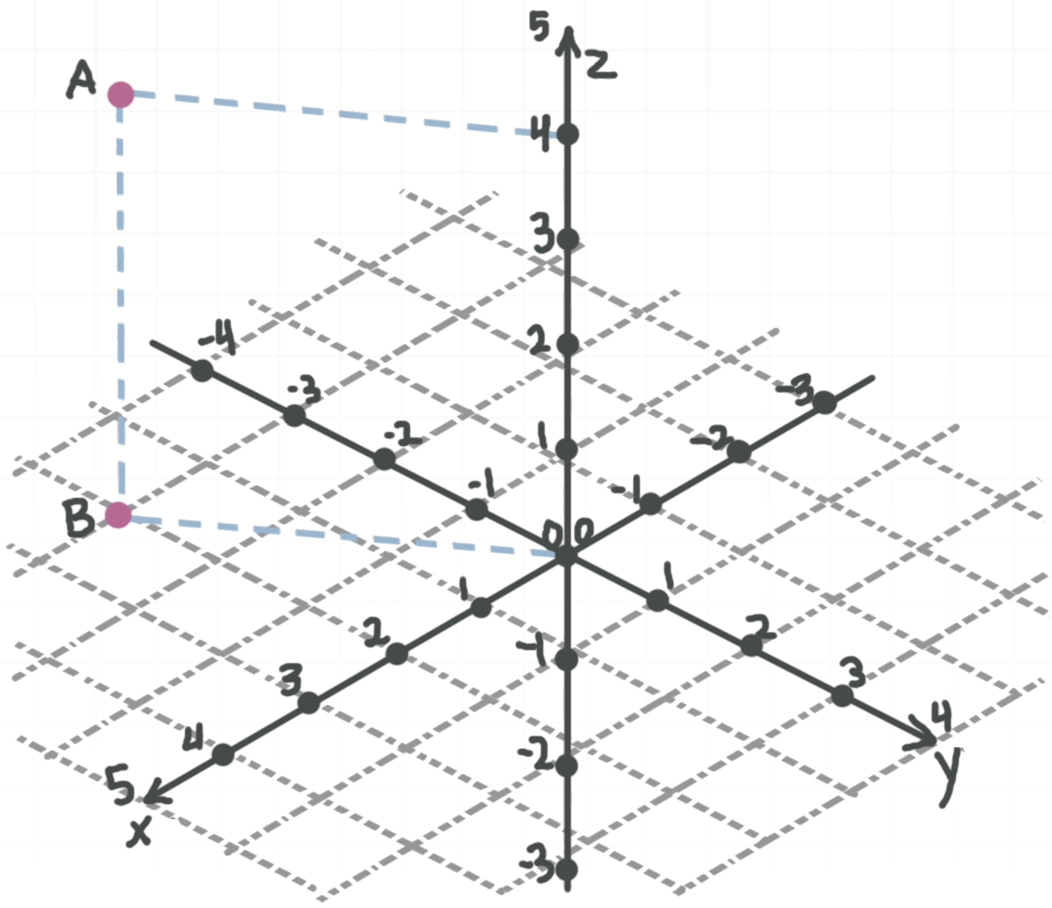
Calculus 3 Workbook Solutions

Three-dimensional coordinate systems

krista king MATH

PLOTTING POINTS IN THREE DIMENSIONS

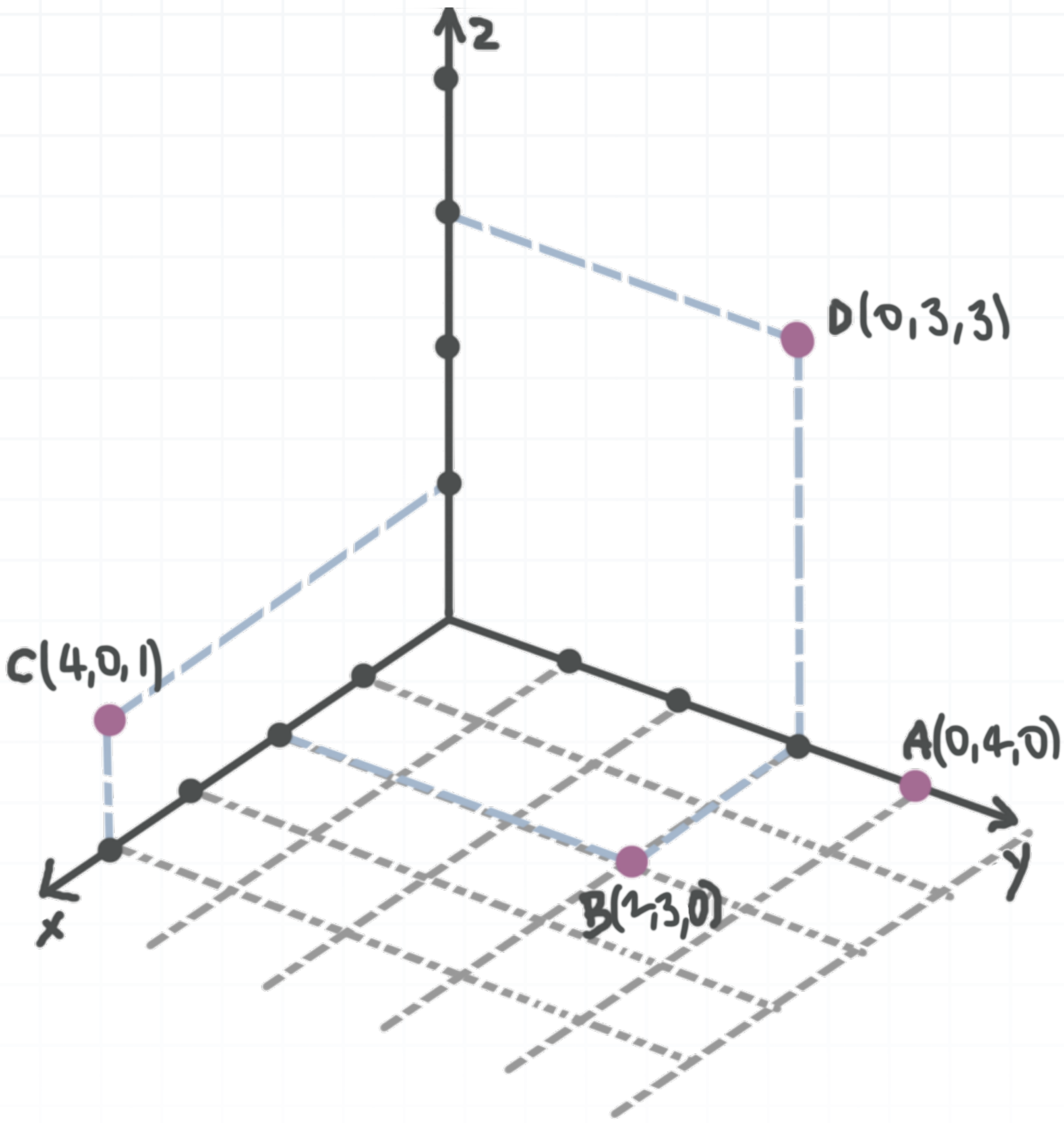
■ 1. What are the coordinates of point A?



Solution:

The coordinates of any point in three-dimensional rectangular coordinates can be given by the ordered triple (x, y, z). Using B as a reference, A is at $x = 2$, $y = -3$, and $z = 4$, which is $(x, y, z) = (2, -3, 4)$.

