Chain rule for multivariable functions

Previously we've been asked to find partial derivatives of multivariable functions, like

$$f(x, y, z) = xyz$$

For a multivariable function like this one, f is the dependent variable, and x, y, and z are the independent variables. When we take partial derivatives of a function like this one, we need one partial derivative with respect to each of the independent variables. Since there are three independent variables, we'll have three partial derivatives.

$$\frac{\partial f}{\partial x} = yz$$

$$\frac{\partial f}{\partial y} = xz$$

$$\frac{\partial f}{\partial z} = xy$$

But now we need to introduce a new type of multivariable function, one in which we insert *intermediate* variables in between the dependent and independent variables. For example, if we defined parametric equations for x, y, and z from the function above, we might have something like

$f(x, y, z) = xyz$, where

$$x = t$$

$$y = t^2$$

$$z = t^3$$

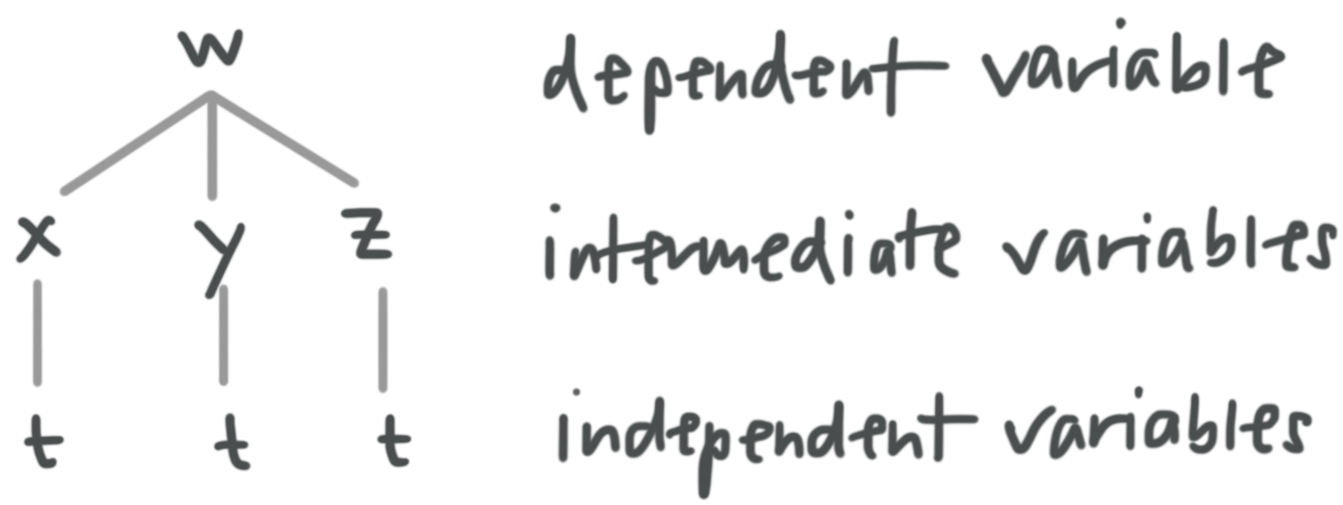
In this case, f is the dependent variable, x, y, and z are intermediate variables, and the parameter t is the independent variable. Just as before, the number of partial derivatives we'll find for this function depends on the number of independent variables. Since we have just one independent variable, we'll only have one derivative. And since we only have one derivative, it'll be a "normal" derivative, instead of a partial derivative. This is called a Case I function.

Let's look in more detail at what we call Case I and Case II functions.

Case I

A Case I scenario is when we have **one independent variable**.

More specifically, the function is defined for one dependent variable in terms of multiple intermediate variables, which are all in terms of one independent variable, like this:



Case I always results in only one derivative, which is the derivative of the dependent variable with respect to the independent variable. For the case described in the tree diagram above, the formula for the partial derivative would be

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Notice how we multiply the **partial** derivatives of the dependent variable with respect to the intermediate variables, by the “**normal**” derivatives of the intermediate variables with respect to the independent variable, and then add those products together.

