Calculus 3 Workbook Solutions

Change of variables

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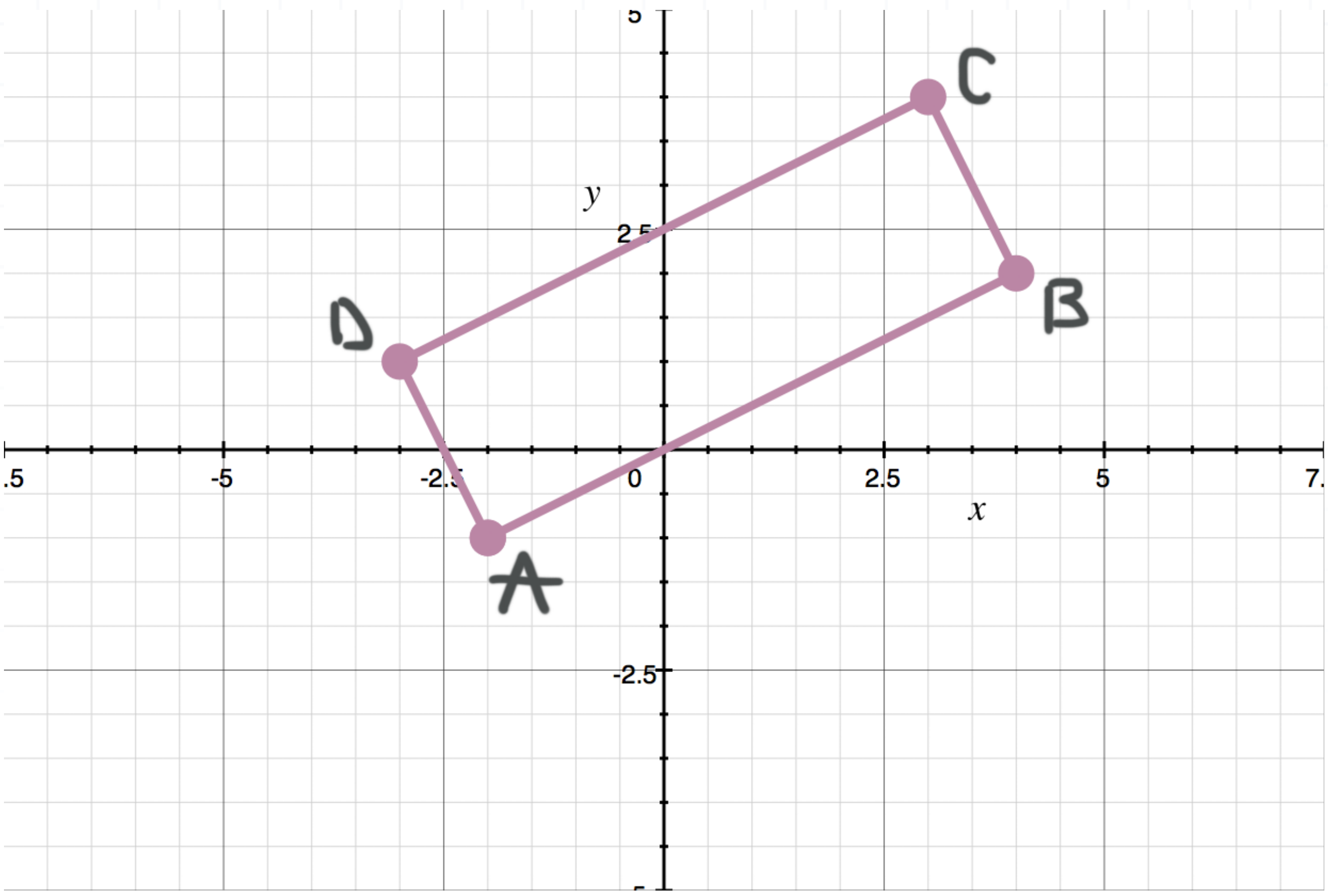
JACOBIAN FOR TWO VARIABLES

■ 1. Find the Jacobian of the transformation that rotates the rectangle $ABCD$, given by $A(-2,-1)$, $B(4,2)$, $C(3,4)$, and $D(-3,1)$, clockwise about the origin in such a way that AB will lie on the x-axis.



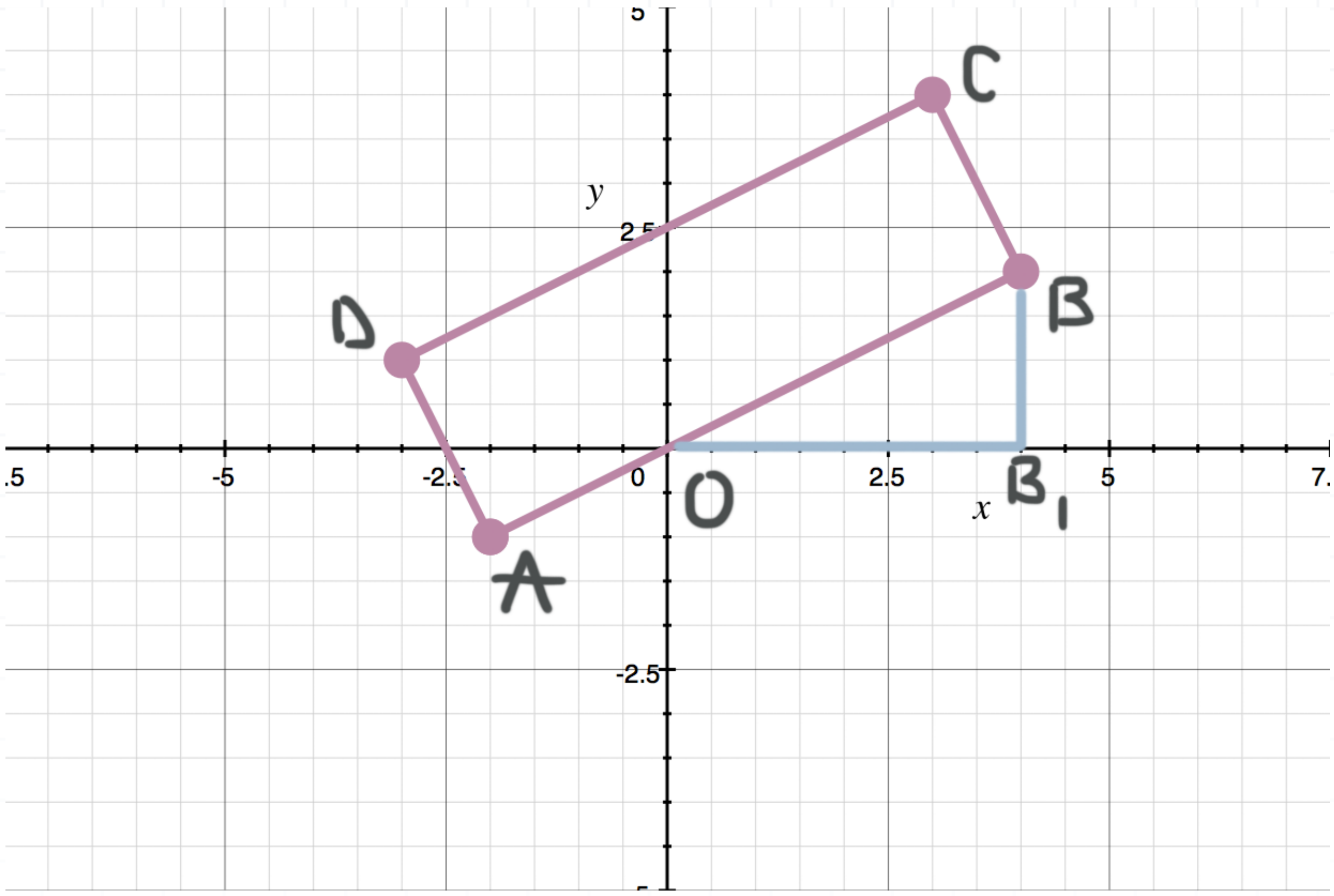
Solution:

Let (u, v) be the coordinates after the transformation. The formula of rotation by an angle ϕ is given by

$$x = u\cos\phi - v\sin\phi$$

$$y = u\sin\phi + v\cos\phi$$

Let $B_1(4,0)$ be the projection of the point B onto the x-axis.



