Green's theorem for one region

Green's theorem gives us a way to change a line integral into a double integral. If a line integral is particularly difficult to evaluate, then using Green's theorem to change it to a double integral might be a good way to approach the problem.

If we want to find the area of a simple region, and the original line integral has the form

$$\oint_c P\ dx + Q\ dy$$

then we can apply Green's theorem to change the line integral into a double integral in the form

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

where

$\frac{\partial Q}{\partial x}$ is the partial derivative of Q with respect to x

$\frac{\partial P}{\partial y}$ is the partial derivative of P with respect to y

If we choose to use Green's theorem and change the line integral to a double integral, we'll need to find limits of integration for both x and y so that we can evaluate the double integral as an iterated integral. Often the limits for x and y will be given to us in the problem.

Example

Solve the line integral for the region $(\pm 1, \pm 1)$.

$$\oint_c \left(2x^2 + 4y\right)\ dx + \left(x^2 - 5y^3\right)\ dy$$

Since the integral we were given matches the form

$$\oint_c P\ dx + Q\ dy$$

we know we can use Green's theorem to change it to

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

We'll start by finding partial derivatives.

Since $Q(x, y) = x^2 - 5y^3$,

$$\frac{\partial Q}{\partial x} = 2x$$

Since $P(x, y) = 2x^2 + 4y$,

$$\frac{\partial P}{\partial y} = 4$$

We were told in the problem that the region would be given by the interval $(\pm 1, \pm 1)$. Plugging everything we have into the converted formula, we get

$$\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \ dA$$

$$\int_{-1}^{1} \int_{-1}^{1} 2x - 4 \ dy \ dx$$

Integrating with respect to y and evaluating over the associated interval, we get

$$\int_{-1}^{1} 2xy - 4y \Big|_{y=-1}^{y=1} \ dx$$

$$\int_{-1}^{1} 2x(1) - 4(1) - \left[2x(-1) - 4(-1)\right] \ dx$$

$$\int_{-1}^{1} 2x - 4 - (-2x + 4) \ dx$$

$$\int_{-1}^{1} 2x - 4 + 2x - 4 \ dx$$

$$\int_{-1}^{1} 4x - 8 \ dx$$

Integrating with respect to x and evaluating over its interval, we get

$$2x^2 - 8x \Big|_{-1}^{1}$$

$$2(1)^2 - 8(1) - \left[2(-1)^2 - 8(-1)\right]$$

$$2 - 8 - (2 + 8)$$

$$2 - 8 - 2 - 8$$

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This is the area of the region.