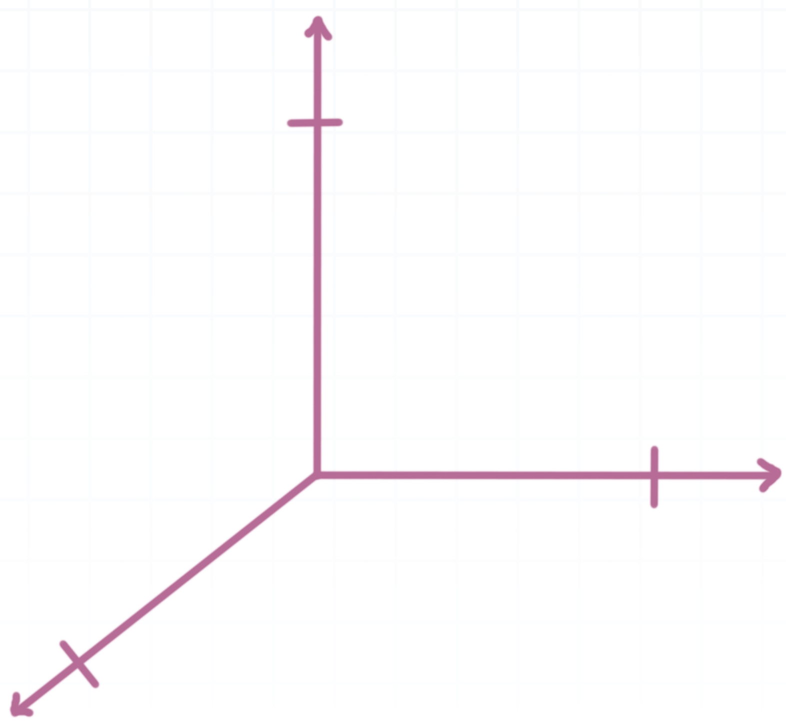
Calculus 3 Notes

krista king MATH

Plotting points in three dimensions

To plot points in three-dimensional coordinate space, we'll start with a three dimensional coordinate system, where the x-axis comes toward us on the left, the y-axis moves out toward the right, and the z-axis is perfectly vertical.



If we need to consider negative values of x, y, or z, then we should know that the negative direction of the x-axis follows the straight line of the positive x-axis away from us, that the negative direction of the y-axis moves out to the left, and that the negative direction of the z-axis is perfectly vertical, extending out below the positive direction of the z-axis.

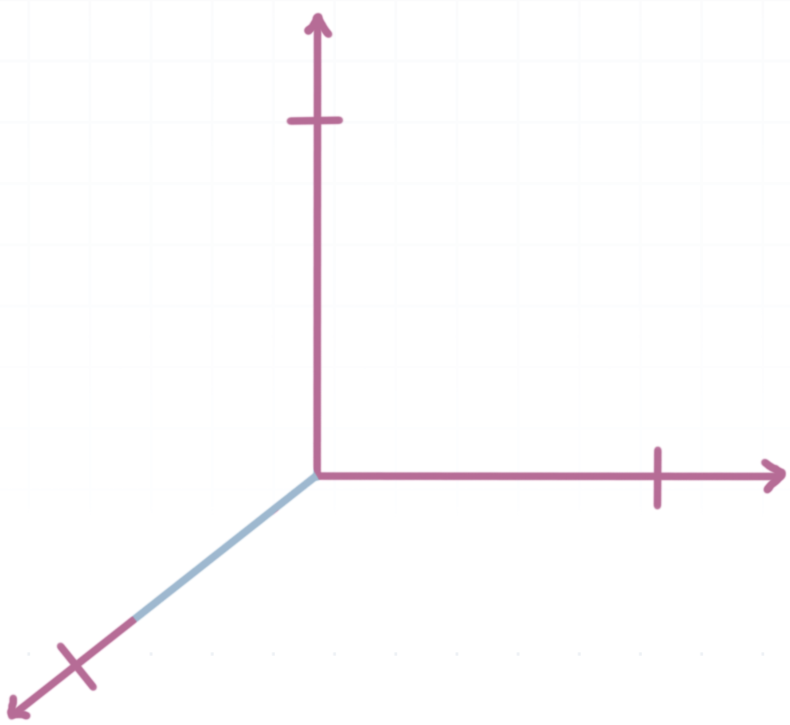
In the same way that we plot points in two-dimensional coordinate space by moving out along the x-axis to our x value, and then moving parallel to the y-axis until we find our point, in three-dimensional space we'll move along the x-axis, then parallel to the y-axis, then parallel to the z-axis until we arrive at our coordinate point.

Example

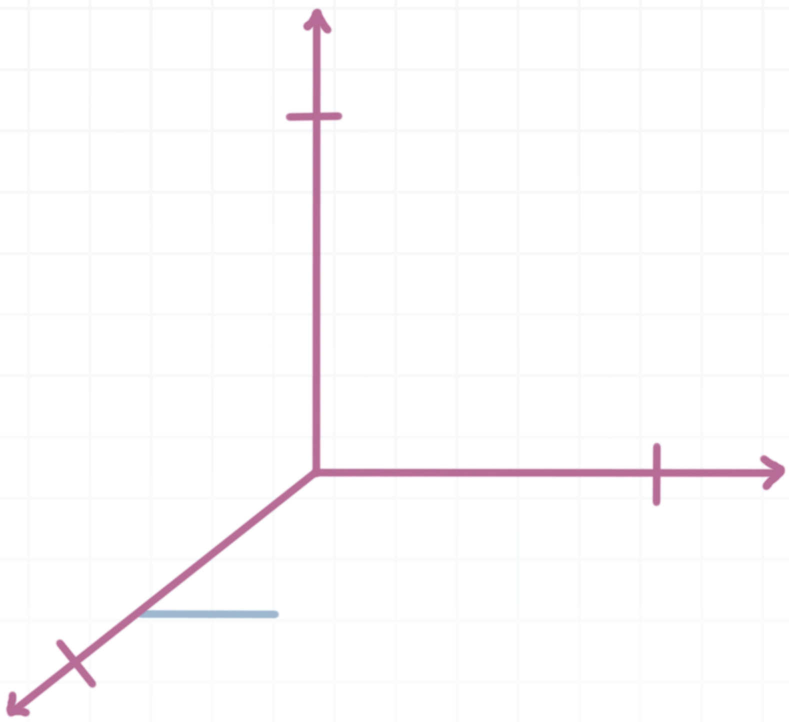
Plot the point in a three-dimensional coordinate system.

$$(4,2,3)$$

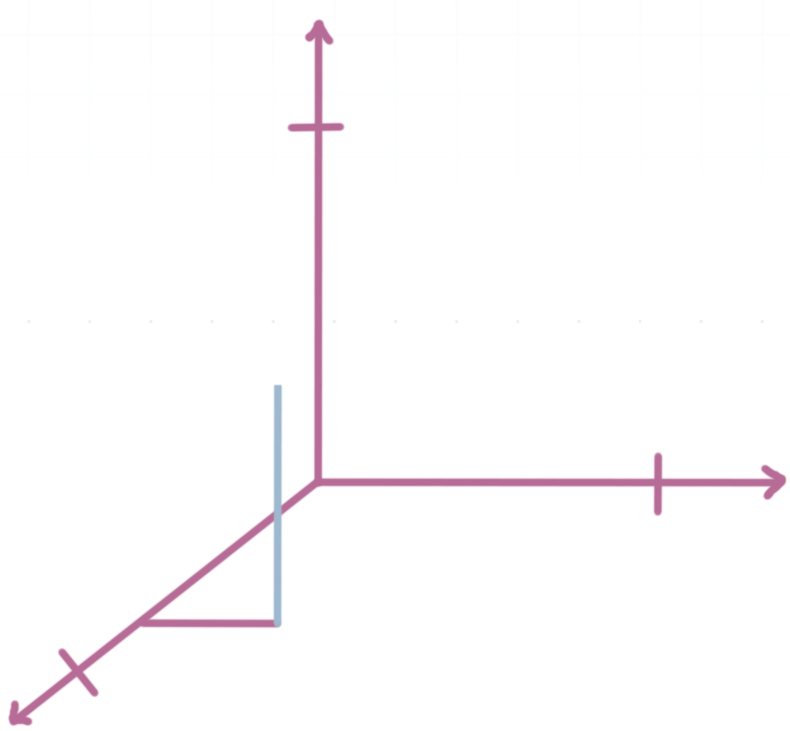
We'll start by drawing our axes, then moving out from the origin along the x-axis until we get to $x = 4$.



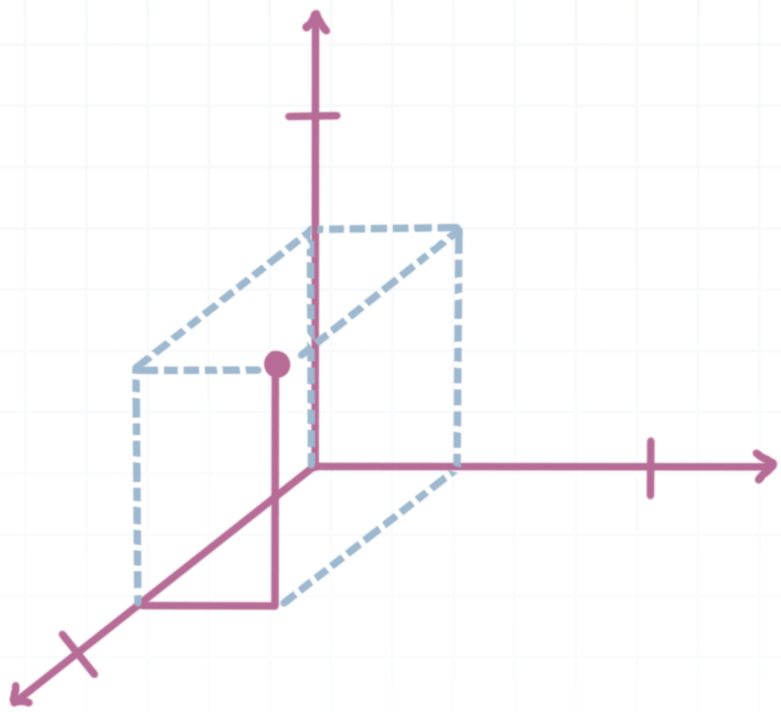
To get to $(4,2)$ in the xy-plane, we'll start where we left off on the x-axis, and move parallel to the y-axis until we get to $y = 2$.



To get to $(4,2,3)$ in three-dimensional space, we'll start where we left off in the xy-coordinate plane, and move parallel to the z-axis until we get to $z = 3$.



If we only plot the point, and nothing else, it can be difficult or impossible to identify the location of a three-dimensional point on a two-dimensional piece of paper. To fix this problem, we can fill in the three-dimensional box, putting one corner of the box at the origin, and the opposite corner at the coordinate point we just plotted.



Even though it's only technically necessary to plot and label the coordinate point, you can see how drawing in the lines we used to get to the point, and the box that connects the origin to the coordinate point, is really helpful in giving us some perspective.

Distance between points in three dimensions

Given two points A and B in three-dimensional space,

$$A(x_1, y_1, z_1)$$

$$B(x_2, y_2, z_2)$$

we can calculate the distance between them using the distance formula.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

It doesn't matter which point is A and which point is B. The fact that the square the differences inside the square root means that all of our values will be positive, which means we'll get a positive value for the distance between the points.

Example

Use the distance formula to

1. Find the distance between $(0,1,3)$ and $(-1,4,5)$.

2. Say which of $(0,1,3)$ and $(-1,4,5)$ lies in the yz-plane.

3. Say which of $(0,1,3)$ and $(-1,4,5)$ is closer to the xy-plane.

For the first part of the question, we'll use the distance formula to calculate the distance between the points.

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$D = \sqrt{(-1 - 0)^2 + (4 - 1)^2 + (5 - 3)^2}$$

$$D = \sqrt{1 + 9 + 4}$$

$$D = \sqrt{14}$$

For the second part of the question, we need to realize that in order for a point to be in the yz-plane, its x-coordinate must be 0. With that in mind, we can say that $(0,1,3)$ lies on the yz-plane, and that $(-1,4,5)$ does not lie in the yz-plane.

For the third part of the question, we need to realize that the z-value of the coordinate point will tell us how far the point is from the xy-plane. So if we just take the absolute value of the z-coordinate for each of our points, we'll be able to say which one is closer.

Point $(0,1,3)$ has $|z| = |3| = 3$

Point $(-1,4,5)$ has $|z| = |5| = 5$

Since the absolute value of z in the point $(0,1,3)$ is less than the absolute value of z in the point $(-1,4,5)$, we can say that $(0,1,3)$ is closer to the xy-plane.

Center, radius, and equation of the sphere

We can calculate the equation of a sphere using the formula

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

where (h,k,l) is the center of the sphere and r is the radius of the sphere.

To calculate the radius of the sphere, we can use the distance formula

$$D=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

where D is the length of the radius, (x_1,y_1,z_1) is one point on the surface of the sphere and (x_2,y_2,z_2) is the center of the sphere.

Let's try an example where we're given a point on the surface and the center of the sphere.

Example

Find the equation of the sphere with center $(1,1,2)$ that passes through the point $(2,4,6)$.

Since we're given the center of the sphere in the question, we can plug it into the equation of the sphere immediately.

$$(x-1)^2+(y-1)^2+(z-2)^2=r^2$$

We'll find the radius of the sphere using the distance formula, plugging the point on the surface of the sphere in for (x_1, y_1, z_1), and plugging the center of the sphere in for (x_2, y_2, z_2).

$$D = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

$$r = \sqrt{(2-1)^2 + (4-1)^2 + (6-2)^2}$$

$$r = \sqrt{1 + 9 + 16}$$

$$r = \sqrt{26}$$

Plugging this into our equation, we get

$$(x-1)^2 + (y-1)^2 + (z-2)^2 = \left(\sqrt{26}\right)^2$$

$$(x-1)^2 + (y-1)^2 + (z-2)^2 = 26$$

This is the equation of the sphere. We can also write it as

$$(x-1)^2 + (y-1)^2 + (z-2)^2 = 26$$

$$x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 - 4z + 4 = 26$$

$$x^2 - 2x + y^2 - 2y + z^2 - 4z = 20$$

Remember, using the distance formula to find the radius, we'll always get a value for r. But we need r^2 in the equation of the sphere. So we can either solve for r, square it, and then substitute for r^2 into the equation, or

solve for r, substitute for r into the equation, then square it to simplify.

Either way will work, so do the steps in whichever order you prefer.

Let's try another example when we're given the expanded form of the equation and we need to find the center and radius.

Example

Find the center and radius of the sphere.

$$x^2 + 2x + y^2 - 2y + z^2 - 6z = 14$$

We know we eventually need to change the equation into the standard form of the equation of a sphere,

$$(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$$

In order to do so, we'll need to complete the square with respect to each variable. Remember that the process of completing the square requires us to use the coefficient on the first degree term. For x that's $2x$ so the coefficient is 2; for y that's $-2y$ so the coefficient is -2; for z that's $-6z$ so the coefficient is -6. Completing the square tells us that we'll divide each of those coefficients by 2, and then take the result that we get and square it. These final values will be what we add into (and subtract out of) the equation of the sphere.

With respect to x:

$$\frac{2}{2} = 1 \qquad 1^2 = 1$$

With respect to y:

$$\frac{-2}{2} = -1 \qquad (-1)^2 = 1$$

With respect to z:

$$\frac{-6}{2} = -3 \qquad (-3)^2 = 9$$

Adding each of these values into our equation, and subtracting them out again, we get

$$\left(x^2 + 2x + 1\right) - 1 + \left(y^2 - 2y + 1\right) - 1 + \left(z^2 - 6z + 9\right) - 9 = 14$$

$$\left(x^2 + 2x + 1\right) + \left(y^2 - 2y + 1\right) + \left(z^2 - 6z + 9\right) = 25$$

$$(x + 1)^2 + (y - 1)^2 + (z - 3)^2 = 25$$

With our equation in standard form, we can pull out the center point. Remember, the standard form of a circle is $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$, which means that we have to include a negative sign if we have $x + x_1$, $y + y_1$, or $z + z_1$. The center is at $(-1,1,3)$.

To find the radius, it's important that we take the square root of the right-hand side, and not just the full value from the right, since the standard form of the equation of a sphere has r^2 on the right-hand side.

$$r^2 = 25$$

$$r = 5$$

To summarize our findings, we can say that the sphere has center $(-1,1,3)$ and radius $r = 5$.

Vector, parametric and symmetric equations of the line

Vector, parametric, and symmetric equations are different types of equations that can be used to represent the same line. We use different equations at different times to tell us information about the line, so we need to know how to find all three types of equations.

The **vector** equation of a line is given by

$$r = r_0 + tv$$

where r_0 is a point on the line and v is a parallel vector

The **parametric** equations of a line are given by

$$x = a$$

$$y = b$$

$$z = c$$

where a, b and c are the coefficients from the vector equation $r = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

The **symmetric** equations of a line are given by

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

where $a(a_1, a_2, a_3)$ are the coordinates from a point on the line and v_1, v_2 and v_3 are the coordinates from a parallel vector.

Example

Find the vector, parametric and symmetric equations of the line that passes through the point $a(2,-1,3)$ and is parallel to the vector $2\mathbf{i}-\mathbf{j}+4\mathbf{k}$.

Before we get started, we can say that the given point $a(2,-1,3)$ can also be represented by $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$. Additionally, we know that the given vector $2\mathbf{i}-\mathbf{j}+4\mathbf{k}$ can be represented by $\langle 2,-1,4\rangle$, or $2\mathbf{i}-\mathbf{j}+4\mathbf{k}$. To summarize what we know, we have

$$(2,-1,3) \qquad 2\mathbf{i}-\mathbf{j}+3\mathbf{k}$$

$$\langle 2,-1,4\rangle \qquad 2\mathbf{i}-\mathbf{j}+4\mathbf{k}$$

To find the vector equation of the line, we'll use $r=r_0+tv$, where r_0 is the point on the line $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and v is the parallel vector $2\mathbf{i}-\mathbf{j}+4\mathbf{k}$.

$$r=(2\mathbf{i}-\mathbf{j}+3\mathbf{k})+t(2\mathbf{i}-\mathbf{j}+4\mathbf{k})$$

$$r=2\mathbf{i}-\mathbf{j}+3\mathbf{k}+2\mathbf{i}t-\mathbf{j}t+4\mathbf{k}t$$

$$r=(2\mathbf{i}+2\mathbf{i}t)+(-\mathbf{j}-\mathbf{j}t)+(3\mathbf{k}+4\mathbf{k}t)$$

$$r=(2+2t)\mathbf{i}+(-1-t)\mathbf{j}+(3+4t)\mathbf{k}$$

With the vector equation of this line in hand, it'll be very easy for us to find the parametric equations of the line, because all we have to do is take the coefficients from the vector equation, and the parametric equations are

$$x=2+2t$$

$$y = -1 - t$$

$$z = 3 + 4t$$

To find the symmetric equations, we'll just plug the given coordinate point in for a_1, a_2 and a_3, plus the coefficients from the perpendicular vector in for v_1, v_2 and v_3.

$$\frac{x-2}{2} = \frac{y-(-1)}{-1} = \frac{z-3}{4}$$

$$\frac{x-2}{2} = -y-1 = \frac{z-3}{4}$$

In conclusion, we've found the following three equations for the same line:

Vector	$r = (2+2t)\mathbf{i} + (-1-t)\mathbf{j} + (3+4t)\mathbf{k}$
Parametric	$x = 2+2t$, $y = -1-t$, $z = 3+4t$
Symmetric	$\frac{x-2}{2} = -y-1 = \frac{z-3}{4}$

There's one other thing you need to be aware of when you're finding symmetric equations. Sometimes v_1, v_2 or v_3 will be equal to 0. In this case, you pull that particular fraction out of the symmetric equation, put it by itself, and don't divide by 0. So, if the formula for symmetric equations is

$$\frac{x-a_1}{v_1} = \frac{y-a_2}{v_2} = \frac{z-a_3}{v_3}$$

and $v_1 = 0$, then the symmetric equations become

$$x - a_1, \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

Parallel, intersecting, skew, and perpendicular lines

To determine whether two lines are parallel, intersecting, skew or perpendicular, we will need to perform a number of tests on the two lines.

Given two lines,

$$L_1: \quad x_1 = a_1 + b_1t \qquad y_1 = c_1 + d_1t \qquad z_1 = e_1 + f_1t$$

$$L_2: \quad x_2 = a_2 + b_2s \qquad y_2 = c_2 + d_2s \qquad z_2 = e_2 + f_2s$$

then the lines are

parallel if the ratio equality is true.

$$\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$$

intersecting if the lines are not parallel or if you can solve them as a system of simultaneous equations.

perpendicular if the lines are intersecting and their dot product is 0.

$$L_1 \cdot L_2 = 0$$

skew if the lines are not parallel and not intersecting.

Example

Say whether the lines are parallel, intersecting, perpendicular or skew.

$$L_1: \quad x_1 = 1 + 5t \qquad y_1 = -3 + 2t \qquad z_1 = 1 + t$$

$$L_2: \quad x_2 = 2 + 3s \qquad y_2 = 3 + 4s \qquad z_2 = 3 - 2s$$

We'll start by testing the lines to see if they're parallel by pulling out the coefficients

$$\frac{b_1}{b_2} = \frac{d_1}{d_2} = \frac{f_1}{f_2}$$

$$\frac{5}{3} = \frac{2}{4} = \frac{1}{-2}$$

$$\frac{5}{3} = \frac{1}{2} = \frac{1}{-2}$$

Since $5/3 \neq 1/2 \neq -1/2$, we know the lines are not parallel.

Because they're not parallel, we'll test to see whether or not they're intersecting. We'll set the equations for x, y, and z from each line equal to each other. If we can find a solution set for the parameter values s and t, and this solution set satisfies all three equations, then we've proven that the lines are intersecting.

Setting $x_1 = x_2$, we get

$$x_1 = x_2$$

$$1 + 5t = 2 + 3s$$

$$5t = 1 + 3s$$

[1] $t = \frac{1}{5} + \frac{3}{5}s$

Setting $y_1 = y_2$, we get

$$y_1 = y_2$$

[2] $-3 + 2t = 3 + 4s$

Plugging [1] into [2] gives

$$-3 + 2\left(\frac{1}{5} + \frac{3}{5}s\right) = 3 + 4s$$

$$-3 + \frac{2}{5} + \frac{6}{5}s = 3 + 4s$$

$$-6 + \frac{2}{5} = 4s - \frac{6}{5}s$$

$$-\frac{28}{5} = \frac{14}{5}s$$

$$s = -\frac{28}{14}$$

[3] $s = -2$

Plugging [3] into [1] gives

$$t = \frac{1}{5} + \frac{3}{5}(-2)$$

$$t = \frac{1}{5} - \frac{6}{5}$$

[4] $t = -1$

Setting $z_1 = z_2$, we get

$$z_1 = z_2$$

[5] $1 + t = 3 - 2s$

Plugging [3] and [4] into [5] gives

$$1 + t = 3 - 2s$$

$$1 + (-1) = 3 - 2(-2)$$

$$0 = 7$$

Since $0 \neq 7$, the lines are not intersecting.

Because L_1 and L_2 are not parallel and not intersecting, by definition they must be skew.

In the previous example, we didn't test for perpendicularity because only intersecting lines can be perpendicular, and we found that the lines were not intersecting. If we had found that L_1 and L_2 were in fact perpendicular, we would have needed to test for perpendicularity by taking the dot product, like this:

$$L_1 \cdot L_2 = (5)(3) + (2)(4) + (1)(-2)$$

$$L_1 \cdot L_2 = 15 + 8 - 2$$

$$L_1 \cdot L_2 = 21$$

Since the dot product isn't 0, we've proven that the lines are not perpendicular.

Equation of a plane

The equation of a plane is given by the formula

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

where $\langle a, b, c\rangle$ are the direction numbers from the normal vector to the plane.

Given three points in the plane $P(P_1, P_2, P_3)$, $Q(Q_1, Q_2, Q_3)$ and $R(R_1, R_2, R_3)$, we can find the equation of the plane by

using the points to generate **two vectors**

$$\overrightarrow{PQ}=\left\langle (Q_1-P_1),(Q_2-P_2),(Q_3-P_3)\right\rangle$$

$$\overrightarrow{PR}=\left\langle (R_1-P_1),(R_2-P_2),(R_3-P_3)\right\rangle,$$

taking the **cross product** of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ to get the normal vector to the plane

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ PQ_1 & PQ_2 & PQ_3\\ PR_1 & PR_2 & PR_3\end{vmatrix}=\mathbf{i}\begin{vmatrix}PQ_2 & PQ_3\\ PR_2 & PR_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}PQ_1 & PQ_3\\ PR_1 & PR_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}PQ_1 & PQ_2\\ PR_1 & PR_2\end{vmatrix}$$

$$=\mathbf{i}\left(PQ_2PR_3-PQ_3PR_2\right)-\mathbf{j}\left(PQ_1PR_3-PQ_3PR_1\right)$$

$$+\mathbf{k}\left(PQ_1PR_2-PQ_2PR_1\right)$$

and then plugging the given points and the normal vector into the **formula** for the equation of the plane.

Example

Find the equation of the plane that passes through the given points.

$$P(1,0,2)$$

$$Q(2,-1,3)$$

$$R(1,-1,2)$$

We'll start by using the given points P, Q and R to find two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ that lie in the plane.

$$\overrightarrow{PQ} = \langle (2-1), (-1-0), (3-2) \rangle$$

$$\overrightarrow{PQ} = \langle 1,-1,1 \rangle$$

and

$$\overrightarrow{PR} = \langle (1-1), (-1-0), (2-2) \rangle$$

$$\overrightarrow{PR} = \langle 0,-1,0 \rangle$$

Taking the cross product of these two vectors, we get

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i}\begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-1)(0) - (1)(-1)\right]\mathbf{i} - \left[(1)(0) - (1)(0)\right]\mathbf{j} + \left[(1)(-1) - (-1)(0)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (0+1)\mathbf{i} - (0-0)\mathbf{j} + (-1-0)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = 1\mathbf{i} - 0\mathbf{j} - 1\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1,0,-1 \rangle$$

Now we'll plug any of the given points, we'll use P, and the direction numbers from the cross product into the formula for the equation of the plane.

$$(1)(x-1) + (0)(y-0) + (-1)(z-2) = 0$$

$$x - 1 - z + 2 = 0$$

$$x - z = -1$$

Intersection of a line and a plane

If a line and a plane intersect one another, the intersection will be a single point.

To find the point of intersection, we'll

> substitute the values of x, y and z from the equation of the line into the equation of the plane and solve for the parameter t
>
> take the value of t and plug it back into the equation of the line

This will give us the coordinates of the point of intersection.

Example

Find the point where the line intersects the plane.

> The line is given by $x = -1 + 2t$, $y = 4 - 5t$, and $z = 1 + t$
>
> The plane is given by $2x - 3y + z = 3$

Our first step is to plug the values for x, y and z given by the equation of the line into the equation of the plane.

$$2(-1 + 2t) - 3(4 - 5t) + (1 + t) = 3$$

$$-2 + 4t - 12 + 15t + 1 + t = 3$$

$$20t = 16$$

$$t=\frac{16}{20}$$

$$t=\frac{4}{5}$$

Now we'll plug the value we found for t back into the equation of the line.

$$x=-1+2\left(\frac{4}{5}\right)$$

$$x=\frac{3}{5}$$

and

$$y=4-5\left(\frac{4}{5}\right)$$

$$y=0$$

and

$$z=1+\left(\frac{4}{5}\right)$$

$$z=\frac{9}{5}$$

Putting these values together, we can say the point of intersection of the line and the plane is the coordinate point

$$\left(\frac{3}{5},0,\frac{9}{5}\right)$$

If we want to double-check ourselves, we can plug this coordinate point back into the equation of the plane.

$$2\left(\frac{3}{5}\right)-3(0)+\left(\frac{9}{5}\right)=3$$

$$\frac{6}{5}+\frac{9}{5}=3$$

$$\frac{15}{5}=3$$

$$3=3$$

Since $3=3$ is true, we know that the point we found is a true intersection point with the plane.

Parallel, perpendicular and angle between planes

Given two planes:

Plane	**Normal vector to the plane**
$a_1x + a_2y + a_3z = c$	$a\langle a_1, a_2, a_3\rangle$
$b_1x + b_2y + b_3z = d$	$b\langle b_1, b_2, b_3\rangle$

they will always be

parallel if the ratio equality is true.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

perpendicular if the dot product of their normal vectors is 0.

$$a \cdot b = 0$$

set at a non-90° angle if the planes are neither parallel nor perpendicular, in which case the **angle** between the planes is given by

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where a and b are the normal vectors to the given planes, $a \cdot b$ is the dot product of the vectors, $|a|$ is the magnitude of the vector a (its length) and $|b|$ is the magnitude of the vector b

(its length). We can find the magnitude of both vectors using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

for a three-dimensional vector where the point (x_1, y_1, z_1) is the origin $(0,0,0)$.

Example

Say whether the planes are parallel, perpendicular, or neither. If the planes are neither parallel nor perpendicular, find the angle between the planes.

$$3x - y + 2z = 5$$

$$x + 4y + 3z = 1$$

First we'll find the normal vectors of the given planes.

Plane	Normal vector to the plane
$3x - y + 2z = 5$	$a\langle 3, -1, 2\rangle$
$x + 4y + 3z = 1$	$b\langle 1, 4, 3\rangle$

To say whether the planes are parallel, we'll set up our ratio inequality using the direction numbers from their normal vectors.

$$\frac{3}{1} = \frac{-1}{4} = \frac{2}{3}$$

Since the ratios are not equal, the planes are not parallel.

To say whether the planes are perpendicular, we'll take the dot product of their normal vectors.

$$a \cdot b = (3)(1) + (-1)(4) + (2)(3)$$

$$a \cdot b = 3 - 4 + 6$$

$$a \cdot b = 5$$

Since the dot product is not 0, the planes are not perpendicular.

Since the planes are not parallel or perpendicular, we know that they are set at a non-90° angle from one another, which is given by the formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

We need to find the dot product of the normal vectors, and the magnitude of each of them. We already know from our perpendicular test that their dot product is

$$a \cdot b = (3)(1) + (-1)(4) + (2)(3)$$

$$a \cdot b = 3 - 4 + 6$$

$$a \cdot b = 5$$

The magnitude of $a\langle 3, -1, 2\rangle$ is

$$|a| = \sqrt{(3-0)^2 + (-1-0)^2 + (2-0)^2}$$

$$|a| = \sqrt{9 + 1 + 4}$$

$$|a| = \sqrt{14}$$

The magnitude of $b\langle 1,4,3\rangle$ is

$$|b| = \sqrt{(1-0)^2 + (4-0)^2 + (3-0)^2}$$

$$|b| = \sqrt{1+16+9}$$

$$|b| = \sqrt{26}$$

Plugging $a \cdot b = 5$, $|a| = \sqrt{14}$, and $|b| = \sqrt{26}$ into our cosine formula gives

$$\cos\theta = \frac{5}{\sqrt{14}\sqrt{26}}$$

$$\cos\theta = \frac{5}{\sqrt{364}}$$

$$\theta = \arccos\frac{5}{\sqrt{364}}$$

$$\theta = 74.8^\circ$$

The angle between the planes is 74.8°.

Parametric equations for the line of intersection of two planes

If two planes intersect each other, the intersection will always be a line.

The vector equation for the line of intersection is given by

$$r = r_0 + tv$$

where r_0 is a point on the line and v is the vector result of the cross product of the normal vectors of the two planes.

The parametric equations for the line of intersection are given by

$x = a$, $y = b$, and $z = c$

where a, b and c are the coefficients from the vector equation $r = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Example

Find the parametric equations for the line of intersection of the planes.

$$2x + y - z = 3$$

$$x - y + z = 3$$

We need to find the vector equation of the line of intersection. In order to get it, we'll need to first find v, the cross product of the normal vectors of the given planes.

The normal vectors for the planes are

Plane	Normal vector to the plane
$2x + y - z = 3$	$a\langle 2,1,-1\rangle$
$x - y + z = 3$	$b\langle 1,-1,1\rangle$

The cross product of the normal vectors is

$$v = |a \times b| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$v = |a \times b| = \mathbf{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$v = |a \times b| = \left[(1)(1) - (-1)(-1)\right]\mathbf{i} - \left[(2)(1) - (-1)(1)\right]\mathbf{j} + \left[(2)(-1) - (1)(1)\right]\mathbf{k}$$

$$v = |a \times b| = (1 - 1)\mathbf{i} - (2 + 1)\mathbf{j} + (-2 - 1)\mathbf{k}$$

$$v = |a \times b| = 0\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$$

$$v = |a \times b| = \langle 0, -3, -3\rangle$$

We also need a point on the line of intersection. To get it, we'll use the equations of the given planes as a system of linear equations. If we set $z = 0$ in both equations, we get

$$2x + y - z = 3$$

$$2x + y - 0 = 3$$

$$2x + y = 3$$

and

$$x-y+z=3$$

$$x-y+0=3$$

$$x-y=3$$

Now we'll add the equations together.

$$(2x+x)+(y-y)=3+3$$

$$3x+0=6$$

$$x=2$$

Plugging $x=2$ back into $x-y=3$, we get

$$2-y=3$$

$$-y=1$$

$$y=-1$$

Putting these values together, the point on the line of intersection is

$$(2,-1,0)$$

$$r_0=2\mathbf{i}-\mathbf{j}+0\mathbf{k}$$

$$r_0=\langle 2,-1,0\rangle$$

Now we'll plug v and r_0 into the vector equation.

$$r=r_0+tv$$

$$r=(2\mathbf{i}-\mathbf{j}+0\mathbf{k})+t(0\mathbf{i}-3\mathbf{j}-3\mathbf{k})$$

$$r=2\mathbf{i}-\mathbf{j}+0\mathbf{k}+0\mathbf{i}t-3\mathbf{j}t-3\mathbf{k}t$$

$$r = 2\mathbf{i} - \mathbf{j} - 3\mathbf{j}t - 3\mathbf{k}t$$

$$r = (2)\mathbf{i} + (-1 - 3t)\mathbf{j} + (-3t)\mathbf{k}$$

With the vector equation for the line of intersection in hand, we can find the parametric equations for the same line. Matching up $r = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with our vector equation $r = (2)\mathbf{i} + (-1 - 3t)\mathbf{j} + (-3t)\mathbf{k}$, we can say that

$$a = 2$$

$$b = -1 - 3t$$

$$c = -3t$$

Therefore, the parametric equations for the line of intersection are

$$x = 2$$

$$y = -1 - 3t$$

$$z = -3t$$

Symmetric equations for the line of intersection of two planes

If two planes intersect, their intersection will always be a line. The only exception is when the planes are identical, in which case they intersect everywhere.

The vector equation for the line of intersection of the planes is given by

$$r = r_0 + tv$$

where r_0 is a point on the line and v is the cross product of the normal vectors of the two planes. If we break this vector equation into parametric equations for the plane, we get

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Solving each of the parametric equations for t gives

$$t = \frac{x - x_0}{a}$$

$$t = \frac{y - y_0}{b}$$

$$t = \frac{z - z_0}{c}$$

Then, because we have three values all equal to t, we know those values must be equal to each other, and we find

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) is a point on the line, and a, b, and c come from the cross product $v = \langle a, b, c \rangle$. These are the symmetric equations for the line of intersection of two planes.

However, realize that any zero value in the cross product vector $v = \langle 0, b, c \rangle$ or $v = \langle a, 0, c \rangle$ or $v = \langle a, b, 0 \rangle$ will result in division by 0 in the symmetric equations. In that case, we'll substitute the corresponding parametric equation in place of the symmetric equation. For instance, given $v = \langle 0, b, c \rangle$, the parametric equation for x will be

$$x = x_0 + at$$

$$x = x_0 + 0t$$

$$x = x_0$$

Then if we wanted to express the equation of the line in terms of its parametric equations, we'd give

$$x = x_0, \ \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Let's do an example so that we can work through a step-by-step process for finding the equation of the line of intersection of two planes.

Example

Find the symmetric equations for the line of intersection of the planes.

$$2x + y - z = 3$$

$$x - y + z = 3$$

The normal vectors for the planes are $a\langle 2,1,-1\rangle$ for the plane $2x + y - z = 3$ and $b\langle 1,-1,1\rangle$ for the plane $x - y + z = 3$. So the cross product of the normal vectors is

$$v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$v = \mathbf{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$v = [(1)(1) - (-1)(-1)]\mathbf{i} - [(2)(1) - (-1)(1)]\mathbf{j} + [(2)(-1) - (1)(1)]\mathbf{k}$$

$$v = (1 - 1)\mathbf{i} - (2 + 1)\mathbf{j} + (-2 - 1)\mathbf{k}$$

$$v = 0\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$$

$$v = \langle 0, -3, -3\rangle$$

To find a point on the line of intersection, we'll set $z = 0$ in both equations.

$$2x + y - z = 3$$

$$2x + y - 0 = 3$$

$$2x + y = 3$$

and

$$x - y + z = 3$$

$$x - y + 0 = 3$$

$$x - y = 3$$

We'll add the equations to find x.

$$(2x + x) + (y - y) = 3 + 3$$

$$3x + 0 = 6$$

$$x = 2$$

Plugging $x = 2$ back into $x - y = 3$, we get

$$2 - y = 3$$

$$-y = 1$$

$$y = -1$$

Putting these values together, the point on the line of intersection is $(2, -1, 0)$. Then with the cross product $v = \langle 0, -3, -3 \rangle$ and $(2, -1, 0)$, we'll build the parametric equations.

$$x = 2 + 0t \qquad y = -1 - 3t \qquad z = 0 - 3t$$

$$x = 2 \qquad z = -3t$$

Solve these equations for the parameter t.

$$x = 2 \qquad t = -\frac{y+1}{3} \qquad t = -\frac{z}{3}$$

The equation $x = 2$ can't be solved for t, so we'll separate it from the other two equations for t, listing the symmetric equations for the line of intersection of the planes as

$$x = 2, \ -\frac{y+1}{3} = -\frac{z}{3}$$

Partial derivatives in two variables

By this point we've already learned how to find derivatives of single-variable functions. After learning derivative rules like power rule, product rule, quotient rule, chain rule and others, we're pretty comfortable handling the derivatives of functions like these:

$$f(x) = x^2 + 5$$

$$f(x) = \frac{\left(x^2+4\right)^3 \sin x}{x^4 + \ln 7x^4}$$

But now it's time to start talking about derivatives of multivariable functions, such as

$$f(x, y) = x^4y^3 + x^3y^2 + \ln xe^y$$

Finding derivatives of a multivariable function like this one may be less challenging than you think, because we're actually only going to take the derivative with respect to one variable at a time. For example, we'll take the derivative with respect to x while we treat y like it's a constant. Then we'll take another derivative of the original function, this one with respect to y, and we'll treat x as a constant.

In that way, we sort of reduce the problem to a single-variable derivative problem, which is a derivative we already know how to handle!

We call these kinds of derivatives "partial derivatives" because we're only taking the derivative of one part (variable) of the function at a time.

Remember the definition of the derivative from single-variable calculus (aka the difference quotient)? Let's adapt that definition so that it works for us for multivariable functions.

We know that, if z is a function defined in terms of x and y, like $z = f(x, y)$, then

The partial derivative of z with respect to x is

$$z_x = f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

The partial derivative of z with respect to y is

$$z_y = f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

The definition as we've written it here gives two different kinds of notation for the partial derivatives of z: z_x or z_y and $f_x(x, y)$ or $f_y(x, y)$. In fact, there are many ways you might see partial derivatives defined.

The partial derivatives of a function z defined in terms of x and y could be written in all of these ways:

The partial derivative of z with respect to x:

$$f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x, y) = f_x = z_x$$

The partial derivative of z with respect to y:

$$f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x, y) = f_y = z_y$$

Let's use what we've learned so far to work through an example using the difference quotient to find the partial derivatives of a multivariable function.

Example

Using the definition, find the partial derivatives of

$$f(x, y) = 2x^2y$$

For the partial derivative of z with respect to x, we'll substitute $x + h$ into the original function for x.

$$f(x+h, y) = 2(x+h)^2y$$

$$f(x+h, y) = 2\left(x^2 + 2xh + h^2\right)y$$

$$f(x+h, y) = 2x^2y + 4xhy + 2h^2y$$

Plugging our values of $f(x, y)$ and $f(x+h, y)$ into the definition, we get

$$f_x(x, y) = \lim_{h \to 0} \frac{2x^2y + 4xhy + 2h^2y - 2x^2y}{h}$$

$$f_x(x, y) = \lim_{h \to 0} \frac{4xhy + 2h^2y}{h}$$

$$f_x(x, y) = \lim_{h \to 0} 4xy + 2hy$$

$$f_x(x, y) = \lim_{h \to 0} 4xy + 2(0)y$$

$$f_x(x, y) = 4xy$$

For the partial derivative of z with respect to y, we'll substitute $y + h$ into the original function for y.

$$f(x, y + h) = 2x^2(y + h)$$

$$f(x, y + h) = 2x^2y + 2x^2h$$

Plugging our values of $f(x, y)$ and $f(x, y + h)$ into the definition, we get

$$f_y(x, y) = \lim_{h \to 0} \frac{2x^2y + 2x^2h - 2x^2y}{h}$$

$$f_y(x, y) = \lim_{h \to 0} \frac{2x^2h}{h}$$

$$f_y(x, y) = \lim_{h \to 0} 2x^2$$

$$f_y(x, y) = 2x^2$$

You'll remember from single-variable calculus that using the definition of the derivative was the "long way" that we learned to take the derivative before we learned the derivative rules that made the process faster. The good news is that we can apply all the same derivative rules to multivariable functions to avoid using the difference quotient! We just have to remember to work with only one variable at a time, treating all other variables as constants.

The next example shows how the power rule provides a faster way to find this function's partial derivatives.

Example

Using the power rule, find the partial derivatives of

$$f(x, y) = 2x^2y$$

For the partial derivative of z with respect to x, we treat y as a constant and use power rule to find the derivative.

$$f_x(x, y) = 2\left(\frac{d}{dx}x^2\right)y$$

$$f_x(x, y) = 2(2x)y$$

$$f_x(x, y) = 4xy$$

For the partial derivative of z with respect to y, we treat x as a constant and use power rule to find the derivative.

$$f_y(x, y) = 2x^2\left(\frac{d}{dy}y\right)$$

$$f_y(x, y) = 2x^2(1)$$

$$f_y(x, y) = 2x^2$$

Partial derivatives in three or more variables

Sometimes we need to find partial derivatives for functions with three or more variables, and we'll do it the same way we found partial derivatives for functions in two variables.

We'll take the derivative of the function with respect to each variable separately, which means we'll end up with one partial derivative for each of our variables.

When we take the derivative with respect to one variable, we'll treat all the other variables as constants.

Example

Find the partial derivatives of the function.

$$f(x,y,z)=3x^5y^2z^3$$

With three variables x, y, and z, we need to find three partial derivatives. When we take the partial derivative with respect to one variable, we'll hold all others constant.

$$\frac{\partial f}{\partial x}=3\left(5x^4\right)y^2z^3$$

$$\frac{\partial f}{\partial x}=15x^4y^2z^3$$

and

$$\frac{\partial f}{\partial y} = 3x^5(2y)z^3$$

$$\frac{\partial f}{\partial y} = 6x^5yz^3$$

and

$$\frac{\partial f}{\partial z} = 3x^5y^2\left(3z^2\right)$$

$$\frac{\partial f}{\partial z} = 9x^5y^2z^2$$

$\partial f/\partial x$ is the partial derivative of the function f with respect to x, $\partial f/\partial y$ is the partial derivative of the function f with respect to y, and $\partial f/\partial z$ is the partial derivative of the function f with respect to z.

Let's try a more complex example with more than three variables.

Example

Find the partial derivatives of the function.

$$f(w,x,y,z) = 2wx^4 - \frac{3y^3}{7z^2} + \sin 2x$$

With four variables w, x, y, and z, we need to find four partial derivatives. When we take the partial derivative with respect to one variable, we'll hold all others constant.

$$\frac{\partial f}{\partial w} = 2x^4$$

and

$$\frac{\partial f}{\partial x} = 2w\left(4x^3\right) + (2)\cos 2x$$

$$\frac{\partial f}{\partial x} = 8wx^3 + 2\cos 2x$$

and

$$\frac{\partial f}{\partial y} = -\frac{3\left(3y^2\right)}{7z^2}$$

$$\frac{\partial f}{\partial y} = -\frac{9y^2}{7z^2}$$

and

$$\frac{\partial f}{\partial z} = -\frac{3y^3}{7}(-2)z^{-3}$$

$$\frac{\partial f}{\partial z} = \frac{6y^3}{7z^3}$$

$\partial f/\partial w$ is the partial derivative of the function f with respect to w, $\partial f/\partial x$ is the partial derivative of the function f with respect to x, $\partial f/\partial y$ is the partial derivative of the function f with respect to y, and $\partial f/\partial z$ is the partial derivative of the function f with respect to z.

Higher order partial derivatives

We already learned in single-variable calculus how to find second derivatives; we just took the derivative of the derivative. Remember how we even used the second derivative to help us with inflection points and concavity when we were learning optimization and sketching graphs?

Here's an example from single variable calculus of what a second derivative looks like:

$$f(x) = 2x^3$$

$$f'(x) = 6x^2$$

$$f''(x) = 12x$$

Well, we can find the second derivative of a multivariable function in the same way. Except, instead of just one function that defines the second derivative (like $f''(x) = 12x$ above), we'll need four functions that define the second derivative! Our second-order partial derivatives will be:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

The derivative with respect to x, of the first-order partial derivative with respect to x

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

The derivative with respect to y, of the first-order partial derivative with respect to y

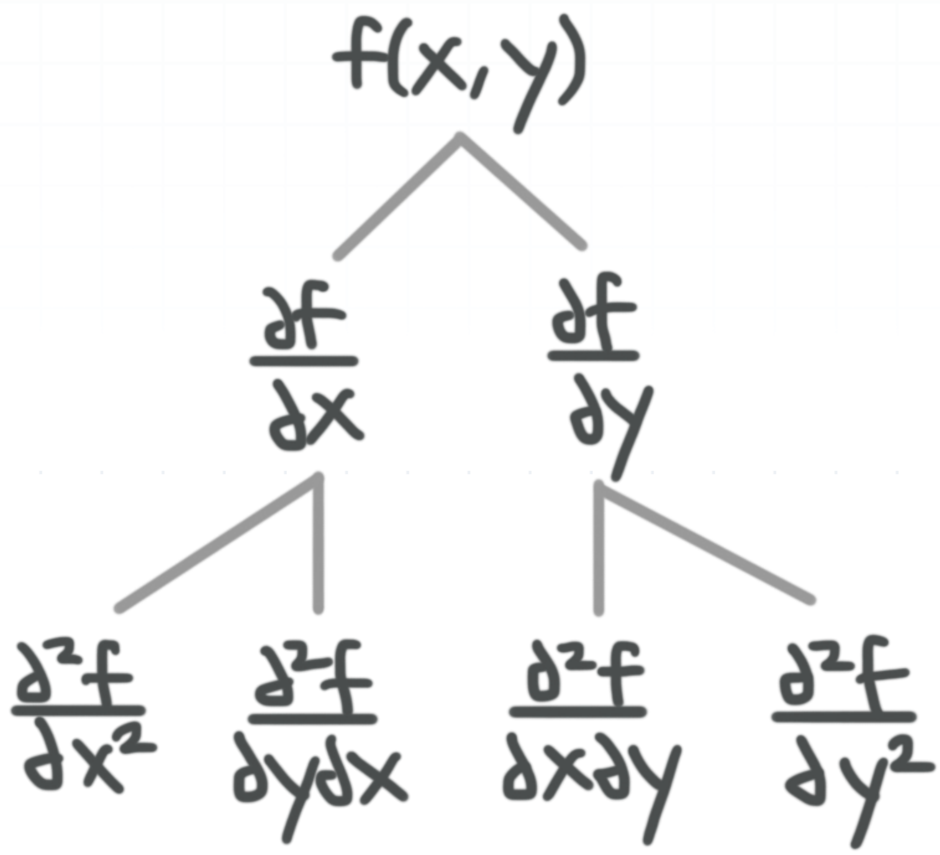
$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

The derivative with respect to y, of the first-order partial derivative with respect to x

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

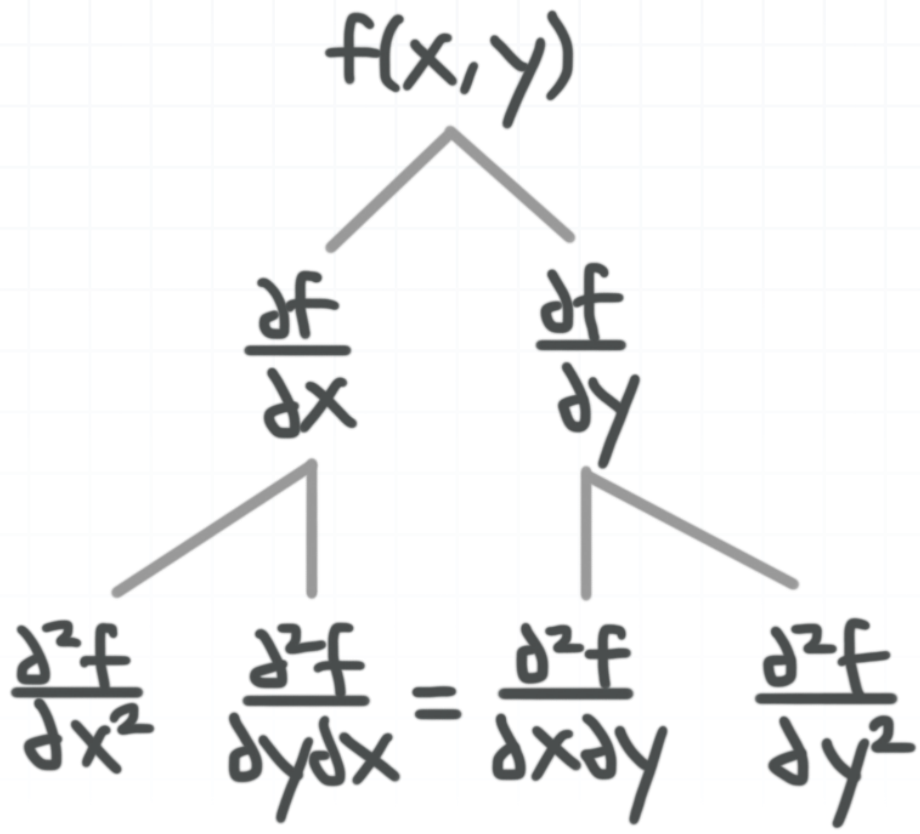
The derivative with respect to x, of the first-order partial derivative with respect to y

That wording is a little bit complicated. We can think about like the illustration below, where we start with the original function in the first row, take first derivatives in the second row, and then second derivatives in the third row.



