Calculus 3 Workbook Solutions

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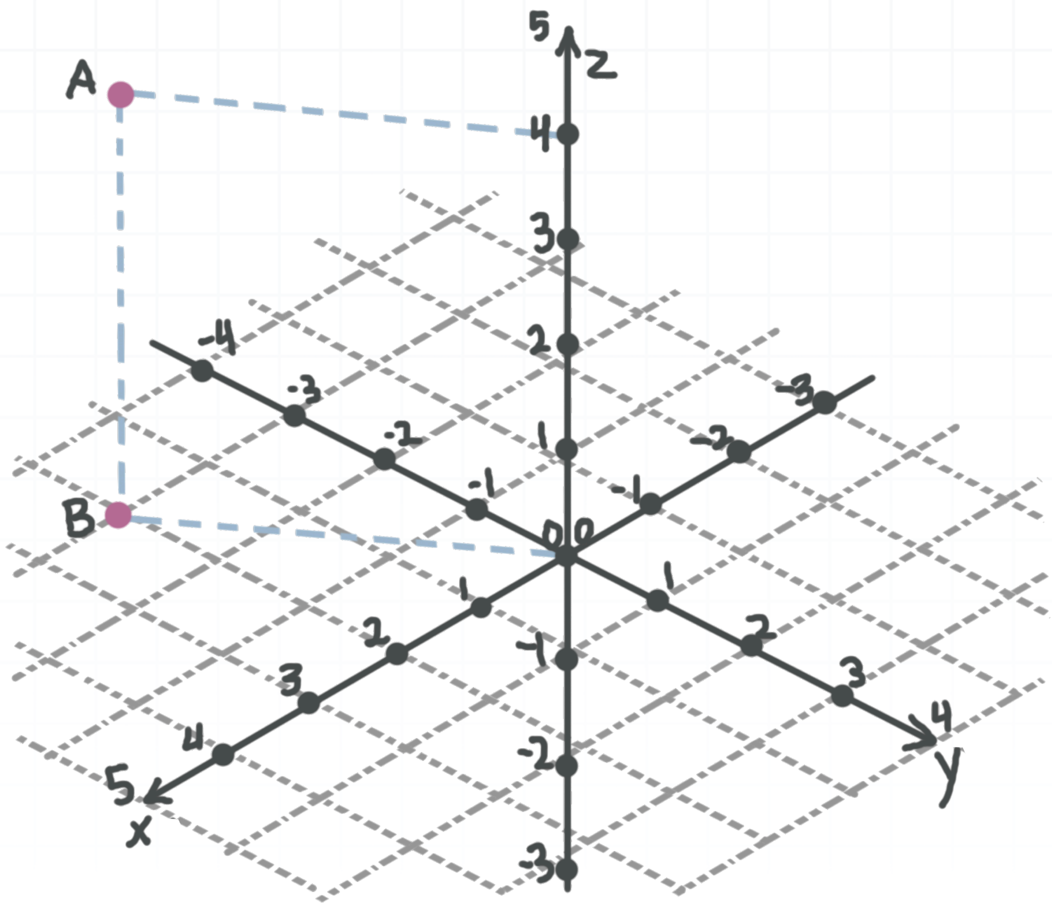
MATH

PLOTTING POINTS IN THREE DIMENSIONS

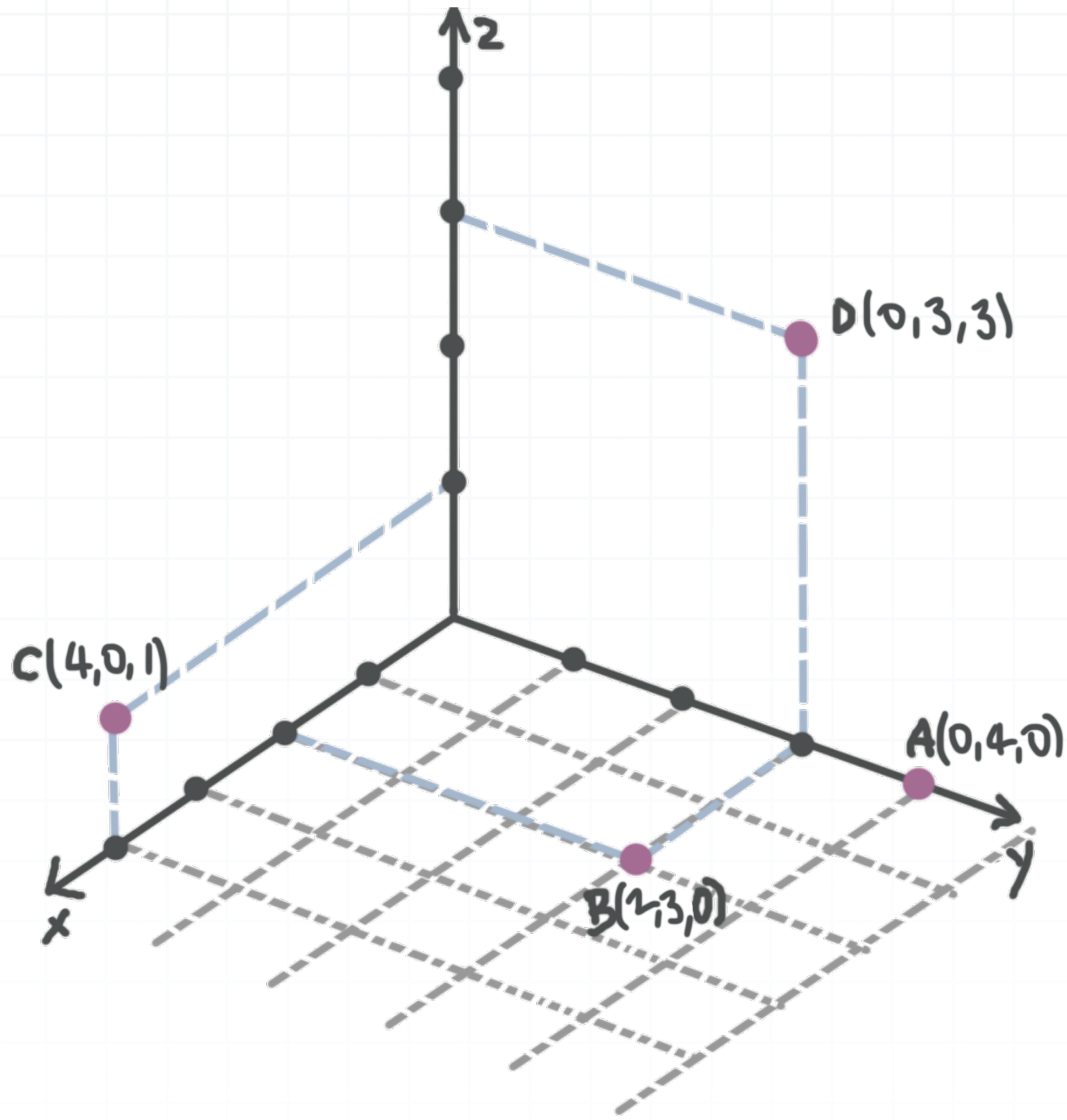
■ 1. What are the coordinates of point A?



Solution:

The coordinates of any point in three-dimensional rectangular coordinates can be given by the ordered triple (x, y, z). Using B as a reference, A is at $x = 2$, $y = -3$, and $z = 4$, which is $(x, y, z) = (2, -3, 4)$.

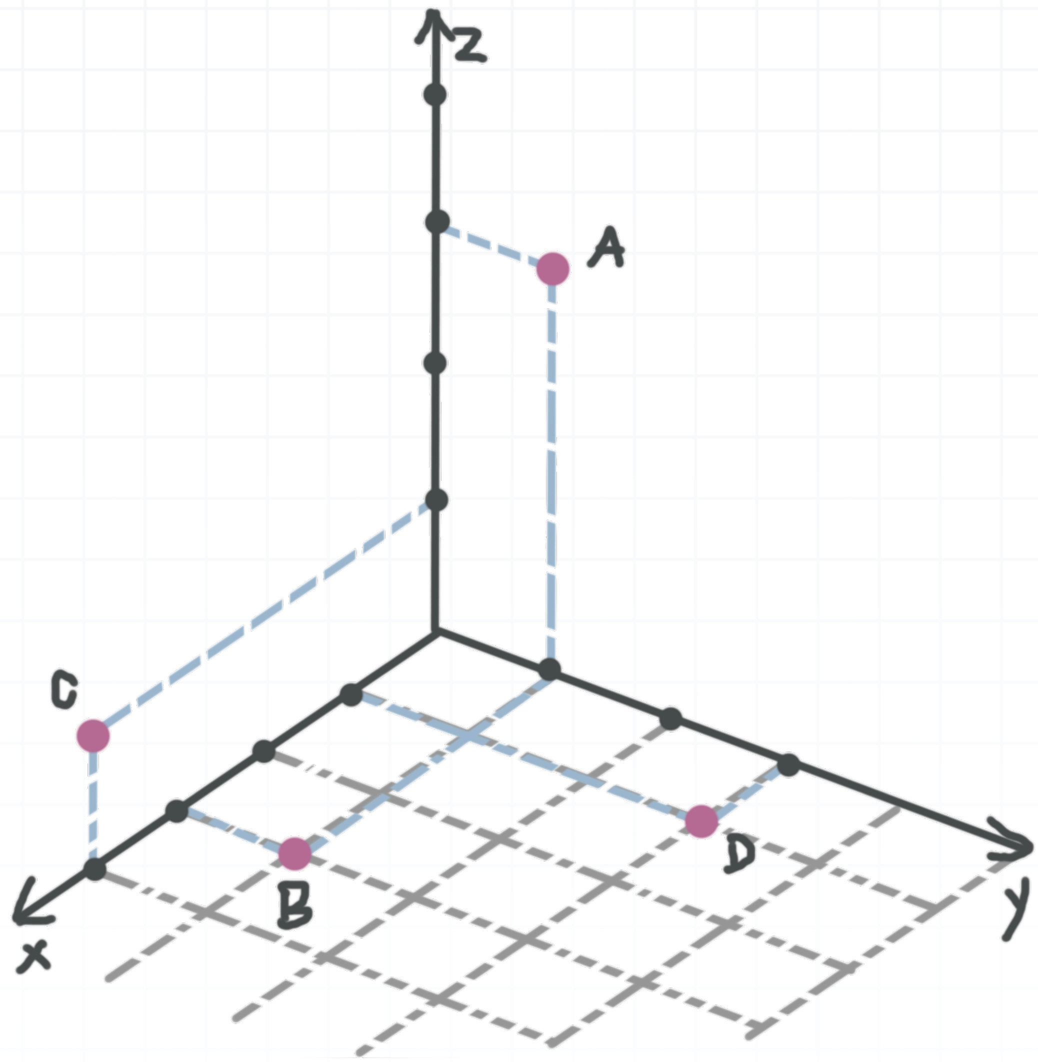
■ 2. Which of the points A, B, C, and D lie on the xz-plane?



Solution:

The point lies on the xz-plane only if its y-coordinate is equal to 0, which means we're looking for a point in the form $(x,0,z)$. Only point C matches that format.

■ 3. Which of the points A, B, C, and D has coordinates $(3,1,0)$?



Solution:

The coordinates of any point in three-dimensional rectangular coordinate space are given by the ordered triple (x, y, z). The points have coordinates $A(0,1,3)$, $B(3,1,0)$, $C(4,0,1)$, and $D(1,3,0)$, which means point B is the correct answer.

DISTANCE BETWEEN POINTS IN THREE DIMENSIONS

■ 1. Find the perimeter P of the triangle ABC given $A(1,0,0)$, $B(2,3,0)$, and $C(3,3,-3)$.

Solution:

To find the perimeter of triangle ABC we need to use the distance formula

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

to find the lengths of sides AB, BC, and AC, then add the three lengths.

Side AB is

$$AB=\sqrt{(2-1)^2+(3-0)^2+(0-0)^2}$$

$$AB=\sqrt{1^2+3^2+0^2}$$

$$AB=\sqrt{1+9+0}$$

$$AB=\sqrt{10}$$

Side BC is

$$BC=\sqrt{(3-2)^2+(3-3)^2+(-3-0)^2}$$

$$BC=\sqrt{1^2+0^2+3^2}$$

$$BC = \sqrt{1 + 0 + 9}$$

$$BC = \sqrt{10}$$

Side AC is

$$AC = \sqrt{(3-1)^2 + (3-0)^2 + (-3-0)^2}$$

$$AC = \sqrt{2^2 + 3^2 + 3^2}$$

$$AC = \sqrt{4 + 9 + 9}$$

$$AC = \sqrt{22}$$

So the perimeter P of triangle ABC is

$$P = \sqrt{10} + \sqrt{10} + \sqrt{22}$$

$$P = 2\sqrt{10} + \sqrt{22}$$

■ 2. Given $A(-1,0,0)$, $B(-2,0,2)$, and $C(0,1,0)$, find the measure of angle BAC in degrees using the law of cosines:

$$\cos(BAC) = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$$

Solution:

To use the law of cosines, we need the distances AB, AC, and BC. Side AB is

$$AB^2 = (-2 - (-1))^2 + (0 - 0)^2 + (2 - 0)^2$$

$$AB^2 = (-1)^2 + 0^2 + 2^2$$

$$AB^2 = 1 + 0 + 4$$

$$AB^2 = 5$$

$$AB = \sqrt{5}$$

Side BC is

$$BC^2 = (0 - (-2))^2 + (1 - 0)^2 + (0 - 2)^2$$

$$BC^2 = 2^2 + 1^2 + 2^2$$

$$BC^2 = 4 + 1 + 4$$

$$BC^2 = 9$$

$$BC = 3$$

Side AC is

$$AC^2 = (0 - (-1))^2 + (1 - 0)^2 + (0 - 0)^2$$

$$AC^2 = 1^2 + 1^2 + 0^2$$

$$AC^2 = 1 + 1 + 0$$

$$AC^2 = 2$$

$$AC = \sqrt{2}$$

Then the law of cosines gives the angle BAC as

$$\cos(BAC) = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$$

$$\cos(BAC) = \frac{5 + 2 - 9}{2\sqrt{5}\sqrt{2}}$$

$$\cos(BAC) = \frac{-2}{2\sqrt{5}\sqrt{2}}$$

$$\cos(BAC) = \frac{-1}{\sqrt{10}}$$

Evaluate.

$$BAC \approx \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right)$$

$$BAC \approx 108.4^\circ$$

■ 3. Find the point on the x-axis that's equidistant from $A(-1,1,0)$ and $B(-2,1,-1)$.

Solution:

Every point on the x-axis can be given by $(x,0,0)$. If we call this the point P, then

$$PA = PB$$

$$PA^2 = PB^2$$

The distance formula gives

$$PA^2 = (x - (-1))^2 + (0 - 1)^2 + (0 - 0)^2$$

$$PA^2 = (x + 1)^2 + 1$$

and

$$PB^2 = (x - (-2))^2 + (0 - 1)^2 + (0 - (-1))^2$$

$$PB^2 = (x + 2)^2 + 2$$

Since $PA^2 = PB^2$, we can solve for x.

$$(x + 1)^2 + 1 = (x + 2)^2 + 2$$

$$x^2 + 2x + 1 + 1 = x^2 + 4x + 4 + 2$$

$$2x = 4x + 4$$

$$2x = -4$$

$$x = -2$$

The the point P on the x-axis that's equidistant from $A(-1,1,0)$ and $B(-2,1,-1)$ is $P(-2,0,0)$.

CENTER, RADIUS, AND EQUATION OF THE SPHERE

■ 1. Find the equation of a sphere with center $(1,3,-2)$ and y-intercept 1.

Solution:

The standard equation of the sphere with the center (h,k,l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

Since the sphere passes through the y-intercept $(0,1,0)$, then the distance formula gives the radius as

$$r^2=(0-1)^2+(1-3)^2+(0-(-2))^2$$

$$r^2=1^2+2^2+2^2$$

$$r^2=1+4+4$$

$$r^2=9$$

$$r=3$$

Substitute the center $(1,3,-2)$ for (h,k,l) and the value of 3 for r into the standard sphere equation.

$$(x-1)^2+(y-3)^2+(z+2)^2=3^2$$

2. Of the points $A(-4,-1,7)$, $B(-5,1,5)$, $C(-6,-6,5)$, and $D(-7,0,3)$, which one does not lie in the interior of the sphere?

$$(x+5)^2+(y+3)^2+(z-4)^2=16$$

Solution:

The point lies in the interior of the sphere if the distance between the point and the center of the sphere, $(-5,-3,4)$, is less than the radius, 4. Or equally, if the square of the distance between the point and the center is less than r^2, 16.

The distance from each point to the center $(-5,-3,4)$ is

For $A(-4,-1,7)$,

$$(-4+5)^2+(-1+3)^2+(7-4)^2$$

$$1^2+2^2+3^2$$

$$1+4+9$$

$$14<16$$

For $B(-5,1,5)$,

$$(-5+5)^2+(1+3)^2+(5-4)^2$$

$$0^2+4^2+1^2$$

$$0+16+1$$

$$17 > 16$$

For $C(-6,-6,5)$,

$$(-6+5)^2+(-6+3)^2+(5-4)^2$$

$$(-1)^2+(-3)^2+1^2$$

$$1+9+1$$

$$11 < 16$$

For $D(-7,0,3)$,

$$(-7+5)^2+(0+3)^2+(3-4)^2$$

$$(-2)^2+3^2+(-1)^2$$

$$4+9+1$$

$$14 < 16$$

The points A, C, and D lie in the interior of the circle, but $B(-5,1,5)$ does not.

■ 3. The endpoints of the diameter of a sphere are $A(2,4,-3)$ and $B(6,0,-1)$. Find the equation of this sphere.

Solution:

The standard equation of a sphere with center (h,k,l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

Since the center is the midpoint of the diameter AB, by the midpoint formula, its coordinates are

$$h=\frac{2+6}{2}=\frac{8}{2}=4$$

$$k=\frac{4+0}{2}=\frac{4}{2}=2$$

$$l=\frac{-3+(-1)}{2}=\frac{-4}{2}=-2$$

The radius of the sphere, r, is equal to the distance between the point A and the center of the circle. So the distance formula gives the radius as

$$r=\sqrt{(4-2)^2+(2-4)^2+(-2-(-3))^2}$$

$$r=\sqrt{2^2+(-2)^2+1^2}$$

$$r=\sqrt{4+4+1}$$

$$r=\sqrt{9}$$

$$r=3$$

So the equation of the sphere with center $(4,2,-2)$ and radius 3 is

$$(x-4)^2+(y-2)^2+(z+2)^2=3^2$$

$$(x-4)^2+(y-2)^2+(z+2)^2=9$$

DESCRIBING A REGION IN THREE DIMENSIONAL SPACE

■ 1. Describe the surface in three-dimensional space.

$$x^2 + 2x + z^2 = 0$$

Solution:

Complete the square with respect to x.

$$x^2 + 2x + 1 - 1 + z^2 = 0$$

$$(x + 1)^2 + z^2 - 1 = 0$$

$$(x + 1)^2 + z^2 = 1^2$$

This equation represents the circle with center $(-1, y, 0)$ and radius 1, so the surface is a cylinder that's parallel to the y-axis and intersects the xz-plane in the circle with center $(-1, 0, 0)$ and radius 1.

■ 2. Describe the surface in three-dimensional space.

$$z = 7$$

Solution:

Since $z = 7$ has no x-variable, the surface is parallel to the x-axis. Similarly, since $z = 7$ has no y-variable, the surface is parallel to the y-axis.

Because it's parallel to both x- and y-axes, $z = 7$ must be a plane parallel to xy-plane, that intersects the z-axis at $(0,0,7)$.

■ 3. Describe the surface in three-dimensional space.

$$xy = 0$$

Solution:

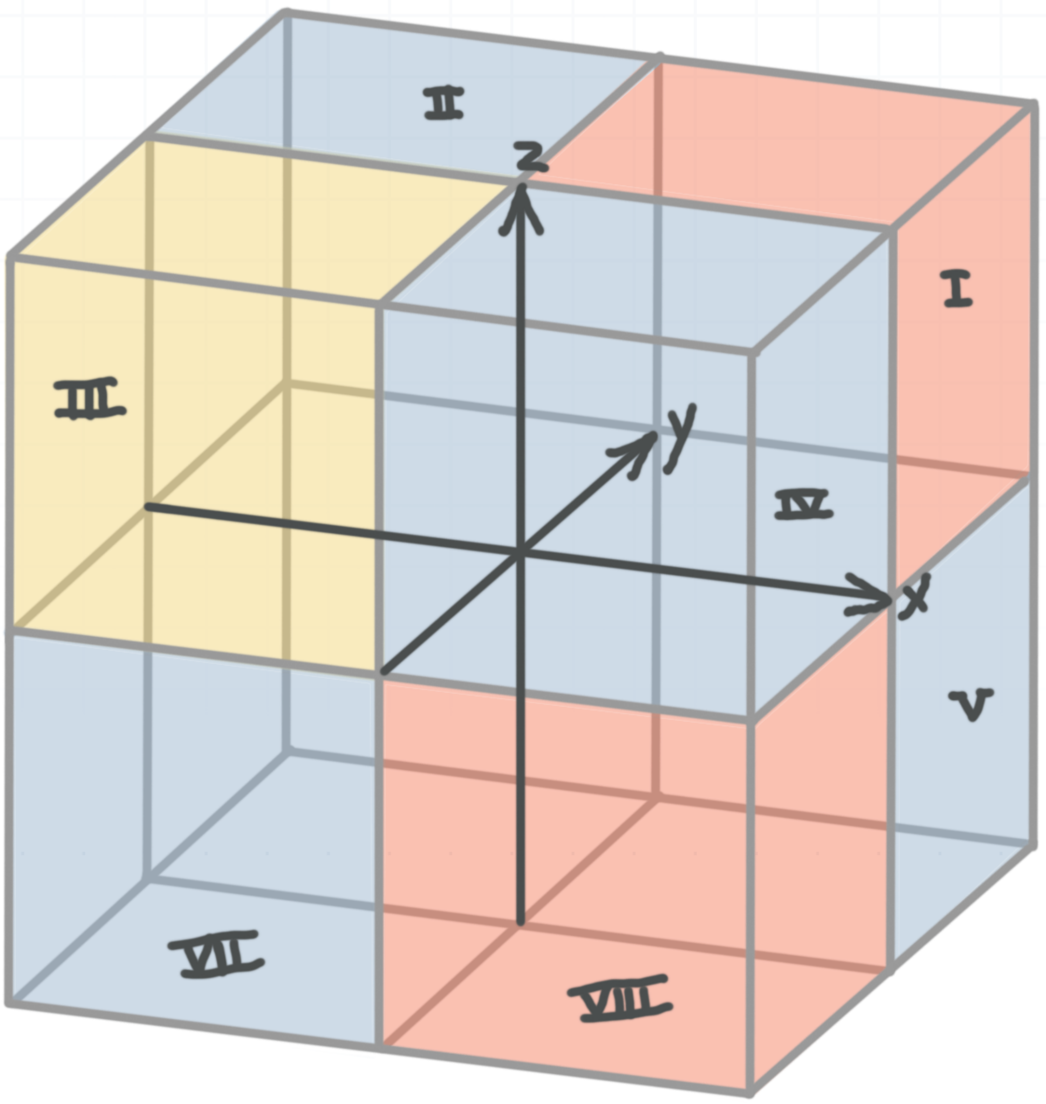
The product xy becomes 0 if $x = 0$, or $y = 0$.

The equation $x = 0$ represents the yz-plane, and the equation $y = 0$ represents the xz-plane. So the equation $xy = 0$ represents the surface which consists of both the yz- and xz-planes.

These planes intersect at the z-axis, which means that also fully belongs to the region.

USING INEQUALITIES TO DESCRIBE THE REGION

1. What set of inequalities describes Octant III? Remember that an "octant" is one of the eight spaces that make up the three-dimensional coordinate system.



Solution:

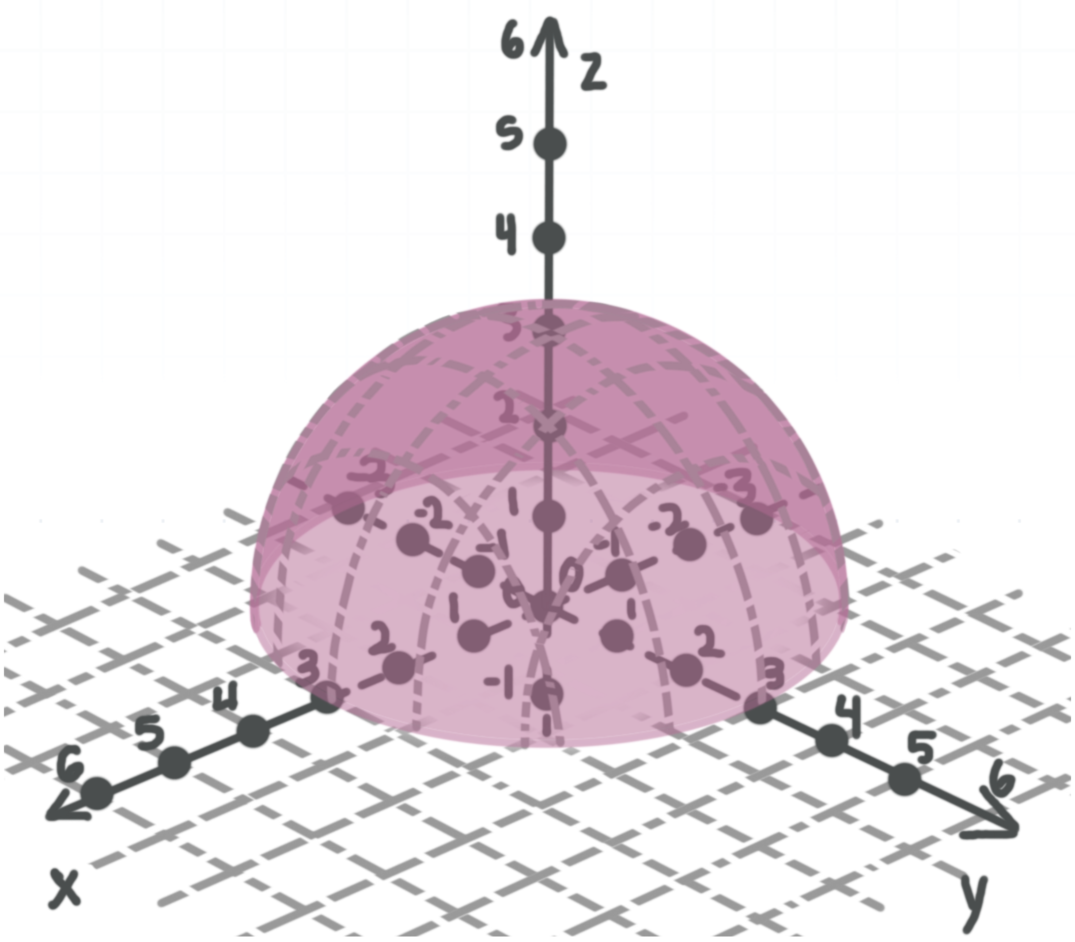
The equation $x = 0$ divides three-dimensional space into two regions, where $x > 0$ contains octants I, IV, V, VIII, and $x < 0$ contains octants II, III, VI, VII.

Similarly, $y > 0$ contains octants I, II, V, VI, and $y < 0$ contains octants III, IV, VII, VIII.

Finally, $z > 0$ contains octants I, II, III, IV, and $z < 0$ contains octants V, VI, VII, VIII.

So octant III is bounded by $x < 0$, $y < 0$, and $z > 0$.

■ 2. What set of inequalities describes the region consisting of all points inside the hemisphere, if the base of the sphere is centered at $(0,0,0)$?



Solution:

The standard equation of a sphere with center (h, k, l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

Because we can see from the figure that the sphere's edge extends to $x=3$ and $y=3$, that means the radius must be 3, so substitute $(0,0,0)$ for (h,k,l) and 3 for r to get the equation of the given sphere.

$$(x-0)^2+(y-0)^2+(z-0)^2=3^2$$

$$x^2+y^2+z^2=3^2$$

So $x^2+y^2+z^2<3^2$ describes the region consisting of all points inside the full sphere. To get only the hemisphere, $z>0$. Putting these together means the hemisphere is described by

When we put our two inequalities together to describe the region, we get

$$x^2+y^2+z^2<9 \text{ and } z>0$$

■ 3. What set of inequalities describes the region consisting of all points which lie at most 5 units from the yz-plane?

Solution:

The x-coordinate of the point tells us how far the point is from the yz-plane. Since the distance from a point to the yz-plane, which is the x-coordinate, can be at most 5, we get the inequality

$$|x|\le 5$$

or, after removing the absolute value form inequality,

$$-5 \le x \le 5$$

In other words, the value of x must fall between $x = -5$ and $x = 5$ in order to stay within 5 units of the xy-plane.

SKETCHING GRAPHS OF MULTIVARIABLE FUNCTIONS

1. Find the range of the function.

$$f(x, y) = x^2 + 2y^2 - 3$$

Solution:

Since $x^2 + 2y^2 \geq 0$, then $x^2 + 2y^2 = 0$ when $x = y = 0$.

$$f(0,0) = 0^2 + 2(0)^2 - 3$$

$$f(0,0) = -3$$

Since $x^2 + 2y^2$ can be arbitrarily large, the range of the $f(x, y)$ is $[-3,\infty)$.

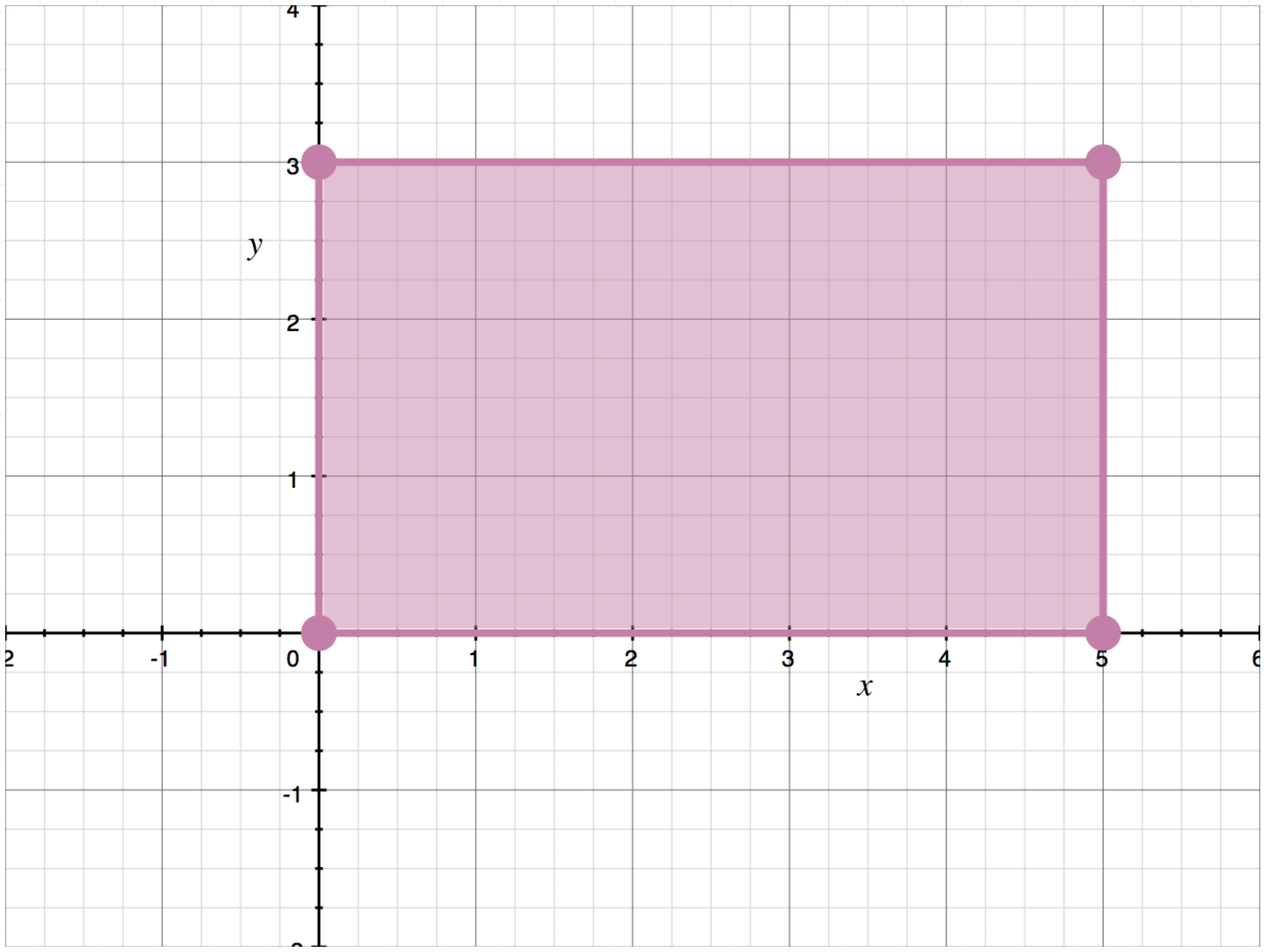
2. Which function's domain is given by the graph, if the left and right sides of the rectangle are included in the domain, but the top and bottom sides are not?

A $\quad f(x, y) = 3y\sqrt{3x - x^2} + 4x\ln(5y - y^2)$

B $\quad f(x, y) = 3y\sqrt{5x - x^2} + 4x\ln(3y - y^2)$

C $\quad f(x, y) = 3x\sqrt{x^2 - 3x} + 4y\ln(y^2 - 5y)$

D $\quad f(x, y) = 3x\sqrt{x - 5x^2} + 4y\ln(y - 3y^2)$



Solution:

Because the left and right sides of the rectangle are included in the domain, but the top and bottom sides are not, the system of inequalities that describes the domain in the graph is $x \geq 0$, $x \leq 5$, $y > 0$, and $y < 3$.

The domain of the function in answer choice A is

$3x - x^2 \geq 0$ so $x(3 - x) \geq 0$, so $x \geq 0$ and $x \leq 3$

$5y - y^2 > 0$ so $y(5 - y) > 0$, so $y > 0$ and $y < 5$

The domain of the function in answer choice B is

$5x - x^2 \geq 0$ so $x(5 - x) \geq 0$, so $x \geq 0$ and $x \leq 5$

$$3y - y^2 > 0 \text{ so } y(3-y) > 0, \text{ so } y > 0 \text{ and } y < 3$$

The domain of the function in answer choice C is

$$x^2 - 3x \geq 0 \text{ so } x(x-3) \geq 0, \text{ so } x \geq 3 \text{ and } x \leq 0$$

$$y^2 - 5y > 0 \text{ so } y(y-5) > 0, \text{ so } y < 0 \text{ and } y > 5$$

The domain of the function in answer choice D is

$$x - 5x^2 \geq 0 \text{ so } x(1-5x) \geq 0, \text{ so } x \geq 0 \text{ and } x \leq 1/5$$

$$y - 3y^2 > 0 \text{ so } y(1-3y) > 0, \text{ so } y > 0 \text{ and } y < 1/3$$

The only matching domain comes from answer choice B.

■ 3. Find the value of the constant a for which $(2,-1,0)$ lies on the graph of the function.

$$f(x,y) = x^2 + 2axy + y^2 - 1$$

Solution:

Substitute $x=2$, $y=-1$, and $f(2,-1)=0$ into the equation.

$$0 = 2^2 + 2a(2)(-1) + (-1)^2 - 1$$

$$0 = 4 - 4a + 1 - 1$$

$$4a = 4$$

$$a = 1$$

■ 4. Find the intersection point of the function and the y-axis.

$$f(x, y) = \sqrt{x^2 - 5y + 15}$$

Solution:

At any intersection point with the y-axis, we know $x = z = 0$. Substitute $x = 0$ and $z = f(x, y) = 0$ into the function.

$$0 = \sqrt{0^2 - 5y + 15}$$

$$0 = \sqrt{-5y + 15}$$

$$-5y + 15 = 0$$

$$y = \frac{15}{5} = 3$$

So the function intersects the y-axis at $(0,3,0)$.

■ 5. Write the equation of the function $f(x, y)$ shifted in a positive direction along the x-axis by 2 units.

$$f(x, y) = x^2y^2 - 2xy - 4y^2 - 4y + 4x$$

Solution:

Shifting a function in a positive direction along the x-axis by a is equivalent to replacing x with $(x-a)$, where a is a constant. So substitute $(x-2)$ for x into the equation, calling it a new function $g(x, y)$.

$$g(x, y) = (x-2)^2y^2 - 2(x-2)y - 4y^2 - 4y + 4(x-2)$$

$$g(x, y) = (x^2 - 4x + 4)y^2 - 2(x-2)y - 4y^2 - 4y + 4(x-2)$$

$$g(x, y) = x^2y^2 - 4xy^2 + 4y^2 - 4y^2 - 2xy + 4y - 4y + 4x - 8$$

$$g(x, y) = x^2y^2 - 4xy^2 - 2xy + 4x - 8$$

■ 6. Which function A, B, C, or D is a reflection of $f(x, y)$ over the xz-plane? Hint: Use the even identity $\cos(-t) = \cos t$ to simplify.

$$f(x, y) = \cos(x^2 - y^2 + 2xy)$$

$$A(x, y) = \cos(-x^2 + y^2 + 2xy)$$

$$B(x, y) = \cos(x^2 - y^2 + 2xy)$$

$$C(x, y) = \cos(x^2 + y^2 - 2xy)$$

$$D(x, y) = \cos(-x^2 - y^2 - 2xy)$$

Solution:

The reflection of a function over the xz-plane is equivalent to replacing y with $-y$. So substitute $-y$ for y into $f(x, y)$.

$$f(x, -y) = \cos(x^2 - (-y)^2 + 2x(-y))$$

$$f(x, -y) = \cos(x^2 - y^2 - 2xy)$$

$$f(x, -y) = \cos(-(y^2 - x^2 + 2xy))$$

Since $\cos t$ is an even function, $\cos(-t) = \cos t$, so

$$f(x, -y) = \cos(y^2 - x^2 + 2xy)$$

This matches

$$A(x, y) = \cos(-x^2 + y^2 + 2xy)$$

■ 7. Find the absolute maximum of the function.

$$f(x, y) = 5 - 2x^2 - 7y^2$$

Solution:

The absolute maximum is the highest point over the entire domain of a function. If (x_0, y_0) is the point on the xy-plane where the absolute maximum occurs, then

$$f(x_0, y_0) \geq f(x, y)$$

for any (x, y) in the domain. Since $-2x^2 - 7y^2 \le 0$, then $-2x^2 - 7y^2 = 0$ when $x = y = 0$.

$$f(0,0) = 5 - 2(0)^2 - 7(0)^2$$

$$f(0,0) = 5$$

So the absolute maximum occurs at $(0,0)$ and has a value of 5.

SKETCHING LEVEL CURVES OF MULTIVARIABLE FUNCTIONS

■ 1. Find the level curve of $f(x, y)$ when $z = 5$.

$$f(x, y) = x^2 - 2xy + 6y - 4$$

Solution:

Substitute $f(x, y) = z = 5$ into the equation and then solve for x.

$$x^2 - 2xy + 6y - 4 = 5$$

$$x^2 - 2xy + 6y - 9 = 0$$

$$-2y(x - 3) + (x - 3)(x + 3) = 0$$

$$(x - 3)(-2y + x + 3) = 0$$

$x = 3$ or $x = 2y - 3$

So for $z = 5$, the level curve includes two lines:

$y = \dfrac{3 + x}{2}$ and $x = 3$

■ 2. Find the level curve of $f(x, y)$ which passes through $(0,1,z)$.

$$f(x, y) = 2x^2 - y + 2$$

Solution:

Substitute $f(x, y) = z$ into the equation, then solve the equation for y.

$$2x^2 - y + 2 = z$$

$$y = 2x^2 + 2 - z$$

Since this is a parabola that passes through $(0,1,z)$, substitute 0 for x and 1 for y, then solve for z.

$$1 = 2(0)^2 + 2 - z$$

$$1 = 0 + 2 - z$$

$$z = 2 - 1$$

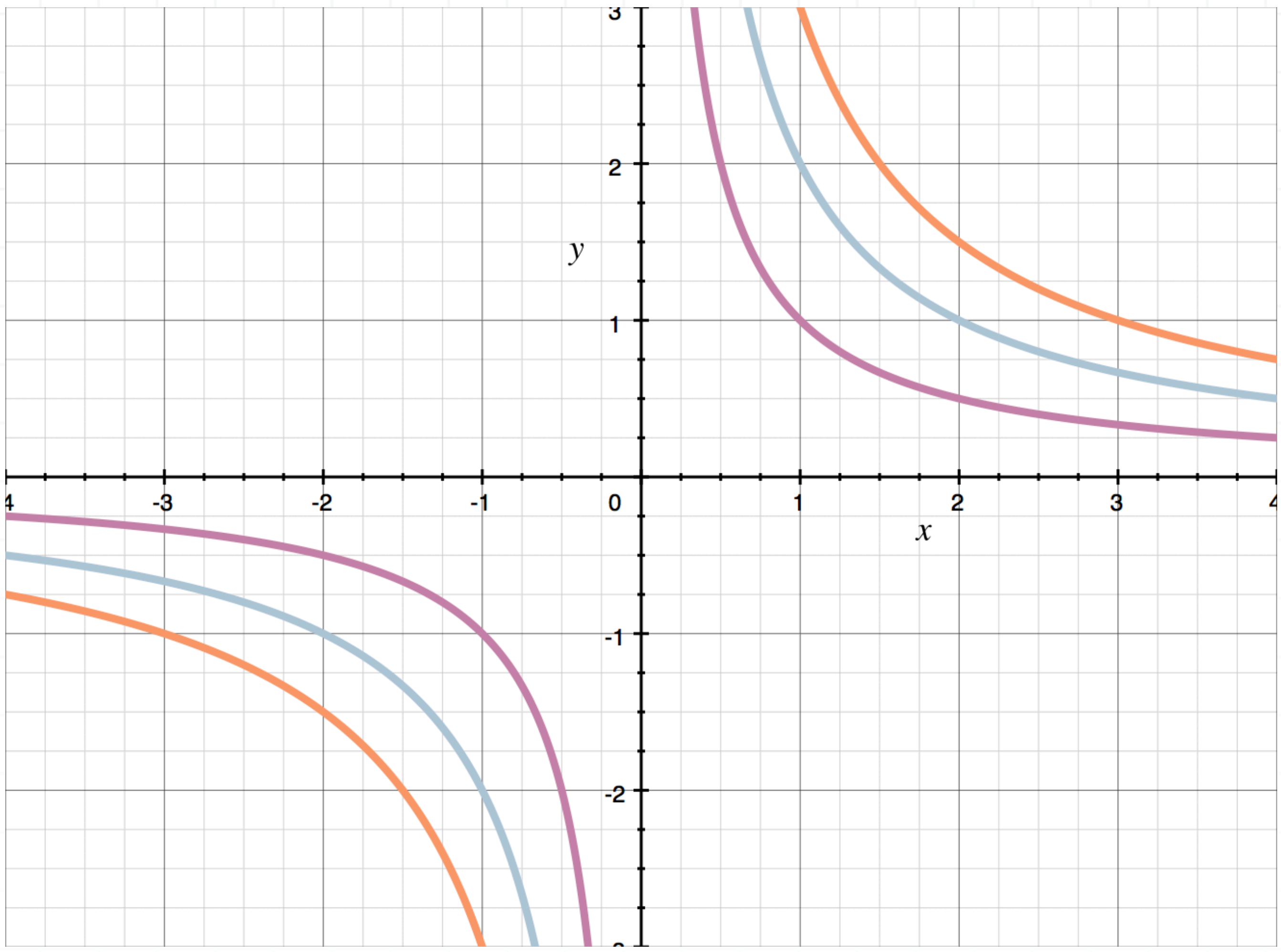
$$z = 1$$

Substitute $z = 1$ back into the level curve equation to get the equation of the level curve at $z = 1$.

$$y = 2x^2 + 2 - 1$$

$$y = 2x^2 + 1$$

■ 3. The graph shows level curves of $f(x, y) = 4xy$. Find the value of z that corresponds to the light blue curve.



Solution:

The light blue curve corresponds to the hyperbola $y = 2/x$, or $xy = 2$. So substitute the 2 for xy into the $f(x, y)$.

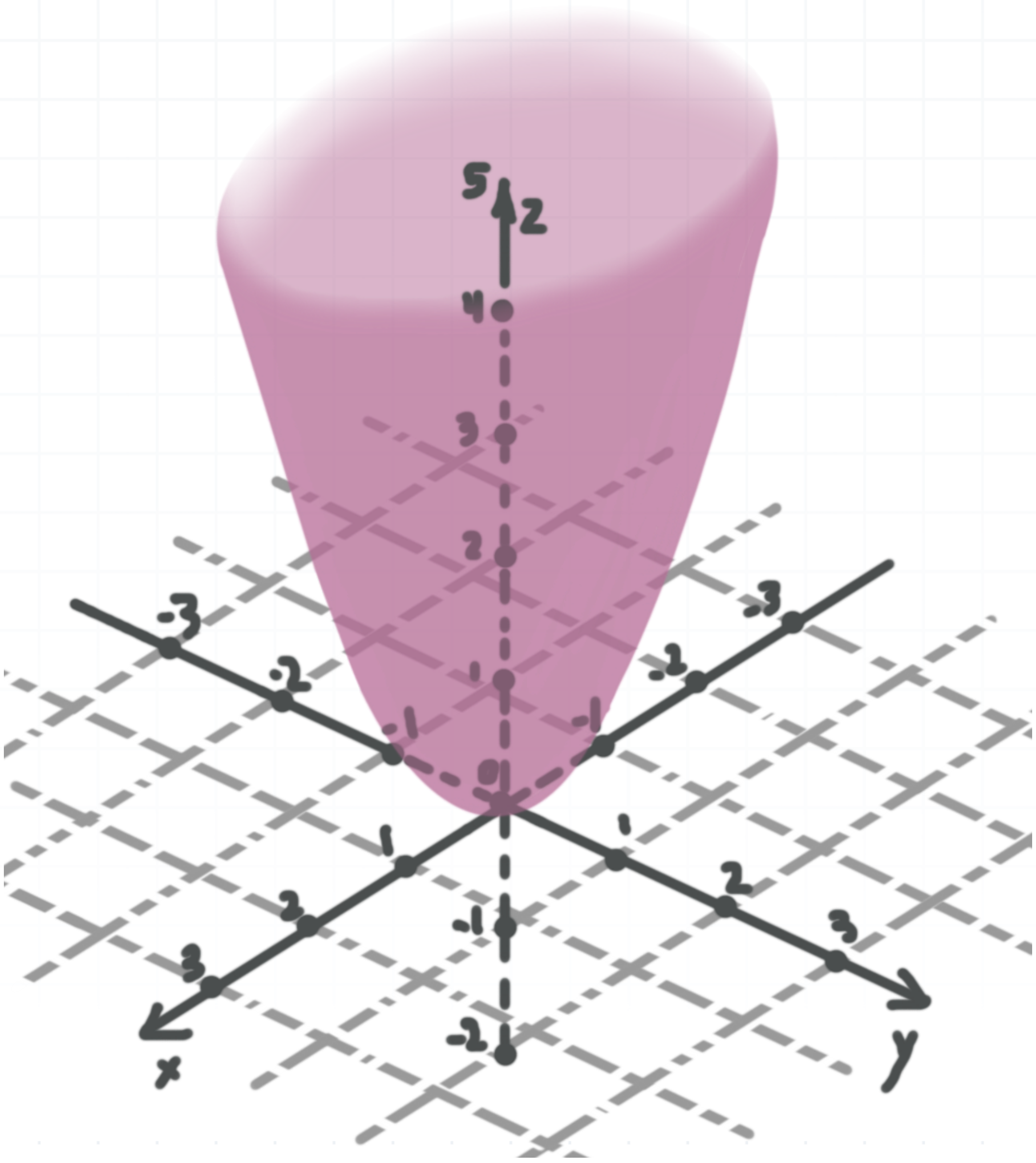
$$f(x, y) = 4(xy)$$

$$f(x, y) = 4(2)$$

$$f(x, y) = 8$$

So the light blue level curve corresponds to $z = 8$.

4. Think about the shape of the level curves of the graph of the elliptic paraboloid. Are they lines, ellipses, parabolas, or hyperbolas?



Solution:

A level curve is a set of points where the function takes a constant value of z. For the graph shown of the elliptic paraboloid, the level curves are ellipses.

MATCHING THE FUNCTION WITH THE GRAPH AND LEVEL CURVES

1. Which statement is true for the graph of the function?

$$x^2 - 2y^2 + z^2 - 8y - 6z = 0$$

A The graph is the hyperboloid centered at $(0,2,-3)$.

B The graph is the hyperboloid centered at $(0,-2,3)$.

C The graph is the ellipsoid centered at $(0,2,-3)$.

D The graph is the ellipsoid centered at $(0,-2,3)$.

Solution:

Transform the equation into standard form by completing the square with respect to y and z.

$$x^2 - 2(y^2 + 4y + 4 - 4) + (z^2 - 6z + 9 - 9) = 0$$

$$x^2 - 2((y+2)^2 - 4) + ((z-3)^2 - 9) = 0$$

$$x^2 - 2(y+2)^2 + 8 + (z-3)^2 - 9 = 0$$

$$x^2 - 2(y+2)^2 + (z-3)^2 = 1$$

$$x^2 - \frac{(y+2)^2}{1/2} + (z-3)^2 = 1$$

The quadric is a hyperboloid in standard form, with its center at $(0,-2,3)$, which is answer choice B.

2. Find the equation of ellipsoid centered at $(2,0,2)$ that has the level curve $(x-2)^2+4y^2=0.75$ for $z=1.5$.

Solution:

An ellipsoid is given in standard form by

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1$$

Substitute $(2,0,2)$ for (h,k,l).

$$\frac{(x-2)^2}{a^2}+\frac{y^2}{b^2}+\frac{(z-2)^2}{c^2}=1$$

To find the level curve for $z=1.5$, substitute into the equation.

$$\frac{(x-2)^2}{a^2}+\frac{y^2}{b^2}+\frac{(1.5-2)^2}{c^2}=1$$

$$\frac{(x-2)^2}{a^2}+\frac{y^2}{b^2}=1-\frac{0.25}{c^2}$$

Since the level curve is $(x-2)^2+4y^2=0.75$, we can create a system of equations.

$$a^2=1$$

$$b^2 = 1/4$$

$$1 - \frac{0.25}{c^2} = 0.75, \text{ so } 0.25c^2 = 0.25$$

so

$$a = 1$$

$$b = 1/2$$

$$c = 1$$

Substitute these values into the ellipsoid's equation.

$$\frac{(x-2)^2}{1^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} + \frac{(z-2)^2}{1^2} = 1$$

$$(x-2)^2 + 4y^2 + (z-2)^2 = 1$$

3. Which of the surfaces has the same level curves for any z?

A The plane $2x + 3y + z = 1$

B The ellipsoid $x^2 + 2y^2 + 4z^2 - 4x - 2y = 1$

C The cylinder $2x^2 + y^2 - 5x + 7y = 1$

D The elliptic cone $x^2 + 3y^2 - z^2 = 0$

Solution:

Only the cylinder $2x^2 + y^2 - 5x + 7y = 1$ has the same level curves for any z. Since z-coordinate is missing in the cylinder equation, when solved for y it'll give the same solution for any z.

We also know that a cylinder is parallel to the z-axis and therefore has to have the same level curves.

VECTOR, PARAMETRIC, AND SYMMETRIC EQUATIONS OF A LINE

■ 1. Find the vector equation of the line that passes through $A(1,0,-2)$ and is parallel to the symmetric equation.

$$\frac{x-3}{2}=-\frac{y}{2}=z+1$$

Solution:

Rewrite the symmetric equation in standard form.

$$\frac{x-3}{2}=\frac{y+0}{-2}=\frac{z+1}{1}$$

So the parallel vector is $\langle 2,-2,1\rangle$. Since we also know the line passes through $A(1,0,-2)$, the equation is

$$r=(1\mathbf{i}+0\mathbf{j}-2\mathbf{k})+t(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$$

$$r=\mathbf{i}+2t\mathbf{i}-2t\mathbf{j}-2\mathbf{k}+t\mathbf{k}$$

$$r=(1+2t)\mathbf{i}+(-2t)\mathbf{j}+(-2+t)\mathbf{k}$$

■ 2. Find the parametric equation of the line that passes through $A(2,3,-2)$ and $B(0,-1,5)$.

Solution:

Vector AB is

$$AB = \langle 0-2, -1-3, 5-(-2)\rangle$$

$$AB = \langle -2, -4, 7\rangle$$

Since the line should be parallel to $AB = \langle -2, -4, 7\rangle$ and passes through $A(2,3,-2)$, the parametric equation is

$$x = 2 - 2t$$

$$y = 3 - 4t$$

$$z = -2 + 7t$$

■ 3. Which line passes through $A(1,0,-4)$?

A $\quad x = 1 - 2t,\ y = -5t,\ z = 3 - 4t$

B $\quad x = 2 - 5t,\ y = 5t,\ z = 2 - 4t$

C $\quad x = 6 - t,\ y = 5 - t,\ z = 6 - 2t$

D $\quad x = 6 + t,\ y = 5 + t,\ z = 6 + t$

Solution:

If the line passes though $A(1,0,-4)$, then there's a value t_0 where

$$x(t_0) = 1$$

$$y(t_0) = 0$$

$$z(t_0) = -4$$

We can use $(1,0,-4)$ to check each of the answer choices.

For answer choice A, start with $y = -5t$. Substituting $y(t_0) = 0$ gives $0 = -5t_0$, and then $t_0 = 0$. Plugging into $x = 1 - 2t$ and $z = 3 - 4t$ gives

$$x(t_0) = 1 - 2t_0 = 1 - 2(0) = 1$$

$$z(t_0) = 3 - 4t_0 = 3$$

Putting these values together gives $(1,0,3)$, which doesn't match $(1,0,-4)$, so let's try answer choice B. Start with $y = 5t$. Substituting $y(t_0) = 0$ gives $0 = 5t_0$, and then $t_0 = 0$. Plugging into $x = 2 - 5t$ and $z = 2 - 4t$ gives

$$x(t_0) = 2 - 5t_0 = 2 - 5(0) = 2$$

$$z(t_0) = 2 - 4t_0 = 2 - 4(0) = 2$$

Putting these values together gives $(2,0,2)$, which doesn't match $(1,0,-4)$, so let's try answer choice C. Start with $y = 5 - t$. Substituting $y(t_0) = 0$ gives $0 = 5 - t_0$, and then $t_0 = 5$. Plugging into $x = 6 - t$ and $z = 6 - 2t$ gives

$$x(t_0) = 6 - t_0 = 6 - 5 = 1$$

$$z(t_0) = 6 - 2t_0 = 6 - 2(5) = -4$$

Putting these together gives $(1,0,-4)$, so answer choice C is correct.

PARALLEL, INTERSECTING, SKEW AND PERPENDICULAR LINES

■ 1. For $A(1,0,-1)$, $B(1,3,0)$, $C(0,0,2)$, and $D(-1,-2,3)$, are lines AB and CD parallel, intersecting, skew, or perpendicular?

Solution:

Find vectors AB and CD and the equations of the corresponding lines. The vector AB is

$$AB = \langle 1-1, 3-0, 0-(-1)\rangle = \langle 0,3,1\rangle$$

Its parametric equation is

$$x = 1 + 0t = 1$$

$$y = 0 + 3t = 3t$$

$$z = -1 + 1t = t - 1$$

The vector CD is

$$CD = \langle -1-0, -2-0, 3-2\rangle = \langle -1,-2,1\rangle$$

Its parametric equation is

$$x = 0 - 1t = -t$$

$$y = 0 - 2t = -2t$$

$$z = 2 + 1t = 2 + t$$

Since $0/(-1)$ isn't equal to $3/(-2)$, AB isn't parallel to CD. To check if the lines intersect, solve the system of equations.

$$1 = -s$$

$$3t = -2s$$

$$t - 1 = 2 + s$$

So

$$s = -1$$

$$3t = 2$$

$$t = 2$$

Since the system has no solutions, the lines don't intersect, so the lines are skew.

■ 2. Find the line L_2 that passes through $A(1,0,1)$ and is perpendicular to L_1.

$$L_1: \quad x = 2 - 2t, \; y = t, \; z = 3 + t$$

Solution:

The vector for L_1 is $a = \langle -2,1,1 \rangle$.

Let B be the point on L_1 where L_1 intersects L_2. Then B has coordinates $(2-2s, s, 3+s)$. So vector AB is

$$AB = \langle 2-2s-1, s-0, 3+s-1 \rangle$$

$$AB = \langle 1-2s, s, 2+s \rangle$$

Since AB is perpendicular to a, we know that $AB \cdot a = 0$, so

$$-2(1-2s) + 1(s) + 1(2+s) = 0$$

$$-2 + 4s + s + 2 + s = 0$$

$$6s = 0$$

$$s = 0$$

So $AB = \langle 1,0,2 \rangle$. Since L_2 passes through $A(1,0,1)$ and is parallel to the vector $AB = \langle 1,0,2 \rangle$, its equation is

$$x = 1 + t$$

$$y = 0$$

$$z = 1 + 2t$$

■ 3. Which line is perpendicular to L_1?

L_1: $\quad x = 2t$, $y = 21 - t$, $z = 6 - t$

A $\quad x = 2 - 3t$, $y = 7 + 5t$, $z = 2t$

B $x = 2 + 3t$, $y = 7 + 5t$, $z = t$

C $x = 2 + 3t$, $y = 7 - 5t$, $z = -t$

D $x = 2 - 3t$, $y = 5 - 5t$, $z = -t$

Solution:

The vector for L_1 is $a = \langle 2, -1, -1 \rangle$, so check each answer choice to see if its vector is perpendicular to a. If we call the vector for each answer choice b, then we can say $b \cdot a = 0$.

For answer choice A, $b = \langle -3,5,2 \rangle$ and $b \cdot a = 2(-3) - 1(5) - 1(2) = -13 \neq 0$.

For answer choice B, $b = \langle 3,5,1 \rangle$ and $b \cdot a = 2(3) - 1(5) - 1(1) = 0$. Let's check if this line intersects L_1.

$$2 + 3t = 2s$$

$$7 + 5t = 21 - s$$

$$t = 6 - s$$

Substitute $6 - s$ for t into the first equation.

$$2 + 3(6 - s) = 2s$$

$$20 - 3s = 2s$$

$$s = 4$$

Then $t = 6 - 4 = 2$ and $7 + 5(2) = 21 - 4$.

For answer choice C, $b = \langle 3, -5, -1\rangle$ and $b \cdot a = 2(3) - 1(-5) - 1(-1) = 12 \neq 0$.

For answer choice D, $b = \langle -3, -5, -1\rangle$ and $b \cdot a = 2(-3) - 1(-5) - 1(-1) = 0$. Let's check if this line intersects L_1.

$$2 - 3t = 2s$$

$$5 - 5t = 21 - s$$

$$-t = 6 - s$$

$$t = s - 6$$

Substitute $s - 6$ for t to the first equation:

$$2 - 3(s - 6) = 2s$$

$$20 - 3s = 2s$$

$$s = 4$$

Then $t = 4 - 6 = -2$ and $5 - 5(-2) = 21 - 4$.

From the given choices, only the line in answer choice B is perpendicular and intersects L_1.

EQUATION OF A PLANE

■ 1. Find the equation of a plane that passes through $A(1,4,-2)$ and is perpendicular to the line.

$$r = \langle 1,3,3\rangle + t\langle -2,3,1\rangle$$

Solution:

Since the plane is perpendicular to the line with vector $\langle -2,3,1\rangle$, it has the same normal vector, $\langle -2,3,1\rangle$.

The equation of the plane that passes through $A(1,4,-2)$ and has the normal vector $\langle -2,3,1\rangle$ is

$$-2(x-1) + 3(y-4) + 1(z+2) = 0$$

$$-2x + 2 + 3y - 12 + z + 2 = 0$$

$$-2x + 3y + z = 8$$

■ 2. Find the equation of a plane that passes through $A(1,4,-2)$ and the line given by the parametric equation.

$$x = 2 - 4t,\ y = 3t,\ z = 1 + t$$

Solution:

Evaluating the parametric equation at $t = 0$ gives $(2,0,1)$. The vector of the parametric equation is $a = \langle -4,3,1 \rangle$.

The normal vector to the plane is given by the cross product $n = AB \times a$ where $AB = \langle 1, -4,3 \rangle$.

$$AB \times a = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 3 \\ -4 & 3 & 1 \end{vmatrix}$$

$$AB \times a = \mathbf{i}((-4)(1) - (3)(3)) - \mathbf{j}((1)(1) - (3)(-4)) + \mathbf{k}((1)(3) - (-4)(-4))$$

$$AB \times a = -13\mathbf{i} - 13\mathbf{j} - 13\mathbf{k}$$

The equation of the plane that passes through $B(2,0,1)$ and has the normal vector $\langle -13, -13, -13 \rangle$ is

$$-13(x-2) - 13(y-0) - 13(z-1) = 0$$

$$(x-2) + (y-0) + (z-1) = 0$$

$$x + y + z - 3 = 0$$

$$x + y + z = 3$$

■ 3. Which of the lines lie in the plane $2x - y + 3z = 1$? Choose as many of the answer choices as are correct.

A $\quad x = 1 + 2t,\ y = 1 - 3t,\ z = -5t$

B $\quad x = 1 - 2t,\ y = 1 - 5t,\ z = 4t$

C $\quad x = 1 + 4t,\ y = 1 + t,\ z = -3t$

D $\quad x = 1 + 2t,\ y = 1 + t,\ z = -t$

Solution:

A line lies in the plane if any two points on the line lie in that plane. So we can check two points on the line of each answer choice to see if they satisfy the plane equation.

All of the lines pass through the point $(1,1,0)$ so let's check that in the plane equation.

$$2(1) - (1) + 3(0) = 1$$

$$2 - 1 + 0 = 1$$

$$1 = 1$$

So the first point on each line lies in the plane. Let's take the second point on each line for $t = 1$ and substitute them into the plane equation.

For answer choice A, $t = 1$ gives $(3, -2, -5)$, and then

$$2(3) - (-2) + 3(-5) = 1$$

$$6 + 2 - 15 = 1$$

$$-7 = 1$$

For answer choice B, $t = 1$ gives $(-1, -4, 4)$, and then

$$2(-1) - (-4) + 3(4) = 1$$

$$-2 + 4 + 12 = 1$$

$$14 = 1$$

For answer choice C, $t = 1$ gives $(5, 2, -3)$, and then

$$2(5) - (2) + 3(-3) = 1$$

$$10 - 2 - 9 = 1$$

$$-1 = 1$$

For answer choice D, $t = 1$ gives $(3, 2, -1)$, and then

$$2(3) - (2) + 3(-1) = 1$$

$$6 - 2 - 3 = 1$$

$$1 = 1$$

Because none of the equations held true for the lines in A, B, and C, none of those lines lie in the plane. But the line in D gave us $1 = 1$, a true equation, which means the line in D is the only line that lies in the plane.

■ 4. Find the equation of a plane that passes through the intersecting lines L_1 and L_2.

$$L_1: \quad \frac{x-2}{2} = \frac{y+3}{3} = \frac{z}{2}$$

$$L_2: \quad \frac{x-2}{-1} = \frac{y+3}{2} = \frac{z}{5}$$

Solution:

The vectors for L_1 and L_2 are $n_1 = \langle 2,3,2 \rangle$ and $n_2 = \langle -1,2,5 \rangle$ respectively. The normal vector to the plane is the cross-product $n = n_1 \times n_2$.

$$n_1 \times n_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ -1 & 2 & 5 \end{vmatrix}$$

$$n_1 \times n_2 = \mathbf{i}((3)(5) - (2)(2)) - \mathbf{j}((2)(5) - (2)(-1)) + \mathbf{k}((2)(2) - (3)(-1))$$

$$n_1 \times n_2 = 11\mathbf{i} - 12\mathbf{j} + 7\mathbf{k}$$

The lines intersect at $(2,-3,0)$. Since the equation of the plane passes through $(2,-3,0)$ and has the normal vector $\langle 11,-12,7 \rangle$, its equation is

$$11(x-2) - 12(y+3) + 7(z-0) = 0$$

$$11x - 22 - 12y - 36 + 7z = 0$$

$$11x - 12y + 7z = 58$$

■ 5. Find the equation of a plane that passes through the parallel lines L_1 and L_2.

$$L_1: \quad r = \langle 1,2,-4\rangle + t\langle 0,1,-1\rangle$$

$$L_2: \quad r = \langle 2,-3,0\rangle + t\langle 0,1,-1\rangle$$

Solution:

$A(1,2,-4)$ is a point on L_1 and $B(2,-3,0)$ is a point on L_2. The vector for both lines is $a = \langle 0,1,-1\rangle$. So the normal vector to the plane is the cross-product $n = AB \times a$, where $AB = \langle 1,-5,4\rangle$.

$$AB \times a = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -5 & 4 \\ 0 & 1 & -1 \end{vmatrix}$$

$$AB \times a = \mathbf{i}((-5)(-1) - (4)(1)) - \mathbf{j}((1)(-1) - (4)(0)) + \mathbf{k}((1)(1) - (-5)(0))$$

$$AB \times a = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

The equation of the plane that passes through $A(1,2,-4)$ and has the normal vector $\langle 1,1,1\rangle$ is

$$1(x-1) + 1(y-2) + 1(z+4) = 0$$

$$x - 1 + y - 2 + z + 4 = 0$$

$$x + y + z = -1$$

INTERSECTION OF A LINE AND A PLANE

■ 1. Find the x-, y-, and z-intercepts of the plane.

$$2x - 3y + z = 6$$

Solution:

The x-intercept will exist where $y = 0$ and $z = 0$.

$$2x - 3(0) + (0) = 6$$

$$2x = 6$$

$$x = 3$$

The y-intercept will exist where $x = 0$ and $z = 0$.

$$2(0) - 3y + (0) = 6$$

$$-3y = 6$$

$$y = -2$$

The z-intercept will exist where $x = 0$ and $y = 0$.

$$2(0) - 3(0) + z = 6$$

$$z = 6$$

The x-, y-, and z-intercepts are $(3,0,0)$, $(0,-2,0)$, and $(0,0,6)$ respectively.

■ 2. Find the intersection of AB and the plane $x + 2y + 4z = 12$, where A and B are the points $A(0,1,-2)$ and $B(-1,2,0)$, or determine that the line segment and the plane do no intersect.

Solution:

The line segment AB intersects the plane if the line that passes through the points A and B intersects the plane at some point C, and this point C lies between the points A and B.

The vector for AB is

$$AB = \langle -1 - 0, 2 - 1, 0 - (-2) \rangle$$

$$AB = \langle -1, 1, 2 \rangle$$

The parametric equation of the line AB that passes through A is

$$x = -t$$

$$y = 1 + t$$

$$z = -2 + 2t$$

Substitute x, y, and z into the plane equation.

$$(-t) + 2(1 + t) + 4(-2 + 2t) = 12$$

$$9t - 6 = 12$$

$$t = 2$$

Plugging $t = 2$ back into the parametric equation gives the intersection point $(-2,3,2)$.

This point on the line AB lies between A and B if its coordinates lie between corresponding coordinates of the points A and B. But looking at the x-coordinates of each point, -2 doesn't lie between 0 and -1, so $(-2,3,2)$ doesn't lie on the interval AB, so AB and the plane do no intersect.

■ 3. Find the value of the constant p for which the line doesn't intersect the plane.

The line $\dfrac{x+1}{2} = \dfrac{y}{3} = z - 1$

The plane $px + 2y + z = 4$

Solution:

Rewrite the line equation in parametric form as

$$x = 2t - 1$$

$$y = 3t$$

$$z = t + 1$$

Substitute x, y, and z into the plane equation.

$$p(2t-1)+2(3t)+(t+1)=4$$

$$2tp+6t+t-p+1-4=0$$

$$t(2p+7)-p-3=0$$

$$t(2p+7)=p+3$$

$$t=\frac{p+3}{2p+7}$$

This equation for t is valid only if

$$2p+7\neq 0$$

$$2p\neq -7$$

$$p\neq -\frac{7}{2}$$

The line doesn't intersect the plane if $p=-7/2$.

PARALLEL, PERPENDICULAR, AND ANGLE BETWEEN PLANES

■ 1. Find the equation of the plane that passes through the points $A(1,0,-1)$ and $B(0,1,-1)$ and is perpendicular to the plane $x+2y+3z=6$.

Solution:

Let $n=\langle n_1,n_2,n_3\rangle$ be the normal vector to the plane we need to find. Since there are an infinite number of such vectors with different length, we can take a vector with an x-coordinate 1, like $n=\langle 1,n_2,n_3\rangle$. Let's find other components of n.

Since n is perpendicular to the plane $x+2y+3z=6$,

$$\langle 1,n_2,n_3\rangle \cdot \langle 1,2,3\rangle = 0$$

$$1+2n_2+3n_3=0$$

Since n is the normal vector to the plane which includes the vector AB, n is also perpendicular to AB.

$$n \cdot AB = 0$$

$$\langle 1,n_2,n_3\rangle \cdot \langle -1,1,0\rangle = 0$$

$$-1+n_2=0$$

We have the system of linear equations in terms of n_2 and n_3.

From the second equation, we get $n_2 = 1$. Then substitute into the first equation.

$$1 + 2(1) + 3n_3 = 0$$

$$3 + 3n_3 = 0$$

$$n_3 = -1$$

So $n = \langle 1,1,-1 \rangle$. Let's find the equation of the plane with normal vector $\langle 1,1,-1 \rangle$ which passes through $A(1,0,-1)$.

$$1(x-1) + 1(y-0) - 1(z+1) = 0$$

$$x + y - z - 2 = 0$$

■ 2. Find the equation of the plane that passes through $A(3,2,-4)$ and is parallel to the plane $-x + 3y - 2z = 4$.

Solution:

Since the plane is parallel to $-x + 3y - 2z = 4$, it has the same normal vector, $\langle -1,3,-2 \rangle$.

Let's find the equation of the plane with normal vector $\langle -1,3,-2 \rangle$ which passes through the point $A(3,2,-4)$.

$$-1(x-3) + 3(y-2) - 2(z+4) = 0$$

$$-x + 3 + 3y - 6 - 2z - 8 = 0$$

$$-x + 3y - 2z - 11 = 0$$

■ 3. Find the equation of the plane a that passes through the point $A(1,-2,-3)$ and form equal angles with all of the coordinate planes, xy, yz, and xz.

Solution:

Let $n = \langle n_1, n_2, n_3 \rangle$ be the normal vector to the plane a. The xy-plane has normal vector $\langle 0,0,1 \rangle$, and cosine of the angle between a and the xy-plane is

$$\frac{n \cdot \langle 0,0,1 \rangle}{|n|} = \frac{n_3}{|n|}$$

Similarly, cosine of the angle between a and the yz-plane is

$$\frac{n \cdot \langle 1,0,0 \rangle}{|n|} = \frac{n_1}{|n|}$$

Cosine of the angle between a and the xz-plane is

$$\frac{n \cdot \langle 0,1,0 \rangle}{|n|} = \frac{n_2}{|n|}$$

Since all three angles are equal,

$$\frac{n_1}{|n|} = \frac{n_2}{|n|} = \frac{n_3}{|n|}$$

$$n_1 = n_2 = n_3$$

We can use a normal vector of any length, so we can choose the vector $n = \langle 1,1,1 \rangle$. Let's find the equation of the plane with normal vector $\langle 1,1,1 \rangle$ which passes through $A(1,-2,-3)$.

$$1(x-1) + 1(y+2) + 1(z+3) = 0$$

$$x - 1 + y + 2 + z + 3 = 0$$

$$x + y + z + 4 = 0$$

PARAMETRIC EQUATIONS FOR THE LINE OF INTERSECTION OF TWO PLANES

1. Find the parametric equations for the line of intersection of the planes with normal vectors $a = \langle 2,0,-1\rangle$ and $b = \langle 1,2,-3\rangle$, and that have the common point $A(1,2,2)$.

Solution:

To find the vector of a line, we need to take the cross product of the normal vectors of the planes.

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & 2 & -3 \end{vmatrix}$$

$$a \times b = \mathbf{i}((0)(-3) - (-1)(2)) - \mathbf{j}((2)(-3) - (-1)(1)) + \mathbf{k}((2)(2) - (0)(1))$$

$$a \times b = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$$

Since point A is common to both planes, it lies on the line of intersection of the planes. Let's find the equation of the line with vector $\langle 2,5,4\rangle$ which passes through the point $A(1,2,2)$.

$$x = 1 + 2t$$

$$y = 2 + 5t$$

$$z = 2 + 4t$$

■ 2. Find the parametric equations for the line of intersection of the plane $2x-3y-4z=2$ with xz-plane.

Solution:

Since the intersection line lies on the xz-plane, $y=0$. Substitute $y=0$ into the plane equation.

$$2x-4z=2$$

$$x-2z=1$$

So we have the equation of a line on the xz-plane. To get a parametric equation, introduce $z=t$, and find x from the equation $x=2t+1$. So the parametric equations for the line of intersection is

$$x=2t+1$$

$$y=0$$

$$z=t$$

■ 3. Find the equations of a plane a that's perpendicular to the plane b, which is $x-3y+z=2$, and intersects b along the line given by the parametric equation.

$$x=2t$$

$$y = 1 + t$$

$$z = 2 - t$$

Solution:

The normal vector of the plane b is $\langle 1, -3, 1 \rangle$. The vector of the line of intersection is $\langle 2, 1, -1 \rangle$.

Since the plane a is perpendicular to the plane b, their normal vectors are perpendicular.

Also, since the line of intersection lies in the plane a, it's perpendicular to the normal vector of a.

So we can find the normal vector of a as a cross product.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$\mathbf{i}((-3)(-1) - (1)(1)) - \mathbf{j}((1)(-1) - (1)(2)) + \mathbf{k}((1)(1) - (-3)(2))$$

$$2\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$$

Let's take the point on the plane a by plugging $t = 0$ into the equation for the line of intersection.

$$x = 2(0) = 0$$

$$y = 1 + 0 = 1$$

$$z = 2 - 0 = 2$$

Let's find the equation of the plane with normal vector $\langle 2,3,7 \rangle$ which passes through $(0,1,2)$.

$$2(x - 0) + 3(y - 1) + 7(z - 2) = 0$$

$$2x + 3y - 3 + 7z - 14 = 0$$

$$2x + 3y + 7z - 17 = 0$$

SYMMETRIC EQUATIONS FOR THE LINE OF INTERSECTION OF TWO PLANES

■ 1. Find the symmetric equations for the line of intersection of the plane a, which is $2x - y + 3z = 12$, and the plane a' that's symmetric to the plane a with respect to the xz-plane.

Solution:

To get the equation of the plane a', just substitute $(-y)$ for y into the equation of the plane a.

$$2x + y + 3z = 12$$

The line of intersection of two symmetric planes with respect to the xz -plane lies in the xz-plane, so $y = 0$.

$$2x + 3z = 12$$

To obtain the symmetric equations for the line of intersection, isolate x and divide each side of the equation by 6 (least common multiple of 2 and 3).

$$2x = -(3z - 12)$$

$$\frac{2x}{6} = -\frac{3z - 12}{6}$$

$$\frac{x}{3} = -\frac{z - 4}{2}$$

So the symmetric equation for the line of intersection of the planes is

$$y = 0,\ \frac{x}{3} = -\frac{z-4}{2}$$

■ 2. Find the symmetric equations for the line of intersection of the plane $6x - 5y + z = 10$ with yz-plane.

Solution:

Since the intersection line lies in the yz-plane, that means $x = 0$. So substitute $x = 0$ into the plane equation.

$$-5y + z = 10$$

So we have the equation of a line in the yz-plane. To get the symmetric equation, isolate z and divide each side of the equation by 5.

$$z = 5y + 10$$

$$\frac{z}{5} = \frac{5y + 10}{5}$$

$$\frac{z}{5} = y + 2$$

So the symmetric equation for the line of intersection of the planes is

$$x = 0,\ \frac{z}{5} = y + 2$$

■ 3. Find the equations of a plane a that's perpendicular to the plane b, which is $2x - y - z = 3$, and intersects b along the line

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z}{2}$$

Solution:

The normal vector of the plane b is $\langle 2, -1, -1 \rangle$. The vector of the line of intersection is $\langle 3,2,2 \rangle$. Since the plane a is perpendicular to the plane b, their normal vectors are perpendicular.

And since the line of intersection lies in the plane a, it's perpendicular to the normal vector of a.

So we can find the normal vector of a as the cross product.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 3 & 2 & 2 \end{vmatrix}$$

$$\mathbf{i}((-1)(2) - (-1)(2)) - \mathbf{j}((2)(2) - (-1)(3)) + \mathbf{k}((2)(2) - (-1)(3))$$

$$0\mathbf{i} - 7\mathbf{j} + 7\mathbf{k}$$

Let's take the point on the plane a by plugging $z = 0$ into the intersection line equation.

$x - 1 = 0$ so $x = 1$

$$y + 2 = 0 \text{ so } y = -2$$

Let's find the equation of the plane with normal vector $\langle 0, -7, 7 \rangle$ which passes through $(1, -2, 0)$.

$$0(x - 1) - 7(y + 2) + 7(z - 0) = 0$$

$$-7y - 14 + 7z = 0$$

$$-y - 2 + z = 0$$

$$-y + z = 2$$

Note that the plane a appears to be parallel to x-axis. The equation of the plane is

$$-y + z = 2$$

DISTANCE BETWEEN A POINT AND A LINE

1. Determine the length of the height of triangle ABC, that's perpendicular to BC, if $A(2,0,-1)$, $B(4,5,2)$, and $C(4,3,0)$.

Solution:

The height of triangle ABC is equal to the distance between the point A and the line passing through B and C. By the distance formula between the line BC and the point A,

$$d = \frac{|AB \times AC|}{|BC|}$$

The numerator is twice the area of the triangle ABC, and the denominator is the length of the base of the triangle, BC. The vectors of the triangle are

$$AB = \langle 2,5,3 \rangle$$

$$AC = \langle 2,3,1 \rangle$$

$$BC = \langle 0,-2,-2 \rangle$$

Then

$$|BC| = \sqrt{(0)^2 + (-2)^2 + (-2)^2} = 2\sqrt{2}$$

and the cross product $AB \times AC$ is

$$AB \times AC = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & 3 \\ 2 & 3 & 1 \end{vmatrix}$$

$$AB \times AC = \mathbf{i}((5)(1) - (3)(3)) - \mathbf{j}((2)(1) - (3)(2)) + \mathbf{k}((2)(3) - (5)(2))$$

$$AB \times AC = -4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

Then

$$|AB \times AC| = \sqrt{(-4)^2 + 4^2 + (-4)^2} = 4\sqrt{3}$$

By the distance formula,

$$d = \frac{4\sqrt{3}}{2\sqrt{2}} = \sqrt{6}$$

■ 2. Find the sum of distances from $A(3,3,-1)$ to all of the coordinate axes, x, y, and z.

Solution:

The distance between A and the x-axis is the distance between $(3,3,-1)$ and its projection on this axis, $(3,0,0)$.

$$\sqrt{(3-3)^2 + (3-0)^2 + (-1-0)^2} = \sqrt{10}$$

The distance between A and the y-axis is the distance between $(3,3,-1)$ and its projection on this axis, $(0,3,0)$.

$$\sqrt{(3-0)^2+(3-3)^2+(-1-0)^2}=\sqrt{10}$$

The distance between A and the z-axis is the distance between $(3,3,-1)$ and its projection on this axis, $(0,0,-1)$.

$$\sqrt{(3-0)^2+(3-0)^2+(-1-(-1))^2}=3\sqrt{2}$$

The total distance is therefore

$$\sqrt{10}+\sqrt{10}+3\sqrt{2}$$

$$2\sqrt{10}+3\sqrt{2}$$

■ 3. Find the distance between $A(0,-2,1)$, and the line.

$$\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z}{2}$$

Solution:

The point on the line $B(-1,2,0)$, and the vector of the line is $a=\langle 2,-1,2\rangle$. By the distance formula between the line and the point A,

$$d=\frac{|AB\times a|}{|a|}$$

where

$$AB = \langle -1, 4, -1 \rangle$$

$$|a| = \sqrt{(2)^2 + (-1)^2 + (2)^2} = 3$$

The cross product $AB \times a$ is

$$AB \times a = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & -1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$AB \times a = \mathbf{i}((4)(2) - (-1)(-1)) - \mathbf{j}((-1)(2) - (-1)(2)) + \mathbf{k}((-1)(-1) - (4)(2))$$

$$AB \times a = 7\mathbf{i} + 0\mathbf{j} - 7\mathbf{k}$$

Then

$$|AB \times a| = \sqrt{(7)^2 + 0^2 + (-7)^2} = 7\sqrt{2}$$

By the distance formula,

$$d = \frac{7\sqrt{2}}{3}$$

DISTANCE BETWEEN A POINT AND A PLANE

■ 1. Determine the length of the height of tetrahedron $ABCD$, that's perpendicular to the plane BCD, if $A(1,0,-1)$, $B(0,1,0)$, $C(2,3,4)$, and $D(2,2,2)$.

Solution:

The height of tetrahedron $ABCD$ is equal to the distance between the point A and the plane passing through B, C, and D. The normal vector to BCD is the cross product $n = BC \times BD$, where $BC = \langle 2,2,4 \rangle$ and $BD = \langle 2,1,2 \rangle$.

$$BC \times BD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 4 \\ 2 & 1 & 2 \end{vmatrix}$$

$$BC \times BD = \mathbf{i}((2)(2) - (4)(1)) - \mathbf{j}((2)(2) - (4)(2)) + \mathbf{k}((2)(1) - (2)(2))$$

$$BC \times BD = 0\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

Then

$$|n| = \sqrt{(0)^2 + 4^2 + (-2)^2} = 2\sqrt{5}$$

By the distance formula between the plane BCD and the point A,

$$d = \frac{|n \cdot AB|}{|n|}$$

with $AB = \langle -1,1,1 \rangle$. So

$$d=\frac{|\langle 0,4,-2\rangle \cdot \langle -1,1,1\rangle|}{2\sqrt{5}}$$

$$d=\frac{0(-1)+4(1)-2(1)}{2\sqrt{5}}$$

$$d=\frac{2}{2\sqrt{5}}$$

$$d=\frac{1}{\sqrt{5}}$$

■ 2. Find the sum of distances from the point $A(2,3,-5)$ to all of the coordinate planes, xy, yz, and xz.

Solution:

The distance between the point A and the xy-plane is the distance between $(2,3,-5)$ and its projection on this plane, $(2,3,0)$. In other words, the distance between the point A and the xy-plane is equal to the absolute value of the z-coordinate of the point A, -5.

$$\sqrt{(2-2)^2+(3-3)^2+(-5-0)^2}=5$$

Similarly, the distance between the point A and yz-plane is equal to the absolute value of the x-coordinate of the point A, 2.

Finally, the distance between the point A and the xz-plane is equal to the absolute value of the y-coordinate of the point A, 3. The total distance is

$$5 + 2 + 3 = 10$$

3. Find the points on the line L_1 that lie at a distance of 6 from plane a.

$$L_1: \qquad x = 1 + t, \ y = 2t, \ z = 3 - 2t$$

$$a: \qquad x + 2y - 2z = 4$$

Solution:

By the distance formula between an arbitrary point on the line L_1 and the plane a,

$$d = \frac{|x + 2y - 2z - 4|}{\sqrt{1 + 2^2 + (-2)^2}}$$

$$d = \frac{|x + 2y - 2z - 4|}{3}$$

Substitute $x = 1 + t$, $y = 2t$, and $z = 3 - 2t$.

$$d = \frac{|(1 + t) + 2(2t) - 2(3 - 2t) - 4|}{3}$$

$$d = \frac{|1 + t + 4t - 6 + 4t - 4|}{3}$$

$$d = \frac{|9t - 9|}{3}$$

$$d = |3t - 3|$$

Since the distance is equal to 6, we get an equation for t.

$$d = |3t - 3| = 6$$

$3t - 3 = 6$ or $3t - 3 = -6$

$3t = 9$ or $3t = -3$

$t = 3$ or $t = -1$

Coordinates of the point for $t = 3$ are $(1 + 3, 2(3), 3 - 2(3)) = (4, 6, -3)$, and the coordinates of the point for $t = -1$ are $(1 - 1, 2(-1), 3 - 2(-1)) = (0, -2, 5)$.

So the points on the line are

$$(4, 6, -3)$$

$$(0, -2, 5)$$

DISTANCE BETWEEN PARALLEL PLANES

■ 1. Determine the length of the height of triangular prism $ABCA_1B_1C_1$ that's perpendicular to the plane ABC if $A(2,0,1)$, $B(1,0,0)$, $C(2,2,3)$, $A_1(3,2,5)$, $B_1(2,2,4)$, and $C_1(3,4,7)$.

Solution:

The height of triangular prism $ABCA_1B_1C_1$ is equal to the distance between the prism bases, ABC and $A_1B_1C_1$, and equal to the distance between parallel planes ABC and $A_1B_1C_1$.

Find the distance between the parallel planes as the distance between $A_1(3,2,5)$ and plane ABC. The normal vector to ABC is the cross-product $n = BA \times BC$, where $BA = \langle 1,0,1 \rangle$ and $BC = \langle 1,2,3 \rangle$.

$$BA \times BC = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$BA \times BC = \mathbf{i}((0)(3) - (1)(2)) - \mathbf{j}((1)(3) - (1)(1)) + \mathbf{k}((1)(2) - (0)(1))$$

$$BA \times BC = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

Then

$$|n| = \sqrt{(-2)^2 + (-2)^2 + (2)^2} = 2\sqrt{3}$$

By the distance formula between the plane ABC and the point A_1,

$$d = \frac{|n \cdot AA_1|}{|n|}$$

$$AA_1 = \langle 1,2,4 \rangle$$

So

$$d = \frac{|\langle -2,-2,2 \rangle \cdot \langle 1,2,4 \rangle|}{2\sqrt{3}}$$

$$d = \frac{-2(1) - 2(2) + 2(4)}{2\sqrt{3}}$$

$$d = \frac{2}{2\sqrt{3}}$$

$$d = \frac{1}{\sqrt{3}}$$

■ 2. Find the equations of the two planes that are parallel to the given plane and that lie at a distance of 2 from it.

$$2x - 2y - z = 3$$

Solution:

Since the parallel planes have the same normal vector, they have the equation

$$2x - 2y - z = k$$

where k is some arbitrary real number. Taking a point on this plane with $x = y = 0$, then $z = -k$. So by the distance between $(0,0,-k)$ and the given plane,

$$d = \frac{|2(0) - 2(0) - (-k) - 3|}{\sqrt{2^2 + (-2)^2 + (-1)^2}}$$

$$d = \frac{|k - 3|}{3}$$

Since the distance is equal to 2, we can get a value for k.

$$d = \frac{|k - 3|}{3} = 2$$

$$|k - 3| = 6$$

$k = -3$ or $k = 9$

So the parallel planes are

$$2x - 2y - z = -3$$

$$2x - 2y - z = 9$$

3. Find the equations of the two parallel planes a and b that pass through $A(2,-6,3)$ and $B(7,10,11)$ respectively, and have the largest possible distance between them.

Solution:

We'll get the largest possible distance between the planes a and b if the interval AB is perpendicular to each plane. So the vector AB is a normal vector to both planes, which gives $AB = \langle 5,16,8 \rangle$.

The equation of the plane that passes through $A(2,-6,3)$ and has the normal vector $AB = \langle 5,16,8 \rangle$ is

$$5(x-2)+16(y+6)+8(z-3)=0$$

$$5x-10+16y+96+8z-24=0$$

$$5x+16y+8z=-62$$

The equation of the plane that passes through $B(7,10,11)$ and has the normal vector $AB = \langle 5,16,8 \rangle$ is

$$5(x-7)+16(y-10)+8(z-11)=0$$

$$5x-35+16y-160+8z-88=0$$

$$5x+16y+8z=283$$

So the parallel planes are

$$5x+16y+8z=-62$$

$$5x + 16y + 8z = 283$$

REDUCING EQUATIONS TO STANDARD FORM

■ 1. What is the standard form and identity of the quadratic surface?

$$16x^2 + 49y^2 + 784z^2 + 128x - 294y - 87 = 0$$

Solution:

Isolate the terms with x, y, and z, then complete the squares.

$$(16x^2 + 128x + 256 - 256) + (49y^2 - 294y + 441 - 441) + 784z^2 - 87 = 0$$

$$(16x^2 + 128x + 256) - 256 + (49y^2 - 294y + 441) - 441 + 784z^2 - 87 = 0$$

$$16(x^2 + 8x + 16) + 49(y^2 - 6y + 9) + 784z^2 - 87 - 256 - 441 = 0$$

$$16(x + 4)^2 + 49(y - 3)^2 + 784z^2 - 784 = 0$$

Divide each term by 784 (the least common multiple of 16, 49, and 784).

$$\frac{16(x + 4)^2}{784} + \frac{49(y - 3)^2}{784} + z^2 - 1 = 0$$

$$\frac{(x + 4)^2}{49} + \frac{(y - 3)^2}{16} + z^2 = 1$$

$$\frac{(x + 4)^2}{7^2} + \frac{(y - 3)^2}{4^2} + z^2 = 1$$

The surface is the ellipsoid centered at $(-4,3,0)$.

2. What is the standard form and identity of the quadratic surface?

$$25y^2 + 9z^2 - 50y + -36z - 225x - 839 = 0$$

Solution:

Isolate the terms with x, y, and z, then complete the squares.

$$(25y^2 - 50y + 25 - 25) + (9z^2 - 36z + 36 - 36) - 225x - 839 = 0$$

$$(25y^2 - 50y + 25) - 25 + (9z^2 - 36z + 36) - 36 - 225x - 839 = 0$$

$$25(y^2 - 2y + 1) + 9(z^2 - 4z + 4) - 225x - 839 - 25 - 36 = 0$$

$$25(y-1)^2 + 9(z-2)^2 - 225x - 900 = 0$$

$$25(y-1)^2 + 9(z-2)^2 - 225(x+4) = 0$$

Divide each term by 225 (the least common multiple of 25 and 9).

$$\frac{25(y-1)^2}{225} + \frac{9(z-2)^2}{225} - (x+4) = 0$$

$$\frac{(y-1)^2}{9} + \frac{(z-2)^2}{25} = x + 4$$

$$x + 4 = \frac{(y-1)^2}{3^2} + \frac{(z-2)^2}{5^2}$$

The surface is the elliptic paraboloid with vertex at $(-4,1,2)$.

■ 3. What is the standard form and identity of the quadratic surface?

$$9x^2 - 9y^2 + 4z^2 + 18x - 36y - 8z = 23$$

Solution:

Isolate the terms with x, y, and z, then complete the squares.

$$(9x^2 + 18x + 9 - 9) - (9y^2 + 36y + 36 - 36) + (4z^2 - 8z + 4 - 4) = 23$$

$$(9x^2 + 18x + 9) - 9 - (9y^2 + 36y + 36) + 36 + (4z^2 - 8z + 4) - 4 = 23$$

$$9(x^2 + 2x + 1) - 9(y^2 + 4y + 4) + 4(z^2 - 2z + 1) - 9 + 36 - 4 = 23$$

$$9(x + 1)^2 - 9(y + 2)^2 + 4(z - 1)^2 + 23 = 23$$

Divide each term by 36 (the least common multiple of 4 and 9).

$$\frac{9(x + 1)^2}{36} - \frac{9(y + 2)^2}{36} + \frac{4(z - 1)^2}{36} = 0$$

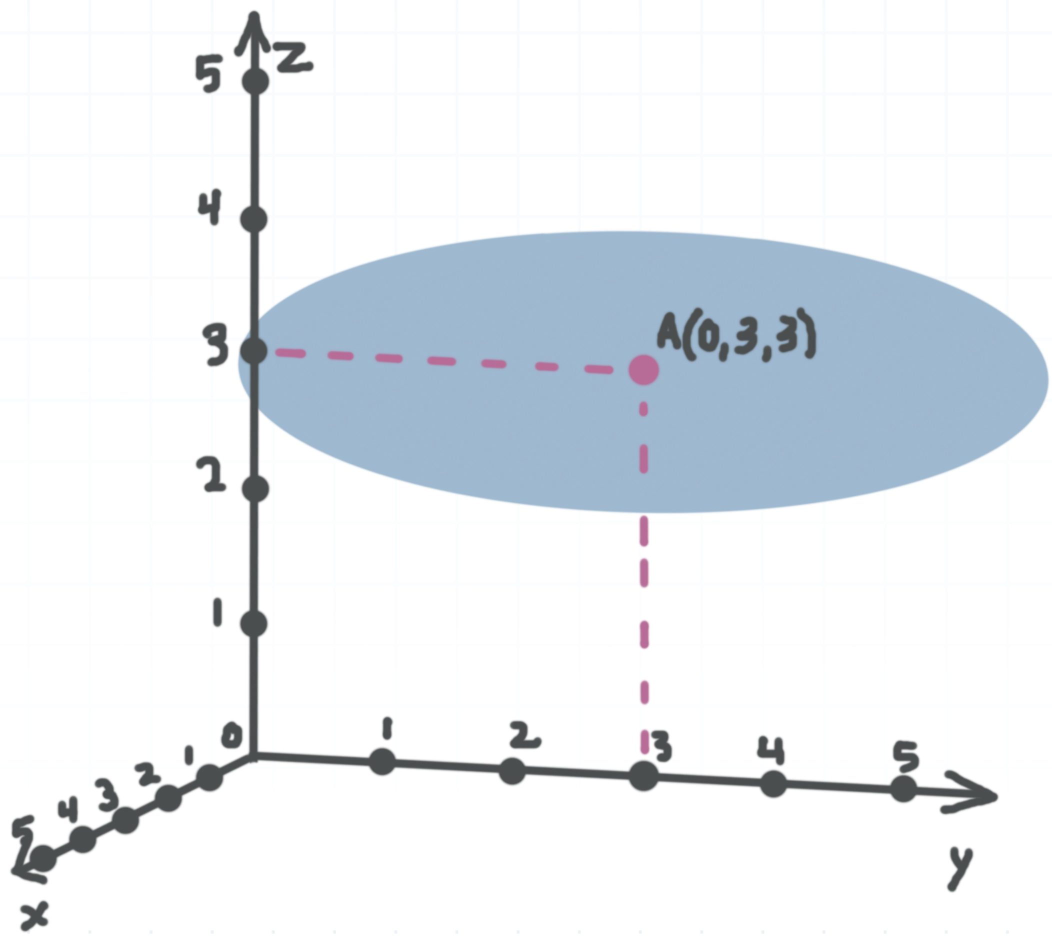
$$\frac{(x + 1)^2}{4} - \frac{(y + 2)^2}{4} + \frac{(z - 1)^2}{9} = 0$$

$$\frac{(x + 1)^2}{2^2} + \frac{(z - 1)^2}{3^2} - \frac{(y + 2)^2}{2^2} = 0$$

The surface is the elliptic cone centered at $(-1, -2,1)$.

SKETCHING THE SURFACE

1. Find the equation of the surface if its x- and z- principal axes have length 4 and 2 respectively.



Solution:

The surface is the ellipsoid with center $(0,3,3)$. The standard equation of an ellipsoid is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1$$

where (h,k,l) is the center, and a, b, and c are the x, y, and z semi-axes.

$a = 4/2 = 2$

$b = 3$

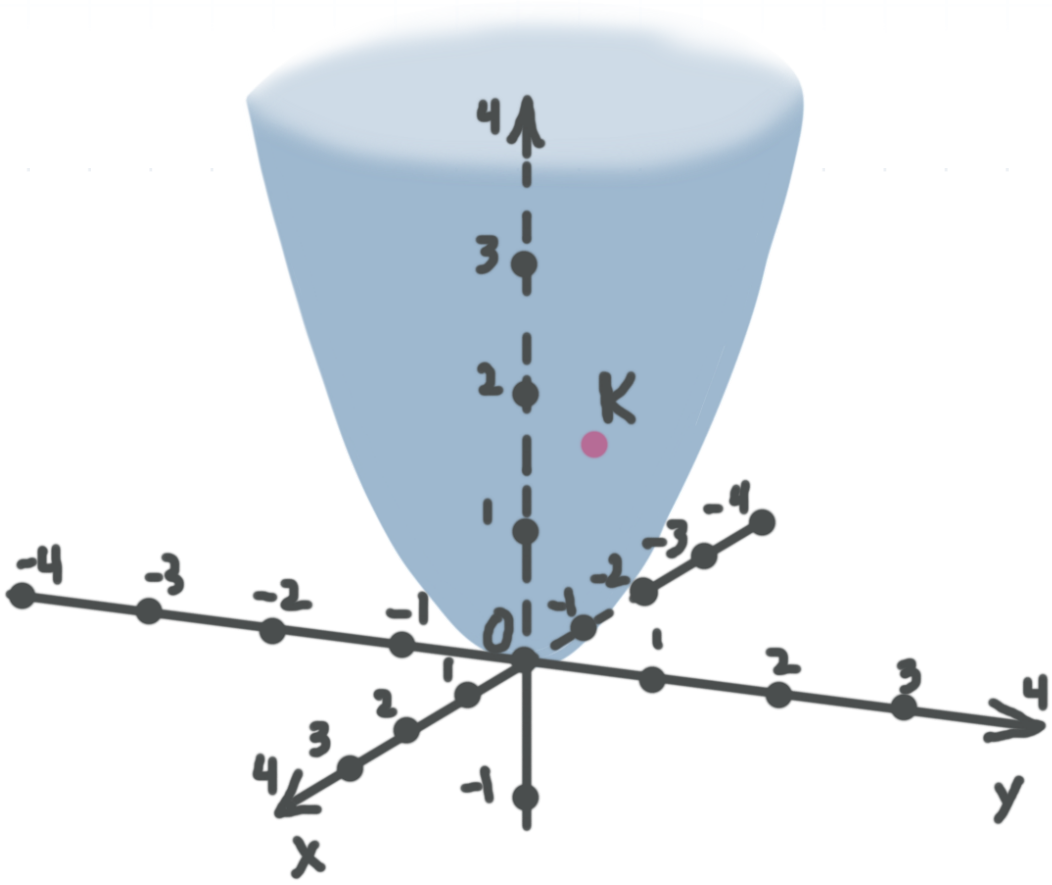
$c = 2/2 = 1$

And the center is $h = 0$, $k = 3$, and $l = 3$. Substitute these into the equation.

$$\frac{(x-0)^2}{2^2} + \frac{(y-3)^2}{3^2} + \frac{(z-3)^2}{1^2} = 1$$

$$\frac{x^2}{2^2} + \frac{(y-3)^2}{3^2} + (z-3)^2 = 1$$

■ 2. Find the equation of the circular paraboloid that passes through $K(1,1,2)$.



Solution:

The standard equation of circular paraboloid with a z-axis of symmetry is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{a^2}=z-l$$

where (h,k,l) is the center. Since from the picture $h=k=l=0$, the equation is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2}=z$$

To find the value of the constant a, substitute $K(1,1,2)$ into the equation.

$$\frac{1^2}{a^2}+\frac{1^2}{a^2}=2$$

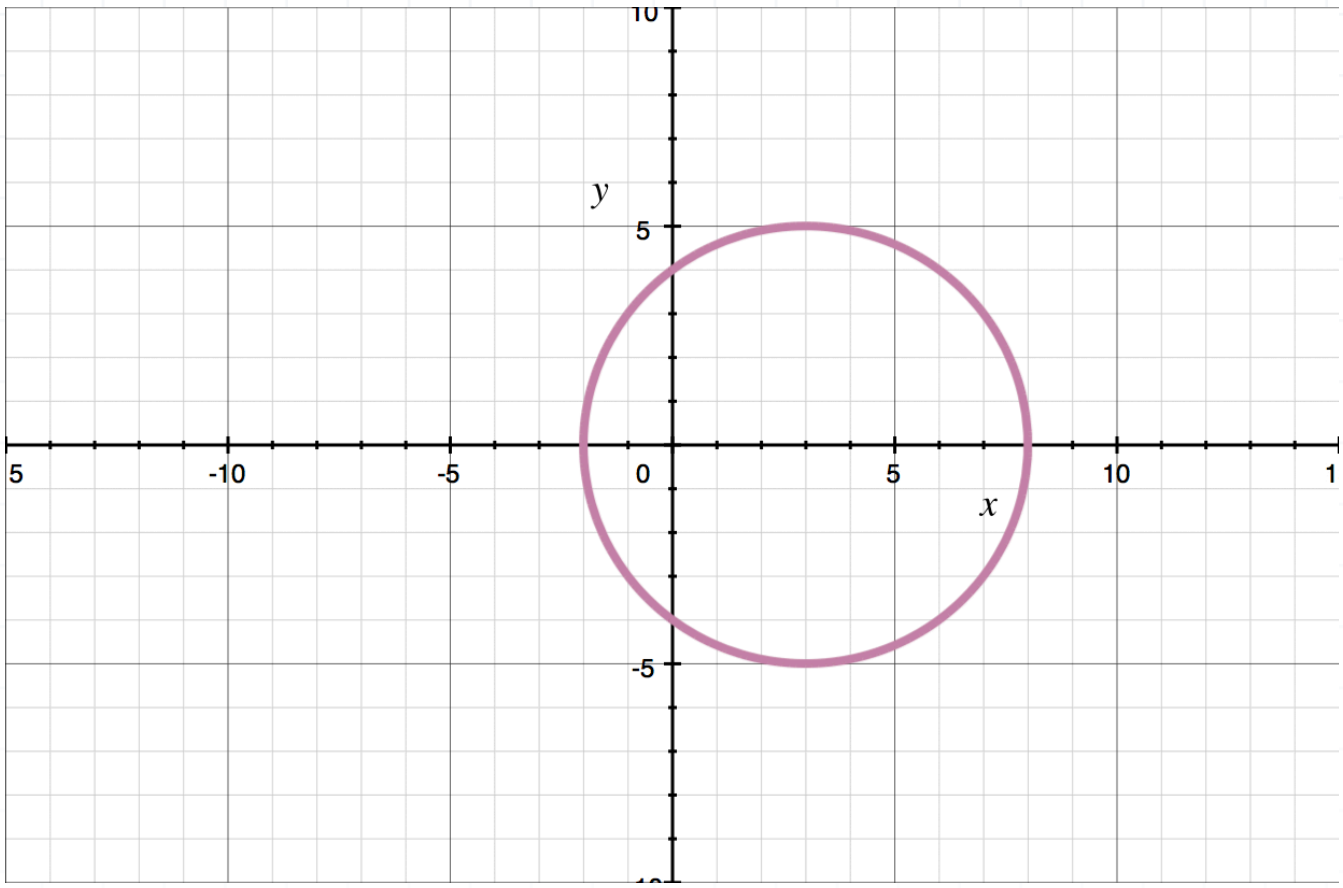
$$\frac{2}{a^2}=2$$

$$a^2=1$$

Since a is positive, $a=1$. So the equation is

$$x^2+y^2=z$$

■ 3. Find the equation of the surface obtained by rotating the circle about the x-axis.



Solution:

The standard equation of a sphere is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

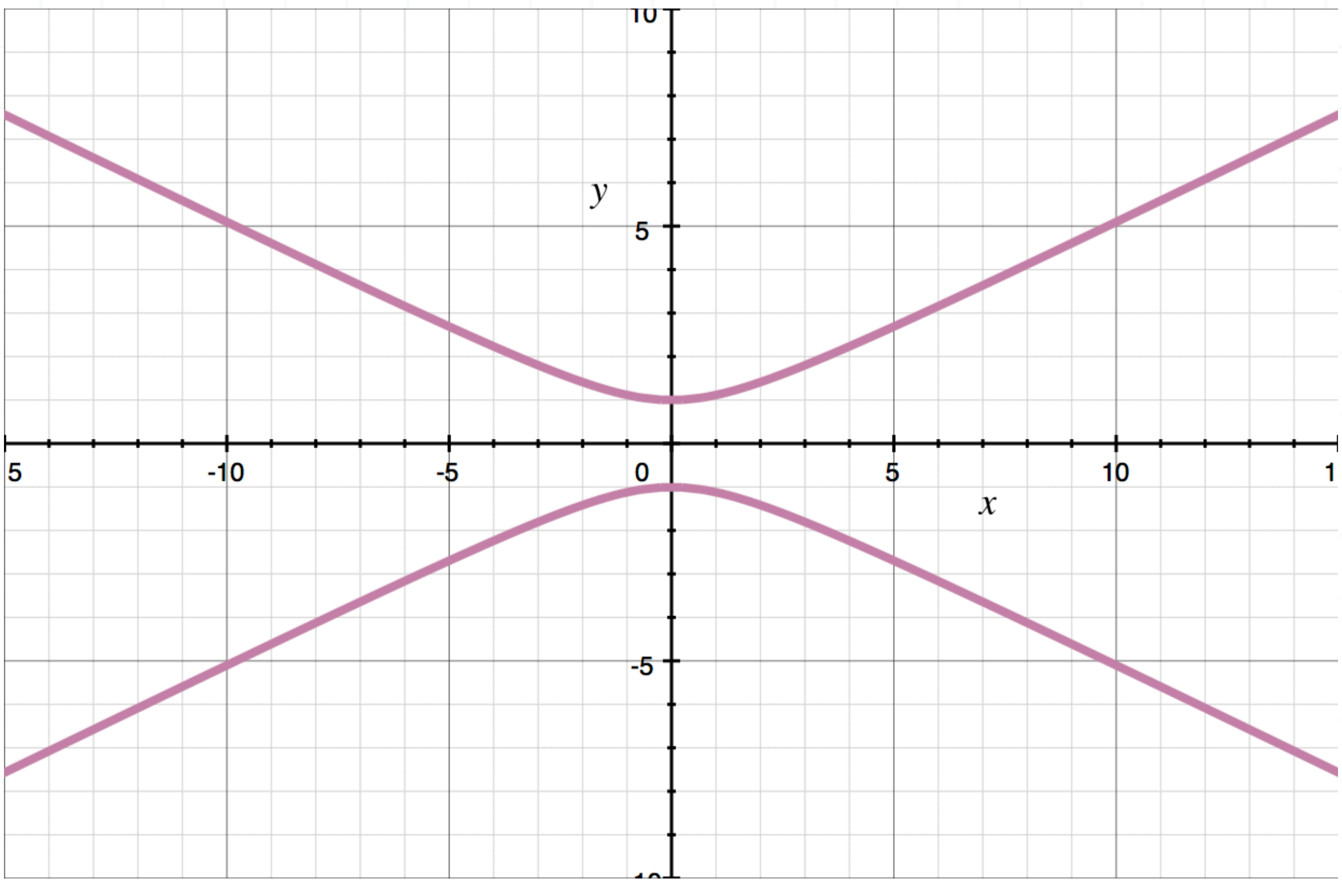
where (h,k,l) is the center and r is the radius.

The radius of the sphere is equal to the radius of the given circle, $r=5$, and the center of the sphere has the same coordinates as the center of the circle in space, $(3,0,0)$, so the equation of the sphere is

$$(x-3)^2+y^2+z^2=5^2$$

■ 4. Determine the identity and the equation of the surface obtained by rotating the hyperbola about the x-axis.

$$\frac{x^2}{2^2} - z^2 = -1$$



Solution:

The surface obtained by rotating this hyperbola about the x-axis is a circular hyperboloid of two sheets (special case of elliptic hyperboloid of two sheets). The standard equation of a circular hyperboloid is

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{a^2} + \frac{(z-l)^2}{c^2} = -1$$

where (h, k, l) is the center, c is the z semi axis (distance from the vertex of one sheet to the center), and a is a constant.

The semi axis of the hyperboloid is equal to the semi axis of hyperbola, $c=1$, and the center of the hyperboloid has the same coordinates as the center of the hyperbola in space, $(0,0,0)$.

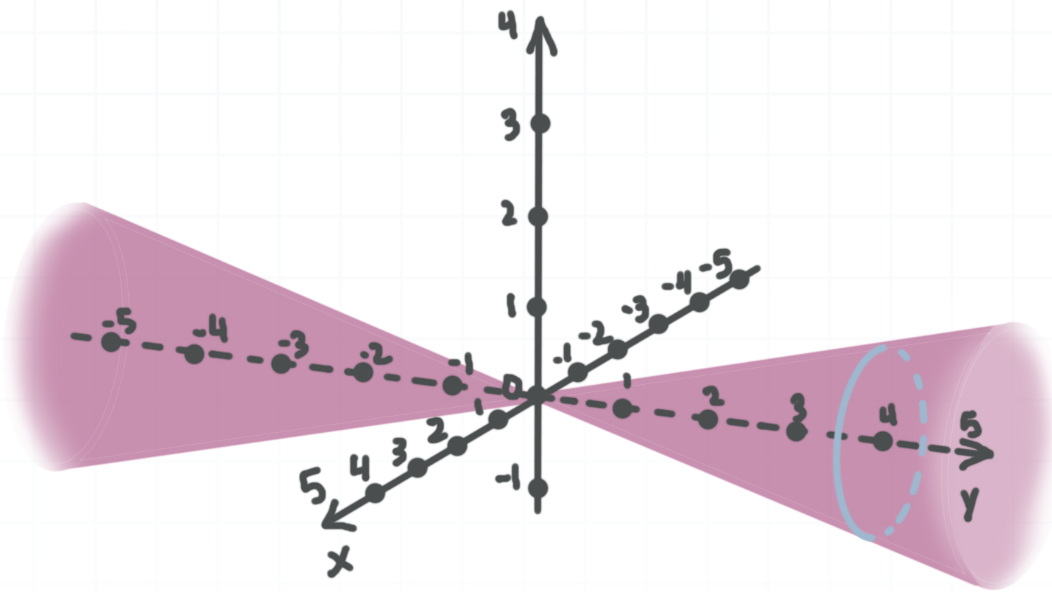
The value of the constant a can be determined from the hyperbola equation. Since the x-term is $x^2/2^2$, then $a=2$.

So the equation of the circular hyperboloid of two sheets is

$$\frac{x^2}{2^2}+\frac{y^2}{2^2}-z^2=-1$$

TRACES TO SKETCH AND IDENTIFY THE SURFACE

■ 1. Find the identity and the equation of the surface that has a trace $x^2 + z^2 = 1$ for $y = 4$.



Solution:

The surface in the picture is the cone with center at $(0,0,0)$. Since it has a circular trace, it is a circular cone.

The standard equation of a circular cone is

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} + \frac{(z-l)^2}{a^2} = 0$$

where (h, k, l) is the center, and a and b are the constants. Since $h = k = l = 0$, we get

$$\frac{(x-0)^2}{a^2}-\frac{(y-0)^2}{b^2}+\frac{(z-0)^2}{a^2}=0$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{a^2}=0$$

Substitute $y=4$ to get a trace.

$$\frac{x^2}{a^2}-\frac{4^2}{b^2}+\frac{z^2}{a^2}=0$$

$$x^2+z^2=\frac{4^2a^2}{b^2}$$

Since the trace is $x^2+z^2=1$,

$$\frac{4^2a^2}{b^2}=1$$

$$b^2=4^2a^2$$

Substitute 4^2a^2 for b^2 into the equation of the cone.

$$\frac{x^2}{a^2}-\frac{y^2}{4^2a^2}+\frac{z^2}{a^2}=0$$

$$x^2-\frac{y^2}{4^2}+z^2=0$$

■ 2. Find the trace of the surface in the plane $y=7$ and identify it.

$$\frac{(x+5)^2}{81}+\frac{(y-3)^2}{4}-\frac{(z+8)^2}{49}=1$$

Solution:

Substitute the value of 7 for y into the surface equation to get a trace.

$$\frac{(x+5)^2}{81}+\frac{(7-3)^2}{4}-\frac{(z+8)^2}{49}=1$$

$$\frac{(x+5)^2}{9^2}+\frac{(4)^2}{4}-\frac{(z+8)^2}{7^2}=1$$

$$\frac{(x+5)^2}{9^2}+4-\frac{(z+8)^2}{7^2}=1$$

$$\frac{(x+5)^2}{9^2}-\frac{(z+8)^2}{7^2}=-3$$

This is the hyperbola with center $(-5,7,-8)$ and equation

$$\frac{(x+5)^2}{9^2}-\frac{(z+8)^2}{7^2}=-3$$

■ 3. Find the traces of the surface in the planes $x=-2$, $y=8$, and $z=-4$ and use them to identify the surface.

$$\frac{(x+2)^2}{49}+\frac{(y-8)^2}{16}=z+5$$

Solution:

For $x = -2$,

$$\frac{(-2+2)^2}{49} + \frac{(y-8)^2}{16} = z + 5$$

$$\frac{(y-8)^2}{16} = z + 5$$

This is a parabola in the plane $x = -2$ with vertex $(-2,8,-5)$.

For $y = 8$,

$$\frac{(x+2)^2}{49} + \frac{(8-8)^2}{16} = z + 5$$

$$\frac{(y-8)^2}{16} = z + 5$$

This is a parabola in the plane $x = -2$ with vertex $(-2,8,-5)$.

For $z = -4$,

$$\frac{(x+2)^2}{49} + \frac{(y-8)^2}{16} = -4 + 5$$

$$\frac{(x+2)^2}{7^2} + \frac{(y-8)^2}{4^2} = 1$$

This is an ellipse in the plane $z = -4$ with center $(-2,8,-4)$ and semi axes 7 and 4.

Since the traces of this surface are an ellipse and two parabolas, the surface is elliptic paraboloid.

DOMAIN OF A MULTIVARIABLE FUNCTION

■ 1. Find the domain of the multivariable function.

$$f(x, y) = \sqrt{\sin(2x + y)}$$

Solution:

An expression under the square root should be nonnegative, so

$$\sin(2x + y) \geq 0$$

The function $\sin t \geq 0$ if $2\pi k \leq t \leq \pi + 2\pi k$ for any integer k. So

$$2\pi k \leq 2x + y \leq \pi + 2\pi k \text{ for any integer } k$$

■ 2. Find the domain of the multivariable function.

$$f(x, y) = (x^2 - y^2)\tan(2x)\cot(y + \pi)$$

Solution:

The argument of a tangent function can't be equal to

$$\frac{\pi}{2} + \pi k \text{ for any integer } k$$

so

$$2x \neq \frac{\pi}{2} + \pi k$$

$$x \neq \frac{\pi}{4} + \frac{\pi k}{2} \text{ for any integer } k$$

The argument of a cotangent function can't be equal to

$$\pi m \text{ for any integer } m$$

so

$$y + \pi \neq \pi m$$

$$y \neq \pi(m-1) \text{ for any integer } m$$

Let $n = m - 1$ where n is also any integer, then

$$y \neq \pi n \text{ for any integer } n$$

So the domain of the function is

$$x \neq \frac{\pi}{4} + \frac{\pi k}{2} \text{ for any integer } k$$

$$y \neq \pi n \text{ for any integer } n$$

■ 3. Find the domain of the multivariable function.

$$f(x, y) = \sin(3x + y)\log_{x-y}(x^2)$$

Solution:

The domain of the logarithmic function $\log_a b$ is $a > 0$, $a \neq 1$, and $b > 0$. So

$$x - y > 0$$

$$x - y \neq 1$$

$$x^2 > 0$$

Since x^2 is always greater than 0 except $x = 0$, we can say $x^2 > 0$ if $x \neq 0$. The domain of the function is

$$x - y > 0$$

$$x - y \neq 1$$

$$x \neq 0$$

■ 4. Find the set of points that lie within the domain of the multivariable function.

$$f(x, y) = 3\sqrt{x^2 + 2x + y^2 - 4y - 4}$$

Solution:

An expression under the square root should be nonnegative, so

$$x^2 + 2x + y^2 - 4y - 4 \geq 0$$

Complete the square with respect to each variable.

$$(x^2 + 2x + 1 - 1) + (y^2 - 4y + 4 - 4) - 4 \geq 0$$

$$(x + 1)^2 - 1 + (y - 2)^2 - 4 - 4 \geq 0$$

$$(x + 1)^2 + (y - 2)^2 - 9 \geq 0$$

$$(x + 1)^2 + (y - 2)^2 \geq 3^2$$

The domain is all points except the inner points of the circle with center at $(-1,2)$ and radius 3.

■ 5. Find the set of points that lie within the domain of the multivariable function.

$$f(x, y) = (2xy)^{-\frac{3}{4}}$$

Solution:

The function can be rewritten as

$$f(x, y) = \frac{1}{(2xy)^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{(2xy)^3}}$$

An expression under the square root should be positive.

$$(2xy)^3 > 0$$

$$2xy > 0$$

$$xy > 0$$

So x and y must be both positive, or both be negative. Which means the domain will be all points in quadrants I and III in the xy-plane.

LIMIT OF A MULTIVARIABLE FUNCTION

■ 1. If the limit exists, find its value.

$$\lim_{(x,y)\to(0,0)} \ln(2x + 3ey + e^2)$$

Solution:

Since the function is continuous at $(0,0)$, just substitute $(0,0)$ for (x, y).

$$\lim_{(x,y)\to(0,0)} \ln(2(0) + 3e(0) + e^2)$$

$$\lim_{(x,y)\to(0,0)} \ln(e^2)$$

$$2$$

■ 2. If the limit exists, find its value.

$$\lim_{(x,y)\to(\pi,\frac{\pi}{2})} \frac{\sin(3x + y)}{\cos(x - 2y)}$$

Solution:

Since the function is continuous at $(\pi, \pi/2)$, just substitute the respective values for (x, y).

$$\lim_{(x,y)\to(\pi,\frac{\pi}{2})} \frac{\sin(3\pi + \frac{\pi}{2})}{\cos(\pi - 2\frac{\pi}{2})}$$

$$\lim_{(x,y)\to(\pi,\frac{\pi}{2})} \frac{\sin\left(\frac{7\pi}{2}\right)}{\cos(0)}$$

$$\frac{-1}{1}$$

$$-1$$

3. If the limit exists, find its value.

$$\lim_{(x,y)\to(-\infty,-\infty)} (x^3 + 4y)(\sin(x^2 + 2y) + 3)$$

Solution:

Since $-1 \le \sin t \le 1$, then

$$-1 \le \sin(x^2 + 2y) \le 1$$

$$-1 + 3 \le \sin(x^2 + 2y) + 3 \le 1 + 3$$

$$2 \le \sin(x^2 + 2y) + 3 \le 4$$

If $x \to -\infty$ and $y \to -\infty$, then $x^3 + 4y \to -\infty$. So

$$\lim_{(x,y)\to(-\infty,-\infty)} (x^3 + 4y)(\sin(x^2 + 2y) + 3) \le \lim_{(x,y)\to(-\infty,-\infty)} 4(x^3 + 4y) = -\infty$$

$$\lim_{(x,y)\to(-\infty,-\infty)} (x^3 + 4y)(\sin(x^2 + 2y) + 3) = -\infty$$

■ 4. If the limit exists, find its value.

$$\lim_{(x,y)\to 0,0)} \frac{4x^4 - y^4}{2x^2 + y^2}$$

Solution:

Rewrite the function as

$$\frac{(2x^2 - y^2)(2x^2 + y^2)}{2x^2 + y^2}$$

$$2x^2 - y^2$$

This function is continuous at all real values of x and y.

$$\lim_{(x,y)\to(0,0)} \frac{4x^4 - y^4}{2x^2 + y^2} = \lim_{(x,y)\to(0,0)} (2x^2 - y^2) = 0$$

■ 5. If the limit exists, find its value.

$$\lim_{(x,y)\to(\infty,\infty)} 2^y - x^2$$

Solution:

In order for this limit to exist, the function must be approaching the same value regardless of the path that we take as we move in towards (∞, ∞).

Consider the path $y = \log_2(x^2)$.

$$\lim_{(x,\log_2(x^2))\to(\infty,\infty)} (2^{\log_2(x^2)} - x^2)$$

$$\lim_{(x,\log_2(x^2))\to(\infty,\infty)} (x^2 - x^2) = 0$$

Then consider the path $y = x$.

$$\lim_{(x,x)\to(\infty,\infty)} (2^x - x^2) = \infty$$

Since the limits from two different paths are not equal, the limit does not exist.

■ 6. If the limit exists, find its value.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + 2x^2y^2 - xy}{2x^3 + y^2}$$

Solution:

In order for this limit to exist, the function must be approaching the same value regardless of the path that we take as we approach $(0,0)$.

Consider the path $y = x$.

$$\lim_{(x,x)\to(0,0)} \frac{x^4 + 2x^2(x)^2 - x(x)}{2x^3 + x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^4 - x^2}{2x^3 + x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^2 - 1}{2x + 1}$$

$$\frac{3(0)^2 - 1}{2(0) + 1} = -1$$

Consider the path $y = -x$.

$$\lim_{(x,x)\to(0,0)} \frac{x^4 + 2x^2(-x)^2 - x(-x)}{2x^3 + (-x)^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^4 + x^2}{2x^3 + x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^2 + 1}{2x + 1}$$

$$\frac{3(0)^2 + 1}{2(0) + 1} = 1$$

Since the limits from two different paths are not equal, the limit does not exist.

PRECISE DEFINITION OF THE LIMIT FOR MULTIVARIABLE FUNCTIONS

■ 1. Which value of δ can be used to apply the precise definition of the limit to $f(x,y)$ with $\epsilon = 0.002$ at the point $(0,0)$?

$$f(x,y) = (x^2 + y^2)(3 - xy)$$

Solution:

We need to find a δ such that $|f(x,y) - f(0,0)| < \epsilon$ whenever $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$.

$$f(0,0) = (0^2 + 0^2)(3 - (0)(0)) = 0$$

$$|3 - xy| \le |3| + |xy| = 3 + |x||y|$$

$$|x| = \sqrt{x^2} \le \sqrt{x^2 + y^2} = \delta$$

Similarly,

$$|y| = \sqrt{y^2} \le \sqrt{x^2 + y^2} = \delta$$

So

$$|3 - xy| \le 3 + |x||y| \le 3 + \delta^2$$

Finally,

$$|f(x,y) - f(0,0)| = |(x^2 + y^2)(3 - xy)| \le \delta^2(3 + \delta^2)$$

Since δ is relatively small, $3 + \delta^2 \le 4$. So

$$|f(x,y) - f(0,0)| \le 4\delta^2$$

Let $\epsilon = 4\delta^2$. Then

$$\delta = \frac{\sqrt{\epsilon}}{2}$$

■ 2. Which value of δ can be used to apply the precise definition of the limit to $f(x,y)$ with $\epsilon = 0.001$ at the point $(0,0)$? Hint: Use the polar form of the function.

$$f(x,y) = \frac{5x^2y}{x^2 + y^2}$$

Solution:

We need to find a δ such that $|f(x,y) - \lim_{(x,y)\to(0,0)} f(x,y)| < \epsilon$ whenever $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$.

Since $f(x,y)$ is not continuous at $(0,0)$, we can switch to polar coordinates to investigate it. Substituting $x^2 + y^2 = r^2$, $x = r\cos\theta$, and $y = r\sin\theta$, we rewrite the function in polar coordinates.

$$f(r,\theta)=\frac{5(r\cos\theta)^2(r\sin\theta)}{r^2}$$

$$f(r,\theta)=\frac{5r^3\cos^2\theta\sin\theta}{r^2}$$

$$f(r,\theta)=5r\cos^2\theta\sin\theta$$

Since $0\le\sqrt{(x-0)^2+(y-0)^2}\le\delta$, then $0\le r\le\delta$. And if $x=0$ and $y=0$, then $r=0$.

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{r\to 0}f(r,\theta)=\lim_{r\to 0}5r\cos^2\theta\sin\theta=0$$

$$|f(r,\theta)-0|=|5r\cos^2\theta\sin\theta|=5r|\cos^2\theta\sin\theta|$$

Since $|\sin\theta|\le 1$ and $|\cos\theta|\le 1$,

$$|f(r,\theta)-0|\le 5r\le 5\delta$$

Let $\epsilon=5\delta$. Then

$$\delta=\frac{1}{5}\epsilon$$

■ 3. We know that $f(x,y)$ is a continuous function, and that for any real $\epsilon>0$, there exists a $\delta>0$ such that $\sqrt{(x-4)^2+(y+3)^2}<\delta$ implies $|f(x,y)-7|<\epsilon$. If the limit exists, find its value.

$$\lim_{(x,y)\to(4,-3)}(f(x,y))^2$$

Solution:

From the given statement, by the precise definition of the limit there exists

$$\lim_{(x,y)\to(4,-3)} f(x,y)$$

$$\lim_{(x,y)\to(4,-3)} f(x,y) = f(4,-3) = 7$$

By properties of limits,

$$\lim_{(x,y)\to(4,-3)} (f(x,y))^2 = \left(\lim_{(x,y)\to(4,-3)} f(x,y)\right)^2 = (7)^2 = 49$$

■ 4. We know that $f(x,y)$ and $g(x,y)$ are continuous functions, and that for any real $\epsilon > 0$, there exists a $\delta > 0$ such that $\sqrt{(x-2)^2 + y^2} < \delta$ implies $|f(x,y) + 3| + |g(x,y) - 5| < \epsilon$. If the limit exists, find its value

$$\lim_{(x,y)\to(2,0)} (3f(x,y) - 2g(x,y))$$

Solution:

From the given statement,

$$|f(x,y) + 3| \le |f(x,y) + 3| + |g(x,y) - 5| < \epsilon$$

So by the precise definition of the limit there exists

$$\lim_{(x,y)\to(2,0)} f(x,y)$$

$$\lim_{(x,y)\to(2,0)} f(x,y) = f(2,0) = -3$$

Similarly, for the function $g(x,y)$,

$$|g(x,y) - 5| \le |f(x,y) + 3| + |g(x,y) - 5| < \epsilon$$

$$\lim_{(x,y)\to(2,0)} g(x,y) = g(2,0) = 5$$

By properties of limits,

$$\lim_{(x,y)\to(2,0)} (3f(x,y) - 2g(x,y))$$

$$3 \lim_{(x,y)\to(2,0)} f(x,y) - 2 \lim_{(x,y)\to(2,0)} g(x,y)$$

$$3(-3) - 2(5) = -19$$

■ 5. We know that for any real $\epsilon > 0$, there exists a $\delta > 0$ such that

for $x > 0$, $\sqrt{x^2 + y^2} < \delta$ implies $|f(x,y) - 4| < \epsilon$

for $x \le 0$, $\sqrt{x^2 + y^2} < \delta$ implies $|f(x,y) + 4| < \epsilon$

If the limit exists, find its value.

$$\lim_{(x,y)\to(0,0)} 3^{f(x,y)}$$

Solution:

From the given statement, by the precise definition of the limit, if (x, y) approaches $(0,0)$ along the path $y = x$ for $x > 0$, then

$$\lim_{(x,x)\to(0,0)} f(x, x) = 4$$

But if (x, y) approaches $(0,0)$ along the path $y = x$ for $x < 0$, then

$$\lim_{(x,x)\to(0,0)} f(x, x) = -4$$

So the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist, and by the properties of limits,

$$\lim_{(x,y)\to(0,0)} 3^{f(x,y)}$$

also does not exist.

■ 6. We know that for any real $\epsilon > 0$, there exists a $\delta > 0$ such that $\sqrt{(x+1)^2 + (y-12)^2} < \delta$ implies $f(x, y) > \epsilon$. If the limit exists, find its value.

$$\lim_{(x,y)\to(-1,12)} (f(x,y) - 13)$$

Solution:

From the given statement, by the precise definition of the limit, there exists

$$\lim_{(x,y)\to(-1,12)} f(x,y)$$

$$\lim_{(x,y)\to(-1,12)} f(x,y) = \infty$$

By the properties of limits,

$$\lim_{(x,y)\to(-1,12)} (f(x,y) - 12)$$

$$\lim_{(x,y)\to(-1,12)} f(x,y) - \lim_{(x,y)\to(-1,12)} 12$$

$$\infty - 12$$

$$\infty$$

DISCONTINUITIES OF MULTIVARIABLE FUNCTIONS

■ 1. Find any discontinuities of the function.

$$f(x, y) = 3^{x^2-2y^2+\sqrt{x^2+5y^2-x+1}}$$

Solution:

The power function 3^t is continuous for every real number t. An expression under the square root should be nonnegative, so

$$x^2 + 5y^2 - x + 1 \ge 0$$

$$x^2 - 2(0.5)x + 0.25 - 0.25 + 5y^2 + 1 \ge 0$$

$$(x - 0.5)^2 - 0.25 + 5y^2 + 1 \ge 0$$

$$(x - 0.5)^2 + 5y^2 + 0.75 \ge 0$$

Since $(x - 0.5)^2 \ge 0$ and $5y^2 \ge 0$ and $0.75 > 0$, the sum of these terms is always positive, so

$$(x - 0.5)^2 + 5y^2 + 0.75 > 0$$

So the given function is continuous for all real numbers x and y.

■ 2. Find any discontinuities of the function.

$$f(x, y) = \sqrt{\sin x \cos y + \sin y \cos x}$$

Solution:

An expression under the square root should be nonnegative, so

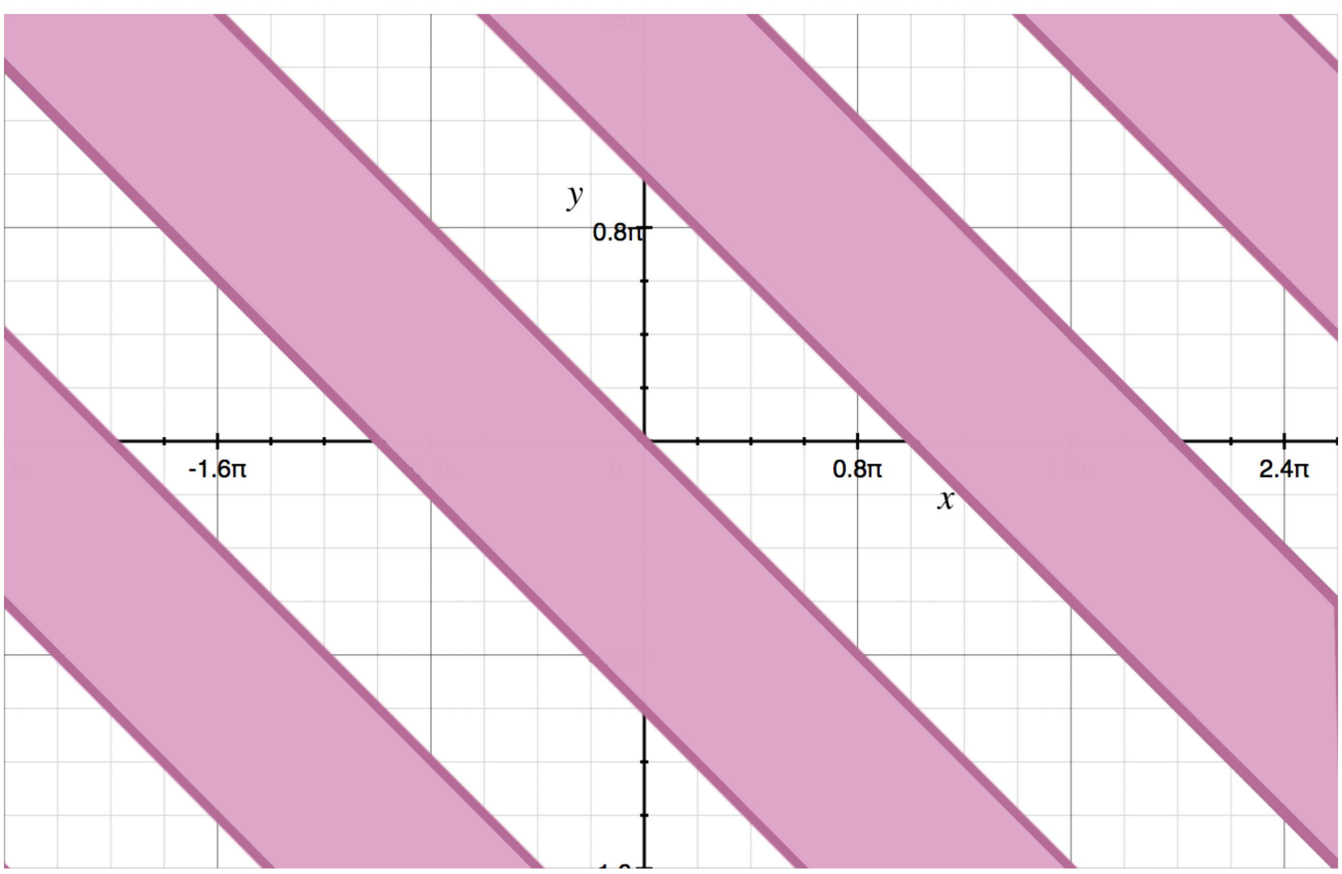
$$\sin x \cos y + \sin y \cos x \geq 0$$

$$\sin(x + y) \geq 0$$

The function is discontinuous if $\sin(x + y) < 0$. Solve the trigonometric inequality

$$2\pi k - \pi < x + y < 2\pi k$$

$$-x - \pi + 2\pi k < y < -x + 2\pi k$$



The function is discontinuous when $-x - \pi + 2\pi k < y < -x + 2\pi k$ for any integer k.

■ 3. Find any discontinuities of the function.

$$f(x, y) = \begin{cases} \frac{4x^2 - y^2}{2x - y} & y \neq 2x \\ 0 & y = 2x \end{cases}$$

Solution:

Simplify the function for $y \neq 2x$.

$$\frac{4x^2 - y^2}{2x - y} = \frac{(2x - y)(2x + y)}{2x - y} = 2x + y$$

So for all of the points $y \neq 2x$ the function $f(x, y)$ is continuous.

For the points $y = 2x$ the function is continuous only at the points (x_0, y_0) where

$$\lim_{(x,y)\to(x_0,y_0)} 2x + y = 0$$

$$2x_0 + y_0 = 0$$

Since $y_0 = 2x_0$, we have

$$2x_0 + 2x_0 = 0$$

$$4x_0 = 0$$

$$x_0 = 0$$

Which gives $y_0 = 0$. Therefore, for the points $y = 2x$, the function is continuous only at $(0,0)$. So the function is continuous for all real numbers x and y, excluding the points $y = 2x$, but including the point $(0,0)$.

■ 4. Find and classify any discontinuities of the function.

$$f(x, y) = \frac{7x - y}{4x^2 + y^2 - 4x + 1}$$

Solution:

The denominator should be nonzero.

$$4x^2 + y^2 - 4x + 1 \neq 0$$

$$(2x - 1)^2 + y^2 \neq 0$$

The denominator equals 0 only at the point where $2x - 1 = 0$ and $y = 0$, or $(1/2,0)$. To classify the discontinuity, investigate the limit.

$$\lim_{(x,y)\to(1/2,0)} \frac{7x - y}{4x^2 + y^2 - 4x + 1}$$

Since the numerator at $(1/2,0)$ is positive, $7(1/2)+0=7/2>0$, and the denominator is positive, the function tends to infinity as (x,y) approaches $(1/2,0)$. So at $(1/2,0)$, the function has an infinite discontinuity.

So the single discontinuity at $(1/2,0)$ is an infinite discontinuity.

■ 5. Find and classify any discontinuities of the function.

$$f(x,y)=\frac{x^2-9y^2-2x+1}{|x-1|+|3y|}$$

Solution:

Simplify the function.

$$f(x,y)=\frac{(x-1)^2-9y^2}{|x-1|+3|y|}$$

$$f(x,y)=\frac{|x-1|^2-9|y|^2}{|x-1|+3|y|}$$

$$f(x,y)=\frac{(|x-1|-3|y|)(|x-1|+3|y|)}{|x-1|+3|y|}$$

$f(x,y)=|x-1|-3|y|$, assuming $|x-1|+3|y|\neq 0$

This function is continuous for all real numbers x and y.

If $|x-1|+3|y|=0$, then $x-1=0$ and $y=0$. So the function is discontinuous at $(1,0)$. Since the function $|x-1|-3|y|$ is continuous and finite at $(1,0)$, the function has a removable discontinuity at this point.

So the single discontinuity at $(1,0)$ is removable.

COMPOSITIONS OF MULTIVARIABLE FUNCTIONS

■ 1. Find $f(g(x,y))$.

$$f(t)=\ln(3t)$$

$$g(x,y)=\frac{x+1}{y+2}$$

Solution:

Substitute $g(x,y)$ for t into $f(t)$.

$$f(x,y)=\ln\left(3\frac{x+1}{y+2}\right)$$

$$f(x,y)=\ln 3+\ln(x+1)-\ln(y+2)$$

■ 2. Find $f(x(t),y(t))$.

$$f(x,y)=x^2-y^2+3$$

$$x(t)=\sqrt{t-5}$$

$$y(t)=2^{t+2}$$

Solution:

Substitute $x(t)$ for x and 2^{t+2} for y into $f(x, y)$.

$$f(t) = (\sqrt{t-5})^2 - (2^{t+2})^2 + 3$$

$$f(t) = t - 5 - (2^{t+2})^2 + 3$$

$$f(t) = t - 5 - 2^{2t+4} + 3$$

$$f(t) = t - 2 - 2^{2t+4}$$

■ 3. Find $f(u(x, y), v(x, y))$.

$$f(u, v) = u^2 + v^2 + \frac{u - v}{\sqrt{2}}$$

$$u(x, y) = \sin(x + y)$$

$$v(x, y) = \cos(x + y)$$

Solution:

Substitute u and v into f.

$$f(u, v) = u^2 + v^2 + \frac{u - v}{\sqrt{2}}$$

$$f(x, y) = (\sin(x + y))^2 + (\cos(x + y))^2 + \frac{\sin(x + y) - \cos(x + y)}{\sqrt{2}}$$

Using the trig identity $\sin^2(a) + \cos^2(a) = 1$ simplifies the equation to

$$f(x, y) = 1 + \frac{\sin(x + y) - \cos(x + y)}{\sqrt{2}}$$

$$f(x, y) = 1 + \frac{\sin(x + y)}{\sqrt{2}} - \frac{\cos(x + y)}{\sqrt{2}}$$

Because

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

the function $f(x, y)$ can be rewritten as

$$f(x, y) = 1 + \cos\frac{\pi}{4}\sin(x + y) - \sin\frac{\pi}{4}\cos(x + y)$$

By the trigonometric identity $\sin(a - b) = \sin a \cos b - \cos a \sin b$, the equation becomes

$$f(x, y) = 1 + \sin\left(x + y - \frac{\pi}{4}\right)$$

PARTIAL DERIVATIVES

■ 1. Find $f_x + f_y$.

$$f(x, y) = \sqrt{\sin(x + y)}$$

Solution:

The partial derivatives f_x and f_y are

$$f_x = \frac{\frac{\partial}{\partial x}\sin(x + y)}{2\sqrt{\sin(x + y)}} = \frac{\cos(x + y)}{2\sqrt{\sin(x + y)}}$$

$$f_y = \frac{\frac{\partial}{\partial y}\sin(x + y)}{2\sqrt{\sin(x + y)}} = \frac{\cos(x + y)}{2\sqrt{\sin(x + y)}}$$

Then the sum is

$$f_x + f_y = 2\frac{\cos(x + y)}{2\sqrt{\sin(x + y)}} = \frac{\cos(x + y)}{\sqrt{\sin(x + y)}}$$

■ 2. Find f_r and f_θ.

$$f(r, \theta) = r^2(\sin 2\theta - \cos 2\theta)$$

Solution:

The partial derivatives are

$$f_r = 2r(\sin 2\theta - \cos 2\theta)$$

$$f_\theta = r^2(2\cos 2\theta + 2\sin 2\theta) = 2r^2(\cos 2\theta + \sin 2\theta)$$

■ 3. Find u_s and u_t.

$$u(t, s) = 2^{\frac{t}{s}}$$

Solution:

The partial derivatives are

$$u_s = \ln(2)2^{\frac{t}{s}}\frac{\partial}{\partial s}\left(\frac{t}{s}\right)$$

$$u_s = \ln(2)2^{\frac{t}{s}}\left(-\frac{t}{s^2}\right)$$

$$u_s = \frac{-t\ln(2)}{s^2}2^{\frac{t}{s}}$$

and

$$u_t = \ln(2)2^{\frac{t}{s}}\frac{\partial}{\partial t}\left(\frac{t}{s}\right)$$

$$u_t = \ln(2)2^{\frac{t}{s}}\left(\frac{1}{s}\right)$$

$$u_t = \frac{\ln(2)}{s}2^{\frac{t}{s}}$$

■ 4. Find the point (x, y) where $f_x = f_y = 0$.

$$f(x, y) = 3x^2 - 2xy + 3y^2 - 4x + 2y - 1$$

Solution:

The partial derivatives are

$$f_x = 6x - 2y - 4 = 0$$

$$f_x = 3x - y - 2 = 0$$

and

$$f_y = -2x + 6y + 2 = 0$$

$$f_y = -x + 3y + 1 = 0$$

That makes the system of linear equations

$$3x - y - 2 = 0$$

$$-x + 3y + 1 = 0$$

The first equation solves for y as $y = 3x - 2$. Then substituting this into $-x + 3y + 1 = 0$ gives

$$-x + 3(3x - 2) + 1 = 0$$

$$8x - 5 = 0$$

$$x = \frac{5}{8}$$

Which means

$$y = 3 \cdot \frac{5}{8} - 2$$

$$y = -\frac{1}{8}$$

Then the point where $f_x = f_y = 0$ is

$$\left(\frac{5}{8}, -\frac{1}{8}\right)$$

PARTIAL DERIVATIVES IN THREE OR MORE VARIABLES

■ 1. Find $f_x^2 + f_y^2 + f_z^2$.

$$f(x, y, z) = \tan(x^2 + y^2 + z^2)$$

Solution:

The partial derivatives of f are

$$f_x = \sec^2(x^2 + y^2 + z^2)\frac{\partial}{\partial x}(x^2 + y^2 + z^2)$$

$$f_x = 2x\sec^2(x^2 + y^2 + z^2)$$

and

$$f_y = \sec^2(x^2 + y^2 + z^2)\frac{\partial}{\partial y}(x^2 + y^2 + z^2)$$

$$f_y = 2y\sec^2(x^2 + y^2 + z^2)$$

and

$$f_z = \sec^2(x^2 + y^2 + z^2)\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$f_z = 2z\sec^2(x^2 + y^2 + z^2)$$

Then the sum is

$$f_x^2 + f_y^2 + f_z^2 = (2x \sec^2(x^2 + y^2 + z^2))^2 + (2y \sec^2(x^2 + y^2 + z^2))^2$$

$$+(2z \sec^2(x^2 + y^2 + z^2))^2$$

$$f_x^2 + f_y^2 + f_z^2 = 4(x^2 + y^2 + z^2)\sec^4(x^2 + y^2 + z^2)$$

■ 2. Find f_u, f_v, and f_w.

$$f(u, v, w) = u^{v^w}$$

Solution:

The partial derivatives are

$$f_u = v^w \cdot u^{v^w - 1}$$

and

$$f_v = u^{v^w} \cdot \ln u \cdot \frac{\partial}{\partial v} v^w$$

$$f_v = u^{v^w} \cdot \ln u \cdot w \cdot v^{w-1}$$

and

$$f_w = u^{v^w} \cdot \ln u \cdot \frac{\partial}{\partial w} v^w$$

$$f_w = u^{v^w} \cdot \ln u \cdot \ln v \cdot v^w$$

■ 3. Find the point (a, b, c, d) where $f_a = f_b = f_c = f_d = 0$.

$$f(a, b, c, d) = a^2 + b^2 - c^2 - d^2 + 4ab - 4cd - 6a + 6c + 8 = 0$$

Solution:

The partial derivatives are

$$f_a = 2a + 4b - 6 = 0$$

$$f_b = 2b + 4a = 0$$

$$f_c = -2c - 4d + 6 = 0$$

$$f_d = -2d - 4c = 0$$

These make a system of linear equations.

$$a + 2b - 3 = 0$$

$$b + 2a = 0$$

$$c + 2d - 3 = 0$$

$$d + 2c = 0$$

The solution to the system is $a = -1$, $b = 2$, $c = -1$, and $d = 2$, or $(a, b, c, d) = (-1,2,-1,2)$.

HIGHER ORDER PARTIAL DERIVATIVES

■ 1. Find f_{uvw}.

$$f(u, v, w) = \sqrt{u^2 + v^2 + w^2}$$

Solution:

Find f_u.

$$f_u = \frac{1}{2\sqrt{u^2 + v^2 + w^2}} \frac{\partial}{\partial u}(u^2 + v^2 + w^2)$$

$$f_u = \frac{2u}{2\sqrt{u^2 + v^2 + w^2}}$$

$$f_u = \frac{u}{\sqrt{u^2 + v^2 + w^2}}$$

Then f_{uv} is

$$f_{uv} = -\frac{u}{2(u^2 + v^2 + w^2)^{3/2}} \frac{\partial}{\partial v}(u^2 + v^2 + w^2)$$

$$f_{uv} = -\frac{2uv}{2(u^2 + v^2 + w^2)^{3/2}}$$

$$f_{uv} = -\frac{uv}{(u^2 + v^2 + w^2)^{3/2}}$$

Then f_{uvw} is

$$f_{uvw} = \frac{3uv}{2(u^2+v^2+w^2)^{5/2}} \frac{\partial}{\partial w}(u^2+v^2+w^2)$$

$$f_{uvw} = \frac{6uvw}{2(u^2+v^2+w^2)^{5/2}}$$

$$f_{uvw} = \frac{3uvw}{(u^2+v^2+w^2)^{5/2}}$$

■ 2. Find and identify the curve for the set of the points (x, y) where $f_{xx} = f_{yy}$.

$$f(x, y) = 3x^3 - 4x^2y + y^3 - x^2 + 5y + 7$$

Solution:

Find f_{xx} and f_{yy}.

$$f_x = 9x^2 - 8xy - 2x$$

$$f_{xx} = 18x - 8y - 2$$

and

$$f_y = -4x^2 + 3y^2 + 5$$

$$f_{yy} = 6y$$

Since $f_{xx} = f_{yy}$, we get the equation

$$18x - 8y - 2 = 6y$$

$$9x - 7y - 1 = 0$$

This is the equation of a line in the xy-plane with the equation $9x - 7y - 1 = 0$.

■ 3. Find and identify the curve(s) for the set of the points (x, y) where $f_{xx} = f_{yy}$.

$$f(x, y) = \sin(x^2 + y^2)$$

Solution:

Find f_{xx}.

$$f_x = \cos(x^2 + y^2)\frac{\partial}{\partial x}(x^2 + y^2)$$

$$f_x = 2x\cos(x^2 + y^2)$$

so

$$f_{xx} = 2\cos(x^2 + y^2) + 2x\frac{\partial}{\partial x}\cos(x^2 + y^2)$$

$$f_{xx} = 2\cos(x^2 + y^2) + 2x \cdot (-2x\sin(x^2 + y^2))$$

$$f_{xx} = 2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2)$$

Because x and y are symmetric,

$$f_{yy} = 2\cos(x^2 + y^2) - 4y^2\sin(x^2 + y^2)$$

So $f_{xx} = f_{yy}$ gives

$$2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2) = 2\cos(x^2 + y^2) - 4y^2\sin(x^2 + y^2)$$

$$(x^2 - y^2)\sin(x^2 + y^2) = 0$$

The solutions are

$$x^2 - y^2 = 0$$

$$(x - y)(x + y) = 0$$

$x = y$ or $x = -y$

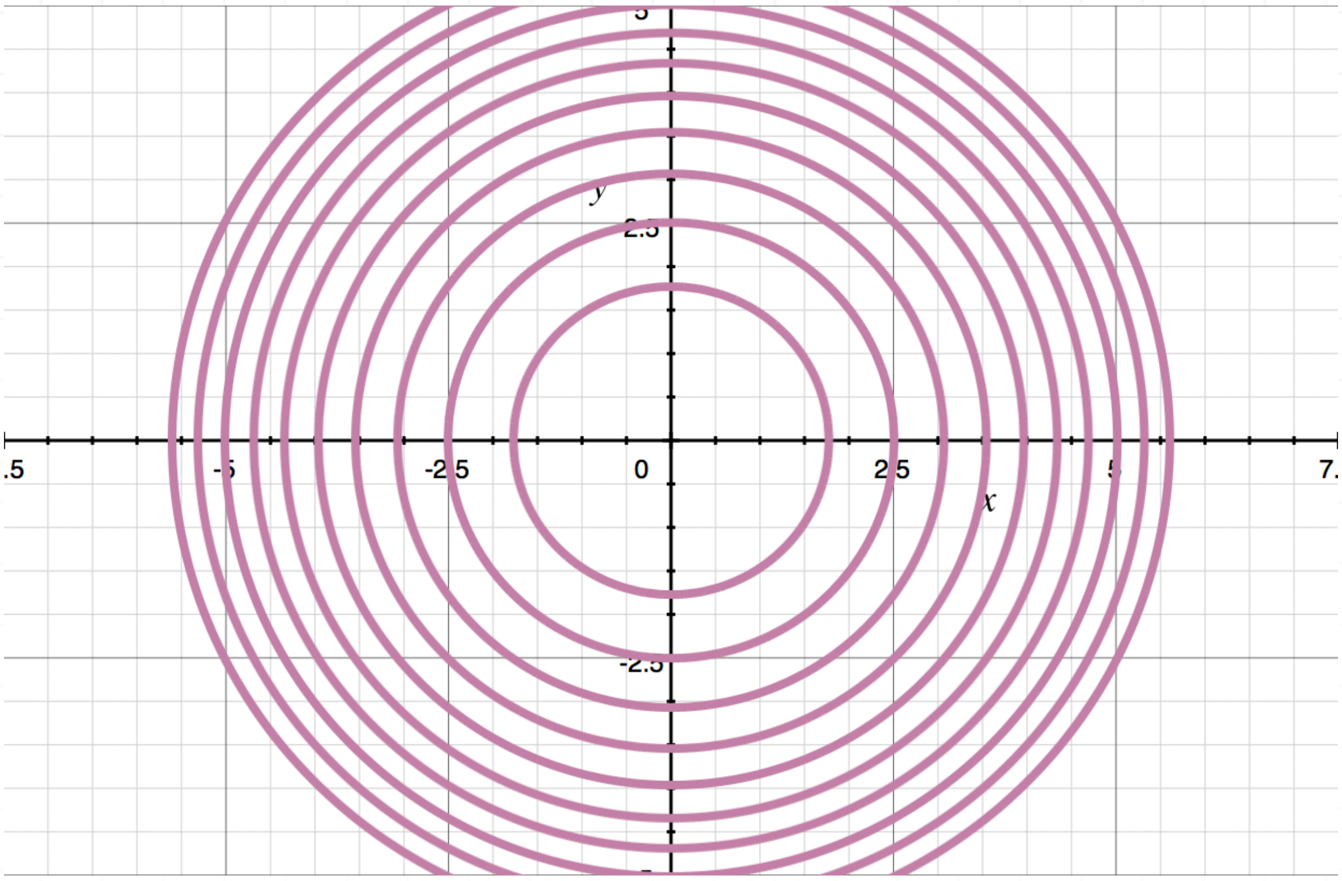
and

$$\sin(x^2 + y^2) = 0$$

$x^2 + y^2 = \pi k$ for any integer k

Since $x^2 + y^2 \geq 0$, $x^2 + y^2 = \pi k$ for any nonnegative integer k. We have a sequence of concentrated circles with radii $\sqrt{\pi k}$.

So the set of points is the pair of intersecting lines, $x = y$ and $x = -y$, and a sequence of concentrated circles with radii $\sqrt{\pi k}$ for any nonnegative integer k.



■ 4. Find all four second-order partial derivatives for the function. Is $f_{ts} = f_{st}$?

$$f(t, s) = e^{ts}$$

Solution:

Find the partial derivatives

$$f_t = se^{ts}$$

$$f_{tt} = s^2e^{ts}$$

$$f_{ts} = tse^{ts} + e^{ts}$$

and

$$f_s = te^{ts}$$

$$f_{ss} = t^2e^{ts}$$

$$f_{st} = tse^{ts} + e^{ts}$$

The four partial derivatives are

$$f_{tt} = s^2e^{ts}$$

$$f_{ss} = t^2e^{ts}$$

$$f_{ts} = f_{st} = tse^{ts} + e^{ts}$$

■ 5. Find the nth-order partial derivatives $\partial^n/\partial x^n$ and $\partial^n/\partial y^n$ by looking for patterns in the partial derivatives with respect to x and y.

$$f(x, y) = 2^{2x+4y}$$

Solution:

Look for a pattern in the partial derivatives with respect to x.

$$f_x = \ln(2)2^{2x+4y}\frac{\partial}{\partial x}(2x + 4y)$$

$$f_x = 2\ln(2)2^{2x+4y} = \ln(2) \cdot 2^{2x+4y+1}$$

$$f_{xx} = \ln^2(2)2^{2x+4y+1}\frac{\partial}{\partial x}(2x + 4y + 1)$$

$$f_{xx} = 2\ln^2(2)2^{2x+4y+1} = \ln^2(2) \cdot 2^{2x+4y+2}$$

$$f_{xxx} = \ln^3(2)2^{2x+4y+2}\frac{\partial}{\partial x}(2x+4y+2)$$

$$f_{xxx} = 2\ln^3(2)2^{2x+4y+2} = \ln^3(2)\cdot 2^{2x+4y+3}$$

...

$$\frac{\partial^n}{\partial x^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+n}$$

Look for a pattern in the partial derivatives with respect to y.

$$f_y = \ln(2)2^{2x+4y}\frac{\partial}{\partial y}(2x+4y)$$

$$f_y = 4\ln(2)2^{2x+4y} = \ln(2)\cdot 2^{2x+4y+2}$$

$$f_{yy} = \ln^2(2)2^{2x+4y+2}\frac{\partial}{\partial y}(2x+4y+2)$$

$$f_{yy} = 4\ln^2(2)2^{2x+4y+2} = \ln^2(2)\cdot 2^{2x+4y+4}$$

...

$$\frac{\partial^n}{\partial y^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+2n}$$

So the nth-order partial derivatives are

$$\frac{\partial^n}{\partial x^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+n}$$

$$\frac{\partial^n}{\partial y^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+2n}$$

DIFFERENTIAL OF A MULTIVARIABLE FUNCTION

■ 1. Find the differential of the multivariable function.

$$f(r,\theta)=\frac{r^2}{\sin 2\theta+\cos 2\theta}$$

Solution:

Find partial derivatives of f with respect to r and θ.

$$f_r=\frac{2r}{\sin 2\theta+\cos 2\theta}$$

and

$$f_\theta=\frac{(0)(\sin 2\theta+\cos 2\theta)-(r^2)(2\cos 2\theta-2\sin 2\theta)}{(\sin 2\theta+\cos 2\theta)^2}$$

$$f_\theta=-\frac{2r^2(\cos 2\theta-\sin 2\theta)}{(\sin 2\theta+\cos 2\theta)^2}$$

So the differential is

$$df=\frac{2r}{\sin 2\theta+\cos 2\theta}\ dr-\frac{2r^2(\cos 2\theta-\sin 2\theta)}{(\sin 2\theta+\cos 2\theta)^2}\ d\theta$$

■ 2. Find the differential of the multivariable function.

$$U(u,v,w)=\frac{(2u+1)^2(3v+4)}{\sqrt{w-2}}$$

Solution:

Rewrite the function by bringing the denominator up into the numerator.

$$U(u,v,w)=(2u+1)^2(3v+4)(w-2)^{-\frac{1}{2}}$$

Find partial derivatives of U with respect to u, v, and w.

$$U_u=2(2)(2u+1)(3v+4)(w-2)^{-\frac{1}{2}}$$

$$U_u=4(2u+1)(3v+4)(w-2)^{-\frac{1}{2}}$$

and

$$U_v=(2u+1)^2(3)(w-2)^{-\frac{1}{2}}$$

$$U_v=3(2u+1)^2(w-2)^{-\frac{1}{2}}$$

and

$$U_w=-\frac{1}{2}(2u+1)^2(3v+4)(w-2)^{-\frac{1}{2}}$$

Then the differential is

$$dU=4(2u+1)(3v+4)(w-2)^{-\frac{1}{2}}\ du+3(2u+1)^2(w-2)^{-\frac{1}{2}}\ dv$$

$$-\frac{1}{2}(2u+1)^2(3v+4)(w-2)^{-\frac{3}{2}}\ dw$$

$$dU = \frac{4(2u+1)(3v+4)}{\sqrt{w-2}}\ du + \frac{3(2u+1)^2}{\sqrt{w-2}}\ dv$$

$$-\frac{(2u+1)^2(3v+4)}{2\sqrt{(w-2)^3}}\ dw$$

■ 3. Find the differential of the multivariable function at $(-6,2)$.

$$f(x,y) = 4\log_2(x^2 - 2xy + y^2)$$

Solution:

Rewrite the function.

$$f(x,y) = 4\log_2(x-y)^2$$

Since for the point $(-6,2)$ the expression $(x-y)$ is negative,

$$\log_2(x-y)^2 = 2\log_2(y-x)$$

So

$$f(x,y) = 8\log_2(y-x)$$

Find partial derivatives.

$$f_x = \frac{8}{\ln(2)(y-x)} \cdot \frac{\partial}{\partial x}(y-x) = -\frac{8}{\ln(2)(y-x)}$$

$$f_x(-6,2) = -\frac{8}{\ln(2)(2-(-6))} = -\frac{1}{\ln(2)}$$

and

$$f_y = \frac{8}{\ln(2)(y-x)} \cdot \frac{\partial}{\partial y}(y-x) = \frac{8}{\ln(2)(y-x)}$$

$$f_y(-6,2) = \frac{8}{\ln(2)(2-(-6))} = \frac{1}{\ln(2)}$$

So the differential is

$$df = -\frac{dx}{\ln 2} + \frac{dy}{\ln 2}$$

■ 4. Find the point(s) where the differential of the multivariable function is equal to 0 (i.e., find the critical points of the function).

$$f(s,t) = 3t^4 - 2t^2s - s^2 + 16t + 5$$

Solution:

Find the partial derivatives.

$$f_s = -2t^2 - 2s = -2(t^2 + s)$$

$$f_t = 3(4t^3) - 2(2t)s + 16 = 12t^3 - 4ts + 16 = 4(3t^3 - ts + 4)$$

Since $df = 0$, we get a system of nonlinear equations.

$$3t^3 - ts + 4 = 0$$

$$t^2 + s = 0$$

The second equation solves for s as $s = -t^2$. Then by by substitution, we get

$$3t^3 - t(-t^2) + 4 = 0$$

$$3t^3 + t^3 + 4 = 0$$

$$4t^3 + 4 = 0$$

$$t^3 = -1$$

$$t = -1$$

Then $s = -(-1)^2 = -1$, and the differential is 0 at $(-1, -1)$.

■ 5. Find and identify the set of point(s) where the differential of the multivariable function $f(x, y)$ doesn't depend on dy.

$$f(x, y) = \cos(e^{x^2+y})$$

Solution:

Since $df = f_x\ dx$ for these points, $f_y = 0$.

$$f_y = -\sin(e^{x^2+y})\frac{\partial}{\partial y}e^{x^2+y}$$

$$f_y = -\sin(e^{x^2+y})e^{x^2+y}\frac{\partial}{\partial y}(x^2+y)$$

$$f_y = -\sin(e^{x^2+y})e^{x^2+y} = 0$$

Since $e^{x^2+y} > 0$ for any (x, y), $\sin(e^{x^2+y}) = 0$ and $e^{x^2+y} = \pi n$ for any integer n.

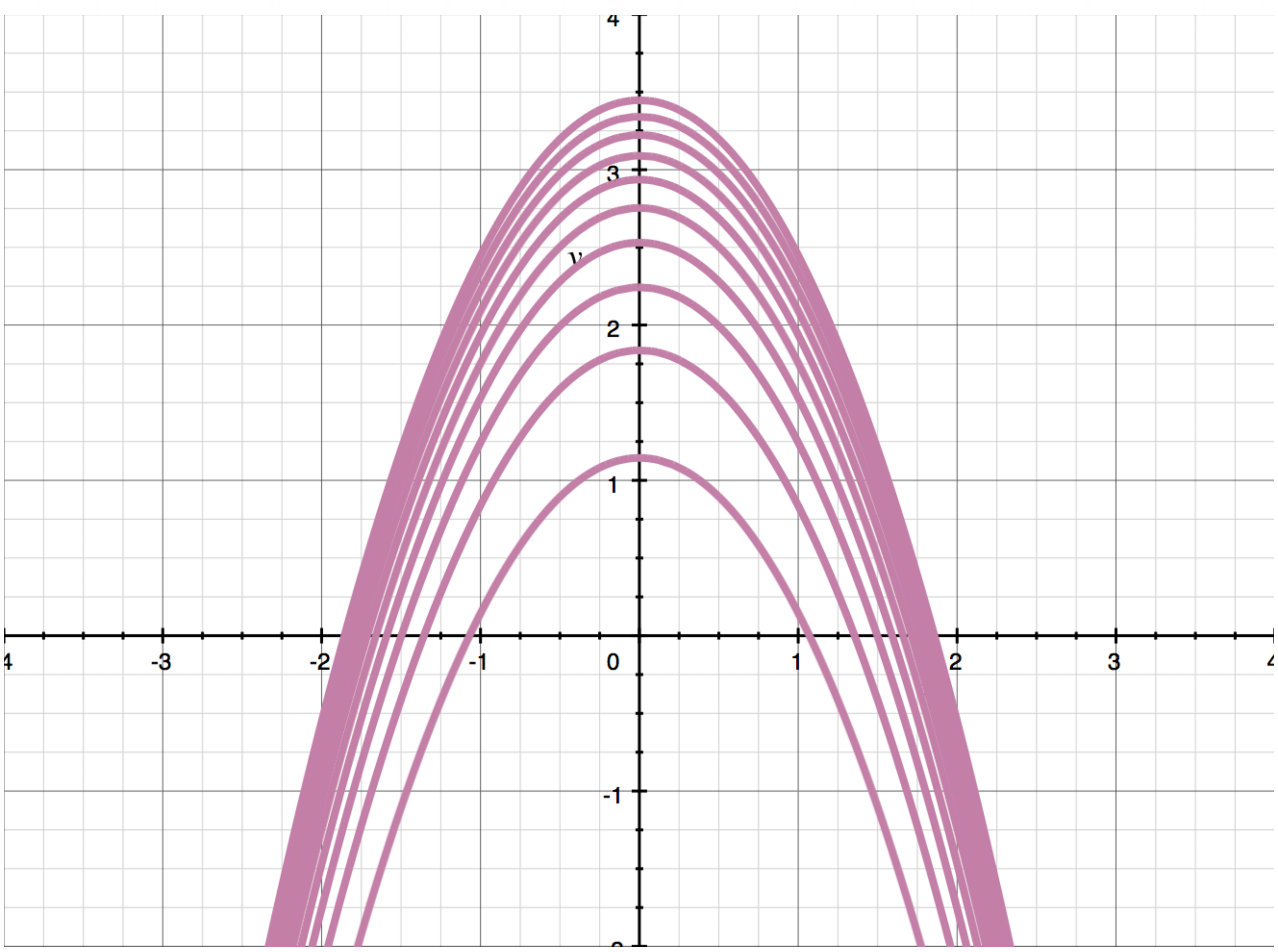
Since $e^{x^2+y} > 0$ for any (x, y), πn should be positive, i.e. $e^{x^2+y} = \pi n$ for any positive integer n.

Solve the equation for y.

$$x^2 + y = \ln(\pi n)$$

$$y = -x^2 + \ln(\pi n) \text{ for integer } n \geq 1$$

This is the sequence of parabolas with vertices at the points $(0, \ln(\pi n))$ for integers $n \geq 1$.



CHAIN RULE FOR MULTIVARIABLE FUNCTIONS

■ 1. If $x = e^t$, $y = t^2 - 3$, and $z = 2t + 1$, use chain rule to find df/dt.

$$f(x, y, z) = xy^2z^3$$

Solution:

Find derivatives of the parametric curve.

$$\frac{dx}{dt} = e^t$$

$$\frac{dy}{dt} = 2t$$

$$\frac{dz}{dt} = 2$$

Find partial derivatives of $f(x, y, z)$.

$$\frac{\partial f}{\partial x} = y^2z^3$$

$$\frac{\partial f}{\partial y} = 2xyz^3$$

$$\frac{\partial f}{\partial z} = 3xy^2z^2$$

Use chain rule to find df/dt.

$$\frac{df}{dt}=\frac{\partial f}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}+\frac{\partial f}{\partial z}\cdot\frac{dz}{dt}$$

$$\frac{df}{dt}=(y^2z^3)e^t+(2xyz^3)2t+(3xy^2z^2)2$$

$$\frac{df}{dt}=((t^2-3)^2(2t+1)^3)e^t+(2e^t(t^2-3)(2t+1)^3)2t+(3e^t(t^2-3)^2(2t+1)^2)2$$

$$\frac{df}{dt}=e^t((t^2-3)^2(2t+1)^3+4t(t^2-3)(2t+1)^3+6(t^2-3)^2(2t+1)^2)$$

$$\frac{df}{dt}=e^t(t^2-3)(2t+1)^2((t^2-3)(2t+1)+4t(2t+1)+6(t^2-3))$$

$$\frac{df}{dt}=e^t(t^2-3)(2t+1)^2(2t^3+15t^2-2t-21)$$

■ 2. If $r=\phi^2$ and $\theta=\phi+\pi$, use chain rule to find $dz/d\phi$ at $\phi=\pi/4$.

$$z(r,\theta)=r^2\sin\theta$$

Solution:

Find partial derivatives of r and θ at $\phi=\pi/4$.

$$\frac{\partial r}{\partial\phi}=2\phi$$

$$\frac{\partial r}{\partial\phi}\left(\frac{\pi}{4}\right)=\frac{\pi}{2}$$

and

$$\frac{\partial \theta}{\partial \phi} = 1$$

$$\frac{\partial \theta}{\partial \phi}\left(\frac{\pi}{4}\right) = 1$$

Find partial derivatives of z at $\phi = \pi/4$.

$$\frac{\partial z}{\partial r} = 2r\sin\theta$$

$$\frac{\partial z}{\partial r} = 2\phi^2\sin(\phi+\pi)$$

$$\frac{\partial z}{\partial r}\left(\frac{\pi}{4}\right) = 2\left(\frac{\pi}{4}\right)^2\sin\left(\frac{\pi}{4}+\pi\right)$$

$$\frac{\partial z}{\partial r}\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8}\cdot\left(-\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial z}{\partial r}\left(\frac{\pi}{4}\right) = -\frac{\pi^2}{8\sqrt{2}}$$

and

$$\frac{\partial z}{\partial \theta} = r^2\cos\theta$$

$$\frac{\partial z}{\partial \theta} = \phi^4\cos(\phi+\pi)$$

$$\frac{\partial z}{\partial \theta}\left(\frac{\pi}{4}\right)=\left(\frac{\pi}{4}\right)^4\cos\left(\frac{\pi}{4}+\pi\right)$$

$$\frac{\partial z}{\partial \theta}\left(\frac{\pi}{4}\right)=\frac{\pi^4}{256}\cdot\left(-\frac{1}{\sqrt{2}}\right)$$

$$\frac{\partial z}{\partial \theta}\left(\frac{\pi}{4}\right)=-\frac{\pi^4}{256\sqrt{2}}$$

Use chain rule to find $dz/d\phi$.

$$\frac{dz}{d\phi}=\frac{\partial z}{\partial r}\cdot\frac{dr}{d\phi}+\frac{\partial z}{\partial \theta}\cdot\frac{d\theta}{d\phi}$$

$$\frac{dz}{d\phi}=-\frac{\pi^2}{8\sqrt{2}}\cdot\frac{\pi}{2}-\frac{\pi^4}{256\sqrt{2}}\cdot 1$$

$$\frac{dz}{d\phi}=-\frac{\pi^3}{16\sqrt{2}}-\frac{\pi^4}{256\sqrt{2}}$$

■ 3. If $u=\ln(3t)$ and $v=\ln t$ with $t>0$, use chain rule to find the global maximum of the function.

$$f(u,v)=3u-2v^2$$

Solution:

Find derivatives of u and v.

$$\frac{du}{dt} = \frac{1}{t}$$

$$\frac{dv}{dt} = \frac{1}{t}$$

Find partial derivatives of $f(u, v)$.

$$\frac{\partial f}{\partial u} = 3$$

$$\frac{\partial f}{\partial v} = -4v$$

Use chain rule to find df/dt.

$$\frac{df}{dt} = \frac{\partial f}{\partial u} \cdot \frac{du}{dt} + \frac{\partial f}{\partial v} \cdot \frac{dv}{dt}$$

$$\frac{df}{dt} = 3\frac{1}{t} - 4v\frac{1}{t}$$

$$\frac{df}{dt} = \frac{3 - 4\ln t}{t} \text{ with } t > 0$$

Solve $f'(t) = 0$ to find critical points.

$$\frac{3 - 4\ln t}{t} = 0$$

$$3 - 4\ln t = 0$$

$$3 = 4\ln t$$

$$\ln t = \frac{3}{4}$$

$$t = e^{3/4}$$

Since $f'(t) > 0$ for $t < e^{3/4}$, and $f'(t) < 0$ for $t > e^{3/4}$, then $t = e^{3/4}$ is the global maximum of the function.

$$f(e^{3/4}) = 3\ln(3e^{3/4}) - 2(\ln(e^{3/4}))^2$$

$$f(e^{3/4}) = \ln(27) + \frac{9}{4} - 2\frac{3^2}{4^2}$$

$$f(e^{3/4}) = \frac{9}{8} + \ln(27)$$

CHAIN RULE FOR MULTIVARIABLE FUNCTIONS AND TREE DIAGRAMS

■ 1. If $x = \sin(t + s)$, $y = 2ts$, and $z = 2t - 5s$, use chain rule to find the partial derivatives f_t and f_s.

$$f(x, y, z) = 7x + 2y^2z$$

Solution:

Find partial derivatives with respect to s and t of x, y, and z.

$$x_s = \cos(t + s) \qquad x_t = \cos(t + s)$$

$$y_s = 2t \qquad y_t = 2s$$

$$z_s = -5 \qquad z_t = 2$$

Find partial derivatives for $f(x, y, z)$.

$$\frac{\partial f}{\partial x} = 7$$

$$\frac{\partial f}{\partial y} = 4yz$$

$$\frac{\partial f}{\partial z} = 2y^2$$

Use chain rule to find $\partial f/\partial t$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot x_s + \frac{\partial f}{\partial y} \cdot y_s + \frac{\partial f}{\partial z} \cdot z_s$$

$$\frac{\partial f}{\partial s} = (7)\cos(t+s) + (4yz)2t + (2y^2)(-5)$$

$$\frac{\partial f}{\partial s} = 7\cos(t+s) + 8t(2ts)(2t-5s) - 10(2ts)^2$$

$$\frac{\partial f}{\partial s} = 7\cos(t+s) + 8st^2(4t-15s)$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot x_t + \frac{\partial f}{\partial y} \cdot y_t + \frac{\partial f}{\partial z} \cdot z_t$$

$$\frac{\partial f}{\partial t} = (7)\cos(t+s) + (4yz)2s + (2y^2)2$$

$$\frac{\partial f}{\partial t} = 7\cos(t+s) + 8s(2ts)(2t-5s) + 4(2ts)^2$$

$$\frac{\partial f}{\partial t} = 7\cos(t+s) + 16s^2t(3t-5s)$$

The partial derivatives of f with respect to s and t are

$$\frac{\partial f}{\partial s} = 7\cos(t+s) + 8st^2(4t-15s)$$

$$\frac{\partial f}{\partial t} = 7\cos(t+s) + 16s^2t(3t-5s)$$

■ 2. If $x = \log_2(ts)$ and $y = \log_3(2t + s)$, use chain rule to find partial derivatives f_s and f_t at $(1,1)$.

$$f(x, y) = x^2 - 2xy - y^2 + x + 3y - 4$$

Solution:

Evaluate x and y at $(1,1)$.

$$x(1,1) = \log_2((1)(1)) = 0$$

$$y(1,1) = \log_3(2(1) + 1) = 1$$

Find partial derivatives of x and y at $(1,1)$.

$$x_s = \frac{1}{s\ln(2)} = \frac{1}{\ln(2)}$$

$$x_t = \frac{1}{t\ln(2)} = \frac{1}{\ln(2)}$$

and

$$y_s = \frac{1}{(2t + s)\ln(3)} = \frac{1}{3\ln(3)}$$

$$y_t = \frac{2}{(2t + s)\ln(3)} = \frac{2}{3\ln(3)}$$

Find partial derivatives f with respect to x and y.

$$\frac{\partial f}{\partial x} = 2x - 2y + 1 = 2(0) - 2(1) + 1 = -1$$

$$\frac{\partial f}{\partial y} = -2x - 2y + 3 = -2(0) - 2(1) + 3 = 1$$

Use chain rule to find $\partial f/\partial s$ and $\partial f/\partial t$ at $(1,1)$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot x_s + \frac{\partial f}{\partial y} \cdot y_s$$

$$\frac{\partial f}{\partial s} = (-1)\frac{1}{\ln(2)} + 1\frac{1}{3\ln(3)}$$

$$\frac{\partial f}{\partial s} = \frac{1}{3\ln(3)} - \frac{1}{\ln(2)}$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot x_t + \frac{\partial f}{\partial y} \cdot y_t$$

$$\frac{\partial f}{\partial t} = (-1)\frac{1}{\ln(2)} + 1\frac{2}{3\ln(3)}$$

$$\frac{\partial f}{\partial t} = \frac{2}{3\ln(3)} - \frac{1}{\ln(2)}$$

The partial derivatives of f at $(1,1)$ with respect to s and t are

$$\frac{\partial f}{\partial s}(1,1) = \frac{1}{3\ln(3)} - \frac{1}{\ln(2)}$$

$$\frac{\partial f}{\partial t}(1,1) = \frac{2}{3\ln(3)} - \frac{1}{\ln(2)}$$

3. If $x = 2t - s$ and $y = t + 2s$, use chain rule to find the point (s, t) where $f_t = f_s = 0$.

$$f(x, y) = 2x^2 - 3xy + y^2 + y + 9$$

Solution:

Find partial derivatives of x and y with respect to s and t.

$$x_s = -1 \qquad\qquad x_t = 2$$

$$y_s = 2 \qquad\qquad y_t = 1$$

Find partial derivatives of f with respect to x and y.

$$\frac{\partial f}{\partial x} = 4x - 3y$$

$$\frac{\partial f}{\partial y} = -3x + 2y + 1$$

Use chain rule to find $\partial f/\partial s$ and $\partial f/\partial t$.

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot x_s + \frac{\partial f}{\partial y} \cdot y_s$$

$$\frac{\partial f}{\partial s} = (4x - 3y)(-1) + (-3x + 2y + 1)(2)$$

$$\frac{\partial f}{\partial s} = -10x + 7y + 2$$

$$\frac{\partial f}{\partial s} = -10(2t - s) + 7(t + 2s) + 2$$

$$\frac{\partial f}{\partial s} = 24s - 13t + 2$$

and

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \cdot x_t + \frac{\partial f}{\partial y} \cdot y_t$$

$$\frac{\partial f}{\partial t} = (4x - 3y)(2) + (-3x + 2y + 1)(1)$$

$$\frac{\partial f}{\partial t} = 5x - 4y + 1$$

$$\frac{\partial f}{\partial t} = 5(2t - s) - 4(t + 2s) + 1$$

$$\frac{\partial f}{\partial t} = -13s + 6t + 1$$

Solve the system of equations.

$$-13s + 6t + 1 = 0$$

$$24s - 13t + 2 = 0$$

We get

$$-13s + 6t = -1$$

$$24s - 13t = -2$$

then

$$-169s + 78t = -13$$

$$144s - 78t = -12$$

Add the equations.

$$-169s + 78t + (144s - 78t) = -13 + (-12)$$

$$-169s + 78t + 144s - 78t = -13 - 12$$

$$-25s = -25$$

$$s = 1$$

Then

$$-13(1) + 6t = -1$$

$$6t = 12$$

$$t = 2$$

So the point we're looking for is $(s, t) = (1,2)$.

IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find the partial derivative dy/dx.

$$\sin(x+y) = x+y$$

Solution:

Implicitly differentiate both sides.

$$\cos(x+y)\left(1+1\frac{dy}{dx}\right) = 1+1\frac{dy}{dx}$$

Solve for dy/dx.

$$\cos(x+y) + \frac{dy}{dx}\cos(x+y) = 1+\frac{dy}{dx}$$

$$\cos(x+y) - 1 = \frac{dy}{dx} - \frac{dy}{dx}\cos(x+y)$$

$$\cos(x+y) - 1 = \frac{dy}{dx}(1-\cos(x+y))$$

$$\frac{dy}{dx} = \frac{\cos(x+y)-1}{1-\cos(x+y)}$$

Simplify.

$$\frac{dy}{dx} = -\frac{1-\cos(x+y)}{1-\cos(x+y)}$$

$$\frac{dy}{dx} = -1$$

■ 2. Use implicit differentiation to find the partial derivative $\partial z/\partial x$ of the multivariable function.

$$y \ln z = 2x - 3y + 2z$$

Solution:

Rewrite the equation as

$$0 = 2x - 3y + 2z - y \ln z$$

$$F(x, y, z) = 2x - 3y + 2z - y \ln z$$

Then the partial derivatives of F are

$$\frac{\partial F}{\partial x} = 2$$

$$\frac{\partial F}{\partial z} = 2 - \frac{y}{z}$$

So the partial derivative $\partial z/\partial x$ is

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{2}{2 - \frac{y}{z}} = -\frac{2z}{2z - y} = \frac{2z}{y - 2z}$$

3. Use implicit differentiation to find the partial derivative $\partial z/\partial y$ of the multivariable function.

$$e^z = x^2 + y + z$$

Solution:

Rewrite the equation as

$$0 = x^2 + y + z - e^z$$

$$F(x, y, z) = x^2 + y + z - e^z$$

Then the partial derivatives of F are

$$\frac{\partial F}{\partial y} = 1$$

$$\frac{\partial F}{\partial y} = 1 - e^z$$

So the partial derivative $\partial z/\partial y$ is

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{1}{1 - e^z} = \frac{1}{e^z - 1}$$

DIRECTIONAL DERIVATIVES

1. Find the directional derivative in the direction of $\vec{v} = \langle 2,2,1 \rangle$.

$$f(x, y, z) = \cos(2x + 3y + z)$$

Solution:

Convert the vector to its unit vector form.

$$\vec{u} = \left\langle \frac{2}{\sqrt{2^2 + 2^2 + 1^2}}, \frac{2}{\sqrt{2^2 + 2^2 + 1^2}}, \frac{1}{\sqrt{2^2 + 2^2 + 1^2}} \right\rangle$$

$$\vec{u} = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \right\rangle$$

Find the partial derivatives of f.

$$\frac{\partial f}{\partial x} = -2\sin(2x + 3y + z)$$

$$\frac{\partial f}{\partial y} = -3\sin(2x + 3y + z)$$

$$\frac{\partial f}{\partial z} = -\sin(2x + 3y + z)$$

Then the directional derivative is

$$D_u f(x, y, z) = -\frac{2}{3} \cdot 2\sin(2x + 3y + z) - \frac{2}{3} \cdot 3\sin(2x + 3y + z)$$

$$-\frac{1}{3} \cdot \sin(2x + 3y + z)$$

$$D_u f(x, y, z) = -\frac{\sin(2x + 3y + z)}{3}(4 + 6 + 1)$$

$$D_u f(x, y, z) = -\frac{11\sin(2x + 3y + z)}{3}$$

■ 2. Find the directional derivative in the direction of $\vec{v} = \langle 0, -3, -4 \rangle$.

$$f(x, y, z) = x^2 \ln(y - z)$$

Solution:

Convert the vector to its unit vector form.

$$\vec{u} = \left\langle \frac{0}{\sqrt{0^2 + (-3)^2 + (-4)^2}}, \frac{-3}{\sqrt{0^2 + (-3)^2 + (-4)^2}}, \frac{-4}{\sqrt{0^2 + (-3)^2 + (-4)^2}} \right\rangle$$

$$\vec{u} = \left\langle 0, -\frac{3}{5}, -\frac{4}{5} \right\rangle$$

Find the partial derivatives of f.

$$\frac{\partial f}{\partial x} = 2x\ln(y - z)$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{y-z}$$

$$\frac{\partial f}{\partial z} = -\frac{x^2}{y-z}$$

Then the directional derivative is

$$D_u f(x,y,z) = 0 \cdot 2x\ln(y-z) - \frac{3x^2}{5(y-z)} + \frac{4x^2}{5(y-z)}$$

$$D_u f(x,y,z) = -\frac{3x^2}{5(y-z)} + \frac{4x^2}{5(y-z)}$$

$$D_u f(x,y,z) = \frac{4x^2 - 3x^2}{5(y-z)}$$

$$D_u f(x,y,z) = \frac{x^2}{5(y-z)}$$

■ 3. Find the directional derivative in the direction of $\vec{v} = \langle 3, -6, 2 \rangle$ at the point $A(\pi/2, 1/2, \pi)$.

$$f(x,y,z) = x\sin(yz)$$

Solution:

We'll start by converting the given vector to its unit vector form.

$$\vec{u} = \left\langle \frac{3}{\sqrt{3^2 + (-6)^2 + 2^2}}, \frac{-6}{\sqrt{3^2 + (-6)^2 + 2^2}}, \frac{2}{\sqrt{3^2 + (-6)^2 + 2^2}} \right\rangle$$

$$\vec{u} = \left\langle \frac{3}{7}, -\frac{6}{7}, \frac{2}{7} \right\rangle$$

Find the partial derivatives of f.

$$\frac{\partial f}{\partial x} = \sin(yz)$$

$$\frac{\partial f}{\partial y} = xz\cos(yz)$$

$$\frac{\partial f}{\partial z} = xy\cos(yz)$$

Then the directional derivative is

$$D_u f(x, y, z) = \frac{3}{7} \cdot \sin(yz) - \frac{6}{7} \cdot xz\cos(yz) + \frac{2}{7} \cdot xy\cos(yz)$$

$$D_u f\left(\frac{\pi}{2}, \frac{1}{2}, \pi\right) = \frac{3}{7} \cdot \sin\frac{\pi}{2} - \frac{6}{7} \cdot \pi^2 \cos\frac{\pi}{2} + \frac{2}{7} \cdot \frac{\pi}{4} \cos\frac{\pi}{2}$$

$$D_u f\left(\frac{\pi}{2}, \frac{1}{2}, \pi\right) = \frac{3}{7}$$

LINEAR APPROXIMATION IN TWO VARIABLES

■ 1. Find the linear approximation of the function at $(1,1)$ and use it to approximate $f(0.99,0.99)$. Compare it to the exact value of $f(0.99,0.99)$.

$$f(t,s)=\sqrt{3t^2+s^2}$$

Solution:

The linearization of f at (a,b) is

$$L(t,s)=f(a,b)+f_t(a,b)(t-a)+f_s(a,b)(s-b)$$

Since $(a,b)=(1,1)$,

$$f(1,1)=\sqrt{3(1)^2+1^2}=\sqrt{4}=2$$

The partial derivatives at $(1,1)$ are

$$f_s(t,s)=\frac{2s}{2\sqrt{3t^2+s^2}}=\frac{s}{\sqrt{3t^2+s^2}}$$

$$f_s(1,1)=\frac{1}{\sqrt{3(1)^2+1^2}}=\frac{1}{2}$$

and

$$f_t(t,s)=\frac{6t}{2\sqrt{3t^2+s^2}}=\frac{3t}{\sqrt{3t^2+s^2}}$$

$$f_t(1,1) = \frac{3}{\sqrt{3(1)^2 + 1^2}} = \frac{3}{2}$$

The linear approximation of f at $(1,1)$ is

$$L(t,s) = 2 + \frac{3}{2}(t-1) + \frac{1}{2}(s-1)$$

$$L(t,s) = 1 + \frac{3}{2}t - \frac{3}{2} + \frac{1}{2}s - \frac{1}{2}$$

$$L(t,s) = \frac{3}{2}t + \frac{1}{2}s$$

The linear approximation at $(0.99,0.99)$ is

$$L(0.99,0.99) = \frac{3}{2} \cdot 0.99 + \frac{1}{2} \cdot 0.99 = 1.98$$

and $f(0.99,0.99)$ is

$$f(0.99,0.99) = \sqrt{3(0.99)^2 + (0.99)^2} = 1.98$$

So at $(0.99,0.99)$,

$$f(0.99,0.99) = L(0.99,0.99) = 1.98$$

■ 2. Calculate the percentage error of the linear approximation of the function at $f(0.9e,0.81)$. Use the initial point $(e,1)$.

$$f(x,y) = \ln\left(\frac{x^2}{y}\right)$$

Solution:

The linearization of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The function f can be rewritten as

$$f(x, y) = \ln(x^2) - \ln y$$

$$f(x, y) = 2\ln x - \ln y$$

Since $(a, b) = (e,1)$,

$$f(e,1) = 2\ln(e) - \ln(1) = 2(1) - 0 = 2$$

The partial derivatives of f are

$$f_x(x, y) = 2 \cdot \frac{1}{x} - 0 = \frac{2}{x}$$

$$f_x(e,1) = \frac{2}{e}$$

and

$$f_y(x, y) = 0 - \frac{1}{y} = -\frac{1}{y}$$

$$f_y(e,1) = -\frac{1}{1} = -1$$

The linear approximation of f at $(e,1)$ is

$$L(x,y) = 2 + \frac{2}{e}(x - e) - 1(y - 1)$$

$$L(x,y) = 2 + \frac{2}{e} \cdot x - 2 - y + 1$$

$$L(x,y) = \frac{2}{e}x - y + 1$$

The linear approximation at $(0.9e,0.81)$ is

$$f(0.9e,0.81) \approx L(0.9e,0.81) = \frac{2}{e} \cdot 0.9e - 0.81 + 1 = 1.99$$

and $f(0.9e,0.81)$ is

$$f(0.9e,0.81) = \ln\left(\frac{(0.9e)^2}{0.81}\right) = \ln\left(\frac{0.81e^2}{0.81}\right) = \ln(e^2) = 2$$

The percentage error is

$$\frac{|1.99 - 2|}{2} \cdot 100\,\% = 0.5\,\%$$

■ 3. Find the linear approximation of the function at $(0,0)$ and use it to approximate $f(0.2,0.01)$. Round to two decimal places.

$$f(u,v) = 3e^{2u-7v}$$

Solution:

The linearization of a function f at (a,b) is

$$L(u,v)=f(a,b)+f_u(a,b)(u-a)+f_v(a,b)(v-b)$$

Since $(a,b)=(0,0)$,

$$f(0,0)=3e^0=3$$

The partial derivatives of f are

$$f_u(u,v)=3\cdot 2e^{2u-7v}=6e^{2u-7v}$$

$$f_u(0,0)=6e^0=6$$

and

$$f_v(u,v)=3\cdot(-7)e^{2u-7v}=-21e^{2u-7v}$$

$$f_v(u,v)=-21e^0=-21$$

The linear approximation of at $(0,0)$ is

$$L(u,v)=3+6(u-0)-21(v-0)$$

$$L(u,v)=3+6u-21v$$

So

$$f(0.2,0.01)\approx L(0.2,0.01)$$

$$f(0.2,0.01)\approx 3+6\cdot 0.2-21\cdot 0.01$$

$$f(0.2,0.01)\approx 3.99$$

■ 4. Find the values of a and b and write down the linear approximation of the function at (a, b), given that $f_x(a, b) = -5$ and $f_y(a, b) = 11$.

$$f(x, y) = x^2 - 3y^2 - 7x - y$$

Solution:

The linearization of f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

Since $f_x(a, b) = -5$ and $f_y(a, b) = 11$,

$$L(x, y) = f(a, b) - 5(x - a) + 11(y - b)$$

Because $f_x(x, y) = 2x - 7$ and $f_x(a, b) = -5$,

$$2a - 7 = -5$$

$$a = 1$$

And because $f_y(x, y) = -6y - 1$ and $f_y(a, b) = 11$,

$$-6b - 1 = 11$$

$$b = -2$$

So

$$f(a, b) = 1^2 - 3(-2)^2 - 7 - (-2) = -16$$

The $a=1$, $b=-2$, and the linear approximation of f at $(1,-2)$ is

$$L(x,y)=-16-5(x-1)+11(y+2)$$

$$L(x,y)=-16-5x+5+11y+22$$

$$L(x,y)=11-5x+11y$$

■ 5. Find $f_x(1,2)$ and $f_y(1,2)$, given that $f(1,2)=5$, $L(1,2.1)=5.5$, and $L(1.1,1.95)=5.4$, where $L(x,y)$ is the linear approximation of $f(x,y)$ at $(1,2)$.

