Calculus 3 Workbook Solutions

Parametric surfaces and areas

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POINTS ON THE SURFACE

1. Find the points of the surface $\vec{r}(u,v)$ that lie on the z-axis.

$$\vec{r}(u,v)=\langle u^2-3v^2-1,4u^2-9v^2-7,e^{u+v}\rangle$$

Solution:

Since the points we're interested in lie on the z-axis, their x- and y-coordinates are 0, so

$$u^2-3v^2-1=0$$

$$4u^2-9v^2-7=0$$

Consider the two equations as a system and solve it for u and v. Make the substitution $t=u^2$ and $s=v^2$ to get a linear system of equations.

$$t-3s-1=0$$

$$4t-9s-7=0$$

The first equation gives $t=3s+1$, so substitute this value into the second equation.

$$4(3s+1)-9s-7=0$$

$$12s+4-9s-7=0$$

$$3s-3=0$$

$$s = 1$$

Plug $s = 1$ back in to get the value of t.

$$t = 3(1) + 1 = 4$$

So

$u^2 = 4$ so $u = \pm 2$

$v^2 = 1$ so $v = \pm 1$

There are four points on the surface which lie on the z-axis. In order to find their z-coordinates, plug in the values we've found for u and v into the third component of the vector function.

$$r_3(2,1) = e^{2+1} = e^3$$

$$r_3(2,-1) = e^{2-1} = e$$

$$r_3(-2,1) = e^{-2+1} = e^{-1}$$

$$r_3(-2,-1) = e^{-2-1} = e^{-3}$$

■ 2. Find the intersection point(s) of the surface $\vec{r}(u,v)$ and the line $x = y + 2 = z - 1$.

$$\vec{r}(u,v) = \langle \sin u + v, \cos u + v - 3, 2v + 7 + \sin u \rangle$$

Solution:

Consider the linear equations as a system of equations.

$$x = y + 2$$

$$x = z - 1$$

Rewrite the vector function $\vec{r}(u, v)$ in parametric form.

$$x(u, v) = \sin u + v$$

$$y(u, v) = \cos u + v - 3$$

$$z(u, v) = 2v + 7 + \sin u$$

Substitute these parametric equations into the equations of the line.

$$\sin u + v = \cos u + v - 3 + 2$$

$$\sin u + v = 2v + 7 + \sin u - 1$$

Solve the system of equations for u and v. We get

$$\sin u = \cos u - 1$$

$$v = 2v + 6$$

and then

$$\cos u - \sin u = 1$$

$$v = -6$$

To solve the first trigonometric equation, use the identity

$$\cos\left(u+\frac{\pi}{4}\right)=\frac{\cos u}{\sqrt{2}}-\frac{\sin u}{\sqrt{2}}$$

$$\cos u-\sin u=\sqrt{2}\cos\left(u+\frac{\pi}{4}\right)$$

in order to get

$$\sqrt{2}\cos\left(u+\frac{\pi}{4}\right)=1$$

$$\cos\left(u+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Since the cosine function is $1/\sqrt{2}$ at the angles $\pi/4$ and $7\pi/4$, we have two options:

$$u+\frac{\pi}{4}=\frac{\pi}{4}+2\pi n_1$$

$$u+\frac{\pi}{4}=\frac{7\pi}{4}+2\pi n_2$$

Simplify.

$$u=2\pi n_1$$

$$u=\frac{3\pi}{2}+2\pi n_2$$

Plug $u=2\pi n_1$ and $v=-6$ into the surface equation.

$$\vec{r}(2\pi n_1,-6)=\langle\sin(2\pi n_1)-6,\cos(2\pi n_1)-6-3,2(-6)+7+\sin(2\pi n_1)\rangle$$

$$\vec{r}(2\pi n_1, -6) = \langle -6, 1-6-3, 2(-6)+7 \rangle$$

$$\vec{r}(2\pi n_1, -6) = \langle -6, -8, -5 \rangle$$

Now plug $u = (3\pi/2) + 2\pi n_2$ and $v = -6$ into the surface equation.

$$\vec{r}\left(\frac{3\pi}{2} + 2\pi n_2, -6\right) = \Bigg\langle \sin\left(\frac{3\pi}{2} + 2\pi n_2\right) - 6,$$

$$\cos\left(\frac{3\pi}{2} + 2\pi n_2\right) - 6 - 3, 2(-6) + 7 + \sin\left(\frac{3\pi}{2} + 2\pi n_2\right)\Bigg\rangle$$

$$\vec{r}\left(\frac{\pi}{2} + 2\pi n_2, -6\right) = \langle -1-6, -6-3, 2(-6)+7-1 \rangle$$

$$\vec{r}\left(\frac{\pi}{2} + 2\pi n_2, -6\right) = \langle -7, -9, -6 \rangle$$

■ 3. Identify the set of points of the surface $\vec{r}(u,v) = \langle u^2 + 2v^2, u, v+2 \rangle$ that lie in the xy-plane.

