Calculus 3 Workbook Solutions

Tangent planes and normal lines

krista king MATH

EQUATION OF THE TANGENT PLANE

1. Find a tangent plane to the surface $f(u,v,w)=0$ at $(3,-1,5)$.

$$f(u,v,w)=\ln\frac{u^2+1}{v^2w^5}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(u,v,w)=\left\langle\frac{\partial f}{\partial u},\frac{\partial f}{\partial v},\frac{\partial f}{\partial w}\right\rangle$$

Expand f using properties of logarithms.

$$f(u,v,w)=\ln(u^2+1)-2\ln v-5\ln w$$

Find partial derivatives of f.

$$\frac{\partial f}{\partial u}=\frac{2u}{u^2+1}$$

$$\frac{\partial f}{\partial v}=-\frac{2}{v}$$

$$\frac{\partial f}{\partial w}=-\frac{5}{w}$$

Evaluate the partial derivatives at $(3,-1,5)$.

$$\frac{\partial f}{\partial u}(3,-1,5)=\frac{2(3)}{(3)^2+1}=0.6$$

$$\frac{\partial f}{\partial v}(3,-1,5)=-\frac{2}{-1}=2$$

$$\frac{\partial f}{\partial w}(3,-1,5)=-\frac{5}{5}=-1$$

The equation of the tangent plane to the surface $f(u,v,w)=0$ at (u_0,v_0,w_0) is

$$\frac{\partial f}{\partial u}(u_0,v_0,w_0)(u-u_0)+\frac{\partial f}{\partial v}(u_0,v_0,w_0)(v-v_0)+\frac{\partial f}{\partial w}(u_0,v_0,w_0)(w-w_0)=0$$

Substitute the values and simplify

$$0.6(u-3)+2(v-(-1))-1(w-5)=0$$

$$0.6u+2v-w+5.2=0$$

■ 2. Find any tangent planes to the surface $f(x,y,z)=0$ that are parallel to the plane $5x-4y+2z+5=0$.

$$f(x,y,z)=x^3-4y^2+z^2+2x+12y+5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y,z)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

Find partial derivatives of f.

$$\frac{\partial f}{\partial x} = 3x^2 + 2$$

$$\frac{\partial f}{\partial y} = -8y + 12$$

$$\frac{\partial f}{\partial z} = 2z$$

Since the tangent plane should be parallel to the given plane $5x - 4y + 2z + 5 = 0$, it should have the same normal vector, $\langle 5, -4, 2 \rangle$. Since the normal vector to the surface is equal to the gradient at the point, we have a system of equations in terms of x, y, and z.

$$3x^2 + 2 = 5$$

$$-8y + 12 = -4$$

$$2z = 2$$

The solutions to the system are $x = -1$ or $x = 1$, $y = 2$, and $z = 1$. So we have two points where the tangent plane is parallel to the given plane, $(1,2,1)$ and $(-1,2,1)$.

The equation of the tangent plane to the surface $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Substitute the values and simplify for $(1,2,1)$.

$$5(x - 1) - 4(y - 2) + 2(z - 1) = 0$$

$$5x - 4y + 2z + 1 = 0$$

Substitute the values and simplify for $(-1,2,1)$.

$$5(x+1) - 4(y-2) + 2(z-1) = 0$$

$$5x - 4y + 2z + 11 = 0$$

There are two tangent planes to f that area parallel to the plane:

$$5x - 4y + 2z + 1 = 0$$

$$5x - 4y + 2z + 11 = 0$$

■ 3. Find a line of intersection of the xy-plane and tangent plane to the surface $f(x, y, z) = 0$ at $(\pi, -1, \sqrt{6})$.

$$f(x, y, z) = 2\cos(x+\pi)(y^2 + y + 5) - 3z^3$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Find partial derivatives of f.

$$\frac{\partial f}{\partial x} = -2\sin(x+\pi)(y^2 + y + 5)$$

$$\frac{\partial f}{\partial y} = 2\cos(x+\pi)(2y+1)$$

$$\frac{\partial f}{\partial z} = -9z^2$$

Evaluate the partial derivatives at $(\pi, -1, \sqrt{6})$.

$$\frac{\partial f}{\partial x}(\pi, -1, \sqrt{6}) = -2\sin(\pi+\pi)((-1)^2 + (-1) + 5) = 0$$

$$\frac{\partial f}{\partial y}(\pi, -1, \sqrt{6}) = 2\cos(\pi+\pi)(2(-1)+1) = -2$$

$$\frac{\partial f}{\partial z}(\pi, -1, \sqrt{6}) = -9(\sqrt{6})^2 = -54$$

The equation of the tangent plane to the surface $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Substitute the values and simplify.

