Calculus 3 Workbook Solutions

Approximating double integrals

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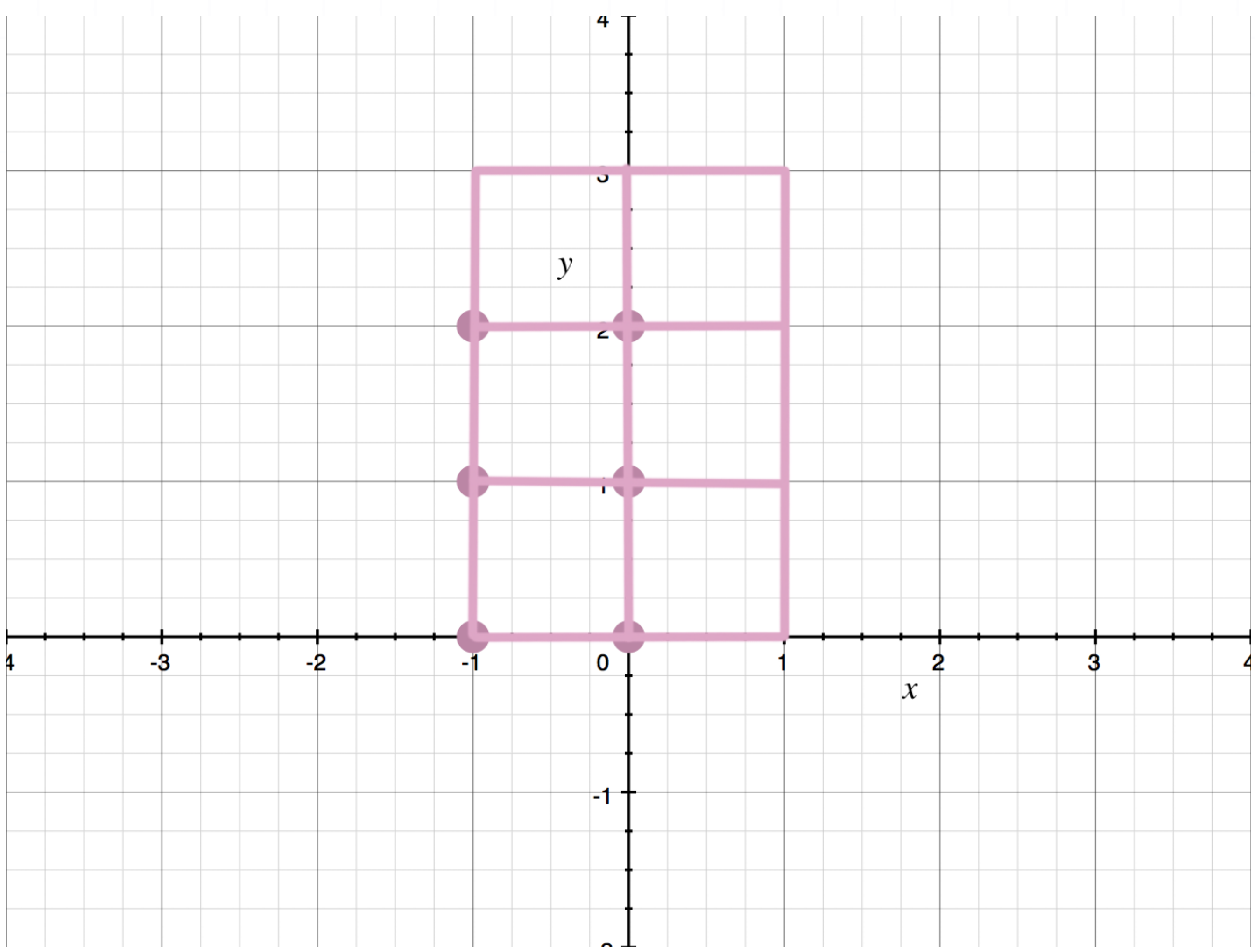
APPROXIMATING DOUBLE INTEGRALS WITH RECTANGLES

■ 1. Estimate the volume between the surface $z = 3(x - 2)e^{y-2}$ and the xy-plane, on the region $-1 \leq x \leq 1$ and $0 \leq y \leq 3$. Use lower-left corners and 1×1 squares, then 0.5×0.5. Round the answers for volume to the nearest tenth. Which square size gives the better approximation if exact volume is 31.