Solution:

The linearization of f at (a,b) is

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

Since $(a,b)=(1,2)$ and $f(1,2)=5$, the linear approximation of f at $(1,2)$ is

$$L(x,y)=5+f_x(1,2)(x-1)+f_y(1,2)(y-2)$$

Since $L(1,2.1)=5.5$, substitute $x=1$ and $y=2.1$ into this linear approximation equation.

$$5+f_x(1-1)+f_y(2.1-2)=5.5$$

$$5+0.1f_y=5.5$$

$$f_y=5$$

So $f_y(1,2)=5$. In the same way, since $L(1.1,1.95)=5.4$, substitute $x=1.1$ and $y=1.95$ into the linear approximation equation.

$$5+f_x(1.1-1)+f_y(1.95-2)=5.4$$

$$5+0.1f_x-0.05f_y=5.4$$

Substitute $f_y=5$ and solve for f_x.

$$0.1f_x-5\cdot 0.05=0.4$$

$$0.1f_x=0.65$$

$$f_x=6.5$$

So $f_x(1,2)=6.5$, and the partial derivatives at $(1,2)$ are

$$f_x(1,2)=6.5$$

$$f_y(1,2)=5$$

LINEARIZATION OF A MULTIVARIABLE FUNCTION

■ 1. Find the percentage error of the linear approximation of the function at $(3.2, -1.1, 0.8)$, if the initial point is $(3, -1, 1)$.

$$f(x, y, z) = 2xy^2z^3$$

Solution:

The linearization of f at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

Since $(a, b, c) = (3, -1, 1)$,

$$f(3, -1, 1) = 2(3)(-1)^2(1)^3$$

$$f(3, -1, 1) = 6$$

The partial derivatives of f at $(3, -1, 1)$ are

$$f_x(x, y, z) = 2y^2z^3$$

$$f_x(3, -1, 1) = 2(-1)^2(1)^3 = 2$$

and

$$f_y(x, y, z) = 2x(2y)z^3 = 4xyz^3$$

$$f_y(3, -1, 1) = 4(3)(-1)(1)^3 = -12$$

and

$$f_z(x, y, z) = 2xy^2(3z^2) = 6xy^2z^2$$

$$f_z(3, -1, 1) = 6(3)(-1)^2(1)^2 = 18$$

So the linear approximation of f at $(3, -1, 1)$ is

$$L(x, y, z) = 6 + 2(x - 3) - 12(y + 1) + 18(z - 1)$$

$$L(x, y, z) = 6 + 2x - 6 - 12y - 12 + 18z - 18$$

$$L(x, y, z) = 2x - 12y + 18z - 30$$

Then the approximation of $(3.2, -1.1, 0.8)$ is

$$L(3.2, -1.1, 0.8) = 2(3.2) - 12(-1.1) + 18(0.8) - 30$$

$$L(3.2, -1.1, 0.8) = 4$$

The exact value of f at $(3.2, -1.1, 0.8)$ is

$$f(3.2, -1.1, 0.8) = 2(3.2)(-1.1)^2(0.8)^3 = 3.964928$$

So the percentage error is

$$\frac{|4 - 3.964928|}{3.964928} \cdot 100\,\% \approx 0.9\,\%$$

■ 2. Find the linear approximation of the function at $(2, \pi/6, -\pi/6)$ and use it to approximate $R(2, 0.5, -0.5)$. Round to two decimal places.

$$R(r, \phi, \theta) = r^2 \sin(2\phi)\cos(\theta + \pi)$$

Solution:

The linearization of R at (a, b, c) is

$$L(r, \phi, \theta) = R(a, b, c) + R_r(a, b, c)(r - a) + R_\phi(a, b, c)(\phi - b) + R_\theta(a, b, c)(\theta - c)$$

Since $(a, b, c) = (2, \pi/6, -\pi/6)$,

$$R\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = (2)^2 \sin\left(2 \cdot \frac{\pi}{6}\right) \cos\left(-\frac{\pi}{6} + \pi\right)$$

$$R\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = -3$$

The partial derivatives of R at $(2, \pi/6, -\pi/6)$ are

$$R_r(r, \phi, \theta) = 2r\sin(2\phi)\cos(\theta + \pi)$$

$$R_r\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = 2(2)\sin\left(2 \cdot \frac{\pi}{6}\right) \cos\left(-\frac{\pi}{6} + \pi\right) = -3$$

and

$$R_\phi(r, \phi, \theta) = 2r^2\cos(2\phi)\cos(\theta + \pi)$$

$$R_\phi\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = 2(2)^2\cos\left(2 \cdot \frac{\pi}{6}\right) \cos\left(-\frac{\pi}{6} + \pi\right) = -2\sqrt{3}$$

and

$$R_\theta(r, \phi, \theta) = -r^2\sin(2\phi)\sin(\theta + \pi)$$

$$R_\theta\left(2,\frac{\pi}{6},-\frac{\pi}{6}\right)=-(2)^2\sin\left(2\cdot\frac{\pi}{6}\right)\sin\left(-\frac{\pi}{6}+\pi\right)=-\sqrt{3}$$

So the linear approximation of R at $(2,\pi/6,-\pi/6)$ is

$$L(r,\phi,\theta)=-3-3(r-2)-2\sqrt{3}\left(\phi-\frac{\pi}{6}\right)-\sqrt{3}\left(\theta+\frac{\pi}{6}\right)$$

$$L(r,\phi,\theta)=-3r-2\sqrt{3}\phi-\sqrt{3}\theta+3+\frac{\sqrt{3}\pi}{6}$$

Then the approximation of $(2,0.5,-0.5)$ is

$$L(2,0.5,-0.5)=-3(2)-2\sqrt{3}(0.5)-\sqrt{3}(-0.5)+3+\frac{\sqrt{3}\pi}{6}$$

$$L(2,0.5,-0.5)=-3-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}\pi}{6}$$

■ 3. Find the values of the first order partial derivatives of $f(x,y,z)$ at $(3,4,-8)$, where $L(x,y,z)$ is the linear approximation of the function $f(x,y,z)$ at $(3,4,-8)$, and $f(3,4,-8)=3$.

$$L(3.1,4.2,-8.1)=3$$

$$L(3.2,3.9,-7.8)=3.4$$

$$L(2.9,4.3,-8.1)=2.8$$

Solution:

The linearization of f at (a,b,c) is

$$L(x,y,z)=f(a,b,c)+f_x(a,b,c)(x-a)+f_y(a,b,c)(y-b)+f_z(a,b,c)(z-c)$$

Since $(a,b,c)=(3,4,-8)$ and $f(3,4,-8)=3$, the linear approximation of f at $(3,4,-8)$ is

$$L(x,y,z)=3+f_x(x-3)+f_y(y-4)+f_z(z+8)$$

Since $L(3.1,4.2,-8.1)=3$, substitute 3.1 for x, 4.2 for y, and -8.1 for z into the equation.

$$3+f_x(3.1-3)+f_y(4.2-4)+f_z(-8.1+8)=3$$

$$3+0.1f_x+0.2f_y-0.1f_z=3$$

In the same way, using $L(3.2,3.9,-7.8)=3.4$, we get

$$3+f_x(3.2-3)+f_y(3.9-4)+f_z(-7.8+8)=3.4$$

$$3+0.2f_x-0.1f_y+0.2f_z=3.4$$

Using $L(2.9,4.3,-8.1)=2.8$, we get

$$3+f_x(2.9-3)+f_y(4.3-4)+f_z(-8.1+8)=2.8$$

$$3-0.1f_x+0.3f_y-0.1f_z=2.8$$

So we have three linear equations in terms of f_x, f_y, and f_z.

$$3+0.1f_x+0.2f_y-0.1f_z=3$$

$$3 + 0.2f_x - 0.1f_y + 0.2f_z = 3.4$$

$$3 - 0.1f_x + 0.3f_y - 0.1f_z = 2.8$$

Simplify, and multiply each equation by 10.

$$f_x + 2f_y - f_z = 0$$

$$2f_x - f_y + 2f_z = 4$$

$$-f_x + 3f_y - f_z = -2$$

Solve the system of equations for f_x, f_y, and f_z.

$$f_x = 1$$

$$f_y = 0$$

$$f_z = 1$$

So the partial derivatives are

$$f_x(3,4,-8) = 1$$

$$f_y(3,4,-8) = 0$$

$$f_z(3,4,-8) = 1$$

GRADIENT VECTORS

■ 1. Find the gradient vector ∇f at $(\sqrt{\pi},0,0)$.

$$f(x,y,z)=\sin(x^2+2y^2-z^2-2xyz)$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y,z)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

$$\nabla f(x,y,z)=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$$

Use the chain rule to calculate partial derivatives.

$$\frac{\partial f}{\partial x}=\cos(x^2+2y^2-z^2-2xyz)\frac{\partial}{\partial x}(x^2+2y^2-z^2-2xyz)$$

$$\frac{\partial f}{\partial x}=(2x-2yz)\cos(x^2+2y^2-z^2-2xyz)$$

$$\frac{\partial f}{\partial x}=2(x-yz)\cos(x^2+2y^2-z^2-2xyz)$$

and

$$\frac{\partial f}{\partial y}=\cos(x^2+2y^2-z^2-2xyz)\frac{\partial}{\partial y}(x^2+2y^2-z^2-2xyz)$$

$$\frac{\partial f}{\partial y} = (4y - 2xz)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

$$\frac{\partial f}{\partial y} = 2(2y - xz)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

and

$$\frac{\partial f}{\partial z} = \cos(x^2 + 2y^2 - z^2 - 2xyz)\frac{\partial}{\partial z}(x^2 + 2y^2 - z^2 - 2xyz)$$

$$\frac{\partial f}{\partial z} = (-2z - 2xy)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

$$\frac{\partial f}{\partial z} = -2(z + xy)\cos(x^2 + 2y^2 - z^2 - 2xyz)$$

Evaluate the partial derivatives at $(\sqrt{\pi},0,0)$.

$$\frac{\partial f}{\partial x}(\sqrt{\pi},0,0) = 2(\sqrt{\pi} - (0)(0))\cos(\pi + 2(0)^2 - (0)^2 - 2\sqrt{\pi}(0)(0))$$

$$\frac{\partial f}{\partial x}(\sqrt{\pi},0,0) = 2\sqrt{\pi}\cos(\pi)$$

$$\frac{\partial f}{\partial x}(\sqrt{\pi},0,0) = -2\sqrt{\pi}$$

and

$$\frac{\partial f}{\partial y}(\sqrt{\pi},0,0) = 2(2(0) - \sqrt{\pi}(0))\cos(\pi + 2(0)^2 - (0)^2 - 2\sqrt{\pi}(0)(0))$$

$$\frac{\partial f}{\partial y}(\sqrt{\pi},0,0) = 0\cos(\pi)$$

$$\frac{\partial f}{\partial y}(\sqrt{\pi},0,0) = 0$$

and

$$\frac{\partial f}{\partial z}(\sqrt{\pi},0,0) = -2((0) + \sqrt{\pi}(0))\cos(\pi + 2(0)^2 - (0)^2 - 2\sqrt{\pi}(0)(0))$$

$$\frac{\partial f}{\partial z}(\sqrt{\pi},0,0) = 0\cos(\pi)$$

$$\frac{\partial f}{\partial z}(\sqrt{\pi},0,0) = 0$$

■ 2. Find unit gradient vector of the function f at $(-2,1)$.

$$f(t,s) = \frac{4t - st^4}{t^2 s}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

The unit gradient vector (gradient vector of magnitude 1) is given by

$$\widehat{\nabla f} = \frac{\nabla f}{\|\nabla f\|}$$

Simplify $f(t,s)$.

$$f(t,s) = \frac{4t}{t^2 s} - \frac{st^4}{t^2 s} = \frac{4}{ts} - t^2$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial t} = -\frac{4}{t^2 s} - 2$$

$$\frac{\partial f}{\partial s} = -\frac{4}{ts^2}$$

Evaluate the partial derivatives at $(-2,1)$.

$$\frac{\partial f}{\partial t}(-2,1) = -\frac{4}{(-2)^2(1)} - 2(-2) = -1 + 4 = 3$$

$$\frac{\partial f}{\partial s}(-2,1) = -\frac{4}{(-2)(1)^2} = 2$$

Calculate the magnitude of ∇f.

$$\| \nabla f \| = \sqrt{3^2 + 2^2} = \sqrt{13}$$

So

$$\widehat{\nabla f} = \left\langle \frac{3}{\sqrt{13}}, \frac{2}{\sqrt{13}} \right\rangle$$

■ 3. Find the point where the gradient vector of the function f is equal to the zero vector.

$$f(x, y) = \ln \frac{(x-2)^2 y}{x-y}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Simplify $f(x, y)$ using laws of logarithms.

$$f(x, y) = \ln((x-2)^2) + \ln y - \ln(x-y)$$

$$f(x, y) = 2\ln(x-2) + \ln y - \ln(x-y)$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2}{x-2} - \frac{1}{x-y}$$

$$\frac{\partial f}{\partial y} = \frac{1}{y} + \frac{1}{x-y}$$

Since the gradient is equal to 0, we get a system of equations in terms of x and y.

$$\frac{2}{x-2} - \frac{1}{x-y} = 0$$

$$\frac{1}{y} + \frac{1}{x-y} = 0$$

Solve the system for x and y.

$$2(x-y)-(x-2)=0$$

$$x-y+y=0$$

$$x-2y+2=0$$

$$x=0$$

So the solution is $x=0$ and $y=1$, or $(0,1)$.

■ 4. Find and identify the set of points where the magnitude of the gradient vector of the function f is equal to 1.

$$f(x,y)=x^2+4y^2-2x+8y-5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x}=2x-2$$

$$\frac{\partial f}{\partial y}=8y+8$$

Calculate the magnitude of ∇f and set it equal to 1.

$$\| \nabla f \| = \sqrt{(2x-2)^2 + (8y+8)^2} = 1$$

Since $(2x-2)^2 + (8y+8)^2 > 0$, square each side of the equation.

$$(2x-2)^2 + (8y+8)^2 = 1$$

This is an ellipse. To find its center and vertices, rewrite the equation in standard form.

$$4(x-1)^2 + 64(y+1)^2 = 1$$

$$\frac{(x-1)^2}{0.5^2} + \frac{(y+1)^2}{0.125^2} = 1$$

So the set of points where $\| \nabla f \| = 1$ is the ellipse

$$\frac{(x-1)^2}{0.5^2} + \frac{(y+1)^2}{0.125^2} = 1$$

with center $(1,-1)$, with semi-major axis 0.5 and semi-minor axis 0.125.

■ 5. Find the directional derivative of the function f in the direction $m = 3\mathbf{i} - 4\mathbf{j}$ at $(0,4)$.

$$f(x,y) = 2^x(y^2-1)$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x}=2^x(y^2-1)\ln 2$$

$$\frac{\partial f}{\partial y}=2y2^x=y2^{x+1}$$

Evaluate the partial derivatives at $(0,4)$.

$$\frac{\partial f}{\partial x}(0,4)=2^0(4^2-1)\ln 2=15\ln 2$$

$$\frac{\partial f}{\partial y}(0,4)=2(4)2^0=8$$

So

$$\nabla f(x,y)=\langle 15\ln 2,8\rangle$$

The directional derivative in the direction of vector m is given by

$$\nabla f\cdot\widehat{m}$$

where $\widehat{m}$ is the unit vector in the direction of m, which is

$$\widehat{m}=\frac{m}{\|\nabla m\|}$$

Find the magnitude of the vector m.

$$\| \nabla m \| = \sqrt{3^2 + 4^2} = 5$$

So

$$\widehat{m} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

Then the directional derivative is

$$\nabla f \cdot \widehat{m} = \langle 15 \ln 2, 8 \rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$$

$$\nabla f \cdot \widehat{m} = \frac{3(15 \ln 2)}{5} - \frac{4(8)}{5}$$

$$\nabla f \cdot \widehat{m} = 9 \ln 2 - \frac{32}{5}$$

GRADIENT VECTORS AND THE TANGENT PLANE

1. Use the gradient vector to find the tangent line to the curve $(y-2)^2 - e^x = 0$ at $(0,3)$.

Solution:

Given the multivariable function,

$$f(x, y) = (y-2)^2 - e^x$$

the tangent line to the curve has the equation

$$a(x - x_0) + b(y - y_0) = 0$$

where (x_0, y_0) is the point on the curve, and

$$\nabla f(x, y) = \langle a, b \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = -e^x$$

$$\frac{\partial f}{\partial y} = 2(y-2)$$

Evaluate the partial derivatives at $(0,3)$.

$$\frac{\partial f}{\partial x}(0,3) = -e^x = -1$$

$$\frac{\partial f}{\partial y}(0,3) = 2(3-2) = 2$$

So the tangent line to the curve is

$$-1(x-0) + 2(y-3) = 0$$

$$x - 2y = -6$$

■ 2. Use the gradient vector to find the tangent plane to the surface $(r+1)\sin(\phi+\pi)\tan(\theta) = 0$ at $(2,\pi/6,\pi/4)$.

Solution:

Given the multivariable function,

$$f(r,\phi,\theta) = (r+1)\sin(\phi+\pi)\tan(\theta)$$

the tangent line to the curve has the equation

$$a(r-r_0) + b(\phi-\phi_0) + c(\theta-\theta_0) = 0$$

where (r_0,ϕ_0,θ_0) is the point on the surface, and

$$\nabla f(r,\phi,\theta) = \langle a,b,c\rangle = \left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial \phi}, \frac{\partial f}{\partial \theta} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial r} = \sin(\phi + \pi)\tan(\theta)$$

$$\frac{\partial f}{\partial \phi} = (r + 1)\cos(\phi + \pi)\tan(\theta)$$

$$\frac{\partial f}{\partial \theta} = (r + 1)\sin(\phi + \pi)\sec^2(\theta)$$

Evaluate the partial derivatives at $(2,\pi/6,\pi/4)$.

$$a = \sin\left(\frac{\pi}{6} + \pi\right)\tan\left(\frac{\pi}{4}\right) = -\frac{1}{2}$$

$$b = (2 + 1)\cos\left(\frac{\pi}{6} + \pi\right)\tan\left(\frac{\pi}{4}\right) = -\frac{3\sqrt{3}}{2}$$

$$c = (2 + 1)\sin\left(\frac{\pi}{6} + \pi\right)\sec^2\left(\frac{\pi}{4}\right) = -3$$

So the tangent plane to the curve is

$$-\frac{1}{2}(r - 2) - \frac{3\sqrt{3}}{2}\left(\phi - \frac{\pi}{6}\right) - 3\left(\theta - \frac{\pi}{4}\right) = 0$$

Multiply by -4 and simplify.

$$2r + 6\sqrt{3}\phi + 12\theta - 4 - 3\pi - \sqrt{3}\pi = 0$$

3. Use the gradient vector to find the tangent plane(s) to the surface $x^2 - 2xy - 3y^2 + z^2 + 4x + 4y - 2z - 3 = 0$ that are parallel to the xy-plane.

Solution:

Given the multivariable function,

$$f(x, y, z) = x^2 - 2xy - 3y^2 + z^2 + 4x + 4y - 2z - 3$$

the tangent plane to the surface has the equation

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where (x_0, y_0, z_0) is the point on the surface, and

$$\nabla f(x, y, z) = \langle a, b, c \rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 2x - 2y + 4$$

$$\frac{\partial f}{\partial y} = -2x - 6y + 4$$

$$\frac{\partial f}{\partial z} = 2z - 2$$

Since the plane is parallel to the xy-plane, its normal vector has 0 for x and y coordinates, which means $a = b = 0$. So we have a system of equations.

$$2x - 2y + 4 = 0$$

$$-2x - 6y + 4 = 0$$

The solution is

$$(-1)^2 - 2(-1)(1) - 3(1)^2 + z^2 + 4(-1) + 4(1) - 2z - 3 = 0$$

Substitute these values into the equation of the surface to find z.

$$(-1)^2 - 2(-1)(1) - 3(1)^2 + z^2 + 4(-1) + 4(1) - 2z - 3 = 0$$

$$z^2 - 2z - 3 = 0$$

The solutions are $z = -1$ or $z = 3$.

So we have two points where the tangent plane to the given surface is parallel to the xy-plane, $(-1,1,-1)$ and $(-1,1,3)$.

The tangent plane at $(-1,1,-1)$ is

$$\frac{\partial f}{\partial z}(-1,1,-1) = 2(-1) - 2 = -4$$

$$-4(z+1) = 0$$

$$z + 1 = 0$$

The tangent plane at $(-1,1,3)$ is

$$\frac{\partial f}{\partial z}(-1,1,3) = 2(3) - 2 = 4$$

$$4(z-3) = 0$$

$$z - 3 = 0$$

So the tangent planes to the surface are

$$z + 1 = 0$$

$$z - 3 = 0$$

MAXIMUM RATE OF CHANGE AND ITS DIRECTION

■ 1. Find the point where the maximum rate of change of the function f is equal to 0.

$$f(x,y)=6x^2-4xy+y^2-12x-6y+4$$

Solution:

The maximum rate of change of the function f is equal to 0 if and only if the gradient vector is equal to the zero vector.

The gradient vector of a multivariable function is given by

$$\nabla f(x,y)=\left\langle \frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x}=12x-4y-12$$

$$\frac{\partial f}{\partial y}=-4x+2y-6$$

Since the gradient is equal to 0, we have the system of equations in terms of x and y.

$$12x-4y-12=0$$

$$-4x + 2y - 6 = 0$$

The solution is $x = 6$ and $y = 15$, or $(6,15)$.

■ 2. Find the maximum rate of change and its direction for the function f at $(3, -\pi/2, 0)$.

$$f(x, y, z) = x^2(2z - 1)\sin^2 y$$

Solution:

The direction of the maximum rate of change is given by the gradient vector. The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 2x(2z - 1)\sin^2(y)$$

$$\frac{\partial f}{\partial y} = 2x(2z - 1)(2\sin(y)\cos(y)) = 2x(2z - 1)\sin(2y)$$

$$\frac{\partial f}{\partial z} = 2x^2\sin^2(y)$$

Evaluate the partial derivatives at $(3, -\pi/2, 0)$.

$$\frac{\partial f}{\partial x}\left(3,-\frac{\pi}{2},0\right)=2(3)(2(0)-1)\sin^2\left(-\frac{\pi}{2}\right)=-6$$

$$\frac{\partial f}{\partial y}\left(3,-\frac{\pi}{2},0\right)=2(3)(2(0)-1)\sin\left(-2\frac{\pi}{2}\right)=0$$

$$\frac{\partial f}{\partial z}\left(3,-\frac{\pi}{2},0\right)=2(3)^2\sin^2\left(-\frac{\pi}{2}\right)=18$$

So

$$\nabla f\left(3,-\frac{\pi}{2},0\right)=\langle -6,0,18\rangle$$

Calculate the magnitude of ∇f.

$$\|\nabla f\|=\sqrt{(-6)^2+0^2+18^2}=6\sqrt{10}$$

So the direction is $\langle -6,0,18\rangle$ and the maximum rate of change is $6\sqrt{10}$.

■ 3. Find the minimum rate of change and its direction for the function f at $(2,1,4)$.

$$f(u,v,w)=\sqrt{2u-4v+6w+1}$$

Solution:

The direction of the minimum rate of change is given by the vector opposite to the gradient vector.

The gradient vector of a multivariable function is given by

$$\nabla f(u,v,w)=\left\langle\frac{\partial f}{\partial u},\frac{\partial f}{\partial v},\frac{\partial f}{\partial w}\right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial u}=\frac{1}{\sqrt{2u-4v+6w+1}}$$

$$\frac{\partial f}{\partial v}=-\frac{2}{\sqrt{2u-4v+6w+1}}$$

$$\frac{\partial f}{\partial w}=\frac{3}{\sqrt{2u-4v+6w+1}}$$

Evaluate the partial derivatives at $(2,1,4)$.

$$\frac{\partial f}{\partial u}=\frac{1}{\sqrt{2(2)-4(1)+6(4)+1}}=\frac{1}{\sqrt{25}}=\frac{1}{5}$$

$$\frac{\partial f}{\partial v}=-\frac{2}{\sqrt{2(2)-4(1)+6(4)+1}}=-\frac{2}{5}$$

$$\frac{\partial f}{\partial w}=\frac{3}{\sqrt{2(2)-4(1)+6(4)+1}}=\frac{3}{5}$$

So

$$\nabla f=\left\langle\frac{1}{5},-\frac{2}{5},\frac{3}{5}\right\rangle$$

So the direction of the minimum rate of change is

$$\left\langle -\frac{1}{5}, \frac{2}{5}, -\frac{3}{5} \right\rangle$$

Calculate the magnitude.

$$\frac{\sqrt{(-1)^2 + (2)^2 + (-3)^2}}{5} = \frac{\sqrt{14}}{5}$$

EQUATION OF THE TANGENT PLANE

■ 1. Find a tangent plane to the surface $f(u,v,w)=0$ at $(3,-1,5)$.

$$f(u,v,w)=\ln\frac{u^2+1}{v^2w^5}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(u,v,w)=\left\langle\frac{\partial f}{\partial u},\frac{\partial f}{\partial v},\frac{\partial f}{\partial w}\right\rangle$$

Expand f using properties of logarithms.

$$f(u,v,w)=\ln(u^2+1)-2\ln v-5\ln w$$

Find partial derivatives of f.

$$\frac{\partial f}{\partial u}=\frac{2u}{u^2+1}$$

$$\frac{\partial f}{\partial v}=-\frac{2}{v}$$

$$\frac{\partial f}{\partial w}=-\frac{5}{w}$$

Evaluate the partial derivatives at $(3,-1,5)$.

$$\frac{\partial f}{\partial u}(3,-1,5)=\frac{2(3)}{(3)^2+1}=0.6$$

$$\frac{\partial f}{\partial v}(3,-1,5)=-\frac{2}{-1}=2$$

$$\frac{\partial f}{\partial w}(3,-1,5)=-\frac{5}{5}=-1$$

The equation of the tangent plane to the surface $f(u,v,w)=0$ at (u_0,v_0,w_0) is

$$\frac{\partial f}{\partial u}(u_0,v_0,w_0)(u-u_0)+\frac{\partial f}{\partial v}(u_0,v_0,w_0)(v-v_0)+\frac{\partial f}{\partial w}(u_0,v_0,w_0)(w-w_0)=0$$

Substitute the values and simplify

$$0.6(u-3)+2(v-(-1))-1(w-5)=0$$

$$0.6u+2v-w+5.2=0$$

■ 2. Find any tangent planes to the surface $f(x,y,z)=0$ that are parallel to the plane $5x-4y+2z+5=0$.

$$f(x,y,z)=x^3-4y^2+z^2+2x+12y+5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x,y,z)=\left\langle\frac{\partial f}{\partial x},\frac{\partial f}{\partial y},\frac{\partial f}{\partial z}\right\rangle$$

Find partial derivatives of f.

$$\frac{\partial f}{\partial x} = 3x^2 + 2$$

$$\frac{\partial f}{\partial y} = -8y + 12$$

$$\frac{\partial f}{\partial z} = 2z$$

Since the tangent plane should be parallel to the given plane $5x - 4y + 2z + 5 = 0$, it should have the same normal vector, $\langle 5, -4, 2 \rangle$. Since the normal vector to the surface is equal to the gradient at the point, we have a system of equations in terms of x, y, and z.

$$3x^2 + 2 = 5$$

$$-8y + 12 = -4$$

$$2z = 2$$

The solutions to the system are $x = -1$ or $x = 1$, $y = 2$, and $z = 1$. So we have two points where the tangent plane is parallel to the given plane, $(1,2,1)$ and $(-1,2,1)$.

The equation of the tangent plane to the surface $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Substitute the values and simplify for $(1,2,1)$.

$$5(x - 1) - 4(y - 2) + 2(z - 1) = 0$$

$$5x - 4y + 2z + 1 = 0$$

Substitute the values and simplify for $(-1,2,1)$.

$$5(x + 1) - 4(y - 2) + 2(z - 1) = 0$$

$$5x - 4y + 2z + 11 = 0$$

There are two tangent planes to f that area parallel to the plane:

$$5x - 4y + 2z + 1 = 0$$

$$5x - 4y + 2z + 11 = 0$$

■ 3. Find a line of intersection of the xy-plane and tangent plane to the surface $f(x, y, z) = 0$ at $(\pi, -1, \sqrt{6})$.

$$f(x, y, z) = 2\cos(x + \pi)(y^2 + y + 5) - 3z^3$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Find partial derivatives of f.

$$\frac{\partial f}{\partial x} = -2\sin(x + \pi)(y^2 + y + 5)$$

$$\frac{\partial f}{\partial y} = 2\cos(x+\pi)(2y+1)$$

$$\frac{\partial f}{\partial z} = -9z^2$$

Evaluate the partial derivatives at $(\pi, -1, \sqrt{6})$.

$$\frac{\partial f}{\partial x}(\pi, -1, \sqrt{6}) = -2\sin(\pi+\pi)((-1)^2 + (-1) + 5) = 0$$

$$\frac{\partial f}{\partial y}(\pi, -1, \sqrt{6}) = 2\cos(\pi+\pi)(2(-1)+1) = -2$$

$$\frac{\partial f}{\partial z}(\pi, -1, \sqrt{6}) = -9(\sqrt{6})^2 = -54$$

The equation of the tangent plane to the surface $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Substitute the values and simplify.

$$0(x-\pi) - 2(y - (-1)) - 54(z - \sqrt{6}) = 0$$

$$-2y - 54z + 54\sqrt{6} - 2 = 0$$

$$y + 27z - 27\sqrt{6} + 1 = 0$$

Since a line of intersection lies in the xy-plane, then $z = 0$.

$$y - 27\sqrt{6} + 1 = 0$$

$$y = 27\sqrt{6} - 1$$

The line of intersection is $y = 27\sqrt{6} - 1$ with $z = 0$, so the line is parallel to x-axis.

■ 4. Find and identify the set of the points where the tangent plane to the surface $f(x, y, z) = 0$ is parallel to z-axis.

$$f(x, y, z) = x^2 + 4y^2 + z^2 + 2x - 8y + 8z + 17 = 0$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

The equation of the tangent plane to the surface $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$\frac{\partial f}{\partial x}(x_0, y_0, z_0)(x - x_0) + \frac{\partial f}{\partial y}(x_0, y_0, z_0)(y - y_0) + \frac{\partial f}{\partial z}(x_0, y_0, z_0)(z - z_0) = 0$$

Since the tangent plane should be parallel to z-axis, $\partial f / \partial z$ must be equal to 0 at the point. Find the partial derivative.

$$\frac{\partial f}{\partial z} = 2z + 8$$

$$2z + 8 = 0$$

$$z = -4$$

Substitute $z = -4$ into the equation of the surface to get the set equation.

$$x^2 + 4y^2 + (-4)^2 + 2x - 8y + 8(-4) + 17 = 0$$

$$x^2 + 2x + 4y^2 - 8y + 1 = 0$$

Complete the squares and transform into standard form.

$$(x + 1)^2 + 4(y - 1)^2 - 4 = 0$$

$$\frac{(x + 1)^2}{2^2} + \frac{(y - 1)^2}{1^2} = 1$$

So the set of points is the ellipse with center at $(-1,1)$, semi-major axis 2, and semi-minor axis 1.

NORMAL LINE TO THE SURFACE

■ 1. Use the gradient vector to find a symmetric equation of the normal line to the curve $f(s,t)=0$ at $(-3,3)$, where

$$f(s,t)=t2^{2t+s-3}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(s,t)=\left\langle\frac{\partial f}{\partial s},\frac{\partial f}{\partial t}\right\rangle$$

Use the product rule to calculate partial derivatives.

$$\frac{\partial f}{\partial s}=t\ln(2)2^{2t+s-3}$$

and

$$\frac{\partial f}{\partial t}=\left(\frac{d}{dt}t\right)2^{2t+s-3}+t\left(\frac{d}{dt}2^{2t+s-3}\right)$$

$$\frac{\partial f}{\partial t}=2^{2t+s-3}+t\ln(2)2^{2t+s-2}$$

Evaluate the partial derivatives at $(-3,3)$.

$$\frac{\partial f}{\partial s}(-3,3)=(3)\ln(2)2^{2(3)+(-3)-3}=3\ln 2$$

$$\frac{\partial f}{\partial t}(-3,3) = 2^{2(3)+(-3)-3} + (3)\ln(2)2^{2(3)+(-3)-2} = 1 + 6\ln 2$$

The formula for the symmetric equation of the normal line to the curve $f(s, t) = 0$ at (s_0, t_0) is

$$\frac{t - t_0}{\frac{\partial f}{\partial t}(s_0, t_0)} = \frac{s - s_0}{\frac{\partial f}{\partial s}(s_0, t_0)}$$

Substitute the values and simplify.

$$\frac{t - 3}{1 + 6\ln 2} = \frac{s + 3}{3\ln 2}$$

■ 2. Use the gradient vector to find a vector equation of the normal line to the surface $f(x, y, z) = 0$ at $(0, -5, 1)$.

$$f(x, y, z) = \frac{2x - y^2}{z}$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2}{z}$$

$$\frac{\partial f}{\partial y} = -\frac{2y}{z}$$

$$\frac{\partial f}{\partial z} = -\frac{2x - y^2}{z^2}$$

Evaluate the partial derivatives at $(0, -5, 1)$.

$$\frac{\partial f}{\partial x}(0, -5, 1) = \frac{2}{1} = 2$$

$$\frac{\partial f}{\partial y}(0, -5, 1) = -\frac{2(-5)}{1} = 10$$

$$\frac{\partial f}{\partial z}(0, -5, 1) = -\frac{2(0) - (-5)^2}{(1)^2} = 25$$

The formula for the vector equation of the normal line to the curve $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$r = \left\langle x_0, y_0, z_0 \right\rangle + t \left\langle \frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0) \right\rangle$$

$$r = \langle 0, -5, 1 \rangle + t \langle 2, 10, 25 \rangle$$

■ 3. Use the gradient vector to find a parametric equation of the normal line to the surface $f(x, y, z) = 0$ at $(2, -3, 0)$.

$$f(x, y, z) = 3x^3 - 2xyz - 2y^2 + 5yz + 1$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 9x^2 - 2yz$$

$$\frac{\partial f}{\partial y} = -2xz - 4y + 5z$$

$$\frac{\partial f}{\partial z} = -2xy + 5y$$

Evaluate the partial derivatives at $(2, -3,0)$.

$$\frac{\partial f}{\partial x}(2, -3,0) = 9(2)^2 - 2(-3)(0) = 36$$

$$\frac{\partial f}{\partial y}(2, -3,0) = -2(2)(0) - 4(-3) + 5(0) = 12$$

$$\frac{\partial f}{\partial z}(2, -3,0) = -2(2)(-3) + 5(-3) = -3$$

The formula for the parametric equation of the normal line to the curve $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$x = x_0 + t\frac{\partial f}{\partial x}(x_0, y_0, z_0)$$

$$y = y_0 + t\frac{\partial f}{\partial y}(x_0, y_0, z_0)$$

$$z = z_0 + t\frac{\partial f}{\partial z}(x_0, y_0, z_0)$$

Then the parametric equation of the normal line is

$$x = 2 + 36t$$

$$y = -3 + 12t$$

$$z = -3t$$

■ 4. Use the gradient vector to find a parametric equation of the normal line to the surface $f(x, y, z) = 0$ that's parallel to the line $r = \langle 2,17,-6\rangle + t\langle 9,1,-6\rangle$.

$$f(x, y, z) = 2x^2 + y^2 + 3z^2 - 3x - 5y + 5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 4x - 3$$

$$\frac{\partial f}{\partial y} = 2y - 5$$

$$\frac{\partial f}{\partial z} = 6z$$

Since the tangent line is parallel to the given line, it has the same direction vector $\langle 9,1,-6 \rangle$, and it's equal to the gradient vector at the point of tangency (x_0, y_0, z_0). So we have a system of equations in terms of x_0, y_0, and z_0.

$$4x_0 - 3 = 9$$

$$2y_0 - 5 = 1$$

$$6z_0 = -6$$

The solution of the system is $x_0 = 3$, $y_0 = 3$, and $z_0 = -1$.

The formula for the parametric equation of the normal line to the curve $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$x = x_0 + t\frac{\partial f}{\partial x}(x_0, y_0, z_0)$$

$$y = y_0 + t\frac{\partial f}{\partial y}(x_0, y_0, z_0)$$

$$z = z_0 + t\frac{\partial f}{\partial z}(x_0, y_0, z_0)$$

Substitute the values we found to get the parametric equation of the normal line to the surface.

$$x = 3 + 9t$$

$$y = 3 + t$$

$$z = -1 - 6t$$

■ 5. Use the gradient vector to find a vector equation of the normal line to the surface $f(x, y, z) = 0$ that's perpendicular to the plane $x + 4y - 8z + 12 = 0$.

$$f(x, y, z) = ye^{2x+6} + z^2 - 5$$

Solution:

The gradient vector of a multivariable function is given by

$$\nabla f(x, y, z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$$

Calculate partial derivatives.

$$\frac{\partial f}{\partial x} = 2ye^{2x+6}$$

$$\frac{\partial f}{\partial y} = e^{2x+6}$$

$$\frac{\partial f}{\partial z} = 2z$$

Since the tangent line is perpendicular to the given plane, the normal vector to the plane is equal to the gradient vector at the point of tangency (x_0, y_0, z_0). So we have a system of equations in terms of x_0, y_0, and z_0.

$$2y_0e^{2x_0+6} = 1$$

$$e^{2x_0+6} = 4$$

$$2z_0 = -8$$

The solution of the system is $x_0 = \ln 2 - 3$, $y_0 = 1/8$, and $z_0 = -4$.

The formula for the vector equation of the normal line to the curve $f(x, y, z) = 0$ at (x_0, y_0, z_0) is

$$r = \left\langle x_0, y_0, z_0 \right\rangle + t\left\langle \frac{\partial f}{\partial x}(x_0, y_0, z_0), \frac{\partial f}{\partial y}(x_0, y_0, z_0), \frac{\partial f}{\partial z}(x_0, y_0, z_0) \right\rangle$$

Substitute the values we found to get the vector equation of the normal line to the surface.

$$r = \left\langle \ln 2 - 3, \frac{1}{8}, -4 \right\rangle + t\langle 1,4, -8 \rangle$$

CRITICAL POINTS

1. Find a set of critical points for $f(t, s)$.

$$f(t, s) = \ln \frac{2t^3 - 12t^2 + 1}{s^2 + 6s + 11}$$

Solution:

Rewrite f using laws of logs.

$$f(t, s) = \ln(2t^3 - 12t^2 + 1) - \ln(s^2 + 6s + 11)$$

Use chain rule to find partial derivatives.

$$\frac{\partial f}{\partial t} = \frac{6t^2 - 24t}{2t^3 - 12t^2 + 1}$$

$$\frac{\partial f}{\partial s} = -\frac{2s + 6}{s^2 + 6s + 11}$$

Set both partial derivatives equal to 0 and use those as a system of equations to find critical points.

$$\frac{6t^2 - 24t}{2t^3 - 12t^2 + 1} = 0$$

$$-\frac{2s + 6}{s^2 + 6s + 11} = 0$$

That gives the system

$$6t^2 - 24t = 0$$

$$2s + 6 = 0$$

and then

$$t(t-4) = 0$$

$$s + 3 = 0$$

The solutions to the system are $(0, -3)$ and $(4, -3)$.

Check if each of these points lie within the domain of f.

$$\frac{2(0)^3 - 12(0)^2 + 1}{(-3)^2 + 6(-3) + 11} = \frac{1}{2} > 0$$

$$\frac{2(4)^3 - 12(4)^2 + 1}{(-3)^2 + 6(-3) + 11} = \frac{-63}{2} < 0$$

So only the point $(0, -3)$ lies within the domain and is a critical point of the function $f(t, s)$.

■ 2. Find and identify a set of critical points for $f(x, y, z)$.

$$f(x, y, z) = x^2 \cos(y + z)$$

Solution:

Use chain rule to find partial derivatives.

$$\frac{\partial f}{\partial x} = 2x\cos(y+z)$$

$$\frac{\partial f}{\partial y} = -x^2\sin(y+z)$$

$$\frac{\partial f}{\partial z} = -x^2\sin(y+z)$$

Set the partial derivatives equal to 0 and use those as a system of equations to find critical points.

$$2x\cos(y+z) = 0$$

$$-x^2\sin(y+z) = 0$$

$$-x^2\sin(y+z) = 0$$

Since the system of equations

$$\sin\theta = 0$$

$$\cos\theta = 0$$

has no real solutions, $\cos(y+z)$ and $\sin(y+z)$ can't be equal to 0 simultaneously. So the solution to the system is $x=0$, or the yz-plane.

■ 3. Find a set of critical points for $f(x_1, x_2, x_3, x_4)$.

$$f(x_1, x_2, x_3, x_4) = x_1^2 - 2x_1x_2 + 2x_2^2 - 4x_3^2 + 4x_1 + 5x_4^2 - 10x_4 + 6$$

Solution:

Use chain rule to find partial derivatives.

$$\frac{\partial f}{\partial x_1} = 2x_1 - 2x_2 + 4$$

$$\frac{\partial f}{\partial x_2} = -2x_1 + 4x_2$$

$$\frac{\partial f}{\partial x_3} = -8x_3$$

$$\frac{\partial f}{\partial x_4} = 10x_4 - 10$$

Setting all the partial derivatives equal to 0 to create a system of equations gives

$$2x_1 - 2x_2 + 4 = 0$$

$$-2x_1 + 4x_2 = 0$$

$$-8x_3 = 0$$

$$10x_4 - 10 = 0$$

The solution to the system is $(x_1, x_2, x_3, x_4) = (-4, -2, 0, 1)$, so this is the critical point of the function.

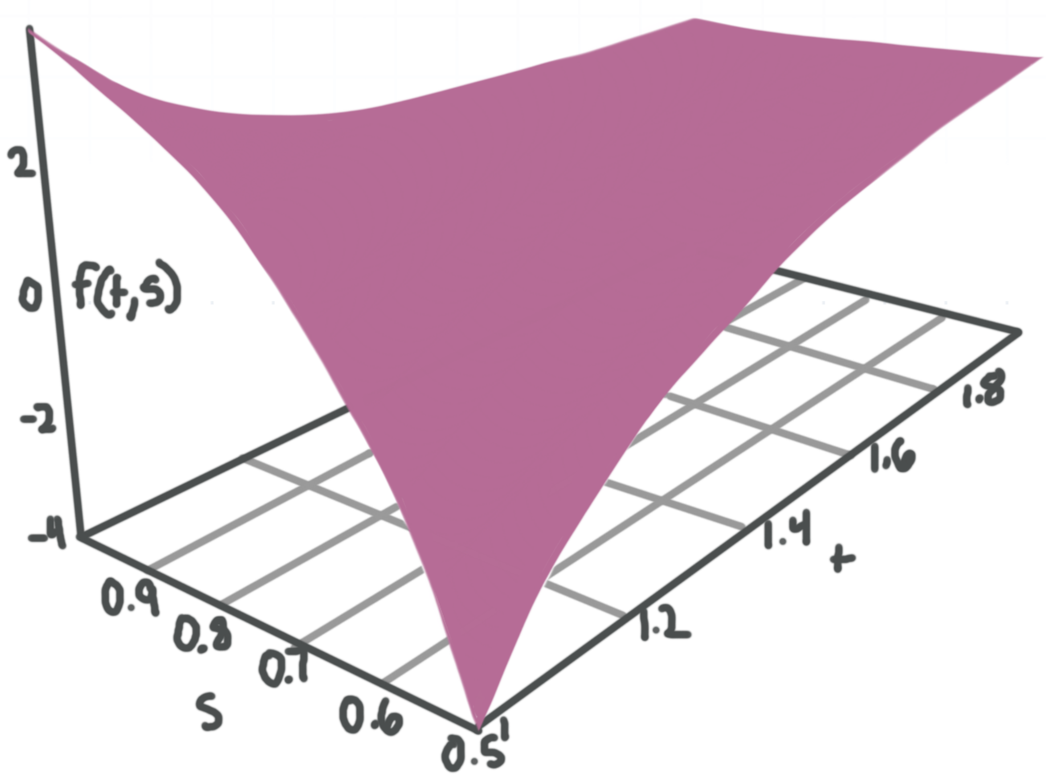
SECOND DERIVATIVE TEST

1. Use the second derivative test to classify the critical points of $f(t, s)$.

$$f(t,s)=\frac{t^5s-3t+6s^2}{s^2t^5}$$

Solution:

A sketch of the function is



Rewrite f.

$$f(t,s)=\frac{t^5s}{s^2t^5}-\frac{3t}{s^2t^5}+\frac{6s^2}{s^2t^5}$$

$$f(t,s) = \frac{1}{s} - \frac{3}{s^2t^4} + \frac{6}{t^5}$$

$$f(t,s) = s^{-1} - 3s^{-2}t^{-4} + 6t^{-5}$$

Use chain rule to find first-order partial derivatives.

$$\frac{\partial f}{\partial t} = 12s^{-2}t^{-5} - 30t^{-6}$$

$$\frac{\partial f}{\partial s} = 6s^{-3}t^{-4} - s^{-2}$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$12s^{-2}t^{-5} - 30t^{-6} = 0$$

$$6s^{-3}t^{-4} - s^{-2} = 0$$

Solve the system for $t \neq 0$ and $s \neq 0$.

$$2t - 5s^2 = 0$$

$$st^4 - 6 = 0$$

$$t = 2.5s^2$$

$$s(2.5s^2)^4 - 6 = 0$$

$$s^9 2.5^4 = 6$$

The values that satisfy the system are $s = 2^{\frac{5}{9}}3^{\frac{1}{9}}5^{-\frac{4}{9}} \approx 0.81$ and $t = 3^{\frac{2}{9}}10^{\frac{1}{9}} \approx 1.65$, so the solution is $(1.65,0.81)$.

Calculate the second-order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial t^2} = -60s^{-2}t^{-6} + 180t^{-7}$$

$$\frac{\partial^2 f}{\partial s^2} = -18s^{-4}t^{-4} + 2s^{-3}$$

$$\frac{\partial^2 f}{\partial t \partial s} = \frac{\partial^2 f}{\partial s \partial t} = -24s^{-3}t^{-5}$$

Substitute $(1.65,0.81)$ for (t, s).

$$\frac{\partial^2 f}{\partial t^2}(1.65,0.81) = -60 \cdot 0.81^{-2}1.65^{-6} + 180 \cdot 1.65^{-7} \approx 0.87$$

$$\frac{\partial^2 f}{\partial s^2}(1.65,0.81) = -18 \cdot 0.81^{-4}1.65^{-4} + 2 \cdot 0.81^{-3} \approx -1.88$$

$$\frac{\partial^2 f}{\partial t \partial s}(1.65,0.81) = -24 \cdot 0.81^{-3}1.65^{-5} \approx -3.69$$

Perform the second derivative test.

$$D(t, s) = \frac{\partial^2 f}{\partial t^2} \cdot \frac{\partial^2 f}{\partial s^2} - \left(\frac{\partial^2 f}{\partial t \partial s}\right)^2$$

$$D(1.65,0.81) = 0.87 \cdot (-1.88) - (-3.69)^2 = -15.2517 < 0$$

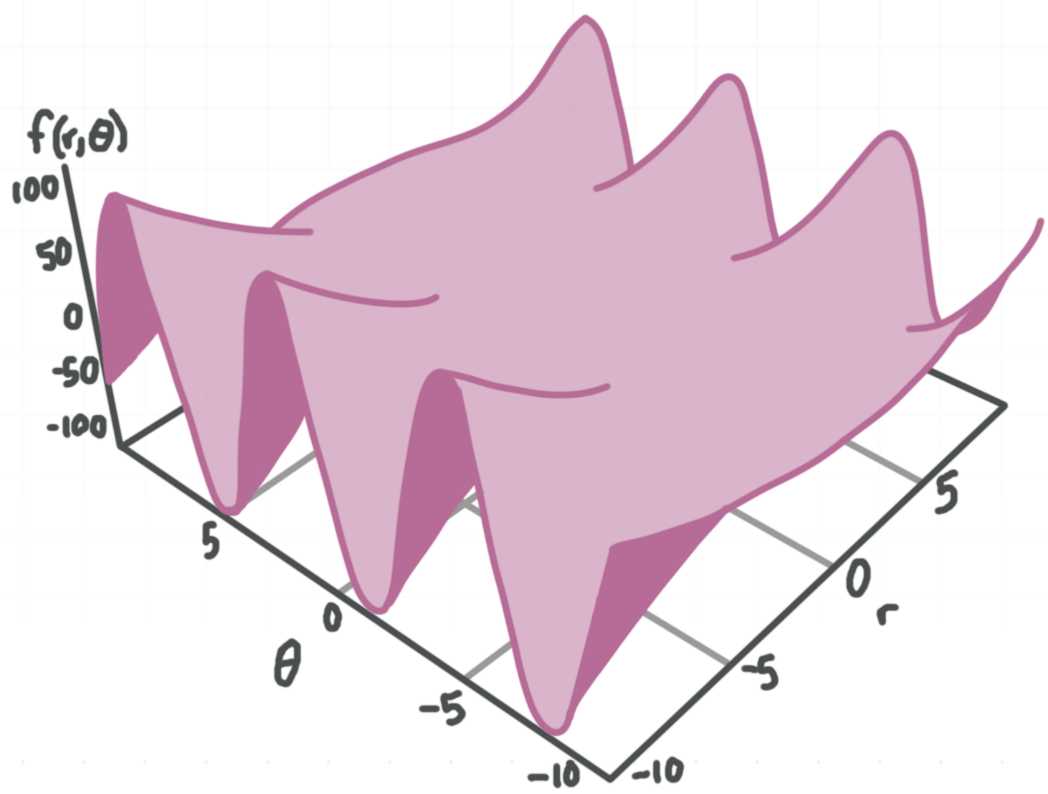
So $(1.65,0.81)$ is a saddle point.

2. Use the second derivative test to classify the critical points of $f(r, \theta)$.

$$f(r, \theta) = (r^2 - 2r - 3)\sin\theta$$

Solution:

A sketch of the function is



Find first-order partial derivatives.

$$\frac{\partial f}{\partial r} = (2r - 2)\sin\theta$$

$$\frac{\partial f}{\partial \theta} = (r^2 - 2r - 3)\cos\theta$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$(2r-2)\sin\theta = 0$$

$$(r^2-2r-3)\cos\theta = 0$$

Solve the system for r and θ.

$$(r-1)\sin\theta = 0$$

$$(r+1)(r-3)\cos\theta = 0$$

Since $\sin\theta$ and $\cos\theta$ can't be equal to 0 simultaneously, we have the three sets of solutions:

1) $r-1=0$ and $\cos\theta = 0$

$r=1$ and $\theta = \dfrac{\pi}{2} + \pi n$ where n is any integer number

2) $r+1=0$ and $\sin\theta = 0$

$r=-1$ and $\theta = \pi m$ where m is any integer number

3) $r-3=0$ and $\sin\theta = 0$

$r=3$ and $\theta = \pi k$ where k is any integer number

Calculate the second-order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial r^2} = 2\sin\theta$$

$$\frac{\partial^2 f}{\partial \theta^2} = -(r^2-2r-3)\sin\theta$$

$$\frac{\partial^2 f}{\partial r \partial \theta} = \frac{\partial^2 f}{\partial \theta \partial r} = (2r - 2)\cos\theta$$

Perform the second derivative test.

$$D(r, \theta) = \frac{\partial^2 f}{\partial r^2} \cdot \frac{\partial^2 f}{\partial \theta^2} - \left(\frac{\partial^2 f}{\partial r \partial \theta}\right)^2$$

The second derivative test for the first set of solutions is

$$\frac{\partial^2 f}{\partial r^2}\left(1, \frac{\pi}{2} + \pi n\right) = 2\sin\left(\frac{\pi}{2} + \pi n\right) = 2(\text{n is even}) \text{ or } -2(\text{n is odd})$$

$$\frac{\partial^2 f}{\partial \theta^2}\left(1, \frac{\pi}{2} + \pi n\right) = -(1^2 - 2(1) - 3)\sin\left(\frac{\pi}{2} + \pi n\right) = 4(\text{n is even}) \text{ or } -4(\text{n is odd})$$

$$\frac{\partial^2 f}{\partial r \partial \theta}\left(1, \frac{\pi}{2} + \pi n\right) = \frac{\partial^2 f}{\partial \theta \partial r} = (2(1) - 2)\cos\left(\frac{\pi}{2} + \pi n\right) = 0$$

For even n:

$$D\left(1, \frac{\pi}{2} + \pi n\right) = 2 \cdot 4 - 0^2 = 8 > 0$$

For odd n:

$$D\left(1, \frac{\pi}{2} + \pi n\right) = (-2) \cdot (-4) - 0^2 = 8 > 0$$

Since $\partial^2 f/\partial r^2 > 0$ for even n, and < 0 otherwise, the set of critical points $r = 1$, $\theta = \pi/2 + \pi n$ where n is even, are the local minima, and the set of critical points $r = 1$, $\theta = \pi/2 + \pi n$ where n is odd, are the local maxima.

Perform the second derivative test for the second set of solutions is

$$\frac{\partial^2 f}{\partial r^2}(-1,\pi m) = 2\sin(\pi m) = 0$$

$$\frac{\partial^2 f}{\partial \theta^2}(-1,\pi m) = -((-1)^2 - 2(-1) - 3)\sin(\pi m) = 0$$

$$\frac{\partial^2 f}{\partial r \partial \theta}(-1,\pi m) = \frac{\partial^2 f}{\partial \theta \partial r} = (2(-1) - 2)\cos(\pi m) = \pm 4$$

Then

$$D(-1,\pi m) = 0 \cdot 0 - (\pm 4)^2 = -16 < 0$$

Since $D < 0$, the set of critical points $r = -1$, $\theta = \pi m$ where m is an integer, are saddle points.

Perform the second derivative test for the third set of solutions is

$$\frac{\partial^2 f}{\partial r^2}(3,\pi k) = 2\sin\left(\pi k\right) = 0$$

$$\frac{\partial^2 f}{\partial \theta^2}(3,\pi k) = -(3^2 - 2(3) - 3)\sin\left(\pi k\right) = 0$$

$$\frac{\partial^2 f}{\partial r \partial \theta}(3,\pi k) = \frac{\partial^2 f}{\partial \theta \partial r} = (2(3) - 2)\cos\left(\pi k\right) = \pm 4$$

Then

$$D(3,\pi k) = 0 \cdot 0 - (\pm 4)^2 = -16 < 0$$

Since $D < 0$, the set of critical points $r = 3$, $\theta = \pi k$ where k is integer, are saddle points.

Then the extrema of the function are

Local minima: $r = 1$, $\theta = \frac{\pi}{2} + \pi n$ where n is even

Local maxima: $r = 1$, $\theta = \frac{\pi}{2} + \pi n$ where n is odd

Saddle points: $r = -1$, $\theta = \pi m$, and $r = 3$, $\theta = \pi k$ where m and k are integers

■ 3. Find the set of all possible values of a for which $f(x, y)$ has only one local minimum.

$$f(x, y) = x^2 + ay^2 - 4x + 8y - 6$$

Solution:

Calculate the first order partial derivatives:

$$\frac{\partial f}{\partial x} = 2x - 4$$

$$\frac{\partial f}{\partial y} = 2ay + 8$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$2x - 4 = 0$$

$$2ay + 8 = 0$$

The solution to the system is $x = 2$, $y = -4/a$ for $a \neq 0$. If $a = 0$, then the function has no critical points.

Calculate the second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2a$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = 0$$

The second derivative test gives

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D\left(2, -\frac{4}{a}\right) = 2 \cdot 2a - 0^2 = 4a$$

Since the function must have the local maximum at this critical point, $D > 0$, and

$$4a > 0$$

$$a > 0$$

If $a > 0$, then $D > 0$ and $\partial^2 f/\partial x^2 = 2 > 0$ at the point $(2, -4/a)$. So the function has only one local minimum when $a > 0$.

LOCAL EXTREMA AND SADDLE POINTS

■ 1. Find the local extrema of $f(t, s)$.

$$f(t, s) = \frac{t^3 + s^3 + 1}{ts}$$

Solution:

Rewrite f.

$$f(t, s) = \frac{t^3}{ts} + \frac{s^3}{ts} + \frac{1}{ts}$$

$$f(t, s) = \frac{t^2}{s} + \frac{s^2}{t} + \frac{1}{ts}$$

$$f(t, s) = t^2s^{-1} + s^2t^{-1} + t^{-1}s^{-1}$$

Find first-order partial derivatives.

$$\frac{\partial f}{\partial t} = 2ts^{-1} - s^2t^{-2} - t^{-2}s^{-1}$$

$$\frac{\partial f}{\partial s} = -t^2s^{-2} + 2st^{-1} - t^{-1}s^{-2}$$

Setting the partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$2ts^{-1} - s^2t^{-2} - t^{-2}s^{-1} = 0$$

$$-t^2s^{-2} + 2st^{-1} - t^{-1}s^{-2} = 0$$

Solve the system for $t \neq 0$ and $s \neq 0$.

$$s^3 - 2t^3 + 1 = 0$$

$$t^3 - 2s^3 + 1 = 0$$

Subtract equations to get

$$3t^3 - 3s^3 = 0$$

$$t^3 = s^3$$

$$t = s$$

and then

$$s^3 - 2s^3 + 1 = 0$$

$$-s^3 + 1 = 0$$

$$s = 1$$

So the solution to the system is $(1,1)$.

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial t^2} = 2s^{-1} + 2s^2t^{-3} + 2s^{-1}t^{-3}$$

$$\frac{\partial^2 f}{\partial s^2} = 2t^2s^{-3} + 2t^{-1} + 2t^{-1}s^{-3}$$

$$\frac{\partial^2 f}{\partial t \partial s} = \frac{\partial^2 f}{\partial s \partial t} = -2ts^{-2} - 2st^{-2} + t^{-2}s^{-2}$$

Evaluate the second order partial derivatives at $(1,1)$.

$$\frac{\partial^2 f}{\partial t^2}(1,1) = 2(1)^{-1} + 2(1)^2(1)^{-3} + 2(1)^{-1}(1)^{-3} = 6$$

$$\frac{\partial^2 f}{\partial s^2}(1,1) = 2(1)^2(1)^{-3} + 2(1)^{-1} + 2(1)^{-1}(1)^{-3} = 6$$

$$\frac{\partial^2 f}{\partial t \partial s}(1,1) = -2(1)(1)^{-2} - 2(1)(1)^{-2} + (1)^{-2}(1)^{-2} = -3$$

Perform the second derivative test.

$$D(t, s) = \frac{\partial^2 f}{\partial t^2} \cdot \frac{\partial^2 f}{\partial s^2} - \left(\frac{\partial^2 f}{\partial t \partial s}\right)^2$$

$$D(1,1) = 6 \cdot 6 - (-3)^2 = 27 > 0$$

Since $D(1,1) > 0$ and $\partial^2 f/\partial t^2(1,1) > 0$, the point $(1,1)$ is a local minima. So find $f(1,1)$.

$$f(1,1) = \frac{(1)^3 + (1)^3 + 1}{(1)(1)} = 3$$

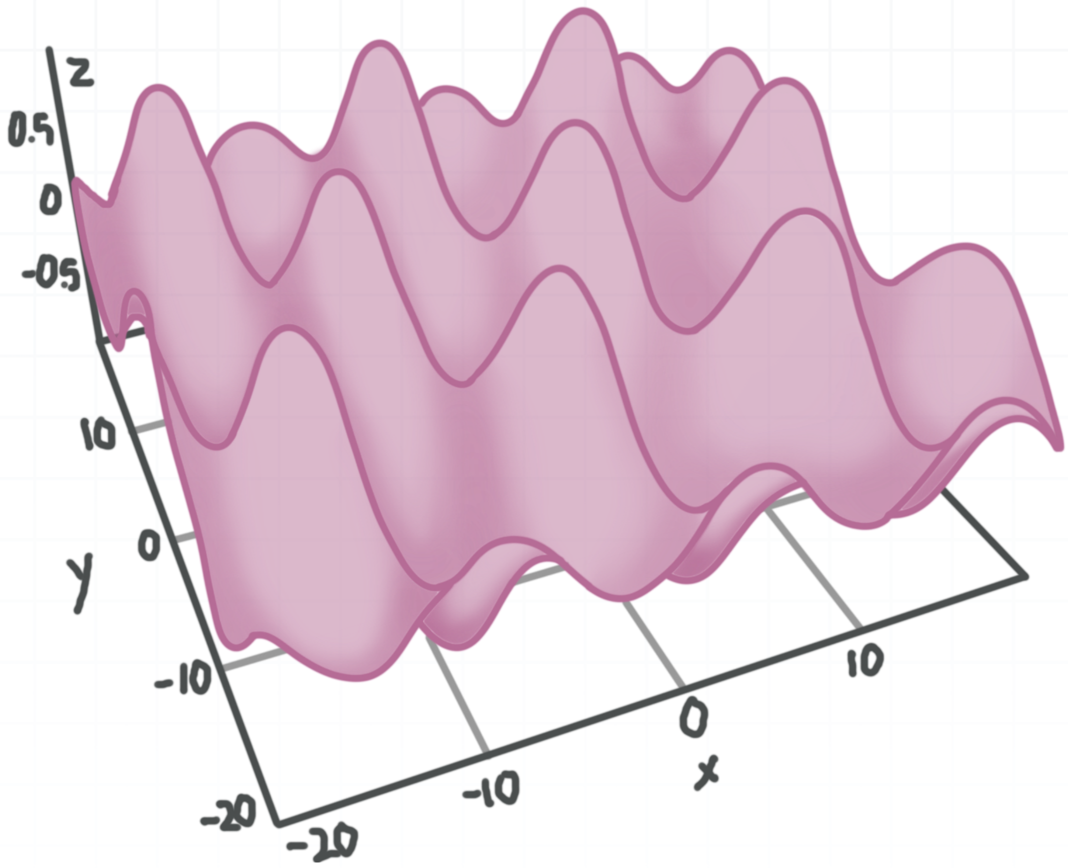
Then $f(1,1) = 3$ is the local minimum.

■ 2. Find the local extrema of $f(x, y)$.

$$f(x, y) = \sin(0.5x)\cos(0.25y)$$

Solution:

A sketch of the surface is



Use chain rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 0.5\cos(0.5x)\cos(0.25y)$$

$$\frac{\partial f}{\partial y} = -0.25\sin(0.5x)\sin(0.25y)$$

Set both partial derivatives equal to 0 to make a system of equations that we can use to find critical points.

$$0.5\cos(0.5x)\cos(0.25y) = 0$$

$$-0.25\sin(0.5x)\sin(0.25y) = 0$$

Solve the system for x and y.

$$\cos(0.5x)\cos(0.25y) = 0$$

$$\sin(0.5x)\sin(0.25y) = 0$$

Since sine and cosine functions of the same angle can't be 0 at the same time, we have two sets of solutions.

1) $\cos(0.5x) = 0$, $\sin(0.25y) = 0$

2) $\cos(0.25y) = 0$, $\sin(0.5x) = 0$

So the solution sets are

1) $0.5x = 0.5\pi + \pi n$, $0.25y = \pi m$

$x = \pi + 2\pi n, y = 4\pi m$ where n and m are integers

2) $0.5x = \pi n$, $0.25y = 0.5\pi + \pi m$

$x = 2\pi n, y = 2\pi + 4\pi m$ where n and m are integers

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2} = -0.5^2 \sin(0.5x)\cos(0.25y)$$

$$\frac{\partial^2 f}{\partial y^2} = -0.5^4 \sin(0.5x)\cos(0.25y)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial s \partial t} = -0.5^3 \cos(0.5x)\sin(0.25y)$$

The second derivative test gives

$$D(x, y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

The second derivative test for the first solution set, substituting $(\pi + 2\pi n, 4\pi m)$ for (x, y) gives

$$\frac{\partial^2 f}{\partial x^2}(\pi + 2\pi n, 4\pi m) = -0.5^2 \sin(0.5(\pi + 2\pi n))\cos(0.25(4\pi m)) = \pm 0.5^2$$

$$\frac{\partial^2 f}{\partial y^2}(\pi + 2\pi n, 4\pi m) = -0.5^4 \sin(0.5(\pi + 2\pi n))\cos(0.25(4\pi m)) = \pm 0.5^4$$

$$\frac{\partial^2 f}{\partial x \partial y}(\pi + 2\pi n, 4\pi m) = -0.5^3 \cos(0.5(\pi + 2\pi n))\sin(0.25(4\pi m)) = 0$$

So the second derivative test gives

$$D(\pi + 2\pi n, 4\pi m) = 0.5^6 \sin^2(0.5(\pi + 2\pi n))\cos^2(0.25(4\pi m)) = 0.5^6 > 0$$

When n and m are both even, or when n and m are both odd, $\partial^2 f/\partial x^2(\pi + 2\pi n, 4\pi m) < 0$, so we have a local maximum at the point. Otherwise the point is a local minimum.

Find the value of f.

$$f(\pi + 2\pi n, 4\pi m) = \sin(0.5(\pi + 2\pi n))\cos(0.25(4\pi m)) = \sin(0.5\pi + \pi n))\cos(\pi m) = 1$$

when n and m are both even, or when n and m are both odd, or equal to -1.

The second derivative test for the second solution set, substituting $(2\pi n, 2\pi + 4\pi m)$ for (x, y) gives

$$\frac{\partial^2 f}{\partial x^2}(2\pi n, 2\pi + 4\pi m) = -0.5^2 \sin(0.5(2\pi n))\cos(0.25(2\pi + 4\pi m)) = 0$$

$$\frac{\partial^2 f}{\partial y^2}(2\pi n, 2\pi + 4\pi m) = -0.5^4 \sin(0.5(2\pi n))\cos(0.25(2\pi + 4\pi m)) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(2\pi n, 2\pi + 4\pi m) = -0.5^3 \cos(0.5(2\pi n))\sin(0.25(2\pi + 4\pi m)) = \pm 0.5^3$$

So the second derivative test gives

$$D(2\pi n, 2\pi + 4\pi m) = 0 \cdot 0 - (\pm 0.5^3)^2 < 0$$

Since $D < 0$ for any n and m, all points from the second solution set are saddle points.

So the extrema of the function are

Local maxima at $(\pi + 2\pi n, 4\pi m, 1)$ when n and m are either both even or both odd.

Local minima at $(\pi + 2\pi n, 4\pi m, -1)$ when n is even and m is odd, or vice versa.

3. Find the equation(s) of the tangent plane to $f(x, y)$ at the function's local maximum.

$$f(x, y) = -x^2 - 2y^2 + 4x - 12y - 9$$

Solution:

Use power rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = -2x + 4$$

$$\frac{\partial f}{\partial y} = -4y - 12$$

Setting both partial derivatives equal to 0 gives us a system of equations that we can use to find critical points.

$$-2x + 4 = 0$$

$$-4y - 12 = 0$$

The solution to the system is $x = 2$ and $y = -3$.

Calculate second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2}(2, -3) = -2$$

$$\frac{\partial^2 f}{\partial y^2}(2, -3) = -4$$

$$\frac{\partial^2 f}{\partial x \partial y}(2, -3) = \frac{\partial^2 f}{\partial s \partial t} = 0$$

Then the second derivative test gives

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(2,-3) = (-2) \cdot (-4) - (0)^2 = 8 > 0$$

Since $D > 0$ and $\partial^2 f/\partial x^2 < 0$, the point $(2,-3)$ is a local maximum.

Since $\partial f/\partial x(2,-3) = 0$ and $\partial f/\partial x(2,-3) = 0$, the equation of the tangent plane to the surface $z = f(x,y)$ at $(2,-3)$ is

$$z = f(2,-3) = -2^2 - 2(-3)^2 + 4(2) - 12(-3) - 9 = 13$$

So the equation of the tangent plane is $z = 13$.

■ 4. Find the values of a and b where $f(x,y)$ has a local minimum at $(5,-3)$.

$$f(x,y) = 4x^2 + 2y^4 - ax - by + 5$$

Solution:

Use power rule to calculate first order partial derivatives.

$$\frac{\partial f}{\partial x} = 8x - a$$

$$\frac{\partial f}{\partial y} = 8y^3 - b$$

Setting both partial derivatives equal to 0 gives us a system of equations we can use to find critical points.

$$8x - a = 0$$

$$8y^3 - b = 0$$

The solution to the system is $x = a/8$ and $y = \sqrt[3]{b}/2$.

Setting x and y equal to 5 and -3 respectively and using these equations as a system of simultaneous equations to find the values of a and b gives

$$\frac{a}{8} = 5$$

$$\frac{\sqrt[3]{b}}{2} = -3$$

The solution to this system is $a = 8(5) = 40$ and $b = (2(-3))^3 = -216$.

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial x^2}(5, -3) = 8$$

$$\frac{\partial^2 f}{\partial y^2}(5, -3) = 24(-3)^2 = 216$$

$$\frac{\partial^2 f}{\partial x \partial y}(5, -3) = \frac{\partial^2 f}{\partial s \partial t} = 0$$

The second derivative test gives

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2$$

$$D(5, -3) = 8 \cdot 216 - (0)^2 = 1728 > 0$$

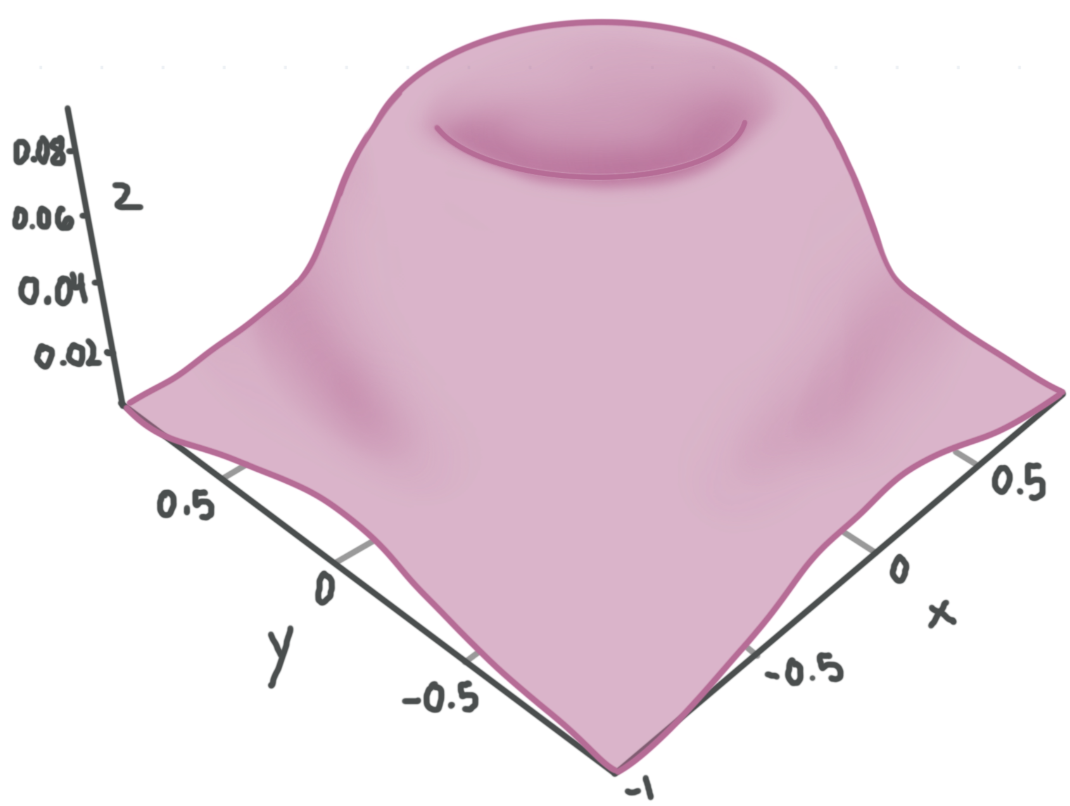
Since $D > 0$ and $\partial^2 f / \partial x^2 > 0$, the point $(5, -3)$ is a local minimum. So the values of a and b are $a = 40$ and $b = -216$.

■ 5. Find and identify the set of local maxima of $f(x,y)$.

$$f(x,y) = e^{-4(x^2+y^2)}(x^2 + y^2)$$

Solution:

A sketch of the surface is



Use product rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = -8xe^{-4(x^2+y^2)}(x^2+y^2) + e^{-4(x^2+y^2)}(2x) = -2xe^{-4(x^2+y^2)}(4x^2+4y^2-1)$$

$$\frac{\partial f}{\partial y} = -8ye^{-4(x^2+y^2)}(x^2+y^2) + e^{-4(x^2+y^2)}(2y) = -2ye^{-4(x^2+y^2)}(4x^2+4y^2-1)$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$-2xe^{-4(x^2+y^2)}(4x^2+4y^2-1) = 0$$

$$-2ye^{-4(x^2+y^2)}(4x^2+4y^2-1) = 0$$

Since $e^a > 0$,

$$x(4x^2+4y^2-1) = 0$$

$$y(4x^2+4y^2-1) = 0$$

Then the solution sets are

1) $4x^2+4y^2-1=0$, or $x^2+y^2=(0.5)^2$, which is a circle with center $(0,0)$ and radius 0.5

2) $x=0$, $y=0$

Since $f(x,y) > 0$ for all (x,y) except $(0,0)$, and $f(0,0)=0$, the point $(0,0)$ is a local (and also global) minimum.

To check if the solution set $(x,y)\,|\,x^2+y^2=(0.5)^2$ are a set of local maxima, we need to perform the second derivative test. Since the points from the set form a continuous curve, all of them may be classified the same way. So we can check any point from the set, for example $(0.5,0)$.

Find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 2e^{-4(x^2+y^2)}(32x^4 + 32x^2y^2 - 20x^2 - 4y^2 + 1)$$

$$\frac{\partial^2 f}{\partial y^2} = 2e^{-4(x^2+y^2)}(32y^4 + 32x^2y^2 - 20y^2 - 4x^2 + 1)$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial s \partial t} = 32xye^{-4(x^2+y^2)}(2x^2 + 2y^2 - 1)$$

Evaluate the second order partial derivatives at $(0.5,0)$.

$$\frac{\partial^2 f}{\partial x^2}(0.5,0) = -\frac{4}{e}$$

$$\frac{\partial^2 f}{\partial y^2}(0.5,0) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(0.5,0) = 0$$

The second derivative test gives

$$D(x,y) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(0.5,0) = -\frac{4}{e} \cdot 0 - (0)^2 = 0$$

So the second derivative test is inconclusive. So consider the arbitrary cross-section through the critical point $(0.5,0)$ along the line $y = t(x - 0.5)$. Substitute $t(x - 0.5)$ for y into the expression for $f(x, y)$.

$$g(x) = f(x, t(x - 0.5)) = e^{-4(x^2+t^2(x-0.5)^2)}(x^2 + t^2(x - 0.5)^2)$$

$$g(x) = e^{(-4(t^2+1)x^2+4t^2x-t^2)}((t^2 + 1)x^2 - t^2x + 0.25t^2)$$

Investigate the sign of $g'(x)$ in the vicinity of the $x = 0.5$.

$$g'(x) = e^{(-4(t^2+1)x^2+4t^2x-t^2)}((-8(t^2 + 1)x + 4t^2)((t^2 + 1)x^2 - t^2x + 0.25t^2) + 2(t^2 + 1)x - t^2)$$

Since $g'(0.5) = 0$ and $g'(0.5^-) > 0$, $g'(0.5+) < 0$, the function $g(x)$ has a local maximum at $x = 0.5$. So since every cross-section through the point $(0.5,0)$ has a local maximum at this point, the function $f(x, y)$ also has a local maximum at $(0.5,0)$.

So the set of local maxima is given by $x^2 + y^2 = (0.5)^2$, which is the circle with center $(0,0)$ and radius 0.5.

■ 6. Find and identify the saddle points and local extrema of $f(x, y)$.

$$f(x, y) = (x - 4)^8 - (y + 7)^{12}$$

Solution:

Use power rule to find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 8(x-4)^7$$

$$\frac{\partial f}{\partial y} = -12(y+7)^{11}$$

Setting both partial derivatives equal to 0 gives a system of equations that we can use to find critical points.

$$8(x-4)^7 = 0$$

$$-12(y+7)^{11} = 0$$

The solution to this system is $x = 4$, $y = -7$.

Find second order partial derivatives.

$$\frac{\partial^2 f}{\partial x^2} = 8 \cdot 7(x-4)^6$$

$$\frac{\partial^2 f}{\partial y^2} = -12 \cdot 11(y+7)^{10}$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Evaluate the second order partial derivatives at $(4, -7)$.

$$\frac{\partial^2 f}{\partial x^2}(4, -7) = 0$$

$$\frac{\partial^2 f}{\partial y^2}(4, -7) = 0$$

$$\frac{\partial^2 f}{\partial x \partial y}(4,-7)=0$$

Since $D(4,-7)=0$, the second derivative test is inconclusive. Then consider two cross-sections through the critical point $(4,-7)$, along the lines $x=4$ and $y=-7$.

Substitute $x=4$ into the expression for $f(x,y)$.

$$g(y)=f(4,y)=-(y+7)^{12}$$

Since $g(-7)=0$ and $g(y)<0$ when $y\neq -7$, the function has a local maximum at the point.

Substitute $y=-7$ into the expression for $f(x,y)$.

$$h(x)=f(x,-7)=(x-4)^8$$

Since $h(4)=0$ and $h(x)>0$ when $x\neq 4$, the function has a local minimum at the point.

So since two cross sections have minimum and maximum at the critical point $(4,-7)$, it's a saddle point.

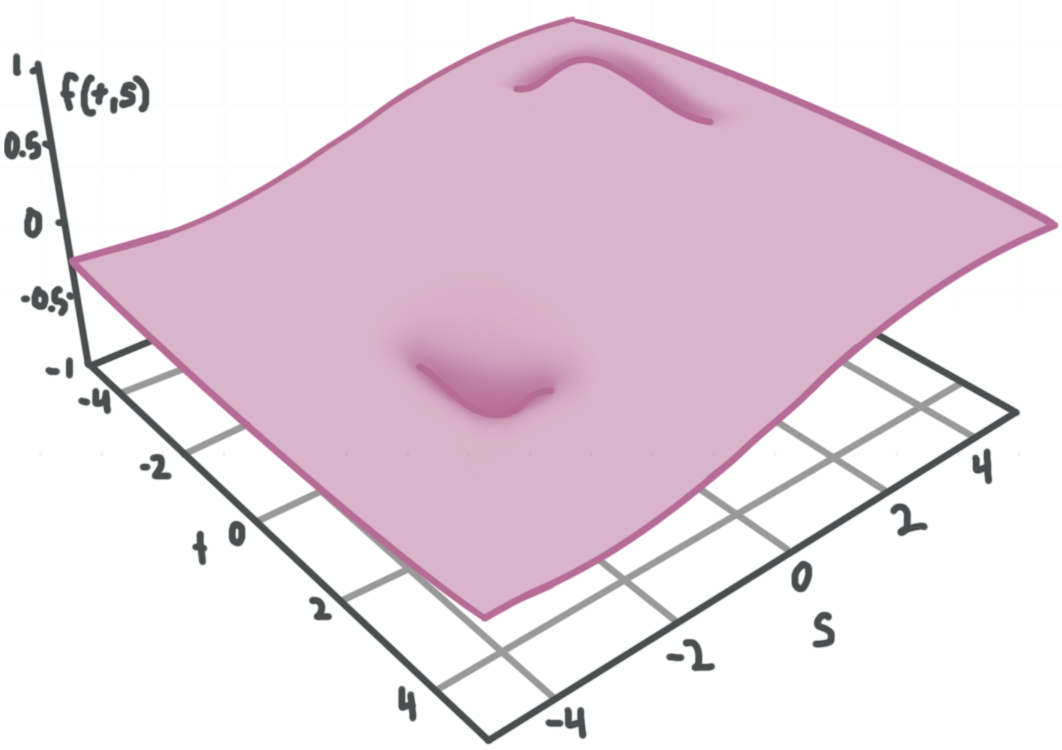
GLOBAL EXTREMA

■ 1. Find the global extrema of $f(t, s)$ over R^2.

$$f(t, s) = \frac{4s}{t^2 + 2s^2 + 2}$$

Solution:

A sketch of the surface is



Use quotient rule to find first order partial derivatives.

$$\frac{\partial f}{\partial t} = \frac{-8ts}{(t^2 + 2s^2 + 2)^2}$$

$$\frac{\partial f}{\partial s} = \frac{4t^2 - 8s^2 + 8}{(t^2 + 2s^2 + 2)^2}$$

Setting both of them equal to 0 and using these equations as a system of simultaneous equations to find critical points gives

$$-8ts = 0$$

$$4t^2 - 8s^2 + 8 = 0$$

Then

$$ts = 0$$

$$t^2 - 2s^2 + 2 = 0$$

If $t = 0$, then

$$(0)^2 - 2s^2 + 2 = 0$$

$$s^2 = 1$$

$$s = \pm 1$$

If $s = 0$, then

$$t^2 - 2(0)^2 + 2 = 0$$

$$t^2 = -2$$

and there are no solutions. So the solutions to the system are $(0,1)$ and $(0, -1)$.

Find second order partial derivatives to perform the second derivative test.

$$\frac{\partial^2 f}{\partial t^2} = \frac{8s(3t^2 - 2s^2 - 2)}{(t^2 + 2s^2 + 2)^3}$$

$$\frac{\partial^2 f}{\partial s^2} = \frac{16s(-3t^2 + 2s^2 - 6)}{(t^2 + 2s^2 + 2)^3}$$

$$\frac{\partial^2 f}{\partial t \partial s} = \frac{\partial^2 f}{\partial s \partial t} = \frac{8t(-t^2 + 6s^2 - 2)}{(t^2 + 2s^2 + 2)^3}$$

Evaluate the second order partial derivatives at $(0,1)$.

$$\frac{\partial^2 f}{\partial t^2}(0,1) = \frac{8(1)(3(0)^2 - 2(1)^2 - 2)}{((0)^2 + 2(1)^2 + 2)^3} = -0.5$$

$$\frac{\partial^2 f}{\partial s^2}(0,1) = \frac{16(1)(-3(0)^2 + 2(1)^2 - 6)}{((0)^2 + 2(1)^2 + 2)^3} = -1$$

$$\frac{\partial^2 f}{\partial t \partial s}(0,1) = \frac{\partial^2 f}{\partial s \partial t} = \frac{8(0)(-(0)^2 + 6(1)^2 - 2)}{((0)^2 + 2(1)^2 + 2)^3} = 0$$

Perform the second derivative test for $(0,1)$.

$$D(t,s) = \frac{\partial^2 f}{\partial t^2} \cdot \frac{\partial^2 f}{\partial s^2} - \left(\frac{\partial^2 f}{\partial t \partial s}\right)^2$$

$$D(0,1) = (-0.5) \cdot (-1) - (0)^2 = 0.5 > 0$$

So the critical point $(0,1)$ is a local extremum. Since $\partial^2 f/\partial t^2(0,1) < 0$, it's a local maximum.

$$f(0,1) = \frac{4(1)}{(0)^2 + 2(1)^2 + 2} = 1$$

Evaluate the second order partial derivatives at $(0, -1)$.

$$\frac{\partial^2 f}{\partial t^2}(0, -1) = \frac{8(-1)(3(0)^2 - 2(-1)^2 - 2)}{((0)^2 + 2(-1)^2 + 2)^3} = 0.5$$

$$\frac{\partial^2 f}{\partial s^2}(0, -1) = \frac{16(-1)(-3(0)^2 + 2(-1)^2 - 6)}{((0)^2 + 2(-1)^2 + 2)^3} = 1$$

$$\frac{\partial^2 f}{\partial t \partial s}(0, -1) = \frac{\partial^2 f}{\partial s \partial t} = \frac{8(0)(-(0)^2 + 6(-1)^2 - 2)}{((0)^2 + 2(-1)^2 + 2)^3} = 0$$

Perform the second derivative test for $(0, -1)$.

$$D(0, -1) = 0.5 \cdot 1 - (0)^2 = 0.5 > 0$$

So the critical point $(0, -1)$ is a local extremum. Since $\partial^2 f/\partial t^2(0, -1) > 0$, it's a local minimum.

$$f(0, -1) = \frac{4(-1)}{(0)^2 + 2(-1)^2 + 2} = -1$$

To check if these local extrema are global, we need to check the values of $f(t, s)$ on the boundaries of the region, in our case on $t \to \infty$, $s \to \infty$.

$$\lim_{t \to \infty, s \to \infty} f(t, s)$$

$$\lim_{t \to \infty, s \to \infty} \frac{4s}{t^2 + 2s^2 + 2}$$

$$\lim_{t \to \infty, s \to \infty} \frac{4}{t^2/s + 2s + 2/s}$$

$$\frac{4}{\infty} = 0$$

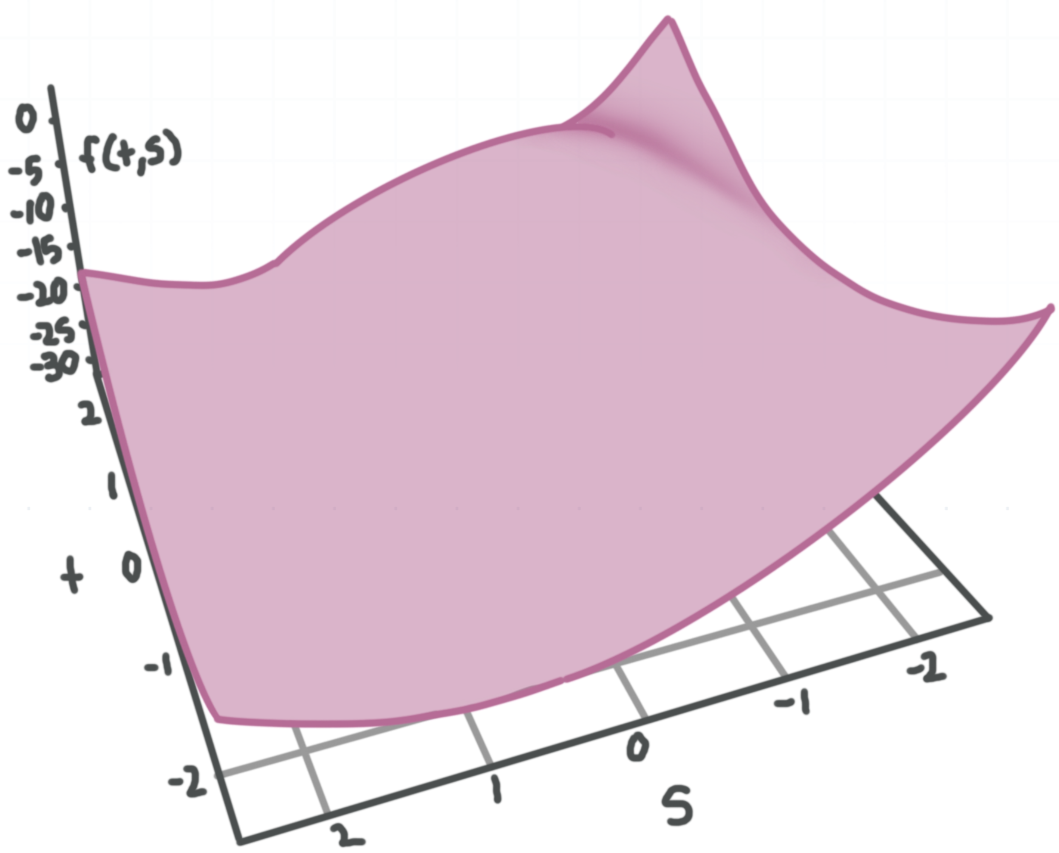
So 1 is the global maximum at $(0,1)$ and -1 is the global minimum at $(0,-1)$.

2. Find the global extrema of $f(t,s)$ over R^2.

$$f(t,s)=t^2s^2-4t^2-4s^2-4s+1$$

Solution:

A sketch of the surface is



Use power rule to find first order partial derivatives.

$$\frac{\partial f}{\partial t}=2ts^2-8t$$

$$\frac{\partial f}{\partial s}=2t^2s-8s-4$$

Setting both partial derivatives equal to 0 gives us a system of equations we can use to find critical points.

$$2ts^2 - 8t = 0$$

$$2t^2s - 8s - 4 = 0$$

and then

$$t(s^2 - 4) = 0$$

$$t^2s - 4s - 2 = 0$$

Consider the first equation. If $t = 0$, then

$$-4s - 2 = 0$$

$$s = -0.5$$

If $s = 2$, then

$$t^2(2) - 4(2) - 2 = 0$$

$$2t^2 - 10 = 0$$

$$t = \pm\sqrt{5}$$

If $s = -2$, then

$$t^2(-2) - 4(-2) - 2 = 0$$

$$-2t^2 + 6 = 0$$

$$t = \pm\sqrt{3}$$

So the solutions to the system are $(0,-0.5)$, $(\pm\sqrt{5},2)$, and $(\pm\sqrt{3},-2)$.

So we have five critical points, and we can perform the second derivative test and get the result that the point $(0,-0.5)$ is a local maximum, and four other points are saddle points. However, it's not necessary here, because we can see that the function has no global minima/maxima because it tends to infinity for big t and s.

For example, consider the path $t=0$, $s\to\infty$.

$$\lim_{t=0,s\to\infty} f(t,s)$$

$$\lim_{t=0,s\to\infty} (t^2s^2-4t^2-4s^2-4s+1)$$

$$\lim_{s\to\infty} (-4s^2-4s+1)=-\infty$$

So the function has no global minimum because it tends to $-\infty$ for some t and s.

As another example, consider the path $t=s$, $s\to\infty$.

$$\lim_{t=s,s\to\infty} f(t,s)$$

$$\lim_{t=s,s\to\infty} (t^2s^2-4t^2-4s^2-4s+1)$$

$$\lim_{s\to\infty} (s^2s^2-4s^2-4s^2-4s+1)$$

$$\lim_{s\to\infty} (s^4-8s^2-4s+1)$$

$$\lim_{s \to \infty} s^4 \left(1 - \frac{8}{s^2} - \frac{4}{s^3} + \frac{1}{s^4}\right)$$

$$\infty \cdot 1 = \infty$$

So the function has no global maximum because it tends to ∞ for some values of t and s. So the global extrema do not exist.

■ 3. Find the global extrema of $f(x, y)$ over R^2.

$$f(x, y) = \frac{\sin(3x)}{x^2 + 3y^2}$$

Solution:

Let's investigate the behavior of the function when either x or y approaches infinity.

$$|f(x, y)| = \left| \frac{\sin(3x)}{x^2 + 3y^2} \right| = \frac{|\sin(3x)|}{|x^2 + 3y^2|} \le \frac{1}{|x^2 + 3y^2|} = \frac{1}{x^2 + 3y^2}$$

So $f(x, y) \to 0$ as x or y approach infinity. Which means the function can have global extrema if it has no infinite discontinuity. But it has one at the point $(0,0)$.

So consider the path $y = 0$, $x \to 0^+$.

$$\lim_{x \to 0^+, y=0} f(x, y)$$

$$\lim_{x\to 0^+, y=0} \frac{\sin(3x)}{x^2+3y^2}$$

$$\lim_{x\to 0^+} \frac{\sin(3x)}{x^2}$$

$$\lim_{x\to 0^+} \frac{\sin(3x)}{x} \lim_{x\to 0^+} \frac{1}{x}$$

$$3 \cdot \infty = \infty$$

So the function has no global maximum because it tends to ∞ for some x and y.

Then consider the path $y=0$, $x \to 0^-$.

$$\lim_{x\to 0^-, y=0} f(x,y)$$

$$\lim_{x\to 0^-, y=0} \frac{\sin(3x)}{x^2+3y^2}$$

$$\lim_{x\to 0^-} \frac{\sin(3x)}{x^2}$$

$$\lim_{x\to 0^-} \frac{\sin(3x)}{x} \lim_{x\to 0^-} \frac{1}{x}$$

$$3 \cdot -\infty = -\infty$$

So the function has no global minimum because it tends to $-\infty$ for some x and y. So the global extrema do not exist.

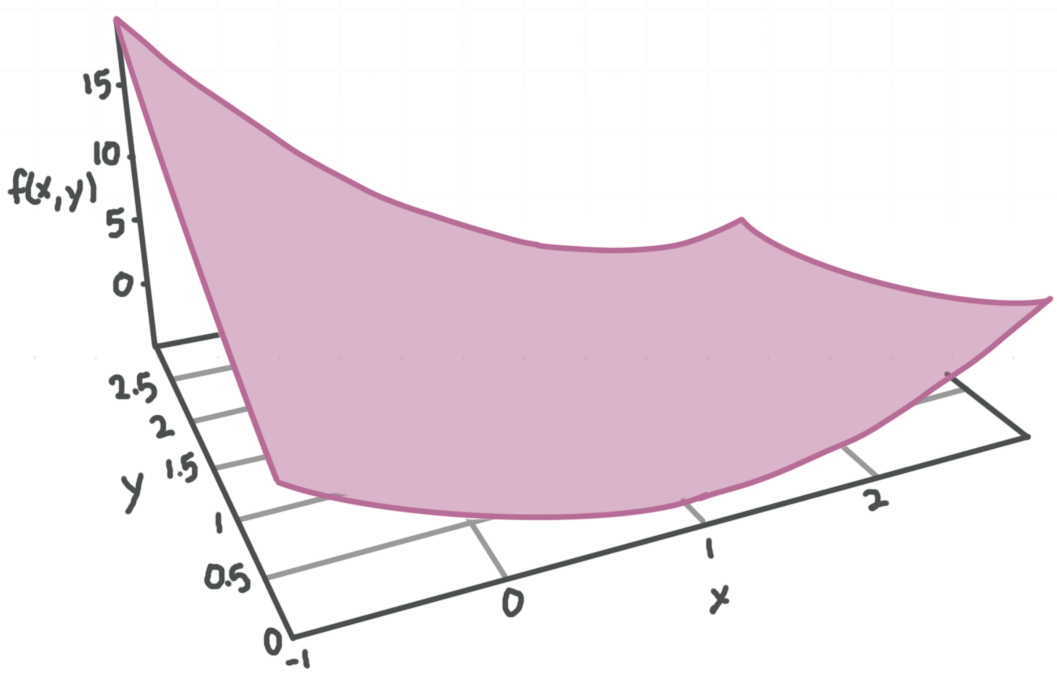
EXTREME VALUE THEOREM

■ 1. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of $f(x,y)$ on the closed rectangle $-1 \leq x \leq 3$, $0 \leq y \leq 3$.

$$f(x,y) = 2x^2 - 2xy + y^2 - 4x - 1$$

Solution:

A sketch of the surface is



Calculate the first order partial derivatives:

$$\frac{\partial f}{\partial x} = 4x - 2y - 4$$

$$\frac{\partial f}{\partial y} = -2x + 2y$$

Setting both partial derivatives equal to 0 and using these as a system of equations to find critical points gives

$$4x - 2y - 4 = 0$$

$$-2x + 2y = 0$$

The solution to the system is $(2,2)$.

Polynomial functions are always continuous on R^2. So we consider the continuous function over a closed region, and by the extreme value theorem it has a global maximum and a global minimum.

So we don't need to perform a secondary derivative test in order to find global extrema. We just need to determine the values of the function in every critical point (including boundaries), and find the greatest/the least of them.

Find critical points on the boundary $x = -1$.

$$g(y) = f(-1,y)$$

$$g(y) = 2(-1)^2 - 2(-1)y + y^2 - 4(-1) - 1$$

$$g(y) = y^2 + 2y + 5$$

$$g'(y) = 2y + 2 = 0$$

$$y = -1$$

The point is out of the region, so we should check only the endpoints $(-1,0)$ and $(-1,3)$.

Find critical points on the boundary $x = 3$.

$$g(y) = f(3,y)$$

$$g(y) = 2(3)^2 - 2(3)y + y^2 - 4(3) - 1$$

$$g(y) = y^2 - 6y + 5$$

$$g'(y) = 2y - 6 = 0$$

$$y = 3$$

The point is the endpoint of the region, so we should check the endpoints $(3,0)$ and $(3,3)$.

Find critical points on the boundary $y = 0$.

$$h(x) = f(x,0)$$

$$h(x) = 2x^2 - 2x(0) + (0)^2 - 4x - 1$$

$$h(x) = 2x^2 - 4x - 1$$

$$h'(x) = 4x - 4 = 0$$

$$x = 1$$

So we should check the critical point $(1,0)$ and the endpoints $(-1,0)$ and $(3,0)$.

Find critical points on the boundary $y = 3$.

$$h(x) = f(x,3)$$

$$h(x) = 2x^2 - 2x(3) + (3)^2 - 4x - 1$$

$$h(x) = 2x^2 - 10x + 8$$

$$h'(x) = 4x - 10 = 0$$

$$x = 2.5$$

So we should check the critical point $(2.5,3)$ and the endpoints $(-1,3)$ and $(3,3)$.

Collect all of the points together.

$$f(2,2) = -5$$

$$f(-1,0) = 5$$

$$f(-1,3) = 20$$

$$f(3,0) = 5$$

$$f(3,3) = -4$$

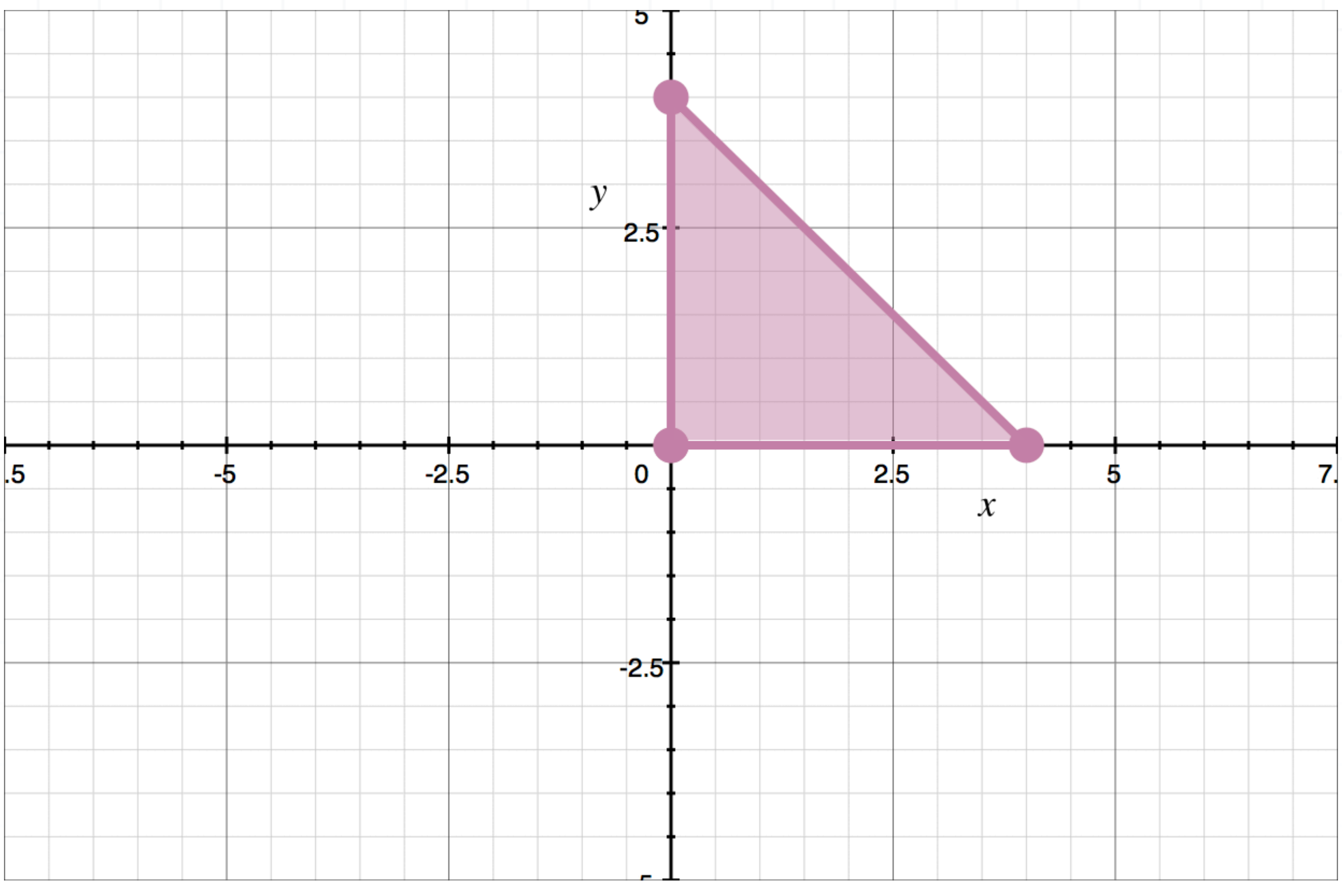
$$f(1,0) = -3$$

$$f(2.5,3) = -4.5$$

Comparing all of these, we can say that 20 is the global maximum at $(-1,3)$, and -5 is the global minimum at $(2,2)$.

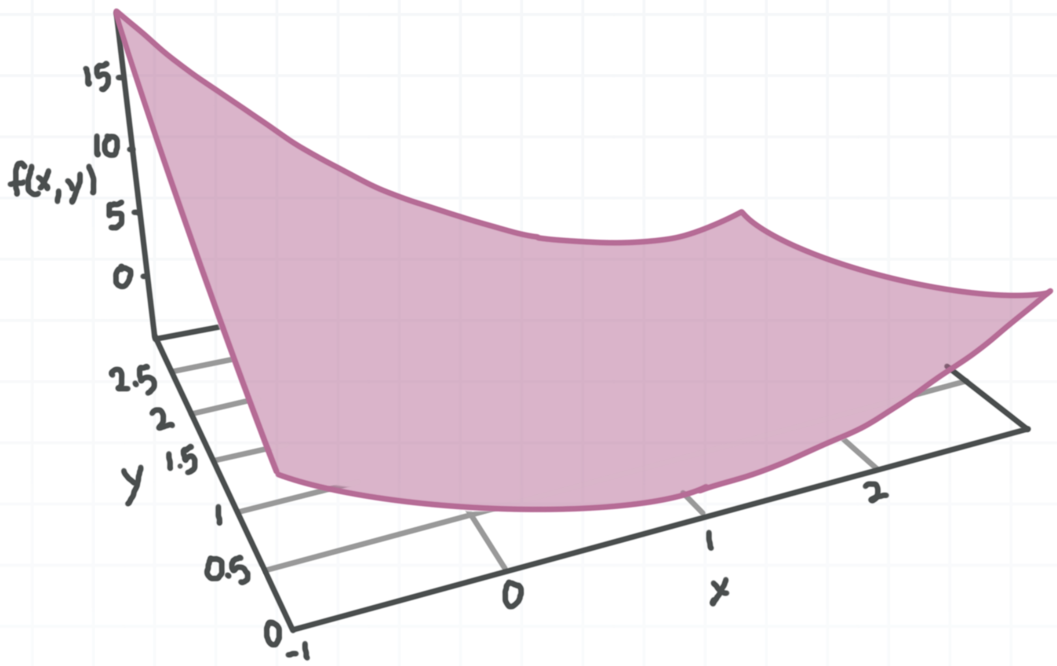
■ 2. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of $f(x, y)$ on a closed triangle bounded by $x = 0$, $y = 0$, and $x + y - 4 = 0$.

$$f(x, y) = \ln(x^2 + y^2 - 2y - x + 2)$$



Solution:

A sketch of the surface is



Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2x - 1}{x^2 + y^2 - 2y - x + 2}$$

$$\frac{\partial f}{\partial y} = \frac{2y - 2}{x^2 + y^2 - 2y - x + 2}$$

Since $x^2 + y^2 - 2y - x + 2$ is always positive, setting both numerators equal to 0 and using these equations as a system of equations to find critical points gives

$$2x - 1 = 0$$

$$2y - 2 = 0$$

The solution to the system is $(0.5,1)$.

The logarithmic function $\ln a$ is continuous on its domain $a > 0$, and $x^2 + y^2 - 2y - x + 2$ is always positive. So we consider the continuous function over a closed region, and by the extreme value theorem it has a global maximum and a global minimum.

So we don't need to perform a secondary derivative test in order to find the global extrema. We just need to determine the values of the function at every critical point (including boundaries), and find the greatest/the least of them.

The vertices of the triangle are $(0,0)$, $(0,4)$, $(4,0)$.

Find critical points on the boundary $x = 0$.

$$g(y) = f(0,y)$$

$$g(y) = \ln((0)^2 + y^2 - 2y - (0) + 2)$$

$$g(y) = \ln(y^2 - 2y + 2)$$

$$g'(y) = \frac{2y - 2}{y^2 - 2y + 2} = 0$$

$$y = 1$$

So we should check the critical point $(0,1)$.

Find critical points on the boundary $y = 0$.

$$h(x) = f(x,0)$$

$$h(x) = \ln(x^2 + (0)^2 - 2(0) - x + 2)$$

$$h(x) = \ln(x^2 - x + 2)$$

$$h'(x) = \frac{2x - 1}{x^2 - x + 2} = 0$$

$$x = 0.5$$

So we should check the critical point $(0.5,0)$.

Find critical points on the boundary $x + y - 4 = 0$, or $y = 4 - x$.

$$h(x) = f(x,4 - x)$$

$$h(x) = \ln(x^2 + (4 - x)^2 - 2(4 - x) - x + 2)$$

$$h(x) = \ln(2x^2 - 7x + 10)$$

$$h'(x) = 4x - 7 = 0$$

$x = 1.75$ and $y = 4 - 1.75 = 2.25$

So we should check the critical point $(1.75,2.25)$.

Collect all of the points together.

$$f(0.5,1) = \ln(0.75)$$

$$f(0,0) = \ln(2)$$

$$f(0,4) = \ln(10)$$

$$f(4,0) = \ln(14)$$

$$f(0,1) = \ln(1)$$

$$f(0.5,0) = \ln(1.75)$$

$$f(1.75,2.25) = \ln(3.875)$$

Since the logarithm is an increasing function, then $\ln(14)$ is a maximum, and $\ln(0.75)$ is a minimum. So $\ln(14)$ is the global maximum at $(4,0)$, and $\ln(0.75)$ is the global minimum at $(0.5,1)$.

■ 3. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of $f(x, y)$ on the closed rectangle $-\pi \leq x \leq \pi$, $-1 \leq y \leq 3$.

$$f(x, y) = y^2 \tan(2x)$$

Solution:

The tangent function is continuous at any point within its domain. The points excluded from domain are

$$\frac{\pi}{2} + \pi n \text{ for any integer } n$$

So we have the inequality

$$2x \neq \frac{\pi}{2} + \pi n \text{ for any integer } n$$

$$x \neq \frac{\pi}{4} + \frac{\pi}{2}n \text{ for any integer } n$$

Consider the point

$$y = 1,\ n = 0,\ x = \frac{\pi}{4} + \frac{\pi}{2}n = \frac{\pi}{4}$$

The point $(\pi/4,1)$ lies within the given closed rectangle, and the function $f(x,y)$ is not continuous in it. So the Extreme Value Theorem does not apply in this case.

And because of this, there's no global extrema on the given rectangle since $f(x,y)$ tends to ∞ or $-\infty$ as x approaches $\pi/4$ for any y.

■ 4. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of the function $f(x,y)$ on the closed rectangle $-\pi \le x \le \pi$, $-2 \le y \le 2$.

$$f(x,y) = (y^2 + 2y + 3)\tan\left(\frac{x}{4}\right)$$

Solution:

The tangent function is continuous at any point within its domain. The points excluded from domain are

$\dfrac{\pi}{2} + \pi n$ for any integer n

So we have the inequality

$\dfrac{x}{4} \neq \dfrac{\pi}{2} + \pi n$ for any integer n

$x \neq 2\pi + 4\pi n$ for any integer n

Since none of these points lie within the given interval $-\pi \le x \le \pi$, the function $\tan(x/4)$ is continuous on it. Also, because the polynomial function $y^2 + 2y + 3$ is continuous for any values of x and y, the function $f(x, y)$ is continuous on the given rectangle.

So Extreme Value Theorem applies in this case, and the function has a global maximum and a global minimum. To find the global extrema, we need to determine the values of function in every critical point (including boundaries), and find the greatest/the least of them.

Find the first order partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{1}{4}(y^2 + 2y + 3)\sec^2\left(\frac{x}{4}\right)$$

$$\frac{\partial f}{\partial y} = (2y + 2)\tan\left(\frac{x}{4}\right)$$

Setting both of them equal to 0 and using these equations as a system of equations to find critical points gives

$$\frac{1}{4}(y^2 + 2y + 3)\sec^2\left(\frac{x}{4}\right) = 0$$

$$(2y + 2)\tan\left(\frac{x}{4}\right) = 0$$

Since $(y^2 + 2y + 3)\sec^2(x/4) > 0$ for any (x, y), the system has no solutions. So there are no critical points inside the rectangle.

Find critical points on the boundary $x = -\pi$.

$$g(y) = f(-\pi, y) = (y^2 + 2y + 3)$$

$$\tan\left(\frac{-\pi}{4}\right) = -(y^2 + 2y + 3)$$

$$g'(y) = -2y - 2 = 0$$

$$y = -1$$

So we should check the critical point $(-\pi, -1)$ and the endpoints $(-\pi, -2)$ and $(-\pi,2)$.

Find critical points on the boundary $x = \pi$.

$$g(y) = f(\pi, y)$$

$$g(y) = (y^2 + 2y + 3)\tan\left(\frac{\pi}{4}\right)$$

$$g(y) = (y^2 + 2y + 3)$$

$$g'(y) = 2y + 2 = 0$$

$$y = -1$$

So we should check the critical point $(\pi,1)$ and the endpoints $(\pi, -2)$ and $(\pi,2)$.

Find critical points on the boundary $y = -2$.

$$h(x) = f(x, -2)$$

$$h(x) = ((-2)^2 + 2(-2) + 3)\tan\left(\frac{x}{4}\right)$$

$$h(x) = 3\tan\left(\frac{x}{4}\right)$$

$$h'(x) = \frac{3}{4}\sec^2\left(\frac{x}{4}\right) > 0$$

There are no critical points on the boundary, so we should check the endpoints $(-\pi, -2)$ and $(\pi, -2)$.

Find critical points on the boundary $y = 2$.

$$h(x) = f(x,2)$$

$$h(x) = ((2)^2 + 2(2) + 3)\tan\left(\frac{x}{4}\right)$$

$$h(x) = 11\tan\left(\frac{x}{4}\right)$$

$$h'(x) = \frac{11}{4}\sec^2\left(\frac{x}{4}\right) > 0$$

There are no critical points on the boundary, so we should check the endpoints $(-\pi,2)$ and $(\pi,2)$.

Collect all of the points together.

$$f(-\pi,2) = -11$$

$$f(\pi,2) = 11$$

$$f(-\pi, -2) = -3$$

$$f(\pi, -2) = 3$$

$$f(-\pi, -1) = -2$$

$$f(\pi,1) = 6$$

Comparing all of these, we can say that 11 is the global maximum at $(\pi,2)$, and -11 is the global minimum at $(-\pi,2)$.

5. Determine whether the Extreme Value Theorem applies. If the theorem applies, find the global extrema of the function $f(x, y)$ on the closed circle with center at the origin and radius 1.

$$f(x, y) = x^2 + y^2 - 2x + 2\sqrt{3}y - 3$$

Solution:

The polynomial function is continuous at any point. So the Extreme Value Theorem applies in this case, and the function has a global maximum and a global minimum.

To find the global extrema, we need to determine the values of the function at every critical point (including boundaries), and find the greatest/the least of them.

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 2x - 2$$

$$\frac{\partial f}{\partial y} = 2y + 2\sqrt{3}$$

Setting both of them equal to 0 and using these equations as a system of equations to find critical points gives

$$2x - 2 = 0$$

$$2y + 2\sqrt{3} = 0$$

The solution to the system is $(1, -\sqrt{3})$. The point lies outside the given circle. To find the critical points on the boundary, consider the function using polar coordinates.

Convert into polar coordinates (r, θ) assuming $r = 1$ on the circle.

$$r \geq 0, \ 0 \leq \theta < 2\pi$$

$$x = r\cos\theta = \cos\theta$$

$$y = r\sin\theta = \sin\theta$$

$r^2 = x^2 + y^2 = 1$ on the circle

Substitute into the function equation.

$$g(\theta) = f(\cos\theta, \sin\theta)$$

$$g(\theta) = 1 - 2\cos\theta + 2\sqrt{3}\sin\theta - 3$$

$$g(\theta) = -2\cos\theta + 2\sqrt{3}\sin\theta - 2$$

$$g'(\theta) = 2\sin\theta + 2\sqrt{3}\cos\theta = 0$$

$$\sin\theta = -\sqrt{3}\cos\theta$$

$$\tan\theta = -\sqrt{3}$$

$$\theta = \frac{2\pi}{3} \text{ or } \theta = \frac{5\pi}{3}$$

So there are two critical points on the circle, $(1,2\pi/3)$ and $(1,5\pi/3)$.

$$g\left(\frac{2\pi}{3}\right) = -2\cos\left(\frac{2\pi}{3}\right) + 2\sqrt{3}\sin\left(\frac{2\pi}{3}\right) - 2 = 2$$

$$g\left(\frac{5\pi}{3}\right) = -2\cos\left(\frac{5\pi}{3}\right) + 2\sqrt{3}\sin\left(\frac{5\pi}{3}\right) - 2 = -6$$

So we have a global maximum of 2 and global minimum of -6. Return back to (x, y) coordinates.

$$x_1 = \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}$$

$$y_1 = \sin\left(\frac{2\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

$$x_2 = \cos\left(\frac{5\pi}{3}\right) = \frac{1}{2}$$

$$y_2 = \sin\left(\frac{5\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$

So the extrema of the function are

A global maximum of 2 at $\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$

A global minimum of -6 at $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$

APPLIED OPTIMIZATION

1. Find the maximum volume of a rectangular box inscribed in a hemisphere with radius 4.

Solution:

Let x and y be the halves of linear dimensions of the base of the box (which lies on the flat base of the hemisphere), and let z be the height of the box. The volume of the box is

$$V = (2x)(2y)z = 4xyz$$

Since the top corner of the box lies on the sphere, we get

$$x^2 + y^2 + z^2 = \text{radius}^2 = 4^2 = 16$$

Solve the equation for z,

$$z = \sqrt{16 - x^2 - y^2}$$

then substitute into the volume equation

$$V = 4xy\sqrt{16 - x^2 - y^2}$$

Since x and y should be positive and the point (x, y) should be within the circle of radius 4, we have an optimization task: find the global maximum for the function

$$V(x, y) = 4xy\sqrt{16 - x^2 - y^2}$$

on the closed circular sector defined by

$$0 \le x$$

$$0 \le y$$

$$x^2 + y^2 \le 16$$

Use product rule to find first order partial derivatives.

$$\frac{\partial V}{\partial x} = \frac{4y(16 - 2x^2 - y^2)}{16 - x^2 - y^2}$$

$$\frac{\partial V}{\partial y} = \frac{4x(16 - x^2 - 2y^2)}{16 - x^2 - y^2}$$

Setting the numerators equal to 0 and using these equations as a system of simultaneous equations to find critical points, we get

$$y(16 - 2x^2 - y^2) = 0$$

$$x(16 - x^2 - 2y^2) = 0$$

Luckily, we don't need to consider the cases where $x = 0$ or $y = 0$ for this task, because in that case the volume would be equal to 0. And we don't need to consider the sector boundary $x^2 + y^2 = 16$. If the point (x, y) would lie on the circle, then $z = 0$, and volume is also equal to 0 in that case.

So solve the system for $x \neq 0$, $y \neq 0$.

$$16 - 2x^2 - y^2 = 0$$

$$16 - x^2 - 2y^2 = 0$$

Subtract the equations.

$$16 - 2x^2 - y^2 - (16 - x^2 - 2y^2) = 0 - 0$$

$$16 - 2x^2 - y^2 - 16 + x^2 + 2y^2 = 0$$

$$-2x^2 + x^2 - y^2 + 2y^2 = 0$$

$$-x^2 + y^2 = 0$$

$$x^2 = y^2$$

Substitute to the first equation:

$$16 - 2y^2 - y^2 = 0$$

$$16 - 3y^2 = 0$$

$$y^2 = \frac{16}{3}$$

$$y = \pm \frac{4}{\sqrt{3}}$$

Since we need only positive solutions,

$$x = y = \frac{4}{\sqrt{3}}$$

Find the volume at this point.

$$V\left(\frac{4}{\sqrt{3}},\frac{4}{\sqrt{3}}\right)=4\frac{4^2}{3}\sqrt{16-\frac{4^2}{3}-\frac{4^2}{3}}=\frac{256}{3\sqrt{3}}$$

Since we have only one critical point in the circular sector, $V(x,y)>0$ at this point, and $V(x,y)=0$ at the boundaries, by the extreme value theorem this is the global maximum of the function.

■ 2. Find the minimum distance from $(2,2,-1)$ to the plane $8x-4y+z+11=0$.

Solution:

Let (x,y,z) be the coordinates of the point on the given plane. Then the square distance from this point to $(2,2,-1)$ is

$$d^2=(x-2)^2+(y-2)^2+(z+1)^2$$

Isolate z in the plane equation and substitute it to the distance formula.

$$z=-8x+4y-11$$

$$d^2=(x-2)^2+(y-2)^2+(-8x+4y-11+1)^2$$

$$d^2=(x-2)^2+(y-2)^2+(-8x+4y-10)^2$$

We have an optimization task: find the global minimum for the function $H(x,y)=(x-2)^2+(y-2)^2+(-8x+4y-10)^2$ on R^2. So calculate the first order partial derivatives using the power rule.

$$\frac{\partial H}{\partial x} = 2(x-2) + 2(-8)(-8x+4y-10) = 2(65x-32y+78)$$

$$\frac{\partial H}{\partial y} = 2(y-2) + 2(4)(-8x+4y-10) = 2(-32x+17y-42)$$

Setting both partial derivatives equal to 0 and using these equations as a system of equations to find critical points gives

$$65x - 32y + 78 = 0$$

$$-32x + 17y - 42 = 0$$

So the solution of the system is $x = 2/9$, $y = 26/9$.

Find second order partial derivatives.

$$\frac{\partial^2 H}{\partial x^2} = 130$$

$$\frac{\partial^2 H}{\partial y^2} = 34$$

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x} = -64$$

The second derivative test gives

$$D(x,y) = \frac{\partial^2 H}{\partial x^2} \cdot \frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 H}{\partial x \partial y}\right)^2$$

$$D\left(\frac{2}{9}, \frac{26}{9}\right) = 130 \cdot 34 - (-64)^2 = 324 > 0$$

Since $D > 0$, and $\partial^2 H/\partial x^2 > 0$, the point $(2/9,26/9)$ is a local minimum. Since $H(x,y)$ tends to infinity for big x and y, this local minimum is a global minimum.

Find the distance.

$$d^2 = \left(\frac{2}{9} - 2\right)^2 + \left(\frac{26}{9} - 2\right)^2 + \left(-8 \cdot \frac{2}{9} + 4 \cdot \frac{26}{9} - 10\right)^2$$

$$d^2 = \left(-\frac{16}{9}\right)^2 + \left(\frac{8}{9}\right)^2 + \left(-\frac{2}{9}\right)^2$$

$$d^2 = \frac{256}{81} + \frac{64}{81} + \frac{4}{81}$$

$$d^2 = 4$$

$$d = 2$$

The minimum distance from the point to the plane is $d = 2$.

■ 3. Find the minimum distance from $(-4,4,0)$ to the cone $3x^2 + y^2 = z^2$.

Solution:

Let (x, y, z) be the coordinates of the point on the cone. Then the square distance from this point to $(-4,4,0)$ is

$$d^2 = (x+4)^2 + (y-4)^2 + z^2$$

Substitute z^2 from the cone equation into the distance formula

$$d^2 = (x+4)^2 + (y-4)^2 + 3x^2 + y^2$$

$$d^2 = 4x^2 + 8x + 2y^2 - 8y + 32$$

We have an optimization task: find the global minimum for the function on R^2.

$$H(x, y) = 4x^2 + 8x + 2y^2 - 8y + 32$$

Find first order partial derivatives.

$$\frac{\partial H}{\partial x} = 8x + 8$$

$$\frac{\partial H}{\partial y} = 4y - 8$$

Setting both partial derivatives equal to 0 and using these equations as a system to find critical points gives

$$8x + 8 = 0$$

$$4y - 8 = 0$$

The solution to the system is $x = -1$, $y = 2$.

Find second order partial derivatives.

$$\frac{\partial^2 H}{\partial x^2} = 8$$

$$\frac{\partial^2 H}{\partial y^2} = 4$$

$$\frac{\partial^2 H}{\partial x \partial y} = \frac{\partial^2 H}{\partial y \partial x} = 0$$

The second derivative test gives

$$D(x,y) = \frac{\partial^2 H}{\partial x^2} \cdot \frac{\partial^2 H}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(-1,2)) = 8 \cdot 4 - (0)^2 = 32 > 0$$

Since $D > 0$, and $\partial^2 H/\partial x^2 > 0$, the point $(-1,2)$ is a local minimum. Since $H(x,y)$ tends to infinity for big x and y, this local minimum is a global minimum.

Find the distance.

$$d^2 = 4(-1)^2 + 8(-1) + 2(2)^2 - 8(2) + 32$$

$$d^2 = 20$$

$$d = \sqrt{20}$$

$$d = 2\sqrt{5}$$

The minimum distance from the point to the cone is $d = 2\sqrt{5}$.

TWO DIMENSIONS, ONE CONSTRAINT

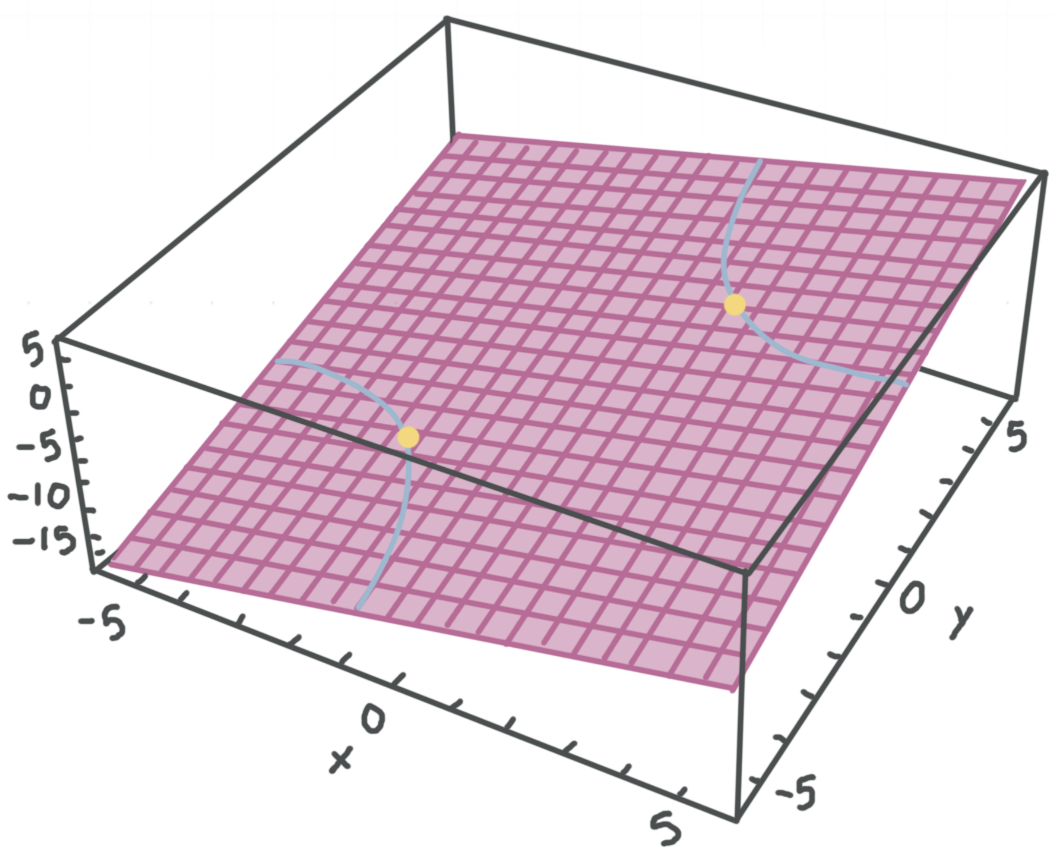
■ 1. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = x + y - 5$$

$$xy = 1$$

Solution:

A sketch of the surface is



Let

$$g(x, y) = xy - 1$$

Create a system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 1 \qquad \frac{\partial g}{\partial x} = y$$

$$\frac{\partial f}{\partial y} = 1 \qquad \frac{\partial g}{\partial y} = x$$

Substitute partial derivatives into the system.

$$1 = y\lambda$$

$$1 = x\lambda$$

Solve the system for λ.

$$x = y = \frac{1}{\lambda} \text{ with } x \neq 0, y \neq 0$$

Plug x into the constraint equation.

$$(y)y = 1$$

$$y^2 = 1$$

$$y = \pm 1$$

Since $x = y$, the solutions are $(1,1,\lambda = 1)$, $(-1, -1,\lambda = -1)$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where $L(x, y) = f(x, y) - \lambda g(x, y)$. The second derivative test for $(1,1)$ with $\lambda = 1$.

$$L(x, y) = f(x, y) - (1)g(x, y)$$

$$L(x, y) = x + y - 5 - (xy - 1)$$

$$L(x, y) = x + y - xy - 4$$

Find second order partial derivatives.

$$L_{xx} = 0$$

$$L_{yy} = 0$$

$$L_{xy} = L_{yx} = -1$$

Substitute $(1,1)$.

$$g_x(1,1) = 1$$

$$g_y(1,1) = 1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{vmatrix}$$

Then $det(HL) = -2 < 0$. Since $det(HL) < 0$ at $(1,1)$, it's a local minimum, and

$$f(1,1) = (1) + (1) - 5 = -3$$

The second derivative test for $(-1, -1)$ with $\lambda = -1$.

$$L(x, y) = f(x, y) - (-1)g(x, y)$$

$$L(x, y) = x + y - 5 + (xy - 1)$$

$$L(x, y) = x + y + xy - 6$$

Find second order partial derivatives.

$$L_{xx} = 0$$

$$L_{yy} = 0$$

$$L_{xy} = L_{yx} = 1$$

Substitute $(-1, -1)$.

$$g_x(-1, -1) = -1$$

$$g_y(-1, -1) = -1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix}$$

Then $det(HL) = 2 > 0$. Since $det(HL) > 0$ at $(-1, -1)$, it's a local maximum, and

$$f(-1, -1) = (-1) + (-1) - 5 = -7$$

Then the extrema of the function are

A local maximum of -7 at $(-1, -1)$

A local minimum of -3 at $(1,1)$

■ 2. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = e^{2x+y}$$

$$x^2 + y^2 = 5$$

Solution:

Let

$$g(x, y) = x^2 + y^2 - 5$$

Create a system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 2e^{2x+y} \qquad \frac{\partial g}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = e^{2x+y} \qquad \frac{\partial g}{\partial y} = 2y$$

Substitute partial derivatives into the system.

$$2e^{2x+y} = 2x\lambda$$

$$e^{2x+y} = 2y\lambda$$

Solve the system for λ.

$$\frac{e^{2x+y}}{x} = \lambda$$

$$\frac{e^{2x+y}}{2y} = \lambda$$

$$\frac{e^{2x+y}}{x} = \frac{e^{2x+y}}{2y}$$

Since $e^{2x+y} > 0$, we can say $x = 2y$. Plug x into the constraint equation.

$$(2y)^2 + y^2 = 5$$

$$4y^2 + y^2 = 5$$

$$y^2 = 1$$

$$y = \pm 1$$

Substitute to find x.

$$x^2 + (\pm 1)^2 = 5$$

$$x^2 = 4$$

$$x = \pm 2$$

Since $x = 2y$, the solutions to the system are $(2,1)$ and $(-2,-1)$. Calculate λ for each solution pair.

$$\lambda(2,1) = \frac{e^{2(2)+(1)}}{2} = \frac{e^5}{2}$$

$$\lambda(-2,-1) = \frac{e^{2(-2)+(-1)}}{-2} = -\frac{e^{-5}}{2}$$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where $L(x,y) = f(x,y) - \lambda g(x,y)$. The second derivative test for $(2,1)$ with $\lambda = e^5/2$.

$$L(x,y) = f(x,y) - \frac{e^5}{2}g(x,y)$$

$$L(x,y) = e^{2x+y} - \frac{e^5}{2}(x^2 + y^2 - 5)$$

Find second order partial derivatives.

$$L_{xx} = 4e^{2x+y} - e^5$$

$$L_{yy} = e^{2x+y} - e^5$$

$$L_{xy} = L_{yx} = 2e^{2x+y}$$

Then at $(2,1)$,

$$g_x(2,1) = 2(2) = 4$$

$$g_y(2,1) = 2(1) = 2$$

$$L_{xx}(2,1) = 4e^{2(2)+(1)} - e^5 = 3e^5$$

$$L_{yy}(2,1) = e^{2(2)+(1)} - e^5 = 0$$

$$L_{xy}(2,1) = L_{yx}(2,1) = 2e^{2(2)+(1)} = 2e^5$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -4 & -2 \\ -4 & 3e^5 & 2e^5 \\ -2 & 2e^5 & 0 \end{vmatrix}$$

Then $det(HL) = 20e^5 > 0$. Since $det(HL) > 0$ at $(2,1)$, it's a local maximum, and $f(2,1) = e^5$.

The second derivative test for $(-2, -1)$ with $\lambda = -e^{-5}/2$,

$$L(x, y) = f(x, y) + \frac{e^{-5}}{2}g(x, y)$$

$$L(x, y) = e^{2x+y} + \frac{e^{-5}}{2}(x^2 + y^2 - 5)$$

Find second order partial derivatives.

$$L_{xx} = 4e^{2x+y} + e^{-5}$$

$$L_{yy} = e^{2x+y} + e^{-5}$$

$$L_{xy} = L_{yx} = 2e^{2x+y}$$

Substitute $(-2, -1)$.

$$g_x(-2, -1) = 2(2) = 4$$

$$g_y(-2, -1) = 2(1) = 2$$

$$L_{xx}(-2, -1) = 4e^{2(-2)+(-1)} + e^{-5} = 5e^{-5}$$

$$L_{yy}(-2, -1) = e^{2(-2)+(-1)} + e^{-5} = 2e^{-5}$$

$$L_{xy}(-2, -1) = L_{yx}(-2, -1) = 2e^{2(-2)+(-1)} = 2e^{-5}$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -4 & -2 \\ -4 & 5e^{-5} & 2e^{-5} \\ -2 & 2e^{-5} & 2e^{-5} \end{vmatrix}$$

Then $det(HL) = -20e^{-5} < 0$. Since $det(HL) < 0$ at $(-2, -1)$, it's a local minimum, and $f(-2, -1) = e^{-5}$.

Since the restriction curve is a closed circle, the function is continuous, and has one local minimum and one local maximum, and the local minima and maxima are global extrema.

The extrema of the function are

A global maximum of e^5 at $(2,1)$

A global minimum of e^{-5} at $(-2,-1)$

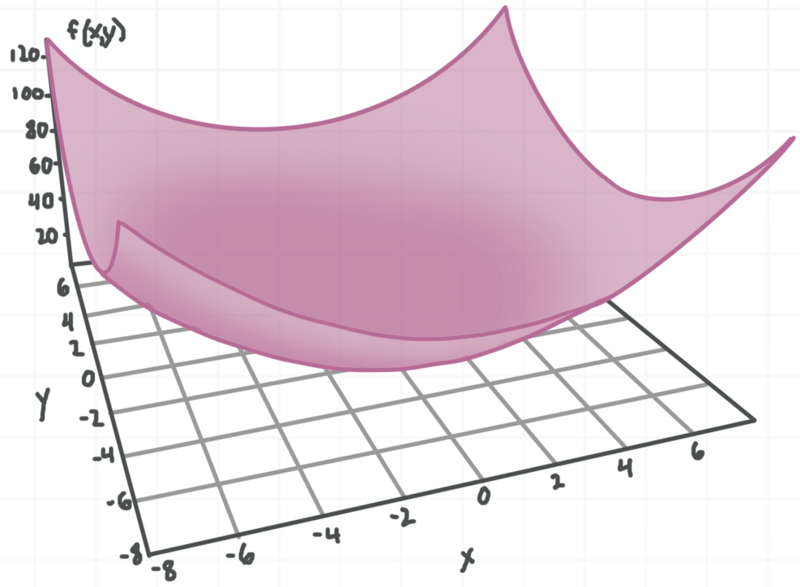
■ 3. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = x^2 + y^2 + 3$$

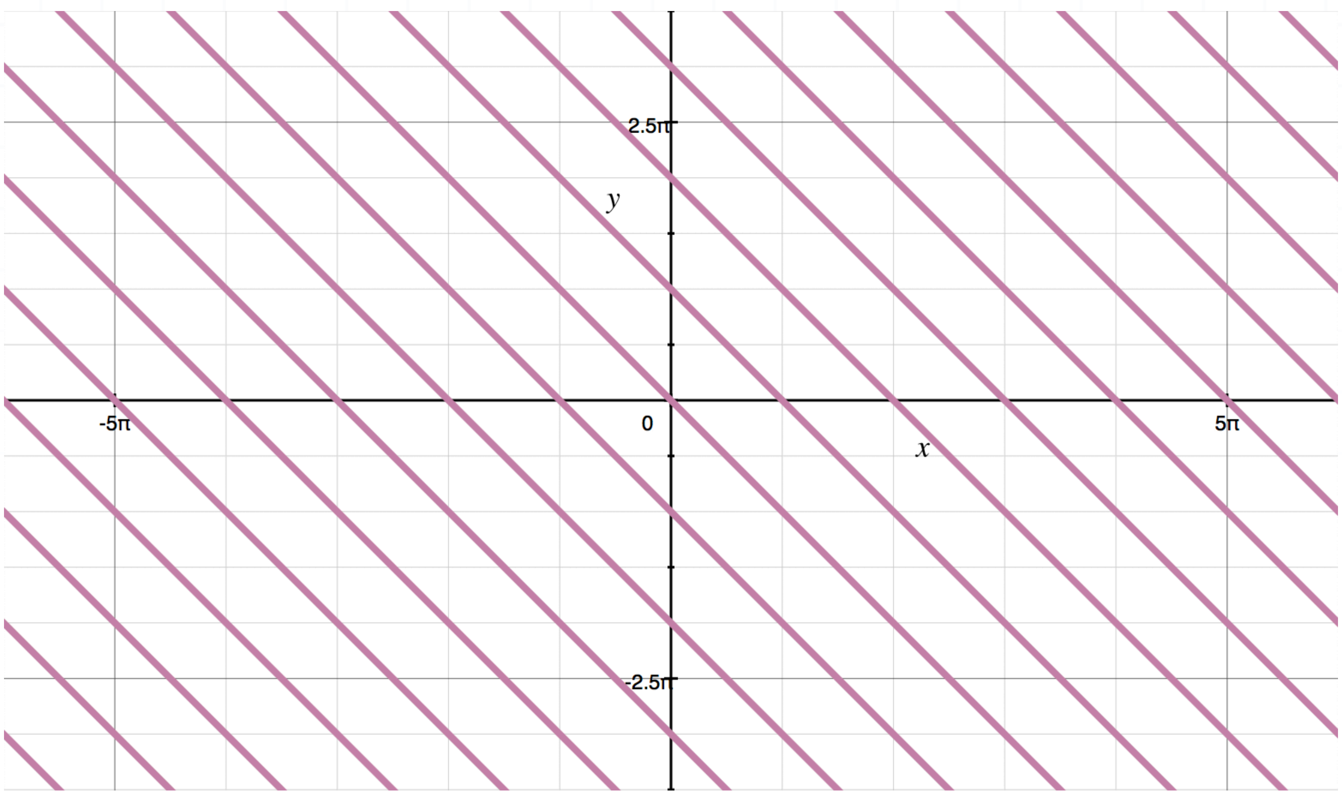
$$\sin(x + y) = 0$$

Solution:

A sketch of the surface is



And it’s subject to the constraint



Let

$$g(x,y)=\sin(x+y)$$

Create a system of equations.

$$\frac{\partial f}{\partial x}=\lambda\frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = 2x \qquad \frac{\partial g}{\partial x} = \cos(x+y)$$

$$\frac{\partial f}{\partial y} = 2y \qquad \frac{\partial g}{\partial y} = \cos(x+y)$$

Substitute partial derivatives into the system.

$$2x = \lambda \cos(x+y)$$

$$2y = \lambda \cos(x+y)$$

Solve the system for λ. If $\cos(x+y)=0$, then $x=0$ and $y=0$, but $\cos(x+y)=\cos(0)=1$, so $\cos(x+y)$ can't be 0. Since $\cos(x+y)\neq 0$,

$$\frac{2x}{\cos(x+y)} = \lambda$$

$$\frac{2y}{\cos(x+y)} = \lambda$$

$$\frac{2x}{\cos(x+y)} = \frac{2y}{\cos(x+y)}$$

So $x=y$. Plug x into the constraint equation.

$$\sin(y+y) = 0$$

$$\sin(2y) = 0$$

We get

$$2y = \pi n, \text{ where } n \text{ is any integer}$$

$$y = 0.5\pi n, \text{ where } n \text{ is any integer}$$

Since $x = y$, $x = 0.5\pi n$, where n is any integer, the solution to the system is $(0.5\pi n, 0.5\pi n)$, where n is any integer number.

Calculate λ for the solution.

$$2(0.5\pi n) = \lambda \cos(0.5\pi n + 0.5\pi n)$$

$$\pi n = \lambda \cos(\pi n)$$

Which gives

$$\lambda = \pm \pi n, \text{ where } n \text{ is any integer}$$

$$\lambda = \pi n, \text{ where } n \text{ is an even integer}$$

$$\lambda = -\pi n, \text{ where } n \text{ is an odd integer}$$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

where $L(x, y) = f(x, y) - \lambda g(x, y)$.

$$L(x, y) = f(x, y) - (\pm \pi n)g(x, y)$$

$$L(x, y) = x^2 + y^2 + 3 - (\pm \pi n)\sin(x + y)$$

Find second order partial derivatives.

$$L_{xx} = 2 + (\pm \pi n)\sin(x + y)$$

$$L_{yy} = 2 + (\pm \pi n)\sin(x + y)$$

$$L_{xy} = L_{yx} = (\pm \pi n)\sin(x + y)$$

Substitute $(0.5\pi n, 0.5\pi n)$.

$g_x(0.5\pi n, 0.5\pi n) = \cos(0.5\pi n + 0.5\pi n) = \pm 1$(the same sign as λ)

$g_y(0.5\pi n, 0.5\pi n) = \cos(0.5\pi n + 0.5\pi n) = \pm 1$(the same sign as λ)

$$L_{xx}(0.5\pi n, 0.5\pi n) = 2 + (\pm \pi n)\sin(0.5\pi n + 0.5\pi n) = 2$$

$$L_{yy}(0.5\pi n, 0.5\pi n) = 2 + (\pm \pi n)\sin(0.5\pi n + 0.5\pi n) = 2$$

$$L_{xy}(0.5\pi n, 0.5\pi n) = L_{xy}(0.5\pi n, 0.5\pi n) = (\pm \pi n)\sin(0.5\pi n + 0.5\pi n) = 0$$

Form the bordered Hessian matrix for even n.

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{vmatrix}$$

Then $det(HL) = -4 < 0$. Form the bordered Hessian matrix for even n.

$$\begin{vmatrix} 0 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{vmatrix}$$

Then $det(HL) = -4 < 0$. Since $det(HL) < 0$ at $(0.5\pi n, 0.5\pi n)$, all of them are local minima, and

$$f(0.5\pi n, 0.5\pi n) = (0.5\pi n)^2 + (0.5\pi n)^2 + 3 = \frac{\pi^2 n^2}{2} + 3$$

where n is any integer number.

So the extremum of the function is

A local minima of $\frac{\pi^2 n^2}{2} + 3$ at $(0.5\pi n, 0.5\pi n)$, where n is any integer number.

■ 4. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y) = \ln \frac{2x - 4}{y^2}$$

$$4x + 8y - 15 = 0$$

Solution:

Expand the logarithm to simplify $f(x, y)$.

$$f(x, y) = \ln(2x - 4) - 2 \ln y$$

Let

$$g(x, y) = 4x + 8y - 15$$

Create the system of equations.

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x}$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y}$$

Find first order partial derivatives.

$$\frac{\partial f}{\partial x} = \frac{2}{2x - 4} = \frac{1}{x - 2} \qquad \frac{\partial g}{\partial x} = 4$$

$$\frac{\partial f}{\partial y} = -\frac{2}{y} \qquad \frac{\partial g}{\partial y} = 8$$

Substitute partial derivatives into the system of equations.

$$\frac{1}{x - 2} = 4\lambda$$

$$-\frac{2}{y} = 8\lambda$$

Solve the system for λ.

$$\frac{1}{4(x - 2)} = \lambda$$

$$-\frac{1}{4y} = \lambda$$

$$\frac{1}{4(x-2)} = -\frac{1}{4y}$$

$$\frac{1}{x-2} = -\frac{1}{y}$$

$$-y = x - 2$$

$y = -x + 2$ for $x \neq 2$ and $y \neq 0$

Plug y into the constraint equation.

$$4x + 8(-x + 2) - 15 = 0$$

$$-4x + 16 - 15 = 0$$

$$x = \frac{1}{4}$$

$$y = -\frac{1}{4} + 2 = \frac{7}{4}$$

So the solution is $x = 1/4$, $y = 7/4$.

Check if the point lies within the domain of $f(x, y)$.

$$\frac{2(0.25) - 4}{(1.75)^2} = -\frac{8}{7} < 0$$

Since there's no critical point inside the domain of $f(x, y)$, it has no extrema.

THREE DIMENSIONS, ONE CONSTRAINT

1. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x, y, z) = x^5 - 160y + 160z$$

$$x + y^2 + z^2 = 0$$

Solution:

Let $g(x, y, z) = x + y^2 + z^2$, and create a system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = 5x^4 \qquad g_x = 1$$

$$f_y = -160 \qquad g_y = 2y$$

$$f_z = 160 \qquad g_z = 2z$$

Substitute partial derivatives into the system.

$$5x^4 = \lambda$$

$$-160 = 2y\lambda$$

$$160 = 2z\lambda$$

Solve the system for λ.

$$5x^4 = \lambda$$

$$-80 = y\lambda$$

$$80 = z\lambda$$

So $x \neq 0$.

$$y = -\frac{80}{\lambda} = -\frac{80}{5x^4} = -\frac{16}{x^4}$$

$$z = \frac{80}{\lambda} = \frac{80}{5x^4} = \frac{16}{x^4}$$

Plug these into the constraint equation.

$$x + \left(-\frac{16}{x^4}\right)^2 + \left(\frac{16}{x^4}\right)^2 = 0$$

$$x + 2 \cdot \frac{256}{x^8} = 0$$

$$x^9 = -512$$

$$x = -2$$

Calculate y, z, and λ.

$$y = -\frac{16}{x^4} = -\frac{16}{(-2)^4} = -1$$

$$z = \frac{16}{x^4} = \frac{16}{(-2)^4} = 1$$

$$\lambda = 5x^4 = 5(-2)^4 = 80$$

So the solution to the system is $(-2, -1, 1)$ with $\lambda = 80$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda\, g(x, y, z)$. The second derivative test for $(-2, -1, 1)$ with $\lambda = 80$.

$$L(x, y, z) = f(x, y, z) - 80g(x, y, z)$$

$$L(x, y, z) = x^5 - 160y + 160z - 80(x + y^2 + z^2)$$

Calculate the second order partial derivatives and substitute $(-2, -1, 1)$.

$$L_{xx} = 20(-2)^3 = -160,\ L_{yy} = -160,\ L_{zz} = -160$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 1,\ g_y = 2(-1) = -2,\ g_z = 2(1) = 2$$

Form the bordered Hessian matrix H_4.

$$\begin{vmatrix} 0 & -1 & 2 & -2 \\ -1 & -160 & 0 & 0 \\ 2 & 0 & -160 & 0 \\ -2 & 0 & 0 & -160 \end{vmatrix}$$

Then $det(H_4) = -230{,}400 < 0$. Since $det(H_4) < 0$ at $(-2,-1,1)$, it's a local extremum.

$$f(-2,-1,1) = (-2)^5 - 160(-1) + 160(1) = 288$$

To check if it's local minimum or maximum, form the Hessian submatrix H_3.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

$$\begin{vmatrix} 0 & -1 & 2 \\ -1 & -160 & 0 \\ 2 & 0 & -160 \end{vmatrix}$$

Then $det(H_3) = 800 > 0$. Since $det(H_3) > 0$ at $(-2,-1,1)$, it's a local maximum.

So the function has a local maximum of 288 at $(-2,-1,1)$.

■ 2. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x,y,z) = x + 2y^2 - 3z^2 - 4$$

$$e^x + y - 3z = -\frac{1}{4}$$

Solution:

Let $g(x, y, z) = e^x + y - 3z + 1/4$, and create a system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = 1 \qquad g_x = e^x$$

$$f_y = 4y \qquad g_y = 1$$

$$f_z = -6z \qquad g_z = -3$$

Substitute partial derivatives into the system of equations.

$$1 = e^x\lambda$$

$$4y = \lambda$$

$$-6z = -3\lambda$$

Solve the system.

$$6z = 3\lambda = 3(4y)$$

$$z = 2y$$

$$1 = e^x\lambda = 4ye^x$$

$$\frac{1}{4e^x} = y$$

$$z = \frac{1}{2e^x}$$

Plug these values for y and z into the constraint equation.

$$e^x + \frac{1}{4e^x} - 3\frac{1}{2e^x} + \frac{1}{4} = 0$$

$$e^x - \frac{5}{4e^x} + \frac{1}{4} = 0$$

$$4(e^x)^2 + e^x - 5 = 0$$

$$(e^x - 1)(4e^x + 5) = 0$$

Since $4e^x + 5 > 0$, the only solution is

$$e^x - 1 = 0$$

$$e^x = 1$$

$$x = 0$$

Calculate y, x, λ.

$$y = \frac{1}{4e^0} = 0.25$$

$$z = \frac{1}{2e^0} = 0.5$$

$$\lambda = 4y = 4(0.25) = 1$$

So the solution is $(0,0.25,0.5)$ with $\lambda = 1$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z)$.

$$L(x, y, z) = f(x, y, z) - g(x, y, z)$$

$$L(x, y, z) = x + 2y^2 - 3z^2 - 4 - (e^x + y - 3z + 0.25)$$

$$L(x, y, z) = x + 2y^2 - 3z^2 - e^x - y + 3z - 4.25)$$

Calculate the second order partial derivatives and substitute $(0,0.25,0.5)$.

$$L_{xx} = -e^x = -e^0 = -1, \ L_{yy} = 4, \ L_{zz} = -6$$

$$L_{xy} = L_{yx} = 0, \ L_{yz} = L_{zy} = 0, \ L_{xz} = L_{zx} = 0$$

$$g_x = e^0 = 1, \ g_y = 1, \ g_z = -3$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -1 & -1 & 3 \\ -1 & -1 & 0 & 0 \\ -1 & 0 & 4 & 0 \\ 3 & 0 & 0 & -6 \end{vmatrix}$$

Then $det(HL) = 54 > 0$. Since $det(HL) > 0$ at $(0,0.5,0.25)$, it's a saddle point.

So the function has no extrema.

3. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x, y, z) = |x^3y^7z^5|$$

$$3x + 7y + 5z = 60$$

Solution:

Since $f(x, y, z) \geq 0$, and $f(x, y, z) = 0$ when $x = 0$, or $y = 0$, or $z = 0$, all these points are global minima by the definition.

Consider the points where $x \neq 0$, $y \neq 0$, and $z \neq 0$. To simplify the calculations, introduce the helper function

$$F(x, y, z) = \ln(f(x, y, z)) = \ln|x^3y^7z^5|$$

$$F(x, y, z) = 3\ln|x| + 7\ln|y| + 5\ln|z|$$

Since the logarithm is a monotonic increasing function, $\ln f$ has the local extrema at the same points as the initial function f.

Let $g(x, y, z) = 3x + 7y + 5z - 60$, then create the system of simultaneous equations.

$$F_x = \lambda g_x$$

$$F_y = \lambda g_y$$

$$F_z = \lambda g_z$$

Calculate first order partial derivatives.

$$F_x = \frac{3}{x} \qquad g_x = 3$$

$$F_y = \frac{7}{y} \qquad g_y = 7$$

$$F_z = \frac{5}{z} \qquad g_z = 5$$

Substitute partial derivatives into the system of equations.

$$\frac{3}{x} = 3\lambda$$

$$\frac{7}{y} = 7\lambda$$

$$\frac{5}{z} = 5\lambda$$

Solve the system for λ.

$$\frac{1}{x} = \lambda$$

$$\frac{1}{y} = \lambda$$

$$\frac{1}{z} = \lambda$$

$$x = y = z$$

Plug in $y = x$ and $z = x$ into the constraint equation.

$$3x + 7(x) + 5(x) - 60 = 0$$

$$15x - 60 = 0$$

$$x = 4$$

Calculate y, x, and λ.

$$y = z = 4$$

$$\frac{1}{(4)} = \lambda$$

$$\lambda = 0.25$$

The solution to the system is $(4,4,4)$ with $\lambda = 0.25$.

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = F(x, y, z) - \lambda g(x, y, z)$. The second derivative test for the point $(4,4,4)$ with $\lambda = 0.25$ is

$$L(x, y, z) = F(x, y, z) - 0.25g(x, y, z)$$

$$L(x, y, z) = 3\ln|x| + 7\ln|y| + 5\ln|z| - 0.25(3x + 7y + 5z - 60)$$

Find second order partial derivatives at $(4,4,4)$.

$$L_{xx} = -\frac{3}{(4)^2} = -\frac{3}{16},\ L_{yy} = -\frac{7}{(4)^2} = -\frac{7}{16},\ L_{zz} = -\frac{5}{(4)^2} = -\frac{5}{16}$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 3,\ g_y = 7,\ g_z = 5$$

Form the bordered Hessian matrix H_4.

$$\begin{vmatrix} 0 & -3 & -7 & -5 \\ -3 & -\frac{3}{16} & 0 & 0 \\ -7 & 0 & -\frac{7}{16} & 0 \\ -5 & 0 & 0 & -\frac{5}{16} \end{vmatrix}$$

Then $det(H_4) \approx -6.15 < 0$. Since $det(H_4) < 0$ at $(4,4,4)$, it's a local extremum, and

$$f(4,4,4) = |(4)^3(4)^5(4)^7| = 4^{15} = 1,073,741,824$$

To check if it's a local minimum or maximum, form the Hessian submatrix H_3.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

$$\begin{vmatrix} 0 & -3 & -7 \\ -3 & -\frac{3}{16} & 0 \\ -7 & 0 & -\frac{7}{16} \end{vmatrix}$$

Then $det(H_3) = 13.125 > 0$. Since $det(H_3) > 0$ at $(4,4,4)$, it's a local maximum.

The extrema of the function are

A local maximum of $1,073,741,824$ at $(4,4,4)$

Local (and also global) minima of all points where $x = 0$, or $y = 0$, or $z = 0$.

■ 4. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x, y, z) = x + yz + 2y$$

$$y^2 + xyz = 2$$

Solution:

Let $g(x, y, z) = y^2 + xyz - 2$, then create a system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = 1 \qquad g_x = yz$$

$$f_y = z + 2 \qquad g_y = 2y + xz$$

$$f_z = y \qquad g_z = xy$$

Substitute partial derivatives into the system.

$$1 = yz\lambda$$

$$z + 2 = (2y + xz)\lambda$$

$$y = xy\lambda$$

Solve the system for λ. Since $y \neq 0$ (or else $1 = (0)z\lambda$, which is impossible),

$$1 = x\lambda$$

$$\frac{1}{x} = \lambda$$

Plug this into the second equation.

$$z + 2 = (2y + xz)\frac{1}{x}$$

$$x(z + 2) = 2y + xz$$

$$2x = 2y$$

$$x = y$$

Substitute into the first equation.

$$1 = yz\frac{1}{x}$$

$$x = yz$$

Since $x = y$ and $x = yz$, then $1 = z$. Plug in $z = 1$ and $y = x$ into the constraint equation.

$$(x)^2 + x(x)(1) = 2$$

$$2x^2 = 2$$

$$x^2 = 1$$

$$x = \pm 1$$

Calculate y and λ.

$$y = x = \pm 1$$

$$\lambda = \frac{1}{\pm 1} = \pm 1$$

The solutions are

$(1,1,1)$ with $\lambda = 1$

$(-1,-1,1)$ with $\lambda = -1$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x,y,z) = f(x,y,z) - \lambda g(x,y,z)$. The second derivative test for $(1,1,1)$ with $\lambda = 1$:

$$L(x,y,z) = f(x,y,z) - 1 \cdot g(x,y,z)$$

$$L(x,y,z) = x + yz + 2y - (y^2 + xyz - 2)$$

$$L(x,y,z) = x + yz + 2y - y^2 - xyz + 2$$

Find second order partial derivatives at $(1,1,1)$.

$$L_{xx} = 0,\ L_{yy} = -2,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = -z = -1,\ L_{yz} = L_{zy} = 1 - x = 0,\ L_{xz} = L_{zx} = -y = -1$$

$$g_x = yz = (1)(1) = 1,\ g_y = 2y + xz = 2(1) + (1)(1) = 3,\ g_z = xy = (1)(1) = 1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -1 & -3 & -1 \\ -1 & 0 & -1 & -1 \\ -3 & -1 & -2 & 0 \\ -1 & -1 & 0 & 0 \end{vmatrix}$$

Then $det(HL) = 8 > 0$. Since $det(HL) > 0$ at $(1,1,1)$, it's a saddle point.

The second derivative test for $(-1, -1,1)$ with $\lambda = -1$.

$$L(x, y, z) = f(x, y, z) + 1 \cdot g(x, y, z)$$

$$L(x, y, z) = x + yz + 2y + (y^2 + xyz - 2)$$

$$L(x, y, z) = x + yz + 2y + y^2 + xyz - 2$$

Find second order partial derivatives at $(-1, -1,1)$.

$$L_{xx} = 0,\ L_{yy} = 2,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = z = 1,\ L_{yz} = L_{zy} = x + 1 = 0,\ L_{xz} = L_{zx} = y = -1$$

$$g_x = yz = (-1)(1) = -1$$

$$g_y = 2y + xz = 2(-1) + (-1)(1) = -3$$

$$g_z = xy = (-1)(-1) = 1$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 1 & 3 & -1 \\ 1 & 0 & 1 & -1 \\ 3 & 1 & 2 & 0 \\ -1 & -1 & 0 & 0 \end{vmatrix}$$

Then $det(HL) = 8 > 0$. Since $det(HL) > 0$ at $(-1, -1, 1)$, it's a saddle point. So the function has no extrema.

■ 5. Use Lagrange multipliers to find the extrema of the function, subject to the given constraint.

$$f(x, y, z) = \ln \frac{xy + 3y}{z - 2}$$

$$x + y + 3z = 6$$

Solution:

Rewrite $f(x, y, z)$ as

$$f(x, y, z) = \ln \frac{(x + 3)y}{z - 2}$$

$$f(x, y, z) = \ln(x + 3) + \ln y - \ln(z - 2)$$

And let

$$g(x, y, z) = x + y + 3z - 6 = 0$$

Create the system of equations.

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$f_z = \lambda g_z$$

Find first order partial derivatives.

$$f_x = \frac{1}{x+3} \qquad g_x = 1$$

$$f_y = \frac{1}{y} \qquad g_y = 1$$

$$f_z = -\frac{1}{z-2} \qquad g_z = 3$$

Substitute partial derivatives into the system of equations.

$$\frac{1}{x+3} = \lambda$$

$$\frac{1}{y} = \lambda$$

$$-\frac{1}{z-2} = 3\lambda$$

Solve the system for λ.

$$y = \frac{1}{\lambda}$$

$$x + 3 = \frac{1}{\lambda}, \ x = \frac{1}{\lambda} - 3 = y - 3$$

$$z - 2 = -\frac{1}{3\lambda}, \ z = -\frac{1}{3\lambda} + 2 = -\frac{1}{3}y + 2$$

Plug these values for x and z into the constraint equation.

$$y-3+y+3\left(-\frac{1}{3}y+2\right)=6$$

$$2y-3-y+6=6$$

$$y-3=0$$

$$y=3$$

Then

$$x=3-3=0$$

$$z=-\frac{1}{3}\cdot 3+2=1$$

The solution to the system is $x=0$, $y=3$, and $z=1$. Check to see if the point lies within the domain of $f(x,y,z)$.

$$\frac{0\cdot 3+3\cdot 3}{1-2}=-9<0$$

Since there is no critical point inside the domain of $f(x,y,z)$, it has no extrema.

■ 6. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraint.

$$f(x,y,z)=\sin^2 x\cdot\sin 2y\cdot\sin z$$

$$2x + 2y + z = \frac{2}{3}\pi \text{ with } x > 0,\ y > 0,\ z > 0$$

Solution:

To simplify the calculations, introduce the helper function.

$$F(x, y, z) = \ln(f(x, y, z))$$

$$F(x, y, z) = \ln(\sin^2 x \cdot \sin 2y \cdot \sin z)$$

$$F(x, y, z) = 2\ln(\sin x) + \ln(\sin 2y) + \ln(\sin z)$$

Since the logarithm is a monotonic increasing function, $\ln f$ has the local extrema at the same points as the initial function f.

And let

$$g(x, y, z) = 2x + 2y + z - \frac{2}{3}\pi$$

Build a system of equations.

$$F_x = \lambda g_x$$

$$F_y = \lambda g_y$$

$$F_z = \lambda g_z$$

Find first order partial derivatives.

$$F_x = 2\frac{\cos x}{\sin x} = 2\cot x \qquad g_x = 2$$

$$F_y = 2\frac{\cos 2y}{\sin 2y} = 2\cot 2y \qquad g_y = 2$$

$$F_z = \frac{\cos zx}{\sin z} = \cot z \qquad g_z = 1$$

Substitute partial derivatives into the system.

$$2\cot x = 2\lambda$$

$$2\cot 2y = 2\lambda$$

$$\cot z = \lambda$$

Solve the system for λ.

$$\cot x = \lambda$$

$$\cot 2y = \lambda$$

$$\cot z = \lambda$$

So $\cot x = \cot 2y = \cot z$. Since $x > 0$, $y > 0$, $z > 0$, and $2x + 2y + z = (2/3)\pi$.

$$0 < x < \frac{2}{3}\pi$$

$$0 < 2y < \frac{2}{3}\pi$$

$$0 < z < \frac{2}{3}\pi$$

Since $\cot x$ is a one to one function on $(0,(2/3)\pi)$ if $\cot x = \cot 2y = \cot z$, then $x = 2y = z$. Plug $x = 2y$ and $z = 2y$ into the constraint equation.

$$2(2y) + 2y + 2y = \frac{2}{3}\pi$$

$$8y = \frac{2}{3}\pi$$

$$y = \frac{\pi}{12}$$

Calculate x, z, λ.

$$x = z = 2 \cdot \frac{\pi}{12} = \frac{\pi}{6}$$

$$\lambda = \cot\frac{\pi}{6} = \sqrt{3}$$

The solution to the system is

$$\left(\frac{\pi}{6}, \frac{\pi}{12}, \frac{\pi}{6}\right) \text{ with } \lambda = \sqrt{3}$$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & -g_x & -g_y & -g_z \\ -g_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = F(x, y, z) - \lambda g(x, y, z)$. The second derivative test for $(\pi/6, \pi/12, \pi/6)$ with $\lambda = \sqrt{3}$.

$$L(x, y, z) = F(x, y, z) - \sqrt{3}g(x, y, z)$$

$$L(x, y, z) = 2\ln(\sin x) + \ln(\sin 2y) + \ln(\sin z) - \sqrt{3}\left(2x + 2y + z - \frac{2}{3}\pi\right)$$

Find second order partial derivatives at $(\pi/6,\pi/12,\pi/6)$.

$$L_{xx} = -2\csc^2\frac{\pi}{6} = -8$$

$$L_{yy} = -4\csc^2\left(2\cdot\frac{\pi}{12}\right) = -16$$

$$L_{zz} = -\csc^2\frac{\pi}{6} = -4$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 2,\ g_y = 2,\ g_z = 1$$

Form the bordered Hessian matrix H_4.

$$\begin{vmatrix} 0 & -2 & -2 & -1 \\ -2 & -8 & 0 & 0 \\ -2 & 0 & -16 & 0 \\ -1 & 0 & 0 & -4 \end{vmatrix}$$

Then $det(H_4) = -512 < 0$. Since $det(H_4) < 0$ at $(\pi/6,\pi/12,\pi/6)$, it's a local extremum,

$$f\left(\frac{\pi}{6},\frac{\pi}{12},\frac{\pi}{6}\right) = \sin^2\frac{\pi}{6}\cdot\sin\left(2\cdot\frac{\pi}{12}\right)\cdot\sin\frac{\pi}{6}$$

$$f\left(\frac{\pi}{6},\frac{\pi}{12},\frac{\pi}{6}\right) = \sin^4\frac{\pi}{6}$$

$$f\left(\frac{\pi}{6},\frac{\pi}{12},\frac{\pi}{6}\right)=\frac{1}{16}$$

To check if it's a local minimum or maximum, form the Hessian submatrix H_3.

$$\begin{vmatrix} 0 & -g_x & -g_y \\ -g_x & L_{xx} & L_{xy} \\ -g_y & L_{yx} & L_{yy} \end{vmatrix}$$

$$\begin{vmatrix} 0 & -2 & -2 \\ -2 & -8 & 0 \\ -2 & 0 & -16 \end{vmatrix}$$

Then $det(H_3)=96>0$. Since $det(H_3)>0$ at $(\pi/6,\pi/12,\pi/6)$, it's a local maximum. So the function has a local maximum of $1/16$ at $(\pi/6,\pi/12,\pi/6)$.

THREE DIMENSIONS, TWO CONSTRAINTS

■ 1. Use Lagrange multipliers to find the shortest distance from the vertex of the elliptic paraboloid $(x-2)^2+2(y+1)^2=3z+6$ to the line that's the intersection of the planes $x+3y+5z=18$ and $3x+5y+z=28$.

Solution:

The vertex of the elliptic paraboloid $(x-2)^2+2(y+1)^2=3(z+2)$ is the point $(2,-1,-2)$.

The square of the distance has a minimum at the same point as the distance itself. So let's minimize the square of the distance from $(2,-1,-2)$. The function to be minimized is $f(x,y,z)=(x-2)^2+(y+1)^2+(z+2)^2$.

Let $g(x,y,z)=x+3y+5z-18$ and $h(x,y,z)=3x+5y+z-28$, then create the system of equations.

$$f_x=\lambda g_x+\mu h_x$$

$$f_y=\lambda g_y+\mu h_y$$

$$f_z=\lambda g_z+\mu h_z$$

To build the system, we'll need first order partial derivatives of f, g, and h.

$f_x=2(x-2)$ $\qquad$ $g_x=1$ $\qquad$ $h_x=3$

$f_y = 2(y+1)$ $\qquad$ $g_y = 3$ $\qquad$ $h_y = 5$

$f_z = 2(z+2)$ $\qquad$ $g_z = 5$ $\qquad$ $h_z = 1$

Plug into the system.

$$2(x-2) = \lambda + 3\mu$$

$$2(y+1) = 3\lambda + 5\mu$$

$$2(z+2) = 5\lambda + \mu$$

Solve the system for x, y, and z.

$$x = \frac{\lambda + 3\mu}{2} + 2$$

$$y = \frac{3\lambda + 5\mu}{2} - 1$$

$$z = \frac{5\lambda + \mu}{2} - 2$$

Plug these into the constraint equations.

$$\frac{\lambda + 3\mu}{2} + 2 + 3\left(\frac{3\lambda + 5\mu}{2} - 1\right) + 5\left(\frac{5\lambda + \mu}{2} - 2\right) = 18$$

$$3\left(\frac{\lambda + 3\mu}{2} + 2\right) + 5\left(\frac{3\lambda + 5\mu}{2} - 1\right) + \frac{5\lambda + \mu}{2} - 2 = 28$$

Simplify both equations to get

$$35\lambda + 23\mu = 58$$

$$23\lambda + 35\mu = 58$$

The solution of the system is $\lambda = 1$, $\mu = 1$, so plug these back into the equations for x, y, and z,

$$x = \frac{1 + 3 \cdot 1}{2} + 2 = 4$$

$$y = \frac{3 \cdot 1 + 5 \cdot 1}{2} - 1 = 3$$

$$z = \frac{5 \cdot 1 + 1}{2} - 2 = 1$$

and the plug these into the equation for $f(x, y, z)$.

$$f(4,3,1) = (4 - 2)^2 + (3 + 1)^2 + (1 + 2)^2 = 29$$

Since the distance function from the point to the line cloud have only one minimum, the critical point $(4,3,1)$ is the global minimum, and the distance is

$$d = \sqrt{29}$$

■ 2. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = x^2 + 2y - 3z^2$$

$4x - y = 0$ and $y + 8z = 0$

Solution:

Let $g(x, y, z) = 4x - y$ and $h(x, y, z) = y + 8z$, then create the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

To build the system, we'll need first order partial derivatives of f, g, and h.

$$f_x = 2x \qquad g_x = 4 \qquad h_x = 0$$

$$f_y = 2 \qquad g_y = -1 \qquad h_y = 1$$

$$f_z = -6z \qquad g_z = 0 \qquad h_z = 8$$

Plug into the system.

$$2x = 4\lambda$$

$$2 = -\lambda + \mu$$

$$-6z = 8\mu$$

So

$$x = 2\lambda$$

$$\lambda = \mu - 2$$

$$z = -\frac{4}{3}\mu$$

Plug these into the constraint equations.

$$4(2\mu - 4) - y = 0$$

$$y + 8\left(-\frac{4}{3}\mu\right) = 0$$

Simplify both equations to get

$$8\mu - 16 = y$$

$$y = \frac{32}{3}\mu$$

So

$$8\mu - 16 = \frac{32}{3}\mu$$

$$\mu = -6$$

And then $\lambda = -6 - 2 = -8$. So plug these back into the equations for x, y, and z.

$$x = 2(-8) = -16$$

$$y = 8 \cdot (-6) - 16 = -64$$

$$z = -\frac{4}{3}(-6) = -8$$

The the solution to the system is $(-16, -64, -8)$, $\lambda = -8$, and $\mu = -6$.

Perform the second derivative test for constrained extrema. The bordered Hessian matrix is

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$. The second derivative test for the solution to the system is

$$L(x, y, z) = f(x, y, z) + 8 \cdot g(x, y, z) + 6 \cdot h(x, y, z)$$

$$L(x, y, z) = x^2 + 2y - 3z^2 + 8(4x - y) + 6(y + 8z)$$

$$L(x, y, z) = x^2 + 32x - 3z^2 + 48z$$

Find second order partial derivatives and evaluate them at $(-16, -64, -8)$.

$$L_{xx} = 2$$

$$L_{yy} = 0$$

$$L_{zz} = -6$$

$$L_{xy} = L_{yx} = 0$$

$$L_{yz} = L_{zy} = 0$$

$$L_{xz} = L_{zx} = 0$$

And we already have $g_x = 4$, $g_y = -1$, $g_z = 0$ and $h_x = 0$, $h_y = 1$, and $h_z = 8$. So the Hessian matrix becomes

$$\begin{vmatrix} 0 & 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & -1 & -8 \\ -4 & 0 & 2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & -6 \end{vmatrix}$$

The determinant is $det(HL) = 32 > 0$. Since $det(HL) > 0$ at $(-16, -64, -8)$, this point is a local minimum.

So the function has a local minimum of -8 at $(-16, -64, -8)$.

■ 3. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = z$$

$4x + 2y + 3z = -2$ and $3x^2 + y^2 - z^2 = 5$

Solution:

Let $g(x, y, z) = 4x + 2y + 3z + 2$ and $h(x, y, z) = 3x^2 + y^2 - z^2 - 5$, then create the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

To build the system, we'll need first order partial derivatives of f, g, and h.

$$f_x = 0 \qquad g_x = 4 \qquad h_x = 6x$$

$$f_y = 0 \qquad g_y = 2 \qquad h_y = 2y$$

$$f_z = 1 \qquad g_z = 3 \qquad h_z = -2z$$

Plug into the system.

$$0 = 4\lambda + 6x\mu$$

$$0 = 2\lambda + 2y\mu$$

$$1 = 3\lambda - 2z\mu$$

Solve the system for x, y, and z.

$$3x\mu = -2\lambda$$

$$y\mu = -\lambda$$

$$2z\mu = 3\lambda - 1$$

Since $\mu \neq 0$ (or else $\lambda = 0$ and $1 = 0 - 0$, which is impossible), we can say

$$x = -\frac{2\lambda}{3\mu}$$

$$y = -\frac{\lambda}{\mu}$$

$$z = \frac{3\lambda - 1}{2\mu}$$

Plug these into the constraint equations.

$$4\left(-\frac{2\lambda}{3\mu}\right) + 2\left(-\frac{\lambda}{\mu}\right) + 3\left(\frac{3\lambda - 1}{2\mu}\right) = -2$$

$$3\left(-\frac{2\lambda}{3\mu}\right)^2 + \left(-\frac{\lambda}{\mu}\right)^2 - \left(\frac{3\lambda - 1}{2\mu}\right)^2 = 5$$

Simplify both equations to get

$$\lambda = 12\mu - 9$$

$$\lambda^2 + 18\lambda - 3 = 60\mu^2$$

Plug $\lambda = 12\mu - 9$ into the second constraint equation.

$$(12\mu - 9)^2 + 18(12\mu - 9) - 3 = 60\mu^2$$

$$84\mu^2 - 84 = 0$$

So $\mu = 1$ or $\mu = -1$. Calculate λ, x, y, z for $\mu = 1$:

$$\lambda = 12 \cdot 1 - 9 = 3$$

$$x = -\frac{2 \cdot 3}{3 \cdot 1} = -2$$

$$y = -\frac{3}{1} = -3$$

$$z = \frac{3 \cdot 3 - 1}{2 \cdot 1} = 4$$

Calculate λ, x, y, z for $\mu = -1$:

$$\lambda = 12 \cdot (-1) - 9 = -21$$

$$x = -\frac{2 \cdot (-21)}{3 \cdot (-1)} = -14$$

$$y = -\frac{(-21)}{(-1)} = -21$$

$$z = \frac{3 \cdot (-21) - 1}{2 \cdot (-1)} = 32$$

The solutions to the system are

$(-2, -3, 4)$ with $\lambda = 3$ and $\mu = 1$

$(-14, -21, 32)$ with $\lambda = -21$ and $\mu = -1$

Perform the second derivative test for constrained extrema using the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$.

Find L at $(-2, -3, 4)$ with $\lambda = 3$ and $\mu = 1$.

$$L(x, y, z) = f(x, y, z) - 3 \cdot g(x, y, z) - 1 \cdot h(x, y, z)$$

$$L(x, y, z) = z - 3(4x + 2y + 3z + 2) - (3x^2 + y^2 - z^2 - 5)$$

$$L(x, y, z) = -3x^2 - y^2 + z^2 - 12x - 6y - 8z - 1$$

Find second order partial derivatives and evaluate at $(-2, -3,4)$.

$$L_{xx} = -6$$

$$L_{yy} = -2$$

$$L_{zz} = 2$$

$$L_{xy} = L_{yx} = 0$$

$$L_{yz} = L_{zy} = 0$$

$$L_{xz} = L_{zx} = 0$$

And we already have $g_x = 4$, $g_y = 2$, $g_z = 3$ and $h_x = -12$, $h_y = -6$, and $h_z = -8$. So the Hessian matrix becomes

$$\begin{vmatrix} 0 & 0 & -4 & -2 & -3 \\ 0 & 0 & 12 & 6 & 8 \\ -4 & 12 & -6 & 0 & 0 \\ -2 & 6 & 0 & -2 & 0 \\ -3 & 8 & 0 & 0 & 2 \end{vmatrix}$$

The determinant is $det(HL) = -56 < 0$. Since $det(HL) < 0$ at $(-2, -3,4)$, this point is a local maximum. So the function has a local maximum of 4 at $(-2, -3,4)$.

Find L at $(-14,-21,32)$ with $\lambda=-21$ and $\mu=-1$.

$$L(x,y,z)=f(x,y,z)+21\cdot g(x,y,z)+1\cdot h(x,y,z)$$

$$L(x,y,z)=z+21(4x+2y+3z+2)+(3x^2+y^2-z^2-5)$$

$$L(x,y,z)=3x^2+y^2-z^2+84x+42y+64z+37$$

Find second order partial derivatives and evaluate at $(-14,-21,32)$.

$$L_{xx}=6$$

$$L_{yy}=2$$

$$L_{zz}=-2$$

$$L_{xy}=L_{yx}=0$$

$$L_{yz}=L_{zy}=0$$

$$L_{xz}=L_{zx}=0$$

And we already have $g_x=4$, $g_y=2$, $g_z=3$ and $h_x=-84$, $h_y=126$, and $h_z=-256$. So the Hessian matrix becomes

$$\begin{vmatrix} 0 & 0 & -4 & -2 & -3 \\ 0 & 0 & 84 & -126 & 256 \\ -4 & 84 & 6 & 0 & 0 \\ -2 & -126 & 0 & 2 & 0 \\ -3 & 256 & 0 & 0 & -2 \end{vmatrix}$$

The determinant is $det(HL) = 5{,}041{,}400 > 0$. Since $det(HL) > 0$ at $(-14, -21, 32)$, this point is a local minimum. So the function has a local maximum of 32 at $(-14, -21, 32)$.

So the extrema of the function are

A local maximum of 4 at $(-2, -3, 4)$

A local minimum of 32 at $(-14, -21, 32)$

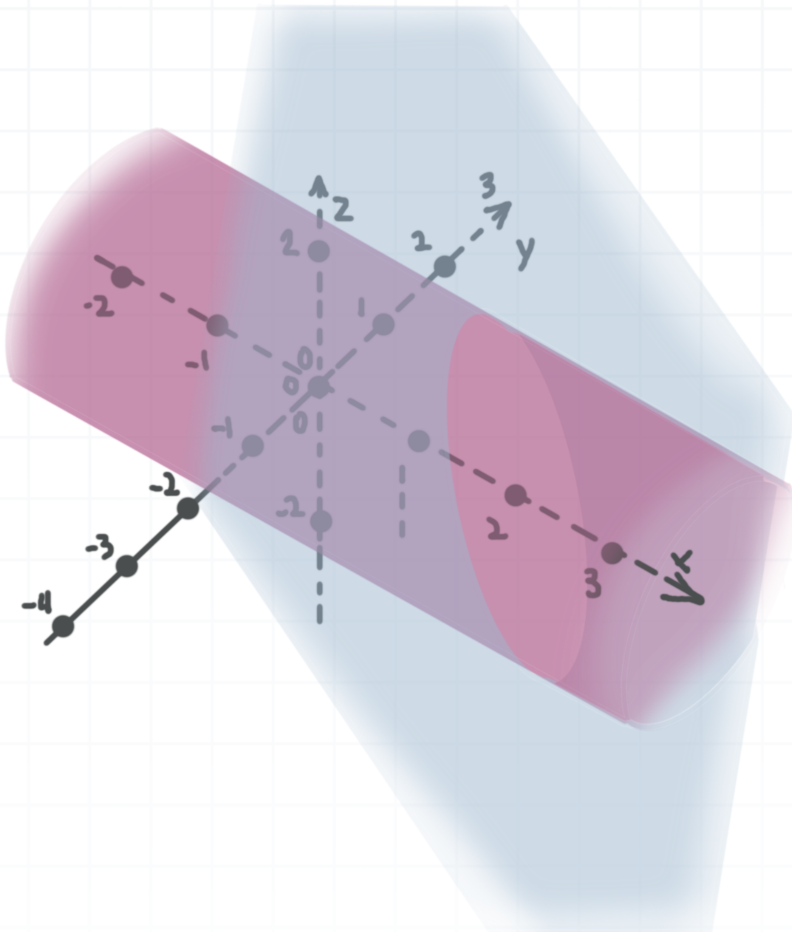
■ 4. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = 2\ln x + z$$

$2x + y + z = 4$ and $y^2 + z^2 = 2$

Solution:

A sketch of the surface is



Let $g(x, y, z) = 2x + y + z - 4$ and $h(x, y, z) = y^2 + z^2 - 2$. Build the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

Calculate first order partial derivatives.

$$f_x = \frac{2}{x} \qquad g_x = 2 \qquad h_x = 0$$

$$f_y = 0 \qquad g_y = 1 \qquad h_y = 2y$$

$$f_z = 1 \qquad g_z = 1 \qquad h_z = 2z$$

Substitute the partial derivatives into the system of equations.

$$\frac{2}{x} = 2\lambda$$

$$0 = \lambda + 2y\mu$$

$$1 = \lambda + 2z\mu$$

Solve the system for x, y, and z.

Since $\lambda \neq 0$ (or else $1/x = 0$, which is impossible) and $\mu \neq 0$ (or else $\lambda = 0$ and $\lambda = 1$, which is impossible),

$$x = \frac{1}{\lambda}$$

$$y = \frac{-\lambda}{2\mu}$$

$$z = \frac{1-\lambda}{2\mu}$$

Plug these into the constraint equations.

$$2\left(\frac{1}{\lambda}\right) + \left(\frac{-\lambda}{2\mu}\right) + \left(\frac{1-\lambda}{2\mu}\right) = 4$$

$$\left(\frac{-\lambda}{2\mu}\right)^2 + \left(\frac{1-\lambda}{2\mu}\right)^2 = 2$$

Simplify the system.

$$-2\lambda^2 + \lambda + 4\mu - 8\lambda\mu = 0$$

$$2\lambda^2 - 2\lambda + 1 - 8\mu^2 = 0$$

Factor the first equation to get

$$(\lambda + 4\mu)(2\lambda - 1) = 0$$

$\lambda = -4\mu$ or $\lambda = \dfrac{1}{2}$

First plug $\mu = -\dfrac{\lambda}{4}$ into the second equation.

$$2\lambda^2 - 2\lambda + 1 - 8\left(-\frac{\lambda}{4}\right)^2 = 0$$

$$3\lambda^2 - 4\lambda + 2 = 0$$

This equation has no solutions.

Then plug $\lambda = 1/2$ into the second equation.

$$2\left(\frac{1}{2}\right)^2 - 2\left(\frac{1}{2}\right) + 1 - 8\mu^2 = 0$$

$\mu = \dfrac{1}{4}$ or $\mu = -\dfrac{1}{4}$

For $\lambda = 1/2$ and $\mu = 1/4$, find x, y, z, and $f(x, y, z)$.

$$x = \frac{1}{\frac{1}{2}} = 2$$

$$y = -\frac{\frac{1}{2}}{2 \cdot \frac{1}{4}} = -1$$

$$z = \frac{1 - \frac{1}{2}}{2 \cdot \frac{1}{4}} = 1$$

$$f(2, -1, 1) = 2 \ln 2 + 1$$

Then for $\lambda = 1/2$ and $\mu = -1/4$, find x, y, z, and $f(x, y, z)$.

$$x = \frac{1}{\frac{1}{2}} = 2$$

$$y = -\frac{\frac{1}{2}}{2 \cdot \left(-\frac{1}{4}\right)} = 1$$

$$z = \frac{1 - \frac{1}{2}}{2 \cdot \left(-\frac{1}{4}\right)} = -1$$

$$f(2, 1, -1) = 2 \ln 2 - 1$$

Since the curve of intersection of the two constraints equations, a cylinder and a plane, is a closed continuous curve, then by the extreme value theorem the function has one global minimum and one global maximum. Since we have only two critical points $(2, -1, 1)$ and $(2, 1, -1)$, one of them is the maximum, and the other is the minimum. Because $f(2, -1, 1) > f(2, 1, -1)$, $(2, -1, 1)$ is the maximum.

So the extrema of the function are

The local and global maximum is $2\ln 2+1$ at $(2,-1,1)$

The local and global minimum is $2\ln 2-1$ at $(2,1,-1)$

5. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x,y,z)=2\ln x-\ln y^4+z^2+5$$

$2x+3z^2=12$ and $4y+z^2=4$

Solution:

Rewrite the function as

$$f(x,y,z)=2\ln x-4\ln|y|+z^2+5$$

The two constraint functions are $g(x,y,z)=2x+3z^2-12$ and $h(x,y,z)=4y+z^2-4$. Create the system of equations.

$$f_x=\lambda g_x+\mu h_x$$

$$f_y=\lambda g_y+\mu h_y$$

$$f_z=\lambda g_z+\mu h_z$$

Calculate first order partial derivatives.

$$f_x = \frac{2}{x} \qquad g_x = 2 \qquad h_x = 0$$

$$f_y = -\frac{4}{y} \qquad g_y = 0 \qquad h_y = 4$$

$$f_z = 2z \qquad g_z = 6z \qquad h_z = 2z$$

Substitute partial derivatives into the system.

$$\frac{2}{x} = 2\lambda$$

$$-\frac{4}{y} = 4\mu$$

$$2z = 6z\lambda + 2z\mu$$

Simplify the system.

$$\frac{1}{x} = \lambda$$

$$\frac{1}{y} = -\mu$$

$$z(3\lambda + \mu - 1) = 0$$

From the third equation, $z = 0$ or $3\lambda + \mu - 1 = 0$. First, plug $z = 0$ into the constraint equations.

$$2x = 12$$

$$4y = 4$$

So

$$x = 6$$

$$y = 1$$

$$\lambda = \frac{1}{6}$$

$$\mu = -1$$

So the solution to the system is $(6,1,0)$, with $\lambda = 1/6$ and $\mu = -1$.

Second, if $3\lambda + \mu - 1 = 0$, since $\lambda \neq 0$ and $\mu \neq 0$, then

$$\mu = 1 - 3\lambda$$

$$x = \frac{1}{\lambda}$$

$$y = -\frac{1}{1 - 3\lambda}$$

Plug in these values for x and y into the constraint equations.

$$2\left(\frac{1}{\lambda}\right) + 3z^2 = 12$$

$$4\left(-\frac{1}{1 - 3\lambda}\right) + z^2 = 4$$

Simplify the system.

$$\lambda(3z^2 - 12) = -2$$

$$\lambda(3z^2 - 12) = z^2 - 8$$

So $z^2 - 8 = -2$, $z = \sqrt{6}$ or $z = -\sqrt{6}$.

From the first equation,

$$\lambda(3 \cdot 6 - 12) = -2$$

$$\lambda = -\frac{1}{3}$$

Calculate μ, x, and y for $z^2 = 6$ and $\lambda = -1/3$.

$$\mu = 1 - 3\left(-\frac{1}{3}\right) = 2$$

$$x = \frac{1}{-\frac{1}{3}} = -3$$

$$y = -\frac{1}{2}$$

So the solutions are

$\left(-3, -\frac{1}{2}, \sqrt{6}\right)$ with $\lambda = -\frac{1}{3}$ and $\mu = 2$

$\left(-3, -\frac{1}{2}, -\sqrt{6}\right)$ with $\lambda = -\frac{1}{3}$ and $\mu = 2$

Check to see if the points lie within the domain of $f(x, y, z)$. $x = -3 < 0$, so neither of these points are critical points.

Perform the second derivative test for constrained extrema at the point $(6,1,0)$. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$. The second derivative test for the point $(6,1,0)$ with $\lambda = 1/6$ and $\mu = -1$ is

$$L(x, y, z) = f(x, y, z) - \frac{1}{6} \cdot g(x, y, z) + h(x, y, z)$$

$$L(x, y, z) = 2 \ln x - \ln y^4 + z^2 + 5 - \frac{1}{6}(2x + 3z^2 - 12) + 4y + z^2 - 4$$

$$L(x, y, z) = 2 \ln x - \ln y^4 - \frac{x}{3} + 4y + \frac{3z^2}{2} + 3$$

Calculate the second order partial derivatives and substitute $(6,1,0)$.

$$L_{xx} = -\frac{1}{18}$$

$$L_{yy} = 4$$

$$L_{zz} = 3$$

$$L_{xy} = L_{yx} = 0$$

$$L_{yz} = L_{zy} = 0$$

$$L_{xz} = L_{zx} = 0$$

And we already know $g_x = 2$, $g_y = 0$, $g_z = 0$, and $h_x = 0$, $h_y = 4$, and $h_z = 0$. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ -2 & 0 & -\frac{1}{18} & 0 & 0 \\ 0 & -4 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{vmatrix}$$

The determinant is $det(HL) = 192 > 0$. Since $det(HL) > 0$ at $(6,1,0)$, this point is a local minimum. So the function has a local minimum of $2\ln 6 + 5$ at $(6,1,0)$.

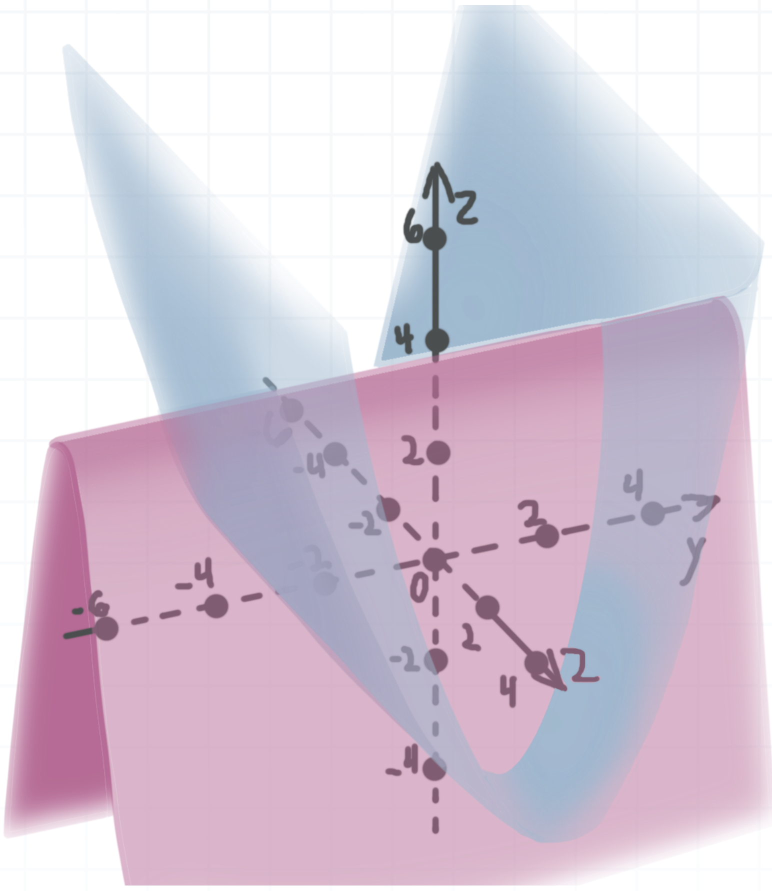
■ 6. Use Lagrange multipliers to find the local extrema of the function, subject to the given constraints.

$$f(x, y, z) = 4x^3 + 2y^3 - 4z + 1$$

$3y^2 - 4z = 12$ and $6x^2 + 4z = 15$

Solution:

A sketch of the surface is



The two constraint functions are $g(x, y, z) = 3y^2 - 4z - 12$ and $h(x, y, z) = 6x^2 + 4z - 15$.

Create the system of equations.

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

Calculate the first order partial derivatives.

