Calculus 3 Workbook Solutions

Curl and divergence

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CURL AND DIVERGENCE OF A VECTOR FIELD

1. Find the set of points in R^3 where the curl of the vector field $\vec{F}(x,y,z)$ is parallel to the vector $\vec{a} = \langle 2,1,2 \rangle$.

$$\vec{F}(x,y,z) = \left\langle \frac{z}{2}, \ln(xyz), z^2 \right\rangle$$

Solution:

The curl of a vector field in three dimensions is given by

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

If we calculate partial derivatives, we get

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(z^2) = 0$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(\ln(xyz)) = \frac{1}{z}$$

and

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{2}\right) = \frac{1}{2}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(z^2) = 0$$

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and

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(\ln(xyz)) = \frac{1}{x}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{2}\right) = 0$$

So the curl of the vector field $\vec{F}(x, y, z)$ is

$$\text{curl } \vec{F} = \left\langle 0 - \frac{1}{z}, \frac{1}{2} - 0, \frac{1}{x} - 0 \right\rangle$$

$$\text{curl } \vec{F} = \left\langle -\frac{1}{z}, \frac{1}{2}, \frac{1}{x} \right\rangle$$

Since the curl is parallel to the vector $\vec{a} = \langle 2, 1, 2 \rangle$, there's a constant k such that curl $\vec{F} = k\vec{a}$. Therefore,

$$\left\langle -\frac{1}{z}, \frac{1}{2}, \frac{1}{x} \right\rangle = k\langle 2, 1, 2 \rangle$$

$$\left\langle -\frac{1}{z}, \frac{1}{2}, \frac{1}{x} \right\rangle = \langle 2k, k, 2k \rangle$$

From the second component,

$$k = \frac{1}{2}$$

So

$$-\frac{1}{z} = 2 \cdot \frac{1}{2}$$

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$$\frac{1}{x} = 2 \cdot \frac{1}{2}$$

Therefore, $x = 1$ and $z = -1$. Therefore, the curl of the vector field $\overrightarrow{F}(x, y, z)$ is parallel to the vector $\overrightarrow{a} = \langle 2, 1, 2 \rangle$ for any point with coordinates $(1, y, -1)$, in other words, for any point on the line $x = 1$ and $z = -1$.

■ 2. Find the set of points in R^3, where the divergence of the vector field $\overrightarrow{F}(x, y, z) = \langle x^3 + 12xy, y^3 + 3z^2y - 9y, 3z^2 - 6xz \rangle$ is 0.

Solution:

The divergence of a vector field in three dimensions is given by

$$\text{div } \overrightarrow{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Calculate partial derivatives.

$$\frac{\partial F_x}{\partial x} = \frac{\partial}{\partial x}(x^3 + 12xy) = 3x^2 + 12y$$

$$\frac{\partial F_y}{\partial y} = \frac{\partial}{\partial y}(y^3 + 3z^2y - 9y) = 3y^2 + 3z^2 - 9$$

$$\frac{\partial F_z}{\partial z} = \frac{\partial}{\partial z}(3z^2 - 6xz) = 6z - 6x$$

So the divergence of the vector field $\overrightarrow{F}(x, y, z)$ is

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$$\text{div } \vec{F} = 3x^2 + 12y + 3y^2 + 3z^2 - 9 + 6z - 6x$$

Since the divergence of the vector field $\vec{F}(x, y, z)$ is 0,

$$3x^2 + 12y + 3y^2 + 3z^2 - 9 + 6z - 6x = 0$$

$$x^2 + 4y + y^2 + z^2 - 3 + 2z - 2x = 0$$

Complete the square with respect to each variable.

$$x^2 - 2x + 1 - 1 + y^2 + 4y + 4 - 4 + z^2 + 2z + 1 - 1 - 3 = 0$$

$$(x - 1)^2 + (y + 2)^2 + (z + 1)^2 = 9$$

Therefore, the divergence of the vector field $\vec{F}(x, y, z)$ is 0 on the sphere centered at $(1, -2, -1)$ with radius 3.

■ 3. Find the maximum value of the divergence of the vector field $\vec{F}(x, y, z)$.

$$\vec{F}(x, y, z) = \langle \ln(x^2 + 4), -e^{y+2}, -z\,e^{-y} - z^3 \rangle$$

Solution:

The divergence of a vector field in three dimensions is given by

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Calculate the partial derivatives.

