Normal line to the surface

The normal line to the surface of the tangent plane is the line which is set at 90° from the surface.

In order to find the equation of the normal line to a tangent plane, we aren't required to first find the equation of the tangent plane, but both the normal line and the tangent plane use the partial derivatives of the original function and the point of tangency. Therefore, once you find the partial derivatives, you can very quickly find the equation of the tangent plane as well as the equation of the normal line to the surface.

The tangent plane is given by

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

(x_0, y_0, z_0) is the point of tangency

$\frac{\partial f}{\partial x}(x_0, y_0, z_0)$ is the partial derivative of f with respect to x at (x_0, y_0, z_0)

$\frac{\partial f}{\partial y}(x_0, y_0, z_0)$ is the partial derivative of f with respect to y at (x_0, y_0, z_0)

$\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ is the partial derivative of f with respect to z at (x_0, y_0, z_0)

The normal line to the surface is given by

$$\frac{x - x_0}{\frac{\partial f}{\partial x}} = \frac{y - y_0}{\frac{\partial f}{\partial y}} = \frac{z - z_0}{\frac{\partial f}{\partial z}}$$

(x_0, y_0, z_0) is the point of tangency

$\frac{\partial f}{\partial x}(x_0, y_0, z_0)$ is the partial derivative of f with respect to x at (x_0, y_0, z_0)

$\frac{\partial f}{\partial y}(x_0, y_0, z_0)$ is the partial derivative of f with respect to y at (x_0, y_0, z_0)

$\frac{\partial f}{\partial z}(x_0, y_0, z_0)$ is the partial derivative of f with respect to z at (x_0, y_0, z_0)

Example

For the function $x^2 + y^2 + 2z^2 = 36$,

a) find the equation of the tangent plane at the point $P(-1,2,4)$

b) find the normal line to the tangent plane at the point $P(-1,2,4)$

In order to find the equation of the tangent plane, we have to find the partial derivatives of the function f with respect to each variable at the point $P(-1,2,4)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial x}(-1,2,4) = 2(-1)$$

$$\frac{\partial f}{\partial x}(-1,2,4) = -2$$

and

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial y}(-1,2,4) = 2(2)$$

$$\frac{\partial f}{\partial y}(-1,2,4) = 4$$

and

$$\frac{\partial f}{\partial z} = 4z$$

$$\frac{\partial f}{\partial z}(-1,2,4) = 4(4)$$

$$\frac{\partial f}{\partial z}(-1,2,4) = 16$$

Plugging the slope in each direction and the point $P(-1,2,4)$ into the equation of the tangent plane gives

$$-2\left[x-(-1)\right] + 4(y-2) + 16(z-4) = 0$$

$$-2(x+1)+4(y-2)+16(z-4)=0$$

$$-2x-2+4y-8+16z-64=0$$

$$-2x+4y+16z=74$$

$$-x+2y+8z=37$$

To find the normal line to the tangent plane $-x+2y+8z=37$ at the point $P(-1,2,4)$, we'll plug the partial derivatives we found earlier and the point $P(-1,2,4)$ into the symmetric formula

$$\frac{x-x_0}{\frac{\partial f}{\partial x}}=\frac{y-y_0}{\frac{\partial f}{\partial y}}=\frac{z-z_0}{\frac{\partial f}{\partial z}}$$

$$\frac{x-(-1)}{-2}=\frac{y-2}{4}=\frac{z-4}{16}$$

$$\frac{x+1}{-2}=\frac{y-2}{4}=\frac{z-4}{16}$$

We'll summarize our findings.

Tangent plane	$-x+2y+8z=37$
Normal line to the plane	$\frac{x+1}{-2}=\frac{y-2}{4}=\frac{z-4}{16}$