$r(t) = \int v(t)\ dt$$

$$r(t) = \int 4t\mathbf{i} + 4\mathbf{j} + 3t^2\mathbf{k}\ dt$$

Integrate the velocity function to get position. We need to include a constant of integration in each coefficient.

$$r(t) = (2t^2 + m)\mathbf{i} + (4t + n)\mathbf{j} + (t^3 + p)\mathbf{k}$$

We were told that $r(0) = \mathbf{k}$, which tells us that

$$r(0) = 0\mathbf{i} + 0\mathbf{j} + \mathbf{k}$$

Now we can substitute.

$$0\mathbf{i} + 0\mathbf{j} + \mathbf{k} = (2(0)^2 + m)\mathbf{i} + (4(0) + n)\mathbf{j} + ((0)^3 + p)\mathbf{k}$$

$$0\mathbf{i} + 0\mathbf{j} + \mathbf{k} = (m)\mathbf{i} + (n)\mathbf{j} + (p)\mathbf{k}$$

Equating coefficients gives us $m = 0$, $n = 0$ and $p = 1$. We can substitute these values into $r(t) = (2t^2 + m)\mathbf{i} + (4t + n)\mathbf{j} + (t^3 + p)\mathbf{k}$.

$$r(t) = (2t^2 + 0)\mathbf{i} + (4t + 0)\mathbf{j} + (t^3 + 1)\mathbf{k}$$

$$r(t) = 2t^2\mathbf{i} + 4t\mathbf{j} + (t^3 + 1)\mathbf{k}$$

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Topic: Velocity and position given acceleration and initial conditions

Question: Find the velocity and position of the acceleration function.

$$a(t) = 6\mathbf{j} + 8\mathbf{k}$$

when $v(0) = -\mathbf{i}$ and $r(0) = \mathbf{j}$

Answer choices:

A $\quad v(t) = 6t\mathbf{j} + 8t^2\mathbf{k}$ and $r(t) = (3t^2 + 1)\mathbf{j} + 4t^3\mathbf{k}$

B $\quad v(t) = 6t\mathbf{j} + 8t\mathbf{k}$ and $r(t) = (3t^2 + 1)\mathbf{j} + 4t^2\mathbf{k}$

C $\quad v(t) = \mathbf{i} + 6t\mathbf{j} + 8t^2\mathbf{k}$ and $r(t) = t\mathbf{i} + (3t^2 + 1)\mathbf{j} + 4t^3\mathbf{k}$

D $\quad v(t) = -\mathbf{i} + 6t\mathbf{j} + 8t\mathbf{k}$ and $r(t) = -t\mathbf{i} + (3t^2 + 1)\mathbf{j} + 4t^2\mathbf{k}$

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Solution: D

To find velocity, we integrate the acceleration function. Remember that $\mathbf{i}$ is missing, but we should include it as $0\mathbf{i}$.

$$v(t)=\int a(t)\ dt$$

$$v(t)=\int 6\mathbf{j}+8\mathbf{k}\ dt$$

$$v(t)=\int \mathbf{i}+6\mathbf{j}+8\mathbf{k}\ dt$$

Integrate the acceleration function to get velocity. We need to include a constant of integration in each coefficient.

$$v(t)=(b)\mathbf{i}+(6t+c)\mathbf{j}+(8t+d)\mathbf{k}$$

We were told that $v(0)=-\mathbf{i}$, which tells us that

$$v(0)=-\mathbf{i}+0\mathbf{j}+0\mathbf{k}$$

Now we can substitute.

$$-\mathbf{i}+0\mathbf{j}+0\mathbf{k}=(b)\mathbf{i}+(6(0)+c)\mathbf{j}+(8(0)+d)\mathbf{k}$$

$$-\mathbf{i}+0\mathbf{j}+0\mathbf{k}=(b)\mathbf{i}+(c)\mathbf{j}+(d)\mathbf{k}$$

Equating coefficients gives us $b=-1$, $c=0$ and $d=0$. We can substitute these values into $v(t)=(b)\mathbf{i}+(6t+c)\mathbf{j}+(8t+d)\mathbf{k}$.

$$v(t)=(-1)\mathbf{i}+(6t+0)\mathbf{j}+(8t+0)\mathbf{k}$$

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$$v(t) = -\mathbf{i} + 6t\mathbf{j} + 8t\mathbf{k}$$

To find position, we integrate the velocity function.

$$r(t) = \int v(t)\ dt$$

$$r(t) = \int -\mathbf{i} + 6t\mathbf{j} + 8t\mathbf{k}\ dt$$

Integrate the velocity function to get position. We need to include a constant of integration in each coefficient.

$$r(t) = (-t + m)\mathbf{i} + (3t^2 + n)\mathbf{j} + (4t^2 + p)\mathbf{k}$$

We were told that $r(0) = \mathbf{j}$, which tells us that

$$r(0) = 0\mathbf{i} + \mathbf{j} + 0\mathbf{k}$$

Now we can substitute.

$$0\mathbf{i} + \mathbf{j} + 0\mathbf{k} = (-(0) + m)\mathbf{i} + (3(0)^2 + n)\mathbf{j} + (4(0)^2 + p)\mathbf{k}$$

$$0\mathbf{i} + \mathbf{j} + 0\mathbf{k} = (m)\mathbf{i} + (n)\mathbf{j} + (p)\mathbf{k}$$

Equating coefficients gives us $m = 0$, $n = 1$ and $p = 0$. We can substitute these values into $r(t) = (-t + m)\mathbf{i} + (3t^2 + n)\mathbf{j} + (4t^2 + p)\mathbf{k}$.

$$r(t) = (-t + 0)\mathbf{i} + (3t^2 + 1)\mathbf{j} + (4t^2 + 0)\mathbf{k}$$

$$r(t) = -t\mathbf{i} + (3t^2 + 1)\mathbf{j} + 4t^2\mathbf{k}$$

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Topic: Tangential and normal components of acceleration

Question: Find the tangential and normal components of acceleration.

$$r(t) = t^2\mathbf{i} + 2t\mathbf{j} - 3t^2\mathbf{k}$$

Answer choices:

A $\quad a_T = \dfrac{1}{\sqrt{10}} \qquad a_N = \dfrac{2\sqrt{2}}{3t}$

B $\quad a_T = \sqrt{10} \qquad a_N = \dfrac{2\sqrt{2}}{3t}$

C $\quad a_T = \dfrac{2t\sqrt{10t^2+1}}{t^2+1} \qquad a_N = \dfrac{3\sqrt{50t^2+1}}{50t^2+1}$

D $\quad a_T = \dfrac{20t\sqrt{10t^2+1}}{10t^2+1} \qquad a_N = \dfrac{2\sqrt{10\left(10t^2+1\right)}}{10t^2+1}$

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Solution: D

Find the first derivative of the position vector

$$r(t) = t^2\mathbf{i} + 2t\mathbf{j} - 3t^2\mathbf{k}$$

in order to get velocity,

$$r'(t) = 2t\mathbf{i} + 2\mathbf{j} - 6t\mathbf{k}$$

and the second derivative in order to get acceleration.

$$r''(t) = 2\mathbf{i} + 0\mathbf{j} - 6\mathbf{k}$$

Find the dot product of the first and second derivatives.

$$r'(t) \cdot r''(t) = (2t)(2) + (2)(0) + (-6t)(-6)$$

$$r'(t) \cdot r''(t) = 4t + 0 + 36t$$

$$r'(t) \cdot r''(t) = 40t$$

Find the magnitude of the first derivative, $r'(t) = 2t\mathbf{i} + 2\mathbf{j} - 6t\mathbf{k}$.

$$|\, r'(t) \,| = \sqrt{(r'(t)_1)^2 + (r'(t)_2)^2 + (r'(t)_3)^2}$$

$$|\, r'(t) \,| = \sqrt{(2t)^2 + (2)^2 + (-6t)^2}$$

$$|\, r'(t) \,| = \sqrt{4t^2 + 4 + 36t^2}$$

$$|\, r'(t) \,| = \sqrt{40t^2 + 4}$$

$$|\, r'(t) \,| = \sqrt{4(10t^2 + 1)}$$

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$$|r'(t)| = 2\sqrt{10t^2+1}$$

Find the tangential component of the acceleration vector.

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

$$a_T = \frac{40t}{2\sqrt{10t^2+1}}$$

$$a_T = \frac{20t}{\sqrt{10t^2+1}}$$

Rationalize the denominator.

$$a_T = \frac{20t}{\sqrt{10t^2+1}}\left(\frac{\sqrt{10t^2+1}}{\sqrt{10t^2+1}}\right)$$

$$a_T = \frac{20t\sqrt{10t^2+1}}{10t^2+1}$$

This is the tangential component of acceleration.

To find the normal component of acceleration, we'll find the cross product $r'(t) \times r''(t)$, using the cross product formula:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

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$$r'(t) \times r''(t) = \mathbf{i}\left[(2)(-6) - (-6t)(0)\right] - \mathbf{j}\left[(2t)(-6) - (-6t)(2)\right] + \mathbf{k}\left[(2t)(0) - (2)(2)\right]$$

$$r'(t) \times r''(t) = \mathbf{i}(-12) - \mathbf{j}(-12t + 12t) + \mathbf{k}(-4)$$

$$r'(t) \times r''(t) = -12\mathbf{i} + 0\mathbf{j} - 4\mathbf{k}$$

Find the magnitude of the cross product.

$$|r'(t) \times r''(t)| = \sqrt{(-12)^2 + (0)^2 + (-4)^2}$$

$$|r'(t) \times r''(t)| = \sqrt{144 + 16}$$

$$|r'(t) \times r''(t)| = \sqrt{160}$$

$$|r'(t) \times r''(t)| = 4\sqrt{10}$$

Find the normal component of the acceleration vector.

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$$a_N = \frac{4\sqrt{10}}{2\sqrt{10t^2 + 1}}$$

$$a_N = \frac{2\sqrt{10}}{\sqrt{10t^2 + 1}}$$

Rationalize the denominator.

$$a_N = \frac{2\sqrt{10}}{\sqrt{10t^2 + 1}}\left(\frac{\sqrt{10t^2 + 1}}{\sqrt{10t^2 + 1}}\right)$$

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$$a_N = \frac{2\sqrt{10}\sqrt{10t^2 + 1}}{10t^2 + 1}$$

$$a_N = \frac{2\sqrt{10\left(10t^2 + 1\right)}}{10t^2 + 1}$$

Topic: Tangential and normal components of acceleration

Question: Find the tangential and normal components of acceleration.

$$r(t) = 2\mathbf{i} + 4t^2\mathbf{j} + 2t\mathbf{k}$$

Answer choices:

A $\quad a_T = \dfrac{64t\sqrt{16t^2+1}}{16t^2+1} \qquad a_N = \dfrac{16\sqrt{16t^2+1}}{16t^2+1}$

B $\quad a_T = \dfrac{32t\sqrt{16t^2+1}}{16t^2+1} \qquad a_N = \dfrac{8\sqrt{16t^2+1}}{16t^2+1}$

C $\quad a_T = \dfrac{9t\sqrt{16t^2+1}}{16t^2+1} \qquad a_N = \dfrac{2\sqrt{16t^2+1}}{16t^2+1}$

D $\quad a_T = \dfrac{18t\sqrt{16t^2+1}}{16t^2+1} \qquad a_N = \dfrac{4\sqrt{16t^2+1}}{16t^2+1}$

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Solution: B

Find the first derivative of the position vector

$$r(t) = 2\mathbf{i} + 4t^2\mathbf{j} + 2t\mathbf{k}$$

in order to get velocity,

$$r'(t) = 0\mathbf{i} + 8t\mathbf{j} + 2\mathbf{k}$$

and the second derivative in order to get acceleration.

$$r''(t) = 0\mathbf{i} + 8\mathbf{j} + 0\mathbf{k}$$

Find the dot product of the first and second derivatives.

$$r'(t) \cdot r''(t) = (0)(0) + (8t)(8) + (2)(0)$$

$$r'(t) \cdot r''(t) = 0 + 64t + 0$$

$$r'(t) \cdot r''(t) = 64t$$

Find the magnitude of the first derivative, $r'(t) = 0\mathbf{i} + 8t\mathbf{j} + 2\mathbf{k}$.

$$|r'(t)| = \sqrt{(r'(t)_1)^2 + (r'(t)_2)^2 + (r'(t)_3)^2}$$

$$|r'(t)| = \sqrt{(0)^2 + (8t)^2 + (2)^2}$$

$$|r'(t)| = \sqrt{64t^2 + 4}$$

$$|r'(t)| = 2\sqrt{16t^2 + 1}$$

Find the tangential component of the acceleration vector.

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$$a_T = \frac{r'(t) \cdot r''(t)}{|\, r'(t)\, |}$$

$$a_T = \frac{64t}{2\sqrt{16t^2 + 1}}$$

$$a_T = \frac{32t}{\sqrt{16t^2 + 1}}$$

Rationalize the denominator.

$$a_T = \frac{32t}{\sqrt{16t^2 + 1}} \left(\frac{\sqrt{16t^2 + 1}}{\sqrt{16t^2 + 1}} \right)$$

$$a_T = \frac{32t\sqrt{16t^2 + 1}}{16t^2 + 1}$$

This is the tangential component of acceleration.

To find the normal component of acceleration, we'll find the cross product $r'(t) \times r''(t)$, using the cross product formula:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \mathbf{i} \left(a_2 b_3 - a_3 b_2 \right) - \mathbf{j} \left(a_1 b_3 - a_3 b_1 \right) + \mathbf{k} \left(a_1 b_2 - a_2 b_1 \right)$$

$$r'(t) \times r''(t) = \mathbf{i} \left[(8t)(0) - (2)(8) \right] - \mathbf{j} \left[(0)(0) - (2)(0) \right] + \mathbf{k} \left[(0)(8) - (8t)(0) \right]$$

$$r'(t) \times r''(t) = \mathbf{i}(0 - 16) - \mathbf{j}(0 - 0) + \mathbf{k}(0 - 0)$$

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$$r'(t) \times r''(t) = -16\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

Find the magnitude of the cross product.

$$|r'(t) \times r''(t)| = \sqrt{(-16)^2 + (0)^2 + (0)^2}$$

$$|r'(t) \times r''(t)| = \sqrt{256 + 0 + 0}$$

$$|r'(t) \times r''(t)| = 16$$

Find the normal component of the acceleration vector.

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$$a_N = \frac{16}{2\sqrt{16t^2 + 1}}$$

$$a_N = \frac{8}{\sqrt{16t^2 + 1}}$$

Rationalize the denominator.

$$a_N = \frac{8}{\sqrt{16t^2 + 1}}\left(\frac{\sqrt{16t^2 + 1}}{\sqrt{16t^2 + 1}}\right)$$

$$a_N = \frac{8\sqrt{16t^2 + 1}}{16t^2 + 1}$$

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Topic: Tangential and normal components of acceleration

Question: Find the tangential and normal components of the acceleration.

$$r(t) = \sin t\mathbf{i} + 2t\mathbf{j} + \cos t\mathbf{k}$$

Answer choices:

A $\quad a_T = \sqrt{5} \quad a_N = \frac{\sqrt{5}}{5}$

B $\quad a_T = \frac{\sqrt{5}}{5} \quad a_N = \sqrt{5}$

C $\quad a_T = 0 \quad a_N = 1$

D $\quad a_T = 1 \quad a_N = 0$

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Solution: C

Find the first derivative of the position vector

$$r(t) = \sin t\mathbf{i} + 2t\mathbf{j} + \cos t\mathbf{k}$$

in order to get velocity,

$$r'(t) = \cos t\mathbf{i} + 2\mathbf{j} - \sin t\mathbf{k}$$

and the second derivative in order to get acceleration.

$$r'(t) = -\sin t\mathbf{i} + 0\mathbf{j} - \cos t\mathbf{k}$$

Find the dot product of the first and second derivatives.

$$r'(t) \cdot r''(t) = (\cos t)(-\sin t) + (2)(0) + (-\sin t)(-\cos t)$$

$$r'(t) \cdot r''(t) = -\sin t\cos t + 0 + \sin t\cos t$$

$$r'(t) \cdot r''(t) = 0$$

Find the magnitude of the first derivative, $r'(t) = -\sin t\mathbf{i} + 0\mathbf{j} - \cos t\mathbf{k}$.

$$|r'(t)| = \sqrt{(r'(t)_1)^2 + (r'(t)_2)^2 + (r'(t)_3)^2}$$

$$|r'(t)| = \sqrt{(\cos t)^2 + (2)^2 + (-\sin t)^2}$$

$$|r'(t)| = \sqrt{\cos^2 t + 4 + \sin^2 t}$$

$$|r'(t)| = \sqrt{\sin^2 t + \cos^2 t + 4}$$

Use the identity $\sin^2 t + \cos^2 t = 1$.

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$$|r'(t)| = \sqrt{1+4}$$

$$|r'(t)| = \sqrt{5}$$

Find the tangential component of the acceleration vector.

$$a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|}$$

$$a_T = \frac{0}{\sqrt{5}}$$

$$a_T = 0$$

This is the tangential component of acceleration.

To find the normal component of acceleration, we'll find the cross product $r'(t) \times r''(t)$, using the cross product formula:

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

$$r'(t) \times r''(t) = \mathbf{i}\left[(2)(-\cos t) - (-\sin t)(0)\right] - \mathbf{j}\left[(\cos t)(-\cos t) - (-\sin t)(-\sin t)\right]$$

$$+\mathbf{k}\left[(\cos t)(0) - (2)(-\sin t)\right]$$

$$r'(t) \times r''(t) = (-2\cos t + 0)\mathbf{i} - \left(-\cos^2 t - \sin^2 t\right)\mathbf{j} + \left(0 + 2\sin t\right)\mathbf{k}$$

$$r'(t) \times r''(t) = -2\cos t\mathbf{i} + \left(\cos^2 t + \sin^2 t\right)\mathbf{j} + 2\sin t\mathbf{k}$$

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Use the identity $\sin^2 x + \cos^2 x = 1$.

$$r'(t) \times r''(t) = -2\cos t\mathbf{i} + (1)\mathbf{j} + 2\sin t\mathbf{k}$$

$$r'(t) \times r''(t) = -2\cos t\mathbf{i} + \mathbf{j} + 2\sin t\mathbf{k}$$

Find the magnitude of the cross product.

$$|r'(t) \times r''(t)| = \sqrt{(-2\cos t)^2 + (1)^2 + (2\sin t)^2}$$

$$|r'(t) \times r''(t)| = \sqrt{4\cos^2 t + 1 + 4\sin^2 t}$$

$$|r'(t) \times r''(t)| = \sqrt{4\left(\cos^2 t + \sin^2 t\right) + 1}$$

Use the identity $\sin^2 x + \cos^2 x = 1$.

$$|r'(t) \times r''(t)| = \sqrt{4(1) + 1}$$

$$|r'(t) \times r''(t)| = \sqrt{5}$$

Find the normal component of the acceleration vector.

$$a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}$$

$$a_N = \frac{\sqrt{5}}{\sqrt{5}}$$

$$a_N = 1$$

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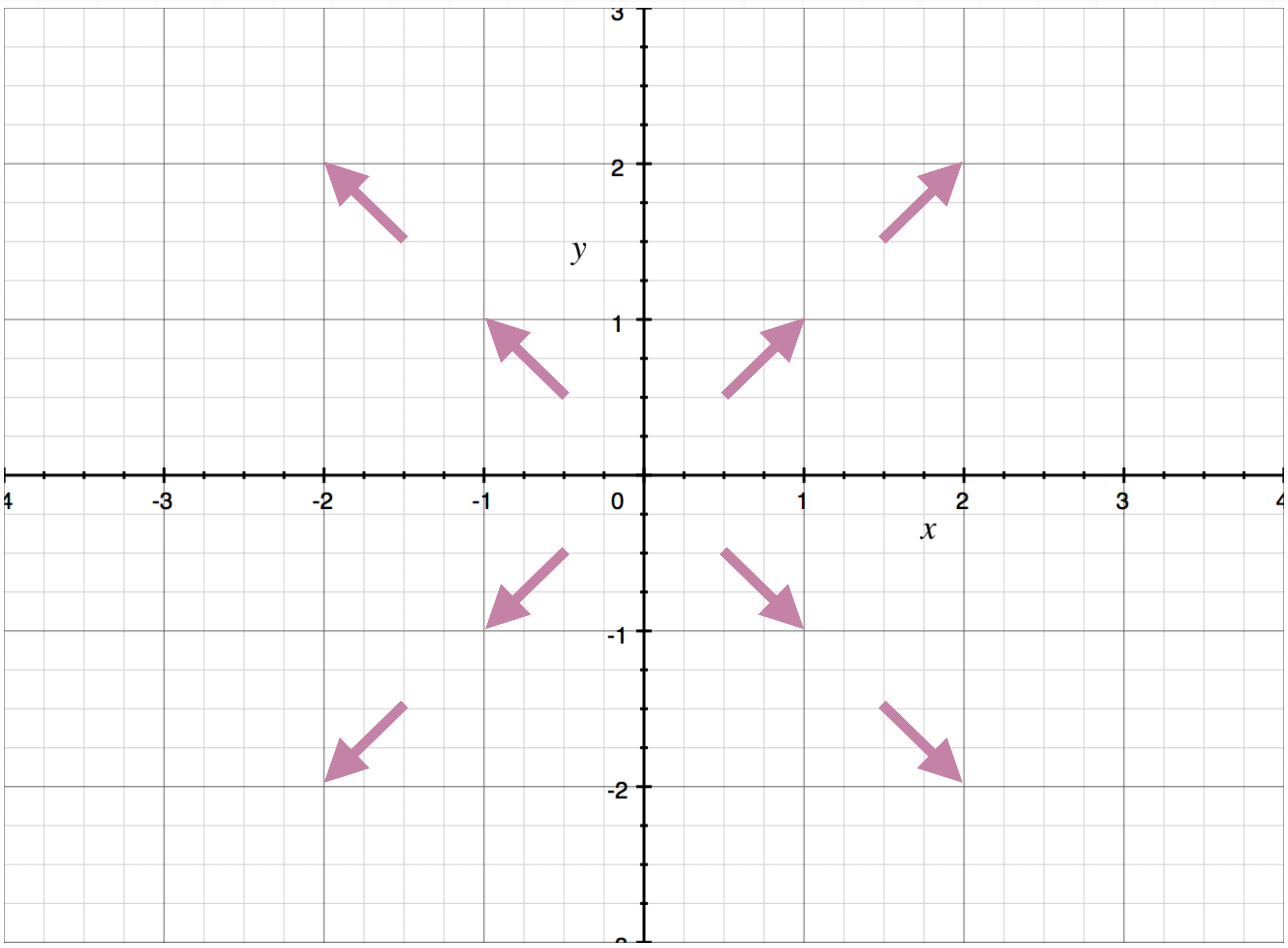
Topic: Sketching the vector field

Question: Which vector field is shown in the sketch?



Answer choices:

A $F(x, y) = -x\mathbf{i} + y\mathbf{j}$

B $F(x, y) = x\mathbf{i} - y\mathbf{j}$

C $F(x, y) = x\mathbf{i} + y\mathbf{j}$

D $F(x, y) = -x\mathbf{i} - y\mathbf{j}$

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Solution: C

For this sketch, we can see that the following points were used:

$(1,1)$ $\qquad$ $(2,-2)$

$(1,-1)$ $\qquad$ $(-2,-2)$

$(-1,-1)$ $\qquad$ $(-2,2)$

$(-1,1)$ $\qquad$ $(2,2)$

The vectors at each point are simple and easy to calculate.

- For the point $(1,1)$, the vector is $F(x,y)=\mathbf{i}+\mathbf{j}$.
- For the point $(1,-1)$, the vector is $F(x,y)=\mathbf{i}-\mathbf{j}$.
- For the point $(-1,-1)$, the vector is $F(x,y)=-\mathbf{i}-\mathbf{j}$.
- For the point $(-1,1)$, the vector is $F(x,y)=-\mathbf{i}+\mathbf{j}$.

From these four points we can develop a general vector for this field.

$$F(x,y)=x\mathbf{i}+y\mathbf{j}$$

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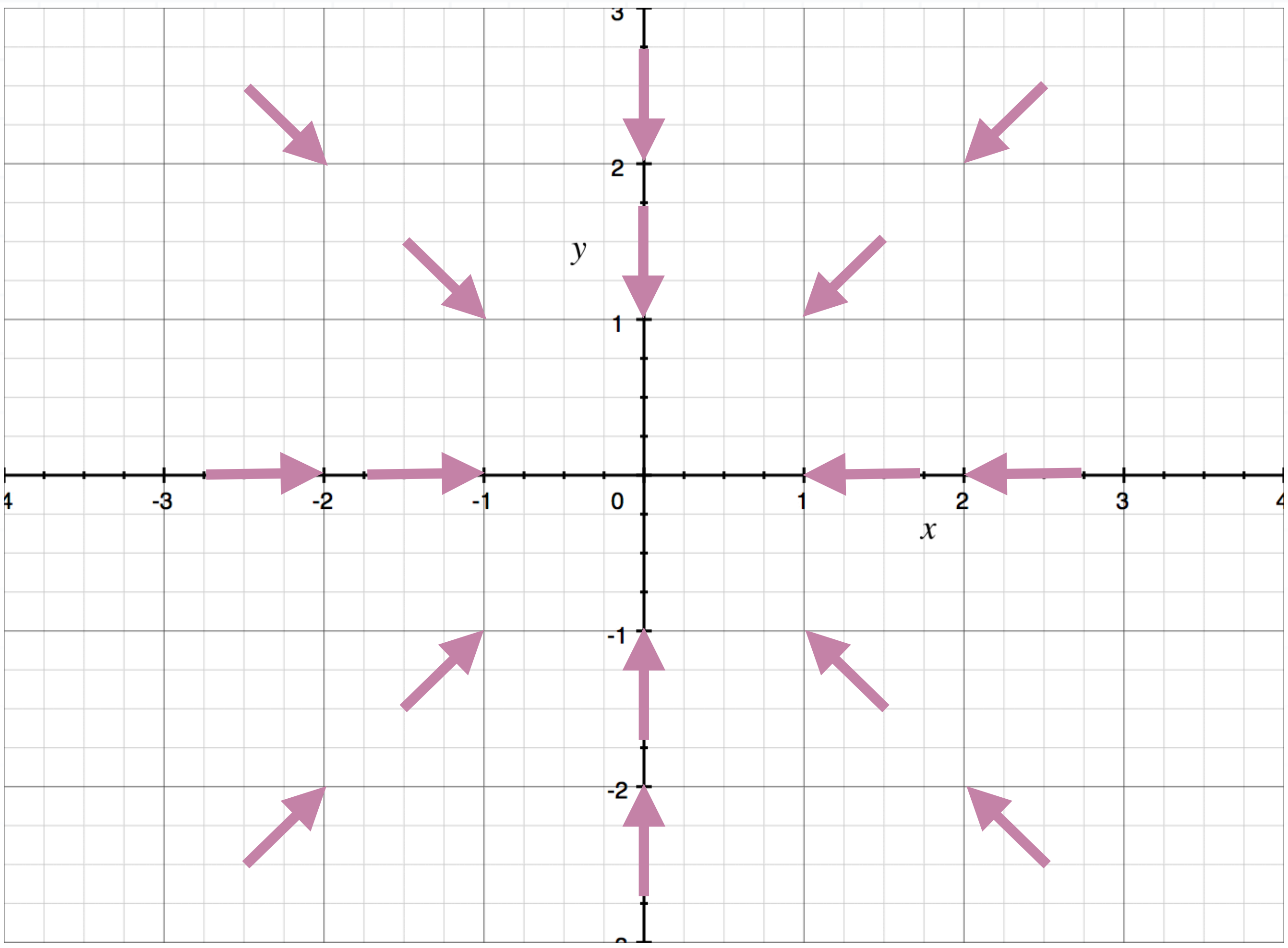
Topic: Sketching the vector field

Question: Which vector field is shown in the sketch?



Answer choices:

A $\quad F(x, y) = -x\mathbf{i} + y\mathbf{j}$

B $\quad F(x, y) = x\mathbf{i} - y\mathbf{j}$

C $\quad F(x, y) = x\mathbf{i} + y\mathbf{j}$

D $\quad F(x, y) = -x\mathbf{i} - y\mathbf{j}$

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Solution: D

For this sketch, we can see that the following points were used:

$(1,1)$ $\qquad$ $(2,-2)$

$(1,-1)$ $\qquad$ $(-2,-2)$

$(-1,-1)$ $\qquad$ $(-2,2)$

$(-1,1)$ $\qquad$ $(2,2)$

The vectors at each point are simple and easy to calculate.

- For the point $(1,1)$, the vector is $F(x,y)=-\mathbf{i}-\mathbf{j}$.
- For the point $(1,-1)$, the vector is $F(x,y)=-\mathbf{i}+\mathbf{j}$.
- For the point $(-1,-1)$, the vector is $F(x,y)=\mathbf{i}+\mathbf{j}$.
- For the point $(-1,1)$, the vector is $F(x,y)=\mathbf{i}-\mathbf{j}$.

