Calculus 3 Workbook Solutions

Gradient vectors

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GRADIENT VECTORS

■ 1. Find the gradient vector ∇f at $(\sqrt{\pi},0,0)$.

$$f(x,y,z)=\sin(x^2+2y^2-z^2-2xyz)$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y,z)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

$$\nabla f(x,y,z)=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$$

Use the chain rule to calculate partial derivatives.

$$\frac{\partial f}{\partial x}=\cos(x^2+2y^2-z^2-2xyz)\frac{\partial}{\partial x}(x^2+2y^2-z^2-2xyz)$$

$$\frac{\partial f}{\partial x}=(2x-2yz)\cos(x^2+2y^2-z^2-2xyz)$$

$$\frac{\partial f}{\partial x}=2(x-yz)\cos(x^2+2y^2-z^2-2xyz)$$

and

$$\frac{\partial f}{\partial y}=\cos(x^2+2y^2-z^2-2xyz)\frac{\partial}{\partial y}(x^2+2y^2-z^2-2xyz)$$

$$\frac{\partial f}{\partial y} = (4y - 2xz)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

$$\frac{\partial f}{\partial y} = 2(2y - xz)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

and

$$\frac{\partial f}{\partial z} = \cos(x^2 + 2y^2 - z^2 - 2xyz)\frac{\partial}{\partial z}(x^2 + 2y^2 - z^2 - 2xyz)$$

$$\frac{\partial f}{\partial z} = (-2z - 2xy)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

$$\frac{\partial f}{\partial z} = -2(z + xy)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

Evaluate the partial derivatives at $(\sqrt{\pi},0,0)$.

$$\frac{\partial f}{\partial x}(\sqrt{\pi},0,0) = 2(\sqrt{\pi} - (0)(0))\cos(\pi + 2(0)^2 - (0)^2 - 2\sqrt{\pi}(0)(0))$$

$$\frac{\partial f}{\partial x}(\sqrt{\pi},0,0) = 2\sqrt{\pi}\cos(\pi)$$

$$\frac{\partial f}{\partial x}(\sqrt{\pi},0,0) = -2\sqrt{\pi}$$

and

$$\frac{\partial f}{\partial y}(\sqrt{\pi},0,0) = 2(2(0) - \sqrt{\pi}(0))\cos(\pi + 2(0)^2 - (0)^2 - 2\sqrt{\pi}(0)(0))$$

$$\frac{\partial f}{\partial y}(\sqrt{\pi},0,0) = 0\cos(\pi)$$

$$\frac{\partial f}{\partial y}(\sqrt{\pi},0,0) = 0$$

and

$$\frac{\partial f}{\partial z}(\sqrt{\pi},0,0) = -2((0) + \sqrt{\pi}(0))\cos(\pi + 2(0)^2 - (0)^2 - 2\sqrt{\pi}(0)(0))$$

$$\frac{\partial f}{\partial z}(\sqrt{\pi},0,0) = 0\cos(\pi)$$

$$\frac{\partial f}{\partial z}(\sqrt{\pi},0,0) = 0$$

■ 2. Find unit gradient vector of the function f at $(-2,1)$.

$$f(t, s) = \frac{4t - st^4}{t^2 s}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

The unit gradient vector (gradient vector of magnitude 1) is given by

$$\widehat{\nabla f} = \frac{\nabla f}{\| \nabla f \|}$$

Simplify $f(t,s)$.

$$f(t,s)=\frac{4t}{t^2s}-\frac{st^4}{t^2s}=\frac{4}{ts}-t^2$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial t}=-\frac{4}{t^2s}-2$$

$$\frac{\partial f}{\partial s}=-\frac{4}{ts^2}$$

Evaluate the partial derivatives at $(-2,1)$.

$$\frac{\partial f}{\partial t}(-2,1)=-\frac{4}{(-2)^2(1)}-2(-2)=-1+4=3$$

$$\frac{\partial f}{\partial s}(-2,1)=-\frac{4}{(-2)(1)^2}=2$$

Calculate the magnitude of ∇f.

$$\|\nabla f\|=\sqrt{3^2+2^2}=\sqrt{13}$$

So

$$\widehat{\nabla f}=\left\langle\frac{3}{\sqrt{13}},\frac{2}{\sqrt{13}}\right\rangle$$

