Calculus 3 Workbook Solutions

Chain rule

krista king MATH

CHAIN RULE FOR MULTIVARIABLE FUNCTIONS

■ 1. If $x = e^t$, $y = t^2 - 3$, and $z = 2t + 1$, use chain rule to find df/dt.

$$f(x, y, z) = xy^2z^3$$

Solution:

Find derivatives of the parametric curve.

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dz}{dt} = 2$$

Find partial derivatives of $f(x, y, z)$.

$$\frac{\partial f}{\partial x} = y^2z^3$$

$$\frac{\partial f}{\partial y} = 2xyz^3$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2$$

Use chain rule to find df/dt.

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial f}{\partial z} \cdot \frac{dz}{dt}$$

$$\frac{df}{dt} = (y^2z^3)e^t + (2xyz^3)2t + (3xy^2z^2)2$$

$$\frac{df}{dt} = ((t^2-3)^2(2t+1)^3)e^t + (2e^t(t^2-3)(2t+1)^3)2t + (3e^t(t^2-3)^2(2t+1)^2)2$$

$$\frac{df}{dt} = e^t((t^2-3)^2(2t+1)^3 + 4t(t^2-3)(2t+1)^3 + 6(t^2-3)^2(2t+1)^2)$$

$$\frac{df}{dt} = e^t(t^2-3)(2t+1)^2((t^2-3)(2t+1) + 4t(2t+1) + 6(t^2-3))$$

$$\frac{df}{dt} = e^t(t^2-3)(2t+1)^2(2t^3 + 15t^2 - 2t - 21)$$

■ 2. If $r = \phi^2$ and $\theta = \phi + \pi$, use chain rule to find $dz/d\phi$ at $\phi = \pi/4$.

$$z(r, \theta) = r^2 \sin\theta$$

Solution:

Find partial derivatives of r and θ at $\phi = \pi/4$.

$$\frac{\partial r}{\partial \phi} = 2\phi$$

$$\frac{\partial r}{\partial \phi}\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

and

$$\frac{\partial \theta}{\partial \phi} = 1$$

$$\frac{\partial \theta}{\partial \phi}\left(\frac{\pi}{4}\right) = 1$$

Find partial derivatives of z at $\phi = \pi/4$.

$$\frac{\partial z}{\partial r} = 2r\sin\theta$$

$$\frac{\partial z}{\partial r} = 2\phi^2\sin\left(\phi + \pi\right)$$

$$\frac{\partial z}{\partial r}\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right)^2\sin\left(\frac{\pi}{4} + \pi\right)$$

$$\frac{\partial z}{\partial r}\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8}\cdot\left(-\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial z}{\partial r}\left(\frac{\pi}{4}\right) = -\frac{\pi^2}{8\sqrt{2}}$$

and

$$\frac{\partial z}{\partial \theta} = r^2\cos\theta$$

$$\frac{\partial z}{\partial \theta} = \phi^4\cos\left(\phi + \pi\right)$$

$$\frac{\partial z}{\partial \theta}\left(\frac{\pi}{4}\right) = \left(\frac{\pi}{4}\right)^4 \cos\left(\frac{\pi}{4} + \pi\right)$$

$$\frac{\partial z}{\partial \theta}\left(\frac{\pi}{4}\right) = \frac{\pi^4}{256} \cdot \left(-\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial z}{\partial \theta}\left(\frac{\pi}{4}\right) = -\frac{\pi^4}{256\sqrt{2}}$$

Use chain rule to find $dz/d\phi$.

$$\frac{dz}{d\phi} = \frac{\partial z}{\partial r} \cdot \frac{dr}{d\phi} + \frac{\partial z}{\partial \theta} \cdot \frac{d\theta}{d\phi}$$

$$\frac{dz}{d\phi} = -\frac{\pi^2}{8\sqrt{2}} \cdot \frac{\pi}{2} - \frac{\pi^4}{256\sqrt{2}} \cdot 1$$

$$\frac{dz}{d\phi} = -\frac{\pi^3}{16\sqrt{2}} - \frac{\pi^4}{256\sqrt{2}}$$

■ 3. If $u = \ln(3t)$ and $v = \ln t$ with $t > 0$, use chain rule to find the global maximum of the function.

