Calculus 3 Workbook Solutions

Stokes’ and divergence theorem

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STOKES' THEOREM

1. Use Stokes' theorem to evaluate the surface integral where S is the part of the elliptic paraboloid $z+x^2+y^2-3=0$ above the plane $z=-1$. Assume that S has a positive orientation.

$$\iint_S \text{curl } \overrightarrow{F} \cdot d\overrightarrow{S}$$

$$\overrightarrow{F} = \langle y+2, -z^2, 2xy \rangle$$

Solution:

In order to find C, the curve of intersection of the elliptic paraboloid $z+x^2+y^2-3=0$ and the plane $z=-1$, plug $z=-1$ into the equation of the paraboloid.

$$(-1)+x^2+y^2-3=0$$

$$x^2+y^2=4$$

So C is the circle that lies in the plane $z=-1$, centered at $(0,0,-1)$ with radius 2.

Since the surface S is positively oriented, the normal vectors point outward from the surface, and therefore, by the right-hand rule, the circle C has counterclockwise direction. Therefore, its parametrization is

$$x(t)=2\cos t$$

$$y(t) = 2\sin t$$

$$z(t) = -1$$

Which means the vector function is $\vec{r}(t) = \langle 2\cos t, 2\sin t, -1 \rangle$, and we can take the derivative of $\vec{r}$ to get $\vec{r}'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$. The function is

$$\vec{F}(t) = \langle 2\sin t + 2, -0^2, 2(2\cos t)(2\sin t) \rangle$$

$$\vec{F}(t) = \langle 2\sin t + 2, 0, 8\cos t \sin t \rangle$$

So the line integral is

$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\ dt = \int_0^{2\pi} \langle 2\sin t + 2, 0, 8\cos t \sin t \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle\ dt$$

The integral on the right side of the equation simplifies to

$$\int_0^{2\pi} (2\sin t + 2) \cdot (-2\sin t) + 0 \cdot 2\cos t + 8\cos t \sin t \cdot 0\ dt$$

$$-2\int_0^{2\pi} 2\sin^2 t + 2\sin t\ dt$$

$$-2\int_0^{2\pi} 1 - \cos 2t + 2\sin t\ dt$$

Since the integral of sine and cosine functions over a 2π-period is 0,

$$-2\int_0^{2\pi} 1\ dt$$

$$-2 \cdot (2\pi - 0) = -4\pi$$

■ 2. Use Stokes' theorem to evaluate the line integral, where C is the rectangle $KMNO$ with vertices $K(0,0,0)$, $M(0,6,0)$, $N(3,6,0)$ and $O(3,0,0)$. Assume that C has a clockwise orientation as viewed from the positive z-axis.

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = \langle 2xyz, x^2 + y^2, 2xyz \rangle$$

Solution:

The parametrization of the points inside the rectangle $KMNO$ is

$$x(u, v) = u$$

$$y(u, v) = v$$

$$z(u, v) = 0$$

So the vector function is $\vec{r}(u, v) = \langle u, v, 0 \rangle$. Take the partial derivatives of $\vec{r}$.

$$\vec{r}_u(t) = \langle 1,0,0 \rangle$$

$$\vec{r}_v(t) = \langle 0,1,0 \rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0 \cdot 0 - 0 \cdot 1, \ -1 \cdot 0 + 0 \cdot 0, 1 \cdot 1 - 0 \cdot 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 0,0,1 \rangle$$

Since the normal vector of the surface must point in the negative direction of the z-axis, $\vec{n} = \langle 0,0,-1 \rangle$. Evaluate curl $\vec{F}$. The curl of a vector field in three dimensions is given by

$$\text{curl}\ \vec{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

So

$$\text{curl}\ \vec{F} = \text{curl}\ \langle 2xyz, x^2 + y^2, 2xyz \rangle$$

$$\text{curl}\ \vec{F} = \langle 2xz - 0, 2xy - 2yz, 2x - 2xz \rangle$$

$$\text{curl}\ \vec{F} = \langle 2xz, 2xy - 2yz, 2x - 2xz \rangle$$

Plug in the parametrization $\vec{r}(u, v) = \langle u, v, 0 \rangle$.

$$\langle 2u \cdot 0, 2uv - 2v \cdot 0, 2u - 2u \cdot 0 \rangle$$

$$\langle 0, 2uv, 2u \rangle$$

So the surface integral is

$$\iint_S \text{curl}\ \vec{F} \cdot \vec{n}\ dA = \int_0^3 \int_0^6 \langle 0, 2uv, 2u \rangle \cdot \langle 0,0,-1 \rangle\ dv\ du$$

