Type I and type II regions

We already know that we can use double integrals to find the volume below a surface over some region $R = [a, b] \times [c, d]$.

We can define the region R as Type I, Type II, or a mix of both. Type I curves are curves that can be defined for y in terms of x and lie more or less "above and below" each other. On the other hand, Type II curves are curves that can be defined for x in terms of y and lie more or less "left and right" of each other.

Type I regions can be broken up into vertical slices, and Type II regions can be broken up into horizontal slices.

Sometimes a region can be considered both Type I and Type II, in which case you can choose to evaluate it either way.

Example

Define the region D as Type I or Type II, then find volume below the surface over the region D.

$$\int\int_D x^2 + 6y - 20 \ dA$$

where D is the triangle bounded by $y = 1$, $y = 3x$, and $y = 4 - x$

The first thing we'll do is sketch the region D. It'll be easy if we solve for the intersection points of the three lines.

We'll find the intersection of $y = 1$ and $y = 3x$.

$$3x = 1$$

$$x = \frac{1}{3}$$

Pairing $x = \frac{1}{3}$ with $y = 1$, the intersection point is $\left(\frac{1}{3},1\right)$.

We'll find the intersection of $y = 1$ and $y = 4 - x$.

$$4 - x = 1$$

$$-x = -3$$

$$x = 3$$

Pairing $x = 3$ with $y = 1$, the intersection point is $(3,1)$.

We'll find the intersection of $y = 3x$ and $y = 4 - x$.

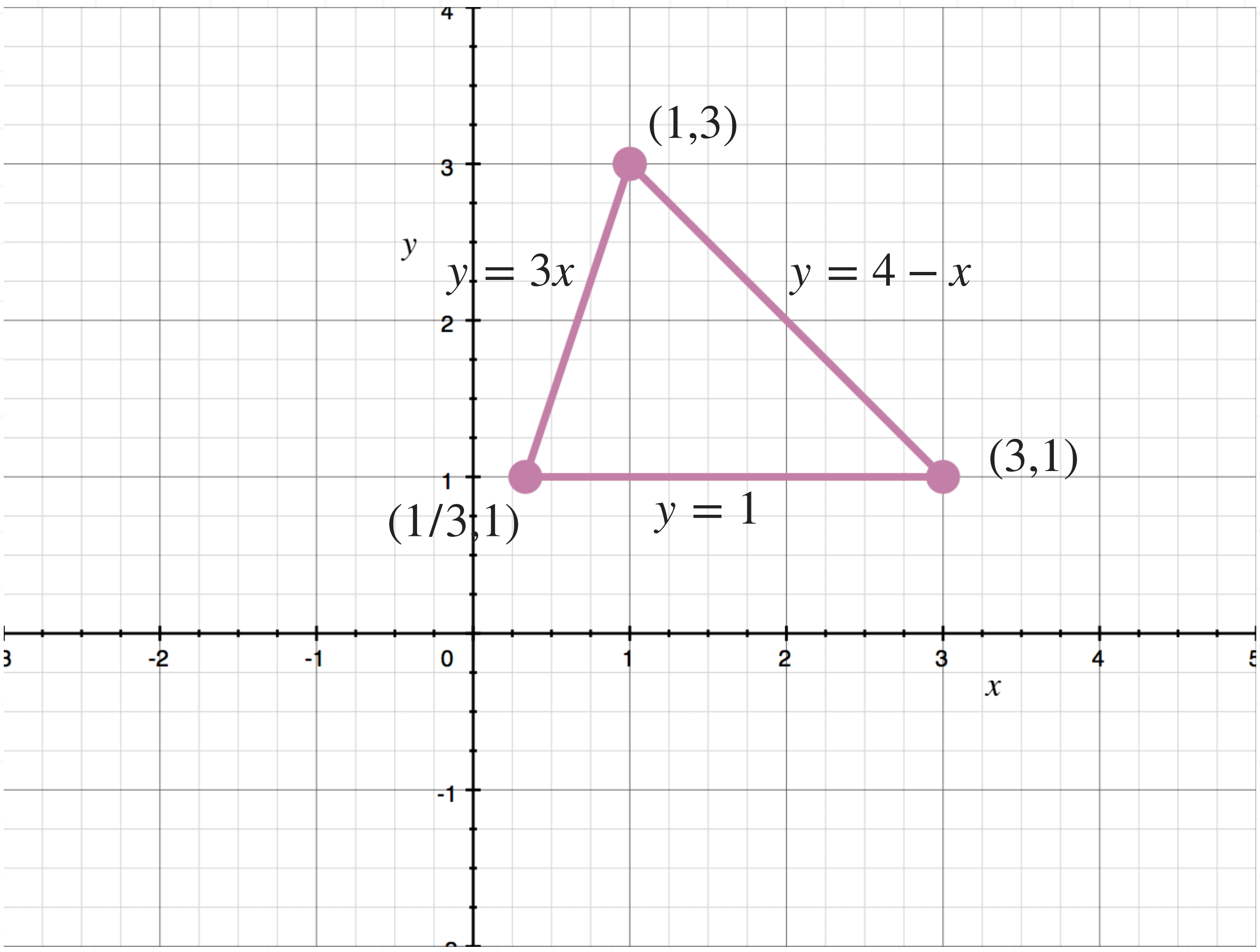
$$3x = 4 - x$$

$$4x = 4$$

$$x = 1$$

Plugging $x = 1$ into $y = 3x$ to see that $y = 3$, the intersection point is $(1,3)$.

If we plot these points and sketch the lines that connect them, we see the triangular region D.