$$f(u, v) = 3u - 2v^2$$

Solution:

Find derivatives of u and v.

$$\frac{du}{dt} = \frac{1}{t}$$

$$\frac{dv}{dt} = \frac{1}{t}$$

Find partial derivatives of $f(u, v)$.

$$\frac{\partial f}{\partial u} = 3$$

$$\frac{\partial f}{\partial v} = -4v$$

Use chain rule to find df/dt.

$$\frac{df}{dt} = \frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt}$$

$$\frac{df}{dt} = 3\frac{1}{t} - 4v\frac{1}{t}$$

$$\frac{df}{dt} = \frac{3 - 4\ln t}{t} \text{ with } t > 0$$

Solve $f'(t) = 0$ to find critical points.

$$\frac{3 - 4\ln t}{t} = 0$$

$$3 - 4\ln t = 0$$

$$3 = 4\ln t$$

$$\ln t = \frac{3}{4}$$

$$t = e^{3/4}$$

Since $f'(t) > 0$ for $t < e^{3/4}$, and $f'(t) < 0$ for $t > e^{3/4}$, then $t = e^{3/4}$ is the global maximum of the function.

$$f(e^{3/4}) = 3\ln(3e^{3/4}) - 2(\ln(e^{3/4}))^2$$

$$f(e^{3/4}) = \ln(27) + \frac{9}{4} - 2\frac{3^2}{4^2}$$

$$f(e^{3/4}) = \frac{9}{8} + \ln(27)$$

CHAIN RULE FOR MULTIVARIABLE FUNCTIONS AND TREE DIAGRAMS

■ 1. If $x = \sin(t + s)$, $y = 2ts$, and $z = 2t - 5s$, use chain rule to find the partial derivatives f_t and f_s.

$$f(x, y, z) = 7x + 2y^2z$$

Solution:

Find partial derivatives with respect to s and t of x, y, and z.

$$x_s = \cos(t + s) \qquad x_t = \cos(t + s)$$

$$y_s = 2t \qquad y_t = 2s$$

$$z_s = -5 \qquad z_t = 2$$

Find partial derivatives for $f(x, y, z)$.

$$\frac{\partial f}{\partial x} = 7$$

$$\frac{\partial f}{\partial y} = 4yz$$

$$\frac{\partial f}{\partial z} = 2y^2$$

Use chain rule to find $\partial f/\partial t$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot x_s + \frac{\partial f}{\partial y} \cdot y_s + \frac{\partial f}{\partial z} \cdot z_s$$

$$\frac{\partial f}{\partial s} = (7)\cos(t+s) + (4yz)2t + (2y^2)(-5)$$

$$\frac{\partial f}{\partial s} = 7\cos(t+s) + 8t(2ts)(2t-5s) - 10(2ts)^2$$

$$\frac{\partial f}{\partial s} = 7\cos(t+s) + 8st^2(4t-15s)$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot x_t + \frac{\partial f}{\partial y} \cdot y_t + \frac{\partial f}{\partial z} \cdot z_t$$

$$\frac{\partial f}{\partial t} = (7)\cos(t+s) + (4yz)2s + (2y^2)2$$

$$\frac{\partial f}{\partial t} = 7\cos(t+s) + 8s(2ts)(2t-5s) + 4(2ts)^2$$

$$\frac{\partial f}{\partial t} = 7\cos(t+s) + 16s^2t(3t-5s)$$

The partial derivatives of f with respect to s and t are

$$\frac{\partial f}{\partial s} = 7\cos(t+s) + 8st^2(4t-15s)$$

$$\frac{\partial f}{\partial t} = 7\cos(t+s) + 16s^2t(3t-5s)$$

■ 2. If $x = \log_2(ts)$ and $y = \log_3(2t + s)$, use chain rule to find partial derivatives f_s and f_t at $(1,1)$.

$$f(x, y) = x^2 - 2xy - y^2 + x + 3y - 4$$

Solution:

Evaluate x and y at $(1,1)$.

$$x(1,1) = \log_2((1)(1)) = 0$$

$$y(1,1) = \log_3(2(1) + 1) = 1$$

Find partial derivatives of x and y at $(1,1)$.