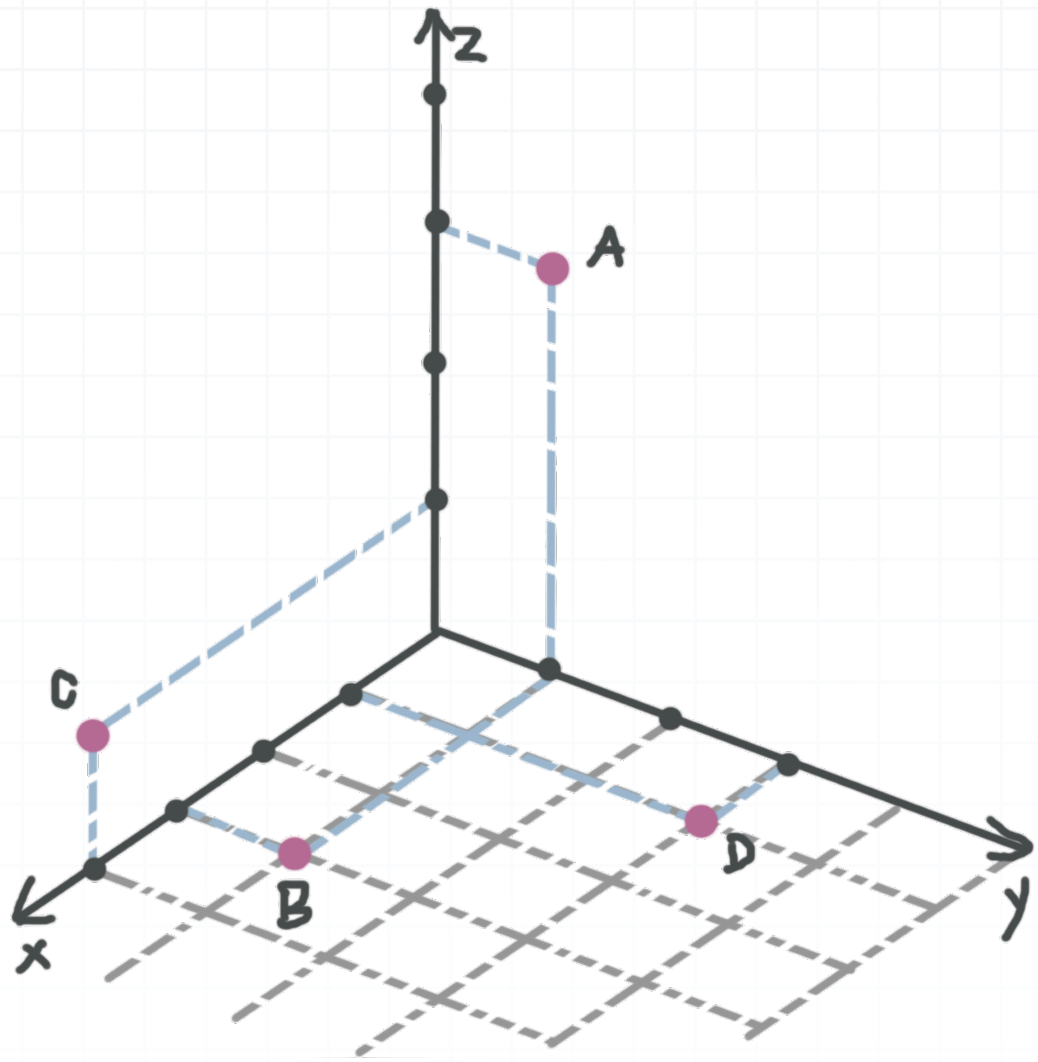
■ 2. Which of the points A, B, C, and D lie on the xz-plane?



Solution:

The point lies on the xz-plane only if its y-coordinate is equal to 0, which means we're looking for a point in the form $(x,0,z)$. Only point C matches that format.

■ 3. Which of the points A, B, C, and D has coordinates $(3,1,0)$?



Solution:

The coordinates of any point in three-dimensional rectangular coordinate space are given by the ordered triple (x, y, z). The points have coordinates $A(0,1,3)$, $B(3,1,0)$, $C(4,0,1)$, and $D(1,3,0)$, which means point B is the correct answer.

DISTANCE BETWEEN POINTS IN THREE DIMENSIONS

■ 1. Find the perimeter P of the triangle ABC given $A(1,0,0)$, $B(2,3,0)$, and $C(3,3,-3)$.

Solution:

To find the perimeter of triangle ABC we need to use the distance formula

$$D=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

to find the lengths of sides AB, BC, and AC, then add the three lengths.

Side AB is

$$AB=\sqrt{(2-1)^2+(3-0)^2+(0-0)^2}$$

$$AB=\sqrt{1^2+3^2+0^2}$$

$$AB=\sqrt{1+9+0}$$

$$AB=\sqrt{10}$$

Side BC is

$$BC=\sqrt{(3-2)^2+(3-3)^2+(-3-0)^2}$$

$$BC=\sqrt{1^2+0^2+3^2}$$

$$BC = \sqrt{1+0+9}$$

$$BC = \sqrt{10}$$

Side AC is

$$AC = \sqrt{(3-1)^2 + (3-0)^2 + (-3-0)^2}$$

$$AC = \sqrt{2^2 + 3^2 + 3^2}$$

$$AC = \sqrt{4+9+9}$$

$$AC = \sqrt{22}$$

So the perimeter P of triangle ABC is

$$P = \sqrt{10} + \sqrt{10} + \sqrt{22}$$

$$P = 2\sqrt{10} + \sqrt{22}$$

■ 2. Given $A(-1,0,0)$, $B(-2,0,2)$, and $C(0,1,0)$, find the measure of angle BAC in degrees using the law of cosines:

$$\cos(BAC) = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$$

Solution:

To use the law of cosines, we need the distances AB, AC, and BC. Side AB is

$$AB^2 = (-2-(-1))^2 + (0-0)^2 + (2-0)^2$$

$$AB^2 = (-1)^2 + 0^2 + 2^2$$

$$AB^2 = 1 + 0 + 4$$

$$AB^2 = 5$$

$$AB = \sqrt{5}$$

Side BC is

$$BC^2 = (0-(-2))^2 + (1-0)^2 + (0-2)^2$$

$$BC^2 = 2^2 + 1^2 + 2^2$$

$$BC^2 = 4 + 1 + 4$$

$$BC^2 = 9$$

$$BC = 3$$

Side AC is

$$AC^2 = (0-(-1))^2 + (1-0)^2 + (0-0)^2$$

$$AC^2 = 1^2 + 1^2 + 0^2$$

$$AC^2 = 1 + 1 + 0$$

$$AC^2 = 2$$

$$AC = \sqrt{2}$$

Then the law of cosines gives the angle BAC as

$$\cos(BAC) = \frac{AB^2 + AC^2 - BC^2}{2 \cdot AB \cdot AC}$$

$$\cos(BAC) = \frac{5 + 2 - 9}{2\sqrt{5}\sqrt{2}}$$

$$\cos(BAC) = \frac{-2}{2\sqrt{5}\sqrt{2}}$$

$$\cos(BAC) = \frac{-1}{\sqrt{10}}$$

Evaluate.

$$BAC \approx \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right)$$

$$BAC \approx 108.4^\circ$$

■ 3. Find the point on the x-axis that's equidistant from $A(-1,1,0)$ and $B(-2,1,-1)$.

Solution:

Every point on the x-axis can be given by $(x,0,0)$. If we call this the point P, then

$$PA = PB$$

$$PA^2 = PB^2$$

The distance formula gives

$$PA^2 = (x - (-1))^2 + (0 - 1)^2 + (0 - 0)^2$$

$$PA^2 = (x + 1)^2 + 1$$

and

$$PB^2 = (x - (-2))^2 + (0 - 1)^2 + (0 - (-1))^2$$

$$PB^2 = (x + 2)^2 + 2$$

Since $PA^2 = PB^2$, we can solve for x.

$$(x + 1)^2 + 1 = (x + 2)^2 + 2$$

$$x^2 + 2x + 1 + 1 = x^2 + 4x + 4 + 2$$

$$2x = 4x + 4$$

$$2x = -4$$

$$x = -2$$

The the point P on the x-axis that's equidistant from $A(-1,1,0)$ and $B(-2,1,-1)$ is $P(-2,0,0)$.

CENTER, RADIUS, AND EQUATION OF THE SPHERE

■ 1. Find the equation of a sphere with center $(1,3,-2)$ and y-intercept 1.

Solution:

The standard equation of the sphere with the center (h,k,l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

Since the sphere passes through the y-intercept $(0,1,0)$, then the distance formula gives the radius as

$$r^2=(0-1)^2+(1-3)^2+(0-(-2))^2$$

$$r^2=1^2+2^2+2^2$$

$$r^2=1+4+4$$

$$r^2=9$$

$$r=3$$

Substitute the center $(1,3,-2)$ for (h,k,l) and the value of 3 for r into the standard sphere equation.

$$(x-1)^2+(y-3)^2+(z+2)^2=3^2$$

2. Of the points $A(-4,-1,7)$, $B(-5,1,5)$, $C(-6,-6,5)$, and $D(-7,0,3)$, which one does not lie in the interior of the sphere?

$$(x+5)^2+(y+3)^2+(z-4)^2=16$$

Solution:

The point lies in the interior of the sphere if the distance between the point and the center of the sphere, $(-5,-3,4)$, is less than the radius, 4. Or equally, if the square of the distance between the point and the center is less than r^2, 16.

