Calculus 3 Workbook Solutions

Cross products

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CROSS PRODUCT OF TWO VECTORS

■ 1. Find the vector $\vec{a}$ given that $\vec{a} \times \vec{b} = \vec{c}$, where $\vec{a} = \langle 1, a_2, a_3 \rangle$, $\vec{b} = \langle 3,1,1 \rangle$, and $\vec{c} = \langle 1,2,-5 \rangle$.

Solution:

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle b_1, b_2, b_3 \rangle = \langle 3,1,1 \rangle$ and $a_1 = 1$.

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2 - a_3) - \mathbf{j}(1 - 3a_3) + \mathbf{k}(1 - 3a_2)$$

Since the cross product is $\vec{c} = \mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, we get a system of equations.

$$a_2 - a_3 = 1$$

$$-(1 - 3a_3) = 2$$

$$1 - 3a_2 = -5$$

Solve the system.

$$1 + 5 = 3a_2,\ 6 = 3a_2,\ a_2 = 2$$

$$1 - 3a_3 = -2,\ \ -3a_3 = -3,\ a_3 = 1$$

$$a_2 - a_3 = 2 - 1 = 1$$

In the general case, the vector equation $\vec{x} \times \vec{b} = \vec{c}$ has an infinite number of solutions for $\vec{x}$.

■ 2. Find the cross product $\vec{a} \times \vec{a}$ for an arbitrary vector $\vec{a}$.

Solution:

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Plug in $\langle b_1, b_2, b_3 \rangle = \langle a_1, a_2, a_3 \rangle$.

$$\vec{a} \times \vec{a} = \mathbf{i}(a_2a_3 - a_3a_2) - \mathbf{j}(a_1a_3 - a_3a_1) + \mathbf{k}(a_1a_2 - a_2a_1)$$

$$\vec{a} \times \vec{a} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0)$$

So the cross product of the vector by itself is equal to the zero vector.

$$\vec{O} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}$$

In the general case, the cross product of any vector by a vector with the same direction is always equal to the zero vector.

■ 3. Find the cross product $\vec{a} \times \text{proj}_{xy}\vec{a}$, where $\vec{a} = \langle 4,5,-3 \rangle$ and $\text{proj}_{xy}\vec{a}$ is the vector projection of the vector $\vec{a}$ onto the xy-plane.

Solution:

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

The vector projection of the vector onto the xy-plane is the vector with the same x and y coordinates, but a $z = 0$ coordinate.

$$\text{proj}_{xy}\vec{a} = \langle 4,5,0 \rangle$$

Calculate the cross product.

$$\vec{a} \times \text{proj}_{xy}\vec{a} = \mathbf{i}(5 \cdot 0 - (-3) \cdot 5) - \mathbf{j}(4 \cdot 0 - (-3) \cdot 4) + \mathbf{k}(4 \cdot 5 - 5 \cdot 4)$$

$$\vec{a} \times \text{proj}_{xy}\vec{a} = 15\mathbf{i} - 12\mathbf{j} + 0\mathbf{k}$$

In the general case, the cross product of any vector by its projection onto the plane is always a vector that lies in the plane.

VECTOR ORTHOGONAL TO THE PLANE

■ 1. Find the vector orthogonal to the plane which passes through the point $A(2,3,1)$ and the z-axis.

Solution:

In order to find the vector orthogonal to the plane, we can compute the cross product of any two nonparallel vectors lying in the plane. For simplicity, let's take the unit vector along the z-axis, $\vec{u_z} = \langle 0,0,1 \rangle$, and the vector from the origin to the point A, $\vec{a} = \langle 2,3,1 \rangle$.

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{u_z} = \langle 2,3,1 \rangle \times \langle 0,0,1 \rangle$$

$$\vec{a} \times \vec{u_z} = \mathbf{i}(3 \cdot 1 - 1 \cdot 0) - \mathbf{j}(2 \cdot 1 - 1 \cdot 0) + \mathbf{k}(2 \cdot 0 - 3 \cdot 0)$$

$$\vec{a} \times \vec{u_z} = 3\mathbf{i} - 2\mathbf{j} + 0\mathbf{k}$$

There are infinite number of vectors orthogonal to a plane; all of them may have different magnitudes, but the same (or opposite) direction.