$$0(x-\pi) - 2(y-(-1)) - 54(z-\sqrt{6}) = 0$$

$$-2y - 54z + 54\sqrt{6} - 2 = 0$$

$$y + 27z - 27\sqrt{6} + 1 = 0$$

Since a line of intersection lies in the xy-plane, then $z = 0$.

$$y - 27\sqrt{6} + 1 = 0$$

$$y = 27\sqrt{6} - 1$$

The line of intersection is $y = 27\sqrt{6} - 1$ with $z = 0$, so the line is parallel to x-axis.

■ 4. Find and identify the set of the points where the tangent plane to the surface $f(x, y, z) = 0$ is parallel to z-axis.

$$f(x, y, z) = x^2 + 4y^2 + z^2 + 2x - 8y + 8z + 17 = 0$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The equation of the tangent plane to the surface $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Since the tangent plane should be parallel to z-axis, $\partial f/\partial z$ must be equal to 0 at the point. Find the partial derivative.

$$\frac{\partial f}{\partial z} = 2z + 8$$

$$2z + 8 = 0$$

$$z = -4$$

Substitute $z=-4$ into the equation of the surface to get the set equation.

$$x^2+4y^2+(-4)^2+2x-8y+8(-4)+17=0$$

$$x^2+2x+4y^2-8y+1=0$$

Complete the squares and transform into standard form.

$$(x+1)^2+4(y-1)^2-4=0$$

$$\frac{(x+1)^2}{2^2}+\frac{(y-1)^2}{1^2}=1$$

So the set of points is the ellipse with center at $(-1,1)$, semi-major axis 2, and semi-minor axis 1.

NORMAL LINE TO THE SURFACE

■ 1. Use the gradient vector to find a symmetric equation of the normal line to the curve $f(s,t)=0$ at $(-3,3)$, where

$$f(s,t)=t2^{2t+s-3}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(s,t)=\left\langle \frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle$$

Use the product rule to calculate partial derivatives.

$$\frac{\partial f}{\partial s}=t\ln(2)2^{2t+s-3}$$

and

$$\frac{\partial f}{\partial t}=\left(\frac{d}{dt}t\right)2^{2t+s-3}+t\left(\frac{d}{dt}2^{2t+s-3}\right)$$

$$\frac{\partial f}{\partial t}=2^{2t+s-3}+t\ln(2)2^{2t+s-2}$$

Evaluate the partial derivatives at $(-3,3)$.

$$\frac{\partial f}{\partial s}(-3,3)=(3)\ln(2)2^{2(3)+(-3)-3}=3\ln 2$$

$$\frac{\partial f}{\partial t}(-3,3) = 2^{2(3)+(-3)-3} + (3)\ln(2)2^{2(3)+(-3)-2} = 1 + 6\ln 2$$

The formula for the symmetric equation of the normal line to the curve $f(s, t) = 0$ at (s_0, t_0) is

$$\frac{t - t_0}{\frac{\partial f}{\partial t}(s_0, t_0)} = \frac{s - s_0}{\frac{\partial f}{\partial s}(s_0, t_0)}$$

Substitute the values and simplify.

$$\frac{t - 3}{1 + 6\ln 2} = \frac{s + 3}{3\ln 2}$$

■ 2. Use the gradient vector to find a vector equation of the normal line to the surface $f(x, y, z) = 0$ at $(0, -5, 1)$.

$$f(x, y, z) = \frac{2x - y^2}{z}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2}{z}$$

$$\frac{\partial f}{\partial y} = -\frac{2y}{z}$$

$$\frac{\partial f}{\partial z} = -\frac{2x - y^2}{z^2}$$

Evaluate the partial derivatives at $(0,-5,1)$.

$$\frac{\partial f}{\partial x}(0,-5,1) = \frac{2}{1} = 2$$

$$\frac{\partial f}{\partial y}(0,-5,1) = -\frac{2(-5)}{1} = 10$$

$$\frac{\partial f}{\partial z}(0,-5,1) = -\frac{2(0)-(-5)^2}{(1)^2} = 25$$

The formula for the vector equation of the normal line to the curve $f(x,y,z)=0$ at (x_0,y_0,z_0) is

$$r = \langle x_0, y_0, z_0 \rangle + t\left\langle \frac{\partial f}{\partial x}(x_0,y_0,z_0), \frac{\partial f}{\partial y}(x_0,y_0,z_0), \frac{\partial f}{\partial z}(x_0,y_0,z_0) \right\rangle$$

$$r = \langle 0,-5,1 \rangle + t\langle 2,10,25 \rangle$$

■ 3. Use the gradient vector to find a parametric equation of the normal line to the surface $f(x,y,z)=0$ at $(2,-3,0)$.