$f_x = 12x^2$	$g_x = 0$	$h_x = 12x$
$f_y = 6y^2$	$g_y = 6y$	$h_y = 0$
$f_z = -4$	$g_z = -4$	$h_z = 4$

Substitute partial derivatives into the system.

$$12x^2 = 12x\mu$$

$$6y^2 = 6y\lambda$$

$$-4 = -4\lambda + 4\mu$$

Simplify the system.

$$x(x - \mu) = 0$$

$$y(y - \lambda) = 0$$

$$\lambda = \mu + 1$$

So from the first equation, $x = 0$ or $x = \mu$. First, plug $x = 0$ into the constraint equations.

$$3y^2 - 4z = 12$$

$$4z = 15$$

$$z = \frac{15}{4}$$

and then

$$y^2 = \frac{12 + 4 \cdot \frac{15}{4}}{3} = 9$$

$y = 3$ or $y = -3$

Calculate λ and μ for $(0,3,15/4)$.

$$\lambda = 3$$

$$\mu = 3 - 1 = 2$$

Calculate λ and μ for the point $(0, -3, 15/4)$.

$$\lambda = -3$$

$$\mu = -3 - 1 = -4$$

So the solutions are

$$\left(0, 3, \frac{15}{4}\right) \text{ with } \lambda = 3 \text{ and } \mu = 2$$

$$\left(0, -3, \frac{15}{4}\right) \text{ with } \lambda = -3 \text{ and } \mu = -4$$

Second, if $x = \mu$, then from the second equation, $y = 0$ or $y = \lambda$. Plug in $y = 0$ into the constraint equations.

$$-4z = 12$$

$$6x^2 + 4z = 15$$

$$z = -3$$

Then

$$x^2 = \frac{15 - 4(-3)}{6} = \frac{9}{2}$$

$$x = \frac{3}{\sqrt{2}} \text{ or } x = -\frac{3}{\sqrt{2}}$$

Calculate λ and μ for the point $(3/\sqrt{2}, 0, -3)$.

$$\mu = \frac{3}{\sqrt{2}}$$

$$\lambda = \frac{3}{\sqrt{2}} + 1$$

Calculate λ and μ for the point $(-3/\sqrt{2}, 0, -3)$.

$$\mu = -\frac{3}{\sqrt{2}}$$

$$\lambda = -\frac{3}{\sqrt{2}} + 1$$

So the solutions are

$$\left(-\frac{3}{\sqrt{2}}, 0, -3\right) \text{ with } \lambda = -\frac{3}{\sqrt{2}} + 1 \text{ and } \mu = -\frac{3}{\sqrt{2}}$$

$$\left(\frac{3}{\sqrt{2}}, 0, -3\right) \text{ with } \lambda = \frac{3}{\sqrt{2}} + 1 \text{ and } \mu = \frac{3}{\sqrt{2}}$$

Last, if $x = \mu$ and $y = \lambda$ then, from the third equation, $y = x + 1$. Plug $y = x + 1$ into the constraint equations.

$$3(x+1)^2 - 4z = 12$$

$$6x^2 + 4z = 15$$

$$4z = 3(x+1)^2 - 12 = 15 - 6x^2$$

Solve the equation $3(x+1)^2 - 12 = 15 - 6x^2$ for x.

$$9x^2 + 6x - 24 = 0$$

$$x = -2 \text{ or } x = \frac{4}{3}$$

Calculate y, z, λ, and μ for $x = -2$.

$$y = -2 + 1 = -1$$

$$z = \frac{15 - 6(-2)^2}{4} = -\frac{9}{4}$$

$$\mu = -2$$

$$\lambda = -1$$

Calculate y, z, λ, and μ for $x = 4/3$.

$$y = \frac{4}{3} + 1 = \frac{7}{3}$$

$$z = \frac{15 - 6(\frac{4}{3})^2}{4} = \frac{13}{12}$$

$$\mu = \frac{4}{3}$$

$$\lambda = \frac{7}{3}$$

So the solutions are

$$\left(-2,-1,-\frac{9}{4}\right) \text{ with } \lambda=-1 \text{ and } \mu=-2$$

$$\left(\frac{4}{3},\frac{7}{3},\frac{13}{12}\right) \text{ with } \lambda=\frac{7}{3} \text{ and } \mu=\frac{4}{3}$$

So we have six critical points.

1) $\left(0,3,\frac{15}{4}\right)$ with $\lambda=3$ and $\mu=2$

2) $\left(0,-3,\frac{15}{4}\right)$ with $\lambda=-3$ and $\mu=-4$

3) $\left(-\frac{3}{\sqrt{2}},0,-3\right)$ with $\lambda=-\frac{3}{\sqrt{2}}+1$ and $\mu=-\frac{3}{\sqrt{2}}$

4) $\left(\frac{3}{\sqrt{2}},0,-3\right)$ with $\lambda=\frac{3}{\sqrt{2}}+1$ and $\mu=\frac{3}{\sqrt{2}}$

5) $\left(-2,-1,-\frac{9}{4}\right)$ with $\lambda=-1$ and $\mu=-2$

6) $\left(\frac{4}{3},\frac{7}{3},\frac{13}{12}\right)$ with $\lambda=\frac{7}{3}$ and $\mu=\frac{4}{3}$

Perform the second derivative test for constrained extrema. Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & -g_x & -g_y & -g_z \\ 0 & 0 & -h_x & -h_y & -h_z \\ -g_x & -h_x & L_{xx} & L_{xy} & L_{xz} \\ -g_y & -h_y & L_{yx} & L_{yy} & L_{yz} \\ -g_z & -h_z & L_{zx} & L_{zy} & L_{zz} \end{vmatrix}$$

where $L(x, y, z) = f(x, y, z) - \lambda g(x, y, z) - \mu h(x, y, z)$.

1) The second derivative test for $(0,3,15/4)$ with $\lambda = 3$ and $\mu = 2$:

$$L(x, y, z) = f(x, y, z) - 3 \cdot g(x, y, z) - 2 \cdot h(x, y, z)$$

$$L(x, y, z) = 4x^3 + 2y^3 - 4z + 1 - 3(3y^2 - 4z - 12) - 2(6x^2 + 4z - 15)$$

$$L(x, y, z) = 4x^3 - 12x^2 + 2y^3 - 9y^2 + 67$$

Find second order partial derivatives at $(0,3,15/4)$.

$$L_{xx} = -24, \ L_{yy} = 18, \ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0, \ L_{yz} = L_{zy} = 0, \ L_{xz} = L_{zx} = 0$$

$$g_x = 0, \ g_y = 18, \ g_z = -4$$

$$h_x = 0, \ h_y = 0, \ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & -18 & 4 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & -24 & 0 & 0 \\ -18 & 0 & 0 & 18 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) = -124{,}416 < 0$. Since $det(HL) < 0$ at $(0,3,15/4)$, it's a local maximum, and $f(0,3,15/4) = 40$.

2) The second derivative test for $(0,-3,15/4)$ with $\lambda = -3$ and $\mu = -4$:

$$L(x,y,z) = f(x,y,z) + 3 \cdot g(x,y,z) + 4 \cdot h(x,y,z)$$

$$L(x,y,z) = 4x^3 + 2y^3 - 4z + 1 + 3(3y^2 - 4z - 12) + 4(6x^2 + 4z - 15)$$

$$L(x,y,z) = 4x^3 + 24x^2 + 2y^3 + 9y^2 - 95$$

Find second order partial derivatives at $(0,-3,15/4)$.

$$L_{xx} = 48,\ L_{yy} = -18,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = -18,\ g_z = -4$$

$$h_x = 0,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 18 & 4 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 48 & 0 & 0 \\ 18 & 0 & 0 & -18 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL)=248{,}832>0$. Since $det(HL)>0$ at $(0,-3,15/4)$, it's a local maximum, and $f(0,-3,15/4)=-68$.

3) The second derivative test for $(-3/\sqrt{2},0,-3)$ with $\lambda=-3/\sqrt{2}+1$ and $\mu=-3/\sqrt{2}$:

We don't need high accuracy here because we only need to estimate the determinant value. So to simplify calculations, let's consider the values rounded to one decimal place.

$$L(x,y,z)=f(x,y,z)+1.1\cdot g(x,y,z)+2.1\cdot h(x,y,z)$$

$$L(x,y,z)=4x^3+2y^3-4z+1+1.1(3y^2-4z-12)+2.1(6x^2+4z-15)$$

$$L(x,y,z)=4x^3+12.6x^2+2y^3+3.3y^2-43.7$$

Find second order partial derivatives at $(-2.1,0,-3)$.

$$L_{xx}=-25.2,\ L_{yy}=6.6,\ L_{zz}=0$$

$$L_{xy}=L_{yx}=0,\ L_{yz}=L_{zy}=0,\ L_{xz}=L_{zx}=0$$

$$g_x=0,\ g_y=0,\ g_z=-4$$

$$h_x=-25.2,\ h_y=0,\ h_z=4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 25.2 & 0 & -4 \\ 0 & 25.2 & -25.2 & 0 & 0 \\ 0 & 0 & 0 & 6.6 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) \approx 67{,}060 > 0$. Since $det(HL) > 0$ at $(-3/\sqrt{2},0,-3)$, it's a local minimum, and $f(-3/\sqrt{2},0,-3) = 13 - 27\sqrt{2}$.

4) The second derivative test for $(3/\sqrt{2},0,-3)$ with $\lambda = 3/\sqrt{2} + 1$ and $\mu = 3/\sqrt{2}$:

$$L(x,y,z) = f(x,y,z) - 3.1 \cdot g(x,y,z) - 2.1 \cdot h(x,y,z)$$

$$L(x,y,z) = 4x^3 + 2y^3 - 4z + 1 - 3.1(3y^2 - 4z - 12) - 2.1(6x^2 + 4z - 15)$$

$$L(x,y,z) = 4x^3 - 12.6x^2 + 2y^3 - 9.3y^2 + 69.7$$

Find second order partial derivatives at $(3/\sqrt{2},0,-3)$.

$$L_{xx} = 25.2,\ L_{yy} = -18.6,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = 0,\ g_z = -4$$

$$h_x = 25.2,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & -25.2 & 0 & -4 \\ 0 & -25.2 & 25.2 & 0 & 0 \\ 0 & 0 & 0 & -18.6 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) \approx -188{,}988 < 0$. Since $det(HL) < 0$ at $(3/\sqrt{2},0,-3)$, it's a local maximum, and $f(3/\sqrt{2},0,-3) = 13 + 27\sqrt{2}$.

5) The second derivative test for $(-2,-1,-2.25)$ with $\lambda=-1$ and $\mu=-2$.

$$L(x,y,z)=f(x,y,z)+1\cdot g(x,y,z)+2\cdot h(x,y,z)$$

$$L(x,y,z)=4x^3+2y^3-4z+1+(3y^2-4z-12)+2(6x^2+4z-15)$$

$$L(x,y,z)=4x^3+12x^2+2y^3+3y^2-41$$

Find second order partial derivatives at $(-2,-1,-2.25)$.

$$L_{xx}=-24,\ L_{yy}=-6,\ L_{zz}=0$$

$$L_{xy}=L_{yx}=0,\ L_{yz}=L_{zy}=0,\ L_{xz}=L_{zx}=0$$

$$g_x=0,\ g_y=-6,\ g_z=-4$$

$$h_x=-24,\ h_y=0,\ h_z=4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & 6 & 4 \\ 0 & 0 & 24 & 0 & -4 \\ 0 & 24 & -24 & 0 & 0 \\ 6 & 0 & 0 & -6 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL)=-69{,}120<0$. Since $det(HL)<0$ at $(-2,-1,-2.25)$, it's a local maximum, and $f(-2,-1,-2.25)=-24$.

6) The second derivative test for $(4/3,7/3,13/12)$ with $\lambda=7/3$ and $\mu=4/3$.

$$L(x,y,z)=f(x,y,z)-2.3\cdot g(x,y,z)-1.3\cdot h(x,y,z)$$

$$L(x, y, z) = 4x^3 + 2y^3 - 4z + 1 - 2.3(3y^2 - 4z - 12) - 1.3(6x^2 + 4z - 15)$$

$$L(x, y, z) = 4x^3 - 7.8x^2 + 2y^3 - 6.9y^2 + 48.1$$

Find second order partial derivatives at $(4/3,7/3,13/12)$.

$$L_{xx} = 15.6,\ L_{yy} = 13.8,\ L_{zz} = 0$$

$$L_{xy} = L_{yx} = 0,\ L_{yz} = L_{zy} = 0,\ L_{xz} = L_{zx} = 0$$

$$g_x = 0,\ g_y = 13.8,\ g_z = -4$$

$$h_x = 15.6,\ h_y = 0,\ h_z = 4$$

Form the bordered Hessian matrix.

$$\begin{vmatrix} 0 & 0 & 0 & -13.8 & 4 \\ 0 & 0 & -15.6 & 0 & -4 \\ 0 & -15.6 & 15.6 & 0 & 0 \\ -13.8 & 0 & 0 & 13.8 & 0 \\ 4 & -4 & 0 & 0 & 0 \end{vmatrix}$$

The determinant is $det(HL) \approx 101{,}268 > 0$. Since $det(HL) > 0$ at $(4/3,7/3,13/12)$, it's a local minimum, and $f(4/3,7/3,13/12) = 284/9$.

So the extrema of the function are

A local maximum of 40 at $(0,3,15/4)$

A local minimum of -68 at $(0,-3,15/4)$

A local minimum of $13 - 27\sqrt{2}$ at $(-3/\sqrt{2},0,-3)$

A local maximum of $13 + 27\sqrt{2}$ at $(3/\sqrt{2}, 0, -3)$

A local maximum of -24 at $(-2, -1, -2.25)$

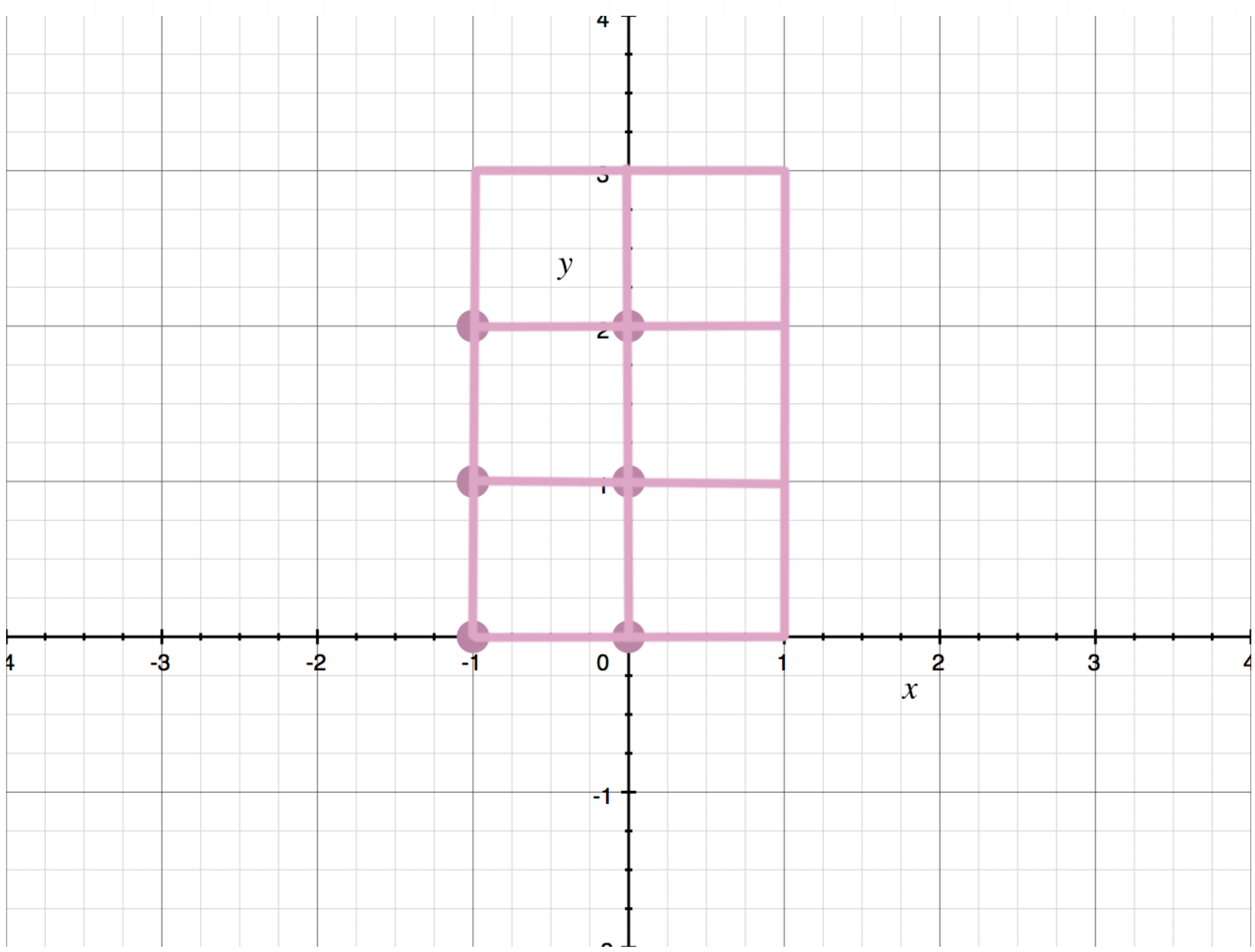
A local minimum of $284/9$ at $(4/3, 7/3, 13/12)$

APPROXIMATING DOUBLE INTEGRALS WITH RECTANGLES

■ 1. Estimate the volume between the surface $z = 3(x - 2)e^{y-2}$ and the xy-plane, on the region $-1 \leq x \leq 1$ and $0 \leq y \leq 3$. Use lower-left corners and 1×1 squares, then 0.5×0.5. Round the answers for volume to the nearest tenth. Which square size gives the better approximation if exact volume is 31.

Solution:

Sketch the region, including the 1×1 squares and the lower-left corners.



The lower-left corners are at $(-1,0)$, $(0,0)$, $(-1,1)$, $(0,1)$, $(-1,2)$, and $(0,2)$, so we'll evaluate z at these points.

$$z(-1,0) = 3((-1) - 2)e^{0-2} \approx -1.22$$

$$z(0,0) = 3((0) - 2)e^{0-2} \approx -0.81$$

$$z(-1,1) = 3((-1) - 2)e^{1-2} \approx -3.31$$

$$z(0,1) = 3((0) - 2)e^{1-2} \approx -2.21$$

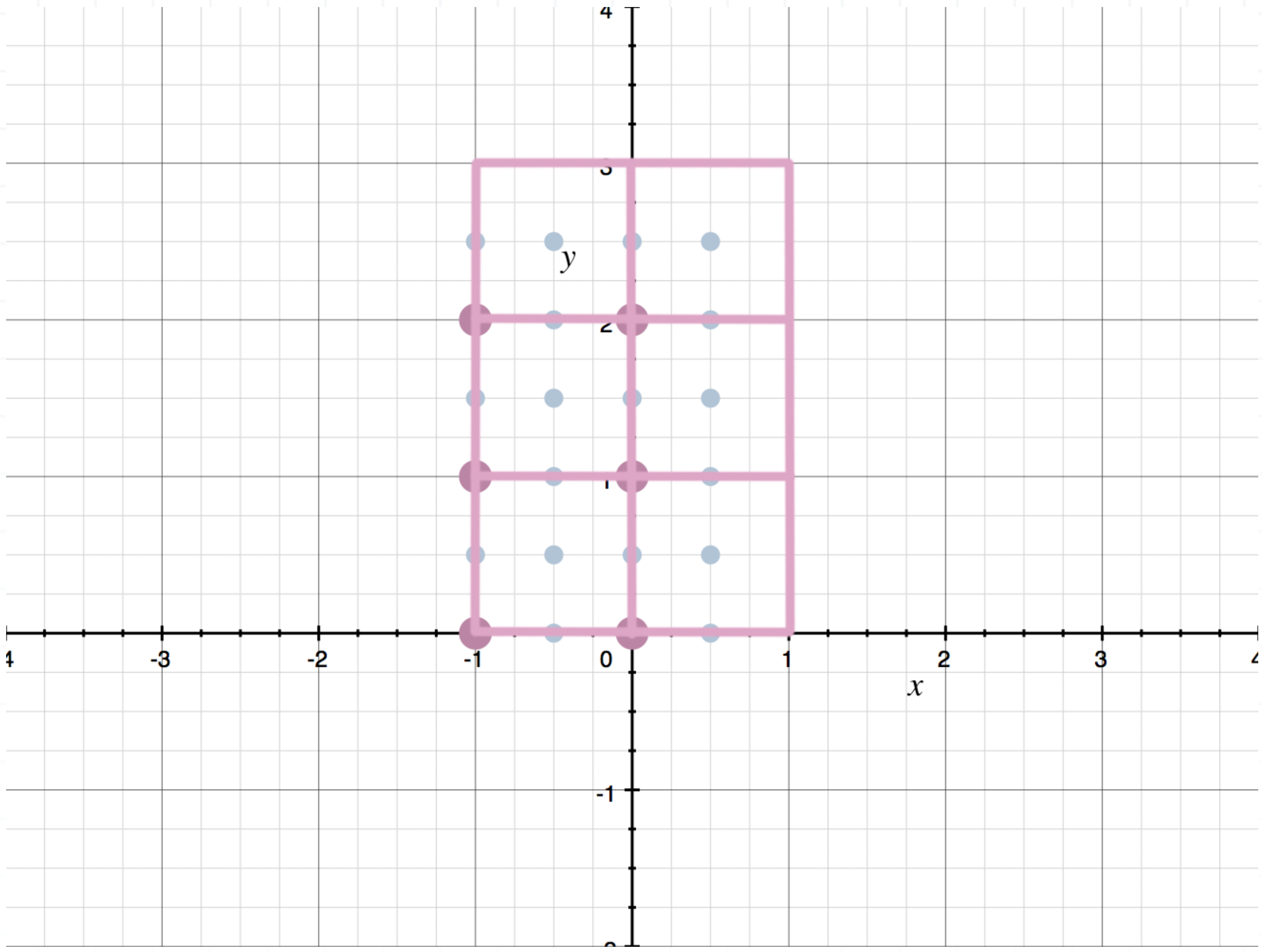
$$z(-1,2) = 3((-1) - 2)e^{2-2} \approx -9.00$$

$$z(0,2) = 3((0) - 2)e^{2-2} \approx -6.00$$

The area of a 1×1 square is $A = 1$, so the volume approximation is

$$1 \cdot |-1.22 - 0.81 - 3.31 - 2.21 - 9 - 6| \approx 22.6$$

If we use 0.5×0.5 squares, we'll need all the points from the 1×1 squares, as well as these new points in blue:



The values of z at these new blue points are

$$z(-0.5,0) = 3((-0.5) - 2)e^{0-2} \approx -1.02$$

$$z(0.5,0) = 3((0.5) - 2)e^{0-2} \approx -0.61$$

$$z(-1,0.5) = 3((-1) - 2)e^{0.5-2} \approx -2.01$$

$$z(-0.5,0.5) = 3((-0.5) - 2)e^{0.5-2} \approx -1.67$$

$$z(0,0.5) = 3((0) - 2)e^{0.5-2} \approx -1.34$$

$$z(0.5,0.5) = 3((0.5) - 2)e^{0.5-2} \approx -1$$

$$z(-0.5,1) = 3((-0.5) - 2)e^{1-2} \approx -2.76$$

$$z(0.5,1) = 3((0.5) - 2)e^{1-2} \approx -1.66$$

$$z(-1,1.5) = 3((-1) - 2)e^{1.5-2} \approx -5.46$$

$$z(-0.5,1.5) = 3((-0.5) - 2)e^{1.5-2} \approx -4.55$$

$$z(0,1.5) = 3((0) - 2)e^{1.5-2} \approx -3.64$$

$$z(0.5,1.5) = 3((0.5) - 2)e^{1.5-2} \approx -2.73$$

$$z(-0.5,2) = 3((-0.5) - 2)e^{2-2} \approx -7.5$$

$$z(0.5,2) = 3((0.5) - 2)e^{2-2} \approx -4.5$$

$$z(-1,2.5) = 3((-1) - 2)e^{2.5-2} \approx -14.84$$

$$z(-0.5,2.5) = 3((-0.5) - 2)e^{2.5-2} \approx -12.37$$

$$z(0,2.5) = 3((0) - 2)e^{2.5-2} \approx -9.89$$

$$z(0.5,2.5) = 3((0.5) - 2)e^{2.5-2} \approx -7.42$$

The area of a 0.5×0.5 square is $A = 0.25$, so the volume approximation, using these points and the ones from the 1×1 calculation, is

$$0.25 \cdot | -1.22 - 0.81 - 3.31 - 2.21 - 9 - 6 - 1.02 - 0.61$$

$$-2.01 - 1.67 - 1.34 - 1 - 2.76 - 1.66 - 5.46 - 4.55$$

$$-3.64 - 2.73 - 7.5 - 4.5 - 14.84 - 12.37 - 9.89 - 7.42 | \approx 26.9$$

The exact area is 31, the 0.5×0.5 squares approximate the volume at 26.9, and the 1×1 squares approximate the volume at 22.6, so the 0.5×0.5 squares give a more accurate approximation.

2. Estimate the volume below the surface $z = 3 + x + y^2$, over the triangular region bounded by the x- and y-axes and the line $2x + 3y = 6$. Use lower-left corners and 1×1 squares. For squares that lie partially within the region, divide area by 2.

Solution:

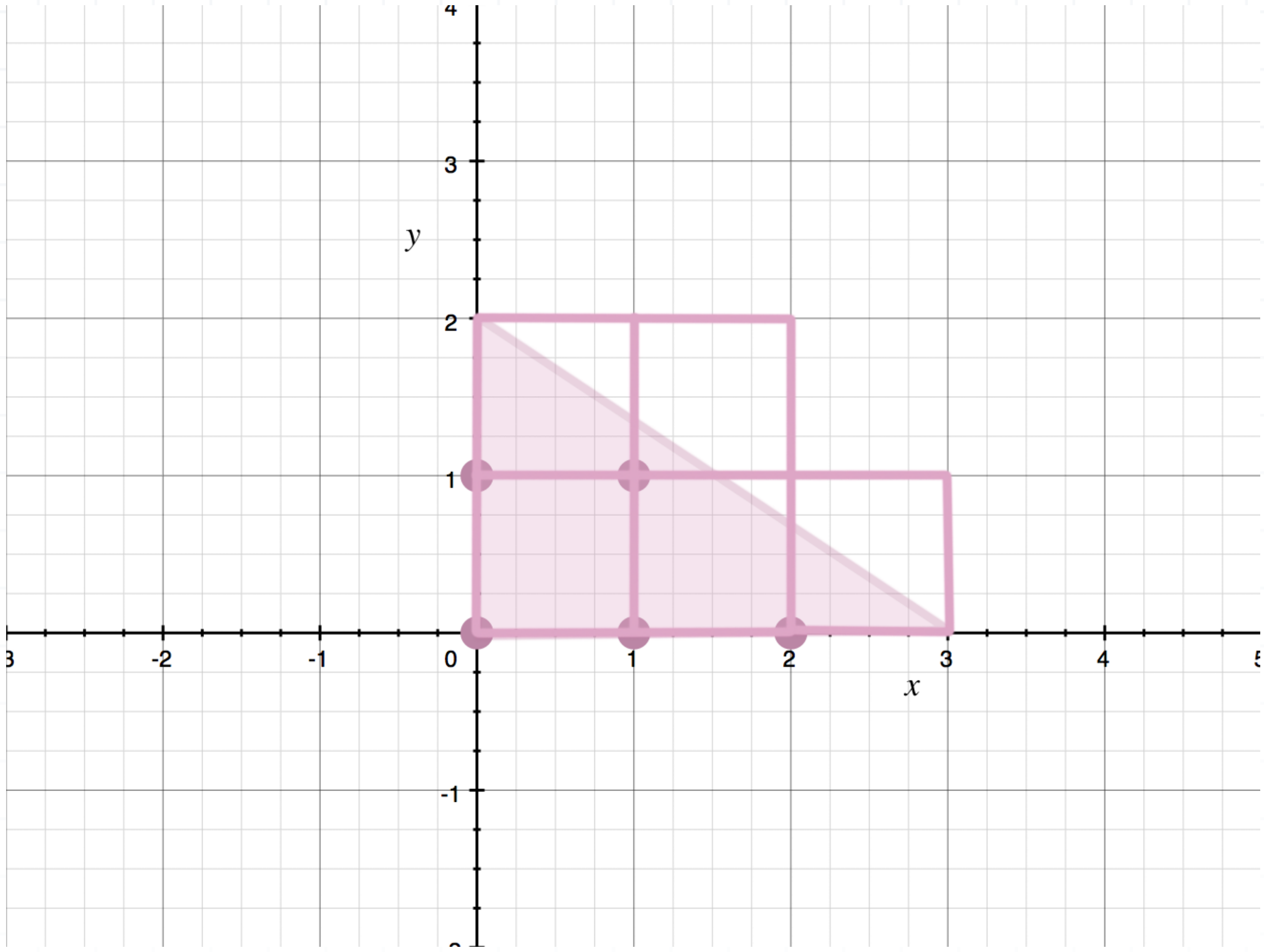
Rewrite the line in slope-intercept form.

$$2x + 3y = 6$$

$$3y = -2x + 6$$

$$y = -\frac{2}{3}x + 2$$

The line intersects the major axes at $(3,0)$ and $(0,2)$. If we sketch the triangular region, the 1×1 squares, and the lower-left corners, we get



Evaluate z at each lower-left point.

$$z(0,0) = 3 + 0 + 0^2 = 3$$

$$z(1,0) = 3 + 1 + 0^2 = 4$$

$$z(2,0) = 3 + 2 + 0^2 = 5$$

$$z(0,1) = 3 + 0 + 1^2 = 4$$

$$z(1,1) = 3 + 1 + 1^2 = 5$$

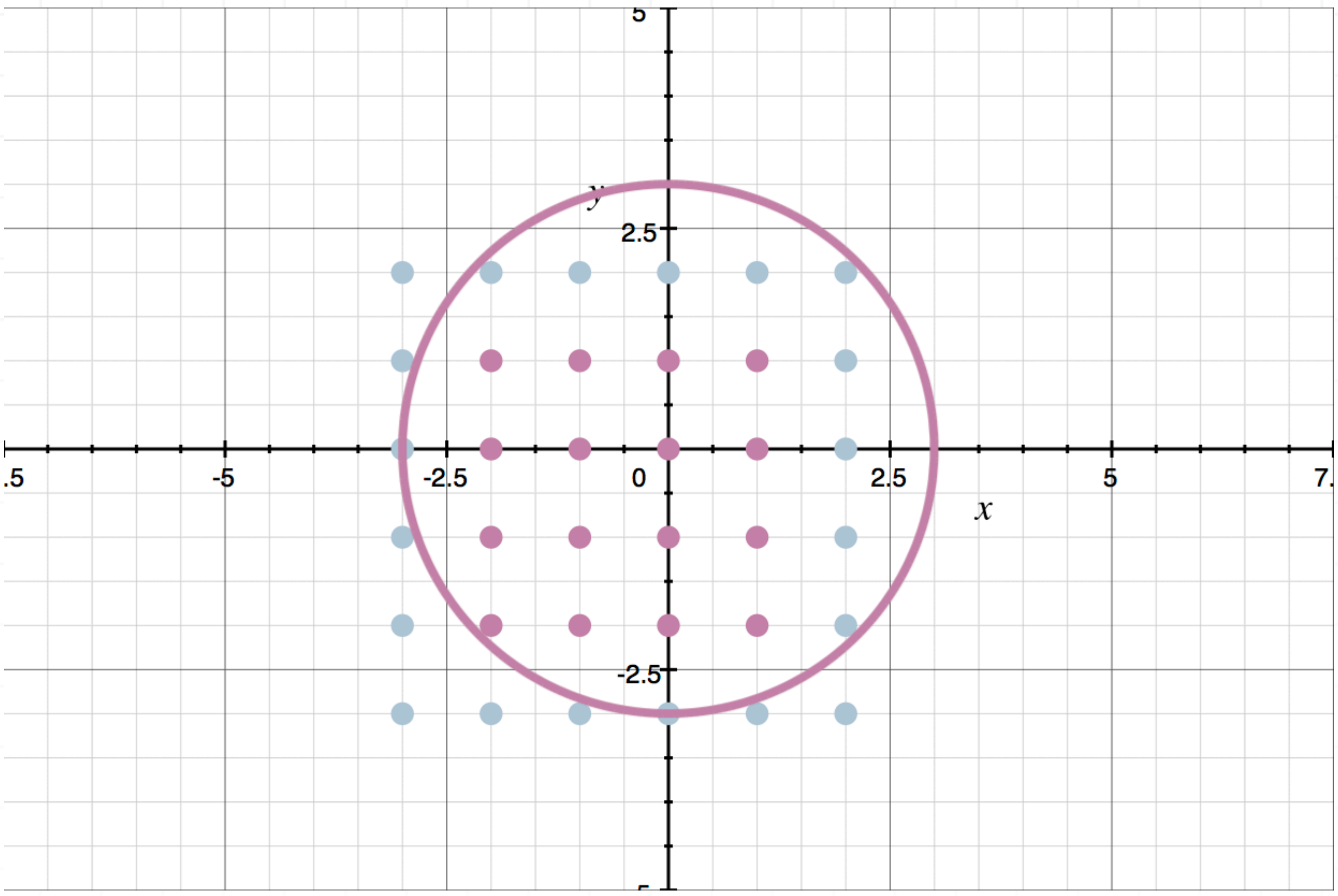
The only square that lies fully within the region is the square associated with $(0,0)$. The squares associated with $(1,0)$, $(2,0)$, $(0,1)$, and $(1,1)$ lie partially within the region, so we'll multiply the sum of those areas by 0.5. The area of a 1×1 square is $A = 1$, so the volume approximation is

$$1 \cdot |3 + 0.5(4 + 5 + 4 + 5)| = 12$$

3. Assume the base of a right circular cylinder with radius 3 and height 5 lies in the xy-plane with its center at the origin. Use 1×1 squares and lower-left corners to estimate the volume of the cylinder. If an approximating square lies only partially within the circle, divide its area in half as part of the estimation. Calculate exact volume of the cylinder using $V = \pi r^2 h$, then find the percentage error of the approximation.

Solution:

Sketch the base of the cylinder in the xy-plane, along with the lower-left points of each square. Purple points represent squares that lie entirely within the region, while blue points represent squares that lie only partially within the region.



Because the cylinder has height 5, the value of z at every point is $z = 5$. There are 16 purple points and 20 blue points (whose area should be divided in half), so the volume is approximated by

$$5 \cdot |\, 16 + 0.5(20) \,| = 130$$

The cylinder's exact volume is

$$V = \pi r^2 h$$

$$V = \pi \cdot 3^2 \cdot 5$$

$$V = 45\pi$$

$$V \approx 141.4$$

Then the percentage error in the approximation is

$$E_P = \frac{|\text{Exact volume} - \text{Approximate volume}|}{\text{Exact volume}} \cdot 100\,\%$$

$$E_P = \frac{|45\pi - 130|}{45\pi} \cdot 100\,\%$$

$$E_P \approx 8.04\,\%$$

MIDPOINT RULE FOR DOUBLE INTEGRALS

1. Assuming the integral's exact value is 12, say which estimation is more accurate if we estimate volume of the integral below using Midpoint Rule and rectangles of dimensions $\pi/2 \times 1$, and then rectangles of dimensions $\pi/3 \times 2/3$.

$$\int_0^\pi \int_{-1}^1 (y+3)\sin x \ dy \ dx$$

Solution:

The region of integration given by the bounds of the double integral is $x = [0,\pi]$ and $y = [-1,1]$. If we use rectangles of dimensions $\pi/2 \times 1$, then the midpoints of the rectangles will be

$$\left(\frac{\pi}{4}, -0.5\right), \left(\frac{3\pi}{4}, -0.5\right), \left(\frac{\pi}{4}, 0.5\right), \left(\frac{3\pi}{4}, 0.5\right)$$

Evaluate $z = (y+3)\sin x$ at these four midpoints.

$$z\left(\frac{\pi}{4}, -0.5\right) = (-0.5+3)\sin\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{4}$$

$$z\left(\frac{3\pi}{4}, -0.5\right) = (-0.5+3)\sin\left(\frac{3\pi}{4}\right) = \frac{5\sqrt{2}}{4}$$

$$z\left(\frac{\pi}{4},0.5\right)=(0.5+3)\sin\left(\frac{\pi}{4}\right)=\frac{7\sqrt{2}}{4}$$

$$z\left(\frac{3\pi}{4},0.5\right)=(0.5+3)\sin\left(\frac{3\pi}{4}\right)=\frac{7\sqrt{2}}{4}$$

The area of a $\pi/2\times 1$ rectangle is $\pi/2$, so the Midpoint Rule volume estimate with rectangles of these dimensions is

$$V=\frac{\pi}{2}\left[z\left(\frac{\pi}{4},-0.5\right)+z\left(\frac{3\pi}{4},-0.5\right)+z\left(\frac{\pi}{4},0.5\right)+z\left(\frac{3\pi}{4},0.5\right)\right]$$

$$V\approx\frac{\pi}{2}\left(\frac{5\sqrt{2}}{4}+\frac{5\sqrt{2}}{4}+\frac{7\sqrt{2}}{4}+\frac{7\sqrt{2}}{4}\right)$$

$$V\approx 3\pi\sqrt{2}$$

$$V\approx 13.33$$

If we use rectangles of dimensions $\pi/3\times 2/3$, then the midpoints of the rectangles will be

$$\left(\frac{\pi}{6},-\frac{2}{3}\right),\left(\frac{\pi}{2},-\frac{2}{3}\right),\left(\frac{5\pi}{6},-\frac{2}{3}\right),\left(\frac{\pi}{6},0\right),\left(\frac{\pi}{2},0\right)$$

$$\left(\frac{5\pi}{6},0\right),\left(\frac{\pi}{6},\frac{2}{3}\right),\left(\frac{\pi}{2},\frac{2}{3}\right),\left(\frac{5\pi}{6},\frac{2}{3}\right)$$

Evaluate $z=(y+3)\sin x$ at these nine midpoints.

$$z\left(\frac{\pi}{6},-\frac{2}{3}\right)=\left(-\frac{2}{3}+3\right)\sin\left(\frac{\pi}{6}\right)=\frac{7}{6}$$

$$z\left(\frac{\pi}{2},-\frac{2}{3}\right)=\left(-\frac{2}{3}+3\right)\sin\left(\frac{\pi}{2}\right)=\frac{7}{3}$$

$$z\left(\frac{5\pi}{6},-\frac{2}{3}\right)=\left(-\frac{2}{3}+3\right)\sin\left(\frac{5\pi}{6}\right)=\frac{7}{6}$$

$$z\left(\frac{\pi}{6},0\right)=(0+3)\sin\left(\frac{\pi}{6}\right)=\frac{3}{2}$$

$$z\left(\frac{\pi}{2},0\right)=(0+3)\sin\left(\frac{\pi}{2}\right)=3$$

$$z\left(\frac{5\pi}{6},0\right)=(0+3)\sin\left(\frac{5\pi}{6}\right)=\frac{3}{2}$$

$$z\left(\frac{\pi}{6},\frac{2}{3}\right)=\left(\frac{2}{3}+3\right)\sin\left(\frac{\pi}{6}\right)=\frac{11}{6}$$

$$z\left(\frac{\pi}{2},\frac{2}{3}\right)=\left(\frac{2}{3}+3\right)\sin\left(\frac{\pi}{2}\right)=\frac{11}{3}$$

$$z\left(\frac{5\pi}{6},\frac{2}{3}\right)=\left(\frac{2}{3}+3\right)\sin\left(\frac{5\pi}{6}\right)=\frac{11}{6}$$

The area of a $\pi/3\times 2/3$ rectangle is $2\pi/9$, so the Midpoint Rule volume estimate with rectangles of these dimensions is

$$V=\frac{2\pi}{9}\left[z\left(\frac{\pi}{6},-\frac{2}{3}\right)+z\left(\frac{\pi}{2},-\frac{2}{3}\right)+z\left(\frac{5\pi}{6},-\frac{2}{3}\right)\right.$$

$$\left.+z\left(\frac{\pi}{6},0\right)+z\left(\frac{\pi}{2},0\right)+z\left(\frac{5\pi}{6},0\right)\right.$$

$$+z\left(\frac{\pi}{6},\frac{2}{3}\right)+z\left(\frac{\pi}{2},\frac{2}{3}\right)+z\left(\frac{5\pi}{6},\frac{2}{3}\right)\Bigg]$$

$$V \approx \frac{2\pi}{9}\left(\frac{7}{6}+\frac{7}{3}+\frac{7}{6}+\frac{3}{2}+3+\frac{3}{2}+\frac{11}{6}+\frac{11}{3}+\frac{11}{6}\right)$$

$$V \approx \frac{2\pi}{9}\left(\frac{36}{6}+\frac{18}{3}+\frac{6}{2}+3\right)$$

$$V \approx \frac{2\pi}{9}(6+6+3+3)$$

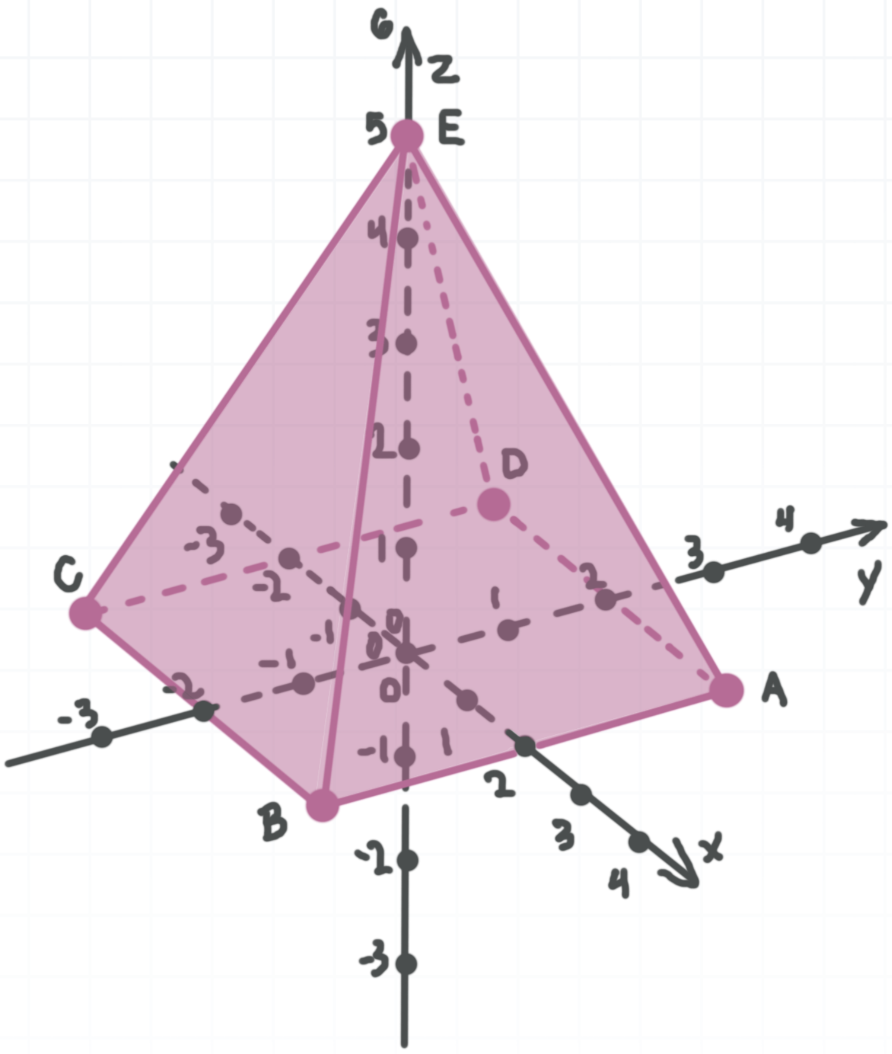
$$V \approx \frac{2\pi}{9}(18)$$

$$V \approx 4\pi$$

$$V \approx 12.57$$

If exact volume is 12, rectangles with dimensions $\pi/2 \times 1$ give an approximation of $V \approx 13.33$, and rectangles with dimensions $\pi/3 \times 2/3$ give an approximation of $V \approx 12.57$, we can say that the $\pi/3 \times 2/3$ rectangles give a better approximation.

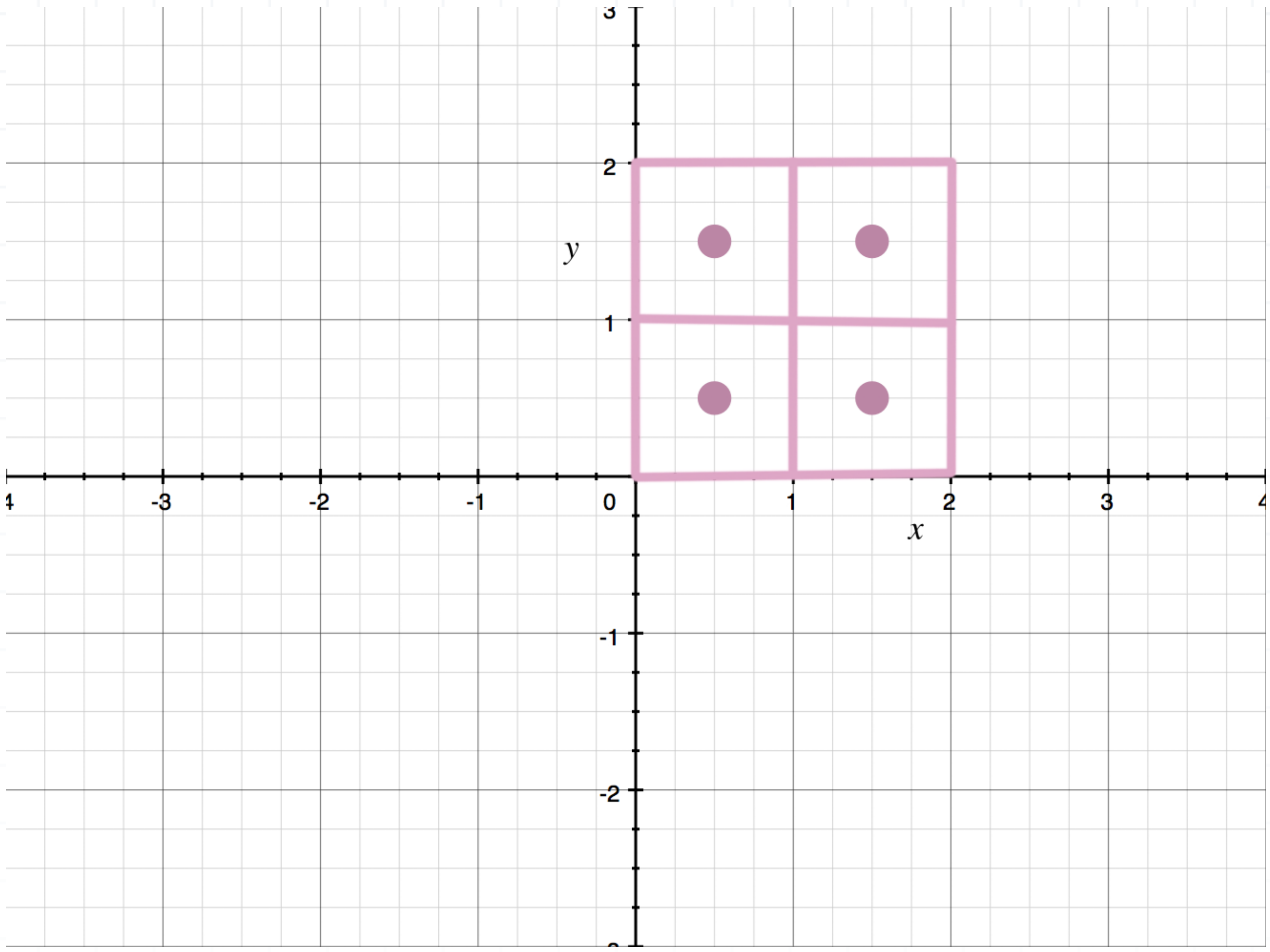
■ 2. Use Midpoint Rule and 1×1 squares to estimate the volume of the right square pyramid $ABCDE$ with base side length 4 and height 5, assuming its base lies in the xy-plane with its sides parallel to the major coordinate axes, and the vertex of the pyramid lies on z-axis. If the pyramid's exact volume is $V = 80/3$, find the percentage error of the approximation.



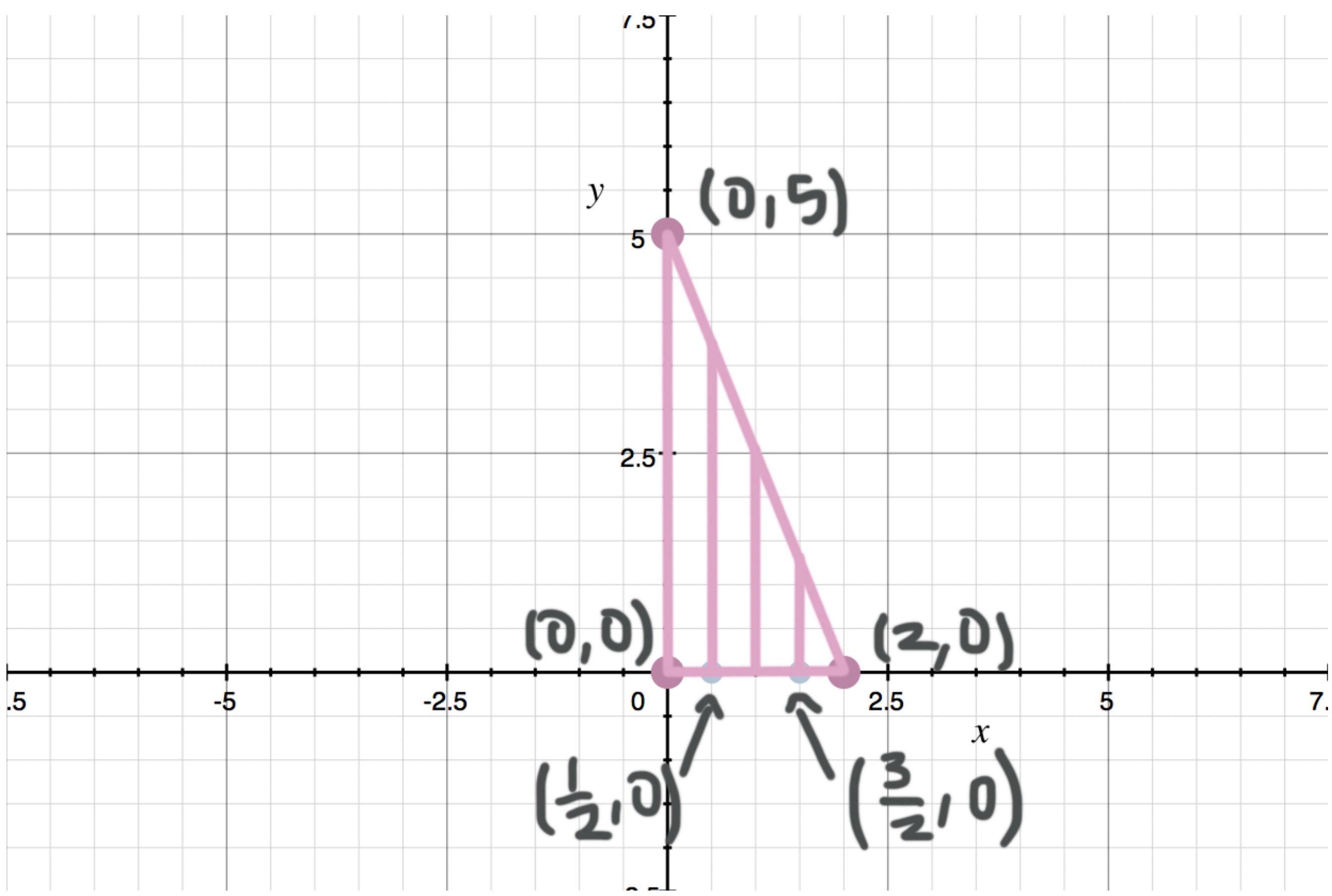
Solution:

Since a right square pyramid is symmetric and can be split into four equal parts, we can approximate the volume of the pyramid lying in the first quadrant, then multiply that volume by 4 to get an approximation of total volume.

Sketch the region of the base in the first quadrant only, the 1×1 squares, and their midpoints.



The midpoints are $(0.5,0.5)$, $(1.5,0.5)$, $(0.5,1.5)$, and $(1.5,1.5)$. To find z for each of these midpoints, we look at the triangle OAE.



Because $(0.5,0.5)$ and $(1.5,1.5)$ lie on the segment OA, the value of z at both points can be given by the equation of AE. Since $(2,0)$ and $(0,5)$ define the endpoints of AE, the equation of AE is

$$y = -\frac{5}{2}x + 5$$

So at $(0.5,0.5)$, z is

$$z = y = -\frac{5}{2}(0.5) + 5$$

$$z = y = 3.75$$

And at $(1.5,1.5)$, z is

$$z = y = -\frac{5}{2}(1.5) + 5$$

$$z = y = 1.25$$

This isn't one of the midpoints, but the point $(1,1)$ is also along OA, which means the value at that point can be given by the equation of AE.

$$z = y = -\frac{5}{2}(1) + 5$$

$$z = y = 2.5$$

We need this value in order to calculate the z-values for the midpoints $(1.5,0.5)$ and $(0.5,1.5)$. We can get the z-value of $(1.5,0.5)$ by recognizing that $(1.5,0.5)$ is the midpoint of $(2,0,0)$ and $(1,1,2.5)$.

$$z = \frac{0 + 2.5}{2} = 1.25$$

Similarly, we can get the z-value of $(0.5,1.5)$ by recognizing that $(0.5,1.5)$ is the midpoint of $(0,2,0)$ and $(1,1,2.5)$.

$$z = \frac{0 + 2.5}{2} = 1.25$$

Since the area of a 1×1 square is $A = 1$, the volume approximation is

$$1 \cdot |3.75 + 1.25 + 1.25 + 1.25| = 7.5$$

This is the approximation of the pyramid's volume in the first quadrant only, so the approximation of its total volume is $4 \cdot 7.5 = 30$.

Because exact volume is given as $V = 80/3$, the percentage error is

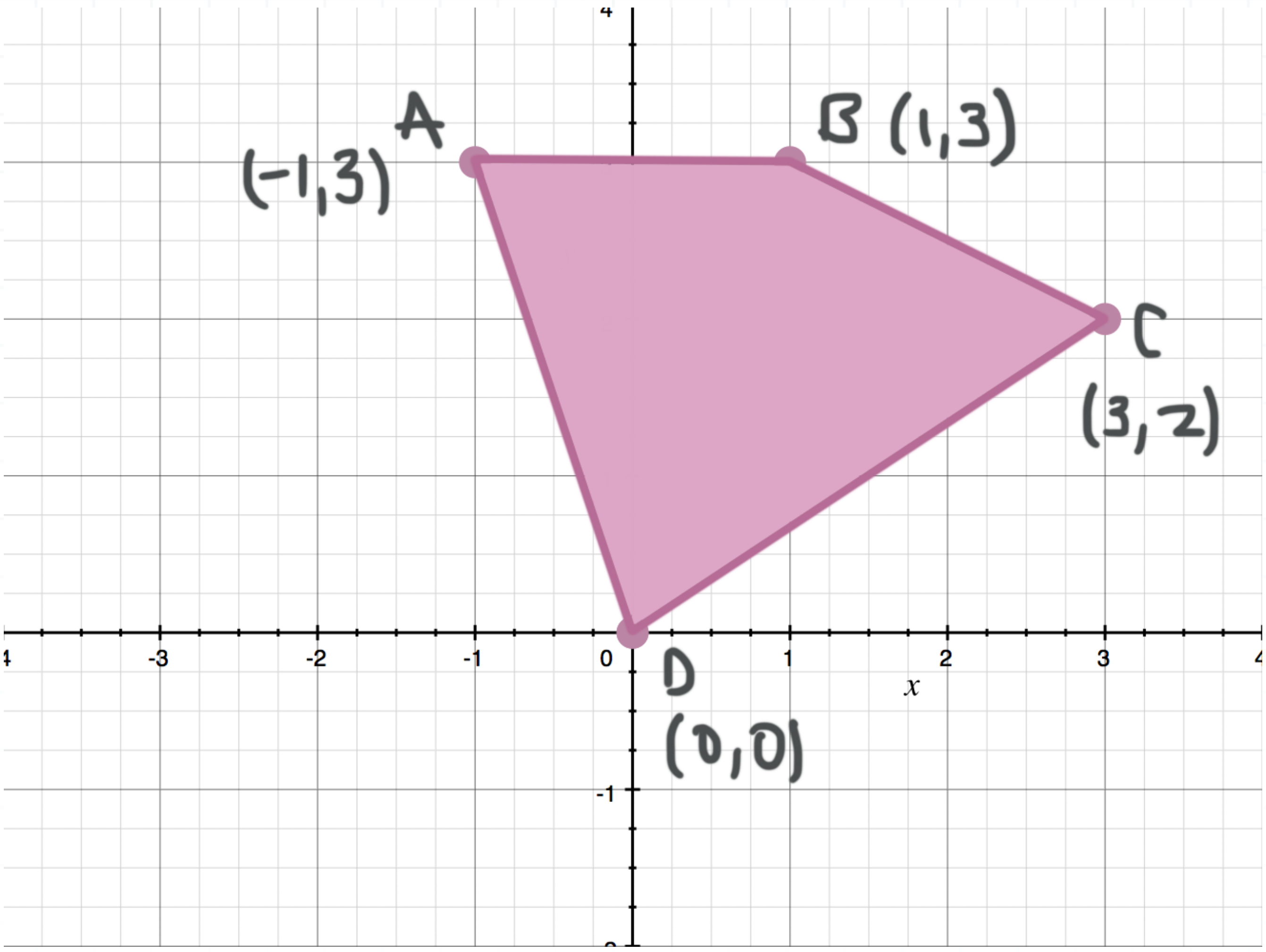
$$E_P = \frac{|\text{Exact volume} - \text{Approximate volume}|}{\text{Exact volume}} \cdot 100\,\%$$

$$E_P = \frac{|80/3 - 30|}{80/3} \cdot 100\,\%$$

$$E_P = 12.5\,\%$$

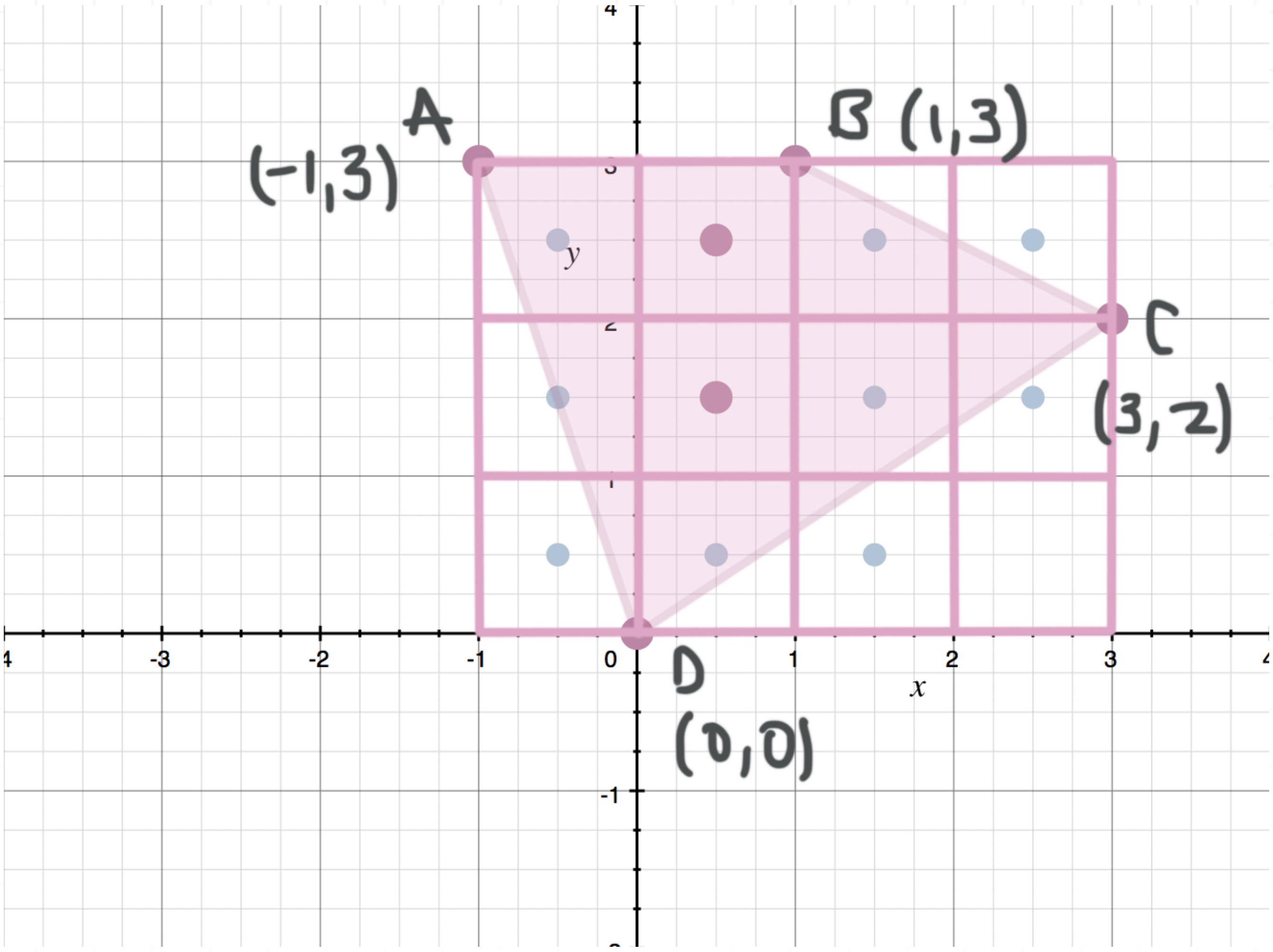
■ 3. Use Midpoint Rule with 1×1 squares to estimate the value of the given integral, where $ABCD$ is the quadrilateral shown in the xy-plane below. For any square that lie only partially inside the region, divide the area in half, then round the final approximation to the nearest tenth.

$$\iint_{ABCD} x + \sqrt{y+3}\ dy\ dx$$



Solution:

Sketch the region, the 1×1 squares, and each of their midpoints. We'll use purple dots for midpoints of squares that lie completely within the region, and blue dots for midpoints of squares that lie only partially within the region.



Evaluate $z = x + \sqrt{y+3}$ at each midpoint, rounding each value to the nearest hundredth.

$$z(0.5,1.5) = 0.5 + \sqrt{1.5+3} = 2.62$$

$$z(0.5,2.5) = 0.5 + \sqrt{2.5+3} = 2.85$$

$$z(-0.5,0.5) = -0.5 + \sqrt{0.5+3} = 1.37$$

$$z(0.5,0.5) = 0.5 + \sqrt{0.5+3} = 2.37$$

$$z(1.5,0.5) = 1.5 + \sqrt{0.5+3} = 3.37$$

$$z(-0.5,1.5) = -0.5 + \sqrt{1.5+3} = 1.62$$

$$z(1.5,1.5) = 1.5 + \sqrt{1.5+3} = 3.62$$

$$z(2.5,1.5) = 2.5 + \sqrt{1.5 + 3} = 4.62$$

$$z(-0.5,2.5) = -0.5 + \sqrt{2.5 + 3} = 1.85$$

$$z(1.5,2.5) = 1.5 + \sqrt{2.5 + 3} = 3.85$$

$$z(2.5,2.5) = 2.5 + \sqrt{2.5 + 3} = 4.85$$

A 1×1 square has area $A = 1$, and if we divide area of partially contained squares by 2, then the integral approximation is

$$A = 1 \cdot (2.62 + 2.85)$$

$$+0.5(1.37 + 2.37 + 3.37 + 1.62 + 3.62 + 4.62 + 1.85 + 3.85 + 4.85)$$

$$A = 5.47 + 0.5(27.52)$$

$$A \approx 19.2$$

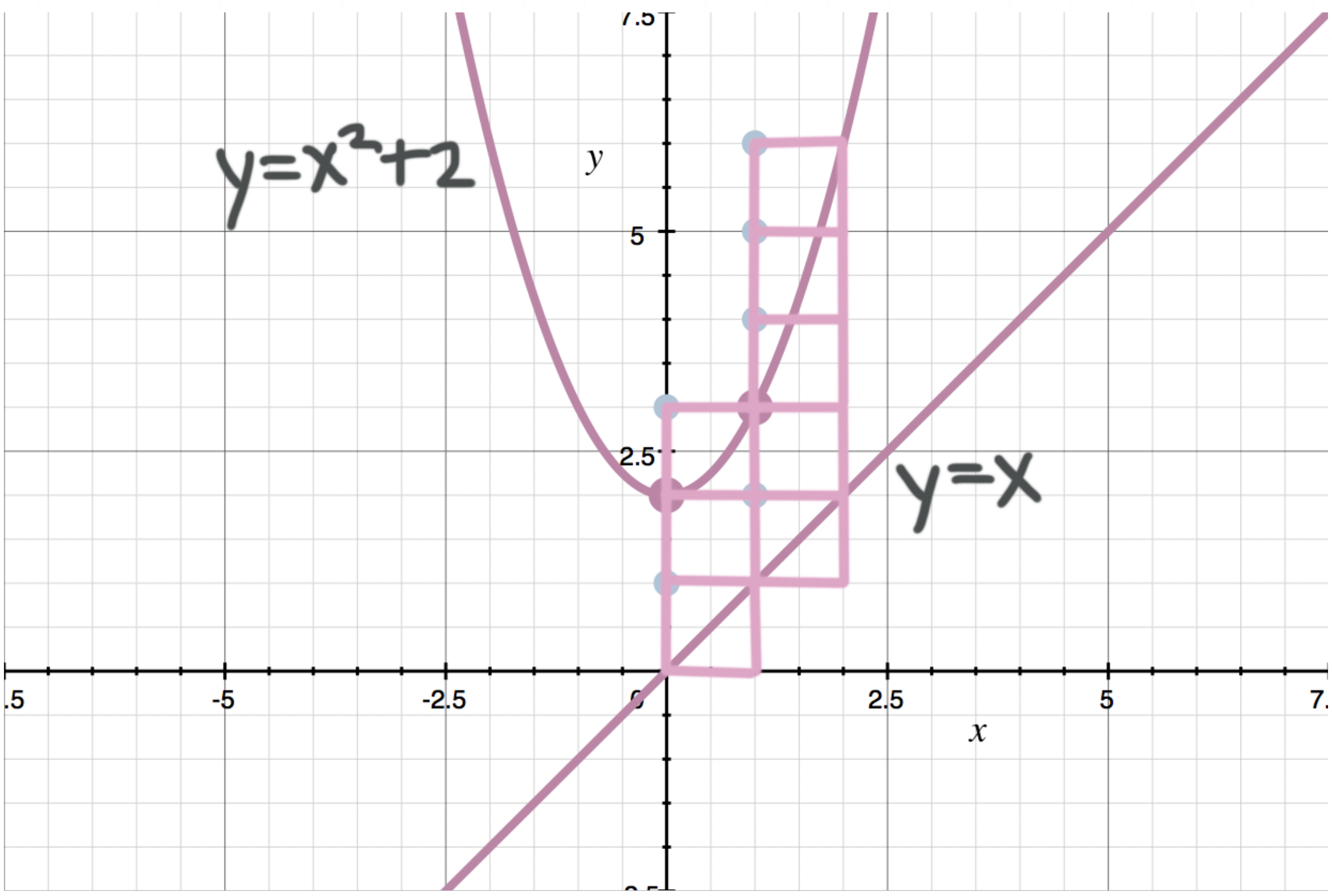
RIEMANN SUMS FOR DOUBLE INTEGRALS

■ 1. Use Riemann sums to approximate the integral, using squares with sides of length 1, and the upper-left corner of the squares. If the square partially lies within the domain, then divide its area in half.

$$\int_0^2 \int_x^{x^2+2} x^2 + 2x - 3y + 1 \ dy \ dx$$

Solution:

Sketch in the upper-left corner of each square.



Find the value of z for each point. The purple points are for squares that lie fully in the domain,

$$z(0,2) = 0^2 + 2(0) - 3(2) + 1 = -5$$

$$z(1,3) = 1^2 + 2(1) - 3(3) + 1 = -5$$

and the blue points are for squares partially in the domain.

$$z(0,1) = 0^2 + 2(0) - 3(1) + 1 = -2$$

$$z(0,3) = 0^2 + 2(0) - 3(3) + 1 = -8$$

$$z(1,2) = 1^2 + 2(1) - 3(2) + 1 = -2$$

$$z(1,4) = 1^2 + 2(1) - 3(4) + 1 = -8$$

$$z(1,5) = 1^2 + 2(1) - 3(5) + 1 = -11$$

$$z(1,6) = 1^2 + 2(1) - 3(6) + 1 = -14$$

The area of a square with side 1 is 1. So the integral approximation is

$$1\left(-5 - 5 + \frac{1}{2}(-2 - 8 - 2 - 8 - 11 - 14)\right)$$

$$-10 + \frac{1}{2}(-45)$$

$$-\frac{20}{2} - \frac{45}{2}$$

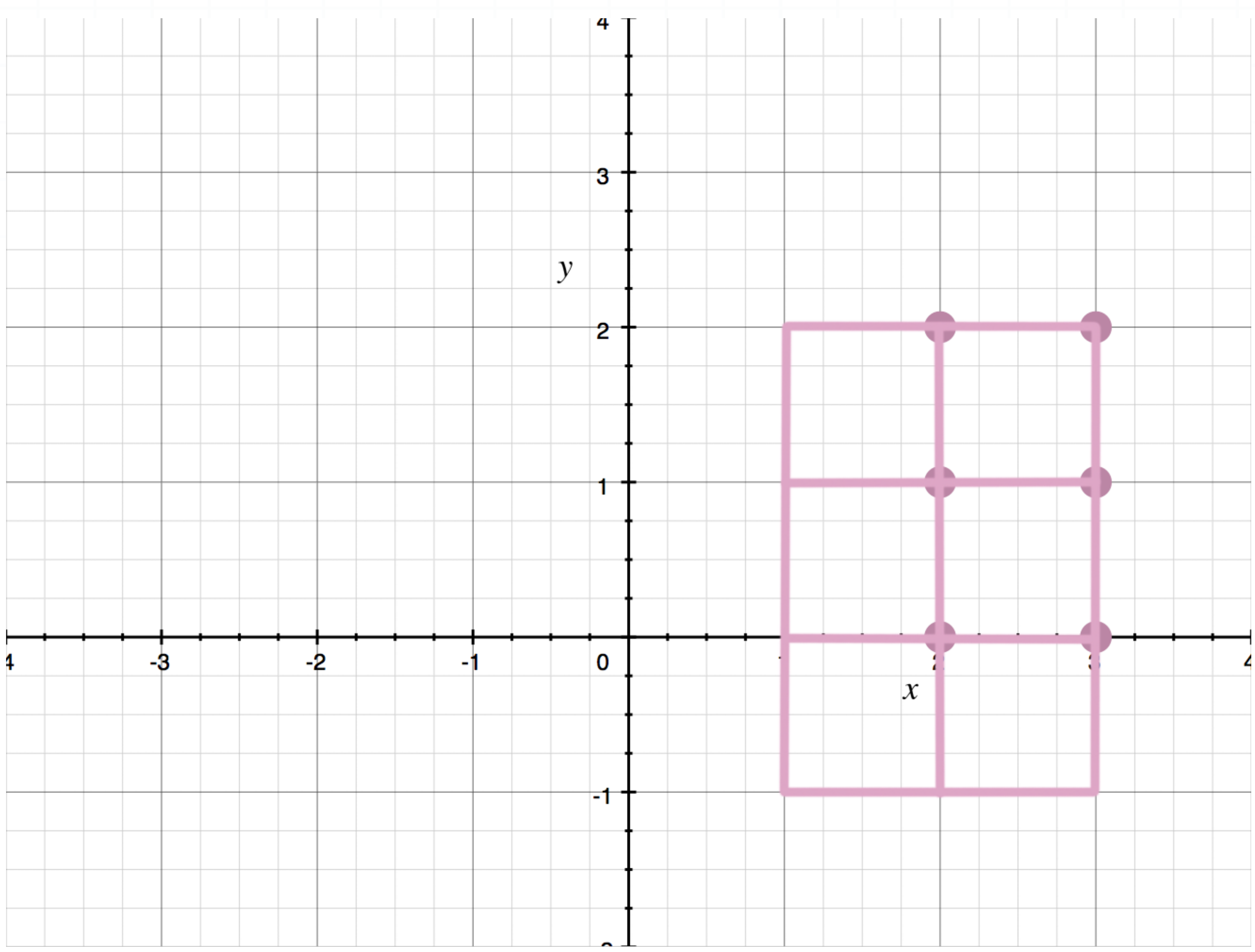
$$-\frac{65}{2}$$

■ 2. Use Riemann sums to approximate the integral over the rectangle $R = [1,3] \times [-1,2]$, using squares with sides of length 1, and the upper-right corners of the squares. Round your answer to the nearest tenth.

$$\iint_R \ln(x^2 + y + 2)\ dy\ dx$$

Solution:

Sketch in the upper-right corner of each square.



Find the value of z for each point, rounding the values to the nearest hundredth.

$$z(2,0) = \ln(2^2 + 0 + 2) \approx 1.79$$

$$z(3,0) = \ln(3^2 + 0 + 2) \approx 2.40$$

$$z(2,1) = \ln(2^2 + 1 + 2) \approx 1.95$$

$$z(3,1) = \ln(3^2 + 1 + 2) \approx 2.48$$

$$z(2,2) = \ln(2^2 + 2 + 2) \approx 2.08$$

$$z(3,2) = \ln(3^2 + 2 + 2) \approx 2.56$$

The area of a square with side 1 is 1. So the integral approximation, rounded to the nearest tenth, is

$$1(1.79 + 2.40 + 1.95 + 2.48 + 2.08 + 2.56)$$

$$1.79 + 2.40 + 1.95 + 2.48 + 2.08 + 2.56$$

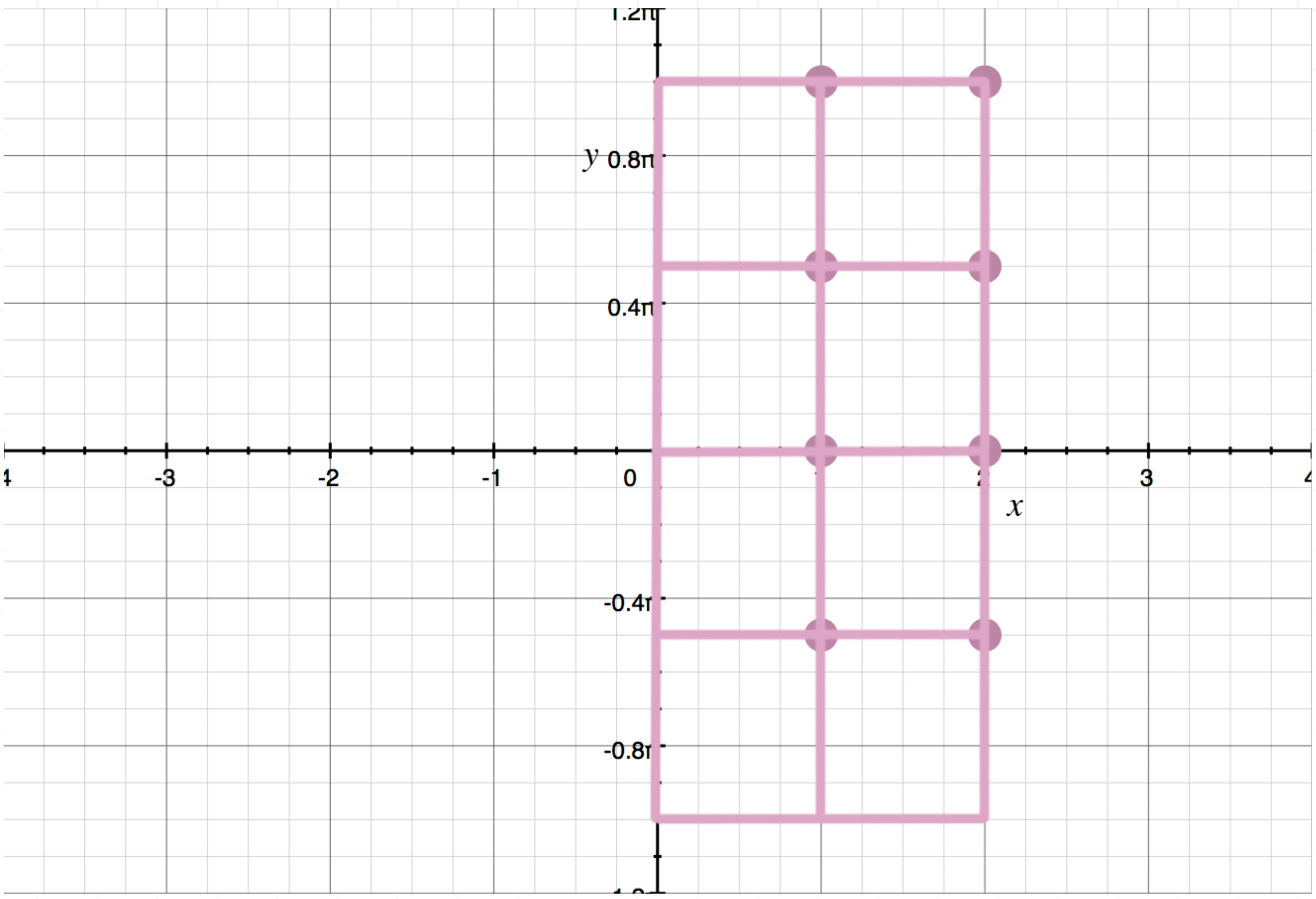
$$13.3$$

■ 3. Use Riemann sums to approximate the integral, using rectangles with sides $1 \times \pi/2$, and the upper-right corners of the rectangles. If the exact value is 14π, find the percentage error of the approximation.

$$\int_{-\pi}^{\pi} \int_{0}^{2} 2x + \sin^2 y + 1 \ dx \ dy$$

Solution:

Sketch in the upper-right corner of each rectangle.



Find the value of z for each point.

$$z(1, -\pi/2) = 2(1) + \sin^2(-\pi/2) + 1 = 4$$

$$z(2, -\pi/2) = 2(2) + \sin^2(-\pi/2) + 1 = 6$$

$$z(1,0) = 2(1) + \sin^2(0) + 1 = 3$$

$$z(2,0) = 2(2) + \sin^2(0) + 1 = 5$$

$$z(1,\pi/2) = 2(1) + \sin^2(\pi/2) + 1 = 4$$

$$z(2,\pi/2) = 2(2) + \sin^2(\pi/2) + 1 = 6$$

$$z(1,\pi) = 2(1) + \sin^2(\pi) + 1 = 3$$

$$z(2,\pi) = 2(2) + \sin^2(\pi) + 1 = 5$$

The area of a rectangle with dimensions $1 \times \pi/2$ is $\pi/2$. So the integral approximation is

$$\frac{\pi}{2} \cdot (4 + 6 + 3 + 5 + 4 + 6 + 3 + 5)$$

$$\frac{\pi}{2}(36)$$

$$18\pi$$

The percentage error is given by

$$\% \text{ error} = \frac{|\text{Exact value} - \text{Approximate value}|}{\text{Exact value}} \cdot 100\,\%$$

$$\% \text{ error} = \frac{|14\pi - 18\pi|}{14\pi} \cdot 100\,\%$$

$$\% \text{ error} = \frac{|-4\pi|}{14\pi} \cdot 100\,\%$$

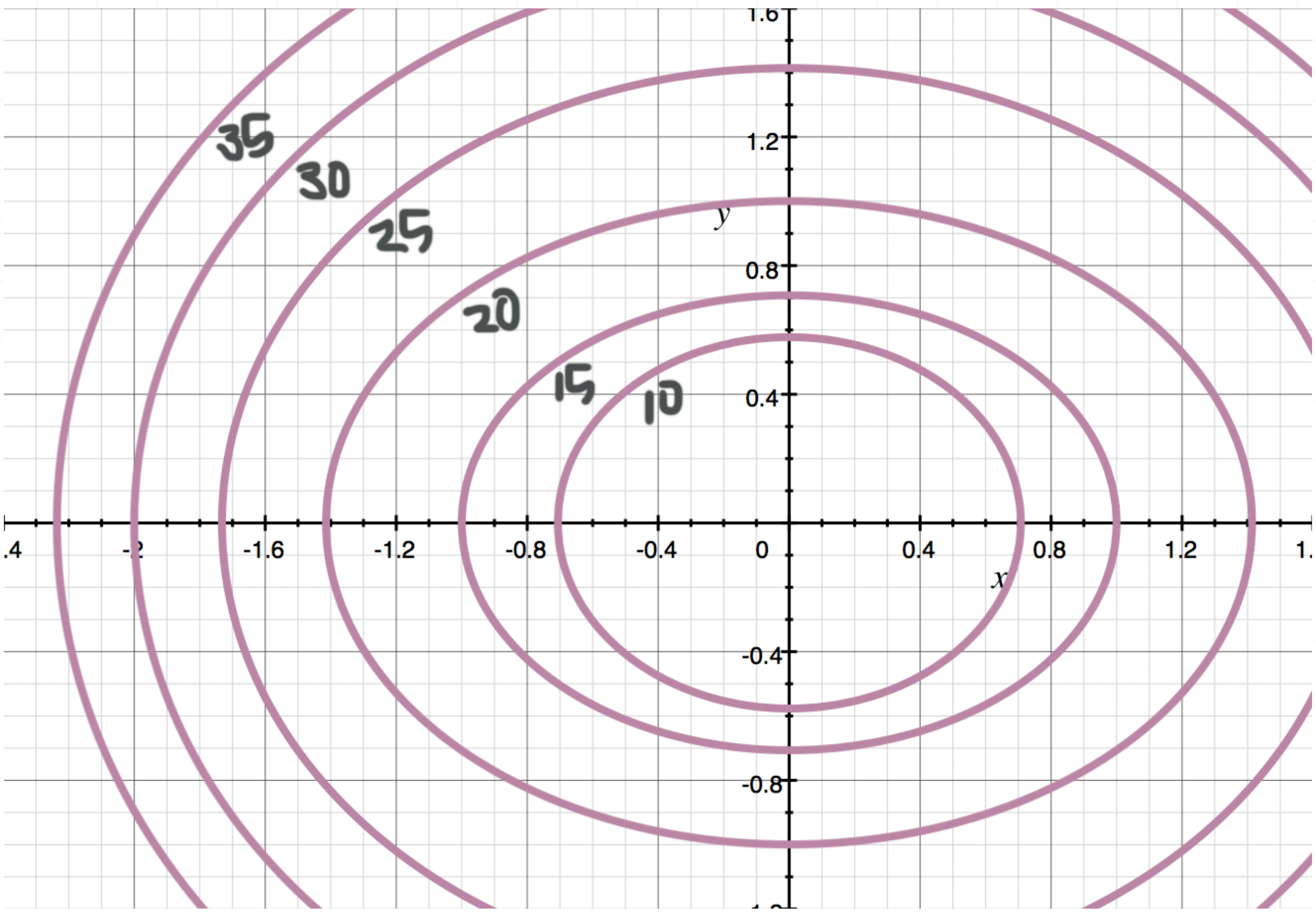
$$\% \text{ error} = \frac{4\pi}{14\pi} \cdot 100\,\%$$

$$\% \text{ error} = \frac{2}{7} \cdot 100\,\%$$

$$\% \text{ error} = 28.6\,\%$$

AVERAGE VALUE

■ 1. Use midpoints of squares with side lengths 1 to estimate the average value of the region $R = [-2,1] \times [-2,2]$, given the sketch of level curves.



Solution:

We can find the area of the rectangle, and the area of the smaller squares.

$$A(R) = (3)(4) = 12$$

$$\Delta A = (1)(1) = 1$$

Find estimates of the function's value at each midpoint.

$$f(-1.5,-1.5)=28 \qquad f(-0.5,-1.5)=26 \qquad f(0.5,-1.5)=26$$

$$f(-1.5,-0.5)=18 \qquad f(-0.5,-0.5)=12 \qquad f(0.5,-0.5)=12$$

$$f(-1.5,0.5)=18 \qquad f(-0.5,0.5)=12 \qquad f(0.5,0.5)=12$$

$$z(-1.5,1.5)=28 \qquad f(-0.5,1.5)=26 \qquad f(0.5,1.5)=26$$

So the average value is

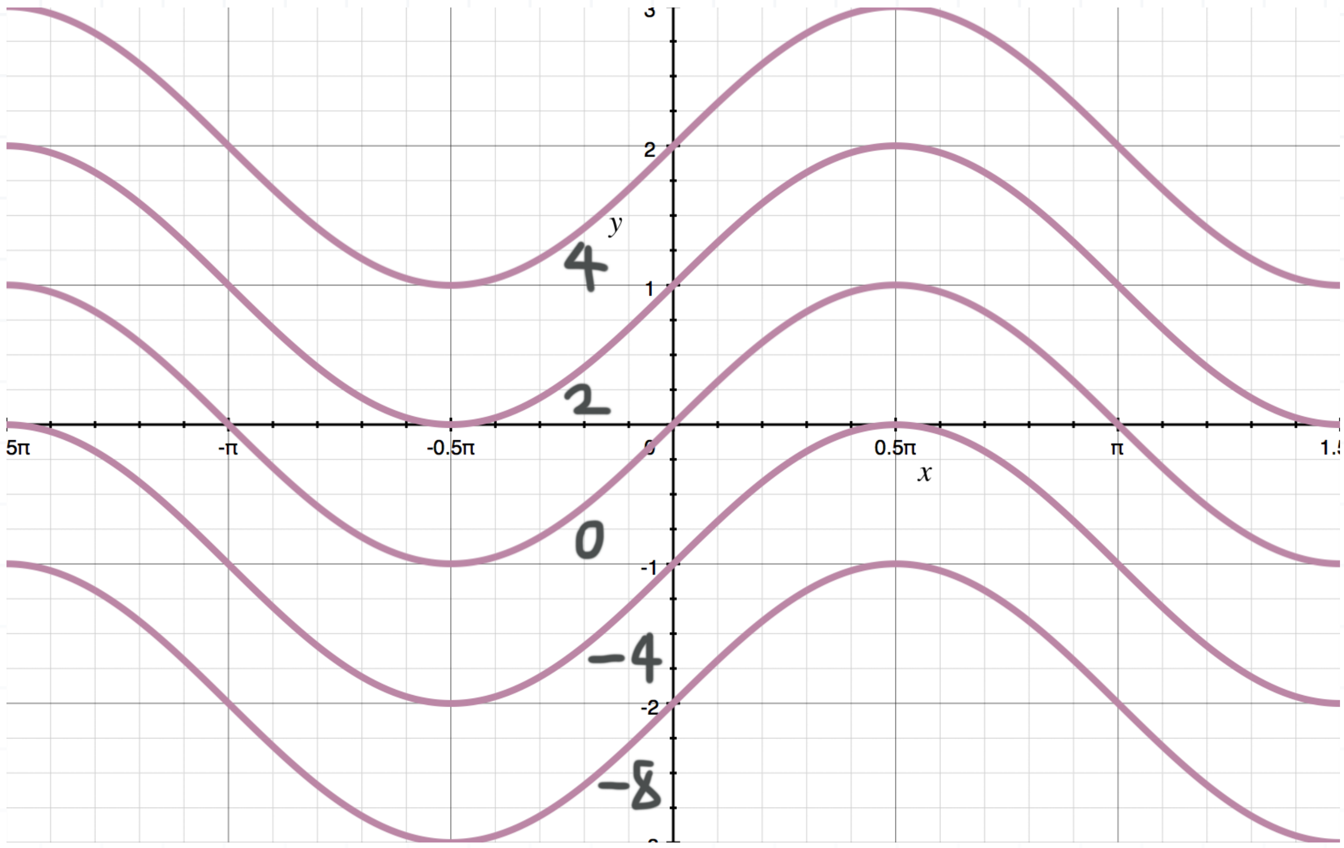
$$f_{avg}=\frac{1}{A(R)}\iint_R f(x,y)\ \Delta A$$

$$f_{avg}=\frac{1}{12}\cdot 1\cdot(28+26+26+18+12+12+18+12+12+28+26+26)$$

$$f_{avg}=\frac{1}{12}(244)$$

$$f_{avg}=\frac{61}{3}$$

■ 2. Use midpoints of rectangles with dimensions $\pi\times 1$ to estimate the average value of the region $R=[-\pi,\pi]\times[-2,2]$, given the sketch of level curves.



Solution:

We can find the area of the rectangle, and the area of the smaller squares.

$$A(R) = (2\pi)(4) = 8\pi$$

$$\Delta A = (\pi)(1) = \pi$$

Find estimates of the function's value at each midpoint.

$$f(-\pi/2, -1.5) = -2 \qquad f(\pi/2, -1.5) = -10$$

$$f(-\pi/2, -0.5) = 1 \qquad f(\pi/2, -0.5) = -6$$

$$f(-\pi/2, 0.5) = 3 \qquad f(\pi/2, 0.5) = -2$$

$$f(-\pi/2, 1.5) = 5 \qquad f(\pi/2, 1.5) = 1$$

So the average value is

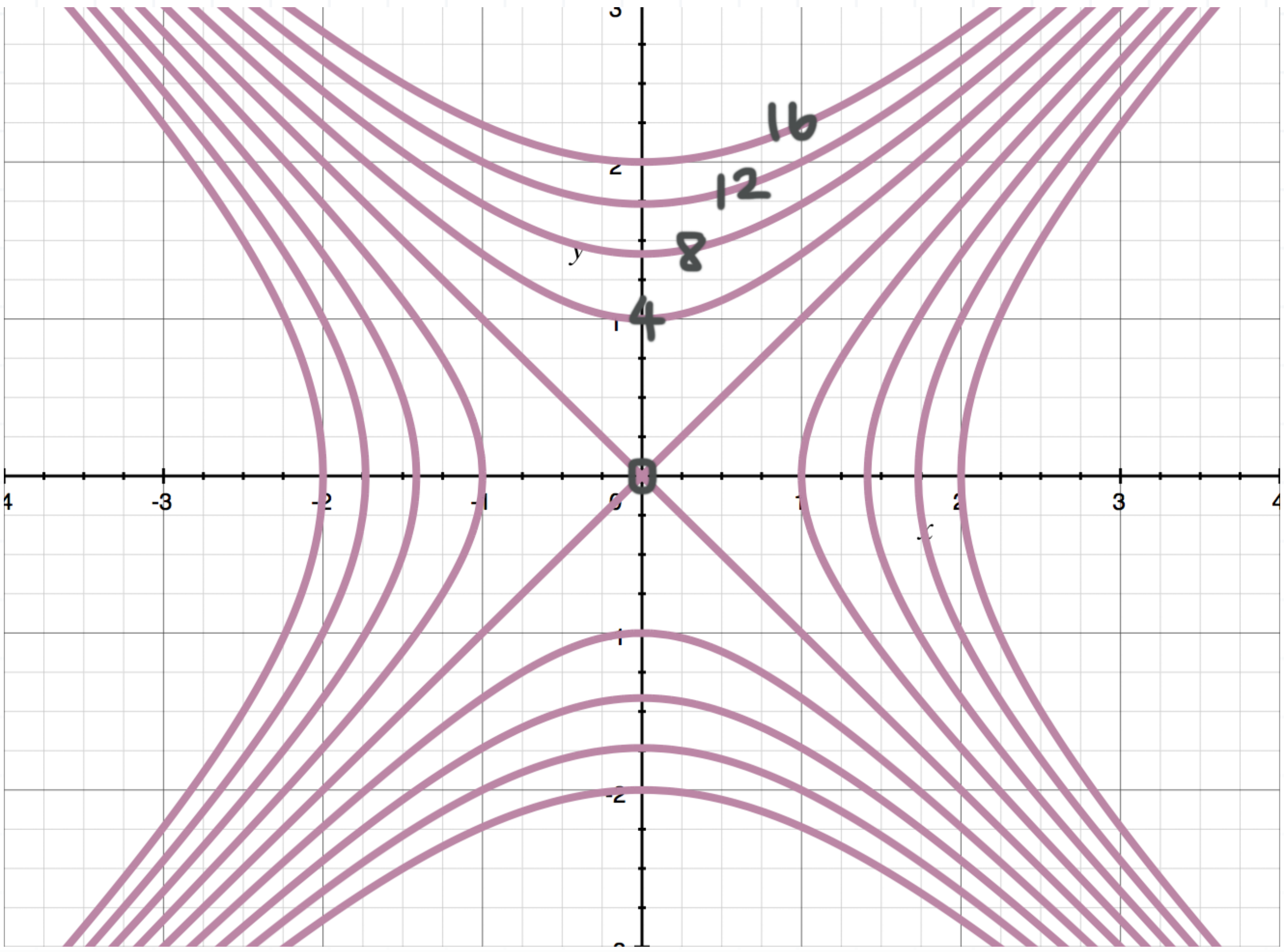
$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)\ \Delta A$$

$$f_{avg} = \frac{1}{8\pi} \cdot \pi \cdot (-2 - 10 + 1 - 6 + 3 - 2 + 5 + 1)$$

$$f_{avg} = \frac{1}{8}(-10)$$

$$f_{avg} = -\frac{5}{4}$$

■ 3. Use midpoints of rectangles with dimensions 2×1 to estimate the average value of the region $R = [-2,2] \times [-2,2]$, given the sketch of level curves.



Solution:

We can find the area of the rectangle, and the area of the smaller squares.

$$A(R) = (4)(4) = 16$$

$$\Delta A = (2)(1) = 2$$

Find estimates of the function's value at each midpoint. Since the level curves are symmetric about the origin, we can estimate the function values for the midpoints of the small rectangles in the first quadrant only.

$$f(1,0.5) = 3$$

$$f(1,1.5) = 5$$

So the average value is

$$f_{avg} = \frac{1}{A(R)} \iint_R f(x, y)\ \Delta A$$

$$f_{avg} = \frac{1}{16} \cdot 2 \cdot (3 + 5 + 3 + 5 + 3 + 5 + 3 + 5)$$

$$f_{avg} = \frac{1}{8}(32)$$

$$f_{avg} = 4$$

ITERATED INTEGRALS

1. Evaluate the iterated integral.

$$\int_{2}^{4}\int_{1}^{2}\log_2\frac{y^2}{x^4}\ dx\ dy$$

Solution:

Use laws of logs to rewrite the integrand.

$$\int_{2}^{4}\int_{1}^{2}2\log_2 y-4\log_2 x\ dx\ dy$$

Use the integration formula.

$$\int\log_a x\ dx=\frac{x\ln x-x}{\ln a}+C$$

Integrate with respect to x, treating y as a constant.

$$\int_{2}^{4}(2\log_2 y)x-\frac{4x\ln x-4x}{\ln 2}\Bigg|_{x=1}^{x=2}\ dy$$

$$\int_{2}^{4}(2\log_2 y)(2)-\frac{4(2)\ln 2-4(2)}{\ln 2}-\left((2\log_2 y)(1)-\frac{4(1)\ln 1-4(1)}{\ln 2}\right)\ dy$$

$$\int_{2}^{4}4\log_2 y-\frac{8\ln 2-8}{\ln 2}-\left(2\log_2 y-\frac{4\ln 1-4}{\ln 2}\right)\ dy$$

$$\int_{2}^{4} 4\log_2 y - \frac{8\ln 2 - 8}{\ln 2} - \left(2\log_2 y - \frac{4(0) - 4}{\ln 2}\right)\ dy$$

$$\int_{2}^{4} 4\log_2 y - \frac{8\ln 2 - 8}{\ln 2} - 2\log_2 y - \frac{4}{\ln 2}\ dy$$

$$\int_{2}^{4} 2\log_2 y - \left(\frac{8\ln 2 - 8}{\ln 2} + \frac{4}{\ln 2}\right)\ dy$$

$$\int_{2}^{4} 2\log_2 y - \frac{8\ln 2 - 4}{\ln 2}\ dy$$

$$\int_{2}^{4} 2\log_2 y - \frac{8\ln 2}{\ln 2} + \frac{4}{\ln 2}\ dy$$

$$\int_{2}^{4} 2\log_2 y - 8 + \frac{4}{\ln 2}\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\frac{2y\ln y - 2y}{\ln 2} + \frac{4}{\ln 2}y - 8y\Big|_{2}^{4}$$

$$\frac{2(4)\ln(4) - 2(4)}{\ln 2} + \frac{4}{\ln 2}(4) - 8(4) - \left(\frac{2(2)\ln(2) - 2(2)}{\ln 2} + \frac{4}{\ln 2}(2) - 8(2)\right)$$

$$\frac{8\ln 4 - 8}{\ln 2} + \frac{16}{\ln 2} - 32 - \frac{4\ln 2 - 4}{\ln 2} - \frac{8}{\ln 2} + 16$$

$$\frac{8\ln 4 - 8 + 8 - 4\ln 2 + 4}{\ln 2} - 16$$

$$\frac{8\ln 4 - 4\ln 2 + 4}{\ln 2} - \frac{16\ln 2}{\ln 2}$$

$$\frac{8\ln 4 - 4\ln 2 + 4 - 16\ln 2}{\ln 2}$$

$$\frac{8\ln 4 - 20\ln 2 + 4}{\ln 2}$$

$$\frac{8(\ln 2 + \ln 2) - 20\ln 2 + 4}{\ln 2}$$

$$\frac{8\ln 2 + 8\ln 2 - 20\ln 2 + 4}{\ln 2}$$

$$\frac{4 - 4\ln 2}{\ln 2}$$

■ 2. Evaluate the iterated integral.

$$\int_{-5}^{5}\int_{0}^{\pi}(3x^2 - 4x + 10)\sin(y + \pi)\ dy\ dx$$

Solution:

Integrate with respect to y, treating x as a constant.

$$\int_{-5}^{5} -(3x^2 - 4x + 10)\cos(y + \pi)\Big|_{y=0}^{y=\pi}\ dx$$

$$\int_{-5}^{5} -(3x^2-4x+10)\cos(\pi+\pi)+(3x^2-4x+10)\cos(0+\pi)\ dx$$

$$\int_{-5}^{5} -(3x^2-4x+10)(1)+(3x^2-4x+10)(-1)\ dx$$

$$\int_{-5}^{5} -3x^2+4x-10-3x^2+4x-10\ dx$$

$$\int_{-5}^{5} -6x^2+8x-20\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$-2x^3+4x^2-20x\Big|_{-5}^{5}$$

$$-2(5)^3+4(5)^2-20(5)-(-2(-5)^3+4(-5)^2-20(-5))$$

$$-2(125)+4(25)-20(5)+2(-125)-4(25)+20(-5)$$

$$-250+100-100-250-100-100$$

$$-700$$

■ 3. Evaluate the iterated integral.

$$\int_{-1}^{1}\int_{0}^{2} xe^{x^2-3y+1}\ dx\ dy$$

Solution:

Rewrite the integrand using laws of exponents.

$$\int_{-1}^{1}\int_{0}^{2} xe^{x^2-3y+1}\ dx\ dy$$

$$\int_{-1}^{1}\int_{0}^{2} xe^{x^2}\cdot e^{-3y}\cdot e^{1}\ dx\ dy$$

$$e\int_{-1}^{1}\int_{0}^{2} xe^{x^2}\cdot e^{-3y}\ dx\ dy$$

Since e^{-3y} is a constant for the inner integral,

$$e\int_{-1}^{1} e^{-3y}\int_{0}^{2} xe^{x^2}\ dx\ dy$$

$$e\int_{-1}^{1} e^{-3y}\ dy\cdot\int_{0}^{2} xe^{x^2}\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$-\frac{1}{3}e\cdot e^{-3y}\Big|_{-1}^{1}\cdot\int_{0}^{2} xe^{x^2}\ dx$$

$$-\frac{1}{3}e(e^{-3(1)}-e^{-3(-1)})\cdot\int_{0}^{2} xe^{x^2}\ dx$$

$$-\frac{1}{3}e(e^{-3}-e^{3})\cdot\int_{0}^{2} xe^{x^2}\ dx$$

$$\left(-\frac{1}{3}e^{-2}+\frac{1}{3}e^{4}\right)\int_0^2 xe^{x^2}\ dx$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\int_0^2 xe^{x^2}\ dx$$

Use u-substitution,

$$u=x^2$$

$$dx=\frac{1}{2x}du$$

to rewrite the integrand.

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\int_{x=0}^{x=2} xe^{u}\cdot\frac{1}{2x}du$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\cdot\frac{1}{2}\int_{x=0}^{x=2} e^{u}\ du$$

Integrate, then back-substitute, and evaluate over the interval.

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\cdot\frac{1}{2}e^{u}\Big|_{x=0}^{x=2}$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\cdot\frac{1}{2}e^{x^2}\Big|_{x=0}^{x=2}$$

$$\left(\frac{1}{3}e^{4}-\frac{1}{3e^2}\right)\left(\frac{1}{2}e^{2^2}-\frac{1}{2}e^{0^2}\right)$$

$$\left(\frac{1}{3}e^4-\frac{1}{3e^2}\right)\left(\frac{1}{2}e^4-\frac{1}{2}(1)\right)$$

$$\left(\frac{1}{3}e^4-\frac{1}{3e^2}\right)\left(\frac{1}{2}e^4-\frac{1}{2}\right)$$

$$\left(\frac{e^4}{3}-\frac{e^{-2}}{3}\right)\left(\frac{e^4-1}{2}\right)$$

$$\left(\frac{e^4-e^{-2}}{3}\right)\left(\frac{e^4-1}{2}\right)$$

$$\frac{(e^4-e^{-2})(e^4-1)}{6}$$

DOUBLE INTEGRALS

1. Evaluate the double integral, where R is the rectangle $[0,\pi] \times [0,1]$.

$$\iint_R \cos(x - \pi y)\ dx\ dy$$

Solution:

Since we're integrating over R, we'll add bounds to the integrals.

$$\int_0^1 \int_0^\pi \cos(x - \pi y)\ dx\ dy$$

Integrate with respect to x, treating y as a constant.

$$\int_0^1 \sin(x - \pi y) \Big|_{x=0}^{x=\pi}\ dy$$

$$\int_0^1 \sin(\pi - \pi y) - \sin(0 - \pi y)\ dy$$

$$\int_0^1 \sin(\pi - \pi y) - \sin(-\pi y)\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\frac{1}{\pi}\cos(\pi - \pi y) - \frac{1}{\pi}\cos(-\pi y)\Big|_0^1$$

$$\frac{1}{\pi}\cos(\pi-\pi(1))-\frac{1}{\pi}\cos(-\pi(1))-\left(\frac{1}{\pi}\cos(\pi-\pi(0))-\frac{1}{\pi}\cos(-\pi(0))\right)$$

$$\frac{1}{\pi}\cos(\pi-\pi)-\frac{1}{\pi}\cos(-\pi)-\frac{1}{\pi}\cos(\pi)+\frac{1}{\pi}\cos(0)$$

$$\frac{1}{\pi}(1)-\frac{1}{\pi}(-1)-\frac{1}{\pi}(-1)+\frac{1}{\pi}(1)$$

$$\frac{1}{\pi}+\frac{1}{\pi}+\frac{1}{\pi}+\frac{1}{\pi}$$

$$\frac{4}{\pi}$$

■ 2. Evaluate the double integral, where R is the rectangle $[1,3]\times[1,5]$.

$$\int\int_R \frac{1}{(x+y)^2}\ dx\ dy$$

Solution:

Since we're integrating over R, we'll add bounds to the integrals.

$$\int_1^5\int_1^3 \frac{1}{(x+y)^2}\ dx\ dy$$

Integrate with respect to x, treating y as a constant.

$$\int_1^5 -\frac{1}{x+y}\Big|_{x=1}^{x=3}\ dy$$

$$\int_1^5 -\frac{1}{3+y}-\left(-\frac{1}{1+y}\right)\ dy$$

$$\int_1^5 -\frac{1}{3+y}+\frac{1}{1+y}\ dy$$

$$\int_1^5 \frac{1}{1+y}-\frac{1}{3+y}\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\ln|1+y|-\ln|3+y|\Big|_1^5$$

$$\ln|1+5|-\ln|3+5|-\ln|1+1|+\ln|3+1|$$

$$\ln 6-\ln 8-\ln 2+\ln 4$$

$$\ln\frac{6}{8}-\ln 2+\ln 4$$

$$\ln\frac{\frac{6}{8}}{2}+\ln 4$$

$$\ln\frac{6}{8\cdot 2}+\ln 4$$

$$\ln\frac{6\cdot 4}{8\cdot 2}$$

$$\ln\frac{3}{2}$$

■ 3. Evaluate the double integral, where R is the rectangle $[x, y] = [-\pi/2, \pi/2] \times [0, \pi]$.

$$\iint_R \cos(x+y) - x\sin(x+y)\ dx\ dy$$

Solution:

Since we're integrating over R, we'll add bounds to the integrals.

$$\int_0^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x+y) - x\sin(x+y)\ dx\ dy$$

Integrate with respect to x, treating y as a constant. Use integration by parts on the second integral.

$$\int_0^2 \left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos(x+y)\ dx - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(x+y)\ dx \right]\ dy$$

$$\int_0^2 \left[\sin(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin(x+y)\ dx \right]\ dy$$

$$u = x$$

$$du = dx$$

$$dv = \sin(x+y)\ dx$$

$$v = -\cos(x+y)$$

$$\int_0^2 \left[\sin(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \left(-x\cos(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} -\cos(x+y)\ dx \right) \right]\ dy$$

$$\int_0^2 \left[\sin(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} - \left(-x\cos(x+y) + \sin(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \right) \right]\ dy$$

$$\int_0^2 \left[\sin(x+y) + x\cos(x+y) - \sin(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \right]\ dy$$

$$\int_0^2 \left[x\cos(x+y)\Big|_{x=-\frac{\pi}{2}}^{x=\frac{\pi}{2}} \right]\ dy$$

$$\int_0^2 \left[\frac{\pi}{2}\cos\left(\frac{\pi}{2}+y\right) - \left(-\frac{\pi}{2}\right)\cos\left(-\frac{\pi}{2}+y\right) \right]\ dy$$

$$\int_0^2 \frac{\pi}{2}\cos\left(\frac{\pi}{2}+y\right) + \frac{\pi}{2}\cos\left(-\frac{\pi}{2}+y\right)\ dy$$

$$\frac{\pi}{2}\int_0^2 \cos\left(\frac{\pi}{2}+y\right) + \cos\left(-\frac{\pi}{2}+y\right)\ dy$$

Integrate with respect to y, then evaluate over the interval.

$$\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+y\right) + \sin\left(-\frac{\pi}{2}+y\right)\right)\Big|_0^2$$

$$\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+2\right) + \sin\left(-\frac{\pi}{2}+2\right)\right) - \frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+0\right) + \sin\left(-\frac{\pi}{2}+0\right)\right)$$

$$\frac{\pi}{2}\left(\sin\left(\frac{\pi}{2}+2\right)+\sin\left(-\frac{\pi}{2}+2\right)\right)-\frac{\pi}{2}(1+(-1))$$

$$\frac{\pi}{2}\sin\left(\frac{\pi}{2}+2\right)+\frac{\pi}{2}\sin\left(-\frac{\pi}{2}+2\right)$$

TYPE I AND II REGIONS

■ 1. Evaluate the double integral if D is the circle centered at the origin with radius 4.

$$\iint_D 4x^2y + 3\ dA$$

Solution:

Since the area of integration is symmetric, the region can be treated either as Type I or Type II. Let's do this as a Type I region.

The equation of the circle centered at the origin with radius 4 is

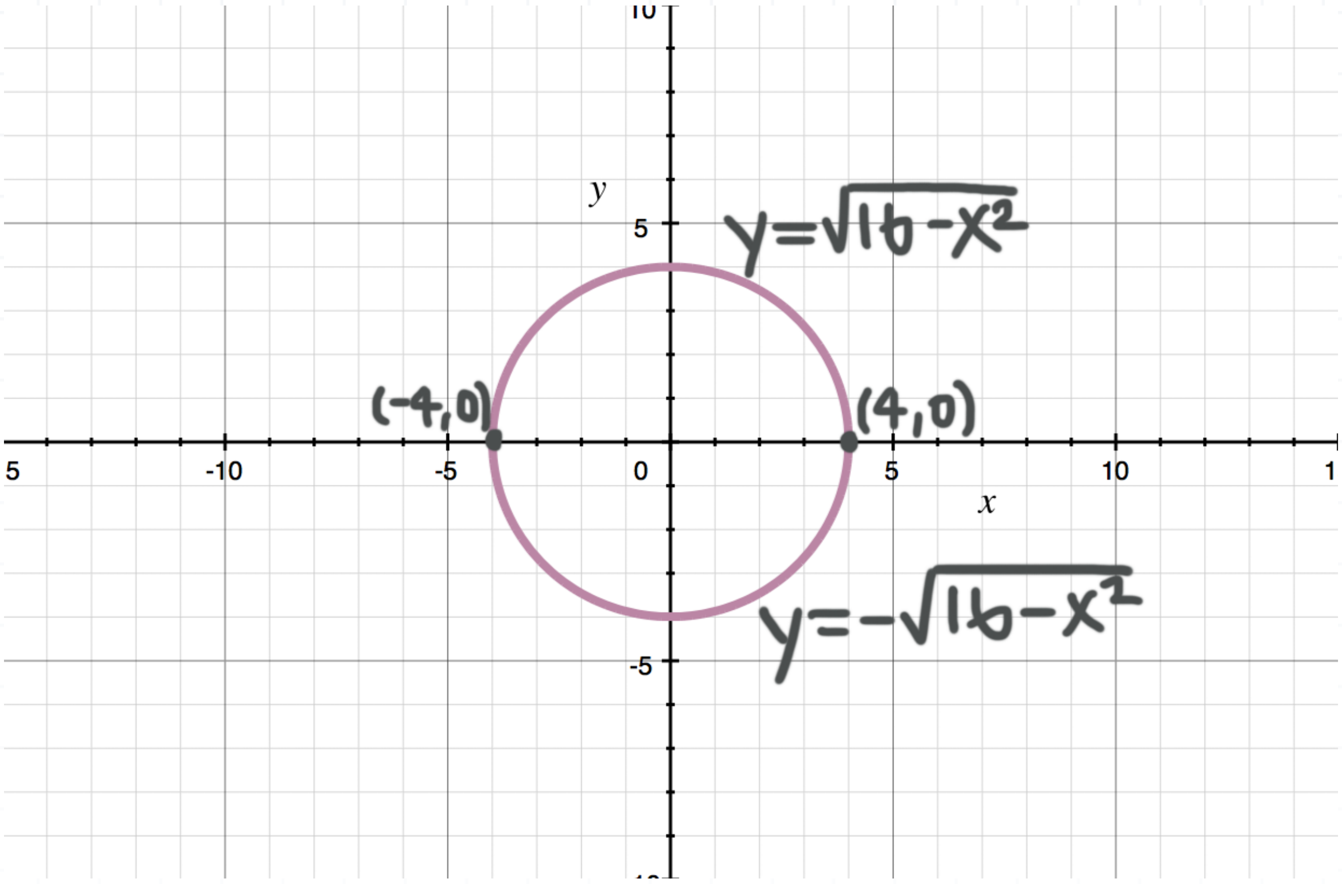
$$x^2 + y^2 = 4^2$$

Solve for y.

$$y^2 = 16 - x^2$$

$$y = \pm\sqrt{16 - x^2}$$

A sketch of the region is



So for every x from -4 to 4, y changes from $-\sqrt{16-x^2}$ to $\sqrt{16-x^2}$. Therefore, the given integral is equivalent to

$$\int_{-4}^{4}\int_{-\sqrt{16-x^2}}^{\sqrt{16-x^2}} 4x^2y+3\ dy\ dx$$

Integrate with respect to y.

$$\int_{-4}^{4} 2x^2y^2+3y\Big|_{y=-\sqrt{16-x^2}}^{y=\sqrt{16-x^2}}\ dx$$

$$\int_{-4}^{4} 2x^2(\sqrt{16-x^2})^2+3(\sqrt{16-x^2})-(2x^2(-\sqrt{16-x^2})^2+3(-\sqrt{16-x^2}))\ dx$$

$$\int_{-4}^{4} 2x^2(16-x^2)+3\sqrt{16-x^2}-(2x^2(16-x^2)-3\sqrt{16-x^2})\ dx$$

$$\int_{-4}^{4} 32x^2 - 2x^4 + 3\sqrt{16 - x^2} - 32x^2 + 2x^4 + 3\sqrt{16 - x^2}\ dx$$

$$\int_{-4}^{4} 6\sqrt{16 - x^2}\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$4(16 - x^2)^{\frac{3}{2}}\Big|_{-4}^{4}$$

$$4(16 - 4^2)^{\frac{3}{2}} - 4(16 - (-4)^2)^{\frac{3}{2}}$$

$$4(16 - 16)^{\frac{3}{2}} - 4(16 - 16)^{\frac{3}{2}}$$

$$0$$

■ 2. Evaluate the double integral if D is the region bounded by the curves $y + x^2 - 4 = 0$ and $y + 2x^2 - 8 = 0$.

$$\iint_D 462y\sqrt{x + 2}\ dA$$

Solution:

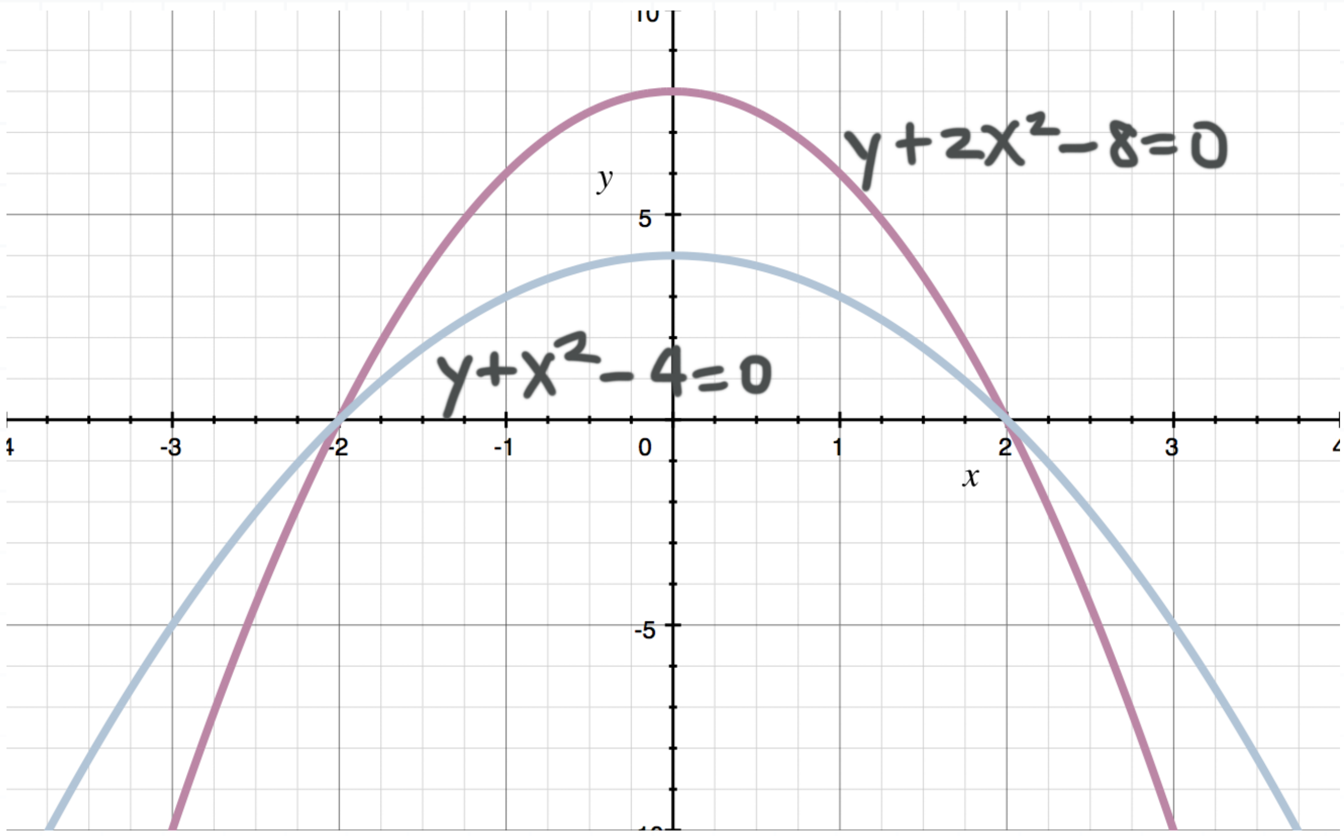
Find the points of intersection of $y = -x^2 + 4$ and $y = -2x^2 + 8$.

$$-x^2 + 4 = -2x^2 + 8$$

$$x^2 = 4$$

$$x = \pm 2$$

A sketch of the curves is



So we have a Type I region, and for every x from -2 to 2, y changes from $-x^2+4$ to $-2x^2+8$. Therefore, the given integral is equivalent to

$$\int_{-2}^{2} \int_{-x^2+4}^{-2x^2+8} 462y\sqrt{x+2}\ dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$\int_{-2}^{2} 231y^2\sqrt{x+2} \Big|_{y=-x^2+4}^{y=-2x^2+8} dx$$

$$\int_{-2}^{2} 231(-2x^2+8)^2\sqrt{x+2} - 231(-x^2+4)^2\sqrt{x+2}\ dx$$

$$231\int_{-2}^{2}(4x^4-32x^2+64)\sqrt{x+2}-(x^4-8x^2+16)\sqrt{x+2}\ dx$$

$$231\int_{-2}^{2}4x^4\sqrt{x+2}-32x^2\sqrt{x+2}+64\sqrt{x+2}$$

$$-x^4\sqrt{x+2}+8x^2\sqrt{x+2}-16\sqrt{x+2}\ dx$$

$$231\int_{-2}^{2}3x^4\sqrt{x+2}-24x^2\sqrt{x+2}+48\sqrt{x+2}\ dx$$

$$693\int_{-2}^{2}x^4\sqrt{x+2}-8x^2\sqrt{x+2}+16\sqrt{x+2}\ dx$$

Substitute $u=x+2$, $x=u-2$, and $du=dx$.

$$693\int_{x=-2}^{x=2}(u-2)^4\sqrt{u}-8(u-2)^2\sqrt{u}+16\sqrt{u}\ du$$

$$693\int_{x=-2}^{x=2}(u^4-8u^3+24u^2-32u+16)\sqrt{u}$$

$$-(8u^2-32u+32)\sqrt{u}+16\sqrt{u}\ du$$

$$693\int_{x=-2}^{x=2}u^{\frac{9}{2}}-8u^{\frac{7}{2}}+24u^{\frac{5}{2}}-8u^{\frac{5}{2}}-32u^{\frac{3}{2}}+32u^{\frac{3}{2}}+16u^{\frac{1}{2}}-32u^{\frac{1}{2}}+16u^{\frac{1}{2}}\ du$$

$$693\int_{x=-2}^{x=2}u^{\frac{9}{2}}-8u^{\frac{7}{2}}+16u^{\frac{5}{2}}\ du$$

Integrate with respect to x, back-substitute, then evaluate over the interval.

$$693\left(\frac{2}{11}u^{\frac{11}{2}}-\frac{16}{9}u^{\frac{9}{2}}+\frac{32}{7}u^{\frac{7}{2}}\right)\Bigg|_{x=-2}^{x=2}$$

$$693\left(\frac{2}{11}(x+2)^{\frac{11}{2}}-\frac{16}{9}(x+2)^{\frac{9}{2}}+\frac{32}{7}(x+2)^{\frac{7}{2}}\right)\Bigg|_{-2}^{2}$$

$$693\left(\frac{2}{11}(2+2)^{\frac{11}{2}}-\frac{16}{9}(2+2)^{\frac{9}{2}}+\frac{32}{7}(2+2)^{\frac{7}{2}}\right)$$

$$-693\left(\frac{2}{11}(-2+2)^{\frac{11}{2}}-\frac{16}{9}(-2+2)^{\frac{9}{2}}+\frac{32}{7}(-2+2)^{\frac{7}{2}}\right)$$

$$693\left(\frac{2}{11}(4)^{\frac{11}{2}}-\frac{16}{9}(4)^{\frac{9}{2}}+\frac{32}{7}(4)^{\frac{7}{2}}\right)$$

$$693\left(\frac{2}{11}2^{11}-\frac{16}{9}2^{9}+\frac{32}{7}2^{7}\right)$$

$$693\left(\frac{2^{12}}{11}-\frac{2^{13}}{9}+\frac{2^{12}}{7}\right)$$

$$693(2^{12})\left(\frac{1}{11}-\frac{2}{9}+\frac{1}{7}\right)$$

Find a common denominator.

$$693(2^{12})\left(\frac{63}{693}-\frac{154}{693}+\frac{99}{693}\right)$$

$$2^{12}(63-154+99)$$

$$2^{12}(8)$$

$$2^{15}$$

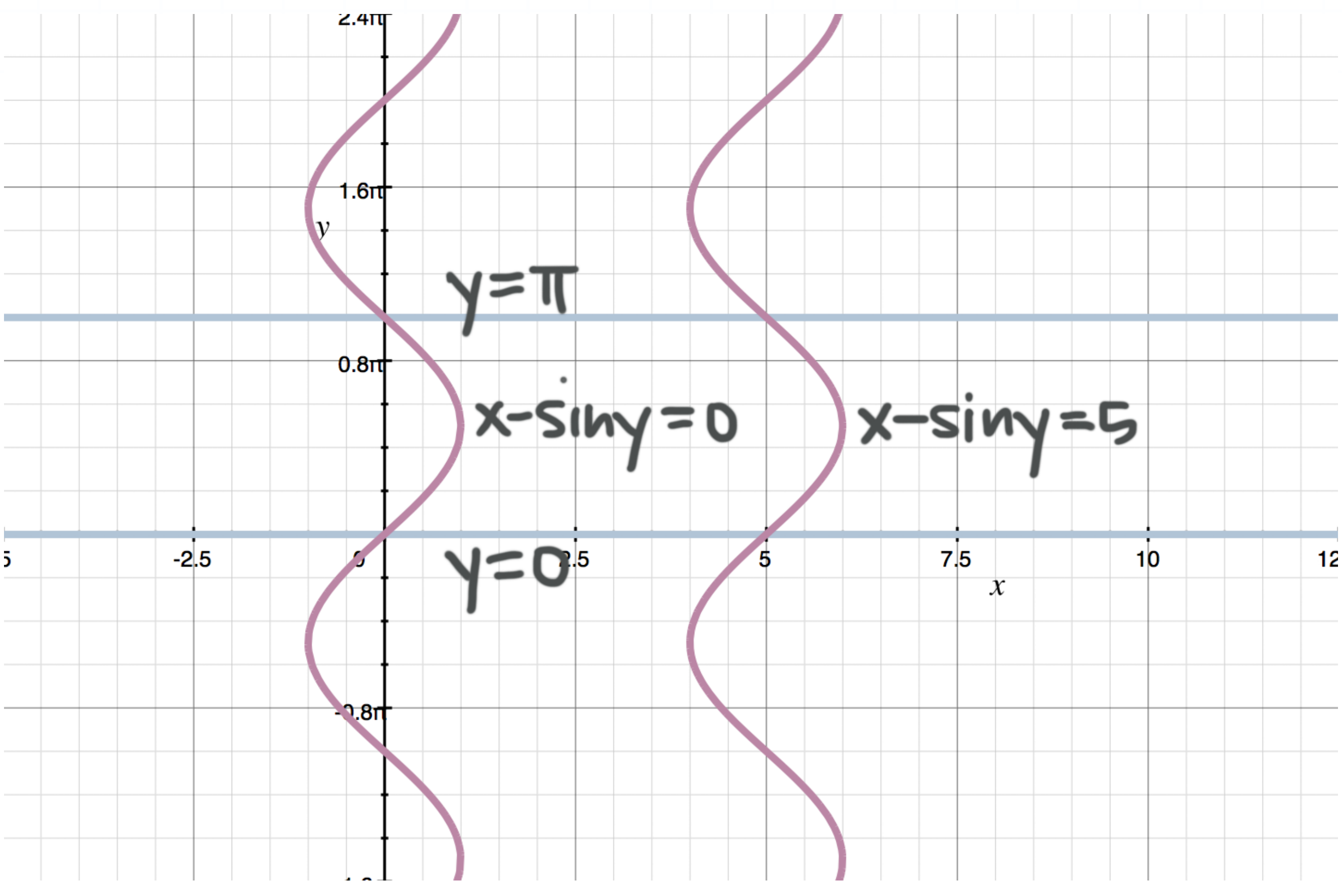
$$32,768$$

■ 3. Evaluate the double integral if D is the region bounded by $x - \sin y = 0$, $x - \sin y = 5$, $y = 0$, and $y = \pi$.

$$\iint_D 2x \ dA$$

Solution:

A sketch of the region is



We have a Type II region, and for every y from 0 to π, x changes from $\sin y$ to $\sin y + 5$. Therefore, the given integral is equivalent to

$$\int_0^{\pi} \int_{\sin y}^{\sin y+5} 2x \ dx \ dy$$

Integrate with respect to x, then evaluate over the interval.

$$\int_0^{\pi} x^2 \Big|_{x=\sin y}^{x=\sin y+5} \ dy$$

$$\int_0^{\pi} (\sin y + 5)^2 - (\sin y)^2 \ dy$$

$$\int_0^{\pi} \sin^2 y + 10 \sin y + 25 - \sin^2 y \ dy$$

$$\int_0^{\pi} 10 \sin y + 25 \ dy$$

Integrate with respect to y, then evaluate over the interval.

$$25y - 10 \cos y \Big|_0^{\pi}$$

$$25\pi - 10 \cos \pi - (25(0) - 10 \cos 0)$$

$$25\pi - 10 \cos \pi + 10 \cos 0$$

$$25\pi - 10(-1) + 10(1)$$

$$25\pi + 10 + 10$$

$$25\pi + 20$$

FINDING SURFACE AREA

■ 1. Find area of the surface $z = \sqrt{3}x + y^2 + 1$ inside the rectangle $-1 \le x \le 1$, $0 \le y \le 1$.

Solution:

The partial derivatives of z are

$$z_x = \sqrt{3}$$

$$z_y = 2y$$

Then the area of the surface is

$$A = \iint_D \sqrt{1 + z_x^2 + z_y^2} \ dA$$

$$A = \int_{-1}^{1} \int_0^1 \sqrt{1 + z_x^2 + z_y^2} \ dy \ dx$$

$$A = \int_{-1}^{1} \int_0^1 \sqrt{1 + 3 + 4y^2} \ dy \ dx$$

$$A = 2\int_{-1}^{1} \int_0^1 \sqrt{1 + y^2} \ dy \ dx$$

Use a substitution with $y = \tan u$ and $dy = \sec^2 u \ du$. The bounds for y change from $[0,1]$ to $[0,\pi/4]$.

$$A = 2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sqrt{1+\tan^2 u}\sec^2 u\ du\ dx$$

$$A = 2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sqrt{\sec^2 u}\sec^2 u\ du\ dx$$

$$A = 2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sec u\sec^2 u\ du\ dx$$

$$A = 2\int_{-1}^{1}\int_{0}^{\frac{\pi}{4}}\sec^3 u\ du\ dx$$

The integral of $\sec^3 x$ can be found with the reduction formula

$$\int \sec^3 x\ dx = \frac{1}{2}\tan x\sec x + \frac{1}{2}\int \sec x\ dx$$

Substitute from the reduction formula into the double integral.

$$A = 2\int_{-1}^{1}\left[\frac{1}{2}\tan u\sec u\Big|_{0}^{\frac{\pi}{4}} + \frac{1}{2}\int_{0}^{\frac{\pi}{4}}\sec u\ du\right]\ dx$$

$$A = 2\int_{-1}^{1}\left[\frac{1}{2}\tan u\sec u\Big|_{0}^{\frac{\pi}{4}} + \frac{1}{2}\ln|\sec u+\tan u|\Big|_{0}^{\frac{\pi}{4}}\right]\ dx$$

$$A = 2\int_{-1}^{1}\frac{1}{2}\tan u\sec u + \frac{1}{2}\ln|\sec u+\tan u|\Big|_{0}^{\frac{\pi}{4}}\ dx$$

$$A = 2\int_{-1}^{1}\frac{1}{2}\tan\frac{\pi}{4}\sec\frac{\pi}{4} + \frac{1}{2}\ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|$$

$$-\left(\frac{1}{2}\tan(0)\sec(0)+\frac{1}{2}\ln|\sec(0)+\tan(0)|\right)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{2}(1)\frac{2}{\sqrt{2}}+\frac{1}{2}\ln\left|\frac{2}{\sqrt{2}}+1\right|-\left(\frac{1}{2}(0)(1)+\frac{1}{2}\ln|1+0|\right)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}+\frac{1}{2}\ln\left(\frac{2}{\sqrt{2}}+1\right)-\frac{1}{2}\ln 1\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}+\frac{1}{2}\ln\left(\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}}+1\right)-\frac{1}{2}(0)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}+\frac{1}{2}\ln(\sqrt{2}+1)\ dx$$

$$A=2\int_{-1}^{1}\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{\sqrt{2}}\right)+\frac{1}{2}\ln(\sqrt{2}+1)\ dx$$

$$A=2\int_{-1}^{1}\frac{\sqrt{2}}{2}+\frac{1}{2}\ln(\sqrt{2}+1)\ dx$$

$$A=\int_{-1}^{1}\sqrt{2}+\ln(\sqrt{2}+1)\ dx$$

The integrand is now a constant (it doesn't include any x variables), so the integral is

$$A=(\sqrt{2}+\ln(\sqrt{2}+1))x\Big|_{-1}^{1}$$

$$A = (\sqrt{2} + \ln(\sqrt{2} + 1))(1) - (\sqrt{2} + \ln(\sqrt{2} + 1))(-1)$$

$$A = (\sqrt{2} + \ln(\sqrt{2} + 1)) + (\sqrt{2} + \ln(\sqrt{2} + 1))$$

$$A = 2(\sqrt{2} + \ln(\sqrt{2} + 1))$$

$$A = 2\sqrt{2} + 2\ln(\sqrt{2} + 1)$$

■ 2. Find area of the surface $z = \ln(\sin(3x)) + 2\sqrt{2}y - 5$ inside the rectangle $\pi/6 \le x \le \pi/4$, $0 \le y \le 1$.

Solution:

The partial derivatives of z are

$$z_x = \frac{1}{\sin(3x)}(\cos(3x))(3)$$

$$z_x = 3\cot(3x)$$

and

$$z_y = 2\sqrt{2}$$

Then the area of the surface is

$$A = \iint_D \sqrt{1 + z_x^2 + z_y^2}\ dA$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}\sqrt{1+z_x^2+z_y^2}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}\sqrt{1+(3\cot(3x))^2+(2\sqrt{2})^2}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}\sqrt{1+9\cot^2(3x)+8}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}3\sqrt{1+\cot^2(3x)}\ dx\ dy$$

$$A=\int_0^1\int_{\pi/6}^{\pi/4}3\sqrt{\csc^2(3x)}\ dx\ dy$$

Since $\csc(3x)\geq 0$ inside the given region,

$$A=\int_0^1\int_{\pi/6}^{\pi/4}3\csc(3x)\ dx\ dy$$

Integrate with respect to x.

$$A=\int_0^1 -3\left(\frac{1}{3}\right)\ln|\csc(3x)+\cot(3x)|\Big|_{x=\pi/6}^{x=\pi/4}\ dy$$

$$A=\int_0^1 -\ln|\csc(3x)+\cot(3x)|\Big|_{x=\pi/6}^{x=\pi/4}\ dy$$

$$A=\int_0^1 -\ln\left|\csc\left(3\cdot\frac{\pi}{4}\right)+\cot\left(3\cdot\frac{\pi}{4}\right)\right|-\left(-\ln\left|\csc\left(3\cdot\frac{\pi}{6}\right)+\cot\left(3\cdot\frac{\pi}{6}\right)\right|\right)\ dy$$

$$A=\int_0^1 -\ln\left|\csc\frac{3\pi}{4}+\cot\frac{3\pi}{4}\right|+\ln\left|\csc\frac{\pi}{2}+\cot\frac{\pi}{2}\right|\ dy$$

$$A=\int_0^1 -\ln\left|\frac{2}{\sqrt{2}}+(-1)\right|+\ln|1+0|\ dy$$

$$A=\int_0^1 -\ln\left|\frac{\sqrt{2}\sqrt{2}}{\sqrt{2}}-1\right|+0\ dy$$

$$A=\int_0^1 -\ln(\sqrt{2}-1)\ dy$$

The integrand is now a constant (it doesn't include any y variables), so the integral is

$$A=(-\ln(\sqrt{2}-1))y\Big|_0^1$$

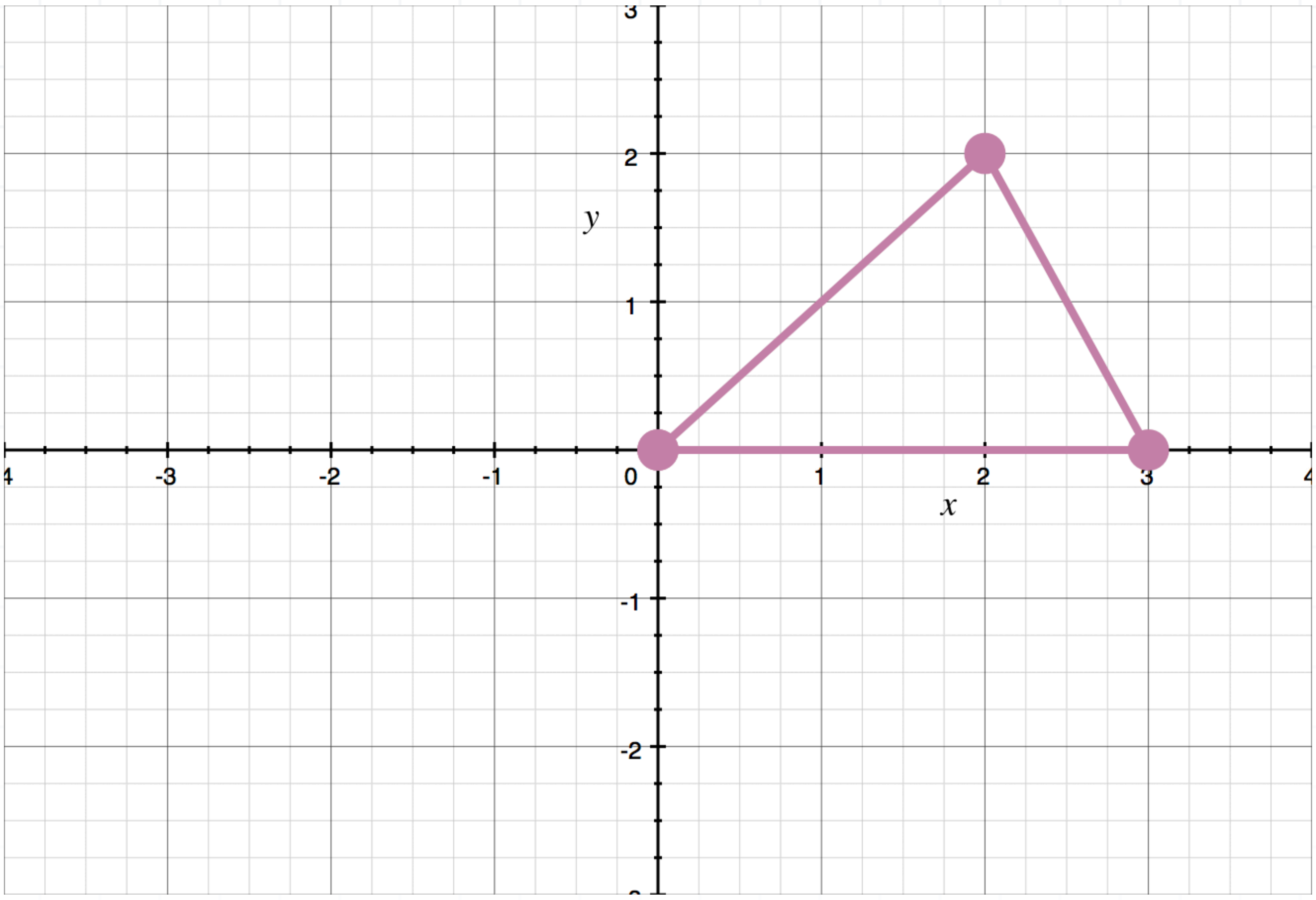
$$A=(-\ln(\sqrt{2}-1))(1)-((-\ln(\sqrt{2}-1))(0))$$

$$A=-\ln(\sqrt{2}-1)$$

■ 3. Find area of the surface $z=2(x+3)^{3/2}+5^{3/2}y-6$ inside the triangle OAB, if O is the origin and A and B are at $A(3,0)$ and $B(2,2)$.

Solution:

Sketch the triangle.



Based on the sketch, treat the triangle as a Type II region. The equation of line OB is $y = x$, and the equation of line AB is $2x + y = 6$, or

$$x = -\frac{y}{2} + 3$$

Find the partial derivatives of $z = 2(x+3)^{3/2} + 5^{3/2}y - 6$.

$$z_x = 2 \cdot \frac{3}{2}(x+3)^{1/2}$$

$$z_x = 3\sqrt{x+3}$$

and

$$z_y = 5^{3/2}$$

Then the area of the surface is

$$A=\iint_{OAB}\sqrt{1+z_x^2+z_y^2}\ dA$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+z_x^2+z_y^2}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+(3\sqrt{x+3})^2+(5^{3/2})^2}\ dx\ dy$$

since $x+3\geq 0$ inside the triangle OAB,

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+9(x+3)+125}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{1+9x+27+125}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}\sqrt{9x+153}\ dx\ dy$$

$$A=\int_0^2\int_y^{-\frac{y}{2}+3}3\sqrt{x+17}\ dx\ dy$$

Integrate with respect to x.

$$A=\int_0^2 3\cdot\frac{2}{3}(x+17)^{\frac{3}{2}}\Big|_{x=y}^{x=-\frac{y}{2}+3}\ dy$$

$$A=\int_0^2 2(x+17)^{\frac{3}{2}}\Big|_{x=y}^{x=-\frac{y}{2}+3}\ dy$$

$$A=\int_0^2 2\left(-\frac{y}{2}+3+17\right)^{\frac{3}{2}}-2(y+17)^{\frac{3}{2}}\ dy$$

$$A=\int_0^2 2\left(-\frac{y}{2}+20\right)^{\frac{3}{2}}-2(y+17)^{\frac{3}{2}}\ dy$$

Integrate with respect to y.

$$A=2\cdot\frac{2}{5}(-2)\left(-\frac{y}{2}+20\right)^{\frac{5}{2}}-2\cdot\frac{2}{5}(y+17)^{\frac{5}{2}}\Big|_0^2$$

$$A=-\frac{8}{5}\left(-\frac{y}{2}+20\right)^{\frac{5}{2}}-\frac{4}{5}(y+17)^{\frac{5}{2}}\Big|_0^2$$

$$A=-\frac{8}{5}\left(-\frac{2}{2}+20\right)^{\frac{5}{2}}-\frac{4}{5}(2+17)^{\frac{5}{2}}-\left[-\frac{8}{5}\left(-\frac{0}{2}+20\right)^{\frac{5}{2}}-\frac{4}{5}(0+17)^{\frac{5}{2}}\right]$$

$$A=-\frac{8}{5}(-1+20)^{\frac{5}{2}}-\frac{4}{5}(19)^{\frac{5}{2}}-\left[-\frac{8}{5}(0+20)^{\frac{5}{2}}-\frac{4}{5}(17)^{\frac{5}{2}}\right]$$

$$A=-\frac{8}{5}(19)^{\frac{5}{2}}-\frac{4}{5}(19)^{\frac{5}{2}}-\left[-\frac{8}{5}(20)^{\frac{5}{2}}-\frac{4}{5}(17)^{\frac{5}{2}}\right]$$

$$A=-\frac{12}{5}(19)^{\frac{5}{2}}+\frac{8}{5}(20)^{\frac{5}{2}}+\frac{4}{5}(17)^{\frac{5}{2}}$$

$$A\approx 38.9$$

FINDING VOLUME

1. Use a double integral to find the volume of the solid that's bounded by the surface and the xy-plane, on $0 \le x \le 2$ and $0 \le y \le \pi/2$.

$$z = \frac{\sin(2y)}{(x+1)^2}$$

Solution:

Treating the region as Type I, the volume is

$$V = \int_0^2 \int_0^{\frac{\pi}{2}} \frac{\sin(2y)}{(x+1)^2} \ dy \ dx$$

Integrate with respect to y by treating x as a constant.

$$V = \int_0^2 \frac{1}{(x+1)^2} \int_0^{\frac{\pi}{2}} \sin(2y) \ dy \ dx$$

$$V = \int_0^2 \frac{1}{(x+1)^2} \left[-\frac{1}{2}\cos(2y) \Big|_{y=0}^{y=\frac{\pi}{2}} \right] \ dx$$

$$V = \int_0^2 \frac{1}{(x+1)^2} \left[-\frac{1}{2}\cos\left(2 \cdot \frac{\pi}{2}\right) - \left(-\frac{1}{2}\cos(2 \cdot 0)\right) \right] \ dx$$

$$V = \int_0^2 \frac{1}{(x+1)^2} \left[-\frac{1}{2}\cos\pi + \frac{1}{2}\cos(0) \right] \ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}\left(-\frac{1}{2}(-1)+\frac{1}{2}(1)\right)\ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}\left(\frac{1}{2}+\frac{1}{2}\right)\ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}(1)\ dx$$

$$V=\int_0^2 \frac{1}{(x+1)^2}\ dx$$

Integrate with respect to x.

$$V=\int_0^2 (x+1)^{-2}\ dx$$

$$V=-(x+1)^{-1}\Big|_0^2$$

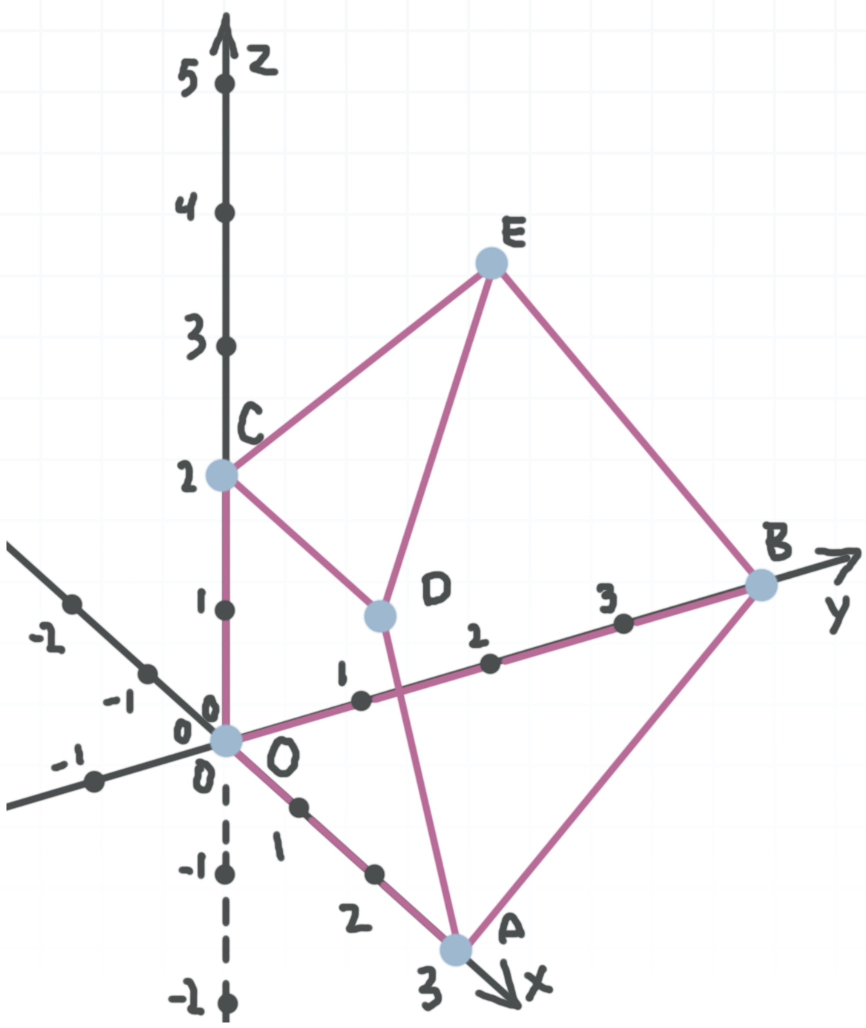
$$V=-\frac{1}{x+1}\Big|_0^2$$

$$V=-\frac{1}{2+1}-\left(-\frac{1}{0+1}\right)$$

$$V=-\frac{1}{3}+\frac{1}{1}$$

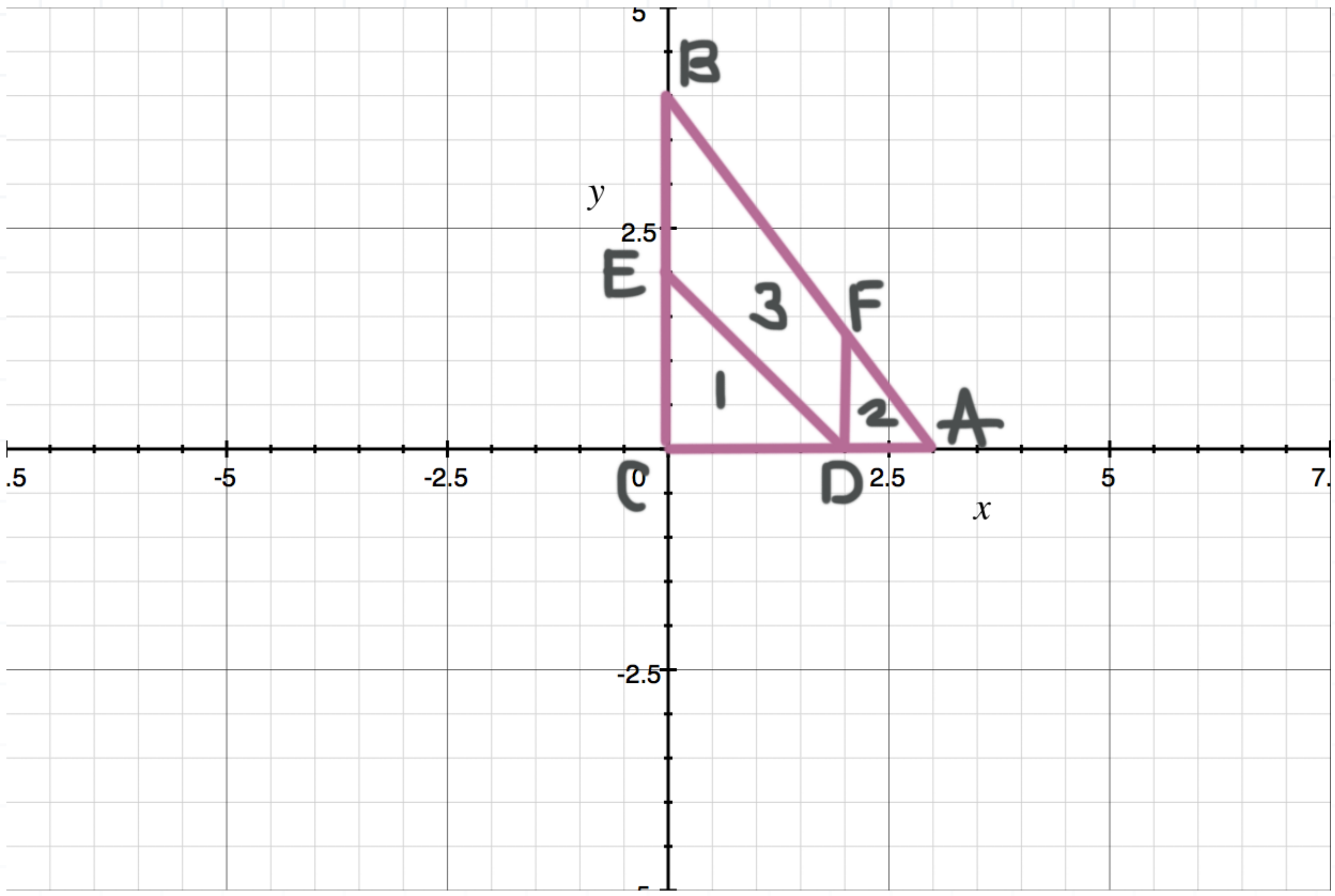
$$V=\frac{2}{3}$$

■ 2. Use a double integral to find the volume of the irregular hexagon $OABCDE$, where O is the origin, and the hexagon's other vertices are $A(3,0,0)$, $B(0,4,0)$, $C(0,0,2)$, $D(2,0,2)$, and $E(0,2,3)$.



Solution:

The hexagon's projection onto the xy-plane is



Let's add a helper point at $F(2,4/3,0)$. Then the hexagon's volume can be calculated as the sum of three volumes: below the triangle $CED(1)$, below the triangle $ADF(2)$, and below the quadrilateral $BEDF(3)$. We can treat all of them as Type I regions.

(1) The equation of the plane CDE that passes through $C(0,0,2)$, $D(2,0,2)$, and $E(0,2,3)$ is

$$-y + 2z - 4 = 0$$

$$2z = 4 + y$$

$$z = 2 + 0.5y$$

The equation of the line that's the projection of ED is $x + y = 2$. So when x is changing from 0 to 2, y is changing from 0 to $2 - x$. Therefore, the volume is

$$V_{CDE} = \int_0^2 \int_0^{2-x} 2 + 0.5x \ dy \ dx$$

Integrate with respect to y, while treating x as a constant.

$$V_{CDE} = \int_0^2 2y + 0.5xy \Big|_{y=0}^{y=2-x} dx$$

$$V_{CDE} = \int_0^2 2(2-x) + 0.5x(2-x) - (2(0) + 0.5x(0)) \ dx$$

$$V_{CDE} = \int_0^2 (4-2x) + (x - 0.5x^2) - (0+0) \ dx$$

$$V_{CDE} = \int_0^2 4 - 2x + x - 0.5x^2 \ dx$$

$$V_{CDE} = \int_0^2 4 - x - 0.5x^2 \ dx$$

Integrate with respect to x.

$$V_{CDE} = 4x - 0.5x^2 - \frac{0.5}{3}x^3 \Big|_0^2$$

$$V_{CDE} = 4(2) - 0.5(2)^2 - \frac{0.5}{3}(2)^3 - \left(4(0) - 0.5(0)^2 - \frac{0.5}{3}(0)^3\right)$$

$$V_{CDE} = 4(2) - 0.5(4) - \frac{0.5}{3}(8)$$

$$V_{CDE} = 8 - 2 - \frac{4}{3}$$

$$V_{CDE} = \frac{14}{3}$$

(2) The equation of the plane $ABED$ that passes through $A(3,0,0)$, $B(0,4,0)$, $E(0,2,3)$, and $D(2,0,2)$ is

$$4x + 3y + 2z - 12 = 0$$

$$z = -2x - 1.5y + 6$$

The equation of the line AB is

$$4x + 3y = 12$$

$$3y = -4x + 12$$

$$y = -\frac{4}{3}x + 4$$

So when x is changing from 2 to 3, y is changing from 0 to $(-4/3)x + 4$. Therefore, the volume is

$$V_{AB} = \int_2^3 \int_0^{-\frac{4}{3}x+4} -2x - 1.5y + 6 \ dy \ dx$$

Integrate with respect to y by treating x as a constant.

$$V_{AB} = \int_2^3 -2xy - 0.75y^2 + 6y \Big|_{y=0}^{y=-\frac{4}{3}x+4} \ dx$$

$$V_{AB} = \int_2^3 -2x\left(-\frac{4}{3}x + 4\right) - 0.75\left(-\frac{4}{3}x + 4\right)^2 + 6\left(-\frac{4}{3}x + 4\right)$$

$$-(-2x(0)-0.75(0)^2+6(0))\ dx$$

$$V_{AB}=\int_2^3 \frac{8}{3}x^2-8x-\frac{3}{4}\left(\frac{16}{9}x^2-\frac{32}{3}x+16\right)-8x+24\ dx$$

$$V_{AB}=\int_2^3 \frac{8}{3}x^2-8x-\frac{4}{3}x^2+8x-12-8x+24\ dx$$

$$V_{AB}=\int_2^3 \frac{4}{3}x^2-8x+12\ dx$$

Integrate with respect to x.

$$V_{AB}=\frac{4}{9}x^3-4x^2+12x\Big|_2^3$$

$$V_{AB}=\frac{4}{9}(3)^3-4(3)^2+12(3)-\left(\frac{4}{9}(2)^3-4(2)^2+12(2)\right)$$

$$V_{AB}=12-36+36-\frac{32}{9}+16-24$$

$$V_{AB}=\frac{4}{9}$$

(3) The equation of the plane $ABED$ that passes through $A(3,0,0)$, $B(0,4,0)$, $E(0,2,3)$, and $D(2,0,2)$ is

$$z=-2x-1.5y+6$$

The equation of the line AB is

$$y=-\frac{4}{3}x+4$$

The equation of the line that's the projection of ED is

$$y = 2 - x$$

So when x is changing from 0 to 2, y is changing from $2 - x$ to $(-4/3)x + 4$. Therefore, the volume is

$$V_{ED} = \int_0^2 \int_{2-x}^{-\frac{4}{3}x+4} -2x - 1.5y + 6 \ dy \ dx$$

Integrate with respect to y by treating x as a constant.

$$V_{ED} = \int_0^2 -2xy - 0.75y^2 + 6y \Big|_{y=2-x}^{y=-\frac{4}{3}x+4} \ dx$$

$$V_{ED} = \int_0^2 -2x\left(-\frac{4}{3}x+4\right) - 0.75\left(-\frac{4}{3}x+4\right)^2 + 6\left(-\frac{4}{3}x+4\right)$$

$$-(-2x(2-x) - 0.75(2-x)^2 + 6(2-x)) \ dx$$

$$V_{ED} = \int_0^2 \frac{8}{3}x^2 - 8x - \frac{3}{4}\left(\frac{16}{9}x^2 - \frac{32}{3}x + 16\right) - 8x + 24$$

$$-\left(-4x + 2x^2 - 3 + 3x - \frac{3}{4}x^2 + 12 - 6x\right) \ dx$$

$$V_{ED} = \int_0^2 \frac{8}{3}x^2 - 8x - \frac{4}{3}x^2 + 8x - 12 - 8x + 24$$

$$+4x - 2x^2 + 3 - 3x + \frac{3}{4}x^2 - 12 + 6x \ dx$$

$$V_{ED} = \int_0^2 \frac{8}{3}x^2 - \frac{4}{3}x^2 - 2x^2 + \frac{3}{4}x^2$$

$$-8x + 8x - 8x + 4x - 3x + 6x - 12 + 24 + 3 - 12 \ dx$$

$$V_{ED} = \int_0^2 \frac{1}{12}x^2 - x + 3 \ dx$$

Integrate with respect to x.

$$V_{ED} = \frac{1}{36}x^3 - \frac{1}{2}x^2 + 3x \Big|_0^2$$

$$V_{ED} = \frac{1}{36}(2)^3 - \frac{1}{2}(2)^2 + 3(2) - \left(\frac{1}{36}(0)^3 - \frac{1}{2}(0)^2 + 3(0)\right)$$

$$V_{ED} = \frac{2}{9} - 2 + 6$$

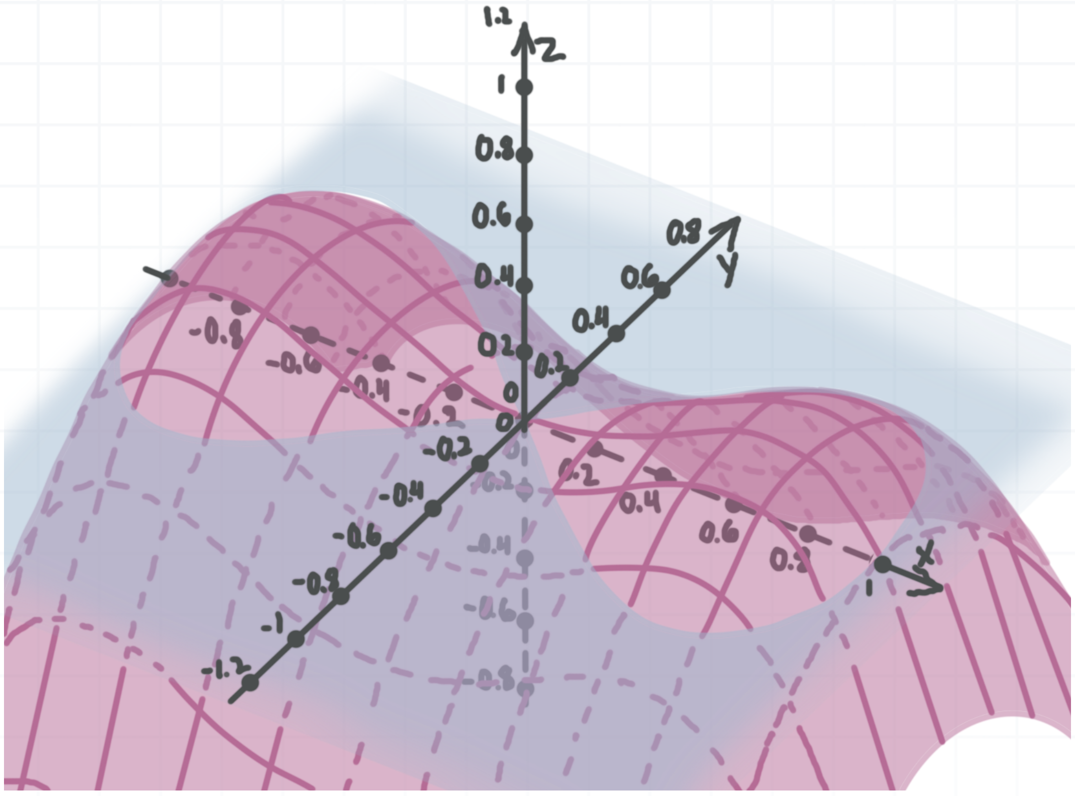
$$V_{ED} = \frac{38}{9}$$

Adding the volumes we found in (1), (2), and (3) gives

$$\frac{14}{3} + \frac{4}{9} + \frac{38}{9}$$

$$\frac{28}{3}$$

■ 3. Use a double integral to find the volume of the solid that's bounded by the surface $z = -x^4 + x^2 - y^2$ and the xy-plane.



Solution:

Since the solid is symmetric, we can calculate its volume within the first quadrant, then multiply the result by 4, to get the total volume from all four quadrants.

We'll consider the intersection of the curve and the xy-plane.

$$-x^4 + x^2 - y^2 = 0$$

$$y^2 = x^2 - x^4$$

The region in the first quadrant is bounded by

$$y = \sqrt{x^2 - x^4} = x\sqrt{1 - x^2}$$

We'll treat the region as Type I, so when x is changing from 0 to 1, y is changing from 0 to $x\sqrt{1 - x^2}$. Therefore, the volume is

$$V=\int_0^1\int_0^{x\sqrt{1-x^2}}-x^4+x^2-y^2\ dy\ dx$$

Integrate with respect to y by treating x as a constant.

$$V=\int_0^1 -x^4y+x^2y-\frac{1}{3}y^3\Big|_{y=0}^{y=x\sqrt{1-x^2}}\ dx$$

$$V=\int_0^1 -x^4(x\sqrt{1-x^2})+x^2(x\sqrt{1-x^2})-\frac{1}{3}(x\sqrt{1-x^2})^3$$

$$-\left(-x^4(0)+x^2(0)-\frac{1}{3}(0)^3\right)\ dx$$

$$V=\int_0^1 -x^5(1-x^2)^{\frac{1}{2}}+x^3(1-x^2)^{\frac{1}{2}}-\frac{1}{3}x^3(1-x^2)^{\frac{3}{2}}\ dx$$

$$V=\int_0^1 (1-x^2)^{\frac{1}{2}}\left(-x^5+x^3-\frac{1}{3}x^3(1-x^2)\right)\ dx$$

$$V=\int_0^1 (1-x^2)^{\frac{1}{2}}\left(-x^5+x^3-\frac{1}{3}x^3+\frac{1}{3}x^5\right)\ dx$$

$$V=\int_0^1 (1-x^2)^{\frac{1}{2}}\left(-\frac{2}{3}x^5+\frac{2}{3}x^3\right)\ dx$$

$$V=\int_0^1 \frac{2}{3}x^3(1-x^2)^{\frac{1}{2}}(1-x^2)\ dx$$

$$V=\int_0^1 \frac{2}{3}x^3(1-x^2)^{\frac{3}{2}}\ dx$$

Use substitution to integrate with respect to x, setting

$$u = 1 - x^2$$

$x^2 = 1 - u$ and $x = (1 - u)^{\frac{1}{2}}$

$\frac{du}{dx} = -2x$, so $du = -2x\ dx$ and $dx = -\frac{du}{2x}$

The bounds on x of $[0,1]$ becomes bounds on u of $[1,0]$.

$$V = \int_1^0 \frac{2}{3}((1-u)^{\frac{1}{2}})^3 u^{\frac{3}{2}} \left(-\frac{du}{2x}\right)$$

$$V = -\frac{1}{3}\int_1^0 \frac{(1-u)^{\frac{3}{2}}}{x} u^{\frac{3}{2}}\ du$$

$$V = -\frac{1}{3}\int_1^0 \frac{(1-u)^{\frac{3}{2}}}{(1-u)^{\frac{1}{2}}} u^{\frac{3}{2}}\ du$$

$$V = -\frac{1}{3}\int_1^0 u^{\frac{3}{2}}(1-u)\ du$$

$$V = -\frac{1}{3}\int_1^0 u^{\frac{3}{2}} - u^{\frac{5}{2}}\ du$$

$$V = -\frac{1}{3}\left(\frac{2}{5}u^{\frac{5}{2}} - \frac{2}{7}u^{\frac{7}{2}}\right)\Bigg|_1^0$$

$$V = -\frac{1}{3}\left(\frac{2}{5}(0)^{\frac{5}{2}} - \frac{2}{7}(0)^{\frac{7}{2}}\right) + \frac{1}{3}\left(\frac{2}{5}(1)^{\frac{5}{2}} - \frac{2}{7}(1)^{\frac{7}{2}}\right)$$

$$V=\frac{1}{3}\left(\frac{2}{5}-\frac{2}{7}\right)$$

$$V=\frac{1}{3}\left(\frac{4}{35}\right)$$

$$V=\frac{4}{105}$$

This is the volume in the first quadrant. Multiply by 4 to get the volume in all four quadrants.

$$V=4\cdot\frac{4}{105}$$

$$V=\frac{16}{105}$$

CHANGING THE ORDER OF INTEGRATION

■ 1. Change the order of integration of the iterated integral.

$$\int_{-3}^{0}\int_{-\frac{2}{3}\sqrt{9-x^2}}^{\frac{2}{3}\sqrt{9-x^2}} z(x,y)\ dy\ dx$$

Solution:

The double integral represents a Type I region where x is changing from -3 to 0. The bounds on y are

$$y=-\frac{2}{3}\sqrt{9-x^2}$$

$$y=\frac{2}{3}\sqrt{9-x^2}$$

If we square both sides of either equation, we get

$$y^2=\frac{4}{9}(9-x^2)$$

$$9y^2=36-4x^2$$

$$9y^2+4x^2=36$$

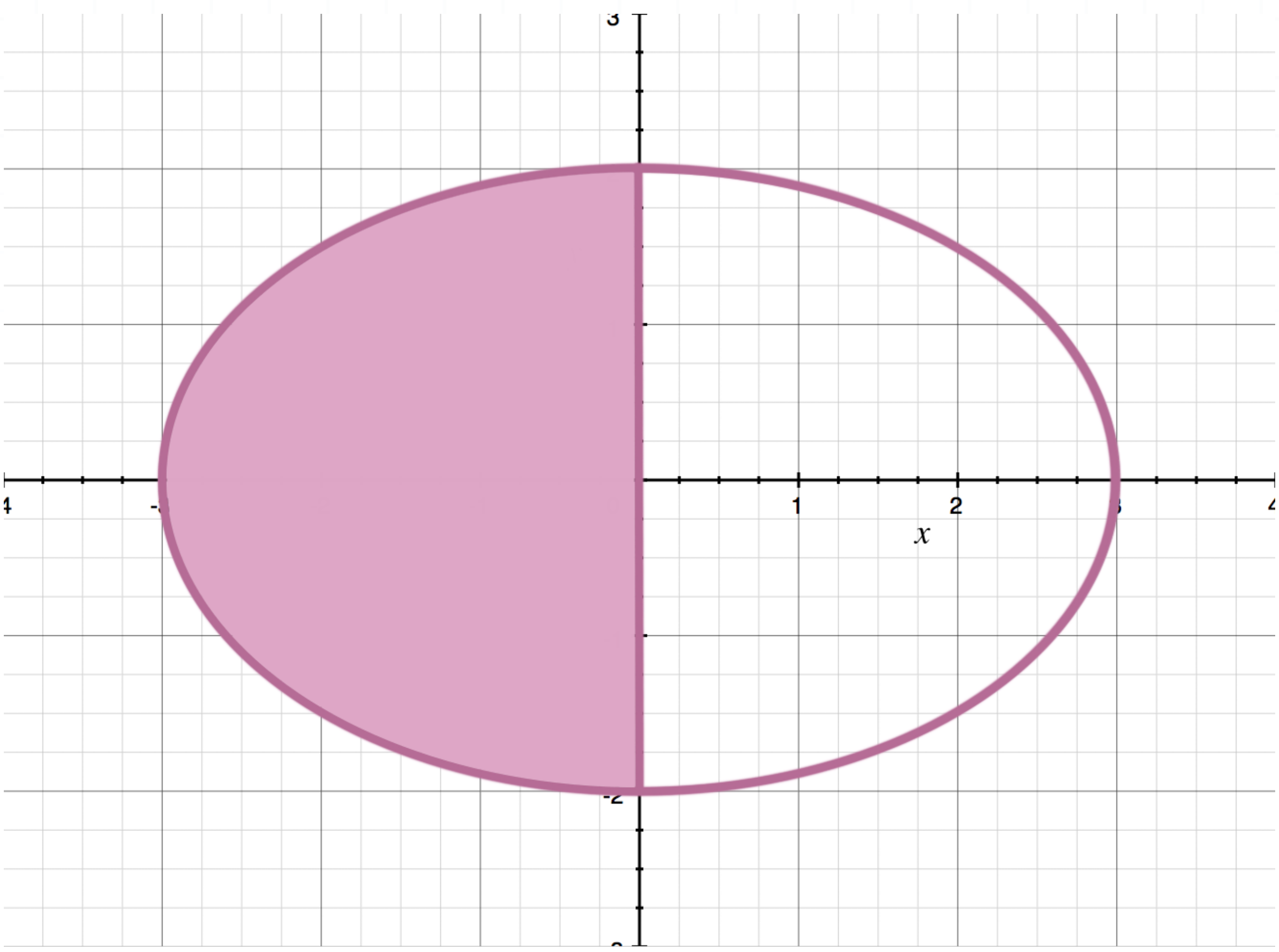
If we divide through by 36, we get an ellipse in standard form.

$$\frac{9}{36}y^2 + \frac{4}{36}x^2 = \frac{36}{36}$$

$$\frac{1}{4}y^2 + \frac{1}{9}x^2 = 1$$

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

Given that the limits given on x are given as $x = [-3,0]$, the region of integration given by the original double integral is the left half of the ellipse with center at the origin, bounded by $x = [-3,3]$ and y bounded by $[-2,2]$.



To change the order of integration, solve the equation of the ellipse for x.

$$9y^2 + 4x^2 = 36$$

$$4x^2 = 36 - 9y^2$$

$$2x = \pm\sqrt{36 - 9y^2}$$

$$2x = \pm\sqrt{9(4 - y^2)}$$

$$2x = \pm 3\sqrt{4 - y^2}$$

$$x = \pm\frac{3}{2}\sqrt{4 - y^2}$$

So while y changes from $y = -2$ to $y = 2$, x changes from

$$x = -\frac{3}{2}\sqrt{4 - y^2}$$

to $x = 0$. Therefore, we can rewrite the original integral as

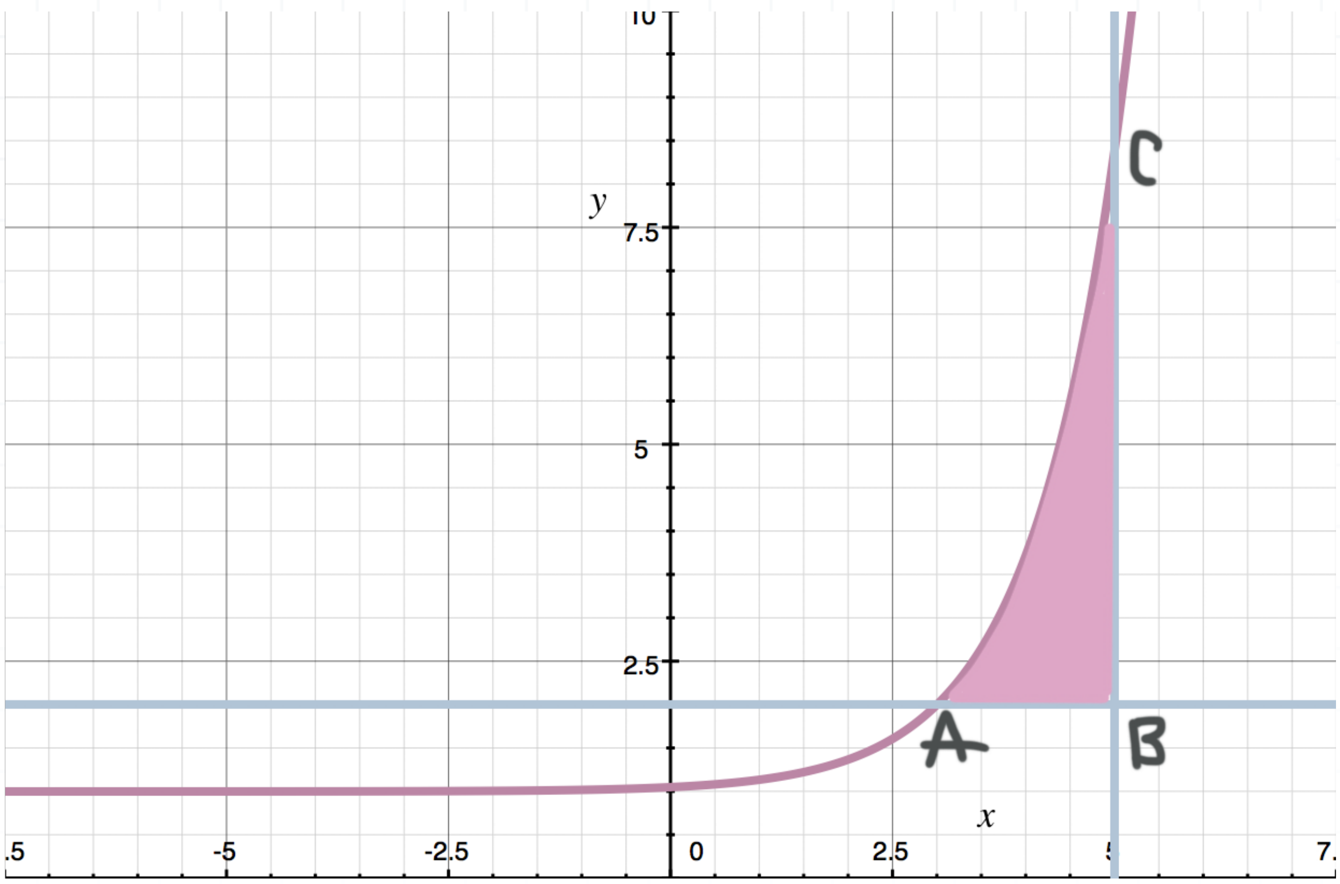
$$\int_{-2}^{2}\int_{-\frac{3}{2}\sqrt{4-y^2}}^{0} z(x, y)\ dx\ dy$$

■ 2. Change the order of integration of the iterated integral.

$$\int_{3}^{5}\int_{2}^{e^{x-3}+1} z(x, y)\ dy\ dx$$

Solution:

The double integral represents a Type I region where x is changing from 3 to 5, and y is changing from 2 to $e^{x-3}+1$. So the region is ABC, where AB and BA are lines, and AC is the curve $y=e^{x-3}+1$.



To find the coordinates of C, plug $x=5$ into the equation of the curve.

$$y(5)=e^{5-3}+1=e^2+1\approx 8.4$$

Solve $y=e^{x-3}+1$ for x.

$$y=e^{x-3}+1$$

$$y-1=e^{x-3}$$

$$x-3=\ln(y-1)$$

$$x=3+\ln(y-1)$$

So when y is changing from $y = 2$ to $y = e^2 + 1$, x is changing from $x = 3 + \ln(y - 1)$ to $x = 5$. Therefore, we can rewrite the original integral as

$$\int_{2}^{e^2+1} \int_{3+\ln(y-1)}^{5} z(x, y)\ dx\ dy$$

■ 3. Change the order of integration of the iterated integral.

$$\int_{-2}^{2} \int_{\frac{1}{4}x^4-x^2}^{0} z(x, y)\ dy\ dx$$

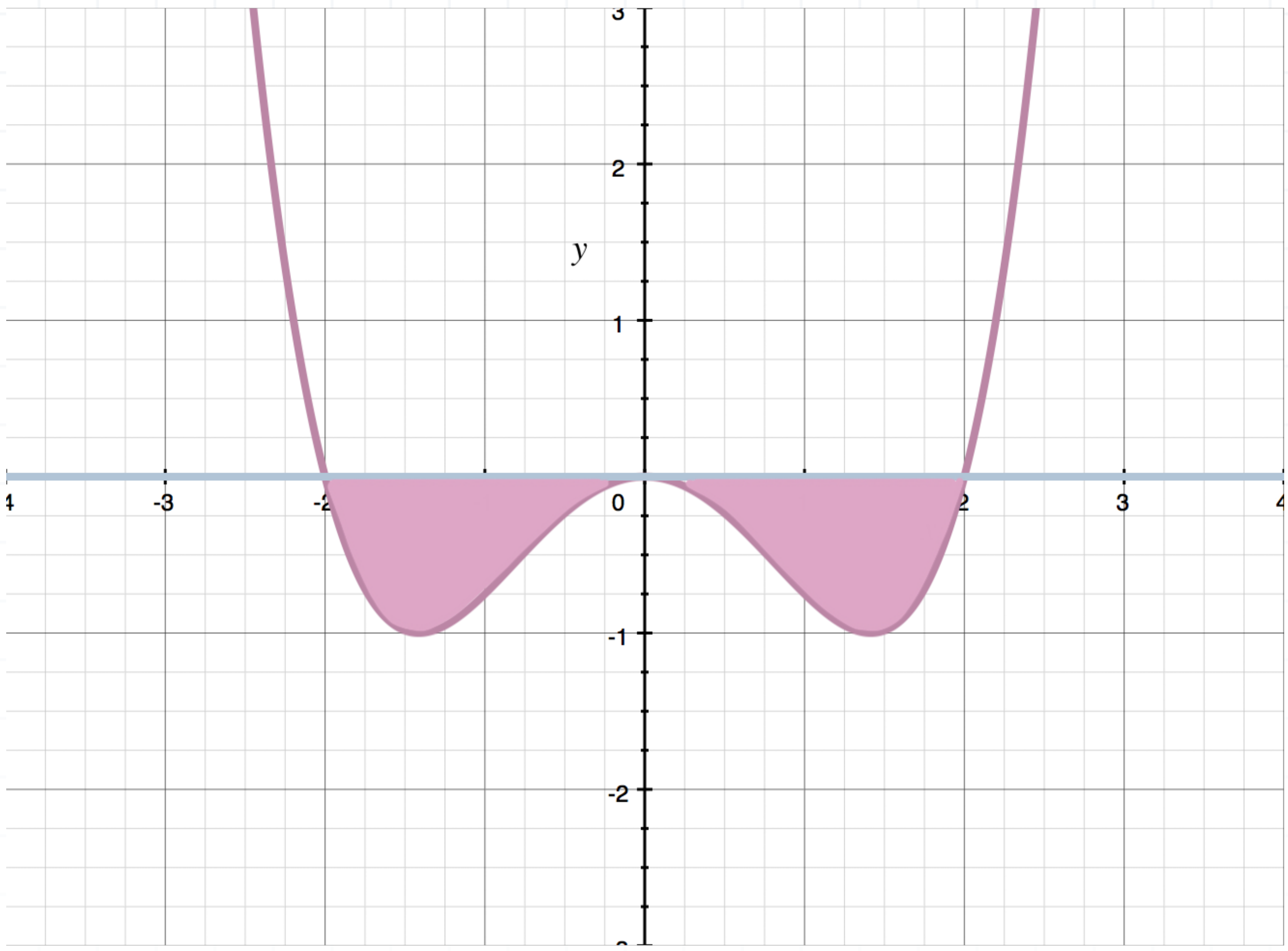
Solution:

The double integral represents a Type I region where x is changing from -2 to 2, and y is changing from $(1/4)x^4 - x^2$ to 0. So the region consists of two parts:

(1) the region on $x = [-2,0]$, and

(2) the region on $x = [0,2]$.

A sketch of both parts is



The area of region (1) is given by

$$\int_{-2}^{0}\int_{\frac{1}{4}x^4-x^2}^{0} z(x,y)\ dy\ dx$$

To find the bounds on y, we need to find the local minimum of $(1/4)x^4 - x^2$.

$$y' = x^3 - 2x$$

$$y' = x(x^2 - 2)$$

$$x(x^2 - 2) = 0$$

$$x = 0,\ \ \pm\sqrt{2}$$

These are the critical points of the curve, which we can see in the sketch of the curve. The local minimum exists in the first region at $x=-\sqrt{2}$, and exists in the second region at $x=\sqrt{2}$.

$$y(-\sqrt{2})=\frac{(-\sqrt{2})^4}{4}-(-\sqrt{2})^2$$

$$y(-\sqrt{2})=\frac{4}{4}-2$$

$$y(-\sqrt{2})=1-2$$

$$y(-\sqrt{2})=-1$$

So y is changing from -1 to 0. To find the bounds on x, solve the equation $y=(1/4)x^4-x^2$ for x.

$$y=\frac{1}{4}x^4-x^2$$

$$4y=x^4-4x^2$$

$$x^4-4x^2-4y=0$$

Use the quadratic equation to find the values of x that satisfy the equation.

$$x^2=2\pm 2\sqrt{y+1}$$

$$x=\pm\sqrt{2\pm 2\sqrt{y+1}}$$

This gives four solutions in total. For the first region (the region on the left), the value of x changes from

$$x=-\sqrt{2+2\sqrt{y+1}} \text{ to } x=-\sqrt{2-2\sqrt{y+1}}$$

Therefore, the first region (the region on the left), can be represented by

$$\int_{-1}^{0}\int_{-\sqrt{2+2\sqrt{y+1}}}^{-\sqrt{2-2\sqrt{y+1}}} z(x,y)\ dx\ dy$$

For the second region (the region on the right), the value of x changes from

$$x=\sqrt{2-2\sqrt{y+1}} \text{ to } x=\sqrt{2+2\sqrt{y+1}}$$

Therefore, the second region (the region on the right), can be represented by

$$\int_{-1}^{0}\int_{\sqrt{2-2\sqrt{y+1}}}^{\sqrt{2+2\sqrt{y+1}}} z(x,y)\ dx\ dy$$

Then both regions together can be represented as

$$\int_{-1}^{0}\int_{-\sqrt{2-2\sqrt{y+1}}}^{-\sqrt{2+2\sqrt{y+1}}} z(x,y)\ dx\ dy+\int_{-1}^{0}\int_{\sqrt{2-2\sqrt{y+1}}}^{\sqrt{2+2\sqrt{y+1}}} z(x,y)\ dx\ dy$$

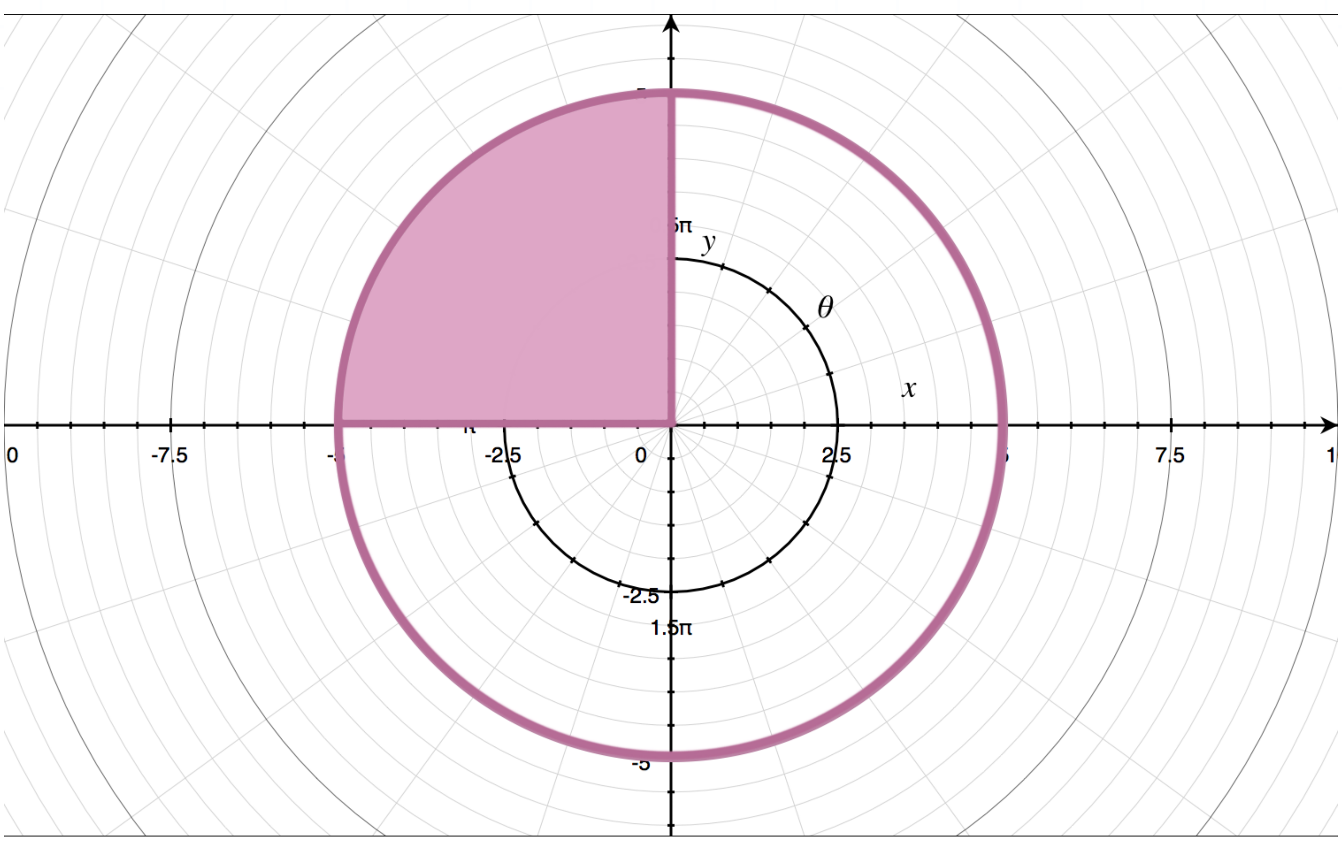
CHANGING ITERATED INTEGRALS TO POLAR COORDINATES

■ 1. Convert the iterated integral to polar coordinates, then find its value.

$$\int_{-5}^{0}\int_{0}^{\sqrt{25-x^2}} xy\ dy\ dx$$

Solution:

The double integral shows that x is bounded on $x=[-5,0]$ while y is bounded on 0 to $\sqrt{25-x^2}$. So the area of integration is the part of the circle with center at the origin and radius 5, that lies in the second quadrant.



On this region, r is bounded on $r=[0,5]$ and θ is bounded on $\theta=[\pi/2,\pi]$. Converting the function to polar coordinates gives

$$xy$$

$$r\cos\theta \cdot r\sin\theta$$

$$r^2\cos\theta\sin\theta$$

So the integral in polar coordinates is

$$\int_{\pi/2}^{\pi}\int_{0}^{5} r^2\cos\theta\sin\theta\ r\ dr\ d\theta$$

$$\int_{\pi/2}^{\pi}\int_{0}^{5} r^3\cos\theta\sin\theta\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_{\pi/2}^{\pi} \frac{1}{4}r^4\cos\theta\sin\theta\Big|_{r=0}^{r=5}\ d\theta$$

$$\int_{\pi/2}^{\pi} \frac{1}{4}(5)^4\cos\theta\sin\theta - \frac{1}{4}(0)^4\cos\theta\sin\theta\ d\theta$$

$$\int_{\pi/2}^{\pi} \frac{625}{4}\cos\theta\sin\theta\ d\theta$$

Integrate with respect to θ, using a substitution with

$$u = \sin\theta$$

$\dfrac{du}{d\theta} = \cos\theta$, so $du = \cos\theta\ d\theta$ and $d\theta = \dfrac{du}{\cos\theta}$

The bounds $\theta = [\pi/2,\pi]$ become $u = [1,0]$, so the integral can be rewritten as

$$\int_1^0 \frac{625}{4}(\cos\theta)u\left(\frac{du}{\cos\theta}\right)$$

$$\int_1^0 \frac{625}{4}u\ du$$

Integrate and evaluate.

$$\frac{625}{8}u^2\Big|_1^0$$

$$\frac{625}{8}(0)^2 - \frac{625}{8}(1)^2$$

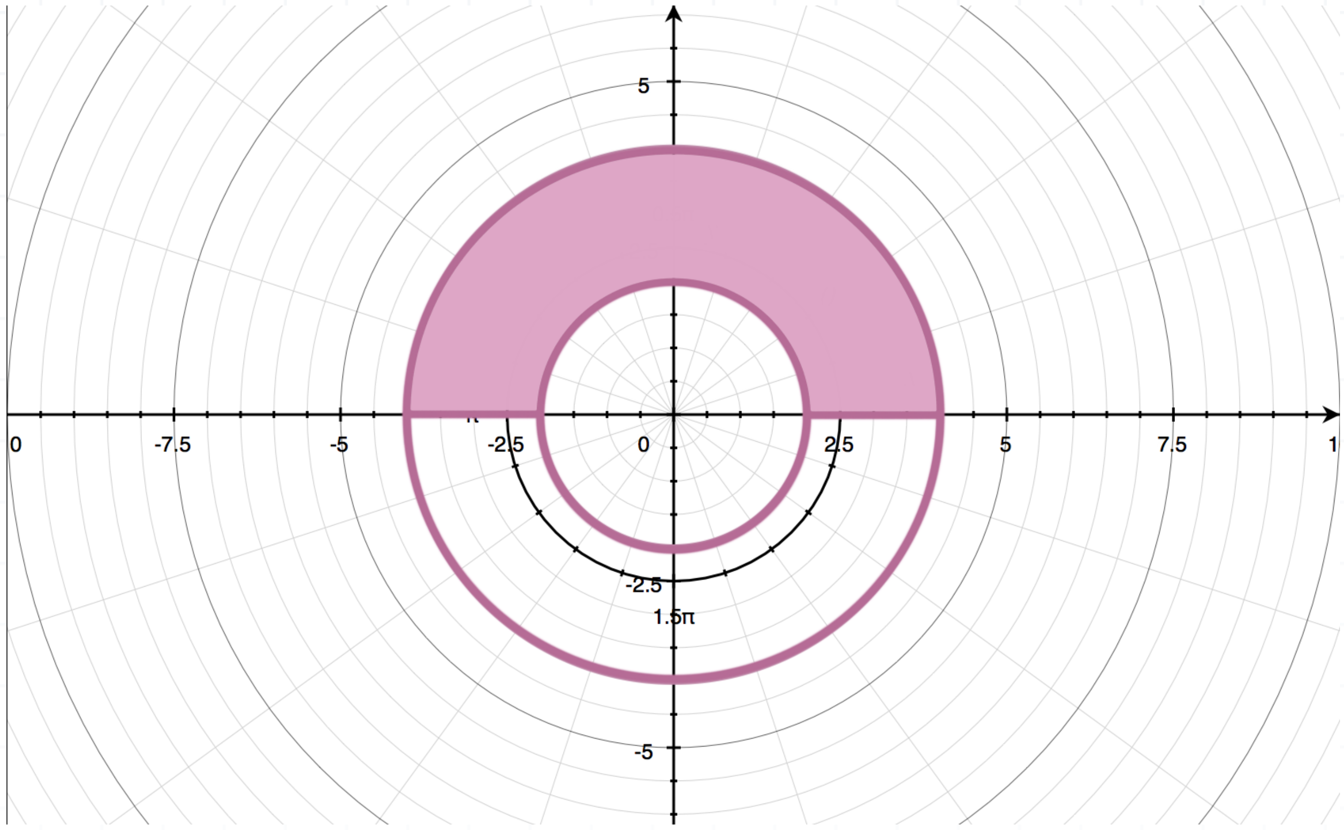
$$-\frac{625}{8}$$

■ 2. Convert the sum of iterated integrals to polar coordinates, then find its value.

$$\int_{-4}^{-2}\int_0^{\sqrt{16-x^2}} x - y\ dy\ dx + \int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{16-x^2}} x - y\ dy\ dx + \int_2^4\int_0^{\sqrt{16-x^2}} x - y\ dy\ dx$$

Solution:

If we consider the bounds on x and y from each integral, we can see that the region of integration is the area between the circles with centers at the origin and radii 2 and 4, in only the first and second quadrants.