From these four points we can develop a general vector for this field.

$$F(x,y)=-x\mathbf{i}-y\mathbf{j}$$

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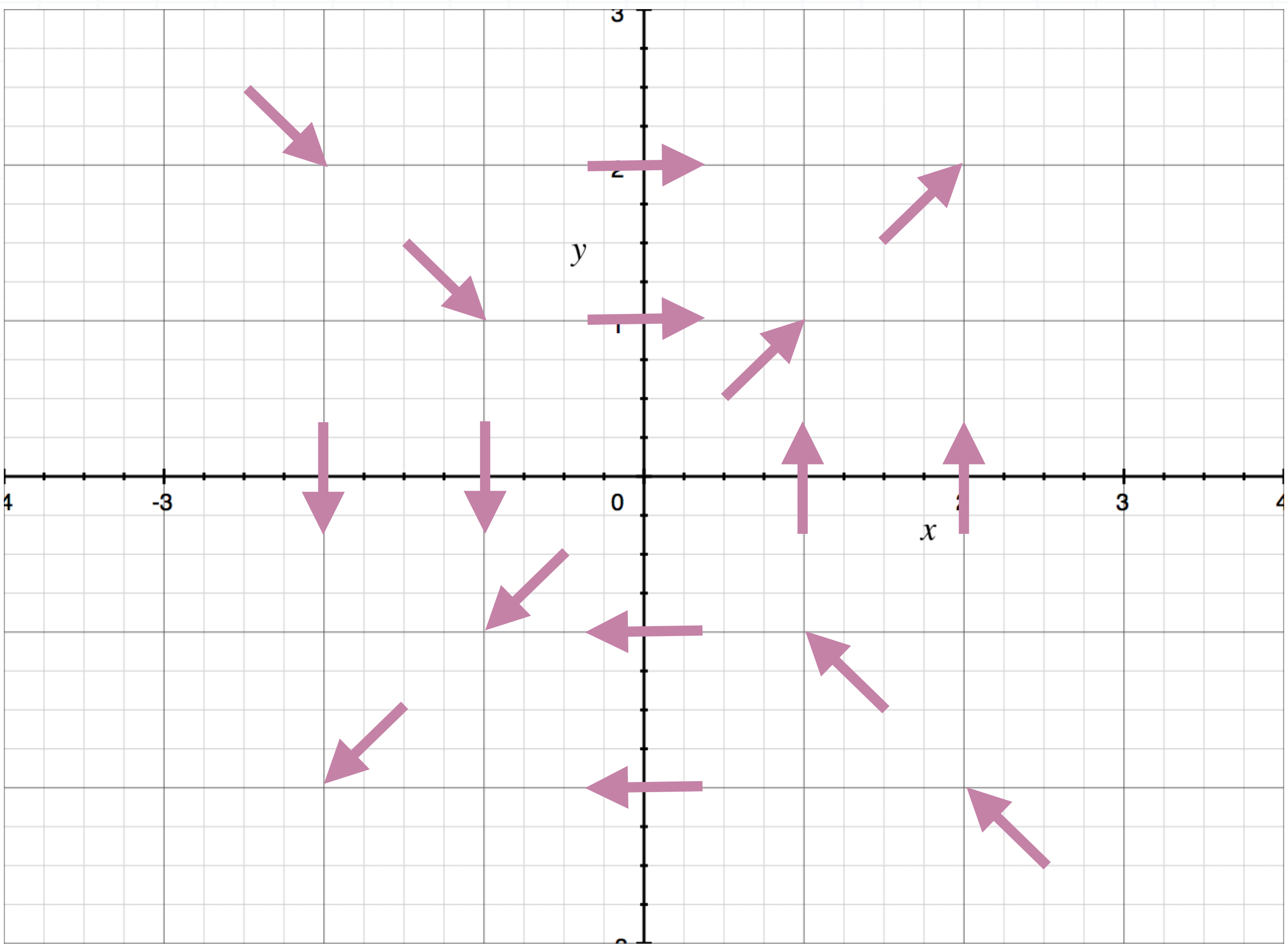
Topic: Sketching the vector field

Question: Which vector field is shown in the sketch?



Answer choices:

A $F(x, y) = x\mathbf{i} + y\mathbf{j}$

B $F(x, y) = y\mathbf{i} + x\mathbf{j}$

C $F(x, y) = -x\mathbf{i} - y\mathbf{j}$

D $F(x, y) = -y\mathbf{i} - x\mathbf{j}$

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Solution: B

For this sketch, we can see that the following points were used:

$(1,1)$	$(2,-2)$
$(1,-1)$	$(-2,-2)$
$(-1,-1)$	$(-2,2)$
$(-1,1)$	$(2,2)$

The vectors at each point are simple and easy to calculate.

- For the point $(1,1)$, the vector is $F(x,y)=\mathbf{i}+\mathbf{j}$.
- For the point $(1,-1)$, the vector is $F(x,y)=-\mathbf{i}+\mathbf{j}$.
- For the point $(-1,-1)$, the vector is $F(x,y)=-\mathbf{i}-\mathbf{j}$.
- For the point $(-1,1)$, the vector is $F(x,y)=\mathbf{i}-\mathbf{j}$.
- For the point $(2,-2)$, the vector is $F(x,y)=-2\mathbf{i}+2\mathbf{j}$.
- For the point $(-2,-2)$, the vector is $F(x,y)=-2\mathbf{i}-2\mathbf{j}$.
- For the point $(-2,2)$, the vector is $F(x,y)=2\mathbf{i}-2\mathbf{j}$.
- For the point $(2,2)$, the vector is $F(x,y)=2\mathbf{i}+2\mathbf{j}$.

From these eight points we can develop a general vector for this field.

$$F(x,y)=y\mathbf{i}+x\mathbf{j}$$

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Topic: Gradient vector field

Question: Find the gradient vector field of the function.

$$f(xy, x^2e^y)$$

Answer choices:

A $\nabla f = \langle x, 2xe^y \rangle$

B $\nabla f = \left\langle y, x^2 \right\rangle$

C $\nabla f = \left\langle xy, x^2e^y \right\rangle$

D $\nabla f = \left\langle y, x^2e^y \right\rangle$

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Solution: D

To find the gradient vector field of the function $f(xy, x^2e^y)$, we can use the formula $\nabla f = \langle f_x, f_y \rangle$ where f_x is the partial derivative of the function $f(x, y)$ with respect to x and f_y is the partial derivative of the function $f(x, y)$ with respect to y.

We'll find the partial derivative of the x-value with respect to x. The x-value from $f(xy, x^2e^y)$ is xy, and the derivative of that with respect to x is

$$f_x = y$$

We'll find the partial derivative of the y-value with respect to y. The y-value from $f(xy, x^2e^y)$ is x^2e^y, and the derivative of that with respect to y is

$$f_y = x^2e^y$$

Now we'll use these values to find the gradient vector.

$$\nabla f = \langle f_x, f_y \rangle$$

$$\nabla f = \left\langle y, x^2e^y \right\rangle$$

Topic: Gradient vector field

Question: Find the gradient vector field of the function.

$$f\left(z \sin x, xyz, z^2 \ln x\right)$$

Answer choices:

A $\quad \nabla f = \langle -z \sin x, xz, 2z \ln x \rangle$

B $\quad \nabla f = \langle z \cos x, xz, 2z \ln x \rangle$

C $\quad \nabla f = \left\langle z \cos x, xz, \dfrac{z^2}{x} \right\rangle$

D $\quad \nabla f = \left\langle z \sin x, xz, \dfrac{z^2}{x} \right\rangle$

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Solution: B

To find the gradient vector field of the function $f\left(z\sin x, xyz, z^2\ln x\right)$, we can use the formula $\nabla f=\langle f_x, f_y, f_z\rangle$ where f_x is the partial derivative of the function $f(x,y,z)$ with respect to x, f_y is the partial derivative of the function $f(x,y,z)$ with respect to y, and f_z is the partial derivative of the function $f(x,y,z)$ with respect to z.

We'll find the partial derivative of the x-value with respect to x. The x-value from $f\left(z\sin x, xyz, z^2\ln x\right)$ is $z\sin x$, and the derivative of that with respect to x is

$$f_x=z\cos x$$

We'll find the partial derivative of the y-value with respect to y. The y-value from $f\left(z\sin x, xyz, z^2\ln x\right)$ is xyz, and the derivative of that with respect to y is

$$f_y=xz$$

We'll find the partial derivative of the z-value with respect to z. The z-value from $f\left(z\sin x, xyz, z^2\ln x\right)$ is $z^2\ln x$, and the derivative of that with respect to z is

$$f_z=2z\ln x$$

Now we'll use these values to find the gradient vector.

$$\nabla f=\langle f_x, f_y, f_z\rangle$$

$$\nabla f=\langle z\cos x, xz, 2z\ln x\rangle$$

Topic: Gradient vector field

Question: Find the gradient vector field of the function.

$$f\left(e^{-xyz}, \sin x \cos z, y^3 2^z\right)$$

Answer choices:

A $\quad \nabla f = \left\langle -xyze^{-xyz}, -\sin x \sin z, 3y^2 2^z \right\rangle$

B $\quad \nabla f = \left\langle -yze^{-xyz}, -\sin x \sin z, 3y^2 2^z \right\rangle$

C $\quad \nabla f = \left\langle -yze^{-xyz}, 0, \ln(2) y^3 2^z \right\rangle$

D $\quad \nabla f = \left\langle -xyze^{-xyz}, 0, y^3 2^z \right\rangle$

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Solution: C

To find the gradient vector field of the function $f\left(e^{-xyz}, \sin x \cos z, y^3 2^z\right)$, we can use the formula $\nabla f = \langle f_x, f_y, f_z \rangle$ where f_x is the partial derivative of the function $f(x, y, z)$ with respect to x, f_y is the partial derivative of the function $f(x, y, z)$ with respect to y, and f_z is the partial derivative of the function $f(x, y, z)$ with respect to z.

We'll find the partial derivative of the x-value with respect to x. The x-value from $f\left(e^{-xyz}, \sin x \cos z, y^3 2^z\right)$ is e^{-xyz}, and the derivative of that with respect to x is

$$f_x = -yze^{-xyz}$$

We'll find the partial derivative of the y-value with respect to y. The y-value from $f\left(e^{-xyz}, \sin x \cos z, y^3 2^z\right)$ is $\sin x \cos z$, and the derivative of that with respect to y is

$$f_y = 0$$

We'll find the partial derivative of the z-value with respect to z. The z-value from $f\left(e^{-xyz}, \sin x \cos z, y^3 2^z\right)$ is $y^3 2^z$, and the derivative of that with respect to z is

$$f_z = \ln(2) y^3 2^z$$

Now we'll use these values to find the gradient vector.

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

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$$\nabla f = \left\langle -yze^{-xyz}, 0, \ln(2)y^3 2^z \right\rangle$$

Topic: Line integral of a curve

Question: Calculate the line integral if c is the line segment from $(0,0,0)$ to $(1,1,1)$.

$$\int_c xyz\ ds$$

Answer choices:

A $\quad \frac{\sqrt{3}}{4}$

B $\quad \frac{3}{4}$

C $\quad \frac{\sqrt{3}}{2}$

D $\quad \frac{3}{2}$

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Solution: A

We've been asked to look at the line integral for the line segment connecting $(0,0,0)$ and $(1,1,1)$. We'll find the parametric equations for the line segment.

$$x = (x_2 - x_1)t$$

$$y = (y_2 - y_1)t$$

$$z = (z_2 - z_1)t$$

We get

$$x = (1 - 0)t \qquad y = (1 - 0)t \qquad z = (1 - 0)t$$

$$x = t \qquad y = t \qquad z = t$$

Find the derivatives of these parametric equations.

$$\frac{dx}{dt} = 1 \qquad \frac{dy}{dt} = 1 \qquad \frac{dz}{dt} = 1$$

Now we'll take the function from the integral $f(x, y, z) = xyz$ and rewrite it in terms of t using $x = t$, $y = t$ and $z = t$.

$$f(x(t), y(t), z(t)) = (t)(t)(t)$$

$$f(x(t), y(t), z(t)) = t^3$$

Now we can find the line integral of the curve.

$$\int_c f(x, y, z)\ ds$$

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$$\int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\ dt$$

$$\int_0^1 t^3\sqrt{(1)^2+(1)^2+(1)^2}\ dt$$

$$\int_0^1 t^3\sqrt{3}\ dt$$

$$\sqrt{3}\int_0^1 t^3\ dt$$

Integrate, then evaluate over the interval to get the line integral.

$$\sqrt{3}\left(\frac{1}{4}t^4\right)\Big|_0^1$$

$$\sqrt{3}\left(\frac{1}{4}(1)^4-\frac{1}{4}(0)^4\right)$$

$$\sqrt{3}\left(\frac{1}{4}\right)$$

$$\frac{\sqrt{3}}{4}$$

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Topic: Line integral of a curve

Question: Calculate the line integral if c is the line segment from $(0,0,0)$ to $(3,4,0)$.

$$\int_c x^2ye^z \ ds$$

Answer choices:

A $\quad 32$

B $\quad 160$

C $\quad 45$

D $\quad 8$

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Solution: C

We've been asked to look at the line integral for the line segment connecting $(0,0,0)$ and $(3,4,0)$. We'll find the parametric equations for the line segment.

$$x = (x_2 - x_1)t$$

$$y = (y_2 - y_1)t$$

$$z = (z_2 - z_1)t$$

We get

$$x = (3 - 0)t \qquad y = (4 - 0)t \qquad z = (0 - 0)t$$

$$x = 3t \qquad y = 4t \qquad z = 0$$

Find the derivatives of these parametric equations.

$$\frac{dx}{dt} = 3 \qquad \frac{dy}{dt} = 4 \qquad \frac{dz}{dt} = 0$$

Now we'll take the function from the integral $f(x, y, z) = x^2ye^z$ and rewrite it in terms of t using $x = 3t$, $y = 4t$ and $z = 0$.

$$f(x(t), y(t), z(t)) = (3t)^2(4t)e^0$$

$$f(x(t), y(t), z(t)) = 36t^3$$

Now we can find the line integral of the curve.

$$\int_c f(x, y, z)\ ds$$

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$$\int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$\int_0^1 36t^3\sqrt{(3)^2 + (4)^2 + (0)^2}\ dt$$

$$\int_0^1 36t^3\sqrt{9 + 16 + 0}\ dt$$

$$\int_0^1 36t^3\sqrt{25}\ dt$$

$$\int_0^1 36(5)t^3\ dt$$

$$180\int_0^1 t^3\ dt$$

Integrate, then evaluate over the interval to get the line integral.

$$180\left(\frac{1}{4}t^4\right)\Big|_0^1$$

$$180\left(\frac{1}{4}(1)^4 - \frac{1}{4}(0)^4\right)$$

$$180\left(\frac{1}{4}\right)$$

$$45$$

Topic: Line integral of a curve

Question: Calculate the line integral if c is the line segment from $(0,0,0)$ to $(2,1,4)$.

$$\int_c ye^{xz}\ ds$$

Answer choices:

A $\quad \dfrac{\sqrt{21}}{16}(e^{64}-1)$

B $\quad \dfrac{\sqrt{21}}{16}(e^{8}-1)$

C $\quad \dfrac{\sqrt{21}}{8}(e^{64}-1)$

D $\quad \dfrac{\sqrt{21}}{8}(e^{8}-1)$

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Solution: B

We've been asked to look at the line integral for the line segment connecting $(0,0,0)$ and $(2,1,4)$. We'll find the parametric equations for the line segment.

$$x = (x_2 - x_1)t$$

$$y = (y_2 - y_1)t$$

$$z = (z_2 - z_1)t$$

We get

$$x = (2 - 0)t \qquad y = (1 - 0)t \qquad z = (4 - 0)t$$

$$x = 2t \qquad y = t \qquad z = 4t$$

Find the derivatives of these parametric equations.

$$\frac{dx}{dt} = 2 \qquad \frac{dy}{dt} = 1 \qquad \frac{dz}{dt} = 4$$

Now we'll take the function from the integral $f(x, y, z) = ye^{xz}$ and rewrite it in terms of t using $x = 2t$, $y = t$ and $z = 4t$.

$$f(x(t), y(t), z(t)) = (t)e^{(2t)(4t)}$$

$$f(x(t), y(t), z(t)) = te^{8t^2}$$

Now we can find the line integral of the curve.

$$\int_c f(x, y, z)\ ds$$

$$\int_a^b f(x(t), y(t), z(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$\int_0^1 te^{8t^2}\sqrt{(2)^2 + (1)^2 + (4)^2}\ dt$$

$$\int_0^1 te^{8t^2}\sqrt{4 + 1 + 16}\ dt$$

$$\int_0^1 te^{8t^2}\sqrt{21}\ dt$$

$$\sqrt{21}\int_0^1 te^{8t^2}\ dt$$

Integrate using u-substitution.

$$u = 8t^2$$

$$du = 16t\ dt$$

$$dt = \frac{1}{16t}\ du$$

Substitute into the integral.

$$\sqrt{21}\int_{t=0}^{t=1} te^u\left(\frac{1}{16t}\ du\right)$$

$$\frac{\sqrt{21}}{16}\int_{t=0}^{t=1} e^u\ du$$

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Integrate, and then back-substitute to get the value back in terms of t.

$$\frac{\sqrt{21}}{16}e^{u}\Big|_{t=0}^{t=1}$$

$$\frac{\sqrt{21}}{16}e^{8t^2}\Big|_{0}^{1}$$

Evaluate over the interval.

$$\frac{\sqrt{21}}{16}e^{8(1)^2}-\frac{\sqrt{21}}{16}e^{8(0)^2}$$

$$\frac{\sqrt{21}}{16}e^{8}-\frac{\sqrt{21}}{16}e^{0}$$

$$\frac{\sqrt{21}}{16}e^{8}-\frac{\sqrt{21}}{16}$$

$$\frac{\sqrt{21}}{16}(e^{8}-1)$$

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Topic: Line integral of a vector function

Question: Find the line integral of the vector function.

$$F(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k}$$

at $r(t) = t\mathbf{i} - t\mathbf{j} + 2t\mathbf{k}$

on $0 \leq t \leq 1$

Answer choices:

A $\frac{17}{6}$

B $\frac{35}{6}$

C $\frac{13}{6}$

D $\frac{15}{6}$

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Solution: C

We'll start by writing out the parametric equations for the vector function $r(t) = t\mathbf{i} - t\mathbf{j} + 2t\mathbf{k}$.

$$x = t$$

$$y = -t$$

$$z = 2t$$

Next we can find $F(r(t))$ using $F(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + z^2\mathbf{k}$ and these parametric equations.

$$F(r(t)) = (t)\mathbf{i} - 2(-t)\mathbf{j} + (2t)^2\mathbf{k}$$

$$F(r(t)) = t\mathbf{i} + 2t\mathbf{j} + 4t^2\mathbf{k}$$

Now we can find $r'(t)$ using $r(t) = t\mathbf{i} - t\mathbf{j} + 2t\mathbf{k}$.

$$r'(t) = \mathbf{i} - \mathbf{j} + 2\mathbf{k}$$

The limits are given in the question as $0 \le t \le 1$, which means $a = 0$ and $b = 1$. Next we can find the line integral.

$$\int_a^b F(r(t))r'(t)\ dt$$

$$\int_0^1 (t\mathbf{i} + 2t\mathbf{j} + 4t^2\mathbf{k})(\mathbf{i} - \mathbf{j} + 2\mathbf{k})\ dt$$

Simplify, remembering that only the like terms multiply (the $\mathbf{i}$ term to the other $\mathbf{i}$ term and so on).

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$$\int_0^1 t\mathbf{i} - 2t\mathbf{j} + 8t^2\mathbf{k} \ dt$$

At this point we can drop the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$\int_0^1 t - 2t + 8t^2 \ dt$$

$$\int_0^1 -t + 8t^2 \ dt$$

Integrate, then evaluate over the interval.

$$-\frac{1}{2}t^2 + \frac{8}{3}t^3 \Big|_0^1$$

$$-\frac{1}{2}(1)^2 + \frac{8}{3}(1)^3 - \left(-\frac{1}{2}(0)^2 + \frac{8}{3}(0)^3\right)$$

$$-\frac{1}{2} + \frac{8}{3}$$

$$-\frac{3}{6} + \frac{16}{6}$$

$$\frac{13}{6}$$

This the line line integral of the vector function.

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Topic: Line integral of a vector function

Question: Find the line integral of the vector function.

$$F(x, y, z) = \cos x\mathbf{i} + e^{2y}\mathbf{j} - yz\mathbf{k}$$

at $r(t) = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k}$

on $0 \le t \le 1$

Answer choices:

A $\quad \sin(1) + \frac{1}{2}e^2 + \frac{1}{3}$

B $\quad \sin(1) + \frac{1}{2}e^2 - \frac{1}{3}$

C $\quad \sin(1) + \frac{1}{2}e^2 - \frac{1}{6}$

D $\quad \sin(1) + \frac{1}{2}e^2 - \frac{5}{6}$

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Solution: D

We'll start by writing out the parametric equations for the vector function $r(t) = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k}$.

$$x = t^2$$

$$y = t$$

$$z = t$$

Next we can find $F(r(t))$ using $F(x, y, z) = \cos x\mathbf{i} + e^{2y}\mathbf{j} - yz\mathbf{k}$ and these parametric equations.

$$F(r(t)) = \cos(t^2)\mathbf{i} + e^{2t}\mathbf{j} - (t)(t)\mathbf{k}$$

$$F(r(t)) = \cos(t^2)\mathbf{i} + e^{2t}\mathbf{j} - t^2\mathbf{k}$$

Now we can find $r'(t)$ using $r(t) = t^2\mathbf{i} + t\mathbf{j} + t\mathbf{k}$.

$$r'(t) = 2t\mathbf{i} + \mathbf{j} + \mathbf{k}$$

The limits are given in the question as $0 \le t \le 1$, which means $a = 0$ and $b = 1$. Next we can find the line integral.

$$\int_a^b F(r(t))r'(t)\ dt$$

$$\int_0^1 \left(\cos(t^2)\mathbf{i} + e^{2t}\mathbf{j} - t^2\mathbf{k}\right)\left(2t\mathbf{i} + \mathbf{j} + \mathbf{k}\right)\ dt$$

Simplify, remembering that only the like terms multiply (the $\mathbf{i}$ term to the other $\mathbf{i}$ term and so on).

$$\int_0^1 2t\cos(t^2)\mathbf{i} + e^{2t}\mathbf{j} - t^2\mathbf{k}\ dt$$

At this point we can drop the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$\int_0^1 2t\cos(t^2) + e^{2t} - t^2\ dt$$

$$\int_0^1 2t\cos(t^2)\ dt + \int_0^1 e^{2t}\ dt + \int_0^1 -t^2\ dt$$

We'll use u-substitution to handle the first integral, setting $u = t^2$, and $du = 2t\ dt$, so $dt = du/2t$.

$$\int_{t=0}^{t=1} 2t\cos u\left(\frac{du}{2t}\right) + \int_0^1 e^{2t}\ dt + \int_0^1 -t^2\ dt$$

$$\int_{t=0}^{t=1} \cos u\ du + \int_0^1 e^{2t}\ dt + \int_0^1 -t^2\ dt$$

Integrate, then back substitute.

$$\sin u\Big|_{t=0}^{t=1} + \int_0^1 e^{2t}\ dt + \int_0^1 -t^2\ dt$$

$$\sin(t^2)\Big|_0^1 + \int_0^1 e^{2t}\ dt + \int_0^1 -t^2\ dt$$

Integrate the other integrals.

$$\sin(t^2)\Big|_0^1 + \frac{1}{2}e^{2t}\Big|_0^1 - \frac{1}{3}t^3\Big|_0^1$$

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$$\sin(t^2) + \frac{1}{2}e^{2t} - \frac{1}{3}t^3 \Big|_0^1$$

Evaluate over the interval.

$$\sin(1^2) + \frac{1}{2}e^{2(1)} - \frac{1}{3}(1)^3 - \left(\sin(0^2) + \frac{1}{2}e^{2(0)} - \frac{1}{3}(0)^3\right)$$

$$\sin(1) + \frac{1}{2}e^2 - \frac{1}{3} - \left(\sin(0) + \frac{1}{2}e^0 - 0\right)$$

$$\sin(1) + \frac{1}{2}e^2 - \frac{1}{3} - \frac{1}{2}$$

$$\sin(1) + \frac{1}{2}e^2 - \frac{5}{6}$$

This the line line integral of the vector function.

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Topic: Line integral of a vector function

Question: Find the line integral of the vector function.

$$F(x,y,z) = e^x\mathbf{i} + y\mathbf{j} - \frac{1}{z}\mathbf{k}$$

at $r(t) = t\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$

on $1 \le t \le 2$

Answer choices:

A $e^2 - e + 6 - \ln 2$

B $e + 6 + \ln\left(\frac{1}{2}\right)$

C $e^2 - e + 6 + \ln 2$

D $e + 6 + \ln 2$

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Solution: A

We'll start by writing out the parametric equations for the vector function $r(t) = t\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$.

$$x = t$$

$$y = -2t$$

$$z = t$$

Next we can find $F(r(t))$ using $F(x, y, z) = e^x\mathbf{i} + y\mathbf{j} - (1/z)\mathbf{k}$ and these parametric equations.

$$F(r(t)) = e^t\mathbf{i} - 2t\mathbf{j} - \frac{1}{t}\mathbf{k}$$

Now we can find $r'(t)$ using $r(t) = t\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$.

$$r'(t) = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

The limits are given in the question as $1 \leq t \leq 2$, which means $a = 1$ and $b = 2$. Next we can find the line integral.

$$\int_a^b F(r(t))r'(t)\ dt$$

$$\int_1^2 \left(e^t\mathbf{i} - 2t\mathbf{j} - \frac{1}{t}\mathbf{k}\right)(\mathbf{i} - 2\mathbf{j} + \mathbf{k})\ dt$$

Simplify, remembering that only the like terms multiply (the $\mathbf{i}$ term to the other $\mathbf{i}$ term and so on).

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$$\int_1^2 e^t\mathbf{i} + 4t\mathbf{j} - \frac{1}{t}\mathbf{k}\ dt$$

At this point we can drop the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$\int_1^2 e^t + 4t - \frac{1}{t}\ dt$$

Integrate, then evaluate over the interval.

$$e^t + 2t^2 - \ln t\Big|_1^2$$

$$e^2 + 2(2)^2 - \ln 2 - \left(e^1 + 2(1)^2 - \ln 1\right)$$

$$e^2 + 8 - \ln 2 - (e + 2 - 0)$$

$$e^2 + 8 - \ln 2 - e - 2$$

$$e^2 - e + 6 - \ln 2$$

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Topic: Potential function of a conservative vector field

Question: Is the vector field conservative?

$$F(x, y) = \frac{4}{3}x\sqrt{y^3}\mathbf{i} + x^2\sqrt{y}\mathbf{j}$$

Answer choices:

A Yes when $y = 0$

B Yes

C No

D Yes when $x = 0$

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Solution: C

Given the vector field

$$F(x, y) = \frac{4}{3}x\sqrt{y^3}\mathbf{i} + x^2\sqrt{y}\mathbf{j}$$

we know that $F_2 = x^2\sqrt{y}$, so

$$\frac{\partial F_2}{\partial x} = 2xy^{\frac{1}{2}}$$

We know also that

$$F_1 = \frac{4}{3}x\sqrt{y^3}\mathbf{i}$$

so

$$\frac{\partial F_1}{\partial y} = \frac{4}{3}\left(\frac{3}{2}\right)xy^{\frac{1}{2}}$$

$$\frac{\partial F_1}{\partial y} = 2xy^{\frac{1}{2}}$$

Now we can calculate the scalar curl.

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 0$$

$$2xy^{\frac{1}{2}} - 2xy^{\frac{1}{2}} = 0$$

$$0 = 0$$

Since the scalar curl of the vector field is 0, it might be conservative.

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Next we need to check if there are any undefined values in the domain. The vector field contains radicals in the coefficients, and negative values beneath the roots will create undefined values in the domain.

The vector field has a scalar curl of 0 but it also has undefined values in its domain. Therefore, it's not conservative.