Solution:

Since the points we're interested in lie in the xy-plane, their z-coordinates are 0, so $v + 2 = 0$, or $v = -2$.

Plug $v = -2$ into the surface equation and consider it the parametric equation of the curve in two-dimensional space.

$$\vec{r}(u,-2)=\langle u^2+2(-2)^2,u,-2+2\rangle$$

$$\vec{r}(u,-2)=\langle u^2+8,u,0\rangle$$

Consider the parametric equation of the curve in the xy-plane.

$$x=u^2+8$$

$$y=u$$

Substitute $u=y$ into the first equation to get rid of the parameter u.

$$x=y^2+8$$

So the curve is the parabola with vertex at $(8,0)$.

SURFACE OF THE VECTOR EQUATION

1. Identify the quadratic surface given as a vector function, where $u \in [0,2\pi]$ and $v \in (-\infty, \infty)$.

$$\vec{r}(u, v) = \langle 3 \sin u, 2v - 3, 5 \cos u \rangle$$

Solution:

Since the x- and z-coordinates are independent of v and y, and since $y(v) = 2v - 3$ has a range of $(-\infty, \infty)$, the surface is a right cylinder that's parallel to the y-axis.

To identify the cylinder's type, consider its section by the xz-plane, which is the plane for $y = 0$, or $2v - 3 = 0$, where $u \in [0,2\pi]$.

$$x(u) = 3 \sin u$$

$$z(u) = 5 \cos u$$

These equations represent the ellipse in the xz-plane, with center at the origin, x-semi-axis of 3, and z-semi-axis of 5.

Therefore, the surface is the right cylinder such that the cylinder's axis coincides with the y-axis, an x-semi-axis of 3, and the z-semi-axis of 5.

2. Identify the quadratic surface given as a vector function, where $u \in [0,\pi]$ and $v \in [0,2\pi]$.

$$\vec{r}(u,v) = \langle -3 + 2\cos u, 2 + 2\sin u \cos v, 2\sin u \sin v \rangle$$

Solution:

Consider the formulas that we use to convert rectangular coordinates to spherical coordinates.

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

Plug in $\rho = 2$ and change the axes as $x_1 \to z$, $y_1 \to x$, and

$$z_1 \to y$$

$$x_1 = 2\cos\phi$$

$$y_1 = 2\sin\phi\cos\theta$$

$$z_1 = 2\sin\phi\sin\theta$$

Since the changing order of axes is a rotation transformation which doesn't change the form of the sphere, the new equations for x_1, y_1, and z_1 represent the parametrization of the sphere with center at the origin and radius 2.

Move the sphere by -3 in the x-direction, and by 2 in the y-direction. Also, rename $u = \phi$ and $v = \theta$.

$$x_2 = -3 + 2\cos u$$

$$y_2 = 2 + 2\sin u \cos v$$

$$z_2 = 2\sin u \sin v$$

The new equations for x_2, y_2, and z_2 coincide with the given vector function and represent the parametrization of the sphere centered at $(-3,2,0)$ with radius 2.

■ 3. Identify the quadratic surface given as a vector function, where $u^2 + v^2 \leq 9$.

$$\vec{r}(u, v) = \langle v + 1, 5 + \sqrt{9 - u^2 - v^2}, u - 2\rangle$$

Solution:

Rewrite the vector function as a set of parametric equations.

$$x(u, v) = v + 1$$

$$y(u, v) = 5 + \sqrt{9 - u^2 - v^2}$$

$$z(u, v) = u - 2$$

Solve for u and v in the first and third equations, then plug those values into the second equation.

$$v = x - 1$$

$$u = z + 2$$

$$y = 5 + \sqrt{9 - (z + 2)^2 - (x - 1)^2}$$

$$y - 5 = \sqrt{9 - (z + 2)^2 - (x - 1)^2}$$

Since the square root is non-negative, $y - 5 \geq 0$, or $y \geq 5$. Square both sides of the equation.

$$(y - 5)^2 = 9 - (z + 2)^2 - (x - 1)^2$$

$$(x - 1)^2 + (y - 5)^2 + (z + 2)^2 = 9$$

This equation represents the sphere centered at $(1,5,-2)$ with radius 3. Since $y \geq 5$, the surface is the part of the sphere that lies above the plane $y = 5$, which means the surface is a hemisphere.