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$$\frac{\partial F_x}{\partial x} = \frac{\partial}{\partial x}(\ln(x^2+4)) = \frac{2x}{x^2+4}$$

$$\frac{\partial F_y}{\partial y} = \frac{\partial}{\partial y}(-e^{y+2}) = -e^{y+2}$$

$$\frac{\partial F_z}{\partial z} = \frac{\partial}{\partial z}(-ze^{-y} - z^3) = -e^{-y} - 3z^2$$

So the divergence of the vector field $\vec{F}(x, y, z)$ is

$$\text{div } \vec{F} = \frac{2x}{x^2+4} - e^{y+2} - e^{-y} - 3z^2$$

It's possible to rewrite the maximized function $f(x, y, z) = \text{div } \vec{F}$ as a sum of three independent functions.

$$f(x, y, z) = f_1(x) + f_2(y) + f_3(z) = \left(\frac{2x}{x^2+4}\right) + (-e^{y+2} - e^{-y}) + (-3z^2)$$

So we can find the points x_0, y_0, and z_0 where each of these three functions reach their global maximum, and the maximum value of the sum of the function will be

$$f(x_0, y_0, z_0) = f_1(x_0) + f_2(y_0) + f_3(z_0)$$

To maximize $f_1(x) = 2x/(x^2+4)$, find critical points, take the derivative, and set it equal to 0.

$$\frac{2(4-x^2)}{(x^2+4)^2} = 0$$

$$4 - x^2 = 0$$

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$$x = \pm 2$$

Since $f_1'(x) > 0$ for $x \in (-2,2)$, and $f_1'(x) < 0$ for $x < -2$ or $x > 2$, and since $f_1(x)$ tends to 0 when x approaches $\pm\infty$, $x = -2$ is a global minimum and $x_0 = 2$ is a global maximum. To maximize $f_2(y) = -e^{y+2} - e^{-y}$, find critical points, take the derivative, and set it equal to 0.

$$-e^{y+2} + e^{-y} = 0$$

$$e^{y+2} = e^{-y}$$

$$e^{2y+2} = 1$$

$$2y + 2 = 0$$

$$y = -1$$

Since $f_2'(y) > 0$ for $y < -1$, and $f_2'(y) < 0$ for $y > -1$, $y_0 = -1$ is a global maximum. To maximize $f_3(z) = -3z^2$, find critical points, take the derivative, and set it equal to 0.

$$f_3'(z) = -6z = 0$$

$$z = 0$$

Since $f_3'(z) > 0$ for $z < 0$, and $f_3'(z) < 0$ for $z > 0$, $z_0 = 0$ is a global maximum.

So the function div $\vec{F}$ reaches its global maximum at $(2,-1,0)$, and its value is

$$\text{div } \vec{F}(2,-1,0) = \frac{2(2)}{2^2+4} - e^{-1+2} - e^{-(-1)} - 3(0)^2 = \frac{1}{2} - 2e$$

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POTENTIAL FUNCTION OF THE CONSERVATIVE VECTOR FIELD, THREE DIMENSIONS

■ 1. Find the potential function of the conservative vector field.

$$\vec{F}(x,y,z)=\left\langle \frac{2x}{z},\frac{1}{z},-\frac{x^2+y}{z^2}\right\rangle$$

Solution:

A potential function $f(x,y,z)$ of a vector field $\vec{F}(x,y,z)$ satisfies $\nabla f=\vec{F}$, or

$$\frac{\partial f}{\partial x}(x,y,z)=F_x(x,y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z)=F_y(x,y,z)$$

$$\frac{\partial f}{\partial z}(x,y,z)=F_z(x,y,z)$$

where F_x, F_y, and F_z are the components of the vector field.

$$\frac{\partial f}{\partial x}(x,y,z)=\frac{2x}{z}$$

$$\frac{\partial f}{\partial y}(x,y,z)=\frac{1}{z}$$

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$$\frac{\partial f}{\partial z}(x,y,z) = -\frac{x^2+y}{z^2}$$

