Calculus 1 Workbook Solutions

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MATH

IDEA OF THE LIMIT

1. The table below shows some values of a function $g(x)$. What does the table show for the value of $\lim_{x\to 4} g(x)$?

x	$g(x)$
3.9	1.9748
3.99	1.9975
3.999	1.9997
4.001	2.0002
4.01	2.0025
4.1	2.0248

Solution:

We see that when x approaches 4 both from the left and right sides, $g(x)$ approaches 2. Then $\lim_{x\to 4} g(x) = 2$.

2. How would we express, mathematically, the limit of the function $f(x) = x^2 - x + 2$ as x approaches 3?

Solution:

When a is the value that x approaches, and $f(x)$ is the given function, the limit is written as

$$\lim_{x \to a} f(x)$$

In this case x approaches 3 so $a = 3$, and the function is $f(x) = x^2 - x + 2$. So we'd write the limit as

$$\lim_{x \to 3} (x^2 - x + 2)$$

■ 3. How would you write the limit of $g(x)$ as x approaches ∞, using correct mathematical notation?

$$g(x) = \frac{5x^2 - 7}{3x^2 + 8}$$

Solution:

When a is the value that x approaches, and $g(x)$ is the given function, the limit is written as

$$\lim_{x \to a} g(x)$$

In this case x approaches ∞ so $a = \infty$, and the function is

$$g(x) = \frac{5x^2 - 7}{3x^2 + 8}$$

So we'd write the limit as

$$\lim_{x\to\infty} \frac{5x^2 - 7}{3x^2 + 8}$$

4. Explain what is meant by the equation.

$$\lim_{x\to-2} (x^3 + 2) = -6$$

Solution:

Break down the given limit into its component parts.

- x approaches -2
- the function is $f(x) = x^3 + 2$
- the value of the limit is -6

Putting these pieces together gives a full statement about the limit:

"The limit as x approaches -2 of the function $f(x) = x^3 + 2$ is equal to -6."

5. Evaluate the limit.

$$\lim_{x\to-1} \frac{-x^2 + 3x - 1}{5}$$

Solution:

To evaluate the limit,

$$\lim_{x \to -1} \frac{-x^2 + 3x - 1}{5}$$

plug the value that's being approached into the function, then simplify the result.

$$\frac{-(-1)^2 + 3(-1) - 1}{5}$$

$$\frac{-1 - 3 - 1}{5}$$

$$-\frac{5}{5}$$

$$-1$$

■ 6. Evaluate the limit.

$$\lim_{x \to 0} \frac{x^2 - 5}{2}$$

Solution:

To evaluate the limit,

$$\lim_{x \to 0} \frac{x^2 - 5}{2}$$

plug the value that's being approached into the function, then simplify the answer.

$$\frac{0^2 - 5}{2}$$

$$\frac{-5}{2}$$

$$-\frac{5}{2}$$

ONE-SIDED LIMITS

1. Find the limit.

$$\lim_{x\to -7^+} x^2\sqrt{x+7}$$

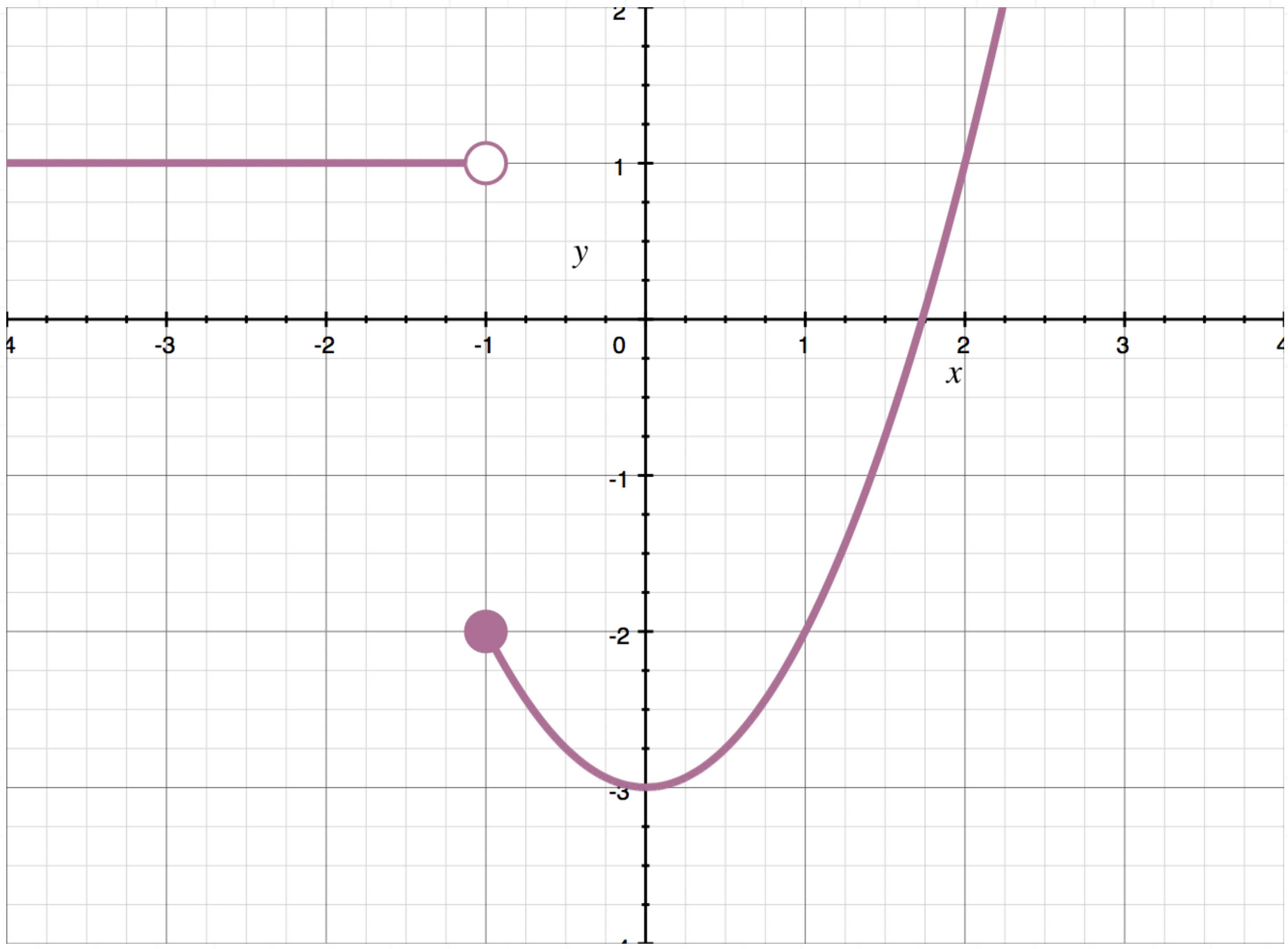
Solution:

The value of the limit is 0.

x	-6.09	-6.9	-6.99	-6.999	-6.9999	-7
Value	35.38	15.056	4.886	1.5481	0.48999	0

We see that as x approaches -7 from the right, the value of the function approaches 0. Then $\lim_{x\to -7^+} x^2\sqrt{x+7} = 0$. We could also graph the function to visually analyze its limit.

2. What does the graph of $f(x)$ say about the value of $\lim_{x\to -1^+} f(x)$?



Solution:

The positive sign after the -1 indicates that we're talking about the limit as we approach -1 from the positive, or right side of -1. From the graph, we see that the limit is

$$\lim_{x\to -1^+} f(x) = -2$$

■ 3. The table shows values of $k(x)$. What is $\lim_{x\to -5^-} k(x)$?

x	-5.1	-5.01	-5.0001	-5	-4.999	-4.99	-4.9
k(x)	-392.1	-3,812	-38,012	?	37,988	3,788	368.1

Solution:

The negative sign after the -5 indicates that we're talking about the limit as we approach -5 from the negative, or left side. From the table, we see that as we get very close to $x = -5$ on the left side, the function's value is trending toward $-\infty$, but as we get very close to $x = -5$ on the right side, the function's value is trending toward ∞. So the left-hand limit is

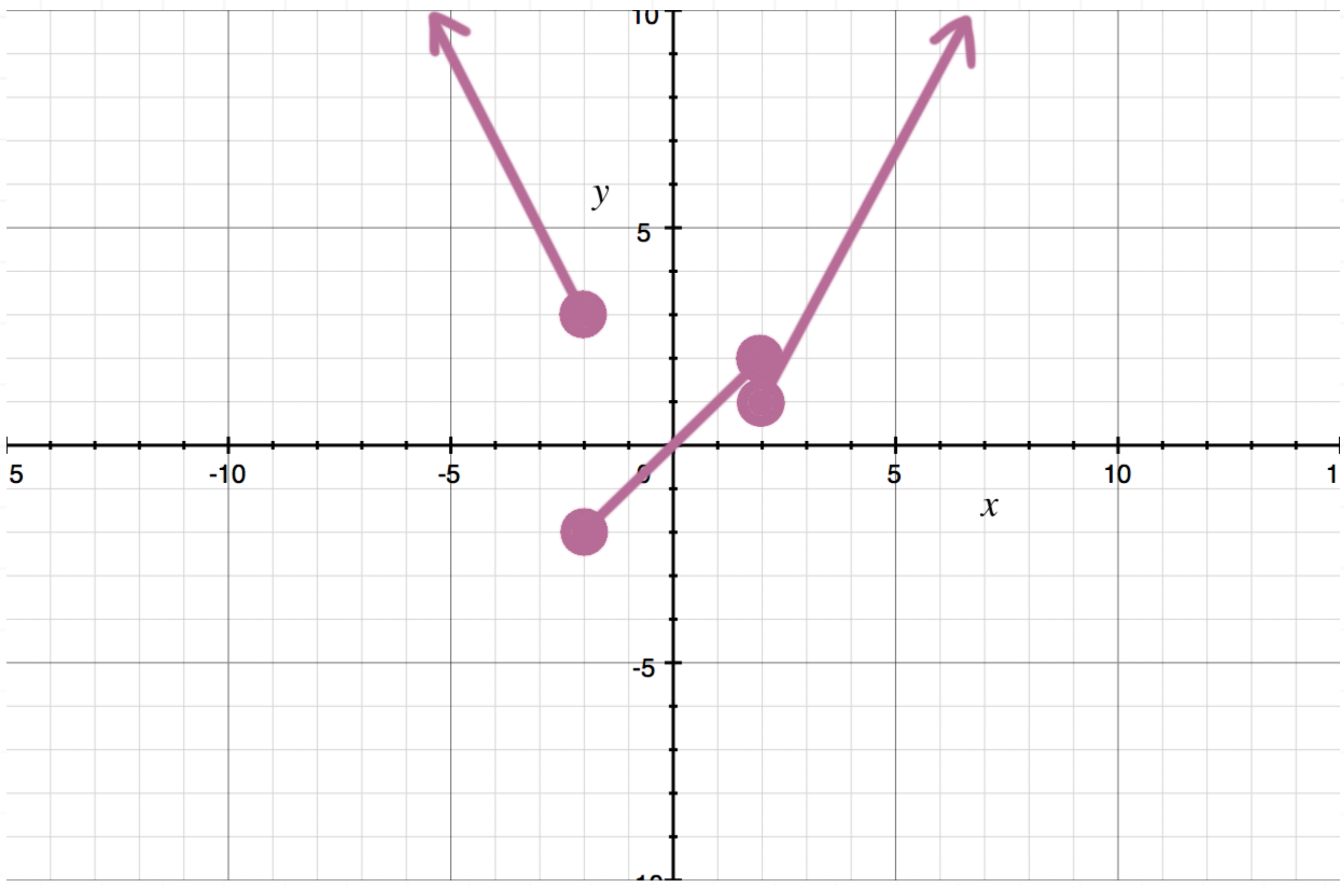
$$\lim_{x \to -5^-} k(x) = -\infty$$

■ 4. What is $\lim_{x \to -2^-} h(x)$?

$$h(x) = \begin{cases} -2x - 1 & x < -2 \\ x & -2 \leq x < 2 \\ 2x - 3 & x \geq 2 \end{cases}$$

Solution:

The graph of $h(x)$ is



Based on the graph, the limit is 3. Or we could plug into the first piece of the function, which is the piece that approaches $x = -2$ from the left side.

$$\lim_{x \to -2^-} h(x) = \left[-2(-2) - 1\right] = 3$$

■ 5. What is $\lim_{x \to 6^+} g(x)$?

$$g(x) = \frac{x^2 + x - 42}{x - 6}$$

Solution:

We could tell that the limit is 13 by making a table,

x	6	6.001	6.01	6.1
g(x)	?	13.001	13.01	13.1

Alternatively, we could have factored the numerator, canceled like terms, and then evaluated at the limit.

$$g(x) = \frac{x^2 + x - 42}{x - 6}$$

$$g(x) = \frac{(x + 7)(x - 6)}{x - 6}$$

$$g(x) = x + 7$$

Then the limit is

$$\lim_{x \to 6^+} x + 7$$

$$6 + 7$$

$$13$$

■ 6. Find the left- and right-hand limits of the function at $x = 3$.

$$f(x) = \frac{|x - 3|}{x - 3}$$

Solution:

This function includes $|x-3|$, which is the absolute value of $x-3$. When $x<3$, $|x-3|=-(x-3)$, so the left-hand limit is

$$\lim_{x \to 3^-} \frac{-(x-3)}{x-3}$$

$$\frac{-1}{1}$$

$$-1$$

When $x>3$, $|x-3|=x-3$, so the right-hand limit is

$$\lim_{x \to 3^+} \frac{x-3}{x-3}$$

$$1$$

PROVING THAT THE LIMIT DOES NOT EXIST

1. Prove that the limit does not exist.

$$\lim_{x\to 0}\frac{-2|3x|}{3x}$$

Solution:

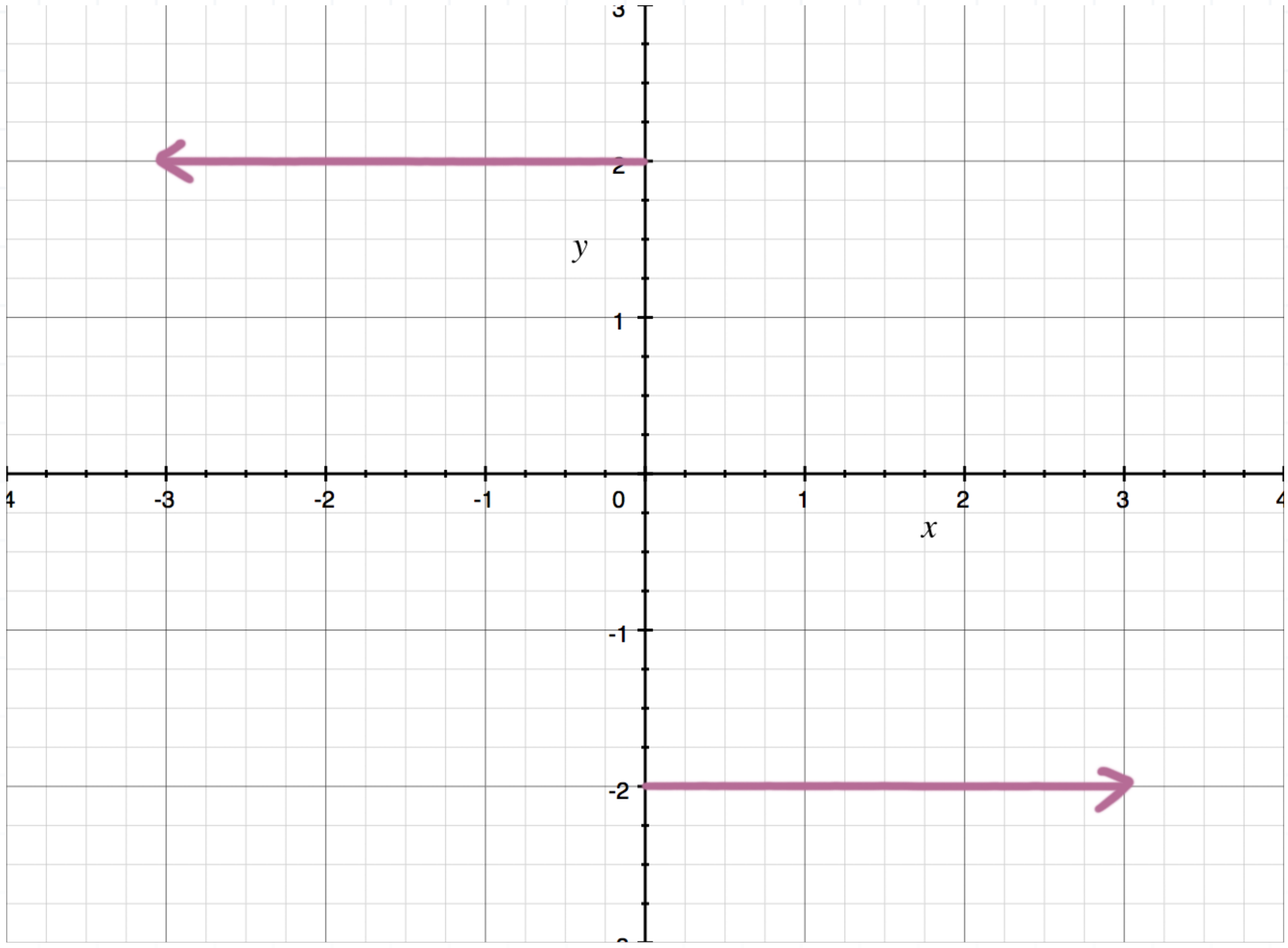
The left-hand limit is

$$\lim_{x\to 0^-}\frac{-2|3x|}{3x}=\lim_{x\to 0^-}\frac{-2(-3x)}{3x}=\frac{6x}{3x}=2$$

The right-hand limit is

$$\lim_{x\to 0^+}\frac{-2|3x|}{3x}=\lim_{x\to 0^+}\frac{-2(3x)}{3x}=\frac{-6x}{3x}=-2$$

Since the left- and right-hand limits aren't equal, the limit does not exist. The graph of the function would also prove that the limit doesn't exist.



2. Prove that the limit does not exist.

$$\lim_{x\to -5} \frac{x^2+7x+9}{x^2-25}$$

Solution:

The left-hand limit is

$$\lim_{x\to -5.001} \frac{x^2+7x+9}{x^2-25} = \frac{(-5.001)^2+7(-5.001)+9}{(-5.001)^2-25} = -99.69$$

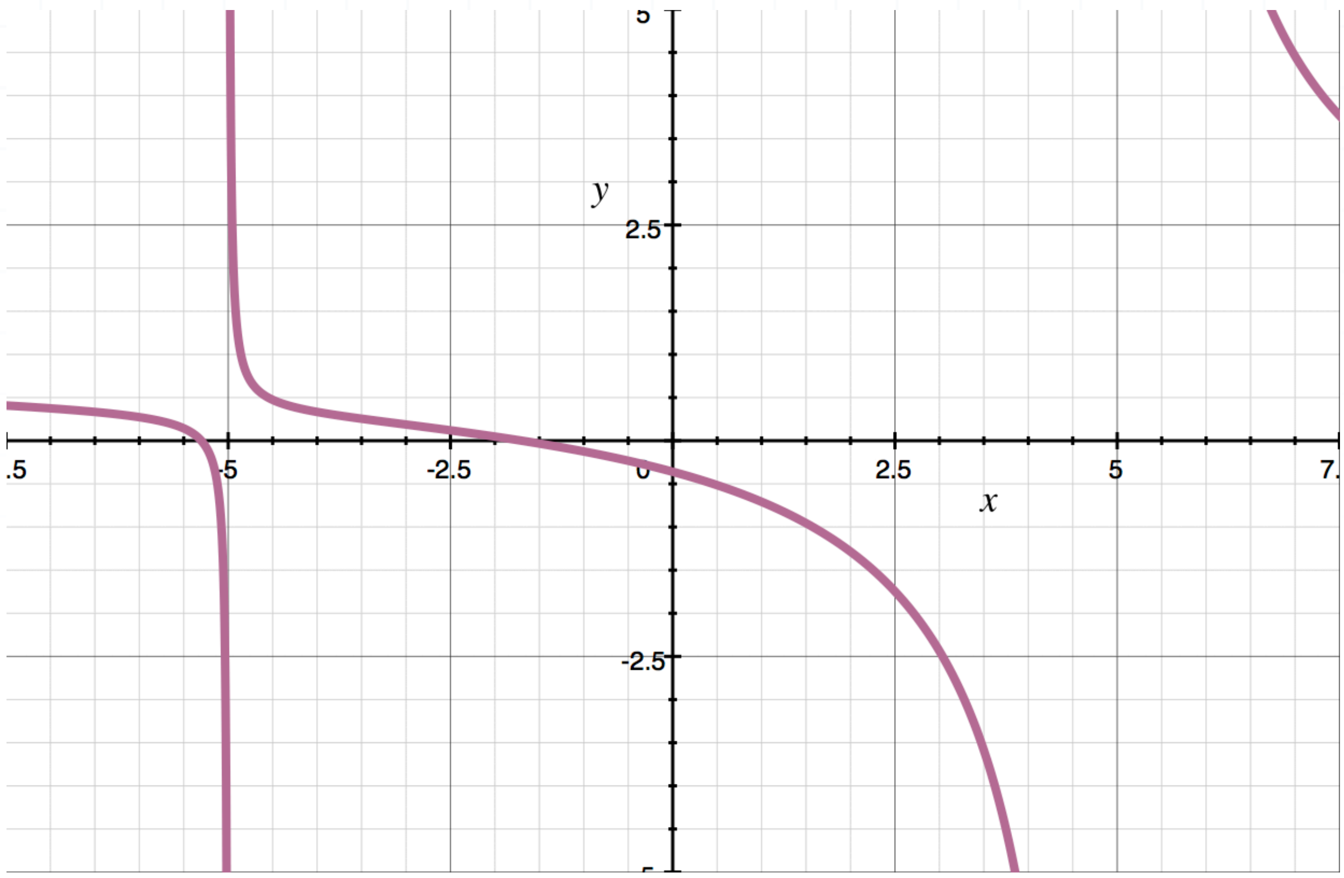
$$\lim_{x\to -5^-} \frac{x^2+7x+9}{x^2-25} = -\infty$$

The right-hand limit is

$$\lim_{x \to -4.999} \frac{x^2 + 7x + 9}{x^2 - 25} = \frac{(-4.999)^2 + 7(-4.999) + 9}{(-4.999)^2 - 25} = 100.31$$

$$\lim_{x \to -5^+} \frac{x^2 + 7x + 9}{x^2 - 25} = \infty$$

Since the left- and right-hand limits aren't equal, the limit does not exist. The graph of the function would also prove that the limit doesn't exist.



■ 3. Prove that $\lim_{x \to 1} f(x)$ does not exist.

$$f(x) = \begin{cases} -3x + 2 & x < 1 \\ 3x - 2 & x \ge 1 \end{cases}$$

Solution:

The left-hand limit is

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (-3x + 2) = \left[-3(1) + 2\right] = -1$$

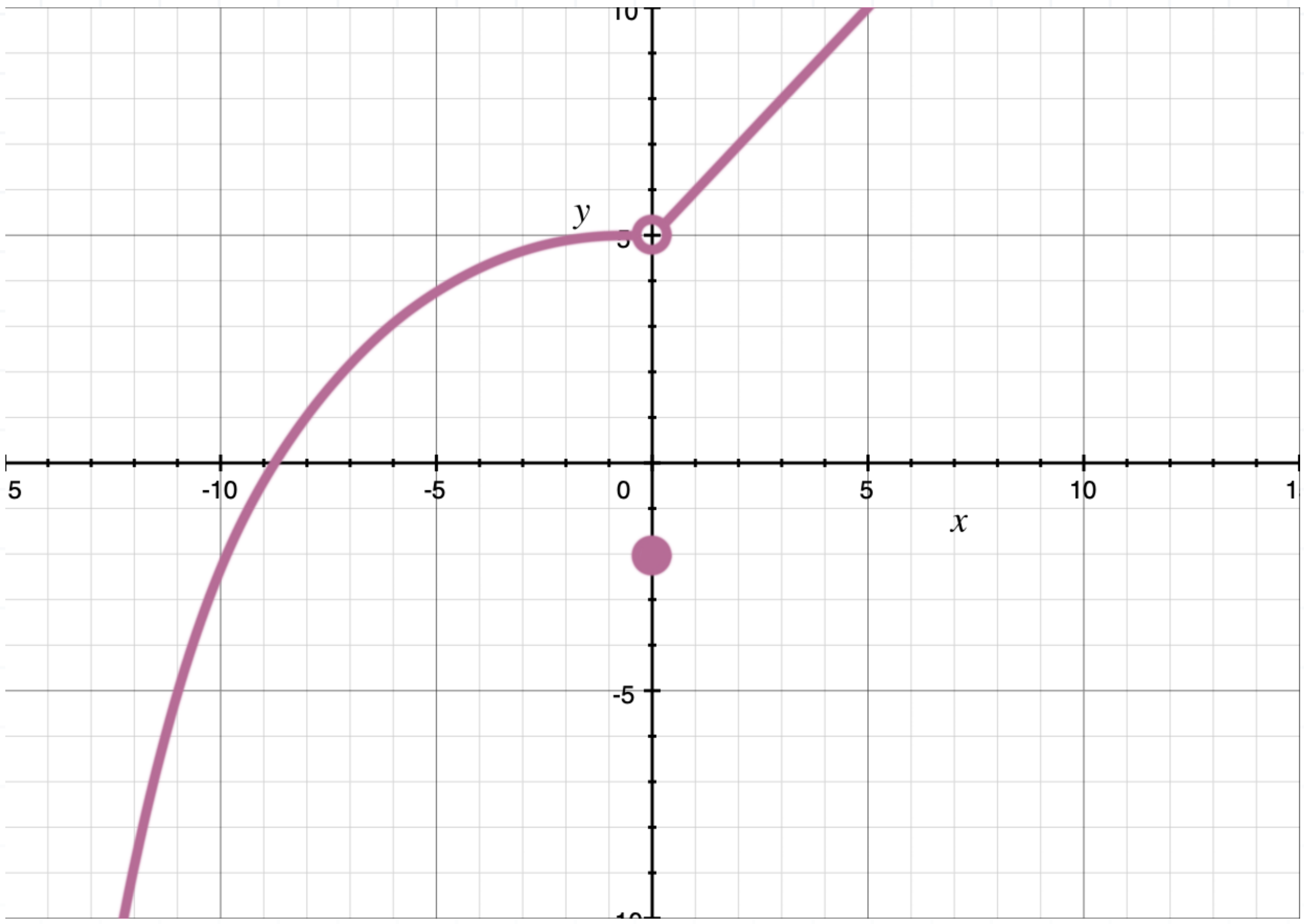
The right-hand limit is

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (3x - 2) = \left[3(1) - 2\right] = 1$$

Because the left- and right-hand limits aren't equal, the limit does not exist.

$$\lim_{x \to 1^-} f(x) \neq \lim_{x \to 1^+} f(x)$$

■ 4. Use the graph to determine whether or not the limit exists at $x = 0$.



Solution:

At $x = 0$, the function is approaching 5 from the left side and approaching 5 from the right side. So if we say that the graph represents the function $f(x)$, then the one-sided limits are

$$\lim_{x \to 0^-} f(x) = 5$$

$$\lim_{x \to 0^+} f(x) = 5$$

Because the left- and right-hand limits are equal, we've proven that the general limit of the function exists at $x = 0$ and is equal to 5.

$$\lim_{x \to 0} f(x) = 5$$

■ 5. Suppose we know that $\lim_{x \to 5} f(x) = 12$. If possible, determine the values of the one-sided limits.

$$\lim_{x \to 5^-} f(x)$$

$$\lim_{x \to 5^+} f(x)$$

Solution:

If the general limit exists at a point $x = c$, then the left- and right-hand limits exist at $x = c$ and are equal to one another. Because the general limit exists, we know that the one-sided limits also exist, and they must both be equal to the value of the general limit. Therefore,

$$\lim_{x \to 5^-} f(x) = \lim_{x \to 5^+} f(x) = 12$$

■ 6. Prove that the limit does not exist.

$$\lim_{x \to -2} \frac{x^2 - 4}{(x+2)^2}$$

Solution:

The left-hand limit is

$$\lim_{x \to -2.001} \frac{x^2 - 4}{(x+2)^2} = \frac{(-2.001)^2 - 4}{(-2.001+2)^2} = 4{,}001$$

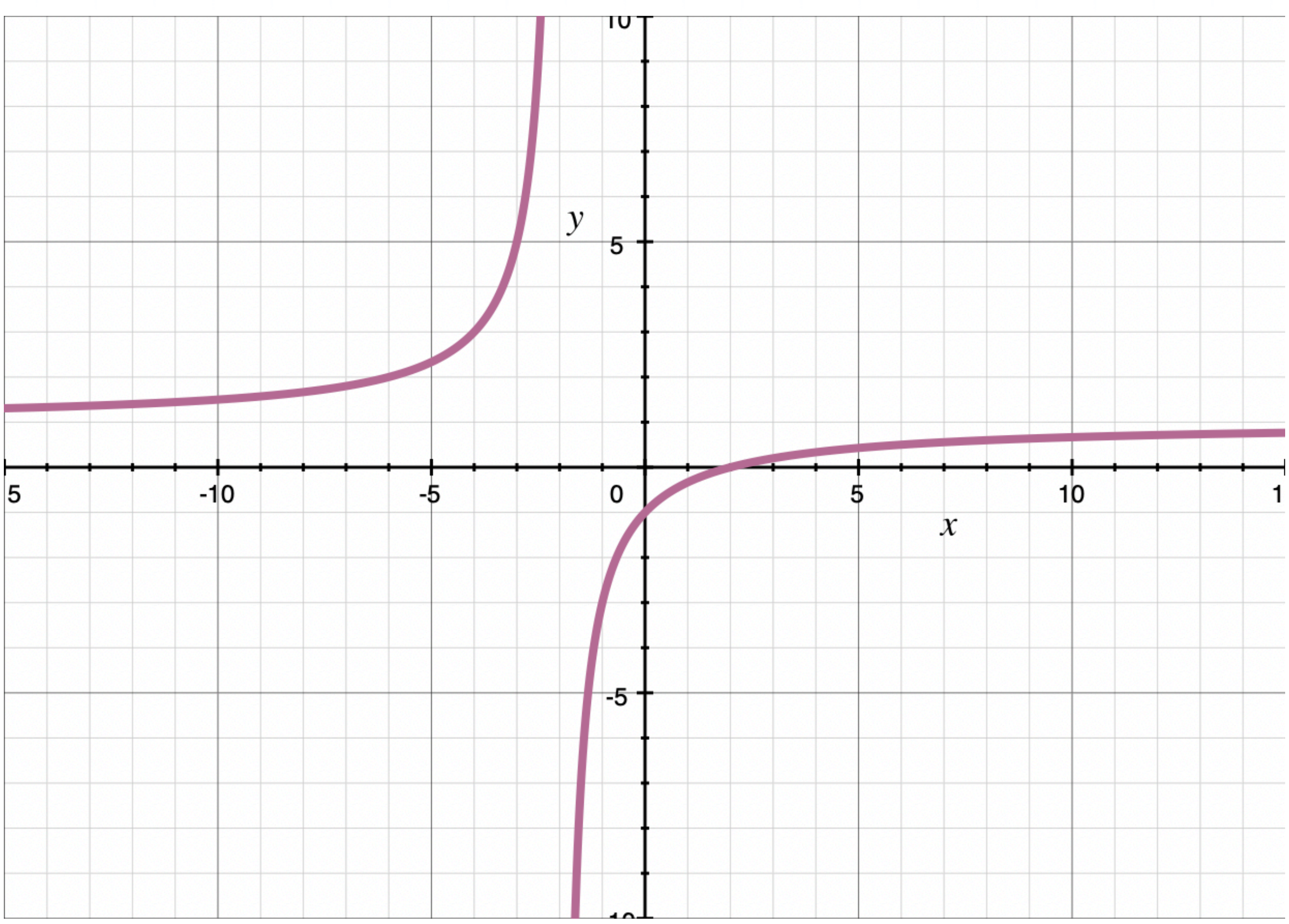
$$\lim_{x \to -2^-} \frac{x^2 - 4}{(x+2)^2} = \infty$$

The right-hand limit is

$$\lim_{x \to -1.999} \frac{x^2 - 4}{(x+2)^2} = \frac{(-1.999)^2 - 4}{(-1.999+2)^2} = -3{,}999$$

$$\lim_{x \to -2^+} \frac{x^2 - 4}{(x+2)^2} = -\infty$$

Since the left- and right-hand limits aren't equal, the limit does not exist. Graphing the function shows the unequal one-sided limits.



PRECISE DEFINITION OF THE LIMIT

■ 1. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to 4}(5x - 16) = 4$$

Solution:

If $0 < |x - 4| < \delta$, then $|(5x - 16) - 4| < \epsilon$. So,

$$|5x - 20| < \epsilon$$

$$|5(x - 4)| < \epsilon$$

$$|5| \cdot |x - 4| < \epsilon$$

$$5 \cdot |x - 4| < \epsilon$$

$$|x - 4| < \frac{\epsilon}{5}$$

Now if $|x - 4| < \epsilon/5$ and $0 < |x - 4| < \delta$, then if $\epsilon > 0$ then $\delta = \epsilon/5$. Therefore, the limit equation is true.

■ 2. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to -7}(-2x + 15) = 29$$

Solution:

If $0 < |x - (-7)| < \delta$ then $| - 2x + 15 - 29| < \epsilon$. Or we could rewrite this as $0 < |x + 7| < \delta$ and $| - 2x - 14| < \epsilon$. So,

$$|(-2)(x + 7)| < \epsilon$$

$$| - 2| \cdot |x + 7| < \epsilon$$

$$2 \cdot |x + 7| < \epsilon$$

$$|x + 7| < \frac{\epsilon}{2}$$

Now if $|x + 7| < \epsilon/2$ and $0 < |x + 7| < \delta$, then if $\epsilon > 0$ then $\delta = \epsilon/2$. Therefore, the limit equation is true.

■ 3. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x \to 16} \left(\frac{2}{5}x - \frac{17}{5} \right) = 3$$

Solution:

If $0 < |x - 16| < \delta$ then $\left| \left((2/5)x - (17/5) \right) - 3 \right| < \epsilon$. Or we could rewrite this as $0 < |x - 16| < \delta$ and

$$\left|\left(\frac{2}{5}x-\frac{17}{5}\right)-\frac{15}{5}\right|<\epsilon$$

$$\left|\frac{2}{5}x-\frac{32}{5}\right|<\epsilon$$

$$\left|\frac{2}{5}(x-16)\right|<\epsilon$$

$$\left|\frac{2}{5}\right||x-16|<\epsilon$$

$$|x-16|<\frac{5}{2}\epsilon$$

Now if $|x-16|<(5/2)\epsilon$ and $0<|x-16|<\delta$, then if $\epsilon>0$, then $\delta=(5/2)\epsilon$. Therefore, the limit equation is true.

■ 4. Use the precise definition of the limit to prove the value of the limit.

$$\lim_{x\to 7}\frac{x^2-15x+56}{x-7}=-1$$

Solution:

We'll apply the precise definition to the given limit.

If $0 < |x-7| < \delta$, then $\left| \left(\frac{x^2 - 15x + 56}{x-7} \right) - (-1) \right| < \epsilon$.

If $0 < |x-7| < \delta$, then $\left| \left(\frac{x^2 - 15x + 56}{x-7} \right) - \frac{-1(x-7)}{x-7} \right| < \epsilon$.

So,

$$\left| \left(\frac{x^2 - 15x + 56}{x-7} \right) + \frac{x-7}{x-7} \right| < \epsilon$$

$$\left| \frac{x^2 - 14x + 49}{x-7} \right| < \epsilon$$

$$\left| \frac{(x-7)(x-7)}{x-7} \right| < \epsilon$$

$$|x-7| < \varepsilon$$

Now, if $|x-7| < \epsilon$ and $0 < |x-7| < \delta$, then if $\epsilon > 0$ and $\delta = \epsilon$. Therefore, the limit equation is true.

■ 5. Find δ when $f(x) = 2x - 5$, such that if $0 < |x-1| < \delta$ then $|f(x) + 3| < 0.1$.

Solution:

We want to use the value for ϵ to determine the δ value by remembering from the precise definition of the limit that

if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

If $0 < |x - 1| < \delta$ then $|2x - 5 + 3| < \epsilon = 0.1$, and we can rewrite this second inequality as

$$|2x - 2| < 0.1$$

$$|2| \cdot |x - 1| < 0.1$$

$$2 \cdot |x - 1| < 0.1$$

$$|x - 1| < \frac{0.1}{2}$$

$$|x - 1| < 0.05$$

So,

$$\delta = 0.05$$

■ 6. Find a value of δ given $\epsilon = 0.04$.

$$\lim_{x \to 2}(x - 2)^2 = 0$$

Solution:

We want to use the value for ϵ to determine the δ value by remembering from the precise definition of the limit that

if $0 < |x - a| < \delta$ then $|f(x) - L| < \epsilon$

If $0 < |x - 2| < \delta$, then $|(x-2)^2 - 0| < \epsilon$, and we can rewrite this second inequality as

$$|(x-2)^2| < \epsilon$$

$$|x - 2| < \sqrt{\epsilon}$$

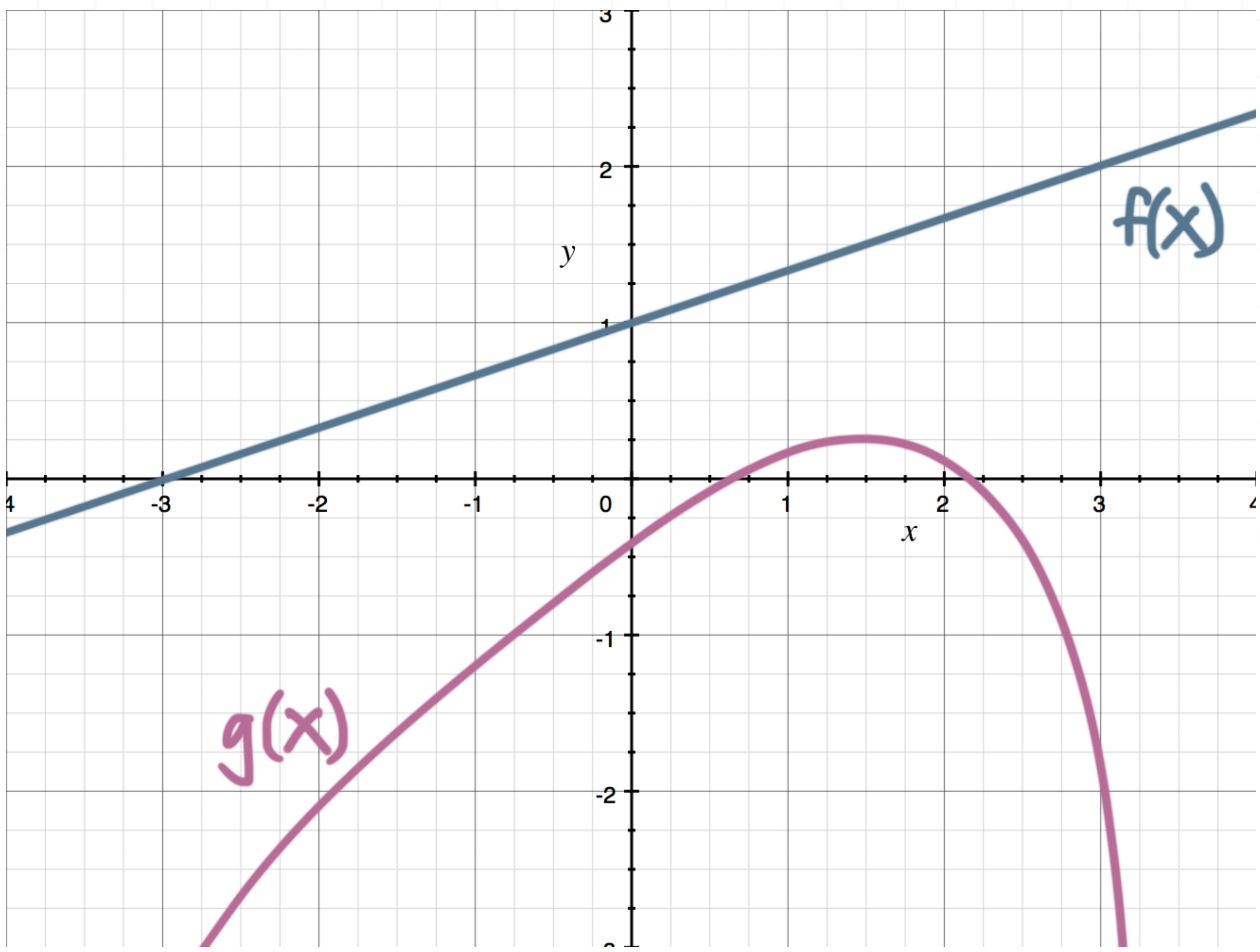
So,

$$\delta = \sqrt{\epsilon} = \sqrt{0.04} = 0.2$$

LIMITS OF COMBINATIONS

1. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x \to 3} \left[4f(x) - 3g(x)\right]$$



Solution:

We can simplify the limit, and then evaluate both functions at $x = 3$.

$$\lim_{x \to 3} \left[4f(x) - 3g(x)\right]$$

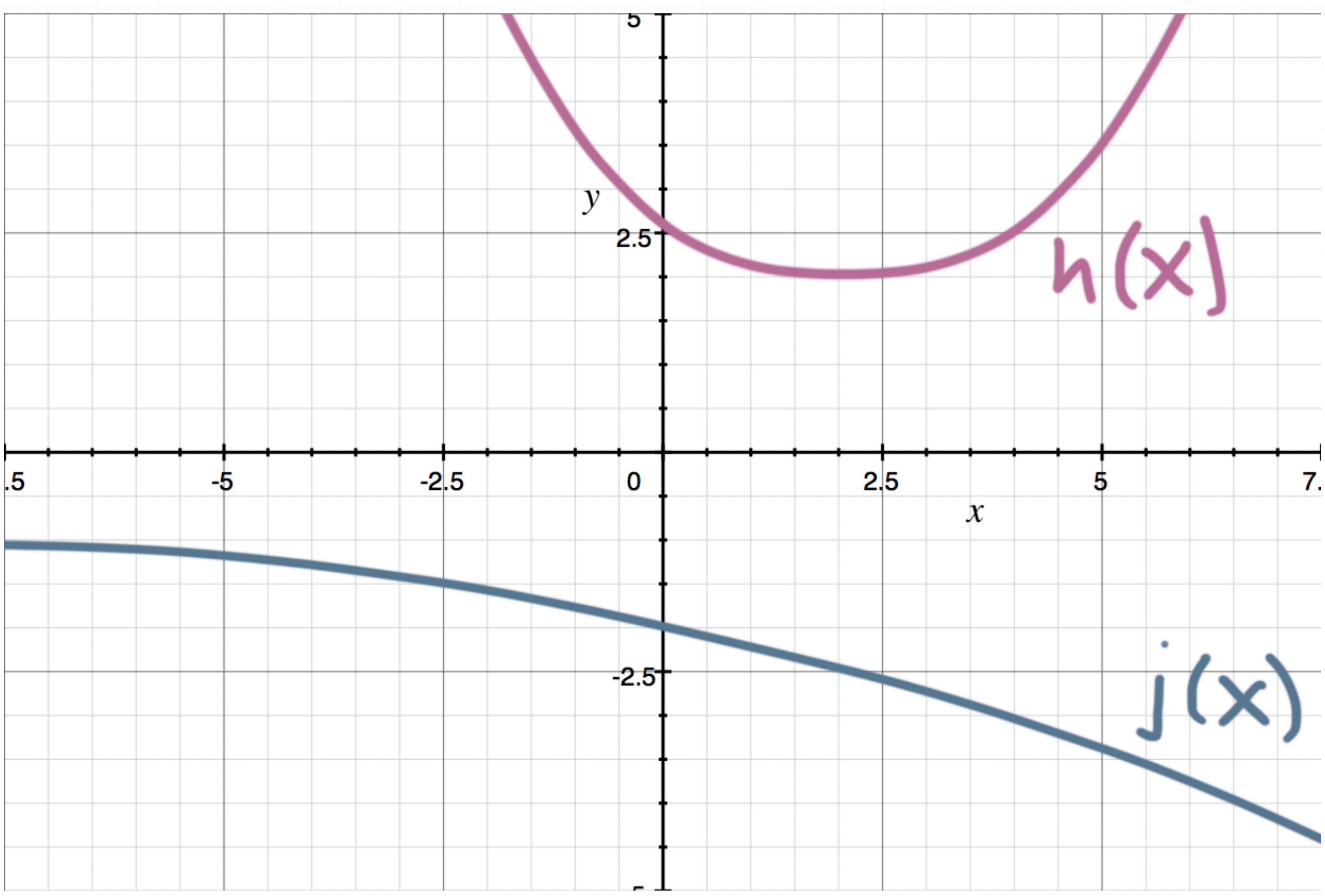
$$4\lim_{x\to 3} f(x) - 3\lim_{x\to 3} g(x)$$

$$4(2) - 3(-2)$$

$$14$$

■ 2. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x\to 4} \frac{h(x)}{j(x)}$$



Solution:

We can simplify the limit, and then evaluate both functions at $x = 4$.

$$\lim_{x\to 4} \frac{h(x)}{j(x)}$$

$$\frac{\lim_{x\to 4} h(x)}{\lim_{x\to 4} j(x)}$$

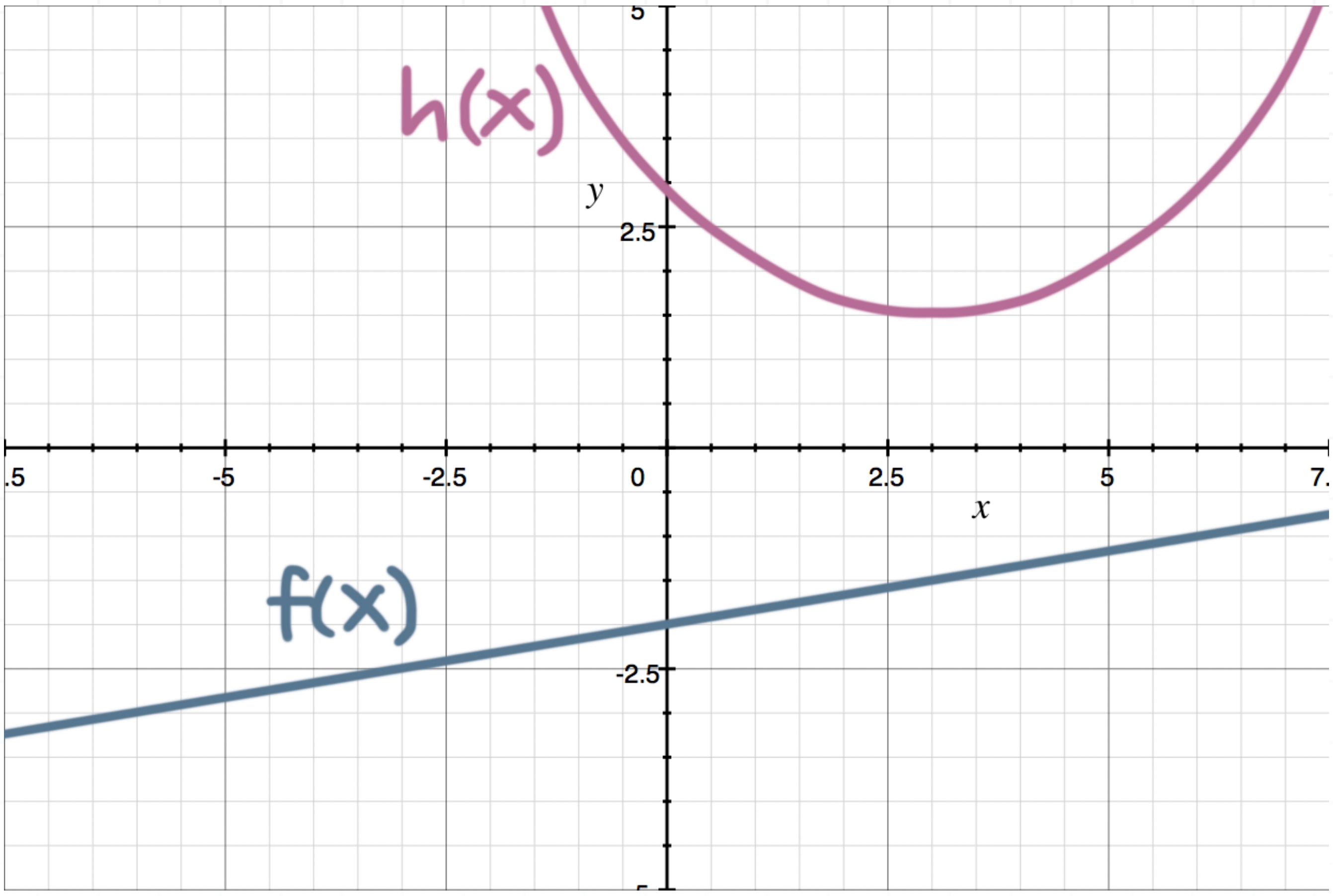
$$\frac{\frac{5}{2}}{-3}$$

$$\frac{5}{2} \cdot \frac{1}{-3}$$

$$-\frac{5}{6}$$

■ 3. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x\to 0} \left[2f(x) \cdot 3h(x)\right]$$



Solution:

We can simplify the limit, and then evaluate both functions at $x = 0$.

$$\lim_{x\to 0}\left[2f(x)\cdot 3h(x)\right]$$

$$2\lim_{x\to 0} f(x)\cdot 3\lim_{x\to 0} h(x)$$

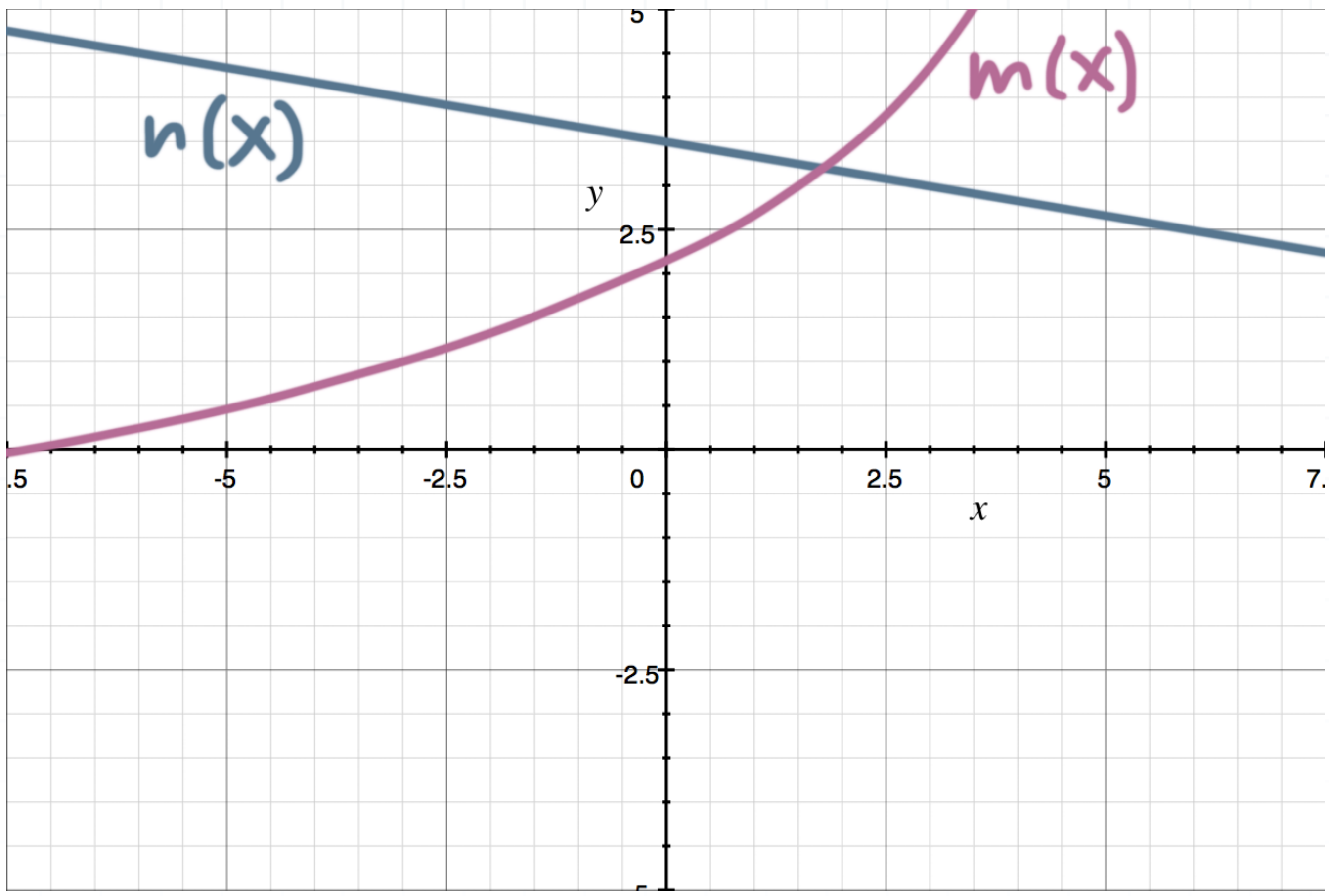
$$6\lim_{x\to 0} f(x)\cdot \lim_{x\to 0} h(x)$$

$$6(-2)(3)$$

$$-36$$

4. Use limit laws and the graph below to evaluate the limit.

$$\lim_{x \to -3} \left[\frac{5m(x)}{n(x)} - \frac{4m(x)}{n(x)} \right]$$



Solution:

We can simplify the limit, and then evaluate both functions at $x = -3$.

$$\lim_{x \to -3} \left[\frac{5m(x)}{n(x)} - \frac{4m(x)}{n(x)} \right]$$

$$\frac{5 \lim_{x \to -3} m(x)}{\lim_{x \to -3} n(x)} - \frac{4 \lim_{x \to -3} m(x)}{\lim_{x \to -3} n(x)}$$

$$\frac{5(1)}{4} - \frac{4(1)}{4}$$

$$\frac{1}{4}$$

■ 5. Evaluate the limit.

$$\lim_{x\to 6}\left(\sqrt{x-2}+\frac{e^x}{2x+3}-x^2-12\right)$$

Solution:

We’ll start by distributing the limit across the combination.

$$\lim_{x\to 6}\sqrt{x-2}+\lim_{x\to 6}\frac{e^x}{2x+3}-\lim_{x\to 6}(x^2+12)$$

$$\sqrt{\lim_{x\to 6}(x-2)}+\frac{\lim_{x\to 6}e^x}{\lim_{x\to 6}(2x+3)}-\lim_{x\to 6}(x^2+12)$$

Now we’ll substitute the value we’re approaching into each function.

$$\sqrt{4}+\frac{e^6}{2(6)+3}-(6^2+12)$$

$$2+\frac{e^6}{15}-48$$

$$\frac{e^6}{15}-46$$

6. If $f(x) = x^2 + 4$, $g(x) = x - 5$, and $h(x) = -5x$, evaluate the limit.

$$\lim_{x \to 1} \sqrt{\frac{f(x)g(x)}{h(x)}}$$

Solution:

We'll start by distributing the limit across the combination.

$$\sqrt{\lim_{x \to 1} \frac{f(x)g(x)}{h(x)}}$$

$$\sqrt{\frac{\lim_{x \to 1} f(x) \lim_{x \to 1} g(x)}{\lim_{x \to 1} h(x)}}$$

$$\sqrt{\frac{\lim_{x \to 1}(x^2 + 4) \lim_{x \to 1}(x - 5)}{\lim_{x \to 1}(-5x)}}$$

Now we'll substitute the value we're approaching into each function.

$$\sqrt{\frac{(1^2 + 4)(1 - 5)}{(-5(1))}}$$

$$\sqrt{\frac{5(-4)}{-5}}$$

$$\sqrt{\frac{-20}{-5}}$$

$$\sqrt{4}$$

$$2$$

LIMITS OF COMPOSITES

■ 1. What is $\lim_{x\to 3} f(g(x))$ if $f(x)=4x$ and $g(x)=6x-9$?

Solution:

If f is continuous at $x=3$, then

$$\lim_{x\to 3} f(g(x)) = f\left(\lim_{x\to 3} g(x)\right)$$

$$\lim_{x\to 3} f(g(x)) = f\left(\lim_{x\to 3}(6x-9)\right)$$

$$\lim_{x\to 3} f(g(x)) = f(6(3)-9) = f(9) = 4(9) = 36$$

■ 2. What is $\lim_{x\to -4} f(g(x))$ if $f(x)=2x^2$ and $g(x)=2x-1$?

Solution:

If f is continuous at $x=-4$, then

$$\lim_{x\to -4} f(g(x)) = f\left(\lim_{x\to -4} g(x)\right)$$

$$\lim_{x \to -4} f(g(x)) = f\left(\lim_{x \to -4} (2x - 1)\right)$$

$$\lim_{x \to -4} f(g(x)) = f(2(-4) - 1) = f(-9) = 2(-9)^2 = 162$$

■ 3. What is $\lim_{x \to \frac{\pi}{2}} f(g(x))$ if $f(x) = \sin x$ and $g(x) = x/2$?

Solution:

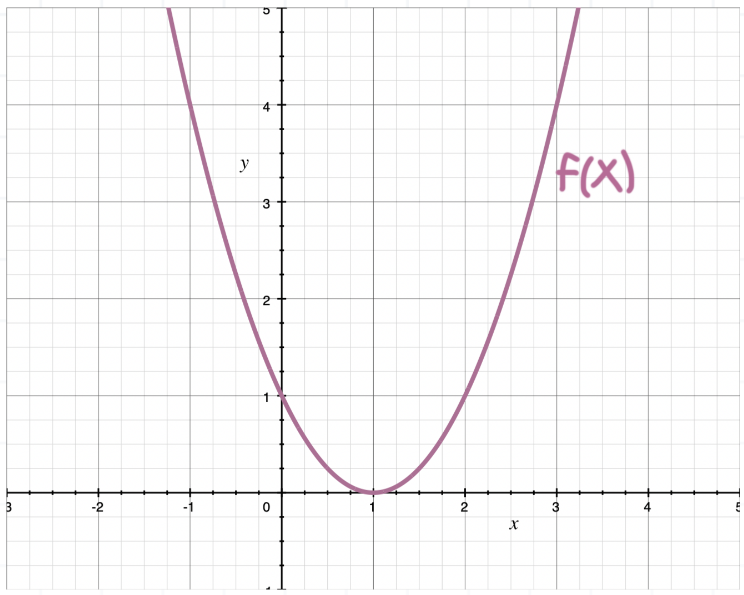
If f is continuous at $x = \pi/2$, then

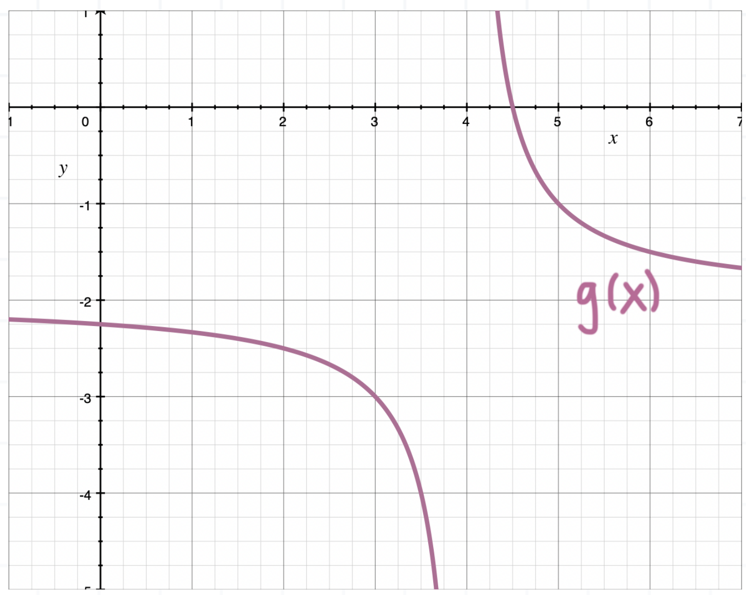
$$\lim_{x \to \frac{\pi}{2}} f(g(x)) = f\left(\lim_{x \to \frac{\pi}{2}} g(x)\right)$$

$$\lim_{x \to \frac{\pi}{2}} f(g(x)) = f\left(\lim_{x \to \frac{\pi}{2}} \frac{x}{2}\right)$$

$$\lim_{x \to \frac{\pi}{2}} f(g(x)) = f\left(\frac{\frac{\pi}{2}}{2}\right) = f\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

■ 4. If $f(x)$ and $g(x)$ are graphed below, find $\lim_{x \to 3} g(f(x))$.





Solution:

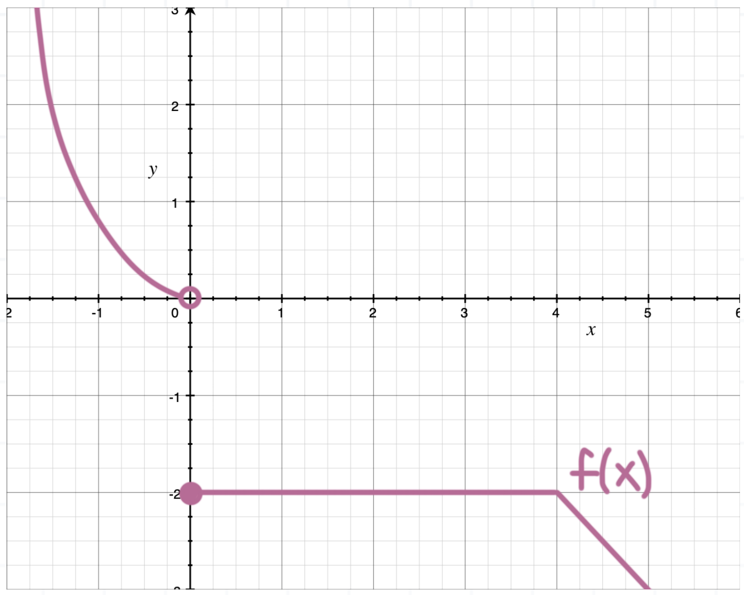
If f is continuous at $x = 3$, then

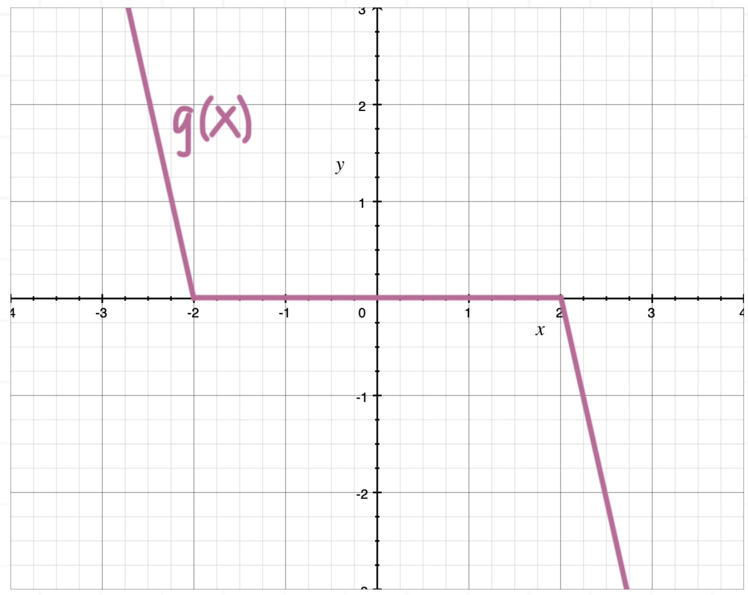
$$\lim_{x \to 3} g(f(x)) = g\left(\lim_{x \to 3} f(x)\right)$$

Because $\lim_{x \to 3} f(x) = 4$,

$$\lim_{x \to 3} g(f(x)) = g(4) = \text{DNE}$$

■ 5. If $f(x)$ and $g(x)$ are graphed below, find $\lim_{x \to 2} g(f(x))$.





Solution:

If f is continuous at $x = 2$, then

$$\lim_{x \to 2} g(f(x)) = g\left(\lim_{x \to 2} f(x)\right)$$

Because $\lim_{x \to 2} f(x) = -2$,

$$\lim_{x \to 2} g(f(x)) = g(-2) = 0$$

■ 6. If $f(x) = 2x + 1$ and $\lim_{x \to 3} h(x) = -2$, find $\lim_{x \to 3} f(h(x))$.

Solution:

If f is continuous at $x = 3$, then

$$\lim_{x\to 3} f(h(x)) = f\left(\lim_{x\to 3} h(x)\right)$$

$$\lim_{x\to 3} f(h(x)) = f(-2) = 2(-2) + 1 = -4 + 1 = -3$$

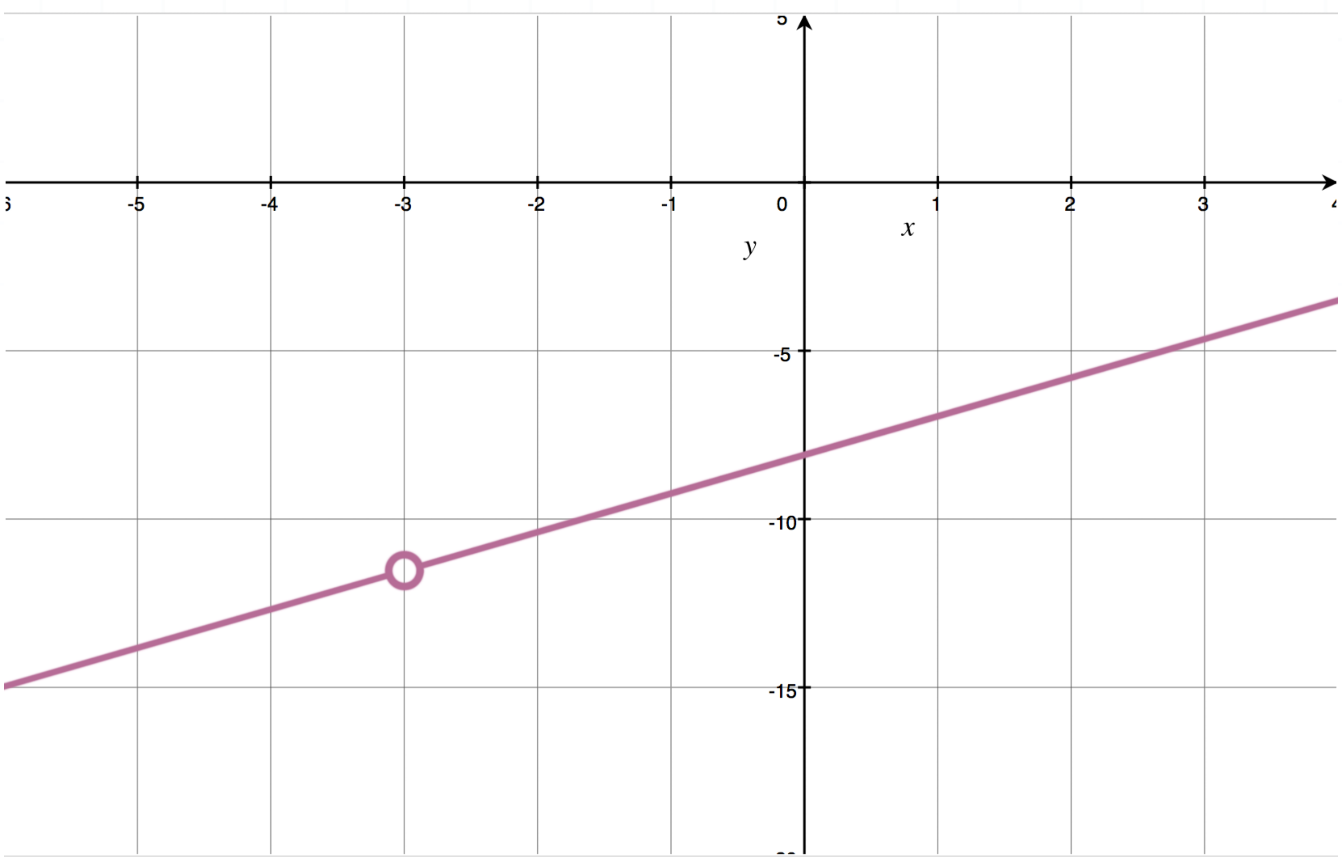
POINT DISCONTINUITIES

1. Redefine the function as a continuous piecewise function.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

Solution:

The function is discontinuous at $x = -3$.



Factor and reduce to remove the discontinuity.

$$f(x) = \frac{x^2 - 6x - 27}{x + 3}$$

$$f(x) = \frac{(x+3)(x-9)}{x+3}$$

$$f(x) = x - 9$$

Evaluate $f(x)$ at $x = -3$.

$$f(-3) = -3 - 9 = -12$$

Therefore, to make the function continuous, we have to redefine it as

$$f(x) = \begin{cases} \frac{x^2 - 6x - 27}{x+3} & x \neq -3 \\ -12 & x = -3 \end{cases}$$

We can see whether or not this function is continuous at $x = -3$ by looking at the limit as x approaches -3.

$$\lim_{x \to -3} \frac{x^2 - 6x - 27}{x+3} = \lim_{x \to -3} \frac{(x+3)(x-9)}{x+3}) = -12$$

Since -12 is also the value of the function at $x = -3$, we see that this function is continuous.

■ 2. Identify the non-removable discontinuities of the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

Solution:

Factor the function.

$$k(x) = \frac{x^3 + 3x^2 - 25x - 75}{x^2 + x - 12}$$

$$k(x) = \frac{(x+5)(x-5)(x+3)}{(x+4)(x-3)}$$

No factors can be canceled. Which means the function has discontinuities at $x = -4$ and $x = 3$, both of which are non-removable.

3. What is the set of removable discontinuities of the function?

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta}$$

Solution:

We can rewrite the function as

$$j(\theta) = \frac{\cos^2\theta \cdot \sin^2\theta}{\tan^2\theta} = \frac{\cos^2\theta \cdot \sin^2\theta}{\frac{\sin^2\theta}{\cos^2\theta}} = \frac{\cos^2\theta \cdot \sin^2\theta \cdot \cos^2\theta}{\sin^2\theta} = \cos^4\theta$$

The removable discontinuities are the values of θ that make the sine function equal to 0, which are all the multiples of π.

$$\theta = \pm 0, \ \pm\pi, \ \pm 2\pi, \ \pm 3\pi, \ \pm 4\pi, \ \ldots$$

$\theta = n\pi$, where n is the set of all integers

■ 4. Examine whether or not the function is continuous at $x = 0$.

$$g(x) = \begin{cases} 2 - x^2 & x \le 0 \\ x - 2 & x > 0 \end{cases}$$

Solution:

We can say that $f(x)$ is continuous at $x = a$ if $f(a)$ is defined and $\lim_{x \to a} f(x) = f(a)$.

Evaluate $f(x)$ at $x = 0$.

$$f(0) = 2 - 0^2 = 2$$

The right-hand limit at $x = 0$ is

$$\lim_{x \to 0^+} (x - 2) = 0 - 2 = -2$$

The right-hand limit at $x = 0$ isn't equivalent to the function's value at $x = 0$, so the function is not continuous there.

■ 5. Where is the removable discontinuity in the graph of the function?

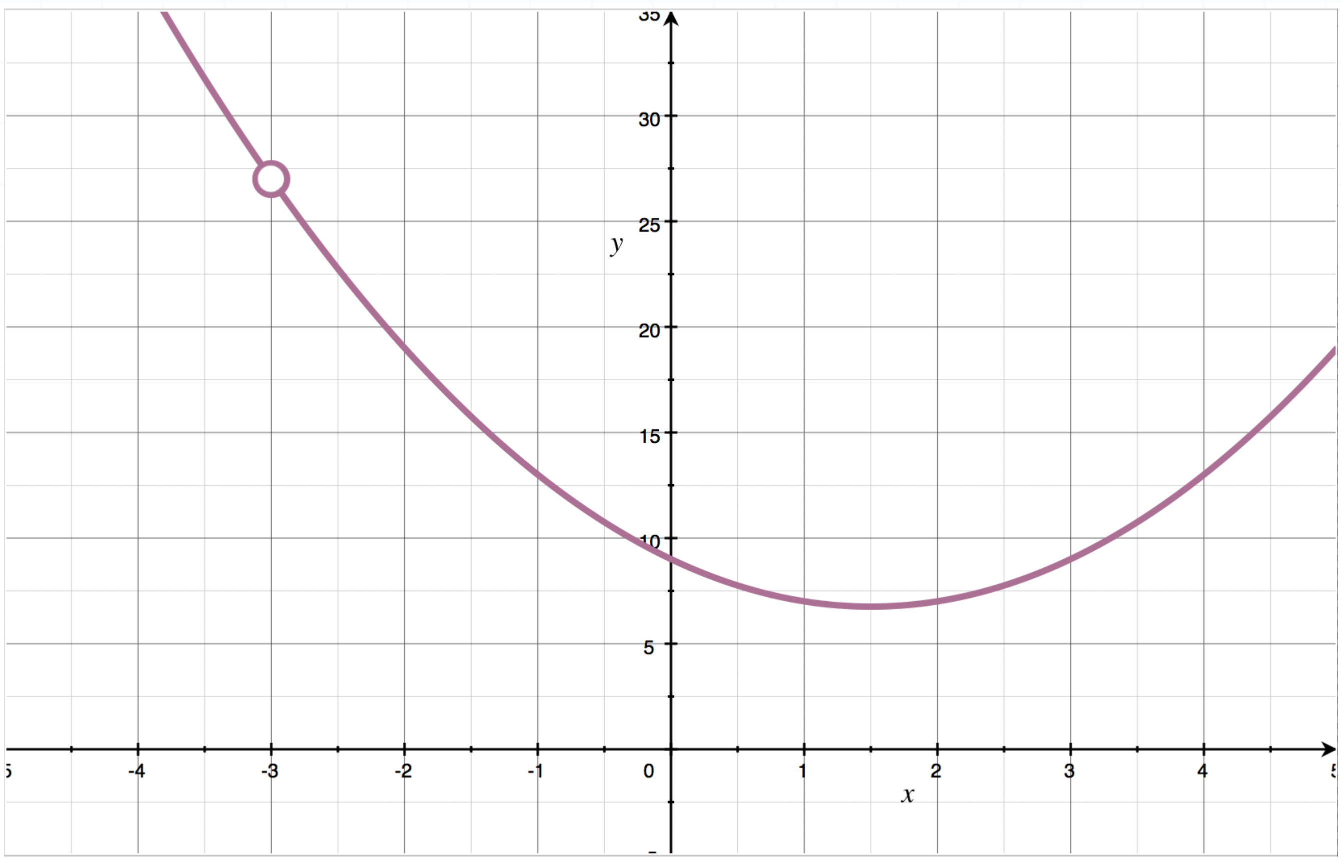
$$f(x) = \frac{x^3 + 27}{x + 3}$$

Solution:

If we factor the function, we can cancel a factor of $x+3$. We could have also simplified the function by using polynomial long division to find the quotient.

$$f(x)=\frac{x^3+27}{x+3}=\frac{(x+3)(x^2-3x+9)}{x+3}=x^2-3x+9$$

Because the factor of $x+3$ cancels, the removable discontinuity is at $x+3=0$, or $x=-3$.



■ 6. Identify the removable discontinuities in the function.

$$k(x)=\frac{x^4-2x^3-16x^2+2x+15}{x^2-2x-15}$$

Solution:

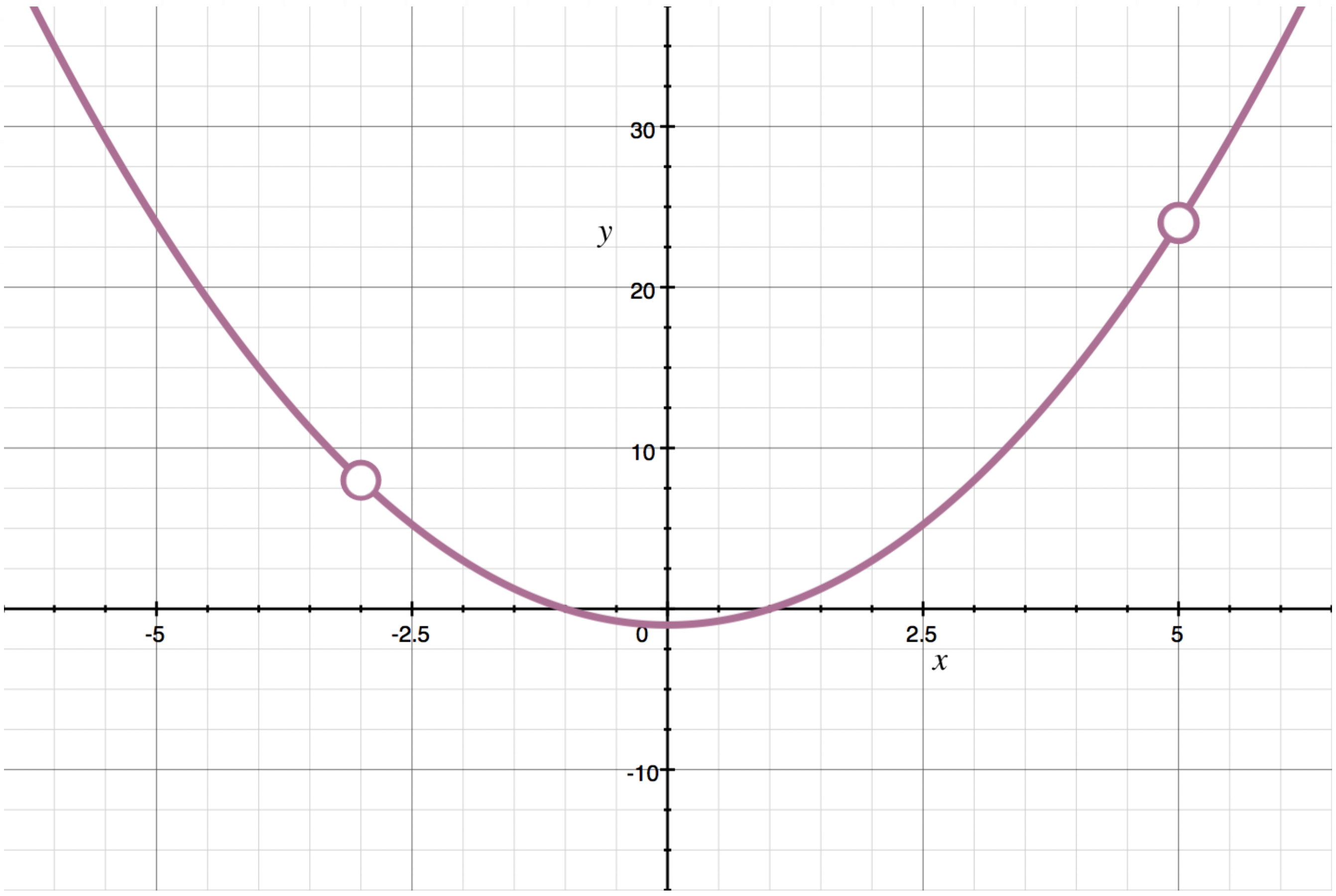
The function $k(x)$ has removable discontinuities at $x = -3$ and $x = 5$ because the function factors as

$$k(x) = \frac{(x+3)(x-5)(x+1)(x-1)}{(x+3)(x-5)}$$

and both factors from the denominator can be cancelled.

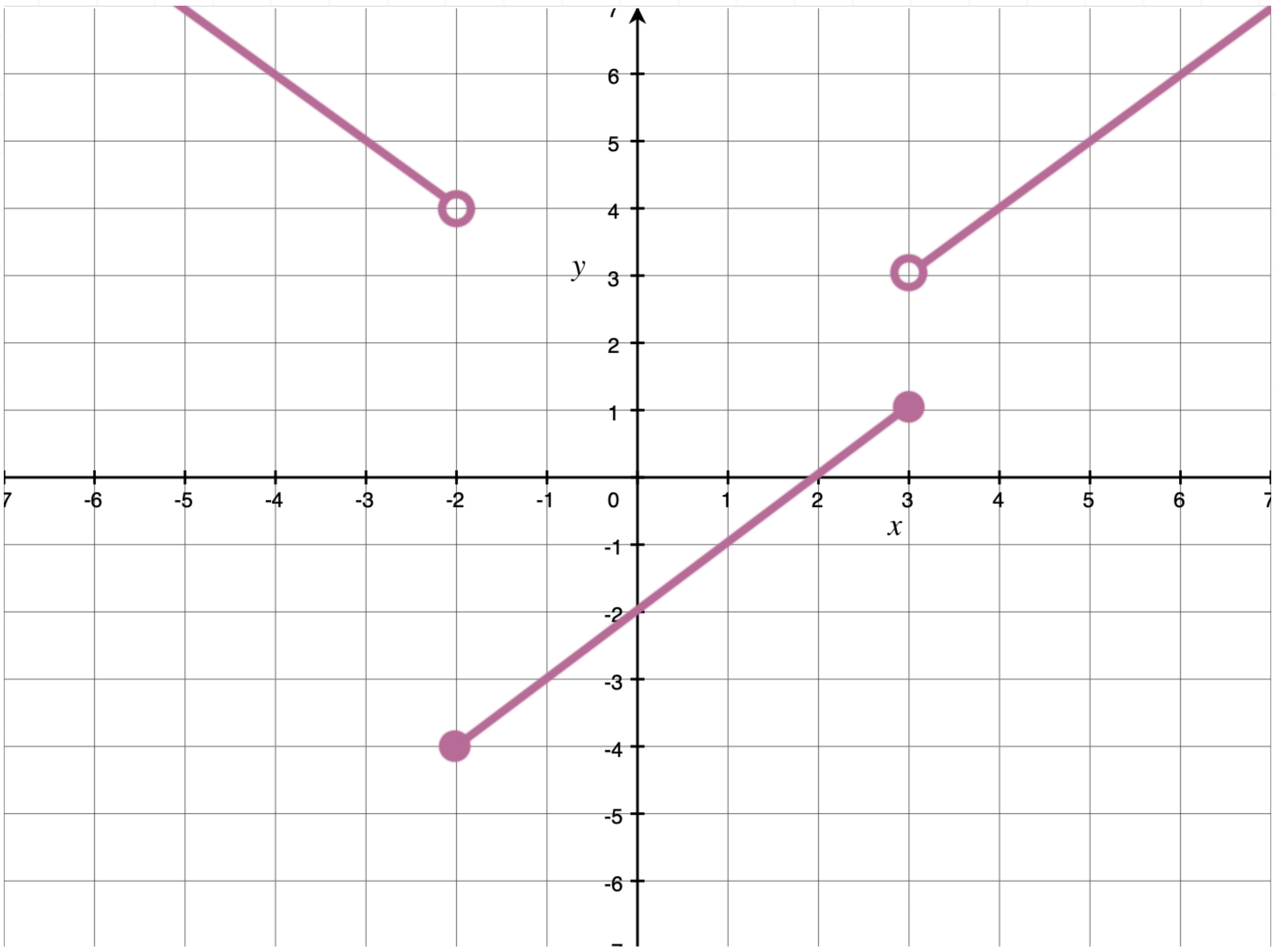
$$k(x) = (x+1)(x-1)$$

The graph is shown below.



JUMP DISCONTINUITIES

■ 1. What are the x-values where the graph of $f(x)$, shown below, has jump discontinuities?



Solution:

The function $f(x)$ has jump discontinuities at $x=-2$ and $x=3$ because the left- and right-hand limits aren't equal at $x=-2$,

$$\lim_{x\to -2^-} f(x)=4 \quad \neq \quad \lim_{x\to -2^+} f(x)=-4$$

and they aren't equal at $x=3$.

$$\lim_{x\to 3^-} f(x) = 1 \quad \neq \quad \lim_{x\to 3^+} f(x) = 3$$

■ 2. Where are the jump discontinuities in the graph of the function?

$$h(x) = \begin{cases} -\frac{1}{3}x^2 + 2 & x < 0 \\ 3 & 0 \le x \le 1 \\ \frac{1}{3}x^2 + 4 & x > 1 \end{cases}$$

Solution:

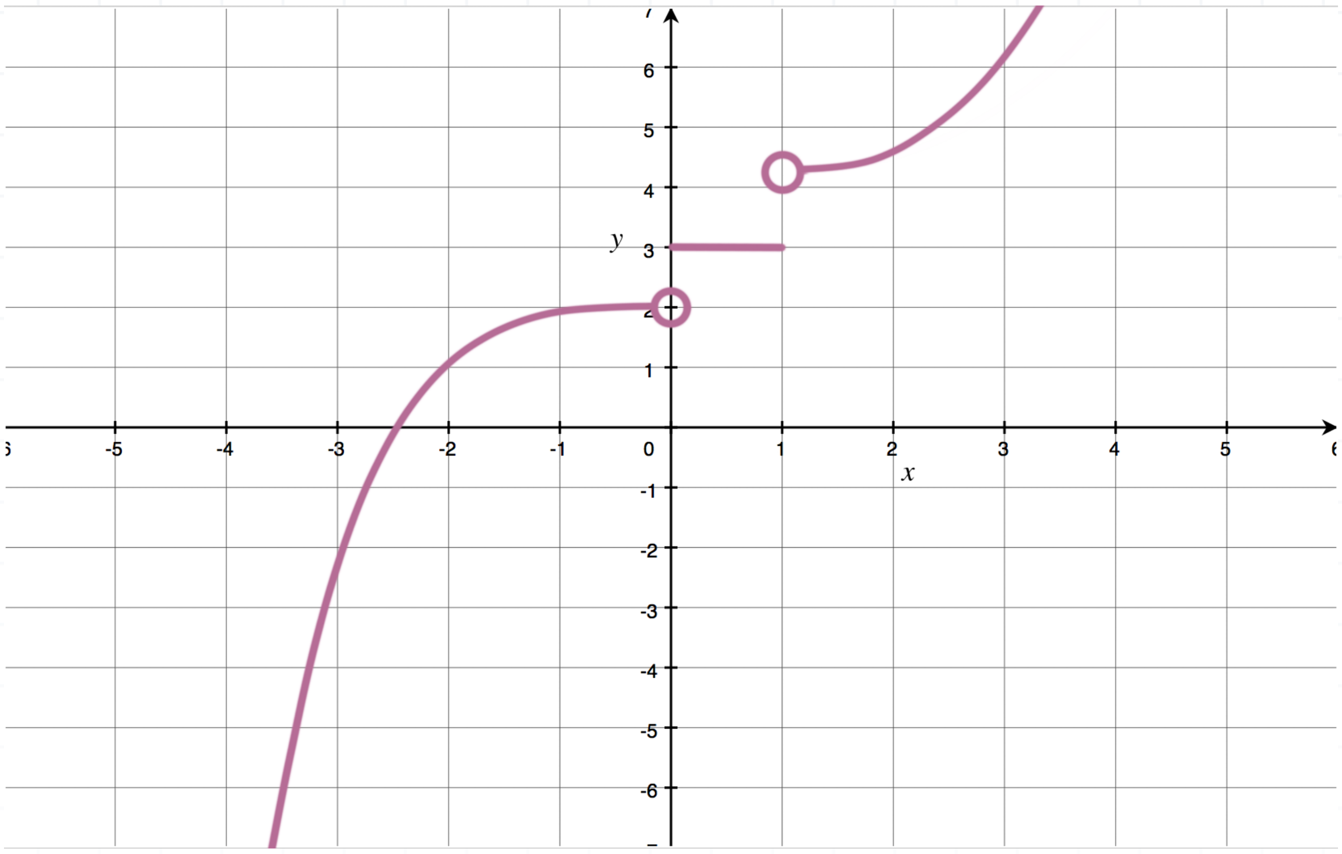
The function $h(x)$ has jump discontinuities at $x = 0$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = 0$,

$$\lim_{x\to 0^-} f(x) = 2 \quad \neq \quad \lim_{x\to 0^+} f(x) = 3$$

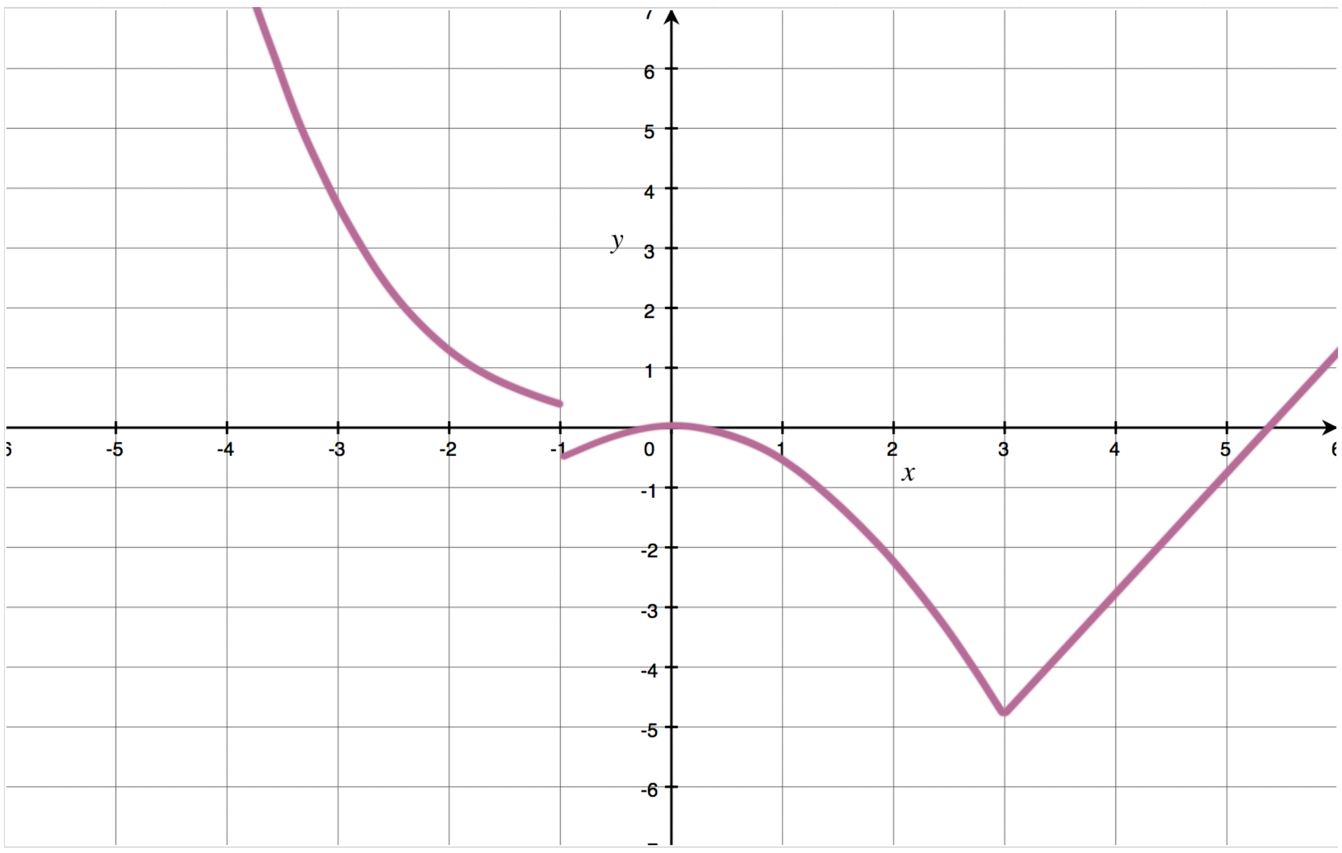
or at $x = 1$.

$$\lim_{x\to 1^-} f(x) = 3 \quad \neq \quad \lim_{x\to 1^+} f(x) = \frac{13}{3}$$

We can see the discontinuities in the function's graph, as well.