Let’s try a Case I example.

Example

Use chain rule to find the partial derivatives of the multivariable function.

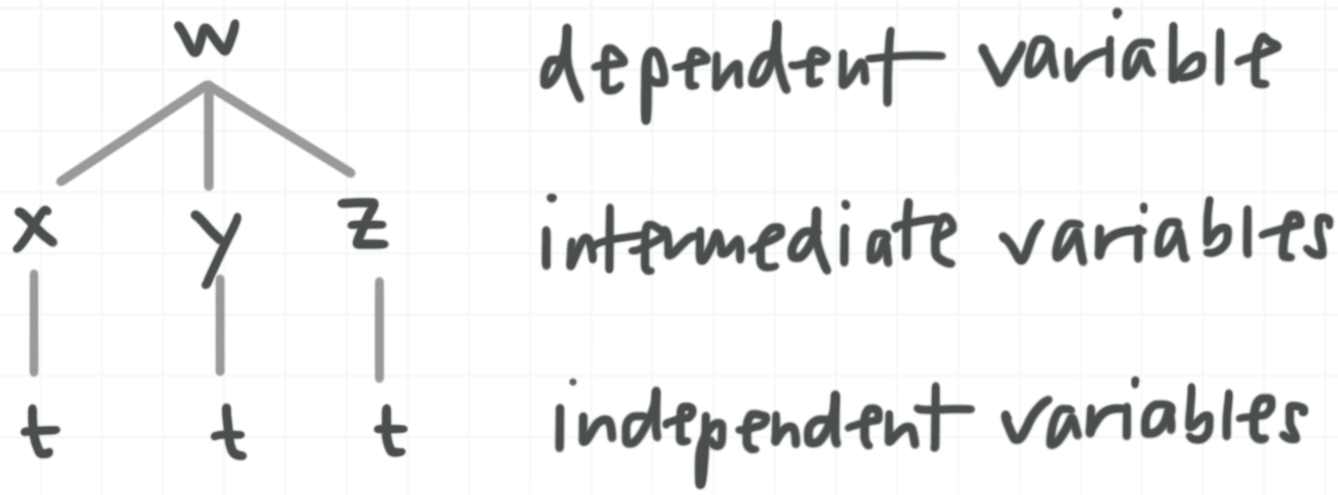
$w = x^2y - 6y^3\sqrt{z}$, where

$$x = 5t^2$$

$$y = 4t + 1$$

$$z = t^3 - 5t$$

Since w is defined in terms of x, y, and z, and x, y, and z are all defined in terms of t, we have the following tree diagram:



The number of independent variables dictates the number of derivatives we need to find. In this case, with only have one independent variable, we'll only have one derivative, which will be the derivative of the dependent variable w with respect to the independent variable t.

$$\frac{dw}{dt}=\frac{\partial w}{\partial x}\frac{dx}{dt}+\frac{\partial w}{\partial y}\frac{dy}{dt}+\frac{\partial w}{\partial z}\frac{dz}{dt}$$

We need to find each component of the formula for $\partial w/\partial t$, and we'll start with the partial derivatives of w with respect to the intermediate variables x, y, and z.

$$\frac{\partial w}{\partial x}=(2x)y$$

$$\frac{\partial w}{\partial x}=2xy$$

and

$$\frac{\partial w}{\partial y}=x^2(1)-6(3y^2)\sqrt{z}$$

$$\frac{\partial w}{\partial y}=x^2-18y^2\sqrt{z}$$

and

$$\frac{\partial w}{\partial z} = -6y^3\left(\frac{1}{2}\right)z^{-\frac{1}{2}}$$

$$\frac{\partial w}{\partial z} = -3y^3z^{-\frac{1}{2}}$$

Now we just need to find the derivatives of the intermediate variables x, y, and z with respect to the independent variable t.

$$\frac{dx}{dt} = 10t$$

$$\frac{dy}{dt} = 4$$

$$\frac{dz}{dt} = 3t^2 - 5$$

Plug everything we found into the equation for $\partial w/\partial t$.

