Topic: Extreme value theorem

Question: Which of these statements is true?

Answer choices:

A The Extreme Value Theorem shows you how to calculate the absolute extrema points in a function $f(x, y)$.

B The Extreme Value Theorem confirms the existence of absolute extrema in a function $f(x, y)$.

C The Extreme Value Theorem states that the absolute extrema can only exist in the interior of the designated bounded set.

D The Extreme Value Theorem states that the absolute extrema can only exist on the boundaries of the designated bounded set.

Solution: B

The Extreme Value Theorem (EVT) states that if $f(x, y)$ is continuous in some closed, bounded set D, then there are points in D, (x_1, y_1) and (x_2, y_2), such that $f(x_1, y_1)$ is the absolute maximum and $f(x_2, y_2)$ is the absolute minimum.

This EVT doesn't tell us where the absolute extrema will occur, just that they'll exist.

Answer choice A is incorrect because the theorem does not tell you how to calculate the absolute extrema points in a function $f(x, y)$.

Answer choice B is correct because the theorem confirms the existence of absolute extrema in a function $f(x, y)$.

Option C and D are incorrect because the absolute extrema can occur within the bounded set or on the boundary's edge.

Topic: Extreme value theorem

Question: Find the absolute minima and absolute maxima of the function.

$$f(x, y) = 3x^2 - 3y^2 + xy + 2$$

for $D = \{(x, y) \mid -1 \le x \le 1, -1 \le y \le 1\}$

Answer choices:

A Absolute maxima at $(-1,1/6)$ and $(-1,1/6)$

Absolute minima at $(-1,1)$ and $(1,-1)$

B Absolute maxima at $(1,1/6)$ and $(-1,1/6)$

Absolute minima at $(-1,1)$ and $(1,-1)$

C Absolute maxima at $(1,1/6)$ and $(-1,-1/6)$

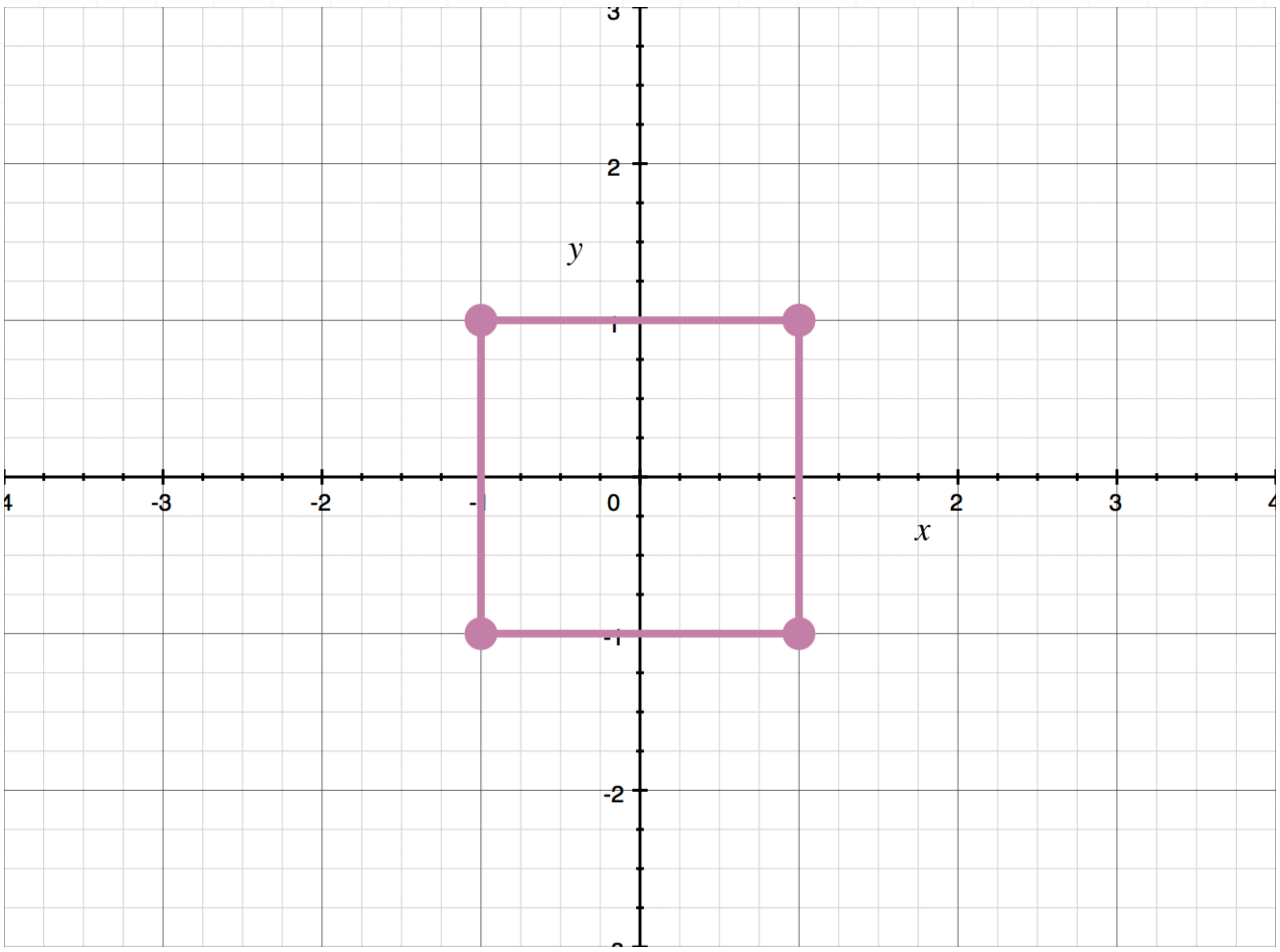
Absolute minima at $(-1/6,1)$ and $(1/6,-1)$

