Topic: Second derivative test

Question: Find the critical points of the function.

$$f(x,y) = 2x^2 + 3xy^2 + y^2 - 2$$

Answer choices:

A $(0,0)$ $\left(-\frac{1}{3},\frac{2}{3}\right)$ $\left(\frac{1}{3},-\frac{2}{3}\right)$

B $(0,0)$ $\left(-\frac{1}{3},\frac{2}{3}\right)$ $\left(-\frac{1}{3},-\frac{2}{3}\right)$

C $(0,0)$ $\left(-\frac{1}{3},\frac{2}{3}\right)$ $\left(\frac{1}{3},\frac{2}{3}\right)$

D $(0,0)$ $\left(-\frac{1}{3},\frac{2}{3}\right)$ $\left(-\frac{1}{3},\frac{2}{3}\right)$

Solution: B

To find the critical points of the multivariable function, we'll set the first order partial derivatives equal to 0.

$$\frac{\partial f}{\partial x} = 4x + 3y^2$$

$$4x + 3y^2 = 0$$

$$4x = -3y^2$$

$$x = -\frac{3}{4}y^2$$

and

$$\frac{\partial f}{\partial y} = 6xy + 2y$$

$$6xy + 2y = 0$$

Plugging this value for x into $6xy + 2y = 0$ gives

$$6\left(-\frac{3}{4}y^2\right)y + 2y = 0$$

$$-\frac{9}{2}y^3 + 2y = 0$$

$$y\left(-\frac{9}{2}y^2 + 2\right) = 0$$

$$y = 0$$

and

$$-\frac{9}{2}y^2+2=0$$

$$-\frac{9}{2}y^2=-2$$

$$y^2=\frac{4}{9}$$

$$y=\pm\frac{2}{3}$$

So we know that $y=0$, $y=2/3$ and $y=-2/3$ are all critical points. Now we just need to solve for their associated x values.

For $y=0$,

$$x=-\frac{3}{4}(0)^2$$

$$x=0$$

The first critical point is $(0,0)$.

For $y=\frac{2}{3}$,

$$x=-\frac{3}{4}\left(\frac{2}{3}\right)^2$$

$$x=-\frac{3}{4}\left(\frac{4}{9}\right)$$

$$x = -\frac{1}{3}$$

The second critical point is $\left(-\frac{1}{3}, \frac{2}{3}\right)$.

For $y = -\frac{2}{3}$,

$$x = -\frac{3}{4}\left(-\frac{2}{3}\right)^2$$

$$x = -\frac{3}{4}\left(\frac{4}{9}\right)$$

$$x = -\frac{1}{3}$$

The third critical point is $\left(-\frac{1}{3}, -\frac{2}{3}\right)$.

Topic: Second derivative test

Question: Evaluate the critical points of the function.

$$f(x, y) = 4x^2 - 2xy + 2y^2 - 18$$

Answer choices:

A Local maximum at $(0,0)$

B Saddle point at $(0,0)$

C Local minimum at $(0,0)$

D The test is inconclusive

Solution: C

Our first step is to find critical points. We'll find the first order partial derivatives of the function, set them equal to 0, and then solve the resulting system of equations for critical points.

$$\frac{\partial f}{\partial x} = 8x - 2y$$

$$8x - 2y = 0$$

$$-2y = -8x$$

$$y = 4x$$

and

$$\frac{\partial f}{\partial y} = -2x + 4y$$

$$-2x + 4y = 0$$

Plugging this value for y into $-2x + 4y = 0$ gives

$$-2x + 4(4x) = 0$$

$$-2x + 16x = 0$$

$$14x = 0$$

$$x = 0$$

So we know that $x = 0$ is a critical point. Now we just need to solve for its associated y value.

For $x = 0$,

$$y = 4(0)$$

$$y = 0$$

The critical point is $(0,0)$.