$$f(x,y,z) = 3x^3 - 2xyz - 2y^2 + 5yz + 1$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 9x^2 - 2yz$$

$$\frac{\partial f}{\partial y} = -2xz - 4y + 5z$$

$$\frac{\partial f}{\partial z} = -2xy + 5y$$

Evaluate the partial derivatives at $(2, -3,0)$.

$$\frac{\partial f}{\partial x}(2, -3,0) = 9(2)^2 - 2(-3)(0) = 36$$

$$\frac{\partial f}{\partial y}(2, -3,0) = -2(2)(0) - 4(-3) + 5(0) = 12$$

$$\frac{\partial f}{\partial z}(2, -3,0) = -2(2)(-3) + 5(-3) = -3$$

The formula for the parametric equation of the normal line to the curve $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$x = x_0 + t\frac{\partial f}{\partial x}(x_0, y_0, z_0)$$

$$y = y_0 + t\frac{\partial f}{\partial y}(x_0, y_0, z_0)$$

$$z = z_0 + t\frac{\partial f}{\partial z}(x_0, y_0, z_0)$$

Then the parametric equation of the normal line is

$$x = 2 + 36t$$

$$y = -3 + 12t$$

$$z = -3t$$

■ 4. Use the gradient vector to find a parametric equation of the normal line to the surface $f(x, y, z) = 0$ that's parallel to the line $r = \langle 2,17,-6\rangle + t\langle 9,1,-6\rangle$.

$$f(x, y, z) = 2x^2 + y^2 + 3z^2 - 3x - 5y + 5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 4x - 3$$

$$\frac{\partial f}{\partial y} = 2y - 5$$

$$\frac{\partial f}{\partial z} = 6z$$

Since the tangent line is parallel to the given line, it has the same direction vector $\langle 9,1,-6 \rangle$, and it's equal to the gradient vector at the point of tangency (x_0, y_0, z_0). So we have a system of equations in terms of x_0, y_0, and z_0.

$$4x_0 - 3 = 9$$

$$2y_0 - 5 = 1$$

$$6z_0 = -6$$

The solution of the system is $x_0 = 3$, $y_0 = 3$, and $z_0 = -1$.

The formula for the parametric equation of the normal line to the curve $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$x = x_0 + t\frac{\partial f}{\partial x}(x_0, y_0, z_0)$$

$$y = y_0 + t\frac{\partial f}{\partial y}(x_0, y_0, z_0)$$

$$z = z_0 + t\frac{\partial f}{\partial z}(x_0, y_0, z_0)$$

Substitute the values we found to get the parametric equation of the normal line to the surface.

$$x = 3 + 9t$$

$$y = 3 + t$$

$$z = -1 - 6t$$

■ 5. Use the gradient vector to find a vector equation of the normal line to the surface $f(x, y, z) = 0$ that's perpendicular to the plane $x + 4y - 8z + 12 = 0$.

$$f(x, y, z) = ye^{2x+6} + z^2 - 5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 2ye^{2x+6}$$

$$\frac{\partial f}{\partial y} = e^{2x+6}$$

$$\frac{\partial f}{\partial z} = 2z$$

Since the tangent line is perpendicular to the given plane, the normal vector to the plane is equal to the gradient vector at the point of tangency (x_0, y_0, z_0). So we have a system of equations in terms of x_0, y_0, and z_0.

$$2y_0e^{2x_0+6} = 1$$

$$e^{2x_0+6} = 4$$

$$2z_0 = -8$$

The solution of the system is $x_0 = \ln 2 - 3$, $y_0 = 1/8$, and $z_0 = -4$.

The formula for the vector equation of the normal line to the curve $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$r = \langle x_0, y_0, z_0 \rangle + t \left\langle \frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0) \right\rangle$$

Substitute the values we found to get the vector equation of the normal line to the surface.

$$r = \left\langle \ln 2 - 3, \frac{1}{8}, -4 \right\rangle + t\langle 1,4,-8 \rangle$$

W W W . K R I S T A K I N G M A T H . C O M