Curvature

To find the curvature $\kappa(t)$ of a vector function $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, we'll use the equation

$$\kappa(t) = \frac{\left|T'(t)\right|}{\left|r'(t)\right|}$$

where $\left|T'(t)\right|$ is the magnitude of the derivative of the unit tangent vector $T(t)$, which we can find using

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

where $T(t)$ is the unit tangent vector, which we can find using

$$T(t) = \frac{r'(t)}{\left|r'(t)\right|}$$

where $r'(t)$ is the derivative of the vector function and where $\left|r'(t)\right|$ is the magnitude of the derivative of the vector function, which we can find using

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

In other words, in order to find $\kappa(t)$, we'll

1. Find $r'(t)$, and use it to

2. Find $\left|r'(t)\right|$, and then use $r'(t)$ and $\left|r'(t)\right|$ to

3. Find $T(t)$, and then use it to

4. Find $T'(t)$, and then use it to

5. Find $\left|T'(t)\right|$, and then use $\left|r'(t)\right|$ and $\left|T'(t)\right|$ to

6. Find $\kappa(t)$

Example

Find the curvature of the vector function.

$$r(t) = 4t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$$

We'll start by calculating the derivative of the vector function.

$$r'(t) = 4\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$$

Then we'll find $\left|r'(t)\right|$.

$$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left|r'(t)\right| = \sqrt{(4)^2 + (2t)^2 + (2)^2}$$

$$\left|r'(t)\right| = \sqrt{16 + 4t^2 + 4}$$

$$\left|r'(t)\right| = \sqrt{4t^2 + 20}$$

$$\left|r'(t)\right| = \sqrt{4\left(t^2 + 5\right)}$$

$$\left|r'(t)\right| = 2\sqrt{t^2 + 5}$$

Now we'll use the derivative and its magnitude to find an equation for the unit tangent vector $T(t)$.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

$$T(t) = \frac{4\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}}{2\sqrt{t^2 + 5}}$$

$$T(t) = \frac{4}{2\sqrt{t^2 + 5}}\mathbf{i} + \frac{2t}{2\sqrt{t^2 + 5}}\mathbf{j} + \frac{2}{2\sqrt{t^2 + 5}}\mathbf{k}$$

$$T(t) = \frac{2}{\sqrt{t^2 + 5}}\mathbf{i} + \frac{t}{\sqrt{t^2 + 5}}\mathbf{j} + \frac{1}{\sqrt{t^2 + 5}}\mathbf{k}$$

Now we can find the derivative of the unit tangent vector $T'(t)$. We'll need to use quotient rule to find the derivatives of the coefficients on $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

$$T'(t) = \frac{(0)\sqrt{t^2 + 5} - (2)\left[\frac{1}{2}\left(t^2 + 5\right)^{-\frac{1}{2}}(2t)\right]}{\left(\sqrt{t^2 + 5}\right)^2}\mathbf{i} + \frac{(1)\sqrt{t^2 + 5} - (t)\left[\frac{1}{2}\left(t^2 + 5\right)^{-\frac{1}{2}}(2t)\right]}{\left(\sqrt{t^2 + 5}\right)^2}\mathbf{j}$$

$$+ \frac{(0)\sqrt{t^2 + 5} - (1)\left[\frac{1}{2}\left(t^2 + 5\right)^{-\frac{1}{2}}(2t)\right]}{\left(\sqrt{t^2 + 5}\right)^2}\mathbf{k}$$

$$T'(t) = \frac{-2t\left(t^2 + 5\right)^{-\frac{1}{2}}}{t^2 + 5}\mathbf{i} + \frac{\sqrt{t^2 + 5} - t^2\left(t^2 + 5\right)^{-\frac{1}{2}}}{t^2 + 5}\mathbf{j} + \frac{-t\left(t^2 + 5\right)^{-\frac{1}{2}}}{t^2 + 5}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\sqrt{t^2+5}-\frac{t^2}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\frac{t^2+5}{\sqrt{t^2+5}}-\frac{t^2}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\frac{t^2+5-t^2}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{\frac{5}{\sqrt{t^2+5}}}{t^2+5}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

$$T'(t)=-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{i}+\frac{5}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{j}-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\mathbf{k}$$

Then we'll find the magnitude of the derivative of the unit tangent vector $\left|T'(t)\right|$.

$$\left|T'(t)\right|=\sqrt{\left[T'(t)_1\right]^2+\left[T'(t)_2\right]^2+\left[T'(t)_3\right]^2}$$

$$\left|T'(t)\right|=\sqrt{\left[-\frac{2t}{\left(t^2+5\right)^{\frac{3}{2}}}\right]^2+\left[\frac{5}{\left(t^2+5\right)^{\frac{3}{2}}}\right]^2+\left[-\frac{t}{\left(t^2+5\right)^{\frac{3}{2}}}\right]^2}$$

$$\left|T'(t)\right|=\sqrt{\frac{4t^2}{\left(t^2+5\right)^3}+\frac{25}{\left(t^2+5\right)^3}+\frac{t^2}{\left(t^2+5\right)^3}}$$

$$\left|T'(t)\right| = \sqrt{\frac{5t^2+25}{\left(t^2+5\right)^3}}$$

$$\left|T'(t)\right| = \sqrt{\frac{5\left(t^2+5\right)}{\left(t^2+5\right)^3}}$$

$$\left|T'(t)\right| = \sqrt{\frac{5}{\left(t^2+5\right)^2}}$$

$$\left|T'(t)\right| = \frac{\sqrt{5}}{t^2+5}$$

Finally we can solve for the curvature $\kappa(t)$ of the vector function

$$\kappa(t) = \frac{\left|T'(t)\right|}{\left|r'(t)\right|}$$

$$\kappa(t) = \frac{\frac{\sqrt{5}}{t^2+5}}{2\sqrt{t^2+5}}$$

$$\kappa(t) = \frac{\sqrt{5}}{2\left(t^2+5\right)^{\frac{3}{2}}}$$

This is the curvature of the vector function.