Find sine and cosine of ϕ from the triangle OBB_1. Since $OB = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$,

$$\sin \phi = \frac{BB_1}{OB} = \frac{2}{2\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\cos \phi = \frac{OB_1}{OB} = \frac{4}{2\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

Plug these values into the rotation formulas.

$$x = \frac{2\sqrt{5}}{5}u - \frac{\sqrt{5}}{5}v$$

$$y = \frac{\sqrt{5}}{5}u + \frac{2\sqrt{5}}{5}v$$

The partial derivatives are

$$\frac{\partial x}{\partial u}=\frac{\partial}{\partial u}\left(\frac{2\sqrt{5}}{5}u-\frac{\sqrt{5}}{5}v\right)=\frac{2\sqrt{5}}{5}$$

$$\frac{\partial x}{\partial v}=\frac{\partial}{\partial v}\left(\frac{2\sqrt{5}}{5}u-\frac{\sqrt{5}}{5}v\right)=-\frac{\sqrt{5}}{5}$$

$$\frac{\partial y}{\partial u}=\frac{\partial}{\partial u}\left(\frac{\sqrt{5}}{5}u+\frac{2\sqrt{5}}{5}v\right)=\frac{\sqrt{5}}{5}$$

$$\frac{\partial y}{\partial v}=\frac{\partial}{\partial v}\left(\frac{\sqrt{5}}{5}u+\frac{2\sqrt{5}}{5}v\right)=\frac{2\sqrt{5}}{5}$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial x}{\partial v}\\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}=\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v}-\frac{\partial x}{\partial v}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)}=\frac{2\sqrt{5}}{5}\cdot\frac{2\sqrt{5}}{5}-\left(-\frac{\sqrt{5}}{5}\right)\cdot\frac{\sqrt{5}}{5}=1$$

■ 2. Find the Jacobian of the transformation which converts the ellipse $5\sqrt{2}x^2+6\sqrt{2}xy+8x+5\sqrt{2}y^2-8y=0$ into the ellipse with center at the origin, and x- and $y$semi-axes 2 and 1 respectively. Use a rotation counterclockwise by $\pi/4$, and then move it by 2 to the positive direction of the x-axis.

Solution:

Let (u, v) be the coordinates after the transformation. The formula of rotation by the angle ϕ is given by

$$x = u\cos\phi - v\sin\phi$$

$$y = u\sin\phi + v\cos\phi$$

Since we rotate by a counterclockwise angle of $\pi/4$, we'll plug in $\phi = -\pi/4$.

$$x = u\cos\left(-\frac{\pi}{4}\right) - v\sin\left(-\frac{\pi}{4}\right)$$

$$y = u\sin\left(-\frac{\pi}{4}\right) + v\cos\left(-\frac{\pi}{4}\right)$$

and we get

$$x = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v$$

$$y = -\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v$$

Next, to move the ellipse by 2 to the positive direction of x-axis, we need to subtract 2 from the u coordinate. Finally, the transformation equations are

$$x = \frac{\sqrt{2}}{2}(u-2) + \frac{\sqrt{2}}{2}v = \frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}$$

$$y = -\frac{\sqrt{2}}{2}(u-2) + \frac{\sqrt{2}}{2}v = -\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}$$

To check the transformation, plug these values for x and y into the equation of the ellipse.

$$5\sqrt{2}x^2 + 6\sqrt{2}xy + 8x + 5\sqrt{2}y^2 - 8y = 0$$

$$5\sqrt{2}\left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}\right)^2 + 6\sqrt{2}\left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}\right)\left(-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}\right)$$

$$+8\left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}\right) + 5\sqrt{2}\left(-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}\right)^2$$

$$-8\left(-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}\right) = 0$$

We get

$$\frac{5\sqrt{2}}{2}(u+v-2)^2 + 3\sqrt{2}(v^2-(u-2)^2) + \frac{5\sqrt{2}}{2}(-u+v+2)^2 + 8\sqrt{2}(u-2) = 0$$

$$5(u+v-2)^2 + 6(v^2-(u-2)^2) + 5(-u+v+2)^2 + 16(u-2) = 0$$

$$4(u^2+4v^2-4) = 0$$

$$\frac{u^2}{4} + v^2 = 1$$

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v - \sqrt{2}\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}\right) = -\frac{\sqrt{2}}{2}$$

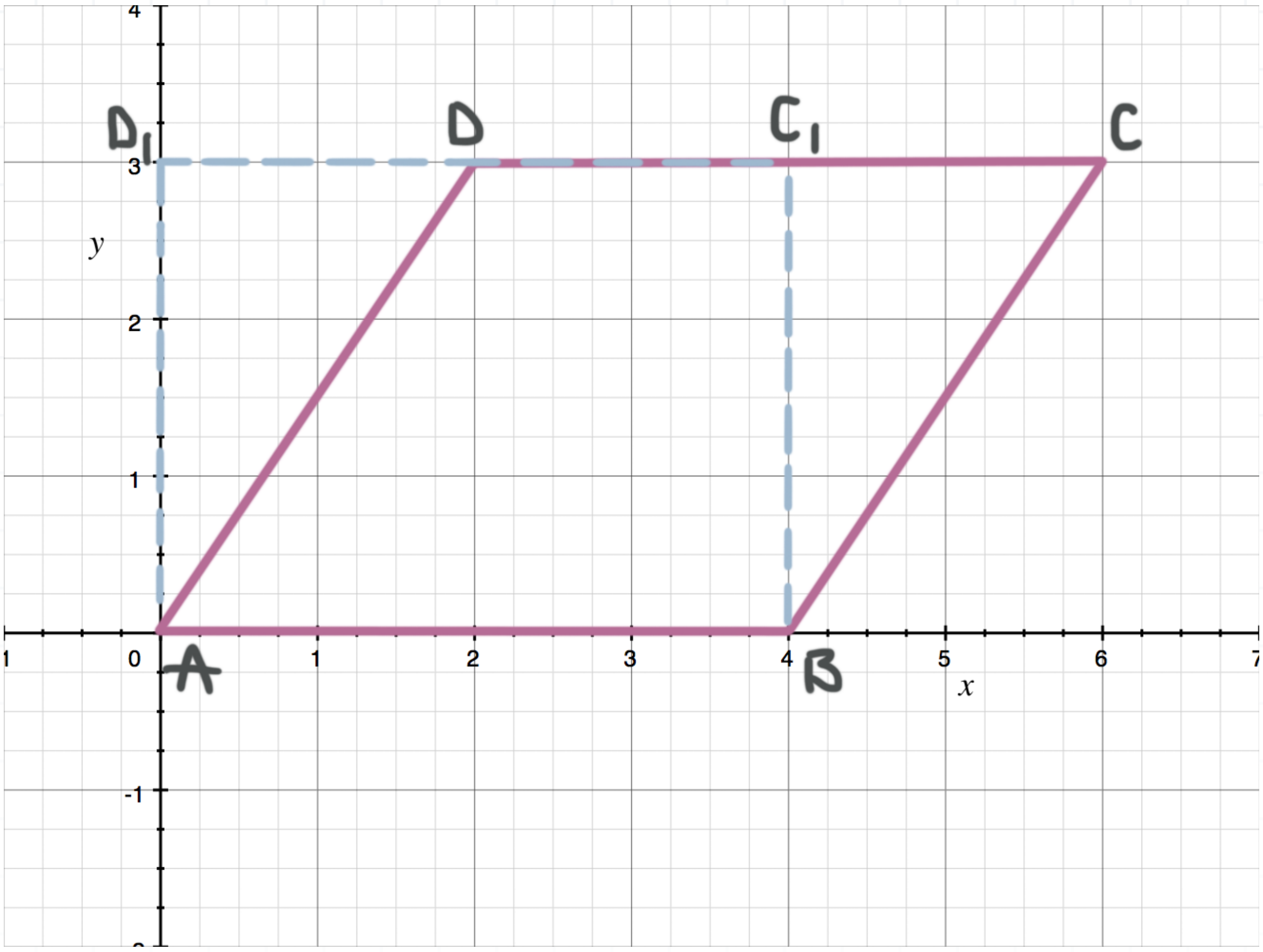
$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(-\frac{\sqrt{2}}{2}u + \frac{\sqrt{2}}{2}v + \sqrt{2}\right) = \frac{\sqrt{2}}{2}$$