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Topic: Potential function of a conservative vector field

Question: Find the function for the conservative vector field.

$$F(x, y) = \frac{1}{2}xy^4\mathbf{i} + x^2y^3\mathbf{j}$$

$$F = \nabla f$$

Answer choices:

A $\quad f(x, y) = \frac{1}{4}xy^4 + k$

B $\quad f(x, y) = xy^3 + k$

C $\quad f(x, y) = \frac{1}{4}x^2y^4 + k$

D $\quad f(x, y) = x^2y^3 + k$

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Solution: C

From

$$F(x, y) = \frac{1}{2}xy^4\mathbf{i} + x^2y^3\mathbf{j}$$

we know that

$$F_1 = \frac{1}{2}xy^4$$

First, we'll calculate

$$\int f_x(x, y) = \int F_1\ dx$$

We get

$$\int f_x(x, y) = \int \frac{1}{2}xy^4\ dx$$

$$\int f_x(x, y) = \frac{1}{4}x^2y^4 + g(y)$$

We'll find the partial derivative of the right side of this with respect to y.

$$f(x, y) = \frac{1}{4}x^2y^4 + g(y)$$

$$f_y(x, y) = x^2y^3 + g'(y)$$

Remember $f_y(x, y) = F_2$ which is $f_y(x, y) = x^2y^3$. We'll set the two partial derivatives with respect to y equal to each other.

$$x^2y^3 + g'(y) = x^2y^3$$

$$g'(y) = 0$$

Now we need to integrate both sides to find $g(y)$.

$$g(y) = \int g'(y) = \int 0 \ dy$$

$$g(y) = k$$

We'll now plug in $g(y) = k$.

$$f(x, y) = \frac{1}{4}x^2y^4 + k$$

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Topic: Potential function of a conservative vector field

Question: Find the function for the conservative vector field.

$$F(x, y) = (ye^x + 2xy^3)\mathbf{i} + (e^x + 3x^2y^2)\mathbf{j}$$

$$F = \nabla f$$

Answer choices:

A $f(x, y) = ye^x + x^2y^2 + k$

B $f(x, y) = ye^x + 2xy^2 + k$

C $f(x, y) = ye^x + 2xy^3 + k$

D $f(x, y) = ye^x + x^2y^3 + k$

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Solution: D

From

$$F(x, y) = (ye^x + 2xy^3)\mathbf{i} + (e^x + 3x^2y^2)\mathbf{j}$$

we know that

$$F_1 = ye^x + 2xy^3$$

First, we'll calculate

$$\int f_x(x, y) = \int F_1 \ dx$$

We get

$$\int f_x(x, y) = \int ye^x + 2xy^3 \ dx$$

$$\int f_x(x, y) = ye^x + x^2y^3 + g(y)$$

We'll find the partial derivative of the right side of this with respect to y.

$$f(x, y) = ye^x + x^2y^3 + g(y)$$

$$f_y(x, y) = e^x + 3x^2y^2 + +g'(y)$$

Remember $f_y(x, y) = F_2$ which is $f_y(x, y) = e^x + 3x^2y^2$. We'll set the two partial derivatives with respect to y equal to each other.

$$e^x + 3x^2y^2 + g'(y) = e^x + 3x^2y^2$$

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$$g'(y) = 0$$

Now we need to integrate both sides to find $g(y)$.

$$g(y) = \int g'(y) = \int 0 \ dy$$

$$g(y) = k$$

We'll now plug in $g(y) = k$.

$$f(x, y) = ye^x + x^2y^3 + k$$

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Topic: Potential function of a conservative vector field to evaluate a line integral

Question: Find the line integral when $F = \nabla f$, where c is the line segment from $(1,2,0)$ to $(2,2,1)$.

$$F(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

Answer choices:

A $\quad 6$

B $\quad 1$

C $\quad 2$

D $\quad 4$

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Solution: D

Given

$$F(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$$

and remembering that

$$F(x, y, z) = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

we'll get

$$f_x = yz$$

$$f_y = xz$$

$$f_z = xy$$

We can integrate $f_x = yz$ to find f. This means $\int f_x \ dx = f$.

$$f = xyz + g(y, z)$$

To find $g(y, z)$, we can find the partial derivatives with respect to y and z of $f = xyz + g(y, z)$.

$$f_y = xz + g_y(y, z)$$

$$f_z = xy + g_z(y, z)$$

Remember we already calculated $f_y = xz$ and $f_z = xy$. We can now equate these partial derivatives.

$$xz + g_y(y, z) = xz$$

$$g_y(y, z) = 0$$

and

$$xy + g_z(y, z) = xy$$

$$g_z(y, z) = 0$$

Next we can integrate $g_y(y, z) = 0$ and $g_z(y, z) = 0$ to get $g(y, z)$.

$$g(y, z) = h(z)$$

and

$$g(y, z) = k(y)$$

Now we equate $g(y, z) = h(z)$ and $g(y, z) = k(y)$.

$$h(z) = k(y)$$

Since there are no z terms $h(z) = 0$, and since there are no y terms $k(y) = 0$. Therefore we can say that $g(y, z) = 0$. This means that $f = xyz + g(y, z)$ is actually $f = xyz$.

Using this information and the endpoints of the line segment $(1,2,0)$ and $(2,2,1)$, we can find the line integral.

$$\int_c \nabla f \cdot dr = f_2(x_2, y_2, z_2) - f_1(x_1, y_1, z_1)$$

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$$\int_c \nabla f \cdot dr = (2)(2)(1) - (1)(2)(0)$$

$$\int_c \nabla f \cdot dr = 4 - 0$$

$$\int_c \nabla f \cdot dr = 4$$

This is the line integral.

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Topic: Potential function of a conservative vector field to evaluate a line integral

Question: Find the line integral when $F = \nabla f$, where c is the line segment from $(3,2,2)$ to $(4,5,2)$.

$$F(x, y, z) = 4xyz^3\mathbf{i} + 2x^2z^3\mathbf{j} + 6x^2yz^2\mathbf{k}$$

Answer choices:

A $\quad 1{,}280$

B $\quad 992$

C $\quad 288$

D $\quad 1{,}568$

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Solution: B

Given

$$F(x, y, z) = 4xyz^3\mathbf{i} + 2x^2z^3\mathbf{j} + 6x^2yz^2\mathbf{k}$$

and remembering that

$$F(x, y, z) = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

we'll get

$$f_x = 4xyz^3$$

$$f_y = 2x^2z^3$$

$$f_z = 6x^2yz^2$$

We can integrate $f_x = 4xyz^3$ to find f. This means $\int f_x\ dx = f$.

$$f = 2x^2yz^3 + g(y, z)$$

To find $g(y, z)$, we can find the partial derivatives with respect to y and z of $f = 2x^2yz^3 + g(y, z)$.

$$f_y = 2x^2z^3 + g_y(y, z)$$

$$f_z = 6x^2yz^2 + g_z(y, z)$$

Remember we already calculated $f_y = 2x^2z^3$ and $f_z = 6x^2yz^2$. We can now equate these partial derivatives.

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$$2x^2z^3 + g_y(y, z) = 2x^2z^3$$

$$g_y(y, z) = 0$$

and

$$6x^2yz^2 + g_z(y, z) = 6x^2yz^2$$

$$g_z(y, z) = 0$$

Next we can integrate $g_y(y, z) = 0$ and $g_z(y, z) = 0$ to get $g(y, z)$.

$$g(y, z) = h(z)$$

and

$$g(y, z) = k(y)$$

Now we equate $g(y, z) = h(z)$ and $g(y, z) = k(y)$.

$$h(z) = k(y)$$

Since there are no z terms $h(z) = 0$, and since there are no y terms $k(y) = 0$. Therefore we can say that $g(y, z) = 0$. This means that $f = 2x^2yz^3 + g(y, z)$ is actually $f = 2x^2yz^3$.

Using this information and the endpoints of the line segment $(3,2,2)$ and $(4,5,2)$, we can find the line integral.

$$\int_c \nabla f \cdot dr = f_2(x_2, y_2, z_2) - f_1(x_1, y_1, z_1)$$

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$$\int_c \nabla f \cdot dr = 2(4)^2(5)(2)^3 - 2(3)^2(2)(2)^3$$

$$\int_c \nabla f \cdot dr = 1{,}280 - 288$$

$$\int_c \nabla f \cdot dr = 992$$

This is the line integral.

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Topic: Potential function of a conservative vector field to evaluate a line integral

Question: Find the line integral when $F = \nabla f$, where c is the line segment from $(1,1,3)$ to $(4,2,6)$.

$$F(x, y, z) = 2z\mathbf{i} - 3y^2\mathbf{j} + 2x\mathbf{k}$$

Answer choices:

A $\quad 45$

B $\quad 35$

C $\quad 42$

D $\quad 54$

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Solution: B

Given

$$F(x, y, z) = 2z\mathbf{i} - 3y^2\mathbf{j} + 2x\mathbf{k}$$

and remembering that

$$F(x, y, z) = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k}$$

we'll get

$$f_x = 2z$$

$$f_y = -3y^2$$

$$f_z = 2x$$

We can integrate $f_x = 2z$ to find f. This means $\int f_x \, dx = f$.

$$f = 2xz + g(y, z)$$

To find $g(y, z)$, we can find the partial derivatives with respect to y and z of $f = 2xz + g(y, z)$.

$$f_y = g_y(y, z)$$

$$f_z = 2x + g_z(y, z)$$

Remember we already calculated $f_y = -3y^2$ and $f_z = 2x$. We can now equate these partial derivatives.

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$$g_y(y,z) = -3y^2$$

and

$$2x + g_z(y,z) = 2x$$

$$g_z(y,z) = 0$$

Next we can integrate $g_y(y,z) = -3y^2$ and $g_z(y,z) = 0$ to get $g(y,z)$.

$$g(y,z) = -y^3 + h(z)$$

and

$$g(y,z) = k(y)$$

Now we equate $g(y,z) = -y^3 + h(z)$ and $g(y,z) = k(y)$.

$$-y^3 + h(z) = k(y)$$

Since there are no z terms $h(z) = 0$, and since there is a y term $k(y) = -y^3$. Therefore we can say that $g(y,z) = -y^3$. This means that $f = 2xz + g(y,z)$ is actually $f = 2xz - y^3$.

Using this information and the endpoints of the line segment $(1,1,3)$ and $(4,2,6)$, we can find the line integral.

$$\int_c \nabla f \cdot dr = f_2(x_2, y_2, z_2) - f_1(x_1, y_1, z_1)$$

$$\int_c \nabla f \cdot dr = 2(4)(6) - (2)^3 - (2(1)(3) - (1)^3)$$

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$$\int_c \nabla f \cdot dr = 40 - 5$$

$$\int_c \nabla f \cdot dr = 35$$

This is the line integral.

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Topic: Independence of path

Question: Is the integral independent of path?

$$\int_c x^2y^2\ dx + 2x^3y\ dy$$

Answer choices:

A Yes, since the vector field of the integral is conservative.

B No, since the vector field of the integral is conservative.

C Yes, since the vector field of the integral is not conservative.

D No, since the vector field of the integral is not conservative.

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Solution: D

If an integral

$$\int_c P\ dx + Q\ dy$$

is independent of path, its vector field is conservative. A conservative vector field has a scalar curl equal to 0 and a completely defined domain.

To determine whether or not the scalar curl is 0, use the formula

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

where $\partial Q/\partial x$ is the partial derivative with respect to x of the coefficient on $\mathbf{j}$ from the vector field equation and $\partial P/\partial y$ is the partial derivative with respect to y of the coefficient on $\mathbf{i}$ from the vector field equation.

To verify if the domain can be defined at all values, check for radicals, rationals and logarithmic functions which all potentially contain undefined values.

Given

$$\int_c x^2y^2\ dx + 2x^3y\ dy$$

we'll find the derivative of Q with respect to x of $Q = 2x^3y$.

$$\frac{\partial Q}{\partial x} = 6x^2y$$

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We'll find the derivative of P with respect to y of $P = x^2y^2$.

$$\frac{\partial P}{\partial y} = 2x^2y$$

Now we can calculate the scalar curl.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

$$6x^2y - 2x^2y = 0$$

$$4x^2y \neq 0$$

Since the scalar curl does not equal 0, it can't be conservative. Therefore the integral is not independent of path.

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Topic: Independence of path

Question: Is the integral independent of path?

$$\int_C \frac{3}{2}x^2e^{2y}\ dx + x^3e^{2y}\ dy$$

Answer choices:

A Yes, since the vector field of the integral is conservative.

B No, since the vector field of the integral is conservative.

C Yes, since the vector field of the integral is not conservative.

D No, since the vector field of the integral is not conservative.

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Solution: A

If an integral

$$\int_c P\ dx + Q\ dy$$

is independent of path, its vector field is conservative. A conservative vector field has a scalar curl equal to 0 and a completely defined domain.

To determine whether or not the scalar curl is 0, use the formula

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

where $\partial Q/\partial x$ is the partial derivative with respect to x of the coefficient on $\mathbf{j}$ from the vector field equation and $\partial P/\partial y$ is the partial derivative with respect to y of the coefficient on $\mathbf{i}$ from the vector field equation.

To verify if the domain can be defined at all values, check for radicals, rationals and logarithmic functions which all potentially contain undefined values.

Given

$$\int_c \frac{3}{2}x^2e^{2y}\ dx + x^3e^{2y}\ dy$$

we'll find the derivative of Q with respect to x of $Q = x^3e^{2y}$.

$$\frac{\partial Q}{\partial x} = 3x^2e^{2y}$$

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We'll find the derivative of P with respect to y of $P = (3/2)x^2e^{2y}$.

$$\frac{\partial P}{\partial y} = \frac{3}{2}x^2e^{2y}(2)$$

$$\frac{\partial P}{\partial y} = 3x^2e^{2y}$$

Now we can calculate the scalar curl.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

$$3x^2e^{2y} - 3x^2e^{2y} = 0$$

$$0 = 0$$

Since the scalar curl does equal 0, it might be conservative. Therefore the integral is not independent of path.

Next we need to check to see whether there are any undefined values that restrict the domain. But the vector field contains no undefined values in its domain.

The vector field has a scalar curl of 0 and no undefined values in its domain, therefore it's conservative. So the integral is independent of path.

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Topic: Independence of path

Question: Find the value of the integral when c is any path from $(1,1)$ to $(3,1)$.

$$\int_c 6y^2e^x\ dx + 12ye^x\ dy$$

Answer choices:

A $\quad 6e^3 - 1$

B $\quad 6e^3 + 1$

C $\quad 6e(e^2 - 1)$

D $\quad 6e(e^2 + 1)$

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Solution: C

To solve a line integral that is independent of path, find f given

$$\int_c P\ dx + Q\ dy$$

where $F(x, y) = P\mathbf{i} + Q\mathbf{j}$. Calculate the partial derivatives of f and then take the integral of one of the partial derivatives.

To evaluate the line integral, remember that $F = \nabla f$ so

$$\int_c F \cdot dr \text{ can become } \int_c \nabla f \cdot dr$$

We'll use the formula

$$\int_c \nabla f \cdot dr = f_2(x_2, y_2) - f_1(x_1, y_1)$$

where $f_2(x_2, y_2)$ is f at the terminal point of the path of c and $f_1(x_1, y_1)$ is f at the origin point of the path of c.

Given the integral

$$\int_c 6y^2e^x\ dx + 12ye^x\ dy$$

we can say that $P = 6y^2e^x$ and $Q = 12ye^x$. P is the partial derivative of f with respect to x, and Q is the partial derivative of f with respect to y.

$$f_x = 6y^2e^x$$

$$f_y = 12ye^x$$

We can use $f_x = 6y^2e^x$ to find f by integrating it. This means

$$\int f_x \ dx = f$$

$$\int 6y^2e^x \ dx = f$$

$$f = 6y^2e^x + g(y)$$

Now we need to find $g(y)$. We'll take the partial derivative with respect to y.

$$f_y = 12ye^x + g_y(y)$$

Remember that $f_y = Q$, and $f_y = 12ye^x$. Now we can equate the two partial derivatives with respect to y.

$$12ye^x + g_y(y) = 12ye^x$$

$$g_y(y) = 0$$

Now we need to integrate both sides to find $g(y)$.

$$g(y) = \int g_y(y) = \int 0 \ dy$$

$$g(y) = k$$

Finally we can take $f = 6y^2e^x + g(y)$ and $g(y) = k$ to get

$$f = 6y^2e^x + k$$

Plug all this, and the points $c_1(1,1)$ and $c_2(3,1)$ into the line integral formula.

$$\int_c \nabla f \cdot dr = f_2(x_2, y_2) - f_1(x_1, y_1)$$

$$\int_c \nabla f \cdot dr = 6(1)^2 e^3 + k - (6(1)^2 e^1 + k)$$

$$\int_c \nabla f \cdot dr = 6e^3 + k - (6e + k)$$

$$\int_c \nabla f \cdot dr = 6e^3 + k - 6e - k$$

$$\int_c \nabla f \cdot dr = 6e^3 - 6e$$

$$\int_c \nabla f \cdot dr = 6e(e^2 - 1)$$

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Topic: Work done by the force field

Question: Find the work done by the force field to move an object from $A(1,1)$ to $B(3,2)$.

$$F(x,y)=\frac{1}{2}xy^2\mathbf{i}+\frac{1}{2}x^2y\mathbf{j}$$

Answer choices:

A $\frac{4}{35}$

B $\frac{35}{4}$

C $\frac{37}{36}$

D $\frac{34}{35}$

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Solution: B

From the function

$$F(x, y) = \frac{1}{2}xy^2\mathbf{i} + \frac{1}{2}x^2y\mathbf{j}$$

we can identify P and Q.

$$P = \frac{1}{2}xy^2$$

$$Q = \frac{1}{2}x^2y$$

Then we'll take the partial derivative of P with respect to y, and of Q with respect to x.

$$\frac{\partial P}{\partial y} = xy$$

$$\frac{\partial Q}{\partial x} = xy$$

Then the scalar curl is

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0$$

$$xy - xy = 0$$

$$0 = 0$$

Since the scalar curl of the original function does in fact equal 0, it might be conservative.

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Next we need to check to see whether there are any undefined values in the domain. But the vector field contains no undefined places in its domain.

The vector field has a scalar curl of 0 and no undefined values in its domain, which means it's conservative. So the function $F(x, y)$ is independent of path.

Next, we say that $P = f_x$, and that $Q = f_y$.

$$f_x = \frac{1}{2}xy^2$$

$$f_y = \frac{1}{2}x^2y$$

We'll integrate f_x to find f.

$$\int f_x \ dx = f$$

$$\int \frac{1}{2}xy^2 \ dx = f$$

$$f = \frac{1}{4}x^2y^2 + g(y)$$

To find $g(y)$, we'll take the partial derivative of f with respect to y.

$$f_y = \frac{1}{2}x^2y + g_y(y)$$

Remembering that

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$$f_y = Q$$

$$f_y = \frac{1}{2}x^2y$$

we can equate the two values for f_y.

$$\frac{1}{2}x^2y + g_y(y) = \frac{1}{2}x^2y$$

$$g_y(y) = 0$$

Now we need to integrate both sides to find $g(y)$. The integral of 0 will always be a constant, so we'll get

$$g(y) = \int g_y(y) = \int 0 \ dy$$

$$g(y) = k$$

Plugging $g(y) = k$ into the equation we found before for f gives

$$f = \frac{1}{4}x^2y^2 + g(y)$$

$$f = \frac{1}{4}x^2y^2 + k$$

Now we'll find the work done by the force field using the points $A(1,1)$, $B(3,2)$ and the function f.

$$\int_c F \cdot dr$$

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$$\int_c \nabla f \cdot dr = f_2(x_2, y_2) - f_1(x_1, y_1)$$

$$\int_c \nabla f \cdot dr = \frac{1}{4}(3)^2(2)^2 + k - \left(\frac{1}{4}(1)^2(1)^2 + k\right)$$

$$\int_c \nabla f \cdot dr = 9 + k - \frac{1}{4} - k$$

$$\int_c \nabla f \cdot dr = \frac{35}{4}$$

This is the work done by the force field to move the object from point A to point B.

Topic: Work done by the force field

Question: Find the work done by the force field to move an object from $A(1,0)$ to $B(3,1)$.

$$F(x,y) = x^2y^2\mathbf{i} + \frac{2}{3}x^3y\mathbf{j}$$

Answer choices:

A 18

B 3

C 27

D 9

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Solution: D

From the function

$$F(x, y) = x^2y^2\mathbf{i} + \frac{2}{3}x^3y\mathbf{j}$$

we can identify P and Q.

$$P = x^2y^2$$

$$Q = \frac{2}{3}x^3y$$

Then we'll take the partial derivative of P with respect to y, and of Q with respect to x.

$$\frac{\partial P}{\partial y} = 2x^2y$$

$$\frac{\partial Q}{\partial x} = 2x^2y$$

Then the scalar curl is

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0$$

$$2x^2y - 2x^2y = 0$$

$$0 = 0$$

Since the scalar curl of the original function does in fact equal 0, it might be conservative.

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Next we need to check to see whether there are any undefined values in the domain. But the vector field contains no undefined places in its domain.

The vector field has a scalar curl of 0 and no undefined values in its domain, which means it's conservative. So the function $F(x, y)$ is independent of path.

Next, we say that $P = f_x$, and that $Q = f_y$.

$$f_x = x^2y^2$$

$$f_y = \frac{2}{3}x^3y$$

We'll integrate f_x to find f.

$$\int f_x \ dx = f$$

$$\int x^2y^2 \ dx = f$$

$$f = \frac{1}{3}x^3y^2 + g(y)$$

To find $g(y)$, we'll take the partial derivative of f with respect to y.

$$f_y = \frac{2}{3}x^3y + g_y(y)$$

Remembering that

$$f_y = Q$$

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$$f_y = \frac{2}{3}x^3y$$

we can equate the two values for f_y.

$$\frac{2}{3}x^3y + g_y(y) = \frac{2}{3}x^3y$$

$$g_y(y) = 0$$

Now we need to integrate both sides to find $g(y)$. The integral of 0 will always be a constant, so we'll get

$$g(y) = \int g_y(y) = \int 0 \ dy$$

$$g(y) = k$$

Plugging $g(y) = k$ into the equation we found before for f gives

$$f = \frac{1}{3}x^3y^2 + g(y)$$

$$f = \frac{1}{3}x^3y^2 + k$$

Now we'll find the work done by the force field using the points $A(1,0)$, $B(3,1)$ and the function f.

$$\int_c F \cdot dr$$

$$\int_c \nabla f \cdot dr = f_2(x_2, y_2) - f_1(x_1, y_1)$$

$$\int_c \nabla f \cdot dr = \frac{1}{3}(3)^3(1)^2 + k - \left(\frac{1}{3}(1)^3(0)^2 + k\right)$$

$$\int_c \nabla f \cdot dr = 9 + k - 0 - k$$

$$\int_c \nabla f \cdot dr = 9$$

This is the work done by the force field to move the object from point A to point B.

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Topic: Work done by the force field

Question: Find the work done by the force field to move an object from $A(2,2)$ to $B(3,3)$.

$$F(x,y) = 4xe^y\mathbf{i} + 2x^2e^y\mathbf{j}$$

Answer choices:

A $6e^3 - 4e^2$

B $2e$

C $18e^3 - 8e^2$

D $10e$

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Solution: C

From the function

$$F(x, y) = 4xe^y\mathbf{i} + 2x^2e^y\mathbf{j}$$

we can identify P and Q.

$$P = 4xe^y$$

$$Q = 2x^2e^y$$

Then we'll take the partial derivative of P with respect to y, and of Q with respect to x.

$$\frac{\partial P}{\partial y} = 4xe^y$$

$$\frac{\partial Q}{\partial x} = 4xe^y$$

Then the scalar curl is

$$\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial y} = 0$$

$$4xe^y - 4xe^y = 0$$

$$0 = 0$$

Since the scalar curl of the original function does in fact equal 0, it might be conservative.

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Next we need to check to see whether there are any undefined values in the domain. But the vector field contains no undefined places in its domain.

The vector field has a scalar curl of 0 and no undefined values in its domain, which means it's conservative. So the function $F(x, y)$ is independent of path.

Next, we say that $P = f_x$, and that $Q = f_y$.

$$f_x = 4xe^y$$

$$f_y = 2x^2e^y$$

We'll integrate f_x to find f.

$$\int f_x \ dx = f$$

$$\int 4xe^y \ dx = f$$

$$f = 2x^2e^y + g(y)$$

To find $g(y)$, we'll take the partial derivative of f with respect to y.

$$f_y = 2x^2e^y + g_y(y)$$

Remembering that

$$f_y = Q$$

$$f_y = 2x^2e^y$$

we can equate the two values for f_y.

$$2x^2e^y + g_y(y) = 2x^2e^y$$

$$g_y(y) = 0$$

Now we need to integrate both sides to find $g(y)$. The integral of 0 will always be a constant, so we'll get

$$g(y) = \int g_y(y) = \int 0 \ dy$$

$$g(y) = k$$

Plugging $g(y) = k$ into the equation we found before for f gives

$$f = 2x^2e^y + g(y)$$

$$f = 2x^2e^y + k$$

Now we'll find the work done by the force field using the points $A(2,2)$, $B(3,3)$ and the function f.

$$\int_c F \cdot dr$$

$$\int_c \nabla f \cdot dr = f_2(x_2, y_2) - f_1(x_1, y_1)$$

$$\int_c \nabla f \cdot dr = 2(3)^2e^{(3)} + k - \left(2(2)^2e^{(2)} + k\right)$$

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$$\int_c \nabla f \cdot dr = 18e^3 + k - 8e^2 - k$$

$$\int_c \nabla f \cdot dr = 18e^3 - 8e^2$$

This is the work done by the force field to move the object from point A to point B.

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Topic: Open, connected, and simply-connected

Question: Fill in the blank.

A region D is ________________ if you can join up any two points in the region with a path that lies completely in D.

Answer choices:

A connected

B simply-connected

C open

D none of the above

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Solution: A

A region D is open if it doesn’t contain any of its boundary points.

A region D is connected if you can join up any two points in the region with a path that lies completely in D.

A region D is simply-connected if it is connected (you can connect any two points in the region with a path that lies completely in D) and it contains no holes.

The answer is:

A region D is **connected** if you can join up any two points in the region with a path that lies completely in D.

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Topic: Open, connected, and simply-connected

Question: Fill in the blank.

A region D is _______________ if it doesn't contain any of its boundary points.

Answer choices:

A connected

B simply-connected

C open

D none of the above

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Solution: C

A region D is open if it doesn't contain any of its boundary points.

A region D is connected if you can join up any two points in the region with a path that lies completely in D.

A region D is simply-connected if it is connected (you can connect any two points in the region with a path that lies completely in D) and it contains no holes.

The answer is:

A region D is **open** if it doesn't contain any of its boundary points.

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Topic: Open, connected, and simply-connected

Question: Fill in the blanks.

A region D is ______________ if it is ______________ and it contains no holes.

Answer choices:

A connected, simply-connected

B simply-connected, connected

C open, connected

D connected, open

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Solution: B

A region D is open if it doesn't contain any of its boundary points.

A region D is connected if you can join up any two points in the region with a path that lies completely in D.

A region D is simply-connected if it is connected (you can connect any two points in the region with a path that lies completely in D) and it contains no holes.

The answer is:

A region D is **simply-connected** if it is **connected** and it contains no holes.

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Topic: Green's theorem for one region

Question: Use Green's theorem to calculate the integral.

$$\oint_c (x+2y)\ dx + (x^2 - y)\ dy$$

over the interval $(\pm 1, \pm 1)$

Answer choices:

A $\quad -4$

B $\quad -8$

C $\quad 4$

D $\quad 8$

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Solution: B

We'll change the given line integral

$$\oint_c (x+2y)\ dx + (x^2 - y)\ dy$$

from the form

$$\oint_c P\ dx + Q\ dy$$

to the form

$$\int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

We can see from the given integral that $Q(x, y) = x^2 - y$. We'll take the partial derivative with respect to x.

$$\frac{\partial Q}{\partial x} = 2x$$

We also know that $P(x, y) = x + 2y$. We'll take the partial derivative with respect to y.

$$\frac{\partial P}{\partial y} = 2$$

We'll sketch the interval given by $(\pm 1, \pm 1)$.

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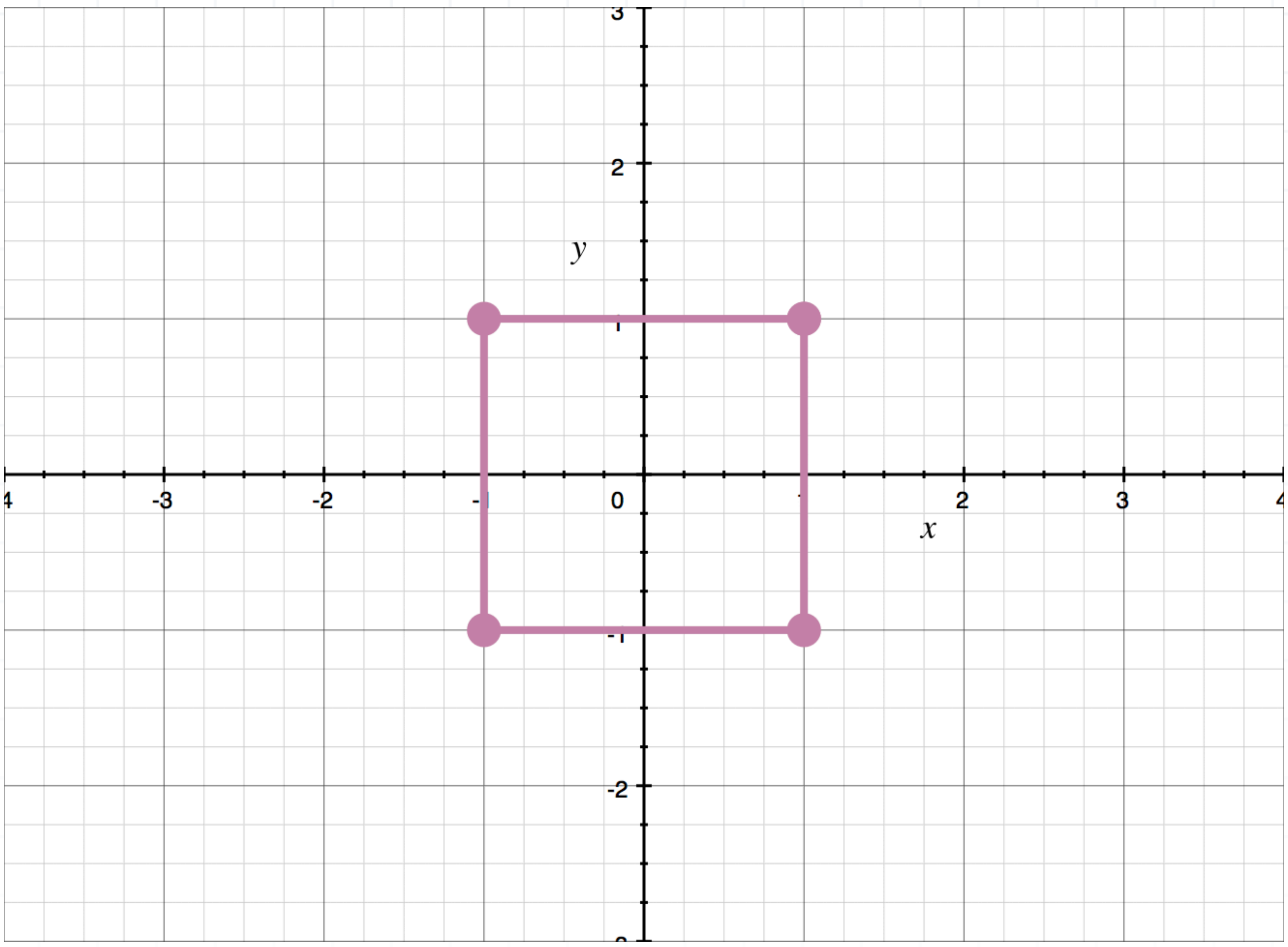

From the graph, we know that the object has one region, that y is defined between -1 and 1, and that x is defined between -1 and 1.