The good news is that, even though this looks like four second-order partial derivatives, it's actually only three. That's because the two second-order partial derivatives in the middle of the third row will always come out to be the same.

Whether you start with the first-order partial derivative with respect to x, and then take the partial derivative of that with respect to y; or if you start with the first-order partial derivative with respect to y, and then take the partial derivative of that with respect to x; you'll get the same answer in both cases. Which means our tree actually looks like this:



Example

Find the second-order partial derivatives of the multivariable function.

$$f(x, y) = 2x^2y$$

We found the first-order partial derivatives of this function in a previous section, and they were

$$f_x(x, y) = 4xy$$

$$f_y(x,y) = 2x^2$$

The four second order partial derivatives are:

$$f_{xx} = \frac{\partial}{\partial x}(4xy) = 4y$$

$$f_{xy} = \frac{\partial}{\partial x}\left(2x^2\right) = 4x$$

$$f_{yx} = \frac{\partial}{\partial y}(4xy) = 4x$$

$$f_{yy} = \frac{\partial}{\partial y}\left(2x^2\right) = 0$$

Notice that the mixed second-order partial derivative is the same, regardless of whether you take the partial derivative first with respect to x and then y, or vice versa.

Differential of the function

The differential of a multivariable function is given by

$$dz = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

$\frac{\partial z}{\partial x}$ is the partial derivative of f with respect to x

$\frac{\partial z}{\partial y}$ is the partial derivative of f with respect to y

Example

Find the differential of the multivariable function.

$$z = 6x^2y - 4\ln y$$

Before we can use the formula for the differential, we need to find the partial derivatives of the function with respect to each variable.

$$\frac{\partial z}{\partial x} = 6(2x)y$$

$$\frac{\partial z}{\partial x} = 12xy$$

and

$$\frac{\partial z}{\partial y} = 6x^2 - 4\left(\frac{1}{y}\right)$$

$$\frac{\partial z}{\partial y} = 6x^2 - \frac{4}{y}$$

We'll plug the partial derivatives into the formula for the differential.

$$dz = (12xy)dx + \left(6x^2 - \frac{4}{y}\right) dy$$

$$dz = 12xy\ dx + 6x^2\ dy - \frac{4}{y}\ dy$$

This is the differential of the function.

Chain rule for multivariable functions

Previously we've been asked to find partial derivatives of multivariable functions, like

$$f(x, y, z) = xyz$$

For a multivariable function like this one, f is the dependent variable, and x, y, and z are the independent variables. When we take partial derivatives of a function like this one, we need one partial derivative with respect to each of the independent variables. Since there are three independent variables, we'll have three partial derivatives.

$$\frac{\partial f}{\partial x} = yz$$

$$\frac{\partial f}{\partial y} = xz$$

$$\frac{\partial f}{\partial z} = xy$$

But now we need to introduce a new type of multivariable function, one in which we insert *intermediate* variables in between the dependent and independent variables. For example, if we defined parametric equations for x, y, and z from the function above, we might have something like

$f(x, y, z) = xyz$, where

$$x = t$$

$$y = t^2$$

$$z = t^3$$

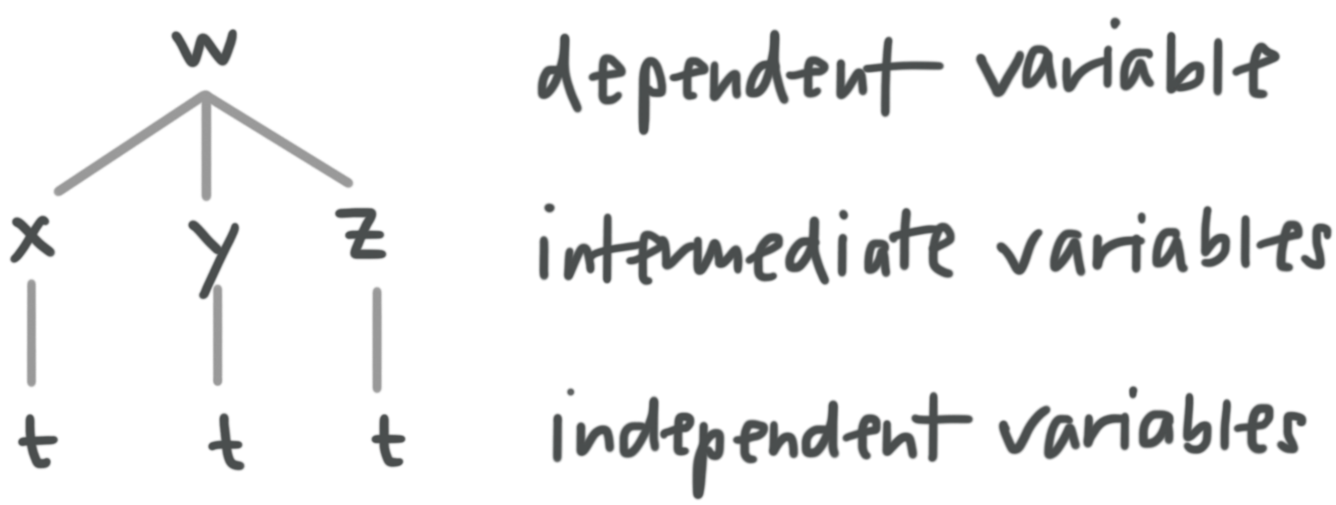
In this case, f is the dependent variable, x, y, and z are intermediate variables, and the parameter t is the independent variable. Just as before, the number of partial derivatives we'll find for this function depends on the number of independent variables. Since we have just one independent variable, we'll only have one derivative. And since we only have one derivative, it'll be a "normal" derivative, instead of a partial derivative. This is called a Case I function.

Let's look in more detail at what we call Case I and Case II functions.

Case I

A Case I scenario is when we have **one independent variable**.

More specifically, the function is defined for one dependent variable in terms of multiple intermediate variables, which are all in terms of one independent variable, like this:



Case I always results in only one derivative, which is the derivative of the dependent variable with respect to the independent variable. For the case described in the tree diagram above, the formula for the partial derivative would be

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

Notice how we multiply the **partial** derivatives of the dependent variable with respect to the intermediate variables, by the “**normal**” derivatives of the intermediate variables with respect to the independent variable, and then add those products together.

Let’s try a Case I example.

Example

Use chain rule to find the partial derivatives of the multivariable function.

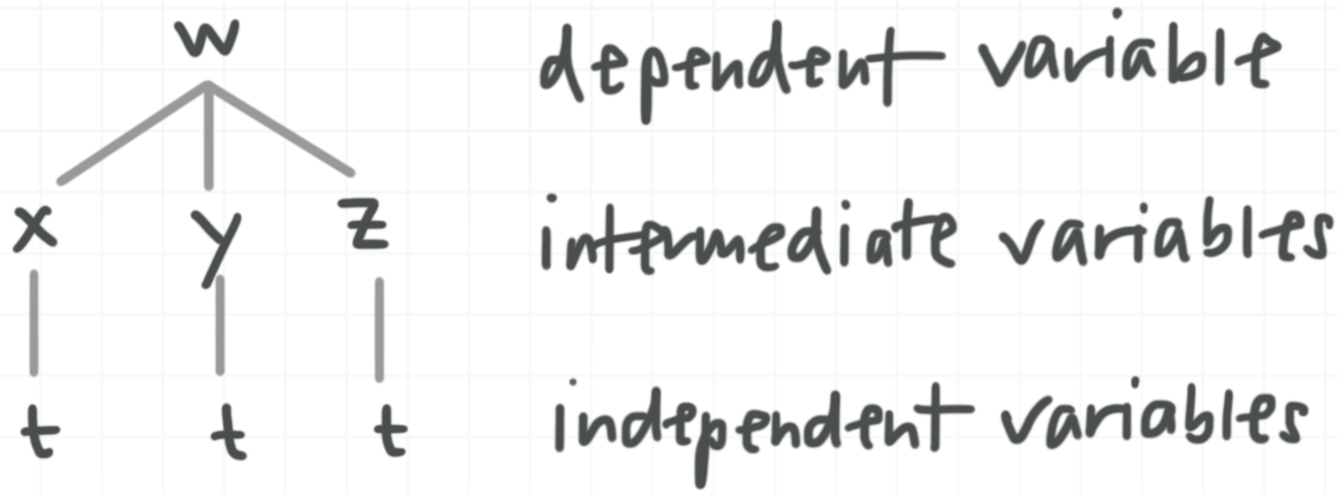
$w=x^2y-6y^3\sqrt{z}$, where

$$x=5t^2$$

$$y=4t+1$$

$$z=t^3-5t$$

Since w is defined in terms of x, y, and z, and x, y, and z are all defined in terms of t, we have the following tree diagram:



The number of independent variables dictates the number of derivatives we need to find. In this case, with only have one independent variable, we'll only have one derivative, which will be the derivative of the dependent variable w with respect to the independent variable t.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

We need to find each component of the formula for $\partial w/\partial t$, and we'll start with the partial derivatives of w with respect to the intermediate variables x, y, and z.

$$\frac{\partial w}{\partial x} = (2x)y$$

$$\frac{\partial w}{\partial x} = 2xy$$

and

$$\frac{\partial w}{\partial y} = x^2(1) - 6(3y^2)\sqrt{z}$$

$$\frac{\partial w}{\partial y} = x^2 - 18y^2\sqrt{z}$$

and

$$\frac{\partial w}{\partial z} = -6y^3\left(\frac{1}{2}\right)z^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial z} = -3y^3z^{-\frac{1}{2}}$$

Now we just need to find the derivatives of the intermediate variables x, y, and z with respect to the independent variable t.

$$\frac{dx}{dt} = 10t$$

$$\frac{dy}{dt} = 4$$

$$\frac{dz}{dt} = 3t^2 - 5$$

Plug everything we found into the equation for $\partial w/\partial t$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = (2xy)(10t) + \left(x^2 - 18y^2\sqrt{z}\right)(4) + \left(-3y^3z^{-\frac{1}{2}}\right)\left(3t^2 - 5\right)$$

$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} - 9y^3z^{-\frac{1}{2}}t^2 + 15y^3z^{-\frac{1}{2}}$$

$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} - \frac{9y^3t^2}{\sqrt{z}} + \frac{15y^3}{\sqrt{z}}$$

$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} + \frac{15y^3 - 9y^3t^2}{\sqrt{z}}$$

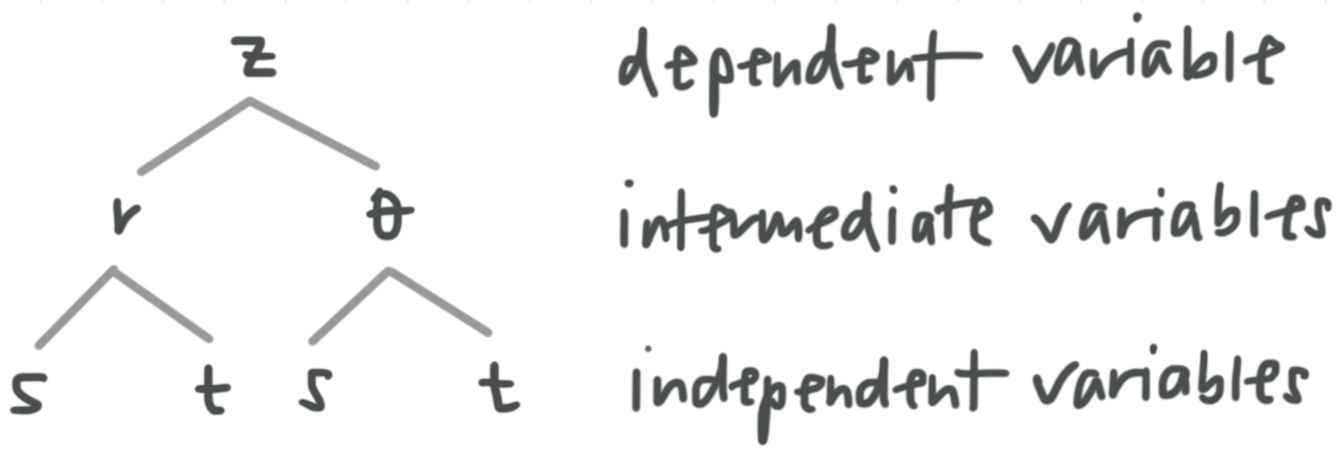
$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} + \frac{3y^3\left(5 - 3t^2\right)}{\sqrt{z}}$$

This is the derivative of w with respect to t.

Case II

A Case II scenario is when we have **multiple independent variables**.

More specifically, the function is defined for one dependent variable in terms of multiple intermediate variables, which are all in terms of multiple independent variables, like this:



Case II always results in one partial derivative for each of the independent variables, and they'll be the partial derivatives of the dependent variable with respect to each independent variable. For the tree diagram above, the formulas for the partial derivatives would be

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

Notice how we multiply the **partial** derivatives of the dependent variable with respect to the intermediate variables, by the **partial** derivatives of the intermediate variables with respect to the independent variables, and then add the products together.

For Case II, we just have to make sure that we separate the independent variables into different partial derivatives.

Let's try a Case II example.

Example

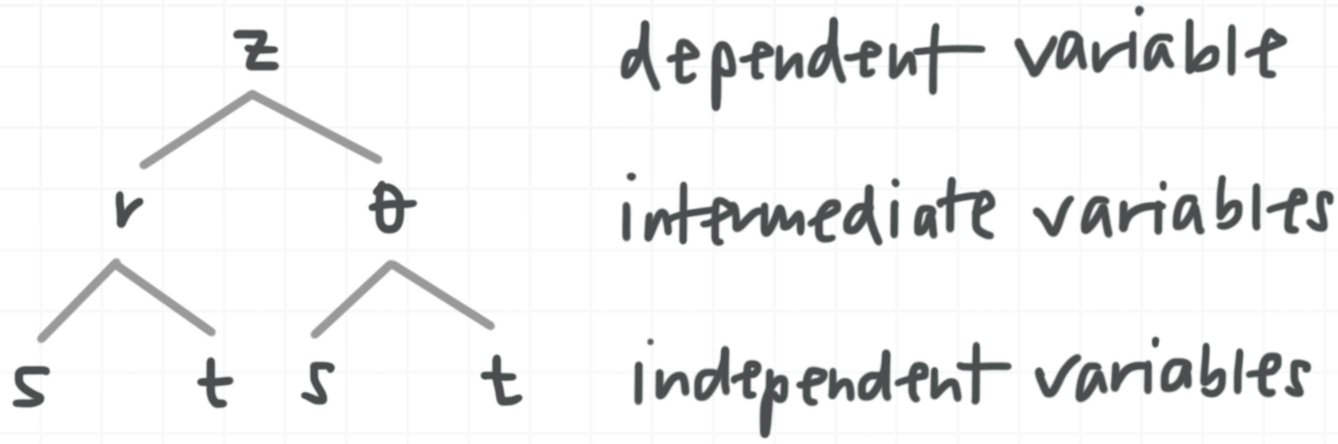
Use chain rule to find the partial derivatives of the multivariable function.

$$z = \ln r + r^2 \sin \theta$$

$$r = 3s^2 - t$$

$$\theta = 2t^2 - \frac{4}{s^2}$$

Since z is defined in terms of r and θ, and r and θ are all defined in terms of s and t, we have the following tree diagram:



The number of independent variables dictates the number of partial derivatives we need to find. In this case, with have two independent variables, so we'll have two partial derivatives, which will be the partial derivatives of the dependent variable z with respect to the independent variables s and t.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

We need to find each component of the formulas for $\partial z/\partial s$ and $\partial z/\partial t$, and we'll start with the partial derivatives of z with respect to the intermediate variables r and θ.

Now we can solve for our four partial derivatives.

$$\frac{\partial z}{\partial r} = \frac{1}{r} + 2r\sin\theta$$

and

$$\frac{\partial z}{\partial \theta} = r^2\cos\theta$$

Now we just need to find the partial derivatives of the intermediate variables r and θ with respect to the independent variables s and t.

With respect to s:

$$\frac{\partial r}{\partial s} = 6s$$

and

$$\frac{\partial \theta}{\partial s} = \frac{8}{s^3}$$

With respect to t:

$$\frac{\partial r}{\partial t} = -1$$

and

$$\frac{\partial \theta}{\partial t} = 4t$$

Plug everything we found into the equations for $\partial z/\partial s$ and $\partial z/\partial t$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial s} = \left(\frac{1}{r} + 2r\sin\theta\right)(6s) + \left(r^2\cos\theta\right)\left(\frac{8}{s^3}\right)$$

$$\frac{\partial z}{\partial s} = \frac{6s}{r} + 12rs\sin\theta + \frac{8r^2\cos\theta}{s^3}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$\frac{\partial z}{\partial t} = \left(\frac{1}{r} + 2r\sin\theta\right)(-1) + \left(r^2\cos\theta\right)(4t)$$

$$\frac{\partial z}{\partial t} = -\frac{1}{r} - 2r\sin\theta + 4r^2t\cos\theta$$

These are the partial derivatives of z with respect to s and t.

Implicit differentiation for multivariable functions

When we want to use implicit differentiation to find partial derivatives of multivariable functions, we'll use the following formulas.

For multivariable functions **in two variables**:

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

For multivariable functions **in three variables**:

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Before we can use these formulas to find derivatives or partial derivatives of the original function, we'll need to rearrange it so that it's equal to 0, and then rename it $F(x, y)$ (for two variable functions) or $F(x, y, z)$ (for three variable functions).

Let's try an example with a multivariable function in two variables.

Example

Use implicit differentiation to find the partial derivative of the multivariable function.

$$xe^{2y} = x^2 - y^3$$

We'll start by rearranging the function so that it's equal to 0, then we'll call it $F(x, y)$.

$$0 = x^2 - y^3 - xe^{2y}$$

$$F(x, y) = x^2 - y^3 - xe^{2y}$$

Now we'll take partial derivatives of F with respect to x and y.

$$\frac{\partial F}{\partial x} = 2x - e^{2y}$$

and

$$\frac{\partial F}{\partial y} = -3y^2 - xe^{2y}(2)$$

$$\frac{\partial F}{\partial y} = -3y^2 - 2xe^{2y}$$

We'll plug the partial derivatives into the formula for the derivative of a multivariable function with two variables.

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$\frac{dy}{dx} = -\frac{2x - e^{2y}}{-3y^2 - 2xe^{2y}}$$

$$\frac{dy}{dx} = \frac{2x - e^{2y}}{3y^2 + 2xe^{2y}}$$

This is the partial derivative of $xe^{2y} = x^2 - y^3$.

Now we'll try an example with a multivariable function in three variables.

Example

Use implicit differentiation to find the partial derivatives of the multivariable function.

$$x^2 \sin z = 3yz + 2x^3$$

We'll start by rearranging the function so that it's equal to 0, then we'll call it $F(x, y, z)$.

$$0 = 3yz + 2x^3 - x^2 \sin z$$

$$F(x, y, z) = 3yz + 2x^3 - x^2 \sin z$$

Now we'll take partial derivatives of F with respect to x, y, and z.

$$\frac{\partial F}{\partial x} = 6x^2 - 2x \sin z$$

$$\frac{\partial F}{\partial y} = 3z$$

$$\frac{\partial F}{\partial z} = 3y - x^2 \cos z$$

We'll plug the partial derivatives into the formula for the derivative of a multivariable function with three variables.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial x} = -\frac{6x^2 - 2x \sin z}{3y - x^2 \cos z}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{3z}{3y - x^2 \cos z}$$

These are the partial derivatives of the $x^2 \sin z = 3yz + 2x^3$.

Directional derivatives

The directional derivative of a multivariable function takes into account the direction (given by the unit vector $\vec{u}$) as well as the partial derivatives of the function with respect to each of the variables.

In a **two variable function**, the formula for the directional derivative is

$$D_u f(x, y) = a\left(\frac{\partial f}{\partial x}\right) + b\left(\frac{\partial f}{\partial y}\right)$$

where

a and b come from the unit vector $\vec{u} = \langle a, b \rangle$

If asked to find the directional derivative in the direction of $\vec{v} = \langle c, d \rangle$, we'll need to convert $\vec{v} = \langle c, d \rangle$ to the unit vector using

$$\vec{u} = \left\langle \frac{c}{\sqrt{c^2 + d^2}}, \frac{d}{\sqrt{c^2 + d^2}} \right\rangle$$

$\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x

$\frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y

In a **three variable function**, the formula for the directional derivative is

$$D_u f(x, y, z) = a\left(\frac{\partial f}{\partial x}\right) + b\left(\frac{\partial f}{\partial y}\right) + c\left(\frac{\partial f}{\partial z}\right)$$

where

a, b and c come from the unit vector $\vec{u} = \langle a, b, c \rangle$

If asked to find the directional derivative in the direction of $\vec{v} = \langle d, e, f \rangle$, we'll need to convert $\vec{v} = \langle d, e, f \rangle$ to the unit vector using

$$\vec{u} = \left\langle \frac{d}{\sqrt{d^2 + e^2 + f^2}}, \frac{e}{\sqrt{d^2 + e^2 + f^2}}, \frac{f}{\sqrt{d^2 + e^2 + f^2}} \right\rangle$$

$\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x

$\frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y

$\frac{\partial f}{\partial z}$ is the partial derivative of f with respect to z

Let's try an example with a two variable function.

Example

Find the directional derivative of the function.

$$f(x, y) = 2x^3 + 3x^2y + y^2$$

in the direction $\vec{v} = \langle 1,2 \rangle$

at the point $P(1, -2)$

We'll start by converting the given vector to its unit vector form.

$$\vec{u} = \left\langle \frac{c}{\sqrt{c^2 + d^2}}, \frac{d}{\sqrt{c^2 + d^2}} \right\rangle$$

$$\vec{u} = \left\langle \frac{1}{\sqrt{(1)^2 + (2)^2}}, \frac{2}{\sqrt{(1)^2 + (2)^2}} \right\rangle$$

$$\vec{u} = \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

Now we'll find the partial derivatives of f with respect to x and y.

$$\frac{\partial f}{\partial x} = 6x^2 + 6xy$$

and

$$\frac{\partial f}{\partial y} = 3x^2 + 2y$$

With the unit vector and the partial derivatives, we have everything we need to plug into our formula for the directional derivative.

$$D_u f(x, y) = \frac{1}{\sqrt{5}} \left(6x^2 + 6xy\right) + \frac{2}{\sqrt{5}} \left(3x^2 + 2y\right)$$

We want to find the directional derivative at the point $P(1, -2)$, so we'll plug this into the equation we just found for the directional derivative, and we'll get

$$D_uf(1,-2)=\frac{1}{\sqrt{5}}\left[6(1)^2+6(1)(-2)\right]+\frac{2}{\sqrt{5}}\left[3(1)^2+2(-2)\right]$$

$$D_uf(1,-2)=\frac{-6}{\sqrt{5}}+\frac{-2}{\sqrt{5}}$$

$$D_uf(1,-2)=\frac{-8}{\sqrt{5}}$$

This is the directional derivative of the function $f(x,y)=2x^3+3x^2y+y^2$ in the direction $\vec{v}=\langle 1,2\rangle$ at the point $P(1,-2)$.

Linear approximation in two variables

We can use the linear approximation formula

$$L(x, y) = f(a, b) + \frac{\partial f}{\partial x}(a, b)(x - a) + \frac{\partial f}{\partial y}(a, b)(y - b)$$

(a, b) is the given point

$f(a, b)$ is the value of the function at (a, b)

$\frac{\partial f}{\partial x}(a, b)$ is the partial derivative of f with respect to x at (a, b)

$\frac{\partial f}{\partial y}(a, b)$ is the partial derivative of f with respect to y at (a, b)

to find an approximation of the function at the given point (a, b).

Example

Find the linear approximation of the multivariable function at the given point.

$$f(x, y) = 6x^3 - 2xy^2$$

at $(1,2)$

The problem tells us that $(a, b) = (1,2)$, so we need to find $f(a, b) = f(1,2)$.

$$f(1,2) = 6(1)^3 - 2(1)(2)^2$$

$$f(1,2) = 6 - 2(4)$$

$$f(1,2) = -2$$

Then we need to find the partial derivatives of the function with respect to x and y.

$$\frac{\partial f}{\partial x} = 18x^2 - 2y^2$$

$$\frac{\partial f}{\partial x}(1,2) = 18(1)^2 - 2(2)^2$$

$$\frac{\partial f}{\partial x}(1,2) = 10$$

and

$$\frac{\partial f}{\partial y} = -4xy$$

$$\frac{\partial f}{\partial y}(1,2) = -4(1)(2)$$

$$\frac{\partial f}{\partial y}(1,2) = -8$$

Plugging the slope in each direction, (a, b), and $f(a, b)$ into the linear approximation formula, we get

$$L(x, y) = -2 + (10)(x - 1) + (-8)(y - 2)$$

$$L(x, y) = -2 + 10x - 10 - 8y + 16$$

$$L(x, y) = 10x - 8y + 4$$

This is the linear approximation of the given function at the given point.

Gradient vectors

The find the gradient (also called the gradient vector) of a two variable function, we'll use the formula

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

This gives a vector-valued function that describes the function's gradient everywhere. If we want to find the gradient at a particular point, we just evaluate at that point.

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}(x, y), \frac{\partial f}{\partial y}(x, y) \right\rangle$$

The maximal directional derivative is given by the magnitude of the gradient.

$$\| \nabla f \| = \| a, b \| = \sqrt{a^2 + b^2}$$

where a and b come from $\nabla f(x, y) = \langle a, b \rangle$

The gradient ∇f always points in the direction of the maximal directional derivative.

Remember that the gradient is not limited to two variable functions. We can modify the two variable formula to accommodate more than two variables as needed.

Example

Find the maximal directional derivative and the direction in which it occurs.

$$f(x, y) = x^3 + 2x^2y + 4y^2$$

at $P(1,1)$

We'll start with the partial derivatives of the given function f.

$$\frac{\partial f}{\partial x} = 3x^2 + 4xy$$

$$\frac{\partial f}{\partial y} = 2x^2 + 8y$$

The gradient of the function in general is

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f = \left\langle 3x^2 + 4xy, 2x^2 + 8y \right\rangle$$

To find the gradient at the point we're interested in, we'll plug in $P(1,1)$.

$$\nabla f(1,1) = \left\langle 3(1)^2 + 4(1)(1), 2(1)^2 + 8(1) \right\rangle$$

$$\nabla f(1,1) = \langle 7,10 \rangle$$

To find the maximal directional derivative, we take the magnitude of the gradient that we found.

$$\| \nabla f \| = \| a, b \| = \sqrt{a^2 + b^2}$$

$$\| 7,10 \| = \sqrt{(7)^2 + (10)^2}$$

$$\| 7,10 \| = \sqrt{149}$$

The maximal directional derivative always points in the direction of the gradient. So the maximal directional derivative is $\| 7,10 \| = \sqrt{149}$, and it points toward $\nabla f(1,1) = \langle 7,10 \rangle$.

Derivative rules and the gradient

The gradient can also be found for the product and quotient of functions. To find the gradient of the product of two functions f and g, we extend the product rule for derivatives to say that the gradient of the product is

$$\nabla (fg) = f \nabla g + g \nabla f$$

Or to find the gradient of the quotient of two functions f and g, we extend the quotient rule for derivatives to say that the gradient of the quotient is

$$\nabla \left(\frac{f}{g} \right) = \frac{g \nabla f - f \nabla g}{g^2}$$

Let's work through an example using a derivative rule.

Example

Find $\nabla (f/g)$.

$$f(x, y) = 3x^2y$$

$$g(x, y) = x^3 + 2x^2y + x$$

First we'll find $\nabla f(x, y)$ and $\nabla g(x, y)$.

$$\nabla f(x, y) = \frac{\partial\left(3x^2y\right)}{\partial x}\mathbf{i} + \frac{\partial\left(3x^2y\right)}{\partial y}\mathbf{j}$$

$$\nabla f(x, y) = 6xy\mathbf{i} + 3x^2\mathbf{j}$$

and

$$\nabla g(x, y) = \frac{\partial\left(x^3 + 2x^2y + x\right)}{\partial x}\mathbf{i} + \frac{\partial\left(x^3 + 2x^2y + x\right)}{\partial y}\mathbf{j}$$

$$\nabla g(x, y) = \left(3x^2 + 4xy + 1\right)\mathbf{i} + 2x^2\mathbf{j}$$

Plug into the formula.

$$\nabla\left(\frac{f}{g}\right) = \frac{g\,\nabla f - f\,\nabla g}{g^2}$$

$$\nabla\left(\frac{f}{g}\right) = \frac{(x^3 + 2x^2y + x)(6xy\mathbf{i} + 3x^2\mathbf{j}) - (3x^2y)((3x^2 + 4xy + 1)\mathbf{i} + 2x^2\mathbf{j})}{(x^3 + 2x^2y + x)^2}$$

$$\nabla\left(\frac{f}{g}\right) = \frac{6xy(x^3 + 2x^2y + x)\mathbf{i} + 3x^2(x^3 + 2x^2y + x)\mathbf{j} - 3x^2y(3x^2 + 4xy + 1)\mathbf{i} - 3x^2y(2x^2)\mathbf{j}}{(x^3 + 2x^2y + x)^2}$$

$$\nabla\left(\frac{f}{g}\right) = \frac{(6x^4y + 12x^3y^2 + 6x^2y)\mathbf{i} + (3x^5 + 6x^4y + 3x^3)\mathbf{j} + (-9x^4y - 12x^3y^2 - 3x^2y)\mathbf{i} - 6x^4y\mathbf{j}}{(x^3 + 2x^2y + x)^2}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{(6x^4y+12x^3y^2+6x^2y-9x^4y-12x^3y^2-3x^2y)\mathbf{i}+(3x^5+6x^4y+3x^3-6x^4y)\mathbf{j}}{(x^3+2x^2y+x)^2}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{(-3x^4y+3x^2y)\mathbf{i}+(3x^5+3x^3)\mathbf{j}}{(x^3+2x^2y+x)^2}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{-3x^4y+3x^2y}{(x^3+2x^2y+x)^2}\mathbf{i}+\frac{3x^5+3x^3}{(x^3+2x^2y+x)^2}\mathbf{j}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{-3x^2y(x^2-1)}{x^2(x^2+2xy+1)^2}\mathbf{i}+\frac{3x^3(x^2+1)}{x^2(x^2+2xy+1)^2}\mathbf{j}$$

$$\nabla\left(\frac{f}{g}\right)=\frac{-3y(x^2-1)}{(x^2+2xy+1)^2}\mathbf{i}+\frac{3x(x^2+1)}{(x^2+2xy+1)^2}\mathbf{j}$$

Gradient vectors and the tangent plane

We previously learned how to find the gradient vector at a specific point. We just use the formula

$$\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

where (x,y,z) is the point we're interested in. If the result of evaluating the gradient vector at the point (x,y,z) gives us

$$\nabla f(x,y,z) = \langle a,b,c \rangle$$

then a, b, and c represent the slope of the original function in the x, y, and z directions, respectively. Therefore, if we're interested in finding the equation of the tangent plane at $P(x_0,y_0,z_0)$, then we can plug the values of a, b, and c, along with the point $P(x_0,y_0,z_0)$, into the formula for the equation of the tangent plane

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

where a, b, and c come from $\nabla f(x,y,z) = \langle a,b,c \rangle$, and x_0, y_0, and z_0 come from the given point $P(x_0,y_0,z_0)$.

Example

Find the gradient vector of the function and use it to find the equation of the tangent plane at $P(3,4,-287)$.

$$x^4 - 5x^3y - y^2 + 3y^4 + z = 6$$

Rearrange the function.

$$f(x, y, z) = x^4 - 5x^3y - y^2 + 3y^4 + z - 6$$

Now we'll start with the partial derivatives of the given function f.

$$\frac{\partial f}{\partial x} = 4x^3 - 15x^2y$$

$$\frac{\partial f}{\partial y} = -5x^3 - 2y + 12y^3$$

$$\frac{\partial f}{\partial z} = 1$$

To find the gradient vector at the point we're interested in, we'll plug the partial derivatives in to the formula for the gradient vector, and then evaluate at the point of interest.

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\nabla f(x, y, z) = \langle 4x^3 - 15x^2y, -5x^3 - 2y + 12y^3, 1 \rangle$$

Evaluating at $P(3,4,-287)$, we get

$$\nabla f(3,4,-287) = \langle 4(3)^3 - 15(3)^2(4), -5(3)^3 - 2(4) + 12(4)^3, 1 \rangle$$

$$\nabla f(3,4,-287) = \langle -432, 625, 1 \rangle$$

Now we have everything we need to find the equation of the tangent plane.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-432(x - 3) + 625(y - 4) + 1(z + 287) = 0$$

$$-432x + 1{,}296 + 625y - 2{,}500 + z + 287 = 0$$

$$-432x + 625y + z = 917$$

Equation of the tangent plane

The tangent plane is the 3D version of the tangent line. Remember how we learned that the tangent line was a line that just barely skims against the graph of a function, intersecting it at only one point? Well, put that function in 3D space, and now the tangent line becomes a tangent plane.

Think about it as a flat surface, like a sheet of paper, that comes to balance on top of the graph, intersecting the graph at only one point. The equation of the tangent plane then gives us the slope of the function at the point where they intersect each other, and we can use the equation of the tangent plane to figure out how fast the function is increasing or decreasing at that intersection point.

To find the equation of the tangent plane to a three-dimensional function at a specific point (x_1, y_1, z_1), we'll use the formula

$$z - z_1 = \frac{\partial z}{\partial x}(x_1, y_1, z_1)(x - x_1) + \frac{\partial z}{\partial y}(x_1, y_1, z_1)(y - y_1)$$

where

$\frac{\partial z}{\partial x}(x_1, y_1, z_1)$ is the partial derivative of z with respect to x at the point (x_1, y_1, z_1), and

$\frac{\partial z}{\partial y}(x_1, y_1, z_1)$ is the partial derivative of z with respect to y at the point (x_1, y_1, z_1)

We want to tackle tangent plane problems in three steps:

1. Find the partial derivatives of the function with respect to each variable

2. Evaluate the partial derivatives at the given point to find the slope in each direction

3. Plug the slopes and the given point into the formula for the equation of the tangent plane

$$z - z_1 = \frac{\partial z}{\partial x}(x_1, y_1, z_1)(x - x_1) + \frac{\partial z}{\partial y}(x_1, y_1, z_1)(y - y_1)$$

Example

Find the equation of the tangent plane of the function at the given point.

$$z = 3 + \frac{x^2}{16} + \frac{y^2}{9}$$

at $P(-4,3,5)$

We'll start by finding partial derivatives of the function z with respect to x and y.

$$\frac{\partial z}{\partial x} = \frac{2x}{16}$$

$$\frac{\partial z}{\partial x} = \frac{x}{8}$$

and

$$\frac{\partial z}{\partial y} = \frac{2y}{9}$$

Plugging the given point $P(-4,3,5)$ into the partial derivatives, we'll get values for the slope in each direction.

$$\frac{\partial z}{\partial x}P(-4,3,5) = \frac{-4}{8}$$

$$\frac{\partial z}{\partial x}P(-4,3,5) = -\frac{1}{2}$$

and

$$\frac{\partial z}{\partial y}(-4,3,5) = \frac{2(3)}{9}$$

$$\frac{\partial z}{\partial y}(-4,3,5) = \frac{2}{3}$$

Now we'll plug the slope in each direction and the point $P(-4,3,5)$ into the formula for the equation of the tangent plane.

$$z - z_1 = \frac{\partial z}{\partial x}(x_1, y_1, z_1)(x - x_1) + \frac{\partial z}{\partial y}(x_1, y_1, z_1)(y - y_1)$$

$$z - 5 = -\frac{1}{2}\left[x - (-4)\right] + \frac{2}{3}(y - 3)$$

$$z - 5 = -\frac{1}{2}(x + 4) + \frac{2}{3}y - 2$$

$$z - 5 = -\frac{1}{2}x - 2 + \frac{2}{3}y - 2$$

$$z = -\frac{1}{2}x + \frac{2}{3}y + 1$$

Normal line to the surface

The normal line to the surface of the tangent plane is the line which is set at 90° from the surface.

In order to find the equation of the normal line to a tangent plane, we aren't required to first find the equation of the tangent plane, but both the normal line and the tangent plane use the partial derivatives of the original function and the point of tangency. Therefore, once you find the partial derivatives, you can very quickly find the equation of the tangent plane as well as the equation of the normal line to the surface.

The tangent plane is given by

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

(x_0, y_0, z_0) is the point of tangency

$\frac{\partial f}{\partial x}(x_0, y_0, z_0)$ is the partial derivative of f with respect to x at (x_0, y_0, z_0)

$\frac{\partial f}{\partial y}(x_0, y_0, z_0)$ is the partial derivative of f with respect to y at (x_0, y_0, z_0)

$\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ is the partial derivative of f with respect to z at (x_0, y_0, z_0)

The normal line to the surface is given by

$$\frac{x-x_0}{\frac{\partial f}{\partial x}}=\frac{y-y_0}{\frac{\partial f}{\partial y}}=\frac{z-z_0}{\frac{\partial f}{\partial z}}$$

(x_0, y_0, z_0) is the point of tangency

$\frac{\partial f}{\partial x}(x_0, y_0, z_0)$ is the partial derivative of f with respect to x at (x_0, y_0, z_0)

$\frac{\partial f}{\partial y}(x_0, y_0, z_0)$ is the partial derivative of f with respect to y at (x_0, y_0, z_0)

$\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ is the partial derivative of f with respect to z at (x_0, y_0, z_0)

Example

For the function $x^2+y^2+2z^2=36$,

a) find the equation of the tangent plane at the point $P(-1,2,4)$

b) find the normal line to the tangent plane at the point $P(-1,2,4)$

In order to find the equation of the tangent plane, we have to find the partial derivatives of the function f with respect to each variable at the point $P(-1,2,4)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial x}(-1,2,4) = 2(-1)$$

$$\frac{\partial f}{\partial x}(-1,2,4) = -2$$

and

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial y}(-1,2,4) = 2(2)$$

$$\frac{\partial f}{\partial y}(-1,2,4) = 4$$

and

$$\frac{\partial f}{\partial z} = 4z$$

$$\frac{\partial f}{\partial z}(-1,2,4) = 4(4)$$

$$\frac{\partial f}{\partial z}(-1,2,4) = 16$$

Plugging the slope in each direction and the point $P(-1,2,4)$ into the equation of the tangent plane gives

$$-2\left[x-(-1)\right] + 4(y-2) + 16(z-4) = 0$$

$$-2(x+1)+4(y-2)+16(z-4)=0$$

$$-2x-2+4y-8+16z-64=0$$

$$-2x+4y+16z=74$$

$$-x+2y+8z=37$$

To find the normal line to the tangent plane $-x+2y+8z=37$ at the point $P(-1,2,4)$, we'll plug the partial derivatives we found earlier and the point $P(-1,2,4)$ into the symmetric formula

$$\frac{x-x_0}{\frac{\partial f}{\partial x}}=\frac{y-y_0}{\frac{\partial f}{\partial y}}=\frac{z-z_0}{\frac{\partial f}{\partial z}}$$

$$\frac{x-(-1)}{-2}=\frac{y-2}{4}=\frac{z-4}{16}$$

$$\frac{x+1}{-2}=\frac{y-2}{4}=\frac{z-4}{16}$$

We'll summarize our findings.

Tangent plane	$-x+2y+8z=37$
Normal line to the plane	$\frac{x+1}{-2}=\frac{y-2}{4}=\frac{z-4}{16}$

Second derivative test

Just as we did with single variable functions, we can use the second derivative test with multivariable functions to classify any critical points that the function might have.

To use the second derivative test, we'll need to take partial derivatives of the function with respect to each variable. Once we have the partial derivatives, we'll set them equal to 0 and use these as a system of simultaneous equations to solve for the coordinates of all possible critical points.

In order to classify the critical points we find using the second derivative test, we'll need to find the second-order partial derivatives and plug them into the formula,

$$D(x,y)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $D(x,y)<0$, then f has a **saddle point** at (x,y)

If $D(x,y)=0$, then the test is **inconclusive**

If $D(x,y)>0$ and

$\frac{\partial^2 f}{\partial x^2}(x,y)>0$, then f has a **local minimum** at (x,y)

$\frac{\partial^2 f}{\partial x^2}(x,y)<0$, then f has a **local maximum** at (x,y)

Let's try an example to see how we can use the second derivative test to classify the critical points of a multivariable function.

Example

Find and classify the critical points of the function.

$$f(x, y) = 6 + x^3 + y^3 - 3xy$$

We'll start by finding the first-order partial derivatives.

$$\frac{\partial f}{\partial x} = 3x^2 - 3y$$

and

$$\frac{\partial f}{\partial y} = 3y^2 - 3x$$

Setting both of them equal to 0 and using these as a system of simultaneous equations to find critical points gives

$$3x^2 - 3y = 0$$

$$3x^2 = 3y$$

$$y = x^2$$

$$3y^2 - 3x = 0$$

$$3\left(x^2\right)^2 - 3x = 0$$

$$3x^4 - 3x = 0$$

$$3x\left(x^3 - 1\right) = 0$$

$x = 0$ or $x = 1$

$$y = (0)^2 \qquad y = (1)^2$$

$$y = 0 \qquad y = 1$$

$$(0,0) \qquad (1,1)$$

To test these critical points, we'll use the second derivative test by calculating the second-derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial y^2} = 6y$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3$$

Plugging these into the second derivative test formula, we get

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (6x)(6y) - (-3)^2$$

$$D(x, y) = 36xy - 9$$

Now we can use the equation for $D(x, y)$ to test our critical points one at a time.

For $(0,0)$:

$$D(0,0) = 36(0)(0) - 9$$

$$D(0,0) = -9$$

Since $-9 < 0$, the function has a saddle point at $(0,0)$.

For $(1,1)$:

$$D(1,1) = 36(1)(1) - 9$$

$$D(1,1) = 27$$

Since $27 > 0$, we have to look at $\partial^2 f/\partial x^2$ to evaluate $(1,1)$ further.

$$\frac{\partial^2 f}{\partial x^2} = 6x$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 6(1)$$

$$\frac{\partial^2 f}{\partial x^2}(1,1) = 6$$

Since $6 > 0$, the function has a local minimum at $(1,1)$.

To summarize our findings, we can say that the function has

a saddle point at $(0,0)$

a local minimum at $(1,1)$

Extreme value theorem

In order to find global extrema of the function which is defined for a specific set of points, follow these steps:

1. Find first-order partial derivatives of the function

2. Use the first-order partial derivatives to find critical points, and eliminate any critical points that lie outside the set

3. Plug the remaining critical points into the original function to find the value of the function at those points

4. Find single-variable equations for the line segments that define the edges of the set

5. Take first-order partial derivatives of these line-segment equations with respect to the changing variable

6. Use the first-order partial derivatives of the line-segment equations to find critical points of each line segment

7. Plug the critical points from Step 6 into the original function to find the value of function at those points

8. Identify "corners" of the set and treat the coordinate points of the corners as critical points, plugging them into the original function to find the value of the function at those points

9. Compare the values of all the critical points (inside the set, along the edges, at the corners) to find the maximum and minimum values that the function attains in the set

Lagrange multipliers

We already know how to find critical points of a multivariable function and use the second derivative test to classify those critical points.

But sometimes we're asked to find and classify the critical points of a multivariable function that's subject to a secondary constraint equation.

Multivariable function	$z = f(x, y)$
Constraint equation	$g(x, y) = c$

To find the critical points of a function like this one, we can use the Lagrange Multiplier λ (lambda) to develop a system of simultaneous equations that will allow us to solve for critical points. We'll follow these steps:

1. Bring the constant c in $g(x, y)$ over to the left-hand side, so that you end up with two functions in the same form, $f(x, y) = \dots$ and $g(x, y) = \dots$.

2. Take the partial derivatives of $f(x, y)$ and $g(x, y)$.

3. Multiply the partial derivatives of $g(x, y)$ by λ.

4. Create the system of simultaneous equations $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}\lambda$ and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}\lambda$

5. Solve both equations for λ, then set them equal to each other.

6. Solve for one variable in terms of the other, then plug into the original constraint equation to find values for both x and y.

7. Use the second derivative test to classify the critical point.

$$D(x, y, \lambda) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

If $D(x, y) < 0$, then f has a **saddle point** at (x, y)

If $D(x, y) = 0$, then the test is **inconclusive**

If $D(x, y) > 0$ and

$\frac{\partial^2 f}{\partial x^2}(x, y) > 0$, then f has a **local minimum** at (x, y)

$\frac{\partial^2 f}{\partial x^2}(x, y) < 0$, then f has a **local maximum** at (x, y)

Example

Find the maximum and minimum of the function with the given constraint.

$$f(x, y) = x^2 + 2y^2 - 120$$

if $2y - 4x = 24$

$f(x, y)$ is the function, and it's subject to the constraint $2y - 4x = 24$.

We need to set the constraint equation equal to 0 so that we can get it into the same form as our function, setting it equal to $g(x, y)$.

$$2y - 4x - 24 = 0$$

$$g(x, y) = 2y - 4x - 24$$

Now we can find the partial derivatives of the function $f(x, y)$ and the constraint equation $g(x, y)$.