■ 3. Find the point where the gradient vector of the function f is equal to the zero vector.

$$f(x, y) = \ln \frac{(x-2)^2 y}{x-y}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Simplify $f(x, y)$ using laws of logarithms.

$$f(x, y) = \ln((x-2)^2) + \ln y - \ln(x-y)$$

$$f(x, y) = 2\ln(x-2) + \ln y - \ln(x-y)$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2}{x-2} - \frac{1}{x-y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{y} + \frac{1}{x-y}$$

Since the gradient is equal to 0, we get a system of equations in terms of x and y.

$$\frac{2}{x-2} - \frac{1}{x-y} = 0$$

$$\frac{1}{y} + \frac{1}{x-y} = 0$$

Solve the system for x and y.

$$2(x-y)-(x-2)=0$$

$$x-y+y=0$$

$$x-2y+2=0$$

$$x=0$$

So the solution is $x=0$ and $y=1$, or $(0,1)$.

■ 4. Find and identify the set of points where the magnitude of the gradient vector of the function f is equal to 1.

$$f(x,y)=x^2+4y^2-2x+8y-5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x}=2x-2$$

$$\frac{\partial f}{\partial y}=8y+8$$

Calculate the magnitude of ∇f and set it equal to 1.

$$\| \nabla f \| = \sqrt{(2x-2)^2 + (8y+8)^2} = 1$$

Since $(2x-2)^2 + (8y+8)^2 > 0$, square each side of the equation.

$$(2x-2)^2 + (8y+8)^2 = 1$$

This is an ellipse. To find its center and vertices, rewrite the equation in standard form.

$$4(x-1)^2 + 64(y+1)^2 = 1$$

$$\frac{(x-1)^2}{0.5^2} + \frac{(y+1)^2}{0.125^2} = 1$$

So the set of points where $\| \nabla f \| = 1$ is the ellipse

$$\frac{(x-1)^2}{0.5^2} + \frac{(y+1)^2}{0.125^2} = 1$$

with center $(1,-1)$, with semi-major axis 0.5 and semi-minor axis 0.125.

■ 5. Find the directional derivative of the function f in the direction $m = 3\mathbf{i} - 4\mathbf{j}$ at $(0,4)$.

$$f(x,y) = 2^x(y^2-1)$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 2^x(y^2 - 1)\ln 2$$

$$\frac{\partial f}{\partial y} = 2y2^x = y2^{x+1}$$

Evaluate the partial derivatives at $(0,4)$.

$$\frac{\partial f}{\partial x}(0,4) = 2^0(4^2 - 1)\ln 2 = 15\ln 2$$

$$\frac{\partial f}{\partial y}(0,4) = 2(4)2^0 = 8$$

So

$$\nabla f(x, y) = \langle 15\ln 2, 8 \rangle$$

The directional derivative in the direction of vector m is given by

$$\nabla f \cdot \widehat{m}$$

where $\widehat{m}$ is the unit vector in the direction of m, which is

$$\widehat{m} = \frac{m}{\| \nabla m \|}$$

Find the magnitude of the vector m.

$$\| \nabla m \| = \sqrt{3^2 + 4^2} = 5$$

So

$$\widehat{m} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

Then the directional derivative is

$$\nabla f \cdot \widehat{m} = \langle 15 \ln 2, 8 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

$$\nabla f \cdot \widehat{m} = \frac{3(15 \ln 2)}{5} - \frac{4(8)}{5}$$

$$\nabla f \cdot \widehat{m} = 9 \ln 2 - \frac{32}{5}$$

GRADIENT VECTORS AND THE TANGENT PLANE

■ 1. Use the gradient vector to find the tangent line to the curve $(y-2)^2 - e^x = 0$ at $(0,3)$.

