Calculus 3 Workbook Solutions

Triple integrals in spherical coordinates

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SPHERICAL COORDINATES

■ 1. Evaluate the triple integral given in the spherical coordinates, where $f(\rho, \theta, \phi) = 2\rho \sin\theta \cos\phi$.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{3}} \int_{1}^{5} f(\rho, \theta, \phi)\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

Solution:

Set up the integral.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{3}} \int_{1}^{5} (2\rho \sin\theta \cos\phi)\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{3}} \int_{1}^{5} 2\rho^3 \sin\theta \cos\phi \sin\phi\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{3}} \frac{1}{2}\rho^4 \sin\theta \cos\phi \sin\phi \Big|_{\rho=1}^{\rho=5}\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_{0}^{\frac{\pi}{3}} \frac{1}{2}(5)^4 \sin\theta \cos\phi \sin\phi - \frac{1}{2}(1)^4 \sin\theta \cos\phi \sin\phi\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{3}}\frac{625}{2}\sin\theta\cos\phi\sin\phi-\frac{1}{2}\sin\theta\cos\phi\sin\phi\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{3}}312\sin\theta\cos\phi\sin\phi\ d\theta\ d\phi$$

Integrate with respect to θ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}-312\cos\theta\cos\phi\sin\phi\Big|_{\theta=0}^{\theta=\frac{\pi}{3}}\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}-312\cos\frac{\pi}{3}\cos\phi\sin\phi+312\cos(0)\cos\phi\sin\phi\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}-156\cos\phi\sin\phi+312\cos\phi\sin\phi\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}156\cos\phi\sin\phi\ d\phi$$

Integrate with respect to ϕ, using a substitution with $u=\sin\phi$ and $d\phi=du/\cos\phi$, where u changes from $\sqrt{2}/2$ to 1.

$$\int_{\frac{\sqrt{2}}{2}}^{1}156u\cos\phi\left(\frac{du}{\cos\phi}\right)$$

$$\int_{\frac{\sqrt{2}}{2}}^{1}156u\ du$$

$$78u^2\Big|_{\frac{\sqrt{2}}{2}}^{1}$$

$$78(1)^2 - 78\left(\frac{\sqrt{2}}{2}\right)^2$$

$$78 - 78\left(\frac{1}{2}\right)$$

$$78 - 39$$

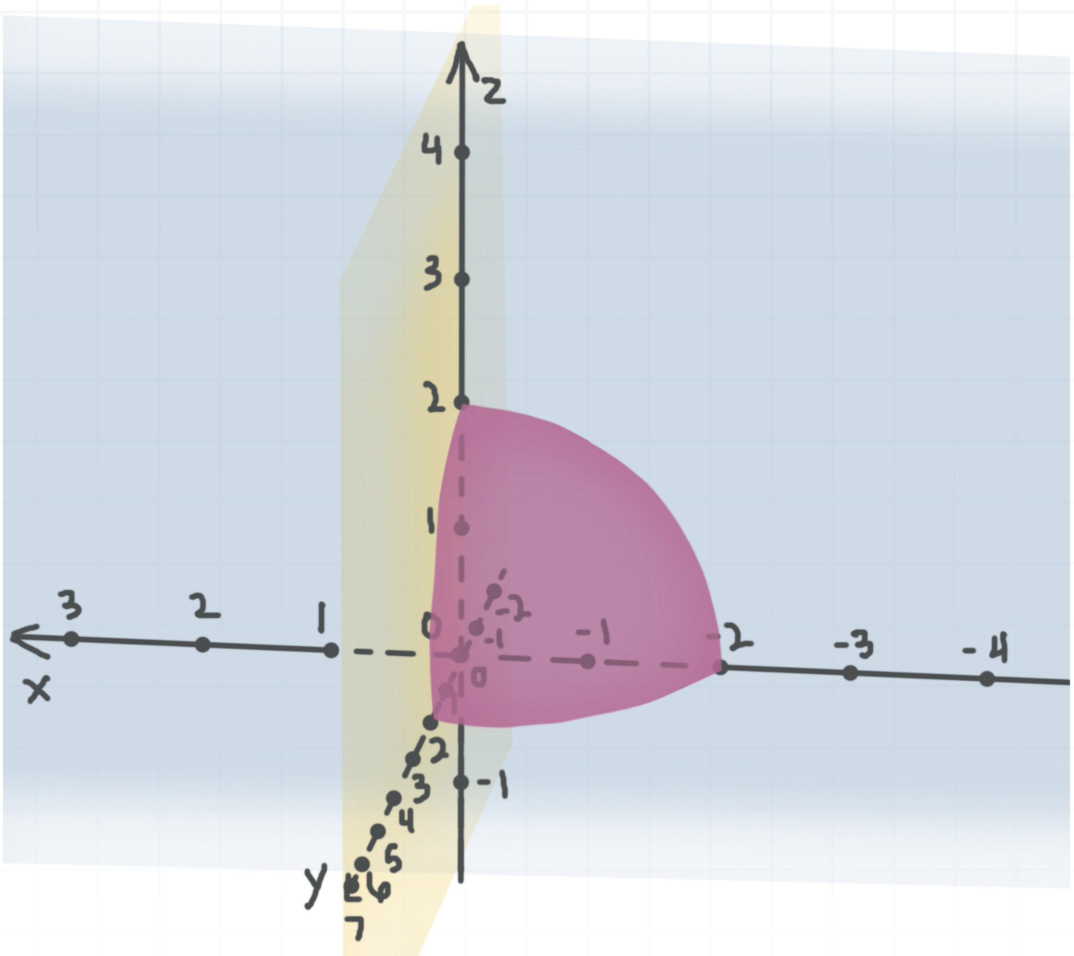
$$39$$

■ 2. Identify the solid given by the iterated integral in spherical coordinates.

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi}\int_0^2 f(\rho,\theta,\phi)\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

Solution:

The value of ρ changes from 0 to 2. So the solid is the part of the sphere with center at the origin and radius 2. The value of θ changes from $\pi/2$ to π and ϕ changes from 0 to $\pi/2$. Therefore, we consider the part of the sphere that lies in the second octant ($x \leq 0$, $y \geq 0$).

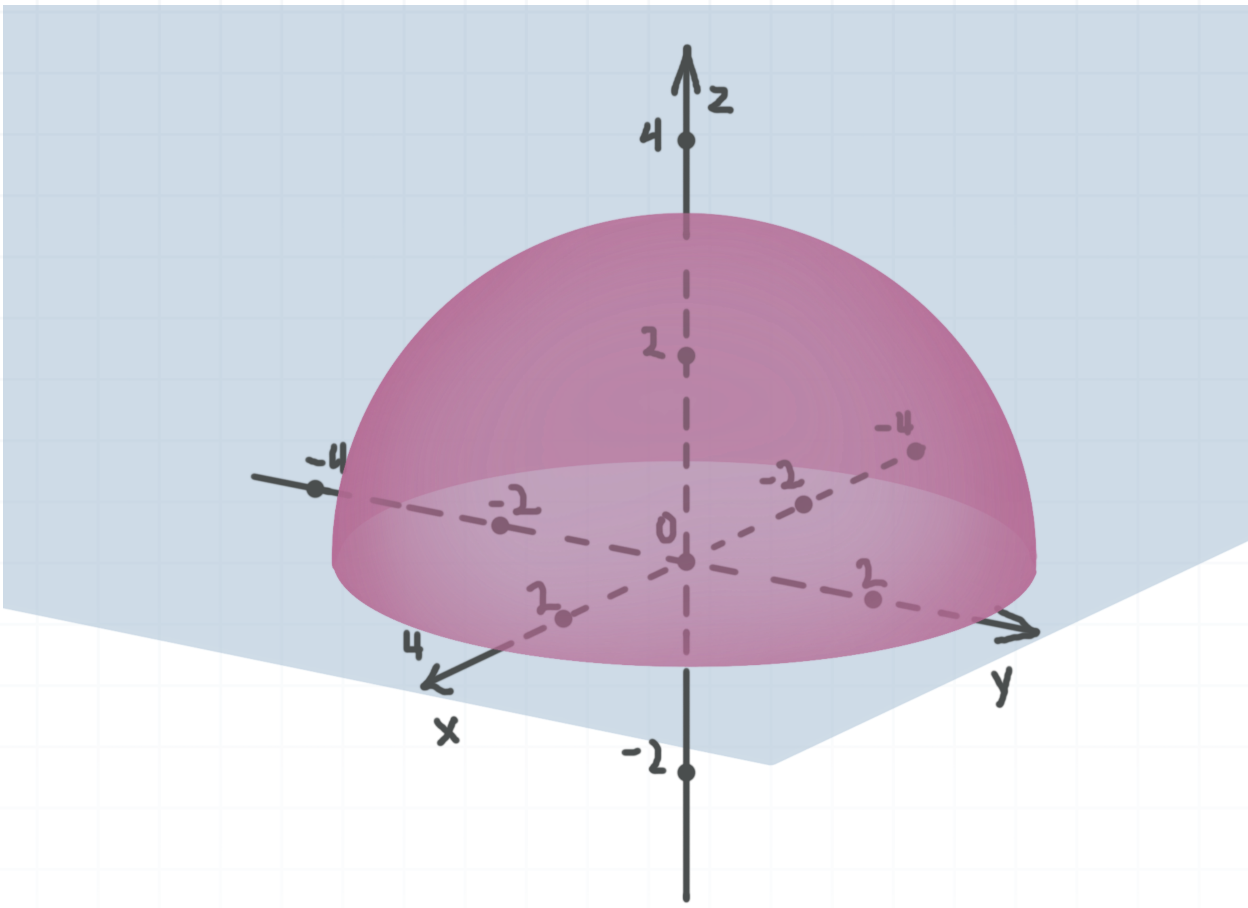


■ 3. Identify the solid given by the iterated improper integral in the spherical coordinates.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\pi} \int_{\pi}^{\infty} f(\rho, \theta, \phi)\ \rho^2 \sin \phi\ d\rho\ d\theta\ d\phi$$