For $f(x, y)$:

$$\frac{\partial f}{\partial x} = 2x$$

and

$$\frac{\partial f}{\partial y} = 4y$$

For $g(x, y)$:

$$\frac{\partial g}{\partial x} = -4$$

and

$$\frac{\partial g}{\partial y} = 2$$

We'll multiply the partial derivatives of the constraint equation by λ, and get

$$\frac{\partial g}{\partial x}\lambda = -4\lambda$$

and

$$\frac{\partial g}{\partial y}\lambda = 2\lambda$$

Next, we create this system of simultaneous equations:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}\lambda$$

$$2x = -4\lambda$$

$$\lambda = -\frac{x}{2}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}\lambda$$

$$4y = 2\lambda$$

$$\lambda = 2y$$

With two equations for λ, we can set them equal to one another, and then solve for y in terms of x.

$$2y = -\frac{x}{2}$$

$$y = -\frac{x}{4}$$

Plugging this y-value into the constraint equation and solving for x, we get

$$2y - 4x = 24$$

$$2\left(-\frac{x}{4}\right) - 4x = 24$$

$$-\frac{x}{2} - 4x = 24$$

$$-x - 8x = 48$$

$$-9x = 48$$

$$x = -\frac{48}{9}$$

Now we can plug this real-number value of x into the equation to find a real-number value of y.

$$2y - 4\left(-\frac{48}{9}\right) = 24$$

$$2y + \frac{192}{9} = 24$$

$$2y + \frac{64}{3} = 24$$

$$6y + 64 = 72$$

$$6y = 8$$

$$y = \frac{4}{3}$$

This tells us that our single critical point is

$$\left(-\frac{48}{9},\frac{4}{3}\right)$$

Now we need to use the second derivative test to classify this critical point. To do so, we'll find the second-order partial derivatives of f.

$$\frac{\partial^2 f}{\partial x^2}=2$$

$$\frac{\partial^2 f}{\partial y^2}=4$$

$$\frac{\partial^2 f}{\partial x \partial y}=0$$

Plug the second-order partial derivatives into the second derivative test formula.

$$D(x,y,\lambda)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x,y,\lambda)=(2)(4)-(0)^2$$

$$D(x,y,\lambda)=8$$

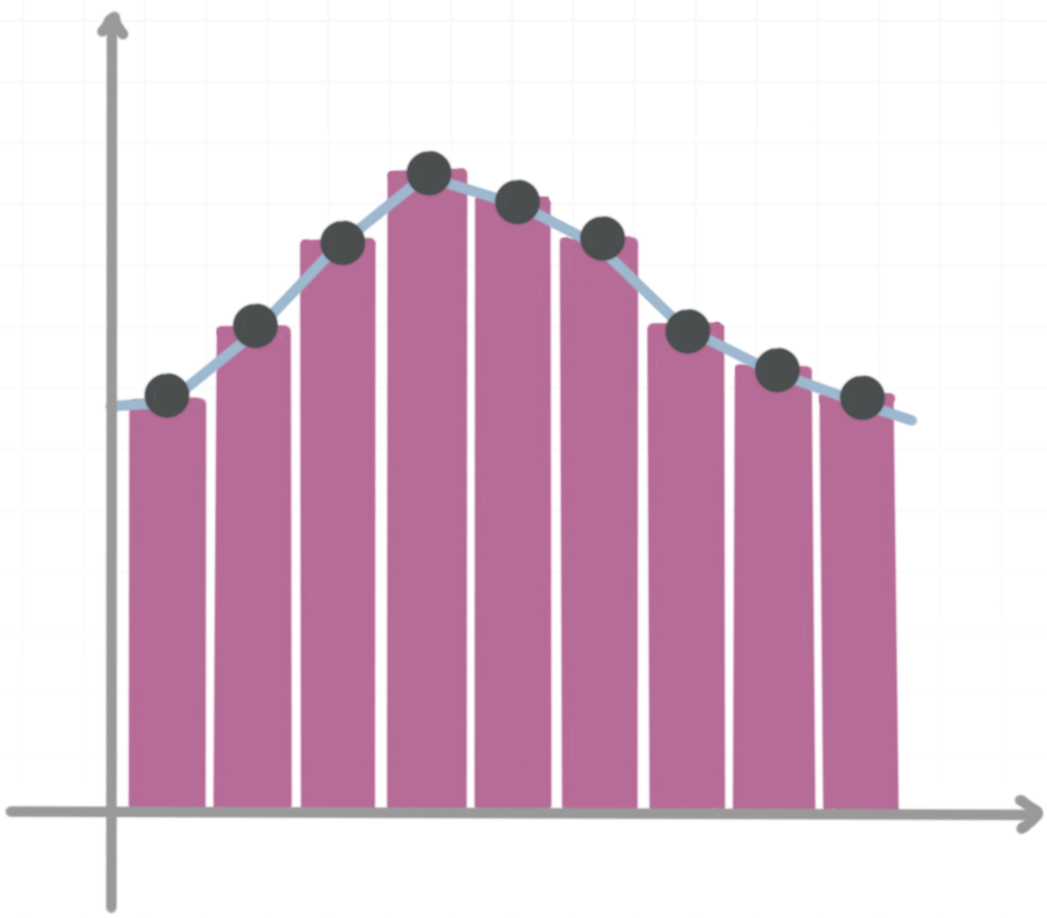
Since we have no variables remaining in $D(x,y,\lambda)$, this means we'll have the same result for all possible points. Because $D>0$, we have to look at $\partial^2 f/\partial x^2$.

$$\frac{\partial^2 f}{\partial x^2}\left(-\frac{48}{9},\frac{4}{3}\right)=2$$

Since $2 > 0$, the function has a local minimum at this critical point.

Midpoint rule for double integrals

In the past, we used midpoint rule to estimate the area under a single variable function. We'd draw rectangles under the curve so that the midpoint at the top of each rectangle touched the graph of the function.



Then we'd add the area of each rectangle together to find an approximation of the area under the curve.

When we translate this into three-dimensional space, it means that we use three-dimensional rectangular prisms, instead of two-dimensional rectangles, to approximate the volume under a multivariable function.

And instead of defining an x-interval $[a, b]$ over which we want to find area, we have to define an xy-interval $R = [a, b] \times [c, d]$ over which we want to find volume.

The volume under a multivariable function $f(x,y)$ over the region $R=[a,b]\times[c,d]$ can be approximated using a double Riemann sum with midpoints.

$$\int\int_R f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x_i},\overline{y_j}\right)\Delta A$$

m is the number of prisms in the x-direction and n is the number of prisms in the y-direction within the region R. The larger the values of m and n, the more accurate the approximation will be. Once we know R, m and n, we can find the midpoint of each prism. ΔA is the area of the base of one prism, in the same way that Δx was the width of one rectangle when we were using midpoint rule to find area under single variable functions.

Let's try an example to get a better idea of how to use the midpoint rule formula with a multivariable function.

Example

Use the midpoint rule to approximate the volume under the curve.

$$f(x,y)=x+y^2+2$$

$$R=[0,2]\times[0,4]$$

$$m=n=2$$

If we plug the values we've been given into the midpoint rule formula, we get

$$\iint_R f(x,y)\ dA \approx \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(\overline{x_i}, \overline{y_j}\right)\Delta A$$

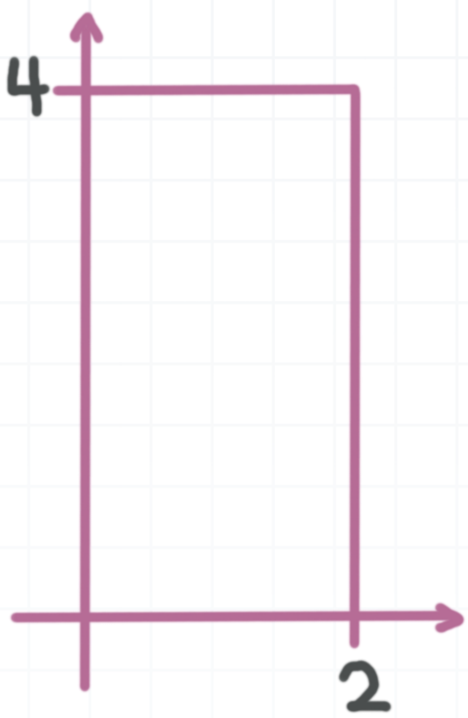
$$\int_0^4\int_0^2 x + y^2 + 2\ dx\ dy \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x_i}, \overline{y_j}\right)\Delta A$$

Notice the double integral on the left side of the equation. We turned it into an iterated integral (where we can integrate with respect to one variable at a time) by attaching the x-interval to the inside integral and the y-interval to the outside integral. Since we put the limits of integration for x on the inside interval, that means we also have to put dx on the inside before dy, which comes second since the limits of integration for y are on the outside integral.

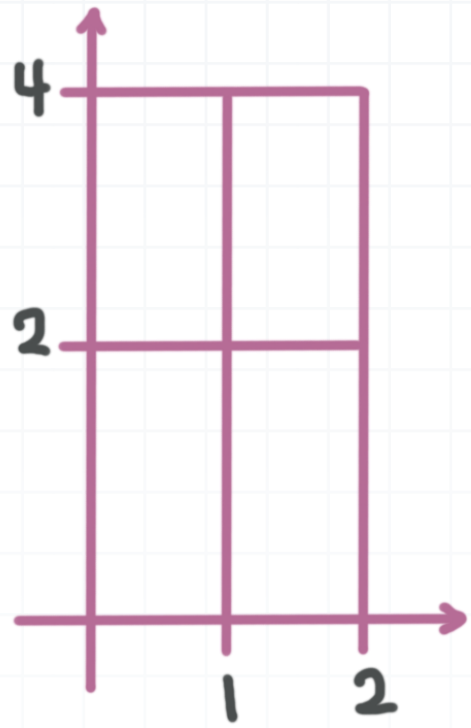
We could evaluate this iterated integral to find *exact* volume under the curve over the rectangle $R = [0,2] \times [0,4]$, but we've been asked to use midpoint rule to *approximate* volume, so instead we'll use the right side of the formula.

We'll need to define the rectangle R, then divide it into smaller rectangles based on m and n, and then find the midpoint of each rectangle so that we can plug the midpoints into our function $f(x,y) = x + y^2 + 2$, then plug the sum of the results into the approximation formula for $f\left(\overline{x_i}, \overline{y_j}\right)$.

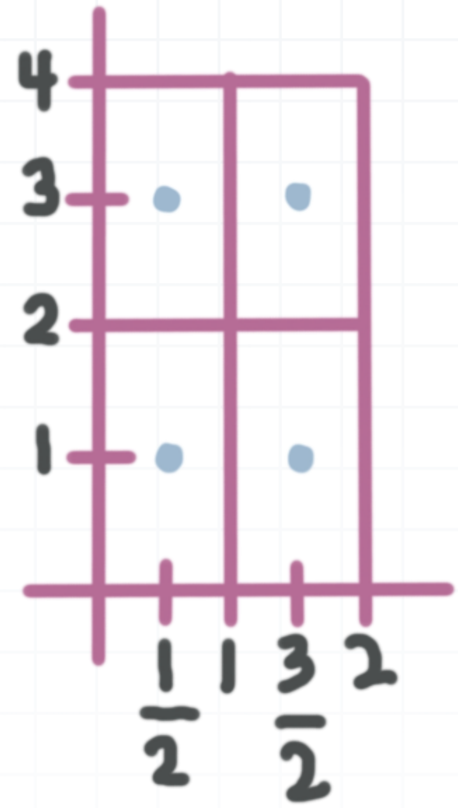
The rectangle $R = [0,2] \times [0,4]$ means that we want to integrate over the x -interval $[0,2]$ and over the y-interval $[0,4]$.



$R = [0,2] \times [0,4]$



Divide the x-interval into $m = 2$ sections; divide the y-interval into $n = 2$ sections



The midpoints of each rectangle lie halfway between its edges

The midpoints of the four smaller rectangles are

$$\left(\frac{1}{2},1\right) \quad \left(\frac{3}{2},1\right) \quad \left(\frac{1}{2},3\right) \quad \left(\frac{3}{2},3\right)$$

We also need to find ΔA, the area of the smaller rectangles. Since the dimensions of the smaller rectangles are 1×2, we can say

$$\Delta A = (1)(2)$$

$$\Delta A = 2$$

Plugging each midpoint into the original function $f(x, y) = x + y^2 + 2$ and summing the results, we get

$$\left[\frac{1}{2}+(1)^2+2\right]+\left[\frac{3}{2}+(1)^2+2\right]+\left[\frac{1}{2}+(3)^2+2\right]+\left[\frac{3}{2}+(3)^2+2\right]$$

We want to plug this sum into the approximation formula for

$$\sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x_i},\overline{y_j}\right)$$

and also plug in $\Delta A = 2$.

$$\int_0^4\int_0^2 x+y^2+2\ dx\ dy \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\overline{x_i},\overline{y_j}\right)\Delta A$$

$$\int_0^4\int_0^2 x+y^2+2\ dx\ dy \approx$$

$$\left[\left(\frac{1}{2}+(1)^2+2\right)+\left(\frac{3}{2}+(1)^2+2\right)+\left(\frac{1}{2}+(3)^2+2\right)+\left(\frac{3}{2}+(3)^2+2\right)\right](2)$$

$$\left[\left(\frac{1}{2}+1+2\right)+\left(\frac{3}{2}+1+2\right)+\left(\frac{1}{2}+9+2\right)+\left(\frac{3}{2}+9+2\right)\right](2)$$

$$\left[\left(\frac{1}{2}+3\right)+\left(\frac{3}{2}+3\right)+\left(\frac{1}{2}+11\right)+\left(\frac{3}{2}+11\right)\right](2)$$

$$\left[\left(\frac{1}{2}+3\cdot\frac{2}{2}\right)+\left(\frac{3}{2}+3\cdot\frac{2}{2}\right)+\left(\frac{1}{2}+11\cdot\frac{2}{2}\right)+\left(\frac{3}{2}+11\cdot\frac{2}{2}\right)\right](2)$$

$$\left[\left(\frac{1}{2}+\frac{6}{2}\right)+\left(\frac{3}{2}+\frac{6}{2}\right)+\left(\frac{1}{2}+\frac{22}{2}\right)+\left(\frac{3}{2}+\frac{22}{2}\right)\right](2)$$

$$\left[\left(\frac{7}{2}\right)+\left(\frac{9}{2}\right)+\left(\frac{23}{2}\right)+\left(\frac{25}{2}\right)\right](2)$$

$$\frac{64}{2}(2)$$

$$64$$

So we can say that the volume under the function $f(x, y) = x + y^2 + 2$ over the region $R = [0,2] \times [0,4]$ is approximately 64 cubic units.

Average value

We can estimate the average value of a region of level curves by using the formula

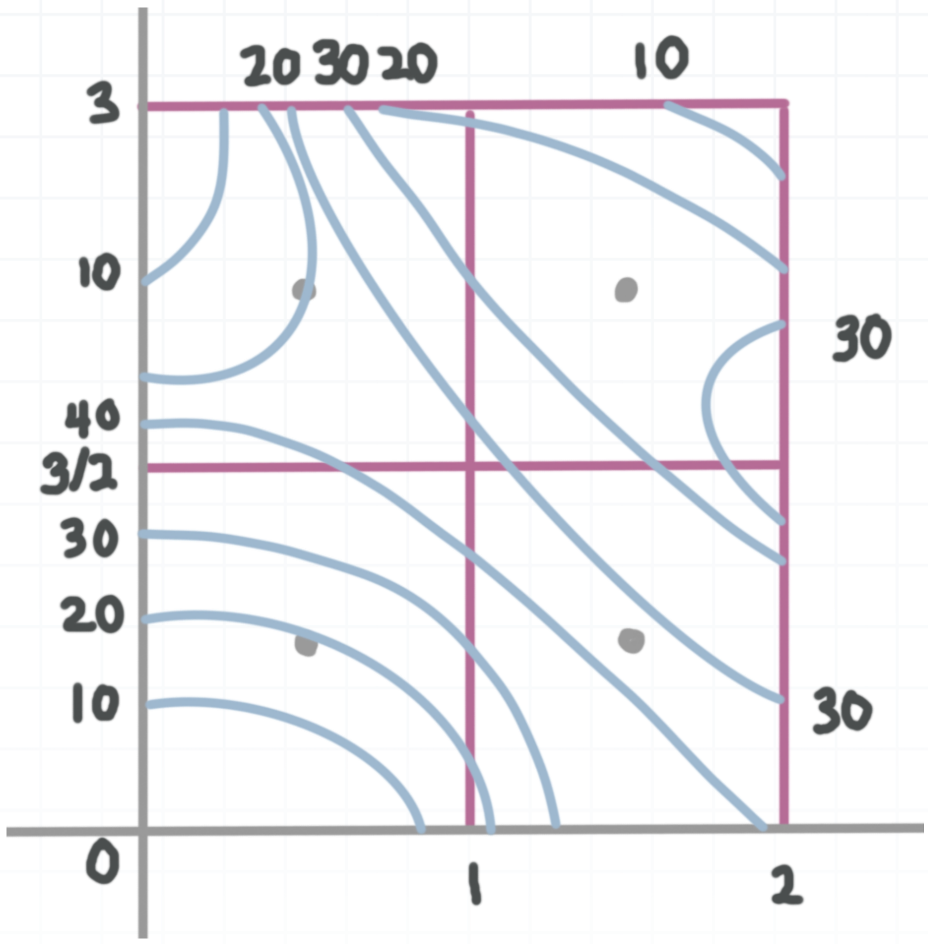
$$f_{ave} = \frac{1}{A(R)} \int \int_R f(x, y) \Delta A$$

where $A(R)$ is the area of the rectangle defined by $R = [x_1, x_2] \times [y_1, y_2]$, and where the double integral gives the volume under the surface $f(x, y)$ over the region R.

If an equation is not given, we can use the provided diagram to estimate the value of the function at the midpoints.

Example

Use the sketch of level curves to estimate the region's average value where $m = n = 2$ and $R = [0,2] \times [0,3]$.



The question is asking us to use a sketch of the level curves that represent our function to estimate the average value of the region depicted and defined by $R = [0,2] \times [0,3]$. We'll need to use the formula for average value:

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x, y) \Delta A$$

We can start by solving for $A(R)$, the total area of the region R.

$$A(R) = (2)(3)$$

$$A(R) = 6$$

Next, we can solve for ΔA, which is the area of one of the subrectangles. Based on the figure we were given, the width of each subrectangle is 1 and the height is $3/2$. Which means that the area of each subrectangle is

$$\Delta A = (1)\left(\frac{3}{2}\right)$$

$$\Delta A = \frac{3}{2}$$

Next we need the midpoints of each subrectangle.

Using $R = [0,2] \times [0,3]$, $m = n = 2$ and our diagram, we can see that the midpoints of the four smaller rectangles are

$$\left(\frac{1}{2},\frac{3}{4}\right), \left(\frac{3}{2},\frac{3}{4}\right), \left(\frac{1}{2},\frac{9}{4}\right) \text{ and } \left(\frac{3}{2},\frac{9}{4}\right)$$

Since we don't have an equation for our function, we can use the diagram of its level curves to estimate the functions values at each midpoint.

At $\left(\frac{1}{2},\frac{3}{4}\right)$, the function is approximately 19.

At $\left(\frac{3}{2},\frac{3}{4}\right)$, the function is approximately 35.

At $\left(\frac{1}{2},\frac{9}{4}\right)$, the function is approximately 20.

At $\left(\frac{3}{2},\frac{9}{4}\right)$, the function is approximately 26.

Now we have everything we need to find an estimate for the function's average value over the region. We'll use the formula

$$f_{ave}=\frac{1}{A(R)}\int\int_R f(x,y)\Delta A$$

but since we don't have the function, we'll use the midpoint rule formula instead of the double integral, such that the formula becomes

$$f_{ave}=\frac{1}{A(R)}\left[\Delta A\left(f(x_1,y_1)+f(x_2,y_2)+f(x_3,y_3)+f(x_4,y_4)\right)\right]$$

$$f_{ave}=\frac{1}{6}\left[\frac{3}{2}\left(19+35+20+26\right)\right]$$

$$f_{ave}=\frac{1}{6}\left[\frac{3}{2}(100)\right]$$

$$f_{ave}=\frac{300}{12}$$

$$f_{ave}=25$$

Using midpoint rule, this is our estimate for the function's average value over the region.

Iterated and double integrals

A double integral has no explicitly defined limits of integration. Instead, the interval is some region R, like

$$\iint_R f(x, y)\ dA$$

An iterated integral is one in which limits of integration have been clearly defined for each variable, like

$$\int_0^4 \int_0^2 x + y^2 + 2\ dx\ dy$$

Whenever we're given a double integral, we want to turn it into an iterated integral, because with iterated integrals, we can easily evaluate one integral at a time, like we would in single variable calculus.

When we evaluate iterated integrals, we always work from the inside out. In the iterated integral above, because dx is on the inside and dy is on the outside, that means that the limits of integration $[0,2]$ on the inside integral correspond to x and the limits of integration $[0,4]$ on the outside integral correspond to y.

Working from the inside out, we'll integrate first with respect to x, treating y as a constant, and we'll evaluate over the x-interval $[0,2]$. Then we'll integrate with respect to y and evaluate over the y-interval $[0,4]$.

Example

Evaluate the iterated integral.

$$\int_0^2 \int_1^3 x^2y^3 - xe^y \ dy \ dx$$

Since dy is on the inside, we have to integrate with respect to y first. When we integrate with respect to y, we treat x as a constant, similarly to the way we hold one variable constant when we take a partial derivative with respect to the other variable.

$$\int_0^2 \int_1^3 x^2y^3 - xe^y \ dy \ dx$$

$$\int_0^2 x^2\left(\frac{1}{4}y^4\right) - x\left(e^y\right)\Big|_{y=1}^{y=3} \ dx$$

$$\int_0^2 \frac{x^2y^4}{4} - xe^y\Big|_{y=1}^{y=3} \ dx$$

Notice how we've indicated that we're evaluating over the interval $y = 1$ to $y = 3$. It's helpful to designate the variable that the interval applies to so that you remember to plug in for the right variable.

Now we can evaluate the interval. Since we took the integral with respect to y, we're evaluating the integral with respect to y.

$$\int_0^2 \left[\frac{x^2(3)^4}{4} - xe^3\right] - \left[\frac{x^2(1)^4}{4} - xe^1\right] \ dx$$

$$\int_0^2 \left(\frac{81x^2}{4} - xe^3\right) - \left(\frac{x^2}{4} - xe\right) \ dx$$

$$\int_0^2 \frac{81x^2}{4} - xe^3 - \frac{x^2}{4} + xe \ dx$$

$$\int_0^2 \frac{80x^2}{4} - xe^3 + xe \ dx$$

$$\int_0^2 20x^2 - xe^3 + xe \ dx$$

Now we'll take the integral with respect to x.

$$\frac{20x^3}{3} - \frac{1}{2}x^2e^3 + \frac{1}{2}x^2e \Big|_0^2$$

Now, we can evaluate the interval. Remember we took the integral with respect to x so we are evaluating the the integral also with respect to x.

$$\frac{20(2)^3}{3} - \frac{1}{2}(2)^2e^3 + \frac{1}{2}(2)^2e - \left[\frac{20(0)^3}{3} - \frac{1}{2}(0)^2e^3 + \frac{1}{2}(0)^2e\right]$$

$$\frac{20(8)}{3} - \frac{1}{2}(4)e^3 + \frac{1}{2}(4)e - (0 - 0 + 0)$$

$$\frac{160}{3} - 2e^3 + 2e$$

This is the value of the iterated integral, which means it's the volume under the function $f(x, y) = x^2y^3 - xe^y$ over the region $R = [0,2] \times [1,3]$.

Now let's try an example with a double integral, where the intervals for x and y aren't already incorporated into the integral.

Example

Evaluate the double integral.

$$\iint_R y\sin(3x) - y^3\cos x \ dA$$

$$R = \left\{ (x,y) \middle| 0 \le x \le \frac{\pi}{2}, -1 \le y \le 2 \right\}$$

The question is asking us to evaluate a double integral, and they're giving us R. The values in R correspond to the x and y intervals for our double integral, so we can go ahead and insert this information into the double integral to turn it into an iterated integral.

$$\iint_R y\sin(3x) - y^3\cos x \ dA$$

$$\int_0^{\frac{\pi}{2}}\int_{-1}^{2} y\sin(3x) - y^3\cos x \ dy \ dx$$

It doesn't matter if we put dx on the inside and dy on the outside, or vice versa. But we need to make sure that the limits of integration on each integral match the order of dx and dy. Since we put dy on the inside, the limits of integration for y have to get attached to the inside integral. And since dx is on the outside, we put the limits of integration for x on the outside integral.

Since dy is on the inside, and we always work our way inside out, we'll integrate first with respect to y, treating x as a constant.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}y^2\sin(3x) - \frac{1}{4}y^4\cos x \Big|_{y=-1}^{y=2} \ dx$$

Now we'll evaluate over the interval for y.

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}(2)^2\sin(3x) - \frac{1}{4}(2)^4\cos x - \left[\frac{1}{2}(-1)^2\sin(3x) - \frac{1}{4}(-1)^4\cos x\right] \ dx$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}(4)\sin(3x) - \frac{1}{4}(16)\cos x - \left[\frac{1}{2}(1)\sin(3x) - \frac{1}{4}(1)\cos x\right] \ dx$$

$$\int_0^{\frac{\pi}{2}} 2\sin(3x) - 4\cos x - \frac{1}{2}\sin(3x) + \frac{1}{4}\cos x \ dx$$

$$\int_0^{\frac{\pi}{2}} \frac{3}{2}\sin(3x) - \frac{15}{4}\cos x \ dx$$

Next we'll integrate with respect to x, then evaluate over the x-interval.

$$-\frac{3}{2}\left(\frac{1}{3}\right)\cos(3x) - \frac{15}{4}\sin x\Big|_0^{\frac{\pi}{2}}$$

$$-\frac{1}{2}\cos(3x) - \frac{15}{4}\sin x\Big|_0^{\frac{\pi}{2}}$$

$$-\frac{1}{2}\cos\left(3\cdot\frac{\pi}{2}\right)-\frac{15}{4}\sin\frac{\pi}{2}-\left[-\frac{1}{2}\cos(3\cdot 0)-\frac{15}{4}\sin 0\right]$$

$$-\frac{1}{2}\cos\frac{3\pi}{2}-\frac{15}{4}\sin\frac{\pi}{2}+\frac{1}{2}\cos 0+\frac{15}{4}\sin 0$$

$$-\frac{1}{2}(0)-\frac{15}{4}(1)+\frac{1}{2}\cos 0+\frac{15}{4}\sin 0$$

$$-0-\frac{15}{4}+\frac{1}{2}(1)+\frac{15}{4}(0)$$

$$0-\frac{15}{4}+\frac{1}{2}+0$$

$$-\frac{15}{4}+\frac{2}{4}$$

$$-\frac{13}{4}$$

The value of the double integral is $-13/4$, which means the volume under the function $f(x,y)=y\sin(3x)-y^3\cos x$ over the region $R=[0,\pi/2]\times[-1,2]$ is $-13/4$.

Type I and type II regions

We already know that we can use double integrals to find the volume below a surface over some region $R = [a, b] \times [c, d]$.

We can define the region R as Type I, Type II, or a mix of both. Type I curves are curves that can be defined for y in terms of x and lie more or less "above and below" each other. On the other hand, Type II curves are curves that can be defined for x in terms of y and lie more or less "left and right" of each other.

Type I regions can be broken up into vertical slices, and Type II regions can be broken up into horizontal slices.

Sometimes a region can be considered both Type I and Type II, in which case you can choose to evaluate it either way.

Example

Define the region D as Type I or Type II, then find volume below the surface over the region D.

$$\int\int_D x^2 + 6y - 20 \ dA$$

where D is the triangle bounded by $y = 1$, $y = 3x$, and $y = 4 - x$

The first thing we'll do is sketch the region D. It'll be easy if we solve for the intersection points of the three lines.

We'll find the intersection of $y = 1$ and $y = 3x$.

$$3x = 1$$

$$x = \frac{1}{3}$$

Pairing $x = \frac{1}{3}$ with $y = 1$, the intersection point is $\left(\frac{1}{3}, 1\right)$.

We'll find the intersection of $y = 1$ and $y = 4 - x$.

$$4 - x = 1$$

$$-x = -3$$

$$x = 3$$

Pairing $x = 3$ with $y = 1$, the intersection point is $(3,1)$.

We'll find the intersection of $y = 3x$ and $y = 4 - x$.

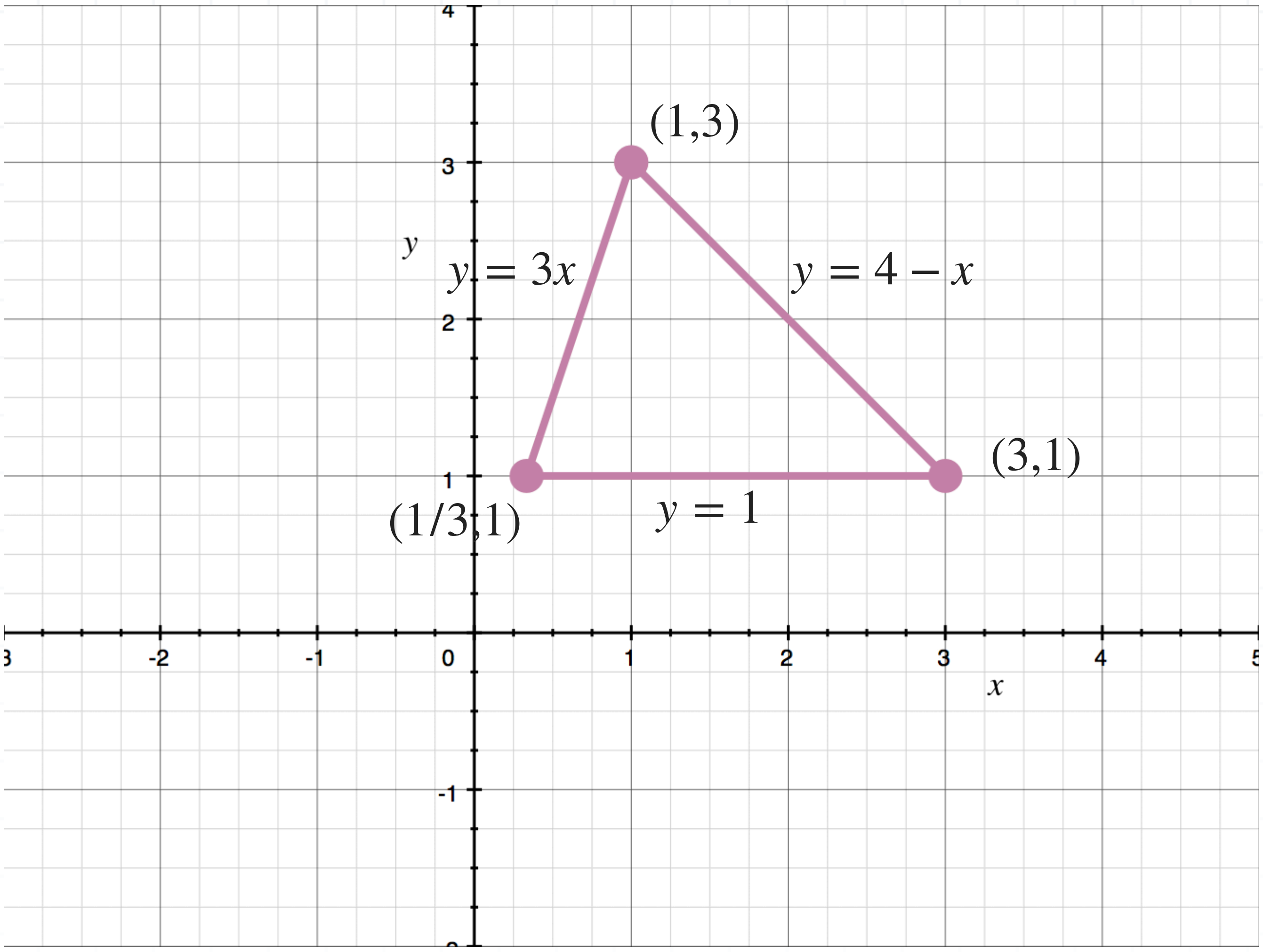
$$3x = 4 - x$$

$$4x = 4$$

$$x = 1$$

Plugging $x = 1$ into $y = 3x$ to see that $y = 3$, the intersection point is $(1,3)$.

If we plot these points and sketch the lines that connect them, we see the triangular region D.



Now we need to decide if D is a Type I or Type II region. Since we can get uniform slices either way (vertical slices if we treat it as a Type I region, or horizontal slices if we treat it as a Type II region), we can choose which type we want to use, and we'll get the same answer with both methods. We'll solve it as a Type II region, which means we'll use horizontal slices of region D.

To solve as Type II, the equations that define the lines that are the edges of region D need to be defined for x in terms of y. If we solve each of them for x we get

$$y = 3x$$

$$x = \frac{1}{3}y$$

and

$$y = 4 - x$$

$$x = 4 - y$$

For a Type II region, we'll integrate first with respect to x, then with respect to y. The upper limit of integration for x is given by $x = 4 - y$ (because that line defines the top of each of our horizontal slices), and the lower limit of integration for x is given by $x = (1/3)y$ (because that line defines the bottom of each of our horizontal slices). For the region D, y is defined between $y = 1$ and $y = 3$ ($y = 3$ comes from the intersection point $(1,3)$). So the original integral becomes

$$\iint_D x^2 + 6y - 20\ dA$$

$$\int_1^3 \int_{\frac{y}{3}}^{4-y} x^2 + 6y - 20\ dx\ dy$$

Evaluating from the inside out and therefore integrating with respect to x first, we get

$$\int_1^3 \frac{1}{3}x^3 + 6xy - 20x \Big|_{x=\frac{y}{3}}^{x=4-y}\ dy$$

$$\int_1^3 \frac{1}{3}(4-y)^3 + 6(4-y)y - 20(4-y) - \left[\frac{1}{3}\left(\frac{y}{3}\right)^3 + 6\left(\frac{y}{3}\right)y - 20\left(\frac{y}{3}\right)\right]\ dy$$

$$\int_1^3 \frac{1}{3}(4-y)(4-y)(4-y) + 6y(4-y) - 80 + 20y - \left[\frac{1}{3}\left(\frac{y^3}{27}\right) + 6y\left(\frac{y}{3}\right) - \frac{20y}{3}\right]\ dy$$

$$\int_1^3 \frac{1}{3}\left(16-8y+y^2\right)(4-y)+24y-6y^2-80+20y-\left(\frac{y^3}{81}+\frac{6y^2}{3}-\frac{20y}{3}\right)\ dy$$

$$\int_1^3 \frac{1}{3}\left(64-16y-32y+8y^2+4y^2-y^3\right)+24y-6y^2-80+20y-\frac{1}{81}y^3-2y^2+\frac{20}{3}y\ dy$$

$$\int_1^3 \frac{1}{3}\left(64-48y+12y^2-y^3\right)-\frac{1}{81}y^3-8y^2+44y+\frac{20}{3}y-80\ dy$$

$$\int_1^3 \frac{64}{3}-16y+4y^2-\frac{1}{3}y^3-\frac{1}{81}y^3-8y^2+\frac{132}{3}y+\frac{20}{3}y-80\ dy$$

$$\int_1^3 -\frac{1}{3}y^3-\frac{1}{81}y^3+4y^2-8y^2-16y+\frac{132}{3}y+\frac{20}{3}y+\frac{64}{3}-80\ dy$$

$$\int_1^3 -\frac{27}{81}y^3-\frac{1}{81}y^3-4y^2-\frac{48}{3}y+\frac{132}{3}y+\frac{20}{3}y+\frac{64}{3}-\frac{240}{3}\ dy$$

$$\int_1^3 -\frac{28}{81}y^3-4y^2+\frac{104}{3}y-\frac{176}{3}\ dy$$

Integrating with respect to y, we get

$$-\frac{28}{81}\left(\frac{1}{4}\right)y^4-4\left(\frac{1}{3}\right)y^3+\frac{104}{3}\left(\frac{1}{2}\right)y^2-\frac{176}{3}y\Big|_1^3$$

$$-\frac{28}{324}y^4-\frac{4}{3}y^3+\frac{104}{6}y^2-\frac{176}{3}y\Big|_1^3$$

$$-\frac{7}{81}y^4-\frac{4}{3}y^3+\frac{52}{3}y^2-\frac{176}{3}y\Big|_1^3$$

$$-\frac{7}{81}(3)^4-\frac{4}{3}(3)^3+\frac{52}{3}(3)^2-\frac{176}{3}(3)-\left[-\frac{7}{81}(1)^4-\frac{4}{3}(1)^3+\frac{52}{3}(1)^2-\frac{176}{3}(1)\right]$$

$$-\frac{7}{81}(81)-\frac{4}{3}(27)+\frac{52}{3}(9)-\frac{176}{3}(3)+\frac{7}{81}+\frac{4}{3}-\frac{52}{3}+\frac{176}{3}$$

$$-\frac{7}{1}(1)-\frac{4}{1}(9)+\frac{52}{1}(3)-\frac{176}{1}(1)+\frac{7}{81}+\frac{4}{3}-\frac{52}{3}+\frac{176}{3}$$

$$-7-36+156-176+\frac{7}{81}+\frac{128}{3}$$

$$-63+\frac{7}{81}+\frac{128}{3}$$

$$-\frac{5,103}{81}+\frac{7}{81}+\frac{3,456}{81}$$

$$-\frac{1,640}{81}$$

We can say that the volume under the curve $z=x^2+6y-20$ over the region D is

$$-\frac{1,640}{81}$$

The fact that the volume is negative means that there is more volume enclosed by the curve and the xy-plane that lies below the xy-plane than the amount that lies above the xy-plane.

Finding surface area

Most often with these kinds of surface area problems, we'll be asked to find the surface area of "the part of function A that lies inside function B". When this is the case, let's think about the function that's "inside" as the primary function, and the function that's "outside" as the secondary function.

To find surface area of the primary, multivariable function, we can use the formula

$$A(s) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \, dA$$

where D represents the secondary function and $\partial z/\partial x$ and $\partial z/\partial y$ are partial derivatives of the primary function. Whether we integrate first with respect to x or first with respect to y depends on the object D.

Example

Find the surface area of the part of $z = xy$ that lies inside $x^2 + y^2 = 4$.

Because we want to find the surface area of the part of $z = xy$ that's inside $x^2 + y^2 = 4$, we'll call $z = xy$ the primary function and $x^2 + y^2 = 4$ the secondary function.

Then we'll find first-order partial derivatives of the primary function, $z = xy$.

$$\frac{\partial z}{\partial x} = y$$

$$\frac{\partial z}{\partial y} = x$$

We need to figure out if the region we're interested in is type I or type II. Since the region for which we're finding surface area is defined by the secondary function $x^2 + y^2 = 4$, and this is a simple circle, we can treat the region as type I. We could choose either order, but we'll decide to integrate first with respect to y and then x, which means that so far, the integral looks like

$$A(s) = \int\int_D \sqrt{1 + y^2 + x^2}\ dy\ dx$$

All we need now are the limits of integration, which will come from $x^2 + y^2 = 4$. Solving this equation for y in terms of x in order to get limits of integration with respect to y gives

$$x^2 + y^2 = 4$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4 - x^2}$$

Remember, we had to find limits of integration for y with respect to x because we're integrating first with respect to y, and after we do, we need to have x variables left over so that we can then integrate with respect to x. Now the integral is

$$A(s) = \int \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+y^2+x^2}\ dy\ dx$$

Next we can solve for the limits of integration with respect to x. We want these limits to be constants so that, after we integrate with respect to x and evaluate over the interval, we end up with a constant for the surface area of the region. Since we're looking for the surface area inside the circle with radius 2, we know that x is defined over the interval $[-2,2]$. So the integral becomes

$$A(s) = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+y^2+x^2}\ dy\ dx$$

Now that we have all the pieces in place, we can work on solving the integral. The easiest way to solve it will be converting it to polar form. If we remember our conversion formulas, we know that $r^2 = x^2 + y^2$ and that $dy\ dx = r\ dr\ d\theta$. Making these substitutions gives

$$A(s) = \int_{x=-2}^{x=2} \int_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}} \sqrt{1+r^2}\ r\ dr\ d\theta$$

Next we need to change the limits of integration into polar form. The limits on the inner integral need to match dr, which means they have to be with respect to r. Since we're dealing with the circle centered at the origin with radius 2, r is defined over $[0,2]$. So

$$A(s) = \int_{x=-2}^{x=2} \int_{0}^{2} \sqrt{1+r^2}\ r\ dr\ d\theta$$

The limits on the outer integral need to match $d\theta$, which means they have to be with respect to θ. Since we're dealing with the circle centered at the origin with radius 2, θ is defined over $[0,2\pi]$. So

$$A(s) = \int_0^{2\pi} \int_0^2 \sqrt{1+r^2}\ r\ dr\ d\theta$$

To evaluate the resulting integral, we'll use a u-substitution with

$$u = 1 + r^2$$

$$du = 2r\ dr$$

$$dr = \frac{du}{2r}$$

Making these substitutions into the double integral gives

$$A(s) = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \sqrt{u}\ r \left(\frac{du}{2r}\right)\ d\theta$$

$$A(s) = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \frac{1}{2}\sqrt{u}\ du\ d\theta$$

$$A(s) = \int_{\theta=0}^{\theta=2\pi} \int_{r=0}^{r=2} \frac{1}{2}u^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u.

$$A(s) = \int_{\theta=0}^{\theta=2\pi} \frac{1}{3}u^{\frac{3}{2}}\Big|_{r=0}^{r=2}\ d\theta$$

Now, we can substitute back in for u using $u = 1 + r^2$.

$$A(s)=\int_{\theta=0}^{\theta=2\pi}\frac{1}{3}\left(1+r^2\right)^{\frac{3}{2}}\Big|_{r=0}^{r=2}\ d\theta$$

Evaluate the inner integral.

$$A(s)=\int_0^{2\pi}\frac{1}{3}\left[\left(1+2^2\right)^{\frac{3}{2}}-\left(1+0^2\right)^{\frac{3}{2}}\right]\ d\theta$$

$$A(s)=\int_0^{2\pi}\frac{1}{3}\left(5^{\frac{3}{2}}-1^{\frac{3}{2}}\right)\ d\theta$$

$$A(s)=\int_0^{2\pi}\frac{\sqrt{125}}{3}-\frac{1}{3}d\theta$$

Integrate the outer integral.

$$A(s)=\frac{\sqrt{125}}{3}\theta-\frac{1}{3}\theta\Big|_0^{2\pi}$$

Evaluate the outer integral

$$A(s)=\left(\frac{\sqrt{125}}{3}(2\pi)-\frac{1}{3}(2\pi)\right)-\left(\frac{\sqrt{125}}{3}(0)-\frac{1}{3}(0)\right)$$

$$A(s)=\frac{2\pi\sqrt{125}}{3}-\frac{2\pi}{3}$$

$$A(s)=\frac{2\pi\sqrt{125}-2\pi}{3}$$

$$A(s)=\frac{2\pi}{3}\left(\sqrt{125}-1\right)$$

$$A(s)=\frac{2\pi}{3}\left(5\sqrt{5}-1\right)$$

This is the surface area of the part of $z=xy$ that lies inside $x^2+y^2=4$.

Finding volume

We already know that we can use double integrals to find the volume below a function over some region $R = [a,b] \times [c,d]$.

We use the double integral formula

$$V = \iint_D f(x,y)\ dA$$

to find volume, where D represents the region over which we're integrating, and $f(x,y)$ is the curve below which we want to find volume. We need to turn the double integral into an iterated integral by finding limits of integration for x and y.

Example

Find volume below the function over the region D.

$$z = 2xy$$

where D is the triangle bounded by the lines $y = 1$, $x = 1$, and $y = 3 - x$

The first thing we'll do is sketch the region D. It'll be easy if we solve for the intersection points of the three lines.

We'll find the intersection of $y = 1$ and $x = 1$.

Pairing $x = 1$ with $y = 1$, the intersection point is $(1,1)$.

We'll find the intersection of $y=1$ and $y=3-x$.

$$3-x=1$$

$$-x=-2$$

$$x=2$$

Pairing $x=2$ with $y=1$, the intersection point is $(2,1)$.

We'll find the intersection of $x=1$ and $y=3-x$.

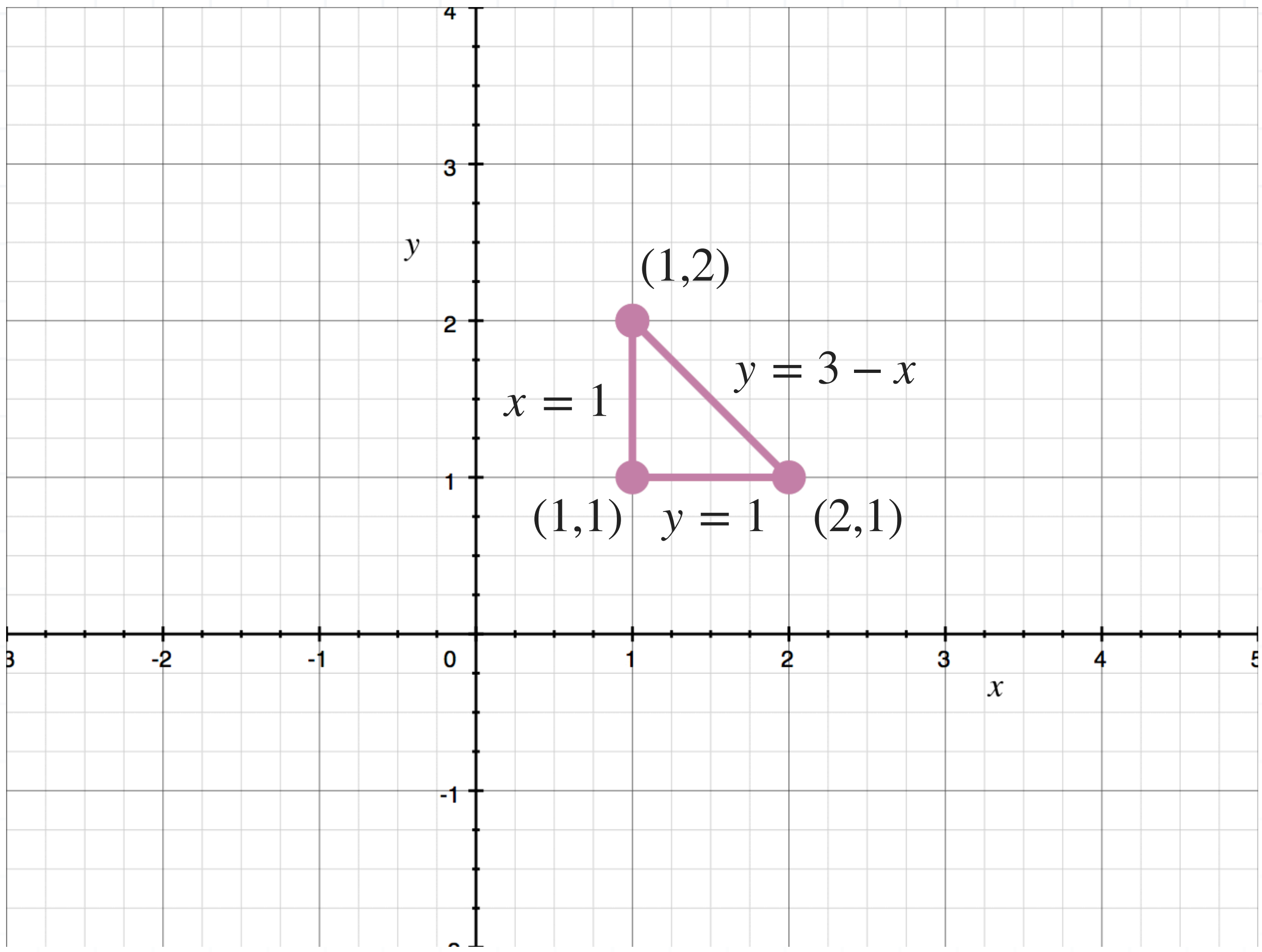
$$y=3-x$$

$$y=3-1$$

$$y=2$$

Pairing $x=1$ with $y=2$, the intersection point is $(1,2)$.

If we plot these points and sketch the lines that connect them, we see the triangular region D.



Since we only have one complex equation, $y = 3 - x$ and it's solved for y, we'll integrate with respect to y first, which means we'll treat this as a Type I integral, and so the inner integral will have limits of integration for y.

If we divide the triangular region D into vertical slices, the tops of those slices are defined by the line $y = 3 - x$, and the bottoms are defined by $y = 1$. Looking at the sketch of region D, we can see that x is defined on $[1,2]$. Therefore, we'll get

$$V = \iint_D f(x, y)\ dA$$

$$V = \int_1^2 \int_1^{3-x} 2xy\ dy\ dx$$

We'll integrate first with respect to y.

$$V=\int_1^2 xy^2\Big|_{y=1}^{y=3-x}\ dx$$

$$V=\int_1^2 x(3-x)^2-x(1)^2\ dx$$

$$V=\int_1^2 x\left(9-6x+x^2\right)-x\ dx$$

$$V=\int_1^2 9x-6x^2+x^3-x\ dx$$

$$V=\int_1^2 8x-6x^2+x^3\ dx$$

Then we'll integrate with respect to x.

$$V=4x^2-2x^3+\frac{1}{4}x^4\Big|_1^2$$

$$V=4(2)^2-2(2)^3+\frac{1}{4}(2)^4-\left[4(1)^2-2(1)^3+\frac{1}{4}(1)^4\right]$$

$$V=16-16+4-\left(4-2+\frac{1}{4}\right)$$

$$V=2-\frac{1}{4}$$

$$V=\frac{7}{4}$$

We can say that the volume under the curve $z=2xy$ over the region D is 7/4.

Changing iterated and double integrals to polar coordinates

To change an iterated integral to polar coordinates we'll need to convert the function itself, the limits of integration, and the differential.

