Vector function for the curve of intersection of two surfaces

When two three-dimensional surfaces intersect each other, the intersection is a curve. We can find the vector equation of that intersection curve using these steps:

1. Set the curves equal to each other and solve for one of the remaining variables in terms of the other

2. Define each of the variables in terms of the parameter t to get parametric equations for the intersection curve,

$$x = r(t)_1$$

$$y = r(t)_2$$

$$z = r(t)_3$$

3. Generate the vector function that describes the intersection curve using the formulas

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

$$r(t) = \langle r(t)_1, r(t)_2, r(t)_3 \rangle$$

Example

Find the vector function for the curve of intersection of the surfaces.

The ellipsoid $z = \sqrt{1 + x^2 - y^2}$

The plane $z = 2 + x$

Since both of the curves have z on the left-hand side, we can set the right-hand sides equal to one another and solve for one variable of the remaining variables in terms of the other.

$$\sqrt{1 + x^2 - y^2} = 2 + x$$

$$1 + x^2 - y^2 = (2 + x)^2$$

$$1 + x^2 - y^2 = 4 + 4x + x^2$$

$$-3 - y^2 = 4x$$

$$x = -\frac{3}{4} - \frac{1}{4}y^2$$

We want to define each variable in terms of the parameter t, so we'll set $y = t$.

$$x = -\frac{3}{4} - \frac{1}{4}t^2$$

To find z in terms of t, we'll plug x in terms of t into $z = 2 + x$.

$$z = 2 + x$$

$$z = 2 - \frac{3}{4} - \frac{1}{4}t^2$$

$$z = \frac{5}{4} - \frac{1}{4}t^2$$

Now we have parametric equations for the curve of intersection, defined by

$$x = -\frac{3}{4} - \frac{1}{4}t^2$$

$$y = t$$

$$z = \frac{5}{4} - \frac{1}{4}t^2$$

With the parametric equations in hand, we can plug each of them into the formula for the vector function.

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

$$r(t) = \left(-\frac{3}{4} - \frac{1}{4}t^2\right)\mathbf{i} + t\mathbf{j} + \left(\frac{5}{4} - \frac{1}{4}t^2\right)\mathbf{k}$$

This is the vector function for the curve of intersection. You can also write it as

$$r(t) = \left\langle -\frac{3}{4} - \frac{1}{4}t^2, t, \frac{5}{4} - \frac{1}{4}t^2 \right\rangle$$