PARAMETRIC REPRESENTATION OF THE SURFACE

1. Consider the right circular cylinder with radius 5 and a cylindrical axis that's parallel to the z-axis and passes through $(2,-4,5)$. Find the parametrization of the part of the cylinder that lies above the xy-plane.

Solution:

There exist an infinite number of parameterizations of a cylinder. The most common parametrization of the cylinder with radius r that has a cylindrical axis parallel to the z-axis is

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$z = z$$

Plug in $r = 5$ and move the cylindrical axis by 2 in the x-direction, and by -4 in the y-direction.

$$x = 2 + 5\cos\phi$$

$$y = -4 + 5\sin\phi$$

$$z = z$$

So we get the parametrization of the cylinder we're interested in. For the part of the cylinder that lies above the xy-plane, z changes from 0 to ∞. Rename the parameters as $\phi \to u$ and $v \to z$.

$$x(u, v) = 2 + 5\cos u$$

$$y(u, v) = -4 + 5\sin u$$

$$z(u, v) = v$$

Rewrite the parametrization as a vector function.

$$\vec{r}(u, v) = \langle 2 + 5\cos u, -4 + 5\sin u, v \rangle$$

■ 2. Consider the plane $2x - 3y + z - 1 = 0$. Find the parametrization of the part of the plane that lies between the planes $y = -3$ and $y = 3$.

Solution:

There are an infinite number of parameterizations of a plane. Let the parameter u be x, and the parameter v be y. Plug u and v into the plane equation in order to get the expression for z.

$$2u - 3v + z - 1 = 0$$

$$z = 3v - 2u + 1$$

So we get the parametrization of the plane we're interested in. Rewrite the parametrization as a vector function.

$$\vec{r}(u,v) = \langle u, v, 3v - 2u + 1 \rangle$$

Within the part of the plane that lies between the planes $y = -3$ and $y = 3$, u changes from $-\infty$ to ∞, and v changes from -3 to 3.

■ 3. Consider the elliptic paraboloid $2(y+3)^2 + 4(z-2)^2 - x - 1 = 0$. Find the parametrization of the paraboloid for $x \le 3$.

Solution:

There are an infinite number of parameterizations of a paraboloid. Let the parameter u be y, and the parameter v be z. Plug u and v into the paraboloid's equation to get the expression for x.

$$2(u+3)^2 + 4(v-2)^2 - x - 1 = 0$$

$$x = 2(u+3)^2 + 4(v-2)^2 - 1$$

So we get the parametrization of the paraboloid we're interested in. Rewrite the parametrization as a vector function.

$$\vec{r}(u,v) = \langle 2(u+3)^2 + 4(v-2)^2 - 1, u, v \rangle$$

For the whole paraboloid, the parameters u and v are any real numbers. To get the parametrization of the paraboloid for $x \le 3$, consider the inequality

$$2(u+3)^2 + 4(v-2)^2 - 1 \le 3$$

$$2(u+3)^2 + 4(v-2)^2 \le 4$$

$$(u+3)^2 + 2(v-2)^2 \le 2$$

Let's find the widest possible range for v, then express the bounds for u in terms of v. When $(u+3)^2 = 0$, or $u = -3$, the expression $2(v-2)^2$ reaches its maximum, so

$$2(v-2)^2 \le 2$$

$$(v-2)^2 \le 1$$

$$-1 \le v-2 \le 1$$

$$1 \le v \le 3$$

So the parameter v changes from 1 to 3. In order to get the bounds for u, solve the inequality for u.

$$(u+3)^2 + 2(v-2)^2 \le 2$$

$$(u+3)^2 \le 2 - 2(v-2)^2$$

$$-\sqrt{2-2(v-2)^2} \le u+3 \le \sqrt{2-2(v-2)^2}$$

$$-3-\sqrt{2-2(v-2)^2} \le u \le -3+\sqrt{2-2(v-2)^2}$$

So the parameter u changes from $-3-\sqrt{2-2(v-2)^2}$ to $-3+\sqrt{2-2(v-2)^2}$.