■ 3. What are the x-values where the graph of $g(x)$ has jump discontinuities?

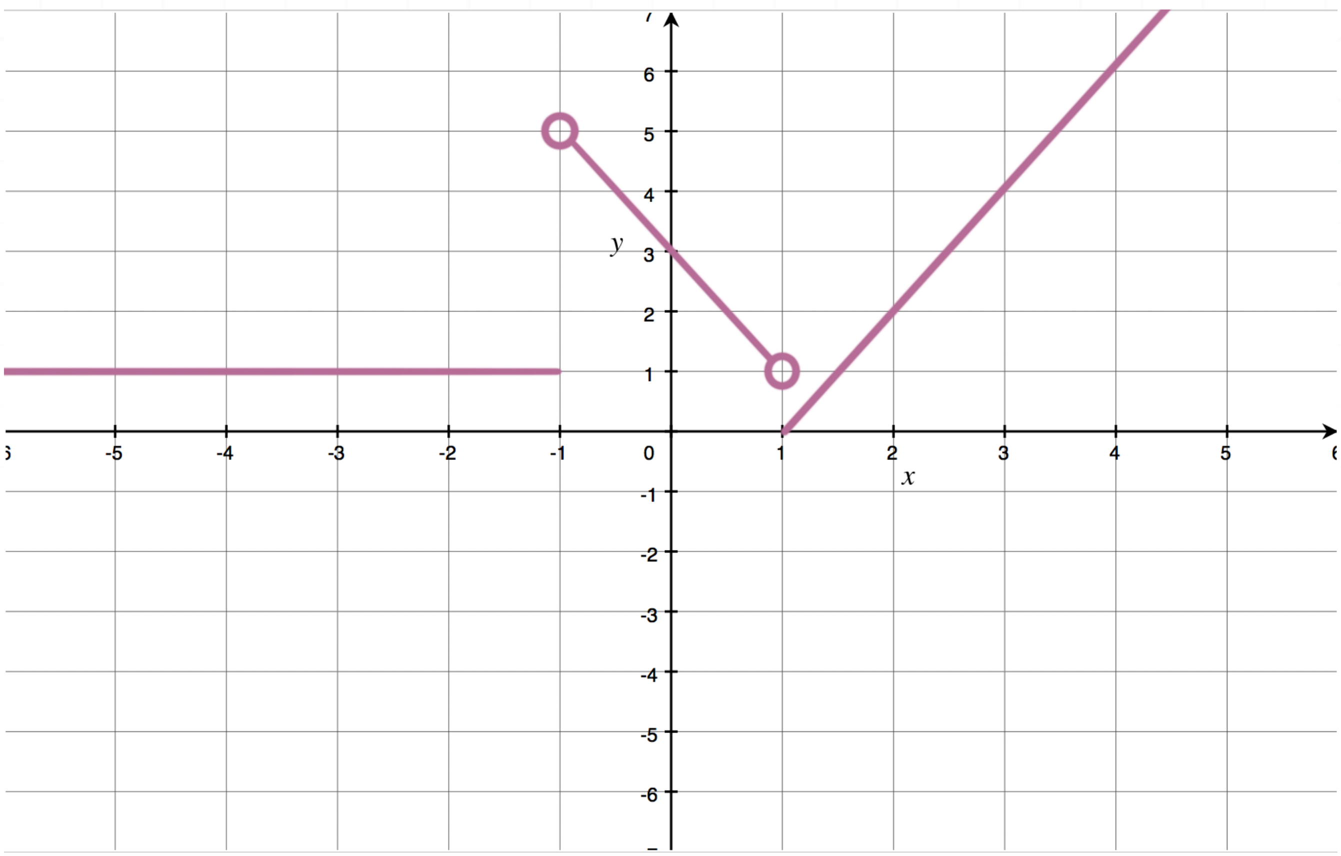


Solution:

The function $g(x)$ has a jump discontinuity at $x = -1$ because the left- and right-hand limits aren't equal there.

$$\lim_{x\to-1^-} f(x) = \frac{1}{3} \quad \neq \quad \lim_{x\to-1^+} f(x) = -\frac{1}{3}$$

■ 4. Show that $f(x)$ has jump discontinuity at $x = -1$ and $x = 1$.



Solution:

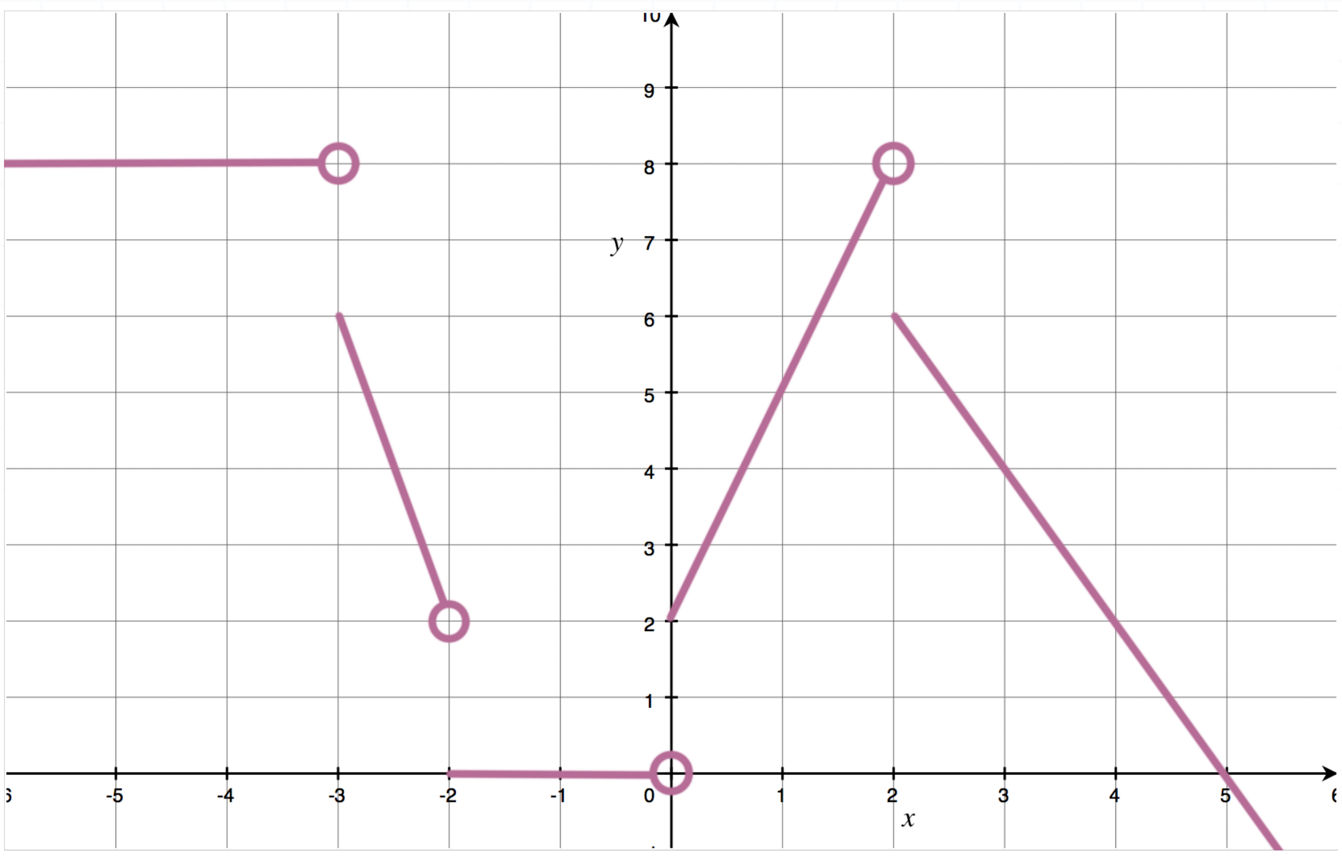
The function $f(x)$ has jump discontinuities at $x = -1$ and $x = 1$ because the left- and right-hand limits aren't equal at $x = -1$

$$\lim_{x\to-1^-} f(x) = 1 \quad \neq \quad \lim_{x\to-1^+} f(x) = 5$$

or at $x = 1$.

$$\lim_{x \to 1^-} f(x) = 1 \quad \neq \quad \lim_{x \to 1^+} f(x) = 0$$

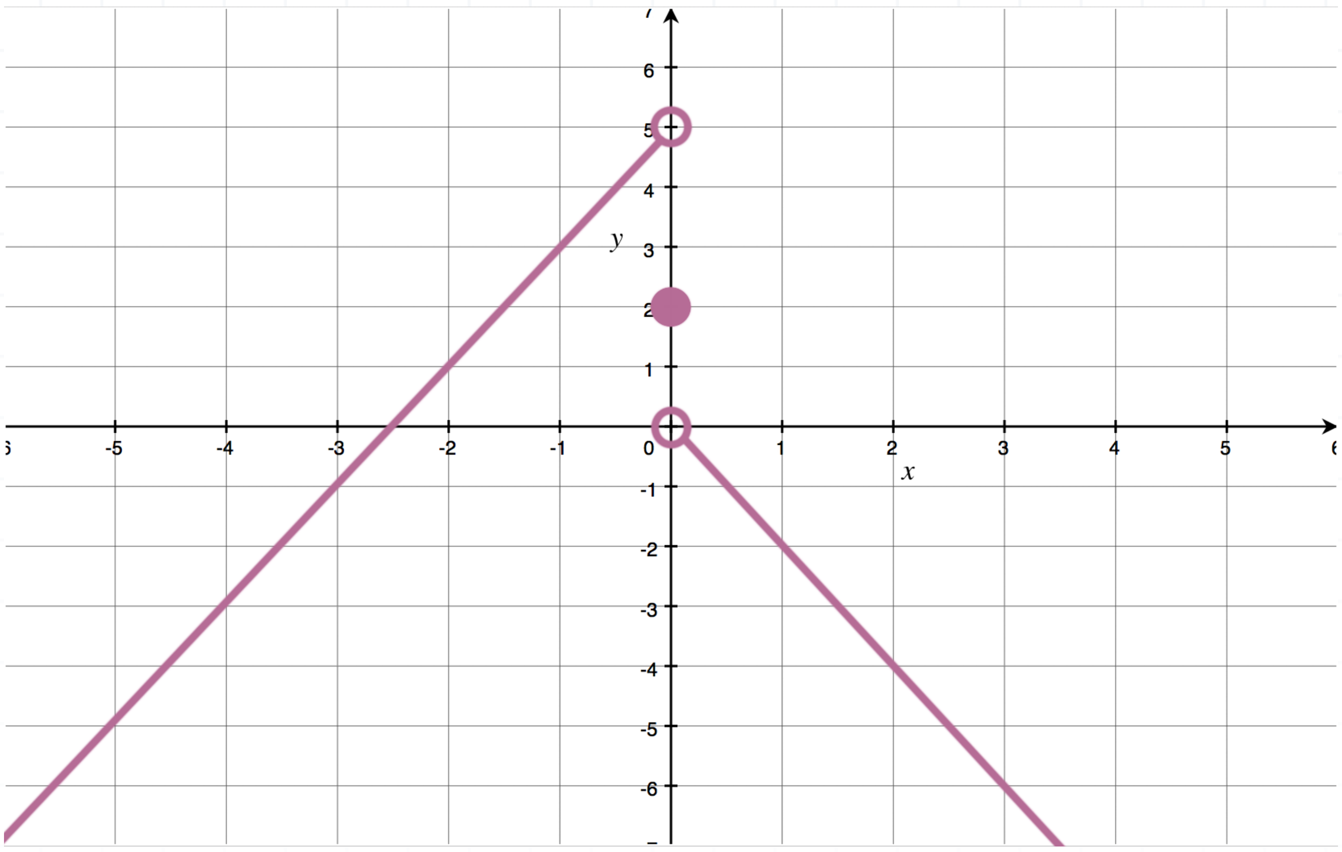
■ 5. Where are the jump discontinuities in the graph of the function shown below?



Solution:

The function has jump discontinuities at $x = -3$, $x = -2$, $x = 0$, and $x = 2$, because at each x-value, the left- and right-hand limits aren't equal.

■ 6. What are the x-values where the graph of $h(x)$, shown below, has jump discontinuities?



Solution:

The function $h(x)$ has a jump discontinuity at $x = 0$ because the left- and right-hand limits aren't equal there.

$$\lim_{x \to 0^-} f(x) = 5 \quad \neq \quad \lim_{x \to 0^+} f(x) = 0$$

INFINITE DISCONTINUITIES

1. At what x-values does the function have infinite discontinuities?

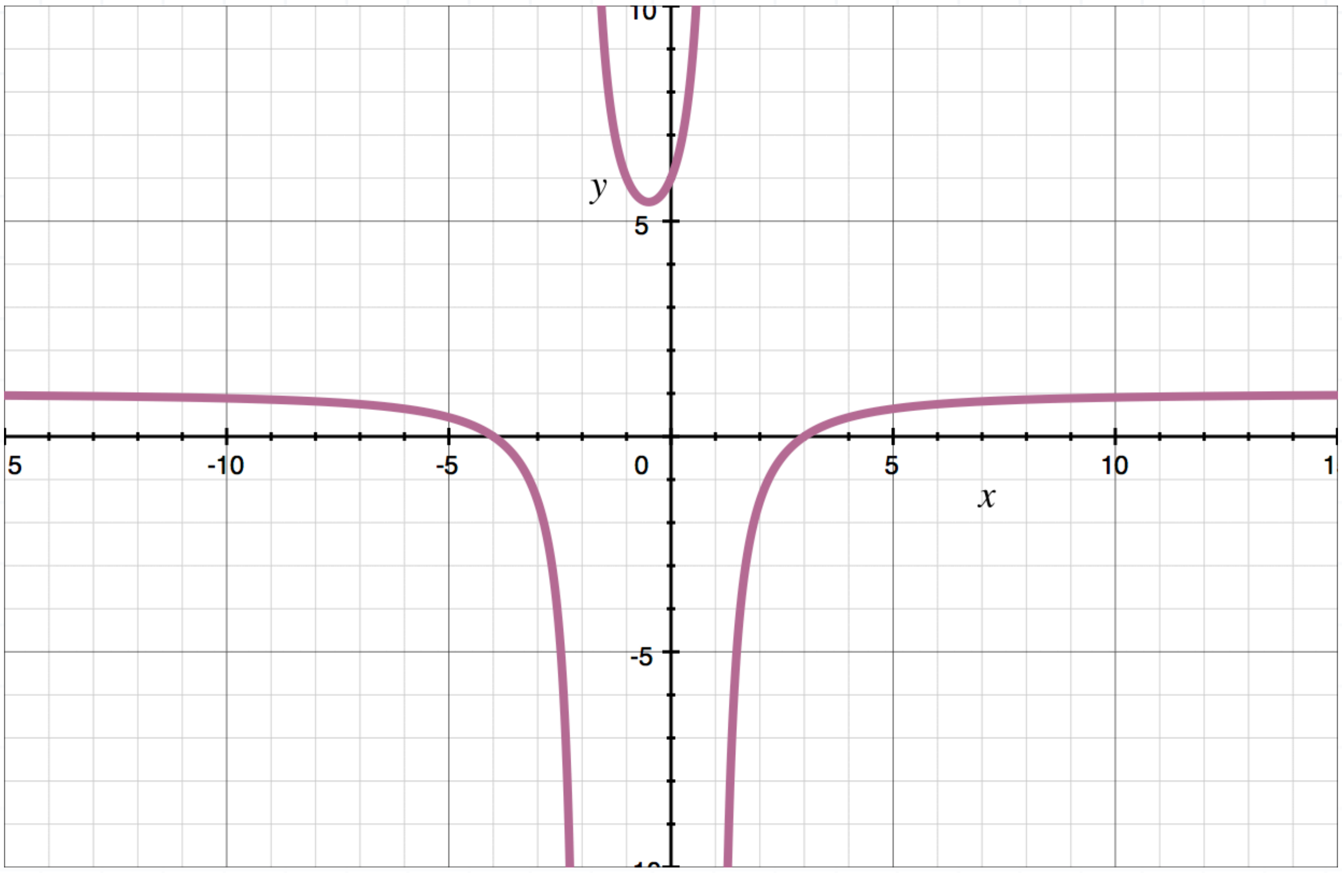
$$f(x)=\frac{x^2+x-12}{x^2+x-2}$$

Solution:

Factor the function.

$$f(x)=\frac{x^2+x-12}{x^2+x-2}=\frac{(x+4)(x-3)}{(x+2)(x-1)}$$

None of these factors cancel, which means that $x+2=0$ and $x-1=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-2$ and $x=1$.



■ 2. Where are the infinite discontinuities of the function?

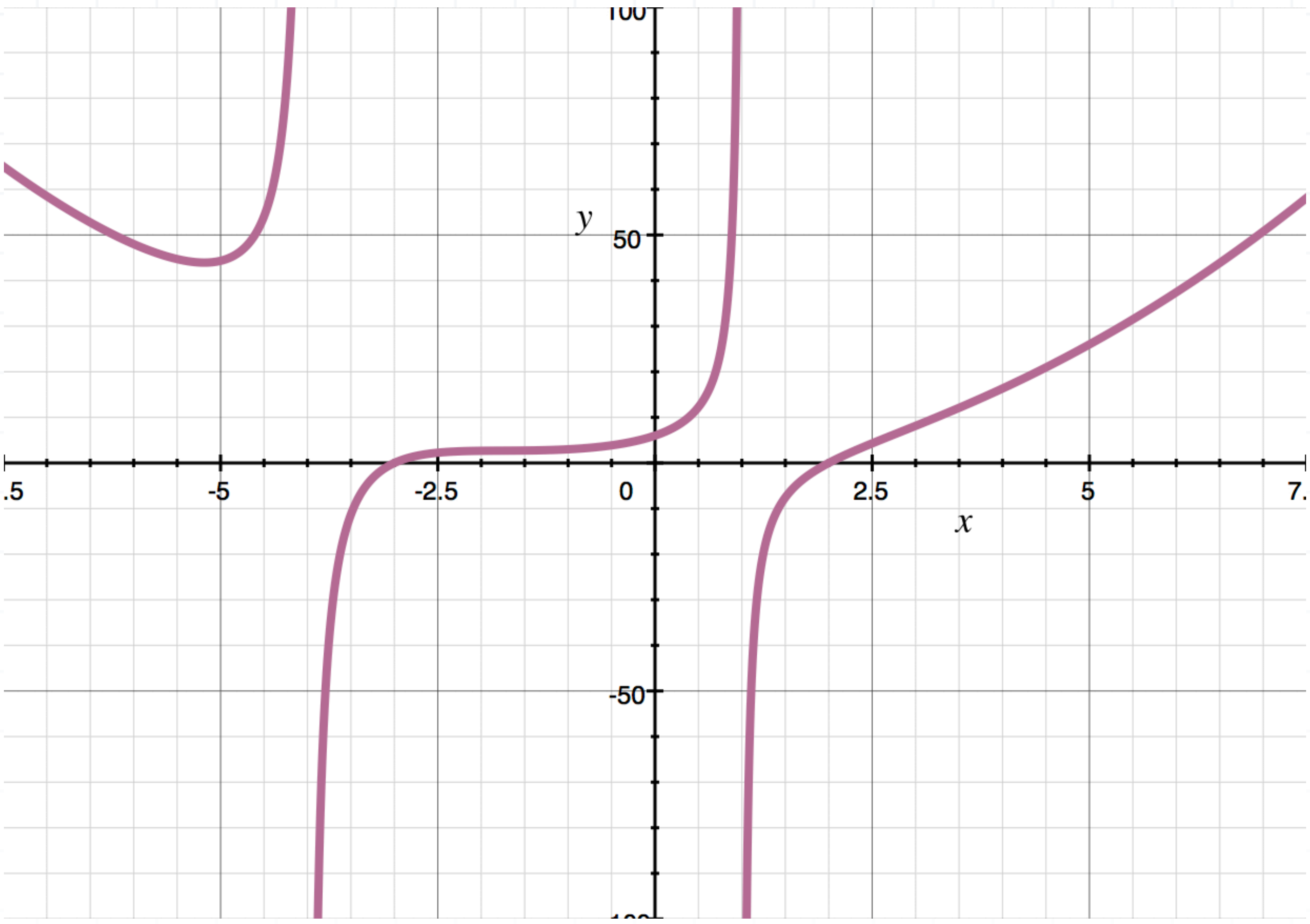
$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4}$$

Solution:

Factor the function.

$$h(x) = \frac{x^4 + 3x^3 - 8x - 24}{x^2 + 3x - 4} = \frac{(x-2)(x^2+2x+4)(x+3)}{(x+4)(x-1)}$$

None of these factors cancel, which means that $x + 4 = 0$ and $x - 1 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -4$ and $x = 1$.



■ 3. At what x-values does the function have infinite discontinuities?

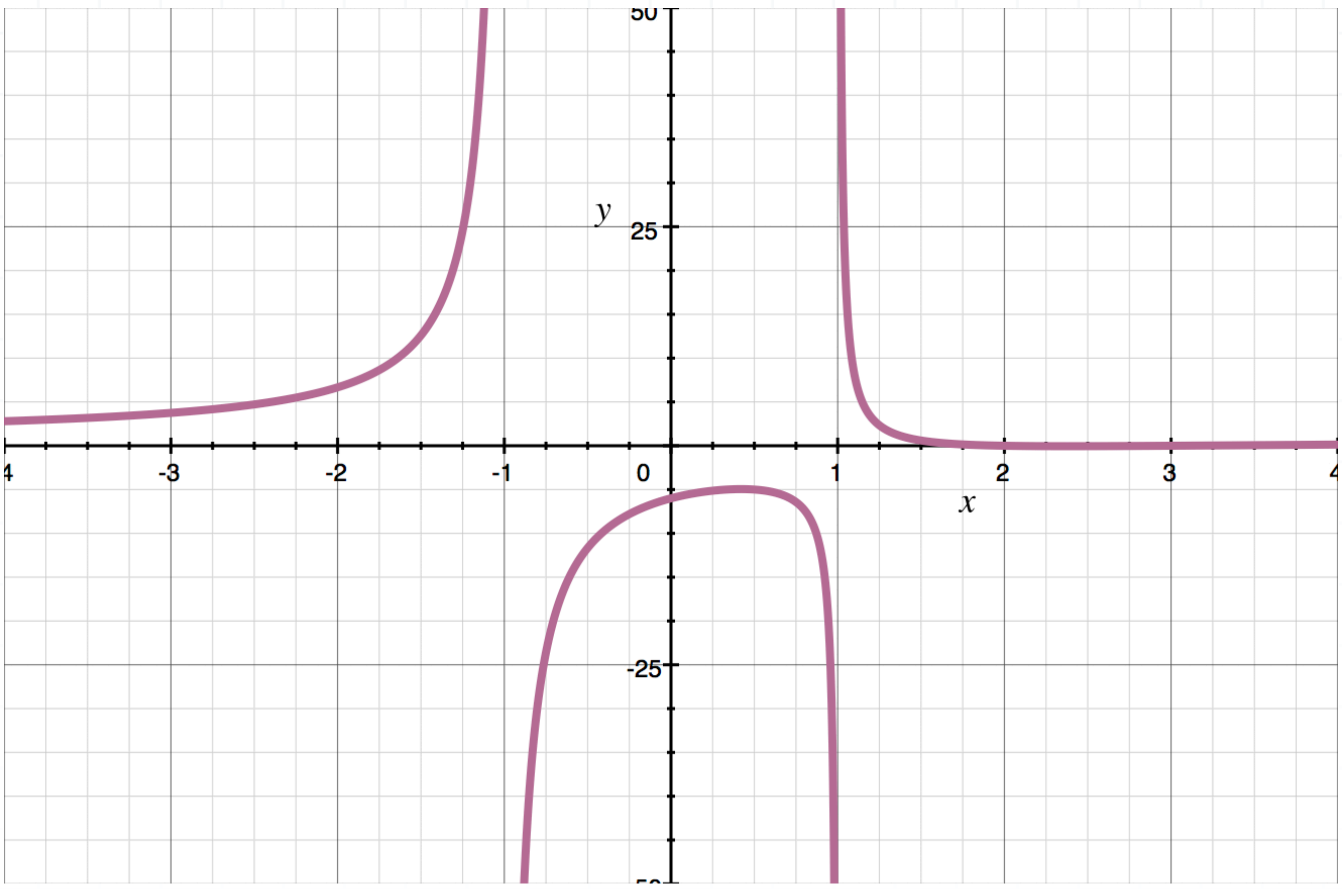
$$g(x)=\frac{x^2-5x+6}{x^2-1}$$

Solution:

Factor the function.

$$g(x)=\frac{x^2-5x+6}{x^2-1}=\frac{(x-3)(x-2)}{(x+1)(x-1)}$$

None of these factors cancel, which means that $x+1=0$ and $x-1=0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x=-1$ and $x=1$.



■ 4. Where are the infinite discontinuities of the function?

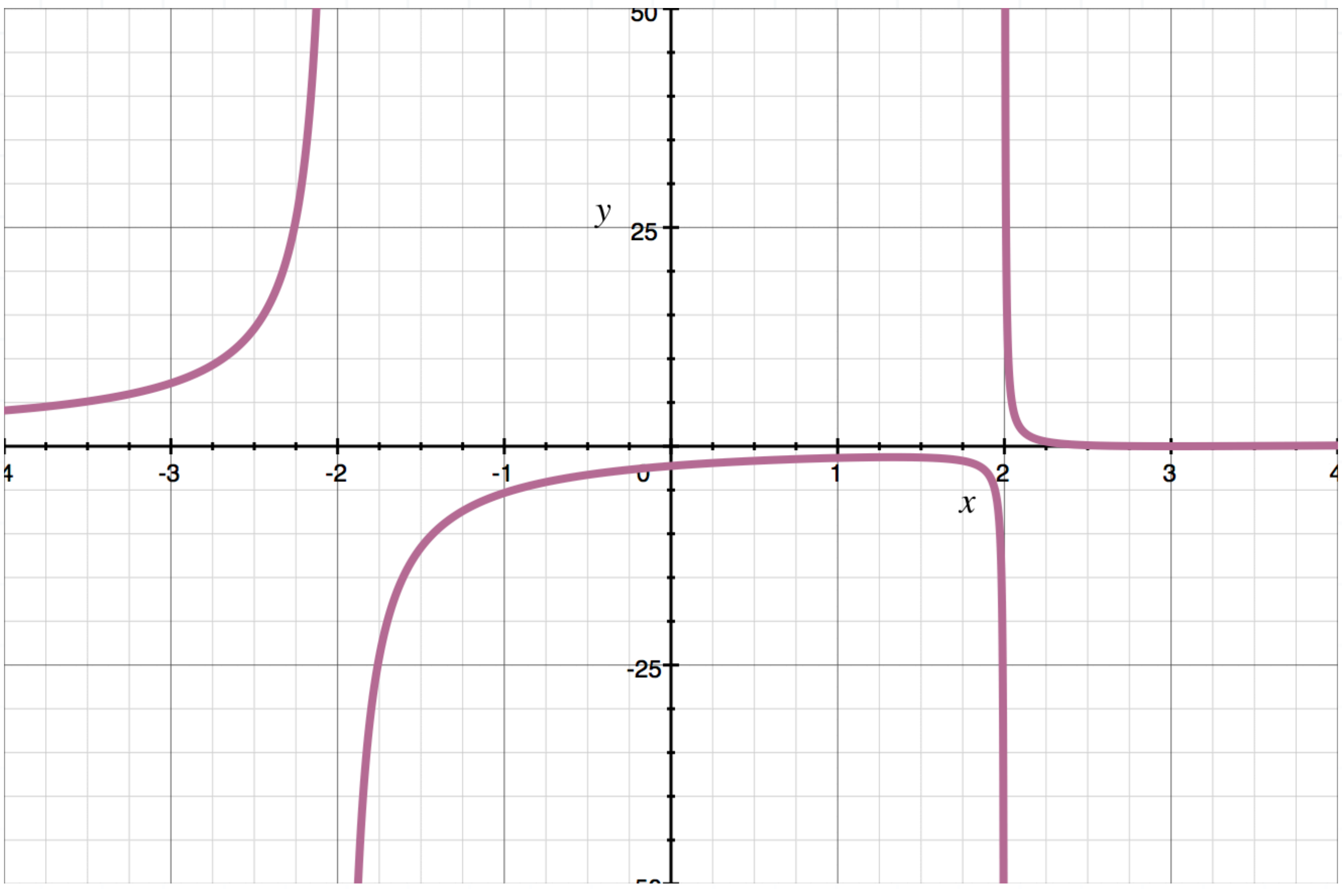
$$h(x)=\frac{x^2-6x+9}{x^2-4}$$

Solution:

Factor the function.

$$h(x)=\frac{x^2-6x+9}{x^2-4}=\frac{(x-3)^2}{(x+2)(x-2)}$$

None of these factors cancel, which means that $x + 2 = 0$ and $x - 2 = 0$ will both make the denominator equal to 0. Which means there are infinite discontinuities at $x = -2$ and $x = 2$.



■ 5. At what x-values does the function have infinite discontinuities?

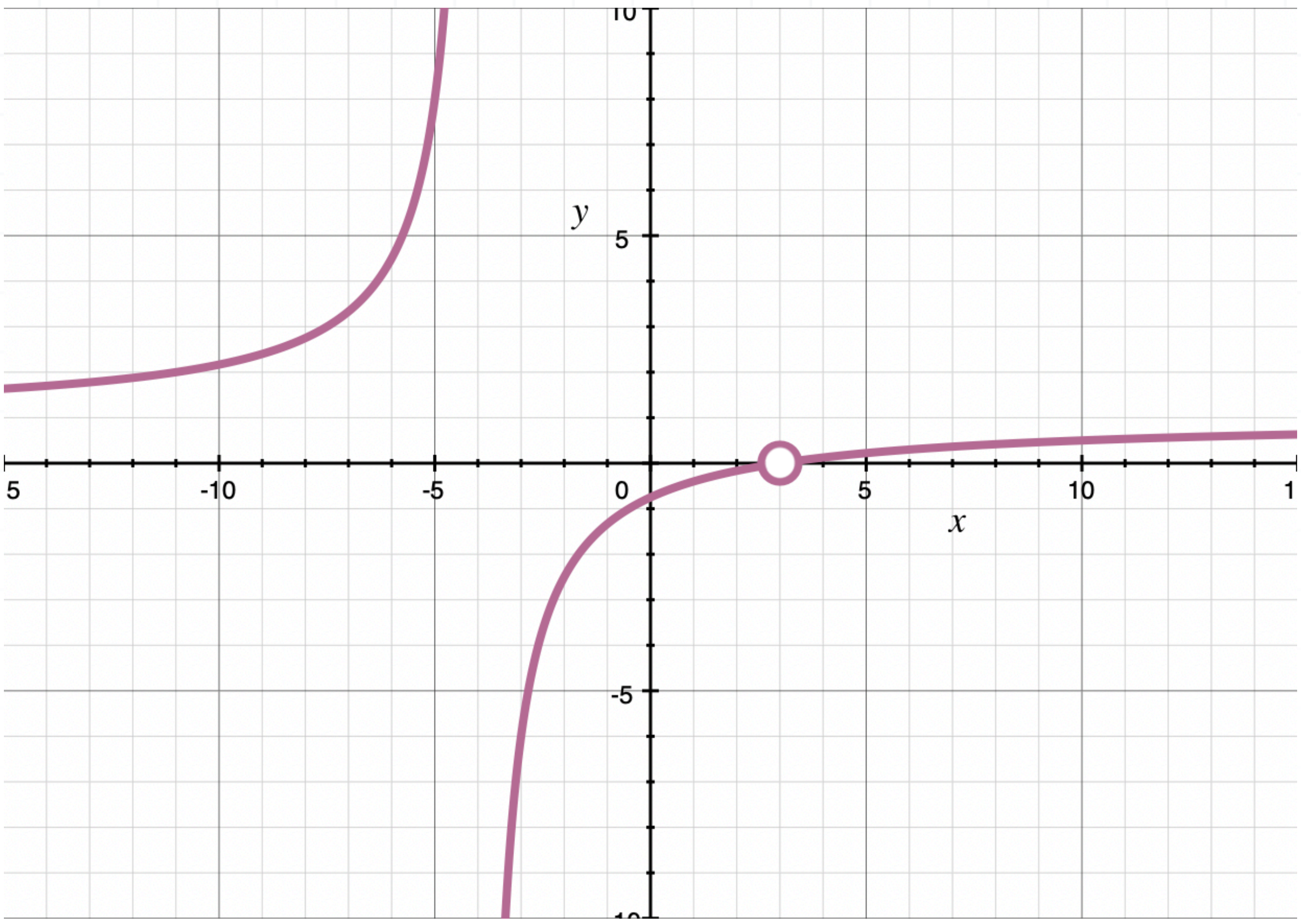
$$h(x) = \frac{x^2 - 6x + 9}{x^2 + x - 12}$$

Solution:

Factor the function.

$$h(x) = \frac{x^2 - 6x + 9}{x^2 + x - 12} = \frac{(x-3)^2}{(x+4)(x-3)} = \frac{x-3}{x+4}$$

In this form, we can see that the denominator is 0 at both $x = 3$ and $x = -4$. Because the $x - 3$ can be canceled, there's a point discontinuity at $x = 3$. Since the $x + 4$ can't be canceled, and, no matter how much we simplify the fraction, $x = -4$ will always make the denominator 0, that tells us there's a vertical asymptote at $x = -4$, and therefore an infinite discontinuity there.



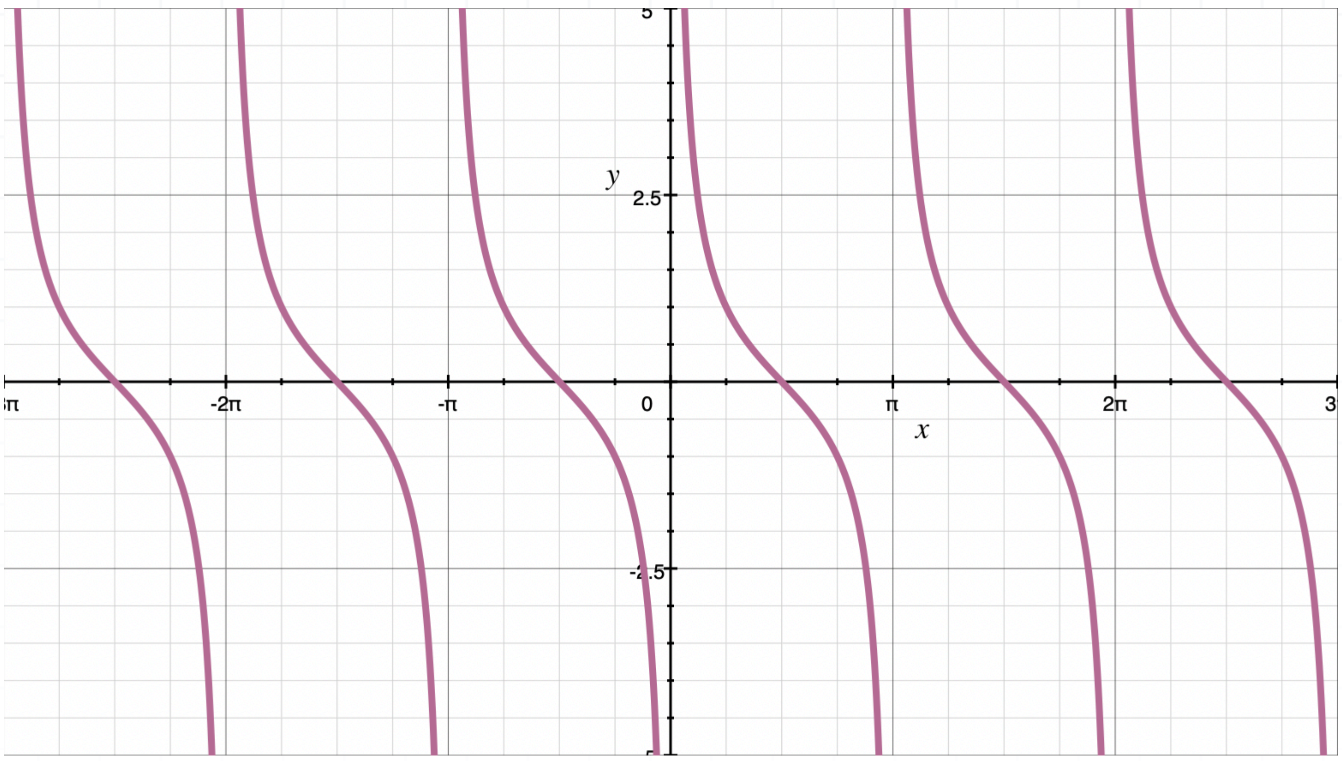
■ 6. Classify the discontinuities of $f(x) = \cot x$ on the interval $[0,2\pi]$.

Solution:

Factor the function.

$$f(x) = \cot x = \frac{\cos x}{\sin x}$$

None of these factors cancel, which means that the denominator will be 0 whenever $\sin x = 0$. The value of $\sin x$ is 0 at all integer multiplies of π, so $f(x)$ has an infinite discontinuity at three points in the interval $[0,2\pi]$, which are at $x = 0$, $x = \pi$, and $x = 2\pi$.



ENDPOINT DISCONTINUITIES

1. What is the value of the limit on the interval $[0,3]$?

$$\lim_{x\to 3} -\sqrt{x+5}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 3$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x\to 3^-} -\sqrt{x+5} = -2\sqrt{2} \quad \neq \quad \lim_{x\to 3^+} -\sqrt{x+5} = \text{DNE}$$

2. What is the value of the limit on the interval $[\pi,2\pi]$?

$$\lim_{x\to \pi} \sin x$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = \pi$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to \pi^+} \sin x = 0 \quad \neq \quad \lim_{x \to \pi^-} \sin x = \text{DNE}$$

■ 3. What is the value of the limit on the interval $[4,\infty)$?

$$\lim_{x \to 4} -\frac{x+7}{x^2-6x+15}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = 4$. The left-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to 4^+} -\frac{x+7}{x^2-6x+15} = -\frac{11}{7} \quad \neq \quad \lim_{x \to 4^-} -\frac{x+7}{x^2-6x+15} = \text{DNE}$$

■ 4. What is the value of the limit on the interval $[-9/2,5/2]$?

$$\lim_{x \to \frac{5}{2}} \frac{x+3}{x^2+x+1}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = 5/2$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to \frac{5}{2}^-} \frac{x+3}{x^2+x+1} = \frac{22}{39} \quad \neq \quad \lim_{x \to \frac{5}{2}^+} \frac{x+3}{x^2+x+1} = \text{DNE}$$

■ 5. What is the value of the limit on the interval $(-2,2]$?

$$\lim_{x \to -2} \sqrt{2x+4}$$

Solution:

The limit does not exist because only the right-hand limit exists at $x = -2$. The left-hand limit does not exist, which means that the one-sided limits are not equal.

$$\lim_{x \to -2^+} \sqrt{2x+4} = 0 \quad \neq \quad \lim_{x \to -2^-} \sqrt{2x+4} = \text{DNE}$$

■ 6. What is the value of the limit on the interval $[-\pi, \pi]$?

$$\lim_{x \to \pi} -\frac{5 \cos x}{2}$$

Solution:

The limit does not exist because only the left-hand limit exists at $x = \pi$. The right-hand limit does not exist, which means the one-sided limits are not equal.

$$\lim_{x \to \pi^-} -\frac{5\cos x}{2} = \frac{5}{2} \quad \neq \quad \lim_{x \to \pi^+} -\frac{5\cos x}{2} = \text{DNE}$$

INTERMEDIATE VALUE THEOREM WITH AN INTERVAL

■ 1. The value $c=-1$ satisfies the conditions of the Intermediate Value Theorem for the function on the interval $[-3,5]$ because $f(c)$ equals what value?

$$f(x)=\frac{1}{4}(2x+5)(x-3)^2$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y=f(x)$ is continuous on a closed interval $[a,b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0=f(c)$ for some c in $[a,b]$. In this problem, $f(a)=f(-3)=-9$ and $f(b)=f(5)=15$. Then,

$$f(c)=f(-1)=\frac{1}{4}(2(-1)+5)(-1-3)^2=12$$

The IVT requires that $f(a)\le f(c)\le f(b)$ and $-9\le 12\le 15$.

■ 2. The value $c=2$ does not satisfy the conditions of the Intermediate Value Theorem for $g(x)=2x^2-11x+4$ on the interval $[-2,4]$ because $g(c)$ equals what value?

Solution:

The Intermediate Value Theorem (IVT) states that a function $y=f(x)$ is continuous on a closed interval $[a,b]$ takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0=f(c)$ for some c in $[a,b]$. In this problem, $g(a)=g(-2)=34$ and $g(b)=g(4)=-8$. However, $g(c)=g(2)=2(2)^2-11(2)+4=-10$. The IVT requires that $g(a)\le g(c)\le g(b)$ or $g(b)\le g(c)\le g(a)$, but -10 is not between -8 and 34.

■ 3. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-3,3]$ if $h(x)=3(x+1)^3$ and $h(c)=24$?

Solution:

The Intermediate Value Theorem (IVT) states that a function $y=f(x)$ is continuous on a closed interval $[a,b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0=f(c)$ for some c in $[a,b]$. In this problem, $h(a)=h(-3)=-24$ and $h(b)=h(3)=192$. Thus, if $h(c)=24$, the IVT requires that since $f(a)\le f(c)\le f(b)$, $a\le c\le b$. Thus, since $h(c)=24$, we get

$$3(c+1)^3=24$$

$$(c+1)^3=8$$

$$c+1=2$$

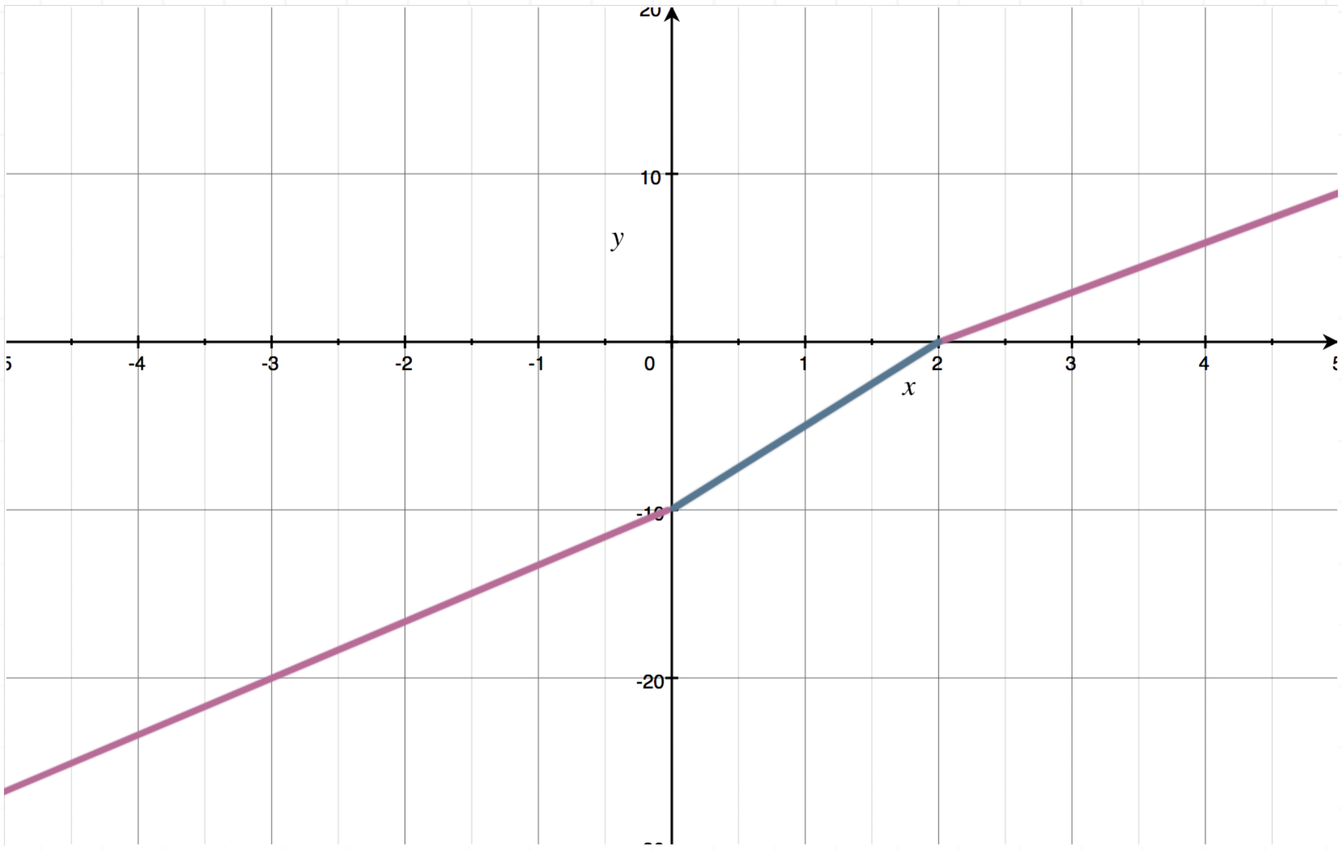
$$c=1$$

■ 4. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[-5,6]$ if $f(c) = -6$ and

$$f(x) = \begin{cases} 3x - 10 & \text{if } x \le 0 \\ x^2 + 3x - 10 & \text{if } 0 < x < 2 \\ 3x - 6 & \text{if } x \ge 2 \end{cases}$$

Solution:

The Intermediate Value Theorem (IVT) states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, first confirm that the function $f(x)$ is continuous on the interval $[-5,6]$ by evaluating the function on both sides of $x = 0$ and $x = 2$. The function is continuous, as shown in the graph below.



$f(a) = f(-5) = -25$ and $f(b) = f(6) = 12$. Thus, if $f(c) = -6$, the IVT requires that since $f(a) \leq f(c) \leq f(b)$, $a \leq c \leq b$. Thus, since $f(c) = -6$,

$$c^2 + 3c - 10 = -6$$

$$c^2 + 3c - 4 = 0$$

$$(c + 4)(c - 1) = 0$$

$c = -4$ and $c = 1$

when using $x^2 + 3x - 10$, but we consider $x^2 + 3x - 10$ on the interval $(0,2)$, so $c = -4$ does not satisfy the conditions. Because $f(x)$ is defined piecewise, there can be other values of c that might satisfy the IVT, but there are no other values of c that satisfy the conditions.

■ 5. Show that the function has a zero in the interval $[2,9]$ and find the solution.

$$g(x) = \frac{x^2 - 9}{x + 3}$$

Solution:

The IVT states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$. In this problem, $g(a) = g(2) = -1$ and $g(b) = g(9) = 6$.

Because the function is below the x-axis at the left edge of the interval, and above the x-axis at the right edge of the interval, we can say $g(a) < g(c) < g(b)$, or more specifically, $-1 < g(c) < 6$, where $g(c) = 0$.

Therefore, by the IVT, it must be true that the function has a root on the interval $[2,9]$. To find the root, which is the point where the graph of the function crosses the x-axis, we'll set the function equal to 0.

$$\frac{x^2 - 9}{x + 3} = 0$$

$$\frac{(x + 3)(x - 3)}{x + 3} = 0$$

$$x - 3 = 0$$

$$x = 3$$

Therefore, the root in the interval $[2,9]$ is at $x = 3$, or the point $(3,0)$.

■ 6. What value of c is guaranteed by the Intermediate Value Theorem on the interval $[3,6]$ if c is a root of $h(x)$.

$$h(x) = \frac{x^3 - 4x^2 - 11x + 30}{x^2 - 4}$$

Solution:

The IVT states that a function $y = f(x)$ is continuous on a closed interval $[a, b]$ and takes on every value between $f(a)$ and $f(b)$. In other words, if y_0 is between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.

In this problem, $h(a) = h(3) = -12/5$ and $h(b) = h(6) = 9/8$. Thus, if $h(c)$ is a root of $h(x)$, then $h(c) = 0$. The IVT requires that since $f(a) \le f(c) \le f(b)$, $a \le c \le b$. Thus, since $h(c) = 0$, then

$$\frac{c^3 - 4c^2 - 11c + 30}{c^2 - 4} = 0$$

Solving this equation gives $c = 5$ and $c = -3$, but -3 is not in the interval $[3,6]$. Note that although $h(x)$, as defined, contains discontinuities at $x = -2$ and $x = 2$, the function is continuous in the given interval, therefore satisfying the IVT.

INTERMEDIATE VALUE THEOREM WITHOUT AN INTERVAL

■ 1. Use the Intermediate Value Theorem to prove that the equation $2e^x = 3\cos x$ has at least one positive solution. In what interval is that solution?

Solution:

Let $f(x) = 2e^x - 3\cos x$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[0,1]$. Then,

$$f(0) = 2e^0 - 3\cos 0 = 2 - 3 = -1$$

$$f(1) = 2e^1 - 3\cos 1 = 2e - 1.6209$$

which is approximately 3.8157. Since the function's value changes sign in the interval $[0,1]$, and since $f(x)$ is continuous in the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 2. Use the Intermediate Value Theorem to prove that the equation $3\sin x + 7 = x^2 - 2x - 2$ has at least one positive solution. In what interval is that solution?

Solution:

Let $g(x) = 3\sin x + 7 - (x^2 - 2x - 2)$. The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[3,4]$. Then,

$$g(3) = 3\sin(3) + 7 - (3^2 - 2(3) - 2) = 6.4234$$

$$g(4) = 3\sin(4) + 7 - (4^2 - 2(4) - 2) = -1.2704$$

Since the function's value changes sign in the interval $[3,4]$, and $g(x)$ is continuous on the interval, the function has a zero in that interval.

■ 3. Use the Intermediate Value Theorem to prove that the equation $x^6 - 9x^4 + 7 = x^5 - 8x^3 - 9$ has at least one positive solution. In what interval is that solution?

Solution:

Let $h(x) = (x^6 - 9x^4 + 7) - (x^5 - 8x^3 - 9)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value of changes sign in that closed interval.

Consider the interval $[1,2]$. Then,

$$h(1) = ((1)^6 - 9(1)^4 + 7) - ((1)^5 - 8(1)^3 - 9) = 15$$

$$h(2) = ((2)^6 - 9(2)^4 + 7) - ((2)^5 - 8(2)^3 - 9) = -32$$

Since the function's value changes sign in the interval $[1,2]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 4. Use the Intermediate Value Theorem to prove that the equation $4e^{x-3} = 2(x^3 - 5x + 9)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $f(x) = 4e^{x-3} - 2(x^3 - 5x + 9)$. The root of $f(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $f(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-3,-2]$. Then,

$$f(-3) = 4e^{-3-3} - 2((-3)^3 - 5(-3) + 9) = 6.0099$$

$$f(-2) = 4e^{-2-3} - 2((-2)^3 - 5(-2) + 9) = -21.97$$

Since the function's value changes sign in the interval $[-3, -2]$, and $f(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 5. Use the Intermediate Value Theorem to show that the equation has at least one positive solution. In what interval is that solution?

$$6e^{-x} = -\left(\frac{1}{5}x^2 - 4x + 9\right)$$

Solution:

Let

$$g(x) = 6e^{-x} + \frac{1}{5}x^2 - 4x + 9$$

The root of $g(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $g(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[2,3]$. Then,

$$g(2) = 6e^{-2} + \frac{1}{5}(2)^2 - 4(2) + 9 = 2.612$$

$$g(3) = 6e^{-3} + \frac{1}{5}(3)^2 - 4(3) + 9 = -0.9013$$

Since the function's value changes sign in the interval $[2,3]$, and $g(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

■ 6. Use the Intermediate Value Theorem to show that the equation $2\sin(4x-1)=\cos(2x-3)$ has at least one negative solution. In what interval is that solution?

Solution:

Let $h(x)=2\sin(4x-1)-\cos(2x-3)$. The root of $h(x)$ is a solution to the given equation. The Intermediate Value Theorem guarantees that the function $h(x)$ has a root in a certain closed interval if the function's value changes sign in that closed interval.

Consider the interval $[-2,-1]$. Then,

$$h(-2)=2\sin(4(-2)-1)-\cos(2(-2)-3)=-1.578$$

$$h(-1)=2\sin(4(-1)-1)-\cos(2(-1)-3)=1.634$$

Since the function's value changes sign in the interval $[-2,-1]$, and $h(x)$ is continuous on the interval, by the Intermediate Value Theorem the function has a zero in that interval.

SOLVING WITH SUBSTITUTION

■ 1. What is the value of the limit?

$$\lim_{x \to 3} (-x^4 + x^3 + 2x^2)$$

Solution:

Use substitution and plug $x = 3$ into the function.

$$\lim_{x \to 3} (-x^4 + x^3 + 2x^2)$$

$$-(3)^4 + (3)^3 + 2(3)^2$$

$$-36$$

■ 2. What is the value of the limit?

$$\lim_{x \to 7} \frac{x^2 - 5}{x^2 + 5}$$

Solution:

Use substitution and plug $x = 7$ into the function.

$$\lim_{x \to 7} \frac{x^2 - 5}{x^2 + 5}$$

$$\frac{7^2 - 5}{7^2 + 5}$$

$$\frac{44}{54} = \frac{22}{27}$$

■ 3. What is the value of the limit.

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 4x - 6}{x^2 + 7x + 6}$$

Solution:

Use substitution and plug $x = -2$ into the function.

$$\lim_{x \to -2} \frac{x^3 - 5x^2 + 4x - 6}{x^2 + 7x + 6}$$

$$\frac{(-2)^3 - 5(-2)^2 + 4(-2) - 6}{(-2)^2 + 7(-2) + 6}$$

$$\frac{21}{2}$$

■ 4. Evaluate the limit.

$$\lim_{y \to -2} \frac{|y-5|}{y+1}$$

Solution:

Use substitution and plug $y = -2$ into the function.

$$\lim_{y \to -2} \frac{|y-5|}{y+1}$$

$$\frac{|-2-5|}{-2+1}$$

$$\frac{|-7|}{(-1)}$$

$$\frac{7}{(-1)}$$

$$-7$$

■ 5. Evaluate the limit.

$$\lim_{x \to 2} \left(\sin \left(\frac{\pi x}{4} \right) + \ln \left(\frac{2e}{x} \right) \right)$$

Solution:

Use substitution and plug $x = 2$ into the function.

$$\lim_{x \to 2} \left(\sin \left(\frac{\pi x}{4} \right) + \ln \left(\frac{2e}{x} \right) \right)$$

$$\sin \left(\frac{2\pi}{4} \right) + \ln \left(\frac{2e}{2} \right)$$

$$\sin \left(\frac{\pi}{2} \right) + \ln e$$

$$1 + 1$$

$$2$$

■ 6. Evaluate the limits $\lim_{x \to -1} f(x)$ and $\lim_{x \to 2} f(x)$.

$$f(x) = \begin{cases} -3x + 5 & x < -1 \\ \frac{1}{2}x^2 - 3x + 1 & x \geq -1 \end{cases}$$

Solution:

Since -1 is the value where the definition of the function changes, the substitution rule is not valid. We inspect each one-sided limit.

$$\lim_{x \to -1^-} f(x) = \lim_{x \to -1^-} (-3x + 5) = -3(-1) + 5 = 3 + 5 = 8$$

$$\lim_{x\to-1^+} f(x) = \lim_{x\to-1^+}\left(\frac{1}{2}x^2 - 3x + 1\right) = \frac{1}{2}(-1)^2 - 3(-1) + 1 = \frac{1}{2} + 3 + 1 = \frac{9}{2}$$

Since $\lim_{x\to-1^-} f(x) \neq \lim_{x\to-1^+} f(x)$, the general limit $\lim_{x\to-1} f(x)$ does not exist at $x = -1$.

To find the limit as x approaches 2, we'll substitute $x = 2$ into the second piece of the function, since that's the piece that defines the function when $x \geq -1$, which includes $x = 2$.

$$\lim_{x\to2} f(x) = \lim_{x\to2}\left(\frac{1}{2}x^2 - 3x + 1\right) = \frac{1}{2}(2)^2 - 3(2) + 1 = \frac{1}{2}(4) - 6 + 1 = 2 - 5 = -3$$

SOLVING WITH FACTORING

1. What is the value of the limit?

$$\lim_{x \to -7} \frac{6x^3 + 42x^2}{2x^2 + 26x + 84}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to -7} \frac{6x^3 + 42x^2}{2x^2 + 26x + 84}$$

$$\lim_{x \to -7} \frac{6x^2(x + 7)}{2(x + 6)(x + 7)}$$

$$\lim_{x \to -7} \frac{6x^2}{2(x + 6)}$$

Now we can evaluate the limit at $x = -7$ using substitution.

$$\frac{6(-7)^2}{2\left(-7 + 6\right)}$$

$$-147$$

2. What is the value of the limit?

$$\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{x - 1}$$

$$\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{(\sqrt[3]{x} - 1)(\sqrt[3]{x^2} + \sqrt[3]{x} + 1)}$$

$$\lim_{x \to 1} \frac{1}{\sqrt[3]{x^2} + \sqrt[3]{x} + 1}$$

Now we can evaluate the limit at $x = 1$ using substitution.

$$\frac{1}{\sqrt[3]{1^2} + \sqrt[3]{1} + 1}$$

$$\frac{1}{1 + 1 + 1}$$

$$\frac{1}{3}$$

3. What is the value of the limit?

$$\lim_{x\to 0}\frac{(x+3)^2-9}{x}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x\to 0}\frac{(x+3)^2-9}{x}$$

$$\lim_{x\to 0}\frac{x^2+6x+9-9}{x}$$

$$\lim_{x\to 0}\frac{x^2+6x}{x}$$

$$\lim_{x\to 0}\frac{x(x+6)}{x}$$

$$\lim_{x\to 0}(x+6)$$

Now we can evaluate the limit at $x=0$ using substitution.

$$0+6$$

$$6$$

■ 4. What is the value of the limit?

$$\lim_{x\to 7} \frac{x^3 - x^2 - 42x}{2x^2 - 20x + 42}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x\to 7} \frac{x^3 - x^2 - 42x}{2x^2 - 20x + 42}$$

$$\lim_{x\to 7} \frac{x(x-7)(x+6)}{2(x-3)(x-7)}$$

$$\lim_{x\to 7} \frac{x(x+6)}{2(x-3)}$$

Now we can evaluate the limit at $x = 7$ using substitution.

$$\frac{7(7+6)}{2(7-3)}$$

$$\frac{91}{8}$$

■ 5. What is the value of the limit?

$$\lim_{x\to 8} \frac{x^2 + 2x - 80}{2x^3 - 24x^2 + 64x}$$

Solution:

If the limit is evaluated using substitution, the limit is undefined. However, we can factor it.

$$\lim_{x\to 8} \frac{x^2+2x-80}{2x^3-24x^2+64x}$$

$$\lim_{x\to 8} \frac{(x+10)(x-8)}{2x(x-8)(x-4)}$$

$$\lim_{x\to 8} \frac{x+10}{2x(x-4)}$$

Now we can evaluate the limit at $x = 8$ using substitution.

$$\frac{8+10}{2(8)(8-4)}$$

$$\frac{18}{64} = \frac{9}{32}$$

■ 6. What is the value of the limit?

$$\lim_{x\to 0} \frac{1}{x}\left(1-\frac{16}{(x-4)^2}\right)$$

Solution:

If we use substitution, we get an undefined value. But we can factor the function instead.

$$\lim_{x\to 0} \frac{1}{x}\left(1 - \frac{16}{(x-4)^2}\right)$$

$$\lim_{x\to 0} \frac{1}{x}\left(\frac{(x-4)^2 - 16}{(x-4)^2}\right)$$

$$\lim_{x\to 0} \frac{1}{x}\left(\frac{x^2 - 8x + 16 - 16}{(x-4)^2}\right)$$

$$\lim_{x\to 0} \frac{1}{x}\left(\frac{x^2 - 8x}{(x-4)^2}\right)$$

$$\lim_{x\to 0} \frac{x-8}{(x-4)^2}$$

Now we can evaluate the limit at $x = 0$ using substitution.

$$\frac{0-8}{(0-4)^2}$$

$$\frac{-8}{(-4)^2}$$

$$\frac{-8}{16}$$

$$-\frac{1}{2}$$

SOLVING WITH CONJUGATE METHOD

■ 1. Use conjugate method to evaluate the limit.

$$\lim_{x\to 16}\frac{3(x-16)}{\sqrt{x}-4}$$

Solution:

We'll apply conjugate method by multiplying both the numerator and denominator by the conjugate of the denominator.

$$\lim_{x\to 16}\frac{3(x-16)(\sqrt{x}+4)}{(\sqrt{x}-4)(\sqrt{x}+4)}$$

$$\lim_{x\to 16}\frac{3(x-16)(\sqrt{x}+4)}{x-16}$$

$$\lim_{x\to 16}3(\sqrt{x}+4)$$

Then use substitution to evaluate the limit.

$$3(\sqrt{16}+4)$$

$$3(4+4)$$

$$24$$

2. What is the value of the limit?

$$\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x\to 0}\frac{\sqrt{x+3}-\sqrt{3}}{x}\left(\frac{\sqrt{x+3}+\sqrt{3}}{\sqrt{x+3}+\sqrt{3}}\right)$$

$$\lim_{x\to 0}\frac{x+3-\sqrt{3}\sqrt{x+3}+\sqrt{3}\sqrt{x+3}-3}{x(\sqrt{x+3}+\sqrt{3})}$$

$$\lim_{x\to 0}\frac{x+3-3}{x(\sqrt{x+3}+\sqrt{3})}$$

$$\lim_{x\to 0}\frac{x}{x(\sqrt{x+3}+\sqrt{3})}$$

$$\lim_{x\to 0}\frac{1}{\sqrt{x+3}+\sqrt{3}}$$

Then use substitution to evaluate the limit.

$$\frac{1}{\sqrt{0+3}+\sqrt{3}}$$

$$\frac{1}{\sqrt{3}+\sqrt{3}}$$

$$\frac{1}{2\sqrt{3}}$$

■ 3. What is the value of the limit?

$$\lim_{x\to 0}\frac{\sqrt{x^2+16}-4}{x^2}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x\to 0}\frac{\sqrt{x^2+16}-4}{x^2}$$

$$\lim_{x\to 0}\frac{\sqrt{x^2+16}-4}{x^2}\left(\frac{\sqrt{x^2+16}+4}{\sqrt{x^2+16}+4}\right)$$

$$\lim_{x\to 0}\frac{(\sqrt{x^2+16})^2-4^2}{x^2(\sqrt{x^2+16}+4)}$$

$$\lim_{x\to 0}\frac{x^2+16-16}{x^2(\sqrt{x^2+16}+4)}$$

$$\lim_{x\to 0}\frac{x^2}{x^2(\sqrt{x^2+16}+4)}$$

$$\lim_{x\to 0}\frac{1}{\sqrt{x^2+16}+4}$$

Then use substitution to evaluate the limit.

$$\frac{1}{\sqrt{0^2+16}+4}$$

$$\frac{1}{\sqrt{16}+4}$$

$$\frac{1}{8}$$

■ 4. Use conjugate method to evaluate the limit.

$$\lim_{x\to 49}\frac{x-49}{3(\sqrt{x}-7)}$$

Solution:

We'll apply conjugate method by multiplying both the numerator and denominator by the conjugate of the denominator.

$$\lim_{x \to 49} \frac{(x-49)(\sqrt{x}+7)}{3(\sqrt{x}-7)(\sqrt{x}+7)}$$

$$\lim_{x \to 49} \frac{(x-49)(\sqrt{x}+7)}{3(x-49)}$$

$$\lim_{x \to 49} \frac{\sqrt{x}+7}{3}$$

Then use substitution to evaluate the limit.

$$\frac{\sqrt{49}+7}{3}$$

$$\frac{14}{3}$$

5. What is the value of the limit?

$$\lim_{x \to 1} \frac{4-\sqrt{x+15}}{2(x-1)}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, use conjugate method.

$$\lim_{x \to 1} \frac{4-\sqrt{x+15}}{2(x-1)}$$

$$\lim_{x\to 1} \frac{4-\sqrt{x+15}}{2(x-1)}\left(\frac{4+\sqrt{x+15}}{4+\sqrt{x+15}}\right)$$

$$\lim_{x\to 1} \frac{4^2-(\sqrt{x+15})^2}{2(x-1)(4+\sqrt{x+15})}$$

$$\lim_{x\to 1} \frac{16-(x+15)}{2(x-1)(4+\sqrt{x+15})}$$

$$\lim_{x\to 1} \frac{1-x}{2(x-1)(4+\sqrt{x+15})}$$

$$\lim_{x\to 1} -\frac{1}{2(4+\sqrt{x+15})}$$

Then use substitution to evaluate the limit.

$$-\frac{1}{2(4+\sqrt{1+15})}$$

$$-\frac{1}{2(4+\sqrt{16})}$$

$$-\frac{1}{2(4+4)}$$

$$-\frac{1}{2(8)}$$

$$-\frac{1}{16}$$

6. What is the value of the limit?

$$\lim_{x \to 2} \frac{\sqrt{11-x}-3}{\sqrt{6-x}-2}$$

Solution:

Since the limit can't be evaluated using substitution or factoring, apply conjugate method, first using the conjugate of the numerator,

$$\lim_{x \to 2} \frac{\sqrt{11-x}-3}{\sqrt{6-x}-2}$$

$$\lim_{x \to 2} \frac{\sqrt{11-x}-3}{\sqrt{6-x}-2}\left(\frac{\sqrt{11-x}+3}{\sqrt{11-x}+3}\right)$$

$$\lim_{x \to 2} \frac{(\sqrt{11-x})^2-3^2}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}$$

$$\lim_{x \to 2} \frac{11-x-9}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}$$

$$\lim_{x \to 2} \frac{2-x}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}$$

and then the conjugate of the denominator.

$$\lim_{x\to 2}\frac{2-x}{(\sqrt{6-x}-2)(\sqrt{11-x}+3)}\left(\frac{(\sqrt{6-x}+2)}{(\sqrt{6-x}+2)}\right)$$

$$\lim_{x\to 2}\frac{(2-x)(\sqrt{6-x}+2)}{((\sqrt{6-x})^2-2^2)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{(2-x)(\sqrt{6-x}+2)}{(6-x-4)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{(2-x)(\sqrt{6-x}+2)}{(2-x)(\sqrt{11-x}+3)}$$

$$\lim_{x\to 2}\frac{\sqrt{6-x}+2}{\sqrt{11-x}+3}$$

Then use substitution to evaluate the limit.

$$\frac{\sqrt{6-2}+2}{\sqrt{11-2}+3}$$

$$\frac{\sqrt{4}+2}{\sqrt{9}+3}$$

$$\frac{4}{6}$$

$$\frac{2}{3}$$

INFINITE LIMITS AND VERTICAL ASYMPTOTES

■ 1. What is the value of the limit?

$$\lim_{x \to 2} \frac{x^2 - x - 6}{-3x^2 - 3x + 18}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to 2} \frac{x^2 - x - 6}{-3x^2 - 3x + 18}$$

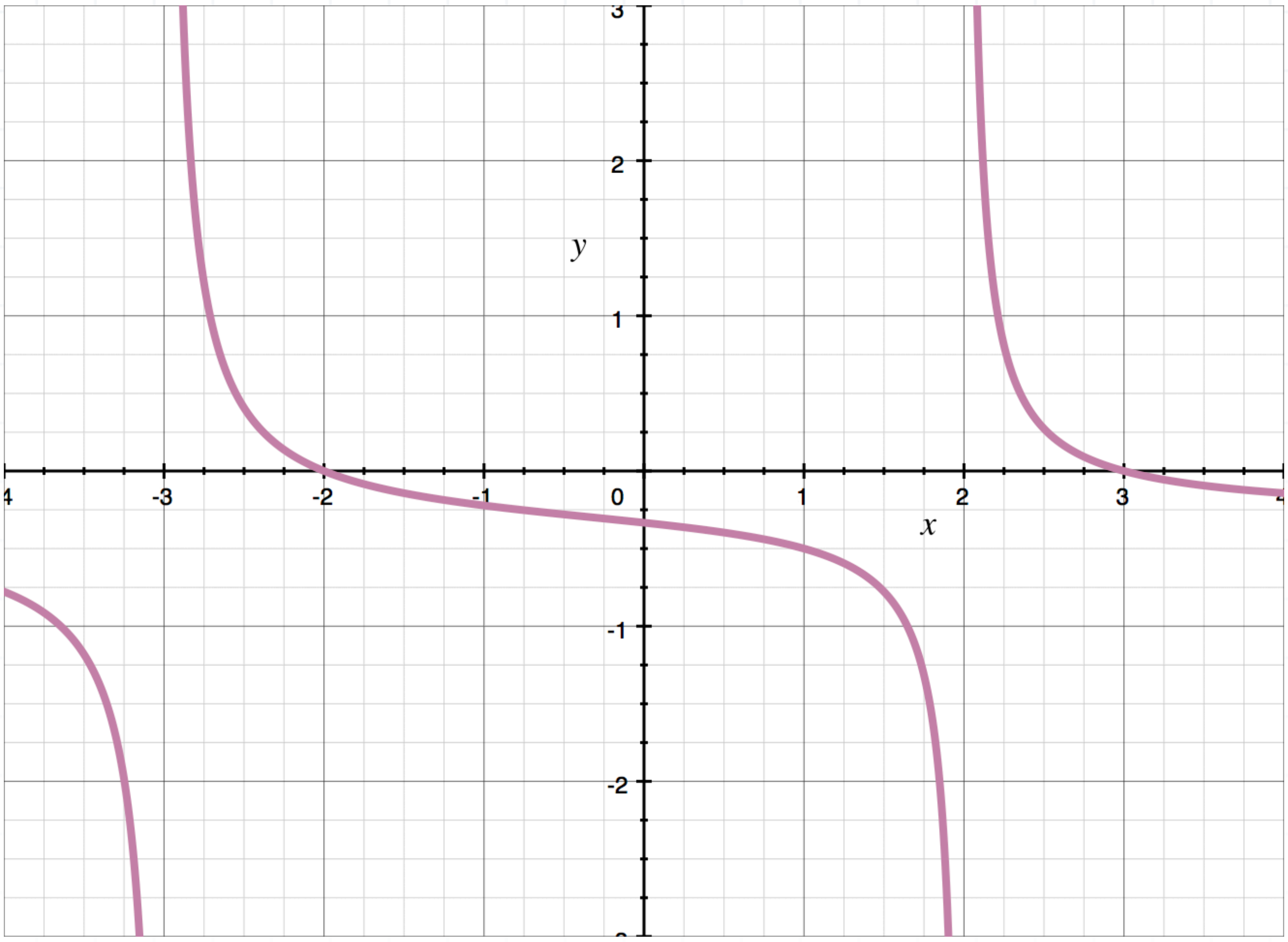
$$\lim_{x \to 2} \frac{(x-3)(x+2)}{-3(x+3)(x-2)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to 2^-} \frac{(x-3)(x+2)}{-3(x+3)(x-2)} = -\infty$$

$$\lim_{x \to 2^+} \frac{(x-3)(x+2)}{-3(x+3)(x-2)} = \infty$$

and they are not the same. Therefore, the limit does not exist (DNE). The graph is shown below.



2. What is the value of the limit?

$$\lim_{x\to -1}\frac{x^2+x-6}{4x^2+16x+12}$$

Solution:

Factor to simplify the limit.

$$\lim_{x\to -1}\frac{x^2+x-6}{4x^2+16x+12}$$

$$\lim_{x\to -1}\frac{(x+3)(x-2)}{4(x+3)(x+1)}$$

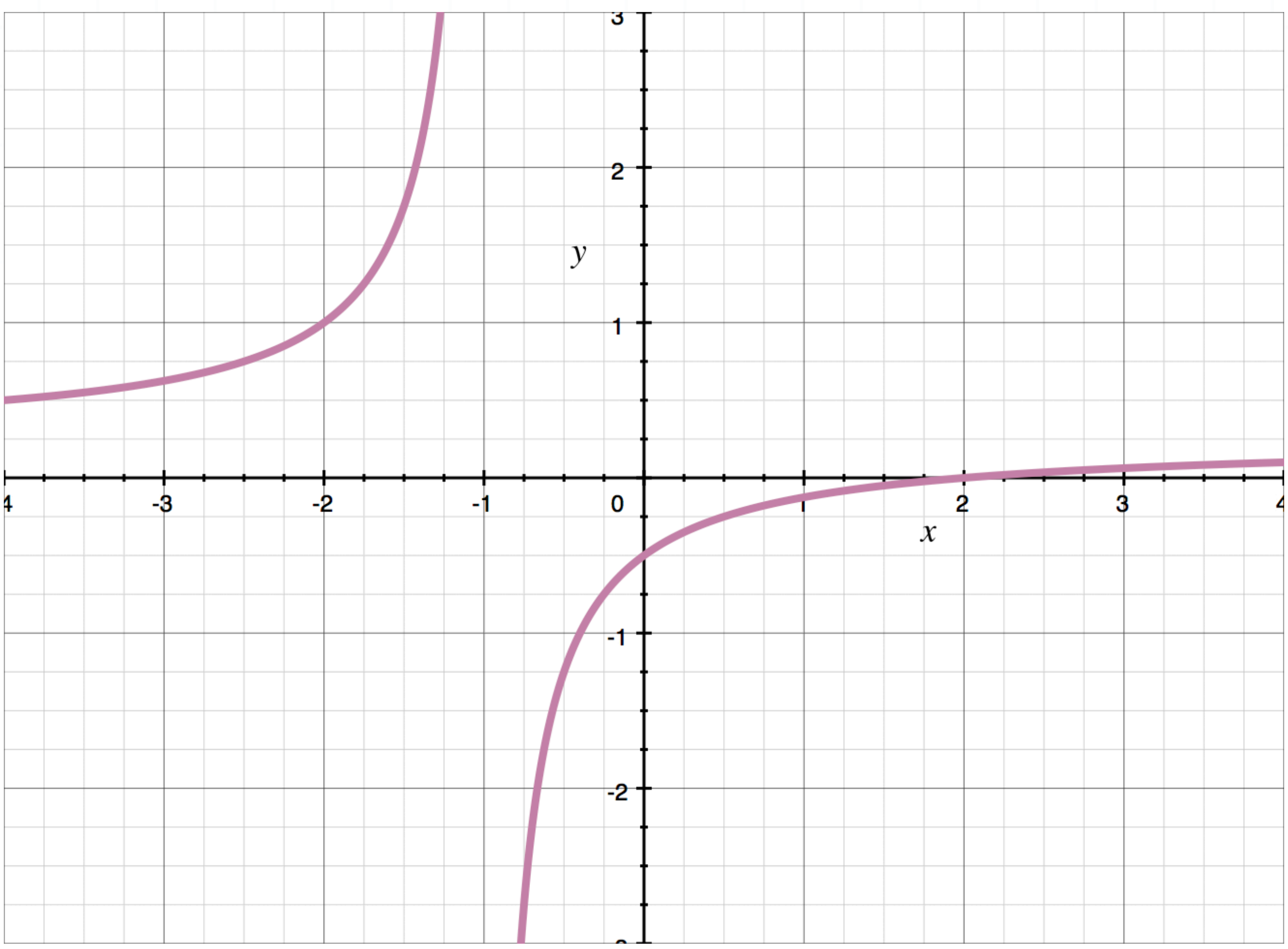
$$\lim_{x \to -1} \frac{x - 2}{4(x + 1)}$$

No other factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to -1^-} \frac{x - 2}{4(x + 1)} = \infty$$

$$\lim_{x \to -1^+} \frac{x - 2}{4(x + 1)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



■ 3. What is the value of the limit?

$$\lim_{x \to 3} \frac{1}{|x - 3|}$$

Solution:

No factors can be cancelled. The left- and right-hand limits are

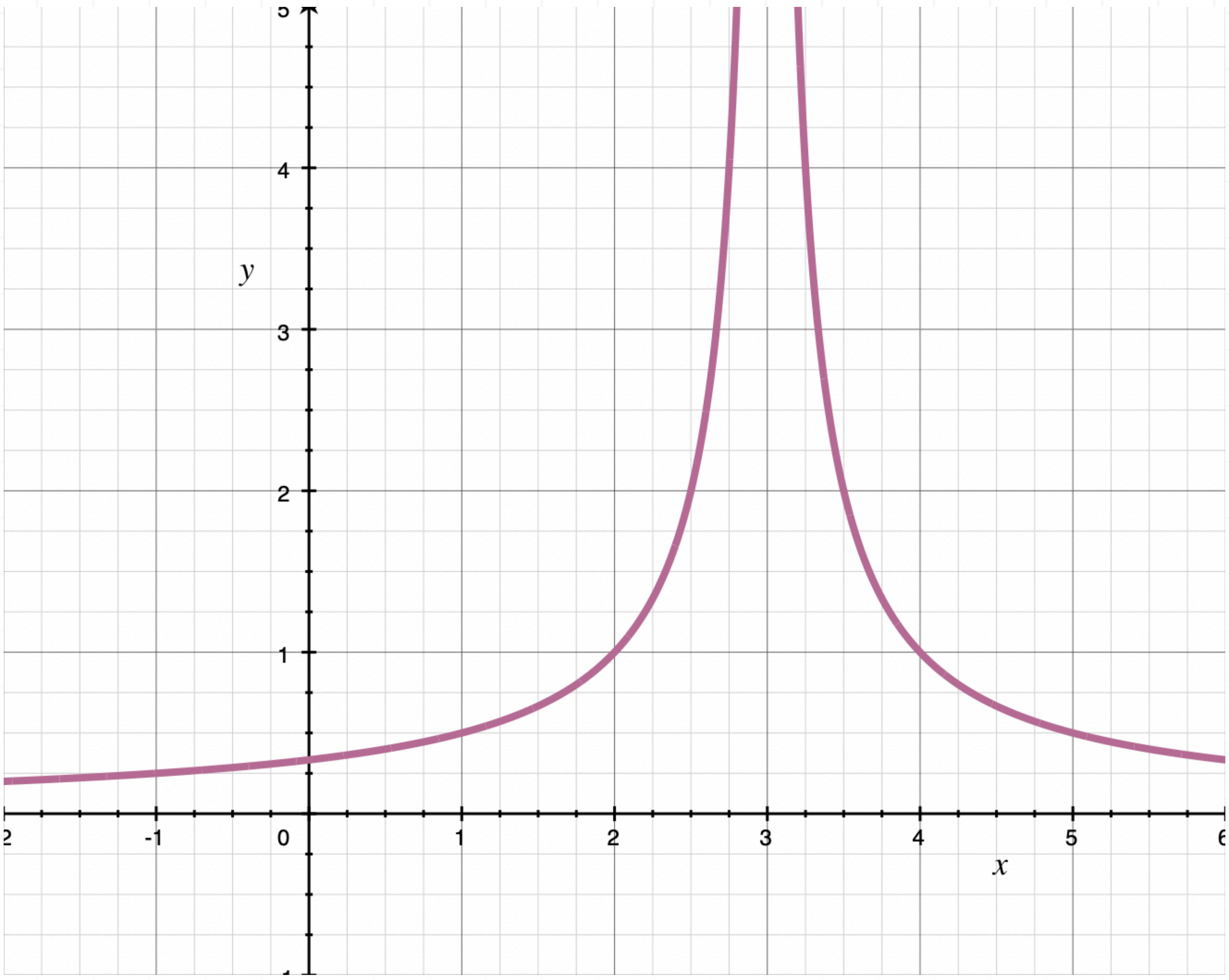
$$\lim_{x \to 3^-} \frac{1}{|x - 3|} = \infty$$

$$\lim_{x \to 3^+} \frac{1}{|x - 3|} = \infty$$

and they are the same. Therefore,

$$\lim_{x \to 3} \frac{1}{|x - 3|} = \infty$$

The graph is shown below.



■ 4. What is the value of the limit?

$$\lim_{x \to -4} \frac{\sqrt{x^2 - 1}}{x + 4}$$

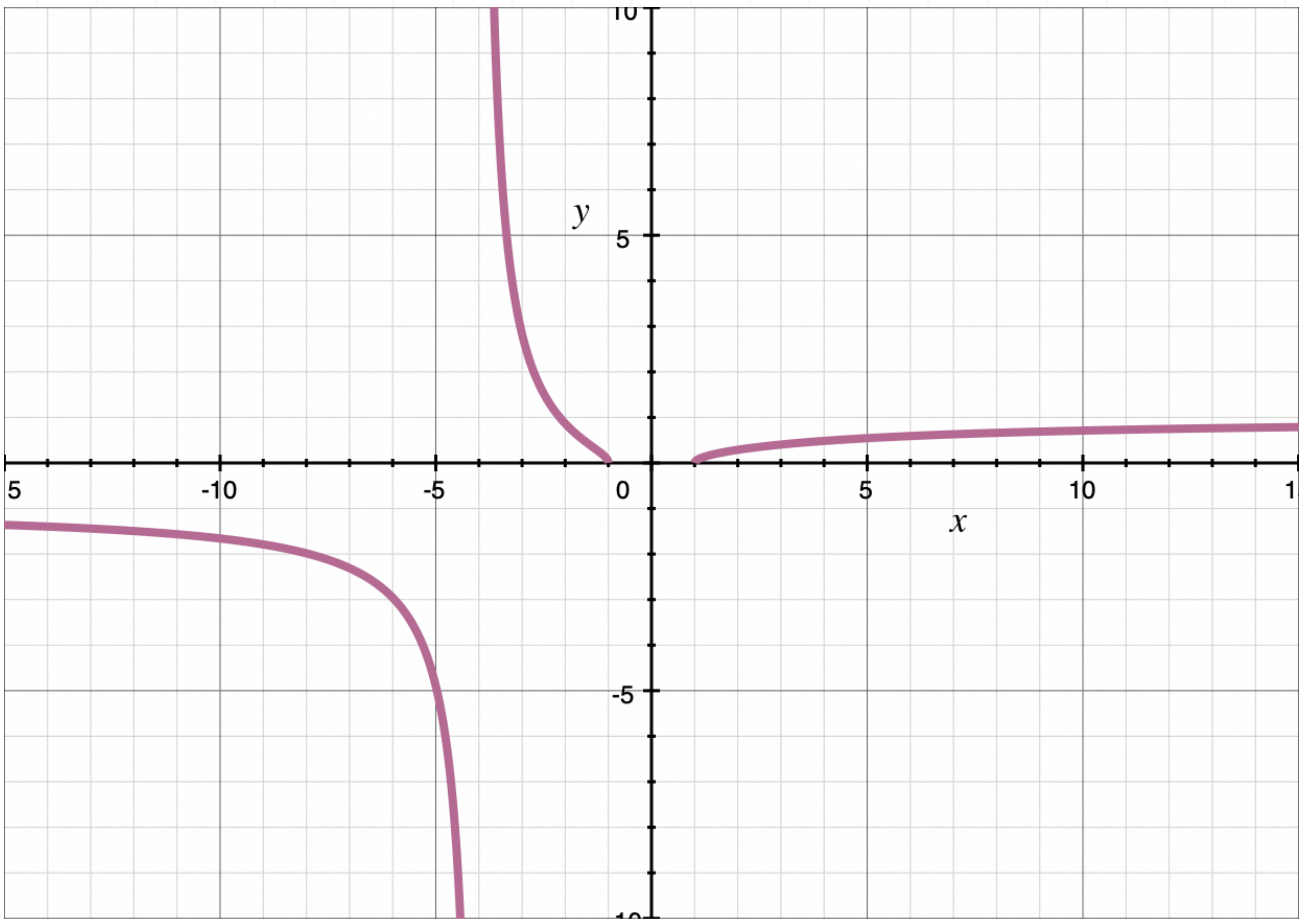
Solution:

No factors can be canceled. The left- and right-hand limits are

$$\lim_{x \to -4^-} \frac{\sqrt{x^2 - 1}}{x + 4} = -\infty$$

$$\lim_{x \to -4^+} \frac{\sqrt{x^2 - 1}}{x + 4} = \infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



5. What is the value of the limit?

$$\lim_{x \to 3} \frac{x^2 - 4x}{x^2 - 2x - 3}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to 3} \frac{x^2 - 4x}{x^2 - 2x - 3}$$

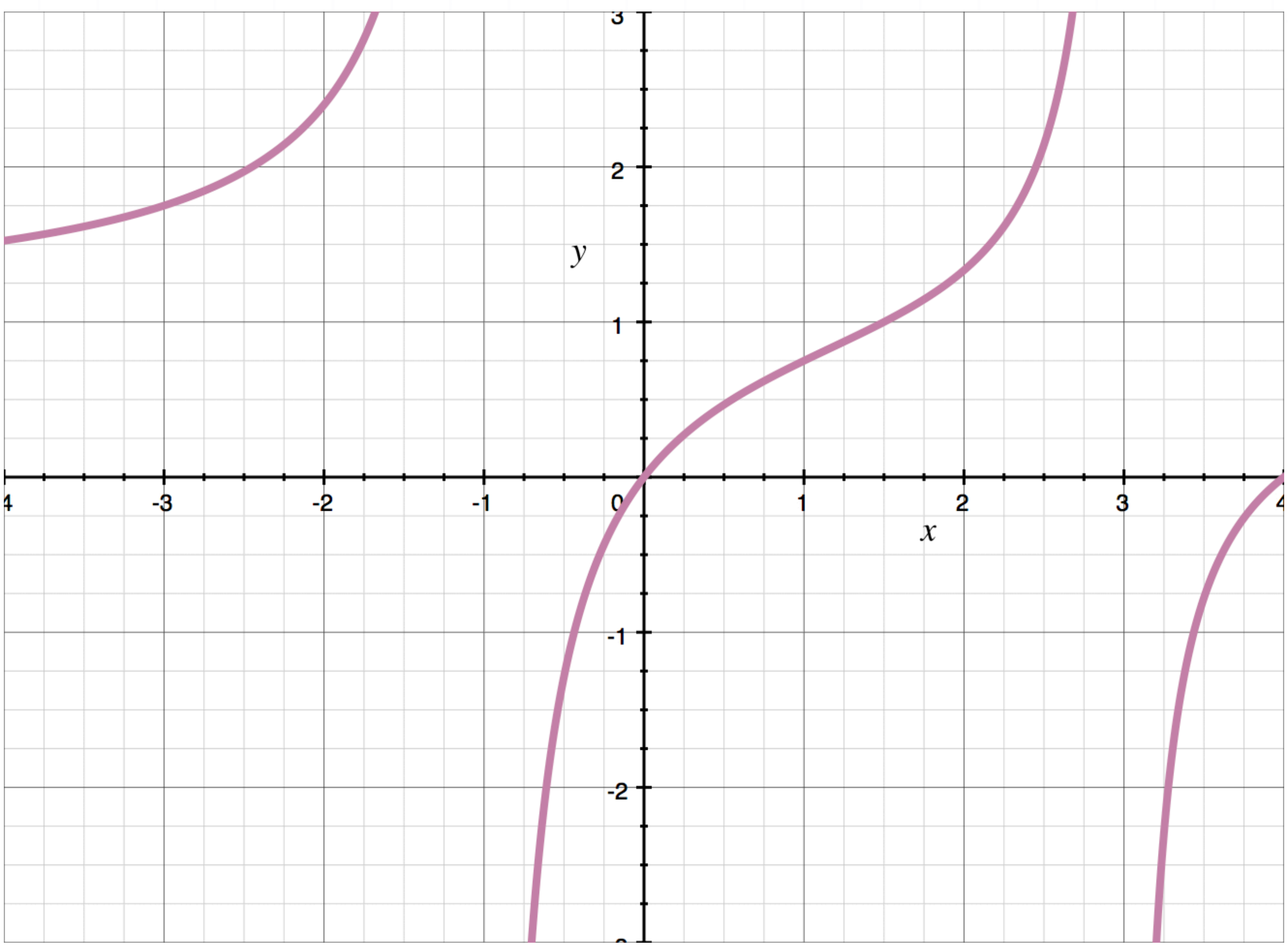
$$\lim_{x \to 3} \frac{x(x-4)}{(x-3)(x+1)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to 3^-} \frac{x(x-4)}{(x-3)(x+1)} = \infty$$

$$\lim_{x \to 3^+} \frac{x(x-4)}{(x-3)(x+1)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



6. What is the value of the limit?

$$\lim_{x \to -2} \frac{x^2 - 16}{-x^2 + x + 6}$$

Solution:

Factor to simplify the limit.

$$\lim_{x \to -2} \frac{x^2 - 16}{-x^2 + x + 6}$$

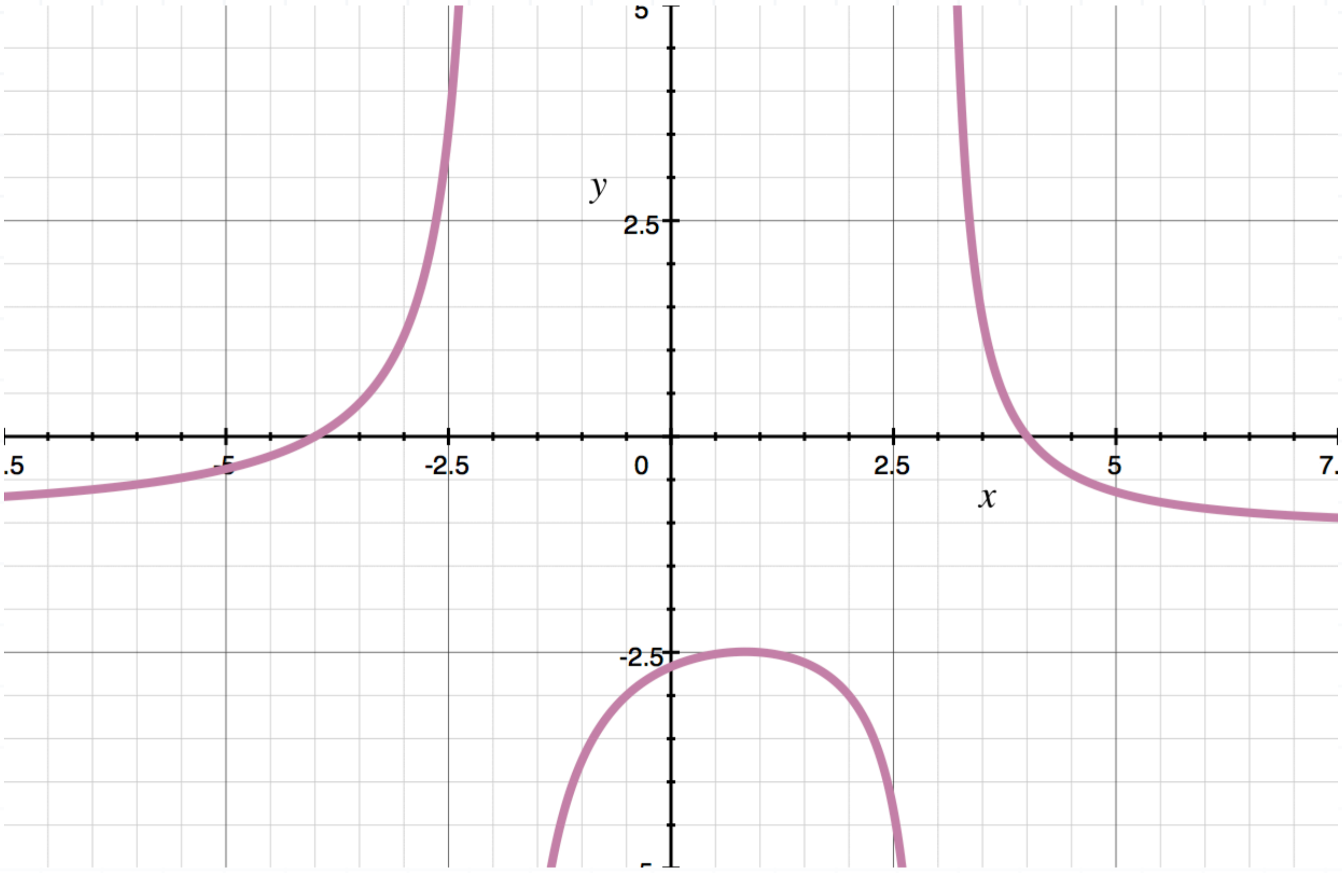
$$\lim_{x \to -2} \frac{(x+4)(x-4)}{-(x-3)(x+2)}$$

No factors can be cancelled. The left- and right-hand limits are

$$\lim_{x \to -2^-} \frac{(x+4)(x-4)}{-(x-3)(x+2)} = \infty$$

$$\lim_{x \to -2^+} \frac{(x+4)(x-4)}{-(x-3)(x+2)} = -\infty$$

and they are not the same. Therefore, the limit does not exist. The graph is shown below.



y
2.5
-2.5
.5
-5
-2.5
0
2.5
5
7.
x

LIMITS AT INFINITY AND HORIZONTAL ASYMPTOTES

■ 1. What is the value of the limit?

$$\lim_{x\to\infty}\frac{3x^3-5x+2}{9x^3+7x^2-x}$$

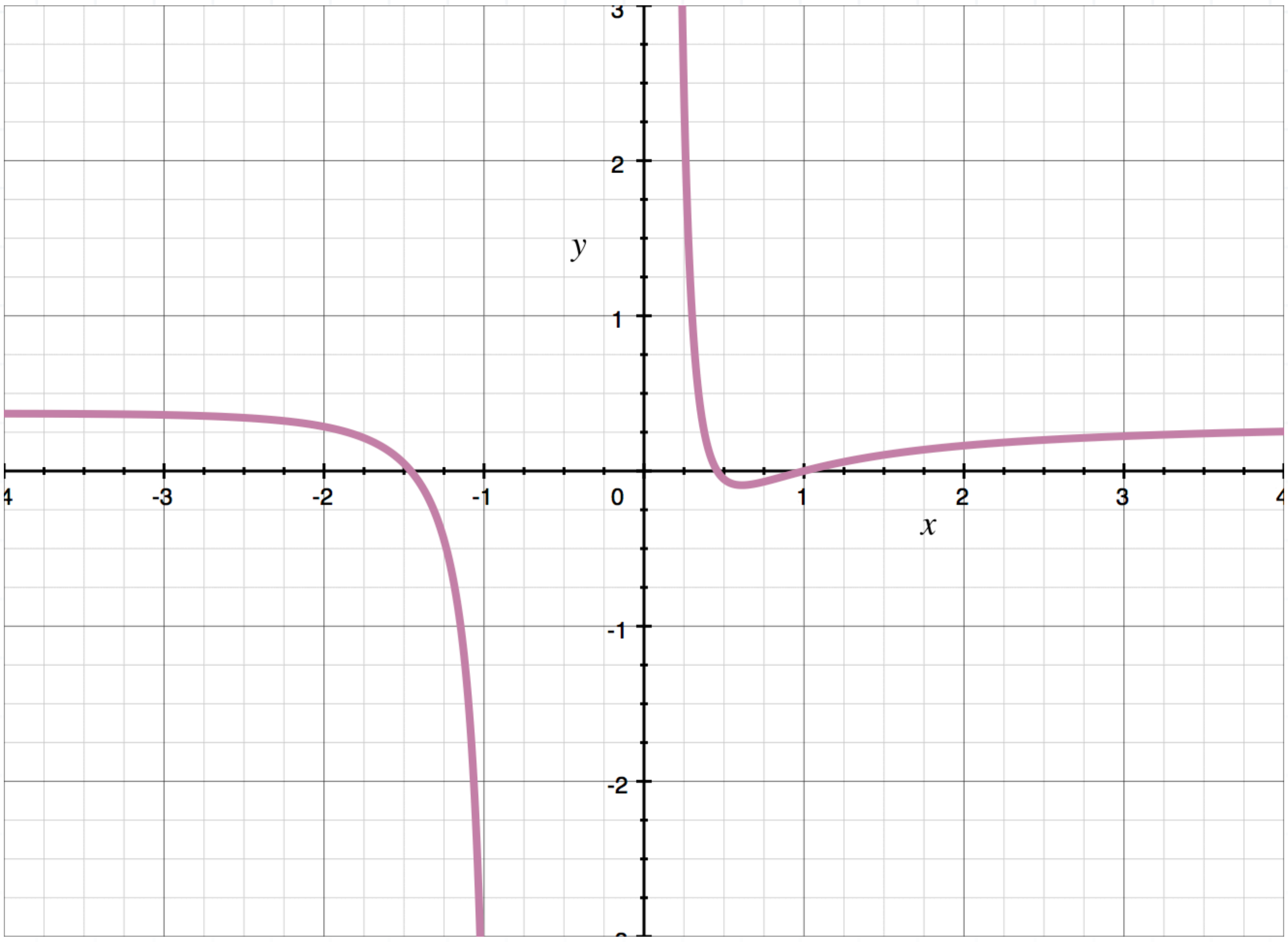
Solution:

Since this is a limit as $x\to\infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behavior of the function.