D Absolute maxima at $(-1,1/6)$ and $(-1,-1/6)$

Absolute minima at $(-1,1)$ and $(1,-1)$

Solution: C

The set D is defined over $x = [-1,1]$ and $y = [-1,1]$, which means the region looks like this:



We'll start with the critical points of the function, which we'll find by taking first order partial derivatives of the function, setting them equal to 0, and solving the resulting system of equations. Since $f(x, y) = 3x^2 - 3y^2 + xy + 2$, we get

$$\frac{\partial f}{\partial x} = 6x + y$$

$$6x + y = 0$$

[1] $y = -6x$

and

$$\frac{\partial f}{\partial y} = -6y + x$$

[2] $-6y + x = 0$

Plugging [1] into [2] gives

$$-6(-6x) + x = 0$$

$$36x + x = 0$$

$$37x = 0$$

$$x = 0$$

The fact that we only found one value tells us that the only critical point is at $x = 0$. Now we just need to find the corresponding value of y, which we'll do by plugging $x = 0$ into $y = -6x$.

$$y = -6(0)$$

$$y = 0$$

Putting $y = 0$ together with $x = 0$, we can say that the only critical point in the region D is $(0,0)$.

To find the value of the function at this critical point, we'll plug $(0,0)$ into $f(x, y)$.

$$f(x, y) = 3x^2 - 3y^2 + xy + 2$$

$$f(0,0) = 3(0)^2 - 3(0)^2 + (0)(0) + 2$$

$$f(0,0) = 2$$

Extrema can only occur at critical points and at the edges of the region. We now know that the value of the function at the only critical point is $f(0,0) = 2$. Our next step is to check the value of the function everywhere along the edges of the region. Comparing the value of the function along the edges to the value at the critical point, we can say that the largest value we find is the global maximum and that the smallest value we find is the global minimum.

Let's start by looking at the right edge of the region, which corresponds to the line $x = 1$, where $-1 \leq y \leq 1$. Since $x = 1$ everywhere along this edge, the function that defines the edge is

$$g(y) = f(1,y) = 3(1)^2 - 3y^2 + (1)y + 2$$

$$g(y) = f(1,y) = 3 - 3y^2 + y + 2$$

$$g(y) = f(1,y) = -3y^2 + y + 5$$

To figure out the value of the function along the edge, we'll look for critical points of the function that defines the edge, which means we need to take the derivative of the new function.

$$g'(y) = f'(1,y) = -6y + 1$$

Now we'll set the derivative equal to 0 and solve for y.

$$-6y + 1 = 0$$

$$-6y = -1$$

$$y = \frac{1}{6}$$

With just one critical point along the edge, we know that the highest and lowest values along the edge have to occur at the critical point $y = 1/6$, or at the ends of the interval, $y = [-1,1]$.

