Calculus 3 Workbook Solutions

Surface integrals

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MATH

SURFACE INTEGRALS

1. Evaluate the surface integral of the scalar vector field $f(x,y,z)=\ln(x+y+z)$ over the surface $\vec{r}=\langle 3u-7v+1, u+5v+2, -3u+v-1\rangle$, where u changes from 0 to 4 and v changes from -1 to 1.

Solution:

Take partial derivatives.

$$\vec{r}_u=\langle 3,1,-3\rangle$$

$$\vec{r}_v=\langle -7,5,1\rangle$$

Take the cross product of these vectors.

$$\vec{r}_u\times\vec{r}_v=\langle 3,1,-3\rangle\times\langle -7,5,1\rangle$$

$$\vec{r}_u\times\vec{r}_v=\langle 1\cdot 1-(-3)\cdot 5, -3\cdot 1-3\cdot(-7), 3\cdot 5-1\cdot(-7)\rangle$$

$$\vec{r}_u\times\vec{r}_v=\langle 16,18,22\rangle$$

The magnitude of the cross product is

$$|\vec{r}_u\times\vec{r}_v|=\sqrt{16^2+18^2+22^2}=2\sqrt{266}$$

The function is

$$f(x,y,z)=\ln((3u-7v+1)+(u+5v+2)+(-3u+v-1))$$

$$f(x,y,z)=\ln(u-v+2)$$

So the surface integral is

$$\int_0^4\int_{-1}^1 2\sqrt{266}\ln(u-v+2)\ dv\ du$$

Integrate with respect to v, treating u as a constant.

$$2\sqrt{266}\int_0^4 (v-u-2)\ln(u-v+2)-v\Big|_{v=-1}^{v=1}\ du$$

$$2\sqrt{266}\int_0^4 (1-u-2)\ln(u-1+2)-1-((-1-u-2)\ln(u+1+2)+1)\ du$$

$$2\sqrt{266}\int_0^4 (-u-1)\ln(u+1)+(u+3)\ln(u+3)-2\ du$$

$$2\sqrt{266}\int_0^4 (u+3)\ln(u+3)-(u+1)\ln(u+1)-2\ du$$

Integrate with respect to u using integration by parts.

$$2\sqrt{266}\left[\frac{1}{2}(u+3)^2\ln(u+3)-\frac{(u+3)^2}{4}-\frac{1}{2}(u+1)^2\ln(u+1)+\frac{(u+1)^2}{4}-2u\right]\Bigg|_0^4$$

$$2\sqrt{266}\left(\frac{1}{2}(4+3)^2\ln(4+3)-\frac{(4+3)^2}{4}-\frac{1}{2}(4+1)^2\ln(4+1)+\frac{(4+1)^2}{4}-2\cdot 4\right)$$

$$-2\sqrt{266}\left(\frac{1}{2}(0+3)^2\ln(0+3)-\frac{(0+3)^2}{4}-\frac{1}{2}(0+1)^2\ln(0+1)+\frac{(0+1)^2}{4}-2\cdot 0\right)$$

$$2\sqrt{266}\left(\frac{49}{2}\ln 7-\frac{49}{4}-\frac{25}{2}\ln 5+\frac{25}{4}-8-\frac{9}{2}\ln 3+\frac{9}{4}+\frac{1}{2}\ln 1-\frac{1}{4}\right)$$

$$\sqrt{266}(49\ln 7-25\ln 5-9\ln 3-24)$$

■ 2. Evaluate the surface integral of the scalar vector field $f(x,y,z)=x^2+y^2+4z^2$ over the part of the cylinder $x^2+y^2=9$, where $-2\le z\le 5$.

Solution:

The standard parametrization of the cylinder with radius r and cylindrical axis parallel to the z-axis is

$$x=r\cos\phi$$

$$y=r\sin\phi$$

$$z=z$$

Since the given cylinder has radius 3, plug in $r=3$ and rename the parameters as $\phi\to u$ and $z\to v$.

$$x(u,v)=3\cos u$$

$$y(u,v)=3\sin u$$

$$z(u,v)=v$$

So we get the parametrization of the part of the cylinder we're interested in. Take partial derivatives.

$$\vec{r_u} = \langle -3\sin u, 3\cos u, 0\rangle$$

$$\vec{r_v} = \langle 0,0,1\rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle -3\sin u, 3\cos u, 0\rangle \times \langle 0,0,1\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 3\cos u \cdot 1 - 0 \cdot 0, -(-3\sin u)\cdot 1 + 0 \cdot 0, -3\sin u \cdot 0 - 3\cos u \cdot 0\rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 3\cos u, 3\sin u, 0\rangle$$

The magnitude of the cross product is

$$|\vec{r_u} \times \vec{r_v}| = \sqrt{(3\cos u)^2 + (3\sin u)^2 + 0^2}$$

$$|\vec{r_u} \times \vec{r_v}| = \sqrt{9\cos^2 u + 9\sin^2 u}$$

$$|\vec{r_u} \times \vec{r_v}| = \sqrt{9}$$

$$|\vec{r_u} \times \vec{r_v}| = 3$$

The function is

$$f(x,y,z) = (3\cos u)^2 + (3\sin u)^2 + 4v^2$$

$$f(x,y,z) = 9\cos^2 u + 9\sin^2 u + 4v^2$$

$$f(x,y,z) = 9 + 4v^2$$

So the surface integral is

$$\int_0^{2\pi}\int_{-2}^{5} 3(9+4v^2)\ dv\ du$$

$$\int_0^{2\pi}\int_{-2}^{5} 27+12v^2\ dv\ du$$

Integrate with respect to v.