If the highest power in the numerator is the same as the highest power in the denominator, then the limit as $x\to\infty$ is the ratio of the leading coefficients.

$$\lim_{x\to\infty}\frac{3x^3-5x+2}{9x^3+7x^2-x}=\lim_{x\to\infty}\frac{3x^3}{9x^3}=\lim_{x\to\infty}\frac{3}{9}=\frac{3}{9}=\frac{1}{3}$$

The graph of the function shows this end behavior.



■ 2. What is the value of the limit?

$$\lim_{x\to\infty} \frac{\sqrt[3]{x}+1}{\sqrt{x}-1}$$

Solution:

Since this is a limit as $x \to \infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behavior of the function.

The highest-degree term in the numerator is $\sqrt[3]{x}$ or $x^{\frac{1}{3}}$, which has a degree of $1/3$. The highest-degree term in the denominator is $\sqrt{x}$ or $x^{\frac{1}{2}}$, which has a degree of $1/2$.

If the highest power in the numerator is smaller than the highest power in the denominator, then the limit as $x \to \infty$ is 0.

$$\lim_{x\to\infty} \frac{\sqrt[3]{x}+1}{\sqrt{x}-1} = \lim_{x\to\infty} \frac{\sqrt[3]{x}}{\sqrt{x}} = \lim_{x\to\infty} x^{\frac{1}{3}-\frac{1}{2}} = \lim_{x\to\infty} x^{-\frac{1}{6}} = \lim_{x\to\infty} \frac{1}{x^{\frac{1}{6}}} = 0$$

■ 3. What is the value of the limit?

$$\lim_{x\to\infty} \frac{x^3+x^2+1}{1+2x}$$

Solution:

Since this is a limit as $x \to \infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behavior of the function.

If the highest power in the numerator is greater than the highest power in the denominator, then the limit as $x \to \infty$ does not exist.

We can divide the numerator and the denominator by the x with the greatest power. Let's divide all terms by x^3.

$$\lim_{x\to\infty} \frac{x^3+x^2+1}{1+2x} = \lim_{x\to\infty} \frac{\frac{x^3}{x^3}+\frac{x^2}{x^3}+\frac{1}{x^3}}{\frac{1}{x^3}+\frac{2x}{x^3}} = \lim_{x\to\infty} \frac{1+\frac{1}{x}+\frac{1}{x^3}}{\frac{1}{x^3}+\frac{2}{x^2}}$$

Evaluating the limit gives

$$\lim_{x\to\infty}\frac{1+\frac{1}{x}+\frac{1}{x^3}}{\frac{1}{x^3}+\frac{2}{x^2}}=\frac{1+0+0}{0+0}=\frac{1}{0}$$

Division by zero is undefined, so this limit does not exist.

■ 4. What is the value of the limit?

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+5}}{x+4}$$

Solution:

We can divide the numerator and the denominator by the greatest power of x that we find in the fraction, which in this case is x.

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+5}}{x+4}=\lim_{x\to-\infty}\frac{\frac{\sqrt{x^2+5}}{x}}{\frac{x+4}{x}}=\lim_{x\to-\infty}\frac{\pm\sqrt{\frac{x^2+5}{x^2}}}{\frac{x+4}{x}}=\lim_{x\to-\infty}\frac{\pm\sqrt{1+\frac{5}{x^2}}}{1+\frac{4}{x}}$$

Evaluating the limit gives

$$\lim_{x\to-\infty}\frac{\pm\sqrt{1+\frac{5}{x^2}}}{1+\frac{4}{x}}=\frac{\pm\sqrt{1+0}}{1+0}=\pm 1$$

Now we could pick test values, like $x=-100$ and $x=100$, and we'd find that the function approaches 1 as $x\to\infty$, and that the function approaches -1 as $x\to-\infty$.

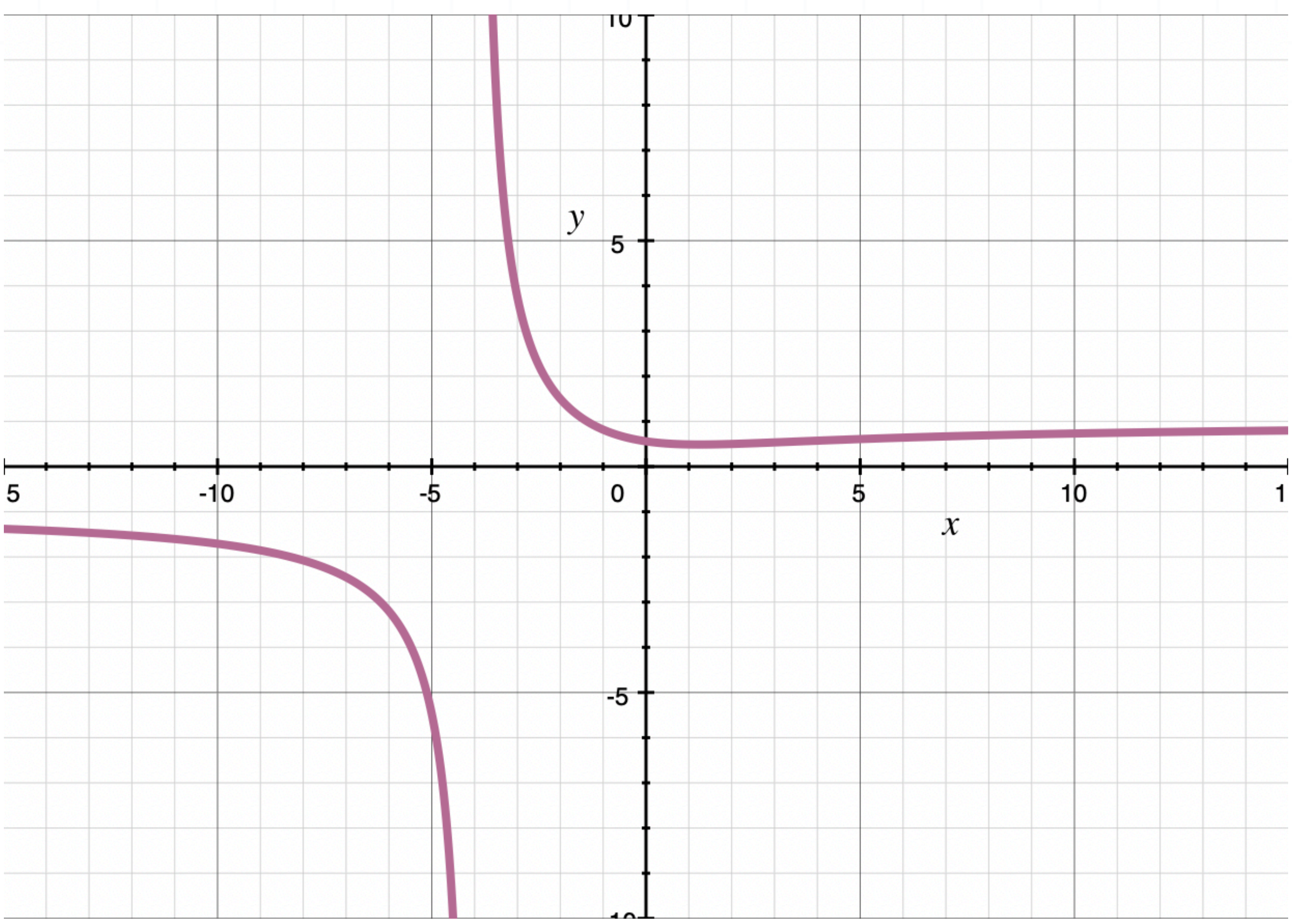
$$f(-100) = \frac{\sqrt{(-100)^2 + 5}}{-100 + 4} = \frac{\sqrt{10{,}005}}{-96} \approx -1.04$$

$$f(100) = \frac{\sqrt{100^2 + 5}}{100 + 4} = \frac{\sqrt{10{,}005}}{104} \approx 0.96$$

Putting these two tests together, we can verify that

$$\lim_{x \to -\infty} \frac{\sqrt{x^2 + 5}}{x + 4} = -1$$

The graph of the function shows this end behavior.



■ 5. What is the value of the limit?

$$\lim_{x \to -\infty} \frac{19x + 21}{x^3 + 15x + 11}$$

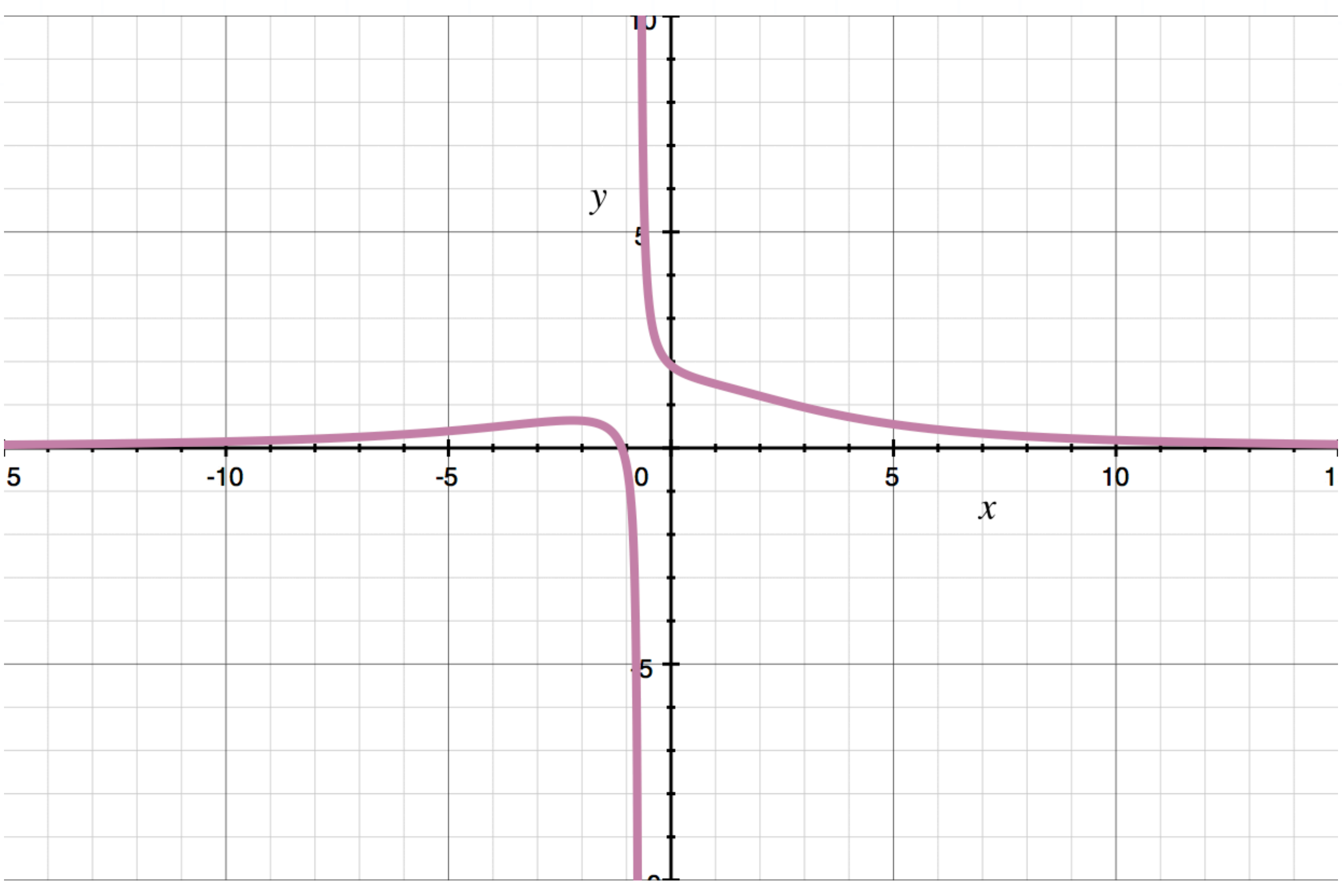
Solution:

Since this is a limit as $x \to -\infty$, use the powers of the leading terms and their coefficients, since these terms dominate the end behaviors.

If the highest power in the numerator is smaller than the highest power in the denominator, then the limit as $x \to -\infty$ is 0.

$$\lim_{x \to -\infty} \frac{19x + 21}{x^3 + 15x + 11} = \lim_{x \to -\infty} \frac{19x}{x^3} = \lim_{x \to -\infty} \frac{19}{x^2} = 0$$

The graph of the function shows this end behavior.



6. What is the value of the limit?

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-x)$$

Solution:

To find the limit we need to apply conjugate method.

$$\lim_{x\to\infty}(\sqrt{x^2+2x}-x)\left(\frac{\sqrt{x^2+2x}+x}{\sqrt{x^2+2x}+x}\right)$$

$$\lim_{x\to\infty}\frac{x^2+2x+x\sqrt{x^2+2x}-x\sqrt{x^2+2x}-x^2}{\sqrt{x^2+2x}+x}$$

$$\lim_{x\to\infty}\frac{x^2+2x-x^2}{\sqrt{x^2+2x}+x}$$

$$\lim_{x\to\infty}\frac{2x}{\sqrt{x^2+2x}+x}$$

Now we can divide the numerator and the denominator by the greatest power of x. Let's divide all terms by x.

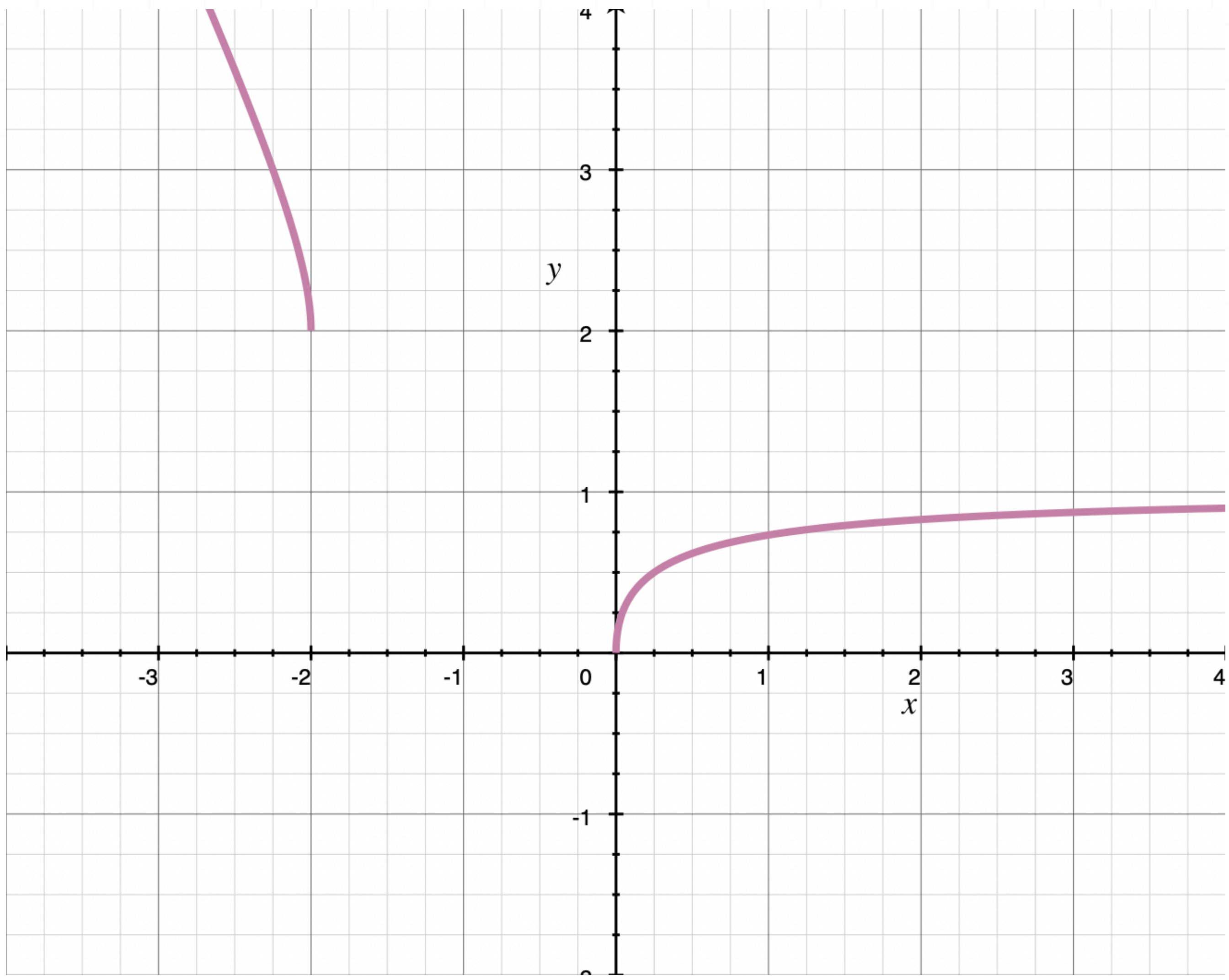
$$\lim_{x\to\infty}\frac{\frac{2x}{x}}{\frac{\sqrt{x^2+2x}+x}{x}}=\lim_{x\to\infty}\frac{2}{\frac{\sqrt{x^2+2x}}{x}+\frac{x}{x}}=\lim_{x\to\infty}\frac{2}{\frac{\sqrt{x^2+2x}}{\sqrt{x^2}}+1}=\lim_{x\to\infty}\frac{2}{\sqrt{\frac{x^2+2x}{x^2}}+1}$$

$$\lim_{x\to\infty}\frac{2}{\sqrt{\frac{x^2}{x^2}+\frac{2x}{x^2}}+1}=\lim_{x\to\infty}\frac{2}{\sqrt{1+\frac{2}{x}}+1}$$

Evaluating the limit gives

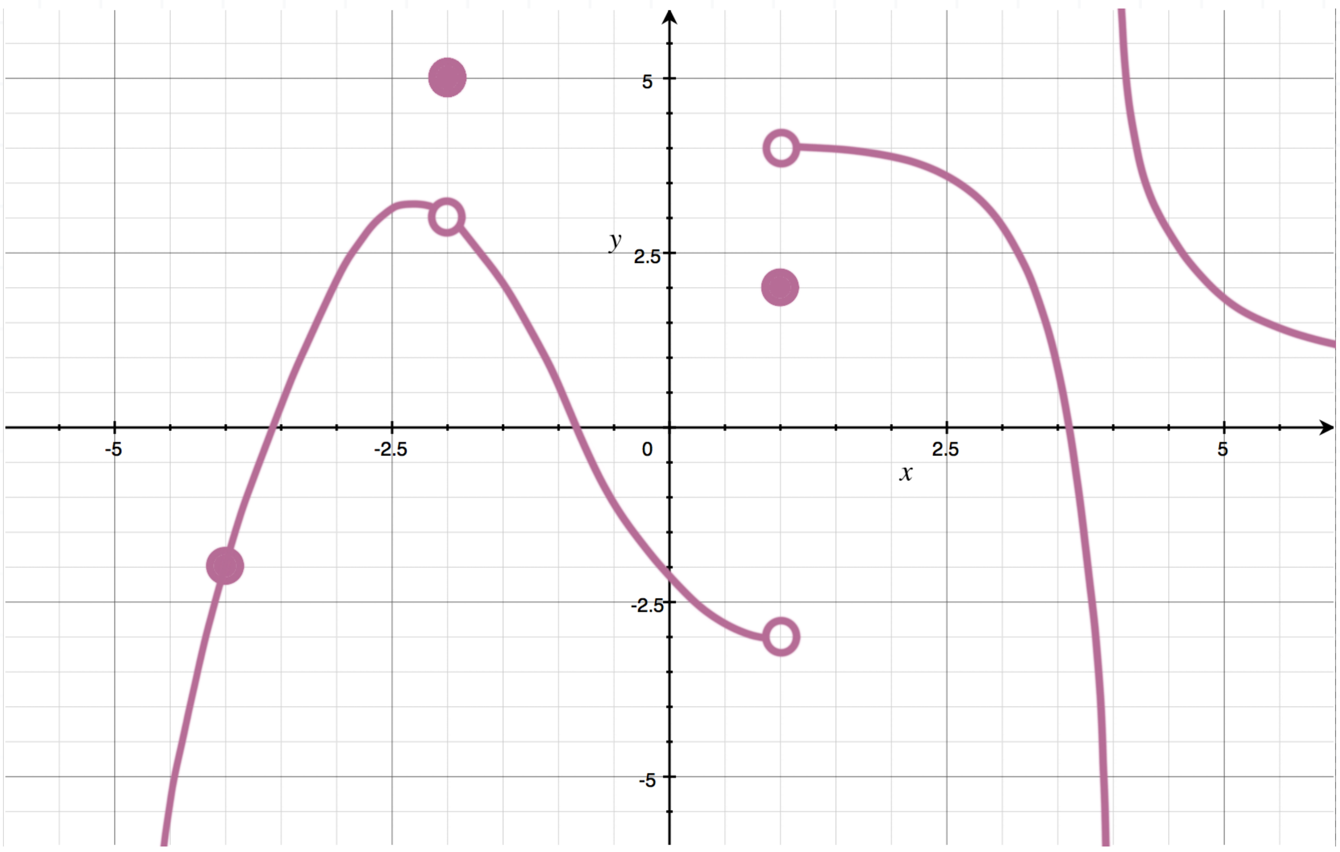
$$\frac{2}{\sqrt{1+0}+1}=\frac{2}{1+1}=\frac{2}{2}=1$$

The graph of the function shows this end behavior.



CRAZY GRAPHS

1. Use the graph to find the value of $\lim_{x \to 1} f(x)$.



Solution:

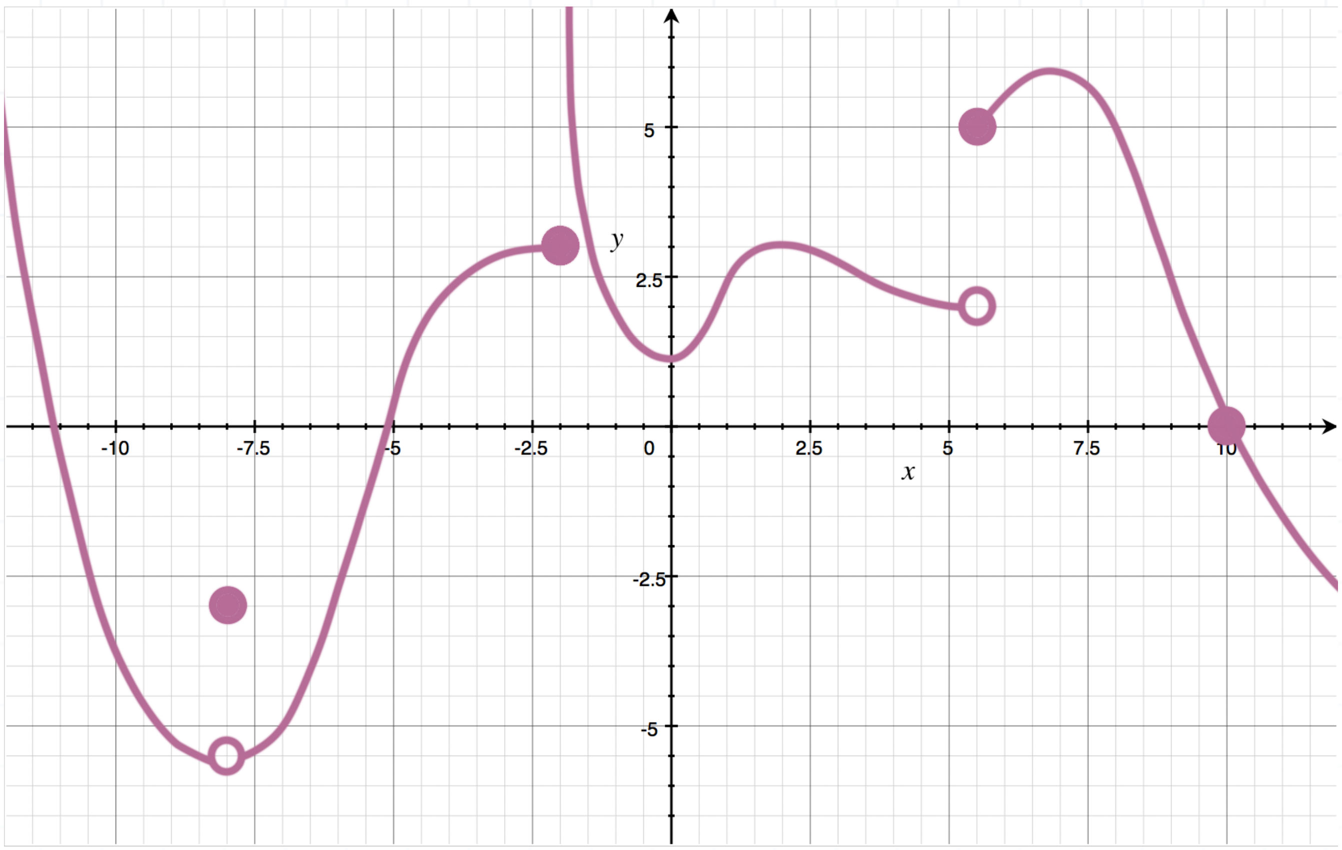
Notice in the graph that $f(x)$ has a jump discontinuity at $x = 1$.

$$\lim_{x \to 1^-} f(x) = -3$$

$$\lim_{x \to 1^+} f(x) = 4$$

Because these limits are unequal, the limit does not exist (DNE).

■ 2. Use the graph to find the value of $\lim_{x \to 5.5} g(x)$.



Solution:

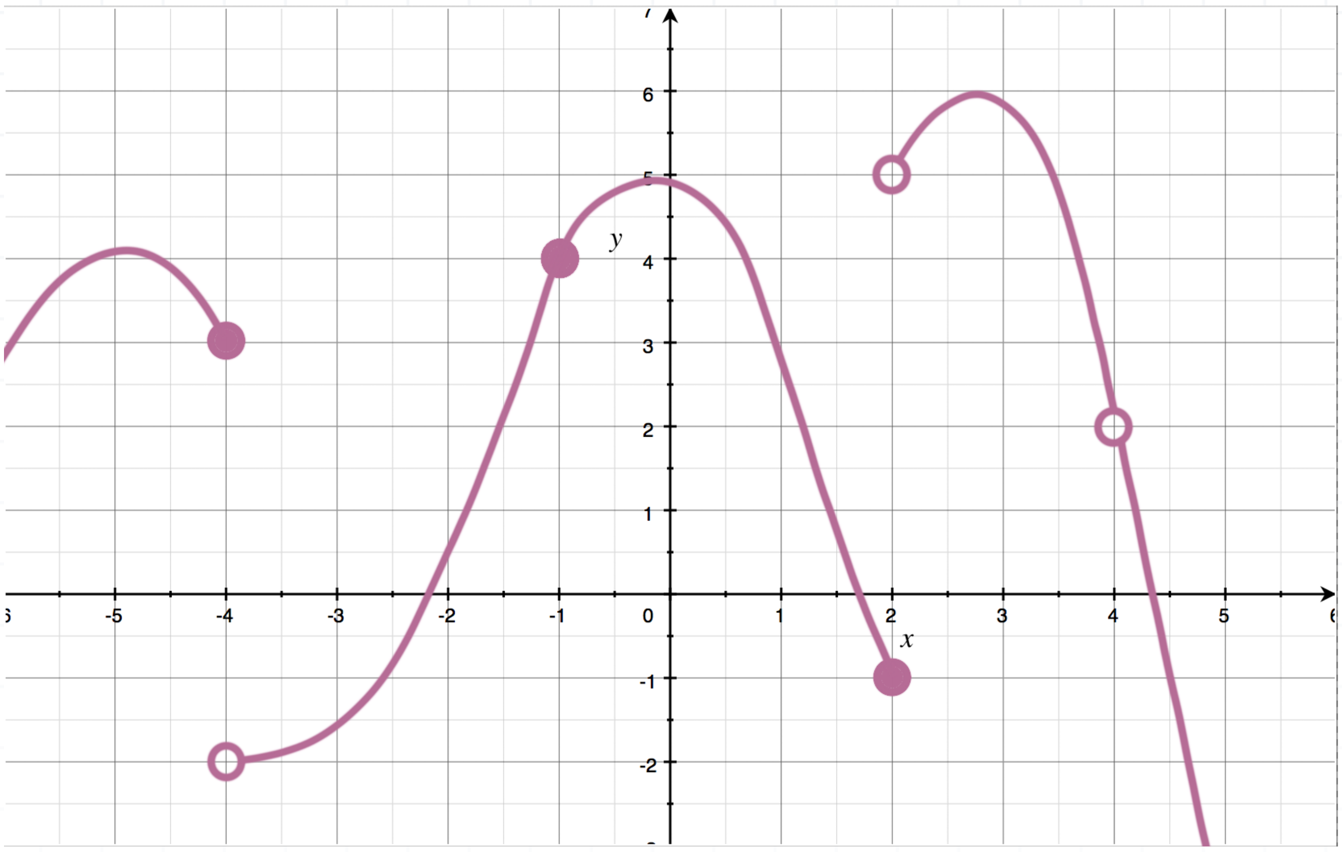
Notice in the graph that $g(x)$ has a jump discontinuity at $x = 5.5$.

$$\lim_{x \to 5.5^-} g(x) = 2$$

$$\lim_{x \to 5.5^+} g(x) = 5$$

Because these limits are unequal, the limit does not exist (DNE).

■ 3. Use the graph to find the value of $\lim_{x \to 4} h(x)$.



Solution:

Notice in the graph that $h(x)$ has a discontinuity at $x = 4$.

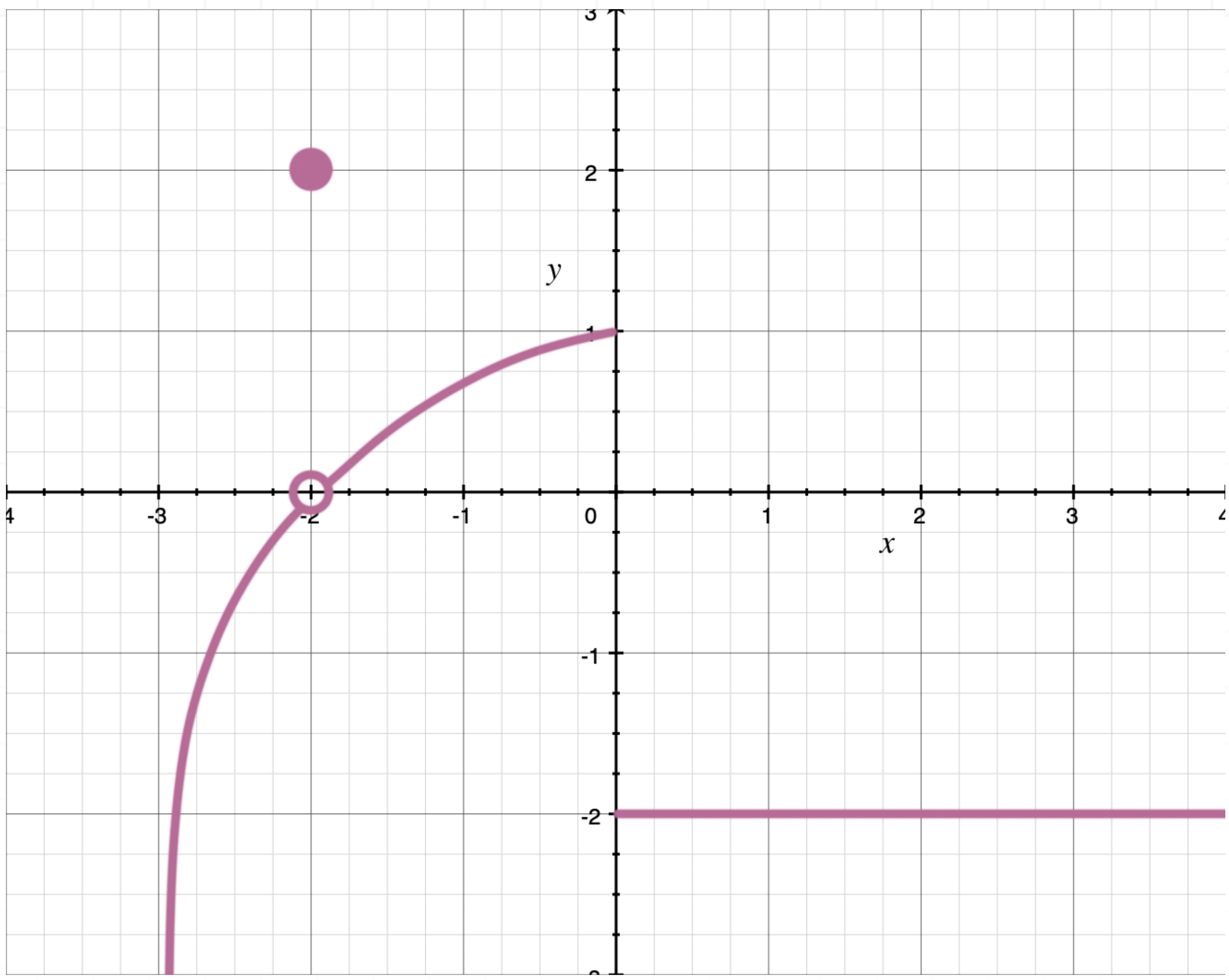
$$\lim_{x \to 4^-} h(x) = 2$$

$$\lim_{x \to 4^+} h(x) = 2$$

These limits are the same, which means

$$\lim_{x \to 4} h(x) = 2$$

■ 4. Use the graph to determine whether or not the limit exists at $x = 0$.



Solution:

At $x = 0$, the function is approaching 1 from the left side. But from the right side, the function is approaching -2. So if we say that the graph represents the function $f(x)$, then the one-sided limits are

$$\lim_{x\to 0^-} f(x) = 1$$

$$\lim_{x\to 0^+} f(x) = -2$$

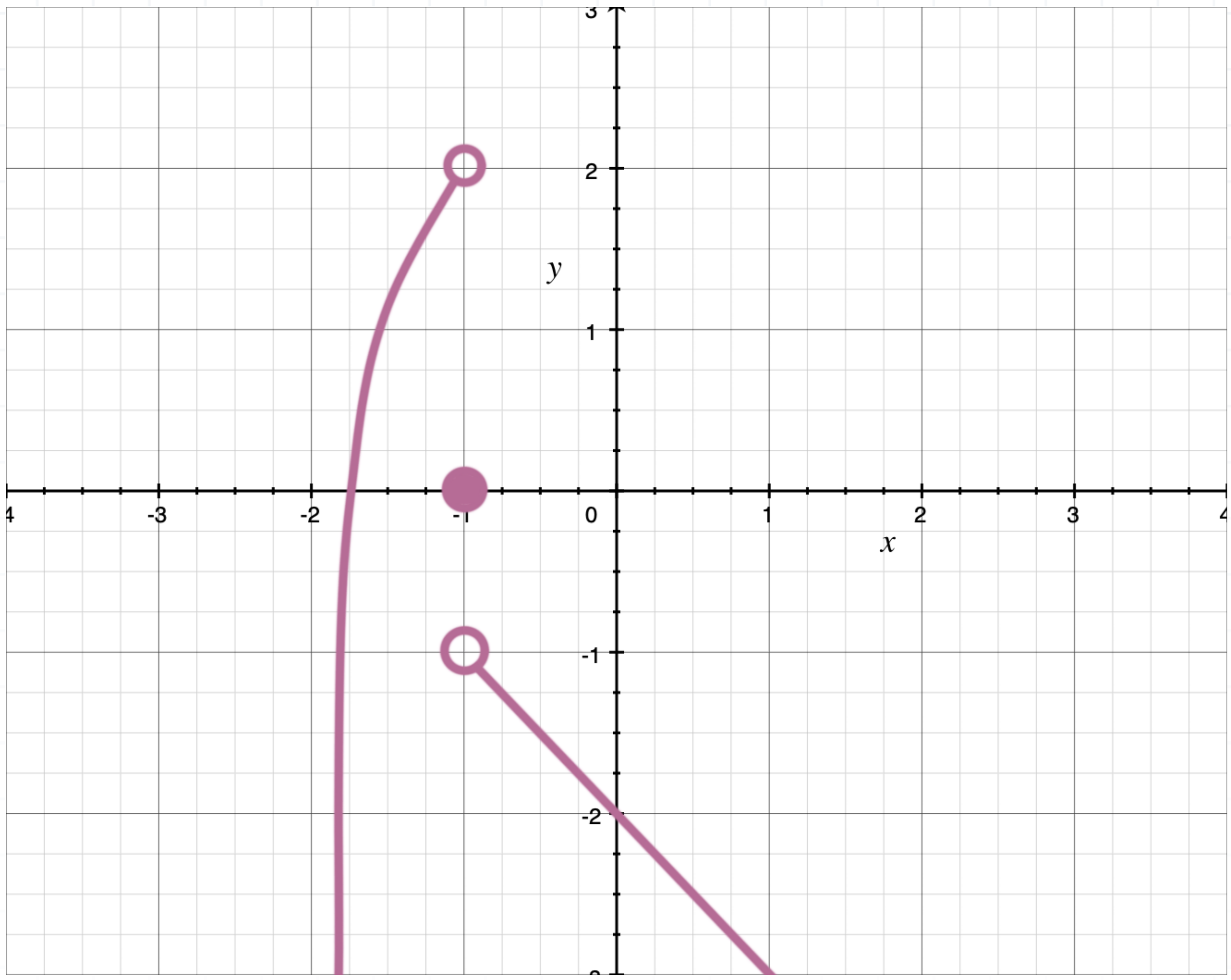
Because the left- and right-hand limits aren't equal, we've proven that the general limit of this function does not exist at $x = 0$.

■ 5. Sketch the graph of a function that satisfies each of the following conditions.

$$\lim_{x \to -1^-} f(x) = 2 \qquad \lim_{x \to -1^+} f(x) = -1 \qquad f(-1) = 0$$

Solution:

There are an infinite number of graphs that could satisfy these conditions, but the function's value at $x = -1$ is 0, so we plot a point at $(-1,0)$. Then we need another piece of the function that approaches 2 as x gets close to -1 from the negative side, and finally another piece of the function that approaches -1 as x gets close to -1 from the positive side.



6. Sketch the graph of a function that satisfies each of the following conditions.

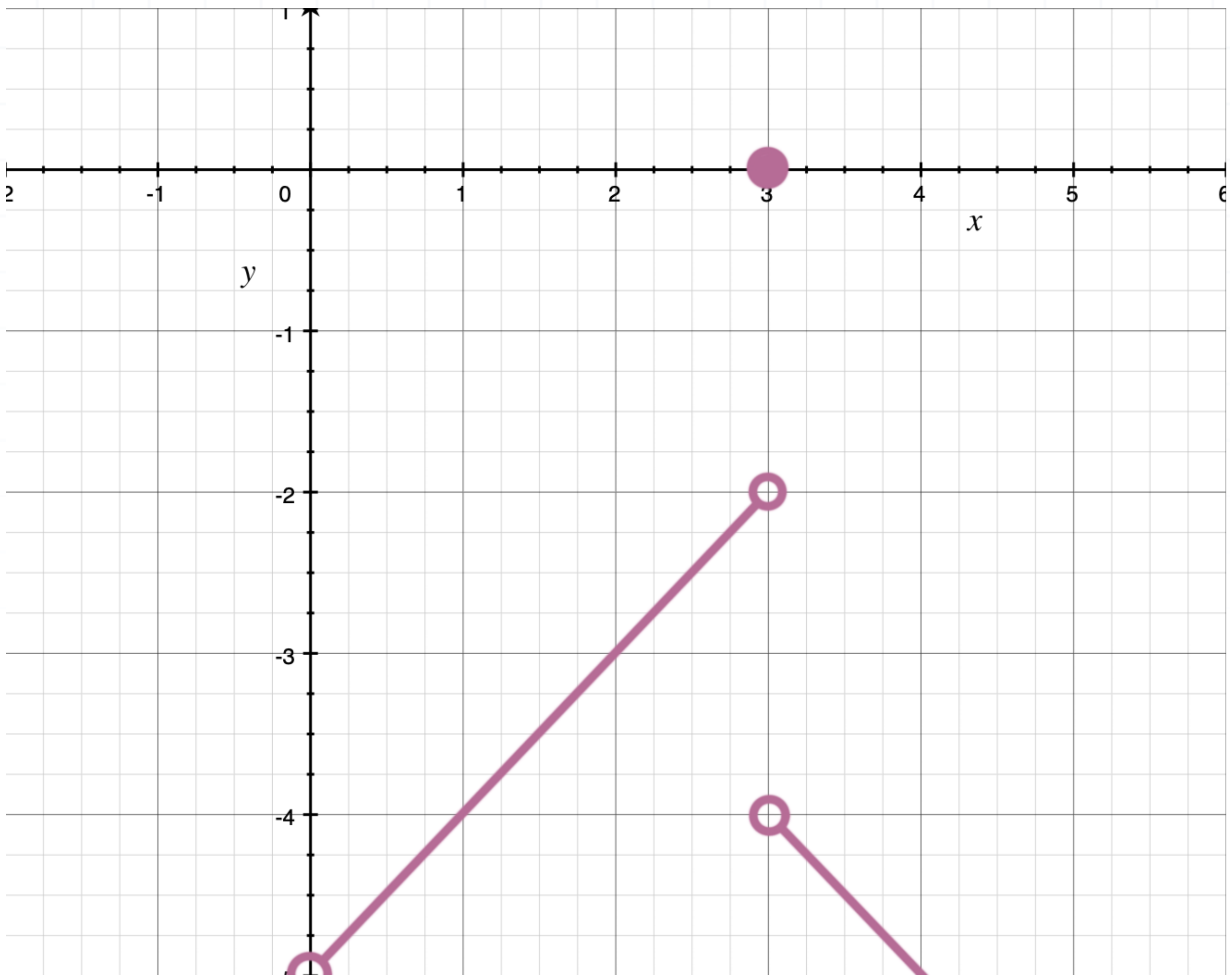
$$\lim_{x\to 0} f(x) = -5 \qquad f(0) \text{ does not exist}$$

$$\lim_{x\to 3^-} f(x) = -2 \qquad \lim_{x\to 3^+} f(x) = -4 \qquad f(3) = 0$$

Solution:

There are an infinite number of graphs that could satisfy these conditions, but the function's value at $x = 3$ is 0, so we plot a point at $(3,0)$. Then we need another piece of the function that approaches -2 as x gets close to 3 from the negative side, and then another piece of the function that approaches -4 as x gets close to 3 from the positive side.

Because the function does not exist when $x = 0$, we'll add a point discontinuity there, and then make sure that the function's limit is -5 as x gets close to 0.



TRIGONOMETRIC LIMITS

■ 1. Find $\lim_{x \to \pi} f(x)$ if $f(x) = 3\cos x - 2$.

Solution:

The one-sided limits are

$$\lim_{x \to \pi^-} 3\cos x - 2 = -5$$

$$\lim_{x \to \pi^+} 3\cos x - 2 = -5$$

Therefore, $\lim_{x \to \pi} f(x) = -5$.

■ 2. Evaluate the limit.

$$\lim_{x \to 0} \frac{\sin(8x)}{x}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{\sin(8 \cdot 0)}{0}$$

$$\frac{0}{0}$$

But if we rewrite the limit as

$$\lim_{x \to 0} \frac{\sin 8x}{x} = \lim_{x \to 0} \frac{\sin 8x}{x\frac{8}{8}} = \lim_{x \to 0} \frac{\sin 8x}{8x \cdot \frac{1}{8}} = \lim_{x \to 0} \frac{\sin 8x}{8x} \cdot 8 = 8 \lim_{x \to 0} \frac{\sin 8x}{8x}$$

and we know the value of the trig limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

then we can evaluate the limit as

$$8 \cdot 1$$

$$8$$

■ 3. Evaluate the limit.

$$\lim_{x \to \frac{\pi}{4}} \frac{1 - \cot x}{\cos x - \sin x}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{1-\cot\frac{\pi}{4}}{\cos\frac{\pi}{4}-\sin\frac{\pi}{4}}$$

$$\frac{1-1}{\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}$$

$$\frac{0}{0}$$

We can rewrite the limit as

$$\lim_{x\to\frac{\pi}{4}}\frac{1-\cot x}{\cos x-\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{1-\frac{\cos x}{\sin x}}{\cos x-\sin x}=\lim_{x\to\frac{\pi}{4}}\frac{\frac{\sin x-\cos x}{\sin x}}{\cos x-\sin x}$$

$$\lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{(\cos x-\sin x)\sin x}=\lim_{x\to\frac{\pi}{4}}\left(-\frac{\cos x-\sin x}{(\cos x-\sin x)\sin x}\right)=\lim_{x\to\frac{\pi}{4}}\left(-\frac{1}{\sin x}\right)$$

Now we can evaluate the limit using direct substitution.

$$-\frac{1}{\sin\frac{\pi}{4}}$$

$$-\frac{1}{\frac{\sqrt{2}}{2}}$$

$$-\sqrt{2}$$

■ 4. Evaluate the limit.

$$\lim_{x\to 0}\frac{\tan(4x)}{\sin(2x)}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{\tan(0)}{\sin(0)}$$

$$\frac{0}{0}$$

We can rewrite the limit as

$$\lim_{x\to 0}\frac{\tan(4x)}{\sin(2x)}=\lim_{x\to 0}\frac{4\frac{\tan(4x)}{4x}}{2\frac{\sin(2x)}{2x}}=2\lim_{x\to 0}\frac{\frac{\tan(4x)}{4x}}{\frac{\sin(2x)}{2x}}$$

We know the value of the trig limit

$$\lim_{x\to 0}\frac{\sin x}{x}=1$$

Therefore,

$$2\lim_{x\to 0}\frac{\frac{\tan(4x)}{4x}}{\frac{\sin(2x)}{2x}}=2\lim_{x\to 0}\frac{\frac{\tan(4x)}{4x}}{1}=2\lim_{x\to 0}\frac{\tan(4x)}{4x}$$

$$2\lim_{x\to 0}\frac{\tan(4x)}{4x} = 2\lim_{x\to 0}\frac{\frac{\sin(4x)}{\cos(4x)}}{4x} = 2\lim_{x\to 0}\frac{\sin(4x)}{4x\cos(4x)} = 2\lim_{x\to 0}\frac{\sin(4x)}{4x}\cdot\lim_{x\to 0}\frac{1}{\cos(4x)}$$

We know the value of the trig limit

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

We get

$$2\cdot 1\cdot\lim_{x\to 0}\frac{1}{\cos(4x)}$$

Now we can evaluate the limit using direct substitution.

$$2\cdot 1\cdot\frac{1}{\cos(4\cdot 0)}$$

$$2\cdot 1\cdot\frac{1}{1}$$

$$2$$

■ 5. Evaluate the limit.

$$\lim_{x\to 0}\frac{x}{\sin\frac{x}{3}}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\lim_{x\to 0} \frac{0}{\sin 0}$$

$$\frac{0}{0}$$

We can rewrite the limit as

$$\lim_{x\to 0} \frac{x}{\sin \frac{x}{3}} = \lim_{x\to 0} \frac{\frac{x}{x}}{\frac{\sin \frac{x}{3}}{3\frac{x}{3}}} = 3\lim_{x\to 0} \frac{1}{\frac{\sin \frac{x}{3}}{\frac{x}{3}}}$$

We know the value of the trig limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

We get

$$3 \cdot \frac{1}{1}$$

$$3$$

■ 6. Evaluate the limit.

$$\lim_{x\to 0} \frac{\sin^2 x}{1 - \cos x}$$

Solution:

If we use direct substitution to evaluate the limit, we get the indeterminate form $0/0$.

$$\frac{\sin^2 0}{1-\cos 0}$$

$$\frac{0}{1-1}$$

$$\frac{0}{0}$$

Substitution doesn't work, and there's nothing to factor, but since we have exactly two terms in the numerator, we can actually use the conjugate method for the first step of this problem.

$$\lim_{x\to 0}\frac{\sin^2 x}{1-\cos x}\left(\frac{1+\cos x}{1+\cos x}\right)$$

$$\lim_{x\to 0}\frac{\sin^2 x(1+\cos x)}{1-\cos^2 x}$$

Applying the Pythagorean identity $1-\cos^2 x=\sin^2 x$ to the denominator gives

$$\lim_{x\to 0}\frac{\sin^2 x(1+\cos x)}{\sin^2 x}$$

$$\lim_{x\to 0}(1+\cos x)$$

Now we can evaluate the limit using direct substitution.

$$1 + \cos 0$$

$$1 + 1$$

$$2$$

MAKING THE FUNCTION CONTINUOUS

■ 1. What value of c makes the function $h(x)$ continuous if c is a constant?

$$h(x) = \begin{cases} x^2 & x \le 4 \\ 3x + c & x > 4 \end{cases}$$

Solution:

Since $h(x)$ is defined as a piecewise function, $x^2 = 3x + c$ at $x = 4$.

$$4^2 = 3(4) + c$$

$$16 = 12 + c$$

$$c = 4$$

■ 2. What value of k makes the function $f(x)$ continuous if k is a constant?

$$f(x) = \begin{cases} kx^2 - 2x + 1 & x \le 3 \\ kx + 1 & x > 3 \end{cases}$$

Solution:

Since $f(x)$ is defined as a piecewise function, the function will be continuous at $x = 3$ when $kx^2 - 2x + 1 = kx + 1$.

$$kx^2 - 2x + 1 = kx + 1$$

$$k(3)^2 - 2(3) + 1 = k(3) + 1$$

$$9k - 6 + 1 = 3k + 1$$

$$6k = 6$$

$$k = 1$$

■ 3. What values of a and b make the function $g(x)$ continuous if a and b are constant?

$$g(x) = \begin{cases} 3 & x \leq -2 \\ ax - b & -2 < x < 2 \\ -2 & x \geq 2 \end{cases}$$

Solution:

Since $g(x)$ is defined as a piecewise function, the function will be continuous at $x = -2$ when $3 = ax - b$, and continuous at $x = 2$ when $ax - b = -2$. So we need to solve these two equations as a system.

$$3 = -2a - b$$

$$2a - b = -2$$

Add the equations.

$$3 + (-2) = -2a - b + (2a - b)$$

$$3 - 2 = -2a - b + 2a - b$$

$$-2b = 1$$

$$b = -\frac{1}{2}$$

Substitute $b = -1/2$ into the first equation to solve for a.

$$3 = -2a - \left(-\frac{1}{2}\right)$$

$$\frac{5}{2} = -2a$$

$$a = -\frac{5}{4}$$

So the function is continuous when $a = -5/4$ and $b = -1/2$.

■ 4. What value of c makes the function $f(x)$ continuous if c is a constant?

$$f(x) = \begin{cases} 2x^3 - 6x^2 + 8x + 3 & x \le 1 \\ cx + 9 & x > 1 \end{cases}$$

Solution:

Since $f(x)$ is defined as a piecewise function, $2x^3 - 6x^2 + 8x + 3 = cx + 9$ at $x = 1$.

$$2(1)^3 - 6(1)^2 + 8(1) + 3 = c(1) + 9$$

$$2 - 6 + 8 + 3 = c + 9$$

$$7 = c + 9$$

$$c = -2$$

■ 5. What value of c makes the function $g(x)$ continuous if c is a constant?

$$g(x) = \begin{cases} \sqrt{x} + 18 & x \le 16 \\ x - 2c & x > 16 \end{cases}$$

Solution:

Since $g(x)$ is defined as a piecewise function, the function will be continuous at $x = 16$ when $\sqrt{x} + 18 = x - 2c$.

$$\sqrt{16} + 18 = 16 - 2c$$

$$4 + 18 = 16 - 2c$$

$$22 = 16 - 2c$$

$$6 = -2c$$

$$c = -3$$

6. What values of a and b make the function $h(x)$ continuous if a and b are constant?

$$h(x) = \begin{cases} ax^2 & x \leq -1 \\ ax + b & -1 < x < 3 \\ bx + 2 & x \geq 3 \end{cases}$$

Solution:

Since $h(x)$ is defined as a piecewise function, the function will be continuous at $x = -1$ when $ax^2 = ax + b$, and continuous at $x = 3$ when $ax + b = bx + 2$. So we need to solve these two equations as a system.

$$a = -a + b$$

$$3a + b = 3b + 2$$

Simplify the system.

$$2a = b$$

$$3a - 2b = 2$$

Substitute the first equation into the second equation to solve for a.

$$3a - 2(2a) = 2$$

$$3a - 4a = 2$$

$$-a = 2$$

$$a = -2$$

Then, $b = 2(-2) = -4$. So the function is continuous when $a = -2$ and $b = -4$.

SQUEEZE THEOREM

1. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} \left(x^2 \sin\left(\frac{1}{x}\right) - 2 \right)$$

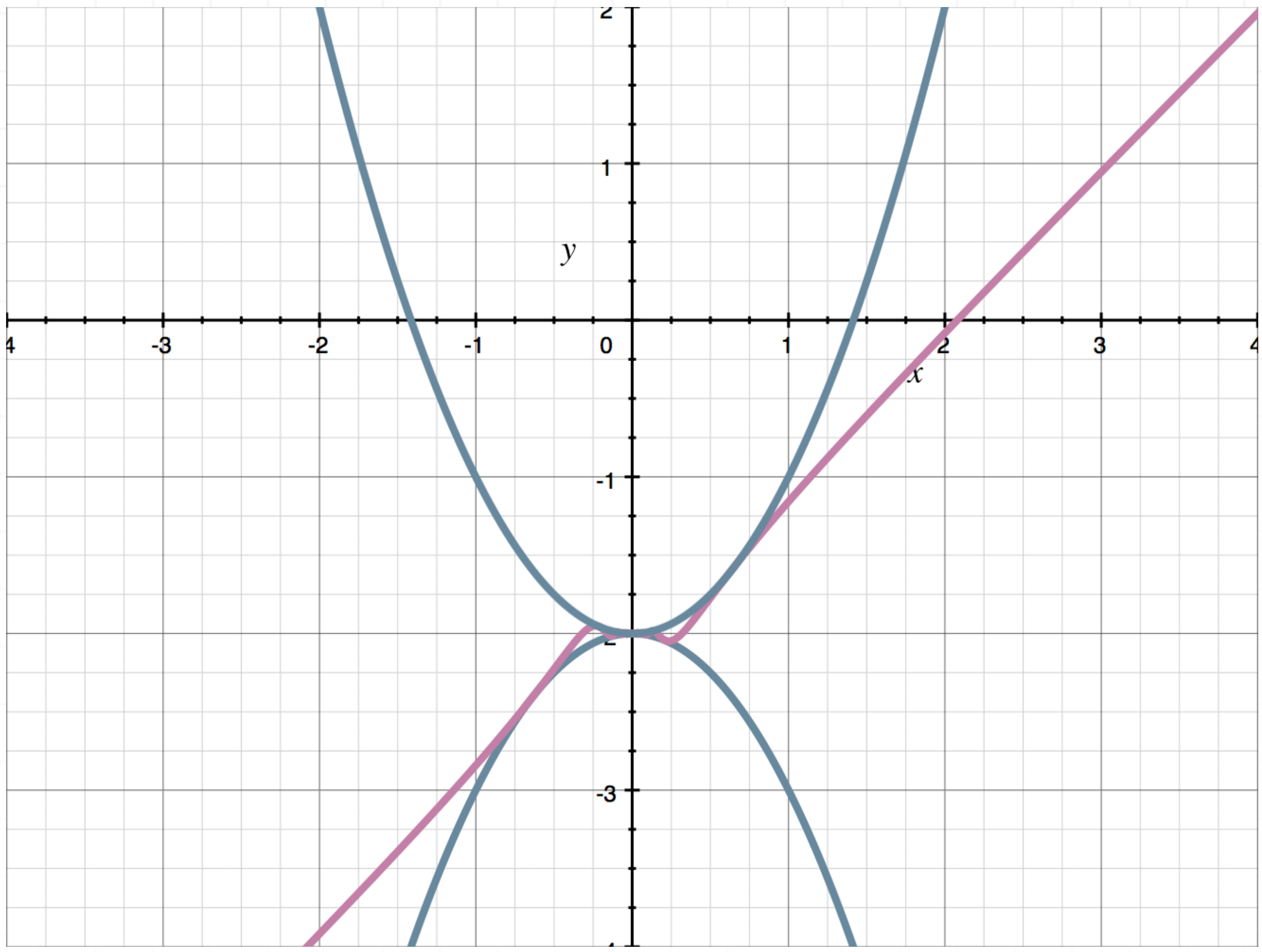
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 - 2$$

$$g(x) = x^2 \sin\left(\frac{1}{x}\right) - 2$$

$$h(x) = x^2 - 2$$



Notice that $f(x) \le g(x) \le h(x)$. Therefore,

$$\lim_{x \to 0} f(x) \le \lim_{x \to 0} g(x) \le \lim_{x \to 0} h(x)$$

$$\lim_{x \to 0}(-x^2 - 2) \le \lim_{x \to 0}\left(x^2 \sin\left(\frac{1}{x}\right) - 2\right) \le \lim_{x \to 0}(x^2 - 2)$$

We can evaluate the limits on the left and right sides.

$$-0^2 - 2 \le \lim_{x \to 0}\left(x^2 \sin\left(\frac{1}{x}\right) - 2\right) \le 0^2 - 2$$

$$-2 \le \lim_{x \to 0}\left(x^2 \sin\left(\frac{1}{x}\right) - 2\right) \le -2$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be -2.

2. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{3 \sin x}{4x}$$

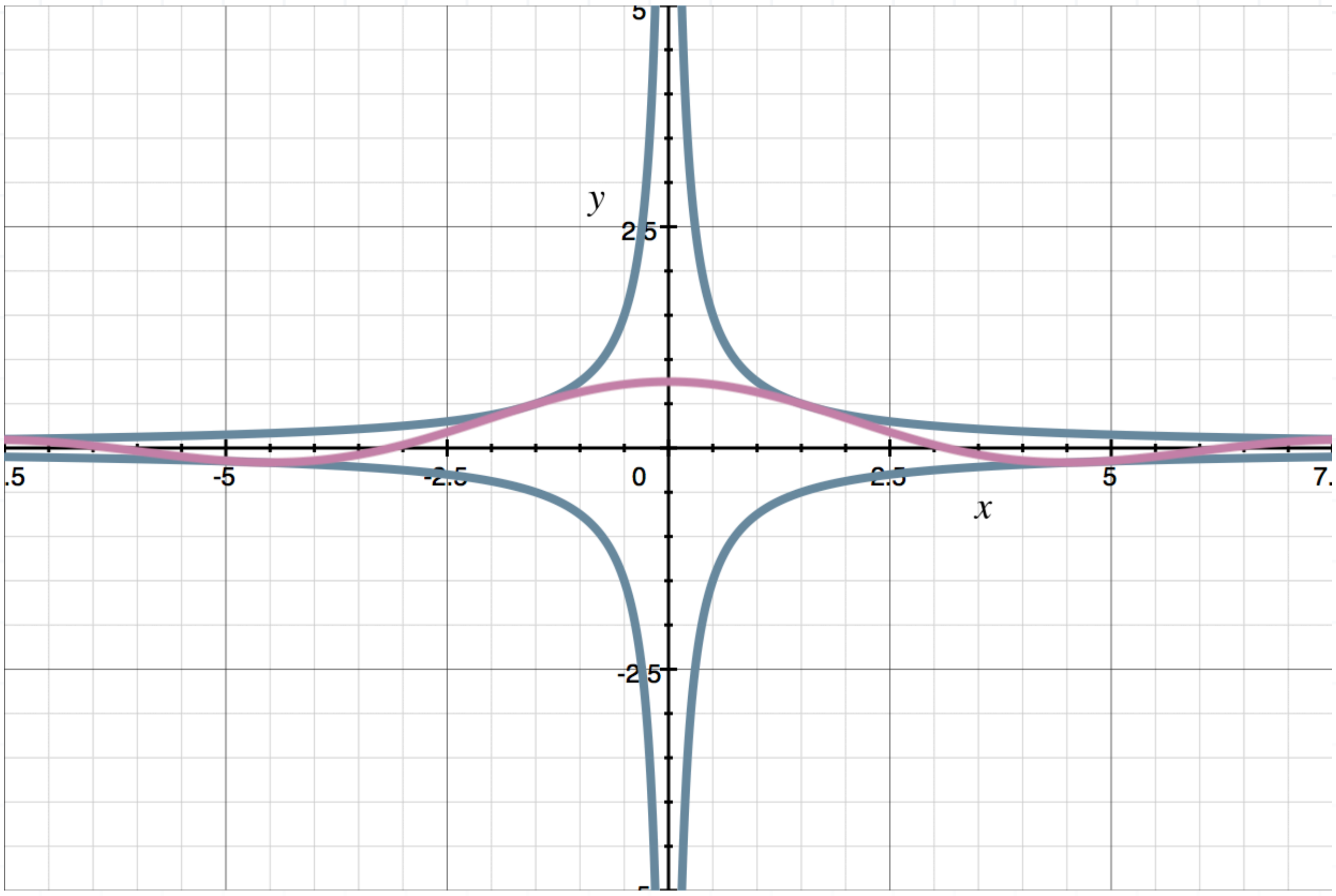
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -\frac{3}{4x}$$

$$g(x) = \frac{3 \sin x}{4x}$$

$$h(x) = \frac{3}{4x}$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x\to\infty} f(x) \leq \lim_{x\to\infty} g(x) \leq \lim_{x\to\infty} h(x)$$

$$\lim_{x\to\infty}\left(-\frac{3}{4x}\right) \leq \lim_{x\to\infty}\frac{3\sin x}{4x} \leq \lim_{x\to\infty}\left(\frac{3}{4x}\right)$$

We can evaluate the limits on the left and right sides.

$$0 \leq \lim_{x\to\infty}\frac{3\sin x}{4x} \leq 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 3. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to 0}\left(x^2\cos\left(\frac{1}{x^2}\right)+1\right)$$

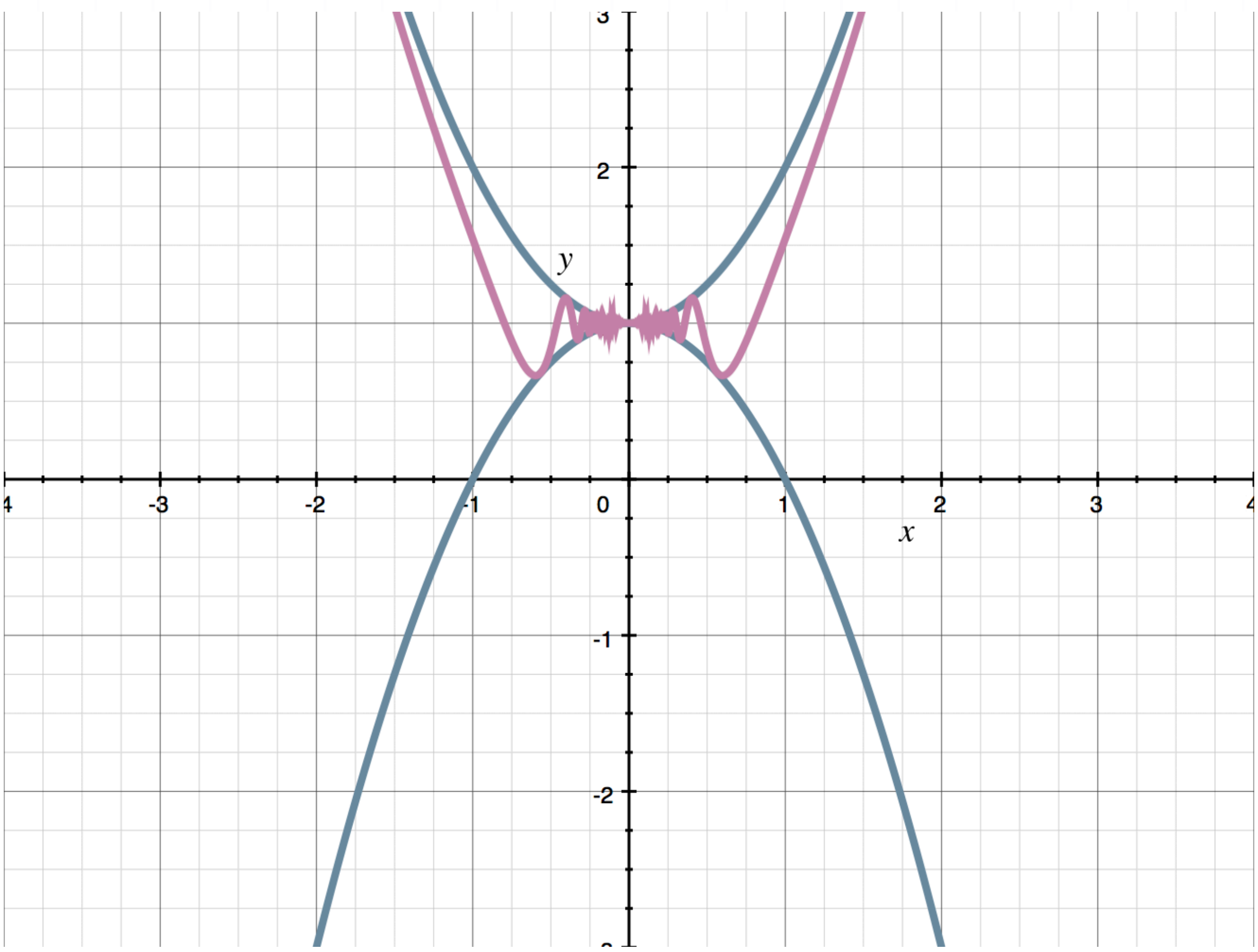
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -x^2 + 1$$

$$g(x) = x^2\cos\left(\frac{1}{x^2}\right) + 1$$

$$h(x) = x^2 + 1$$



Notice that $f(x) \le g(x) \le h(x)$. Therefore,

$$\lim_{x\to0} f(x) \le \lim_{x\to0} g(x) \le \lim_{x\to0} h(x)$$

$$\lim_{x\to0} -x^2+1 \le \lim_{x\to0}\left(x^2\cos\left(\frac{1}{x^2}\right)+1\right) \le \lim_{x\to0} x^2+1$$

We can evaluate the limits on the left and right sides.

$$-0^2+1 \le \lim_{x\to0}\left(x^2\cos\left(\frac{1}{x^2}\right)+1\right) \le 0^2+1$$

$$1 \le \lim_{x\to0}\left(x^2\cos\left(\frac{1}{x^2}\right)+1\right) \le 1$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 1.

■ 4. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to\infty}\frac{e^{-x}}{x}$$

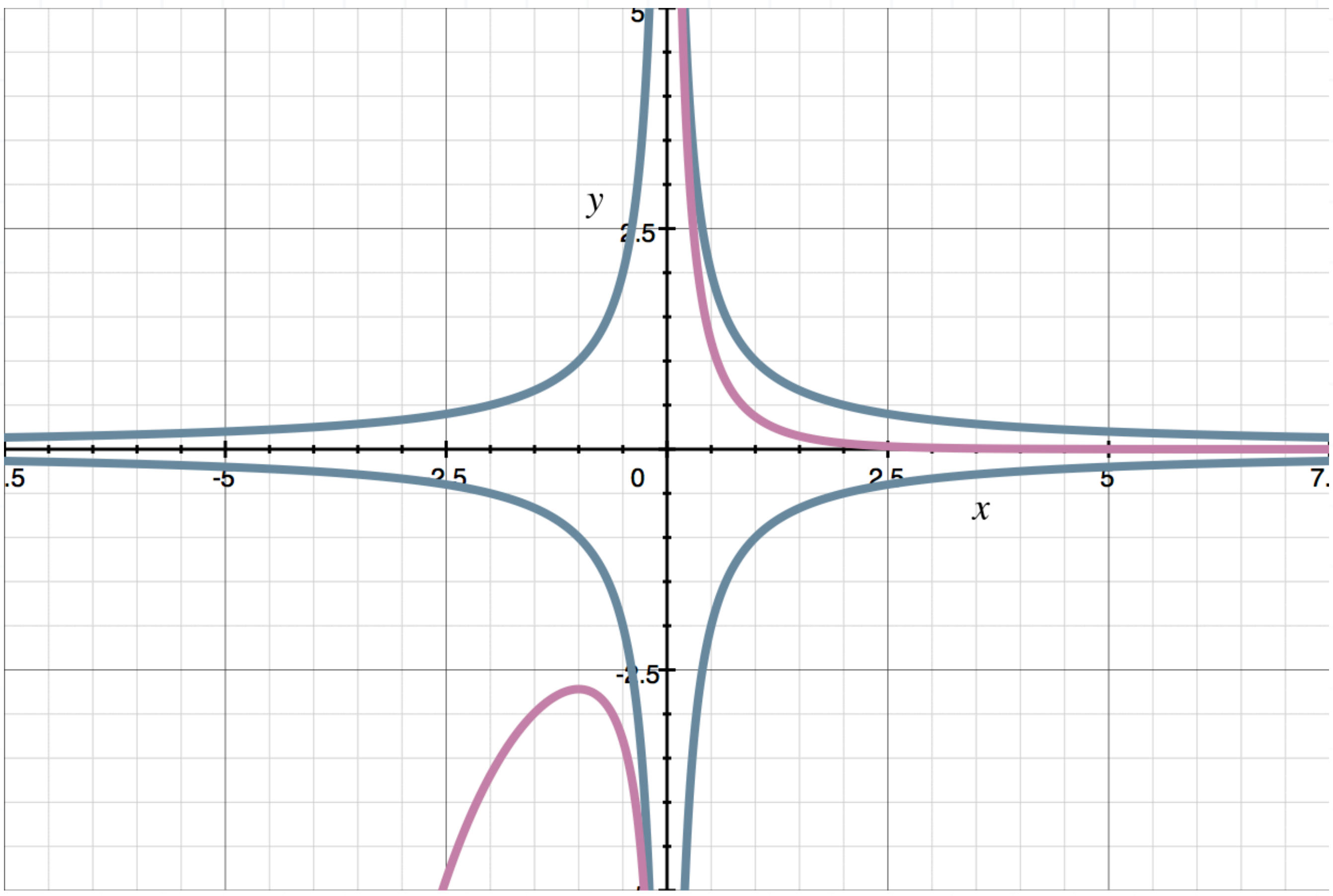
Solution:

Consider the graphs of the three functions shown below.

$$f(x) = -\frac{1}{x}$$

$$g(x) = \frac{e^{-x}}{x}$$

$$h(x) = \frac{1}{x}$$



Notice that $f(x) \leq g(x) \leq h(x)$. Therefore,

$$\lim_{x \to \infty} f(x) \leq \lim_{x \to \infty} g(x) \leq \lim_{x \to \infty} h(x)$$

$$\lim_{x \to \infty} \left(-\frac{1}{x}\right) \leq \lim_{x \to \infty} \frac{e^{-x}}{x} \leq \lim_{x \to \infty} \left(\frac{1}{x}\right)$$

We can evaluate the limits on the left and right sides.

$$0 \leq \lim_{x \to \infty} \frac{e^{-x}}{x} \leq 0$$

Therefore, by the Squeeze Theorem, we know that the value of the limit must be 0.

■ 5. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{x^2 + x \sin \sqrt{x}}{4x^2 + 7}$$

Solution:

We know that the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \leq \sin \sqrt{x} \leq 1$$

Multiply each part of the inequality by x.

$$-x \leq x \sin \sqrt{x} \leq x$$

Add x^2 to each part of the inequality.

$$x^2 - x \leq x^2 + x \sin \sqrt{x} \leq x + x^2$$

Divide through the inequality by $4x^2 + 7$ to get the function at the center of the inequality to match the one we were given.

$$\frac{x^2 - x}{4x^2 + 7} \leq \frac{x^2 + x \sin \sqrt{x}}{4x^2 + 7} \leq \frac{x + x^2}{4x^2 + 7}$$

Apply the limit throughout the inequality.

$$\lim_{x\to\infty}\frac{x^2-x}{4x^2+7}\le\lim_{x\to\infty}\frac{x^2+x\sin\sqrt{x}}{4x^2+7}\le\lim_{x\to\infty}\frac{x+x^2}{4x^2+7}$$

$$\frac{1}{4}\le\lim_{x\to\infty}\frac{x^2+x\sin\sqrt{x}}{4x^2+7}\le\frac{1}{4}$$

Because we were able to squeeze the limit between the same $1/4$ value, the value of the limit is

$$\lim_{x\to\infty}\frac{x^2+x\sin\sqrt{x}}{4x^2+7}=\frac{1}{4}$$

■ 6. Use the Squeeze Theorem to evaluate the limit.

$$\lim_{x\to\infty}\frac{\sin x}{x}$$

Solution:

We know that the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1\le\sin x\le 1$$

Divide through the inequality by x.

$$-\frac{1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$$

Apply the limit throughout the inequality.

$$\lim_{x\to\infty} -\frac{1}{x} \le \lim_{x\to\infty} \frac{\sin x}{x} \le \lim_{x\to\infty} \frac{1}{x}$$

$$0 \le \lim_{x\to\infty} \frac{\sin x}{x} \le 0$$

Because we were able to squeeze the limit between the same 0 value, the value of the limit is

$$\lim_{x\to\infty} \frac{\sin x}{x} = 0$$

DEFINITION OF THE DERIVATIVE

■ 1. Use the definition of the derivative to find the derivative of $f(x)=2x^2+2x-12$ at $(4,28)$.

Solution:

At $(4,28)$, the derivative will be

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{f(4+h)-f(4)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{(2(4+h)^2+2(4+h)-12)-(2(4)^2+2(4)-12)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{(2(16+8h+h^2)+8+2h-12)-(32+8-12)}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{32+16h+2h^2+8+2h-12-32-8+12}{h}$$

$$f'(4)=\lim_{h\to 0}\frac{18h+2h^2}{h}$$

$$f'(4)=\lim_{h\to 0}(18+2h)$$

Evaluate the limit to find the derivative at $(4,28)$.

$$f'(4) = 18 + 2(0)$$

$$f'(4) = 18$$

■ 2. Use the definition of the derivative to find the derivative of $g(x) = 3x^3 - 4x + 7$ at $(-2, -9)$.

Solution:

At $(-2, -9)$, the derivative will be

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$g'(-2) = \lim_{h \to 0} \frac{g(-2+h) - g(-2)}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{(3(-2+h)^3 - 4(-2+h) + 7) - (3(-2)^3 - 4(-2) + 7)}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{(3(-8 + 4h + 8h - 4h^2 - 2h^2 + h^3) + 8 - 4h + 7) - (-24 + 8 + 7)}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{-24 + 12h + 24h - 12h^2 - 6h^2 + 3h^3 + 8 - 4h + 7 + 24 - 8 - 7}{h}$$

$$g'(-2) = \lim_{x \to 0} \frac{3h^3 - 18h^2 + 32h}{h}$$

$$g'(-2) = \lim_{x \to 0} (3h^2 - 18h + 32)$$

Evaluate the limit to find the derivative at $(-2,-9)$.

$$g'(-2)=3(0)^2-18(0)+32$$

$$g'(-2)=32$$

■ 3. Use the definition of the derivative to find the derivative at $(-1,-1)$.

$$f(x)=\frac{x}{x+2}$$

Solution:

At $(-1,-1)$, the derivative will be

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f'(-1)=\lim_{h\to 0}\frac{f(-1+h)-f(-1)}{h}$$

$$f'(-1)=\lim_{h\to 0}\frac{\frac{-1+h}{-1+h+2}-\frac{-1}{-1+2}}{h}$$

$$f'(-1)=\lim_{h\to 0}\frac{\frac{-1+h}{h+1}-\frac{-1}{1}}{h}$$

$$f'(-1)=\lim_{h\to 0}\frac{\frac{-1+h}{h+1}+1}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{\frac{-1+h+h+1}{h+1}}{h}$$

$$f'(-1) = \lim_{h \to 0} \frac{2h}{h(h+1)}$$

$$f'(-1) = \lim_{h \to 0} \frac{2}{h+1}$$

Evaluate the limit to find the derivative at $(-1, -1)$.

$$f'(-1) = \frac{2}{0+1}$$

$$f'(-1) = 2$$

■ 4. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{5x-4}$ at $x = 4$.

Solution:

At $x = 4$, the derivative will be

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{f(4+h) - f(4)}{h}$$

$$f'(4) = \lim_{h \to 0} \frac{\sqrt{5(4+h)-4} - \sqrt{5(4)-4}}{h}$$

$$f'(4) = \lim_{h\to 0} \frac{\sqrt{20 + 5h - 4} - \sqrt{20 - 4}}{h}$$

$$f'(4) = \lim_{h\to 0} \frac{\sqrt{16 + 5h} - \sqrt{16}}{h}$$

$$f'(4) = \lim_{h\to 0} \frac{\sqrt{16 + 5h} - 4}{h}$$

Use conjugate method.

$$f'(4) = \lim_{h\to 0} \frac{\sqrt{16 + 5h} - 4}{h} \cdot \frac{\sqrt{16 + 5h} + 4}{\sqrt{16 + 5h} + 4}$$

$$f'(4) = \lim_{h\to 0} \frac{16 + 5h - 16}{h(\sqrt{16 + 5h} + 4)}$$

$$f'(4) = \lim_{h\to 0} \frac{5h}{h(\sqrt{16 + 5h} + 4)}$$

$$f'(4) = \lim_{h\to 0} \frac{5}{\sqrt{16 + 5h} + 4}$$

Evaluate the limit to find the derivative at $x = 4$.

$$f'(4) = \frac{5}{\sqrt{16 + 5(0)} + 4}$$

$$f'(4) = \frac{5}{4 + 4}$$

$$f'(4) = \frac{5}{8}$$

■ 5. Use the definition of the derivative to find the derivative of $g(x) = \cos(x - 1)$ at $x = \pi/2$.

Solution:

At $x = \pi/2$, the derivative will be

$$g'(x) = \lim_{h \to 0} \frac{g(x + h) - g(x)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{\cos(x + h - 1) - \cos(x - 1)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{\cos(x - 1 + h) - \cos(x - 1)}{h}$$

Apply the sum-identity for cosine.

$$g'(x) = \lim_{h \to 0} \frac{\cos(x - 1)\cos h - \sin(x - 1)\sin h - \cos(x - 1)}{h}$$

$$g'(x) = \lim_{h \to 0} \frac{\cos(x - 1)(\cos h - 1) - \sin(x - 1)\sin h}{h}$$

$$g'(x) = \lim_{h \to 0} \left(\frac{\cos(x - 1)(\cos h - 1)}{h} - \frac{\sin(x - 1)\sin h}{h} \right)$$

Pull the expressions with x out in front of the limits, which only apply to h.

$$g'(x) = \cos(x-1)\lim_{h \to 0}\frac{\cos h - 1}{h} - \sin(x-1)\lim_{h \to 0}\frac{\sin h}{h}$$

Applying formulas for the limits of these common trigonometric functions, we get

$$g'(x) = \cos(x-1)(0) - \sin(x-1)(1)$$

$$g'(x) = -\sin(x-1)$$

Evaluate the limit to find the derivative at $x = \pi/2$.

$$g'\left(\frac{\pi}{2}\right) = -\sin\left(\frac{\pi}{2} - 1\right)$$

If we choose, we could apply the cofunction identity for cosine,

$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$$

Using that identity allows us to simplify the answer to

$$g'\left(\frac{\pi}{2}\right) = -\cos(1)$$

■ 6. Use the definition of the derivative to find the derivative of $g(x) = |x|$ at $x = 0$.

Solution:

At $x = 0$, the derivative will be

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$g'(0) = \lim_{h \to 0} \frac{g(0+h) - g(0)}{h}$$

$$g'(0) = \lim_{h \to 0} \frac{|h| - |0|}{h}$$

$$g'(0) = \lim_{h \to 0} \frac{|h|}{h}$$

We know that

$$|h| = \begin{cases} -h & h < 0 \\ h & h \geq 0 \end{cases}$$

$$\lim_{h \to 0^-} \frac{|h|}{h} = \lim_{h \to 0^-} \frac{-h}{h} = -1$$

$$\lim_{h \to 0^+} \frac{|h|}{h} = \lim_{h \to 0^+} \frac{h}{h} = 1$$

Since the left- and right-hand limits aren't equal, this limit doesn't exist, which means the function isn't differentiable at $x = 0$, and we can't find its derivative there.

POWER RULE

■ 1. Find the derivative of $f(x) = 7x^3 - 17x^2 + 51x - 25$ using the power rule.

Solution:

Differentiating $f(x) = 7x^3 - 17x^2 + 51x - 25$ term-by-term gives

$$f'(x) = 7(3)x^{3-1} - 17(2)x^{2-1} + 51(1)x^{1-1} - 25(0)x^{0-1}$$

$$f'(x) = 21x^2 - 34x^1 + 51x^0 - 0x^{-1}$$

$$f'(x) = 21x^2 - 34x + 51(1) - 0$$

$$f'(x) = 21x^2 - 34x + 51$$

■ 2. Find the derivative of $g(x) = 2x^4 + 8x^3 + 6x^2 - 32x + 16$ using the power rule.

Solution:

Differentiating $g(x) = 2x^4 + 8x^3 + 6x^2 - 32x + 16$ term-by-term gives

$$g'(x) = 2(4)x^{4-1} + 8(3)x^{3-1} + 6(2)x^{2-1} - 32(1)x^{1-1} + 16(0)x^{0-1}$$

$$g'(x) = 8x^3 + 24x^2 + 12x^1 - 32x^0 + 0x^{-1}$$

$$g'(x) = 8x^3 + 24x^2 + 12x - 32(1) + 0$$

$$g'(x) = 8x^3 + 24x^2 + 12x - 32$$

■ 3. Find the derivative of $h(x) = 22x^3 - 19x^2 + 13x - 17$ using the power rule.

Solution:

Differentiating $h(x) = 22x^3 - 19x^2 + 13x - 17$ term-by-term gives

$$h'(x) = 22(3)x^{3-1} - 19(2)x^{2-1} + 13(1)x^{1-1} - 17(0)x^{0-1}$$

$$h'(x) = 66x^2 - 38x^1 + 13x^0 - 0x^{-1}$$

$$h'(x) = 66x^2 - 38x + 13(1) - 0$$

$$h'(x) = 66x^2 - 38x + 13$$

■ 4. Find the derivative of $h(s) = s^4 - s^3 + 3s - 7$ using the power rule.

Solution:

Differentiating $h(s) = s^4 - s^3 + 3s - 7$ term-by-term gives

$$h'(s) = 4s^{4-1} - 3s^{3-1} + 3(1)s^{1-1} - 7(0)x^{0-1}$$

$$h'(s) = 4s^3 - 3s^2 + 3s^0 - 0$$

$$h'(s) = 4s^3 - 3s^2 + 3$$

■ 5. Find the derivative using the power rule.

$$g(t) = \frac{2}{3}t^3 - \frac{5}{2}t^6$$

Solution:

Differentiating the function term-by-term gives

$$g'(t) = \frac{2}{3}(3)t^{3-1} - \frac{5}{2}(6)t^{6-1}$$

$$g'(t) = 2t^2 - 15t^5$$

■ 6. Find the derivative of $f(x) = 20x^{100} + 5x^{21} - 3x - 1$ using the power rule.