Solution:

The value of ρ changes from π to ∞. So the solid is the set of outer points of the sphere with center at the origin and radius π. The value of θ changes from 0 to 2π and ϕ changes from $-\pi/2$ to $\pi/2$. Therefore, we consider the set of outer points of the sphere for $z \geq 0$.


z
4
2
-4
-2
0
-2
-4
2
2
4
x
y
-2

CHANGING TRIPLE INTEGRALS TO SPHERICAL COORDINATES

1. Evaluate the triple integral by changing it to spherical coordinates, if E is the sphere with center at the origin and radius 3.

$$\iiint_E 5x^2 - 2\ dV$$

Solution:

Coordinates x, y, and z change within the sphere with center at the origin and radius 3. The value of ρ changes from 0 to 3, the value of θ changes from 0 to 2π, and the value of ϕ changes from 0 to π. The function is

$$5x^2 - 2$$

$$5(\rho \sin\phi \cos\theta)^2 - 2$$

$$5\rho^2 \sin^2\phi \cos^2\theta - 2$$

Then the integral in spherical coordinates is

$$\int_0^{\pi} \int_0^{2\pi} \int_0^3 (5\rho^2 \sin^2\phi \cos^2\theta - 2)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi} \int_0^{2\pi} \int_0^3 5\rho^4 \sin^3\phi \cos^2\theta - 2\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^3 5\rho^4\sin^3\phi\cos^2\theta\ d\rho\ d\theta\ d\phi-\int_0^{\pi}\int_0^{2\pi}\int_0^3 2\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\sin^3\phi\ d\phi\cdot\int_0^{2\pi}\cos^2\theta\ d\theta\cdot\int_0^3 5\rho^4\ d\rho-\int_0^{\pi}\sin\phi\ d\phi\cdot\int_0^{2\pi}d\theta\cdot\int_0^3 2\rho^2\ d\rho$$

Evaluate each integral.

$$\left(\frac{1}{12}\cos(3\phi)-\frac{3}{4}\cos\phi\Big|_0^{\pi}\right)\left(\frac{1}{2}x+\frac{1}{2}\sin x\cos x\Big|_0^{2\pi}\right)\left(\rho^5\Big|_0^3\right)$$

$$-\left(-\cos\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(\frac{2}{3}\rho^3\Big|_0^3\right)$$

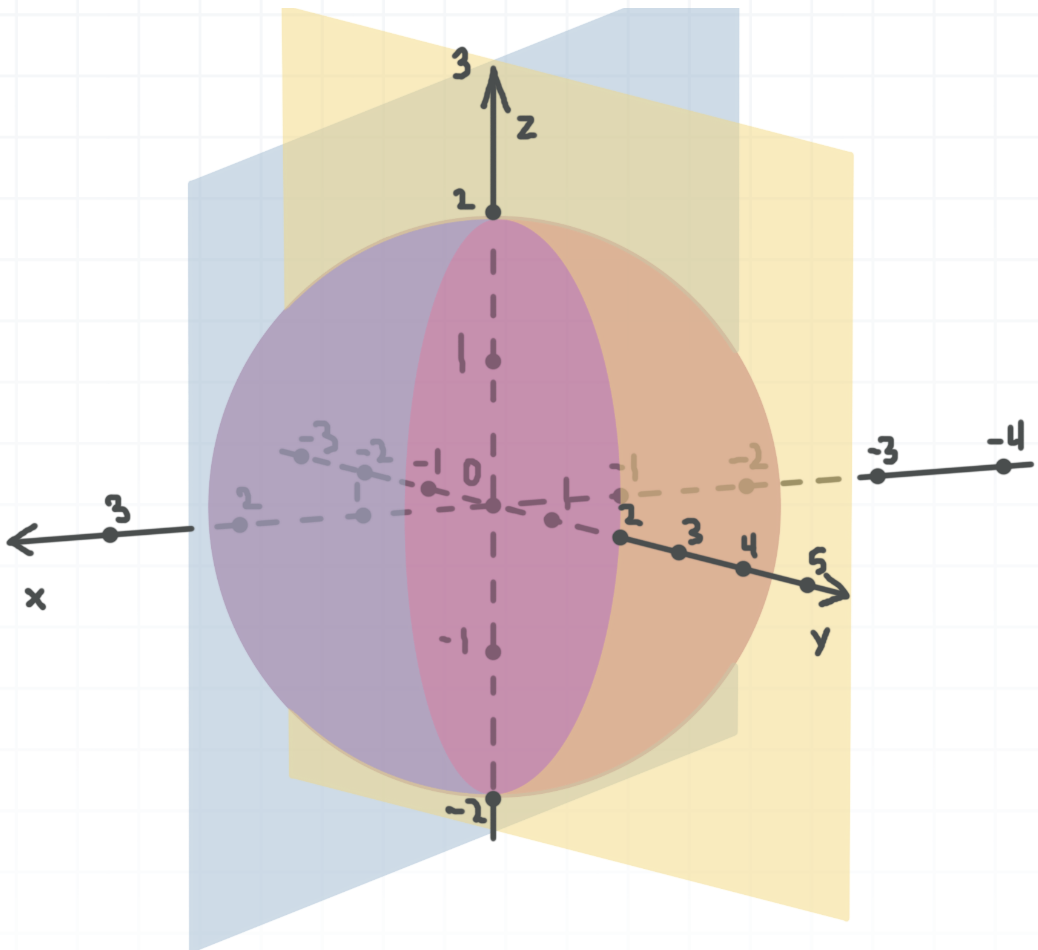
$$\left(\frac{4}{3}\right)(\pi)(243)-(2)(2\pi)(18)$$

$$324\pi-72\pi$$

$$252\pi$$

2. Write down the triple integral by converting it to spherical coordinates, if E is the part of the sphere with center at the origin, radius 2, that lies between the planes $x=0$ and $y=x$, and in the space $y>0$.

$$\iiint_E x^2+y^2+2z\ dV$$



Solution:

The values of x, y, and z change within part of the sphere with center at the origin and radius 2. Therefore, the value of ρ changes from 0 to 2 and the value of ϕ changes from 0 to π.

Change the equation of the plane $x = y$ into spherical coordinates. For any ρ and θ,

$$\rho \sin \phi \cos \theta = \rho \sin \phi \sin \theta$$

$$\cos \theta = \sin \theta$$

$$\theta = \pi/4$$

Change the equation of the plane $x = 0$ into spherical coordinates. For any ρ and θ,

$$\rho \sin\phi \cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \pi/2$$

The equation of the plane $x = y$ corresponds to the equation $\theta = \pi/4$ in spherical coordinates and the equation of the plane $x = 0$ corresponds to the equation $\theta = \pi/2$. So the value of θ changes from $\pi/4$ to $\pi/2$.