From the first equation,

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{2x}{z}$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x,y,z) = \int \frac{2x}{z}\ dx$$

$$f(x,y,z) = \frac{2}{z} \cdot \frac{x^2}{2} + C(y,z)$$

$$f(x,y,z) = \frac{x^2}{z} + C(y,z)$$

Differentiate $f(x,y,z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{x^2}{z} + C(y,z)\right)$$

$$\frac{\partial f}{\partial y}(x,y,z) = 0 + \frac{\partial C}{\partial y}(y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial C}{\partial y}(y,z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{1}{z}$$

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So

$$\frac{\partial C}{\partial y}(y,z)=\frac{1}{z}$$

Integrate both sides with respect to y, treating z as a constant.

$$C(y,z)=\int \frac{1}{z}\ dy$$

$$C(y,z)=\frac{y}{z}+C_1(z)$$

So

$$f(x,y,z)=\frac{x^2}{z}+\frac{y}{z}+C_1(z)$$

$$f(x,y,z)=\frac{x^2+y}{z}+C_1(z)$$

Next, differentiate $f(x,y,z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x,y,z)=\frac{\partial}{\partial z}\left(\frac{x^2+y}{z}+C_1(z)\right)$$

$$\frac{\partial f}{\partial z}(x,y,z)=-\frac{x^2+y}{z^2}+\frac{\partial}{\partial z}C_1(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x,y,z)=-\frac{x^2+y}{z^2}$$

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So

$$-\frac{x^2+y}{z^2}+\frac{\partial}{\partial z}C_1(z)=-\frac{x^2+y}{z^2}$$

$$\frac{\partial}{\partial z}C_1(z)=0$$

Which means $C_1(z)$ is a constant c. Therefore,

$$f(x,y,z)=\frac{x^2+y}{z}+c$$

For any conservative vector field, there are an infinite number of possible potential functions, which vary by an additive constant.

■ 2. Find the value of a such that the vector field $\vec{F}$ has a potential function, then find that potential function.

$$\vec{F}(x,y,z)=\langle 4x^ay^3z^2, 3x^4y^2z^2, 2x^4y^3z\rangle$$

Solution:

A potential function $f(x,y,z)$ of a vector field $\vec{F}(x,y,z)$ satisfies the equality $\nabla f=\vec{F}$, or

$$\frac{\partial f}{\partial x}(x,y,z)=F_x(x,y,z)$$

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$$\frac{\partial f}{\partial y}(x,y,z) = F_y(x,y,z)$$

$$\frac{\partial f}{\partial z}(x,y,z) = F_z(x,y,z)$$

where F_x, F_y, and F_z are the components of the vector field.

$$\frac{\partial f}{\partial x}(x,y,z) = 4x^a y^3 z^2$$

$$\frac{\partial f}{\partial y}(x,y,z) = 3x^4 y^2 z^2$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2x^4 y^3 z$$

Integrate both sides of the first equation with respect to x, treating y and z as constants.

$$f(x,y,z) = \int 4x^a y^3 z^2 \ dx$$

$$f(x,y,z) = 4y^3 z^2 \cdot \frac{x^{a+1}}{a+1} + C(y,z)$$

$$f(x,y,z) = \frac{4}{a+1} x^{a+1} y^3 z^2 + C(y,z)$$

Differentiate $f(x,y,z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{4}{a+1} x^{a+1} y^3 z^2 + C(y,z)\right)$$

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$$\frac{\partial f}{\partial y}(x,y,z) = \frac{4}{a+1}x^{a+1}z^2 \cdot (3y^2) + \frac{\partial C}{\partial y}(y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{12}{a+1}x^{a+1}z^2y^2 + \frac{\partial C}{\partial y}(y,z)$$

From the second equation, we get

$$\frac{12}{a+1}x^{a+1}z^2y^2 + \frac{\partial C}{\partial y}(y,z) = 3x^4y^2z^2$$

From this equation we can conclude that

$$\frac{12}{a+1}x^{a+1} = 3x^4$$

$$\frac{\partial C}{\partial y}(y,z) = 0$$

which means that $a = 3$, and that $C(y,z)$ is a constant in terms of y, i.e. $C(y,z) = C(z)$. So

$$f(x,y,z) = x^4y^3z^2 + C(z)$$

Next, differentiate $f(x,y,z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}(x^4y^3z^2 + C(z))$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2x^4y^3z + C'(z)$$

From the third equation, we get

$$2x^4y^3z + C'(z) = 2x^4y^3z$$

$$C'(z) = 0$$

which means $C(z)$ is a constant, $C(z) = c$. Therefore,

$$f(x, y, z) = x^4y^3z^2 + c$$

For any conservative vector field, there are an infinite number of possible potential functions, each of which vary by an additive constant.