Solution:

Differentiating $f(x) = 20x^{100} + 5x^{21} - 3x - 1$ term-by-term gives

$$f'(x) = 20(100)x^{100-1} + 5(21)x^{21-1} - 3(1)x^{1-1} - 1(0)x^{0-1}$$

$$f'(x) = 2{,}000x^{99} + 105x^{20} - 3 - 0$$

$$f'(x) = 2{,}000x^{99} + 105x^{20} - 3$$

POWER RULE FOR NEGATIVE POWERS

■ 1. Find the derivative of the function using the power rule.

$$f(x) = \frac{7}{x^2} - \frac{5}{x^4} + \frac{2}{x}$$

Solution:

Rearrange $f(x)$ to use the power rule.

$$f(x) = 7x^{-2} - 5x^{-4} + 2x^{-1}$$

Differentiating term-by-term gives

$$f'(x) = 7(-2)x^{-2-1} - 5(-4)x^{-4-1} + 2(-1)x^{-1-1}$$

$$f'(x) = -14x^{-3} + 20x^{-5} - 2x^{-2}$$

Move the variables back to the denominator to make positive exponents.

$$f'(x) = -\frac{14}{x^3} + \frac{20}{x^5} - \frac{2}{x^2}$$

■ 2. Find the derivative of the function using the power rule.

$$g(x) = \frac{1}{9x^4} + \frac{2}{3x^5} - \frac{1}{x}$$

Solution:

Rearrange $g(x)$ to use the power rule.

$$g(x) = \frac{1}{9}x^{-4} + \frac{2}{3}x^{-5} - x^{-1}$$

Differentiating term-by-term gives

$$g'(x) = \frac{1}{9}(-4)x^{-4-1} + \frac{2}{3}(-5)x^{-5-1} - (-1)x^{-1-1}$$

$$g'(x) = -\frac{4}{9}x^{-5} - \frac{10}{3}x^{-6} + x^{-2}$$

Move the variables back to the denominator to make positive exponents.

$$g'(x) = -\frac{4}{9x^5} - \frac{10}{3x^6} + \frac{1}{x^2}$$

■ 3. Find the derivative of the function using the power rule.

$$h(x) = -\frac{7}{6x^6} - \frac{1}{4x^4} + \frac{9}{2x^2}$$

Solution:

Rearrange $h(x)$ to use the power rule.

$$h(x) = -\frac{7}{6}x^{-6} - \frac{1}{4}x^{-4} + \frac{9}{2}x^{-2}$$

Differentiating term-by-term gives

$$h'(x) = -\frac{7}{6}(-6)x^{-6-1} - \frac{1}{4}(-4)x^{-4-1} + \frac{9}{2}(-2)x^{-2-1}$$

$$h'(x) = 7x^{-7} + x^{-5} - 9x^{-3}$$

Move the variables back to the denominator to make positive exponents.

$$h'(x) = \frac{7}{x^7} + \frac{1}{x^5} - \frac{9}{x^3}$$

■ 4. Find the derivative of the function using the power rule.

$$g(x) = \frac{3}{x^2} + \frac{3}{2x^4} + \frac{1}{2}$$

Solution:

Rearrange $g(x)$ to use the power rule.

$$g(x) = 3x^{-2} + \frac{3}{2}x^{-4} + \frac{1}{2}$$

Differentiating term-by-term gives

$$g'(x) = 3(-2)x^{-2-1} + \frac{3}{2}(-4)x^{-4-1} + 0$$

$$g'(x) = -6x^{-3} - 6x^{-5}$$

Move the variables back to the denominator to make positive exponents.

$$g'(x) = -\frac{6}{x^3} - \frac{6}{x^5}$$

5. Find the derivative of the function using the power rule.

$$f(x) = -2x^{-4} + \frac{1}{x^2} + 7x$$

Solution:

Rearrange $f(x)$ to use the power rule.

$$f(x) = -2x^{-4} + x^{-2} + 7x$$

Differentiating term-by-term gives

$$f'(x) = -2(-4)x^{-4-1} + (-2)x^{-2-1} + 7x^{1-1}$$

$$f'(x) = 8x^{-5} - 2x^{-3} + 7$$

Move the variables back to the denominator to make positive exponents.

$$f'(x) = \frac{8}{x^5} - \frac{2}{x^3} + 7$$

6. Find the derivative of the function using the power rule, if a, b, and c are constants.

$$f(x) = 2ax^{-3a} + \frac{b}{cx^{2c}} - 2a$$

Solution:

Rearrange $f(x)$ to use the power rule.

$$f(x) = 2ax^{-3a} + \frac{b}{c}x^{-2c} - 2a$$

Differentiating term-by-term gives

$$f'(x) = 2a(-3a)x^{-3a-1} + \frac{b}{c}(-2c)x^{-2c-1} - 2a(0)x^{0-1}$$

$$f'(x) = -6a^2x^{-3a-1} - 2bx^{-2c-1}$$

Move the variables back to the denominator to make positive exponents.

$$f'(x) = -\frac{6a^2}{x^{3a+1}} - \frac{2b}{x^{2c+1}}$$

POWER RULE FOR FRACTIONAL POWERS

■ 1. Find the derivative of the function using the power rule.

$$f(x) = 4x^{\frac{3}{2}} - 6x^{\frac{5}{3}}$$

Solution:

Differentiating the function term-by-term gives

$$f'(x) = 4\left(\frac{3}{2}\right)x^{\frac{3}{2}-1} - 6\left(\frac{5}{3}\right)x^{\frac{5}{3}-1}$$

$$f'(x) = 6x^{\frac{3}{2}-\frac{2}{2}} - 10x^{\frac{5}{3}-\frac{3}{3}}$$

$$f'(x) = 6x^{\frac{1}{2}} - 10x^{\frac{2}{3}}$$

■ 2. Find the derivative of the function using the power rule.

$$g(x) = 6x^{\sqrt{3}} - 4x^{\sqrt{5}}$$

Solution:

Differentiating the function term-by-term gives

$$g'(x) = 6\sqrt{3}x^{\sqrt{3}-1} - 4\sqrt{5}x^{\sqrt{5}-1}$$

■ 3. Find the derivative of the function using the power rule.

$$h(x)=\frac{1}{3}x^{\frac{6}{5}}+\frac{1}{4}x^{\frac{8}{3}}-\frac{1}{5}x^{\frac{5}{2}}$$

Solution:

Differentiating the function term-by-term gives

$$h'(x)=\frac{1}{3}\left(\frac{6}{5}\right)x^{\frac{6}{5}-1}+\frac{1}{4}\left(\frac{8}{3}\right)x^{\frac{8}{3}-1}-\frac{1}{5}\left(\frac{5}{2}\right)x^{\frac{5}{2}-1}$$

$$h'(x)=\frac{2}{5}x^{\frac{6}{5}-\frac{5}{5}}+\frac{2}{3}x^{\frac{8}{3}-\frac{3}{3}}-\frac{1}{2}x^{\frac{5}{2}-\frac{2}{2}}$$

$$h'(x)=\frac{2}{5}x^{\frac{1}{5}}+\frac{2}{3}x^{\frac{5}{3}}-\frac{1}{2}x^{\frac{3}{2}}$$

■ 4. Find the derivative of the function using the power rule.

$$h(x)=\sqrt{x}+2\sqrt[3]{x}-3\sqrt[5]{x^2}$$

Solution:

Rewrite the function with fractional powers.

$$h(x)=x^{\frac{1}{2}}+2x^{\frac{1}{3}}-3x^{\frac{2}{5}}$$

Then differentiating term-by-term gives

$$h'(x) = \frac{1}{2}x^{\frac{1}{2}-1} + 2\left(\frac{1}{3}\right)x^{\frac{1}{3}-1} - 3\left(\frac{2}{5}\right)x^{\frac{2}{5}-1}$$

$$h'(x) = \frac{1}{2}x^{\frac{1}{2}-\frac{2}{2}} + \frac{2}{3}x^{\frac{1}{3}-\frac{3}{3}} - \frac{6}{5}x^{\frac{2}{5}-\frac{5}{5}}$$

$$h'(x) = \frac{1}{2}x^{-\frac{1}{2}} + \frac{2}{3}x^{-\frac{2}{3}} - \frac{6}{5}x^{-\frac{3}{5}}$$

$$h'(x) = \frac{1}{2x^{\frac{1}{2}}} + \frac{2}{3x^{\frac{2}{3}}} - \frac{6}{5x^{\frac{3}{5}}}$$

Change back from fractional powers to roots.

$$h'(x) = \frac{1}{2\sqrt{x}} + \frac{2}{3\sqrt[3]{x^2}} - \frac{6}{5\sqrt[5]{x^3}}$$

■ 5. Find the derivative of the function using the power rule.

$$f(z) = \frac{3}{\sqrt{z^5}} + \frac{5}{4z^4} - 2z^{-2}$$

Solution:

Rewrite the function.

$$f(z) = 3z^{-\frac{5}{2}} + \frac{5}{4}z^{-4} - 2z^{-2}$$

Then differentiating term-by-term gives

$$f'(z)=3\left(-\frac{5}{2}\right)z^{-\frac{5}{2}-1}+\frac{5}{4}(-4)z^{-4-1}-2(-2)z^{-2-1}$$

$$f'(z)=-\frac{15}{2}z^{-\frac{5}{2}-\frac{2}{2}}-5z^{-5}+4z^{-3}$$

$$f'(z)=-\frac{15}{2}z^{-\frac{7}{2}}-5z^{-5}+4z^{-3}$$

$$f'(z)=-\frac{15}{2z^{\frac{7}{2}}}-\frac{5}{z^5}+\frac{4}{z^3}$$

Change back from fractional powers to roots.

$$f'(z)=-\frac{15}{2\sqrt{z^7}}-\frac{5}{z^5}+\frac{4}{z^3}$$

■ 6. Find the derivative of the function using the power rule.

$$h(t)=\frac{2}{3t^6}+\frac{t^4}{4}-9t^3+\sqrt{t^3}+\frac{1}{2\sqrt[3]{t^2}}$$

Solution:

Rewrite the function.

$$h(t)=\frac{2}{3}t^{-6}+\frac{1}{4}t^4-9t^3+t^{\frac{3}{2}}+\frac{1}{2}t^{-\frac{2}{3}}$$

Then differentiating term-by-term gives

$$h'(t) = \frac{2}{3}(-6)t^{-6-1} + \frac{1}{4}(4)t^{4-1} - 9(3)t^{3-1} + \frac{3}{2}t^{\frac{3}{2}-1} + \frac{1}{2}\left(-\frac{2}{3}\right)t^{-\frac{2}{3}-1}$$

$$h'(t) = -4t^{-7} + t^3 - 27t^2 + \frac{3}{2}t^{\frac{3}{2}-\frac{2}{2}} - \frac{1}{3}t^{-\frac{2}{3}-\frac{3}{3}}$$

$$h'(t) = -4t^{-7} + t^3 - 27t^2 + \frac{3}{2}t^{\frac{1}{2}} - \frac{1}{3}t^{-\frac{5}{3}}$$

Change back from fractional powers to roots.

$$h'(t) = -\frac{4}{t^7} + t^3 - 27t^2 + \frac{3}{2}\sqrt{t} - \frac{1}{3\sqrt[3]{t^5}}$$

PRODUCT RULE WITH TWO FUNCTIONS

■ 1. Use the product rule to find the derivative of the function.

$$h(x) = (3x + 5)(2x^2 - 3x + 1)$$

Solution:

Let

$$f(x) = 3x + 5$$

$$f'(x) = 3$$

$$g(x) = 2x^2 - 3x + 1$$

$$g'(x) = 4x - 3$$

By product rule, the derivative is

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = (3x + 5)(4x - 3) + 3(2x^2 - 3x + 1)$$

Expand the derivative, then collect like terms.

$$h'(x) = 12x^2 - 9x + 20x - 15 + 6x^2 - 9x + 3$$

$$h'(x) = 18x^2 + 2x - 12$$

■ 2. Use the product rule to find the derivative of the function.

$$h(x) = 8x^3\sqrt[3]{x^2}$$

Solution:

Let

$$f(x) = 8x^3$$

$$f'(x) = 24x^2$$

$$g(x) = \sqrt[3]{x^2}$$

$$g'(x) = \frac{2}{3}x^{-\frac{1}{3}}$$

Then by product rule, the derivative is

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = 8x^3 \cdot \frac{2}{3}x^{-\frac{1}{3}} + 24x^2 \cdot \sqrt[3]{x^2}$$

$$h'(x) = \frac{16}{3}x^{\frac{8}{3}} + 24x^{\frac{8}{3}}$$

$$h'(x) = \frac{88}{3}x^{\frac{8}{3}}$$

■ 3. Use the product rule to find the derivative of the function.

$$h(x) = (5x^2 - x)\left(\frac{1}{x^4} - 6\right)$$

Solution:

Let

$$f(x) = 5x^2 - x$$

$$f'(x) = 10x - 1$$

$$g(x) = \frac{1}{x^4} - 6$$

$$g'(x) = -4x^{-5}$$

Then by product rule, the derivative is

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = (5x^2 - x)(-4x^{-5}) + (10x - 1)\left(\frac{1}{x^4} - 6\right)$$

$$h'(x) = -20x^{-3} + 4x^{-4} + \frac{10}{x^3} - \frac{1}{x^4} - 60x + 6$$

$$h'(x) = -\frac{20}{x^3} + \frac{4}{x^4} + \frac{10}{x^3} - \frac{1}{x^4} - 60x + 6$$

$$h'(x) = -\frac{10}{x^3} + \frac{3}{x^4} - 60x + 6$$

4. Use the product rule to find the derivative of the function.

$$h(x) = (1 + \sqrt{x^3})(x^{-2} - 3\sqrt[3]{x})$$

Solution:

Let

$$f(x) = 1 + \sqrt{x^3}$$

$$f'(x) = \frac{3}{2}\sqrt{x}$$

$$g(x) = x^{-2} - 3\sqrt[3]{x}$$

$$g'(x) = -2x^{-3} - x^{-\frac{2}{3}}$$

Then by product rule, the derivative is

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = (1 + \sqrt{x^3})(-2x^{-3} - x^{-\frac{2}{3}}) + \left(\frac{3}{2}\sqrt{x}\right)(x^{-2} - 3\sqrt[3]{x})$$

$$h'(x) = -2x^{-3} - x^{-\frac{2}{3}} - 2x^{-\frac{3}{2}} - x^{\frac{5}{6}} + \frac{3}{2}x^{-\frac{3}{2}} - \frac{9}{2}x^{\frac{5}{6}}$$

$$h'(x) = -2x^{-3} - x^{-\frac{2}{3}} - \frac{1}{2}x^{-\frac{3}{2}} - \frac{11}{2}x^{\frac{5}{6}}$$

$$h'(x) = -\frac{2}{x^3} - \frac{1}{\sqrt[3]{x^2}} - \frac{1}{2\sqrt{x^3}} - \frac{11}{2}\sqrt[6]{x^5}$$

■ 5. If $f(3) = -4$, $f'(3) = 2$, $g(3) = -1$, and $g'(3) = 3$, determine the value of $(fg)'(3)$.

Solution:

Evaluating the product rule formula at $x = 3$ gives

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(3) = f(3)g'(3) + f'(3)g(3)$$

$$h'(3) = (-4)(3) + 2(-1)$$

$$h'(3) = -12 - 2$$

$$h'(3) = -14$$

■ 6. If $h(x) = 2x^3g(x)$, $g(-4) = -5$, and $g'(-4) = 1$, determine the value of $h'(-4)$.

Solution:

We're given $g(-4) = -5$ and $g'(-4) = 1$, but let's also set $f(x) = 2x^3$ and $f'(x) = 6x^2$. Then

$$f(-4) = 2(-4)^3 = -128$$

$$f'(-4) = 6(-4)^2 = 96$$

So evaluating the product rule formula at $x = -4$ gives

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(-4) = f(-4)g'(-4) + f'(-4)g(-4)$$

$$h'(-4) = (-128)(1) + 96(-5)$$

$$h'(-4) = -128 - 480$$

$$h'(-4) = -608$$

PRODUCT RULE WITH THREE OR MORE FUNCTIONS

■ 1. Use the product rule to find the derivative of the function.

$$y = 5x^4(2x - x^2)\left(\frac{1}{x^2} - 5\right)$$

Solution:

Let

$$f(x) = 5x^4 \qquad g(x) = 2x - x^2 \qquad h(x) = \frac{1}{x^2} - 5$$

$$f'(x) = 20x^3 \qquad g'(x) = 2 - 2x \qquad h'(x) = -\frac{2}{x^3}$$

Then by product rule, the derivative is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$y' = (20x^3)(2x - x^2)\left(\frac{1}{x^2} - 5\right) + (5x^4)(2 - 2x)\left(\frac{1}{x^2} - 5\right) + (5x^4)(2x - x^2)\left(-\frac{2}{x^3}\right)$$

$$y' = (40x^4 - 20x^5)\left(\frac{1}{x^2} - 5\right) + (10x^4 - 10x^5)\left(\frac{1}{x^2} - 5\right) + (10x^5 - 5x^6)\left(-\frac{2}{x^3}\right)$$

$$y' = 40x^2 - 200x^4 - 20x^3 + 100x^5 + 10x^2 - 50x^4 - 10x^3 + 50x^5 - 20x^2 + 10x^3$$

$$y' = 150x^5 - 250x^4 - 20x^3 + 30x^2$$

■ 2. Use the product rule to find the derivative of the function.

$$y = 30\left(\frac{1}{x^3} + x^2\right)(2x^4 - x^2 - x)$$

Solution:

Let

$$f(x) = 30 \qquad g(x) = \frac{1}{x^3} + x^2 \qquad h(x) = 2x^4 - x^2 - x$$

$$f'(x) = 0 \qquad g'(x) = -\frac{3}{x^4} + 2x \qquad h'(x) = 8x^3 - 2x - 1$$

Then by product rule, the derivative is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$y' = (0)\left(\frac{1}{x^3} + x^2\right)(2x^4 - x^2 - x) + (30)\left(-\frac{3}{x^4} + 2x\right)(2x^4 - x^2 - x)$$

$$+(30)\left(\frac{1}{x^3} + x^2\right)(8x^3 - 2x - 1)$$

$$y' = 0 + (30)\left(-6 + \frac{3}{x^2} + \frac{3}{x^3} + 4x^5 - 2x^3 - 2x^2\right)$$

$$+(30)\left(8 - \frac{2}{x^2} - \frac{1}{x^3} + 8x^5 - 2x^3 - x^2\right)$$

$$y' = -180 + \frac{90}{x^2} + \frac{90}{x^3} + 120x^5 - 60x^3 - 60x^2$$

$$+240 - \frac{60}{x^2} - \frac{30}{x^3} + 240x^5 - 60x^3 - 30x^2$$

$$y' = 360x^5 - 120x^3 - 90x^2 + \frac{30}{x^2} + \frac{60}{x^3} + 60$$

■ 3. Use the product rule to find the derivative of the function.

$$y = (x^2 - 3x + 5)(7 + 2x - 5x^2)(2 - 2\sqrt{x})$$

Solution:

Let

$$f(x) = x^2 - 3x + 5 \qquad g(x) = 7 + 2x - 5x^2 \qquad h(x) = 2 - 2\sqrt{x}$$

$$f'(x) = 2x - 3 \qquad g'(x) = 2 - 10x \qquad h'(x) = -\frac{1}{\sqrt{x}}$$

Then by product rule, the derivative is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$y' = (2x - 3)(7 + 2x - 5x^2)(2 - 2\sqrt{x}) + (x^2 - 3x + 5)(2 - 10x)(2 - 2\sqrt{x})$$

$$+(x^2 - 3x + 5)(7 + 2x - 5x^2)\left(-\frac{1}{\sqrt{x}}\right)$$

$$y' = (14x + 4x^2 - 10x^3 - 21 - 6x + 15x^2)(2 - 2\sqrt{x})$$

$$+(2x^2 - 10x^3 - 6x + 30x^2 + 10 - 50x)(2 - 2\sqrt{x})$$

$$+(7x^2 + 2x^3 - 5x^4 - 21x - 6x^2 + 15x^3 + 35 + 10x - 25x^2)\left(-\frac{1}{\sqrt{x}}\right)$$

$$y' = (-10x^3 + 19x^2 + 8x - 21)(2 - 2\sqrt{x}) + (-10x^3 + 32x^2 - 56x + 10)(2 - 2\sqrt{x})$$

$$+(-5x^4 + 17x^3 - 24x^2 - 11x + 35)(-\frac{1}{\sqrt{x}})$$

$$y' = -40x^3 + 102x^2 - 96x + 45x^{\frac{7}{2}} - 119x^{\frac{5}{2}} + 120x^{\frac{3}{2}} + 33\sqrt{x} - \frac{35}{\sqrt{x}} - 22$$

■ 4. Use the product rule to find the derivative of the function.

$$y = \left(x - \frac{3}{x}\right)(x^2 + 4x)(7x^4)\left(-5x^2 - \frac{1}{2}\right)$$

Solution:

Let

$$f(x) = x - \frac{3}{x} \qquad g(x) = x^2 + 4x \qquad h(x) = 7x^4 \qquad k(x) = -5x^2 - \frac{1}{2}$$

$$f'(x) = 1 + \frac{3}{x^2} \qquad g'(x) = 2x + 4 \qquad h'(x) = 28x^3 \qquad k'(x) = -10x$$

Then by product rule, the derivative is

$$y' = f'(x)g(x)h(x)k(x) + f(x)g'(x)h(x)k(x) + f(x)g(x)h'(x)k(x) + f(x)g(x)h(x)k'(x)$$

$$y' = \left(1 + \frac{3}{x^2}\right)(x^2 + 4x)(7x^4)\left(-5x^2 - \frac{1}{2}\right) + \left(x - \frac{3}{x}\right)(2x + 4)(7x^4)\left(-5x^2 - \frac{1}{2}\right)$$

$$+\left(x - \frac{3}{x}\right)(x^2 + 4x)(28x^3)\left(-5x^2 - \frac{1}{2}\right) + \left(x - \frac{3}{x}\right)(x^2 + 4x)(7x^4)(-10x)$$

$$y' = \left(1 + \frac{3}{x^2}\right)(7x^6 + 28x^5)\left(-5x^2 - \frac{1}{2}\right) + \left(x - \frac{3}{x}\right)(14x^5 + 28x^4)\left(-5x^2 - \frac{1}{2}\right)$$

$$+\left(x - \frac{3}{x}\right)(28x^5 + 112x^4)\left(-5x^2 - \frac{1}{2}\right) + \left(x - \frac{3}{x}\right)(7x^6 + 28x^5)(-10x)$$

$$y' = (7x^6 + 28x^5 + 21x^4 + 84x^3)\left(-5x^2 - \frac{1}{2}\right)$$

$$+(14x^6 + 28x^5 - 42x^4 - 84x^3)\left(-5x^2 - \frac{1}{2}\right)$$

$$+(28x^6 + 112x^5 - 84x^4 - 336x^3)\left(-5x^2 - \frac{1}{2}\right)$$

$$+(7x^7 + 28x^6 - 21x^5 - 84x^4)(-10x)$$

$$y' = -35x^8 - \frac{7}{2}x^6 - 140x^7 - 14x^5 - 105x^6 - \frac{21}{2}x^4 - 420x^5 - 42x^3$$

$$-70x^8 - 7x^6 - 140x^7 - 14x^5 + 210x^6 + 21x^4 + 420x^5 + 42x^3$$

$$-140x^8 - 14x^6 - 560x^7 - 56x^5 + 420x^6 + 42x^4 + 1{,}680x^5 + 168x^3$$

$$-70x^8 - 280x^7 + 210x^6 + 840x^5$$

$$y' = -35x^8 - 70x^8 - 140x^8 - 70x^8$$

$$-140x^7 - 140x^7 - 560x^7 - 280x^7$$

$$-\frac{7}{2}x^6 - 105x^6 - 7x^6 + 210x^6 - 14x^6 + 420x^6 + 210x^6$$

$$-14x^5 - 420x^5 - 14x^5 + 420x^5 - 56x^5 + 1{,}680x^5 + 840x^5$$

$$-\frac{21}{2}x^4 + 21x^4 + 42x^4$$

$$-42x^3 + 42x^3 + 168x^3$$

$$y' = -315x^8 - 1{,}120x^7 + \frac{1{,}421}{2}x^6 + 2{,}436x^5 + \frac{105}{2}x^4 + 168x^3$$

■ 5. Use $f(-2) = 5$, $f'(-2) = -7$, $g(-2) = -8$, $g'(-2) = -3$, $h(-2) = 1$ and $h'(-2) = 0$ to determine the value of $(fgh)'(-2)$.

Solution:

Evaluate the product rule formula for three functions at $x = -2$.

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$(fgh)'(2) = f'(-2)g(-2)h(-2) + f(-2)g'(-2)h(-2) + f(-2)g(-2)h'(-2)$$

$$(fgh)'(2) = (-7)(-8)(1) + (5)(-3)(1) + (5)(-8)(0)$$

$$(fgh)'(2) = 56 - 15 + 0$$

$$(fgh)'(2) = 41$$

■ 6. Use $f(5) = 4$, $f'(5) = 2$, $g(5) = -2$, $g'(5) = 3$, $h(5) = -3$, and $h'(5) = -8$ if $y = [x^2 - f(x)]g(x)h(x)$, to determine the value of $y'(5)$.

Solution:

Evaluate the product rule formula at $x = 5$.

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

$$y' = 2xg(x)h(x) + x^2g'(x)h(x) + x^2g(x)h'(x) - f'(x)g(x)h(x)$$

$$-f(x)g'(x)h(x) - f(x)g(x)h'(x)$$

$$y'(5) = 2(5)g(5)h(5) + 5^2g'(5)h(5) + 5^2g(5)h'(5) - f'(5)g(5)h(5)$$

$$-f(5)g'(5)h(5) - f(5)g(5)h'(5)$$

$$y'(5) = 10g(5)h(5) + 25g'(5)h(5) + 25g(5)h'(5) - f'(5)g(5)h(5)$$

$$-f(5)g'(5)h(5) - f(5)g(5)h'(5)$$

Substitute $f(5) = 4$, $f'(5) = 2$, $g(5) = -2$, $g'(5) = 3$, $h(5) = -3$, and $h'(5) = -8$.

$$y'(5) = 10(-2)(-3) + 25(3)(-3) + 25(-2)(-8) - 2(-2)(-3)$$

$$-4(3)(-3) - 4(-2)(-8)$$

$$y'(5) = 10(6) + 25(-9) + 25(16) - 2(6) - 4(-9) - 4(16)$$

$$y'(5) = 60 - 225 + 400 - 12 + 36 - 64$$

$$y'(5) = 195$$

QUOTIENT RULE

■ 1. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{2x+6}{7x+5}$$

Solution:

Let

$$f(x) = 2x + 6$$

$$f'(x) = 2$$

$$g(x) = 7x + 5$$

$$g'(x) = 7$$

Then the derivative is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(x) = \frac{2 \cdot (7x+5) - (2x+6) \cdot 7}{(7x+5)^2}$$

$$h'(x) = \frac{14x + 10 - 14x - 42}{(7x+5)^2}$$

$$h'(x) = -\frac{32}{(7x+5)^2}$$

■ 2. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{\sqrt[3]{x}}{1+2x^2}$$

Solution:

Let

$$f(x) = \sqrt[3]{x}$$

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

$$g(x) = 1 + 2x^2$$

$$g'(x) = 4x$$

Then the derivative is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(x) = \frac{\frac{1}{3\sqrt[3]{x^2}} \cdot (1+2x^2) - (\sqrt[3]{x}) \cdot 4x}{(1+2x^2)^2}$$

$$h'(x) = \frac{\frac{1}{3\sqrt[3]{x^2}} + \frac{2}{3}\sqrt[3]{x^4} - 4\sqrt[3]{x^4}}{(1 + 2x^2)^2}$$

$$h'(x) = \frac{\frac{1}{3\sqrt[3]{x^2}} - \frac{10}{3}\sqrt[3]{x^4}}{(1 + 2x^2)^2}$$

■ 3. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{-8x}{5x + 2}$$

Solution:

Let

$$f(x) = -8x$$

$$f'(x) = -8$$

$$g(x) = 5x + 2$$

$$g'(x) = 5$$

Then the derivative is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(x) = \frac{-8 \cdot (5x+2) - (-8x) \cdot 5}{(5x+2)^2}$$

$$h'(x) = \frac{-40x - 16 + 40x}{(5x+2)^2}$$

$$h'(x) = -\frac{16}{(5x+2)^2}$$

■ 4. Use the quotient rule to find the derivative of the function.

$$h(x) = \frac{2 - 4x + 5x^2}{5x + x^3}$$

Solution:

Let

$$f(x) = 2 - 4x + 5x^2$$

$$f'(x) = -4 + 10x$$

$$g(x) = 5x + x^3$$

$$g'(x) = 5 + 3x^2$$

Then the derivative is

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(x)=\frac{(-4+10x)(5x+x^3)-(2-4x+5x^2)(5+3x^2)}{(5x+x^3)^2}$$

$$h'(x)=\frac{(-20x-4x^3+50x^2+10x^4)-(10+6x^2-20x-12x^3+25x^2+15x^4)}{(5x+x^3)^2}$$

$$h'(x)=\frac{-20x-4x^3+50x^2+10x^4-10-6x^2+20x+12x^3-25x^2-15x^4}{(5x+x^3)^2}$$

$$h'(x)=\frac{-5x^4+8x^3+19x^2-10}{(5x+x^3)^2}$$

5. Use the quotient rule to find the derivative of the function.

$$k(x)=\frac{(2-3x)(1+x)}{2+3x^2}$$

Solution:

This function is given in the form

$$k(x)=\frac{f(x)h(x)}{g(x)}$$

which means the combination of quotient rule and product rule will be

$$k'(x)=\frac{[f(x)h(x)]'g(x)-f(x)h(x)g'(x)}{[g(x)]^2}$$

$$k'(x) = \frac{[f'(x)h(x) + f(x)h'(x)]g(x) - f(x)h(x)g'(x)}{[g(x)]^2}$$

Let

$$f(x) = 2 - 3x$$

$$f'(x) = -3$$

$$h(x) = 1 + x$$

$$h'(x) = 1$$

$$g(x) = 2 + 3x^2$$

$$g'(x) = 6x$$

Then the derivative is

$$k'(x) = \frac{[(-3)(1 + x) + (2 - 3x)(1)](2 + 3x^2) - (2 - 3x)(1 + x)(6x)}{(2 + 3x^2)^2}$$

$$k'(x) = \frac{(-3 - 3x + 2 - 3x)(2 + 3x^2) - (2 - 3x)(6x + 6x^2)}{(2 + 3x^2)^2}$$

$$k'(x) = \frac{(-1 - 6x)(2 + 3x^2) - (2 - 3x)(6x + 6x^2)}{(2 + 3x^2)^2}$$

$$k'(x) = \frac{(-2 - 3x^2 - 12x - 18x^3) - (12x + 12x^2 - 18x^2 - 18x^3)}{(2 + 3x^2)^2}$$

$$k'(x) = \frac{-2 - 3x^2 - 12x - 18x^3 - 12x - 12x^2 + 18x^2 + 18x^3}{(2 + 3x^2)^2}$$

$$k'(x) = \frac{3x^2 - 24x - 2}{(2 + 3x^2)^2}$$

■ 6. Use $f(5) = 4$, $f'(5) = 2$, $g(5) = -2$, $g'(5) = 3$, $h(5) = -3$, and $h'(5) = -8$ to determine the value of $k'(5)$.

$$k'(5) = \left(\frac{fg}{h}\right)'(5)$$

Solution:

This function is given in the form

$$k(x) = \frac{f(x)g(x)}{h(x)}$$

which means the combination of quotient rule and product rule will be

$$k'(x) = \frac{[f(x)g(x)]'h(x) - f(x)g(x)h'(x)}{[h(x)]^2}$$

$$k'(x) = \frac{[f'(x)g(x) + f(x)g'(x)]h(x) - f(x)g(x)h'(x)}{[h(x)]^2}$$

$$k'(5) = \frac{[f'(5)g(5) + f(5)g'(5)]h(5) - f(5)g(5)h'(5)}{[h(5)]^2}$$

Substitute the values we were given.

$$k'(5)=\frac{[2(-2)+4(3)](-3)-4(-2)(-8)}{(-3)^2}$$

$$k'(5)=\frac{(-4+12)(-3)-64}{9}$$

$$k'(5)=\frac{8(-3)-64}{9}$$

$$k'(5)=-\frac{88}{9}$$

TRIGONOMETRIC DERIVATIVES

■ 1. Find $f'(x)$ if $f(x) = 3x^{-4} + x^2 \cot x$.

Solution:

Let's look at one term at a time. The derivative of $3x^{-4}$ is

$$-12x^{-5}$$

To find the derivative of $x^2 \cot x$, we'll need to use product rule. If we set $f(x) = x^2$, $f'(x) = 2x$, $g(x) = \cot x$, and $g'(x) = -\csc^2 x$, then we can plug directly into the product rule formula.

$$f(x)g'(x) + f'(x)g(x)$$

$$(x^2)(-\csc^2 x) + (2x)(\cot x)$$

$$-x^2 \csc^2 x + 2x \cot x$$

Putting these derivatives together, we get

$$f'(x) = -12x^{-5} - x^2 \csc^2 x + 2x \cot x$$

■ 2. Find $h'(x)$.

$$h(x) = \frac{\sin x}{5 - 2\cos x}$$

Solution:

We'll need to use quotient rule. If we set $f(x) = \sin x$, $f'(x) = \cos x$, $g(x) = 5 - 2\cos x$, and $g'(x) = 2\sin x$, then we can plug directly into the quotient rule formula.

$$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$h'(x) = \frac{\cos x(5 - 2\cos x) - \sin x(2\sin x)}{(5 - 2\cos x)^2}$$

$$h'(x) = \frac{5\cos x - 2\cos^2 x - 2\sin^2 x}{(5 - 2\cos x)^2}$$

$$h'(x) = \frac{5\cos x - 2}{(5 - 2\cos x)^2}$$

■ 3. Find $h'(x)$ if $h(x) = 3\sin x\cos x + 5\sec x$.

Solution:

To find the derivative of $3\sin x\cos x$, we'll need to use product rule. If we set $f(x) = 3\sin x$, $f'(x) = 3\cos x$, $g(x) = \cos x$, and $g'(x) = -\sin x$, then we can plug directly into the product rule formula.

$$f(x)g'(x) + f'(x)g(x)$$

$$(3\sin x)(-\sin x) + (3\cos x)(\cos x)$$

$$-3\sin^2 x + 3\cos^2 x$$

The derivative of $5\sec x$ is

$$5\sec x\tan x$$

Putting these derivatives together, we get

$$h'(x) = -3\sin^2 x + 3\cos^2 x + 5\sec x\tan x$$

■ 4. Find the derivative of the trigonometric function.

$$y = 3 - 2\sqrt{x}\csc x$$

Solution:

The derivative of 3 is 0. To find the derivative of $-2\sqrt{x}\csc x$, we'll need to use product rule. If we set

$$f(x) = -2\sqrt{x}$$

$$f'(x) = -\frac{1}{\sqrt{x}}$$

$$g(x) = \csc x$$

$$g'(x) = -\csc x\cot x$$

then we can plug directly into the product rule formula.

$$f(x)g'(x) + f'(x)g(x)$$

$$(-2\sqrt{x})(-\csc x \cot x) + \left(-\frac{1}{\sqrt{x}}\right)(\csc x)$$

$$2\sqrt{x}\csc x \cot x - \frac{\csc x}{\sqrt{x}}$$

Putting these derivatives together, we get

$$y' = 2\sqrt{x}\csc x \cot x - \frac{\csc x}{\sqrt{x}}$$

■ 5. Find the derivative of the trigonometric function.

$$y = \frac{2}{4\cos x - 5\sin x}$$

Solution:

We can use reciprocal rule,

$$y' = \frac{-ag'(x)}{[g(x)]^2}$$

to find the derivative. If $a = 2$, $-a = -2$, $g(x) = 4\cos x - 5\sin x$, and $g'(x) = -4\sin x - 5\cos x$, then we can plug directly into the reciprocal rule formula.

$$y' = \frac{-2(-4\sin x - 5\cos x)}{(4\cos x - 5\sin x)^2}$$

$$y' = \frac{8\sin x + 10\cos x}{(4\cos x - 5\sin x)^2}$$

■ 6. Find the derivative of y.

$$y = 2x^4 + \frac{x\tan x}{x^2 + 1}$$

Solution:

Let's look at one term at a time. The derivative of $2x^4$ is

$$8x^3$$

To find the derivative of the fraction, we'll need to use the quotient and product rules.

$$\frac{(x\sec^2 x + \tan x)(x^2 + 1) - x\tan x(2x)}{(x^2 + 1)^2}$$

$$\frac{x^3\sec^2 x + x^2\tan x + x\sec^2 x + \tan x - 2x^2\tan x}{(x^2 + 1)^2}$$

$$\frac{x^3\sec^2 x - x^2\tan x + x\sec^2 x + \tan x}{(x^2+1)^2}$$

Putting these derivatives together, we get

$$y' = 8x^3 + \frac{x^3\sec^2 x - x^2\tan x + x\sec^2 x + \tan x}{(x^2+1)^2}$$

EXPONENTIAL DERIVATIVES

■ 1. Find $f'(x)$ if $f(x) = (x^3 - x)e^x$.

Solution:

Use product rule to take the derivative.

$$f'(x) = \frac{d}{dx}(x^3 - x) \cdot e^x + (x^3 - x) \cdot \frac{d}{dx}e^x$$

$$f'(x) = (3x^2 - 1) \cdot e^x + (x^3 - x) \cdot e^x$$

Factor to simplify.

$$f'(x) = e^x(3x^2 - 1 + x^3 - x)$$

$$f'(x) = e^x(x^3 + 3x^2 - x - 1)$$

■ 2. Find $g'(x)$ if $g(x) = 5^x(x^2 - 7x + 1)$.

Solution:

Use product rule to take the derivative.

$$g'(x) = \frac{d}{dx}(5^x) \cdot (x^2 - 7x + 1) + 5^x \cdot \frac{d}{dx}(x^2 - 7x + 1)$$

$$g'(x) = 5^x \ln 5(x^2 - 7x + 1) + 5^x(2x - 7)$$

3. Find $h'(x)$ if $h(x) = \sin xe^x - x^2 \cos x$.

Solution:

Use product rule to take the derivative of the first and second term.

$$h'(x) = \frac{d}{dx} \sin x \cdot e^x + \sin x \cdot \frac{d}{dx} e^x - \left(\frac{d}{dx} x^2 \cdot \cos x + x^2 \cdot \frac{d}{dx} \cos x \right)$$

$$h'(x) = \cos x \cdot e^x + \sin x \cdot e^x - (2x \cdot \cos x + x^2 \cdot (-\sin x))$$

$$h'(x) = e^x \cos x + e^x \sin x - 2x \cos x + x^2 \sin x$$

Factor to simplify.

$$h'(x) = \cos x(e^x - 2x) + \sin x(e^x + x^2)$$

4. Find $f'(x)$.

$$f(x) = \frac{4e^x}{3e^x - 1}$$

Solution:

Use quotient rule to take the derivative.

$$f'(x) = \frac{\frac{d}{dx}(4e^x) \cdot (3e^x - 1) - 4e^x \cdot \frac{d}{dx}(3e^x - 1)}{(3e^x - 1)^2}$$

$$f'(x) = \frac{4e^x \cdot (3e^x - 1) - 4e^x \cdot (3e^x)}{(3e^x - 1)^2}$$

$$f'(x) = \frac{12e^{2x} - 4e^x - 12e^{2x}}{(3e^x - 1)^2}$$

$$f'(x) = -\frac{4e^x}{(3e^x - 1)^2}$$

■ 5. Find $g'(x)$ if $g(x) = 8^x + 3e^x \cot x$.

Solution:

Use product rule to take the derivative of the second term.

$$g'(x) = 8^x \ln 8 + \frac{d}{dx}(3e^x) \cdot \cot x + 3e^x \cdot \frac{d}{dx}(\cot x)$$

$$g'(x) = 8^x \ln 8 + 3e^x \cdot \cot x + 3e^x \cdot (-\csc^2 x)$$

$$g'(x) = 8^x \ln 8 + 3e^x \cot x - 3e^x \csc^2 x$$

■ 6. Find $h'(x)$ if $h(x) = \dfrac{x^3 e^x}{x + 3^x}$.

Solution:

Use quotient rule to take the derivative.

$$h'(x) = \frac{\frac{d}{dx}(x^3e^x) \cdot (x + 3^x) - x^3e^x \cdot \frac{d}{dx}(x + 3^x)}{(x + 3^x)^2}$$

Use product rule to take the derivative x^3e^x.

$$\frac{d}{dx}(x^3) \cdot (e^x) + x^3 \cdot \frac{d}{dx}(e^x)$$

$$3x^2e^x + x^3e^x$$

Then the derivative of the function is

$$h'(x) = \frac{(3x^2e^x + x^3e^x)(x + 3^x) - x^3e^x(1 + 3^x \ln 3)}{(x + 3^x)^2}$$

LOGARITHMIC DERIVATIVES

■ 1. Find $f'(x)$.

$$f(x) = 2\log_5 x - 11\log_{13} x$$

Solution:

Differentiate by applying the derivative formula for $f(x) = \log_a x$,

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a}$$

So the derivative is

$$f'(x) = 2\left(\frac{1}{x\ln 5}\right) - 11\left(\frac{1}{x\ln 13}\right)$$

$$f'(x) = \frac{2}{x\ln 5} - \frac{11}{x\ln 13}$$

■ 2. Find $g'(x)$.

$$g(x) = \log_4 x - x^6\ln x$$

Solution:

We need to take the derivative one term at a time, applying the derivative formulas for the log and natural log. We'll also need to apply product rule to the second term.

$$g'(x) = \frac{1}{x \ln 4} - \left[(6x^5)(\ln x) + x^6\left(\frac{1}{x}\right)\right]$$

$$g'(x) = \frac{1}{x \ln 4} - 6x^5 \ln x - x^5$$

■ 3. Find $h'(x)$.

$$h(x) = \log_7 x \ln x$$

Solution:

We need to take the derivative by applying the derivative formulas for the log and natural log. We'll also need to apply product rule.

$$h'(x) = \frac{1}{x \ln 7}(\ln x) + \log_7 x\left(\frac{1}{x}\right)$$

$$h'(x) = \frac{\ln x}{x \ln 7} + \frac{\log_7 x}{x}$$

Using properties of logs, we get

$$h'(x) = \frac{\log_7 x}{x} + \frac{\log_7 x}{x}$$

$$h'(x) = \frac{2\log_7 x}{x}$$

4. Find $y'(x)$.

$$y = \frac{1 + 7\ln x}{6x^4}$$

Solution:

We need to take the derivative by applying the derivative formulas for the natural log. We'll also need to apply quotient rule.

$$y' = \frac{\frac{7}{x}(6x^4) - (1 + 7\ln x)(24x^3)}{(6x^4)^2}$$

$$y' = \frac{42x^3 - 24x^3 - 168x^3\ln x}{36x^8}$$

$$y' = \frac{18x^3 - 168x^3\ln x}{36x^8}$$

$$y' = \frac{3 - 28\ln x}{6x^5}$$

5. Find $y'(x)$.

$$y = \frac{x^3 + \log_5 x}{5^x}$$

Solution:

We need to take the derivative by applying the derivative formulas for the log. We'll also need to apply quotient rule.

$$y' = \frac{\left(3x^2 + \frac{1}{x \ln 5}\right)(5^x) - (x^3 + \log_5 x)(5^x \ln 5)}{(5^x)^2}$$

$$y' = \frac{3x^2 + \frac{1}{x \ln 5} - (x^3 + \log_5 x)(\ln 5)}{5^x}$$

$$y' = \frac{3x^2 + \frac{1}{x \ln 5} - x^3 \ln 5 - \log_5 x \ln 5}{5^x}$$

■ 6. Find $y'(x)$.

$$y = \frac{x^7 e^x}{\ln x}$$

Solution:

We need to take the derivative by applying the derivative formulas for the natural log. We'll also need to apply product and quotient rule.

$$y' = \frac{(7x^6 e^x + x^7 e^x)(\ln x) - (x^7 e^x)\left(\frac{1}{x}\right)}{(\ln x)^2}$$

$$y' = \frac{7x^6e^x \ln x + x^7e^x \ln x - x^6e^x}{(\ln x)^2}$$

$$y' = \frac{x^6e^x(7 \ln x + x \ln x - 1)}{(\ln x)^2}$$

CHAIN RULE WITH POWER RULE

■ 1. Find $h'(x)$ if $h(x) = (3x^2 - 7)^4$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $3x^2 - 7$, and that the derivative of that inside function is $6x$. Therefore, the derivative is

$$h'(x) = 4(3x^2 - 7)^3(6x)$$

$$h'(x) = 24x(3x^2 - 7)^3$$

■ 2. Find $h'(x)$ if $h(x) = \sqrt{2 - 4x^2}$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $2 - 4x^2$, and that the derivative of that inside function is $-8x$. Therefore, the derivative is

$$h'(x) = \frac{1}{2}(2 - 4x^2)^{-\frac{1}{2}}(-8x)$$

$$h'(x) = -4x(2 - 4x^2)^{-\frac{1}{2}}$$

$$h'(x) = -\frac{4x}{\sqrt{2 - 4x^2}}$$

■ 3. Find $h'(x)$ if $h(x) = (2x^2 - 6x + 5)^7$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $2x^2 - 6x + 5$, and that the derivative of that inside function is $4x - 6$. Therefore, the derivative is

$$h'(x) = 7(2x^2 - 6x + 5)^6(4x - 6)$$

$$h'(x) = 7(4x - 6)(2x^2 - 6x + 5)^6$$

■ 4. Find $h'(x)$ if $h(x) = 2(x^3 + 4x^2 - 2x)^{-5}$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $x^3 + 4x^2 - 2x$, and that the derivative of that inside function is $3x^2 + 8x - 2$. Therefore, the derivative is

$$h'(x) = 2(-5)(x^3 + 4x^2 - 2x)^{-6}(3x^2 + 8x - 2)$$

$$h'(x) = -10(3x^2 + 8x - 2)(x^3 + 4x^2 - 2x)^{-6}$$

$$h'(x) = -\frac{10(3x^2 + 8x - 2)}{(x^3 + 4x^2 - 2x)^6}$$

■ 5. Find $f'(x)$ if $f(x) = 3(5x^2 + \sin x)^4$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $5x^2 + \sin x$, and that the derivative of that inside function is $10x + \cos x$. Therefore, the derivative is

$$f'(x) = 3(4)(5x^2 + \sin x)^3(10x + \cos x)$$

$$f'(x) = 12(10x + \cos x)(5x^2 + \sin x)^3$$

■ 6. Find $g'(y)$ if $g(y) = \sqrt{3y + (2y + y^2)^2}$.

Solution:

To find the derivative, we have to apply chain rule two times. We'll say that the inside function is $3y + (2y + y^2)^2$, and that the derivative of that inside function is $3 + 2(2y + y^2)(2 + 2y)$ or $3 + (4 + 4y)(2y + y^2)$. Therefore, the derivative is

$$g'(y) = \frac{1}{2}(3y + (2y + y^2)^2)^{-\frac{1}{2}}(3 + (4 + 4y)(2y + y^2))$$

$$g'(y) = \frac{3 + (4 + 4y)(2y + y^2)}{2\sqrt{3y + (2y + y^2)^2}}$$

CHAIN RULE WITH TRIG, LOG, AND EXPONENTIAL FUNCTIONS

■ 1. Find $f'(x)$.

$$f(x) = \ln(x^2 + 6x + 9)$$

Solution:

Take the derivative, remembering to apply chain rule.

$$f'(x) = \frac{1}{x^2 + 6x + 9} \cdot (2x + 6)$$

$$f'(x) = \frac{2x + 6}{x^2 + 6x + 9}$$

$$f'(x) = \frac{2(x + 3)}{(x + 3)(x + 3)}$$

$$f'(x) = \frac{2}{x + 3}$$

■ 2. Find $g'(x)$ if $g(x) = 3\sin(4x^3) - 4\cos(6x) + 3\sec(2x^4)$.

Solution:

Differentiate one term at a time, remembering to apply chain rule.

$$g'(x) = 3\cos(4x^3)(12x^2) - 4(-\sin(6x))(6) + 3\sec(2x^4)\tan(2x^4)(8x^3)$$

$$g'(x) = 36x^2\cos(4x^3) + 24\sin(6x) + 24x^3\tan(2x^4)\sec(2x^4)$$

$$g'(x) = 12(2x^3\tan(2x^4)\sec(2x^4) + 3x^2\cos(4x^3) + 2\sin(6x))$$

■ 3. Find $h'(x)$ if $h(x) = \cos(\sin x + 3x^3)$.

Solution:

To find the derivative, we have to apply chain rule. We'll say that the inside function is $\sin x + 3x^3$, and that the derivative of that inside function is $\cos x + 9x^2$. Therefore, the derivative is

$$h'(x) = -\sin(\sin x + 3x^3)(\cos x + 9x^2)$$

$$h'(x) = -(\cos x + 9x^2)\sin(\sin x + 3x^3)$$

■ 4. Find $f'(y)$ if $f(y) = e^{y+\ln y} + 8^{\cos y}$.

Solution:

To find the derivative, we have to apply chain rule two times. In the first term, $e^{y+\ln y}$, the inside function is $y + \ln y$, and the derivative of that inside function is $1 + (1/y)$. In the second term, $8^{\cos y}$, the inside function is $\cos y$, and the derivative of that inside function is $-\sin y$. Therefore, the derivative is

$$f'(y) = e^{y+\ln y}\left(1 + \frac{1}{y}\right) + 8^{\cos y}\ln 8(-\sin y)$$

$$f'(y) = e^{y+\ln y}\left(1 + \frac{1}{y}\right) - \ln 8\sin(y)8^{\cos y}$$

Use properties of exponential functions to simplify.

$$f'(y) = ye^{y}\left(1 + \frac{1}{y}\right) - \ln 8\sin(y)8^{\cos y}$$

$$f'(y) = ye^{y}\left(\frac{y+1}{y}\right) - \ln 8\sin(y)8^{\cos y}$$

$$f'(y) = e^{y}(y+1) - \ln 8\sin(y)8^{\cos y}$$

■ 5. Find $f'(x)$ if $f(x) = \tan^5 x + \tan x^5$.

Solution:

To find the derivative, we have to apply chain rule two times. In the first term, $\tan^5 x$, the inside function is $\tan x$, and the derivative of that inside function is $\sec^2 x$. In the second term, $\tan x^5$, the inside function is x^5, and the derivative of that inside function is $5x^4$. Therefore, the derivative is

$$f'(x) = 5\tan^4 x(\sec^2 x) + \sec^2(x^5)(5x^4)$$

$$f'(x) = 5\tan^4 x \sec^2 x + 5x^4 \sec^2(x^5)$$

■ 6. Find $g'(x)$ if $g(x) = \ln(e^{\sin x} - \sin^2 x)$.

Solution:

To find the derivative, we have to apply chain rule three times. We'll say that the inside function is $e^{\sin x} - \sin^2 x$, and to find the derivative of that inside function we need to use chain rule for each term.

The derivative of that inside function is $e^{\sin x}(\cos x) - 2\sin x(\cos x)$, so the derivative $g'(x)$ is

$$g'(x) = \frac{1}{e^{\sin x} - \sin^2 x}(e^{\sin x}(\cos x) - 2\sin x(\cos x))$$

$$g'(x) = \frac{e^{\sin x}(\cos x) - 2\sin x(\cos x)}{e^{\sin x} - \sin^2 x}$$

$$g'(x) = \frac{\cos x(e^{\sin x} - 2\sin x)}{e^{\sin x} - \sin^2 x}$$

CHAIN RULE WITH PRODUCT RULE

■ 1. Find $y'(x)$ if $y(x) = (3x - 2)(5x^3)^5$.

Solution:

To find the derivative, we have to apply product rule.

$$y'(x) = \frac{d}{dx}(3x - 2) \cdot (5x^3)^5 + (3x - 2) \cdot \frac{d}{dx}(5x^3)^5$$

To find each derivative, we have to apply chain rule.

$$y'(x) = 3 \cdot (5x^3)^5 + (3x - 2) \cdot 5(5x^3)^4(15x^2)$$

$$y'(x) = 3(5x^3)^5 + 75x^2(3x - 2)(5x^3)^4$$

■ 2. Find $h'(x)$ if $h(x) = (x^2 - 5x)^2(2x^3 - 3x^2)^5$.

Solution:

To find the derivative, we have to apply product rule.

$$h'(x) = \frac{d}{dx}(x^2 - 5x)^2 \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot \frac{d}{dx}(2x^3 - 3x^2)^5$$

To find each derivative, we have to apply chain rule.

$$h'(x) = 2(x^2 - 5x)(2x - 5) \cdot (2x^3 - 3x^2)^5 + (x^2 - 5x)^2 \cdot 5(2x^3 - 3x^2)^4(6x^2 - 6x)$$

$$h'(x) = 2(2x - 5)(x^2 - 5x)(2x^3 - 3x^2)^5 + 5(6x^2 - 6x)(x^2 - 5x)^2(2x^3 - 3x^2)^4$$

■ 3. Find the derivative of the function.

$$y = (\sin(7x))(7e^{4x})(2x^6 + 1)$$

Solution:

The derivative of a function $y = f(x)g(x)h(x)$ using the product rule is

$$y' = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$$

To find each derivative, we have to apply chain rule. We'll let

$$f(x) = \sin(7x)$$

$$f'(x) = 7\cos(7x)$$

$$g(x) = 7e^{4x}$$

$$g'(x) = 28e^{4x}$$

$$h(x) = 2x^6 + 1$$

$$h'(x) = 12x^5$$

Then by product rule, the derivative is

$$y' = (7\cos(7x))(7e^{4x})(2x^6 + 1) + (\sin(7x))(28e^{4x})(2x^6 + 1) + (\sin(7x))(7e^{4x})(12x^5)$$

$$y' = 49e^{4x}(2x^6 + 1)\cos(7x) + 28e^{4x}(2x^6 + 1)\sin(7x) + 84x^5e^{4x}\sin(7x)$$

■ 4. Find $h'(x)$ if $h(x) = \sin(4x)e^{3x^2+4}$.

Solution:

Use product rule to take the derivative.

$$h'(x) = \frac{d}{dx}\sin(4x) \cdot e^{3x^2+4} + \sin(4x) \cdot \frac{d}{dx}e^{3x^2+4}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \cos(4x)(4) \cdot e^{3x^2+4} + \sin(4x) \cdot e^{3x^2+4}(6x)$$

$$h'(x) = 4e^{3x^2+4}\cos(4x) + 6xe^{3x^2+4}\sin(4x)$$

Factor to simplify.

$$h'(x) = 2e^{3x^2+4}(3x\sin(4x) + 2\cos(4x))$$

■ 5. Find the derivative of the function.

$$y = \sin(x^2e^{x^2})$$

Solution:

If we use substitution with $u = x^2 e^{x^2}$, then we can rewrite the function as

$$y = \sin u$$

$$y' = \cos(u)u'$$

We need to plug in for u and u', so let's find u' using product rule and chain rule with

$$f(x) = x^2$$

$$f'(x) = 2x$$

$$g(x) = e^{x^2}$$

$$g'(x) = e^{x^2}(2x) = 2xe^{x^2}$$

Then u' is

$$u' = (x^2)(2xe^{x^2}) + (2x)(e^{x^2})$$

$$u' = 2x^3e^{x^2} + 2xe^{x^2}$$

$$u' = 2xe^{x^2}(x^2 + 1)$$

Now we can back-substitute into the equation we found for y', and then simplify.

$$y' = \cos(x^2e^{x^2}) \cdot 2xe^{x^2}(x^2 + 1)$$

$$y' = 2xe^{x^2}(x^2 + 1)\cos(x^2e^{x^2})$$

■ 6. Find $h'(x)$ if $h(x) = \ln(x^3\sqrt{3x^4 - 2x^2 + 3})$.

Solution:

If we use substitution with $u = x^3\sqrt{3x^4 - 2x^2 + 3}$, then we can rewrite the function and its derivative as

$$y = \ln u$$

$$y' = \frac{1}{u} \cdot u'$$

We need to plug in for u and u', so let's find u' using product rule and chain rule with

$$f(x) = x^3$$

$$f'(x) = 3x^2$$

$$g(x) = \sqrt{3x^4 - 2x^2 + 3}$$

$$g'(x) = \frac{1}{2}(3x^4 - 2x^2 + 3)^{-\frac{1}{2}}(12x^3 - 4x)$$

$$g'(x) = \frac{6x^3 - 2x}{\sqrt{3x^4 - 2x^2 + 3}}$$

Then

$$u' = (x^3)\left(\frac{6x^3 - 2x}{\sqrt{3x^4 - 2x^2 + 3}}\right) + (3x^2)(\sqrt{3x^4 - 2x^2 + 3})$$

$$u' = \frac{6x^6 - 2x^4}{\sqrt{3x^4 - 2x^2 + 3}} + 3x^2\sqrt{3x^4 - 2x^2 + 3}$$

Now we can back-substitute into the equation we found for y', and then simplify.

$$y' = \frac{1}{x^3\sqrt{3x^4 - 2x^2 + 3}}\left(\frac{6x^6 - 2x^4}{\sqrt{3x^4 - 2x^2 + 3}} + 3x^2\sqrt{3x^4 - 2x^2 + 3}\right)$$

$$y' = \frac{6x^3 - 2x}{3x^4 - 2x^2 + 3} + \frac{3}{x}$$

CHAIN RULE WITH QUOTIENT RULE

■ 1. Find $h'(x)$.

$$h(x) = \frac{(2x+1)^3}{(3x-2)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(2x+1)^3 \cdot (3x-2)^2 - (2x+1)^3 \cdot \frac{d}{dx}(3x-2)^2}{((3x-2)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{3(2x+1)^2(2) \cdot (3x-2)^2 - (2x+1)^3 \cdot 2(3x-2)(3)}{((3x-2)^2)^2}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2)^2 - 6(2x+1)^3(3x-2)}{(3x-2)^4}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2) - 6(2x+1)^3}{(3x-2)^3}$$

Factor the numerator.

$$h'(x) = \frac{6(2x+1)^2(3x-2-(2x+1))}{(3x-2)^3}$$

$$h'(x) = \frac{6(2x+1)^2(3x-2-2x-1)}{(3x-2)^3}$$

$$h'(x) = \frac{6(2x+1)^2(x-3)}{(3x-2)^3}$$

■ 2. Find $h'(x)$.

$$h(x) = \frac{(4x+5)^5}{(x+3)^2}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(4x+5)^5 \cdot (x+3)^2 - (4x+5)^5 \cdot \frac{d}{dx}(x+3)^2}{((x+3)^2)^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{5(4x+5)^4(4) \cdot (x+3)^2 - (4x+5)^5 \cdot 2(x+3)(1)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3)^2 - 2(4x+5)^5(x+3)}{(x+3)^4}$$

$$h'(x) = \frac{20(4x+5)^4(x+3) - 2(4x+5)^5}{(x+3)^3}$$

Factor the numerator.

$$h'(x) = \frac{2(4x+5)^4(10(x+3)-(4x+5))}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(10x+30-4x-5)}{(x+3)^3}$$

$$h'(x) = \frac{2(4x+5)^4(6x+25)}{(x+3)^3}$$

■ 3. Find $h'(x)$.

$$h(x) = \ln\left(\frac{x^3}{x^2+3}\right)$$

Solution:

If we use substitution with

$$u = \frac{x^3}{x^2+3}$$

then we can rewrite the function and take its derivative.

$$y = \ln u$$

$$y' = \frac{1}{u} \cdot u'$$

We need to plug in for u and u', so let's find u' using quotient rule, where $f(x)$ is the numerator and $g(x)$ is the denominator.

$$u' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$u' = \frac{(3x^2)(x^2+3) - (x^3)(2x)}{(x^2+3)^2}$$

Now we can back-substitute into the equation we found for y'.

$$h'(x) = \frac{1}{\frac{x^3}{x^2+3}} \cdot \frac{3x^2(x^2+3) - x^3(2x)}{(x^2+3)^2}$$

$$h'(x) = \frac{x^2+3}{x^3} \cdot \frac{3x^2(x^2+3) - x^3(2x)}{(x^2+3)^2}$$

$$h'(x) = \frac{1}{x} \cdot \frac{3(x^2+3) - x(2x)}{x^2+3}$$

$$h'(x) = \frac{3x^2+9-2x^2}{x(x^2+3)}$$

$$h'(x) = \frac{x^2+9}{x(x^2+3)}$$

■ 4. Find $h'(x)$.

$$h(x) = \frac{\sec(2-x)}{2x+e^{-x}}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}\sec(2-x)\cdot(2x+e^{-x}) - \sec(2-x)\cdot\frac{d}{dx}(2x+e^{-x})}{(2x+e^{-x})^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{\sec(2-x)\tan(2-x)(-1)\cdot(2x+e^{-x}) - \sec(2-x)\cdot(2+e^{-x}(-1))}{(2x+e^{-x})^2}$$

$$h'(x) = \frac{-(2x+e^{-x})\sec(2-x)\tan(2-x) - (2-e^{-x})\sec(2-x)}{(2x+e^{-x})^2}$$

$$h'(x) = -\frac{(2x+e^{-x})\sec(2-x)\tan(2-x) + (2-e^{-x})\sec(2-x)}{(2x+e^{-x})^2}$$

■ 5. Find $h'(x)$.

$$h(x) = \frac{2+\ln(3x)}{x+\cot(2x)}$$

Solution:

To find the derivative, we have to apply quotient rule.

$$h'(x) = \frac{\frac{d}{dx}(2+\ln(3x)) \cdot (x+\cot(2x)) - (2+\ln(3x)) \cdot \frac{d}{dx}((x+\cot(2x))}{(x+\cot(2x))^2}$$

To find each derivative, we have to apply chain rule.

$$h'(x) = \frac{\frac{1}{3x}(3) \cdot (x+\cot(2x)) - (2+\ln(3x)) \cdot (1-\csc^2(2x)(2))}{(x+\cot(2x))^2}$$

$$h'(x) = \frac{\frac{x+\cot(2x)}{x} - (2+\ln(3x))(1-2\csc^2(2x))}{(x+\cot(2x))^2}$$

$$h'(x) = \frac{\frac{x+\cot(2x)}{x}}{(x+\cot(2x))^2} - \frac{(2+\ln(3x))(1-2\csc^2(2x))}{(x+\cot(2x))^2}$$

$$h'(x) = \frac{1}{x(x+\cot(2x))} - \frac{(2+\ln(3x))(1-2\csc^2(2x))}{(x+\cot(2x))^2}$$

■ 6. Find $h'(x)$.

$$h(x) = x^2 \sin\left(\frac{x^3+4x}{\sqrt{x^4-2}}\right)$$

Solution:

Use product rule with $h'(x) = f(x)g'(x) + f'(x)g(x)$ and let

$$f(x) = x^2$$

$$f'(x) = 2x$$

and

$$g(x) = \sin\left(\frac{x^3 + 4x}{\sqrt{x^4 - 2}}\right)$$

To find the derivative $g'(x)$, we have to apply chain rule. If we use substitution with

$$u = \frac{x^3 + 4x}{\sqrt{x^4 - 2}}$$

then we can rewrite $g(x)$ and its derivative as

$$g(u) = \sin u$$

$$g'(u) = \cos u \cdot u'$$

We need to plug in for u and u', so let's find u' using quotient rule.

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2}) - (x^3 + 4x)\left(\frac{1}{2\sqrt{x^4 - 2}}\right)(4x^3)}{(\sqrt{x^4 - 2})^2}$$

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2}) - 2x^3(x^3 + 4x)\left(\frac{1}{\sqrt{x^4 - 2}}\right)}{x^4 - 2}$$

$$u' = \frac{(3x^2 + 4)(\sqrt{x^4 - 2})}{x^4 - 2} - \frac{2x^3(x^3 + 4x)\left(\frac{1}{\sqrt{x^4 - 2}}\right)}{x^4 - 2}$$

$$u' = \frac{3x^2+4}{\sqrt{x^4-2}} - \frac{2x^3(x^3+4x)}{\sqrt{(x^4-2)^3}}$$

Now we can back-substitute into $g'(u) = \cos u \cdot u'$, and then simplify.

$$g'(x) = \cos\left(\frac{x^3+4x}{\sqrt{x^4-2}}\right) \cdot \left(\frac{3x^2+4}{\sqrt{x^4-2}} - \frac{2x^3(x^3+4x)}{\sqrt{(x^4-2)^3}}\right)$$

Now plug into the product rule formula.

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = x^2\cos\left(\frac{x^3+4x}{\sqrt{x^4-2}}\right)\left(\frac{3x^2+4}{\sqrt{x^4-2}} - \frac{2x^3(x^3+4x)}{\sqrt{(x^4-2)^3}}\right) + 2x\sin\left(\frac{x^3+4x}{\sqrt{x^4-2}}\right)$$

INVERSE TRIGONOMETRIC DERIVATIVES

1. Find $f'(t)$.

$$f(t)=4\sin^{-1}\left(\frac{t}{4}\right)$$

Solution:

The derivative of inverse sine is given by

$$\frac{d}{dt}a\sin^{-1}(y(t))=a\cdot\frac{y'(t)}{\sqrt{1-[y(t)]^2}}$$

If $a=4$ and $y(t)=t/4$, then $y'(t)=1/4$. Then the derivative is

$$f'(t)=4\cdot\frac{\frac{1}{4}}{\sqrt{1-\left(\frac{t}{4}\right)^2}}=\frac{1}{\sqrt{\frac{16}{16}-\frac{t^2}{16}}}=\frac{1}{\sqrt{\frac{16-t^2}{16}}}=\frac{1}{\frac{\sqrt{16-t^2}}{4}}=\frac{4}{\sqrt{16-t^2}}$$

2. Find $g'(t)$.

$$g(t)=-6\cos^{-1}(2t+3)$$

Solution:

The derivative of inverse cosine is given by

$$\frac{d}{dt}a\cos^{-1}(y(t)) = a \cdot \left(-\frac{y'(t)}{\sqrt{1-[y(t)]^2}}\right)$$

If $a = -6$ and $y(t) = 2t + 3$, then $y'(t) = 2$, and the derivative is

$$g'(t) = (-6) \cdot \left(-\frac{2}{\sqrt{1-(2t+3)^2}}\right)$$

$$g'(t) = \frac{12}{\sqrt{1-4t^2-12t-9}}$$

$$g'(t) = \frac{12}{\sqrt{-4t^2-12t-8}}$$

$$g'(t) = \frac{12}{2\sqrt{-t^2-3t-2}}$$

$$g'(t) = \frac{6}{\sqrt{-(t+1)(t+2)}}$$

■ 3. Find $h'(t)$.

$$h(t) = 2\sec^{-1}(6t^2+3) - 8\cot^{-1}\left(\frac{t^3}{3}\right)$$

Solution:

We differentiate one term at a time. The derivative of inverse secant is given by

$$\frac{d}{dt}a\sec^{-1}(y(t))=a\cdot\frac{y'(t)}{|y(t)|\sqrt{[y(t)]^2-1}}$$

If $a=2$ and $y(t)=6t^2+3$, then $y'(t)=12t$, and the derivative of the first term is

$$2\cdot\frac{12t}{|6t^2+3|\sqrt{(6t^2+3)^2-1}}$$

$$\frac{24t}{|6t^2+3|\sqrt{36t^4+36t^2+9-1}}$$

$$\frac{24t}{|6t^2+3|\sqrt{36t^4+36t^2+8}}$$

$$\frac{12t}{|6t^2+3|\sqrt{9t^4+9t^2+2}}$$

$$\frac{4t}{|2t^2+1|\sqrt{9t^4+9t^2+2}}$$

The derivative of inverse cotangent is given by

$$\frac{d}{dt}a\cot^{-1}(y(t))=a\cdot\left(-\frac{y'(t)}{1+[y(t)]^2}\right)$$

If $a=-8$ and $y(t)=t^3/3$, then $y'(t)=t^2$ and the derivative is

$$-8 \cdot \left(-\frac{t^2}{1 + \left(\frac{t^3}{3}\right)^2} \right) = \frac{8t^2}{1 + \frac{t^6}{9}} = \frac{72t^2}{9 + t^6}$$

Then

$$h'(t) = \frac{4t}{|2t^2 + 1|\sqrt{9t^4 + 9t^2 + 2}} + \frac{72t^2}{9 + t^6}$$

4. Find the derivative.

$$y = (x^4 + x^2)\csc^{-1} x + \sin(5x^3)$$

Solution:

We'll need to use product rule for the first term, $(x^4 + x^2)\csc^{-1} x$, as well as the formula for the derivative of inverse cosecant.

$$\frac{d}{dt} a \csc^{-1}(y(t)) = a \cdot \left(-\frac{y'(t)}{|y(t)|\sqrt{[y(t)]^2 - 1}} \right)$$

Then the derivative is

$$y' = (4x^3 + 2x)(\csc^{-1} x) + (x^4 + x^2)\left(-\frac{1}{|x|\sqrt{x^2 - 1}} \right) + \cos(5x^3)(15x^2)$$

$$y' = (4x^3 + 2x)(\csc^{-1} x) - \frac{x^4 + x^2}{|x|\sqrt{x^2 - 1}} + 15x^2 \cos(5x^3)$$

■ 5. Find the derivative.

$$y = \frac{\sin^{-1}\left(x + \frac{x^2}{2}\right)}{1 + x}$$

Solution:

We'll need to use quotient rule and the formula for the derivative of the inverse sine function.

$$\frac{d}{dt} a \sin^{-1}(y(t)) = a \cdot \frac{y'(t)}{\sqrt{1 - [y(t)]^2}}$$

With

$$y(t) = x + \frac{x^2}{2}$$

$$y'(t) = 1 + \frac{2x}{2} = 1 + x$$

the derivative is

$$y' = \frac{\frac{d}{dx} \sin^{-1}\left(x + \frac{x^2}{2}\right) \cdot (1 + x) - \frac{d}{dx}(1 + x) \cdot \sin^{-1}\left(x + \frac{x^2}{2}\right)}{(1 + x)^2}$$

$$y' = \frac{\frac{1+x}{\sqrt{1-\left(x+\frac{x^2}{2}\right)^2}} \cdot (1+x) - 1 \cdot \sin^{-1}\left(x+\frac{x^2}{2}\right)}{(1+x)^2}$$

$$y' = \frac{\frac{(1+x)^2}{\sqrt{1-\left(x+\frac{x^2}{2}\right)^2}} - \sin^{-1}\left(x+\frac{x^2}{2}\right)}{(1+x)^2}$$

$$y' = \frac{\frac{(1+x)^2}{\sqrt{1-\left(x+\frac{x^2}{2}\right)^2}}}{(1+x)^2} - \frac{\sin^{-1}\left(x+\frac{x^2}{2}\right)}{(1+x)^2}$$

$$y' = \frac{1}{\sqrt{1-\left(x+\frac{x^2}{2}\right)^2}} - \frac{\sin^{-1}\left(x+\frac{x^2}{2}\right)}{(1+x)^2}$$

■ 6. Find the derivative.

$$y = \frac{1-\sin^{-1}(2x)}{1+\cos^{-1}(2x)}$$

Solution:

We'll need to use quotient rule and the formulas for the derivatives of the inverse sine and cosine functions.

$$\frac{d}{dt}a\sin^{-1}(y(t)) = a \cdot \frac{y'(t)}{\sqrt{1-[y(t)]^2}}$$

$$\frac{d}{dt}a\cos^{-1}(y(t)) = a \cdot \left(-\frac{y'(t)}{\sqrt{1-[y(t)]^2}}\right)$$

If $y(t) = 2x$, then $y'(t) = 2$. Then the derivative is

$$y' = \frac{\frac{d}{dx}(1-\sin^{-1}(2x)) \cdot (1+\cos^{-1}(2x)) - \frac{d}{dx}(1+\cos^{-1}(2x)) \cdot (1-\sin^{-1}(2x))}{(1+\cos^{-1}(2x))^2}$$

$$y' = \frac{\left(-\frac{2}{\sqrt{1-(2x)^2}}\right) \cdot (1+\cos^{-1}(2x)) - \left(-\frac{2}{\sqrt{1-(2x)^2}}\right) \cdot (1-\sin^{-1}(2x))}{(1+\cos^{-1}(2x))^2}$$

$$y' = \frac{\left(-\frac{2}{\sqrt{1-(2x)^2}}\right)[(1+\cos^{-1}(2x)) - (1-\sin^{-1}(2x))]}{(1+\cos^{-1}(2x))^2}$$

$$y' = -\frac{2(1+\cos^{-1}(2x) - 1 + \sin^{-1}(2x))}{\sqrt{1-(2x)^2}(1+\cos^{-1}(2x))^2}$$

$$y' = -\frac{2(\cos^{-1}(2x) + \sin^{-1}(2x))}{\sqrt{1-4x^2}(1+\cos^{-1}(2x))^2}$$

HYPERBOLIC DERIVATIVES

■ 1. Find $f'(\theta)$ if $f(\theta) = 3\sinh(2\theta^2 - 5\theta + 2)$.

Solution:

The derivative of hyperbolic sine is given by

$$\frac{d}{d\theta}a\sinh(y(\theta)) = a \cdot \cosh(y(\theta)) \cdot y'(\theta)$$

If $a = 3$ and $y(\theta) = 2\theta^2 - 5\theta + 2$, then $y'(\theta) = 4\theta - 5$. Then the derivative is

$$f'(\theta) = 3\cosh(2\theta^2 - 5\theta + 2)(4\theta - 5)$$

$$f'(\theta) = 3(4\theta - 5)\cosh(2\theta^2 - 5\theta + 2)$$

■ 2. Find $g'(\theta)$ if $g(\theta) = 2\cosh(5\theta^{\frac{3}{2}} + 6\theta)$.

Solution:

The derivative of hyperbolic cosine is given by

$$\frac{d}{d\theta}a\cosh(y(\theta)) = a \cdot \sinh(y(\theta)) \cdot y'(\theta)$$

If $a = 2$ and $y(\theta) = 5\theta^{\frac{3}{2}} + 6\theta$, then $y'(\theta) = 5(3/2)\theta^{\frac{1}{2}} + 6$. Then the derivative is

$$g'(\theta) = 2\sinh(5\theta^{\frac{3}{2}} + 6\theta)\left(\frac{15}{2}\theta^{\frac{1}{2}} + 6\right)$$

$$g'(\theta) = (15\theta^{\frac{1}{2}} + 12)\sinh(5\theta^{\frac{3}{2}} + 6\theta)$$

$$g'(\theta) = 3(5\theta^{\frac{1}{2}} + 4)\sinh(5\theta^{\frac{3}{2}} + 6\theta)$$

■ 3. Find $h'(\theta)$ if $h(\theta) = 9\tanh(3\theta^2 - \theta^{\sqrt{3}})$.

Solution:

The derivative of hyperbolic tangent is given by

$$\frac{d}{d\theta}a\tanh(y(\theta)) = a \cdot \text{sech}^2(y(\theta)) \cdot y'(\theta)$$

If $a = 9$ and $y(\theta) = 3\theta^2 - \theta^{\sqrt{3}}$, then $y'(\theta) = 6\theta - \sqrt{3} \cdot \theta^{\sqrt{3}-1}$. Then the derivative is

$$h'(\theta) = 9(6\theta - \sqrt{3} \cdot \theta^{\sqrt{3}-1})\text{sech}^2(3\theta^2 - \theta^{\sqrt{3}})$$

■ 4. Find the derivative of the hyperbolic function.

$$y = \coth(x^2 + 3x) - x^4\text{csch}(x^2)$$

Solution:

Let's work on one term at a time. The derivative of the first term can be found by applying the formula for the derivative of hyperbolic cotangent with $g(x) = x^2 + 3x$. The derivative of that first term will be given by

$$-\text{csch}^2[g(x)][g'(x)]$$

$$-\text{csch}^2(x^2 + 3x)(2x + 3)$$

$$-(2x + 3)\text{csch}^2(x^2 + 3x)$$

To find the derivative of the second term, we'll use product rule and the formula for the derivative of hyperbolic cosecant.

$$(x^4)(-\text{csch}(x^2)\coth(x^2))(2x) + (4x^3)(\text{csch}(x^2))$$

$$-2x^5\text{csch}(x^2)\coth(x^2) + 4x^3\text{csch}(x^2)$$

Putting these derivatives together gives the derivative for the original function.

$$y' = -(2x + 3)\text{csch}^2(x^2 + 3x) - (-2x^5\text{csch}(x^2)\coth(x^2) + 4x^3\text{csch}(x^2))$$

$$y' = -(2x + 3)\text{csch}^2(x^2 + 3x) + 2x^5\text{csch}(x^2)\coth(x^2) - 4x^3\text{csch}(x^2)$$

■ 5. Find the derivative of the hyperbolic function.

$$y = \frac{2x + 3e^x}{\cosh(x^{-5})}$$

Solution:

To find the derivative, we'll use quotient rule, and apply the formula for the derivative of hyperbolic cosine.

$$y' = \frac{(2+3e^x)(\cosh(x^{-5})) - (2x+3e^x)(\sinh(x^{-5}))(-5x^{-6})}{(\cosh(x^{-5}))^2}$$

$$y' = \frac{(2+3e^x)\cosh(x^{-5}) + 5x^{-6}(2x+3e^x)\sinh(x^{-5})}{(\cosh(x^{-5}))^2}$$

■ 6. Find the derivative of the hyperbolic function.

$$y = \tanh(x^2)\tan(x^2)$$

Solution:

To find the derivative, we'll use product rule and the formula for the derivative of hyperbolic tangent. Following the format of product rule,

$$y' = f(x)g'(x) + f'(x)g(x)$$

we'll identify values for the product rule formula.

$$f(x) = \tanh(x^2)$$

$$f'(x) = \text{sech}^2(x^2)(2x) = 2x\text{sech}^2(x^2)$$

and

$$g(x) = \tan(x^2)$$

$$g'(x) = \sec^2(x^2)(2x) = 2x\sec^2(x^2)$$

Then the derivative of the original function is

$$y' = (\tanh(x^2))(2x\sec^2(x^2)) + (2x\text{sech}^2(\text{x}^2))(\tan(x^2))$$

$$y' = 2x\tanh(x^2)\sec^2(x^2) + 2x\tan(x^2)\text{sech}^2(x^2)$$

INVERSE HYPERBOLIC DERIVATIVES

■ 1. Find $f'(t)$ if $f(t) = 7\sinh^{-1}(5t^4)$.

Solution:

The derivative of inverse hyperbolic sine is given by

$$\frac{d}{dt}a\sinh^{-1}(y(t)) = a \cdot \frac{y'(t)}{\sqrt{[y(t)]^2 + 1}}$$

If $a = 7$ and $y(t) = 5t^4$, then $y'(t) = 20t^3$. Then the derivative is

$$f'(t) = 7 \cdot \frac{20t^3}{\sqrt{(5t^4)^2 + 1}} = \frac{140t^3}{\sqrt{25t^8 + 1}}$$

■ 2. Find $g'(t)$ if $g(t) = 4\cosh^{-1}(2t - 3)$.

Solution:

The derivative of inverse hyperbolic cosine is given by

$$\frac{d}{dt}a\cosh^{-1}(y(t)) = a \cdot \frac{y'(t)}{\sqrt{[y(t)]^2 - 1}}$$

If $a=4$ and $y(t)=2t-3$, then $y'(t)=2$. Then the derivative is

$$g'(t)=4\cdot\frac{2}{\sqrt{(2t-3)^2-1}}$$

$$g'(t)=\frac{8}{\sqrt{4t^2-12t+9-1}}$$

$$g'(t)=\frac{8}{\sqrt{4t^2-12t+8}}$$

$$g'(t)=\frac{8}{\sqrt{4(t-1)(t-2)}}$$

$$g'(t)=\frac{4}{\sqrt{(t-1)(t-2)}}$$

■ 3. Find $h'(t)$ if $h(t)=9\tanh^{-1}(-7t+2)$.

Solution:

The derivative of inverse hyperbolic tangent is given by

$$\frac{d}{dt}a\tanh^{-1}(y(t))=a\cdot\frac{y'(t)}{1-\left[y(t)\right]^2}$$

If $a=9$ and $y(t)=-7t+2$, then $y'(t)=-7$. Then the derivative is

$$h'(t) = 9 \cdot \frac{-7}{1-(-7t+2)^2}$$

$$h'(t) = -\frac{63}{1-(49t^2-28t+4)}$$

$$h'(t) = -\frac{63}{1-49t^2+28t-4}$$

$$h'(t) = -\frac{63}{-49t^2+28t-3}$$

$$h'(t) = \frac{63}{49t^2-28t+3}$$

■ 4. Find the derivative of the inverse hyperbolic function.

$$y = \cosh^{-1}(3x^3+4x^2) - x^2\sinh^{-1}(e^x)$$

Solution:

Apply the formula for the derivative of inverse hyperbolic cosine with $g(x) = 3x^3+4x^2$ and $g'(x) = 9x^2+8x$, and the formula for the derivative of inverse hyperbolic sine with $g(x) = e^x$ and $g'(x) = e^x$. We'll also need to use product rule for the second term.

$$y' = \left(\frac{1}{\sqrt{(3x^3+4x^2)^2-1}}\right)(9x^2+8x)$$

$$-\left[(2x)(\sinh^{-1}(e^x))+(x^2)\left(\frac{1}{\sqrt{(e^x)^2+1}}\right)(e^x)\right]$$

$$y'=\frac{9x^2+8x}{\sqrt{(3x^3+4x^2)^2-1}}-2x\sinh^{-1}(e^x)-\frac{x^2e^x}{\sqrt{e^{2x}+1}}$$

■ 5. Find the derivative of the inverse hyperbolic function.

$$y=\left(\operatorname{csch}^{-1}\left(\frac{x^2}{3x^4+1}\right)\right)^5$$

Solution:

Use a substitution with

$$u=\operatorname{csch}^{-1}\left(\frac{x^2}{3x^4+1}\right)$$

and apply the formula for the derivative of inverse hyperbolic cosecant. With

$$g(x)=\frac{x^2}{3x^4+1}$$

$$g'(x)=\frac{2x-6x^5}{(3x^4+1)^2}$$

we get

$$u' = -\frac{1}{\left|\frac{x^2}{3x^4+1}\right|\sqrt{\left(\frac{x^2}{3x^4+1}\right)^2+1}} \cdot \frac{2x-6x^5}{(3x^4+1)^2}$$

$$u' = -\frac{\left|\frac{3x^4+1}{x^2}\right| \cdot \frac{2x-6x^5}{(3x^4+1)^2}}{\sqrt{\left(\frac{x^2}{3x^4+1}\right)^2+1}}$$

$$u' = -\frac{\frac{2-6x^4}{x(3x^4+1)}}{\sqrt{\left(\frac{x^2}{3x^4+1}\right)^2+1}}$$

$$u' = -\frac{2-6x^4}{x(3x^4+1)\sqrt{\left(\frac{x^2}{3x^4+1}\right)^2+1}}$$

$$u' = -\frac{2-6x^4}{x(3x^4+1)\sqrt{\frac{x^4}{(3x^4+1)^2}+1}}$$

$$u' = -\frac{2-6x^4}{x(3x^4+1)\sqrt{\frac{x^4+(3x^4+1)^2}{(3x^4+1)^2}}}$$

$$u' = -\frac{2-6x^4}{x(3x^4+1)\frac{\sqrt{x^4+(3x^4+1)^2}}{3x^4+1}}$$

$$u' = -\frac{2-6x^4}{x\sqrt{x^4+(3x^4+1)^2}}$$

Then the function is

$$y = u^5$$

and its derivative is

$$y'(x) = 5u^4 \cdot u'$$

$$y'(x) = 5\left(\text{csch}^{-1}\left(\frac{x^2}{3x^4+1}\right)\right)^4 \cdot \left(-\frac{2-6x^4}{x\sqrt{x^4+(3x^4+1)^2}}\right)$$

$$y'(x) = -\frac{10-30x^4}{x\sqrt{x^4+(3x^4+1)^2}}\left(\text{csch}^{-1}\left(\frac{x^2}{3x^4+1}\right)\right)^4$$

■ 6. Find the derivative of the inverse hyperbolic function.

$$y = -\frac{\coth^{-1} x}{\tanh^{-1}(2x^4)}$$

Solution:

Apply the formulas for the derivative of inverse hyperbolic cotangent and inverse hyperbolic tangent. We'll also need to use quotient rule.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

First, let's list out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = \coth^{-1} x$$

$$f'(x) = \frac{1}{1 - x^2}$$

and

$$g(x) = \tanh^{-1}(2x^4)$$

$$g'(x) = \frac{1}{1 - (2x^4)^2} \cdot (8x^3) = \frac{8x^3}{1 - 4x^8}$$

Now we can plug these values directly into the quotient rule formula.

$$y' = -\frac{\left(\frac{1}{1 - x^2}\right)(\tanh^{-1}(2x^4)) - (\coth^{-1} x)\left(\frac{8x^3}{1 - 4x^8}\right)}{(\tanh^{-1}(2x^4))^2}$$

$$y' = -\frac{\left(\frac{1}{1 - x^2}\right)(\tanh^{-1}(2x^4))}{(\tanh^{-1}(2x^4))^2} + \frac{(\coth^{-1} x)\left(\frac{8x^3}{1 - 4x^8}\right)}{(\tanh^{-1}(2x^4))^2}$$

$$y' = -\frac{\frac{1}{1 - x^2}}{\tanh^{-1}(2x^4)} + \frac{\coth^{-1} x\left(\frac{8x^3}{1 - 4x^8}\right)}{(\tanh^{-1}(2x^4))^2}$$

$$y' = -\frac{1}{(1 - x^2)\tanh^{-1}(2x^4)} + \frac{8x^3 \coth^{-1} x}{(1 - 4x^8)(\tanh^{-1}(2x^4))^2}$$

LOGARITHMIC DIFFERENTIATION

1. Use logarithmic differentiation to find dy/dx.

$$y = (\ln x)^{\ln(x^2)}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln((\ln x)^{\ln(x^2)})$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln(x^2)\ln(\ln x)$$

Differentiate, remembering to apply product and chain rule, then solve for dy/dx.

$$\frac{1}{y} \cdot y' = \frac{2x}{x^2} \cdot \ln(\ln x) + \ln x^2 \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$\frac{1}{y} \cdot y' = \frac{2\ln(\ln x)}{x} + \frac{\ln x^2}{x \ln x}$$

$$y' = y\left(\frac{2\ln(\ln x)}{x} + \frac{\ln x^2}{x \ln x}\right)$$

Substitute for y.

$$y' = (\ln x)^{\ln(x^2)}\left(\frac{2\ln(\ln x)}{x} + \frac{\ln x^2}{x\ln x}\right)$$

■ 2. Use logarithmic differentiation to find dy/dx.

$$y = 5x^4e^{3x}\sqrt[4]{x}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln\left(5x^4e^{3x}\sqrt[4]{x}\right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln 5x^4 + \ln e^{3x} + \ln\sqrt[4]{x}$$

$$\ln y = \ln 5 + \ln x^4 + 3x + \ln x^{\frac{1}{4}}$$

$$\ln y = \ln 5 + 4\ln x + 3x + \frac{1}{4}\ln x$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y}\cdot\frac{dy}{dx} = 0 + \frac{4}{x} + 3 + \frac{1}{4x}$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \frac{4}{x} + 3 + \frac{1}{4x}$$

$$\frac{dy}{dx} = y\left(\frac{4}{x} + 3 + \frac{1}{4x}\right)$$

Substitute for y.

$$\frac{dy}{dx} = 5x^4e^{3x}\sqrt[4]{x}\left(\frac{4}{x} + 3 + \frac{1}{4x}\right)$$

You could leave the answer like this, or try to simplify.

$$\frac{dy}{dx} = 5x^4e^{3x}\sqrt[4]{x}\left(\frac{16}{4x} + \frac{12x}{4x} + \frac{1}{4x}\right)$$

$$\frac{dy}{dx} = 5x^4e^{3x}\sqrt[4]{x}\left(\frac{12x + 17}{4x}\right)$$

$$\frac{dy}{dx} = \frac{5x^3e^{3x}\sqrt[4]{x}(12x + 17)}{4}$$

■ 3. Use logarithmic differentiation to find dy/dx.

$$y = (7 - 4x^3)^{x^2+9}\sqrt[3]{1 - \cos(3x)}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln[(7 - 4x^3)^{x^2+9}\sqrt[3]{1 - \cos(3x)}]$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln(7-4x^3)^{x^2+9} + \ln(\sqrt[3]{1-\cos(3x)})$$

$$\ln y = (x^2+9)\ln(7-4x^3) + \ln(1-\cos(3x))^{\frac{1}{3}}$$

$$\ln y = (x^2+9)\ln(7-4x^3) + \frac{1}{3}\ln(1-\cos(3x))$$

Differentiate, remembering to apply product and chain rule, then solve for dy/dx.

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2x\ln(7-4x^3) + (x^2+9)\left(\frac{-12x^2}{7-4x^3}\right) + \frac{1}{3(1-\cos(3x))}(3\sin(3x))$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = 2x\ln(7-4x^3) - \frac{12x^2(x^2+9)}{7-4x^3} + \frac{\sin(3x)}{1-\cos(3x)}$$

$$\frac{dy}{dx} = y\left(2x\ln(7-4x^3) - \frac{12x^2(x^2+9)}{7-4x^3} + \frac{\sin(3x)}{1-\cos(3x)}\right)$$

Substitute for y.

$$\frac{dy}{dx} = (7-4x^3)^{x^2+9}\sqrt[3]{1-\cos(3x)}\left(2x\ln(7-4x^3) - \frac{12x^2(x^2+9)}{7-4x^3} + \frac{\sin(3x)}{1-\cos(3x)}\right)$$

■ 4. Use logarithmic differentiation to find dy/dx.

$$y = \frac{(2e)^{\cos x}}{(3e)^{\sin x}}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln\left(\frac{(2e)^{\cos x}}{(3e)^{\sin x}}\right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln(2e)^{\cos x} - \ln(3e)^{\sin x}$$

$$\ln y = \cos x \ln(2e) - \sin x \ln(3e)$$

$$\ln y = (\cos x)(\ln 2 + \ln e) - (\sin x)(\ln 3 + \ln e)$$

$$\ln y = (\cos x)(\ln 2 + 1) - (\sin x)(\ln 3 + 1)$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y} \cdot \frac{dy}{dx} = (-\sin x)(\ln 2 + 1) + (\cos x)(0) - \left[(\cos x)(\ln 3 + 1) + (\sin x)(0)\right]$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = -(\ln 2 + 1)\sin x - (\ln 3 + 1)\cos x$$

$$\frac{dy}{dx} = -y\left[(\ln 2 + 1)\sin x + (\ln 3 + 1)\cos x\right]$$

Substitute for y.

$$\frac{dy}{dx} = -\frac{(2e)^{\cos x}}{(3e)^{\sin x}}\left[(\ln 2 + 1)\sin x + (\ln 3 + 1)\cos x\right]$$

■ 5. Use logarithmic differentiation to find dy/dx.

$$y = e^x(2e)^{\sin x}(3e)^{\cos x}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln\left(e^x(2e)^{\sin x}(3e)^{\cos x}\right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln e^x + \ln(2e)^{\sin x} + \ln(3e)^{\cos x}$$

$$\ln y = x + \sin x\ln(2e) + \cos x\ln(3e)$$

$$\ln y = x + \sin x(\ln 2 + \ln e) + \cos x(\ln 3 + \ln e)$$

$$\ln y = x + \sin x(\ln 2 + 1) + \cos x(\ln 3 + 1)$$

Differentiate, remembering to apply chain rule, then solve for dy/dx.

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1 + (\ln 2 + 1)\cos x - (\ln 3 + 1)\sin x$$

$$\frac{dy}{dx} = y\left[1 + (\ln 2 + 1)\cos x - (\ln 3 + 1)\sin x\right]$$

Substitute for y.

$$\frac{dy}{dx} = e^x(2e)^{\sin x}(3e)^{\cos x}\left[1 + (\ln 2 + 1)\cos x - (\ln 3 + 1)\sin x\right]$$

■ 6. Use logarithmic differentiation to find dy/dx.

$$y = \frac{(1-2x)^{\sin x}}{(x^3-2x)^{5x+7}}$$

Solution:

Take the natural log of both sides.

$$\ln y = \ln\left(\frac{(1-2x)^{\sin x}}{(x^3-2x)^{5x+7}}\right)$$

Use properties of logarithms to rewrite the equation.

$$\ln y = \ln(1-2x)^{\sin x} - \ln(x^3-2x)^{5x+7}$$

$$\ln y = \sin x \ln(1-2x) - (5x+7)\ln(x^3-2x)$$

Differentiate, remembering to apply product and chain rule, then solve for dy/dx.