The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = 1$$

■ 3. Find the Jacobian of the linear transformation that converts the parallelogram $ABCD$, given by $A(0,0)$, $B(4,0)$, $C(6,3)$, $D(2,3)$, into the rectangle ABC_1D_1 with the same base and height.



Solution:

For the linear transformation in two dimensions, we get the formula by moving two different points, and then just apply it to all other points of the shape. Let's choose the points B and C.

Let (u, v) be the coordinates after the transformation. The standard form for the linear transformation is

$$x = au + bv$$

$$y = cu + dv$$

where a, b, c, and d are real numbers. Since the y coordinate is not changing, we can simplify the formula for y.

$$x = au + bv$$

$$y = v$$

Since $B(4,0)$ transforms to $B(4,0)$, we can plug in $u = 4$, $x = 4$, and $v = 0$ into the first equation.

$$4 = a(4) + b(0)$$

$$a = 1$$

Since $C(6,3)$ transforms to $C_1(4,3)$, we can plug in $u = 4$, $x = 6$, and $v = 3$ to the first equation.

$$6 = 1(4) + b(3)$$

$$3b = 2$$

$$b = \frac{2}{3}$$

So the transformation formulas are

$$x = u + \frac{2}{3}v$$

$$y = v$$

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(u + \frac{2}{3}v\right) = 1$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(u + \frac{2}{3}v\right) = \frac{2}{3}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(v) = 0$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}(v) = 1$$

Then the Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = 1 \cdot 1 - \frac{2}{3} \cdot 0 = 1$$

JACOBIAN FOR THREE VARIABLES

1. Find the Jacobian of the transformation that rotates the space clockwise about the y-axis by $\pi/6$.

Solution:

Since the y-coordinate remains unchanged, we can use a two-dimensional formula of rotation in the xz-plane. Let u, v, and w be the coordinates after the transformation. The rotation by angle ϕ is given by

$$x = u\cos\phi - w\sin\phi$$

$$z = u\sin\phi + w\cos\phi$$

Plug in $\sin(\pi/6) = 1/2$ and $\cos(\pi/6) = \sqrt{3}/2$ to get the final transformation equations.

$$x = \frac{\sqrt{3}}{2}u - \frac{1}{2}w$$

$$y = v$$

$$z = \frac{1}{2}u + \frac{\sqrt{3}}{2}w$$

The partial derivatives of these are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{3}}{2}u - \frac{1}{2}w\right) = \frac{\sqrt{3}}{2}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{3}}{2}u - \frac{1}{2}w\right) = 0$$

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}\left(\frac{\sqrt{3}}{2}u - \frac{1}{2}w\right) = -\frac{1}{2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial v}{\partial u} = 0$$

$$\frac{\partial y}{\partial v} = \frac{\partial v}{\partial v} = 1$$

$$\frac{\partial y}{\partial w} = \frac{\partial v}{\partial w} = 0$$

$$\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}w\right) = \frac{1}{2}$$

$$\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}w\right) = 0$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}\left(\frac{1}{2}u + \frac{\sqrt{3}}{2}w\right) = \frac{\sqrt{3}}{2}$$

The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\partial x}{\partial u}\left(\frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial v}\right) - \frac{\partial x}{\partial v}\left(\frac{\partial y}{\partial u}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial u}\right)$$

$$+\frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial u}\cdot\frac{\partial z}{\partial v} - \frac{\partial y}{\partial v}\cdot\frac{\partial z}{\partial u}\right)$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{\sqrt{3}}{2}\left(1\cdot\frac{\sqrt{3}}{2} - 0\cdot 0\right) - 0\cdot\left(0\cdot\frac{\sqrt{3}}{2} - 0\cdot\frac{1}{2}\right) - \frac{1}{2}\left(0\cdot 0 - 1\cdot\frac{1}{2}\right) = 1$$

■ 2. Find the Jacobian of the transformation to spherical coordinates that converts the ellipsoid to the unit sphere (a sphere with center at the origin and radius 1).

$$\frac{x^2}{4} + \frac{y^2}{25} + \frac{z^2}{9} = 1$$

Solution:

The transformation formulas to spherical coordinates are

$$x = \rho\sin\phi\cos\theta$$

$$y = \rho\sin\phi\sin\theta$$

$$z = \rho \cos \phi$$

Since the ellipsoid has its center at the origin, we can use the standard spherical coordinates, but scaled on all three directions. Since the semi-axis is the x-direction has length 2, the semi-axis in the y-direction has length 5, and the semi-axis in the z-direction has length 3, we need to scale the x, y, and z directions by 2, 5, and 3 respectively. So the transformation formulas are

$$x = 2\rho \sin \phi \cos \theta$$

$$y = 5\rho \sin \phi \sin \theta$$

$$z = 3\rho \cos \phi$$

Let's check that the formulas are correct by plugging x, y, and z into the equation of the ellipsoid.

$$\frac{(2\rho \sin \phi \cos \theta)^2}{4} + \frac{(5\rho \sin \phi \sin \theta)^2}{25} + \frac{(3\rho \cos \phi)^2}{9} = 1$$

$$(\rho \sin \phi \cos \theta)^2 + (\rho \sin \phi \sin \theta)^2 + (\rho \cos \phi)^2 = 1$$

$$\rho^2 \sin^2 \phi + \rho^2 \cos^2 \phi = 1$$

$$\rho^2 = 1$$

$$\rho = 1$$

This is the equation of the unit sphere in spherical coordinates. The partial derivatives are

$$\frac{\partial x}{\partial \rho}=\frac{\partial}{\partial \rho}(2\rho\sin\phi\cos\theta)=2\sin\phi\cos\theta$$

$$\frac{\partial x}{\partial \theta}=\frac{\partial}{\partial \theta}(2\rho\sin\phi\cos\theta)=-2\rho\sin\phi\sin\theta$$

$$\frac{\partial x}{\partial \phi}=\frac{\partial}{\partial \phi}(2\rho\sin\phi\cos\theta)=2\rho\cos\phi\cos\theta$$

$$\frac{\partial y}{\partial \rho}=\frac{\partial}{\partial \rho}(5\rho\sin\phi\sin\theta)=5\sin\phi\sin\theta$$

$$\frac{\partial y}{\partial \theta}=\frac{\partial}{\partial \theta}(5\rho\sin\phi\sin\theta)=5\rho\sin\phi\cos\theta$$

