Calculus 3 Workbook Solutions

Applied optimization

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APPLIED OPTIMIZATION

1. Find the maximum volume of a rectangular box inscribed in a hemisphere with radius 4.

Solution:

Let x and y be the halves of linear dimensions of the base of the box (which lies on the flat base of the hemisphere), and let z be the height of the box. The volume of the box is

$$V = (2x)(2y)z = 4xyz$$

Since the top corner of the box lies on the sphere, we get

$$x^2 + y^2 + z^2 = \text{radius}^2 = 4^2 = 16$$

Solve the equation for z,

$$z = \sqrt{16 - x^2 - y^2}$$

then substitute into the volume equation

$$V = 4xy\sqrt{16 - x^2 - y^2}$$

Since x and y should be positive and the point (x, y) should be within the circle of radius 4, we have an optimization task: find the global maximum for the function

$$V(x,y) = 4xy\sqrt{16 - x^2 - y^2}$$

on the closed circular sector defined by

$$0 \le x$$

$$0 \le y$$

$$x^2 + y^2 \le 16$$

Use product rule to find first order partial derivatives.

$$\frac{\partial V}{\partial x} = \frac{4y(16 - 2x^2 - y^2)}{16 - x^2 - y^2}$$

$$\frac{\partial V}{\partial y} = \frac{4x(16 - x^2 - 2y^2)}{16 - x^2 - y^2}$$

Setting the numerators equal to 0 and using these equations as a system of simultaneous equations to find critical points, we get

$$y(16 - 2x^2 - y^2) = 0$$

$$x(16 - x^2 - 2y^2) = 0$$

Luckily, we don't need to consider the cases where $x = 0$ or $y = 0$ for this task, because in that case the volume would be equal to 0. And we don't need to consider the sector boundary $x^2 + y^2 = 16$. If the point (x, y) would lie on the circle, then $z = 0$, and volume is also equal to 0 in that case.

So solve the system for $x \neq 0$, $y \neq 0$.

$$16 - 2x^2 - y^2 = 0$$

$$16 - x^2 - 2y^2 = 0$$

Subtract the equations.

$$16 - 2x^2 - y^2 - (16 - x^2 - 2y^2) = 0 - 0$$

$$16 - 2x^2 - y^2 - 16 + x^2 + 2y^2 = 0$$

$$-2x^2 + x^2 - y^2 + 2y^2 = 0$$

$$-x^2 + y^2 = 0$$

$$x^2 = y^2$$

Substitute to the first equation:

$$16 - 2y^2 - y^2 = 0$$

$$16 - 3y^2 = 0$$

$$y^2 = \frac{16}{3}$$

$$y = \pm\frac{4}{\sqrt{3}}$$

Since we need only positive solutions,

$$x = y = \frac{4}{\sqrt{3}}$$

Find the volume at this point.

$$V\left(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)=4\frac{4^2}{3}\sqrt{16-\frac{4^2}{3}-\frac{4^2}{3}}=\frac{256}{3\sqrt{3}}$$

Since we have only one critical point in the circular sector, $V(x,y)>0$ at this point, and $V(x,y)=0$ at the boundaries, by the extreme value theorem this is the global maximum of the function.

■ 2. Find the minimum distance from $(2,2,-1)$ to the plane $8x-4y+z+11=0$.

Solution:

Let (x,y,z) be the coordinates of the point on the given plane. Then the square distance from this point to $(2,2,-1)$ is

$$d^2=(x-2)^2+(y-2)^2+(z+1)^2$$

Isolate z in the plane equation and substitute it to the distance formula.

$$z=-8x+4y-11$$

$$d^2=(x-2)^2+(y-2)^2+(-8x+4y-11+1)^2$$

$$d^2=(x-2)^2+(y-2)^2+(-8x+4y-10)^2$$

We have an optimization task: find the global minimum for the function $H(x,y)=(x-2)^2+(y-2)^2+(-8x+4y-10)^2$ on R^2. So calculate the first order partial derivatives using the power rule.

