Line integral of a curve

Single variable integrals

In single variable calculus we learned how to evaluate an integral over an interval $[a, b]$ in order to calculate the area under the curve on that interval. We could approximate the area under the curve using a Riemann sum, or calculate the area exactly using an integral.

The Riemann sum might have been

$$A = \sum_{i=1}^{n} f(x_i^*)\Delta x$$

where A is the area underneath the function $f(x)$ and above the x-axis, n is the number of rectangles we use to approximate the area, and Δx is the width of our approximating rectangles.

We learned that our approximation became more and more accurate as we used a larger and larger number of approximating rectangles, and so we knew that to find *exact* area, we had to use an *infinite* number of rectangles. To translate that into our area approximation equation above, we took the limit of the sum as $n \to \infty$ to get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x$$

The limit and sum notation $\lim_{n\to\infty}\sum_{i=1}^{n}$ becomes the integral, $f(x_i^*)$ becomes $f(x)$, and Δx becomes dx when we translate the approximating sum into an integral, and we get

$$A = \int_a^b f(x)\ dx$$

where A is the area underneath the function $f(x)$ and above the x-axis.

Line integrals

In contrast, when we find a line integral, we take the integral over a curve C, instead of over the interval $[a,b]$. Where we used to divide the interval into n rectangles, each with a width of Δx, now we'll divide the interval into n sub-arcs, each with a width of Δs. Which means the Riemann sum representing the line integral is

$$A = \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Letting $n \to \infty$ to find *exact* area, we get a formula for the line integral:

If f is defined on a smooth curve C, then the line integral of f along C is

$$\int_C f(x,y)\ ds = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$$

Since ds represents arc length, we can replace it in our integral with the arc length formula, and get this formula for the line integral:

$$\int_C f(x,y)\ ds = \int_a^b f\left(x(t),y(t)\right)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\ dt$$

Where the value of a normal integral of a single-variable function is the area underneath the curve, the value of the line integral is the area of one side of the "curtain", or "fence" or "wall" whose base is the curve C and whose height is given by the function $f(x,y)$.