At $y = \frac{1}{6}$,

$$g\left(\frac{1}{6}\right) = -3\left(\frac{1}{6}\right)^2 + \frac{1}{6} + 5$$

$$g\left(\frac{1}{6}\right) = -\frac{3}{36} + \frac{6}{36} + \frac{180}{36}$$

$$g\left(\frac{1}{6}\right) \approx 5.08$$

At $y = 1$,

$$g(1) = -3(1)^2 + 1 + 5$$

$$g(1) = -3 + 1 + 5$$

$$g(1) = 3$$

At $y = -1$,

$$g(-1) = -3(-1)^2 + (-1) + 5$$

$$g(-1) = -3 - 1 + 5$$

$$g(-1) = 1$$

We'll repeat this same process for all four edges of the region. For the left edge of the region, which corresponds to the line $x = -1$, where $-1 \le y \le 1$, the function that defines the edge is

$$h(y) = f(-1,y) = 3(-1)^2 - 3y^2 + (-1)y + 2$$

$$h(y) = f(-1,y) = 3 - 3y^2 - y + 2$$

$$h(y) = f(-1,y) = -3y^2 - y + 5$$

Taking the derivative gives

$$h'(y) = f'(-1,y) = -6y - 1$$

We'll set the derivative equal to 0 and solve for y.

$$-6y - 1 = 0$$

$$-6y = 1$$

$$y = -\frac{1}{6}$$

Evaluating the function at the critical point and at the endpoints of the interval $y = [-1,1]$, we get

At $y = -\frac{1}{6}$,

$$h\left(-\frac{1}{6}\right) = -3\left(-\frac{1}{6}\right)^2 - \left(-\frac{1}{6}\right) + 5$$

$$h\left(-\frac{1}{6}\right) = -\frac{3}{36} + \frac{6}{36} + \frac{180}{36}$$

$$h\left(-\frac{1}{6}\right) \approx 5.08$$

At $y = 1$,

$$h(1) = -3(1)^2 - (1) + 5$$

$$h(1) = -3 - 1 + 5$$

$$h(1) = 1$$

At $y = -1$,

$$h(-1) = -3(-1)^2 - (-1) + 5$$

$$h(-1) = -3 + 1 + 5$$

$$h(-1) = 3$$

For the top edge of the region, which corresponds to the line $y = 1$, where $-1 \le x \le 1$, the function that defines the edge is

$$m(x) = f(x,1) = 3x^2 - 3(1)^2 + x(1) + 2$$

$$m(x) = f(x,1) = 3x^2 - 3 + x + 2$$

$$m(x) = f(x,1) = 3x^2 + x - 1$$

Taking the derivative gives

$$m'(x) = f'(x,1) = 6x + 1$$

We'll set the derivative equal to 0 and solve for x.

$$6x + 1 = 0$$

$$6x = -1$$

$$x = -\frac{1}{6}$$

Evaluating the function at the critical point and at the endpoints of the interval $x = [-1,1]$, we get

At $x = -\frac{1}{6}$,

$$m\left(-\frac{1}{6}\right) = 3\left(-\frac{1}{6}\right)^2 + \left(-\frac{1}{6}\right) - 1$$

$$m\left(-\frac{1}{6}\right) = \frac{3}{36} - \frac{6}{36} - \frac{36}{36}$$

$$m\left(-\frac{1}{6}\right) \approx -1.08$$

At $x = 1$,

$$m(1) = 3(1)^2 + (1) - 1$$

$$m(1) = 3 + 1 - 1$$

$$m(1) = 3$$

At $x = -1$,

$$m(-1) = 3(-1)^2 + (-1) - 1$$

$$m(-1) = 3 - 1 - 1$$

$$m(-1) = 1$$

For the bottom edge of the region, which corresponds to the line $y = -1$, where $-1 \le x \le 1$, the function that defines the edge is

$$n(x) = f(x, -1) = 3x^2 - 3(-1)^2 + x(-1) + 2$$

$$n(x) = f(x, -1) = 3x^2 - 3 - x + 2$$

$$n(x) = f(x, -1) = 3x^2 - x - 1$$

Taking the derivative gives

$$n'(x) = f'(x, -1) = 6x - 1$$

We'll set the derivative equal to 0 and solve for x.