$$\frac{\partial y}{\partial \phi}=\frac{\partial}{\partial \phi}(5\rho\sin\phi\sin\theta)=5\rho\cos\phi\sin\theta$$

$$\frac{\partial z}{\partial \rho}=\frac{\partial}{\partial \rho}(3\rho\cos\phi)=3\cos\phi$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial}{\partial \theta}(3\rho\cos\phi)=0$$

$$\frac{\partial z}{\partial \phi}=\frac{\partial}{\partial \phi}(3\rho\cos\phi)=-3\rho\sin\phi$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi}\\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi}\\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi}\end{vmatrix}$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\frac{\partial x}{\partial \rho}\left(\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\cdot\frac{\partial z}{\partial \theta}\right)-\frac{\partial x}{\partial \theta}\left(\frac{\partial y}{\partial \rho}\cdot\frac{\partial z}{\partial \phi}-\frac{\partial y}{\partial \phi}\cdot\frac{\partial z}{\partial \rho}\right)$$

$$+\frac{\partial x}{\partial \phi}\left(\frac{\partial y}{\partial \rho}\cdot\frac{\partial z}{\partial \theta}-\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial \rho}\right)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=2\sin\phi\cos\theta(-5\rho\sin\phi\cos\theta\cdot 3\rho\sin\phi-5\rho\cos\phi\sin\theta\cdot 0)$$

$$+2\rho\sin\phi\sin\theta(-5\sin\phi\sin\theta\cdot 3\rho\sin\phi-5\rho\cos\phi\sin\theta\cdot 3\cos\phi)$$

$$+2\rho\cos\phi\cos\theta(5\sin\phi\sin\theta\cdot 0-5\rho\sin\phi\cos\theta\cdot 3\cos\phi)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-30\rho^2\sin^3\phi\cos^2\theta-30\rho^2\sin\phi\sin^2\theta-30\rho^2\sin\phi\cos^2\phi\cos^2\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-30\rho^2\sin\phi$$

■ 3. Find the following transformations:

1. the transformation that converts the given cylinder to a circular cylinder with radius 1 and an axis parallel to the z-axis

$$\frac{x^2}{5}+\frac{z^2}{4}=1$$

2. the transformation that converts to cylindrical coordinates

Then write down the composition of these two transformations and find its Jacobian.

Solution:

1. Let u, v, and w be the transformed coordinates. To change the axis of the cylinder, we can just change the order of the coordinates y and z as follows.

$$x = u$$

$$y = w$$

$$z = v$$

To scale the cylinder by $\sqrt{5}$ in the x direction and by 2 in z direction, we can multiply the transformation equations by $\sqrt{5}$ and 2 respectively. So

$$x = \sqrt{5}u$$

$$y = w$$

$$z = 2v$$

Let's check that the formulas are correct by plugging x, y, and z into the equation of the cylinder.

$$\frac{(\sqrt{5}u)^2}{5} + \frac{(2v)^2}{4} = 1$$

$$u^2 + v^2 = 1$$

Therefore, we get the circular cylinder with radius 1 and axis along the w-axis in u, v, and w coordinates.

2. We need to use the standard conversion formulas for cylindrical coordinates:

$$u = r\cos\theta$$

$$v = r\sin\theta$$

$$w = w$$

Then the composition of transformations is given by

$$x = \sqrt{5}r\cos\theta$$

$$y = w$$

$$z = 2r\sin\theta$$

We can check that the composition is correct by plugging x, y, and z into the equation of the cylinder.

$$\frac{(\sqrt{5}r\cos\theta)^2}{5} + \frac{(2r\sin\theta)^2}{4} = 1$$

$$r^2\cos\theta + r^2\sin\theta = 1$$

$$r^2 = 1$$

$$r = 1$$

This is the correct equation of the cylinder with radius 1. The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}\left(\sqrt{5}r\cos\theta\right) = \sqrt{5}\cos\theta$$

$$\frac{\partial x}{\partial \theta} = \frac{\partial}{\partial \theta}(\sqrt{5}r\cos\theta) = -\sqrt{5}r\sin\theta$$

$$\frac{\partial x}{\partial w} = \frac{\partial}{\partial w}(\sqrt{5}r\cos\theta) = 0$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(w) = 0$$

$$\frac{\partial y}{\partial \theta} = \frac{\partial}{\partial \theta}(w) = 0$$

$$\frac{\partial y}{\partial w} = \frac{\partial}{\partial w}(w) = 1$$

$$\frac{\partial z}{\partial r} = \frac{\partial}{\partial r}(2r\sin\theta) = 2\sin\theta$$

$$\frac{\partial z}{\partial \theta} = \frac{\partial}{\partial \theta}(2r\sin\theta) = 2r\cos\theta$$

$$\frac{\partial z}{\partial w} = \frac{\partial}{\partial w}(2r\sin\theta) = 0$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(r,\theta,w)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$= \frac{\partial x}{\partial r}\left(\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial \theta}\right) - \frac{\partial x}{\partial \theta}\left(\frac{\partial y}{\partial r}\cdot\frac{\partial z}{\partial w} - \frac{\partial y}{\partial w}\cdot\frac{\partial z}{\partial r}\right)$$

$$+\frac{\partial x}{\partial w}\left(\frac{\partial y}{\partial r}\cdot\frac{\partial z}{\partial \theta}-\frac{\partial y}{\partial \theta}\cdot\frac{\partial z}{\partial r}\right)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\sqrt{5}\cos\theta(0\cdot 0-1\cdot 2r\cos\theta)+\sqrt{5}r\sin\theta(0\cdot 0-1\cdot 2\sin\theta)$$

$$+0\cdot(0\cdot 2r\cos\theta-0\cdot 2\sin\theta)$$

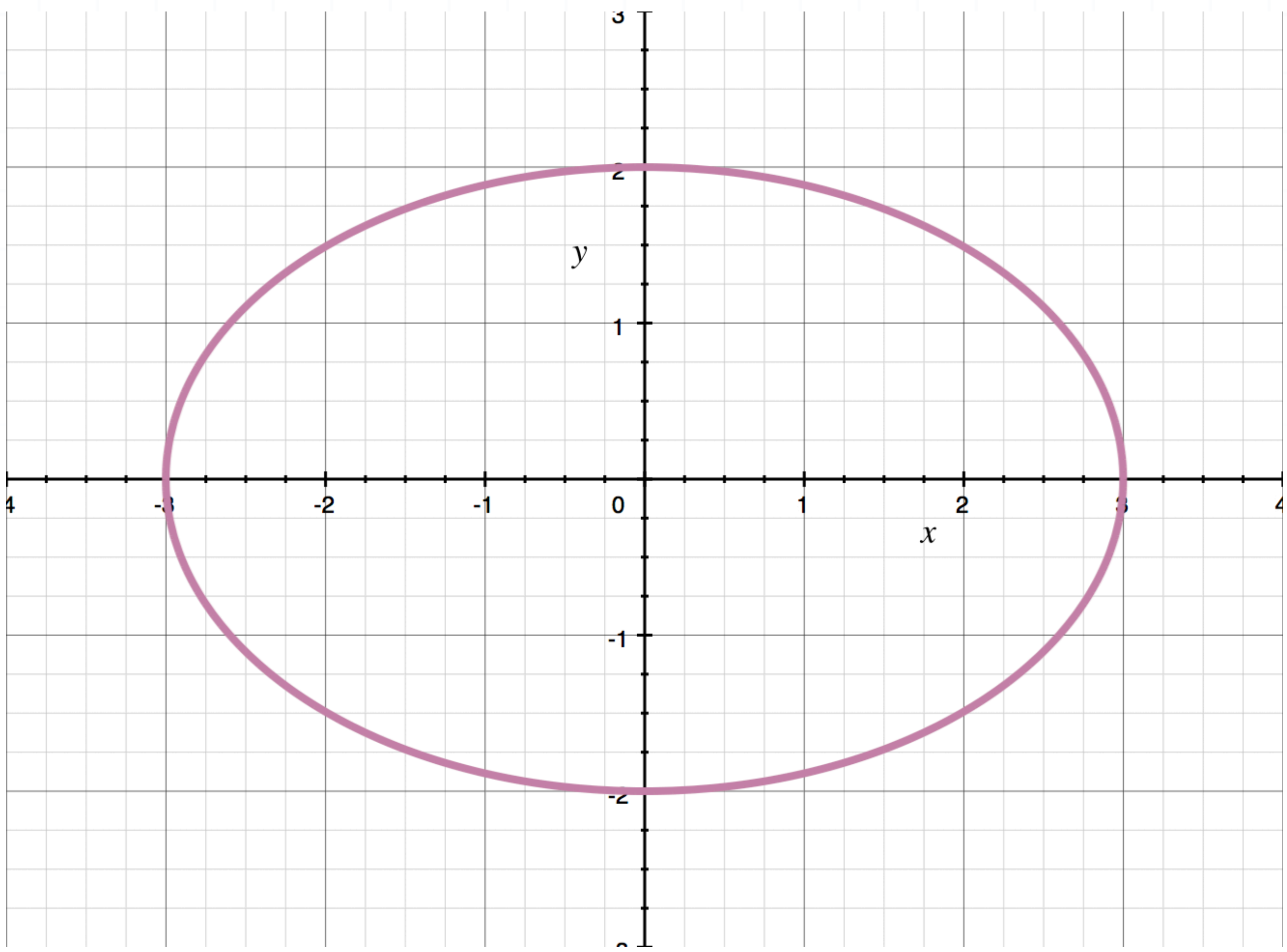
$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-2\sqrt{5}r$$