To change the function and limits of integration from rectangular coordinates to polar coordinates, we'll use the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

Remember also that when you convert dA or $dy\ dx$ to polar coordinates, it converts as

$$dA = dy\ dx = r\ dr\ d\theta$$

If we start with a double integral, we'll need to first evaluate it, select which region type it is and then set up our limits of integration. After this, our double integral is an iterated integral so we just use the same techniques to solve for the polar coordinates.

Example

Convert the double integral from rectangular coordinates to polar coordinates.

$$\iint_D \cos\left(x^2 + y^2\right)\ dA$$

D is bounded by $y = \pm\sqrt{25 - x^2}$

First we'll need to convert our double integral into an iterated integral. The first step in this is to evaluate D to determine what region type it is. If we rearrange D we'll get $x^2 + y^2 = 25$, which means that the region is a circle that we can treat as a type I region.

This means that our outer integral will be with respect to x and our inner integral will be with respect to y. We can start by finding the limits of integration of our inner integral. We have actually been given this information which is $y = \pm\sqrt{25 - x^2}$. We can put this into our integral.

$$\int \int_{-\sqrt{25 - x^2}}^{\sqrt{25 - x^2}} \cos\left(x^2 + y^2\right)\ dy\ dx$$

Next we can find the limits of integration of our outer integral. Since our function is a circle with radius 5, our limits of integration will be ± 5. We can put this into our integral.

$$\int_{-5}^{5} \int_{-\sqrt{25 - x^2}}^{\sqrt{25 - x^2}} \cos\left(x^2 + y^2\right)\ dy\ dx$$

Now we're ready to convert it to polar form. Using the conversion formulas, we get

$$\int_{-5}^{5} \int_{-\sqrt{25 - x^2}}^{\sqrt{25 - x^2}} \cos\left(r^2\right) r\ dr\ d\theta$$

Next we need to change the limits of integration into polar form. We can start with the inner integral which is with respect to r. The lowest value that r can have in our function is 0 and the highest value r can have is the radius of the circle which is 5. We can put these into our integral now.

$$\int_{-5}^{5}\int_{0}^{5}\cos\left(r^2\right)r\ dr\ d\theta$$

Now, we can change the limits of our outer integral. The outer integral is now with respect to θ. Since our function is a circle, the lowest angle it can have is 0 and the highest angle it can have is 2π. We can place these into our integral.

$$\int_{0}^{2\pi}\int_{0}^{5}r\cos\left(r^2\right)\ dr\ d\theta$$

This is the same double integral we started with, except that we've converted it from rectangular coordinates to polar coordinates.

Sketching area

To sketch the area of integration of a double polar integral, you'll need to analyze the function and evaluate both sets of limits separately. Remember, you'll need to sketch the polar function on polar coordinate axes, where the r values represent the radius of a circle and the θ values will produce straight lines.

Example

Sketch the area given by the double polar integral.

$$\int_{\frac{\pi}{3}}^{\frac{3\pi}{4}}\int_{1}^{3} r\ dr\ d\theta$$

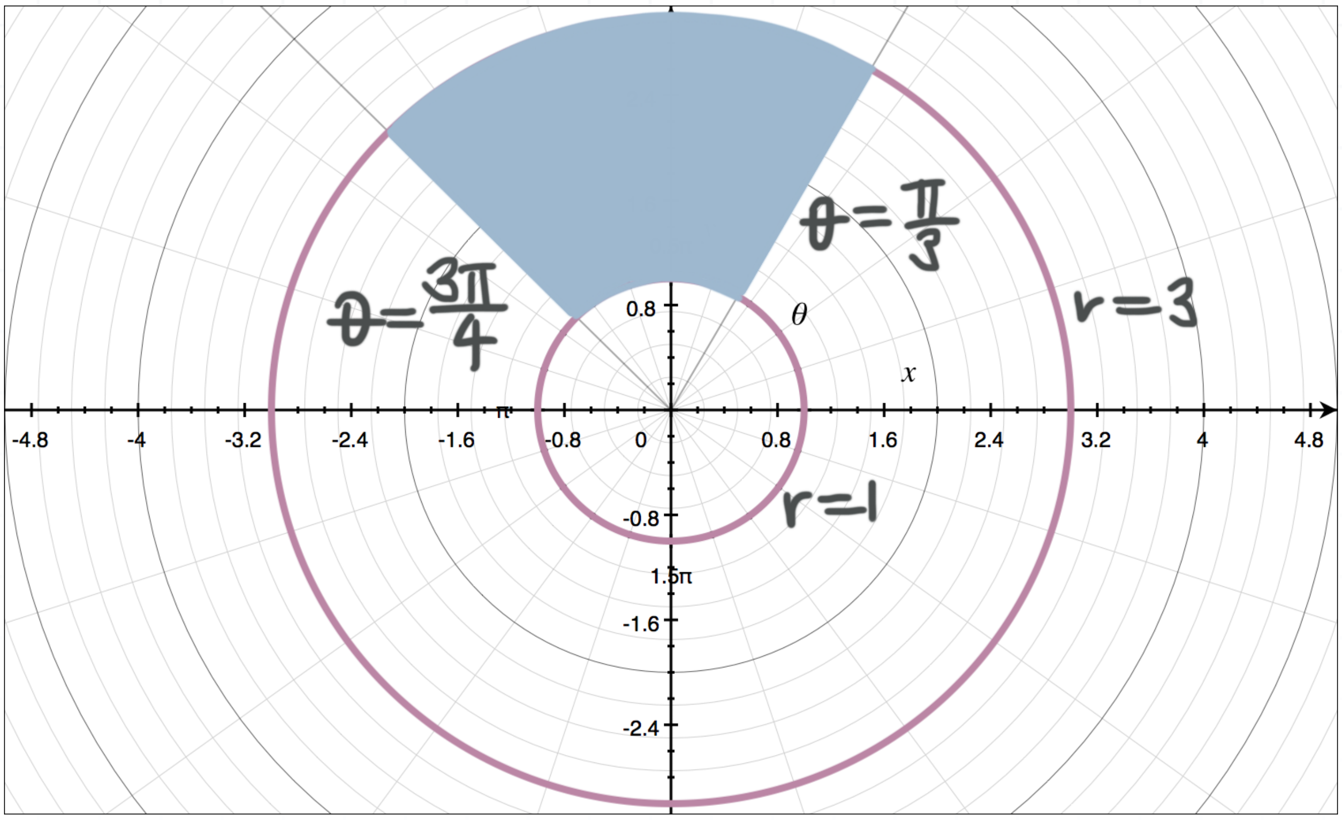
To sketch the area defined by the double polar integral, we'll have to look at the function and both sets of limits of integration.

Let's first look at our inner integral. These are the limits with respect to r which means that on a polar graph these limits are simply the radius of two separate circles. This means that our function exists in the space between the two circles. The inner circle has a radius of 1 and the outer circle has a radius of 3.

Next we can analyze the outer integral. These limits are with respect to θ. On a polar graph, the angular points can be represented by straight lines from the origin.

Looking at our integral we can see that we'll have one line going through $\pi/3$ and the second line going through $3\pi/4$.

This means that our area of integration is bounded by the two circles and the two lines. The graph of this area looks like this



Finding area

You can use a double integral to find the area inside a polar curve. Assuming the function itself and the limits of integration are already in polar form, you'll be able to evaluate the iterated integral directly. Otherwise, if either the function and/or the limits of integration are in rectangular form, you'll need to convert to polar before you evaluate.

If you don't have limits of integration, often the best way to find them is to sketch the function so that you can identify the intervals for r and θ over which the function is defined.

Example

Find the area given by the double polar integral.

$$\int_0^{\frac{\pi}{2}}\int_1^3 r\ dr\ d\theta$$

Since the function and the limits of integration are already in terms of polar coordinates, we just need to evaluate the iterated integral. First we'll integrate with respect to r.

$$\int_0^{\frac{\pi}{2}}\int_1^3 r\ dr\ d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}r^2\Big|_1^3\ d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{2}(3)^2 - \frac{1}{2}(1)^2 \ d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{9}{2} - \frac{1}{2} \ d\theta$$

$$\int_0^{\frac{\pi}{2}} \frac{8}{2} \ d\theta$$

$$\int_0^{\frac{\pi}{2}} 4 \ d\theta$$

Now we'll integrate with respect to θ.

$$4\theta\Big|_0^{\frac{\pi}{2}}$$

$$4\left(\frac{\pi}{2}\right) - 4(0)$$

$$2\pi$$

This is the area defined by the double polar integral.

Finding volume

To evaluate a double polar integral, we'll need to make sure that both function and the limits of integration are in terms of polar coordinates (r, θ).

If we're given a double integral in rectangular coordinates and asked to evaluate it as a double polar integral, we'll need to convert the function and the limits of integration from rectangular coordinates (x, y) to polar coordinates (r, θ), and then evaluate the integral.

To change the function and limits of integration from rectangular coordinates to polar coordinates, we'll use the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$r^2 = x^2 + y^2$$

Remember also that when you convert dA or $dy\ dx$ to polar coordinates, it converts as

$$dA = dy\ dx = r\ dr\ d\theta$$

Example

Use a double polar integral to find the volume of the solid enclosed by the given curves.

$$z = \sqrt{x^2 + y^2 + 4}$$

$$z = 4$$

First, we'll convert both functions to polar coordinates and get

$$z = \sqrt{r^2 + 4}$$

$$z = 4$$

Now we want to set the equations equal to each other to get points of intersection, so that we can figure out the limits of integration.

$$\sqrt{r^2 + 4} = 4$$

$$r^2 + 4 = 16$$

$$r^2 = 12$$

$$r = \pm\sqrt{12}$$

$$r = \pm 2\sqrt{3}$$

Since r represents distance from the origin in the polar system, and radius can't be negative, we change the $-2\sqrt{3}$ to 0 and we can say that the limits of integration for r are $0 \le r \le 2\sqrt{3}$.

Next we can solve for the limits of integration with respect to θ. Since we got $r^2 = 12$, which is the same as $x^2 + y^2 = 12$, we know the region is a circle, which means the limits of integration for θ will be a full 360°, or $0 \le \theta \le 2\pi$.

Now we need to build our function. We need to figure out which function is further from the origin (has the outer radius) and which one is closer to the origin (has the inner radius). We know that $z=4$ will always stay the same, no matter what the value of r. But the value of $z=\sqrt{r^2+4}$ can change, depending on the value of r that we choose. If we plug in the upper and lower limits of integration, we get

$$z=\sqrt{\left(\sqrt{12}\right)^2+4}$$

$$z=\sqrt{16}$$

$$z=4$$

and

$$z=\sqrt{(0)^2+4}$$

$$z=\sqrt{4}$$

$$z=2$$

Since $z=\sqrt{r^2+4}$ will sometimes have a value smaller than $z=4$ (since we just found that $z=2$ when we substitute $r=0$), we'll treat $z=4$ as the outer function and $z=\sqrt{r^2+4}$ as the inner function. Therefore, we get

$$V=\int\int_D\left(4-\sqrt{r^2+4}\right)r\ dr\ d\theta$$

Adding the limits of integration, we get

$$V=\int_0^{2\pi}\int_0^{2\sqrt{3}}\left(4-\sqrt{r^2+4}\right)r\ dr\ d\theta$$

$$V=\int_0^{2\pi}\int_0^{2\sqrt{3}}4r-r\sqrt{r^2+4}\ dr\ d\theta$$

Now we'll evaluate the integral by splitting it into two separate integrals so that we can use u-substitution on the second one.

$$V=\int_0^{2\pi}\int_0^{2\sqrt{3}}4r\ dr\ d\theta-\int_0^{2\pi}\int_0^{2\sqrt{3}}r\sqrt{r^2+4}\ dr\ d\theta$$

$$u=r^2+4$$

$$du=2r\ dr$$

$$dr=\frac{du}{2r}$$

$$V=\int_0^{2\pi}2r^2\Big|_{r=0}^{r=2\sqrt{3}}\ d\theta-\int_0^{2\pi}\int_{r=0}^{r=2\sqrt{3}}r\sqrt{u}\left(\frac{du}{2r}\right)\ d\theta$$

$$V=\int_0^{2\pi}2r^2\Big|_{r=0}^{r=2\sqrt{3}}\ d\theta-\int_0^{2\pi}\int_{r=0}^{r=2\sqrt{3}}\frac{1}{2}u^{\frac{1}{2}}\ du\ d\theta$$

$$V=\int_0^{2\pi}2r^2\Big|_{r=0}^{r=2\sqrt{3}}\ d\theta-\int_0^{2\pi}\frac{1}{2}\left(\frac{2}{3}\right)u^{\frac{3}{2}}\Big|_{r=0}^{r=2\sqrt{3}}\ d\theta$$

$$V=\int_0^{2\pi}2r^2\Big|_{r=0}^{r=2\sqrt{3}}\ d\theta-\int_0^{2\pi}\frac{1}{3}(r^2+4)^{\frac{3}{2}}\Big|_{r=0}^{r=2\sqrt{3}}\ d\theta$$

$$V=\int_0^{2\pi}2\left(2\sqrt{3}\right)^2-2(0)^2\ d\theta-\int_0^{2\pi}\frac{1}{3}\left[\left(2\sqrt{3}\right)^2+4\right]^{\frac{3}{2}}-\frac{1}{3}\left[(0)^2+4\right]^{\frac{3}{2}}\ d\theta$$

$$V=\int_0^{2\pi}2(12)\ d\theta-\int_0^{2\pi}\frac{1}{3}(12+4)^{\frac{3}{2}}-\frac{1}{3}(4)^{\frac{3}{2}}\ d\theta$$

$$V=\int_0^{2\pi}24\ d\theta-\int_0^{2\pi}\frac{1}{3}\left(16\right)^{\frac{3}{2}}-\frac{1}{3}(4)^{\frac{3}{2}}\ d\theta$$

$$V=\int_0^{2\pi}24\ d\theta-\int_0^{2\pi}\frac{1}{3}\left(64\right)-\frac{1}{3}(8)\ d\theta$$

$$V=\int_0^{2\pi}24\ d\theta-\int_0^{2\pi}\frac{64}{3}-\frac{8}{3}\ d\theta$$

$$V=\int_0^{2\pi}24\ d\theta-\int_0^{2\pi}\frac{56}{3}\ d\theta$$

Now we'll put the integrals back together.

$$V=\int_0^{2\pi}24-\frac{56}{3}\ d\theta$$

$$V=\int_0^{2\pi}\frac{72}{3}-\frac{56}{3}\ d\theta$$

$$V=\int_0^{2\pi}\frac{16}{3}\ d\theta$$

And then we'll integrate with respect to θ.

$$V=\frac{16}{3}\theta\Big|_0^{2\pi}$$

$$V = \frac{16}{3}(2\pi) - \frac{16}{3}(0)$$

$$V = \frac{32\pi}{3}$$

This is the volume of the solid enclosed by the curves.

Midpoint rule for triple integrals

We can approximate the value of a triple integral using midpoint rule for triple integrals. Similarly to the way we used midpoints to approximate single integrals by taking the midpoint at the top of each approximating rectangle, and to the way we used midpoints to approximate double integrals by taking the midpoint at the top of each approximating prism, we can use midpoints to approximate a triple integral by taking the midpoint of each sub-cube.

The midpoint rule for triple integrals is

$$\iiint_B f(x,y,z)\ dV \approx \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{o} f(\overline{x_i},\overline{y_j},\overline{z_k})\ \Delta V$$

where B is the region for which we want to find volume, m is the number of subintervals in the x direction, n is the number of subintervals in the y direction, o is the number of subintervals in the z direction, and ΔV is the volume of one of the sub-cubes.

Once we define the boundaries of each sub-cube, we can take the midpoints of each of them, and then plug them into the midpoint formula, along with ΔV.

Example

Use midpoint rule with 8 equal sub-cubes to estimate the mass of the triple integral.

$$\iiint_B \sin(2xyz)\ dV$$

where $B = \{(x, y, z)\,|\,0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1\}$

B is the cube with one corner at the origin, and side length 1, such that $B = [0,1] \times [0,1] \times [0,1]$. Since we need to use 8 sub-cubes, that means we'll have to divide each side of the cube in half to find the edges of the sub-boxes.

For example, the sub-cube whose corner is on the origin is defined by

$$B_1 = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$$

If we head in the positive direction of the x-axis and take the second sub-cube, it's defined by

$$B_2 = \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right]$$

Our 8 sub-cubes can be defined as

$$B_1 = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \qquad B_5 = \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]$$

$$B_2 = \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \qquad B_6 = \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right]$$

$$B_3 = \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right] \qquad B_7 = \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right] \times \left[\frac{1}{2}, 1\right]$$

$$B_4=\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right] \qquad B_8=\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]\times\left[\frac{1}{2},1\right]$$

To find midpoint of each cube, we just take the value halfway between the edges of each interval, such that the midpoints are

$$B_1\left(\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) \qquad B_5\left(\frac{1}{4},\frac{1}{4},\frac{3}{4}\right)$$

$$B_2\left(\frac{3}{4},\frac{1}{4},\frac{1}{4}\right) \qquad B_6\left(\frac{3}{4},\frac{1}{4},\frac{3}{4}\right)$$

$$B_3\left(\frac{1}{4},\frac{3}{4},\frac{1}{4}\right) \qquad B_7\left(\frac{1}{4},\frac{3}{4},\frac{3}{4}\right)$$

$$B_4\left(\frac{3}{4},\frac{3}{4},\frac{1}{4}\right) \qquad B_8\left(\frac{3}{4},\frac{3}{4},\frac{3}{4}\right)$$

Next we need to solve for ΔV. We'll use the dimensions of one of the smaller cubes to find ΔV.

$$V_{B_1}=\Delta V=\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)$$

$$\Delta V=\frac{1}{8}$$

Plugging everything into the midpoint formula, we find that the mass is approximated by

$$\frac{1}{8}\left\{\sin\left[2\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\right]+\sin\left[2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\right]\right.$$

$$+\sin\left[2\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\right]+\sin\left[2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\right]$$

$$+\sin\left[2\left(\frac{1}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\right]+\sin\left[2\left(\frac{3}{4}\right)\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\right]$$

$$+\sin\left[2\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right]+\sin\left[2\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\left(\frac{3}{4}\right)\right]\Bigg\}$$

$$\frac{1}{8}\Bigg[\sin\left(\frac{1}{32}\right)+\sin\left(\frac{3}{32}\right)+\sin\left(\frac{3}{32}\right)+\sin\left(\frac{9}{32}\right)$$

$$+\sin\left(\frac{3}{32}\right)+\sin\left(\frac{9}{32}\right)+\sin\left(\frac{9}{32}\right)+\sin\left(\frac{27}{32}\right)\Bigg]$$

$$\frac{1}{8}\left[\sin\left(\frac{1}{32}\right)+3\sin\left(\frac{3}{32}\right)+3\sin\left(\frac{9}{32}\right)+\sin\left(\frac{27}{32}\right)\right]$$

$$\frac{1}{8}(0.0312+0.2808+0.8327+0.7471)$$

$$\frac{1}{8}(1.8918)$$

$$0.2365$$

This is the mass within the cube B, which we found using midpoint rule for triple integrals.

Iterated and triple integrals

Iterated integrals are double or triple integrals whose limits of integration are already specified. An iterated triple integral might look like

$$\int_0^1 \int_1^3 \int_0^2 x^2y^3 \ dx \ dy \ dz$$

In this case, because the integral ends in $dx \ dy \ dz$ and we always integrate "inside out", we'd integrate first with respect to x, then with respect to y, and lastly with respect to z, evaluating over the associated interval after each integration.

You'll also see triple integrals in which the limits of integration have not yet been specified, like

$$\iiint_E 7xy^2 \ dV$$

Instead of a *triple iterated integral* (where the word *iterated* indicates the presence of the limits of integration), we just call this a *triple integral.* Since we can't solve the triple integral without finding limits of integration, we'll calculate limits of integration for each variable, then add them into the triple integral to turn it into a triple iterated integral.

Let's try solving a triple iterated integral.

Example

Evaluate the iterated integral.

$$\int_0^1 \int_1^3 \int_0^2 x^2y^3 \ dx \ dy \ dz$$

We always work our way “inside out” in order to evaluate iterated integrals. Since dx is listed closest to the “inside”, we know we have to integrate with respect to x first, and that the limits of integration $[2,0]$ on the innermost integral are associated with x.

Integrating with respect to x (keeping y and z constant), and then evaluating over the interval $[2,0]$ gives

$$\int_0^1 \int_1^3 \left(\frac{1}{3}x^3y^3 \Big|_{x=0}^{x=2} \right) \ dy \ dz$$

$$\int_0^1 \int_1^3 \left[\frac{1}{3}(2)^3y^3 - \frac{1}{3}(0)^3y^3 \right] \ dy \ dz$$

$$\int_0^1 \int_1^3 \frac{8}{3}y^3 \ dy \ dz$$

Since dy is listed next, we know we have to integrate with respect to y (keeping z constant), and then evaluate over the interval $[1,3]$.

$$\int_0^1 \left(\frac{2}{3}y^4 \Big|_{y=1}^{y=3} \right) \ dz$$

$$\int_0^1 \left[\frac{2}{3}(3)^4 - \frac{2}{3}(1)^4 \right] \ dz$$

$$\int_0^1 \left(\frac{162}{3} - \frac{2}{3}\right)\ dz$$

$$\int_0^1 \frac{160}{3}\ dz$$

Since dz is listed last, we know we have to integrate with respect to z, and then evaluate over the interval $[0,1]$.

$$\frac{160}{3}z\Big|_0^1$$

$$\frac{160}{3}(1) - \frac{160}{3}(0)$$

$$\frac{160}{3}$$

This is the value of the triple iterated integral.

Now let's do a triple integral without limits of integration to see how it's different.

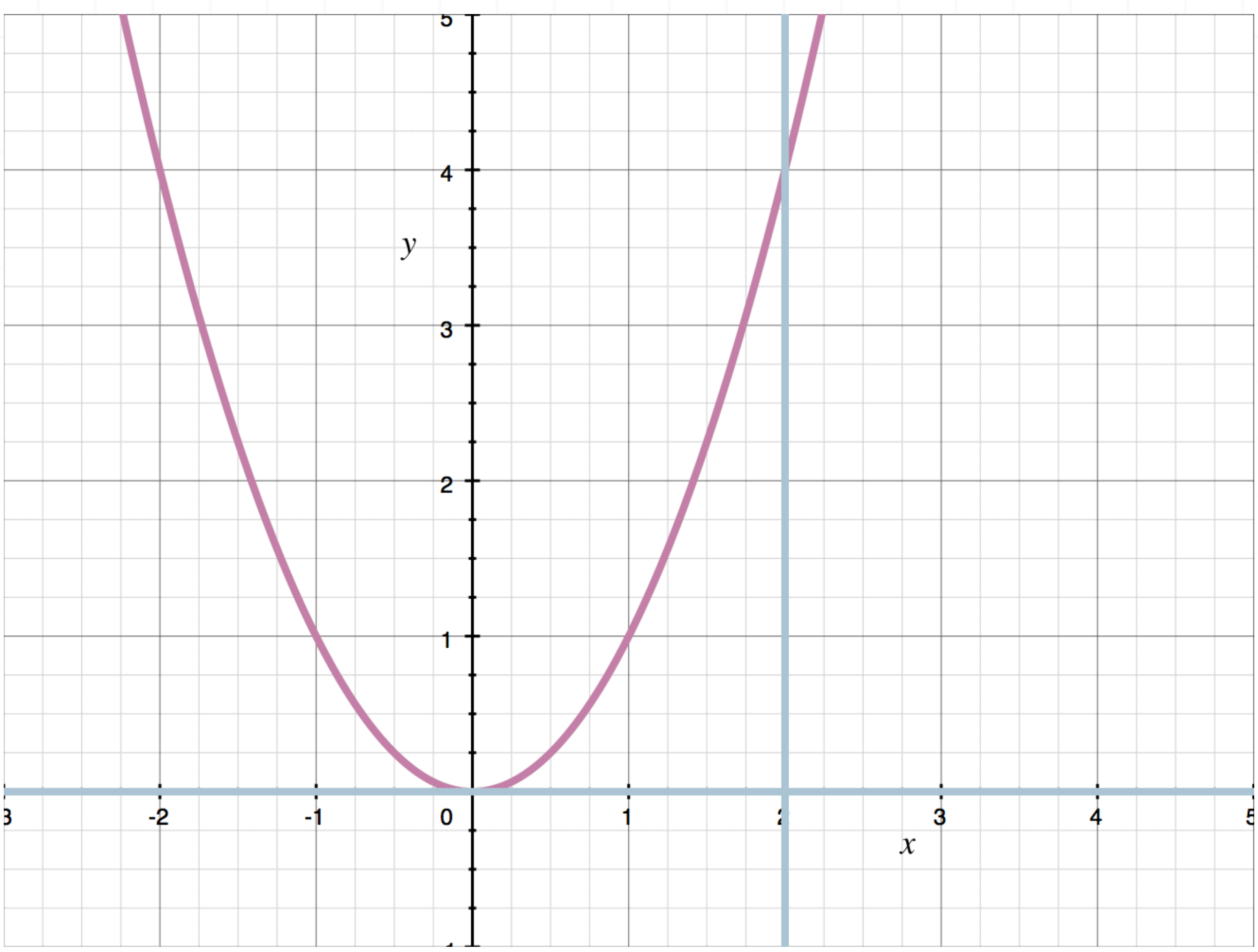
Example

Evaluate the triple integral if E is the region below $z = x + y - 1$ but above the region bounded by $y = x^2$, $y = 0$ and $x = 2$.

$$\iiint_E 7xy^2\ dV$$

We know that the region E lies below $z = x + y - 1$ but above the region bounded by $y = x^2$, $y = 0$ and $x = 2$.

If we graph the region bounded by $y = x^2$, $y = 0$ and $x = 2$, we can see that it lies in the xy-plane.



Since we're looking for the volume directly above the that planar region and not outside it, we can use the planar region to define the limits of integration for x and y.

Based on the graph, we can see that x is defined on the interval $[0,2]$, and that y is defined on the interval $\left[0,x^2\right]$.

Since the volume we're solving for sits on top of the region graphed above in the xy-plane, we know that the lower limit of integration for z is 0. Since the volume lies under $z = x + y - 1$, the upper limit of integration will be $x + y - 1$, which means that z is defined on the interval $[0, x + y - 1]$.

Generally speaking, we put the most complicated limits of integration on the innermost integral, and the simplest limits of integration on the outermost integral. Since the limits of integration for z are defined in terms of two variables, we'll put those on the innermost integral. The limits of integration for y are defined in terms of one variable, so those will come next. Since the limits of integration for x are constants, those will come last on the outermost integral.

$$\iiint_E 7xy^2 \, dV = \int_0^2 \int_0^{x^2} \int_0^{x+y-1} 7xy^2 \, dz \, dy \, dx$$

Now we're dealing with a triple iterated integral, which we already know how to solve. We'll start by integrating with respect to z (holding x and y constant), and then we'll evaluate over the interval for z.

$$\int_0^2 \int_0^{x^2} \left[7xy^2 z \Big|_{z=0}^{z=x+y-1} \right] \, dy \, dx$$

$$\int_0^2 \int_0^{x^2} 7xy^2(x + y - 1) - 7xy^2(0) \, dy \, dx$$

$$\int_0^2 \int_0^{x^2} 7x^2y^2 + 7xy^3 - 7xy^2 \, dy \, dx$$

Integrating with respect to y (holding x constant), and then evaluating over the interval for y, we get

$$\int_0^2 \left[\frac{7}{3}x^2y^3 + \frac{7}{4}xy^4 - \frac{7}{3}xy^3\Big|_{y=0}^{y=x^2}\right] \ dx$$

$$\int_0^2 \frac{7}{3}x^2\left(x^2\right)^3 + \frac{7}{4}x\left(x^2\right)^4 - \frac{7}{3}x\left(x^2\right)^3 - \left[\frac{7}{3}x^2(0)^3 + \frac{7}{4}x(0)^4 - \frac{7}{3}x(0)^3\right] \ dx$$

$$\int_0^2 \frac{7}{3}x^2\left(x^6\right) + \frac{7}{4}x\left(x^8\right) - \frac{7}{3}x\left(x^6\right) \ dx$$

$$\int_0^2 \frac{7}{3}x^8 + \frac{7}{4}x^9 - \frac{7}{3}x^7 \ dx$$

Finally, integrating with respect to x and then evaluating over the interval for x, we get

$$\frac{7}{27}x^9 + \frac{7}{40}x^{10} - \frac{7}{24}x^8\Big|_0^2$$

$$\frac{7}{27}(2)^9 + \frac{7}{40}(2)^{10} - \frac{7}{24}(2)^8 - \left[\frac{7}{27}(0)^9 + \frac{7}{40}(0)^{10} - \frac{7}{24}(0)^8\right]$$

$$\frac{7}{27}(512) + \frac{7}{40}(1{,}024) - \frac{7}{24}(256)$$

$$7(256)\left[\frac{1}{27}(2) + \frac{1}{40}(4) - \frac{1}{24}\right]$$

$$1{,}792\left(\frac{2}{27} + \frac{1}{10} - \frac{1}{24}\right)$$

$$1{,}792\left[\frac{2}{27}\left(\frac{80}{80}\right)+\frac{1}{10}\left(\frac{216}{216}\right)-\frac{1}{24}\left(\frac{90}{90}\right)\right]$$

$$1{,}792\left(\frac{160}{2{,}160}+\frac{216}{2{,}160}-\frac{90}{2{,}160}\right)$$

$$1{,}792\left(\frac{286}{2{,}160}\right)$$

$$1{,}792\left(\frac{143}{1{,}080}\right)$$

$$896\left(\frac{143}{540}\right)$$

$$\frac{128{,}128}{540}$$

$$\frac{32{,}032}{135}$$

This is the value of the triple integral.

Average value

To find the average value of a function over some object E, we'll use the formula

$$f_{avg} = \frac{1}{V(E)} \iiint_E f(x, y, z) \ dV$$

where $V(E)$ is the volume of the object E.

In order to use the formula, we'll have find the volume of the object, plus the domain of x, y, and z so that we can set limits of integration, turn the triple integral into an iterated integral, and replace dV with $dz \ dy \ dx$.

Example

Find the average value of the function over a cube with side length 2, lying in the first octant with one corner at the origin $(0,0,0)$ and three sides lying in the coordinate planes.

$$f(x, y, z) = 3xyz^2$$

We'll start by finding the volume of the cube. Since we're dealing with a cube with side length 2, the volume will be

$$V(E) = (2)(2)(2)$$

$$V(E) = 8$$

To find the limits of integration, we have to look at the object we've been given. In this case, it's a cube whose corner is sitting at $(0,0,0)$ on the origin. Since the cube has side length 2, the limits of integration are $x=[0,2]$, $y=[0,2]$ and $z=[0,2]$.

Plugging everything we've found into the triple integral formula for average value, including the function itself, we get

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2\int_0^2 3xyz^2\ dz\ dy\ dx$$

$$f_{avg}=\frac{3}{8}\int_0^2\int_0^2\int_0^2 xyz^2\ dz\ dy\ dx$$

Integrating with respect to z, we get

$$f_{avg}=\frac{3}{8}\int_0^2\int_0^2 \frac{1}{3}xyz^3\Big|_{z=0}^{z=2}\ dy\ dx$$

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 xyz^3\Big|_{z=0}^{z=2}\ dy\ dx$$

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 xy(2)^3-xy(0)^3\ dy\ dx$$

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 8xy\ dy\ dx$$

Now we'll integrate with respect to y.

$$f_{avg} = \frac{1}{8}\int_0^2 8\left(\frac{1}{2}\right)xy^2\Big|_{y=0}^{y=2}\ dx$$

$$f_{avg} = \frac{1}{2}\int_0^2 xy^2\Big|_{y=0}^{y=2}\ dx$$

$$f_{avg} = \frac{1}{2}\int_0^2 x(2)^2 - x(0)^2\ dx$$

$$f_{avg} = \frac{1}{2}\int_0^2 4x\ dx$$

Finally we'll integrate with respect to x.

$$f_{avg} = \frac{1}{2}\left(\frac{4}{2}x^2\right)\Big|_0^2$$

$$f_{avg} = x^2\Big|_0^2$$

$$f_{avg} = (2)^2 - (0)^2$$

$$f_{avg} = 4$$

The average value of the function $f(x, y, z) = 3xyz^2$ over the cube E is 4.

Finding volume

We can use triple integrals to solve for the volume of a solid three-dimensional object. The volume formula is

$$V = \iiint_E f(x, y, z) \ dV$$

where E represents the solid object. We'll wind up replacing dV with dx, dy and dz. We can integrate in any order, so we'll try to integrate in whichever order is easiest, depending on the limits of integration, which we'll find by analyzing the object E.

Keep in mind that, when it comes to limits of integration, you want limits that are in terms of the variables left to be integrated. As an example, if we choose to integrate first with respect to z, then y, then x, then our volume integral will look like

$$V = \iiint_E f(x, y, z) \ dz \ dy \ dx$$

Since we'd be integrating z first, that means we won't have integrated with respect to y or x yet, which means we want our limits of integration for z to be in terms of x and y, or constants.

Similarly, since we won't have integrated yet with respect to x when we do the integration for y, we want our limits of integration for y to be in terms of x, or constants.

Since x is the last variable to be integrated, we want our limits of integration for x to be constants.

Example

Use a triple integral to find the volume of the tetrahedron enclosed by $3x + 2y + z = 6$ and the coordinate planes.

The most traditional order of integration is z, then y, then x, so that's what we'll do here. That means we'll need limits of integration as follows:

Limits of integration for z:	in terms of x and y
Limits of integration for y:	in terms of x
Limits of integration for x:	constants

Since the tetrahedron rests on the coordinate planes, we know that the lower limit of integration for x, y and z will be 0.

To find the upper limit of integration for x (a constant), we'll set $y = 0$ and $z = 0$ in $3x + 2y + z = 6$ and then solve for x.

$$3x + 2y + z = 6$$

$$3x + 2(0) + 0 = 6$$

$$3x = 6$$

$$x = 2$$

This means that the limits of integration for x are $[0,2]$.

To find the upper limit of integration for y (a value in terms of x), we'll set $z=0$ and then rearrange $3x+2y+z=6$ so that it's solved for y in terms of x.

$$3x+2y+z=6$$

$$3x+2y+0=6$$

$$3x+2y=6$$

$$2y=6-3x$$

$$y=3-\frac{3}{2}x$$

This means that the limits of integration for y are $[0,3-(3/2)x]$.

To find the upper limit of integration for z (a value in terms of x and y), we'll solve $3x+2y+z=6$ for z.

$$3x+2y+z=6$$

$$z=6-3x-2y$$

This means that the limits of integration for z are $[0,6-3x-2y]$.

Plugging everything into the triple integral, we get

$$V=\int_0^2\int_0^{3-\frac{3}{2}x}\int_0^{6-3x-2y}1\ dz\ dy\ dx$$

Since we're finding the volume of the shape itself and our limits of integration are in terms of the shape, the function we are integrating will simply be $f(x, y, z) = 1$.

We always integrate from the inside out, so we'll integrate first with respect to z, treating all other variables as constants.

$$V = \int_0^2 \int_0^{3-\frac{3}{2}x} z\Big|_{z=0}^{z=6-3x-2y} \ dy \ dx$$

$$V = \int_0^2 \int_0^{3-\frac{3}{2}x} 6 - 3x - 2y - 0 \ dy \ dx$$

$$V = \int_0^2 \int_0^{3-\frac{3}{2}x} 6 - 3x - 2y \ dy \ dx$$

Now we'll integrate with respect to y, treating all other variables as constants.

$$V = \int_0^2 6y - 3xy - y^2\Big|_{y=0}^{y=3-\frac{3}{2}x} \ dx$$

$$V = \int_0^2 6\left(3 - \frac{3}{2}x\right) - 3x\left(3 - \frac{3}{2}x\right) - \left(3 - \frac{3}{2}x\right)^2 - \left[6(0) - 3x(0) - (0)^2\right] \ dx$$

$$V = \int_0^2 18 - 9x - 9x + \frac{9x^2}{2} - \left(9 - 9x + \frac{9x^2}{4}\right) \ dx$$

$$V = \int_0^2 18 - 9x - 9x + \frac{9x^2}{2} - 9 + 9x - \frac{9x^2}{4} \ dx$$

$$V=\int_0^2 9-9x+\frac{18x^2}{4}-\frac{9x^2}{4}\ dx$$

$$V=\int_0^2 9-9x+\frac{9x^2}{4}\ dx$$

Now we'll integrate with respect to x.

$$V=9x-\frac{9}{2}x^2+\frac{9x^3}{4(3)}\Big|_0^2$$

$$V=9x-\frac{9x^2}{2}+\frac{3x^3}{4}\Big|_0^2$$

$$V=9(2)-\frac{9(2)^2}{2}+\frac{3(2)^3}{4}-\left[9(0)-\frac{9(0)^2}{2}+\frac{3(0)^3}{4}\right]$$

$$V=18-\frac{36}{2}+\frac{24}{4}$$

$$V=18-18+6$$

$$V=6$$

The volume of the solid is 6 cubic units.

Expressing the integral six ways

There are six ways to express an iterated triple integral. While the function $f(x, y, z)$ inside the integral always stays the same, the order of integration will change, and the limits of integration will change to match the order.

With order of integration x, y, z $\qquad \iiint_E f(x, y, z)\ dx\ dy\ dz$

With order of integration x, z, y $\qquad \iiint_E f(x, y, z)\ dx\ dz\ dy$

With order of integration y, x, z $\qquad \iiint_E f(x, y, z)\ dy\ dx\ dz$

With order of integration y, z, x $\qquad \iiint_E f(x, y, z)\ dy\ dz\ dx$

With order of integration z, x, y $\qquad \iiint_E f(x, y, z)\ dz\ dx\ dy$

With order of integration z, y, x $\qquad \iiint_E f(x, y, z)\ dz\ dy\ dx$

The only hard part of these problems is finding the limits of integration for each of the three individual integrals in each of the six triple iterated integrals.

Remember that, with all iterated integrals, you work your way from the inside toward the outside. So, if the integral ends in $dx\ dy\ dz$, it means you integrate from the inside out, from left to right, first with respect to x, then with respect to y, and then with respect to z.

$$\iiint_E f(x,y,z)\ dx\ dy\ dz$$

Since you're integrating with respect to x first, and you'll need to integrate with respect to y and z later, you need to have y and z variables left over after you integrate with respect to x and evaluate over the associated limits of integration. This means that the inner integral needs to have limits of integration in terms of y and z.

$$\int\int\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$$

Once you've integrated with respect to x, you'll have only y and z variables remaining. You'll integrate with respect to y, plugging in the limits of integration associated with y. Since you need to leave z variables in the function in order to later integrate with respect to z, that means your limits of integration for y need to be in terms of z.

$$\int\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$$

Finally, once you've eliminated y and have only z variables remaining, the limits of integration associated with z should be constants, so that your final answer is a constant.

$$\int_z^z\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$$

Notice in the chart below how all of the innermost integrals have limits of integration in terms of two variables, the second integral has limits of integration in terms of one variable, and the outermost integral has constant limits of integration.

With order of integration x, y, z $\quad \int_{z}^{z}\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$

With order of integration x, z, y $\quad \int_{y}^{y}\int_{z(y)}^{z(y)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dz\ dy$

With order of integration y, x, z $\quad \int_{z}^{z}\int_{x(z)}^{x(z)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dx\ dz$

With order of integration y, z, x $\quad \int_{x}^{x}\int_{z(x)}^{z(x)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dz\ dx$

With order of integration z, x, y $\quad \int_{y}^{y}\int_{x(y)}^{x(y)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dx\ dy$

With order of integration z, y, x $\quad \int_{x}^{x}\int_{y(x)}^{y(x)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dy\ dx$

It's best to find all of the limits of integration you'll need before you start writing down all of the integrals. The easiest way to keep the limits of integration organized is with the chart below.

	x	y	z	Multivariable
x	**Constants** Set $y=0$, $z=0$ Solve for x	$x(y)$ Set $z=0$ Solve for x	$x(z)$ Set $y=0$ Solve for x	$x(y,z)$ Solve for x
y	$y(x)$ Set $z=0$ Solve for y	**Constants** Set $x=0$, $z=0$ Solve for y	$y(z)$ Set $x=0$ Solve for y	$y(x,z)$ Solve for y
z	$z(x)$ Set $y=0$ Solve for z	$z(y)$ Set $x=0$ Solve for z	**Constants** Set $x=0$, $y=0$ Solve for z	$z(x,y)$ Solve for z

Once you've got the chart completely filled in, you'll be able to go straight to the limits of integration you need for each integral.

Example

Express the triple iterated integral

$$\iiint_E f(x,y,z)\ dV$$

six ways for the volume bounded by the given curves.

$$y = x^2 + 15z^2 - 9$$

$$y = 0$$

We'll start by creating the chart for the limits of integration.

	x	y	z	Multivariable
x	$0=x^2+15(0)^2-9$ $x^2=9$ $x=\pm 3$	$y=x^2+15(0)^2-9$ $x^2=y+9$ $x=\pm\sqrt{y+9}$	$0=x^2+15z^2-9$ $x^2=9-15z^2$ $x=\pm\sqrt{9-15z^2}$	$y=x^2+15z^2-9$ $x^2=y-15z^2+9$ $x=\pm\sqrt{y-15z^2+9}$
y	$y=x^2+15(0)^2-9$ $y=x^2-9$	$y=(0)^2+15(0)^2-9$ $y=-9$	$y=(0)^2+15z^2-9$ $y=15z^2-9$	$y=x^2+15z^2-9$
z	$0=x^2+15z^2-9$ $15z^2=9-x^2$ $z=\pm\sqrt{(9-x^2)/15}$	$y=(0)^2+15z^2-9$ $15z^2=y+9$ $z=\pm\sqrt{(y+9)/15}$	$0=(0)^2+15z^2-9$ $15z^2=9$ $z=\pm\sqrt{3/5}$	$15z^2=y-x^2+9$ $z=\pm\sqrt{\dfrac{y-x^2+9}{15}}$

Before we go on, notice that we've found two limits of integration in every section of the chart above, except in the row for y. It'll often be the case with these types of problems, that you have one variable for which you're only able to find one limit of integration.

Usually you can find the other one by going back to the question we were asked at the beginning. In this case, we were told that $y=0$, so we can use $y=0$ as the second limit of integration. We'll revise the chart by adding this in.

	x	y	z	Multivariable
x	$0=x^2+15(0)^2-9$ $x^2=9$ $x=\pm 3$	$y=x^2+15(0)^2-9$ $x^2=y+9$ $x=\pm\sqrt{y+9}$	$0=x^2+15z^2-9$ $x^2=9-15z^2$ $x=\pm\sqrt{9-15z^2}$	$y=x^2+15z^2-9$ $x^2=y-15z^2+9$ $x=\pm\sqrt{y-15z^2+9}$
y	$y=x^2+15(0)^2-9$ $y=x^2-9$ $y=0$	$y=(0)^2+15(0)^2-9$ $y=-9$ $y=0$	$y=(0)^2+15z^2-9$ $y=15z^2-9$ $y=0$	$y=x^2+15z^2-9$ $y=0$
z	$0=x^2+15z^2-9$ $15z^2=9-x^2$ $z=\pm\sqrt{(9-x^2)/15}$	$y=(0)^2+15z^2-9$ $15z^2=y+9$ $z=\pm\sqrt{(y+9)/15}$	$0=(0)^2+15z^2-9$ $15z^2=9$ $z=\pm\sqrt{3/5}$	$15z^2=y-x^2+9$ $z=\pm\sqrt{\dfrac{y-x^2+9}{15}}$

Now we can pull the limits of integration we've found into each of the six triple iterated integrals.

$$\int_{z}^{z}\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz \qquad \int_{-\sqrt{3/5}}^{\sqrt{3/5}}\int_{15z^2-9}^{0}\int_{-\sqrt{y-15z^2+9}}^{\sqrt{y-15z^2+9}} f(x,y,z)\ dx\ dy\ dz$$

$$\int_{y}^{y}\int_{z(y)}^{z(y)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dz\ dy \qquad \int_{-9}^{0}\int_{-\sqrt{(y+9)/15}}^{\sqrt{(y+9)/15}}\int_{-\sqrt{y-15z^2+9}}^{\sqrt{y-15z^2+9}} f(x,y,z)\ dx\ dz\ dy$$

$$\int_{z}^{z}\int_{x(z)}^{x(z)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dx\ dz \qquad \int_{-\sqrt{3/5}}^{\sqrt{3/5}}\int_{-\sqrt{9-15z^2}}^{\sqrt{9-15z^2}}\int_{x^2+15z^2-9}^{0} f(x,y,z)\ dy\ dx\ dz$$

$$\int_{x}^{x}\int_{z(x)}^{z(x)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dz\ dx \qquad \int_{-3}^{3}\int_{-\sqrt{(9-x^2)/15}}^{\sqrt{(9-x^2)/15}}\int_{x^2+15z^2-9}^{0} f(x,y,z)\ dy\ dz\ dx$$

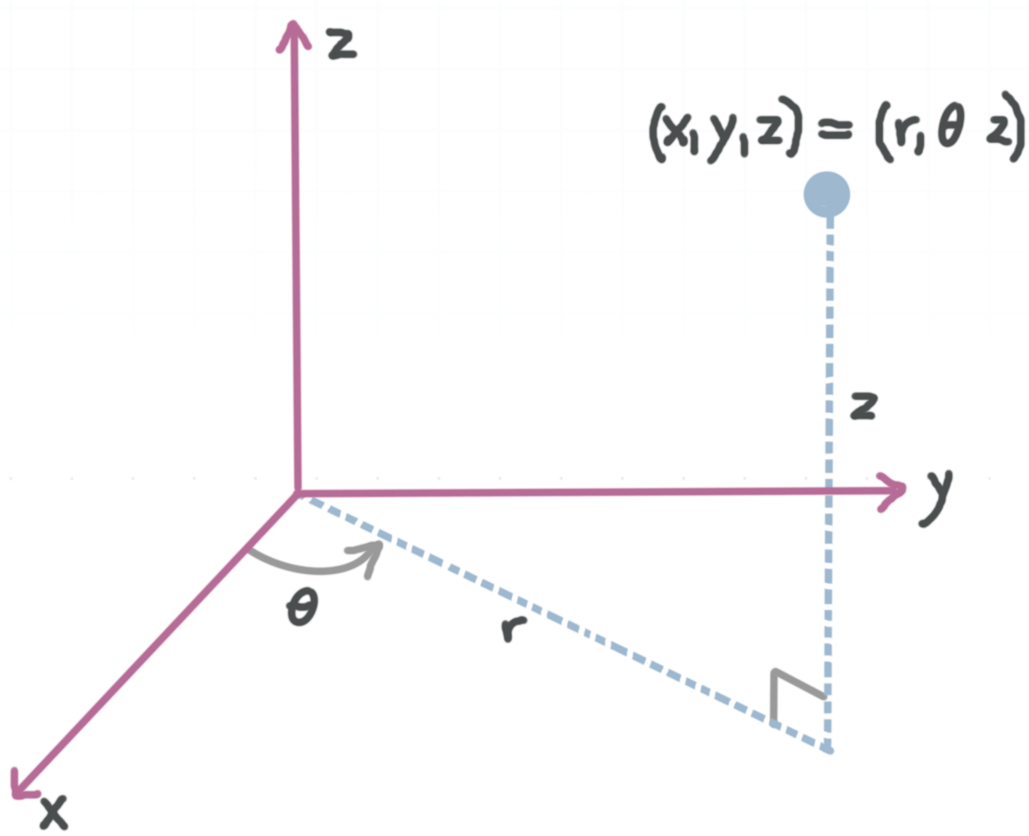
$$\int_{y}^{y}\int_{x(y)}^{x(y)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dx\ dy \qquad \int_{-9}^{0}\int_{-\sqrt{y+9}}^{\sqrt{y+9}}\int_{-\sqrt{(y-x^2+9)/15}}^{\sqrt{(y-x^2+9)/15}} f(x,y,z)\ dz\ dx\ dy$$

$$\int_{x}^{x}\int_{y(x)}^{y(x)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dy\ dx \qquad \int_{-3}^{3}\int_{x^2-9}^{0}\int_{-\sqrt{(y-x^2+9)/15}}^{\sqrt{(y-x^2+9)/15}} f(x,y,z)\ dz\ dy\ dx$$

Cylindrical coordinates

Like cartesian (or rectangular) coordinates and polar coordinates, cylindrical coordinates are just another way to describe points in three-dimensional space.

Cylindrical coordinates are like polar coordinates, just in three-dimensional space instead of two-dimensional space. Since polar coordinates in two dimensions are given as (r, θ), cylindrical coordinates have us add a value for z to account for the third dimension, so cylindrical coordinates are given as (r, θ, z).