In polar coordinates, that means the region is defined on $r=[2,4]$ and $\theta=[0,\pi]$. And the function converts to

$$x-y$$

$$r\cos\theta-r\sin\theta$$

$$r(\cos\theta-\sin\theta)$$

Then the integral in polar coordinates is

$$\int_0^{\pi}\int_2^4 r(\cos\theta-\sin\theta)r\ dr\ d\theta$$

$$\int_0^{\pi}\int_2^4 r^2(\cos\theta-\sin\theta)\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{\pi} \frac{1}{3}r^3(\cos\theta - \sin\theta)\Big|_{r=2}^{r=4}\ d\theta$$

$$\int_0^{\pi} \frac{1}{3}(4)^3(\cos\theta - \sin\theta) - \frac{1}{3}(2)^3(\cos\theta - \sin\theta)\ d\theta$$

$$\int_0^{\pi} \frac{64}{3}(\cos\theta - \sin\theta) - \frac{8}{3}(\cos\theta - \sin\theta)\ d\theta$$

$$\frac{56}{3}\int_0^{\pi} \cos\theta - \sin\theta\ d\theta$$

Integrate with respect to θ.

$$\frac{56}{3}(\sin\theta + \cos\theta)\Big|_0^{\pi}$$

$$\frac{56}{3}(\sin\pi + \cos\pi) - \frac{56}{3}(\sin(0) + \cos(0))$$

$$\frac{56}{3}(0 + (-1)) - \frac{56}{3}(0 + 1)$$

$$-\frac{56}{3} - \frac{56}{3}$$

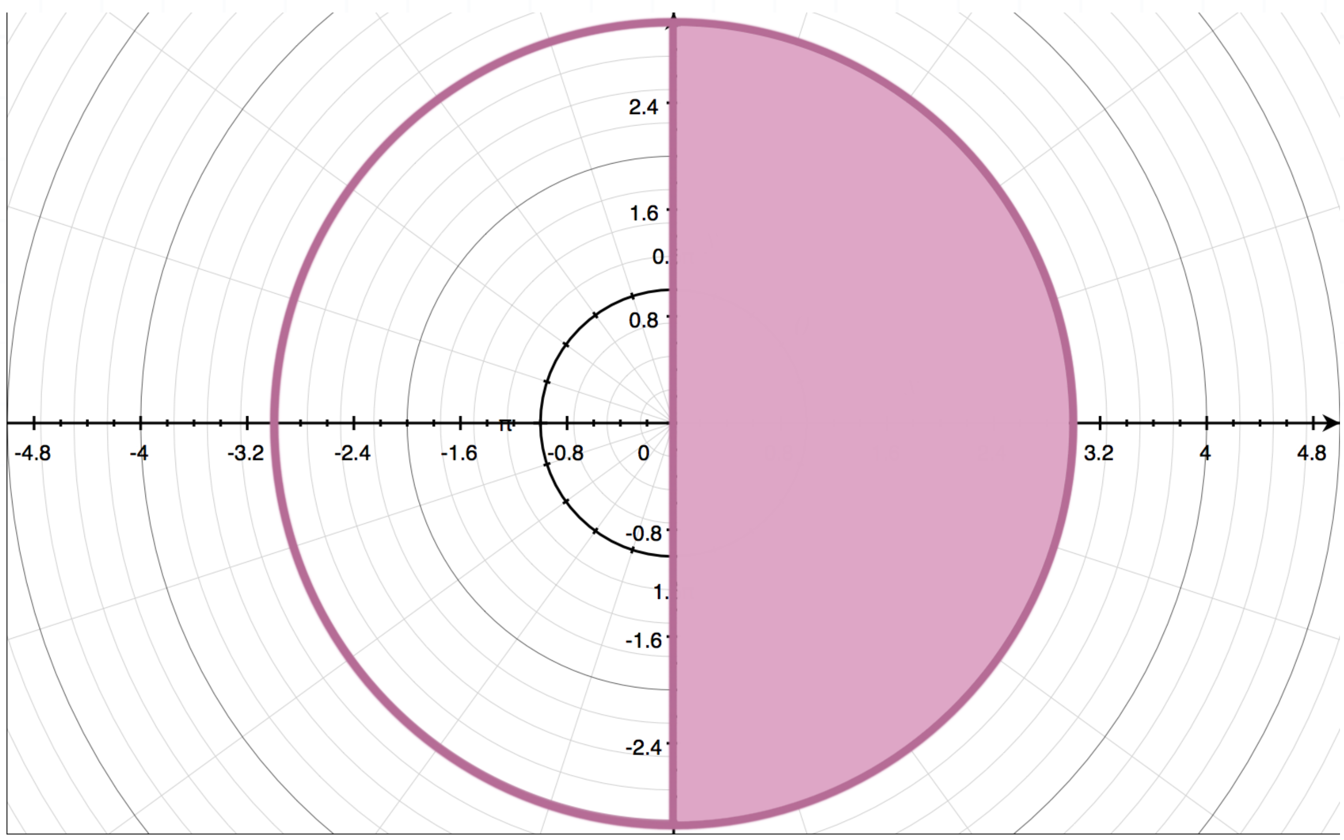
$$-\frac{112}{3}$$

■ 3. Convert the iterated integral to polar coordinates, then find its value.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} \ln(x^2+y^2)\ dx\ dy$$

Solution:

If we consider the bounds on x and y from the double integral, we can see that the region of integration is the part of the circle with center at the origin and radius 3, that lies in only the first and fourth quadrants.



In polar coordinates, that means the region is defined on $r=[0,3]$ and $\theta=[-\pi/2,\pi/2]$. And the function converts to

$$\ln(x^2+y^2)=\ln(r^2)=2\ln r$$

Then the integral in polar coordinates is

$$\int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\ln r)\ r\ d\theta\ dr$$

$$\int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r\ln r\ d\theta\ dr$$

Integrate with respect to θ, treating r as a constant.

$$\int_0^3 2r\theta\ln r\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\ dr$$

$$\int_0^3 2r\left(\frac{\pi}{2}\right)\ln r - 2r\left(-\frac{\pi}{2}\right)\ln r\ dr$$

$$\int_0^3 r\pi\ln r + r\pi\ln r\ dr$$

$$\int_0^3 2\pi r\ln r\ dr$$

Integrate with respect to r, using integration by parts.

$u = \ln r$ with $du = \frac{1}{r}\ dr$

$dv = 2\pi r$ with $v = \pi r^2$

Then the integral becomes

$$\pi r^2\ln r\Big|_0^3 - \int_0^3 \pi r^2\left(\frac{1}{r}\ dr\right)$$

$$\pi r^2 \ln r \Big|_0^3 - \int_0^3 \pi r \ dr$$

$$\pi r^2 \ln r \Big|_0^3 - \left[\frac{1}{2} \pi r^2 \Big|_0^3 \right]$$

$$\pi r^2 \ln r - \frac{1}{2} \pi r^2 \Big|_0^3$$

$$\pi(3)^2 \ln 3 - \frac{1}{2}\pi(3)^2 - \left(\pi(0)^2 \ln 0 - \frac{1}{2}\pi(0)^2 \right)$$

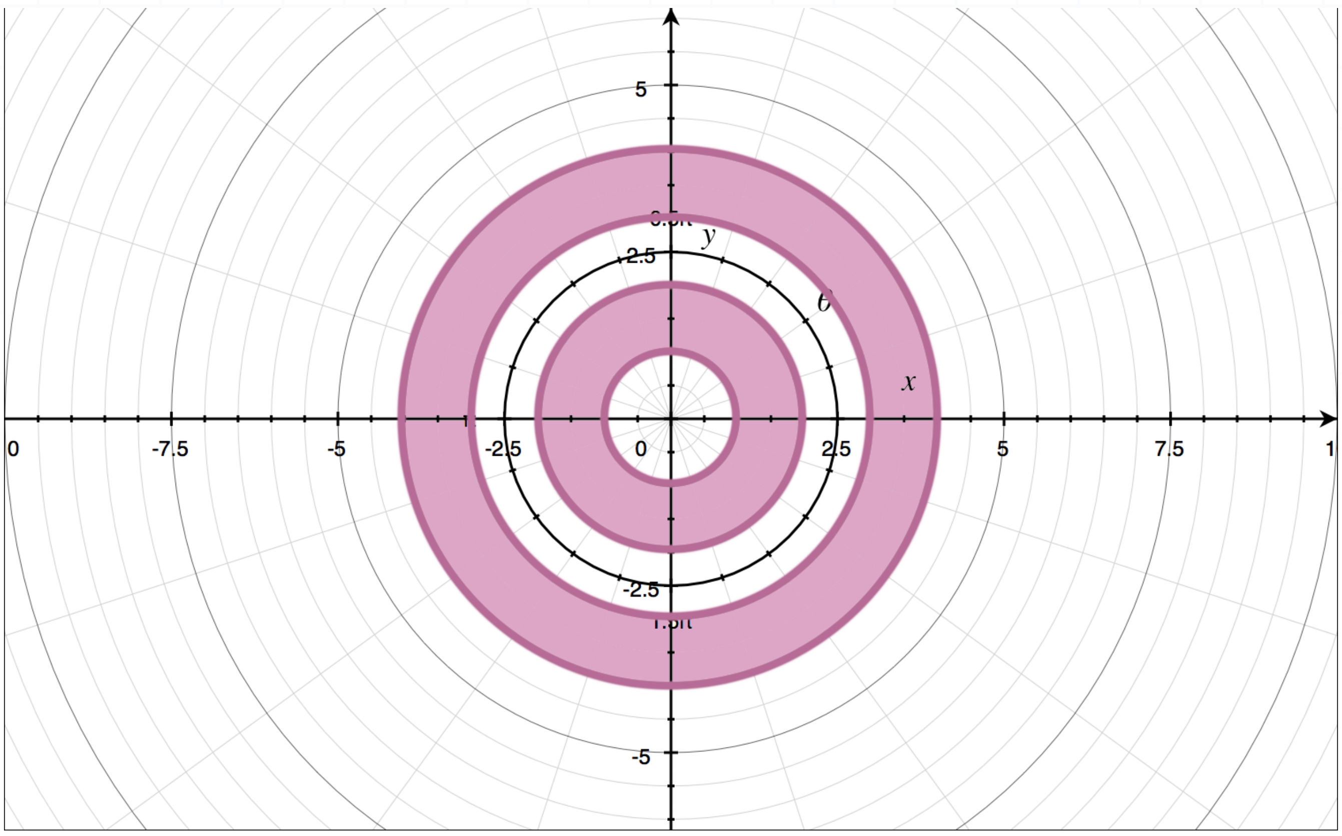
$$9\pi \ln 3 - \frac{9}{2}\pi - (0 - 0)$$

$$9\pi \ln 3 - \frac{9}{2}\pi$$

CHANGING DOUBLE INTEGRALS TO POLAR COORDINATES

■ 1. The region D consists of two rings centered at the origin, where the inner ring is defined on $r = [1,2]$ and the outer ring is defined on $r = [3,4]$. Convert the double integral to polar coordinates, then find its value.

$$\iint_D x + 2y \ dA$$



Solution:

We'll use separate integrals for each ring, but first, we'll convert the function.

$$x + 2y$$

$$r\cos\theta + 2r\sin\theta$$

$$r(\cos\theta + 2\sin\theta)$$

For each ring, θ is defined on $\theta = [0,2\pi]$, so the entire region can be integrated by

$$\int_0^{2\pi}\int_1^2 r(\cos\theta + 2\sin\theta)r\ dr\ d\theta + \int_0^{2\pi}\int_3^4 r(\cos\theta + 2\sin\theta)r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_1^2 r^2(\cos\theta + 2\sin\theta)\ dr\ d\theta + \int_0^{2\pi}\int_3^4 r^2(\cos\theta + 2\sin\theta)\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{2\pi} \frac{1}{3}r^3(\cos\theta + 2\sin\theta)\Big|_{r=1}^{r=2}\ d\theta + \int_0^{2\pi} \frac{1}{3}r^3(\cos\theta + 2\sin\theta)\Big|_{r=3}^{r=4}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{3}(2)^3(\cos\theta + 2\sin\theta) - \frac{1}{3}(1)^3(\cos\theta + 2\sin\theta)\ d\theta$$

$$+\int_0^{2\pi} \frac{1}{3}(4)^3(\cos\theta + 2\sin\theta) - \frac{1}{3}(3)^3(\cos\theta + 2\sin\theta)\ d\theta$$

$$\int_0^{2\pi} \frac{8}{3}(\cos\theta + 2\sin\theta) - \frac{1}{3}(\cos\theta + 2\sin\theta)\ d\theta$$

$$+\int_0^{2\pi} \frac{64}{3}(\cos\theta + 2\sin\theta) - \frac{27}{3}(\cos\theta + 2\sin\theta)\ d\theta$$

$$\int_0^{2\pi} \frac{7}{3}(\cos\theta + 2\sin\theta)\ d\theta + \int_0^{2\pi} \frac{37}{3}(\cos\theta + 2\sin\theta)\ d\theta$$

$$\int_0^{2\pi} \frac{7}{3}\cos\theta + \frac{14}{3}\sin\theta\ d\theta + \int_0^{2\pi} \frac{37}{3}\cos\theta + \frac{74}{3}\sin\theta\ d\theta$$

$$\int_0^{2\pi} \frac{7}{3}\cos\theta + \frac{14}{3}\sin\theta + \frac{37}{3}\cos\theta + \frac{74}{3}\sin\theta\ d\theta$$

$$\int_0^{2\pi} \frac{88}{3}\sin\theta + \frac{44}{3}\cos\theta\ d\theta$$

Integrate with respect to θ.

$$-\frac{88}{3}\cos\theta + \frac{44}{3}\sin\theta\Big|_0^{2\pi}$$

$$\frac{44}{3}\sin\theta - \frac{88}{3}\cos\theta\Big|_0^{2\pi}$$

$$\frac{44}{3}\sin(2\pi) - \frac{88}{3}\cos(2\pi) - \left(\frac{44}{3}\sin(0) - \frac{88}{3}\cos(0)\right)$$

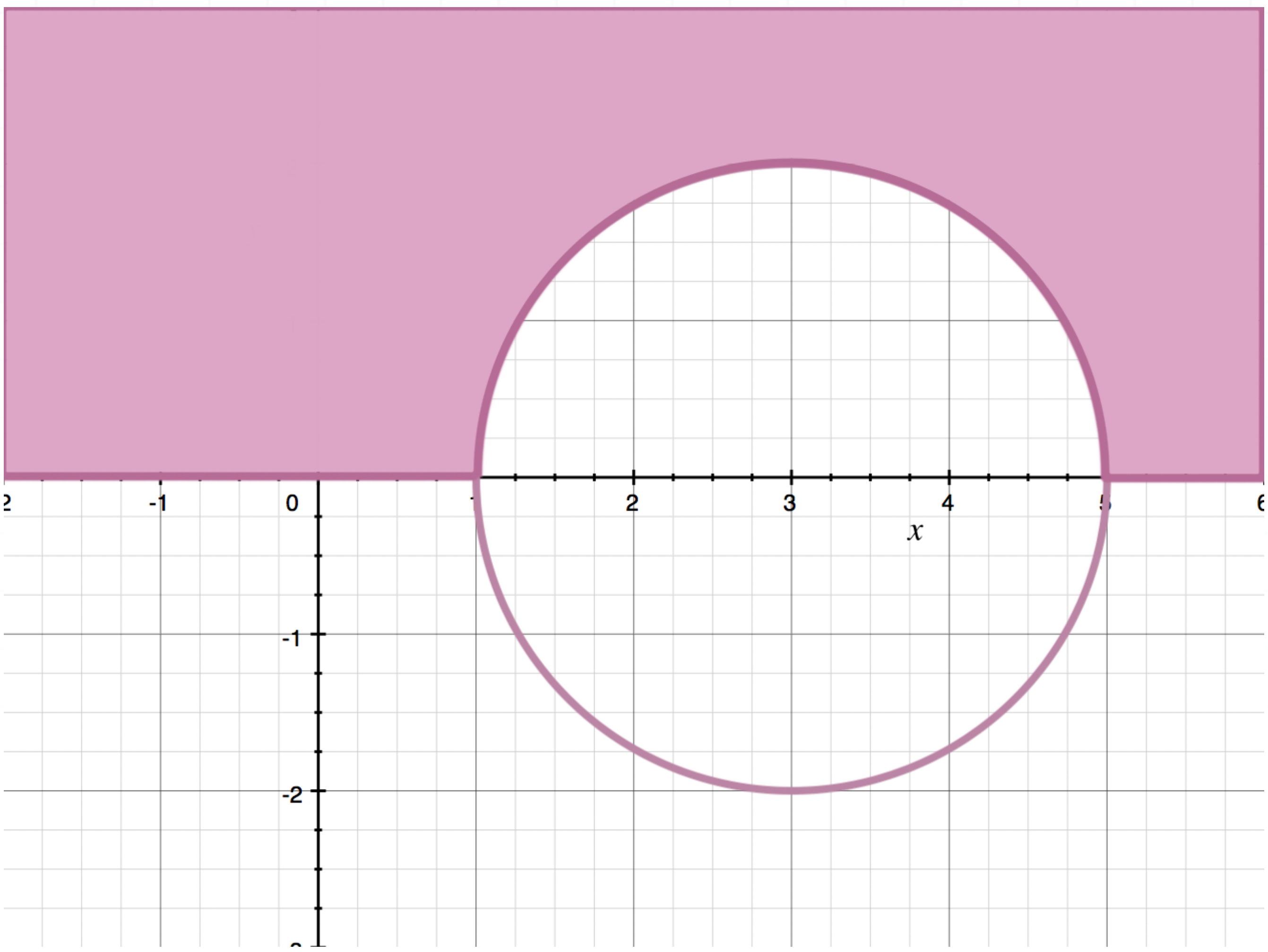
$$\frac{44}{3}(0) - \frac{88}{3}(1) - \left(\frac{44}{3}(0) - \frac{88}{3}(1)\right)$$

$$-\frac{88}{3} + \frac{88}{3}$$

$$0$$

■ 2. The region D consists of all the points in the first and second quadrants outside the circle centered at $(3,0)$ with radius $r = 2$. Convert the double integral to polar coordinates, using the conversion formulas $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$ for a circle shifted off the origin, then find its value.

$$\int\int_D \frac{1}{((x-3)^2 + y^2)^2}\ dA$$



Solution:

Applying $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$, the conversion formulas become $x = 3 + r\cos\theta$ and $y = r\sin\theta$. Using these, we'll convert the function to polar coordinates.

$$\frac{1}{((x-3)^2 + y^2)^2}$$

$$\frac{1}{((3+r\cos\theta-3)^2+(r\sin\theta)^2)^2}$$

$$\frac{1}{((r\cos\theta)^2+(r\sin\theta)^2)^2}$$

$$\frac{1}{(r^2\cos^2\theta+r^2\sin^2\theta)^2}$$

$$\frac{1}{(r^2(\cos^2\theta+\sin^2\theta))^2}$$

$$\frac{1}{(r^2(1))^2}$$

$$\frac{1}{(r^2)^2}$$

$$\frac{1}{r^4}$$

The integral in polar coordinates is therefore

$$\int_0^{\pi}\int_2^{\infty}\frac{1}{r^4}r\ dr\ d\theta$$

$$\int_0^{\pi}\int_2^{\infty}\frac{1}{r^3}\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{\pi}-\frac{1}{2r^2}\Bigg|_{r=2}^{r=\infty}\ d\theta$$

$$\int_0^\pi -\frac{1}{2(\infty)^2} - \left(-\frac{1}{2(2)^2}\right)\ d\theta$$

$$\int_0^\pi 0 + \frac{1}{2(4)}\ d\theta$$

$$\int_0^\pi \frac{1}{8}\ d\theta$$

Integrate with respect to θ.

$$\frac{1}{8}\theta\Big|_0^\pi$$

$$\frac{1}{8}\pi - \frac{1}{8}(0)$$

$$\frac{\pi}{8}$$

■ 3. The region D consists of all the points inside the circle centered at $(-2,2)$ with radius $r=1$. Convert the double integral to polar coordinates, using the conversion formulas $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$ for a circle shifted off the origin, then find its value.

$$\iint_D x^2 + y^2\ dA$$

Solution:

Applying $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$, the conversion formulas become $x = -2 + r\cos\theta$ and $y = 2 + r\sin\theta$. Using these, we'll convert the function to polar coordinates.

$$x^2 + y^2$$

$$(-2 + r\cos\theta)^2 + (2 + r\sin\theta)^2$$

$$4 - 4r\cos\theta + r^2\cos^2\theta + 4 + 4r\sin\theta + r^2\sin^2\theta$$

$$8 + 4r(\sin\theta - \cos\theta) + r^2(\cos^2\theta + \sin^2\theta)$$

$$8 + 4r(\sin\theta - \cos\theta) + r^2(1)$$

$$8 + 4r(\sin\theta - \cos\theta) + r^2$$

Therefore, the integral in polar coordinates is:

$$\int_0^{2\pi}\int_0^1 (8 + 4r(\sin\theta - \cos\theta) + r^2)r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 8r + 4r^2(\sin\theta - \cos\theta) + r^3\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{2\pi} 4r^2 + \frac{4}{3}r^3(\sin\theta - \cos\theta) + \frac{1}{4}r^4\Big|_{r=0}^{r=1}\ d\theta$$

$$\int_0^{2\pi} 4(1)^2 + \frac{4}{3}(1)^3(\sin\theta - \cos\theta) + \frac{1}{4}(1)^4$$

$$-\left(4(0)^2+\frac{4}{3}(0)^3(\sin\theta-\cos\theta)+\frac{1}{4}(0)^4\right)\ d\theta$$

$$\int_0^{2\pi} 4+\frac{4}{3}(\sin\theta-\cos\theta)+\frac{1}{4}\ d\theta$$

$$\int_0^{2\pi} \frac{17}{4}+\frac{4}{3}\sin\theta-\frac{4}{3}\cos\theta\ d\theta$$

Integrate with respect to θ.

$$\frac{17}{4}\theta-\frac{4}{3}\cos\theta-\frac{4}{3}\sin\theta\Big|_0^{2\pi}$$

$$\frac{17}{4}(2\pi)-\frac{4}{3}\cos(2\pi)-\frac{4}{3}\sin(2\pi)-\left(\frac{17}{4}(0)-\frac{4}{3}\cos(0)-\frac{4}{3}\sin(0)\right)$$

$$\frac{17\pi}{2}-\frac{4}{3}(1)-\frac{4}{3}(0)+\frac{4}{3}(1)+\frac{4}{3}(0)$$

$$\frac{17\pi}{2}-\frac{4}{3}+\frac{4}{3}$$

$$\frac{17\pi}{2}$$

SKETCHING AREA

1. Identify the region of integration given by the double integral.

$$\int_0^{2\pi}\int_0^{\frac{3}{\sqrt{1+1.25\cos^2\theta}}} f(r,\theta)\ dr\ d\theta$$

Solution:

Rewrite the upper bound on r.

$$r=\frac{3}{\sqrt{1+1.25\cos^2\theta}}$$

$$r^2=\frac{9}{1+1.25\cos^2\theta}$$

$$r^2(1+1.25\cos^2\theta)=9$$

$$r^2+1.25r^2\cos^2\theta=9$$

$$r^2+1.25(r\cos\theta)^2=9$$

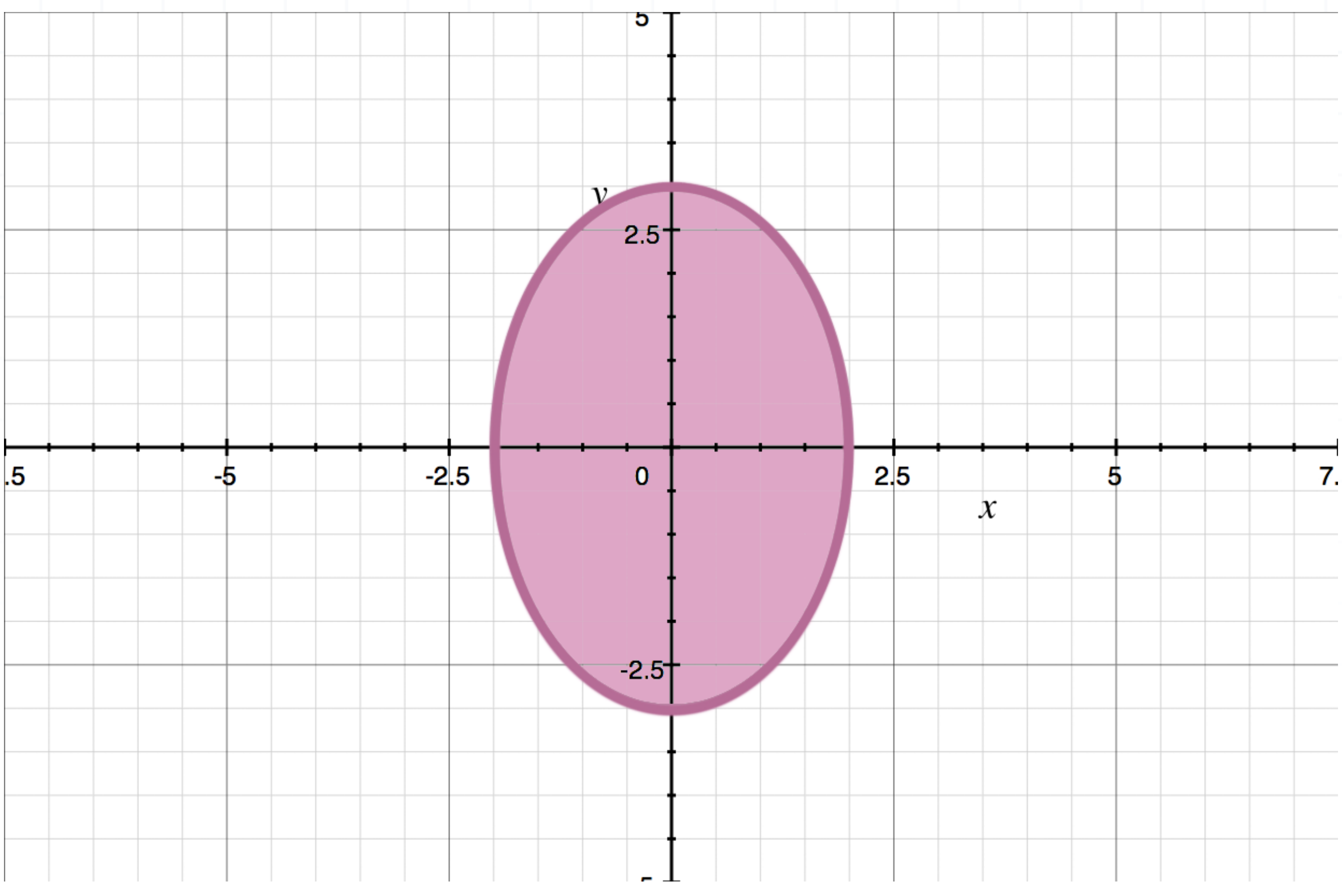
Substitute $r^2=x^2+y^2$ and $x=r\cos\theta$ to convert this upper bound into rectangular coordinates.

$$x^2+y^2+\frac{5}{4}x^2=9$$

$$\frac{9}{4}x^2 + y^2 = 9$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

We see now that the upper bound on r is the ellipse centered at the origin with horizontal radius 2 and vertical radius 3.



Going back to the original integral, r is bounded below at 0 and above by this ellipse, and θ is bounded on $[0,2\pi]$, which means the region of integration is this entire ellipse.

■ 2. Identify the region of integration given by the double integral.

$$\int_0^5 \int_0^{\frac{1}{3}\cos^{-1}\left(\frac{r}{5}\right)} f(r,\theta)\ d\theta\ dr$$

Solution:

Rewrite the upper bound on θ into rectangular coordinates.

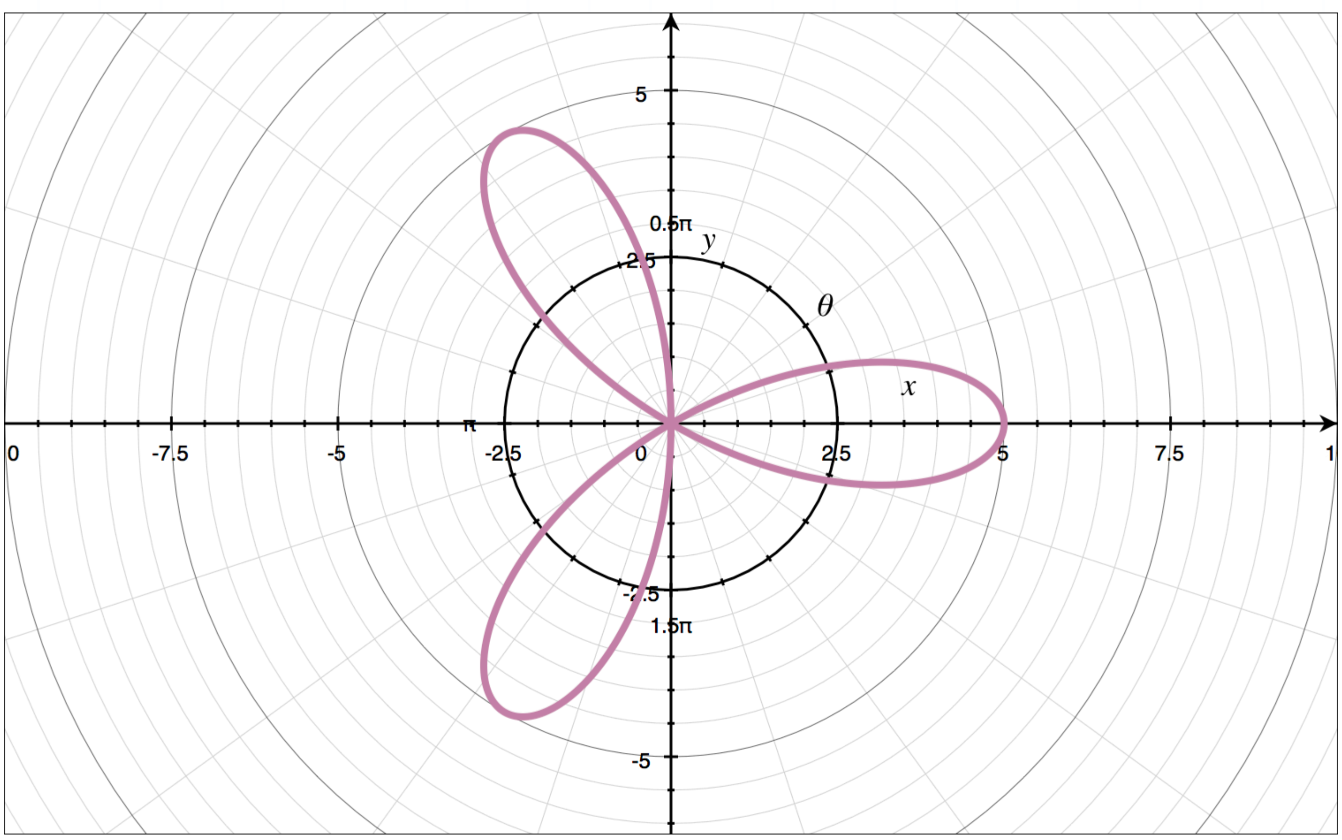
$$\theta = \frac{1}{3}\cos^{-1}\left(\frac{r}{5}\right)$$

$$3\theta = \cos^{-1}\left(\frac{r}{5}\right)$$

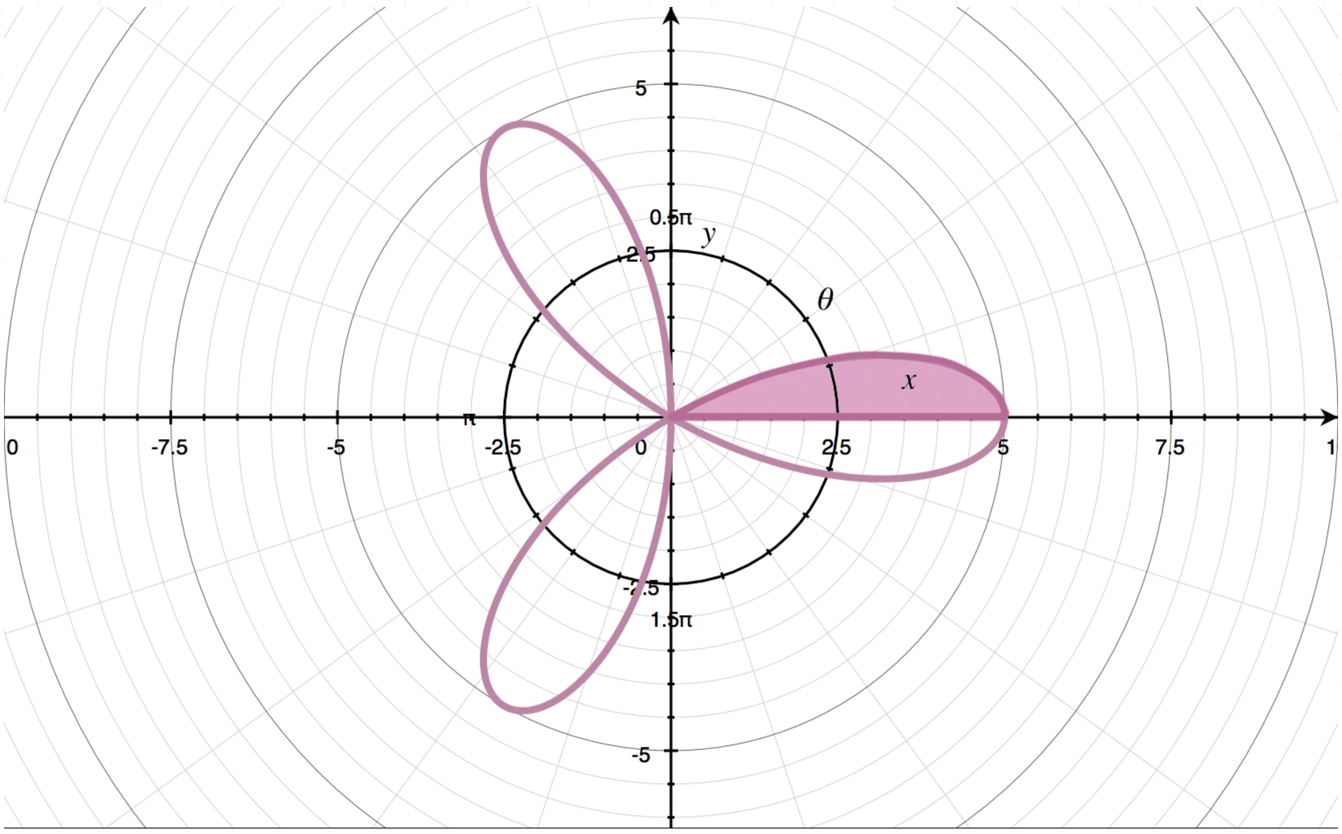
$$\cos(3\theta) = \frac{r}{5}$$

$$r = 5\cos(3\theta)$$

This polar curve is a three-petaled rose whose petals extend to $r = 5$.



Because the interval over which θ is defined is $\theta = [0,5\cos(3\theta)]$, we'll start at the angle $\theta = 0$ and rotate counterclockwise until we run into the curve. Therefore, we can say that the region of integration is defined as just the upper-half of the first petal.



■ 3. Identify the region of integration given by the double integral.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} f(r,\theta)\ dr\ d\theta$$

Solution:

The region is defined on $r = 2\cos\theta$ to $r = 4\cos\theta$, so we'll convert these bounds using $x^2 + y^2 = r^2$ and $x = r\cos\theta$.

$$r = 2\cos\theta$$

$$\sqrt{x^2+y^2} = 2\frac{x}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 2x$$

$$x^2-2x+y^2 = 0$$

Complete the square.

$$x^2-2x+1-1+y^2 = 0$$

$$(x-1)^2-1+y^2 = 0$$

$$(x-1)^2+y^2 = 1$$

So the lower bound on r is the circle centered at $(1,0)$ with radius 1. Now convert the upper bound.

$$r = 4\cos\theta$$

$$\sqrt{x^2+y^2} = 4\frac{x}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 4x$$

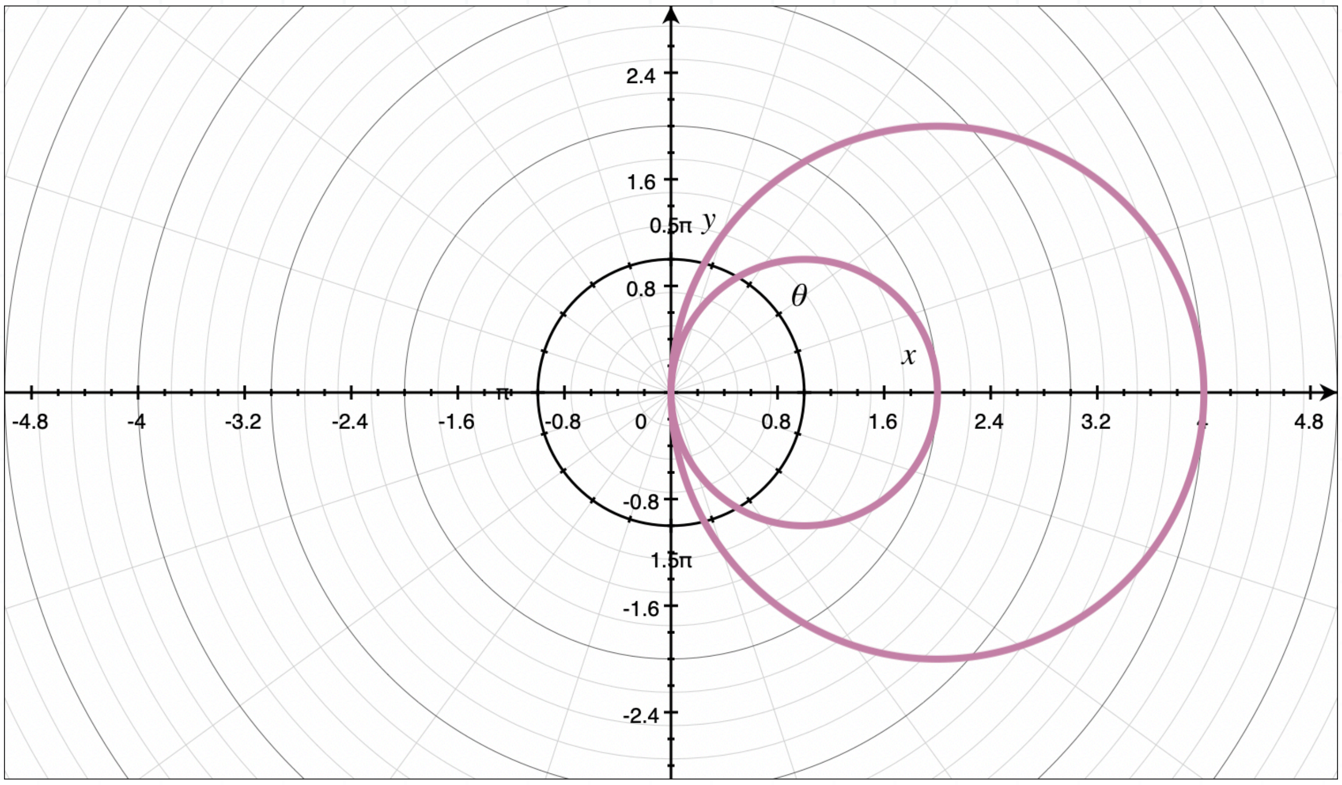
$$x^2-4x+y^2 = 0$$

$$x^2-4x+4-4+y^2 = 0$$

$$(x-2)^2-4+y^2 = 0$$

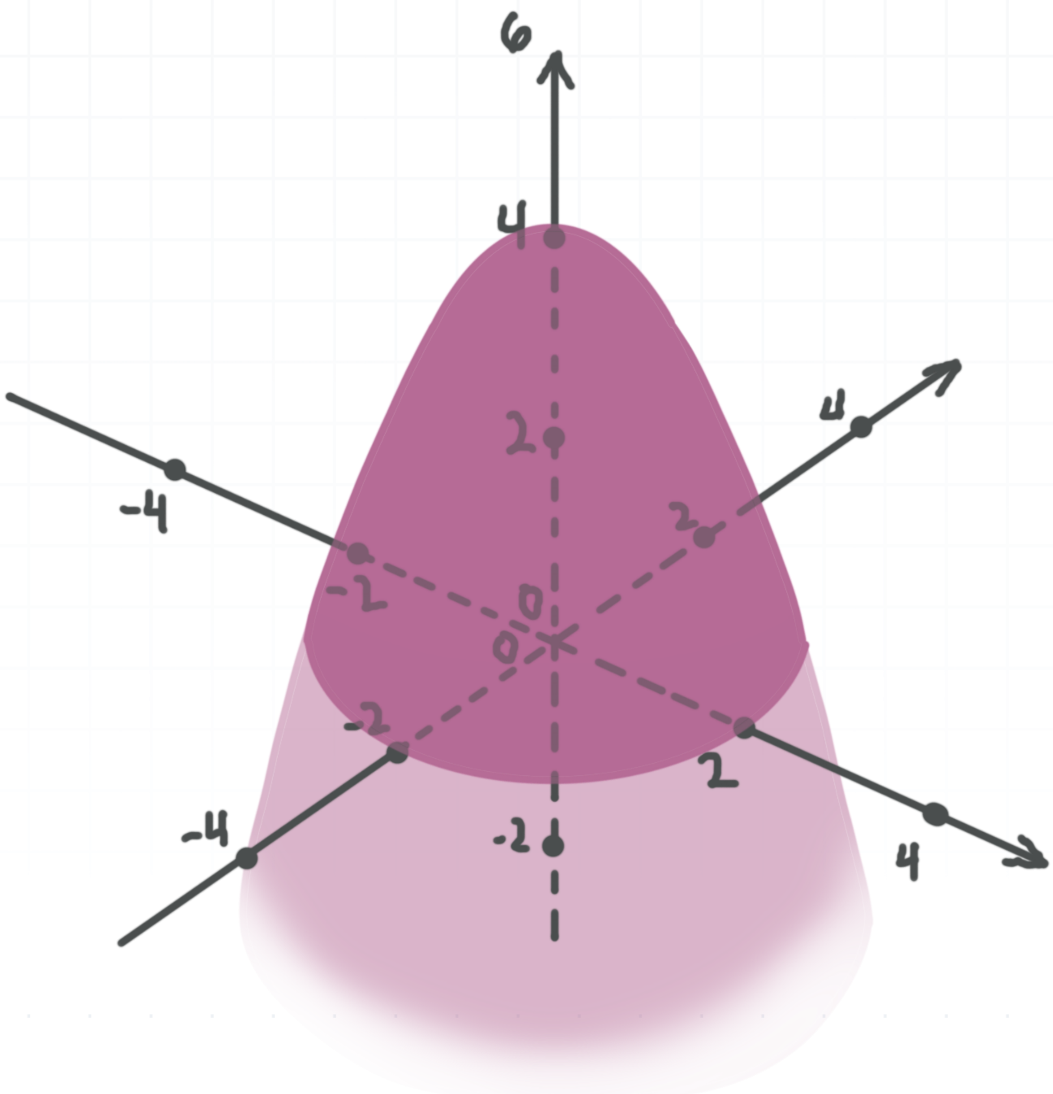
$$(x-2)^2+y^2 = 4$$

So the upper bound on r is the circle centered at $(2,0)$ with radius 2, and the region of integration is the area between $(x-1)^2+y^2=1$ and $(x-2)^2+y^2=4$.



FINDING AREA

■ 1. Find area of the surface $x^2+y^2+z-4=0$ above the xy-plane.



Solution:

Determinate the curve of intersection of the surface $x^2+y^2+z-4=0$ and the xy-plane $z=0$.

$$x^2+y^2+z-4=0$$

$$x^2+y^2+0=4$$

$$x^2+y^2=2^2$$

The curve of intersection is the circle with the center at the origin and radius 2, which means x is defined on $x=[-2,2]$ and y is defined on $y=[-\sqrt{4-x^2},\sqrt{4-x^2}]$. The function is $z=4-x^2-y^2$ and its partial derivatives are

$$z_x=-2x$$

$$z_y=-2y$$

So the area of the surface is

$$\iint_D \sqrt{1+z_x^2+z_y^2}\ dA$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+4x^2+4y^2}\ dy\ dx$$

$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \sqrt{1+4(x^2+y^2)}\ dy\ dx$$

Convert to polar coordinates, remembering that the bounds define the circle $x^2+y^2=2^2$, which means r is defined on $r=[0,2]$ and θ is defined on $\theta=[0,2\pi]$.

$$\int_0^{2\pi}\int_0^2 \sqrt{1+4r^2}\ r\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant, by using a substitution.

$$u = 4r^2$$

$$\frac{du}{dr} = 8r, \text{ so } du = 8r\ dr \text{ and } dr = \frac{du}{8r}$$

The bounds $r = [0,2]$ convert to $u = [0,16]$.

$$\int_0^{2\pi} \int_0^{16} \sqrt{1+u}\ r \left(\frac{du}{8r}\right)\ d\theta$$

$$\int_0^{2\pi} \int_0^{16} \frac{1}{8}\sqrt{1+u}\ du\ d\theta$$

$$\int_0^{2\pi} \int_0^{16} \frac{1}{8}(1+u)^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u.

$$\int_0^{2\pi} \frac{1}{8} \cdot \frac{2}{3}(1+u)^{\frac{3}{2}}\Big|_0^{16}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{12}(1+u)^{\frac{3}{2}}\Big|_0^{16}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{12}(1+16)^{\frac{3}{2}} - \frac{1}{12}(1+0)^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{12}(17)^{\frac{3}{2}} - \frac{1}{12}(1)^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} \frac{17^{\frac{3}{2}}}{12} - \frac{1}{12}\ d\theta$$

$$\int_0^{2\pi} \frac{17^{\frac{3}{2}} - 1}{12} \ d\theta$$

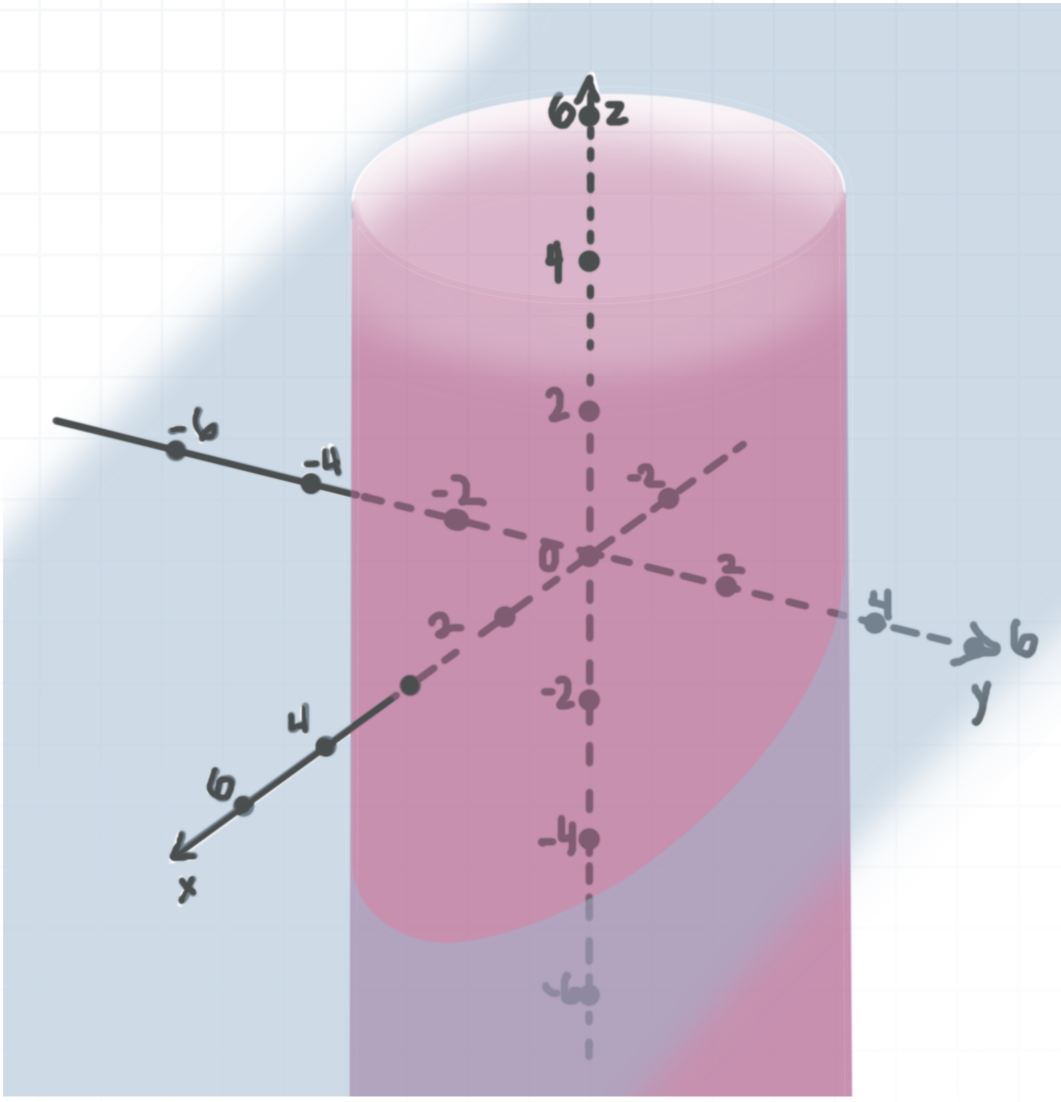
Integrate with respect to θ.

$$\frac{17^{\frac{3}{2}} - 1}{12}\theta \Big|_0^{2\pi}$$

$$\frac{17^{\frac{3}{2}} - 1}{12}(2\pi) - \frac{17^{\frac{3}{2}} - 1}{12}(0)$$

$$\frac{(17^{\frac{3}{2}} - 1)\pi}{6}$$

■ 2. Find area of the part of the plane $2x - y + 3z - 3 = 0$ that lies within the cylinder $(x - 3)^2 + (y - 2)^2 = 3^2$.



Solution:

The plane is

$$2x - y + 3z - 3 = 0$$

$$z = -\frac{2}{3}x + \frac{1}{3}y + 1$$

and its partial derivatives are

$$z_x = -\frac{2}{3}$$

$$z_y = \frac{1}{3}$$

So the area of the surface is

$$\iint_D \sqrt{1 + z_x^2 + z_y^2}\ dA$$

$$\iint_D \sqrt{1 + \frac{4}{9} + \frac{1}{9}}\ dA$$

$$\iint_D \sqrt{\frac{14}{9}}\ dA$$

$$\iint_D \frac{\sqrt{14}}{3}\ dA$$

The region of integration is the circle centered at $(3,2)$ with radius 3, so r is defined on $r = [0,3]$ and θ is defined on $[0,2\pi]$. Then the integral in polar coordinates is

$$\frac{\sqrt{14}}{3}\int_0^{2\pi}\int_0^3 r\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\frac{\sqrt{14}}{3}\int_0^{2\pi} \frac{1}{2}r^2\Big|_{r=0}^{r=3}\ d\theta$$

$$\frac{\sqrt{14}}{3}\int_0^{2\pi} \frac{1}{2}(3)^2 - \frac{1}{2}(0)^2\ d\theta$$

$$\frac{\sqrt{14}}{3}\int_0^{2\pi}\frac{9}{2}\ d\theta$$

Integrate with respect to θ.

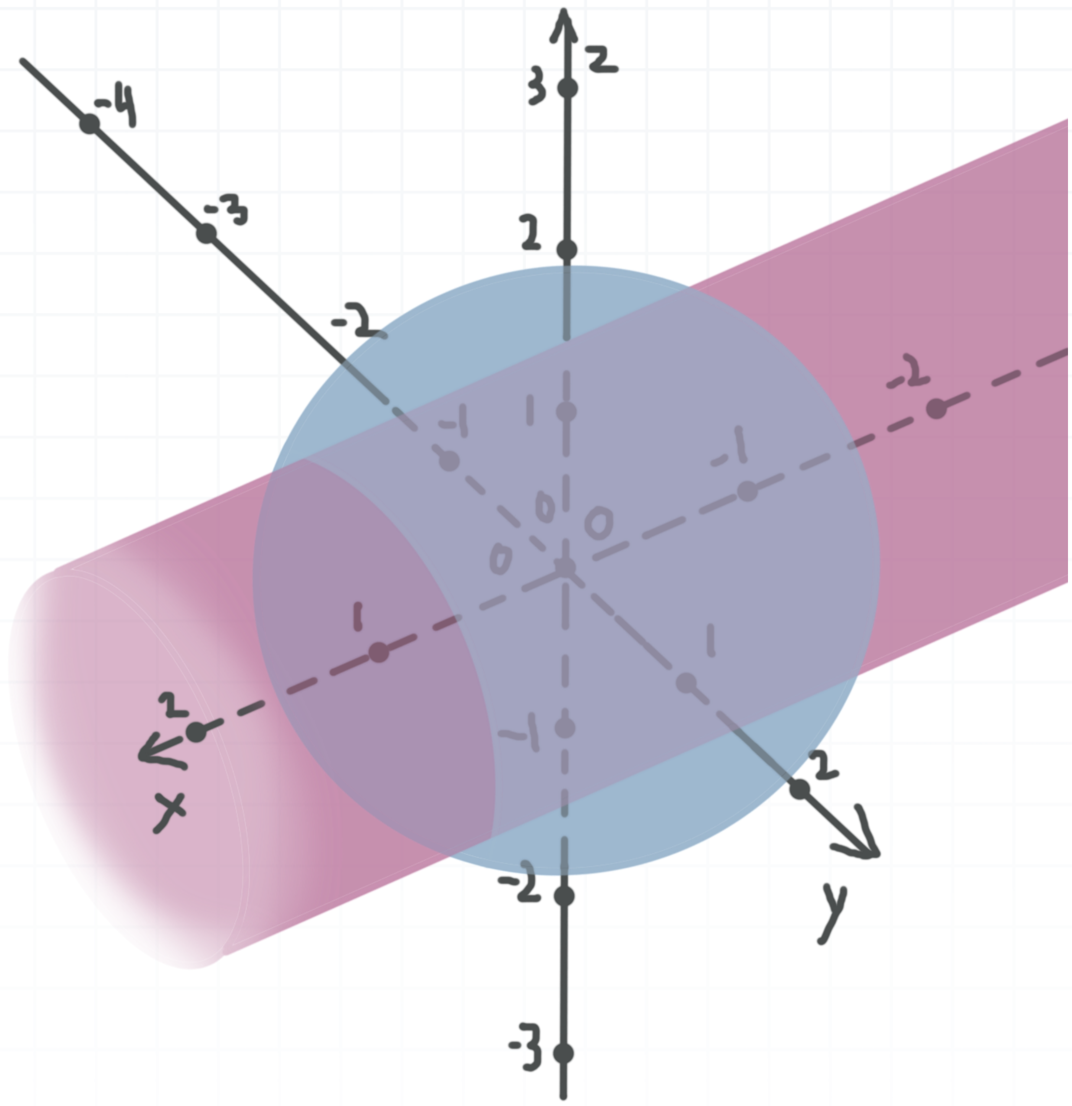
$$\frac{\sqrt{14}}{3}\left(\frac{9}{2}\theta\right)\Big|_0^{2\pi}$$

$$\frac{3\sqrt{14}}{2}\theta\Big|_0^{2\pi}$$

$$\frac{3\sqrt{14}}{2}(2\pi)-\frac{3\sqrt{14}}{2}(0)$$

$$3\sqrt{14}\pi$$

■ 3. Find area of the sphere $x^2+y^2+z^2-2=0$ that lies within the cylinder $y^2+z^2=1$.



Solution:

Since the area consists of two equal parts, lying on either side of the yz-plane, we'll calculate the area for $x > 0$, then double the result. Because the cylinder is parallel to the x-axis, we should use the function $x = f(y, z)$.

$$x^2 + y^2 + z^2 - 2 = 0$$

$$x^2 = 2 - y^2 - z^2$$

$$x = \sqrt{2 - y^2 - z^2}$$

Then the partial derivatives are

$$x_y = \frac{-y}{\sqrt{2 - y^2 - z^2}}$$

$$x_z = \frac{-z}{\sqrt{2 - y^2 - z^2}}$$

So the area of the surface is

$$\iint_D \sqrt{1 + z_x^2 + z_y^2}\ dA$$

$$\iint_D \sqrt{1 + \frac{y^2}{2 - y^2 - z^2} + \frac{z^2}{2 - y^2 - z^2}}\ dA$$

$$\iint_D \frac{\sqrt{(2 - y^2 - z^2) + y^2 + z^2}}{\sqrt{2 - y^2 - z^2}}\ dA$$

$$\iint_D \frac{\sqrt{2}}{\sqrt{2 - y^2 - z^2}}\ dA$$

$$\sqrt{2}\iint_D \frac{1}{\sqrt{2 - y^2 - z^2}}\ dA$$

If we use $r^2 = y^2 + z^2$ and $dy\ dz = r\ dr\ d\theta$ to convert the function, we get

$$\sqrt{2}\iint_D \frac{1}{\sqrt{2 - r^2}}\ dA$$

The region of integration is the circle centered at the origin with radius 1, so r is defined on $r=[0,1]$ and θ is defined on $[0,2\pi]$. Then the integral in polar coordinates is

$$\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{2-r^2}}r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{2-r^2}}\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant, using a substitution with

$$u=r^2$$

$$\frac{du}{dr}=2r\text{, so } du=2r\ dr \text{ and } dr=\frac{du}{2r}$$

If r is defined on $r=[0,1]$, then u is defined on $u=[0,1]$.

$$\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{2-u}}\left(\frac{du}{2r}\right)\ d\theta$$

$$\int_0^{2\pi}\int_0^1 \frac{1}{2\sqrt{2-u}}\ du\ d\theta$$

$$\int_0^{2\pi}\int_0^1 \frac{1}{2}(2-u)^{-\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u, treating θ as a constant.

$$\int_0^{2\pi} \frac{1}{2} \cdot \frac{2}{-1}(2-u)^{\frac{1}{2}} \Big|_{u=0}^{u=1} \ d\theta$$

$$\int_0^{2\pi} -\sqrt{2-u} \Big|_{u=0}^{u=1} \ d\theta$$

$$\int_0^{2\pi} -\sqrt{2-1} - (-\sqrt{2-0}) \ d\theta$$

$$\int_0^{2\pi} -\sqrt{1} + \sqrt{2} \ d\theta$$

$$\int_0^{2\pi} \sqrt{2} - 1 \ d\theta$$

Integrate with respect to θ.

$$(\sqrt{2}-1)\theta \Big|_0^{2\pi}$$

$$(\sqrt{2}-1)(2\pi) - (\sqrt{2}-1)(0)$$

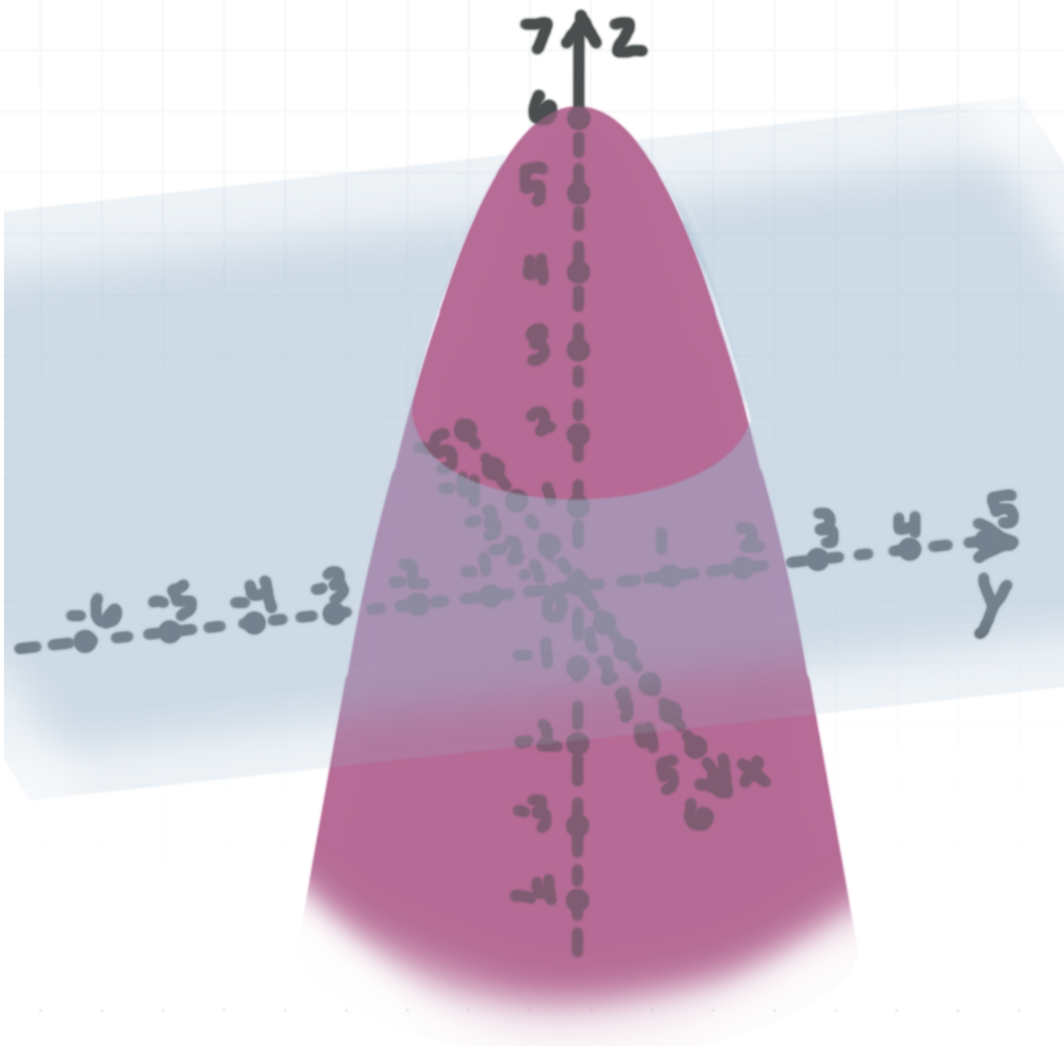
$$2\pi(\sqrt{2}-1)$$

This is exactly half the area, so we'll double this to find total area.

$$4\pi(\sqrt{2}-1)$$

FINDING VOLUME

■ 1. Find the volume of the region bounded $x^2 + y^2 + z - 6 = 0$ and $z = 2$.



Solution:

Finding the volume bounded by $x^2 + y^2 + z - 6 = 0$ and $z = 2$ is equivalent to finding the volume bounded by $x^2 + y^2 + z - 4 = 0$ and $z = 0$. Find the curve of intersection of these last two surfaces.

$$x^2 + y^2 + 0 - 4 = 0$$

$$x^2+y^2=2^2$$

So the curve of intersection is the circle centered at the origin with radius 2. And we can rewrite $x^2+y^2+z-4=0$ as the function $z=4-x^2-y^2$, which means the volume can be defined by

$$\iint_D 4-x^2-y^2\ dA$$

Since the region of integration is the circle centered at the origin with radius 2, the value of r is defined on $r=[0,2]$, and the value of θ is defined on $[0,2\pi]$. Using the conversion formula $r^2=x^2+y^2$, the integral in the polar coordinates is

$$\int_0^{2\pi}\int_0^2 (4-r^2)r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2 4r-r^3\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{2\pi} 2r^2-\frac{1}{4}r^4\Big|_{r=0}^{r=2}\ d\theta$$

$$\int_0^{2\pi} 2(2)^2-\frac{1}{4}(2)^4-\left(2(0)^2-\frac{1}{4}(0)^4\right)\ d\theta$$

$$\int_0^{2\pi} 2(4)-\frac{1}{4}(16)\ d\theta$$

$$\int_0^{2\pi} 8 - 4\ d\theta$$

$$\int_0^{2\pi} 4\ d\theta$$

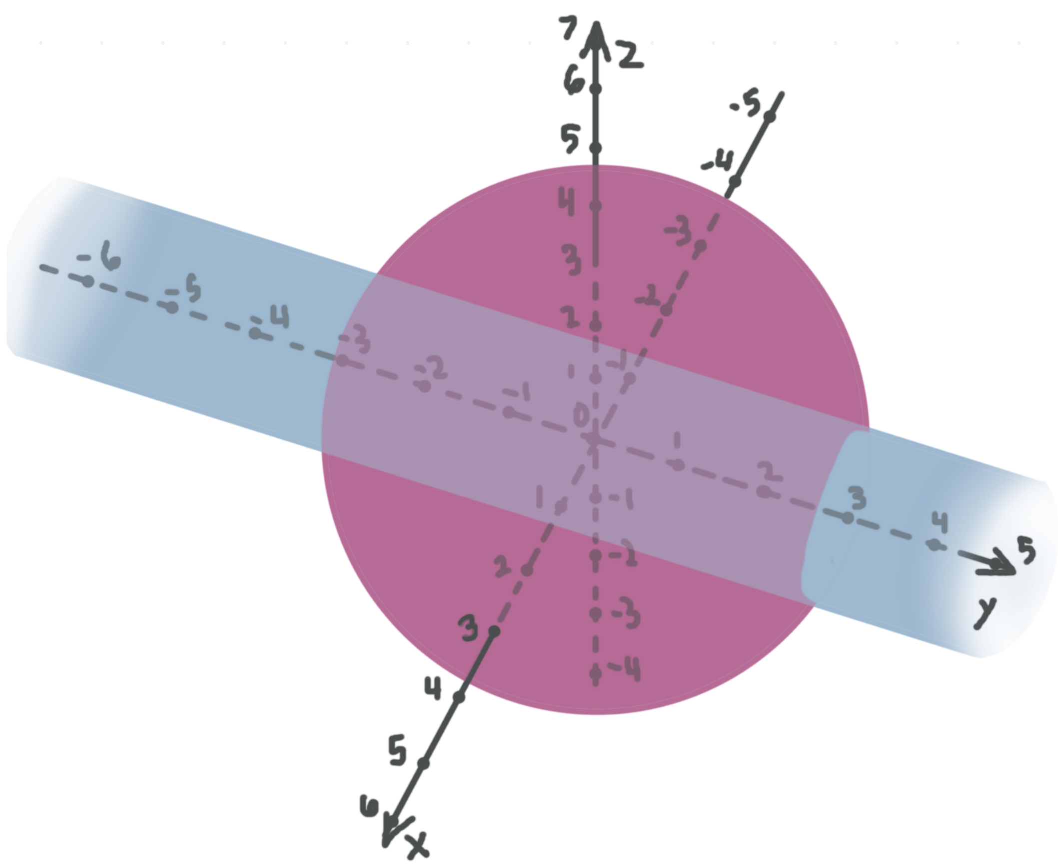
Integrate with respect to θ.

$$4\theta \Big|_0^{2\pi}$$

$$4(2\pi) - 4(0)$$

$$8\pi$$

■ 2. Find volume of the sphere $x^2 + y^2 + z^2 - 9 = 0$ that lies within the cylinder $x^2 + z^2 = 1$.



Solution:

Since the region consists of two equal parts on either side of the xz-plane, we'll calculate the volume for $y > 0$, then double the result.

The cylinder is parallel to the y-axis, so we'll use the function $y = f(x, z)$.

$$x^2 + y^2 + z^2 - 9 = 0$$

$$y^2 = 9 - x^2 - z^2$$

$$y = \sqrt{9 - x^2 - z^2}$$

The volume of the region is

$$\int\int_D \sqrt{9 - x^2 - z^2}\ dA$$

Since the region of integration is the circle centered at the origin with radius 1, the value of r is defined on $r = [0,1]$ and the value of θ is defined on $\theta = [0,2\pi]$. Then using the conversion formulas $r^2 = x^2 + z^2$ and $dx\ dz = r\ dr\ d\theta$, the integral in polar coordinates is

$$\int_0^{2\pi} \int_0^1 \sqrt{9 - r^2}\ r\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant, using a substitution.

$$u = r^2$$

$$\frac{du}{dr} = 2r, \text{ so } du = 2r\ dr \text{ and } dr = \frac{du}{2r}$$

If r is defined on $r = [0,1]$, then u is defined on $u = [0,1]$.

$$\int_0^{2\pi} \int_0^1 \sqrt{9-u}\ r \left(\frac{du}{2r}\right)\ d\theta$$

$$\int_0^{2\pi} \int_0^1 \frac{1}{2}(9-u)^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u, treating θ as a constant.

$$\int_0^{2\pi} -\frac{1}{2} \cdot \frac{2}{3}(9-u)^{\frac{3}{2}}\Big|_{u=0}^{u=1}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(9-u)^{\frac{3}{2}}\Big|_{u=0}^{u=1}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(9-1)^{\frac{3}{2}} + \frac{1}{3}(9-0)^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}8^{\frac{3}{2}} + \frac{1}{3}9^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(8^{\frac{1}{2}})^3 + \frac{1}{3}(9^{\frac{1}{2}})^3\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(2\sqrt{2})^3 + \frac{1}{3}(3)^3\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}8(2)\sqrt{2}+9\ d\theta$$

$$\int_0^{2\pi} 9-\frac{16\sqrt{2}}{3}\ d\theta$$

Integrate with respect to θ.

$$\left(9-\frac{16\sqrt{2}}{3}\right)\theta\Big|_0^{2\pi}$$

$$\left(9-\frac{16\sqrt{2}}{3}\right)(2\pi)-\left(9-\frac{16\sqrt{2}}{3}\right)(0)$$

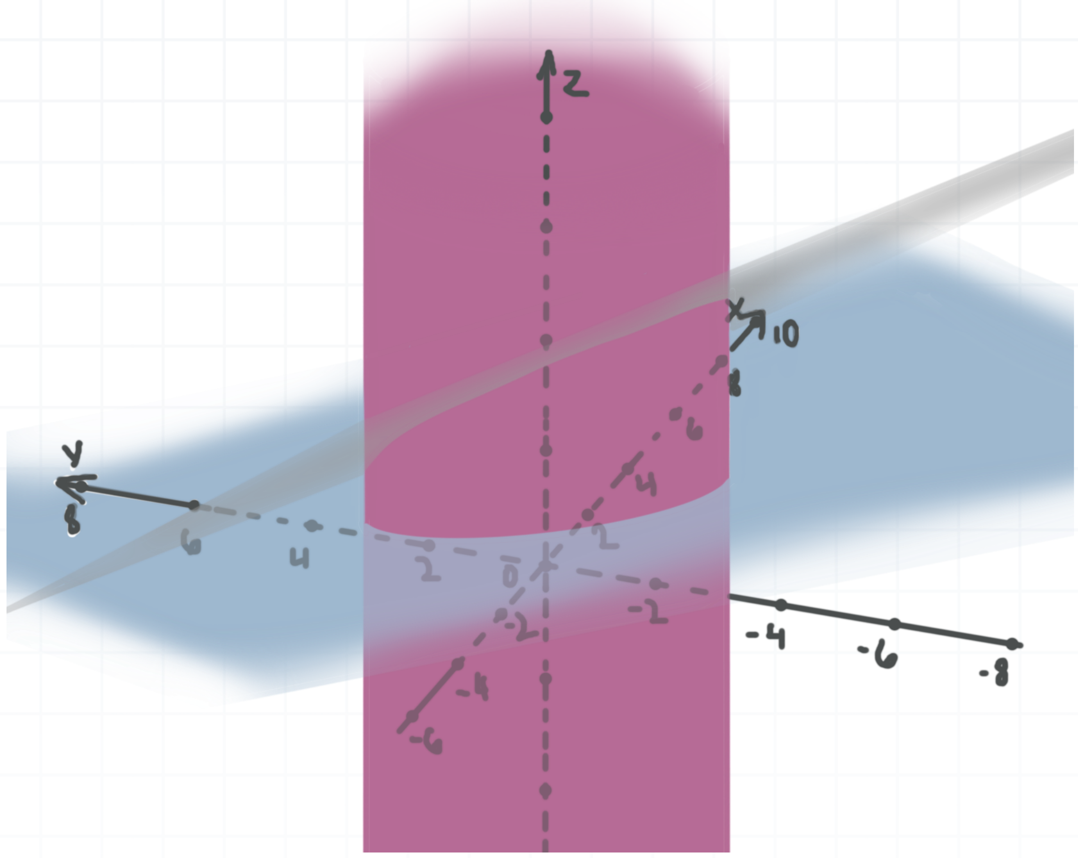
$$2\pi\left(9-\frac{16\sqrt{2}}{3}\right)$$

$$18\pi-\frac{32\sqrt{2}\pi}{3}$$

This is half of the total area, so we'll double it to get total area.

$$36\pi-\frac{64\sqrt{2}\pi}{3}$$

■ 3. Find the volume bounded by the cylinder $x^2+y^2=9$, the plane $x+y+5z-6=0$, and the plane $x+2y+3z-10=0$.



Solution:

We need to find the volume below the plane $x + y + 5z - 6 = 0$, then find the volume below the plane $x + 2y + 3z - 10 = 0$, and then calculate the difference.

The equation of the first plane is

$$x + y + 5z - 6 = 0$$

$$z = \frac{1}{5}(6 - x - y)$$

Then the volume is

$$\iint_D \frac{1}{5}(6 - x - y)\ dA$$

Since the region of integration is the circle centered at the origin with radius 3, we know that r is defined on $r = [0,3]$ and θ is defined on $\theta = [0,2\pi]$. Then the integral in polar coordinates is

$$\frac{1}{5}\int_0^{2\pi}\int_0^3 (6 - r\cos\theta - r\sin\theta)r\ dr\ d\theta$$

$$\frac{1}{5}\int_0^{2\pi}\int_0^3 6r - r^2\cos\theta - r^2\sin\theta\ dr\ d\theta$$

Let's change the order of integration to simplify the integration.

$$\frac{1}{5}\int_0^{3}\int_0^{2\pi} 6r - r^2\cos\theta - r^2\sin\theta\ d\theta\ dr$$

Integrate with respect to θ, treating r as a constant.

$$\frac{1}{5}\int_0^3 6r\theta - r^2\sin\theta + r^2\cos\theta\Big|_{\theta=0}^{\theta=2\pi}\ dr$$

$$\frac{1}{5}\int_0^3 6r(2\pi) - r^2\sin(2\pi) + r^2\cos(2\pi) - (6r(0) - r^2\sin(0) + r^2\cos(0))\ dr$$

$$\frac{1}{5}\int_0^3 12\pi r - r^2(0) + r^2(1) - (-r^2(0) + r^2(1))\ dr$$

$$\frac{1}{5}\int_0^3 12\pi r + r^2 - r^2\ dr$$

$$\frac{1}{5}\int_0^3 12\pi r\ dr$$

Integrate with respect to r.

$$\frac{1}{5} \cdot 6\pi r^2 \Big|_0^3$$

$$\frac{6}{5}\pi r^2 \Big|_0^3$$

$$\frac{6}{5}\pi(3)^2 - \frac{6}{5}\pi(0)^2$$

$$\frac{54}{5}\pi$$

The equation of the second plane is

$$x + 2y + 3z - 10 = 0$$

$$z = \frac{1}{3}(10 - x - 2y)$$

Then the volume is

$$\int\int_D \frac{1}{3}(10 - x - 2y)\ dA$$

$$\frac{1}{3}\int_0^{2\pi}\int_0^3 (10 - r\cos\theta - 2r\sin\theta)r\ dr\ d\theta$$

$$\frac{1}{3}\int_0^{2\pi}\int_0^3 10r - r^2\cos\theta - 2r^2\sin\theta\ dr\ d\theta$$

Let's change the order of integration to simplify the integration.

$$\frac{1}{3}\int_0^3\int_0^{2\pi} 10r - r^2\cos\theta - 2r^2\sin\theta\ d\theta\ dr$$

Integrate with respect to θ, treating r as a constant.

$$\frac{1}{3}\int_0^3 10r\theta - r^2\sin\theta + 2r^2\cos\theta\Big|_{\theta=0}^{\theta=2\pi}\ dr$$

$$\frac{1}{3}\int_0^3 10r(2\pi) - r^2\sin(2\pi) + 2r^2\cos(2\pi) - (10r(0) - r^2\sin(0) + 2r^2\cos(0))\ dr$$

$$\frac{1}{3}\int_0^3 20\pi r - r^2(0) + 2r^2(1) - (-r^2(0) + 2r^2(1))\ dr$$

$$\frac{1}{3}\int_0^3 20\pi r + 2r^2 - 2r^2\ dr$$

$$\frac{1}{3}\int_0^3 20\pi r\ dr$$

Integrate with respect to r.

$$\frac{1}{3}\cdot 10\pi r^2\Big|_0^3$$

$$\frac{10}{3}\pi r^2\Big|_0^3$$

$$\frac{10}{3}\pi(3)^2 - \frac{10}{3}\pi(0)^2$$

$$\frac{90}{3}\pi$$

$$30\pi$$

Then the difference is

$$30\pi - \frac{54\pi}{5}$$

$$\frac{150\pi}{5} - \frac{54\pi}{5}$$

$$\frac{96\pi}{5}$$

DOUBLE INTEGRALS TO FIND MASS AND CENTER OF MASS

■ 1. The circular disk with radius 12 has density $\delta = 1/(r+4)$, where r is the distance to the center of disk. Find the mass and center of mass of the disk.

Solution:

The mass of the disk is given by the double integral

$$\iint_D \delta(x,y)\ dA$$

but we'll need to convert it to polar coordinates using

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx\ dy = r\ dr\ d\theta$$

The area of integration is the circle with center at the origin and radius 12. So the polar coordinate r changes from 0 to 12, and θ changes from 0 to 2π. So the integral in polar coordinates is

$$\int_0^{2\pi}\int_0^{12} \frac{1}{r+4} r\ dr\ d\theta$$

$$\int_0^{2\pi} \int_0^{12} \frac{r}{r+4} \ dr \ d\theta$$

$$\int_0^{2\pi} \int_0^{12} \frac{r+4-4}{r+4} \ dr \ d\theta$$

$$\int_0^{2\pi} \int_0^{12} 1 - \frac{4}{r+4} \ dr \ d\theta$$

Integrate with respect to r.

$$\int_0^{2\pi} r - 4\ln(r+4) \Big|_0^{12} \ d\theta$$

$$\int_0^{2\pi} 12 - 4\ln(12+4) - [0 - 4\ln(0+4)] \ d\theta$$

$$\int_0^{2\pi} 12 - 4\ln(12+4) - 0 + 4\ln(0+4) \ d\theta$$

$$\int_0^{2\pi} 12 - 4\ln(16) + 4\ln(4) \ d\theta$$

Integrate with respect to θ.

$$12\theta - 4\ln(16)\theta + 4\ln(4)\theta \Big|_0^{2\pi}$$

$$12(2\pi) - 4\ln(16)(2\pi) + 4\ln(4)(2\pi) - [12(0) - 4\ln(16)(0) + 4\ln(4)(0)]$$

$$24\pi - 8\pi\ln(16) + 8\pi\ln(4)$$

Use laws of logs to simplify.

$$24\pi - 8\pi \ln(4^2) + 8\pi \ln(4)$$

$$24\pi - 8(2)\pi \ln(4) + 8\pi \ln(4)$$

$$24\pi - 16\pi \ln(4) + 8\pi \ln(4)$$

$$24\pi - 8\pi \ln(4)$$

Since the disk is symmetric and has symmetric density about x-axis and y-axis, its center of mass is the center of the disk.

■ 2. The rectangular plate with length 4 m and width 2 m has density $\delta = 2d$ kg/m^2, where d is the distance from its left 2 m side. Find the mass and center of mass of the plate.

Solution:

The mass of the plate is given by the double integral:

$$\int\int_D \delta(x, y)\ dA$$

Let's place the origin at the bottom left corner of the plate, and put the edges of the plate along the x- and y-axes. So the plate is defined on $x = [0,4]$ and $y = [0,2]$, and its density is $2x$. So the double integral is

$$\int_0^2 \int_0^4 2x \ dx \ dy$$

Integrate with respect to x.

$$\int_0^2 x^2 \Big|_0^4 \ dy$$

$$\int_0^2 4^2 - 0^2 \ dy$$

$$\int_0^2 16 \ dy$$

Integrate with respect to y.

$$16y \Big|_0^2$$

$$16(2) - 16(0)$$

32 kg

Find the x-coordinate of the center of mass.

$$\bar{x} = \frac{1}{M} \int \int_D x\delta(x, y) \ dA$$

$$\bar{x} = \frac{1}{32} \int_0^2 \int_0^4 2x^2 \ dx \ dy$$

$$\bar{x} = \frac{1}{32}\int_0^2 \frac{2}{3}x^3 \Big|_0^4 \ dy$$

$$\bar{x} = \frac{1}{32}\int_0^2 \frac{2}{3}(4)^3 - \frac{2}{3}(0)^3 \ dy$$

$$\bar{x} = \frac{1}{32}\int_0^2 \frac{128}{3} \ dy$$

$$\bar{x} = \frac{4}{3}\int_0^2 \ dy$$

$$\bar{x} = \frac{4}{3}y\Big|_0^2$$

$$\bar{x} = \frac{4}{3}(2) - \frac{4}{3}(0)$$

$$\bar{x} = \frac{8}{3}$$

Find the y-coordinate of the center of mass.

$$\bar{y} = \frac{1}{M}\iint_D y\delta(x, y) \ dA$$

$$\bar{y} = \frac{1}{32}\int_0^2 \int_0^4 2xy \ dx \ dy$$

$$\bar{y} = \frac{1}{32}\int_0^2 x^2y \Big|_{x=0}^{x=4} \ dy$$

$$\bar{y} = \frac{1}{32}\int_0^2 4^2y - 0^2y\ dy$$

$$\bar{y} = \frac{1}{32}\int_0^2 16y\ dy$$

$$\bar{y} = \frac{1}{2}\int_0^2 y\ dy$$

$$\bar{y} = \frac{1}{2}\left(\frac{1}{2}y^2\right)\Big|_0^2$$

$$\bar{y} = \frac{1}{4}y^2\Big|_0^2$$

$$\bar{y} = \frac{1}{4}(2)^2 - \frac{1}{4}(0)^2$$

$$\bar{y} = 1$$

Therefore, the mass is $M = 32$ kg, and the center of mass is at $(\bar{x}, \bar{y}) = (8/3, 1)$.

■ 3. Some gas is distributed above the line with density $\delta = e^{-ad^2}$, where d is the distance to point A on the line, and a is a constant. Find the total mass of the gas and its center of mass.

Solution:

The mass of the gas is given by the double integral

$$\iint_D \delta(x, y)\ dA$$

Let's place the origin at the point A on the line, and the x-axis on this line. Then x is defined from $-\infty$ to ∞ and y is defined from 0 to ∞. The density is $\delta = e^{-a(x^2+y^2)}$. Then the mass is

$$\int_0^\infty \int_{-\infty}^\infty e^{-a(x^2+y^2)}\ dx\ dy$$

$$\int_0^\infty \int_{-\infty}^\infty e^{-ax^2}e^{-ay^2}\ dx\ dy$$

$$\int_0^\infty e^{-ay^2}\ dy \cdot \int_{-\infty}^\infty e^{-ax^2}\ dx$$

These integrals can't be calculated directly, but we know their values.

$$\int_{-\infty}^\infty e^{-ax^2}\ dx = \sqrt{\frac{\pi}{a}}$$

Since the function is symmetric,

$$\int_0^\infty e^{-ay^2}\ dy = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

So the expression becomes

$$\frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot \sqrt{\frac{\pi}{a}}$$

$$\frac{1}{2}\left(\frac{\pi}{a}\right)$$

$$\frac{\pi}{2a}$$

Since the region and density are symmetric about y-axis, $\bar{x} = 0$. Calculate $\bar{y}$.

$$\bar{y} = \frac{1}{M}\iint_D y\delta(x, y)\ dA$$

$$\bar{y} = \frac{2a}{\pi}\int_0^{\infty}\int_{-\infty}^{\infty} ye^{-a(x^2+y^2)}\ dx\ dy$$

$$\bar{y} = \frac{2a}{\pi}\int_0^{\infty} ye^{-ay^2}dy \cdot \int_{-\infty}^{\infty} e^{-ax^2}dx$$

$$\bar{y} = \frac{2a}{\pi}\int_0^{\infty} ye^{-ay^2}dy \cdot \sqrt{\frac{\pi}{a}}$$

$$\bar{y} = \frac{\sqrt{a}\sqrt{a}\sqrt{\pi}}{\sqrt{a}\sqrt{\pi}\sqrt{\pi}}\int_0^{\infty} 2ye^{-ay^2}dy$$

$$\bar{y} = \frac{\sqrt{a}}{\sqrt{\pi}}\int_0^{\infty} 2ye^{-ay^2}dy$$

$$\bar{y} = \sqrt{\frac{a}{\pi}}\int_0^{\infty} 2ye^{-ay^2}dy$$

Use substitution with $u = y^2$, $du = 2y\ dy$, and where u changes from 0 to ∞.

$$\bar{y} = \sqrt{\frac{a}{\pi}}\int_0^{\infty} 2ye^{-ay^2}dy$$

$$\bar{y} = \sqrt{\frac{a}{\pi}}\int_0^{\infty} e^{-au}\ du$$

$$\bar{y}=\sqrt{\frac{a}{\pi}}\left(\frac{1}{-a}e^{-au}\right)\Big|_0^{\infty}$$

$$\bar{y}=\left[\lim_{u\to\infty}\sqrt{\frac{a}{\pi}}\left(\frac{1}{-a}e^{-au}\right)\right]-\sqrt{\frac{a}{\pi}}\left(\frac{1}{-a}e^{-a(0)}\right)$$

$$\bar{y}=\sqrt{\frac{a}{\pi}}\left(\frac{1}{-a}(0)\right)+\sqrt{\frac{a}{\pi}}\left(\frac{1}{a}\right)$$

$$\bar{y}=\frac{1}{\sqrt{a\pi}}$$

MIDPOINT RULE FOR TRIPLE INTEGRALS

1. Use the midpoint rule to approximate the value of the triple integral, using boxes with sides $2 \times 2 \times \pi$.

$$\int_{-2}^{2}\int_{0}^{4}\int_{-2\pi}^{2\pi} x^2 y \cos z \ dz \ dy \ dx$$

Solution:

The volume of integration consists of eight boxes, so the Riemann sum estimate is given by

$$\int_{-2}^{2}\int_{0}^{4}\int_{-2\pi}^{2\pi} x^2 y \cos(z) \ dz \ dy \ dx \approx \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V$$

where $\Delta V = 2 \times 2 \times \pi = 4\pi$ is the volume of each box. The midpoints of these boxes are

$$(-1,1,-\pi) \qquad (-1,1,\pi) \qquad (-1,3,-\pi) \qquad (-1,3,\pi)$$

$$(1,1,-\pi) \qquad (1,1,\pi) \qquad (1,3,-\pi) \qquad (1,3,\pi)$$

Find $f(x, y, z)$ for each point.

$$f(-1,1,-\pi) = (-1)^2 \cdot (1) \cdot \cos(-\pi) = -1$$

$$f(-1,1,\pi) = (-1)^2 \cdot (1) \cdot \cos(\pi) = -1$$

$$f(-1,3,-\pi)=(-1)^2\cdot(3)\cdot\cos(-\pi)=-3$$

$$f(-1,3,\pi)=(-1)^2\cdot(3)\cdot\cos(\pi)=-3$$

$$f(1,1,-\pi)=(1)^2\cdot(1)\cdot\cos(-\pi)=-1$$

$$f(1,1,\pi)=(1)^2\cdot(1)\cdot\cos(\pi)=-1$$

$$f(1,3,-\pi)=(1)^2\cdot(3)\cdot\cos(-\pi)=-3$$

$$f(1,3,\pi)=(1)^2\cdot(3)\cdot\cos(\pi)=-3$$

Then the Riemann sum is

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}f(x_i,y_j,z_k)\Delta V=4\pi(-1-1-3-3-1-1-3-3)$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}f(x_i,y_j,z_k)\Delta V=4\pi(-16)$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}f(x_i,y_j,z_k)\Delta V=-64\pi$$

■ 2. Use the midpoint rule to approximate the value of the triple integral, where D is the cube with opposite corners $(0,1,-1)$ and $(4,5,3)$. Use cubes with side length 2.

$$\iiint_D \log_2((x+1)^5y^2(z+2))\ dV$$

Solution:

Use laws of logs to simplify the integrand.

$$\iiint_D 5\log_2(x+1) + 2\log_2 y + \log_2(z+2)\ dV$$

The volume of integration consists of eight cubes. The Riemann sum estimate is given by

$$\int_0^4 \int_1^5 \int_{-1}^3 (5\log_2(x+1) + 2\log_2 y + \log_2(z+2))\ dz\ dy\ dx$$

$$\approx \sum_{i=1}^{2} \sum_{j=1}^{2} \sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V$$

where $\Delta V = 2 \times 2 \times 2 = 8$ is the volume of each cube. The midpoints of these boxes are

$(1,2,0)$	$(1,2,2)$	$(1,4,0)$	$(1,4,2)$
$(3,2,0)$	$(3,2,2)$	$(3,4,0)$	$(3,4,2)$

Find $f(x, y, z)$ for each point.

$$f(1,2,0) = 5\log_2(1+1) + 2\log_2 2 + \log_2(0+2) = 5 \cdot 1 + 2 \cdot 1 + 1 = 8$$

$$f(1,2,2) = 5\log_2(1+1) + 2\log_2 2 + \log_2(2+2) = 5 \cdot 1 + 2 \cdot 1 + 2 = 9$$

$$f(1,4,0) = 5\log_2(1+1) + 2\log_2 4 + \log_2(0+2) = 5 \cdot 1 + 2 \cdot 2 + 1 = 10$$

$$f(1,4,2) = 5\log_2(1+1) + 2\log_2 4 + \log_2(2+2) = 5 \cdot 1 + 2 \cdot 2 + 2 = 11$$

$$f(3,2,0) = 5\log_2(3+1) + 2\log_2 2 + \log_2(0+2) = 5 \cdot 2 + 2 \cdot 1 + 1 = 13$$

$$f(3,2,2) = 5\log_2(3+1) + 2\log_2 2 + \log_2(2+2) = 5 \cdot 2 + 2 \cdot 1 + 2 = 14$$

$$f(3,4,0) = 5\log_2(3+1) + 2\log_2 4 + \log_2(0+2) = 5 \cdot 2 + 2 \cdot 2 + 1 = 15$$

$$f(3,4,2) = 5\log_2(3+1) + 2\log_2 4 + \log_2(2+2) = 5 \cdot 2 + 2 \cdot 2 + 2 = 16$$

Then the Riemann sum is

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V = (8)(8+9+10+11+13+14+15+16)$$

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2} f(x_i, y_j, z_k)\Delta V = 768$$

■ 3. Use the midpoint rule to approximate the value of the improper triple integral. Use cubes with side length 1.

$$\int_0^1 \int_0^1 \int_0^\infty \log_4(x)\frac{(y-1)^3}{z^2}\ dz\ dy\ dx$$

Solution:

The volume of integration consists of an infinite number of boxes. The Riemann sum estimate is therefore given by

$$\int_0^1 \int_0^1 \int_0^\infty \log_4(x)\frac{(y-1)^3}{z^2}\ dz\ dy\ dx \approx \sum_{k=1}^{\infty} f\left(\frac{1}{2}, \frac{1}{2}, z_k\right)\Delta V$$

where $\Delta V = 1 \times 1 \times 1 = 1$ is the volume of each cube, and $z_k = (2k-1)/2$ for k from 1 to ∞. Find $f(1/2,1/2,z_k)$.

$$\log_4\left(\frac{1}{2}\right) \cdot \frac{\left(\frac{1}{2}-1\right)^3}{\left(\frac{2k-1}{2}\right)^2}$$

$$-\frac{1}{2} \cdot \left(-\frac{1}{8}\right) \cdot 4 \cdot \frac{1}{(2k-1)^2}$$

$$\frac{1}{4} \cdot \frac{1}{(2k-1)^2}$$

So the Riemann sum is

$$\sum_{k=1}^{\infty} f\left(\frac{1}{2}, \frac{1}{2}, z_k\right)\Delta V$$

$$\sum_{k=1}^{\infty} \frac{1}{4} \cdot \frac{1}{(2k-1)^2}(1)$$

$$\frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

$$\frac{1}{4}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}\right)$$

$$\frac{1}{4}\left(\frac{\pi^2}{8}\right)$$

$$\frac{\pi^2}{32}$$

ITERATED INTEGRALS

1. Evaluate the iterated integral.

$$\int_{-2}^{3}\int_{0}^{\pi}\int_{-4}^{-2}\frac{2x^3}{x^2+1}\sin y(3z^2-4z+3)\ dz\ dy\ dx$$

Solution:

Since $f(x,y,z)$ can be factored as $a(x)b(y)c(z)$, we can rewrite the triple integral as a product of three single integrals.

$$\int_{-2}^{3}\frac{2x^3}{x^2+1}\ dx\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

Use substitution with $u=x^2$, $du=2x\ dx$, and u changing from 4 to 0 to evaluate the first integral.

$$\int_{4}^{0}\frac{u}{u+1}\ du\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

$$\int_{4}^{0}\frac{u+1-1}{u+1}\ du\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

$$\int_{4}^{0}1-\frac{1}{u+1}\ du\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

$$u - \ln|u+1| \Big|_4^0 \cdot \int_0^{\pi} \sin y \ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

$$[0 - \ln|0+1| - (4 - \ln|4+1|)] \cdot \int_0^{\pi} \sin y \ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

$$(-\ln 1 - 4 + \ln 5) \cdot \int_0^{\pi} \sin y \ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

$$(\ln 5 - 4) \cdot \int_0^{\pi} \sin y \ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

Evaluate the second integral.

$$(\ln 5 - 4) \cdot \left(-\cos y \Big|_0^{\pi} \right) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

$$(\ln 5 - 4) \cdot (-\cos \pi - (-\cos 0)) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

$$(\ln 5 - 4) \cdot (-(-1) + 1) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

$$2(\ln 5 - 4) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3 \ dz$$

Evaluate the third integral.

$$2(\ln 5 - 4) \cdot \left(z^3 - 2z^2 + 3z \Big|_{-4}^{-2} \right)$$

$$2(\ln 5 - 4) \cdot [(-2)^3 - 2(-2)^2 + 3(-2) - ((-4)^3 - 2(-4)^2 + 3(-4))]$$

$$2(\ln 5 - 4) \cdot [-8 - 8 - 6 - (-64 - 32 - 12)]$$

$$2(\ln 5 - 4) \cdot (-8 - 8 - 6 + 64 + 32 + 12)$$

$$2(\ln 5 - 4) \cdot 86$$

$$172(\ln 5 - 4)$$

2. Evaluate the iterated improper integral.

$$\int_0^\infty \int_0^\infty \int_1^\infty \frac{1}{(x+2y+z)^5}\ dz\ dy\ dx$$

Solution:

Integrate with respect to z, treating x and y as constants.

$$\int_0^\infty \int_0^\infty \int_1^\infty (x+2y+z)^{-5}\ dz\ dy\ dx$$

$$\int_0^\infty \int_0^\infty \frac{1}{-4}(x+2y+z)^{-4}\Big|_{z=1}^{z=\infty}\ dy\ dx$$

$$\int_0^\infty \int_0^\infty -\frac{1}{4(x+2y+z)^4}\Big|_{z=1}^{z=\infty}\ dy\ dx$$

$$\int_0^\infty \int_0^\infty \lim_{z\to\infty}\left[-\frac{1}{4(x+2y+z)^4}\right] - \left(-\frac{1}{4(x+2y+1)^4}\right)\ dy\ dx$$

$$\int_0^\infty \int_0^\infty 0 + \frac{1}{4(x+2y+1)^4}\ dy\ dx$$

$$\int_0^\infty \int_0^\infty \frac{1}{4(x+2y+1)^4}\ dy\ dx$$

Integrate with respect to y, treating x as a constant.

$$\int_0^\infty \int_0^\infty \frac{1}{4}(x+2y+1)^{-4}\ dy\ dx$$

$$\int_0^\infty \frac{1}{-12}\left(\frac{1}{2}\right)(x+2y+1)^{-3}\Big|_{y=0}^{y=\infty}\ dx$$

$$\int_0^\infty -\frac{1}{24(x+2y+1)^3}\Big|_{y=0}^{y=\infty}\ dx$$

$$\int_0^\infty \lim_{y\to\infty}\left[-\frac{1}{24(x+2y+1)^3}\right] - \left(-\frac{1}{24(x+2(0)+1)^3}\right)\ dx$$

$$\int_0^\infty 0 + \frac{1}{24(x+1)^3}\ dx$$

$$\int_0^\infty \frac{1}{24(x+1)^3}\ dx$$

Integrate with respect to x.

$$\int_0^\infty \frac{1}{24}(x+1)^{-3}\ dx$$

$$\frac{1}{-48}(x+1)^{-2}\Big|_0^\infty$$

$$-\frac{1}{48(x+1)^2}\Big|_0^{\infty}$$

$$\lim_{x\to\infty}\left[-\frac{1}{48(x+1)^2}\right]-\left(-\frac{1}{48(0+1)^2}\right)$$

$$0+\frac{1}{48}$$

$$\frac{1}{48}$$

■ 3. Evaluate the iterated integral.

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\int_{2y}^{x+\frac{\pi}{2}}\cos(x-2y+z)\ dz\ dy\ dx$$

Solution:

Integrate with respect to z, treating x any y as constants.

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\sin(x-2y+z)\Big|_{z=2y}^{z=x+\frac{\pi}{2}}\ dy\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\sin\left(x-2y+x+\frac{\pi}{2}\right)-\sin(x-2y+2y)\ dy\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\sin\left(2x-2y+\frac{\pi}{2}\right)-\sin x\ dy\ dx$$

Integrate with respect to y, treating x as a constant.

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos\left(2x-2y+\frac{\pi}{2}\right)-(\sin x)y\Big|_{y=x}^{y=\frac{\pi}{4}}\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos\left(2x-2\left(\frac{\pi}{4}\right)+\frac{\pi}{2}\right)-(\sin x)\frac{\pi}{4}-\left[-\frac{1}{2}\cos\left(2x-2x+\frac{\pi}{2}\right)-(\sin x)x\right]\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos(2x)-(\sin x)\frac{\pi}{4}-\left[-\frac{1}{2}\cos\left(\frac{\pi}{2}\right)-x\sin x\right]\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos(2x)-(\sin x)\frac{\pi}{4}+x\sin x\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos(2x)+\left(x-\frac{\pi}{4}\right)\sin x\ dx$$

Integrate with respect to x.

$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}} x\sin x\ dx$$

Use integration by parts with $u=x$, $du=dx$, $dv=\sin x$, and $v=-\cos x$ to rewrite the remaining integral.

$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x\Big|_0^{\frac{\pi}{2}}+\left[-x\cos x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}} -\cos x\ dx\right]$$

$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x-x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\ dx$$

$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x-x\cos x+\sin x\Big|_0^{\frac{\pi}{2}}$$

Evaluate over the interval.

$$-\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)+\frac{\pi}{4}\cos\frac{\pi}{2}-\frac{\pi}{2}\cos\frac{\pi}{2}+\sin\frac{\pi}{2}$$

$$-\left(-\frac{1}{4}\sin(2(0))+\frac{\pi}{4}\cos(0)-0\cos(0)+\sin(0)\right)$$

$$-\frac{1}{4}(0)+\frac{\pi}{4}(0)-\frac{\pi}{2}(0)+1-\left(-\frac{1}{4}(0)+\frac{\pi}{4}(1)-0+0\right)$$

$$1-\frac{\pi}{4}$$

TRIPLE INTEGRALS

1\. Evaluate the triple integral, where D is the box with opposite corners $(5,0,1)$ and $(14,2,10)$.

$$\iiint_D y\log\left(\frac{z^4}{(x-4)^2\cdot 10^{y^2}}\right)\ dV$$

Solution:

Based on the coordinates of the opposite corners, x is defined on $[5,14]$, y is defined on $[0,2]$, and z is defined on $[1,10]$.

$$\int_1^{10}\int_0^2\int_5^{14} y\log\left(\frac{z^4}{(x-4)^2\cdot 10^{y^2}}\right)\ dx\ dy\ dz$$

Use laws of logs to simplify the integrand.

$$\int_1^{10}\int_0^2\int_5^{14} y[-2\log(x-4)-\log 10^{y^2}+4\log z]\ dx\ dy\ dz$$

$$\int_1^{10}\int_0^2\int_5^{14} y[-2\log(x-4)-y^2+4\log z]\ dx\ dy\ dz$$

$$\int_1^{10}\int_0^2\int_5^{14} -2y\log(x-4)-y^3+4y\log z\ dx\ dy\ dz$$

$$\int_1^{10}\int_0^2\int_5^{14} -2y\log(x-4)\ dx\ dy\ dz - \int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz$$

$$+\int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

Work on the first integral.

$$\int_5^{14}\log(x-4)\ dx\cdot\int_0^2 -2y\ dy\cdot\int_1^{10}\ dz - \int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz$$

$$+\int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\left(\frac{(x-4)(\ln(x-4)-1)}{\ln 10}\Bigg|_5^{14}\right)\left(-y^2\Big|_0^2\right)\left(z\Big|_1^{10}\right) - \int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz$$

$$+\int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\left(\frac{(14-4)(\ln(14-4)-1)}{\ln 10} - \left(\frac{(5-4)(\ln(5-4)-1)}{\ln 10}\right)\right)(-2^2-(-0^2))(10-1)$$

$$-\int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz + \int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\left(\frac{10(\ln(10)-1)}{\ln 10} + \frac{1}{\ln 10}\right)(-4)(9)$$

$$-\int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz + \int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2\int_5^{14}y^3\ dx\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

Work on the second integral.

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2 xy^3\Big|_{x=5}^{x=14}\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2 14y^3-5y^3\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2 9y^3\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\frac{9}{4}y^4\Big|_{y=0}^{y=2}\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\frac{9}{4}(2)^4-\frac{9}{4}(0)^4\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}36\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-36z\Big|_1^{10}+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-(36(10)-36(1))+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

Work on the third integral.

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 4xy\log z\Big|_{x=5}^{x=14}\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 4(14)y\log z-4(5)y\log z\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 56y\log z-20y\log z\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 36y\log z\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10} 18y^2\log z\Big|_{y=0}^{y=2}\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10} 18(2)^2\log z-18(0)^2\log z\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10} 72\log z\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\left[\frac{72z(\ln z-1)}{\ln 10}\Big|_1^{10}\right]$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\frac{72(10)(\ln 10-1)}{\ln 10}-\frac{72(1)(\ln 1-1)}{\ln 10}$$

$$\frac{324-360\ln 10}{\ln 10}-324+\frac{720\ln 10-720}{\ln 10}+\frac{72}{\ln 10}$$

$$\frac{324 - 360\ln 10 + 720\ln 10 - 720 + 72}{\ln 10} - 324$$

$$\frac{360\ln 10 - 324}{\ln 10} - 324$$

Find a common denominator.

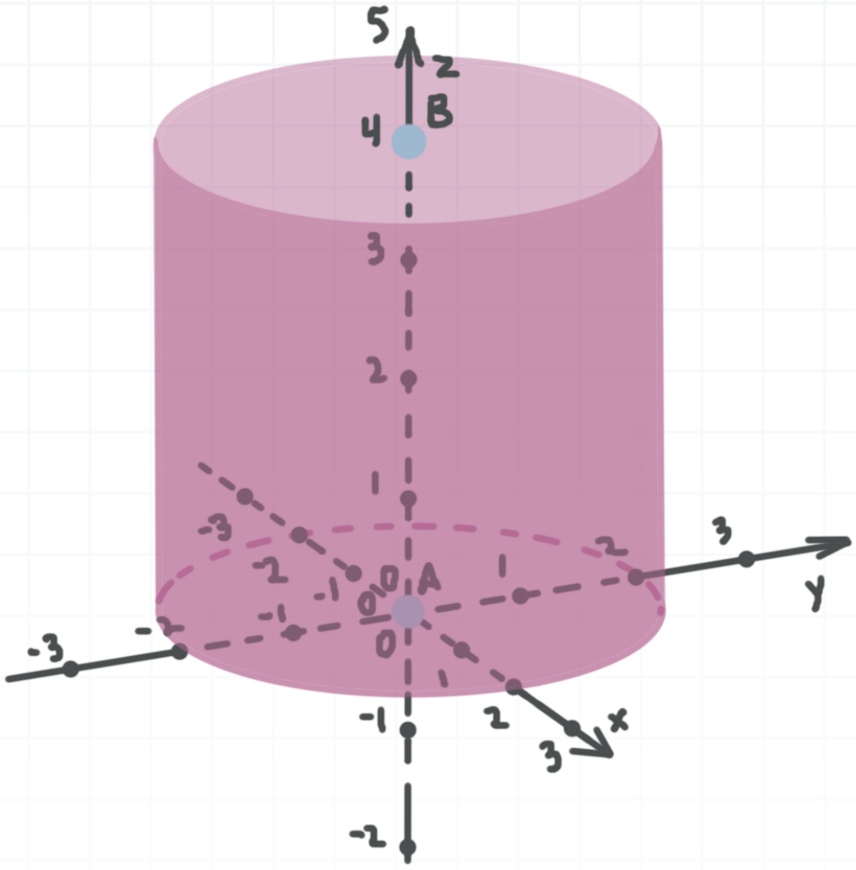
$$\frac{360\ln 10 - 324}{\ln 10} - \frac{324\ln 10}{\ln 10}$$

$$\frac{360\ln 10 - 324\ln 10 - 324}{\ln 10}$$

$$\frac{36\ln 10 - 324}{\ln 10}$$

■ 2. Evaluate the triple integral, where D is the right circular cylinder with radius 2, height 4, and a base that lies in the xy-plane with center at the origin.

$$\iiint_D e^{0.5z}\sqrt{x^2 + y^2}\ dV$$



Solution:

The value of z is defined on $[0,4]$, and x and y change within the circle C with radius 2 that lies in the xy-plane with center at the origin. Rewrite the triple integral as a product of single and double integrals.

$$\iiint_D e^{0.5z}\sqrt{x^2+y^2}\ dV$$

$$\int_0^4 \iint_C e^{0.5z}\sqrt{x^2+y^2}\ dA\ dz$$

$$\int_0^4 e^{0.5z}\ dz \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

Work on the first integral.

$$\frac{1}{0.5}e^{0.5z}\Big|_0^4 \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

$$(2e^{0.5(4)} - 2e^{0.5(0)}) \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

$$(2e^2 - 2) \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

Convert the second integral to polar coordinates, then evaluate it.

$$(2e^2 - 2) \cdot \int_0^2 \int_0^{2\pi} r \cdot r\ d\theta\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 \int_0^{2\pi} r^2\ d\theta\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 r^2\theta\Big|_{\theta=0}^{\theta=2\pi}\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 r^2(2\pi) - r^2(0)\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 2\pi r^2\ dr$$

$$(2e^2 - 2) \cdot \left[\frac{2}{3}\pi r^3\Big|_0^2\right]$$

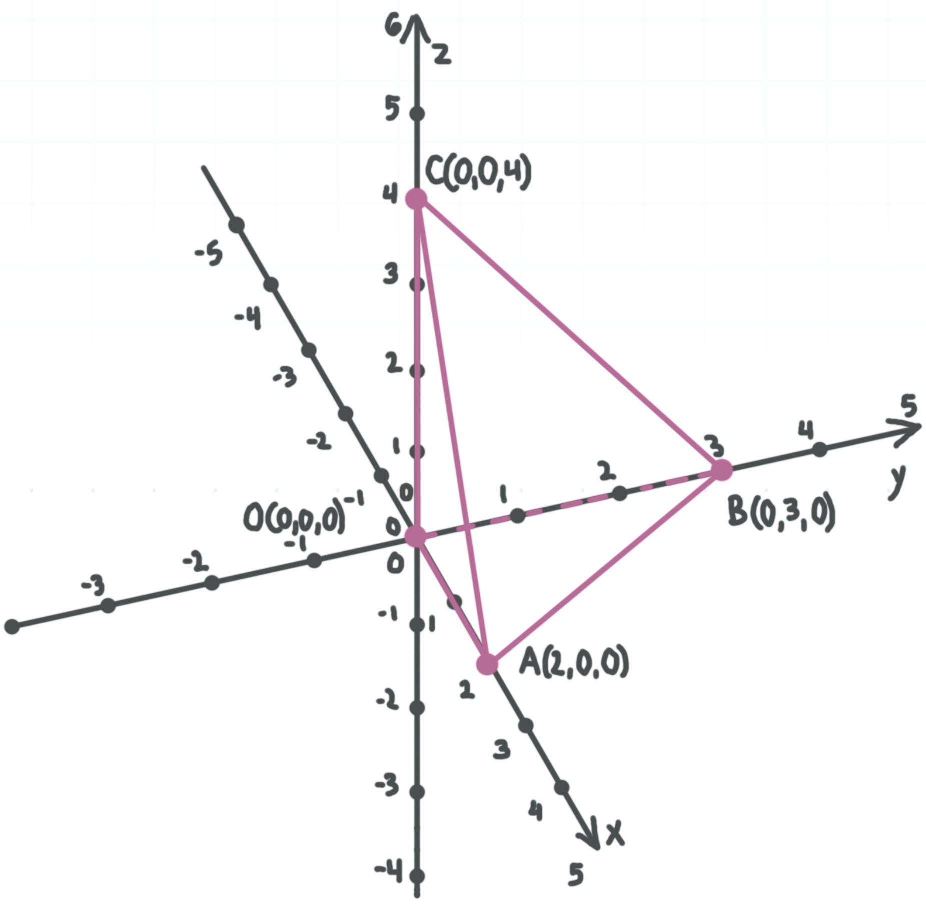
$$(2e^2 - 2) \cdot \left[\frac{2}{3}\pi(2)^3 - \frac{2}{3}\pi(0)^3\right]$$

$$(2e^2 - 2) \cdot \frac{16}{3}\pi$$

$$\frac{32\pi(e^2 - 1)}{3}$$

■ 3. Evaluate the triple integral, where $ABCO$ is the irregular pyramid such that O is the origin and the vertices are $A(2,0,0)$, $B(0,3,0)$, and $C(0,0,4)$.

$$\iiint_{ABCO} 72xy\ dV$$



Solution:

The equation of the line AB in the xy-plane is $y = -1.5x + 3$. So when x changes from 0 to 2, y changes from 0 to $-1.5x + 3$.

The equation of the plane ABC is $z = -2x - (4/3)y + 4$. So when x and y change within the triangle OAB, z changes from 0 to $-2x - (4/3)y + 4$.

Then we can write the triple integral as an iterated integral.

$$\int_0^2 \int_0^{-1.5x+3} \int_0^{-2x-\frac{4}{3}y+4} 72xy\ dz\ dy\ dx$$

Integrate with respect to z, treating x and y as constants.

$$\int_0^2 \int_0^{-1.5x+3} 72xyz\Big|_{z=0}^{z=-2x-\frac{4}{3}y+4}\ dy\ dx$$

$$\int_0^2 \int_0^{-1.5x+3} 72xy\left(-2x - \frac{4}{3}y + 4\right) - 72xy(0)\ dy\ dx$$

$$\int_0^2 \int_0^{-1.5x+3} -144x^2y - 96xy^2 + 288xy\ dy\ dx$$

Integrate with respect to y, treating x as a constant.

$$\int_0^2 -72x^2y^2 - 32xy^3 + 144xy^2\Big|_{y=0}^{y=-1.5x+3}\ dx$$

$$\int_0^2 -72x^2(-1.5x+3)^2 - 32x(-1.5x+3)^3 + 144x(-1.5x+3)^2$$

$$-(-72x^2(0)^2 - 32x(0)^3 + 144x(0)^2)\ dx$$

$$\int_0^2 -72x^2(-1.5x+3)^2 - 32x(-1.5x+3)^3 + 144x(-1.5x+3)^2 \ dx$$

$$\int_0^2 -72x^2\left(\frac{9}{4}x^2 - \frac{9}{2}x + 9\right) - 32x\left(-\frac{27}{8}x^3 + \frac{27}{2}x^2 - 27x + 27\right)$$

$$+144x\left(\frac{9}{4}x^2 - \frac{9}{2}x + 9\right) \ dx$$

$$\int_0^2 -162x^4 + 324x^3 - 648x^2 + 108x^4 - 432x^3 + 864x^2 - 864x$$

$$+324x^3 - 648x^2 + 1{,}296x \ dx$$

$$\int_0^2 -54x^4 + 216x^3 - 432x^2 + 432x \ dx$$

Integrate with respect to x.

$$-\frac{54}{5}x^5 + 54x^4 - \frac{432}{3}x^3 + 216x^2\Big|_0^2$$

$$-\frac{54}{5}(2)^5 + 54(2)^4 - \frac{432}{3}(2)^3 + 216(2)^2$$

$$-\left(-\frac{54}{5}(0)^5 + 54(0)^4 - \frac{432}{3}(0)^3 + 216(0)^2\right)$$

$$-\frac{54}{5}(32) + 54(16) - \frac{432}{3}(8) + 216(4)$$

$$-\frac{1{,}728}{5} + 864 - \frac{3{,}456}{3} + 864$$

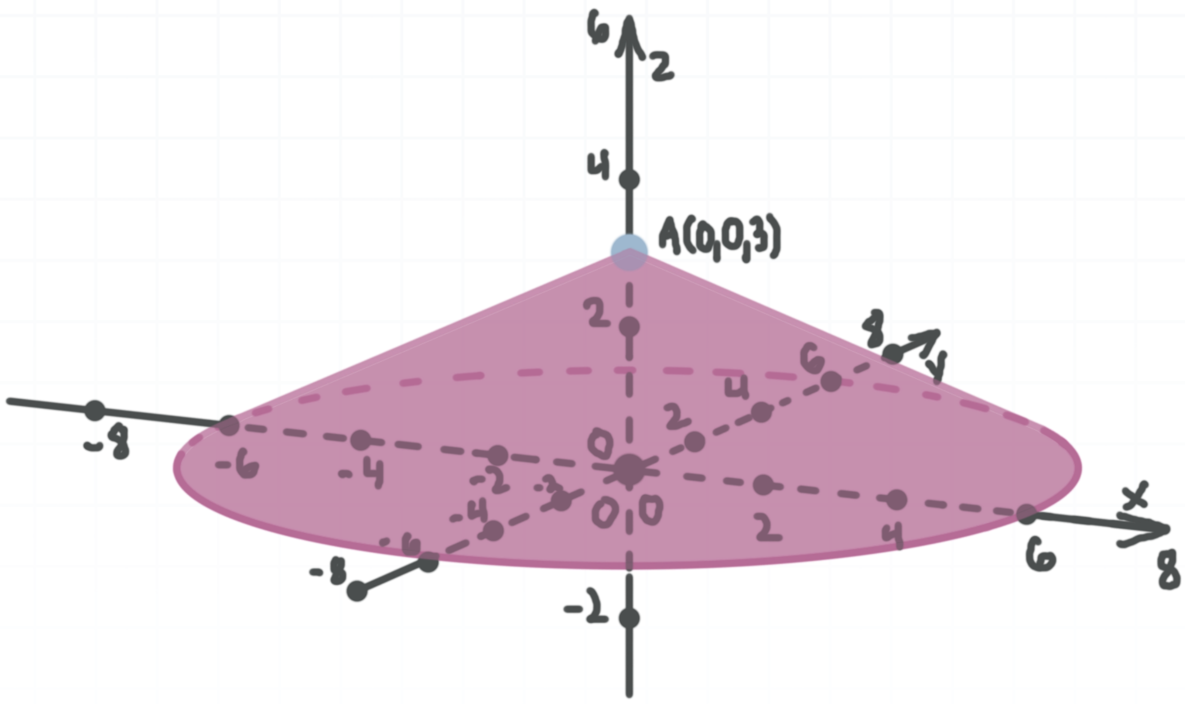
$$-\frac{5{,}184}{15}-\frac{17{,}280}{15}+\frac{25{,}920}{15}$$

$$\frac{3{,}456}{15}$$

$$\frac{1{,}152}{5}$$

AVERAGE VALUE

1. Use triple integrals to find the average value of the function $f(x, y, z) = x^2 + y^2 + z^2$ over a right circular cone with radius $R = 6$, height $h = 3$, and a base that lies in the xy-plane with center at the origin.

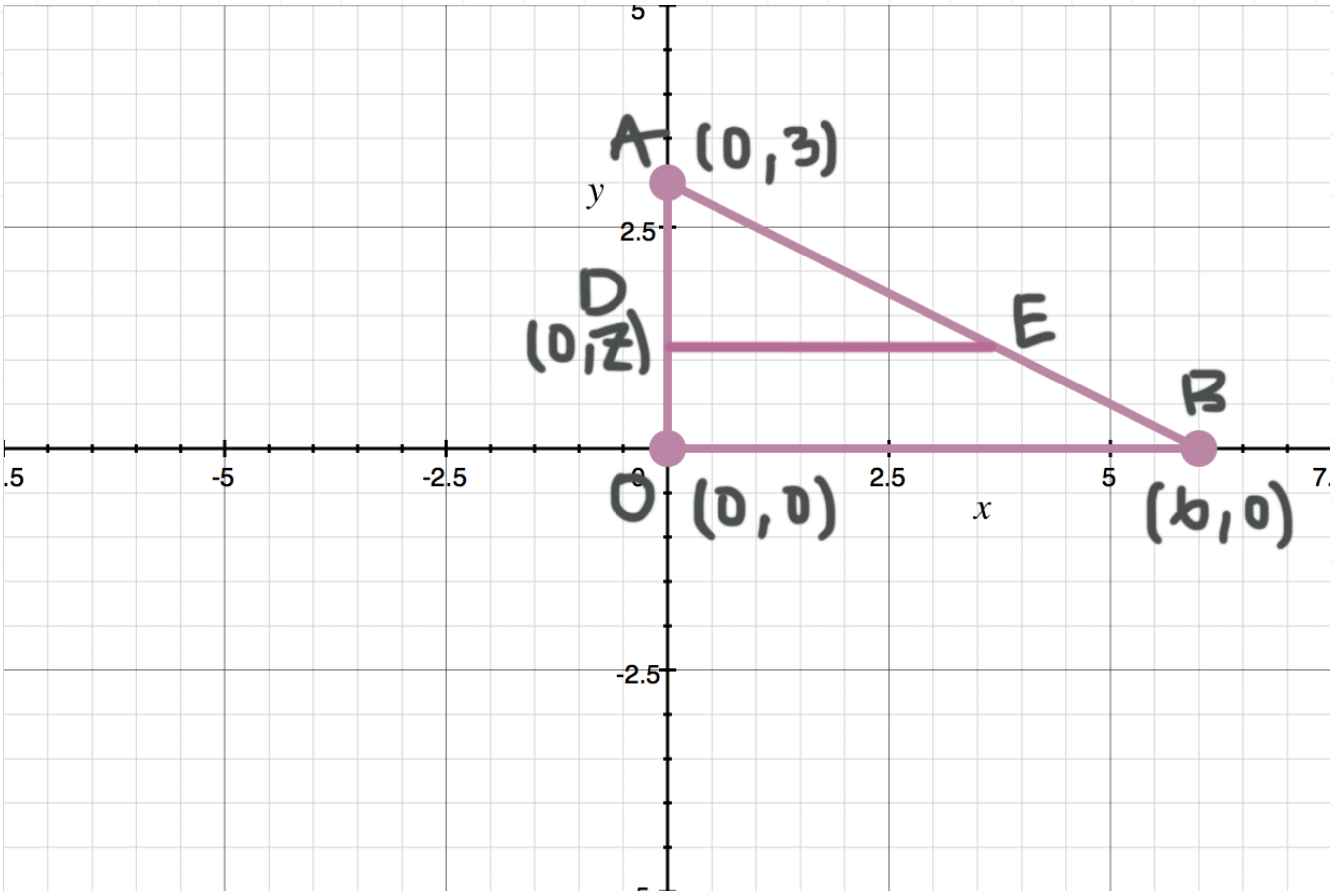


Solution:

The volume of the right circular cone is

$$V(E) = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi \cdot (6)^2 \cdot 3 = 36\pi$$

The value of z in the cone is defined from 0 to 3, and x and y change within the circle C with radius r and center $(0,0,z)$ that lies in the plane parallel to xy-plane. Find r using the section through OA and the x-axis.



In the triangle OAB, $OB = R = 6$, $OA = h = 3$, $OD = z$, and $AD = h - z = 3 - z$. Since DE is parallel to OB, the triangles OAB and DAE are similar, so

$$\frac{DE}{OB} = \frac{DA}{OA}$$

$$\frac{DE}{6} = \frac{3-z}{3}$$

$$DE = 2(3-z) = 6 - 2z$$

Then the triple integral will be

$$\iiint_E x^2 + y^2 + z^2 \ dV$$

$$\int_0^3 \iint_C x^2 + y^2 + z^2 \ dA \ dz$$

Convert the inner integral to polar coordinates.

$$\int_0^3 \int_0^{6-2z} \int_0^{2\pi} (r^2 + z^2) \cdot r \ d\theta \ dr \ dz$$

$$\int_0^3 \int_0^{6-2z} \int_0^{2\pi} r^3 + rz^2 \ d\theta \ dr \ dz$$

$$\int_0^3 \left[\int_0^{6-2z} r^3 + rz^2 \ dr \cdot \int_0^{2\pi} \ d\theta \right] \ dz$$

Integrate with respect to θ.

$$\int_0^3 \left[\int_0^{6-2z} r^3 + rz^2 \ dr \cdot \theta \Big|_0^{2\pi} \right] \ dz$$

$$\int_0^3 \left[\int_0^{6-2z} r^3 + rz^2 \ dr \cdot (2\pi - 0) \right] \ dz$$

$$\int_0^3 2\pi \int_0^{6-2z} r^3 + rz^2 \ dr \ dz$$

Integrate with respect to r.

$$\int_0^3 2\pi \left(\frac{1}{4}r^4 + \frac{1}{2}r^2z^2 \right) \Big|_{r=0}^{r=6-2z} \ dz$$

$$\int_0^3 2\pi \left(\frac{1}{4}(6-2z)^4 + \frac{1}{2}(6-2z)^2z^2 \right) - 2\pi \left(\frac{1}{4}(0)^4 + \frac{1}{2}(0)^2z^2 \right) \ dz$$

$$\int_0^3 \frac{\pi}{2}(6-2z)^4 + \pi(6-2z)^2 z^2 \ dz$$

$$\int_0^3 \frac{\pi}{2}(36-24z+4z^2)^2 + \pi(36-24z+4z^2)z^2 \ dz$$

$$\int_0^3 \frac{\pi}{2}(16z^4 - 192z^3 + 288z^2 - 1{,}152z + 1{,}296) + 36\pi z^2 - 24\pi z^3 + 4\pi z^4 \ dz$$

$$\int_0^3 8\pi z^4 - 96\pi z^3 + 144\pi z^2 - 576\pi z + 648\pi + 36\pi z^2 - 24\pi z^3 + 4\pi z^4 \ dz$$

$$\int_0^3 12\pi z^4 - 120\pi z^3 + 180\pi z^2 - 576\pi z + 648\pi \ dz$$

Integrate with respect to z.

$$\frac{12\pi}{5}z^5 - 30\pi z^4 + 60\pi z^3 - 288\pi z^2 + 648\pi z \Big|_0^3$$

$$\frac{12\pi}{5}(3)^5 - 30\pi(3)^4 + 60\pi(3)^3 - 288\pi(3)^2 + 648\pi(3)$$

$$-\left(\frac{12\pi}{5}(0)^5 - 30\pi(0)^4 + 60\pi(0)^3 - 288\pi(0)^2 + 648\pi(0)\right)$$

$$\frac{2{,}916\pi}{5} - 2{,}430\pi + 1{,}620\pi - 2{,}592\pi + 1{,}944\pi$$

$$-\frac{4{,}374\pi}{5}$$

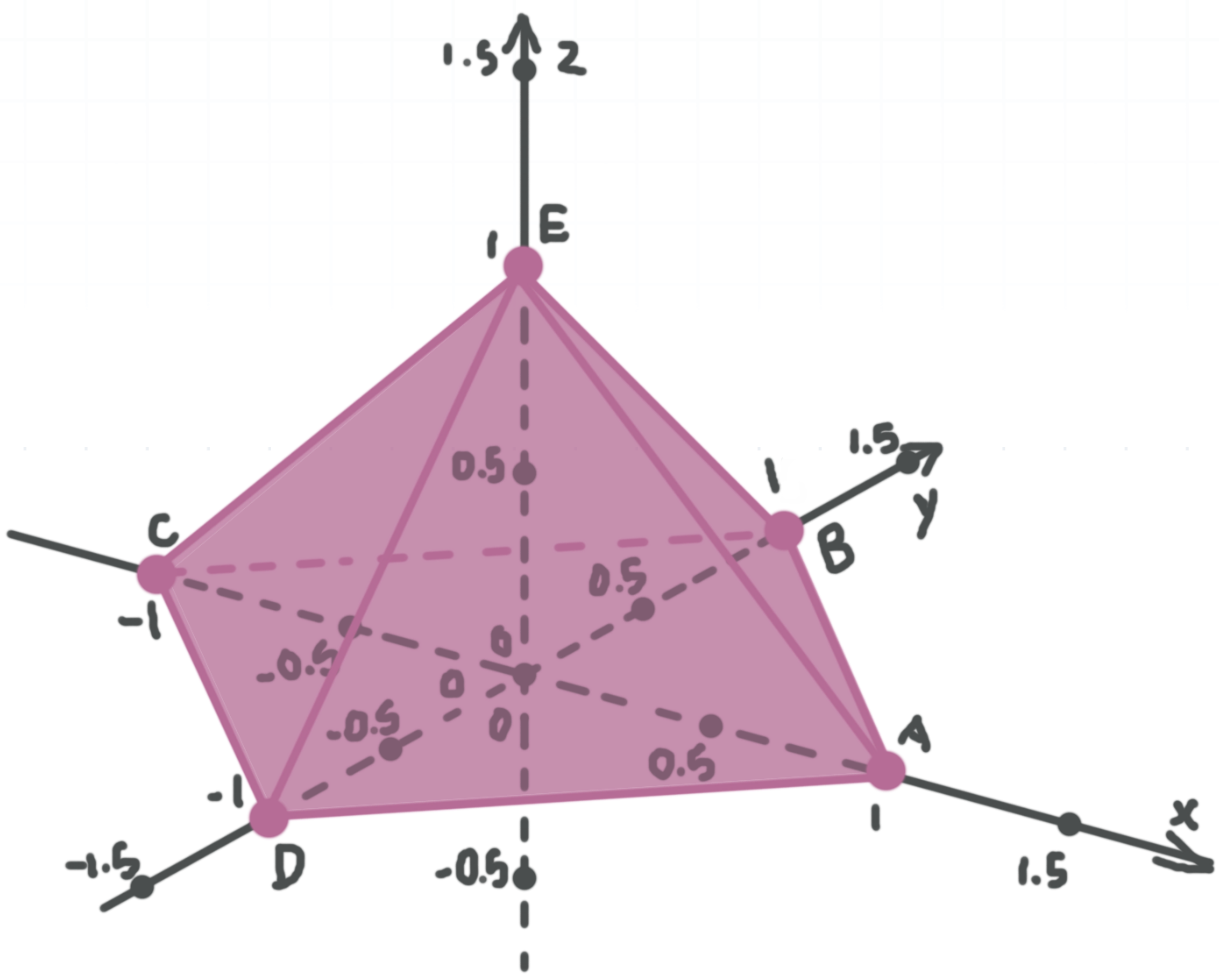
Then the average value is

$$f_{avg}(x, y, z) = \frac{1}{V(E)} \iiint_E f(x, y, z)\ dV$$

$$f_{avg}(x, y, z) = \frac{1}{36\pi}\left(-\frac{4{,}374\pi}{5}\right)$$

$$f_{avg}(x, y, z) = -24.3$$

■ 2. Use triple integrals to find the average value of the function $f(x, y, z) = |xyz|$ over a regular pyramid $ABCDE$, where $A(1,0,0)$, $B(0,1,0)$, $C(-1,0,0)$, $D(0,-1,0)$, and $E(0,0,1)$.



Solution:

The volume of the regular pyramid is

$$V(G) = \frac{1}{3}a^2h = \frac{1}{3}(\sqrt{2})^2 \cdot 1 = \frac{2}{3}$$

Because the pyramid is regular and centered at the origin, it has the same volume in each octant. Therefore, we can calculate the volume in just the first octant, and multiply the result by 4 to get total volume.

The equation of the line AB in the xy-plane is $y = 1 - x$. So when x changes from 0 to 1, y changes from 0 to $1 - x$. The equation of the plane ABE is $z = -x - y + 1$. So when x and y change within the triangle OAB, z changes from 0 to $-x - y + 1$. Therefore, the triple integral can be rewritten as an iterated integral.

$$4\iiint_{G_1} xyz\ dV$$

$$4\int_0^1 \int_0^{1-x} \int_0^{-x-y+1} xyz\ dz\ dy\ dx$$

Integrate with respect to z.

$$4\int_0^1 \int_0^{1-x} \frac{1}{2}xyz^2\Big|_{z=0}^{z=-x-y+1}\ dy\ dx$$

$$4\int_0^1 \int_0^{1-x} \frac{1}{2}xy(-x-y+1)^2 - \frac{1}{2}xy(0)^2\ dy\ dx$$

$$4\int_0^1 \int_0^{1-x} \frac{1}{2}xy(x^2 - xy - x + xy + y^2 - y - x - y + 1)\ dy\ dx$$

$$4\int_0^1 \int_0^{1-x} \frac{1}{2}x^3y - x^2y + \frac{1}{2}xy - xy^2 + \frac{1}{2}xy^3 \ dy \ dx$$

$$\int_0^1 \int_0^{1-x} 2x^3y - 4x^2y + 2xy - 4xy^2 + 2xy^3 \ dy \ dx$$

Integrate with respect to y.

$$\int_0^1 x^3y^2 - 2x^2y^2 + xy^2 - \frac{4}{3}xy^3 + \frac{1}{2}xy^4 \Bigg|_{y=0}^{y=1-x} \ dx$$

$$\int_0^1 x^3(1-x)^2 - 2x^2(1-x)^2 + x(1-x)^2 - \frac{4}{3}x(1-x)^3 + \frac{1}{2}x(1-x)^4$$

$$-\left(x^3(0)^2 - 2x^2(0)^2 + x(0)^2 - \frac{4}{3}x(0)^3 + \frac{1}{2}x(0)^4\right) \ dx$$

$$\int_0^1 (x^3 - 2x^2 + x)(1-x)^2 + x(1-x)^3\left(-\frac{4}{3} + \frac{1}{2}(1-x)\right) \ dx$$

$$\int_0^1 (x^3 - 2x^2 + x)(1-x)^2 + \left(-\frac{4}{3}x + \frac{1}{2}x - \frac{1}{2}x^2\right)(1-x)^3 \ dx$$

$$\int_0^1 (x^3 - 2x^2 + x)(1-x)^2 - \left(\frac{1}{2}x^2 + \frac{5}{6}x\right)(1-x)^3 \ dx$$

$$\int_0^1 x^5 - 4x^4 + 6x^3 - 4x^2 + x - \left(\frac{1}{2}x^2 + \frac{5}{6}x\right)(1 - 3x + 3x^2 - x^3) \ dx$$

$$\int_0^1 x^5 - 4x^4 + 6x^3 - 4x^2 + x - \left(-\frac{1}{2}x^5 + \frac{2}{3}x^4 + x^3 - 2x^2 + \frac{5}{6}x\right) \ dx$$

$$\int_0^1 \frac{3}{2}x^5 - \frac{14}{3}x^4 + 5x^3 - 2x^2 + \frac{1}{6}x \ dx$$

Integrate with respect to x.

$$\frac{1}{4}x^6 - \frac{14}{15}x^5 + \frac{5}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{12}x^2 \Big|_0^1$$

$$\frac{1}{4}(1)^6 - \frac{14}{15}(1)^5 + \frac{5}{4}(1)^4 - \frac{2}{3}(1)^3 + \frac{1}{12}(1)^2$$

$$-\left(\frac{1}{4}(0)^6 - \frac{14}{15}(0)^5 + \frac{5}{4}(0)^4 - \frac{2}{3}(0)^3 + \frac{1}{12}(0)^2\right)$$

$$\frac{1}{4} - \frac{14}{15} + \frac{5}{4} - \frac{2}{3} + \frac{1}{12}$$

$$\frac{15}{60} - \frac{56}{60} + \frac{75}{60} - \frac{40}{60} + \frac{5}{60}$$

$$-\frac{1}{60}$$

Then the average value is

$$f_{avg}(x, y, z) = \frac{1}{V(G)} \iiint_G f(x, y, z) \ dV$$

$$f_{avg}(x, y, z) = \frac{1}{\frac{2}{3}}\left(-\frac{1}{60}\right)$$

$$f_{avg}(x, y, z) = -\frac{1}{40}$$

■ 3. Use triple integrals to find the average value of the function $f(x,y,z)=2x-3y+z$ over a layer bounded by the planes $z=2$ and $z=4$.

Solution:

Since the region of integration is an infinite layer, the average value of the function $f(x,y,z)$ over the region E could be calculated using a limit.

$$f_{avg}(x,y,z)=\lim_{t\to\infty}\frac{1}{V(E(t))}\iiint_{E(t)} f(x,y,z)\ dV$$

The box $E(t)$ has dimensions $x\in[-t,t]$, $y\in[-t,t]$, and $z\in[2,4]$, so the volume of $E(t)$ is

$$V(E(t))=(2)(2t)(2t)=8t^2$$

Then the triple iterated integral is

$$\int_{-t}^{t}\int_{-t}^{t}\int_{2}^{4} 2x-3y+z\ dz\ dy\ dx$$

Integrate with respect to z.

$$\int_{-t}^{t}\int_{-t}^{t} 2xz-3yz+\frac{1}{2}z^2\Big|_{z=2}^{z=4}\ dy\ dx$$

$$\int_{-t}^{t}\int_{-t}^{t} 2x(4)-3y(4)+\frac{1}{2}(4)^2-\left(2x(2)-3y(2)+\frac{1}{2}(2)^2\right)\ dy\ dx$$

$$\int_{-t}^{t}\int_{-t}^{t} 8x-12y+8-4x+6y-2\ dy\ dx$$

$$\int_{-t}^{t}\int_{-t}^{t} 4x-6y+6\ dy\ dx$$

Integrate with respect to y.

$$\int_{-t}^{t} 4xy-3y^2+6y\Big|_{y=-t}^{y=t}\ dx$$

$$\int_{-t}^{t} 4x(t)-3(t)^2+6(t)-(4x(-t)-3(-t)^2+6(-t))\ dx$$

$$\int_{-t}^{t} 4xt-3t^2+6t+4xt+3t^2+6t\ dx$$

$$\int_{-t}^{t} 8xt+12t\ dx$$

Integrate with respect to x.

$$4x^2t+12xt\Big|_{x=-t}^{x=t}$$

$$4t^2t+12(t)t-(4(-t)^2t+12(-t)t)$$

$$4t^3+12t^2-4t^3+12t^2$$

$$24t^2$$

Then the average value is

$$f_{avg}(x, y, z) = \lim_{t \to \infty} \frac{1}{8t^2}(24t^2)$$

$$f_{avg}(x, y, z) = \lim_{t \to \infty} 3$$

$$f_{avg}(x, y, z) = 3$$

FINDING VOLUME

1. Find the volume given by the triple integral.

$$\int_{-4}^{6}\int_{3-2x^2}^{10}\int_{2x-y}^{12-y} dz\ dy\ dx$$

Solution:

Integrate with respect to z.

$$\int_{-4}^{6}\int_{3-2x^2}^{10} z\Big|_{z=2x-y}^{z=12-y} dy\ dx$$

$$\int_{-4}^{6}\int_{3-2x^2}^{10} 12 - y - (2x - y)\ dy\ dx$$

$$\int_{-4}^{6}\int_{3-2x^2}^{10} 12 - 2x\ dy\ dx$$

Integrate with respect to y.

$$\int_{-4}^{6} 12y - 2xy\Big|_{y=3-2x^2}^{y=10} dx$$

$$\int_{-4}^{6} 12(10) - 2x(10) - (12(3 - 2x^2) - 2x(3 - 2x^2))\ dx$$

$$\int_{-4}^{6} 120 - 20x - (36 - 24x^2 - 6x + 4x^3)\ dx$$

$$\int_{-4}^{6} 120 - 20x - 36 + 24x^2 + 6x - 4x^3\ dx$$

$$\int_{-4}^{6} 84 - 14x + 24x^2 - 4x^3\ dx$$

Integrate with respect to x.

$$84x - 7x^2 + 8x^3 - x^4 \Big|_{-4}^{6}$$

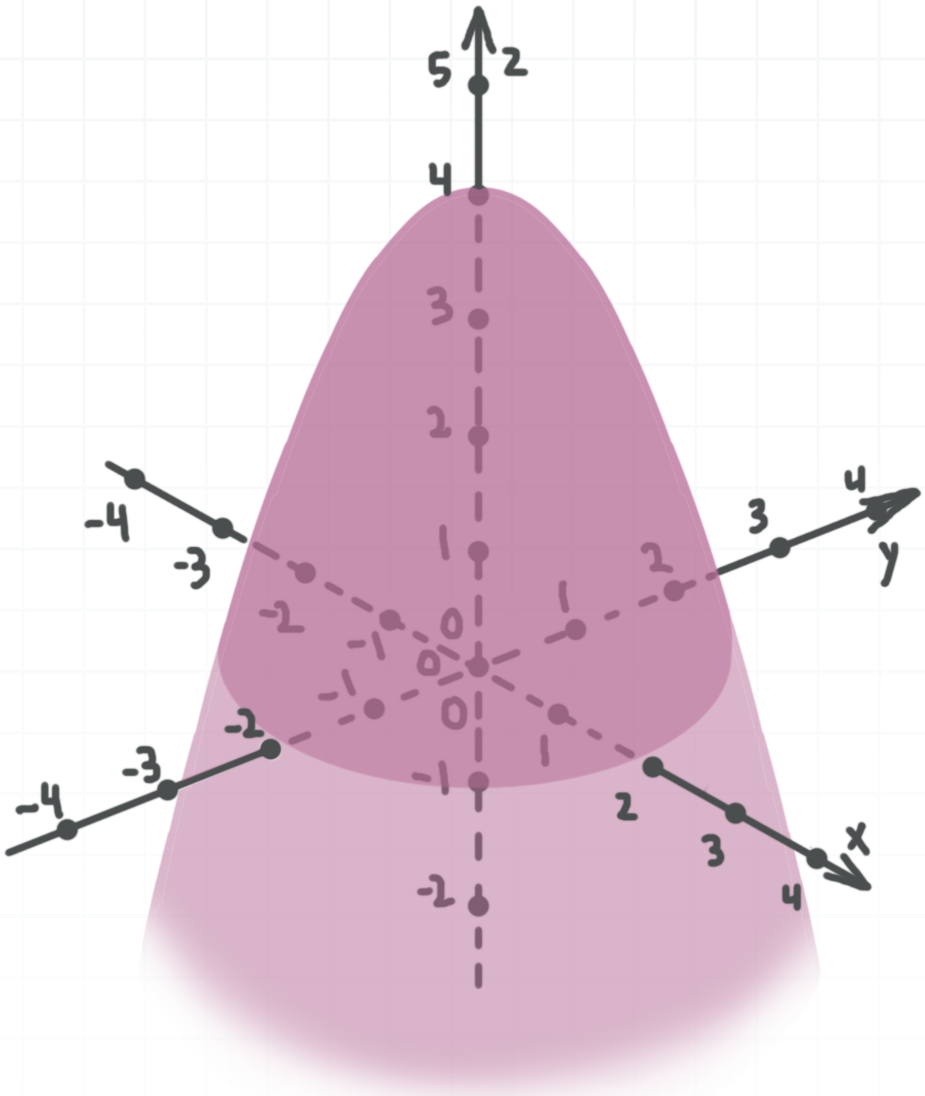
$$84(6) - 7(6)^2 + 8(6)^3 - 6^4 - (84(-4) - 7(-4)^2 + 8(-4)^3 - (-4)^4)$$

$$504 - 252 + 1{,}728 - 1{,}296 - (-336 - 112 - 512 - 256)$$

$$504 - 252 + 1{,}728 - 1{,}296 + 336 + 112 + 512 + 256$$

$$1{,}900$$

■ 2. Use a triple integral to find the volume of the solid bounded by the circular paraboloid $4 - x^2 - y^2 - z = 0$ and the xy-plane.



Solution:

The value of z is defined from 0 to 4, x and y change within the circle C with radius $4 - z$ and center $(0,0,z)$ that lies in the plane parallel to the xy-plane. So the volume is given by

$$V = \int_0^4 \iint_C \, dA \, dz$$

$$V = \int_0^4 \pi(4 - z)^2 \, dz$$

$$V = \int_0^4 \pi z^2 - 8\pi z + 16\pi \ dz$$

Integrate with respect to z.

$$V = \frac{\pi}{3}z^3 - 4\pi z^2 + 16\pi z \Big|_0^4$$

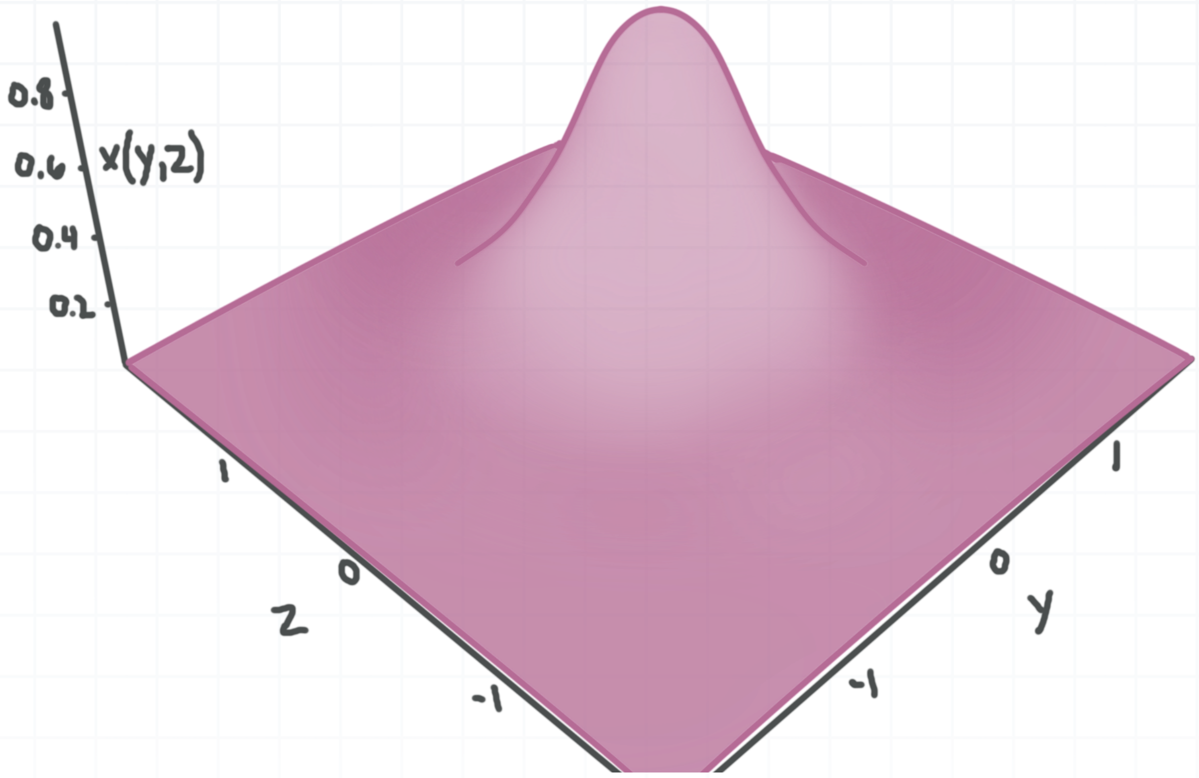
$$V = \frac{\pi}{3}(4)^3 - 4\pi(4)^2 + 16\pi(4) - \left(\frac{\pi}{3}(0)^3 - 4\pi(0)^2 + 16\pi(0)\right)$$

$$V = \frac{64\pi}{3} - 64\pi + 64\pi$$

$$V = \frac{64\pi}{3}$$

■ 3. Use a triple integral to find the volume of the solid bounded by the surface $x = g(y, z)$ and the yz-plane.

$$x = \frac{1}{(y^2 + z^2 + 1)^2}$$



Solution:

The volume of a region E is equal to the triple integral with $f(x, y, z) = 1$, i.e.

$$V = \iiint_E dV$$

Since $g(y, z) \geq 0$, $g(y, z) \to 0$ when $y, z \to \infty$, $g(y, z) \leq 1$, and $g(y, z) = 1$ when $y = z = 0$, the value of x changes from 0 to 1, y and z change within the given region.

$$x = \frac{1}{(y^2 + z^2 + 1)^2}$$

$$(y^2 + z^2 + 1)^2 = \frac{1}{x}$$

$$y^2 + z^2 + 1 = \frac{1}{\sqrt{x}}$$

$$y^2 + z^2 = \frac{1}{\sqrt{x}} - 1$$

$$y^2 + z^2 = x^{-\frac{1}{2}} - 1$$

So y and z change within the circle C with radius $\sqrt{x^{-\frac{1}{2}} - 1}$ and center $(x,0,0)$ that lies in the plane parallel to the yz-plane. So the volume is given by

$$V = \int_0^1 \iint_C \ dA \ dx$$

$$V = \int_0^1 \pi(x^{-\frac{1}{2}} - 1) \ dx$$

$$V = \int_0^1 \pi x^{-\frac{1}{2}} - \pi \ dx$$

Integrate with respect to x.

$$V = 2\pi x^{\frac{1}{2}} - \pi x \Big|_0^1$$

$$V = 2\pi(1)^{\frac{1}{2}} - \pi(1) - (2\pi(0)^{\frac{1}{2}} - \pi(0))$$

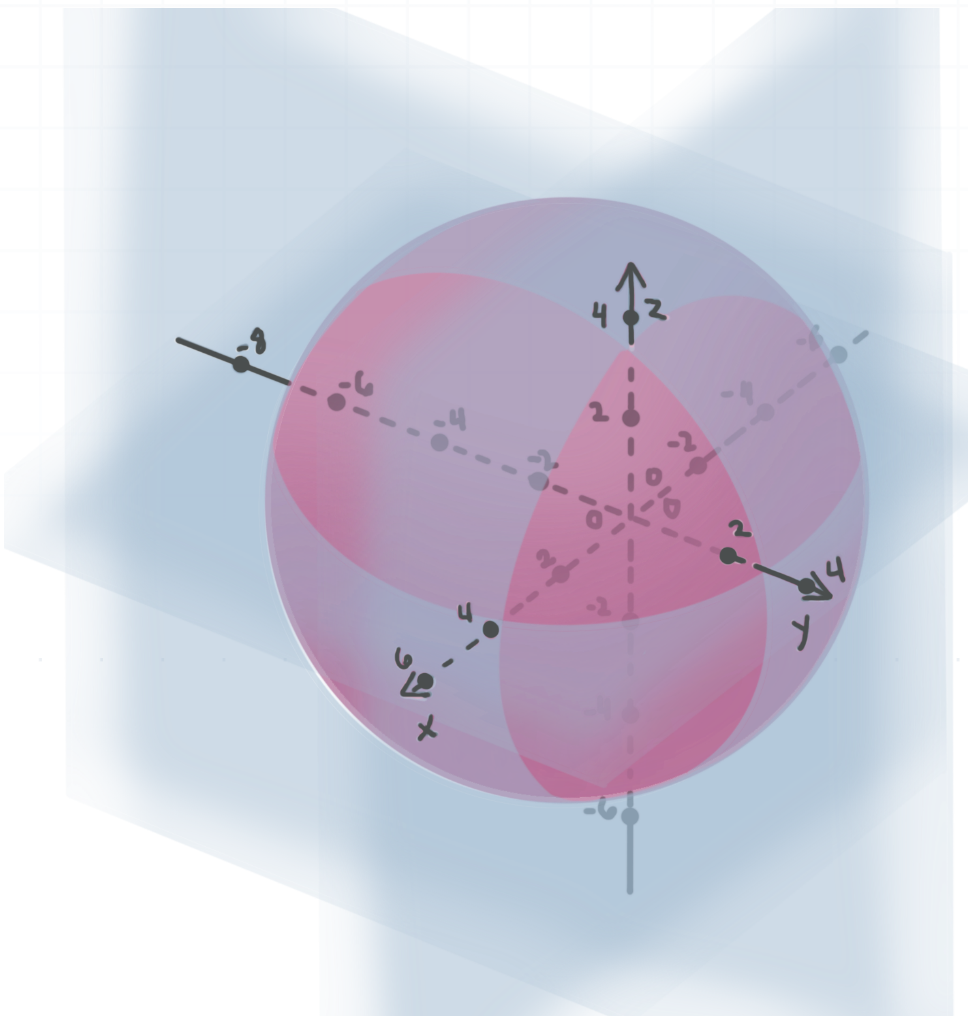
$$V = 2\pi - \pi$$

$$V = \pi$$

EXPRESSING THE INTEGRAL SIX WAYS

■ 1. Represent the triple integral as an iterated integral in which the order of integration is $dx\ dz\ dy$, where E is the part of the sphere with center at $(-1,-2,-1)$ and radius 25, lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$).

$$\iiint_E f(x,y,z)\ dV$$



Solution:

The equation of the sphere is

$$(x+1)^2+(y+2)^2+(z+1)^2=25$$

To find the interval over which y is defined, plug in $x=z=0$.

$$(0+1)^2+(y+2)^2+(0+1)^2=25$$

$$(y+2)^2=23$$

$$y=-2\pm\sqrt{23}$$

In the first octant, y is defined from 0 to $-2+\sqrt{23}$.

The innermost integral is in terms of x, so we can find the limits of integration on x by solving the equation of the sphere.

$$(x+1)^2+(y+2)^2+(z+1)^2=25$$

$$(x+1)^2=25-(y+2)^2-(z+1)^2$$

$$x=-1\pm\sqrt{25-(y+2)^2-(z+1)^2}$$

So in the first octant x is defined from 0 to $-1+\sqrt{25-(y+2)^2-(z+1)^2}$.

Next we'll try to find bounds for the middle integral. We'll set $x=0$ in the sphere equation, since by this point in the integral we would have evaluated for z. We'll solve this equation for z.

$$(0+1)^2+(y+2)^2+(z+1)^2=25$$

$$1+(y+2)^2+(z+1)^2=25$$

$$(z+1)^2=24-(y+2)^2$$

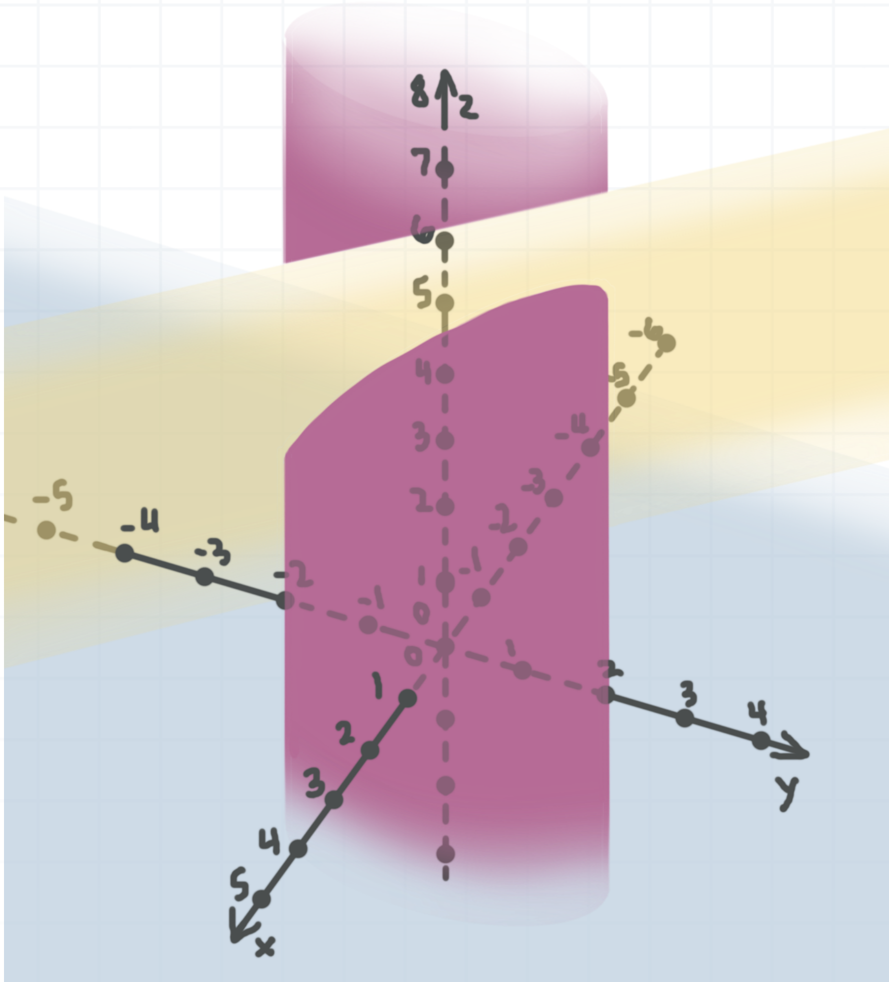
$$z = -1 \pm \sqrt{24 - (y+2)^2}$$

So in the first octant z is defined from 0 to $-1 + \sqrt{24 - (y+2)^2}$. Therefore, the iterated integral is given by

$$\int_0^{-2+\sqrt{23}} \int_0^{-1+\sqrt{24-(y+2)^2}} \int_0^{-1+\sqrt{25-(y+2)^2-(z+1)^2}} f(x, y, z)\ dx\ dz\ dy$$

■ 2. Represent the triple integral as an iterated integral using the order of integration $dz\ dy\ dx$, where E is the part of the cylinder $4x^2 + y^2 = 4$, between the planes $z = -3$ and $x + y - z + 4 = 0$.

$$\iiint_E f(x, y, z)\ dV$$



Solution:

To find an upper bound for z, solve the plane equation for z.

$$x + y - z + 4 = 0$$

$$z = x + y + 4$$

So z is defined from -3 to $x + y + 4$. The values of x and y for each z change within the ellipse $4x^2 + y^2 = 4$.

In the outermost integral, x changes from -1 to 1, and to find the bounds for y in the middle integral we'll solve the ellipse equation for y.

$$4x^2 + y^2 = 4$$

$$y^2 = 4 - 4x^2$$

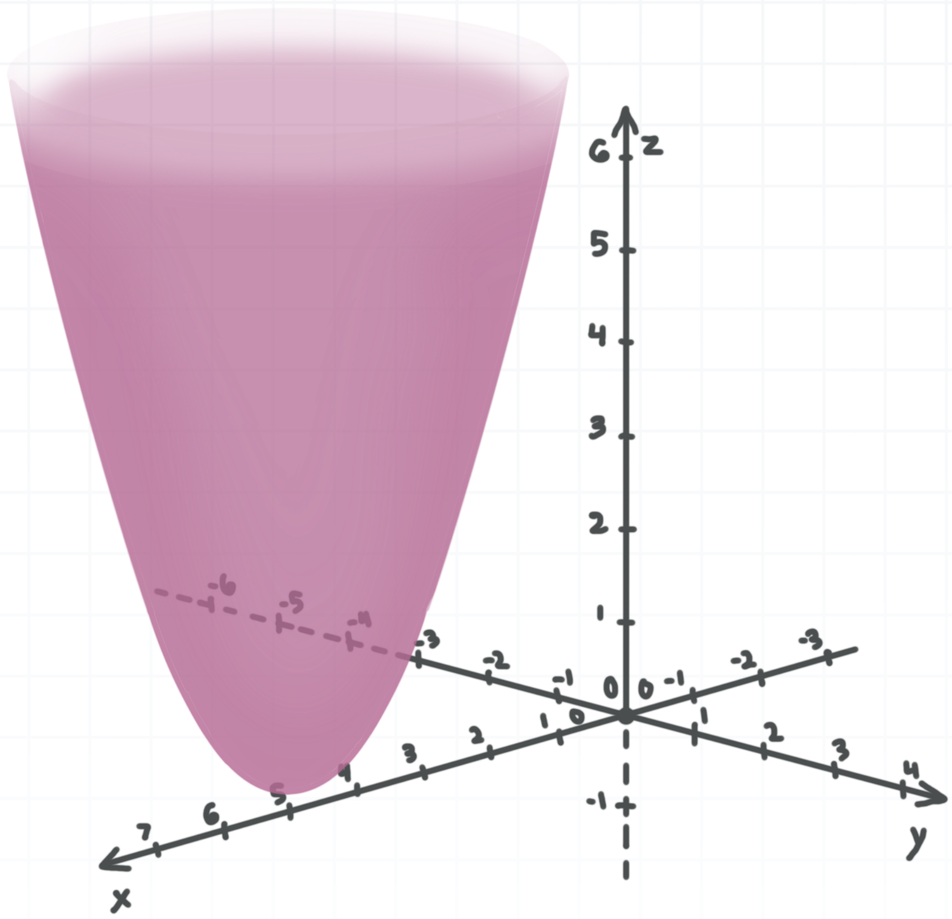
$$y = \pm 2\sqrt{1 - x^2}$$

Therefore, y changes from $-2\sqrt{1-x^2}$ to $2\sqrt{1-x^2}$, and the iterated integral is

$$\int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \int_{-3}^{x+y+4} f(x, y, z) \ dz \ dy \ dx$$

■ 3. Represent the triple integral as an improper iterated integral using the order $dx \ dy \ dz$, where E is interior of the circular paraboloid $x^2 - 4x + y^2 + 6y - z + 12 = 0$.

$$\iiint_E f(x, y, z) \ dV$$



Solution:

Rewrite the paraboloid equation in standard form.

$$x^2 - 4x + y^2 + 6y - z + 12 = 0$$

$$(x^2 - 4x + 4 - 4) + (y^2 + 6y + 9 - 9) - z + 12 = 0$$

$$(x - 2)^2 - 4 + (y + 3)^2 - 9 - z + 12 = 0$$

$$(x - 2)^2 + (y + 3)^2 - z - 1 = 0$$

$$(x - 2)^2 + (y + 3)^2 = z + 1$$

Since the vertex is at $(2, -3, -1)$, the value of z changes from -1 to ∞.

The values of x and y change for each z within the circle $(x-2)^2+(y+3)^2=z+1$ with center at $(2,-3,z)$ and radius $r=\sqrt{z+1}$ that lies in the plane parallel to the xy-plane.

So y changes from $-3-r$ to $-3+r$, or from $-3-\sqrt{z+1}$ to $-3+\sqrt{z+1}$. To find the bounds on x, solve the equation of the circle equation for x.

$$(x-2)^2+(y+3)^2=z+1$$

$$(x-2)^2=z+1-(y+3)^2$$

$$x-2=\pm\sqrt{z+1-(y+3)^2}$$

$$x=2\pm\sqrt{z+1-(y+3)^2}$$

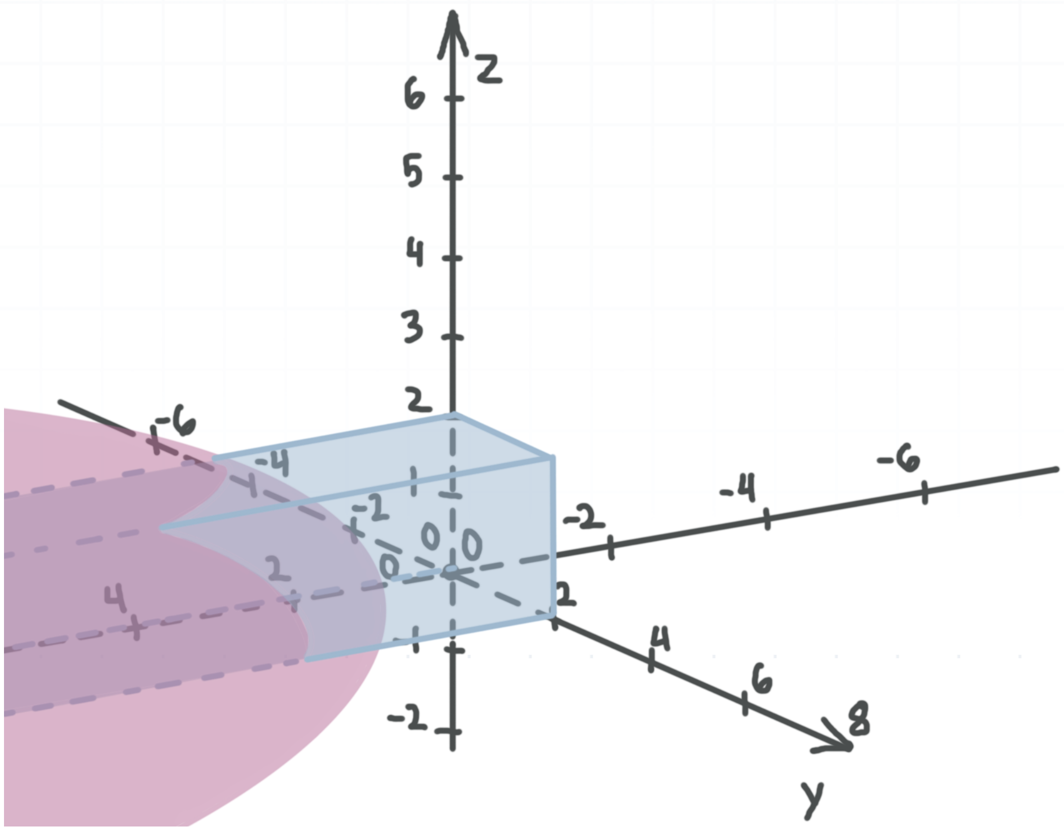
Then x changes from $2-\sqrt{z+1-(y+3)^2}$ to $2+\sqrt{z+1-(y+3)^2}$, and the iterated integral is

$$\int_{-1}^{\infty}\int_{-3-\sqrt{z+1}}^{-3+\sqrt{z+1}}\int_{2-\sqrt{z+1-(y+3)^2}}^{2+\sqrt{z+1-(y+3)^2}} f(x,y,z)\ dx\ dy\ dz$$

TYPE I, II, AND III REGIONS

■ 1. Evaluate the triple integral, where E is the region that lies in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$), and is bounded by the surfaces $y = 2$, $z = 2$, and $x - 0.5y^2 - 0.5z^2 - 1 = 0$.

$$\iiint_E 4x + 2y - 2z \ dV$$



Solution:

The value of y changes from 0 to 2, z changes from 0 to 2, and x changes from 0 to $x = 0.5y^2 + 0.5z^2 + 1$. Since the function needs to be integrated with respect to x first, we can treat E as a type I region. So we can rewrite the triple integral as an iterated integral.

$$\int_0^2 \int_0^2 \int_0^{0.5y^2+0.5z^2+1} 4x + 2y - 2z \ dx \ dy \ dz$$

Integrate with respect to x.

$$\int_0^2 \int_0^2 2x^2 + 2xy - 2xz \Big|_{x=0}^{x=0.5y^2+0.5z^2+1} \ dy \ dz$$

$$\int_0^2 \int_0^2 2(0.5y^2 + 0.5z^2 + 1)^2 + 2(0.5y^2 + 0.5z^2 + 1)y - 2(0.5y^2 + 0.5z^2 + 1)z$$

$$-(2(0)^2 + 2(0)y - 2(0)z) \ dy \ dz$$

$$\int_0^2 \int_0^2 \frac{1}{2}y^4 + 2y^2 + 2 + y^2z^2 + 2z^2 + \frac{1}{2}z^4 + y^3 + yz^2 + 2y - y^2z - z^3 - 2z \ dy \ dz$$

Integrate with respect to y.

$$\int_0^2 \frac{1}{10}y^5 + \frac{2}{3}y^3 + 2y + \frac{1}{3}y^3z^2 + 2yz^2 + \frac{1}{2}yz^4$$

$$+\frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + y^2 - \frac{1}{3}y^3z - yz^3 - 2yz \Big|_{y=0}^{y=2} \ dz$$

$$\int_0^2 \frac{1}{10}y^5 + \frac{2}{3}y^3 + 2y + \frac{1}{3}y^3z^2 + 2yz^2 + \frac{1}{2}yz^4$$

$$+\frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + y^2 - \frac{1}{3}y^3z - yz^3 - 2yz \Big|_{y=0}^{y=2} \ dz$$

$$\int_0^2 \frac{1}{10}(2)^5 + \frac{2}{3}(2)^3 + 2(2) + \frac{1}{3}(2)^3 z^2$$

$$+2(2)z^2 + \frac{1}{2}(2)z^4 + \frac{1}{4}(2)^4 + \frac{1}{2}(2)^2 z^2 + (2)^2 - \frac{1}{3}(2)^3 z - (2)z^3 - 2(2)z$$

$$-\left(\frac{1}{10}(0)^5 + \frac{2}{3}(0)^3 + 2(0) + \frac{1}{3}(0)^3 z^2\right.$$

$$\left.+2(0)z^2 + \frac{1}{2}(0)z^4 + \frac{1}{4}(0)^4 + \frac{1}{2}(0)^2 z^2 + (0)^2 - \frac{1}{3}(0)^3 z - (0)z^3 - 2(0)z\right) dz$$

$$\int_0^2 \frac{16}{5} + \frac{16}{3} + 4 + \frac{8}{3}z^2 + 4z^2 + z^4 + 4 + 2z^2 + 4 - \frac{8}{3}z - 2z^3 - 4z \ dz$$

$$\int_0^2 z^4 - 2z^3 + \frac{26}{3}z^2 - \frac{20}{3}z + \frac{308}{15} \ dz$$

Integrate with respect to z.

$$\frac{1}{5}z^5 - \frac{1}{2}z^4 + \frac{26}{9}z^3 - \frac{20}{6}z^2 + \frac{308}{15}z \Big|_0^2$$

$$\frac{1}{5}(2)^5 - \frac{1}{2}(2)^4 + \frac{26}{9}(2)^3 - \frac{20}{6}(2)^2 + \frac{308}{15}(2)$$

$$-\left(\frac{1}{5}(0)^5 - \frac{1}{2}(0)^4 + \frac{26}{9}(0)^3 - \frac{20}{6}(0)^2 + \frac{308}{15}(0)\right)$$

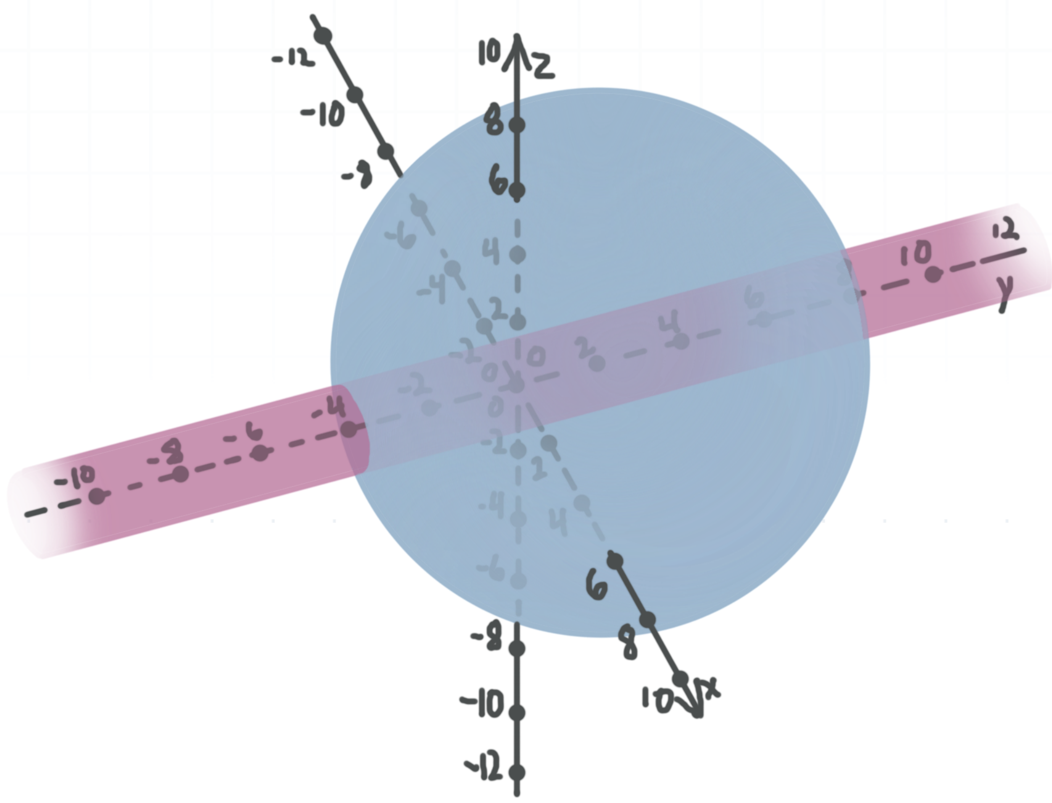
$$\frac{32}{5} - 8 + \frac{208}{9} - \frac{40}{3} + \frac{616}{15}$$

$$\frac{288}{45} - \frac{360}{45} + \frac{1,040}{45} - \frac{600}{45} + \frac{1,848}{45}$$

$$\frac{288}{45} - \frac{360}{45} + \frac{1,040}{45} - \frac{600}{45} + \frac{1,848}{45}$$

$$\frac{2,216}{45}$$

■ 2. Use a triple integral to find the volume of the region E that's bounded by the cylinder $x^2 + z^2 = 1$ and the sphere $x^2 + (y-2)^2 + z^2 = 36$.



Solution:

The values of x and z change within the circle with the canter at the origin and the radius 1. To find the bounds for y, let's solve the equation of the sphere for y.

$$x^2 + (y-2)^2 + z^2 = 36$$

$$(y-2)^2 = 36 - x^2 - z^2$$

$$y - 2 = \pm\sqrt{36 - x^2 - z^2}$$

$$y = 2 \pm \sqrt{36 - x^2 - z^2}$$

So y changes from $2 - \sqrt{36 - x^2 - z^2}$ to $2 + \sqrt{36 - x^2 - z^2}$. Since the function need to be integrated with respect to y first, we can treat E as a type II region. So we can rewrite the triple integral as an iterated integral.

$$\iint_C \int_{2-\sqrt{36-x^2-z^2}}^{2+\sqrt{36-x^2-z^2}} dy\ dA$$

Integrate with respect to y.

$$\iint_C y\Big|_{2-\sqrt{36-x^2-z^2}}^{2+\sqrt{36-x^2-z^2}} dA$$

$$\iint_C 2 + \sqrt{36 - x^2 - z^2} - (2 - \sqrt{36 - x^2 - z^2})\ dA$$

$$\iint_C 2 + \sqrt{36 - x^2 - z^2} - 2 + \sqrt{36 - x^2 - z^2}\ dA$$

$$\iint_C \sqrt{36 - x^2 - z^2} + \sqrt{36 - x^2 - z^2}\ dA$$

$$\iint_C 2\sqrt{36 - x^2 - z^2}\ dA$$

Convert the integral into polar coordinates.

$$\int_0^1 \int_0^{2\pi} 2\sqrt{36 - r^2} \cdot r \ d\theta \ dr$$

$$\int_0^1 2r\sqrt{36 - r^2} \ dr \cdot \int_0^{2\pi} \ d\theta$$

$$2\pi \int_0^1 2r\sqrt{36 - r^2} \ dr$$

Integrate with respect to r, using a substitution with $r^2 = u$, $du = 2r \ dr$, and where u changes from 0 to 1.

$$2\pi \int_0^1 \sqrt{36 - u} \ du$$

$$2\pi \left(-\frac{2(36 - u)^{\frac{3}{2}}}{3} \right) \Bigg|_0^1$$

$$-\frac{4\pi(36 - u)^{\frac{3}{2}}}{3} \Bigg|_0^1$$

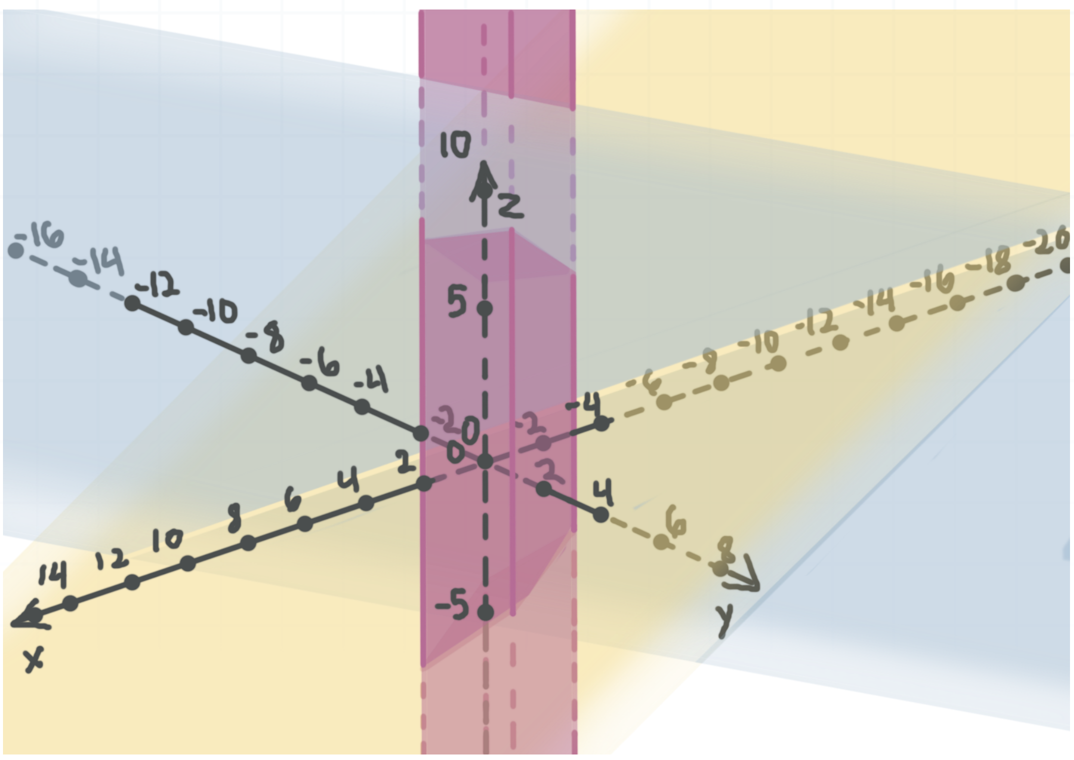
$$-\frac{4\pi(36 - 1)^{\frac{3}{2}}}{3} - \left(-\frac{4\pi(36 - 0)^{\frac{3}{2}}}{3} \right)$$

$$-\frac{4\pi(35)^{\frac{3}{2}}}{3} + \frac{4\pi(36)^{\frac{3}{2}}}{3}$$

$$\frac{4\pi(6^3) - 4\pi(35)^{\frac{3}{2}}}{3}$$

$$\frac{864\pi - 140\pi\sqrt{35}}{3}$$

■ 3. Use a triple integral to find the volume of the region E that lies in the first and fifth octants ($x \geq 0$, $y \geq 0$), and is bounded by the planes $x = 2$, $y = 3$, $2x + y - 2z + 12 = 0$, and $x - y + z + 4 = 0$.



Solution:

The value of x changes from 0 to 2, y changes from 0 to 3. To find the bounds for z, we'll solve the plane equations for z.

$$z = -x + y - 4$$

$$z = x + 0.5y + 6$$

So z changes from $-x+y-4$ to $x+0.5y+6$. Since the function needs to be integrated with respect to z first, we can treat E as a type III region. So we can rewrite the triple integral as an iterated integral.

$$\int_0^2 \int_0^3 \int_{-x+y-4}^{x+0.5y+6} dz\ dy\ dx$$

Integrate with respect to z.

$$\int_0^2 \int_0^3 z \Big|_{z=-x+y-4}^{z=x+0.5y+6} dy\ dx$$

$$\int_0^2 \int_0^3 x+0.5y+6-(-x+y-4)\ dy\ dx$$

$$\int_0^2 \int_0^3 2x-0.5y+10\ dy\ dx$$

Integrate with respect to y.

$$\int_0^2 2xy-0.25y^2+10y \Big|_{y=0}^{y=3} dx$$

$$\int_0^2 2x(3)-0.25(3)^2+10(3)-(2x(0)-0.25(0)^2+10(0))\ dx$$

$$\int_0^2 6x-2.25+30\ dx$$

$$\int_0^2 6x+27.75\ dx$$

Integrate with respect to x.

$$3x^2 + 27.75x \Big|_0^2$$

$$3(2)^2 + 27.75(2) - (3(0)^2 + 27.75(0))$$

$$12 + 55.5$$

$$67.5$$

CYLINDRICAL COORDINATES

1. Evaluate the triple integral given in cylindrical coordinates, where $f(r, \theta, z) = (3r - 12z^2)\cos\theta$.

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{2}^{3} f(r, \theta, z)\ r\ dr\ d\theta\ dz$$

Solution:

Set up the integral.

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{2}^{3} r(3r - 12z^2)\cos\theta\ dr\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_{2}^{3} (3r^2 - 12rz^2)\cos\theta\ dr\ d\theta\ dz$$

Integrate with respect to r.

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (r^3 - 6r^2z^2)\cos\theta\Big|_{r=2}^{r=3}\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (3^3 - 6(3)^2z^2)\cos\theta - ((2^3 - 6(2)^2z^2)\cos\theta)\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (27-54z^2)\cos\theta-(8-24z^2)\cos\theta\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} [(27-54z^2)-(8-24z^2)]\cos\theta\ d\theta\ dz$$

$$\int_{-1}^{1}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (19-30z^2)\cos\theta\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_{-1}^{1} (19-30z^2)\sin\theta\Big|_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}}\ dz$$

$$\int_{-1}^{1} (19-30z^2)\sin\frac{\pi}{4}-(19-30z^2)\sin\left(-\frac{\pi}{4}\right)\ dz$$

$$\int_{-1}^{1} (19-30z^2)\left(\frac{\sqrt{2}}{2}\right)-(19-30z^2)\left(-\frac{\sqrt{2}}{2}\right)\ dz$$

$$\int_{-1}^{1} (19-30z^2)\left(\frac{\sqrt{2}}{2}\right)+(19-30z^2)\left(\frac{\sqrt{2}}{2}\right)\ dz$$

$$\int_{-1}^{1} 2(19-30z^2)\left(\frac{\sqrt{2}}{2}\right)\ dz$$

$$\int_{-1}^{1} 19\sqrt{2}-30\sqrt{2}z^2\ dz$$

Integrate with respect to z.

$$19\sqrt{2}z - 10\sqrt{2}z^3 \Big|_{-1}^{1}$$

$$19\sqrt{2}(1) - 10\sqrt{2}(1)^3 - (19\sqrt{2}(-1) - 10\sqrt{2}(-1)^3)$$

$$19\sqrt{2} - 10\sqrt{2} - 10\sqrt{2} + 19\sqrt{2}$$

$$38\sqrt{2} - 20\sqrt{2}$$

$$18\sqrt{2}$$

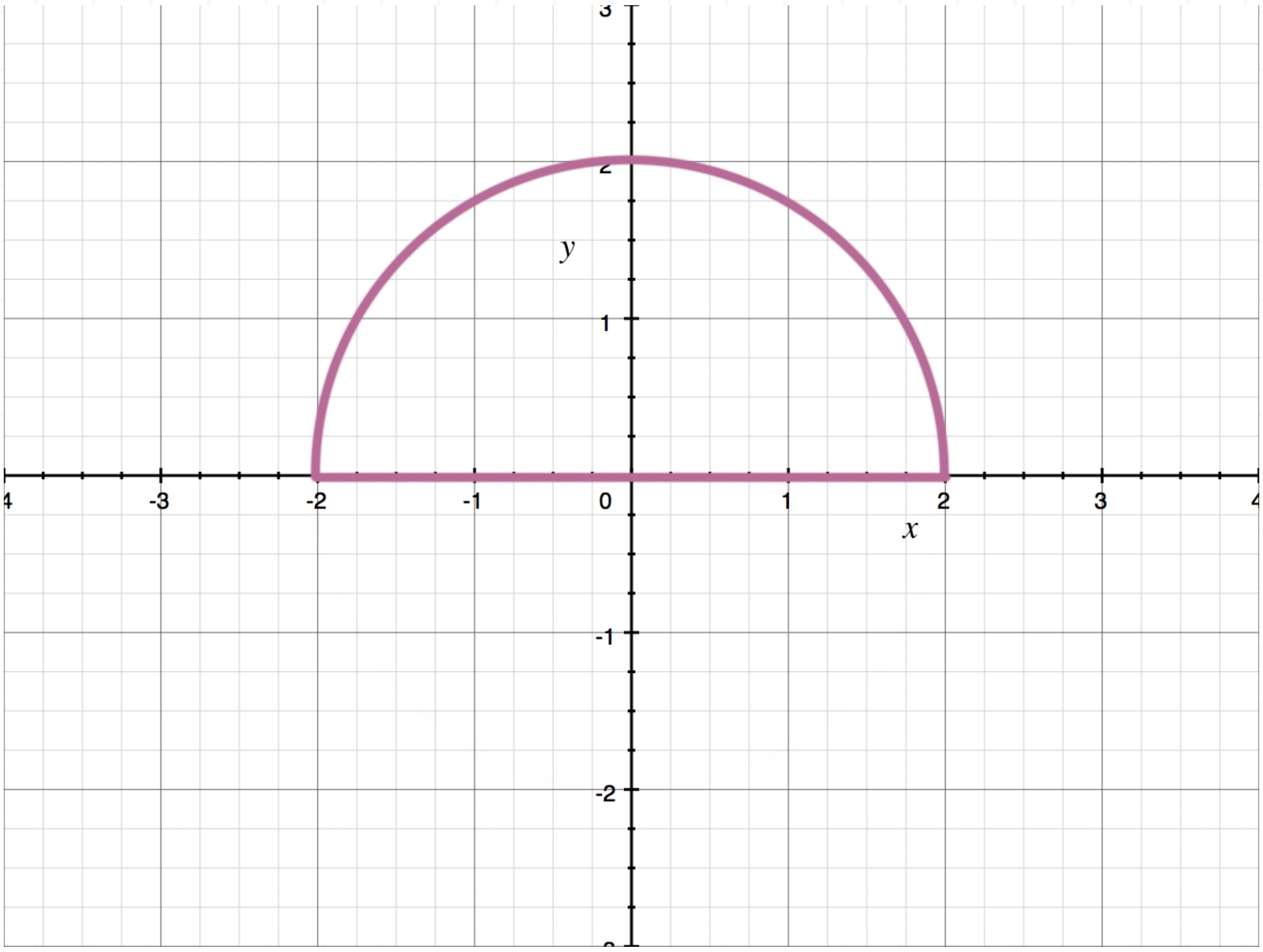
■ 2. Identify the solid given by the following iterated integral in cylindrical coordinates.

$$\int_{-4}^{6}\int_{0}^{\pi}\int_{0}^{2} f(r,\theta,x)\ r\ dr\ d\theta\ dx$$

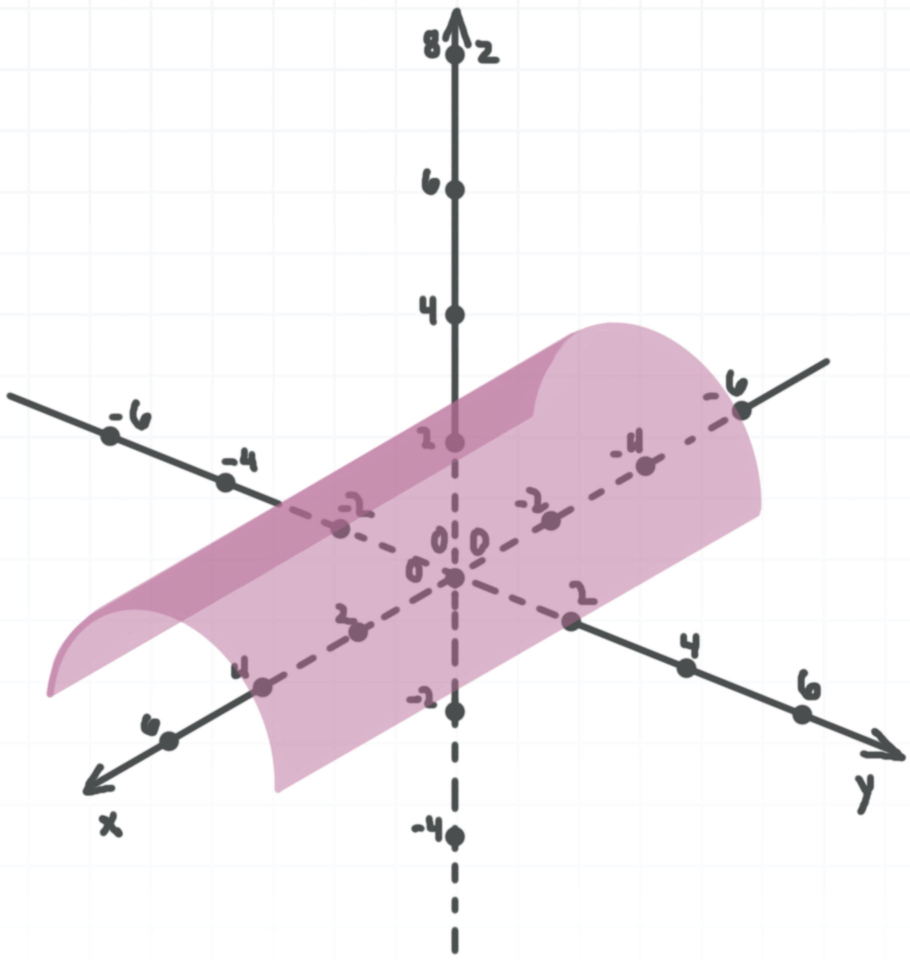
Solution:

The value of x changes from -4 to 6. The polar coordinates r and θ change within the semicircle with center at the x-axis and radius 2, that lies in the plane parallel to yz-plane.

Since the angular coordinate θ changes from 0 to π, the region of integration includes only the points within the semicircle for $z > 0$.



So the region of integration in three dimensions is the half of the cylinder with radius 2 and height 10, with the cylinder's axis parallel to the x-axis, and bases lying in the planes $x = -4$ and $x = 6$, and only the points of the cylinder where $z > 0$.



3. Identify the solid given by the iterated improper integral in cylindrical coordinates.

$$\int_{2}^{\infty}\int_{0}^{2\pi}\int_{0}^{\sqrt{2y-4}} f(r,\theta,y)\ r\ dr\ d\theta\ dy$$

Solution:

The value of y changes from 2 to ∞, and the values of r and θ change within the circle with center at the y-axis that lies in the plane parallel to the xz-plane.