$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \ln(1-2x) + \sin x\left(\frac{-2}{1-2x}\right) - \left[5\ln(x^3-2x) + (5x+7)\left(\frac{3x^2-2}{x^3-2x}\right)\right]$$

$$\frac{1}{y}\cdot\frac{dy}{dx} = \cos x \ln(1-2x) - \frac{2\sin x}{1-2x} - 5\ln(x^3-2x) - \frac{(3x^2-2)(5x+7)}{x^3-2x}$$

Substitute for y.

$$\frac{dy}{dx}=y\left(\cos x\ln(1-2x)-\frac{2\sin x}{1-2x}-5\ln(x^3-2x)-\frac{(3x^2-2)(5x+7)}{x^3-2x}\right)$$

$$\frac{dy}{dx}=\frac{(1-2x)^{\sin x}}{(x^3-2x)^{5x+7}}\left(\cos x\ln(1-2x)-\frac{2\sin x}{1-2x}-5\ln(x^3-2x)-\frac{(3x^2-2)(5x+7)}{x^3-2x}\right)$$

TANGENT LINES

■ 1. Find the equation of the tangent line to the graph of the equation at $(1/2,\pi)$.

$$f(x)=4\arctan 2x$$

Solution:

The derivative of $\arctan x$ is given by

$$\frac{d}{dx}\arctan x=\frac{1}{1+x^2}$$

So the derivative is

$$f'(x)=\frac{4}{1+(2x)^2}\cdot 2$$

$$f'(x)=\frac{8}{1+4x^2}$$

Evaluating the derivative at $(1/2,\pi)$, we get

$$f'\left(\frac{1}{2}\right)=\frac{8}{1+4\left(\frac{1}{2}\right)^2}=\frac{8}{1+1}=\frac{8}{2}=4$$

Now we can find the equation of the tangent line by plugging the slope $f'(1/2) = 4$ and the point $(1/2,\pi)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f\left(\frac{1}{2}\right) + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4\left(x - \frac{1}{2}\right)$$

$$y = \pi + 4x - 2$$

$$y = 4x + \pi - 2$$

■ 2. Find the equation of the tangent line to the graph of the equation at $(-1, -9)$.

$$g(x) = x^3 - 2x^2 + x - 5$$

Solution:

The derivative is

$$g'(x) = 3x^2 - 4x + 1$$

Evaluating the derivative at $(-1, -9)$, we get

$$g'(-1) = 3(-1)^2 - 4(-1) + 1$$

$$g'(-1) = 3(1) + 4(1) + 1$$

$$g'(-1) = 3 + 4 + 1$$

$$g'(-1) = 8$$

Now we can find the equation of the tangent line by plugging the slope $g'(-1) = 8$ and the point $(-1, -9)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = g(-1) + 8(x - (-1))$$

$$y = -9 + 8(x + 1)$$

$$y = -9 + 8x + 8$$

$$y = 8x - 1$$

■ 3. Find the equation of the tangent line to the graph of the equation at $(0, -4)$.

$$h(x) = -4e^{-x} + 3x$$

Solution:

The derivative is

$$h'(x) = -4(-1)e^{-x} + 3$$

$$h'(x) = 4e^{-x} + 3$$

Evaluating the derivative at $(0, -4)$, we get

$$h'(0) = 4e^{-0} + 3$$

$$h'(0) = 4(1) + 3$$

$$h'(0) = 7$$

Now we can find the equation of the tangent line by plugging the slope $h'(0) = 7$ and the point $(0, -4)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = h(0) + 7(x - 0)$$

$$y = -4 + 7(x - 0)$$

$$y = -4 + 7x$$

$$y = 7x - 4$$

■ 4. Find the equation of the tangent line to the graph of the equation at $(1,1)$.

$$f(x) = -6x^4 + 4x^3 - 3x^2 + 5x + 1$$

Solution:

The derivative is

$$f'(x) = -24x^3 + 12x^2 - 6x + 5$$

Evaluating the derivative at $(1,1)$, we get

$$f'(1) = -24(1)^3 + 12(1)^2 - 6(1) + 5$$

$$f'(1) = -24 + 12 - 6 + 5$$

$$f'(1) = -13$$

Now we can find the equation of the tangent line by plugging the slope $f'(1) = -13$ and the point $(1,1)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f(1) - 13(x - 1)$$

$$y = 1 - 13x + 13$$

$$y = -13x + 14$$

■ 5. At what point(s) is the tangent line of $f(x) = 2x(3 - x)^2$ horizontal?

Solution:

A line is horizontal when it's slope is 0, so we need to find the point(s) at which the slope of the tangent line is 0.

Take the derivative of the function using the product rule.

$$f'(x) = 2(3 - x)^2 + (2x)(-2)(3 - x)$$

$$f'(x) = 2(9 - 6x + x^2) - 12x + 4x^2$$

$$f'(x) = 18 - 12x + 2x^2 - 12x + 4x^2$$

$$f'(x) = 6x^2 - 24x + 18$$

$$f'(x) = 6(x^2 - 4x + 3)$$

$$f'(x) = 6(x - 3)(x - 1)$$

Set the derivative equal to 0 and solve for x.

$$6(x - 3)(x - 1) = 0$$

$x = 3$ and $x = 1$

Plug these x-values into $f(x)$ to find the y-values where tangent line is horizontal.

$$f(1) = 2(1)(3 - 1)^2 = 2(2^2) = 8$$

$$f(3) = 2(3)(3 - 3)^2 = 6(0) = 0$$

So the function has horizontal tangent lines at $(1,8)$ and $(3,0)$.

■ 6. Find the constants a, b, and c such that the function $f(x) = ax^2 + bx + c$ intersects the point $(-2,5)$ and has a horizontal tangent line at $(0, -3)$.

Solution:

We know the function must intersect $(-2,5)$ and $(0,-3)$. We'll start by plugging $(0,-3)$ into the function,

$$-3 = a(0)^2 + b(0) + c$$

$$-3 = c$$

then we'll plug in $(-2,5)$.

$$5 = a(-2)^2 + b(-2) - 3$$

$$5 = 4a - 2b - 3$$

$$8 = 4a - 2b$$

$$4 = 2a - b$$

When the derivative is zero, the tangent line is horizontal, so we'll take the derivative,

$$f'(x) = 2ax + b$$

and then set it equal to 0 and solve for x.

$$0 = 2a(0) + b$$

$$b = 0$$

Substitute $b = 0$ into $4 = 2a - b$ to find the value of a.

$$4 = 2a - 0$$

$$a = 2$$

So the function $f(x) = ax^2 + bx + c$ will intersect $(-2,5)$ and $(0, -3)$ and have a horizontal tangent line at $(0, -3)$ when $a = 2$, $b = 0$, and $c = -3$, such that the function is

$$f(x) = 2x^2 + 0x - 3$$

$$f(x) = 2x^2 - 3$$

VALUE THAT MAKES TWO TANGENT LINES PARALLEL

■ 1. Find the value of a such that the tangent lines to $f(x) = 2x^3 + 2$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $f(x)$.

$$f'(x) = 6x^2$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 6a^2$$

$$f'(a + 1) = 6a^2 + 12a + 6$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$6a^2 = 6a^2 + 12a + 6$$

$$0 = 12a + 6$$

$$-12a = 6$$

$$a = -\frac{1}{2}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = -\frac{1}{2} + 1$$

$$a + 1 = \frac{1}{2}$$

Therefore, the function has parallel tangent lines one unit apart at $x = -1/2$ and $x = 1/2$.

■ 2. Find the value of a such that the tangent lines to $g(x) = x^3 + x^2 + 7$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $g(x)$.

$$g'(x) = 3x^2 + 2x$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$g'(a) = 3a^2 + 2a$$

$$g'(a + 1) = 3a^2 + 8a + 5$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 2a = 3a^2 + 8a + 5$$

$$-6a = 5$$

$$a = -\frac{5}{6}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = -\frac{5}{6} + 1$$

$$a + 1 = \frac{1}{6}$$

Therefore, the function has parallel tangent lines one unit apart at $x = -5/6$ and $x = 1/6$.

■ 3. Find the value of a such that the tangent lines to $h(x) = \tan^{-1} x$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $h(x)$.

$$h'(x) = \frac{1}{1 + x^2}$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$h'(a) = \frac{1}{1 + a^2}$$

$$h'(a + 1) = \frac{1}{a^2 + 2a + 2}$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$\frac{1}{1+a^2}=\frac{1}{a^2+2a+2}$$

$$1+a^2=a^2+2a+2$$

$$-1=2a$$

$$a=-\frac{1}{2}$$

If this is the value of a, then $a+1$ is

$$a+1=-\frac{1}{2}+1$$

$$a+1=\frac{1}{2}$$

Therefore, the function has parallel tangent lines one unit apart at $x=-1/2$ and $x=1/2$.

■ 4. Find parallel tangent lines to $f(x)=4x^3-6x+7$ at $x=a$ and $x=a+1$.

Solution:

We want to find the equation of the tangent lines at $x = a$ and $x = a + 1$. We'll start by working on the line at $x = a$. We'll need $f(a) = 4a^3 - 6a + 7$, and the value of the derivative at $x = a$.

$$f'(x) = 12x^2 - 6$$

$$f'(a) = 12a^2 - 6$$

So the equation of the tangent line at $x = a$ is

$$y = f(a) + f'(a)(x - a)$$

$$y = 4a^3 - 6a + 7 + (12a^2 - 6)(x - a)$$

Now we'll do the same thing at $x = a + 1$. We know that

$$f(a + 1) = 4(a + 1)^3 - 6(a + 1) + 7$$

$$f(a + 1) = 4a^3 + 12a^2 + 6a + 5$$

and the derivative at $x = a + 1$ will be

$$f'(x) = 12x^2 - 6$$

$$f'(a + 1) = 12a^2 + 24a + 6$$

So the equation of the tangent line at $x = a + 1$ is

$$y = f(a + 1) + f'(a + 1)(x - (a + 1))$$

$$y = 4a^3 + 12a^2 + 6a + 5 + (12a^2 + 24a + 6)(x - (a + 1))$$

For the two tangent lines to be parallel, we'll set their slopes equal to each other and solve for a.

$$12a^2 - 6 = 12a^2 + 24a + 6$$

$$-12 = 24a$$

$$a = -\frac{1}{2}$$

The slope of $f(x)$ at $x = -1/2$ is -3, and the slope of $f(x)$ at $x = 1/2$ is -3.

So the equation of the tangent line at $x = a$ is

$$y = 4a^3 - 6a + 7 + (12a^2 - 6)(x - a)$$

$$y = 4\left(-\frac{1}{2}\right)^3 - 6\left(-\frac{1}{2}\right) + 7 + \left(12\left(-\frac{1}{2}\right)^2 - 6\right)\left(x - \left(-\frac{1}{2}\right)\right)$$

$$y = 4\left(-\frac{1}{8}\right) + 3 + 7 + \left(12\left(\frac{1}{4}\right) - 6\right)\left(x + \frac{1}{2}\right)$$

$$y = -3x + 8$$

and the equation of the tangent line at $x = a + 1$ is

$$y = 4a^3 + 12a^2 + 6a + 5 + (12a^2 + 24a + 6)(x - (a + 1))$$

$$y = 4\left(-\frac{1}{2}\right)^3 + 12\left(-\frac{1}{2}\right)^2 + 6\left(-\frac{1}{2}\right) + 5$$

$$+\left(12\left(-\frac{1}{2}\right)^2 + 24\left(-\frac{1}{2}\right) + 6\right)\left(x - \left(-\frac{1}{2}\right) + 1)\right)$$

$$y = -3x + 6$$

■ 5. Find the value of a such that the tangent lines to $g(x) = (x-2)^3 + x^2 + 3$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $g(x)$.

$$g'(x) = 3(x-2)^2 + 2x$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$g'(a) = 3(a-2)^2 + 2a$$

$$g'(a+1) = 3(a-1)^2 + 2(a+1)$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3(a-2)^2 + 2a = 3(a-1)^2 + 2(a+1)$$

$$3a^2 - 10a + 12 = 3a^2 - 4a + 5$$

$$7 = 6a$$

$$a = \frac{7}{6}$$

If this is the value of a, then $a + 1$ is

$$a + 1 = \frac{7}{6} + 1$$

$$a + 1 = \frac{13}{6}$$

Therefore, the function has parallel tangent lines one unit apart at $x = 7/6$ and $x = 13/6$.

■ 6. Find the approximate value of a, rounded to the nearest hundredth, such that the tangent lines to $h(x) = e^x - 3x^2$ at $x = a$ and $x = a + 1$ are parallel.

Solution:

Start by finding the derivative of $h(x)$.

$$h'(x) = e^x - 6x$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$h'(a) = e^a - 6a$$

$$h'(a + 1) = e^{a+1} - 6(a + 1)$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$e^a - 6a = e^{a+1} - 6(a + 1)$$

$$e^a - 6a = e^{a+1} - 6a - 6$$

$$e^a = e^{a+1} - 6$$

$$e^{a+1} - e^a - 6 = 0$$

$$e^a e^1 + (-e^a) - 6 = 0$$

$$(-e^a)e^1 - (-e^a) + 6 = 0$$

Substitute $x = -e^a$, then solve for x.

$$xe - x + 6 = 0$$

$$x(e-1) + 6 = 0$$

$$x(e-1) = -6$$

$$x = -\frac{6}{e-1}$$

Back-substitute.

$$-e^a = -\frac{6}{e-1}$$

$$e^a = \frac{6}{e-1}$$

Take the natural log of both sides.

$$\ln(e^a) = \ln\left(\frac{6}{e-1}\right)$$

$$a = \ln\left(\frac{6}{e-1}\right)$$

$$a \approx 1.25$$

Therefore, the function has parallel tangent lines one unit apart at about $x \approx 1.25$ and $x \approx 2.25$.

VALUES THAT MAKE THE FUNCTION DIFFERENTIABLE

1. What value of a and b will make the function differentiable?

$$f(x)=\begin{cases} x^2 & x \le 3 \\ ax-b & x>3 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x=3$,

$$\lim_{x\to 3^-}(x^2)=\lim_{x\to 3^+}(ax-b)$$

$$3^2=a(3)-b$$

$$9=3a-b$$

$$b=3a-9$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x=3$ must be equal to each other. So $2x=a$, and when $x=3$, and $a=6$. Therefore, $a=6$ and

$$b=3(6)-9$$

$$b=9$$

■ 2. What value of a and b will make the function differentiable?

$$g(x) = \begin{cases} ax + b & x \le -1 \\ bx^2 - 1 & x > -1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x = -1$,

$$\lim_{x \to -1^-} (ax + b) = \lim_{x \to -1^+} (bx^2 - 1)$$

$$a(-1) + b = b(-1)^2 - 1$$

$$-a + b = b - 1$$

$$a = 1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x = -1$ must be equal to each other. So $a = 2bx$ when $x = -1$, and $a = 1$.

$$1 = 2b(-1)$$

$$b = -\frac{1}{2}$$

Therefore, $a = 1$ and $b = -1/2$.

■ 3. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^3 & x \le 2 \\ x^2 - b & x > 2 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x = 2$,

$$\lim_{x \to 2^-} (ax^3) = \lim_{x \to 2^+} (x^2 - b)$$

$$a(2)^3 = (2)^2 - b$$

$$8a = 4 - b$$

$$8a - 4 = -b$$

$$b = 4 - 8a$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x = 2$ must be equal to each other. So $3ax^2 = 2x$ when $x = 2$, and

$$3a(2)^2 = 2(2)$$

$$12a = 4$$

$$a = \frac{1}{3}$$

To get b, we'll plug in $a = 1/3$.

$$b = 4 - 8\left(\frac{1}{3}\right) = \frac{4}{3}$$

Therefore, $a = 1/3$ and $b = 4/3$.

■ 4. What value of a and b will make the function differentiable?

$$f(x) = \begin{cases} 3 - x & x \le 1 \\ ax^2 - bx & x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $f(x)$ continuous at $x = 1$,

$$\lim_{x \to 1^-} (3 - x) = \lim_{x \to 1^+} (ax^2 - bx)$$

$$3 - (1) = a(1)^2 - b(1)$$

$$2 = a - b$$

$$a = 2 + b$$

If $f(x)$ is differentiable, then the derivatives of $f(x)$ at $x = 1$ must be equal to each other. So

$$-1 = 2ax - b$$

$$-1 = 2a(1) - b$$

$$-1 = 2a - b$$

$$-1 - 2a = -b$$

$$b = 2a + 1$$

Now, since $a = 2 + b$ and $b = 2a + 1$,

$$a = 2 + 2a + 1$$

$$-a = 3$$

$$a = -3$$

Then

$$b = 2a + 1$$

$$b = 2(-3) + 1$$

$$b = -5$$

The answer is $a = -3$ and $b = -5$.

■ 5. What value of a and b will make the function differentiable?

$$g(x) = \begin{cases} x^3 & x \le 1 \\ a(x-2)^2 - b & x > 1 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $g(x)$ continuous at $x = 1$,

$$\lim_{x \to 1^-} (x^3) = \lim_{x \to 1^+} (a(x-2)^2 - b)$$

$$(1)^3 = a(1-2)^2 - b$$

$$1 = a - b$$

$$-b = 1 - a$$

$$b = a - 1$$

If $g(x)$ is differentiable, then the derivatives of $g(x)$ at $x = 1$ must be equal to each other. So

$$3x^2 = 2a(x-2)$$

$$3(1) = 2a(1-2)$$

$$3 = 2a(-1)$$

$$a = -\frac{3}{2}$$

So $a = -3/2$ and $b = -3/2 - 1 = -5/2$.

■ 6. What value of a and b will make the function differentiable?

$$h(x) = \begin{cases} ax^2 + b & x \leq 3 \\ bx + 4 & x > 3 \end{cases}$$

Solution:

To be differentiable, the function has to be continuous. To make $h(x)$ continuous at $x=3$,

$$\lim_{x\to 3^-}(ax^2+b)=\lim_{x\to 3^+}(bx+4)$$

$$a(3)^2+b=b(3)+4$$

$$9a+b=3b+4$$

$$9a=2b+4$$

If $h(x)$ is differentiable, then the derivatives of $h(x)$ at $x=3$ must be equal to each other. So

$$2ax=b$$

$$2a(3)=b$$

$$b=6a$$

Plugging $b=6a$ into the equation for a gives

$$9a=2(6a)+4$$

$$9a=12a+4$$

$$-3a=4$$

$$a=-\frac{4}{3}$$

Then, $b = 6(-4/3) = -8$. The answer is $a = -4/3$ and $b = -8$.

NORMAL LINES

■ 1. Find the equation of the normal line to the graph of $f(x) = 5x^4 + 3e^x$ at $(0,3)$.

Solution:

Begin by finding the slope of the tangent line at $(0,3)$, starting with taking the derivative. Then evaluate the derivative at $(0,3)$.

$$f'(x) = 20x^3 + 3e^x$$

$$f'(0) = 20(0)^3 + 3e^0$$

$$f'(0) = 0 + 3(1)$$

$$f'(0) = 3$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/f'(a) = -1/3$, so the equation of the normal line is

$$y = f(a) - \frac{1}{f'(a)}(x - a)$$

$$y = 3 - \frac{1}{3}(x - 0)$$

$$y = -\frac{1}{3}x + 3$$

■ 2. Find the equation of the normal line to the graph of $g(x) = \ln e^{4x} + 2x^3$ at $(2,24)$.

Solution:

Begin by finding the slope of the tangent line at $(2,24)$, starting with taking the derivative. Then evaluate the derivative at $(2,24)$.

$$g'(x) = 4 + 6x^2$$

$$g'(2) = 4 + 6(2)^2$$

$$g'(2) = 4 + 24$$

$$g'(2) = 28$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/g'(a) = -1/28$, so the equation of the normal line is

$$y = g(a) - \frac{1}{g'(a)}(x - a)$$

$$y = 24 - \frac{1}{28}(x - 2)$$

$$y = -\frac{1}{28}x + \frac{337}{14}$$

3. Find the equation of the normal line to the graph of $h(x) = 5\cos x + 5\sin x$ at $(\pi/2, 5)$.

Solution:

Begin by finding the slope of the tangent line at $(\pi/2, 5)$, starting with taking the derivative. Then evaluate the derivative at $(\pi/2, 5)$.

$$h'(x) = -5\sin x + 5\cos x$$

$$h'\left(\frac{\pi}{2}\right) = -5\sin\left(\frac{\pi}{2}\right) + 5\cos\left(\frac{\pi}{2}\right)$$

$$h'\left(\frac{\pi}{2}\right) = -5(1) + 5(0)$$

$$h'\left(\frac{\pi}{2}\right) = -5$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/h'(a) = 1/5$, so the equation of the normal line is

$$y = h(a) - \frac{1}{h'(a)}(x - a)$$

$$y = 5 + \frac{1}{5}\left(x - \frac{\pi}{2}\right)$$

4. Find the equation of the normal line to the graph of $f(x) = 7x^3 + 2x^2 - 5x + 9$ at $(2,63)$.

Solution:

Begin by finding the slope of the tangent line at $(2,63)$, starting with taking the derivative. Then evaluate the derivative at $(2,63)$.

$$f'(x) = 21x^2 + 4x - 5$$

$$f'(2) = 21(2)^2 + 4(2) - 5$$

$$f'(2) = 84 + 8 - 5$$

$$f'(2) = 87$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/f'(a) = -1/87$, so the equation of the normal line is

$$y = f(a) - \frac{1}{f'(a)}(x - a)$$

$$y = 63 - \frac{1}{87}(x - 2)$$

$$y = -\frac{1}{87}x + \frac{5{,}483}{87}$$

5. Find the equation of the normal line to the graph of $g(x) = 5\sqrt{x^2 - 14x + 49}$ at $(2,25)$.

Solution:

Begin by finding the slope of the tangent line at $(2,25)$, starting with taking the derivative. Then evaluate the derivative at $(2,25)$.

$$g'(x) = \frac{5}{2\sqrt{x^2 - 14x + 49}} \cdot (2x - 14)$$

$$g'(x) = \frac{5x - 35}{\sqrt{x^2 - 14x + 49}}$$

$$g'(x) = \frac{5(x - 7)}{|x - 7|}$$

$$g'(2) = \frac{5(2 - 7)}{|2 - 7|}$$

$$g'(2) = \frac{-25}{5}$$

$$g'(2) = -5$$

Since the normal line is the line that's perpendicular to the function at the same point, the slope of the normal line is $-1/g'(a) = 1/5$, so the equation of the normal line is

$$y = g(a) - \frac{1}{g'(a)}(x - a)$$

$$y = 25 + \frac{1}{5}(x - 2)$$

$$y = \frac{1}{5}x + \frac{123}{5}$$

■ 6. Find the equations of the tangent and normal lines of $g(x) = (2x^2 - 5x + 3)^2$ at $(0,9)$.

Solution:

Differentiate the function,

$$g'(x) = 2(2x^2 - 5x + 3)(4x - 5)$$

$$g'(x) = (2x^2 - 5x + 3)(8x - 10)$$

then evaluate at $(0,9)$.

$$g'(0) = (2(0)^2 - 5(0) + 3)(8(0) - 10)$$

$$g'(0) = 3(-10)$$

$$g'(0) = -30$$

With $g'(0) = -30$ and $(a, g(a)) = (0,9)$, the equation of the tangent line is

$$y = g(a) + g'(a)(x - a)$$

$$y = 9 - 30(x - 0)$$

$$y = 9 - 30x$$

$$y = -30x + 9$$

Since the normal line is perpendicular to the tangent line at the same point, the slope of the normal line is $-1/g'(a) = 1/30$, so the equation of the normal line is

$$y = g(a) - \frac{1}{g'(a)}(x - a)$$

$$y = 9 + \frac{1}{30}(x - 0)$$

$$y = \frac{x}{30} + 9$$

AVERAGE RATE OF CHANGE

■ 1. Find the average rate of change of the function over the interval $[4,9]$.

$$f(x)=\frac{5\sqrt{x}-2}{3}$$

Solution:

The values of $f(9)$ and $f(4)$ are

$$f(9)=\frac{5\sqrt{9}-2}{3}=\frac{5(3)-2}{3}=\frac{13}{3}$$

$$f(4)=\frac{5\sqrt{4}-2}{3}=\frac{5(2)-2}{3}=\frac{8}{3}$$

Therefore, average rate of change on $[a,b]=[4,9]$ is given by

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{\frac{13}{3}-\frac{8}{3}}{9-4}=\frac{\frac{5}{3}}{\frac{5}{1}}=\frac{5}{3}\cdot\frac{1}{5}=\frac{1}{3}$$

■ 2. Find the average rate of change of the function over the interval $[16,25]$.

$$g(x) = \frac{2x - 8}{\sqrt{x} - 2}$$

Solution:

The values of $g(25)$ and $g(16)$ are

$$g(25) = \frac{2(25) - 8}{\sqrt{25} - 2} = \frac{42}{3} = 14$$

$$g(16) = \frac{2(16) - 8}{\sqrt{16} - 2} = \frac{24}{2} = 12$$

Therefore, average rate of change on $[a, b] = [16,25]$ is given by

$$\frac{g(b) - g(a)}{b - a}$$

$$\frac{14 - 12}{25 - 16} = \frac{2}{9}$$

■ 3. Find the average rate of change of the function over the interval $[0,4]$.

$$h(x) = \frac{x^3 - 8}{x^2 - 4x - 5}$$

Solution:

In this question, $g(4)$ and $g(0)$ are

$$h(4) = \frac{4^3 - 8}{4^2 - 4(4) - 5} = \frac{64 - 8}{16 - 16 - 5} = \frac{56}{-5} = -\frac{56}{5}$$

$$h(0) = \frac{0^3 - 8}{0^2 - 4(0) - 5} = \frac{0 - 8}{0 - 0 - 5} = \frac{-8}{-5} = \frac{8}{5}$$

Therefore, average rate of change on $[a, b] = [0,4]$ is given by

$$\frac{h(b) - h(a)}{b - a}$$

$$\frac{\frac{-56}{5} - \frac{8}{5}}{4 - 0} = \frac{-\frac{64}{5}}{\frac{4}{1}} = -\frac{64}{5} \cdot \frac{1}{4} = -\frac{16}{5}$$

■ 4. Find the average rate of change of the function over the interval $[-2, -3/2]$.

$$f(x) = -\frac{1}{4 - x}$$

Solution:

The values of $f(-2)$ and $f(-3/2)$ are

$$f(-2) = -\frac{1}{4 - (-2)} = -\frac{1}{4 + 2} = -\frac{1}{6}$$

$$f\left(-\frac{3}{2}\right) = -\frac{1}{4-\left(-\frac{3}{2}\right)} = -\frac{1}{4+\frac{3}{2}} = -\frac{1}{\frac{8}{2}+\frac{3}{2}} = -\frac{1}{\frac{11}{2}} = -\frac{2}{11}$$

Therefore, average rate of change on $[a, b] = [-2, -3/2]$ is given by

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{-\frac{2}{11}-\left(-\frac{1}{6}\right)}{-\frac{3}{2}-(-2)} = \frac{-\frac{2}{11}+\frac{1}{6}}{-\frac{3}{2}+2} = \frac{-\frac{12}{66}+\frac{11}{66}}{-\frac{3}{2}+\frac{4}{2}} = \frac{-\frac{1}{66}}{\frac{1}{2}} = -\frac{1}{66}\cdot\frac{2}{1} = -\frac{1}{33}$$

■ 5. On Thursday, the price of a gallon of gas was $\$3.24$. What was the price of a gallon of gas on Sunday, if the average rate of change of the price of a gallon of gas from Thursday to Sunday is $\$0.09$ per day?

Solution:

From the problem above, we know x_1 is Thursday and x_2 is Sunday, so $x_2 - x_1$ is 3 days.

Therefore, average rate of change on $[x_1, x_2]$ is given by

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}$$

$$\frac{f(x_2)-3.24}{3} = 0.09$$

$$f(x_2) - 3.24 = 0.09(3)$$

$$f(x_2) - 3.24 = 0.27$$

$$f(x_2) = 0.27 + 3.24$$

$$f(x_2) = 3.51$$

The price of a gallon of gas on Sunday was $\$3.51$.

■ 6. Find an expression in terms of a that models the average rate of change of the function $f(x) = 2x^2 + 5x - 4$ over the interval $[0,2a]$.

Solution:

The values are $f(2a)$ and $f(0)$ are

$$f(2a) = 2(2a)^2 + 5(2a) - 4$$

$$f(2a) = 2(4a^2) + 10a - 4$$

$$f(2a) = 8a^2 + 10a - 4$$

and

$$f(0) = 2(0)^2 + 5(0) - 4$$

$$f(0) = 0 + 0 - 4$$

$$f(0) = -4$$

Therefore, the average rate of change on $[a,b]=[0,2a]$ is given by

$$\frac{f(b)-f(a)}{b-a}$$

$$\frac{8a^2+10a-4-(-4)}{2a-0}=\frac{8a^2+10a}{2a}=4a+5$$

IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find dy/dx at $(3,4)$.

$$4x^3 - 3xy^2 + y^3 = 28$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$12x^2 - 3y^2 - 6xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} = 0$$

$$(3y^2 - 6xy)\frac{dy}{dx} = 3y^2 - 12x^2$$

$$\frac{dy}{dx} = \frac{3y^2 - 12x^2}{3y^2 - 6xy}$$

$$\frac{dy}{dx} = \frac{y^2 - 4x^2}{y^2 - 2xy}$$

Evaluate dy/dx at $(3,4)$.

$$\frac{dy}{dx}(3,4) = \frac{(4)^2 - 4(3)^2}{(4)^2 - 2(3)(4)} = \frac{16 - 36}{16 - 24} = \frac{5}{2}$$

■ 2. Use implicit differentiation to find dy/dx.

$$5x^3 + xy^2 = 4x^3y^3$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$5x^3 + xy^2 = 4x^3y^3$$

$$xy^2 - 4x^3y^3 = -5x^3$$

Use implicit differentiation to take the derivative of both sides.

$$y^2 + 2xy\frac{dy}{dx} - 12x^2y^3 - 12x^3y^2\frac{dy}{dx} = -15x^2$$

$$2xy\frac{dy}{dx} - 12x^3y^2\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$(2xy - 12x^3y^2)\frac{dy}{dx} = 12x^2y^3 - 15x^2 - y^2$$

$$\frac{dy}{dx} = \frac{12x^2y^3 - 15x^2 - y^2}{2xy - 12x^3y^2}$$

■ 3. Use implicit differentiation to find dy/dx.

$$3x^2 = (3xy - 1)^2$$

Solution:

Rearrange the function. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$3x^2 = (3xy - 1)^2$$

$$3x^2 = 9x^2y^2 - 6xy + 1$$

Use implicit differentiation to take the derivative of both sides.

$$6x = 18xy^2 + 18x^2y\frac{dy}{dx} - 6y - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = 18x^2y\frac{dy}{dx} - 6x\frac{dy}{dx}$$

$$6x - 18xy^2 + 6y = (18x^2y - 6x)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x - 18xy^2 + 6y}{18x^2y - 6x}$$

$$\frac{dy}{dx} = \frac{x - 3xy^2 + y}{3x^2y - x}$$

■ 4. Use implicit differentiation to find dy/dx.

$$\sin(2x + 5y) = \cos^2 x + \cos^2 y$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$\cos(2x+5y)\left(2+5\frac{dy}{dx}\right)=2\cos x(-\sin x)+2\cos y(-\sin y)\frac{dy}{dx}$$

$$2\cos(2x+5y)+5\cos(2x+5y)\frac{dy}{dx}=-2\sin x\cos x-2\sin y\cos y\frac{dy}{dx}$$

$$2\sin y\cos y\frac{dy}{dx}+5\cos(2x+5y)\frac{dy}{dx}=-2\sin x\cos x-2\cos(2x+5y)$$

$$(2\sin y\cos y+5\cos(2x+5y))\frac{dy}{dx}=-2\sin x\cos x-2\cos(2x+5y)$$

$$\frac{dy}{dx}=\frac{-2\sin x\cos x-2\cos(2x+5y)}{2\sin y\cos y+5\cos(2x+5y)}$$

$$\frac{dy}{dx}=-\frac{2\sin x\cos x+2\cos(2x+5y)}{2\sin y\cos y+5\cos(2x+5y)}$$

$$\frac{dy}{dx}=-\frac{\sin(2x)+2\cos(2x+5y)}{\sin(2y)+5\cos(2x+5y)}$$

■ 5. Use implicit differentiation to find dy/dx.

$$e^{2xy}=3x^3-\ln(xy^2)$$

Solution:

Rearrange the equation. We'll do this to get all the terms that include y on one side of the equation, which will make it easier to solve for dy/dx later on.

$$e^{2xy} = 3x^3 - \ln(xy^2)$$

$$e^{2xy} + \ln(xy^2) = 3x^3$$

Use implicit differentiation to take the derivative of both sides.

$$e^{2xy}\left(2y + 2x\frac{dy}{dx}\right) + \frac{1}{xy^2}\left(y^2 + 2xy\frac{dy}{dx}\right) = 9x^2$$

$$2ye^{2xy} + 2xe^{2xy}\frac{dy}{dx} + \frac{y^2}{xy^2} + \frac{2xy}{xy^2}\left(\frac{dy}{dx}\right) = 9x^2$$

$$2ye^{2xy} + 2xe^{2xy}\frac{dy}{dx} + \frac{1}{x} + \frac{2}{y}\left(\frac{dy}{dx}\right) = 9x^2$$

$$2xe^{2xy}\frac{dy}{dx} + \frac{2}{y}\left(\frac{dy}{dx}\right) = 9x^2 - 2ye^{2xy} - \frac{1}{x}$$

$$\left(2xe^{2xy} + \frac{2}{y}\right)\frac{dy}{dx} = 9x^2 - 2ye^{2xy} - \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{9x^2 - 2ye^{2xy} - \frac{1}{x}}{2xe^{2xy} + \frac{2}{y}}$$

■ 6. Use implicit differentiation to find dy/dx at $(0,2)$.

$$\frac{2x - y^3}{y + x^2} = 5x - 4$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$\frac{\left(2 - 3y^2\frac{dy}{dx}\right)(y + x^2) - (2x - y^3)\left(\frac{dy}{dx} + 2x\right)}{(y + x^2)^2} = 5$$

$$\frac{2y + 2x^2 - 3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - \left(2x\frac{dy}{dx} + 4x^2 - y^3\frac{dy}{dx} - 2xy^3\right)}{(y + x^2)^2} = 5$$

$$\frac{2y + 2x^2 - 3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 4x^2 + y^3\frac{dy}{dx} + 2xy^3}{(y + x^2)^2} = 5$$

$$2y + 2x^2 - 3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - 2x\frac{dy}{dx} - 4x^2 + y^3\frac{dy}{dx} + 2xy^3 = 5(y + x^2)^2$$

$$-3y^3\frac{dy}{dx} - 3x^2y^2\frac{dy}{dx} - 2x\frac{dy}{dx} + y^3\frac{dy}{dx} = 5(y + x^2)^2 - 2y - 2x^2 + 4x^2 - 2xy^3$$

$$(-3y^3 - 3x^2y^2 - 2x + y^3)\frac{dy}{dx} = 5(y + x^2)^2 - 2y + 2x^2 - 2xy^3$$

$$(-2y^3 - 3x^2y^2 - 2x)\frac{dy}{dx} = 5(y + x^2)^2 - 2y + 2x^2 - 2xy^3$$

$$\frac{dy}{dx} = \frac{5(y + x^2)^2 - 2y + 2x^2 - 2xy^3}{-2y^3 - 3x^2y^2 - 2x}$$

$$\frac{dy}{dx} = -\frac{5(y + x^2)^2 - 2y + 2x^2 - 2xy^3}{2y^3 + 3x^2y^2 + 2x}$$

Evaluate dy/dx at $(0,2)$.

$$\frac{dy}{dx}(0,2) = -\frac{5(2 + 0^2)^2 - 2(2) + 2(0)^2 - 2(0)(2)^3}{2(2)^3 + 3(0)^2(2)^2 + 2(0)}$$

$$\frac{dy}{dx}(0,2) = -\frac{5(2)^2 - 2(2)}{2(8)}$$

$$\frac{dy}{dx}(0,2) = -\frac{20 - 4}{16}$$

$$\frac{dy}{dx}(0,2) = -\frac{16}{16}$$

$$\frac{dy}{dx}(0,2) = -1$$

EQUATION OF THE TANGENT LINE WITH IMPLICIT DIFFERENTIATION

■ 1. Use implicit differentiation to find the equation of the tangent line to $5y^2 = 2x^3 - 5y + 6$ at $(3,3)$.

Solution:

Rearrange the function.

$$5y^2 = 2x^3 - 5y + 6$$

$$5y^2 + 5y = 2x^3 + 6$$

Use implicit differentiation to take the derivative of both sides.

$$10y\frac{dy}{dx} + 5\frac{dy}{dx} = 6x^2$$

$$(10y + 5)\frac{dy}{dx} = 6x^2$$

$$\frac{dy}{dx} = \frac{6x^2}{10y + 5}$$

Evaluate dy/dx at $(3,3)$.

$$\frac{dy}{dx}(3,3) = \frac{6(3)^2}{10(3) + 5} = \frac{54}{35}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - 3 = \frac{54}{35}(x - 3)$$

$$y = \frac{54}{35}(x - 3) + 3$$

■ 2. Use implicit differentiation to find the equation of the tangent line to $5x^3 = -3xy + 4$ at $(2, -6)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$15x^2 = -3y - 3x\frac{dy}{dx}$$

$$15x^2 + 3y = -3x\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{15x^2 + 3y}{-3x}$$

$$\frac{dy}{dx} = -\frac{5x^2 + y}{x}$$

Evaluate dy/dx at $(2, -6)$.

$$\frac{dy}{dx}(2, -6) = -\frac{5(2)^2 + (-6)}{2} = -\frac{20 - 6}{2} = -7$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 6 = -7(x - 2)$$

$$y = -7(x - 2) - 6$$

$$y = -7x + 8$$

■ 3. Use implicit differentiation to find the equation of the tangent line to $4y^2 + 8 = 3x^2$ at $(6, -5)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8y\frac{dy}{dx} = 6x$$

$$\frac{dy}{dx} = \frac{6x}{8y} = \frac{3x}{4y}$$

Evaluate dy/dx at $(6, -5)$.

$$\frac{dy}{dx}(6, -5) = \frac{3(6)}{4(-5)} = -\frac{18}{20} = -\frac{9}{10}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 5 = -\frac{9}{10}(x - 6)$$

$$y = -\frac{9}{10}(x - 6) - 5$$

$$y = -\frac{9}{10}x + \frac{54}{10} - 5$$

$$y = -\frac{9}{10}x + \frac{27}{5} - \frac{25}{5}$$

$$y = -\frac{9}{10}x + \frac{2}{5}$$

■ 4. Use implicit differentiation to find the equation of the tangent line to $2x + 3y - 5 = \ln(x^5 + y^5)$ at $(1,0)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$2 + 3y' = \frac{1}{x^5 + y^5}(5x^4 + 5y^4y')$$

$$2 + 3y' = \frac{5x^4}{x^5 + y^5} + \frac{5y^4y'}{x^5 + y^5}$$

$$3y' - \frac{5y^4y'}{x^5 + y^5} = \frac{5x^4}{x^5 + y^5} - 2$$

$$y'\left(3 - \frac{5y^4}{x^5 + y^5}\right) = \frac{5x^4}{x^5 + y^5} - 2$$

$$y'\left(\frac{3(x^5 + y^5) - 5y^4}{x^5 + y^5}\right) = \frac{5x^4 - 2(x^5 + y^5)}{x^5 + y^5}$$

$$y' = \frac{5x^4 - 2(x^5 + y^5)}{x^5 + y^5}\left(\frac{x^5 + y^5}{3(x^5 + y^5) - 5y^4}\right)$$

$$y' = \frac{5x^4 - 2(x^5 + y^5)}{3(x^5 + y^5) - 5y^4}$$

Evaluate y' at $(1,0)$ to find the slope of the tangent line.

$$\frac{dy}{dx}(1,0) = \frac{5(1)^4 - 2(1^5 + 0^5)}{3(1^5 + 0^5) - 5(0)^4} = \frac{5(1) - 2(1 + 0)}{3(1 + 0) - 0} = \frac{5 - 2}{3} = 1$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y + 0 = 1(x - 1)$$

$$y = x - 1$$

■ 5. Use implicit differentiation to find the equations of the tangent and normal line to $\cos x = \sin(2y) + 9$ at $(\pi/2,\pi)$.

Solution:

Use implicit differentiation to take the derivative of both sides.

$$-\sin x = \cos(2y)(2y')$$

$$-\sin x = 2y' \cos(2y)$$

$$y' = -\frac{\sin x}{2\cos(2y)}$$

Evaluate y' at $(\pi/2,\pi)$ to find the slope of the tangent line.

$$y'\left(\frac{\pi}{2},\pi\right) = -\frac{\sin\frac{\pi}{2}}{2\cos(2\pi)} = -\frac{1}{2(1)} = -\frac{1}{2}$$

Then the equation of the tangent line is

$$y - y_1 = m(x - x_1)$$

$$y - \pi = -\frac{1}{2}\left(x - \frac{\pi}{2}\right)$$

$$y - \pi = -\frac{1}{2}x + \frac{\pi}{4}$$

$$y = -\frac{1}{2}x + \frac{5\pi}{4}$$

Since the normal line is perpendicular to the tangent line at the point of tangency, the slope of the normal line is 2 (the negative reciprocal of $-1/2$), so the equation of the normal line is

$$y - \pi = 2\left(x - \frac{\pi}{2}\right)$$

$$y - \pi = 2x - \pi$$

$$y = 2x$$

■ 6. Use implicit differentiation to find the equation of the tangent line to $4x^2 - xy + y^2 = 6$ at the points in the second and third quadrant when $x = -1$.

Solution:

First we need to find points on the curve in second and third quadrant at $x = -1$.

$$4(-1)^2 - (-1)y + y^2 = 6$$

$$4 + y + y^2 = 6$$

$$y^2 + y - 2 = 0$$

$$(y + 2)(y - 1) = 0$$

$y = -2$ and $y = 1$

Therefore, we get two points $(-1, -2)$ and $(-1, 1)$.

Now use implicit differentiation to take the derivative of both sides of $4x^2 - xy + y^2 = 6$.

$$8x - y - xy' + 2yy' = 0$$

$$2yy' - xy' = y - 8x$$

$$y'(2y - x) = y - 8x$$

$$y' = \frac{y - 8x}{2y - x}$$

Evaluate y' at $(-1, -2)$.

$$y'(-1, -2) = \frac{-2 - 8(-1)}{2(-2) - (-1)} = \frac{-2 + 8}{-4 + 1} = \frac{6}{-3} = -2$$

Then the equation of the tangent line at $(-1, -2)$ is

$$y - y_1 = m(x - x_1)$$

$$y + 2 = -2(x + 1)$$

$$y + 2 = -2x - 2$$

$$y = -2x - 4$$

Evaluate y' at $(-1,1)$.

$$y'(-1,1) = \frac{1 - 8(-1)}{2(1) - (-1)} = \frac{1 + 8}{2 + 1} = \frac{9}{3} = 3$$

Then the equation of the tangent line at $(-1,1)$ is

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 3(x + 1)$$

$$y - 1 = 3x + 3$$

$$y = 3x + 4$$

HIGHER-ORDER DERIVATIVES

■ 1. Find the second and third derivatives of the function at $x = -1$.

$$y = 2x^5 - 3x^4 + x^3 + x^2 - 7$$

Solution:

Find the first derivative.

$$y = 2x^5 - 3x^4 + x^3 + x^2 - 7$$

$$y' = 10x^4 - 12x^3 + 3x^2 + 2x$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (10x^4 - 12x^3 + 3x^2 + 2x)'$$

$$y'' = 40x^3 - 36x^2 + 6x + 2$$

Evaluate the second derivative at $x = -1$.

$$y''(-1) = 40(-1)^3 - 36(-1)^2 + 6(-1) + 2$$

$$y''(-1) = -40 - 36 - 6 + 2$$

$$y''(-1) = -80$$

Find the third derivative by taking the derivative of the second derivative.

$$y''' = (y'')' = (40x^3 - 36x^2 + 6x + 2)'$$

$$y''' = 120x^2 - 72x + 6$$

Evaluate the third derivative at $x = -1$.

$$y'''(-1) = 120(-1)^2 - 72(-1) + 6$$

$$y'''(-1) = 120 + 72 + 6$$

$$y'''(-1) = 198$$

■ 2. Find the second derivative of the function $y = -3x^{\frac{2}{3}} + x^{-\frac{1}{2}}$.

Solution:

Find the first derivative.

$$y = -3x^{\frac{2}{3}} + x^{-\frac{1}{2}}$$

$$y' = -3\left(\frac{2}{3}x^{-\frac{1}{3}}\right) - \frac{1}{2}x^{-\frac{3}{2}}$$

$$y' = -2x^{-\frac{1}{3}} - \frac{1}{2}x^{-\frac{3}{2}}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = \left(-2x^{-\frac{1}{3}} - \frac{1}{2}x^{-\frac{3}{2}}\right)'$$

$$y'' = -2\left(-\frac{1}{3}\right)x^{-\frac{4}{3}} - \frac{1}{2}\left(-\frac{3}{2}\right)x^{-\frac{5}{2}}$$

$$y'' = \frac{2}{3}x^{-\frac{4}{3}} + \frac{3}{4}x^{-\frac{5}{2}}$$

■ 3. Find the second derivative of the function.

$$y = -3x^7 \sin x$$

Solution:

Find the first derivative.

$$y = -3x^7 \sin x$$

$$y' = -21x^6 \sin x + (-3x^7 \cos x)$$

$$y' = -21x^6 \sin x - 3x^7 \cos x$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (-21x^6 \sin x - 3x^7 \cos x)'$$

$$y'' = -126x^5 \sin x + (-21x^6 \cos x) - (21x^6 \cos x + 3x^7(-\sin x))$$

$$y'' = -126x^5 \sin x - 21x^6 \cos x - (21x^6 \cos x - 3x^7 \sin x)$$

$$y'' = -126x^5 \sin x - 21x^6 \cos x - 21x^6 \cos x + 3x^7 \sin x$$

$$y'' = -126x^5 \sin x - 42x^6 \cos x + 3x^7 \sin x$$

$$y'' = 3x^7 \sin x - 42x^6 \cos x - 126x^5 \sin x$$

$$y'' = 3x^5(x^2 \sin x - 14x \cos x - 42 \sin x)$$

■ 4. Find the second and the third derivatives of the function.

$$y = \ln(x^5\sqrt{x})$$

Solution:

Find the first derivative.

$$y = \ln(x^5\sqrt{x})$$

$$\frac{dy}{dx} = \frac{1}{x^5\sqrt{x}}\left(5x^4\sqrt{x} + x^5\frac{1}{2\sqrt{x}}\right)$$

$$\frac{dy}{dx} = \frac{5x^4\sqrt{x}}{x^5\sqrt{x}} + \frac{\frac{x^5}{2\sqrt{x}}}{x^5\sqrt{x}}$$

$$\frac{dy}{dx} = \frac{5}{x} + \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{10}{2x} + \frac{1}{2x}$$

$$\frac{dy}{dx} = \frac{11}{2x}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{11}{2x}\right)$$

$$\frac{d^2y}{dx^2} = \frac{11}{2}(-1)x^{-2}$$

$$\frac{d^2y}{dx^2} = -\frac{11}{2x^2}$$

Find the third derivative by taking the derivative of the second derivative.

$$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right) = \frac{d}{dx}\left(-\frac{11}{2x^2}\right)$$

$$\frac{d^3y}{dx^3} = -\frac{11}{2}(-2)x^{-3}$$

$$\frac{d^3y}{dx^3} = 11x^{-3}$$

$$\frac{d^3y}{dx^3} = \frac{11}{x^3}$$

■ 5. Find the second derivative of the function.

$$y = \frac{2x}{\sin(x^2)}$$

Solution:

Find the first derivative.

$$y = \frac{2x}{\sin(x^2)}$$

$$y' = \frac{2\sin(x^2) - 2x\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$y' = \frac{2\sin(x^2) - 4x^2\cos(x^2)}{\sin^2(x^2)}$$

$$y' = \frac{2\sin(x^2)}{\sin^2(x^2)} - \frac{4x^2\cos(x^2)}{\sin^2(x^2)}$$

$$y' = \frac{2}{\sin(x^2)} - \frac{4x^2\cot(x^2)}{\sin(x^2)}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = \left(\frac{2}{\sin(x^2)} - \frac{4x^2\cot(x^2)}{\sin(x^2)}\right)'$$

$$y'' = \frac{(0)\sin(x^2) - 2\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$-\frac{(8x\cot(x^2) + 4x^2(-\csc^2(x^2))(2x))\sin(x^2) - 4x^2\cot(x^2)\cos(x^2)(2x)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cos(x^2)}{\sin^2(x^2)} - \frac{(8x\cot(x^2) - 8x^3\csc^2(x^2))\sin(x^2) - 8x^3\cot(x^2)\cos(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cot(x^2)}{\sin(x^2)} - \frac{8x\sin(x^2)\cot(x^2) - 8x^3\sin(x^2)\csc^2(x^2) - 8x^3\cos(x^2)\cot(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cot(x^2)}{\sin(x^2)} - \frac{8x\sin(x^2)\cot(x^2)}{\sin^2(x^2)} + \frac{8x^3\sin(x^2)\csc^2(x^2)}{\sin^2(x^2)} + \frac{8x^3\cos(x^2)\cot(x^2)}{\sin^2(x^2)}$$

$$y'' = -\frac{4x\cot(x^2)}{\sin(x^2)} - \frac{8x\cot(x^2)}{\sin(x^2)} + \frac{8x^3\csc^2(x^2)}{\sin(x^2)} + \frac{8x^3\cot^2(x^2)}{\sin(x^2)}$$

$$y'' = -\frac{12x\cot(x^2)}{\sin(x^2)} + \frac{8x^3\csc^2(x^2)}{\sin(x^2)} + \frac{8x^3\cot^2(x^2)}{\sin(x^2)}$$

$$y'' = \frac{8x^3\cot^2(x^2) + 8x^3\csc^2(x^2) - 12x\cot(x^2)}{\sin(x^2)}$$

■ 6. Find the second derivative of the function at $x = 0$.

$$y = \frac{e^x}{4x - 9}$$

Solution:

Find the first derivative.

$$y = \frac{e^x}{4x-9}$$

$$\frac{dy}{dx} = \frac{e^x(4x-9) - e^x(4)}{(4x-9)^2}$$

$$\frac{dy}{dx} = \frac{e^x(4x-9-4)}{(4x-9)^2}$$

$$\frac{dy}{dx} = \frac{e^x(4x-13)}{(4x-9)^2}$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{e^x(4x-13)}{(4x-9)^2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x-13) + e^x(4))(4x-9)^2 - e^x(4x-13)2(4x-9)(4)}{(4x-9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x-13) + 4e^x)(4x-9)^2 - 8e^x(4x-13)(4x-9)}{(4x-9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{(e^x(4x-13) + 4e^x)(4x-9)^2}{(4x-9)^4} - \frac{8e^x(4x-13)(4x-9)}{(4x-9)^4}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x-13) + 4e^x}{(4x-9)^2} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x-13+4)}{(4x-9)^2} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x(4x-9)}{(4x-9)^2} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

$$\frac{d^2y}{dx^2} = \frac{e^x}{4x-9} - \frac{8e^x(4x-13)}{(4x-9)^3}$$

Then evaluate the derivative at $x = 0$.

$$\frac{d^2y}{dx^2}(0) = \frac{e^0}{4(0)-9} - \frac{8e^0(4(0)-13)}{(4(0)-9)^3}$$

$$\frac{d^2y}{dx^2}(0) = \frac{1}{-9} - \frac{8(-13)}{(-9)^3}$$

$$\frac{d^2y}{dx^2}(0) = -\frac{1}{9} - \frac{104}{729}$$

$$\frac{d^2y}{dx^2}(0) = -\frac{185}{729}$$

SECOND DERIVATIVES WITH IMPLICIT DIFFERENTIATION

1. Use implicit differentiation to find d^2y/dx^2.

$$2x^3 = 2y^2 + 4$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$6x^2 = 4y\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{6x^2}{4y}$$

$$\frac{dy}{dx} = \frac{3x^2}{2y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(6x)(2y) - (3x^2)\left(2 \cdot \frac{dy}{dx}\right)}{(2y)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - 6x^2\frac{dy}{dx}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{6xy - 3x^2 \frac{dy}{dx}}{2y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{6xy - 3x^2 \left(\frac{3x^2}{2y}\right)}{2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{6xy - \frac{9x^4}{2y}}{2y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

$$\frac{d^2y}{dx^2} = \frac{12xy - \frac{9x^4}{y}}{4y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = \frac{12xy^2 - 9x^4}{4y^3}$$

■ 2. Use implicit differentiation to find d^2y/dx^2.

$$4x^2 = 2y^3 + 4y - 2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$8x = 6y^2\frac{dy}{dx} + 4\frac{dy}{dx}$$

$$8x = (6y^2 + 4)\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{8x}{6y^2 + 4}$$

$$\frac{dy}{dx} = \frac{4x}{3y^2 + 2}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = \frac{(4)(3y^2 + 2) - (4x)\left(6y \cdot \frac{dy}{dx}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\frac{dy}{dx}}{(3y^2 + 2)^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - 24xy\left(\frac{4x}{3y^2 + 2}\right)}{(3y^2 + 2)^2}$$

$$\frac{d^2y}{dx^2} = \frac{12y^2 + 8 - \frac{96x^2y}{3y^2 + 2}}{(3y^2 + 2)^2}$$

Multiply through the numerator and denominator by $3y^2+2$ to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2}=\frac{12y^2(3y^2+2)+8(3y^2+2)-96x^2y}{(3y^2+2)^3}$$

$$\frac{d^2y}{dx^2}=\frac{(12y^2+8)(3y^2+2)-96x^2y}{(3y^2+2)^3}$$

$$\frac{d^2y}{dx^2}=\frac{4(3y^2+2)(3y^2+2)-96x^2y}{(3y^2+2)^3}$$

$$\frac{d^2y}{dx^2}=\frac{4(3y^2+2)^2-96x^2y}{(3y^2+2)^3}$$

■ 3. Use implicit differentiation to find d^2y/dx^2 at $(0,3)$.

$$3x^2+3y^2=27$$

Solution:

Rewrite the equation.

$$3x^2+3y^2=27$$

$$x^2+y^2=9$$

Use implicit differentiation to take the derivative of both sides.

$$2x + 2y\frac{dy}{dx} = 0$$

$$2y\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -\frac{2x}{2y}$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2} = -\frac{(1)(y) - (x)(1)\left(\frac{dy}{dx}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y - x\frac{dy}{dx}}{y^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2} = -\frac{y - x\left(-\frac{x}{y}\right)}{y^2}$$

$$\frac{d^2y}{dx^2} = -\frac{y + \frac{x^2}{y}}{y^2}$$

Multiply through the numerator and denominator by y to get rid of the fraction in the numerator.

$$\frac{d^2y}{dx^2} = -\frac{y^2 + x^2}{y^3}$$

$$\frac{d^2y}{dx^2} = \frac{-x^2 - y^2}{y^3}$$

Evaluate the second derivative at $(0,3)$.

$$\frac{d^2y}{dx^2}(0,3) = \frac{-0^2 - 3^2}{3^3} = \frac{-9}{27} = -\frac{1}{3}$$

■ 4. Use implicit differentiation to find d^2y/dx^2 at $(2,1)$.

$$e^{x-2y} = 2x - y$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$e^{x-2y}\left(1 - 2\frac{dy}{dx}\right) = 2 - \frac{dy}{dx}$$

$$e^{x-2y} - 2e^{x-2y}\frac{dy}{dx} = 2 - \frac{dy}{dx}$$

$$\frac{dy}{dx} - 2e^{x-2y}\frac{dy}{dx} = 2 - e^{x-2y}$$

$$\frac{dy}{dx}(1 - 2e^{x-2y}) = 2 - e^{x-2y}$$

$$\frac{dy}{dx}=\frac{2-e^{x-2y}}{1-2e^{x-2y}}$$

Use implicit differentiation again on both sides to find the second derivative.

$$\frac{d^2y}{dx^2}=\frac{-e^{x-2y}\left(1-2\frac{dy}{dx}\right)(1-2e^{x-2y})-(2-e^{x-2y})(-2e^{x-2y})\left(1-2\frac{dy}{dx}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2\frac{dy}{dx}\right)e^{x-2y}[-(1-2e^{x-2y})-(2-e^{x-2y})(-2)]}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2\frac{dy}{dx}\right)e^{x-2y}(-1+2e^{x-2y}+4-2e^{x-2y})}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{\left(1-2\frac{dy}{dx}\right)e^{x-2y}(3)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(1-2\frac{dy}{dx}\right)}{(1-2e^{x-2y})^2}$$

Substitute the first derivative for dy/dx and then simplify.

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(1-2\left(\frac{2-e^{x-2y}}{1-2e^{x-2y}}\right)\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(\frac{1-2e^{x-2y}-2(2-e^{x-2y})}{1-2e^{x-2y}}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(\frac{1-2e^{x-2y}-4+2e^{x-2y}}{1-2e^{x-2y}}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{3e^{x-2y}\left(\frac{-3}{1-2e^{x-2y}}\right)}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=\frac{-\frac{9e^{x-2y}}{1-2e^{x-2y}}}{(1-2e^{x-2y})^2}$$

$$\frac{d^2y}{dx^2}=-\frac{9e^{x-2y}}{(1-2e^{x-2y})^3}$$

Evaluate the second derivative at $(2,1)$.

$$\frac{d^2y}{dx^2}(2,1)=-\frac{9e^{2-2(1)}}{(1-2e^{2-2(1)})^3}=-\frac{9e^0}{(1-2e^0)^3}=-\frac{9}{(1-2)^3}=-\frac{9}{-1}=9$$

■ 5. Use implicit differentiation to find y''.

$$y\sin x=7-2y^2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$y'\sin x+y\cos x=-4yy'$$

$$y'\sin x+4yy'=-y\cos x$$

$$y'(\sin x + 4y) = -y\cos x$$

$$y' = -\frac{y\cos x}{\sin x + 4y}$$

Use implicit differentiation again on both sides to find the second derivative.

$$y'' = -\frac{[y'\cos x + y(-\sin x)](\sin x + 4y) - (y\cos x)(\cos x + 4y')}{(\sin x + 4y)^2}$$

$$y'' = -\frac{(y'\cos x - y\sin x)(\sin x + 4y) - (y\cos x)(\cos x + 4y')}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y'\sin x\cos x + 4yy'\cos x - y\sin^2 x - 4y^2\sin x - y\cos^2 x - 4yy'\cos x}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y'\sin x\cos x - y(\sin^2 x + \cos^2 x) - 4y^2\sin x}{(\sin x + 4y)^2}$$

$$y'' = -\frac{y'\sin x\cos x - y - 4y^2\sin x}{(\sin x + 4y)^2}$$

Substitute the first derivative for y' and then simplify.

$$y'' = -\frac{\left(-\frac{y\cos x}{\sin x + 4y}\right)\sin x\cos x - y - 4y^2\sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\left(\frac{y\cos x}{\sin x + 4y}\right)\sin x\cos x + y + 4y^2\sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\left(\frac{y \cos x}{\sin x + 4y}\right) \sin x \cos x}{(\sin x + 4y)^2} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{\frac{y \sin x \cos^2 x}{\sin x + 4y}}{(\sin x + 4y)^2} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

$$y'' = \frac{y \sin x \cos^2 x}{(\sin x + 4y)^3} + \frac{y + 4y^2 \sin x}{(\sin x + 4y)^2}$$

■ 6. Use implicit differentiation to find y'' at $(0,3)$.

$$e^{2y} - 2x = y^4 - 2$$

Solution:

Use implicit differentiation to take the derivative of both sides.

$$e^{2y}(2y') - 2 = 4y^3y'$$

$$2e^{2y}y' - 4y^3y' = 2$$

$$y'(2e^{2y} - 4y^3) = 2$$

$$y' = \frac{2}{2e^{2y} - 4y^3}$$

$$y' = \frac{1}{e^{2y} - 2y^3}$$

Use implicit differentiation again on both sides to find the second derivative.

$$y'' = \frac{(0)(e^{2y} - 2y^3) - (1)(e^{2y}(2y') - 6y^2y')}{(e^{2y} - 2y^3)^2}$$

$$y'' = \frac{-2e^{2y}y' + 6y^2y'}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{2y'(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^2}$$

Substitute the first derivative for y' and then simplify.

$$y'' = -\frac{2\left(\frac{1}{e^{2y} - 2y^3}\right)(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{\left(\frac{2(e^{2y} - 3y^2)}{e^{2y} - 2y^3}\right)}{(e^{2y} - 2y^3)^2}$$

$$y'' = -\frac{2(e^{2y} - 3y^2)}{(e^{2y} - 2y^3)^3}$$

CRITICAL POINTS AND THE FIRST DERIVATIVE TEST

■ 1. Identify the critical point(s) of the function on the interval $[-3,2]$.

$$f(x) = x^{\frac{2}{3}}(x+2)$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x) = 0$ or is undefined.

Rewrite the function.

$$f(x) = x^{\frac{2}{3}}(x+2)$$

$$f(x) = x^{\frac{5}{3}} + 2x^{\frac{2}{3}}$$

Find the derivative.

$$f'(x) = \frac{5}{3}x^{\frac{2}{3}} + 2 \cdot \frac{2}{3}x^{-\frac{1}{3}}$$

$$f'(x) = \frac{5}{3}\sqrt[3]{x^2} + \frac{4}{3\sqrt[3]{x}}$$

When $x = 0$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

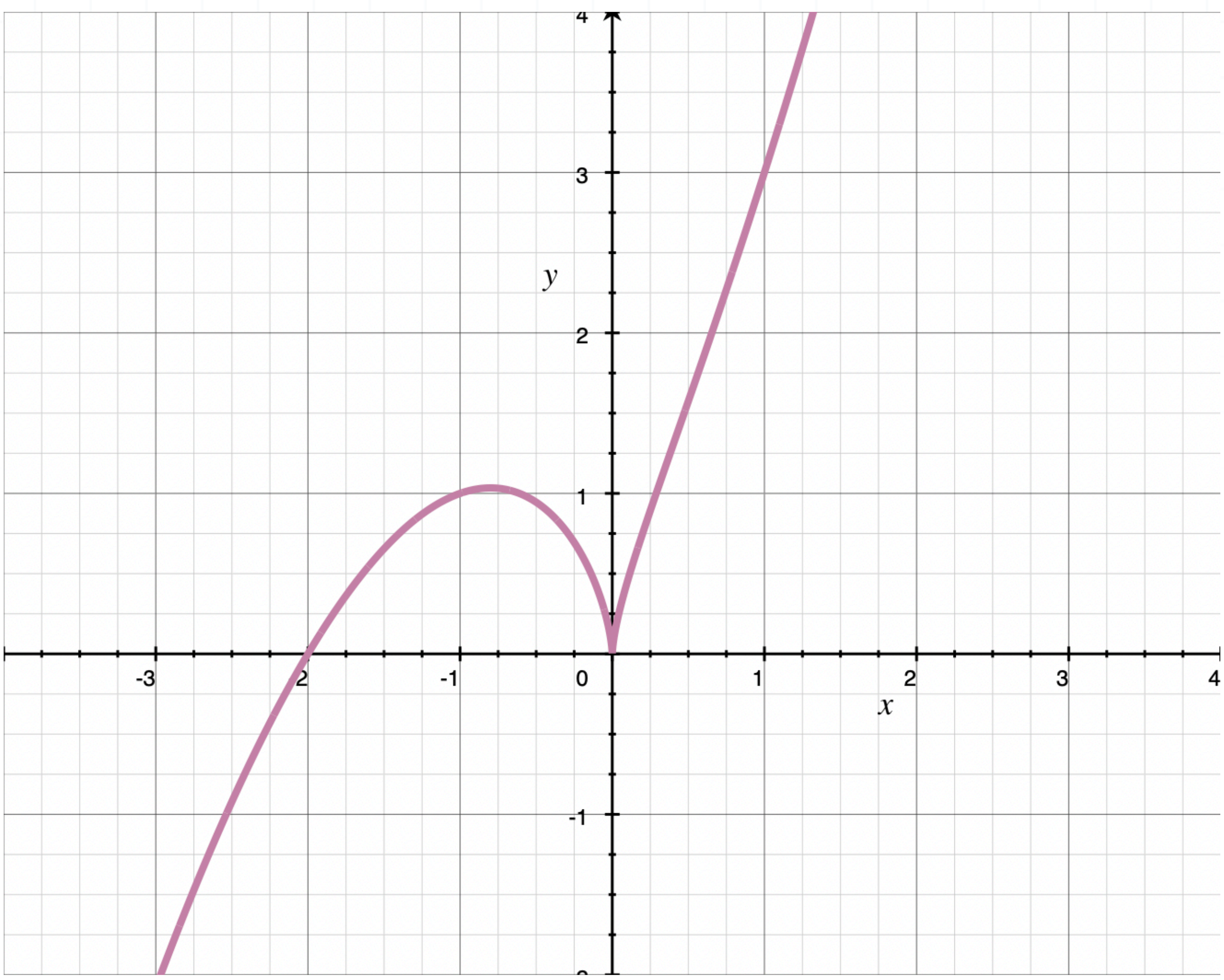
$$\frac{5}{3}\sqrt[3]{x^2} + \frac{4}{3\sqrt[3]{x}} = 0$$

$$\frac{5}{3}\sqrt[3]{x^2} = -\frac{4}{3\sqrt[3]{x}}$$

$$5x = -4$$

$$x = -\frac{4}{5}$$

Because both $x = -4/5$ and $x = 0$ are in the interval $[-3,2]$, the critical numbers are $x = -4/5, 0$.



■ 2. Identify the critical point(s) of the function on the interval $[-2,2]$.

$$g(x) = x\sqrt{4 - x^2}$$

Solution:

Find $g'(x)$ and the x-values inside the given interval for which $g'(x) = 0$ or is undefined.

Find the derivative.

$$g'(x) = (1)\sqrt{4 - x^2} + (x)\left(\frac{1}{2}\right)(4 - x^2)^{-\frac{1}{2}}(-2x)$$

$$g'(x) = \sqrt{4 - x^2} - x^2(4 - x^2)^{-\frac{1}{2}}$$

$$g'(x) = \sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}}$$

When $x = \pm 2$, the denominator of the second fraction will be 0, which will make the derivative undefined. The derivative will also be equal to 0:

$$\sqrt{4 - x^2} - \frac{x^2}{\sqrt{4 - x^2}} = 0$$

$$\sqrt{4 - x^2} = \frac{x^2}{\sqrt{4 - x^2}}$$

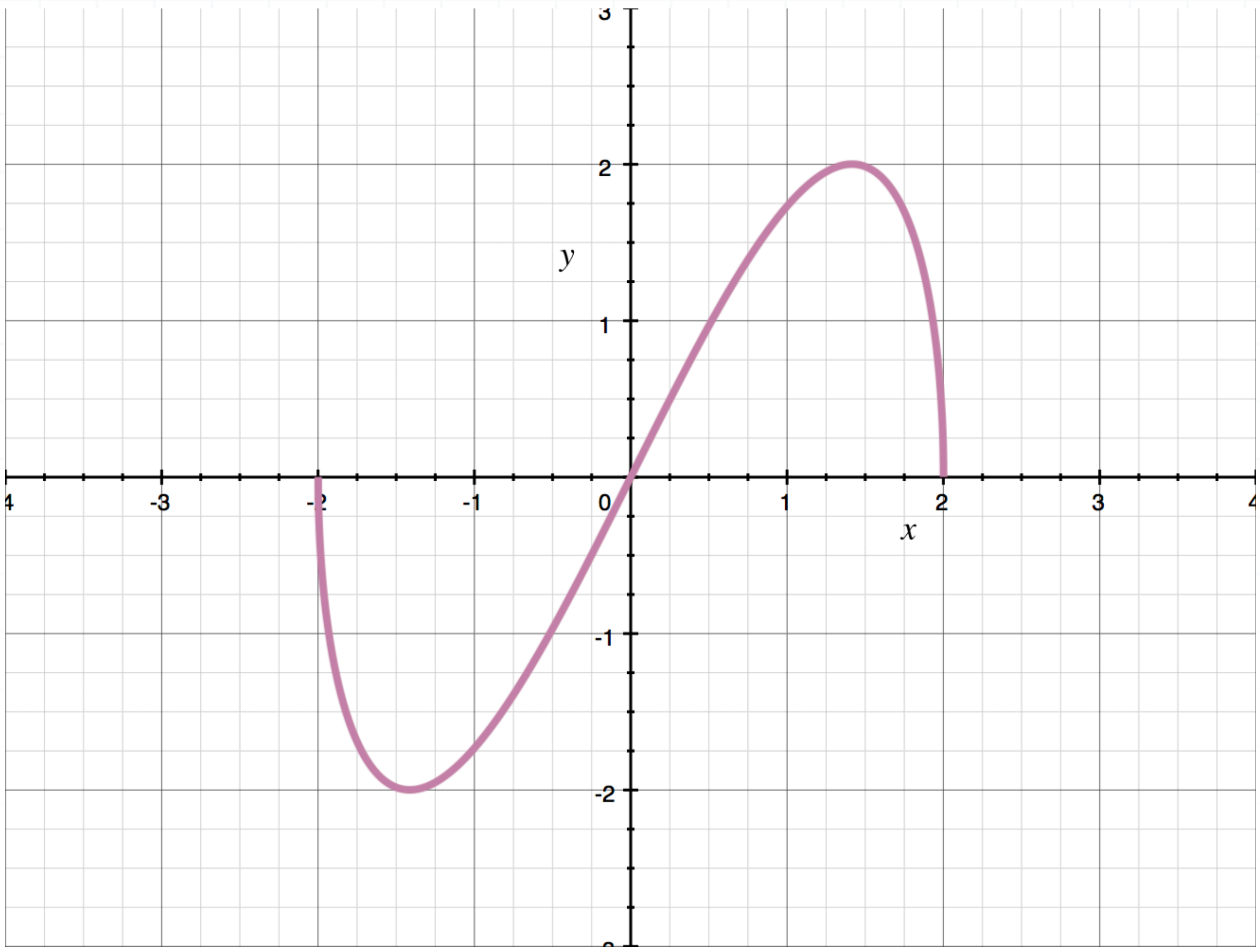
$$4 - x^2 = x^2$$

$$4 = 2x^2$$

$$2 = x^2$$

$$x = \pm\sqrt{2}$$

The critical numbers are therefore $x = - \pm \sqrt{2}, \ \pm 2$.



3. Determine the intervals in which the function is increasing and decreasing.

$$f(x) = \frac{5}{4}x^4 - 10x^2$$

Solution:

Find the derivative $f'(x) = 5x^3 - 20x$, then identify the critical points where $f'(x) = 0$ or $f'(x)$ is undefined. The derivative exists everywhere.

$$5x^3 - 20x = 0$$

$$5x(x^2 - 4) = 0$$

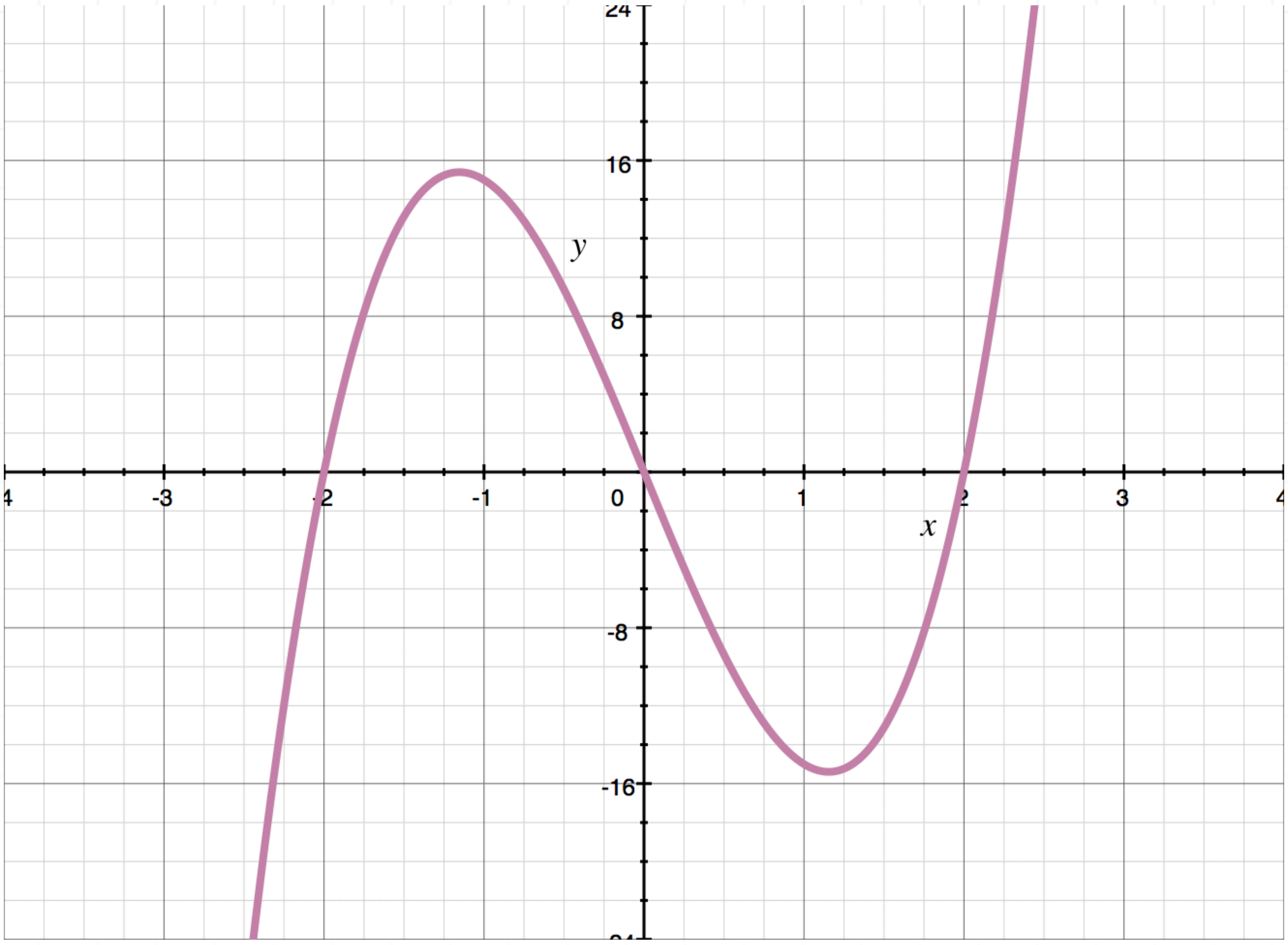
$$5x(x+2)(x-2) = 0$$

$$x = -2,\ 0,\ 2$$

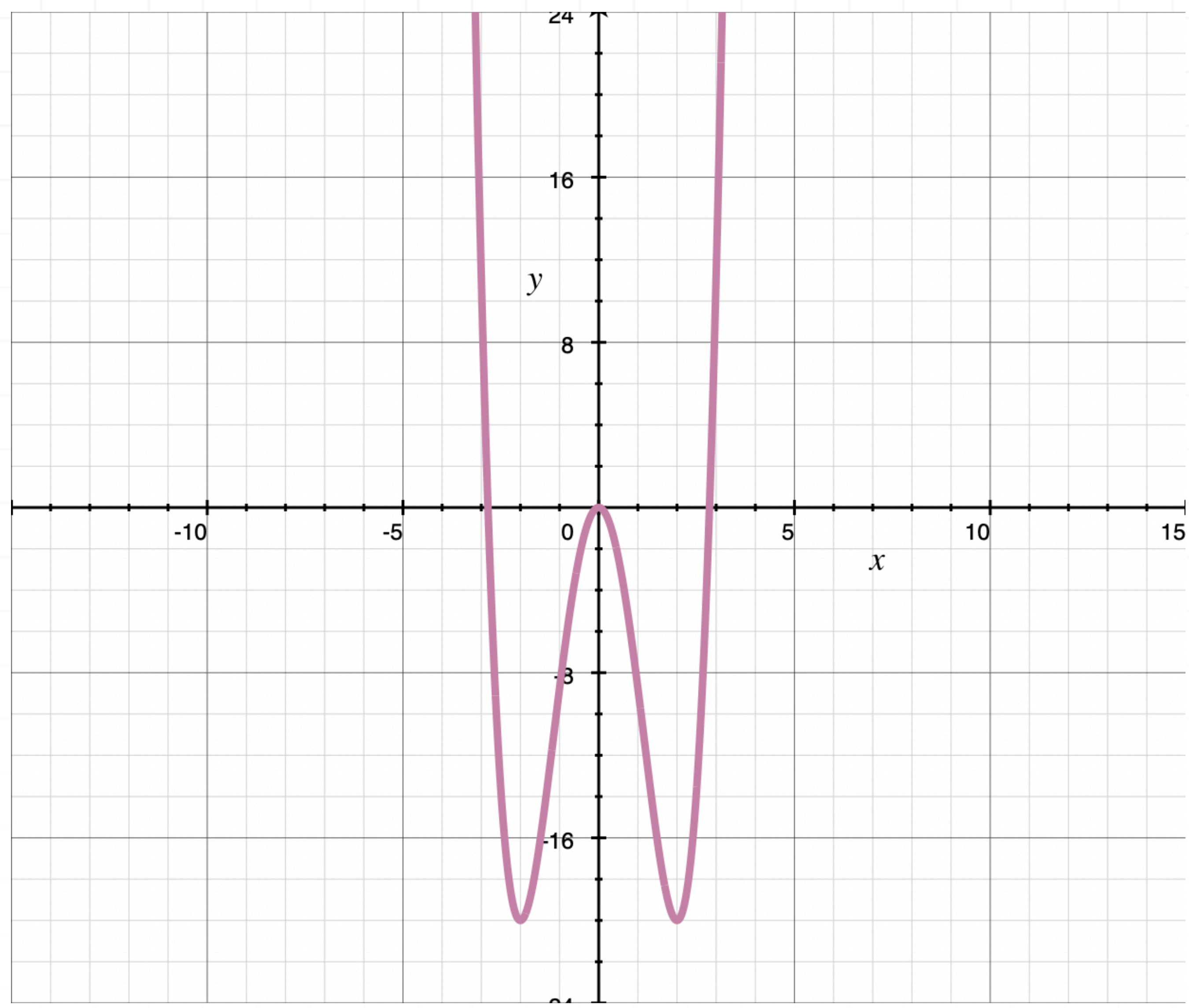
Determine where $f'(x) > 0$ or $f'(x) < 0$ by selecting a value between each critical number.

Interval	x<-2	-2<x<0	0<x<2	x>2
x	-3	-1	1	3
f'(x)	<0	>0	<0	>0
f(x)	Decreasing	Increasing	Decreasing	Increasing

The graph of $f'(x)$ shows that $f'(x) < 0$ on $(-\infty, -2) \cup (0,2)$ and $f'(x) > 0$ on $(-2,0) \cup (2,\infty)$.



The graph of $f(x)$ shows that the function is decreasing on $(-\infty, -2) \cup (0,2)$ and increasing on $(-2,0) \cup (2,\infty)$.



4. Determine the intervals in which the function is increasing and decreasing.

$$f(x) = (4 - 3x)e^x$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x) = 0$ or is undefined.

Find the derivative using product rule.

$$f'(x) = -3e^x + (4 - 3x)e^x$$

$$f'(x) = e^x(-3 + 4 - 3x)$$

$$f'(x) = e^x(1 - 3x)$$

This derivative exists everywhere. The derivative will also be equal to 0 when

$$e^x(1 - 3x) = 0$$

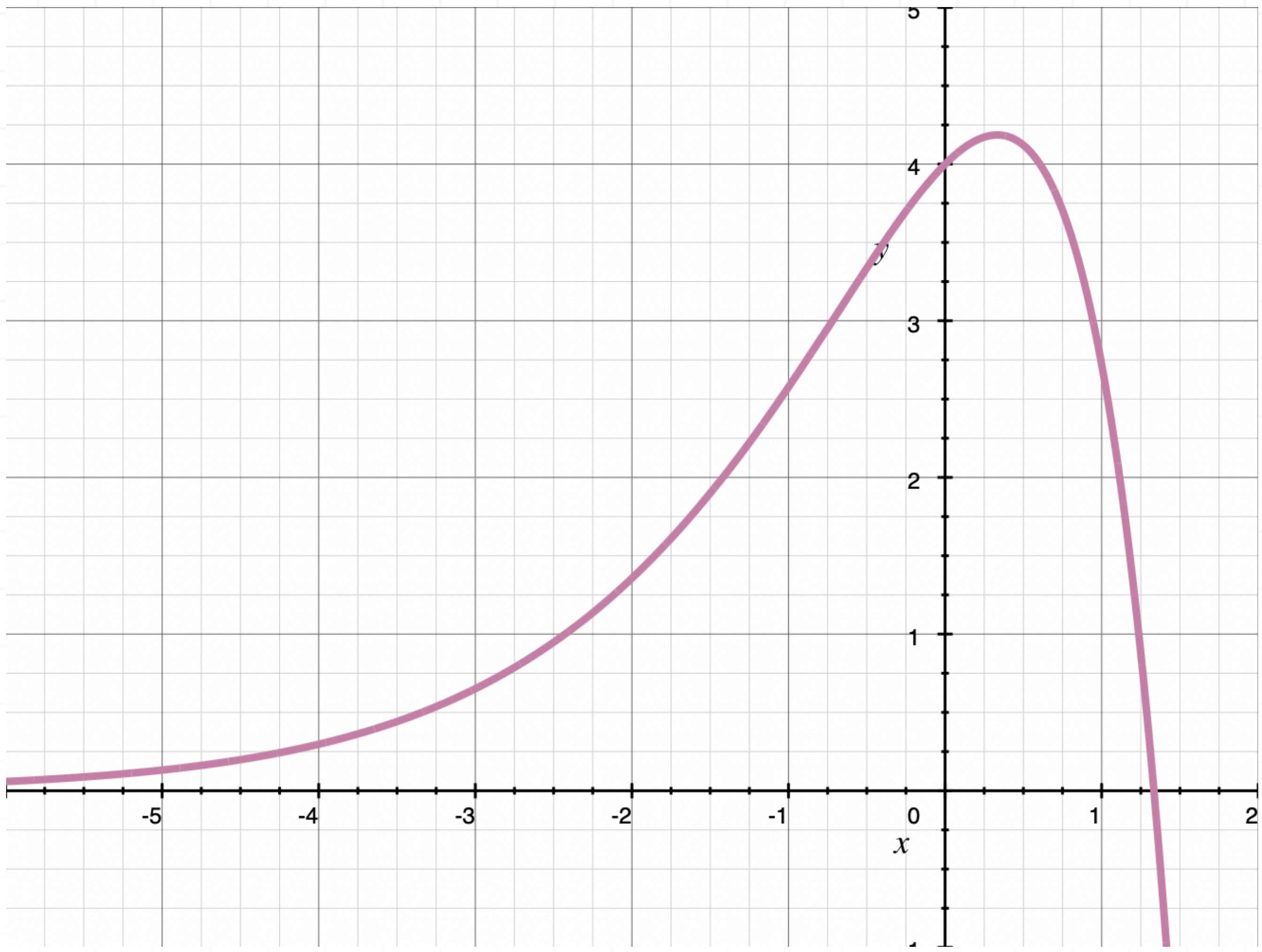
$$1 - 3x = 0$$

$$x = \frac{1}{3}$$

Determine where $f'(x) > 0$ or $f'(x) < 0$ by selecting a value between each critical number.

Interval	x<1/3	x>1/3
x	0	1
f'(x)	>0	<0
f(x)	Increasing	Decreasing

The graph of $f(x)$ shows that the function is decreasing on $(1/3,\infty)$ and increasing on $(-\infty,1/3)$.



5. Identify the critical point(s) of the function.

$$f(x) = x + 3\ln(2x + 3)$$

Solution:

Find $f'(x)$ and the x-values inside the given interval for which $f'(x) = 0$ or is undefined.

Find the derivative.

$$f'(x) = 1 + \frac{3}{2x + 3} \cdot 2$$

$$f'(x) = 1 + \frac{6}{2x+3}$$

When $x = -3/2$, the denominator of the second fraction will be 0, which will make the derivative undefined, but $x = -3/2$ is not in the domain of function, so $x = -3/2$ isn't a critical point. The derivative will also be equal to 0 when

$$1 + \frac{6}{2x+3} = 0$$

$$\frac{6}{2x+3} = -1$$

$$6 = -2x - 3$$

$$-2x = 9$$

$$x = -\frac{9}{2}$$

In the same way, $x = -9/2$ is not in the domain of the function, so $x = -9/2$ isn't critical point. Therefore, the function does not have critical points.

■ 6. Find the values a and b such that $f(x) = x^3 + ax^2 + b$ will have a critical point at $(-1,5)$.

Solution:

Since $(-1,5)$ is a critical point, we have

$$5 = (-1)^3 + a(-1)^2 + b$$

$$5 = -1 + a + b$$

$$a + b = 6$$

The derivative is $f'(x) = 3x^2 + 2ax$, so we can identify the critical points where $f'(-1) = 0$. This derivative exists everywhere.

$$3x^2 + 2ax = 0$$

$$3(-1)^2 + 2a(-1) = 0$$

$$3 - 2a = 0$$

$$a = \frac{3}{2}$$

Substitute $a = 3/2$ into $a + b = 6$ and solve for b.

$$\frac{3}{2} + b = 6$$

$$b = \frac{12}{2} - \frac{3}{2}$$

$$b = \frac{9}{2}$$

So the function is

$$f(x) = x^3 + \frac{3}{2}x^2 + \frac{9}{2}$$

INFLECTION POINTS AND THE SECOND DERIVATIVE TEST

■ 1. Find the inflection points of the function.

$$f(x) = \frac{1}{3}x^3 + x^2$$

Solution:

The function will have inflection points where its second derivative is equal to 0, so we'll find the second derivative,

$$f'(x) = x^2 + 2x$$

$$f''(x) = 2x + 2$$

then set it equal to 0 and solve for x.

$$2x + 2 = 0$$

$$x + 1 = 0$$

$$x = -1$$

So the function has an inflection point at $x = -1$, which means that it's either concave down to the left of $x = -1$ and concave up to the right of $x = -1$, or its concave up to the left of $x = -1$ and concave down to the right of $x = -1$.

■ 2. For $g(x) = -x^3 + 2x^2 + 3$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$g'(x) = -3x^2 + 4x$$

$$g''(x) = -6x + 4$$

The function has an inflection point when $g''(x) = 0$.

$$-6x + 4 = 0$$

$$-6x = -4$$

$$x = \frac{2}{3}$$

Check values around $x = 2/3$.

Interval	x<2/3	x=2/3	x>2/3
x	-1	2/3	1
g''(x)	+	0	-
Concavity	Up	Inflection	Down

The inflection point is at $x = 2/3$ and $g(2/3) = 97/27$, so the inflection point is $(2/3,97/27)$. The function is concave up on $(-\infty,2/3)$ and concave down on $(2/3,\infty)$.

■ 3. For $h(x) = x^4 + x^3 - 3x^2 + 2$, find inflection points and identify where the function is concave up and concave down.

Solution:

Find the first and second derivatives.

$$h'(x) = 4x^3 + 3x^2 - 6x$$

$$h''(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$$

The function has an inflection point when $h''(x) = 0$.

$$6(2x - 1)(x + 1) = 0$$

$$x = -1, \frac{1}{2}$$

Check values around these inflection points.

Interval	x<-1	x=-1	-1<x<1/2	x=1/2	x>1/2
x	-2	-1	0	1/2	1
h”(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

An inflection point is at $x = -1$ and $h(-1) = -1$, so an inflection point is $(-1, -1)$. Another inflection point is at $x = 1/2$ and $h(1/2) = 23/16$, so an inflection point is $(1/2,23/16)$. The function is concave up on $(-\infty,-1)\cup(1/2,\infty)$ and concave down on $(-1,1/2)$.

■ 4. Use the second derivative test to identify the extrema of $f(x)=x^3-12x-2$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$f'(x)=3x^2-12$$

$$f''(x)=6x$$

The function has critical points when $f'(x)=0$.

$$3x^2-12=0$$

$$(x+2)(x-2)=0$$

$$x=-2,\ 2$$

Plug these values into the second derivative.

$$f''(-2)=6(-2)=-12$$

$$f''(2)=6(2)=12$$

By the second derivative test, the function has a local maximum at $x = -2$. Since $f(-2) = 14$, $(-2,14)$ is a maximum. The function has a local minimum at $x = 2$. Since $f(2) = -18$, $(2, -18)$ is a minimum.

■ 5. Use the second derivative test to identify the extrema of $g(x) = -4xe^{-\frac{x}{2}}$ as maxima or minima.

Solution:

Find the first and second derivatives.

$$g'(x) = -4e^{-\frac{x}{2}} - 4x\left(-\frac{1}{2}\right)e^{-\frac{x}{2}}$$

$$g'(x) = -4e^{-\frac{x}{2}} + 2xe^{-\frac{x}{2}}$$

$$g'(x) = e^{-\frac{x}{2}}(2x - 4)$$

and

$$g''(x) = e^{-\frac{x}{2}}(4 - x)$$

The function has critical points when $g'(x) = 0$.

$$e^{-\frac{x}{2}}(2x - 4) = 0$$

$$2x - 4 = 0$$

$$x = 2$$

Plug this value into the second derivative.

$$g''(2) = e^{-\frac{2}{2}}(4 - 2) = \frac{2}{e}$$

The second derivative is positive at $x = 2$, which means the function has a minimum at $(2, -8/e)$.

$$g(2) = -\frac{8}{e}$$

■ 6. Use the second derivative test to identify the extrema of $h(x) = 2x^4 - 4x^2 + 1$ as maximum values or minimum values.

Solution:

Find the first and second derivatives.

$$h'(x) = 8x^3 - 8x$$

$$h''(x) = 24x^2 - 8$$

The function has critical points when $h'(x) = 0$.

$$8x^3 - 8x = 0$$

$$x(x + 1)(x - 1) = 0$$

$$x = -1, 0, 1$$

Plug these values into the second derivative.

$$h''(-1) = 24(-1)^2 - 8 = 16$$

$$h''(0) = 24(0)^2 - 8 = -8$$

$$h''(1) = 24(1)^2 - 8 = 16$$

By the second derivative test, the function has a local minimum at $x = -1$. Since $h(-1) = -1$, $(-1, -1)$ is a minimum. The function has a local maximum at $x = 0$. Since $h(0) = 1$, $(0,1)$ is a maximum. The function has a local minimum at $x = 1$. Since $h(1) = -1$, $(1, -1)$ is a minimum.

INTERCEPTS AND VERTICAL ASYMPTOTES

■ 1. Find the x-intercepts and any vertical asymptote(s) of the function.

$$f(x) = \frac{-x^2 + 16x - 63}{x^2 - 2x - 35}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{-(x-7)(x-9)}{(x-7)(x+5)}$$

The denominator is equal to 0 if $x = 7$ or $x = -5$, which means the function has two discontinuities. However, the function simplifies to

$$f(x) = \frac{-(x-9)}{x+5}$$

$$f(x) = \frac{9-x}{x+5}$$

So, the function has an x-intercept at $(9,0)$. Therefore, the function has a removable discontinuity at $x = 7$ and a vertical asymptote at $x = -5$, which means the domain of the function is $(-\infty, -5) \cup (-5,7) \cup (7,\infty)$.

■ 2. Find any vertical asymptote(s) of the function.

$$g(x) = \frac{x^2 - 3x - 10}{x^2 + x - 2}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$g(x) = \frac{(x-5)(x+2)}{(x-1)(x+2)}$$

The denominator is equal to 0 if $x = -2$ or $x = 1$, which means the function has two discontinuities. However, the function simplifies to

$$g(x) = \frac{x-5}{x-1}$$

Therefore, the function has a removable discontinuity at $x = -2$ and a vertical asymptote at $x = 1$, which means the domain of the function is $(-\infty, -2) \cup (-2,1) \cup (1,\infty)$.