■ 3. Find a potential function of the conservative vector field $\vec{F}(x, y, z)$, then use this function to calculate the line integral of $\vec{F}$ over the curve $\vec{r}(t)$ between the parameter values $t = -2$ and $t = 2$.

$$\vec{F}(x, y, z) = \langle 2(x+1), 2(z-y), 2(y-1) \rangle$$

$$\vec{r}(t) = \left\langle e^{t^2-4}, \sin\frac{\pi t}{4}, e^{-t^2+4} \right\rangle$$

Solution:

The initial point of the curve for $t = -2$ is

$$\vec{r}(-2) = \left\langle e^{(-2)^2-4}, \sin\frac{\pi(-2)}{4}, e^{-(-2)^2+4} \right\rangle$$

$$\vec{r}(-2) = \langle 1, -1, 1 \rangle$$

The terminal point of the curve for $t = 2$ is

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$$\vec{r}(2) = \left\langle e^{(2)^2-4}, \sin\frac{\pi(2)}{4}, e^{-(2)^2+4} \right\rangle$$

$$\vec{r}(2) = \langle 1,1,1 \rangle$$

A potential function $f(x, y, z)$ of a vector field $\vec{F}(x, y, z)$ satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y, z) = F_x(x, y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

where F_x, F_y, and F_z are the components of the vector field.

$$\frac{\partial f}{\partial x}(x, y, z) = 2(x+1)$$

$$\frac{\partial f}{\partial y}(x, y, z) = 2(z-y)$$

$$\frac{\partial f}{\partial z}(x, y, z) = 2(y-1)$$

Integrate both sides of the first equation with respect to x, treating y and z as constants.

$$f(x, y, z) = \int 2(x+1)\ dx$$

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$$f(x,y,z) = 2\left(\frac{x^2}{2} + x\right) + C(y,z)$$

$$f(x,y,z) = x^2 + 2x + C(y,z)$$

Differentiate $f(x,y,z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial}{\partial y}(x^2 + 2x + C(y,z))$$

$$\frac{\partial f}{\partial y}(x,y,z) = 0 + \frac{\partial C}{\partial y}(y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial C}{\partial y}(y,z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x,y,z) = 2(z-y)$$

So

$$\frac{\partial C}{\partial y}(y,z) = 2(z-y)$$

Integrate both sides with respect to y, treating z as a constant.

$$C(y,z) = \int 2(z-y)\ dy$$

$$C(y,z) = 2\left(yz - \frac{y^2}{2}\right) + C_1(z)$$

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$$C(y,z) = 2yz - y^2 + C_1(z)$$

So

$$f(x,y,z) = x^2 + 2x + 2yz - y^2 + C_1(z)$$

Finally, differentiate $f(x,y,z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}(x^2 + 2x + 2yz - y^2 + C_1(z))$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2y + \frac{\partial}{\partial z}C_1(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x,y,z) = 2(y-1)$$

So

$$2y + \frac{\partial}{\partial z}C_1(z) = 2(y-1)$$

$$\frac{\partial}{\partial z}C_1(z) = -2$$

Integrate both parts with respect to z.

$$C_1(z) = \int -2\ dz$$

$$C_1(z) = -2z + c$$

Therefore,

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$$f(x, y, z) = x^2 + 2x + 2yz - y^2 - 2z + c$$

So the line integral is

$$\int_c \vec{F} \cdot d\vec{r} = f(1,1,1) - f(1,-1,1)$$

$$\int_c \vec{F} \cdot d\vec{r} = 1^2 + 2 \cdot 1 + 2 \cdot 1 \cdot 1 - 1^2 - 2 \cdot 1 + c$$

$$-(1^2 + 2 \cdot 1 + 2 \cdot (-1) \cdot 1 - (-1)^2 - 2 \cdot 1 + c)$$

$$\int_c \vec{F} \cdot d\vec{r} = 4$$

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