TANGENT PLANE TO THE PARAMETRIC SURFACE

■ 1. Find the equation of the tangent plane to the surface $\vec{r}(u, v) = \langle u + 2\cos v, u - 2\cos v, uv\rangle$ at the point $(4,0,\pi)$.

Solution:

In order to find the values of the parameters u and v that correspond to $(4,0,\pi)$, solve the system of equations $\vec{r}(u, v) = \langle 4,0,\pi\rangle$ for u and v.

$$u + 2\cos v = 4$$

$$u - 2\cos v = 0$$

$$uv = \pi$$

Take the sum of the first and second equations.

$$2u = 4$$

$$u = 2$$

From the third equation, we get

$$2v = \pi$$

$$v = \frac{\pi}{2}$$

It's easy to check that, for the values of the parameters $u = 2$ and $v = \pi/2$, all three of the equations hold, so these values correspond to $(4,0,\pi)$ on the surface.

To get the tangent plane to the surface, we need to find the normal vector to the surface as the cross product of two tangent vectors at the given point,

$$\vec{r_u} = \frac{\partial \vec{r}(u,v)}{\partial u} \text{ and } \vec{r_v} = \frac{\partial \vec{r}(u,v)}{\partial u}$$

We find

$$\vec{r_u} = \langle 1,1,v \rangle$$

$$\vec{r_u}\left(2,\frac{\pi}{2}\right) = \left\langle 1,1,\frac{\pi}{2} \right\rangle$$

and

$$\vec{r_v} = \langle -2\sin v, 2\sin v, u \rangle$$

$$\vec{r_v}\left(2,\frac{\pi}{2}\right) = \langle -2,2,2 \rangle$$

Evaluate the cross product $\vec{r_u} \times \vec{r_v}$, if the cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle a_1, a_2, a_3 \rangle = \langle 1,1,\pi/2 \rangle$ and $\langle b_1, b_2, b_3 \rangle = \langle -2,2,2 \rangle$.

$$\vec{r_u} \times \vec{r_v} = \mathbf{i}\left(1 \cdot 2 - \frac{\pi}{2} \cdot 2\right) - \mathbf{j}\left(1 \cdot 2 - \frac{\pi}{2} \cdot (-2)\right) + \mathbf{k}(1 \cdot 2 - 1 \cdot (-2))$$

$$\vec{r_u} \times \vec{r_v} = (2 - \pi)\mathbf{i} - (2 + \pi)\mathbf{j} + 4\mathbf{k}$$

So the normal vector to the surface at the given point is

$$\vec{n} = \langle 2 - \pi, -2 - \pi, 4 \rangle$$

The standard equation of the plane that passes through (x_0, y_0, z_0), and with normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Plug in $(x_0, y_0, z_0) = (4, 0, \pi)$ and $\vec{n} = \langle 2 - \pi, -2 - \pi, 4 \rangle$.

$$(2 - \pi)(x - 4) + (-2 - \pi)(y - 0) + 4(z - \pi) = 0$$

$$(\pi - 2)x + (\pi + 2)y - 4z + 8 = 0$$

■ 2. Find the equation of the tangent plane(s) to the parametric surface $\vec{r}(u, v) = \langle u^2 + 2v, u - 2v, uv + 1 \rangle$ such that its normal vector $\vec{n}$ is parallel to the y-axis.

Solution:

Find the normal vector to the surface in general form as the cross product of two tangent vectors,

$$\vec{r}_u = \frac{\partial \vec{r}(u,v)}{\partial u} \text{ and } \vec{r}_v = \frac{\partial \vec{r}(u,v)}{\partial u}$$

We find

$$\vec{r}_u = \langle 2u, 1, v \rangle$$

$$\vec{r}_v = \langle 2, -2, u \rangle$$

Evaluate the cross product $\vec{r}_u \times \vec{r}_v$, if the cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle a_1, a_2, a_3 \rangle = \langle 2u, 1, v \rangle$ and $\langle b_1, b_2, b_3 \rangle = \langle 2, -2, u \rangle$.