Now we can set up the double integral.

$$\oint_c P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

$$\oint_c (x + 2y)\ dx + (x^2 - y)\ dy = \int_{-1}^{1} \int_{-1}^{1} 2x - 2\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$\int_{-1}^{1} 2xy - 2y \Big|_{y=-1}^{y=1}\ dx$$

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$$\int_{-1}^{1} 2x(1) - 2(1) - (2x(-1) - 2(-1))\ dx$$

$$\int_{-1}^{1} 2x - 2 - (-2x + 2)\ dx$$

$$\int_{-1}^{1} 2x - 2 + 2x - 2\ dx$$

$$\int_{-1}^{1} 4x - 4\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$2x^2 - 4x\Big|_{-1}^{1}$$

$$2(1)^2 - 4(1) - (2(-1)^2 - 4(-1))$$

$$2 - 4 - (2(1) + 4)$$

$$2 - 4 - 2 - 4$$

$$-8$$

By Green's theorem, this is the area of the region.

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Topic: Green's theorem for one region

Question: Use Green's theorem to calculate the integral.

$$\oint_c (x^2 - 3y)\ dx + (-x + y^2)\ dy$$

over the interval $(\pm 1, \pm 1)$

Answer choices:

A -4

B -8

C 4

D 8

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Solution: D

We'll change the given line integral

$$\oint_c (x^2 - 3y)\ dx + (-x + y^2)\ dy$$

from the form

$$\oint_c P\ dx + Q\ dy$$

to the form

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

We can see from the given integral that $Q(x, y) = -x + y^2$. We'll take the partial derivative with respect to x.

$$\frac{\partial Q}{\partial x} = -1$$

We also know that $P(x, y) = x^2 - 3y$. We'll take the partial derivative with respect to y.

$$\frac{\partial P}{\partial y} = -3$$

We'll sketch the interval given by $(\pm 1, \pm 1)$.

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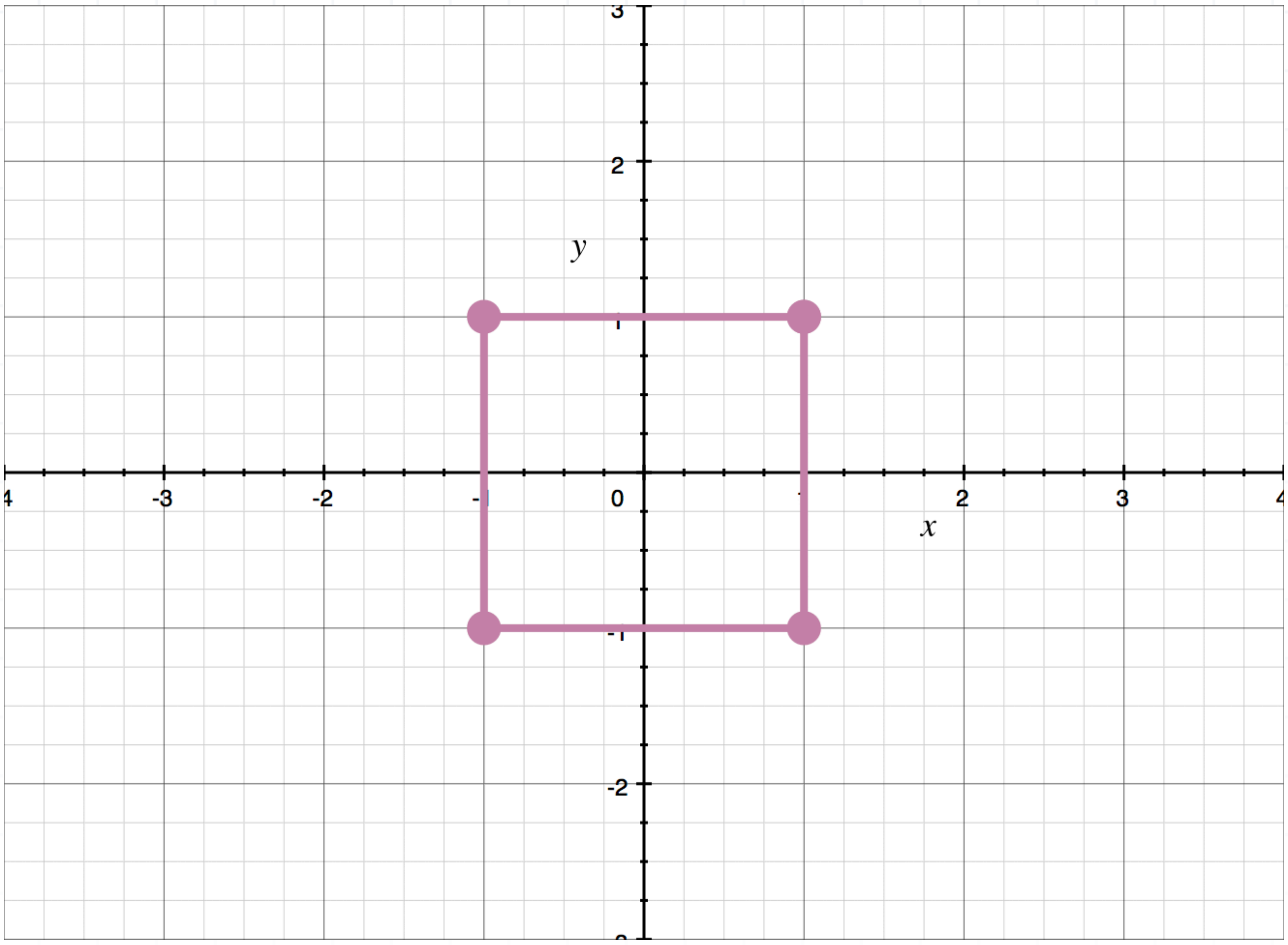

From the graph, we know that the object has one region, that y is defined between -1 and 1, and that x is defined between -1 and 1.

Now we can set up the double integral.

$$\oint_C P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

$$\oint_C (x^2 - 3y)\ dx + (-x + y^2)\ dy = \int_{-1}^{1} \int_{-1}^{1} -1 - (-3)\ dy\ dx$$

$$\int_{-1}^{1} \int_{-1}^{1} 2\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

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$$\int_{-1}^{1} 2y \Big|_{y=-1}^{y=1} dx$$

$$\int_{-1}^{1} 2(1) - 2(-1)\ dx$$

$$\int_{-1}^{1} 2 + 2\ dx$$

$$\int_{-1}^{1} 4\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$4x \Big|_{-1}^{1}$$

$$4(1) - 4(-1)$$

$$4(1) + 4(1)$$

$$8$$

By Green's theorem, this is the area of the region.

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Topic: Green's theorem for one region

Question: Use Green's theorem to calculate the integral.

$$\oint_c (3x^3 + 5y)\ dx + (2x^2 - 2y^2)\ dy$$

on the interval $(\pm 1, \pm 1)$

Answer choices:

A $\quad -20$

B $\quad -12$

C $\quad 20$

D $\quad 12$

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Solution: A

We'll change the given line integral

$$\oint_c (3x^3 + 5y)\ dx + (2x^2 - 2y^2)\ dy$$

from the form

$$\oint_c P\ dx + Q\ dy$$

to the form

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

We can see from the given integral that $Q(x, y) = 2x^2 - 2y^2$. We'll take the partial derivative with respect to x.

$$\frac{\partial Q}{\partial x} = 4x$$

We also know that $P(x, y) = 3x^3 + 5y$. We'll take the partial derivative with respect to y.

$$\frac{\partial P}{\partial y} = 5$$

We'll sketch the interval given by $(\pm 1, \pm 1)$.

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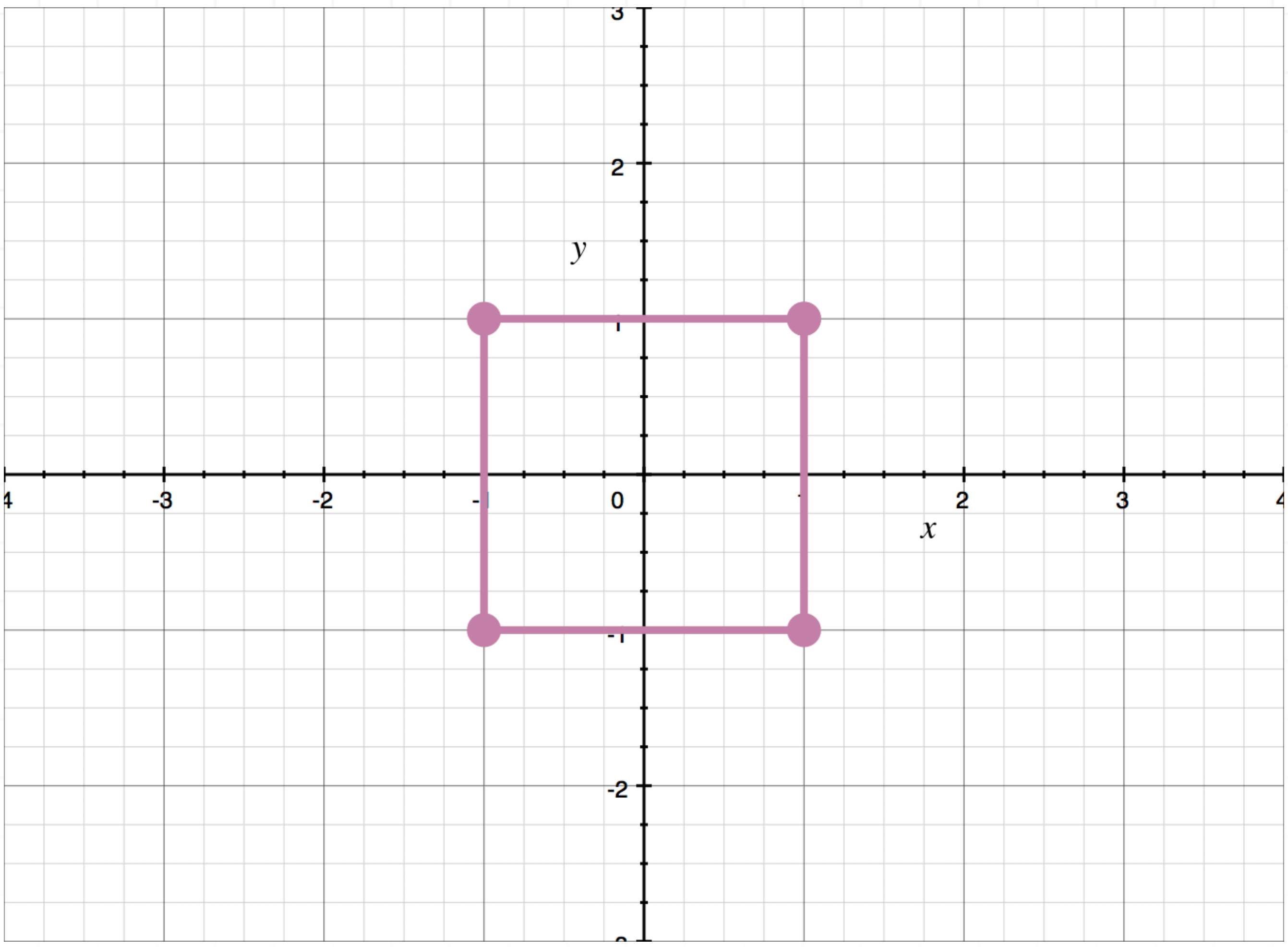

From the graph, we know that the object has one region, that y is defined between -1 and 1, and that x is defined between -1 and 1.

Now we can set up the double integral.

$$\oint_C P\ dx + Q\ dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

$$\oint_C (3x^3 + 5y)\ dx + (2x^2 - 2y^2)\ dy = \int_{-1}^{1} \int_{-1}^{1} 4x - 5\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$\int_{-1}^{1} 4xy - 5y\Big|_{y=-1}^{y=1}\ dx$$

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$$\int_{-1}^{1} 4x(1) - 5(1) - (4x(-1) - 5(-1))\ dx$$

$$\int_{-1}^{1} 4x - 5 - (-4x + 5)\ dx$$

$$\int_{-1}^{1} 4x - 5 + 4x - 5\ dx$$

$$\int_{-1}^{1} 8x - 10\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$4x^2 - 10x \Big|_{-1}^{1}$$

$$4(1)^2 - 10(1) - (4(-1)^2 - 10(-1))$$

$$4 - 10 - (4(1) + 10)$$

$$4 - 10 - 4 - 10$$

$$-20$$

By Green's theorem, this is the area of the region.

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Topic: Green's Theorem for two regions

Question: What is the value of the integral for the region bounded by $(0,0)$, $(1,1)$, and $(2,0)$?

$$\oint_C (3x - y)\ dx + (x^2 + 2y)\ dy$$

Answer choices:

A $-\frac{26}{3}$

B -3

C $\frac{26}{3}$

D 3

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Solution: D

The given line integral

$$\oint_c (3x-y)\ dx+(x^2+2y)\ dy$$

is in the form

$$\oint_c P\ dx+Q\ dy$$

We'll change it to the form

$$\int\int_{R1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\ dA+\int\int_{R2}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\ dA$$

From the line integral, we know that $Q(x,y)=x^2+2y$, so

$$\frac{\partial Q}{\partial x}=2x$$

And we know that $P(x,y)=3x-y$, so

$$\frac{\partial P}{\partial y}=-1$$

A sketch of the region bounded by $(0,0)$, $(1,1)$, and $(2,0)$ is

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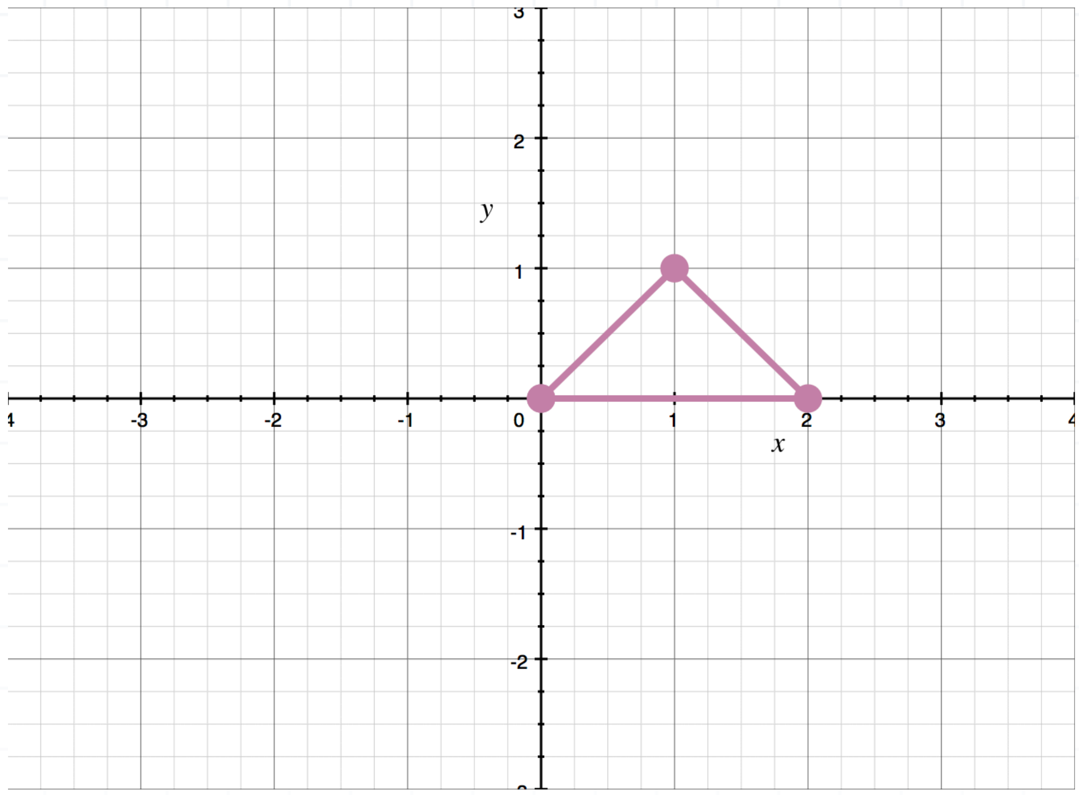
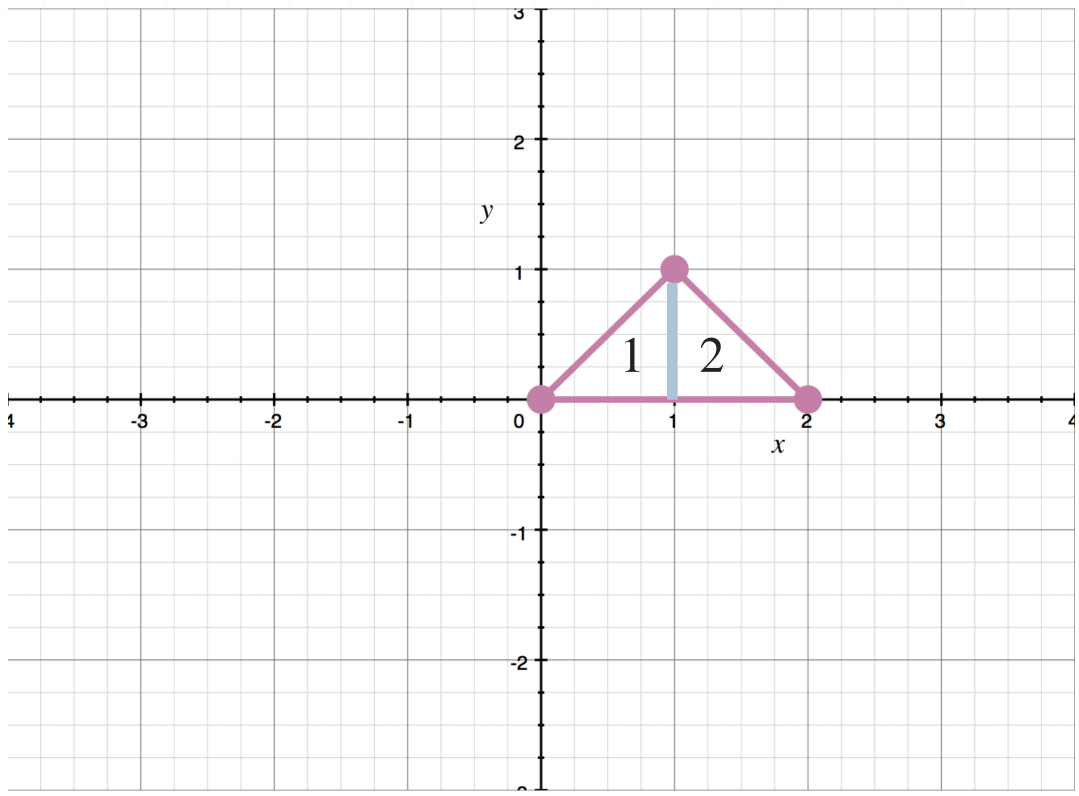


The region needs to be divided in two.



To calculate the limits for each region, we'll need to know the equations for the lines that form the three sides of the triangle. Just by looking at the graph, we can see that two of the lines are $y = 0$ and $y = x$. We can use the points $(1,1)$, and $(2,0)$ to solve for the third line using this formula for the equation of a line:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 1 = \frac{0 - 1}{2 - 1}(x - 1)$$

$$y - 1 = -(x - 1)$$

$$y = -x + 1 + 1$$

$$y = -x + 2$$

This means that region 1 on the graph is defined for y on $[0,x]$ and for x on $[0,1]$. Region 2 is defined for y on $[0, -x+2]$ and for x on $[1,2]$.

Using Green's theorem to convert the line integral into double integrals gives

$$\oint_c P\ dx + Q\ dy = \iint_{R1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA + \iint_{R2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

$$\oint_c (3x - y)\ dx + (x^2 + 2y)\ dy = \int_0^1 \int_0^x 2x + 1\ dy\ dx + \int_1^2 \int_0^{-x+2} 2x + 1\ dy\ dx$$

Integrate both integrals with respect to y, then evaluate over each interval.

$$\oint_c (3x - y)\ dx + (x^2 + 2y)\ dy = \int_0^1 2xy + y\Big|_{y=0}^{y=x} dx + \int_1^2 2xy + y\Big|_{y=0}^{y=-x+2} dx$$

$$\int_0^1 2x(x) + x - (2x(0) + 0)\ dx + \int_1^2 2x(-x+2) + (-x+2) - (2x(0) + (0))\ dx$$

$$\int_0^1 2x^2 + x\ dx + \int_1^2 -2x^2 + 4x - x + 2\ dx$$

$$\int_0^1 2x^2 + x\ dx + \int_1^2 -2x^2 + 3x + 2\ dx$$

Integrate both integrals with respect to x, then evaluate over each interval.

$$\frac{2}{3}x^3 + \frac{1}{2}x^2 \Big|_0^1 + -\frac{2}{3}x^3 + \frac{3}{2}x^2 + 2x \Big|_1^2$$

$$\frac{2}{3}(1)^3 + \frac{1}{2}(1)^2 - \left(\frac{2}{3}(0)^3 + \frac{1}{2}(0)^2\right) + -\frac{2}{3}(2)^3 + \frac{3}{2}(2)^2 + 2(2) - \left(-\frac{2}{3}(1)^3 + \frac{3}{2}(1)^2 + 2(1)\right)$$

$$\frac{2}{3} + \frac{1}{2} + -\frac{2}{3}(8) + \frac{3}{2}(4) + 4 - \left(-\frac{2}{3} + \frac{3}{2} + 2\right)$$

$$\frac{2}{3} + \frac{1}{2} - \frac{16}{3} + 6 + 4 + \frac{2}{3} - \frac{3}{2} - 2$$

$$\frac{4}{3} + \frac{1}{2} - \frac{16}{3} + 10 - \frac{3}{2} - 2$$

$$-\frac{2}{2} - \frac{12}{3} + 8$$

$$3$$

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Topic: Green's Theorem for two regions

Question: What is the value of the integral for the region bounded by $(0,0)$, $(1,1)$, and $(2,0)$?

$$\oint_C \left(e^x + y^2\right)\ dx + \left(3x - \cos y\right)\ dy$$

Answer choices:

A $\frac{7}{3}$

B $\frac{5}{3}$

C $-\frac{7}{3}$

D $-\frac{5}{3}$

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Solution: A

The given line integral

$$\oint_c \left(e^x + y^2\right)\ dx + \left(3x - \cos y\right)\ dy$$

is in the form

$$\oint_c P\ dx + Q\ dy$$

We'll change it to the form

$$\int\int_{R1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA + \int\int_{R2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

From the line integral, we know that $Q(x, y) = 3x - \cos y$, so

$$\frac{\partial Q}{\partial x} = 3$$

And we know that $P(x, y) = e^x + y^2$, so

$$\frac{\partial P}{\partial y} = 2y$$

A sketch of the region bounded by $(0,0)$, $(1,1)$, and $(2,0)$ is

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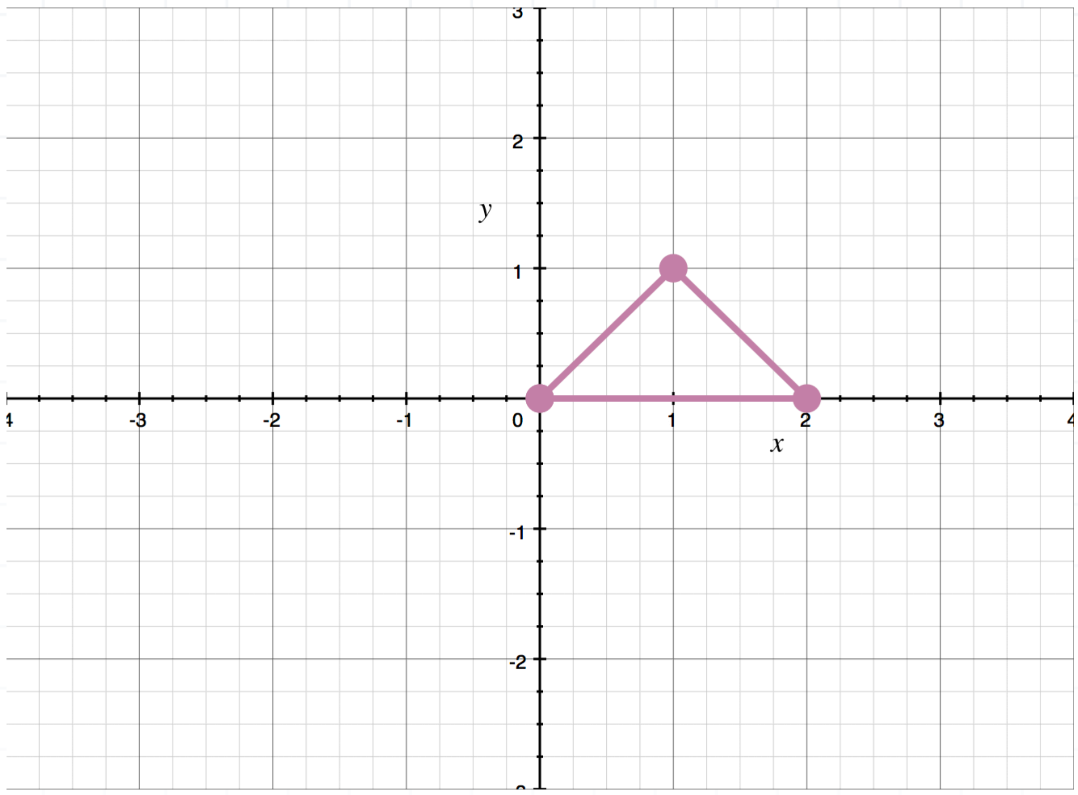
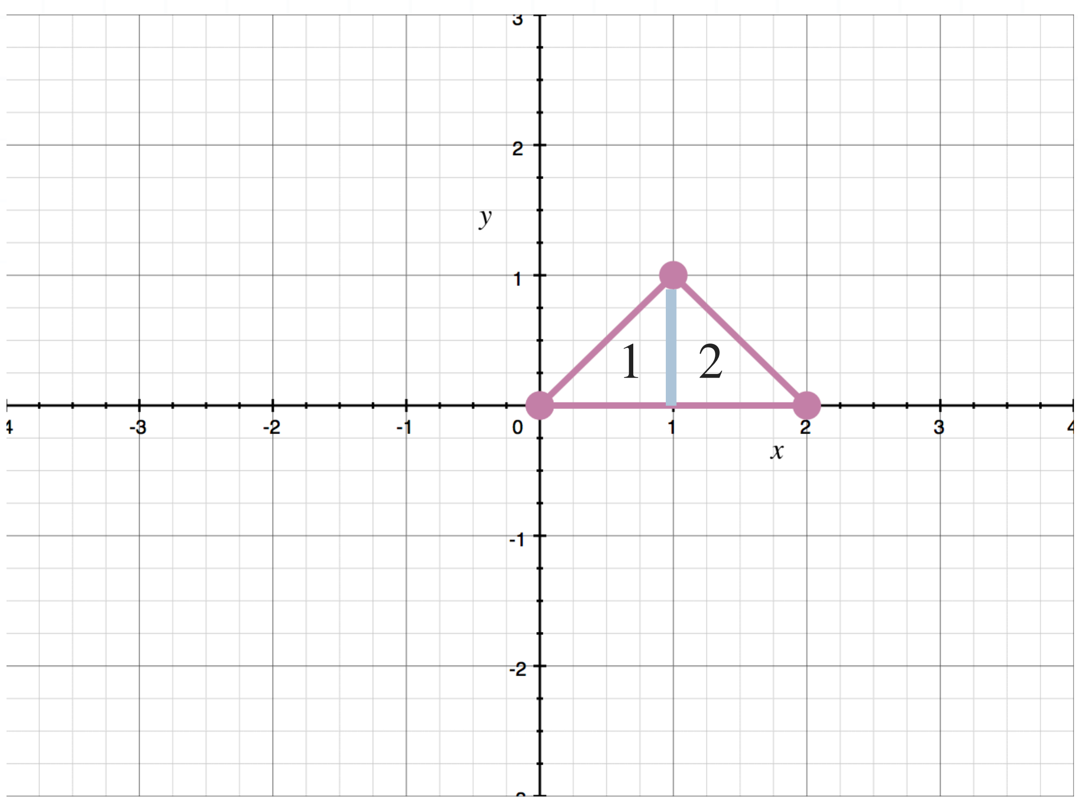


The region needs to be divided in two.