■ 3. Find any vertical asymptote(s) of the function.

$$h(x) = \frac{40 - 27x - 12x^2 - x^3}{9x^2 + 63x - 72}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$h(x) = \frac{-(x+8)(x+5)(x-1)}{9(x-1)(x+8)}$$

Cancel common factors from the numerator and denominator, then simplify.

$$h(x) = \frac{-(x+5)}{9}$$

$$h(x) = -\frac{x+5}{9}$$

There are no values of x that make this denominator 0, so the function has no vertical asymptotes. But it does have removable discontinuities for the factors we canceled, at $x = -8$ and $x = 1$.

■ 4. Find the y-intercepts and any vertical asymptote(s) of the function.

$$f(x) = \frac{x^2 + -2x - 8}{x^2 - 9x + 20}$$

Solution:

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{(x+2)(x-4)}{(x-4)(x-5)}$$

Cancel common factors from the numerator and denominator, then simplify.

$$f(x) = \frac{x+2}{x-5}$$

Therefore, the function has a removable discontinuity at $x = 4$ and a vertical asymptote at $x = 5$. Substitute $x = 0$ into the function,

$$f(x) = \frac{0+2}{0-5} = -\frac{2}{5}$$

So the function has a y-intercept at $(0, -2/5)$.

■ 5. Find any vertical asymptote(s) of the function.

$$g(x) = \ln(x^2 + 5x)$$

Solution:

The logarithmic function is undefined where the argument of the function is equal to zero.

$$x^2 + 5x = 0$$

$$x(x+5) = 0$$

$x = 0$ and $x = -5$

Therefore, the function has vertical asymptotes at $x = -5$ and $x = 0$.

■ 6. Find any vertical asymptote(s) of the function.

$$h(x) = \sec\left(x + \frac{\pi}{2}\right)$$

Solution:

We know that

$$h(x) = \sec\left(x + \frac{\pi}{2}\right) = \frac{1}{\cos\left(x + \frac{\pi}{2}\right)}$$

The function is undefined whenever the denominator is equal to 0.

$$\cos\left(x + \frac{\pi}{2}\right) = 0$$

The cosine function is equal to 0 when the argument is equal to

$\dfrac{\pi}{2} + k\pi$, where k is any integer

Therefore, the function has vertical asymptotes at

$$x + \frac{\pi}{2} = \frac{\pi}{2} + k\pi$$

$x = k\pi$, where k is any integer

HORIZONTAL AND SLANT ASYMPTOTES

■ 1. Find the horizontal asymptote(s) of the function.

$$f(x) = \frac{8x^4 - x^2 + 1}{4x^4 - 1}$$

Solution:

The degree of the numerator is 4, and the degree of the denominator is 4. So the degree of the numerator is equal to the degree of the denominator, which means the ratio of the coefficients on these highest-degree terms is the equation of the horizontal asymptote,

$$y = \frac{8}{4} = 2$$

so the function has a horizontal asymptote at $y = 2$.

■ 2. Find the horizontal asymptote(s) of the function.

$$g(x) = \frac{2x^2 - 5x + 12}{3x^2 - 11x - 4}$$

Solution:

The degree of the numerator is 2, and the degree of the denominator is 2. So the degree of the numerator is equal to the degree of the denominator, which means the ratio of the coefficients on these highest-degree terms is the equation of the horizontal asymptote,

$$y = \frac{2}{3}$$

so the function has the horizontal asymptote at $y = 2/3$.

■ 3. Find the horizontal asymptote(s) of the function.

$$h(x) = \frac{x^3 - x^2 + 6x - 1}{7x^4 - 1}$$

Solution:

The x^3 term is the highest-degree term in the numerator, and the x^4 term is the highest-degree term in the denominator.

Because the degree of the numerator is less than the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

■ 4. Find the slant asymptote of the function.

$$f(x) = \frac{3x^4 - x^3 + x^2 - 4}{x^3 - x^2 + 1}$$

Solution:

The degree of the numerator is exactly one greater than the degree of the denominator, so the function has a slant asymptote.

Use polynomial long division to rewrite the function as

$$f(x) = 3x + 2 + \frac{3x^2 - 3x - 6}{x^3 - x^2 + 1}$$

The slant asymptote is what we get when we remove the remainder from this rewritten function. If we remove the remainder, we get

$$y = 3x + 2$$

Therefore, the equation of the slant asymptote is $y = 3x + 2$.

■ 5. Find the slant asymptote of the function.

$$g(x) = \frac{8x^2 + 14x - 7}{4x - 1}$$

Solution:

The degree of the numerator is exactly one greater than the degree of the denominator, $2 > 1$, so the function has a slant asymptote.

Use polynomial long division to rewrite the function as

$$2x + 4 - \frac{3}{4x - 1}$$

The slant asymptote is what we get when we remove the remainder from this rewritten function. If we remove the remainder, we get

$$y = 2x + 4$$

Therefore, the equation of the slant asymptote is $y = 2x + 4$.

■ 6. Determine whether the function has a horizontal asymptote, slant asymptote, or neither.

$$h(x) = \frac{x^4 - x^3 - 8}{x^2 - 5x + 6}$$

Solution:

The degree of the numerator is greater than the degree of the denominator, but we know that the function has a slant asymptote if the degree of the numerator is exactly one greater than the degree of the denominator. So the function doesn't have a horizontal asymptote or a slant asymptote.

SKETCHING GRAPHS

1. Sketch the graph of the function.

$$f(x) = x^3 - 4x^2 + 8$$

Solution:

First, let's find the y-intercepts by substituting $x = 0$.

$$y = 0^3 - 4(0)^2 + 8 = 8$$

So the function has a y-intercept at $(0,8)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x^3 - 4x^2 + 8$$

$$(x - 2)(x^2 - 2x - 4) = 0$$

$$x = 2, 1 - \sqrt{5}, 1 + \sqrt{5}$$

So the function has x-intercepts at $(1 - \sqrt{5}, 0)$, $(1 + \sqrt{5}, 0)$, and $(2,0)$.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = 3x^2 - 8x$$

$$3x^2 - 8x = 0$$

$$x(3x - 8) = 0$$

$$x = 0, \ \frac{8}{3}$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0	x=0	0<x<8/3	x=8/3	x>8/3
x	-2	0	1	8/3	4
f'(x)	+	0	-	0	+
Direction	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $(-\infty,0)$,
- has a local maximum at $x = 0$,
- decreases on the interval $(0,8/3)$,
- has a local minimum at $x = 8/3$, and then
- increases on the interval $(8/3,\infty)$.

Evaluate the function at the extrema.

$$f(0) = (0)^3 - 4(0)^2 + 8 = 8$$

$$f\left(\frac{8}{3}\right) = \left(\frac{8}{3}\right)^3 - 4\left(\frac{8}{3}\right)^2 + 8 = -\frac{40}{27}$$

There's a local maximum at $(0,8)$ and a local minimum at $(8/3, -40/27)$. Now use the second derivative to determine concavity.

$$f''(x) = 6x - 8$$

$$6x - 8 = 0$$

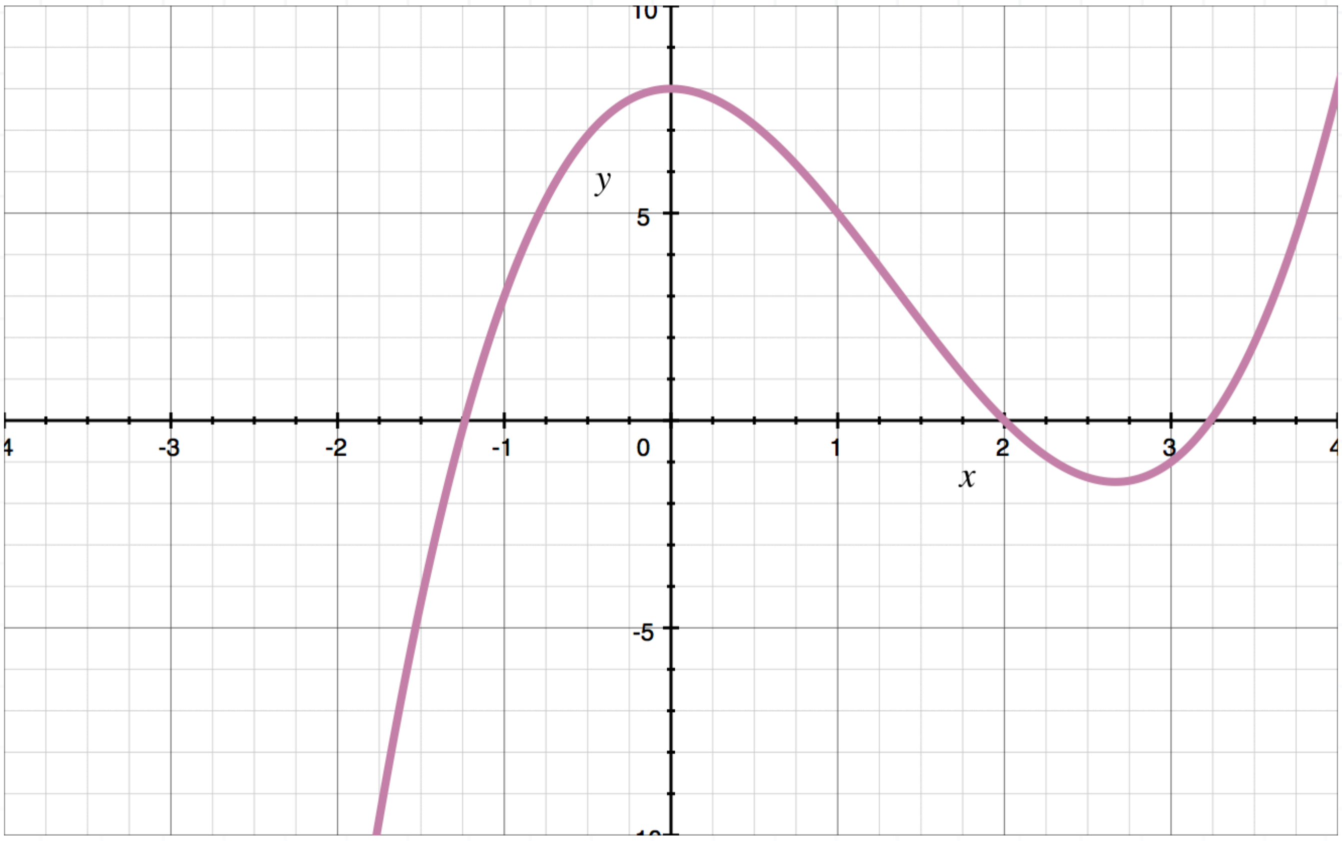
$$x = \frac{4}{3}$$

Test values around the inflection point $x = 4/3$.

Interval	x<4/3	x=4/3	x>4/3
x	0	4/3	3
f”(x)	-	0	+
Concavity	Down	Inflection	Up

We can see that $f(x)$ is concave down on the interval $(-\infty,4/3)$ and concave up on the interval $(4/3,\infty)$. Because $f(4/3) = 88/27$, $f(x)$ has an inflection point at $(4/3,88/27)$. Since $f(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



■ 2. Sketch the graph of the function.

$$g(x) = \frac{1}{4}x^4 - \frac{1}{3}x^3 - 3x^2 + 1$$

Solution:

First, let's find the y-intercepts by substituting $x = 0$.

$$y = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - 3(0)^2 + 1 = 1$$

So the function has a y-intercept at $(0,1)$. To find x-intercepts, we'll substitute $y = 0$. We get four x-intercepts, but it's not easy to find them.

Take the derivative, then set it equal to 0 to find critical points.

$$g'(x) = x^3 - x^2 - 6x$$

$$x^3 - x^2 - 6x = 0$$

$$x(x-3)(x+2) = 0$$

$$x = -2, 0, 3$$

Use the first derivative test to see where $g(x)$ is increasing and decreasing.

Interval	x<-2	x=-2	-2<x<0	x=0	0<x<3	x=3	x>3
x	-4	-2	-1	0	2	3	4
g'(x)	-	0	+	0	-	0	+
Direction	Decreasing	Minimum	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $g(x)$

- decreases on the interval $(-\infty, -2)$,
- has a local minimum at $x = -2$,
- increases on the interval $(-2,0)$,
- has a local maximum at $x = 0$
- decreases on the interval $(0,3)$
- has a local minimum at $x = 3$, and then
- increases on the interval $(3,\infty)$.

Evaluate the function at the extrema.

$$g(-2) = \frac{1}{4}(-2)^4 - \frac{1}{3}(-2)^3 - 3(-2)^2 + 1 = -\frac{13}{3}$$

$$g(0) = \frac{1}{4}(0)^4 - \frac{1}{3}(0)^3 - 3(0)^2 + 1 = 1$$

$$g(3) = \frac{1}{4}(3)^4 - \frac{1}{3}(3)^3 - 3(3)^2 + 1 = -\frac{59}{4}$$

There's a local minimum at $(-2, -13/3)$, a local maximum at $(0,1)$, and a local minimum at $(3, -59/4)$. Now use the second derivative to determine concavity.

$$g''(x) = 3x^2 - 2x - 6$$

$$3x^2 - 2x - 6 = 0$$

$$x = \frac{2 \pm \sqrt{4 + 72}}{6} = \frac{1 \pm \sqrt{19}}{3}$$

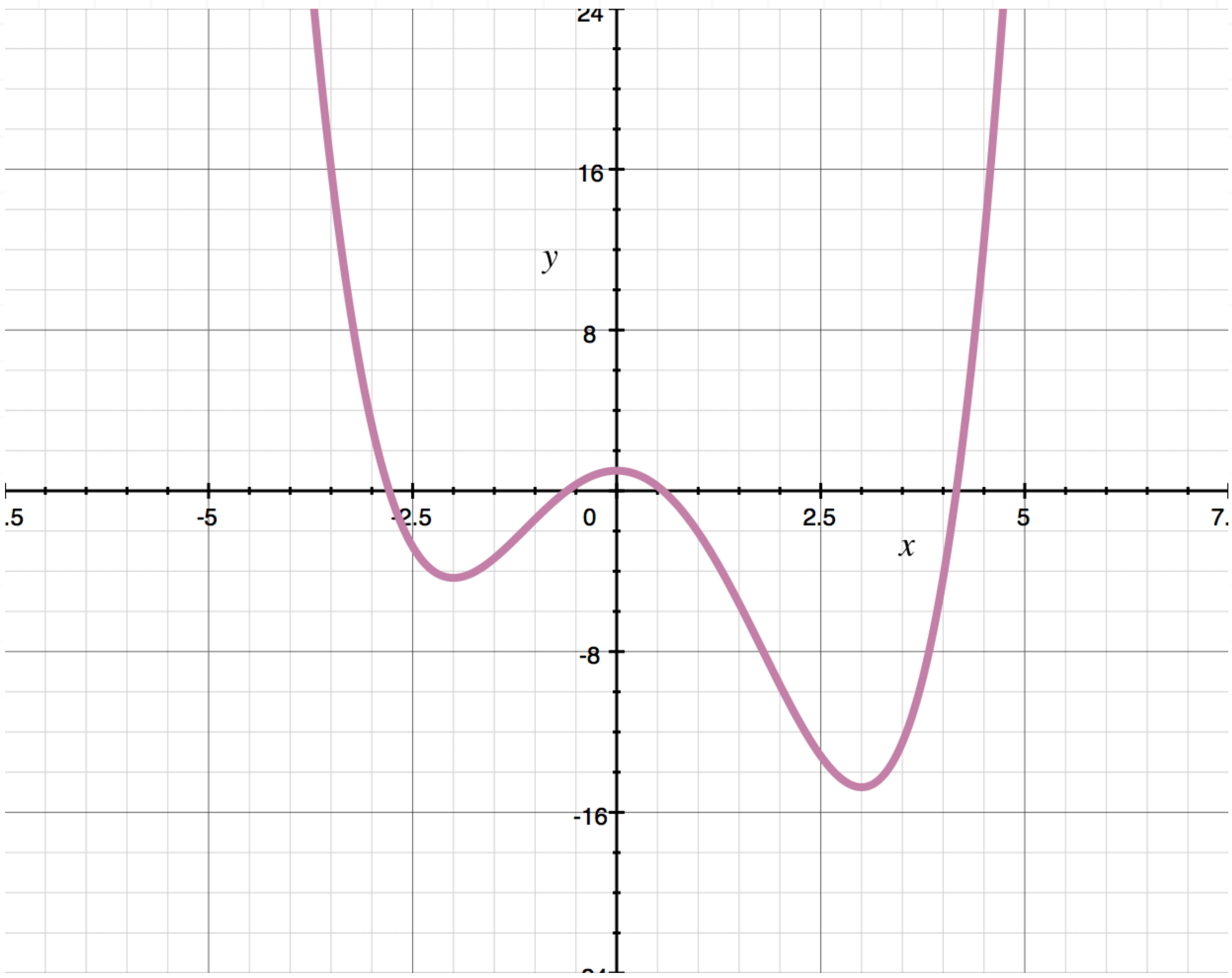
Test values around the inflection points using their approximate values.

Interval	x<-1.12	x=-1.12	-1.12<x<1.79	x=1.79	x>1.79
x	-2	-1.2	1	1.79	4
g''(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

We can see that $g(x)$ is concave up on the interval $(-\infty, (1 - \sqrt{19})/3)$, concave down on the interval $((1 - \sqrt{19})/3, (1 + \sqrt{19})/3)$, and concave up on the interval $((1 + \sqrt{19})/3, \infty)$. Because $g((1 - \sqrt{19})/3) \approx -1.9$, $g(x)$ has an inflection point at approximately $(-1.12, -1.9)$. Because

$g((1+\sqrt{19})/3) \approx -7.96$, $g(x)$ has an inflection point at approximately $(1.79,-7.96)$. Since $g(x)$ is a polynomial function, its graph has no asymptotes.

Putting all this together, the graph is



■ 3. Sketch the graph of the function.

$$h(x)=\frac{x^2+x-6}{4x^2+16x+12}$$

Solution:

Factor the numerator and denominator, then cancel common factors.

$$h(x) = \frac{x^2 + x - 6}{4x^2 + 16x + 12} = \frac{(x+3)(x-2)}{4(x+3)(x+1)} = \frac{x-2}{4(x+1)}$$

So the domain of the function is all real numbers except $x = -3$ and $x = -1$.

Take the derivative, then set it equal to 0 to find critical points.

$$h'(x) = \frac{12(x^2 + 6x + 9)}{(4x^2 + 16x + 12)^2} = \frac{12(x+3)^2}{[4(x+1)(x+3)]^2} = \frac{12(x+3)^2}{16(x+1)^2(x+3)^2} = \frac{3}{4(x+1)^2}$$

There are no values for which $h'(x) = 0$, so there are no critical points. The derivative is undefined when $x = -1$, but this is not a critical point since it's not in the domain of the function. Now use the second derivative to determine concavity.

$$h''(x) = -\frac{3}{2(x+1)^3}$$

There are no values for which $h''(x) = 0$, but $h''(x)$ is undefined when $x = -1$. Test values around the inflection point $x = -1$.

Interval	x<-1	x=-1	x>-1
x	-3	-1	3
h''(x)	+	DNE	-
Concavity	Up	Inflection	Down

We can see that $h(x)$ is concave up on the interval $(-\infty, -1)$ and concave down on the interval $(-1, \infty)$. Since $h(x)$ is a rational function, we need to look for asymptotes.

The behavior of the function is dominated by the highest degree terms in the numerator and denominator, which means the horizontal asymptote is

$$\lim_{x \to \pm\infty} \frac{x^2}{4x^2} = \lim_{x \to \pm\infty} \frac{1}{4} = \frac{1}{4}$$

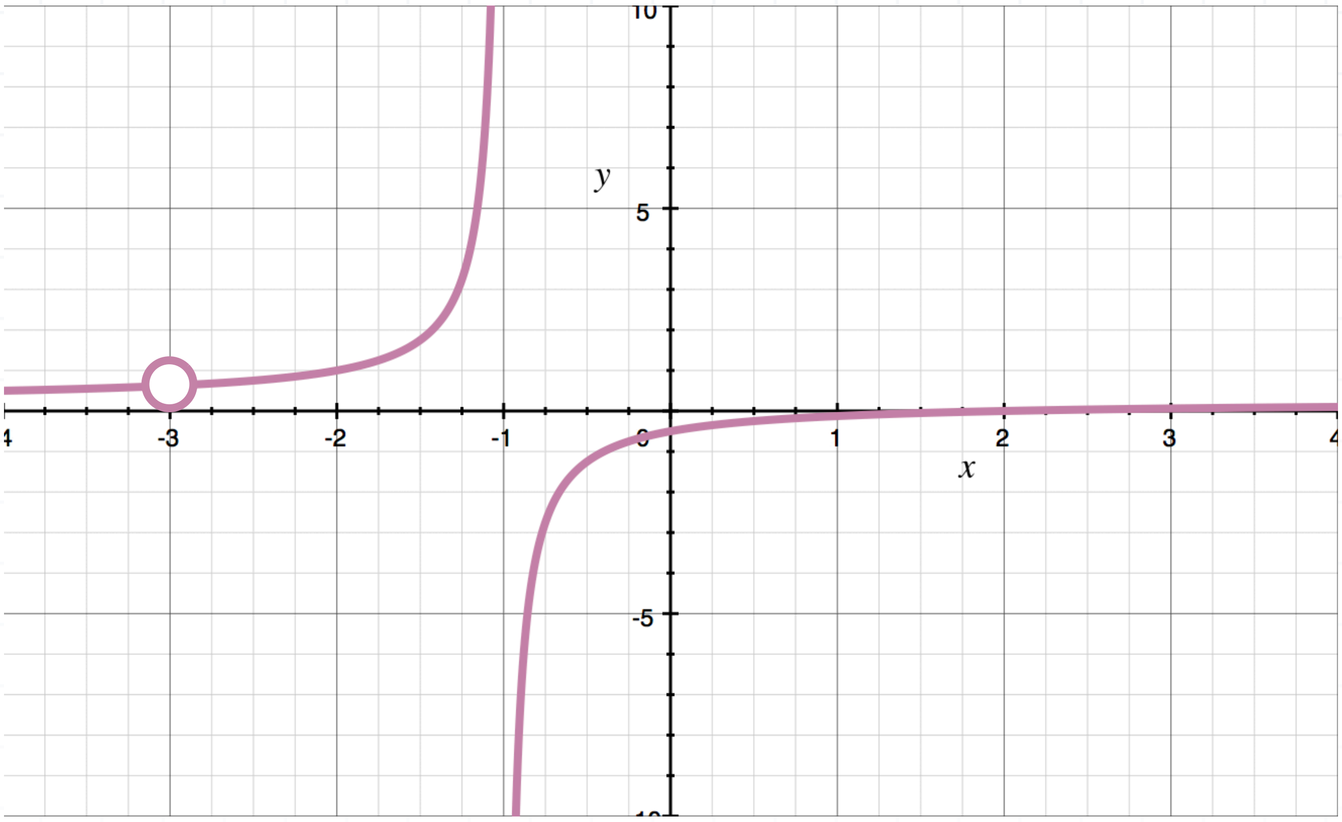
The denominator of $h(x)$ is 0 when $x = -1$, so the function has a vertical asymptote there and the removable discontinuity at $x = -3$. To determine the behavior of the function near $x = -1$, find

$$\lim_{x \to -1^-} h(x) = \infty$$

$$\lim_{x \to -1^+} h(x) = -\infty$$

Lastly, the function crosses the x-axis when the numerator equals 0, which occurs at $x = 2$, and the function crosses the y-axis when $x = 0$, which gives a y-intercept of $(0, -1/2)$.

Putting all this together, the graph is



4. Sketch the graph of the function.

$$f(x) = \frac{4}{1 + x^2}$$

Solution:

First, let's find the y-intercepts by substituting $x = 0$.

$$f(0) = \frac{4}{1 + 0^2} = 4$$

So the function has a y-intercept at $(0,4)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = \frac{4}{1 + x^2}$$

Because there's no value of x that makes this equation true, the function has no x-intercepts.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = -\frac{8x}{(1+x^2)^2}$$

$$-8x = 0$$

$$x = 0$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0	x=0	x>0
x	-1	0	1
f'(x)	+	0	-
Direction	Increasing	Maximum	Decreasing

We can see that $f(x)$

- increases on the interval $(-\infty, 0)$,
- has a local maximum at $x = 0$,
- decreases on the interval $(0, \infty)$,

Evaluate the function at the extrema.

$$f(0) = \frac{4}{1+0^2} = 4$$

There’s a local maximum at $(0,4)$. Now use the second derivative to determine concavity.

$$f''(x) = \frac{8(3x^2 - 1)}{(x^2 + 1)^3}$$

$$3x^2 - 1 = 0$$

$$x = \pm \frac{1}{\sqrt{3}}$$

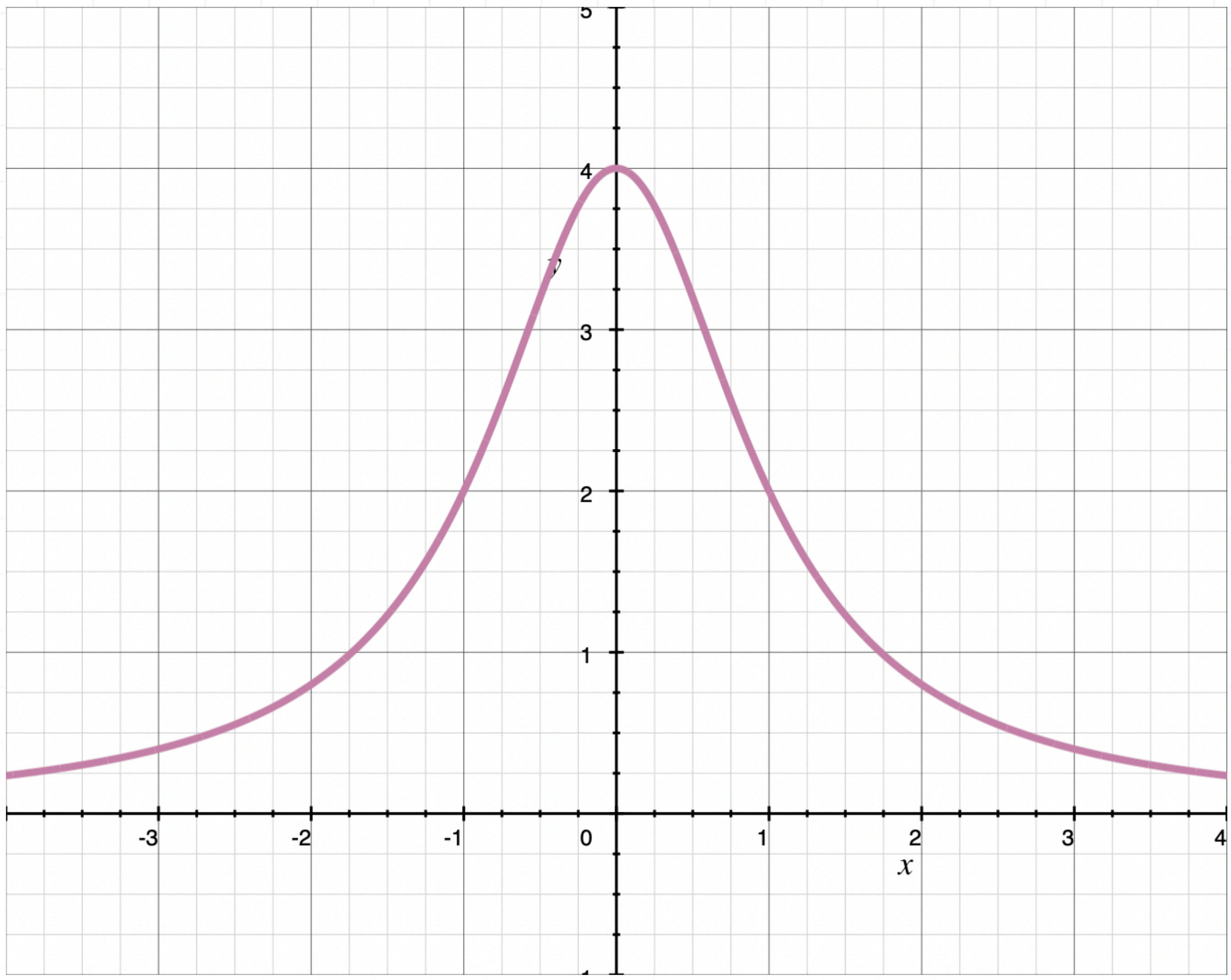
Test values around the inflection point $x = \pm 1/\sqrt{3}$.

Interval	x<-0.6	x=-0.6	-0.6<x<0.6	x=0.6	x>0.6
x	-1	-0.6	0	0.6	1
f”(x)	+	0	-	0	+
Concavity	Up	Inflection	Down	Inflection	Up

We can see that $f(x)$ is concave up on the intervals $(-\infty, -1/\sqrt{3})$ and $(1/\sqrt{3}, \infty)$, and concave down on the interval $(-1/\sqrt{3}, 1/\sqrt{3})$. Because $f(-1/\sqrt{3}) = 3$ and $f(1/\sqrt{3}) = 3$, $f(x)$ has inflection points at $(-1/\sqrt{3}, 3)$ and $(1/\sqrt{3}, 3)$.

The function has no vertical asymptotes, but because the degree of the numerator is less than the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

Putting all this together, the graph is



■ 5. Sketch the graph of the function.

$$f(x) = 2x \ln x$$

Solution:

The domain of the function is $x > 0$. The function has no y-intercepts. To find x-intercepts, we'll substitute $y = 0$.

$$0 = 2x \ln x$$

$$x = 1$$

So the function has an x-intercept at $(1,0)$.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = 2\ln x + 2$$

$$0 = 2\ln x + 2$$

$$\ln x = -1$$

$$x = \frac{1}{e}$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	x<0.37	x=0.37	x>0.37
x	0.1	0.37	1
f'(x)	-	0	+
Direction	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $\left(\frac{1}{e}, \infty\right)$,
- has a local minimum at $x = \frac{1}{e}$,
- decreases on the interval $\left(0, \frac{1}{e}\right)$,

Evaluate the function at the extrema.

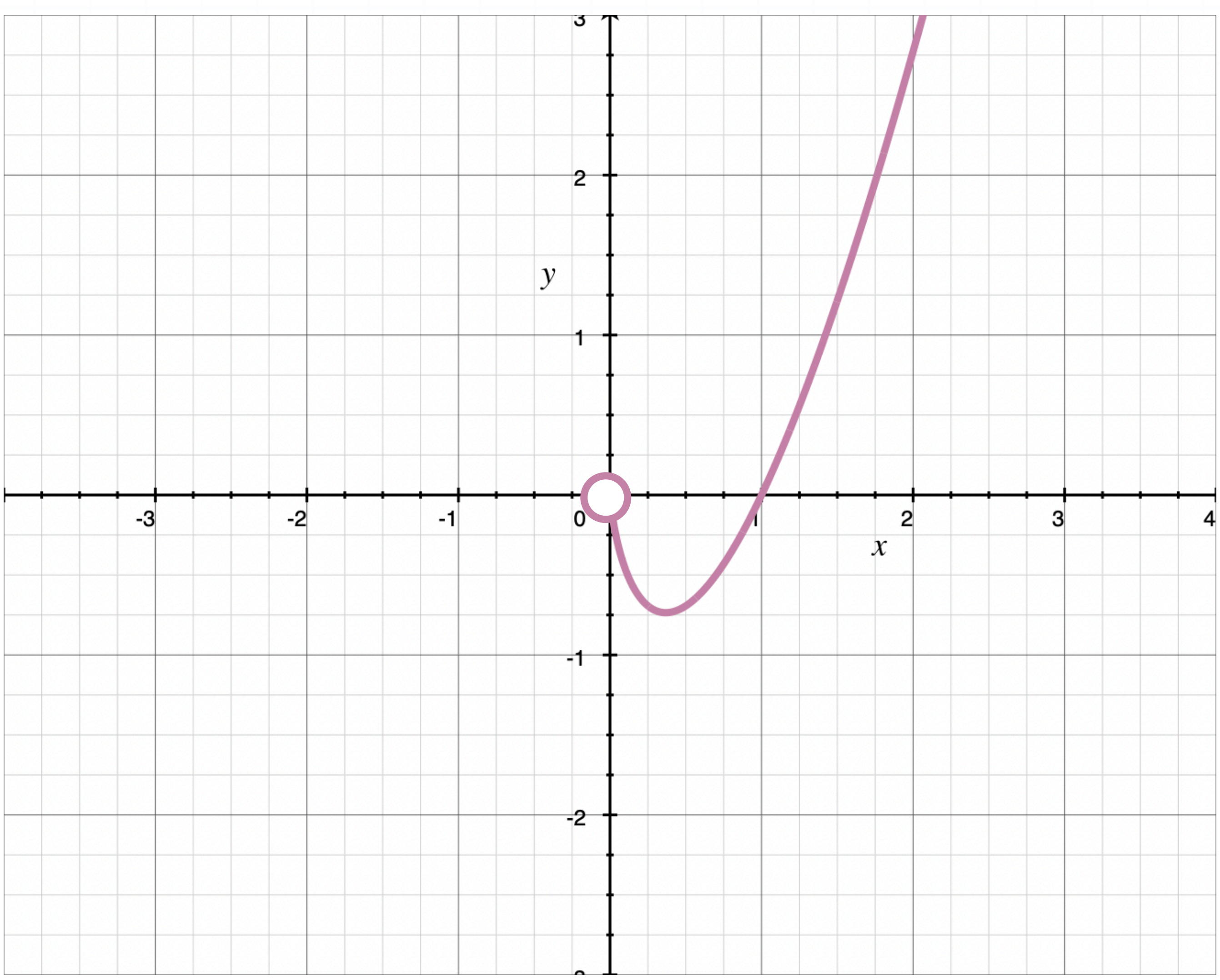
$$f\left(\frac{1}{e}\right) = 2\frac{1}{e}\ln\frac{1}{e} = -\frac{2}{e}$$

There's a local minimum at $(1/e, -2/e)$. Now use the second derivative to determine concavity.

$$f''(x) = \frac{2}{x}$$

$$\frac{2}{x} = 0$$

The function has no inflection points. Since $f''(x) > 0$ for all x in the domain, the function is concave up everywhere. And the function $f(x)$ has no asymptotes. Putting all this together, the graph is



■ 6. Sketch the graph of the function.

$$f(x) = x^2\sqrt{x+4}$$

Solution:

The domain of the function is $x \geq -4$. First, let's find the y-intercepts by substituting $x = 0$.

$$f(0) = 0^2\sqrt{0+4} = 0$$

So the function has a y-intercepts at $(0,0)$. To find x-intercepts, we'll substitute $y = 0$.

$$0 = x^2\sqrt{x+4}$$

$$x = -4,0$$

So the function has x-intercepts at $(-4,0)$ and $(0,0)$.

Take the derivative, then set it equal to 0 to find critical points.

$$f'(x) = \frac{5x^2+16x}{2\sqrt{x+4}}$$

$$0 = \frac{5x^2+16x}{2\sqrt{x+4}}$$

$$5x^2 + 16x = 0$$

$$x = -4, \ -\frac{16}{5}, 0$$

Use the first derivative test to see where $f(x)$ is increasing and decreasing.

Interval	-4<x<-16/5	x=-16/5	-16/5<x<0	x=0	x>0
x	-3.5	-16/5	-1	0	1
f'(x)	+	0	-	0	+
Direction	Increasing	Maximum	Decreasing	Minimum	Increasing

We can see that $f(x)$

- increases on the interval $\left(-4,-\frac{16}{5}\right)$ and $(0,\infty)$,
- has a local maximum at $x = -\frac{16}{5}$,
- decreases on the interval $\left(-\frac{16}{5},0\right)$,
- has a local minimum at $x = 0$,

Evaluate the function at the extrema.

$$f(0) = 0^2\sqrt{0+4} = 0$$

$$f\left(-\frac{16}{5}\right) = \left(-\frac{16}{5}\right)^2\sqrt{-\frac{16}{5}+4} = \frac{512}{25\sqrt{5}} \approx 9$$

There's a local maximum at $(-3.2,9)$ and local minimum at $(0,0)$. Now use the second derivative to determine concavity.

$$f''(x) = \frac{15x^2 + 96x + 128}{4(x+4)^{\frac{3}{2}}}$$

$$0 = \frac{15x^2 + 96x + 128}{4(x+4)^{\frac{3}{2}}}$$

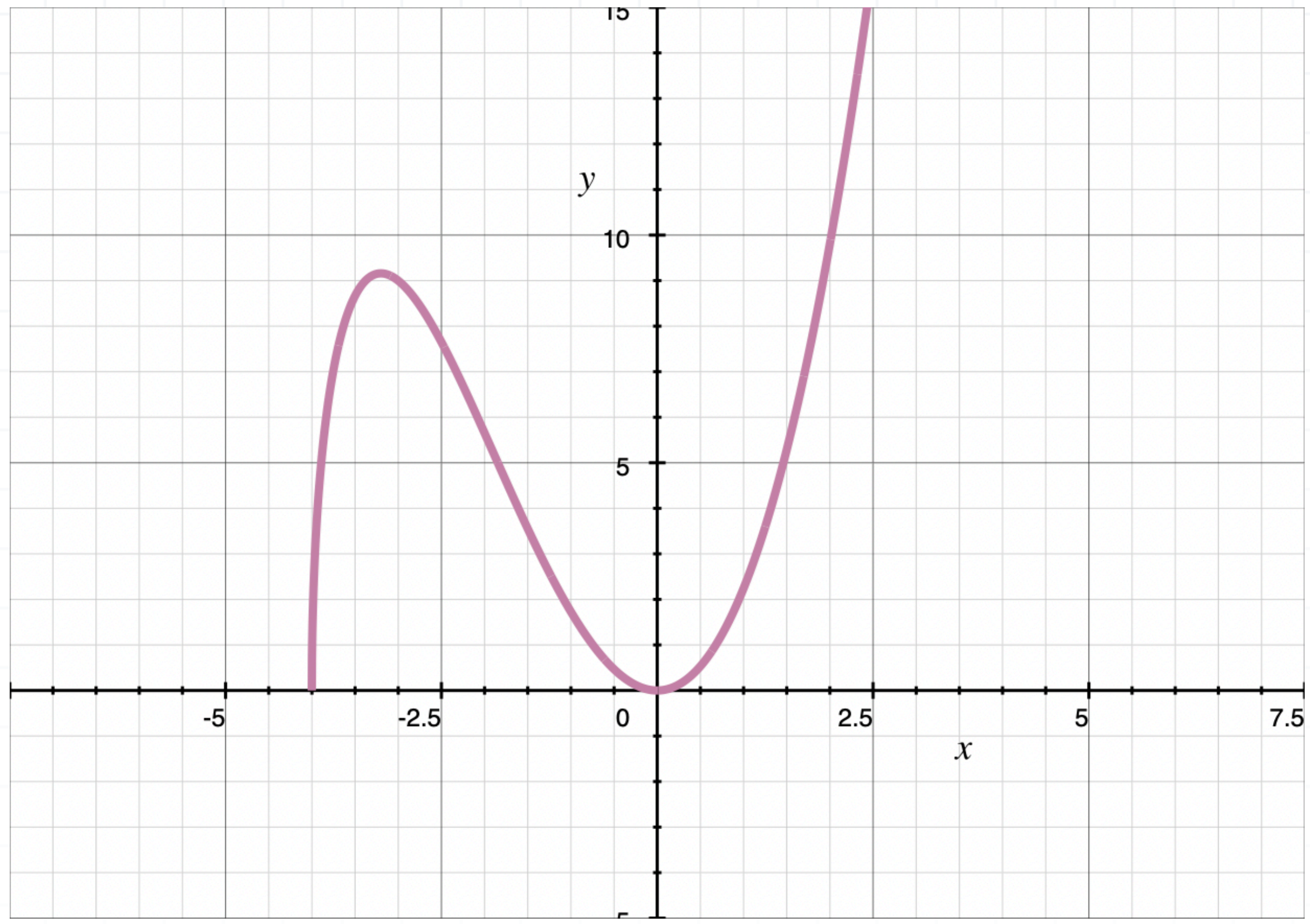
$$15x^2 + 96x + 128 = 0$$

$$x \approx -4.5, \ -1.9$$

Test values around the inflection point $x = -1.9$. The value $x = -4.5$ does not represent an inflection point, because it's not in domain.

Interval	-4<x<-1.9	x=-1.9	x>-1.9
x	-2	-1.9	0
f''(x)	-	0	+
Concavity	Down	Inflection	Up

We can see that $f(x)$ is concave up on the interval $(-1.9,\infty)$ and concave down on the interval $(-4,-1.9)$. Because $f(-1.9)=5.2$, $f(x)$ has an inflection point at $(-1.9,5.2)$. The function has no asymptotes, so putting all this together, the graph is


y
10
5
-5
-2.5
0
2.5
5
7.5
x

EXTREMA ON A CLOSED INTERVAL

1. Find the extrema of $f(x)=x^3-3x^2+5$ over the closed interval $[-3,4]$.

Solution:

Find the critical points of the function.

$$f'(x)=3x^2-6x$$

$$3x^2-6x=0$$

$$3x(x-2)=0$$

$$x=0,\ 2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x=-3$, $\quad (-3)^3-3(-3)^2+5=-27-27+5=-49$

For $x=0$, $\quad (0)^3-3(0)^2+5=0-0+5=5$

For $x=2$, $\quad (2)^3-3(2)^2+5=8-12+5=1$

For $x=4$, $\quad (4)^3-3(4)^2+5=64-48+5=21$

The results show that $f(x)$ has a global minimum at $(-3,-49)$, a local maximum at $(0,5)$, a local minimum at $(2,1)$, and a global maximum at $(4,21)$.

■ 2. Find the extrema of $g(x)=\sqrt[3]{2x^2+3}$ over the closed interval $[-1,5]$.

Solution:

Find the critical points of the function.

$$g'(x)=\frac{1}{3}(2x^2+3)^{-\frac{2}{3}}(4x)=\frac{4x}{3\sqrt[3]{(2x^2+3)^2}}$$

$$4x=0$$

$$x=0$$

Evaluate the function at the endpoints and at the critical numbers.

For $x=-1$, $\sqrt[3]{2(-1)^2+3}=\sqrt[3]{5}\approx 1.71$

For $x=0$, $\sqrt[3]{2(0)^2+3}=\sqrt[3]{3}\approx 1.44$

For $x=5$, $\sqrt[3]{2(5)^2+3}=\sqrt[3]{53}\approx 3.76$

The results show that $g(x)$ has a global minimum at $(0,\sqrt[3]{3})$, a local maximum at $(0,\sqrt[3]{5})$, and a global maximum at $(5,\sqrt[3]{53})$.

■ 3. Find the extrema of $h(x)=-4x^3+6x^2-3x-2$ over the closed interval $[-4,6]$.

Solution:

Find the critical points of the function.

$$h'(x) = -12x^2 + 12x - 3$$

$$-12x^2 + 12x - 3 = 0$$

$$-3(4x^2 - 4x + 1) = 0$$

$$-3(2x - 1)(2x - 1) = 0$$

$$x = 1/2$$

Evaluate the function at the endpoints and at the critical numbers.

For $x = -4$, $\quad -4(-4)^3 + 6(-4)^2 - 3(-4) - 2 = 362$

For $x = 1/2$, $\quad -4(1/2)^3 + 6(1/2)^2 - 3(1/2) - 2 = -5/2$

For $x = 6$, $\quad -4(6)^3 + 6(6)^2 - 3(6) - 2 = -668$

The results show that $h(x)$ has a global maximum at $(-4,362)$, a horizontal tangent line at $(1/2, -5/2)$, and a global minimum at $(6, -668)$.

■ 4. Find the extrema of the function over the closed interval $[-1,3]$.

$$f(x) = \frac{x^2}{x^2 + 7}$$

Solution:

Find the critical points of the function.

$$f'(x) = \frac{14x}{(x^2 + 7)^2}$$

$$14x = 0$$

$$x = 0$$

Evaluate the function at the endpoints of the interval and at the critical points.

For $x = -1$, $\quad \dfrac{(-1)^2}{(-1)^2 + 7} = \dfrac{1}{8}$

For $x = 0$, $\quad \dfrac{(0)^2}{(0)^2 + 7} = 0$

For $x = 3$, $\quad \dfrac{(3)^2}{(3)^2 + 7} = \dfrac{9}{16}$

The results show that $f(x)$ has a global minimum at $(0,0)$, a local maximum at $(-1,1/8)$, and a global maximum at $(3,9/16)$.

■ 5. Find the extrema of $g(x) = e^{2x^3+4x^2-8x+3}$ over the closed interval $[-4,0]$.

Solution:

Find the critical points of the function.

$$g'(x) = (6x^2 + 8x - 8)e^{2x^3+4x^2-8x+3}$$

$$0 = (6x^2 + 8x - 8)e^{2x^3+4x^2-8x+3}$$

$$(6x^2 + 8x - 8) = 0$$

$$2(3x - 2)(x + 2) = 0$$

$$x = -2, \frac{2}{3}$$

The critical point $x = 2/3$ is outside the interval $[-4,0]$, so we'll ignore it. Evaluate the function at the endpoints of the interval and at the critical points.

For $x = -4$, $\quad e^{2(-4)^3+4(-4)^2-8(-4)+3} = \frac{1}{e^{29}}$

For $x = -2$, $\quad e^{2(-2)^3+4(-2)^2-8(-2)+3} = e^{19}$

For $x = 0$, $\quad e^{2(0)^3+4(0)^2-8(0)+3} = e^3$

The results show that $g(x)$ has a global minimum at $(-4,1/e^{29})$, a local minimum at $(0,e^3)$, and a global maximum at $(-2,e^{19})$.

■ 6. Find the extrema of $h(x) = x - \cos x$ over the closed interval $[0,\pi]$.

Solution:

Find the critical points of the function.

$$f'(x) = 1 + \sin x$$

$$1 + \sin x = 0$$

$$\sin x = -1$$

$$x = -\frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

All critical points are outside the interval $[0,\pi]$, so we'll ignore them. Evaluate the function at the endpoints of the interval.

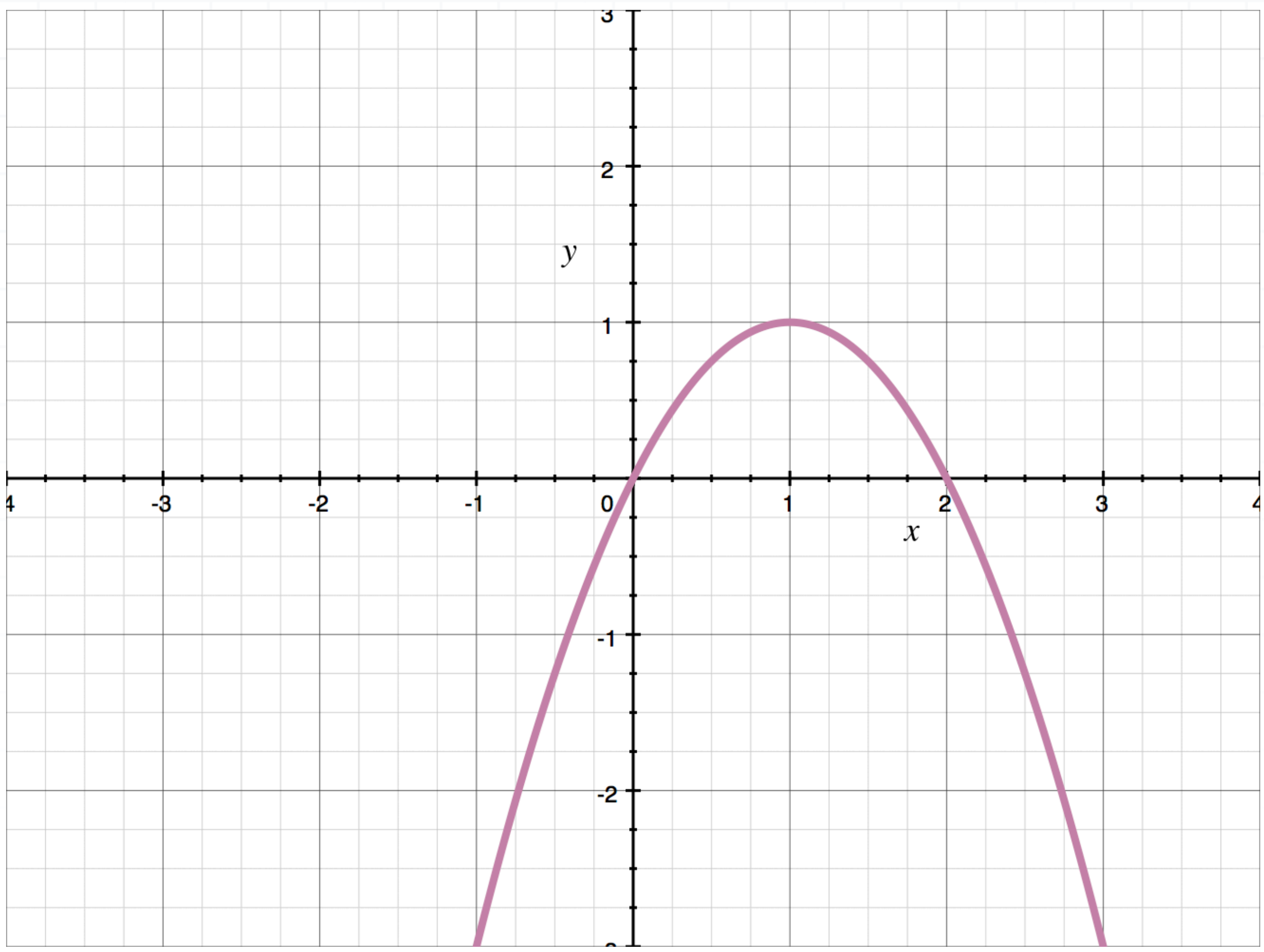
For $x = 0$, $\quad 0 - \cos 0 = -1$

For $x = \pi$, $\quad \pi - \cos \pi = \pi + 1$

The results show that $h(x)$ has a global minimum at $(0, -1)$ and a global maximum at $(\pi, \pi + 1)$.

SKETCHING F(X) FROM F'(X)

■ 1. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.



Solution:

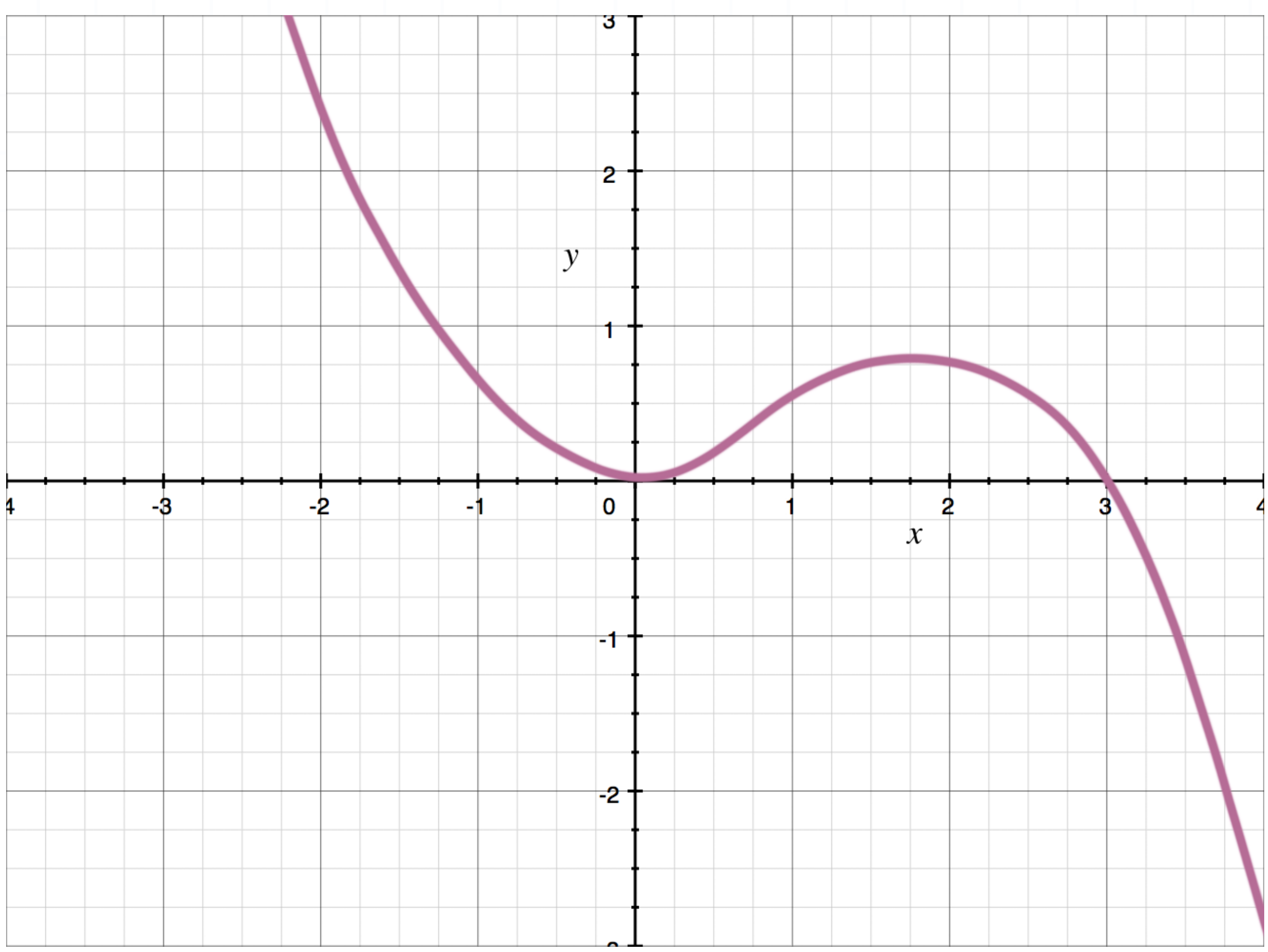
The graph of $f'(x)$ is below the x-axis on the intervals $(-\infty,0)$ and $(2,\infty)$, which means the function $f(x)$ has a negative slope and is decreasing on these intervals.

Additionally, the graph of $f'(x)$ is above the x-axis on the interval $(0,2)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

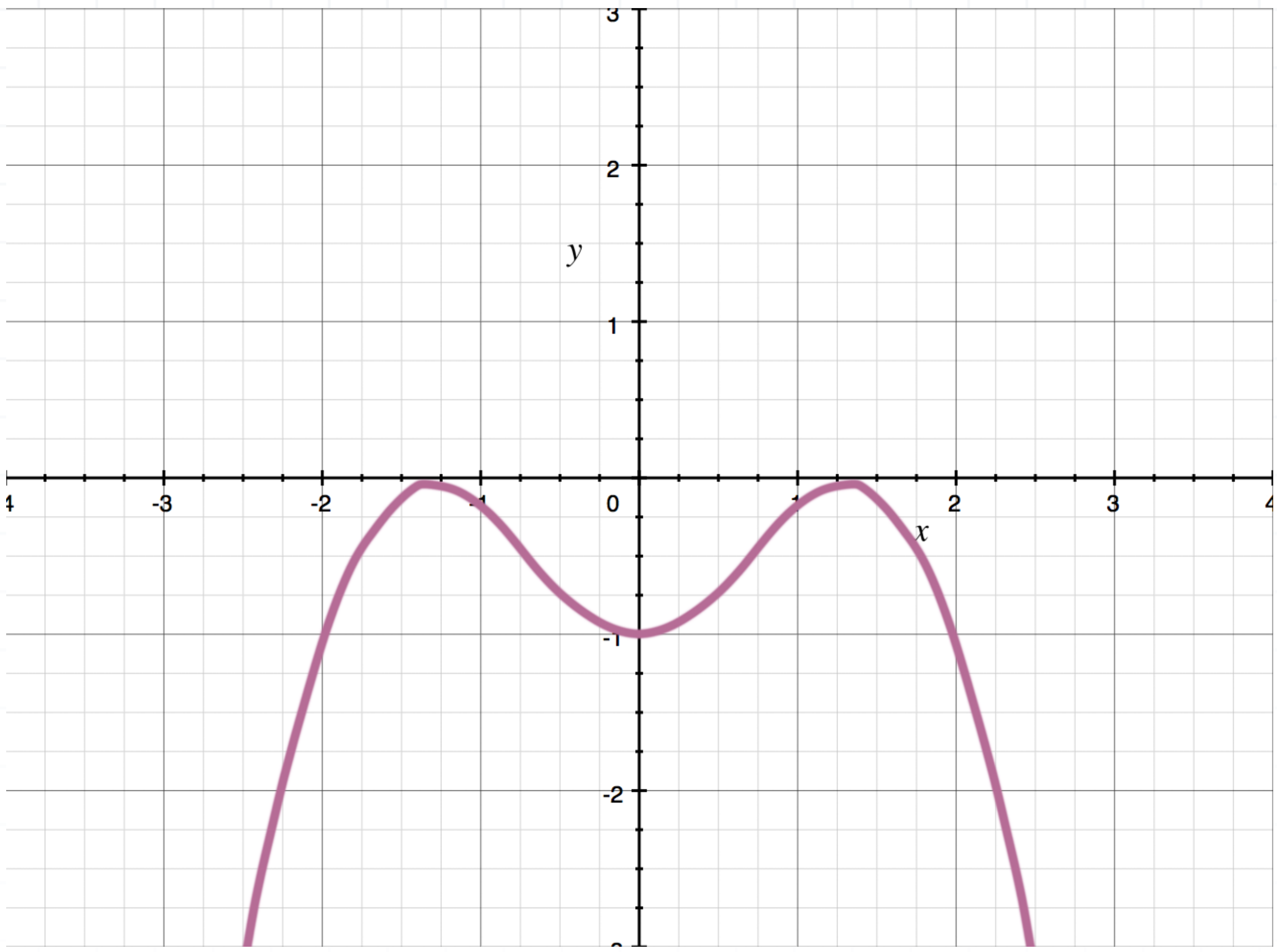
The graph of $f'(x)$ passes through the x-axis and changes sign from negative to positive at $x = 0$, which means that the graph of $f(x)$ has a minimum value at $x = 0$, and the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 2$, which means that the graph of $f(x)$ has a maximum value at $x = 2$.

The graph of $f'(x)$ has a maximum value at $x = 1$, and its slope changes from positive to negative at that point. This means that the graph of $f(x)$ is concave up to the left of $x = 1$, has an inflection point at $x = 1$, and is concave down to the right of $x = 1$.

Putting these facts together, and based on the "assumption" that $f(x)$ contains the point $(0,0)$, this is a possible graph of $f(x)$:



■ 2. Sketch a possible graph of $g'(x)$ given the graph below of $g(x)$.



Solution:

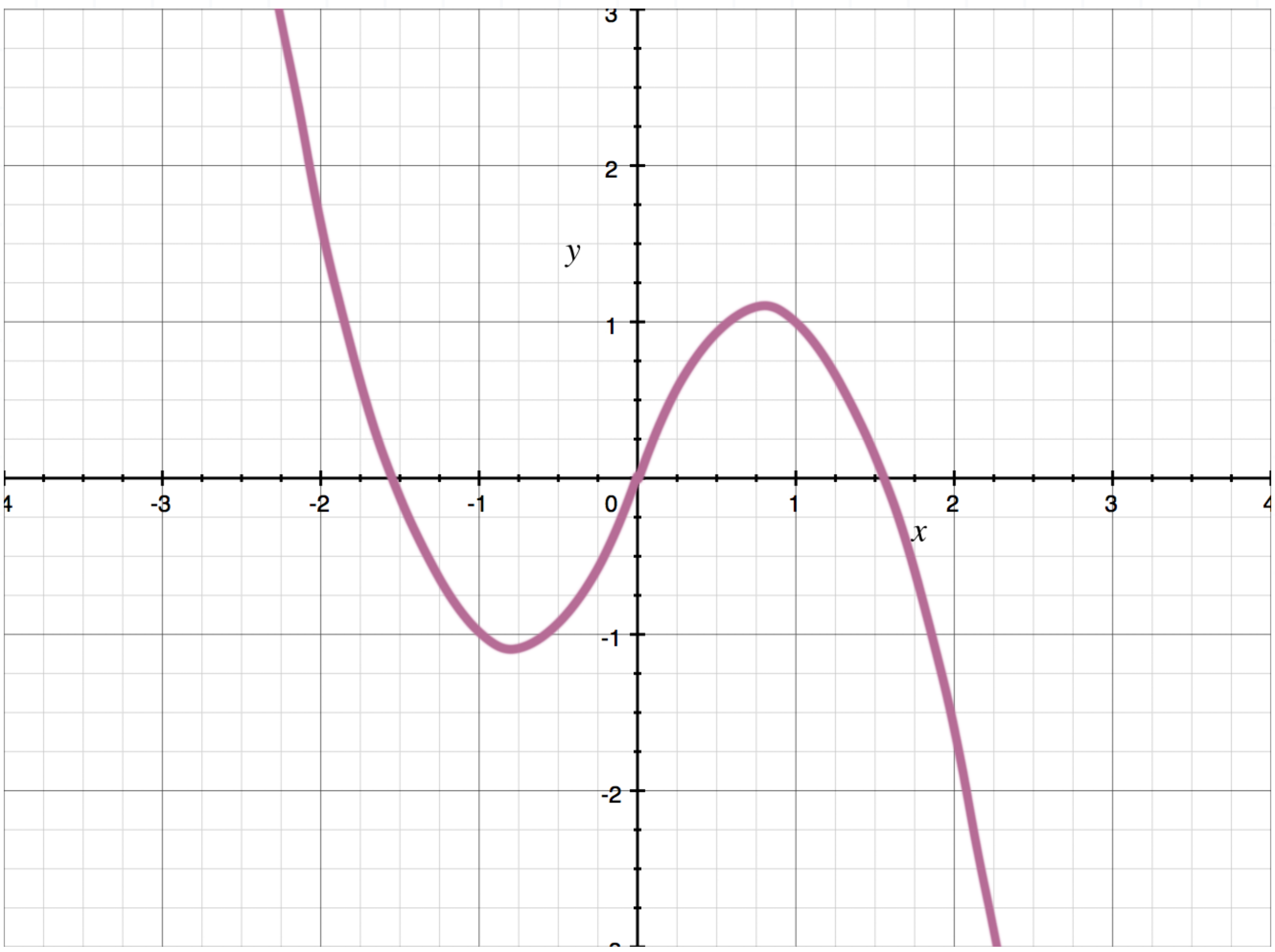
The graph of $g(x)$ has a positive slope on the intervals $(-\infty, -1.5)$ and $(0,1.5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the intervals $(-1.5,0)$ and $(1.5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on these intervals.

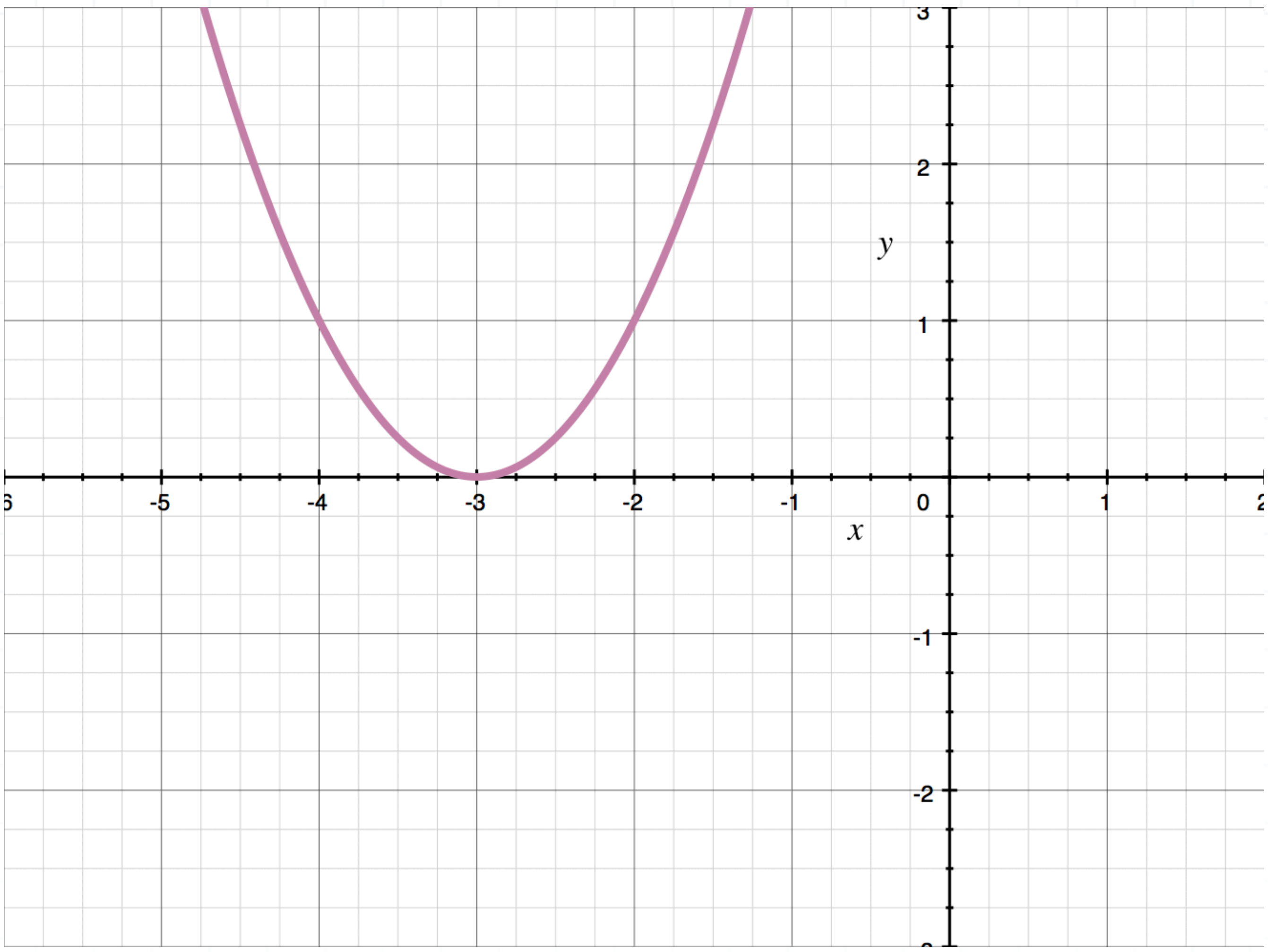
The graph of $g(x)$ has a maximum value at $x = -1.5$ and $x = 1.5$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x = -1.5$ and $x = 1.5$.

The graph of $g(x)$ has a minimum value at $x = 0$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x = 0$, and changes from negative to positive.

It appears that the graph of $g(x)$ has an inflection point at $x = -0.75$ and $x = 0.75$, so the graph of $g'(x)$ has extrema at those points. Putting these facts together, this is a possible graph of $g'(x)$:



3. Sketch a possible graph of $h(x)$ given the graph below of $h'(x)$.

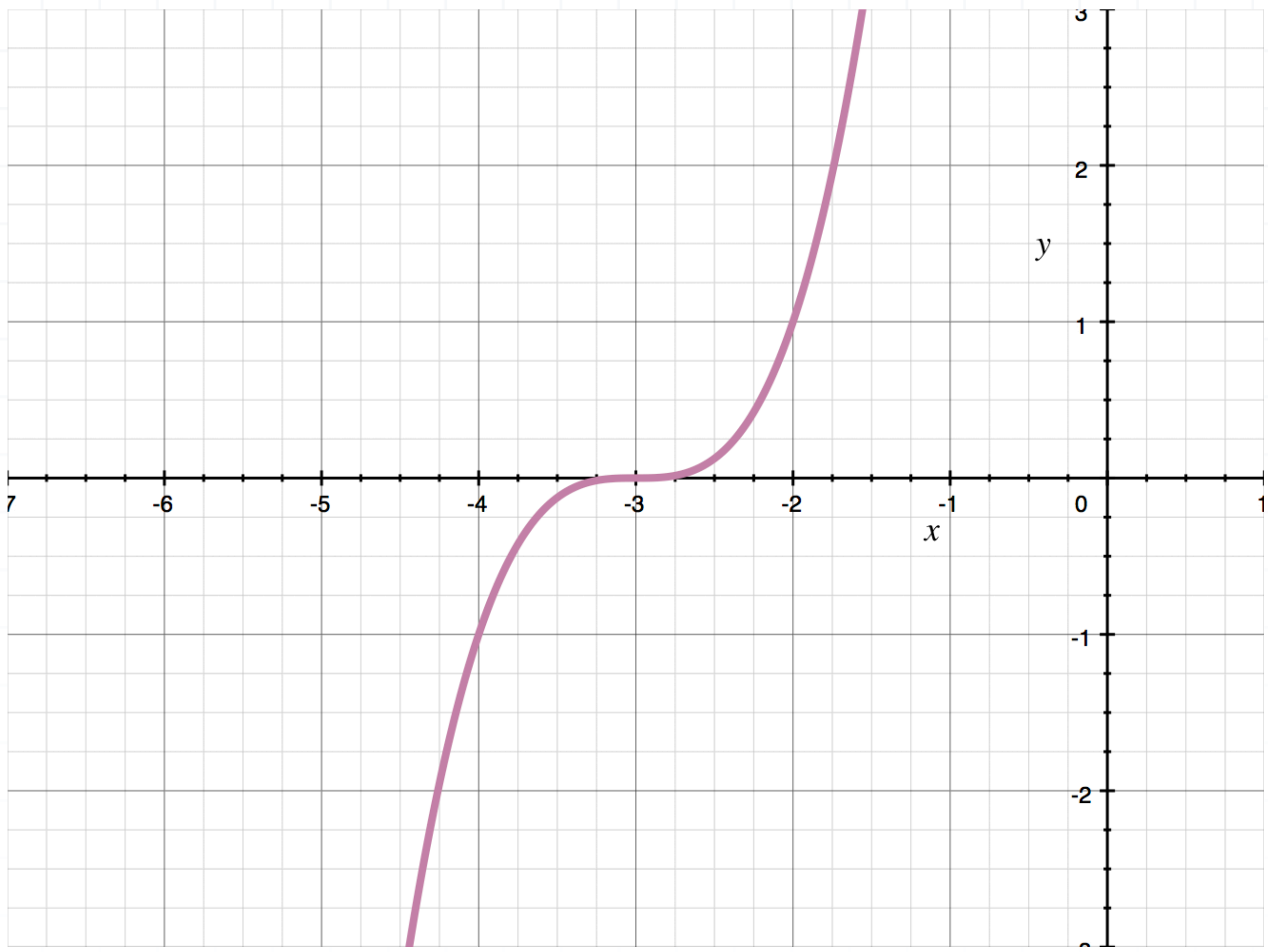


Solution:

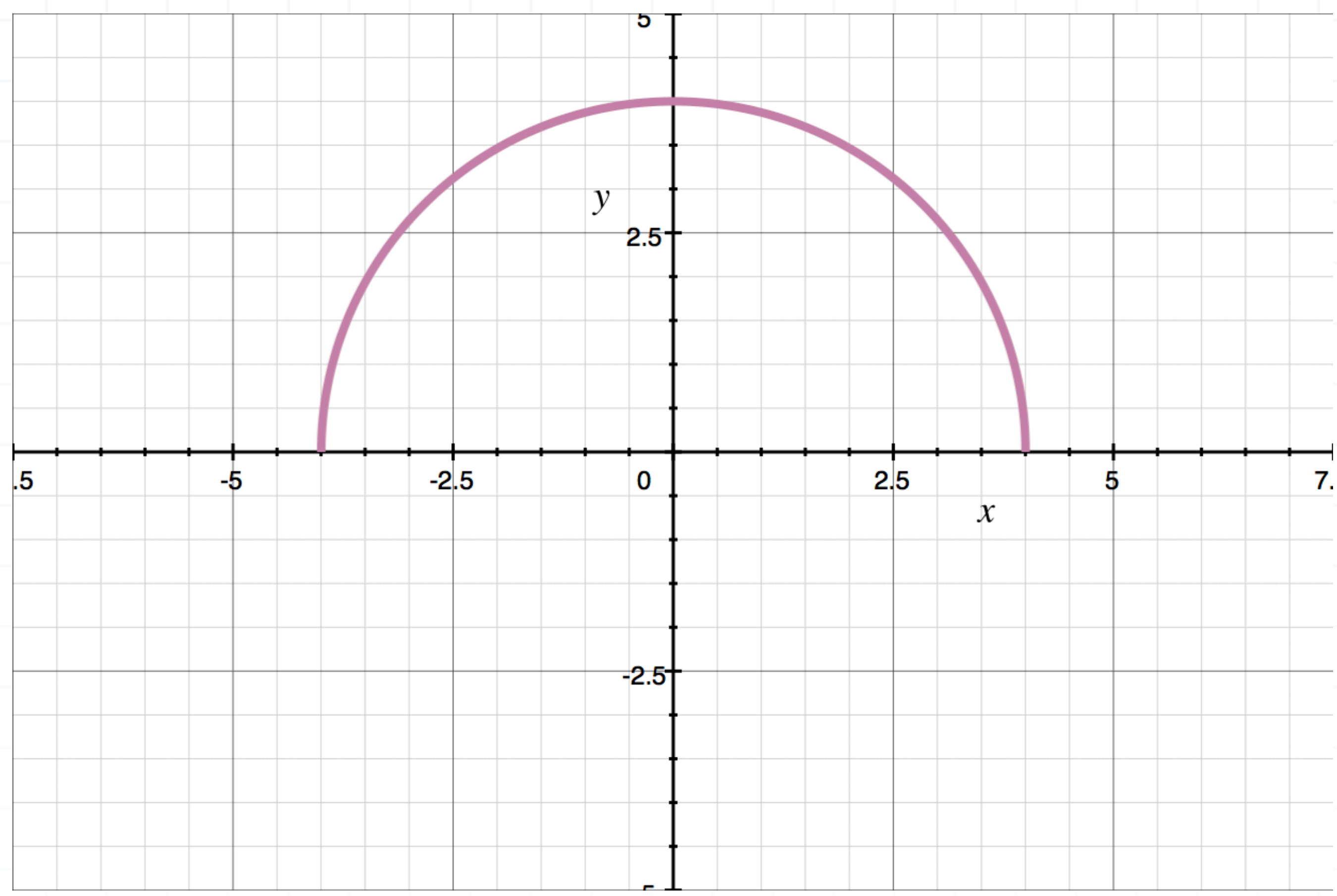
The graph of $h'(x)$ is above the x-axis on the intervals $(-\infty, -3)$ and $(-3,\infty)$, which means the function $h(x)$ has positive slopes and is increasing on these intervals. Since we're only excluding the single point $x = -3$, that means the function is essentially increasing everywhere.

The graph of $h'(x)$ is on the x-axis at $x = -3$, which means the function $h(x)$ has a horizontal tangent at $x = -3$ and is increasing on both sides of this point.

The graph of $h'(x)$ has a minimum value at $x = -3$, and its slope changes from positive to negative at that point. This means that the graph of $h(x)$ is concave down to the left of $x = -3$, has an inflection point at $x = -3$, and is concave up to the right of $x = -3$. Putting these facts together, this is a possible graph of $h(x)$:



■ 4. Sketch a possible graph of $f'(x)$ given the graph below of $f(x)$.



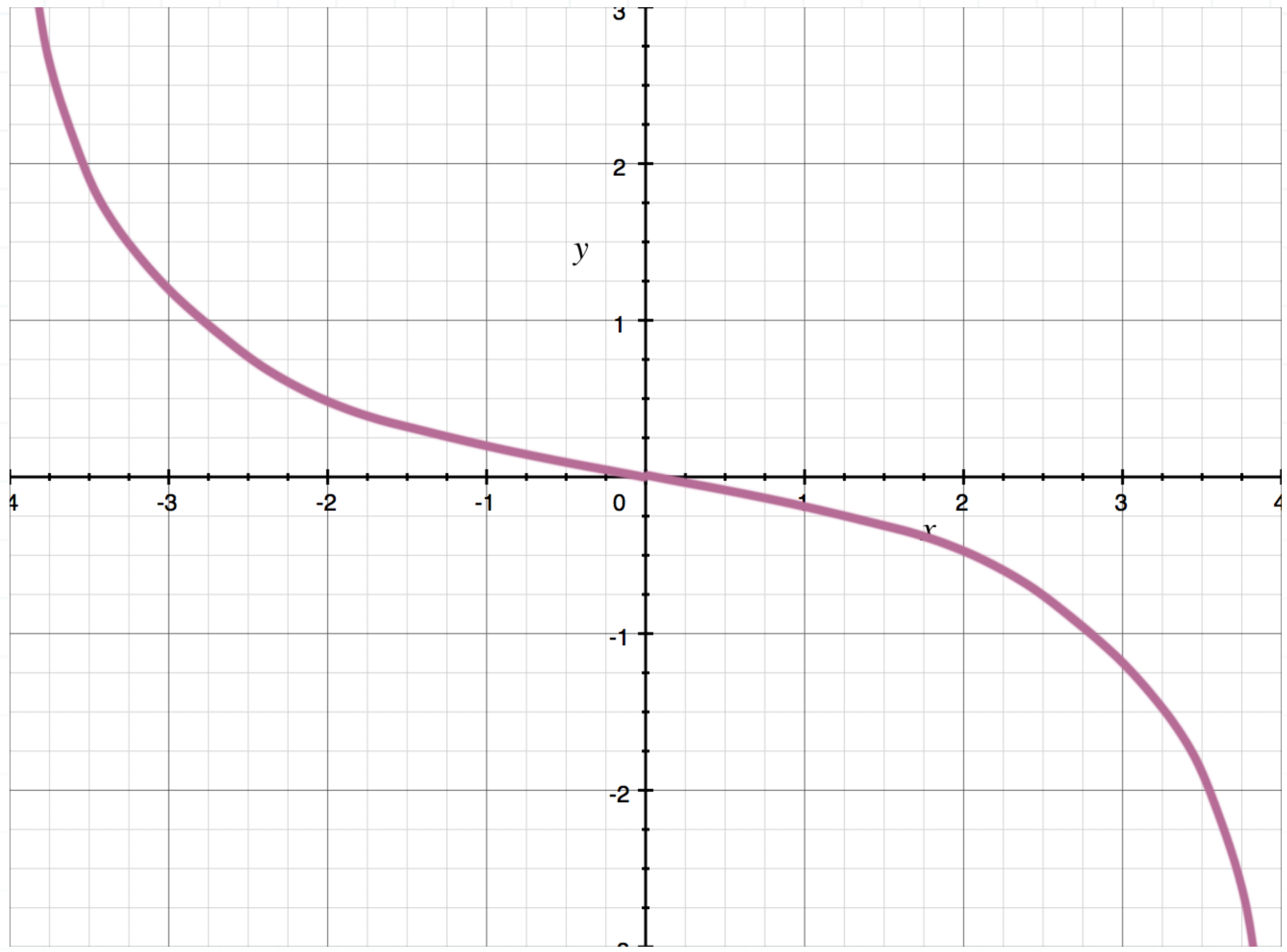
Solution:

The graph of $f(x)$ has a positive slope on the interval $(-4,0)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is above the x-axis on this interval.

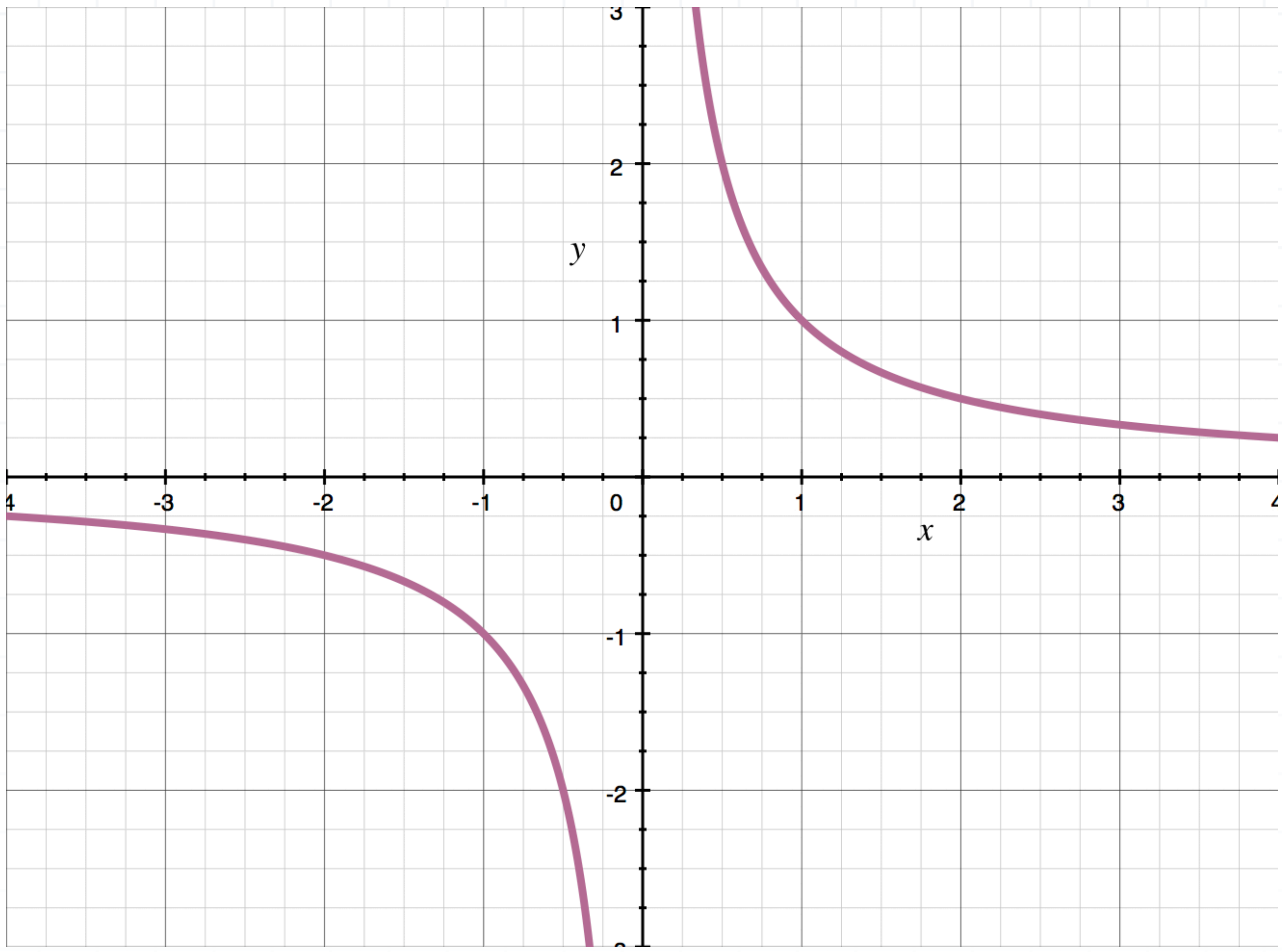
The graph of $f(x)$ has a negative slope on the interval $(0,4)$. Since $f'(x)$ is the derivative of $f(x)$, the graph of $f'(x)$ is below the x-axis on this interval.

The graph of $f(x)$ has a maximum value at $x = 0$ and its slope is 0, so the graph of $f'(x)$ passes through the x-axis and changes sign from positive to negative at $x = 0$.

The graph of $f(x)$ has no inflection points so the graph of $f'(x)$ has no extrema in the interval $(-4,4)$. Putting these facts together, this is a possible graph of $f'(x)$:



■ 5. Sketch a possible graph of $f(x)$ given the graph below of $f'(x)$.



Solution:

The graph of $f'(x)$ is below the x-axis on the interval $(-\infty,0)$, which means the function $f(x)$ has a negative slope and is decreasing on this interval.

The graph of $f'(x)$ is above the x-axis on the interval $(0,\infty)$, which means the function $f(x)$ has a positive slope and is increasing on this interval.

The graph of the $f'(x)$ does not pass through the x-axis, which means that the graph of $f(x)$ does not have any extrema.

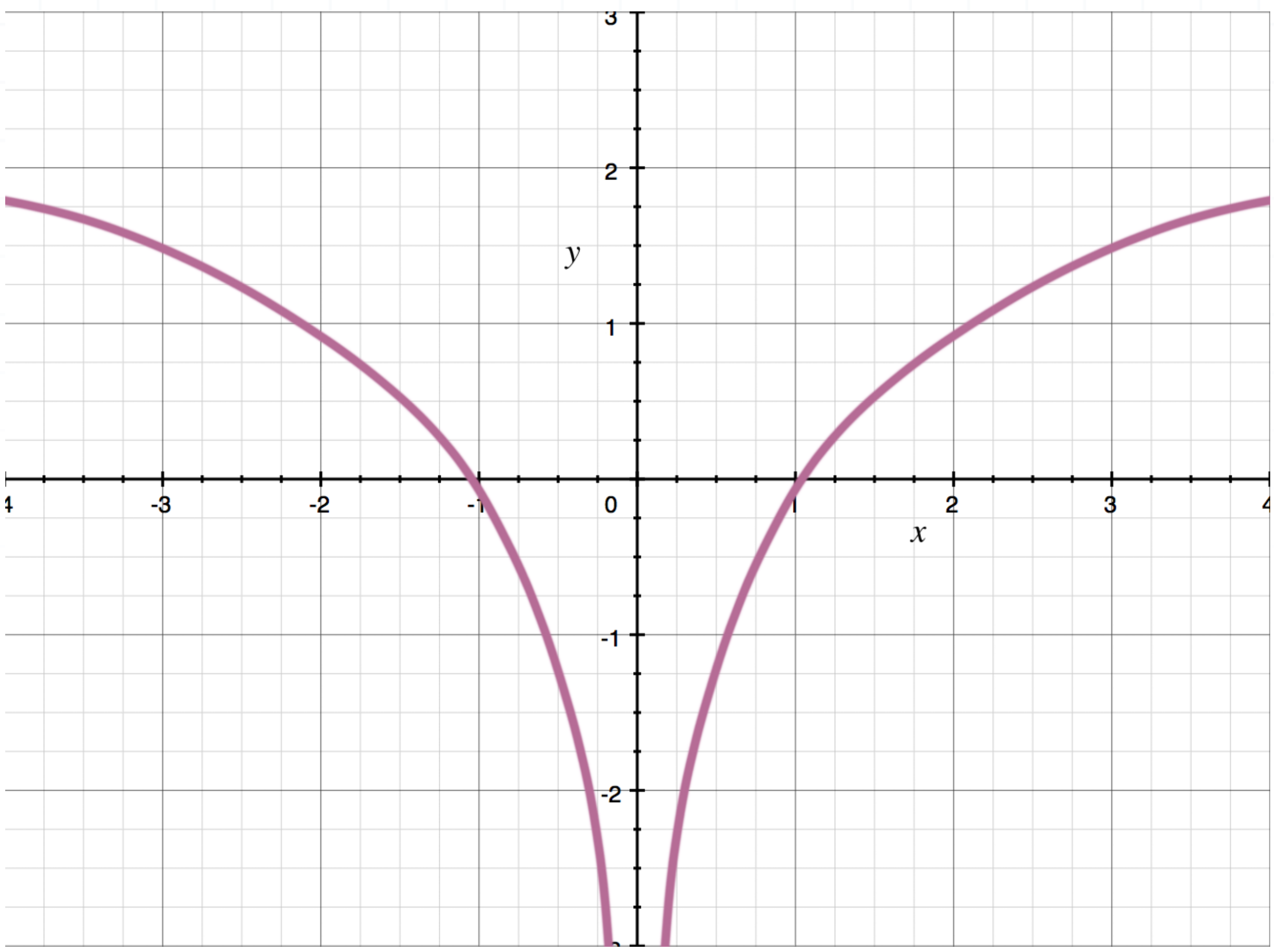
The slope of the graph of $f'(x)$ is negative on $(-\infty,0)$ and $(0,\infty)$. This means that the graph of $f(x)$ is concave down to the left and to the right of the y-axis.

The graph of $f'(x)$ has these limits:

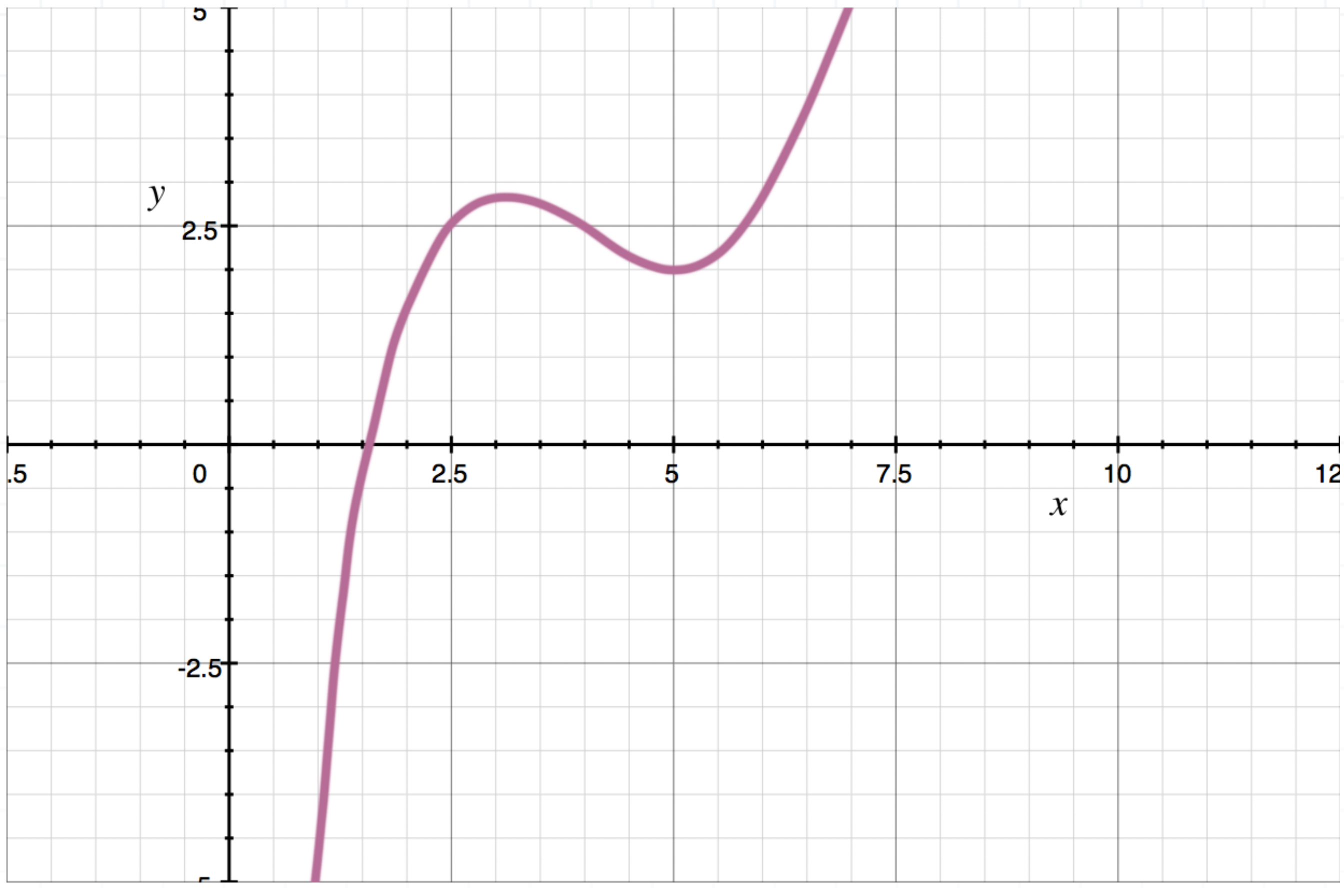
$$\lim_{x\to 0^-} f'(x) = -\infty$$

$$\lim_{x\to 0^+} f'(x) = \infty$$

This means the graph of $f(x)$ has an asymptote on the y-axis. Putting these facts together, this is a possible graph of $f(x)$:



■ 6. Sketch a possible graph of $g'(x)$ and $g''(x)$ given the graph below of $g(x)$.