The function is

$$x^2 + y^2 + 2z$$

$$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + \rho \cos\phi$$

$$\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho \cos\phi$$

$$\rho^2 \sin^2\phi + \rho \cos\phi$$

So the triple integral is then

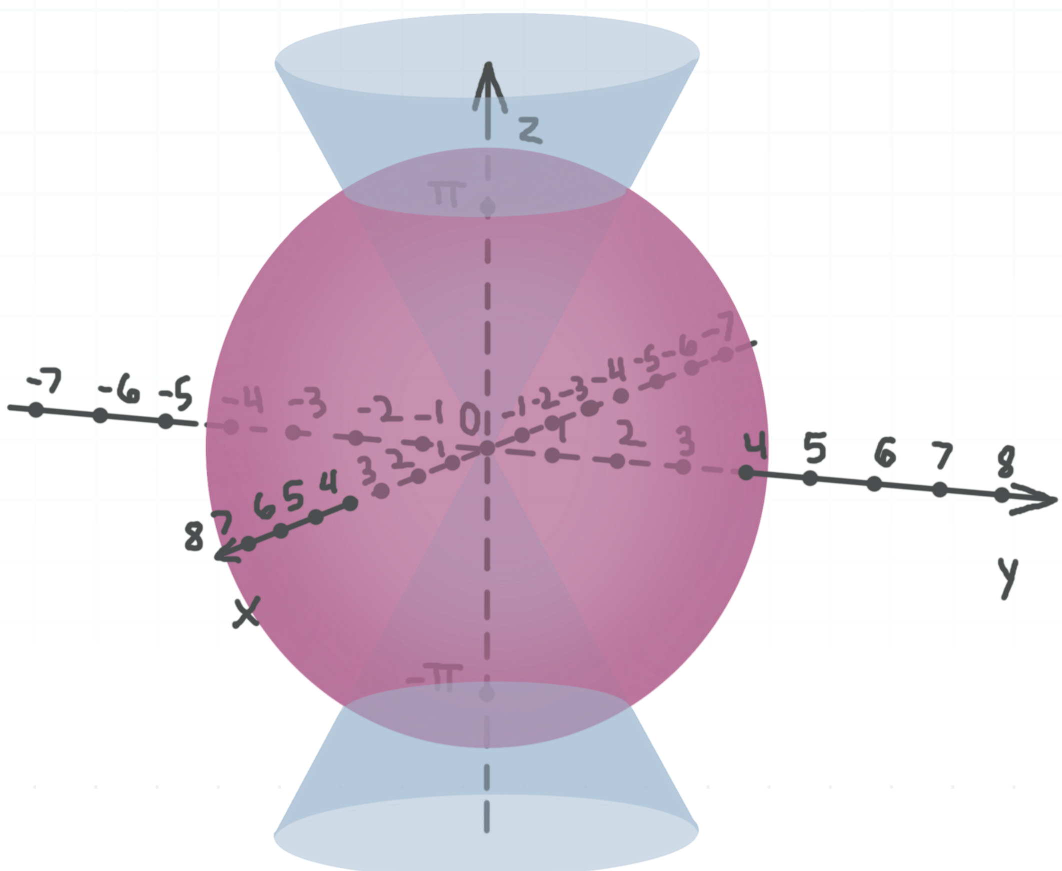
$$\iiint_E x^2 + y^2 + 2z \ dV$$

$$\int_0^{\pi} \int_{\pi/4}^{\pi/2} \int_0^2 (\rho^2 \sin^2\phi + \rho \cos\phi)\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \int_{\pi/4}^{\pi/2} \int_0^2 (\rho^4 \sin^3\phi + \rho^3 \cos\phi \sin\phi) \ d\rho \ d\theta \ d\phi$$

■ 3. Convert the triple integral to spherical coordinates, where E is the solid bounded by the sphere $x^2+y^2+z^2=16$ and the cone $x^2+y^2=z^2/3$, that lies in the half-space $z>0$.

$$\iiint_E \ln(x^2y^2z^2+1)\ dV$$



Solution:

The values of x, y, and z change within the part of the sphere with center at the origin and radius 4. Therefore, the value of ρ changes from 0 to 4. Since the solid has circular symmetry about the z-axis, the value of θ changes from 0 to 2π.

Convert the equation of the cone to spherical coordinates.

$$x^2 + y^2 = \frac{z^2}{3}$$

$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = \frac{1}{3}(\rho \cos \phi)^2$$

$$\sin^2 \phi(\cos^2 \theta + \sin^2 \theta) = \frac{1}{3}\cos^2 \phi$$

$$\tan^2 \phi = \frac{1}{3}$$

Since $z > 0$, we can say $0 \leq \phi \leq \pi/2$, and

$$\tan \phi = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

Therefore, the value of ϕ changes within the cone from 0 to $\pi/6$. The function is

$$\ln(x^2y^2z^2 + 1)$$

$$\ln((\rho \sin \phi \cos \theta)^2(\rho \sin \phi \sin \theta)^2(\rho \cos \phi)^2 + 1)$$

$$\ln(\rho^6 \sin^4 \phi \cos^2 \phi \sin^2 \theta \cos^2 \theta + 1)$$

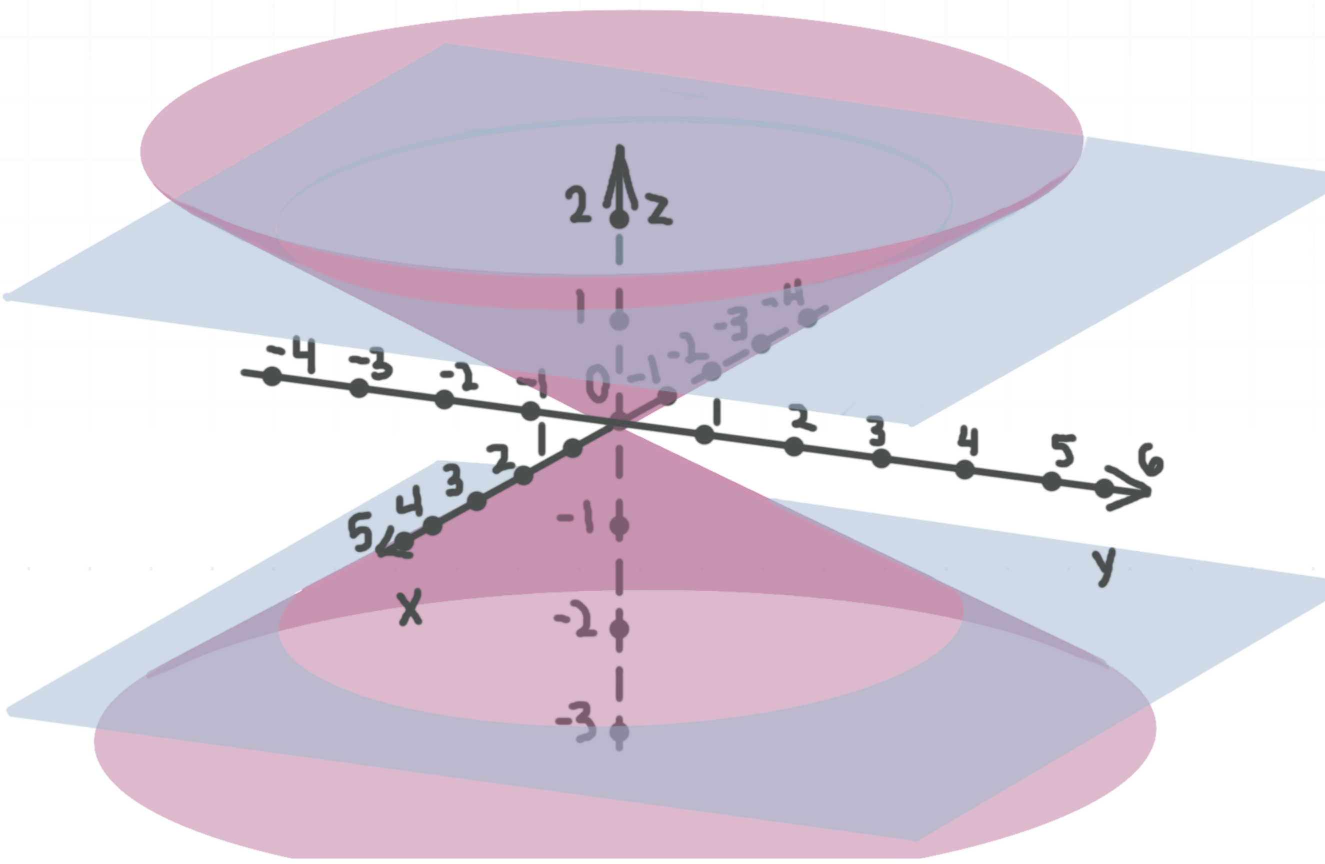
So the triple integral is

$$\iiint_E \ln(x^2y^2z^2 + 1)\ dV$$

$$\int_0^{\pi/6}\int_0^{2\pi}\int_0^4 \ln(\rho^6\sin^4\phi\cos^2\phi\sin^2\theta\cos^2\theta+1)\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

■ 4. If E is the solid bounded by the cone $x^2+y^2=3z^2$ and the planes $z=2$ and $z=-2$, evaluate the triple integral by changing it to spherical coordinates.

$$\iiint_E \sqrt{x^2+y^2+z^2}\ dV$$



Solution:

Since the solid and function are symmetric around the xy-plane,

$$\iiint_E \sqrt{x^2 + y^2 + z^2}\ dV = 2\iiint_{E_1} \sqrt{x^2 + y^2 + z^2}\ dV$$

where E_1 is the part of E that lies in the half-space $z \geq 0$. Since the solid has circular symmetry around the z-axis, θ changes from 0 to 2π. Convert the equation of the plane $z = 2$ into spherical coordinates.

$$\rho \cos \phi = 2$$

$$\rho = \frac{2}{\cos \phi}$$

Therefore, ρ changes from 0 to $2/\cos \phi$. Convert the equation of the cone $x^2 + y^2 = 3z^2$ into spherical coordinates.

