Arc length of a vector function

To find the arc length of the vector function, we will need to use the formula

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\ dt$$

where L is the arc length of the vector function, $[a, b]$ is the interval that defines the arc, and dx/dt, dy/dt, and dz/dt are the derivatives of the parametric equations of x, y and z respectively.

To solve for arc length, we'll need the parametric equations of the vector function. Whether our vector function is given as $r(t) = \left\langle r(t)_1, r(t)_2, r(t)_3 \right\rangle$ or $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, the parametric equations are

$$x = r(t)_1$$

$$y = r(t)_2$$

$$z = r(t)_3$$

Once we have these parametric equations, we'll take the derivative of each one to get dx/dt, dy/dt and dz/dt. Assuming we're given $[a, b]$, we'll have everything we need to use the formula for arc length.

Example

Find the arc length of the vector function over the given interval.

$$r(t) = \langle \sin(2t), \cos(2t), 2t \rangle$$

on $0 \leq t \leq 2$

We'll pull the parametric equations out of the vector function as

$$x = \sin(2t)$$

$$y = \cos(2t)$$

$$z = 2t$$

Now we'll take the derivative of each of these.

$$\frac{dx}{dt} = 2\cos(2t)$$

$$\frac{dy}{dt} = -2\sin(2t)$$

$$\frac{dz}{dt} = 2$$

Plugging the derivatives and the given interval $0 \leq t \leq 2$ into the formula for arc length, we get

$$L = \int_0^2 \sqrt{\left[2\cos(2t)\right]^2 + \left[-2\sin(2t)\right]^2 + (2)^2}\ dt$$

$$L = \int_0^2 \sqrt{4\cos^2(2t) + 4\sin^2(2t) + 4}\ dt$$

$$L = \int_0^2 \sqrt{4\left[\cos^2(2t) + \sin^2(2t)\right] + 4}\ dt$$

Since $\cos^2 x + \sin^2 x = 1$, we can simplify the integral to

$$L = \int_0^2 \sqrt{4(1) + 4}\ dt$$

$$L = \int_0^2 \sqrt{8}\ dt$$

$$L = \int_0^2 \sqrt{4 \cdot 2}\ dt$$

$$L = \int_0^2 2\sqrt{2}\ dt$$

$$L = 2\sqrt{2}t \Big|_0^2$$

Evaluating over the interval, we get

$$L = 2\sqrt{2}(2) - 2\sqrt{2}(0)$$

$$L = 4\sqrt{2}$$

The arc length of the vector function over the interval $0 \le t \le 2$ is $L = 4\sqrt{2}$.