$$\frac{\partial H}{\partial x} = 2(x-2) + 2(-8)(-8x+4y-10) = 2(65x-32y+78)$$

$$\frac{\partial H}{\partial y} = 2(y-2) + 2(4)(-8x+4y-10) = 2(-32x+17y-42)$$

Setting both partial derivatives equal to 0 and using these equations as a system of equations to find critical points gives

$$65x - 32y + 78 = 0$$

$$-32x + 17y - 42 = 0$$

So the solution of the system is $x = 2/9$, $y = 26/9$.

Find second order partial derivatives.

$$\frac{\partial^2 H}{\partial x^2} = 130$$

$$\frac{\partial^2 H}{\partial y^2} = 34$$

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x} = -64$$

The second derivative test gives

$$D(x,y) = \frac{\partial^2 H}{\partial x^2} \cdot \frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2$$

$$D\left(\frac{2}{9}, \frac{26}{9}\right) = 130 \cdot 34 - (-64)^2 = 324 > 0$$

Since $D > 0$, and $\partial^2 H/\partial x^2 > 0$, the point $(2/9,26/9)$ is a local minimum. Since $H(x, y)$ tends to infinity for big x and y, this local minimum is a global minimum.

Find the distance.

$$d^2 = \left(\frac{2}{9} - 2\right)^2 + \left(\frac{26}{9} - 2\right)^2 + \left(-8 \cdot \frac{2}{9} + 4 \cdot \frac{26}{9} - 10\right)^2$$

$$d^2 = \left(-\frac{16}{9}\right)^2 + \left(\frac{8}{9}\right)^2 + \left(-\frac{2}{9}\right)^2$$

$$d^2 = \frac{256}{81} + \frac{64}{81} + \frac{4}{81}$$

$$d^2 = 4$$

$$d = 2$$

The minimum distance from the point to the plane is $d = 2$.

■ 3. Find the minimum distance from $(-4,4,0)$ to the cone $3x^2 + y^2 = z^2$.

Solution:

Let (x, y, z) be the coordinates of the point on the cone. Then the square distance from this point to $(-4,4,0)$ is

$$d^2 = (x + 4)^2 + (y - 4)^2 + z^2$$

Substitute z^2 from the cone equation into the distance formula

$$d^2 = (x+4)^2 + (y-4)^2 + 3x^2 + y^2$$

$$d^2 = 4x^2 + 8x + 2y^2 - 8y + 32$$

We have an optimization task: find the global minimum for the function on R^2.

$$H(x, y) = 4x^2 + 8x + 2y^2 - 8y + 32$$

Find first order partial derivatives.

$$\frac{\partial H}{\partial x} = 8x + 8$$

$$\frac{\partial H}{\partial y} = 4y - 8$$

Setting both partial derivatives equal to 0 and using these equations as a system to find critical points gives

$$8x + 8 = 0$$

$$4y - 8 = 0$$

The solution to the system is $x = -1$, $y = 2$.

Find second order partial derivatives.

$$\frac{\partial^2 H}{\partial x^2} = 8$$

$$\frac{\partial^2 H}{\partial y^2} = 4$$

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x} = 0$$

The second derivative test gives

$$D(x, y) = \frac{\partial^2 H}{\partial x^2} \cdot \frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(-1,2)) = 8 \cdot 4 - (0)^2 = 32 > 0$$

Since $D > 0$, and $\partial^2 H/\partial x^2 > 0$, the point $(-1,2)$ is a local minimum. Since $H(x, y)$ tends to infinity for big x and y, this local minimum is a global minimum.

Find the distance.

$$d^2 = 4(-1)^2 + 8(-1) + 2(2)^2 - 8(2) + 32$$

$$d^2 = 20$$

$$d = \sqrt{20}$$

$$d = 2\sqrt{5}$$

The minimum distance from the point to the cone is $d = 2\sqrt{5}$.

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