■ 2. Find the equation of the plane that passes through the point D and is parallel to the plane ABC, if $A(1,2,-2)$, $B(1,4,3)$, $C(-5,3,-1)$, and $D(2,-4,7)$.

Solution:

Since the two planes are parallel, they have equal normal vectors. In order to find the vector orthogonal to the plane ABC, we can calculate the cross product of any two nonparallel vectors lying in the plane, like $\overrightarrow{b}=\overrightarrow{AB}$ and $\overrightarrow{c}=\overrightarrow{AC}$.

$$\overrightarrow{b}=\langle 1-1,4-2,3-(-2)\rangle=\langle 0,2,5\rangle$$

$$\overrightarrow{c}=\langle -5-1,3-2,-1-(-2)\rangle=\langle -6,1,1\rangle$$

The cross product of two vectors $\overrightarrow{c}$ and $\overrightarrow{b}$ is given by

$$\overrightarrow{a}\times\overrightarrow{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Calculate the cross product.

$$\overrightarrow{c}\times\overrightarrow{b}=\langle -6,1,1\rangle\times\langle 0,2,5\rangle$$

$$\overrightarrow{c}\times\overrightarrow{b}=\mathbf{i}(1\cdot 5-1\cdot 2)-\mathbf{j}(-6\cdot 5-1\cdot 0)+\mathbf{k}(-6\cdot 2-1\cdot 0)$$

$$\overrightarrow{c}\times\overrightarrow{b}=3\mathbf{i}+30\mathbf{j}-12\mathbf{k}$$

Plug in the vector $\overrightarrow{n}$ and the point D into the equation of the plane in general form.

$$n_1(x-x_0)+n_2(y-y_0)+n_3(z-z_0)=0$$

$$3(x-2)+30(y+4)-12(z-7)=0$$

$$(x-2)+10(y+4)-4(z-7)=0$$

$$x+10y-4z+66=0$$

■ 3. Find the equation of the line that passes through the point $A(-2,3,4)$ and is orthogonal to the plane that includes the vectors $\vec{a}=\langle 2,4,0\rangle$ and $\vec{b}=\langle -1,1,2\rangle$.

Solution:

Since the line is orthogonal to the plane, its direction vector is equal to the normal vector of the plane. In order to find the vector orthogonal to the plane, we can compute the cross product $\vec{a}\times\vec{b}$.

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}\times\vec{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Calculate the cross product.

$$\vec{a}\times\vec{b}=\langle 2,4,0\rangle\times\langle -1,1,2\rangle$$

$$\vec{a}\times\vec{b}=\mathbf{i}(4\cdot 2-0\cdot 1)-\mathbf{j}(2\cdot 2-0\cdot(-1))+\mathbf{k}(2\cdot 1-4\cdot(-1))$$

$$\vec{a}\times\vec{b}=8\mathbf{i}-4\mathbf{j}+6\mathbf{k}$$

So the normal vector to the plane is $\vec{n}=\langle 8,-4,6\rangle$.

The parametric equations of a line in general form are

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Plug the vector $\vec{n} = \langle 8, -4, 6 \rangle$ and the point $A(-2,3,4)$ into these equations.

$$x = -2 + 8t$$

$$y = 3 - 4t$$

$$z = 4 + 6t$$

VOLUME OF THE PARALLELEPIPED FROM VECTORS

1. Find the height of the parallelepiped given that its volume is 670, and that the vectors $\vec{a} = \langle 1,0,-1 \rangle$ and $\vec{b} = \langle 2,3,5 \rangle$ are the adjacent edges of its base.

Solution:

The volume of the parallelepiped is equal to the product of its height, and the area of its base. So its height is equal to the volume, divided by the base area.