Solution:

Given the multivariable function,

$$f(x, y) = (y-2)^2 - e^x$$

the tangent line to the curve has the equation

$$a(x - x_0) + b(y - y_0) = 0$$

where (x_0, y_0) is the point on the curve, and

$$\nabla f(x, y) = \langle a, b \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = -e^x$$

$$\frac{\partial f}{\partial y} = 2(y-2)$$

Evaluate the partial derivatives at $(0,3)$.

$$\frac{\partial f}{\partial x}(0,3) = -e^x = -1$$

$$\frac{\partial f}{\partial y}(0,3) = 2(3-2) = 2$$

So the tangent line to the curve is

$$-1(x-0) + 2(y-3) = 0$$

$$x - 2y = -6$$

■ 2. Use the gradient vector to find the tangent plane to the surface $(r+1)\sin(\phi+\pi)\tan(\theta) = 0$ at $(2,\pi/6,\pi/4)$.

Solution:

Given the multivariable function,

$$f(r,\phi,\theta) = (r+1)\sin(\phi+\pi)\tan(\theta)$$

the tangent line to the curve has the equation

$$a(r-r_0) + b(\phi-\phi_0) + c(\theta-\theta_0) = 0$$

where (r_0,ϕ_0,θ_0) is the point on the surface, and

$$\nabla f(r,\phi,\theta) = \langle a,b,c\rangle = \left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial \theta} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial r} = \sin(\phi + \pi)\tan(\theta)$$

$$\frac{\partial f}{\partial \phi} = (r + 1)\cos(\phi + \pi)\tan(\theta)$$

$$\frac{\partial f}{\partial \theta} = (r + 1)\sin(\phi + \pi)\sec^2(\theta)$$

Evaluate the partial derivatives at $(2,\pi/6,\pi/4)$.

$$a = \sin\left(\frac{\pi}{6} + \pi\right)\tan\left(\frac{\pi}{4}\right) = -\frac{1}{2}$$

$$b = (2 + 1)\cos\left(\frac{\pi}{6} + \pi\right)\tan\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{3}}{2}$$

$$c = (2 + 1)\sin\left(\frac{\pi}{6} + \pi\right)\sec^2\left(\frac{\pi}{4}\right) = -3$$

So the tangent plane to the curve is

$$-\frac{1}{2}(r - 2) - \frac{3\sqrt{3}}{2}\left(\phi - \frac{\pi}{6}\right) - 3\left(\theta - \frac{\pi}{4}\right) = 0$$

Multiply by -4 and simplify.

$$2r + 6\sqrt{3}\phi + 12\theta - 4 - 3\pi - \sqrt{3}\pi = 0$$

3. Use the gradient vector to find the tangent plane(s) to the surface $x^2 - 2xy - 3y^2 + z^2 + 4x + 4y - 2z - 3 = 0$ that are parallel to the xy-plane.

Solution:

Given the multivariable function,

$$f(x, y, z) = x^2 - 2xy - 3y^2 + z^2 + 4x + 4y - 2z - 3$$

the tangent plane to the surface has the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is the point on the surface, and

$$\nabla f(x, y, z) = \langle a, b, c \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 2x - 2y + 4$$

$$\frac{\partial f}{\partial y} = -2x - 6y + 4$$

$$\frac{\partial f}{\partial z} = 2z - 2$$

Since the plane is parallel to the xy-plane, its normal vector has 0 for x and y coordinates, which means $a = b = 0$. So we have a system of equations.

$$2x - 2y + 4 = 0$$

$$-2x - 6y + 4 = 0$$

The solution is

$$(-1)^2 - 2(-1)(1) - 3(1)^2 + z^2 + 4(-1) + 4(1) - 2z - 3 = 0$$

Substitute these values into the equation of the surface to find z.

$$(-1)^2 - 2(-1)(1) - 3(1)^2 + z^2 + 4(-1) + 4(1) - 2z - 3 = 0$$

$$z^2 - 2z - 3 = 0$$

The solutions are $z = -1$ or $z = 3$.

So we have two points where the tangent plane to the given surface is parallel to the xy-plane, $(-1,1,-1)$ and $(-1,1,3)$.