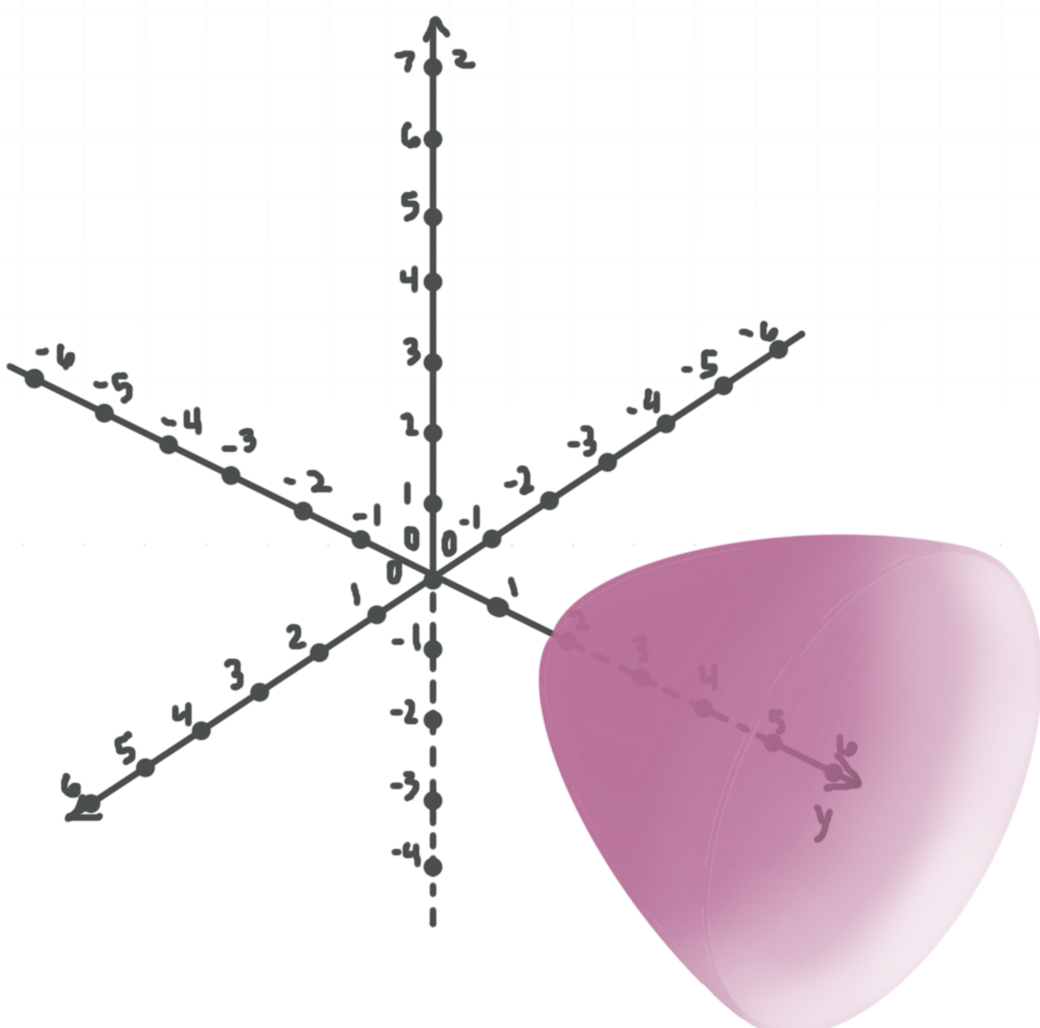
Since r changes from 0 to $\sqrt{2y-4}$, we can find the upper bound for r by substituting $r^2=x^2+z^2$ into the equation $r=\sqrt{2y-4}$.

$$r^2=2y-4$$

$$x^2+z^2=2y-4$$

$$\frac{x^2}{2}+\frac{z^2}{2}=y-2$$

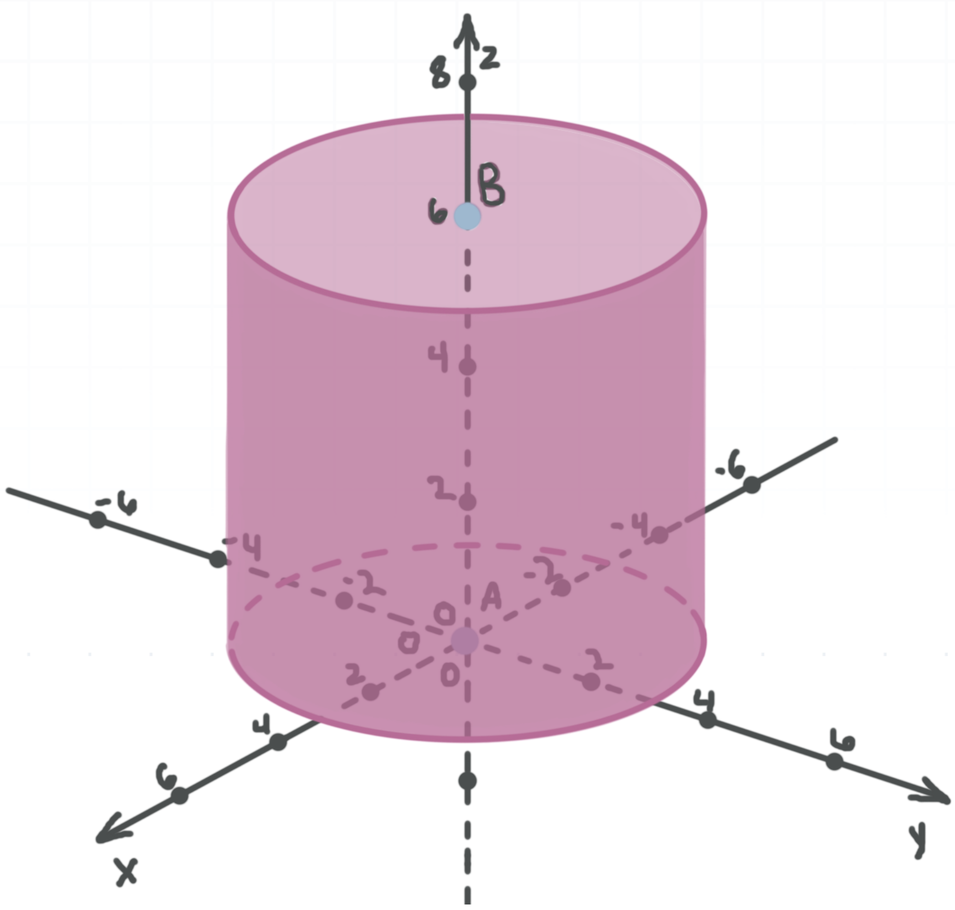
So the upper bound for r is the circular paraboloid with center at the point $(0,2,0)$, whose axis is parallel to the y-axis.



CHANGING TRIPLE INTEGRALS TO CYLINDRICAL COORDINATES

■ 1. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the right circular cylinder with radius 3, height 6, and a base that lies in the xy-plane with center at the origin.

$$\iiint_E (x^2 + y^2)2^z \ dV$$



Solution:

The value of z changes from 0 to 6, x and y change within the circle C with radius 3 that lies in the xy-plane with center at the origin. The value of r

changes from 0 to 3, and θ changes from 0 to 2π. Then the integral in cylindrical coordinates is

$$\int_0^6 \int_0^{2\pi} \int_0^3 r^2 2^z \cdot r \ dr \ d\theta \ dz$$

$$\int_0^6 \int_0^{2\pi} \int_0^3 r^3 2^z \ dr \ d\theta \ dz$$

$$\int_0^6 2^z \ dz \cdot \int_0^{2\pi} d\theta \cdot \int_0^3 r^3 \ dr$$

Evaluate each integral.

$$\frac{2^z}{\ln 2}\Bigg|_0^6 \cdot \theta\Bigg|_0^{2\pi} \cdot \frac{1}{4}r^4\Bigg|_0^3$$

$$\left(\frac{2^6}{\ln 2} - \frac{2^0}{\ln 2}\right)(2\pi - 0)\left(\frac{1}{4}(3)^4 - \frac{1}{4}(0)^4\right)$$

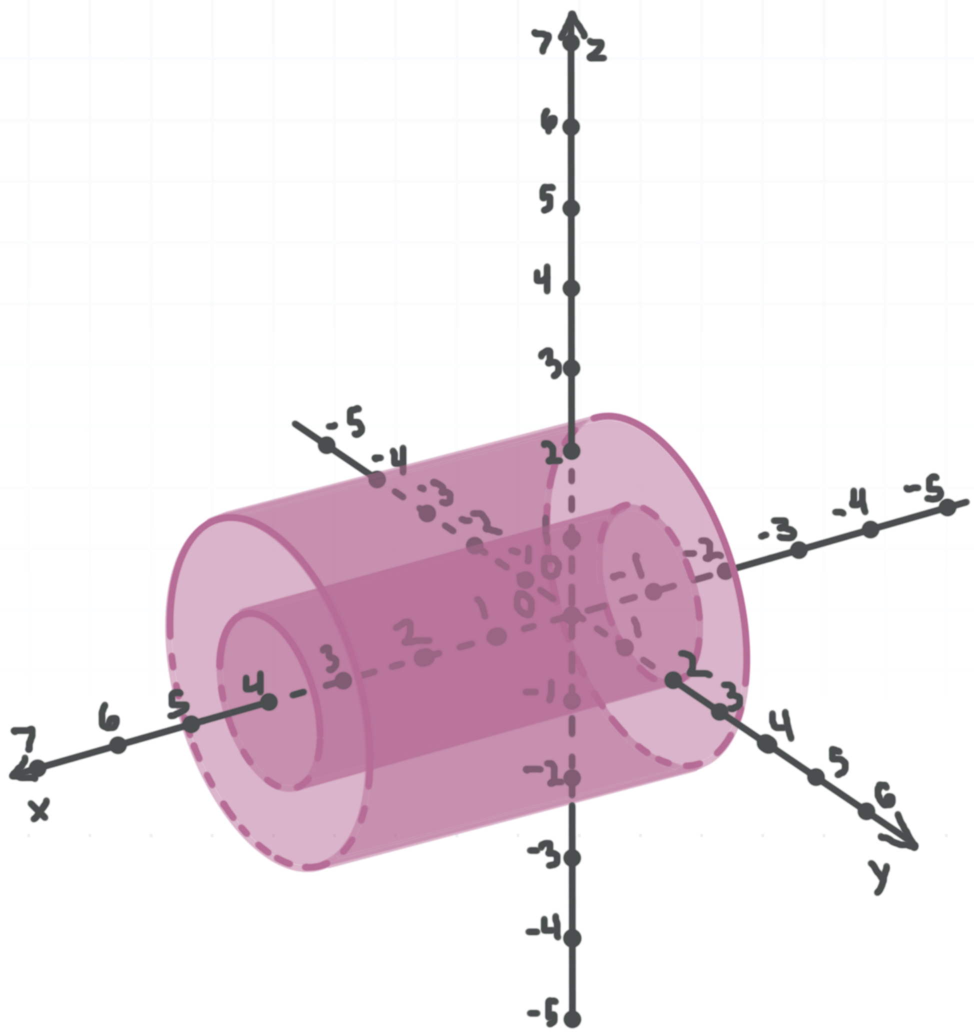
$$\left(\frac{64}{\ln 2} - \frac{1}{\ln 2}\right)(2\pi)\left(\frac{81}{4}\right)$$

$$\frac{63(81)(2\pi)}{4\ln 2}$$

$$\frac{5{,}103\pi}{2\ln 2}$$

■ 2. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the set of points between right circular cylinders with radius 1 and 2, height 5, cylinder axes are x -axis, and bases that lie in the planes $x=-1$ and $x=4$.

$$\iiint_E \frac{x+y+z}{y^2+z^2}\ dV$$



Solution:

The value of x changes from -1 to 4, and x and y change between the circles C_1 and C_2 with centers at the x-axis, radii 1 and 2 respectively, that lie in the plane parallel to the yz-plane. The value of r changes from 1 to 2, and θ changes from 0 to 2π, so the integral in cylindrical coordinates is

$$\int_{-1}^{4}\int_{0}^{2\pi}\int_{1}^{2}\left(\frac{x}{r^2}+\frac{\cos\theta}{r}+\frac{\sin\theta}{r}\right)\cdot r\ dr\ d\theta\ dx$$

$$\int_{-1}^{4}\int_{0}^{2\pi}\int_{1}^{2}\frac{x}{r}+\cos\theta+\sin\theta\ dr\ d\theta\ dx$$

Integrate with respect to r.

$$\int_{-1}^{4}\int_{0}^{2\pi}x\ln r+r\cos\theta+r\sin\theta\Big|_{r=1}^{r=2}\ d\theta\ dx$$

$$\int_{-1}^{4}\int_{0}^{2\pi}x\ln 2+2\cos\theta+2\sin\theta-(x\ln 1+\cos\theta+\sin\theta)\ d\theta\ dx$$

$$\int_{-1}^{4}\int_{0}^{2\pi}x\ln 2+\cos\theta+\sin\theta\ d\theta\ dx$$

Integrate with respect to θ.

$$\int_{-1}^{4}\theta x\ln 2+\sin\theta-\cos\theta\Big|_{\theta=0}^{\theta=2\pi}\ dx$$

$$\int_{-1}^{4}2\pi x\ln 2+\sin(2\pi)-\cos(2\pi)-(0x\ln 2+\sin(0)-\cos(0))\ dx$$

$$\int_{-1}^{4}2\pi x\ln 2-1+1\ dx$$

$$\int_{-1}^{4}2\pi x\ln 2\ dx$$

Integrate with respect to x.

$$\pi x^2 \ln 2 \Big|_{-1}^{4}$$

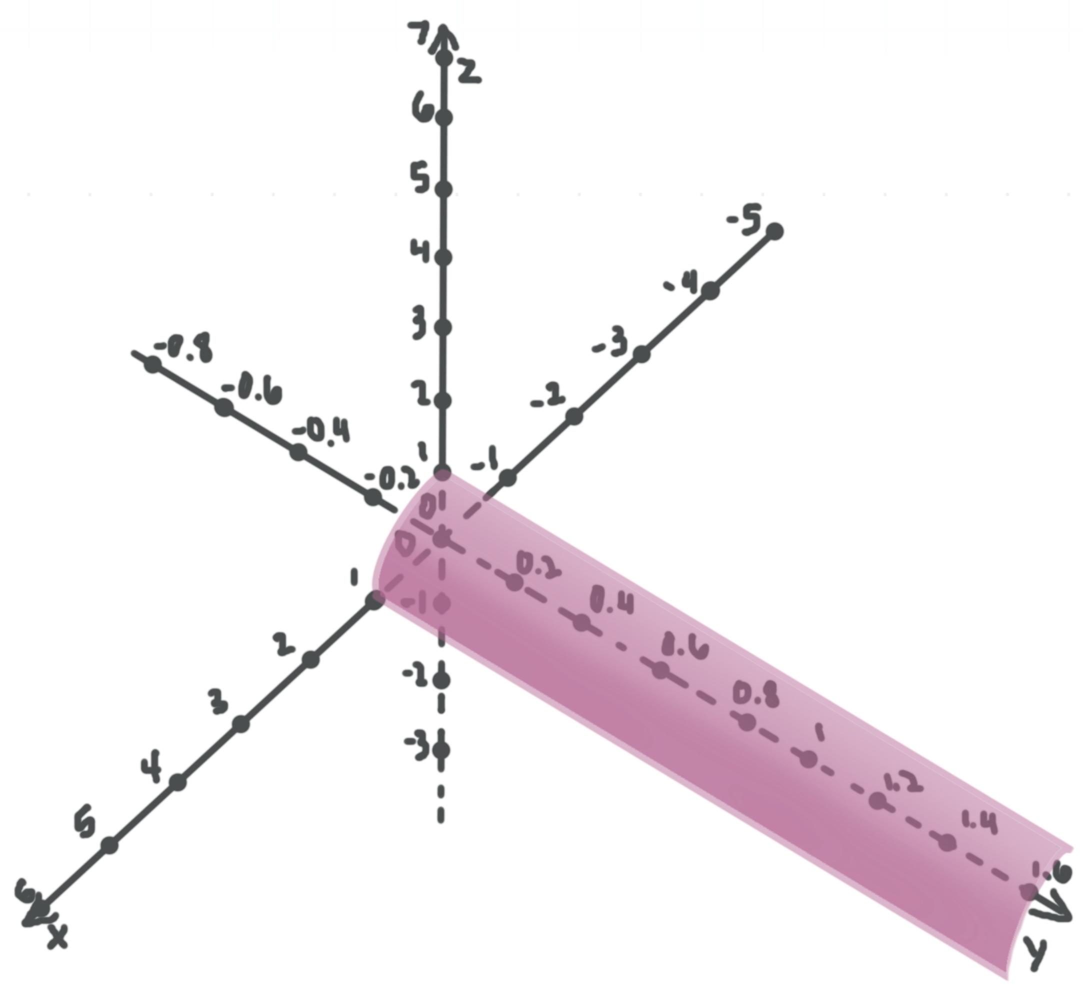
$$\pi(4)^2 \ln 2 - \pi(-1)^2 \ln 2$$

$$16\pi \ln 2 - \pi \ln 2$$

$$15\pi \ln 2$$

■ 3. Evaluate the improper triple integral by changing it to cylindrical coordinates, where E is part of the cylinder $x^2 + z^2 = 1$ that lies in the first octant.

$$\iiint_E 2e^{-x^2-y^2-z^2} \, dV$$



Solution:

The value of y changes from 0 to ∞, x and z change in the first quarter of the circle C with center at the y-axis, radius 1, that lies in the plane parallel to the xz-plane. The values of r change from 0 to 1, and θ changes from 0 to $\pi/2$. So the integral in cylindrical coordinates is

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^1 2e^{-r^2-y^2} \cdot r \ dr \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^1 2re^{-r^2-y^2} \ dr \ d\theta \ dy$$

Integrate with respect to r, using a substitution with $u = r^2$, $du = 2r \ dr$, and where u changes from 0 to 1.

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \int_0^1 e^{-u-y^2} \ du \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \frac{1}{-1} e^{-u-y^2} \Big|_{u=0}^{u=1} \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -e^{-u-y^2} \Big|_{u=0}^{u=1} \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -e^{-1-y^2} - (-e^{-0-y^2}) \ d\theta \ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -e^{-1-y^2}+e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -e^{-1}e^{-y^2}+e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} -\frac{1}{e}e^{-y^2}+e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \left(1-\frac{1}{e}\right)e^{-y^2}\ d\theta\ dy$$

$$\int_0^\infty \int_0^{\frac{\pi}{2}} \frac{e-1}{e}e^{-y^2}\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_0^\infty \frac{e-1}{e}e^{-y^2}\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\ dy$$

$$\int_0^\infty \frac{e-1}{e}e^{-y^2}\left(\frac{\pi}{2}\right)-\frac{e-1}{e}e^{-y^2}(0)\ dy$$

$$\int_0^\infty \frac{\pi e-\pi}{2e}e^{-y^2}\ dy$$

Integrating with respect to y, remembering that

$$\int_{-\infty}^\infty e^{-t^2}\ dt=\sqrt{\pi}$$

gives us

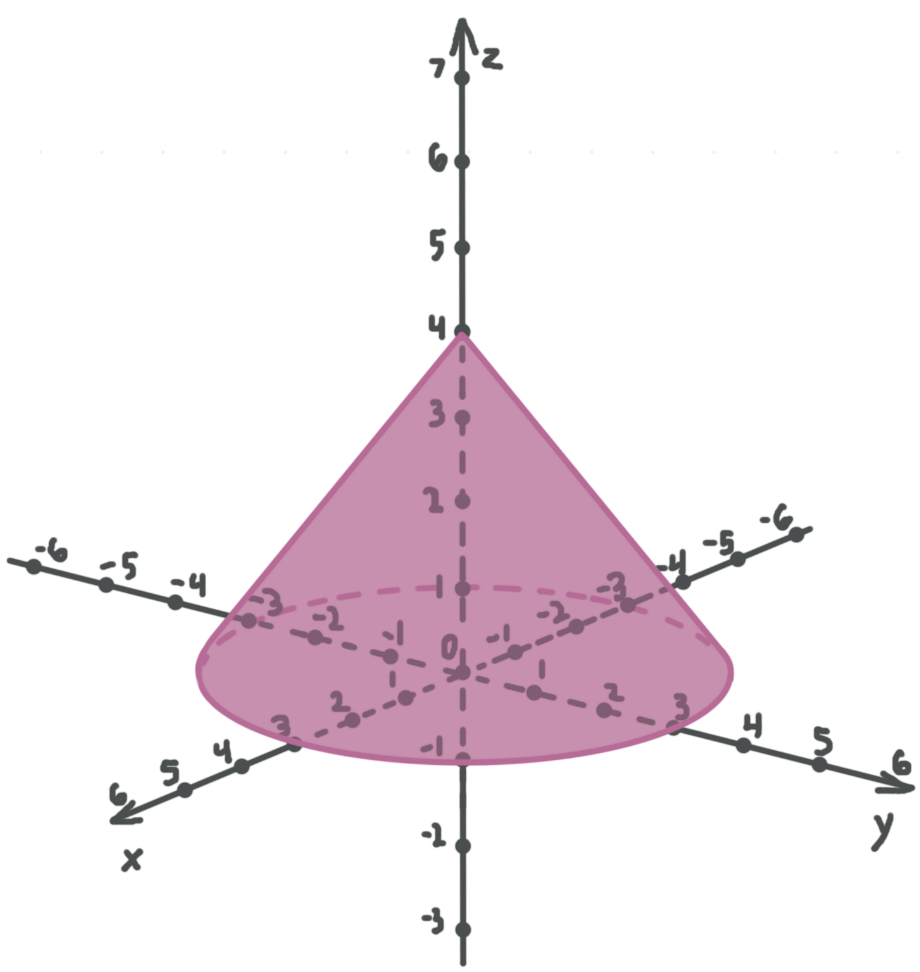
$$\left(\frac{\pi e-\pi}{2e}\right)\left(\frac{\sqrt{\pi}}{2}\right)$$

$$\frac{\pi e\sqrt{\pi}-\pi\sqrt{\pi}}{4e}$$

$$\frac{\pi\sqrt{\pi}(e-1)}{4e}$$

■ 4. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the right circular cone with radius 3, vertex at the point $(0,0,4)$, and a base that lies in the xy-plane with its center at the origin.

$$\iiint_E 48x^2y^2(z+2)\ dV$$



Solution:

The value of z changes from 0 to 4, x and y changes within the circle centered at the origin that lies in the plane parallel to the xy-plane. The value of θ changes from 0 to 2π, and r changes linearly on z, i.e. $r = az + b$. When $z = 0$, $r = 3$, and when $z = 4$, $r = 0$. Substitute these values into the linear equation and solve the resulting system for a and b.

$$3 = a(0) + b$$

$$0 = a(4) + b$$

Then $b = 3$, so $4a + 3 = 0$ and $a = -3/4$. Therefore, $r = -(3/4)z + 3$. The function is

$$48x^2y^2(z+2)$$

$$48r^4 \cdot \cos^2\theta \cdot \sin^2\theta\ (z+2)$$

$$12r^4(2\cos\theta \cdot \sin\theta)^2(z+2)$$

$$12r^4\sin^2(2\theta)(z+2)$$

$$6r^4(1-\cos(4\theta))(z+2)$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^4 \int_0^{2\pi} \int_0^{-\frac{3}{4}z+3} 6r^4(1-\cos(4\theta))(z+2) \cdot r\ dr\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^{-\frac{3}{4}z+3} 6r^5(1-\cos(4\theta))(z+2)\ dr\ d\theta\ dz$$

Integrate with respect to r.

$$\int_0^4 \int_0^{2\pi} r^6(1-\cos(4\theta))(z+2)\Big|_{r=0}^{r=-\frac{3}{4}z+3}\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \left(-\frac{3}{4}z+3\right)^6(1-\cos(4\theta))(z+2)-0^6(1-\cos(4\theta))(z+2)\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \left(-\frac{3}{4}z+3\right)^6(1-\cos(4\theta))(z+2)\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_0^4 \left(-\frac{3}{4}z+3\right)^6\left(\theta-\frac{1}{4}\sin(4\theta)\right)(z+2)\Big|_{\theta=0}^{\theta=2\pi}\ dz$$

$$\int_0^4 \left(-\frac{3}{4}z+3\right)^6\left(2\pi-\frac{1}{4}\sin(4(2\pi))\right)(z+2)$$

$$-\left(-\frac{3}{4}z+3\right)^6\left(0-\frac{1}{4}\sin(4(0))\right)(z+2)\ dz$$

$$\int_0^4 2\pi\left(-\frac{3}{4}z+3\right)^6(z+2)\ dz$$

Rewrite the function.

$$2\pi\int_0^4\left(\frac{729}{4{,}096}z^6-\frac{2{,}187}{512}z^5+\frac{10{,}935}{256}z^4-\frac{3{,}645}{16}z^3\right.$$

$$\left.+\frac{10{,}935}{16}z^2-\frac{2{,}187}{2}z+729\right)(z+2)\ dz$$

$$2\pi\int_0^4\frac{729}{4{,}096}z^7-\frac{2{,}187}{512}z^6+\frac{10{,}935}{256}z^5-\frac{3{,}645}{16}z^4+\frac{10{,}935}{16}z^3-\frac{2{,}187}{2}z^2+729z$$

$$+\frac{729}{2{,}048}z^6-\frac{2{,}187}{256}z^5+\frac{10{,}935}{128}z^4-\frac{3{,}645}{8}z^3+\frac{10{,}935}{8}z^2-2{,}187z+1{,}458\ dz$$

$$2\pi\int_0^4\frac{729}{4{,}096}z^7-\frac{8{,}019}{2{,}048}z^6+\frac{8{,}748}{256}z^5-\frac{18{,}225}{128}z^4+\frac{3{,}645}{16}z^3$$

$$+\frac{2{,}187}{8}z^2-1{,}458z+1{,}458\ dz$$

Integrate with respect to z.

$$2\pi\left(\frac{729}{32{,}768}z^8-\frac{8{,}019}{14{,}336}z^7+\frac{729}{128}z^6-\frac{3{,}645}{128}z^5+\frac{3{,}645}{64}z^4\right.$$

$$\left.+\frac{729}{8}z^3-729z^2+1{,}458z\right)\Bigg|_0^4$$

$$2\pi\left(\frac{729}{32{,}768}(4)^8-\frac{8{,}019}{14{,}336}(4)^7+\frac{729}{128}(4)^6-\frac{3{,}645}{128}(4)^5+\frac{3{,}645}{64}(4)^4\right.$$

$$+\frac{729}{8}(4)^3 - 729(4)^2 + 1{,}458(4)\Bigg)$$

$$2\pi\Bigg(729(2) - \frac{8{,}019}{7}(8) + 729(32) - 3{,}645(8) + 3{,}645(4)$$

$$+729(8) - 729(16) + 1{,}458(4)\Bigg)$$

$$2\pi\left(1{,}458 - \frac{64{,}152}{7} + 23{,}328 - 29{,}160 + 14{,}580 + 5{,}832 - 11{,}664 + 5{,}832\right)$$

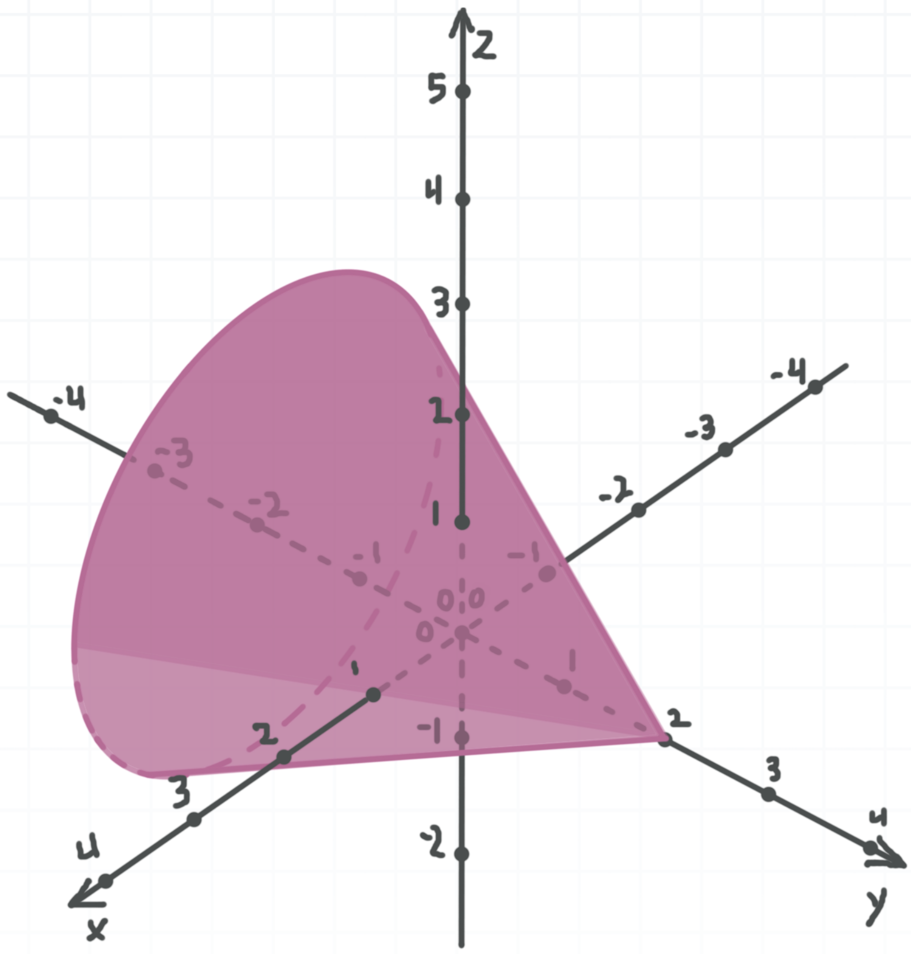
$$2\pi\left(10{,}206 - \frac{64{,}152}{7}\right)$$

$$2\pi\left(\frac{71{,}442}{7} - \frac{64{,}152}{7}\right)$$

$$\frac{14{,}580\pi}{7}$$

■ 5. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the right circular cone with radius 2, vertex at the point $(0,2,0)$, and a base that lies in the plane $y = -2$ with its center at the point $(0,-2,0)$.

$$\iiint_E \frac{(x+z)^2}{(y+4)}\ dV$$



Solution:

The value of y changes from -2 to 2, x and z change within the circle C with center at the y-axis, that lies in the plane parallel to the xz-plane. The value of θ changes from 0 to 2π, and r changes linearly on y, i.e. $r = ay + b$. When $y = -2$, $r = 2$, and when $y = 2$, $r = 0$. Substitute the values into the linear equation and solve the resulting system for a and b.

$$2 = a(-2) + b$$

$$0 = a(2) + b$$

$b = 1$ and $a = -0.5$

So $r = -0.5y + 1$. The function is

$$\frac{(x+z)^2}{(y+4)}$$

$$\frac{x^2+z^2+2xz}{(y+4)}$$

$$\frac{r^2+2r^2\cos\theta\sin\theta}{y+4}$$

$$\frac{r^2+r^2\sin 2\theta}{y+4}$$

Therefore, the integral in cylindrical coordinates is

$$\int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{-0.5y+1}\frac{r^2+r^2\sin 2\theta}{y+4}\cdot r\ dr\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\int_{0}^{-0.5y+1}\frac{r^3(1+\sin 2\theta)}{y+4}\ dr\ d\theta\ dy$$

Integrate with respect to r.

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{r^4(1+\sin 2\theta)}{4(y+4)}\Big|_{r=0}^{r=-0.5y+1}\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{(-0.5y+1)^4(1+\sin 2\theta)}{4(y+4)}-\frac{0^4(1+\sin 2\theta)}{4(y+4)}\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{(-0.5y+1)^4(1+\sin 2\theta)}{4(y+4)}\ d\theta\ dy$$

$$\int_{-2}^{2}\int_{0}^{2\pi}\frac{(-0.5y+1)^4}{4(y+4)}(1+\sin 2\theta)\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(\theta-\frac{1}{2}\cos(2\theta)\right)\Bigg|_{\theta=0}^{\theta=2\pi}\ dy$$

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(2\pi-\frac{1}{2}\cos(2(2\pi))\right)-\frac{(-0.5y+1)^4}{4(y+4)}\left(0-\frac{1}{2}\cos(2(0))\right)\ dy$$

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(2\pi-\frac{1}{2}\right)+\frac{(-0.5y+1)^4}{8(y+4)}\ dy$$

$$\int_{-2}^{2}\frac{(-0.5y+1)^4}{4(y+4)}\left(2\pi-\frac{1}{2}+\frac{1}{2}\right)\ dy$$

$$\int_{-2}^{2}\frac{\pi(-0.5y+1)^4}{2(y+4)}\ dy$$

Simplify the function.

$$\int_{-2}^{2}\frac{\pi\left(\frac{1}{16}y^4-\frac{2}{4}y^3+\frac{3}{2}y^2-2y+1\right)}{2y+8}\ dy$$

$$\int_{-2}^{2}\frac{\frac{\pi}{16}(y^4-8y^3+48y^2-32y+16)}{2(y+4)}\ dy$$

$$\frac{\pi}{32}\int_{-2}^{2}\frac{y^4-8y^3+48y^2-32y+16}{y+4}\ dy$$

Use long division of polynomials.

$$\frac{\pi}{32}\int_{-2}^{2} y^3-12y^2+96y-416+\frac{1{,}680}{y+4}\ dy$$

Integrate with respect to y.

$$\frac{\pi}{32}\left(\frac{1}{4}y^4-4y^3+48y^2-416y+1{,}680\ln(y+4)\right)\Big|_{-2}^{2}$$

$$\frac{\pi}{32}\left(\frac{1}{4}(2)^4-4(2)^3+48(2)^2-416(2)+1{,}680\ln(2+4)\right)$$

$$-\frac{\pi}{32}\left(\frac{1}{4}(-2)^4-4(-2)^3+48(-2)^2-416(-2)+1{,}680\ln(-2+4)\right)$$

$$\frac{\pi}{32}(4-32+192-832+1{,}680\ln 6)-\frac{\pi}{32}(4+32+192+832+1{,}680\ln 2)$$

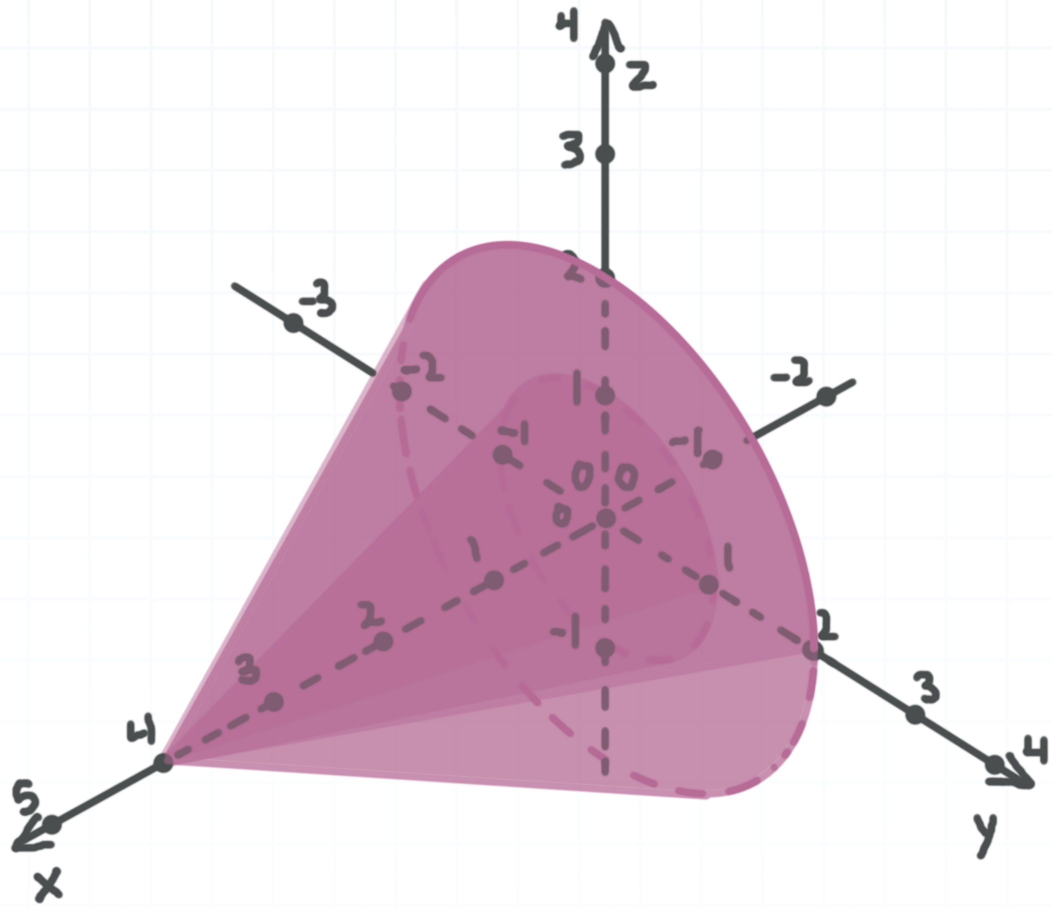
$$\frac{\pi}{32}(4-32+192-832+1{,}680\ln 6-4-32-192-832-1{,}680\ln 2)$$

$$\frac{\pi}{32}(1{,}680\ln 6-1{,}680\ln 2-1{,}728)$$

$$\frac{3\pi}{2}(35\ln 6-35\ln 2-36)$$

■ 6. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the set of points between two right circular cones with radii 1 and 2, vertexes at the point $(4,0,0)$, and bases that lie in the yz-plane with center at the origin.

$$\iiint_E x^2 - y^2 - z^2 \, dV$$



Solution:

The value of x changes from 0 to 4, and y and z change between the circles with centers at the x-axis that lie in the plane parallel to the yz-plane. The value of θ changes from 0 to 2π, and r changes between $r_1(x)$ and $r_2(x)$.

Let's find $r_2 = ax + b$. When $x = 0$, $r = 2$, and when $x = 4$, $r = 0$. Substitute the values into the linear equation and solve the resulting system for a and b.

$$2 = a(0) + b$$

$$0 = a(4) + b$$

$$b = 2 \text{ and } a = -0.5$$

So the upper bound is $r_2 = -0.5x + 2$. Let's find $r_1 = ax + b$. When $x = 0$, $r = 1$, and when $x = 4$, $r = 0$. Substitute the values into the linear equation and solve the resulting system for a and b.

$$1 = a(0) + b$$

$$0 = a(4) + b$$

$$b = 1 \text{ and } a = -0.25$$

So lower bound is $r_1 = -0.25x + 1$, and the integral in cylindrical coordinates is

$$\int_0^4 \int_0^{2\pi} \int_{-0.25x+1}^{-0.5x+2} (x^2 - r^2) \cdot r \ dr \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \int_{-0.25x+1}^{-0.5x+2} x^2 r - r^3 \ dr \ d\theta \ dx$$

Integrate with respect to r.

$$\int_0^4 \int_0^{2\pi} \frac{1}{2}x^2 r^2 - \frac{1}{4}r^4 \Big|_{r=-0.25x+1}^{r=-0.5x+2} \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \frac{(-0.5x+2)^2 x^2}{2} - \frac{(-0.5x+2)^4}{4}$$

$$-\frac{x^2(-0.25x+1)^2}{2} + \frac{(-0.25x+1)^4}{4} \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \frac{(-0.5x+2)^2x^2 - x^2(-0.25x+1)^2}{2} + \frac{(-0.25x+1)^4 - (-0.5x+2)^4}{4} \ d\theta \ dx$$

$$\int_0^4 \int_0^{2\pi} \frac{81}{1{,}024}x^4 - \frac{33}{64}x^3 + \frac{3}{32}x^2 + \frac{15}{4}x - \frac{15}{4} \ d\theta \ dx$$

Integrate with respect to θ.

$$\int_0^4 \frac{81}{1{,}024}x^4\theta - \frac{33}{64}x^3\theta + \frac{3}{32}x^2\theta + \frac{15}{4}x\theta - \frac{15}{4}\theta \Big|_{\theta=0}^{\theta=2\pi} \ dx$$

$$\int_0^4 \frac{81}{1{,}024}x^4(2\pi) - \frac{33}{64}x^3(2\pi) + \frac{3}{32}x^2(2\pi) + \frac{15}{4}x(2\pi) - \frac{15}{4}(2\pi) \ dx$$

$$\int_0^4 \frac{81\pi}{512}x^4 - \frac{33\pi}{32}x^3 + \frac{3\pi}{16}x^2 + \frac{15\pi}{2}x - \frac{15\pi}{2} \ dx$$

Integrate with respect to x.

$$\frac{81\pi}{2{,}560}x^5 - \frac{33\pi}{128}x^4 + \frac{3\pi}{48}x^3 + \frac{15\pi}{4}x^2 - \frac{15\pi}{2}x \Big|_0^4$$

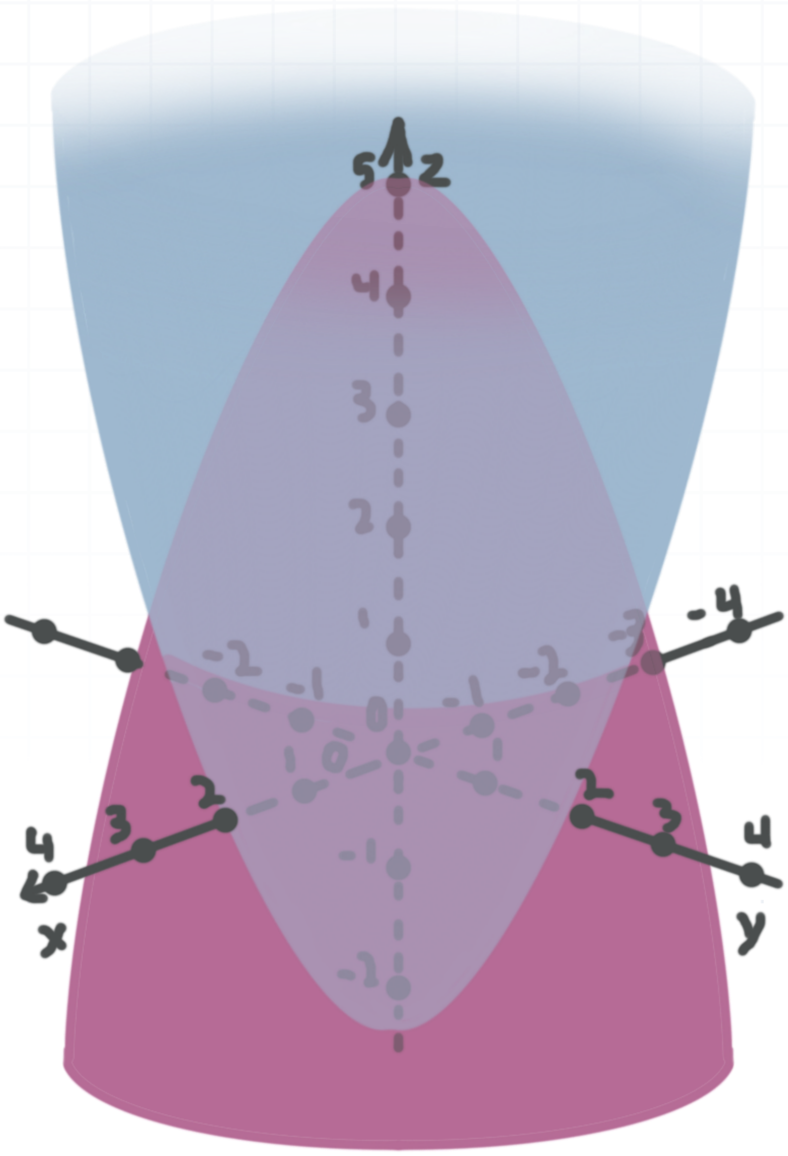
$$\frac{81\pi}{2{,}560}(4)^5 - \frac{33\pi}{128}(4)^4 + \frac{3\pi}{48}(4)^3 + \frac{15\pi}{4}(4)^2 - \frac{15\pi}{2}(4)$$

$$\frac{162\pi}{5} - 66\pi + 4\pi + 60\pi - 30\pi$$

$$\frac{2\pi}{5}$$

■ 7. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the solid bounded by the surfaces $x^2 + y^2 + z - 5 = 0$ and $x^2 + y^2 - z - 3 = 0$.

$$\iiint_E 3\sqrt{x^2 + y^2}\, dV$$



Solution:

The given surfaces are circular paraboloids, concave up and down, with vertexes at the points $(0,0,5)$ and $(0,0,-3)$, respectively.

To find the set of points where the paraboloids intersect, solve their equations as a system.

$$x^2 + y^2 + z - 5 = 0$$

$$x^2 + y^2 - z - 3 = 0$$

The curve of intersection of these two paraboloids are the circle with center $(0,0,1)$ and radius 2, that lies in the plane $z = 1$.

The value of z changes from -3 to 5, and θ changes from 0 to 2π. Since r has different upper bounds for $z \le 1$ and $z \ge 1$, we need to split the triple integral into the two iterated integrals.

For z from -3 to 1, we get

$$x^2 + y^2 - z - 3 = 0$$

$$r^2 - z - 3 = 0$$

$$r = \sqrt{3 + z}$$

And for z from 1 to 5, we get

$$x^2 + y^2 + z - 5 = 0$$

$$r^2 + z - 5 = 0$$

$$r = \sqrt{5 - z}$$

Therefore, the integral in cylindrical coordinates is the sum of the two iterated integrals.

$$\int_{-3}^{1}\int_{0}^{2\pi}\int_{0}^{\sqrt{3+z}} 3r\cdot r\ dr\ d\theta\ dz+\int_{1}^{5}\int_{0}^{2\pi}\int_{0}^{\sqrt{5-z}} 3r\cdot r\ dr\ d\theta\ dz$$

$$\int_{-3}^{1}\int_{0}^{2\pi}\int_{0}^{\sqrt{3+z}} 3r^2\ dr\ d\theta\ dz+\int_{1}^{5}\int_{0}^{2\pi}\int_{0}^{\sqrt{5-z}} 3r^2\ dr\ d\theta\ dz$$

Integrate with respect to r.

$$\int_{-3}^{1}\int_{0}^{2\pi}(\sqrt{3+z})^3\ d\theta\ dz+\int_{1}^{5}\int_{0}^{2\pi}(\sqrt{5-z})^3\ d\theta\ dz$$

$$\int_{-3}^{1}\int_{0}^{2\pi}(3+z)^{\frac{3}{2}}\ d\theta\ dz+\int_{1}^{5}\int_{0}^{2\pi}(5-z)^{\frac{3}{2}}\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_{-3}^{1}(3+z)^{\frac{3}{2}}\theta\Big|_{\theta=0}^{\theta=2\pi}\ dz+\int_{1}^{5}(5-z)^{\frac{3}{2}}\theta\Big|_{\theta=0}^{\theta=2\pi}\ dz$$

$$\int_{-3}^{1}(3+z)^{\frac{3}{2}}(2\pi)\ dz+\int_{1}^{5}(5-z)^{\frac{3}{2}}(2\pi)\ dz$$

$$\int_{-3}^{1}2\pi(3+z)^{\frac{3}{2}}\ dz+\int_{1}^{5}2\pi(5-z)^{\frac{3}{2}}\ dz$$

Integrate with respect to z.

$$\frac{4\pi}{5}(3+z)^{\frac{5}{2}}\Big|_{-3}^{1}-\frac{4\pi}{5}(5-z)^{\frac{5}{2}}\Big|_{1}^{5}$$

$$\frac{4\pi}{5}(3+1)^{\frac{5}{2}}-\frac{4\pi}{5}(3-3)^{\frac{5}{2}}-\left(\frac{4\pi}{5}(5-5)^{\frac{5}{2}}-\frac{4\pi}{5}(5-1)^{\frac{5}{2}}\right)$$

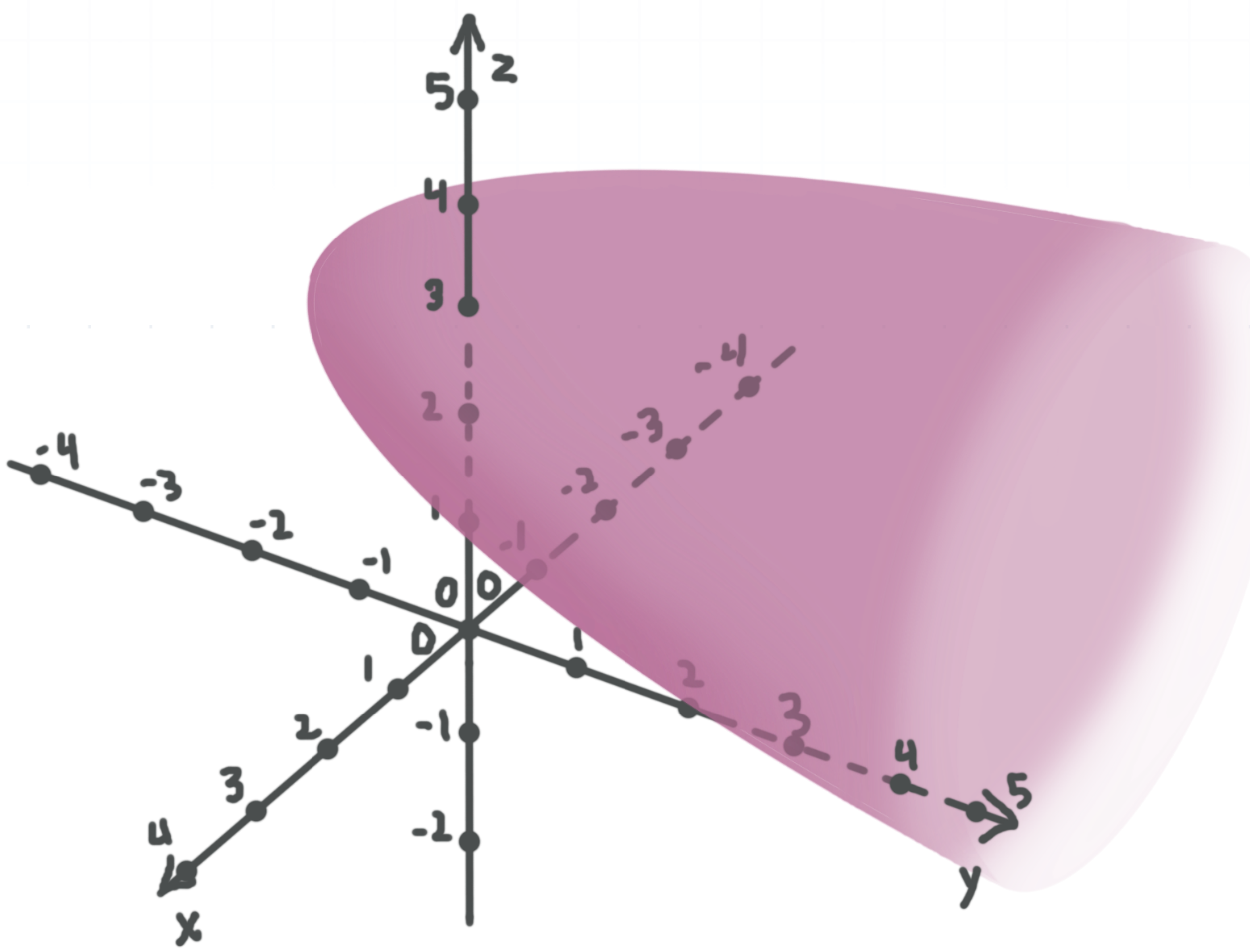
$$\frac{4\pi}{5}(4)^{\frac{5}{2}}+\frac{4\pi}{5}(4)^{\frac{5}{2}}$$

$$\frac{128\pi}{5}+\frac{128\pi}{5}$$

$$\frac{256\pi}{5}$$

■ 8. Evaluate the triple improper integral by changing it to cylindrical coordinates, where E is the interior of the surface $(x+1)^2+(z-2)^2=y+2$.

$$\iiint_E xz10^{-y}\ dV$$



Solution:

The surface is a circular paraboloid with vertex $(-1,-2,2)$, and an axis parallel to the y-axis. Use conversion formulas to move the vertex to the origin.

$$x_1 = x + 1 \text{ so } x = x_1 - 1$$

$$y_1 = y + 2 \text{ so } y = y_1 - 2$$

$$z_1 = z - 2 \text{ so } z = z_1 + 2$$

The function transforms to

$$xz10^{-y} = (x_1 - 1)(z_1 + 2)10^{-y_1+2}$$

$$xz10^{-y} = 100(x_1 - 1)(z_1 + 2)10^{-y_1}$$

The surface transforms to

$$x_1^2 + z_1^2 = y_1$$

Therefore, the given triple integral is equal to

$$\iiint_{E_1} 100(x-1)(z+2)10^{-y}\ dV$$

Inside the surface $x^2 + z^2 = y$, the value of y changes from 0 to ∞, x and z change within the circle C with center at the y-axis, that lies in the plane parallel to the xz-plane. The value of θ changes from 0 to 2π, and since the upper bound is $r^2 = y$, r changes from 0 to $\sqrt{y}$. The function is

$$100(x-1)(z+2)10^{-y}$$

$$100(r\sin\theta-1)(r\cos\theta+2)10^{-y}$$

$$100(r^2\sin\theta\cos\theta+2r\sin\theta-r\cos\theta-2)10^{-y}$$

$$(50r^2\sin 2\theta+200r\sin\theta-100r\cos\theta-200)10^{-y}$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{\sqrt{y}}(50r^2\sin 2\theta+200r\sin\theta-100r\cos\theta-200)10^{-y}\cdot r\ dr\ d\theta\ dy$$

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{\sqrt{y}}(50r^3\sin 2\theta+200r^2\sin\theta-100r^2\cos\theta-200r)10^{-y}\ dr\ d\theta\ dy$$

Since the integrals of $\cos\theta$, $\sin\theta$, and $\sin 2\theta$ over $[0,2\pi]$ are 0, the integral simplifies to

$$\int_0^{\infty}\int_0^{2\pi}\int_0^{\sqrt{y}}-200(10^{-y})r\ dr\ d\theta\ dy$$

Integrate with respect to r.

$$\int_0^{\infty}\int_0^{2\pi}-100(10^{-y})r^2\Big|_{r=0}^{r=\sqrt{y}}\ d\theta\ dy$$

$$\int_0^{\infty}\int_0^{2\pi}-100(10^{-y})(\sqrt{y})^2+100(10^{-y})(0)^2\ d\theta\ dy$$

$$\int_0^{\infty}\int_0^{2\pi}-100y(10^{-y})\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_0^\infty -100y(10^{-y})\theta \Big|_{\theta=0}^{\theta=2\pi} \ dy$$

$$\int_0^\infty -100y(10^{-y})(2\pi) + 100y(10^{-y})(0) \ dy$$

$$\int_0^\infty -200\pi y(10^{-y}) \ dy$$

Integrate with respect to y using integration by parts with $u = y$, $du = dy$, $dv = 10^{-y} \ dy$, and $v = -(10^{-y})/(\ln 10)$.

$$-200\pi \left[-\frac{y10^{-y}}{\ln 10} \Big|_0^\infty + \frac{1}{\ln 10} \int_0^\infty 10^{-y} \ dy \right]$$

$$-200\pi \left[-\frac{y10^{-y}}{\ln 10} - \frac{10^{-y}}{\ln^2 10} \Big|_0^\infty \right]$$

$$200\pi \left[\frac{y10^{-y}}{\ln 10} + \frac{10^{-y}}{\ln^2 10} \Big|_0^\infty \right]$$

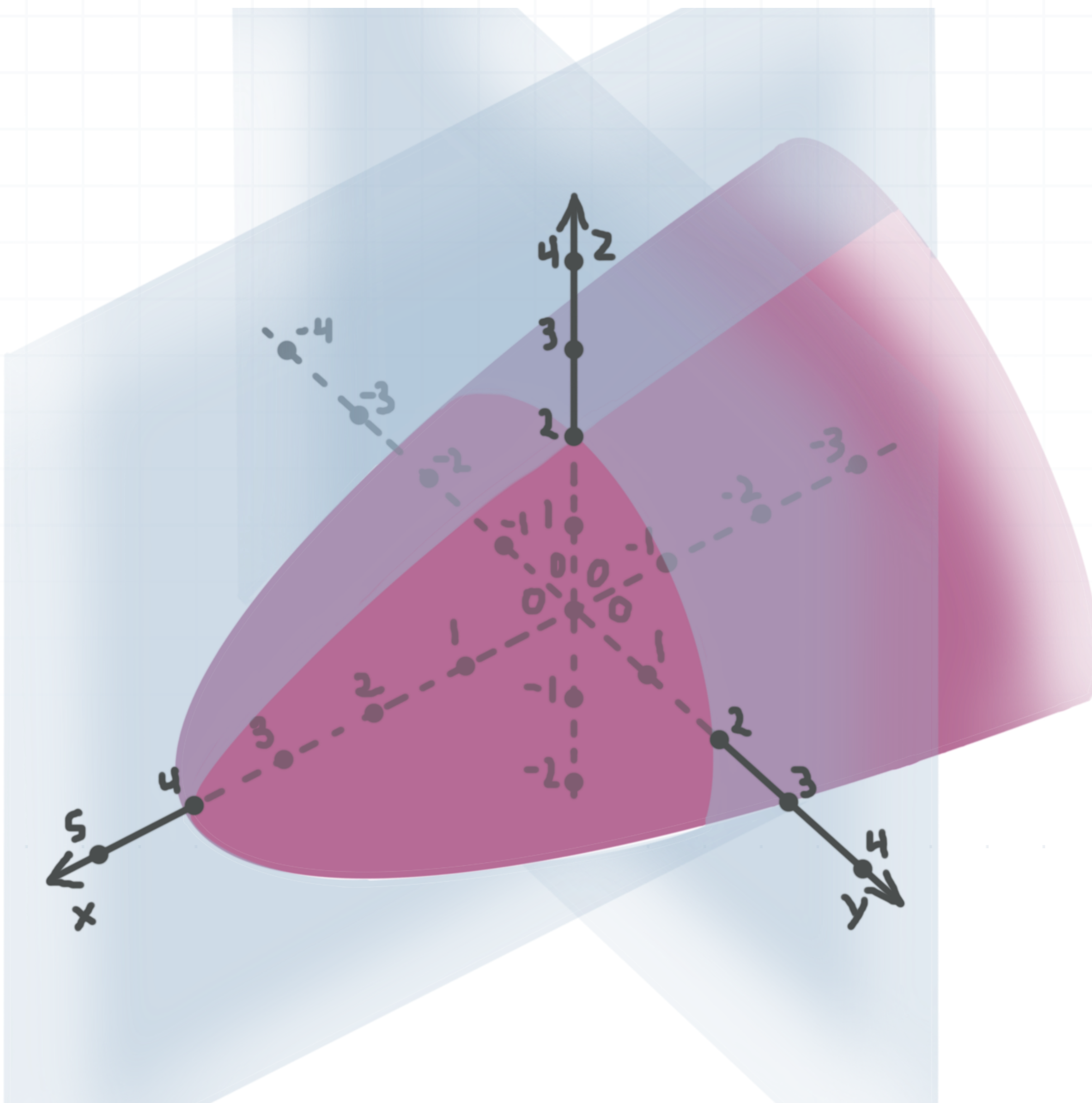
$$200\pi \lim_{t\to\infty} \left[\frac{t10^{-t}}{\ln 10} + \frac{10^{-t}}{\ln^2 10} \right] - 200\pi \left[\frac{(0)10^{-0}}{\ln 10} + \frac{10^{-0}}{\ln^2 10} \right]$$

$$0 - 200\pi \cdot \frac{10^0}{\ln^2 10}$$

$$-\frac{200\pi}{\ln^2 10}$$

■ 9. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the interior of the surface $y^2 + z^2 + x - 4 = 0$ that lies within the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$).

$$\iiint_E x + 8yz \ dV$$



Solution:

The surface is the circular paraboloid with vertex $(4,0,0)$ whose axis is the x-axis. The value of x changes from 0 to 4, y and z change within the circle C with center at the x-axis that lies in the plane parallel to the yz-plane. The value of θ changes from 0 to $\pi/2$, and since the upper bound is $r^2+x-4=0$, r changes from 0 to $\sqrt{4-x}$. The function is

$$x+8yz$$

$$x+8(r\cos\theta)(r\sin\theta)$$

$$x+8r^2\cos\theta\sin\theta$$

$$x+4r^2\sin 2\theta$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^4\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{4-x}}(x+4r^2\sin 2\theta)\cdot r\ dr\ d\theta\ dx$$

$$\int_0^4\int_0^{\frac{\pi}{2}}\int_0^{\sqrt{4-x}}xr+4r^3\sin 2\theta\ dr\ d\theta\ dx$$

Integrate with respect to r.

$$\int_0^4\int_0^{\frac{\pi}{2}}\frac{1}{2}xr^2+r^4\sin 2\theta\Big|_{r=0}^{r=\sqrt{4-x}}\ d\theta\ dx$$

$$\int_0^4\int_0^{\frac{\pi}{2}}\frac{1}{2}x(\sqrt{4-x})^2+(\sqrt{4-x})^4\sin 2\theta-\left(\frac{1}{2}x(0)^2+0^4\sin 2\theta\right)\ d\theta\ dx$$

$$\int_0^4\int_0^{\frac{\pi}{2}}\frac{1}{2}x(4-x)+(4-x)^2\sin 2\theta\ d\theta\ dx$$

Integrate with respect to θ.

$$\int_0^4 \frac{1}{2}x(4-x)\theta - \frac{1}{2}(4-x)^2\cos 2\theta \Big|_{\theta=0}^{\theta=\frac{\pi}{2}} \ dx$$

$$\int_0^4 \frac{\pi x(4-x)}{4} + (4-x)^2 \ dx$$

$$\int_0^4 (4-x)\left(\frac{\pi x}{4} + 4 - x\right) \ dx$$

$$\int_0^4 \pi x + 16 - 4x - \frac{\pi x^2}{4} - 4x + x^2 \ dx$$

$$\int_0^4 \pi x + 16 - 8x - \frac{\pi x^2}{4} + x^2 \ dx$$

Integrate with respect to x.

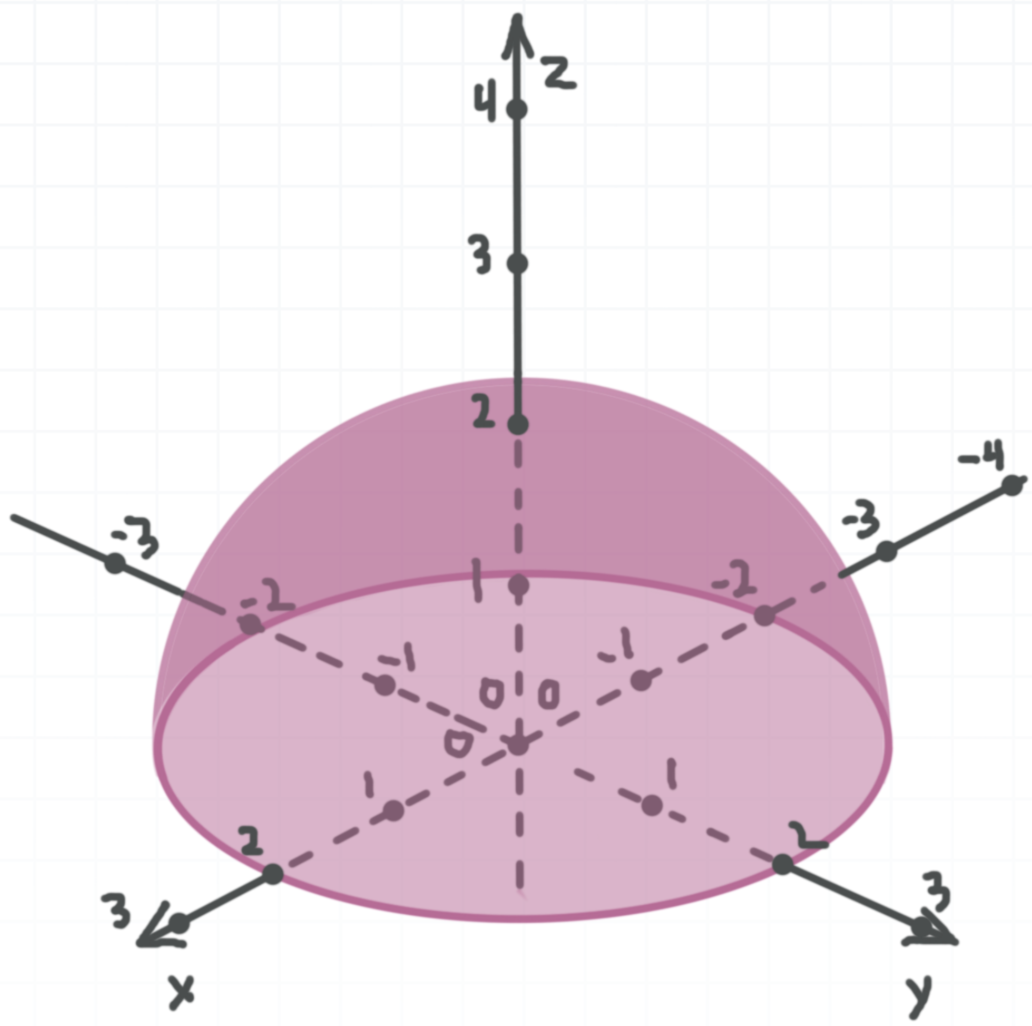
$$\frac{\pi x^2}{2} + 16x - 4x^2 - \frac{\pi x^3}{12} + \frac{1}{3}x^3 \Big|_0^4$$

$$\frac{\pi(4)^2}{2} + 16(4) - 4(4)^2 - \frac{\pi(4)^3}{12} + \frac{1}{3}(4)^3$$

$$\frac{8\pi + 64}{3}$$

■ 10. Evaluate the triple integral by changing it to cylindrical coordinates, where E is the hemisphere with center at the origin, radius 2, and $z \geq 0$.

$$\iiint_E 4x^2 + 4y^2 - 12z^2 \ dV$$



Solution:

The equation of the sphere with center at the origin and radius 2 is

$$x^2 + y^2 + z^2 = 4$$

The value of z changes from 0 to 2, x and y change within the circle C with center at the z-axis, that lies in the plane parallel to the xy-plane. The value of θ changes from 0 to $\pi/2$, and since the upper bound is $r^2 + z^2 = 4$, r changes from 0 to $\sqrt{4 - z^2}$. The function is

$$4x^2 + 4y^2 - 12z^2 = 4r^2 - 12z^2$$

Therefore, the integral in cylindrical coordinates is

$$\int_0^2 \int_0^{2\pi} \int_0^{\sqrt{4-z^2}} (4r^2 - 12z^2) \cdot r \ dr \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} \int_0^{\sqrt{4-z^2}} 4r^3 - 12z^2 r \ dr \ d\theta \ dz$$

Integrate with respect to r.

$$\int_0^2 \int_0^{2\pi} r^4 - 6z^2r^2 \Big|_{r=0}^{r=\sqrt{4-z^2}} \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} 16 - 8z^2 + z^4 - 6z^2(4 - z^2) \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} 16 - 8z^2 + z^4 - 24z^2 + 6z^4 \ d\theta \ dz$$

$$\int_0^2 \int_0^{2\pi} 7z^4 - 32z^2 + 16 \ d\theta \ dz$$

Integrate with respect to θ.

$$\int_0^2 7z^4\theta - 32z^2\theta + 16\theta \Big|_{\theta=0}^{\theta=2\pi} \ dz$$

$$\int_0^2 7z^4(2\pi) - 32z^2(2\pi) + 16(2\pi) - (7z^4(0) - 32z^2(0) + 16(0)) \ dz$$

$$\int_0^2 14\pi z^4 - 64\pi z^2 + 32\pi \ dz$$

Integrate with respect to θ.

$$\frac{14\pi}{5}z^5 - \frac{64\pi}{3}z^3 + 32\pi z \Big|_0^2$$

$$\frac{14\pi}{5}(2)^5 - \frac{64\pi}{3}(2)^3 + 32\pi(2) - \left(\frac{14\pi}{5}(0)^5 - \frac{64\pi}{3}(0)^3 + 32\pi(0)\right)$$

$$\frac{448\pi}{5} - \frac{512\pi}{3} + 64\pi$$

$$\frac{1{,}344\pi}{15} - \frac{2{,}560\pi}{15} + \frac{960\pi}{15}$$

$$-\frac{256\pi}{15}$$

FINDING VOLUME

1. Evaluate the integral.

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} 4rz\ dz\ dr\ d\theta$$

Solution:

Integrate first with respect to z, then evaluate over the interval.

$$\int_0^{2\pi}\int_0^1\int_0^{\sqrt{4-r^2}} 4rz\ dz\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 2rz^2\Big|_{z=0}^{z=\sqrt{4-r^2}}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 2r(\sqrt{4-r^2})^2-2r(0)^2\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 2r(4-r^2)\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 8r-2r^3\ dr\ d\theta$$

Then integrate with respect to r, and evaluate over the interval.

$$\int_0^{2\pi} 4r^2 - \frac{1}{2}r^4 \Big|_{r=0}^{r=1} \ d\theta$$

$$\int_0^{2\pi} 4(1)^2 - \frac{1}{2}(1)^4 - \left(4(0)^2 - \frac{1}{2}(0)^4\right) \ d\theta$$

$$\int_0^{2\pi} \frac{7}{2} \ d\theta$$

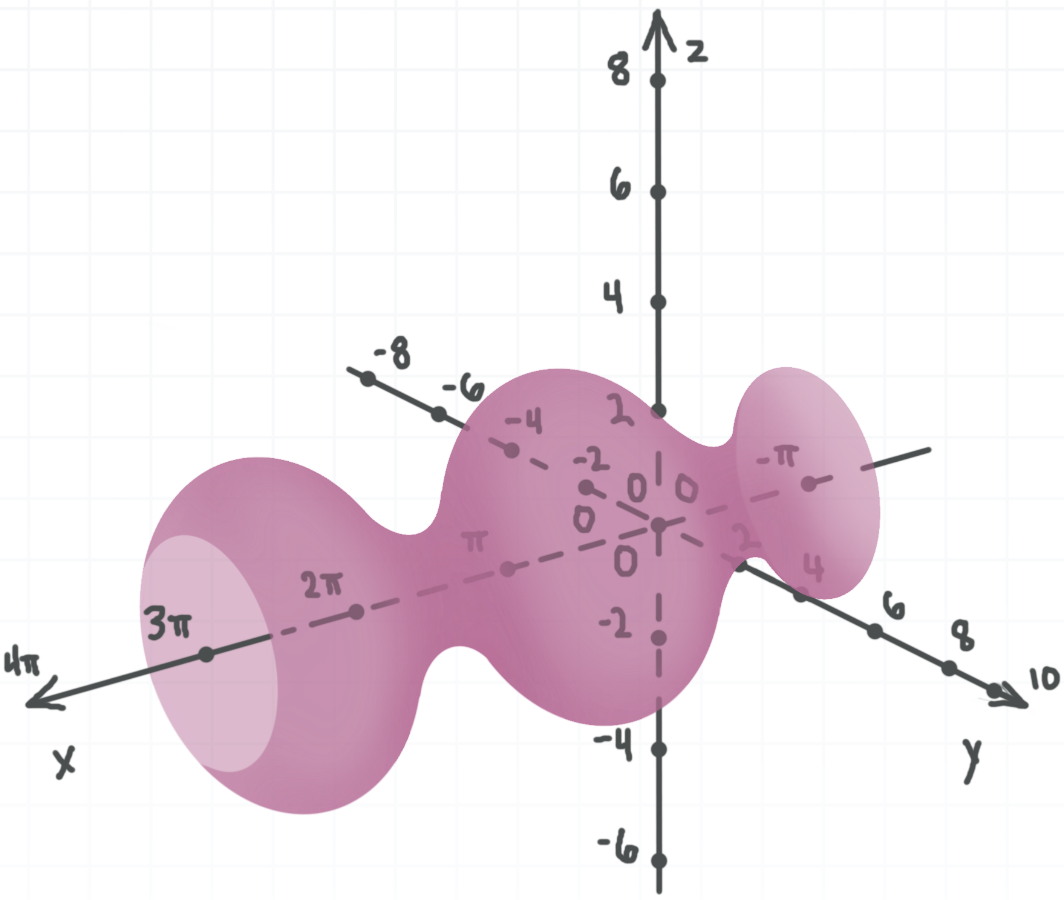
Then integrate with respect to θ, and evaluate over the interval.

$$\frac{7}{2}\theta \Big|_0^{2\pi}$$

$$\frac{7}{2}(2\pi) - \frac{7}{2}(0)$$

$$7\pi$$

■ 2. Use a triple integral in cylindrical coordinates to find the volume of the solid E, where E is the set of points within the surface of revolution created by rotating the curve $z = 2 + \sin x$ around the x-axis, and bounded by the planes $x = -\pi$ and $x = 3\pi$.



Solution:

The value of x changes from $-\pi$ to 3π, y and z change within the circle C with center at the x-axis, that lies in the plane parallel to the yz-plane. The value of θ changes from 0 to 2π, and since the upper bound is $z = 2 + \sin x$, r changes from 0 to $2 + \sin x$.

Then the integral in cylindrical coordinates is

$$\int_{-\pi}^{3\pi} \int_{0}^{2\pi} \int_{0}^{2+\sin x} r \ dr \ d\theta \ dx$$

Integrate with respect to r.

$$\int_{-\pi}^{3\pi} \int_{0}^{2\pi} \frac{1}{2} r^2 \Big|_{0}^{2+\sin x} \ d\theta \ dx$$

$$\int_{-\pi}^{3\pi}\int_{0}^{2\pi}\frac{1}{2}(2+\sin x)^2\ d\theta\ dx$$

$$\int_{-\pi}^{3\pi}\int_{0}^{2\pi}2+2\sin x+\frac{1}{2}\sin^2 x\ d\theta\ dx$$

Integrate with respect to θ.

$$\int_{-\pi}^{3\pi}2\theta+2\theta\sin x+\frac{1}{2}\theta\sin^2 x\Big|_{\theta=0}^{\theta=2\pi}\ dx$$

$$\int_{-\pi}^{3\pi}2(2\pi)+2(2\pi)\sin x+\frac{1}{2}(2\pi)\sin^2 x\ dx$$

$$\int_{-\pi}^{3\pi}4\pi+4\pi\sin x+\pi\sin^2 x\ dx$$

Integrate with respect to x.

$$4\pi x-4\pi\cos x+\frac{\pi\sin^3 x}{3\cos x}\Big|_{-\pi}^{3\pi}$$

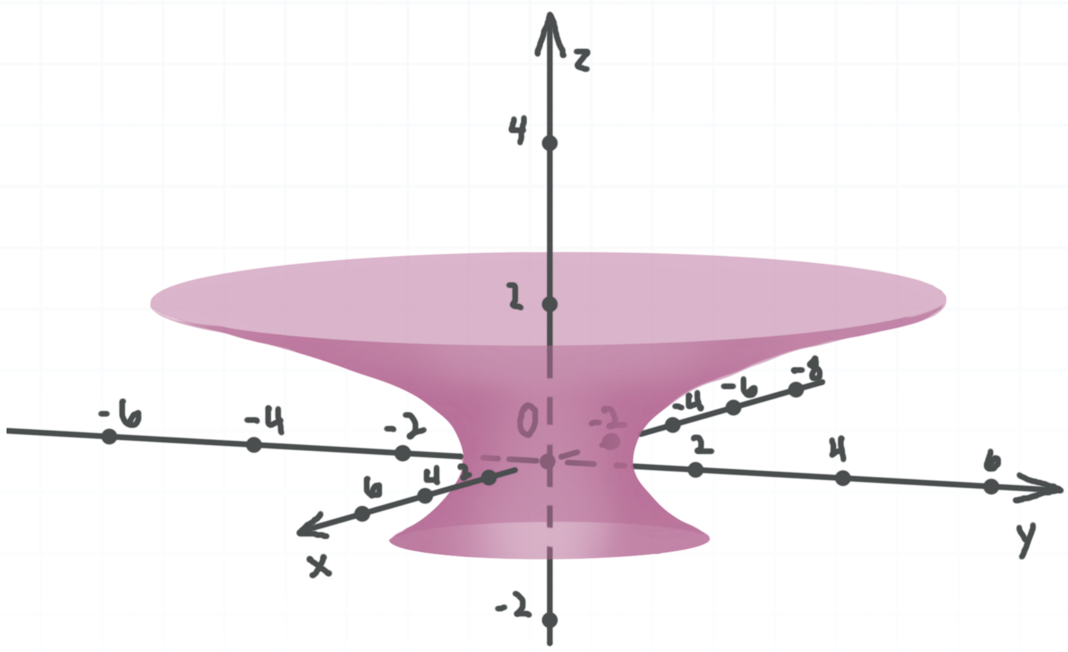
$$4\pi(3\pi)-4\pi\cos(3\pi)+\frac{\pi\sin^3(3\pi)}{3\cos(3\pi)}-\left(4\pi(-\pi)-4\pi\cos(-\pi)+\frac{\pi\sin^3(-\pi)}{3\cos(-\pi)}\right)$$

$$12\pi^2-4\pi(-1)+\frac{\pi(0)}{3(-1)}-\left(-4\pi^2-4\pi(-1)+\frac{\pi(0)}{3(-1)}\right)$$

$$12\pi^2+4\pi+4\pi^2-4\pi$$

$$16\pi^2$$

3. Use a triple integral in cylindrical coordinates to find the volume of the solid E, where E is the set of points within the surface of revolution created by rotating the curve $x = z^2 + 1$ around the z-axis, and bounded by the planes $z = -1$ and $z = 2$.



Solution:

The value of z changes from -1 to 2, x and y change within the circle C with center at the z-axis, that lies in the plane parallel to the xy-plane. The value of θ changes from 0 to 2π, and since the upper bound is $x = z^2 + 1$, r changes from 0 to $z^2 + 1$. Therefore, the integral in cylindrical coordinates is

$$\int_{-1}^{2} \int_{0}^{2\pi} \int_{0}^{z^2+1} r \ dr \ d\theta \ dz$$

Integrate with respect to r.

$$\int_{-1}^{2}\int_{0}^{2\pi}\frac{1}{2}r^2\Big|_0^{z^2+1}\ d\theta\ dz$$

$$\int_{-1}^{2}\int_{0}^{2\pi}\frac{1}{2}(z^2+1)^2\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_{-1}^{2}\frac{1}{2}(z^2+1)^2\theta\Big|_{\theta=0}^{\theta=2\pi}\ dz$$

$$\int_{-1}^{2}\frac{1}{2}(z^2+1)^2(2\pi)\ dz$$

$$\int_{-1}^{2}\pi(z^2+1)^2\ dz$$

$$\int_{-1}^{2}\pi z^4+2\pi z^2+\pi\ dz$$

Integrate with respect to z.

$$\frac{1}{5}\pi z^5+\frac{2}{3}\pi z^3+\pi z\Big|_{-1}^{2}$$

$$\frac{1}{5}\pi(2)^5+\frac{2}{3}\pi(2)^3+\pi(2)-\left(\frac{1}{5}\pi(-1)^5+\frac{2}{3}\pi(-1)^3+\pi(-1)\right)$$

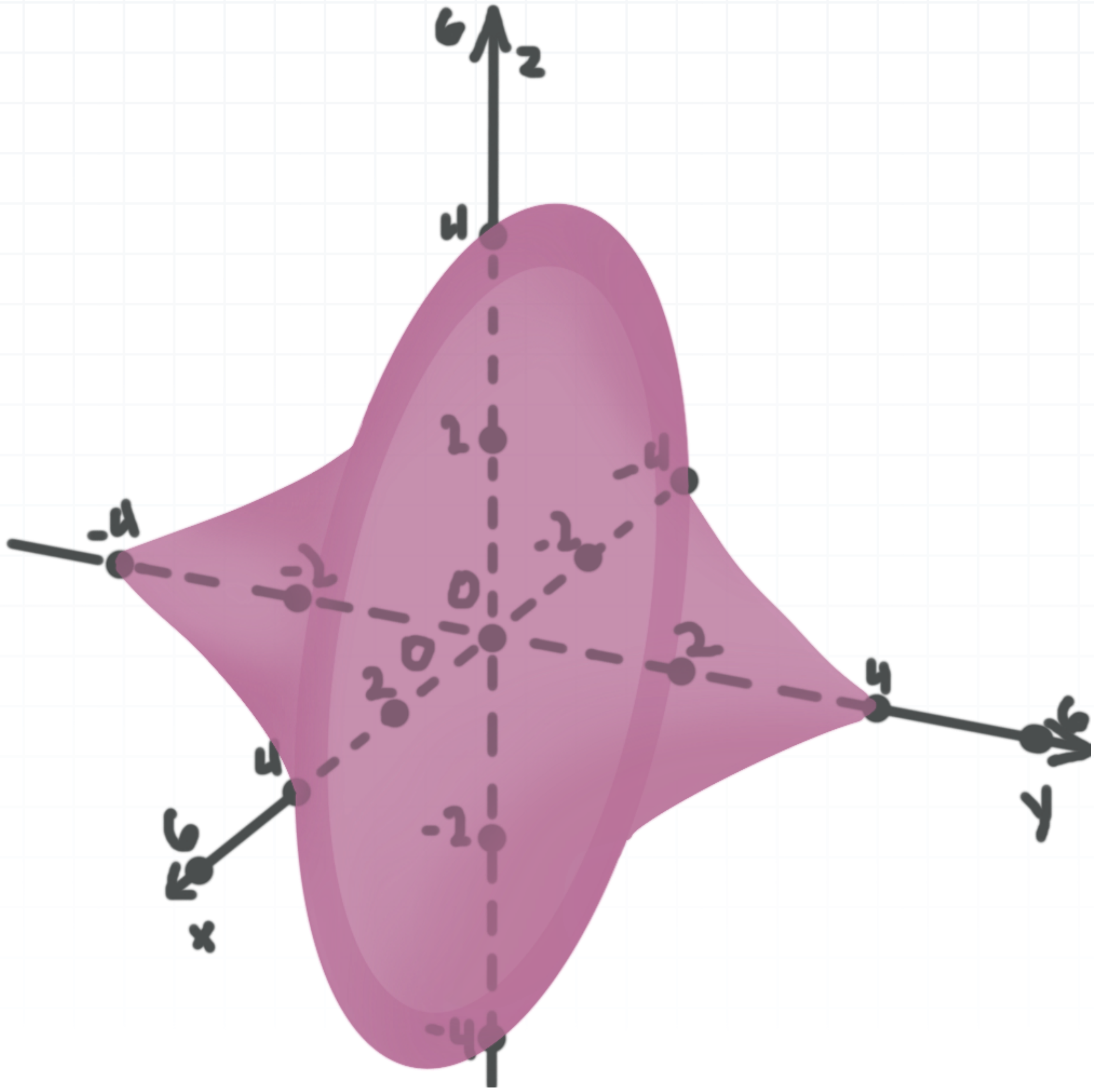
$$\frac{32}{5}\pi+\frac{16}{3}\pi+2\pi-\left(-\frac{1}{5}\pi-\frac{2}{3}\pi-\pi\right)$$

$$\frac{32}{5}\pi + \frac{16}{3}\pi + 2\pi + \frac{1}{5}\pi + \frac{2}{3}\pi + \pi$$

$$\frac{99}{15}\pi + \frac{90}{15}\pi + \frac{45}{15}\pi$$

$$\frac{78}{5}\pi$$

■ 4. Use a triple integral in cylindrical coordinates to find the volume of the solid E, where E is the set of the points within the surface of revolution created by rotating the curve $x = 4 - 2\sqrt{|y|}$ around the y-axis, and bounded by the planes $y = -4$ and $y = 4$.



Solution:

The value of y changes from -4 to 4, x and z change within the circle C with center at the y-axis, that lies in the plane parallel to the xz-plane. The value of θ changes from 0 to 2π, and since the upper bound is $x = 4 - 2\sqrt{|y|}$, r changes from 0 to $4 - 2\sqrt{|y|}$. Therefore, the integral in cylindrical coordinates is

$$\int_{-4}^{4}\int_{0}^{2\pi}\int_{0}^{4-2\sqrt{|y|}} r\ dr\ d\theta\ dy$$

Since the volumes are equal for y from -4 to 0, and from 0 to 4, the volume integral can be simplified to

$$\int_{0}^{4}\int_{0}^{2\pi}\int_{0}^{4-2\sqrt{y}} 2r\ dr\ d\theta\ dy$$

Integrate with respect to r.

$$\int_{0}^{4}\int_{0}^{2\pi} r^2\Big|_{0}^{4-2\sqrt{y}}\ d\theta\ dy$$

$$\int_{0}^{4}\int_{0}^{2\pi} (4-2\sqrt{y})^2\ d\theta\ dy$$

Integrate with respect to θ.

$$\int_{0}^{4} (4-2\sqrt{y})^2\theta\Big|_{\theta=0}^{\theta=2\pi}\ dy$$

$$\int_{0}^{4} 2\pi(4-2\sqrt{y})^2\ dy$$

$$\int_{0}^{4} 2\pi(16-16\sqrt{y}+4y)\ dy$$

$$8\pi\int_{0}^{4} 4-4\sqrt{y}+y\ dy$$

Integrate with respect to y.

$$8\pi\left(4y-\frac{8}{3}y^{\frac{3}{2}}+\frac{1}{2}y^2\right)\Big|_0^4$$

$$8\pi\left(4(4)-\frac{8}{3}(4)^{\frac{3}{2}}+\frac{1}{2}(4)^2\right)$$

$$8\pi\left(16-\frac{64}{3}+8\right)$$

$$64\pi\left(\frac{9}{3}-\frac{8}{3}\right)$$

$$\frac{64\pi}{3}$$

SPHERICAL COORDINATES

1. Evaluate the triple integral given in the spherical coordinates, where $f(\rho, \theta, \phi) = 2\rho \sin\theta \cos\phi$.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_1^5 f(\rho, \theta, \phi)\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

Solution:

Set up the integral.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_1^5 (2\rho \sin\theta \cos\phi)\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \int_1^5 2\rho^3 \sin\theta \cos\phi \sin\phi\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \frac{1}{2}\rho^4 \sin\theta \cos\phi \sin\phi \Big|_{\rho=1}^{\rho=5}\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\frac{\pi}{3}} \frac{1}{2}(5)^4 \sin\theta \cos\phi \sin\phi - \frac{1}{2}(1)^4 \sin\theta \cos\phi \sin\phi\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{3}}\frac{625}{2}\sin\theta\cos\phi\sin\phi-\frac{1}{2}\sin\theta\cos\phi\sin\phi\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_{0}^{\frac{\pi}{3}}312\sin\theta\cos\phi\sin\phi\ d\theta\ d\phi$$

Integrate with respect to θ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}-312\cos\theta\cos\phi\sin\phi\Big|_{\theta=0}^{\theta=\frac{\pi}{3}}\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}-312\cos\frac{\pi}{3}\cos\phi\sin\phi+312\cos(0)\cos\phi\sin\phi\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}-156\cos\phi\sin\phi+312\cos\phi\sin\phi\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}156\cos\phi\sin\phi\ d\phi$$

Integrate with respect to ϕ, using a substitution with $u=\sin\phi$ and $d\phi=du/\cos\phi$, where u changes from $\sqrt{2}/2$ to 1.

$$\int_{\frac{\sqrt{2}}{2}}^{1}156u\cos\phi\left(\frac{du}{\cos\phi}\right)$$

$$\int_{\frac{\sqrt{2}}{2}}^{1}156u\ du$$

$$78u^2\Big|_{\frac{\sqrt{2}}{2}}^{1}$$

$$78(1)^2-78\left(\frac{\sqrt{2}}{2}\right)^2$$

$$78-78\left(\frac{1}{2}\right)$$

$$78-39$$

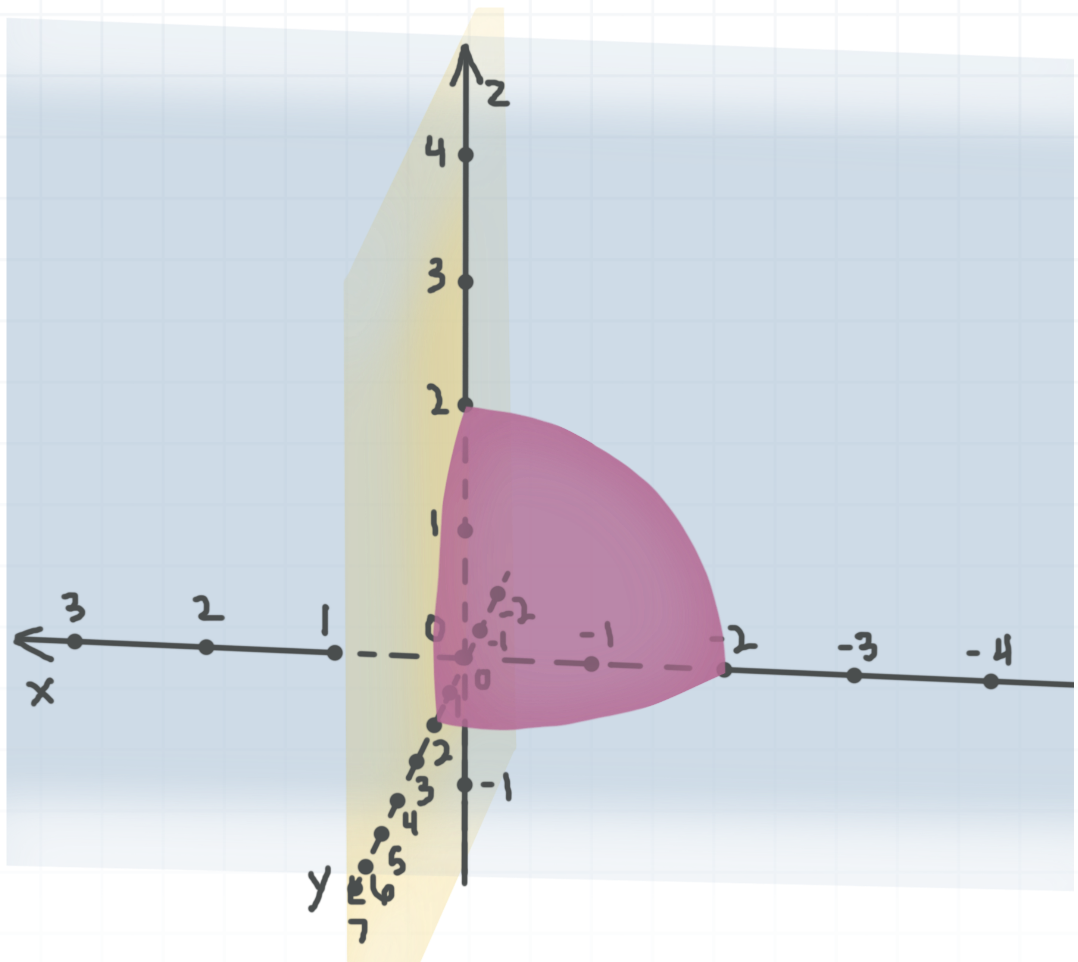
$$39$$

■ 2. Identify the solid given by the iterated integral in spherical coordinates.

$$\int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{2}}^{\pi}\int_0^2 f(\rho,\theta,\phi)\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

Solution:

The value of ρ changes from 0 to 2. So the solid is the part of the sphere with center at the origin and radius 2. The value of θ changes from $\pi/2$ to π and ϕ changes from 0 to $\pi/2$. Therefore, we consider the part of the sphere that lies in the second octant ($x\le 0$, $y\ge 0$).

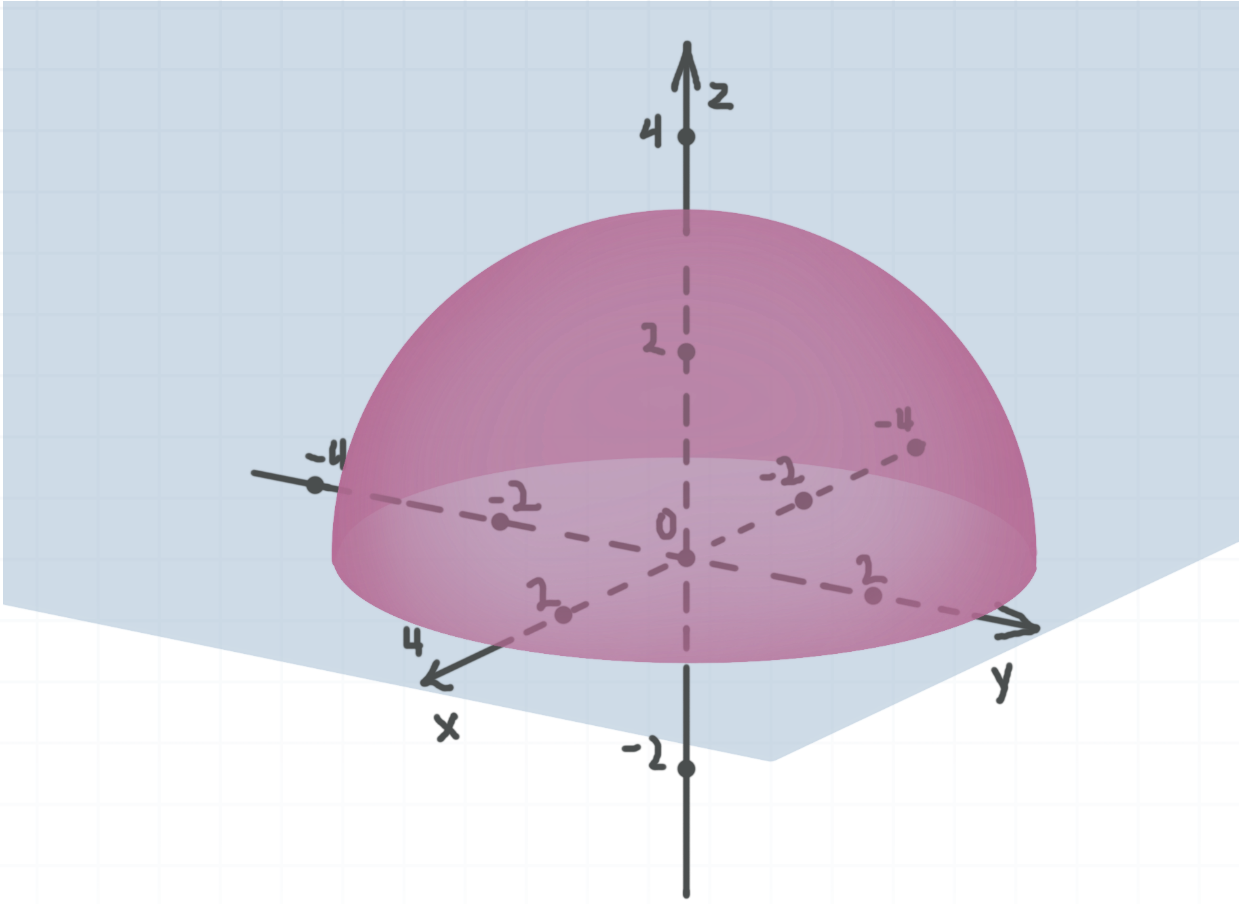


■ 3. Identify the solid given by the iterated improper integral in the spherical coordinates.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\pi} \int_{\pi}^{\infty} f(\rho, \theta, \phi)\ \rho^2 \sin \phi\ d\rho\ d\theta\ d\phi$$

Solution:

The value of ρ changes from π to ∞. So the solid is the set of outer points of the sphere with center at the origin and radius π. The value of θ changes from 0 to 2π and ϕ changes from $-\pi/2$ to $\pi/2$. Therefore, we consider the set of outer points of the sphere for $z \geq 0$.


z
4
2
-4
-2
0
-2
-4
2
2
4
x
y
-2

CHANGING TRIPLE INTEGRALS TO SPHERICAL COORDINATES

1. Evaluate the triple integral by changing it to spherical coordinates, if E is the sphere with center at the origin and radius 3.

$$\iiint_E 5x^2 - 2 \ dV$$

Solution:

Coordinates x, y, and z change within the sphere with center at the origin and radius 3. The value of ρ changes from 0 to 3, the value of θ changes from 0 to 2π, and the value of ϕ changes from 0 to π. The function is

$$5x^2 - 2$$

$$5(\rho \sin \phi \cos \theta)^2 - 2$$

$$5\rho^2 \sin^2 \phi \cos^2 \theta - 2$$

Then the integral in spherical coordinates is

$$\int_0^{\pi} \int_0^{2\pi} \int_0^3 (5\rho^2 \sin^2 \phi \cos^2 \theta - 2)\rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \int_0^{2\pi} \int_0^3 5\rho^4 \sin^3 \phi \cos^2 \theta - 2\rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{3} 5\rho^4 \sin^3\phi \cos^2\theta \ d\rho \ d\theta \ d\phi - \int_0^{\pi}\int_0^{2\pi}\int_0^{3} 2\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \sin^3\phi \ d\phi \cdot \int_0^{2\pi} \cos^2\theta \ d\theta \cdot \int_0^{3} 5\rho^4 \ d\rho - \int_0^{\pi} \sin\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^{3} 2\rho^2 \ d\rho$$

Evaluate each integral.

$$\left(\frac{1}{12}\cos(3\phi) - \frac{3}{4}\cos\phi\Big|_0^{\pi}\right)\left(\frac{1}{2}x + \frac{1}{2}\sin x \cos x\Big|_0^{2\pi}\right)\left(\rho^5\Big|_0^{3}\right)$$

$$-\left(-\cos\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(\frac{2}{3}\rho^3\Big|_0^{3}\right)$$

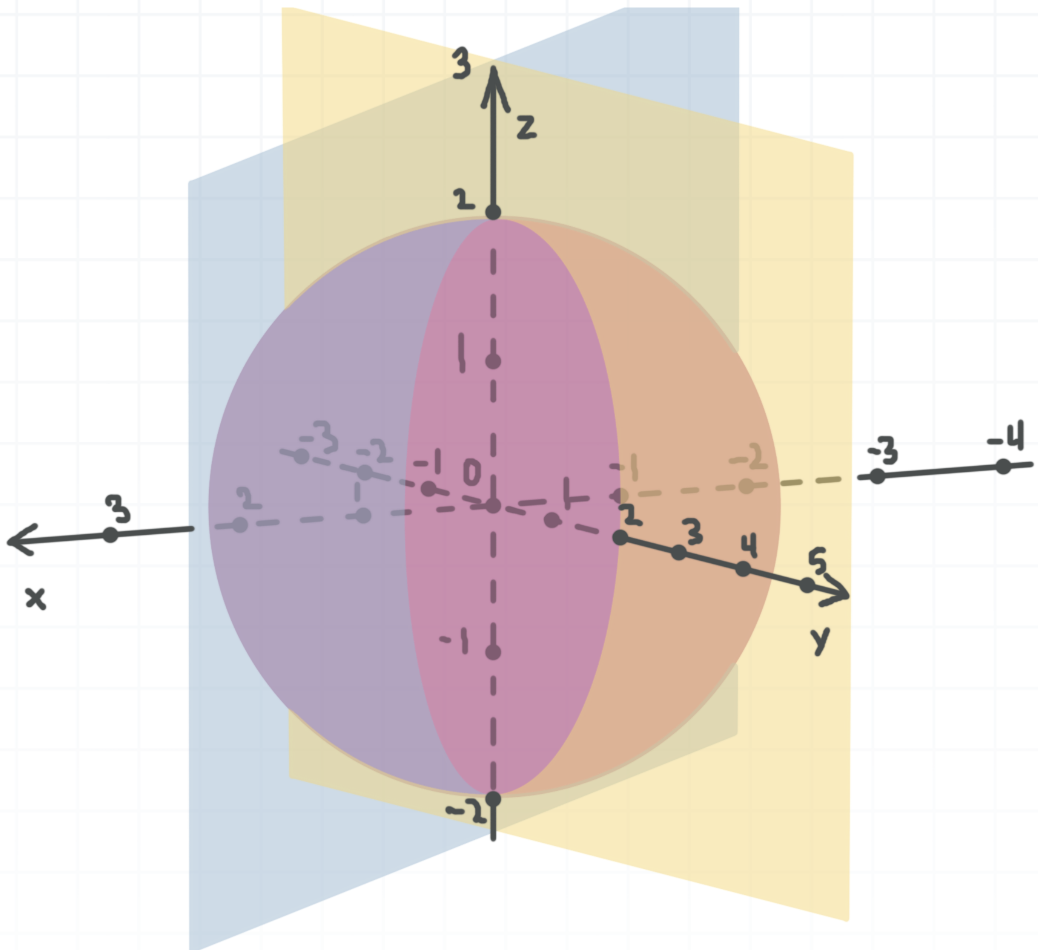
$$\left(\frac{4}{3}\right)(\pi)(243) - (2)(2\pi)(18)$$

$$324\pi - 72\pi$$

$$252\pi$$

■ 2. Write down the triple integral by converting it to spherical coordinates, if E is the part of the sphere with center at the origin, radius 2, that lies between the planes $x = 0$ and $y = x$, and in the space $y > 0$.

$$\iiint_E x^2 + y^2 + 2z \ dV$$



Solution:

The values of x, y, and z change within part of the sphere with center at the origin and radius 2. Therefore, the value of ρ changes from 0 to 2 and the value of ϕ changes from 0 to π.

Change the equation of the plane $x = y$ into spherical coordinates. For any ρ and θ,

$$\rho \sin \phi \cos \theta = \rho \sin \phi \sin \theta$$

$$\cos \theta = \sin \theta$$

$$\theta = \pi/4$$

Change the equation of the plane $x = 0$ into spherical coordinates. For any ρ and θ,

$$\rho \sin\phi \cos\theta = 0$$

$$\cos\theta = 0$$

$$\theta = \pi/2$$

The equation of the plane $x = y$ corresponds to the equation $\theta = \pi/4$ in spherical coordinates and the equation of the plane $x = 0$ corresponds to the equation $\theta = \pi/2$. So the value of θ changes from $\pi/4$ to $\pi/2$.

The function is

$$x^2 + y^2 + 2z$$

$$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 + \rho \cos\phi$$

$$\rho^2 \sin^2\phi(\cos^2\theta + \sin^2\theta) + \rho \cos\phi$$

$$\rho^2 \sin^2\phi + \rho \cos\phi$$

So the triple integral is then

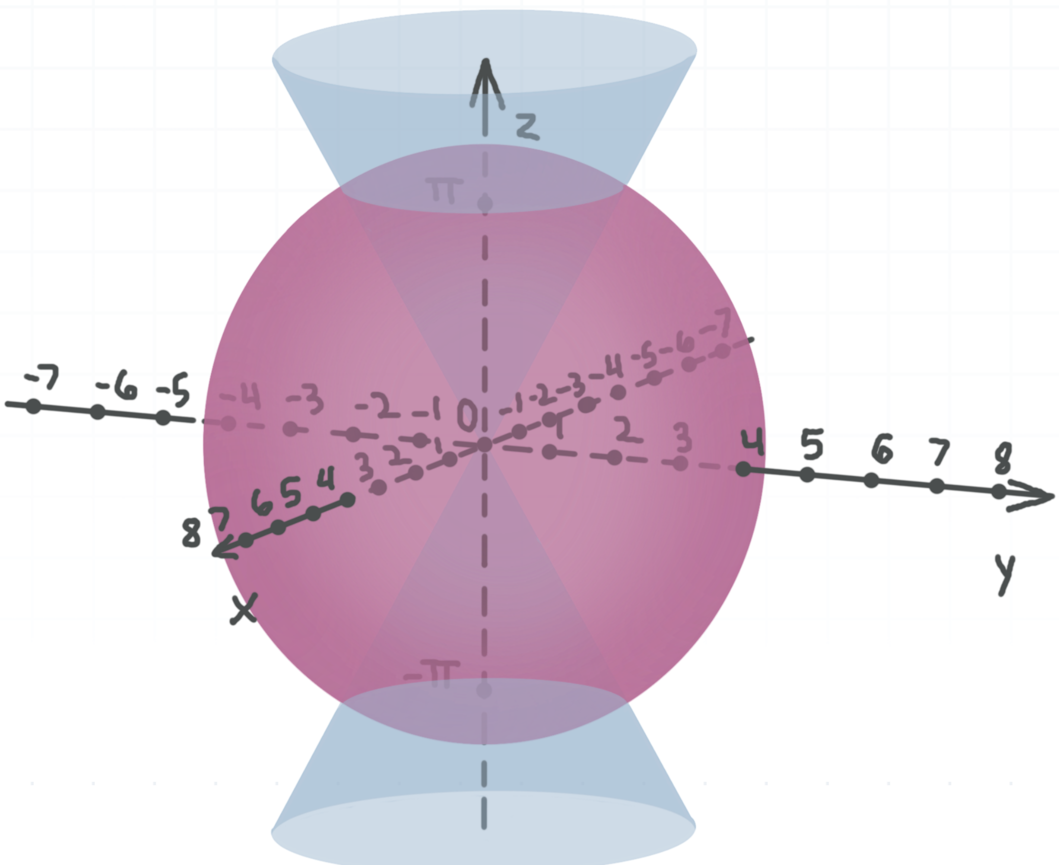
$$\iiint_E x^2 + y^2 + 2z \ dV$$

$$\int_0^{\pi} \int_{\pi/4}^{\pi/2} \int_0^2 (\rho^2 \sin^2\phi + \rho \cos\phi)\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \int_{\pi/4}^{\pi/2} \int_0^2 (\rho^4 \sin^3\phi + \rho^3 \cos\phi \sin\phi) \ d\rho \ d\theta \ d\phi$$

3. Convert the triple integral to spherical coordinates, where E is the solid bounded by the sphere $x^2 + y^2 + z^2 = 16$ and the cone $x^2 + y^2 = z^2/3$, that lies in the half-space $z > 0$.

$$\iiint_E \ln(x^2y^2z^2 + 1)\ dV$$



Solution:

The values of x, y, and z change within the part of the sphere with center at the origin and radius 4. Therefore, the value of ρ changes from 0 to 4. Since the solid has circular symmetry about the z-axis, the value of θ changes from 0 to 2π.

Convert the equation of the cone to spherical coordinates.

$$x^2 + y^2 = \frac{z^2}{3}$$

$$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = \frac{1}{3}(\rho \cos\phi)^2$$

$$\sin^2\phi(\cos^2\theta + \sin^2\theta) = \frac{1}{3}\cos^2\phi$$

$$\tan^2\phi = \frac{1}{3}$$

Since $z > 0$, we can say $0 \le \phi \le \pi/2$, and

$$\tan\phi = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

Therefore, the value of ϕ changes within the cone from 0 to $\pi/6$. The function is

$$\ln(x^2y^2z^2 + 1)$$

$$\ln((\rho \sin\phi \cos\theta)^2(\rho \sin\phi \sin\theta)^2(\rho \cos\phi)^2 + 1)$$

$$\ln(\rho^6 \sin^4\phi \cos^2\phi \sin^2\theta \cos^2\theta + 1)$$

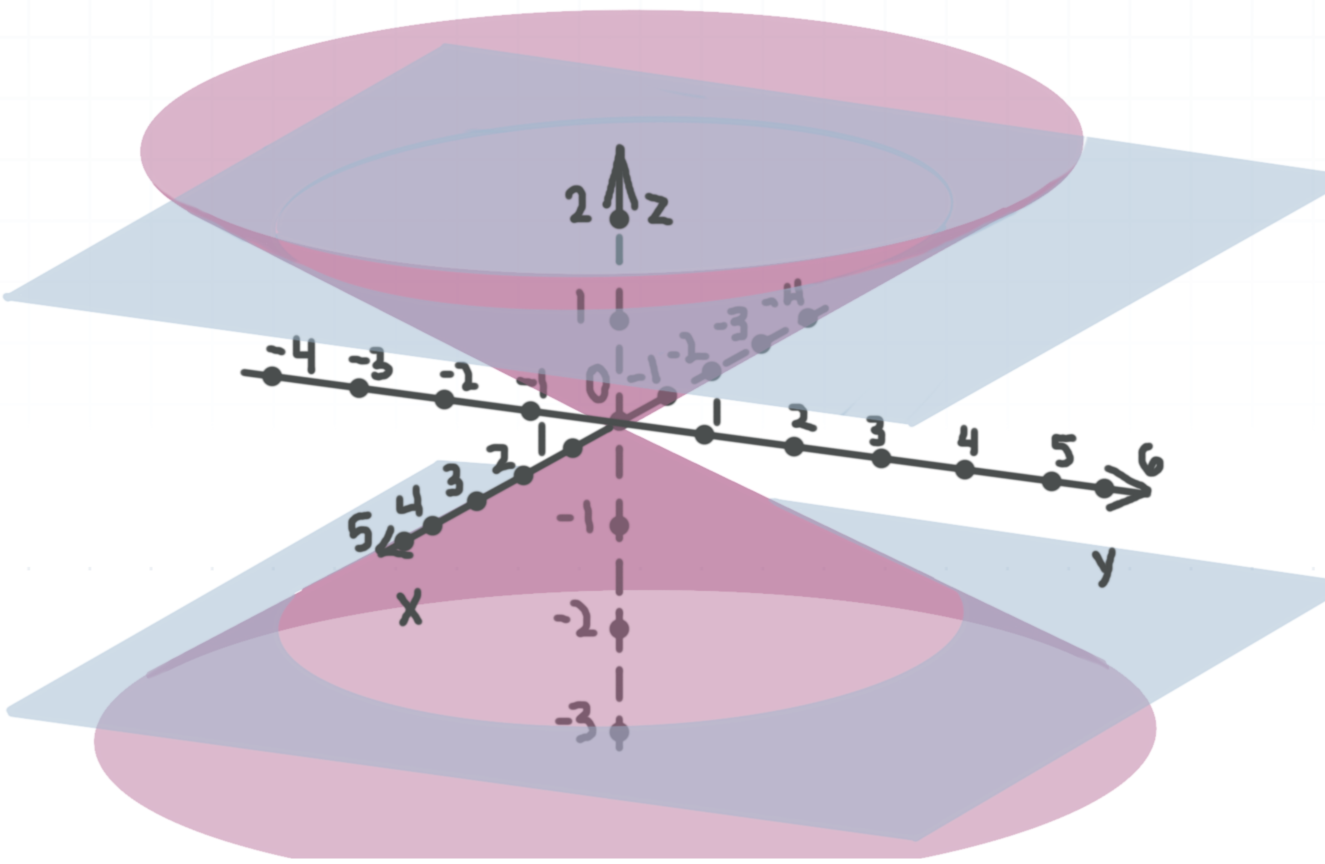
So the triple integral is

$$\iiint_E \ln(x^2y^2z^2 + 1)\ dV$$

$$\int_0^{\pi/6}\int_0^{2\pi}\int_0^4 \ln(\rho^6\sin^4\phi\cos^2\phi\sin^2\theta\cos^2\theta+1)\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

■ 4. If E is the solid bounded by the cone $x^2+y^2=3z^2$ and the planes $z=2$ and $z=-2$, evaluate the triple integral by changing it to spherical coordinates.

$$\iiint_E \sqrt{x^2+y^2+z^2}\ dV$$



Solution:

Since the solid and function are symmetric around the xy-plane,

$$\iiint_E \sqrt{x^2+y^2+z^2}\ dV = 2\iiint_{E_1} \sqrt{x^2+y^2+z^2}\ dV$$

where E_1 is the part of E that lies in the half-space $z \geq 0$. Since the solid has circular symmetry around the z-axis, θ changes from 0 to 2π. Convert the equation of the plane $z = 2$ into spherical coordinates.

$$\rho \cos\phi = 2$$

$$\rho = \frac{2}{\cos\phi}$$

Therefore, ρ changes from 0 to $2/\cos\phi$. Convert the equation of the cone $x^2 + y^2 = 3z^2$ into spherical coordinates.

$$(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2 = 3(\rho\cos\phi)^2$$

$$\sin^2\phi(\cos^2\theta + \sin^2\theta) = 3\cos^2\phi$$

$$\tan^2\phi = 3$$

Since $z > 0$, then $0 \leq \phi \leq \pi/2$, and

$$\tan\phi = \sqrt{3}$$

$$\phi = \frac{\pi}{3}$$

So within the cone, ϕ changes from 0 to $\pi/3$. The function is

$$\sqrt{x^2+y^2+z^2} = \sqrt{\rho^2} = \rho$$

Therefore, the integral in spherical coordinates is

$$\iiint_E \sqrt{x^2+y^2+z^2}\ dV$$

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{\frac{2}{\cos\phi}} \rho\cdot\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi}\int_0^{\frac{2}{\cos\phi}} \rho^3\sin\phi\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{1}{4}\rho^4\sin\phi\Big|_{\rho=0}^{\rho=\frac{2}{\cos\phi}}\ d\theta\ d\phi$$

$$2\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{1}{4}\left(\frac{2}{\cos\phi}\right)^4\sin\phi\ d\theta\ d\phi$$

$$8\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \frac{\sin\phi}{\cos^4\phi}\ d\theta\ d\phi$$

Integrate with respect to θ.

$$8\int_0^{\frac{\pi}{3}} \frac{\sin\phi}{\cos^4\phi}\theta\Big|_{\theta=0}^{\theta=2\pi}\ d\phi$$

$$8\int_0^{\frac{\pi}{3}} \frac{\sin\phi}{\cos^4\phi}(2\pi)\ d\phi$$

$$16\pi\int_0^{\frac{\pi}{3}} \frac{\sin\phi}{\cos^4\phi}\ d\phi$$

Integrate with respect to ϕ, using a substitution with $u = \cos\phi$, $du = -\sin\phi\ d\phi$, and with u changing from 1 to $1/2$.

$$-16\pi\int_{1}^{\frac{1}{2}}\frac{1}{u^4}\ du$$

$$\frac{16\pi}{3u^3}\Bigg|_{1}^{\frac{1}{2}}$$

$$\frac{16\pi}{3\left(\frac{1}{2}\right)^3}-\frac{16\pi}{3(1)^3}$$

$$\frac{128\pi}{3}-\frac{16\pi}{3}$$

$$\frac{112\pi}{3}$$

■ 5. If E is the set of outer points of the sphere with center at the origin and radius 5, evaluate the improper triple integral by changing it to spherical coordinates.

$$\iiint_E \frac{1}{(x^2+y^2+z^2)^3}\ dV$$

Solution:

The values of x, y, and z change outside of the sphere with center at the origin and radius 5. The value of ρ changes from 5 to ∞, θ changes from 0 to 2π, and ϕ changes from 0 to π. The function is

$$\frac{1}{(x^2+y^2+z^2)^3}=\frac{1}{\rho^6}$$

So the integral in spherical coordinates is

$$\int_0^{\pi}\int_0^{2\pi}\int_5^{\infty}\frac{1}{\rho^6}\ \rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_5^{\infty}\frac{1}{\rho^4}\sin\phi\ d\rho\ d\theta\ d\phi$$

Split up the integral as the product of three single integrals.

$$\int_0^{\pi}\sin\phi\ d\phi\cdot\int_0^{2\pi}\ d\theta\cdot\int_5^{\infty}\frac{1}{\rho^4}\ d\rho$$

Evaluate each integral.

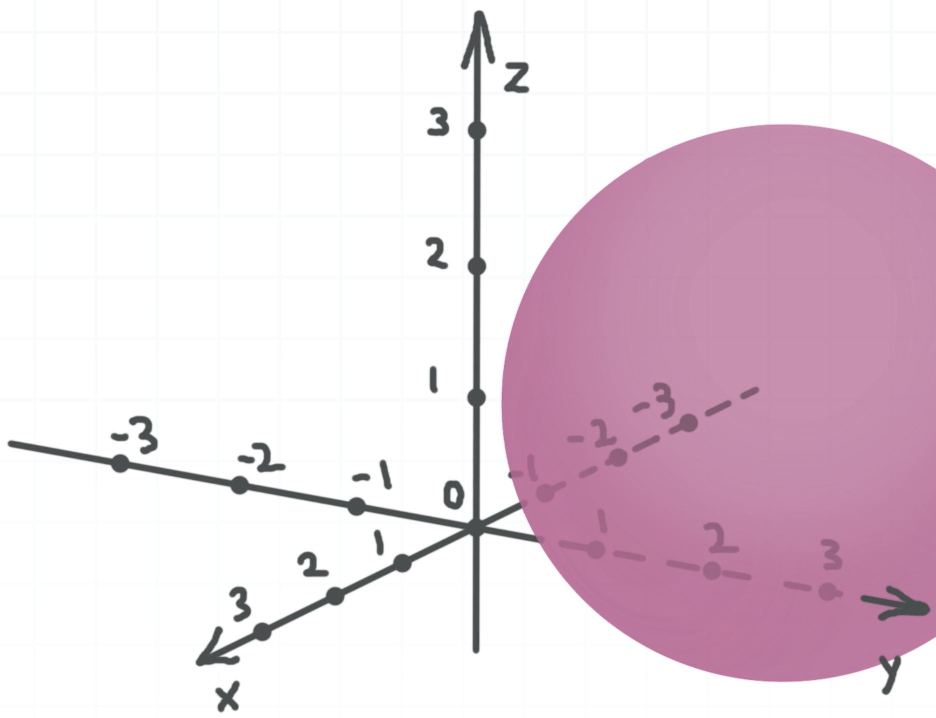
$$\left(-\cos\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\frac{1}{3\rho^3}\Big|_5^{\infty}\right)$$

$$(2)(2\pi)\left(\frac{1}{375}\right)$$

$$\frac{4\pi}{375}$$

6. Evaluate the triple integral by changing it to spherical coordinates, where E is the sphere with center at the point $(-1,2,1)$ and radius 2.

$$\iiint_E 5x + 3y - 2z \ dV$$



Solution:

Use conversion formulas to move the center of the sphere to the origin.

$$x_1 = x + 1, \ x = x_1 - 1$$

$$y_1 = y - 2, \ y = y_1 + 2$$

$$z_1 = z - 1, \ z = z_1 + 1$$

The function transforms to

$$5x + 3y - 2z = 5(x_1 - 1) + 3(y_1 + 2) - 2(z_1 + 1) = 5x_1 + 3y_1 - 2z_1 - 1$$

The surface transforms to the sphere with center at the origin and radius 2, so the given triple integral can be rewritten as

$$\iiint_{E_1} 5x + 3y - 2z - 1 \ dV$$

where E_1 is the interior points of the sphere with center at the origin and radius 2. The value of ρ changes from 0 to 2. The value of θ changes from 0 to 2π and the value of ϕ changes from 0 to π.

The function is

$$5x + 3y - 2z - 1 = 5\rho \sin \phi \cos \theta + 3\rho \sin \phi \sin \theta - 2\rho \cos \phi - 1$$

Therefore, the integral in spherical coordinates is

$$\int_0^{\pi} \int_0^{2\pi} \int_0^2 (5\rho \sin \phi \cos \theta + 3\rho \sin \phi \sin \theta - 2\rho \cos \phi - 1)\rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \int_0^{2\pi} \int_0^2 5\rho^3 \sin^2 \phi \cos \theta + 3\rho^3 \sin^2 \phi \sin \theta - 2\rho^3 \cos \phi \sin \phi - \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

Since the integrals of sine and cosine functions over a 2π period are equal to zero, the integral simplifies to

$$\int_0^{\pi} \int_0^{2\pi} \int_0^2 -2\rho^3 \cos \phi \sin \phi - \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \int_0^{2\pi} \int_0^2 -\rho^3 \sin 2\phi \ d\rho \ d\theta \ d\phi + \int_0^{\pi} \int_0^{2\pi} \int_0^2 -\rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \sin 2\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^2 -\rho^3 \ d\rho + \int_0^{\pi} \sin\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^2 -\rho^2 \ d\rho$$

Evaluate each integral.

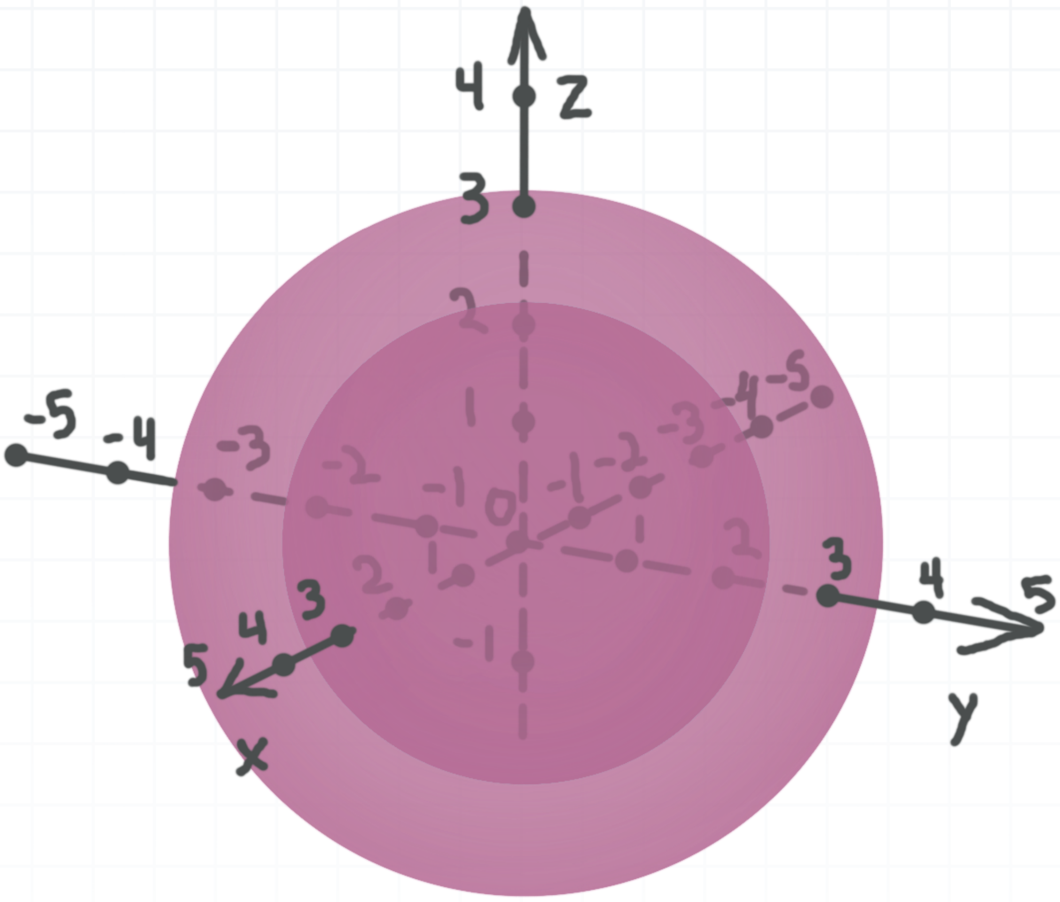
$$\left(-\frac{1}{2}\cos 2\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\frac{1}{4}\rho^4\Big|_0^2\right) + \left(-\cos\phi\Big|_0^{\pi}\right)\left(\theta\Big|_0^{2\pi}\right)\left(-\frac{1}{3}\rho^3\Big|_0^2\right)$$

$$(0)(2\pi)(-4) + (2)(2\pi)\left(-\frac{8}{3}\right)$$

$$-\frac{32\pi}{3}$$

■ 7. Evaluate the triple integral by changing it to spherical coordinates, if E is the set of points between the spheres $x^2 + y^2 + z^2 = 9$ and $x^2 + y^2 + z^2 = 4$.

$$\iiint_E 15y^2 + 4y \ dV$$



Solution:

The values of x, y, and z change within the sphere with center at the origin and radius 3, and outside the sphere with center at the origin and radius 2. The value of ρ changes from 2 to 3, the value of θ changes from 0 to 2π and the value of ϕ changes from 0 to π. The function is

$$15y^2 + 4y$$

$$15(\rho \sin \phi \sin \theta)^2 + 4\rho \sin \phi \sin \theta$$

$$15\rho^2 \sin^2 \phi \sin^2 \theta + 4\rho \sin \phi \sin \theta$$

Therefore, the integral in spherical coordinates is

$$\int_0^{\pi}\int_0^{2\pi}\int_2^3 (15\rho^2\sin^2\phi\sin^2\theta+4\rho\sin\phi\sin\theta)\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_2^3 15\rho^4\sin^3\phi\sin^2\theta+4\rho^3\sin^2\phi\sin\theta\ d\rho\ d\theta\ d\phi$$

Since the integral of the sine function over a 2π period is equal to zero, the integral simplifies to

$$\int_0^{\pi}\int_0^{2\pi}\int_2^3 15\rho^4\sin^3\phi\sin^2\theta\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\sin^3\phi\ d\phi\cdot\int_0^{2\pi}\sin^2\theta\ d\theta\cdot\int_2^3 15\rho^4\ d\rho$$

Evaluate each integral.

$$\left(\frac{1}{12}\cos(3\phi)-\frac{3}{4}\cos\phi\Big|_0^{\pi}\right)\left(\frac{1}{2}\theta-\frac{1}{2}\sin\theta\cos\theta\Big|_0^{2\pi}\right)\left(3\rho^5\Big|_2^3\right)$$

$$\left(\frac{4}{3}\right)(\pi)(633)$$

$$844\pi$$

■ 8. Evaluate the improper triple integral by changing it to spherical coordinates, where E is the first octant ($x\geq 0$, $y\geq 0$, $z\geq 0$).

$$\iiint_E 2^{-\sqrt{(x^2+y^2+z^2)^3}}\ dV$$

Solution:

The values of x, y, and z change within the first octant, so the value of ρ changes from 0 to ∞. The values of θ and ϕ change from 0 to $\pi/2$. The function is

$$2^{-\sqrt{(x^2+y^2+z^2)^3}} = 2^{-\rho^3}$$

Therefore, the integral in spherical coordinates is

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^{\infty} 2^{-\rho^3}\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{\infty} 2^{-\rho^3}\rho^2 \ d\rho$$

Evaluate each integral, using a substitution with $u = -\rho^3$, $du = -3\rho^2 \ d\rho$, and where u changes from 0 to $-\infty$.

$$\int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{-\infty} 2^u\rho^2 \ \frac{du}{-3\rho^2}$$

$$\int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{\frac{\pi}{2}} d\theta \cdot \int_0^{-\infty} -\frac{1}{3}(2^u) \ du$$

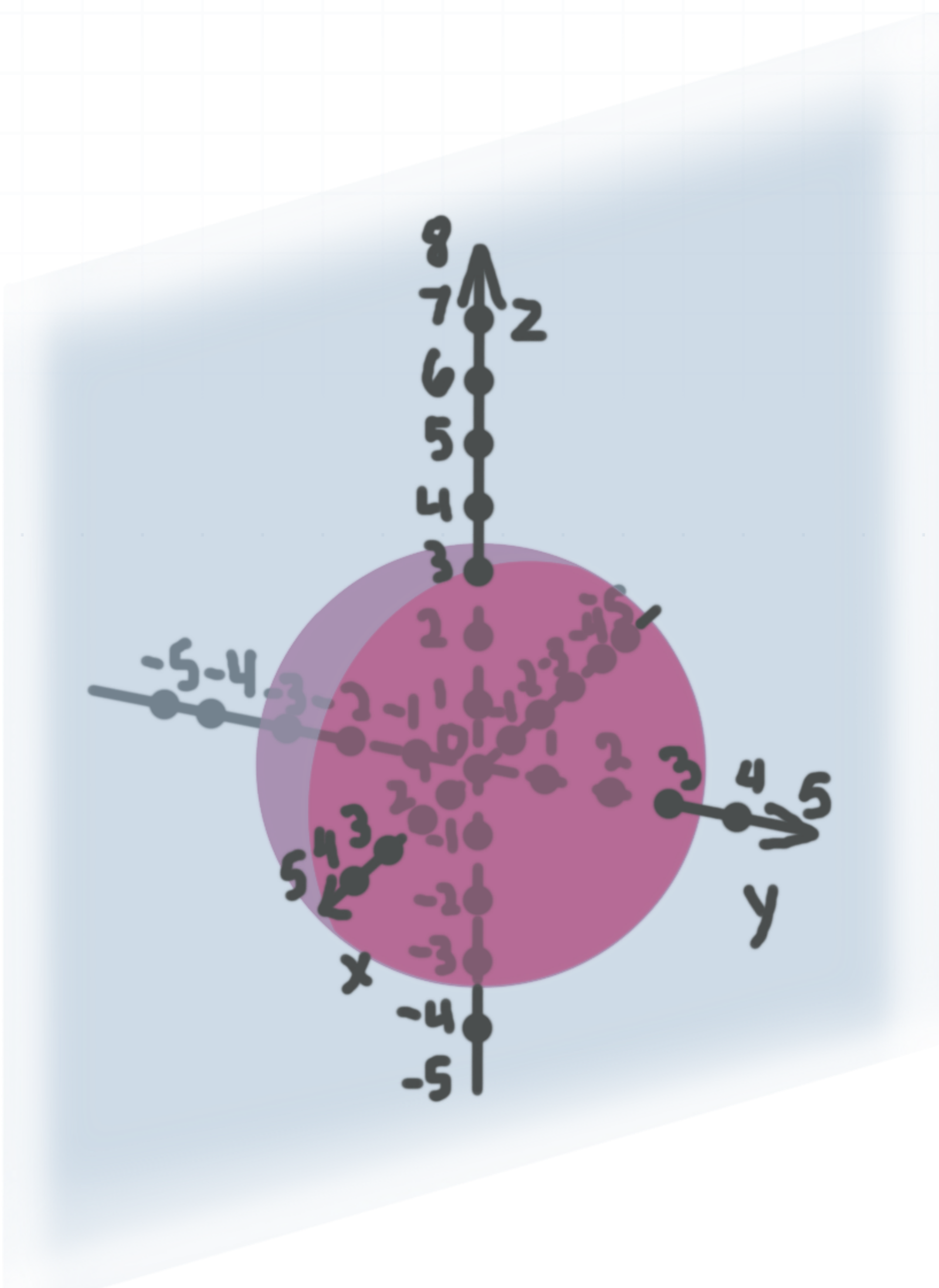
$$\left(-\cos\phi\Big|_0^{\frac{\pi}{2}}\right)\left(\theta\Big|_0^{\frac{\pi}{2}}\right)\left(-\frac{1}{3}\lim_{t\to-\infty}\left(\frac{2^t}{\ln 2}\right)-\frac{2^0}{\ln 2}\right)$$

$$(1)\left(\frac{\pi}{2}\right)\left(\frac{1}{3\ln 2}\right)$$

$$\frac{\pi}{6\ln 2}$$

■ 9. Evaluate the triple integral by changing it to spherical coordinates, if E is the solid bounded by the sphere with center at the origin and radius 3, and the plane $x + \sqrt{3}y = 0$. Consider the hemisphere that includes the points from the first octant.

$$\iiint_E x^4 + y^4 + z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2 \ dV$$



Solution:

The values x, y, and z change within part of the sphere with center at the origin and radius 3. Therefore, the value of ρ changes from 0 to 3 and the value of ϕ changes from 0 to π. Convert the equation of the plane $x+\sqrt{3}y=0$ into spherical coordinates. For any ρ and θ,

$$\rho\sin\phi\cos\theta=-\sqrt{3}\rho\sin\phi\sin\theta$$

So

$$\cos\theta=-\sqrt{3}\sin\theta$$

$$\tan\theta=-\frac{1}{\sqrt{3}}$$

$$\theta_1=-\frac{\pi}{6}\text{ and }\theta_2=\frac{5\pi}{6}$$

So the value of θ changes from $-\pi/6$ to $5\pi/6$. The function is

$$x^4+y^4+z^4+2x^2y^2+2x^2z^2+2y^2z^2=(x^2+y^2+z^2)^2=\rho^4$$

Therefore, the integral in spherical coordinates is

$$\int_0^{\pi}\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_0^3\rho^4\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}}\int_0^3\rho^6\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi} \sin\phi \ d\phi \cdot \int_{-\frac{\pi}{6}}^{\frac{5\pi}{6}} d\theta \cdot \int_0^3 \rho^6 \ d\rho$$

Evaluate each integral.

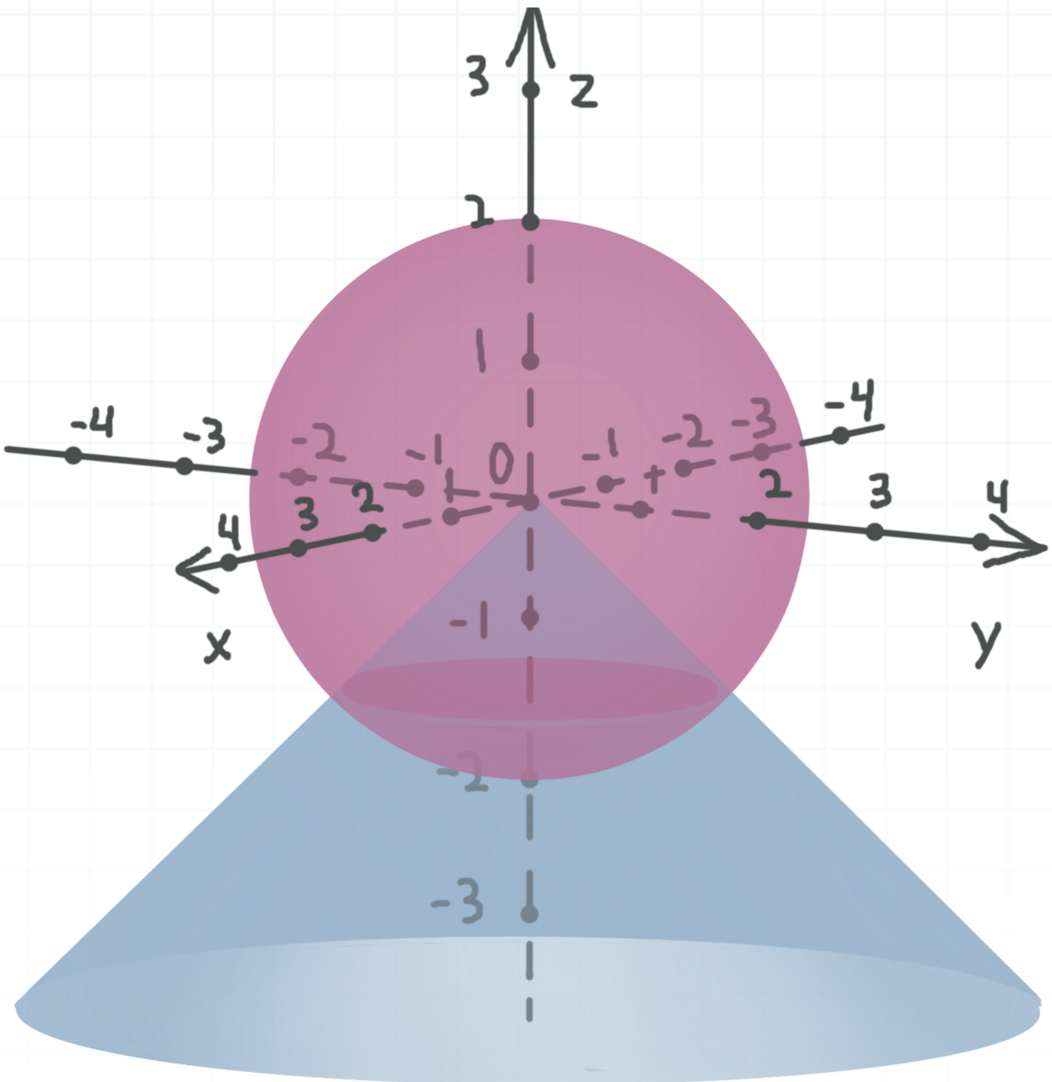
$$\left(-\cos\phi \Big|_0^{\pi}\right)\left(\theta \Big|_{-\frac{\pi}{6}}^{\frac{5\pi}{6}}\right)\left(\frac{1}{7}\rho^7 \Big|_0^3\right)$$

$$(2)(\pi)(2{,}187)$$

$$4{,}374\pi$$

■ 10. Evaluate the improper triple integral by changing it to spherical coordinates, where E is the region that consists of the points inside the cone $x^2 + y^2 = z^2$ and outside the sphere $x^2 + y^2 + z^2 = 4$, that lie in the half-space $z < 0$.

$$\iiint_E \frac{4z + 10}{(x^2 + y^2 + z^2)^4} \ dV$$



Solution:

The values of x, y, and z change outside the sphere with center at the origin and radius 2. Therefore, the value of ρ changes from 2 to ∞. Since the region E has circular symmetry around the z-axis, the coordinate θ changes from 0 to 2π.

Convert the equation of the cone $x^2 + y^2 = z^2$ to spherical coordinates.

$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = (\rho \cos \phi)^2$$

$$\sin^2 \phi(\cos^2 \theta + \sin^2 \theta) = \cos^2 \phi$$

$$\tan^2 \phi = 1$$

$$\tan\phi = \pm 1$$

Since $z < 0$, then $\pi/2 \leq \phi \leq \pi$, and

$$\tan\phi = -1$$

$$\phi = \frac{3\pi}{4}$$

Therefore, the value of ϕ changes inside the cone from $\pi/2$ to $3\pi/4$. The function is

$$\frac{4z+10}{(x^2+y^2+z^2)^4} = \frac{4\rho\cos\phi + 10}{\rho^8}$$

So the integral in spherical coordinates is

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_0^{2\pi} \int_2^{\infty} \frac{4\rho\cos\phi + 10}{\rho^8}\ \rho^2 \sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_0^{2\pi} \int_2^{\infty} \frac{2\rho\sin 2\phi + 10\sin\phi}{\rho^6}\ d\rho\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_0^{2\pi} \int_2^{\infty} \frac{2\sin 2\phi}{\rho^5} + \frac{10\sin\phi}{\rho^6}\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}} \int_0^{2\pi} -\frac{\sin 2\phi}{2\rho^4} - \frac{2\sin\phi}{\rho^5}\Bigg|_{\rho=2}^{\rho=\infty}\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\int_0^{2\pi}\frac{\sin 2\phi+2\sin\phi}{32}\ d\theta\ d\phi$$

Integrate with respect to θ.

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\frac{\sin 2\phi+2\sin\phi}{32}\theta\Big|_{\theta=0}^{\theta=2\pi}\ d\phi$$

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{4}}\frac{\pi\sin 2\phi+2\pi\sin\phi}{16}\ d\phi$$

Integrate with respect to ϕ.

$$\frac{\pi\sin 2\phi+2\pi\sin\phi}{16}\Big|_{\phi=\frac{\pi}{2}}^{\phi=\frac{3\pi}{4}}$$

$$\frac{-\pi+\sqrt{2}\pi}{16}-\frac{2\pi}{16}$$

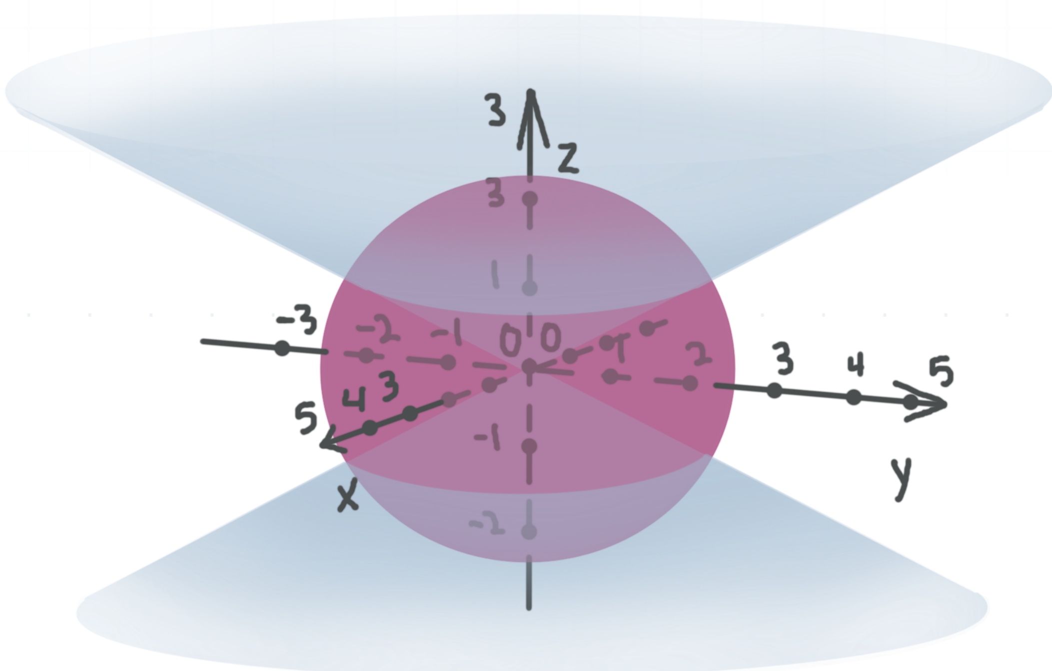
$$\frac{-3\pi+\sqrt{2}\pi}{16}$$

FINDING VOLUME

■ 1. Use a triple integral in spherical coordinates to find the volume of the region E that consists of the points inside the sphere $x^2 + y^2 + z^2 = 6$ and outside the cone $y^2 + z^2 = 3x^2$.

Solution:

Since rotation doesn't change the volume, the volume of the region E is equal to the volume of region E_1 that consists of the points inside the sphere $x^2 + y^2 + z^2 = 6$ and outside the cone $x^2 + y^2 = 3z^2$.



The values of x, y, and z change within the part of the sphere with center at the origin and radius $\sqrt{6}$. Therefore, the value of ρ changes from 0 to $\sqrt{6}$. Since E_1 has circular symmetry around the z-axis, and the value of θ changes from 0 to 2π.

Convert the equation of the cone $x^2 + y^2 = 3z^2$ to spherical coordinates.

$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 = 3(\rho \cos \phi)^2$$

$$\sin^2 \phi(\cos^2 \theta + \sin^2 \theta) = 3 \cos^2 \phi$$

$$\tan^2 \phi = 3$$

$$\tan \phi = \pm\sqrt{3}$$

Since $0 \le \phi \le \pi$,

$$\phi_1 = \frac{\pi}{3} \text{ and } \phi_2 = \frac{2\pi}{3}$$

Therefore, the value of ϕ changes outside the cone from $\pi/3$ to $2\pi/3$. So the integral in spherical coordinates is

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \int_0^{2\pi} \int_0^{\sqrt{6}} \rho^2 \sin \phi \ d\rho \ d\theta \ d\phi$$

$$\int_{\frac{\pi}{3}}^{\frac{2\pi}{3}} \sin \phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\sqrt{6}} \rho^2 \ d\rho$$

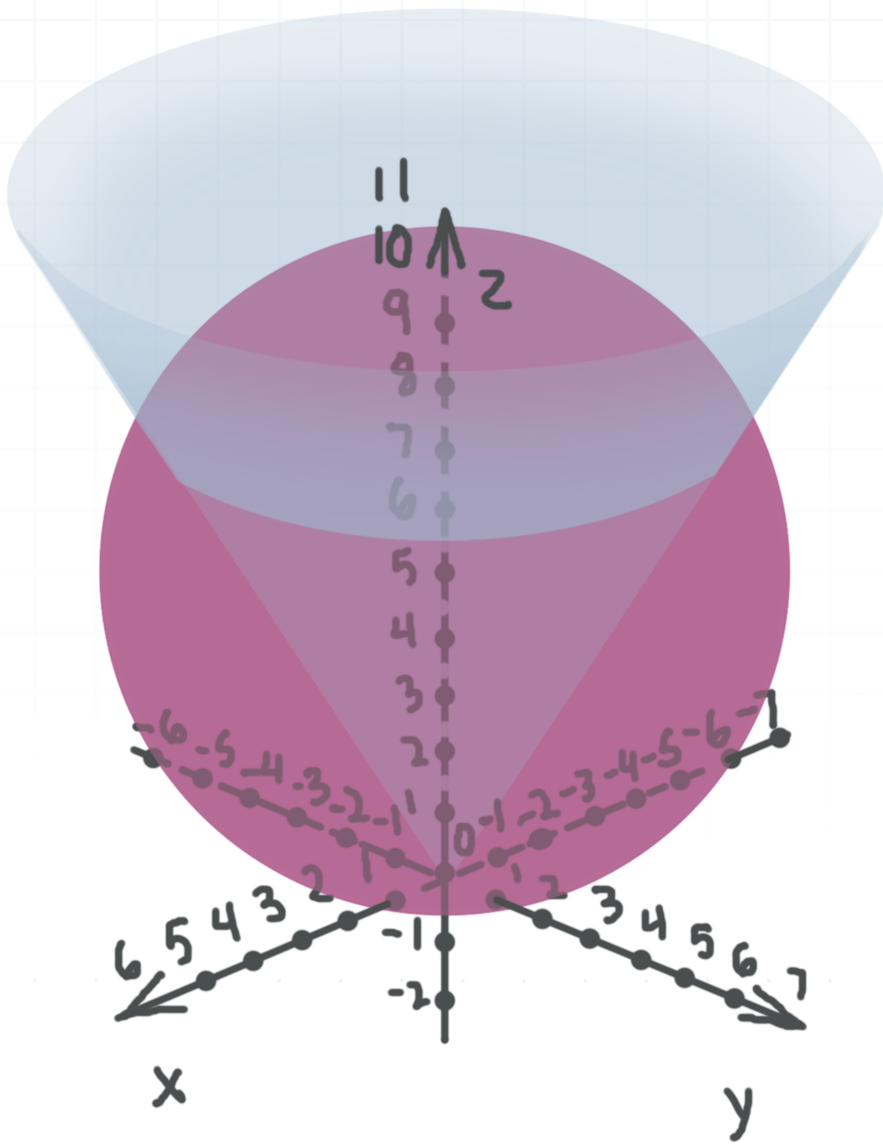
Evaluate each integral.

$$\left(-\cos \phi \Big|_{\frac{\pi}{3}}^{\frac{2\pi}{3}}\right) \cdot \left(\theta \Big|_0^{2\pi}\right)\left(\frac{1}{3}\rho^3 \Big|_0^{\sqrt{6}}\right)$$

$$(1)(2\pi)(2\sqrt{6}))$$

$$4\pi\sqrt{6}$$

■ 2. Use a triple integral in spherical coordinates to find the volume of an ice cream cone formed by the points common to the cone $3x^2 + 3y^2 = z^2$ and the sphere $x^2 + y^2 + z^2 - 10z = 0$.



Solution:

If we put the equation of the sphere in standard form,

$$x^2 + y^2 + (z - 5)^2 = 25$$

we see that it's centered at $(0,0,5)$ and has radius 5. Given that, the value of θ changes from 0 to 2π. Now convert the equation of the sphere into spherical coordinates.

$$x^2 + y^2 + z^2 - 10z = 0$$

$$\rho^2 - 10\rho\cos\phi = 0$$

$$\rho - 10\cos\phi = 0$$

$$\rho = 10\cos\phi$$

So the value of ρ changes from 0 to $10\cos\phi$. Now convert the equation of the cone into spherical coordinates.

$$3x^2 + 3y^2 = z^2$$

$$x^2 + y^2 = \frac{1}{3}z^2$$

$$(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2 = \frac{1}{3}(\rho\cos\phi)^2$$

$$\sin^2\phi(\cos^2\theta + \sin^2\theta) = \frac{1}{3}\cos^2\phi$$

$$\tan^2\phi = \frac{1}{3}$$

Since $z > 0$, then $0 \le \phi \le \pi/2$, and

$$\tan\phi = \frac{1}{\sqrt{3}}$$

$$\phi = \frac{\pi}{6}$$

So the value of ϕ changes from 0 to $\pi/6$, and the volume is

$$\int_0^{\frac{\pi}{6}}\int_0^{2\pi}\int_0^{10\cos\phi} \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

Integrate with respect to ρ.

$$\int_0^{\frac{\pi}{6}}\int_0^{2\pi} \frac{1}{3}\rho^3 \sin\phi \Big|_{\rho=0}^{\rho=10\cos\phi} \ d\theta \ d\phi$$

$$\int_0^{\frac{\pi}{6}}\int_0^{2\pi} \frac{1{,}000}{3} \cos^3\phi \sin\phi \ d\theta \ d\phi$$

Integrate with respect to θ.

$$\int_0^{\frac{\pi}{6}} \frac{1{,}000}{3}\theta \cos^3\phi \sin\phi \Big|_{\theta=0}^{\theta=2\pi} \ d\phi$$

$$\int_0^{\frac{\pi}{6}} \frac{1{,}000}{3}(2\pi)\cos^3\phi \sin\phi \ d\phi$$

$$\int_0^{\frac{\pi}{6}} \frac{2{,}000\pi}{3} \cos^3\phi \sin\phi \ d\phi$$

Integrate with respect to ϕ, using a substitution with $u = \cos\phi$, $du = -\sin\phi \ d\phi$, and u changing from 1 to $\sqrt{3}/2$.

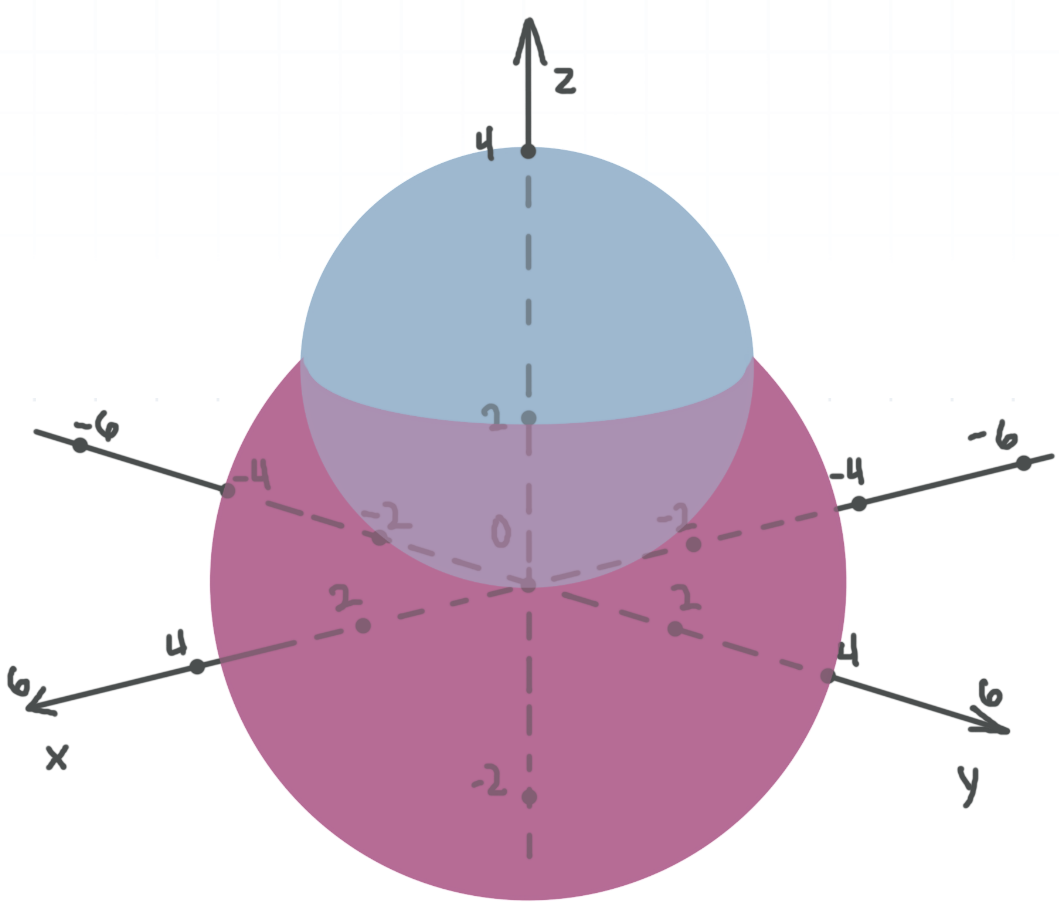
$$\frac{2{,}000\pi}{3}\int_1^{\frac{\sqrt{3}}{2}} -u^3 \ du$$

$$-\frac{500\pi}{3}u^4\Big|_1^{\frac{\sqrt{3}}{2}}$$

$$-\frac{375\pi}{4}+\frac{500\pi}{3}$$

$$\frac{875\pi}{12}$$

■ 3. Use a triple integral in spherical coordinates to find the volume of the three-dimensional lens common to the two spheres $x^2+y^2+z^2-8=0$ and $x^2+y^2+z^2-4z=0$.



Solution:

Rewrite the equations of both spheres in standard form.

$$x^2 + y^2 + z^2 = 8$$

$$x^2 + y^2 + (z - 2)^2 = 4$$

The first sphere has its center at the origin and radius $\sqrt{8}$, and the second sphere has its center at $(0,0,2)$ and radius 2.

To find the equation of the curve of intersection, consider the two sphere equations $x^2 + y^2 + z^2 - 8 = 0$ and $x^2 + y^2 + z^2 - 4z = 0$ as a system, then subtract them to get

$$4z - 8 = 0$$

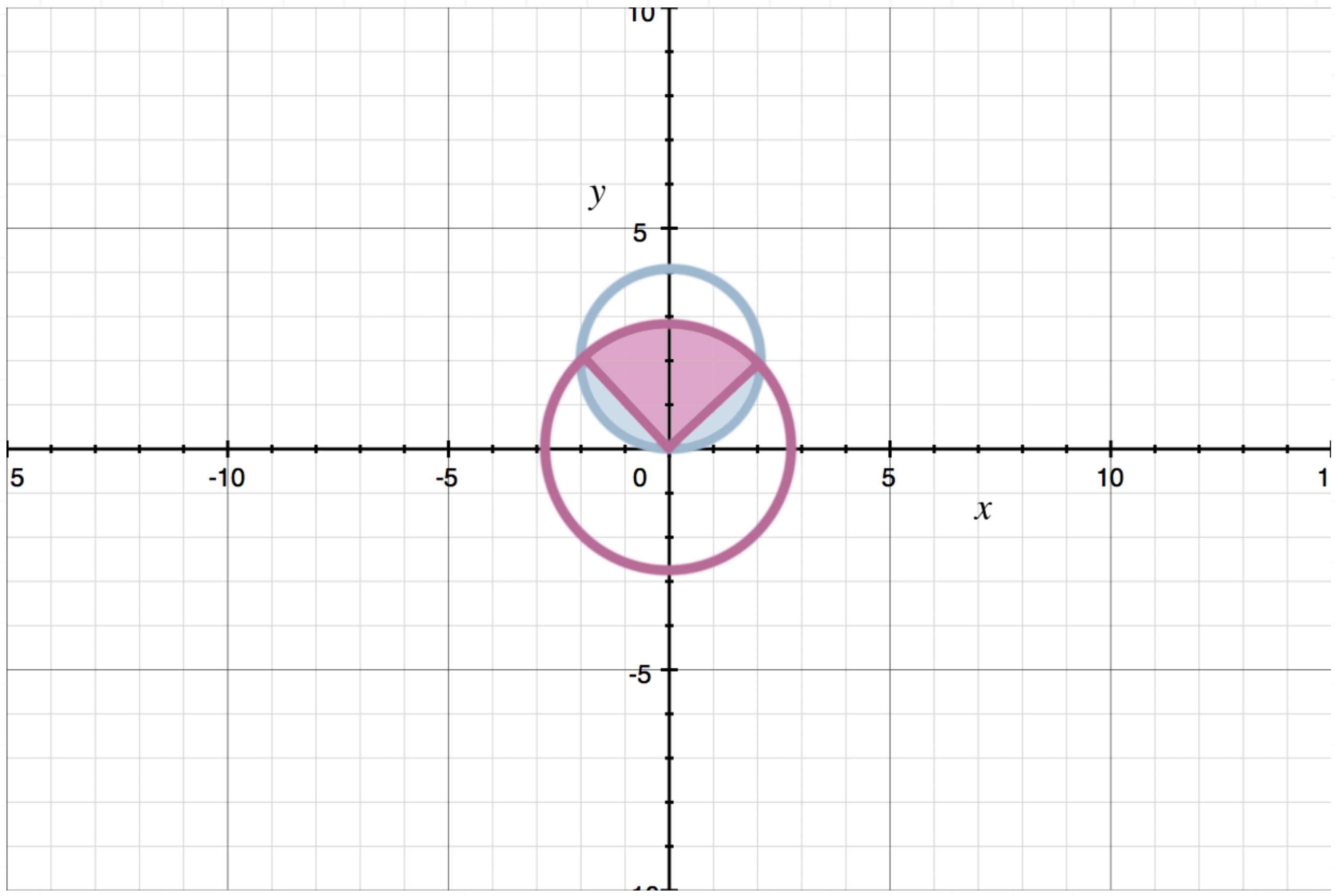
$$z = 2$$

Plug $z = 2$ into the first equation.

$$x^2 + y^2 + (2)^2 - 8 = 0$$

$$x^2 + y^2 = 4$$

So the curve of intersection is a circle with its center at $(0,0,2)$ and radius 2, that lies in the plane $z = 2$. Consider the lens section for $y = 0$.



The circle of intersection corresponds to the angle $\phi = \pi/4$. Split the lens into the two solids E_1 and E_2, where E_1 is defined for ϕ between 0 and $\pi/4$, and E_2 is defined for ϕ between $\pi/4$ and $\pi/2$.

For E_1, the value of ρ changes from 0 to $\sqrt{8}$, the value of θ changes from 0 to 2π, and the value of ϕ changes from 0 to $\pi/4$. So the volume is

$$\int_0^{\frac{\pi}{4}} \int_0^{2\pi} \int_0^{\sqrt{8}} \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\frac{\pi}{4}} \sin\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^{\sqrt{8}} \rho^2 \ d\rho$$

Evaluate each integral.

$$\left(-\cos\phi \Big|_0^{\frac{\pi}{4}} \right) \left(\theta \Big|_0^{2\pi} \right) \left(\frac{1}{3}\rho^3 \Big|_0^{\sqrt{8}} \right)$$

$$\left(-\frac{\sqrt{2}}{2}+1\right)(2\pi)\left(\frac{16\sqrt{2}}{3}\right)$$

$$\frac{32\pi\sqrt{2}-32\pi}{3}$$

For E_2, the value of θ changes from 0 to 2π, and the value of ϕ changes from $\pi/4$ to $\pi/2$. To find the limits for ρ, convert the equation of the sphere into spherical coordinates.

$$x^2+y^2+z^2-4z=0$$

$$\rho^2-4\rho\cos\phi=0$$

$$\rho-4\cos\phi=0$$

$$\rho=4\cos\phi$$

So the value of ρ changes from 0 to $4\cos\phi$, and the volume is

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^{4\cos\phi}\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

Integrate with respect to ρ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{1}{3}\rho^3\sin\phi\Big|_{\rho=0}^{\rho=4\cos\phi}\ d\theta\ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{2\pi}\frac{64}{3}\cos^3\phi\sin\phi\ d\theta\ d\phi$$

Integrate with respect to θ.

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{64}{3}\theta \cos^3\phi \sin\phi \Big|_{\theta=0}^{\theta=2\pi} \ d\phi$$

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{128\pi}{3} \cos^3\phi \sin\phi \ d\phi$$

Integrate with respect to ϕ, using a substitution with $u = \cos\phi$, $du = -\sin\phi \ d\phi$, and u changing from $\sqrt{2}/2$ to 0.

$$\frac{128\pi}{3}\int_{\frac{\sqrt{2}}{2}}^{0} -u^3 \ du$$

$$-\frac{32\pi}{3}u^4\Big|_{\frac{\sqrt{2}}{2}}^{0}$$

$$\frac{8\pi}{3}$$

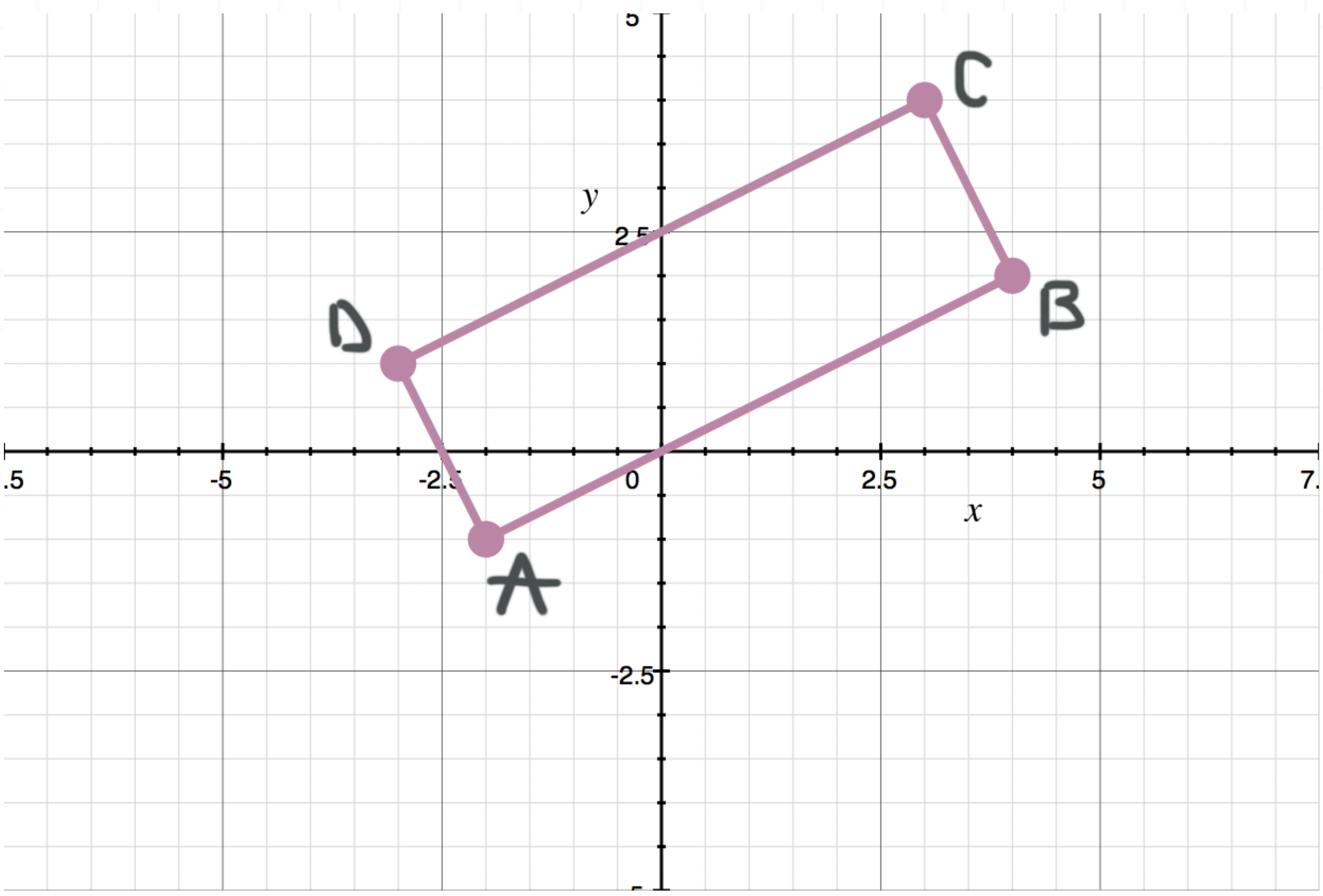
Then the sum of the two volumes is

$$\frac{32\pi\sqrt{2}-32\pi}{3}+\frac{8\pi}{3}$$

$$\frac{32\pi\sqrt{2}-24\pi}{3}$$

JACOBIAN FOR TWO VARIABLES

■ 1. Find the Jacobian of the transformation that rotates the rectangle $ABCD$, given by $A(-2,-1)$, $B(4,2)$, $C(3,4)$, and $D(-3,1)$, clockwise about the origin in such a way that AB will lie on the x-axis.



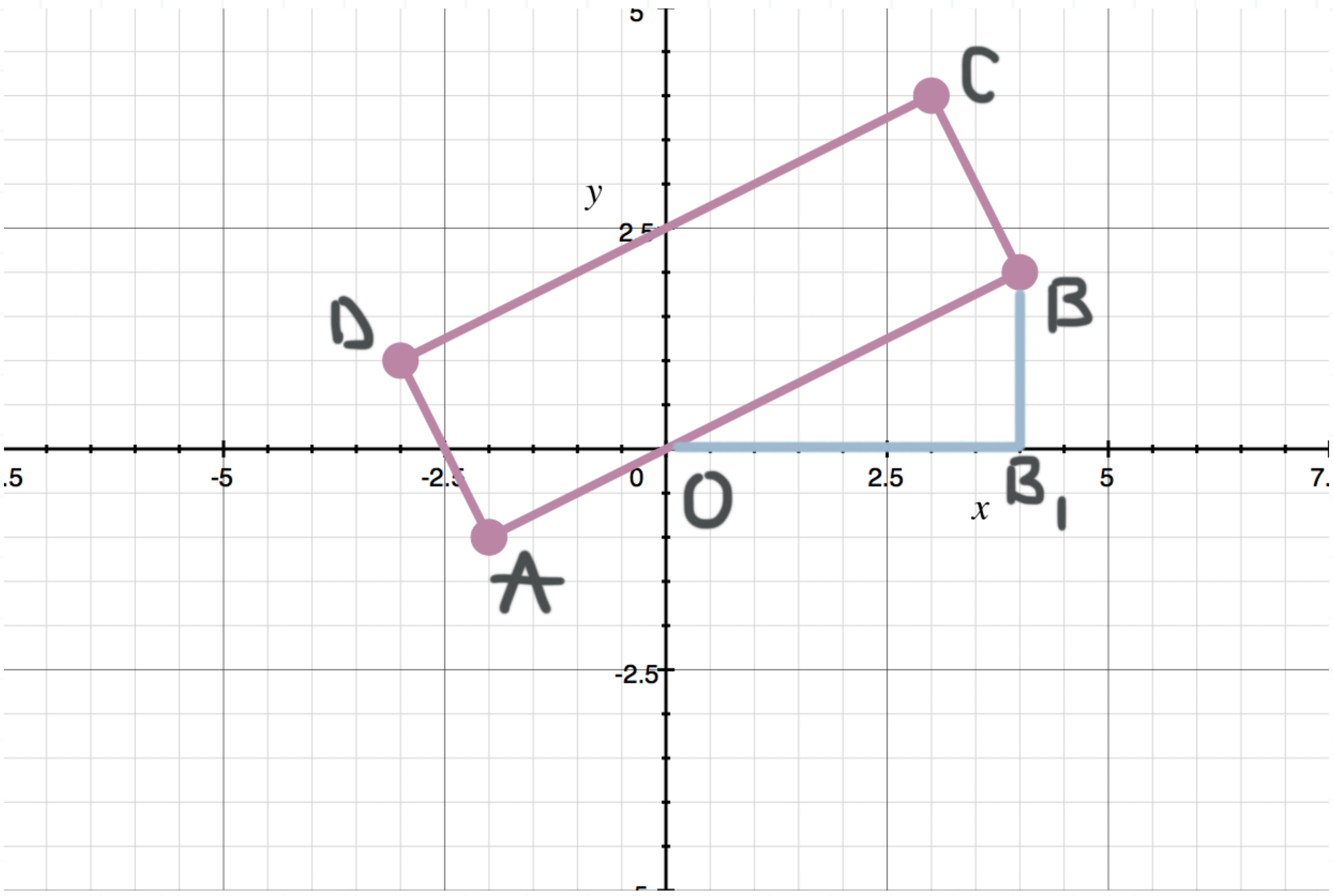
Solution:

Let (u, v) be the coordinates after the transformation. The formula of rotation by an angle ϕ is given by

$$x = u\cos\phi - v\sin\phi$$

$$y = u\sin\phi + v\cos\phi$$

Let $B_1(4,0)$ be the projection of the point B onto the x-axis.



Find sine and cosine of ϕ from the triangle OBB_1. Since $OB = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$,

$$\sin\phi = \frac{BB_1}{OB} = \frac{2}{2\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cos\phi = \frac{OB_1}{OB} = \frac{4}{2\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Plug these values into the rotation formulas.

$$x = \frac{2\sqrt{5}}{5}u - \frac{\sqrt{5}}{5}v$$

$$y = \frac{\sqrt{5}}{5}u + \frac{2\sqrt{5}}{5}v$$

The partial derivatives are

$$\frac{\partial x}{\partial u}=\frac{\partial}{\partial u}\left(\frac{2\sqrt{5}}{5}u-\frac{\sqrt{5}}{5}v\right)=\frac{2\sqrt{5}}{5}$$

$$\frac{\partial x}{\partial v}=\frac{\partial}{\partial v}\left(\frac{2\sqrt{5}}{5}u-\frac{\sqrt{5}}{5}v\right)=-\frac{\sqrt{5}}{5}$$

$$\frac{\partial y}{\partial u}=\frac{\partial}{\partial u}\left(\frac{\sqrt{5}}{5}u+\frac{2\sqrt{5}}{5}v\right)=\frac{\sqrt{5}}{5}$$

$$\frac{\partial y}{\partial v}=\frac{\partial}{\partial v}\left(\frac{\sqrt{5}}{5}u+\frac{2\sqrt{5}}{5}v\right)=\frac{2\sqrt{5}}{5}$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{2\sqrt{5}}{5}\cdot\frac{2\sqrt{5}}{5}-\left(-\frac{\sqrt{5}}{5}\right)\cdot\frac{\sqrt{5}}{5}=1$$

■ 2. Find the Jacobian of the transformation which converts the ellipse $5\sqrt{2}x^2+6\sqrt{2}xy+8x+5\sqrt{2}y^2-8y=0$ into the ellipse with center at the origin, and x- and $y$semi-axes 2 and 1 respectively. Use a rotation counterclockwise by $\pi/4$, and then move it by 2 to the positive direction of the x-axis.

Solution:

Let (u, v) be the coordinates after the transformation. The formula of rotation by the angle ϕ is given by

$$x = u\cos\phi - v\sin\phi$$

$$y = u\sin\phi + v\cos\phi$$

Since we rotate by a counterclockwise angle of $\pi/4$, we'll plug in $\phi = -\pi/4$.

$$x = u\cos\left(-\frac{\pi}{4}\right) - v\sin\left(-\frac{\pi}{4}\right)$$

$$y = u\sin\left(-\frac{\pi}{4}\right) + v\cos\left(-\frac{\pi}{4}\right)$$

and we get

$$x = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v$$

$$y = -\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v$$

Next, to move the ellipse by 2 to the positive direction of x-axis, we need to subtract 2 from the u coordinate. Finally, the transformation equations are

$$x = \frac{\sqrt{2}}{2}(u-2) + \frac{\sqrt{2}}{2}v = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}$$

$$y=-\frac{\sqrt{2}}{2}(u-2)+\frac{\sqrt{2}}{2}v=-\frac{\sqrt{2}}{2}u+\frac{\sqrt{2}}{2}v+\sqrt{2}$$

To check the transformation, plug these values for x and y into the equation of the ellipse.

$$5\sqrt{2}x^2+6\sqrt{2}xy+8x+5\sqrt{2}y^2-8y=0$$

$$5\sqrt{2}\left(\frac{\sqrt{2}}{2}u+\frac{\sqrt{2}}{2}v-\sqrt{2}\right)^2+6\sqrt{2}\left(\frac{\sqrt{2}}{2}u+\frac{\sqrt{2}}{2}v-\sqrt{2}\right)\left(-\frac{\sqrt{2}}{2}u+\frac{\sqrt{2}}{2}v+\sqrt{2}\right)$$

$$+8\left(\frac{\sqrt{2}}{2}u+\frac{\sqrt{2}}{2}v-\sqrt{2}\right)+5\sqrt{2}\left(-\frac{\sqrt{2}}{2}u+\frac{\sqrt{2}}{2}v+\sqrt{2}\right)^2$$

$$-8\left(-\frac{\sqrt{2}}{2}u+\frac{\sqrt{2}}{2}v+\sqrt{2}\right)=0$$

We get

$$\frac{5\sqrt{2}}{2}(u+v-2)^2+3\sqrt{2}(v^2-(u-2)^2)+\frac{5\sqrt{2}}{2}(-u+v+2)^2+8\sqrt{2}(u-2)=0$$

$$5(u+v-2)^2+6(v^2-(u-2)^2)+5(-u+v+2)^2+16(u-2)=0$$

$$4(u^2+4v^2-4)=0$$

$$\frac{u^2}{4}+v^2=1$$

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}\right) = -\frac{\sqrt{2}}{2}$$

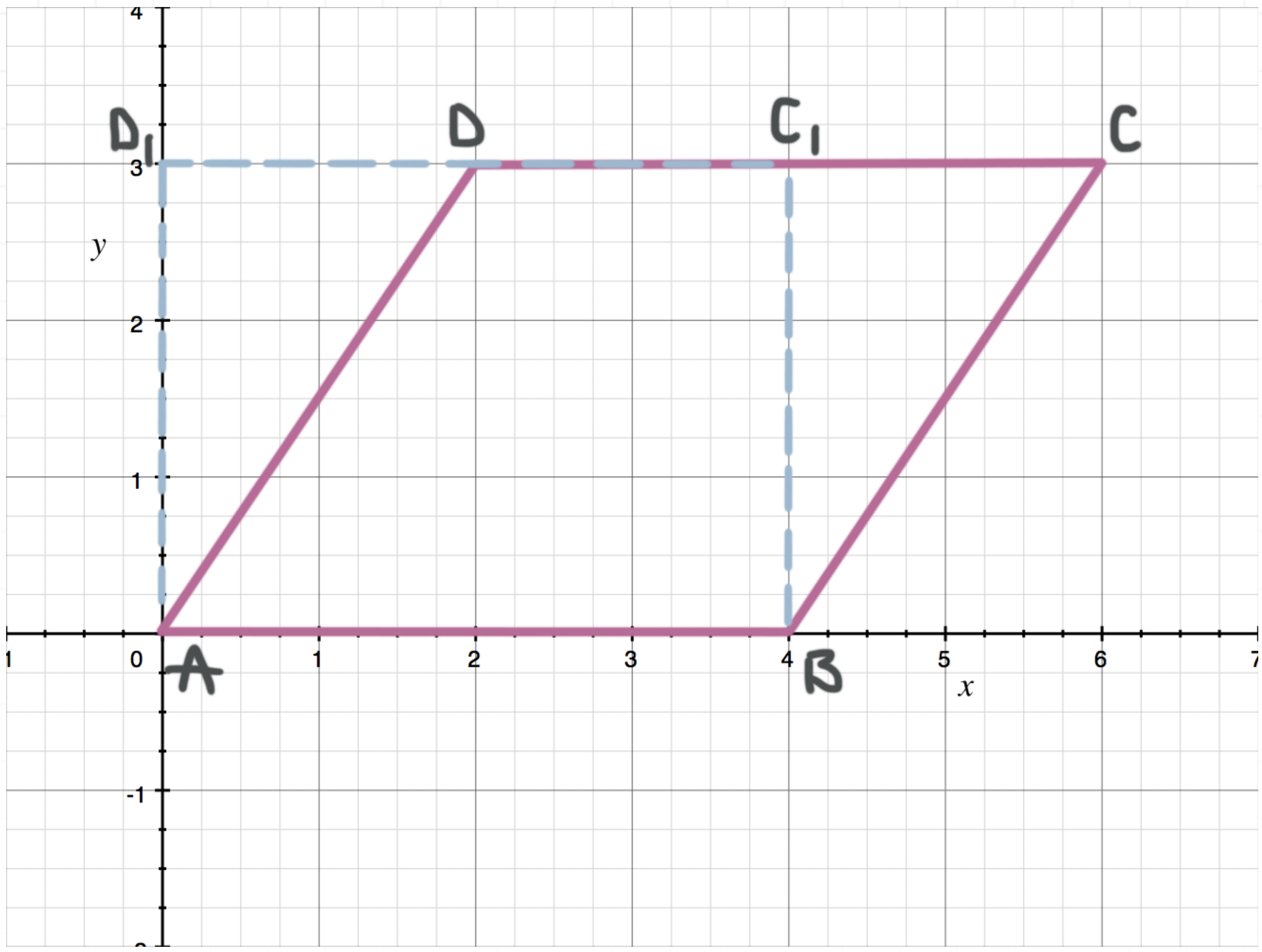
$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}\right) = \frac{\sqrt{2}}{2}$$

The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = 1$$

■ 3. Find the Jacobian of the linear transformation that converts the parallelogram $ABCD$, given by $A(0,0)$, $B(4,0)$, $C(6,3)$, $D(2,3)$, into the rectangle ABC_1D_1 with the same base and height.



Solution:

For the linear transformation in two dimensions, we get the formula by moving two different points, and then just apply it to all other points of the shape. Let's choose the points B and C.

Let (u, v) be the coordinates after the transformation. The standard form for the linear transformation is

$$x = au + bv$$

$$y = cu + dv$$

where a, b, c, and d are real numbers. Since the y coordinate is not changing, we can simplify the formula for y.

$$x = au + bv$$

$$y = v$$

Since $B(4,0)$ transforms to $B(4,0)$, we can plug in $u = 4$, $x = 4$, and $v = 0$ into the first equation.

$$4 = a(4) + b(0)$$

$$a = 1$$

Since $C(6,3)$ transforms to $C_1(4,3)$, we can plug in $u = 4$, $x = 6$, and $v = 3$ to the first equation.

$$6 = 1(4) + b(3)$$

$$3b = 2$$

$$b = \frac{2}{3}$$

So the transformation formulas are

$$x = u + \frac{2}{3}v$$

$$y = v$$

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(u + \frac{2}{3}v\right) = 1$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(u + \frac{2}{3}v\right) = \frac{2}{3}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(v) = 0$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(v) = 1$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = 1 \cdot 1 - \frac{2}{3} \cdot 0 = 1$$

JACOBIAN FOR THREE VARIABLES

■ 1. Find the Jacobian of the transformation that rotates the space clockwise about the y-axis by $\pi/6$.

Solution:

Since the y-coordinate remains unchanged, we can use a two-dimensional formula of rotation in the xz-plane. Let u, v, and w be the coordinates after the transformation. The rotation by angle ϕ is given by

$$x = u\cos\phi - w\sin\phi$$

$$z = u\sin\phi + w\cos\phi$$

Plug in $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$ to get the final transformation equations.

$$x = \frac{\sqrt{3}}{2}u - \frac{1}{2}w$$

$$y = v$$

$$z = \frac{1}{2}u + \frac{\sqrt{3}}{2}w$$

The partial derivatives of these are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{3}}{2}u - \frac{1}{2}w\right) = \frac{\sqrt{3}}{2}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{3}}{2}u - \frac{1}{2}w\right) = 0$$

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}\left(\frac{\sqrt{3}}{2}u - \frac{1}{2}w\right) = -\frac{1}{2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial v}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = \frac{\partial v}{\partial v} = 1$$

$$\frac{\partial y}{\partial w} = \frac{\partial v}{\partial w} = 0$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}w\right) = \frac{1}{2}$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}w\right) = 0$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}w\right) = \frac{\sqrt{3}}{2}$$

The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial u}\right)$$

$$+\frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u}\cdot\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial u}\right)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\sqrt{3}}{2}\left(1\cdot\frac{\sqrt{3}}{2} - 0\cdot 0\right) - 0\cdot\left(0\cdot\frac{\sqrt{3}}{2} - 0\cdot\frac{1}{2}\right) - \frac{1}{2}\left(0\cdot 0 - 1\cdot\frac{1}{2}\right) = 1$$

■ 2. Find the Jacobian of the transformation to spherical coordinates that converts the ellipsoid to the unit sphere (a sphere with center at the origin and radius 1).

$$\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{9} = 1$$

Solution:

The transformation formulas to spherical coordinates are

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho \cos \phi$$

Since the ellipsoid has its center at the origin, we can use the standard spherical coordinates, but scaled on all three directions. Since the semi-axis is the x-direction has length 2, the semi-axis in the y-direction has length 5, and the semi-axis in the z-direction has length 3, we need to scale the x, y, and z directions by 2, 5, and 3 respectively. So the transformation formulas are

$$x = 2\rho \sin \phi \cos \theta$$

$$y = 5\rho \sin \phi \sin \theta$$

$$z = 3\rho \cos \phi$$

Let's check that the formulas are correct by plugging x, y, and z into the equation of the ellipsoid.

$$\frac{(2\rho \sin \phi \cos \theta)^2}{4} + \frac{(5\rho \sin \phi \sin \theta)^2}{25} + \frac{(3\rho \cos \phi)^2}{9} = 1$$

$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 + (\rho \cos \phi)^2 = 1$$

$$\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi = 1$$

$$\rho^2 = 1$$

$$\rho = 1$$

This is the equation of the unit sphere in spherical coordinates. The partial derivatives are

$$\frac{\partial x}{\partial \rho} = \frac{\partial}{\partial \rho}(2\rho \sin\phi \cos\theta) = 2\sin\phi\cos\theta$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(2\rho \sin\phi \cos\theta) = -2\rho\sin\phi\sin\theta$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(2\rho \sin\phi \cos\theta) = 2\rho\cos\phi\cos\theta$$

$$\frac{\partial y}{\partial \rho} = \frac{\partial}{\partial \rho}(5\rho \sin\phi \sin\theta) = 5\sin\phi\sin\theta$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(5\rho \sin\phi \sin\theta) = 5\rho\sin\phi\cos\theta$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(5\rho \sin\phi \sin\theta) = 5\rho\cos\phi\sin\theta$$

$$\frac{\partial z}{\partial \rho} = \frac{\partial}{\partial \rho}(3\rho\cos\phi) = 3\cos\phi$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}(3\rho\cos\phi) = 0$$

$$\frac{\partial z}{\partial \phi} = \frac{\partial}{\partial \phi}(3\rho\cos\phi) = -3\rho\sin\phi$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\frac{\partial x}{\partial \rho}\left(\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\cdot\frac{\partial z}{\partial \theta}\right)-\frac{\partial x}{\partial \theta}\left(\frac{\partial y}{\partial \rho}\cdot\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\cdot\frac{\partial z}{\partial \rho}\right)$$

$$+\frac{\partial x}{\partial \phi}\left(\frac{\partial y}{\partial \rho}\cdot\frac{\partial z}{\partial \theta}-\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial \rho}\right)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=2\sin\phi\cos\theta(-5\rho\sin\phi\cos\theta\cdot 3\rho\sin\phi-5\rho\cos\phi\sin\theta\cdot 0)$$

$$+2\rho\sin\phi\sin\theta(-5\sin\phi\sin\theta\cdot 3\rho\sin\phi-5\rho\cos\phi\sin\theta\cdot 3\cos\phi)$$

$$+2\rho\cos\phi\cos\theta(5\sin\phi\sin\theta\cdot 0-5\rho\sin\phi\cos\theta\cdot 3\cos\phi)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-30\rho^2\sin^3\phi\cos^2\theta-30\rho^2\sin\phi\sin^2\theta-30\rho^2\sin\phi\cos^2\phi\cos^2\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-30\rho^2\sin\phi$$

■ 3. Find the following transformations:

1. the transformation that converts the given cylinder to a circular cylinder with radius 1 and an axis parallel to the z-axis

$$\frac{x^2}{5}+\frac{z^2}{4}=1$$

2. the transformation that converts to cylindrical coordinates

Then write down the composition of these two transformations and find its Jacobian.

Solution:

1. Let u, v, and w be the transformed coordinates. To change the axis of the cylinder, we can just change the order of the coordinates y and z as follows.

$$x = u$$

$$y = w$$

$$z = v$$

To scale the cylinder by $\sqrt{5}$ in the x direction and by 2 in z direction, we can multiply the transformation equations by $\sqrt{5}$ and 2 respectively. So

$$x = \sqrt{5}u$$

$$y = w$$

$$z = 2v$$

Let's check that the formulas are correct by plugging x, y, and z into the equation of the cylinder.

$$\frac{(\sqrt{5}u)^2}{5} + \frac{(2v)^2}{4} = 1$$

$$u^2 + v^2 = 1$$

Therefore, we get the circular cylinder with radius 1 and axis along the w-axis in u, v, and w coordinates.

2. We need to use the standard conversion formulas for cylindrical coordinates:

$$u = r\cos\theta$$

$$v = r\sin\theta$$

$$w = w$$

Then the composition of transformations is given by

$$x = \sqrt{5}r\cos\theta$$

$$y = w$$

$$z = 2r\sin\theta$$

We can check that the composition is correct by plugging x, y, and z into the equation of the cylinder.

$$\frac{(\sqrt{5}r\cos\theta)^2}{5} + \frac{(2r\sin\theta)^2}{4} = 1$$

$$r^2\cos\theta + r^2\sin\theta = 1$$

$$r^2 = 1$$

$$r = 1$$

This is the correct equation of the cylinder with radius 1. The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}\left(\sqrt{5}r\cos\theta\right) = \sqrt{5}\cos\theta$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(\sqrt{5}r\cos\theta) = -\sqrt{5}r\sin\theta$$

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}(\sqrt{5}r\cos\theta) = 0$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(w) = 0$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(w) = 0$$

$$\frac{\partial y}{\partial w} = \frac{\partial}{\partial w}(w) = 1$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(2r\sin\theta) = 2\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}(2r\sin\theta) = 2r\cos\theta$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}(2r\sin\theta) = 0$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,w)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \frac{\partial x}{\partial r}\left(\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial \theta}\right) - \frac{\partial x}{\partial \theta}\left(\frac{\partial y}{\partial r}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial r}\right)$$

$$+\frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial r}\cdot\frac{\partial z}{\partial \theta}-\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial r}\right)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\sqrt{5}\cos\theta(0\cdot 0-1\cdot 2r\cos\theta)+\sqrt{5}r\sin\theta(0\cdot 0-1\cdot 2\sin\theta)$$

$$+0\cdot(0\cdot 2r\cos\theta-0\cdot 2\sin\theta)$$

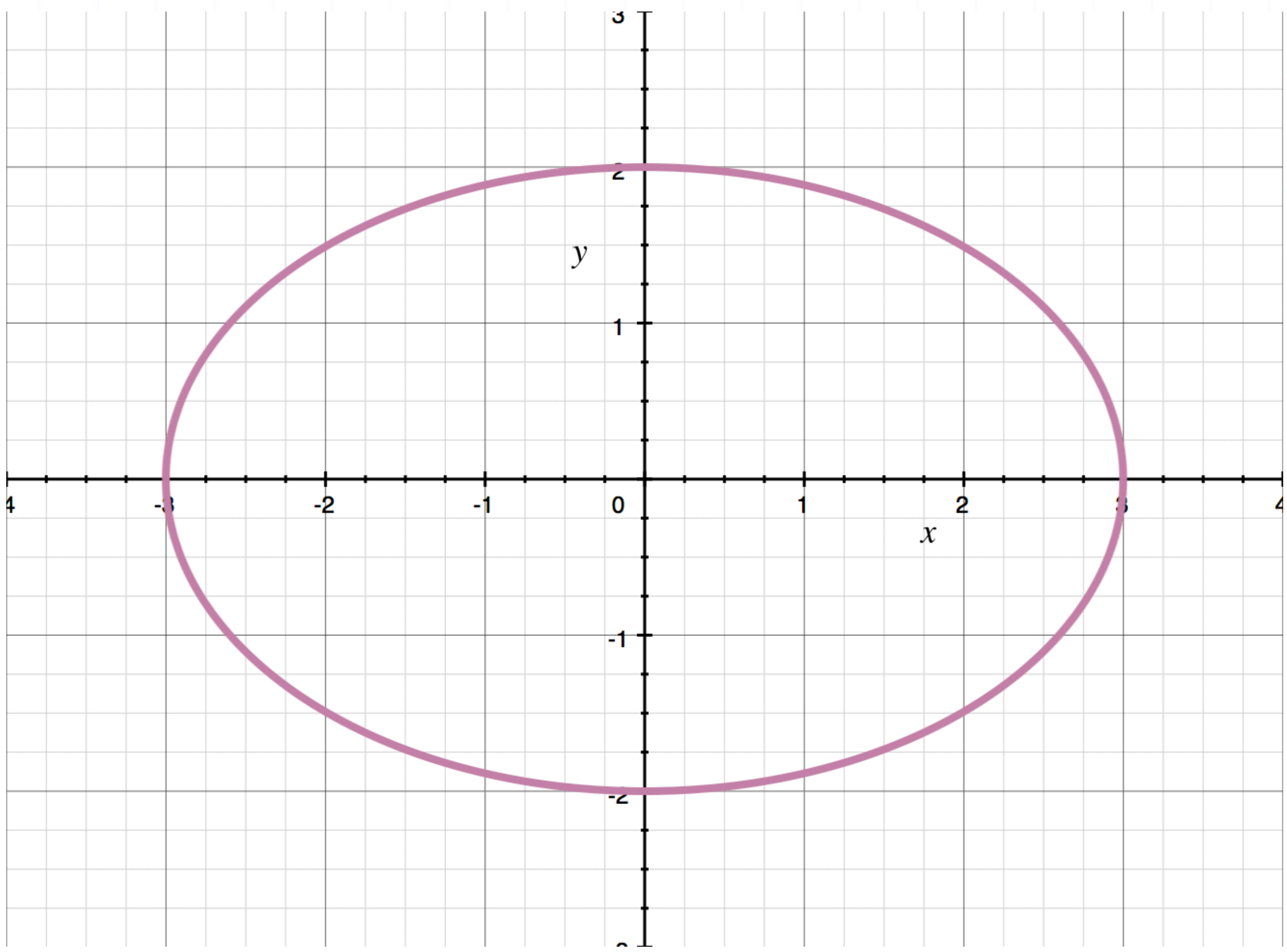
$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-2\sqrt{5}r$$

EVALUATING DOUBLE INTEGRALS

■ 1. Find the Jacobian of the transformation and use it to find the area of the ellipse with center at the origin, semi-major axis along the x-axis with length 3, and semi-minor axis along the y-axis with length 2.

$$x = 3r\cos\phi$$

$$y = 2r\sin\phi$$



Solution:

The equation of the ellipse is

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

Plug in $x = 3r\cos\phi$ and $y = 2r\sin\phi$.

$$\frac{3^2r^2\cos^2\phi}{3^2} + \frac{2^2r^2\sin^2\phi}{2^2} = 1$$

$$r^2\cos^2\phi + r^2\sin^2\phi = 1$$

$$r^2 = 1$$

$$r = 1$$

So the equation of the ellipse in the transformed coordinates is $r = 1$. Therefore, the value of r changes from 0 to 1, and ϕ changes from 0 to 2π. The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(3r\cos\phi) = 3\cos\phi$$

$$\frac{\partial x}{\partial\phi} = \frac{\partial}{\partial\phi}(3r\cos\phi) = -3r\sin\phi$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(2r\sin\phi) = 2\sin\phi$$

$$\frac{\partial y}{\partial\phi} = \frac{\partial}{\partial\phi}(2r\sin\phi) = 2r\cos\phi$$

The Jacobian of the transformation is then

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = 3\cos\phi \cdot 2r\cos\phi - (-3r\sin\phi) \cdot 2\sin\phi$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = 6r$$

The area of the ellipse is

$$\int_0^{2\pi} \int_0^1 6r \ dr \ d\phi$$

$$\int_0^{2\pi} 3r^2 \Big|_0^1 \ d\phi$$

$$\int_0^{2\pi} 3 \ d\phi$$

$$3\phi \Big|_0^{2\pi}$$

$$6\pi$$

■ 2. Find the Jacobian of the transformation and use it to find the double integral of the function $f(x, y) = x^2 + y^2$ over the circle with center at $(-2,3)$ and radius 2.

$$x = -2 + r\cos\phi$$

$$y = 3 + r\sin\phi$$

Solution:

The equation of the circle is

$$(x+2)^2 + (y-3)^2 = 2^2$$

Plug in $x = -2 + r\cos\phi$ and $y = 3 + r\sin\phi$.

$$(-2 + r\cos\phi + 2)^2 + (3 + r\sin\phi - 3)^2 = 2^2$$

$$r^2\cos^2\phi + r^2\sin^2\phi = 2^2$$

$$r^2 = 2^2$$

$$r = 2$$

So the equation of the circle in the transformed coordinates is $r = 2$. Therefore, the value or r changes from 0 to 2, and ϕ changes from 0 to 2π.