To calculate the limits for each region, we'll need to know the equations for the lines that form the three sides of the triangle. Just by looking at the graph, we can see that two of the lines are $y = 0$ and $y = x$. We can use the points $(1,1)$, and $(2,0)$ to solve for the third line using this formula for the equation of a line:

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 1 = \frac{0 - 1}{2 - 1}(x - 1)$$

$$y - 1 = -(x - 1)$$

$$y = -x + 1 + 1$$

$$y = -x + 2$$

This means that region 1 on the graph is defined for y on $[0,x]$ and for x on $[0,1]$. Region 2 is defined for y on $[0,-x+2]$ and for x on $[1,2]$.

Using Green's theorem to convert the line integral into double integrals gives

$$\oint_c P\ dx + Q\ dy = \iint_{R1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA + \iint_{R2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

$$\oint_c \left(e^x + y^2\right)\ dx + \left(3x - \cos y\right)\ dy = \int_0^1 \int_0^x 3 - 2y\ dy\ dx + \int_1^2 \int_0^{-x+2} 3 - 2y\ dy\ dx$$

Integrate both integrals with respect to y, then evaluate over each interval.

$$\oint_c \left(e^x + y^2\right)\ dx + \left(3x - \cos y\right)\ dy = \int_0^1 3y - y^2\Big|_{y=0}^{y=x}\ dx + \int_1^2 3y - y^2\Big|_{y=0}^{y=-x+2}\ dx$$

$$\int_0^1 3x - x^2 - \left(3(0) - (0)^2\right)\ dx + \int_1^2 3(-x+2) - (-x+2)^2 - \left(3(0) - (0)^2\right)\ dx$$

$$\int_0^1 3x - x^2\ dx + \int_1^2 -3x + 6 - \left(x^2 - 4x + 4\right)\ dx$$

$$\int_0^1 3x - x^2\ dx + \int_1^2 -3x + 6 - x^2 + 4x - 4\ dx$$

$$\int_0^1 3x - x^2\ dx + \int_1^2 -x^2 + x + 2\ dx$$

Integrate both integrals with respect to x, then evaluate over each interval.

$$\frac{3}{2}x^2 - \frac{1}{3}x^3\Big|_0^1 + -\frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x\Big|_1^2$$

$$\frac{3}{2}(1)^2 - \frac{1}{3}(1)^3 - \left(\frac{3}{2}(0)^2 - \frac{1}{3}(0)^3\right) + -\frac{1}{3}(2)^3 + \frac{1}{2}(2)^2 + 2(2) - \left(-\frac{1}{3}(1)^3 + \frac{1}{2}(1)^2 + 2(1)\right)$$

$$\frac{3}{2} - \frac{1}{3} + -\frac{1}{3}(8) + \frac{1}{2}(4) + 4 - \left(-\frac{1}{3} + \frac{1}{2} + 2\right)$$

$$\frac{3}{2} - \frac{1}{3} - \frac{8}{3} + \frac{4}{2} + 4 + \frac{1}{3} - \frac{1}{2} - 2$$

$$\frac{3}{2} - \frac{8}{3} + \frac{3}{2} + 2$$

$$3 - \frac{8}{3} + 2$$

$$\frac{7}{3}$$

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Topic: Green's Theorem for two regions

Question: What is the value of the integral for the region bounded by $(0,0)$, $(1,2)$, and $(2,0)$?

$$\oint_c (4\sin(2x)+y)\ dx+(x^2+\cos(3y))\ dy$$

Answer choices:

A $\quad -2$

B $\quad 2$

C $\quad -\frac{4}{3}$

D $\quad \frac{4}{3}$

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Solution: B

The given line integral

$$\oint_c (4\sin(2x)+y)\ dx+(x^2+\cos(3y))\ dy$$

is in the form

$$\oint_c P\ dx+Q\ dy$$

We'll change it to the form

$$\int\int_{R1}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\ dA+\int\int_{R2}\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\ dA$$

From the line integral, we know that $Q(x,y)=x^2+\cos(3y)$, so

$$\frac{\partial Q}{\partial x}=2x$$

And we know that $P(x,y)=4\sin(2x)+y$, so

$$\frac{\partial P}{\partial y}=1$$

A sketch of the region bounded by $(0,0)$, $(1,2)$, and $(2,0)$ is

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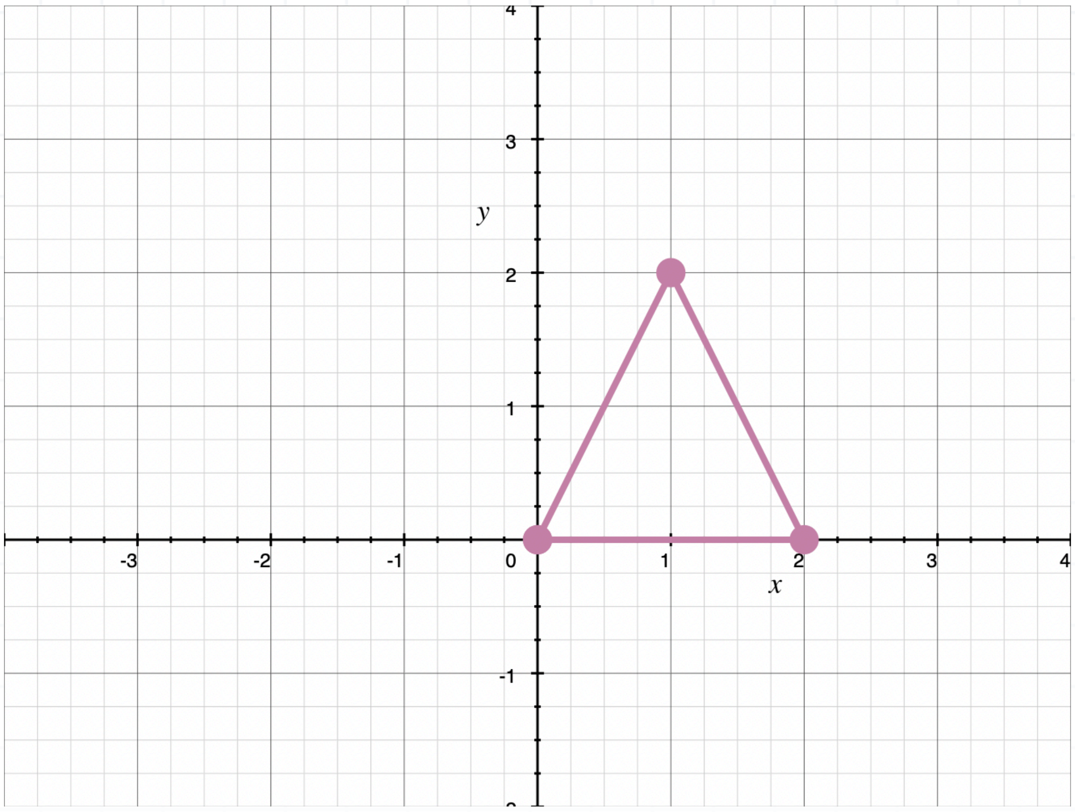
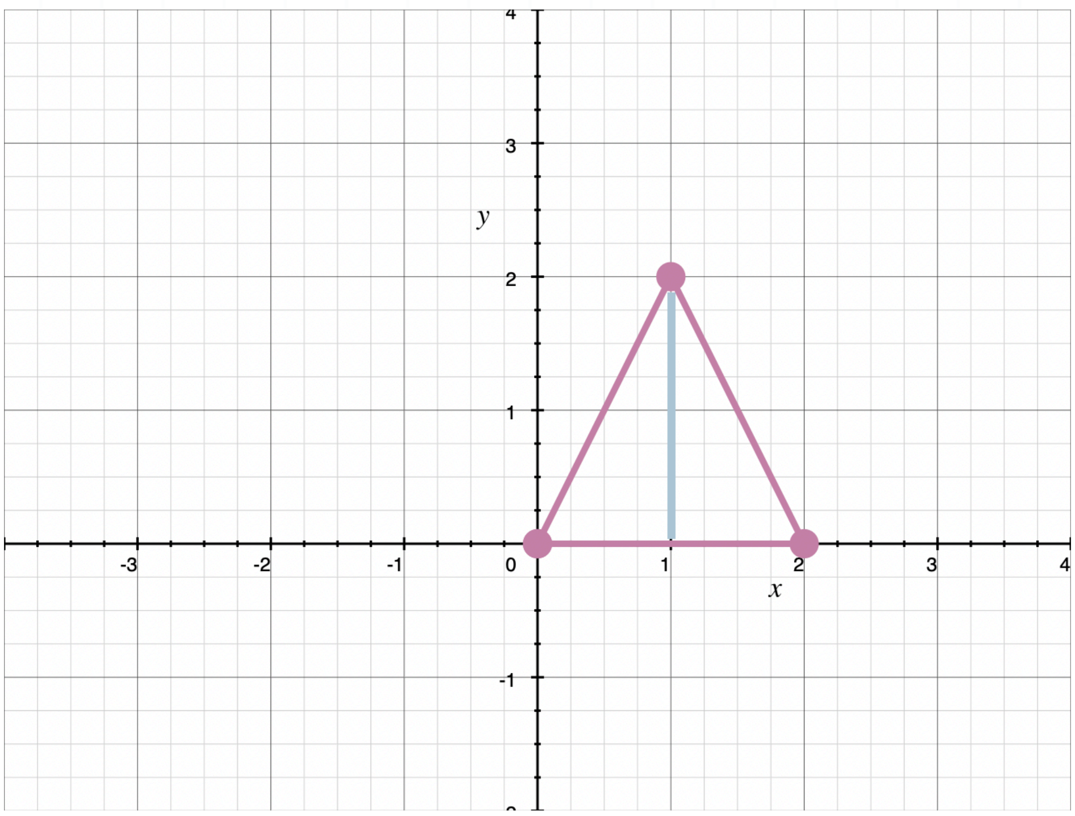


The region needs to be divided in two.



To calculate the limits for each region, we'll need to know the equations for the lines that form the three sides of the triangle. Just by looking at the graph, we can see that two of the lines are $y = 0$ and $y = 2x$. We can use the points $(1,2)$ and $(2,0)$ to solve for the equation of the third line.

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$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

$$y - 2 = \frac{0 - 2}{2 - 1}(x - 1)$$

$$y - 2 = -2(x - 1)$$

$$y = -2x + 2 + 2$$

$$y = -2x + 4$$

This means that the left region is defined for y on $[0,2x]$ and for x on $[0,1]$. The region on the right is defined for y on $[0, -2x + 4]$ and for x on $[1,2]$.

Using Green's Theorem to convert the line integral into double integrals gives

$$\oint_c P\ dx + Q\ dy = \iint_{R1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA + \iint_{R2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

$$\oint_c (4\sin(2x) + y)\ dx + (x^2 + \cos(3y))\ dy$$

$$= \int_0^1 \int_0^{2x} 2x - 1\ dy\ dx + \int_1^2 \int_0^{-2x+4} 2x - 1\ dy\ dx$$

Integrate with respect to y, then evaluate over each interval.

$$\int_0^1 2xy - y \Big|_{y=0}^{y=2x}\ dx + \int_1^2 2xy - y \Big|_{y=0}^{y=-2x+4}\ dx$$

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$$\int_0^1 2x(2x) - (2x) - (2x(0) - 0)\ dx + \int_1^2 2x(-2x+4) - (-2x+4) - (2x(0) - 0)\ dx$$

$$\int_0^1 4x^2 - 2x\ dx + \int_1^2 -4x^2 + 8x + 2x - 4\ dx$$

$$\int_0^1 4x^2 - 2x\ dx + \int_1^2 -4x^2 + 10x - 4\ dx$$

Integrate with respect to x, then evaluate over each interval.

$$\frac{4}{3}x^3 - x^2\Big|_0^1 + -\frac{4}{3}x^3 + 5x^2 - 4x\Big|_1^2$$

$$\frac{4}{3}(1)^3 - (1)^2 - \left(\frac{4}{3}(0)^3 - (0)^2\right) + -\frac{4}{3}(2)^3 + 5(2)^2 - 4(2) - \left(-\frac{4}{3}(1)^3 + 5(1)^2 - 4(1)\right)$$

$$\frac{4}{3} - 1 - \frac{4}{3}(8) + 5(4) - 8 - \left(-\frac{4}{3} + 5 - 4\right)$$

$$\frac{4}{3} - 1 - \frac{32}{3} + 20 - 8 + \frac{4}{3} - 5 + 4$$

$$\frac{4}{3} - \frac{28}{3} + 10$$

$$-\frac{24}{3} + 10$$

$$2$$

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Topic: Curl and divergence of a vector field

Question: Find the curl of the vector field.

$$F(x, y, z) = \left\langle x^2y, yz, xz^3 \right\rangle$$

Answer choices:

A curl $= y\mathbf{i} - z^3\mathbf{j} + x^2\mathbf{k}$

B curl $= -y\mathbf{i} + z^3\mathbf{j} - x^2\mathbf{k}$

C curl $= -y\mathbf{i} - z^3\mathbf{j} - x^2\mathbf{k}$

D curl $= y\mathbf{i} + z^3\mathbf{j} + x^2\mathbf{k}$

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Solution: C

To find the curl of the vector field

$$F(x, y, z) = \langle x^2y, yz, xz^3 \rangle$$

we can use the formula for curl

$$\text{curl} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

In the given vector field,

$$P = x^2y$$

$$Q = yz$$

$$R = xz^3$$

So the partial derivatives we need are

$$R_y = 0 \qquad P_z = 0 \qquad Q_x = 0$$

$$Q_z = y \qquad R_x = z^3 \qquad P_y = x^2$$

Plug these into the formula for curl.

$$\text{curl} = (0 - y)\mathbf{i} + (0 - z^3)\mathbf{j} + (0 - x^2)\mathbf{k}$$

$$\text{curl} = -y\mathbf{i} - z^3\mathbf{j} - x^2\mathbf{k}$$

This is the curl of the vector field.

Topic: Curl and divergence of a vector field

Question: Find the divergence of the vector field.

$$F(x, y, z) = \left\langle e^x z^3, y^2 z, 3xz \right\rangle$$

Answer choices:

A $\quad$ div $= z^3 - 2yz + 3x$

B $\quad$ div $= e^x z^3 - 2yz + 3x$

C $\quad$ div $= z^3 + 2yz + 3x$

D $\quad$ div $= e^x z^3 + 2yz + 3x$

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Solution: D

To find the divergence of the vector field

$$F(x, y, z) = \left\langle e^x z^3, y^2 z, 3xz \right\rangle$$

we can use the formula for divergence

$$\text{div} = P_x + Q_y + R_z$$

where the given vector field is given by either $F(x, y, z) = \langle P, Q, R \rangle$ or $F(x, y, z) = Pi + Qj + Rk$.

In the given vector field,

$$P_x = e^x z^3$$

$$Q_y = 2yz$$

$$R_z = 3x$$

Plug these into the formula for divergence.

$$\text{div} = e^x z^3 + 2yz + 3x$$

This is the divergence of the vector field.

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Topic: Curl and divergence of a vector field

Question: Find the curl and divergence of the vector field.

$$F(x, y, z) = \left\langle y^2 \sin x, ye^z, -e^{xz} \right\rangle$$

Answer choices:

A curl $= -ye^z\mathbf{i} + ze^x\mathbf{j} - 2y\sin x\mathbf{k}$ div $= y^2\cos x + e^z - xe^z$

B curl $= -ye^z\mathbf{i} + ze^{xz}\mathbf{j} - 2y\sin x\mathbf{k}$ div $= y^2\cos x + e^z - xe^{xz}$

C curl $= y^2\cos x\mathbf{i} + e^z\mathbf{j} - xe^z\mathbf{k}$ div $= -ye^z + ze^x - 2y\sin x$

D curl $= y^2\cos x\mathbf{i} + e^z\mathbf{j} - xe^{xz}\mathbf{k}$ div $= -ye^z + ze^{xz} - 2y\sin x$

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Solution: B

To find the curl of the vector field

$$F(x, y, z) = \left\langle y^2 \sin x, ye^z, -e^{xz} \right\rangle$$

we can use the formula for curl

$$\text{curl} = (R_y - Q_z)\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}$$

In the given vector field,

$$P = y^2 \sin x$$

$$Q = ye^z$$

$$R = -e^{xz}$$

So the partial derivatives we need are

$$R_y = 0 \qquad P_z = 0 \qquad Q_x = 0$$

$$Q_z = ye^z \qquad R_x = -ze^{xz} \qquad P_y = 2y \sin x$$

Plug these into the formula for curl.

$$\text{curl} = (0 - ye^z)\mathbf{i} + (0 - (-ze^{xz}))\mathbf{j} + (0 - 2y\sin(x))\mathbf{k}$$

$$\text{curl} = -ye^z\mathbf{i} + ze^{xz}\mathbf{j} - 2y \sin x\mathbf{k}$$

This is the curl of the vector field.

To find the divergence of the vector field

$$F(x, y, z) = \left\langle y^2 \sin x, ye^z, -e^{xz} \right\rangle$$

we can use the formula for divergence

$$\text{div} = P_x + Q_y + R_z$$

where the given vector field is given by either $F(x, y, z) = \langle P, Q, R \rangle$ or $F(x, y, z) = Pi + Qj + Rk$.

In the given vector field,

$$P_x = y^2 \cos x$$

$$Q_y = e^z$$

$$R_z = -xe^{xz}$$

Plug these into the formula for divergence.

$$\text{div} = y^2 \cos x + e^z - xe^{xz}$$

This is the divergence of the vector field.

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Topic: Potential function of the conservative vector field, three dimensions

Question: Which is the potential function for the following conservative vector field?

$$\mathbf{F}(x,y,z) = 2y^2\mathbf{i} + \left(4xy + e^z\right)\mathbf{j} + \left(ye^z\right)\mathbf{k}$$

Answer choices:

A $f(x,y,z) = x^2 - y^2 + C$

B $f(x,y,z) = 3x^2 + 3y^2 + C$

C $f(x,y,z) = 6x^2 - 3y^2 + C$

D $f(x,y,z) = 2xy^2 + ye^z + C$

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Solution: D

Take the derivative of the given vector field. We can say

$$f_x(x, y, z) = 2y^2$$

$$f_y(x, y, z) = 4xy + e^z$$

$$f_z(x, y, z) = ye^z$$

Find the integral of f_x with respect to x.

$$f(x, y, z) = \int f_x(x, y, z)\ dx$$

$$f(x, y, z) = \int 2y^2\ dx$$

$$f(x, y, z) = 2xy^2 + g(y, z)$$

Differentiate this function with respect to y.

$$f_y(x, y, z) = 4xy + g_y(y, z)$$

Comparing these two functions:

$$f_y(x, y, z) = 4xy + g_y(y, z)$$

$$f_y(x, y, z) = 4xy + e^z$$

tells us that $g_y(y, z) = e^z$. So we'll integrate $g_y(y, z) = e^z$ with respect to y.

$$g(y, z) = \int g_y(y, z)\ dy$$

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$$g(y,z) = \int e^z \ dy$$

$$g(y,z) = ye^z + h(z)$$

We found before that $f(x,y,z) = 2xy^2 + g(y,z)$, so now we can substitute $g(y,z) = ye^z + h(z)$ into that equation for $g(y,z)$.

$$f(x,y,z) = 2xy^2 + g(y,z)$$

$$f(x,y,z) = 2xy^2 + ye^z + h(z)$$

$$f(x,y,z) = 2xy^2 + ye^z + C$$

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Topic: Potential function of the conservative vector field, three dimensions

Question: Which of the following vector field functions can have a potential function?

Answer choices:

A $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + (x^2y + e^{5z})\mathbf{j} + 5ye^{5z}\mathbf{k}$

B $\mathbf{F}(x, y, z) = x^2y^2\mathbf{i} + (x^2y - e^{5z})\mathbf{j} + 5ye^{5z}\mathbf{k}$

C $\mathbf{F}(x, y, z) = xy^2\mathbf{i} + (3x^2y + 5e^{5z})\mathbf{j} - 5ye^{5z}\mathbf{k}$

D $\mathbf{F}(x, y, z) = x^2y^2\mathbf{i} + (2xy - 5e^{5z})\mathbf{j} + 5ye^{5z}\mathbf{k}$

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Solution: A

From the given vector field function, we have:

$$M = xy^2$$

$$N = x^2y + e^{5z}$$

$$P = 5ye^{5z}$$

Check the following conditions to see whether vector the field function is conservative:

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$$

$$\frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

For answer choice A, we get

$\frac{\partial P}{\partial y} = 5e^{5z}$ and $\frac{\partial N}{\partial z} = 5e^{5z}$ so $\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}$

$\frac{\partial P}{\partial x} = 0$ and $\frac{\partial M}{\partial z} = 0$ so $\frac{\partial P}{\partial x} = \frac{\partial M}{\partial z}$

$\frac{\partial M}{\partial y} = 2xy$ and $\frac{\partial N}{\partial x} = 2xy$ so $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Therefore, the function given in answer choice A

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$$\mathbf{F}(x, y, z) = xy^2\mathbf{i} + (x^2y + e^{5z})\mathbf{j} + 5ye^{5z}\mathbf{k}$$

is conservative; so it can have a potential function.

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Topic: Potential function of the conservative vector field, three dimensions

Question: Which vector field could result in this potential function?

$$f(x,y,z) = x^2y + y^2z^3 + C$$

Answer choices:

A $\quad \mathbf{F}(x,y,z) = 2xy\mathbf{i} + \left(x^2 + z\right)\mathbf{j} + x^3z\mathbf{k}$

B $\quad \mathbf{F}(x,y,z) = 2xy\mathbf{i} + \left(x^2 + 2yz^3\right)\mathbf{j} + 3y^2z^2\mathbf{k}$

C $\quad \mathbf{F}(x,y,z) = 2x^2y\mathbf{i} + (x + z)\mathbf{j} + \left(xy^3z^2 - 4x\right)\mathbf{k}$

D $\quad \mathbf{F}(x,y,z) = 2xyz\mathbf{i} + \left(x^2 + z\right)\mathbf{j} + \left(xy + z^2\right)\mathbf{k}$

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Solution: B

Choose $\mathbf{F}(x, y, z) = 2xy\mathbf{i} + \left(x^2 + 2yz^3\right)\mathbf{j} + 3y^2z^2\mathbf{k}$ as the vector field function, and consider the following functions as partial derivatives:

$$f_x(x, y, z) = 2xy$$

$$f_y(x, y, z) = x^2 + 2yz^3$$

$$f_z(x, y, z) = 3y^2z^2$$

Find the integrals of these functions with respect to x, y, and z.

$$f(x, y, z) = \int f_x(x, y, z)\ dx = \int 2xy\ dx = x^2y + g(y, z)$$

$$f(x, y, z) = \int f_y(x, y, z)\ dy = \int \left(x^2 + 2yz^3\right)\ dy = x^2y + y^2z^3 + h(x, z) + C$$

$$f(x, y, z) = \int f_z(x, y, z)\ dx = \int 3y^2z^2\ dz = y^2z^3 + k(x, y) + C$$

These are three forms of the function $f(x, y, z)$. Comparing these to the function originally given to us, we get $g(x, y) = y^2z^3$ and $h(x, z) = 0$. Therefore, $f\left(x, y, z\right) = x^2y + y^2z^3 + C$.

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Topic: Points on the surface

Question: Say whether the point $(3,0,5)$ lies on the surface.

$$r(u,v) = \langle u+v, 2u-v, u+2v \rangle$$

Answer choices:

A The point does not lie on the surface ($u=2$ and $v=1$)

B The point does not lie on the surface ($u=1$ and $v=2$)

C The point lies on the surface ($u=2$ and $v=1$)

D The point lies on the surface ($u=1$ and $v=2$)

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Solution: D

We'll convert the vector function

$$r(u,v) = \langle u+v, 2u-v, u+2v \rangle$$

into its parametric equations.

$$x = u + v$$

$$y = 2u - v$$

$$z = u + 2v$$

Putting this together with the point we're interested in, $(3,0,5)$, gives

$$3 = u + v$$

$$0 = 2u - v$$

$$5 = u + 2v$$

Let's solve $3 = u + v$ for u.

$$u = 3 - v$$

Plug this value for u into $0 = 2u - v$.

$$0 = 2u - v$$

$$0 = 2(3 - v) - v$$

$$0 = 6 - 2v - v$$

$$3v = 6$$

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$$v = 2$$

Substitute $v = 2$ back into $u = 3 - v$.

$$u = 3 - v$$

$$u = 3 - 2$$

$$u = 1$$

Now we can plug $u = 1$ and $v = 2$ into $5 = u + 2v$ to verify that our values for u and v are correct.

$$5 = u + 2v$$

$$5 = 1 + 2(2)$$

$$5 = 5$$

Since the values for u and v work in all three equations, the point $(3,0,5)$ lies on the surface of $r(u, v) = \langle u + v, 2u - v, u + 2v \rangle$.

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Topic: Points on the surface

Question: Say whether the point $(2,7,3)$ lies on the surface.

$$r(u, v) = \langle 2 + u + 2v, 3u - v, u + v\rangle$$

Answer choices:

A The point does not lie on the surface ($u = -2$ and $v = 1$)

B The point does not lie on the surface ($u = 2$ and $v = -1$)

C The point lies on the surface ($u - = 2$ and $v = 1$)

D The point lies on the surface ($u = 2$ and $v = -1$)

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Solution: B

We'll convert the vector function

$$r(u, v) = \langle 2 + u + 2v, 3u - v, u + v \rangle$$

into its parametric equations.

$$x = 2 + u + 2v$$

$$y = 3u - v$$

$$z = u + v$$

Putting this together with the point we're interested in, $(2,7,3)$, gives

$$2 = 2 + u + 2v$$

$$7 = 3u - v$$

$$3 = u + v$$

Let's solve $2 = 2 + u + 2v$ for u.

$$u = -2v$$

Plug this value for u into $7 = 3u - v$.

$$7 = 3u - v$$

$$7 = 3(-2v) - v$$

$$7 = -6v - v$$

$$7 = -7v$$

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$$v = -1$$

Substitute $v = -1$ back into $u = -2v$.

$$u = -2v$$

$$u = -2(-1)$$

$$u = 2$$

Now we can plug $u = 2$ and $v = -1$ into $3 = u + v$ to verify that our values for u and v are correct.

$$3 = u + v$$

$$3 = 2 + (-1)$$

$$3 \neq 1$$

Since the values for u and v do not work in all three equations, the point $(2,7,3)$ does not lie on the surface of $r(u, v) = \langle 2 + u + 2v, 3u - v, u + v \rangle$.

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Topic: Points on the surface

Question: Say whether the point $(0,-7,3)$ lies on the surface.

$$r(u,v)=\langle 2u-v,3u+2v,2-u+3v\rangle$$

Answer choices:

A The point does not lie on the surface ($u=-1$ and $v=-2$)

B The point does not lie on the surface ($u=1$ and $v=2$)

C The point lies on the surface ($u=-1$ and $v=-2$)

D The point lies on the surface ($u=1$ and $v=2$)

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Solution: A

We'll convert the vector function

$$r(u, v) = \langle 2u - v, 3u + 2v, 2 - u + 3v \rangle$$

into its parametric equations.

$$x = 2u - v$$

$$y = 3u + 2v$$

$$z = 2 - u + 3v$$

Putting this together with the point we're interested in, $(0, -7, 3)$, gives

$$0 = 2u - v$$

$$-7 = 3u + 2v$$

$$3 = 2 - u + 3v$$

Let's solve $0 = 2u - v$ for u.

$$u = \frac{1}{2}v$$

Plug this value for u into $-7 = 3u + 2v$.

$$-7 = 3u + 2v$$

$$-7 = 3\left(\frac{1}{2}v\right) + 2v$$

$$-7 = \frac{3}{2}v + \frac{4}{2}v$$

$$-7 = \frac{7}{2}v$$

$$-1 = \frac{1}{2}v$$

$$v = -2$$

Substitute $v = -2$ back into $u = (1/2)v$.

$$u = -2v$$

$$u = \frac{1}{2}(-2)$$

$$u = -1$$

Now we can plug $u = -1$ and $v = -2$ into $3 = 2 - u + 3v$ to verify that our values for u and v are correct.

$$3 = 2 - u + 3v$$

$$3 = 2 - (-1) + 3(-2)$$

$$3 \neq -3$$

Since the values for u and v do not work in all three equations, the point $(0, -7,3)$ does not lie on the surface of $r(u, v) = \langle 2u - v, 3u + 2v, 2 - u + 3v \rangle$.