Solution:

Sketch the region, including the 1×1 squares and the lower-left corners.



The lower-left corners are at $(-1,0)$, $(0,0)$, $(-1,1)$, $(0,1)$, $(-1,2)$, and $(0,2)$, so we'll evaluate z at these points.

$$z(-1,0)=3((-1)-2)e^{0-2}\approx-1.22$$

$$z(0,0)=3((0)-2)e^{0-2}\approx-0.81$$

$$z(-1,1)=3((-1)-2)e^{1-2}\approx-3.31$$

$$z(0,1)=3((0)-2)e^{1-2}\approx-2.21$$

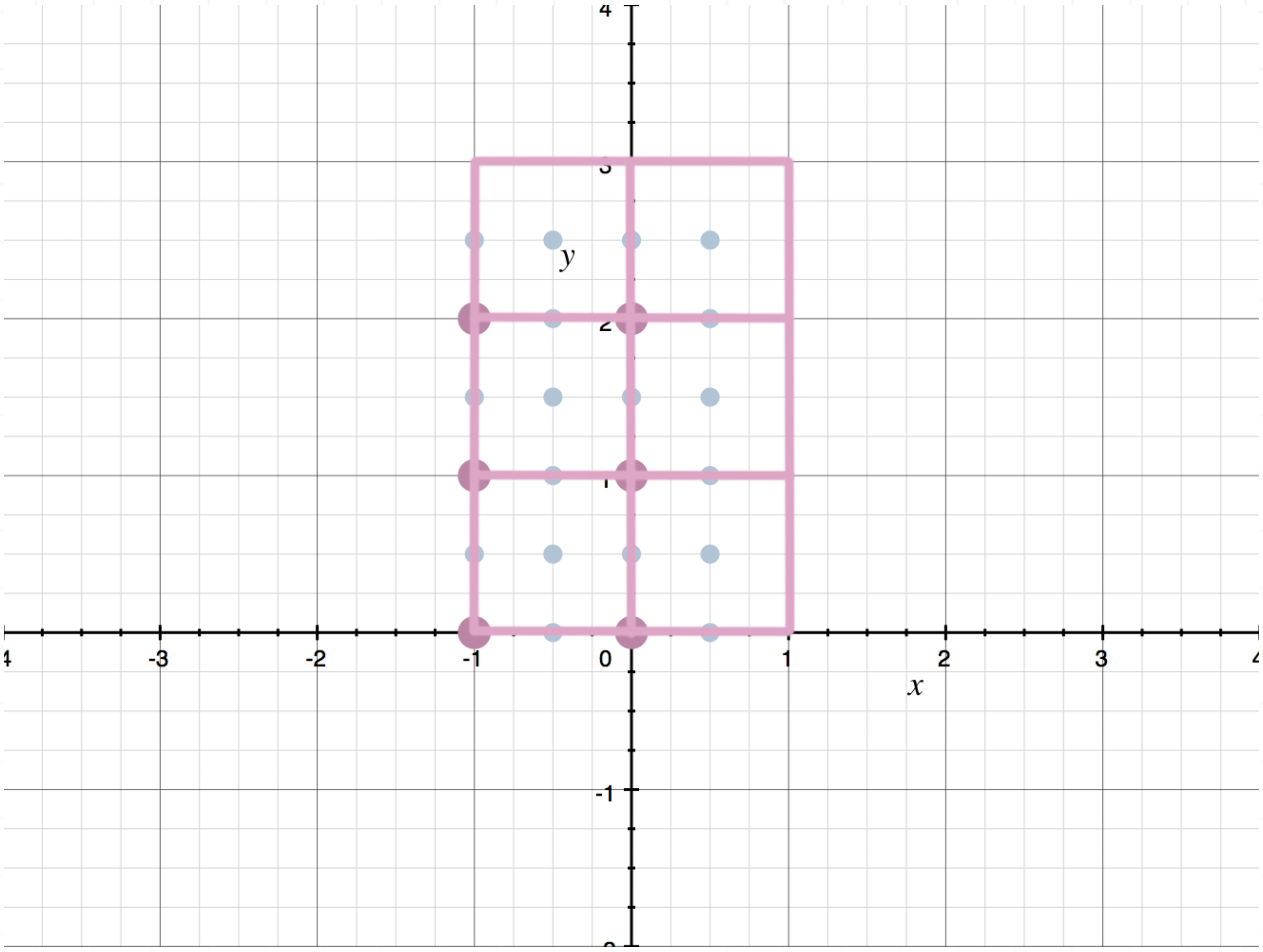
$$z(-1,2)=3((-1)-2)e^{2-2}\approx-9.00$$

