Jacobian for two variables

In the past we've converted multivariable functions defined in terms of cartesian coordinates x and y into functions defined in terms of polar coordinates r and θ.

Similarly, given a region defined in the uv-plane, we can use a Jacobian transformation to redefine it in the xy-plane, or vice versa.

Given two equations $x = f(u, v)$ and $y = g(u, v)$, the Jacobian is

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

Example

Find the Jacobian of the transformation.

$$x = uv$$

$$y = 2u - v^2$$

Our functions tell us that we have a 2×2 set-up, so we'll use the formula

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{\partial x}{\partial u} \cdot \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \cdot \frac{\partial y}{\partial u}$$

We need to start by finding the partial derivatives of x and y with respect to both u and v.

$$\frac{\partial x}{\partial u} = v$$

$$\frac{\partial x}{\partial v} = u$$

and

$$\frac{\partial y}{\partial u} = 2$$

$$\frac{\partial y}{\partial v} = -2v$$

We'll plug the partial derivatives into our formula and get

$$\frac{\partial(x, y)}{\partial(u, v)} = (v)(-2v) - (u)(2)$$

$$\frac{\partial(x, y)}{\partial(u, v)} = -2v^2 - 2u$$

This is the Jacobian of the transformation.