The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(-2 + r\cos\phi) = \cos\phi$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(-2 + r\cos\phi) = -r\sin\phi$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(3 + r\sin\phi) = \sin\phi$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(3 + r\sin\phi) = r\cos\phi$$

The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \cos\phi \cdot r\cos\phi - (-r\sin\phi) \cdot \sin\phi$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = r$$

The function is

$$(-2 + r\cos\phi)^2 + (3 + r\sin\phi)^2$$

$$r^2\cos^2\phi + r^2\sin^2\phi - 4r\cos\phi + 6r\sin\phi + 13$$

$$r^2 - 4r\cos\phi + 6r\sin\phi + 13$$

Therefore, the integral is,

$$\int_0^{2\pi}\int_0^2 (r^2 - 4r\cos\phi + 6r\sin\phi + 13)r\ dr\ d\phi$$

$$\int_0^{2\pi}\int_0^2 r^3 - 4r^2\cos\phi + 6r^2\sin\phi + 13r\ dr\ d\phi$$

Since the integrals of sine and cosine functions over a 2π-period are equal to 0, the double integral simplifies to

$$\int_0^{2\pi}\int_0^2 r^3+13r\ dr\ d\phi$$

Integrate with respect to r.

$$\int_0^{2\pi}\frac{1}{4}r^4+\frac{13}{2}r^2\Big|_0^2\ d\phi$$

$$\int_0^{2\pi}\frac{1}{4}(2)^4+\frac{13}{2}(2)^2\ d\phi$$

$$\int_0^{2\pi}4+26\ d\phi$$

$$\int_0^{2\pi}30\ d\phi$$

Integrate with respect to ϕ.

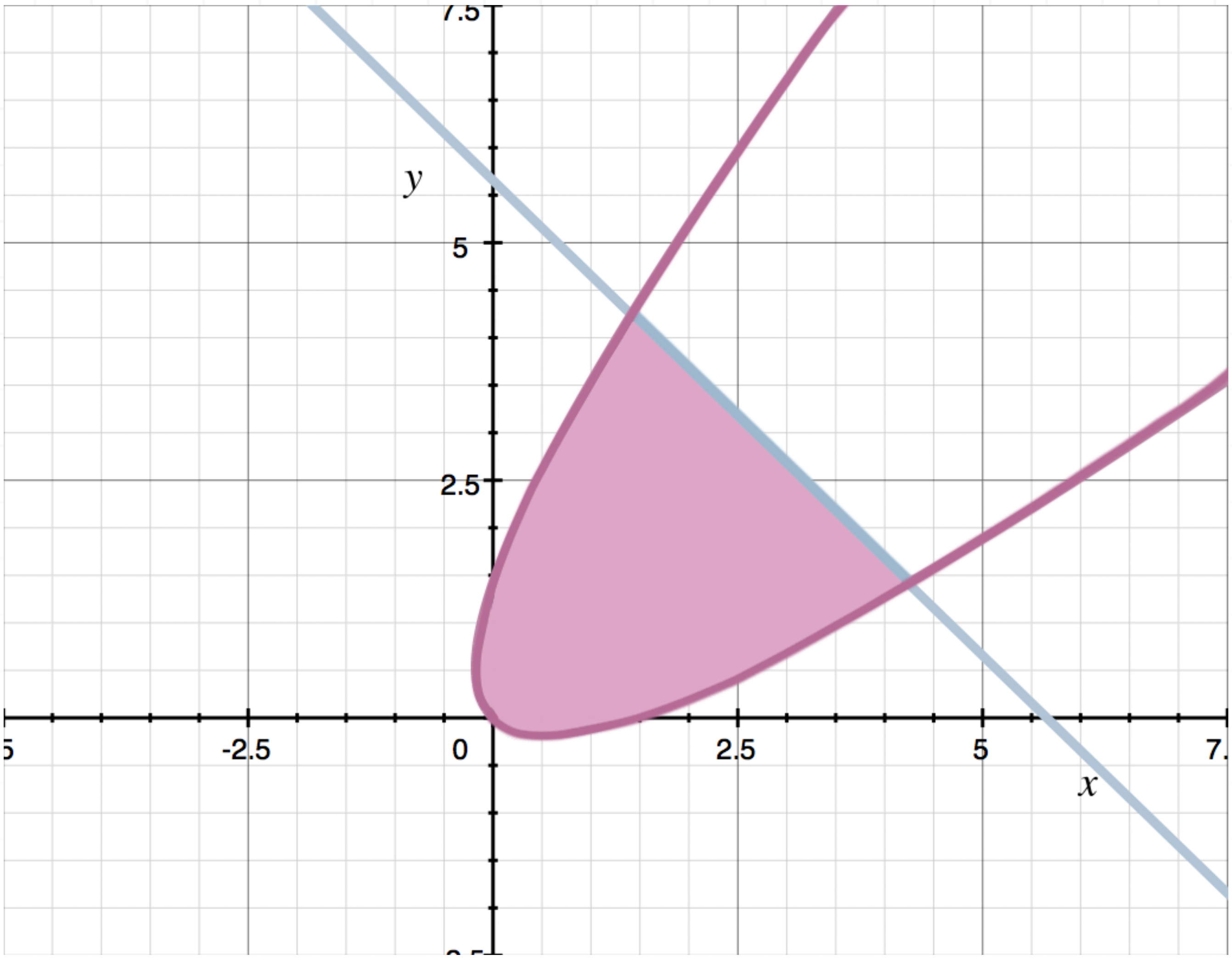
$$30\phi\Big|_0^{2\pi}$$

$$60\pi$$

■ 3. Find the Jacobian of the transformation, and use it to find the area bounded by the curves $x^2-2xy+y^2-\sqrt{2}x-\sqrt{2}y=0$ and $x+y=4\sqrt{2}$.

$$x=\frac{\sqrt{2}}{2}(u+v)$$

$$y = \frac{\sqrt{2}}{2}(v - u)$$



Solution:

The equation of the parabola is

$$x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y = 0$$

Plug in the transformation equations.

$$\frac{2}{4}(u+v)^2 - 2 \cdot \frac{\sqrt{2}}{2}(u+v) \cdot \frac{\sqrt{2}}{2}(v-u) + \frac{2}{4}(v-u)^2 - \sqrt{2} \cdot \frac{\sqrt{2}}{2}(u+v) - \sqrt{2} \cdot \frac{\sqrt{2}}{2}(v-u) = 0$$

$$\frac{1}{2}(u^2 + 2uv + v^2) - (v^2 - u^2) + \frac{1}{2}(v-u)^2 - (u+v) - (v-u) = 0$$

$$\frac{1}{2}u^2 + uv + \frac{1}{2}v^2 - v^2 + u^2 + \frac{1}{2}v^2 - uv + \frac{1}{2}u^2 - 2v = 0$$

$$2u^2 - 2v = 0$$

$$v = u^2$$

So the equation of the parabola in the transformed coordinates is $v = u^2$. The equation of the line is

$$x + y = 4\sqrt{2}$$

Plug in $x = (\sqrt{2}/2)(u + v)$ and $y = (\sqrt{2}/2)(v - u)$.

$$x + y = 4\sqrt{2}$$

$$\frac{\sqrt{2}}{2}(u + v) + \frac{\sqrt{2}}{2}(v - u) = 4\sqrt{2}$$

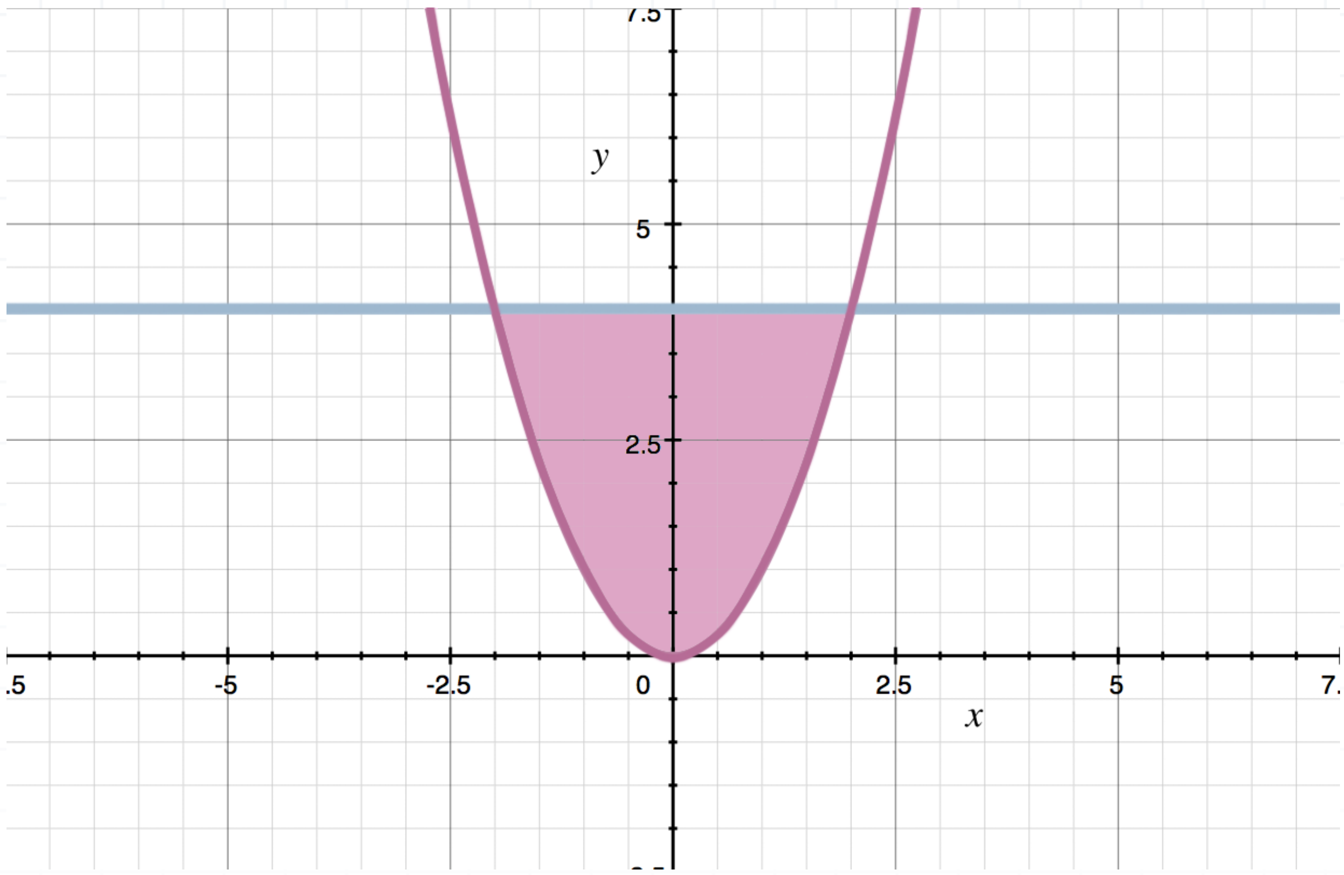
$$v = 4$$

So the equation of the line in the transformed coordinates is $v = 4$. Therefore, we need to find the area between the curves $v = u^2$ and $v = 4$. Find the points of intersection.

$$u^2 = 4$$

$$u = \pm 2$$

So the intersection points are $(-2,4)$ and $(2,4)$.



Therefore, the value of u changes from -2 to 2, and v changes from u^2 to 4.

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{2}}{2}(u+v)\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{2}}{2}(u+v)\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{2}}{2}(v-u)\right) = -\frac{\sqrt{2}}{2}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{2}}{2}(v-u)\right) = \frac{\sqrt{2}}{2}$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} = 1$$

The area of the region E bounded by the curves $v = u^2$ and $v = 4$ is

$$\int_{-2}^{2} \int_{u^2}^{4} dv\ du$$

$$\int_{-2}^{2} v \Big|_{u^2}^{4} du$$

$$\int_{-2}^{2} 4 - u^2\ du$$

$$4u - \frac{1}{3}u^3 \Big|_{-2}^{2}$$

$$4(2) - \frac{1}{3}(2)^3 - \left(4(-2) - \frac{1}{3}(-2)^3\right)$$

$$8 - \frac{8}{3} + 8 - \frac{8}{3}$$

$$16 - \frac{16}{3}$$

$$\frac{32}{3}$$

EQUATIONS OF THE TRANSFORMATION

1. Identify the equation obtained from $f(x, y, z) = x^2 + y^2 + z^2 - 2$ by applying the transformation.

$$x = \sin u + \cos u$$

$$y = \sin u - \cos u$$

$$z = \sqrt{u + v + w}$$

Solution:

Substitute expressions for x, y, and z into the equation of $f(x, y, z)$.

$$f(x, y, z) = x^2 + y^2 + z^2 - 2$$

$$f(u, v, w) = (\sin u + \cos u)^2 + (\sin u - \cos u)^2 + (\sqrt{u + v + w})^2 - 2$$

$$f(u, v, w) = \sin^2 u + 2\sin u \cos u + \cos^2 u + \sin^2 u$$

$$-2\sin u \cos u + \cos^2 u + (u + v + w) - 2$$

$$f(u, v, w) = 2\sin^2 u + 2\cos^2 u + u + v + w - 2$$

$$f(u, v, w) = 2 + u + v + w - 2$$

$$f(u, v, w) = u + v + w$$

■ 2. Find the inverse transformation and determine its Jacobian.

$$u = x - 2y + 1$$

$$v = -3x + y + 2$$

Solution:

Solve the transformation equations as a system of equations.

$$x = u + 2y - 1$$

$$v = -3(u + 2y - 1) + y + 2$$

$$v = -3u - 6y + 3 + y + 2$$

$$5y = -3u - v + 5$$

$$y = -\frac{3u}{5} - \frac{v}{5} + 1$$

Substitute y back to the first equation to find x.

$$x = u + 2\left(-\frac{3u}{5} - \frac{v}{5} + 1\right) - 1$$

$$x = u - \frac{6u}{5} - \frac{2v}{5} + 2 - 1$$

$$x = -\frac{u}{5} - \frac{2v}{5} + 1$$

So the inverse transformation is

$$x = -\frac{u}{5} - \frac{2v}{5} + 1$$

$$y = -\frac{3u}{5} - \frac{v}{5} + 1$$

The partial derivatives of these equations are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(-\frac{u}{5} - \frac{2v}{5} + 1\right) = -\frac{1}{5}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(-\frac{u}{5} - \frac{2v}{5} + 1\right) = -\frac{2}{5}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(-\frac{3u}{5} - \frac{v}{5} + 1\right) = -\frac{3}{5}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(-\frac{3u}{5} - \frac{v}{5} + 1\right) = -\frac{1}{5}$$

The Jacobian of the transformation is the

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right) - \left(-\frac{2}{5}\right)\left(-\frac{3}{5}\right)$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{25} - \frac{6}{25}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = -\frac{1}{5}$$

■ 3. Find the inverse transformation $x(r, \phi)$ and $y(r, \phi)$ and determine its Jacobian.

$$r = \sqrt{\frac{x^2 + y^2}{4}}$$

$$\phi = \arctan \frac{y}{x} \text{ for } x \neq 0$$

Solution:

We need to find expressions for x and y in the form $x(r, \phi)$ and $y(r, \phi)$. So we'll solve the system of transformation equations for x and y. Since the given transformation is almost the same as inverse to the conversion to the polar coordinates, we can substitute $x = kr\cos\phi$ and $y = kr\sin\phi$, where k is an unknown constant.

$$r = \sqrt{\frac{(kr\cos\phi)^2 + (kr\sin\phi)^2}{4}}$$

$$r = \sqrt{\frac{k^2r^2(\cos^2\phi + \sin^2\phi)}{4}}$$

$$r = \sqrt{\frac{k^2r^2}{4}}$$

$$r = \frac{kr}{2}$$

$$k = 2$$

Since $k = 2$, we can substitute $x = 2r\cos\phi$ and $y = 2r\sin\phi$ to check if the second equation holds.

$$\phi = \arctan\left(\frac{2r\sin\phi}{2r\cos\phi}\right)$$

$$\phi = \arctan\left(\frac{\sin\phi}{\cos\phi}\right)$$

$$\phi = \arctan(\tan\phi)$$

$$\phi = \phi$$

So the transformation is

$$x = 2r\cos\phi$$

$$y = 2r\sin\phi$$

The partial derivatives of these are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(2r\cos\phi) = 2\cos\phi$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(2r\cos\phi) = -2r\sin\phi$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(2r\sin\phi) = 2\sin\phi$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(2r \sin \phi) = 2r \cos \phi$$

Then the Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(r, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial(x, y)}{\partial(r, \phi)} = 2 \cos \phi \cdot 2r \cos \phi - (-2r \sin \phi) \cdot 2 \sin \phi$$

$$\frac{\partial(x, y)}{\partial(r, \phi)} = 4r$$

IMAGE OF THE SET UNDER THE TRANSFORMATION

■ 1. Identify the surface obtained from the unit sphere (a sphere with center at the origin and radius 1 by applying the transformation.

$$x = u - 2v + 2$$

$$y = u + v + w$$

$$z = u + v - w + 4$$

Solution:

The equation of the unit sphere is

$$x^2 + y^2 + z^2 = 1$$

Plug in the expressions for x, y, and z.

$$(u - 2v + 2)^2 + (u + v + w)^2 + (u + v - w + 4)^2 = 1$$

$$(u^2 - 4uv + 4u + 4v^2 - 8v + 4) + (u^2 + v^2 + w^2 + 2uv + 2uw + 2vw)$$

$$+(u^2 + 2uv - 2uw + 8u + v^2 - 2vw + 8v + w^2 - 8w + 16) = 1$$

$$3u^2 + 12u + 6v^2 + 2w^2 - 8w + 19 = 0$$

Complete the square with respect to each variable.

$$3(u^2 + 4u + 4 - 4) + 6v^2 + 2(w^2 - 4w + 4 - 4) + 19 = 0$$

$$3(u+2)^2 - 12 + 6v^2 + 2(w-2)^2 - 8 + 19 = 0$$

$$3(u+2)^2 + 6v^2 + 2(w-2)^2 = 1$$

So the sphere moves to the ellipsoid with center at $(-2,0,2)$, and semi-axes $x = 1/\sqrt{3}$, $y = 1/\sqrt{6}$, and $z = 1/\sqrt{2}$.

■ 2. Identify the shape obtained from the parallelogram $ABCD$, where $A(-2,2)$, $B(-2,5)$, $C(-3,6)$, and $D(-3,3)$, by applying the transformation.

$$u = -x - 2$$

$$v = \frac{x+y}{3}$$

Solution:

Since the transformation converts line segments into segments, we can just substitute the coordinates of A, B, C, and D to get a new quadrilateral $A_1B_1C_1D_1$.

Transforming A gives $A_1(0,0)$.

$$u = -(-2) - 2 = 0$$

$$v = \frac{(-2)+(2)}{3} = 0$$

Transforming B gives $B_1(0,1)$.

$$u = -(-2) - 2 = 0$$

$$v = \frac{(-2) + (5)}{3} = 1$$

Transforming C gives $C_1(1,1)$.

$$u = -(-3) - 2 = 1$$

$$v = \frac{(-3) + (6)}{3} = 1$$

Transforming D gives $D_1(1,0)$.

$$u = -(-3) - 2 = 1$$

$$v = \frac{(-3) + (3)}{3} = 0$$

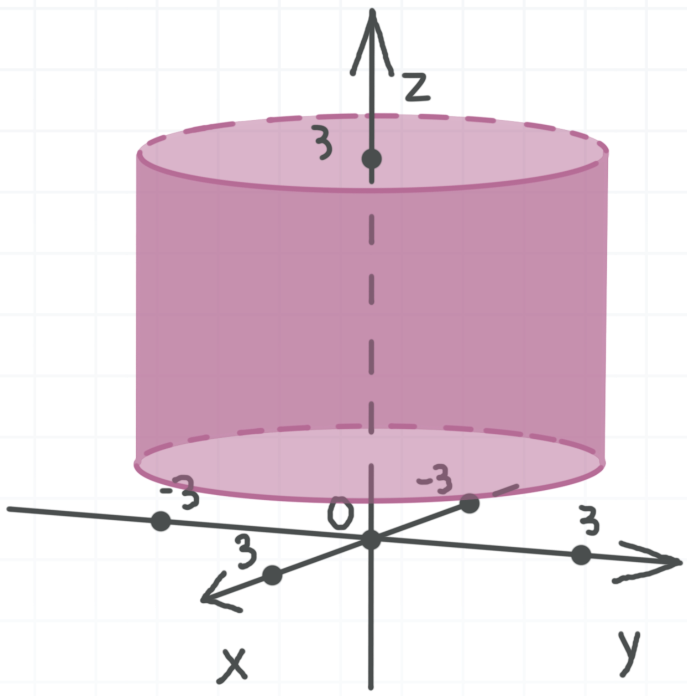
Therefore, parallelogram $ABCD$ transforms into the unit square $A_1B_1C_1D_1$ in the first quadrant, where $A_1(0,0)$, $B_1(0,1)$, $C_1(1,1)$, and $D_1(1,0)$.

■ 3. Identify the solid obtained from the set of interior points of the circular cylinder $x^2 + y^2 = 9$, with $1 \leq z \leq 5$, when the following transformation is applied.

$$u = \sqrt{x^2 + y^2}$$

$$v = \arctan \frac{y}{x}$$

$$w = z - 1$$



Solution:

Since we have the transformation $(x, y, z) \to (u, v, w)$, we need first to find the inverse transformation $(u, v, w) \to (x, y, z)$ by solving the system of equations for x, y, and z. We want to realize that the given transformation is almost an inverse to standard cylindrical coordinates with an exception in z/w pair.

From the third equation, $z = w + 1$. From the second equation, $y/x = \tan v$. Let $u = \sqrt{x^2 + y^2} = r$, $x = r\cos\phi$, and $y = r\sin\phi$, then substitute.

$$\frac{r\sin\phi}{r\cos\phi} = \tan v$$

$$\frac{\sin\phi}{\cos\phi} = \tan v$$

$$\tan\phi = \tan v$$

$$\phi = v$$

So the transformation is

$$x = u\cos v$$

$$y = u\sin v$$

$$z = w + 1$$

To find the borders for u, substitute x and y into the cylinder's equation.

$$x^2 + y^2 = 9$$

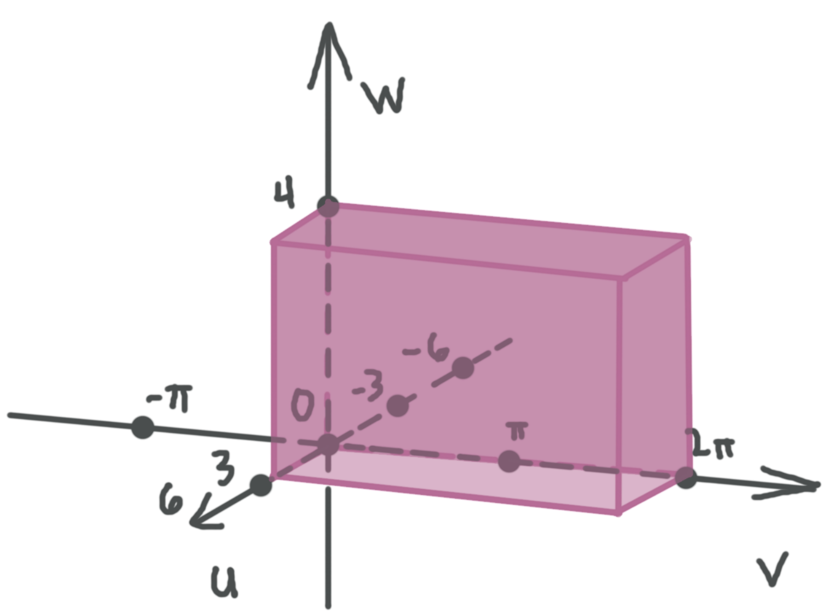
$$(u\cos\phi)^2 + (u\sin\phi)^2 = 3^2$$

$$u^2 = 3^2$$

$$u = 3$$

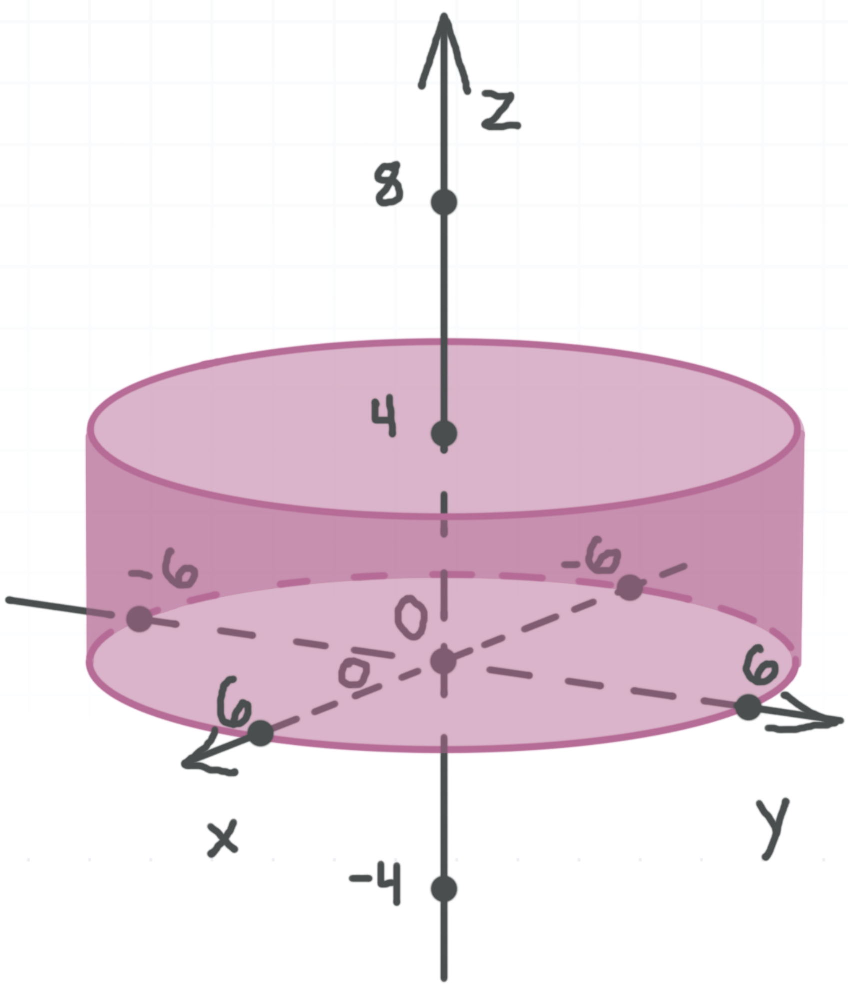
So $0 \le u \le 3$ and $0 \le v < 2\pi$, and since z changes from 1 to 5, $0 \le w \le 4$.

Therefore, the image of the cylinder is the rectangular prism with dimensions $3 \times 2\pi \times 4$.



TRIPLE INTEGRALS TO FIND MASS AND CENTER OF MASS

■ 1. The disk with radius 6 and height 4 has density $\delta = 1/(d+2)$, where d is the distance to the central axis of the disk. Find the mass and center of mass of the disk.



Solution:

Consider the cylinder with radius 6 and height 4, whose base lies in the xy-plane with its center at the origin. Within the cylinder, r changes from 0 to 6, θ changes from 0 to 2π, and z changes from 0 to 4. The density function is

$$\delta = \frac{1}{r+2}$$

Therefore, the integral representing mass in cylindrical coordinates is

$$\int_0^4 \int_0^{2\pi} \int_0^6 \frac{1}{r+2}\ r\ dr\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^6 \frac{r+2-2}{r+2}\ dr\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^6 \frac{r+2}{r+2} - \frac{2}{r+2}\ dr\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^6 1 - \frac{2}{r+2}\ dr\ d\theta\ dz$$

Integrate with respect to r.

$$\int_0^4 \int_0^{2\pi} r - 2\ln|r+2| \Big|_0^6\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} 6 - 2\ln|6+2| - (0 - 2\ln|0+2|)\ d\theta\ dz$$

$$\int_0^4 \int_0^{2\pi} 6 - 2\ln 8 + 2\ln 2\ d\theta\ dz$$

Integrate with respect to θ.

$$\int_0^4 6\theta - 2\theta\ln 8 + 2\theta\ln 2 \Big|_0^{2\pi}\ dz$$

$$\int_0^4 6(2\pi) - 2(2\pi)\ln 8 + 2(2\pi)\ln 2 \ dz$$

$$\int_0^4 12\pi - 4\pi \ln 8 + 4\pi \ln 2 \ dz$$

Integrate with respect to z.

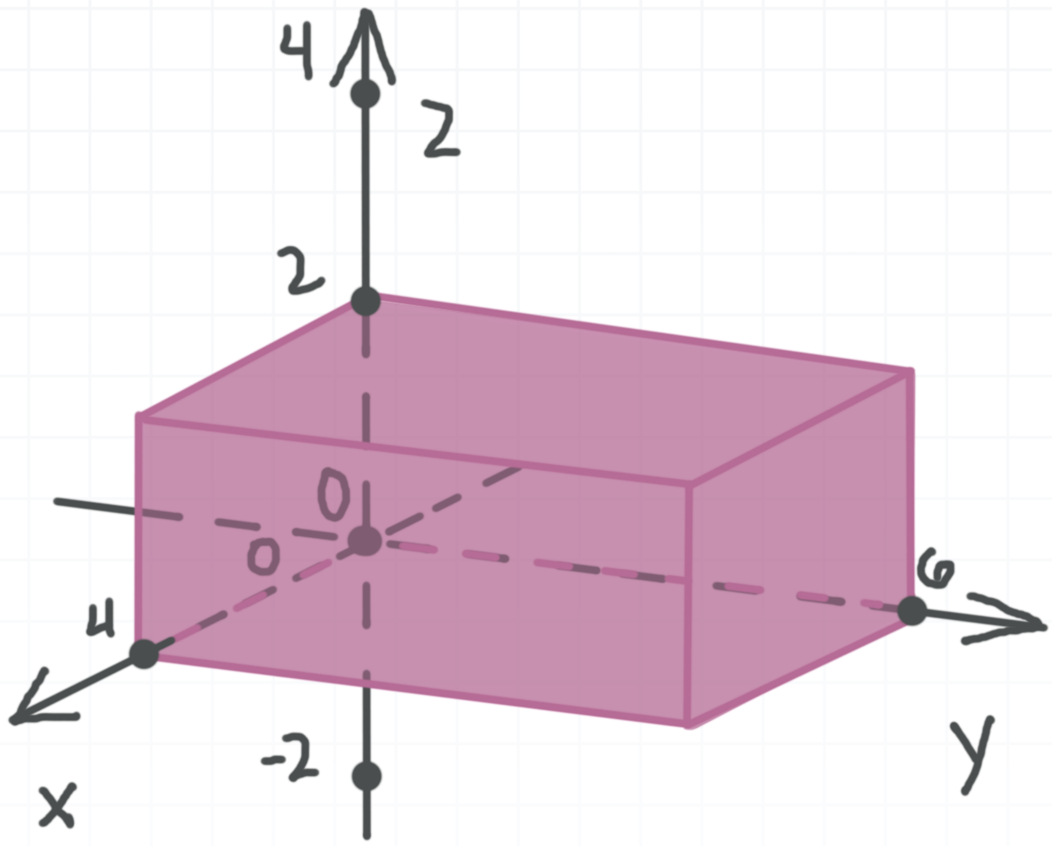
$$12\pi z - 4\pi z \ln 8 + 4\pi z \ln 2 \Big|_0^4$$

$$12\pi(4) - 4\pi(4)\ln 8 + 4\pi(4)\ln 2$$

$$48\pi - 16\pi \ln 8 + 16\pi \ln 2$$

Since the disk is symmetric and has symmetric density about the x- and y-axes, its center of mass lies on its axis. Also, since the mass is equally distributed along the z-axis, its center of mass lies on the half of its axis. Therefore, the center of mass has coordinates $(0,0,2)$.

■ 2. The rectangular plate with base dimensions 4×5 m and height 2 m has density $\delta = 4d$ kg/m^2, where d is the distance from its 4×2 m left face. Find the mass and center of mass of the plate.



Solution:

Place the origin at the bottom left corner of the plate, and the x- and y-axes along the dimensions 4 and 5 respectively. So the value of x changes from 0 to 4, y changes from 0 to 5, and z changes from 0 to 2. The density is equal to $4y$, so the triple integral will be

$$\int_0^2 \int_0^5 \int_0^4 4y \ dx \ dy \ dz$$

$$\int_0^2 \int_0^5 4xy \Big|_0^4 \ dy \ dz$$

$$\int_0^2 \int_0^5 16y \ dy \ dz$$

$$\int_0^2 8y^2\Big|_0^5 \ dz$$

$$\int_0^2 200 \ dz$$

$$200z\Big|_0^2$$

$$400$$

So the mass M of the plate is 400 kg. Find the center of mass. We get

$$\bar{x} = \frac{1}{M}\iiint_V x\delta(x, y, z) \ dV$$

$$\bar{x} = \frac{1}{400}\int_0^2\int_0^5\int_0^4 4xy \ dx \ dy \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2\int_0^5 2x^2y\Big|_{x=0}^{x=4} \ dy \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2\int_0^5 32y \ dy \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2 16y^2\Big|_0^5 \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2 400 \ dz$$

$$\bar{x} = \frac{400z}{400}\Big|_0^2$$

$$\bar{x} = z\Big|_0^2$$

$$\bar{x} = 2$$

and

$$\bar{y} = \frac{1}{M}\iiint_V y\delta(x, y, z)\ dV$$

$$\bar{y} = \frac{1}{400}\int_0^2\int_0^5\int_0^4 4y^2\ dx\ dy\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2\int_0^5 4xy^2\Big|_{x=0}^{x=4}\ dy\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2\int_0^5 16y^2\ dy\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2 \frac{16}{3}y^3\Big|_0^5\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2 \frac{2{,}000}{3}\ dz$$

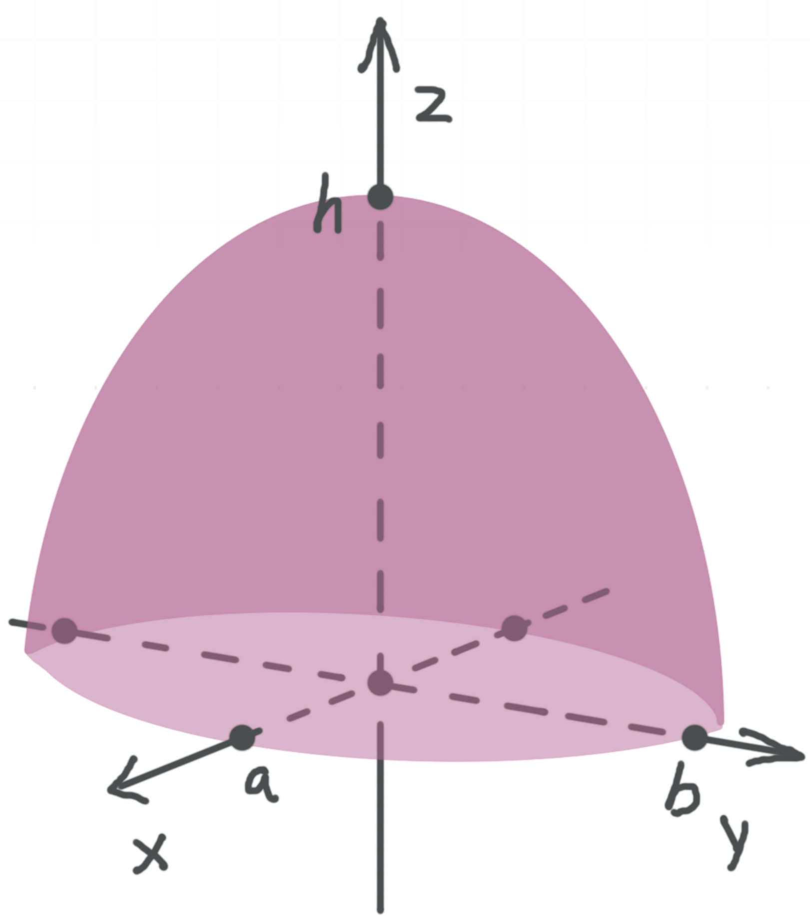
$$\bar{y} = \int_0^2 \frac{5}{3}\ dz$$

$$\bar{y} = \frac{5}{3}z\Big|_0^2$$

$$\bar{y} = \frac{10}{3}$$

Since the mass is equally distributed along the z-axis, the center of mass lies at half-plate height, so $\bar{z} = 2/2 = 1$, and the center of mass is at $(\bar{x}, \bar{y}, \bar{z}) = (2,10/3,1)$.

■ 3. The half ellipsoid has a base with semi-axes a and b, height h, and constant density δ. Find its mass and center of mass.



Solution:

Position the base in the xy-plane with the semi-axes along the x and y axes. Then the equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{h^2}=1$$

Within the half ellipsoid, the value of ρ changes from 0 to 1, θ changes from 0 to 2π, and ϕ changes from 0 to $\pi/2$. The partial derivatives are

$$\frac{\partial x}{\partial \rho}=\frac{\partial}{\partial \rho}(a\rho\sin\phi\cos\theta)=a\sin\phi\cos\theta$$

$$\frac{\partial x}{\partial \theta}=\frac{\partial}{\partial \theta}(a\rho\sin\phi\cos\theta)=-a\rho\sin\phi\sin\theta$$

$$\frac{\partial x}{\partial \phi}=\frac{\partial}{\partial \phi}(a\rho\sin\phi\cos\theta)=a\rho\cos\phi\cos\theta$$

$$\frac{\partial y}{\partial \rho}=\frac{\partial}{\partial \rho}(b\rho\sin\phi\sin\theta)=b\sin\phi\sin\theta$$

$$\frac{\partial y}{\partial \theta}=\frac{\partial}{\partial \theta}(b\rho\sin\phi\sin\theta)=b\rho\sin\phi\cos\theta$$

$$\frac{\partial y}{\partial \phi}=\frac{\partial}{\partial \phi}(b\rho\sin\phi\sin\theta)=b\rho\cos\phi\sin\theta$$

$$\frac{\partial z}{\partial \rho}=\frac{\partial}{\partial \rho}(h\rho\cos\phi)=h\cos\phi$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial}{\partial \theta}(h\rho\cos\phi)=0$$

$$\frac{\partial z}{\partial \phi}=\frac{\partial}{\partial \phi}(h\rho\cos\phi)=-h\rho\sin\phi$$

The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$=\frac{\partial x}{\partial \rho}\left(\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\cdot\frac{\partial z}{\partial \theta}\right)-\frac{\partial x}{\partial \theta}\left(\frac{\partial y}{\partial \rho}\cdot\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\cdot\frac{\partial z}{\partial \rho}\right)$$

$$+\frac{\partial x}{\partial \phi}\left(\frac{\partial y}{\partial \rho}\cdot\frac{\partial z}{\partial \theta}-\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial \rho}\right)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=a\sin\phi\cos\theta(-b\rho\sin\phi\cos\theta\cdot h\rho\sin\phi-b\rho\cos\phi\sin\theta\cdot 0)$$

$$+a\rho\sin\phi\sin\theta(-b\sin\phi\sin\theta\cdot h\rho\sin\phi-b\rho\cos\phi\sin\theta\cdot h\cos\phi)$$

$$+a\rho\cos\phi\cos\theta(b\sin\phi\sin\theta\cdot 0-b\rho\sin\phi\cos\theta\cdot h\cos\phi)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-abh\rho^2\sin^3\phi\cos^2\theta-abh\rho^2\sin\phi\sin^2\theta-abh\rho^2\sin\phi\cos^2\phi\cos^2\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-abh\rho^2\sin\phi$$

Then the mass in spherical coordinates is given by

$$\delta\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 abh\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$abh\delta \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$abh\delta \int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^2 \ d\rho$$

Evaluate each integral.

$$abh\delta \left(-\cos\phi \Big|_0^{\frac{\pi}{2}} \right) \left(\theta \Big|_0^{2\pi} \right) \left(\frac{1}{3}\rho^3 \Big|_0^1 \right)$$

$$abh\delta (1)(2\pi)\left(\frac{1}{3}\right)$$

$$\frac{2\pi abh\delta}{3}$$

Since the half ellipsoid is symmetric about the x- and y-axis, its center of mass lies on the z-axis, and

$$\bar{z} = \frac{1}{M} \iiint_V z\delta(x, y, z) \ dV$$

$$\bar{z} = \frac{3}{2\pi abh\delta} \cdot \delta \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 h\rho\cos\phi \ abh\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\bar{z} = \frac{3h}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^3 \cos\phi \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\bar{z} = \frac{3h}{4\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^3 \sin 2\phi \ d\rho \ d\theta \ d\phi$$

$$\bar{z} = \frac{3h}{4\pi} \int_0^{\frac{\pi}{2}} \sin 2\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^3 \ d\rho$$

Evaluate each integral.

$$\bar{z} = \frac{3h}{4\pi} \left(-\frac{1}{2} \cos 2\phi \Big|_0^{\frac{\pi}{2}} \right) \left(\theta \Big|_0^{2\pi} \right) \left(\frac{1}{4} \rho^4 \Big|_0^1 \right)$$

$$\bar{z} = \frac{3h}{4\pi}(1)(2\pi)\left(\frac{1}{4}\right)$$

$$\bar{z} = \frac{3h}{8}$$

So the center of mass is at $(0,0,3h/8)$.

MOMENTS OF INERTIA

■ 1. The spherical solid object with radius 5 has density $\delta = d^2$, where d is the distance to the center of the sphere. Find the moment of inertia of the object about any line that passes through its center.

Solution:

Since the sphere is symmetric about any line that passes through its center, and has symmetrical density, it has the same moment of inertia about any such line. Find, for example, the moment of inertia about the z-axis.

The moment of inertia about the z-axis is given by the triple integral.

$$\iiint_E (x^2 + y^2)\delta(x, y, z)\ dV$$

The function is

$$(x^2 + y^2)\delta(x, y, z)$$

$$\left[(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2\right] \cdot \rho^2$$

$$\rho^4 \sin^2\phi$$

Therefore, the integral in spherical coordinates representing the moment of inertia about the z-axis is

$$\int_0^{\pi}\int_0^{2\pi}\int_0^5 \rho^4 \sin^2\phi \cdot \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^5 \rho^6 \sin^3\phi \ d\rho \ d\theta \ d\phi$$

$$\int_0^{\pi} \sin^3\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^5 \rho^6 \ d\rho$$

Use the reduction formula

$$\int \sin^m x \ dx = -\frac{\cos x \sin^{m-1} x}{m} + \frac{m-1}{m}\int \sin^{-2+m} x \ dx, \text{ where } m = 3$$

to simplify the first integral.

$$\left(-\frac{1}{3}\sin^2\phi\cos\phi\Big|_0^{\pi} + \frac{2}{3}\int_0^{\pi}\sin\phi \ d\phi\right) \cdot \int_0^{2\pi} d\theta \cdot \int_0^5 \rho^6 \ d\rho$$

Evaluate each integral.

$$\left(-\frac{1}{3}\sin^2\phi\cos\phi - \frac{2}{3}\cos\phi\Big|_0^{\pi}\right)(2\pi - 0)\left(\frac{1}{7}(5)^7 - \frac{1}{7}(0)^7\right)$$

$$\left(-\frac{1}{3}\sin^2\pi\cos\pi - \frac{2}{3}\cos\pi + \frac{1}{3}\sin^2(0)\cos(0) + \frac{2}{3}\cos(0)\right)(2\pi)\left(\frac{1}{7}(5)^7\right)$$

$$\left(-\frac{1}{3}(0)(-1) - \frac{2}{3}(-1) + \frac{1}{3}(0)(1) + \frac{2}{3}(1)\right)(2\pi)\left(\frac{1}{7}(5)^7\right)$$

$$\left(\frac{2}{3} + \frac{2}{3}\right)(2\pi)\left(\frac{78{,}125}{7}\right)$$

$$\frac{8\pi}{3}\left(\frac{78{,}125}{7}\right)$$

$$\frac{625{,}000\pi}{21}$$

■ 2. A box (rectangular cuboid) has length 6, width 4, height 2, and constant density δ. Find the moment of inertia of the box about all of its edges.

Solution:

Although the box has 12 edges, it has only three different moments of inertia, because the moments about the edges of length 6 are equal, the moments about the edges of length 4 are equal, and the moments about the edges of length 2 are equal.

Place one corner of the box at the origin, with one edge each along the three major axes. Then the moment of inertia about the x-axis is given by

$$\iiint_E (y^2+z^2)\delta(x,y,z)\ dV$$

$$\delta\iiint_E y^2+z^2\ dV$$

$$\delta\int_0^2\int_0^4\int_0^6 y^2+z^2\ dx\ dy\ dz$$

Integrate with respect to x.

$$\delta \int_0^2 \int_0^4 xy^2 + xz^2 \Big|_{x=0}^{x=6} \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 6y^2 + 6z^2 \ dy \ dz$$

Integrate with respect to y.

$$\delta \int_0^2 2y^3 + 6yz^2 \Big|_{y=0}^{y=4} \ dz$$

$$\delta \int_0^2 2(4)^3 + 6(4)z^2 - (2(0)^3 + 6(0)z^2) \ dz$$

$$\delta \int_0^2 128 + 24z^2 \ dz$$

Integrate with respect to z.

$$\delta(128z + 8z^3) \Big|_0^2$$

$$\delta(128(2) + 8(2)^3) - \delta(128(0) + 8(0)^3)$$

$$\delta(256 + 64)$$

$$320\delta$$

In the same way, find the moment of inertia about the y-axis.

$$\delta \int_0^2 \int_0^4 \int_0^6 x^2 + z^2 \ dx \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 \frac{1}{3}x^3 + xz^2 \Big|_{x=0}^{x=6} \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 72 + 6z^2 \ dy \ dz$$

$$\delta \int_0^2 72y + 6yz^2 \Big|_{y=0}^{y=4} \ dz$$

$$\delta \int_0^2 288 + 24z^2 \ dz$$

$$\delta(288z + 8z^3) \Big|_0^2$$

$$\delta(288(2) + 8(2)^3)$$

$$\delta(576 + 64)$$

$$640\delta$$

In the same way, find the moment of inertia about the z-axis.

$$\delta \int_0^2 \int_0^4 \int_0^6 x^2 + y^2 \ dx \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 \frac{1}{3}x^3 + xy^2 \Big|_{x=0}^{x=6} \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 \frac{1}{3}(6)^3 + 6y^2 \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 72 + 6y^2 \ dy \ dz$$

$$\delta \int_0^2 72y + 2y^3 \Big|_0^4 \ dz$$

$$\delta \int_0^2 288 + 128 \ dz$$

$$\delta \int_0^2 416 \ dz$$

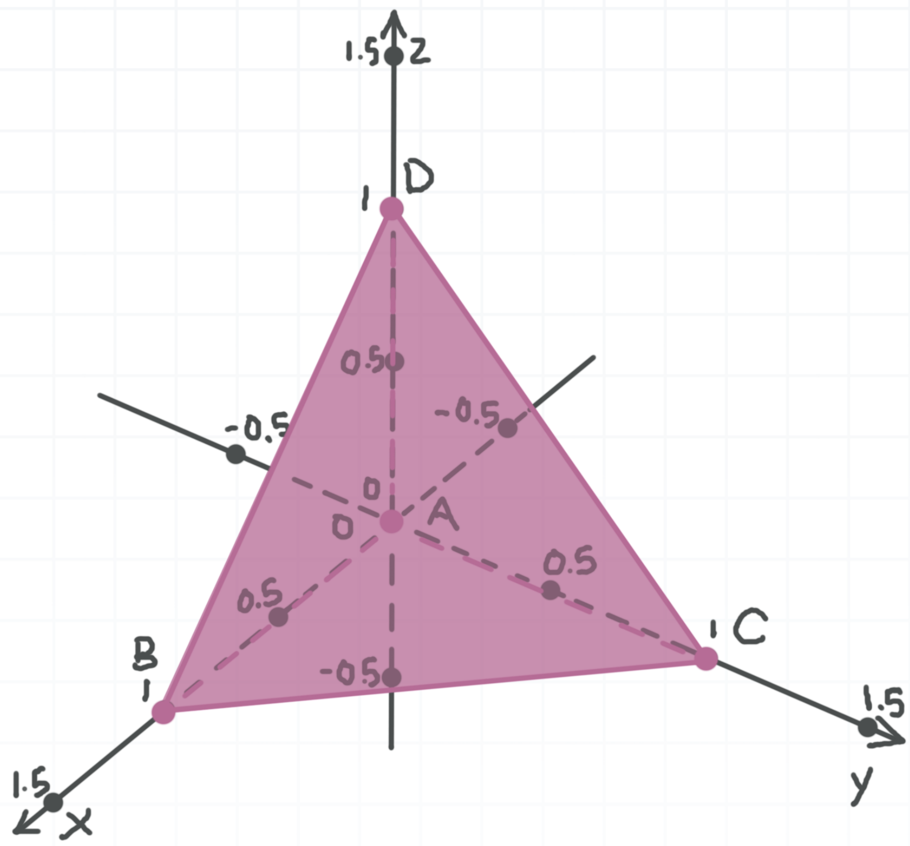
$$\delta(416z)\Big|_0^2$$

$$\delta(416(2))$$

$$832\delta$$

Therefore, the moments of inertia are 320δ, 640δ, and 832δ.

■ 3. The tetrahedron $ABCD$ has constant density δ. Find the moment of inertia of the solid about the line AB, where $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, and $D(0,0,1)$.



Solution:

The moment of inertia about the line AB (x-axis) is given by the triple integral.

$$\iiint_E (y^2 + z^2)\delta(x, y, z)\ dV$$

$$\delta \iiint_E y^2 + z^2\ dV$$

The equation of the line BC in the xy-plane is $y = 1 - x$. So when x changes from 0 to 1, y changes from 0 to $1 - x$. The equation of the plane BCD is $z = 1 - x - y$. So when x and y change within the triangle ABC, z changes from 0 to $1 - x - y$. Therefore, the integral representing the moment of inertia about the x-axis is given by

$$\delta\int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2+z^2\ dz\ dy\ dx$$

Integrate with respect to z.

$$\delta\int_0^1\int_0^{1-x} y^2z+\frac{1}{3}z^3\Big|_{z=0}^{z=1-x-y}\ dy\ dx$$

$$\delta\int_0^1\int_0^{1-x} y^2(1-x-y)+\frac{1}{3}(1-x-y)^3\ dy\ dx$$

$$\delta\int_0^1\int_0^{1-x} y^2-xy^2-y^3+\frac{1}{3}-\frac{1}{3}x^3+x^2-x-x^2y+2xy-xy^2-y+y^2-\frac{1}{3}y^3\ dy\ dx$$

$$\delta\int_0^1\int_0^{1-x} \frac{1}{3}-\frac{1}{3}x^3+x^2-x-x^2y+2xy-2xy^2-y+2y^2-\frac{4}{3}y^3\ dy\ dx$$

Integrate with respect to y.

$$\delta\int_0^1 \frac{1}{3}y-\frac{1}{3}x^3y+x^2y-xy-\frac{1}{2}x^2y^2+xy^2-\frac{2}{3}xy^3-\frac{1}{2}y^2+\frac{2}{3}y^3-\frac{1}{3}y^4\Big|_{y=0}^{y=1-x}\ dx$$

$$\delta\int_0^1 \frac{1}{3}(1-x)-\frac{1}{3}x^3(1-x)+x^2(1-x)-x(1-x)-\frac{1}{2}x^2(1-x)^2$$

$$+x(1-x)^2-\frac{2}{3}x(1-x)^3-\frac{1}{2}(1-x)^2+\frac{2}{3}(1-x)^3-\frac{1}{3}(1-x)^4\ dx$$

$$\delta\int_0^1 \frac{1}{6}-\frac{2}{3}x+x^2-\frac{2}{3}x^3+\frac{1}{6}x^4\ dx$$

Integrate with respect to x.

$$\delta\left(\frac{1}{6}x-\frac{1}{3}x^2+\frac{1}{3}x^3-\frac{1}{6}x^4+\frac{1}{30}x^5\right)\Bigg|_0^1$$

$$\delta\left(\frac{1}{6}(1)-\frac{1}{3}(1)^2+\frac{1}{3}(1)^3-\frac{1}{6}(1)^4+\frac{1}{30}(1)^5\right)$$

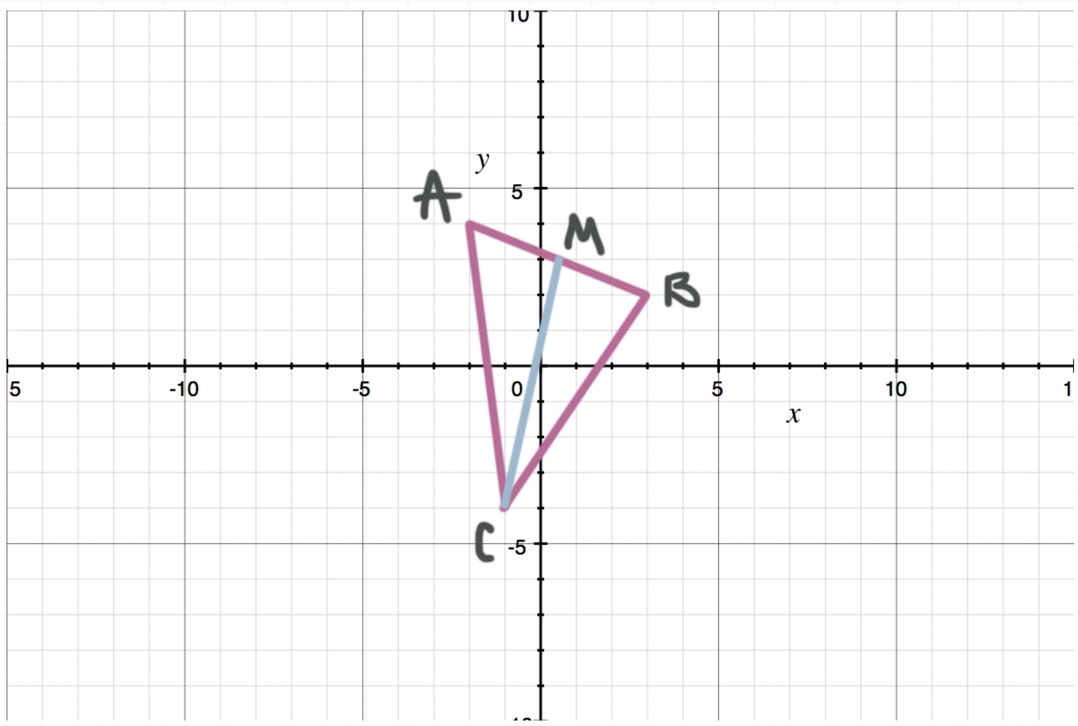
$$\delta\left(\frac{1}{6}-\frac{1}{3}+\frac{1}{3}-\frac{1}{6}+\frac{1}{30}\right)$$

$$\frac{1}{30}\delta$$

Therefore, the moment of inertia about the x-axis is $\delta/30$.

VECTOR FROM TWO POINTS

■ 1. Find the vector $\overrightarrow{CM}$, if M is the midpoint of $\overline{AB}$.



Solution:

The coordinates of the vertices are $A(-2,4)$, $B(3,2)$, and $C(-1,-4)$. Since M is the midpoint of AB, it has coordinates

$$x_M = \frac{x_A + x_B}{2} = \frac{-2+3}{2} = 0.5$$

$$y_M = \frac{y_A + y_B}{2} = \frac{4+2}{2} = 3$$

Since C is the initial point and M is the terminal point of the vector $\overrightarrow{CM}$,

$$\overrightarrow{CM} = \langle x_M - x_C, y_M - y_C \rangle$$

$$\overrightarrow{CM} = \langle 0.5 - (-1), 3 - (-4) \rangle$$

$$\overrightarrow{CM} = \langle 1.5, 7 \rangle$$

■ 2. Find the coordinates of the point P, given $Q(-\sqrt{2}, 0, \sqrt{2})$ and $\overrightarrow{PQ} = \langle \sqrt{2}, 4, \sqrt{2} \rangle$.

Solution:

Since P is the initial point and Q is the terminal point of the vector $\overrightarrow{PQ}$,

$$x_{\overrightarrow{PQ}} = x_Q - x_P$$

$$y_{\overrightarrow{PQ}} = y_Q - y_P$$

$$z_{\overrightarrow{PQ}} = z_Q - z_P$$

So

$$x_P = x_Q - x_{\overrightarrow{PQ}} = -\sqrt{2} - \sqrt{2} = -2\sqrt{2}$$

$$y_P = y_Q - y_{\overrightarrow{PQ}} = 0 - 4 = -4$$

$$z_P = z_Q - z_{\overrightarrow{PQ}} = \sqrt{2} - \sqrt{2} = 0$$

■ 3. Find the coordinates of the point C, given the coordinates of the point $A(-2,3,4)$, and the vectors $\overrightarrow{AB} = \langle 0,5,0 \rangle$ and $\overrightarrow{BC} = \langle 2,-3,6 \rangle$.

Solution:

Since A is the initial point of the vector $\overrightarrow{AB}$, and B is the terminal point,

$$x_{\overrightarrow{AB}} = x_B - x_A$$

$$y_{\overrightarrow{AB}} = y_B - y_A$$

$$z_{\overrightarrow{AB}} = z_B - z_A$$

So

$$x_B = x_A + x_{\overrightarrow{AB}} = -2 + 0 = -2$$

$$y_B = y_A + y_{\overrightarrow{AB}} = 3 + 5 = 8$$

$$z_B = z_A + z_{\overrightarrow{AB}} = 4 + 0 = 4$$

Therefore, the point B has coordinates $(-2,8,4)$. Similarly, since B is the initial point of the vector $\overrightarrow{BC}$, and C is the terminal point,

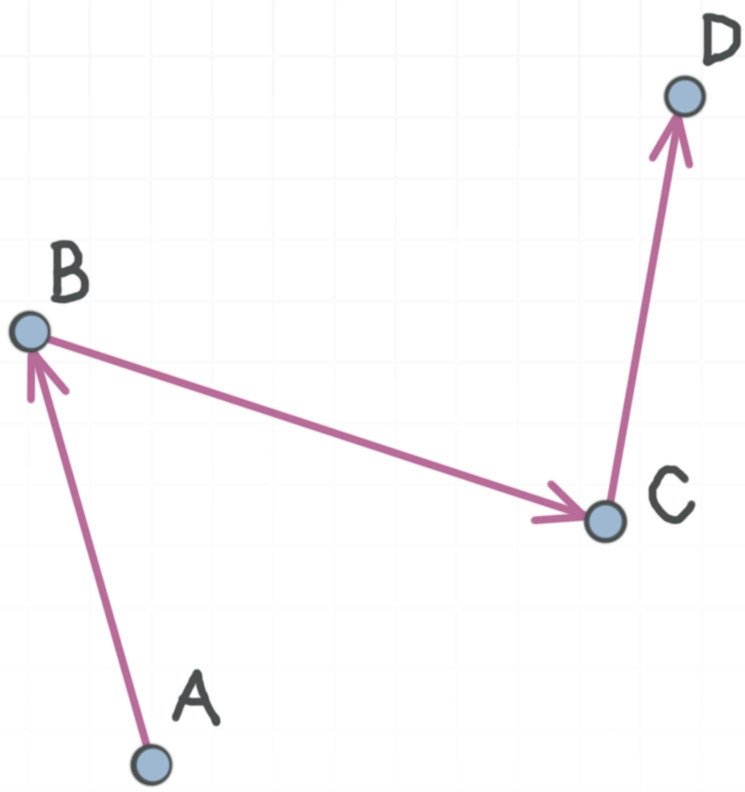
$$x_C = x_B + x_{\overrightarrow{BC}} = -2 + 2 = 0$$

$$y_C = y_B + y_{\overrightarrow{BC}} = 8 - 3 = 5$$

$$z_C = z_B + z_{\overrightarrow{BC}} = 4 + 6 = 10$$

COMBINATIONS OF VECTORS

1. Find the combination $\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CD}$.



Solution:

The combination of two vectors is the vector that connects the initial point of the first with the terminal point of the second.

So $\overrightarrow{AC}$ is the combination of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, or

$$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}$$

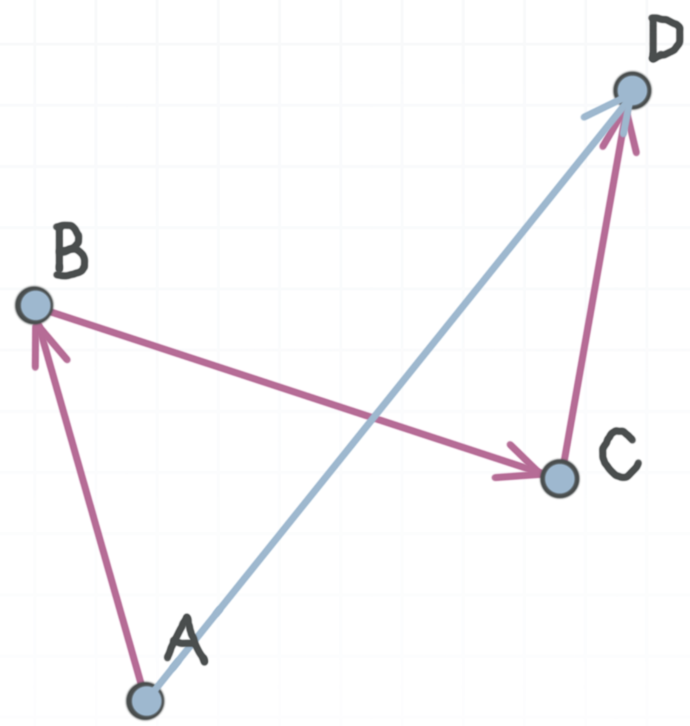
Similarly, $\overrightarrow{AD}$ is the combination of $\overrightarrow{AC}$ and $\overrightarrow{CD}$, or

$$\overrightarrow{AD}=\overrightarrow{AC}+\overrightarrow{CD}$$

Substitute the value of $\overrightarrow{AC}$.

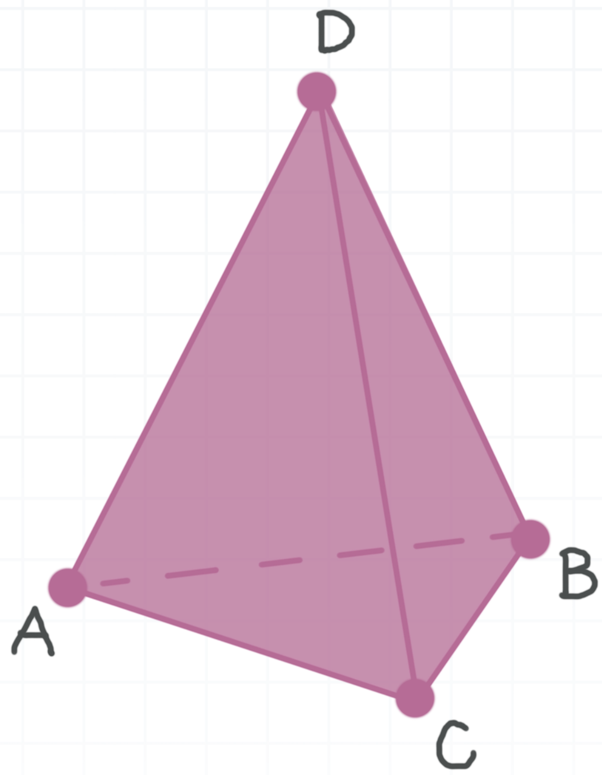
$$\overrightarrow{AD} = (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD}$$

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$$



We can generalize the problem and conclude that combination of a connected sequence of vectors, where the initial point of the next vector is the same as the terminal point of the previous, is the vector that connects the initial point of the first with the terminal point of the last.

■ 2. In the tetrahedron $ABCD$, find the resulting vector $\overrightarrow{DA} - \overrightarrow{DB} - \overrightarrow{BC}$.



Solution:

The combination of two vectors is the vector that connects the initial point of the first with the terminal point of the second. Consider the triangle ABD. Here, $\overrightarrow{DA}$ is the combination of $\overrightarrow{DB}$ and $\overrightarrow{BA}$, or

$$\overrightarrow{DA} = \overrightarrow{DB} + \overrightarrow{BA}$$

Subtract $\overrightarrow{DB}$ from each side

$$\overrightarrow{DA} - \overrightarrow{DB} = \overrightarrow{BA}$$

So

$$\overrightarrow{DA} - \overrightarrow{DB} - \overrightarrow{BC} = \overrightarrow{BA} - \overrightarrow{BC}$$

Similarly, consider the triangle ABC. Here, $\overrightarrow{BA}$ is the combination of $\overrightarrow{BC}$ and $\overrightarrow{CA}$, or

$$\overrightarrow{BA} = \overrightarrow{BC} + \overrightarrow{CA}$$

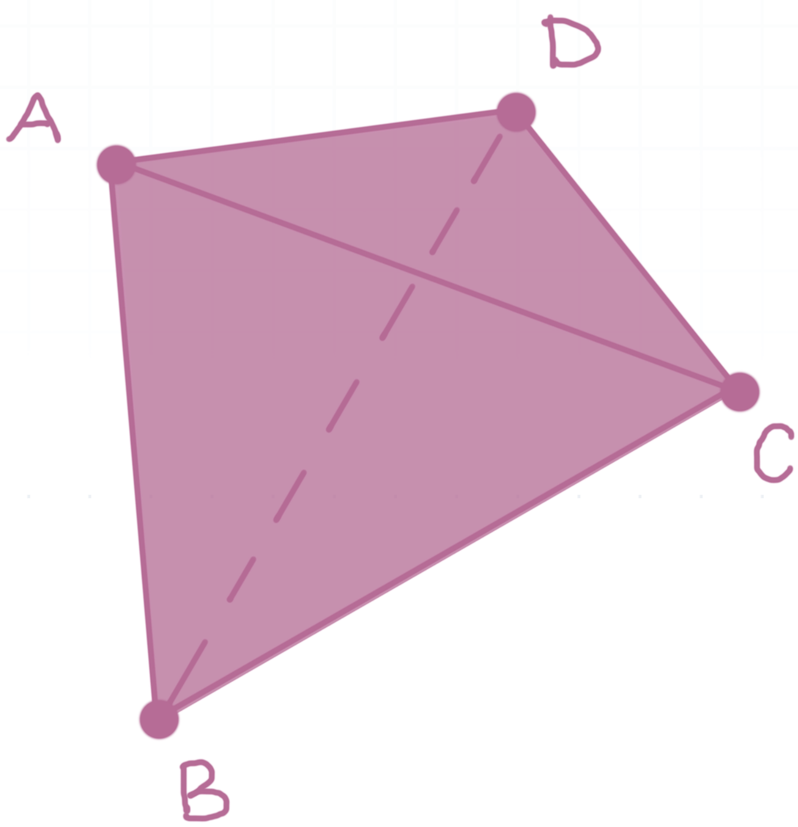
Subtract $\overrightarrow{BC}$ from each side

$$\overrightarrow{BA} - \overrightarrow{BC} = \overrightarrow{CA}$$

So

$$\overrightarrow{DA} - \overrightarrow{DB} - \overrightarrow{BC} = \overrightarrow{CA}$$

■ 3. In tetrahedron $ABCD$, find the vector $\overrightarrow{AB} + \overrightarrow{DC} + \overrightarrow{BD} - \overrightarrow{BC}$.



Solution:

The combination of two vectors is the vector that connects the initial point of the first with the terminal point of the second.

Notice that we can't find the combination $\overrightarrow{AB} + \overrightarrow{DC}$ directly, so change the order of the summation of the first three terms.

$$\overrightarrow{AB} + \overrightarrow{DC} + \overrightarrow{BD} = \overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC}$$

Since the combination of the connected sequence of vectors is the vector that connects the initial point of the first with the terminal point of the last.

$$\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC} = \overrightarrow{AC}$$

So

$$\overrightarrow{AB} + \overrightarrow{DC} + \overrightarrow{BD} - \overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{BC}$$

Consider the triangle ABC. Here $\overrightarrow{AC}$ is the combination of $\overrightarrow{AB}$ and $\overrightarrow{BC}$, or

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

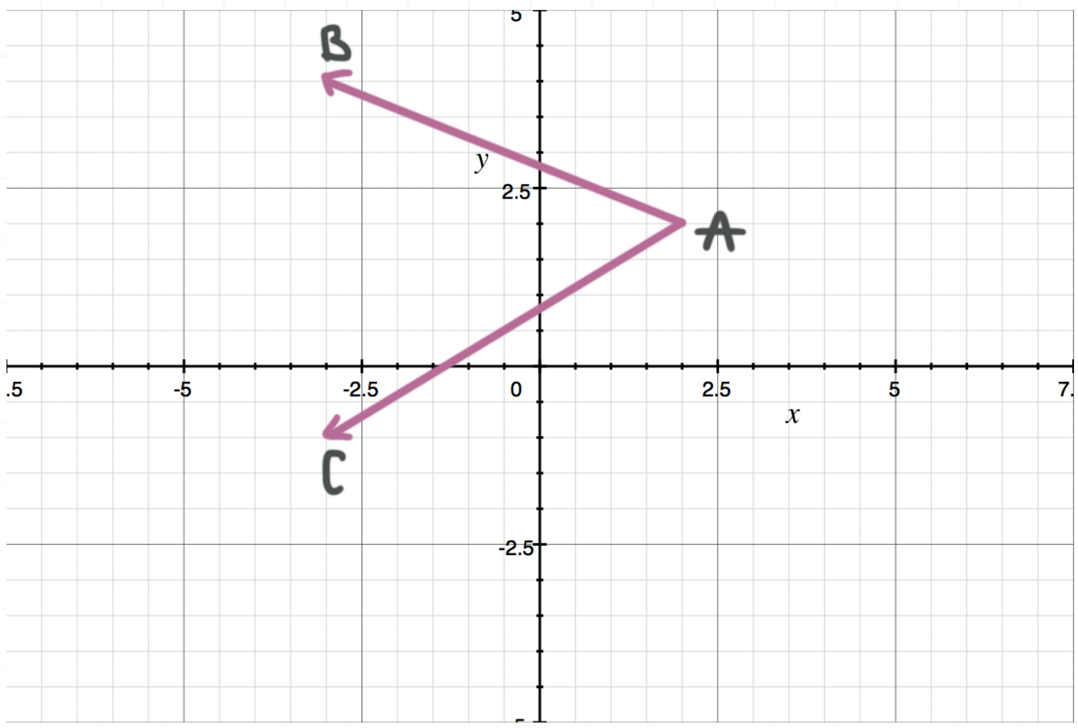
$$\overrightarrow{AC} - \overrightarrow{BC} = \overrightarrow{AB}$$

So

$$\overrightarrow{AB} + \overrightarrow{DC} + \overrightarrow{BD} - \overrightarrow{BC} = \overrightarrow{AB}$$

SUM OF TWO VECTORS

1. Find the sum $\overrightarrow{AB}+\overrightarrow{AC}$.



Solution:

The coordinates of the three points are $A(2,2)$, $B(-3,4)$, and $C(-3,-1)$.

Since A is the initial point and B is the terminal point of the vector $\overrightarrow{AB}$,

$$\overrightarrow{AB}=\langle x_B-x_A, y_B-y_A\rangle$$

$$\overrightarrow{AB}=\langle -3-2, 4-2\rangle$$

$$\overrightarrow{AB}=\langle -5,2\rangle$$

Similarly, since A is the initial point and C is the terminal point of the vector $\overrightarrow{AC}$,

$$\overrightarrow{AC} = \langle x_C - x_A, y_C - y_A \rangle$$

$$\overrightarrow{AC} = \langle -3 - 2, -1 - 2 \rangle$$

$$\overrightarrow{AC} = \langle -5, -3 \rangle$$

Then the sum of the vectors is

$$\overrightarrow{AB} + \overrightarrow{AC} = \langle x_{\overrightarrow{AB}} + x_{\overrightarrow{AC}}, y_{\overrightarrow{AB}} + y_{\overrightarrow{AC}} \rangle$$

$$\overrightarrow{AB} + \overrightarrow{AC} = \langle -5 - 5, 2 - 3 \rangle$$

$$\overrightarrow{AB} + \overrightarrow{AC} = \langle -10, -1 \rangle$$

■ 2. Find the vector $\overrightarrow{a} - \overrightarrow{b} + 2\overrightarrow{c}$, if $\overrightarrow{a} = \langle 0,4,5 \rangle$, $\overrightarrow{b} = \langle -3,2,1 \rangle$, and $\overrightarrow{c} = \langle 6,0,2 \rangle$.

Solution:

The difference $\overrightarrow{a} - \overrightarrow{b}$ is

$$\overrightarrow{a} - \overrightarrow{b} = \langle x_a - x_b, y_a - y_b, z_a - z_b \rangle$$

$$\overrightarrow{a} - \overrightarrow{b} = \langle 0 - (-3), 4 - 2, 5 - 1 \rangle$$

$$\overrightarrow{a} - \overrightarrow{b} = \langle 3,2,4 \rangle$$

The scaled vector $2\overrightarrow{c}$ is

$$2\overrightarrow{c} = \langle 2x_c, 2y_c, 2z_c \rangle$$

$$2\vec{c} = \langle 2(6), 2(0), 2(2) \rangle$$

$$2\vec{c} = \langle 12, 0, 4 \rangle$$

Finally, the sum $\vec{a} - \vec{b} + 2\vec{c}$ is

$$\vec{a} - \vec{b} + 2\vec{c} = \langle 3, 2, 4 \rangle + \langle 12, 0, 4 \rangle$$

$$\vec{a} - \vec{b} + 2\vec{c} = \langle 3 + 12, 2 + 0, 4 + 4 \rangle$$

$$\vec{a} - \vec{b} + 2\vec{c} = \langle 15, 2, 8 \rangle$$

■ 3. Find the sum of the vectors.

$$\sum_{k=1}^{100} \langle 5, k \rangle$$

Solution:

The sum of x_k coordinates is

$$x_1 + x_2 + \ldots + x_{100} = 5 + 5 + \ldots + 5 = 100(5) = 500$$

The coordinates y_k form the arithmetic progression

$$y_1 + y_2 + \ldots + y_{100} = 1 + 2 + 3 + \ldots + 100$$

Its sum is given by the formula

$$S_n = \frac{n(a_1 + a_n)}{2}$$

So

$$y_1 + y_2 + \ldots + y_{100} = S_{100} = 1 + 2 + 3 + \ldots + 100$$

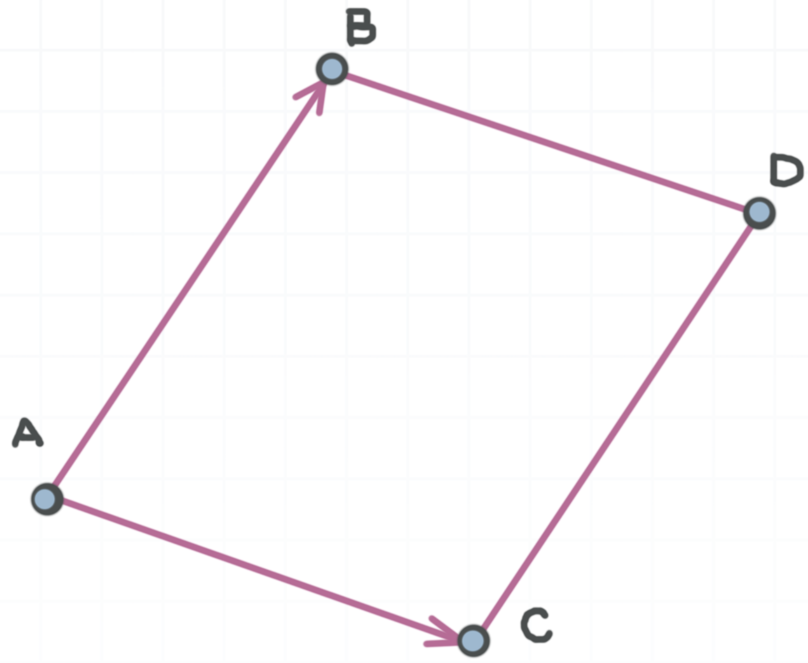
$$y_1 + y_2 + \ldots + y_{100} = S_{100} = \frac{100(1 + 100)}{2} = 50(101) = 5{,}050$$

Therefore,

$$\sum_{k=1}^{100} \langle 5,k \rangle = \langle 500,5{,}050 \rangle$$

COPYING VECTORS AND USING THEM TO DRAW COMBINATIONS

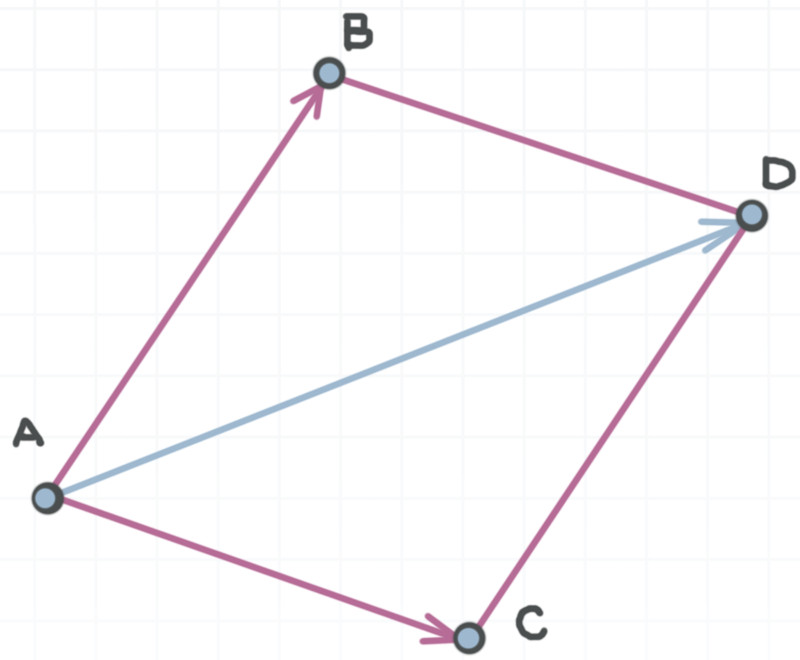
■ 1. In parallelogram $ABDC$, find the combination $\overrightarrow{AB}+\overrightarrow{AC}$



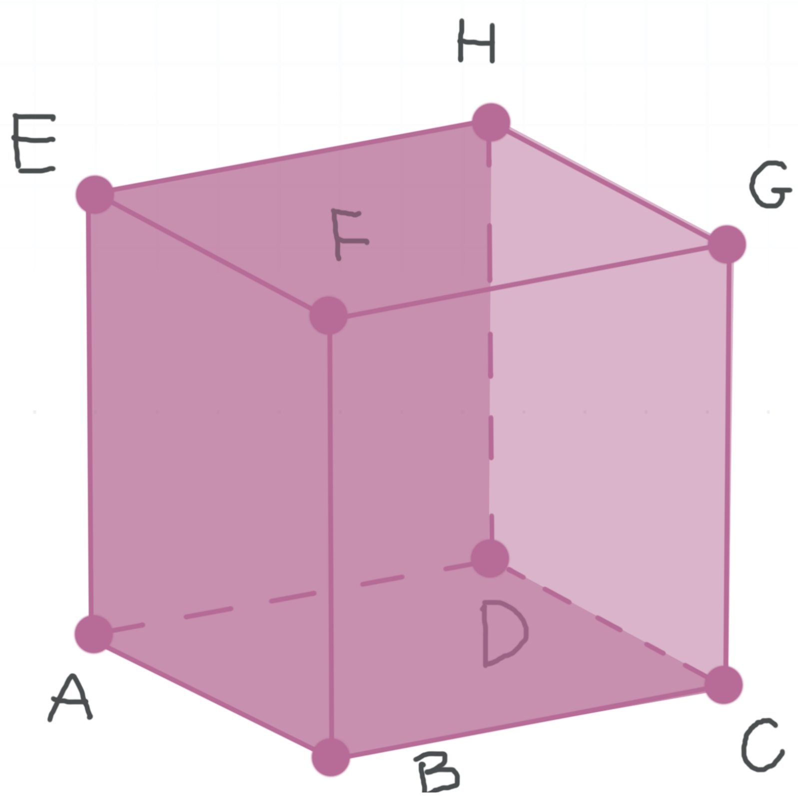
Solution:

Since the vectors $\overrightarrow{AC}$ and $\overrightarrow{BD}$ are parallel and have the same magnitude, they are equal. So we can connect the initial point A of the vector $\overrightarrow{AC}$ to the terminal point B of the vector $\overrightarrow{AB}$. So

$$\overrightarrow{AB}+\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BD}=\overrightarrow{AD}$$



■ 2. In the cube $ABCDEFGH$, find the combination $\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE}$.



Solution:

Since the vectors $\overrightarrow{AD}$ and $\overrightarrow{BC}$ are parallel and have the same magnitude, they are equal. So we can connect the initial point A of the vector $\overrightarrow{AD}$ to the terminal point B of the vector $\overrightarrow{AB}$.

$$\overrightarrow{AB} + \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

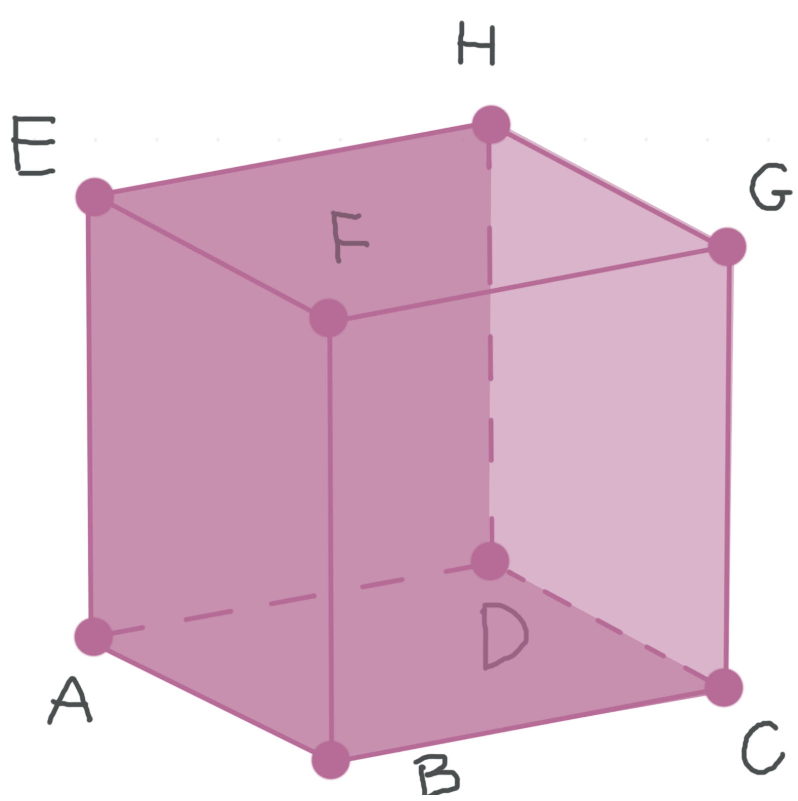
So

$$\overrightarrow{AB} + \overrightarrow{AD} + \overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{AE}$$

Similarly, since the vectors $\overrightarrow{AE}$ and $\overrightarrow{CG}$ are parallel and have the same magnitude, they are equal. So

$$\overrightarrow{AC} + \overrightarrow{AE} = \overrightarrow{AC} + \overrightarrow{CG} = \overrightarrow{AG}$$

■ 3. In the cube $ABCDEFGH$, find the combination $\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AE}$.



Solution:

Consider the triangle ABD. Here, $\overrightarrow{AB}$ is the combination of $\overrightarrow{AD}$ and $\overrightarrow{DB}$.

$$\overrightarrow{AB} = \overrightarrow{AD} + \overrightarrow{DB}$$

$$\overrightarrow{AB} - \overrightarrow{AD} = \overrightarrow{DB}$$

So

$$\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AE} = \overrightarrow{DB} - \overrightarrow{AE}$$

Since the vectors $\overrightarrow{AE}$ and $\overrightarrow{DH}$ are parallel and have the same magnitude, they are equal. So

$$\overrightarrow{DB} - \overrightarrow{AE} = \overrightarrow{DB} - \overrightarrow{DH}$$

Consider the triangle BDH. Here, $\overrightarrow{DB}$ is the combination of $\overrightarrow{DH}$ and $\overrightarrow{HB}$.

$$\overrightarrow{DB} = \overrightarrow{DH} + \overrightarrow{HB}$$

$$\overrightarrow{DB} - \overrightarrow{DH} = \overrightarrow{HB}$$

So

$$\overrightarrow{AB} - \overrightarrow{AD} - \overrightarrow{AE} = \overrightarrow{HB}$$

UNIT VECTOR IN THE DIRECTION OF THE GIVEN VECTOR

■ 1. Find the unit vector in the direction of the combination $\vec{a}+\vec{b}$, where $\vec{a}=\langle -2,-7\rangle$ and $\vec{b}=\langle 5,3\rangle$.

Solution:

Let $\vec{c}$ be the combination $\vec{c}=\vec{a}+\vec{b}$.

$$\vec{c}=\langle x_a+x_b, y_a+y_b\rangle$$

$$\vec{c}=\langle -2+5,-7+3\rangle$$

$$\vec{c}=\langle 3,-4\rangle$$

The magnitude of $\vec{c}$ is

$$|\vec{c}\,|=\sqrt{x_c^2+y_c^2}=\sqrt{(3)^2+(-4)^2}=\sqrt{25}=5$$

The unit vector in the direction of $\vec{c}$ is

$$\vec{u}_c=\frac{\vec{c}}{|\vec{c}\,|}$$

$$\vec{u}_c=\frac{\langle 3,-4\rangle}{5}$$

$$\vec{u}_c=\left\langle \frac{3}{5},-\frac{4}{5}\right\rangle$$

■ 2. The magnitude of the vector $\vec{a}$ is three times larger than the unit vector in the same direction. Find the vector $\vec{a}$.

$$\vec{u}_a = \left\langle \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \right\rangle$$

Solution:

Since magnitude of $\vec{u}_a$ is 1, and since $\vec{a}$ three times larger,

$$|\vec{a}| = 3$$

The unit vector in the direction of $\vec{a}$ is

$$\vec{u}_a = \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}}{3}$$

So

$$\vec{a} = 3\vec{u}_a$$

$$\vec{a} = 3\left\langle \frac{1}{3}, -\frac{2}{3}, -\frac{2}{3} \right\rangle$$

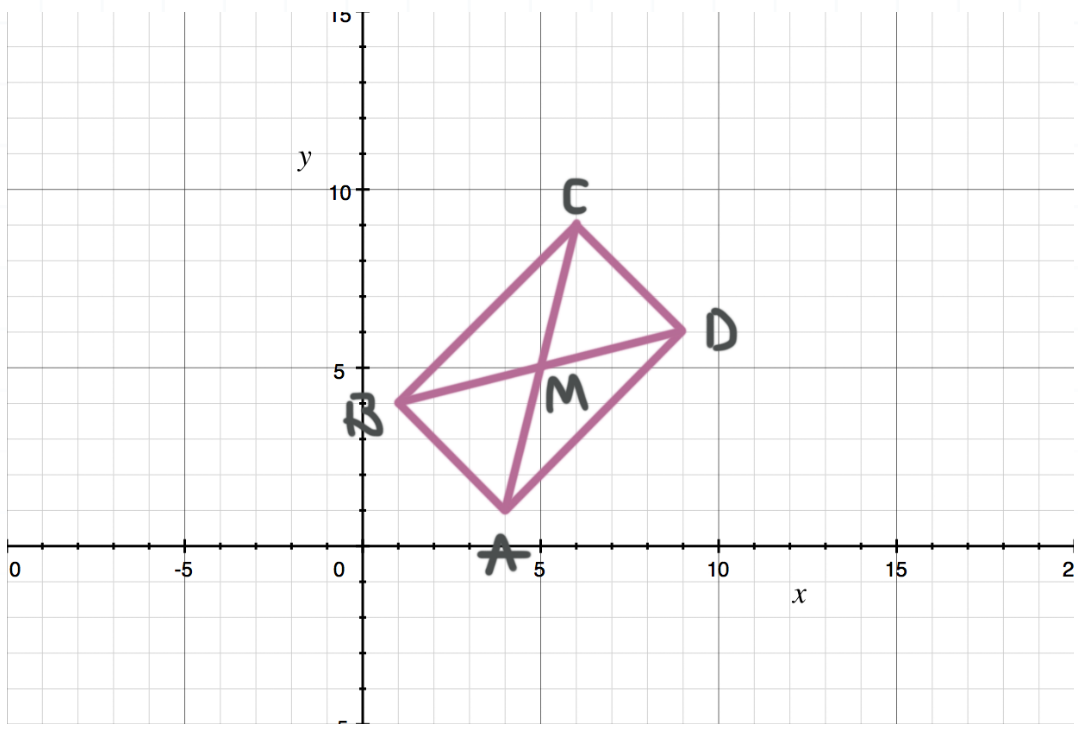
$$\vec{a} = \langle 1, -2, -2 \rangle$$

■ 3. Find the unit vector in the direction of $\overrightarrow{AC}$ in the rectangle $ABCD$, if $A(4,1)$, $B(1,4)$, and $D(9,6)$.

Solution:

We don't know the coordinates of the point C. Use the fact that in a rectangle its diagonal passes through the midpoint of the other diagonal.

Let M be the midpoint of BD. Since $\overrightarrow{AC}$ and $\overrightarrow{AM}$ have the same direction, they have the same unit vector $\overrightarrow{u}$.



Since M is a midpoint of BD, it has coordinates

$$x_M = \frac{x_B + x_D}{2} = \frac{1+9}{2} = 5$$

$$y_M = \frac{y_B + y_D}{2} = \frac{4+6}{2} = 5$$

Since A is the initial point of the vector $\overrightarrow{AM}$, and M is the terminal point,

$$\overrightarrow{AM} = \langle x_M - x_A, y_M - y_A \rangle$$

$$\overrightarrow{AM} = \langle 5 - 4, 5 - 1 \rangle$$

$$\overrightarrow{AM} = \langle 1, 4 \rangle$$

The unit vector in the direction of $\overrightarrow{AM}$ is

$$\vec{u} = \frac{\overrightarrow{AM}}{|\overrightarrow{AM}|}$$

where

$$|\overrightarrow{AM}| = \sqrt{\left(x_{\overrightarrow{AM}}\right)^2 + \left(y_{\overrightarrow{AM}}\right)^2} = \sqrt{(1)^2 + (4)^2} = \sqrt{17}$$

So

$$\vec{u} = \frac{\langle 1, 4 \rangle}{\sqrt{17}}$$

$$\vec{u} = \left\langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \right\rangle$$

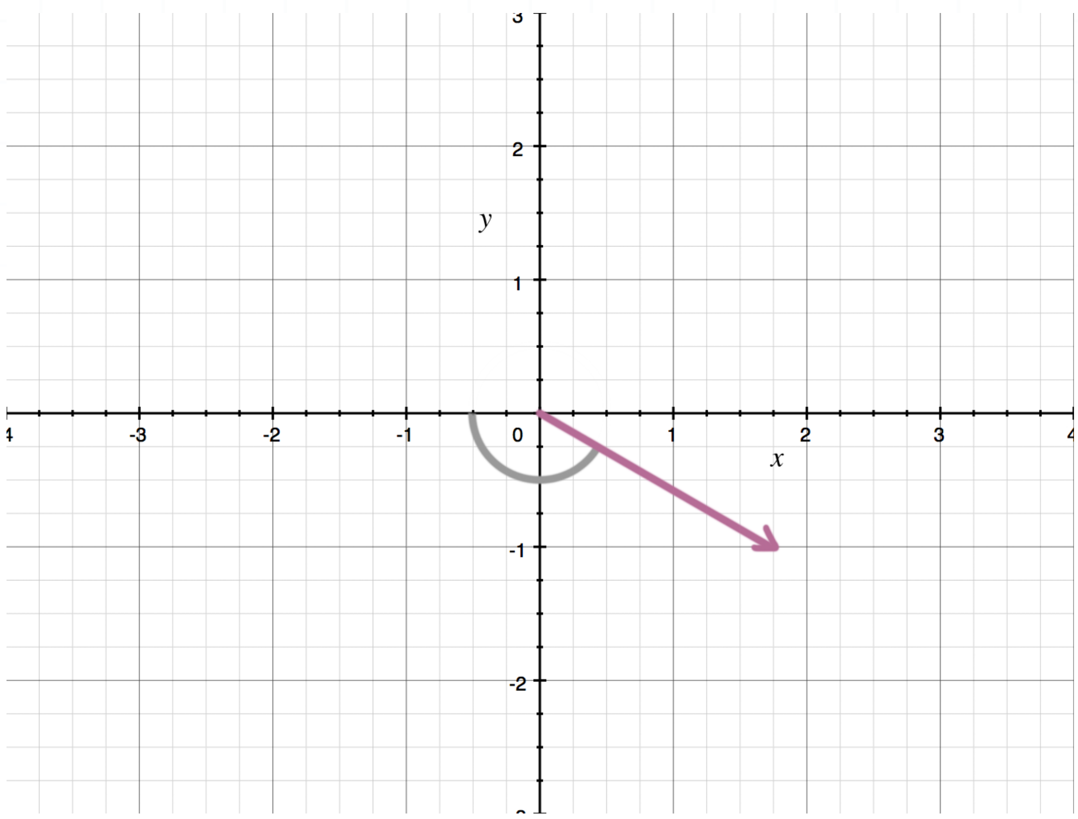
ANGLE BETWEEN A VECTOR AND THE X-AXIS

■ 1. Find the clockwise angle in radians between the vector $\overrightarrow{a} = \langle \sqrt{3}, -1 \rangle$ and the negative direction of the x-axis.

Solution:

Sketch the vector and its angle.

$$\overrightarrow{a} = \langle \sqrt{3}, -1 \rangle \approx \langle 1.7, -1 \rangle$$



Let α be the angle between $\overrightarrow{a}$ and the positive direction of the x-axis. Then the clockwise angle between $\overrightarrow{a}$ and the negative direction of the x-axis is $\pi - \alpha$. Use the formula for the angle.

$$\alpha = \arctan\left|\frac{y_a}{x_a}\right|$$

$$\alpha = \arctan\left|\frac{1}{-\sqrt{3}}\right|$$

$$\alpha = \arctan\frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

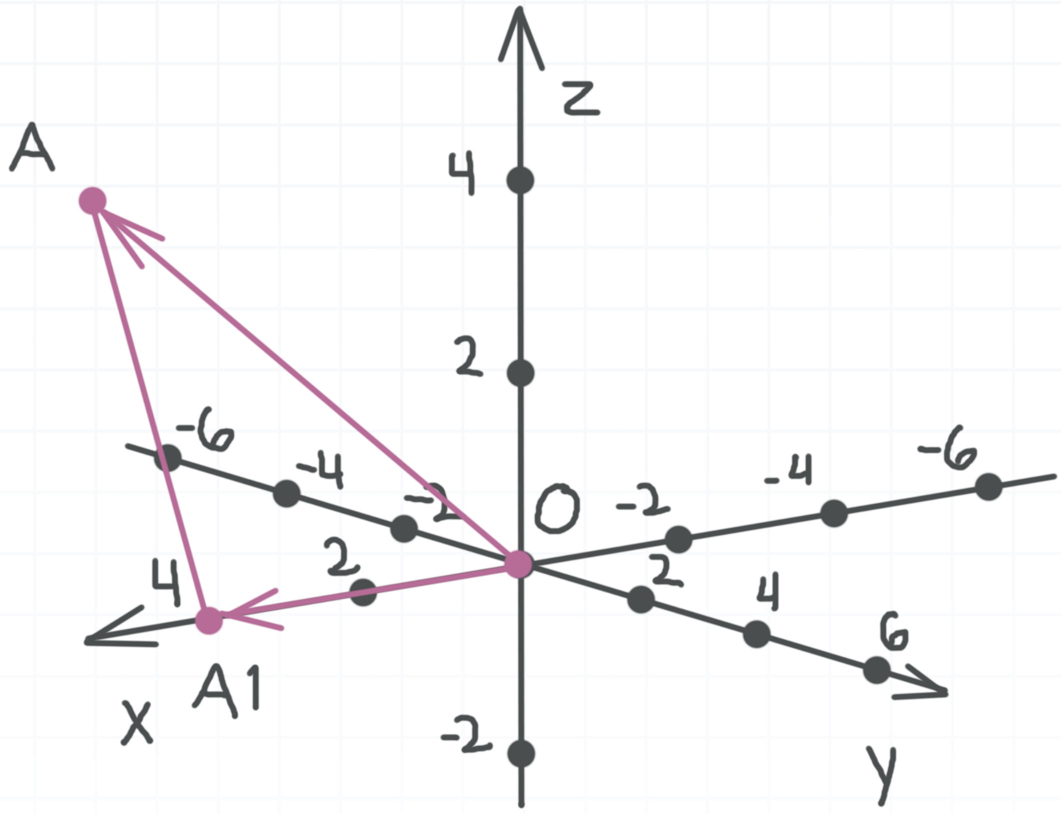
So

$$\pi - \alpha = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

■ 2. Find the angle between the vector $\overrightarrow{OA} = \langle 4, -4, 2\rangle$ and the positive direction of the x-axis.

Solution:

Let $A_1(4,0,0)$ be the projection of the point A onto the x-axis.



In the right triangle OAA_1 we need to find the angle AOA_1. Cosine of this angle is

$$\cos AOA_1 = \frac{OA_1}{OA}$$

Find magnitudes.

$$OA = \sqrt{(4)^2 + (-4)^2 + (2)^2} = \sqrt{36} = 6$$

$$OA_1 = \sqrt{(4)^2 + (0)^2 + (0)^2} = \sqrt{16} = 4$$

Then

$$\cos AOA_1 = \frac{4}{6} = \frac{2}{3}$$

$$AOA_1 = \arccos\frac{2}{3} \approx 0.84$$

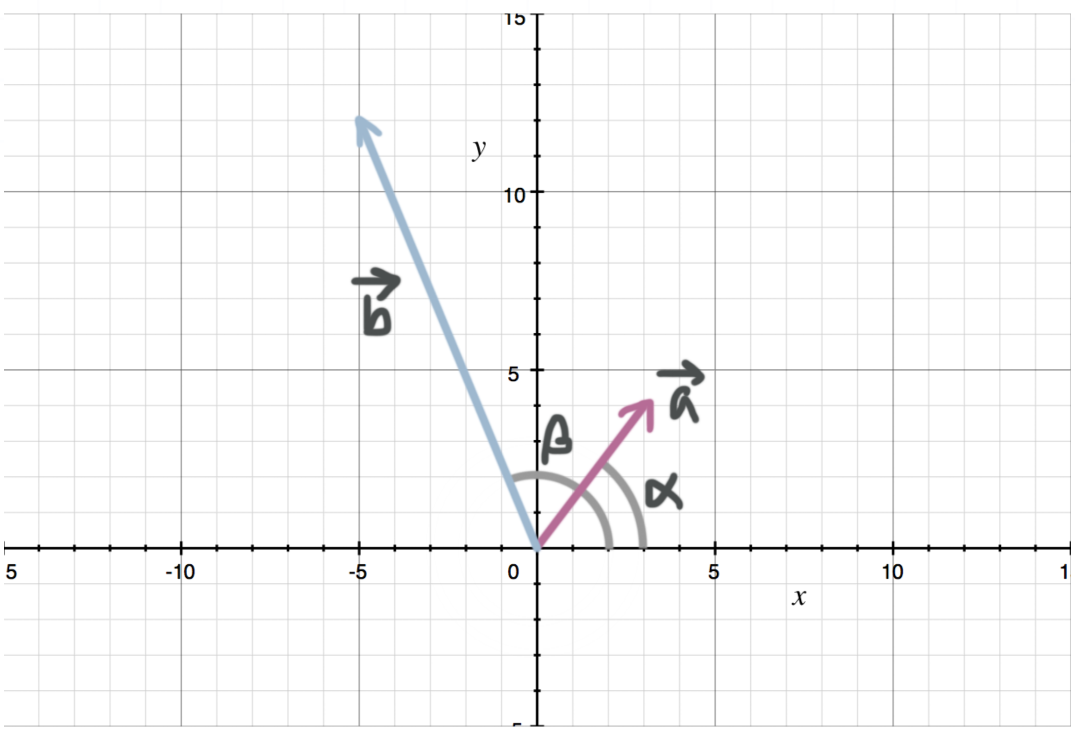
Convert to degrees.

$$\frac{0.84 \cdot 180^\circ}{\pi} \approx 48^\circ$$

■ 3. Find the angle between the vectors $\overrightarrow{a} = \langle 3,4 \rangle$ and $\overrightarrow{b} = \langle -5,12 \rangle$.

Solution:

The main idea is to find the angles α and β between the positive direction of the x-axis, and the vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ respectively, and then compute the difference $\beta - \alpha$.



Use the formula for the angle.

$$\alpha = \arctan \left| \frac{y_a}{x_a} \right|$$

$$\alpha = \arctan \left| \frac{4}{3} \right|$$

$$\alpha = \arctan \frac{4}{3}$$

Find the angle between β and the negative direction of the x-axis, then subtract it from π.

$$\beta = \pi - \arctan \left| \frac{y_b}{x_b} \right|$$

$$\beta = \pi - \arctan \left| \frac{12}{-5} \right|$$

$$\beta = \pi - \arctan \frac{12}{5}$$

Compute the difference.

$$\beta - \alpha = \pi - \arctan \frac{12}{5} - \arctan \frac{4}{3} \approx 1.038$$

Convert to degrees.

$$\frac{1.038 \cdot 180^\circ}{\pi} \approx 59.5^\circ$$

MAGNITUDE AND ANGLE OF THE RESULTANT FORCE

■ 1. Find the magnitude and angle of the resultant force $\vec{f}$ of the vectors $\overrightarrow{a} = \langle 2, -1 \rangle$, $\overrightarrow{b} = \langle 5,1 \rangle$, and $\overrightarrow{c} = \langle -3,3 \rangle$.

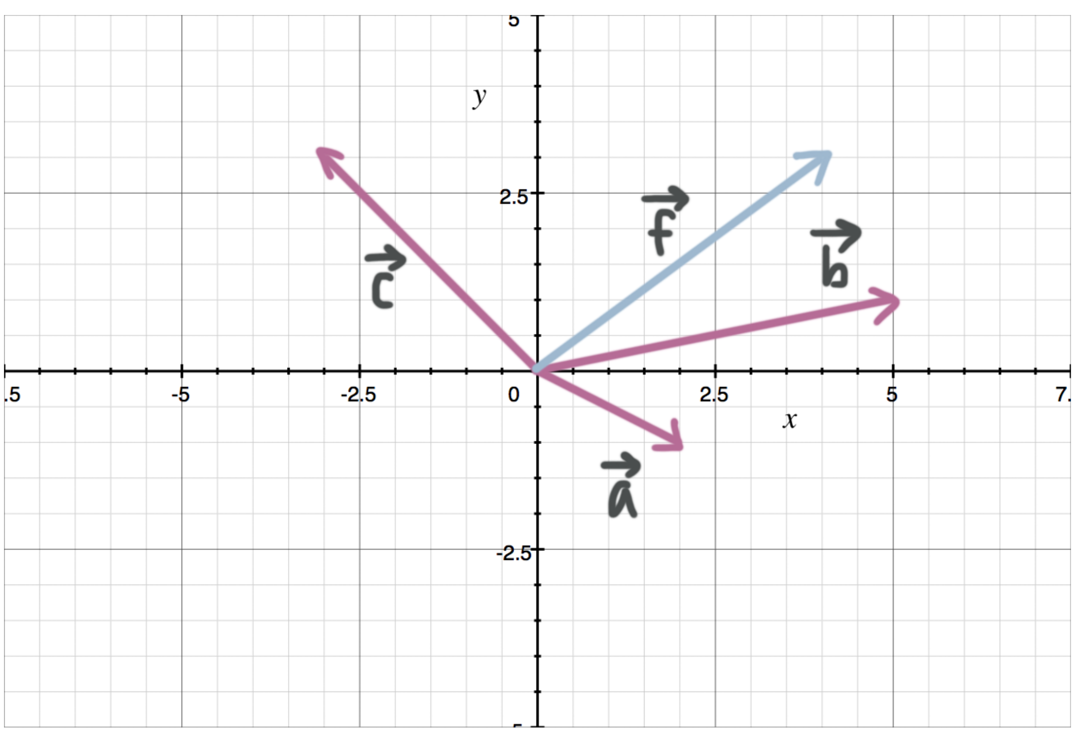
Solution:

Find the resultant force vector $\vec{f}$ as the summation of $\overrightarrow{a} = \langle 2, -1 \rangle$, $\overrightarrow{b} = \langle 5,1 \rangle$, and $\overrightarrow{c} = \langle -3,3 \rangle$.

$$\vec{f} = \langle x_a + x_b + x_c, y_a + y_b + y_c \rangle$$

$$\vec{f} = \langle 2 + 5 - 3, -1 + 1 + 3 \rangle$$

$$\vec{f} = \langle 4,3 \rangle$$



Then the magnitude is

$$|\vec{f}| = \sqrt{(x_f)^2 + (y_f)^2}$$

$$|\vec{f}| = \sqrt{4^2 + 3^2}$$

$$|\vec{f}| = 5$$

Use the formula for the angle of the resulting force vector.

$$\arctan\left|\frac{y_f}{x_f}\right| = \arctan\frac{3}{4} \approx 0.6435$$

Convert to degrees.

$$\frac{0.6435 \cdot 180^\circ}{\pi} \approx 36.9^\circ$$

■ 2. Find the magnitude of the resultant force $\vec{f}$ of the vectors $\vec{a} = \langle 4,0,0 \rangle$, $\vec{b} = \langle 0,4,0 \rangle$, and $\vec{c} = \langle 0,0,2 \rangle$, then find the angles between $\vec{f}$ and each of the major coordinate axes.

Solution:

Find the resultant force vector $\vec{f}$ by adding $\vec{a} = \langle 4,0,0 \rangle$, $\vec{b} = \langle 0,4,0 \rangle$, and $\vec{c} = \langle 0,0,2 \rangle$.

$$\vec{f} = \langle x_a + x_b + x_c, y_a + y_b + y_c, z_a + z_b + z_c \rangle$$

$$\vec{f} = \langle 4 + 0 + 0, 0 + 4 + 0, 0 + 0 + 2 \rangle$$

$$\vec{f} = \langle 4,4,2 \rangle$$

Then the magnitude is

$$|\vec{f}| = \sqrt{(x_f)^2 + (y_f)^2 + (z_f)^2}$$

$$|\vec{f}| = \sqrt{4^2 + 4^2 + 2^2}$$

$$|\vec{f}| = 6$$

Since $\vec{a}$ is the projection of $\vec{f}$ onto the x-axis,

$$\cos \alpha_x = \frac{|\vec{a}|}{|\vec{f}|}$$

$$\cos \alpha_x = \frac{4}{6} = \frac{2}{3}$$

$$\alpha_x = \arccos \frac{2}{3} \approx 0.84$$

$$\frac{0.84 \cdot 180^\circ}{\pi} \approx 48^\circ$$

Since $\vec{b}$ is the projection of $\vec{f}$ onto the y-axis,

$$\cos \alpha_y = \frac{|\vec{b}|}{|\vec{f}|}$$

$$\cos \alpha_y = \frac{4}{6} = \frac{2}{3}$$

$$\alpha_y = \arccos \frac{2}{3} \approx 0.84$$

$$\frac{0.84 \cdot 180^\circ}{\pi} \approx 48^\circ$$

Since $\vec{c}$ is the projection of $\vec{f}$ onto the z-axis,

$$\cos \alpha_z = \frac{|\vec{c}\,|}{|\vec{f}\,|}$$

$$\cos \alpha_z = \frac{2}{6} = \frac{1}{3}$$

$$\alpha_z = \arccos \frac{1}{3} \approx 1.23$$

$$\frac{1.23 \cdot 180^\circ}{\pi} \approx 70.5^\circ$$

■ 3. The resultant force $\vec{f}$ of the vectors $\vec{a}$ and $\vec{b}$ has a magnitude of 12 and an angle of $2\pi/3$. Find vector $\vec{b}$, if $\vec{a} = \langle -8, 5\sqrt{3} \rangle$.

Solution:

Find the coordinates of the vector $\vec{f}$.

$$\vec{f} = \langle |\vec{f}\,| \cdot \cos \alpha_f, |\vec{f}\,| \cdot \sin \alpha_f \rangle$$

$$\vec{f} = \left\langle 12 \cos \frac{2\pi}{3}, 12 \sin \frac{2\pi}{3} \right\rangle$$

$$\vec{f} = \left\langle 12\left(-\frac{1}{2}\right), 12\left(\frac{\sqrt{3}}{2}\right)\right\rangle$$

$$\vec{f} = \langle -6, 6\sqrt{3} \rangle$$

Since $\vec{f} = \vec{a} + \vec{b}$, then $\vec{b} = \vec{f} - \vec{a}$. So

$$\vec{b} = \langle x_f - x_a, y_f - y_a \rangle$$

$$\vec{b} = \langle -6 - (-8), 6\sqrt{3} - 5\sqrt{3} \rangle$$

$$\vec{b} = \langle 2, \sqrt{3} \rangle$$

DOT PRODUCT OF TWO VECTORS

■ 1. Find the dot product $\vec{a} \cdot \vec{b}$, where the vectors $\vec{a}$ and $\vec{b}$ have opposite directions, and $\vec{b}$ has a magnitude two times larger than $\vec{a} = \langle 2, -3, 5 \rangle$.

Solution:

Since $\vec{b}$ has opposite direction to $\vec{a}$, and a magnitude two times larger, $\vec{b} = -2\vec{a}$. So

$$\vec{b} = \langle -2(2), -2(-3), -2(5) \rangle$$

$$\vec{b} = \langle -4, 6, -10 \rangle$$

Then the dot product is

$$\vec{a} \cdot \vec{b} = x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b$$

$$\vec{a} \cdot \vec{b} = (2)(-4) + (-3)(6) + (5)(-10)$$

$$\vec{a} \cdot \vec{b} = -76$$

■ 2. Find the value(s) of the parameter p such that the dot product of the vectors $\vec{a} = \langle p, 2p+1, 3 \rangle$ and $\vec{b} = \langle p-2, 5, -4 \rangle$ is 2.

Solution:

The dot product is

$$\vec{a} \cdot \vec{b} = x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b$$

$$\vec{a} \cdot \vec{b} = p(p-2) + (2p+1)(5) + 3(-4)$$

$$\vec{a} \cdot \vec{b} = p^2 + 8p - 7$$

Since $\vec{a} \cdot \vec{b} = 2$, we can write an equation for p.

$$p^2 + 8p - 7 = 2$$

$$p^2 + 8p - 9 = 0$$

$$(p+9)(p-1) = 0$$

$$p = -9 \text{ and } p = 1$$

3. Find the unit vector(s) $\vec{u}$ such that the dot product $\vec{a} \cdot \vec{u}$ reaches its maximum value, if $\vec{a} = \langle 2,2 \rangle$.

Solution:

Since $\vec{u}$ is the unit vector, its magnitude is 1. Let ϕ be the angle between $\vec{u}$ and the positive direction of the x-axis. So $\vec{u} = \langle \cos\phi, \sin\phi \rangle$, where $0 \le \phi < 2\pi$. The dot product is

$$\vec{a} \cdot \vec{u} = x_a \cdot x_u + y_a \cdot y_u$$

$$\vec{a} \cdot \vec{u} = 2\cos\phi + 2\sin\phi$$

Let the function $f(\phi) = 2\cos\phi + 2\sin\phi$, then find the absolute maximum of $f(\phi)$ on the interval $[0,2\pi]$.

$$f'(\phi) = -2\sin\phi + 2\cos\phi = 0$$

This equation gives $2\sin\phi = 2\cos\phi$, and since $\cos\phi \neq 0$, $\tan\phi = 1$.

The critical points on $[0,2\pi]$ are $\pi/4$ and $5\pi/4$. Substitute these critical points and the bounds of the interval into $f(\phi)$.

$$f(0) = 2\cos(0) + 2\sin(0) = 2$$

$$f\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right) = 2\cos\left(\frac{5\pi}{4}\right) + 2\sin\left(\frac{5\pi}{4}\right) = -2\sqrt{2}$$

$$f(2\pi) = 2\cos(2\pi) + 2\sin(2\pi) = 2$$

So the function $f(\phi)$ has an absolute maximum at $\phi = \pi/4$. The unit vector is

$$\vec{u} = \left\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \right\rangle$$

$$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

The dot product is largest when $\vec{u}$ is the unit vector of $\vec{a}$ with the same direction. That's always true for any $\vec{a}$.

ANGLE BETWEEN TWO VECTORS

■ 1. Use dot products to find the angles between the vector $\vec{a}=\langle -2,4,-4\rangle$ and the positive direction of each major coordinate axis.

Solution:

The magnitude of $\vec{a}$ is

$$|\vec{a}|=\sqrt{(x_a)^2+(y_a)^2+(z_a)^2}$$

$$|\vec{a}|=\sqrt{(-2)^2+4^2+(-4)^2}$$

$$|\vec{a}|=6$$

The angle between $\vec{a}$ and $\vec{u_x}=\langle 1,0,0\rangle$ is

$$\cos\phi_x=\frac{\vec{a}\cdot\vec{u_x}}{|\vec{a}||\vec{u_x}|}$$

$$\cos\phi_x=\frac{(-2)(1)+(4)(0)+(-4)(0)}{(6)(1)}$$

$$\cos\phi_x=-\frac{1}{3}$$

$$\phi_x=\arccos -\frac{1}{3}\approx 1.911$$

$$\phi_x = \frac{1.911 \cdot 180^\circ}{\pi} \approx 109.5^\circ$$

The angle between $\vec{a}$ and $\vec{u}_y = \langle 0,1,0 \rangle$ is

$$\cos \phi_y = \frac{\vec{a} \cdot \vec{u}_y}{|\vec{a}||\vec{u}_y|}$$

$$\cos \phi_y = \frac{(-2)(0) + (4)(1) + (-4)(0)}{(6)(1)}$$

$$\cos \phi_y = \frac{2}{3}$$

$$\phi_y = \arccos \frac{2}{3} \approx 0.841$$

$$\phi_y = \frac{0.841 \cdot 180^\circ}{\pi} \approx 48^\circ$$

The angle between $\vec{a}$ and $\vec{u}_z = \langle 0,0,1 \rangle$ is

$$\cos \phi_z = \frac{\vec{a} \cdot \vec{u}_z}{|\vec{a}||\vec{u}_z|}$$

$$\cos \phi_z = \frac{(-2)(0) + (4)(0) + (-4)(1)}{(6)(1)}$$

$$\cos \phi_z = -\frac{2}{3}$$

$$\phi_z = \arccos \left(-\frac{2}{3}\right) \approx 2.3$$

$$\phi_z = \frac{2.3 \cdot 180^\circ}{\pi} \approx 132^\circ$$

■ 2. Find the angle between the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, if $\vec{a} = \langle 3, -4, 4 \rangle$ and $\vec{b} = \langle -6, 2, -1 \rangle$.

Solution:

The sum of the vectors is

$$\vec{a} + \vec{b} = \langle x_a + x_b, y_a + y_b, z_a + z_b \rangle$$

$$\vec{a} + \vec{b} = \langle 3 - 6, -4 + 2, 4 - 1 \rangle$$

$$\vec{a} + \vec{b} = \langle -3, -2, 3 \rangle$$

The difference of the vectors is

$$\vec{a} - \vec{b} = \langle x_a - x_b, y_a - y_b, z_a - z_b \rangle$$

$$\vec{a} - \vec{b} = \langle 3 - (-6), -4 - 2, 4 - (-1) \rangle$$

$$\vec{a} - \vec{b} = \langle 9, -6, 5 \rangle$$

The angle between the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$ is given by

$$\cos\phi = \frac{(\vec{a} + \vec{b}) \cdot (\vec{a} - \vec{b})}{|\vec{a} + \vec{b}||\vec{a} - \vec{b}|}$$

The dot product is

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=\langle -3,-2,3\rangle\cdot\langle 9,-6,5\rangle$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=(-3)(9)+(-2)(-6)+(3)(5)$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$$

Since the dot product is 0, $\cos\phi=0$ and $\phi=90^\circ$. So the vector $\vec{a}+\vec{b}$ is perpendicular to the vector $\vec{a}-\vec{b}$. The angle between the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is equal to 90° not for any pair of vectors, but only for the vectors which have equal magnitude, i.e. only if $|\vec{a}|=|\vec{b}|$.

■ 3. Find the two vectors $\vec{b_1}$ and $\vec{b_2}$ with magnitude 5 that each have an angle of 30° with $\vec{a}=\langle -2,1\rangle$.

Solution:

The angle between $\vec{a}$ and the positive direction of the x-axis, which we'll represent with $\vec{u_x}=\langle 1,0,0\rangle$, is

$$\cos\phi=\frac{\vec{a}\cdot\vec{u_x}}{|\vec{a}||\vec{u_x}|}$$

$$\cos\phi=\frac{\langle -2,1\rangle\cdot\langle 1,0\rangle}{\sqrt{(-2)^2+1^2}\cdot 1}$$

$$\cos\phi = \frac{(-2)(1)+(1)(0)}{\sqrt{5}}$$

$$\cos\phi = -\frac{2\sqrt{5}}{5}$$

$$\phi = \arccos\left(-\frac{2\sqrt{5}}{5}\right) \approx 2.678$$

$$\phi = \frac{2.678\cdot 180^\circ}{\pi} \approx 153.4^\circ$$

Since $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ have an angle of 30° with $\overrightarrow{a}$,

$$\phi_1 = 153.4^\circ + 30^\circ = 183.4^\circ$$

$$\phi_2 = 153.4^\circ - 30^\circ = 123.4^\circ$$

The formula for the vector $\overrightarrow{c}$ with magnitude M and angle α is given by

$$\overrightarrow{c} = \langle M\cos\alpha, M\sin\alpha\rangle$$

Therefore,

$$\overrightarrow{b_1} = \langle 5\cos 183.4^\circ, 5\sin 183.4^\circ\rangle$$

$$\overrightarrow{b_1} = \langle -5, -0.3\rangle$$

and

$$\overrightarrow{b_2} = \langle 5\cos 123.4^\circ, 5\sin 123.4^\circ\rangle$$

$$\overrightarrow{b_2} = \langle -2.8, 4.2\rangle$$

ORTHOGONAL, PARALLEL, OR NEITHER

■ 1. Find the terminal point B of the vector $\overrightarrow{AB}$ that has initial point $A(2,0,-1)$, magnitude 24, and is parallel to the vector $\vec{c}=\langle -2,4,4\rangle$.

Solution:

Parallel vectors have the same direction. Find the unit vector $\vec{u}$ in the direction of $\vec{c}$ (which is in the same direction as $\overrightarrow{AB}$).

$$\vec{u}=\frac{\vec{c}}{|\vec{c}\,|}$$

$$\vec{u}=\frac{\langle -2,4,4\rangle}{\sqrt{(-2)^2+4^2+4^2}}$$

$$\vec{u}=\frac{\langle -2,4,4\rangle}{6}$$

$$\vec{u}=\left\langle -\frac{1}{3},\frac{2}{3},\frac{2}{3}\right\rangle$$

Since $\overrightarrow{AB}$ has the same direction as $\vec{u}$ and a magnitude of 24,

$$\overrightarrow{AB}=24\vec{u}$$

$$\overrightarrow{AB}=24\left\langle -\frac{1}{3},\frac{2}{3},\frac{2}{3}\right\rangle$$

$$\overrightarrow{AB} = \langle -8,16,16 \rangle$$

Since A is the initial point of $\overrightarrow{AB}$, and B is the terminal point,

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

We know

$x_B - 2 = -8$ so $x_B = -6$

$y_B - 0 = 16$ so $y_B = 16$

$z_B - (-1) = 16$ so $z_B = 15$

■ 2. Find two vectors $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ with magnitude 2, that are orthogonal to $\overrightarrow{a} = \langle 3, -1 \rangle$.

Solution:

Let $\overrightarrow{u}$ be the unit vector orthogonal to $\overrightarrow{a}$. Let ϕ be the angle between $\overrightarrow{u}$ and the positive direction of the x-axis. So $\overrightarrow{u} = \langle \cos\phi, \sin\phi \rangle$. Since $\overrightarrow{u}$ is orthogonal to $\overrightarrow{a}$, the dot product is 0.

$$\overrightarrow{u} \cdot \overrightarrow{a} = 0$$

$$\langle \cos\phi, \sin\phi \rangle \cdot \langle 3, -1 \rangle = 0$$

$$3\cos\phi - \sin\phi = 0$$

$$\sin\phi = 3\cos\phi$$

$$\tan\phi = 3$$

$$\phi_1 = \arctan 3 \approx 1.249$$

$$\phi_2 = \pi + \arctan 3 \approx 4.39$$

The formula for the vector $\overrightarrow{c}$ with magnitude M and angle α is given by

$$\overrightarrow{c} = \langle M\cos\alpha, M\sin\alpha\rangle$$

Plug in $M = 2$ and $\alpha = \phi_1, \phi_2$.

$$\overrightarrow{b_1} = \langle 2\cos(1.249), 2\sin(1.249)\rangle$$

$$\overrightarrow{b_1} = \langle 0.6, 1.9\rangle$$

Similarly,

$$\overrightarrow{b_2} = \langle 2\cos(4.39), 2\sin(4.39)\rangle$$

$$\overrightarrow{b_2} = \langle -0.6, -1.9\rangle$$

■ 3. Find value(s) of the parameter p, such that the vectors $\overrightarrow{a} = \langle p, p+3, 6-p\rangle$ and $\overrightarrow{b} = \langle p-1, 4, 2\rangle$ are (a) parallel, and (b) orthogonal.

Solution:

(a) The vectors are parallel if their respective coordinates are proportional.

$$\frac{p}{p-1}=\frac{p+3}{4}=\frac{6-p}{2}$$

Solve the second equation for p.

$$\frac{p+3}{4}=\frac{6-p}{2}$$

$$p+3=2(6-p)$$

$$3p=9$$

$$p=3$$

Check if the first equation holds.

$$\frac{3}{3-1}=\frac{3+3}{4}=\frac{3}{2}$$

So the vectors $\vec{a}$ and $\vec{b}$ are parallel if $p=3$.

(b) The vectors are orthogonal if their dot product is 0.

$$\vec{a}\cdot\vec{b}=0$$

$$\langle p,p+3,6-p\rangle\cdot\langle p-1,4,2\rangle=0$$

$$(p)(p-1)+(p+3)(4)+(6-p)(2)=0$$

$$p^2+p+24=0$$

Since this equation has no real solutions, the vectors $\vec{a}$ and $\vec{b}$ can't be orthogonal for any p.

ACUTE ANGLE BETWEEN THE LINES

■ 1. Find the acute angle between the lines.

Line 1: $x = 2t + 1,\ y = t - 4,\ z = 6$

Line 2: $\dfrac{x-1}{4} = \dfrac{y+1}{5} = z$

Solution:

The angle between the lines is the same as the angle between their direction vectors.

The direction vector of the first line given in parametric form is $\vec{a} = \langle 2,1,0 \rangle$, and the direction vector of the second line given in symmetric form is $\vec{b} = \langle 4,5,1 \rangle$. Then the angle ϕ between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\phi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\phi = \frac{(2)(4) + (1)(5) + (0)(1)}{\sqrt{2^2 + 1^2 + 0^2} \cdot \sqrt{4^2 + 5^2 + 1^2}}$$

$$\cos\phi = \frac{13}{\sqrt{5} \cdot \sqrt{42}}$$

$$\phi = \arccos \frac{13}{\sqrt{210}} \approx 0.46$$

$$\phi = \frac{0.46 \cdot 180^\circ}{\pi} \approx 26^\circ$$

■ 2. Find the acute angle between the line and the plane.

Line: $x = t + 7,\ y = -2t - 5,\ z = 3t + 6$

Plane: $3x - y - 4z + 15 = 0$

Solution:

Let ϕ be the acute angle between the line and the vector $\vec{n}$, which is the normal (orthogonal) vector to the given plane. Then the angle between the line and the plane is $90^\circ - \phi$.

The direction vector of the line given in parametric form is $\vec{a} = \langle 1, -2, 3 \rangle$, and the normal vector to the given plane is $\vec{n} = \langle 3, -1, -4 \rangle$.

The angle ϕ between $\vec{a}$ and $\vec{n}$ is given by

$$\cos \phi = \frac{\vec{a} \cdot \vec{n}}{|\vec{a}||\vec{n}|}$$

$$\cos \phi = \frac{(1)(3) + (-2)(-1) + (3)(-4)}{\sqrt{1^2 + (-2)^2 + 3^2} \cdot \sqrt{3^2 + (-1)^2 + (-4)^2}}$$

$$\cos\phi = \frac{-7}{\sqrt{14}\cdot\sqrt{26}}$$

$$\phi = \arccos\frac{-7}{\sqrt{364}} \approx 1.9465$$

$$\phi = \frac{1.9465\cdot 180^\circ}{\pi} \approx 111.5^\circ$$

Since $\phi > 90^\circ$, the acute angle between the line and the vector $\vec{n}$ is

$$180^\circ - 111.5^\circ = 68.5^\circ$$

The acute angle between the line and the plane is

$$90^\circ - 68.5^\circ = 21.5^\circ$$

■ 3. Find the acute angle between the planes.

Plane 1: $x - 2y + 1 = 0$

Plane 2: $x + y + 2z + 4 = 0$

Solution:

The angle between the planes is equal to the angle between their normal vectors. The normal vector to the first plane is $\vec{n_1} = \langle 1, -2, 0\rangle$ and the normal vector to the second plane is $\vec{n_2} = \langle 1,1,2\rangle$.

The angle ϕ between $\vec{n_1}$ and $\vec{n_2}$ is given by

$$\cos\phi = \frac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$$

$$\cos\phi = \frac{(1)(1) + (-2)(1) + (0)(2)}{\sqrt{1^2 + (-2)^2 + 0^2} \cdot \sqrt{1^2 + 1^2 + 2^2}}$$

$$\cos\phi = \frac{-1}{\sqrt{5} \cdot \sqrt{6}}$$

$$\phi = \arccos\frac{-1}{\sqrt{30}} \approx 1.7544$$

$$\phi = \frac{1.7544 \cdot 180^\circ}{\pi} \approx 100.5^\circ$$

Therefore, the acute angle between the planes is

$$180^\circ - 100.5\circ = 79.5^\circ$$

ACUTE ANGLES BETWEEN THE CURVES

1. Find the acute angle(s) between the curves.

$$x^2 + y^2 = 4$$

$$x^2 + 4y^2 = 4$$

Solution:

Set the curves equal to each other to find the points where they intersect.

$$x^2 + y^2 - 4 = x^2 + 4y^2 - 4$$

$$3y^2 = 0$$

$$y = 0$$

Substitute $y = 0$ into the first equation in order to solve for x.

$$x^2 + (0)^2 = 4$$

$$x^2 = 4$$

$x_1 = -2$ and $x_2 = 2$

So we have two intersection points, $(-2,0)$ and $(2,0)$. Now differentiate the first equation with respect to x.

$$2x + 2yy' = 0$$

$$x + yy' = 0$$

Substitute $(-2,0)$ for (x, y).

$$(-2) + (0)y' = 0$$

$$-2 = 0$$

So the derivative does not exist. Since the implicit function is differentiable at this point, but the derivative does not exist, the tangent line is vertical.

Similarly, for the point $(2,0)$ the derivative does not exist, and so the tangent line is vertical.

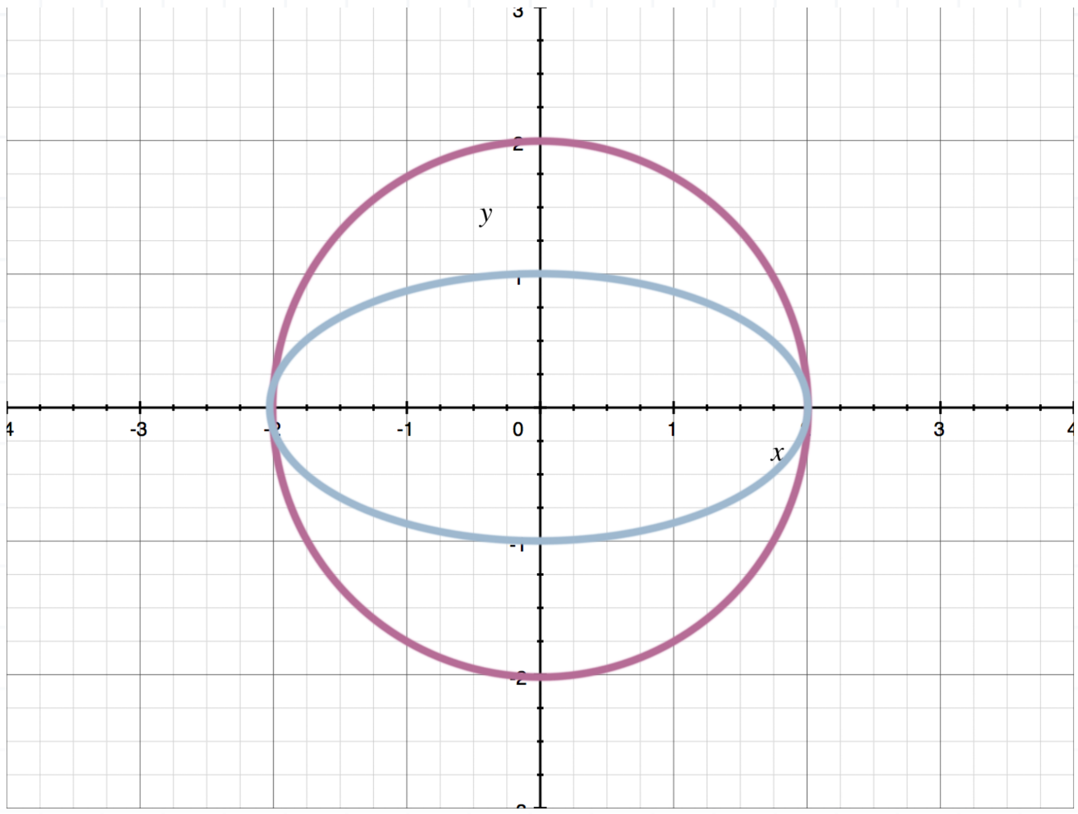
Differentiate the second equation with respect to x.

$$2x + 8yy' = 0$$

$$x + 4yy' = 0$$

The derivative does not exist at both intersection points, so the tangent lines are vertical.

Therefore, the angles between the curves are equal to 0 for both intersection points, $(-2,0)$ and $(2,0)$. We can confirm the result by sketching both curves. The circle $x^2 + y^2 = 4$ is centered at the origin with radius 2, and the ellipse $x^2 + 4y^2 = 4$ is centered at the origin with a horizontal semi-axis of 2 and a vertical semi-axis of 1.



■ 2. Find the acute angle(s) between the curves given in parametric form.

$$x = t^2 + 1,\ y = 2t^2 + t - 3,\ z = t - 1$$

$$x = 2s^2 - 7,\ y = s - 5,\ z = s - 3$$

Solution:

Set the curves equal to one another to find the points where they intersect.

$$t^2 + 1 = 2s^2 - 7$$

$$2t^2 + t - 3 = s - 5$$

$$t - 1 = s - 3$$

Solve the system of equations for t and s. In the third equation, isolate t and substitute it into the first equation.

$$t = s - 2$$

$$(s-2)^2 + 1 = 2s^2 - 7$$

$$s^2 - 4s + 4 + 1 = 2s^2 - 7$$

$$s^2 + 4s - 12 = 0$$

$$(s-2)(s+6) = 0$$

$s = 2$, and then $t = 2 - 2 = 0$

or

$s = -6$, and then $t = -6 - 2 = -8$

Substitute each solution into the second equation

$$2(0)^2 + (0) - 3 = (2) - 5$$

$$2(-8)^2 + (-8) - 3 = (-6) - 5$$

The first equation is true and the second is false, so we have only one solution, which is $t = 0$ and $s = 2$. The point of intersection is

$$x(0) = (0)^2 + 1 = 1$$

$$y(0) = 2(0)^2 + (0) - 3 = -3$$

$$z(0) = (0) - 1 = -1$$

Therefore, the curves intersect at the point $(1, -3, -1)$.

At $t = 0$, the first curve has values

$$x'(t) = 2t, \ x'(0) = 0$$

$$y'(t) = 4t + 1, \ y'(0) = 1$$

$$z'(t) = 1, \ z'(0) = 1$$

So the tangent vector for the first curve is $\vec{a} = \langle 0,1,1 \rangle$.

At $s = 2$, the second curve has values

$$x'(s) = 4s, \ x'(2) = 8$$

$$y'(s) = 1, \ y'(2) = 1$$

$$z'(s) = 1, \ z'(2) = 1$$

So the tangent vector for the second curve is $\vec{b} = \langle 8,1,1 \rangle$.

The angle ϕ between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\phi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\phi = \frac{(0)(8) + (1)(1) + (1)(1)}{\sqrt{0^2 + 1^2 + 1^2} \cdot \sqrt{8^2 + 1^2 + 1^2}}$$

$$\cos\phi = \frac{2}{\sqrt{2} \cdot \sqrt{66}}$$

$$\cos\phi = \frac{1}{\sqrt{33}}$$

$$\phi = \arccos\frac{1}{\sqrt{33}} \approx 1.4$$

$$\phi = \frac{1.4 \cdot 180^\circ}{\pi} \approx 80^\circ$$

■ 3. Find the value of the parameter p such that $f(x) = e^x$ and $g(x) = e^{-x} + 2p$ are orthogonal at the point(s) of intersection.

Solution:

Set the curves equal to each other to find the points where they intersect.

$$e^x = e^{-x} + 2p$$

Make a substitution of $u = e^x$.

$$u = \frac{1}{u} + 2p$$

$$u^2 - 2pu - 1 = 0$$

Use the quadratic formula to solve the equation.

$$u = p \pm \sqrt{p^2 + 1}$$

$$e^x = p \pm \sqrt{p^2+1}$$

Since $p - \sqrt{p^2+1} < 0$, we have only one solution, which is $e^x = p + \sqrt{p^2+1}$. Then the intersection point is given by

$$x = \ln(p + \sqrt{p^2+1}) \text{ and } y = p + \sqrt{p^2+1}$$

Find the slope for $f(x)$ and $g(x)$ at this intersection point.

$$f'(x) = e^x = p + \sqrt{p^2+1}$$

$$g'(x) = -e^{-x} = -\frac{1}{p + \sqrt{p^2+1}}$$

So the tangent vectors for the functions $f(x)$ and $g(x)$ are

$$\overrightarrow{a_f} = \left\langle 1, p + \sqrt{p^2+1} \right\rangle$$

$$\overrightarrow{a_g} = \left\langle 1, -\frac{1}{p + \sqrt{p^2+1}} \right\rangle$$

The curves are orthogonal if the dot product of their tangent vectors at the intersection point is 0.

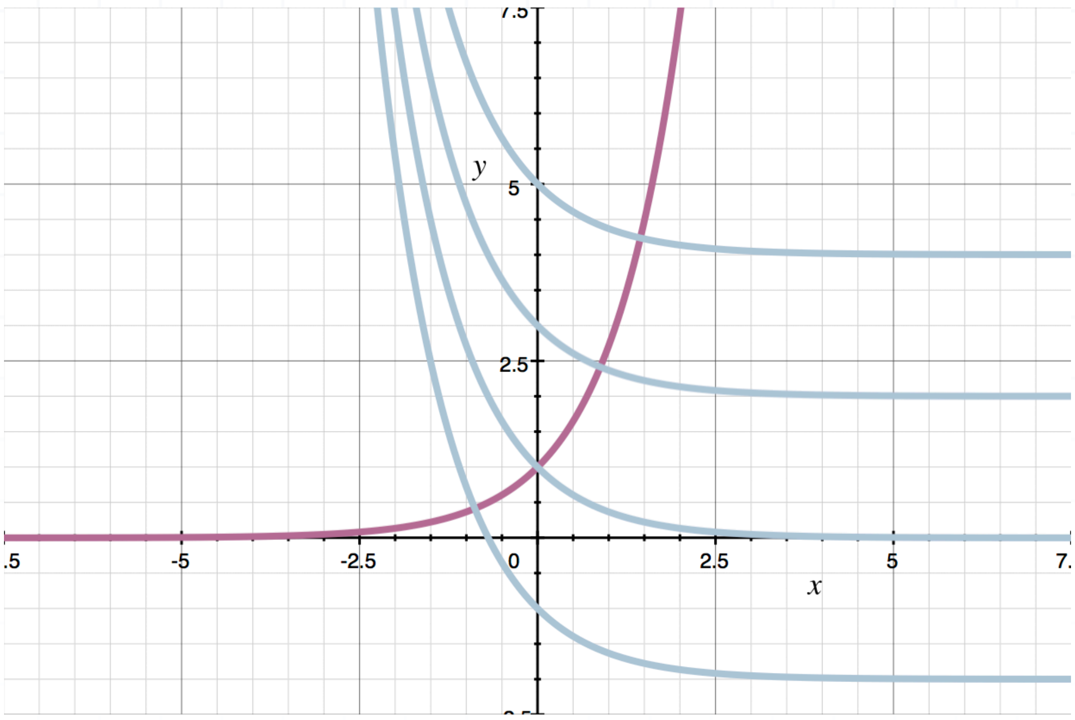
$$\overrightarrow{a_f} \cdot \overrightarrow{a_g} = 0$$

$$\left\langle 1, p + \sqrt{p^2+1} \right\rangle \cdot \left\langle 1, -\frac{1}{p + \sqrt{p^2+1}} \right\rangle = 0$$

$$(1)(1)+(p+\sqrt{p^2+1})\left(-\frac{1}{p+\sqrt{p^2+1}}\right)=0$$

$$1-1=0$$

So these curves are orthogonal for any real value of p.



DIRECTION COSINES AND DIRECTION ANGLES

■ 1. Find the direction angles of the linear combination $\vec{c} = 2\vec{a} - 3\vec{b}$, where $\vec{a} = \langle 3,1,-3 \rangle$ and $\vec{b} = \langle 0,-2,-2 \rangle$.

Solution:

Since $\vec{c}$ is the linear combination of two vectors, we can compute its coordinates as

$$\vec{c} = \langle 2x_a - 3x_b, 2y_a - 3y_b, 2z_a - 3z_b \rangle$$

$$\vec{c} = \langle 2(3) - 3(0), 2(1) - 3(-2), 2(-3) - 3(-2) \rangle$$

$$\vec{c} = \langle 6,8,0 \rangle$$

The magnitude of the vector $\vec{c}$ is given by

$$|\vec{c}| = \sqrt{x_c^2 + y_c^2 + z_c^2}$$

$$|\vec{c}| = \sqrt{6^2 + 8^2 + 0^2}$$

$$|\vec{c}| = \sqrt{100} = 10$$

The direction angle α of the vector $\vec{c}$ with respect to the x-axis is

$$\alpha = \arccos \frac{x_c}{|\vec{c}|}$$

$$\alpha = \arccos \frac{6}{10} = \arccos \frac{3}{5} \approx 0.927$$

$$\alpha = \frac{0.927 \cdot 180^\circ}{\pi} \approx 53^\circ$$

The direction angle with respect to the y-axis is

$$\beta = \arccos \frac{y_c}{|\vec{c}\,|}$$

$$\beta = \arccos \frac{8}{10} = \arccos \frac{4}{5} \approx 0.64$$

$$\beta = \frac{0.64 \cdot 180^\circ}{\pi} \approx 37^\circ$$

The direction angle with respect to the z-axis is

$$\gamma = \arccos \frac{z_c}{|\vec{c}\,|}$$

$$\gamma = \arccos \frac{0}{10} = \arccos 0 = \frac{\pi}{2}$$

$$\gamma = 90^\circ$$

■ 2. Find the vector $\vec{a}$ with magnitude 6 that has direction angles 120°, 45°, and 135° with respect to x, y, and z-axes, respectively.

Solution:

The cosine functions of the direction angles of the vector $\vec{a}$ are given by

$$\cos\alpha = \frac{x_a}{|\vec{a}|}$$

$$\cos\beta = \frac{y_a}{|\vec{a}|}$$

$$\cos\gamma = \frac{z_a}{|\vec{a}|}$$

Solve these equations for x_a, y_a, and z_a.

$$x_a = |\vec{a}|\cos\alpha$$

$$y_a = |\vec{a}|\cos\beta$$

$$z_a = |\vec{a}|\cos\gamma$$

Plug in $|\vec{a}| = 6$, $\alpha = 120^\circ$, $\beta = 45^\circ$, and $\gamma = 135^\circ$.

$$x_a = 6\cos 120^\circ = 6\left(-\frac{1}{2}\right) = -3$$

$$y_a = 6\cos 45^\circ = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$$

$$z_a = 6\cos 135^\circ = 6\left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}$$

■ 3. Find the vector $\vec{a}$ that has an x-coordinate of 2, y-coordinate of -1, and direction angle with respect to the z-axis of $\pi/3$.

Solution:

Let z be the unknown coordinate of the vector $\vec{a}$, so that $\vec{a} = \langle 2, -1, z\rangle$, then consider the direction angle γ of the vector $\vec{a}$ with respect to the z-axis.

$$\cos\gamma = \frac{z}{|\vec{a}|}$$

The magnitude of the vector $\vec{a}$ is

$$|\vec{a}| = \sqrt{x_a^2 + y_a^2 + z_a^2}$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + z^2}$$

$$|\vec{a}| = \sqrt{5 + z^2}$$

Plug $|\vec{a}|$ and $\gamma = \pi/3$ into the expression for $\cos\gamma$, then solve the equation for z.

$$\cos\left(\frac{\pi}{3}\right) = \frac{z}{\sqrt{5 + z^2}}$$

$$\frac{1}{2} = \frac{z}{\sqrt{5 + z^2}}$$

Square both sides, then multiply through by $4(5 + z^2)$.

$$\frac{1}{4} = \frac{z^2}{5 + z^2}$$

$$5 + z^2 = 4z^2$$

$$3z^2 = 5$$

$$z^2 = \frac{5}{3}$$

$$z_1 = -\frac{\sqrt{15}}{3} \text{ and } z_2 = \frac{\sqrt{15}}{3}$$

Since the vector $\vec{a}$ has the positive direction angle with respect to the z-axis, it also has a positive z-coordinate. So

$$z = \frac{\sqrt{15}}{3} \text{ and } \vec{a} = \left\langle 2, -1, \frac{\sqrt{15}}{3} \right\rangle$$

SCALAR EQUATION OF A LINE

■ 1. Find the parametric scalar equations of the line that pass through the points $A(5,4,-3)$ and $B(1,0,3)$.

Solution:

To find the parametric equations of a line, use the formulas

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Take $A(5,4,-3)$ as a point on the line, and $\overrightarrow{AB}$ as a direction vector. Since A is the initial point of the vector $\overrightarrow{AB}$, and B is the terminal point,

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

$$\overrightarrow{AB} = \langle 1-5, 0-4, 3-(-3) \rangle$$

$$\overrightarrow{AB} = \langle -4, -4, 6 \rangle$$

Plug these values into the equations for the line.

$$x = 5 - 4t$$

$$y = 4 - 4t$$

$$z = -3 + 6t$$

■ 2. Find the parametric scalar equations of the line that passes through the point $A(4,-1,0)$ and is orthogonal to the plane $x + 2y - z = 7$.

Solution:

Since the plane has the equation $x + 2y - z = 7$, its normal vector is $\langle 1,2,-1 \rangle$. Also, since the line is orthogonal to the plane, we can use the normal vector of the plane as the direction vector of the line.

To find the parametric equations of a line, use the formulas

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Take $A(4,-1,0)$ as a point on the line, and $\langle 1,2,-1 \rangle$ as a direction vector.

$$x = 4 + (1)t = 4 + t$$

$$y = -1 + (2)t = -1 + 2t$$

$$z = 0 + (-1)t = -t$$

■ 3. Find the parametric scalar equations of the line that forms the intersection of the planes $2x + 3y - z = 1$ and $x - y + 4z = -4$.

Solution:

There are an infinite number of correct answers for this problem, because we can choose any point on the line, and also any direction vector. Let's choose the points at $x = 0$ and $x = 1$.

Substitute $x = 0$ and solve the system for y and z to get

$$3y - z = 1$$

$$-y + 4z = -4$$

and then

$$z = 3y - 1$$

$$-y + 4(3y - 1) = -4$$

So

$$11y - 4 = -4$$

$y = 0$ and $z = -1$.

Substitute $x = 1$ to get

$$2(1) + 3y - z = 1$$

$$(1) - y + 4z = -4$$

and then

$$z = 3y + 1$$

$$1 - y + 4(3y + 1) = -4$$

So

$$11y = -9$$

$$y = -\frac{9}{11} \text{ and } z = -\frac{16}{11}$$

So we have two points on the line, $A(0,0,-1)$ and $B(1,-9/11,-16/11)$. To find the parametric equations of a line, use the formulas

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Take $A(0,0,-1)$ as a point on the line and $\overrightarrow{AB}$ as a direction vector. Since A is the initial point of the vector $\overrightarrow{AB}$, and B is the terminal point,

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

$$\overrightarrow{AB} = \left\langle 1 - 0, -\frac{9}{11} - 0, -\frac{16}{11} - (-1) \right\rangle$$

$$\overrightarrow{AB} = \left\langle 1, -\frac{9}{11}, -\frac{5}{11} \right\rangle$$

Plug these values into the equations for the line.

$$x = 0 + (1)t = t$$

$$y = 0 - \frac{9}{11}t = -\frac{9}{11}t$$

$$z = -1 - \frac{5}{11}t$$

SCALAR EQUATION OF A PLANE

■ 1. Find the scalar equations of the plane, given its vector equation.

$$\langle 1,2,-1\rangle \cdot (\vec{r} - \langle 0,5,-4\rangle) = 0$$

Solution:

The scalar equation of the plane in general form is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

From the vector equation of the plane we can get a normal vector to the plane $\vec{n} = \langle 1,2,-1\rangle$, and a point that lies in the plane $(0,5,-4)$. Plug these values into the scalar equation of the plane.

$$1(x-0)+2(y-5)+(-1)(z-(-4))=0$$

$$x+2y-z-14=0$$

■ 2. Find the scalar equations of the plane that passes through the points $A(2,0,1)$, $B(-1,3,2)$, and $C(1,1,-4)$.

Solution:

The scalar equation of the plane in general form is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Let $\overrightarrow{n} = \langle a, b, c \rangle$ be the unknown normal vector of the plane, then $\overrightarrow{n}$ is orthogonal to any vector in the plane. Since $\overrightarrow{n}$ is orthogonal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$,

$$\overrightarrow{n} \cdot \overrightarrow{AB} = 0$$

$$\overrightarrow{n} \cdot \overrightarrow{AC} = 0$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \langle -1 - 2, 3 - 0, 2 - 1 \rangle = \langle -3, 3, 1 \rangle$$

$$\overrightarrow{AC} = \langle 1 - 2, 1 - 0, -4 - 1 \rangle = \langle -1, 1, -5 \rangle$$

So

$$\langle a, b, c \rangle \cdot \langle -3, 3, 1 \rangle = 0$$

$$\langle a, b, c \rangle \cdot \langle -1, 1, -5 \rangle = 0$$

Therefore, we have a system of equations in terms of a, b, and c.

$$-3a + 3b + c = 0$$

$$-a + b - 5c = 0$$

The system has an infinite number of solutions since there are an infinite number of normal vectors to the plane. Let's choose the vector with $a = 1$, and solve the system for the associated values of b and c. We get

$$-3 + 3b + c = 0$$

$$-1 + b - 5c = 0$$

and then

$$c = 3 - 3b$$

$$-1 + b - 5(3 - 3b) = 0$$

Therefore, $b = 1$ and $c = 0$. So the normal vector is $\vec{n} = \langle 1,1,0 \rangle$. Plug $\vec{n}$ and $A(2,0,1)$ into the scalar equation of the plane.

$$1(x - 2) + 1(y - 0) + 0(z - 1) = 0$$

$$x + y - 2 = 0$$

■ 3. Find the scalar equation of a plane(s) that's 6 units from, and parallel to, the plane $x - 2y + 2z - 2 = 0$.

Solution:

Since the planes are parallel, they have the same normal vector $\vec{n} = \langle 1, -2,2 \rangle$. Let's take any point in the given plane, then find the points that are at a distance of 6 from it in the direction of $\pm \vec{n}$.

Let $x = 0$ and $y = 0$. Then $2z - 2 = 0$ and $z = 1$. So the point $A(0,0,1)$ is in the given plane. The magnitude of $\vec{n}$ is

$$|\vec{n}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

Let O be the origin and B be the point at a distance of 6 from A in the direction of $\vec{n}$. Since $|\vec{n}| = 3$ and $AB = 6$, we can find the coordinates of B using a vector equation.

$$\overrightarrow{OB} = \overrightarrow{OA} + 2\vec{n}$$

$$\overrightarrow{OB} = \langle 0,0,1 \rangle + 2\langle 1, -2,2 \rangle$$

$$\overrightarrow{OB} = \langle 2, -4,5 \rangle$$

So the point B has coordinates $(2, -4,5)$. Similarly, let C be the point at a distance of 6 from A in the direction of $-\vec{n}$.

$$\overrightarrow{OC} = \langle 0,0,1 \rangle - 2\langle 1, -2,2 \rangle$$

$$\overrightarrow{OC} = \langle -2,4, -3 \rangle$$

So the point C has coordinates $(-2,4, -3)$. Plug $\vec{n}$ and $B(2, -4,5)$ into the scalar equation of the plane.

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$1(x - 2) + (-2)(y + 4) + 2(z - 5) = 0$$

$$x - 2y + 2z - 20 = 0$$

Plug $\vec{n}$ and $C(-2,4, -3)$ into the scalar equation of the plane.

$$1(x + 2) + (-2)(y - 4) + 2(z + 3) = 0$$

$$x - 2y + 2z + 16 = 0$$

SCALAR AND VECTOR PROJECTIONS

1. Find the vector sum of projections of the vector $\vec{a} = \langle 13, -8, 9 \rangle$ onto the three coordinate axes.

Solution:

The vector projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is given by

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}$$

While finding the vector projection onto a line, we can choose any direction vector of the line. So let's find the vector projection of $\vec{a}$ onto the three unit vectors, $\vec{x} = \langle 1, 0, 0 \rangle$, $\vec{y} = \langle 0, 1, 0 \rangle$, and $\vec{z} = \langle 0, 0, 1 \rangle$.

Since the magnitude of each unit vector is 1,

$$\text{proj}_{\vec{x}}(\vec{a}) = (\vec{a} \cdot \vec{x})\vec{x} = (13(1) - 8(0) + 9(0))\langle 1, 0, 0 \rangle = \langle 13, 0, 0 \rangle$$

$$\text{proj}_{\vec{y}}(\vec{a}) = (\vec{a} \cdot \vec{y})\vec{y} = (13(0) - 8(1) + 9(0))\langle 0, 1, 0 \rangle = \langle 0, -8, 0 \rangle$$

$$\text{proj}_{\vec{z}}(\vec{a}) = (\vec{a} \cdot \vec{z})\vec{z} = (13(0) - 8(0) + 9(1))\langle 0, 0, 1 \rangle = \langle 0, 0, 9 \rangle$$

The sum of the vector projections is

$$\text{proj}_{\vec{x}}(\vec{a}) + \text{proj}_{\vec{y}}(\vec{a}) + \text{proj}_{\vec{z}}(\vec{a}) = \langle 13, 0, 0 \rangle + \langle 0, -8, 0 \rangle + \langle 0, 0, 9 \rangle = \langle 13, -8, 9 \rangle$$

In fact, the vector sum of the projections of any vector $\vec{a}$ onto the coordinate axes is always equal to the vector $\vec{a}$ itself.

■ 2. Find the projection of the vector $\vec{a}=\langle 4,3,-1\rangle$ onto the plane Q, which is given by $2x-y+2z-7=0$.

Solution:

The vector projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is given by

$$\text{proj}_{\vec{b}}(\vec{a})=\frac{\vec{a}\cdot\vec{b}}{|\vec{b}|^2}\vec{b}$$

The projection of $\vec{a}$ onto a plane can be calculated by subtracting the component of $\vec{a}$ that's orthogonal to the plane, from $\vec{a}$. So

$$\text{proj}_Q(\vec{a})=\vec{a}-\text{proj}_{\vec{n}}(\vec{a})=\vec{a}-\frac{\vec{a}\cdot\vec{n}}{|\vec{n}|^2}\vec{n}$$

Since the plane has equation $2x-y+2z-7=0$, its normal vector is $\vec{n}=\langle 2,-1,2\rangle$. The magnitude of $\vec{n}$ is

$$|\vec{n}|=\sqrt{2^2+(-1)^2+2^2}=\sqrt{9}=3$$

The dot product of $\vec{a}$ and $\vec{n}$ is

$$\vec{a}\cdot\vec{n}=\langle 4,3,-1\rangle\cdot\langle 2,-1,2\rangle$$

$$\vec{a} \cdot \vec{n} = (4)(2) + (3)(-1) + (-1)(2)$$

$$\vec{a} \cdot \vec{n} = 3$$

Plug these values into the formula for a vector projection onto a plane.

$$\text{proj}_Q(\vec{a}) = \langle 4,3,-1 \rangle - \frac{3}{3^2} \langle 2,-1,2 \rangle$$

$$\text{proj}_Q(\vec{a}) = \left\langle 4 - \frac{2}{3}, 3 + \frac{1}{3}, -1 - \frac{2}{3} \right\rangle$$

$$\text{proj}_Q(\vec{a}) = \left\langle \frac{10}{3}, \frac{10}{3}, -\frac{5}{3} \right\rangle$$

■ 3. Find the vector $\vec{a}$ if its scalar projections onto the vectors $\vec{b} = \langle 4,-3 \rangle$ and $\vec{c} = \langle 0,2 \rangle$ are both 3.

Solution:

The scalar projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is given by

$$\text{comp}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Let x and y be the coordinates of the vector $\vec{a}$. The magnitudes of $\vec{b}$ and $\vec{c}$ are

$$|\vec{b}| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$$

$$|\vec{c}| = \sqrt{0^2 + 2^2} = \sqrt{4} = 2$$

Since the scalar projection of $\vec{a}$ onto $\vec{b}$ is 3,

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 3$$

$$\frac{4x - 3y}{5} = 3$$

$$4x - 3y = 15$$

Similarly, since the scalar projection of $\vec{a}$ onto $\vec{c}$ is 3,

$$\frac{\vec{a} \cdot \vec{c}}{|\vec{c}|} = 3$$

$$\frac{0 \cdot x + 2y}{2} = 3$$

$$2y = 6$$

$$y = 3$$

Substitute $y = 3$ into $4x - 3y = 15$ in order to solve for x.

$$4x - 3(3) = 15$$

$$4x = 24$$

$$x = 6$$

Then the vector $\vec{a}$ is given by $\vec{a} = \langle 6,3 \rangle$.

CROSS PRODUCT OF TWO VECTORS

■ 1. Find the vector $\vec{a}$ given that $\vec{a} \times \vec{b} = \vec{c}$, where $\vec{a} = \langle 1, a_2, a_3 \rangle$, $\vec{b} = \langle 3,1,1 \rangle$, and $\vec{c} = \langle 1,2,-5 \rangle$.

Solution:

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle b_1, b_2, b_3 \rangle = \langle 3,1,1 \rangle$ and $a_1 = 1$.

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2 - a_3) - \mathbf{j}(1 - 3a_3) + \mathbf{k}(1 - 3a_2)$$

Since the cross product is $\vec{c} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, we get a system of equations.

$$a_2 - a_3 = 1$$

$$-(1 - 3a_3) = 2$$

$$1 - 3a_2 = -5$$

Solve the system.

$$1 + 5 = 3a_2,\ 6 = 3a_2,\ a_2 = 2$$

$$1 - 3a_3 = -2,\ \ -3a_3 = -3,\ a_3 = 1$$

$$a_2 - a_3 = 2 - 1 = 1$$

In the general case, the vector equation $\overrightarrow{x} \times \overrightarrow{b} = \overrightarrow{c}$ has an infinite number of solutions for $\overrightarrow{x}$.

■ 2. Find the cross product $\overrightarrow{a} \times \overrightarrow{a}$ for an arbitrary vector $\overrightarrow{a}$.

Solution:

The cross product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is given by

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle b_1, b_2, b_3 \rangle = \langle a_1, a_2, a_3 \rangle$.

$$\overrightarrow{a} \times \overrightarrow{a} = \mathbf{i}(a_2a_3 - a_3a_2) - \mathbf{j}(a_1a_3 - a_3a_1) + \mathbf{k}(a_1a_2 - a_2a_1)$$

$$\overrightarrow{a} \times \overrightarrow{a} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0)$$

So the cross product of the vector by itself is equal to the zero vector.

$$\overrightarrow{O} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

In the general case, the cross product of any vector by a vector with the same direction is always equal to the zero vector.

■ 3. Find the cross product $\overrightarrow{a} \times \text{proj}_{xy}\overrightarrow{a}$, where $\overrightarrow{a} = \langle 4,5,-3 \rangle$ and $\text{proj}_{xy}\overrightarrow{a}$ is the vector projection of the vector $\overrightarrow{a}$ onto the xy-plane.

Solution:

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

The vector projection of the vector onto the xy-plane is the vector with the same x and y coordinates, but a $z = 0$ coordinate.

$$\text{proj}_{xy}\vec{a} = \langle 4,5,0 \rangle$$

Calculate the cross product.

$$\vec{a} \times \text{proj}_{xy}\vec{a} = \mathbf{i}(5 \cdot 0 - (-3) \cdot 5) - \mathbf{j}(4 \cdot 0 - (-3) \cdot 4) + \mathbf{k}(4 \cdot 5 - 5 \cdot 4)$$

$$\vec{a} \times \text{proj}_{xy}\vec{a} = 15\mathbf{i} - 12\mathbf{j} + 0\mathbf{k}$$

In the general case, the cross product of any vector by its projection onto the plane is always a vector that lies in the plane.

VECTOR ORTHOGONAL TO THE PLANE

1. Find the vector orthogonal to the plane which passes through the point $A(2,3,1)$ and the z-axis.

Solution:

In order to find the vector orthogonal to the plane, we can compute the cross product of any two nonparallel vectors lying in the plane. For simplicity, let's take the unit vector along the z-axis, $\vec{u_z} = \langle 0,0,1 \rangle$, and the vector from the origin to the point A, $\vec{a} = \langle 2,3,1 \rangle$.

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{u_z} = \langle 2,3,1 \rangle \times \langle 0,0,1 \rangle$$

$$\vec{a} \times \vec{u_z} = \mathbf{i}(3 \cdot 1 - 1 \cdot 0) - \mathbf{j}(2 \cdot 1 - 1 \cdot 0) + \mathbf{k}(2 \cdot 0 - 3 \cdot 0)$$

$$\vec{a} \times \vec{u_z} = 3\mathbf{i} - 2\mathbf{j} + 0\mathbf{k}$$

There are infinite number of vectors orthogonal to a plane; all of them may have different magnitudes, but the same (or opposite) direction.

■ 2. Find the equation of the plane that passes through the point D and is parallel to the plane ABC, if $A(1,2,-2)$, $B(1,4,3)$, $C(-5,3,-1)$, and $D(2,-4,7)$.

Solution:

Since the two planes are parallel, they have equal normal vectors. In order to find the vector orthogonal to the plane ABC, we can calculate the cross product of any two nonparallel vectors lying in the plane, like $\vec{b}=\overrightarrow{AB}$ and $\vec{c}=\overrightarrow{AC}$.

$$\vec{b}=\langle 1-1,4-2,3-(-2)\rangle=\langle 0,2,5\rangle$$

$$\vec{c}=\langle -5-1,3-2,-1-(-2)\rangle=\langle -6,1,1\rangle$$

The cross product of two vectors $\vec{c}$ and $\vec{b}$ is given by

$$\vec{a}\times\vec{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Calculate the cross product.

$$\vec{c}\times\vec{b}=\langle -6,1,1\rangle\times\langle 0,2,5\rangle$$

$$\vec{c}\times\vec{b}=\mathbf{i}(1\cdot 5-1\cdot 2)-\mathbf{j}(-6\cdot 5-1\cdot 0)+\mathbf{k}(-6\cdot 2-1\cdot 0)$$

$$\vec{c}\times\vec{b}=3\mathbf{i}+30\mathbf{j}-12\mathbf{k}$$

Plug in the vector $\vec{n}$ and the point D into the equation of the plane in general form.

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0$$

$$3(x-2)+30(y+4)-12(z-7)=0$$

$$(x-2)+10(y+4)-4(z-7)=0$$

$$x+10y-4z+66=0$$

3. Find the equation of the line that passes through the point $A(-2,3,4)$ and is orthogonal to the plane that includes the vectors $\vec{a}=\langle 2,4,0\rangle$ and $\vec{b}=\langle -1,1,2\rangle$.

Solution:

Since the line is orthogonal to the plane, its direction vector is equal to the normal vector of the plane. In order to find the vector orthogonal to the plane, we can compute the cross product $\vec{a}\times\vec{b}$.

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}\times\vec{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Calculate the cross product.

$$\vec{a}\times\vec{b}=\langle 2,4,0\rangle\times\langle -1,1,2\rangle$$

$$\vec{a}\times\vec{b}=\mathbf{i}(4\cdot 2-0\cdot 1)-\mathbf{j}(2\cdot 2-0\cdot(-1))+\mathbf{k}(2\cdot 1-4\cdot(-1))$$

$$\vec{a}\times\vec{b}=8\mathbf{i}-4\mathbf{j}+6\mathbf{k}$$

So the normal vector to the plane is $\vec{n}=\langle 8,-4,6\rangle$.

The parametric equations of a line in general form are

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Plug the vector $\overrightarrow{n} = \langle 8, -4, 6 \rangle$ and the point $A(-2,3,4)$ into these equations.

$$x = -2 + 8t$$

$$y = 3 - 4t$$

$$z = 4 + 6t$$

VOLUME OF THE PARALLELEPIPED FROM VECTORS

■ 1. Find the height of the parallelepiped given that its volume is 670, and that the vectors $\vec{a}=\langle 1,0,-1\rangle$ and $\vec{b}=\langle 2,3,5\rangle$ are the adjacent edges of its base.

Solution:

The volume of the parallelepiped is equal to the product of its height, and the area of its base. So its height is equal to the volume, divided by the base area.

The area of parallelogram is equal to the magnitude of the cross product of its edge vectors. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}\times\vec{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Calculate the cross product.

$$\vec{a}\times\vec{b}=\langle 1,0,-1\rangle\times\langle 2,3,5\rangle$$

$$\vec{a}\times\vec{b}=\mathbf{i}(0\cdot 5-(-1)\cdot 3)-\mathbf{j}(1\cdot 5-(-1)\cdot 2)+\mathbf{k}(1\cdot 3-0\cdot 2)$$

$$\vec{a}\times\vec{b}=3\mathbf{i}-7\mathbf{j}+3\mathbf{k}$$

The area of the base is

$$|\vec{a}\times\vec{b}|=\sqrt{3^2+(-7)^2+3^2}=\sqrt{67}$$

Therefore, the height of the parallelepiped is

$$\text{Height} = \frac{\text{Volume}}{\text{Area of Base}} = \frac{670}{\sqrt{67}} = 10\sqrt{67}$$

■ 2. Find the volume of the tetrahedron whose adjacent edges are the vectors $\vec{a} = \langle 0,0,3 \rangle$, $\vec{b} = \langle 2,1,4 \rangle$, and $\vec{c} = \langle -1,-2,1 \rangle$.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = [\text{Height}] \cdot [\text{Area of parallelogram}] = |(\vec{a} \times \vec{b}) \cdot \vec{c}\,|$$

The volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{3}[\text{Height}] \cdot \frac{1}{2}[\text{Area of parallelogram}] = \frac{1}{6}V_{\text{parallelepiped}}$$

Therefore,

$$V_{\text{tetrahedron}} = \frac{1}{6}|(\vec{a} \times \vec{b}) \cdot \vec{c}\,|$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 0,0,3 \rangle \times \langle 2,1,4 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(0 \cdot 4 - 3 \cdot 1) - \mathbf{j}(0 \cdot 4 - 3 \cdot 2) + \mathbf{k}(0 \cdot 1 - 0 \cdot 2)$$

$$\vec{a} \times \vec{b} = -3\mathbf{i} + 6\mathbf{j} + 0\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle -3,6,0 \rangle \cdot \langle -1,-2,1 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (-3)(-1) + (6)(-2) + (0)(1) = -9$$

Therefore, the volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{6}|-9| = 1.5$$

■ 3. Find the value of p such that the volume of the parallelepiped with adjacent edges $\vec{a} = \langle 0,2,3 \rangle$, $\vec{b} = \langle 1,-2,1 \rangle$, and $\vec{c} = \langle p,p,p \rangle$ is equal to 63.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 0,2,3 \rangle \times \langle 1,-2,1 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(2 \cdot 1 - 3 \cdot (-2)) - \mathbf{j}(0 \cdot 1 - 3 \cdot 1) + \mathbf{k}(0 \cdot (-2) - 2 \cdot 1)$$

$$\vec{a} \times \vec{b} = 8\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle 8,3,-2 \rangle \cdot \langle p,p,p \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 8p + 3p - 2p = 9p$$

Since the volume of the parallelepiped is equal to 63, we can write an equation for p.

$$9p = 63$$

$$p = 7$$

VOLUME OF THE PARALLELEPIPED FROM ADJACENT EDGES

1. Find the volume of tetrahedron $ABCD$, given $A(2,0,3)$, $B(-1,1,3)$, $C(4,5,-2)$, and $D(2,2,3)$.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = |(\vec{a} \times \vec{b}) \cdot \vec{c}\,|$$

The volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{6} V_{\text{parallelepiped}}$$

Therefore,

$$V_{\text{tetrahedron}} = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}\,|$$

For the scalar triple product, we can choose any three adjacent edges with a common initial point. For example, let's take

$$\vec{a} = \overrightarrow{AB} = \langle -1-2, 1-0, 3-3 \rangle = \langle -3,1,0 \rangle$$

$$\vec{b} = \overrightarrow{AC} = \langle 4-2, 5-0, -2-3 \rangle = \langle 2,5,-5 \rangle$$

$$\vec{c} = \overrightarrow{AD} = \langle 2-2, 2-0, 3-3 \rangle = \langle 0,2,0 \rangle$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle -3,1,0 \rangle \times \langle 2,5,-5 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(1 \cdot (-5) - 0 \cdot 5) - \mathbf{j}(-3 \cdot (-5) - 0 \cdot 2) + \mathbf{k}(-3 \cdot 5 - 1 \cdot 2)$$

$$\vec{a} \times \vec{b} = -5\mathbf{i} - 15\mathbf{j} - 17\mathbf{k}$$

The dot product is

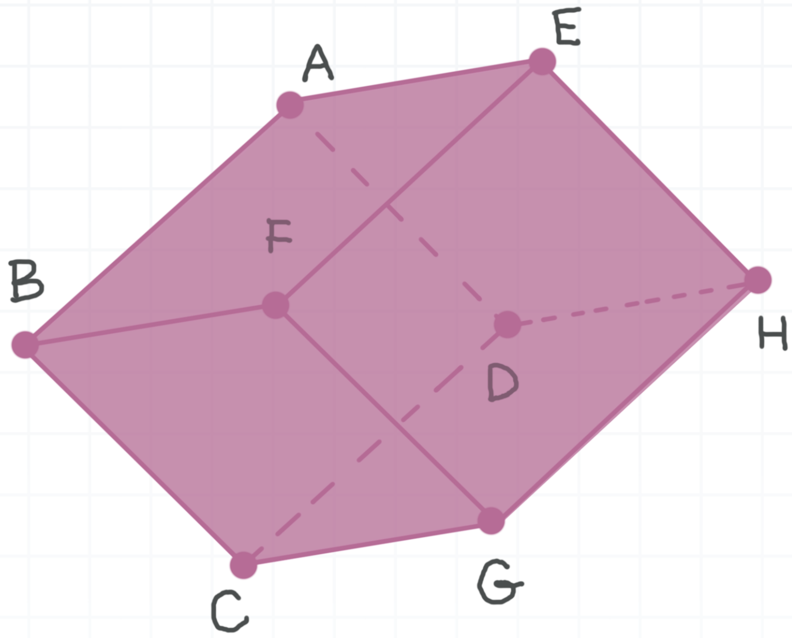
$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle -5,-15,-17 \rangle \cdot \langle 0,2,0 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -5 \cdot 0 + (-15) \cdot 2 + (-17) \cdot 0 = -30$$

Therefore, the volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{6}|-30| = 5$$

■ 2. Find the volume of parallelepiped $ABCDEFGH$, given $A(1,2,2)$, $B(-1,-2,0)$, $F(4,3,-1)$, and $G(5,6,-4)$.



Solution:

We don't have three adjacent edges to use for calculating the scalar triple product. So we first need to find the coordinates of D and E. Let's use the main property of parallelepipeds, which is that any parallelepiped has three quadruples of parallel and equal edges.

Since $\overrightarrow{AE} = \overrightarrow{BF}$, we can find the coordinates of E by adding the vector $\overrightarrow{BF}$ to the coordinates of the point A. So

$$x_E = x_A + (x_F - x_B) = 1 + (4 - (-1)) = 6$$

$$y_E = y_A + (y_F - y_B) = 2 + (3 - (-2)) = 7$$

$$z_E = z_A + (z_F - z_B) = 2 + (-1 - 0) = 1$$

Similarly, since $\overrightarrow{FG} = \overrightarrow{AD}$, we can find the coordinates of the point D by adding the vector $\overrightarrow{FG}$ to the coordinates of the point A. So

$$x_D = x_A + (x_G - x_F) = 1 + (5 - 4) = 2$$

$$y_D = y_A + (y_G - y_F) = 2 + (6 - 3) = 5$$

$$z_D = z_A + (z_G - z_F) = 2 + (-4 - (-1)) = -1$$

So the three adjacent vectors are

$$\overrightarrow{AB} = \langle -1 - 1, -2 - 2, 0 - 2 \rangle = \langle -2, -4, -2 \rangle$$

$$\overrightarrow{AD} = \langle 2 - 1, 5 - 2, -1 - 2 \rangle = \langle 1, 3, -3 \rangle$$

$$\overrightarrow{AE} = \langle 6 - 1, 7 - 2, 1 - 2 \rangle = \langle 5, 5, -1 \rangle$$

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = |(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE}|$$

The cross product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is given by

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\overrightarrow{AB} \times \overrightarrow{AD} = \langle -2, -4, -2 \rangle \times \langle 1, 3, -3 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \mathbf{i}(-4 \cdot (-3) - (-2) \cdot 3) - \mathbf{j}(-2 \cdot (-3) - (-2) \cdot 1) + \mathbf{k}(-2 \cdot 3 - (-4) \cdot 1)$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = 18\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

The dot product is

$$(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE} = \langle 18, -8, -2 \rangle \cdot \langle 5, 5, -1 \rangle$$

$$(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE} = 18 \cdot 5 + (-8) \cdot 5 + (-2) \cdot (-1) = 52$$

■ 3. Find the volume of the parallelepiped with base $ABCD$ and height 5, if $A(3,3,3)$, $B(0,-2,-2)$, and $C(-3,1,0)$.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = [\text{Height}] \cdot [\text{Area of Base}]$$

The area of the base is

$$\text{Area}_{ABCD} = |\vec{a} \times \vec{b}|$$

where $\vec{a}$ and $\vec{b}$ are any two adjacent edges of the base. Let's choose $\vec{a} = \overrightarrow{BA}$ and $\vec{b} = \overrightarrow{BC}$. So the two adjacent vectors are

$$\overrightarrow{BA} = \langle 3-0, 3-(-2), 3-(-2)\rangle = \langle 3,5,5\rangle$$

$$\overrightarrow{BC} = \langle -3-0, 1-(-2), 0-(-2)\rangle = \langle -3,3,2\rangle$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\overrightarrow{BA} \times \overrightarrow{BD} = \langle 3,5,5\rangle \times \langle -3,3,2\rangle$$

$$\overrightarrow{BA} \times \overrightarrow{BD} = \mathbf{i}(5 \cdot 2 - 5 \cdot 3) - \mathbf{j}(3 \cdot 2 - 5 \cdot (-3)) + \mathbf{k}(3 \cdot 3 - 5 \cdot (-3))$$

$$\overrightarrow{BA} \times \overrightarrow{BD} = -5\mathbf{i} - 21\mathbf{j} + 24\mathbf{k}$$

The magnitude of the cross product is

$$|\overrightarrow{BA} \times \overrightarrow{BD}| = \sqrt{(-5)^2 + (-21)^2 + 24^2} = \sqrt{1{,}042}$$

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = [\text{Height}] \cdot [\text{Area of Base}] = 5\sqrt{1{,}042}$$

SCALAR TRIPLE PRODUCT TO PROVE VECTORS ARE COPLANAR

1. Find the value of the parameter p such that the vectors $\vec{a} = \langle 1,3,-1 \rangle$, $\vec{b} = \langle 2,2,2 \rangle$, and $\vec{c} = \langle 0,-1,p \rangle$ are coplanar.

Solution:

The vectors are coplanar if their scalar triple product is 0, i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 1,3,-1 \rangle \times \langle 2,2,2 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(3 \cdot 2 - (-1) \cdot 2) - \mathbf{j}(1 \cdot 2 - (-1) \cdot 2) + \mathbf{k}(1 \cdot 2 - 3 \cdot 2)$$

$$\vec{a} \times \vec{b} = 8\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle 8,-4,-4 \rangle \cdot \langle 0,-1,p \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 8 \cdot 0 + (-4) \cdot (-1) + (-4) \cdot p = 4 - 4p$$

Since the scalar triple product is 0, we can write the equation for p.

$$4 - 4p = 0$$

$$p = 1$$

■ 2. Check if the vectors $\vec{a} = \langle 1,1,0 \rangle$, $\vec{b} = \langle 0,1,1 \rangle$, and $\vec{c} = \langle 1,0,-1 \rangle$ are coplanar. If they are, find the equation of the plane, assuming that the initial point of the vectors is the origin.

Solution:

The vectors are coplanar if their scalar triple product is 0, i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 1,1,0 \rangle \times \langle 0,1,1 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(1 \cdot 1 - 0 \cdot 1) - \mathbf{j}(1 \cdot 1 - 0 \cdot 0) + \mathbf{k}(1 \cdot 1 - 1 \cdot 0)$$

$$\vec{a} \times \vec{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle 1,-1,1 \rangle \cdot \langle 1,0,-1 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 1 \cdot 1 + (-1) \cdot 0 + 1 \cdot (-1) = 0$$

Since the scalar triple product is 0, the vectors are coplanar.

The equation of the plane is

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0$$

Since the vectors $\vec{a}$ and $\vec{b}$ lie in the plane, their cross product is orthogonal to the plane, i.e. $\vec{n}=\vec{a}\times\vec{b}=\langle 1,-1,1\rangle$.

Since the initial point of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the origin, it lies in the plane, i.e. $(x_0,y_0,z_0)=(0,0,0)$. Plug into the plane equation.

$$1(x-0)-1(y-0)+1(z-0)=0$$

$$x-y+z=0$$

■ 3. Check if the points $A(0,0,1)$, $B(2,0,3)$, $C(2,3,0)$, and $D(3,2,2)$ lie in the same plane.

Solution:

The points lie in the same plane if the vectors joining these points are coplanar. Check if the vectors $\vec{a}=\overrightarrow{AB}$, $\vec{b}=\overrightarrow{AC}$, and $\vec{c}=\overrightarrow{AD}$ are coplanar.

$$\overrightarrow{AB}=\langle 2-0,0-0,3-1\rangle=\langle 2,0,2\rangle$$

$$\overrightarrow{AC}=\langle 2-0,3-0,0-1\rangle=\langle 2,3,-1\rangle$$

$$\overrightarrow{AD}=\langle 3-0,2-0,2-1\rangle=\langle 3,2,1\rangle$$

The vectors are coplanar if their scalar triple product is 0, i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 2,0,2 \rangle \times \langle 2,3,-1 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(0 \cdot (-1) - 2 \cdot 3) - \mathbf{j}(2 \cdot (-1) - 2 \cdot 2) + \mathbf{k}(2 \cdot 3 - 0 \cdot 2)$$

$$\vec{a} \times \vec{b} = -6\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle -6,6,6 \rangle \cdot \langle 3,2,1 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -6 \cdot 3 + 6 \cdot 2 + 6 \cdot 1 = 0$$

Since the scalar triple product is 0, the vectors are coplanar and the points lie in the same plane.

DOMAIN OF A VECTOR FUNCTION

■ 1. Find the domain of the vector function.

$$\vec{F}(t,s) = \left\langle \sqrt{ts}, \frac{t}{s}, e^{t^2+s^2} \right\rangle$$

Solution:

The domain of the vector function is the intersection of the domains of all its components.

$$\text{dom } \vec{F} = \text{dom } F_1 \cap \text{dom } F_2 \cap \text{dom } F_3$$

Find the domain of the first component.

$$F_1(t,s) = \sqrt{ts}$$

$$ts \geq 0$$

$$[t \geq 0, s \geq 0] \text{ or } [t \leq 0, s \leq 0]$$

Find the domain of the second component.

$$F_2(t,s) = \frac{t}{s}$$

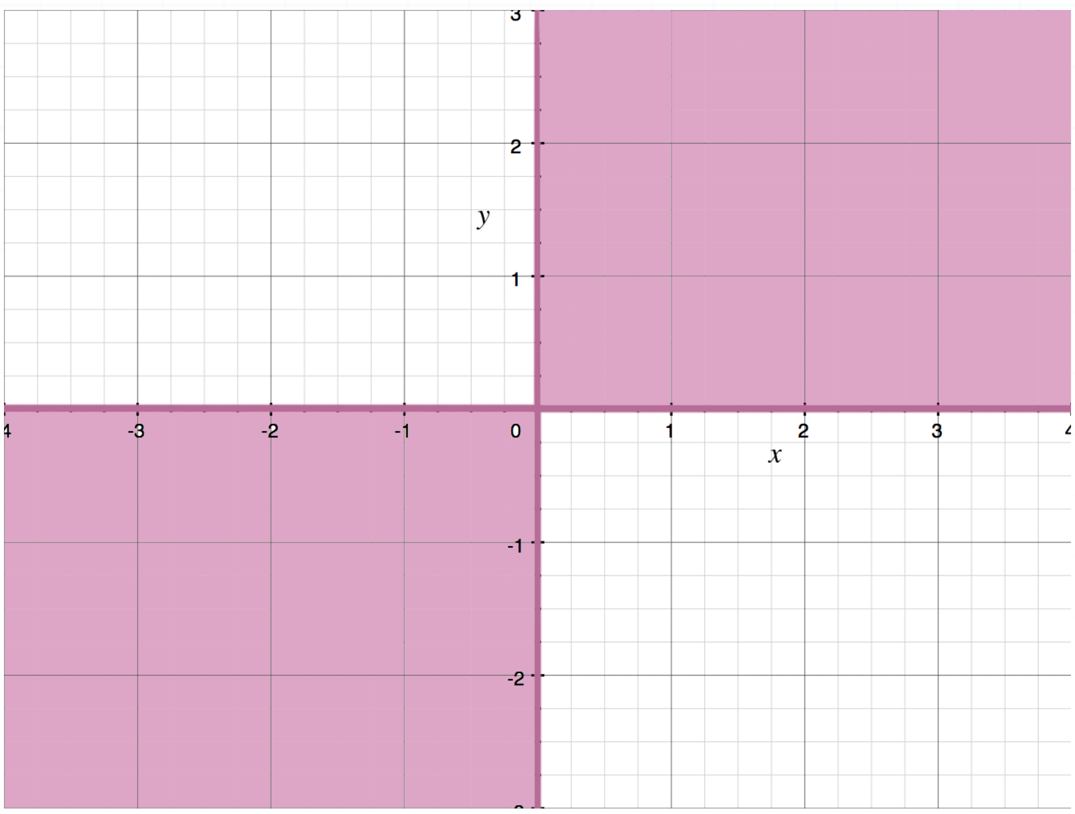
t is any real number, $s \neq 0$

Find the domain of the third component.

$$F_3(t, s) = e^{t^2+s^2}$$

s , t are any real numbers

So the intersection of the domains is $[t \geq 0, s \geq 0]$ or $[t \leq 0, s \leq 0]$, intersected with $[s \neq 0]$. Therefore, $[t \geq 0, s > 0]$ or $[t \leq 0, s < 0]$. So the domain of the vector function is all of the points within the first and third quadrants including the t-axis, but excluding the s-axis and the origin.



■ 2. Find the domain of the vector function.

$$\vec{F}(x, y) = \ln(x + y - 3) \cdot \mathbf{i} + \sqrt{2x - 2} \cdot \mathbf{j} + \sqrt{6 - y} \cdot \mathbf{k}$$

Solution:

The domain of the vector function is the intersection of the domains of all its components.

$$\text{dom } \overrightarrow{F} = \text{dom } F_1 \cap \text{dom } F_2 \cap \text{dom } F_3$$

Find the domain of the first component.

$$F_1(x, y) = \ln(x + y - 3)$$

$$x + y - 3 > 0$$

$$y > -x + 3$$

So the domain of $F_1(x, y)$ is all of the points above the line $y = -x + 3$ (excluding the line).

Find the domain of the second component.

$$F_2(x, y) = \sqrt{2x - 2}$$

$$2x - 2 \geq 0$$

$$x \geq 1$$

So the domain of $F_2(x, y)$ is all of the points to the right of the vertical line $x = 1$ (including the line).

Find the domain of the third component.

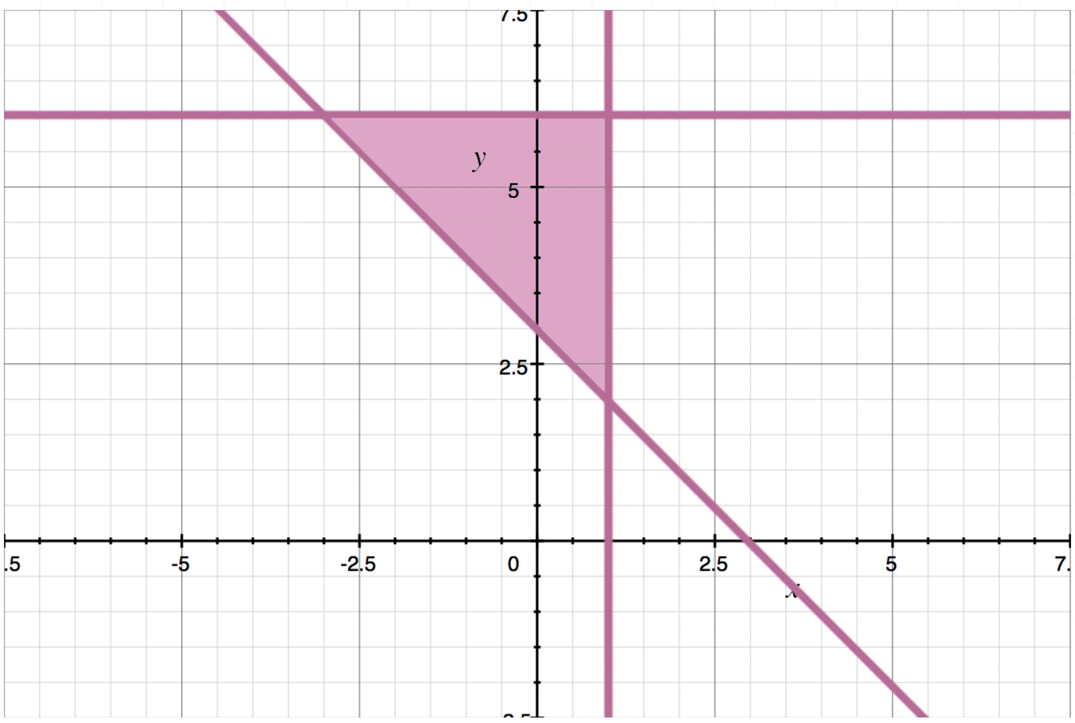
$$F_3(x, y) = \sqrt{6 - y}$$

$$6 - y \geq 0$$

$$y \leq 6$$

So the domain of $F_3(x, y)$ is all of the points below the horizontal line $y = 6$ (including the line).

So the intersection of the domains is all of the points within the triangle bounded by the lines $y = -x + 3$ (excluding), $x = 1$ (including), and $y = 6$ (including).



■ 3. Find the domain of the vector function.

$$\vec{F}(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2} \cdot \mathbf{i} + \frac{2x - y}{x + y + z - 4} \cdot \mathbf{j}$$

Solution:

The domain of the vector function is the intersection of the domains of all its components.

$$\text{dom } \vec{F} = \text{dom } F_1 \cap \text{dom } F_2$$

Find the domain of the first component.

$$F_1(x, y, z) = \sqrt{4 - x^2 - y^2 - z^2}$$

$$4 - x^2 - y^2 - z^2 \geq 0$$

$$x^2 + y^2 + z^2 \leq 4$$

So the domain of $F_1(x, y, z)$ is the set of interior points of the sphere with center at the origin and radius 2 (including sphere).

Find the domain of the second component.

$$F_2(x, y, z) = \frac{2x - y}{x + y + z - 4}$$

$$x + y + z - 4 \neq 0$$

So the domain of $F_2(x, y, z)$ is all of the points in space except the plane $x + y + z - 4 = 0$.

Let's find the intersection of the domains. It seems that the plane doesn't intersect the sphere. To check this let's find the distance from the plane to the center of sphere (which is at the origin).

The distance from the point (x_0, y_0, z_0) to the plane $Ax + By + Cz + D = 0$ is given by

$$d = \frac{|Ax_0 + By_0 + Cz_0 + D|}{\sqrt{A^2 + B^2 + C^2}}$$

Plug in $(x_0, y_0, z_0) = (0,0,0)$, $A = B = C = 1$, and $D = -4$.

$$d = \frac{|1 \cdot 0 + 1 \cdot 0 + 1 \cdot 0 - 4|}{\sqrt{1^2 + 1^2 + 1^2}}$$

$$d = \frac{4}{\sqrt{3}} \approx 2.3$$

Since the distance from the center of the sphere to the plane is larger than the radius, $2.3 > 2$, the sphere and the plane does not intersect.