$$z(0,2)=3((0)-2)e^{2-2}\approx-6.00$$

The area of a 1×1 square is $A=1$, so the volume approximation is

$$1\cdot|-1.22-0.81-3.31-2.21-9-6|\approx22.6$$

If we use 0.5×0.5 squares, we'll need all the points from the 1×1 squares, as well as these new points in blue:



The values of z at these new blue points are

$$z(-0.5,0) = 3((-0.5) - 2)e^{0-2} \approx -1.02$$

$$z(0.5,0) = 3((0.5) - 2)e^{0-2} \approx -0.61$$

$$z(-1,0.5) = 3((-1) - 2)e^{0.5-2} \approx -2.01$$

$$z(-0.5,0.5) = 3((-0.5) - 2)e^{0.5-2} \approx -1.67$$

$$z(0,0.5) = 3((0) - 2)e^{0.5-2} \approx -1.34$$

$$z(0.5,0.5) = 3((0.5) - 2)e^{0.5-2} \approx -1$$

$$z(-0.5,1) = 3((-0.5) - 2)e^{1-2} \approx -2.76$$

$$z(0.5,1) = 3((0.5) - 2)e^{1-2} \approx -1.66$$

$$z(-1,1.5) = 3((-1) - 2)e^{1.5-2} \approx -5.46$$

$$z(-0.5,1.5) = 3((-0.5) - 2)e^{1.5-2} \approx -4.55$$

$$z(0,1.5) = 3((0) - 2)e^{1.5-2} \approx -3.64$$

$$z(0.5,1.5) = 3((0.5) - 2)e^{1.5-2} \approx -2.73$$

$$z(-0.5,2) = 3((-0.5) - 2)e^{2-2} \approx -7.5$$

$$z(0.5,2) = 3((0.5) - 2)e^{2-2} \approx -4.5$$

$$z(-1,2.5) = 3((-1) - 2)e^{2.5-2} \approx -14.84$$

$$z(-0.5,2.5) = 3((-0.5) - 2)e^{2.5-2} \approx -12.37$$

$$z(0,2.5) = 3((0) - 2)e^{2.5-2} \approx -9.89$$

$$z(0.5,2.5) = 3((0.5) - 2)e^{2.5-2} \approx -7.42$$

The area of a 0.5×0.5 square is $A = 0.25$, so the volume approximation, using these points and the ones from the 1×1 calculation, is

$$0.25 \cdot |-1.22 - 0.81 - 3.31 - 2.21 - 9 - 6 - 1.02 - 0.61$$

$$-2.01 - 1.67 - 1.34 - 1 - 2.76 - 1.66 - 5.46 - 4.55$$

$$-3.64 - 2.73 - 7.5 - 4.5 - 14.84 - 12.37 - 9.89 - 7.42| \approx 26.9$$

The exact area is 31, the 0.5×0.5 squares approximate the volume at 26.9, and the 1×1 squares approximate the volume at 22.6, so the 0.5×0.5 squares give a more accurate approximation.

■ 2. Estimate the volume below the surface $z = 3 + x + y^2$, over the triangular region bounded by the x- and y-axes and the line $2x + 3y = 6$. Use lower-left corners and 1×1 squares. For squares that lie partially within the region, divide area by 2.