$$6x - 1 = 0$$

$$6x = 1$$

$$x = \frac{1}{6}$$

Evaluating the function at the critical point and at the endpoints of the interval $x = [-1,1]$, we get

At $x = \frac{1}{6}$,

$$n\left(\frac{1}{6}\right) = 3\left(\frac{1}{6}\right)^2 - \left(\frac{1}{6}\right) - 1$$

$$n\left(\frac{1}{6}\right)=\frac{3}{36}-\frac{6}{36}-\frac{36}{36}$$

$$n\left(\frac{1}{6}\right)\approx-1.08$$

At $x=1$,

$$n(1)=3(1)^2-(1)-1$$

$$n(1)=3-1-1$$

$$n(1)=1$$

At $x=-1$,

$$n(-1)=3(-1)^2-(-1)-1$$

$$n(-1)=3+1-1$$

$$n(-1)=3$$

If we collect all of our values together, we can see the value of the function at various points throughout the region D.

At the critical point	$(0,0)$	$f(x,y)=2$
Along the right edge at	$(1,1/6)$	$f(x,y)\approx 5.08$
	$(1,1)$	$f(x,y)=3$
	$(1,-1)$	$f(x,y)=1$
Along the left edge at	$(-1,-1/6)$	$f(x,y)\approx 5.08$

	$(-1,1)$	$f(x,y) = 1$
	$(-1,-1)$	$f(x,y) = 3$
Along the top edge at	$(-1/6,1)$	$f(x,y) \approx -1.08$
	$(1,1)$	$f(x,y) = 3$
	$(-1,1)$	$f(x,y) = 1$
Along the bottom edge at	$(1/6,-1)$	$f(x,y) \approx -1.08$
	$(1,-1)$	$f(x,y) = 1$
	$(-1,-1)$	$f(x,y) = 3$

The largest of these values is 5.08 and the smallest is -1.08, which means

the global maxima occur at $(1,1/6)$ and $(-1,-1/6)$

the global minima occur at $(-1/6,1)$ and $(1/6,-1)$

Topic: Extreme value theorem

Question: Find the absolute minima and absolute maxima of the function.

$$f(x, y) = x^2 - 2y^2 + 4y$$

for $D = \{(x, y) \mid x^2 + y^2 \leq 4\}$

Answer choices:

A Absolute maxima at $(4\sqrt{2}/3, 2/3)$ and $(-4\sqrt{2}/3, 2/3)$

Absolute minimum at $(0, -2)$

B Absolute maxima at $(4\sqrt{2}/3, 2/3)$ and $(-4\sqrt{2}/3, 2/3)$

Absolute minimum at $(0, 2)$

C Absolute maxima at $(4\sqrt{2}/3, -2/3)$ and $(-4\sqrt{2}/3, -2/3)$

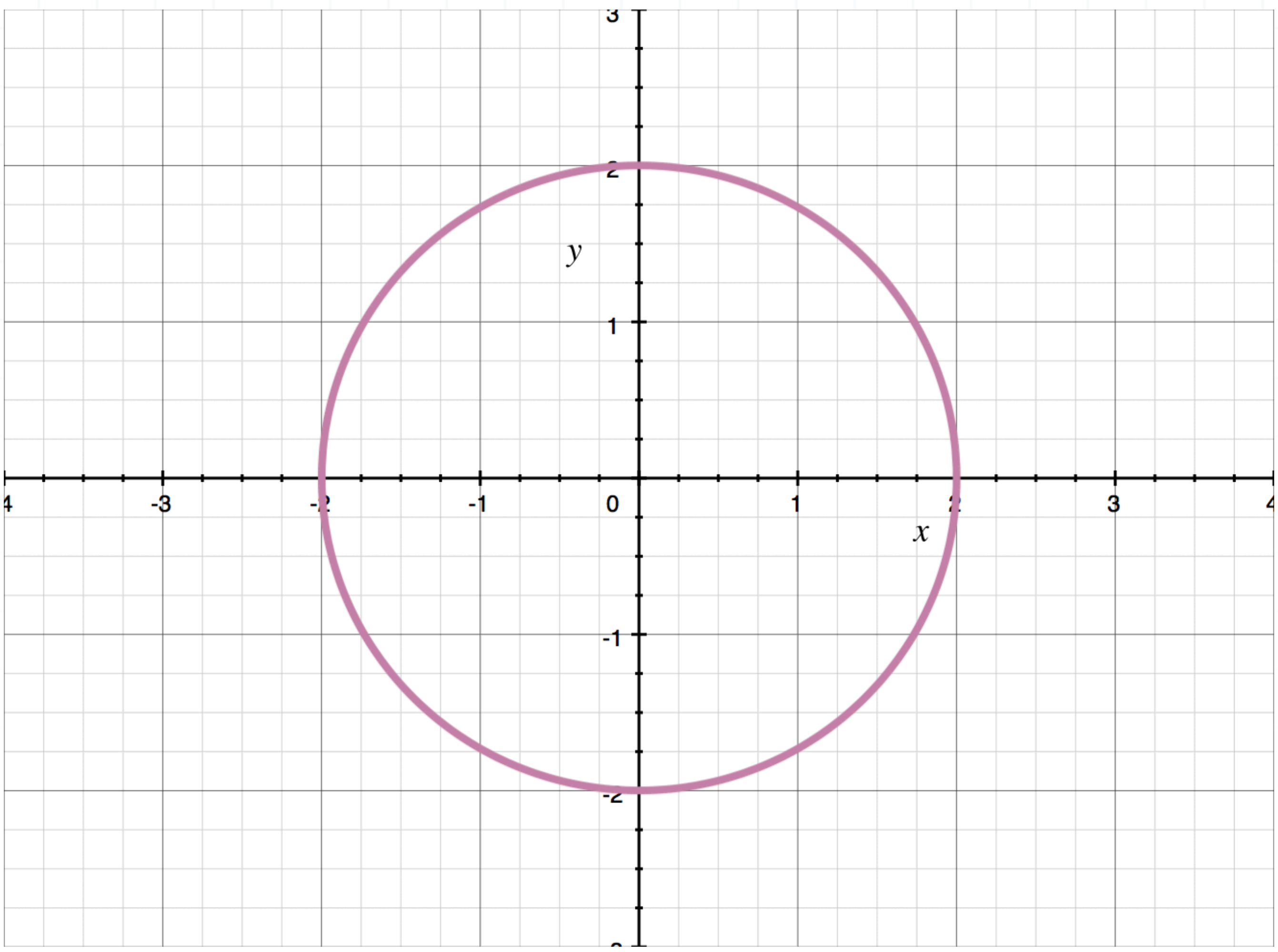
Absolute minimum at $(0, -2)$

D Absolute maxima at $(4\sqrt{2}/3, -2/3)$ and $(-4\sqrt{2}/3, -2/3)$

Absolute minimum at $(0, 2)$

Solution: A

The set D is given by the inequality $x^2 + y^2 \le 4$, which means that D is the circle centered at the origin with radius 2, and includes all of the area inside the circle, including its boundary at $r = 2$. It looks like this:



We'll start with the critical points of the function, which we'll find by taking first order partial derivatives of the function, setting them equal to 0, and solving the resulting system of equations. Since $f(x, y) = x^2 - 2y^2 + 4y$, we get

$$\frac{\partial f}{\partial x} = 2x$$

$$2x = 0$$

$$x = 0$$

and

$$\frac{\partial f}{\partial y} = -4y + 4$$

$$-4y + 4 = 0$$

$$-4y = -4$$

$$y = 1$$

The fact that we only found one value for both x and y tells us that we can put them together and say that the only critical point in the region D is at $(0,1)$.