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(1 \cdot u - v \cdot (-2)) - \mathbf{j}(2u \cdot u - v \cdot 2) + \mathbf{k}(2u \cdot (-2) - 1 \cdot 2)$$

$$\vec{r}_u \times \vec{r}_v = (u + 2v)\mathbf{i} - (-2u^2 + 2v)\mathbf{j} + (-2 - 4u)\mathbf{k}$$

So the normal vector to the surface is

$$\vec{n} = \langle u + 2v, -2u^2 + 2v, -2 - 4u \rangle$$

Since the normal vector is parallel to the y-axis, its x- and z-components are 0, so

$$u + 2v = 0$$

$$-2 - 4u = 0$$

Solve the system of equations for u and v. From the second equation,

$$4u = -2$$

$$u = -0.5$$

Plug $u = -0.5$ into the first equation.

$$-0.5 + 2v = 0$$

$$v = 0.25$$

Find the point on the surface that corresponds to the parameter values $u = -0.5$ and $v = 0.25$.

$$\vec{r}(-0.5,0.25) = \langle(-0.5)^2 + 2 \cdot 0.25, -0.5 - 2 \cdot 0.25, -0.5 \cdot 0.25 + 1\rangle$$

$$\vec{r}(-0.5,0.25) = \langle 0.75, -1, 0.875\rangle$$

The standard equation of the plane that passes through (x_0, y_0, z_0) with normal vector $\vec{n} = \langle n_1, n_2, n_3\rangle$ is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

For the normal vector we can take any vector parallel to the y-axis, so we'll plug $\vec{n} = \langle 0,1,0\rangle$ and $(x_0, y_0, z_0) = (0.75, -1, 0.875)$ into the equation.

$$1 \cdot (y - (-1)) = 0$$

$$y + 1 = 0$$

■ 3. Find the equation of the tangent plane(s) to the parametric surface $\vec{r}(u, v) = \langle v^2, u - v + 2, u^2 - 2\rangle$ such that it's parallel to $3x - 24y + 2z - 1 = 0$.

Solution:

Find the normal vector to the surface as a cross product of the tangent vectors $\vec{r}_u = \langle 0,1,2u \rangle$ and $\vec{r}_v = \langle 2v, -1,0 \rangle$. Evaluate the cross product $\vec{r}_u \times \vec{r}_v$, given that the cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle a_1, a_2, a_3 \rangle = \langle 0,1,2u \rangle$ and $\langle b_1, b_2, b_3 \rangle = \langle 2v, -1,0 \rangle$.

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(1 \cdot 0 - 2u \cdot (-1)) - \mathbf{j}(0 \cdot 0 - 2u \cdot 2v) + \mathbf{k}(0 \cdot (-1) - 1 \cdot 2v)$$

$$\vec{r}_u \times \vec{r}_v = 2u\mathbf{i} + 4uv\mathbf{j} - 2v\mathbf{k}$$

So the normal vector to the surface is

$$\vec{n} = \langle 2u,4uv, -2v \rangle$$

Since the tangent plane we're interested in is parallel to the plane $3x - 24y + 2z - 1 = 0$, it has a normal vector that's parallel to $\langle 3, -24,2 \rangle$, so there's a nonzero real number k such that

$$\langle 2u,4uv, -2v \rangle = k\langle 3, -24,2 \rangle$$

This equation gives a new system.

$$2u = 3k$$

$$4uv = -24k$$

$$-2v = 2k$$

Solve the system of equations for u, v, and k. Solve u and v in the first and third equations.

$$u = 1.5k$$

$$v = -k$$

Plug these values into the second equation.

$$4(1.5k)(-k) = -24k$$

$$-6k^2 = -24k$$

$$k = 4$$

So the solution of the system is $k = 4$, $u = 6$, and $v = -4$. Find the point on the surface that corresponds to the parameter values $u = 6$ and $v = -4$.

$$\vec{r}(6, -4) = \langle (-4)^2, 6 - (-4) + 2, 6^2 - 2 \rangle$$

$$\vec{r}(6, -4) = \langle 16, 12, 34 \rangle$$

The standard equation of the plane that passes through (x_0, y_0, z_0) and with normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Plug in $(x_0, y_0, z_0) = (16, 12, 34)$ and $\vec{n} = \langle 3, -24, 2 \rangle$.

$$(x - 16) - 24(y - 12) + 2(z - 34) = 0$$

$$3x - 24y + 2z + 172 = 0$$

AREA OF A SURFACE

1. Find the area of the part of the surface $z = 2x + 2y - 1$ that lies within the rectangle given by $0 \le x \le \pi$ and $-1 \le y \le 1$.