The distance from each point to the center $(-5,-3,4)$ is

For $A(-4,-1,7)$,

$$(-4+5)^2+(-1+3)^2+(7-4)^2$$

$$1^2+2^2+3^2$$

$$1+4+9$$

$$14<16$$

For $B(-5,1,5)$,

$$(-5+5)^2+(1+3)^2+(5-4)^2$$

$$0^2+4^2+1^2$$

$$0+16+1$$

$$17 > 16$$

For $C(-6,-6,5)$,

$$(-6+5)^2 + (-6+3)^2 + (5-4)^2$$

$$(-1)^2 + (-3)^2 + 1^2$$

$$1 + 9 + 1$$

$$11 < 16$$

For $D(-7,0,3)$,

$$(-7+5)^2 + (0+3)^2 + (3-4)^2$$

$$(-2)^2 + 3^2 + (-1)^2$$

$$4 + 9 + 1$$

$$14 < 16$$

The points A, C, and D lie in the interior of the circle, but $B(-5,1,5)$ does not.

■ 3. The endpoints of the diameter of a sphere are $A(2,4,-3)$ and $B(6,0,-1)$. Find the equation of this sphere.

Solution:

The standard equation of a sphere with center (h,k,l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

Since the center is the midpoint of the diameter AB, by the midpoint formula, its coordinates are

$$h=\frac{2+6}{2}=\frac{8}{2}=4$$

$$k=\frac{4+0}{2}=\frac{4}{2}=2$$

$$l=\frac{-3+(-1)}{2}=\frac{-4}{2}=-2$$

The radius of the sphere, r, is equal to the distance between the point A and the center of the circle. So the distance formula gives the radius as

$$r=\sqrt{(4-2)^2+(2-4)^2+(-2-(-3))^2}$$

$$r=\sqrt{2^2+(-2)^2+1^2}$$

$$r=\sqrt{4+4+1}$$

$$r=\sqrt{9}$$

$$r=3$$

So the equation of the sphere with center $(4,2,-2)$ and radius 3 is

$$(x-4)^2+(y-2)^2+(z+2)^2=3^2$$

$$(x-4)^2+(y-2)^2+(z+2)^2=9$$

DESCRIBING A REGION IN THREE DIMENSIONAL SPACE

■ 1. Describe the surface in three-dimensional space.

$$x^2 + 2x + z^2 = 0$$

Solution:

Complete the square with respect to x.

$$x^2 + 2x + 1 - 1 + z^2 = 0$$

$$(x + 1)^2 + z^2 - 1 = 0$$

$$(x + 1)^2 + z^2 = 1^2$$

This equation represents the circle with center $(-1,y,0)$ and radius 1, so the surface is a cylinder that's parallel to the y-axis and intersects the xz-plane in the circle with center $(-1,0,0)$ and radius 1.

■ 2. Describe the surface in three-dimensional space.

$$z = 7$$

Solution:

Since $z = 7$ has no x-variable, the surface is parallel to the x-axis. Similarly, since $z = 7$ has no y-variable, the surface is parallel to the y-axis.

Because it's parallel to both x- and y-axes, $z = 7$ must be a plane parallel to xy-plane, that intersects the z-axis at $(0,0,7)$.

■ 3. Describe the surface in three-dimensional space.

$$xy = 0$$

Solution:

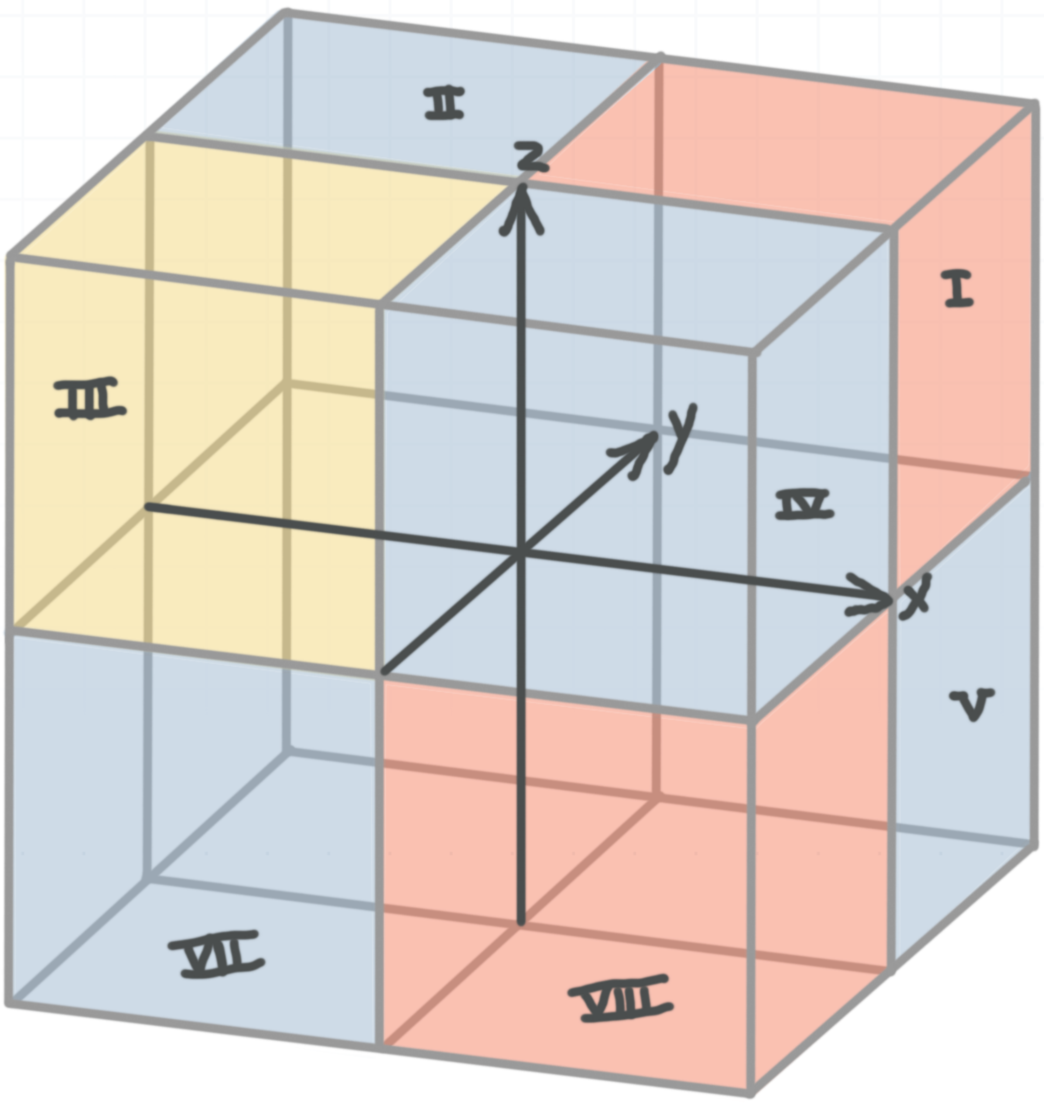
The product xy becomes 0 if $x = 0$, or $y = 0$.

The equation $x = 0$ represents the yz-plane, and the equation $y = 0$ represents the xz-plane. So the equation $xy = 0$ represents the surface which consists of both the yz- and xz-planes.

These planes intersect at the z-axis, which means that also fully belongs to the region.

USING INEQUALITIES TO DESCRIBE THE REGION

■ 1. What set of inequalities describes Octant III? Remember that an "octant" is one of the eight spaces that make up the three-dimensional coordinate system.



Solution:

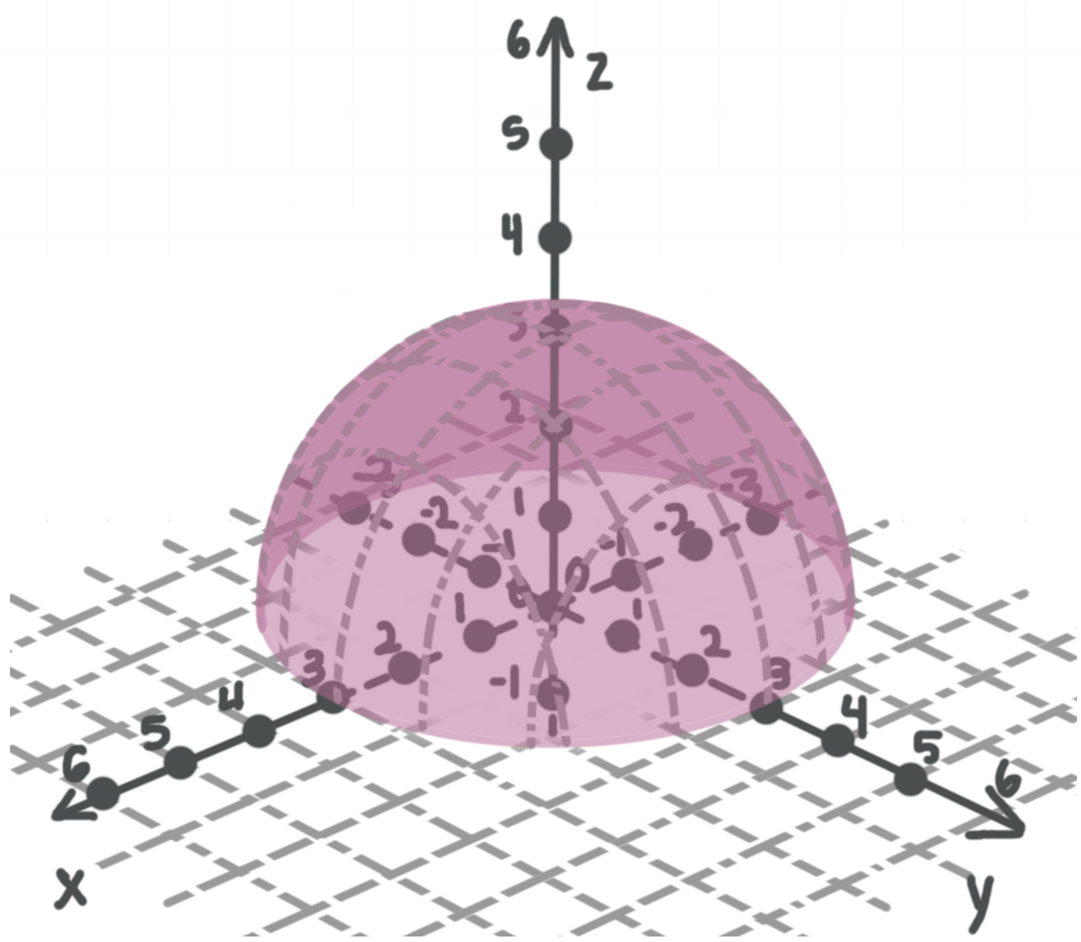
The equation $x = 0$ divides three-dimensional space into two regions, where $x > 0$ contains octants I, IV, V, VIII, and $x < 0$ contains octants II, III, VI, VII.

Similarly, $y > 0$ contains octants I, II, V, VI, and $y < 0$ contains octants III, IV, VII, VIII.

Finally, $z > 0$ contains octants I, II, III, IV, and $z < 0$ contains octants V, VI, VII, VIII.

So octant III is bounded by $x < 0$, $y < 0$, and $z > 0$.

■ 2. What set of inequalities describes the region consisting of all points inside the hemisphere, if the base of the sphere is centered at $(0,0,0)$?



Solution:

The standard equation of a sphere with center (h, k, l) and radius r is

$$(x-h)^2+(y-k)^2+(z-l)^2=r^2$$

Because we can see from the figure that the sphere's edge extends to $x=3$ and $y=3$, that means the radius must be 3, so substitute $(0,0,0)$ for (h,k,l) and 3 for r to get the equation of the given sphere.

$$(x-0)^2+(y-0)^2+(z-0)^2=3^2$$

$$x^2+y^2+z^2=3^2$$

So $x^2+y^2+z^2<3^2$ describes the region consisting of all points inside the full sphere. To get only the hemisphere, $z>0$. Putting these together means the hemisphere is described by

When we put our two inequalities together to describe the region, we get

$$x^2+y^2+z^2<9 \text{ and } z>0$$

■ 3. What set of inequalities describes the region consisting of all points which lie at most 5 units from the yz-plane?

Solution:

The x-coordinate of the point tells us how far the point is from the yz-plane. Since the distance from a point to the yz-plane, which is the x-coordinate, can be at most 5, we get the inequality

$$|x|\le 5$$

or, after removing the absolute value form inequality,

$$-5 \le x \le 5$$

In other words, the value of x must fall between $x = -5$ and $x = 5$ in order to stay within 5 units of the xy-plane.

W W W . K R I S T A K I N G M A T H . C O M