To find the value of the function at this critical point, we'll plug $(0,1)$ into $f(x, y)$.

$$f(x, y) = x^2 - 2y^2 + 4y$$

$$f(0,1) = (0)^2 - 2(1)^2 + 4(1)$$

$$f(0,1) = 2$$

Extrema can only occur at critical points and at the edges of the region. We now know that the value of the function at the only critical point is $f(0,1) = 2$. Our next step is to check the value of the function everywhere along the edges of the region. Comparing the value of the function along the edges to the value at the critical point, we can say that the largest value we find is the global maximum and that the smallest value we find is the global minimum.

In order to look at the boundary, we need to find an equation that describes the value of the original function along the boundary of D. If we solve the inequality that represents D for x^2 we get

$$x^2 = -y^2 + 4$$

Now we'll make a substitution into the original function, defining this as a new function $g(y)$, that describes the value of $f(x, y)$ along the boundary of the region D.

$$f(x, y) = x^2 - 2y^2 + 4y$$

$$g(y) = \left(-y^2 + 4\right) - 2y^2 + 4y$$

$$g(y) = -3y^2 + 4y + 4$$

Since this new function describes the boundary, and since we're looking for extrema along the boundary, our next step is to find critical points of $g(y)$, which we'll do by taking its derivative.

$$g'(y) = -6y + 4$$

Setting the derivative equal to 0 and solving for y gives

$$-6y + 4 = 0$$

$$-6y = -4$$

$$y = \frac{2}{3}$$

With just one critical point along the edge, we know that the highest and lowest values along the edge have to occur at the critical point $y = 2/3$, or at the ends of the interval, $y=[-2,2]$. Remember that $-2 \le y \le 2$ because D is the circle with radius 2.

At $y=2/3$,

$$g\left(\frac{2}{3}\right)=-3\left(\frac{2}{3}\right)^2+4\left(\frac{2}{3}\right)+4$$

$$g\left(\frac{2}{3}\right)=-\frac{4}{3}+\frac{8}{3}+\frac{12}{3}$$

$$g\left(\frac{2}{3}\right)\approx 5.33$$

At $y=2$,

$$g(2)=-3(2)^2+4(2)+4$$

$$g(2)=-12+8+4$$

$$g(2)=0$$

At $y=-2$,

$$g(-2)=-3(-2)^2+4(-2)+4$$

$$g(-2)=-12-8+4$$

$$g(-2)=-16$$

We need to find the values of x that correspond with $y=2/3$, $y=2$ and $y=-2$ so that we can identify the coordinate points these correspond to. Plugging each of them into $x^2=-y^2+4$ gives

For $y = 2/3$,

$$x^2 = -\left(\frac{2}{3}\right)^2 + 4$$

$$x^2 = -\frac{4}{9} + \frac{36}{9}$$

$$x^2 = \frac{32}{9}$$

$$x = \pm\frac{4\sqrt{2}}{3}$$

For $y = 2$,

$$x^2 = -2^2 + 4$$

$$x^2 = -4 + 4$$

$$x^2 = 0$$

$$x = 0$$

For $y = -2$,

$$x^2 = -(-2)^2 + 4$$

$$x^2 = -4 + 4$$

$$x^2 = 0$$

$$x = 0$$

If we collect all of our values together, we can see the value of the function at various points throughout the region D.

At the critical point	$(0,1)$	$f(x,y)=2$
Along the boundary at	$\left(\frac{4\sqrt{2}}{3},\frac{2}{3}\right)$	$f(x,y)\approx 5.33$
	$\left(-\frac{4\sqrt{2}}{3},\frac{2}{3}\right)$	$f(x,y)\approx 5.33$
	$(0,2)$	$f(x,y)=0$
	$(0,-2)$	$f(x,y)=-16$

The largest of these values is 5.33 and the smallest is -16, which means

the global maxima occur at $(4\sqrt{2}/3,2/3)$ and $(-4\sqrt{2}/3,2/3)$

the global minimum occurs at $(0,-2)$