$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = 3(\rho \cos \phi)^2$$

$$\sin^2 \phi(\cos^2 \theta + \sin^2 \theta) = 3 \cos^2 \phi$$

$$\tan^2 \phi = 3$$

Since $z > 0$, then $0 \leq \phi \leq \pi/2$, and

$$\tan \phi = \sqrt{3}$$

$$\phi = \frac{\pi}{3}$$

So within the cone, ϕ changes from 0 to $\pi/3$. The function is

$$\sqrt{x^2 + y^2 + z^2} = \sqrt{\rho^2} = \rho$$

Therefore, the integral in spherical coordinates is

$$\iiint_E \sqrt{x^2+y^2+z^2}\ dV$$

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{\frac{2}{\cos\phi}} \rho\cdot\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{\frac{2}{\cos\phi}} \rho^3\sin\phi\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{1}{4}\rho^4\sin\phi\Big|_{\rho=0}^{\rho=\frac{2}{\cos\phi}}\ d\theta\ d\phi$$

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{1}{4}\left(\frac{2}{\cos\phi}\right)^4\sin\phi\ d\theta\ d\phi$$

$$8\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{\sin\phi}{\cos^4\phi}\ d\theta\ d\phi$$

Integrate with respect to θ.

$$8\int_0^{\frac{\pi}{3}} \frac{\sin\phi}{\cos^4\phi}\theta\Big|_{\theta=0}^{\theta=2\pi}\ d\phi$$

$$8\int_0^{\frac{\pi}{3}} \frac{\sin\phi}{\cos^4\phi}(2\pi)\ d\phi$$

$$16\pi\int_0^{\frac{\pi}{3}} \frac{\sin\phi}{\cos^4\phi}\ d\phi$$

Integrate with respect to ϕ, using a substitution with $u = \cos \phi$, $du = -\sin \phi \ d\phi$, and with u changing from 1 to $1/2$.

$$-16\pi \int_1^{\frac{1}{2}} \frac{1}{u^4} \ du$$

$$\frac{16\pi}{3u^3} \Big|_1^{\frac{1}{2}}$$

$$\frac{16\pi}{3\left(\frac{1}{2}\right)^3} - \frac{16\pi}{3(1)^3}$$

$$\frac{128\pi}{3} - \frac{16\pi}{3}$$

$$\frac{112\pi}{3}$$

■ 5. If E is the set of outer points of the sphere with center at the origin and radius 5, evaluate the improper triple integral by changing it to spherical coordinates.

$$\iiint_E \frac{1}{(x^2 + y^2 + z^2)^3} \ dV$$

Solution:

The values of x, y, and z change outside of the sphere with center at the origin and radius 5. The value of ρ changes from 5 to ∞, θ changes from 0 to 2π, and ϕ changes from 0 to π. The function is

$$\frac{1}{(x^2+y^2+z^2)^3}=\frac{1}{\rho^6}$$

So the integral in spherical coordinates is

$$\int_0^{\pi}\int_0^{2\pi}\int_5^{\infty}\frac{1}{\rho^6}\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_5^{\infty}\frac{1}{\rho^4}\sin\phi\ d\rho\ d\theta\ d\phi$$

Split up the integral as the product of three single integrals.

$$\int_0^{\pi}\sin\phi\ d\phi\cdot\int_0^{2\pi}\ d\theta\cdot\int_5^{\infty}\frac{1}{\rho^4}\ d\rho$$

Evaluate each integral.

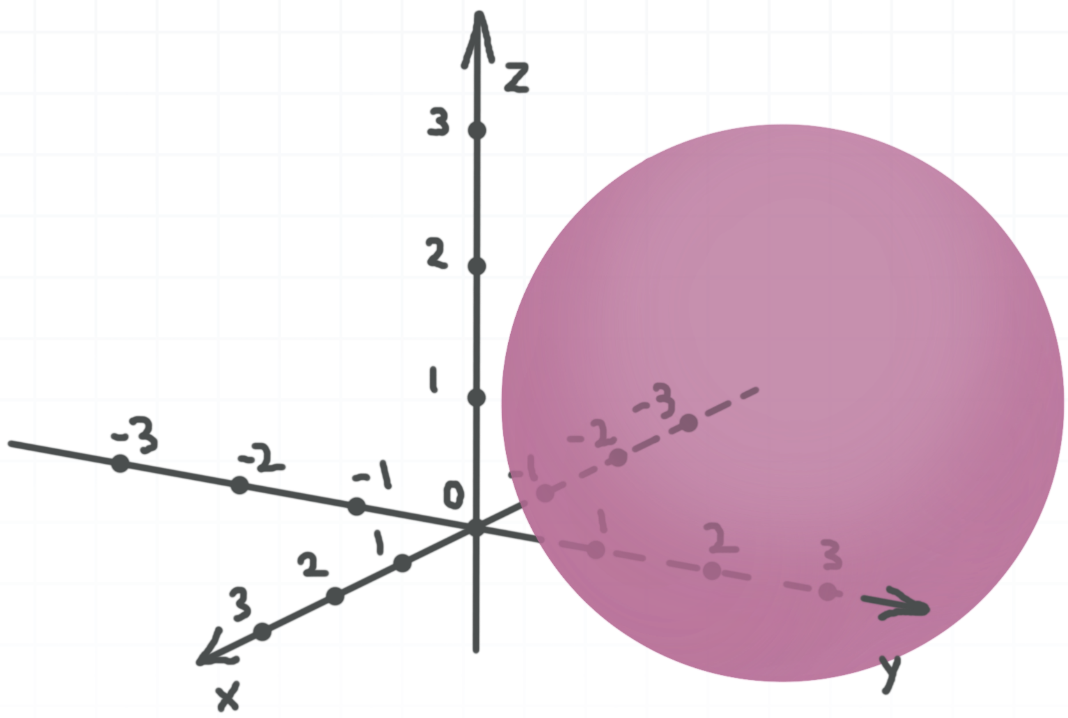
$$\left(-\cos\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\frac{1}{3\rho^3}\Big|_5^{\infty}\right)$$

$$(2)(2\pi)\left(\frac{1}{375}\right)$$

$$\frac{4\pi}{375}$$

6. Evaluate the triple integral by changing it to spherical coordinates, where E is the sphere with center at the point $(-1,2,1)$ and radius 2.

$$\iiint_E 5x + 3y - 2z \ dV$$



Solution:

Use conversion formulas to move the center of the sphere to the origin.

$$x_1 = x + 1, \ x = x_1 - 1$$

$$y_1 = y - 2, \ y = y_1 + 2$$

$$z_1 = z - 1, \ z = z_1 + 1$$

The function transforms to

$$5x + 3y - 2z = 5(x_1 - 1) + 3(y_1 + 2) - 2(z_1 + 1) = 5x_1 + 3y_1 - 2z_1 - 1$$

The surface transforms to the sphere with center at the origin and radius 2, so the given triple integral can be rewritten as

$$\iiint_{E_1} 5x + 3y - 2z - 1\ dV$$

where E_1 is the interior points of the sphere with center at the origin and radius 2. The value of ρ changes from 0 to 2. The value of θ changes from 0 to 2π and the value of ϕ changes from 0 to π.

The function is

$$5x + 3y - 2z - 1 = 5\rho \sin\phi \cos\theta + 3\rho \sin\phi \sin\theta - 2\rho \cos\phi - 1$$

Therefore, the integral in spherical coordinates is

$$\int_0^{\pi}\int_0^{2\pi}\int_0^2 (5\rho \sin\phi \cos\theta + 3\rho \sin\phi \sin\theta - 2\rho \cos\phi - 1)\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^2 5\rho^3 \sin^2\phi \cos\theta + 3\rho^3 \sin^2\phi \sin\theta - 2\rho^3 \cos\phi \sin\phi - \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

Since the integrals of sine and cosine functions over a 2π period are equal to zero, the integral simplifies to

$$\int_0^{\pi}\int_0^{2\pi}\int_0^2 -2\rho^3 \cos\phi \sin\phi - \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^2 -\rho^3 \sin 2\phi\ d\rho\ d\theta\ d\phi + \int_0^{\pi}\int_0^{2\pi}\int_0^2 -\rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi} \sin 2\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^2 -\rho^3 \ d\rho + \int_0^{\pi} \sin\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^2 -\rho^2 \ d\rho$$

Evaluate each integral.

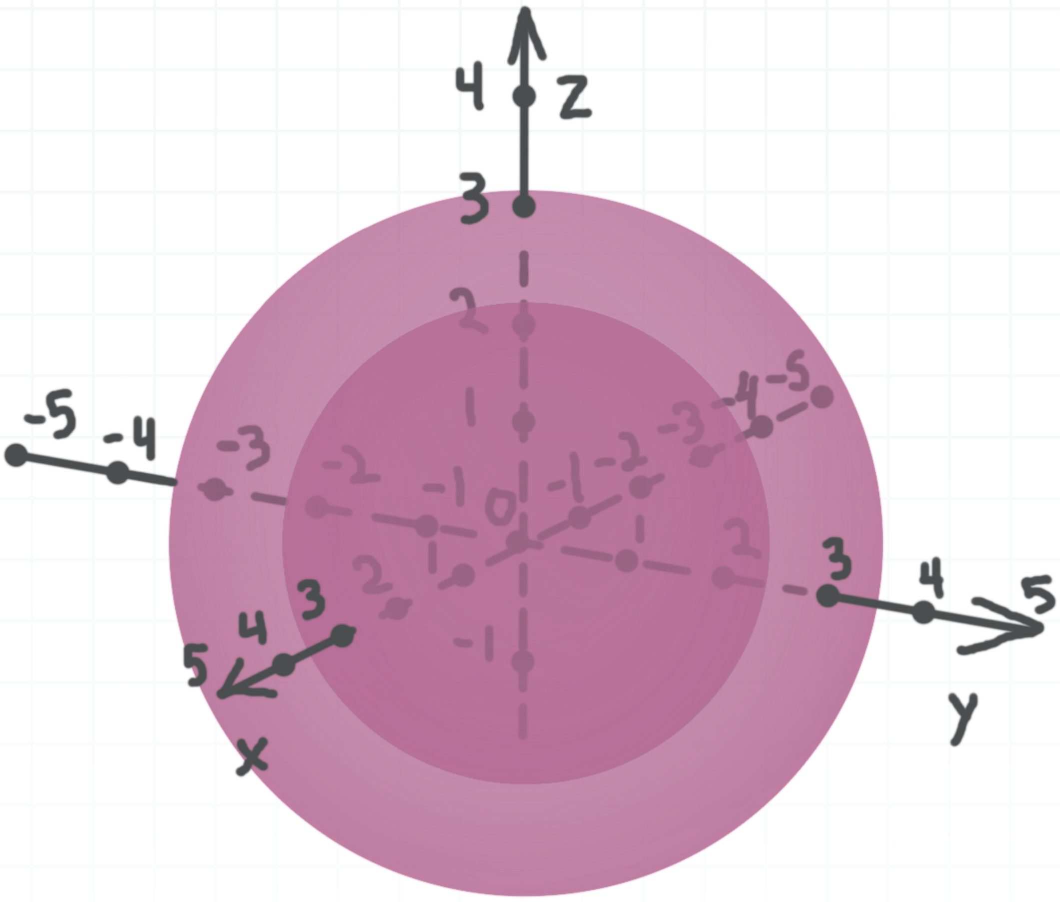
$$\left(-\frac{1}{2}\cos 2\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\frac{1}{4}\rho^4\Big|_0^{2}\right) + \left(-\cos\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\frac{1}{3}\rho^3\Big|_0^{2}\right)$$

$$(0)(2\pi)(-4) + (2)(2\pi)\left(-\frac{8}{3}\right)$$

$$-\frac{32\pi}{3}$$

■ 7. Evaluate the triple integral by changing it to spherical coordinates, if E is the set of points between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 4$.