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Topic: Surface of the vector equation

Question: Which statement is true about the surface of the vector equation?

$$r(u, v) = \langle u^2 + v^2, u, v \rangle$$

Answer choices:

A The vector equation $r(u, v) = \langle u^2 + v^2, u, v \rangle$ is a hyperbolic paraboloid centered at $(0,0,0)$ that opens in the positive direction of the x-axis.

B The vector equation $r(u, v) = \langle u^2 + v^2, u, v \rangle$ is a hyperbolic paraboloid centered at $(0,0,0)$ that opens in the negative direction of the x-axis.

C The vector equation $r(u, v) = \langle u^2 + v^2, u, v \rangle$ is an elliptic paraboloid centered at $(0,0,0)$ that opens in the positive direction of the x-axis.

D The vector equation $r(u, v) = \langle u^2 + v^2, u, v \rangle$ is an elliptic paraboloid centered at $(0,0,0)$ that opens in the negative direction of the x-axis.

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Solution: C

First convert

$$r(u, v) = \langle u^2 + v^2, u, v \rangle$$

into parametric equations.

$$x = u^2 + v^2$$

$$y = u$$

$$z = v$$

Next we can substitute $y = u$ and $z = v$ into $x = u^2 + v^2$ to get an equation without parameters.

$$x = y^2 + z^2$$

This equation most closely resembles the equation for an elliptic paraboloid.

$$\frac{z - z_0}{a} = \frac{(x - x_0)^2}{b} + \frac{(y - y_0)^2}{c}$$

In our equation however, the x has an exponent of 1, which means that the elliptic paraboloid opens in the positive direction of the x-axis. The center of the base is indicated by (x_0, y_0, z_0), which in our case is $(0,0,0)$.

The vector equation $r(u, v) = \langle u^2 + v^2, u, v \rangle$ is an elliptic paraboloid centered at $(0,0,0)$ that opens in the positive direction of the x-axis.

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Topic: Surface of the vector equation

Question: Which statement is true about the surface of the vector equation?

$$r(u,v) = \langle 4 + 2u + v, 1 + u, u - v \rangle$$

Answer choices:

A The vector equation $r(u,v) = \langle 4 + 2u + v, 1 + u, u - v \rangle$ is a plane that passes through the point $(4,0,1)$ and contains the vectors $\langle 2,1,1 \rangle$ and $\langle 1,0,1 \rangle$.

B The vector equation $r(u,v) = \langle 4 + 2u + v, 1 + u, u - v \rangle$ is a plane that passes through the point $(4,0,1)$ and contains the vectors $\langle 2,1,1 \rangle$ and $\langle 1,0,-1 \rangle$.

C The vector equation $r(u,v) = \langle 4 + 2u + v, 1 + u, u - v \rangle$ is a plane that passes through the point $(4,1,0)$ and contains the vectors $\langle 2,1,1 \rangle$ and $\langle 1,0,1 \rangle$.

D The vector equation $r(u,v) = \langle 4 + 2u + v, 1 + u, u - v \rangle$ is a plane that passes through the point $(4,1,0)$ and contains the vectors $\langle 2,1,1 \rangle$ and $\langle 1,0,-1 \rangle$.

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Solution: D

First convert

$$r(u, v) = \langle 4 + 2u + v, 1 + u, u - v \rangle$$

into parametric equations.

$$x = 4 + 2u + v$$

$$y = 1 + u$$

$$z = u - v$$

Next we can solve $y = 1 + u$ for u.

$$y = 1 + u$$

$$u = y - 1$$

Then we'll solve $z = u - v$ for v.

$$z = u - v$$

$$v = u - z$$

Substitute $u = y - 1$ into $v = u - z$.

$$v = (y - 1) - z$$

$$v = y - z - 1$$

Substitute $u = y - 1$ and $v = y - z - 1$ into $x = 4 + 2u + v$.

$$x = 4 + 2(y - 1) + (y - z - 1)$$

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$$x = 4 + 2y - 2 + y - z - 1$$

$$x = 3y - z + 1$$

$$x - 3y + z = 1$$

In this equation, each exponent is raised to the power of 1, which means the equation represents a plane.

We can find out more information from the parametric equations of the original vector function.

$$x = 4 + 2u + v$$

$$y = 1 + u + 0v$$

$$z = 0 + u - v$$

From the parametric equations, we know that the plane passes through the point $(4,1,0)$ and contains the vectors $\langle 2,1,1 \rangle$ and $\langle 1,0,-1 \rangle$.

The vector equation $r(u, v) = \langle 4 + 2u + v, 1 + u, u - v \rangle$ is a plane that passes through the point $(4,1,0)$ and contains the vectors $\langle 2,1,1 \rangle$ and $\langle 1,0,-1 \rangle$.

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Topic: Surface of the vector equation

Question: Which statement is true about the surface of the vector equation?

$$r(u, v) = \langle u, (v-1)^2 - u^2, v \rangle$$

Answer choices:

A The vector equation $r(u, v) = \langle u, (v-1)^2 - u^2, v \rangle$ is a hyperbolic paraboloid centered at $(0,0,-1)$ that opens in the positive direction of the y-axis.

B The vector equation $r(u, v) = \langle u, (v-1)^2 - u^2, v \rangle$ is a hyperbolic paraboloid centered at $(0,0,1)$ that opens in the positive direction of the y-axis.

C The vector equation $r(u, v) = \langle u, (v-1)^2 - u^2, v \rangle$ is an elliptic paraboloid centered at $(0,0,-1)$ that opens in the positive direction of the y-axis.

D The vector equation $r(u, v) = \langle u, (v-1)^2 - u^2, v \rangle$ is an elliptic paraboloid centered at $(0,0,1)$ that opens in the positive direction of the y-axis.

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Solution: B

First convert

$$r(u, v) = \langle u, (v-1)^2 - u^2, v\rangle$$

into parametric equations.

$$x = u$$

$$y = (v-1)^2 - u^2$$

$$z = v$$

Next we can substitute $x = u$ and $z = v$ into $y = (v-1)^2 - u^2$ to get an equation without parameters.

$$y = (z-1)^2 - x^2$$

This equation most closely resembles the equation for a hyperbolic paraboloid.

$$\frac{z - z_0}{a} = \frac{(y - y_0)^2}{b} + \frac{(x - x_0)^2}{c}$$

In our equation however, the y has an exponent of 1, which means that the hyperbolic paraboloid opens in the positive direction of the y-axis. The center of the base is indicated by (x_0, y_0, z_0), which in our case is $(0,0,1)$.

The vector equation $r(u, v) = \langle u, (v-1)^2 - u^2, v\rangle$ is a hyperbolic paraboloid centered at $(0,0,1)$ that opens in the positive direction of the y-axis.

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Topic: Parametric representation of the surface

Question: What is the parametric representation of the surface which is the part of the sphere $x^2 + y^2 + z^2 = 9$ that lies above the cone $z = \sqrt{x^2 + y^2}$?

Answer choices:

A $\quad x = 9 \sin \phi \cos \theta \qquad y = 9 \sin \phi \sin \theta \qquad z = 9 \cos \phi$

with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$

B $\quad x = 3 \sin \phi \cos \theta \qquad y = 3 \sin \phi \sin \theta \qquad z = 3 \cos \phi$

with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$

C $\quad x = 9 \sin \phi \cos \theta \qquad y = 9 \sin \phi \sin \theta \qquad z = 9 \cos \phi$

with $0 \le \theta \le \dfrac{\pi}{2}$ and $0 \le \phi \le \pi$

D $\quad x = 3 \sin \phi \cos \theta \qquad y = 3 \sin \phi \sin \theta \qquad z = 3 \cos \phi$

with $0 \le \theta \le \dfrac{\pi}{2}$ and $0 \le \phi \le \pi$

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Solution: B

First, we'll find ρ. The equation of a sphere is $x^2 + y^2 + z^2 = \rho^2$, and if we match that up to $x^2 + y^2 + z^2 = 9$, we know that

$$\rho^2 = 9$$

$$\rho = 3$$

The parametric equations of a sphere are given by

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

so the parametric equations for this sphere are

$$x = 3\sin\phi \cos\theta$$

$$y = 3\sin\phi \sin\theta$$

$$z = 3\cos\phi$$

We need to find the bounds for the parameters ϕ and θ. Since we're dealing with a full sphere, we know θ will be defined on $0 \le \theta \le 2\pi$. Because the region lies above the cone $z = \sqrt{x^2 + y^2}$, to find the bounds for ϕ we can substitute our parametric equations into the equation for the cone.

$$z = \sqrt{x^2 + y^2}$$

$$3\cos\phi = \sqrt{(3\sin\phi\cos\theta)^2 + (3\sin\phi\sin\theta)^2}$$

$$3\cos\phi = \sqrt{9\sin^2\phi\cos^2\theta + 9\sin^2\phi\sin^2\theta}$$

$$3\cos\phi = \sqrt{9\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)}$$

From the identity $\cos^2\theta + \sin^2\theta = 1$, we can substitute and get

$$3\cos\phi = \sqrt{9\sin^2\phi(1)}$$

$$3\cos\phi = 3\sin\phi$$

$$\cos\phi = \sin\phi$$

The sine and cosine function (we can tell this from the unit circle) are equal equal to each other when $\phi = \pi/4$. So the interval for ϕ will be $0 \leq \phi \leq \pi/4$.

The parametric representation of the surface is

$$x = 3\sin\phi\cos\theta$$

$$y = 3\sin\phi\sin\theta$$

$$z = 3\cos\phi$$

with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \dfrac{\pi}{4}$

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Topic: Parametric representation of the surface

Question: What is the parametric representation of the surface which is the part of the sphere $x^2 + y^2 + z^2 = 16$ that lies above the cone $z = \sqrt{x^2 + y^2}$?

Answer choices:

A $x = 16 \sin \phi \cos \theta$ $y = 16 \sin \phi \sin \theta$ $z = 16 \cos \phi$

with $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \pi$

B $x = 16 \sin \phi \cos \theta$ $y = 16 \sin \phi \sin \theta$ $z = 16 \cos \phi$

with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{4}$

C $x = 4 \sin \phi \cos \theta$ $y = 4 \sin \phi \sin \theta$ $z = 4 \cos \phi$

with $0 \leq \theta \leq \frac{\pi}{2}$ and $0 \leq \phi \leq \pi$

D $x = 4 \sin \phi \cos \theta$ $y = 4 \sin \phi \sin \theta$ $z = 4 \cos \phi$

with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{4}$

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Solution: D

First, we'll find ρ. The equation of a sphere is $x^2+y^2+z^2=\rho^2$, and if we match that up to $x^2+y^2+z^2=16$, we know that

$$\rho^2=16$$

$$\rho=4$$

The parametric equations of a sphere are given by

$$x=\rho\sin\phi\cos\theta$$

$$y=\rho\sin\phi\sin\theta$$

$$z=\rho\cos\phi$$

so the parametric equations for this sphere are

$$x=4\sin\phi\cos\theta$$

$$y=4\sin\phi\sin\theta$$

$$z=4\cos\phi$$

We need to find the bounds for the parameters ϕ and θ. Since we're dealing with a full sphere, we know θ will be defined on $0\le\theta\le 2\pi$. Because the region lies above the cone $z=\sqrt{x^2+y^2}$, to find the bounds for ϕ we can substitute our parametric equations into the equation for the cone.

$$z=\sqrt{x^2+y^2}$$

$$4\cos\phi = \sqrt{(4\sin\phi\cos\theta)^2 + (4\sin\phi\sin\theta)^2}$$

$$4\cos\phi = \sqrt{16\sin^2\phi\cos^2\theta + 16\sin^2\phi\sin^2\theta}$$

$$4\cos\phi = \sqrt{16\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)}$$

From the identity $\cos^2\theta + \sin^2\theta = 1$, we can substitute and get

$$4\cos\phi = \sqrt{16\sin^2\phi(1)}$$

$$4\cos\phi = 4\sin\phi$$

$$\cos\phi = \sin\phi$$

The sine and cosine function (we can tell this from the unit circle) are equal equal to each other when $\phi = \pi/4$. So the interval for ϕ will be $0 \le \phi \le \pi/4$.

The parametric representation of the surface is

$$x = 4\sin\phi\cos\theta$$

$$y = 4\sin\phi\sin\theta$$

$$z = 4\cos\phi$$

with $0 \le \theta \le 2\pi$ and $0 \le \phi \le \dfrac{\pi}{4}$

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Topic: Parametric representation of the surface

Question: What is the parametric representation of the surface which is the part of the sphere $3x^2+3y^2+3z^2=12$ that lies above the cone $z=\sqrt{x^2+y^2}$?

Answer choices:

A $\quad x=\sqrt{12}\sin\phi\cos\theta \qquad y=\sqrt{12}\sin\phi\sin\theta \qquad z=\sqrt{12}\cos\phi$

with $0\le\theta\le 2\pi$ and $0\le\phi\le\frac{\pi}{4}$

B $\quad x=\sqrt{12}\sin\phi\cos\theta \qquad y=\sqrt{12}\sin\phi\sin\theta \qquad z=\sqrt{12}\cos\phi$

with $0\le\theta\le\frac{\pi}{2}$ and $0\le\phi\le\pi$

C $\quad x=2\sin\phi\cos\theta \qquad y=2\sin\phi\sin\theta \qquad z=2\cos\phi$

with $0\le\theta\le 2\pi$ and $0\le\phi\le\frac{\pi}{4}$

D $\quad x=2\sin\phi\cos\theta \qquad y=2\sin\phi\sin\theta \qquad z=2\cos\phi$

with $0\le\theta\le\frac{\pi}{2}$ and $0\le\phi\le\pi$

Solution: C

First, we'll find ρ. The equation of a sphere is $x^2+y^2+z^2=\rho^2$, and if we match that up to

$$3x^2+3y^2+3z^2=12$$

$$x^2+y^2+z^2=4$$

we know that

$$\rho^2=4$$

$$\rho=2$$

The parametric equations of a sphere are given by

$$x=\rho\sin\phi\cos\theta$$

$$y=\rho\sin\phi\sin\theta$$

$$z=\rho\cos\phi$$

so the parametric equations for this sphere are

$$x=2\sin\phi\cos\theta$$

$$y=2\sin\phi\sin\theta$$

$$z=2\cos\phi$$

We need to find the bounds for the parameters ϕ and θ. Since we're dealing with a full sphere, we know θ will be defined on $0\le\theta\le 2\pi$. Because the region lies above the cone $z = \sqrt{x^2 + y^2}$, to find the bounds for ϕ we can substitute our parametric equations into the equation for the cone.

$$z = \sqrt{x^2 + y^2}$$

$$2\cos\phi = \sqrt{(2\sin\phi\cos\theta)^2 + (2\sin\phi\sin\theta)^2}$$

$$2\cos\phi = \sqrt{4\sin^2\phi\cos^2\theta + 4\sin^2\phi\sin^2\theta}$$

$$2\cos\phi = \sqrt{4\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)}$$

From the identity $\cos^2\theta + \sin^2\theta = 1$, we can substitute and get

$$2\cos\phi = \sqrt{4\sin^2\phi(1)}$$

$$2\cos\phi = 2\sin\phi$$

$$\cos\phi = \sin\phi$$

The sine and cosine function (we can tell this from the unit circle) are equal equal to each other when $\phi = \pi/4$. So the interval for ϕ will be $0 \le \phi \le \pi/4$.

The parametric representation of the surface is

$$x = 2\sin\phi\cos\theta$$

$$y = 2\sin\phi\sin\theta$$

$$z = 2\cos\phi$$

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with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \dfrac{\pi}{4}$

Topic: Tangent plane to the parametric surface

Question: What is the equation of the tangent plane to the parametric surface when $u = 1$ and $v = 2$?

$$r(u, v) = uv^2\mathbf{i} + v^3\mathbf{j} + u^2\mathbf{k}$$

Answer choices:

A $\quad -24x + 8y + 48z = -16$

B $\quad -24x + 8y + 48z = 16$

C $\quad -24x + 8y + 32z = 0$

D $\quad -24x + 8y + 32z = 128$

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Solution: B

We'll find the parametric equations for $r(u,v) = uv^2\mathbf{i} + v^3\mathbf{j} + u^2\mathbf{k}$.

$$x = uv^2$$

$$y = v^3$$

$$z = u^2$$

Substitute $u = 1$ and $v = 2$ into these parametric equations.

$$x = (1)(2)^2$$

$$x = 4$$

and

$$y = (2)^3$$

$$y = 8$$

and

$$z = (1)^2$$

$$z = 1$$

This gives us a point on the tangent plane $(4,8,1)$. Now we can find the partial derivatives of $r(u,v) = uv^2\mathbf{i} + v^3\mathbf{j} + u^2\mathbf{k}$ with respect to both u and v.

$$\frac{\partial r(u,v)}{\partial u} = v^2\mathbf{i} + 0\mathbf{j} + 2u\mathbf{k}$$

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$$\frac{\partial r(u,v)}{\partial v} = 2uv\mathbf{i} + 3v^2\mathbf{j} + 0\mathbf{k}$$

Find the cross product of the partial derivatives.

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = \left((0)(0) - (2u)(3v^2)\right)\mathbf{i}$$

$$-\left((v^2)(0) - (2u)(2uv)\right)\mathbf{j} + \left((v^2)(3v^2) - (0)(2uv)\right)\mathbf{k}$$

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = \left(0 - 6uv^2\right)\mathbf{i} - \left(0 - 4u^2v\right)\mathbf{j} + \left(3v^4 - 0\right)\mathbf{k}$$

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = -6uv^2\mathbf{i} + 4u^2v\mathbf{j} + 3v^4\mathbf{k}$$

Substitute $u = 1$ and $v = 2$ into the cross product.

$$m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} = -6(1)(2)^2\mathbf{i} + 4(1)^2(2)\mathbf{j} + 3(2)^4\mathbf{k}$$

$$m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} = -24\mathbf{i} + 8\mathbf{j} + 48\mathbf{k}$$

This gives us $m_1 = -24$, $m_2 = 8$ and $m_3 = 48$.

Now we can find the equation of the tangent plane at the point $(4,8,1)$.

$$m_1(x - r(u,v)_1) + m_2(y - r(u,v)_2) + m_3(z - r(u,v)_3) = 0$$

$$-24(x-4) + 8(y-8) + 48(z-1) = 0$$

$$-24x + 96 + 8y - 64 + 48z - 48 = 0$$

$$-24x + 8y + 48z - 16 = 0$$

$$-24x + 8y + 48z = 16$$

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Topic: Tangent plane to the parametric surface

Question: What is the equation of the tangent plane to the parametric surface when $u = -1$ and $v = 3$?

$$r(u, v) = 3u^2v\mathbf{i} - 2uv\mathbf{j} - v^3\mathbf{k}$$

Answer choices:

A $-162x - 486y - 18z = 972$

B $-162x - 486y - 18z = -972$

C $162x - 486y - 18z = 972$

D $162x - 486y - 18z = -972$

Solution: D

We'll find the parametric equations for $r(u, v) = 3u^2v\mathbf{i} - 2uv\mathbf{j} - v^3\mathbf{k}$.

$$x = 3u^2v$$

$$y = -2uv$$

$$z = -v^3$$

Substitute $u = -1$ and $v = 3$ into these parametric equations.

$$x = 3(-1)^2(3)$$

$$x = 9$$

and

$$y = -2(-1)(3)$$

$$y = 6$$

and

$$z = -(3)^3$$

$$z = -27$$

This gives us a point on the tangent plane $(9,6,-27)$. Now we can find the partial derivatives of $r(u, v) = 3u^2v\mathbf{i} - 2uv\mathbf{j} - v^3\mathbf{k}$ with respect to both u and v.

$$\frac{\partial r(u, v)}{\partial u} = 6uv\mathbf{i} - 2v\mathbf{j} + 0\mathbf{k}$$

$$\frac{\partial r(u,v)}{\partial v} = 3u^2\mathbf{i} - 2u\mathbf{j} - 3v^2\mathbf{k}$$

Find the cross product of the partial derivatives.

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = \left((-2v)(-3v^2) - (0)(-2u)\right)\mathbf{i}$$

$$-\left((6uv)(-3v^2) - (0)(3u^2)\right)\mathbf{j} + \left((6uv)(-2u) - (-2v)(3u^2)\right)\mathbf{k}$$

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = 6v^3\mathbf{i} + 18uv^3\mathbf{j} + \left(-12u^2v + 6u^2v\right)\mathbf{k}$$

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = 6v^3\mathbf{i} + 18uv^3\mathbf{j} - 6u^2v\mathbf{k}$$

Substitute $u = -1$ and $v = 3$ into the cross product.

$$m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} = 6(3)^3\mathbf{i} + 18(-1)(3)^3\mathbf{j} - 6(-1)^2(3)\mathbf{k}$$

$$m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} = 162\mathbf{i} - 486\mathbf{j} - 18\mathbf{k}$$

This gives us $m_1 = 162$, $m_2 = -486$ and $m_3 = -18$.

Now we can find the equation of the tangent plane at the point $(9,6,-27)$.

$$m_1(x - r(u,v)_1) + m_2(y - r(u,v)_2) + m_3(z - r(u,v)_3) = 0$$

$$162(x-9) - 486(y-6) - 18(z-(-27)) = 0$$

$$162x - 1{,}458 - 486y + 2{,}916 - 18z - 486 = 0$$

$$162x - 486y - 18z + 972 = 0$$

$$162x - 486y - 18z = -972$$

Topic: Tangent plane to the parametric surface

Question: What is the equation of the tangent plane to the parametric surface when $u = 2$ and $v = 0$?

$$r(u, v) = -ue^v\mathbf{i} + u^3\mathbf{j} + 4uv^2\mathbf{k}$$

Answer choices:

A $0x + 0y - 24z = -10$

B $0x + 0y + 24z = 6$

C $0x + 0y + 24z = 0$

D $0x + 0y + 24z = -6$

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Solution: C

We'll find the parametric equations for $r(u, v) = -ue^v\mathbf{i} + u^3\mathbf{j} + 4uv^2\mathbf{k}$.

$$x = -ue^v$$

$$y = u^3$$

$$z = 4uv^2$$

Substitute $u = 2$ and $v = 0$ into these parametric equations.

$$x = -(2)e^{(0)}$$

$$x = -2$$

and

$$y = (2)^3$$

$$y = 8$$

and

$$z = 4(2)(0)^2$$

$$z = 0$$

This gives us a point on the tangent plane $(-2,8,0)$. Now we can find the partial derivatives of $r(u, v) = -ue^v\mathbf{i} + u^3\mathbf{j} + 4uv^2\mathbf{k}$ with respect to both u and v.

$$\frac{\partial r(u, v)}{\partial u} = -e^v\mathbf{i} + 3u^2\mathbf{j} + 4v^2\mathbf{k}$$

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$$\frac{\partial r(u,v)}{\partial v} = -ue^v\mathbf{i} + 0\mathbf{j} + 8uv\mathbf{k}$$

Find the cross product of the partial derivatives.

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = \left((3u^2)(8uv) - (4v^2)(0)\right)\mathbf{i}$$

$$-\left((-e^v)(8uv) - (4v^2)(-ue^v)\right)\mathbf{j} + \left((-e^v)(0) - (3u^2)(-ue^v)\right)\mathbf{k}$$

$$\frac{\partial r(u,v)}{\partial u} \times \frac{\partial r(u,v)}{\partial v} = 24u^3v\mathbf{i} - \left(-8uve^v + 4uv^2e^v\right)\mathbf{j} + 3u^3e^v\mathbf{k}$$

Substitute $u = 2$ and $v = 0$ into the cross product.

$$m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} = 24(2)^3(0)\mathbf{i} - \left(-8(2)(0)e^{(0)} + 4(2)(0)^2e^{(0)}\right)\mathbf{j} + 3(2)^3e^{(0)}\mathbf{k}$$

$$m_1\mathbf{i} + m_2\mathbf{j} + m_3\mathbf{k} = 0\mathbf{i} + 0\mathbf{j} + 24\mathbf{k}$$

This gives us $m_1 = 0$, $m_2 = 0$ and $m_3 = 24$.

Now we can find the equation of the tangent plane at the point $(-2,8,0)$.

$$m_1(x - r(u,v)_1) + m_2(y - r(u,v)_2) + m_3(z - r(u,v)_3) = 0$$

$$0(x - (-2)) + 0(y - 8) + 24(z - 0) = 0$$

$$0x + 0y + 24z = 0$$

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Topic: Area of a surface

Question: What is the surface area of the part of $z = xy$ that lies inside the cylinder $x^2 + y^2 = 4$?

Answer choices:

A $\pi\left(5\sqrt{5} - 1\right)$

B $2\pi\left(5\sqrt{5} - 1\right)$

C $\frac{2\pi}{3}\left(5\sqrt{5} - 1\right)$

D $\frac{\pi}{3}\left(5\sqrt{5} - 1\right)$

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Solution: C

We'll find the partial derivatives of the surface $z = xy$.

$$\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = x$$

Because the cylinder $x^2 + y^2 = 4$ defines the boundary of the surface we want, and we know that the cylinder is defined in polar coordinates on $r = [0,2]$. These will be the bounds for r.

The limits for θ are based on the surface $z = xy$ cutting through the cylinder in a circular manner. This will give us the limits $0 \le \theta \le 2\pi$.

Then the area formula is

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$A(s) = \int_0^{2\pi} \int_0^2 \sqrt{1 + y^2 + x^2}\ r\ dr\ d\theta$$

To convert to polar coordinates, we know that $x^2 + y^2 = r^2$.

$$A(s) = \int_0^{2\pi} \int_0^2 \sqrt{1 + r^2}\ r\ dr\ d\theta$$

We can use u-substitution with

$$u = 1 + r^2$$

$$\frac{du}{dr} = 2r$$

$$r \ dr = \frac{du}{2}$$

Substitute into the equation.

$$A(s) = \int_0^{2\pi} \int_{r=0}^{r=2} \sqrt{u} \left(\frac{du}{2} \right) \ d\theta$$

$$A(s) = \frac{1}{2} \int_0^{2\pi} \int_{r=0}^{r=2} u^{\frac{1}{2}} \ du \ d\theta$$

Integrate with respect to u.

$$A(s) = \frac{1}{2} \int_0^{2\pi} \frac{2}{3} u^{\frac{3}{2}} \Big|_{r=0}^{r=2} \ d\theta$$

$$A(s) = \frac{1}{3} \int_0^{2\pi} u^{\frac{3}{2}} \Big|_{r=0}^{r=2} \ d\theta$$

Back-substitute for $u = 1 + r^2$.

$$A(s) = \frac{1}{3} \int_0^{2\pi} \left(1 + r^2\right)^{\frac{3}{2}} \Big|_0^2 \ d\theta$$

Evaluate over the integral.

$$A(s) = \frac{1}{3} \int_0^{2\pi} \left(1 + (2)^2\right)^{\frac{3}{2}} - \left(1 + (0)^2\right)^{\frac{3}{2}} \ d\theta$$

$$A(s) = \frac{1}{3} \int_0^{2\pi} (5)^{\frac{3}{2}} - (1)^{\frac{3}{2}} \ d\theta$$

$$A(s) = \frac{1}{3} \int_0^{2\pi} 5\sqrt{5} - 1 \ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$A(s) = \frac{1}{3} \left(5\sqrt{5}\theta - \theta \right) \Big|_0^{2\pi}$$

$$A(s) = \frac{1}{3} \left[\left(5\sqrt{5}(2\pi) - (2\pi) \right) - \left(5\sqrt{5}(0) - (0) \right) \right]$$

$$A(s) = \frac{1}{3} \left(10\pi\sqrt{5} - 2\pi \right)$$

$$A(s) = \frac{2\pi}{3} \left(5\sqrt{5} - 1 \right)$$

This is the surface area of the part of $z = xy$ that lies inside the cylinder $x^2 + y^2 = 4$.

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Topic: Area of a surface

Question: What is the surface area of the part of $z = xy$ that lies inside the cylinder $x^2 + y^2 = 9$?

Answer choices:

A $\quad \frac{\pi}{3}\left(10\sqrt{10} - 1\right)$

B $\quad \frac{2\pi}{3}\left(10\sqrt{10} - 1\right)$

C $\quad \pi\left(10\sqrt{10} - 1\right)$

D $\quad 2\pi\left(10\sqrt{10} - 1\right)$

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Solution: B

We'll find the partial derivatives of the surface $z = xy$.

$$\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = x$$

Because the cylinder $x^2 + y^2 = 9$ defines the boundary of the surface we want, and we know that the cylinder is defined in polar coordinates on $r = [0,3]$. These will be the bounds for r.

The limits for θ are based on the surface $z = xy$ cutting through the cylinder in a circular manner. This will give us the limits $0 \le \theta \le 2\pi$.