Solution:

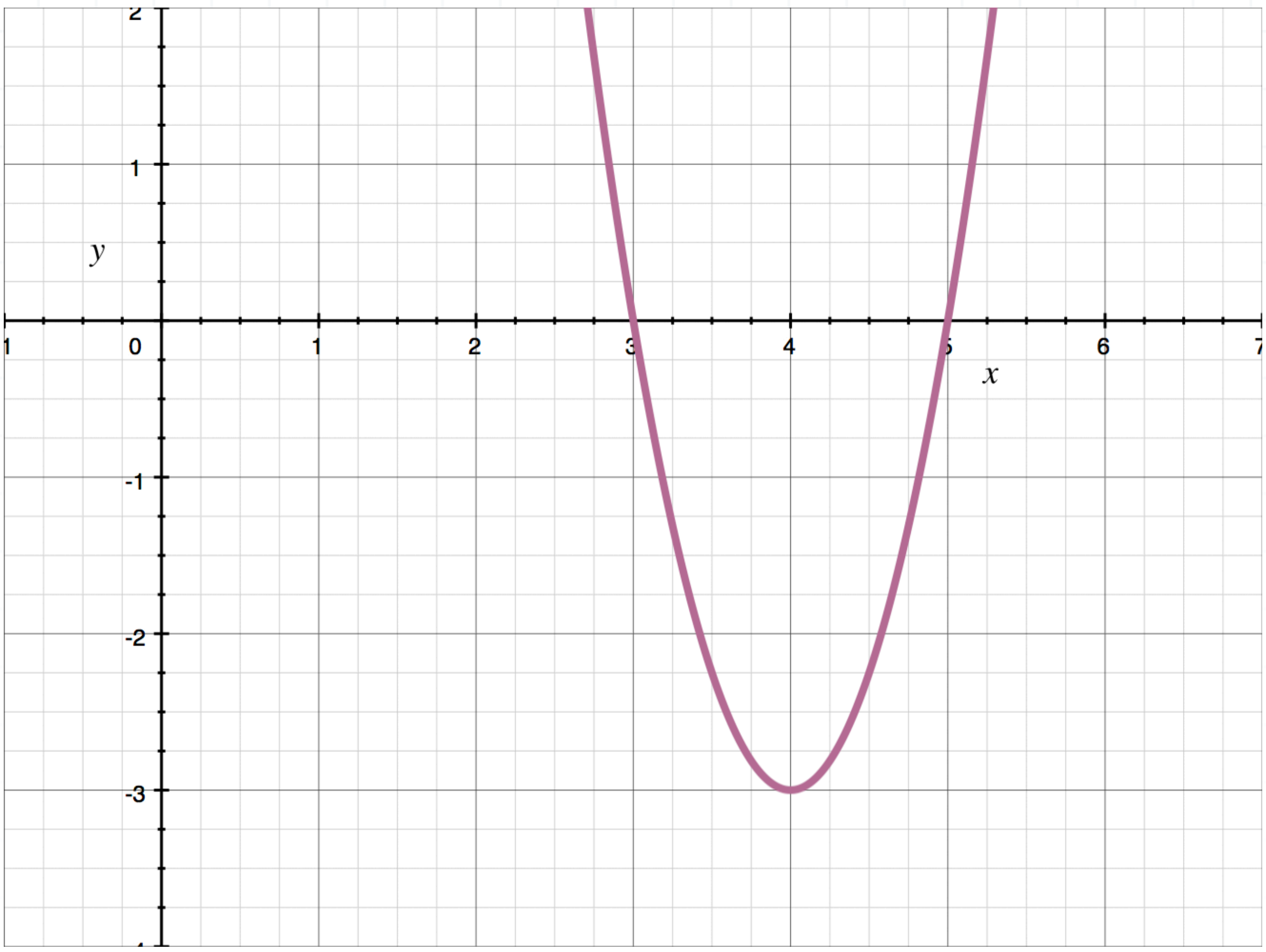
The graph of $g(x)$ has a positive slope on the intervals $(-\infty,3)$ and $(5,\infty)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is above the x-axis on these intervals.

The graph of $g(x)$ has a negative slope on the interval the $(3,5)$. Since $g'(x)$ is the derivative of $g(x)$, the graph of $g'(x)$ is below the x-axis on this interval.

The graph of $g(x)$ has a maximum value at $x=3$ and its slope is 0, so the graph of $g'(x)$ passes through the x-axis and changes sign from positive to negative at $x=3$.

The graph of $g(x)$ has a minimum value at $x=5$, and its slope changes from negative to positive at that point. This means that the graph of $g'(x)$ passes through the x-axis at $x=5$, and changes from negative to positive.

It appears that the graph of $g(x)$ has an inflection point at $x = 4$, so the graph of $g'(x)$ has extrema at $x = 4$. Putting these facts together, this is a possible graph of $g'(x)$:

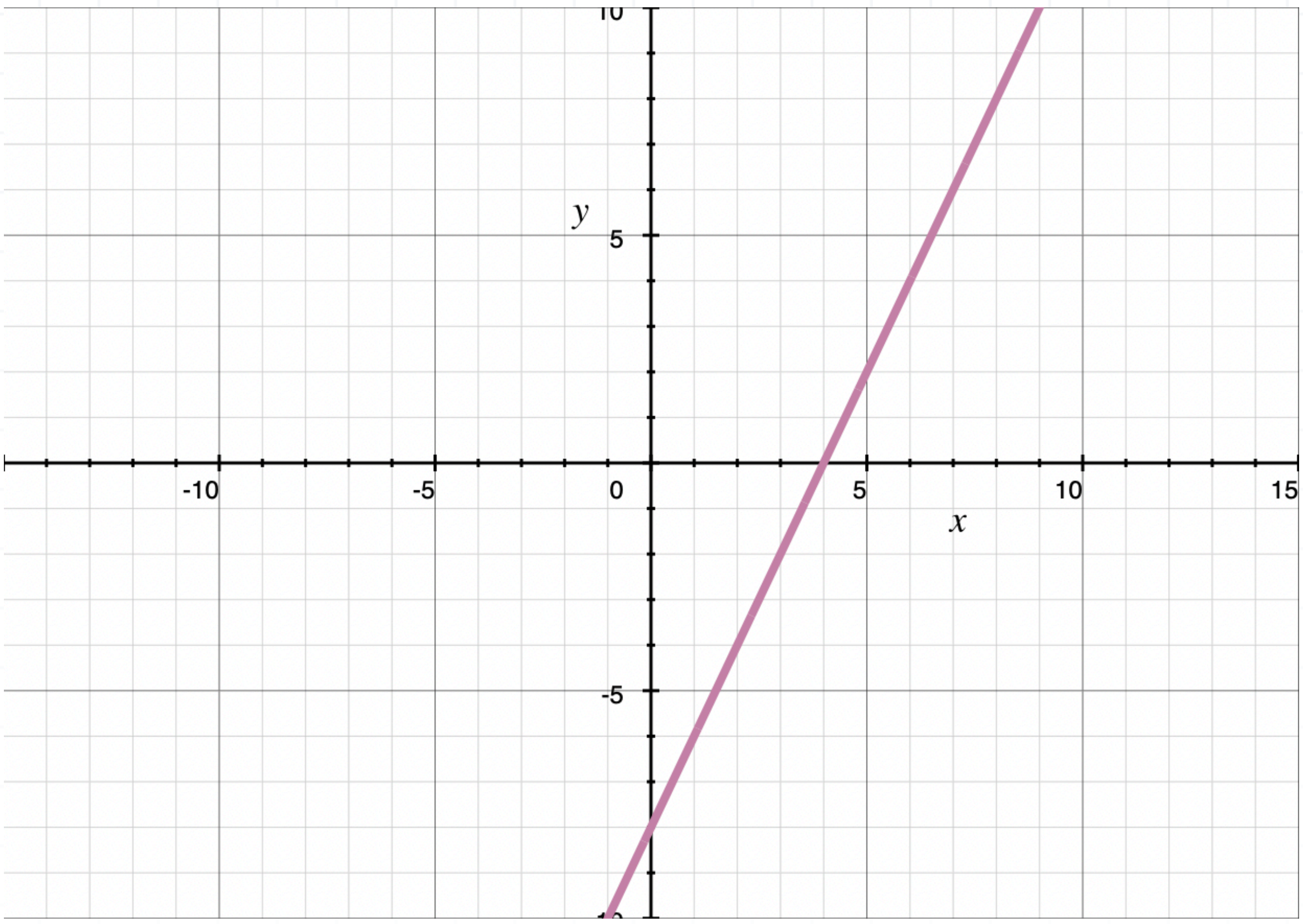


The graph of $g(x)$ has an inflection point at $x = 4$, so the graph of $g''(x)$ has an x-intercept at $x = 4$.

The graph of $g(x)$ is concave down on the interval $(-\infty,4)$, so the graph of $g''(x)$ is above the x-axis on this interval. The graph of $g(x)$ is concave up on the interval $(4,\infty)$, so the graph of $g''(x)$ is below the x-axis on this interval.

The graph of $g'(x)$ is concave up on the interval $(-\infty, \infty)$, so the graph of $g''(x)$ is increasing on this interval.

Putting these facts together, this is a possible graph of $g''(x)$:



y
5
-5
-10
-5
0
5
10
15
x

LINEAR APPROXIMATION

1. Find the linear approximation of $f(x) = x^3 - 4x^2 + 2x - 3$ at $x = 3$ and use it to approximate $f(3.02)$.

Solution:

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 3$, so the linear approximation is $L(x) = f(3) + f'(3)(x - 3)$. Find the pieces that we need for the formula.

$$f(3) = 3^3 - 4(3)^2 + 2(3) - 3 = -6$$

$$f'(x) = 3x^2 - 8x + 2$$

$$f'(3) = 3(3)^2 - 8(3) + 2 = 5$$

Plugging these pieces into the linear approximation gives

$$L(x) = -6 + 5(x - 3)$$

$$L(x) = -6 + 5x - 15$$

$$L(x) = 5x - 21$$

Use this equation to approximate $f(3.02)$.

$$f(3.02) = 5(3.02) - 21 = -5.9$$

2. Find the linear approximation of $g(x)=\sqrt{8x-15}$ at $x=8$ and use it to approximate $f(8.05)$.

Solution:

The linear approximation formula at $x=a$ is $L(x)=g(a)+g'(a)(x-a)$. In this problem, $a=8$, so the linear approximation is $L(x)=g(8)+g'(8)(x-8)$. Find the pieces that we need for the formula.

$$g(8)=\sqrt{8(8)-15}=\sqrt{49}=7$$

$$g'(x)=\frac{8}{2\sqrt{8x-15}}=\frac{4}{\sqrt{8x-15}}$$

$$g'(8)=\frac{4}{\sqrt{8(8)-15}}=\frac{4}{\sqrt{49}}=\frac{4}{7}$$

Plugging these pieces into the linear approximation gives

$$L(x)=7+\frac{4}{7}(x-8)$$

$$L(x)=7+\frac{4}{7}x-\frac{32}{7}$$

$$L(x)=\frac{4}{7}x+\frac{17}{7}$$

Use this equation to approximate $g(8.05)$.

$$g(8.05)=\frac{4}{7}(8.05)+\frac{17}{7}=\frac{246}{35}\approx 7.029$$

■ 3. Find the linear approximation of $h(x) = 2e^{x-4} + 6$ at $x = 5$ and use it to approximate $h(5.1)$.

Solution:

The linear approximation formula at $x = a$ is $L(x) = h(a) + h'(a)(x - a)$. In this problem, $a = 5$, so the linear approximation is $L(x) = h(5) + h'(5)(x - 5)$. Find the pieces that we need for the formula.

$$h(5) = 2e^{5-4} + 6 = 2e + 6$$

$$h'(x) = 2e^{x-4}$$

$$h'(5) = 2e^{5-4} = 2e$$

Plugging these pieces into the linear approximation gives

$$L(x) = 2e + 6 + 2e(x - 5)$$

$$L(x) = 2e + 6 + 2ex - 10e$$

$$L(x) = 2ex - 8e + 6$$

Use this equation to approximate $h(5.1)$.

$$g(5.1) = 2e(5.1) - 8e + 6 = 10.2e - 8e + 6 = 2.2e + 6 \approx 11.98$$

■ 4. Find the linear approximation of $f(x) = \ln(2x - 7)$ at $x = 4$ and use it to approximate $f(3.8)$.

Solution:

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 4$, so the linear approximation is $L(x) = f(4) + f'(4)(x - 4)$. Find the pieces that we need for the formula.

$$f(4) = \ln(2(4) - 7) = \ln(8 - 7) = 0$$

$$f'(x) = \frac{2}{2x - 7}$$

$$f'(4) = \frac{2}{2(4) - 7} = \frac{2}{1} = 2$$

Plugging these pieces into the linear approximation gives

$$L(x) = 0 + 2(x - 4)$$

$$L(x) = 2x - 8$$

Use this equation to approximate $f(3.8)$.

$$f(3.8) = 2(3.8) - 8 = -0.4$$

■ 5. Use linear approximation to estimate $f(3.1)$.

$$f(x) = \sin(3x)$$

Solution:

The first thing we want to realize is that finding $f(3.1)$ gets pretty messy. If we substitute $x = 3.1$ into the function, we get $\sin 9.3$. That's not necessarily an easy value to find. However, $\sin 9.3$ is pretty close to $\sin 9.42 = \sin(3\pi)$, which is a very easy value to find.

Therefore, instead of trying to find $f(3.1)$, let's use a linear approximation equation and $a = \pi$ to get an approximation for $f(3.1)$.

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = \pi$, so the linear approximation is $L(x) = f(\pi) + f'(\pi)(x - \pi)$. Find the pieces that we need for the formula.

$$f(\pi) = \sin(3\pi) = 0$$

$$f'(x) = 3\cos(3x)$$

$$f'(\pi) = 3\cos(3\pi) = -3$$

Plugging these pieces into the linear approximation gives

$$L(x) = 0 - 3(x - \pi)$$

$$L(x) = -3x + 3\pi$$

Use this equation to approximate $f(3.1)$.

$$f(3.1) = -3(3.1) + 3\pi = 0.125$$

6. Use linear approximation to estimate $f(6.1)$.

$$f(x) = e^{\cos x}$$

Solution:

The first thing we want to realize is that finding $f(6.1)$ gets pretty messy. If we substitute $x = 6.1$ into the function, we get $e^{\cos 6.1}$. That's not necessarily an easy value to find. However, $e^{\cos 6.1}$ is pretty close to $e^{\cos 6.28} = e^{\cos(2\pi)}$, which is a very easy value to find.

Therefore, instead of trying to find $f(6.1)$, let's use a linear approximation equation and $a = 2\pi$ to get an approximation for $f(6.1)$.

The linear approximation formula at $x = a$ is $L(x) = f(a) + f'(a)(x - a)$. In this problem, $a = 2\pi$, so the linear approximation is $L(x) = f(2\pi) + f'(2\pi)(x - 2\pi)$. Find the pieces that we need for the formula.

$$f(2\pi) = e^{\cos 2\pi} = e^1 = e$$

$$f'(x) = -\sin x e^{\cos x}$$

$$f'(2\pi) = -\sin(2\pi)e^{\cos(2\pi)} = -0 \cdot e = 0$$

Plugging these pieces into the linear approximation gives

$$L(x) = e + 0(x - 2\pi)$$

$$L(x) = e$$

Use this equation to approximate $f(6.1)$.

$$f(6.1) = e = 2.72$$

ESTIMATING A ROOT

1. Use linear approximation to estimate $\sqrt[5]{34}$.

Solution:

Let $f(x)=\sqrt[5]{x}$ and $a=32$. The linear approximation would be given by

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x)=f(32)+f'(32)(x-32)$$

Find the pieces needed for the formula.

$$f(32)=\sqrt[5]{32}=2$$

$$f(x)=\sqrt[5]{x}=x^{\frac{1}{5}}$$

$$f'(x)=\frac{1}{5}(x)^{-\frac{4}{5}}=\frac{1}{5\cdot\sqrt[5]{x^4}}$$

$$f'(32)=\frac{1}{5\cdot\sqrt[5]{32^4}}=\frac{1}{80}$$

Then the linear approximation is

$$L(x)=2+\frac{1}{80}(x-32)$$

$$L(x) = \frac{1}{80}x + \frac{8}{5}$$

Use this approximation to estimate $\sqrt[5]{34}$.

$$f(34) = \frac{1}{80}(34) + \frac{8}{5} = \frac{81}{40}$$

■ 2. Use linear approximation to estimate $\sqrt[8]{260}$.

Solution:

Let $f(x) = \sqrt[8]{x}$ and $a = 256$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(256) + f'(256)(x - 256)$$

Find the pieces needed for the formula.

$$f(256) = \sqrt[8]{256} = 2$$

$$f(x) = \sqrt[8]{x} = x^{\frac{1}{8}}$$

$$f'(x) = \frac{1}{8}(x)^{-\frac{7}{8}} = \frac{1}{8\sqrt[8]{x^7}}$$

$$f'(256) = \frac{1}{8\sqrt[8]{256^7}} = \frac{1}{1{,}024}$$

Then the linear approximation is

$$L(x) = 2 + \frac{1}{1{,}024}(x - 256)$$

$$L(x) = \frac{1}{1{,}024}x + \frac{7}{4}$$

Use this approximation to estimate $\sqrt[8]{260}$.

$$f(260) = \frac{1}{1{,}024}(260) + \frac{7}{4} = \frac{513}{256} \approx 2.0039$$

■ 3. Use linear approximation to estimate $\sqrt[4]{85}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 81$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(81) + f'(81)(x - 81)$$

Find the pieces needed for the formula.

$$f(81) = \sqrt[4]{81} = 3$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(81) = \frac{1}{4\sqrt[4]{81^3}} = \frac{1}{108}$$

Then the linear approximation is

$$L(x) = 3 + \frac{1}{108}(x - 81)$$

$$L(x) = \frac{1}{108}x + \frac{9}{4}$$

Use this approximation to estimate $\sqrt[4]{85}$.

$$f(85) = \frac{1}{108}(85) + \frac{9}{4} = \frac{82}{27} \approx 3.037$$

■ 4. Use linear approximation to estimate $\sqrt[4]{615}$.

Solution:

Let $f(x) = \sqrt[4]{x}$ and $a = 625$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(625) + f'(625)(x - 625)$$

Find the pieces needed for the formula.

$$f(625) = \sqrt[4]{625} = 5$$

$$f(x) = \sqrt[4]{x} = x^{\frac{1}{4}}$$

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4\sqrt[4]{x^3}}$$

$$f'(625) = \frac{1}{4\sqrt[4]{625^3}} = \frac{1}{500}$$

Then the linear approximation is

$$L(x) = 5 + \frac{1}{500}(x - 625)$$

$$L(x) = \frac{1}{500}x + \frac{15}{4}$$

Use this approximation to estimate $\sqrt[4]{615}$.

$$f(615) = \frac{1}{500}(615) + \frac{15}{4} = \frac{249}{50} \approx 4.98$$

■ 5. Use linear approximation to estimate $\sqrt{95}$.

Solution:

Let $f(x) = \sqrt{x}$ and $a = 100$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(100) + f'(100)(x - 100)$$

Find the pieces needed for the formula.

$$f(100) = \sqrt{100} = 10$$

$$f(x) = \sqrt{x} = x^{\frac{1}{2}}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$$

Then the linear approximation is

$$L(x) = 10 + \frac{1}{20}(x - 100)$$

$$L(x) = \frac{1}{20}x + 5$$

Use this approximation to estimate $\sqrt{95}$.

$$f(95) = \frac{1}{20}(95) + 5 = \frac{39}{4} \approx 9.75$$

■ 6. Use linear approximation to estimate $\sqrt[3]{700}$.

Solution:

Let $f(x) = \sqrt[3]{x}$ and $a = 729$. The linear approximation would be given by

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = f(729) + f'(729)(x - 729)$$

Find the pieces needed for the formula.

$$f(729) = \sqrt[3]{729} = 9.$$

$$f(x) = \sqrt[3]{x} = x^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$$

$$f'(729) = \frac{1}{3\sqrt[3]{729^2}} = \frac{1}{243}$$

Then the linear approximation is

$$L(x) = 9 + \frac{1}{243}(x - 729)$$

$$L(x) = \frac{1}{243}x + 6$$

Use this approximation to estimate $\sqrt[3]{700}$.

$$f(700) = \frac{1}{243}(700) + 6 = \frac{2,158}{243} \approx 8.88$$

ABSOLUTE, RELATIVE, AND PERCENTAGE ERROR

■ 1. Use a linear approximation to estimate the value of $e^{0.002}$, then find the absolute error of the estimate.

Solution:

We need to realize here that $e^{0.002}$ is a difficult value to find. But it's very close to e^0, which we already know is 1. So instead of thinking specifically about $e^{0.002}$, let's think about e^x, and therefore use the function $f(x) = e^x$. We'll differentiate it,

$$f'(x) = e^x$$

then evaluate the derivative at $x = 0$.

$$f'(0) = e^0$$

$$f'(0) = 1$$

So the linear approximation intersects $f(x) = e^x$ at the point of tangency $(0,1)$, and has a slope of $m = 1$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + 1(x - 0)$$

$$L(x) = 1 + x$$

Then the linear approximation at $x = 0.002$ is

$$L(0.002) = 1 + 0.002$$

$$L(0.002) = 1.002$$

In comparison, the actual value of $e^{0.002}$ is

$$f(x) = e^x$$

$$f(0.002) = e^{0.002}$$

$$f(0.002) \approx 1.002002$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(0.002) = |f(0.002) - L(0.002)|$$

$$E_A(0.002) \approx |1.002002 - 1.002|$$

$$E_A(0.002) \approx |0.000002|$$

$$E_A(0.002) = 0.000002$$

■ 2. Use linear approximation to estimate $f(2.15)$, then find the relative error of the estimate.

$$f(x) = 4xe^{3x-6}$$

Solution:

Instead of trying to find $f(2.15)$, let's use a linear approximation equation and $a=2$ to get an approximation for $f(2.15)$.

$$f(x)=4xe^{3x-6}$$

$$f(2)=4(2)e^{3(2)-6}$$

$$f(2)=8$$

Differentiate the function,

$$f'(x)=4e^{3x-6}+4xe^{3x-6}(3)$$

$$f'(x)=4e^{3x-6}+12xe^{3x-6}$$

$$f'(x)=4e^{3x-6}(1+3x)$$

then evaluate the derivative at $x=2$.

$$f'(2)=4e^{3(2)-6}(1+3(2))$$

$$f'(2)=4e^{0}(7)$$

$$f'(2)=28$$

So the linear approximation intersects $f(x)=4xe^{3x-6}$ at the point of tangency $(2,8)$, and has a slope of $m=28$. Substitute these values into the linear approximation equation.

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x)=8+28(x-2)$$

$$L(x) = 8 + 28x - 56$$

$$L(x) = 28x - 48$$

Then the linear approximation at $x = 2.15$ is

$$L(2.15) = 28(2.15) - 48$$

$$L(2.15) = 60.2 - 48$$

$$L(2.15) = 12.2$$

In comparison, the actual value of $f(2.15)$ is

$$f(2.15) = 4(2.15)e^{(3(2.15)-6)}$$

$$f(2.15) \approx 13.4875$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(2.15) = |f(2.15) - L(2.15)|$$

$$E_A(2.15) \approx |13.4875 - 12.2|$$

$$E_A(2.15) \approx |1.2875|$$

$$E_A(2.15) \approx 1.2875$$

and the relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(2.15) = \frac{E_A(2.15)}{f(2.15)}$$

$$E_R(2.15) \approx \frac{1.2875}{13.4875}$$

$$E_R(2.15) \approx 0.095459$$

■ 3. Use linear approximation to estimate $f(1.2)$, then find the percentage error of the estimate.

$$f(x) = \sqrt[3]{x+1}$$

Solution:

Instead of trying to find $f(1.2)$, let's use a linear approximation equation and $a = 0$ to get an approximation for $f(1.2)$.

$$f(x) = \sqrt[3]{x+1}$$

$$f(0) = \sqrt[3]{0+1}$$

$$f(0) = 1$$

Differentiate the function,

$$f'(x) = \frac{1}{3}(x+1)^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3(x+1)^{\frac{2}{3}}}$$

then evaluate the derivative at $x = 0$.

$$f'(0) = \frac{1}{3(0+1)^{\frac{2}{3}}}$$

$$f'(0) = \frac{1}{3(1)}$$

$$f'(0) = \frac{1}{3}$$

So the linear approximation intersects $f(x) = \sqrt[3]{x+1}$ at the point of tangency $(0,1)$, and has a slope of $m = 1/3$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + \frac{1}{3}(x - 0)$$

$$L(x) = 1 + \frac{1}{3}x$$

Then the linear approximation at $x = 1.2$ is

$$L(1.2) = 1 + \frac{1}{3}(1.2)$$

$$L(1.2) = 1 + 0.4$$

$$L(1.2) = 1.4$$

In comparison, the actual value of $f(1.2)$ is

$$f(x) = \sqrt[3]{x+1}$$

$$f(1.2) = \sqrt[3]{1.2+1}$$

$$f(1.2) \approx 1.30059$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(1.2) = |f(1.2) - L(1.2)|$$

$$E_A(1.2) \approx |1.30059 - 1.4|$$

$$E_A(1.2) \approx |-0.09941|$$

$$E_A(1.2) \approx 0.09941$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(1.2) = \frac{E_A(1.2)}{f(1.2)}$$

$$E_R(1.2) \approx \frac{0.09941}{1.30059}$$

$$E_R(1.2) \approx 0.076435$$

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(1.2) = 100\% \cdot E_R(1.2)$$

$$E_P(1.2) \approx 100\% \cdot 0.076435$$

$$E_P(1.2) \approx 7.6435\,\%$$

■ 4. Use a linear approximation to estimate the value of $\sqrt[3]{30}$, then find the relative error of the estimate.

Solution:

We need to realize here that $\sqrt[3]{30}$ is a difficult value to find. But it's very close to $\sqrt[3]{27}$, which we already know is 3. So instead of thinking specifically about $\sqrt[3]{30}$, let's think about $\sqrt[3]{x}$, and therefore use the function $f(x) = \sqrt[3]{x}$.

Differentiate the function,

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

$$f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

then evaluate the derivative at $x = 27$.

$$f'(27) = \frac{1}{3\sqrt[3]{27^2}}$$

$$f'(27) = \frac{1}{3(9)}$$

$$f'(27) = \frac{1}{27}$$

So the linear approximation intersects $f(x) = \sqrt[3]{x}$ at the point of tangency $(27,3)$, and has a slope of $m = 1/27$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$L(x) = 3 + \frac{1}{27}x - 1$$

$$L(x) = \frac{1}{27}x + 2$$

Then the linear approximation at $x = 30$ is

$$L(30) = \frac{1}{27}(30) + 2$$

$$L(30) = \frac{30}{27} + 2$$

$$L(30) = \frac{30}{27} + \frac{54}{27}$$

$$L(30) = \frac{84}{27}$$

$$L(30) \approx 3.111111$$

In comparison, the actual value of $\sqrt[3]{30}$ is

$$f(x) = \sqrt[3]{x}$$

$$f(30) = \sqrt[3]{30}$$

$$f(30) \approx 3.107233$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(30) = |f(30) - L(30)|$$

$$E_A(30) \approx |3.107233 - 3.111111|$$

$$E_A(30) \approx |-0.003878|$$

$$E_A(30) \approx 0.003878$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(30) = \frac{E_A(30)}{f(30)}$$

$$E_R(30) \approx \frac{0.003878}{3.107233}$$

$$E_R(30) \approx 0.001248$$

■ 5. Find the absolute, relative, and percentage error of the approximation 2.7 to the value of e.

Solution:

The absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(a) = |e - 2.7|$$

$$E_A(a) = |2.718282 - 2.7|$$

$$E_A(a) = |0.018282|$$

$$E_A(a) = 0.018282$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(a) \approx \frac{0.018282}{2.718282}$$

$$E_R(a) \approx 0.0067256$$

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(a) \approx 100\% \cdot 0.0067256$$

$$E_P(a) \approx 0.67256\,\%$$

■ 6. Use a linear approximation to estimate the value of $\sin(93^\circ)$, then find the absolute error of the estimate.

Solution:

We need to realize here that $\sin(93^\circ)$ is a difficult value to find. But it's very close to $\sin(90^\circ)$, which we already know is 1. So instead of thinking specifically about $\sin(93^\circ)$, let's think about $\sin x$, and therefore use the function $f(x) = \sin x$.

$$f(x) = \sin x$$

$$f(90^\circ) = \sin 90^\circ$$

$$f(90^\circ) = 1$$

Differentiate the function,

$$f'(x) = \cos x$$

then evaluate the derivative at $x = 90^\circ$.

$$f'(90^\circ) = \cos 90^\circ$$

$$f'(90^\circ) = 0$$

So the linear approximation intersects $f(x) = \sin x$ at the point of tangency $(90^\circ, 1)$, and has a slope of $m = 0$. Substitute these values into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + 0(x - 90^\circ)$$

$$L(x) = 1$$

Then the linear approximation at $x = 93^\circ$ is

$$L(93^\circ) = 1$$

In comparison, the actual value of $\sin(93^\circ)$ is

$$f(x) = \sin x$$

$$f(93^\circ) = \sin 93^\circ$$

$$f(93^\circ) \approx 0.998629$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(93^\circ) = |f(93^\circ) - L(93^\circ)|$$

$$E_A(93^\circ) \approx |0.998629 - 1|$$

$$E_A(93^\circ) \approx |-0.001371|$$

$$E_A(93^\circ) \approx 0.001371$$

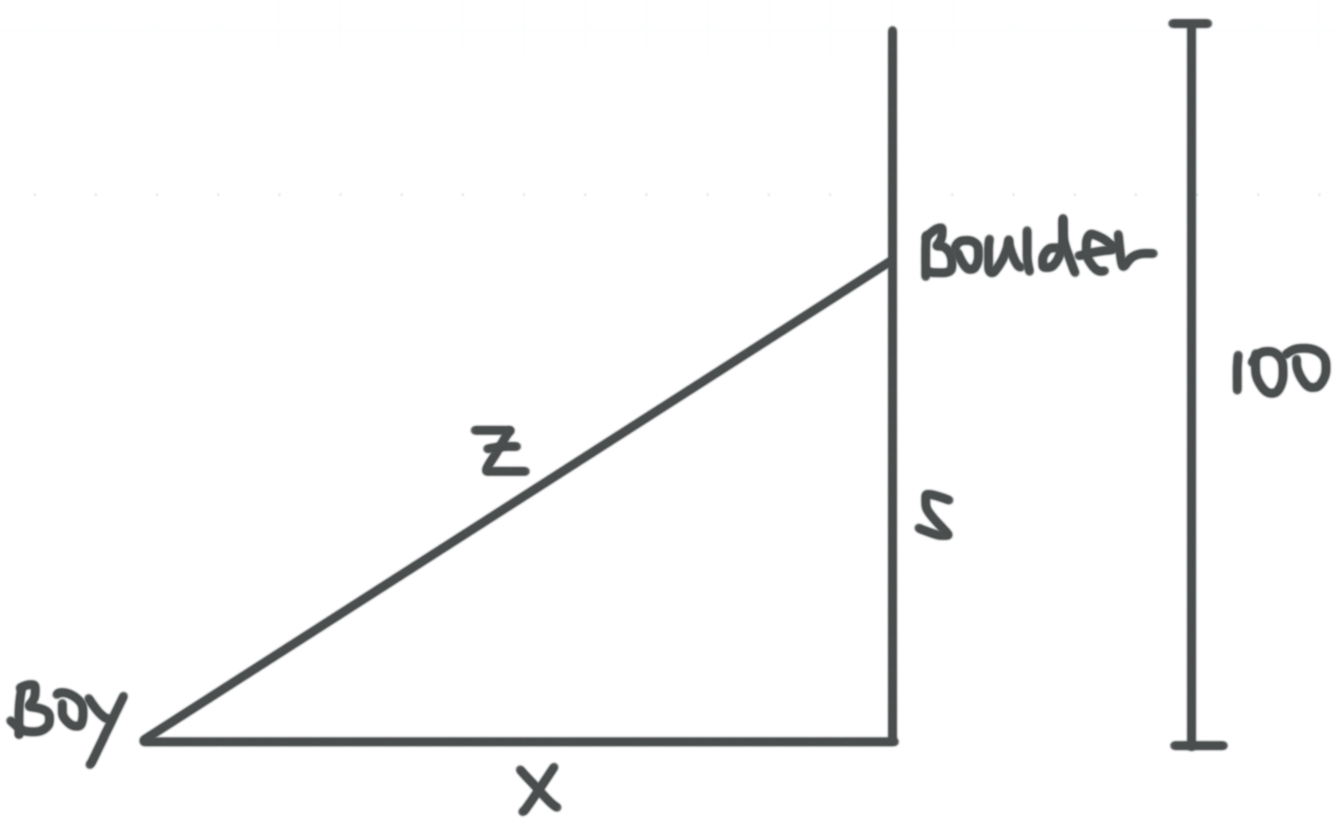
RELATED RATES

■ 1. A boy is standing 15 feet from the base of a 100 feet cliff. As a boulder rolls off the cliff, the boy begins running away at 8 ft/s. At what rate is the distance between the boy and the boulder changing after 2 seconds?

The height of the falling boulder is modeled by the position function $s = -16t^2 + v_0t + s_0$, where s_0 is the initial height and v_0 is the initial velocity of the boulder.

Solution:

Draw a diagram.



In this case, $s_0 = 100$ and $v_0 = 0$, so we'll plug these into the position function.

$$s = -16t^2 + v_0t + s_0$$

$$s = -16t^2 + 100$$

Applying the Pythagorean Theorem to the diagram we drew, we get

$$x^2 + s^2 = z^2$$

$$x^2 + (-16t^2 + 100)^2 = z^2$$

Differentiate with respect to time.

$$2x\frac{dx}{dt} + 2(-16t^2 + 100)(-32t) = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + (-16t^2 + 100)(-32t) = z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + 512t^3 + -3200t = z\frac{dz}{dt}$$

Since the boy initially starts at $x = 15$, after 2 seconds the value of x is

$$x = 15 + 8t$$

$$x = 15 + 8(2)$$

$$x = 31$$

And the height of the boulder after 2 seconds is

$$s(2) = -16(2)^2 + 100$$

$$s(2) = 36$$

We can use the Pythagorean Theorem to find the distance between the boy and the boulder, z, at the time when $x = 31$ and $s = 36$.

$$x^2 + s^2 = z^2$$

$$31^2 + 36^2 = z^2$$

$$961 + 1{,}296 = z^2$$

$$2{,}257 = z^2$$

$$z \approx 47.5$$

Substitute what we know into the derivative, then solve for dz/dt.

$$31(8) + 512(2)^3 + -3{,}200(2) = 47.5\frac{dz}{dt}$$

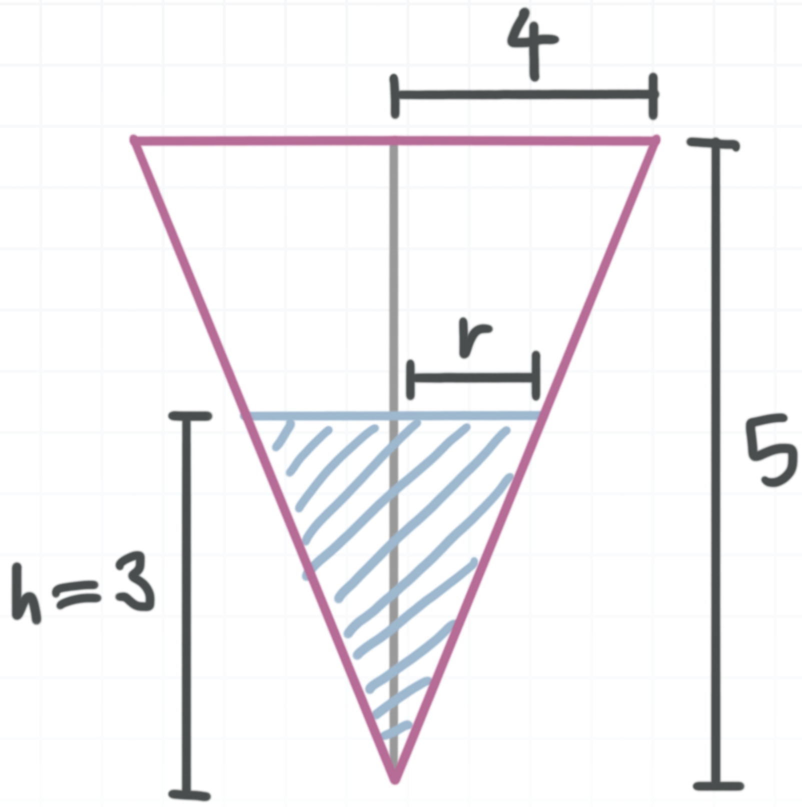
$$-2{,}056 = 47.5\frac{dz}{dt}$$

$$\frac{dz}{dt} = -\frac{2{,}056}{47.5} \approx -43.3$$

■ 2. Water is flowing out of a cone-shaped tank at a rate of 6 cubic inches per second. If the cone has a height of 5 inches and a base radius of 4 inches, how fast is the water level falling when the water is 3 inches deep?

Solution:

Draw a diagram.



The volume of a cone is $V = (1/3)\pi r^2 h$. We want to express volume as a function of h only.

$$\frac{r}{h} = \frac{4}{5}$$

$$r = \frac{4}{5}h$$

Then the volume of the cone of water is

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{4}{5}h\right)^2 h$$

$$V = \frac{1}{3}\pi \left(\frac{16}{25}h^2\right) h$$

$$V = \frac{16}{75}\pi h^3$$

Differentiate the volume equation with respect to t.

$$\frac{dV}{dt} = \frac{16}{75}\pi(3h^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{16}{25}\pi h^2\frac{dh}{dt}$$

The problem states that $dV/dt = -6$. Substitute this and $h = 3$ into the derivative equation and solve for dh/dt.

$$-6 = \frac{16}{25}\pi(3)^2\frac{dh}{dt}$$

$$-6 = \frac{144}{25}\pi\frac{dh}{dt}$$

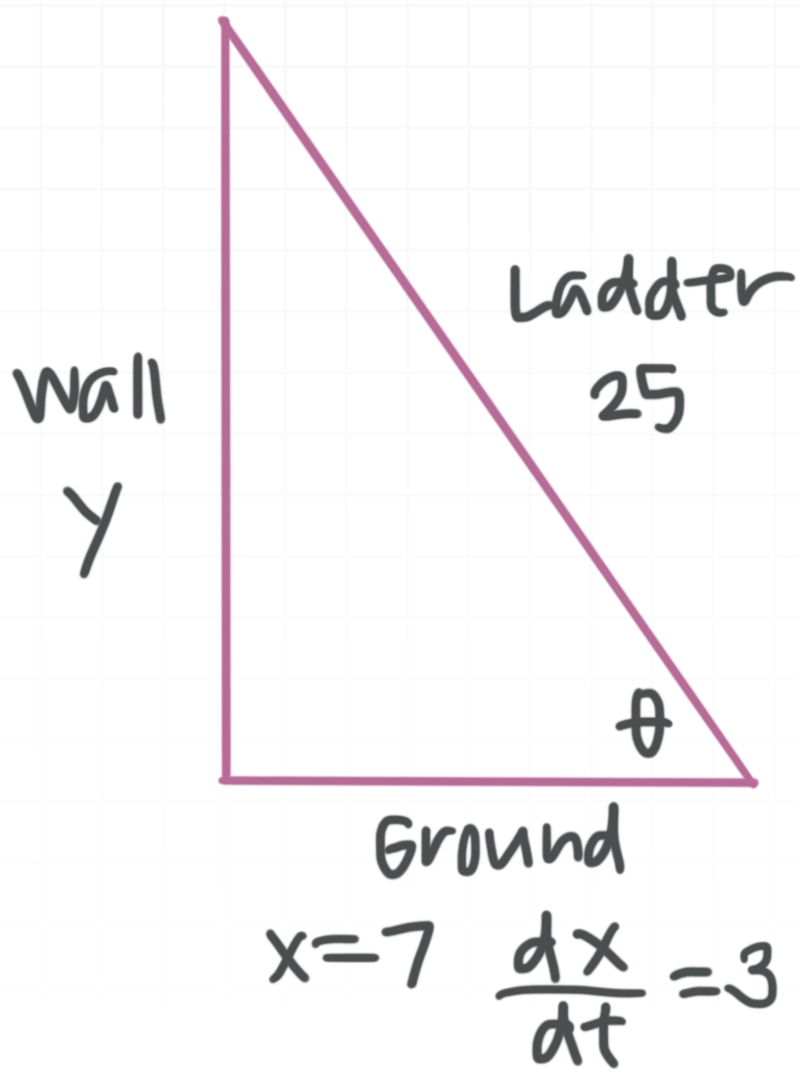
$$\frac{dh}{dt} = -6\frac{25}{144\pi}$$

$$\frac{dh}{dt} = -\frac{25}{24\pi} \approx -0.33 \text{ in/s}$$

■ 3. A ladder 25 feet long leans against a vertical wall of a building. If the bottom of the ladder is pulled away horizontally from the building at 3 feet per second, how fast is the angle formed by the ladder and the horizontal ground decreasing when the bottom of the ladder is 7 feet from the base of the wall?

Solution:

Draw a diagram.



Find y using the Pythagorean theorem.

$$y^2 + 7^2 = 25^2$$

$$y = 24$$

Use the cosine function, which gives the equation

$$\cos\theta = \frac{x}{25}$$

Differentiate with respect to t.

$$-\sin\theta\frac{d\theta}{dt}=\frac{1}{25}\frac{dx}{dt}$$

Use the sine function, which gives the equation

$$\sin\theta=\frac{y}{25}=\frac{24}{25}$$

Substitute what we know.

$$-\frac{24}{25}\cdot\frac{d\theta}{dt}=\frac{1}{25}\cdot 3$$

$$\frac{d\theta}{dt}=\frac{3}{25}\cdot\left(-\frac{25}{24}\right)=-\frac{3}{24}=-\frac{1}{8}\text{ ft/s}$$

■ 4. The radius of a spherical balloon is increasing at a rate of 4.5 ft/hr. At what rate are the sphere's surface area and volume increasing when the surface area is 36π ft^2?

Solution:

The formula for the surface area of a sphere is

$$S=4\pi r^2$$

First we need to find the radius of the ballon when the surface area is 36π ft^2.

$$36\pi=4\pi r^2$$

$$r^2 = 9$$

$$r = 3$$

Use implicit differentiation to take the derivative of both sides of the standard surface area formula.

$$(1)\frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

From the question, we know that $dr/dt = 4.5$ and that $r = 3$, so we'll plug those in.

$$\frac{dS}{dt} = 8\pi(3)(4.5)$$

$$\frac{dS}{dt} = 108\pi$$

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dr/dt = 4.5$ and that $r = 3$, so we'll plug those in.

$$\frac{dV}{dt} = 4\pi(3^2)(4.5)$$

$$\frac{dV}{dt} = 162\pi$$

Therefore, at the moment when the surface area is 36π ft^2, the surface area is increasing at 108π ft/hr, while the volume is increasing at 162π ft/hr.

■ 5. A price p and demand q for a product are related by $q^2 - 2qp + 30p^2 = 10{,}125$. If the price is increasing at a rate of 2.5 dollars per month when the price is 15 dollars, find the rate of change of the demand.

Solution:

First we need to find the value of demand. Substitute $p = 15$ into the equation, then solve for q.

$$q^2 - 2qp + 30p^2 = 10125$$

$$q^2 - 2q(15) + 30(15)^2 = 10125$$

$$q^2 - 30q + 6750 = 10125$$

$q = -45$ and $q = 75$

The value we select has to be greater than 0, so $q = 75$.

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q^2 - 2qp + 30p^2 = 10125$$

$$2q\frac{dq}{dt} - 2p\frac{dq}{dt} - 2q\frac{dp}{dt} + 60p\frac{dp}{dt} = 0$$

$$(2q - 2p)\frac{dq}{dt} + (60p - 2q)\frac{dp}{dt} = 0$$

From the question, we know that $p = 15$, $q = 75$ and $dp/dt = 2.5$, so we'll plug those in.

$$(2(75) - 2(15))\frac{dq}{dt} + (60(15) - 2(75))(2.5) = 0$$

$$(150 - 30)\frac{dq}{dt} + (900 - 150)(2.5) = 0$$

$$120\frac{dq}{dt} + 1875 = 0$$

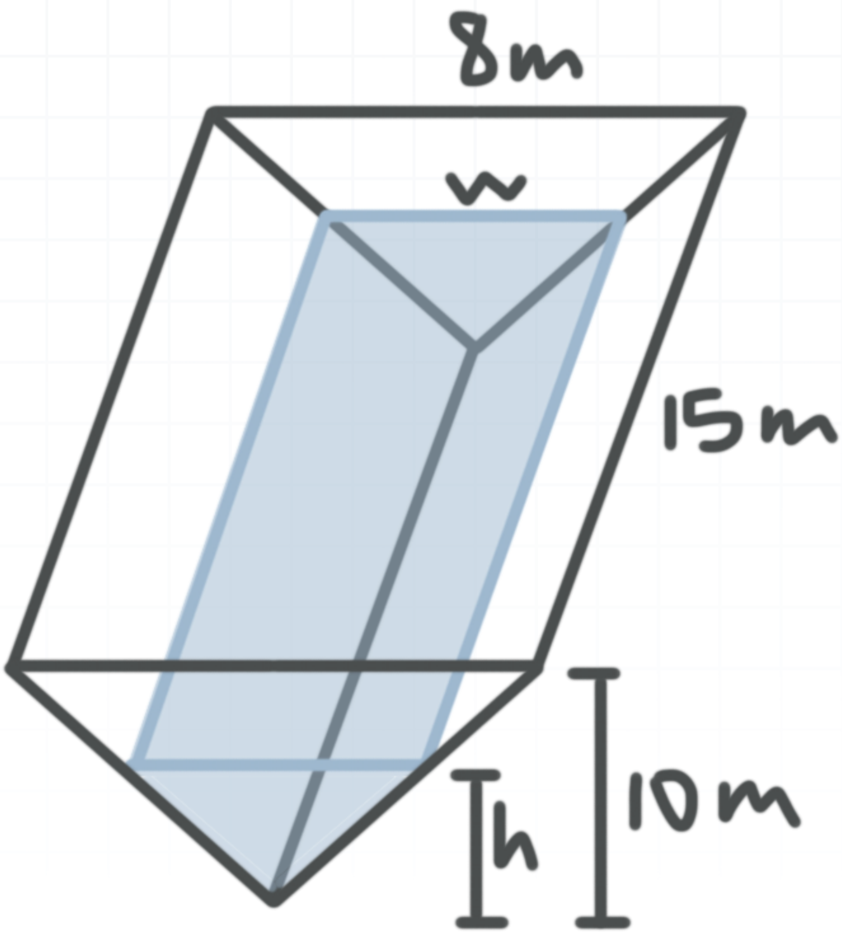
$$\frac{dq}{dt} = -\frac{1875}{120}$$

$$\frac{dq}{dt} \approx -15.6$$

■ 6. A trough of water 15 meters long, 8 meters wide, and 10 meters high has ends shaped like isosceles triangles. If water is being pumped in at a constant rate of 6 m^3/s, how fast are the height and width of the water changing when the water has a height of 250 cm?

Solution:

Sketch a diagram.



The volume of a triangular prism is $V=(1/2)whl$. To find the width of the water, we'll use similar triangles.

$$\frac{w}{h}=\frac{8}{10}$$

$$w=\frac{4}{5}h$$

The volume of the triangular prism of water is

$$V=\frac{1}{2}whl$$

We know that $l = 15$, so

$$V = \frac{1}{2}wh(15)$$

$$V = \frac{15}{2}wh$$

$$V = \frac{15}{2}\left(\frac{4}{5}h\right)h$$

$$V = 6h^2$$

Differentiate the volume equation with respect to t.

$$\frac{dV}{dt} = 6(2h)\frac{dh}{dt}$$

$$\frac{dV}{dt} = 12h\frac{dh}{dt}$$

The problem states that $dV/dt = 6$ and $h = 2.5$, so we'll substitute these into the derivative equation and solve for dh/dt, the rate at which the height of the water is changing when the water has a height of 250 cm.

$$6 = 12(2.5)\frac{dh}{dt}$$

$$6 = 30\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{6}{30}$$

$$\frac{dh}{dt} = \frac{1}{5} \text{ m/s}$$

We can find the relationship between w and h,

$$w = \frac{4}{5}h$$

and then differentiate with respect to t.

$$\frac{dw}{dt} = \frac{4}{5}\frac{dh}{dt}$$

Substitute $dh/dt = 1/5$ into the derivative equation and solve for dw/dt, the rate at which the width of the water is changing when the water has a height of 250 cm.

$$\frac{dw}{dt} = \frac{4}{5} \cdot \frac{1}{5}$$

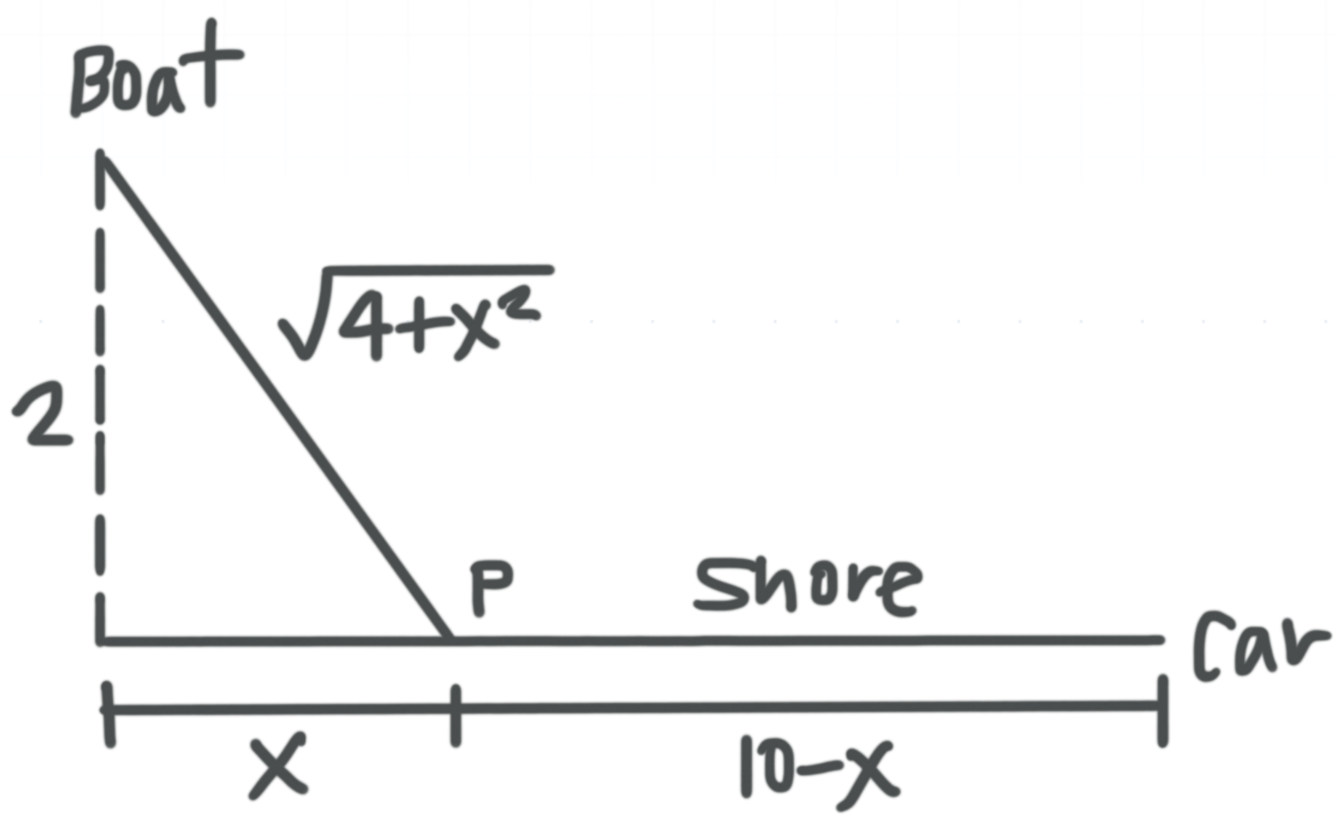
$$\frac{dw}{dt} = \frac{4}{25} \text{ m/s}$$

APPLIED OPTIMIZATION

1. A boater finds herself 2 miles from the nearest point to a straight shoreline, which is 10 miles down the shore from where she parked her car. She plans to row to shore and then walk to her car. If she can walk 4 miles per hour but only row 3 miles per hour, toward what point on the shore should she row in order to reach her car in in the least amount of time?

Solution:

Draw a diagram.



From the diagram, the distance to point P is $\sqrt{4+x^2}$ and the distance from point P to the car is $10-x$. So the total time to reach the car is

$$T=\frac{\sqrt{4+x^2}}{3}+\frac{10-x}{4} \text{ with } 0\le x\le 10$$

We set $0\le x\le 10$ because x will be 0 if she rows directly to the shore, and x will be 10 if she rows directly to the car. Find the derivative of T.

$$dT=\frac{x}{3\sqrt{4+x^2}}-\frac{1}{4}$$

Set $dT=0$ and solve for x.

$$\frac{x}{3\sqrt{4+x^2}}=\frac{1}{4}$$

$$\frac{4x}{3}=\sqrt{4+x^2}$$

$$\frac{16x^2}{9}=4+x^2$$

$$\frac{7}{9}x^2=4$$

$$x^2=\frac{36}{7}$$

$$x=\frac{6}{\sqrt{7}}\approx 2.2678$$

If she rows to point P, where $x=2.2678$ miles down the shoreline, it will take her

$$T = \frac{\sqrt{4 + (2.2678)^2}}{3} + \frac{10 - (2.2678)}{4} \approx 2.9409 \text{ hours}$$

If she rows directly to the shore, where $x = 0$, it will take her

$$T = \frac{2}{3} + \frac{10}{4} \approx 3.167 \text{ hours}$$

Find the distance directly to her car using the Pythagorean Theorem.

$$d^2 = 2^2 + 10^2 = 104$$

$$d = 2\sqrt{26} \text{ miles}$$

If she rows directly to her car, where $x = 10$,

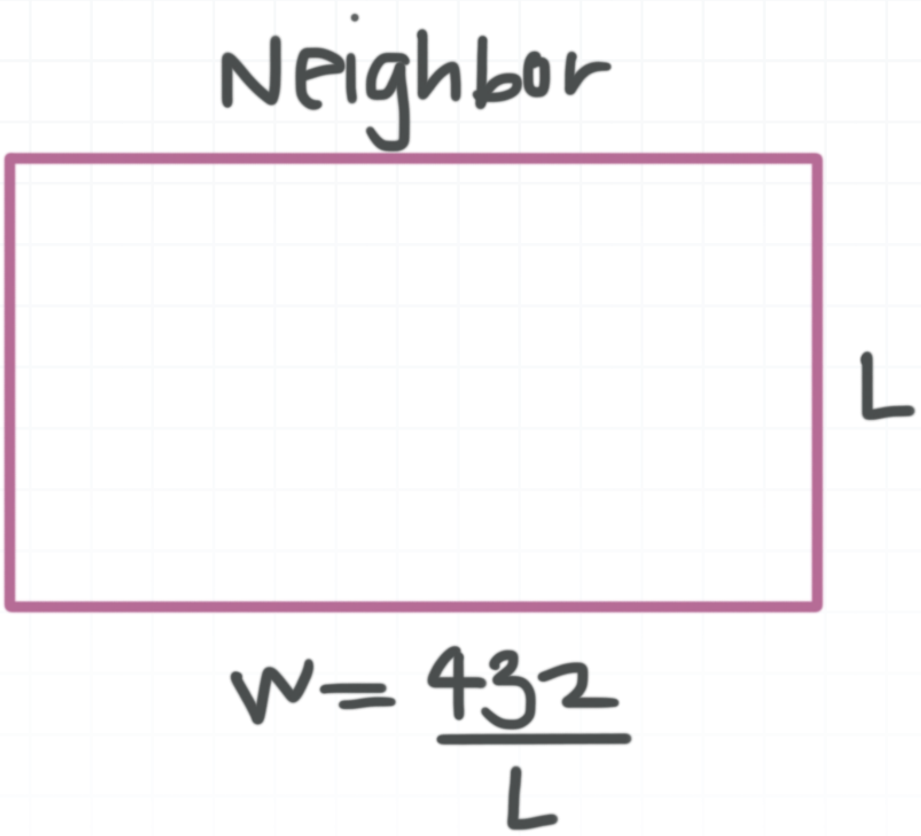
$$T = \frac{2\sqrt{26}}{3} = 3.399 \text{ hours}$$

She will minimize her time by rowing to a point that is 2.2678 miles down shore toward her car.

■ 2. Mr. Quizna wants to build in a completely fenced-in rectangular garden. The fence will be built so that one side is adjacent to his neighbor's property. The neighbor agrees to pay for half of that part of the fence because it borders his property. The garden will contain 432 square meters. What dimensions should Mr. Quizna select for his garden in order to minimize his cost?

Solution:

Draw a diagram.



The area is

$$A = L \cdot W$$

$$432 = L \cdot W$$

$$W = \frac{432}{L}$$

Let C be the total cost and y be the cost per meter. Then,

$$C = 2L \cdot y + \frac{432}{L} \cdot y + \frac{432}{L} \cdot \frac{y}{2} = 2yL + 648yL^{-1} = y(2L + 648L^{-1})$$

Take the derivative of the cost equation.

$$dC = y(2 - 648L^{-2})$$

Set the derivative equal to 0 and solve for L.

$$2 = \frac{648}{L^2}$$

$$2L^2 = 648$$

$$L^2 = 324$$

$$L = 18$$

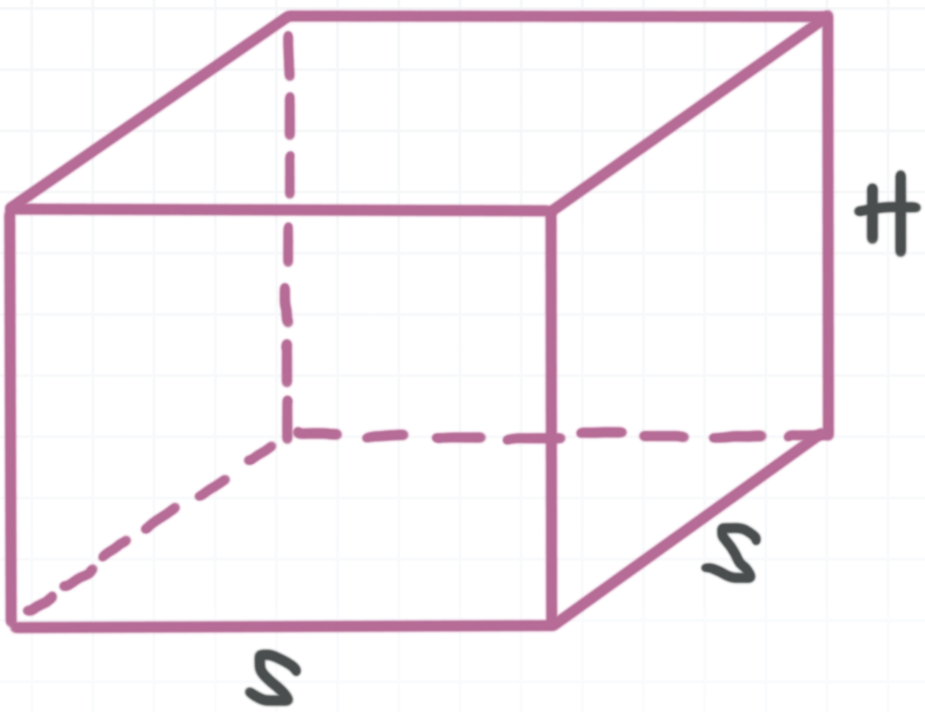
The length of the garden should be $L = 18$ meters and the width of the garden should be

$$W = \frac{432}{L} = \frac{432}{18} = 24 \text{ meters}$$

■ 3. A company is designing shipping crates and wants the volume of each crate to be 6 cubic feet, and each crate's base to be a square between 1.5 feet and 2.0 feet per side. The material for the bottom of the crate costs $\$5$ per square foot, the sides $\$3$ per square foot, and the top $\$1$ per square foot. What dimensions will minimize the cost of the shipping crates?

Solution:

Draw a diagram.



Based on the given information,

$$V = S \cdot S \cdot H$$

$$6 = S \cdot S \cdot H$$

$$H = \frac{6}{S^2}$$

The surface area of the bottom is S^2, the surface area of the top is S^2, and the surface area of the four sides is

$$4 \cdot S \cdot H$$

$$4 \cdot S \cdot \frac{6}{S^2} = \frac{24}{S}$$

Create a cost function.

$$C = 5 \cdot S^2 + 1 \cdot S^2 + 3 \cdot \frac{24}{S} = 6S^2 + \frac{72}{S} = 6S^2 + 72S^{-1}$$

Differentiate the cost function.

$$dC = 12S - \frac{72}{S^2}$$

Set the derivative equal to 0 and solve for S.

$$12S = \frac{72}{S^2}$$

$$12S^3 = 72$$

$$S^3 = 6$$

$$S = \sqrt[3]{6}$$

$$S \approx 1.82$$

Use the first derivative test with test values of $S = 1$ and $S = 2$ to confirm that the critical point represents a minimum.

$$C'(1) = 12(1) - \frac{72}{1^2}$$

$$C'(1) = 12 - 72$$

$$C'(1) = -60$$

and

$$C'(2) = 12(2) - \frac{72}{2^2}$$

$$C'(2) = 24 - 18$$

$$C'(2) = 6$$

Since we get a negative value to the left of the critical point and a positive value to the right of it, the function has a minimum at the critical point. The dimensions that will give the minimum cost are $S = 1.82$ feet and $H = 1.81$ feet.

■ 4. We want to construct a cylindrical can with a bottom and no top, that has a volume of 50 cm^3. Find the dimensions of the can that minimize its surface area.

Solution:

The volume of the cylinder is

$$V = \pi r^2 h$$

$$50 = \pi r^2 h$$

$$h = \frac{50}{\pi r^2}$$

Its surface area is

$$A = 2\pi rh + \pi r^2$$

$$A = 2\pi r \frac{50}{\pi r^2} + \pi r^2$$

$$A = \frac{100}{r} + \pi r^2$$

Differentiate the surface area function.

$$dA = -\frac{100}{r^2} + 2\pi r$$

Set the derivative equal to 0 and solve for r.

$$0 = -\frac{100}{r^2} + 2\pi r$$

$$2\pi r = \frac{100}{r^2}$$

$$2\pi r^3 = 100$$

$$r = \sqrt[3]{\frac{50}{\pi}}$$

$$r \approx 2.52$$

Use the first derivative test with test values of $r = 2$ and $r = 3$ to confirm that the critical point represents a minimum.

$$A'(2) = -\frac{100}{2^2} + 2\pi(2)$$

$$A'(2) = -25 + 4\pi$$

$$A'(2) = -12.4$$

and

$$A'(3) = -\frac{100}{3^2} + 2\pi(3)$$

$$A'(2) = 7.7$$

Since we get a negative value to the left of the critical point and a positive value to the right of it, the function has a minimum at the critical point.

$$h = \frac{50}{\pi(2.52)^2}$$

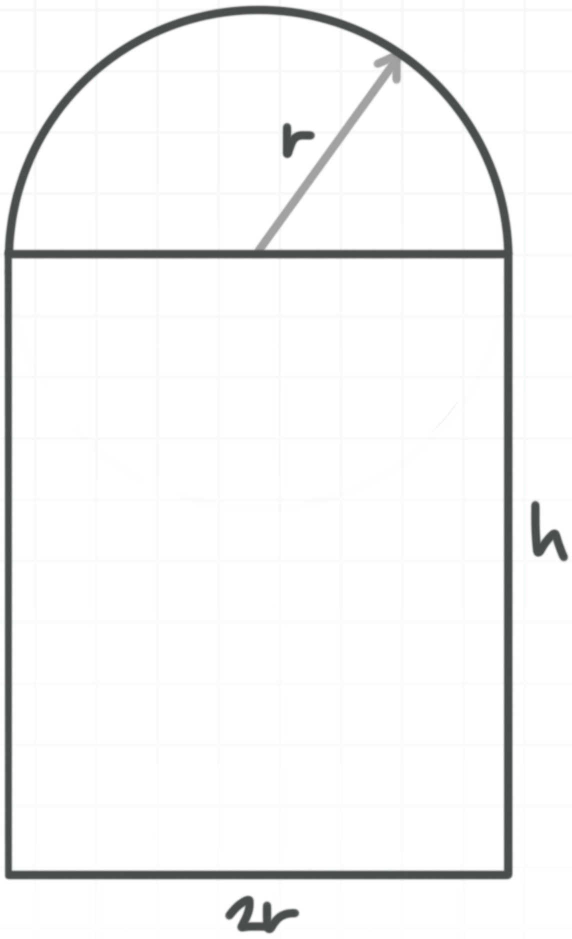
$$h \approx 2.51$$

So $r = 2.52$ ft and $h = 2.51$ ft will minimize the can's surface area.

■ 5. We're building a rectangular window with a a semicircular top. If we have 16 meters of framing material, what dimensions should we use in order to maximize the size of the window in order to let in the most light?

Solution:

Draw a diagram.



The perimeter of the window is the length of the three sides of the rectangular portion, plus half the circumference of a circle of radius r.

$$P = h + h + 2r + \frac{2\pi r}{2}$$

$$P = 2h + 2r + \pi r$$

The perimeter needs to be made with 16 meters of framing material.

$$16 = 2h + 2r + \pi r$$

$$16 - 2r - \pi r = 2h$$

$$h = 8 - r - \frac{\pi r}{2}$$

The area is the area of the rectangle plus the area of the half circle.

$$A = 2rh + \frac{1}{2}\pi r^2$$

$$A = 2r\left(8 - r - \frac{\pi r}{2}\right) + \frac{1}{2}\pi r^2$$

$$A = 16r - 2r^2 - \pi r^2 + \frac{1}{2}\pi r^2$$

$$A = 16r - 2r^2 - \frac{1}{2}\pi r^2$$

Differentiate the area function.

$$dA = 16 - 4r - \pi r$$

$$dA = 16 - r(4 + \pi)$$

Set the derivative equal to 0 and solve for r.

$$0 = 16 - r(4 + \pi)$$

$$r = \frac{16}{4 + \pi}$$

$$r \approx 2.24$$

Use the first derivative test with test values of $r = 2$ and $r = 3$ to confirm that the critical point represents a minimum.

$$A'(2) = 16 - 2(4 + \pi)$$

$$A'(2) = 1.72$$

and

$$A'(3) = 16 - 3(4 + \pi)$$

$$A'(3) = -5.4$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

$$h = 8 - 2.24 - \frac{2.24\pi}{2}$$

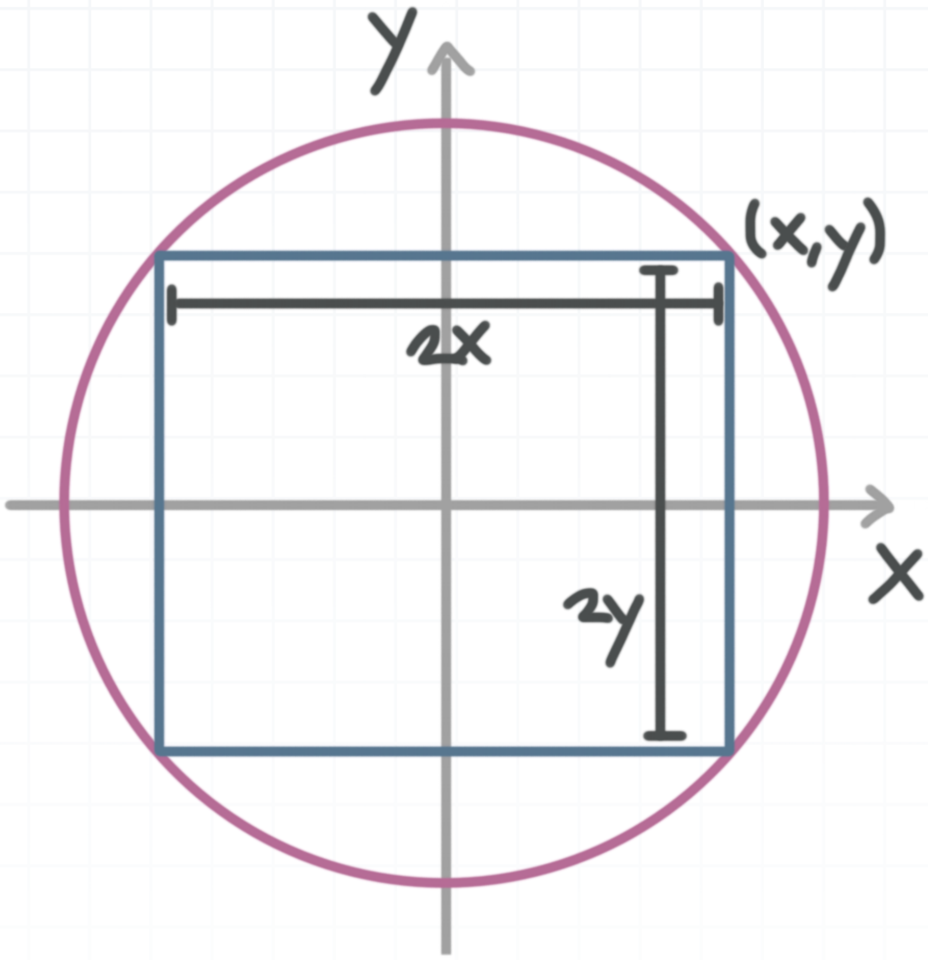
$$h \approx 2.24$$

In order to maximize the area of the window, the semicircle must have a radius of $r = 2.24$ and the rectangle must have dimensions 2.24×4.48 meters.

■ 6. Determine the dimensions of the rectangle with maximum area that can be inscribed in a circle of radius 3 cm.

Solution:

Draw a diagram.



The equation of the circle is

$$x^2 + y^2 = 9$$

$$y = \sqrt{9 - x^2}$$

with $-3 \leq x \leq 3$ and $-3 \leq y \leq 3$. The area of the rectangle is

$$A = (2x)(2y)$$

$$A = 4xy$$

$$A = 4x\sqrt{9 - x^2}$$

Differentiate the area function.

$$dA = 4\sqrt{9 - x^2} + 4x(-2x)\frac{1}{2\sqrt{9 - x^2}}$$

$$dA = 4\sqrt{9 - x^2} - \frac{4x^2}{\sqrt{9 - x^2}}$$

Set the derivative equal to 0 and solve for x.

$$0 = 4\sqrt{9 - x^2} - \frac{4x^2}{\sqrt{9 - x^2}}$$

$$4\sqrt{9 - x^2} = \frac{4x^2}{\sqrt{9 - x^2}}$$

$$9 - x^2 = x^2$$

$$9 = 2x^2$$

$$x^2 = \frac{9}{2}$$

$$x = \sqrt{\frac{9}{2}}$$

$$x \approx 2.12$$

Use the first derivative test with test values of $x = 2$ and $x = 3$ to confirm that the critical point represents a maximum.

$$A'(2) = 4\sqrt{9 - 2^2} - \frac{4(2)^2}{\sqrt{9 - 2^2}}$$

$$A'(2) = 4\sqrt{5} - \frac{16}{\sqrt{5}}$$

$$A'(2) \approx 1.79$$

and

$$A'(2.5) = 4\sqrt{9 - 2.5^2} - \frac{4(2.5)^2}{\sqrt{9 - 2.5^2}}$$

$$A'(2.5) \approx -8.44$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

$$y = \sqrt{9 - x^2}$$

$$y = \sqrt{9 - \left(\sqrt{\frac{9}{2}}\right)^2}$$

$$y = \sqrt{9 - \frac{9}{2}}$$

$$y = \sqrt{\frac{9}{2}}$$

$$y \approx 2.12$$

We defined the dimensions of the rectangle originally as $2x \times 2y$, so the rectangle will have maximum area when its dimensions are

$$2x \times 2y = 2(2.12) \times 2(2.12) = 4.24 \times 4.24$$

MEAN VALUE THEOREM

1. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,5]$.

$$f(x) = x^3 - 9x^2 + 24x - 18$$

Solution:

First, $f(x)$ is continuous and differentiable on the interval $[1,5]$. The problem says to find c in the interval such that

$$f'(c) = \frac{f(5) - f(1)}{5 - 1}$$

Find the values you need for the numerator.

$$f(5) = 5^3 - 9(5)^2 + 24(5) - 18 = 2$$

$$f(1) = 1^3 - 9(1)^2 + 24(1) - 18 = -2$$

Then

$$\frac{f(5) - f(1)}{5 - 1} = \frac{2 - (-2)}{4} = 1$$

Take the derivative $f'(x) = 3x^2 - 18x + 24$, so $f'(c) = 3c^2 - 18c + 24$, then set $f'(c) = 1$ and solve for c.

$$3c^2 - 18c + 24 = 1$$

$$3c^2 - 18c + 23 = 0$$

$$c = \frac{18 \pm \sqrt{18^2 - 4(3)(23)}}{2(3)} = \frac{18 \pm \sqrt{48}}{6} = \frac{18 \pm 4\sqrt{3}}{6} = \frac{9 \pm 2\sqrt{3}}{3}$$

Verify that the slope of the tangent line at these two x-values is 1.

$$f'\left(\frac{9 - 2\sqrt{3}}{3}\right) = 3\left(\frac{9 - 2\sqrt{3}}{3}\right)^2 - 18\left(\frac{9 - 2\sqrt{3}}{3}\right) + 24 = 1$$

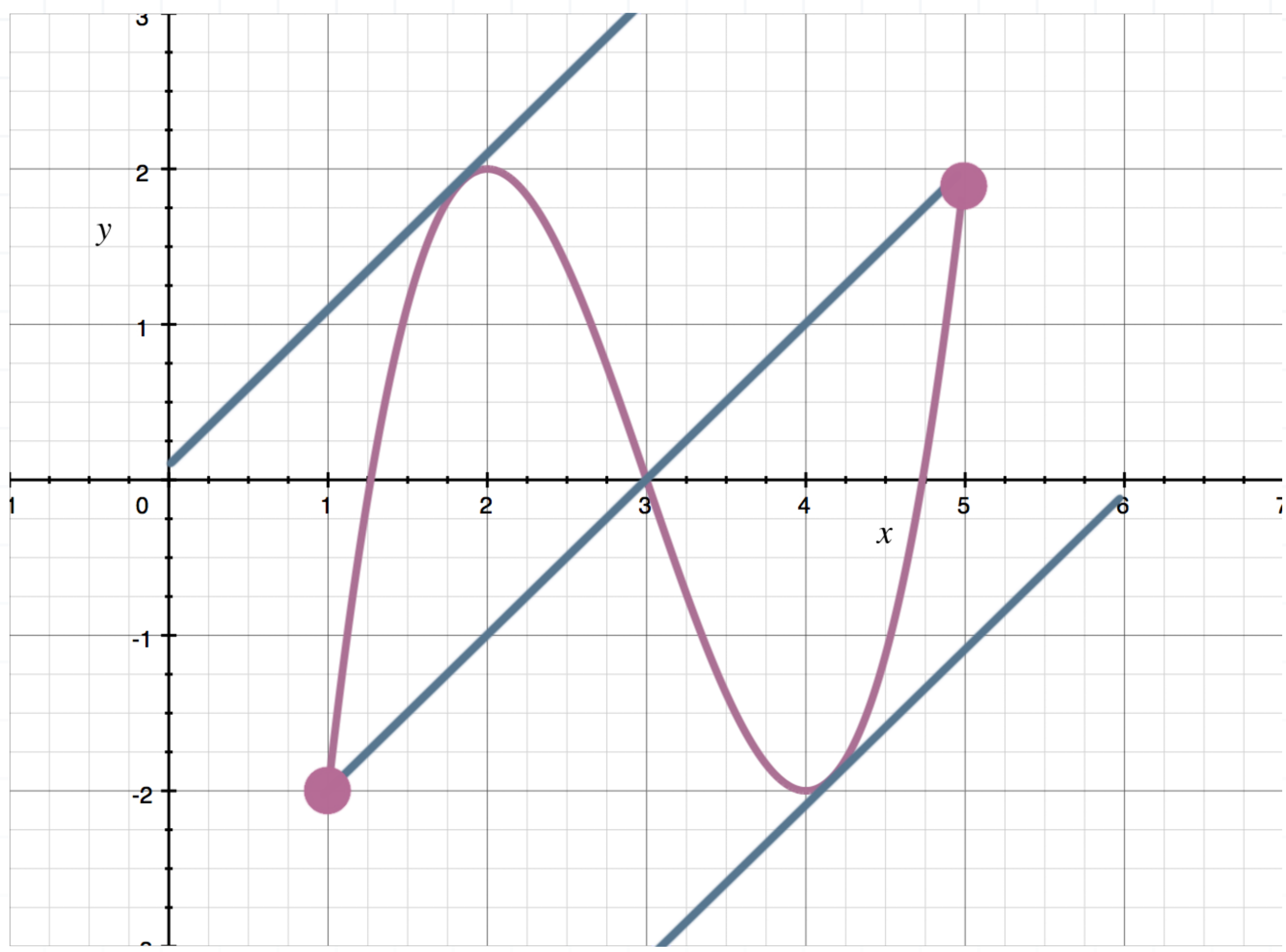
$$f'\left(\frac{9 + 2\sqrt{3}}{3}\right) = 3\left(\frac{9 + 2\sqrt{3}}{3}\right)^2 - 18\left(\frac{9 + 2\sqrt{3}}{3}\right) + 24 = 1$$

And both values are in the interval $[1,5]$.

$$c = \frac{9 + 2\sqrt{3}}{3} \approx 4.2$$

$$c = \frac{9 - 2\sqrt{3}}{3} \approx 1.8$$

Therefore, the values of c are $(9 \pm 2\sqrt{3})/3$. The figure illustrates how these two points satisfy the Mean Value Theorem.



■ 2. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[1,4]$.

$$g(x) = \frac{x^2 - 9}{3x}$$

Solution:

First, $g(x)$ is continuous and differentiable on the interval $[1,4]$. The problem says to find c in the interval such that

$$g'(c) = \frac{g(4) - g(1)}{4 - 1}$$

Find the values you need for the numerator.

$$g(4) = \frac{4^2 - 9}{3(4)} = \frac{16 - 9}{12} = \frac{7}{12}$$

$$g(1) = \frac{1^2 - 9}{3(1)} = \frac{1 - 9}{3} = -\frac{8}{3}$$

Then

$$\frac{g(4) - g(1)}{4 - 1} = \frac{\frac{7}{12} - \left(-\frac{8}{3}\right)}{3} = \frac{\frac{13}{4}}{3} = \frac{13}{4} \cdot \frac{1}{3} = \frac{13}{12}$$

Take the derivative,

$$g'(x) = \frac{(3x)(2x) - (x^2 - 9)(3)}{(3x)^2} = \frac{6x^2 - 3x^2 + 27}{9x^2} = \frac{3x^2 + 27}{9x^2} = \frac{x^2 + 9}{3x^2}$$

$$g'(c) = \frac{c^2 + 9}{3c^2}$$

then set $g'(c) = 13/12$ and solve for c.

$$\frac{c^2 + 9}{3c^2} = \frac{13}{12}$$

$$12c^2 + 108 = 39c^2$$

$$27c^2 = 108$$

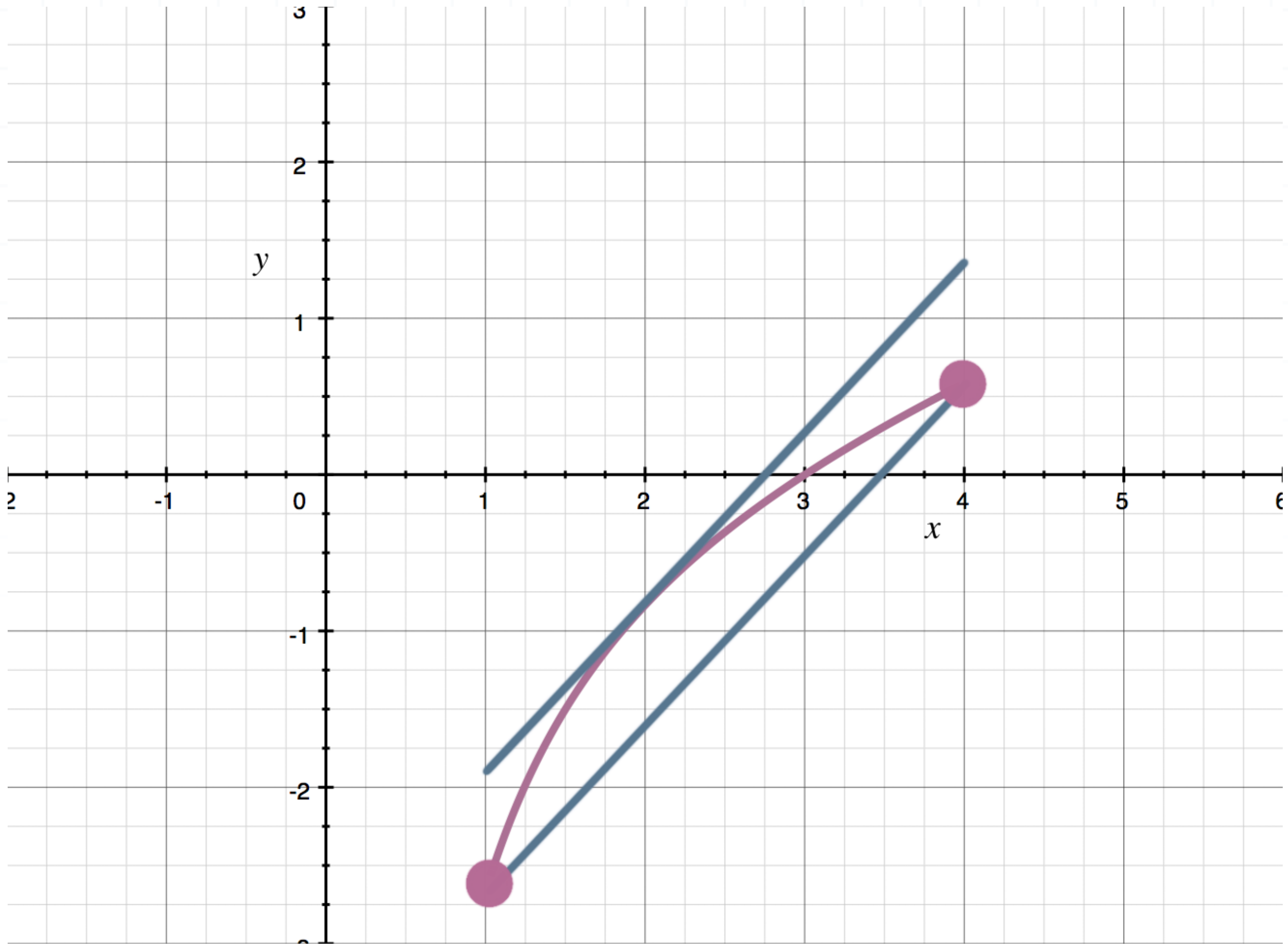
$$c^2 = 4$$

$$c = \pm 2$$

Only $c=2$ is in the given interval. Verify that the slope of the tangent line at this value is $13/12$.

$$g'(2)=\frac{2^2+9}{3(2)^2}=\frac{4+9}{3(4)}=\frac{13}{12}$$

Therefore, the value of c is 2. The figure illustrates how this point satisfies the Mean Value Theorem.



■ 3. Find the value(s) of c that satisfy the Mean Value Theorem for the function in the interval $[0,5]$.

$$h(x)=-\sqrt{25-5x}$$

Solution:

First, $h(x)$ is continuous and differentiable on the interval $[0,5]$. The problem says to find c in the interval such that

$$h'(c) = \frac{h(5) - h(0)}{5 - 0}$$

Find the values you need for the numerator.

$$h(5) = -\sqrt{25 - 5(5)} = -\sqrt{0} = 0$$

$$h\,(0) = -\sqrt{25 - 5(0)} = -\sqrt{25} = -5$$

Then

$$\frac{h(5) - h(0)}{5 - 0} = \frac{0 - (-5)}{5} = 1$$

Take the derivative,

$$h'(x) = -\frac{-5}{2\sqrt{25 - 5x}} = \frac{5}{2\sqrt{25 - 5x}}$$

$$h'(c) = \frac{5}{2\sqrt{25 - 5c}}$$

then set $h'(c) = 1$ and solve for c.

$$\frac{5}{2\sqrt{25 - 5c}} = 1$$

$$5 = 2\sqrt{25 - 5c}$$

$$\frac{5}{2} = \sqrt{25 - 5c}$$

$$\frac{25}{4} = 25 - 5c$$

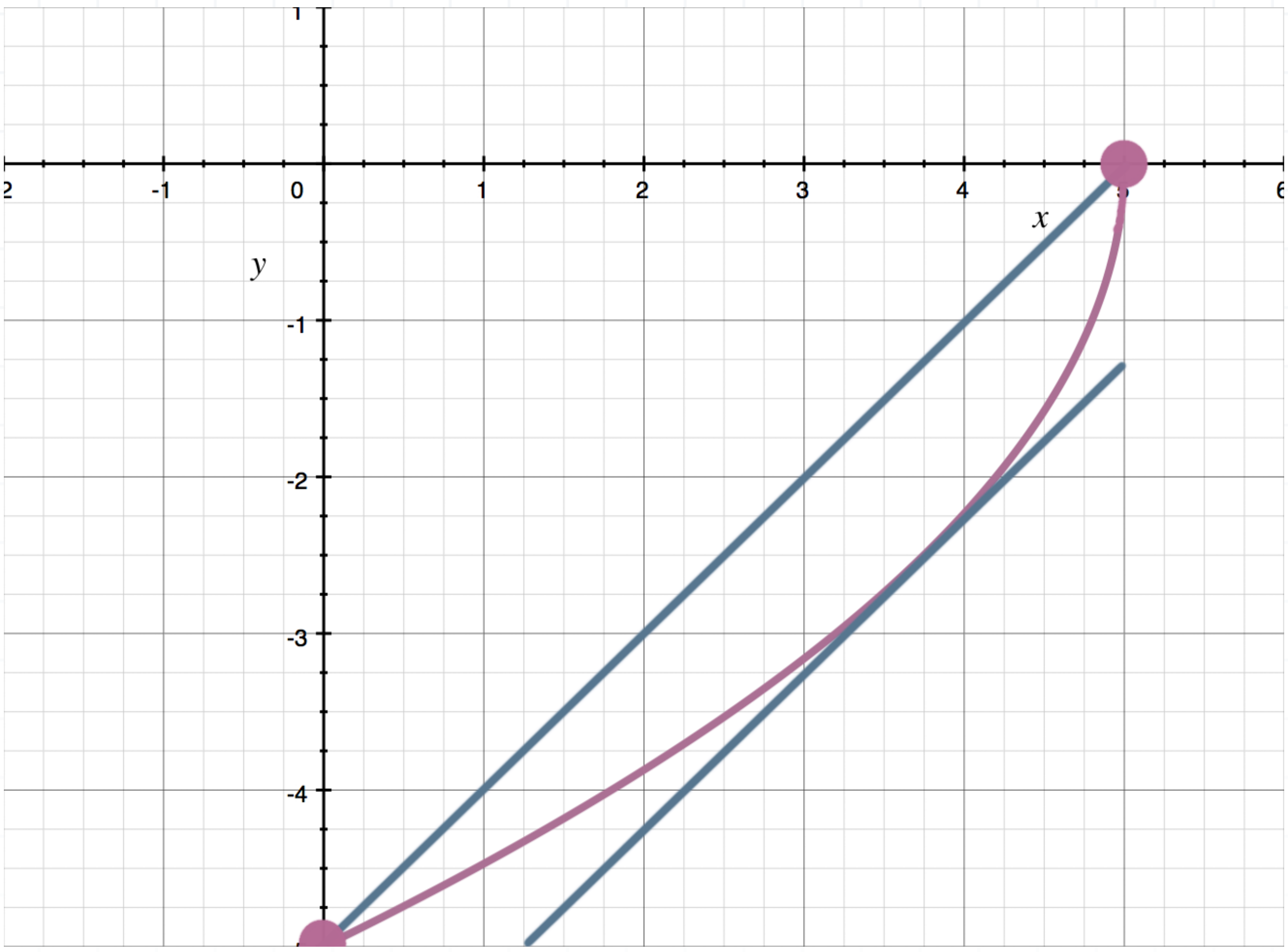
$$c = \left(\frac{25}{4} - 25\right) \div -5$$

$$c = \frac{15}{4}$$

Verify that the slope of the tangent line at this value is 1.

$$h'\left(\frac{15}{4}\right) = \frac{5}{2\sqrt{25 - 5\left(\frac{15}{4}\right)}} = \frac{5}{2\sqrt{\frac{25}{4}}} = \frac{5}{2\left(\frac{5}{2}\right)} = \frac{5}{5} = 1$$

Therefore, the value of c is $15/4$. The figure illustrates how this point satisfies the Mean Value Theorem.



■ 4. If we know that $g(x)$ is continuous and differentiable on $[2,7]$, $g(2) = -5$ and $g'(x) \leq 15$, find the largest possible value for $g(7)$.

Solution:

Use the Mean Value Theorem

$$g'(c) = \frac{g(7) - g(2)}{7 - 2}$$

or

$$g(7) - g(2) = g'(c)(7 - 2)$$

$$g(7) = g'(c)(7-2) + g(2)$$

$$g(7) = 5g'(c) - 5$$

Now we know that $g'(x) \le 15$, so we know that $g'(c) \le 15$.

$$g(7) = 5g'(c) - 5 \le 5(15) - 5 \le 70$$

This means that the largest possible value for $g(7)$ is 70.

■ 5. If we know that $f(x)$ is continuous and differentiable on $[-4,3]$, $f(3) = 12$ and $f'(x) \le 4$, find the smallest possible value for $f(-4)$.

Solution:

Use the Mean Value Theorem

$$f'(c) = \frac{f(3) - f(-4)}{3 - (-4)}$$

or

$$f(3) - f(-4) = f'(c)(3 - (-4))$$

$$12 - f(-4) = 7f'(c)$$

Now we know that $f'(x) \le 4$, so we know that $f'(c) \le 4$.

$$12 - f(-4) = 7f'(c)$$

$$12 - f(-4) \le 7(4)$$

$$12 - f(-4) \le 28$$

$$-f(-4) \le 16$$

$$f(-4) \ge -16$$

This means that the smallest possible value for $f(-4)$ is -16.