Solution:

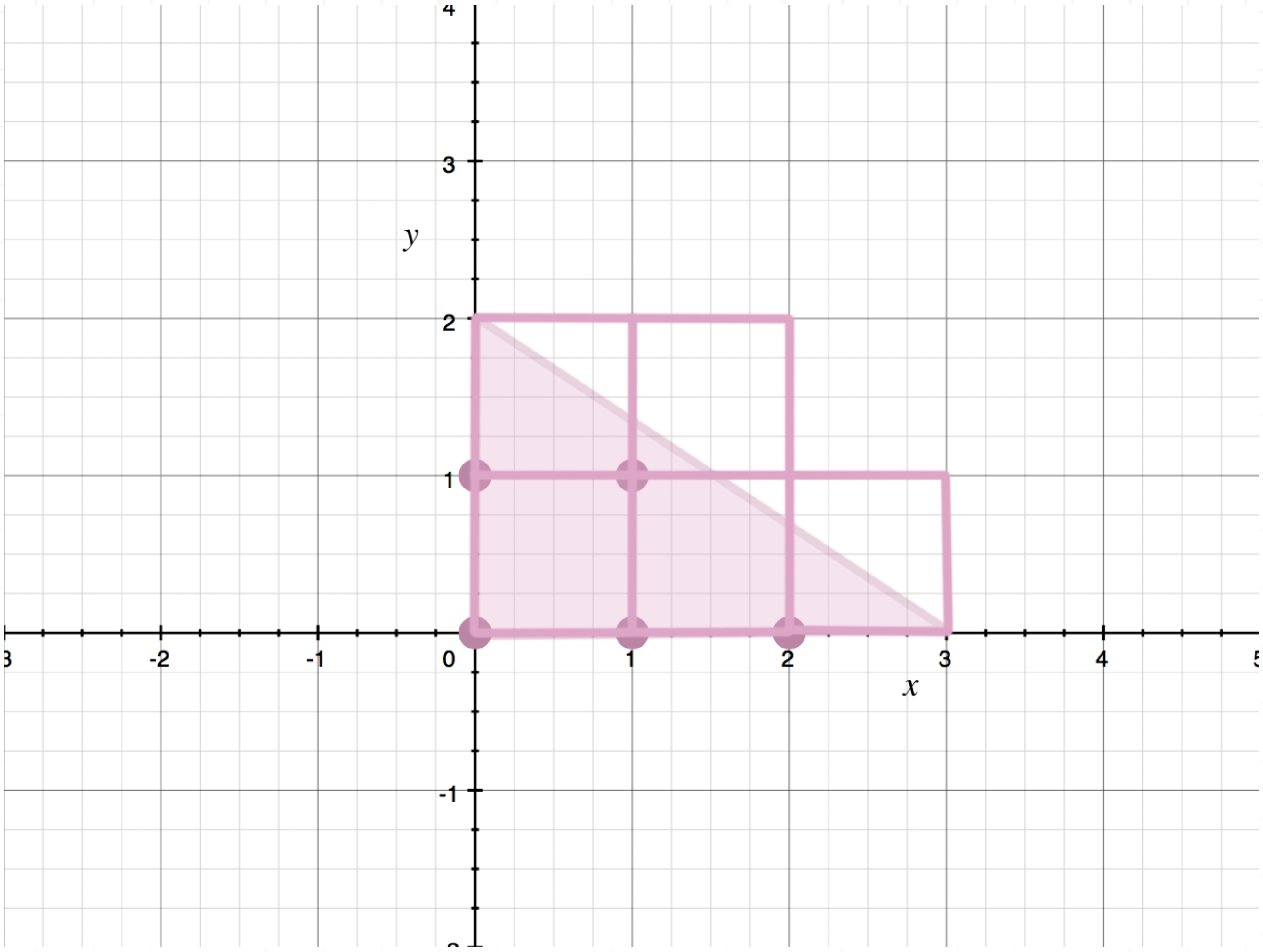
Rewrite the line in slope-intercept form.

$$2x + 3y = 6$$

$$3y = -2x + 6$$

$$y = -\frac{2}{3}x + 2$$

The line intersects the major axes at $(3,0)$ and $(0,2)$. If we sketch the triangular region, the 1×1 squares, and the lower-left corners, we get



Evaluate z at each lower-left point.

