Topic: Linear approximation in two variables

Question: At the point $(2,1)$, find the linear approximation of $f(x, y) = x^3 - y^3$ and use it to approximate $f(2.05,0.95)$.

Answer choices:

A $L(2.05,0.95) = 7$

B $L(2.05,0.95) = 7.75$

C $L(2.05,0.95) = 7.9$

D $L(2.05,0.95) = 7.75775$

Solution: B

The linear approximation of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Since we know point $(a, b) = (2,1)$, we can easily find $f(a, b) = f(2,1) = 2^3 - 1^3 = 8 - 1 = 7$. Next, we need to find values for $f_x(a, b) = f_x(2,1)$ and $f_y(a, b) = f_y(2,1)$.

To find the value for $f_x(2,1)$, we have to find the partial derivative $f_x(x, y)$ by treating y (and therefore y^3) as a constant, while differentiating $f(x, y)$ with respect to x, and then plug the given point into the partial derivative.

$$f(x, y) = x^3 - y^3$$

$$f_x(x, y) = 3x^2 - 0 = 3x^2$$

$$f_x(2,1) = 3(2^2) = 12$$

Similarly, we can find the partial derivative $f_y(x, y)$ by treating x (and therefore x^3) as a constant, while differentiating $f(x, y)$ with respect to y. Then we'll plug the given point into the partial derivative.

$$f(x, y) = x^3 - y^3$$

$$f_y(x, y) = 0 - 3y^2 = -3y^2$$

$$f_y(2,1) = -3(1^2) = -3$$

Using these values for $f_x(2,1)$ and $f_y(2,1)$, we can now find the linear approximation of f.

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

$$L(x,y) = f(2,1) + f_x(2,1)(x-2) + f_y(2,1)(y-1)$$

$$L(x,y) = 7 + 12(x-2) - 3(y-1)$$

$$L(x,y) = 7 + 12x - 24 - 3y + 3$$

$$L(x,y) = 12x - 3y - 14$$

Since we know that the value of $f(x,y)$ near point (a,b) is approximately equal to the linear approximation of f at (a,b), we can use it to find an approximate value for $f(2.1,0.95)$.

$$f(x,y) \approx L(x,y)$$

$$f(x,y) \approx 12x - 3y - 14$$

$$f(2.05,0.95) \approx 12(2.05) - 3(0.95) - 14$$

$$f(2.05,0.95) \approx 24.6 - 2.85 - 14$$

$$f(2.05,0.95) \approx 7.75$$

Therefore, the linear approximation allows us to approximate the value of $f(2.05,0.95)$ as 7.75. If we compare this to the actual value of $f(2.05,0.95) = 2.05^3 - 0.95^3 = 7.75775$, we see that the linear approximation is actually pretty close to the actual value!

Topic: Linear approximation in two variables

Question: At the point $(0,0)$, find the linear approximation of $f(x, y) = 2\cos x \sin y + 1$ and use it to approximate $f(0.1,0.1)$.

Answer choices:

A $L(0.1,0.1) = 1.1987$

B $L(0.1,0.1) = 1$

C $L(0.1,0.1) = 1.25$

D $L(0.1,0.1) = 1.2$

Solution: D

The linearization of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Since we know point $(a, b) = (0,0)$, we can easily find $f(a, b) = f(0,0) = 2\cos 0 \sin 0 + 1 = 1$. Next, we need to find values for $f_x(a, b) = f_x(0,0)$ and $f_y(a, b) = f_y(0,0)$.