The area of parallelogram is equal to the magnitude of the cross product of its edge vectors. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 1,0,-1 \rangle \times \langle 2,3,5 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(0 \cdot 5 - (-1) \cdot 3) - \mathbf{j}(1 \cdot 5 - (-1) \cdot 2) + \mathbf{k}(1 \cdot 3 - 0 \cdot 2)$$

$$\vec{a} \times \vec{b} = 3\mathbf{i} - 7\mathbf{j} + 3\mathbf{k}$$

The area of the base is

$$|\vec{a} \times \vec{b}| = \sqrt{3^2 + (-7)^2 + 3^2} = \sqrt{67}$$

Therefore, the height of the parallelepiped is

$$\text{Height} = \frac{\text{Volume}}{\text{Area of Base}} = \frac{670}{\sqrt{67}} = 10\sqrt{67}$$

■ 2. Find the volume of the tetrahedron whose adjacent edges are the vectors $\vec{a} = \langle 0,0,3 \rangle$, $\vec{b} = \langle 2,1,4 \rangle$, and $\vec{c} = \langle -1,-2,1 \rangle$.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = [\text{Height}] \cdot [\text{Area of parallelogram}] = |(\vec{a} \times \vec{b}) \cdot \vec{c}\,|$$

The volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{3}[\text{Height}] \cdot \frac{1}{2}[\text{Area of parallelogram}] = \frac{1}{6}V_{\text{parallelepiped}}$$

Therefore,

$$V_{\text{tetrahedron}} = \frac{1}{6}|(\vec{a} \times \vec{b}) \cdot \vec{c}\,|$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 0,0,3 \rangle \times \langle 2,1,4 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(0 \cdot 4 - 3 \cdot 1) - \mathbf{j}(0 \cdot 4 - 3 \cdot 2) + \mathbf{k}(0 \cdot 1 - 0 \cdot 2)$$

$$\vec{a} \times \vec{b} = -3\mathbf{i} + 6\mathbf{j} + 0\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle -3,6,0 \rangle \cdot \langle -1,-2,1 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = (-3)(-1) + (6)(-2) + (0)(1) = -9$$

Therefore, the volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{6}|-9| = 1.5$$

■ 3. Find the value of p such that the volume of the parallelepiped with adjacent edges $\vec{a} = \langle 0,2,3 \rangle$, $\vec{b} = \langle 1,-2,1 \rangle$, and $\vec{c} = \langle p,p,p \rangle$ is equal to 63.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 0,2,3 \rangle \times \langle 1,-2,1 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(2 \cdot 1 - 3 \cdot (-2)) - \mathbf{j}(0 \cdot 1 - 3 \cdot 1) + \mathbf{k}(0 \cdot (-2) - 2 \cdot 1)$$

$$\vec{a} \times \vec{b} = 8\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle 8,3,-2 \rangle \cdot \langle p,p,p \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 8p + 3p - 2p = 9p$$

Since the volume of the parallelepiped is equal to 63, we can write an equation for p.

$$9p = 63$$

$$p = 7$$

VOLUME OF THE PARALLELEPIPED FROM ADJACENT EDGES

■ 1. Find the volume of tetrahedron $ABCD$, given $A(2,0,3)$, $B(-1,1,3)$, $C(4,5,-2)$, and $D(2,2,3)$.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

The volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{6} V_{\text{parallelepiped}}$$

Therefore,

$$V_{\text{tetrahedron}} = \frac{1}{6} |(\vec{a} \times \vec{b}) \cdot \vec{c}|$$