$$z(0,0) = 3 + 0 + 0^2 = 3$$

$$z(1,0) = 3 + 1 + 0^2 = 4$$

$$z(2,0) = 3 + 2 + 0^2 = 5$$

$$z(0,1) = 3 + 0 + 1^2 = 4$$

$$z(1,1) = 3 + 1 + 1^2 = 5$$

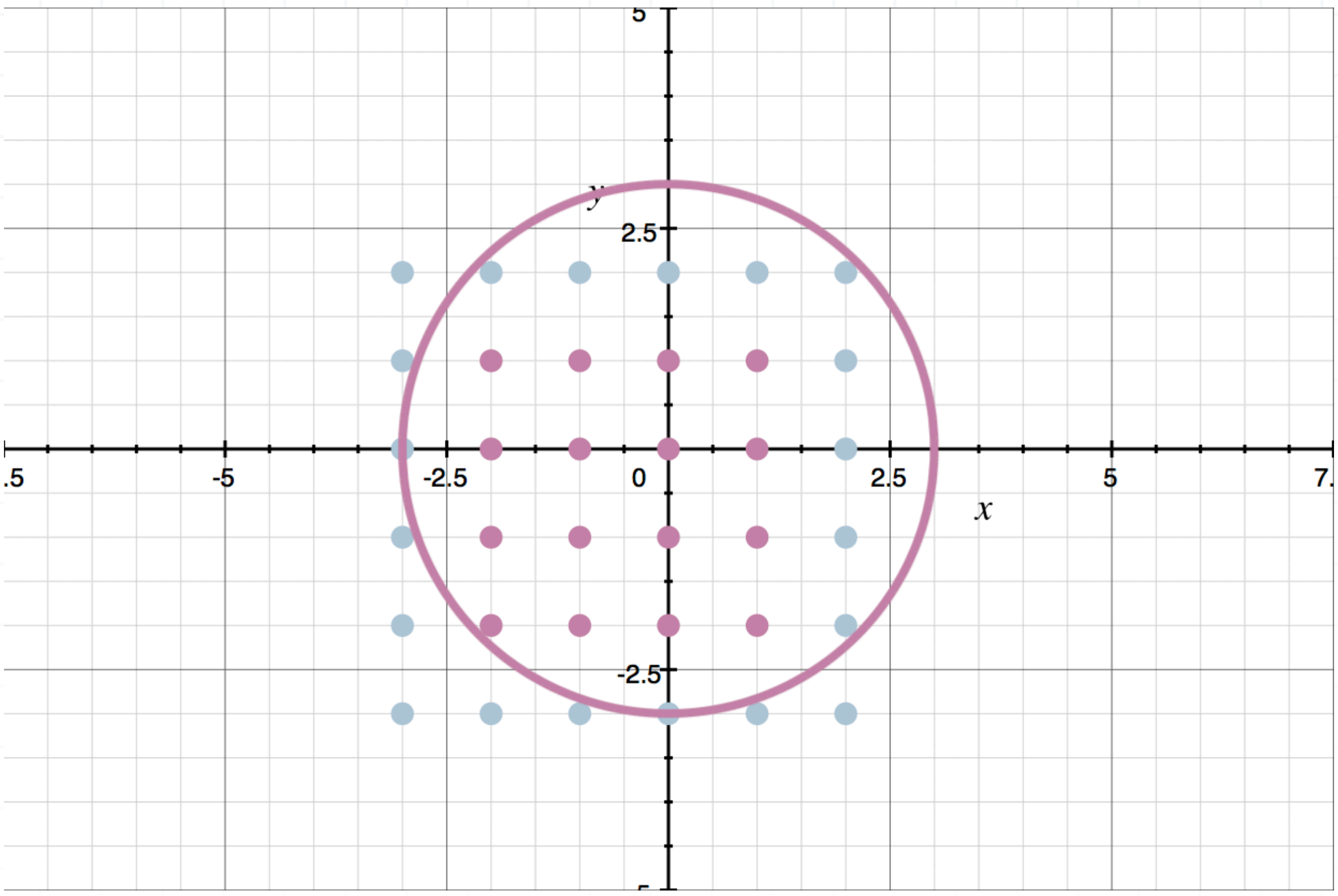
The only square that lies fully within the region is the square associated with $(0,0)$. The squares associated with $(1,0)$, $(2,0)$, $(0,1)$, and $(1,1)$ lie partially within the region, so we'll multiply the sum of those areas by 0.5. The area of a 1×1 square is $A = 1$, so the volume approximation is

$$1 \cdot |3 + 0.5(4 + 5 + 4 + 5)| = 12$$

■ 3. Assume the base of a right circular cylinder with radius 3 and height 5 lies in the xy-plane with its center at the origin. Use 1×1 squares and lower-left corners to estimate the volume of the cylinder. If an approximating square lies only partially within the circle, divide its area in half as part of the estimation. Calculate exact volume of the cylinder using $V = \pi r^2 h$, then find the percentage error of the approximation.