Simplify the integral on the right side of the equation.

$$\int_0^3 \int_0^6 0 \cdot 0 + 2uv \cdot 0 + 2u \cdot (-1)\ dv\ du$$

$$\int_0^3 \int_0^6 -2u \ dv \ du$$

Integrate with respect to v, treating u as a constant.

$$\int_0^3 -2u(6-0) \ du$$

$$\int_0^3 -12u \ du$$

Integrate with respect to u.

$$-6u^2 \Big|_0^3$$

$$-6 \cdot 3^2 - (-6 \cdot 0^2) = -54$$

■ 3. Use Stokes' theorem to evaluate the line integral, where C is the boundary curve of the semicircle centered at the origin with radius 4 that lies in the xz-plane, and with $z \geq 0$. Assume that C has a counterclockwise orientation as viewed from the positive y-axis.

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = \langle x + 3y - z + 2, x - 5y + 9z - 7, -5x - y + 2z + 6 \rangle$$

Solution:

The parametrization of the points inside the semicircle is

$$x(u, v) = v\cos u$$

$$y(u, v) = 0$$

$$z(u, v) = v\sin u$$

The vector form of these parametric equations is

$$\vec{r}(u, v) = \langle v\cos u, 0, v\sin u\rangle$$

Take the partial derivatives of $\vec{r}$.

$$\vec{r}_u(t) = \langle -v\sin u, 0, v\cos u\rangle$$

$$\vec{r}_v(t) = \langle \cos u, 0, \sin u\rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle -v\sin u, 0, v\cos u\rangle \times \langle \cos u, 0, \sin u\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0 \cdot \sin u - v\cos u \cdot 0,\ v\cos u \cdot \cos u + v\sin u \cdot \sin u,\ -v\sin u \cdot 0 - 0 \cdot \cos u\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0, v\cos^2 u + v\sin^2 u, 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0, v, 0\rangle$$

Since the normal vector of the surface must point in the positive direction of the y-axis, and since v is always positive within the semicircle, $\vec{n} = \langle 0, v, 0\rangle$. Evaluate curl $\vec{F}$. The curl of a vector field in three dimensions is given by

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

So

$$\text{curl } \vec{F} = \text{curl } \langle x + 3y - z + 2, x - 5y + 9z - 7, -5x - y + 2z + 6 \rangle$$

$$\text{curl } \vec{F} = \langle -1 - 9, -1 - (-5), 1 - 3 \rangle$$

$$\text{curl } \vec{F} = \langle -10, 4, -2 \rangle$$

So the surface integral is

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \ dA = \int_0^4 \int_0^\pi \langle -10, 4, -2 \rangle \cdot \langle 0, v, 0 \rangle \ du \ dv$$

The integral on the right simplifies to

$$\int_0^4 \int_0^\pi -10 \cdot 0 + 4 \cdot v + (-2) \cdot 0 \ du \ dv$$

$$\int_0^4 \int_0^\pi 4v \ du \ dv$$

Integrate with respect to u, treating v as a constant.

$$\int_0^4 4v(\pi - 0) \ dv$$

$$\int_0^4 4\pi v \ dv$$

Integrate with respect to v.

$$4\pi \cdot \frac{v^2}{2}\Bigg|_0^4$$

$$2\pi v^2\Big|_0^4$$

$$2\pi \cdot 4^2 - 2\pi \cdot 0^2 = 32\pi$$

DIVERGENCE THEOREM

1. Use the Divergence theorem to evaluate the surface integral, where S is the boundary surface of the box $[-3,4] \times [3,5] \times [-3,0]$. Assume that S has a negative orientation.

$$\iint_S \vec{F} \cdot d\vec{S}$$

$$\vec{F} = \langle x + e^{z^2-y^2}, \ln y + y + x^4, z^2 - \arcsin(x+y) \rangle$$

Solution:

The parametrization of the box is given by $x = x$, $y = y$, and $z = z$, where x changes from -3 to 4, y changes from 3 to 5, and z changes from -3 to 0. Evaluate the divergence of $\vec{F}$.