To find the value for $f_x(0,0)$, we must first find the partial derivative $f_x(x, y)$ by treating y (and therefore $\sin y$) as a constant, while differentiating $f(x, y)$ with respect to x, and then plug the given point into the partial derivative.

$$f(x, y) = 2\cos x \sin y + 1$$

$$f_x(x, y) = 2(-\sin x)\sin y + 0 = -2\sin x \sin y$$

$$f_x(0,0) = -2\sin 0 \sin 0 = -2(0)(0) = 0$$

Similarly, we can find the partial derivative $f_y(x, y)$ by treating x (and therefore $\cos x$) as a constant, while differentiating $f(x, y)$ with respect to y, and use this to find $f_y(0,0)$:

$$f(x, y) = 2\cos x \sin y + 1$$

$$f_y(x, y) = 2\cos x(\cos y) + 0 = 2\cos x \cos y$$

$$f_y(0,0) = 2\cos 0 \cos 0 = 2(1)(1) = 2$$

Using these values for $f_x(0,0)$ and $f_y(0,0)$, we can now find the linear approximation of f.

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$L(x, y) = f(0,0) + f_x(0,0)(x - 0) + f_y(0,0)(y - 0)$$

$$L(x, y) = 1 + 0(x - 0) + 2(y - 0)$$

$$L(x, y) = 1 + 2y - 2(0) = 2y + 1$$

Since we know that the value of $f(x, y)$ near point (a, b) is approximately equal to $L(a, b)$, we can find an approximate value for $f(0.1,0.1)$ using the linear approximation.

$$f(x, y) \approx L(x, y)$$

$$f(x, y) \approx 2y + 1$$

$$f(0.1,0.1) \approx 2(0.1) + 1$$

$$f(0.1,0.1) \approx 1.2$$

Therefore, the linear approximation allows us to approximate the value of $f(0.1,0.1)$ as 1.2. If we compare this to the actual value of $f(0.1,0.1) = 2\cos(0.1)\sin(0.1) + 1 \approx 1.1987$, we can see that the linear approximation is pretty close to the actual value!

Topic: Linear approximation in two variables

Question: Find the linear approximation of the function.

At the point $(1,-3)$, find the linear approximation of $f(x,y)=2x^2y^2-4xy^2-3y$ and use it to approximate $f(1.1,-2.9)$.

Answer choices:

A $\quad L(1.1,-2.9)=-8.1$

B $\quad L(1.1,-2.9)=-8$

C $\quad L(1.1,-2.9)=-7.9518$

D $\quad L(1.1,-2.9)=-9$

Solution: A

The linear approximation of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Since we know point $(a, b) = (1, -3)$, we can easily find $f(a, b) = f(1, -3) = 2(1^2)(-3^2) - 4(1)(-3^2) - 3(-3) = -9$.

Next, we need to find values of the partial derivatives $f_x(a, b) = f_x(1, -3)$ and $f_y(a, b) = f_y(1, -3)$.

$$f(x, y) = 2x^2y^2 - 4xy^2 - 3y$$

$$f_x(x, y) = 2(2x)y^2 - 4(1)y^2 - 0 = 4xy^2 - 4y^2$$

$$f_x(1, -3) = 4(1)(-3)^2 - 4(-3)^2 = 0$$

$$f_y(x, y) = 2x^2(2y) - 4x(2y) - 3(1) = 4x^2y - 8xy - 3$$

$$f_y(1, -3) = 4(1)^2(-3) - 8(1)(-3) - 3 = 9$$

Using these values for $f_x(1, -3)$ and $f_y(1, -3)$, we can now find the linear approximation of f.

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

$$L(x, y) = f(1, -3) + f_x(1, -3)(x - 1) + f_y(1, -3)(y + 3)$$

$$L(x, y) = -9 + 0(x - 1) + 9(y + 3) = 9y + 18$$

Since we know that the value of $f(x, y)$ near point (a, b) is approximately equal to $L(a, b)$, we can find an approximate value for $f(1.1, -2.9)$ using the linear approximation.

$$f(x, y) \approx L(x, y) = 9y + 18$$

$$f(1.1, -2.9) \approx 9(-2.9) + 18 = -8.1$$

Which is fairly close to the actual value of f at this point.

$$f(1.1, -2.9) = 2(1.1)^2(-2.9)^2 - 4(1.1)(-2.9)^2 - 3(-2.9) = -7.9518$$