EVALUATING DOUBLE INTEGRALS

■ 1. Find the Jacobian of the transformation and use it to find the area of the ellipse with center at the origin, semi-major axis along the x-axis with length 3, and semi-minor axis along the y-axis with length 2.

$$x = 3r\cos\phi$$

$$y = 2r\sin\phi$$



Solution:

The equation of the ellipse is

$$\frac{x^2}{3^2} + \frac{y^2}{2^2} = 1$$

Plug in $x = 3r\cos\phi$ and $y = 2r\sin\phi$.

$$\frac{3^2r^2\cos^2\phi}{3^2} + \frac{2^2r^2\sin^2\phi}{2^2} = 1$$

$$r^2\cos^2\phi + r^2\sin^2\phi = 1$$

$$r^2 = 1$$

$$r = 1$$

So the equation of the ellipse in the transformed coordinates is $r = 1$. Therefore, the value of r changes from 0 to 1, and ϕ changes from 0 to 2π. The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(3r\cos\phi) = 3\cos\phi$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(3r\cos\phi) = -3r\sin\phi$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(2r\sin\phi) = 2\sin\phi$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(2r\sin\phi) = 2r\cos\phi$$

The Jacobian of the transformation is then

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = 3\cos\phi \cdot 2r\cos\phi - (-3r\sin\phi) \cdot 2\sin\phi$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = 6r$$

The area of the ellipse is

$$\int_0^{2\pi} \int_0^1 6r \ dr \ d\phi$$

$$\int_0^{2\pi} 3r^2 \Big|_0^1 \ d\phi$$

$$\int_0^{2\pi} 3 \ d\phi$$

$$3\phi \Big|_0^{2\pi}$$

$$6\pi$$

■ 2. Find the Jacobian of the transformation and use it to find the double integral of the function $f(x,y) = x^2 + y^2$ over the circle with center at $(-2,3)$ and radius 2.

$$x = -2 + r\cos\phi$$

$$y = 3 + r\sin\phi$$

Solution:

The equation of the circle is

$$(x+2)^2 + (y-3)^2 = 2^2$$

Plug in $x = -2 + r\cos\phi$ and $y = 3 + r\sin\phi$.

$$(-2 + r\cos\phi + 2)^2 + (3 + r\sin\phi - 3)^2 = 2^2$$

$$r^2\cos^2\phi + r^2\sin^2\phi = 2^2$$

$$r^2 = 2^2$$

$$r = 2$$

So the equation of the circle in the transformed coordinates is $r = 2$. Therefore, the value or r changes from 0 to 2, and ϕ changes from 0 to 2π.

The partial derivatives are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(-2 + r\cos\phi) = \cos\phi$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(-2 + r\cos\phi) = -r\sin\phi$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(3 + r\sin\phi) = \sin\phi$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(3 + r\sin\phi) = r\cos\phi$$

The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = \cos\phi \cdot r\cos\phi - (-r\sin\phi) \cdot \sin\phi$$

$$\frac{\partial(x,y)}{\partial(r,\phi)} = r$$

The function is

$$(-2 + r\cos\phi)^2 + (3 + r\sin\phi)^2$$

$$r^2\cos^2\phi + r^2\sin^2\phi - 4r\cos\phi + 6r\sin\phi + 13$$

$$r^2 - 4r\cos\phi + 6r\sin\phi + 13$$

Therefore, the integral is,

$$\int_0^{2\pi}\int_0^2 (r^2 - 4r\cos\phi + 6r\sin\phi + 13)r \ dr \ d\phi$$

$$\int_0^{2\pi}\int_0^2 r^3 - 4r^2\cos\phi + 6r^2\sin\phi + 13r \ dr \ d\phi$$

Since the integrals of sine and cosine functions over a 2π-period are equal to 0, the double integral simplifies to

$$\int_0^{2\pi}\int_0^2 r^3+13r\ dr\ d\phi$$

Integrate with respect to r.

$$\int_0^{2\pi}\frac{1}{4}r^4+\frac{13}{2}r^2\Big|_0^2\ d\phi$$

$$\int_0^{2\pi}\frac{1}{4}(2)^4+\frac{13}{2}(2)^2\ d\phi$$

$$\int_0^{2\pi}4+26\ d\phi$$

$$\int_0^{2\pi}30\ d\phi$$

Integrate with respect to ϕ.