So if **rectangular coordinates** are given as (x, y, z), where x is the distance of (x, y, z) from the origin along the x-axis, y is the distance of (x, y, z) from the origin along the y-axis, and z is the distance of (x, y, z) from the origin along the z-axis, then **cylindrical coordinates** are given as (r, θ, z), where r is the distance of $(r, \theta, 0)$ from the origin, θ is the angle between r (the line connecting (r, θ, z) to the origin) and the positive direction of the x-axis, and z is the the distance of (r, θ, z) from the origin along the z-axis.

To convert between cylindrical coordinates and rectangular coordinates, we'll use

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Let's try an example where we convert rectangular coordinates to cylindrical coordinates.

Example

Convert the rectangular point $(1,1,1)$ to cylindrical coordinates.

We'll plug $(1,1,1)$ into the conversion formulas to get

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$1 = r\cos\theta \qquad 1 = r\sin\theta \qquad 1 = z$$

$$r = \frac{1}{\cos\theta} \qquad r = \frac{1}{\sin\theta}$$

Because we have two equations both defined for r, we can set them equal to each other.

$$\frac{1}{\cos\theta} = \frac{1}{\sin\theta}$$

$$\cos\theta = \sin\theta$$

$$\theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

Since we found two values for θ, we'll have two cylindrical points that can represent the given rectangular point.

For $\theta = \pi/4$,

$$r = \frac{1}{\cos\frac{\pi}{4}}$$

$$r = \frac{1}{\frac{\sqrt{2}}{2}}$$

$$r = \frac{2}{\sqrt{2}}$$

$$r = \sqrt{2}$$

For $\theta = 5\pi/4$,

$$r = \frac{1}{\cos\frac{5\pi}{4}}$$

$$r = \frac{1}{-\frac{\sqrt{2}}{2}}$$

$$r = -\frac{2}{\sqrt{2}}$$

$$r = -\sqrt{2}$$

So the rectangular point $(1,1,1)$ is equivalent to the cylindrical points

$$\left(\sqrt{2},\frac{\pi}{4},1\right)$$

$$\left(-\sqrt{2},\frac{5\pi}{4},1\right)$$

Let's try an example where we convert cylindrical coordinates to rectangular coordinates.

Example

Convert the cylindrical point $(2,\pi,3)$ to a rectangular point.

From the cylindrical coordinate $(r,\theta,z)=(2,\pi,3)$, we know $r=2$, $\theta=\pi$, and $z=3$, so from our conversion formulas we find

$$x=r\cos\theta \qquad y=r\sin\theta \qquad z=z$$

$$x=2\cos\pi \qquad y=2\sin\pi \qquad z=3$$

$$x=2(-1) \qquad y=2(0)$$

$$x=-2 \qquad y=0$$

Putting these values together, we can say that the cylindrical point $(2,\pi,3)$ is the same as the rectangular point $(-2,0,3)$.

Changing triple integrals to cylindrical coordinates

To change a triple integral like

$$\iiint_B f(x, y, z)\ dV$$

into cylindrical coordinates, we'll need to convert both the limits of integration, the function itself, and dV from rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z). To do so, we'll use the conversion formulas

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

and

$$r^2 = x^2 + y^2$$

to convert the limits of integration and the function $f(x, y, z)$. dV will be converted using the formula

$$dV = r\ dz\ dr\ d\theta$$

Example

Evaluate the triple integral in cylindrical coordinates.

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{3} xz\ dz\ dx\ dy$$

Let's start by converting the limits of integration from rectangular coordinates to cylindrical coordinates, starting with the innermost integral. These will be the limits of integration for z, which means they need to be solved for z once we get them to cylindrical coordinates. The upper limit 3 can stay the same since $z = z$ when we go from rectangular to cylindrical coordinates, but the lower limit needs to be converted using the conversion formulas.

$$z = \sqrt{x^2 + y^2}$$

$$z = \sqrt{\left[r\cos\theta\right]^2 + \left[r\sin\theta\right]^2}$$

$$z = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta}$$

$$z = \sqrt{r^2\left(\cos^2\theta + \sin^2\theta\right)}$$

Using the trigonometric identity $\sin^2 x + \cos^2 x = 1$, we can simplify to

$$z = \sqrt{r^2(1)}$$

$$z = r$$

This means that the limits of integration with respect to z in cylindrical coordinates are $[r,3]$.

Next we'll do the limits of integration for the middle integral. These will be the limits of integration for x, which means they need to be solved for r once we get them to cylindrical coordinates.

The lower limit is given by

$$x = -\sqrt{9 - y^2}$$

$$r\cos\theta = -\sqrt{9 - (r\sin\theta)^2}$$

$$r^2\cos^2\theta = 9 - r^2\sin^2\theta$$

$$r^2\sin^2\theta + r^2\cos^2\theta = 9$$

$$r^2\left(\sin^2\theta + \cos^2\theta\right) = 9$$

Since $\sin^2 x + \cos^2 x = 1$,

$$r^2(1) = 9$$

$$r = \pm 3$$

The upper limit is given by

$$x = \sqrt{9 - y^2}$$

$$r\cos\theta = \sqrt{9 - (r\sin\theta)^2}$$

$$r^2\cos^2\theta = 9 - r^2\sin^2\theta$$

$$r^2\sin^2\theta + r^2\cos^2\theta = 9$$

$$r^2\left(\sin^2\theta + \cos^2\theta\right) = 9$$

Since $\sin^2 x + \cos^2 x = 1$,

$$r^2(1) = 9$$

$$r = \pm 3$$

It looks like the limits of integration for r in cylindrical coordinates will be given by $[-3,3]$. However, remember that r represents the radius, or distance from the origin. It doesn't make sense to say that we're -3 units away from the origin. Instead, we always say that the lower bound for r is 0, such that 0 is the closest we can be to the origin (right on the origin), and 3 is the furthest we can be from the origin. So the limits of integration for r will be $[0,3]$.

Finally, we'll do the limits of integration for the outer integral. These will be the limits of integration for y, which means they need to be solved for θ once we get them to cylindrical coordinates. But since we're going to θ, we can just assume that the interval is $[0,2\pi]$, because that interval represents the full set of values for θ, which is just the angle between any point and the positive direction of the x-axis.

Next we'll use the conversion formulas to convert the function itself into cylindrical coordinates.

$$xz = r\cos\theta z$$

$$xz = rz\cos\theta$$

Putting all of this, plus $dV = r\ dz\ dr\ d\theta$ into the integral gives

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}}\int_{\sqrt{x^2+y^2}}^{3} xz\ dz\ dx\ dy$$

$$\int_{0}^{2\pi}\int_{0}^{3}\int_{r}^{3} rz\cos\theta\left(r\ dz\ dr\ d\theta\right)$$

$$\int_{0}^{2\pi}\int_{0}^{3}\int_{r}^{3} r^2z\cos\theta\ dz\ dr\ d\theta$$

We always integrate from the inside out, which means we'll integrate first with respect to z, treating all other variables as constants.

$$\int_{0}^{2\pi}\int_{0}^{3}\frac{1}{2}r^2z^2\cos\theta\Big|_{z=r}^{z=3}\ dr\ d\theta$$

$$\int_{0}^{2\pi}\int_{0}^{3}\frac{1}{2}r^2(3)^2\cos\theta-\frac{1}{2}r^2(r)^2\cos\theta\ dr\ d\theta$$

$$\int_{0}^{2\pi}\int_{0}^{3}\frac{9}{2}r^2\cos\theta-\frac{1}{2}r^4\cos\theta\ dr\ d\theta$$

Now we'll integrate with respect to r, treating all other variables as constants.

$$\int_{0}^{2\pi}\frac{9}{2(3)}r^3\cos\theta-\frac{1}{2(5)}r^5\cos\theta\Big|_{r=0}^{r=3}\ d\theta$$

$$\int_{0}^{2\pi}\frac{3}{2}r^3\cos\theta-\frac{1}{10}r^5\cos\theta\Big|_{r=0}^{r=3}\ d\theta$$

$$\int_0^{2\pi} \frac{3}{2}(3)^3\cos\theta - \frac{1}{10}(3)^5\cos\theta - \left[\frac{3}{2}(0)^3\cos\theta - \frac{1}{10}(0)^5\cos\theta\right]\ d\theta$$

$$\int_0^{2\pi} \frac{81}{2}\cos\theta - \frac{243}{10}\cos\theta\ d\theta$$

$$\int_0^{2\pi} \frac{405}{10}\cos\theta - \frac{243}{10}\cos\theta\ d\theta$$

$$\int_0^{2\pi} \frac{162}{10}\cos\theta\ d\theta$$

$$\int_0^{2\pi} \frac{81}{5}\cos\theta\ d\theta$$

Now we'll integrate with respect to θ.

$$\frac{81}{5}\sin\theta\Big|_0^{2\pi}$$

$$\frac{81}{5}\sin(2\pi) - \frac{81}{5}\sin(0)$$

$$\frac{81}{5}(0) - \frac{81}{5}(0)$$

$$0$$

This means that the volume given by this triple integral is 0.

Finding volume

We can use triple integrals and cylindrical coordinates to solve for the mass of a solid cylinder,

$$\iiint_B f(x, y, z)\ dV$$

where B represents the solid cylinder, f is a function that models density, and dV can be defined in cylindrical coordinates as

$$dV = r\ dz\ dr\ d\theta$$

Remember, rectangular coordinates are given as (x, y, z), and cylindrical coordinates are given as (r, θ, z), and to convert from rectangular to cylindrical coordinates, we can use the formulas, $x = r\cos\theta$, $y = r\sin\theta$, $z = z$, and $r^2 = x^2 + y^2$.

Let's try an example where we use a triple integral to find mass.

Example

Use cylindrical coordinates to find the mass given by the triple integral, where E is the solid that lies within the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$ and below the cone $z^2 = 9x^2 + 9y^2$.

$$\iiint_E 3x^2\ dV$$

First, we'll convert the function we were given into cylindrical coordinates using the conversion formulas.

$$3x^2$$

$$3(r\cos\theta)^2$$

$$3r^2\cos^2\theta$$

Replacing the original function with this one, and substituting for dV, the integral becomes

$$\iiint_E 3r^2\cos^2\theta(r\ dz\ dr\ d\theta)$$

$$\iiint_E 3r^3\cos^2\theta\ dz\ dr\ d\theta$$

Now we just need to find limits of integration. We've been told that we're interested in the solid that lies inside the cylinder $x^2+y^2=4$. If we convert this to cylindrical coordinates using $r^2=x^2+y^2$, we get

$$r^2=4$$

$$r=2$$

Since r represents radius, and radius can only be positive, we can say that the limits of integration for r are $[0,2]$, and therefore

$$\int\int_0^2\int 3r^3\cos^2\theta\ dz\ dr\ d\theta$$

We've also been told that the solid lies above the plane $z = 0$. Since no conversion is required for z when we're moving from rectangular to cylindrical coordinates, we can leave this as-is. We also know that the solid lies below $z^2 = 9x^2 + 9y^2$. We'll factor out 9 and get $z^2 = 9(x^2 + y^2)$. Using $r^2 = x^2 + y^2$ to convert this to cylindrical coordinates, we get

$$z^2 = 9(x^2 + y^2)$$

$$z^2 = 9(r^2)$$

$$z^2 = 9r^2$$

$$\sqrt{z^2} = \sqrt{9r^2}$$

$$z = 3r$$

Putting these two pieces of information together, we can say that the limits of integration for z are $[0,3r]$, and therefore

$$\int \int_0^2 \int_0^{3r} 3r^3 \cos^2\theta \ dz \ dr \ d\theta$$

For all full cylinders, the limits of integration for θ will be $[0,2\pi]$, therefore

$$\int_0^{2\pi} \int_0^2 \int_0^{3r} 3r^3 \cos^2\theta \ dz \ dr \ d\theta$$

We always integrate from the inside out, so we'll integrate with respect to z first, treating all other variables as constants.

$$\int_0^{2\pi} \int_0^2 3r^3 z \cos^2\theta \Big|_{z=0}^{z=3r} \ dr \ d\theta$$

$$\int_0^{2\pi}\int_0^2 3r^3(3r)\cos^2\theta - 3r^3(0)\cos^2\theta\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2 9r^4\cos^2\theta\ dr\ d\theta$$

Now we'll integrate with respect to r, treating all other variables as constants.

$$\int_0^{2\pi} 9\left(\frac{1}{5}\right) r^5\cos^2\theta\Big|_{r=0}^{r=2}\ d\theta$$

$$\int_0^{2\pi} \frac{9}{5}r^5\cos^2\theta\Big|_{r=0}^{r=2}\ d\theta$$

$$\int_0^{2\pi} \frac{9}{5}(2)^5\cos^2\theta - \frac{9}{5}(0)^5\cos^2\theta\ d\theta$$

$$\int_0^{2\pi} \frac{288}{5}\cos^2\theta\ d\theta$$

We'll make a substitution using the trigonometric identity

$$\cos^2\theta = \frac{1}{2}(1+\cos(2\theta))$$

So the integral becomes

$$\int_0^{2\pi} \frac{288}{5}\left\{\frac{1}{2}(1+\cos(2\theta))\right\}\ d\theta$$

$$\int_0^{2\pi} \frac{144}{5}(1+\cos(2\theta))\ d\theta$$

$$\int_0^{2\pi} \frac{144}{5}+\frac{144}{5}\cos(2\theta)\ d\theta$$

$$\frac{144}{5}\theta+\frac{144}{5(2)}\sin(2\theta)\Big|_0^{2\pi}$$

$$\frac{144}{5}\theta+\frac{72}{5}\sin(2\theta)\Big|_0^{2\pi}$$

$$\frac{144}{5}(2\pi)+\frac{72}{5}\sin(2(2\pi))-\left(\frac{144}{5}(0)+\frac{72}{5}\sin(2(0))\right)$$

$$\frac{144}{5}(2\pi)+\frac{72}{5}\sin(4\pi)-\frac{72}{5}\sin 0$$

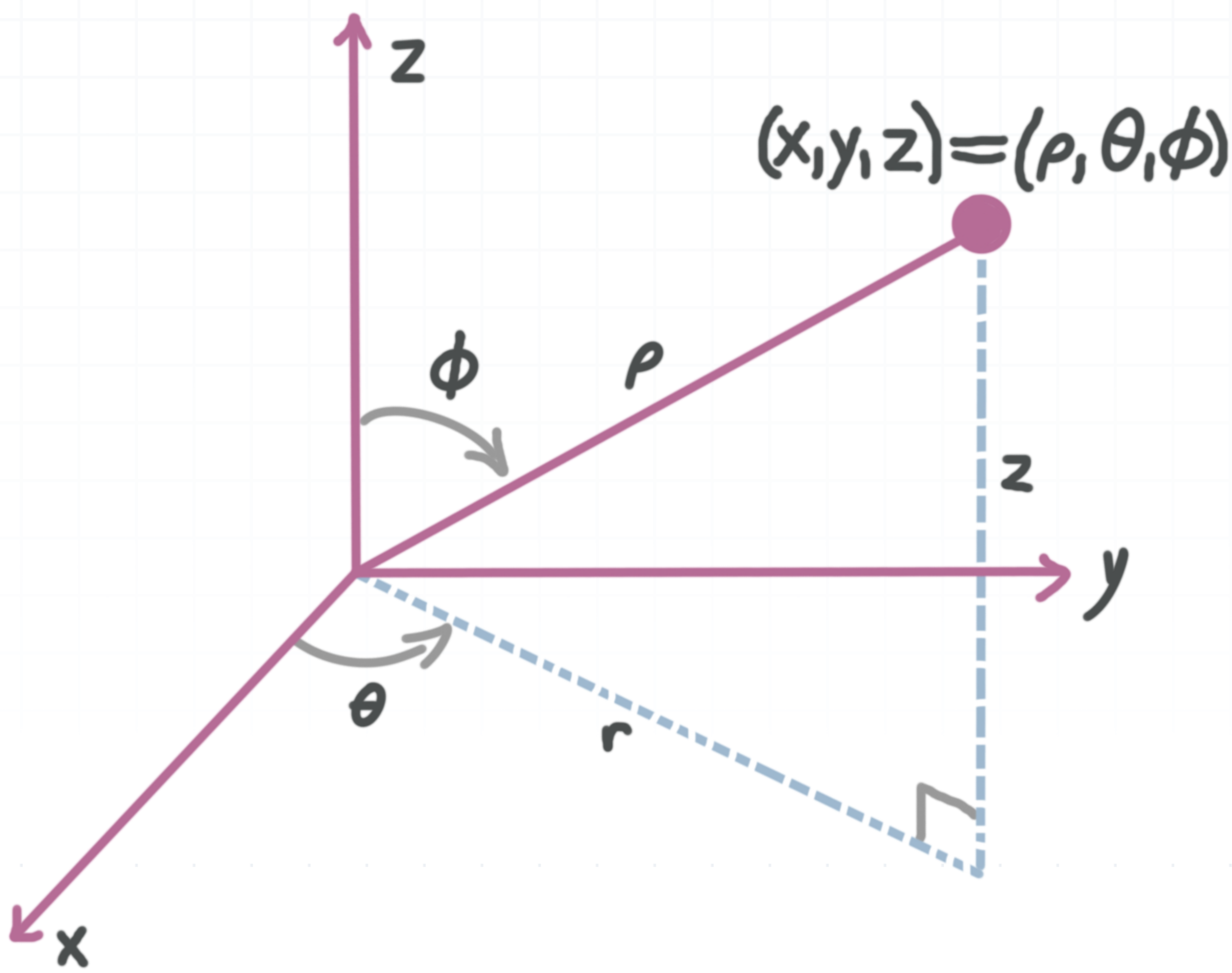
$$\frac{288\pi}{5}+\frac{72}{5}(0)-\frac{72}{5}(0)$$

$$\frac{288\pi}{5}$$

This is the mass of the cylinder.

Spherical coordinates

Like cartesian (or rectangular) coordinates and polar coordinates, spherical coordinates are just another way to describe points in three-dimensional space.



Rectangular coordinates are given as (x, y, z)

where x is the distance of (x, y, z) from the origin along the x-axis

y is the distance of (x, y, z) from the origin along the y-axis

z is the distance of (x, y, z) from the origin along the z -axis

Spherical coordinates are given as (ρ, θ, ϕ)

where ρ is the distance of (ρ, θ, ϕ) from the origin, $p \geq 0$

θ is the angle between r (the shadow of the line connecting (ρ, θ, ϕ) to the origin) and the positive direction of the x-axis

ϕ is the angle between the line connecting (ρ, θ, ϕ) to the origin and the positive direction of the z-axis, $0 \leq \phi \leq \pi$

To convert between spherical coordinates and rectangular coordinates, we will need to use the formulas

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

and

$$\rho^2 = x^2 + y^2 + z^2$$

Let's try an example where we convert rectangular coordinates to spherical coordinates.

Example

Convert the rectangular coordinate point to a spherical coordinate point.

$$(0,0,1)$$

We'll start by plugging $(0,0,1)$ into $\rho^2 = x^2 + y^2 + z^2$.

$$\rho^2 = 0^2 + 0^2 + 1^2$$

$$\rho^2 = 1$$

Since $p \geq 0$, $\rho = 1$

We'll plug $(0,0,1)$ and $\rho = 1$ into $z = \rho \cos \phi$ to solve for ϕ.

$$z = \rho \cos \phi$$

$$1 = (1)\cos \phi$$

$$\cos \phi = 1$$

$$\phi = 0,\ 2\pi$$

Since $0 \leq \phi \leq \pi$, $\phi = 0$

We'll plug $(0,0,1)$, $\rho = 1$, and $\phi = 0$ into $y = \rho \sin \phi \sin \theta$ to solve for θ.

$$y = \rho \sin \phi \sin \theta$$

$$0 = (1)\sin(0)\sin \theta$$

$$0 = (1)(0)\sin \theta$$

Since we have 0 on the right-hand side, θ could be any value and the equation would still be true. This makes sense, since the given point is on the z-axis, and θ is the angle between r (the shadow of the line connecting (ρ, θ, ϕ) to the origin) and the positive direction of the x-axis.

Since it can be any value, let's just choose $\theta = 0$.

Putting these values together, we can say that the spherical coordinate $(1,0,0)$ is the same as the rectangular coordinate $(0,0,1)$.

Let's try an example where we convert spherical coordinates to rectangular coordinates.

Example

Convert the spherical coordinate point to a rectangular coordinate point.

$$\left(1,\pi,\frac{\pi}{2}\right)$$

We know that

$$\rho = 1$$

$$\phi = \frac{\pi}{2}$$

$$\theta = \pi$$

Plugging these into the conversion formulas, we get

$$x = \rho \sin\phi \cos\theta$$

$$x = (1)\sin\frac{\pi}{2}\cos\pi$$

$$x = (1)(1)(-1)$$

$$x = -1$$

and

$$y = \rho\sin\phi\sin\theta$$

$$y = (1)\sin\frac{\pi}{2}\sin\pi$$

$$y = (1)(1)(0)$$

$$y = 0$$

and

$$z = \rho\cos\phi$$

$$z = (1)\cos\frac{\pi}{2}$$

$$z = (1)(0)$$

$$z = 0$$

The rectangular coordinate $(-1,0,0)$ is the same as the spherical coordinate $\left(1,\pi,\frac{\pi}{2}\right)$.

Finding volume

We can use triple integrals and spherical coordinates to solve for the volume of a solid sphere. The volume formula in rectangular coordinates is

$$V = \iiint_B f(x, y, z)\ dV$$

where B represents the solid sphere and dV can be defined in spherical coordinates as

$$dV = \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

To convert in general from rectangular to spherical coordinates, we can use the formulas

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

and

$$\rho^2 = x^2 + y^2 + z^2$$

Remember, rectangular coordinates are given as (x, y, z), and spherical coordinates are given as (ρ, θ, ϕ).

In order to find limits of integration for the triple integral, we'll say that ϕ is defined on the interval $[0,\pi]$ and that θ is defined on the interval $[0,2\pi]$. Then we only have to find an interval for ρ.

Example

Use spherical coordinates to find the volume of the triple integral, where B is a sphere with center $(0,0,0)$ and radius 4.

$$\iiint_B x^2+y^2+z^2\ dV$$

Using the conversion formula $\rho^2=x^2+y^2+z^2$, we can change the given function into spherical notation.

$$\iiint_B x^2+y^2+z^2\ dV=\iiint_B \rho^2\ dV$$

Then we'll use $dV=\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$ to make a substitution for dV.

$$\iiint_B \rho^2\left(\rho^2\sin\phi\ d\rho\ d\theta\ d\phi\right)$$

$$\iiint_B \rho^4\sin\phi\ d\rho\ d\theta\ d\phi$$

Now we'll find limits of integration. We already know the limits of integration for ϕ and θ, since they are always the same if we're dealing with a full sphere, so we get

$$\int_0^{\pi}\int_0^{2\pi}\int \rho^4\sin\phi\ d\rho\ d\theta\ d\phi$$

Since ρ defines the radius of the sphere, and we're told that this sphere has its center at $(0,0,0)$ and radius 4, ρ is defined on $[0,4]$, so

$$\int_0^{\pi}\int_0^{2\pi}\int_0^4 \rho^4 \sin\phi \ d\rho \ d\theta \ d\phi$$

We always integrate inside out, so we'll integrate with respect to ρ first, treating all other variables as constants.

$$V=\int_0^{\pi}\int_0^{2\pi} \frac{1}{5}\rho^5 \sin\phi \Big|_{\rho=0}^{\rho=4} \ d\theta \ d\phi$$

$$V=\int_0^{\pi}\int_0^{2\pi} \frac{1}{5}(4)^5 \sin\phi - \frac{1}{5}(0)^5 \sin\phi \ d\theta \ d\phi$$

$$V=\int_0^{\pi}\int_0^{2\pi} \frac{1{,}024}{5} \sin\phi \ d\theta \ d\phi$$

Now we'll integrate with respect to θ, treating all other variables as constants.

$$V=\frac{1{,}024}{5}\int_0^{\pi} \theta \sin\phi \Big|_{\theta=0}^{\theta=2\pi} \ d\phi$$

$$V=\frac{1{,}024}{5}\int_0^{\pi} 2\pi \sin\phi - 0 \sin\phi \ d\phi$$

$$V=\frac{2{,}048\pi}{5}\int_0^{\pi} \sin\phi \ d\phi$$

Finally, we'll integrate with respect to ϕ.

$$V = \frac{2{,}048\pi}{5}\left(-\cos\phi\right)\Big|_0^{\pi}$$

$$V = -\frac{2{,}048\pi\cos\phi}{5}\Bigg|_0^{\pi}$$

$$V = -\frac{2{,}048\pi}{5}\cos\pi - \left[-\frac{2{,}048\pi}{5}\cos 0\right]$$

$$V = -\frac{2{,}048\pi}{5}(-1) + \frac{2{,}048\pi}{5}(1)$$

$$V = \frac{2{,}048\pi}{5} + \frac{2{,}048\pi}{5}$$

$$V = \frac{4{,}096\pi}{5}$$

This is the volume of the region bounded beneath the surface $x^2 + y^2 + z^2$ and above the sphere defined by B.

Jacobian for two variables

In the past we've converted multivariable functions defined in terms of cartesian coordinates x and y into functions defined in terms of polar coordinates r and θ.

Similarly, given a region defined in the uv-plane, we can use a Jacobian transformation to redefine it in the xy-plane, or vice versa.

Given two equations $x = f(u, v)$ and $y = g(u, v)$, the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

Example

Find the Jacobian of the transformation.

$$x = uv$$

$$y = 2u - v^2$$

Our functions tell us that we have a 2×2 set-up, so we'll use the formula

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

We need to start by finding the partial derivatives of x and y with respect to both u and v.

$$\frac{\partial x}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u$$

and

$$\frac{\partial y}{\partial u} = 2$$

$$\frac{\partial y}{\partial v} = -2v$$

We'll plug the partial derivatives into our formula and get

$$\frac{\partial(x, y)}{\partial(u, v)} = (v)(-2v) - (u)(2)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -2v^2 - 2u$$

This is the Jacobian of the transformation.

Jacobian for three variables

In the past we've converted multivariable functions defined in terms of cartesian coordinates x and y into functions defined in terms of polar coordinates r and θ.

Similarly, given a region defined in uvw-space, we can use a Jacobian transformation to redefine it in xyz-space, or vice versa.

Given three equations $x = f(u, v, w)$, $y = g(u, v, w)$, and $z = h(u, v, w)$, the Jacobian is

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u} \begin{vmatrix} \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} - \frac{\partial x}{\partial v} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial w} \end{vmatrix} + \frac{\partial x}{\partial w} \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{vmatrix}$$

$$= \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial u}\right) + \frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial u}\right)$$

Example

Find the Jacobian of the transformation.

$$x = uw^2$$

$$y = v^3 - 3w$$

$$z = \frac{2uv}{w}$$

Our functions tell us that we have a 3×3 set-up, so we'll use the formula

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial w} - \frac{\partial y}{\partial w} \cdot \frac{\partial z}{\partial u}\right)$$

$$+\frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u} \cdot \frac{\partial z}{\partial v} - \frac{\partial y}{\partial v} \cdot \frac{\partial z}{\partial u}\right)$$

We need to start by finding the partial derivatives of x, y and z with respect to u, v and w.

$$\frac{\partial x}{\partial u} = w^2$$

$$\frac{\partial x}{\partial v} = 0$$

$$\frac{\partial x}{\partial w} = 2uw$$

and

$$\frac{\partial y}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = 3v^2$$

$$\frac{\partial y}{\partial w} = -3$$

and

$$\frac{\partial z}{\partial u}=\frac{2v}{w}$$

$$\frac{\partial z}{\partial v}=\frac{2u}{w}$$

$$\frac{\partial z}{\partial w}=-\frac{2uv}{w^2}$$

We'll plug the partial derivatives into our formula and get

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=w^2\left[3v^2\left(-\frac{2uv}{w^2}\right)-(-3)\left(\frac{2u}{w}\right)\right]-0\left[0\left(-\frac{2uv}{w^2}\right)-(-3)\left(\frac{2v}{w}\right)\right]$$

$$+2uw\left[0\left(\frac{2u}{w}\right)-3v^2\left(\frac{2v}{w}\right)\right]$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=w^2\left(-\frac{6uv^3}{w^2}+\frac{6u}{w}\right)+2uw\left(-\frac{6v^3}{w}\right)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=-\frac{6uv^3w^2}{w^2}+\frac{6uw^2}{w}-\frac{12uv^3w}{w}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=-6uv^3+6uw-12uv^3$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=-18uv^3+6uw$$

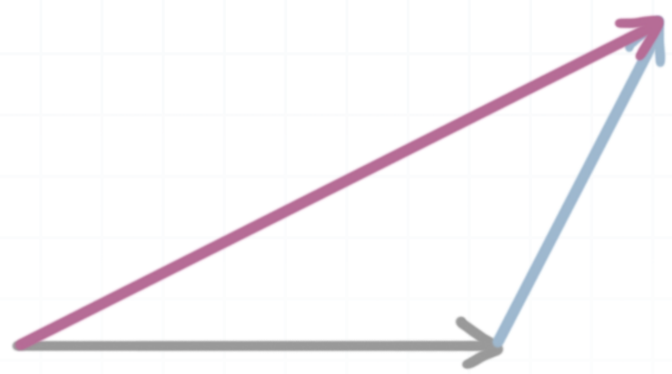
$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=6uw-18uv^3$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = 6u\left(w - 3v^3\right)$$

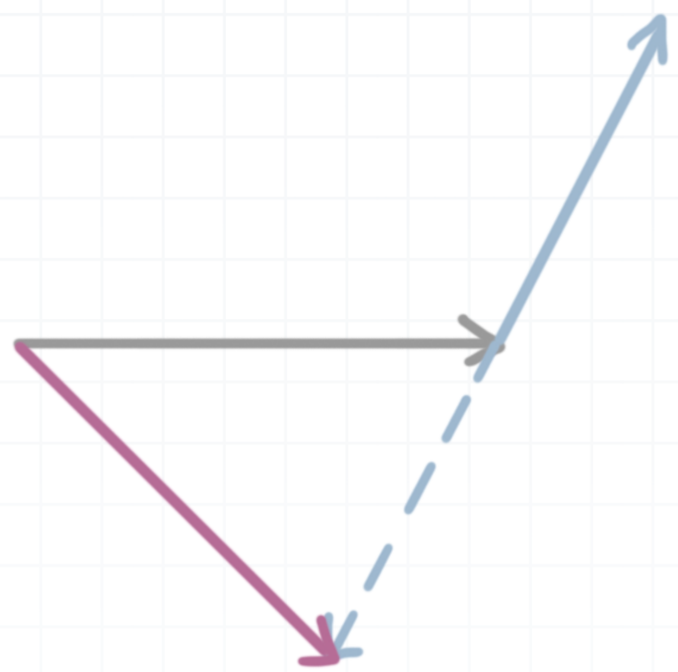
This is the Jacobian of the transformation.

Combinations of vectors

When we want to find the combination of two vectors, we take just match up the initial point of the second vector with the terminal point of the first vector, and then we draw a new third vector from the initial point of the first to the terminal point of the second. In other words, the combination of gray and blue is purple:



Essentially, combining two vectors gives us the same result as adding the vectors. In the example above, gray + blue = purple. We can also subtract vectors. If a vector is being subtracted, we move in exactly the opposite direction of the original vector. In the example below, gray - blue = purple. The solid blue vector is the original vector, but since we're subtracting, we move in the opposite direction.



Let's do an example where we find different vector combinations.

Example

Find the vector combinations.

$$\overrightarrow{AB} + \overrightarrow{BC}$$

$$\overrightarrow{BC} - \overrightarrow{AC}$$

$$\overrightarrow{AC} + \overrightarrow{CB} - \overrightarrow{DB}$$

For $\overrightarrow{AB} + \overrightarrow{BC}$:

The initial point of $\overrightarrow{AB}$ is A, and its terminal point is B. The initial point of $\overrightarrow{BC}$ is B (the terminal point of A), and its terminal point is C. Therefore, the combination of these two vectors, from the starting point of the first to the ending point of the last, is

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

For $\overrightarrow{BC} - \overrightarrow{AC}$:

When two vectors are subtracted, we can change the negative vector to positive by flipping the direction of the vector. In this case, $-\overrightarrow{AC}$ becomes $+\overrightarrow{CA}$. Then we just combine them as normal.

$$\overrightarrow{BC} - \overrightarrow{AC} = \overrightarrow{BC} + \overrightarrow{CA}$$

$$\overrightarrow{BC} - \overrightarrow{AC} = \overrightarrow{BA}$$

For $\overrightarrow{AC} + \overrightarrow{CB} - \overrightarrow{DB}$:

We'll start by changing the negative vector to a positive by flipping the direction of the vector. In this case, $-\overrightarrow{DB}$ becomes $+\overrightarrow{BD}$. We'll get rid of the negative and then combine the vectors two at a time.

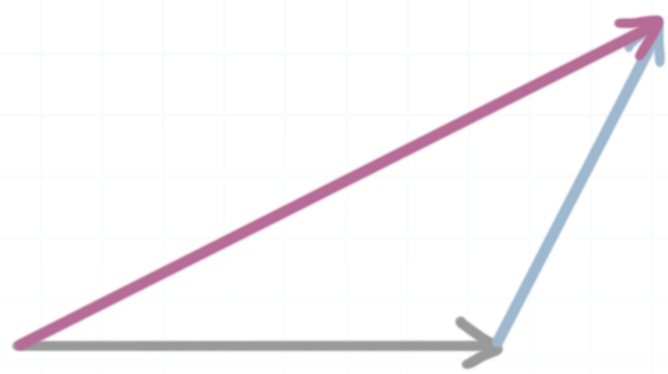
$$\overrightarrow{AC} + \overrightarrow{CB} - \overrightarrow{DB} = \overrightarrow{AC} + \overrightarrow{CB} + \overrightarrow{BD}$$

$$\overrightarrow{AC} + \overrightarrow{CB} - \overrightarrow{DB} = \overrightarrow{AB} + \overrightarrow{BD}$$

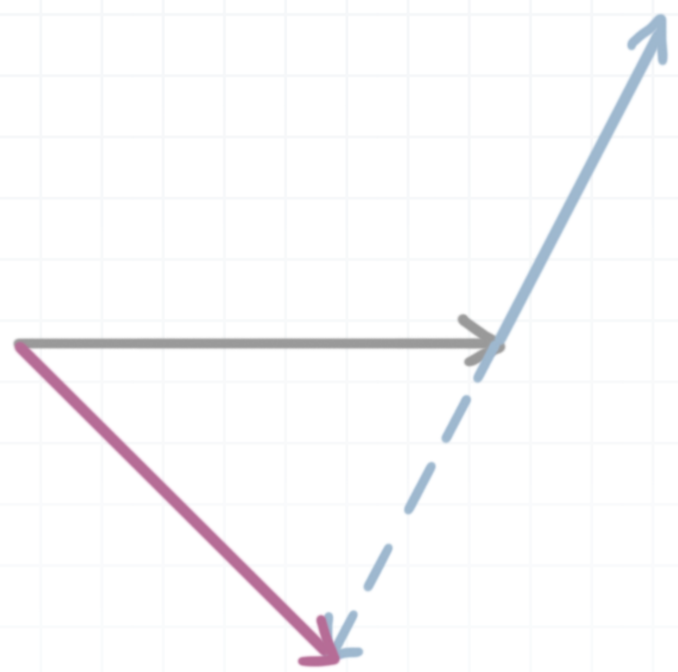
$$\overrightarrow{AC} + \overrightarrow{CB} - \overrightarrow{DB} = \overrightarrow{AD}$$

Sum of two vectors

When we want to find the combination of two vectors, we take just match up the initial point of the second vector with the terminal point of the first vector, and then we draw a new third vector from the initial point of the first to the terminal point of the second. In other words, the combination of gray and blue is purple:



Essentially, combining two vectors gives us the same result as adding the vectors. In the example above, gray + blue = purple. We can also subtract vectors. If a vector is being subtracted, we move in exactly the opposite direction of the original vector. In the example below, gray - blue = purple. The solid blue vector is the original vector, but since we're subtracting, we move in the opposite direction.



When we're given numerical values for the vectors, we'll just sum the x-coordinates to get a new x-coordinate, and sum the y-coordinates to get a new y-coordinate.

Example

Find the sum of the vectors.

$u = \langle 2,1 \rangle$ and $v = \langle -1,5 \rangle$

$u = 2i - 3j$ and $v = 6i + 2j$

For $u = \langle 2,1 \rangle$ and $v = \langle -1,5 \rangle$:

To sum the vectors $u = \langle 2,1 \rangle$ and $v = \langle -1,5 \rangle$, we just sum the x-coordinates to get a new x-coordinate, and then we do the same for the y-coordinates. We can call our new vector w.

$$w = \langle 2 + (-1), 1 + 5 \rangle$$

$$w = \langle 1,6 \rangle$$

For $u = 2i - 3j$ and $v = 6i + 2j$:

To sum the vectors $u = 2i - 3j$ and $v = 6i + 2j$, we'll take the coefficients from our i and j terms, and add them together to find the coefficients on these terms for the vector w.

$$w = (2 + 6)i + (-3 + 2)j$$

$$w = 8i - j$$

We could also write the vector $w = \langle 8, -1 \rangle$.

Magnitude and angle of the resultant force

When we're given two vectors with the same initial point, and they're different lengths and pointing in different directions, we can think about each of them as a force. The longer the vector, the more force it pulls in its direction.

Oftentimes we want to be able to find the net force of the two vectors, which will be a third vector that counterbalances the force and direction of the first two. Think about the resultant vector as representing the amount and direction of force that cancels out the first two vectors, leaving the whole system in balance.

In order to define this third vector, we need to find

its **magnitude** (its length), which will be force, in Newtons N, and

its **angle**, from the positive direction of the x-axis.

To find the magnitude and angle of a resultant force, we

create vector equations for each of the given forces

add the vector equations together to get the vector equation of the resultant force

find **magnitude** of the resultant force using the new vector equation and the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

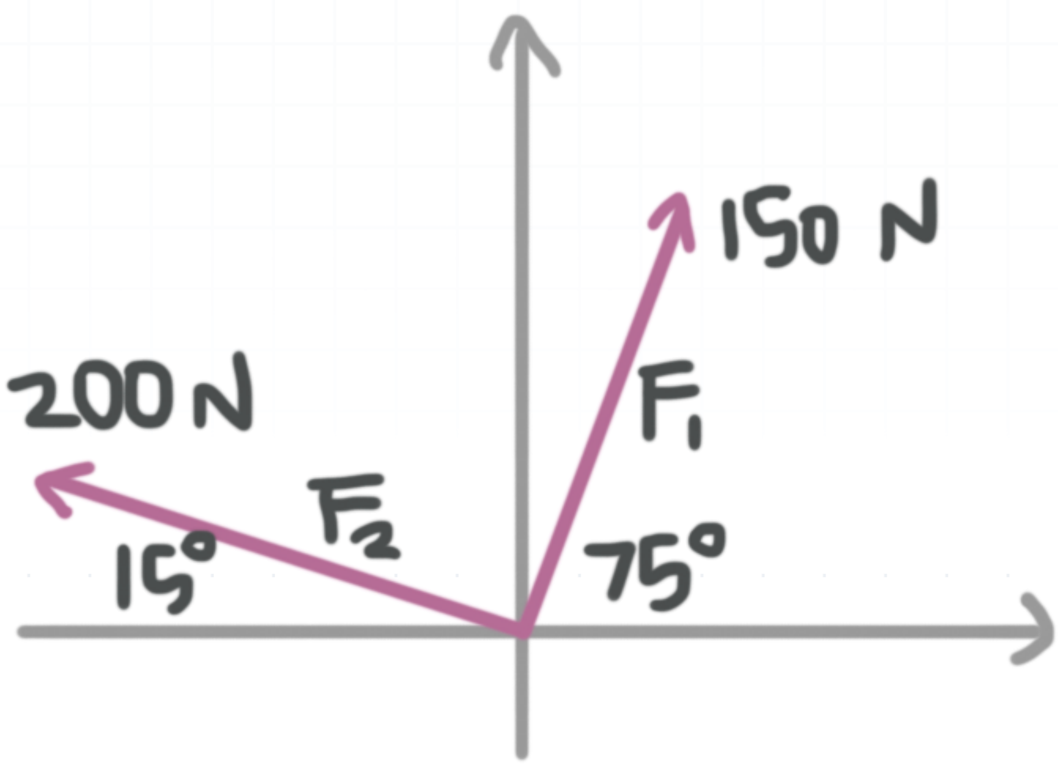
find the **angle** of the resultant force using the new vector equation and the formula

$$\theta_R = 180^\circ - \arctan\frac{|y|}{|x|}$$

Let's do an example in which we're given two forces in the xy-plane.

Example

Find the magnitude and angle of the resultant force.



We'll use the forces and the angles to find vector equations for F_1 and F_2.

$$F_1 = 150\cos 75^\circ\ \mathbf{i} + 150\sin 75^\circ\ \mathbf{j}$$

$$F_1 = 38.82\ \mathbf{i} + 144.89\ \mathbf{j}$$

$$F_1 = \langle 38.82,\ 144.89\rangle$$

and

$$F_2 = -200\cos 15^\circ\ \mathbf{i} + 200\sin 15^\circ\ \mathbf{j}$$

$$F_2 = -193.19\ \mathbf{i} + 51.76\ \mathbf{j}$$

$$F_2 = \langle -193.19,\ 51.76\rangle$$

We'll add our forces together to find the vector equation of the resultant force.

$$F_R = F_1 + F_2$$

$$F_R = 38.82\ \mathbf{i} + 144.89\ \mathbf{j} - 193.19\ \mathbf{i} + 51.76\ \mathbf{j}$$

$$F_R = -154.37\ \mathbf{i} + 196.65\ \mathbf{j}$$

$$F_R = \langle -154.37,\ 196.65\rangle$$

Now we can plug the vector equation into the distance formula to find the length of the resultant force vector. Because both F_1 and F_2 have their initial point at the origin, (x_1, y_1) will be $(0,0)$.

$$D_R = \sqrt{(x_2 - 0)^2 + (y_2 - 0)^2}$$

$$D_R = \sqrt{(-154.37 - 0)^2 + (196.65 - 0)^2}$$

$$D_R = \sqrt{23{,}830.10 + 38{,}671.22}$$

$$D_R = 250.00$$

To find the angle of the resultant force, we'll use the formula

$$\theta_R = 180^\circ - \arctan \frac{|y|}{|x|}$$

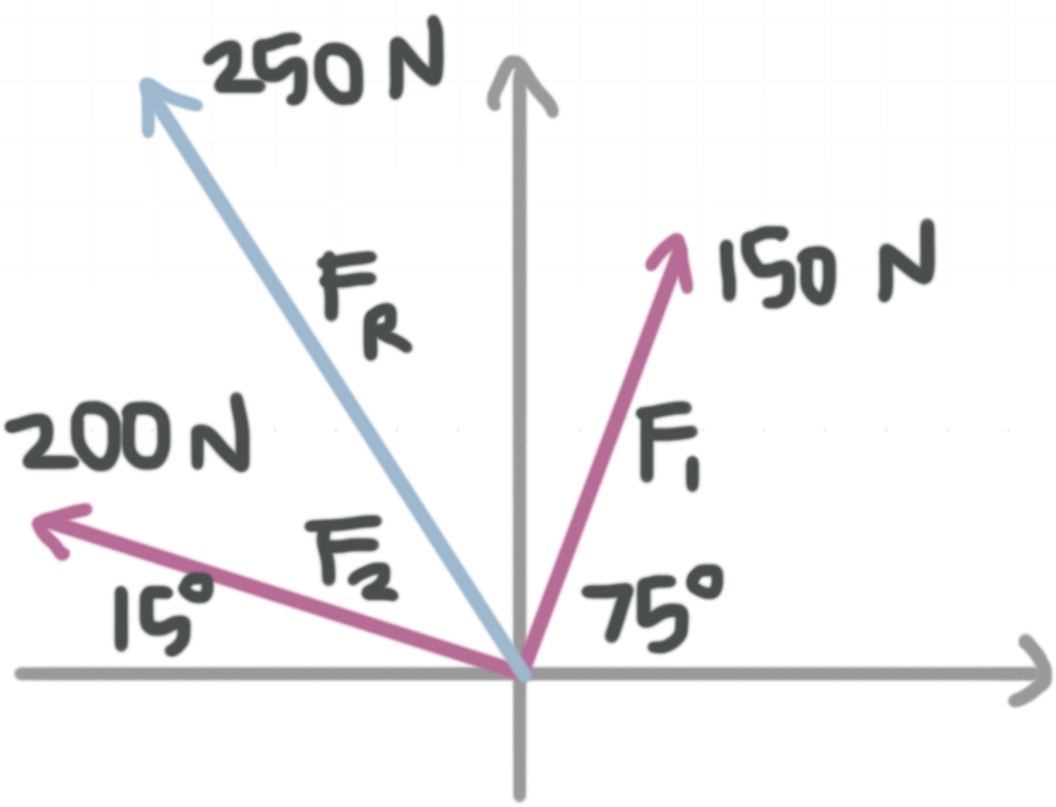
where $F_R = \langle x, y \rangle$. Plugging in x and y from the resultant force, we get

$$\theta_R = 180^\circ - \arctan \frac{196.65}{154.37}$$

$$\theta_R = 180^\circ - 51.87^\circ$$

$$\theta_R = 128.13^\circ$$

The magnitude of the resultant force is 250 N and the angle of the resultant force is 128.13°.



Notice how we built the vector equations for F_1 and F_2 in this last example.

When we measure the angle of the vector, we always measure it from the horizontal axis, which means we'll measure the angles of vectors in the first and fourth quadrants from the positive direction of the horizontal axis, but measure the angles of vectors in the second and third quadrants from the negative direction of the horizontal axis.

Because F_1 fell in the first quadrant, we measured its angle from the positive horizontal axis as 75°. But F_2 fell in the second quadrant, which means we measured its angle from the negative direction of the horizontal axis as 15°.

And we'll always treat the angle between the vector and the horizontal axis as a positive angle. So even for vectors in the third and fourth quadrants, we'll still measure a positive angle from the horizontal axis.

So, while we always keep the angles positive, we do need to change the signs of the coefficients on $\mathbf{i}$ and $\mathbf{j}$, depending on the quadrant of the vector. Consider a generic vector,

$$F_V = F_{V_x} \cos\theta \ \mathbf{i} + F_{V_y} \sin\theta \ \mathbf{j}$$

The signs we use for F_{V_x} and F_{V_y} depend on the quadrant.

In the first quadrant, F_{V_x} is positive, and F_{V_y} is positive

In the second quadrant, F_{V_x} is negative, and F_{V_y} is positive

In the third quadrant, F_{V_x} is negative, and F_{V_y} is negative

In the fourth quadrant, F_{V_x} is positive, and F_{V_y} is negative

That's why, in the previous example, F_1 in the first quadrant got two positive signs, $F_1 = 150\cos 75^\circ\ \mathbf{i} + 150\sin 75^\circ\ \mathbf{j}$, and F_2 in the second quadrant got a negative sign on the first term and a positive sign on the second term, $F_2 = -200\cos 15^\circ\ \mathbf{i} + 200\sin 15^\circ\ \mathbf{j}$.

Dot product of two vectors

To take the dot product of two vectors a and b, we multiply the vectors' like coordinates and then add the products together. In other words, we multiply the x coordinates of the two vectors, then add this to the product of the y coordinates. If we have vectors in three-dimensional space, we'll add the product of the z coordinates as well.

If we're given the vectors $a\langle a_1, a_2\rangle$ and $b\langle b_1, b_2\rangle$, then the dot product of a and b will be

$$a \cdot b = a_1b_1 + a_2b_2$$

Example

Find the dot product the vectors.

$$a = \langle 2,1\rangle$$

$$b = \langle 6, -2\rangle$$

To find the dot product of the vectors a and b, we'll multiply like coordinates and then add the products together.

$$a \cdot b = (2)(6) + (1)(-2)$$

$$a \cdot b = 12 - 2$$

$$a \cdot b = 10$$

The dot product of the vectors a and b is $a \cdot b = 10$.

Example

Find the dot product the vectors.

$$c = 2i - 6j + k$$

$$d = 2j - 3k$$

Converting our vectors into standard form, we get

$$c = \langle 2, -6, 1 \rangle$$

$$d = \langle 0, 2, -3 \rangle$$

To find the dot product of the vectors c and d, we'll multiply like coordinates and then add the products together.

$$c \cdot d = (2)(0) + (-6)(2) + (1)(-3)$$

$$c \cdot d = 0 - 12 - 3$$

$$c \cdot d = -15$$

The dot product of the vectors c and d is $c \cdot d = -15$.