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{div } \vec{F} = 1 + \frac{1}{y} + 1 + 2z$$

$$\text{div } \vec{F} = 2 + \frac{1}{y} + 2z$$

So the triple integral is

$$\iiint_E \text{div } \vec{F} \ dV = \int_{-3}^{0} \int_{3}^{5} \int_{-3}^{4} (2 + \frac{1}{y} + 2z) \ dx \ dy \ dz$$

Integrate with respect to x, treating y and z as constants.

$$\iiint_E \text{div } \vec{F} \ dV = \int_{-3}^{0} \int_{3}^{5} \left(2 + \frac{1}{y} + 2z\right) \cdot (x) \Big|_{-3}^{4} \ dy \ dz$$

Simplify the integral on the right side of this equation.

$$\int_{-3}^{0} \int_{3}^{5} \left(2 + \frac{1}{y} + 2z\right) \cdot (4 - (-3)) \ dy \ dz$$

$$7 \int_{-3}^{0} \int_{3}^{5} 2 + \frac{1}{y} + 2z \ dy \ dz$$

Integrate with respect to y, treating z and as a constant.

$$7 \int_{-3}^{0} 2y + \ln y + 2yz \Big|_{y=3}^{y=5} \ dz$$

$$7 \int_{-3}^{0} 2 \cdot 5 + \ln 5 + 2 \cdot 5 \cdot z - (2 \cdot 3 + \ln 3 + 2 \cdot 3 \cdot z) \ dz$$

$$7 \int_{-3}^{0} 4 + \ln \frac{5}{3} + 4z \ dz$$

Integrate with respect to z.

$$7 \left(4z + \ln \frac{5}{3} \cdot z + 2z^2\right) \Big|_{-3}^{0}$$

$$7\left(4\cdot 0+\ln\frac{5}{3}\cdot 0+2\cdot 0^2\right)-7\left(4\cdot(-3)+\ln\frac{5}{3}\cdot(-3)+2(-3)^2\right)$$

$$-7\left(-12-3\ln\frac{5}{3}+18\right)=-42+21\ln\frac{5}{3}$$

Since the surface has a negative orientation, the sign of the answer needs to be reversed.

$$\iint_S \vec{F}\cdot d\vec{S}=42-21\ln\frac{5}{3}$$

■ 2. Use the Divergence theorem to evaluate the surface integral where S is the boundary surface of the part of the cylinder $y^2+z^2=25$ with $-2\le x\le 4$. Assume that S has a positive orientation.

$$\iint_S \vec{F}\cdot d\vec{S}$$

$$\vec{F}=\langle x^3+y^3, y^3+z^3, z^3+x^3\rangle$$

Solution:

To parametrize the region inside the cylinder, use standard cylindrical coordinates for the cylinder with radius 5 and axis that coincides with the x-axis.

$$x=x$$

$$y = r\cos\phi$$

$$z = r\sin\phi$$

Evaluate the divergence of $\vec{F}$.

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{div } \vec{F} = 3x^2 + 3y^2 + 3z^2$$

Plug in the parametrization for x, y, and z.

$$\text{div } \vec{F}(x, r, \phi) = 3x^2 + 3(r\cos\phi)^2 + 3(r\sin\phi)^2$$

$$\text{div } \vec{F}(x, r, \phi) = 3x^2 + 3r^2\cos^2\phi + 3r^2\sin^2\phi$$

$$\text{div } \vec{F}(x, r, \phi) = 3x^2 + 3r^2$$

So the triple integral is

$$\iiint_E \text{div } \vec{F}\ dV = \int_{-2}^{4}\int_0^5\int_0^{2\pi} 3x^2 + 3r^2\ d\phi\ dr\ dx$$

Integrate with respect to ϕ treating r and x as a constants.

$$\int_{-2}^{4}\int_0^5 (3x^2 + 3r^2)\cdot(2\pi - 0)\ dr\ dx$$

$$2\pi\int_{-2}^{4}\int_0^5 3x^2 + 3r^2\ dr\ dx$$

Integrate with respect to r, treating x as a constant.

$$2\pi \int_{-2}^{4} 3x^2r + r^3 \Big|_0^5 \ dx$$

$$2\pi \int_{-2}^{4} 3x^2 \cdot 5 + 5^3 - (3x^2 \cdot 0 + 0^3) \ dx$$

$$2\pi \int_{-2}^{4} 15x^2 + 125 \ dx$$

Integral with respect to x.

$$2\pi(5x^3 + 125x)\Big|_{-2}^{4}$$

$$2\pi(5 \cdot 4^3 + 125 \cdot 4) - 2\pi(5 \cdot (-2)^3 + 125 \cdot (-2)) = 2{,}220\pi$$

Since the surface has a positive orientation, the sign of the answer is correct.

■ 3. Use the Divergence theorem to evaluate the triple integral where E is the sphere centered at the origin with radius 4.

$$\iiint_E \text{div } \vec{F} \ dV$$

$$\vec{F} = \left\langle \frac{x^2 + y^2 + z^2}{4}, -6y, 6 \right\rangle$$

Solution:

To parametrize the sphere with radius ρ, use a parametrization in spherical coordinates.