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

$$\frac{dw}{dt} = (2xy)(10t) + \left(x^2 - 18y^2\sqrt{z}\right)(4) + \left(-3y^3z^{-\frac{1}{2}}\right)\left(3t^2 - 5\right)$$

$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} - 9y^3z^{-\frac{1}{2}}t^2 + 15y^3z^{-\frac{1}{2}}$$

$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} - \frac{9y^3t^2}{\sqrt{z}} + \frac{15y^3}{\sqrt{z}}$$

$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} + \frac{15y^3 - 9y^3t^2}{\sqrt{z}}$$

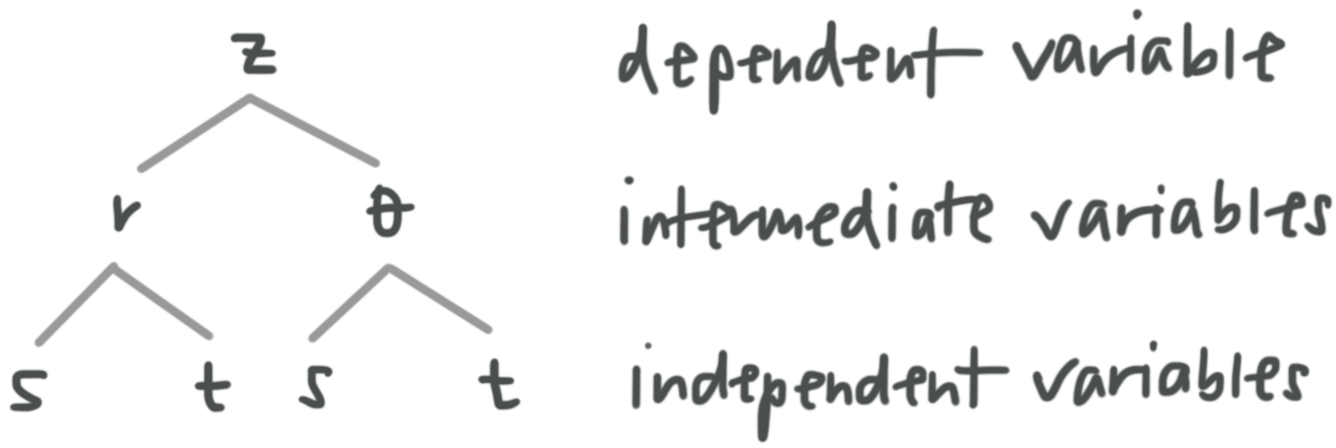
$$\frac{dw}{dt} = 20xyt + 4x^2 - 72y^2\sqrt{z} + \frac{3y^3\left(5 - 3t^2\right)}{\sqrt{z}}$$

This is the derivative of w with respect to t.

Case II

A Case II scenario is when we have **multiple independent variables**.

More specifically, the function is defined for one dependent variable in terms of multiple intermediate variables, which are all in terms of multiple independent variables, like this:



Case II always results in one partial derivative for each of the independent variables, and they'll be the partial derivatives of the dependent variable with respect to each independent variable. For the tree diagram above, the formulas for the partial derivatives would be

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

Notice how we multiply the **partial** derivatives of the dependent variable with respect to the intermediate variables, by the **partial** derivatives of the intermediate variables with respect to the independent variables, and then add the products together.

For Case II, we just have to make sure that we separate the independent variables into different partial derivatives.

Let's try a Case II example.