Angle between two vectors

To find the angle between two vectors, we'll use the formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where a and b are the given vectors, $a \cdot b$ is the dot product of the vectors, $|a|$ is the length of a, and $|b|$ is the length of b. To find the length of the vectors, we'll use the distance formula

$$D = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

Example

Find the angle between the vectors.

$$a = \langle 4, -2, 3 \rangle$$

$$b = \langle 1, 3, -1 \rangle$$

We'll start by finding the dot product of a and b, $a \cdot b$.

$$a \cdot b = (4)(1) + (-2)(3) + (3)(-1)$$

$$a \cdot b = 4 - 6 - 3$$

$$a \cdot b = -5$$

Now we'll use the distance formula to find the length of each vector, remembering that the initial point of both vectors is the origin, and the terminal points are given by $(4,-2,3)$ and $(1,3,-1)$.

$$D=\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2+\left(z_2-z_1\right)^2}$$

$$|a|=D_a=\sqrt{(4-0)^2+(-2-0)^2+(3-0)^2}$$

$$|a|=D_a=\sqrt{16+4+9}$$

$$|a|=D_a=\sqrt{29}$$

and

$$|b|=D_b=\sqrt{(1-0)^2+(3-0)^2+(-1-0)^2}$$

$$|b|=D_b=\sqrt{1+9+1}$$

$$|b|=D_b=\sqrt{11}$$

Plugging everything we've calculated into our formula for the angle between two vectors, we get

$$\cos\theta=\frac{a\cdot b}{|a||b|}$$

$$\cos\theta=\frac{-5}{\left|\sqrt{29}\right|\left|\sqrt{11}\right|}$$

$$\cos\theta=\frac{-5}{\left|\sqrt{319}\right|}$$

$$\theta = 106.3^\circ$$

Orthogonal, parallel, or neither

We say that two vectors a and b are

orthogonal if they are perpendicular (set at 90° from each other)

$$a \cdot b = 0$$

parallel if they point in exactly the same or opposite directions, and never cross each other

after factoring out any common factors, the remaining direction numbers will be equal

neither

Since it's easy to take a dot product, it's a good idea to get in the habit of testing the vectors to see whether they're orthogonal, and then if they're not, testing to see whether they're parallel.

Example

Say whether the following vectors are orthogonal, parallel or neither.

$a = \langle 2,1 \rangle$ and $b = \langle -1,2 \rangle$

$a = 2i + 3j + 5k$ and $b = i + 4j - 2k$

$a = \langle 1, -2, 3 \rangle$ and $b = \langle -2, 4, -6 \rangle$

For $a = \langle 2,1 \rangle$ and $b = \langle -1,2 \rangle$:

We'll take the dot product of our vectors to see whether they're orthogonal to one another.

$$a \cdot b = (2)(-1) + (1)(2)$$

$$a \cdot b = -2 + 2$$

$$a \cdot b = 0$$

Since the dot product is 0, we can say that $a = \langle 2,1 \rangle$ and $b = \langle -1,2 \rangle$ are orthogonal. If we know that they're orthogonal, then by definition they can't be parallel, so we're done with our testing.

For $a = 2i + 3j + 5k$ and $b = i + 4j - 2k$:

First we'll put the vectors in standard form.

$$a = 2i + 3j + 5k$$

$$a = \langle 2,3,5 \rangle$$

and

$$b = i + 4j - 2k$$

$$b = \langle 1,4,-2 \rangle$$

Now we'll take the dot product of our vectors to see whether they're orthogonal to one another.

$$a \cdot b = (2)(1) + (3)(4) + (5)(-2)$$

$$a \cdot b = 2 + 12 - 10$$

$$a \cdot b = 4$$

Since the dot product is not 0, we can say that $a = 2i + 3j + 5k$ and $b = i + 4j - 2k$ are not orthogonal.

To say whether or not the vectors are parallel, we want to look for a common factor in the direction numbers of either vector, and pull it out until both vectors are irreducible.

$a = \langle 2,3,5 \rangle$ is already irreducible because 2, 3 and 5 have no common factors. $b = \langle 1,4,-2 \rangle$ is also irreducible because 1, 4 and -2 have no common factors either.

Therefore, we can say that $a = 2i + 3j + 5k$ and $b = i + 4j - 2k$ are neither orthogonal nor parallel.

For $a = \langle 1,-2,3 \rangle$ and $b = \langle -2,4,-6 \rangle$:

We'll take the dot product of our vectors to see whether they're orthogonal to one another.

$$a \cdot b = (1)(-2) + (-2)(4) + (3)(-6)$$

$$a \cdot b = -2 - 8 - 18$$

$$a \cdot b = -28$$

Since the dot product is not 0, we can say that $a = \langle 1,-2,3 \rangle$ and $b = \langle -2,4,-6 \rangle$ are not orthogonal.

To say whether or not the vectors are parallel, we want to look for a common factor in the direction numbers of either vector, and pull it out until both vectors are irreducible.

$a = \langle 1, -2, 3 \rangle$ is already irreducible because 1, -2 and 3 have no common factors. On the other hand, $b = \langle -2, 4, -6 \rangle$ has a common factor of -2 that can be factored out of the vector.

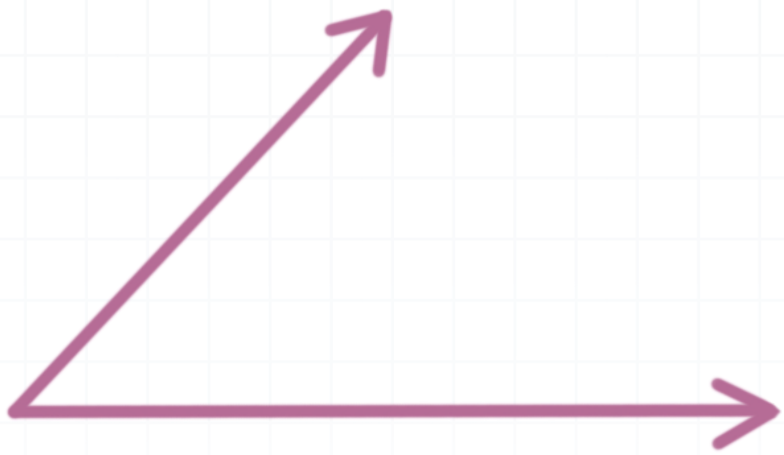
$$b = \langle -2, 4, -6 \rangle$$

$$b = -2 \langle 1, -2, 3 \rangle$$

Now the direction numbers of a and b are equal, so we can say that a and b are parallel.

Acute angle between the lines

An acute angle is an angle that's less than 90°, like this:



If we want to find the acute angle between two lines, we can convert the lines to standard vector form and then use the formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where a and b are the given vectors, $a \cdot b$ is the dot product of the vectors, $|a|$ is the magnitude of the vector a (its length) and $|b|$ is the magnitude of the vector b (its length). We can find the magnitude of both vectors using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

for a two-dimensional vector where the point (x_1, y_1) is the origin $(0,0)$.

If the formula above gives a result that's greater than 90°, then we've found the obtuse angle between the lines. To find the acute angle, we just subtract the obtuse angle from 180°, and we'll get the acute angle.

Example

Find the acute angle between the lines.

$$x + 3y = 2$$

$$3x - 6y = 5$$

First we'll convert the lines to standard vector form.

$$x + 3y = 2$$

$$a = \langle 1,3 \rangle$$

and

$$3x - 6y = 5$$

$$b = \langle 3, -6 \rangle$$

Before we can use our formula, we need to find the dot product of a and b.

$$a \cdot b = (1)(3) + (3)(-6)$$

$$a \cdot b = 3 - 18$$

$$a \cdot b = -15$$

Now we need to find the length of each vector using the distance formula.

$$|a| = \sqrt{(1-0)^2 + (3-0)^2}$$

$$|a| = \sqrt{1+9}$$

$$|a| = \sqrt{10}$$

and

$$|b| = \sqrt{(3-0)^2 + (-6-0)^2}$$

$$|b| = \sqrt{9+36}$$

$$|b| = \sqrt{45}$$

Plugging $a \cdot b = -15$, $|a| = \sqrt{10}$, and $|b| = \sqrt{45}$ into the formula, we get

$$\cos\theta = \frac{-15}{\sqrt{10}\sqrt{45}}$$

$$\cos\theta = \frac{-15}{\sqrt{450}}$$

$$\cos\theta = \frac{-15}{\sqrt{450}}$$

$$\cos\theta = \frac{-15}{\sqrt{225 \cdot 2}}$$

$$\cos\theta = \frac{-15}{15\sqrt{2}}$$

$$\cos\theta = \frac{-1}{\sqrt{2}}$$

Rationalize the denominator.

$$\cos\theta = \frac{-1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)$$

$$\cos\theta = \frac{-\sqrt{2}}{2}$$

Looking at the top half of the unit circle, we can see that

$$\theta = \frac{3\pi}{4} = 135^\circ$$

Since the answer is greater than 90°, we've found the obtuse angle between the lines. To find the acute angle, we'll just subtract this value from 180°.

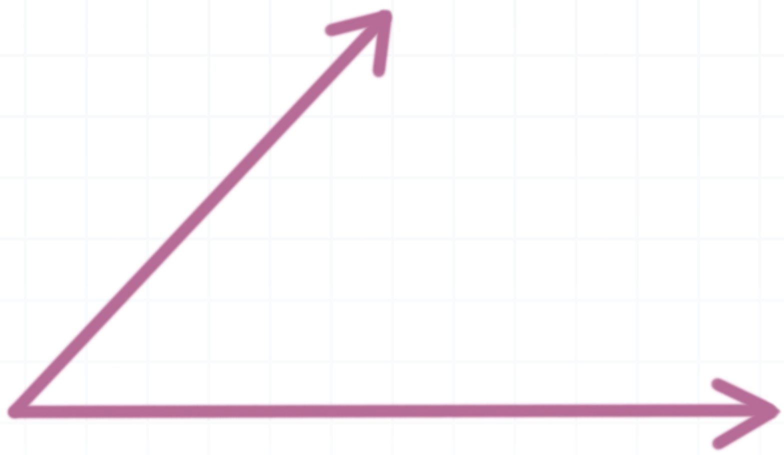
$$\theta = 180^\circ - 135^\circ$$

$$\theta = 45^\circ$$

The acute angle between the lines is 45°.

Acute angle between the curves

An acute angle is angle that's less than 90°, like this:



If we want to find the acute angle between two curves, we'll find the tangent lines to both curves at their point(s) of intersection, convert the tangent lines to standard vector form and then use the formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where a and b are the given vectors, $a \cdot b$ is the dot product of the vectors, $|a|$ is the magnitude of the vector a (its length) and $|b|$ is the magnitude of the vector b (its length). We can find the magnitude of both vectors using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

for a two-dimensional vector where the point (x_1, y_1) is the origin $(0,0)$.

If the formula above gives a result that's greater than 90°, then we've found the obtuse angle between the lines. To find the acute angle, we just subtract the obtuse angle from 180°, and we'll get the acute angle.

Example

Find the acute angle between the curves.

$$y = x^2$$

$$y = 2x^2 - 1$$

We'll start by setting the curves equal to each other and solving for x, in order to find the point(s) where the curves intersect each other.

$$x^2 = 2x^2 - 1$$

$$-x^2 = -1$$

$$x^2 = 1$$

$$x = \pm 1$$

Since we have two points of intersection, we'll need to find two acute angles, one for each of the points of intersection.

We'll plug both values of x into $y = x^2$ to find the corresponding y-values. We can use either curve; they should both return the same y-values.

For $x = 1$:

$$y = x^2$$

$$y = (1)^2$$

$$y = 1$$

$$(1,1)$$

For $x = -1$:

$$y = x^2$$

$$y = (-1)^2$$

$$y = 1$$

$$(-1,1)$$

We need to find the tangent lines for both curves at each of the points of intersection. Remember that to find a tangent line, we'll take the derivative of the function, then evaluate the derivative at the point of intersection to find the slope of the tangent line there. Then we'll plug the slope and the tangent point into the point-slope formula to find the equation of the tangent line.

At $(1,1)$,

for $y = x^2$:

$$y' = 2x$$

$$y'(1,1) = 2(1)$$

$$y'(1,1) = 2$$

for $y = 2x^2 - 1$:

$$y' = 4x$$

$$y' = 4(1)$$

$$y'(1,1) = 4$$

At $(-1,1)$,

for $y = x^2$:

for $y = 2x^2 - 1$:

$$y' = 2x$$

$$y'(-1,1) = 2(-1)$$

$$y'(-1,1) = -2$$

$$y' = 4x$$

$$y' = 4(-1)$$

$$y'(-1,1) = -4$$

Plugging the slopes and the intersection points into the point-slope formula for the equation of a line, we get

At $(1,1)$,

for $y = x^2$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 1$$

for $y = 2x^2 - 1$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 4(x - 1)$$

$$y - 1 = 4x - 4$$

$$y = 4x - 3$$

At $(-1,1)$,

for $y = x^2$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2\left[x - (-1)\right]$$

$$y - 1 = -2(x + 1)$$

$$y - 1 = -2x - 2$$

$$y = -2x - 1$$

for $y = 2x^2 - 1$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4\left[x - (-1)\right]$$

$$y - 1 = -4(x + 1)$$

$$y - 1 = -4x - 4$$

$$y = -4x - 3$$

We need to convert our tangent line equations to standard vector form.

At $(1,1)$,

for $y = x^2$:

$$y = 2x - 1$$

$$-2x + y = -1$$

$$\langle -2,1 \rangle$$

for $y = 2x^2 - 1$:

$$y = 4x - 3$$

$$-4x + y = -3$$

$$\langle -4,1 \rangle$$

At $(-1,1)$,

for $y = x^2$:

$$y = -2x - 1$$

$$2x + y = -1$$

$$\langle 2,1 \rangle$$

for $y = 2x^2 - 1$:

$$y = -4x - 3$$

$$4x + y = -3$$

$$\langle 4,1 \rangle$$

To summarize our findings so far, we can say that we need to find the acute angle

between the vectors $a = \langle -2,1 \rangle$ and $b = \langle -4,1 \rangle$ at the point $(1,1)$

between the vectors $c = \langle 2,1 \rangle$ and $d = \langle 4,1 \rangle$ at the point $(-1,1)$

Before we can use the cosine formula to find the acute angle, we need to find the dot products $a \cdot b$ and $c \cdot d$ and the magnitude of each vector.

For $a = \langle -2,1 \rangle$ and $b = \langle -4,1 \rangle$ at the point $(1,1)$:

For $a \cdot b$:

$$a \cdot b = (-2)(-4) + (1)(1)$$

$$a \cdot b = 8 + 1$$

$$a \cdot b = 9$$

For $|a|$:

$$|a| = \sqrt{(-2-0)^2 + (1-0)^2}$$

$$|a| = \sqrt{4+1}$$

$$|a| = \sqrt{5}$$

For $|b|$:

$$|b| = \sqrt{(-4-0)^2 + (1-0)^2}$$

$$|b| = \sqrt{16+1}$$

$$|b| = \sqrt{17}$$

For $c = \langle 2,1 \rangle$ and $d = \langle 4,1 \rangle$ at the point $(-1,1)$:

For $c \cdot d$:

$$c \cdot d = (2)(4) + (1)(1)$$

$$c \cdot d = 8 + 1$$

$$c \cdot d = 9$$

For $|c|$:

$$|c| = \sqrt{(2-0)^2 + (1-0)^2}$$

$$|c| = \sqrt{4+1}$$

$$|c| = \sqrt{5}$$

For $|d|$:

$$|d| = \sqrt{(4-0)^2 + (1-0)^2}$$

$$|d| = \sqrt{16+1}$$

$$|d| = \sqrt{17}$$

Finally, plug the dot products and magnitudes we've found into our formula.

For $a = \langle -2,1 \rangle$ and $b = \langle -4,1 \rangle$ at the point $(1,1)$:

$$\cos\theta = \frac{9}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{9}{\sqrt{85}}$$

$$\theta = \arccos\frac{9}{\sqrt{85}}$$

$$\theta = 12.5^\circ$$

For $c = \langle 2,1 \rangle$ and $d = \langle 4,1 \rangle$ at the point $(-1,1)$:

$$\cos\theta = \frac{9}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{9}{\sqrt{85}}$$

$$\theta = \arccos\frac{9}{\sqrt{85}}$$

$$\theta = 12.5^\circ$$

In conclusion, we can say that

the acute angle between the tangent lines $y = 2x - 1$ and $y = 4x - 3$ at the tangent point $(1,1)$ is 12.5°

the acute angle between the tangent lines $y = -2x - 1$ and $y = -4x - 3$ at the tangent point $(-1,1)$ is 12.5°

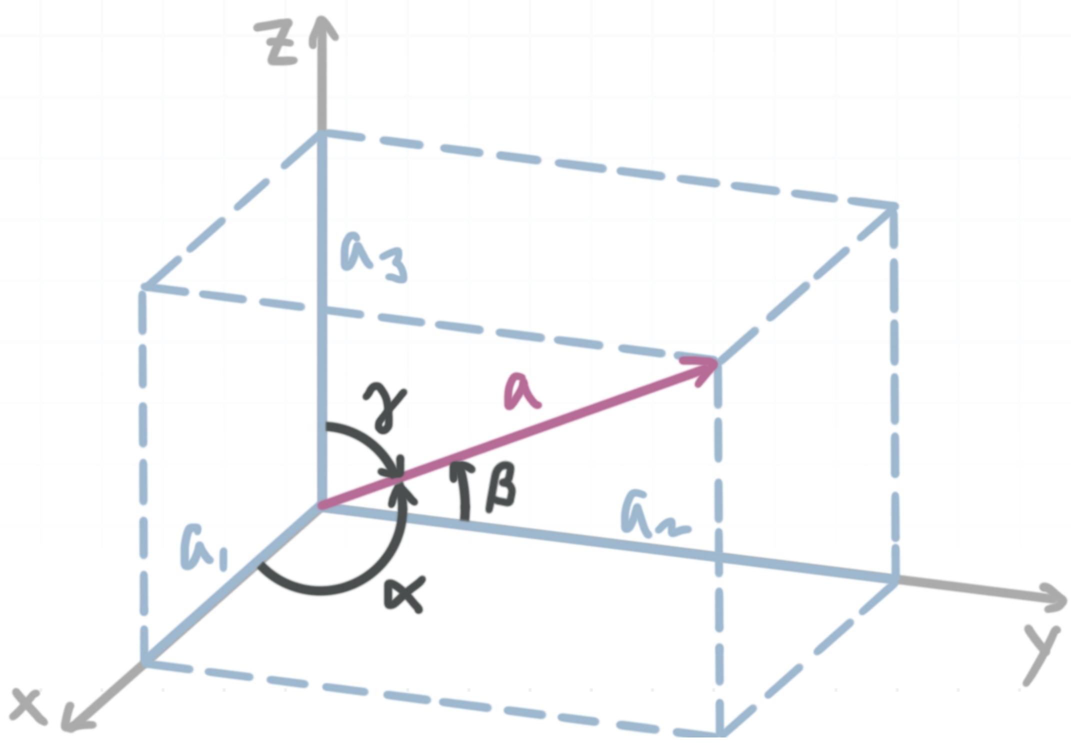
And therefore, we can say that

the acute angle between the curves $y = x^2$ and $y = 2x^2 - 1$ at the intersection point $(1,1)$ is 12.5°

the acute angle between the curves $y = x^2$ and $y = 2x^2 - 1$ at the intersection point $(-1,1)$ is 12.5°

Direction cosines and direction angles

The **direction angles** of a non-zero vector $a = \langle a_1, a_2, a_3 \rangle$ are the angles α (alpha), β (beta), and γ (gamma) that the vector a makes with the positive x-, y-, and z-axes, respectively. In other words, α is the direction angle between a and the positive x-axis, β is the direction angle between a and the positive y-axis, and γ is the direction angle between a and the positive z-axis.



If the length (magnitude) of a is

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

then we take the cosine of each direction angle to get the **direction cosines** of the vector a.

$$\cos\alpha = \frac{a_1}{|a|} \qquad \cos\beta = \frac{a_2}{|a|} \qquad \cos\gamma = \frac{a_3}{|a|}$$

Once we have the direction cosines, we can find the direction angles by applying the inverse cosine to both sides of each of these direction cosine equations.

$$\alpha = \arccos \frac{a_1}{|a|} \qquad \beta = \arccos \frac{a_2}{|a|} \qquad \gamma = \arccos \frac{a_3}{|a|}$$

It can also be helpful to know that the direction angles of a will satisfy

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

and that the direction cosines of a are the components of the unit vector in the direction of a.

$$\frac{1}{|a|}a = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$$

Let's do an example where we find the direction cosines and direction angles of a vector.

Example

Find the direction angles of the vector $a = \langle 5, -3, 1 \rangle$.

We find the magnitude of the vector a using the distance formula.

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|a| = \sqrt{5^2 + (-3)^2 + 1^2}$$

$$|a| = \sqrt{25 + 9 + 1}$$

$$|a| = \sqrt{35}$$

Plugging the vector's components and magnitude into the direction cosine formulas, we get

$$\cos\alpha = \frac{5}{\sqrt{35}} \qquad \cos\beta = \frac{-3}{\sqrt{35}} \qquad \cos\gamma = \frac{1}{\sqrt{35}}$$

Now that we have the direction cosines, we can apply the inverse cosine to both sides of each equation to find the direction angles.

$$\alpha = \arccos\frac{5}{\sqrt{35}} \qquad \beta = \arccos\frac{-3}{\sqrt{35}} \qquad \gamma = \arccos\frac{1}{\sqrt{35}}$$

$$\alpha \approx 32.3^\circ \qquad \beta \approx 120.5^\circ \qquad \gamma \approx 80.3^\circ$$

Let's do one more example, this time with the vector given in terms of the standard unit vectors.

Example

A vector b has direction angles $\alpha = \pi/3$ and $\beta = \pi/6$. Find the third direction angle γ.

Since we already know two of the direction angles of the vector b, we can use $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ to find γ.

$$\cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{6}\right) + \cos^2\gamma = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \cos^2\gamma = 1$$

$$\frac{1}{4} + \frac{3}{4} + \cos^2\gamma = 1$$

Rearrange the equation to solve for γ.

$$\cos^2\gamma = 0$$

$$\cos\gamma = 0$$

$$\gamma = \frac{\pi}{2}$$

The direction angles are $\alpha = \pi/3$, $\beta = \pi/6$, and $\gamma = \pi/2$, or $\alpha = 60^\circ$, $\beta = 30^\circ$, and $\gamma = 90^\circ$.

Scalar equation of a line

To find the scalar equation of a line, we'll use the formulas

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

where $P_0(x_0, y_0, z_0)$ is a given point and $v = \langle a, b, c \rangle$ is the given vector. The vector may also be in the format $v = ai + bj + ck$.

Example

Find the scalar equation of the line.

$$P(2,3,1)$$

$$\langle 2, -1, 5 \rangle$$

Plugging the given point and the given vector into our formulas for x, y and z, we get

$$x = 2 + 2t$$

$$y = 3 - t$$

$$z = 1 + 5t$$

The scalar equation of the line is given by $x = 2 + 2t$, $y = 3 - t$ and $z = 1 + 5t$.

Scalar equation of a plane

To find the scalar equation of a plane, we'll use the formula

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where $P_0(x_0, y_0, z_0)$ is a given point and $v = \langle a, b, c \rangle$ is the normal vector to the plane. The vector may also be in the format $v = ai + bj + ck$.

Example

Find the scalar equation of the plane.

$$P(1,4,-8)$$

$$\langle 3,6,2 \rangle$$

Plugging the given point and the given vector into our formula, we get

$$3(x-1) + 6(y-4) + 2\left[z - (-8)\right] = 0$$

$$3(x-1) + 6(y-4) + 2(z+8) = 0$$

$$3x - 3 + 6y - 24 + 2z + 16 = 0$$

$$3x + 6y + 2z = 11$$

The scalar equation of the plane is given by $3x + 6y + 2z = 11$.

Scalar and vector projections

Scalar projections

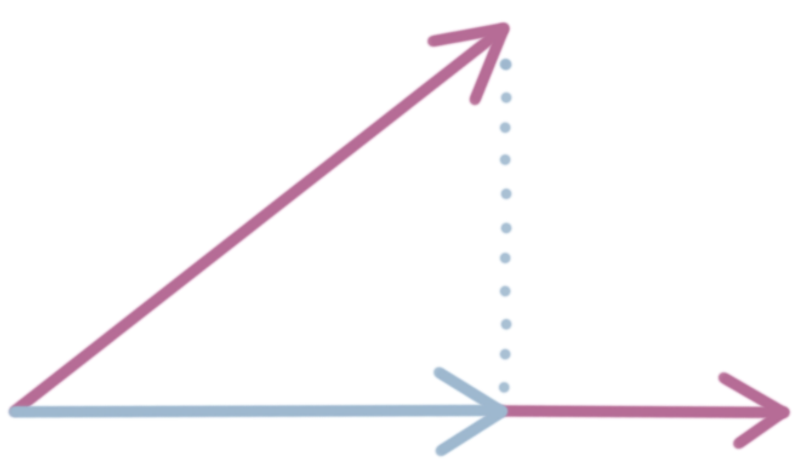
The scalar projection of one vector onto another (also called the component of one vector along another), is

$$\text{comp}_a b = \frac{a \cdot b}{|a|}$$

where $a \cdot b$ is the dot product of the vectors a and b, and $|a|$ is the length of a (also called the magnitude of a).

Vector projections

The vector projection of one vector onto another is like a shadow that one vector casts on another vector. For example, the projection of the top purple vector onto the bottom purple vector is the blue vector:



$$\text{proj}_a b = \left(\frac{a \cdot b}{|a|}\right)\frac{a}{|a|}$$

where $a \cdot b$ is the dot product of the vectors a and b, and $|a|$ is the length of a (also called the magnitude of a).

Example

Find the scalar and vector projections of b onto a.

$$a = i + 2j - 3k$$

$$b = 6i + j$$

Since we use the value of the scalar projection in the formula for the vector projection, we'll start by finding the scalar projection. We'll need the dot product of a and b and the magnitude of a.

We'll convert the given vector equations into the form

$$a = \langle 1,2,-3 \rangle$$

$$b = \langle 6,1,0 \rangle$$

We'll take the dot product.

$$a \cdot b = (1)(6) + (2)(1) + (-3)(0)$$

$$a \cdot b = 6 + 2 + 0$$

$$a \cdot b = 8$$

We'll find the magnitude (length) of a using the distance formula. Remember, the terminal point of a is $(1,2,-3)$ and the initial point of a is $(0,0,0)$.

$$|a| = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2 + \left(z_2 - z_1\right)^2}$$

$$|a| = \sqrt{(1-0)^2 + (2-0)^2 + (-3-0)^2}$$

$$|a| = \sqrt{1+4+9}$$

$$|a| = \sqrt{14}$$

We'll plug $a \cdot b$ and $|a|$ into the formula for the scalar projection.

$$\text{comp}_a b = \frac{8}{\sqrt{14}}$$

Since we have the scalar projection, we already have everything we need to find the vector projection.

$$\text{proj}_a b = \left(\frac{8}{\sqrt{14}}\right)\frac{i + 2j - 3k}{\sqrt{14}}$$

$$\text{proj}_a b = \frac{8i + 16j - 24k}{14}$$

$$\text{proj}_a b = \frac{4i + 8j - 12k}{7}$$

$$\text{proj}_a b = \frac{4}{7}i + \frac{8}{7}j - \frac{12}{7}k$$

To summarize our findings, we'll say that

the scalar projection of b onto a is

$$\text{comp}_a b = \frac{8}{\sqrt{14}}$$

the vector projection of b onto a is

$$\text{proj}_a b = \frac{4}{7}i + \frac{8}{7}j - \frac{12}{7}k$$

Cross product of two vectors

To take the cross product of two vectors

$$a\langle a_1, a_2, a_3\rangle$$

$$b\langle b_1, b_2, b_3\rangle$$

we'll create a matrix in the form

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

As always, we'll use the sign matrix

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

to determine the signs for our top row. We'll expand the matrix to

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$= \mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

and then take the coefficients on $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to form the cross product vector $c\langle c_1, c_2, c_3\rangle$, where

$$c_1 = a_2b_3 - a_3b_2$$

$$c_2 = a_1b_3 - a_3b_1$$

$$c_3 = a_1b_2 - a_2b_1$$

If you can remember the formula for

$$\mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

then you can skip the matrices and go straight to this step. If not, just use the matrix approach.

Example

Find the cross product of the vectors.

$$a\langle 2, -4, 1\rangle$$

$$b\langle -2, 5, 7\rangle$$

For the sake of this example, we'll assume we can't remember the formula for

$$\mathbf{i}\left(a_2b_3 - a_3b_2\right) - \mathbf{j}\left(a_1b_3 - a_3b_1\right) + \mathbf{k}\left(a_1b_2 - a_2b_1\right)$$

and use the matrix. Plugging the values from the given vectors into our 3×3 matrix, we get

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 1 \\ -2 & 5 & 7 \end{vmatrix}$$

$$\vec{a}\times\vec{b}=\mathbf{i}\begin{vmatrix}-4 & 1\\ 5 & 7\end{vmatrix}-\mathbf{j}\begin{vmatrix}2 & 1\\ -2 & 7\end{vmatrix}+\mathbf{k}\begin{vmatrix}2 & -4\\ -2 & 5\end{vmatrix}$$

$$\vec{a}\times\vec{b}=\mathbf{i}\left[(-4)(7)-(1)(5)\right]-\mathbf{j}\left[(2)(7)-(1)(-2)\right]+\mathbf{k}\left[(2)(5)-(-4)(-2)\right]$$

$$\vec{a}\times\vec{b}=\mathbf{i}(-28-5)-\mathbf{j}(14+2)+\mathbf{k}(10-8)$$

$$\vec{a}\times\vec{b}=-33\mathbf{i}-16\mathbf{j}+2\mathbf{k}$$

$$\vec{a}\times\vec{b}=\langle -33,-16,2\rangle$$

This is the cross product of the vectors a and b.

Vector orthogonal to the plane

To find the vector orthogonal to a plane, we need to start with two vectors that lie in the plane.

Sometimes our problem will give us these vectors, in which case we can use them to find the orthogonal vector.

Other times, we'll only be given three points in the plane. If we only have the three points, then we need to use them to find the two vectors that lie in the plane, which we'll do using these formulas:

Given points $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$, and $C(c_1, c_2, c_3)$

$$\overrightarrow{AB} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AB} = (AB_1)\mathbf{i} + (AB_2)\mathbf{j} + (AB_3)\mathbf{k}$$

$$\overrightarrow{AB} = AB\langle AB_1, AB_2, AB_3\rangle$$

and

$$\overrightarrow{AC} = (c_1 - a_1)\mathbf{i} + (c_2 - a_2)\mathbf{j} + (c_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AC} = (AC_1)\mathbf{i} + (AC_2)\mathbf{j} + (AC_3)\mathbf{k}$$

$$\overrightarrow{AC} = AC\langle AC_1, AC_2, AC_3\rangle$$

Once we have our vectors, whether they were given or whether we calculated them using three points in the plane, we'll take their cross product.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ AB_1 & AB_2 & AB_3 \\ AC_1 & AC_2 & AC_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} AB_2 & AB_3 \\ AC_2 & AC_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} AB_1 & AB_3 \\ AC_1 & AC_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} AB_1 & AB_2 \\ AC_1 & AC_2 \end{vmatrix}$$

$$= \mathbf{i}\left(AB_2AC_3 - AB_3AC_2\right) - \mathbf{j}\left(AB_1AC_3 - AB_3AC_1\right) + \mathbf{k}\left(AB_1AC_2 - AB_2AC_1\right)$$

The result is the vector orthogonal to the plane.

Example

Find the vector orthogonal to the plane that includes the given points.

$$A(1,3,2)$$

$$B(-2,4,1)$$

$$C(3,0,-2)$$

We need to use these three points to find two vectors that lie in the plane, so that we can then find the cross product of those vectors.

$$\overrightarrow{AB} = (b_1 - a_1)\mathbf{i} + (b_2 - a_2)\mathbf{j} + (b_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AB} = (-2 - 1)\mathbf{i} + (4 - 3)\mathbf{j} + (1 - 2)\mathbf{k}$$

$$\overrightarrow{AB} = -3\mathbf{i} + \mathbf{j} - \mathbf{k}$$

$$\overrightarrow{AB} = AB\langle -3,1,-1\rangle$$

and

$$\overrightarrow{AC} = (c_1 - a_1)\mathbf{i} + (c_2 - a_2)\mathbf{j} + (c_3 - a_3)\mathbf{k}$$

$$\overrightarrow{AC} = (3 - 1)\mathbf{i} + (0 - 3)\mathbf{j} + (-2 - 2)\mathbf{k}$$

$$\overrightarrow{AC} = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}$$

$$\overrightarrow{AC} = AC\langle 2, -3, -4\rangle$$

Now we'll take the cross product of $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 1 & -1 \\ 2 & -3 & -4 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \mathbf{i}\begin{vmatrix} 1 & -1 \\ -3 & -4 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -3 & -1 \\ 2 & -4 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -3 & 1 \\ 2 & -3 \end{vmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \left[(1)(-4) - (-1)(-3)\right]\mathbf{i} - \left[(-3)(-4) - (-1)(2)\right]\mathbf{j} + \left[(-3)(-3) - (1)(2)\right]\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = (-4 - 3)\,\mathbf{i} - (12 + 2)\,\mathbf{j} + (9 - 2)\,\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = -7\mathbf{i} - 14\mathbf{j} + 7\mathbf{k}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \langle -7, -14, 7\rangle$$

This is the vector which is orthogonal to the plane that includes the given points.

Volume of the parallelepiped from vectors

If we need to find the volume of a parallelepiped and we're given three vectors, all we have to do is find the scalar triple product of the three vectors:

$$\left| a \cdot (b \times c) \right|$$

where the given vectors are $a\langle a_1, a_2, a_3 \rangle$, $b\langle b_1, b_2, b_3 \rangle$ and $c\langle c_1, c_2, c_3 \rangle$. $b \times c$ is the cross product of b and c, and we'll find it using the 3×3 matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i} \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \mathbf{i} \left(b_2 c_3 - b_3 c_2 \right) - \mathbf{j} \left(b_1 c_3 - b_3 c_1 \right) + \mathbf{k} \left(b_1 c_2 - b_2 c_1 \right)$$

We'll convert the result of the cross product into standard vector form, and then take the dot product of $a\langle a_1, a_2, a_3 \rangle$ and the vector result of $b \times c$. The final answer is the value of the scalar triple product, which is the volume of the parallelepiped.

Example

Find the volume of the parallelepiped given by the vectors.

$$a\langle 2, -1, 3 \rangle$$

$$b\langle 3, 2, -4 \rangle$$

$$c\langle -2, 0, 1 \rangle$$

We'll start by taking the cross product of b and c.

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -4 \\ -2 & 0 & 1 \end{vmatrix}$$

$$b \times c = \mathbf{i} \begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 3 & -4 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix}$$

$$b \times c = \left[(2)(1) - (-4)(0)\right] \mathbf{i} - \left[(3)(1) - (-4)(-2)\right] \mathbf{j} + \left[(3)(0) - (2)(-2)\right] \mathbf{k}$$

$$b \times c = (2 + 0)\mathbf{i} - (3 - 8)\mathbf{j} + (0 + 4)\mathbf{k}$$

$$b \times c = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$$

$$b \times c = \langle 2,5,4 \rangle$$

Now we'll take the dot product of $a\langle 2, -1,3 \rangle$ and $b \times c = \langle 2,5,4 \rangle$.

$$\left| a \cdot (b \times c) \right| = (2)(2) + (-1)(5) + (3)(4)$$

$$\left| a \cdot (b \times c) \right| = 4 - 5 + 12$$

$$\left| a \cdot (b \times c) \right| = 11$$

The volume of the parallelepiped is 11.

Volume of the parallelepiped from adjacent edges

If we need to find the volume of a parallelepiped and we're given three adjacent edges of it, all we have to do is find the scalar triple product of the three vectors that define the edges:

$$\left|\overrightarrow{PS}\cdot\left(\overrightarrow{PQ}\times\overrightarrow{PR}\right)\right|$$

where $\overrightarrow{PS}$, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are the three adjacent edges.

First we'll find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$, then we'll find the cross product $\overrightarrow{PQ}\times\overrightarrow{PR}$ using the 3×3 matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ PQ_1 & PQ_2 & PQ_3 \\ PR_1 & PR_2 & PR_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} PQ_2 & PQ_3 \\ PR_2 & PR_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} PQ_1 & PQ_3 \\ PR_1 & PR_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} PQ_1 & PQ_2 \\ PR_1 & PR_2 \end{vmatrix}$$

$$= \left(PQ_2PR_3 - PQ_3PR_2\right)\mathbf{i} - \left(PQ_1PR_3 - PQ_3PR_1\right)\mathbf{j} + \left(PQ_1PR_2 - PQ_2PR_1\right)\mathbf{k}$$

We'll convert the result of the cross product into standard vector form, and then take the dot product of $\overrightarrow{PS}\langle PS_1, PS_2, PS_3\rangle$ and the vector result of $\overrightarrow{PQ}\times\overrightarrow{PR}$. The final answer is the value of the scalar triple product, which is the volume of the parallelepiped.

Example

Find the volume of the parallelepiped given by the adjacent edges $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$.

$$P(5,1,-2)$$

$$Q(0,-1,3)$$

$$R(3,2,-4)$$

$$S(1,-2,0)$$

We need to start by using the four points to find the vectors $\overrightarrow{PQ}$, $\overrightarrow{PR}$ and $\overrightarrow{PS}$, since these are the three adjacent edges of the parallelepiped.

$$\overrightarrow{PQ} = \langle 0-5,-1-1,3-(-2)\rangle$$

$$\overrightarrow{PQ} = \langle -5,-2,5\rangle$$

and

$$\overrightarrow{PR} = \langle 3-5,2-1,-4-(-2)\rangle$$

$$\overrightarrow{PR} = \langle -2,1,-2\rangle$$

and

$$\overrightarrow{PS} = \langle 1-5,-2-1,0-(-2)\rangle$$

$$\overrightarrow{PS} = \langle -4,-3,2\rangle$$

Now we need to take the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$.

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5 & -2 & 5 \\ -2 & 1 & -2 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i}\begin{vmatrix} -2 & 5 \\ 1 & -2 \end{vmatrix} - \mathbf{j}\begin{vmatrix} -5 & 5 \\ -2 & -2 \end{vmatrix} + \mathbf{k}\begin{vmatrix} -5 & -2 \\ -2 & 1 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-2)(-2) - (5)(1)\right]\mathbf{i} - \left[(-5)(-2) - (5)(-2)\right]\mathbf{j} + \left[(-5)(1) - (-2)(-2)\right]\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (4 - 5)\mathbf{i} - (10 + 10)\mathbf{j} + (-5 - 4)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = -\mathbf{i} - 20\mathbf{j} - 9\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, -20, -9\rangle$$

Taking the dot product of $\overrightarrow{PS} = \langle -4, -3, 2\rangle$ and $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, -20, -9\rangle$, we get

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = (-4)(-1) + (-3)(-20) + (2)(-9)$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = 4 + 60 - 18$$

$$\left|\overrightarrow{PS} \cdot \left(\overrightarrow{PQ} \times \overrightarrow{PR}\right)\right| = 46$$

The volume of the parallelepiped is 46.

Scalar triple product to prove vectors are coplanar

The scalar triple product $\left|a \cdot (b \times c)\right|$ of three vectors $a\langle a_1, a_2, a_3\rangle$, $b\langle b_1, b_2, b_3\rangle$ and $c\langle c_1, c_2, c_3\rangle$ will be equal to 0 when the vectors are coplanar, which means that the vectors all lie in the same plane.

a, b, and c are coplanar if $\left|a \cdot (b \times c)\right| = 0$

$b \times c$ is the cross product of b and c, and we'll find it using the 3×3 matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

We'll convert the result of the cross product into standard vector form, and then take the dot product of $a\langle a_1, a_2, a_3\rangle$ and the vector result of $b \times c$.

$$\left|a \cdot (b \times c)\right|$$

The final answer is the scalar triple product. If it's equal to 0, then we've proven that the vectors are coplanar.

Example

Prove that the vectors are coplanar.

$a\langle 3,3,-3\rangle$

$b\langle 1,0,-2\rangle$

$$c\langle 2,3,-1\rangle$$

We'll use the scalar triple product, and we'll start by calculating the cross product of b and c, $b\times c$.

$$b\times c=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 0 & -2\\ 2 & 3 & -1\end{vmatrix}$$

$$b\times c=\mathbf{i}\begin{vmatrix}0 & -2\\ 3 & -1\end{vmatrix}-\mathbf{j}\begin{vmatrix}1 & -2\\ 2 & -1\end{vmatrix}+\mathbf{k}\begin{vmatrix}1 & 0\\ 2 & 3\end{vmatrix}$$

$$b\times c=\mathbf{i}\left[(0)(-1)-(-2)(3)\right]-\mathbf{j}\left[(1)(-1)-(-2)(2)\right]+\mathbf{k}\left[(1)(3)-(0)(2)\right]$$

$$b\times c=\mathbf{i}(0+6)-\mathbf{j}(-1+4)+\mathbf{k}(3-0)$$

$$b\times c=6\mathbf{i}-3\mathbf{j}+3\mathbf{k}$$

$$b\times c=\langle 6,-3,3\rangle$$

Next we'll take the dot product of $a\langle 3,3,-3\rangle$ and $b\times c=\langle 6,-3,3\rangle$.

$$\left|a\cdot(b\times c)\right|=(3)(6)+(3)(-3)+(-3)(3)$$

$$\left|a\cdot(b\times c)\right|=18-9-9$$

$$\left|a\cdot(b\times c)\right|=0$$

Since the scalar triple product of the vectors $a\langle 3,3,-3\rangle$, $b\langle 1,0,-2\rangle$ and $c\langle 2,3,-1\rangle$ is equal to 0,

$$\left|a\cdot(b\times c)\right|=0$$

the vectors a, b, and c are coplanar.

Domain of a vector function

To find the domain of a vector function,

$$r(t) = \langle a, b, c \rangle$$

or

$$r(t) = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

we'll need to find the domain of the individual components a, b and c. Then the domain of the vector function is the values for which the domains of a, b and c overlap.

Example

Find the domain of the vector function.

$$r(t) = \left\langle \ln(t-1), t^2, \frac{1}{9-t^2} \right\rangle$$

We need to find the domain of each component individually.

The domain of $\ln(t-1)$ is

$$t - 1 > 0$$

$$t > 1$$

The domain of t^2 is

all real numbers

The domain of $\frac{1}{9-t^2}$ is

$$9 - t^2 \neq 0$$

$$-t^2 \neq -9$$

$$t^2 \neq 9$$

$$t \neq \pm 3$$

The first component restricts the domain of the entire vector function to $t > 1$. The second component doesn't restrict the domain at all. The third component tells us that the domain can't include $t = \pm 3$. $t = -3$ is outside of the domain $t > 1$, so we can ignore it. But $t = 3$ is inside the domain $t > 1$, so we have to include it.

The domain of the vector function is

$t > 1$ but $t \neq 3$

Limit of a vector function

To find the limit of a vector function,

$$r(t)=a(t)\mathbf{i}+b(t)\mathbf{j}+c(t)\mathbf{k}$$

we'll need to take the limit of each term separately.

$$\lim_{t\to x}\left[a(t)\mathbf{i}+b(t)\mathbf{j}+c(t)\mathbf{k}\right]$$

$$\lim_{t\to x}a(t)\mathbf{i}+\lim_{t\to x}b(t)\mathbf{j}+\lim_{t\to x}c(t)\mathbf{k}$$

$$a(x)\mathbf{i}+b(x)\mathbf{j}+c(x)\mathbf{k}$$

Example

Find the limit of the vector function.

$$\lim_{t\to 0}\left[(t^2-2)\mathbf{i}+\ln(t+e)\mathbf{j}+\frac{4t}{\sin t}\mathbf{k}\right]$$

We'll take the limit of each term separately.

$$\lim_{t\to 0}\left(t^2-2\right)\mathbf{i}+\lim_{t\to 0}\ln(t+e)\mathbf{j}+\lim_{t\to 0}\frac{4t}{\sin t}\mathbf{k}$$

Evaluating the first two terms as $t\to 0$, we get

$$\left(0^2-2\right)\mathbf{i}+\ln(0+e)\mathbf{j}+\lim_{t\to 0}\frac{4t}{\sin t}\mathbf{k}$$

$$-2\mathbf{i} + \ln(e)\mathbf{j} + \lim_{t \to 0} \frac{4t}{\sin t}\mathbf{k}$$

$$-2\mathbf{i} + 1\mathbf{j} + \lim_{t \to 0} \frac{4t}{\sin t}\mathbf{k}$$

$$-2\mathbf{i} + \mathbf{j} + \lim_{t \to 0} \frac{4t}{\sin t}\mathbf{k}$$

Because the third term gives $0/0$ when $t \to 0$, we have to use L'Hospital's rule, replacing the numerator and denominator with their derivatives.

$$-2\mathbf{i} + \mathbf{j} + \lim_{t \to 0} \frac{4}{\cos t}\mathbf{k}$$

Evaluating as $t \to 0$, we get

$$-2\mathbf{i} + \mathbf{j} + \frac{4}{\cos 0}\mathbf{k}$$

$$-2\mathbf{i} + \mathbf{j} + \frac{4}{1}\mathbf{k}$$

$$-2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$$

This is the limit of the vector function.

Projections of the curve

Sometimes the easiest way to sketch a three-dimensional curve is to sketch its projections on the xy-, xz-, and yz-coordinate planes.

Think about the projections of a curve as the shadows they cast against the coordinate planes. You can also think about them as the view of the curve from the coordinate planes. In other words, if you're standing squarely parallel to the xy-coordinate plane, what you see of the curve is the projection of the curve on the xy-coordinate plane.

Once we have the projections of the curve on each of the coordinate planes, we can use them to draw the three-dimensional graph.

Example

Sketch the projections of the curve and use them to sketch the three-dimensional curve.

$$r(t) = \left\langle t, t^2, t^2 + 1 \right\rangle$$

We'll convert the vector function to three parametric equations.

$$x = t$$

$$y = t^2$$

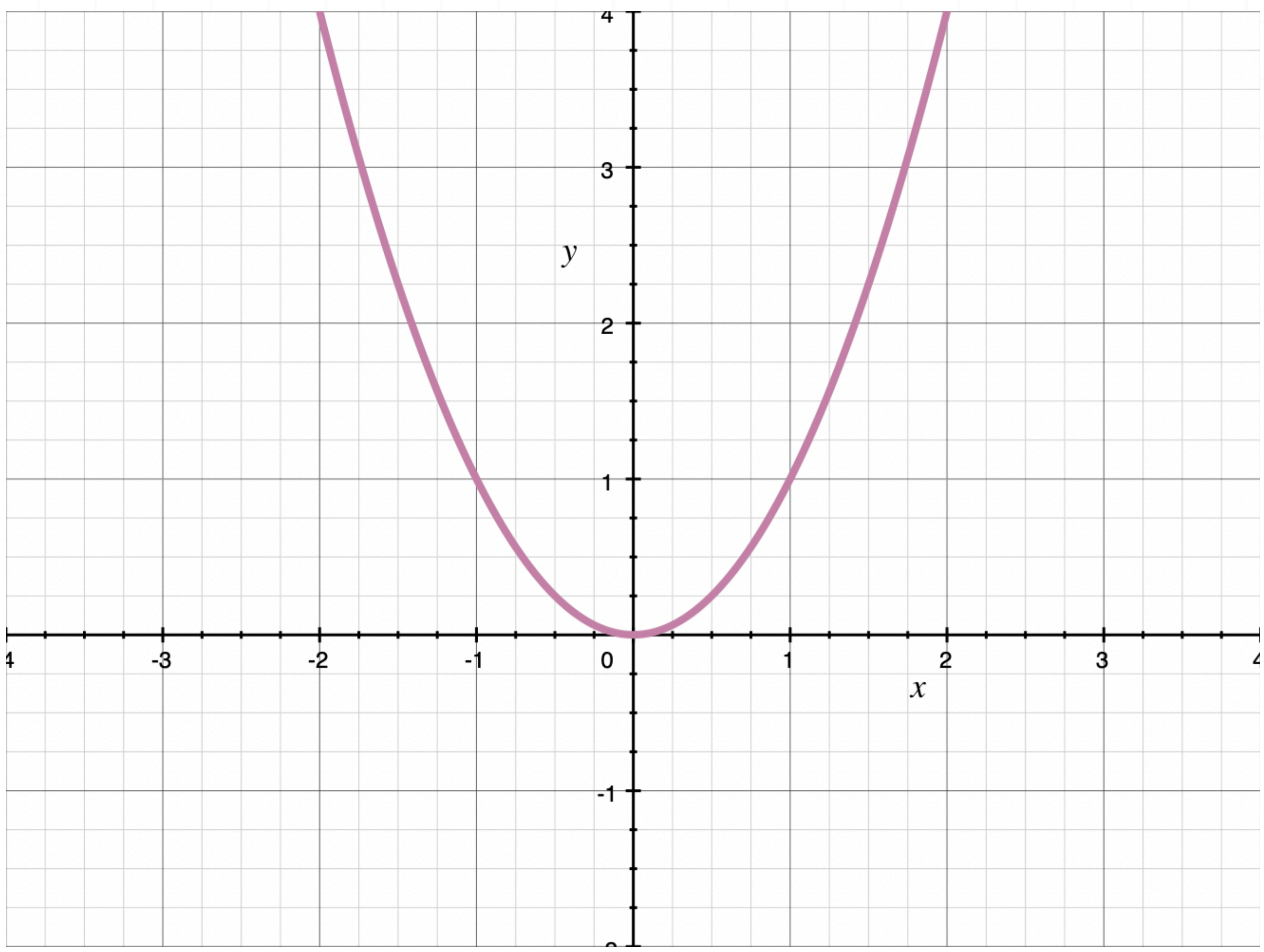
$$z = t^2 + 1$$

To find the projection on the xy-coordinate plane, we need to find an equation in terms of only x and y, which we'll do by plugging $x = t$ into $y = t^2$.

$$y = t^2$$

$$y = x^2$$

We'll sketch this curve in the xy-coordinate plane.