$$\iiint_E 15y^2 + 4y \ dV$$



Solution:

The values of x, y, and z change within the sphere with center at the origin and radius 3, and outside the sphere with center at the origin and radius 2. The value of ρ changes from 2 to 3, the value of θ changes from 0 to 2π and the value of ϕ changes from 0 to π. The function is

$$15y^2 + 4y$$

$$15(\rho \sin \phi \sin \theta)^2 + 4\rho \sin \phi \sin \theta$$

$$15\rho^2 \sin^2 \phi \sin^2 \theta + 4\rho \sin \phi \sin \theta$$

Therefore, the integral in spherical coordinates is

$$\int_0^\pi \int_0^{2\pi} \int_2^3 (15\rho^2 \sin^2\phi \sin^2\theta + 4\rho \sin\phi \sin\theta)\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^\pi \int_0^{2\pi} \int_2^3 15\rho^4 \sin^3\phi \sin^2\theta + 4\rho^3 \sin^2\phi \sin\theta \ d\rho \ d\theta \ d\phi$$

Since the integral of the sine function over a 2π period is equal to zero, the integral simplifies to

$$\int_0^\pi \int_0^{2\pi} \int_2^3 15\rho^4 \sin^3\phi \sin^2\theta \ d\rho \ d\theta \ d\phi$$

$$\int_0^\pi \sin^3\phi \ d\phi \cdot \int_0^{2\pi} \sin^2\theta \ d\theta \cdot \int_2^3 15\rho^4 \ d\rho$$

Evaluate each integral.

$$\left(\frac{1}{12}\cos(3\phi) - \frac{3}{4}\cos\phi \Big|_0^\pi \right) \left(\frac{1}{2}\theta - \frac{1}{2}\sin\theta\cos\theta \Big|_0^{2\pi} \right) \left(3\rho^5 \Big|_2^3 \right)$$

$$\left(\frac{4}{3}\right)(\pi)(633)$$

$$844\pi$$

■ 8. Evaluate the improper triple integral by changing it to spherical coordinates, where E is the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$).

$$\iiint_E 2^{-\sqrt{(x^2+y^2+z^2)^3}} \ dV$$

Solution:

The values of x, y, and z change within the first octant, so the value of ρ changes from 0 to ∞. The values of θ and ϕ change from 0 to $\pi/2$. The function is

$$2^{-\sqrt{(x^2+y^2+z^2)^3}} = 2^{-\rho^3}$$

Therefore, the integral in spherical coordinates is

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty} 2^{-\rho^3}\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{\infty} 2^{-\rho^3}\rho^2 \ d\rho$$

Evaluate each integral, using a substitution with $u = -\rho^3$, $du = -3\rho^2 \ d\rho$, and where u changes from 0 to $-\infty$.

$$\int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{-\infty} 2^u\rho^2 \ \frac{du}{-3\rho^2}$$

$$\int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{-\infty} -\frac{1}{3}(2^u) \ du$$

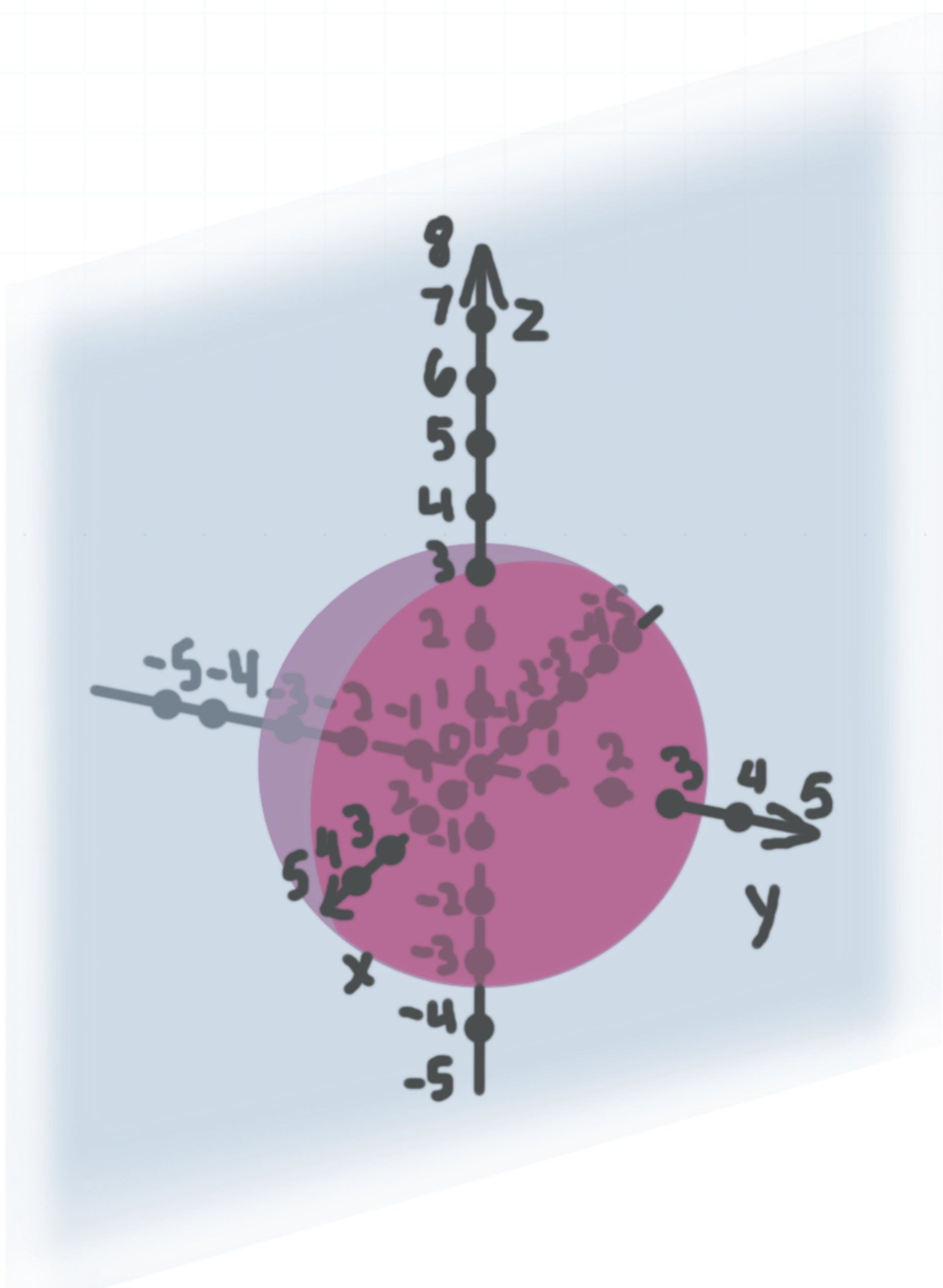
$$\left(-\cos\phi\Big|_0^{\frac{\pi}{2}}\right)\left(\theta\Big|_0^{\frac{\pi}{2}}\right)\left(-\frac{1}{3}\lim_{t\to-\infty}\left(\frac{2^t}{\ln 2}\right)-\frac{2^0}{\ln 2}\right)$$

$$(1)\left(\frac{\pi}{2}\right)\left(\frac{1}{3\ln 2}\right)$$

$$\frac{\pi}{6 \ln 2}$$

■ 9. Evaluate the triple integral by changing it to spherical coordinates, if E is the solid bounded by the sphere with center at the origin and radius 3, and the plane $x + \sqrt{3}y = 0$. Consider the hemisphere that includes the points from the first octant.

$$\iiint_E x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 \ dV$$



Solution:

The values x, y, and z change within part of the sphere with center at the origin and radius 3. Therefore, the value of ρ changes from 0 to 3 and the value of ϕ changes from 0 to π. Convert the equation of the plane $x+\sqrt{3}y=0$ into spherical coordinates. For any ρ and θ,

$$\rho\sin\phi\cos\theta=-\sqrt{3}\rho\sin\phi\sin\theta$$

So

$$\cos\theta=-\sqrt{3}\sin\theta$$

$$\tan\theta=-\frac{1}{\sqrt{3}}$$

$$\theta_1=-\frac{\pi}{6} \text{ and } \theta_2=\frac{5\pi}{6}$$

So the value of θ changes from $-\pi/6$ to $5\pi/6$. The function is

$$x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2=(x^2+y^2+z^2)^2=\rho^4$$

Therefore, the integral in spherical coordinates is

$$\int_0^{\pi}\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_0^3\rho^4\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_0^3\rho^6\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi} \sin\phi \ d\phi \cdot \int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}} d\theta \cdot \int_0^3 \rho^6 \ d\rho$$

Evaluate each integral.