Example

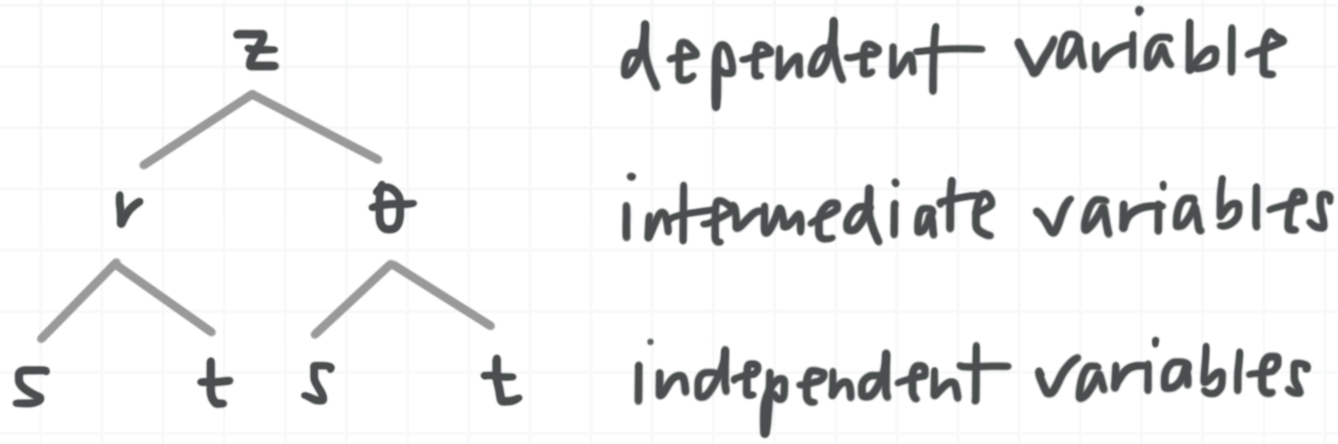
Use chain rule to find the partial derivatives of the multivariable function.

$$z = \ln r + r^2 \sin\theta$$

$$r = 3s^2 - t$$

$$\theta = 2t^2 - \frac{4}{s^2}$$

Since z is defined in terms of r and θ, and r and θ are all defined in terms of s and t, we have the following tree diagram:



The number of independent variables dictates the number of partial derivatives we need to find. In this case, with have two independent variables, so we'll have two partial derivatives, which will be the partial derivatives of the dependent variable z with respect to the independent variables s and t.

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial s}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial t}=\frac{\partial z}{\partial r}\frac{\partial r}{\partial t}+\frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

We need to find each component of the formulas for $\partial z/\partial s$ and $\partial z/\partial t$, and we'll start with the partial derivatives of z with respect to the intermediate variables r and θ.

Now we can solve for our four partial derivatives.

$$\frac{\partial z}{\partial r}=\frac{1}{r}+2r\sin\theta$$

and

$$\frac{\partial z}{\partial \theta}=r^2\cos\theta$$

Now we just need to find the partial derivatives of the intermediate variables r and θ with respect to the independent variables s and t.

With respect to s:

$$\frac{\partial r}{\partial s} = 6s$$

and

$$\frac{\partial \theta}{\partial s} = \frac{8}{s^3}$$

With respect to t:

$$\frac{\partial r}{\partial t} = -1$$

and

$$\frac{\partial \theta}{\partial t} = 4t$$

Plug everything we found into the equations for $\partial z/\partial s$ and $\partial z/\partial t$.

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s}$$

$$\frac{\partial z}{\partial s} = \left(\frac{1}{r} + 2r\sin\theta\right)(6s) + \left(r^2\cos\theta\right)\left(\frac{8}{s^3}\right)$$

$$\frac{\partial z}{\partial s} = \frac{6s}{r} + 12rs\sin\theta + \frac{8r^2\cos\theta}{s^3}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t}$$

$$\frac{\partial z}{\partial t} = \left(\frac{1}{r} + 2r\sin\theta\right)(-1) + \left(r^2\cos\theta\right)(4t)$$

$$\frac{\partial z}{\partial t} = -\frac{1}{r} - 2r\sin\theta + 4r^2t\cos\theta$$

These are the partial derivatives of z with respect to s and t.