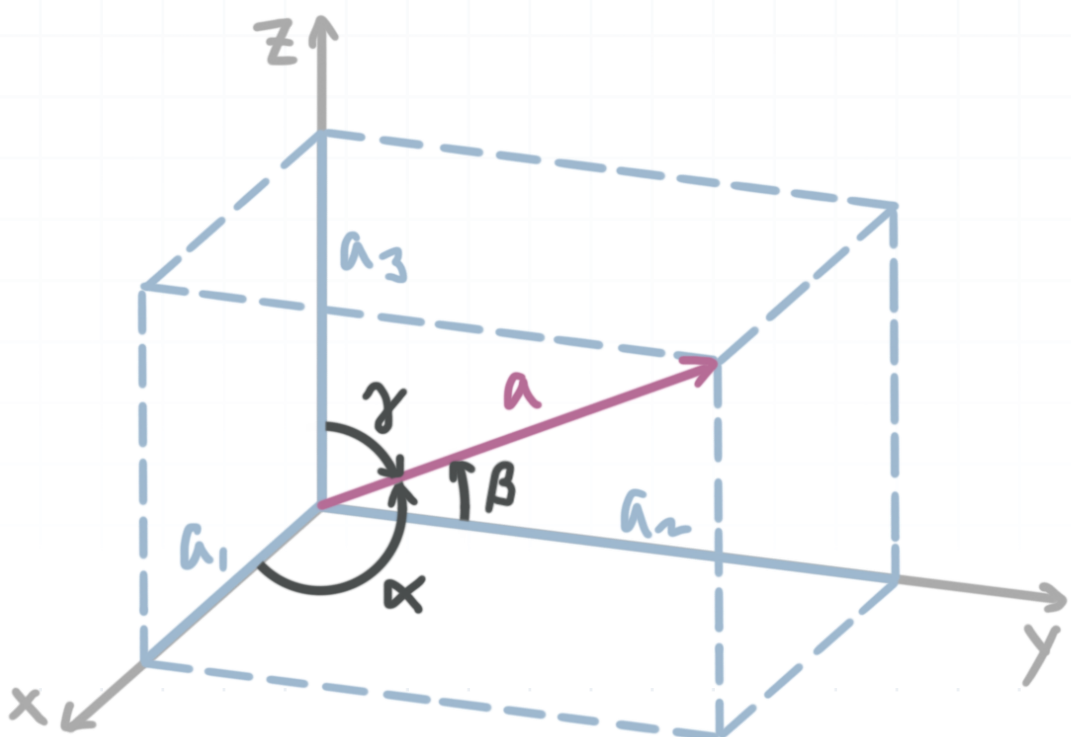
Direction cosines and direction angles

The **direction angles** of a non-zero vector $a = \langle a_1, a_2, a_3 \rangle$ are the angles α (alpha), β (beta), and γ (gamma) that the vector a makes with the positive x-, y-, and z-axes, respectively. In other words, α is the direction angle between a and the positive x-axis, β is the direction angle between a and the positive y-axis, and γ is the direction angle between a and the positive z-axis.



If the length (magnitude) of a is

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

then we take the cosine of each direction angle to get the **direction cosines** of the vector a.

$$\cos\alpha = \frac{a_1}{|a|} \qquad \cos\beta = \frac{a_2}{|a|} \qquad \cos\gamma = \frac{a_3}{|a|}$$

Once we have the direction cosines, we can find the direction angles by applying the inverse cosine to both sides of each of these direction cosine equations.

$$\alpha = \arccos \frac{a_1}{|a|} \qquad \beta = \arccos \frac{a_2}{|a|} \qquad \gamma = \arccos \frac{a_3}{|a|}$$

It can also be helpful to know that the direction angles of a will satisfy

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

and that the direction cosines of a are the components of the unit vector in the direction of a.

$$\frac{1}{|a|}a = \langle \cos \alpha, \cos \beta, \cos \gamma \rangle$$

Let's do an example where we find the direction cosines and direction angles of a vector.

Example

Find the direction angles of the vector $a = \langle 5, -3, 1 \rangle$.

We find the magnitude of the vector a using the distance formula.

$$|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$|a| = \sqrt{5^2 + (-3)^2 + 1^2}$$

$$|a| = \sqrt{25+9+1}$$

$$|a| = \sqrt{35}$$

Plugging the vector's components and magnitude into the direction cosine formulas, we get

$$\cos\alpha = \frac{5}{\sqrt{35}} \qquad \cos\beta = \frac{-3}{\sqrt{35}} \qquad \cos\gamma = \frac{1}{\sqrt{35}}$$

Now that we have the direction cosines, we can apply the inverse cosine to both sides of each equation to find the direction angles.

$$\alpha = \arccos\frac{5}{\sqrt{35}} \qquad \beta = \arccos\frac{-3}{\sqrt{35}} \qquad \gamma = \arccos\frac{1}{\sqrt{35}}$$

$$\alpha \approx 32.3^\circ \qquad \beta \approx 120.5^\circ \qquad \gamma \approx 80.3^\circ$$

Let's do one more example, this time with the vector given in terms of the standard unit vectors.

Example

A vector b has direction angles $\alpha = \pi/3$ and $\beta = \pi/6$. Find the third direction angle γ.

Since we already know two of the direction angles of the vector b, we can use $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$ to find γ.

$$\cos^2\left(\frac{\pi}{3}\right) + \cos^2\left(\frac{\pi}{6}\right) + \cos^2\gamma = 1$$

$$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 + \cos^2\gamma = 1$$

$$\frac{1}{4} + \frac{3}{4} + \cos^2\gamma = 1$$

Rearrange the equation to solve for γ.

$$\cos^2\gamma = 0$$

$$\cos\gamma = 0$$

$$\gamma = \frac{\pi}{2}$$

The direction angles are $\alpha = \pi/3$, $\beta = \pi/6$, and $\gamma = \pi/2$, or $\alpha = 60^\circ$, $\beta = 30^\circ$, and $\gamma = 90^\circ$.