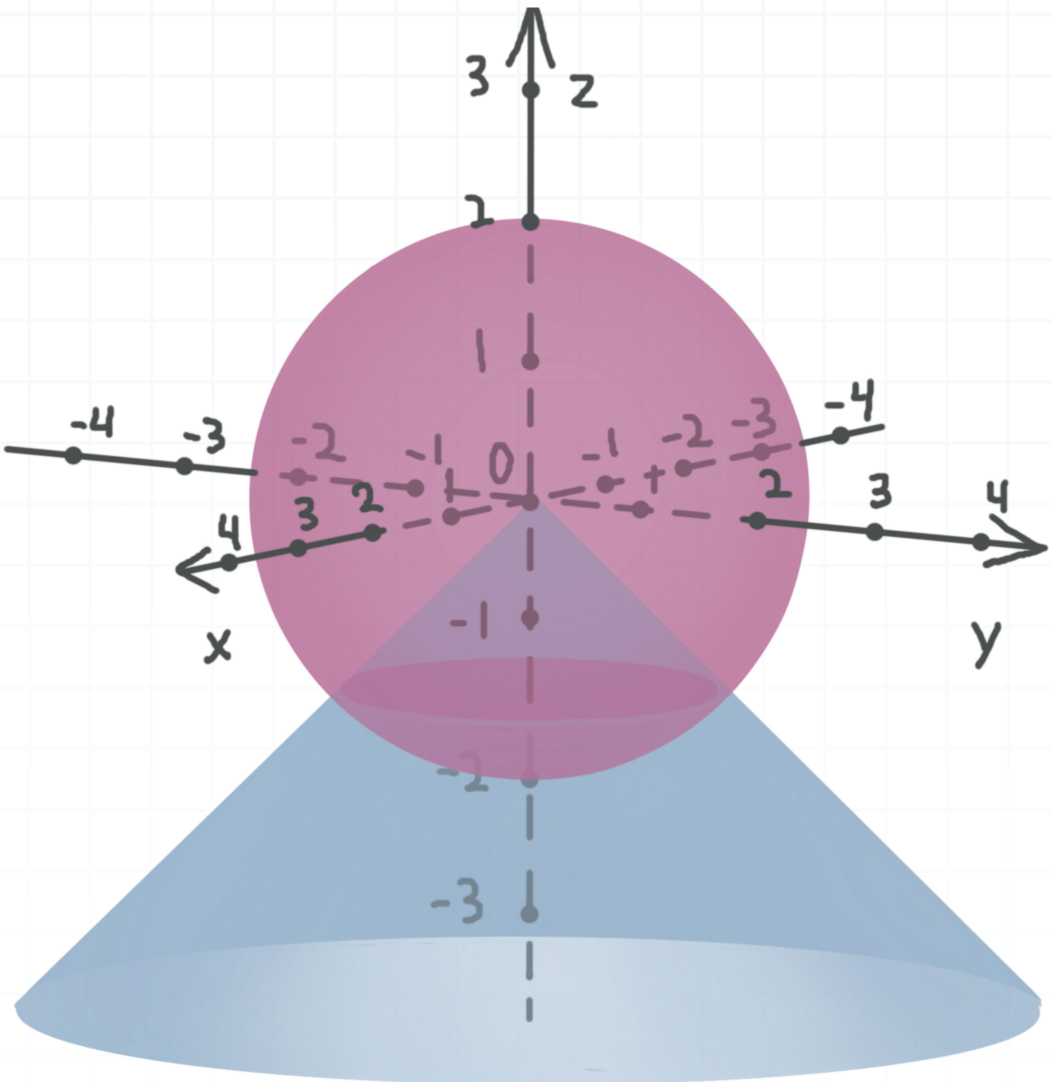
$$\left(-\cos\phi \Big|_0^{\pi}\right)\left(\theta \Big|_{-\frac{\pi}{6}}^{\frac{5\pi}{6}}\right)\left(\frac{1}{7}\rho^7 \Big|_0^3\right)$$

$$(2)(\pi)(2{,}187)$$

$$4{,}374\pi$$

■ 10. Evaluate the improper triple integral by changing it to spherical coordinates, where E is the region that consists of the points inside the cone $x^2 + y^2 = z^2$ and outside the sphere $x^2 + y^2 + z^2 = 4$, that lie in the half-space $z < 0$.

$$\iiint_E \frac{4z + 10}{(x^2 + y^2 + z^2)^4} \ dV$$



Solution:

The values of x, y, and z change outside the sphere with center at the origin and radius 2. Therefore, the value of ρ changes from 2 to ∞. Since the region E has circular symmetry around the z-axis, the coordinate θ changes from 0 to 2π.

Convert the equation of the cone $x^2 + y^2 = z^2$ to spherical coordinates.

$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = (\rho \cos \phi)^2$$

$$\sin^2 \phi(\cos^2 \theta + \sin^2 \theta) = \cos^2 \phi$$

$$\tan^2 \phi = 1$$

$$\tan\phi = \pm 1$$

Since $z < 0$, then $\pi/2 \le \phi \le \pi$, and

$$\tan\phi = -1$$

$$\phi = \frac{3\pi}{4}$$

Therefore, the value of ϕ changes inside the cone from $\pi/2$ to $3\pi/4$. The function is

$$\frac{4z+10}{(x^2+y^2+z^2)^4} = \frac{4\rho\cos\phi + 10}{\rho^8}$$

So the integral in spherical coordinates is

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\int_0^{2\pi}\int_2^{\infty} \frac{4\rho\cos\phi+10}{\rho^8}\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\int_0^{2\pi}\int_2^{\infty} \frac{2\rho\sin 2\phi+10\sin\phi}{\rho^6}\ d\rho\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\int_0^{2\pi}\int_2^{\infty} \frac{2\sin 2\phi}{\rho^5}+\frac{10\sin\phi}{\rho^6}\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\int_0^{2\pi} -\frac{\sin 2\phi}{2\rho^4}-\frac{2\sin\phi}{\rho^5}\Bigg|_{\rho=2}^{\rho=\infty}\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_{0}^{2\pi} \frac{\sin 2\phi + 2\sin\phi}{32}\ d\theta\ d\phi$$

Integrate with respect to θ.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{\sin 2\phi + 2\sin\phi}{32}\theta\Big|_{\theta=0}^{\theta=2\pi}\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \frac{\pi\sin 2\phi + 2\pi\sin\phi}{16}\ d\phi$$

Integrate with respect to ϕ.

$$\frac{\pi\sin 2\phi + 2\pi\sin\phi}{16}\Big|_{\phi=\frac{\pi}{2}}^{\phi=\frac{3\pi}{4}}$$

$$\frac{-\pi+\sqrt{2}\pi}{16} - \frac{2\pi}{16}$$

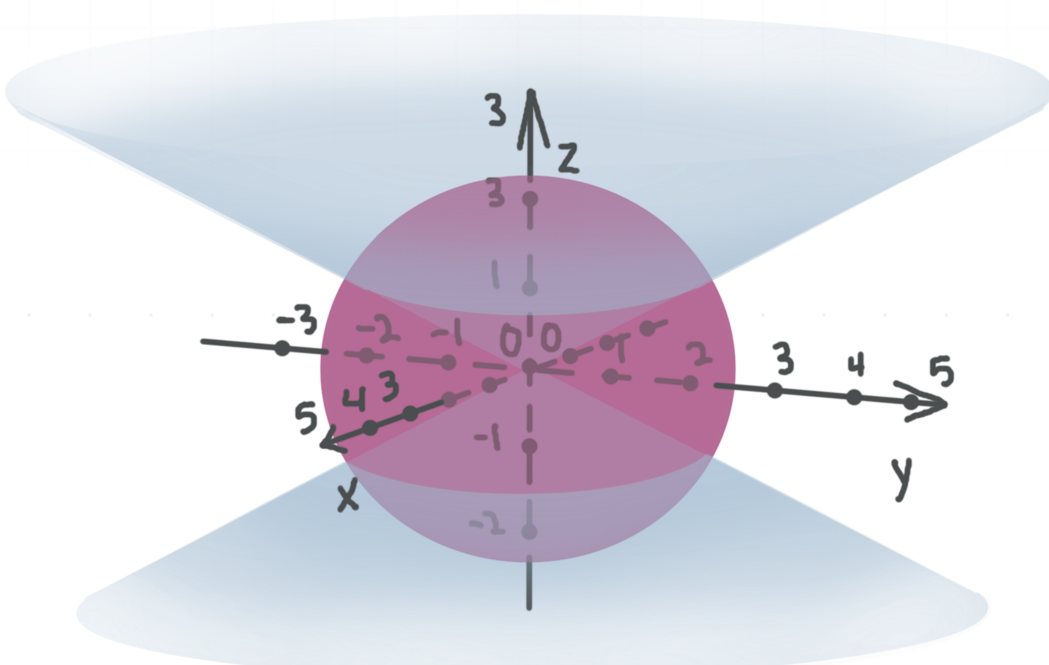
$$\frac{-3\pi+\sqrt{2}\pi}{16}$$

FINDING VOLUME

1. Use a triple integral in spherical coordinates to find the volume of the region E that consists of the points inside the sphere $x^2+y^2+z^2=6$ and outside the cone $y^2+z^2=3x^2$.