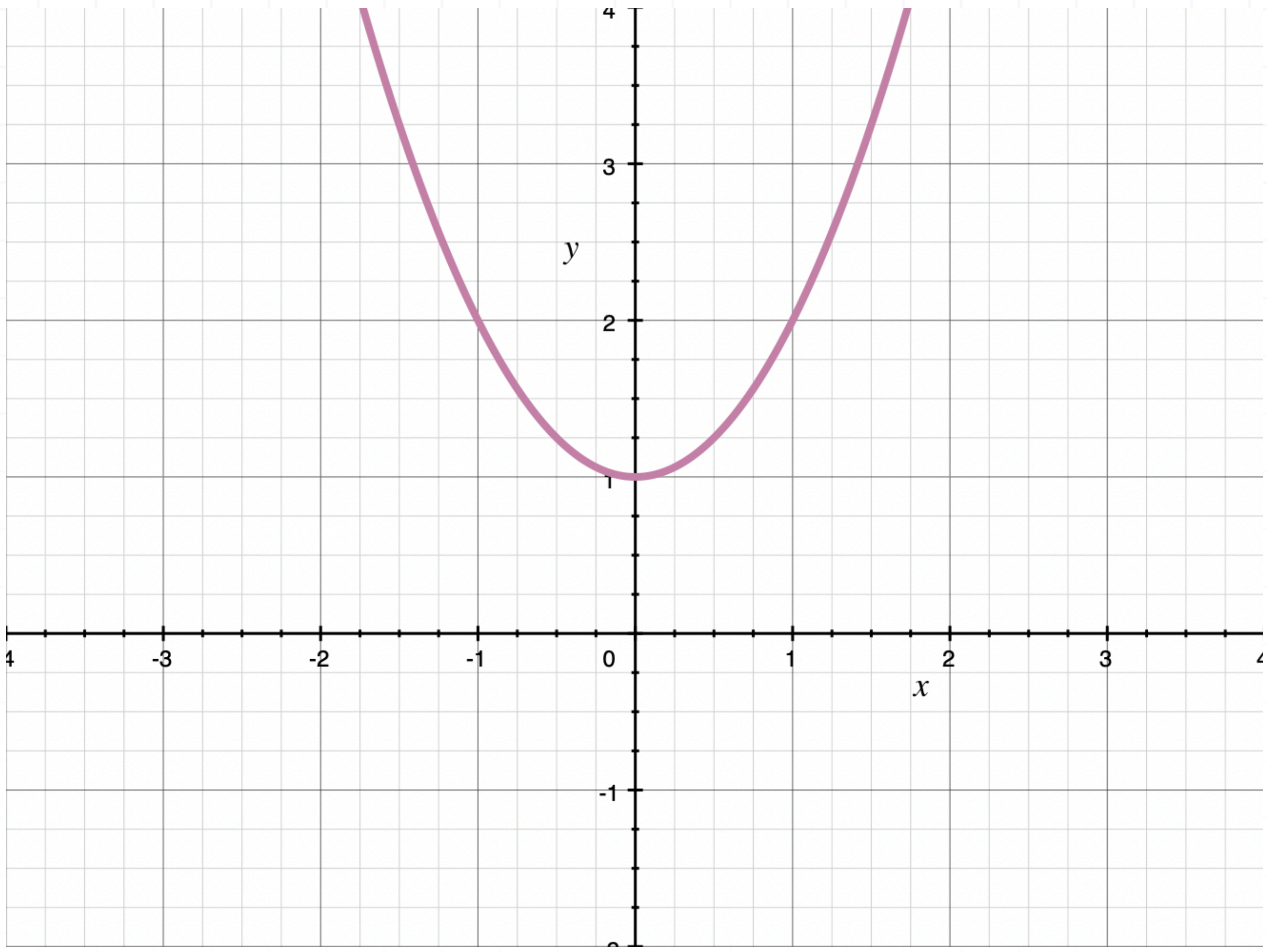


To find the projection on the xz-coordinate plane, we need to find an equation in terms of only x and z, which we'll do by plugging $x = t$ into $z = t^2 + 1$.

$$z = t^2 + 1$$

$$z = x^2 + 1$$

We'll sketch this curve in the xz-coordinate plane.

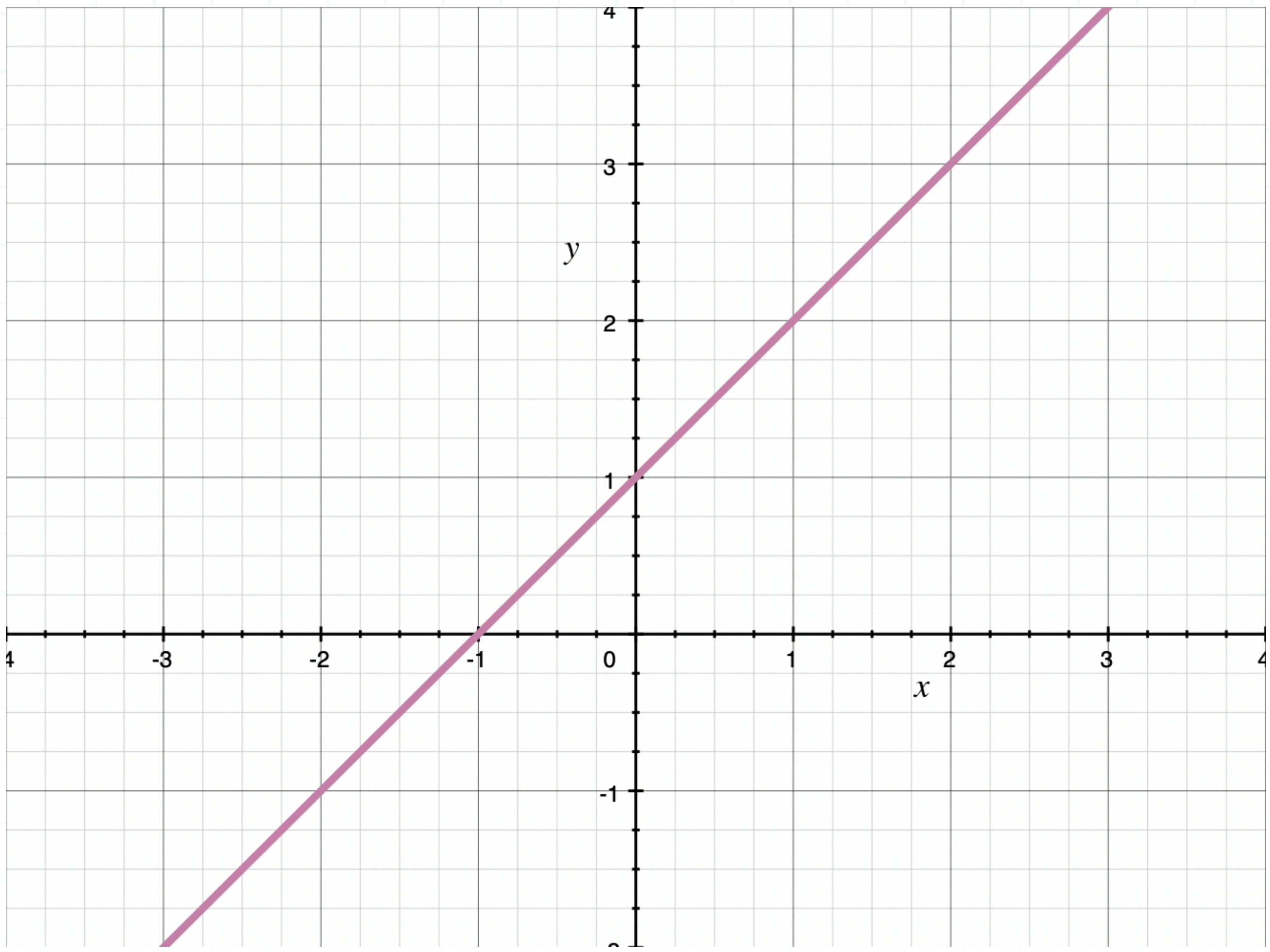


To find the projection on the yz-coordinate plane, we need to find an equation in terms of only y and z, which we'll do by plugging $y = t^2$ into $z = t^2 + 1$.

$$z = t^2 + 1$$

$$z = y + 1$$

We'll sketch this curve in the yz-coordinate plane.



Our final step is to use the projections to sketch the three-dimensional curve. We need a starting point. To find it, we'll set $t = 0$ in our parametric equations, and get

$$x = t$$

$$x = 0$$

and

$$y = t^2$$

$$y = 0^2$$

$$y = 0$$

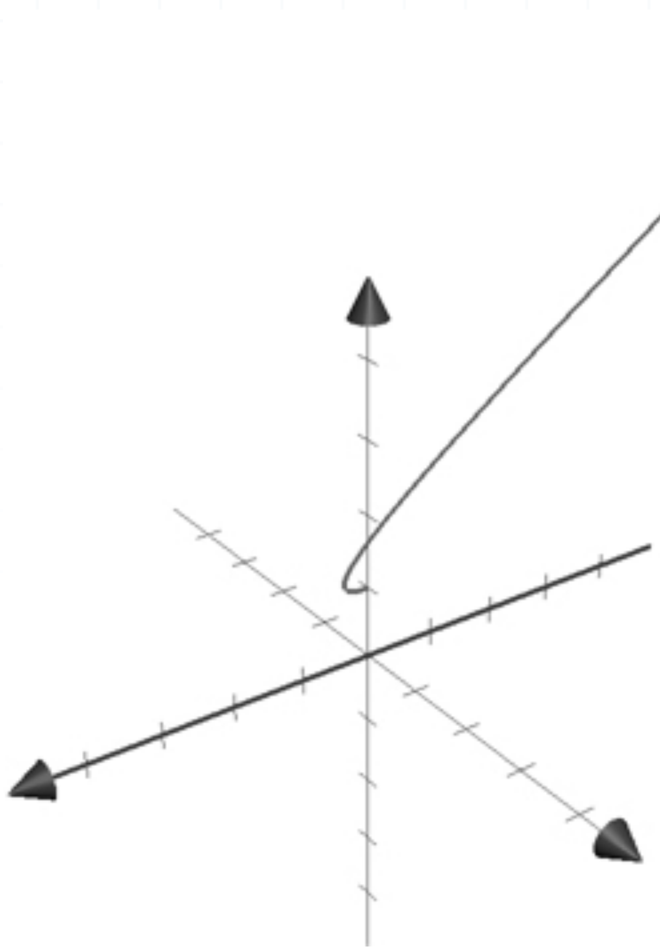
and

$$z = t^2 + 1$$

$$z = 0^2 + 1$$

$$z = 1$$

Putting these values together, we get the point $(0,0,1)$. This means that our graph starts at $(0,0,1)$ and travels upwards in a parabolic shape from the xy- and xz-planar perspective. The three-dimensional graph is



Vector and parametric equations of a line segment

Sometimes we need to find the equation of a line segment when we only have the endpoints of the line segment.

The **vector equation** of the line segment is given by

$$r(t) = (1 - t)r_0 + tr_1$$

where $0 \le t \le 1$ and r_0 and r_1 are the vector equivalents of the endpoints.

The **parametric equations** of the line segment are given by

$$x = r(t)_1$$

$$y = r(t)_2$$

$$z = r(t)_3$$

where $r(t)_1$, $r(t)_2$ and $r(t)_3$ come from the vector function

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

$$r(t) = \langle r(t)_1, r(t)_2, r(t)_3 \rangle$$

Example

Find the vector and parametric equations of the line segment defined by its endpoints.

$$P(1,2,-1)$$

$Q(1,0,3)$

To find the vector equation of the line segment, we'll convert its endpoints to their vector equivalents.

$P(1,2,-1)$ becomes $r_0 = \langle 1,2,-1 \rangle$

$Q(1,0,3)$ becomes $r_1 = \langle 1,0,3 \rangle$

Plugging these into the vector formula for the equation of the line segment gives

$$r(t) = (1-t)\langle 1,2,-1 \rangle + t\langle 1,0,3 \rangle$$

$$r(t) = \langle 1-t, 2-2t, -1+t \rangle + \langle t, 0, 3t \rangle$$

$$r(t) = \langle 1-t+t, 2-2t+0, -1+t+3t \rangle$$

$$r(t) = \langle 1, 2-2t, -1+4t \rangle$$

We can also write the vector equation as

$$r(t) = 1\mathbf{i} + (2-2t)\mathbf{j} + (-1+4t)\mathbf{k}$$

$$r(t) = \mathbf{i} + (2-2t)\mathbf{j} + (-1+4t)\mathbf{k}$$

Now that we have the vector equation of the line segment, we just take its direction numbers, or the coefficients on $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to get the parametric equations of the line segment.

$$x = 1$$

$$y = 2 - 2t$$

$z = -1 + 4t$

We'll summarize our findings.

Vector equation	$r(t) = \langle 1, 2 - 2t, -1 + 4t \rangle$
Parametric equations	$x = 1$, $y = 2 - 2t$ and $z = -1 + 4t$

Vector function for the curve of intersection of two surfaces

When two three-dimensional surfaces intersect each other, the intersection is a curve. We can find the vector equation of that intersection curve using these steps:

1. Set the curves equal to each other and solve for one of the remaining variables in terms of the other

2. Define each of the variables in terms of the parameter t to get parametric equations for the intersection curve,

$$x = r(t)_1$$

$$y = r(t)_2$$

$$z = r(t)_3$$

3. Generate the vector function that describes the intersection curve using the formulas

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

$$r(t) = \langle r(t)_1, r(t)_2, r(t)_3 \rangle$$

Example

Find the vector function for the curve of intersection of the surfaces.

The ellipsoid $z = \sqrt{1 + x^2 - y^2}$

The plane $z = 2 + x$

Since both of the curves have z on the left-hand side, we can set the right-hand sides equal to one another and solve for one variable of the remaining variables in terms of the other.

$$\sqrt{1 + x^2 - y^2} = 2 + x$$

$$1 + x^2 - y^2 = (2 + x)^2$$

$$1 + x^2 - y^2 = 4 + 4x + x^2$$

$$-3 - y^2 = 4x$$

$$x = -\frac{3}{4} - \frac{1}{4}y^2$$

We want to define each variable in terms of the parameter t, so we'll set $y = t$.

$$x = -\frac{3}{4} - \frac{1}{4}t^2$$

To find z in terms of t, we'll plug x in terms of t into $z = 2 + x$.

$$z = 2 + x$$

$$z = 2 - \frac{3}{4} - \frac{1}{4}t^2$$

$$z = \frac{5}{4} - \frac{1}{4}t^2$$

Now we have parametric equations for the curve of intersection, defined by

$$x = -\frac{3}{4} - \frac{1}{4}t^2$$

$$y = t$$

$$z = \frac{5}{4} - \frac{1}{4}t^2$$

With the parametric equations in hand, we can plug each of them into the formula for the vector function.

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

$$r(t) = \left(-\frac{3}{4} - \frac{1}{4}t^2\right)\mathbf{i} + t\mathbf{j} + \left(\frac{5}{4} - \frac{1}{4}t^2\right)\mathbf{k}$$

This is the vector function for the curve of intersection. You can also write it as

$$r(t) = \left\langle -\frac{3}{4} - \frac{1}{4}t^2, t, \frac{5}{4} - \frac{1}{4}t^2 \right\rangle$$

Derivative of a vector function

To find the derivative of a vector function, we just need to find the derivatives of the coefficients when the vector function is in the form

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

With the vector function in this form, the derivative is

$$r'(t) = r'(t)_1\mathbf{i} + r'(t)_2\mathbf{j} + r'(t)_3\mathbf{k}$$

If the vector function is in the form

$$r(t) = \langle r(t)_1, r(t)_2, r(t)_3 \rangle$$

we can just attach each of the direction numbers to $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ to transform it into the form

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

and then take the derivatives of the coefficients. Alternately, we can just take the derivatives of each direction number, leaving the function in its original form. Make sure to give an answer that matches the form of the original vector function. In other words,

Given,	**the answer should be:**
$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$	$r'(t) = r'(t)_1\mathbf{i} + r'(t)_2\mathbf{j} + r'(t)_3\mathbf{k}$
$r(t) = \langle r(t)_1, r(t)_2, r(t)_3 \rangle$	$r'(t) = \langle r'(t)_1, r'(t)_2, r'(t)_3 \rangle$

Let's do an example so that we can practice finding the derivative of a vector function.

Example

Find the derivative of the vector function.

$$r(t) = \langle t^2, e^{3t}, t^3 \sin(4t) \rangle$$

The given function is the same as

$$r(t) = t^2\mathbf{i} + e^{3t}\mathbf{j} + t^3 \sin(4t)\mathbf{k}$$

No matter which way we write it, we just need to replace each coefficient with its derivative. In this particular example, we'll need to use product rule to find the derivative of the coefficient on $\mathbf{k}$.

$$r'(t) = 2t\mathbf{i} + e^{3t}(3)\mathbf{j} + [3t^2(\sin(4t)) + t^3(\cos(4t)(4))]\mathbf{k}$$

$$r'(t) = 2t\mathbf{i} + 3e^{3t}\mathbf{j} + (3t^2 \sin(4t) + 4t^3 \cos(4t))\mathbf{k}$$

Since the question gave the original vector function in terms of its direction numbers, we'll give the derivative in that form in our answer.

$$r'(t) = \langle 2t,\ 3e^{3t},\ 3t^2 \sin(4t) + 4t^3 \cos(4t) \rangle$$

Unit tangent vector

To find the unit tangent vector for a vector function, we use the formula

$$T(t) = \frac{r'(t)}{||r'(t)||}$$

where $r'(t)$ is the derivative of the vector function $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$ and t is given.

Remember that $||r'(t)||$ is the magnitude of the derivative of the vector function at time t. We can find $\left|r'(t)\right|$ using the formula

$$||r'(t)|| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

Example

Find the unit tangent vector of the vector function at $t = 1$.

$$r(t) = 4t^3\mathbf{i} + 6t\mathbf{j} + 4t\ln(t)\mathbf{k}$$

We'll start by finding the derivative of the vector function $r(t) = 4t^3\mathbf{i} + 6t\mathbf{j} + 4t\ln(t)\mathbf{k}$ at time $t = 1$ so that we can plug it into the formula for the unit tangent vector. To find the derivative, we'll just replace each of the coefficients with their derivatives. The derivative of $4t^3$ is $12t^2$; the derivative of $6t$ is 6; the derivative of $4t\ln(t)$ using product rule is $(4)(\ln(t)) + (4t)(1/t)$.

$$r'(t) = 12t^2\mathbf{i} + 6\mathbf{j} + \left[(4)(\ln(t)) + (4t)\left(\frac{1}{t}\right)\right]\mathbf{k}$$

$$r'(t) = 12t^2\mathbf{i} + 6\mathbf{j} + \left[4\ln(t) + 4\right]\mathbf{k}$$

Now we'll find the value of the derivative at $t = 1$.

$$r'(1) = 12(1)^2\mathbf{i} + 6\mathbf{j} + \left[4\ln(1) + 4\right]\mathbf{k}$$

$$r'(1) = 12\mathbf{i} + 6\mathbf{j} + \left[4(0) + 4\right]\mathbf{k}$$

$$r'(1) = 12\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$$

Now we'll use the values from the derivative to find the magnitude of the vector function at $t = 1$ so that we can plug it into the formula for the unit tangent vector.

$$||r'(t)|| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$||r'(1)|| = \sqrt{12^2 + 6^2 + 4^2}$$

$$||r'(1)|| = \sqrt{144 + 36 + 16}$$

$$||r'(1)|| = \sqrt{196}$$

$$||r'(1)|| = 14$$

Plugging everything into the formula for the unit tangent vector, we get

$$T(1) = \frac{12\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}}{14}$$

$$T(1) = \frac{12}{14}\mathbf{i} + \frac{6}{14}\mathbf{j} + \frac{4}{14}\mathbf{k}$$

$$T(1) = \frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$$

which is the equation of the unit tangent vector for $r(t) = 4t^3\mathbf{i} + 6t\mathbf{j} + 4t\ln(t)\mathbf{k}$.

Parametric equations of the tangent line

The parametric equations of the tangent line of a vector function $r(t) = \left\langle r(t)_1, r(t)_2, r(t)_3 \right\rangle$ are

$$x = x_1 + r'(t_0)_1 t$$

$$y = y_1 + r'(t_0)_2 t$$

$$z = z_1 + r'(t_0)_3 t$$

x_1, y_1 and z_1 come from the point $P(x_1, y_1, z_1)$, which is the point of tangency.

You find $r'(t)_1$, $r'(t)_2$ and $r'(t)_3$ by taking the derivative of the vector $r(t) = \left\langle r(t)_1, r(t)_2, r(t)_3 \right\rangle$ or $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$.

You find t_0 by plugging $P(x_1, y_1, z_1)$ into the vector function.

Then you find $r'(t_0)_1$, $r'(t_0)_2$ and $r'(t_0)_3$ by plugging t_0 into the derivative of the vector function.

Example

Find the parametric equations of the tangent line to the vector at the point P.

$$x = e^t$$

$$y = -t\cos t$$

$$z = \sin t$$

at $P(1,0,0)$

First, since the point of tangency is $P(1,0,0)$, we can plug that point into the formulas for the parametric equation of the tangent line from above, and they become

$$x = 1 + r'(t_0)_1 t$$

and

$$y = 0 + r'(t_0)_2 t$$

$$y = r'(t_0)_2 t$$

and

$$z = 0 + r'(t_0)_3 t$$

$$z = r'(t_0)_3 t$$

Now we'll find a value for t_0. We'll use $x = e^t$, change t to t_0 and plug $x = 1$ (from $P(1,0,0)$) into the equation and get

$$1 = e^{t_0}$$

$$\ln 1 = \ln e^{t_0}$$

$$t_0 = 0$$

Plugging $t_0 = 0$ and $y = 0$ (from $P(1,0,0)$) into $y = -t\cos t$ and get

$$0 = -0\cos 0$$

$$0 = 0$$

Since this equation is true, $t_0 = 0$ works for $y = -t\cos t$ as well as $x = e^t$. Now we'll plug $t_0 = 0$ and $z = 0$ (from $P(1,0,0)$) into $z = \sin t$ and get

$$0 = \sin 0$$

$$0 = 0$$

Since this equation is true, we've now shown that $t_0 = 0$ satisfies $x = e^t$, $y = -t\cos t$ and $z = \sin t$, so 0 is the value we want to use for t_0. Therefore, the parametric equations of the tangent line become

$$x = 1 + r'(0)_1 t$$

$$y = r'(0)_2 t$$

$$z = r'(0)_3 t$$

Next we need to find the derivative of the vector function. The original function is

$$r(t) = \left\langle e^t, -t\cos t, \sin t \right\rangle$$

so its derivative is

$$r'(t) = \left\langle e^t, (-1)(\cos t) + (-t)(-\sin t), \cos(t) \right\rangle$$

$$r'(t) = \left\langle e^t, -\cos t + t\sin t, \cos t \right\rangle$$

$$r'(t) = \left\langle e^t, t\sin t - \cos t, \cos t \right\rangle$$

Plugging $t_0 = 0$ into the derivative, we get

$$r'(0) = \left\langle e^0, 0\sin 0 - \cos 0, \cos 0 \right\rangle$$

$$r'(0) = \langle 1, 0 - 1, 1 \rangle$$

$$r'(0) = \langle 1, -1, 1 \rangle$$

We'll take these three values, plug them into our parametric equations, and the parametric equations become

$$x = 1 + 1t$$

$$y = -1t$$

$$z = 1t$$

and these simplify to

$$x = 1 + t$$

$$y = -t$$

$$z = t$$

Integral of a vector function

To find the integral of a vector function $r(t)=r(t)_1\mathbf{i}+r(t)_2\mathbf{j}+r(t)_3\mathbf{k}$, we simply replace each coefficient with its integral. In other words, the integral of the vector function is

$$\int r(t)\ dt=\mathbf{i}\int r(t)_1\ dt+\mathbf{j}\int r(t)_2\ dt+\mathbf{k}\int r(t)_3\ dt$$

If the vector function is given as $r(t)=\left\langle r(t)_1,r(t)_2,r(t)_3\right\rangle$, then its integral is

$$\int r(t)=\left\langle\int r(t)_1\ dt,\int r(t)_2\ dt,\int r(t)_3\ dt\right\rangle$$

Example

Find the integral of the vector function over the interval $[0,\pi]$.

$$r(t)=\sin(2t)\mathbf{i}+2e^{2t}\mathbf{j}+4t^3\mathbf{k}$$

Remember that we're only taking the integrals of the coefficients, which means $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ will be left alone.

$$\int_0^\pi r(t)\ dt=\frac{-\cos(2t)}{2}\Bigg|_0^\pi\mathbf{i}+\frac{2e^{2t}}{2}\Bigg|_0^\pi\mathbf{j}+\frac{4t^4}{4}\Bigg|_0^\pi\mathbf{k}$$

$$\int_0^\pi r(t)\ dt=\frac{-\cos(2t)}{2}\Bigg|_0^\pi\mathbf{i}+e^{2t}\Bigg|_0^\pi\mathbf{j}+t^4\Bigg|_0^\pi\mathbf{k}$$

Evaluating over the interval $[0,\pi]$, we get

$$\int_0^\pi r(t)\ dt = \left[\frac{-\cos(2\pi)}{2} - \frac{-\cos(2(0))}{2}\right]\mathbf{i} + \left[e^{2\pi} - e^{2(0)}\right]\mathbf{j} + \left[\pi^4 - 0^4\right]\mathbf{k}$$

$$\int_0^\pi r(t)\ dt = \left[\frac{-\cos(2\pi)}{2} + \frac{\cos 0}{2}\right]\mathbf{i} + \left(e^{2\pi} - 1\right)\mathbf{j} + \left(\pi^4 - 0\right)\mathbf{k}$$

$$\int_0^\pi r(t)\ dt = \left(\frac{-1}{2} + \frac{1}{2}\right)\mathbf{i} + \left(e^{2\pi} - 1\right)\mathbf{j} + \pi^4\mathbf{k}$$

$$\int_0^\pi r(t)\ dt = 0\mathbf{i} + \left(e^{2\pi} - 1\right)\mathbf{j} + \pi^4\mathbf{k}$$

$$\int_0^\pi r(t)\ dt = \left(e^{2\pi} - 1\right)\mathbf{j} + \pi^4\mathbf{k}$$

This is the integral of the vector function. We could also write it in the form

$$\int_0^\pi r(t)\ dt = \left\langle 0,e^{2\pi} - 1,\pi^4\right\rangle$$

Arc length of a vector function

To find the arc length of the vector function, we will need to use the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

where L is the arc length of the vector function, $[a, b]$ is the interval that defines the arc, and dx/dt, dy/dt, and dz/dt are the derivatives of the parametric equations of x, y and z respectively.

To solve for arc length, we'll need the parametric equations of the vector function. Whether our vector function is given as $r(t) = \left\langle r(t)_1, r(t)_2, r(t)_3 \right\rangle$ or $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, the parametric equations are

$$x = r(t)_1$$

$$y = r(t)_2$$

$$z = r(t)_3$$

Once we have these parametric equations, we'll take the derivative of each one to get dx/dt, dy/dt and dz/dt. Assuming we're given $[a, b]$, we'll have everything we need to use the formula for arc length.

Example

Find the arc length of the vector function over the given interval.

$$r(t) = \langle \sin(2t), \cos(2t), 2t \rangle$$

on $0 \le t \le 2$

We'll pull the parametric equations out of the vector function as

$$x = \sin(2t)$$

$$y = \cos(2t)$$

$$z = 2t$$

Now we'll take the derivative of each of these.

$$\frac{dx}{dt} = 2\cos(2t)$$

$$\frac{dy}{dt} = -2\sin(2t)$$

$$\frac{dz}{dt} = 2$$

Plugging the derivatives and the given interval $0 \le t \le 2$ into the formula for arc length, we get

$$L = \int_0^2 \sqrt{\left[2\cos(2t)\right]^2 + \left[-2\sin(2t)\right]^2 + (2)^2}\ dt$$

$$L = \int_0^2 \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 4}\ dt$$

$$L = \int_0^2 \sqrt{4\left[\cos^2(2t) + \sin^2(2t)\right] + 4}\ dt$$

Since $\cos^2 x + \sin^2 x = 1$, we can simplify the integral to

$$L = \int_0^2 \sqrt{4(1) + 4}\ dt$$

$$L = \int_0^2 \sqrt{8}\ dt$$

$$L = \int_0^2 \sqrt{4 \cdot 2}\ dt$$

$$L = \int_0^2 2\sqrt{2}\ dt$$

$$L = 2\sqrt{2}t\Big|_0^2$$

Evaluating over the interval, we get

$$L = 2\sqrt{2}(2) - 2\sqrt{2}(0)$$

$$L = 4\sqrt{2}$$

The arc length of the vector function over the interval $0 \le t \le 2$ is $L = 4\sqrt{2}$.

Reparametrizing the curve

When we reparametrize a curve, it means that we rewrite it in terms of an independent variable. There's not a specific variable that's always used in reparametrization, but it's common to see s used. Given a vector function

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

the reparametrized curve will be

$$r\left[t(s)\right] = r\left[t(s)\right]_1\mathbf{i} + r\left[t(s)\right]_2\mathbf{j} + r\left[t(s)\right]_3\mathbf{k}$$

To reparametrize the curve with respect to arc length, we'll have to find the arc length of the vector function using

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

where L is the arc length of the vector function, $[a, b]$ is the interval, and dx/dt, dy/dt and dz/dt are the derivatives of the parametric equations for x, y and z, respectively.

Whether the vector function is given as $r(t) = \langle r(t)_1, r(t)_2, r(t)_3 \rangle$ or $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, the parametric equations of the vector are

$$x = r(t)_1$$

$$y = r(t)_2$$

$$z = r(t)_3$$

Once we have the parametric equations, we can find their derivatives and plug them into the arc length formula. We'll evaluate the integral to find arc length, then take the arc length and set it equal to s, such that $L(t) = s$. We'll solve $L(t) = s$ for t, and then plug the value we found for t back into the original vector function. This will give us the reparametrized curve

$$r\left[t(s)\right] = r\left[t(s)\right]_1 \mathbf{i} + r\left[t(s)\right]_2 \mathbf{j} + r\left[t(s)\right]_3 \mathbf{k}$$

Example

Reparametrize the curve with respect to arc length from $t = 0$ in the direction of increasing t.

$$r(t) = -2t\mathbf{i} + 4t\mathbf{j} + (2 - 4t)\mathbf{k}$$

First, we'll use the coefficients from the vector function to generate parametric equations of the vector function.

$$x = -2t$$

$$y = 4t$$

$$z = 2 - 4t$$

Now we'll take the derivatives of these.

$$\frac{dx}{dt} = -2$$

$$\frac{dy}{dt} = 4$$

$$\frac{dz}{dt} = -4$$

Next, we'll plug the derivatives into the arc length formula. We also know that the limits of integration will be $[0,t]$, since the problem indicated that the arc length should start at 0 and head in the direction of increasing t.

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

$$L = \int_0^t \sqrt{(-2)^2 + (4)^2 + (-4)^2}\ dt$$

$$L = \int_0^t \sqrt{4 + 16 + 16}\ dt$$

$$L = \int_0^t \sqrt{36}\ dt$$

$$L = \int_0^t 6\ dt$$

$$L = 6t\Big|_0^t$$

$$L = 6(t) - 6(0)$$

$$L = 6t$$

Now we'll set arc length equal to the independent variable s, and we'll get

$$s = 6t$$

Then we'll solve for t.

$$t = \frac{s}{6}$$

We’ll substitute the value of t into the original vector function, and we’ll get the reparametrized version.

$$r\left[t(s)\right] = -2\left(\frac{s}{6}\right)\mathbf{i} + 4\left(\frac{s}{6}\right)\mathbf{j} + \left[2 - 4\left(\frac{s}{6}\right)\right]\mathbf{k}$$

$$r\left[t(s)\right] = -\frac{1}{3}s\mathbf{i} + \frac{2}{3}s\mathbf{j} + \left(2 - \frac{2}{3}s\right)\mathbf{k}$$

This is the reparametrized curve in terms of arc length.

Unit tangent and unit normal vectors

The unit tangent vector $T(t)$ of a vector function

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

is the vector that is 1 unit long and tangent to the vector function at the point t.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

Remember that $\left| r'(t) \right|$ is the magnitude of the derivative of the vector function at time t, and we can find it using the formula

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

The unit normal vector $N(t)$ of the same vector function is the vector that is 1 unit long and perpendicular to the unit tangent vector at the same point t.

$$N(t) = \frac{T'(t)}{\left| T'(t) \right|}$$

Remember that $\left| T'(t) \right|$ is the magnitude of the derivative of the unit tangent vector at time t, and we can find it using the formula

$$\left| T'(t) \right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

Example

Find the unit normal vector of the vector function at $t = 1$.

$$r(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}$$

In order to find the unit normal vector, we'll have to start by finding the unit tangent vector, given by

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

We'll take the derivative of the vector function to get $r'(t)$. Remember, we only have to take the derivatives of the coefficients, leaving $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ alone.

$$r(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}$$

$$r'(t) = 1\mathbf{i} + 2t\mathbf{j} + 0\mathbf{k}$$

$$r'(t) = \mathbf{i} + 2t\mathbf{j}$$

Now we can use $r'(t)$ to find $\left| r'(t) \right|$.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1 \right]^2 + \left[r'(t)_2 \right]^2 + \left[r'(t)_3 \right]^2}$$

$$\left| r'(t) \right| = \sqrt{(1)^2 + (2t)^2 + (0)^2}$$

$$\left| r'(t) \right| = \sqrt{1 + 4t^2}$$

Plugging these into the formula for the unit tangent vector, we get

$$T(t)=\frac{r'(t)}{\left|r'(t)\right|}$$

$$T(t)=\frac{\mathbf{i}+2t\mathbf{j}}{\sqrt{1+4t^2}}$$

Now that we have an equation for the unit tangent vector, we can use it to find the unit normal vector, given by

$$N(t)=\frac{T'(t)}{\left|T'(t)\right|}$$

We'll have to start by taking the derivative of the unit tangent vector to get $T'(t)$. We'll need quotient rule to do this.

$$T(t)=\frac{\mathbf{i}+2t\mathbf{j}}{\sqrt{1+4t^2}}$$

$$T'(t)=\frac{(0\mathbf{i}+2\mathbf{j})\sqrt{1+4t^2}-(\mathbf{i}+2t\mathbf{j})\frac{1}{2}(1+4t^2)^{-\frac{1}{2}}(8t)}{\left(\sqrt{1+4t^2}\right)^2}$$

$$T'(t)=\frac{2\mathbf{j}\sqrt{1+4t^2}-4t(\mathbf{i}+2t\mathbf{j})(1+4t^2)^{-\frac{1}{2}}}{1+4t^2}$$

$$T'(t)=\frac{2\mathbf{j}\sqrt{1+4t^2}-\frac{4t\left(\mathbf{i}+2t\mathbf{j}\right)}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t)=\frac{\frac{2\mathbf{j}\sqrt{1+4t^2}\sqrt{1+4t^2}}{\sqrt{1+4t^2}}-\frac{4t\left(\mathbf{i}+2t\mathbf{j}\right)}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t)=\frac{\frac{2\mathbf{j}(1+4t^2)}{\sqrt{1+4t^2}}-\frac{4t(\mathbf{i}+2t\mathbf{j})}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t)=\frac{\frac{2\mathbf{j}(1+4t^2)-4t(\mathbf{i}+2t\mathbf{j})}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t)=\frac{2\mathbf{j}\left(1+4t^2\right)-4t\left(\mathbf{i}+2t\mathbf{j}\right)}{\left(1+4t^2\right)^{\frac{3}{2}}}$$

$$T'(t)=\frac{2\mathbf{j}+8t^2\mathbf{j}-4t\mathbf{i}-8t^2\mathbf{j}}{\left(1+4t^2\right)^{\frac{3}{2}}}$$

$$T'(t)=\frac{-4t\mathbf{i}+2\mathbf{j}}{\left(1+4t^2\right)^{\frac{3}{2}}}$$

Next, we'll solve for $T'(1)$ by plugging $t=1$ into the derivative we just found.

$$T'(1)=\frac{-4(1)\mathbf{i}+2\mathbf{j}}{\left(1+4(1)^2\right)^{\frac{3}{2}}}$$

$$T'(1)=\frac{-4\mathbf{i}+2\mathbf{j}}{5^{\frac{3}{2}}}$$

$$T'(1)=-\frac{4}{5\sqrt{5}}\mathbf{i}+\frac{2}{5\sqrt{5}}\mathbf{j}$$

Now we'll find the magnitude $\left|T'(t)\right|$.

$$\left|T'(t)\right|=\sqrt{\left[T'(t)_1\right]^2+\left[T'(t)_2\right]^2+\left[T'(t)_3\right]^2}$$

$$|T'(1)| = \sqrt{\left(-\frac{4}{5\sqrt{5}}\right)^2 + \left(\frac{2}{5\sqrt{5}}\right)^2 + (0)^2}$$

$$|T'(1)| = \sqrt{\frac{16}{25(5)} + \frac{4}{25(5)}}$$

$$|T'(1)| = \sqrt{\frac{20}{125}}$$

$$|T'(1)| = \sqrt{\frac{4}{25}}$$

$$|T'(1)| = \frac{2}{5}$$

Finally, we'll plug everything we've found into the formula for the unit normal vector, and we'll get

$$N(t) = \frac{T'(t)}{|T'(t)|}$$

$$N(1) = \frac{-\frac{4}{5\sqrt{5}}\mathbf{i} + \frac{2}{5\sqrt{5}}\mathbf{j}}{\frac{2}{5}}$$

$$N(1) = -\frac{4(5)}{5(2)\sqrt{5}}\mathbf{i} + \frac{2(5)}{5(2)\sqrt{5}}\mathbf{j}$$

$$N(1) = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

Since multiplying a vector by a constant scalar doesn't change its value, we can multiply by $\sqrt{5}$, and simplify the equation of the unit normal vector to

$$N(1) = -2\mathbf{i} + \mathbf{j}$$

This is the unit normal vector of the vector function $r(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}$ at the point $t = 1$.

Curvature

To find the curvature $\kappa(t)$ of a vector function $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, we'll use the equation

$$\kappa(t) = \frac{\left| T'(t) \right|}{\left| r'(t) \right|}$$

where $\left| T'(t) \right|$ is the magnitude of the derivative of the unit tangent vector $T(t)$, which we can find using

$$\left| T'(t) \right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

where $T(t)$ is the unit tangent vector, which we can find using

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

where $r'(t)$ is the derivative of the vector function and where $\left| r'(t) \right|$ is the magnitude of the derivative of the vector function, which we can find using

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

In other words, in order to find $\kappa(t)$, we'll

1. Find $r'(t)$, and use it to

2. Find $\left| r'(t) \right|$, and then use $r'(t)$ and $\left| r'(t) \right|$ to

3. Find $T(t)$, and then use it to

4. Find $T'(t)$, and then use it to

5. Find $|T'(t)|$, and then use $|r'(t)|$ and $|T'(t)|$ to

6. Find $\kappa(t)$

Example

Find the curvature of the vector function.

$$r(t) = 4t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$$

We'll start by calculating the derivative of the vector function.

$$r'(t) = 4\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$$

Then we'll find $|r'(t)|$.

$$|r'(t)| = \sqrt{[r'(t)_1]^2 + [r'(t)_2]^2 + [r'(t)_3]^2}$$

$$|r'(t)| = \sqrt{(4)^2 + (2t)^2 + (2)^2}$$

$$|r'(t)| = \sqrt{16 + 4t^2 + 4}$$

$$|r'(t)| = \sqrt{4t^2 + 20}$$

$$|r'(t)| = \sqrt{4\left(t^2 + 5\right)}$$

$$|r'(t)| = 2\sqrt{t^2 + 5}$$

Now we'll use the derivative and its magnitude to find an equation for the unit tangent vector $T(t)$.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(t) = \frac{4\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}}{2\sqrt{t^2 + 5}}$$

$$T(t) = \frac{4}{2\sqrt{t^2 + 5}}\mathbf{i} + \frac{2t}{2\sqrt{t^2 + 5}}\mathbf{j} + \frac{2}{2\sqrt{t^2 + 5}}\mathbf{k}$$

$$T(t) = \frac{2}{\sqrt{t^2 + 5}}\mathbf{i} + \frac{t}{\sqrt{t^2 + 5}}\mathbf{j} + \frac{1}{\sqrt{t^2 + 5}}\mathbf{k}$$

Now we can find the derivative of the unit tangent vector $T'(t)$. We'll need to use quotient rule to find the derivatives of the coefficients on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$T'(t) = \frac{(0)\sqrt{t^2 + 5} - (2)\left[\frac{1}{2}\left(t^2 + 5\right)^{-\frac{1}{2}}(2t)\right]}{\left(\sqrt{t^2 + 5}\right)^2}\mathbf{i} + \frac{(1)\sqrt{t^2 + 5} - (t)\left[\frac{1}{2}\left(t^2 + 5\right)^{-\frac{1}{2}}(2t)\right]}{\left(\sqrt{t^2 + 5}\right)^2}\mathbf{j}$$

$$+\frac{(0)\sqrt{t^2 + 5} - (1)\left[\frac{1}{2}\left(t^2 + 5\right)^{-\frac{1}{2}}(2t)\right]}{\left(\sqrt{t^2 + 5}\right)^2}\mathbf{k}$$

$$T'(t) = \frac{-2t\left(t^2 + 5\right)^{-\frac{1}{2}}}{t^2 + 5}\mathbf{i} + \frac{\sqrt{t^2 + 5} - t^2\left(t^2 + 5\right)^{-\frac{1}{2}}}{t^2 + 5}\mathbf{j} + \frac{-t\left(t^2 + 5\right)^{-\frac{1}{2}}}{t^2 + 5}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\sqrt{t^2+5}-\frac{t^2}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\frac{t^2+5}{\sqrt{t^2+5}}-\frac{t^2}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\frac{t^2+5-t^2}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\frac{5}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{5}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

Then we'll find the magnitude of the derivative of the unit tangent vector $\left|T'(t)\right|$.

$$\left|T'(t)\right|=\sqrt{\left[T'(t)_1\right]^2+\left[T'(t)_2\right]^2+\left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right|=\sqrt{\left[-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\right]^2+\left[\frac{5}{\left(t^2+5\right)^{\frac{3}{2}}}\right]^2+\left[-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\right]^2}$$

$$\left|T'(t)\right|=\sqrt{\frac{4t^2}{\left(t^2+5\right)^3}+\frac{25}{\left(t^2+5\right)^3}+\frac{t^2}{\left(t^2+5\right)^3}}$$

$$\left|T'(t)\right| = \sqrt{\frac{5t^2+25}{\left(t^2+5\right)^3}}$$

$$\left|T'(t)\right| = \sqrt{\frac{5\left(t^2+5\right)}{\left(t^2+5\right)^3}}$$

$$\left|T'(t)\right| = \sqrt{\frac{5}{\left(t^2+5\right)^2}}$$

$$\left|T'(t)\right| = \frac{\sqrt{5}}{t^2+5}$$

Finally we can solve for the curvature $\kappa(t)$ of the vector function

$$\kappa(t) = \frac{\left|T'(t)\right|}{\left|r'(t)\right|}$$

$$\kappa(t) = \frac{\frac{\sqrt{5}}{t^2+5}}{2\sqrt{t^2+5}}$$

$$\kappa(t) = \frac{\sqrt{5}}{2\left(t^2+5\right)^{\frac{3}{2}}}$$

This is the curvature of the vector function.

Maximum curvature

Before we can find maximum curvature of a vector function $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, we first have to find curvature $\kappa(t)$. To find the curvature $\kappa(t)$ of a vector function $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, we'll use the equation

$$\kappa(t) = \frac{\left|T'(t)\right|}{\left|r'(t)\right|}$$

where $\left|T'(t)\right|$ is the magnitude of the derivative of the unit tangent vector $T(t)$, which we can find using

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

where $T(t)$ is the unit tangent vector, which we can find using

$$T(t) = \frac{r'(t)}{\left|r'(t)\right|}$$

where $r'(t)$ is the derivative of the vector function and where $\left|r'(t)\right|$ is the magnitude of the derivative of the vector function, which we can find using

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

In other words, in order to find $\kappa(t)$, we'll

1. Find $r'(t)$, and use it to

2. Find $\left|r'(t)\right|$, and then use $r'(t)$ and $\left|r'(t)\right|$ to

3. Find $T(t)$, and then use it to

4. Find $T'(t)$, and then use it to

5. Find $\left|T'(t)\right|$, and then use $\left|r'(t)\right|$ and $\left|T'(t)\right|$ to

6. Find $\kappa(t)$

Once we have curvature, we'll take its derivative $\kappa'(t)$. We'll set the derivative equal to 0 and solve for t. If there's only one value for t, that value is the one associated with maximum curvature. If there's more than one value for t, we'll use the second derivative test to determine which one represents maximum curvature.

Example

Find maximum curvature of the vector function with the given curvature.

$$\kappa(t) = 8t^2 - 4t$$

First, we'll find the derivative of $\kappa(t)$.

$$\kappa(t) = 8t^2 - 4t$$

$$\kappa'(t) = 16t - 4$$

Next we'll set $\kappa'(t) = 0$ and solve for t.

$$0 = 16t - 4$$

$$-16t = -4$$

$$t = \frac{-4}{-16}$$

$$t = \frac{1}{4}$$

Since we found just one value for t, we know that maximum curvature occurs when $t = 1/4$.

Normal and osculating planes

Equation of the normal plane

To find the equation of the normal plane of a vector function $r(t)=r(t)_1\mathbf{i}+r(t)_2\mathbf{j}+r(t)_3\mathbf{k}$ at some point P, we'll use the equation

$$r'(t)_1(x-x_0)+r'(t)_2(y-y_0)+r'(t)_3(z-z_0)=0$$

where x_0, y_0 and z_0 come from the point $P(x_0,y_0,z_0)$, and where $r'(t)_1$, $r'(t)_2$ and $r'(t)_3$ come from the derivative of the vector function $r'(t)=\langle r'(t)_1,r'(t)_2,r'(t)_3\rangle$ at t.

In order to find a value for t, we'll need to compare the vector function $r(t)$ to the point P.

Equation of the osculating plane

If we want to find the equation of the osculating plane of the same vector function $r(t)$ at the same point P, we'll use the equation

$$B(t)_1(x-x_0)+B(t)_2(y-y_0)+B(t)_3(z-z_0)=0$$

where x_0, y_0 and z_0 come from the point $P(x_0,y_0,z_0)$, and where $B(t)_1$, $B(t)_2$ and $B(t)_3$ come from the binormal unit vector $B(t)=\langle B(t)_1,B(t)_2,B(t)_3\rangle$ at t.

In order to find a value for t, we'll need to compare the vector function $r(t)$ to the point P.

To get to the binormal unit vector, we'll first have to find the unit tangent vector $T(t)$, and the unit normal vector $N(t)$.

The unit tangent vector $T(t)$ is given by

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

where $r'(t)$ is the derivative of $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, and where $\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$.

The unit normal vector $N(t)$ is given by

$$N(t) = \frac{T'(t)}{\left| T'(t) \right|}$$

where $T'(t)$ is the derivative of $T(t) = T(t)_1\mathbf{i} + T(t)_2\mathbf{j} + T(t)_3\mathbf{k}$ and where $\left| T'(t) \right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$.