Now we need to decide if D is a Type I or Type II region. Since we can get uniform slices either way (vertical slices if we treat it as a Type I region, or horizontal slices if we treat it as a Type II region), we can choose which type we want to use, and we'll get the same answer with both methods. We'll solve it as a Type II region, which means we'll use horizontal slices of region D.

To solve as Type II, the equations that define the lines that are the edges of region D need to be defined for x in terms of y. If we solve each of them for x we get

$$y = 3x$$

$$x = \frac{1}{3}y$$

and

$$y = 4 - x$$

$$x = 4 - y$$

For a Type II region, we'll integrate first with respect to x, then with respect to y. The upper limit of integration for x is given by $x = 4 - y$ (because that line defines the top of each of our horizontal slices), and the lower limit of integration for x is given by $x = (1/3)y$ (because that line defines the bottom of each of our horizontal slices). For the region D, y is defined between $y = 1$ and $y = 3$ ($y = 3$ comes from the intersection point $(1,3)$). So the original integral becomes

$$\iint_D x^2 + 6y - 20 \ dA$$

$$\int_1^3 \int_{\frac{y}{3}}^{4-y} x^2 + 6y - 20 \ dx \ dy$$

Evaluating from the inside out and therefore integrating with respect to x first, we get

$$\int_1^3 \frac{1}{3}x^3 + 6xy - 20x \Big|_{x=\frac{y}{3}}^{x=4-y} \ dy$$

$$\int_1^3 \frac{1}{3}(4-y)^3 + 6(4-y)y - 20(4-y) - \left[\frac{1}{3}\left(\frac{y}{3}\right)^3 + 6\left(\frac{y}{3}\right)y - 20\left(\frac{y}{3}\right)\right] \ dy$$

$$\int_1^3 \frac{1}{3}(4-y)(4-y)(4-y) + 6y(4-y) - 80 + 20y - \left[\frac{1}{3}\left(\frac{y^3}{27}\right) + 6y\left(\frac{y}{3}\right) - \frac{20y}{3}\right] \ dy$$

$$\int_1^3 \frac{1}{3}\left(16-8y+y^2\right)(4-y)+24y-6y^2-80+20y-\left(\frac{y^3}{81}+\frac{6y^2}{3}-\frac{20y}{3}\right)\ dy$$

$$\int_1^3 \frac{1}{3}\left(64-16y-32y+8y^2+4y^2-y^3\right)+24y-6y^2-80+20y-\frac{1}{81}y^3-2y^2+\frac{20}{3}y\ dy$$

$$\int_1^3 \frac{1}{3}\left(64-48y+12y^2-y^3\right)-\frac{1}{81}y^3-8y^2+44y+\frac{20}{3}y-80\ dy$$

$$\int_1^3 \frac{64}{3}-16y+4y^2-\frac{1}{3}y^3-\frac{1}{81}y^3-8y^2+\frac{132}{3}y+\frac{20}{3}y-80\ dy$$

$$\int_1^3 -\frac{1}{3}y^3-\frac{1}{81}y^3+4y^2-8y^2-16y+\frac{132}{3}y+\frac{20}{3}y+\frac{64}{3}-80\ dy$$

$$\int_1^3 -\frac{27}{81}y^3-\frac{1}{81}y^3-4y^2-\frac{48}{3}y+\frac{132}{3}y+\frac{20}{3}y+\frac{64}{3}-\frac{240}{3}\ dy$$

$$\int_1^3 -\frac{28}{81}y^3-4y^2+\frac{104}{3}y-\frac{176}{3}\ dy$$

Integrating with respect to y, we get

$$-\frac{28}{81}\left(\frac{1}{4}\right)y^4-4\left(\frac{1}{3}\right)y^3+\frac{104}{3}\left(\frac{1}{2}\right)y^2-\frac{176}{3}y\Bigg|_1^3$$

$$-\frac{28}{324}y^4-\frac{4}{3}y^3+\frac{104}{6}y^2-\frac{176}{3}y\Bigg|_1^3$$

$$-\frac{7}{81}y^4-\frac{4}{3}y^3+\frac{52}{3}y^2-\frac{176}{3}y\Bigg|_1^3$$

$$-\frac{7}{81}(3)^4-\frac{4}{3}(3)^3+\frac{52}{3}(3)^2-\frac{176}{3}(3)-\left[-\frac{7}{81}(1)^4-\frac{4}{3}(1)^3+\frac{52}{3}(1)^2-\frac{176}{3}(1)\right]$$

$$-\frac{7}{81}(81)-\frac{4}{3}(27)+\frac{52}{3}(9)-\frac{176}{3}(3)+\frac{7}{81}+\frac{4}{3}-\frac{52}{3}+\frac{176}{3}$$

$$-\frac{7}{1}(1)-\frac{4}{1}(9)+\frac{52}{1}(3)-\frac{176}{1}(1)+\frac{7}{81}+\frac{4}{3}-\frac{52}{3}+\frac{176}{3}$$

$$-7-36+156-176+\frac{7}{81}+\frac{128}{3}$$

$$-63+\frac{7}{81}+\frac{128}{3}$$

$$-\frac{5,103}{81}+\frac{7}{81}+\frac{3,456}{81}$$

$$-\frac{1,640}{81}$$

We can say that the volume under the curve $z=x^2+6y-20$ over the region D is

$$-\frac{1,640}{81}$$

The fact that the volume is negative means that there is more volume enclosed by the curve and the xy-plane that lies below the xy-plane than the amount that lies above the xy-plane.