Therefore, the domain of the vector function $\vec{F}(x, y, z)$ is the set of interior points of the sphere with center at the origin and radius 2 (including the sphere itself).

LIMIT OF A VECTOR FUNCTION

■ 1. Find the limit of the vector function.

$$\lim_{t\to 0,\, s\to 1} \vec{F}(t,s)$$

$$\vec{F}(t,s) = \left\langle \sqrt{s^2 - t^2}, \frac{\sin 3t}{t} + 3s^2, \frac{(t^2 - 2t - 3)(s^2 - 1)}{s - 1} \right\rangle$$

Solution:

Let's find the limit of each of the function's component separately, then evaluate at $t = 0$, $s = 1$.

$$\lim_{t\to 0,\, s\to 1} F_1(t,s) = \lim_{t\to 0,\, s\to 1} \sqrt{s^2 - t^2}$$

$$= \sqrt{1^2 - 0^2} = 1$$

$$\lim_{t\to 0,\, s\to 1} F_2(t,s) = \lim_{t\to 0,\, s\to 1} \left(\frac{\sin 3t}{t} + 3s^2 \right)$$

$$= \lim_{t\to 0,\, s\to 1} \frac{\sin 3t}{t} + \lim_{t\to 0,\, s\to 1} 3s^2$$

$$= \lim_{t\to 0} \frac{\sin 3t}{t} + \lim_{s\to 1} 3s^2$$

$$= \lim_{t\to 0} \frac{\sin 3t}{t} + 3$$

$$= 3 + 3 = 6$$

$$\lim_{t \to 0,\, s \to 1} F_3(t, s) = \lim_{t \to 0,\, s \to 1} \frac{(t^2 - 2t - 3)(s^2 - 1)}{s - 1}$$

$$= \lim_{t \to 0,\, s \to 1} \frac{(t^2 - 2t - 3)(s - 1)(s + 1)}{s - 1}$$

$$= \lim_{t \to 0,\, s \to 1} (t^2 - 2t - 3)(s + 1)$$

$$= (0^2 - 2 \cdot 0 - 3)(1 + 1) = -6$$

So the limit is $\langle 1,6,-6 \rangle$.

■ 2. Find the limit of the vector function.

$$\lim_{x \to \infty,\, y \to \infty} \vec{F}(t, s)$$

$$\vec{F}(x, y) = xy\, e^{-(x^2+y^2)} \cdot \mathbf{i} + \frac{\sin(x + y)}{x + y} \cdot \mathbf{j} + \frac{x}{y^4} \cdot \mathbf{k}$$

Solution:

Find the limit for each of the function's component separately.

$$\lim_{x \to \infty,\, y \to \infty} F_1(x, y) = \lim_{x \to \infty,\, y \to \infty} xy\, e^{-(x^2+y^2)}$$

$$= \lim_{x \to \infty} xe^{-x^2} \cdot \lim_{y \to \infty} ye^{-y^2}$$

$$= 0 \cdot 0 = 0$$

For the second component, consider

$$\lim_{x\to\infty, y\to\infty} |F_2(x,y)| = \lim_{x\to\infty, y\to\infty} \left| \frac{\sin(x+y)}{x+y} \right| \le \lim_{x\to\infty, y\to\infty} \left| \frac{1}{x+y} \right| = 0$$

Since $\lim_{x\to\infty, y\to\infty} |F_2(x,y)|$ exists and is 0, $\lim_{x\to\infty, y\to\infty} F_2(x,y)$ also exists and is 0.

For the third component,

$$\lim_{x\to\infty, y\to\infty} F_3(x,y) = \lim_{x\to\infty, y\to\infty} \frac{x}{y^4}$$

The limit does not exist. For example, approach (∞, ∞) along the curve $y = x$. In this case,

$$\lim_{x\to\infty, y\to\infty} \frac{x}{y^4} = \lim_{x\to\infty} \frac{x}{x^4} = \lim_{x\to\infty} \frac{1}{x^3} = 0$$

Next, approach (∞, ∞) along the curve $y = x^{1/8}$. In this case,

$$\lim_{x\to\infty, y\to\infty} \frac{x}{y^4} = \lim_{x\to\infty} \frac{x}{x^{4/8}} = \lim_{x\to\infty} \sqrt{x} = \infty$$

Since the function approaches different values, the limit does not exist.

Therefore, although the limits of the first two components of the function exist, the overall limit does not exist because the limit of the third component does not exist.

3. Find the limit of the vector function.

$$\lim_{x\to 3,\, y\to\infty,\, z\to 1} \vec{F}(x,y,z)$$

$$\vec{F}(x,y,z) = (x^2y - 3xyz + z^2 - x + 3y - 3z + 5)\mathbf{i} + \ln\frac{x+y}{z+y}\mathbf{j}$$

Solution:

Find the limit for each of the function's component separately.

$$\lim_{x\to 3,\, y\to\infty,\, z\to 1} F_1(x,y,z) = \lim_{x\to 3,\, y\to\infty,\, z\to 1} (x^2y - 3xyz + z^2 - x + 3y - 3z + 5)$$

$$= \lim_{x\to 3,\, y\to\infty,\, z\to 1} (y(x^2 - 3xz + 3) + (z^2 - x - 3z + 5))$$

$$= \lim_{x\to 3,\, y\to\infty,\, z\to 1} F_1(x,y,z) = \infty$$

$$\lim_{x\to 3,\, y\to\infty,\, z\to 1} F_1(x,y,z) = \lim_{x\to 3,\, y\to\infty,\, z\to 1} \ln\frac{x+y}{z+y}$$

$$= \lim_{x\to 3,\, y\to\infty,\, z\to 1} \ln\frac{1+\frac{x}{y}}{1+\frac{z}{y}}$$

$$= \ln\frac{1+0}{1+0} = \ln(1) = 0$$

SKETCHING THE VECTOR EQUATION

■ 1. Identify and sketch the curve that represents $\vec{r}(t)=\langle 3-5t,2t+1,-3t\rangle$.

Solution:

Write the vector function in parametric form.

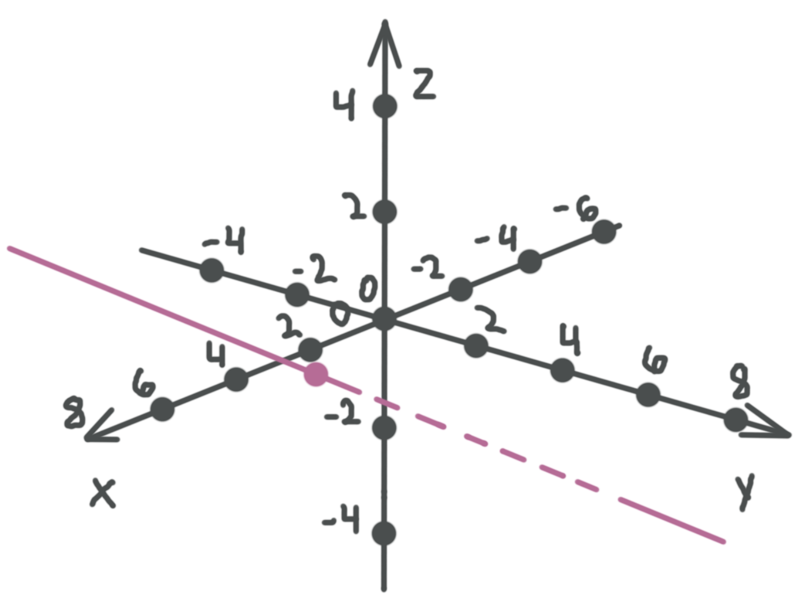
$$x(t)=3-5t$$

$$y(t)=1+2t$$

$$z(t)=-3t$$

These equations represent the line in three dimensions that passes through the point $(3,1,0)$ and has a direction vector of $\langle -5,2,-3\rangle$, so we can rewrite the equation of the line in vector form.

$$\vec{r}(t)=\langle 3,1,0\rangle+t\langle -5,2,-3\rangle$$



■ 2. Identify and sketch the curve representing the graph of the vector function $\vec{r}(t) = \langle 5\sin t, 3\cos t, -2\rangle$.

Solution:

Write the vector function in parametric form.

$$x(t) = 5\sin t$$

$$y(t) = 3\cos t$$

$$z(t) = -2$$

Use the trigonometric identity $\sin^2\phi + \cos^2\phi = 1$ to relate x and y.

$$(3x)^2 + (5y)^2 = (3 \cdot 5\sin t)^2 + (5 \cdot 3\cos t)^2$$

$$15^2\sin^2 t + 15^2\cos^2 t$$

$$15^2(\sin^2 t + \cos^2 t)$$

$$15^2$$

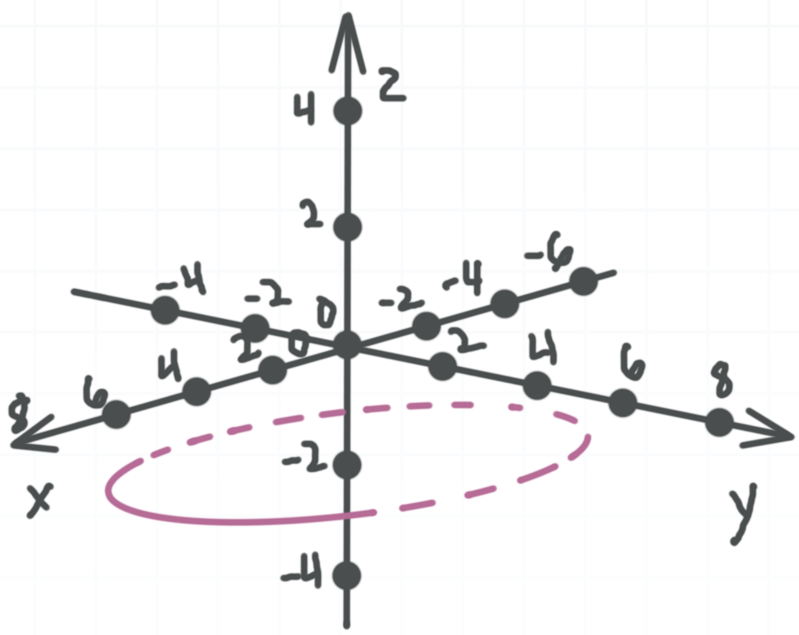
So

$$(3x)^2 + (5y)^2 = 15^2$$

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

$$z = -2$$

So the curve is the ellipse that lies in the plane $z = -2$, with center at the point $(0,0,-2)$, semi-axis of 5 in the x-direction, and semi-axis of 3 in the y-direction.



3. Identify and sketch the surface representing the graph of the vector function.

$$\vec{r}(t,s) = \langle 4\sin t\cos s, 4\sin t\sin s, 4\cos t\rangle$$

Solution:

Write the vector function in parametric form.

$$x(t,s) = 4\sin t\cos s$$

$$y(t,s) = 4\sin t\sin s$$

$$z(t, s) = 4\cos t$$

These equations are really similar to the formulas we use to convert to spherical coordinates, $x = \rho \sin \phi \cos \theta$, $y = \rho \sin \phi \sin \theta$, and $z = \rho \cos \phi$, but in our equations, it looks like $\rho = 4$, $\phi = t$, and $\theta = s$. Use the trigonometric identity $\sin^2 \alpha + \cos^2 \alpha = 1$ to build a relationship between x, y, and z.

$$x^2 + y^2 + z^2 = (4\sin t \cos s)^2 + (4\sin t \sin s)^2 + (4\cos t)^2$$

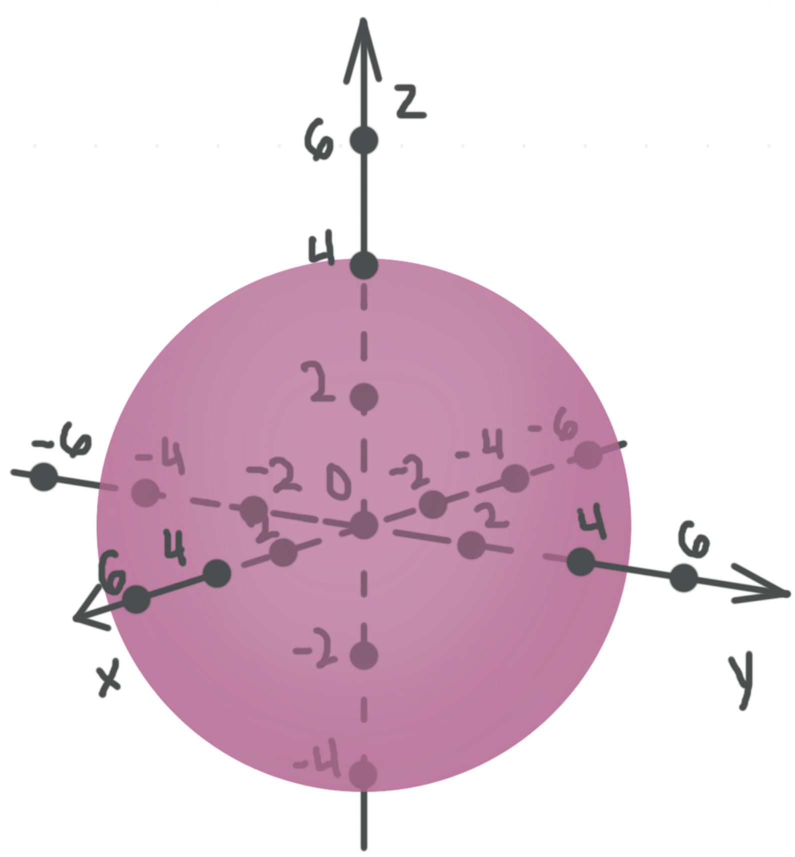
$$x^2 + y^2 + z^2 = 4^2(\sin^2 t \cos^2 s + \sin^2 t \sin^2 s + \cos^2 t)$$

$$x^2 + y^2 + z^2 = 4^2(\sin^2 t(\cos^2 s + \sin^2 s) + \cos^2 t)$$

$$x^2 + y^2 + z^2 = 4^2(\sin^2 t + \cos^2 t)$$

$$x^2 + y^2 + z^2 = 4^2$$

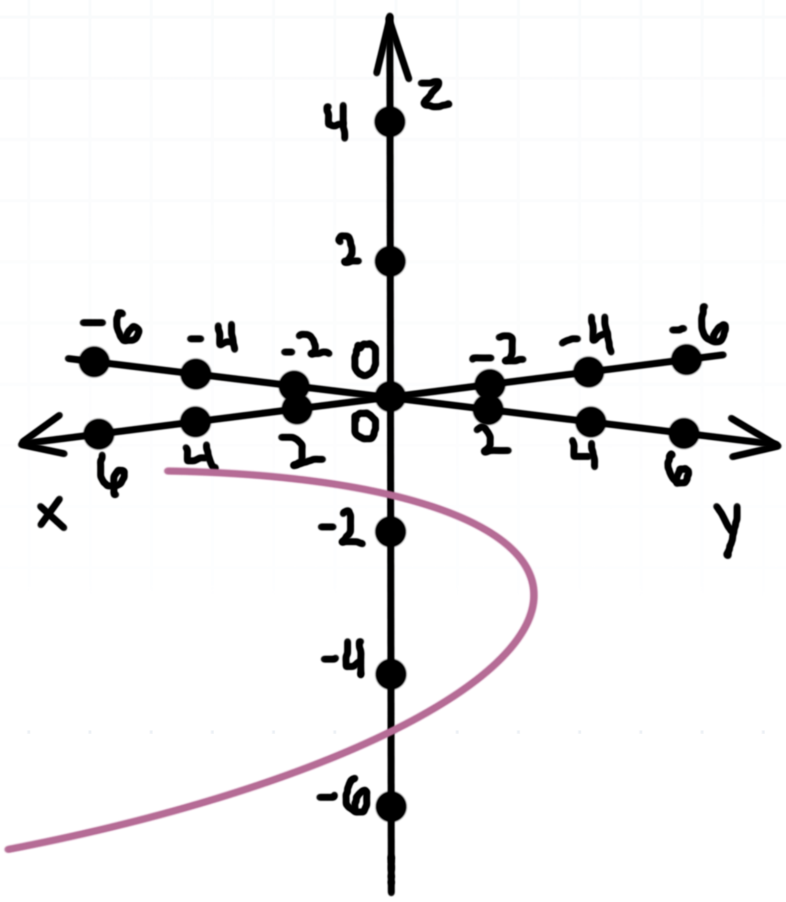
So the surface is the sphere with center at the origin and radius 4.



PROJECTIONS OF THE CURVE

■ 1. Identify and sketch the projections of the curve onto each of the major coordinate planes.

$$\vec{r}(t) = \left\langle t^2 - 1, \frac{t+4}{2}, t - 3 \right\rangle$$



Solution:

Write the vector function in parametric form.

$$x(t) = t^2 - 1$$

$$y(t) = \frac{t+4}{2}$$

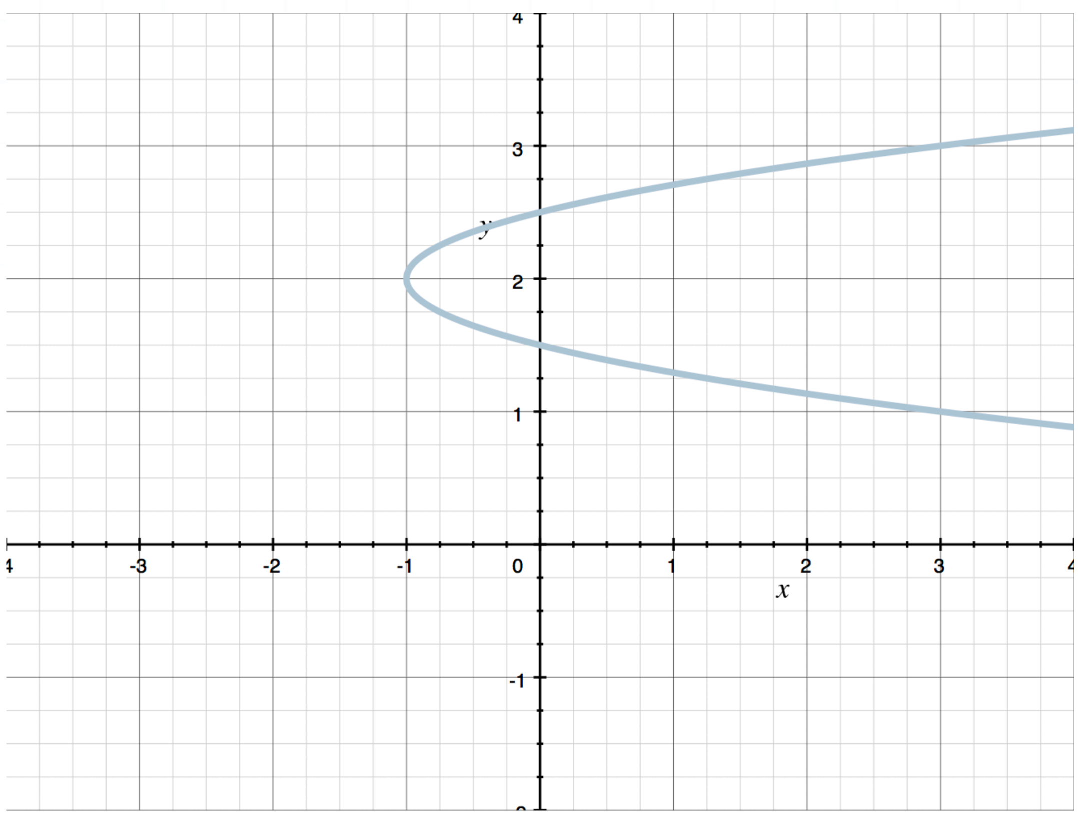
$$z(t) = t - 3$$

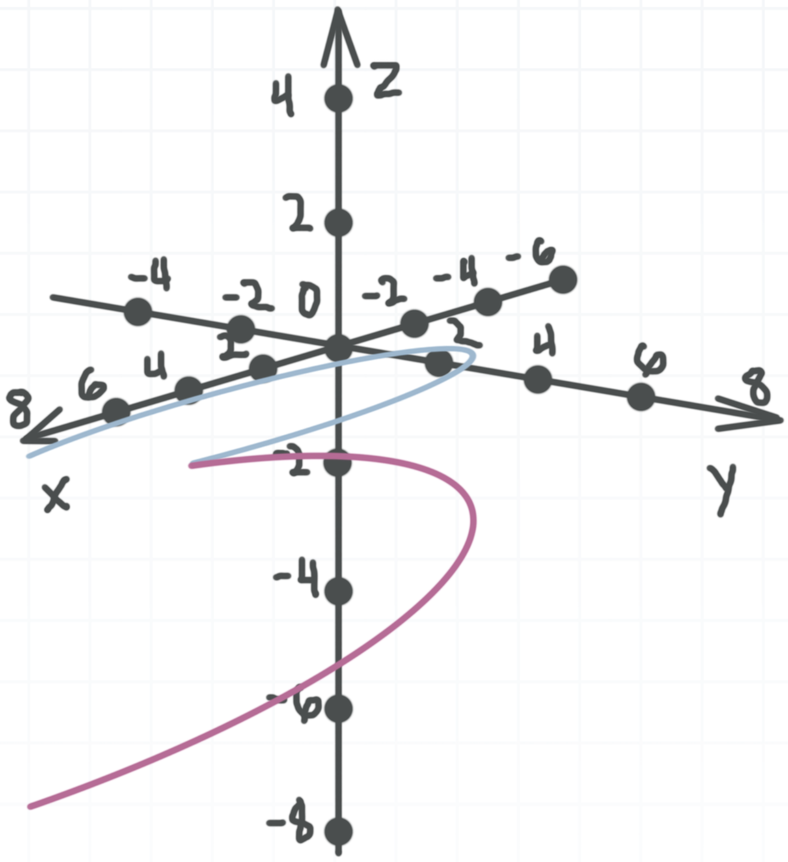
To get the projection onto the xy-plane, solve $y(t)$ for t, then plug the result into $x(t)$.

$$t = 2y - 4$$

$$x = (2y - 4)^2 - 1 = 4(y - 2)^2 - 1$$

So the curve's projection onto the xy-plane is the parabola $x = 4(y - 2)^2 - 1$ that has its vertex at $(-1,2,0)$.



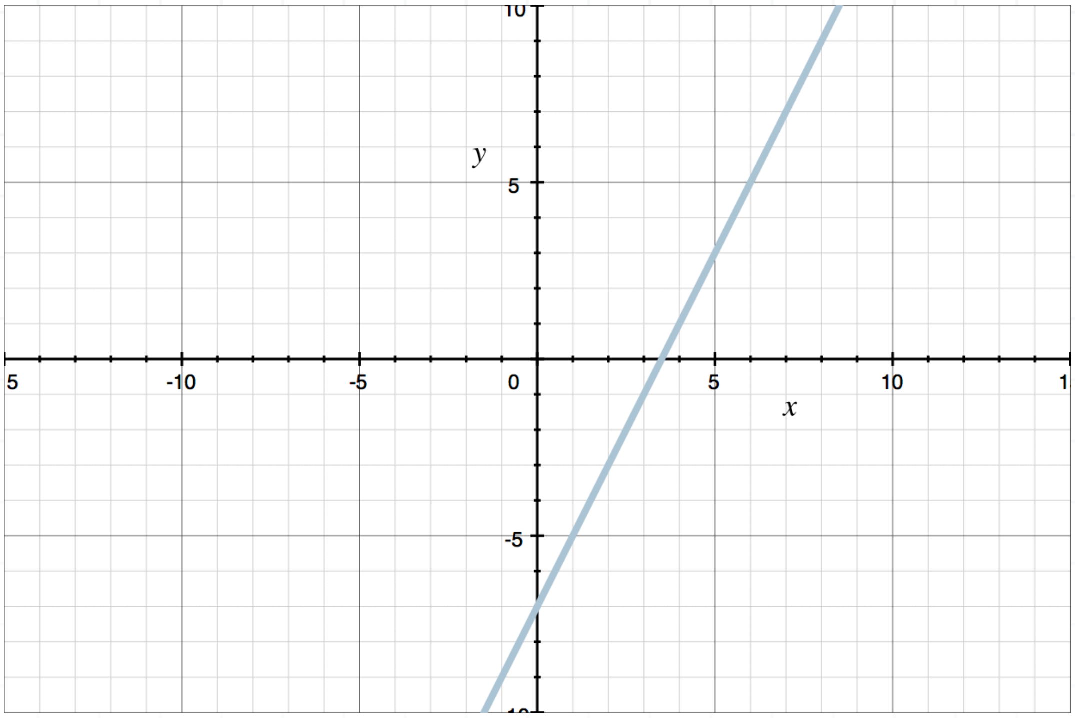


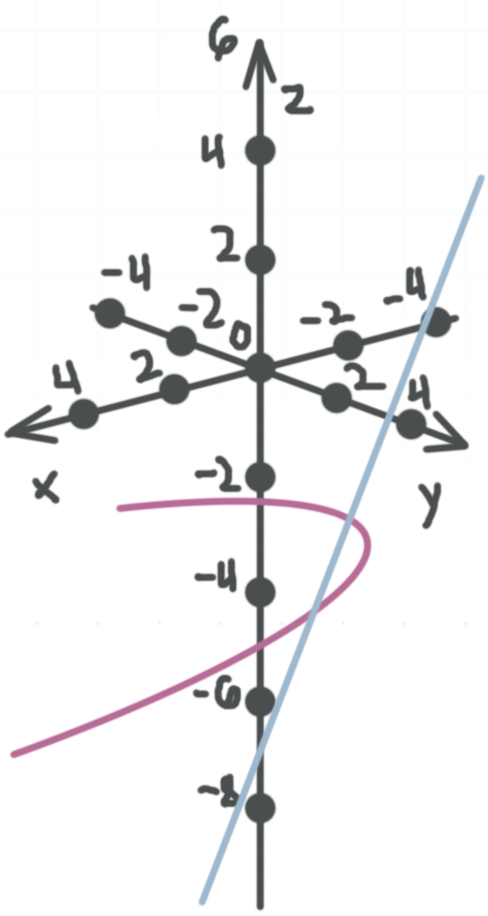
To get the projection onto the yz-plane, solve $y(t)$ for t, then plug the result into $z(t)$.

$$t = 2y - 4$$

$$z = 2y - 4 - 3 = 2y - 7$$

So the curve's projection onto the yz-plane is the line $z = 2y - 7$.



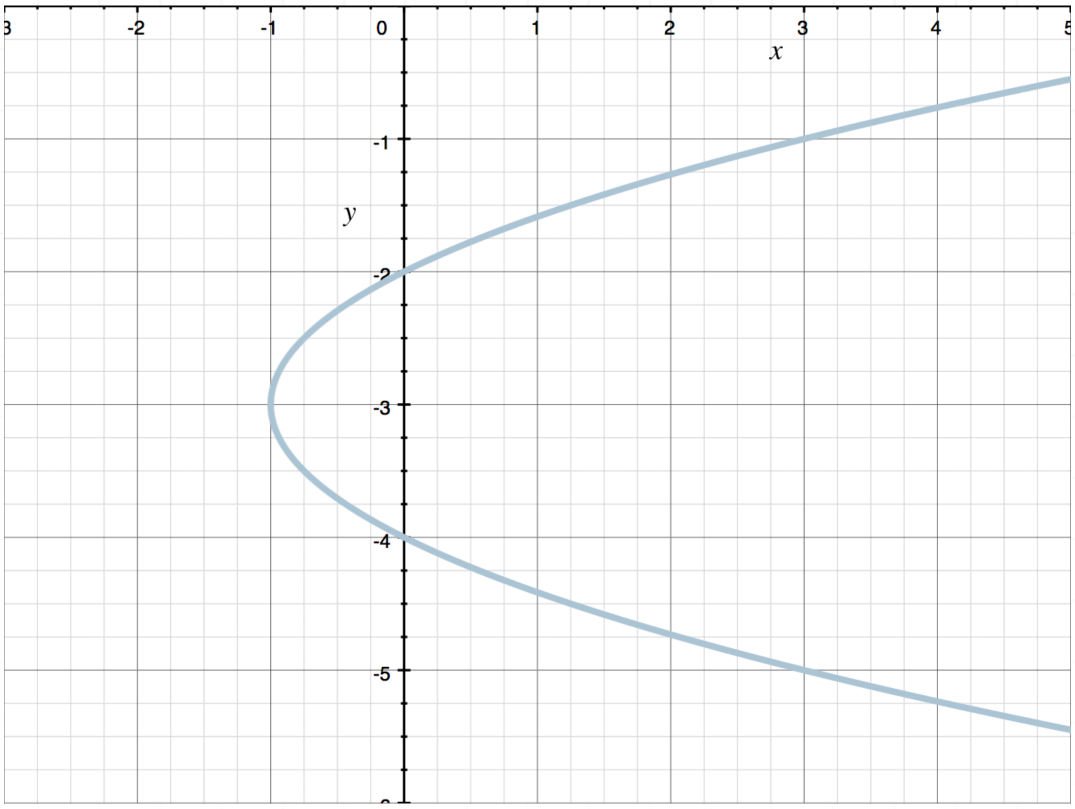


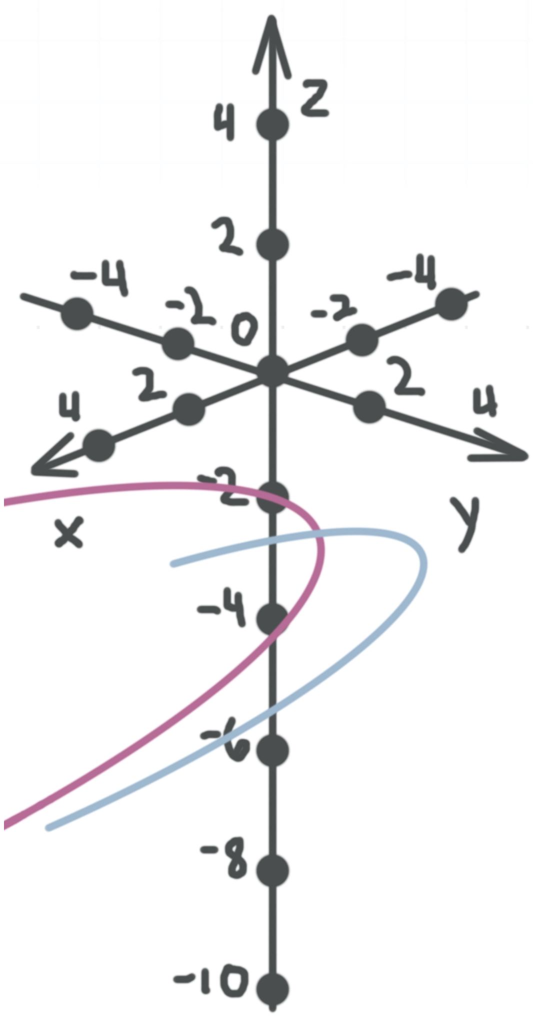
Finally, to get the projection onto the xz-plane, solve $z(t)$ for t, then plug the result into $x(t)$.

$$t = z + 3$$

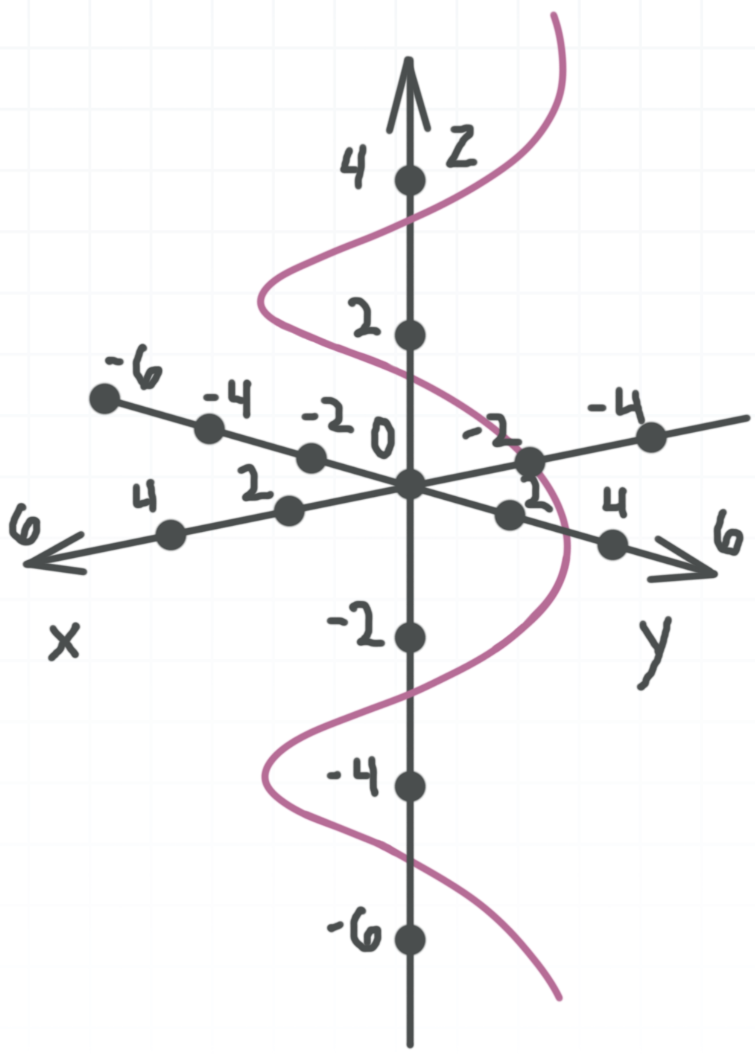
$$x = (z + 3)^2 - 1$$

So the curve projection onto the xz-plane is the parabola $x = (z+3)^2 - 1$ that has its vertex at $(-1,0,-3)$.





■ 2. Identify and sketch the projections of the curve $\vec{r}(t) = \langle 2\cos t, 2\sin t, t+\pi \rangle$ onto each of the coordinate planes.



Solution:

Write the vector function in parametric form.

$$x(t) = 2\cos t$$

$$y(t) = 2\sin t$$

$$z(t) = t + \pi$$

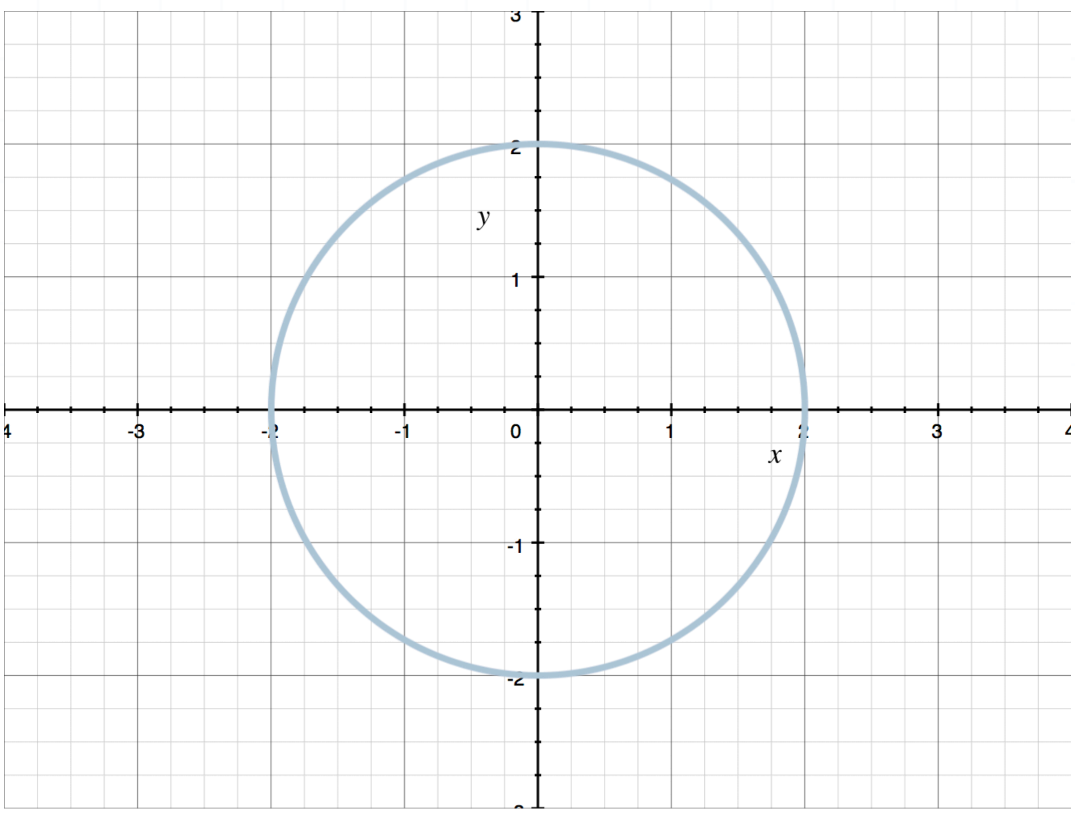
To get the projection onto the xy-plane, use $x(t)$ and $y(t)$ with the trigonometric identity $\sin^2\phi + \cos^2\phi = 1$ to get a relationship between x and y.

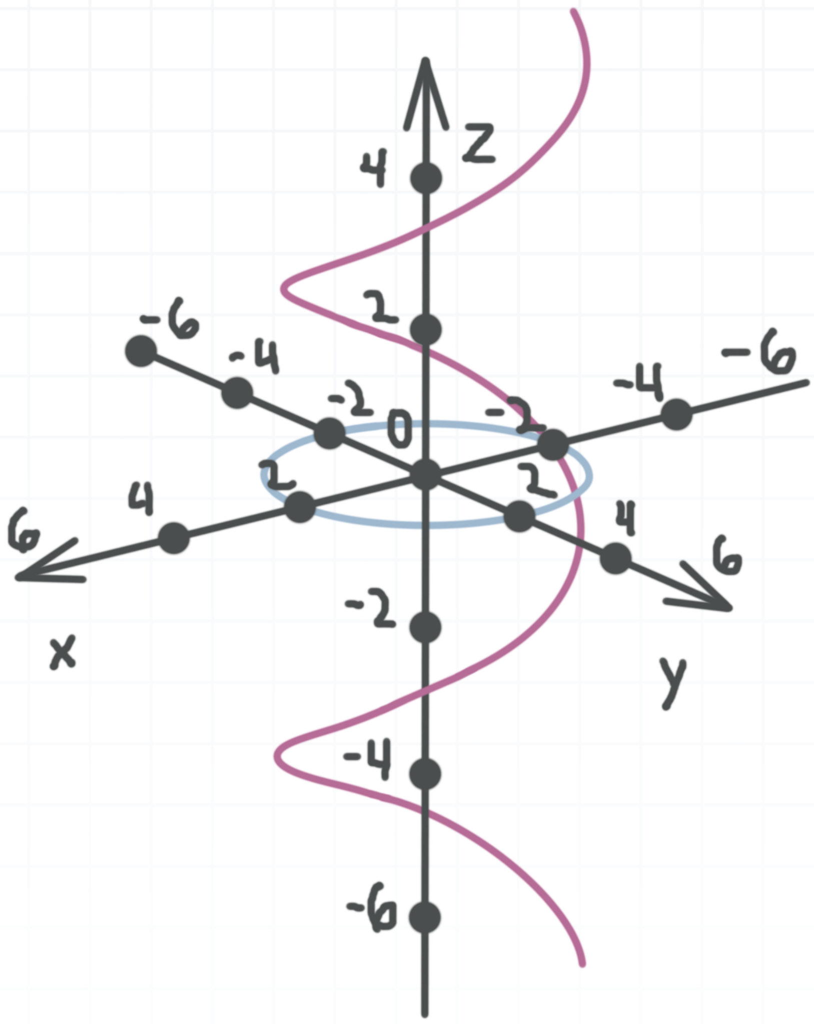
$$x^2 + y^2 = (2\cos t)^2 + (2\sin t)^2$$

$$x^2 + y^2 = 2^2(\cos^2 t + \sin^2 t)$$

$$x^2 + y^2 = 2^2$$

Therefore, the curve's projection onto the xy-plane is the circle with center at the origin and radius 2.



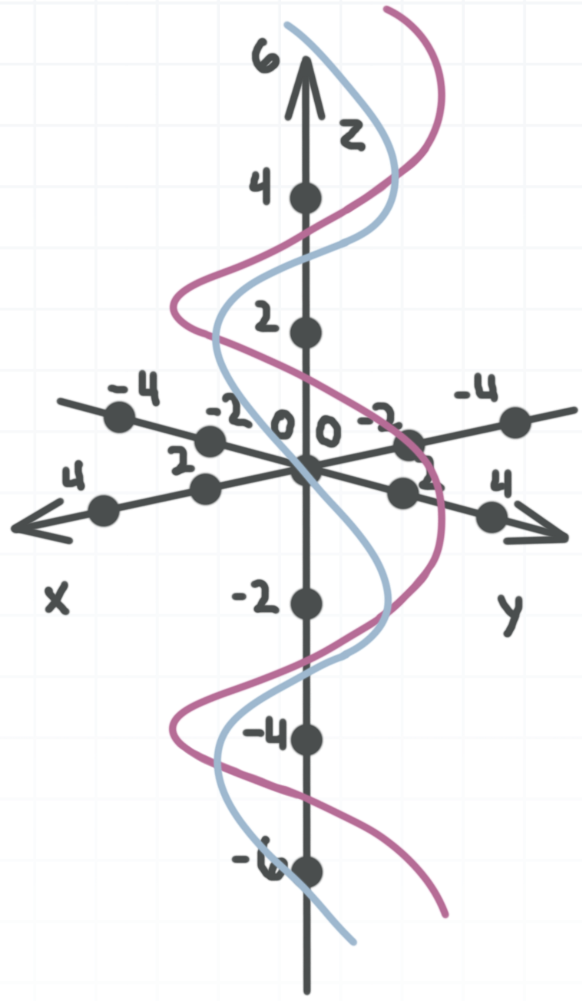


To get the projection onto the yz-plane, solve $z(t)$ for t, then plug the result into $y(t)$.

$$t = z - \pi$$

$$y = 2\sin(z - \pi) = -2\sin z$$

So the curve's projection onto the yz-plane is the sinusoid $y = -2\sin z$.

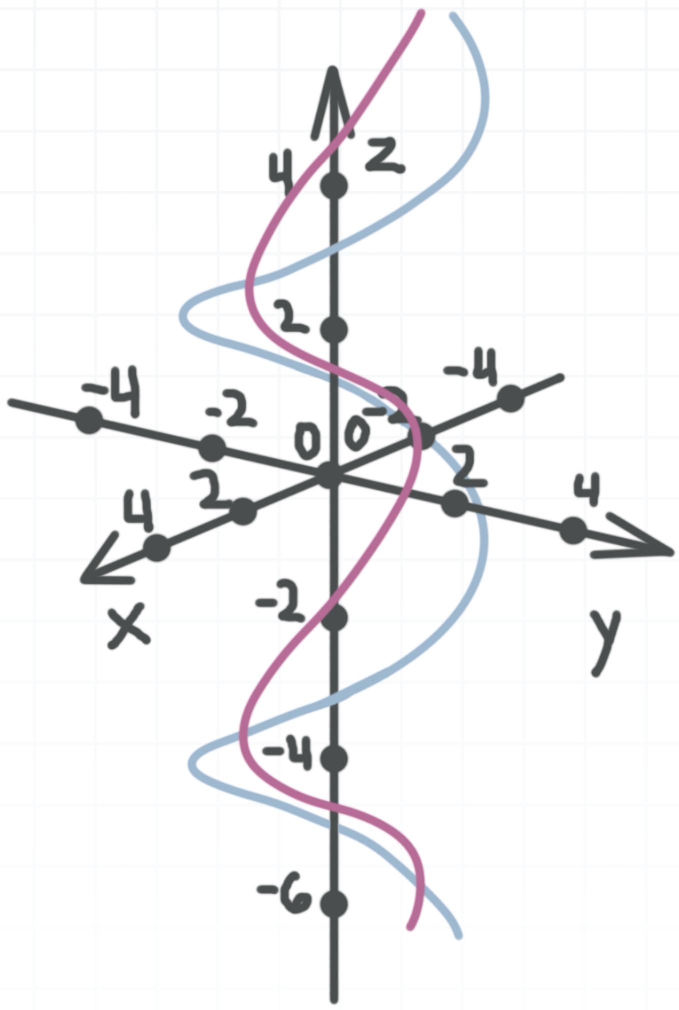


Finally, to get the projection onto the xz-plane, solve $z(t)$ for t, then plug the result into $x(t)$.

$$t = z - \pi$$

$$x = 2\cos(z - \pi) = -2\cos z$$

So the curve's projection onto the yz-plane is the cosinusoid $y = -2\cos z$.



3. Identify and sketch the projections of the surface onto each of the coordinate planes. Using the projections, identify the surface.

$$\vec{r}(u,v)=\left\langle 3\cos u, 3\sin u, \frac{v}{2}\right\rangle$$

Solution:

Write the vector function in parametric form.

$$x(u,v)=3\cos u$$

$$y(u,v)=3\sin u$$

$$z(u,v) = \frac{v}{2}$$

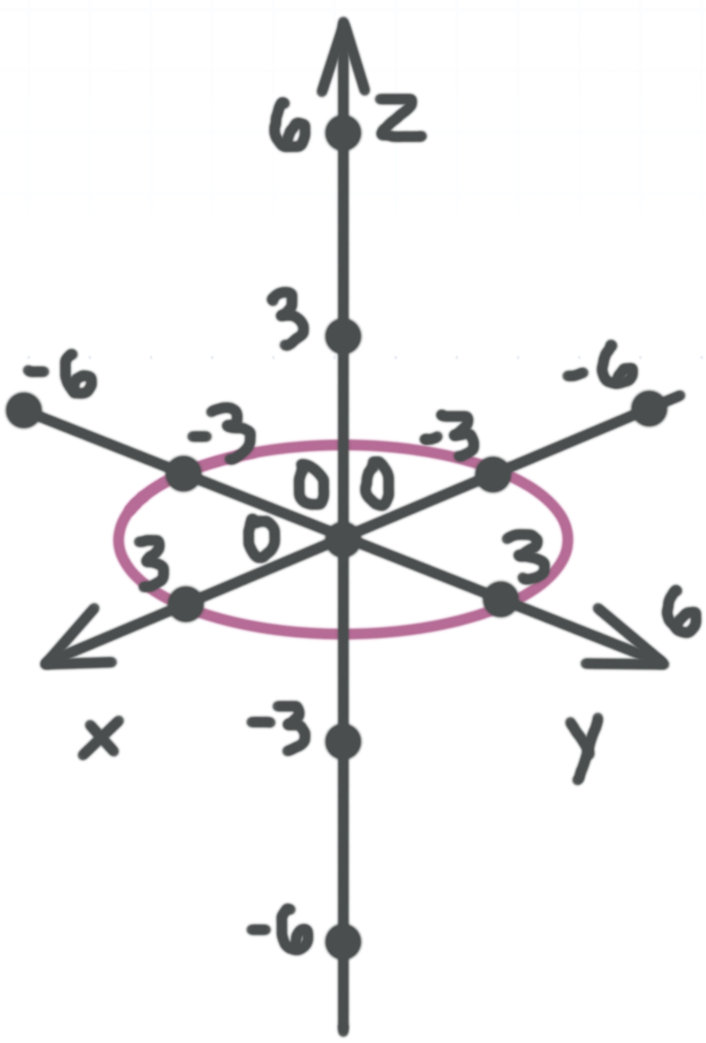
To get the projection onto the xy-plane, use $x(u,v)$ and $y(u,v)$ and the trigonometric identity $\sin^2\phi + \cos^2\phi = 1$ to get a relationship between x and y.

$$x^2 + y^2 = (3\cos u)^2 + (3\sin u)^2$$

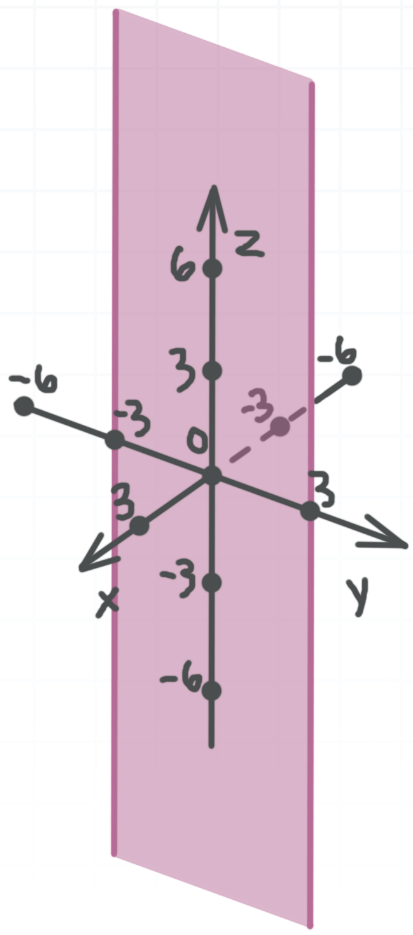
$$x^2 + y^2 = 3^2(\cos^2 u + \sin^2 u)$$

$$x^2 + y^2 = 3^2$$

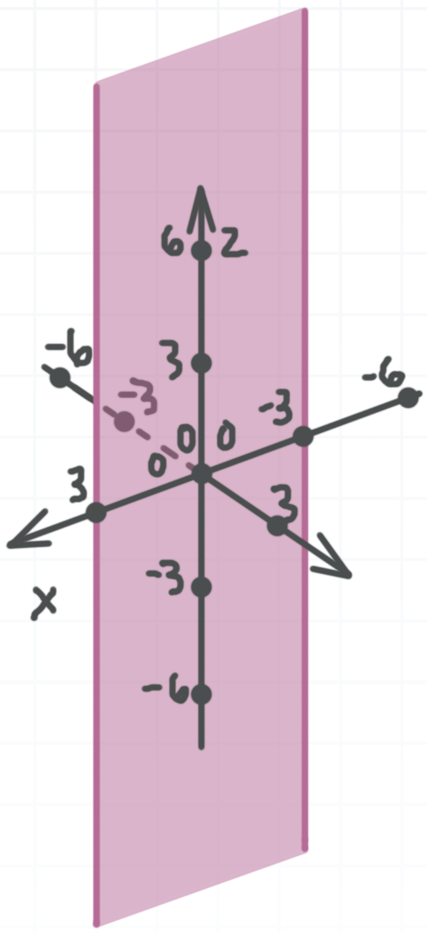
Therefore, the projection onto the xy-plane is the circle with center at the origin and radius 3.



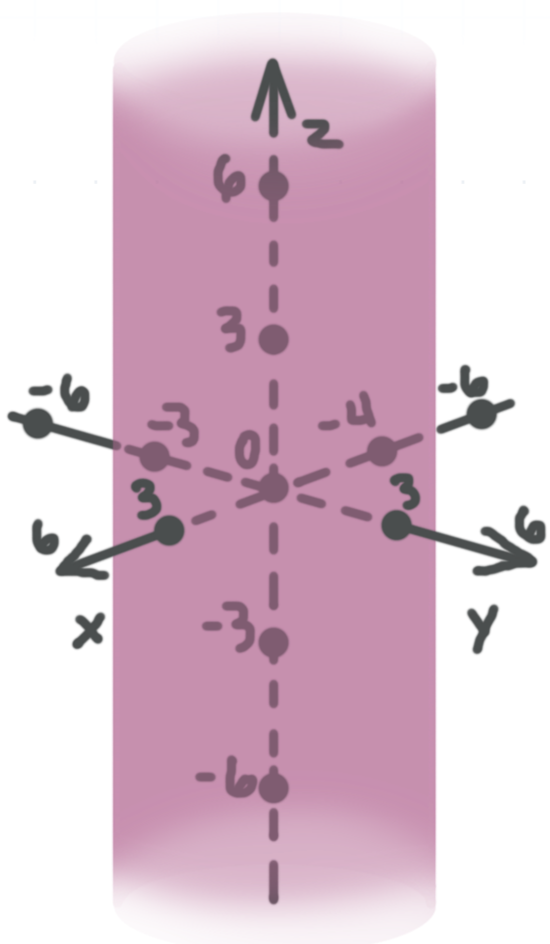
To get the projection onto the yz-plane, use $y(u, v)$ and $z(u, v)$. Since u is any real number, y changes from -3 to 3. Since v is the any real number, z changes from $-\infty$ to ∞. So the projection onto the yz-plane is the infinite bar bounded by $-3 \leq y \leq 3$ and $-\infty < z < \infty$.



Finally, to get the projection onto the xz-plane, use $x(u, v)$ and $z(u, v)$. Similarly to the yz-plane, the surface projection onto the xz-plane is the infinite bar bounded by $-3 \leq x \leq 3$ and $-\infty < z < \infty$.



Using the projections, we can conclude that the surface is the cylinder $x^2 + y^2 = 3^2$.



VECTOR AND PARAMETRIC EQUATIONS OF A LINE SEGMENT

■ 1. Find the vector and parametric equation of the line segment AB, given $A(-4,2)$ and $B(1,5)$.

Solution:

The vector equation of a line segment is given by

$$\vec{r}(t) = (1-t)\vec{r_0} + t\vec{r_1} \text{ with } 0 \leq t \leq 1$$

where $\vec{r_0}$ and $\vec{r_1}$ are vectors with initial points at the origin, and terminal points at the endpoints of the line segment.

Plug in $\vec{r_0} = \langle -4,2 \rangle$ and $\vec{r_1} = \langle 1,5 \rangle$.

$$\vec{r}(t) = (1-t)\langle -4,2 \rangle + t\langle 1,5 \rangle$$

$$\vec{r}(t) = \langle -4(1-t) + t, 2(1-t) + 5t \rangle$$

$$\vec{r}(t) = \langle -4 + 5t, 2 + 3t \rangle \text{ with } 0 \leq t \leq 1$$

To get parametric equations, write down each component of the vector equation.

$$x(t) = -4 + 5t$$

$$y(t) = 2 + 3t$$

$$0 \leq t \leq 1$$

■ 2. Find the vector equation of the line segment AB, if $A(2,-1,3)$, $\overrightarrow{AB}$ is parallel to $\langle -2,2,1\rangle$, and B is the intersection point of the line AB with the xz-plane.

Solution:

Find the coordinates of the point B. The vector equation of the line that passes through the point $A(2,-1,3)$ and has direction vector $\langle -2,2,1\rangle$ is

$$\vec{r}(t)=\langle 2,-1,3\rangle+t\langle -2,2,1\rangle$$

In parametric form, the y component is

$$y(t)=-1+2t$$

For the intersection point of the line AB with the xz-plane, we know $y=0$, which means

$$-1+2t=0$$

$$t=0.5$$

Plug $t=0.5$ into the vector equation of the line to get the coordinates of the point B.

$$\vec{r}(0.5)=\langle 2,-1,3\rangle+0.5\langle -2,2,1\rangle$$

$$\vec{r}(0.5)=\langle 2+0.5\cdot(-2),-1+0.5\cdot 2,3+0.5\cdot 1\rangle$$

$$\vec{r}(0.5) = \langle 1, 0, 3.5 \rangle$$

So we need to find the vector equation of the line segment connecting $A(2, -1,3)$ to $B(1, 0, 3.5)$. The vector equation of a line segment is given by

$$\vec{r}(t) = (1 - t)\vec{r_0} + t\vec{r_1}, \ 0 \le t \le 1$$

Plug in $\overrightarrow{OA}$ for $\vec{r_0}$ and $\overrightarrow{OB}$ for $\vec{r_1}$.

$$\vec{r}(t) = (1 - t)\langle 2, -1,3 \rangle + t\langle 1, 0, 3.5 \rangle$$

$$\vec{r}(t) = \langle 2(1 - t) + t, \ -1(1 - t) + 0 \cdot t, 3(1 - t) + 3.5 \cdot t \rangle$$

$$\vec{r}(t) = \langle 2 - t, \ -1 + t, 3 + 0.5t \rangle \text{ with } 0 \le t \le 1$$

■ 3. Find the endpoints, midpoint, and the length of the line segment for $\vec{r}(t) = \langle 2 - 3t, 4 + t, 2 - 5t \rangle$ with $0 \le t \le 1$.

Solution:

Let A and B be the endpoints of the line segment, and M be the midpoint. The coordinates of A correspond to the parameter value $t = 0$.

$$\vec{r}(0) = \langle 2 - 3 \cdot 0, 4 + 0, 2 - 5 \cdot 0 \rangle$$

$$\vec{r}(0) = \langle 2, 4, 2 \rangle$$

The coordinates of B correspond to the parameter value $t = 1$.

$$\vec{r}(1) = \langle 2 - 3 \cdot 1, 4 + 1, 2 - 5 \cdot 1 \rangle$$

$$\vec{r}(1) = \langle -1, 5, -3 \rangle$$

The coordinates of M correspond to the parameter value $t = 0.5$.

$$\vec{r}(0.5) = \langle 2 - 3 \cdot 0.5, 4 + 0.5, 2 - 5 \cdot 0.5 \rangle$$

$$\vec{r}(0.5) = \langle 0.5, 4.5, -0.5 \rangle$$

So the length of the line segment is the distance between A and B.

$$AB = \sqrt{(x_B - x_A)^2 + (y_B - y_A)^2 + (z_B - z_A)^2}$$

$$AB = \sqrt{(-1 - 2)^2 + (5 - 4)^2 + (-3 - 2)^2}$$

$$AB = \sqrt{35}$$

VECTOR FUNCTION FOR THE CURVE OF INTERSECTION OF TWO SURFACES

■ 1. Find the vector function for the line of intersection of the two planes.

$$2x - y + 3z - 5 = 0$$

$$x + y - 2z + 1 = 0$$

Solution:

Let $x = t$. Substitute $x = t$ into the system of equations and solve it for y and z, treating t as a constant.

$$2t - y + 3z - 5 = 0$$

$$t + y - 2z + 1 = 0$$

Add the equations.

$$3t + z - 4 = 0$$

$$z = 4 - 3t$$

Substitute this value into the second equation to find y.

$$t + y - 2(4 - 3t) + 1 = 0$$

$$t + y - 8 + 6t + 1 = 0$$

$$y = 7 - 7t$$

The parametric equation of the line is

$$x(t) = t$$

$$y(t) = 7 - 7t$$

$$z(t) = 4 - 3t$$

The vector equation is

$$\vec{r}(t) = \langle t, 7 - 7t, 4 - 3t \rangle$$

In standard form, the equation is

$$\vec{r}(t) = \langle 0,7,4 \rangle + t\langle 1, -7, -3 \rangle$$

■ 2. Find the vector function for the curve of intersection of two spheres.

$$x^2 + y^2 + z^2 = 5^2$$

$$(x - 3)^2 + y^2 + z^2 = 4^2$$

Solution:

Try to solve the system for x, y, and z. Subtract the equations

$$x^2 - (x - 3)^2 = 25 - 16$$

$$x^2 - x^2 + 6x - 9 = 9$$

$$6x - 18 = 0$$

$$x = 3$$

Substitute $x = 3$ back into the first equation.

$$3^2 + y^2 + z^2 = 5^2$$

$$y^2 + z^2 = 5^2 - 3^2 = 16 = 4^2$$

So far we have two equations:

$$x = 3$$

$$y^2 + z^2 = 4^2$$

The parametrization of the circle $u^2 + v^2 = R^2$ is given by the standard trigonometric substitution $u = R\cos t$ and $v = R\sin t$. So since $R = 4$,

$$x = 3$$

$$y = 4\cos t$$

$$z = 4\sin t$$

Verify that $y^2 + z^2 = 4^2$.

$$y^2 + z^2 = (4\cos t)^2 + (4\sin t)^2$$

$$y^2 + z^2 = 4^2(\cos^2 t + \sin^2 t)$$

$$y^2 + z^2 = 4^2$$

■ 3. Find the vector function for the curve of intersection of the elliptic cylinder and the plane.

$$\frac{(x-2)^2}{3^2}+\frac{(y+1)^2}{4^2}=1$$

$$2x-3y-z-4=0$$

Solution:

The parametrization of an ellipse in the form

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}=1$$

is given by the standard trigonometric substitution $x(t)=x_0+a\cos t$ and $y(t)=y_0+b\sin t$. In this case $a=3$, $b=4$, and $(x_0,y_0)=(2,-1)$. So

$$x(t)=2+3\cos t$$

$$y(t)=-1+4\sin t$$

Verify that the initial equation holds.

$$\frac{(x-2)^2}{3^2}+\frac{(y+1)^2}{4^2}=\frac{(2+3\cos t-2)^2}{3^2}+\frac{(-1+4\sin t+1)^2}{4^2}$$

$$\frac{(x-2)^2}{3^2}+\frac{(y+1)^2}{4^2}=\frac{(3\cos t)^2}{3^2}+\frac{(4\sin t)^2}{4^2}$$

$$\frac{(x-2)^2}{3^2}+\frac{(y+1)^2}{4^2}=\cos^2 t+\sin^2 t$$

$$\frac{(x-2)^2}{3^2} + \frac{(y+1)^2}{4^2} = 1$$

Substitute $x(t)$ and $y(t)$ into the equation of the plane, then solve it for z.

$$2x - 3y - z - 4 = 0$$

$$z = 2x - 3y - 4$$

$$z = 2(2 + 3\cos t) - 3(-1 + 4\sin t) - 4$$

$$z(t) = 6\cos t - 12\sin t + 3$$

DERIVATIVE OF A VECTOR FUNCTION

1. Find the second order derivative of the vector function.

$$\vec{r}(t) = \left\langle \sqrt{t}, \frac{2}{t}, e^{t+3} \right\rangle$$

Solution:

Differentiate each component individually with respect to t.

$$r_1'(t) = (t^{1/2})' = \frac{1}{2}t^{-1/2}$$

$$r_1''(t) = \left(\frac{1}{2}t^{-1/2}\right)' = -\frac{1}{4}t^{-3/2} = -\frac{1}{4t^{3/2}}$$

$$r_2'(t) = (2t^{-1})' = -2t^{-2}$$

$$r_2''(t) = (-2t^{-2})' = 4t^{-3} = \frac{4}{t^3}$$

$$r_3'(t) = (e^{t+3})' = e^{t+3}$$

$$r_3''(t) = (e^{t+3})' = e^{t+3}$$

2. Find the Jacobian matrix of the vector function at $(u, v) = (1,2)$.

$$\vec{r}(u,v) = \langle 2uv + 1, u^2 + v^2 \rangle$$

Solution:

The Jacobian is given by

$$\frac{\partial \vec{r}(u,v)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial r_1}{\partial u} & \frac{\partial r_1}{\partial v} \\ \frac{\partial r_2}{\partial u} & \frac{\partial r_2}{\partial v} \end{bmatrix}$$

$$\frac{\partial \vec{r}(u,v)}{\partial(u,v)} = \begin{bmatrix} 2v & 2u \\ 2u & 2v \end{bmatrix}$$

Evaluate at $u = 1$ and $v = 2$.

$$\frac{\partial \vec{r}(1,2)}{\partial(u,v)} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

■ 3. Find the Jacobian matrix for the vector function.

$$\vec{r}(t,s) = \langle \ln(ts), 3t + 2s - 1, \ \sin(t + s) \rangle$$

Solution:

The Jacobian is given by

$$\frac{\partial \vec{r}(t,s)}{\partial(t,s)} = \begin{bmatrix} \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial s} \\ \frac{\partial r_2}{\partial t} & \frac{\partial r_2}{\partial s} \\ \frac{\partial r_3}{\partial t} & \frac{\partial r_3}{\partial s} \end{bmatrix}$$

$$\frac{\partial \vec{r}(t,s)}{\partial(t,s)} = \begin{bmatrix} \frac{1}{t} & \frac{1}{s} \\ 3 & 2 \\ \cos(t+s) & \cos(t+s) \end{bmatrix}$$

UNIT TANGENT VECTOR

■ 1. Find the unit tangent vector to the function that that sits at a 30° angle.

$$\vec{r}(t) = \langle t^2 + 4, 2t^3 - 3 \rangle$$

Solution:

If the vector $\langle u, v \rangle$ has an angle of $\phi = 30^\circ$, then

$$\tan\phi = \tan 30^\circ$$

$$\frac{v}{u} = \frac{1}{\sqrt{3}}$$

To find the components of the tangent vector, differentiate $\vec{r}(t)$ with respect to t.

$$r_1'(t) = 2t$$

$$r_2'(t) = 6t^2$$

Since

$$\frac{r_2'(t)}{r_1'(t)} = \frac{1}{\sqrt{3}}$$

for some $t = t_0$,

$$\frac{6t_0^2}{2t_0} = \frac{1}{\sqrt{3}}$$

$$3t_0 = \frac{1}{\sqrt{3}}$$

$$t_0 = \frac{1}{3\sqrt{3}}$$

To find the tangent vector, plug $t_0 = 1/3\sqrt{3}$ into $\vec{r}\,'(t)$.

$$r'_1\left(\frac{1}{3\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$$

$$r'_2\left(\frac{1}{3\sqrt{3}}\right) = 6 \cdot \frac{1}{3^2 \cdot 3} = \frac{2}{9}$$

So the tangent vector is

$$\vec{r}\,'(t_0) = \left\langle \frac{2}{3\sqrt{3}}, \frac{2}{9} \right\rangle$$

The magnitude of the tangent vector is

$$|\,\vec{r}\,'(t_0)\,| = \sqrt{\left(\frac{2}{3\sqrt{3}}\right)^2 + \left(\frac{2}{9}\right)^2} = \sqrt{\frac{16}{81}} = \frac{4}{9}$$

Finally, the unit tangent vector is

$$\frac{\vec{r}'(t_0)}{|\vec{r}'(t_0)|} = \left\langle \frac{2}{3\sqrt{3}} \cdot \frac{9}{4}, \frac{2}{9} \cdot \frac{9}{4} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

■ 2. Find the tangent vector at the point $(-1,0,1)$.

$$\vec{r}(t) = \langle 2t^3 - 3t^2 + 5t - 5,\ \sin(\pi t), e^{t-1} \rangle$$

Solution:

Identify the value of t that corresponds to $(-1,0,1)$. We could use $r_1(t) = -1$, $r_2(t) = 0$, or $r_3(t) = 1$. The first and the second equations will probably give us several solutions, so let's use the third one.

$$e^{t-1} = 1$$

$$t - 1 = 0$$

$$t = 1$$

Check that the other equations hold, $r_1(1) = -1$ and $r_2(1) = 0$.

$$2(1)^3 - 3(1)^2 + 5(1) - 5 = -1$$

$$\sin(\pi(1)) = \sin 0 = 0$$

In order to find the tangent vector, differentiate each term individually with respect to t.

$$r_1'(t) = 6t^2 - 6t + 5$$

$$r_2'(t) = \pi \cos(\pi t)$$

$$r_3'(t) = e^{t-1}$$

Plug in $t = 1$.

$$r_1'(1) = 6(1)^2 - 6(1) + 5 = 5$$

$$r_2'(1) = \pi \cos(\pi \cdot 1) = -\pi$$

$$r_3'(1) = e^{t-1} = 1$$

■ 3. Find the point(s) where the unit tangent vector to the curve is orthogonal to the xz-plane

$$\vec{r}(t) = \langle t^3 + 2, 5t^2 - 3t + 8, t^2 + 5 \rangle$$

Solution:

There are only two unit vectors orthogonal to the xz-plane, which are $\langle 0,1,0 \rangle$ and $\langle 0,-1,0 \rangle$. Identify the value of t that corresponds to these tangent vectors by differentiating each term of the vector function with respect to t.

$$r_1'(t) = 3t^2$$

$$r_2'(t) = 10t - 3$$

$$r_3'(t) = 2t$$

Since $r_1'(t) = 0$ and $r_3'(t) = 0$, we can conclude that $t = 0$, so $r_2'(0) = 10 \cdot 0 - 3 = -3$. Since the tangent vector is $\langle 0, -3, 0 \rangle$, the unit tangent vector is $\langle 0, -1, 0 \rangle$. Find the point for $t = 0$.

$$\vec{r}(0) = \langle 0^3 + 2, 5 \cdot 0^2 - 3 \cdot 0 + 8, 0^2 + 5 \rangle = \langle 2, 8, 5 \rangle$$

PARAMETRIC EQUATIONS OF THE TANGENT LINE

■ 1. Find the parametric equation of the tangent line to $\vec{r}(u)$ at $u=-2$.

$$\vec{r}(u)=\langle e^{u+3}, \ln(1-u)\rangle$$

Solution:

Plug in $u=-2$ to find the coordinates of the point.

$$\vec{r}(-2)=\langle e^{-2+3}, \ln[1-(-2)]\rangle$$

$$\vec{r}(-2)=\langle e, \ln 3\rangle$$

Find the tangent vector at $u=-2$.

$$\vec{r}\,'(u)=\left\langle e^{u+3}, \frac{1}{u-1}\right\rangle$$

$$\vec{r}\,'(-2)=\left\langle e^{-2+3}, \frac{1}{-2-1}\right\rangle=\left\langle e, -\frac{1}{3}\right\rangle$$

The vector equation of the line with this direction vector, which passes through the point $(e, \ln 3)$, is

$$\vec{L}(u)=\langle e, \ln 3\rangle+t\left\langle e, -\frac{1}{3}\right\rangle$$

So the parametric equation is

$$x = e + te$$

$$y = \ln 3 - \frac{t}{3}$$

■ 2. Find the parametric equation(s) of the tangent line to the function $\vec{r}(t)$ that passes through the origin.

$$\vec{r}(t) = \langle 2t^2, 3t + 3, t + 1 \rangle$$

Solution:

The origin doesn't lie on the given curve, so we don't know the point where the tangent line touches the curve. Let T be the value of parameter t such that the tangent line touches the curve at $t = T$, then find the equation of the tangent line at this point. The coordinates of the point are

$$\vec{r}(T) = \langle 2T^2, 3T + 3, T + 1 \rangle$$

The direction vector is $\vec{r}\,'(t) = \langle 4t, 3, 1 \rangle$, so at the point $t = T$, $\vec{r}\,'(T) = \langle 4T, 3, 1 \rangle$.

The vector equation of the line with the direction vector $\langle 4T, 3, 1 \rangle$, which passes through the point $(2T^2, 3T + 3, T + 1)$, is

$$\vec{L}(t) = \langle 2T^2, 3T + 3, T + 1 \rangle + t\langle 4T, 3, 1 \rangle$$

So the parametric equations are

$$x = 2T^2 + 4Tt$$

$$y = 3T + 3 + 3t$$

$$z = T + 1 + t$$

Since this line passes through the origin, there exist values of t and T such that $x(t) = 0$, $y(t) = 0$, and $z(t) = 0$, so we need to solve the system of equations for t and T.

$$2T^2 + 4Tt = 0$$

$$3T + 3 + 3t = 0$$

$$T + 1 + t = 0$$

The second and third equations are equivalent, so solve the third equation for t and substitute the result into the first equation.

$$2T^2 + 4T(-T - 1) = 0$$

$$t = -T - 1$$

This gives

$$2T^2 - 4T^2 - 4T = 0$$

$$t = -T - 1$$

and then

$$T(T + 2) = 0$$

$$t = -T - 1$$

So we have two solutions, which are $T=0$ with $t=-1$, and $T=-2$ with $t=1$. So there are two tangent lines at different points on the curve which pass through the origin. Plug the values of T into the parametric equation of the line.

At the first tangent line for $T=0$,

$$x=0$$

$$y=3+3t$$

$$z=1+t$$

At the second tangent line for $T=-2$,

$$x=2(-2)^2+4(-2)t=8-8t$$

$$y=3(-2)+3+3t=-3+3t$$

$$z=-2+1+t=-1+t$$

■ 3. Find the equation of the tangent plane to the surface $\vec{r}(t,s)$ at the point $t=1$ and $s=4$.

$$\vec{r}(t,s)=\langle t^2+s^2,-3t+5,2s+1\rangle$$

Solution:

First, we need to find any two tangent vectors to the plane $\vec{a}$ and $\vec{b}$ at the given point, then we can find the normal vector to the plane as the cross product $\vec{a} \times \vec{b}$. The simplest way to find two tangent vectors is

(a) keep $s = 4$, consider $\vec{r}(t,4)$ as a function of one variable t, and find the tangent vector at $t = 1$, or

(b) vise versa, keeping $t = 1$, considering $\vec{r}(1,s)$ as a function of one variable s, and find the tangent vector at $s = 4$.

(a) Set $s = 4$:

$$\vec{r}(t,4) = \langle t^2 + 16, -3t + 5, 9 \rangle$$

$$\vec{r}\,'(t,4) = \langle 2t, -3, 0 \rangle$$

Plug in $t = 1$ to get the tangent vector.

$$\vec{r}\,'(1,4) = \langle 2 \cdot 1, -3, 0 \rangle$$

$$\vec{a} = \langle 2, -3, 0 \rangle$$

(b) Set $t = 1$:

$$\vec{r}(1,s) = \langle s^2 + 1, 2, 2s + 1 \rangle$$

$$\vec{r}\,'(1,s) = \langle 2s, 0, 2 \rangle$$

Plug in $s = 4$ to get the tangent vector.

$$\vec{r}\,'(1,4) = \langle 2 \cdot 4, 0, 2 \rangle$$

$$\vec{b} = \langle 8, 0, 2 \rangle$$

Next, find the normal vector $\vec{n} = \vec{a} \times \vec{b}$ to the plane. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\vec{a} = \langle 2, -3, 0 \rangle$ and $\vec{b} = \langle 8, 0, 2 \rangle$.

$$\vec{a} \times \vec{b} = \mathbf{i}(-3 \cdot 2 - 0 \cdot 0) - \mathbf{j}(2 \cdot 2 - 0 \cdot 8) + \mathbf{k}(2 \cdot 0 - (-3) \cdot 8)$$

Therefore,

$$\vec{n} = -6\mathbf{i} - 4\mathbf{j} + 24\mathbf{k}$$

Plug in $t = 1$ and $s = 4$ to find the coordinates.

$$\vec{r}(1,4) = \langle 1^2 + 4^2, -3 \cdot 1 + 5, 2 \cdot 4 + 1 \rangle$$

$$\vec{r}(1,4) = \langle 17, 2, 9 \rangle$$

The plane with normal vector $\vec{n} = \langle -6, -4, 24 \rangle$ which passes through the point $(17,2,9)$ has the equation

$$-6(x - 17) - 4(y - 2) + 24(z - 9) = 0$$

$$3x + 2y - 12z + 53 = 0$$

INTEGRAL OF A VECTOR FUNCTION

1. Find the integral of the vector function.

$$\int \langle e^{3u-2}, e^{5-u}, \sin^2(u-\pi) \rangle \ du$$

Solution:

Integrate each component individually.

$$\int e^{3u-2} \ du = \frac{e^{3u-2}}{3} + C_1$$

$$\int e^{5-u} \ du = -e^{5-u} + C_2$$

$$\int \sin^2(u-\pi) \ du = \int \frac{1}{2} - \frac{1}{2}\cos(2u-2\pi) \ du$$

$$= \int \frac{1}{2} \ du - \int \frac{1}{2}\cos(2u) \ du$$

$$= \frac{u}{2} - \frac{\sin(2u)}{4} + C_3$$

2. Find the improper integral of the vector function.

$$\int_2^\infty \left\langle \frac{t-2}{t^3-8}, 2^{-t+1} \right\rangle \ dt$$

Solution:

Integrate each component individually, starting with the first component.

$$\int_2^\infty \frac{t-2}{t^3-8} \ dt$$

$$= \int_2^\infty \frac{t-2}{(t-2)(t^2+2t+4)} \ dt$$

$$= \int_2^\infty \frac{1}{t^2+2t+4} \ dt$$

$$= \int_2^\infty \frac{1}{(t+1)^2+3} \ dt$$

Substitute $u = t + 1$, $du = dt$, and u changing from 3 to ∞.

$$\int_3^\infty \frac{1}{u^2+3} \ du$$

$$\left. \frac{\arctan \frac{u}{\sqrt{3}}}{\sqrt{3}} \right|_3^\infty$$

$$\lim_{u\to\infty} \frac{\arctan \frac{u}{\sqrt{3}}}{\sqrt{3}} - \frac{\arctan \frac{3}{\sqrt{3}}}{\sqrt{3}}$$

$$\frac{\pi}{2\sqrt{3}} - \frac{\pi}{3\sqrt{3}} = \frac{\pi}{6\sqrt{3}}$$

The integral of the second component is

$$\int_2^\infty 2^{-t+1}\ dt = \left[-\frac{2^{-t+1}}{\ln 2}\right]\Bigg|_2^\infty$$

$$= \lim_{t\to\infty} -\frac{2^{-t+1}}{\ln 2} + \frac{2^{-2+1}}{\ln 2}$$

$$= 0 + \frac{1}{2\ln 2} = \frac{1}{\ln 4}$$

■ 3. Find the double integral of the vector function, where R is the square $[0,\pi] \times [0,\pi]$.

$$\int\int_R \langle ts,\ \sin(t-s)\rangle\ \ dA$$

Solution:

Integrate the first component.

$$\int\int_R ts\ dA$$

$$\int_0^\pi t\ dt \cdot \int_0^\pi s\ ds$$

$$\frac{t^2}{2}\bigg|_0^{\pi}\cdot\frac{s^2}{2}\bigg|_0^{\pi}$$

$$\left(\frac{\pi^2}{2}-\frac{0^2}{2}\right)\left(\frac{\pi^2}{2}-\frac{0^2}{2}\right)$$

$$\frac{\pi^4}{4}$$

Integrate the second component.

$$\iint_R \sin(t-s)\ dA$$

$$\int_0^{\pi}\int_0^{\pi}\sin(t-s)\ dt\ ds$$

Integrate with respect to t, treating s as a constant.

$$\int_0^{\pi}\sin(t-s)\ dt$$

$$-\cos(t-s)\bigg|_0^{\pi}$$

$$-\cos(\pi-s)+\cos(0-s)=2\cos s$$

Integrate with respect to s.

$$\int_0^{\pi}2\cos s\ ds$$

$$2\sin s\Big|_0^{\pi}$$

$$2\sin\pi - 2\sin 0 = 0$$

ARC LENGTH OF A VECTOR FUNCTION

1. Confirm the formula for the arc length $L = 2\pi R$ around the circle by considering the circle's equation as the vector function in polar coordinates, where R is the radius of the circle.

$$\vec{r}(\phi) = \langle R\cos\phi, R\sin\phi\rangle \text{ with } 0 \le \phi \le 2\pi$$

Solution:

Consider the circle centered at the origin with radius R. Rewrite the vector equation in parametric form.

$$x(\phi) = R\cos\phi$$

$$y(\phi) = R\sin\phi$$

Find derivatives.

$$x'(\phi) = -R\sin\phi$$

$$y'(\phi) = R\cos\phi$$

Arc length is given by

$$\int_a^b \sqrt{(x'(\phi))^2 + (y'(\phi))^2}\ d\phi$$

Substitute into the arc length formula.

$$L = \int_0^{2\pi} \sqrt{(-R\sin\phi)^2 + (R\cos\phi)^2}\ d\phi$$

$$L = \int_0^{2\pi} \sqrt{R^2\sin^2\phi + R^2\cos^2\phi}\ d\phi$$

$$L = \int_0^{2\pi} \sqrt{R^2(\sin^2\phi + \cos^2\phi)}\ d\phi$$

$$L = \int_0^{2\pi} \sqrt{R^2}\ d\phi$$

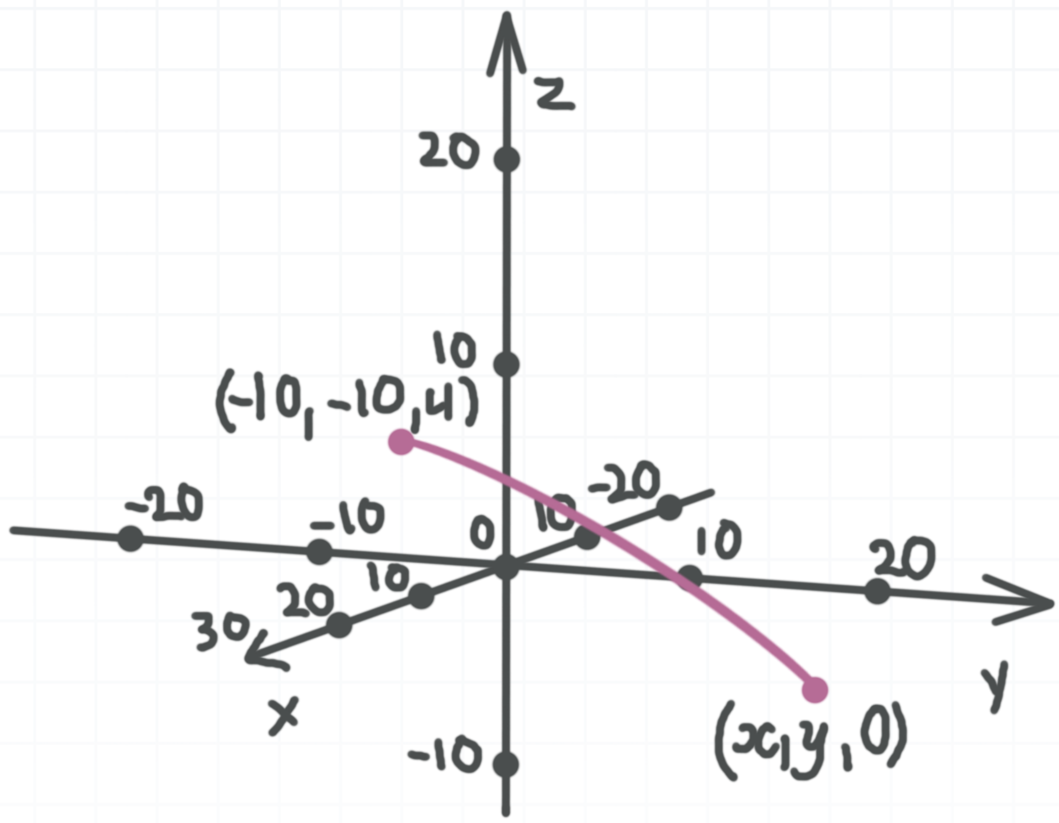
$$L = \int_0^{2\pi} R\ d\phi$$

$$L = R\int_0^{2\pi}\ d\phi$$

$$L = R(2\pi) = 2\pi R$$

■ 2. A cannon ball is shot from the point $A(-10,-10,4)$. Its trajectory can be modeled by the vector function, where $t \geq 0$ is the time. Find the arc length of the ball's trajectory before it hits the ground $z = 0$.

$$\vec{r}(t) = \left\langle t-10, t-10, \frac{-t^2+20t+800}{200} \right\rangle$$



Solution:

Rewrite the vector equation in parametric form.

$$x(t) = t - 10$$

$$y(t) = t - 10$$

$$z(t) = \frac{-t^2 + 20t + 800}{200}$$

Find the value of t when the ball hits the ground by solving the equation $z(t) = 0$ for t.

$$\frac{-t^2 + 20t + 800}{200} = 0$$

$$t^2 - 20t - 800 = 0$$

$$(t-40)(t+20)=0$$

$$t=-20 \text{ or } t=40$$

It’s impossible for $t \geq 0$. So t changes from 0 to 40. Arc length is given by

$$\int_0^{40} \sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$$

Find derivatives.

$$x'(t)=1$$

$$y'(t)=1$$

$$z'(t)=\frac{-2t+20}{200}=\frac{10-t}{100}$$

Substitute the derivatives into the arc length formula.

$$L=\int_0^{40} \sqrt{1^2+1^2+\frac{(10-t)^2}{10{,}000}}\ dt$$

$$L=\int_0^{40} \sqrt{2+\frac{(t-10)^2}{10{,}000}}\ dt$$

Make the substitution $x=t-10$, with $dx=dt$, where x changes from -10 to 30.

$$L=\int_{-10}^{30} \sqrt{2+\frac{x^2}{10{,}000}}\ dx$$

$$L=\frac{1}{100}\int_{-10}^{30}\sqrt{20{,}000+x^2}\ dx$$

Use a trigonometric substitution with the tangent substitution $u=a\tan\theta$, where $a=\sqrt{20{,}000}=100\sqrt{2}$ and $u=x$.

$$L=\frac{1}{100}\int_{x=-10}^{x=30}\sqrt{20{,}000+(100\sqrt{2}\tan\theta)^2}\ (100\sqrt{2}\sec^2\theta\ d\theta)$$

$$L=\int_{x=-10}^{x=30}\sqrt{2}\sec^2\theta\sqrt{20{,}000+20{,}000\tan^2\theta}\ d\theta$$

$$L=\int_{x=-10}^{x=30}\sqrt{2}\sec^2\theta\sqrt{20{,}000(1+\tan^2\theta)}\ d\theta$$

$$L=\int_{x=-10}^{x=30}\sqrt{2}\sec^2\theta\sqrt{20{,}000\sec^2\theta}\ d\theta$$

$$L=\int_{x=-10}^{x=30}\sqrt{2}\sec^2\theta(100\sqrt{2}\sec\theta)\ d\theta$$

$$L=200\int_{x=-10}^{x=30}\sec^3\theta\ d\theta$$

Use integration by parts with $s=\sec\theta$ and $dv=\sec^2\theta\ d\theta$. Then $ds=\sec\theta\tan\theta\ d\theta$, and $v=\tan\theta$.

$$\int_{x=-10}^{x=30}\sec^3\theta\ d\theta=\sec\theta\tan\theta\Big|_{x=-10}^{x=30}-\int_{x=-10}^{x=30}\tan\theta\sec\theta\tan\theta\ d\theta$$

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} \tan^2\theta\sec\theta \ d\theta$$

Use the Pythagorean identity $\tan^2\theta = \sec^2\theta - 1$ to rewrite the integral.

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} (\sec^2\theta - 1)\sec\theta \ d\theta$$

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} \sec^3\theta - \sec\theta \ d\theta$$

$$\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} - \int_{x=-10}^{x=30} \sec^3\theta \ d\theta + \int_{x=-10}^{x=30} \sec\theta \ d\theta$$

$$2\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta\Big|_{x=-10}^{x=30} + \int_{x=-10}^{x=30} \sec\theta \ d\theta$$

$$2\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = \sec\theta\tan\theta + \ln\left|\sec\theta + \tan\theta\right|\Big|_{x=-10}^{x=30}$$

$$200\int_{x=-10}^{x=30} \sec^3\theta \ d\theta = 100\sec\theta\tan\theta + 100\ln\left|\sec\theta + \tan\theta\right|\Big|_{x=-10}^{x=30}$$

Now back-substitute into the equation for L.

$$L = 100\sec\theta\tan\theta + 100\ln\left|\sec\theta + \tan\theta\right|\Big|_{x=-10}^{x=30}$$

Back-substitute to put the expression back in terms of x.

$$L = 100\frac{\sqrt{20{,}000+x^2}}{100\sqrt{2}}\frac{x}{100\sqrt{2}} + 100\ln\left|\frac{\sqrt{20{,}000+x^2}}{100\sqrt{2}} + \frac{x}{100\sqrt{2}}\right|\Bigg|_{-10}^{30}$$

$$L = \frac{x\sqrt{20{,}000+x^2}}{200} + 100\ln\left|\frac{x+\sqrt{20{,}000+x^2}}{100\sqrt{2}}\right|\Bigg|_{-10}^{30}$$

Evaluate over the interval.

$$L = \frac{30\sqrt{20{,}000+30^2}}{200} + 100\ln\left|\frac{30+\sqrt{20{,}000+30^2}}{100\sqrt{2}}\right|$$

$$-\left(\frac{-10\sqrt{20{,}000+(-10)^2}}{200} + 100\ln\left|\frac{-10+\sqrt{20{,}000+(-10)^2}}{100\sqrt{2}}\right|\right)$$

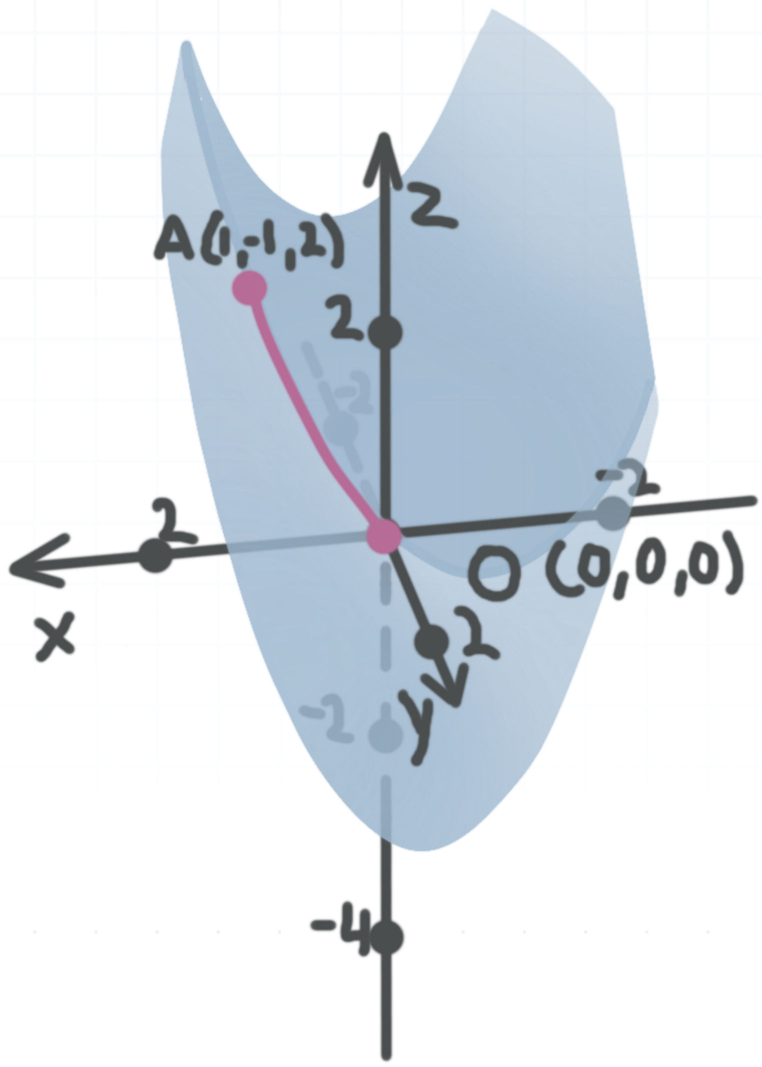
$$L = \frac{3\sqrt{20{,}900}}{20} + 100\ln\left|\frac{30+\sqrt{20{,}900}}{100\sqrt{2}}\right| + \frac{\sqrt{20{,}100}}{20} - 100\ln\left|\frac{-10+\sqrt{20{,}100}}{100\sqrt{2}}\right|$$

$$L = \frac{10\sqrt{201}+30\sqrt{209}}{20} + 100\ln\left|\frac{30+10\sqrt{209}}{100\sqrt{2}}\right| - 100\ln\left|\frac{10\sqrt{201}-10}{100\sqrt{2}}\right|$$

$$L = \frac{\sqrt{201}+3\sqrt{209}}{2} + 100\ln\left|\frac{3+\sqrt{209}}{10\sqrt{2}}\right| - 100\ln\left|\frac{\sqrt{201}-1}{10\sqrt{2}}\right|$$

$$L \approx 56.8964$$

■ 3. Find the arc length of the curve that's the intersection of the cylinder $x^2 - y - z = 0$ and the plane $x + y = 0$, between $O(0,0,0)$ and $A(1, -1,2)$.



Solution:

Let x be t, then

$$t^2 - y - z = 0$$

$$t + y = 0$$

$$y = -t$$

$$z = t^2 - y = t^2 + t$$

So the parametrization of the curve is

$$x(t) = t$$

$$y(t) = -t$$

$$z(t) = t^2 + t$$

Find the limits for t which correspond to O and A.

If $t = 0$, then $x(0) = 0$, $y(0) = 0$, and $z(0) = 0$

If $t = 1$, then $x(1) = 1$, $y(1) = -1$, and $z(1) = 2$

So $0 \leq t \leq 1$. The arc length is given by

$$\int_0^1 \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

Find derivatives.

$$x'(t) = 1$$

$$y'(t) = -1$$

$$z'(t) = 2t + 1$$

Substitute the derivatives into the arc length formula.

$$L=\int_0^1 \sqrt{1^2+(-1)^2+(2t+1)^2}\ dt$$

$$L=\int_0^1 \sqrt{(2t+1)^2+2}\ dt$$

Make the substitution $u=2t+1$, with $du=2\ dt$, and u changing from 1 to 3.

$$L=\frac{1}{2}\int_1^3 \sqrt{u^2+2}\ du$$

$$L=\frac{1}{2}\left[\frac{u\sqrt{u^2+2}}{2}+\ln\left(u+\sqrt{u^2+2}\right)\right]\Bigg|_1^3$$

$$L=\frac{1}{2}\left[\frac{3\sqrt{3^2+2}}{2}+\ln\left(3+\sqrt{3^2+2}\right)\right]-\frac{1}{2}\left[\frac{1\sqrt{1^2+2}}{2}+\ln\left(1+\sqrt{1^2+2}\right)\right]$$

$$L\approx 2.47$$

REPARAMETRIZING THE CURVE

■ 1. Reparametrize $\vec{r}(t) = \langle -3 + t, 2 + 2t, 6 - 2t \rangle$ in terms of the arc length measured from $(-3,2,6)$ in the direction of increasing t.

Solution:

To reparametrize a curve $\vec{r}(t)$ in terms of arc length, we need to modify the curve so that the path is the same, but increasing the argument by 1 results in increasing the arc length by 1. This way, inputting a value of s for the curve will result in the curve having arc length s.

Rewrite $\vec{r}(t)$ as

$$\vec{r}(t) = \langle -3, 2, 6 \rangle + t \langle 1, 2, -2 \rangle$$

Since $\vec{r}(t)$ is a linear curve, each unit increase in t corresponds to an increase of arc length by

$$|\langle 1,2,-2 \rangle| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$$

So

$$3t = s$$

$$t = \frac{s}{3}$$

Substitute $t = s/3$ into $\vec{r}(t)$.

$$\vec{r}(s) = \left\langle -3 + \frac{s}{3}, 2 + 2 \cdot \frac{s}{3}, 6 - 2 \cdot \frac{s}{3} \right\rangle = \left\langle -3 + \frac{s}{3}, 2 + \frac{2}{3}s, 6 - \frac{2}{3}s \right\rangle$$

■ 2. Reparametrize $\vec{r}(t) = \langle 4\cos 3t, -2t, 4\sin 3t \rangle$ in terms of arc length, measured from $(-4,2\pi,0)$.

Solution:

To reparametrize a curve $\vec{r}(t)$ in terms of arc length, we need to modify the curve so that the path is the same, but so that increasing the argument by 1 results in increasing the arc length by 1. This way, inputting a value of s for the curve will result in the curve having arc length s.

Reparametrizing $s(t)$ is given by

$$s(t) = \int_a^t \sqrt{(x'(u))^2 + (y'(u))^2 + (z'(u))^2}\ du$$

In order to find the initial value of t that corresponds to $(-4,2\pi,0)$, solve the system of equations for t.

$$4\cos 3t = -4$$

$$-2t = 2\pi$$

$$4\sin 3t = 0$$

From the second equation, $t = -\pi$. Check that the other equations hold.

$$4\cos(-3\pi) = 4(-1) = -4$$

$$4\sin(-3\pi) = 0$$

So the initial value of t is $a = -\pi$.

Find the derivatives of each component of the vector function.

$$x'(t) = -12\sin 3t$$

$$y'(t) = -2$$

$$z'(t) = 12\cos 3t$$

Substitute into the formula for arc length.

$$s(t) = \int_{-\pi}^{t} \sqrt{(-12\sin 3u)^2 + (-2)^2 + (12\cos 3u)^2}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{144\ \sin^2 3u + 4 + 144\ \cos^2 3u}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{144(\sin^2 3u + \cos^2 3u) + 4}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{144 + 4}\ du$$

$$s(t) = \int_{-\pi}^{t} \sqrt{148}\ du$$

$$s(t) = 2\sqrt{37}\int_{-\pi}^{t} du$$

$$s(t) = 2\sqrt{37}\,(t + \pi)$$

Solve for t.

$$s = 2\sqrt{37}\,(t + \pi)$$

$$t = \frac{s}{2\sqrt{37}} - \pi$$

Substitute t into $\vec{r}(t)$.

$$\vec{r}(s) = \left\langle 4\cos\left(3\left(\frac{s}{2\sqrt{37}} - \pi\right)\right), -2\left(\frac{s}{2\sqrt{37}} - \pi\right), 4\sin\left(3\left(\frac{s}{2\sqrt{37}} - \pi\right)\right)\right\rangle$$

$$\vec{r}(s) = \left\langle -4\cos\left(\frac{3s}{2\sqrt{37}}\right), -\frac{s}{\sqrt{37}} + 2\pi, -4\sin\left(\frac{3s}{2\sqrt{37}}\right)\right\rangle$$

■ 3. Reparametrize the curve $\vec{r}(t) = \langle 2e^{2t}, e^{2t}\rangle$ in terms of arc length measured from $t = 0$. Use the parametrization to find the position after traveling 5 units.

Solution:

To reparametrize a curve $\vec{r}(t)$ in terms of arc length, we need to modify the curve so that the path is the same, but so that increasing the argument by 1 results in increasing the arc length by 1. This way, inputting a value of s for the curve will result in the curve having arc length s.

The reparametrizing function $s(t)$ is given by

$$s(t) = \int_a^t \sqrt{(r_1'(u))^2 + (r_2'(u))^2}\ du$$

Find the derivatives of each component of the vector function.

$$r_1'(t) = 4e^{2t}$$

$$r_2'(t) = 2e^{2t}$$

Substitute the derivatives into the arc length formula.

$$s(t) = \int_0^t \sqrt{(4e^{2u})^2 + (2e^{2u})^2}\ du$$

$$s(t) = \int_0^t \sqrt{16e^{4u} + 4e^{4u}}\ du$$

$$s(t) = \int_0^t \sqrt{20e^{4u}}\ du$$

$$s(t) = \int_0^t 2\sqrt{5}e^{2u}\ du$$

$$s(t) = \sqrt{5}e^{2u}\Big|_0^t$$

$$s(t) = \sqrt{5}e^{2t} - \sqrt{5}e^0$$

$$s(t) = \sqrt{5}e^{2t} - \sqrt{5}$$

Solve for e^{2t} (since we only need e^{2t} for the initial equation).

$$s = \sqrt{5}e^{2t} - \sqrt{5}$$

$$\sqrt{5}e^{2t} = s + \sqrt{5}$$

$$e^{2t} = \frac{s}{\sqrt{5}} + 1$$

Substitute e^{2t} into the vector function.

$$\vec{r}(s) = \left\langle 2\left(\frac{s}{\sqrt{5}} + 1\right), \frac{s}{\sqrt{5}} + 1 \right\rangle$$

$$\vec{r}(s) = \left\langle \frac{2s}{\sqrt{5}} + 2, \frac{s}{\sqrt{5}} + 1 \right\rangle$$

Plug in $s = 5$ to find the position after traveling 5 units.

$$\vec{r}(5) = \left\langle \frac{2 \cdot 5}{\sqrt{5}} + 2, \frac{5}{\sqrt{5}} + 1 \right\rangle$$

$$\vec{r}(5) = \left\langle 2\sqrt{5} + 2, \sqrt{5} + 1 \right\rangle$$

CURVATURE

■ 1. Find the curvature of $f(x) = 2x^2 - 4$ at $x = 1$.

Solution:

The curvature of the vector function is given by

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite $f(x)$ in parametric form for $x = t$ and $y = f(t)$.

$$x(t) = t$$

$$y(t) = 2t^2 - 4$$

Find the derivatives of these functions.

$$x'(t) = 1$$

$$y'(t) = 4t$$

So $\vec{r}'(t) = \langle 1, 4t \rangle$, then find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}\,'(t)| = \sqrt{(4t)^2 + 1^2} = \sqrt{16t^2 + 1}$$

Therefore,

$$\vec{T}(t) = \frac{\langle 1, 4t \rangle}{\sqrt{16t^2 + 1}}$$

$$\vec{T}(t) = \left\langle \frac{1}{\sqrt{16t^2 + 1}}, \frac{4t}{\sqrt{16t^2 + 1}} \right\rangle$$

Find $\vec{T}\,'(t)$ for each component individually.

$$\vec{T}_1{}'(t) = \left(\frac{1}{\sqrt{16t^2 + 1}} \right)' = -\frac{16t}{(16t^2 + 1)^{3/2}}$$

$$\vec{T}_2{}'(t) = \left(\frac{4t}{\sqrt{16t^2 + 1}} \right)' = \frac{4}{(16t^2 + 1)^{3/2}}$$

Plug $t = 1$ into the expressions for $|\vec{r}\,'(t)|$, and $\vec{T}\,'(t)$.

$$|\vec{r}\,'(1)| = \sqrt{16 \cdot 1^2 + 1} = \sqrt{17}$$

$$\vec{T}_1{}'(1) = -\frac{16 \cdot 1}{(16 \cdot 1^2 + 1)^{3/2}} = -\frac{16}{17\sqrt{17}}$$

$$\vec{T}_2{}'(1) = \frac{4}{(16 \cdot 1^2 + 1)^{3/2}} = \frac{4}{17\sqrt{17}}$$

Find the magnitude of $\vec{T}\,'(1)$.

$$|\vec{T}'(1)| = \sqrt{\left(-\frac{16}{17\sqrt{17}}\right)^2 + \left(\frac{4}{17\sqrt{17}}\right)^2} = \frac{4}{17}$$

Plug the values we've found into the formula for $k(t)$.

$$k(1) = \frac{|\vec{T}'(1)|}{|\vec{r}'(1)|}$$

$$k(1) = \frac{\frac{4}{17}}{\sqrt{17}} = \frac{4\sqrt{17}}{289}$$

■ 2. Find the curvature of the vector function at $t = 0$.

$$\vec{r}(t) = \left\langle 2(2+t)^{3/2}, 6t, 2(2-t)^{3/2} \right\rangle$$

Solution:

The curvature of the vector function is given by the formula

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function $\vec{r}(t)$ in parametric form.

$$x(t) = 2(2+t)^{3/2}$$

$$y(t) = 6t$$

$$z(t) = 2(2-t)^{3/2}$$

Find the derivatives of these functions.

$$x'(t) = 3(2+t)^{1/2} = 3\sqrt{2+t}$$

$$y'(t) = 6$$

$$z'(t) = 3(2-t)^{1/2} = 3\sqrt{2-t}$$

So

$$\vec{r}'(t) = \left\langle 3\sqrt{2+t}, 6, 3\sqrt{2-t} \right\rangle$$

Find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{\left(3\sqrt{2+t}\right)^2 + 6^2 + \left(3\sqrt{2-t}\right)^2}$$

$$|\vec{r}'(t)| = \sqrt{9(2+t) + 36 + 9(2-t)} = \sqrt{72} = 6\sqrt{2}$$

Therefore,

$$\vec{T}(t) = \frac{\left\langle 3\sqrt{2+t}, 6, 3\sqrt{2-t} \right\rangle}{6\sqrt{2}}$$

$$\vec{T}(t) = \left\langle \frac{\sqrt{2+t}}{2\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{\sqrt{2-t}}{2\sqrt{2}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{\sqrt{2+t}}{2\sqrt{2}}\right)' = \frac{1}{4\sqrt{2}\sqrt{2+t}}$$

$$\vec{T}_2'(t) = \left(\frac{1}{\sqrt{2}}\right)' = 0$$

$$\vec{T}_3'(t) = \left(\frac{\sqrt{2-t}}{2\sqrt{2}}\right)' = -\frac{1}{4\sqrt{2}\sqrt{2-t}}$$

Plug $t = 0$ into the expressions for $\vec{T}'(t)$.

$$\vec{T}_1'(0) = \frac{1}{4\sqrt{2}\sqrt{2+0}} = \frac{1}{8}$$

$$\vec{T}_2'(0) = 0$$

$$\vec{T}_3'(0) = -\frac{1}{4\sqrt{2}\sqrt{2-0}} = -\frac{1}{8}$$

Find the magnitude of $\vec{T}'(0)$.

$$|\vec{T}'(0)| = \sqrt{\left(\frac{1}{8}\right)^2 + 0^2 + \left(-\frac{1}{8}\right)^2} = \frac{1}{4\sqrt{2}}$$

Plug the values we've found into the formula for $k(t)$.

$$k(0) = \frac{|\vec{T}'(0)|}{|\vec{r}'(0)|}$$

$$k(0) = \frac{\frac{1}{4\sqrt{2}}}{6\sqrt{2}} = \frac{1}{48}$$

■ 3. Find the value(s) of t_0 such that the curvature of $\vec{r}(t) = \langle e^t + 5, 2e^t, -2e^t \rangle$ is 0 at $t = t_0$.

Solution:

The curvature of the vector function is given by the formula

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function $\vec{r}(t)$ in parametric form.

$$x(t) = e^t + 5$$

$$y(t) = 2e^t$$

$$z(t) = -2e^t$$

Find the derivatives of these functions.

$$x'(t) = e^t$$

$$y'(t) = 2e^t$$

$$z'(t) = -2e^t$$

So $\vec{r}'(t) = \langle e^t, 2e^t, -2e^t \rangle$. Find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{(e^t)^2 + (2e^t)^2 + (-2e^t)^2}$$

$$|\vec{r}'(t)| = \sqrt{9(e^t)^2} = 3e^t$$

Therefore,

$$\vec{T}(t) = \frac{\langle e^t, 2e^t, -2e^t \rangle}{3e^t}$$

$$\vec{T}(t) = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3} \right\rangle$$

Since $\vec{T}(t)$ is a constant vector, $\vec{T}'(t) = \langle 0,0,0 \rangle$ and $|\vec{T}'(t)| = 0$. So for any t,

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = 0$$

Since the curvature of the function is 0 at any point, the graph of this function is a line.

MAXIMUM CURVATURE

■ 1. Find the absolute maximum curvature $k(t)$ of $\vec{r}(t) = \langle 2+\sin t, \cos(t+\pi)\rangle$ on the interval $[0,2\pi]$.

Solution:

The curvature of the vector function is given by

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Use the trigonometric identity $\cos(\phi+\pi) = -\cos\phi$ to simplify the function.

$$\vec{r}(t) = \langle 2+\sin t, -\cos t\rangle$$

Rewrite the function in parametric form.

$$x(t) = 2+\sin t$$

$$y(t) = -\cos t$$

Find the derivatives of these equations.

$$x'(t) = \cos t$$

$$y'(t) = \sin t$$

So $\vec{r}\,'(t) = \langle \cos t, \sin t \rangle$, now find the magnitude of $\vec{r}\,'(t)$.

$$|\vec{r}\,'(t)| = \sqrt{(\cos t)^2 + (\sin t)^2} = 1$$

Therefore,

$$\vec{T}(t) = \frac{\langle \cos t, \sin t \rangle}{1}$$

$$\vec{T}(t) = \langle \cos t, \sin t \rangle$$

Find $\vec{T}\,'(t)$.

$$\vec{T}\,'(t) = \langle -\sin t, \cos t \rangle$$

Find the magnitude of $\vec{T}\,'(t)$.

$$|\vec{T}\,'(t)| = \sqrt{(-\sin t)^2 + (\cos t)^2} = 1$$

So the curvature $k(t)$ is

$$k(t) = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$$

$$k(t) = \frac{1}{1} = 1$$

Since the curvature is a constant function, it reaches the maximum value of 1 at any point on the interval $[0,2\pi]$.

■ 2. Find the absolute minimum and maximum curvature $k(x)$ of the function $f(x) = \ln(6x)$ on the interval $(0,1]$.

Solution:

The curvature of the vector function is given by

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function $f(x)$ in parametric form for $x = t$ and $y = f(t)$.

$$x(t) = t$$

$$y(t) = \ln(6t)$$

Find the derivatives of these equations.

$$x'(t) = 1$$

$$y'(t) = \frac{1}{t}$$

So

$$\vec{r}'(t) = \left\langle 1, \frac{1}{t} \right\rangle$$

Find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{1^2 + \left(\frac{1}{t}\right)^2} = \frac{\sqrt{t^2+1}}{t}$$

Therefore,

$$\vec{T}(t) = \frac{\left\langle 1, \frac{1}{t} \right\rangle}{\frac{\sqrt{t^2+1}}{t}}$$

$$\vec{T}(t) = \left\langle \frac{t}{\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{t}{\sqrt{t^2+1}}\right)' = \frac{1}{(t^2+1)^{3/2}}$$

$$\vec{T}_2'(t) = \left(\frac{1}{\sqrt{t^2+1}}\right)' = -\frac{t}{(t^2+1)^{3/2}}$$

Find the magnitude of $\vec{T}'(t)$.

$$|\vec{T}'(t)| = \sqrt{\left(\frac{1}{(t^2+1)^{3/2}}\right)^2 + \left(-\frac{t}{(t^2+1)^{3/2}}\right)^2}$$

$$|\vec{T}'(t)| = \sqrt{\frac{1}{(t^2+1)^3} + \frac{t^2}{(t^2+1)^3}}$$

$$|\vec{T}'(t)| = \frac{1}{t^2+1}$$

So the curvature is

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$k(t) = \frac{\frac{1}{t^2+1}}{\frac{\sqrt{t^2+1}}{t}} = \frac{t}{(t^2+1)^{3/2}}$$

To find the minimum of the function over $(0,1]$, let's investigate the critical points. Take the derivative.

$$k'(t) = \frac{1-2t^2}{(t^2+1)^{5/2}}$$

Solve the equation $k'(t) = 0$ in order to find the critical points.

$$\frac{1-2t^2}{(t^2+1)^{5/2}} = 0$$

$$1 - 2t^2 = 0$$

$$t^2 = \frac{1}{2}$$

Since $0 < t \leq 1$,

$$t = \frac{1}{\sqrt{2}}$$

Since $k'(t) > 0$ for $0 < t < 1/\sqrt{2}$ and $k'(t) < 0$ for $1/\sqrt{2} < t \le 1$, the point $t = 1/\sqrt{2}$ is the local maximum.

$$k\left(\frac{1}{\sqrt{2}}\right) = \frac{\frac{1}{\sqrt{2}}}{\left(\left(\frac{1}{\sqrt{2}}\right)^2 + 1\right)^{3/2}} = \frac{2\sqrt{3}}{9} \approx 0.38$$

So to find the absolute maximum, we need to compare the values of the function at the endpoints. Since the function isn't defined at $t = 0$, we need to consider the limit of the function when t approaches 0.

$$\lim_{t\to 0} k(t) = \lim_{t\to 0} \frac{t}{(t^2+1)^{3/2}} = \frac{0}{(0^2+1)^{3/2}} = 0$$

$$k(1) = \frac{1}{(1^2+1)^{3/2}} = \frac{\sqrt{2}}{4} \approx 0.35$$

So the absolute maximum is $2\sqrt{3}/9$ at $x = 1/\sqrt{2} = \sqrt{2}/2$, and the absolute minimum does not exist (it exists only as a limit when x approaches 0). So to summarize, the absolute maximum is $2\sqrt{3}/9$ at $x = \sqrt{2}/2$, and the absolute minimum does not exist.

■ 3. Find the absolute maximum curvature $k(t)$ of $\vec{r}(t) = \langle 3t+1, 2.5t^2 - 3, 4 - 4t\rangle$ on the interval $(-\infty, \infty)$.

Solution:

The curvature of the vector function is given by

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

where

$$\vec{T} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

Rewrite the function in parametric form.

$$x(t) = 3t + 1$$

$$y(t) = 2.5t^2 - 3$$

$$z(t) = 4 - 4t$$

Find the derivatives of these equations.

$$x(t) = 3$$

$$y(t) = 5t$$

$$z(t) = -4$$

So $\vec{r}'(t) = \langle 3, 5t, -4 \rangle$, and we can find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{3^2 + (5t)^2 + (-4)^2} = \sqrt{25t^2 + 25} = 5\sqrt{t^2 + 1}$$

Therefore,

$$\vec{T}(t) = \frac{\langle 3, 5t, -4 \rangle}{5\sqrt{t^2+1}}$$

$$\vec{T}(t) = \left\langle \frac{3}{5\sqrt{t^2+1}}, \frac{t}{\sqrt{t^2+1}}, -\frac{4}{5\sqrt{t^2+1}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{3}{5\sqrt{t^2+1}}\right)' = -\frac{3t}{5(t^2+1)^{3/2}}$$

$$\vec{T}_2'(t) = \left(\frac{t}{\sqrt{t^2+1}}\right)' = \frac{1}{(t^2+1)^{3/2}}$$

$$\vec{T}_3'(t) = \left(-\frac{4}{5\sqrt{t^2+1}}\right)' = \frac{4t}{5(t^2+1)^{3/2}}$$

Find the magnitude of $\vec{T}'(t)$.

$$|\vec{T}'(t)| = \sqrt{\frac{(3t)^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{(4t)^2}{25(t^2+1)^3}}$$

$$|\vec{T}'(t)| = \sqrt{\frac{1}{(t^2+1)^2}}$$

$$|\vec{T}'(t)| = \frac{1}{t^2+1}$$

So the curvature $k(t)$ is

$$k(t) = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$$

$$k(t) = \frac{\frac{1}{t^2+1}}{5\sqrt{t^2+1}} = \frac{1}{5(t^2+1)^{3/2}}$$

To find the maximum of the function over $(-\infty, \infty)$, let's investigate the critical points. Take the derivative of the curvature function.

$$k'(t) = -\frac{3t}{5(t^2+1)^{5/2}}$$

Solve the equation $k'(t) = 0$ in order to find the critical points.

$$-\frac{3t}{5(t^2+1)^{5/2}} = 0$$

$$t = 0$$

To find the absolute maximum, we need to compare the values of the function at $t = 0$ and when t approaches $-\infty$ and ∞.

$$\lim_{t\to\infty} k(t) = \lim_{t\to\infty} \frac{1}{5(t^2+1)^{3/2}} = 0$$

$$\lim_{t\to-\infty} k(t) = \lim_{t\to-\infty} \frac{1}{5(t^2+1)^{3/2}} = 0$$

$$k(0) = \frac{1}{5(1^2+1)^{3/2}} = \frac{1}{5(2)^{3/2}} = \frac{1}{10\sqrt{2}} = \frac{\sqrt{2}}{20}$$

So the curvature reaches its absolute maximum of $\sqrt{2}/20$ at $t = 0$.

NORMAL AND OSCULATING PLANES

■ 1. Find the point(s) at which the normal plane to the curve $\vec{r}(t)$ is parallel to the y-axis, then find the equation(s) of the normal plane at each point.

$$\vec{r}(t) = \langle 3t^3 - 10t, t^3 - 6t^2 - 15t, 4t + 1 \rangle$$

Solution:

The normal plane is the plane perpendicular to the tangent vector $\vec{r}\,'(t)$ of a space curve. The equation of the normal plane at the point $(x_0, y_0, z_0) = (r_1(t_0), r_2(t_0), r_3(t_0))$ is given by

$$r_1'(t_0)(x - x_0) + r_2'(t_0)(y - y_0) + r_3'(t_0)(z - z_0) = 0$$

Rewrite the function in parametric form.

$$r_1(t) = 3t^3 - 10t$$

$$r_2(t) = t^3 - 6t^2 - 15t$$

$$r_3(t) = 4t + 1$$

Find the derivatives of these equations.

$$r_1'(t) = 9t^2 - 10$$

$$r_2'(t) = 3t^2 - 12t - 15$$

$$r_3'(t) = 4$$

Since the normal plane is parallel to the y-axis at the point t_0, $r_2'(t_0) = 0$. So $t = t_0$ is the solution of the following equation:

$$3t^2 - 12t - 15 = 0$$

$$3(t + 1)(t - 5) = 0$$

So the normal plane is parallel to the y-axis at $t_0 = -1$ or $t_0 = 5$. For $t_0 = -1$,

$$r_1(-1) = 3(-1)^3 - 10(-1) = 7$$

$$r_2(-1) = (-1)^3 - 6(-1)^2 - 15(-1) = 8$$

$$r_3(-1) = 4(-1) + 1 = -3$$

$$r_1'(-1) = 9(-1)^2 - 10 = -1$$

$$r_2'(-1) = 3(-1)^2 - 12(-1) - 15 = 0$$

$$r_3'(-1) = 4$$

So the equation of the normal plane at the point $(7, 8, -3)$ is

$$-1(x - 7) + 4(z + 3) = 0$$

$$-x + 4z + 19 = 0$$

For $t_0 = 5$,

$$r_1(5) = 3(5)^3 - 10(5) = 325$$

$$r_2(5) = (5)^3 - 6(5)^2 - 15(5) = -100$$

$$r_3(5) = 4(5) + 1 = 21$$

$$r_1'(5) = 9(5)^2 - 10 = 215$$

$$r_2'(5) = 3(5)^2 - 12(5) - 15 = 0$$

$$r_3'(5) = 4$$

So the equation of the normal plane at the point $(325, -100, 21)$ is

$$215(x - 325) + 4(z - 21) = 0$$

$$215x + 4z - 69{,}959 = 0$$

■ 2. Find the equation of the osculating plane to $\vec{r}(t) = \langle 12 - 6t, 5t^2 - 10, 7 - 8t \rangle$ at the point $(0,10, -9)$.

Solution:

The equation of the osculating plane at $(x_0, y_0, z_0) = (r_1(t_0), r_2(t_0), r_3(t_0))$ is given by

$$B_1(t_0)(x - x_0) + B_2(t_0)(y - y_0) + B_3(t_0)(z - z_0) = 0$$

where $\vec{B}(t)$ is the binormal vector such that

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

The unit tangent vector $\vec{T}(t)$ is equal to

$$\frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

and the unit normal vector $\vec{N}(t)$ is equal to

$$\frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

To find the value of t_0 which corresponds to $(0,10,-9)$, we can solve $r_1(t)=0$, $r_2(t)=10$, or $r_3(t)=-9$ for t. From the first equation,

$$12-6t=0$$

$$t=2$$

We can also check that the other equations hold for $t_0=2$.

$$r_2(2)=5(2)^2-10=10$$

$$r_3(2)=7-8(2)=-9$$

To find the unit tangent vector $\vec{T}(t)$, rewrite the function in parametric form.

$$x(t)=12-6t$$

$$y(t)=5t^2-10$$

$$z(t)=7-8t$$

Find the derivatives of these equations.

$$x'(t)=-6$$

$$y'(t) = 10t$$

$$z'(t) = -8$$

So $\vec{r}'(t) = \langle -6, 10t, -8 \rangle$, and we can find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{(-6)^2 + (10t)^2 + (-8)^2} = \sqrt{100t^2 + 100} = 10\sqrt{t^2 + 1}$$

Therefore,

$$\vec{T}(t) = \frac{\langle -6, 10t, -8 \rangle}{10\sqrt{t^2 + 1}}$$

$$\vec{T}(t) = \left\langle -\frac{3}{5\sqrt{t^2 + 1}}, \frac{t}{\sqrt{t^2 + 1}}, -\frac{4}{5\sqrt{t^2 + 1}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(-\frac{3}{5\sqrt{t^2 + 1}} \right)' = \frac{3t}{5(t^2 + 1)^{3/2}}$$

$$\vec{T}_2'(t) = \left(\frac{t}{\sqrt{t^2 + 1}} \right)' = \frac{1}{(t^2 + 1)^{3/2}}$$

$$\vec{T}_3'(t) = \left(-\frac{4}{5\sqrt{t^2 + 1}} \right)' = \frac{4t}{5(t^2 + 1)^{3/2}}$$

Find the magnitude of $\vec{T}'(t)$.

$$|\vec{T}'(t)| = \sqrt{\frac{(3t)^2}{25(t^2+1)^3} + \frac{1}{(t^2+1)^3} + \frac{(4t)^2}{25(t^2+1)^3}}$$

$$|\vec{T}'(t)| = \sqrt{\frac{1}{(t^2+1)^2}}$$

$$|\vec{T}'(t)| = \frac{1}{t^2+1}$$

So the unit normal vector $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{\left\langle \frac{3t}{5(t^2+1)^{3/2}}, \frac{1}{(t^2+1)^{3/2}}, \frac{4t}{5(t^2+1)^{3/2}} \right\rangle}{\frac{1}{t^2+1}}$$

$$\vec{N}(t) = \left\langle \frac{3t}{5\sqrt{t^2+1}}, \frac{1}{\sqrt{t^2+1}}, \frac{4t}{5\sqrt{t^2+1}} \right\rangle$$

Plug $t = 2$ into $\vec{T}(t)$ in order to find the unit tangent vector at that point.

$$\vec{T}(2) = \left\langle -\frac{3}{5\sqrt{2^2+1}}, \frac{2}{\sqrt{2^2+1}}, -\frac{4}{5\sqrt{2^2+1}} \right\rangle$$

$$\vec{T}(2) = \left\langle -\frac{3}{5\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{4}{5\sqrt{5}} \right\rangle$$

Plug $t = 2$ into $\vec{N}(t)$ in order to find the unit normal vector at that point.

$$\vec{N}(2) = \left\langle \frac{3 \cdot 2}{5\sqrt{2^2+1}}, \frac{1}{\sqrt{2^2+1}}, \frac{4 \cdot 2}{5\sqrt{2^2+1}} \right\rangle$$

$$\vec{N}(2) = \left\langle \frac{6}{5\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{8}{5\sqrt{5}} \right\rangle$$

Find the binormal vector using the cross product.

$$\vec{B}(2) = \vec{T}(2) \times \vec{N}(2)$$

$$\vec{B}(2) = \left\langle -\frac{3}{5\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{4}{5\sqrt{5}} \right\rangle \times \left\langle \frac{6}{5\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{8}{5\sqrt{5}} \right\rangle$$

The cross product of the two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in the vectors $\vec{a}$ and $\vec{b}$,

$$\langle a_1, a_2, a_3 \rangle = \left\langle -\frac{3}{5\sqrt{5}}, \frac{2}{\sqrt{5}}, -\frac{4}{5\sqrt{5}} \right\rangle$$

$$\langle b_1, b_2, b_3 \rangle = \left\langle \frac{6}{5\sqrt{5}}, \frac{1}{\sqrt{5}}, \frac{8}{5\sqrt{5}} \right\rangle$$

to get

$$\vec{B}(2) = \mathbf{i}\left(\frac{2}{\sqrt{5}} \cdot \frac{8}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} \cdot \frac{1}{\sqrt{5}}\right) - \mathbf{j}\left(-\frac{3}{5\sqrt{5}} \cdot \frac{8}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} \cdot \frac{6}{5\sqrt{5}}\right)$$

$$+\mathbf{k}\left(-\frac{3}{5\sqrt{5}} \cdot \frac{1}{\sqrt{5}} - \frac{2}{\sqrt{5}} \cdot \frac{6}{5\sqrt{5}}\right)$$

$$\vec{B}(2) = \left\langle \frac{4}{5}, 0, -\frac{3}{5} \right\rangle$$

The equation of the plane through the point $(0,10,-9)$ and with the normal vector $\vec{B}(2)$ is

$$\frac{4}{5}(x-0) + 0(y-10) - \frac{3}{5}(z+9) = 0$$

$$4x - 3z - 27 = 0$$

■ 3. Use the binormal vector to prove that the graph of the vector function $\vec{r}(t)$ is a planar curve (a curve that lies in a single plane), then find the equation of the plane.

$$\vec{r}(t) = \langle 2\sin t - 2,\ \cos t + 1, 2\cos t + 5 \rangle$$

Solution:

The curve is planar if its binormal vector is constant for any t. In this case the binormal vector is orthogonal to this plane. The binormal vector $\vec{B}(t)$ is given by

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

where the unit tangent vector $\vec{T}(t)$ is

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

and the unit normal vector $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

To find the unit tangent vector $\vec{T}(t)$, rewrite the function in parametric form.

$$x(t) = 2\sin t - 2$$

$$y(t) = \cos t + 1$$

$$z(t) = 2\cos t + 5$$

Find derivatives of these equations.

$$x'(t) = 2\cos t$$

$$y'(t) = -\sin t$$

$$z'(t) = -2\sin t$$

So $\vec{r}'(t) = \langle 2\cos t, -\sin t, -2\sin t\rangle$, and we can find the magnitude of $\vec{r}'(t)$.

$$|\vec{r}'(t)| = \sqrt{(2\cos t)^2 + (-\sin t)^2 + (-2\sin t)^2}$$

$$|\vec{r}'(t)| = \sqrt{5\sin^2 t + 4\cos^2 t}$$

$$|\vec{r}'(t)| = \sqrt{4 + \sin^2 t}$$

Therefore,

$$\vec{T}(t) = \frac{\langle 2\cos t, -\sin t, -2\sin t\rangle}{\sqrt{4+\sin^2 t}}$$

$$\vec{T}(t) = \left\langle \frac{2\cos t}{\sqrt{4+\sin^2 t}}, -\frac{\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\sin t}{\sqrt{4+\sin^2 t}} \right\rangle$$

Find $\vec{T}'(t)$ for each component individually.

$$\vec{T}_1'(t) = \left(\frac{2\cos t}{\sqrt{4+\sin^2 t}}\right)' = -\frac{10\sin t}{(4+\sin^2 t)^{3/2}}$$

$$\vec{T}_2'(t) = \left(-\frac{\sin t}{\sqrt{4+\sin^2 t}}\right)' = -\frac{4\cos t}{(4+\sin^2 t)^{3/2}}$$

$$\vec{T}_3'(t) = \left(-\frac{2\sin t}{\sqrt{4+\sin^2 t}}\right)' = -\frac{8\cos t}{(4+\sin^2 t)^{3/2}}$$

Find the magnitude of $\vec{T}'(t)$.

$$|\vec{T}'(t)| = \sqrt{\left(-\frac{10\sin t}{(4+\sin^2 t)^{3/2}}\right)^2 + \left(-\frac{4\cos t}{(4+\sin^2 t)^{3/2}}\right)^2 + \left(-\frac{8\cos t}{(4+\sin^2 t)^{3/2}}\right)^2}$$

$$|\vec{T}'(t)| = \sqrt{\frac{20}{(4+\sin^2 t)^2}}$$

$$|\vec{T}'(t)| = \frac{2\sqrt{5}}{4+\sin^2 t}$$

So the unit normal vector $\vec{N}(t)$ is

$$\vec{N}(t) = \frac{\left\langle -\frac{10\sin t}{(4+\sin^2 t)^{3/2}}, -\frac{4\cos t}{(4+\sin^2 t)^{3/2}}, -\frac{8\cos t}{(4+\sin^2 t)^{3/2}} \right\rangle}{\frac{2\sqrt{5}}{4+\sin^2 t}}$$

$$\vec{N}(t) = \left\langle -\frac{10\sin t}{2\sqrt{5}(4+\sin^2 t)^{1/2}}, -\frac{4\cos t}{2\sqrt{5}(4+\sin^2 t)^{1/2}}, -\frac{8\cos t}{2\sqrt{5}(4+\sin^2 t)^{1/2}} \right\rangle$$

$$\vec{N}(t) = \left\langle -\frac{\sqrt{5}\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}}, -\frac{4\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}} \right\rangle$$

Find the unit binormal vector using the cross product.

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

$$\vec{B}(t) = \left\langle \frac{2\cos t}{\sqrt{4+\sin^2 t}}, -\frac{\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\sin t}{\sqrt{4+\sin^2 t}} \right\rangle$$

$$\times \left\langle -\frac{\sqrt{5}\sin t}{\sqrt{4+\sin^2 t}}, -\frac{2\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}}, -\frac{4\cos t}{\sqrt{5}\sqrt{4+\sin^2 t}} \right\rangle$$

Factor the denominator out of each vector.

$$\vec{B}(t) = \frac{\langle 2\cos t, -\sin t, -2\sin t\rangle \times \left\langle -\sqrt{5}\sin t, -\frac{2\cos t}{\sqrt{5}}, -\frac{4\cos t}{\sqrt{5}} \right\rangle}{4+\sin^2 t}$$

The cross product of the two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug $\vec{a}$ and $\vec{b}$,

$$\langle a_1, a_2, a_3\rangle = \langle 2\cos t, -\sin t, -2\sin t\rangle$$

$$\langle b_1, b_2, b_3\rangle = \left\langle -\sqrt{5}\sin t, -\frac{2\cos t}{\sqrt{5}}, -\frac{4\cos t}{\sqrt{5}}\right\rangle$$

into the cross product formula.

$$\vec{B}(t) = \frac{1}{4+\sin^2 t}\left[\mathbf{i}\left(\sin t\cdot\frac{4\cos t}{\sqrt{5}} - 2\sin t\cdot\frac{2\cos t}{\sqrt{5}}\right)\right.$$

$$-\mathbf{j}\left(-2\cos t\cdot\frac{4\cos t}{\sqrt{5}} - 2\sin t\cdot\sqrt{5}\sin t\right)$$

$$\left.+\vec{k}\left(-2\cos t\cdot\frac{2\cos t}{\sqrt{5}} - \sin t\cdot\sqrt{5}\sin t\right)\right]$$

$$\vec{B}(t) = \frac{\left\langle 0, \frac{8\cos^2 t}{\sqrt{5}} + 2\sqrt{5}\sin^2 t, -\frac{4\cos^2 t}{\sqrt{5}} - \sqrt{5}\sin^2 t\right\rangle}{4+\sin^2 t}$$

$$\vec{B}(t) = \left\langle 0, \frac{8\cos^2 t + 10\sin^2 t}{\sqrt{5}(4+\sin^2 t)}, -\frac{4\cos^2 t + 5\sin^2 t}{\sqrt{5}(4+\sin^2 t)}\right\rangle$$

$$\vec{B}(t) = \left\langle 0, \frac{8 + 2\sin^2 t}{\sqrt{5}(4+\sin^2 t)}, -\frac{4+\sin^2 t}{\sqrt{5}(4+\sin^2 t)}\right\rangle$$

$$\vec{B}(t) = \left\langle 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right\rangle$$

So since $\vec{B}(t)$ is the same for any point on the curve, the curve is planar. And since $\vec{B}(t)$ is orthogonal to the plane which contains the curve, $\vec{B}(t)$ is the normal vector to this plane.

Let's take any point on the curve, for example $t = 0$,

$$x(0) = 2\sin 0 - 2 = -2$$

$$y(0) = \cos 0 + 1 = 2$$

$$z(0) = 2\cos 0 + 5 = 7$$

Then the equation of the plane through $(-2,2,7)$ and with the normal vector

$$\vec{N}(t) = \left\langle 0, \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \right\rangle$$

is

$$0(x+2) + \frac{2}{\sqrt{5}}(y-2) - \frac{1}{\sqrt{5}}(z-7) = 0$$

$$2y - z + 3 = 0$$

EQUATION OF THE OSCULATING CIRCLE

■ 1. Find the equation of the osculating circle to the curve $\vec{r}(t) = \langle 2 + 5\sin t, 5\cos t - 1 \rangle$ at an arbitrary point.

Solution:

For the parametric curve $\vec{r}(t)$ in two-dimensional space, the signed curvature is given by

$$k(t) = \frac{r_1'(t) \cdot r_2''(t) - r_1''(t) \cdot r_2'(t)}{(r_1'(t)^2 + r_2'(t)^2)^{3/2}}$$

The unit normal vector is

$$\vec{N}(t) = \frac{\left\langle -r_2'(t), r_1'(t) \right\rangle}{|\vec{r}\,'(t)|}$$

The radius of curvature (equal to the radius of the osculating circle) is

$$R(t) = \frac{1}{|k(t)|}$$

and the vector to the center of the osculating circle is

$$\vec{Q}(t) = \vec{r}(t) + \frac{1}{k(t)}\vec{N}(t)$$

Rewrite the function in parametric form.

$$r_1(t) = 2 + 5\sin t$$

$$r_2(t) = 5\cos t - 1$$

The first-order derivatives are

$$r_1'(t) = 5\cos t$$

$$r_2'(t) = -5\sin t$$

The second-order derivatives are

$$r_1''(t) = -5\sin t$$

$$r_2''(t) = -5\cos t$$

Find the magnitude of $\vec{r}\,'(t)$/

$$|\vec{r}\,'(t)| = \sqrt{(5\cos t)^2 + (-5\sin t)^2} = \sqrt{25\cos^2 t + 25\sin^2 t} = \sqrt{25} = 5$$

The unit normal vector is

$$\vec{N}(t) = \frac{\left\langle -r_2'(t), r_1'(t)\right\rangle}{|\vec{r}\,'(t)|}$$

$$\vec{N}(t) = \frac{\langle 5\sin t, 5\cos t\rangle}{5}$$

$$\vec{N}(t) = \langle \sin t,\ \cos t\rangle$$

The signed curvature is

$$k(t) = \frac{r_1'(t) \cdot r_2''(t) - r_1''(t) \cdot r_2'(t)}{(r_1'(t)^2 + r_2'(t)^2)^{3/2}}$$

$$k(t) = \frac{5\cos t \cdot (-5\cos t) - (-5\sin t) \cdot (-5\sin t)}{((5\cos t)^2 + (-5\sin t)^2)^{3/2}}$$

$$k(t) = \frac{-25}{25^{3/2}} = -\frac{1}{5}$$

The radius of curvature is

$$R(t) = \frac{1}{|k(t)|}$$

$$R(t) = \frac{1}{\left|-\frac{1}{5}\right|} = 5$$

The vector to the center of the osculating circle is

$$\vec{Q}(t) = \vec{r}(t) + \frac{1}{k(t)}\vec{N}(t)$$

$$\vec{Q}(t) = \langle 2 + 5\sin t, 5\cos t - 1 \rangle - 5\langle \sin t, \cos t \rangle$$

$$\vec{Q}(t) = \langle 2, -1 \rangle$$

So the osculating circle is the circle with center $(2, -1)$ and radius 5. The osculating curve is this circle itself, and so the equation of the osculating circle is

$$(x-2)^2 + (y+1)^2 = 25$$

■ 2. Find the center and radius of the osculating circle to the curve $\vec{r}(t)$ at the point $(7,6)$.

$$\vec{r}(t) = \langle 4(5-t)^{5/2} + 3, 24t - 90 \rangle$$

Solution:

In order to find the value of t that corresponds to $(7,6)$, solve the system of equations for t.

$$4(5-t)^{5/2} + 3 = 7$$

$$24t - 90 = 6$$

From the first equation, we get

$$4(5-t)^{5/2} = 4$$

$$(5-t)^{5/2} = 1$$

$$t = 4$$

Check if the second equation holds for $t = 4$.

$$24(4) - 90 = 6$$

For the parametric curve $\vec{r}(t)$ in two-dimensional space, the signed curvature is given by

$$k(t) = \frac{r_1'(t) \cdot r_2''(t) - r_1''(t) \cdot r_2'(t)}{(r_1'(t)^2 + r_2'(t)^2)^{3/2}}$$

The unit normal vector is

$$\vec{N}(t) = \frac{\langle -r_2'(t), r_1'(t) \rangle}{|\vec{r}\,'(t)|}$$

The radius of curvature (equal to the radius of the osculating circle) is

$$R(t) = \frac{1}{|k(t)|}$$

and the vector to the center of the osculating circle is

$$\vec{Q}(t) = \vec{r}(t) + \frac{1}{k(t)}\vec{N}(t)$$

Rewrite the function in parametric form.

$$r_1(t) = 4(5 - t)^{5/2} + 3$$

$$r_2(t) = 24t - 90$$

The first-order derivatives are

$$r_1'(t) = -10(5 - t)^{3/2}$$

$$r_2'(t) = 24$$

The second-order derivatives are

$$r_1''(t) = 15(5 - t)^{1/2} = 15\sqrt{5 - t}$$

$$r_2''(t) = 0$$

Since we don't need the curvature and other parameters in general form, we'll use their values at $t = 4$.

$$r_1'(4) = -10,\ r_2'(4) = 24$$

$$r_1''(4) = 15,\ r_2''(4) = 0$$

Find the magnitude of $\vec{r}\,'(4)$.

$$|\vec{r}\,'(4)| = \sqrt{(-10)^2 + 24^2} = 26$$

The unit normal vector is

$$\vec{N}(4) = \frac{\langle -r_2'(4), r_1'(4)\rangle}{|\vec{r}\,'(4)|}$$

$$\vec{N}(4) = \frac{\langle -24, -10\rangle}{26}$$

$$\vec{N}(4) = \left\langle -\frac{12}{13}, -\frac{5}{13}\right\rangle$$

The signed curvature is

$$k(4) = \frac{r_1'(4)\cdot r_2''(4) - r_1''(4)\cdot r_2'(4)}{(r_1'(4)^2 + r_2'(4)^2)^{3/2}}$$

$$k(4) = \frac{-10\cdot 0 - 15\cdot 24}{((-10)^2 + 24^2)^{3/2}}$$

$$k(4) = -\frac{360}{26^3}$$

$$k(4) = -\frac{45}{2{,}197}$$

The radius of curvature is

$$R(t) = \frac{1}{|k(t)|}$$

$$R(4) = \frac{1}{\left|-\frac{45}{2{,}197}\right|} = \frac{2{,}197}{45} \approx 48.8$$

The vector to the center of the osculating circle is

$$\vec{Q}(4) = \vec{r}(4) + \frac{1}{k(4)}\vec{N}(4)$$

$$\vec{Q}(4) = \langle 7{,}6\rangle - \frac{2{,}197}{45}\left\langle -\frac{12}{13}, -\frac{5}{13}\right\rangle$$

$$\vec{Q}(4) = \left\langle \frac{781}{15}, \frac{232}{9}\right\rangle \approx \langle 52.1, 25.8\rangle$$

So the osculating circle has its center at $(52.1, 25.8)$ and a radius of 48.8.

■ 3. Find the point(s) on the curve $\vec{r}(t) = \langle t^2 + 3, 2t - 5\rangle$ where the osculating circle has a radius of 2.

Solution:

For the parametric curve $\vec{r}(t)$ in two-dimensional space, the signed curvature is given by

$$k(t) = \frac{r_1'(t) \cdot r_2''(t) - r_1''(t) \cdot r_2'(t)}{(r_1'(t)^2 + r_2'(t)^2)^{3/2}}$$

The radius of curvature (equal to the radius of the osculating circle) is

$$R(t) = \frac{1}{|k(t)|}$$

Since the radius is 2, we need to solve the equation for t.

$$|k(t)| = \frac{1}{2}$$

Rewrite the function in parametric form.

$$r_1(t) = t^2 + 3$$

$$r_2(t) = 2t - 5$$

The first-order derivatives are

$$r_1'(t) = 2t$$

$$r_2'(t) = 2$$

The second-order derivatives are

$$r_1''(t) = 2$$

$$r_2''(t) = 0$$

The signed curvature is

$$k(t)=\frac{r_1'(t)\cdot r_2''(t)-r_1''(t)\cdot r_2'(t)}{(r_1'(t)^2+r_2'(t)^2)^{3/2}}$$

$$k(t)=\frac{2t\cdot 0-2\cdot 2}{((2t)^2+2^2)^{3/2}}$$

$$k(t)=\frac{-4}{(4t^2+4)^{3/2}}$$

$$k(t)=\frac{-4}{8(t^2+1)^{3/2}}$$

$$k(t)=-\frac{1}{2(t^2+1)^{3/2}}$$

Solve the equation $|k(t)|=1/2$. Since $k(t)$ is always negative,

$$-\frac{1}{2(t^2+1)^{3/2}}=-\frac{1}{2}$$

$$(t^2+1)^{3/2}=1$$

$$(t^2+1)^3=1$$

$$t^2+1=1$$

$$t=0$$

The coordinates of the points on the curve for $t=0$ are

$r_1(0)=0^2+3=3$ and $r_2(0)=2(0)-5=-5$

VELOCITY AND ACCELERATION VECTORS

1. Find the value of t such that the velocity of the vector function $\vec{r}(t)$ is 0.

$$\vec{r}(t) = \langle 4t^3 - 5t^2 - 28t, 2e^{t-1} + e^{-2t+5}, \cos(\pi t)\rangle$$

Solution:

Since the velocity $\vec{r}\,'(t)$ of the vector function is 0, we know $\vec{r}\,'(t) = \vec{O}$.

Find derivatives for each component individually.

$$r_1'(t) = 12t^2 - 10t - 28$$

$$r_2'(t) = 2e^{t-1} - 2e^{-2t+5}$$

$$r_3'(t) = -\pi\sin(\pi t)$$

Solve the system of equations.

$$12t^2 - 10t - 28 = 0$$

$$2e^{t-1} - 2e^{-2t+5} = 0$$

$$-\pi\sin(\pi t) = 0$$

Since we have three equations and one variable, we can find the value of t from any equation, and then check to see whether the other equations hold.

From the first equation

$$6t^2 - 5t - 14 = 0$$

$$(t-2)(6t+7) = 0$$

$$t = 2 \text{ or } t = -\frac{7}{6}$$

Plug $t = 2$ into the second and third equations.

$$2e^{2-1} - 2e^{-2(2)+5} = 2e - 2e = 0$$

$$-\pi \sin(\pi \cdot 2) = -\pi \cdot 0 = 0$$

So $t = 2$ is a solution to the system. Plug $t = -7/6$ into the second equation.

$$2e^{(-7/6)-1} - 2e^{-2(-7/6)+5} = 2e^{-13/6} - 2e^{22/3} \neq 0$$

So $t = -7/6$ is not a solution.

■ 2. Find the point on the curve such that the velocity along the x-axis reaches its maximum value.

$$\vec{r}(t) = \left\langle \frac{t}{t^2+2}, 3\tan(3t^2), \ln 2t \right\rangle$$

Solution:

Find the velocity along the x-axis.

$$r_1(t) = \frac{t}{t^2+2}$$

$$r_1'(t) = \left(\frac{t}{t^2+2}\right)' = \frac{1 \cdot (t^2+2) - t \cdot 2t}{(t^2+2)^2} = \frac{2-t^2}{(t^2+2)^2}$$

Find the critical points of the function.

$$r_1'(t) = 0$$

$$\frac{2-t^2}{(t^2+2)^2} = 0$$

$$2 - t^2 = 0$$

$$t = \pm\sqrt{2}$$

Check the sign of the derivative.

$r_1'(t) > 0$ for t between $-\sqrt{2}$ and $\sqrt{2}$

$r_1'(t) < 0$ for $t \in (-\infty, -\sqrt{2}) \cup (\sqrt{2}, \infty)$

So $t = \sqrt{2}$ is the local and global maximum. To find the point on the curve, plug $t = \sqrt{2}$ into the vector function.

$$\vec{r}(\sqrt{2}) = \left\langle \frac{\sqrt{2}}{(\sqrt{2})^2+2}, 3\tan(3(\sqrt{2})^2), \ln 2(\sqrt{2}) \right\rangle$$

$$\vec{r}(\sqrt{2}) = \left\langle \frac{\sqrt{2}}{2+2}, 3\tan 6, \ln 2^{3/2} \right\rangle$$

$$\vec{r}(\sqrt{2}) = \left\langle \frac{\sqrt{2}}{4}, 3\tan 6, \frac{3}{2}\ln 2 \right\rangle$$

■ 3. Find the values of the parameters p and q such that the absolute value of acceleration of the non-constant function $\vec{r}(t) = \langle p\sin 3t, 4\cos qt \rangle$ is a constant for any value of t.

Solution:

To find the acceleration, compute the second-order derivative of the vector function for each component individually.

$$r_1(t) = p\sin 3t$$

$$r_1'(t) = 3p\cos 3t$$

$$r_1''(t) = -9p\sin 3t$$

and

$$r_2(t) = 4\cos qt$$

$$r_2'(t) = -4q\sin qt$$

$$r_2''(t) = -4q^2\cos qt$$

The absolute value (magnitude) of acceleration is

$$|\vec{r}\,''(t)| = \sqrt{(r_1''(t))^2 + (r_2''(t))^2}$$

$$|\vec{r}\,''(t)| = \sqrt{(-9p\sin 3t)^2 + (-4q^2\cos qt)^2}$$

$$|\vec{r}\,''(t)| = \sqrt{81p^2\sin^2 3t + 16q^4\cos^2 qt}$$

Since the absolute value of acceleration of $\vec{r}(t)$ is a constant for any value of t,

$$|\vec{r}\,''(t)| = \sqrt{81p^2\sin^2 3t + 16q^4\cos^2 qt} = \text{constant}$$

Therefore

$$81p^2\sin^2 3t + 16q^4\cos^2 qt = \text{constant}$$

So

$$(81p^2\sin^2 3t + 16q^4\cos^2 qt)' = 0$$

$$486p^2\sin 3t\cos 3t - 32q^5\sin qt\cos qt = 0$$

$$243p^2\sin 6t - 16q^5\sin 2qt = 0$$

$$243p^2\sin 6t = 16q^5\sin 2qt$$

Since p and q can't both be 0 (otherwise $\vec{r}$ is a constant vector), we get a system of equations for p and q.

$$243p^2 = 16q^5$$

$$6 = 2q$$

Solve the system.

$$q = 3$$

$$243p^2 = 16(3)^5$$

$$p^2 = 16$$

$$p = \pm 4$$

Therefore, we have two possible pairs of the parameters p and q such that the absolute value of acceleration is a constant, namely $p = -4$ and $q = 3$, or $p = 4$ and $q = 3$.

VELOCITY, ACCELERATION, AND SPEED, GIVEN POSITION

1. Find the point where the speed is 0, given the position function.

$$\vec{r}(t) = \left\langle \ln(2t^2 + 8t + 50), t^4 + 32t + 17, \arctan t - \frac{t}{5} \right\rangle$$

Solution:

Since the speed $|\vec{r}\,'(t)|$ of the vector function is 0, we have the equation for t.

$$\sqrt{(r_1'(t))^2 + (r_2'(t))^2 + (r_3'(t))^2} = 0$$

The sum of squares is 0 if and only if each term is 0, so

$$r_1'(t) = 0, \ r_2'(t) = 0, \ r_3'(t) = 0$$

Find derivatives.

$$r_1'(t) = \frac{(2t^2 + 8t + 50)'}{2t^2 + 8t + 50} = \frac{4t + 8}{2t^2 + 8t + 50} = 0$$

$$r_2'(t) = 4t^3 + 32 = 0$$

$$r_3'(t) = \frac{1}{t^2 + 1} - \frac{1}{5} = 0$$

Since we have three equations and one variable, we can find the value of t from any equation, and then verify that the other equations hold. From the first equation, we get

$$4t + 8 = 0$$

$$t = -2$$

Plug $t = -2$ into the second and third equations.

$$4(-2)^3 + 32 = 0$$

$$\frac{1}{(-2)^2 + 1} - \frac{1}{5} = 0$$

So $t = -2$ is a solution of the system. Plug $t = -2$ into $\vec{r}(t)$ to find the point where the speed is 0.

$$\vec{r}(-2) = \left\langle \ln(2(-2)^2 + 8(-2) + 50),\ (-2)^4 + 32(-2) + 17,\ \arctan(-2) - \frac{(-2)}{5} \right\rangle$$

$$\vec{r}(-2) = \left\langle \ln(42), -31, \frac{2}{5} - \arctan(2) \right\rangle$$

■ 2. Find the interval(s) of t values where the acceleration along the z-axis is negative for the position function.

$$\vec{r}(t) = \langle 2\sin(2t), e^{t^2+1}, t^4 - 10t^3 - 36t^2 - 5t + 45 \rangle$$

Solution:

Since the acceleration along the z-axis, $r_3''(t)$, is negative, we need to solve $r_3''(t) < 0$. First, find the derivatives.

$$r_3'(t) = 4t^3 - 30t^2 - 72t - 5$$

$$r_3''(t) = 12t^2 - 60t - 72$$

Solve the inequality.

$$12t^2 - 60t - 72 < 0$$

$$12(t - 6)(t + 1) < 0$$

$$t \in (-1,6)$$

■ 3. Find the velocity, speed, and acceleration of the position function at the point(s) where the trajectory intersects the xy-plane.

$\vec{r}(t) = \langle \sin 4t, 2\cos(t + \pi), 2 + 2\sin t \rangle$, where $t \in [0,2\pi]$

Solution:

In order to find the point(s) where the trajectory $\vec{r}(t)$ intersects the xy-plane, solve $r_3(t) = 0$ for t over the interval $[0,2\pi]$.

$$2 + 2\sin t = 0$$

$$\sin t = -1$$

$$t = -\frac{\pi}{2} + 2\pi k, \text{ where } k \text{ is any integer}$$

Therefore, inside the interval $[0,2\pi]$, the only possible value of t is $t = 3\pi/2$.

Rewrite the position function in parametric form.

$$r_1(t) = \sin 4t$$

$$r_2(t) = 2\cos(t + \pi)$$

$$r_3(t) = 2 + 2\sin t$$

To find velocity, find the first-order derivatives of the position function for each component individually.

$$r_1'(t) = 4\cos 4t$$

$$r_2'(t) = -2\sin(t + \pi)$$

$$r_3'(t) = 2\cos t$$

Plug $t = 3\pi/2$ into the derivative equations.

$$r_1'\left(\frac{3\pi}{2}\right) = 4\cos\left(4 \cdot \frac{3\pi}{2}\right) = 4$$

$$r_2'\left(\frac{3\pi}{2}\right) = -2\sin\left(\frac{3\pi}{2} + \pi\right) = -2$$

$$r_3'\left(\frac{3\pi}{2}\right) = 2\cos\left(\frac{3\pi}{2}\right) = 0$$

Find the speed at $t = 3\pi/2$.

$$\sqrt{\left[r_1'\left(\frac{3\pi}{2}\right)\right]^2+\left[r_2'\left(\frac{3\pi}{2}\right)\right]^2+\left[r_3'\left(\frac{3\pi}{2}\right)\right]^2}$$

$$\sqrt{4^2+(-2)^2+0^2}$$

$$\sqrt{20}$$

$$2\sqrt{5}$$

To find the acceleration, find the second-order derivatives of the position function for each component individually.

$$r_1''(t)=-16\sin 4t$$

$$r_2''(t)=-2\cos(t+\pi)$$

$$r_3''(t)=-2\sin t$$

Evaluate the second derivatives at $t=3\pi/2$.

$$r_1''\left(\frac{3\pi}{2}\right)=-16\sin\left(4\cdot\frac{3\pi}{2}\right)=0$$

$$r_2''\left(\frac{3\pi}{2}\right)=-2\cos\left(\frac{3\pi}{2}+\pi\right)=0$$

$$r_3''\left(\frac{3\pi}{2}\right)=-2\sin\left(\frac{3\pi}{2}\right)=2$$

VELOCITY AND POSITION GIVEN ACCELERATION AND INITIAL CONDITIONS

■ 1. Find the velocity and position of the acceleration function $\vec{a}(t)=\langle 4\sin^2 t,-\cos t\rangle$ if $\vec{r}(\pi)=\langle -2,1\rangle$, and $\vec{v}(\pi)=\langle 0,0\rangle$.

Solution:

Rewrite the acceleration function in parametric form.

$$a_1(t)=4\sin^2 t$$

$$a_2(t)=-\cos t$$

To find the velocity, integrate each component of the acceleration function with respect to t.

$$v_1(t)=\int a_1(t)\ dt=\int 4\sin^2 t\ dt=\int 2-2\cos 2t\ dt=2t-\sin 2t+C_1$$

$$v_2(t)=\int a_2(t)\ dt=\int -\cos t\ dt=-\sin t+C_2$$

Use the initial condition $\vec{v}(\pi)=\langle 0,0\rangle$ to find the values of the constants C_1 and C_2.

$$v_1(\pi)=2\pi-\sin 2\pi+C_1=0$$

$$v_2(\pi)=-\sin\pi+C_2=0$$

$$C_1 = -2\pi + \sin 2\pi = -2\pi$$

$$C_2 = \sin \pi = 0$$

So the velocity function is

$$\vec{v}(t) = \langle 2t - \sin 2t - 2\pi, -\sin t \rangle$$

To find the position function, integrate each component of the velocity function with respect to t.

$$r_1(t) = \int v_1(t)\ dt = \int (2t - \sin 2t - 2\pi)\ dt = t^2 + \frac{1}{2}\cos 2t - 2\pi t + C_3$$

$$r_2(t) = \int v_2(t)\ dt = \int -\sin t\ dt = \cos t + C_4$$

Use the initial condition $\vec{r}(\pi) = \langle -2,1 \rangle$ to find the values of the constants C_3 and C_4.

$$r_1(\pi) = \pi^2 + \frac{1}{2}\cos 2\pi - 2\pi \cdot \pi + C_3 = -2$$

$$r_2(\pi) = \cos \pi + C_4 = 1$$

$$C_3 = -2 - \pi^2 - \frac{1}{2}\cos 2\pi + 2\pi^2 = \pi^2 - \frac{5}{2}$$

$$C_4 = 1 - \cos \pi = 2$$

So the position function is

$$\vec{r}(t) = \left\langle t^2 + \frac{1}{2}\cos 2t - 2\pi t + \pi^2 - \frac{5}{2}, \cos t + 2 \right\rangle$$

■ 2. Find the speed function given the acceleration function $\vec{a}(t)=\langle 4t^3-1,6t^2+2t,2e^{2t}\rangle$ if $\vec{v}(0)=\langle 1,-3,1\rangle$.

Solution:

Rewrite the acceleration function in parametric form.

$$a_1(t)=4t^3-1$$

$$a_2(t)=6t^2+2t$$

$$a_3(t)=2e^{2t}$$

To find the velocity, integrate each component of the acceleration function with respect to t.

$$v_1(t)=\int a_1(t)\ dt=\int 4t^3-1\ dt=t^4-t+C_1$$

$$v_2(t)=\int a_2(t)\ dt=\int 6t^2+2t\ dt=2t^3+t^2+C_2$$

$$v_3(t)=\int a_3(t)\ dt=\int 2e^{2t}\ dt=e^{2t}+C_3$$

Use the initial condition $\vec{v}(0)=\langle 1,-3,1\rangle$ to find the values of the constants C_1, C_2, and C_3.

$$v_1(0)=0^4-0+C_1=1$$

$$v_2(0) = 2 \cdot 0^3 + 0^2 + C_2 = -3$$

$$v_3(0) = e^{2 \cdot 0} + C_3 = 1$$

So $C_1 = 1$, $C_2 = -3$, and $C_3 = 1 - e^0 = 0$, which means the velocity function is

$$\vec{v}(t) = \langle t^4 - t + 1, 2t^3 + t^2 - 3, e^{2t} \rangle$$

The speed function is given by

$$s(t) = \sqrt{(v_1(t))^2 + (v_2(t))^2 + (v_3(t))^2}$$

$$s(t) = \sqrt{(t^4 - t + 1)^2 + (2t^3 + t^2 - 3)^2 + (e^{2t})^2}$$

$$s(t) = \sqrt{t^8 + 4t^6 + 2t^5 + 3t^4 - 12t^3 - 5t^2 - 2t + e^{4t} + 10}$$

■ 3. Find the distance travelled by a particle during the first 10 seconds, given its acceleration function $\vec{a}(t) = \langle 2 \sin t, 2 \cos t \rangle$, where t is the time in seconds and the initial velocity is $\vec{v}(0) = \langle -2,0 \rangle$.

Solution:

Rewrite the acceleration function in parametric form.

$$a_1(t) = 2 \sin t$$

$$a_2(t) = 2 \cos t$$

To find the velocity, integrate each component of the acceleration function with respect to t.

$$v_1(t) = \int a_1(t)\ dt = \int 2\sin t\ dt = -2\cos t + C_1$$

$$v_2(t) = \int a_2(t)\ dt = \int 2\cos t\ dt = 2\sin t + C_2$$

Use the initial condition $\vec{v}(0) = \langle -2,0 \rangle$ to find the values of the constants C_1 and C_2.

$$v_1(0) = -2\cos 0 + C_1 = -2$$

$$v_2(0) = 2\sin 0 + C_2 = 0$$

We get $C_1 = 0$ and $C_2 = 0$, so the velocity function is

$$\vec{v}(t) = \langle -2\cos t, 2\sin t \rangle$$

We don't need the position function to find the arc length, which is equal to the distance travelled by the particle. The arc length is given by

$$\int_a^b \sqrt{(r_1'(t))^2 + (r_2'(t))^2}\ dt$$

Plug in $a = 0$, $b = 10$, and $\vec{r}\,'(t) = \vec{v}(t)$.

$$\int_0^{10} \sqrt{(-2\cos t)^2 + (2\sin t)^2}\ dt$$

$$\int_0^{10} \sqrt{4\cos^2 t + 4\sin^2 t}\ dt$$

$$\int_0^{10} \sqrt{4}\ dt$$

$$2\int_0^{10} dt$$

$$2 \cdot 10$$

$$20$$

TANGENTIAL AND NORMAL COMPONENTS OF ACCELERATION

■ 1. Find the tangential and normal components of acceleration for the vector function $\vec{r}(t) = \langle e^{2t} + 1, e^{-2t} - 1, t - 4 \rangle$ at the point $(2,0,-4)$.

Solution:

In order to find the value of t which corresponds to the point $(2,0,-4)$, solve the system of equations for t.

$$e^{2t} + 1 = 2$$

$$e^{-2t} - 1 = 0$$

$$t - 4 = -4$$

From the third equation, we get $t = 0$. Check to see whether the other equations hold.

$$e^{2(0)} + 1 = 2$$

$$e^{-2(0)} - 1 = 0$$

So the value of $t = 0$ is the solution of the system.

The tangential component of acceleration is given by

$$a_T = \frac{\vec{r}'(t) \cdot \vec{r}''(t)}{|\vec{r}'(t)|}$$

$$a_T = \frac{\vec{v}(t) \cdot \vec{a}(t)}{s(t)}$$

where $\vec{v}(t)$ is the velocity, $\vec{a}(t)$ is the acceleration, and $s(t)$ is the speed.

Rewrite the vector function in parametric form.

$$r_1(t) = e^{2t} + 1$$

$$r_2(t) = e^{-2t} - 1$$

$$r_3(t) = t - 4$$

To find the velocity, take the first-order derivatives.

$$v_1(t) = 2e^{2t}$$

$$v_2(t) = -2e^{-2t}$$

$$v_3(t) = 1$$

Plug in $t = 0$.

$$v_1(0) = 2e^0 = 2$$

$$v_2(0) = -2e^0 = -2$$

$$v_3(0) = 1$$

Find the speed at $t = 0$.

$$s(0) = \sqrt{[v_1(0)]^2 + [v_2(0)]^2 + [v_3(0)]^2} = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{9} = 3$$

To find the acceleration, take the second-order derivative.

$$a_1(t) = 4e^{2t}$$

$$a_2(t) = 4e^{-2t}$$

$$a_3(t) = 0$$

Plug in $t = 0$.

$$a_1(0) = 4e^0 = 4$$

$$a_2(0) = 4e^0 = 4$$

$$a_3(0) = 0$$

Plug in the values we've found to the tangential component of acceleration formula.

$$a_T(0) = \frac{\vec{v}(0) \cdot \vec{a}(0)}{s(0)}$$

$$a_T(0) = \frac{\langle 2, -2, 1 \rangle \cdot \langle 4, 4, 0 \rangle}{3}$$

$$a_T(0) = \frac{2 \cdot 4 + (-2) \cdot 4 + 1 \cdot 0}{3} = 0$$

The normal component of acceleration function is given by

$$a_N = |\vec{r}\,'(t) \times \vec{r}\,''(t)|$$

$$a_N = |\vec{v}(t) \times \vec{a}(t)|$$

where $\vec{v}(t)$ is the velocity and $\vec{a}(t)$ is the acceleration.

$$a_N(0) = |\vec{v}(0) \times \vec{a}(0)|$$

$$a_N(0) = |\langle 2, -2, 1\rangle \times \langle 4, 4, 0\rangle|$$

The cross product of two vectors $\vec{v}$ and $\vec{a}$ is given by

$$\vec{v} \times \vec{a} = \mathbf{i}(v_2a_3 - v_3a_2) - \mathbf{j}(v_1a_3 - v_3a_1) + \mathbf{k}(v_1a_2 - v_2a_1)$$

Plug in $\langle v_1, v_2, v_3\rangle = \langle 2, -2, 1\rangle$ and $\langle a_1, a_2, a_3\rangle = \langle 4, 4, 0\rangle$.

$$\vec{v} \times \vec{a} = \mathbf{i}(-2 \cdot 0 - 1 \cdot 4) - \mathbf{j}(2 \cdot 0 - 1 \cdot 4) + \mathbf{k}(2 \cdot 4 + 2 \cdot 4)$$

$$\vec{v} \times \vec{a} = -4\mathbf{i} + 4\mathbf{j} + 16\mathbf{k}$$

$$a_N(0) = |\langle -4, 4, 16\rangle|$$

$$a_N(0) = \sqrt{(-4)^2 + 4^2 + 16^2} = \sqrt{288} = 12\sqrt{2}$$

■ 2. Find the point(s) where the tangential component of acceleration for the vector function $\vec{r}(t) = \langle 2\cos t - 2, 3\sin t + 5, 4t - 1\rangle$ is 0.

Solution:

The tangential component of acceleration is given by

$$a_T = \frac{\vec{r}\,'(t) \cdot \vec{r}\,''(t)}{|\vec{r}\,'(t)|}$$

$$a_T = \frac{\vec{v}(t) \cdot \vec{a}(t)}{s(t)}$$

where $\vec{v}(t)$ is velocity, $\vec{a}(t)$ is acceleration, and $s(t)$ is speed. Since $a_T = 0$, we have the equation for t.

$$\frac{\vec{v}(t) \cdot \vec{a}(t)}{s(t)} = 0$$

$$\vec{v}(t) \cdot \vec{a}(t) = 0$$

Rewrite the vector function in parametric form.

$$r_1(t) = 2\cos t - 2$$

$$r_2(t) = 3\sin t + 5$$

$$r_3(t) = 4t - 1$$

To find velocity, take the first-order derivative.

$$v_1(t) = -2\sin t$$

$$v_2(t) = 3\cos t$$

$$v_3(t) = 4$$

To find acceleration, take the second-order derivative.

$$a_1(t) = -2\cos t$$

$$a_2(t) = -3\sin t$$

$$a_3(t) = 0$$

So the equation is

$$\langle -2\sin t, 3\cos t, 4\rangle \cdot \langle -2\cos t, -3\sin t, 0\rangle = 0$$

$$(-2\sin t)(-2\cos t) + (3\cos t)(-3\sin t) + 4 \cdot 0 = 0$$

$$4\sin t\cos t - 9\sin t\cos t = 0$$

$$-5\sin t\cos t = 0$$

Use the trigonometric identity $\sin 2\phi = 2\sin\phi\cos\phi$.

$$-\frac{5}{2}\sin 2t = 0$$

$$\sin 2t = 0$$

So $2t = \pi n$, where n is any integer, or $t = (\pi/2)n$, where n is any integer.

■ 3. Find the values of parameters p and q, such that the normal components of acceleration for $\vec{r}(t) = \langle 2t^2, 3pt, t^2 - 4t + qt\rangle$ are 0 at the origin.

Solution:

The normal component of acceleration function is given by

$$a_N = |\vec{r}'(t) \times \vec{r}''(t)|$$

$$a_N = |\vec{v}(t) \times \vec{a}(t)|$$

where $\vec{v}(t)$ is velocity and $\vec{a}(t)$ is acceleration. The origin corresponds to the value of $t = 0$. Since $a_N = 0$ at $t = 0$, we can get an equation for p and q.

$$|\vec{v}(0) \times \vec{a}(0)| = 0$$

Rewrite the vector function in parametric form.

$$r_1(t) = 2t^2$$

$$r_2(t) = 3pt$$

$$r_3(t) = t^2 - 4t + qt$$

To find the velocity, take the first-order derivative.

$$v_1(t) = 4t$$

$$v_2(t) = 3p$$

$$v_3(t) = 2t - 4 + q$$

Evaluate these derivatives at $t = 0$.

$$v_1(0) = 0$$

$$v_2(0) = 3p$$

$$v_3(0) = -4 + q$$

To find the acceleration, take the second-order derivative.

$$a_1(t) = 4$$

$$a_2(t) = 0$$

$$a_3(t) = 2$$

So the equation is

$$|\langle 0, 3p, -4+q \rangle \times \langle 4, 0, 2 \rangle| = 0$$

The cross product of two vectors $\vec{v}$ and $\vec{a}$ is given by

$$\vec{v} \times \vec{a} = \mathbf{i}(v_2a_3 - v_3a_2) - \mathbf{j}(v_1a_3 - v_3a_1) + \mathbf{k}(v_1a_2 - v_2a_1)$$

Plug in $\langle v_1, v_2, v_3 \rangle = \langle 0, 3p, -4+q \rangle$ and $\langle a_1, a_2, a_3 \rangle = \langle 4,0,2 \rangle$.

$$\vec{v} \times \vec{a} = \mathbf{i}(3p \cdot 2 - (-4+q) \cdot 0) - \mathbf{j}(0 \cdot 2 - (-4+q) \cdot 4) + \mathbf{k}(0 \cdot 0 - 3p \cdot 4)$$

$$\vec{v} \times \vec{a} = 6p\mathbf{i} + (-16+4q)\mathbf{j} - 12p\mathbf{k}$$

$$|\langle 6p, -16+4q, -12p \rangle| = 0$$

The vector magnitude is 0 if and only if each of its component is 0. So we have the system of equations for p and q.

$$6p = 0$$

$$-16 + 4q = 0$$

$$-12p = 0$$

Since we get $p = 0$, we also find

$$4q = 16$$

$$q = 4$$

SKETCHING THE VECTOR FIELD

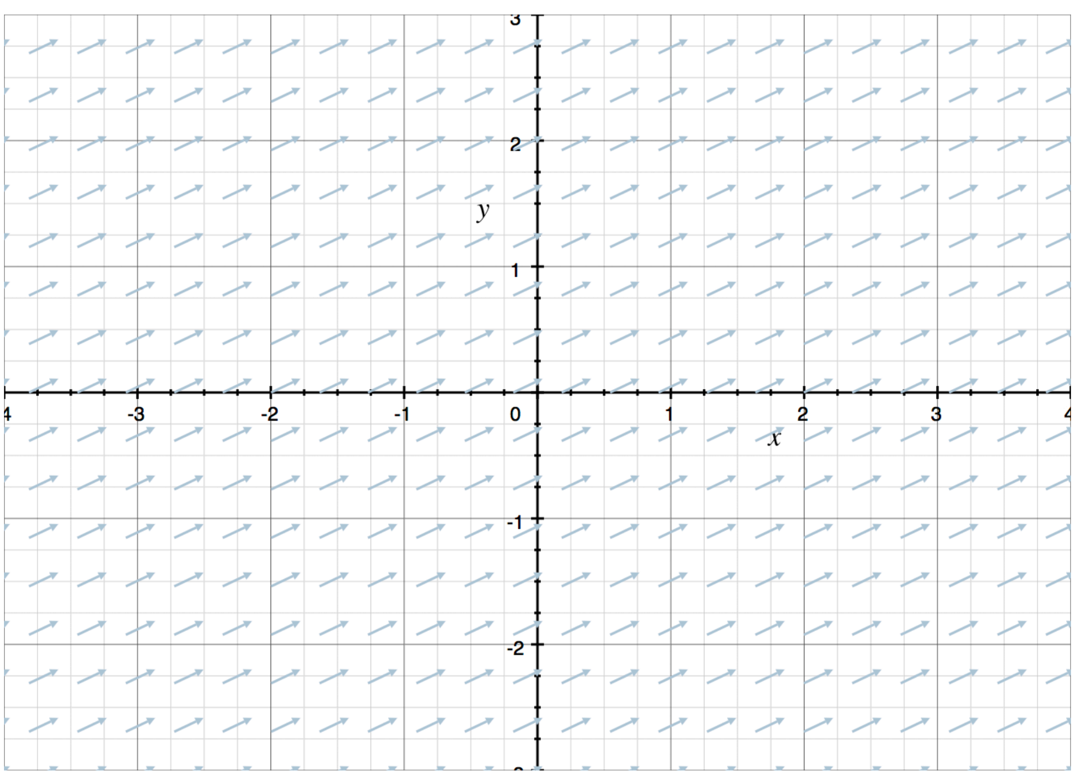
■ 1. Find a two-dimensional vector field in which all of the vectors are orthogonal to $y = -2x$, then sketch the vector field.

Solution:

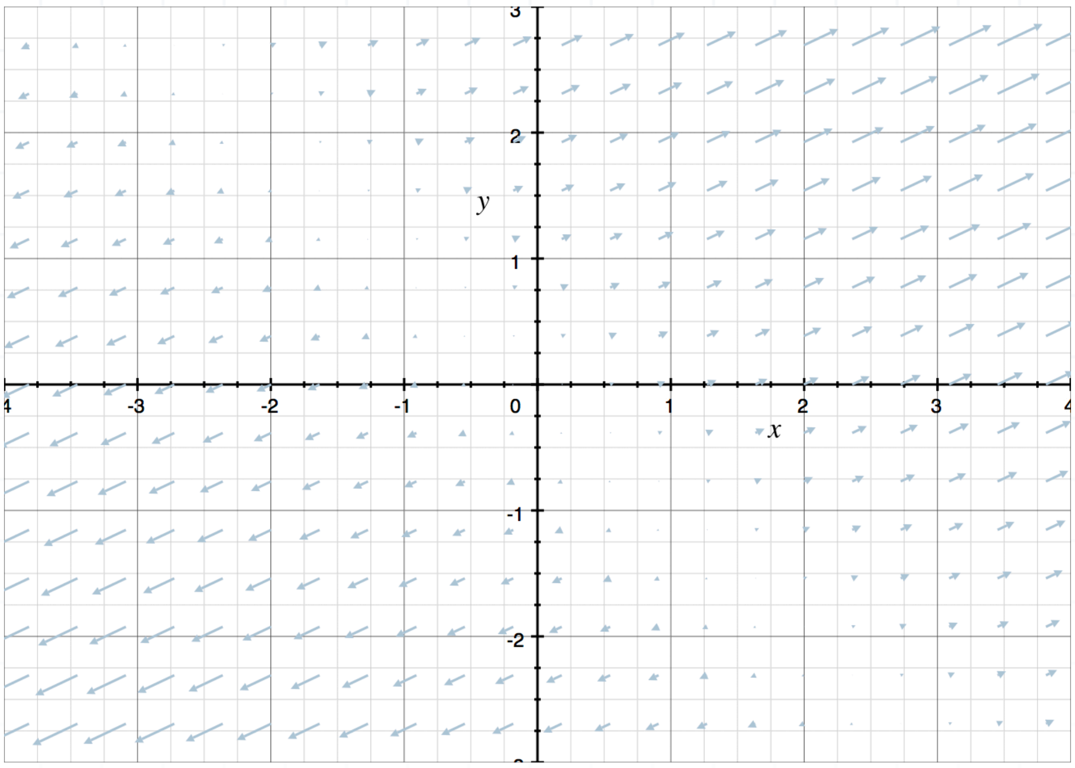
There are an infinite number of solutions to this problem. Since the line is $2x + y = 0$, the vector orthogonal to the line is $\langle 2,1 \rangle$. So the general form of the vector field with all vectors orthogonal to the line is

$F(x, y) = f(x, y)(2\mathbf{i} + \mathbf{j})$.

where $f(x, y)$ is an arbitrary scalar-valued function. For example, if $f(x, y) = 1$, then the vector field is $F(x, y) = 2\mathbf{i} + \mathbf{j}$.



For another example, consider $f(x, y) = x + y$. The vector field is $F(x, y) = 2(x + y)\mathbf{i} + (x + y)\mathbf{j}$.



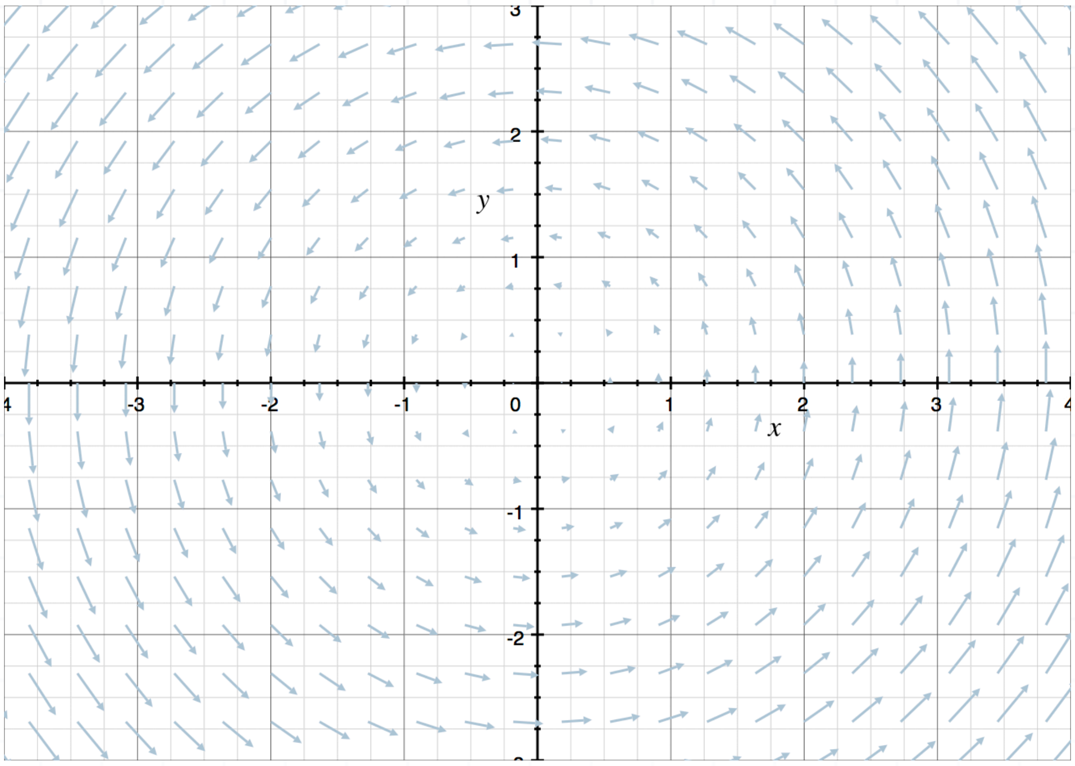
2. Find a two-dimensional vector field such that each vector is tangent to some circle centered at the origin, then sketch the vector field.

Solution:

There are infinite number of solutions to this problem. Let (x, y) be an arbitrary point. Consider the circle with center at the origin that passes through (x, y).

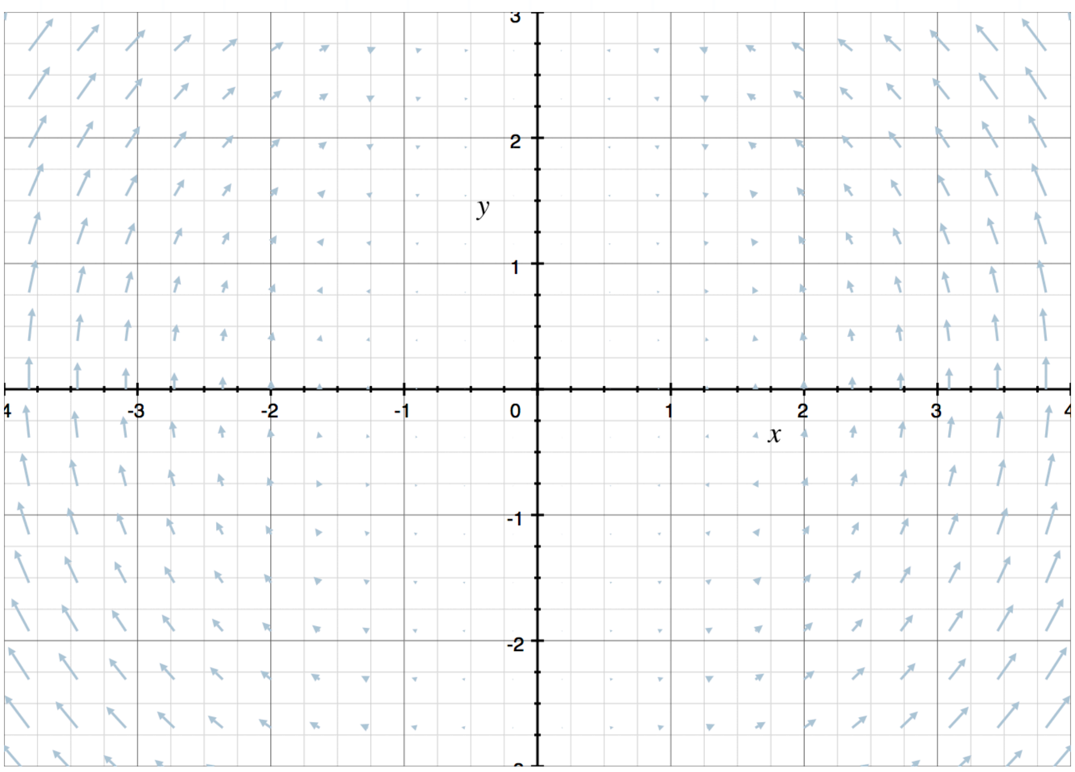
Since the vector $\langle x, y\rangle$ joins the center of the circle (the origin), and the point on the circle, the tangent vector must be orthogonal to it. The vector orthogonal to $\langle x, y\rangle$ is $\langle -y, x\rangle$. So the general form of the vector field is $F(x, y) = f(x, y)(-y\mathbf{i} + x\mathbf{j})$ where $f(x, y)$ is an arbitrary scalar-valued function.

For example, if $f(x, y) = 1$, then the vector field is $F(x, y) = -y\mathbf{i} + x\mathbf{j}$.



For another example, consider $f(x, y) = x$. The vector field is

$F(x, y) = -yx\mathbf{i} + x^2\mathbf{j}$.



■ 3. Find a three-dimensional unit radial vector field (a vector field where all the vectors have magnitude 1, and point straight towards or away from the origin).

Solution:

Let (x, y, z) be the arbitrary point. The vector $\langle x, y, z\rangle$ points away from the origin, and the vector $\langle -x, -y, -z\rangle$ points toward the origin. The magnitude of $\langle x, y, z\rangle$ is

$$|\langle x, y, z\rangle| = \sqrt{x^2 + y^2 + z^2}$$

So the unit radial vector field where all vectors point straight away from the origin is

$$F(x, y) = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} + \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} + \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k}$$

and the unit radial vector field where all vectors point straight toward the origin is

$$F(x, y) = -\frac{x}{\sqrt{x^2 + y^2 + z^2}}\mathbf{i} - \frac{y}{\sqrt{x^2 + y^2 + z^2}}\mathbf{j} - \frac{z}{\sqrt{x^2 + y^2 + z^2}}\mathbf{k}$$

GRADIENT VECTOR FIELD

■ 1. Sketch the gradient vector field of $f(x, y) = x^2 + y^2$.

Solution:

The gradient vector field for $f(x, y)$ is defined by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

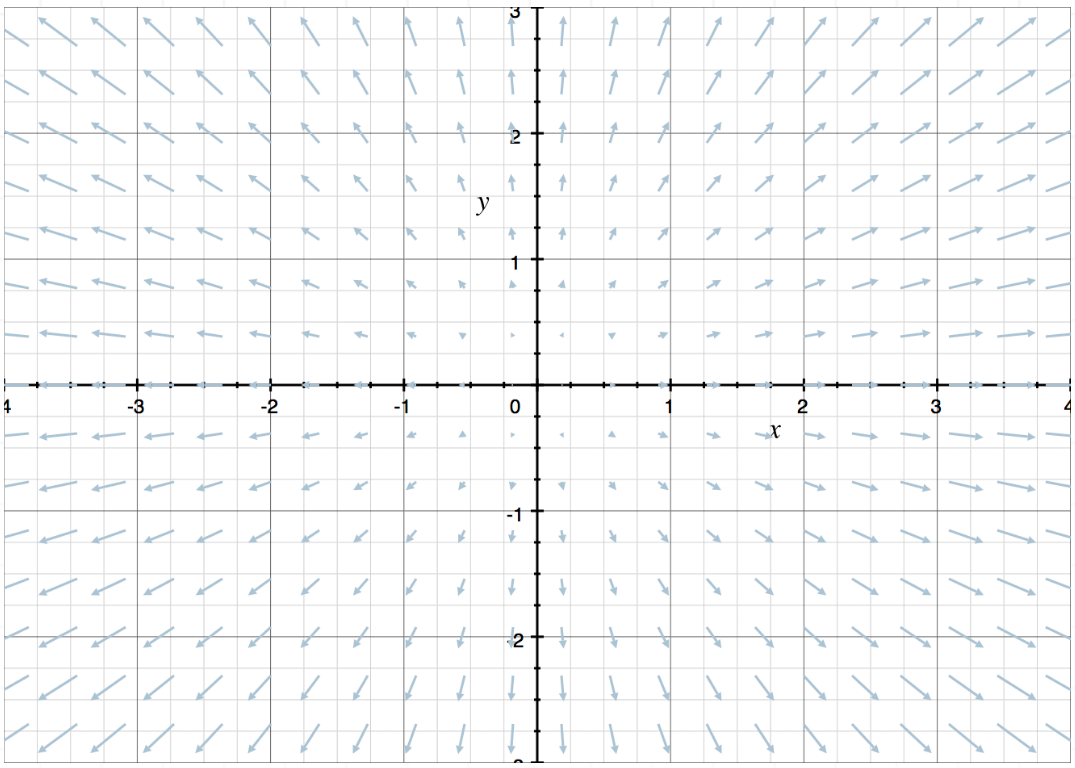
Find the partial derivatives, then the equation of the vector field is

$$\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$$

To sketch the gradient field, consider (x, y) and the vector $\langle 2x, 2y \rangle$. This vector points away from the origin, and has magnitude

$$\sqrt{4x^2 + 4y^2} = 2\sqrt{x^2 + y^2}$$

So the gradient vector field of the function is the radial vector field.



■ 2. Sketch the gradient vector field of $f(x,y)=\ln x+\ln y$.

Solution:

The domain of $f(x,y)$ is the interior points of the first quadrant $(x>0,\ y>0)$, so the gradient vector field ∇f is also defined only within the first quadrant.

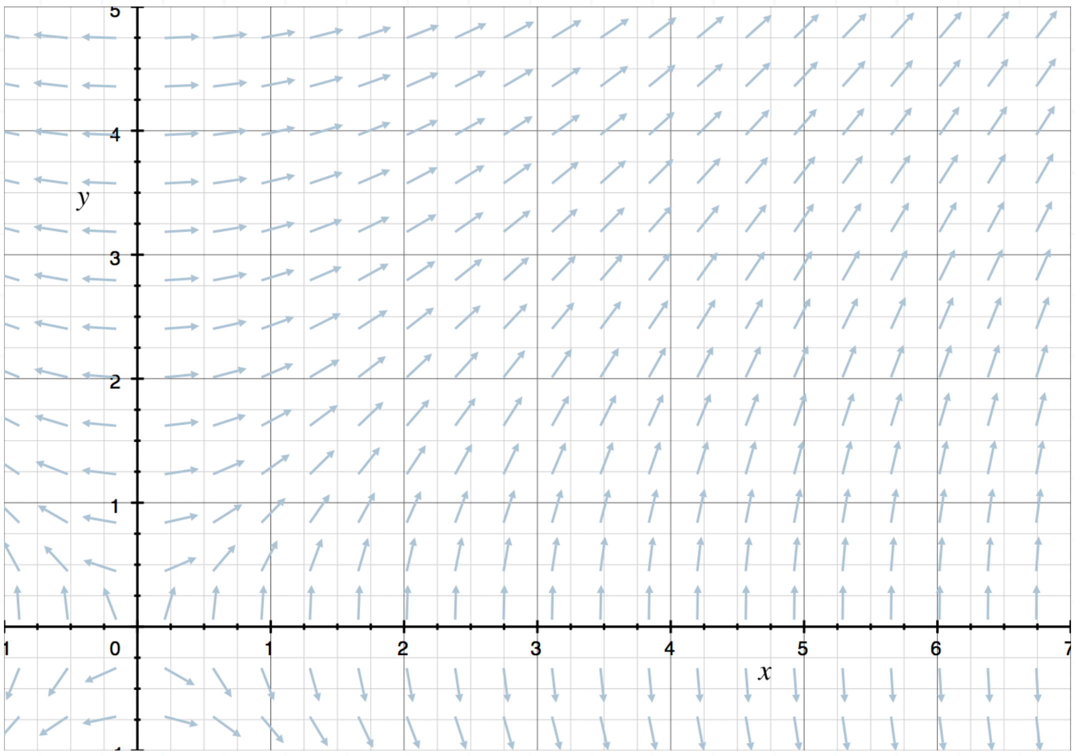
The gradient vector field for $f(x,y)$ is defined by

$$\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}$$

Find the partial derivatives, then the equation of the vector field is

$$\nabla f=\frac{1}{x}\mathbf{i}+\frac{1}{y}\mathbf{j}\text{ with }x>0\text{ and }y>0$$

To sketch the gradient field, consider the point (x, y), and the vector $\langle 1/x, 1/y \rangle$.



■ 3. Find the point(s) such that the gradient vector of the function $f(x, y, z)$ is equal to the zero vector $\vec{O}$.

$$f(x, y, z) = x^2 + y^2 + z^2 - 4xyz - 2x - 2y - 2z + 5$$

Solution:

The gradient vector field for the function $f(x, y, z)$ is defined by

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Find the partial derivatives, then the equation of the vector field is

$$\nabla f = (2x - 4yz - 2)\mathbf{i} + (2y - 4xz - 2)\mathbf{j} + (2z - 4xy - 2)\mathbf{k}$$

Since $\nabla f = \overrightarrow{0}$, we have the system of equations for x, y, and z.

$$2x - 4yz - 2 = 0$$

$$2y - 4xz - 2 = 0$$

$$2z - 4xy - 2 = 0$$

Divide each equation by 2.

$$x - 2yz - 1 = 0$$

$$y - 2xz - 1 = 0$$

$$z - 2xy - 1 = 0$$

Solve the first equation for x, then substitute its value into the second and third equations.

$$x = 2yz + 1$$

$$y - 2(2yz + 1)z - 1 = 0$$

$$z - 2(2yz + 1)y - 1 = 0$$

Then we get

$$x = 2yz + 1$$

$$y - 2z - 4yz^2 - 1 = 0$$

$$z - 2y - 4y^2z - 1 = 0$$

Subtract the third equation from the second.

$$(y - 2z - 4yz^2 - 1) - (z - 2y - 4y^2z - 1) = 0$$

$$3y - 3z - 4yz^2 + 4y^2z = 0$$

$$3(y - z) + 4yz(y - z) = 0$$

$$(y - z)(3 + 4yz) = 0$$

So (1) $y - z = 0$ or (2) $3 + 4yz = 0$.

(1) Substitute $z = y$ into the second equation.

$$y - 2(y) - 4y(y)^2 - 1 = 0$$

$$-4y^3 - y - 1 = 0$$

$$4y^3 + y + 1 = 0$$

$$(2y + 1)(2y^2 - y + 1) = 0$$

Since $2y^2 - y + 1 = 0$ has no solutions,

$$2y + 1 = 0$$

$$y = -0.5$$

$$z = -0.5$$

$$x = 2(-0.5)(-0.5) + 1 = 1.5$$

So the first point has coordinates $(1.5, -0.5, -0.5)$.

(2)

$$3 + 4yz = 0$$

$$z = -\frac{3}{4y}$$

Substitute this value of z into the second equation.

$$y - 2\left(-\frac{3}{4y}\right) - 4y\left(-\frac{3}{4y}\right)^2 - 1 = 0$$

$$y - \frac{3}{4y} - 1 = 0$$

$$4y^2 - 4y - 3 = 0$$

$$(2y - 3)(2y + 1) = 0$$

So $y = -0.5$ or $y = 1.5$.