Next we'll use the second derivative test to evaluate the critical point, which means we'll need to find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 8$$

$$\frac{\partial^2 f}{\partial y^2} = 4$$

$$\frac{\partial^2 f}{\partial x \partial y} = -2$$

Plugging the second derivatives into the second derivative test gives

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (8)(4) - (-2)^2$$

$$D(x, y) = 32 - 4$$

$$D(x, y) = 28$$

Now we'll evaluate at the critical point $(0,0)$.

$$D(0,0) = 28$$

The second derivative test tells us that,

if $D(x, y) < 0$	then the critical point is a saddle point
if $D(x, y) = 0$	then the second derivative test is inconclusive
if $D(x, y) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$	then the critical point is a local minimum
if $D(x, y) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$	then the critical point is a local maximum

Since

$$D(0,0) = 28 > 0$$

and

$$\frac{\partial^2 f}{\partial x^2}(0,0) = 8 > 0$$

then $(0,0)$ is a local minimum.

Topic: Second derivative test

Question: Evaluate the critical points of the function.

$$f(x, y) = 8x^3 - xy + 2y^3 - 4$$

Answer choices:

A Saddle point at $(0,0)$ Local minimum at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$

B Saddle point at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$ Local minimum at $(0,0)$

C Saddle point at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$ Local maximum at $(0,0)$

D Saddle point at $(0,0)$ Local maximum at $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$

Solution: A

The first step is to find critical points. We'll find the first order partial derivatives of the function, set them equal to 0, and then solve the resulting system of equations for critical points.

$$\frac{\partial f}{\partial x} = 24x^2 - y$$

$$24x^2 - y = 0$$

$$-y = -24x^2$$

$$y = 24x^2$$

and

$$\frac{\partial f}{\partial y} = -x + 6y^2$$

$$-x + 6y^2 = 0$$

Plugging this value for y into $-x + 6y^2 = 0$ gives

$$-x + 6\left(24x^2\right)^2 = 0$$

$$-x + 3{,}456x^4 = 0$$

$$x\left(-1 + 3{,}456x^3\right) = 0$$

$$x = 0$$

and

$$-1 + 3{,}456x^3 = 0$$

$$3{,}456x^3 = 1$$

$$x^3 = \frac{1}{3{,}456}$$

$$x = \frac{1}{12\sqrt[3]{2}}$$

So we know that $x = 0$ and $x = 1/12\sqrt[3]{2}$ are critical points. Now we just need to solve for their associated y values.

For $x = 0$,

$$y = 24(0)^2$$

$$y = 0$$

The first critical point is $(0,0)$.

For $x = \frac{1}{12\sqrt[3]{2}}$,

$$y = 24\left(\frac{1}{12\sqrt[3]{2}}\right)^2$$

$$y = \frac{24}{144\sqrt[3]{4}}$$

$$y = \frac{1}{6\sqrt[3]{4}}$$

The second critical point is $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$.

Next we'll use the second derivative test to evaluate the critical point, which means we'll need to find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 48x$$

$$\frac{\partial^2 f}{\partial y^2} = 12y$$

$$\frac{\partial^2 f}{\partial x \partial y} = -1$$

Plugging the second derivatives into the second derivative test gives

$$D(x, y) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y) = (48x)(12y) - (-1)^2$$

$$D(x, y) = 576xy - 1$$

Now we'll evaluate at both critical points.

For $(0,0)$,

$$D(0,0) = 576(0)(0) - 1$$

$$D(0,0) = -1$$

For $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$,

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 576\left(\frac{1}{12\sqrt[3]{2}}\right)\left(\frac{1}{6\sqrt[3]{4}}\right) - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{576}{72\sqrt[3]{2}\sqrt[3]{4}} - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{8}{\sqrt[3]{8}} - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{8}{2} - 1$$

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 3$$

The second derivative test tells us that,

if $D(x,y) < 0$ then the critical point is a saddle point

if $D(x,y) = 0$ then the second derivative test is inconclusive

if $D(x,y) > 0$ and $\frac{\partial^2 f}{\partial x^2} > 0$ then the critical point is a local minimum

if $D(x,y) > 0$ and $\frac{\partial^2 f}{\partial x^2} < 0$ then the critical point is a local maximum

Since

$$D(0,0) = -1 < 0$$

then $(0,0)$ is a saddle point.

Since

$$D\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 3 > 0$$

and

$$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = 48\left(\frac{1}{12\sqrt[3]{2}}\right)$$

$$\frac{\partial^2 f}{\partial x^2}\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right) = \frac{4}{\sqrt[3]{2}} > 0$$

then $\left(\frac{1}{12\sqrt[3]{2}}, \frac{1}{6\sqrt[3]{4}}\right)$ is a local minimum.