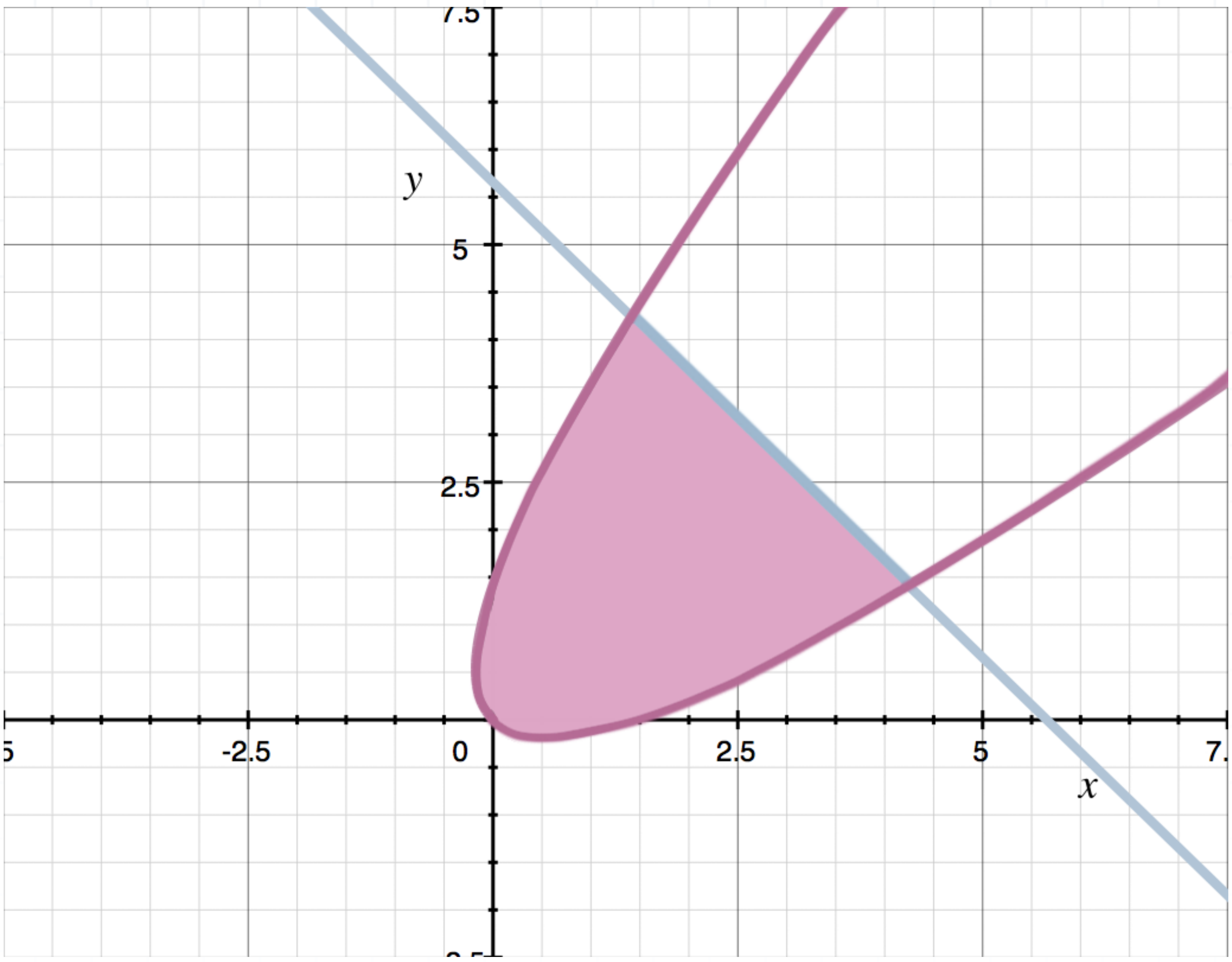
$$30\phi\Big|_0^{2\pi}$$

$$60\pi$$

■ 3. Find the Jacobian of the transformation, and use it to find the area bounded by the curves $x^2-2xy+y^2-\sqrt{2}x-\sqrt{2}y=0$ and $x+y=4\sqrt{2}$.

$$x=\frac{\sqrt{2}}{2}(u+v)$$

$$y = \frac{\sqrt{2}}{2}(v - u)$$



Solution:

The equation of the parabola is

$$x^2 - 2xy + y^2 - \sqrt{2}x - \sqrt{2}y = 0$$

Plug in the transformation equations.

$$\frac{2}{4}(u+v)^2 - 2 \cdot \frac{\sqrt{2}}{2}(u+v) \cdot \frac{\sqrt{2}}{2}(v-u) + \frac{2}{4}(v-u)^2 - \sqrt{2} \cdot \frac{\sqrt{2}}{2}(u+v) - \sqrt{2} \cdot \frac{\sqrt{2}}{2}(v-u) = 0$$

$$\frac{1}{2}(u^2 + 2uv + v^2) - (v^2 - u^2) + \frac{1}{2}(v-u)^2 - (u+v) - (v-u) = 0$$

$$\frac{1}{2}u^2 + uv + \frac{1}{2}v^2 - v^2 + u^2 + \frac{1}{2}v^2 - uv + \frac{1}{2}u^2 - 2v = 0$$

$$2u^2 - 2v = 0$$

$$v = u^2$$

So the equation of the parabola in the transformed coordinates is $v = u^2$. The equation of the line is

$$x + y = 4\sqrt{2}$$

Plug in $x = (\sqrt{2}/2)(u + v)$ and $y = (\sqrt{2}/2)(v - u)$.

$$x + y = 4\sqrt{2}$$

$$\frac{\sqrt{2}}{2}(u + v) + \frac{\sqrt{2}}{2}(v - u) = 4\sqrt{2}$$

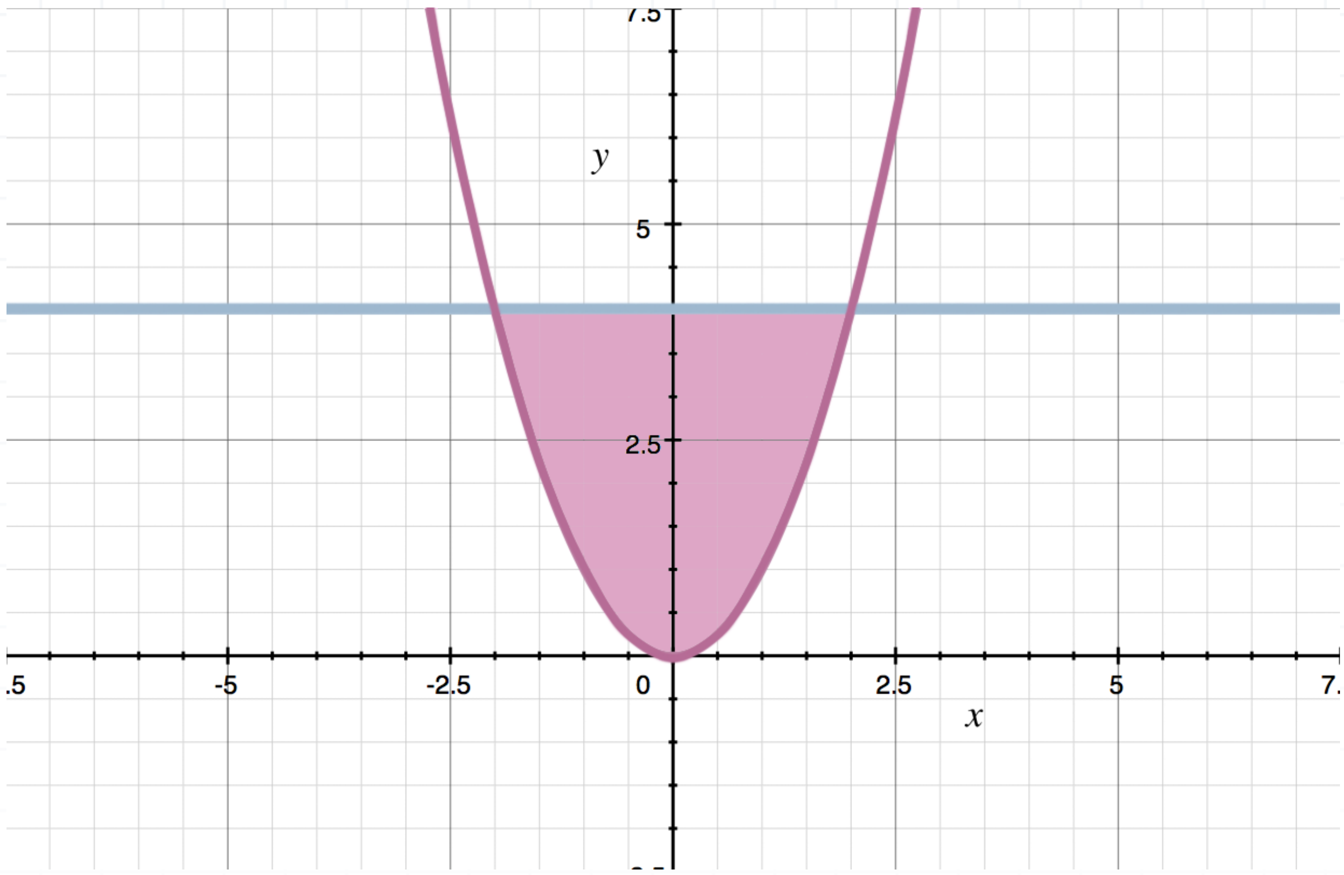
$$v = 4$$

So the equation of the line in the transformed coordinates is $v = 4$. Therefore, we need to find the area between the curves $v = u^2$ and $v = 4$. Find the points of intersection.

$$u^2 = 4$$

$$u = \pm 2$$

So the intersection points are $(-2,4)$ and $(2,4)$.