The tangent plane at $(-1,1,-1)$ is

$$\frac{\partial f}{\partial z}(-1,1,-1) = 2(-1) - 2 = -4$$

$$-4(z + 1) = 0$$

$$z + 1 = 0$$

The tangent plane at $(-1,1,3)$ is

$$\frac{\partial f}{\partial z}(-1,1,3) = 2(3) - 2 = 4$$

$$4(z - 3) = 0$$

$$z - 3 = 0$$

So the tangent planes to the surface are

$$z + 1 = 0$$

$$z - 3 = 0$$

MAXIMUM RATE OF CHANGE AND ITS DIRECTION

1. Find the point where the maximum rate of change of the function f is equal to 0.

$$f(x, y) = 6x^2 - 4xy + y^2 - 12x - 6y + 4$$

Solution:

The maximum rate of change of the function f is equal to 0 if and only if the gradient vector is equal to the zero vector.

The gradient vector of a multivariable function is given by

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 12x - 4y - 12$$

$$\frac{\partial f}{\partial y} = -4x + 2y - 6$$

Since the gradient is equal to 0, we have the system of equations in terms of x and y.

$$12x - 4y - 12 = 0$$

$$-4x + 2y - 6 = 0$$

The solution is $x = 6$ and $y = 15$, or $(6,15)$.

2. Find the maximum rate of change and its direction for the function f at $(3, -\pi/2, 0)$.

$$f(x, y, z) = x^2(2z - 1)\sin^2 y$$

Solution:

The direction of the maximum rate of change is given by the gradient vector. The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 2x(2z - 1)\sin^2(y)$$

$$\frac{\partial f}{\partial y} = 2x(2z - 1)(2\sin(y)\cos(y)) = 2x(2z - 1)\sin(2y)$$

$$\frac{\partial f}{\partial z} = 2x^2\sin^2(y)$$

Evaluate the partial derivatives at $(3, -\pi/2, 0)$.

$$\frac{\partial f}{\partial x}\left(3,-\frac{\pi}{2},0\right)=2(3)(2(0)-1)\sin^2\left(-\frac{\pi}{2}\right)=-6$$

$$\frac{\partial f}{\partial y}\left(3,-\frac{\pi}{2},0\right)=2(3)(2(0)-1)\sin\left(-2\frac{\pi}{2}\right)=0$$

$$\frac{\partial f}{\partial z}\left(3,-\frac{\pi}{2},0\right)=2(3)^2\sin^2\left(-\frac{\pi}{2}\right)=18$$

So

$$\nabla f\left(3,-\frac{\pi}{2},0\right)=\langle -6,0,18\rangle$$

Calculate the magnitude of ∇f.

$$\|\nabla f\|=\sqrt{(-6)^2+0^2+18^2}=6\sqrt{10}$$

So the direction is $\langle -6,0,18\rangle$ and the maximum rate of change is $6\sqrt{10}$.

■ 3. Find the minimum rate of change and its direction for the function f at $(2,1,4)$.

$$f(u,v,w)=\sqrt{2u-4v+6w+1}$$

Solution:

The direction of the minimum rate of change is given by the vector opposite to the gradient vector.

The gradient vector of a multivariable function is given by

$$\nabla f(u, v, w) = \left\langle \frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}, \frac{\partial f}{\partial w} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial u} = \frac{1}{\sqrt{2u - 4v + 6w + 1}}$$

$$\frac{\partial f}{\partial v} = -\frac{2}{\sqrt{2u - 4v + 6w + 1}}$$

$$\frac{\partial f}{\partial w} = \frac{3}{\sqrt{2u - 4v + 6w + 1}}$$

Evaluate the partial derivatives at $(2,1,4)$.

$$\frac{\partial f}{\partial u} = \frac{1}{\sqrt{2(2) - 4(1) + 6(4) + 1}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$$

$$\frac{\partial f}{\partial v} = -\frac{2}{\sqrt{2(2) - 4(1) + 6(4) + 1}} = -\frac{2}{5}$$

$$\frac{\partial f}{\partial w} = \frac{3}{\sqrt{2(2) - 4(1) + 6(4) + 1}} = \frac{3}{5}$$

So

$$\nabla f = \left\langle \frac{1}{5}, -\frac{2}{5}, \frac{3}{5} \right\rangle$$

So the direction of the minimum rate of change is

$$\left\langle -\frac{1}{5}, \frac{2}{5}, -\frac{3}{5} \right\rangle$$

Calculate the magnitude.

$$\frac{\sqrt{(-1)^2+(2)^2+(-3)^2}}{5}=\frac{\sqrt{14}}{5}$$

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