Topic: Gradient vectors and the tangent plane

Question: Find the gradient vector of the function at $P(1,-1)$.

$$f(x,y)=3x^2y-y^3+2x^2-xy-12$$

Answer choices:

A $\nabla f(1,-1)=\langle 1,-1\rangle$

B $\nabla f(1,-1)=\langle -1,1\rangle$

C $\nabla f(1,-1)=\langle -1,-1\rangle$

D $\nabla f(1,-1)=\langle 1,1\rangle$

Solution: C

We'll find the partial derivatives of the function so that we can plug them into the formula for the gradient vector.

$$\frac{\partial f}{\partial x} = 6xy + 4x - y$$

$$\frac{\partial f}{\partial y} = 3x^2 - 3y^2 - x$$

Plugging these into the formula for the gradient vector, we get

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x, y) = \langle 6xy + 4x - y,\ 3x^2 - 3y^2 - x \rangle$$

Evaluating the gradient vector at $P(1, -1)$ gives

$$\nabla f(1, -1) = \langle 6(1)(-1) + 4(1) - (-1),\ 3(1)^2 - 3(-1)^2 - (1) \rangle$$

$$\nabla f(1, -1) = \langle -1, -1 \rangle$$

This is the gradient vector of the function at $P(1, -1)$.

Topic: Gradient vectors and the tangent plane

Question: Use the gradient vector to find the equation of the tangent plane at $P(2,-3)$.

$$f(x,y)=6x^3-4xy^4+3xy+2y^3+9$$

Answer choices:

A $\quad z=261x-924y+2{,}772$

B $\quad z=261x-924y+2{,}631$

C $\quad z=-261x+924y+2{,}772$

D $\quad z=-261x+924y+2{,}631$

Solution: D

We'll find the partial derivatives of the function so that we can plug them into the formula for the gradient vector.

$$\frac{\partial f}{\partial x} = 18x^2 - 4y^4 + 3y$$

$$\frac{\partial f}{\partial y} = -16xy^3 + 3x + 6y^2$$

Plugging these into the formula for the gradient vector, we get

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\nabla f(x, y) = \langle 18x^2 - 4y^4 + 3y, -16xy^3 + 3x + 6y^2 \rangle$$

Evaluating the gradient vector at $P(2, -3)$ gives

$$\nabla f(2, -3) = \langle 18(2)^2 - 4(-3)^4 + 3(-3), -16(2)(-3)^3 + 3(2) + 6(-3)^2 \rangle$$

$$\nabla f(2, -3) = \langle 72 - 324 - 9, 864 + 6 + 54 \rangle$$

$$\nabla f(2, -3) = \langle -261, 924 \rangle$$

This is the gradient vector of the function at $P(2, -3)$. Now that we've got it, we can find the equation of the tangent plane. If we use the formula

$$a(x - x_0) + b(y - y_0) - (z - z_0) = 0$$

for the tangent plane, then we can take $a = -261$ and $b = 924$ from the gradient vector.

$$-261(x-2)+924(y-(-3))-(z-z_0)=0$$

$$z-z_0=-261x+522+924y+2{,}772$$

$$z-z_0=-261x+924y+3{,}294$$

We get z_0 by plugging $P(2,-3)$ into $f(x,y)$.

$$f(2,-3)=6(2)^3-4(2)(-3)^4+3(2)(-3)+2(-3)^3+9$$

$$f(2,-3)=6(8)-4(2)(81)+3(2)(-3)+2(-27)+9$$

$$f(2,-3)=48-648-18-54+9$$

$$f(2,-3)=-663$$

Plugging this missing value into the equation we left off with, we get

$$z-(-663)=-261x+924y+3{,}294$$

$$z+663=-261x+924y+3{,}294$$

$$z=-261x+924y+2{,}631$$

This is the equation of the tangent plane at $P(2,-3)$.

Topic: Gradient vectors and the tangent plane

Question: Use the gradient vector to find the equation of the tangent plane at $P(1,0,-2)$.

$$f(x,y,z) = 8z^2 - 12xy^2 + 34x^3 - 12xz^2 - 3xyz - 56$$

Answer choices:

A $27x + 3y + 8z = 11$

B $27x + 3y + 8z = -11$

C $3x + 8y = -13$

D $3x + 8y = 13$

Solution: A

The partial derivatives of the function are

$$\frac{\partial f}{\partial x} = -12y^2 + 102x^2 - 12z^2 - 3yz$$

$$\frac{\partial f}{\partial y} = -24xy - 3xz$$

$$\frac{\partial f}{\partial z} = 16z - 24xz - 3xy$$

Plugging these into the formula for the gradient vector, we get

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

$$\nabla f(x, y, z) = \langle -12y^2 + 102x^2 - 12z^2 - 3yz,\ -24xy - 3xz,\ 16z - 24xz - 3xy \rangle$$

Evaluating the gradient vector at $P(1,0,-2)$ gives

$$\nabla f(x, y, z) = \langle -12(0)^2 + 102(1)^2 - 12(-2)^2 - 3(0)(-2),$$

$$-24(1)(0) - 3(1)(-2),\ 16(-2) - 24(1)(-2) - 3(1)(0) \rangle$$

$$\nabla f(x, y, z) = \langle 0 + 102 - 48 - 0,\ 0 + 6,\ -32 + 48 - 0 \rangle$$

$$\nabla f(x, y, z) = \langle 54,\ 6,\ 16 \rangle$$

This is the gradient vector of the function at $P(1,0,-2)$. Now that we've got it, we can find the equation of the tangent plane. If we use the formula

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

for the tangent plane, then we can take $a = 54$ and $b = 6$ and $c = 16$ from the gradient vector.

$$54(x-1)+6(y-0)+16(z-(-2))=0$$

$$54x-54+6y+16z+32=0$$

$$54x+6y+16z=22$$

$$27x+3y+8z=11$$

This is the equation of the tangent plane at $P(1,0,-2)$.