Topic: Linearization of a multivariable function

Question: At which point is the linearization of $f(x, y) = x^2 - 2xy + y^2 - 3$ defined by $L(x_0, y_0) = -6x + 6y - 12$?

Answer choices:

A $(2,5)$

B $(-2,5)$

C $(3,5)$

D $(5,-3)$

Solution: A

If we evaluate $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ at the point $(2,5)$, we get

$$f(x, y) = x^2 - 2xy + y^2 - 3$$

$$f(2,5) = (2)^2 - 2(2)(5) + (5)^2 - 3$$

$$f(2,5) = 4 - 20 + 25 - 3$$

$$f(2,5) = 6$$

For the partial derivative with respect to x,

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 - 2xy + y^2 - 3\right)$$

$$f_x(x, y) = 2x - 2y$$

$$f_x(2,5) = \left[2(2) - 2(5)\right]$$

$$f_x(2,5) = -6$$

For the partial derivative with respect to y,

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2 - 2xy + y^2 - 3\right)$$

$$f_y(x, y) = -2x + 2y$$

$$f_y(2,5) = \left[-2(2) + 2(5)\right]$$

$$f_y(2,5) = 6$$

Plug these values into the linearization formula.

$$L(x_0, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(x_0, y_0) = f(2,5) + f_x(2,5)(x - 2) + f_y(2,5)(y - 5)$$

$$L(x_0, y_0) = 6 + (-6)(x - 2) + (6)(y - 5)$$

$$L(x_0, y_0) = 6 - 6x + 12 + 6y - 30$$

$$L(x_0, y_0) = -6x + 6y - 12$$

Topic: Linearization of a multivariable function

Question: What is the linearization of $f(x, y, z) = x^2 + y^2 - xyz$ at $P_0(1,2,-1)$?

Answer choices:

A $L(x, y, z) = 4x + y - 10z + 8$

B $L(x, y, z) = 8x + 10y - z + 12$

C $L(x, y, z) = 4x + 5y - 2z - 9$

D $L(x, y, z) = 8x - 10y - z + 12$

Solution: C

If we evaluate $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ at the point $P_0(1,2, -1)$, we get

$$f(x, y, z) = x^2 + y^2 - xyz$$

$$f(1,2, -1) = (1)^2 + (2)^2 - (1)(2)(-1)$$

$$f(1,2, -1) = 1 + 4 + 2$$

$$f(1,2, -1) = 7$$

For the partial derivative with respect to x,

$$f_x(x, y, z) = \frac{\partial}{\partial x}(x^2 + y^2 - xyz)$$

$$f_x(x, y, z) = 2x - yz$$

$$f_x(1,2, -1) = 2(1) - (2)(-1)$$

$$f_x(1,2, -1) = 4$$

For the partial derivative with respect to y,

$$f_y(x, y, z) = \frac{\partial}{\partial y}(x^2 + y^2 - xyz)$$

$$f_y(x, y, z) = 2y - xz$$

$$f_y(1,2, -1) = [2(2) - (1)(-1)]$$

$$f_y(1,2, -1) = 5$$

For the partial derivative with respect to z,

$$f_z(x, y, z) = \frac{\partial}{\partial z}(x^2 + y^2 - xyz)$$

$$f_z(x, y, z) = (-xy)$$

$$f_z(1,2,-1) = [-(1)(2)]$$

$$f_z(1,2,-1) = -2$$

Plug these values into the linearization formula.

$$L(x_0, y_0, z_0) = f(x_0, y_0, z_0) + f_x(x_0, y_0, z_0)(x - x_0) + f_y(x_0, y_0, z_0)(y - y_0)$$

$$+f_z(x_0, y_0, z_0)(z - z_0)$$

$$L(1,2,-1) = f(1,2,-1) + f_x(1,2,-1)(x-1) + f_y(1,2,-1)(y-2) + f_z(1,2,-1)(z+1)$$

$$L(1,2,-1) = 7 + (4)(x-1) + (5)(y-2) + (-2)(z+1)$$

$$L(1,2,-1) = 7 + 4x - 4 + 5y - 10 - 2z - 2$$

$$L(1,2,-1) = 4x + 5y - 2z - 9$$

Topic: Linearization of a multivariable function

Question: What is the upper bound of the error?

Given the linearization of $f(x, y) = x^2 + xy - y^2 - 1$ at $P_0(3,1)$, find the upper bound of the error $|E|$ over the rectangle defined by $|x - 3| \leq 0.2$ and $|y - 1| \leq 0.2$?

Answer choices:

A 1.06

B 0.06

C 1.16

D 0.16

Solution: D

If we evaluate $f(x, y)$, $f_x(x, y)$, and $f_y(x, y)$ at the point $(3,1)$, we get

$$f(x, y) = x^2 + xy - y^2 - 1$$

$$f(3,1) = (3)^2 + (3)(1) - (1)^2 - 1$$

$$f(3,1) = 9 + 3 - 1 - 1$$

$$f(3,1) = 10$$

For the partial derivative with respect to x,

$$f_x(x, y) = \frac{\partial}{\partial x}\left(x^2 + xy - y^2 - 1\right)$$

$$f_x(x, y) = 2x + y$$

$$f_x(3,1) = \left[2(3) + 1\right]$$

$$f_x(3,1) = 7$$

For the partial derivative with respect to y,

$$f_y(x, y) = \frac{\partial}{\partial y}\left(x^2 + xy - y^2 - 1\right)$$

$$f_y(x, y) = x - 2y$$

$$f_y(3,1) = \left[3 - 2(1)\right]$$

$$f_y(3,1) = 1$$

Plug these values into the linearization formula.

$$L(x_0, y_0) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$L(3,1) = f(3,1) + f_x(3,1)(x - 3) + f_y(3,1)(y - 1)$$

$$L(3,1) = 10 + (7)(x - 3) + (1)(y - 1)$$

$$L(3,1) = 10 + 7x - 21 + y - 1$$

$$L(3,1) = 7x + y - 12$$

We have $f_{xx} = 2$, $f_{yy} = -2$, and $f_{xy} = 1$. The largest of these three values is 2. Therefore, we can consider a common upper bound 2 on the rectangle R. Then

$$\left| E(x_0, y_0) \right| \le \frac{1}{2}(2)\left(|x - 3| + |y - 1| \right)^2$$

$$\left| E(x_0, y_0) \right| \le \frac{2}{2}(0.2 + 0.2)^2$$

$$\left| E(x_0, y_0) \right| = 0.16$$