$$\int_0^{2\pi} 27v+4v^3\Big|_{v=-2}^{v=5}\ du$$

$$\int_0^{2\pi} 27(5)+4(5)^3-(27(-2)+4(-2)^3)\ du$$

$$\int_0^{2\pi} 721\ du$$

Integrate with respect to u.

$$721(2\pi-0)$$

$$1{,}442\pi$$

■ 3. Evaluate the surface integral of the scalar vector field $f(x,y,z)=x^2+y^2+z+1$ over the sphere centered at $(2,-1,-3)$ with radius 2.

Solution:

The standard parametrization of the sphere with radius ρ and center (x_0, y_0, z_0) is

$$x = x_0 + \rho \sin \phi \cos \theta$$

$$y = y_0 + \rho \sin \phi \sin \theta$$

$$z = z_0 + \rho \cos \phi$$

Plug in $\rho = 2$ and $(x_0, y_0, z_0) = (2, -1, -3)$ and rename the parameters $\phi \to u$ and $\theta \to v$.

$$x(u, v) = 2 + 2 \sin u \cos v$$

$$y(u, v) = -1 + 2 \sin u \sin v$$

$$z(u, v) = -3 + 2 \cos u$$

So we get the parametrization of the sphere we're interested in. Take partial derivatives.

$$\vec{r_u} = \langle 2 \cos u \cos v, 2 \cos u \sin v, -2 \sin u \rangle$$

$$\vec{r_v} = \langle -2 \sin u \sin v, 2 \sin u \cos v, 0 \rangle$$

Take the cross product.

$$\vec{r_u} \times \vec{r_v} = \langle 2 \cos u \cos v, 2 \cos u \sin v, -2 \sin u \rangle \times \langle -2 \sin u \sin v, 2 \sin u \cos v, 0 \rangle$$

$$\vec{r_u} \times \vec{r_v} = \langle 2 \cos u \sin v \cdot 0 - (-2 \sin u) \cdot 2 \sin u \cos v$$

$$-2 \cos u \cos v \cdot 0 - 2 \sin u \cdot (-2 \sin u \sin v)$$

$$+2 \cos u \cos v \cdot 2 \sin u \cos v - 2 \cos u \sin v \cdot (-2 \sin u \sin v) \rangle$$

$$\langle 4\sin^2 u\cos v, 4\sin^2 u\sin v, 4\sin u\cos u\rangle$$

The magnitude of the cross product is

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(4\sin^2 u\cos v)^2 + (4\sin^2 u\sin v)^2 + (4\sin u\cos u)^2}$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{16\sin^2 u}$$

$$|\vec{r}_u \times \vec{r}_v| = 4\sin u$$

The function is

$$f(x,y,z) = (2 + 2\sin u\cos v)^2 + (-1 + 2\sin u\sin v)^2 + (-3 + 2\cos u) + 1$$

$$f(x,y,z) = 8\sin u\cos v - 4\sin u\sin v + 2\sin^2 u - 2\cos^2 u + 2\cos u + 5$$

$$f(x,y,z) = 8\sin u\cos v - 4\sin u\sin v - 2\cos 2u + 2\cos u + 5$$

So the surface integral is

$$\int_0^{\pi}\int_0^{2\pi} (8\sin u\cos v - 4\sin u\sin v - 2\cos 2u + 2\cos u + 5) \cdot 4\sin u \ dv \ du$$

$$4\int_0^{\pi}\int_0^{2\pi} 8\sin^2 u\cos v - 4\sin^2 u\sin v - 2\cos 2u\sin u + 2\cos u\sin u + 5\sin u \ dv \ du$$

Since the integral of sine and cosine functions over a 2π-period is 0,

$$4\int_0^{\pi}\int_0^{2\pi} -2\cos 2u\sin u + 2\cos u\sin u + 5\sin u \ dv \ du$$

Integrate with respect to v.

$$4\int_0^{\pi} 2\pi(-2\cos 2u \sin u + \sin 2u + 5\sin u)\ du$$

$$8\pi\int_0^{\pi} -2\cos 2u \sin u + \sin 2u + 5\sin u\ \ du$$

$$8\pi\int_0^{\pi} -\sin 3u + \sin u + \sin 2u + 5\sin u\ \ du$$

$$8\pi\int_0^{\pi} -\sin 3u + \sin 2u + 6\sin u\ \ du$$

$$8\pi\left(\frac{1}{3}\cos 3u - \frac{1}{2}\cos 2u - 6\cos u\right)\Bigg|_0^{\pi}$$

$$8\pi\left(\frac{1}{3}\cos 3\pi - \frac{1}{2}\cos 2\pi - 6\cos\pi\right) - 8\pi\left(\frac{1}{3}\cos 0 - \frac{1}{2}\cos 0 - 6\cos 0\right)$$

$$8\pi\left(-\frac{1}{3} - \frac{1}{2} + 6\right) - 8\pi\left(\frac{1}{3} - \frac{1}{2} - 6\right) = \frac{272\pi}{3}$$

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