Topic: Two dimensions, one constraint

Question: Find the extrema of the function, subject to the given constraint.

$$f(x, y) = x^2 + y^2 + 100$$

when $y + 2x = 6$

Answer choices:

A Local maximum at $\left(\frac{12}{5}, \frac{6}{5}\right)$

B Local minimum at $\left(\frac{6}{5}, \frac{12}{5}\right)$

C Local minimum at $\left(\frac{12}{5}, \frac{6}{5}\right)$

D Local maximum at $\left(\frac{6}{5}, \frac{12}{5}\right)$

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Solution: C

We'll start by moving all terms in the constraint equation to one side, until the equation is equal to 0. Then we'll replace the 0 with $g(x, y)$.

$$y + 2x = 6$$

$$y + 2x - 6 = 0$$

$$g(x, y) = y + 2x - 6$$

Next we'll find the first-order partial derivatives of $f(x, y)$ and $g(x, y)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

and

$$\frac{\partial g}{\partial x} = 2$$

$$\frac{\partial g}{\partial y} = 1$$

We'll multiply the partial derivatives of g by the Lagrange multiplier λ.

$$\frac{\partial g}{\partial x} = 2\lambda$$

$$\frac{\partial g}{\partial y} = \lambda$$

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Then we'll set the partial derivatives of f equal to the corresponding partial derivatives from g, making sure to use the equations that include the Lagrange multiplier.

For the partial derivatives with respect to x we get

$$2x = 2\lambda$$

$$\lambda = x$$

For the partial derivatives with respect to y we get

$$2y = \lambda$$

$$\lambda = 2y$$

Now that we have two equations that are solved for λ, we can set them equal to each other, and then solve this equation for y in terms of x.

$$x = 2y$$

$$y = \frac{x}{2}$$

Plug this value for y back into the constraint equation.

$$y + 2x = 6$$

$$\left(\frac{x}{2}\right) + 2x = 6$$

$$x + 4x = 12$$

$$5x = 12$$

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$$x = \frac{12}{5}$$

Now plug this back into the constraint equation to solve for y.

$$y + 2x = 6$$

$$y + 2\left(\frac{12}{5}\right) = 6$$

$$y + \frac{24}{5} = 6$$

$$y = 6 - \frac{24}{5}$$

$$y = \frac{30 - 24}{5}$$

$$y = \frac{6}{5}$$

Putting these values for x and y together, the critical point is

$$\left(\frac{12}{5}, \frac{6}{5}\right)$$

To say whether this critical point is a maximum or minimum, we'll find second-order partial derivatives of $f(x, y)$.

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

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$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Then we'll plug these into the formula for D.

$$D(x, y, \lambda) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y, \lambda) = (2)(2) - (0)^2$$

$$D(x, y, \lambda) = 4$$

At this point, if we still had variables remaining on the right side of D, we'd evaluate D at the critical point. In this case, since there are no variables, plugging in the critical point won't change the value.

$$D\left(\frac{12}{5}, \frac{6}{5}, \lambda\right) = 4$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

If $D = 0$, then the second derivative test is inconclusive

If $D > 0$,

and $\frac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\frac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

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In this problem, $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so the critical point is a local minimum.

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Topic: Two dimensions, one constraint

Question: Find the extrema of the function, subject to the given constraint.

$$f(x, y) = 4x^2 + 6y^2 - 35$$

when $-10y + 5x = 25$

Answer choices:

A Local maximum at $\left(-\frac{20}{11}, \frac{15}{11}\right)$

B Local minimum at $\left(\frac{15}{11}, -\frac{20}{11}\right)$

C Local minimum at $\left(-\frac{20}{11}, \frac{15}{11}\right)$

D Local maximum at $\left(\frac{15}{11}, -\frac{20}{11}\right)$

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Solution: B

We'll start by moving all terms in the constraint equation to one side, until the equation is equal to 0. Then we'll replace the 0 with $g(x,y)$.

$$-10y + 5x = 25$$

$$5x - 10y - 25 = 0$$

$$g(x,y) = 5x - 10y - 25$$

Next we'll find the first-order partial derivatives of $f(x,y)$ and $g(x,y)$.

$$\frac{\partial f}{\partial x} = 8x$$

$$\frac{\partial f}{\partial y} = 12y$$

and

$$\frac{\partial g}{\partial x} = 5$$

$$\frac{\partial g}{\partial y} = -10$$

We'll multiply the partial derivatives of g by the Lagrange multiplier λ.

$$\frac{\partial g}{\partial x} = 5\lambda$$

$$\frac{\partial g}{\partial y} = -10\lambda$$

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Then we'll set the partial derivatives of f equal to the corresponding partial derivatives from g, making sure to use the equations that include the Lagrange multiplier.