Solution:

Since rotation doesn't change the volume, the volume of the region E is equal to the volume of region E_1 that consists of the points inside the sphere $x^2+y^2+z^2=6$ and outside the cone $x^2+y^2=3z^2$.



The values of x, y, and z change within the part of the sphere with center at the origin and radius $\sqrt{6}$. Therefore, the value of ρ changes from 0 to $\sqrt{6}$. Since E_1 has circular symmetry around the z-axis, and the value of θ changes from 0 to 2π.

Convert the equation of the cone $x^2+y^2=3z^2$ to spherical coordinates.

$$(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2=3(\rho\cos\phi)^2$$

$$\sin^2\phi(\cos^2\theta+\sin^2\theta)=3\cos^2\phi$$

$$\tan^2\phi=3$$

$$\tan\phi=\pm\sqrt{3}$$

Since $0\le\phi\le\pi$,

$$\phi_1=\frac{\pi}{3} \text{ and } \phi_2=\frac{2\pi}{3}$$

Therefore, the value of ϕ changes outside the cone from $\pi/3$ to $2\pi/3$. So the integral in spherical coordinates is

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\int_0^{2\pi}\int_0^{\sqrt{6}}\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\sin\phi\ d\phi\cdot\int_0^{2\pi}d\theta\cdot\int_0^{\sqrt{6}}\rho^2\ d\rho$$

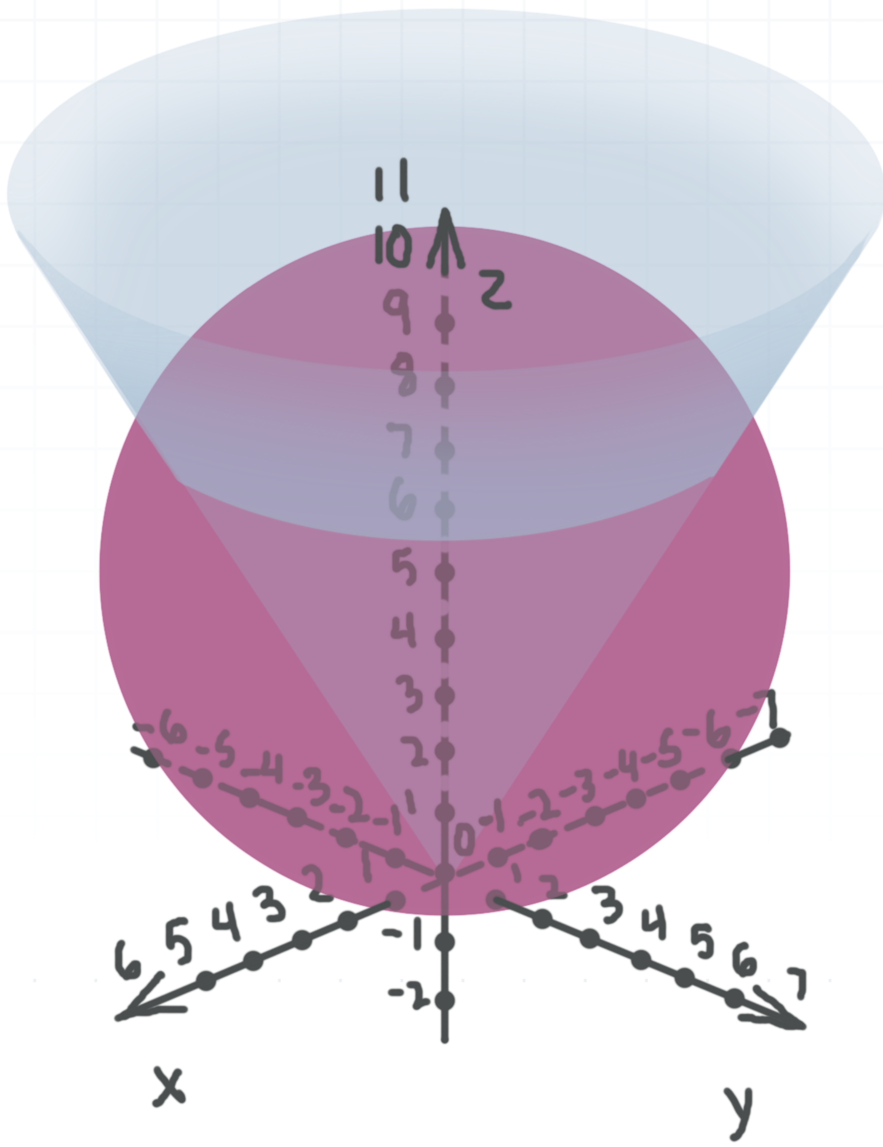
Evaluate each integral.

$$\left(-\cos\phi\Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right)\cdot\left(\theta\Big|_0^{2\pi}\right)\left(\frac{1}{3}\rho^3\Big|_0^{\sqrt{6}}\right)$$

$$(1)(2\pi)(2\sqrt{6}))$$

$$4\pi\sqrt{6}$$

2. Use a triple integral in spherical coordinates to find the volume of an ice cream cone formed by the points common to the cone $3x^2 + 3y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 - 10z = 0$.



Solution:

If we put the equation of the sphere in standard form,

$$x^2 + y^2 + (z - 5)^2 = 25$$

we see that it's centered at $(0,0,5)$ and has radius 5. Given that, the value of θ changes from 0 to 2π. Now convert the equation of the sphere into spherical coordinates.

$$x^2 + y^2 + z^2 - 10z = 0$$

$$\rho^2 - 10\rho\cos\phi = 0$$

$$\rho - 10\cos\phi = 0$$

$$\rho = 10\cos\phi$$

So the value of ρ changes from 0 to $10\cos\phi$. Now convert the equation of the cone into spherical coordinates.

$$3x^2 + 3y^2 = z^2$$

$$x^2 + y^2 = \frac{1}{3}z^2$$

$$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \frac{1}{3}(\rho\cos\phi)^2$$

$$\sin^2\phi(\cos^2\theta + \sin^2\theta) = \frac{1}{3}\cos^2\phi$$

$$\tan^2\phi = \frac{1}{3}$$

Since $z > 0$, then $0 \le \phi \le \pi/2$, and

$$\tan\phi = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

So the value of ϕ changes from 0 to $\pi/6$, and the volume is

$$\int_0^{\frac{\pi}{6}} \int_0^{2\pi} \int_0^{10\cos\phi} \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

Integrate with respect to ρ.

$$\int_0^{\frac{\pi}{6}} \int_0^{2\pi} \frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=10\cos\phi} \ d\theta \ d\phi$$

$$\int_0^{\frac{\pi}{6}} \int_0^{2\pi} \frac{1,000}{3} \cos^3\phi \sin\phi \ d\theta \ d\phi$$

Integrate with respect to θ.

$$\int_0^{\frac{\pi}{6}} \frac{1,000}{3}\theta \cos^3\phi \sin\phi \Big|_{\theta=0}^{\theta=2\pi} \ d\phi$$

$$\int_0^{\frac{\pi}{6}} \frac{1,000}{3}(2\pi)\cos^3\phi \sin\phi \ d\phi$$

$$\int_0^{\frac{\pi}{6}} \frac{2,000\pi}{3} \cos^3\phi \sin\phi \ d\phi$$

Integrate with respect to ϕ, using a substitution with $u = \cos\phi$, $du = -\sin\phi \ d\phi$, and u changing from 1 to $\sqrt{3}/2$.

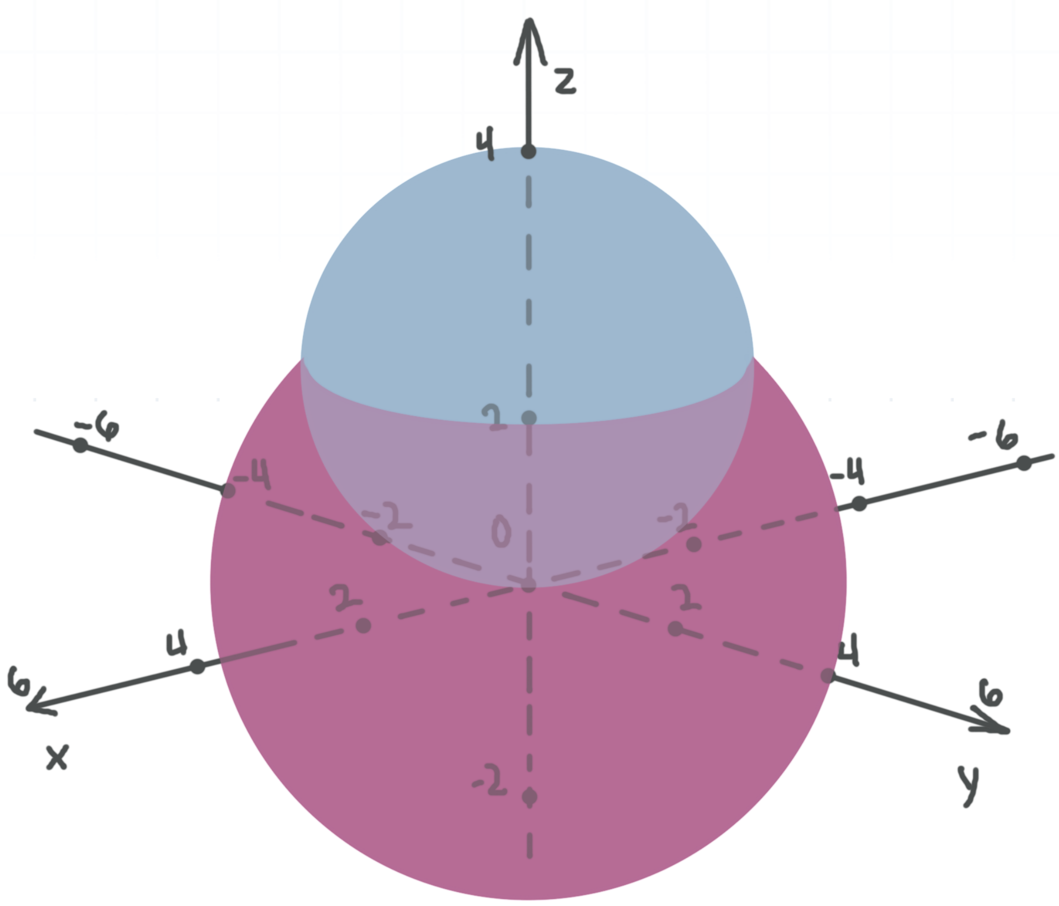
$$\frac{2,000\pi}{3} \int_1^{\frac{\sqrt{3}}{2}} -u^3 \ du$$