Then the area formula is

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \ dA$$

$$A(s) = \int_0^{2\pi} \int_0^3 \sqrt{1 + y^2 + x^2} \ r \ dr \ d\theta$$

To convert to polar coordinates, we know that $x^2 + y^2 = r^2$.

$$A(s) = \int_0^{2\pi} \int_0^3 \sqrt{1 + r^2} \ r \ dr \ d\theta$$

We can use u-substitution with

$$u = 1 + r^2$$

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$$\frac{du}{dr} = 2r$$

$$r\ dr = \frac{du}{2}$$

Substitute into the equation.

$$A(s) = \int_0^{2\pi} \int_{r=0}^{r=3} \sqrt{u} \left(\frac{du}{2} \right)\ d\theta$$

$$A(s) = \frac{1}{2} \int_0^{2\pi} \int_{r=0}^{r=3} u^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u.

$$A(s) = \frac{1}{2} \int_0^{2\pi} \frac{2}{3} u^{\frac{3}{2}} \Big|_{r=0}^{r=3}\ d\theta$$

$$A(s) = \frac{1}{3} \int_0^{2\pi} u^{\frac{3}{2}} \Big|_{r=0}^{r=3}\ d\theta$$

Back-substitute for $u = 1 + r^2$.

$$A(s) = \frac{1}{3} \int_0^{2\pi} \left(1 + r^2\right)^{\frac{3}{2}} \Big|_0^3\ d\theta$$

Evaluate over the integral.

$$A(s) = \frac{1}{3} \int_0^{2\pi} \left(1 + (3)^2\right)^{\frac{3}{2}} - \left(1 + (0)^2\right)^{\frac{3}{2}}\ d\theta$$

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$$A(s) = \frac{1}{3}\int_0^{2\pi} (10)^{\frac{3}{2}} - (1)^{\frac{3}{2}}\ d\theta$$

$$A(s) = \frac{1}{3}\int_0^{2\pi} 10\sqrt{10} - 1\ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$A(s) = \frac{1}{3}\left(10\sqrt{10}\theta - \theta\right)\Big|_0^{2\pi}$$

$$A(s) = \frac{1}{3}\left[\left(10\sqrt{10}(2\pi) - (2\pi)\right) - \left(10\sqrt{10}(0) - (0)\right)\right]$$

$$A(s) = \frac{1}{3}\left(20\pi\sqrt{10} - 2\pi\right)$$

$$A(s) = \frac{2\pi}{3}\left(10\sqrt{10} - 1\right)$$

This is the surface area of the part of $z = xy$ that lies inside the cylinder $x^2 + y^2 = 9$.

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Topic: Area of a surface

Question: What is the surface area of the part of $z = 2xy$ that lies inside the cylinder $x^2 + y^2 = 1$?

Answer choices:

A $\frac{\pi}{12}\left(5\sqrt{5} - 1\right)$

B $\frac{2\pi}{3}\left(5\sqrt{5} - 1\right)$

C $\frac{\pi}{3}\left(5\sqrt{5} - 1\right)$

D $\frac{\pi}{6}\left(5\sqrt{5} - 1\right)$

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Solution: D

We'll find the partial derivatives of the surface $z = 2xy$.

$$\frac{\partial z}{\partial x} = 2y$$

$$\frac{\partial z}{\partial y} = 2x$$

Because the cylinder $x^2 + y^2 = 1$ defines the boundary of the surface we want, and we know that the cylinder is defined in polar coordinates on $r = [0,1]$. These will be the bounds for r.

The limits for θ are based on the surface $z = 2xy$ cutting through the cylinder in a circular manner. This will give us the limits $0 \le \theta \le 2\pi$.

Then the area formula is

$$A(s) = \int\int_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$A(s) = \int_0^{2\pi}\int_0^1 \sqrt{1 + (2y)^2 + (2x)^2}\ r\ dr\ d\theta$$

$$A(s) = \int_0^{2\pi}\int_0^1 \sqrt{1 + 4y^2 + 4x^2}\ r\ dr\ d\theta$$

$$A(s) = \int_0^{2\pi}\int_0^1 \sqrt{1 + 4\left(y^2 + x^2\right)}\ r\ dr\ d\theta$$

To convert to polar coordinates, we know that $x^2 + y^2 = r^2$.

$$A(s) = \int_0^{2\pi} \int_0^1 \sqrt{1 + 4r^2}\ r\ dr\ d\theta$$

We can use u-substitution with

$$u = 1 + 4r^2$$

$$\frac{du}{dr} = 8r$$

$$r\ dr = \frac{du}{8}$$

Substitute into the equation.

$$A(s) = \int_0^{2\pi} \int_{r=0}^{r=1} \sqrt{u} \left(\frac{du}{8} \right)\ d\theta$$

$$A(s) = \frac{1}{8} \int_0^{2\pi} \int_{r=0}^{r=1} u^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u.

$$A(s) = \frac{1}{8} \int_0^{2\pi} \frac{2}{3} u^{\frac{3}{2}} \Big|_{r=0}^{r=1}\ d\theta$$

$$A(s) = \frac{1}{12} \int_0^{2\pi} u^{\frac{3}{2}} \Big|_{r=0}^{r=1}\ d\theta$$

Back-substitute for $u = 1 + 4r^2$.

$$A(s) = \frac{1}{12} \int_0^{2\pi} \left(1 + 4r^2\right)^{\frac{3}{2}} \Big|_0^1\ d\theta$$

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Evaluate over the integral.

$$A(s) = \frac{1}{12}\int_0^{2\pi} \left(1+4(1)^2\right)^{\frac{3}{2}} - \left(1+4(0)^2\right)^{\frac{3}{2}} \ d\theta$$

$$A(s) = \frac{1}{12}\int_0^{2\pi} (5)^{\frac{3}{2}} - 1 \ d\theta$$

$$A(s) = \frac{1}{12}\int_0^{2\pi} 5\sqrt{5} - 1 \ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$A(s) = \frac{1}{12}\left(5\sqrt{5}\theta - \theta\right)\Big|_0^{2\pi}$$

$$A(s) = \frac{1}{12}\left(5\sqrt{5}(2\pi) - 2\pi\right) - \frac{1}{12}\left(5\sqrt{5}(0) - 0\right)$$

$$A(s) = \frac{1}{12}\left(10\pi\sqrt{5} - 2\pi\right)$$

$$A(s) = \frac{\pi}{6}\left(5\sqrt{5} - 1\right)$$

This is the surface area of the part of $z = 2xy$ that lies inside the cylinder $x^2 + y^2 = 1$.

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Topic: Surface integrals

Question: What is the surface integral equal to when S is the part of the plane $z = x + 2y$ that lies above the rectangle $[0,1] \times [0,1]$?

$$\iint_S xyz \ dS$$

Answer choices:

A $\frac{\sqrt{6}}{6}$

B $\frac{\sqrt{6}}{2}$

C $\sqrt{3}$

D $\sqrt{2}$

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Solution: B

We'll find the partial derivatives of the surface $z = x + 2y$.

$$\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial z}{\partial y} = 2$$

From the given surface integral, we know that $f(x, y, z) = xyz$, and we already have the surface $z = x + 2y$, so for $f(x, y, g(x, y))$ we get

$$f(x, y, g(x, y)) = xy(x + 2y)$$

$$f(x, y, g(x, y)) = x^2y + 2xy^2$$

Now we can find the surface integral.

$$\iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\int_0^1 \int_0^1 \left(x^2y + 2xy^2\right)\sqrt{1 + (1)^2 + (2)^2}\ dy\ dx$$

$$\int_0^1 \int_0^1 \left(x^2y + 2xy^2\right)\sqrt{1 + 1 + 4}\ dy\ dx$$

$$\sqrt{6}\int_0^1 \int_0^1 x^2y + 2xy^2\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$\sqrt{6}\int_0^1 \frac{1}{2}x^2y^2+\frac{2}{3}xy^3\Big|_{y=0}^{y=1}\ dx$$

$$\sqrt{6}\int_0^1 \frac{1}{2}x^2(1)^2+\frac{2}{3}x(1)^3-\left(\frac{1}{2}x^2(0)^2+\frac{2}{3}x(0)^3\right)\ dx$$

$$\sqrt{6}\int_0^1 \frac{1}{2}x^2+\frac{2}{3}x\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$\sqrt{6}\left(\frac{1}{6}x^3+\frac{1}{3}x^2\right)\Big|_0^1$$

$$\sqrt{6}\left(\frac{1}{6}(1)^3+\frac{1}{3}(1)^2\right)-\sqrt{6}\left(\frac{1}{6}(0)^3+\frac{1}{3}(0)^2\right)$$

$$\sqrt{6}\left(\frac{1}{6}+\frac{1}{3}\right)$$

$$\sqrt{6}\left(\frac{1}{6}+\frac{2}{6}\right)$$

$$\sqrt{6}\left(\frac{3}{6}\right)$$

$$\sqrt{6}\left(\frac{1}{2}\right)$$

$$\frac{\sqrt{6}}{2}$$

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Topic: Surface integrals

Question: What is the surface integral equal to when S is the part of the plane $z = 2x + y$ that lies above the rectangle $[0,1] \times [0,1]$?

$$\iint_S 2xyz \ dS$$

Answer choices:

A $\frac{\sqrt{6}}{3}$

B $\frac{\sqrt{6}}{2}$

C $\sqrt{3}$

D $\sqrt{6}$

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Solution: D

We'll find the partial derivatives of the surface $z = 2x + y$.

$$\frac{\partial z}{\partial x} = 2$$

$$\frac{\partial z}{\partial y} = 1$$

From the given surface integral, we know that $f(x, y, z) = 2xyz$, and we already have the surface $z = 2x + y$, so for $f(x, y, g(x, y))$ we get

$$f(x, y, g(x, y)) = 2xy(2x + y)$$

$$f(x, y, g(x, y)) = 4x^2y + 2xy^2$$

Now we can find the surface integral.

$$\iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\int_0^1 \int_0^1 \left(4x^2y + 2xy^2\right)\sqrt{1 + (2)^2 + (1)^2}\ dy\ dx$$

$$\int_0^1 \int_0^1 \left(4x^2y + 2xy^2\right)\sqrt{1 + 4 + 1}\ dy\ dx$$

$$\sqrt{6}\int_0^1 \int_0^1 4x^2y + 2xy^2\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$\sqrt{6}\int_0^1 2x^2y^2+\frac{2}{3}xy^3\Big|_{y=0}^{y=1}\ dx$$

$$\sqrt{6}\int_0^1 2x^2(1)^2+\frac{2}{3}x(1)^3-\left(2x^2(0)^2+\frac{2}{3}x(0)^3\right)\ dx$$

$$\sqrt{6}\int_0^1 2x^2+\frac{2}{3}x\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$\sqrt{6}\left(\frac{2}{3}x^3+\frac{1}{3}x^2\right)\Big|_0^1$$

$$\sqrt{6}\left(\frac{2}{3}(1)^3+\frac{1}{3}(1)^2\right)-\sqrt{6}\left(\frac{2}{3}(0)^3+\frac{1}{3}(0)^2\right)$$

$$\sqrt{6}\left(\frac{2}{3}+\frac{1}{3}\right)$$

$$\sqrt{6}(1)$$

$$\sqrt{6}$$

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Topic: Surface integrals

Question: What is the surface integral equal to when S is the part of the plane $z=2x+y$ that lies above the rectangle $[0,2]\times[0,1]$?

$$\iint_S x^2yz\ dS$$

Answer choices:

A $\dfrac{44\sqrt{6}}{9}$

B $\dfrac{13\sqrt{6}}{36}$

C $\dfrac{44\sqrt{2}}{3}$

D $\dfrac{13}{6}$

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Solution: A

We'll find the partial derivatives of the surface $z = 2x + y$.

$$\frac{\partial z}{\partial x} = 2$$

$$\frac{\partial z}{\partial y} = 1$$

From the given surface integral, we know that $f(x, y, z) = x^2yz$, and we already have the surface $z = 2x + y$, so for $f(x, y, g(x, y))$ we get

$$f(x, y, g(x, y)) = x^2y(2x + y)$$

$$f(x, y, g(x, y)) = 2x^3y + x^2y^2$$

Now we can find the surface integral.

$$\iint_D f(x, y, g(x, y))\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\int_0^2 \int_0^1 \left(2x^3y + x^2y^2\right)\sqrt{1 + (2)^2 + (1)^2}\ dy\ dx$$

$$\int_0^2 \int_0^1 \left(2x^3y + x^2y^2\right)\sqrt{1 + 4 + 1}\ dy\ dx$$

$$\sqrt{6}\int_0^2 \int_0^1 2x^3y + x^2y^2\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

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$$\sqrt{6}\int_0^2 x^3y^2+\frac{1}{3}x^2y^3\Big|_{y=0}^{y=1}\ dx$$

$$\sqrt{6}\int_0^2 x^3(1)^2+\frac{1}{3}x^2(1)^3-\left(x^3(0)^2+\frac{1}{3}x^2(0)^3\right)\ dx$$

$$\sqrt{6}\int_0^2 x^3+\frac{1}{3}x^2\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$\sqrt{6}\left(\frac{1}{4}x^4+\frac{1}{9}x^3\right)\Big|_0^2$$

$$\sqrt{6}\left(\frac{1}{4}(2)^4+\frac{1}{9}(2)^3\right)-\sqrt{6}\left(\frac{1}{4}(0)^4+\frac{1}{9}(0)^3\right)$$

$$\sqrt{6}\left(\frac{1}{4}(16)+\frac{1}{9}(8)\right)$$

$$\sqrt{6}\left(4+\frac{8}{9}\right)$$

$$\sqrt{6}\left(\frac{36}{9}+\frac{8}{9}\right)$$

$$\sqrt{6}\left(\frac{44}{9}\right)$$

$$\frac{44\sqrt{6}}{9}$$

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Topic: Surface integrals of oriented surfaces

Question: What is the surface integral equal to when S is the hemisphere $x^2+y^2+z^2=1$ and $z\geq 0$?

$$\iint_S x^2z+y^2z\ dS$$

Answer choices:

A π

B $\frac{\pi}{2}$

C $\frac{\pi}{3}$

D $\frac{\pi}{4}$

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Solution: B

We'll start by finding the partial derivatives of $x^2 + y^2 + z^2 = 1$. We'll solve for z^2 and get $z^2 = 1 - x^2 - y^2$ and then differentiate using implicit differentiation.

$$2z\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

and

$$2z\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

We need to find the bounds for the double integral we'll be setting up. In polar coordinates, we know that r is bounded by $[0,1]$ for the hemisphere $x^2 + y^2 + z^2 = 1$.

The limits for θ are based on the surface $x^2 + y^2 + z^2 = 1$ and the restriction $z \geq 0$. This will give us the limits $0 \leq \theta \leq 2\pi$. Remember $dA = r\ dr\ d\theta$.

At this point we can either take $x^2 + y^2 + z^2 = 1$, solve for z and substitute that value into $f(x, y, z) = x^2z + y^2z$, or we can substitute our known and calculated elements into the formula and simplify in hopes that the z variables will be eliminated. We'll try the second method.

Let's set up the surface integral.

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$$\iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\int_0^{2\pi}\int_0^1 (x^2z+y^2z)\sqrt{1+\left(-\frac{x}{z}\right)^2+\left(-\frac{y}{z}\right)^2}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 (x^2z+y^2z)\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 (x^2z+y^2z)\sqrt{\frac{z^2}{z^2}+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 (x^2z+y^2z)\frac{\sqrt{z^2+x^2+y^2}}{\sqrt{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 (x^2z+y^2z)\frac{\sqrt{z^2+x^2+y^2}}{z}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 (x^2+y^2)\sqrt{x^2+y^2+z^2}\ r\ dr\ d\theta$$

Using the equation $x^2+y^2+z^2=1$, we'll substitute.

$$\int_0^{2\pi}\int_0^1 (x^2+y^2)\sqrt{1}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 (x^2+y^2)\ r\ dr\ d\theta$$

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To convert from rectangular to polar coordinates, we know that $x^2 + y^2 = r^2$.

$$\int_0^{2\pi} \int_0^1 \left(r^2\right) \ r \ dr \ d\theta$$

$$\int_0^{2\pi} \int_0^1 r^3 \ dr \ d\theta$$

Now that everything is in polar coordinates, we'll integrate with respect to r, and then evaluate over the interval.

$$\int_0^{2\pi} \frac{1}{4}r^4 \Big|_{r=0}^{r=1} \ d\theta$$

$$\int_0^{2\pi} \frac{1}{4}(1)^4 - \frac{1}{4}(0)^4 \ d\theta$$

$$\int_0^{2\pi} \frac{1}{4} \ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$\frac{1}{4}\theta \Big|_0^{2\pi}$$

$$\frac{1}{4}(2\pi) - \frac{1}{4}(0)$$

$$\frac{\pi}{2}$$

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Topic: Surface integrals of oriented surfaces

Question: What is the surface integral equal to when S is the hemisphere $x^2 + y^2 + z^2 = 9$ and $z \geq 0$?

$$\iint_S 2x^2z + 2y^2z \ dS$$

Answer choices:

A $\quad 243\pi$

B $\quad 81\pi$

C $\quad \dfrac{243\pi}{2}$

D $\quad \dfrac{81\pi}{2}$

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Solution: A

We'll start by finding the partial derivatives of $x^2 + y^2 + z^2 = 9$. We'll solve for z^2 and get $z^2 = 9 - x^2 - y^2$ and then differentiate using implicit differentiation.

$$2z\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

and

$$2z\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

We need to find the bounds for the double integral we'll be setting up. In polar coordinates, we know that r is bounded by $[0,3]$ for the hemisphere $x^2 + y^2 + z^2 = 9$.

The limits for θ are based on the surface $x^2 + y^2 + z^2 = 9$ and the restriction $z \geq 0$. This will give us the limits $0 \leq \theta \leq 2\pi$. Remember $dA = r\ dr\ d\theta$.

At this point we can either take $x^2 + y^2 + z^2 = 9$, solve for z and substitute that value into $f(x, y, z) = 2x^2z + 2y^2z$, or we can substitute our known and calculated elements into the formula and simplify in hopes that the z variables will be eliminated. We'll try the second method.

Let's set up the surface integral.

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$$\iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\int_0^{2\pi}\int_0^3 (2x^2z+2y^2z)\sqrt{1+\left(-\frac{x}{z}\right)^2+\left(-\frac{y}{z}\right)^2}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 (2x^2z+2y^2z)\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 (2x^2z+2y^2z)\sqrt{\frac{z^2}{z^2}+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 (2x^2z+2y^2z)\frac{\sqrt{z^2+x^2+y^2}}{\sqrt{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 (2x^2z+2y^2z)\frac{\sqrt{z^2+x^2+y^2}}{z}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 (2x^2+2y^2)\sqrt{z^2+x^2+y^2}\ r\ dr\ d\theta$$

Using the equation $x^2+y^2+z^2=9$, we'll substitute.

$$\int_0^{2\pi}\int_0^3 (2x^2+2y^2)\sqrt{9}\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 3\left(2x^2+2y^2\right)\ r\ dr\ d\theta$$

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To convert from rectangular to polar coordinates, we know that $x^2 + y^2 = r^2$.

$$\int_0^{2\pi}\int_0^3 3\left(2r^2\right)\ r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 6r^3\ dr\ d\theta$$

Now that everything is in polar coordinates, we'll integrate with respect to r, and then evaluate over the interval.

$$\int_0^{2\pi} \frac{3}{2}r^4\Big|_{r=0}^{r=3}\ d\theta$$

$$\int_0^{2\pi} \frac{3}{2}(3)^4 - \frac{3}{2}(0)^4\ d\theta$$

$$\int_0^{2\pi} \frac{243}{2}\ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$\frac{243}{2}\theta\Big|_0^{2\pi}$$

$$\frac{243}{2}(2\pi) - \frac{243}{2}(0)$$

$$243\pi$$

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Topic: Surface integrals of oriented surfaces

Question: What is the surface integral equal to when S is the hemisphere $x^2 + y^2 + z^2 = 4$ with $x \geq 0$ and $z \geq 0$?

$$\iint_S 3x^2z + 3y^2z \ dS$$

Answer choices:

A $\quad 4\pi$

B $\quad 6\pi$

C $\quad 24\pi$

D $\quad 48\pi$

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Solution: C

We'll start by finding the partial derivatives of $x^2 + y^2 + z^2 = 4$. We'll solve for z^2 and get $z^2 = 4 - x^2 - y^2$ and then differentiate using implicit differentiation.

$$2z\frac{\partial z}{\partial x} = -2x$$

$$\frac{\partial z}{\partial x} = -\frac{x}{z}$$

and

$$2z\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

We need to find the bounds for the double integral we'll be setting up. In polar coordinates, we know that r is bounded by $[0,2]$ for the hemisphere $x^2 + y^2 + z^2 = 4$.

The limits for θ are based on the surface $x^2 + y^2 + z^2 = 1$ and the restrictions $x \ge 0$ and $z \ge 0$. This will give us the limits $0 \le \theta \le \pi$. Remember $dA = r\ dr\ d\theta$.

At this point we can either take $x^2 + y^2 + z^2 = 4$, solve for z and substitute that value into $f(x, y, z) = 3x^2z + 3y^2z$, or we can substitute our known and calculated elements into the formula and simplify in hopes that the z variables will be eliminated. We'll try the second method.

Let's set up the surface integral.

$$\iint_D f(x,y,g(x,y))\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\ dA$$

$$\int_0^{\pi}\int_0^2 \left(3x^2z+3y^2z\right)\sqrt{1+\left(-\frac{x}{z}\right)^2+\left(-\frac{y}{z}\right)^2}\ r\ dr\ d\theta$$

$$\int_0^{\pi}\int_0^2 \left(3x^2z+3y^2z\right)\sqrt{1+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{\pi}\int_0^2 \left(3x^2z+3y^2z\right)\sqrt{\frac{z^2}{z^2}+\frac{x^2}{z^2}+\frac{y^2}{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{\pi}\int_0^2 \left(3x^2z+3y^2z\right)\frac{\sqrt{z^2+x^2+y^2}}{\sqrt{z^2}}\ r\ dr\ d\theta$$

$$\int_0^{\pi}\int_0^2 \left(3x^2z+3y^2z\right)\frac{\sqrt{z^2+x^2+y^2}}{z}\ r\ dr\ d\theta$$

$$\int_0^{\pi}\int_0^2 3\left(x^2+y^2\right)\sqrt{x^2+y^2+z^2}\ r\ dr\ d\theta$$

Using the equation $x^2+y^2+z^2=4$, we'll substitute.

$$\int_0^{\pi}\int_0^2 3\left(x^2+y^2\right)\sqrt{4}\ r\ dr\ d\theta$$

$$\int_0^{\pi}\int_0^2 6\left(x^2+y^2\right)\ r\ dr\ d\theta$$

To convert from rectangular to polar coordinates, we know that $x^2 + y^2 = r^2$.

$$\int_0^{\pi} \int_0^2 6\left(r^2\right)\ r\ dr\ d\theta$$

$$\int_0^{\pi} \int_0^2 6r^3\ dr\ d\theta$$

Now that everything is in polar coordinates, we'll integrate with respect to r, and then evaluate over the interval.

$$\int_0^{\pi} \frac{3}{2}r^4 \Big|_{r=0}^{r=2}\ d\theta$$

$$\int_0^{\pi} \frac{3}{2}(2)^4 - \frac{3}{2}(0)^4\ d\theta$$

$$\int_0^{\pi} \frac{48}{2}\ d\theta$$

$$\int_0^{\pi} 24\ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$24\theta\Big|_0^{\pi}$$

$$24\pi - 24(0)$$

$$24\pi$$

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Topic: Flux across the surface

Question: What is the flux of the vector field where S is the part of the paraboloid $z = x^2 + y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$ and has an upward orientation?

$$F(x, y, z) = e^y\mathbf{i} + ye^x\mathbf{j} + x^2y\mathbf{k}$$

Answer choices:

A $\quad \frac{5}{3}e + \frac{11}{6}$

B $\quad \frac{5}{3}e + \frac{1}{6}$

C $\quad -\frac{5}{3}e + \frac{11}{6}$

D $\quad -\frac{5}{3}e - \frac{1}{6}$

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Solution: C

The surface S is part of the paraboloid $z = x^2 + y^2$, so we'll find its partial derivatives.

$$r_x = 2x$$

$$r_y = 2y$$

The surface is oriented upward, so

$$\iint_S F \cdot dS = \int_{y_1}^{y_2} \int_{x_1}^{x_2} F(r(x,y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k}) \ dx \ dy$$

In order to find $F(r(x,y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k})$ we'll use $F(r(x,y)) = e^y\mathbf{i} + ye^x\mathbf{j} + x^2y\mathbf{k}$, $r_x = 2x$ and $r_y = 2y$.

$$F(r(x,y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k}) = (e^y\mathbf{i} + ye^x\mathbf{j} + x^2y\mathbf{k}) \cdot (-2x\mathbf{i} - 2y\mathbf{j} + \mathbf{k})$$

$$F(r(x,y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k}) = -2xe^y - 2y^2e^x + x^2y$$

The bounds for the double integral will come from the square bounded by $0 \le x \le 1$ and $0 \le y \le 1$.

Now we can say that flux is given by

$$\iint_S F \cdot dS = \int_{y_1}^{y_2} \int_{x_1}^{x_2} F(r(x,y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k}) \ dx \ dy$$

$$\iint_S F \cdot dS = \int_0^1 \int_0^1 -2xe^y - 2y^2e^x + x^2y \ dx \ dy$$

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Integrate with respect to x, then evaluate over the interval.

$$\int_0^1 -x^2e^y - 2y^2e^x + \frac{1}{3}x^3y\Big|_{x=0}^{x=1}\ dy$$

$$\int_0^1 -(1)^2e^y - 2y^2e^{(1)} + \frac{1}{3}(1)^3y - \left(-(0)^2e^y - 2y^2e^{(0)} + \frac{1}{3}(0)^3y\right)\ dy$$

$$\int_0^1 -e^y - 2y^2e + \frac{1}{3}y - \left(-2y^2(1)\right)\ dy$$

$$\int_0^1 -e^y - 2y^2e + \frac{1}{3}y + 2y^2\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$-e^y - \frac{2}{3}y^3e + \frac{1}{6}y^2 + \frac{2}{3}y^3\Big|_0^1$$

$$-e^{(1)} - \frac{2}{3}(1)^3e + \frac{1}{6}(1)^2 + \frac{2}{3}(1)^3 - \left(-e^{(0)} - \frac{2}{3}(0)^3e + \frac{1}{6}(0)^2 + \frac{2}{3}(0)^3\right)$$

$$-e - \frac{2}{3}e + \frac{1}{6} + \frac{2}{3} - (-1)$$

$$-e - \frac{2}{3}e + \frac{1}{6} + \frac{2}{3} + 1$$

$$-\frac{3}{3}e - \frac{2}{3}e + \frac{1}{6} + \frac{4}{6} + \frac{6}{6}$$

$$-\frac{5}{3}e + \frac{11}{6}$$

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Topic: Flux across the surface

Question: What is the flux of the vector field where S is the part of the paraboloid $z = x^2 + y^2$ that lies above the square $0 \le x \le 1$, $0 \le y \le 1$ and has a downward orientation?

$$F(x, y, z) = x^2y\mathbf{i} + x^3\mathbf{j} + xy\mathbf{k}$$

Answer choices:

A $\quad -\frac{1}{4}$

B $\quad \frac{1}{4}$

C $\quad -\frac{1}{2}$

D $\quad \frac{1}{2}$

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Solution: B

The surface S is part of the paraboloid $z = x^2 + y^2$, so we'll find its partial derivatives.

$$r_x = 2x$$

$$r_y = 2y$$

The surface is oriented downward, so

$$\iint_S F \cdot dS = \int_{x_1}^{x_2} \int_{y_1}^{y_2} F(r(x,y)) \cdot (r_x \mathbf{i} + r_y \mathbf{j} - \mathbf{k}) \ dx \ dy$$

In order to find $F(r(x,y)) \cdot (r_x \mathbf{i} + r_y \mathbf{j} - \mathbf{k})$ we'll use $F(x,y,z) = x^2 y \mathbf{i} + x^3 \mathbf{j} + xy \mathbf{k}$, $r_x = 2x$ and $r_y = 2y$.

$$F(r(x,y)) \cdot (r_x \mathbf{i} + r_y \mathbf{j} - \mathbf{k}) = (x^2 y \mathbf{i} + x^3 \mathbf{j} + xy \mathbf{k}) \cdot (2x \mathbf{i} + 2y \mathbf{j} - \mathbf{k})$$

$$F(r(x,y)) \cdot (r_x \mathbf{i} + r_y \mathbf{j} - \mathbf{k}) = 2x^3 y + 2x^3 y - xy$$

$$F(r(x,y)) \cdot (r_x \mathbf{i} + r_y \mathbf{j} - \mathbf{k}) = 4x^3 y - xy$$

The bounds for the double integral will come from the square bounded by $0 \le x \le 1$ and $0 \le y \le 1$.