Therefore, the value of u changes from -2 to 2, and v changes from u^2 to 4.

The partial derivatives are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{2}}{2}(u+v)\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{2}}{2}(u+v)\right) = \frac{\sqrt{2}}{2}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(\frac{\sqrt{2}}{2}(v-u)\right) = -\frac{\sqrt{2}}{2}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(\frac{\sqrt{2}}{2}(v-u)\right) = \frac{\sqrt{2}}{2}$$

Then the Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} - \left(-\frac{\sqrt{2}}{2}\right) \cdot \frac{\sqrt{2}}{2} = 1$$

The area of the region E bounded by the curves $v = u^2$ and $v = 4$ is

$$\int_{-2}^{2} \int_{u^2}^{4} \, dv \, du$$

$$\int_{-2}^{2} v \Big|_{u^2}^{4} \, du$$

$$\int_{-2}^{2} 4 - u^2 \, du$$

$$4u - \frac{1}{3}u^3 \Big|_{-2}^{2}$$

$$4(2) - \frac{1}{3}(2)^3 - \left(4(-2) - \frac{1}{3}(-2)^3\right)$$

$$8 - \frac{8}{3} + 8 - \frac{8}{3}$$

$$16 - \frac{16}{3}$$

$$\frac{32}{3}$$

EQUATIONS OF THE TRANSFORMATION

1. Identify the equation obtained from $f(x, y, z) = x^2 + y^2 + z^2 - 2$ by applying the transformation.

$$x = \sin u + \cos u$$

$$y = \sin u - \cos u$$

$$z = \sqrt{u + v + w}$$

Solution:

Substitute expressions for x, y, and z into the equation of $f(x, y, z)$.

$$f(x, y, z) = x^2 + y^2 + z^2 - 2$$

$$f(u, v, w) = (\sin u + \cos u)^2 + (\sin u - \cos u)^2 + (\sqrt{u + v + w})^2 - 2$$

$$f(u, v, w) = \sin^2 u + 2\sin u\cos u + \cos^2 u + \sin^2 u$$

$$-2\sin u\cos u + \cos^2 u + (u + v + w) - 2$$

$$f(u, v, w) = 2\sin^2 u + 2\cos^2 u + u + v + w - 2$$

$$f(u, v, w) = 2 + u + v + w - 2$$

$$f(u, v, w) = u + v + w$$

■ 2. Find the inverse transformation and determine its Jacobian.

$$u = x - 2y + 1$$

$$v = -3x + y + 2$$

Solution:

Solve the transformation equations as a system of equations.

$$x = u + 2y - 1$$

$$v = -3(u + 2y - 1) + y + 2$$

$$v = -3u - 6y + 3 + y + 2$$

$$5y = -3u - v + 5$$

$$y = -\frac{3u}{5} - \frac{v}{5} + 1$$

Substitute y back to the first equation to find x.

$$x = u + 2\left(-\frac{3u}{5} - \frac{v}{5} + 1\right) - 1$$

$$x = u - \frac{6u}{5} - \frac{2v}{5} + 2 - 1$$

$$x = -\frac{u}{5} - \frac{2v}{5} + 1$$

So the inverse transformation is

$$x = -\frac{u}{5} - \frac{2v}{5} + 1$$

$$y = -\frac{3u}{5} - \frac{v}{5} + 1$$

The partial derivatives of these equations are

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}\left(-\frac{u}{5} - \frac{2v}{5} + 1\right) = -\frac{1}{5}$$

$$\frac{\partial x}{\partial v} = \frac{\partial}{\partial v}\left(-\frac{u}{5} - \frac{2v}{5} + 1\right) = -\frac{2}{5}$$

$$\frac{\partial y}{\partial u} = \frac{\partial}{\partial u}\left(-\frac{3u}{5} - \frac{v}{5} + 1\right) = -\frac{3}{5}$$

$$\frac{\partial y}{\partial v} = \frac{\partial}{\partial v}\left(-\frac{3u}{5} - \frac{v}{5} + 1\right) = -\frac{1}{5}$$

The Jacobian of the transformation is the

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \left(-\frac{1}{5}\right)\left(-\frac{1}{5}\right) - \left(-\frac{2}{5}\right)\left(-\frac{3}{5}\right)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{25} - \frac{6}{25}$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -\frac{1}{5}$$

■ 3. Find the inverse transformation $x(r, \phi)$ and $y(r, \phi)$ and determine its Jacobian.

$$r = \sqrt{\frac{x^2 + y^2}{4}}$$

$$\phi = \arctan \frac{y}{x} \text{ for } x \neq 0$$

Solution:

We need to find expressions for x and y in the form $x(r, \phi)$ and $y(r, \phi)$. So we'll solve the system of transformation equations for x and y. Since the given transformation is almost the same as inverse to the conversion to the polar coordinates, we can substitute $x = kr\cos\phi$ and $y = kr\sin\phi$, where k is an unknown constant.

$$r = \sqrt{\frac{(kr\cos\phi)^2 + (kr\sin\phi)^2}{4}}$$

$$r = \sqrt{\frac{k^2r^2(\cos^2\phi + \sin^2\phi)}{4}}$$

$$r = \sqrt{\frac{k^2r^2}{4}}$$

$$r = \frac{kr}{2}$$

$$k = 2$$

Since $k = 2$, we can substitute $x = 2r\cos\phi$ and $y = 2r\sin\phi$ to check if the second equation holds.

$$\phi = \arctan\left(\frac{2r\sin\phi}{2r\cos\phi}\right)$$

$$\phi = \arctan\left(\frac{\sin\phi}{\cos\phi}\right)$$

$$\phi = \arctan(\tan\phi)$$

$$\phi = \phi$$

So the transformation is

$$x = 2r\cos\phi$$

$$y = 2r\sin\phi$$

The partial derivatives of these are

$$\frac{\partial x}{\partial r} = \frac{\partial}{\partial r}(2r\cos\phi) = 2\cos\phi$$

$$\frac{\partial x}{\partial \phi} = \frac{\partial}{\partial \phi}(2r\cos\phi) = -2r\sin\phi$$

$$\frac{\partial y}{\partial r} = \frac{\partial}{\partial r}(2r\sin\phi) = 2\sin\phi$$

$$\frac{\partial y}{\partial \phi} = \frac{\partial}{\partial \phi}(2r \sin \phi) = 2r \cos \phi$$

Then the Jacobian of the transformation is

$$\frac{\partial(x, y)}{\partial(r, \phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} \end{vmatrix} = \frac{\partial x}{\partial r} \cdot \frac{\partial y}{\partial \phi} - \frac{\partial x}{\partial \phi} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial(x, y)}{\partial(r, \phi)} = 2 \cos \phi \cdot 2r \cos \phi - (-2r \sin \phi) \cdot 2 \sin \phi$$

$$\frac{\partial(x, y)}{\partial(r, \phi)} = 4r$$

IMAGE OF THE SET UNDER THE TRANSFORMATION

■ 1. Identify the surface obtained from the unit sphere (a sphere with center at the origin and radius 1 by applying the transformation.

