Topic: Three dimensions, one constraint

Question: What is the maximum value of the function subject to the given constraint?

$$f(x, y, z) = 8xyz$$

$$\text{subject to } \frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} = 1$$

Answer choices:

A 48

B 44

C 38

D 32

Solution: A

To maximize

$$f(x, y, z) = 8xyz$$

subject to the constraint

$$g(x, y, z) = \frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} - 1 = 0$$

we'll use the formulas

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$

$$f_y(x, y, z) = \lambda g_y(x, y, z)$$

$$f_z(x, y, z) = \lambda g_z(x, y, z)$$

Applying these formulas to

$$g(x, y, z) = \frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} - 1 = 0$$

gives

$$8yz = \frac{2\lambda x}{3}$$

Multiplying this equation by x gives

$$8xyz = \frac{2\lambda x^2}{3}$$

$$8xz = \frac{2\lambda y}{12}$$

Multiplying this equation by y gives

$$8xyz = \frac{2\lambda y^2}{12}$$

$$8xy = \frac{2\lambda z}{27}$$

Multiplying this equation by z gives

$$8xyz = \frac{2\lambda z^2}{27}$$

The left sides of all the three conclusions are equal. Hence, their right sides are equal as well:

$$\frac{2\lambda x^2}{3} = \frac{2\lambda y^2}{12} = \frac{2\lambda z^2}{27}$$

$$\frac{x^2}{3} = \frac{y^2}{12} = \frac{z^2}{27}$$

Solve the following system of equations:

$$\frac{x^2}{3} = \frac{y^2}{12} = \frac{z^2}{27}$$

$$\frac{x^2}{3} + \frac{y^2}{12} + \frac{z^2}{27} - 1 = 0$$

$$4x^2 = y^2$$

$$9y^2 = 4z^2$$

$$9x^2 = z^2$$

The solutions of this system of equations are $x = 1$, $y = 2$, and $z = 3$. Then the maximum of the function is

$$f(x, y, z) = 8xyz$$

$$f(x, y, z) = 8(1)(2)(3)$$

$$f(x, y, z) = 48$$

Topic: Three dimensions, one constraint

Question: What is the minimum value of the function subject to the given constraint?

$$f(x, y, z) = x^2 + y^2 + z^2$$

subject to $3x - 4y - 5z = -16$

Answer choices:

A $\frac{256}{25}$

B $\frac{128}{25}$

C $\frac{64}{25}$

D $\frac{32}{25}$

Solution: B

Let

$$g(x, y, z) = 3x - 4y - 5z + 16$$

Apply the formulas

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$

$$f_y(x, y, z) = \lambda g_y(x, y, z)$$

$$f_z(x, y, z) = \lambda g_z(x, y, z)$$

Then,

$$2x = 3\lambda$$

$$2y = -4\lambda$$

$$2z = -5\lambda$$

$$3x - 4y - 5z = -16$$

We'll solve each of the first three equations for λ.

$$\lambda = \frac{2}{3}x$$

$$\lambda = -\frac{2}{4}y = -\frac{1}{2}y$$

$$\lambda = -\frac{2}{5}z$$

Therefore, we can say

$$\frac{2}{3}x = -\frac{1}{2}y = -\frac{2}{5}z$$

If we solve this for x, we get

$$x = -\frac{3}{4}y = -\frac{3}{5}z$$

If we solve this for y, we get

$$-\frac{4}{3}x = y = \frac{4}{5}z$$

If we solve this for z, we get

$$-\frac{5}{3}x = \frac{5}{4}y = z$$

Now we can make substitutions into the constraint equation $3x - 4y - 5z = -16$. We'll replace x and y with values in terms of z in order to solve for z.

$$3x - 4y - 5z = -16$$

$$3\left(-\frac{3}{5}z\right) - 4\left(\frac{4}{5}z\right) - 5z = -16$$

$$-\frac{9}{5}z - \frac{16}{5}z - 5z = -16$$

$$-9z - 16z - 25z = -80$$

$$-50z = -80$$

$$z = \frac{80}{50}$$

$$z = \frac{8}{5}$$

Now we'll replace x and z with values in terms of y in order to solve for y.

$$3x - 4y - 5z = -16$$

$$3\left(-\frac{3}{4}y\right) - 4y - 5\left(\frac{5}{4}y\right) = -16$$

$$-\frac{9}{4}y - 4y - \frac{25}{4}y = -16$$

$$-9y - 16y - 25y = -64$$

$$-50y = -64$$

$$y = \frac{64}{50}$$

$$y = \frac{32}{25}$$

Now we'll replace y and z with values in terms of x in order to solve for x.