Now we can say that flux is given by

$$\iint_S F \cdot dS = \int_{x_1}^{x_2} \int_{y_1}^{y_2} F(r(x,y)) \cdot (r_x \mathbf{i} + r_y \mathbf{j} - \mathbf{k}) \ dx \ dy$$

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$$\iint_S F\cdot dS=\int_0^1\int_0^1 4x^3y-xy\ dx\ dy$$

Integrate with respect to x, then evaluate over the interval.

$$\int_0^1\int_0^1 4x^3y-xy\ dx\ dy$$

$$\int_0^1 x^4y-\frac{1}{2}x^2y-\left(x^4y-\frac{1}{2}x^2y\right)\Bigg|_{x=0}^{x=1}\ dy$$

$$\int_0^1 (1)^4y-\frac{1}{2}(1)^2y-\left((0)^4y-\frac{1}{2}(0)^2y\right)\ dy$$

$$\int_0^1 y-\frac{1}{2}y\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\frac{1}{2}y^2-\frac{1}{4}y^2\Big|_0^1$$

$$\frac{1}{2}(1)^2-\frac{1}{4}(1)^2-\left(\frac{1}{2}(0)^2-\frac{1}{4}(0)^2\right)$$

$$\frac{1}{2}-\frac{1}{4}$$

$$\frac{1}{4}$$

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Topic: Flux across the surface

Question: What is the flux of the vector field where S is the part of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$ and has an upward orientation?

$$F(x, y, z) = -x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$$

Answer choices:

A $\frac{101\pi}{12}$

B $\frac{101\pi}{6}$

C $\frac{73\pi}{12}$

D $\frac{73\pi}{6}$

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Solution: D

The surface S is part of the cone $z = \sqrt{x^2 + y^2}$, so we'll find its partial derivatives.

$$r_x = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2x)$$

$$r_x = \frac{x}{\sqrt{x^2 + y^2}}$$

and

$$r_y = \frac{1}{2}\left(x^2 + y^2\right)^{-\frac{1}{2}}(2y)$$

$$r_y = \frac{y}{\sqrt{x^2 + y^2}}$$

The surface is oriented upward, so

$$\int\int_S F \cdot dS = \int_{y_1}^{y_2}\int_{x_1}^{x_2} F(r(x, y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k})\ dx\ dy$$

In order to find $F(r(x, y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k})$ we'll use $F(x, y, z) = -x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$ and the partial derivatives.

$$F(r(x, y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k}) = \left(-x\mathbf{i} - y\mathbf{j} + \left(x^2 + y^2\right)\right)\left(-\frac{x}{\sqrt{x^2 + y^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2}}\mathbf{j} + \mathbf{k}\right)$$

$$F(r(x, y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k}) = \frac{x^2 + y^2}{\sqrt{x^2 + y^2}} + x^2 + y^2$$

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$$F(r(x,y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k}) = \sqrt{x^2+y^2} + x^2 + y^2$$

If we use the conversion formula $x^2 + y^2 = r^2$ to switch this to polar coordinates, we get

$$\sqrt{r^2} + r^2$$

$$r + r^2$$

The bounds for the double integral will come from the fact that the surface is bounded by the planes $z = 1$ and $z = 2$. That means the cone is bounded by

$$1 = \sqrt{x^2+y^2}$$

$$1 = x^2 + y^2$$

and

$$4 = x^2 + y^2$$

If we switch these bounds to polar coordinates, we get

$1 = r^2$ and $4 = r^2$

$1 = r$ and $2 = r$

Now we can say that flux is given by

$$\iint_S F \cdot dS = \int_{y_1}^{y_2} \int_{x_1}^{x_2} F(r(x,y)) \cdot (-r_x\mathbf{i} - r_y\mathbf{j} + \mathbf{k})\ dx\ dy$$

$$\iint_S F \cdot dS = \int_0^{2\pi} \int_1^2 \left(r + r^2\right) r \ dr \ d\theta$$

$$\int_0^{2\pi} \int_1^2 r^2 + r^3 \ dr \ d\theta$$

Integrate with respect to r, then evaluate over the interval.

$$\int_0^{2\pi} \frac{1}{3}r^3 + \frac{1}{4}r^4 \Big|_{r=1}^{r=2} \ d\theta$$

$$\int_0^{2\pi} \frac{1}{3}(2)^3 + \frac{1}{4}(2)^4 - \left(\frac{1}{3}(1)^3 + \frac{1}{4}(1)^4\right) \ d\theta$$

$$\int_0^{2\pi} \frac{1}{3}(8) + \frac{1}{4}(16) - \frac{1}{3} - \frac{1}{4} \ d\theta$$

$$\int_0^{2\pi} \frac{8}{3} + 4 - \frac{1}{3} - \frac{1}{4} \ d\theta$$

$$\int_0^{2\pi} \frac{7}{3} + 4 - \frac{1}{4} \ d\theta$$

$$\int_0^{2\pi} \frac{28}{12} + \frac{48}{12} - \frac{3}{12} \ d\theta$$

$$\int_0^{2\pi} \frac{73}{12} \ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

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$$\frac{73}{12}\theta\Big|_0^{2\pi}$$

$$\frac{73}{12}(2\pi)-\left(\frac{73}{12}(0)\right)$$

$$\frac{73\pi}{6}$$

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Topic: Stokes' theorem

Question: If you use Stokes' Theorem

$$\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = 0$$

where the curve C is defined by $x^2 + y^2 = 1$ and $z = 5y^5$, and the result is 0, which of the following vector fields could you have used?

Answer choices:

A $\mathbf{F}(x, y, z) = 4yz\mathbf{i} + 4xz\mathbf{j} + 4xy\mathbf{k}$

B $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + 4yz\mathbf{j} + 4xyz\mathbf{k}$

C $\mathbf{F}(x, y, z) = 4xy\mathbf{i} + 4xyz\mathbf{j} + 4xz\mathbf{k}$

D $\mathbf{F}(x, y, z) = 4xyz\mathbf{i} + 4yz\mathbf{j} + 4xz\mathbf{k}$

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Solution: A

Answer choice A is the correct vector field. To test it, find $\textbf{curl}\,\mathbf{F}(x, y, z)$.

$$\textbf{curl}\ \mathbf{F}(x, y, z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4yz & 4xz & 4xy \end{vmatrix}$$

$$\textbf{curl}\ \mathbf{F}(x, y, z) = (4x - 4x)\,\mathbf{i}\ - \left(4y - 4y\right)\ \mathbf{j} + (4z - 4z)\ \mathbf{k}$$

$$\textbf{curl}\ \mathbf{F}(x, y, z) = (0)\,\mathbf{i}\ - (0)\ \mathbf{j} + (0)\ \mathbf{k}$$

$$\textbf{curl}\ \mathbf{F}(x, y, z) = 0$$

Now applying Stokes' Theorem yields

$$\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \iint_C \textbf{curl}\,\mathbf{F} \cdot \mathbf{n}\ \ dS$$

$$\oint_C \mathbf{F}(x, y, z) \cdot\ d\mathbf{r} = \iint_C (0) \cdot \mathbf{n}\ dS$$

$$\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = 0$$

Topic: Stokes' theorem

Question: A solid is defined by these parametric functions on the interval $0 \leq t \leq 2\pi$:

$$x = 3\sin t$$

$$y = 3\cos t$$

$$z = 4t$$

Which vector function below did we use for Stokes' theorem if

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = 40\pi^2$$

Answer choices:

A $\mathbf{F}(x, y, z) = t\sin t$

B $\mathbf{F}(x, y, z) = t\cos t$

C $\mathbf{F}(x, y, z) = \sin t$

D $\mathbf{F}(x, y, z) = 4t$

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Solution: D

The derivatives of the parametric functions are

$$dx = 3\cos t$$

$$dy = -3\sin t$$

$$dz = 4$$

Then

$$(dx)^2 + (dy)^2 + (dz)^2 = (3\cos t)^2 + (-3\sin t)^2 + 4^2$$

$$(dx)^2 + (dy)^2 + (dz)^2 = 9\sin^2 t + 9\cos^2 t + 16$$

$$(dx)^2 + (dy)^2 + (dz)^2 = 9\left(\sin^2 t + \cos^2 t\right) + 16$$

$$(dx)^2 + (dy)^2 + (dz)^2 = 9(1) + 16$$

$$(dx)^2 + (dy)^2 + (dz)^2 = 25$$

Now, by Stokes' Theorem, with answer choice D,

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = \int_0^{2\pi} (4t)\sqrt{25}\ dt$$

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = \int_0^{2\pi} 20t\ dt$$

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = 10t^2 \Big|_0^{2\pi}$$

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$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = 10\,(2\pi)^2 - 10\,(0)^2$$

$$\int_C \mathbf{F}(x, y, z) \cdot d\mathbf{s} = 40\pi^2$$

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Topic: Stokes' theorem

Question: The circular disk for the function $\mathbf{F}(x, y, z) = y^2\mathbf{i} + x^2\mathbf{j}$ is defined by $x^2 + y^2 \leq 1$ and $z = 0$. Which of the following equations can be verified by Stokes' Theorem?

Answer choices:

A $\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = -\frac{5}{3}$

B $\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \frac{5\pi}{3}$

C $\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = 0$

D $\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \frac{\pi}{3}$

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Solution: C

The circle $x^2 + y^2 \le 1$ at $z = 0$ for $0 \le t \le 2\pi$ can be defined by the boundary curve $x^2 + y^2 = 1$ with $z = 0$. We can parametrize that circle as $x = \cos t$, $y = \sin t$, and $z = 0$. Then we can write these parametric equations as the vector equation

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 0\mathbf{k}$$

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$$

Then the derivative of the vector equation is

$$r'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j}$$

and $\mathbf{F}(r(t))$ is

$$\mathbf{F}(r(t)) = (\sin t)^2\mathbf{i} + (\cos t)^2\mathbf{j}$$

$$\mathbf{F}(r(t)) = \sin^2 t\mathbf{i} + \cos^2 t\mathbf{j}$$

Then by Stokes' Theorem,

$$\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \oint_0^{2\pi} \mathbf{F}(r(t)) \cdot \mathbf{r}'(t)\ dt$$

$$\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \oint_0^{2\pi} (\sin^2 t\mathbf{i} + \cos^2 t\mathbf{j}) \cdot (-\sin t\mathbf{i} + \cos t\mathbf{j})\ dt$$

$$\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = \oint_0^{2\pi} (\sin^2 t)(-\sin t) + (\cos^2 t)(\cos t)\ dt$$

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$$\oint_C \mathbf{F}(x,y,z) \cdot d\mathbf{r} = \oint_0^{2\pi} -\sin^3 t + \cos^3 t \ dt$$

Integrate each term separately, starting with the integral of $\cos^3 t$.

$$\int_0^{2\pi} \cos^3 t \ dt - \int_0^{2\pi} \sin^3 t \ dt$$

$$\int_0^{2\pi} (\cos^2 t)\cos t \ dt - \int_0^{2\pi} \sin^3 t \ dt$$

$$\int_0^{2\pi} (1 - \sin^2 t)\cos t \ dt - \int_0^{2\pi} \sin^3 t \ dt$$

$u = \sin t$ and $du = \cos t \ dt$

$$\int_{t=0}^{t=2\pi} 1 - u^2 \ du - \int_0^{2\pi} \sin^3 t \ dt$$

$$u - \frac{1}{3}u^3 \Big|_{t=0}^{t=2\pi} - \int_0^{2\pi} \sin^3 t \ dt$$

$$\sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} - \int_0^{2\pi} \sin^3 t \ dt$$

Integrate $\sin^3 t$.

$$\sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} - \int_0^{2\pi} (\sin^2 t)\sin t \ dt$$

$$\sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} - \int_0^{2\pi} (1 - \cos^2 t)\sin t \ dt$$

$u = \cos t$ and $du = -\sin t\ dt$

$$\sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} - \int_{t=0}^{t=2\pi} -(1-u^2)\ du$$

$$\sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} + \int_{t=0}^{t=2\pi} 1-u^2\ du$$

$$\sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} + u - \frac{1}{3}u^3 \Big|_{t=0}^{t=2\pi}$$

$$\sin t - \frac{1}{3}\sin^3 t \Big|_0^{2\pi} + \cos t - \frac{1}{3}\cos^3 t \Big|_0^{2\pi}$$

Then the integrated value is

$$\oint_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \sin t + \cos t - \frac{1}{3}\left(\sin^3 t + \cos^3 t\right)\Big|_0^{2\pi}$$

$$\oint_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = \sin(2\pi) + \cos(2\pi) - \frac{1}{3}\left(\sin^3(2\pi) + \cos^3(2\pi)\right)$$

$$-\left[\sin 0 + \cos 0 - \frac{1}{3}\left(\sin^3 0 + \cos^3 0\right)\right]$$

$$\oint_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = 0 + 1 - \frac{1}{3}(0+1) - \left[0 + 1 - \frac{1}{3}(0+1)\right]$$

$$\oint_C \mathbf{F}(x,y,z)\cdot d\mathbf{r} = 1 - \frac{1}{3} - 1 + \frac{1}{3}$$

$$\oint_C \mathbf{F}(x, y, z) \cdot d\mathbf{r} = 0$$

Topic: Divergence theorem for surface integrals

Question: What is the surface integral of the vector field where S is the surface of a cube with with one corner at the origin and opposite corner at $(1,1,1)$?

$$F(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$$

Answer choices:

A $\quad 2$

B $\quad 1$

C $\quad 4$

D $\quad 3$

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Solution: A

We'll calculate the divergence of $F(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + z^2\mathbf{k}$.

$$\text{div } F = \frac{\partial F(x, y, z)_1}{dx} + \frac{\partial F(x, y, z)_1}{dy} + \frac{\partial F(x, y, z)_1}{dz}$$

$$\text{div } F = y + z + 2z$$

$$\text{div } F = y + 3z$$

S is the surface of the cube sitting in the first quadrant between $(0,0,0)$ and $(1,1,1)$, which means the limits for each variable will be

$$0 \le x \le 1$$

$$0 \le y \le 1$$

$$0 \le z \le 1$$

So the surface integral is given by

$$\iint_S F \cdot dS = \iiint_E \text{div } F \ dV$$

$$\iint_S F \cdot dS = \int_0^1 \int_0^1 \int_0^1 y + 3z \ dx \ dy \ dz$$

Integrate with respect to x, then evaluate over the interval.

$$\int_0^1 \int_0^1 xy + 3xz \Big|_{x=0}^{x=1} \ dy \ dz$$

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$$\int_0^1 \int_0^1 (1)y + 3(1)z - ((0)y + 3(0)z)\ dy\ dz$$

$$\int_0^1 \int_0^1 y + 3z\ dy\ dz$$

Integrate with respect to y, then evaluate over the interval.

$$\int_0^1 \frac{1}{2}y^2 + 3yz\Big|_{y=0}^{y=1}\ dz$$

$$\int_0^1 \frac{1}{2}(1)^2 + 3(1)z - \left(\frac{1}{2}(0)^2 + 3(0)z\right)\ dz$$

$$\int_0^1 \frac{1}{2} + 3z\ dz$$

Integrate with respect to z, then evaluate over the interval.

$$\frac{1}{2}z + \frac{3}{2}z^2\Big|_0^1$$

$$\frac{1}{2}(1) + \frac{3}{2}(1)^2 - \left(\frac{1}{2}(0) + \frac{3}{2}(0)^2\right)$$

$$\frac{1}{2} + \frac{3}{2}$$

$$2$$

Topic: Divergence theorem for surface integrals

Question: What is the surface integral of the vector field where S is the surface of a rectangular box bounded by the planes $x=0$, $x=1$, $y=0$, $y=2$, $z=0$ and $z=3$?

$$F(x,y,z)=x^2\mathbf{i}-y^2\mathbf{j}+z^2\mathbf{k}$$

Answer choices:

A 3

B 12

C 15

D 24

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Solution: B

We'll calculate the divergence of $F(x,y,z) = x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}$.

$$\text{div } F = \frac{\partial F(x,y,z)_1}{dx} + \frac{\partial F(x,y,z)_1}{dy} + \frac{\partial F(x,y,z)_1}{dz}$$

$$\text{div } F = 2x - 2y + 2z$$

S is the surface of the rectangular box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$ and $z = 3$, which means the limits for each variable will be

$$0 \le x \le 1$$

$$0 \le y \le 2$$

$$0 \le z \le 3$$

So the surface integral is given by

$$\iint_S F \cdot dS = \iiint_E \text{div } F \ dV$$

$$\iint_S F \cdot dS = \int_0^3 \int_0^2 \int_0^1 2x - 2y + 2z \ dx \ dy \ dz$$

Integrate with respect to x, then evaluate over the interval.

$$\int_0^3 \int_0^2 x^2 - 2xy + 2xz \Big|_{x=0}^{x=1} \ dy \ dz$$

$$\int_0^3 \int_0^2 (1)^2 - 2(1)y + 2(1)z - \left((0)^2 - 2(0)y + 2(0)z\right) \ dy \ dz$$

$$\int_0^3 \int_0^2 1 - 2y + 2z \ dy \ dz$$

Integrate with respect to y, then evaluate over the interval.

$$\int_0^3 y - y^2 + 2yz \Big|_{y=0}^{y=2} \ dz$$

$$\int_0^3 2 - (2)^2 + 2(2)z - \left(0 - (0)^2 + 2(0)z\right) \ dz$$

$$\int_0^3 2 - 4 + 4z \ dz$$

$$\int_0^3 -2 + 4z \ dz$$

Integrate with respect to z, then evaluate over the interval.

$$-2z + 2z^2 \Big|_0^3$$

$$-2(3) + 2(3)^2 - \left(-2(0) + 2(0)^2\right)$$

$$-6 + 18$$

$$12$$

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Topic: Divergence theorem for surface integrals

Question: What is the surface integral of the vector field where S is the surface of a rectangular box bounded by the planes $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=2$?

$$F(x,y,z)=2xy^2\mathbf{i}+3y^3\mathbf{j}+y^2z\mathbf{k}$$

Answer choices:

A $\quad 1$

B $\quad 2$

C $\quad 4$

D $\quad 8$

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Solution: D

We'll calculate the divergence of $F(x, y, z) = 2xy^2\mathbf{i} + 3y^3\mathbf{j} + y^2z\mathbf{k}$.

$$\text{div } F = \frac{\partial F(x, y, z)_1}{dx} + \frac{\partial F(x, y, z)_1}{dy} + \frac{\partial F(x, y, z)_1}{dz}$$

$$\text{div } F = 2y^2 + 9y^2 + y^2$$

$$\text{div } F = 12y^2$$

S is the surface of the rectangular box bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$ and $z = 2$, which means the limits for each variable will be

$$0 \le x \le 1$$

$$0 \le y \le 1$$

$$0 \le z \le 2$$

So the surface integral is given by

$$\iint_S F \cdot dS = \iiint_E \text{div } F \ dV$$

$$\iint_S F \cdot dS = \int_0^2 \int_0^1 \int_0^1 12y^2 \ dx \ dy \ dz$$

Integrate with respect to x, then evaluate over the interval.

$$\int_0^2 \int_0^1 12xy^2 \Big|_{x=0}^{x=1} \ dy \ dz$$

$$\int_0^2 \int_0^1 12(1)y^2 - 12(0)y^2 \, dy \, dz$$

$$\int_0^2 \int_0^1 12y^2 \, dy \, dz$$

Integrate with respect to y, then evaluate over the interval.

$$\int_0^2 4y^3 \Big|_{y=0}^{y=1} \, dz$$

$$\int_0^2 4(1)^3 - 4(0)^3 \, dz$$

$$\int_0^2 4 \, dz$$

Integrate with respect to z, then evaluate over the interval.

$$4z \Big|_0^2$$

$$4(2) - 4(0)$$

$$8$$

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Topic: Divergence theorem and flux

Question: What is the flux of the vector field where S is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the xy-plane?

$$F(x, y, z) = x^3\mathbf{i} + 2xz^2\mathbf{j} + 3y^2z\mathbf{k}$$

Answer choices:

A 2π

B π

C $\frac{\pi}{2}$

D $\frac{\pi}{4}$

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Solution: C

We'll calculate the divergence of the vector field $F(x, y, z) = x^3\mathbf{i} + 2xz^2\mathbf{j} + 3y^2z\mathbf{k}$.

$$\text{div } F = \frac{\partial F(x, y, z)_1}{dx} + \frac{\partial F(x, y, z)_1}{dy} + \frac{\partial F(x, y, z)_1}{dz}$$

$$\text{div } F = 3x^2 + 0 + 3y^2$$

$$\text{div } F = 3\left(x^2 + y^2\right)$$

To find the bounds we'll use for the triple integral, we'll convert the paraboloid to $z = 1 - x^2 - y^2$ to polar coordinates using the conversion formula $x^2 + y^2 = r^2$.

$$z = 1 - \left(x^2 + y^2\right)$$

$$z = 1 - r^2$$

The limits for z will be $0 \le z \le 1 - r^2$. Which means the limit for r will be $0 \le r \le 1$ using $z = 1 - r^2$ when $z = 0$ (the xy-plane). The limit for θ is $0 \le \theta \le 2\pi$ since we're dealing with the full paraboloid.

We'll also convert div $F = 3\left(x^2 + y^2\right)$ and get div $F = 3r^2$.

Now we can solve for the flux.

$$\iint_S F \cdot dS = \iiint_E \text{div } F \, dV$$

$$\iint_S F \cdot dS = \int_0^{2\pi} \int_0^1 \int_0^{1-r^2} 3r^2 r \, dz \, dr \, d\theta$$

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$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} 3r^3\ dz\ dr\ d\theta$$

Integrate with respect to z, then evaluate over the interval.

$$\int_0^{2\pi}\int_0^1 3r^3z\Big|_{z=0}^{z=1-r^2}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 3r^3\left(1-r^2\right)-3r^3(0)\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 3r^3-3r^5\ dr\ d\theta$$

Integrate with respect to r, then evaluate over the interval.

$$\int_0^{2\pi}\frac{3}{4}r^4-\frac{1}{2}r^6\Big|_{r=0}^{r=1}\ d\theta$$

$$\int_0^{2\pi}\frac{3}{4}(1)^4-\frac{1}{2}(1)^6-\left(\frac{3}{4}(0)^4-\frac{1}{2}(0)^6\right)\ d\theta$$

$$\int_0^{2\pi}\frac{3}{4}-\frac{1}{2}\ d\theta$$

$$\int_0^{2\pi}\frac{3}{4}-\frac{2}{4}\ d\theta$$

$$\int_0^{2\pi}\frac{1}{4}\ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

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$$\frac{1}{4}\theta\Big|_0^{2\pi}$$

$$\frac{1}{4}(2\pi)-\frac{1}{4}(0)$$

$$\frac{\pi}{2}$$

Topic: Divergence theorem and flux

Question: What is the flux of the vector field where S is the surface of the solid bounded by the paraboloid $z = 9 - x^2 - y^2$ and the xy-plane?

$$F(x, y, z) = x^3\mathbf{i} + 4xz^2\mathbf{j} + 3y^2z\mathbf{k}$$

Answer choices:

A $\frac{729\pi}{2}$

B $\frac{729\pi}{4}$

C $\frac{2{,}187\pi}{2}$

D $\frac{2{,}187\pi}{4}$

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Solution: A

We'll calculate the divergence of the vector field
$F(x, y, z) = x^3\mathbf{i} + 4xz^2\mathbf{j} + 3y^2z\mathbf{k}$.

$$\text{div } F = \frac{\partial F(x, y, z)_1}{dx} + \frac{\partial F(x, y, z)_1}{dy} + \frac{\partial F(x, y, z)_1}{dz}$$

$$\text{div } F = 3x^2 + 0 + 3y^2$$

$$\text{div } F = 3\left(x^2 + y^2\right)$$

To find the bounds we'll use for the triple integral, we'll convert the paraboloid to $z = 9 - x^2 - y^2$ to polar coordinates using the conversion formula $x^2 + y^2 = r^2$.

$$z = 9 - \left(x^2 + y^2\right)$$

$$z = 9 - r^2$$

The limits for z will be $0 \le z \le 9 - r^2$. Which means the limit for r will be $0 \le r \le 3$ using $z = 9 - r^2$ when $z = 0$ (the xy-plane). The limit for θ is $0 \le \theta \le 2\pi$ since we're dealing with the full paraboloid.

We'll also convert div $F = 3\left(x^2 + y^2\right)$ and get div $F = 3r^2$.

Now we can solve for the flux.

$$\iint_S F \cdot dS = \iiint_E \text{div } F \ dV$$

$$\iint_S F \cdot dS = \int_0^{2\pi} \int_0^3 \int_0^{9-r^2} 3r^2r \ dz \ dr \ d\theta$$

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$$\int_0^{2\pi} \int_0^3 \int_0^{9-r^2} 3r^3\ dz\ dr\ d\theta$$

Integrate with respect to z, then evaluate over the interval.

$$\int_0^{2\pi} \int_0^3 3r^3 z \Big|_{z=0}^{z=9-r^2}\ dr\ d\theta$$

$$\int_0^{2\pi} \int_0^3 3r^3 \left(9 - r^2\right) - 3r^3(0)\ dr\ d\theta$$

$$\int_0^{2\pi} \int_0^3 27r^3 - 3r^5\ dr\ d\theta$$

Integrate with respect to r, then evaluate over the interval.

$$\int_0^{2\pi} \frac{27}{4}r^4 - \frac{1}{2}r^6 \Big|_{r=0}^{r=3}\ d\theta$$

$$\int_0^{2\pi} \frac{27}{4}(3)^4 - \frac{1}{2}(3)^6 - \left(\frac{27}{4}(0)^4 - \frac{1}{2}(0)^6\right)\ d\theta$$

$$\int_0^{2\pi} \frac{27}{4}(81) - \frac{1}{2}(729)\ d\theta$$

$$\int_0^{2\pi} \frac{2{,}187}{4} - \frac{729}{2}\ d\theta$$

$$\int_0^{2\pi} \frac{2{,}187}{4} - \frac{1{,}458}{4}\ d\theta$$

$$\int_0^{2\pi} \frac{729}{4}\ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$\frac{729}{4}\theta\Big|_0^{2\pi}$$

$$\frac{729}{4}(2\pi) - \frac{729}{4}(0)$$

$$\frac{729\pi}{2}$$

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Topic: Divergence theorem and flux

Question: What is the flux of the vector field where S is the surface of the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ and between the planes $z = 1$ and $z = 4$?

$$F(x, y, z) = xy^2\mathbf{i} + yz\mathbf{j} + zx^2\mathbf{k}$$

Answer choices:

A 11π

B 45π

C 66π

D 99π

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Solution: B

We'll calculate the divergence of the vector field $F(x, y, z) = xy^2\mathbf{i} + yz\mathbf{j} + zx^2\mathbf{k}$.

$$\text{div } F = \frac{\partial F(x, y, z)_1}{dx} + \frac{\partial F(x, y, z)_1}{dy} + \frac{\partial F(x, y, z)_1}{dz}$$

$$\text{div } F = y^2 + z + x^2$$

$$\text{div } F = z + x^2 + y^2$$

Because the surface is bounded by the planes between the planes $z = 1$ and $z = 4$, we can say $1 \leq z \leq 4$. Because the surface also lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$, we'll convert this to polar coordinates using $x^2 + y^2 = r^2$, and see that the surface is between $r^2 = 1$, or $r = 1$, and $r^2 = 4$, or $r = 2$. Since we're dealing with the entire cylinder, the limit for θ is $0 \leq \theta \leq 2\pi$.

We'll also convert div $F = z + x^2 + y^2$ to polar form using $x^2 + y^2 = r^2$ and get div $F = z + r^2$.

Now we can solve for the flux.

$$\iint_S F \cdot dS = \iiint_E \text{div } F \, dV$$

$$\iint_S F \cdot dS = \int_0^{2\pi} \int_1^2 \int_1^4 \left(z + r^2\right) r \ dz \ dr \ d\theta$$

$$\int_0^{2\pi} \int_1^2 \int_1^4 zr + r^3 \ dz \ dr \ d\theta$$

Integrate with respect to z, then evaluate over the interval.

$$\int_0^{2\pi}\int_1^2 \frac{1}{2}z^2r + r^3z\Big|_{z=1}^{z=4} \ dr\ d\theta$$

$$\int_0^{2\pi}\int_1^2 \frac{1}{2}(4)^2r + r^3(4) - \left(\frac{1}{2}(1)^2r + r^3(1)\right) \ dr\ d\theta$$

$$\int_0^{2\pi}\int_1^2 8r + 4r^3 - \left(\frac{1}{2}r + r^3\right) \ dr\ d\theta$$

$$\int_0^{2\pi}\int_1^2 8r + 4r^3 - \frac{1}{2}r - r^3 \ dr\ d\theta$$

$$\int_0^{2\pi}\int_1^2 3r^3 + \frac{15}{2}r \ dr\ d\theta$$

Integrate with respect to r, then evaluate over the interval.

$$\int_0^{2\pi} \frac{3}{4}r^4 + \frac{15}{4}r^2\Big|_{r=1}^{r=2} \ d\theta$$

$$\int_0^{2\pi} \frac{3}{4}(2)^4 + \frac{15}{4}(2)^2 - \left(\frac{3}{4}(1)^4 + \frac{15}{4}(1)^2\right) \ d\theta$$

$$\int_0^{2\pi} \frac{3}{4}(16) + \frac{15}{4}(4) - \left(\frac{3}{4} + \frac{15}{4}\right) \ d\theta$$

$$\int_0^{2\pi} \frac{3}{4}(16) + \frac{15}{4}(4) - \frac{3}{4} - \frac{15}{4} \ d\theta$$

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$$\int_0^{2\pi} \frac{48}{4} + \frac{60}{4} - \frac{3}{4} - \frac{15}{4} \ d\theta$$

$$\int_0^{2\pi} \frac{90}{4} \ d\theta$$

$$\int_0^{2\pi} \frac{45}{2} \ d\theta$$

Integrate with respect to θ, then evaluate over the interval.

$$\frac{45}{2}\theta \Big|_0^{2\pi}$$

$$\frac{45}{2}(2\pi) - \frac{45}{2}(0)$$

$$45\pi$$

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