If $y = -0.5$, $z = -\frac{3}{4(-0.5)} = 1.5$, and $x = 2(-0.5)(1.5) + 1 = -0.5$

If $y = 1.5$, $z = -\frac{3}{4(1.5)} = -0.5$, and $x = 2(1.5)(-0.5) + 1 = -0.5$

LINE INTEGRAL OF A CURVE

■ 1. Calculate the line integral over c, where c is the circle that lies in the plane $z=3$, with center on the z-axis and radius 4.

$$\int_c x^2+y^2+z^2\ ds$$

Solution:

The line integral over the curve for the function $f(x,y,z)$ is given by the formula

$$\int_c f(x,y,z)\ ds=\int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$$

Consider the parametrization of the circle.

$$x(t)=4\cos t$$

$$y(t)=4\sin t$$

$$z(t)=3$$

Over the whole circle, t changes from 0 to 2π, so $a=0$ and $b=2\pi$. Find the first-order derivatives.

$$x'(t)=-4\sin t$$

$$y'(t) = 4\cos t$$

$$z'(t) = 0$$

The function is

$$f(x(t), y(t), z(t)) = (4\cos t)^2 + (4\sin t)^2 + 3^2$$

$$f(x(t), y(t), z(t)) = 16(\cos^2 t + \sin^2 t) + 9$$

$$f(x(t), y(t), z(t)) = 16 + 9$$

$$f(x(t), y(t), z(t)) = 25$$

Therefore, the line integral over the curve is

$$\int_c f(x, y, z)\ ds = \int_0^{2\pi} 25\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 0^2}\ dt$$

$$\int_c f(x, y, z)\ ds = \int_0^{2\pi} 25\sqrt{16\sin^2 t + 16\cos^2 t}\ dt$$

$$\int_c f(x, y, z)\ ds = \int_0^{2\pi} 25\sqrt{16}\ dt$$

$$\int_c f(x, y, z)\ ds = 100\int_0^{2\pi}\ dt$$

$$\int_c f(x, y, z)\ ds = 100\,(2\pi) = 200\pi$$

■ 2. Calculate the line integral P over c, where c is the part of the graph of the vector function $\vec{r}(t)$ between the points $(-2,6,-2)$ and $(4,9,1)$.

$$\vec{r}(t) = \langle 2t, t^2+5, t-1 \rangle$$

$$P = \int_c (y - z^2)\sqrt{5 + x^2}\ ds$$

Solution:

The line integral over the curve for the function $f(x, y, z)$ is given by

$$\int_c f(x, y, z)\ ds = \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

In order to find the value of $t = a$ that corresponds to $(-2,6,-2)$, solve the system of equations for t.

$$2t = -2$$

$$t^2 + 5 = 6$$

$$t - 1 = -2$$

By solving the system, we get $a = t = -1$. In order to find the value of $t = b$ that corresponds to $(4,9,1)$, solve the system of equations for t.

$$2t = 4$$

$$t^2 + 5 = 9$$

$$t - 1 = 1$$

By solving the system, we get $b = t = 2$. Find the first-order derivatives.

$$x'(t) = 2$$

$$y'(t) = 2t$$

$$z'(t) = 1$$

The function is

$$f(x(t), y(t), z(t)) = (y(t) - z(t)^2)\sqrt{5 + x(t)^2}$$

$$f(x(t), y(t), z(t)) = (t^2 + 5 - (t-1)^2)\sqrt{5 + (2t)^2}$$

$$f(x(t), y(t), z(t)) = (2t + 4)\sqrt{5 + 4t^2}$$

Therefore, the line integral over the curve is

$$\int_c f(x, y, z)\ ds = \int_{-1}^{2} (2t + 4)\sqrt{5 + 4t^2}\sqrt{2^2 + (2t)^2 + 1^2}\ dt$$

$$\int_c f(x, y, z)\ ds = \int_{-1}^{2} (2t + 4)\sqrt{5 + 4t^2}\sqrt{5 + 4t^2}\ dt$$

Since $5 + 4t^2 > 0$,

$$\int_c f(x, y, z)\ ds = \int_{-1}^{2} (2t + 4)(5 + 4t^2)\ dt$$

$$\int_c f(x,y,z)\ ds = \int_{-1}^{2} 8t^3 + 16t^2 + 10t + 20\ dt$$

$$\int_c f(x,y,z)\ ds = 2t^4 + \frac{16}{3}t^3 + 5t^2 + 20t\Big|_{-1}^{2}$$

$$\int_c f(x,y,z)\ ds = \left[2(2)^4 + \frac{16}{3}(2)^3 + 5(2)^2 + 20(2)\right] - \left[2(-1)^4 + \frac{16}{3}(-1)^3 + 5(-1)^2 + 20(-1)\right]$$

$$\int_c f(x,y,z)\ ds = \frac{404}{3} + \frac{55}{3} = 153$$

■ 3. Calculate the improper line integral over c, where c is the line of intersection of the surfaces $z - x^2 - y^2 + 2y + 1 = 0$ and $x - y - 1 = 0$.

$$\int_c \frac{1}{(1 + 8(x-1)y)^2}\ ds$$

Solution:

The line integral over the curve for the function $f(x,y,z)$ is given by

$$\int_c f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

Find the parametrization for $f(x(t),y(t),z(t))$. Let $y(t) = t$, then since $x - y - 1 = 0$,

$$x(t) = y(t) + 1 = t + 1$$

Substitute $x(t)$ and $y(t)$ into the equation $z - x^2 - y^2 + 2y + 1 = 0$.

$$z(t) - (t+1)^2 - t^2 + 2t + 1 = 0$$

$$z(t) = (t+1)^2 + t^2 - 2t - 1$$

$$z(t) = 2t^2$$

So the parametrization is

$$x(t) = t + 1$$

$$y(t) = t$$

$$z(t) = 2t^2$$

where t changes from $-\infty$ to ∞. Find the first-order derivatives.

$$x'(t) = 1$$

$$y'(t) = 1$$

$$z'(t) = 4t$$

The function is

$$f(x(t), y(t), z(t)) = \frac{1}{(1 + 8(x(t) - 1)y(t))^2}$$

$$f(x(t), y(t), z(t)) = \frac{1}{(1 + 8(t + 1 - 1)t)^2}$$

$$f(x(t), y(t), z(t)) = \frac{1}{(1 + 8t^2)^2}$$

Therefore, the line integral over the curve is

$$\int_c f(x,y,z)\ ds = \int_{-\infty}^{\infty} \frac{1}{(1+8t^2)^2}\sqrt{1^2+1^2+(4t)^2}\ dt$$

$$\int_c f(x,y,z)\ ds = \int_{-\infty}^{\infty} \frac{\sqrt{2+16t^2}}{(1+8t^2)^2}\ dt$$

$$\int_c f(x,y,z)\ ds = \int_{-\infty}^{\infty} \frac{\sqrt{2}\sqrt{1+8t^2}}{(1+8t^2)^2}\ dt$$

$$\int_c f(x,y,z)\ ds = \sqrt{2}\int_{-\infty}^{\infty} \frac{\sqrt{1+8t^2}}{(1+8t^2)^2}\ dt$$

Since the function under the integral is even,

$$\int_c f(x,y,z)\ ds = 2\sqrt{2}\int_{0}^{\infty} \frac{\sqrt{1+8t^2}}{(1+8t^2)^2}\ dt$$

Make trigonometric substitution with

$$t = \frac{\tan u}{\sqrt{8}} \text{ and } dt = \frac{\sec^2 u}{\sqrt{8}}\ du$$

So $1+8t^2 = 1+\tan^2 u = \sec^2 u$, and u changes from 0 to $\pi/2$.

$$\int_c f(x,y,z)\ ds = 2\sqrt{2}\int_{0}^{\pi/2} \frac{\sqrt{\sec^2 u}}{(\sec^2 u)^2}\left(\frac{\sec^2 u}{\sqrt{8}}\right)\ du$$

$$\int_c f(x,y,z)\ ds = \int_{0}^{\pi/2} \frac{\sec u}{\sec^4 u}\sec^2 u\ du$$

$$\int_c f(x,y,z)\ ds = \int_0^{\pi/2} \frac{1}{\sec u}\ du$$

$$\int_c f(x,y,z)\ ds = \int_0^{\pi/2} \cos u\ du$$

$$\int_c f(x,y,z)\ ds = (\sin u)\Big|_0^{\pi/2} = 1$$

LINE INTEGRAL OF A VECTOR FUNCTION

■ 1. Calculate the line integral of the vector function $\overrightarrow{F}(x, y) = \langle x + y, x - y \rangle$ over the curve $\overrightarrow{r}(t) = \langle t^2 - 1, t^2 + 1 \rangle$ for $-2 \le t \le 3$.

Solution:

The line integral of the vector function $\overrightarrow{F}(x, y)$ over the curve $\overrightarrow{r}(t)$ is given by

$$\int_c \overrightarrow{F}(x, y)\ ds = \int_a^b \overrightarrow{F}(x(t), y(t)) \cdot \overrightarrow{r}'(t)\ dt$$

Plug $x(t) = t^2 - 1$ and $y(t) = t^2 + 1$ into the expression for $F(x, y)$.

$$\overrightarrow{F}(x(t), y(t)) = \langle t^2 - 1 + t^2 + 1, t^2 - 1 - (t^2 + 1) \rangle$$

$$\overrightarrow{F}(x(t), y(t)) = \langle 2t^2, -2 \rangle$$

Find the first-order derivative.

$$\overrightarrow{r}'(t) = \langle 2t, 2t \rangle$$

The dot product is

$$\overrightarrow{F}(x(t), y(t)) \cdot \overrightarrow{r}'(t) = \langle 2t^2, -2 \rangle \cdot \langle 2t, 2t \rangle$$

$$\overrightarrow{F}(x(t), y(t)) \cdot \overrightarrow{r}'(t) = 2t^2(2t) - 2(2t)$$

$$\overrightarrow{F}(x(t), y(t)) \cdot \overrightarrow{r}'(t) = 4t^3 - 4t$$

Therefore, the line integral over the curve is

$$\int_c F(x,y)\,ds = \int_{-2}^{3} 4t^3 - 4t\ dt$$

$$\int_c F(x,y)\,ds = t^4 - 2t^2\Big|_{-2}^{3}$$

$$\int_c F(x,y)\,ds = [3^4 - 2(3)^2] - [(-2)^4 - 2(-2)^2] = 55$$

■ 2. Calculate the line integral of the vector function $\vec{F}(x,y,z) = \langle xyz, -z, y\rangle$ over c, where c is the ellipse that lies in the plane $x = -4$ with the center on the x-axis, a semi-axis of 2 in the y-direction, and a semi-axis of 5 in the z -direction.

Solution:

The line integral of the vector function $\vec{F}(x,y,z)$ over the curve $\vec{r}(t)$ is given by the formula

$$\int_c \vec{F}(x,y,z)\ ds = \int_a^b \vec{F}(x(t),y(t),z(t)) \cdot \vec{r}\,'(t)\ dt$$

where $\vec{F}(x(t),y(t),z(t)) \cdot \vec{r}\,'(t)$ is the dot product of the two vectors.

Parametrize the ellipse.

$$x(t) = -4$$

$$y(t) = 2\cos t$$

$$z(t) = 5\sin t$$

Over the whole ellipse, t changes from 0 to 2π, so $a = 0$ and $b = 2\pi$.

Plug $x(t) = -4$, $y(t) = 2\cos t$, and $z(t) = 5\sin t$ into the expression for $F(x, y, z)$.

$$\vec{F}(x(t), y(t), z(t)) = \langle -4 \cdot 2\cos t \cdot 5\sin t, -5\sin t, 2\cos t \rangle$$

$$\vec{F}(x(t), y(t), z(t)) = \langle -40\cos t \sin t, -5\sin t, 2\cos t \rangle$$

$$\vec{F}(x(t), y(t), z(t)) = \langle -20\sin 2t, -5\sin t, 2\cos t \rangle$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}\,'(t) = \langle 0, -2\sin t, 5\cos t \rangle$$

The dot product is

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t) = \langle -20\sin 2t, -5\sin t, 2\cos t \rangle \cdot \langle 0, -2\sin t, 5\cos t \rangle$$

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t) = 10\sin^2 t + 10\cos^2 t$$

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t) = 10$$

Therefore, the line integral over the curve is

$$\int_C F(x, y, z)\ ds = \int_0^{2\pi} 10\ dt = 10(2\pi) = 20\pi$$

■ 3. Calculate the improper line integral of the vector function $\vec{F}(x, y, z)$ over the curve $\vec{r}(t) = \langle e^t, -e^{-t}, 2t \rangle$ for $t \geq 0$.

$$\vec{F}(x, y, z) = \left\langle y^2, \frac{3}{x^2}, 2xy^2z \right\rangle$$

Solution:

The line integral of the vector function $\vec{F}(x, y, z)$ over the curve $\vec{r}(t)$ is given by

$$\int_C \vec{F}(x, y, z)\ ds = \int_a^b \vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)\ dt$$

where $\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t)$ is the dot product of the two vectors.

Plug $x(t) = e^t$, $y(t) = -e^{-t}$, and $z(t) = 2t$ into the expression for $F(x, y, z)$.

$$\vec{F}(x(t), y(t), z(t)) = \left\langle (-e^{-t})^2, \frac{3}{(e^t)^2}, 2e^t \cdot (-e^{-t})^2 \cdot 2t \right\rangle$$

$$\vec{F}(x(t), y(t), z(t)) = \left\langle e^{-2t}, \frac{3}{e^{2t}}, 4te^{-t} \right\rangle$$

$$\vec{F}(x(t), y(t), z(t)) = \langle e^{-2t}, 3e^{-2t}, 4te^{-t} \rangle$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}\,'(t) = \langle e^t, e^{-t}, 2 \rangle$$

The dot product is

$$\vec{F}(x(t),y(t),z(t))\cdot\vec{r}'(t)=\langle e^{-2t},3e^{-2t},4te^{-t}\rangle\cdot\langle e^{t},e^{-t},2\rangle$$

$$\vec{F}(x(t),y(t),z(t))\cdot\vec{r}'(t)=e^{-2t}e^{t}+3e^{-2t}e^{-t}+8te^{-t}$$

$$\vec{F}(x(t),y(t),z(t))\cdot\vec{r}'(t)=e^{-t}+3e^{-3t}+8te^{-t}$$

Therefore, the line integral over the curve is

$$\int_c F(x,y,z)\ ds=\int_0^\infty e^{-t}+3e^{-3t}+8te^{-t}\ dt$$

$$\int_c F(x,y,z)\ ds=\int_0^\infty e^{-t}\ dt+3\int_0^\infty e^{-3t}\ dt+8\int_0^\infty te^{-t}\ dt$$

Take the integral of te^{-t} using integration by parts with $u=t$, $du=dt$, $dv=e^{-t}\ dt$, and $v=-e^{-t}$.

$$\int te^{-t}\ dt=-te^{-t}+\int e^{-t}\ dt$$

So

$$\int_0^\infty e^{-t}\ dt+3\int_0^\infty e^{-3t}\ dt+8\int_0^\infty te^{-t}\ dt$$

$$-e^{-t}\Big|_0^\infty+(-e^{-3t})\Big|_0^\infty+8(-te^{-t})\Big|_0^\infty+8(-e^{-t})\Big|_0^\infty$$

$$9(-e^{-t})\Big|_0^\infty+(-e^{-3t})\Big|_0^\infty+8(-te^{-t})\Big|_0^\infty$$

$$9\lim_{t\to\infty}(-e^{-t})-9(-e^{0})+\lim_{t\to\infty}(-e^{-3t})-(-e^{0})+8\lim_{t\to\infty}(-te^{-t})-8(-0\cdot e^{0})$$

$$9 \cdot 0 + 9 + 0 + 1 + 8 \cdot 0 - 8 \cdot 0 = 10$$

POTENTIAL FUNCTION OF A CONSERVATIVE VECTOR FIELD

■ 1. Determine whether or not the vector field is conservative.

$$\vec{F}(x, y, z) = \left\langle \ln(2y + z), \frac{2x}{2y + z}, \frac{x}{2y + z} \right\rangle$$

Solution:

If a vector field $F: R^3 \to R^3$ is continuously differentiable in a simply-connected domain $W \in R^3$ and its curl is zero, then F is conservative within the domain W.

$$\vec{F}(x, y, z) = \langle F_x, F_y, F_z \rangle$$

Recall that the curl of a vector field in three dimensions is given by

$$\text{curl } F = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

In other words, curl $F = \vec{0}$ if

$$\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z}$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

The domain of the vector field is the half-space $2y + z > 0$, which is a simply-connected space.

Check if the curl is zero. We get

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{2y+z}\right) = -\frac{2x}{(2y+z)^2}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{2x}{2y+z}\right) = -\frac{2x}{(2y+z)^2}$$

and

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(\ln(2y+z)) = \frac{1}{2y+z}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{2y+z}\right) = \frac{1}{2y+z}$$

and

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2x}{2y+z}\right) = \frac{2}{2y+z}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(\ln(2y+z)) = \frac{2}{2y+z}$$

Because each set of partial derivatives is equivalent, the curl is 0.

2. Find the potential function of the vector field.

$$\vec{F}(x, y) = \langle \cos(x - 3y) + 5, -3\cos(x - 3y) - 8 \rangle$$

Solution:

A potential function $f(x, y)$ of a vector field $\vec{F}(x, y)$ satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y) = F_x(x, y) \text{ and } \frac{\partial f}{\partial y}(x, y) = F_y(x, y)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x, y) = \cos(x - 3y) + 5$$

Integrate both sides with respect to x, treating y as a constant.

$$f(x, y) = \int \cos(x - 3y) + 5 \ dx$$

$$f(x, y) = \sin(x - 3y) + 5x + C(y)$$

Differentiate $f(x, y)$ with respect to y, treating x as a constant.

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}\left(\sin(x - 3y) + 5x + C(y)\right) = -3\cos(x - 3y) + C'(y)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y) = -3\cos(x - 3y) - 8$$

$$-3\cos(x-3y)+C'(y)=-3\cos(x-3y)-8$$

$$C'(y)=-8$$

Integrate both sides with respect to y.

$$C(y)=\int -8\ dy=-8y+c$$

Therefore,

$$f(x,y)=\sin(x-3y)+5x-8y+c$$

For any conservative vector field, there exist an infinite number of possible potential functions, which each vary by an additive constant c.

■ 3. Find the potential function of the vector field.

$$\vec{F}(x,y,z)=\langle z^2\,2^{x+4y}\ln 2,\ z^2\,2^{x+4y+2}\ln 2,\ z\,2^{x+4y+1}-6z^2\rangle$$

Solution:

A potential function $f(x,y,z)$ of a vector field $\vec{F}(x,y,z)$ satisfies the equality $\nabla f=\vec{F}$, or

$$\frac{\partial f}{\partial x}(x,y,z)=F_x(x,y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z)=F_y(x,y,z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x, y, z) = z^2\, 2^{x+4y} \ln 2$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x, y, z) = \int z^2\, 2^{x+4y} \ln 2 \ dx$$

$$f(x, y, z) = z^2\, 2^{4y} \int 2^x \ln 2 \ dx$$

$$f(x, y, z) = z^2\, 2^{4y} \cdot 2^x + C(y, z)$$

So

$$f(x, y, z) = z^2\, 2^{x+4y} + C(y, z)$$

Differentiate $f(x, y, z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}(z^2\, 2^{x+4y} + C(y, z))$$

$$\frac{\partial f}{\partial y}(x, y, z) = 4z^2\, 2^{x+4y} \ln 2 + \frac{\partial C}{\partial y}(y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = z^2\, 2^{x+4y+2} \ln 2 + \frac{\partial C}{\partial y}(y, z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x,y,z) = z^2\, 2^{x+4y+2} \ln 2$$

$$z^2\, 2^{x+4y+2} \ln 2 + \frac{\partial C}{\partial y}(y,z) = z^2\, 2^{x+4y+2} \ln 2$$

$$\frac{\partial C}{\partial y}(y,z) = 0$$

Therefore, $C(y,z)$ is a constant in terms of y, or

$$C(y,z) = C(z)$$

So

$$f(x,y,z) = z^2\, 2^{x+4y} + C(z)$$

Similarly, differentiate $f(x,y,z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}(z^2\, 2^{x+4y} + C(z))$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2z\, 2^{x+4y} + C'(z)$$

$$\frac{\partial f}{\partial z}(x,y,z) = z\, 2^{x+4y+1} + C'(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x,y,z) = z\, 2^{x+4y+1} - 6z^2$$

So

$$z\,2^{x+4y+1} + C'(z) = z\,2^{x+4y+1} - 6z^2$$

$$C'(z) = -6z^2$$

Integrate both sides with respect to z.

$$C(z) = \int -6z^2\ dz$$

$$C(z) = -2z^3 + c$$

Therefore,

$$f(x, y, z) = z^2\,2^{x+4y} - 2z^3 + c$$

For any conservative vector field, there exist an infinite number of possible potential functions, which vary by an additive constant c.

POTENTIAL FUNCTION OF A CONSERVATIVE VECTOR FIELD TO EVALUATE A LINE INTEGRAL

■ 1. Calculate the line integral of the conservative vector field $\vec{F}(x, y)$ over the curve $\vec{r}(t) = \langle 9 \arctan^2 t, t^4 - 2t^2 + 2\rangle$ between $(0,2)$ and $(\pi^2,5)$.

$$\vec{F}(x, y) = \left\langle \frac{y}{\sqrt{x}}, 2(y + \sqrt{x}) \right\rangle$$

Solution:

The line integral of the conservative vector field is independent of the curve, and can be calculated as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f(\vec{r_2}) - f(\vec{r_1})$$

Find the potential function $f(x, y)$ of a vector field $\vec{F}(x, y)$ that satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y) = F_x(x, y) \text{ and } \frac{\partial f}{\partial y}(x, y) = F_y(x, y)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x, y) = \frac{y}{\sqrt{x}}$$

Integrate both sides with respect to x, treating y as a constant.

$$f(x, y) = \int \frac{y}{\sqrt{x}}\ dx$$

$$f(x, y) = 2y\sqrt{x} + C(y)$$

Differentiate $f(x, y)$ with respect to y, treating x as a constant.

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}(2y\sqrt{x} + C(y)) = 2\sqrt{x} + C'(y)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y) = 2(y + \sqrt{x})$$

So

$$2\sqrt{x} + C'(y) = 2(y + \sqrt{x})$$

$$C'(y) = 2y$$

Integrate both sides with respect to y.

$$C(y) = \int 2y\ dy = y^2 + c$$

Therefore,

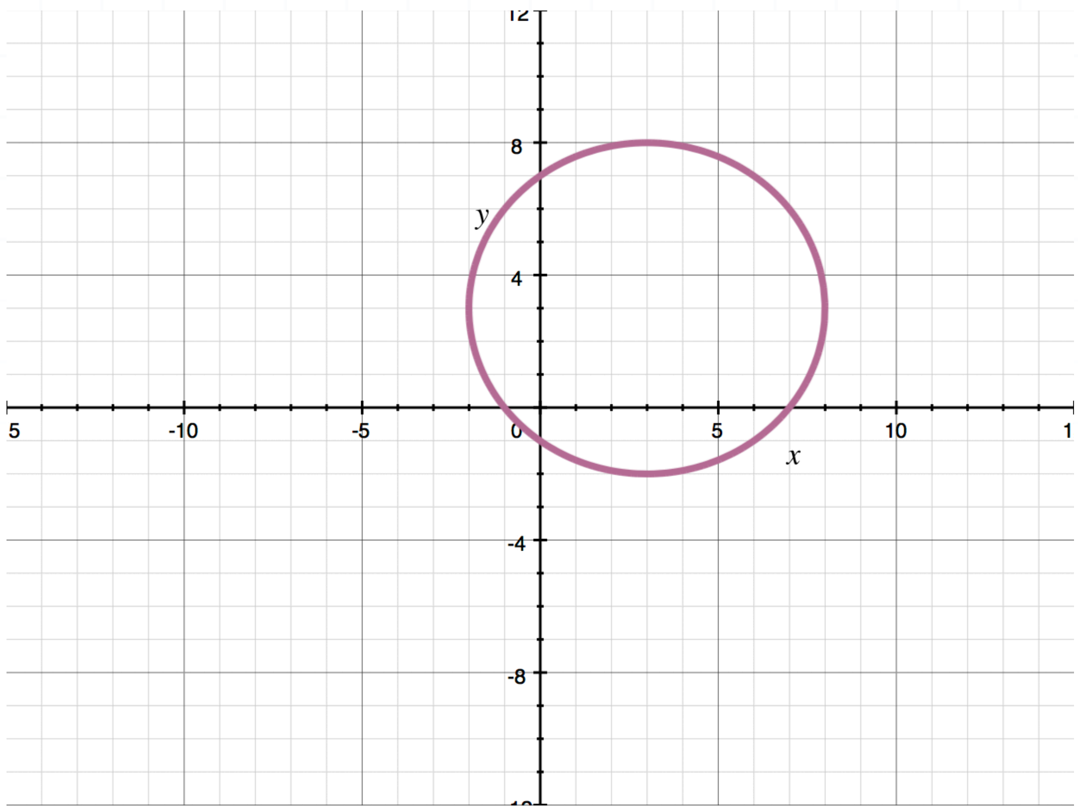
$$f(x, y) = 2y\sqrt{x} + y^2 + c$$

So the line integral is

$$\int_c \vec{F} \cdot d\vec{r} = f(\pi^2,5) - f(0,2)$$

$$\int_c \vec{F} \cdot d\vec{r} = 2 \cdot 5\sqrt{\pi^2} + 5^2 + c - (2 \cdot 2\sqrt{0} + 2^2 + c) = 10\pi + 21$$

■ 2. Calculate the line integral of the conservative vector field $\vec{F}(x,y) = \langle x^2 + y^2, 2xy + 1 \rangle$ over the part of the circle with center at $(3,3)$ and radius 5, that lies in the first quadrant, with clockwise rotation.



Solution:

The line integral of the conservative vector field is independent of the curve, and can be calculated as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f(\vec{r_2}) - f(\vec{r_1})$$

Let's find the potential function $f(x, y)$ of a vector field $\vec{F}(x, y)$ that satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y) = F_x(x, y) \text{ and } \frac{\partial f}{\partial y}(x, y) = F_y(x, y)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x, y) = x^2 + y^2$$

Integrate both sides with respect to x, treating y as a constant.

$$f(x, y) = \int x^2 + y^2 \ dx$$

$$f(x, y) = \frac{x^3}{3} + xy^2 + C(y)$$

Differentiate $f(x, y)$ with respect to y, treating x as a constant.

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}\left(\frac{x^3}{3} + xy^2 + C(y)\right) = 2xy + C'(y)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y) = 2xy + 1$$

So

$$2xy + C'(y) = 2xy + 1$$

$$C'(y) = 1$$

Integrate both sides with respect to y.

$$C(y) = \int 1 \ dy = y + c$$

Therefore,

$$f(x, y) = \frac{x^3}{3} + xy^2 + y + c$$

The initial point of the curve is $(0,7)$, and the terminal point is $(7,0)$, so the line integral is

$$\int_c \vec{F} \cdot d\vec{r} = f(7,0) - f(0,7)$$

$$\int_c \vec{F} \cdot d\vec{r} = \frac{7^3}{3} + 7 \cdot 0^2 + 0 + c - \left(\frac{0^3}{3} + 0 \cdot 7^2 + 7 + c\right) = \frac{322}{3}$$

■ 3. Calculate the line integral of the conservative vector field $\vec{F}(x, y, z) = \langle y^2, 2xy, (1 + z)^{-1} \rangle$ over the curve $\vec{r}(t) = \langle \sin(\pi t^2), t^3 e^{t-1}, (t - 2)^2 \rangle$ for $1 \le t \le 2$.

Solution:

The line integral of the conservative vector field is independent of the curve, and can be calculated as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f(\vec{r_2}) - f(\vec{r_1})$$

Find the potential function $f(x, y, z)$ of a vector field $\vec{F}(x, y, z)$ that satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y, z) = F_x(x, y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x, y, z) = y^2$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x, y, z) = \int y^2\ dx = xy^2 + C(y, z)$$

Differentiate $f(x, y, z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}(xy^2 + C(y, z))$$

$$\frac{\partial f}{\partial y}(x, y, z) = 2xy + \frac{\partial C}{\partial y}(y, z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y, z) = 2xy$$

So

$$2xy + \frac{\partial C}{\partial y}(y, z) = 2xy$$

$$\frac{\partial C}{\partial y}(y, z) = 0$$

Therefore, $C(y, z)$ is a constant in terms of y, or

$$C(y, z) = C(z)$$

So

$$f(x, y, z) = xy^2 + C(z)$$

Similarly, differentiate $f(x, y, z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{\partial}{\partial z}(xy^2 + C(z)) = C'(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x, y, z) = (1 + z)^{-1}$$

So

$$C'(z) = (1 + z)^{-1}$$

Integrate both sides with respect to z.

$$C(z) = \int (1+z)^{-1}\ dz$$

$$C(z) = \ln(1+z) + c$$

Therefore,

$$f(x, y, z) = xy^2 + \ln(1+z) + c$$

Calculate the endpoints of the curve.

$$\vec{r}(1) = \langle \sin(\pi \cdot 1^2), 1^3e^{1-1}, (1-2)^2 \rangle = \langle 0,1,1 \rangle$$

$$\vec{r}(2) = \langle \sin(\pi \cdot 2^2), 2^3e^{2-1}, (2-2)^2 \rangle = \langle 0,8e,0 \rangle$$

So the line integral is

$$\int_c \vec{F} \cdot d\vec{r} = f(0,8e,0) - f(0,1,1)$$

$$\int_c \vec{F} \cdot d\vec{r} = 0 \cdot (8e)^2 + \ln(1+0) + c - (0 \cdot 1^2 + \ln(1+1) + c)$$

$$\int_c \vec{F} \cdot d\vec{r} = -\ln 2$$

INDEPENDENCE OF PATH

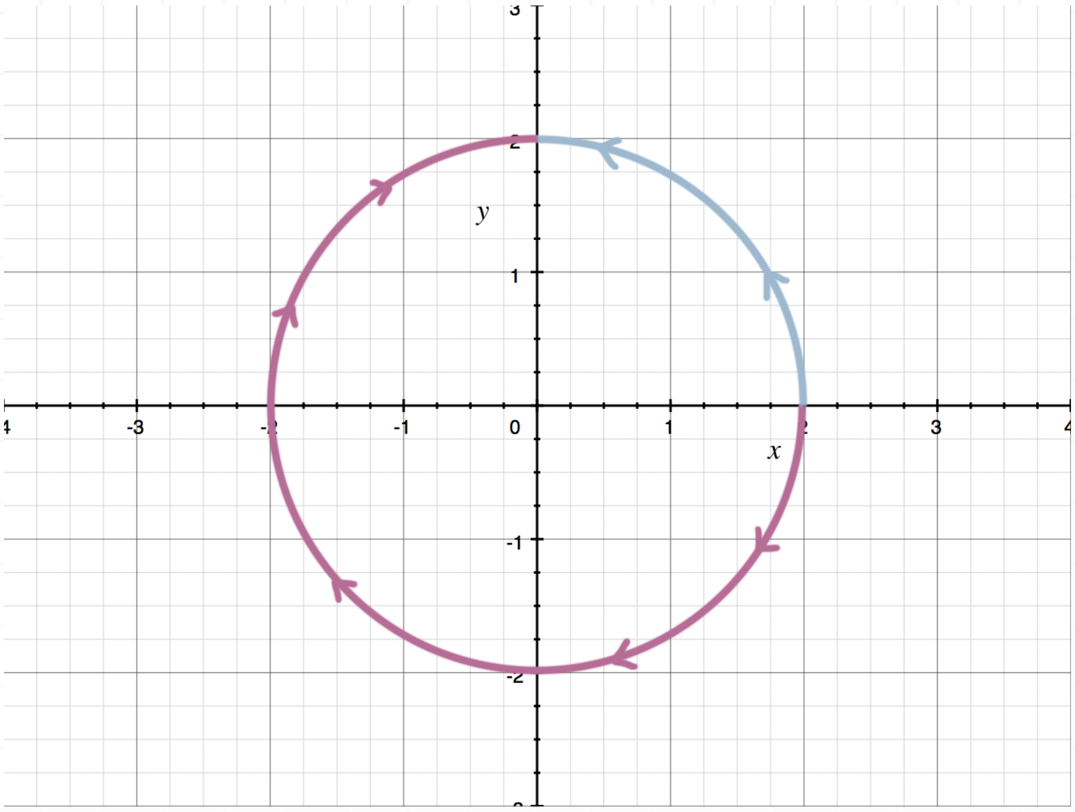
■ 1. Check if the line integral of the vector field $\vec{F}(x,y)$ is independent of path for any curve connecting the points $(2,0)$ and $(0,2)$. If it *is* independent of path, then prove it. If not, give a counterexample.

$$\vec{F}(x,y)=\left\langle\frac{4y}{x^2+y^2},\frac{-4x}{x^2+y^2}\right\rangle$$

Solution:

It's easy to check that curl $F=0$, but since the domain of the vector field has a hole at $(0,0)$, this vector field more probably is not conservative.

To give a counterexample, we can compute the line integrals over two curves that lie on different sides of the hole at $(0,0)$. Consider the circle with center at $(0,0)$ and radius 2, which passes through $(2,0)$ and $(0,2)$. Then, consider two paths between these points along the circle, clockwise and counterclockwise.



Consider the parametrization of the circle.

$$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$$

For the clockwise path, t changes from 0 to $-3\pi/2$, and for the counterclockwise path, t changes from 0 to $\pi/2$.

The line integral of the vector function $\vec{F}(x, y)$ over the curve $\vec{r}(t)$ is given by

$$\int_C \vec{F}(x, y)\ ds = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\ dt$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}\,'(t) = \langle -2\sin t, 2\cos t \rangle$$

The function is

$$F(x(t),y(t)) = \left\langle \frac{4(2\sin t)}{(2\cos t)^2 + (2\sin t)^2}, -\frac{4(2\cos t)}{(2\cos t)^2 + (2\sin t)^2} \right\rangle$$

$$F(x(t),y(t)) = \left\langle \frac{8\sin t}{4(\cos^2 t + \sin^2 t)}, -\frac{8\cos t}{4(\cos^2 t + \sin^2 t)} \right\rangle$$

$$F(x(t),y(t)) = \left\langle \frac{8\sin t}{4}, -\frac{8\cos t}{4} \right\rangle$$

$$F(x(t),y(t)) = \langle 2\sin t, -2\cos t \rangle$$

The dot product is

$$\vec{F}(x(t),y(t)) \cdot \vec{r}'(t) = \langle 2\sin t, -2\cos t \rangle \cdot \langle -2\sin t, 2\cos t \rangle$$

$$\vec{F}(x(t),y(t)) \cdot \vec{r}'(t) = -4\sin^2 t - 4\cos^2 t$$

$$\vec{F}(x(t),y(t)) \cdot \vec{r}'(t) = -4$$

So the line integral over the clockwise circle is

$$\int_{cw} F(x,y)\ ds = \int_0^{-3\pi/2} -4\ dt = -4\left(-\frac{3\pi}{2}\right) = 6\pi$$

and the line integral over the counterclockwise circle is

$$\int_{ccw} F(x,y)\ ds = \int_0^{\pi/2} -4\ dt = -4\left(\frac{\pi}{2}\right) = -2\pi$$

Since these integrals over different paths aren't equal, the line integral of the vector field is dependent of path.

■ 2. Check if the line integral of the vector field $\vec{F}(x, y)$ is independent of path for any curve that lies within the rectangle given by $1 < x < 5$ and $1 < y < 5$, and that connects the points $(2,4)$ and $(4,2)$.

$$\vec{F}(x, y) = \left\langle \frac{2(x-1)}{(x^2 - 2x + y^2 + 1)^2}, \frac{2y}{(x^2 - 2x + y^2 + 1)^2} \right\rangle$$

Solution:

Find the domain of the vector field.

$$(x^2 - 2x + y^2 + 1)^2 \neq 0$$

$$x^2 - 2x + y^2 + 1 \neq 0$$

$$(x-1)^2 + y^2 \neq 0$$

So the domain of the vector field is all points except the point $(1,0)$. Since the domain is not simply-connected, the vector field is not conservative over the domain, and therefore the line integral of the vector field is dependent of path. But since the point $(1,0)$ lies outside the rectangle given by $1 < x < 5$ and $1 < y < 5$, which is an open simply-connected set, the vector field is likely conservative within the rectangle. Let's prove that.

We know that if a vector field $F : R^2 \to R^2$ is continuously differentiable in a simply-connected domain $W \in R^2$ (or differentiable in an open simply-connected domain) and its curl is 0, then F is conservative within the domain W.

The vector field is differentiable on the open rectangle, so we just need to check its curl. Recall that curl of a vector field in two dimensions is given by

$$\text{curl}\, F = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial t}$$

Find partial derivatives.

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2y}{(x^2 - 2x + y^2 + 1)^2}\right) = -\frac{8(x-1)y}{(x^2 - 2x + y^2 + 1)^3}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2(x-1)}{(x^2 - 2x + y^2 + 1)^2}\right) = -\frac{8(x-1)y}{(x^2 - 2x + y^2 + 1)^3}$$

So the vector field is conservative in the given rectangle and therefore its line integral is independent of path.

■ 3. Determine whether the line integral of the vector field $\overrightarrow{F}(x, y, z)$ is independent of path for any curve that connects any two points within the vector field's domain.

$$\overrightarrow{F}(x, y, z) = \langle x \ln(x^2 + y^2 + z^2 - 9), y \ln(x^2 + y^2 + z^2 - 9), z \ln(x^2 + y^2 + z^2 - 9)\rangle$$

Solution:

Find the domain of the vector field.

$$x^2 + y^2 + z^2 - 9 > 0$$

$$x^2 + y^2 + z^2 > 3^2$$

So the domain of the vector field $\overrightarrow{F}$ is the set of all points outside the sphere centered at the origin with radius 3.

We know that if a vector field $F: R^3 \to R^3$ is continuously differentiable in a simply-connected domain $W \in R^3$ (or differentiable in an open simply-connected domain) and its curl is 0, then F is conservative within the domain W.

The set is called simply-connected if every path between any two points can be continuously transformed staying within the set into any other such path between the points. So the domain of the vector field $\overrightarrow{F}$ is simply-connected.

The set R^2 other than the circle, *is not* simply connected, but for any $n > 2$, the set R^n other than the sphere, *is* simply connected.

The domain is an open set, and the vector field is differentiable over the entire domain. Let's determine whether or not the curl of $\overrightarrow{F}$ is 0.

Remember that the curl of a vector field in three dimensions is given by

$$\text{curl}\, F = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

In other words, curl $F = \overrightarrow{0}$ if

$$\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z}$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

If we verify that each of these partial derivative equations are true, we find

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(z \ln(x^2 + y^2 + z^2 - 9)) = \frac{2yz}{x^2 + y^2 + z^2 - 9}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(y \ln(x^2 + y^2 + z^2 - 9)) = \frac{2yz}{x^2 + y^2 + z^2 - 9}$$

and

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(x \ln(x^2 + y^2 + z^2 - 9)) = \frac{2xz}{x^2 + y^2 + z^2 - 9}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(z \ln(x^2 + y^2 + z^2 - 9)) = \frac{2xz}{x^2 + y^2 + z^2 - 9}$$

and

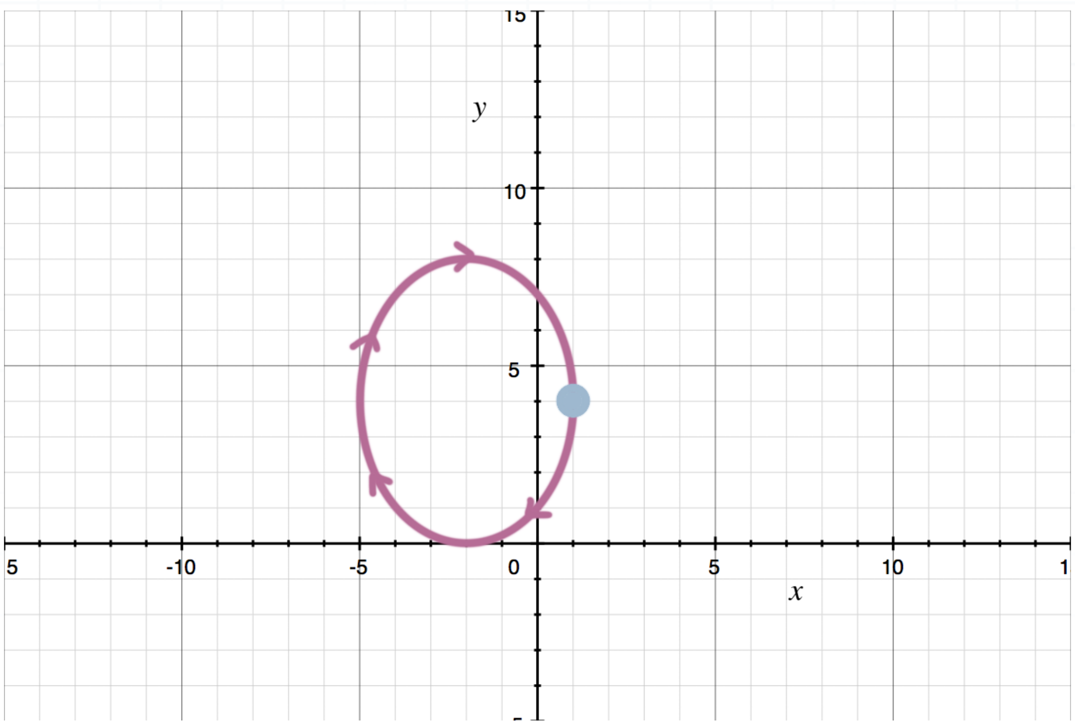
$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(y \ln(x^2 + y^2 + z^2 - 9)) = \frac{2xy}{x^2 + y^2 + z^2 - 9}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(x \ln(x^2 + y^2 + z^2 - 9)) = \frac{2xy}{x^2 + y^2 + z^2 - 9}$$

Therefore, since the vector field is differentiable in the open simply-connected domain and its curl is 0, it's conservative and therefore not dependent of path.

WORK DONE BY A FORCE FIELD

■ 1. Calculate the work done by the force field $\vec{F}(x, y) = \langle 25x^2 + 9y^2 + 1, x - y - 3 \rangle$ to move an object clockwise along the ellipse centered at $(-2,4)$ with semi-axis of 3 in the x-direction and semi-axis of 5 in the y-direction.



Solution:

The work done by the force field $\vec{F}(x, y)$ to move an object along the curve $\vec{r}(t)$ is equal to the line integral of the vector function $\vec{F}(x, y)$ over the curve $\vec{r}(t)$.

$$\int_C \vec{F}(x, y)\ ds = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\ dt$$

The work done by the conservative force field along the closed curve is always 0. Since the given force field $\vec{F}(x,y)$ is not conservative, we need to calculate the work done directly by definition.

The standard parametrization of the ellipse with the center at (x_0, y_0), semi-axis of a in the x-direction, and semi-axis of b in the y-direction, is

$$\vec{r}(t) = \langle x_0 + a\cos t, y_0 + b\sin t\rangle$$

Plug in the values $(x_0, y_0) = (-2,4)$, $a = 3$, and $b = 5$.

$$\vec{r}(t) = \langle -2 + 3\cos t, 4 + 5\sin t\rangle$$

Since the work done by the force field to move an object along the closed curve is independent of the initial point, let's choose the point $(1,4)$, which corresponds to the value $t = 0$. Since the object is moved clockwise, t changes from 0 to -2π.

Plug $x(t) = -2 + 3\cos t$ and $y(t) = 4 + 5\sin t$ into the expression for $F(x,y)$.

$$\vec{F}(x(t), y(t)) = \langle 25(-2 + 3\cos t)^2 + 9(4 + 5\sin t)^2 + 1, -2 + 3\cos t - (4 + 5\sin t) - 3\rangle$$

$$\vec{F}(x(t), y(t)) = \langle -300\cos t + 360\sin t + 470, 3\cos t - 5\sin t - 9\rangle$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}\,'(t) = \langle -3\sin t, 5\cos t\rangle$$

The dot product is

$$\vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t) = \langle -300\cos t + 360\sin t + 470, 3\cos t - 5\sin t - 9\rangle$$

$$\cdot \langle -3\sin t, 5\cos t\rangle$$

$$\vec{F}(x(t),y(t)) \cdot \vec{r}'(t) = -3\sin t(-300\cos t + 360\sin t + 470)$$

$$+5\cos t(3\cos t - 5\sin t - 9)$$

$$\vec{F}(x(t),y(t)) \cdot \vec{r}'(t) = -1{,}080\sin^2 t + 15\cos^2 t - 1{,}410\sin t - 45\cos t + 875\cos t\sin t$$

Therefore, the line integral over the curve is

$$\int_0^{-2\pi} -1{,}080\sin^2 t + 15\cos^2 t - 1{,}410\sin t - 45\cos t + 875\cos t\sin t\ dt$$

$$\int_0^{-2\pi} -540(1-\cos 2t) + \frac{15}{2}(\cos 2t + 1) - 1{,}410\sin t - 45\cos t + \frac{875}{2}\sin 2t\ dt$$

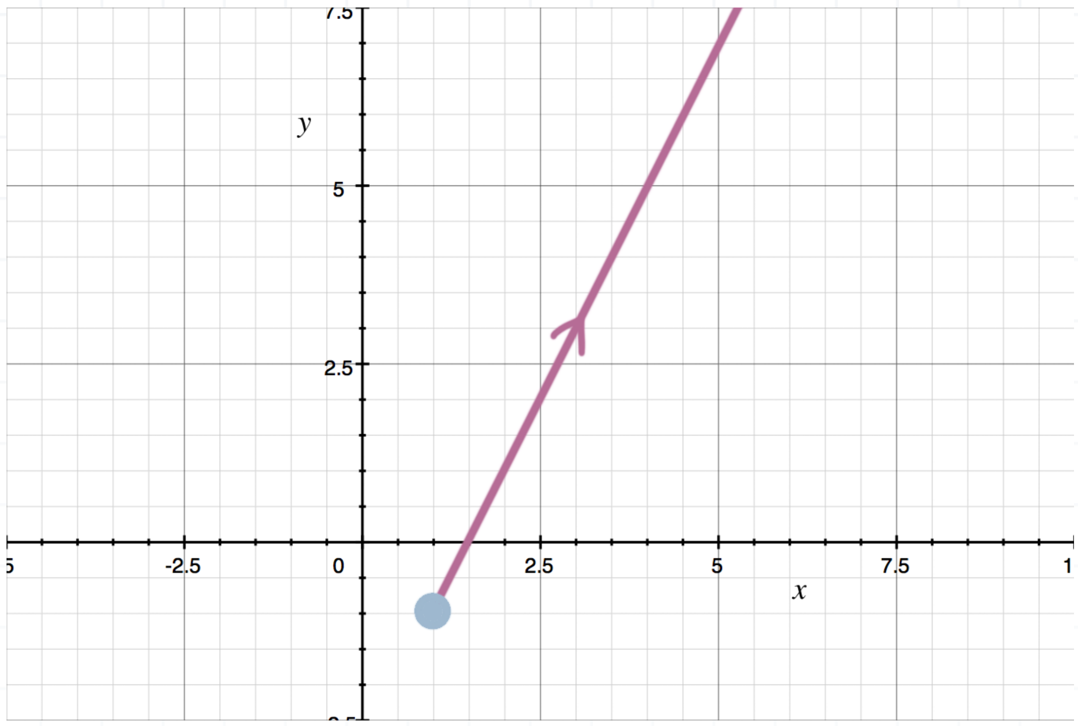
$$\int_0^{-2\pi} -1{,}410\sin t + \frac{875}{2}\sin 2t - 45\cos t + \frac{1{,}095}{2}\cos 2t - \frac{1{,}065}{2}\ dt$$

Since the integral of sine and cosine functions over a 2π period is always 0,

$$\int_0^{-2\pi} -\frac{1{,}065}{2}\ dt = -\frac{1{,}065}{2}(-2\pi) = 1{,}065\pi$$

■ 2. Find the work done by the force field $\vec{F}(x,y)$ to move an object infinitely along the line $y = 2x - 3$, starting from $(1,-1)$, in the positive direction of x.

$$\vec{F}(x,y) = \left\langle xe^{-y}, \frac{y+2}{x^3} \right\rangle$$



Solution:

The work done by the force field $\vec{F}(x, y)$ to move an object along the curve $\vec{r}(t)$ is equal to the line integral of the vector function $\vec{F}(x, y)$ over the curve $\vec{r}(t)$.

$$\int_C \vec{F}(x, y)\ ds = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t)\ dt$$

Consider the parametrization of the line $y = 2x - 3$ as $\vec{r}(t) = \langle t, 2t - 3 \rangle$.

Plug $x(t) = t$ and $y(t) = 2t - 3$ into the expression for $F(x, y)$.

$$\vec{F}(x(t), y(t)) = \left\langle te^{-(2t-3)}, \frac{2t - 3 + 2}{t^3} \right\rangle$$

$$\vec{F}(x(t), y(t)) = \left\langle te^{-2t+3}, \frac{2t - 1}{t^3} \right\rangle$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}'(t) = \langle 1, 2 \rangle$$

The dot product is

$$\vec{F}(x(t), y(t)) \cdot \vec{r}'(t) = \left\langle te^{-2t+3}, \frac{2t-1}{t^3} \right\rangle \cdot \langle 1, 2 \rangle$$

$$\vec{F}(x(t), y(t)) \cdot \vec{r}'(t) = 1 \cdot te^{-2t+3} + 2 \cdot \frac{2t-1}{t^3}$$

$$\vec{F}(x(t), y(t)) \cdot \vec{r}'(t) = te^{-2t+3} + 4t^{-2} - 2t^{-3}$$

Therefore, the line integral over the half-line is

$$\int_c F(x, y)\ ds = \int_1^\infty te^{-2t+3} + 4t^{-2} - 2t^{-3}\ dt$$

$$\int_c F(x, y)\ ds = \int_1^\infty te^{-2t+3}\ dt + \int_1^\infty 4t^{-2} - 2t^{-3}\ dt$$

Take the integral of te^{-2t+3} using integration by parts with $u = t$, $du = dt$, $dv = e^{-2t+3}\ dt$, and $v = -\frac{1}{2}e^{-2t+3}$.

$$\int te^{-2t+3}\ dt = -\frac{t}{2}e^{-2t+3} + \int \frac{1}{2}e^{-2t+3}\ dt$$

So

$$\int_1^\infty te^{-2t+3}\ dt + \int_1^\infty 4t^{-2} - 2t^{-3}\ dt$$

$$-\frac{t}{2}e^{-2t+3}\Big|_1^{\infty}+\int_1^{\infty}\frac{1}{2}e^{-2t+3}\ dt+\int_1^{\infty}4t^{-2}-2t^{-3}\ dt$$

$$-\frac{t}{2}e^{-2t+3}\Big|_1^{\infty}+\left(-\frac{1}{4}e^{-2t+3}\right)\Big|_1^{\infty}+\left(-\frac{4}{t}+\frac{1}{t^2}\right)\Big|_1^{\infty}$$

$$\lim_{t\to\infty}\left(-\frac{t}{2}e^{-2t+3}\right)-\left(-\frac{1}{2}e^{-2+3}\right)+\lim_{t\to\infty}\left(-\frac{1}{4}e^{-2t+3}\right)-\left(-\frac{1}{4}e^{-2+3}\right)$$

$$+\lim_{t\to\infty}\left(-\frac{4}{t}+\frac{1}{t^2}\right)-\left(-\frac{4}{1}+\frac{1}{1^2}\right)$$

$$0+\frac{e}{2}+0+\frac{e}{4}+0+3=3+\frac{3e}{4}$$

■ 3. Find the work done by the conservative force field $\vec{F}(x,y,z)$ to move an object between the four points $A(0,-1,2)$, $B(1,1,3)$, $C(2,3,0)$, and $D(0,2,1)$ (starting from A to B, then to C, and finally to D).

$$\vec{F}(x,y,z)=\langle 1+4x+yz+3z^2,xz-1,x(y+6z)\rangle$$

Solution:

The work done by the force field $\vec{F}$ to move an object along the path is equal to the line integral of the vector function $\vec{F}$ along this path. Since the given vector field is conservative, the line integral is independent of path, and therefore the work done to move the object between the points A, B,

C, and D is equal to the work done to move the object from point A to point D.

The line integral of the conservative vector field over the segment AD can be calculated as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f_D - f_A = f(0,2,1) - f(0,-1,2)$$

where f is the potential function of the vector field, $\nabla f = \vec{F}$. In other words,

$$\frac{\partial f}{\partial x}(x, y, z) = F_x(x, y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

Find the potential function of given vector field. From the first equation,

$$\frac{\partial f}{\partial x}(x, y, z) = 1 + 4x + yz + 3z^2$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x, y, z) = \int 1 + 4x + yz + 3z^2 \ dx = x + 2x^2 + xyz + 3xz^2 + C(y, z)$$

Differentiate $f(x, y, z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}(x + 2x^2 + xyz + 3xz^2 + C(y, z))$$

$$\frac{\partial f}{\partial y}(x, y, z) = xz + \frac{\partial C}{\partial y}(y, z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y, z) = xz - 1$$

So

$$xz + \frac{\partial C}{\partial y}(y, z) = xz - 1$$

$$\frac{\partial C}{\partial y}(y, z) = -1$$

Integrate both sides with respect to y, treating z as a constant.

$$C(y, z) = \int -1 \ dy$$

$$C(y, z) = -y + C(z)$$

So

$$f(x, y, z) = x + 2x^2 + xyz + 3xz^2 - y + C(z)$$

Similarly, differentiate $f(x, y, z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{\partial}{\partial z} x + 2x^2 + xyz + 3xz^2 - y + C(z) = xy + 6xz + C'(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x, y, z) = x(y + 6z)$$

So

$$xy + 6xz + C'(z) = xy + 6xz$$

$$C'(z) = 0$$

Therefore, $C(z)$ is a constant,

$$C(y, z) = c$$

So

$$f(x, y, z) = xyz + 2x^2 + 3xz^2 + x - y + c$$

So the line integral is equal to

$$\int_c \vec{F} \cdot d\vec{r} = f(0,2,1) - f(0, -1,2)$$

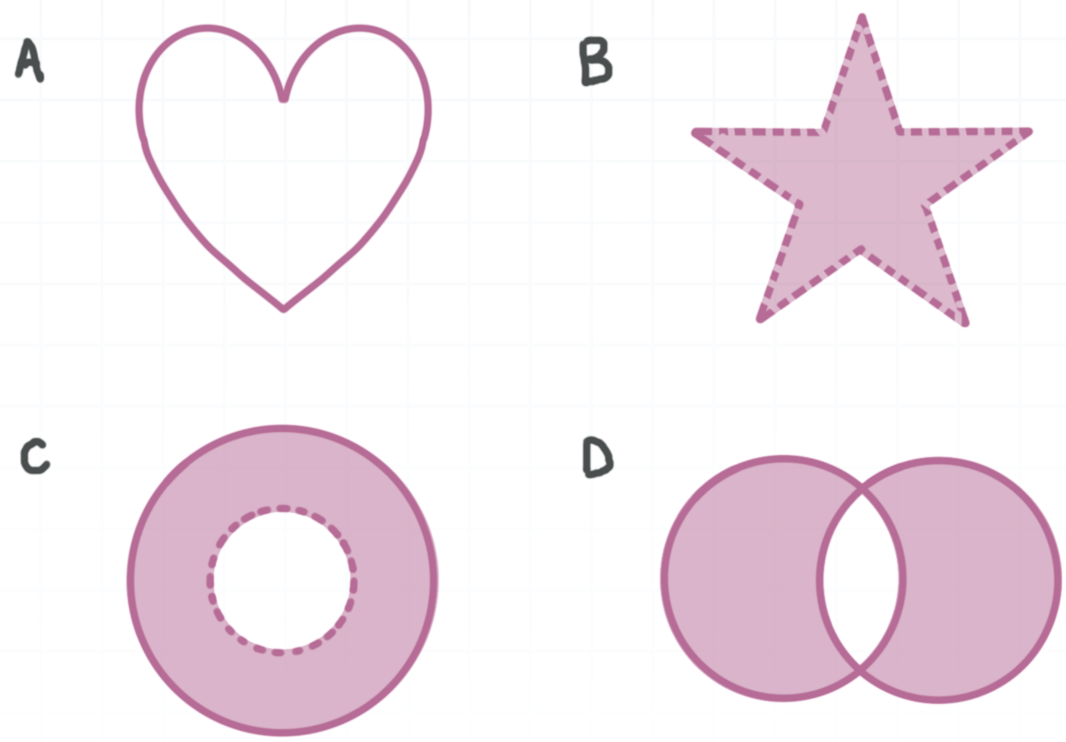
$$\int_c \vec{F} \cdot d\vec{r} = 0 \cdot 2 \cdot 1 + 2 \cdot 0^2 + 3 \cdot 0 \cdot 1^2 + 0 - 2 + c$$

$$-(0 \cdot (-1) \cdot 2 + 2 \cdot 0^2 + 3 \cdot 0 \cdot 2^2 + 0 - (-1) + c)$$

$$\int_c \vec{F} \cdot d\vec{r} = -3$$

OPEN, CONNECTED, AND SIMPLY CONNECTED

■ 1. Determine whether each set is open, closed, connected, or simply-connected.



Solution:

The set A is not open since it includes boundary points along heart's border, it's closed since it contains all of its boundary points, it's connected since it's possible to connect any two points by a path, but it's not simply-connected.

The set B is open since it doesn't contain any boundary points, it's not closed since it doesn't contain any boundary points, it's connected since it's possible to connect any two points by a path, and it's simply connected.

The set C is not open since it contains its boundary points along the outer circle, it's not closed since it doesn't contain its boundary points along the inner circle, it's connected since it's possible to connect any two points by a path, but it's not simply-connected because it has a hole.

The set D is not open since it contains all of its boundary points, it's closed since it contains all of its boundary points, it's connected since it's possible to connect any two points by a path, but it's not simply-connected because it has a hole.

■ 2. Find the domain D of the vector field $\vec{F}$, then determine whether it's open, closed, connected, or simply-connected.

$$\vec{F}(u, v) = \left\langle \sqrt{36 - 9u^2 - 4v^2}, \log_2(uv - v) \right\rangle$$

Solution:

Let's find the domains D_1 and D_2 of each component of the vector field individually, then find their intersection. Find the domain of the first component.

$$36 - 9u^2 - 4v^2 \geq 0$$

$$9u^2 + 4v^2 \leq 36$$

$$\frac{u^2}{2^2} + \frac{v^2}{3^2} \leq 1$$

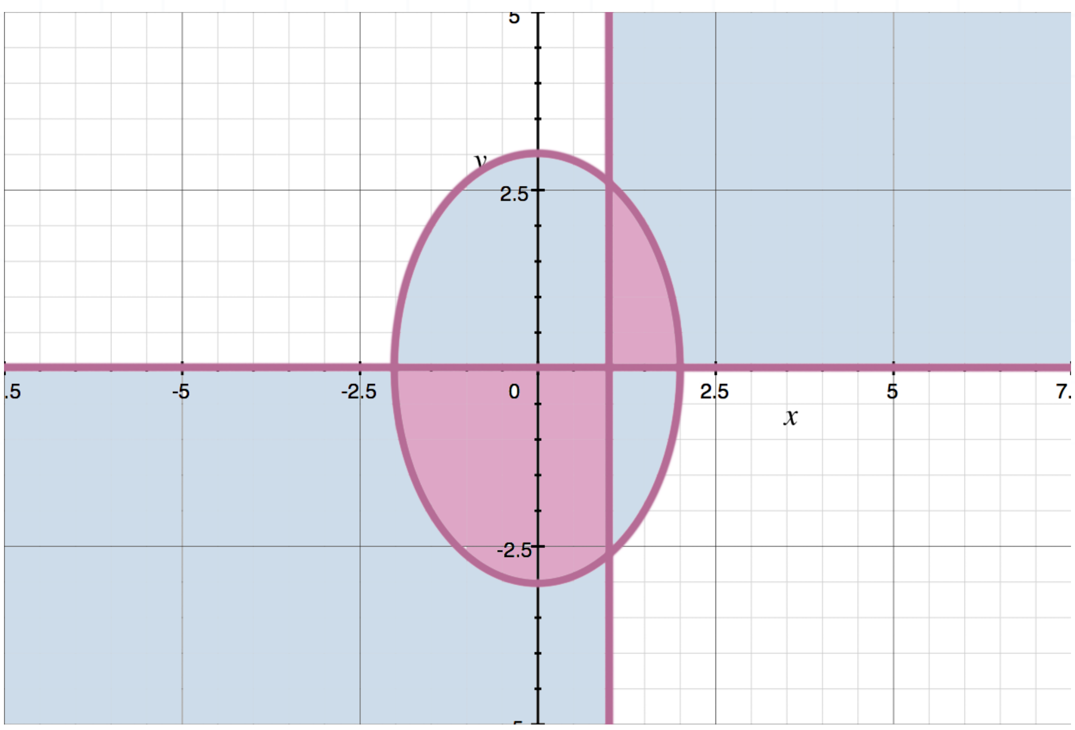
So the domain D_1 is the set of inner points of the ellipse centered at $(0,0)$ with u-semi-axis 2, and v-semi-axis 3.

Find the domain of the second component.

$$(u-1)v > 0$$

So D_2 consists of two sets, $u > 1$, $v > 0$ (above the line $v = 0$, right of the line $u = 1$) and $u < 1$, $v < 0$ (below the line $v = 0$ and left of the line $u = 1$).

Then the intersection of D_1 and D_2 is shown looks like this:



D is not open since it contains boundary points along ellipse border, it’s not closed since its borders along the lines $u = 1$ and $v = 0$ are open, it’s not connected since it's impossible to connect any two points from the two parts of D_2, and it’s not simply-connected since it's not connected.

■ 3. Find the domain D of the vector field $\vec{F}$, then determine whether it’s open, closed, connected, or simply-connected.

$$\vec{F}(x,y,z) = \left\langle \ln(4x - x^2 - y^2 - z^2), \frac{3x}{y^2 + z^2}, \frac{y}{x+8} \right\rangle$$

Solution:

Let's find the domains D_1, D_2, and D_3 of each component of the vector field individually, then find their intersection.

The domain of the first component is

$$4x - x^2 - y^2 - z^2 > 0$$

$$x^2 - 4x + y^2 + z^2 < 0$$

$$x^2 - 4x + 4 - 4 + y^2 + z^2 < 0$$

$$(x-2)^2 + y^2 + z^2 < 2^2$$

So the domain D_1 is the set of inner points of the sphere centered at $(2,0,0)$ with radius 2.

The domain of the second component is

$$y^2 + z^2 \neq 0$$

so y and z can't be 0 simultaneously, which means the domain D_2 is the set of all points except the x-axis.

The domain of the third component is

$$x + 8 \neq 0$$

So the domain D_3 is the set of all points except the plane $x = -8$.

Since the plane $x = -8$ has no common points with the sphere, the intersection of D_1, D_2, and D_3 is the set of inner points of the sphere centered at $(2,0,0)$ with radius 2, except the points that lie on the x-axis.

D is open since it doesn't contain any of its boundary points, and so D is not closed. It's connected since it's possible to connect any two points in D with a path that lies completely in D, and D isn't simply-connected.

GREEN'S THEOREM FOR ONE REGION

■ 1. Use Green's theorem to calculate the line integral of the vector field $\vec{F}(x, y)$ over the circle with the center at the origin and radius 4.

$$\vec{F}(x, y) = \left\langle \ln(x^2 + y^2 + 20) - 2y - 3x, \sqrt{x^2 + y^2 + 9} \right\rangle$$

Solution:

Let P and Q be the components of the vector field.

$$P(x, y) = \ln(x^2 + y^2 + 20) - 2y - 3x$$

$$Q(x, y) = \sqrt{x^2 + y^2 + 9}$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = \frac{2y}{x^2 + y^2 + 20} - 2$$

$$\frac{\partial Q}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + 9}}$$

In polar coordinates, parametrize the region bounded by the circle centered at the origin with radius 4.

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$r^2 = x^2 + y^2$$

$$dx\ dy = r\ dr\ d\phi$$

Then we get

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{x}{\sqrt{x^2+y^2+9}} + 2 - \frac{2y}{x^2+y^2+20}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{r\cos\phi}{\sqrt{r^2+9}} + 2 - \frac{2r\sin\phi}{r^2+20}$$

The line integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA = \int_0^4 \int_0^{2\pi} \left(\frac{r\cos\phi}{\sqrt{r^2+9}} + 2 - \frac{2r\sin\phi}{r^2+20}\right) r\ d\phi\ dr$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA = \int_0^4 \int_0^{2\pi} \frac{r^2\cos\phi}{\sqrt{r^2+9}} + 2r - \frac{2r^2\sin\phi}{r^2+20}\ d\phi\ dr$$

Since the integral of sine and cosine functions over a 2π-period is 0, the integral simplifies to

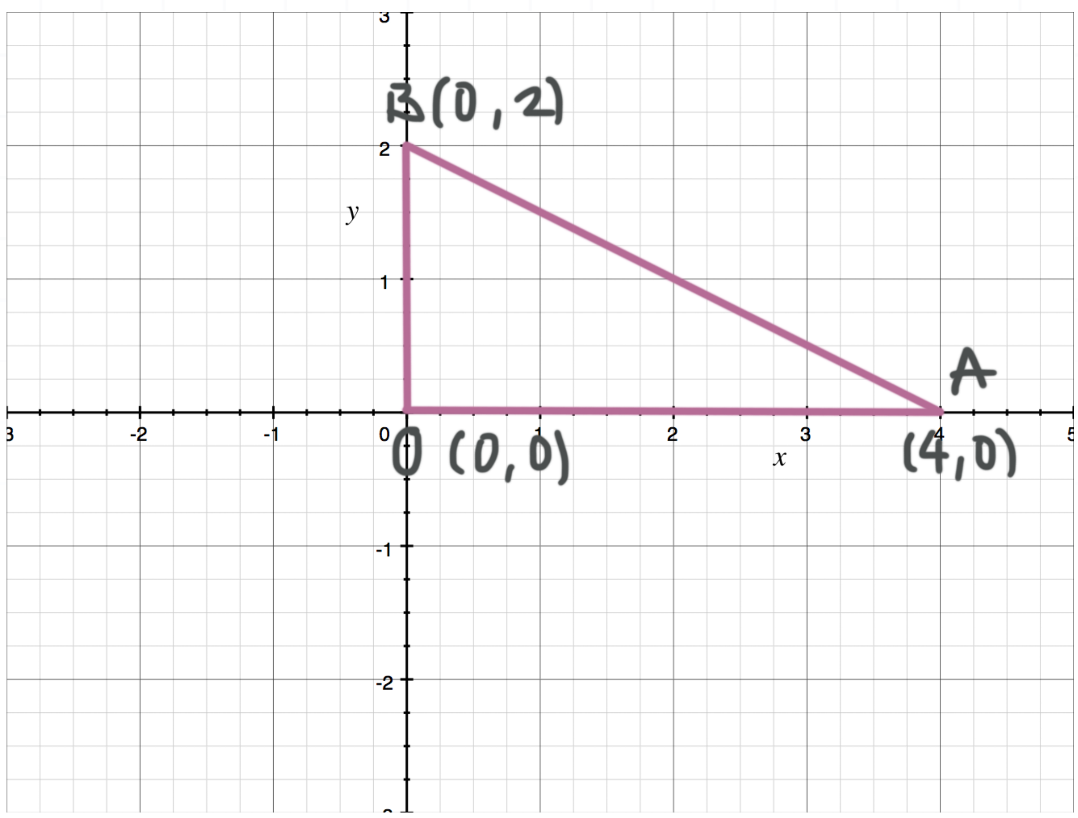
$$\int_0^4 \int_0^{2\pi} 2r\ d\phi\ dr$$

$$\int_0^4 2r\ dr \cdot \int_0^{2\pi}\ d\phi$$

$$r^2 \Big|_0^4 \cdot 2\pi$$

$$(4^2 - 0^2)(2\pi) = 32\pi$$

■ 2. Use Green's theorem to calculate the line integral of the vector field $\vec{F}(x, y) = \langle y(y^2 + \sin x), y^2 - \cos x \rangle$ over the triangle OAB, where $O(0,0)$, $A(4,0)$, and $B(0,2)$.



Solution:

Let P and Q be the components of the vector field.

$$P(x, y) = y(y^2 + \sin x)$$

$$Q(x, y) = y^2 - \cos x$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = 3y^2 + \sin x$$

$$\frac{\partial Q}{\partial x} = \sin x$$

Then the difference of partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin x - 3y^2 - \sin x = -3y^2$$

In the triangle OAB, the value of x changes from 0 to 4 and the value of y changes from 0 to the line AB. Find the equation of the line through the points $(4,0)$ and $(0,2)$. The equation of the line with slope k which passes through (x_0, y_0) is

$$y - y_0 = k(x - x_0)$$

The line AB has a slope of -0.5, and passes through the point $(0,2)$. So its equation is

$$y - 2 = -0.5\,(x - 0)$$

$$y = -0.5x + 2$$

Therefore, in the triangle OAB, y changes from 0 to $-0.5x + 2$, and the line integral is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_0^4 \int_0^{-0.5x+2} -3y^2\ dy\ dx$$

Then the integral on the right simplifies to

$$\int_0^4 -y^3\Big|_0^{-0.5x+2} \ dx$$

$$\int_0^4 -(-0.5x+2)^3 - (-0^3) \ dx$$

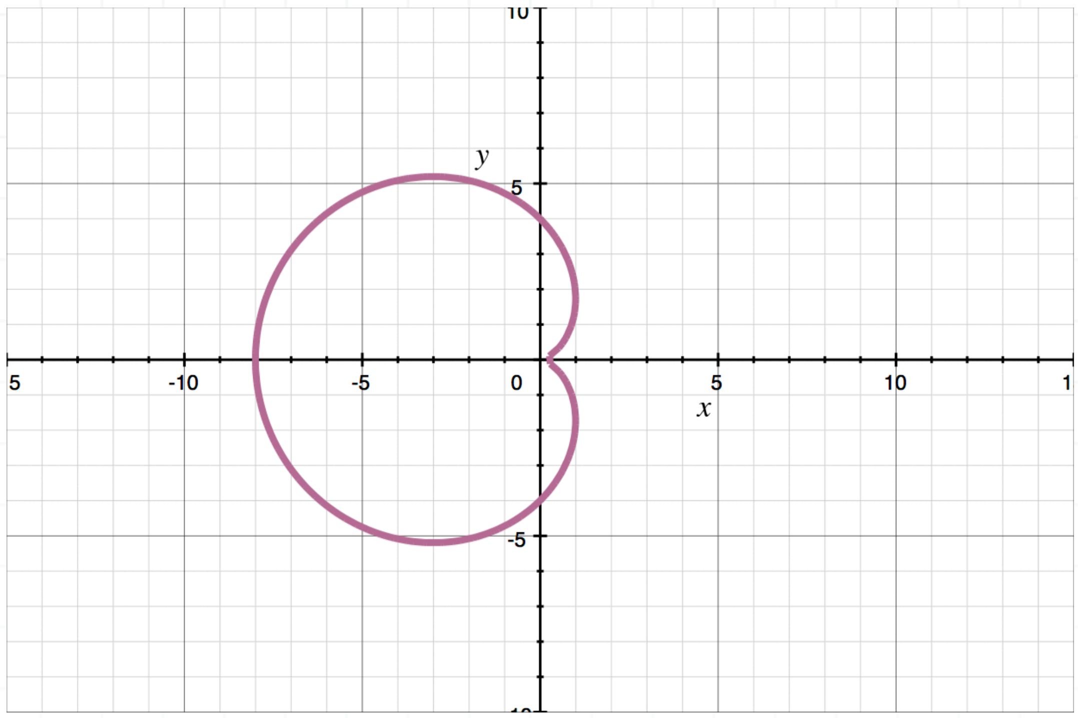
$$\int_0^4 (0.5x-2)^3 \ dx$$

$$\frac{1}{2}(0.5x-2)^4\Big|_0^4$$

$$\frac{1}{2}(0.5 \cdot 4 - 2)^4 - \frac{1}{2}(0.5 \cdot 0 - 2)^4$$

$$\frac{0^4}{2} - \frac{2^4}{2} = -8$$

■ 3. Use Green's theorem to calculate the line integral of the vector field $\overrightarrow{F}(x,y) = \langle x^3 - y^3, x^3 + y^3 \rangle$ over the cardioid $(x^2+y^2)^2 + 8x(x^2+y^2) - 16y^2 = 0$.



Solution:

Consider the standard equation of the cardioid.

$$(x^2+y^2)^2+4ax(x^2+y^2)-4a^2y^2=0$$

The equation of the cardioid in polar coordinates is

$$r(\phi)=2a(1-\cos\phi) \text{ with } 0\le\phi\le 2\pi$$

In this case, $a=2$, so

$$r(\phi)=4(1-\cos\phi) \text{ with } 0\le\phi\le 2\pi$$

Consider the conversion to polar coordinates.

$$x=r\cos\phi$$

$$y=r\sin\phi$$

$$0 \le \phi \le 2\pi$$

$$r^2 = x^2 + y^2$$

$$dx\ dy = r\ dr\ d\phi$$

Within the region enclosed by the cardioid, ϕ changes from 0 to 2π, and r changes from 0 to $4(1 - \cos\phi)$. Now let P and Q be the components of the vector field.

$$P(x, y) = x^3 - y^3$$

$$Q(x, y) = x^3 + y^3$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = -3y^2$$

$$\frac{\partial Q}{\partial x} = 3x^2$$

Then the difference of the partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 - (-3y^2) = 3(y^2 + x^2) = 3r^2$$

Then the line integral is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_0^{2\pi} \int_0^{4(1-\cos\phi)} (3r^2)r\ dr\ d\phi$$

and the right side of this equation simplifies to

$$\int_0^{2\pi}\int_0^{4(1-\cos\phi)} 3r^3\ dr\ d\phi$$

$$\int_0^{2\pi} \frac{3}{4}r^4\Big|_0^{4(1-\cos\phi)}\ d\phi$$

$$\int_0^{2\pi} \frac{3}{4}\cdot 4^4(1-\cos\phi)^4 - \frac{3}{4}\cdot 0^4\ d\phi$$

$$\int_0^{2\pi} 192(1-\cos\phi)^4\ d\phi$$

Expand the integrand.

$$192\int_0^{2\pi} \cos^4\phi - 4\cos^3\phi + 6\cos^2\phi - 4\cos\phi + 1\ d\phi$$

$$192\int_0^{2\pi} \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi + \frac{3}{8} - (3\cos\phi + \cos 3\phi)$$

$$+3\cos 2\phi + 3 - 4\cos\phi + 1\ d\phi$$

Since the integral of cosine functions over a 2π-period is 0, the integral becomes

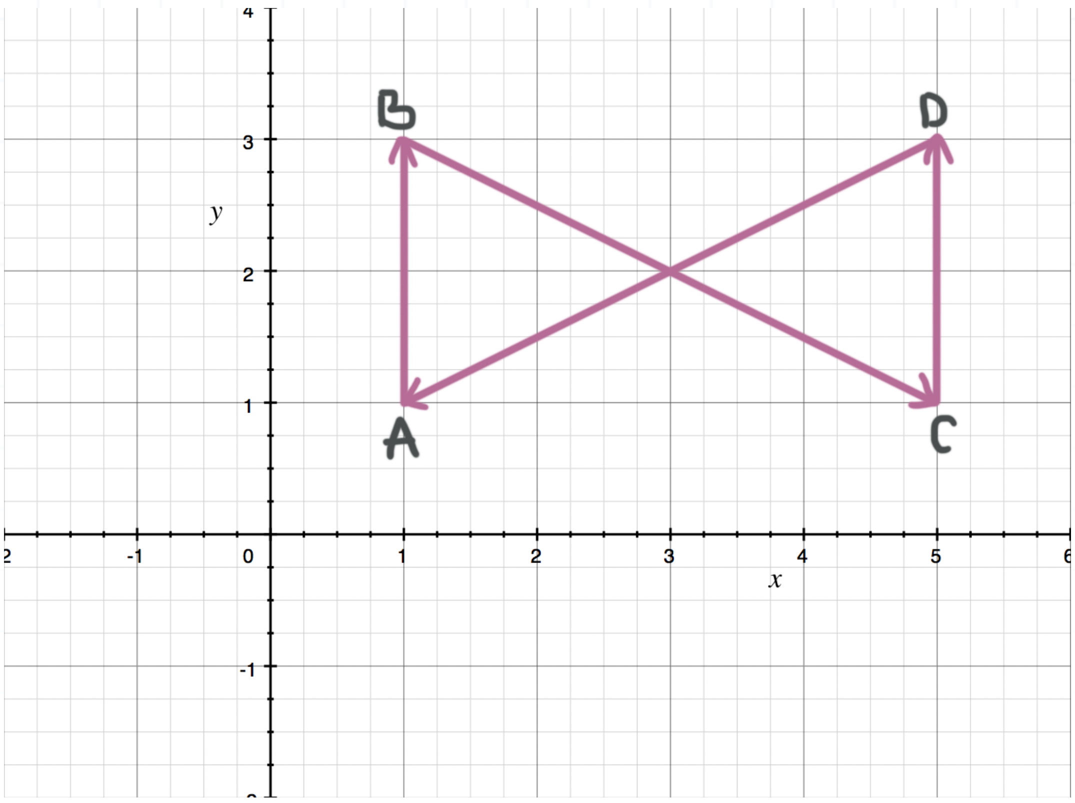
$$192\int_0^{2\pi} \frac{3}{8} + 3 + 1\ d\phi$$

$$192\cdot\frac{35}{8}\int_0^{2\pi}\ d\phi$$

$$840\cdot 2\pi = 1{,}680\pi$$

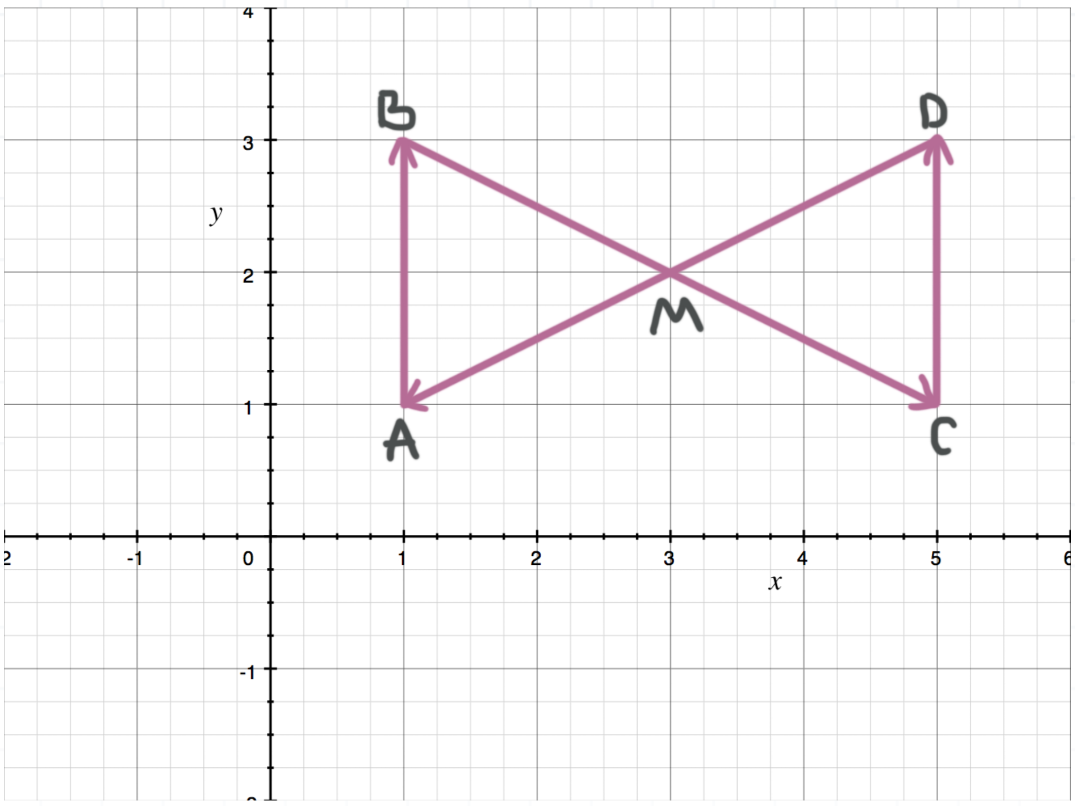
GREEN'S THEOREM FOR TWO REGIONS

1. Use Green's theorem to calculate the line integral of the vector field $\vec{F}(x,y)=\langle x^2,x^3y\rangle$ over the piecewise linear closed curve $ABCDA$, where $A(1,1)$, $B(1,3)$, $C(5,1)$, and $D(5,3)$.



Solution:

Since the given curve crosses itself, we can't apply Green's theorem to it directly. Let $M(3,2)$ be the point of intersection of DA and BC.



So the line integral over the curve $ABCDA$ is

$$\int_{ABCDA} \vec{F}(x,y)\ ds = \int_{ABM} \vec{F}(x,y)\ ds + \int_{MCDM} \vec{F}(x,y)\ ds + \int_{MA} \vec{F}(x,y)\ ds$$

$$\int_{ABCDA} \vec{F}(x,y)\ ds = \int_{ABMA} \vec{F}(x,y)\ ds + \int_{MCDM} \vec{F}(x,y)\ ds$$

So the line integral over the self-intersecting curve $ABCD$ is equal to the sum of two line integrals, over two closed curves $ABMA$ and $MCDM$, and we can apply Green's theorem to each curve separately.

Consider the triangle ABM. In this triangle x changes from 1 to 3, and y changes from the line DA to the line BC.

Let's find the equation of the line DA that passes through the points $(5,3)$ and $(1,1)$. The equation of the line with slope k which passes through the point (x_0, y_0) is

$$y - y_0 = k(x - x_0)$$

The line AD has a slope of 0.5, and passes through $(1,1)$. So its equation is

$$y - 1 = 0.5(x - 1)$$

$$y = 0.5x + 0.5$$

Similarly, let's find the equation of the line BC which passes through the points $(1,3)$ and $(5,1)$. The line BC has a slope of -0.5, and passes through $(1,3)$. So its equation is

$$y - 3 = -0.5(x - 1)$$

$$y = -0.5x + 3.5$$

So in the triangle ABM, y changes from $0.5x + 0.5$ to $-0.5x + 3.5$. For the triangle ABM, the components of the vector field will be

$$P(x, y) = x^2$$

$$Q(x, y) = x^3y$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = 0$$

$$\frac{\partial Q}{\partial x} = 3x^2y$$

Then the difference of the partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2y - 0 = 3x^2y$$

Therefore, the line integral over the triangle ABM is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_1^3 \int_{0.5x+0.5}^{-0.5x+3.5} 3x^2y\ dy\ dx$$

$$\int_1^3 3x^2 \left(\int_{0.5x+0.5}^{-0.5x+3.5} y\ dy \right) dx$$

$$3\int_1^3 x^2 \left(\frac{y^2}{2} \Big|_{0.5x+0.5}^{-0.5x+3.5} \right) dx$$

$$3\int_1^3 x^2 \left(\frac{(-0.5x+3.5)^2}{2} - \frac{(0.5x+0.5)^2}{2} \right) dx$$

$$3\int_1^3 x^2(-2)(x-3)\ dx$$

$$6\int_1^3 3x^2 - x^3\ dx$$

$$6\left(x^3 - \frac{x^4}{4} \right)\Big|_1^3$$

$$6\left(3^3 - \frac{3^4}{4} \right) - 6\left(1^3 - \frac{1^4}{4} \right) = 36$$

Consider the triangle MCD. In this triangle, x changes from 3 to 5, and y changes from the line BC to the line DA, from $-0.5x + 3.5$ to $0.5x + 0.5$. Therefore, the line integral over the triangle MCD is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_3^5 \int_{-0.5x+3.5}^{0.5x+0.5} 3x^2y\ dy\ dx$$

Then the integral on the right simplifies to

$$\int_3^5 3x^2 \left(\int_{-0.5x+3.5}^{0.5x+0.5} y\ dy \right)\ dx$$

$$3\int_3^5 x^2 \left(\frac{y^2}{2} \Big|_{-0.5x+3.5}^{0.5x+0.5} \right)\ dx$$

$$3\int_3^5 x^2 \left(\frac{(0.5x+0.5)^2}{2} - \frac{(-0.5x+3.5)^2}{2} \right)\ dx$$

$$3\int_3^5 x^2 \cdot 2(x-3)\ dx$$

$$6\int_3^5 x^3 - 3x^2\ dx$$

$$6\left(\frac{x^4}{4} - x^3 \right)\Big|_3^5$$

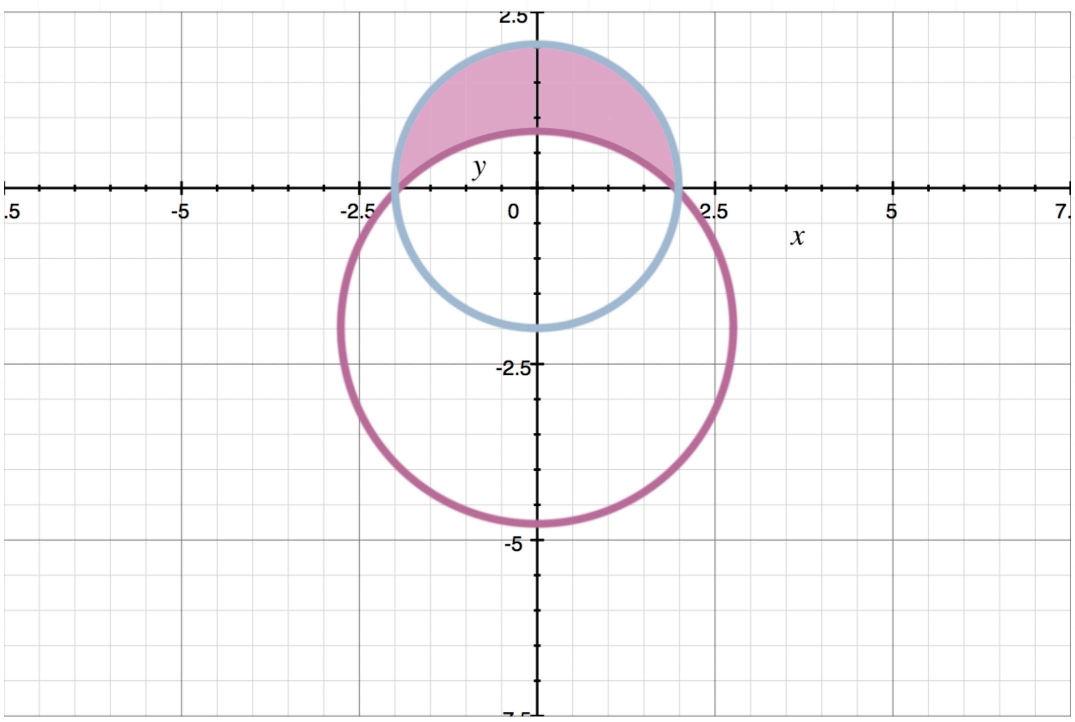
$$6\left(\frac{5^4}{4} - 5^3 \right) - 6\left(\frac{3^4}{4} - 3^3 \right) = 228$$

In total, the integral over the path $ABCDA$ is

$$36 + 228 = 264$$

■ 2. Use Green's theorem (in reverse order) to calculate the double integral over the region D inside the circle $C_1: x^2+y^2=4$, but outside the circle $C_2: x^2+(y+2)^2=8$.

$$\iint_D 3x^2\ dA$$



Solution:

In order to apply Green's theorem, we need to find any vector field $\vec{F}(x,y)=\langle P(x,y),Q(x,y)\rangle$, such that

$$\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}=3x^2$$

For simplicity, let

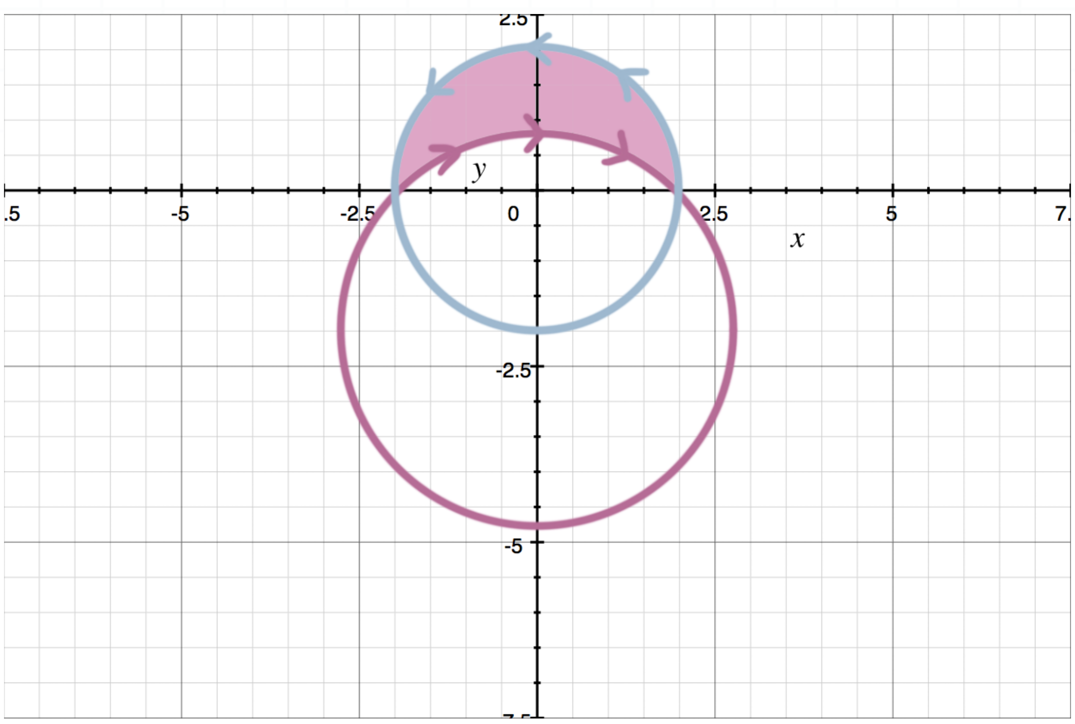
$$\frac{\partial Q}{\partial x} = 3x^2$$

$$\frac{\partial P}{\partial y} = 0$$

After integration, we have $\vec{F}(x, y) = \langle 0, x^3 \rangle$. Apply Green's theorem in reverse order.

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \ dA = \oint_c P \ dx + Q \ dy$$

where D is the region enclosed by curve c. In other words, the double integral over D is equal to the line integral over the closed curve.



Calculate the line integrals over the top and bottom parts of the closed curve individually. In order to find the line integral over the part of the circle C_1, consider the polar parametrization for the circle centered at the origin with radius 2.

$$x = 2\cos t$$

$$y = 2\sin t$$

where t changes from 0 to π. So

$$\vec{F}(x, y) = \langle 0,(2\cos t)^3 \rangle = \langle 0,8\cos^3 t \rangle$$

$$dy = 2\cos t$$

Therefore, the line integral over the bound of D which lies on C_1 is

$$\int_0^\pi (8\cos^3 t)(2\cos t)\ dt$$

$$\int_0^\pi 16\cos^4 t\ dt$$

$$\int_0^\pi 16 \cdot \frac{1}{8}(4\cos 2t + \cos 4t + 3)\ dt$$

$$2\int_0^\pi 4\cos 2t + \cos 4t + 3\ dt$$

Since the integral of cosine functions over a 2π-period is 0, we can ignore the first and second terms of the integral.

$$2\int_0^\pi 3\ dt$$

$$6\int_0^\pi dt = 6\pi$$

In order to find the line integral over the part of the circle C_2, consider the polar parametrization for the circle centered at $(0, -2)$ with radius $\sqrt{8}$.

$$x = \sqrt{8}\cos t$$

$$y = -2 + \sqrt{8}\ \sin t$$

where t changes from $3\pi/4$ to $\pi/4$. So

$$\vec{F}(x, y) = \langle 0, (\sqrt{8}\cos t)^3 \rangle = \langle 0, 8\sqrt{8}\cos^3 t \rangle$$

$$dy = \sqrt{8}\cos t$$

Therefore, the line integral over the bound of D which lies on C_2 is

$$\int_{3\pi/4}^{\pi/4} (8\sqrt{8}\cos^3 t)(\sqrt{8}\cos t)\ dt$$

$$\int_{3\pi/4}^{\pi/4} 64\cos^4 t\ dt$$

$$\int_{3\pi/4}^{\pi/4} 64 \cdot \frac{1}{8}(4\cos 2t + \cos 4t + 3)\ dt$$

$$8\int_{3\pi/4}^{\pi/4} 4\cos 2t + \cos 4t + 3\ dt$$

Since the integral of cosine functions over a 2π-period is

$$8\int_{3\pi/4}^{\pi/4} 4\cos 2t + 3\ dt$$

$$8(2\sin(2t) + 3t)\Big|_{3\pi/4}^{\pi/4}$$

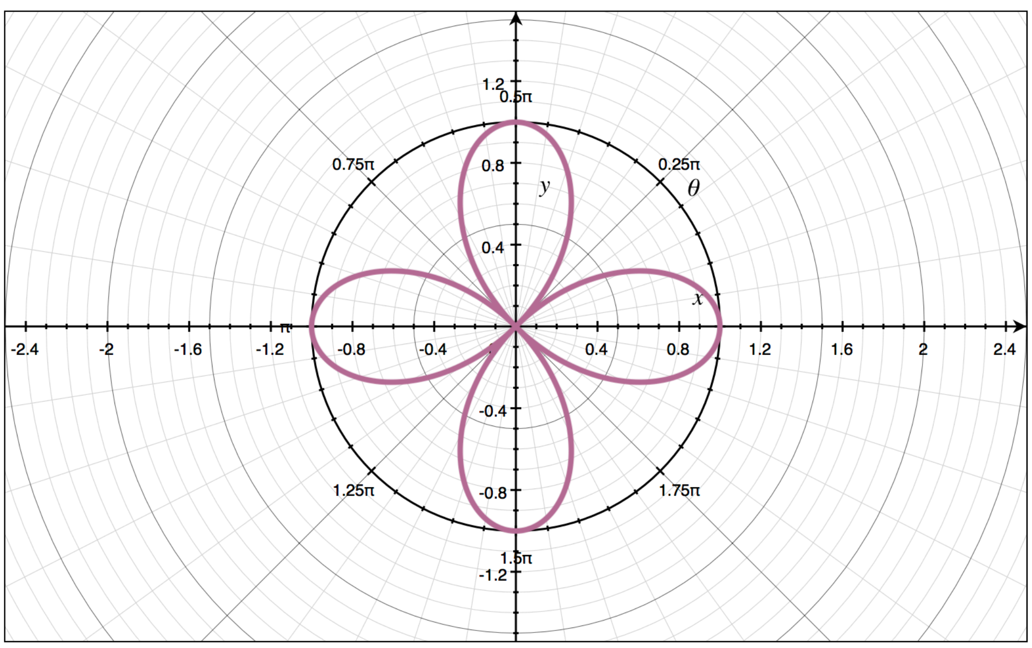
$$8\left[2\sin\left(2\cdot\frac{\pi}{4}\right)+3\cdot\frac{\pi}{4}\right]-8\left[2\sin\left(2\cdot\frac{3\pi}{4}\right)+3\cdot\frac{3\pi}{4}\right]$$

$$8\left(2+\frac{3\pi}{4}\right)-8\left[2\cdot(-1)+\frac{9\pi}{4}\right]=32-12\pi$$

In total, the integral over the region D is

$$6\pi+32-12\pi=32-6\pi$$

■ 3. Use Green's theorem to calculate the line integral of the vector field $\vec{F}(x,y)=\langle e^{x^2}-2y,y^2+2x\rangle$ over the four-petaled rose $r=\cos 2\phi$.



Solution:

The polar equation of the polar four-petaled rose is $r = \cos 2\phi$, where $-\pi/4 \leq \phi \leq \pi/4$ for the right petal, $\pi/4 \leq \phi \leq 3\pi/4$ for the bottom petal, $3\pi/4 \leq \phi \leq 5\pi/4$ for the left petal, and $5\pi/4 \leq \phi \leq 7\pi/4$ for the top petal.

Consider the standard conversion to polar coordinates.

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$0 \leq \phi \leq 2\pi$$

$$r^2 = x^2 + y^2$$

$$dx\ dy = r\ dr\ d\phi$$

Apply Green's theorem for each petal individually. Let P and Q be the components of the vector field.

$$P(x, y) = e^{x^2} - 2y$$

$$Q(x, y) = y^2 + 2x$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = -2$$

$$\frac{\partial Q}{\partial x} = 2$$

Then the difference between the partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - (-2) = 4$$

Within the region enclosed by the right rose petal, ϕ changes from $-\pi/4$ to $\pi/4$, and r changes from 0 to $\cos 2\phi$. Therefore, the line integral is

$$\iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Within the region enclosed by the bottom rose petal, ϕ changes from $\pi/4$ to $3\pi/4$, so the line integral is

$$\iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{\pi/4}^{3\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Within the region enclosed by the left rose petal, ϕ changes from $3\pi/4$ to $5\pi/4$, so the line integral is

$$\iint_{D_3} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{3\pi/4}^{5\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Within the region enclosed by the top rose petal, ϕ changes from $5\pi/4$ to $7\pi/4$, so the line integral is

$$\iint_{D_4} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{5\pi/4}^{7\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Sum the four integrals.

$$\int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi + \int_{\pi/4}^{3\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi + \int_{3\pi/4}^{5\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

$$+\int_{5\pi/4}^{7\pi/4}\int_{0}^{\cos 2\phi} 4r\ dr\ d\phi$$

$$\int_{-\pi/4}^{7\pi/4}\int_{0}^{\cos 2\phi} 4r\ dr\ d\phi$$

Calculate the inner integral by treating ϕ as a constant.

$$\int_{0}^{\cos 2\phi} 4r\ dr$$

$$2r^2\Big|_0^{\cos 2\phi}$$

$$2\cos^2 2\phi - (2 \cdot 0^2)$$

$$2\cos^2 2\phi$$

Calculate the outer integral.

$$\int_{-\pi/4}^{7\pi/4} 2\cos^2 2\phi\ d\phi$$

$$\int_{-\pi/4}^{7\pi/4} \cos 4\phi + 1\ d\phi$$

Since the integral of cosine functions over a 2π-period is

$$\int_{-\pi/4}^{7\pi/4} 1\ d\phi$$

$$\phi \Big|_{-\pi/4}^{7\pi/4}$$

$$\frac{7\pi}{4} - \left(-\frac{\pi}{4}\right) = 2\pi$$

CURL AND DIVERGENCE OF A VECTOR FIELD

■ 1. Find the set of points in R^3 where the curl of the vector field $\vec{F}(x,y,z)$ is parallel to the vector $\vec{a} = \langle 2, 1, 2\rangle$.

$$\vec{F}(x,y,z) = \left\langle \frac{z}{2}, \ln(xyz), z^2 \right\rangle$$

Solution:

The curl of a vector field in three dimensions is given by

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

If we calculate partial derivatives, we get

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(z^2) = 0$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(\ln(xyz)) = \frac{1}{z}$$

and

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}\left(\frac{z}{2}\right) = \frac{1}{2}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(z^2) = 0$$

and

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(\ln(xyz)) = \frac{1}{x}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{z}{2}\right) = 0$$

So the curl of the vector field $\overrightarrow{F}(x, y, z)$ is

$$\text{curl } \overrightarrow{F} = \left\langle 0 - \frac{1}{z}, \frac{1}{2} - 0, \frac{1}{x} - 0 \right\rangle$$

$$\text{curl } \overrightarrow{F} = \left\langle -\frac{1}{z}, \frac{1}{2}, \frac{1}{x} \right\rangle$$

Since the curl is parallel to the vector $\overrightarrow{a} = \langle 2, 1, 2 \rangle$, there's a constant k such that curl $\overrightarrow{F} = k\overrightarrow{a}$. Therefore,

$$\left\langle -\frac{1}{z}, \frac{1}{2}, \frac{1}{x} \right\rangle = k\langle 2, 1, 2 \rangle$$

$$\left\langle -\frac{1}{z}, \frac{1}{2}, \frac{1}{x} \right\rangle = \langle 2k, k, 2k \rangle$$

From the second component,

$$k = \frac{1}{2}$$

So

$$-\frac{1}{z} = 2 \cdot \frac{1}{2}$$

$$\frac{1}{x} = 2 \cdot \frac{1}{2}$$

Therefore, $x = 1$ and $z = -1$. Therefore, the curl of the vector field $\vec{F}(x, y, z)$ is parallel to the vector $\vec{a} = \langle 2, 1, 2 \rangle$ for any point with coordinates $(1, y, -1)$, in other words, for any point on the line $x = 1$ and $z = -1$.

■ 2. Find the set of points in R^3, where the divergence of the vector field $\vec{F}(x, y, z) = \langle x^3 + 12xy, y^3 + 3z^2y - 9y, 3z^2 - 6xz \rangle$ is 0.

Solution:

The divergence of a vector field in three dimensions is given by

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Calculate partial derivatives.

$$\frac{\partial F_x}{\partial x} = \frac{\partial}{\partial x}(x^3 + 12xy) = 3x^2 + 12y$$

$$\frac{\partial F_y}{\partial y} = \frac{\partial}{\partial y}(y^3 + 3z^2y - 9y) = 3y^2 + 3z^2 - 9$$

$$\frac{\partial F_z}{\partial z} = \frac{\partial}{\partial z}(3z^2 - 6xz) = 6z - 6x$$

So the divergence of the vector field $\vec{F}(x, y, z)$ is

$$\text{div } \vec{F} = 3x^2 + 12y + 3y^2 + 3z^2 - 9 + 6z - 6x$$

Since the divergence of the vector field $\vec{F}(x, y, z)$ is 0,

$$3x^2 + 12y + 3y^2 + 3z^2 - 9 + 6z - 6x = 0$$

$$x^2 + 4y + y^2 + z^2 - 3 + 2z - 2x = 0$$

Complete the square with respect to each variable.

$$x^2 - 2x + 1 - 1 + y^2 + 4y + 4 - 4 + z^2 + 2z + 1 - 1 - 3 = 0$$

$$(x - 1)^2 + (y + 2)^2 + (z + 1)^2 = 9$$

Therefore, the divergence of the vector field $\vec{F}(x, y, z)$ is 0 on the sphere centered at $(1, -2, -1)$ with radius 3.

■ 3. Find the maximum value of the divergence of the vector field $\vec{F}(x, y, z)$.

$$\vec{F}(x, y, z) = \langle \ln(x^2 + 4), -e^{y+2}, -z e^{-y} - z^3 \rangle$$

Solution:

The divergence of a vector field in three dimensions is given by

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

Calculate the partial derivatives.

$$\frac{\partial F_x}{\partial x} = \frac{\partial}{\partial x}(\ln(x^2+4)) = \frac{2x}{x^2+4}$$

$$\frac{\partial F_y}{\partial y} = \frac{\partial}{\partial y}(-e^{y+2}) = -e^{y+2}$$

$$\frac{\partial F_z}{\partial z} = \frac{\partial}{\partial z}(-ze^{-y} - z^3) = -e^{-y} - 3z^2$$

So the divergence of the vector field $\vec{F}(x, y, z)$ is

$$\text{div } \vec{F} = \frac{2x}{x^2+4} - e^{y+2} - e^{-y} - 3z^2$$

It's possible to rewrite the maximized function $f(x, y, z) = \text{div } \vec{F}$ as a sum of three independent functions.

$$f(x, y, z) = f_1(x) + f_2(y) + f_3(z) = \left(\frac{2x}{x^2+4}\right) + (-e^{y+2} - e^{-y}) + (-3z^2)$$

So we can find the points x_0, y_0, and z_0 where each of these three functions reach their global maximum, and the maximum value of the sum of the function will be

$$f(x_0, y_0, z_0) = f_1(x_0) + f_2(y_0) + f_3(z_0)$$

To maximize $f_1(x) = 2x/(x^2+4)$, find critical points, take the derivative, and set it equal to 0.

$$\frac{2(4-x^2)}{(x^2+4)^2} = 0$$

$$4 - x^2 = 0$$

$$x = \pm 2$$

Since $f_1'(x) > 0$ for $x \in (-2,2)$, and $f_1'(x) < 0$ for $x < -2$ or $x > 2$, and since $f_1(x)$ tends to 0 when x approaches $\pm\infty$, $x = -2$ is a global minimum and $x_0 = 2$ is a global maximum. To maximize $f_2(y) = -e^{y+2} - e^{-y}$, find critical points, take the derivative, and set it equal to 0.

$$-e^{y+2} + e^{-y} = 0$$

$$e^{y+2} = e^{-y}$$

$$e^{2y+2} = 1$$

$$2y + 2 = 0$$

$$y = -1$$

Since $f_2'(y) > 0$ for $y < -1$, and $f_2'(y) < 0$ for $y > -1$, $y_0 = -1$ is a global maximum. To maximize $f_3(z) = -3z^2$, find critical points, take the derivative, and set it equal to 0.

$$f_3'(z) = -6z = 0$$

$$z = 0$$

Since $f_3'(z) > 0$ for $z < 0$, and $f_3'(z) < 0$ for $z > 0$, $z_0 = 0$ is a global maximum.

So the function div $\vec{F}$ reaches its global maximum at $(2,-1,0)$, and its value is

$$\text{div } \vec{F}(2,-1,0) = \frac{2(2)}{2^2+4} - e^{-1+2} - e^{-(-1)} - 3(0)^2 = \frac{1}{2} - 2e$$

POTENTIAL FUNCTION OF THE CONSERVATIVE VECTOR FIELD, THREE DIMENSIONS

■ 1. Find the potential function of the conservative vector field.

$$\vec{F}(x, y, z) = \left\langle \frac{2x}{z}, \frac{1}{z}, -\frac{x^2 + y}{z^2} \right\rangle$$

Solution:

A potential function $f(x, y, z)$ of a vector field $\vec{F}(x, y, z)$ satisfies $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y, z) = F_x(x, y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

where F_x, F_y, and F_z are the components of the vector field.

$$\frac{\partial f}{\partial x}(x, y, z) = \frac{2x}{z}$$

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{1}{z}$$

$$\frac{\partial f}{\partial z}(x,y,z) = -\frac{x^2+y}{z^2}$$

From the first equation,

$$\frac{\partial f}{\partial x}(x,y,z) = \frac{2x}{z}$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x,y,z) = \int \frac{2x}{z}\ dx$$

$$f(x,y,z) = \frac{2}{z} \cdot \frac{x^2}{2} + C(y,z)$$

$$f(x,y,z) = \frac{x^2}{z} + C(y,z)$$

Differentiate $f(x,y,z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial}{\partial y}\left(\frac{x^2}{z} + C(y,z)\right)$$

$$\frac{\partial f}{\partial y}(x,y,z) = 0 + \frac{\partial C}{\partial y}(y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial C}{\partial y}(y,z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{1}{z}$$

So

$$\frac{\partial C}{\partial y}(y,z) = \frac{1}{z}$$

Integrate both sides with respect to y, treating z as a constant.

$$C(y,z) = \int \frac{1}{z}\ dy$$

$$C(y,z) = \frac{y}{z} + C_1(z)$$

So

$$f(x,y,z) = \frac{x^2}{z} + \frac{y}{z} + C_1(z)$$

$$f(x,y,z) = \frac{x^2+y}{z} + C_1(z)$$

Next, differentiate $f(x,y,z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}\left(\frac{x^2+y}{z} + C_1(z)\right)$$

$$\frac{\partial f}{\partial z}(x,y,z) = -\frac{x^2+y}{z^2} + \frac{\partial}{\partial z}C_1(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x,y,z) = -\frac{x^2+y}{z^2}$$

So

$$-\frac{x^2+y}{z^2}+\frac{\partial}{\partial z}C_1(z)=-\frac{x^2+y}{z^2}$$

$$\frac{\partial}{\partial z}C_1(z)=0$$

Which means $C_1(z)$ is a constant c. Therefore,

$$f(x,y,z)=\frac{x^2+y}{z}+c$$

For any conservative vector field, there are an infinite number of possible potential functions, which vary by an additive constant.

■ 2. Find the value of a such that the vector field $\overrightarrow{F}$ has a potential function, then find that potential function.

$$\overrightarrow{F}(x,y,z)=\langle 4x^ay^3z^2,3x^4y^2z^2,2x^4y^3z\rangle$$

Solution:

A potential function $f(x,y,z)$ of a vector field $\overrightarrow{F}(x,y,z)$ satisfies the equality $\nabla f=\overrightarrow{F}$, or

$$\frac{\partial f}{\partial x}(x,y,z)=F_x(x,y,z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

where F_x, F_y, and F_z are the components of the vector field.

$$\frac{\partial f}{\partial x}(x, y, z) = 4x^a y^3 z^2$$

$$\frac{\partial f}{\partial y}(x, y, z) = 3x^4 y^2 z^2$$

$$\frac{\partial f}{\partial z}(x, y, z) = 2x^4 y^3 z$$

Integrate both sides of the first equation with respect to x, treating y and z as constants.

$$f(x, y, z) = \int 4x^a y^3 z^2 \, dx$$

$$f(x, y, z) = 4y^3 z^2 \cdot \frac{x^{a+1}}{a+1} + C(y, z)$$

$$f(x, y, z) = \frac{4}{a+1} x^{a+1} y^3 z^2 + C(y, z)$$

Differentiate $f(x, y, z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}\left(\frac{4}{a+1} x^{a+1} y^3 z^2 + C(y, z)\right)$$

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{4}{a+1}x^{a+1}z^2 \cdot (3y^2) + \frac{\partial C}{\partial y}(y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{12}{a+1}x^{a+1}z^2y^2 + \frac{\partial C}{\partial y}(y, z)$$

From the second equation, we get

$$\frac{12}{a+1}x^{a+1}z^2y^2 + \frac{\partial C}{\partial y}(y, z) = 3x^4y^2z^2$$

From this equation we can conclude that

$$\frac{12}{a+1}x^{a+1} = 3x^4$$

$$\frac{\partial C}{\partial y}(y, z) = 0$$

which means that $a = 3$, and that $C(y, z)$ is a constant in terms of y, i.e. $C(y, z) = C(z)$. So

$$f(x, y, z) = x^4y^3z^2 + C(z)$$

Next, differentiate $f(x, y, z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{\partial}{\partial z}(x^4y^3z^2 + C(z))$$

$$\frac{\partial f}{\partial z}(x, y, z) = 2x^4y^3z + C'(z)$$

From the third equation, we get

$$2x^4y^3z + C'(z) = 2x^4y^3z$$

$$C'(z) = 0$$

which means $C(z)$ is a constant, $C(z) = c$. Therefore,

$$f(x, y, z) = x^4y^3z^2 + c$$

For any conservative vector field, there are an infinite number of possible potential functions, each of which vary by an additive constant.

■ 3. Find a potential function of the conservative vector field $\vec{F}(x, y, z)$, then use this function to calculate the line integral of $\vec{F}$ over the curve $\vec{r}(t)$ between the parameter values $t = -2$ and $t = 2$.

$$\vec{F}(x, y, z) = \langle 2(x+1), 2(z-y), 2(y-1) \rangle$$

$$\vec{r}(t) = \left\langle e^{t^2-4}, \sin\frac{\pi t}{4}, e^{-t^2+4} \right\rangle$$

Solution:

The initial point of the curve for $t = -2$ is

$$\vec{r}(-2) = \left\langle e^{(-2)^2-4}, \sin\frac{\pi(-2)}{4}, e^{-(-2)^2+4} \right\rangle$$

$$\vec{r}(-2) = \langle 1, -1, 1 \rangle$$

The terminal point of the curve for $t = 2$ is

$$\vec{r}(2) = \left\langle e^{(2)^2-4}, \sin\frac{\pi(2)}{4}, e^{-(2)^2+4} \right\rangle$$

$$\vec{r}(2) = \langle 1,1,1 \rangle$$

A potential function $f(x, y, z)$ of a vector field $\vec{F}(x, y, z)$ satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y, z) = F_x(x, y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

where F_x, F_y, and F_z are the components of the vector field.

$$\frac{\partial f}{\partial x}(x, y, z) = 2(x + 1)$$

$$\frac{\partial f}{\partial y}(x, y, z) = 2(z - y)$$

$$\frac{\partial f}{\partial z}(x, y, z) = 2(y - 1)$$

Integrate both sides of the first equation with respect to x, treating y and z as constants.

$$f(x, y, z) = \int 2(x + 1)\ dx$$

$$f(x, y, z) = 2\left(\frac{x^2}{2} + x\right) + C(y, z)$$

$$f(x, y, z) = x^2 + 2x + C(y, z)$$

Differentiate $f(x, y, z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}(x^2 + 2x + C(y, z))$$

$$\frac{\partial f}{\partial y}(x, y, z) = 0 + \frac{\partial C}{\partial y}(y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial C}{\partial y}(y, z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y, z) = 2(z - y)$$

So

$$\frac{\partial C}{\partial y}(y, z) = 2(z - y)$$

Integrate both sides with respect to y, treating z as a constant.

$$C(y, z) = \int 2(z - y)\ dy$$

$$C(y, z) = 2\left(yz - \frac{y^2}{2}\right) + C_1(z)$$

$$C(y,z) = 2yz - y^2 + C_1(z)$$

So

$$f(x,y,z) = x^2 + 2x + 2yz - y^2 + C_1(z)$$

Finally, differentiate $f(x,y,z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}(x^2 + 2x + 2yz - y^2 + C_1(z))$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2y + \frac{\partial}{\partial z}C_1(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x,y,z) = 2(y-1)$$

So

$$2y + \frac{\partial}{\partial z}C_1(z) = 2(y-1)$$

$$\frac{\partial}{\partial z}C_1(z) = -2$$

Integrate both parts with respect to z.

$$C_1(z) = \int -2\ dz$$

$$C_1(z) = -2z + c$$

Therefore,

$$f(x, y, z) = x^2 + 2x + 2yz - y^2 - 2z + c$$

So the line integral is

$$\int_c \vec{F} \cdot d\vec{r} = f(1,1,1) - f(1,-1,1)$$

$$\int_c \vec{F} \cdot d\vec{r} = 1^2 + 2 \cdot 1 + 2 \cdot 1 \cdot 1 - 1^2 - 2 \cdot 1 + c$$

$$-(1^2 + 2 \cdot 1 + 2 \cdot (-1) \cdot 1 - (-1)^2 - 2 \cdot 1 + c)$$

$$\int_c \vec{F} \cdot d\vec{r} = 4$$

POINTS ON THE SURFACE

1. Find the points of the surface $\vec{r}(u, v)$ that lie on the z-axis.

$$\vec{r}(u, v) = \langle u^2 - 3v^2 - 1, 4u^2 - 9v^2 - 7, e^{u+v} \rangle$$

Solution:

Since the points we're interested in lie on the z-axis, their x- and y-coordinates are 0, so

$$u^2 - 3v^2 - 1 = 0$$

$$4u^2 - 9v^2 - 7 = 0$$

Consider the two equations as a system and solve it for u and v. Make the substitution $t = u^2$ and $s = v^2$ to get a linear system of equations.

$$t - 3s - 1 = 0$$

$$4t - 9s - 7 = 0$$

The first equation gives $t = 3s + 1$, so substitute this value into the second equation.

$$4(3s + 1) - 9s - 7 = 0$$

$$12s + 4 - 9s - 7 = 0$$

$$3s - 3 = 0$$

$$s = 1$$

Plug $s = 1$ back in to get the value of t.

$$t = 3(1) + 1 = 4$$

So

$u^2 = 4$ so $u = \pm 2$

$v^2 = 1$ so $v = \pm 1$

There are four points on the surface which lie on the z-axis. In order to find their z-coordinates, plug in the values we've found for u and v into the third component of the vector function.

$$r_3(2,1) = e^{2+1} = e^3$$

$$r_3(2, -1) = e^{2-1} = e$$

$$r_3(-2,1) = e^{-2+1} = e^{-1}$$

$$r_3(-2, -1) = e^{-2-1} = e^{-3}$$

■ 2. Find the intersection point(s) of the surface $\vec{r}(u, v)$ and the line $x = y + 2 = z - 1$.

$$\vec{r}(u, v) = \langle \sin u + v, \cos u + v - 3, 2v + 7 + \sin u \rangle$$

Solution:

Consider the linear equations as a system of equations.

$$x = y + 2$$

$$x = z - 1$$

Rewrite the vector function $\overrightarrow{r}(u, v)$ in parametric form.

$$x(u, v) = \sin u + v$$

$$y(u, v) = \cos u + v - 3$$

$$z(u, v) = 2v + 7 + \sin u$$

Substitute these parametric equations into the equations of the line.

$$\sin u + v = \cos u + v - 3 + 2$$

$$\sin u + v = 2v + 7 + \sin u - 1$$

Solve the system of equations for u and v. We get

$$\sin u = \cos u - 1$$

$$v = 2v + 6$$

and then

$$\cos u - \sin u = 1$$

$$v = -6$$

To solve the first trigonometric equation, use the identity

$$\cos\left(u+\frac{\pi}{4}\right)=\frac{\cos u}{\sqrt{2}}-\frac{\sin u}{\sqrt{2}}$$

$$\cos u-\sin u=\sqrt{2}\cos\left(u+\frac{\pi}{4}\right)$$

in order to get

$$\sqrt{2}\cos\left(u+\frac{\pi}{4}\right)=1$$

$$\cos\left(u+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}$$

Since the cosine function is $1/\sqrt{2}$ at the angles $\pi/4$ and $7\pi/4$, we have two options:

$$u+\frac{\pi}{4}=\frac{\pi}{4}+2\pi n_1$$

$$u+\frac{\pi}{4}=\frac{7\pi}{4}+2\pi n_2$$

Simplify.

$$u=2\pi n_1$$

$$u=\frac{3\pi}{2}+2\pi n_2$$

Plug $u=2\pi n_1$ and $v=-6$ into the surface equation.

$$\vec{r}(2\pi n_1,-6)=\langle\sin(2\pi n_1)-6,\cos(2\pi n_1)-6-3,2(-6)+7+\sin(2\pi n_1)\rangle$$

$$\vec{r}(2\pi n_1, -6) = \langle -6, 1-6-3, 2(-6)+7 \rangle$$

$$\vec{r}(2\pi n_1, -6) = \langle -6, -8, -5 \rangle$$

Now plug $u = (3\pi/2) + 2\pi n_2$ and $v = -6$ into the surface equation.

$$\vec{r}\left(\frac{3\pi}{2} + 2\pi n_2, -6\right) = \left\langle \sin\left(\frac{3\pi}{2} + 2\pi n_2\right) - 6, \right.$$

$$\left. \cos\left(\frac{3\pi}{2} + 2\pi n_2\right) - 6 - 3, 2(-6) + 7 + \sin\left(\frac{3\pi}{2} + 2\pi n_2\right) \right\rangle$$

$$\vec{r}\left(\frac{\pi}{2} + 2\pi n_2, -6\right) = \langle -1-6, -6-3, 2(-6)+7-1 \rangle$$

$$\vec{r}\left(\frac{\pi}{2} + 2\pi n_2, -6\right) = \langle -7, -9, -6 \rangle$$

■ 3. Identify the set of points of the surface $\vec{r}(u,v) = \langle u^2 + 2v^2, u, v+2 \rangle$ that lie in the xy-plane.

Solution:

Since the points we're interested in lie in the xy-plane, their z-coordinates are 0, so $v + 2 = 0$, or $v = -2$.

Plug $v = -2$ into the surface equation and consider it the parametric equation of the curve in two-dimensional space.

$$\vec{r}(u, -2) = \langle u^2 + 2(-2)^2, u, -2 + 2 \rangle$$

$$\vec{r}(u, -2) = \langle u^2 + 8, u, 0 \rangle$$

Consider the parametric equation of the curve in the xy-plane.

$$x = u^2 + 8$$

$$y = u$$

Substitute $u = y$ into the first equation to get rid of the parameter u.

$$x = y^2 + 8$$

So the curve is the parabola with vertex at $(8,0)$.

SURFACE OF THE VECTOR EQUATION

■ 1. Identify the quadratic surface given as a vector function, where $u \in [0,2\pi]$ and $v \in (-\infty, \infty)$.

$$\overrightarrow{r}(u, v) = \langle 3 \sin u, 2v - 3, 5 \cos u \rangle$$

Solution:

Since the x- and z-coordinates are independent of v and y, and since $y(v) = 2v - 3$ has a range of $(-\infty, \infty)$, the surface is a right cylinder that's parallel to the y-axis.

To identify the cylinder's type, consider its section by the xz-plane, which is the plane for $y = 0$, or $2v - 3 = 0$, where $u \in [0,2\pi]$.

$$x(u) = 3 \sin u$$

$$z(u) = 5 \cos u$$

These equations represent the ellipse in the xz-plane, with center at the origin, x-semi-axis of 3, and z-semi-axis of 5.

Therefore, the surface is the right cylinder such that the cylinder's axis coincides with the y-axis, an x-semi-axis of 3, and the z-semi-axis of 5.

2. Identify the quadratic surface given as a vector function, where $u \in [0,\pi]$ and $v \in [0,2\pi]$.

$$\vec{r}(u,v) = \langle -3 + 2\cos u, 2 + 2\sin u \cos v, 2\sin u \sin v \rangle$$

Solution:

Consider the formulas that we use to convert rectangular coordinates to spherical coordinates.

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

Plug in $\rho = 2$ and change the axes as $x_1 \to z$, $y_1 \to x$, and

$$z_1 \to y$$

$$x_1 = 2\cos\phi$$

$$y_1 = 2\sin\phi\cos\theta$$

$$z_1 = 2\sin\phi\sin\theta$$

Since the changing order of axes is a rotation transformation which doesn't change the form of the sphere, the new equations for x_1, y_1, and z_1 represent the parametrization of the sphere with center at the origin and radius 2.

Move the sphere by -3 in the x-direction, and by 2 in the y-direction. Also, rename $u = \phi$ and $v = \theta$.

$$x_2 = -3 + 2\cos u$$

$$y_2 = 2 + 2\sin u \cos v$$

$$z_2 = 2\sin u \sin v$$

The new equations for x_2, y_2, and z_2 coincide with the given vector function and represent the parametrization of the sphere centered at $(-3,2,0)$ with radius 2.

■ 3. Identify the quadratic surface given as a vector function, where $u^2 + v^2 \leq 9$.

$$\vec{r}(u,v) = \langle v + 1, 5 + \sqrt{9 - u^2 - v^2}, u - 2 \rangle$$

Solution:

Rewrite the vector function as a set of parametric equations.

$$x(u,v) = v + 1$$

$$y(u,v) = 5 + \sqrt{9 - u^2 - v^2}$$

$$z(u,v) = u - 2$$

Solve for u and v in the first and third equations, then plug those values into the second equation.

$$v = x - 1$$

$$u = z + 2$$

$$y = 5 + \sqrt{9 - (z + 2)^2 - (x - 1)^2}$$

$$y - 5 = \sqrt{9 - (z + 2)^2 - (x - 1)^2}$$

Since the square root is non-negative, $y - 5 \geq 0$, or $y \geq 5$. Square both sides of the equation.

$$(y - 5)^2 = 9 - (z + 2)^2 - (x - 1)^2$$

$$(x - 1)^2 + (y - 5)^2 + (z + 2)^2 = 9$$

This equation represents the sphere centered at $(1,5, -2)$ with radius 3. Since $y \geq 5$, the surface is the part of the sphere that lies above the plane $y = 5$, which means the surface is a hemisphere.

PARAMETRIC REPRESENTATION OF THE SURFACE

■ 1. Consider the right circular cylinder with radius 5 and a cylindrical axis that's parallel to the z-axis and passes through $(2,-4,5)$. Find the parametrization of the part of the cylinder that lies above the xy-plane.

Solution:

There exist an infinite number of parameterizations of a cylinder. The most common parametrization of the cylinder with radius r that has a cylindrical axis parallel to the z-axis is

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$z = z$$

Plug in $r = 5$ and move the cylindrical axis by 2 in the x-direction, and by -4 in the y-direction.

$$x = 2 + 5\cos\phi$$

$$y = -4 + 5\sin\phi$$

$$z = z$$

So we get the parametrization of the cylinder we're interested in. For the part of the cylinder that lies above the xy-plane, z changes from 0 to ∞. Rename the parameters as $\phi \to u$ and $v \to z$.

$$x(u, v) = 2 + 5\cos u$$

$$y(u, v) = -4 + 5\sin u$$

$$z(u, v) = v$$

Rewrite the parametrization as a vector function.

$$\vec{r}(u, v) = \langle 2 + 5\cos u, -4 + 5\sin u, v\rangle$$

■ 2. Consider the plane $2x - 3y + z - 1 = 0$. Find the parametrization of the part of the plane that lies between the planes $y = -3$ and $y = 3$.

Solution:

There are an infinite number of parameterizations of a plane. Let the parameter u be x, and the parameter v be y. Plug u and v into the plane equation in order to get the expression for z.

$$2u - 3v + z - 1 = 0$$

$$z = 3v - 2u + 1$$

So we get the parametrization of the plane we're interested in. Rewrite the parametrization as a vector function.

$$\vec{r}(u, v) = \langle u, v, 3v - 2u + 1 \rangle$$

Within the part of the plane that lies between the planes $y = -3$ and $y = 3$, u changes from $-\infty$ to ∞, and v changes from -3 to 3.

■ 3. Consider the elliptic paraboloid $2(y+3)^2 + 4(z-2)^2 - x - 1 = 0$. Find the parametrization of the paraboloid for $x \le 3$.

Solution:

There are an infinite number of parameterizations of a paraboloid. Let the parameter u be y, and the parameter v be z. Plug u and v into the paraboloid's equation to get the expression for x.

$$2(u+3)^2 + 4(v-2)^2 - x - 1 = 0$$

$$x = 2(u+3)^2 + 4(v-2)^2 - 1$$

So we get the parametrization of the paraboloid we're interested in. Rewrite the parametrization as a vector function.

$$\vec{r}(u, v) = \langle 2(u+3)^2 + 4(v-2)^2 - 1, u, v \rangle$$

For the whole paraboloid, the parameters u and v are any real numbers. To get the parametrization of the paraboloid for $x \le 3$, consider the inequality

$$2(u+3)^2 + 4(v-2)^2 - 1 \le 3$$

$$2(u+3)^2 + 4(v-2)^2 \le 4$$

$$(u+3)^2 + 2(v-2)^2 \le 2$$

Let's find the widest possible range for v, then express the bounds for u in terms of v. When $(u+3)^2 = 0$, or $u = -3$, the expression $2(v-2)^2$ reaches its maximum, so

$$2(v-2)^2 \le 2$$

$$(v-2)^2 \le 1$$

$$-1 \le v - 2 \le 1$$

$$1 \le v \le 3$$

So the parameter v changes from 1 to 3. In order to get the bounds for u, solve the inequality for u.

$$(u+3)^2 + 2(v-2)^2 \le 2$$

$$(u+3)^2 \le 2 - 2(v-2)^2$$

$$-\sqrt{2-2(v-2)^2} \le u + 3 \le \sqrt{2-2(v-2)^2}$$

$$-3 - \sqrt{2-2(v-2)^2} \le u \le -3 + \sqrt{2-2(v-2)^2}$$

So the parameter u changes from $-3 - \sqrt{2-2(v-2)^2}$ to $-3 + \sqrt{2-2(v-2)^2}$.

TANGENT PLANE TO THE PARAMETRIC SURFACE

■ 1. Find the equation of the tangent plane to the surface $\vec{r}(u,v)=\langle u+2\cos v, u-2\cos v, uv\rangle$ at the point $(4,0,\pi)$.

Solution:

In order to find the values of the parameters u and v that correspond to $(4,0,\pi)$, solve the system of equations $\vec{r}(u,v)=\langle 4,0,\pi\rangle$ for u and v.

$$u+2\cos v=4$$

$$u-2\cos v=0$$

$$uv=\pi$$

Take the sum of the first and second equations.

$$2u=4$$

$$u=2$$

From the third equation, we get

$$2v=\pi$$

$$v=\frac{\pi}{2}$$

It's easy to check that, for the values of the parameters $u = 2$ and $v = \pi/2$, all three of the equations hold, so these values correspond to $(4,0,\pi)$ on the surface.

To get the tangent plane to the surface, we need to find the normal vector to the surface as the cross product of two tangent vectors at the given point,

$$\vec{r}_u = \frac{\partial \vec{r}(u,v)}{\partial u} \text{ and } \vec{r}_v = \frac{\partial \vec{r}(u,v)}{\partial u}$$

We find

$$\vec{r}_u = \langle 1,1,v \rangle$$

$$\vec{r}_u\left(2,\frac{\pi}{2}\right) = \left\langle 1,1,\frac{\pi}{2} \right\rangle$$

and

$$\vec{r}_v = \langle -2\sin v, 2\sin v, u \rangle$$

$$\vec{r}_v\left(2,\frac{\pi}{2}\right) = \langle -2,2,2 \rangle$$

Evaluate the cross product $\vec{r}_u \times \vec{r}_v$, if the cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle a_1, a_2, a_3 \rangle = \langle 1,1,\pi/2 \rangle$ and $\langle b_1, b_2, b_3 \rangle = \langle -2,2,2 \rangle$.

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}\left(1 \cdot 2 - \frac{\pi}{2} \cdot 2\right) - \mathbf{j}\left(1 \cdot 2 - \frac{\pi}{2} \cdot (-2)\right) + \mathbf{k}(1 \cdot 2 - 1 \cdot (-2))$$

$$\vec{r}_u \times \vec{r}_v = (2-\pi)\mathbf{i} - (2+\pi)\mathbf{j} + 4\mathbf{k}$$

So the normal vector to the surface at the given point is

$$\vec{n} = \langle 2-\pi, -2-\pi, 4 \rangle$$

The standard equation of the plane that passes through (x_0, y_0, z_0), and with normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Plug in $(x_0, y_0, z_0) = (4, 0, \pi)$ and $\vec{n} = \langle 2-\pi, -2-\pi, 4 \rangle$.

$$(2-\pi)(x-4) + (-2-\pi)(y-0) + 4(z-\pi) = 0$$

$$(\pi - 2)x + (\pi + 2)y - 4z + 8 = 0$$

■ 2. Find the equation of the tangent plane(s) to the parametric surface $\vec{r}(u, v) = \langle u^2 + 2v, u - 2v, uv + 1 \rangle$ such that its normal vector $\vec{n}$ is parallel to the y-axis.

Solution:

Find the normal vector to the surface in general form as the cross product of two tangent vectors,

$$\vec{r}_u = \frac{\partial \vec{r}(u,v)}{\partial u} \text{ and } \vec{r}_v = \frac{\partial \vec{r}(u,v)}{\partial u}$$

We find

$$\vec{r}_u = \langle 2u, 1, v \rangle$$

$$\vec{r}_v = \langle 2, -2, u \rangle$$

Evaluate the cross product $\vec{r}_u \times \vec{r}_v$, if the cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle a_1, a_2, a_3 \rangle = \langle 2u, 1, v \rangle$ and $\langle b_1, b_2, b_3 \rangle = \langle 2, -2, u \rangle$.

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(1 \cdot u - v \cdot (-2)) - \mathbf{j}(2u \cdot u - v \cdot 2) + \mathbf{k}(2u \cdot (-2) - 1 \cdot 2)$$

$$\vec{r}_u \times \vec{r}_v = (u + 2v)\mathbf{i} - (-2u^2 + 2v)\mathbf{j} + (-2 - 4u)\mathbf{k}$$

So the normal vector to the surface is

$$\vec{n} = \langle u + 2v, -2u^2 + 2v, -2 - 4u \rangle$$

Since the normal vector is parallel to the y-axis, its x- and z-components are 0, so

$$u + 2v = 0$$

$$-2 - 4u = 0$$

Solve the system of equations for u and v. From the second equation,

$$4u = -2$$

$$u = -0.5$$

Plug $u = -0.5$ into the first equation.

$$-0.5 + 2v = 0$$

$$v = 0.25$$

Find the point on the surface that corresponds to the parameter values $u = -0.5$ and $v = 0.25$.

$$\vec{r}(-0.5,0.25) = \langle (-0.5)^2 + 2 \cdot 0.25, -0.5 - 2 \cdot 0.25, -0.5 \cdot 0.25 + 1 \rangle$$

$$\vec{r}(-0.5,0.25) = \langle 0.75, -1, 0.875 \rangle$$

The standard equation of the plane that passes through (x_0, y_0, z_0) with normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

For the normal vector we can take any vector parallel to the y-axis, so we'll plug $\vec{n} = \langle 0,1,0 \rangle$ and $(x_0, y_0, z_0) = (0.75, -1,0.875)$ into the equation.

$$1 \cdot (y - (-1)) = 0$$

$$y + 1 = 0$$

■ 3. Find the equation of the tangent plane(s) to the parametric surface $\vec{r}(u, v) = \langle v^2, u - v + 2, u^2 - 2 \rangle$ such that it's parallel to $3x - 24y + 2z - 1 = 0$.

Solution:

Find the normal vector to the surface as a cross product of the tangent vectors $\vec{r}_u = \langle 0,1,2u \rangle$ and $\vec{r}_v = \langle 2v, -1,0 \rangle$. Evaluate the cross product $\vec{r}_u \times \vec{r}_v$, given that the cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle a_1, a_2, a_3 \rangle = \langle 0,1,2u \rangle$ and $\langle b_1, b_2, b_3 \rangle = \langle 2v, -1,0 \rangle$.

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(1 \cdot 0 - 2u \cdot (-1)) - \mathbf{j}(0 \cdot 0 - 2u \cdot 2v) + \mathbf{k}(0 \cdot (-1) - 1 \cdot 2v)$$

$$\vec{r}_u \times \vec{r}_v = 2u\mathbf{i} + 4uv\mathbf{j} - 2v\mathbf{k}$$

So the normal vector to the surface is

$$\vec{n} = \langle 2u, 4uv, -2v \rangle$$

Since the tangent plane we're interested in is parallel to the plane $3x - 24y + 2z - 1 = 0$, it has a normal vector that's parallel to $\langle 3, -24,2 \rangle$, so there's a nonzero real number k such that

$$\langle 2u, 4uv, -2v \rangle = k\langle 3, -24,2 \rangle$$

This equation gives a new system.

$$2u = 3k$$

$$4uv = -24k$$

$$-2v = 2k$$

Solve the system of equations for u, v, and k. Solve u and v in the first and third equations.

$$u = 1.5k$$

$$v = -k$$

Plug these values into the second equation.

$$4(1.5k)(-k) = -24k$$

$$-6k^2 = -24k$$

$$k = 4$$

So the solution of the system is $k = 4$, $u = 6$, and $v = -4$. Find the point on the surface that corresponds to the parameter values $u = 6$ and $v = -4$.

$$\vec{r}(6,-4) = \langle (-4)^2, 6-(-4)+2, 6^2-2 \rangle$$

$$\vec{r}(6,-4) = \langle 16,12,34 \rangle$$

The standard equation of the plane that passes through (x_0, y_0, z_0) and with normal vector $\vec{n} = \langle n_1, n_2, n_3 \rangle$ is

$$n_1(x-x_0) + n_2(y-y_0) + n_3(z-z_0) = 0$$

Plug in $(x_0, y_0, z_0) = (16,12,34)$ and $\vec{n} = \langle 3,-24,2 \rangle$.

$$(x-16) - 24(y-12) + 2(z-34) = 0$$

$$3x - 24y + 2z + 172 = 0$$

AREA OF A SURFACE

1. Find the area of the part of the surface $z = 2x + 2y - 1$ that lies within the rectangle given by $0 \le x \le \pi$ and $-1 \le y \le 1$.

Solution:

The area of the surface $z = f(x, y)$ inside the region D is given by

$$A = \iint_D \sqrt{1 + (f'_x)^2 + (f'_y)^2}\ dA$$

Take the partial derivatives of $f(x, y) = 2x + 2y - 1$.

$$f'_x = 2$$

$$f'_y = 2$$

So the area of the part of the surface is given by

$$A = \int_0^\pi \int_{-1}^1 \sqrt{1 + 2^2 + 2^2}\ dy\ dx$$

$$A = \int_0^\pi \int_{-1}^1 3\ dy\ dx$$

Evaluate the inner integral.

$$A = \int_0^{\pi} 3y \Big|_{y=-1}^{y=1} \ dx$$

$$A = \int_0^{\pi} 3(1) - 3(-1) \ dx$$

$$A = \int_0^{\pi} 3 + 3 \ dx$$

$$A = \int_0^{\pi} 6 \ dx$$

Evaluate the outer integral.

$$A = 6x \Big|_0^{\pi}$$

$$A = 6\pi - 6(0)$$

$$A = 6\pi$$

■ 2. Find the area of the part of the surface $\vec{r}(u, v)$ that lies within the values of the parameters $-1 \leq u \leq 1$ and $0 \leq v \leq \sqrt{5}$.

$$\vec{r}(u, v) = \langle 2u - 3v + 1, 5u - v + 4, -u + 4v - 11 \rangle$$

Solution:

The area of the surface $\vec{r}(u, v)$ inside the region D is given by

$$A = \iint_D |\vec{r}_u \times \vec{r}_v| \ dA$$

Take partial derivatives.

$$\vec{r}_u = \langle 2,5, -1 \rangle$$

$$\vec{r}_v = \langle -3, -1,4 \rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle 2,5, -1 \rangle \times \langle -3, -1,4 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(5 \cdot 4 - (-1) \cdot (-1)) - \mathbf{j}(2 \cdot 4 - (-1) \cdot (-3)) + \mathbf{k}(2 \cdot (-1) - 5 \cdot (-3))$$

$$\vec{r}_u \times \vec{r}_v = 19\mathbf{i} - 5\mathbf{j} + 13\mathbf{k}$$

$$\vec{r}_u \times \vec{r}_v = \langle 19, -5,13 \rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{19^2 + (-5)^2 + 13^2} = \sqrt{555}$$

So the area of the part of the surface is

$$A = \int_{-1}^{1} \int_{0}^{\sqrt{5}} \sqrt{555} \ dv \ du$$

Evaluate the inner integral.

$$A = \int_{-1}^{1} \sqrt{5} \cdot \sqrt{555} \ du$$

$$A = \int_{-1}^{1} 5\sqrt{111}\ du$$

Evaluate the outer integral.

$$A = 2(5\sqrt{111}) = 10\sqrt{111}$$

■ 3. Find the area of the part of the surface $\vec{r}(u, v) = \langle 2\cos u, 5v + 3, 2\sin u \rangle$ that lies within the values of the parameters $\pi/6 \le u \le \pi/3$ and $0 \le v \le 3$.

Solution:

The area of the surface $\vec{r}(u, v)$ inside the region D is given by

$$A = \iint_D |\vec{r}_u \times \vec{r}_v|\ dA$$

Take partial derivatives.

$$\vec{r}_u = \langle -2\sin u, 0, 2\cos u \rangle$$

$$\vec{r}_v = \langle 0, 5, 0 \rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle -2\sin u, 0, 2\cos u \rangle \times \langle 0, 5, 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \mathbf{i}(0 \cdot 0 - 2\cos u \cdot 5) - \mathbf{j}(-2\sin u \cdot 0 - 2\cos u \cdot 0) + \mathbf{k}(-2\sin u \cdot 5 - 0 \cdot 0)$$

$$\vec{r}_u \times \vec{r}_v = -10\cos u\mathbf{i} - 0\mathbf{j} - 10\sin u\mathbf{k}$$

$$\vec{r}_u \times \vec{r}_v = \langle -10\cos u, 0, -10\sin u \rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(-10\cos u)^2 + 0^2 + (-10\sin u)^2}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{100\cos^2 u + 100\sin^2 u}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{100}$$

$$|\vec{r}_u \times \vec{r}_v| = 10$$

Then the area of this part of the surface is

$$A = \int_{\pi/6}^{\pi/3} \int_0^3 10 \ dv \ du$$

Integrate with respect to v.

$$A = \int_{\pi/6}^{\pi/3} 10(3-0) \ du$$

$$A = \int_{\pi/6}^{\pi/3} 30 \ du$$

Integrate with respect to u.

$$A = 30\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = 30 \cdot \frac{\pi}{6} = 5\pi$$

SURFACE INTEGRALS

■ 1. Evaluate the surface integral of the scalar vector field $f(x, y, z) = \ln(x + y + z)$ over the surface $\vec{r} = \langle 3u - 7v + 1, u + 5v + 2, -3u + v - 1 \rangle$, where u changes from 0 to 4 and v changes from -1 to 1.

Solution:

Take partial derivatives.

$$\vec{r}_u = \langle 3,1,-3 \rangle$$

$$\vec{r}_v = \langle -7,5,1 \rangle$$

Take the cross product of these vectors.

$$\vec{r}_u \times \vec{r}_v = \langle 3,1,-3 \rangle \times \langle -7,5,1 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 1 \cdot 1 - (-3) \cdot 5, -3 \cdot 1 - 3 \cdot (-7), 3 \cdot 5 - 1 \cdot (-7) \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 16,18,22 \rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{16^2 + 18^2 + 22^2} = 2\sqrt{266}$$

The function is

$$f(x, y, z) = \ln((3u - 7v + 1) + (u + 5v + 2) + (-3u + v - 1))$$

$$f(x, y, z) = \ln(u - v + 2)$$

So the surface integral is

$$\int_0^4 \int_{-1}^1 2\sqrt{266}\ln(u - v + 2)\ dv\ du$$

Integrate with respect to v, treating u as a constant.

$$2\sqrt{266}\int_0^4 (v - u - 2)\ln(u - v + 2) - v\Big|_{v=-1}^{v=1}\ du$$

$$2\sqrt{266}\int_0^4 (1 - u - 2)\ln(u - 1 + 2) - 1 - ((-1 - u - 2)\ln(u + 1 + 2) + 1)\ du$$

$$2\sqrt{266}\int_0^4 (-u - 1)\ln(u + 1) + (u + 3)\ln(u + 3) - 2\ du$$

$$2\sqrt{266}\int_0^4 (u + 3)\ln(u + 3) - (u + 1)\ln(u + 1) - 2\ du$$

Integrate with respect to u using integration by parts.

$$2\sqrt{266}\left[\frac{1}{2}(u + 3)^2\ln(u + 3) - \frac{(u + 3)^2}{4} - \frac{1}{2}(u + 1)^2\ln(u + 1) + \frac{(u + 1)^2}{4} - 2u\right]\Bigg|_0^4$$

$$2\sqrt{266}\left(\frac{1}{2}(4 + 3)^2\ln(4 + 3) - \frac{(4 + 3)^2}{4} - \frac{1}{2}(4 + 1)^2\ln(4 + 1) + \frac{(4 + 1)^2}{4} - 2 \cdot 4\right)$$

$$-2\sqrt{266}\left(\frac{1}{2}(0 + 3)^2\ln(0 + 3) - \frac{(0 + 3)^2}{4} - \frac{1}{2}(0 + 1)^2\ln(0 + 1) + \frac{(0 + 1)^2}{4} - 2 \cdot 0\right)$$

$$2\sqrt{266}\left(\frac{49}{2}\ln 7-\frac{49}{4}-\frac{25}{2}\ln 5+\frac{25}{4}-8-\frac{9}{2}\ln 3+\frac{9}{4}+\frac{1}{2}\ln 1-\frac{1}{4}\right)$$

$$\sqrt{266}(49\ln 7-25\ln 5-9\ln 3-24)$$

■ 2. Evaluate the surface integral of the scalar vector field $f(x,y,z)=x^2+y^2+4z^2$ over the part of the cylinder $x^2+y^2=9$, where $-2\le z\le 5$.

Solution:

The standard parametrization of the cylinder with radius r and cylindrical axis parallel to the z-axis is

$$x=r\cos\phi$$

$$y=r\sin\phi$$

$$z=z$$

Since the given cylinder has radius 3, plug in $r=3$ and rename the parameters as $\phi\to u$ and $z\to v$.

$$x(u,v)=3\cos u$$

$$y(u,v)=3\sin u$$

$$z(u,v)=v$$

So we get the parametrization of the part of the cylinder we're interested in. Take partial derivatives.

$$\vec{r}_u = \langle -3\sin u, 3\cos u, 0\rangle$$

$$\vec{r}_v = \langle 0,0,1\rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle -3\sin u, 3\cos u, 0\rangle \times \langle 0,0,1\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 3\cos u \cdot 1 - 0 \cdot 0, -(-3\sin u)\cdot 1 + 0 \cdot 0, -3\sin u \cdot 0 - 3\cos u \cdot 0\rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 3\cos u, 3\sin u, 0\rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(3\cos u)^2 + (3\sin u)^2 + 0^2}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{9\cos^2 u + 9\sin^2 u}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{9}$$

$$|\vec{r}_u \times \vec{r}_v| = 3$$

The function is

$$f(x,y,z) = (3\cos u)^2 + (3\sin u)^2 + 4v^2$$

$$f(x,y,z) = 9\cos^2 u + 9\sin^2 u + 4v^2$$

$$f(x,y,z) = 9 + 4v^2$$

So the surface integral is

$$\int_0^{2\pi}\int_{-2}^{5} 3(9+4v^2)\ dv\ du$$

$$\int_0^{2\pi}\int_{-2}^{5} 27+12v^2\ dv\ du$$

Integrate with respect to v.

$$\int_0^{2\pi} 27v+4v^3\Big|_{v=-2}^{v=5}\ du$$

$$\int_0^{2\pi} 27(5)+4(5)^3-(27(-2)+4(-2)^3)\ du$$

$$\int_0^{2\pi} 721\ du$$

Integrate with respect to u.

$$721(2\pi-0)$$

$$1{,}442\pi$$

■ 3. Evaluate the surface integral of the scalar vector field $f(x,y,z)=x^2+y^2+z+1$ over the sphere centered at $(2,-1,-3)$ with radius 2.

Solution:

The standard parametrization of the sphere with radius ρ and center (x_0, y_0, z_0) is

$$x = x_0 + \rho \sin \phi \cos \theta$$

$$y = y_0 + \rho \sin \phi \sin \theta$$

$$z = z_0 + \rho \cos \phi$$

Plug in $\rho = 2$ and $(x_0, y_0, z_0) = (2, -1, -3)$ and rename the parameters $\phi \to u$ and $\theta \to v$.

$$x(u, v) = 2 + 2 \sin u \cos v$$

$$y(u, v) = -1 + 2 \sin u \sin v$$

$$z(u, v) = -3 + 2 \cos u$$

So we get the parametrization of the sphere we're interested in. Take partial derivatives.

$$\vec{r_u} = \langle 2 \cos u \cos v, 2 \cos u \sin v, -2 \sin u \rangle$$

$$\vec{r_v} = \langle -2 \sin u \sin v, 2 \sin u \cos v, 0 \rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 2 \cos u \cos v, 2 \cos u \sin v, -2 \sin u \rangle \times \langle -2 \sin u \sin v, 2 \sin u \cos v, 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 2 \cos u \sin v \cdot 0 - (-2 \sin u) \cdot 2 \sin u \cos v$$

$$-2 \cos u \cos v \cdot 0 - 2 \sin u \cdot (-2 \sin u \sin v)$$

$$+2 \cos u \cos v \cdot 2 \sin u \cos v - 2 \cos u \sin v \cdot (-2 \sin u \sin v) \rangle$$

$$\langle 4\sin^2 u\cos v, 4\sin^2 u\sin v, 4\sin u\cos u\rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(4\sin^2 u\cos v)^2 + (4\sin^2 u\sin v)^2 + (4\sin u\cos u)^2}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{16\sin^2 u}$$

$$|\vec{r}_u \times \vec{r}_v| = 4\sin u$$

The function is

$$f(x, y, z) = (2 + 2\sin u\cos v)^2 + (-1 + 2\sin u\sin v)^2 + (-3 + 2\cos u) + 1$$

$$f(x, y, z) = 8\sin u\cos v - 4\sin u\sin v + 2\sin^2 u - 2\cos^2 u + 2\cos u + 5$$

$$f(x, y, z) = 8\sin u\cos v - 4\sin u\sin v - 2\cos 2u + 2\cos u + 5$$

So the surface integral is

$$\int_0^{\pi}\int_0^{2\pi} (8\sin u\cos v - 4\sin u\sin v - 2\cos 2u + 2\cos u + 5)\cdot 4\sin u \ dv \ du$$

$$4\int_0^{\pi}\int_0^{2\pi} 8\sin^2 u\cos v - 4\sin^2 u\sin v - 2\cos 2u\sin u + 2\cos u\sin u + 5\sin u \ dv \ du$$

Since the integral of sine and cosine functions over a 2π-period is 0,

$$4\int_0^{\pi}\int_0^{2\pi} -2\cos 2u\sin u + 2\cos u\sin u + 5\sin u \ dv \ du$$

Integrate with respect to v.

$$4\int_0^{\pi} 2\pi(-2\cos 2u \sin u + \sin 2u + 5\sin u)\ du$$

$$8\pi\int_0^{\pi} -2\cos 2u \sin u + \sin 2u + 5\sin u\ \ du$$

$$8\pi\int_0^{\pi} -\sin 3u + \sin u + \sin 2u + 5\sin u\ \ du$$

$$8\pi\int_0^{\pi} -\sin 3u + \sin 2u + 6\sin u\ \ du$$

$$8\pi\left(\frac{1}{3}\cos 3u - \frac{1}{2}\cos 2u - 6\cos u\right)\Big|_0^{\pi}$$

$$8\pi\left(\frac{1}{3}\cos 3\pi - \frac{1}{2}\cos 2\pi - 6\cos\pi\right) - 8\pi\left(\frac{1}{3}\cos 0 - \frac{1}{2}\cos 0 - 6\cos 0\right)$$

$$8\pi\left(-\frac{1}{3} - \frac{1}{2} + 6\right) - 8\pi\left(\frac{1}{3} - \frac{1}{2} - 6\right) = \frac{272\pi}{3}$$

SURFACE INTEGRALS OF ORIENTED SURFACES

■ 1. Evaluate the surface integral of the vector field $\vec{F} = \langle x^2, y^2, x + y + z\rangle$ over S, where S is the surface of the cube $[0,2] \times [0,2] \times [0,2]$. Assume that S has a positive orientation.

Solution:

Since the given surface has six sides, we need to evaluate the surface integral for each side individually. Since the surface S is closed and has a positive orientation, we need to choose the set of normal vectors that point outward from the cube.

For the side that sits in the plane $x = 0$, y and z both change from 0 to 2. The parametrization of the surface is $\vec{r} = \langle 0, u, v\rangle$. Take partial derivatives.

$$\vec{r_u} = \langle 0,1,0\rangle$$

$$\vec{r_v} = \langle 0,0,1\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 0,1,0\rangle \times \langle 0,0,1\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 1 \cdot 1 - 0 \cdot 0,\ -0 \cdot 1 + 0 \cdot 0, 0 \cdot 0 - 1 \cdot 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 1,0,0\rangle$$

Since the normal vector must point outward from the region, for this side of the cube $\vec{n} = \langle -1,0,0 \rangle$. The function is

$$\vec{F} = \langle 0^2, u^2, 0 + u + v \rangle = \langle 0, u^2, u + v \rangle$$

so the surface integral is

$$\iint_D \vec{F}(u, v) \cdot \vec{n} \ dA = \int_0^2 \int_0^2 \langle 0, u^2, u + v \rangle \cdot \langle -1,0,0 \rangle \ dv \ du$$

The integral on the right simplifies to

$$\int_0^2 \int_0^2 0 \cdot (-1) + u^2 \cdot 0 + (u + v) \cdot 0 \ dv \ du$$

$$\int_0^2 \int_0^2 0 \ dv \ du = 0$$

So the surface integral over this first side is 0.

For the side that sits in the plane $x = 2$, y and z both change from 0 to 2. The parametrization of the surface is $\vec{r} = \langle 2, u, v \rangle$, and the normal vector for this side of the cube is $\vec{n} = \langle 1,0,0 \rangle$. The function is

$$\vec{F} = \langle 2^2, u^2, 2 + u + v \rangle = \langle 4, u^2, 2 + u + v \rangle$$

so the surface integral is

$$\iint_D \vec{F}(u, v) \cdot \vec{n} \ dA = \int_0^2 \int_0^2 \langle 4, u^2, 2 + u + v \rangle \cdot \langle 1,0,0 \rangle \ dv \ du$$

The integral on the right simplifies to

$$\int_0^2 \int_0^2 4 \cdot 1 + u^2 \cdot 0 + (2 + u + v) \cdot 0 \ dv \ du$$

$$\int_0^2 \int_0^2 4 \ dv \ du$$

Integrate with respect to v, treating u as a constant.

$$\int_0^2 4(2 - 0) \ du$$

$$\int_0^2 8 \ du$$

Integrate with respect to u.

$$8(2 - 0) = 16$$

So the surface integral over this second side is 16.

If we follow this same set of steps for the remaining four sides, we get

- a surface integral of 0 for the side that lies in the plane $y = 0$,
- a surface integral of 16 for the side that lies in the plane $y = 2$
- a surface integral of -8 for the side that lies in the plane $z = 0$
- a surface integral of 16 for the side that lies in the plane $z = 2$

Then the surface integral over the cube's whole surface is

$$\iint_S \vec{F} \cdot dS = 0 + 16 + 0 + 16 - 8 + 16 = 40$$

■ 2. Evaluate the surface integral of the vector field $\vec{F} = \langle x + y, y + z, x + z \rangle$ over the surface S which is the part of the right elliptic cylinder with an axis that coincides with the y-axis, an x-semi-axis of 3, a z-semi-axis of 9, and $-3 \leq y \leq 3$. Assume that S has a positive orientation.

Solution:

The standard parametrization of the right elliptic cylinder with x-semi-axis of a, z-semi-axis of b, and axis that coincides with the y-axis is

$$x = a \cos \phi$$

$$y = y$$

$$z = b \sin \phi$$

Plug in $a = 3$, $b = 9$, and rename parameters the $\phi \to u$ and $y \to v$.

$$x(u, v) = 3 \cos u$$

$$y(u, v) = v$$

$$z(u, v) = 9 \sin u$$

or in vector form,

$$\vec{r}(u,v) = \langle 3\cos u, v, 9\sin u\rangle$$

So we get the parametrization of the part of the cylinder we're interested in. Take partial derivatives.

$$\vec{r_u} = \langle -3\sin u, 0, 9\cos u\rangle$$

$$\vec{r_v} = \langle 0,1,0\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle -3\sin u, 0, 9\cos u\rangle \times \langle 0,1,0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 0\cdot 0 - 9\cos u\cdot 1, -(-3\sin u)\cdot 0 + 9\cos u\cdot 0, -3\sin u\cdot 1 - 0\cdot 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle -9\cos u, 0, -3\sin u\rangle$$

Since the normal vector must point outward from the region, $\vec{n} = \langle 9\cos u, 0, 3\sin u\rangle$. The function is

$$\vec{F} = \langle 3\cos u + v, v + 9\sin u, 3\cos u + 9\sin u\rangle$$

So the surface integral is

$$\iint_D \vec{F}(u,v)\cdot\vec{n}\ dA = \int_{-3}^{3}\int_0^{2\pi} \langle 3\cos u + v, v + 9\sin u, 3\cos u + 9\sin u\rangle$$

$$\cdot\langle 9\cos u, 0, 3\sin u\rangle\ du\ dv$$

Simplify the right side.

$$\int_{-3}^{3}\int_0^{2\pi} (3\cos u + v)(9\cos u) + (v + 9\sin u)(0) + (3\cos u + 9\sin u)(3\sin u)\ du\ dv$$

$$\int_{-3}^{3}\int_{0}^{2\pi} 9v\cos u + 9\sin u\cos u + 27\ du\ dv$$

Integrate with respect to u, treating v as a constant.

$$\int_{-3}^{3}\int_{0}^{2\pi} 9v\cos u + \frac{9}{2}\sin 2u + 27\ du\ dv$$

Since the integral of sine and cosine functions over a 2π-period is 0,

$$\int_{-3}^{3}\int_{0}^{2\pi} 27\ du\ dv$$

$$\int_{-3}^{3} 27(2\pi - 0)\ dv$$

$$\int_{-3}^{3} 54\pi\ dv$$

Integrate with respect to v.

$$54\pi(3-(-3)) = 324\pi$$

■ 3. Evaluate the surface integral of the vector field $\vec{F} = \langle x-2, y+1, z-3\rangle$ over the surface S, where S is the surface of revolution generated by rotating the function $y = x^2 + 1$ around the x-axis for $-2 \le x \le 2$. Assume that S has a negative orientation.

Solution:

The standard parametrization of the surface of revolution where the function $y = f(x)$ is rotated around the x-axis is

$$\vec{r} = \langle u, f(u)\sin v, f(u)\cos v\rangle$$

Plug in $f(x) = x^2 + 1$ with $-2 \le x \le 2$.

$$\vec{r} = \langle u, (u^2 + 1)\sin v, (u^2 + 1)\cos v\rangle$$

So we get the parametrization of the surface we're interested in.

Take the partial derivatives.

$$\vec{r_u} = \langle 1, 2u\sin v, 2u\cos v\rangle$$

$$\vec{r_v} = \langle 0, (u^2 + 1)\cos v, -(u^2 + 1)\sin v\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 1, 2u\sin v, 2u\cos v\rangle \times \left\langle 0,\ \left(u^2 + 1\right)\cos v, -\left(u^2 + 1\right)\sin v\right\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 2u\sin v \cdot (-(u^2 + 1)\sin v) - 2u\cos v \cdot (u^2 + 1)\cos v,$$

$$-1 \cdot (-(u^2 + 1)\sin v) + 2u\cos v \cdot 0,$$

$$1 \cdot (u^2 + 1)\cos v - 2u\sin v \cdot 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle -2u - 2u^3, (u^2 + 1)\sin v, (u^2 + 1)\cos v\rangle$$

Since the surface has a negative orientation, the normal vector must point inward to the region. So the normal vector is

$$\vec{n} = \langle 2u + 2u^3, -(u^2+1)\sin v, -(u^2+1)\cos v \rangle$$

The function is

$$\vec{F} = \langle u - 2, (u^2+1)\sin v + 1, (u^2+1)\cos v - 3 \rangle$$

So the surface integral is

$$\iint_D \vec{F}(u,v) \cdot \vec{n}\ dA = \int_{-2}^{2} \int_0^{2\pi} \langle u - 2, (u^2+1)\sin v + 1, (u^2+1)\cos v - 3 \rangle$$

The integral on the right simplifies to

$$\int_{-2}^{2} \int_0^{2\pi} \langle u - 2, (u^2+1)\sin v + 1, (u^2+1)\cos v - 3 \rangle$$

$$\cdot \langle 2u + 2u^3, -(u^2+1)\sin v, -(u^2+1)\cos v \rangle\ dv\ du$$

$$\int_{-2}^{2} \int_0^{2\pi} (u-2)(2u+2u^3) + ((u^2+1)\sin v + 1)(-(u^2+1)\sin v)$$

$$+((u^2+1)\cos v - 3)(-(u^2+1)\cos v)\ dv\ du$$

$$\int_{-2}^{2} \int_0^{2\pi} (u^2+1)(u^2 - 4u - 1 - \sin v + 3\cos v)\ dv\ du$$

Integrate with respect to v, treating u as a constant. Since the integral of sine and cosine functions over a 2π-period is 0, the integral simplifies to.

$$\int_{-2}^{2} \int_0^{2\pi} (u^2+1)(u^2 - 4u - 1)\ dv\ du$$

$$\int_{-2}^{2} 2\pi(u^2+1)(u^2-4u-1)du$$

Integrate with respect to u.

$$2\pi \int_{-2}^{2} u^4 - 4u^3 - 4u - 1 \ du$$

$$2\pi \left(\frac{u^5}{5} - u^4 - 2u^2 - u \right) \Bigg|_{-2}^{2}$$

$$2\pi \left(\frac{2^5}{5} - 2^4 - 2 \cdot 2^2 - 2 \right) - 2\pi \left(\frac{(-2)^5}{5} - (-2)^4 - 2 \cdot (-2)^2 - (-2) \right)$$

$$2\pi \left(\frac{32}{5} - 16 - 8 - 2 \right) - 2\pi \left(-\frac{32}{5} - 16 - 8 + 2 \right) = \frac{88\pi}{5}$$

FLUX ACROSS THE SURFACE

1. Find the flux of the vector field $\vec{F}$ across the part the plane $x+y+z-2=0$ that lies within the rectangle defined by $-1 \le x \le 1$ and $-1 \le y \le 1$. Assume that S has an upward orientation.

$$\vec{F} = \left\langle \frac{1}{x^2+4}, \frac{1}{4y^2+1}, 0 \right\rangle$$

Solution:

To parametrize the plane, choose $x=u$, $y=v$, and $z=2-u-v$. So

$$\vec{r}(u,v) = \langle u, v, 2-u-v \rangle$$

Take partial derivatives.

$$\vec{r_u} = \langle 1,0,-1 \rangle$$

$$\vec{r_v} = \langle 0,1,-1 \rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 1,0,-1 \rangle \times \langle 0,1,-1 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 0 \cdot (-1) - (-1) \cdot 1, (-1) \cdot 0 - 1 \cdot (-1), 1 \cdot 1 - 0 \cdot 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 1,1,1 \rangle$$

Since the normal vector has an upward orientation, its z-coordinate should be positive, so $\vec{n} = \langle 1,1,1 \rangle$. The function is

$$\vec{F} = \left\langle \frac{1}{u^2+4}, \frac{1}{4v^2+1}, 0 \right\rangle$$

So the flux across the surface is

$$\iint_D \vec{F}(u,v) \cdot \vec{n}\ dA = \int_{-1}^{1} \int_{-1}^{1} \left\langle \frac{1}{u^2+4}, \frac{1}{4v^2+1}, 0 \right\rangle \cdot \langle 1,1,1 \rangle\ dv\ du$$

The right side simplifies to

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{u^2+4} \cdot 1 + \frac{1}{4v^2+1} \cdot 1 + 0 \cdot 1\ dv\ du$$

$$\int_{-1}^{1} \int_{-1}^{1} \frac{1}{u^2+4} + \frac{1}{4v^2+1}\ dv\ du$$

Integrate with respect to v, treating u as a constant.

$$\int_{-1}^{1} \frac{v}{u^2+4} + \frac{1}{2} \arctan 2v \Big|_{-1}^{1}\ du$$

$$\int_{-1}^{1} \frac{1}{u^2+4} + \frac{1}{2} \arctan 2(1) - \left(\frac{-1}{u^2+4} + \frac{1}{2} \arctan 2(-1) \right)\ du$$

$$\int_{-1}^{1} \frac{2}{u^2+4} + \arctan 2\ du$$

Integrate with respect to u.

$$\arctan\frac{u}{2} + u\arctan 2\Big|_{-1}^{1}$$

$$\arctan\frac{1}{2} + \arctan 2 - \left(\arctan\frac{-1}{2} - \arctan 2\right)$$

$$2\arctan\frac{1}{2} + 2\arctan 2 = \pi$$

■ 2. Find the flux of the vector field $\vec{F} = \langle x^2 + y^2 + z^2, 3y, 3\rangle$ across the sphere with radius 4 and center at the origin. Assume that S has a positive orientation.

Solution:

To parametrize the sphere with radius ρ, use a parametrization in spherical coordinates.

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho\cos\phi$$

Plug in $\rho = 4$ and rename the parameters $\phi \to u$ and $\theta \to v$.

$$x(u, v) = 4\sin u\cos v$$

$$y(u, v) = 4\sin u\sin v$$

$$z(u,v) = 4\cos u$$

We get the parametrization of the sphere we're interested in.

Take partial derivatives.

$$\vec{r_u} = \langle 4\cos u\cos v, 4\cos u\sin v, -4\sin u\rangle$$

$$\vec{r_v} = \langle -4\sin u\sin v, 4\sin u\cos v, 0\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 4\cos u\cos v, 4\cos u\sin v, -4\sin u\rangle \times \langle -4\sin u\sin v, 4\sin u\cos v, 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 4\cos u\sin v \cdot 0 - (-4\sin u)\cdot 4\sin u\cos v,$$

$$(-4\sin u)\cdot(-4\sin u\sin v) - 4\cos u\cos v \cdot 0,$$

$$4\cos u\cos v \cdot 4\sin u\cos v - 4\cos u\sin v \cdot (-4\sin u\sin v)\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 16\sin^2 u\cos v, 16\sin^2 u\sin v, 8\sin 2u\rangle$$

Since the surface has a positive orientation, the normal vector must point outward from the sphere, so the sign of the cross product above is correct.

Since $x^2 + y^2 + z^2 = r^2 = 4^2 = 16$, the function is

$$\vec{F} = \langle 16, 3\cdot 4\sin u\sin v, 3\rangle$$

$$\vec{F} = \langle 16, 12\sin u\sin v, 3\rangle$$

So the flux across the sphere is

$$\int_0^\pi \int_0^{2\pi} \langle 16\sin^2 u\cos v, 16\sin^2 u \sin v, 8\sin 2u\rangle \cdot \langle 16, 12\sin u \sin v, 3\rangle \ dv \ du$$

$$\int_0^\pi \int_0^{2\pi} 16\sin^2 u\cos v \cdot 16 + 16\sin^2 u\sin v \cdot 12\sin u\sin v + 8\sin 2u \cdot 3 \ dv \ du$$

$$\int_0^\pi \int_0^{2\pi} 256\sin^2 u\cos v + 192\sin^3 u\sin^2 v + 24\sin 2u \ dv \ du$$

Integrate with respect to v, treating u as a constant. Since the integral of sine and cosine functions over a 2π-period is 0, the integral simplifies to

$$\int_0^\pi \int_0^{2\pi} 192\sin^3 u\sin^2 v + 24\sin 2u \ dv \ du$$

$$24\int_0^\pi \int_0^{2\pi} 8\sin^3 u \cdot \frac{1}{2}(1-\cos 2v) + \sin 2u \ dv \ du$$

$$24\int_0^\pi \int_0^{2\pi} 4\sin^3 u - 4\sin^3 u\cos 2v + \sin 2u \ dv \ du$$

$$24\int_0^\pi 4v\sin^3 u - 2\sin^3 u\sin 2v + v\sin 2u \Big|_{v=0}^{v=2\pi} \ du$$

$$24\int_0^\pi 4\cdot 2\pi\sin^3 u - 2\sin^3 u\sin 4\pi + 2\pi\sin 2u$$

$$-(4\cdot 0\cdot\sin^3 u - 2\sin^3 u\sin 0 + 0\cdot\sin 2u) \ du$$

$$24\int_0^\pi 8\pi\sin^3 u + 2\pi\sin 2u \ du$$

$$48\pi \int_0^{\pi} 4\sin^3 u + \sin 2u \ du$$

Integrate with respect to u, using the following trigonometric identity.

$$\sin^3 \alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$$

$$48\pi \int_0^{\pi} 3\sin u - \sin 3u + \sin 2u \ du$$

$$48\pi \left(-3\cos u + \frac{1}{3}\cos 3u - \frac{1}{2}\cos 2u\right)\Big|_0^{\pi}$$

$$48\pi \left(-3\cos\pi + \frac{1}{3}\cos 3\pi - \frac{1}{2}\cos 2\pi\right) - 48\pi\left(-3\cos 0 + \frac{1}{3}\cos 0 - \frac{1}{2}\cos 0\right)$$

$$48\pi\left(3 - \frac{1}{3} - \frac{1}{2}\right) - 48\pi\left(-3 + \frac{1}{3} - \frac{1}{2}\right) = 256\pi$$

■ 3. Suppose the velocity of a fluid in three-dimensional space is described by the vector field $\overrightarrow{F} = \langle x^2+1, y^2+1, z^2+1\rangle$. Find the volume of fluid crossing the disk S defined by $(x+1)^2 + (y-2)^2 \le 4$ in the xy-plane per 10 units of time. Assume that S has an upward orientation.

Solution:

In order to parametrize the points inside the given circle in the xy-plane, with center at $(-1,2)$ and radius $r \le 2$, use standard polar coordinates.

$$x = -1 + r\cos\phi$$

$$y = 2 + r\sin\phi$$

$$z = 0$$

Rename the parameters as $\phi \to u$ and $r \to v$. Then

$$\vec{r} = \langle -1 + v\cos u, 2 + v\sin u, 0\rangle$$

This is the parametrization of the disk we're interested in. Take partial derivatives.

$$\vec{r_u} = \langle -v\sin u, v\cos u, 0\rangle$$

$$\vec{r_v} = \langle \cos u, \sin u, 0\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle -v\sin u, v\cos u, 0\rangle \times \langle \cos u, \sin u, 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle v\cos u \cdot 0 - 0 \cdot \sin u, 0 \cdot \cos u - (-v\sin u) \cdot 0,$$

$$-v\sin u \cdot \sin u - v\cos u \cdot \cos u\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 0, 0, -v\rangle$$

Since the normal vector has an upward orientation, its z-coordinate must be positive. So since v is positive inside the circle, and $\vec{n} = \langle 0,0,v\rangle$. The function is

$$\vec{F} = \langle (-1 + v\cos u)^2 + 1, (2 + v\sin u)^2 + 1, 0^2 + 1\rangle$$

$$\vec{F} = \langle (-1 + v\cos u)^2 + 1, (2 + v\sin u)^2 + 1, 1\rangle$$

So the flux across the disk is

$$\iint_D \vec{F}(u,v)\cdot\vec{n}\ dA=\int_0^2\int_0^{2\pi}\langle(-1+v\cos u)^2+1,(2+v\sin u)^2+1,1\rangle$$

$$\cdot\langle 0,0,v\rangle\ du\ dv$$

$$\int_0^2\int_0^{2\pi}\langle(-1+v\cos u)^2+1,(2+v\sin u)^2+1,1\rangle\cdot\langle 0,0,v\rangle\ du\ dv$$

$$\int_0^2\int_0^{2\pi}((-1+v\cos u)^2+1)\cdot 0+((2+v\sin u)^2+1)\cdot 0+1\cdot(v)\ du\ dv$$

$$\int_0^2\int_0^{2\pi} v\ du\ dv$$

Integrate with respect to u, treating v as a constant.

$$\int_0^2 v(2\pi-0)\ dv$$

$$\int_0^2 2\pi v\ dv$$

Integrate with respect to v.

$$\pi v^2\Big|_0^2$$

$$\pi\cdot 2^2-\pi\cdot 0^2=4\pi$$

So the volume of fluid crossing the disk per 10 units of time is $10\cdot 4\pi=40\pi$.

STOKES' THEOREM

1. Use Stokes' theorem to evaluate the surface integral where S is the part of the elliptic paraboloid $z+x^2+y^2-3=0$ above the plane $z=-1$. Assume that S has a positive orientation.

$$\iint_S \text{curl}\ \vec{F}\cdot d\vec{S}$$

$$\vec{F}=\langle y+2,\ -z^2,\ 2xy\rangle$$

Solution:

In order to find C, the curve of intersection of the elliptic paraboloid $z+x^2+y^2-3=0$ and the plane $z=-1$, plug $z=-1$ into the equation of the paraboloid.

$$(-1)+x^2+y^2-3=0$$

$$x^2+y^2=4$$

So C is the circle that lies in the plane $z=-1$, centered at $(0,0,-1)$ with radius 2.

Since the surface S is positively oriented, the normal vectors point outward from the surface, and therefore, by the right-hand rule, the circle C has counterclockwise direction. Therefore, its parametrization is

$$x(t)=2\cos t$$

$$y(t) = 2\sin t$$

$$z(t) = -1$$

Which means the vector function is $\vec{r}(t) = \langle 2\cos t, 2\sin t, -1\rangle$, and we can take the derivative of $\vec{r}$ to get $\vec{r}'(t) = \langle -2\sin t, 2\cos t, 0\rangle$. The function is

$$\vec{F}(t) = \langle 2\sin t + 2, -(-1)^2, 2(2\cos t)(2\sin t)\rangle$$

$$\vec{F}(t) = \langle 2\sin t + 2, -1, 8\cos t\sin t\rangle$$

So the line integral is

$$\int_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\ dt = \int_0^{2\pi} \langle 2\sin t + 2, -1, 8\cos t\sin t\rangle \cdot \langle -2\sin t, 2\cos t, 0\rangle\ dt$$

The integral on the right side of the equation simplifies to

$$\int_0^{2\pi} (2\sin t + 2)(-2\sin t) + (-1)(2\cos t) + (8\cos t\sin t)(0)\ dt$$

$$\int_0^{2\pi} -4\sin^2 t - 4\sin t - 2\cos t\ dt$$

$$-2\int_0^{2\pi} 2\sin^2 t + 2\sin t + \cos t\ dt$$

$$-2\int_0^{2\pi} 2\left(\frac{1}{2} - \frac{1}{2}\cos(2t)\right) + 2\sin t + \cos t\ dt$$

$$-2\int_0^{2\pi} 1 - \cos(2t) + 2\sin t + \cos t\ dt$$

Integrate, then evaluate over the interval.

$$-2\left(t - \frac{1}{2}\sin(2t) - 2\cos t + \sin t\right)\Bigg|_0^{2\pi}$$

$$\sin(2t) - 2\sin t + 4\cos t - 2t\Big|_0^{2\pi}$$

$$\sin(2(2\pi)) - 2\sin(2\pi) + 4\cos(2\pi) - 2(2\pi) - (\sin(2(0)) - 2\sin(0) + 4\cos(0) - 2(0))$$

$$\sin(4\pi) - 2\sin(2\pi) + 4\cos(2\pi) - 4\pi - \sin(0) + 2\sin(0) - 4\cos(0) + 2(0)$$

$$0 - 2(0) + 4(1) - 4\pi - 0 + 2(0) - 4(1)$$

$$4 - 4\pi - 4$$

$$-4\pi$$

■ 2. Use Stokes' theorem to evaluate the line integral, where C is the rectangle $KMNO$ with vertices $K(0,0,0)$, $M(0,6,0)$, $N(3,6,0)$ and $O(3,0,0)$. Assume that C has a clockwise orientation as viewed from the positive z-axis.

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = \langle 2xyz, x^2 + y^2, 2xyz \rangle$$

Solution:

The parametrization of the points inside the rectangle $KMNO$ is

$$x(u, v) = u$$

$$y(u, v) = v$$

$$z(u, v) = 0$$

So the vector function is $\vec{r}(u, v) = \langle u, v, 0 \rangle$. Take the partial derivatives of $\vec{r}$.

$$\vec{r}_u(t) = \langle 1,0,0 \rangle$$

$$\vec{r}_v(t) = \langle 0,1,0 \rangle$$

Take the cross product.

$$\vec{r}_u \times \vec{r}_v = \langle 1,0,0 \rangle \times \langle 0,1,0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0 \cdot 0 - 0 \cdot 1, \ -1 \cdot 0 + 0 \cdot 0, 1 \cdot 1 - 0 \cdot 0 \rangle$$

$$\vec{r}_u \times \vec{r}_v = \langle 0,0,1 \rangle$$

Since the normal vector of the surface must point in the negative direction of the z-axis, $\vec{n} = \langle 0,0,-1 \rangle$. Evaluate curl $\vec{F}$. The curl of a vector field in three dimensions is given by

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

So

$$\text{curl } \vec{F} = \text{curl } \langle 2xyz, x^2 + y^2, 2xyz \rangle$$

$$\text{curl } \vec{F} = \langle 2xz - 0, 2xy - 2yz, 2x - 2xz \rangle$$

$$\text{curl } \vec{F} = \langle 2xz, 2xy - 2yz, 2x - 2xz \rangle$$

Plug in the parametrization $\vec{r}(u, v) = \langle u, v, 0 \rangle$.

$$\langle 2u \cdot 0, 2uv - 2v \cdot 0, 2u - 2u \cdot 0 \rangle$$

$$\langle 0, 2uv, 2u \rangle$$

So the surface integral is

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \ dA = \int_0^3 \int_0^6 \langle 0, 2uv, 2u \rangle \cdot \langle 0, 0, -1 \rangle \ dv \ du$$

Simplify the integral on the right side of the equation.

$$\int_0^3 \int_0^6 0 \cdot 0 + 2uv \cdot 0 + 2u \cdot (-1) \ dv \ du$$

$$\int_0^3 \int_0^6 -2u \ dv \ du$$

Integrate with respect to v, treating u as a constant.

$$\int_0^3 -2u(6 - 0) \ du$$

$$\int_0^3 -12u \ du$$

Integrate with respect to u.

$$-6u^2\Big|_0^3$$

$$-6 \cdot 3^2 - (-6 \cdot 0^2) = -54$$

■ 3. Use Stokes' theorem to evaluate the line integral, where C is the boundary curve of the semicircle centered at the origin with radius 4 that lies in the xz-plane, and with $z \geq 0$. Assume that C has a counterclockwise orientation as viewed from the positive y-axis.

$$\int_C \vec{F} \cdot d\vec{r}$$

$$\vec{F} = \langle x + 3y - z + 2, x - 5y + 9z - 7, -5x - y + 2z + 6 \rangle$$

Solution:

The parametrization of the points inside the semicircle is

$$x(u, v) = v \cos u$$

$$y(u, v) = 0$$

$$z(u, v) = v \sin u$$

The vector form of these parametric equations is

$$\vec{r}(u, v) = \langle v \cos u, 0, v \sin u \rangle$$

Take the partial derivatives of $\vec{r}$.

$$\vec{r_u}(t) = \langle -v\sin u, 0, v\cos u\rangle$$

$$\vec{r_v}(t) = \langle \cos u, 0, \sin u\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle -v\sin u, 0, v\cos u\rangle \times \langle \cos u, 0, \sin u\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 0 \cdot \sin u - v\cos u \cdot 0,\ v\cos u \cdot \cos u + v\sin u \cdot \sin u,\ -v\sin u \cdot 0 - 0 \cdot \cos u\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 0, v\cos^2 u + v\sin^2 u, 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 0, v, 0\rangle$$

Since the normal vector of the surface must point in the positive direction of the y-axis, and since v is always positive within the semicircle, $\vec{n} = \langle 0, v, 0\rangle$. Evaluate curl $\vec{F}$. The curl of a vector field in three dimensions is given by

$$\text{curl } \vec{F} = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

So

$$\text{curl } \vec{F} = \text{curl } \langle x + 3y - z + 2, x - 5y + 9z - 7, -5x - y + 2z + 6\rangle$$

$$\text{curl } \vec{F} = \langle -1 - 9, -1 - (-5), 1 - 3\rangle$$

$$\text{curl } \vec{F} = \langle -10, 4, -2\rangle$$

So the surface integral is

$$\iint_S \text{curl } \vec{F} \cdot \vec{n} \ dA = \int_0^4 \int_0^\pi \langle -10,4,-2 \rangle \cdot \langle 0,v,0 \rangle \ du \ dv$$

The integral on the right simplifies to

$$\int_0^4 \int_0^\pi -10 \cdot 0 + 4 \cdot v + (-2) \cdot 0 \ du \ dv$$

$$\int_0^4 \int_0^\pi 4v \ du \ dv$$

Integrate with respect to u, treating v as a constant.

$$\int_0^4 4v(\pi - 0) \ dv$$

$$\int_0^4 4\pi v \ dv$$

Integrate with respect to v.

$$4\pi \cdot \frac{v^2}{2} \Big|_0^4$$

$$2\pi v^2 \Big|_0^4$$

$$2\pi \cdot 4^2 - 2\pi \cdot 0^2 = 32\pi$$

DIVERGENCE THEOREM

■ 1. Use the Divergence theorem to evaluate the surface integral, where S is the boundary surface of the box $[-3,4] \times [3,5] \times [-3,0]$. Assume that S has a negative orientation.

$$\iint_S \vec{F} \cdot d\vec{S}$$

$$\vec{F} = \langle x + e^{z^2 - y^2}, \ln y + y + x^4, z^2 - \arcsin(x + y) \rangle$$

Solution:

The parametrization of the box is given by $x = x$, $y = y$, and $z = z$, where x changes from -3 to 4, y changes from 3 to 5, and z changes from -3 to 0. Evaluate the divergence of $\vec{F}$.

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{div } \vec{F} = 1 + \frac{1}{y} + 1 + 2z$$

$$\text{div } \vec{F} = 2 + \frac{1}{y} + 2z$$

So the triple integral is

$$\iiint_E \text{div } \vec{F} \ dV = \int_{-3}^{0}\int_{3}^{5}\int_{-3}^{4} (2 + \frac{1}{y} + 2z) \ dx \ dy \ dz$$

Integrate with respect to x, treating y and z as constants.

$$\iiint_E \text{div } \vec{F} \ dV = \int_{-3}^{0}\int_{3}^{5} \left(2 + \frac{1}{y} + 2z\right) \cdot (x)\Big|_{-3}^{4} \ dy \ dz$$

Simplify the integral on the right side of this equation.

$$\int_{-3}^{0}\int_{3}^{5} \left(2 + \frac{1}{y} + 2z\right) \cdot (4 - (-3)) \ dy \ dz$$

$$7\int_{-3}^{0}\int_{3}^{5} 2 + \frac{1}{y} + 2z \ dy \ dz$$

Integrate with respect to y, treating z and as a constant.

$$7\int_{-3}^{0} 2y + \ln y + 2yz\Big|_{y=3}^{y=5} \ dz$$

$$7\int_{-3}^{0} 2 \cdot 5 + \ln 5 + 2 \cdot 5 \cdot z - (2 \cdot 3 + \ln 3 + 2 \cdot 3 \cdot z) \ dz$$

$$7\int_{-3}^{0} 4 + \ln\frac{5}{3} + 4z \ dz$$

Integrate with respect to z.

$$7\left(4z + \ln\frac{5}{3} \cdot z + 2z^2\right)\Big|_{-3}^{0}$$

$$7\left(4\cdot 0+\ln\frac{5}{3}\cdot 0+2\cdot 0^2\right)-7\left(4\cdot(-3)+\ln\frac{5}{3}\cdot(-3)+2(-3)^2\right)$$

$$-7\left(-12-3\ln\frac{5}{3}+18\right)=-42+21\ln\frac{5}{3}$$

Since the surface has a negative orientation, the sign of the answer needs to be reversed.

$$\iint_S \vec{F}\cdot d\vec{S}=42-21\ln\frac{5}{3}$$

■ 2. Use the Divergence theorem to evaluate the surface integral where S is the boundary surface of the part of the cylinder $y^2+z^2=25$ with $-2\le x\le 4$. Assume that S has a positive orientation.

$$\iint_S \vec{F}\cdot d\vec{S}$$

$$\vec{F}=\langle x^3+y^3,y^3+z^3,z^3+x^3\rangle$$

Solution:

To parametrize the region inside the cylinder, use standard cylindrical coordinates for the cylinder with radius 5 and axis that coincides with the x-axis.

$$x=x$$

$$y = r\cos\phi$$

$$z = r\sin\phi$$

Evaluate the divergence of $\vec{F}$.

$$\text{div } \vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\text{div } \vec{F} = 3x^2 + 3y^2 + 3z^2$$

Plug in the parametrization for x, y, and z.

$$\text{div } \vec{F}(x, r, \phi) = 3x^2 + 3(r\cos\phi)^2 + 3(r\sin\phi)^2$$

$$\text{div } \vec{F}(x, r, \phi) = 3x^2 + 3r^2\cos^2\phi + 3r^2\sin^2\phi$$

$$\text{div } \vec{F}(x, r, \phi) = 3x^2 + 3r^2$$

So the triple integral is

$$\iiint_E \text{div } \vec{F} \ dV = \int_{-2}^{4}\int_{0}^{5}\int_{0}^{2\pi} 3x^2 + 3r^2 \ d\phi \ dr \ dx$$

Integrate with respect to ϕ treating r and x as a constants.

$$\int_{-2}^{4}\int_{0}^{5} (3x^2 + 3r^2) \cdot (2\pi - 0) \ dr \ dx$$

$$2\pi\int_{-2}^{4}\int_{0}^{5} 3x^2 + 3r^2 \ dr \ dx$$

Integrate with respect to r, treating x as a constant.

$$2\pi \int_{-2}^{4} 3x^2r + r^3 \Big|_0^5 \ dx$$

$$2\pi \int_{-2}^{4} 3x^2 \cdot 5 + 5^3 - (3x^2 \cdot 0 + 0^3) \ dx$$

$$2\pi \int_{-2}^{4} 15x^2 + 125 \ dx$$

Integral with respect to x.

$$2\pi(5x^3 + 125x) \Big|_{-2}^{4}$$

$$2\pi(5 \cdot 4^3 + 125 \cdot 4) - 2\pi(5 \cdot (-2)^3 + 125 \cdot (-2)) = 2{,}220\pi$$

Since the surface has a positive orientation, the sign of the answer is correct.

■ 3. Use the Divergence theorem to evaluate the triple integral where E is the sphere centered at the origin with radius 4.

$$\iiint_E \text{div } \vec{F} \ dV$$

$$\vec{F} = \left\langle \frac{x^2 + y^2 + z^2}{4}, -6y, 6 \right\rangle$$

Solution:

To parametrize the sphere with radius ρ, use a parametrization in spherical coordinates.

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

Plug in $\rho = 4$ and rename parameters the $\phi \to u$ and $\theta \to v$.

$$x(u, v) = 4\sin u \cos v$$

$$y(u, v) = 4\sin u \sin v$$

$$z(u, v) = 4\cos u$$

In vector form, these equations are

$$\vec{r} = \langle 4\sin u \cos v, 4\sin u \sin v, 4\cos u\rangle$$

Take partial derivatives of $\vec{r}$.

$$\vec{r_u} = \langle 4\cos u \cos v, 4\cos u \sin v, -4\sin u\rangle$$

$$\vec{r_v} = \langle -4\sin u \sin v, 4\sin u \cos v, 0\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 4\cos u \cos v, 4\cos u \sin v, -4\sin u\rangle \times \langle -4\sin u \sin v, 4\sin u \cos v, 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 4\cos u \sin v \cdot 0 - (-4\sin u) \cdot 4\sin u \cos v,$$

$$(-4\sin u)\cdot(-4\sin u\sin v) - 4\cos u\cos v\cdot 0,$$

$$4\cos u\cos v\cdot 4\sin u\cos v - 4\cos u\sin v\cdot(-4\sin u\sin v)\rangle$$

$$\vec{r_u}\times\vec{r_v} = \langle 16\sin^2 u\cos v, 16\sin^2 u\sin v, 8\sin 2u\rangle$$

Since the surface has positive orientation, the normal vector must point outward from the sphere, so the sign of the cross product is correct.

The function is

$$\vec{F}(u,v) = \left\langle \frac{x^2(u,v)+y^2(u,v)+z^2(u,v)}{4}, -6y(u,v), 6\right\rangle$$

Since $x^2+y^2+z^2=\rho^2=4^2=16$,

$$\vec{F}(u,v) = \left\langle \frac{16}{4}, -6\cdot 4\sin u\sin v, 6\right\rangle$$

$$\vec{F}(u,v) = \langle 4, -24\sin u\sin v, 6\rangle$$

So the surface integral is

$$\int_0^{\pi}\int_0^{2\pi} \langle 4, -24\sin u\sin v, 6\rangle\cdot\langle 16\sin^2 u\cos v, 16\sin^2 u\sin v, 8\sin 2u\rangle\ dv\ du$$

$$\int_0^{\pi}\int_0^{2\pi} 4\cdot 16\sin^2 u\cos v - 24\sin u\sin v\cdot 16\sin^2 u\sin v + 6\cdot 8\sin 2u\ dv\ du$$

$$16\int_0^{\pi}\int_0^{2\pi} 4\sin^2 u\cos v - 24\sin^3 u\sin^2 v + 3\sin 2u\ dv\ du$$

Integrate with respect to v, treating u as a constant. Remember that the integral of cosine functions over a 2π-period is 0.

$$16\int_0^{\pi}\int_0^{2\pi} -24\sin^3 u\sin^2 v + 3\sin 2u\ dv\ du$$

$$48\int_0^{\pi}\int_0^{2\pi} -4\sin^3 u\cdot(1-\cos 2v) + \sin 2u\ dv\ du$$

$$48\int_0^{\pi}\int_0^{2\pi} -4\sin^3 u + 4\sin^3 u\cos 2v + \sin 2u\ dv\ du$$

$$48\int_0^{\pi}\int_0^{2\pi} -4\sin^3 u + \sin 2u\ dv\ du$$

$$48\int_0^{\pi} -4\sin^3 u + \sin 2u\ du\cdot\int_0^{2\pi}\ dv$$

$$48\int_0^{\pi} -4\sin^3 u + \sin 2u\ du\cdot 2\pi$$

Integrate with respect to u, using the trigonometric identity

$$\sin^3\alpha = \frac{1}{4}(3\sin\alpha - \sin 3\alpha)$$

The integral becomes

$$96\pi\int_0^{\pi} -3\sin u + \sin 3u + \sin 2u\ du$$

$$96\pi\left(3\cos u - \frac{1}{3}\cos 3u - \frac{1}{2}\cos 2u\right)\Bigg|_0^{\pi}$$

$$96\pi\left(3\cos\pi-\frac{1}{3}\cos 3\pi-\frac{1}{2}\cos 2\pi\right)-96\pi\left(3\cos 0-\frac{1}{3}\cos 0-\frac{1}{2}\cos 0\right)$$

$$96\pi\left(-3+\frac{1}{3}-\frac{1}{2}\right)-96\pi\left(3-\frac{1}{3}-\frac{1}{2}\right)=-512\pi$$

W W W . K R I S T A K I N G M A T H . C O M