$$-\frac{500\pi}{3}u^4\Big|_1^{\frac{\sqrt{3}}{2}}$$

$$-\frac{375\pi}{4}+\frac{500\pi}{3}$$

$$\frac{875\pi}{12}$$

■ 3. Use a triple integral in spherical coordinates to find the volume of the three-dimensional lens common to the two spheres $x^2+y^2+z^2-8=0$ and $x^2+y^2+z^2-4z=0$.



Solution:

Rewrite the equations of both spheres in standard form.

$$x^2 + y^2 + z^2 = 8$$

$$x^2 + y^2 + (z - 2)^2 = 4$$

The first sphere has its center at the origin and radius $\sqrt{8}$, and the second sphere has its center at $(0,0,2)$ and radius 2.

To find the equation of the curve of intersection, consider the two sphere equations $x^2 + y^2 + z^2 - 8 = 0$ and $x^2 + y^2 + z^2 - 4z = 0$ as a system, then subtract them to get

$$4z - 8 = 0$$

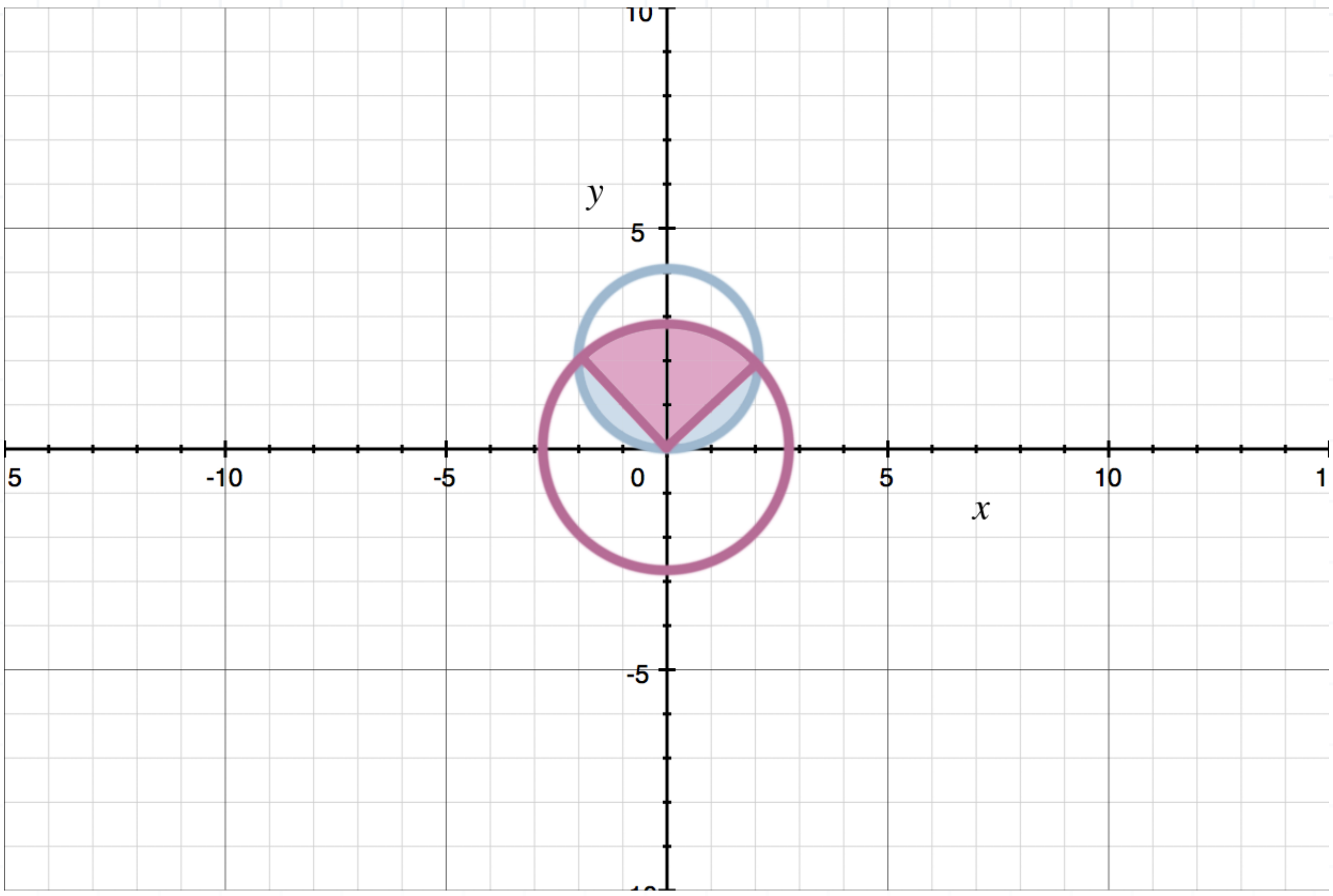
$$z = 2$$

Plug $z = 2$ into the first equation.

$$x^2 + y^2 + (2)^2 - 8 = 0$$

$$x^2 + y^2 = 4$$

So the curve of intersection is a circle with its center at $(0,0,2)$ and radius 2, that lies in the plane $z = 2$. Consider the lens section for $y = 0$.



The circle of intersection corresponds to the angle $\phi = \pi/4$. Split the lens into the two solids E_1 and E_2, where E_1 is defined for ϕ between 0 and $\pi/4$, and E_2 is defined for ϕ between $\pi/4$ and $\pi/2$.

For E_1, the value of ρ changes from 0 to $\sqrt{8}$, the value of θ changes from 0 to 2π, and the value of ϕ changes from 0 to $\pi/4$. So the volume is

$$\int_0^{\frac{\pi}{4}}\int_0^{2\pi}\int_0^{\sqrt{8}} \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\frac{\pi}{4}} \sin\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\sqrt{8}} \rho^2 \ d\rho$$

Evaluate each integral.

$$\left(-\cos\phi\Big|_0^{\frac{\pi}{4}}\right)\left(\theta\Big|_0^{2\pi}\right)\left(\frac{1}{3}\rho^3\Big|_0^{\sqrt{8}}\right)$$

$$\left(-\frac{\sqrt{2}}{2}+1\right)(2\pi)\left(\frac{16\sqrt{2}}{3}\right)$$

$$\frac{32\pi\sqrt{2}-32\pi}{3}$$

For E_2, the value of θ changes from 0 to 2π, and the value of ϕ changes from $\pi/4$ to $\pi/2$. To find the limits for ρ, convert the equation of the sphere into spherical coordinates.

$$x^2+y^2+z^2-4z=0$$

$$\rho^2-4\rho\cos\phi=0$$

$$\rho-4\cos\phi=0$$

$$\rho=4\cos\phi$$

So the value of ρ changes from 0 to $4\cos\phi$, and the volume is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{4\cos\phi}\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{1}{3}\rho^3\sin\phi\Big|_{\rho=0}^{\rho=4\cos\phi}\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{64}{3}\cos^3\phi\sin\phi\ d\theta\ d\phi$$

Integrate with respect to θ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64}{3}\theta \cos^3\phi \sin\phi \Big|_{\theta=0}^{\theta=2\pi} \ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{128\pi}{3} \cos^3\phi \sin\phi \ d\phi$$

Integrate with respect to ϕ, using a substitution with $u = \cos\phi$, $du = -\sin\phi \ d\phi$, and u changing from $\sqrt{2}/2$ to 0.

$$\frac{128\pi}{3}\int_{\frac{\sqrt{2}}{2}}^{0} -u^3 \ du$$

$$-\frac{32\pi}{3}u^4 \Big|_{\frac{\sqrt{2}}{2}}^{0}$$

$$\frac{8\pi}{3}$$

Then the sum of the two volumes is

$$\frac{32\pi\sqrt{2} - 32\pi}{3} + \frac{8\pi}{3}$$

$$\frac{32\pi\sqrt{2} - 24\pi}{3}$$

W W W . K R I S T A K I N G M A T H . C O M