Then the binormal unit vector $B(t)$ is given by

$$B(t) = T(t) \times N(t)$$

where $T(t) \times N(t)$ is the cross product of the unit tangent and unit normal vectors.

Example

Find the equations of the normal and osculating planes of the vector function at $P(1,1,0)$.

$$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$$

We'll need to find the following values, in this order:

The equation of the normal plane

$r'(t)_1(x - x_0) + r'(t)_2(y - y_0) + r'(t)_3(z - z_0) = 0$ using

$$t$$

$$r'(t) = \langle r'(t)_1, r'(t)_2, r'(t)_3 \rangle$$

The equation of the osculating plane

$B(t)_1(x - x_0) + B(t)_2(y - y_0) + B(t)_3(z - z_0) = 0$ using

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T'(t) = \langle T'(t)_1, T'(t)_2, T'(t)_3 \rangle$$

$$\left| T'(t) \right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

$$N(t) = \frac{T'(t)}{\left| T'(t) \right|}$$

$$B(t) = T(t) \times N(t)$$

Finding t

Let's start by finding a value for t. Since the coefficient on $\mathbf{k}$ is t itself, we can just take the value of z from the coordinate point and say that $t=0$.

Finding $r'(t)=\langle r'(t)_1, r'(t)_2, r'(t)_3\rangle$

Now we can move on to the equation of the normal plane. Our first step is to find the derivative of the vector function. In order to take the derivative of a vector function, we ignore $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ and just take the derivative of each of the coefficients.

$$r(t)=\cos t\mathbf{i}+\sin t\mathbf{j}+t\mathbf{k}$$

$$r'(t)=-\sin t\mathbf{i}+\cos t\mathbf{j}+\mathbf{k}$$

$$r'(t)=\langle -\sin t, \cos t, 1\rangle$$

Since $t=0$, the derivative becomes

$$r'(0)=\langle -\sin 0, \cos 0, 1\rangle$$

$$r'(0)=\langle 0,1,1\rangle$$

$$r'(0)=0\mathbf{i}+1\mathbf{j}+1\mathbf{k}$$

$$r'(0)=\mathbf{j}+\mathbf{k}$$

Finding the equation of the normal plane

Plugging in the values from the derivative and the point $P(1,1,0)$ to the formula for the equation of the normal plane, we get

$$r'(t)_1(x - x_0) + r'(t)_2(y - y_0) + r'(t)_3(z - z_0) = 0$$

$$0(x - 1) + 1(y - 1) + 1(z - 0) = 0$$

$$(y - 1) + z = 0$$

$$y + z = 1$$

$$y = 1 - z$$

So the equation of the normal plane is $y = 1 - z$.

Finding $\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$

Now we'll work on the equation of the osculating plane. Our first step is to find the unit tangent vector $T(t)$, but since

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

we'll need to find the magnitude of the derivative first, so that we can plug it into the denominator. We already found $r'(t)$ when we were working on the equation of the normal plane.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left| r'(t) \right| = \sqrt{\left[-\sin t\right]^2 + [\cos t]^2 + [1]^2}$$

$$\left| r'(t) \right| = \sqrt{\sin^2 t + \cos^2 t + 1}$$

Given the trigonometric identity $\sin^2 x + \cos^2 x = 1$, we can say

$$\left| r'(t) \right| = \sqrt{1 + 1}$$

$$\left| r'(t) \right| = \sqrt{2}$$

Since there's no t variable on the right side, evaluating the magnitude of the derivative at $t = 0$ doesn't change its value, so

$$\left| r'(0) \right| = \sqrt{2}$$

Finding $T(t) = \frac{r'(t)}{\left| r'(t) \right|}$

Therefore we can say that the unit tangent vector is

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(t) = \frac{-\sin t\mathbf{i} + \cos t\mathbf{j} + \mathbf{k}}{\sqrt{2}}$$

$$T(t) = -\frac{1}{\sqrt{2}}\sin t\mathbf{i} + \frac{1}{\sqrt{2}}\cos t\mathbf{j} + \frac{1}{\sqrt{2}}\mathbf{k}$$

$$T(t)=\frac{\sqrt{2}}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\sin t\mathbf{i}+\frac{1}{\sqrt{2}}\cos t\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}\right)$$

$$T(t)=-\frac{\sqrt{2}}{2}\sin t\mathbf{i}+\frac{\sqrt{2}}{2}\cos t\mathbf{j}+\frac{\sqrt{2}}{2}\mathbf{k}$$

$$T(t)=\left\langle -\frac{\sqrt{2}}{2}\sin t,\frac{\sqrt{2}}{2}\cos t,\frac{\sqrt{2}}{2}\right\rangle$$

Since $t=0$, we get

$$T(0)=\left\langle -\frac{\sqrt{2}}{2}\sin 0,\frac{\sqrt{2}}{2}\cos 0,\frac{\sqrt{2}}{2}\right\rangle$$

$$T(0)=\left\langle 0,\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right\rangle$$

$$T(0)=\frac{\sqrt{2}}{2}\mathbf{j}+\frac{\sqrt{2}}{2}\mathbf{k}$$

Finding $T'(t)=\langle T'(t)_1,T'(t)_2,T'(t)_3\rangle$

Our next step is to find the unit normal vector $N(t)$, but since

$$N(t)=\frac{T'(t)}{\left|T'(t)\right|}$$

we'll need to find the derivative of the unit tangent vector $T'(t)$ and its magnitude first, so that we can plug them into the formula for the unit normal vector.

$$T(t) = -\frac{\sqrt{2}}{2}\sin t\mathbf{i} + \frac{\sqrt{2}}{2}\cos t\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

$$T'(t) = -\frac{\sqrt{2}}{2}\cos t\mathbf{i} - \frac{\sqrt{2}}{2}\sin t\mathbf{j} + 0\mathbf{k}$$

$$T'(t) = -\frac{\sqrt{2}}{2}\cos t\mathbf{i} - \frac{\sqrt{2}}{2}\sin t\mathbf{j}$$

$$T'(t) = \left\langle -\frac{\sqrt{2}}{2}\cos t, -\frac{\sqrt{2}}{2}\sin t, 0\right\rangle$$

Evaluating the derivative at $t = 0$, we get

$$T'(0) = \left\langle -\frac{\sqrt{2}}{2}\cos 0, -\frac{\sqrt{2}}{2}\sin 0, 0\right\rangle$$

$$T'(0) = \left\langle -\frac{\sqrt{2}}{2}, 0, 0\right\rangle$$

$$T'(0) = -\frac{\sqrt{2}}{2}\mathbf{i}$$

Finding $\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$

Now we'll find the magnitude of the derivative.

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right| = \sqrt{\left(-\frac{\sqrt{2}}{2}\cos t\right)^2 + \left(-\frac{\sqrt{2}}{2}\sin t\right)^2 + (0)^2}$$

$$\left|T'(t)\right| = \sqrt{\frac{2}{4}\cos^2 t + \frac{2}{4}\sin^2 t}$$

$$\left|T'(t)\right| = \sqrt{\frac{1}{2}\left(\cos^2 t + \sin^2 t\right)}$$

Again we'll use the identity $\sin^2 x + \cos^2 x = 1$ to get

$$\left|T'(t)\right| = \sqrt{\frac{1}{2}(1)}$$

$$\left|T'(t)\right| = \sqrt{\frac{1}{2}}$$

$$\left|T'(t)\right| = \frac{\sqrt{1}}{\sqrt{2}}$$

$$\left|T'(t)\right| = \frac{1}{\sqrt{2}}$$

$$\left|T'(t)\right| = \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$$

$$\left|T'(t)\right| = \frac{\sqrt{2}}{2}$$

Since there's no t variable on the right side, evaluating the magnitude of the derivative at $t = 0$ doesn't change its value, so

$$\left|T'(0)\right| = \frac{\sqrt{2}}{2}$$

Finding $N(t) = \frac{T'(t)}{\left|T'(t)\right|}$

Therefore we can say that the unit normal vector is

$$N(t) = \frac{T'(t)}{\left|T'(t)\right|}$$

$$N(t) = \frac{-\frac{\sqrt{2}}{2}\cos t\mathbf{i} - \frac{\sqrt{2}}{2}\sin t\mathbf{j}}{\frac{\sqrt{2}}{2}}$$

$$N(t) = \frac{-1\cos t\mathbf{i} - 1\sin t\mathbf{j}}{1}$$

$$N(t) = -\cos t\mathbf{i} - \sin t\mathbf{j}$$

$$N(t) = \langle -\cos t, -\sin t, 0\rangle$$

Since $t = 0$, we get

$$N(0) = \langle -\cos 0, -\sin 0, 0 \rangle$$

$$N(0) = \langle -1, 0, 0 \rangle$$

$$N(0) = -\mathbf{i}$$

Finding $B(t) = T(t) \times N(t)$

Now that we have the unit tangent and unit normal vectors, we can find the binormal unit vector.

$$B(t) = T(t) \times N(t)$$

$$B(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\frac{\sqrt{2}}{2}\sin t & \frac{\sqrt{2}}{2}\cos t & \frac{\sqrt{2}}{2} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$B(t) = \begin{vmatrix} \frac{\sqrt{2}}{2}\cos t & \frac{\sqrt{2}}{2} \\ -\sin t & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -\frac{\sqrt{2}}{2}\sin t & \frac{\sqrt{2}}{2} \\ -\cos t & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -\frac{\sqrt{2}}{2}\sin t & \frac{\sqrt{2}}{2}\cos t \\ -\cos t & -\sin t \end{vmatrix} \mathbf{k}$$

$$B(t) = \left[\left(\frac{\sqrt{2}}{2}\cos t\right)(0) - (-\sin t)\left(\frac{\sqrt{2}}{2}\right)\right]\mathbf{i}$$

$$-\left[\left(-\frac{\sqrt{2}}{2}\sin t\right)(0) - (-\cos t)\left(\frac{\sqrt{2}}{2}\right)\right]\mathbf{j}$$

$$+\left[\left(-\frac{\sqrt{2}}{2}\sin t\right)(-\sin t) - (-\cos t)\left(\frac{\sqrt{2}}{2}\cos t\right)\right]\mathbf{k}$$

$$B(t) = \left[\sin t\left(\frac{\sqrt{2}}{2}\right)\right]\mathbf{i} - \left[\cos t\left(\frac{\sqrt{2}}{2}\right)\right]\mathbf{j} + \left[\sin t\left(\frac{\sqrt{2}}{2}\sin t\right) + \cos t\left(\frac{\sqrt{2}}{2}\cos t\right)\right]\mathbf{k}$$

$$B(t) = \frac{\sqrt{2}}{2}\sin t\mathbf{i} - \frac{\sqrt{2}}{2}\cos t\mathbf{j} + \left[\frac{\sqrt{2}}{2}\sin^2 t + \frac{\sqrt{2}}{2}\cos^2 t\right]\mathbf{k}$$

$$B(t) = \frac{\sqrt{2}}{2}\sin t\mathbf{i} - \frac{\sqrt{2}}{2}\cos t\mathbf{j} + \left[\frac{\sqrt{2}}{2}\left(\sin^2 t + \cos^2 t\right)\right]\mathbf{k}$$

Again we'll use the identity $\sin^2 x + \cos^2 x = 1$ to get

$$B(t) = \frac{\sqrt{2}}{2}\sin t\mathbf{i} - \frac{\sqrt{2}}{2}\cos t\mathbf{j} + \left(\frac{\sqrt{2}}{2}(1)\right)\mathbf{k}$$

$$B(t) = \frac{\sqrt{2}}{2}\sin t\mathbf{i} - \frac{\sqrt{2}}{2}\cos t\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

$$B(t) = \left\langle \frac{\sqrt{2}}{2}\sin t, -\frac{\sqrt{2}}{2}\cos t, \frac{\sqrt{2}}{2} \right\rangle$$

Since $t = 0$, we get

$$B(0) = \left\langle \frac{\sqrt{2}}{2}\sin 0, -\frac{\sqrt{2}}{2}\cos 0, \frac{\sqrt{2}}{2} \right\rangle$$

$$B(0) = \left\langle 0, -\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

$$B(0) = -\frac{\sqrt{2}}{2}\mathbf{j} + \frac{\sqrt{2}}{2}\mathbf{k}$$

Finding the equation of the osculating plane

Plugging in the values from the unit binormal vector and the point $P(1,1,0)$ to the formula for the equation of the osculating plane, we get

$$B(t)_1(x-x_0)+B(t)_2(y-y_0)+B(t)_3(z-z_0)=0$$

$$0(x-1)-\frac{\sqrt{2}}{2}(y-1)+\frac{\sqrt{2}}{2}(z-0)=0$$

$$-\frac{\sqrt{2}}{2}y+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}z=0$$

$$-\frac{\sqrt{2}}{2}y+\frac{\sqrt{2}}{2}z=-\frac{\sqrt{2}}{2}$$

$$-1y+1z=-1$$

$$-y+z=-1$$

$$y=1+z$$

So the equation of the osculating plane is $y=1+z$.

Velocity and acceleration vectors

Given a position function $r(t)$ that models the position of an object over time, velocity $v(t)$ is the derivative of position, and acceleration $a(t)$ is the derivative of velocity, which means that acceleration is also the second derivative of position.

Position vector $\quad r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$

Velocity vector $\quad v(t) = r'(t) = r'(t)_1\mathbf{i} + r'(t)_2\mathbf{j} + r'(t)_3\mathbf{k}$

Acceleration vector $\quad a(t) = v'(t) = r''(t) = r''(t)_1\mathbf{i} + r''(t)_2\mathbf{j} + r''(t)_3\mathbf{k}$

Since we know that the derivative of position is velocity, and the derivative of velocity is acceleration, that means that we can also go the other way and say that the integral of acceleration is velocity, and the integral of velocity is position.

Acceleration vector $\quad a(t) = a(t)_1\mathbf{i} + a(t)_2\mathbf{j} + a(t)_3\mathbf{k}$

Velocity vector $\quad v(t) = \int a(t)\ dt = \mathbf{i}\int a(t)_1\ dt + \mathbf{j}\int a(t)_2\ dt + \mathbf{k}\int a(t)_3\ dt$

$$= v(t)_1\mathbf{i} + v(t)_2\mathbf{j} + v(t)_3\mathbf{k}$$

Position vector $\quad r(t) = \int v(t)\ dt = \mathbf{i}\int v(t)_1\ dt + \mathbf{j}\int v(t)_2\ dt + \mathbf{k}\int v(t)_3\ dt$

$$= r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

Example

Find the position function if acceleration is given by $a(t) = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$, and if $v(0) = \mathbf{j} - \mathbf{k}$ and $r(0) = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

We've been given acceleration, and we need to find position, which means we'll need to do some integration. We'll start by integrating acceleration to get to velocity.

$$v(t) = \int a(t)\ dt = \mathbf{i}\int 2\ dt + \mathbf{j}\int 1\ dt + \mathbf{k}\int 3\ dt$$

$$v(t) = 2t\mathbf{i} + t\mathbf{j} + 3t\mathbf{k} + C$$

Without any other information, we wouldn't be able to solve for the value of C. But since we know that $v(0) = \mathbf{j} - \mathbf{k}$, we can plug this initial condition into the velocity function to find a value for C.

$$\mathbf{j} - \mathbf{k} = 2(0)\mathbf{i} + (0)\mathbf{j} + 3(0)\mathbf{k} + C$$

$$\mathbf{j} - \mathbf{k} = C$$

Plugging this value for C back into the velocity function, we get

$$v(t) = 2t\mathbf{i} + t\mathbf{j} + 3t\mathbf{k} + \mathbf{j} - \mathbf{k}$$

$$v(t) = 2t\mathbf{i} + (t+1)\mathbf{j} + (3t-1)\mathbf{k}$$

Now we'll integrate the velocity function in order to find position.

$$r(t) = \int v(t)\ dt = \mathbf{i}\int 2t\ dt + \mathbf{j}\int t+1\ dt + \mathbf{k}\int 3t-1\ dt$$

$$r(t) = \left(\frac{2}{2}t^2\right)\mathbf{i} + \left(\frac{1}{2}t^2 + t\right)\mathbf{j} + \left(\frac{3}{2}t^2 - t\right)\mathbf{k} + C_2$$

$$r(t) = t^2\mathbf{i} + \left(\frac{1}{2}t^2 + t\right)\mathbf{j} + \left(\frac{3}{2}t^2 - t\right)\mathbf{k} + C_2$$

We also have an initial condition for the position function, so we'll plug that in to find a value for C_2.

$$\mathbf{i} + 2\mathbf{j} - \mathbf{k} = (0)^2\mathbf{i} + \left[\frac{1}{2}(0)^2 + 0\right]\mathbf{j} + \left[\frac{3}{2}(0)^2 - 0\right]\mathbf{k} + C_2$$

$$\mathbf{i} + 2\mathbf{j} - \mathbf{k} = C_2$$

Plugging this value for C_2 back into the position function, we get

$$r(t) = t^2\mathbf{i} + \left(\frac{1}{2}t^2 + t\right)\mathbf{j} + \left(\frac{3}{2}t^2 - t\right)\mathbf{k} + \mathbf{i} + 2\mathbf{j} - \mathbf{k}$$

$$r(t) = \left(t^2 + 1\right)\mathbf{i} + \left(\frac{1}{2}t^2 + t + 2\right)\mathbf{j} + \left(\frac{3}{2}t^2 - t - 1\right)\mathbf{k}$$

This is the position function associated with the given acceleration function and the initial conditions.

Velocity, acceleration and speed given position

Given a position function

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

the velocity function is the derivative of position, and the acceleration function is the derivative of velocity (which means acceleration is also the second derivative of position).

Position vector	$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$
Velocity vector	$v(t) = r'(t) = r'(t)_1\mathbf{i} + r'(t)_2\mathbf{j} + r'(t)_3\mathbf{k}$
Acceleration vector	$a(t) = v'(t) = r''(t) = r''(t)_1\mathbf{i} + r''(t)_2\mathbf{j} + r''(t)_3\mathbf{k}$

We can also find speed by taking the magnitude of the velocity function.

$$s = \left|v(t)\right| = \sqrt{\left[v(t)_1\right]^2 + \left[v(t)_2\right]^2 + \left[v(t)_3\right]^2}$$

Example

Find the velocity and acceleration functions and speed if the position function is given by $r(t) = 4t^2\mathbf{i} + t^3\mathbf{j} + \sin(2t)\mathbf{k}$.

We'll take the derivative of position to find velocity.

$$v(t) = r'(t) = 8t\mathbf{i} + 3t^2\mathbf{j} + 2\cos(2t)\mathbf{k}$$

Now we'll take the derivative of velocity to find acceleration.

$$a(t) = v'(t) = 8\mathbf{i} + 6t\mathbf{j} - 4\sin(2t)\mathbf{k}$$

Finally, we'll go back to the velocity function we found earlier and find its magnitude in order to get speed.

$$s = \left|v(t)\right| = \sqrt{\left[v(t)_1\right]^2 + \left[v(t)_2\right]^2 + \left[v(t)_3\right]^2}$$

$$s = \left|v(t)\right| = \sqrt{\left[8t\right]^2 + \left[3t^2\right]^2 + \left[2\cos(2t)\right]^2}$$

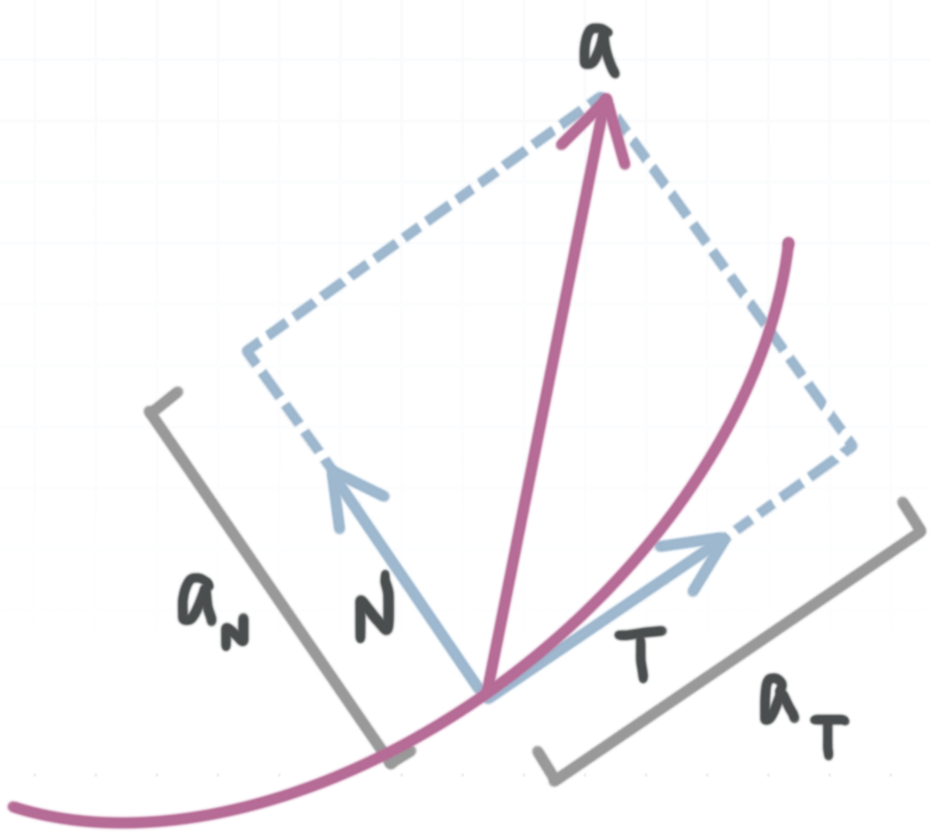
$$s = \sqrt{64t^2 + 9t^4 + 4\cos^2(2t)}$$

We can summarize our findings as

Velocity function	$v(t) = 8t\mathbf{i} + 3t^2\mathbf{j} + 2\cos(2t)\mathbf{k}$
Acceleration function	$a(t) = 8\mathbf{i} + 6t\mathbf{j} - 4\sin(2t)\mathbf{k}$
Speed	$s = \sqrt{64t^2 + 9t^4 + 4\cos^2(2t)}$

Tangential and normal components of the acceleration vector

At any given point along a curve, we can find the acceleration vector a that represents acceleration at that point. If we find the unit tangent vector T and the unit normal vector N at the same point, then the tangential component of acceleration a_T and the normal component of acceleration a_N are shown in the diagram below.



Tangential component of acceleration $$a_T = \frac{r'(t) \cdot r''(t)}{\left| r'(t) \right|}$$

Normal component of acceleration $$a_N = \frac{\left| r'(t) \times r''(t) \right|}{\left| r'(t) \right|}$$

In these formulas for the tangential and normal components,

$r(t)$ is the position vector, $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$

$r'(t)$ is its first derivative, $r'(t) = r'(t)_1\mathbf{i} + r'(t)_2\mathbf{j} + r'(t)_3\mathbf{k}$

$r''(t)$ is its second derivative, $r''(t) = r''(t)_1\mathbf{i} + r''(t)_2\mathbf{j} + r''(t)_3\mathbf{k}$

$r'(t) \cdot r''(t)$ is the dot product of the first and second derivatives, $r'(t) \cdot r''(t) = r'(t)_1 r''(t)_1 + r'(t)_2 r''(t)_2 + r'(t)_3 r''(t)_3$

$\left| r'(t) \right|$ is the magnitude of the first derivative,

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$\left| r'(t) \times r''(t) \right|$ is the magnitude of the cross product of the first and second derivatives, where the cross product is

$$r'(t) \times r''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r'(t)_1 & r'(t)_2 & r'(t)_3 \\ r''(t)_1 & r''(t)_2 & r''(t)_3 \end{vmatrix}$$

We'll start by finding each of the pieces in the list above, and then we'll plug them into the formulas for the tangential and normal components of the acceleration vector.

Example

Find the tangential and normal components of the acceleration vector.

$$r(t) = 2t^2\mathbf{i} + 4t\mathbf{j} + 3t^3\mathbf{k}$$

We'll start by finding $r'(t)$, the derivative of the position function. To find the derivative, we'll just replace the coefficients on $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ with their derivatives.

$$r'(t) = 4t\mathbf{i} + 4\mathbf{j} + 9t^2\mathbf{k}$$

can also be written as $r'(t) = \left\langle 4t, 4, 9t^2 \right\rangle$

We'll repeat the process to find the second derivative.

$$r''(t) = 4\mathbf{i} + 0\mathbf{j} + 18t\mathbf{k}$$

$$r''(t) = 4\mathbf{i} + 18t\mathbf{k}$$

can also be written as $r''(t) = \langle 4, 0, 18t \rangle$

Now we'll find the dot product of the first and second derivatives.

$$r'(t) \cdot r''(t) = (4t)(4) + (4)(0) + (9t^2)(18t)$$

$$r'(t) \cdot r''(t) = 16t + 0 + 162t^3$$

$$r'(t) \cdot r''(t) = 16t + 162t^3$$

$$r'(t) \cdot r''(t) = 162t^3 + 16t$$

Now we'll find the magnitude of the first derivative.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1 \right]^2 + \left[r'(t)_2 \right]^2 + \left[r'(t)_3 \right]^2}$$

$$\left| r'(t) \right| = \sqrt{(4t)^2 + (4)^2 + \left(9t^2\right)^2}$$

$$\left| r'(t) \right| = \sqrt{16t^2 + 16 + 81t^4}$$

$$\left|r'(t)\right| = \sqrt{81t^4 + 16t^2 + 16}$$

Finally, we'll get the cross product of the first and second derivatives, then find its magnitude.

$$r'(t) \times r''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r'(t)_1 & r'(t)_2 & r'(t)_3 \\ r''(t)_1 & r''(t)_2 & r''(t)_3 \end{vmatrix}$$

$$r'(t) \times r''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4t & 4 & 9t^2 \\ 4 & 0 & 18t \end{vmatrix}$$

$$r'(t) \times r''(t) = \begin{vmatrix} 4 & 9t^2 \\ 0 & 18t \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4t & 9t^2 \\ 4 & 18t \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4t & 4 \\ 4 & 0 \end{vmatrix} \mathbf{k}$$

$$r'(t) \times r''(t) = \left[(4)(18t) - (0)\left(9t^2\right)\right] \mathbf{i} - \left[(4t)(18t) - (4)\left(9t^2\right)\right] \mathbf{j} + \left[(4t)(0) - (4)(4)\right] \mathbf{k}$$

$$r'(t) \times r''(t) = (72t - 0)\,\mathbf{i} - \left(72t^2 - 36t^2\right) \mathbf{j} + \left(0 - 16\right) \mathbf{k}$$

$$r'(t) \times r''(t) = 72t\mathbf{i} - 36t^2\mathbf{j} - 16\mathbf{k}$$

$$r'(t) \times r''(t) = 4\left(18t\mathbf{i} - 9t^2\mathbf{j} - 4\mathbf{k}\right)$$

can also be written as $r'(t) \times r''(t) = 4\left\langle 18t, -9t^2, -4\right\rangle$

Now we just need the magnitude of the cross product.

$$\left|r'(t) \times r''(t)\right| = 4\sqrt{(18t)^2 + \left(-9t^2\right)^2 + (-4)^2}$$

$$\left|r'(t) \times r''(t)\right| = 4\sqrt{324t^2 + 81t^4 + 16}$$

$$\left|r'(t) \times r''(t)\right| = 4\sqrt{81t^4 + 324t^2 + 16}$$

We've finally found everything we need to solve for the tangential and normal components of acceleration. Plugging in what we know, we get

The tangential component of acceleration

$$a_T = \frac{r'(t) \cdot r''(t)}{\left|r'(t)\right|}$$

$$a_T = \frac{162t^3 + 16t}{\sqrt{81t^4 + 16t^2 + 16}}$$

The normal component of acceleration

$$a_N = \frac{\left|r'(t) \times r''(t)\right|}{\left|r'(t)\right|}$$

$$a_N = \frac{4\sqrt{81t^4 + 324t^2 + 16}}{\sqrt{81t^4 + 16t^2 + 16}}$$

$$a_N = 4\sqrt{\frac{81t^4 + 324t^2 + 16}{81t^4 + 16t^2 + 16}}$$

These are the tangential and normal components of the acceleration vector.

Line integral of a curve

Single variable integrals

In single variable calculus we learned how to evaluate an integral over an interval $[a, b]$ in order to calculate the area under the curve on that interval. We could approximate the area under the curve using a Riemann sum, or calculate the area exactly using an integral.

The Riemann sum might have been

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where A is the area underneath the function $f(x)$ and above the x-axis, n is the number of rectangles we use to approximate the area, and Δx is the width of our approximating rectangles.

We learned that our approximation became more and more accurate as we used a larger and larger number of approximating rectangles, and so we knew that to find *exact* area, we had to use an *infinite* number of rectangles. To translate that into our area approximation equation above, we took the limit of the sum as $n \to \infty$ to get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The limit and sum notation $\lim_{n\to\infty}\sum_{i=1}^{n}$ becomes the integral, $f(x_i^*)$ becomes $f(x)$, and Δx becomes dx when we translate the approximating sum into an integral, and we get

$$A = \int_a^b f(x)\ dx$$

where A is the area underneath the function $f(x)$ and above the x-axis.

Line integrals

In contrast, when we find a line integral, we take the integral over a curve C, instead of over the interval $[a, b]$. Where we used to divide the interval into n rectangles, each with a width of Δx, now we'll divide the interval into n sub-arcs, each with a width of Δs. Which means the Riemann sum representing the line integral is

$$A = \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Letting $n \to \infty$ to find *exact* area, we get a formula for the line integral:

If f is defined on a smooth curve C, then the line integral of f along C is

$$\int_C f(x, y)\ ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Since ds represents arc length, we can replace it in our integral with the arc length formula, and get this formula for the line integral:

$$\int_C f(x, y)\ ds = \int_a^b f\left(x(t), y(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\ dt$$

Where the value of a normal integral of a single-variable function is the area underneath the curve, the value of the line integral is the area of one side of the “curtain”, or “fence” or “wall” whose base is the curve C and whose height is given by the function $f(x, y)$.

Potential function of a conservative vector field

A vector field $\mathbf{F}$ is called conservative if it's the gradient of some scalar function, that is, if there exists a function f such that

$$\mathbf{F} = \nabla f$$

In this situation f is called a potential function for $\mathbf{F}$.

Showing that a vector field is conservative

Given a vector field

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$$

$\mathbf{F}$ is conservative if

1. The domain of $\mathbf{F}$ is open and simply-connected, and

2. The scalar curl of $\mathbf{F}$ is 0

Open and simply-connected

The domain of $\mathbf{F}$ is open and simply-connected if $\mathbf{F}$ is defined on the entire plane R^2. If the domain of $\mathbf{F}$ is R^2, then the domain of $\mathbf{F}$ is **open**, such that it doesn't contain any of its boundary points, and the domain of $\mathbf{F}$ is **simply-connected**, such that it's connected and contains no holes.

Scalar curl

The scalar curl of $\mathbf{F}$ is 0, which means that

1. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$

    This also implies that

    $$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = 0$$

    and

    $$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$$

2. Because $\partial P/\partial y$ must be equal to $\partial Q/\partial x$, we should be able to subtract P from Q or vice versa and get 0. Because the partial derivatives are interchangeable, you'll sometimes see them written as

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$

but with this notation it's important to remember that we have to take the partial derivative with respect to x of the function Q, and the partial derivative with respect to y of the function P.

Whichever notation we use,

$$\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \quad = \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \quad = \quad \frac{\partial F_1}{\partial y} - \frac{\partial F_2}{\partial x} \quad = \quad \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \quad = \quad 0$$

is called the scalar curl of the vector field $\mathbf{F}$.

Line integrals of conservative vector fields

The value of the line integral over the curve C inside a conservative vector field is always the same, regardless of the path of the curve C. This means that the value of the line integral only depends on the initial and terminal points of C.

This means that we can evaluate the line integral of a conservative vector field using only the endpoints of the curve C, because the line integral of ∇f is just the net change in f.

So the theorem that defines the line integral of a conservative vector field says:

Assume C is a smooth curve defined by the vector function $\mathbf{r}(t)$, with $a \le t \le b$. If f is a differentiable function whose gradient vector ∇f is continuous on C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$= f(x_1, y_1) - f(x_2, y_2)$$

The theorem shows us that, in order to find the value of the line integral of a conservative vector field, we just follow these steps:

1. Show that $\mathbf{F}$ is conservative

2. If $\mathbf{F}$ is conservative, find its potential function f

3. Evaluate f over the interval $[a, b]$, and the answer is the value of the line integral

Independence of path

Independence of path is a property of conservative vector fields. If a conservative vector field contains the entire curve C, then the line integral over the curve C will be independent of path, because every line integral in a conservative vector field is independent of path, since all conservative vector fields are path independent.

We can state the following facts:

1. Conservative vector fields are independent of path

2. Vector fields that are independent of path are conservative

As a result, the value of the line integral depends only on the endpoints of the curve C, and not on the path taken by the integral between the endpoints.

No matter which path you follow between two points in a conservative vector field, whether it's a direct, straight line, or a curvy, winding path, or any other path, the value of the line integral will be the same if the endpoints are the same.

That fact that conservative vector fields are independent of path makes finding the line integral of the vector field easy. All we need is the potential function f of the vector field $\mathbf{F}$, such that

$$\mathbf{F} = \nabla f$$

Once we find f, we simply evaluate it over the interval defined by the endpoints of the curve C, and our answer will be the value of the line integral of the vector field $\mathbf{F}$.

In other words, if the endpoints of the curve C are a and b or (x_1, y_1) and (x_2, y_2), then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r}$$

$$= f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

$$= f(x_2, y_2) - f(x_1, y_1)$$

Green's theorem for one region

Green's theorem gives us a way to change a line integral into a double integral. If a line integral is particularly difficult to evaluate, then using Green's theorem to change it to a double integral might be a good way to approach the problem.

If we want to find the area of a simple region, and the original line integral has the form

$$\oint_c P\ dx + Q\ dy$$

then we can apply Green's theorem to change the line integral into a double integral in the form

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

where

$\frac{\partial Q}{\partial x}$ is the partial derivative of Q with respect to x

$\frac{\partial P}{\partial y}$ is the partial derivative of P with respect to y

If we choose to use Green's theorem and change the line integral to a double integral, we'll need to find limits of integration for both x and y so that we can evaluate the double integral as an iterated integral. Often the limits for x and y will be given to us in the problem.

Example

Solve the line integral for the region $(\pm 1, \pm 1)$.

$$\oint_c \left(2x^2+4y\right)\ dx + \left(x^2-5y^3\right)\ dy$$

Since the integral we were given matches the form

$$\oint_c P\ dx + Q\ dy$$

we know we can use Green's theorem to change it to

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

We'll start by finding partial derivatives.

Since $Q(x,y) = x^2 - 5y^3$,

$$\frac{\partial Q}{\partial x} = 2x$$

Since $P(x,y) = 2x^2 + 4y$,

$$\frac{\partial P}{\partial y} = 4$$

We were told in the problem that the region would be given by the interval $(\pm 1, \pm 1)$. Plugging everything we have into the converted formula, we get

$$\iint_R \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\ dA$$

$$\int_{-1}^{1}\int_{-1}^{1} 2x-4\ dy\ dx$$

Integrating with respect to y and evaluating over the associated interval, we get

$$\int_{-1}^{1} 2xy-4y\Big|_{y=-1}^{y=1}\ dx$$

$$\int_{-1}^{1} 2x(1)-4(1)-\left[2x(-1)-4(-1)\right]\ dx$$

$$\int_{-1}^{1} 2x-4-(-2x+4)\ dx$$

$$\int_{-1}^{1} 2x-4+2x-4\ dx$$

$$\int_{-1}^{1} 4x-8\ dx$$

Integrating with respect to x and evaluating over its interval, we get

$$2x^2-8x\Big|_{-1}^{1}$$

$$2(1)^2-8(1)-\left[2(-1)^2-8(-1)\right]$$

$$2-8-(2+8)$$

$$2-8-2-8$$

$$-16$$

This is the area of the region.

Green's theorem for two regions

Green's theorem gives us a way to change a line integral into a double integral. If a line integral is particularly difficult to evaluate, then using Green's theorem to change it to a double integral might be a good way to approach the problem.

If we want to find the area of a region which is the union of two simple regions, and the original line integral has the form

$$\oint_c P\ dx + Q\ dy$$

then we can apply Green's theorem to change the line integral into a double integral in the form

$$\iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

where

$\frac{\partial Q}{\partial x}$ is the partial derivative of Q with respect to x

$\frac{\partial P}{\partial y}$ is the partial derivative of P with respect to y

If we choose to use Green's theorem and change the line integral to a double integral, we'll need to find limits of integration for both x and y so that we can evaluate the double integral as an iterated integral. Often the limits for x and y will be given to us in the problem.

Example

Solve the line integral for the triangular region with vertices at $(0,0)$, $(1,1)$ and $(2,0)$.

$$\oint_c \left(5\sin x + 5y\right)\ dx + \left(5x^2 - 3y^2\right)\ dy$$

Since the integral we were given matches the form

$$\oint_c P\ dx + Q\ dy$$

we know we can use Green's theorem to change it to

$$\iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

We'll start by finding partial derivatives.

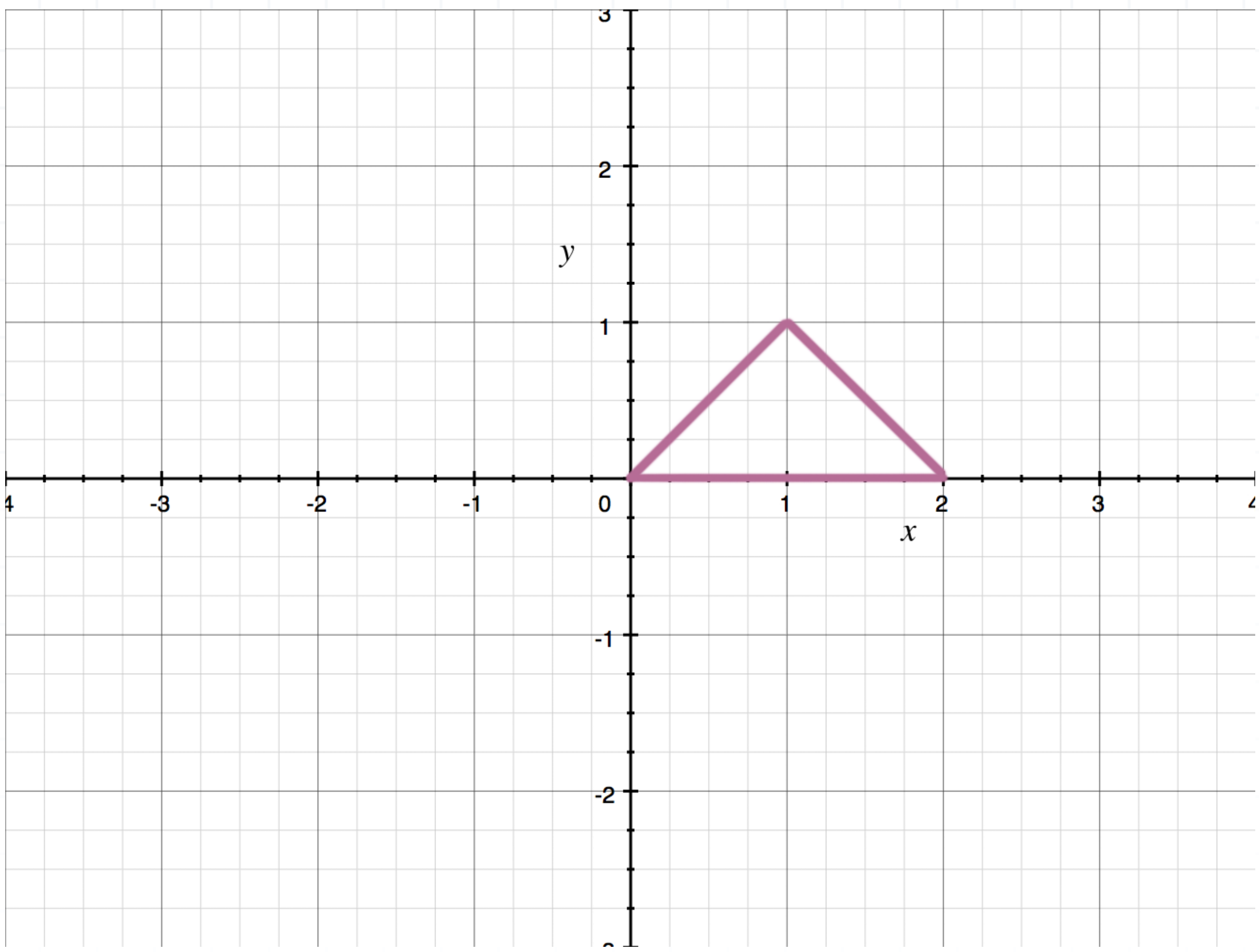
Since $Q(x, y) = 5x^2 - 3y^2$,

$$\frac{\partial Q}{\partial x} = 10x$$

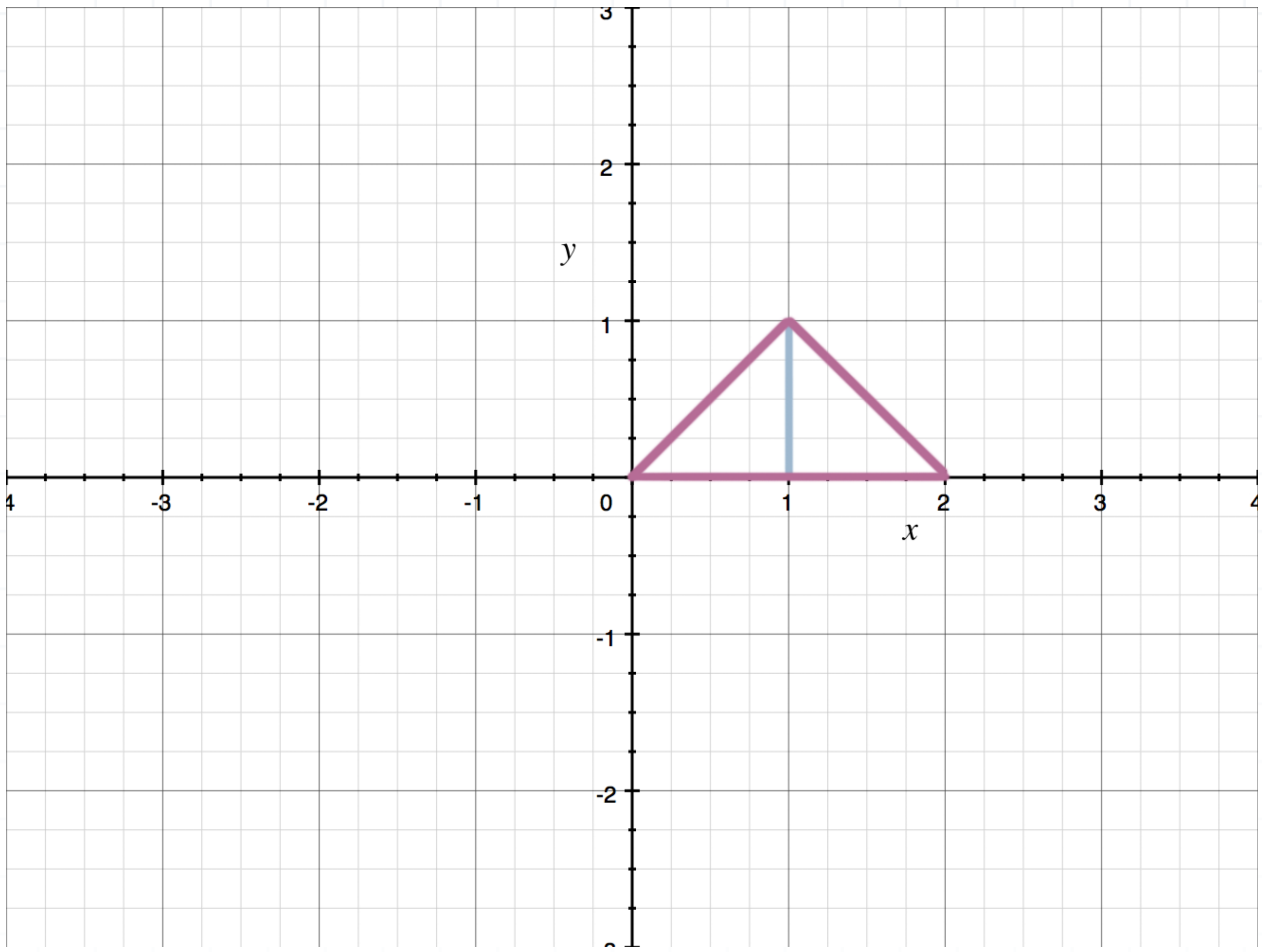
Since $P(x, y) = 5\sin x + 5y$,

$$\frac{\partial P}{\partial y} = 5$$

Now we just need to sketch the region so that we can find limits of integration.



Since the line connecting $(0,0)$ and $(1,1)$ is a different function than the line connecting $(1,1)$ and $(2,0)$, we'll need to divide the region into two parts, separated by the line $x = 1$.



Looking at the sketch of the region, we can say that the region on the left is defined for x on $[0,1]$ and the region on the right is defined for x on $[1,2]$. To find the interval for y for each region, we'll have to find the equation of the lines connecting the points. The equation of the line connecting $(0,0)$ and $(1,1)$ is $y = x$. The equation of the line connecting $(1,1)$ and $(2,0)$ is $y = -x + 2$. Therefore, the equation for area is

$$\int_0^1 \int_0^x 10x - 5\ dy\ dx + \int_1^2 \int_0^{-x+2} 10x - 5\ dy\ dx$$

Now we'll integrate both double integrals with respect to y and evaluate over the associated intervals.

$$\int_0^1 10xy - 5y \Big|_{y=0}^{y=x} dx + \int_1^2 10xy - 5y \Big|_{y=0}^{y=-x+2} dx$$

$$\int_0^1 10x^2 - 5x - \left[10x(0) - 5(0)\right]\ dx$$

$$+\int_1^2 10x(-x+2) - 5(-x+2) - \left[10x(0) - 5(0)\right]\ dx$$

$$\int_0^1 10x^2 - 5x\ dx + \int_1^2 -10x^2 + 20x + 5x - 10\ dx$$

$$\int_0^1 10x^2 - 5x\ dx + \int_1^2 -10x^2 + 25x - 10\ dx$$

Now we'll integrate with respect to x and evaluate over each interval.

$$\frac{10}{3}x^3 - \frac{5}{2}x^2\Big|_0^1 - \frac{10}{3}x^3 + \frac{25}{2}x^2 - 10x\Big|_1^2$$

$$\frac{10}{3}(1)^3 - \frac{5}{2}(1)^2 - \left[\frac{10}{3}(0)^3 - \frac{5}{2}(0)^2\right]$$

$$-\frac{10}{3}(2)^3 + \frac{25}{2}(2)^2 - 10(2) - \left[-\frac{10}{3}(1)^3 + \frac{25}{2}(1)^2 - 10(1)\right]$$

$$\frac{10}{3} - \frac{5}{2} - \frac{80}{3} + \frac{100}{2} - 20 + \frac{10}{3} - \frac{25}{2} + 10$$

$$\frac{10}{3} - \frac{5}{2} - \frac{80}{3} + 50 - 20 + \frac{10}{3} - \frac{25}{2} + 10$$

$$-\frac{60}{3} - \frac{30}{2} + 40$$

$$-20 - 15 + 40$$

$$5$$

This is the area of the region.

Stokes' Theorem

As background for this section, you'll need to remember formulas commonly used to parametrize circles and ellipses.

$x=r\cos(t)$ and $y=r\sin(t)$ will parametrize a circle (where r is the radius), and $x=a\cos(t)$ and $y=b\sin(t)$ will parametrize an ellipse (where a and b come from the equation of the ellipse).

The formulas

You have to figure out which one of the formulas below you'll use when you're trying to use Stokes' theorem to evaluate an integral.

$F \cdot dr$

$= F(r(t)) \cdot r'(t)\ dt$

$= \text{curl } F \cdot dS$

$= \text{curl } F \cdot (\text{grad } f)\ dA$

Remember that all these dots represent dot products, not multiplication.

If you're told that S is the part of the surface in a plane or above a plane, and that surface is given in terms of x, y, and z, then you want to parametrize the surface to make a vector equation for $r(t)$ in terms of t (with i, j, and k), and then use the $F(r(t)) \cdot r'(t)\ dt$ formula. See Example 1.

If S is instead a geometric surface bounded by particular points, vertices, lines, etc., then curl F • (grad f) dA is the formula you want to use. You just need to find one equation that defines the surface, call that g(x,y), and then f(x,y,z)=z-g(x,y). Then you find the gradient of f to substitute for (grad f). See Example 2.