Solution:

The area of the surface $z = f(x, y)$ inside the region D is given by

$$A = \int\int_D \sqrt{1 + (f'_x)^2 + (f'_y)^2}\ dA$$

Take the partial derivatives of $f(x, y) = 2x + 2y - 1$.

$$f'_x = 2$$

$$f'_y = 2$$

So the area of the part of the surface is given by

$$A = \int_0^\pi \int_{-1}^1 \sqrt{1 + 2^2 + 2^2}\ dy\ dx$$

$$A = \int_0^\pi \int_{-1}^1 3\ dy\ dx$$

Evaluate the inner integral.

$$A = \int_0^{\pi} 3y \Big|_{y=-1}^{y=1} \ dx$$

$$A = \int_0^{\pi} 3(1) - 3(-1) \ dx$$

$$A = \int_0^{\pi} 3 + 3 \ dx$$

$$A = \int_0^{\pi} 6 \ dx$$

Evaluate the outer integral.

$$A = 6x \Big|_0^{\pi}$$

$$A = 6\pi - 6(0)$$

$$A = 6\pi$$

■ 2. Find the area of the part of the surface $\vec{r}(u, v)$ that lies within the values of the parameters $-1 \le u \le 1$ and $0 \le v \le \sqrt{5}$.

$$\vec{r}(u, v) = \langle 2u - 3v + 1, 5u - v + 4, -u + 4v - 11 \rangle$$

Solution:

The area of the surface $\vec{r}(u, v)$ inside the region D is given by

$$A = \iint_D |\vec{r}_u \times \vec{r}_v| \ dA$$

Take partial derivatives.

$$\vec{r}_u = \langle 2,5, -1 \rangle$$

$$\vec{r}_v = \langle -3, -1,4 \rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle 2,5, -1 \rangle \times \langle -3, -1,4 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(5 \cdot 4 - (-1) \cdot (-1)) - \mathbf{j}(2 \cdot 4 - (-1) \cdot (-3)) + \mathbf{k}(2 \cdot (-1) - 5 \cdot (-3))$$

$$\vec{r}_u \times \vec{r}_v = 19\mathbf{i} - 5\mathbf{j} + 13\mathbf{k}$$

$$\vec{r}_u \times \vec{r}_v = \langle 19, -5,13 \rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{19^2 + (-5)^2 + 13^2} = \sqrt{555}$$

So the area of the part of the surface is

$$A = \int_{-1}^{1} \int_{0}^{\sqrt{5}} \sqrt{555} \ dv \ du$$

Evaluate the inner integral.

$$A = \int_{-1}^{1} \sqrt{5} \cdot \sqrt{555} \ du$$

$$A = \int_{-1}^{1} 5\sqrt{111}\ du$$

Evaluate the outer integral.

$$A = 2(5\sqrt{111}) = 10\sqrt{111}$$

■ 3. Find the area of the part of the surface $\vec{r}(u, v) = \langle 2\cos u, 5v + 3, 2\sin u \rangle$ that lies within the values of the parameters $\pi/6 \le u \le \pi/3$ and $0 \le v \le 3$.

Solution:

The area of the surface $\vec{r}(u, v)$ inside the region D is given by

$$A = \iint_D |\vec{r}_u \times \vec{r}_v|\ dA$$

Take partial derivatives.

$$\vec{r}_u = \langle -2\sin u, 0, 2\cos u \rangle$$

$$\vec{r}_v = \langle 0, 5, 0 \rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle -2\sin u, 0, 2\cos u \rangle \times \langle 0, 5, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(0 \cdot 0 - 2\cos u \cdot 5) - \mathbf{j}(-2\sin u \cdot 0 - 2\cos u \cdot 0) + \mathbf{k}(-2\sin u \cdot 5 - 0 \cdot 0)$$

$$\vec{r}_u \times \vec{r}_v = -10\cos u\mathbf{i} - 0\mathbf{j} - 10\sin u\mathbf{k}$$

$$\vec{r}_u \times \vec{r}_v = \langle -10\cos u, 0, -10\sin u \rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-10\cos u)^2 + 0^2 + (-10\sin u)^2}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{100\cos^2 u + 100\sin^2 u}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{100}$$

$$|\vec{r}_u \times \vec{r}_v| = 10$$

Then the area of this part of the surface is

$$A = \int_{\pi/6}^{\pi/3} \int_0^3 10 \ dv \ du$$

Integrate with respect to v.

$$A = \int_{\pi/6}^{\pi/3} 10(3-0) \ du$$

$$A = \int_{\pi/6}^{\pi/3} 30 \ du$$

Integrate with respect to u.

$$A = 30\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 30 \cdot \frac{\pi}{6} = 5\pi$$

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