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

Plug in $\rho = 4$ and rename parameters the $\phi \to u$ and $\theta \to v$.

$$x(u, v) = 4 \sin u \cos v$$

$$y(u, v) = 4 \sin u \sin v$$

$$z(u, v) = 4 \cos u$$

In vector form, these equations are

$$\vec{r} = \langle 4 \sin u \cos v, 4 \sin u \sin v, 4 \cos u \rangle$$

Take partial derivatives of $\vec{r}$.

$$\vec{r_u} = \langle 4 \cos u \cos v, 4 \cos u \sin v, -4 \sin u \rangle$$

$$\vec{r_v} = \langle -4 \sin u \sin v, 4 \sin u \cos v, 0 \rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 4 \cos u \cos v, 4 \cos u \sin v, -4 \sin u \rangle \times \langle -4 \sin u \sin v, 4 \sin u \cos v, 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 4 \cos u \sin v \cdot 0 - (-4 \sin u) \cdot 4 \sin u \cos v,$$

$$(-4\sin u)\cdot(-4\sin u\sin v)-4\cos u\cos v\cdot 0,$$

$$4\cos u\cos v\cdot 4\sin u\cos v-4\cos u\sin v\cdot(-4\sin u\sin v))\rangle$$

$$\vec{r_u}\times\vec{r_v}=\langle 16\sin^2 u\cos v, 16\sin^2 u\sin v, 8\sin 2u\rangle$$

Since the surface has positive orientation, the normal vector must point outward from the sphere, so the sign of the cross product is correct.

The function is

$$\vec{F}(u,v)=\left\langle \frac{x^2(u,v)+y^2(u,v)+z^2(u,v)}{4}, -6y(u,v), 6\right\rangle$$

Since $x^2+y^2+z^2=\rho^2=4^2=16$,

$$\vec{F}(u,v)=\left\langle \frac{16}{4}, -6\cdot 4\sin u\sin v, 6\right\rangle$$

$$\vec{F}(u,v)=\langle 4, -24\sin u\sin v, 6\rangle$$

So the surface integral is

$$\int_0^{\pi}\int_0^{2\pi}\langle 4, -24\sin u\sin v, 6\rangle\cdot\langle 16\sin^2 u\cos v, 16\sin^2 u\sin v, 8\sin 2u\rangle\ dv\ du$$

$$\int_0^{\pi}\int_0^{2\pi} 4\cdot 16\sin^2 u\cos v-24\sin u\sin v\cdot 16\sin^2 u\sin v+6\cdot 8\sin 2u\ dv\ du$$

$$16\int_0^{\pi}\int_0^{2\pi} 4\sin^2 u\cos v-24\sin^3 u\sin^2 v+3\sin 2u\ dv\ du$$

Integrate with respect to v, treating u as a constant. Remember that the integral of cosine functions over a 2π-period is 0.

$$16\int_0^{\pi}\int_0^{2\pi} -24\sin^3 u\sin^2 v + 3\sin 2u \ dv \ du$$

$$48\int_0^{\pi}\int_0^{2\pi} -4\sin^3 u\cdot(1-\cos 2v) + \sin 2u \ dv \ du$$

$$48\int_0^{\pi}\int_0^{2\pi} -4\sin^3 u + 4\sin^3 u\cos 2v + \sin 2u \ dv \ du$$

$$48\int_0^{\pi}\int_0^{2\pi} -4\sin^3 u + \sin 2u \ dv \ du$$

$$48\int_0^{\pi} -4\sin^3 u + \sin 2u \ du\cdot\int_0^{2\pi} \ dv$$

$$48\int_0^{\pi} -4\sin^3 u + \sin 2u \ du\cdot 2\pi$$

Integrate with respect to u, using the trigonometric identity

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$$

The integral becomes

$$96\pi\int_0^{\pi} -3\sin u + \sin 3u + \sin 2u \ du$$

$$96\pi\left(3\cos u - \frac{1}{3}\cos 3u - \frac{1}{2}\cos 2u\right)\Bigg|_0^{\pi}$$

$$96\pi\left(3\cos\pi-\frac{1}{3}\cos 3\pi-\frac{1}{2}\cos 2\pi\right)-96\pi\left(3\cos 0-\frac{1}{3}\cos 0-\frac{1}{2}\cos 0\right)$$

$$96\pi\left(-3+\frac{1}{3}-\frac{1}{2}\right)-96\pi\left(3-\frac{1}{3}-\frac{1}{2}\right)=-512\pi$$

W W W . K R I S T A K I N G M A T H . C O M