For the scalar triple product, we can choose any three adjacent edges with a common initial point. For example, let's take

$$\vec{a} = \overrightarrow{AB} = \langle -1-2, 1-0, 3-3 \rangle = \langle -3,1,0 \rangle$$

$$\vec{b} = \overrightarrow{AC} = \langle 4-2, 5-0, -2-3 \rangle = \langle 2,5,-5 \rangle$$

$$\vec{c} = \overrightarrow{AD} = \langle 2-2, 2-0, 3-3 \rangle = \langle 0,2,0 \rangle$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle -3,1,0 \rangle \times \langle 2,5,-5 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(1 \cdot (-5) - 0 \cdot 5) - \mathbf{j}(-3 \cdot (-5) - 0 \cdot 2) + \mathbf{k}(-3 \cdot 5 - 1 \cdot 2)$$

$$\vec{a} \times \vec{b} = -5\mathbf{i} - 15\mathbf{j} - 17\mathbf{k}$$

The dot product is

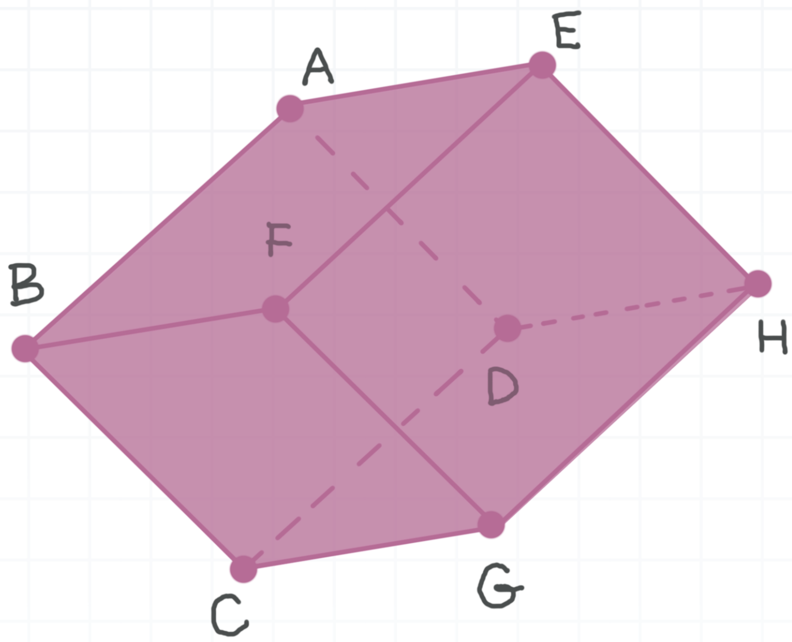
$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle -5,-15,-17 \rangle \cdot \langle 0,2,0 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -5 \cdot 0 + (-15) \cdot 2 + (-17) \cdot 0 = -30$$

Therefore, the volume of the tetrahedron is

$$V_{\text{tetrahedron}} = \frac{1}{6}|-30| = 5$$

■ 2. Find the volume of parallelepiped $ABCDEFGH$, given $A(1,2,2)$, $B(-1,-2,0)$, $F(4,3,-1)$, and $G(5,6,-4)$.



Solution:

We don't have three adjacent edges to use for calculating the scalar triple product. So we first need to find the coordinates of D and E. Let's use the main property of parallelepipeds, which is that any parallelepiped has three quadruples of parallel and equal edges.

Since $\overrightarrow{AE} = \overrightarrow{BF}$, we can find the coordinates of E by adding the vector $\overrightarrow{BF}$ to the coordinates of the point A. So

$$x_E = x_A + (x_F - x_B) = 1 + (4 - (-1)) = 6$$

$$y_E = y_A + (y_F - y_B) = 2 + (3 - (-2)) = 7$$

$$z_E = z_A + (z_F - z_B) = 2 + (-1 - 0) = 1$$

Similarly, since $\overrightarrow{FG} = \overrightarrow{AD}$, we can find the coordinates of the point D by adding the vector $\overrightarrow{FG}$ to the coordinates of the point A. So

$$x_D = x_A + (x_G - x_F) = 1 + (5 - 4) = 2$$

$$y_D = y_A + (y_G - y_F) = 2 + (6 - 3) = 5$$

$$z_D = z_A + (z_G - z_F) = 2 + (-4 - (-1)) = -1$$

So the three adjacent vectors are

$$\overrightarrow{AB} = \langle -1 - 1, -2 - 2, 0 - 2 \rangle = \langle -2, -4, -2 \rangle$$

$$\overrightarrow{AD} = \langle 2 - 1, 5 - 2, -1 - 2 \rangle = \langle 1, 3, -3 \rangle$$

$$\overrightarrow{AE} = \langle 6 - 1, 7 - 2, 1 - 2 \rangle = \langle 5, 5, -1 \rangle$$