$$3x - 4y - 5z = -16$$

$$3x - 4\left(-\frac{4}{3}x\right) - 5\left(-\frac{5}{3}x\right) = -16$$

$$3x + \frac{16}{3}x + \frac{25}{3}x = -16$$

$$9x + 16x + 25x = -48$$

$$50x = -48$$

$$x = -\frac{48}{50}$$

$$x = -\frac{24}{25}$$

These x, y and z values together represent only one critical point, which means that the minimum of the function is

$$f(x, y, z) = x^2 + y^2 + z^2$$

$$f\left(-\frac{24}{25}, \frac{32}{25}, \frac{8}{5}\right) = \left(-\frac{24}{25}\right)^2 + \left(\frac{32}{25}\right)^2 + \left(\frac{8}{5}\right)^2$$

$$\frac{576}{625} + \frac{1{,}024}{625} + \frac{64}{25}$$

$$\frac{576}{625} + \frac{1{,}024}{625} + \frac{1{,}600}{625}$$

$$\frac{3{,}200}{625}$$

$$\frac{128}{25}$$

Topic: Three dimensions, one constraint

Question: To the nearest tenth, what is the minimum value of $f(x, y, z) = 2x^2 + 3y^2 + z^2$ on the plane $x + 2y + 3z = 16$?

Answer choices:

A 23.6

B 20.2

C 19.8

D 19.1

Solution: A

Let

$$g(x, y, z) = x + 2y + 3z - 16 = 0$$

Apply the formulas:

$$f_x(x, y, z) = \lambda g_x(x, y, z)$$

$$f_y(x, y, z) = \lambda g_y(x, y, z)$$

$$f_z(x, y, z) = \lambda g_z(x, y, z)$$

Then,

$$4x = \lambda$$

$$6y = 2\lambda$$

$$2z = 3\lambda$$

$$x + 2y + 3z - 16 = 0$$

We'll solve each of the first three equations for λ.

$$\lambda = 4x$$

$$\lambda = 3y$$

$$\lambda = \frac{2}{3}z$$

Therefore, we can say

$$4x = 3y = \frac{2}{3}z$$

If we solve this for x, we get

$$x = \frac{3}{4}y = \frac{1}{6}z$$

If we solve this for y, we get

$$\frac{4}{3}x = y = \frac{2}{9}z$$

If we solve this for z, we get

$$6x = \frac{9}{2}y = z$$

Now we can make substitutions into the constraint equation $x + 2y + 3z = 16$. We'll replace x and y with values in terms of z in order to solve for z.

$$x + 2y + 3z = 16$$

$$\frac{1}{6}z + 2\left(\frac{2}{9}z\right) + 3z = 16$$

$$\frac{3}{2}z + 4z + 27z = 144$$

$$3z + 8z + 54z = 288$$

$$65z = 288$$

$$z = \frac{288}{65}$$

Now we'll replace x and z with values in terms of y in order to solve for y.

$$x + 2y + 3z = 16$$

$$\frac{3}{4}y + 2y + 3\left(\frac{9}{2}y\right) = 16$$

$$3y + 8y + 54y = 64$$

$$65y = 64$$

$$y = \frac{64}{65}$$

Now we'll replace y and z with values in terms of x in order to solve for x.

$$x + 2y + 3z = 16$$

$$x + 2\left(\frac{4}{3}x\right) + 3(6x) = 16$$

$$3x + 8x + 54x = 48$$

$$65x = 48$$

$$x = \frac{48}{65}$$

These x, y and z values together represent only one critical point, which means that the minimum of the function is

$$f(x, y, z) = 2x^2 + 3y^2 + z^2$$

$$f\left(\frac{48}{65},\frac{64}{65},\frac{288}{65}\right)=2\left(\frac{48}{65}\right)^2+3\left(\frac{64}{65}\right)^2+\left(\frac{288}{65}\right)^2$$

$$\frac{2\cdot 48^2}{65^2}+\frac{3\cdot 64^2}{65^2}+\frac{288^2}{65^2}$$

$$\frac{2\cdot 48^2+3\cdot 64^2+288^2}{65^2}$$

$$\frac{4{,}608+12{,}288+82{,}944}{65^2}$$

$$\frac{99{,}840}{65^2}$$

$$\frac{1{,}536}{65}\approx 23.6$$