For the partial derivatives with respect to x we get

$$8x = 5\lambda$$

$$\lambda = \frac{8}{5}x$$

For the partial derivatives with respect to y we get

$$12y = -10\lambda$$

$$\lambda = -\frac{6}{5}y$$

Now that we have two equations that are solved for λ, we can set them equal to each other, and then solve this equation for y in terms of x.

$$\frac{8}{5}x = -\frac{6}{5}y$$

$$8x = -6y$$

$$y = -\frac{8}{6}x$$

$$y = -\frac{4}{3}x$$

Plug this value for y back into the constraint equation.

$$-10y + 5x = 25$$

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$$-10\left(-\frac{4}{3}x\right)+5x=25$$

$$\frac{40}{3}x+5x=25$$

$$40x+15x=75$$

$$55x=75$$

$$x=\frac{75}{55}$$

$$x=\frac{15}{11}$$

Now plug this back into the constraint equation to solve for y.

$$-10y+5x=25$$

$$-10y+5\left(\frac{15}{11}\right)=25$$

$$-110y+75=275$$

$$y=-\frac{200}{110}$$

$$y=-\frac{20}{11}$$

Putting these values for x and y together, the critical point is

$$\left(\frac{15}{11},-\frac{20}{11}\right)$$

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To say whether this critical point is a maximum or minimum, we'll find second-order partial derivatives of $f(x, y)$.

$$\frac{\partial^2 f}{\partial x^2} = 8$$

$$\frac{\partial^2 f}{\partial y^2} = 12$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Then we'll plug these into the formula for D.

$$D(x, y, \lambda) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y, \lambda) = (8)(12) - (0)^2$$

$$D(x, y, \lambda) = 96$$

At this point, if we still had variables remaining on the right side of D, we'd evaluate D at the critical point. In this case, since there are no variables, plugging in the critical point won't change the value.

$$D\left(\frac{15}{11}, -\frac{20}{11}, \lambda\right) = 96$$

These are the rules for D:

If $D < 0$, then the critical point is a saddle point

If $D = 0$, then the second derivative test is inconclusive

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If $D > 0$,

and $\frac{\partial^2 f}{\partial x^2} > 0$, then the critical point is a local minimum

and $\frac{\partial^2 f}{\partial x^2} < 0$, then the critical point is a local maximum

In this problem, $D > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$, so the critical point is a local minimum.

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Topic: Two dimensions, one constraint

Question: What is the maximum area of the land defined by $f(x, y) = 12xy$, subject to the given constraint?

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

Answer choices:

A 392

B 388

C 380

D 360

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Solution: D

Let

$$g(x, y) = \frac{x^2}{25} + \frac{y^2}{144} - 1$$

Take partial derivatives of the given function $f(x, y) = 12xy$, and set them equal to the corresponding partial derivatives of the constraint equation $g(x, y)$. Don't forget to multiply by λ. Therefore, the equations we're building are

$$f_x(x, y) = \lambda g_x(x, y)$$

$$f_y(x, y) = \lambda g_y(x, y)$$

and we get

$$12y = \frac{2\lambda}{25}x$$

$$12x = \frac{2\lambda}{144}y$$

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

We'll solve the first equation for λ.

$$12y = \frac{2\lambda}{25}x$$

$$\lambda = \frac{150y}{x}$$

Plug this value into the other equation.

$$12x = \frac{2\left(\frac{150y}{x}\right)}{144}y$$

$$x^2 = \left(\frac{25}{144}\right)y^2$$

Plug this value into the constraint equation for x^2.

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

$$\frac{1}{25}\left[\left(\frac{25}{144}\right)y^2\right] + \frac{1}{144}y^2 = 1$$

$$\frac{1}{144}y^2 + \frac{1}{144}y^2 = 1$$

$$y^2 + y^2 = 144$$

$$2y^2 = 144$$

$$y^2 = 72$$

$$y = \pm 6\sqrt{2}$$

Plug these values into the constraint equation to find corresponding values for x.

$$\frac{x^2}{25} + \frac{y^2}{144} = 1$$

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$$\frac{x^2}{25} + \frac{\left(\pm 6\sqrt{2}\right)^2}{144} = 1$$

$$\frac{x^2}{25} + \frac{36(2)}{144} = 1$$

$$\frac{x^2}{25} + \frac{1}{2} = 1$$

$$2x^2 + 25 = 50$$

$$x^2 = \frac{25}{2}$$

$$x = \frac{5\sqrt{2}}{2}$$

Plugging this point into the original function gives

$$f(x, y) = 12xy$$

$$f\left(6\sqrt{2}, \frac{5\sqrt{2}}{2}\right) = 12\left(6\sqrt{2}\right)\left(\frac{5\sqrt{2}}{2}\right)$$

$$f\left(6\sqrt{2}, \frac{5\sqrt{2}}{2}\right) = 12(6)(5)$$

$$f\left(6\sqrt{2}, \frac{5\sqrt{2}}{2}\right) = 360$$

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