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = |(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE}|$$

The cross product of two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ is given by

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\overrightarrow{AB} \times \overrightarrow{AD} = \langle -2, -4, -2 \rangle \times \langle 1, 3, -3 \rangle$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = \mathbf{i}(-4 \cdot (-3) - (-2) \cdot 3) - \mathbf{j}(-2 \cdot (-3) - (-2) \cdot 1) + \mathbf{k}(-2 \cdot 3 - (-4) \cdot 1)$$

$$\overrightarrow{AB} \times \overrightarrow{AD} = 18\mathbf{i} - 8\mathbf{j} - 2\mathbf{k}$$

The dot product is

$$(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE} = \langle 18, -8, -2 \rangle \cdot \langle 5, 5, -1 \rangle$$

$$(\overrightarrow{AB} \times \overrightarrow{AD}) \cdot \overrightarrow{AE} = 18 \cdot 5 + (-8) \cdot 5 + (-2) \cdot (-1) = 52$$

■ 3. Find the volume of the parallelepiped with base $ABCD$ and height 5, if $A(3,3,3)$, $B(0,-2,-2)$, and $C(-3,1,0)$.

Solution:

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = [\text{Height}] \cdot [\text{Area of Base}]$$

The area of the base is

$$\text{Area}_{ABCD} = |\vec{a} \times \vec{b}|$$

where $\vec{a}$ and $\vec{b}$ are any two adjacent edges of the base. Let's choose $\vec{a} = \overrightarrow{BA}$ and $\vec{b} = \overrightarrow{BC}$. So the two adjacent vectors are

$$\overrightarrow{BA} = \langle 3-0, 3-(-2), 3-(-2)\rangle = \langle 3,5,5\rangle$$

$$\overrightarrow{BC} = \langle -3-0, 1-(-2), 0-(-2)\rangle = \langle -3,3,2\rangle$$

The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\overrightarrow{BA} \times \overrightarrow{BD} = \langle 3,5,5\rangle \times \langle -3,3,2\rangle$$

$$\overrightarrow{BA} \times \overrightarrow{BD} = \mathbf{i}(5 \cdot 2 - 5 \cdot 3) - \mathbf{j}(3 \cdot 2 - 5 \cdot (-3)) + \mathbf{k}(3 \cdot 3 - 5 \cdot (-3))$$

$$\overrightarrow{BA} \times \overrightarrow{BD} = -5\mathbf{i} - 21\mathbf{j} + 24\mathbf{k}$$

The magnitude of the cross product is

$$|\overrightarrow{BA} \times \overrightarrow{BD}| = \sqrt{(-5)^2 + (-21)^2 + 24^2} = \sqrt{1{,}042}$$

The volume of the parallelepiped is

$$V_{\text{parallelepiped}} = [\text{Height}] \cdot [\text{Area of Base}] = 5\sqrt{1{,}042}$$

SCALAR TRIPLE PRODUCT TO PROVE VECTORS ARE COPLANAR

■ 1. Find the value of the parameter p such that the vectors $\vec{a} = \langle 1,3,-1 \rangle$, $\vec{b} = \langle 2,2,2 \rangle$, and $\vec{c} = \langle 0,-1,p \rangle$ are coplanar.