■ 6. When a cake is removed from an oven and placed in an environment with an ambient temperature of 20° C, its core temperature is 180° C. Two hours later, the core temperature has fallen to 30° C. Explain why there must exist a time in the interval when the temperature is decreasing at a rate of 75° C per hour.

Solution:

Because the cake cools down for 2 hours, we can set the interval at $t = [0,2]$. If we plug these values into the Mean Value Theorem, we get

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

$$f'(c) = \frac{f(2) - f(0)}{2 - 0}$$

$$f'(c) = \frac{30 - 180}{2 - 0}$$

$$f'(c) = \frac{-150}{2}$$

$$f'(c) = -75$$

This result tells us that, at least at some point, the temperature is decreasing at a rate of 75° C per hour.

ROLLE'S THEOREM

1. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-1,2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = x^3 - 2x^2 - x - 3$$

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[-1,2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-1,2]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(2) = f(-1)$.

$$f(2) = 2^3 - 2(2)^2 - 2 - 3 = -5$$

$$f(-1) = (-1)^3 - 2(-1)^2 - (-1) - 3 = -5$$

Thus, Rolle's Theorem applies. Next, find $f'(x) = 3x^2 - 4x - 1$ and set $f'(c) = 0$ and solve for c using the quadratic formula.

$$3c^2 - 4c - 1 = 0$$

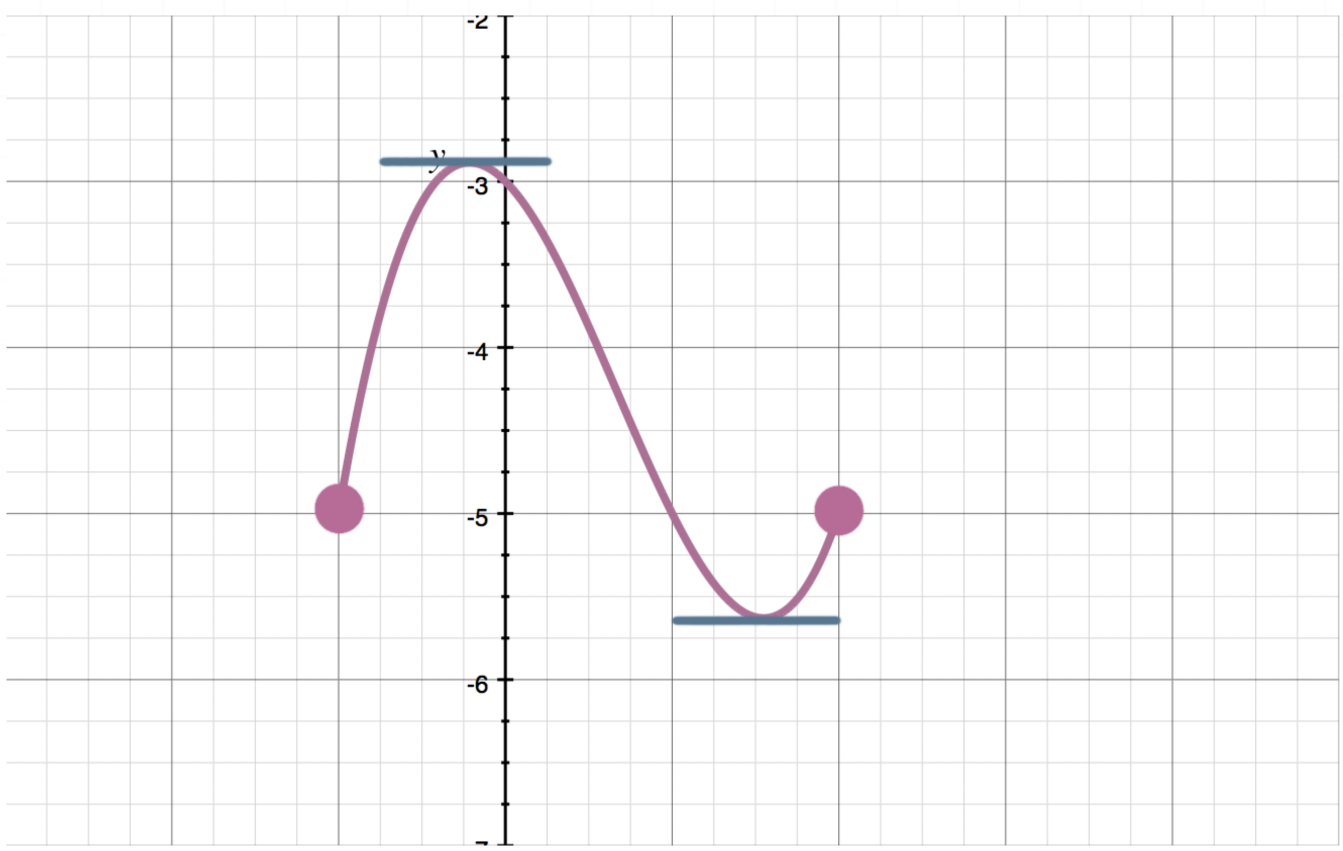
$$c = \frac{4 \pm \sqrt{(-4)^2 - 4(3)(-1)}}{2(3)} = \frac{4 \pm \sqrt{28}}{6} = \frac{4 \pm 2\sqrt{7}}{6} = \frac{2 \pm \sqrt{7}}{3}$$

Verify that the slope of the tangent line at these two x-values is 0.

$$f'\left(\frac{2-\sqrt{7}}{3}\right)=3\left(\frac{2-\sqrt{7}}{3}\right)^2-4\left(\frac{2-\sqrt{7}}{3}\right)-1=0$$

$$f'\left(\frac{2+\sqrt{7}}{3}\right)=3\left(\frac{2+\sqrt{7}}{3}\right)^2-4\left(\frac{2+\sqrt{7}}{3}\right)-1=0$$

Both values are in the interval. Therefore, the values of c such that $f'(c)=0$ are $(2\pm\sqrt{7})/3$. The figure illustrates how these two points satisfy Rolle's Theorem.



■ 2. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-3,5]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$g(x)=\frac{x^2-2x-15}{6-x}$$

Solution:

The function $g(x)$ is continuous and differentiable on the interval $[-3,5]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-3,5]$, such that $g'(c) = 0$.

To use Rolle's Theorem, show that $g(5) = g(-3)$.

$$g(5) = \frac{5^2 - 2(5) - 15}{6 - 5} = \frac{0}{1} = 0$$

$$g(-3) = \frac{(-3)^2 - 2(-3) - 15}{6 - (-3)} = \frac{0}{9} = 0$$

Thus, Rolle's Theorem applies. Next, find

$$g'(x) = \frac{(6 - x)(2x - 2) - (x^2 - 2x - 15)(-1)}{(6 - x)^2} = \frac{-x^2 + 12x - 27}{(6 - x)^2}$$

and set $g'(c) = 0$ and solve for c using the quadratic formula.

$$-c^2 + 12c - 27 = 0$$

$$-(c^2 - 12c + 27) = 0$$

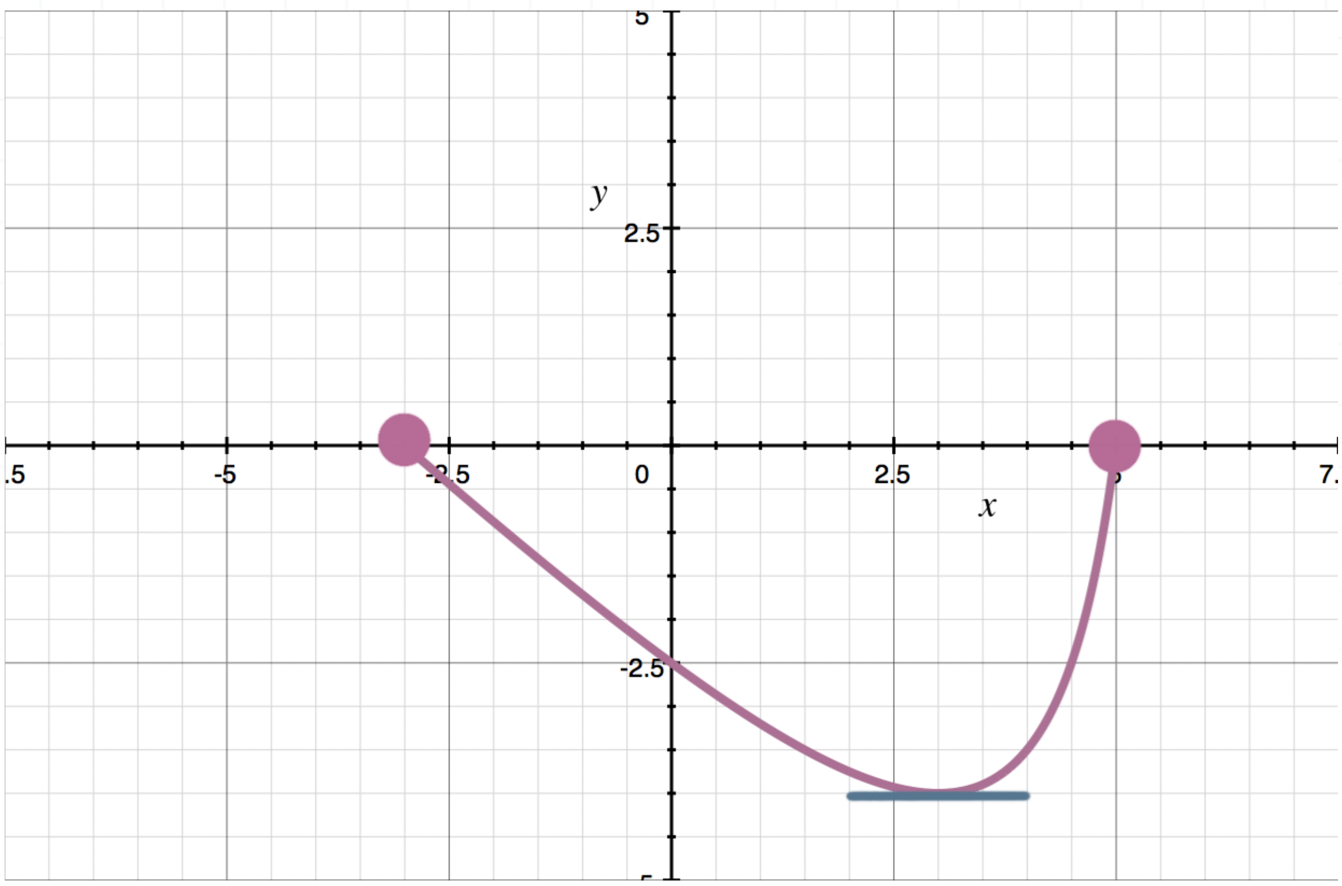
$$-(c - 3)(c - 9) = 0$$

$$c = 3, 9$$

The value $c = 9$ is outside of the given interval. Verify that the slope of the tangent line at $c = 3$ is 0.

$$g'(3) = \frac{-3^2 + 12(3) - 27}{(6-3)^2} = \frac{0}{9} = 0$$

Therefore, the value of c such that $f'(c) = 0$ is 3. The figure illustrates how this point satisfies Rolle's Theorem.



■ 3. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-\pi/2,\pi/2]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$h(x) = \sin(2x)$$

Solution:

The function $h(x)$ is continuous and differentiable on the interval $[-\pi/2,\pi/2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-\pi/2,\pi/2]$, such that $h'(c)=0$.

To use Rolle's Theorem, show that $h(\pi/2)=h(-\pi/2)$.

$$h\left(\frac{\pi}{2}\right)=\sin\left(2\cdot\frac{\pi}{2}\right)=\sin(\pi)=0$$

$$h\left(-\frac{\pi}{2}\right)=\sin\left(2\cdot-\frac{\pi}{2}\right)=\sin(-\pi)=0$$

Thus, Rolle's Theorem applies. Next, find $h'(x)=2\cos(2x)$ and set $h'(c)=0$ and solve for c.

$$2\cos(2c)=0$$

$$\cos(2c)=0$$

$$\arccos(0)=2c$$

$$2c=\pm\frac{\pi}{2}$$

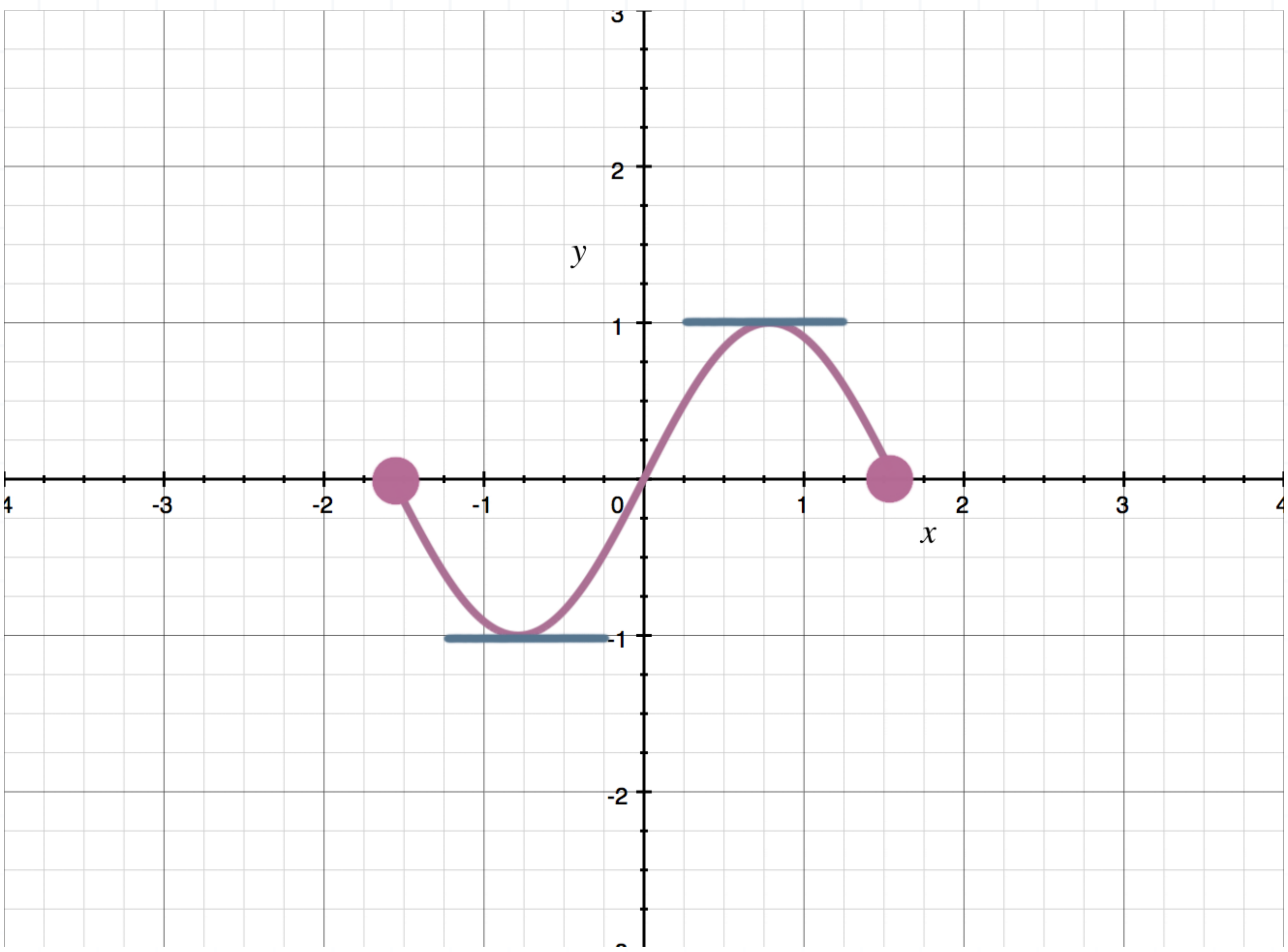
$$c=\pm\frac{\pi}{4}$$

Verify that the slope of the tangent line at these two c-values is 0.

$$h'\left(-\frac{\pi}{4}\right)=2\cos\left(-\frac{\pi}{2}\right)=2\cdot 0=0$$

$$h'\left(\frac{\pi}{4}\right)=2\cos\left(\frac{\pi}{2}\right)=2\cdot 0=0$$

Therefore, the values of c such that $f'(c) = 0$ are $\pm\pi/4$. The figure illustrates how these two points satisfy Rolle's Theorem.



■ 4. Determine whether Rolle's Theorem can be applied to $f(x) = \sqrt{4 - x^2}$ on the interval $[-2,2]$. If Rolle's Theorem applies, find the value(s) of c in the interval such that $f'(c) = 0$.

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[-2,2]$. The problem says to use Rolle's Theorem to find c, in the given interval $[-2,2]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(-2)=f(2)$.

$$f(-2)=\sqrt{4-(-2)^2}=\sqrt{4-4}=\sqrt{0}=0$$

$$f(2)=\sqrt{4-(2)^2}=\sqrt{4-4}=\sqrt{0}=0$$

Because theses values are equivalent, Rolle's Theorem applies. Next, find the derivative and set $f'(c)=0$ to solve for c.

$$f'(x)=-\frac{x}{\sqrt{4-x^2}}$$

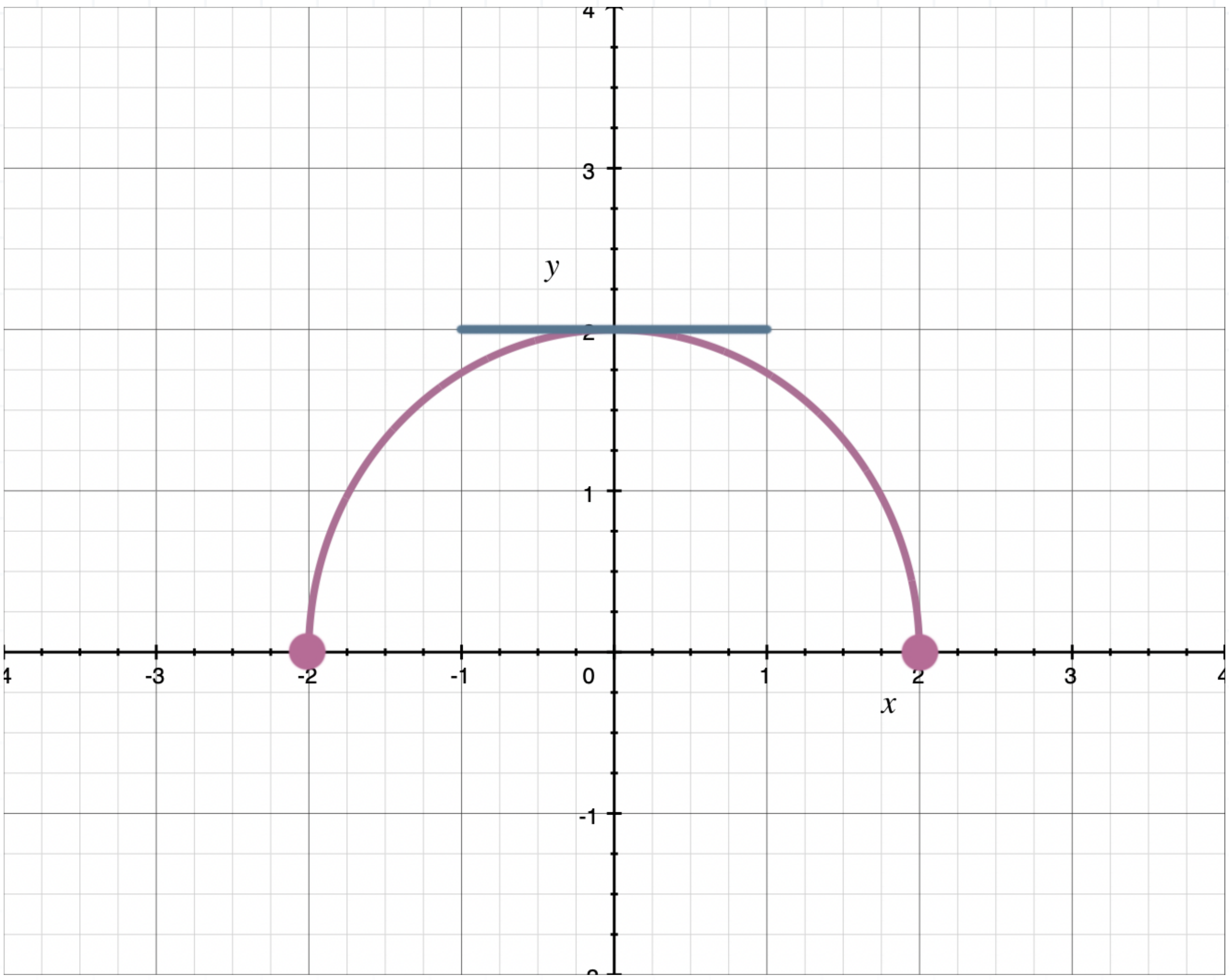
$$-\frac{c}{\sqrt{4-c^2}}=0$$

$$c=0$$

Verify that the slope of the tangent line at $c=0$ is 0.

$$f'(0)=-\frac{0}{\sqrt{4-0^2}}=0$$

Therefore, $c=0$ makes $f'(c)=0$. The figure illustrates how this point satisfies Rolle's Theorem.



■ 5. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[3,5]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = |x - 2|$$

Solution:

The function $f(x)$ is continuous and differentiable on the interval $[3,5]$. The problem says to use Rolle's Theorem to find c in the interval $[3,5]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(3) = f(5)$.

$$f(3) = |3 - 2| = |1| = 1$$

$$f(3) = |5 - 2| = |3| = 3$$

Because the function doesn't have the same value at the endpoints of the interval, Rolle's Theorem can't be applied, at least not on this particular interval.

■ 6. Use Rolle's Theorem to show that the function has a horizontal tangent line in the interval $[-1,1]$. Find the value(s) of c in the interval that satisfy Rolle's Theorem.

$$f(x) = \ln(9 - x^2)$$

Solution:

The domain of the function is $(-3,3)$. Therefore the function $f(x)$ is continuous and differentiable on the interval $[-1,1]$. The problem says to use Rolle's Theorem to find c in the interval $[-1,1]$, such that $f'(c) = 0$.

To use Rolle's Theorem, show that $f(1) = f(-1)$.

$$f(-1) = \ln(9 - (-1)^2) = \ln(9 - 1) = \ln 8$$

$$f(1) = \ln(9 - (1)^2) = \ln(9 - 1) = \ln 8$$

Because theses values are equivalent, Rolle's Theorem applies. Next, find the derivative and set $f'(c) = 0$ to solve for c.

$$f'(x) = -\frac{2x}{9 - x^2}$$

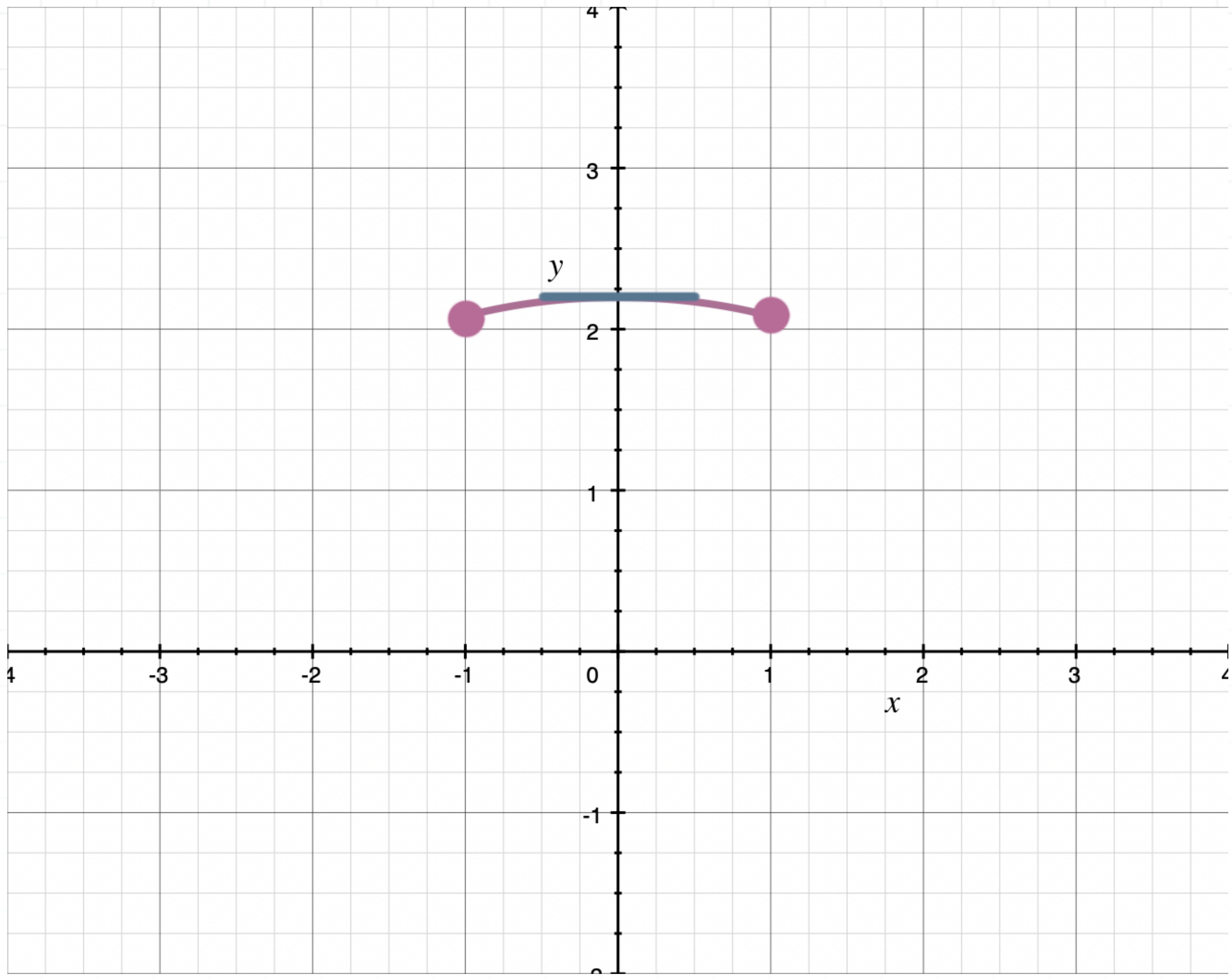
$$-\frac{2c}{9 - c^2} = 0$$

$$c = 0$$

Verify that the slope of the tangent line at $c = 0$ is 0.

$$f'(0) = -\frac{2(0)}{9 - 0^2} = 0$$

Therefore, $c = 0$ makes $f'(c) = 0$. The figure illustrates how this point satisfies Rolle's Theorem.



y
x
4
3
2
1
0
-1
-2
-3
-4

NEWTON'S METHOD

1. Use four iterations of Newton's Method to approximate the root of $g(x) = x^3 - 12$ in the interval $[1,3]$ to the nearest three decimal places.

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^3 - 12 = 0$$

If $g(x) = x^3 - 12$ and $g'(x) = 3x^2$, since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0 = (3 + 1)/2 = 2$. Then $g(2) = -4$ and $g'(2) = 12$. Plug those values into the Newton's Method formula.

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$$

$$x_1 = 2 - \frac{-4}{12} \approx 2.333$$

Next, $g(2.333) = 0.698$ and $g'(2.333) = 16.329$. So

$$x_2 = 2.333 - \frac{0.698}{16.329} = 2.290$$

Next, $g(2.290) = 0.009$ and $g'(2.290) = 15.732$. So

$$x_3 = 2.290 - \frac{0.009}{15.732} = 2.289$$

Next, $g(2.289) = -0.007$ and $g'(2.289) = 15.719$. So

$$x_4 = 2.289 - \frac{-0.007}{15.719} = 2.289$$

■ 2. Use four iterations of Newton's Method to approximate the root of $f(x) = x^4 - 14$ in the interval $[-2, -1]$ to the nearest four decimal places.

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^4 - 14 = 0$$

If $f(x) = x^4 - 14$ and $f'(x) = 4x^3$, since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0 = (-1 - 2)/2 = -3/2$. Then $f(-1.5) = -8.9375$ and $f'(-1.5) = -13.5$. Plug those values into the Newton's Method formula.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = -1.5 - \frac{-8.9375}{-13.5} = -2.1620$$

Next, $f(-2.1620) = 7.8501$ and $f'(-2.1620) = -40.4229$. So

$$x_2 = -2.1620 - \frac{7.8501}{-40.4229} = -1.9678$$

Next, $f(-1.9678) = 0.9957$ and $f'(-1.9678) = -30.4792$. So

$$x_3 = -1.9678 - \frac{0.9957}{-30.4792} = -1.9352$$

Next, $f(-1.9352) = 0.0245$ and $f'(-1.9352) = -28.9893$. So

$$x_4 = -1.9352 - \frac{0.0245}{-28.9893} = -1.9343$$

■ 3. Use four iterations of Newton's Method to approximate the root of $h(x) = 3e^{x-3} - 4 + \sin x$ in the interval $[2,4]$ to the nearest four decimal places.

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$3e^{x-3} - 4 + \sin x = 0$$

If $h(x) = 3e^{x-3} - 4 + \sin x$ and $h'(x) = 3e^{x-3} + \cos x$, since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0 = (4+2)/2 = 3$. Then $h(3) = -0.8589$ and $h'(3) = 2.0100$. Plug those values into the Newton's Method formula.

$$x_{n+1} = x_n - \frac{h(x_n)}{h'(x_n)}$$

$$x_1 = 3 - \frac{-0.8589}{2.0100} = 3.4273$$

Next, $h(3.4273) = 0.3175$ and $h'(3.4273) = 3.6399$. So

$$x_2 = 3.4273 - \frac{0.3175}{3.6399} = 3.3401$$

Next, $h(3.3401) = 0.0181$ and $h'(3.3401) = 3.2349$. So

$$x_3 = 3.3401 - \frac{0.0181}{3.2349} = 3.3345$$

Next, $h(3.3345) = 0.00001$ and $h'(3.3345) = 3.2103$. So

$$x_4 = 3.3345 - \frac{0.00001}{3.2103} = 3.3345$$

■ 4. Use four iterations of Newton's Method to approximate $\sqrt[65]{100}$ to four decimal places.

Solution:

$$\sqrt[65]{100} = x$$

$$100 = x^{65}$$

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^{65} - 100 = 0$$

Take the derivative of the function.

$$f(x_n) = x_n^{65} - 100$$

$$f'(x_n) = 65x_n^{64}$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^{65} - 100}{65x_n^{64}}$$

Let's start with $x_n = 1$, and work our problem with the number of decimal places we were asked for.

$$x_0 = 1$$

$$x_1 = 1 - \frac{1^{65} - 100}{65(1)^{64}} = 2.5231$$

$$x_2 = 2.5231 - \frac{2.5231^{65} - 100}{65(2.5231)^{64}} = 2.4843$$

$$x_3 = 2.4843 - \frac{2.4843^{65} - 100}{65(2.4843)^{64}} = 2.4460$$

$$x_4 = 2.4460 - \frac{2.4460^{65} - 100}{65(2.4460)^{64}} = 2.4084$$

5. Use Newton's Method to approximate to three decimal places the root of the function in the interval $[3,7]$.

$$5x^2+3=e^x$$

Solution:

When we use Newton's Method, the function must be in the form $f(x)=0$.

$$5x^2+3-e^x=0$$

Take the derivative of the function.

$$f(x_n)=5x_n^2+3-e^{x_n}$$

$$f'(x_n)=10x_n-e^{x_n}$$

Then the Newton's Method formula will be

$$x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1}=x_n-\frac{5x_n^2+3-e^{x_n}}{10x_n-e^{x_n}}$$

Since we know the interval where the function has a solution, then we can use the midpoint of the interval as $x_0=(7+3)/2=5$, and work our problem with the number of decimal places we were asked for.

$$x_0=5$$

$$x_1 = 5 - \frac{5(5)^2 + 3 - e^5}{10(5) - e^5} = 4.793$$

$$x_2 = 4.793 - \frac{5(4.793)^2 + 3 - e^{4.793}}{10(4.793) - e^{4.793}} = 4.754$$

$$x_3 = 4.754 - \frac{5(4.754)^2 + 3 - e^{4.754}}{10(4.754) - e^{4.754}} = 4.753$$

$$x_3 = 4.753 - \frac{5(4.753)^2 + 3 - e^{4.753}}{10(4.753) - e^{4.753}} = 4.753$$

Since these last two approximations are identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 4.753$.

■ 6. Use Newton's Method to find an approximation of the root of the function to four decimal places.

$$2 \ln x = \cos x$$

Solution:

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$2 \ln x - \cos x = 0$$

Take the derivative of the function.

$$f(x_n) = 2\ln(x_n) - \cos(x_n)$$

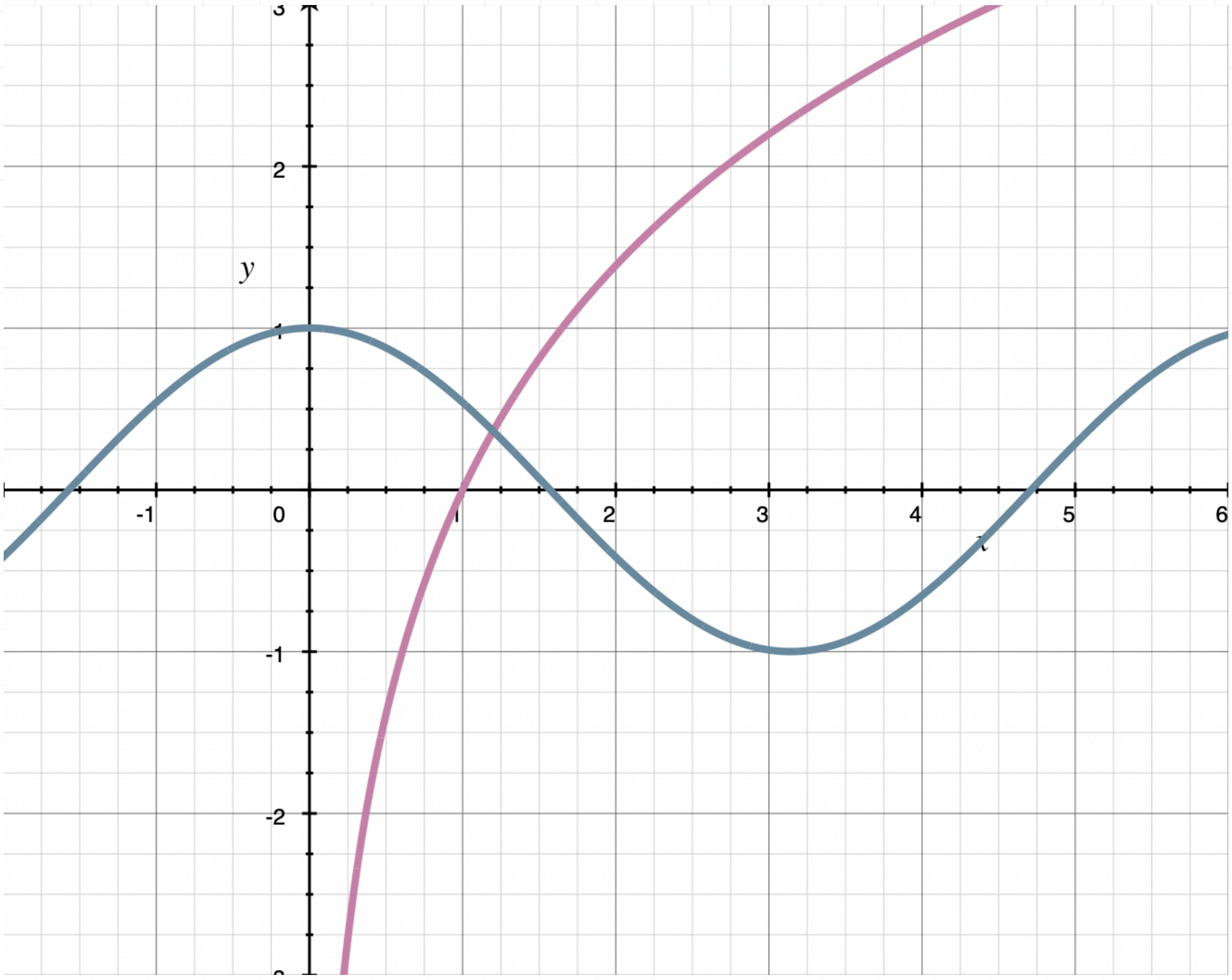
$$f'(x_n) = \frac{2}{x_n} + \sin(x_n)$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2\ln(x_n) - \cos(x_n)}{\frac{2}{x_n} + \sin(x_n)}$$

If we don't know an initial approximation to the solution x_0, we can sketch the graphs of $2\ln x$ and $\cos x$ and use their intersection point to get an estimate of the solution, which we can then use as x_0.



From the graphs, we can see that the two functions intersect one another near $x = 1$, so we can take $x_0 = 1$.

$$x_0 = 1$$

$$x_1 = 1 - \frac{2\ln(1) - \cos(1)}{\frac{2}{1} + \sin(1)} = 1.19$$

$$x_2 = 1.19 - \frac{2\ln(1.19) - \cos(1.19)}{\frac{2}{1.19} + \sin(1.19)} = 1.199$$

$$x_3 = 1.199 - \frac{2\ln(1.199) - \cos(1.199)}{\frac{2}{1.199} + \sin(1.199)} = 1.199$$

Since these last two approximations are identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 1.199$.

L'HOSPITAL'S RULE

■ 1. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to 0} \frac{2\sqrt{x+4} - 4 - \frac{1}{2}x}{x^2}$$

Solution:

Evaluating the limit as $x \to 0$ gives the indeterminate form $0/0$, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to 0} \frac{\frac{1}{\sqrt{x+4}} - \frac{1}{2}}{2x}$$

But evaluating this $x \to 0$ still gives $0/0$, so we'll apply L'Hospital's rule again.

$$\lim_{x \to 0} \frac{-\frac{1}{2\sqrt{(x+4)^3}}}{2} = \lim_{x \to 0} -\frac{1}{4\sqrt{(x+4)^3}}$$

Then we can evaluate as $x \to 0$.

$$-\frac{1}{4\sqrt{(0+4)^3}} = -\frac{1}{4\sqrt{64}} = -\frac{1}{4(8)} = -\frac{1}{32}$$

2. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x}{3+\tan x}$$

Solution:

Evaluating the limit as $x \to \pi/2$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x\to\frac{\pi}{2}} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x\to\frac{\pi}{2}} \frac{\tan x}{\sec x} = \lim_{x\to\frac{\pi}{2}} \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}} = \lim_{x\to\frac{\pi}{2}} \frac{\sin x}{\cos x} \cdot \frac{\cos x}{1} = \lim_{x\to\frac{\pi}{2}} \sin x$$

Then we can evaluate as $x \to \pi/2$.

$$\sin\frac{\pi}{2} = 1$$

3. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x\to\infty} \frac{\ln x}{4\sqrt{x}}$$

Solution:

Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{2}{\sqrt{x}}} = \lim_{x \to \infty} \frac{1}{x} \cdot \frac{\sqrt{x}}{2} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}}$$

Then we can evaluate as $x \to \infty$.

$$\lim_{x \to \infty} \frac{1}{2\sqrt{x}} = 0$$

■ 4. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} \frac{e^x}{x^2}$$

Solution:

Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll use L'Hospital's Rule, and replace both the numerator and denominator with their derivatives.

$$\lim_{x \to \infty} \frac{e^x}{2x}$$

Evaluating the limit as $x \to \infty$ gives the indeterminate form ∞/∞, so we'll apply L'Hospital's Rule again.

$$\lim_{x \to \infty} \frac{e^x}{2}$$

Now evaluating the limit gives ∞, which means that the limit does not exist.

■ 5. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to 0^+} \cos x^{\cot x}$$

Solution:

If we try substitution to evaluate at $x = 0^+$, we get an indeterminate form.

$$1^\infty$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x \to 0^+} \cos x^{\cot x}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x \to 0^+} \ln(\cos x^{\cot x})$$

$$\ln y = \lim_{x \to 0^+} \cot x \ln(\cos x)$$

$$\ln y = \lim_{x\to 0^+} \frac{\ln(\cos x)}{\tan x}$$

With the limit rewritten, we'll apply L'Hospital's Rule to the fraction.

$$\ln y = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{\sec^2 x}$$

$$\ln y = \lim_{x\to 0^+} \frac{\frac{1}{\cos x}(-\sin x)}{\frac{1}{\cos^2 x}}$$

$$\ln y = \lim_{x\to 0^+} -\sin x \cos x$$

Evaluate the limit,

$$\ln y = -(0)(1)$$

$$\ln y = 0$$

then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^0$$

$$y = 1$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x\to 0^+} \cos x^{\cot x}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x \to 0^+} \cos x^{\cot x} = 1$$

■ 6. Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}}$$

Solution:

If we try substitution to evaluate at $x = \infty$, we get an indeterminate form.

$$\infty^0$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x \to \infty} \ln(e^x + 4x)^{\frac{4}{x}}$$

$$\ln y = \lim_{x \to \infty} \frac{4}{x} \ln(e^x + 4x)$$

$$\ln y = \lim_{x \to \infty} \frac{4 \ln(e^x + 4x)}{x}$$

We get the indeterminate form ∞/∞ when we evaluate the limit. With the limit rewritten, we'll apply L'Hospital's Rule to the fraction.

$$\ln y = \lim_{x\to\infty} \frac{\frac{4}{e^x+4x}(e^x+4)}{1}$$

$$\ln y = \lim_{x\to\infty} \frac{4(e^x+4)}{e^x+4x}$$

$$\ln y = \lim_{x\to\infty} \frac{4e^x+16}{e^x+4x}$$

We get an indeterminate form ∞/∞ when we evaluate the limit, so we'll apply L'Hospital's Rule again.

$$\ln y = \lim_{x\to\infty} \frac{4e^x}{e^x+4}$$

We get an indeterminate form ∞/∞ when we evaluate the limit, so we'll apply L'Hospital's Rule one more time and evaluate the limit,

$$\ln y = \lim_{x\to\infty} \frac{4e^x}{e^x}$$

$$\ln y = \lim_{x\to\infty} 4$$

$$\ln y = 4$$

and then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^4$$

$$y = e^4$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x \to \infty} (e^x + 4x)^{\frac{4}{x}} = e^4$$

POSITION, VELOCITY, AND ACCELERATION

■ 1. Find the velocity $v(t)$, speed, and acceleration $a(t)$ at $t = 2$ of the position function.

$$s(t) = -\frac{t^3}{3} + t^2 + 3t - 1$$

Solution:

Velocity is given by the first derivative of the position function.

$$s'(t) = v(t) = -t^2 + 2t + 3$$

$$v(2) = -(2)^2 + 2(2) + 3$$

$$v(2) = 3$$

Acceleration is given by the second derivative of the position function or the first derivative of the velocity function.

$$s''(t) = v'(t) = a(t) = -2t + 2$$

$$a(2) = -2(2) + 2$$

$$a(2) = -2$$

Speed is the absolute value of velocity. So speed is

$$|v(2)| = |3| = 3$$

■ 2.The position of a particle which moves along the x-axis is given by $s(t) = \cos t + \sin t$. What is the acceleration of the particle at the point where the velocity is first equal to zero?

Solution:

Take the derivative of the position function to find the velocity.

$$s(t) = \cos t + \sin t$$

$$v(t) = s'(t) = -\sin t + \cos t$$

We need to find time when velocity is 0.

$$-\sin t + \cos t = 0$$

$$\cos t = \sin t$$

$t = \dfrac{\pi}{4} + \pi k$, where k is any integer

Since we need to find the acceleration of the particle at the point where the velocity is first equal to zero, then $t = \pi/4$.

Take the derivative of the velocity function to find the acceleration.

$$v(t) = -\sin t + \cos t$$

$$a(t) = v'(t) = -\cos t - \sin t$$

$$a\left(\frac{\pi}{4}\right) = -\cos\left(\frac{\pi}{4}\right) - \sin\left(\frac{\pi}{4}\right)$$

$$a\left(\frac{\pi}{4}\right) = -\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$$

$$a\left(\frac{\pi}{4}\right) = -\sqrt{2}$$

■ 3. Find the velocity $v(t)$, speed, and acceleration $a(t)$ at $t = 4$ of the position function.

$$s(t) = \frac{t^2}{2t+4}$$

Solution:

Velocity is given by the first derivative of the position function.

$$s'(t) = v(t) = \frac{(2t)(2t+4) - (t^2)(2)}{(2t+4)^2} = \frac{4t^2 + 8t - 2t^2}{4t^2 + 16t + 16} = \frac{t^2 + 4t}{2t^2 + 8t + 8} = \frac{t(t+4)}{2(t+2)(t+2)}$$

$$v(4) = \frac{4(4+4)}{2(4+2)(4+2)} = \frac{4(8)}{2(6)(6)}$$

$$v(4) = \frac{32}{72} = \frac{4}{9}$$

Acceleration is given by the second derivative of the position function or the first derivative of the velocity function.

$$s''(t) = v'(t) = a(t) = \frac{(2t+4)(2t^2+8t+8) - (t^2+4t)(4t+8)}{(2t^2+8t+8)^2}$$

$$a(t) = \frac{16t+32}{(2t^2+8t+8)^2} = \frac{16(t+2)}{4(t^2+4t+4)^2} = \frac{4(t+2)}{(t+2)^4} = \frac{4}{(t+2)^3}$$

$$a(4) = \frac{4}{(4+2)^3} = \frac{4}{216}$$

$$a(4) = \frac{1}{54}$$

Speed is the absolute value of velocity. So speed is

$$|v(4)| = \left|\frac{4}{9}\right| = \frac{4}{9}$$

■ 4. Let $s(t) = 2t^3 - 12t^2 + 18t + 2$ be the position of a particle. What is the velocity when acceleration is zero? What is the total distance traveled by the particle from $t = 0$ to $t = 2$?

Solution:

Take the derivative of the position function to find the velocity.

$$s(t) = 2t^3 - 12t^2 + 18t + 2$$

$$v(t) = s'(t) = 6t^2 - 24t + 18$$

Take the derivative of the velocity function to find the acceleration.

$$v(t) = 6t^2 - 24t + 18$$

$$a(t) = v'(t) = 12t - 24$$

We need to find time when acceleration is 0.

$$12t - 24 = 0$$

$$12t = 24$$

$$t = 2$$

Then we need to find the velocity when acceleration is zero, $v(2)$.

$$v(2) = 6(2)^2 - 24(2) + 18$$

$$v(2) = 6(4) - 48 + 18$$

$$v(2) = -6$$

To find the total distance traveled by the particle from $t = 0$ to $t = 2$, first we need to find when the particle is at rest.

$$v(t) = 6t^2 - 24t + 18$$

$$6t^2 - 24t + 18 = 0$$

$$6(t - 1)(t - 3) = 0$$

$t = 1$ and $t = 3$

So to find the total distance we need to find the distance from $t = 0$ to $t = 1$ and the distance from $t = 1$ to $t = 2$, then add them.

$$s(0) = 2(0)^3 - 12(0)^2 + 18(0) + 2$$

$$s(0) = 2$$

$$s(1) = 2(1)^3 - 12(1)^2 + 18(1) + 2$$

$$s(1) = 10$$

The distance from $t = 0$ to $t = 1$ is $|10 - 2| = 8$.

$$s(2) = 2(2)^3 - 12(2)^2 + 18(2) + 2$$

$$s(2) = 6$$

The distance from $t = 1$ to $t = 2$ is $|6 - 10| = |-4| = 4$. Therefore the total distance is $8 + 4 = 12$.

■ 5. The position of a particle moving along a line is given. For what values of t is the speed of the particle decreasing?

$$s(t) = \frac{4}{3}t^3 - 12t^2 + 32t - 12 \text{ for } t \geq 0$$

Solution:

Speed is decreasing when velocity and acceleration have opposite signs, such that $v(t) > 0$ with $a(t) < 0$, or $v(t) < 0$ with $a(t) > 0$.

Take the derivative of the position function to find velocity.

$$s(t) = \frac{4}{3}t^3 - 12t^2 + 32t - 12$$

$$v(t) = s'(t) = 4t^2 - 24t + 32$$

We need to find time when acceleration is 0.

$$4t^2 - 24t + 32 = 0$$

$$4(t-4)(t-2) = 0$$

$t = 2$ and $t = 4$

Using the first derivative test with test values of $t = 1$, $t = 3$, and $t = 5$, we can determine that velocity is positive to the left of $t = 2$, negative between $t = 2$ and $t = 4$, and positive again to the right of $t = 4$.

$$v(1) = 4(1)^2 - 24(1) + 32 = 12 > 0$$

$$v(3) = 4(3)^2 - 24(3) + 32 = -4 < 0$$

$$v(5) = 4(5)^2 - 24(5) + 32 = 12 > 0$$

Take the derivative of the velocity function to find the acceleration.

$$v(t) = 4t^2 - 24t + 32$$

$$a(t) = v'(t) = 8t - 24$$

We need to find time when acceleration is 0.

$$8t - 24 = 0$$

$$8t = 24$$

$$t = 3$$

Now we need to determine where acceleration is positive and negative, so we'll use test values of $t = 1$ and $t = 4$.

$$a(1) = 8(1) - 24 = -16 < 0$$

$$a(4) = 8(4) - 24 = 8 > 0$$

Therefore the speed is decreasing on $[0,2]$ and $[3,4]$.

6. A particle moves along the x-axis with its position at time t given by $s(t) = a(t + a)(t - b)$, where a and b are constants and $a \neq b$. Find the values of t when the particle is at rest.

Solution:

To find when the particle is at rest, we first find velocity.

$$s(t) = a(t + a)(t - b)$$

$$s(t) = a(t^2 - bt + at - ab)$$

$$s(t) = at^2 - abt + a^2t - a^2b$$

$$v(t) = 2at - ab + a^2$$

Set velocity equal to zero and solve for t.

$$2at - ab + a^2 = 0$$

$$a(2t - b + a) = 0$$

$$2t - b + a = 0$$

$$2t = b - a$$

$$t = \frac{b - a}{2}$$

So the particle is at rest whenever time satisfies $t = (b - a)/2$.

BALL THROWN UP FROM THE GROUND

■ 1. A ball is thrown straight upward from the ground with an initial velocity of $v_0 = 86$ ft/sec. Assuming constant gravity, find the maximum height, in feet, that the ball attains, the time, in seconds, that it's in the air, as well as the ball's velocity, in ft/sec, when it hits the ground.

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$h(t) = -\frac{1}{2}(32)t^2 + 86t + 0$$

$$h(t) = -16t^2 + 86t$$

When the ball is at its maximum height, velocity is 0, so find $h'(t)$ and set it equal to 0.

$$v(t) = h'(t) = -32t + 86$$

$$-32t + 86 = 0$$

$$32t = 86$$

$$t = \frac{43}{16} \approx 2.69 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -16\left(\frac{43}{16}\right)^2 + 86\left(\frac{43}{16}\right) = \frac{1{,}849}{16} \approx 115.56 \text{ feet}$$

To find the time the ball stays in the air, set the height equal to 0 and solve for t.

$$h(t) = -16t^2 + 86t$$

$$-16t^2 + 86t = 0$$

$$t(43 - 8t) = 0$$

$$t = 0, \frac{43}{8} \approx 5.38 \text{ seconds}$$

Now, find the final velocity of the ball when it hits the ground. Substitute the time the ball lands into the velocity function.

$$v\left(\frac{43}{8}\right) = -32\left(\frac{43}{8}\right) + 86 = -86 \text{ ft/sec}$$

■ 2. A ball is thrown straight upward from the top of a building, which is 56 feet above the ground, with an initial velocity of $v_0 = 48$ ft/sec. Assuming constant gravity, find the maximum height, in feet, that the ball attains, the time, in seconds, that it's in the air, as well as the ball's velocity, in ft/sec, when it hits the ground.

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$h(t) = -\frac{1}{2}(32)t^2 + 48t + 56$$

$$h(t) = -16t^2 + 48t + 56$$

When the ball is at its maximum height, velocity is 0, so find $h'(t)$ and set it equal to 0.

$$h'(t) = v(t) = -32t + 48$$

$$-32t + 48 = 0$$

$$32t = 48$$

$$t = \frac{3}{2} = 1.5 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -16\left(\frac{3}{2}\right)^2 + 48\left(\frac{3}{2}\right) + 56 = 92 \text{ feet}$$

To find the time the ball stays in the air, set the height equal to 0 and solve for t.

$$-16t^2 + 48t + 56 = 0$$

$$2t^2 - 6t - 7 = 0$$

$$t = \frac{3 + \sqrt{23}}{2} \approx 3.90 \text{ seconds}$$

Now, find the final velocity of the ball when it hits the ground. Substitute the time the ball lands into the velocity function.

$$v\left(\frac{3 + \sqrt{23}}{2}\right) = -32\left(\frac{3 + \sqrt{23}}{2}\right) + 48 \approx -76.73 \text{ ft/sec}$$

■ 3. A ball is thrown straight upward from a bridge, which is 24 meters above the water, with an initial velocity of $v_0 = 20$ m/sec. Assuming constant gravity, find the maximum height, in meters, that the ball attains, the time, in seconds, that it's in the air, as well as the ball's velocity, in m/sec, when it hits the water below.

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$h(t) = -\frac{1}{2}(9.8)t^2 + 20t + 24$$

$$h(t) = -4.9t^2 + 20t + 24$$

When the ball is at its maximum height, velocity is 0, so find $h'(t)$ and set it equal to 0.

$$h'(t) = v(t) = -9.8t + 20$$

$$-9.8t + 20 = 0$$

$$9.8t = 20$$

$$t = \frac{20}{9.8} \approx 2.041 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -4.9\left(\frac{100}{49}\right)^2 + 20\left(\frac{100}{49}\right) + 24 \approx 44.41 \text{ meters}$$

To find the time the ball stays in the air, set the height equal to 0 and solve for t.

$$h(t) = -4.9t^2 + 20t + 24$$

$$-4.9t^2 + 20t + 24 = 0$$

$$t \approx 5.05 \text{ seconds}$$

Now, find the final velocity of the ball when it hits the water. Substitute the time the ball lands into the velocity function.

$$v(5.05) = -9.8(5.05) + 20 \approx -29.5 \text{ m/sec}$$

■ 4. A boy needs to jump 2.8 ft in the air in order to dunk a basketball. The height that the boy's feet are above the ground is given by the function $h(t) = -16t^2 + 10t$. What is the maximum height the boy's feet will ever be above the ground, and will he be able to dunk the basketball?

Solution:

The boy's feet will reach their maximum height when velocity is 0, so find $h'(t)$ and set it equal to 0.

$$h(t) = -16t^2 + 10t$$

$$h'(t) = v(t) = -32t + 10$$

$$-32t + 10 = 0$$

$$32t = 10$$

$$t = \frac{10}{32} \approx 0.31 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -16(0.31)^2 + 10(0.31) = 1.56 \text{ ft}$$

The boy won't be able to dunk the ball because he'll never reach the 2.8 ft required.

5. A diver jumps up from a platform and then falls down into a pool. His height as a function of time can be modeled by $h(t) = -16t^2 + 12t + 60$, where t is the time in seconds and h is the height in feet. How long did it take for the diver to reach his maximum height? What was the highest point that he reached? In how many seconds does he hit the water?

Solution:

When the diver reaches his maximum height, velocity is 0, so find $h'(t)$ and set it equal to 0.

$$h(t) = -16t^2 + 12t + 60$$

$$h'(t) = v(t) = -32t + 12$$

$$-32t + 12 = 0$$

$$32t = 12$$

$$t = \frac{12}{32} \approx 0.375 \text{ seconds}$$

Next, find the maximum height.

$$h(t) = -16(0.375)^2 + 12(0.375) + 60 \approx 62.25 \text{ ft}$$

To find the time at which the diver hits the water, set the height equal to 0 and solve for t.

$$h(t) = -16t^2 + 12t + 60$$

$$-16t^2 + 12t + 60 = 0$$

$t \approx 2.35$ seconds

■ 6. An amateur rocketry club is holding a competition. There is cloud cover at 890 ft. If they launch a rocket with an initial velocity of 365 ft/s, determine the amount of time that the rocket is out of site in the cloud cover.

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$h(t) = -\frac{1}{2}(32)t^2 + 365t + 0$$

$$h(t) = -16t^2 + 365t$$

We need to find out when the rocket will reach the height of 890 ft. Substitute $h(t) = 890$ into the formula and solve for t.

$$890 = -16t^2 + 365t$$

$$-16t^2 + 365t - 890 = 0$$

$t \approx 2.78$ seconds and $t \approx 20.04$ seconds

Therefore, the time that the rocket is out of sight is $20.04 - 2.78 = 17.26$ seconds.

COIN DROPPED FROM THE ROOF

1. A rock is dropped from the top of an 800 foot tall cliff, with an initial velocity of $v_0 = 0$ ft/sec. Assuming constant gravity, when does the rock hit the ground, and what is its velocity when it hits the ground?

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(32)t^2 + 0t + 800$$

$$y(t) = -16t^2 + 800$$

The rock hits the ground when its height is 0.

$$-16t^2 + 800 = 0$$

$$16t^2 = 800$$

$$t^2 = 50$$

$t = \sqrt{50} \approx 7$ seconds

To find the velocity of the rock when it hits the ground, find $y'(t)$ and evaluate it at the time the rock hits the ground.

$$y'(t) = -32t$$

$$y'(7.071) = -32(7) \approx -224 \text{ ft/sec}$$

■ 2. A rock is tossed from the top of a 300 foot tall cliff, with an initial velocity of $v_0 = 15$ ft/sec. Assuming constant gravity, when does the rock hit the ground, and what is its velocity when it hits the ground?

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(32)t^2 + 15t + 300$$

$$y(t) = -16t^2 + 15t + 300$$

The rock hits the ground when its height is 0.

$$-16t^2 + 15t + 300 = 0$$

$$t = \frac{-15 \pm \sqrt{(-15)^2 - 4(-16)(300)}}{2(-16)} = \frac{15 \pm 5\sqrt{777}}{32} \approx 4.8242 \text{ seconds}$$

To find the velocity of the rock when it hits the ground, find $y'(t)$ and evaluate it at the time the rock hits the ground.

$$y'(t) = -32t + 15$$

$$y'(4.8242) = -32(4.8242) + 15 \approx -139.37 \text{ ft/sec}$$

■ 3. A coin is tossed downward from the top of a 36 meter tall building, with an initial velocity of $v_0 = 6$ m/sec. Assuming constant gravity, when does the rock hit the ground, and what is its velocity when it hits the ground?

Solution:

Plugging everything we know into the formula for standard projectile motion, we get

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(9.8)t^2 + 6t + 36$$

$$y(t) = -4.9t^2 + 6t + 36$$

The rock hits the ground when its height is 0.

$$-4.9t^2 + 6t + 36 = 0$$

$$t = \frac{-6 \pm \sqrt{(-6)^2 - 4(-4.9)(36)}}{2(-4.9)} = \frac{6 \pm \sqrt{741.6}}{9.8} \approx 3.391 \text{ seconds}$$

To find the velocity of the rock when it hits the ground, find $y'(t)$ and evaluate it at the time the rock hits the ground.

$$y'(t) = -9.8t + 6$$

$$y'(3.391) = -9.8(3.391) + 6 \approx -27.23 \text{ m/sec}$$

■ 4. A raindrop falls from the sky and takes 25 seconds to reach the ground. Assuming constant gravity, what is the raindrop's velocity at impact? How far did it fall? What is its acceleration when $t = 5$ seconds?

Solution:

Given the formula for standard projectile motion,

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

and the formula for velocity,

$$v(t) = x'(t) = -gt + v_0$$

we can find instantaneous velocity at $t = 25$ by substituting into the velocity function.

$$v(25) = -9.8(25) + 0$$

$$v(25) = -245 \text{ m/s}$$

Substitute what we know into the position function to determine how far the raindrop fell.

$$0 = -\frac{1}{2}(9.8)(25)^2 + (0)(15) + y_0$$

$$\frac{1}{2}(9.8)(25)^2 = y_0$$

$$y_0 = 3{,}062.5 \text{ m}$$

Acceleration is relatively constant, $a = g = 9.8$ m/s^2.

5. You throw a rock into the Grand Canyon and it takes 7.55 seconds to hit the ground. Calculate the velocity of the rock at impact in m/s and then find the distance the rock fell in feet.

Solution:

Given the formula for standard projectile motion,

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

and the formula for velocity,

$$v(t) = x'(t) = -gt + v_0$$

we can find instantaneous velocity at $t = 7.55$ by substituting into the velocity function.

$$v(7.55) = -9.8(7.55) + 0$$

$$v(7.55) = -73.99 \text{ m/s}$$

To find the distance the rock fell in feet, substitute what we know into the position formula, using $g = 32$ ft/s^2.

$$0 = -\frac{1}{2}(32)(7.55)^2 + (0)(7.55) + y_0$$

$$\frac{1}{2}(32)(7.55)^2 = y_0$$

$$y_0 = 912.04 \text{ ft}$$

■ 6. A coin is dropped into a very deep wishing well. It hits the water 4.5 s later. How far is it from the top of the well to the water at the bottom? At what velocity does the coin hit the water? How far had the coin fallen when it reached -20m/s?

Solution:

Substitute what we know into the position formula.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$0 = -\frac{1}{2}(9.8)(4.5)^2 + (0)(4.5) + y_0$$

$$\frac{1}{2}(9.8)(4.5)^2 = y_0$$

$$y_0 = 99.225 \text{ m}$$

Find the velocity function by differentiating the position function.

$$v(t) = x'(t) = -gt + v_0$$

To find instantaneous velocity at $t = 4.5$, substitute $t = 4.5$ into the velocity function.

$$v(4.5) = -9.8(4.5) + 0$$

$$v(4.5) = -44.1 \text{ m/s}$$

Now we need to find the time at which the velocity of the coin is -20 m/s.

$$v(t) = -gt + v_0$$

$$-20 = -9.8t + 0$$

$$t = \frac{20}{9.8}$$

$$t \approx 2.04 \text{ s}$$

Therefore,

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(9.8)(2.04)^2 + (0)(2.04) + 99.225$$

$$y(t) = -\frac{1}{2}(9.8)(2.04)^2 + 99.225$$

$$y(t) = 78.83 \text{ m}$$

The coin had fallen $99.225 - 78.83 = 20.39$ m when it reached -20 m/s.

MARGINAL COST, REVENUE, AND PROFIT

■ 1. A company manufactures and sells basketballs for $\$9.50$ each. The company has a fixed cost of $\$395$ per week and a variable cost of $\$2.75$ per basketball. The company can make up to 300 basketballs per week. Find the marginal cost, marginal revenue, and marginal profit, if the company makes 150 basketballs.

Solution:

The cost function is $C(x) = 395 + 2.75x$, where x is the number of basketballs, so marginal cost is $C'(x) = 2.75$, and $C'(150) = \$2.75$.

The revenue function is $R(x) = 9.50x$, where x is the number of basketballs, so marginal revenue is $R'(x) = 9.50$, and $R'(150) = \$9.50$.

The profit function is

$$P(x) = R(x) - C(x)$$

$$P(x) = 9.50x - (395 + 2.75x)$$

$$P(x) = 6.75x - 395$$

Marginal profit is $P'(x) = 6.75$, and $P'(150) = \$6.75$.

2. A company manufactures and sells high end folding tables for $\$250$ each. The company has a fixed cost of $\$3{,}000$ per week and variable costs of $85x + 150\sqrt{x}$, where x is the number of tables manufactured. The company can make up to 200 tables per week. Find the marginal cost, marginal revenue, and marginal profit, if the company makes 64 tables.

Solution:

The cost function is $C(x) = 3{,}000 + 85x + 150\sqrt{x}$, where x is the number of folding tables, so marginal cost is $C'(x) = 85 + 75/\sqrt{x}$, and $C'(64) = 85 + 75/\sqrt{64} = 85 + 9.375 = \$94.375$.

The revenue function is $R(x) = 250x$, where x is the number of folding tables, so marginal revenue is $R'(x) = 250$, and $R'(64) = \$250$.

The profit function is

$$P(x) = R(x) - C(x)$$

$$P(x) = 250x - (3{,}000 + 85x + 150\sqrt{x})$$

$$P(x) = 165x - 150\sqrt{x} - 3{,}000$$

Marginal profit is $P'(x) = 165 - 75/\sqrt{x}$, and

$$P'(64) = 165 - \frac{75}{\sqrt{64}}$$

$$P'(64) = 165 - 9.375$$

$$P'(64) = \$155.63$$

■ 3. A company manufactures and sells electric food mixers for $\$150$ each. The company has a fixed cost of $\$7{,}800$ per week and variable costs of $24x + 0.04x^2$, where x is the number of mixers manufactured. The company can make up to 200 mixers per week. Find the marginal cost, marginal revenue, and marginal profit, if the company makes 75 mixers.

Solution:

The cost function is $C(x) = 7{,}800 + 24x + 0.04x^2$, where x is the number of food mixers, so marginal cost is $C'(x) = 24 + 0.08x$, and $C'(75) = 24 + 0.08(75) = \$30$.

The revenue function is $R(x) = 150x$, where x is the number of food mixers, so marginal revenue is $R'(x) = 150$, and $R'(75) = \$150$.

The profit function is

$$P(x) = R(x) - C(x)$$

$$P(x) = 150x - (7{,}800 + 24x + 0.04x^2)$$

$$P(x) = 126x - 0.04x^2 - 7{,}800$$

Marginal profit is $P'(x) = 126 - 0.08x$, and

$$P'(75) = 126 - 0.08(75)$$

$$P'(75) = 126 - 6$$

$$P'(75) = \$120$$

■ 4. A coffee machine manufacturer determines that the demand function for their coffee machines is given by p, while the cost of producing x coffee machines is given by $C(x) = 25x + 10\sqrt{x^3} + 1{,}250$. What is the marginal cost, marginal revenue, and marginal profit at $x = 25$?

$$p = \frac{750}{\sqrt{x^3}}$$

Solution:

Find marginal cost, then evaluate the marginal cost function at $x = 25$.

$$C'(x) = 25 + \frac{15\sqrt{x^3}}{x}$$

$$C'(25) = 25 + \frac{15\sqrt{25^3}}{25} = \$100$$

Revenue is given by the product of demand and the number of units sold.

$$R(x) = x \cdot p$$

$$R(x) = \frac{750x}{\sqrt{x^3}}$$

Find marginal revenue, then evaluate the marginal revenue function at $x = 25$.

$$R'(x) = -\frac{375}{\sqrt{x^3}}$$

$$R'(25) = -\frac{375}{\sqrt{25^3}} = -\$3$$

Now find profit and marginal profit,

$$P(x) = R(x) - C(x)$$

$$P(x) = \frac{750x}{\sqrt{x^3}} - (25x + 10\sqrt{x^3} + 1{,}250)$$

$$P(x) = \frac{750x}{\sqrt{x^3}} - 10\sqrt{x^3} - 25x - 1{,}250$$

$$P'(x) = \frac{-25\sqrt{x^3} - 15x^2 - 375}{\sqrt{x^3}}$$

then evaluate the marginal profit function at $x = 25$.

$$P'(25) = -\$103$$

The coffee machine manufacturer's marginal cost, revenue, and profit at 25 units are $C'(25) = 100$, $R'(25) = -3$, and $P'(25) = -\$103$, respectively. Therefore, their marginal profit from selling the 26th coffee machine is $-\$103$, meaning that the company should not increase production if their goal is to maximize profit.

■ 5. For the given cost and demand functions, find the number of units the company needs to produce in order to maximize profit.

$$C(x) = 15x + 300$$

$$p = 2x - 250$$

Solution:

Profit will be maximized when marginal profit is 0, or when

$$P'(x) = R'(x) - C'(x)$$

$$0 = R'(x) - C'(x)$$

$$R'(x) = C'(x)$$

Find revenue.

$$R(x) = x \cdot p$$

$$R(x) = x \cdot (2x - 250)$$

$$R(x) = 2x^2 - 250x$$

Then marginal revenue is $R'(x) = 4x - 250$. Using the cost function we've been given, find marginal cost.

$$C'(x) = 15$$

Then profit is maximized when

$$R'(x) = C'(x)$$

$$4x - 250 = 15$$

$$4x = 265$$

$$x = 66.25$$

The profit function is

$$P(x) = R(x) - C(x)$$

$$P(x) = 2x^2 - 250x - (15x + 300)$$

$$P(x) = 2x^2 - 250x - 15x - 300$$

$$P(x) = 2x^2 - 265x - 300$$

Let's determine whether profit is maximized at 66 or 67 units.

$$P(66) = 2(66)^2 - 265(66) - 300$$

$$P(66) = -8{,}478$$

or

$$P(67) = 2(67)^2 - 265(67) - 300$$

$$P(67) = -8{,}477$$

Even though profit is negative for both values (the company is currently losing money), profit is maximized (they lose the least amount of money), when they produce 67 units.

■ 6. A company manufactures and sells kids' toys. The total cost of producing x toys is $C(x) = -0.3x^2 + 25x + 975$, and demand is given by $p(x) = 12 + 3x$. Calculate the marginal profit from selling the 10th toy.

Solution:

Start by finding marginal cost. Since we need to find marginal profit from selling the 10th toy, we'll use $x = 9$ throughout.

$$C(x) = -0.3x^2 + 25x + 975$$

$$C'(x) = -0.6x + 25$$

$$C'(9) = -0.6(9) + 25$$

$$C'(9) = \$19.60$$

The marginal cost of producing the 10th toy is $\$19.60$.

Now find marginal revenue. Since $R(x) = x \cdot p$ and $p = 12 + 3x$, then $R(x) = 12x + 3x^2$. Therefore, $R'(x) = 12 + 6x$.

$$R'(9) = 12 + 6(9)$$

$$R'(9) = 12 + 54$$

$$R'(9) = \$66$$

The marginal revenue from selling the 10th toy is $\$66$.

Now find the marginal profit as the difference of these.

$$P'(x) = R'(x) - C'(x)$$

$$P'(9) = R'(9) - C'(9)$$

$$P'(9) = \$46.40$$

The marginal profit from selling the 10th toy is $\$46.40$.

HALF LIFE

■ 1. Find the half-life of Tritium if its decay constant is 0.0562.

Solution:

Since we're calculating half-life, the exponential decay formula $y = Ce^{-kt}$ can be simplified to

$$\frac{1}{2} = e^{-kt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.0562t}$$

$$\ln\frac{1}{2} = \ln e^{-0.0562t}$$

$$-\ln 2 = -0.0562t$$

$$t = \frac{-\ln 2}{-0.0562} = \frac{\ln 2}{0.0562} \approx 12.33 \text{ years}$$

■ 2. Find the half-life of Cobalt-60 if its decay constant is 0.1315.

Solution:

Since we're calculating half-life, the exponential decay formula $y = Ce^{-kt}$ can be simplified to

$$\frac{1}{2} = e^{-kt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.1315t}$$

$$\ln\frac{1}{2} = \ln e^{-0.1315t}$$

$$-\ln 2 = -0.1315t$$

$$t = \frac{-\ln 2}{-0.1315} = \frac{\ln 2}{0.1315} \approx 5.27 \text{ years}$$

■ 3. Find the half-life of Berkelium-97 if its decay constant is 0.000503.

Solution:

Since we're calculating half-life, the exponential decay formula $y = Ce^{-kt}$ can be simplified to

$$\frac{1}{2} = e^{-kt}$$

Plugging in what we know, we find that half life is

$$\frac{1}{2} = e^{-0.000503t}$$

$$\ln\frac{1}{2} = \ln e^{-0.000503t}$$

$$-\ln 2 = -0.000503t$$

$$t = \frac{-\ln 2}{-0.000503} = \frac{\ln 2}{0.000503} \approx 1{,}378 \text{ years}$$

■ 4. Radium-224 has a half life of 3.66 days. If 3.25 g of Radium-224 remains after 9 days, what was the original mass of Radium-224?

Solution:

Substitute $t = 3.66$ into the half life formula,

$$\frac{1}{2} = e^{-kt}$$

$$\frac{1}{2} = e^{-3.66k}$$

then apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2} = \ln(e^{-3.66k})$$

$$\ln\frac{1}{2}=-3.66k$$

$$k=-\frac{1}{3.66}\ln\frac{1}{2}$$

With a value for k and $t=9$ days and $y=3.25$, we can now solve for the original amount of the substance, C.

$$y=Ce^{-kt}$$

$$3.25=Ce^{-\left(-\frac{1}{3.66}\ln\frac{1}{2}\right)9}$$

$$3.25=Ce^{\frac{9}{3.66}\ln\frac{1}{2}}$$

Solve for C.

$$C=\frac{3.25}{e^{\frac{9}{3.66}\ln\frac{1}{2}}}$$

$$C=\frac{3.25}{e^{\ln\left(\frac{1}{2}\right)^{\frac{9}{3.66}}}}$$

$$C=\frac{3.25}{\left(\frac{1}{2}\right)^{\frac{9}{3.66}}}$$

$C=17.87$ g

The original amount of Radium-224 was 17.87 g

5. The half-life of Potassium-40 is 1.25 billion years. A scientist analyzes a rock that contains only $9.5\,\%$ of the Potassium-40 it contained originally when the rock was formed. How old is the rock?

Solution:

We don't know the original mass of the Potassium-40, but regardless of the size of the mass, we can say that the starting amount was $100\,\%$ of the mass.

Substitute $t = 1.25$ billion years into the half life equation,

$$\frac{1}{2} = e^{-kt}$$

$$\frac{1}{2} = e^{-1.25k}$$

then apply the natural logarithm to both sides in order to solve for k.

$$\ln\frac{1}{2} = \ln(e^{-1.25k})$$

$$\ln\frac{1}{2} = -1.25k$$

$$k = -\frac{1}{1.25}\ln\frac{1}{2}$$

With a value for k, we can now solve for the number of years it'll take for the substance to decay from $100\,\%$ to $9.5\,\%$ of its original mass. We'll substitute $C = 1$ and $y = 0.095$, along with the value we've just found for the decay constant k.

$$y = Ce^{-kt}$$

$$0.095 = 1e^{-\left(-\frac{1}{1.25}\ln\frac{1}{2}\right)t}$$

$$0.095 = e^{\left(\frac{1}{1.25}\ln\frac{1}{2}\right)t}$$

Apply the natural logarithm to both sides in order to solve for t.

$$\ln 0.095 = \ln\left(e^{\left(\frac{1}{1.25}\ln\frac{1}{2}\right)t}\right)$$

$$\ln 0.095 = \left(\frac{1}{1.25}\ln\frac{1}{2}\right)t$$

$$1.25\ln 0.095 = \left(\ln\frac{1}{2}\right)t$$

$$t = \frac{1.25\ln 0.095}{\ln\frac{1}{2}}$$

$t \approx 4.24$ billion years

It would take about 4.24 billion years for the Potassium-40 to decay to 9.5% of its original amount, so the scientist concludes that the rock is 4.24 billion years old.

■ 6. 25 grams of a substance decayed to 13.25 grams in 13 seconds. Determine the half-life of a substance.

Solution:

Substitute $y = 13.25$, $C = 25$, and $t = 13$ into the exponential decay formula,

$$y = Ce^{-kt}$$

$$13.25 = 25e^{-13k}$$

$$0.53 = e^{-13k}$$

then apply the natural logarithm to both sides in order to solve for k.

$$\ln 0.53 = -13k$$

$$k = -\frac{\ln 0.53}{13}$$

$$k \approx 0.0488$$

Now that we have the decay constant, we can substitute it into the half-life formula to find t.

$$\frac{1}{2} = e^{-kt}$$

$$\ln \frac{1}{2} = \ln e^{-0.0488t}$$

$$-\ln 2 = -0.0488t$$

$$t = \frac{-\ln 2}{-0.0488} = \frac{\ln 2}{0.0488} \approx 14.2038 \text{ seconds}$$

NEWTON'S LAW OF COOLING

1. A cup of coffee is 195° F when it's brewed. Room temperature is 74° F. If the coffee is 180° F after 5 minutes, to the nearest degree, how hot is the coffee after 25 minutes?

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T(5) - T_a = (T_0 - T_a)e^{-kt}$$

$$180 - 74 = (195 - 74)e^{-5k}$$

$$106 = 121e^{-5k}$$

$$\frac{106}{121} = e^{-5k}$$

$$\ln\frac{106}{121} = \ln e^{-5k}$$

$$\ln\frac{106}{121} = -5k$$

$$k = -\frac{1}{5}\ln\frac{106}{121} \approx 0.02647$$

Then use k to solve for $T(25)$.

$$T(25) - 74 = (195 - 74)e^{-0.02647(25)}$$

$$T(25) - 74 = 121e^{-0.66175}$$

$$T(25) - 74 = 62.42966$$

$$T(25) = 136.42966$$

The coffee is approximately 136° F after 25 minutes.

■ 2. A boiled egg that's 99° C is placed in a pan of water that's 24° C. If the egg is 62° C after 5 minutes, how much longer, to the nearest minute, will it take the egg to reach 32° C.

Solution:

Use the information given and the temperature after 5 minutes to solve for k in the Newton's Law of Cooling formula.

$$T(5) - T_a = (T_0 - T_a)e^{-kt}$$

$$62 - 24 = (99 - 24)e^{-5k}$$

$$38 = 75e^{-5k}$$

$$\frac{38}{75} = e^{-5k}$$

$$\ln\frac{38}{75} = \ln e^{-5k}$$

$$\ln \frac{38}{75} = -5k$$

$$k = -\frac{1}{5} \ln \frac{38}{75} \approx 0.13598$$

Then use k to solve for t.

$$32 - 24 = (62 - 24)e^{-0.13598t}$$

$$8 = 38e^{-0.13598t}$$

$$\frac{8}{38} = e^{-0.13598t}$$

$$\ln \frac{8}{38} = \ln e^{-0.13598t}$$

$$\ln \frac{8}{38} = -0.13598t$$

$$t = \frac{1}{-0.13598} \ln \frac{8}{38} \approx 11.4586$$

The egg will be 32° C after about 11 and a half more minutes.

■ 3. Suppose a cup of soup cooled from 200° F to 161° F in 10 minutes in a room whose temperature is 68° F. How much longer will it take for the soup to cool to 105° F?

Solution:

Use the information given and the temperature after 10 minutes to solve for k in the Newton's Law of Cooling formula.

$$T(10) - T_a = (T_0 - T_a)e^{-kt}$$

$$161 - 68 = (200 - 68)e^{-10k}$$

$$93 = 132e^{-10k}$$

$$\frac{93}{132} = e^{-10k}$$

$$\ln\frac{93}{132} = \ln e^{-10k}$$

$$\ln\frac{93}{132} = -10k$$

$$k = -\frac{1}{10}\ln\frac{93}{132} \approx 0.03502$$

Then use k to solve for t.

$$105 - 68 = (161 - 68)e^{-0.03502t}$$

$$37 = 93e^{-0.03502t}$$

$$\frac{37}{93} = e^{-0.03502t}$$

$$\ln\frac{37}{93} = \ln e^{-0.03502t}$$

$$\ln\frac{37}{93} = -0.03502t$$

$$t = \frac{1}{-0.03502} \ln \frac{37}{93} \approx 26.3187$$

The egg will be 105° F after about 26 more minutes.

■ 4. A thermometer is measuring 18° C indoors. The thermometer is moved outdoors where the temperature is -5° C, and after 2 minutes the thermometer reads 11° C. How many more minutes will it take for the thermometer to read 0° C?

Solution:

Substitute what we've been given into the Newton's Law of Cooling formula, then solve for k.

$$T(2) - T_a = (T_0 - T_a)e^{-kt}$$

$$11 - (-5) = (18 - (-5))e^{-2k}$$

$$16 = 23e^{-2k}$$

$$\frac{16}{23} = e^{-2k}$$

$$\ln \frac{16}{23} = -2k$$

$$k = -\frac{1}{2} \ln \frac{16}{23} \approx 0.18145$$

Now we'll determine how much longer it'll take for the thermometer to read 0° C. T_0 is now equal to $11^\circ C$.

$$T - T_a = (T_0 - T_a)e^{-kt}$$

$$0 - (-5) = (11 - (-5))e^{-0.18145t}$$

$$5 = 16e^{-0.18145t}$$

$$\frac{5}{16} = e^{-0.18145t}$$

$$\ln\frac{5}{16} = -0.18145t$$

$$t = \frac{1}{-0.18145}\ln\frac{5}{16} \approx 6.41031$$

The thermometer will read 0° C after about 6 more minutes.

■ 5. A cake baking inside an oven currently has a temperature of 220° C. Find the decay constant if the cake's temperature is 168° C 5 minutes after it's removed from the oven, given that the room temperature is 23° C.

Solution:

Substitute what we've been given into the Newton's Law of Cooling formula, then solve for k.

$$T(5) - T_a = (T_0 - T_a)e^{-kt}$$

$$168 - 23 = (220 - 23)e^{-5k}$$

$$145 = 197e^{-5k}$$

$$\frac{145}{197} = e^{-5k}$$

$$\ln \frac{145}{197} = -5k$$

$$k = -\frac{1}{5} \ln \frac{145}{197} \approx 0.06129$$

■ 6. Using the decay constant we calculated in the previous problem, determine the number of minutes that will pass before the cake's temperature will be 50° C.

Solution:

With T_0 now equal to 168° C, we'll calculate time in minutes.

$$T - T_a = (T_0 - T_a)e^{-kt}$$

$$50 - 23 = (168 - 23)e^{-0.06129t}$$

$$27 = 145e^{-0.06129t}$$

$$\frac{27}{145} = e^{-0.06129t}$$

$$\ln \frac{27}{145} = -0.06129t$$

$$t = \frac{1}{-0.06129} \ln \frac{27}{145} \approx 27.4253$$

The cake's temperature will be 50° C after about 27 more minutes.

SALES DECLINE

■ 1. Suppose a pizza company stops a special sale for their three-topping pizza. They will resume the sale if sales drop to 70% of the current sales level. If sales decline to 90% during the first week, when should the company expect to start the special sale again?

Solution:

Use the exponential function $F = Pe^{-rt}$. Plug in what we know.

$$F = Pe^{-rt}$$

$$90 = 100e^{-r(1)}$$

$$\frac{90}{100} = e^{-r}$$

$$\ln \frac{90}{100} = \ln e^{-r}$$

$$\ln 90 - \ln 100 = -r$$

$$r = \ln 100 - \ln 90 \approx 0.10536$$

Find t using a sales level of 70% and $r = 0.10536$.

$$70 = 100e^{-0.10536t}$$

$$\frac{70}{100} = e^{-0.10536t}$$

$$\ln \frac{70}{100} = \ln e^{-0.10536t}$$

$$\ln 70 - \ln 100 = -0.10536t$$

$$t = \frac{\ln 70 - \ln 100}{-0.10536} \approx 3.385$$

Since time t is in weeks, this means the company should expect to start the sale again in about 3 and a half weeks.

■ 2. Suppose a donut store experiments with raising the price of a dozen donuts to see if sales are affected. They'll resume the sale if sales drop to 80% of the current sales level. If sales decline to 90% after two weeks, when should the store change back to the original price?

Solution:

Use the exponential function $F = Pe^{-rt}$. Plug in what we know.

$$F = Pe^{-rt}$$

$$90 = 100e^{-r(2)}$$

$$\frac{90}{100} = e^{-2r}$$

$$\ln \frac{90}{100} = \ln e^{-2r}$$

$$\ln 90 - \ln 100 = -2r$$

$$r = \frac{\ln 90 - \ln 100}{-2} \approx 0.05268$$

Find t using a sales level of 80% and $r = 0.05268$.

$$80 = 100e^{-0.05268t}$$

$$\frac{80}{100} = e^{-0.05268t}$$

$$\ln \frac{80}{100} = \ln e^{-0.05268t}$$

$$\ln 80 - \ln 100 = -0.05268t$$

$$t = \frac{\ln 80 - \ln 100}{-0.05268} \approx 4.2358$$

Since time t is in weeks, this means the store should expect to change back to the original price in about 4 and a quarter weeks.

■ 3. Suppose a flower shop decides to stop ordering roses in the winter time to see if sales are affected. They will resume the sale if sales drop to 90% of the current sales level. If sales decline to 96% after three weeks, when should the shop begin ordering roses again?

Solution:

Use the exponential function $F = Pe^{-rt}$. Plug in what we know.

$$F = Pe^{-rt}$$

$$96 = 100e^{-r(3)}$$

$$\frac{96}{100} = e^{-3r}$$

$$\ln \frac{96}{100} = \ln e^{-3r}$$

$$\ln 96 - \ln 100 = -3r$$

$$r = \frac{\ln 96 - \ln 100}{-3} \approx 0.01361$$

Find t using a sales level of 90% and $r = 0.01361$.

$$90 = 100e^{-0.01361t}$$

$$\frac{90}{100} = e^{-0.01361t}$$

$$\ln \frac{90}{100} = \ln e^{-0.01361t}$$

$$\ln 90 - \ln 100 = -0.01361t$$

$$t = \frac{\ln 90 - \ln 100}{-0.01361} \approx 7.7414$$

Since time t is in weeks, this means the store should begin ordering roses again in about 7 and three-quarter weeks.

■ 4. Mark has been selling lemonade for the last 5 years. Five years ago, he sold $3{,}850$ glasses of lemonade, but this year he's only sold $2{,}985$. Assuming that sales have declined exponentially, what's been the annual rate of decline?

Solution:

Both the sales decline and time have units in years, so with matching units we can plug directly into the sales decline formula to find the rate of decline.

$$F = Pe^{-rt}$$

$$2{,}985 = 3{,}850e^{-5r}$$

$$0.77532 = e^{-5r}$$

Apply the natural log to both sides.

$$\ln(0.77532) = \ln(e^{-5r})$$

$$\ln(0.77532) = -5r$$

$$r = \frac{\ln(0.77532)}{-5}$$

$$r = 0.0509 = 5.09\,\%$$

Over the last 5 years, sales declined by $5.09\,\%$ per year.

■ 5. A bakery sold $5{,}465$ croissants 3 years ago. If the sales declined at a rate of $1.5\,\%$ per month, how many croissants were sold last year?

Solution:

Since the rate of decline and time have different units, let's convert the monthly rate to an annual rate by multiplying the monthly rate by 12.

$$r = 1.5\% \cdot 12 = 18\,\%$$

Substitute what we know into the sales decline formula.

$$F = Pe^{-rt}$$

$$F = 5{,}465e^{-0.18(3)}$$

$$F \approx 3{,}185$$

About $3{,}185$ croissants were sold last year.

■ 6. Suppose a convenience store decides to stop their sale on ice cream in the summer time to see if sales are affected. They will resume the sale if sales drop to $87\,\%$ of their current level. If sales fall to $87\,\%$ of their current level after two weeks, what was the monthly rate of decline?

Solution:

Since we have to find the monthly rate of decline, we need to convert the given time from weeks to months. Two weeks is half a month.

$$t = 2/4 = 0.5$$

Substitute what we know into the sales decline formula.

$$F = Pe^{-rt}$$

$$87 = 100e^{-0.5r}$$

$$0.87 = e^{-0.5r}$$

Apply the natural log to both sides.

$$\ln(0.87) = \ln(e^{-0.5r})$$

$$\ln(0.87) = -0.5r$$

$$r = \frac{\ln(0.87)}{-0.5}$$

$$r \approx 0.2785$$

The rate of decline in sales was $27.85\,\%$.

COMPOUNDING INTEREST

■ 1. Suppose you borrow $\$15{,}000$ with a single payment loan, payable in 2 years, with interest growing exponentially at 1.82% per month, compounded continuously. How much will it cost to pay off the loan after 2 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of months, we'll convert 2 years into 24 months for time t.

$$A = Pe^{rt}$$

$$A = 15{,}000e^{0.0182(24)}$$

$$A = \$23{,}216.20$$

■ 2. Your parents deposit $\$5{,}000$ into a college savings account, with interest growing exponentially at 0.875% per quarter, compounded continuously. How much will be in the account after 18 years?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of quarters, we'll convert 18 years into 72 quarters for time t.

$$A = Pe^{rt}$$

$$A = 5{,}000e^{0.00875(72)}$$

$$A = \$9{,}388.05$$

3. Suppose you win $\$50{,}000$ in a contest and you decide to save it for your retirement. You deposit it into an annuity account that pays $2.4\,\%$ semi-annually, compounded continuously. How much will the account contain after 25 years, when you plan to retire?

Solution:

Plug everything you know into the formula for future value with continuous compounding. Since the given rate is in terms of half-years, we'll convert 25 years into 50 half-years for time t.

$$A = Pe^{rt}$$

$$A = 50{,}000e^{0.024(50)}$$

$$A = \$166{,}005.85$$

4. At a 7.5% interest rate compounded semi-annually, how much money would we have to deposit now to have $\$15{,}500$ after 10 years?

Solution:

Using the compound interest formula, plug what we know. Since the interest is compounded semi-annually, $n = 2$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$15{,}500 = P\left(1 + \frac{0.075}{2}\right)^{2(10)}$$

$$15{,}500 = P\left(1 + \frac{0.075}{2}\right)^{20}$$

$$15{,}500 = P\left(1 + \frac{0.075}{2}\right)^{20}$$

$$P = \frac{15{,}500}{\left(1 + \frac{0.075}{2}\right)^{20}}$$

$$P \approx \$7{,}422.84$$

5. How many years would it take for $\$25{,}000$ to turn into $\$50{,}000$, at an interest rate of 4.75%, compounded annually?

Solution:

Using the compound interest formula, plug in what we know. Since the interest is compounded annually, $n = 1$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$50{,}000 = 25{,}000\left(1 + \frac{0.0475}{1}\right)^{1(t)}$$

$$2 = (1 + 0.0475)^t$$

$$2 = 1.0475^t$$

Apply the natural log to both sides.

$$\ln(2) = \ln(1.0475^t)$$

$$\ln(2) = t\ln(1.0475)$$

$$t = \frac{\ln(2)}{\ln(1.0475)}$$

$$t \approx 14.94$$

It would take almost 15 years for the initial amount of $\$25{,}000$ to grow to $\$50{,}000$, at a interest rate of $4.75\,\%$, compounded annually.

■ 6. At $6.5\,\%$ interest compounded quarterly, how long would it take to triple an initial investment of $\$10{,}000$?

Solution:

Using the compound interest formula, plug in what we know. Since interest is compounded quarterly, $n = 4$. Also, if we want to triple the initial investment, we're looking for a final balance of $\$30{,}000$.

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

$$30{,}000 = 10{,}000\left(1 + \frac{0.065}{4}\right)^{4t}$$

$$3 = (1 + 0.01625)^{4t}$$

$$3 = (1.01625)^{4t}$$

Apply the natural log to both sides.

$$\ln(3) = \ln(1.01625^{4t})$$

$$\ln(3) = 4t\ln(1.01625)$$

$$4t = \frac{\ln(3)}{\ln(1.01625)}$$

$$4t \approx 68.15$$

$$t \approx 17.04$$

It will take about 17 years to triple the initial investment from $\$10,000$ to $\$30,000$, when the $6.5\,\%$ interest rate is compounded quarterly.

W W W . K R I S T A K I N G M A T H . C O M