$$x = u - 2v + 2$$

$$y = u + v + w$$

$$z = u + v - w + 4$$

Solution:

The equation of the unit sphere is

$$x^2 + y^2 + z^2 = 1$$

Plug in the expressions for x, y, and z.

$$(u - 2v + 2)^2 + (u + v + w)^2 + (u + v - w + 4)^2 = 1$$

$$(u^2 - 4uv + 4u + 4v^2 - 8v + 4) + (u^2 + v^2 + w^2 + 2uv + 2uw + 2vw)$$

$$+(u^2 + 2uv - 2uw + 8u + v^2 - 2vw + 8v + w^2 - 8w + 16) = 1$$

$$3u^2 + 12u + 6v^2 + 2w^2 - 8w + 19 = 0$$

Complete the square with respect to each variable.

$$3(u^2 + 4u + 4 - 4) + 6v^2 + 2(w^2 - 4w + 4 - 4) + 19 = 0$$

$$3(u+2)^2 - 12 + 6v^2 + 2(w-2)^2 - 8 + 19 = 0$$

$$3(u+2)^2 + 6v^2 + 2(w-2)^2 = 1$$

So the sphere moves to the ellipsoid with center at $(-2,0,2)$, and semi-axes $x = 1/\sqrt{3}$, $y = 1/\sqrt{6}$, and $z = 1/\sqrt{2}$.

■ 2. Identify the shape obtained from the parallelogram $ABCD$, where $A(-2,2)$, $B(-2,5)$, $C(-3,6)$, and $D(-3,3)$, by applying the transformation.

$$u = -x - 2$$

$$v = \frac{x+y}{3}$$

Solution:

Since the transformation converts line segments into segments, we can just substitute the coordinates of A, B, C, and D to get a new quadrilateral $A_1B_1C_1D_1$.

Transforming A gives $A_1(0,0)$.

$$u = -(-2) - 2 = 0$$

$$v = \frac{(-2)+(2)}{3} = 0$$

Transforming B gives $B_1(0,1)$.

$$u = -(-2) - 2 = 0$$

$$v = \frac{(-2) + (5)}{3} = 1$$

Transforming C gives $C_1(1,1)$.

$$u = -(-3) - 2 = 1$$

$$v = \frac{(-3) + (6)}{3} = 1$$

Transforming D gives $D_1(1,0)$.

$$u = -(-3) - 2 = 1$$

$$v = \frac{(-3) + (3)}{3} = 0$$

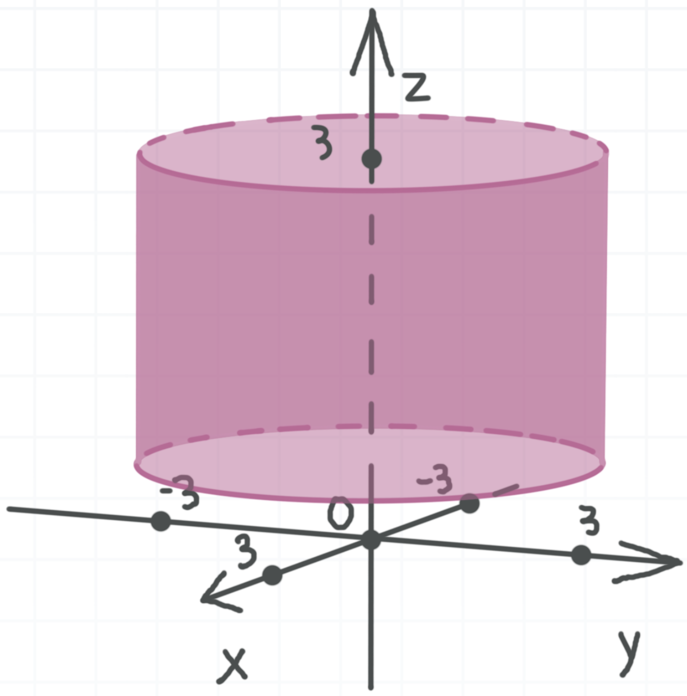
Therefore, parallelogram $ABCD$ transforms into the unit square $A_1B_1C_1D_1$ in the first quadrant, where $A_1(0,0)$, $B_1(0,1)$, $C_1(1,1)$, and $D_1(1,0)$.

■ 3. Identify the solid obtained from the set of interior points of the circular cylinder $x^2 + y^2 = 9$, with $1 \leq z \leq 5$, when the following transformation is applied.

$$u = \sqrt{x^2 + y^2}$$

$$v = \arctan \frac{y}{x}$$

$$w = z - 1$$



Solution:

Since we have the transformation $(x, y, z) \to (u, v, w)$, we need first to find the inverse transformation $(u, v, w) \to (x, y, z)$ by solving the system of equations for x, y, and z. We want to realize that the given transformation is almost an inverse to standard cylindrical coordinates with an exception in z/w pair.

From the third equation, $z = w + 1$. From the second equation, $y/x = \tan v$. Let $u = \sqrt{x^2 + y^2} = r$, $x = r\cos\phi$, and $y = r\sin\phi$, then substitute.

$$\frac{r\sin\phi}{r\cos\phi} = \tan v$$

$$\frac{\sin\phi}{\cos\phi} = \tan v$$

$$\tan\phi = \tan v$$

$$\phi = v$$

So the transformation is

$$x = u\cos v$$

$$y = u\sin v$$

$$z = w + 1$$

To find the borders for u, substitute x and y into the cylinder's equation.

$$x^2 + y^2 = 9$$

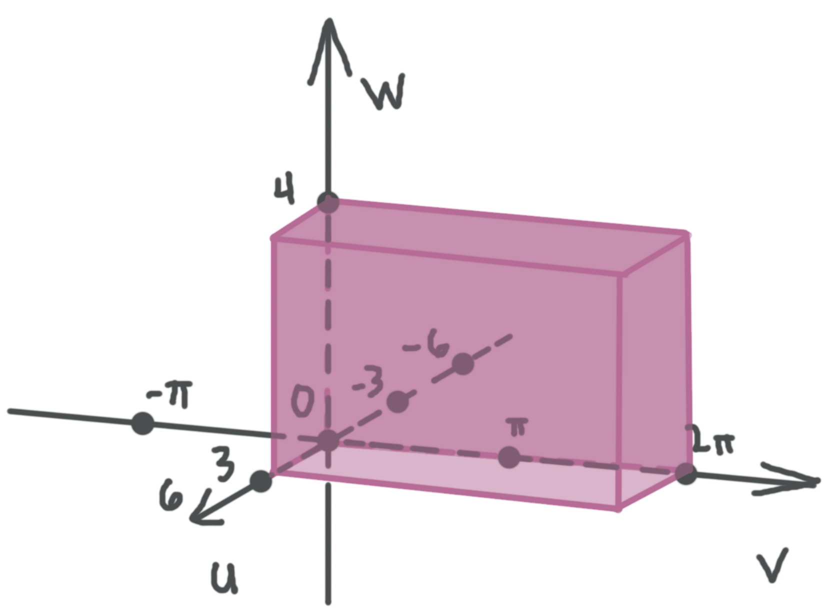
$$(u\cos\phi)^2 + (u\sin\phi)^2 = 3^2$$

$$u^2 = 3^2$$

$$u = 3$$

So $0 \le u \le 3$ and $0 \le v < 2\pi$, and since z changes from 1 to 5, $0 \le w \le 4$.

Therefore, the image of the cylinder is the rectangular prism with dimensions $3 \times 2\pi \times 4$.



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