Solution:

The vectors are coplanar if their scalar triple product is 0, i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 1,3,-1 \rangle \times \langle 2,2,2 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(3 \cdot 2 - (-1) \cdot 2) - \mathbf{j}(1 \cdot 2 - (-1) \cdot 2) + \mathbf{k}(1 \cdot 2 - 3 \cdot 2)$$

$$\vec{a} \times \vec{b} = 8\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle 8,-4,-4 \rangle \cdot \langle 0,-1,p \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 8 \cdot 0 + (-4) \cdot (-1) + (-4) \cdot p = 4 - 4p$$

Since the scalar triple product is 0, we can write the equation for p.

$$4 - 4p = 0$$

$$p = 1$$

■ 2. Check if the vectors $\vec{a} = \langle 1,1,0 \rangle$, $\vec{b} = \langle 0,1,1 \rangle$, and $\vec{c} = \langle 1,0,-1 \rangle$ are coplanar. If they are, find the equation of the plane, assuming that the initial point of the vectors is the origin.

Solution:

The vectors are coplanar if their scalar triple product is 0, i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 1,1,0 \rangle \times \langle 0,1,1 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(1 \cdot 1 - 0 \cdot 1) - \mathbf{j}(1 \cdot 1 - 0 \cdot 0) + \mathbf{k}(1 \cdot 1 - 1 \cdot 0)$$

$$\vec{a} \times \vec{b} = \mathbf{i} - \mathbf{j} + \mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle 1,-1,1 \rangle \cdot \langle 1,0,-1 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = 1 \cdot 1 + (-1) \cdot 0 + 1 \cdot (-1) = 0$$

Since the scalar triple product is 0, the vectors are coplanar.

The equation of the plane is

$$n_1(x - x_0) + n_2(y - y_0) + n_3(z - z_0) = 0$$

Since the vectors $\vec{a}$ and $\vec{b}$ lie in the plane, their cross product is orthogonal to the plane, i.e. $\vec{n} = \vec{a} \times \vec{b} = \langle 1, -1, 1 \rangle$.

Since the initial point of the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ is the origin, it lies in the plane, i.e. $(x_0, y_0, z_0) = (0,0,0)$. Plug into the plane equation.

$$1(x - 0) - 1(y - 0) + 1(z - 0) = 0$$

$$x - y + z = 0$$

■ 3. Check if the points $A(0,0,1)$, $B(2,0,3)$, $C(2,3,0)$, and $D(3,2,2)$ lie in the same plane.

Solution:

The points lie in the same plane if the vectors joining these points are coplanar. Check if the vectors $\vec{a} = \overrightarrow{AB}$, $\vec{b} = \overrightarrow{AC}$, and $\vec{c} = \overrightarrow{AD}$ are coplanar.

$$\overrightarrow{AB} = \langle 2 - 0, 0 - 0, 3 - 1 \rangle = \langle 2,0,2 \rangle$$

$$\overrightarrow{AC} = \langle 2 - 0, 3 - 0, 0 - 1 \rangle = \langle 2,3,-1 \rangle$$

$$\overrightarrow{AD} = \langle 3 - 0, 2 - 0, 2 - 1 \rangle = \langle 3,2,1 \rangle$$

The vectors are coplanar if their scalar triple product is 0, i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a} \times \vec{b} = \mathbf{i}(a_2b_3 - a_3b_2) - \mathbf{j}(a_1b_3 - a_3b_1) + \mathbf{k}(a_1b_2 - a_2b_1)$$

Calculate the cross product.

$$\vec{a} \times \vec{b} = \langle 2,0,2 \rangle \times \langle 2,3,-1 \rangle$$

$$\vec{a} \times \vec{b} = \mathbf{i}(0 \cdot (-1) - 2 \cdot 3) - \mathbf{j}(2 \cdot (-1) - 2 \cdot 2) + \mathbf{k}(2 \cdot 3 - 0 \cdot 2)$$

$$\vec{a} \times \vec{b} = -6\mathbf{i} + 6\mathbf{j} + 6\mathbf{k}$$

The dot product is

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = \langle -6,6,6 \rangle \cdot \langle 3,2,1 \rangle$$

$$(\vec{a} \times \vec{b}) \cdot \vec{c} = -6 \cdot 3 + 6 \cdot 2 + 6 \cdot 1 = 0$$

Since the scalar triple product is 0, the vectors are coplanar and the points lie in the same plane.

W W W . K R I S T A K I N G M A T H . C O M