$$x_s = \frac{1}{s \ln(2)} = \frac{1}{\ln(2)}$$

$$x_t = \frac{1}{t \ln(2)} = \frac{1}{\ln(2)}$$

and

$$y_s = \frac{1}{(2t + s)\ln(3)} = \frac{1}{3 \ln(3)}$$

$$y_t = \frac{2}{(2t + s)\ln(3)} = \frac{2}{3 \ln(3)}$$

Find partial derivatives f with respect to x and y.

$$\frac{\partial f}{\partial x} = 2x - 2y + 1 = 2(0) - 2(1) + 1 = -1$$

$$\frac{\partial f}{\partial y} = -2x - 2y + 3 = -2(0) - 2(1) + 3 = 1$$

Use chain rule to find $\partial f/\partial s$ and $\partial f/\partial t$ at $(1,1)$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot x_s + \frac{\partial f}{\partial y} \cdot y_s$$

$$\frac{\partial f}{\partial s} = (-1)\frac{1}{\ln(2)} + 1\frac{1}{3\ln(3)}$$

$$\frac{\partial f}{\partial s} = \frac{1}{3\ln(3)} - \frac{1}{\ln(2)}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot x_t + \frac{\partial f}{\partial y} \cdot y_t$$

$$\frac{\partial f}{\partial t} = (-1)\frac{1}{\ln(2)} + 1\frac{2}{3\ln(3)}$$

$$\frac{\partial f}{\partial t} = \frac{2}{3\ln(3)} - \frac{1}{\ln(2)}$$

The partial derivatives of f at $(1,1)$ with respect to s and t are

$$\frac{\partial f}{\partial s}(1,1) = \frac{1}{3\ln(3)} - \frac{1}{\ln(2)}$$

$$\frac{\partial f}{\partial t}(1,1) = \frac{2}{3\ln(3)} - \frac{1}{\ln(2)}$$

■ 3. If $x = 2t - s$ and $y = t + 2s$, use chain rule to find the point (s, t) where $f_t = f_s = 0$.

$$f(x, y) = 2x^2 - 3xy + y^2 + y + 9$$

Solution:

Find partial derivatives of x and y with respect to s and t.

$$x_s = -1 \qquad x_t = 2$$

$$y_s = 2 \qquad y_t = 1$$

Find partial derivatives of f with respect to x and y.

$$\frac{\partial f}{\partial x} = 4x - 3y$$

$$\frac{\partial f}{\partial y} = -3x + 2y + 1$$

Use chain rule to find $\partial f/\partial s$ and $\partial f/\partial t$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot x_s + \frac{\partial f}{\partial y} \cdot y_s$$

$$\frac{\partial f}{\partial s} = (4x - 3y)(-1) + (-3x + 2y + 1)(2)$$

$$\frac{\partial f}{\partial s} = -10x + 7y + 2$$

$$\frac{\partial f}{\partial s} = -10(2t - s) + 7(t + 2s) + 2$$

$$\frac{\partial f}{\partial s} = 24s - 13t + 2$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot x_t + \frac{\partial f}{\partial y} \cdot y_t$$

$$\frac{\partial f}{\partial t} = (4x - 3y)(2) + (-3x + 2y + 1)(1)$$

$$\frac{\partial f}{\partial t} = 5x - 4y + 1$$

$$\frac{\partial f}{\partial t} = 5(2t - s) - 4(t + 2s) + 1$$

$$\frac{\partial f}{\partial t} = -13s + 6t + 1$$

Solve the system of equations.

$$-13s + 6t + 1 = 0$$

$$24s - 13t + 2 = 0$$

We get

$$-13s + 6t = -1$$

$$24s - 13t = -2$$

then

$$-169s + 78t = -13$$

$$144s - 78t = -12$$

Add the equations.

$$-169s + 78t + (144s - 78t) = -13 + (-12)$$

$$-169s + 78t + 144s - 78t = -13 - 12$$

$$-25s = -25$$

$$s = 1$$

Then

$$-13(1) + 6t = -1$$

$$6t = 12$$

$$t = 2$$

So the point we're looking for is $(s, t) = (1,2)$.

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