Solution:

Sketch the base of the cylinder in the xy-plane, along with the lower-left points of each square. Purple points represent squares that lie entirely within the region, while blue points represent squares that lie only partially within the region.



Because the cylinder has height 5, the value of z at every point is $z = 5$. There are 16 purple points and 20 blue points (whose area should be divided in half), so the volume is approximated by

$$5 \cdot |16 + 0.5(20)| = 130$$

The cylinder's exact volume is

$$V = \pi r^2 h$$

$$V = \pi \cdot 3^2 \cdot 5$$

$$V = 45\pi$$

$$V \approx 141.4$$

Then the percentage error in the approximation is

$$E_P = \frac{|\text{Exact volume} - \text{Approximate volume}|}{\text{Exact volume}} \cdot 100\,\%$$

$$E_P = \frac{|45\pi - 130|}{45\pi} \cdot 100\,\%$$

$$E_P \approx 8.04\,\%$$

MIDPOINT RULE FOR DOUBLE INTEGRALS

■ 1. Assuming the integral's exact value is 12, say which estimation is more accurate if we estimate volume of the integral below using Midpoint Rule and rectangles of dimensions $\pi/2 \times 1$, and then rectangles of dimensions $\pi/3 \times 2/3$.

$$\int_0^\pi \int_{-1}^1 (y+3)\sin x \ dy \ dx$$

Solution:

The region of integration given by the bounds of the double integral is $x = [0,\pi]$ and $y = [-1,1]$. If we use rectangles of dimensions $\pi/2 \times 1$, then the midpoints of the rectangles will be

$$\left(\frac{\pi}{4}, -0.5\right), \left(\frac{3\pi}{4}, -0.5\right), \left(\frac{\pi}{4}, 0.5\right), \left(\frac{3\pi}{4}, 0.5\right)$$

Evaluate $z = (y+3)\sin x$ at these four midpoints.

$$z\left(\frac{\pi}{4}, -0.5\right) = (-0.5+3)\sin\left(\frac{\pi}{4}\right) = \frac{5\sqrt{2}}{4}$$

$$z\left(\frac{3\pi}{4}, -0.5\right) = (-0.5+3)\sin\left(\frac{3\pi}{4}\right) = \frac{5\sqrt{2}}{4}$$

$$z\left(\frac{\pi}{4},0.5\right)=(0.5+3)\sin\left(\frac{\pi}{4}\right)=\frac{7\sqrt{2}}{4}$$

$$z\left(\frac{3\pi}{4},0.5\right)=(0.5+3)\sin\left(\frac{3\pi}{4}\right)=\frac{7\sqrt{2}}{4}$$

The area of a $\pi/2\times1$ rectangle is $\pi/2$, so the Midpoint Rule volume estimate with rectangles of these dimensions is

$$V=\frac{\pi}{2}\left[z\left(\frac{\pi}{4},-0.5\right)+z\left(\frac{3\pi}{4},-0.5\right)+z\left(\frac{\pi}{4},0.5\right)+z\left(\frac{3\pi}{4},0.5\right)\right]$$

$$V\approx\frac{\pi}{2}\left(\frac{5\sqrt{2}}{4}+\frac{5\sqrt{2}}{4}+\frac{7\sqrt{2}}{4}+\frac{7\sqrt{2}}{4}\right)$$

$$V\approx3\pi\sqrt{2}$$

$$V\approx13.33$$

If we use rectangles of dimensions $\pi/3\times2/3$, then the midpoints of the rectangles will be

$$\left(\frac{\pi}{6},-\frac{2}{3}\right),\left(\frac{\pi}{2},-\frac{2}{3}\right),\left(\frac{5\pi}{6},-\frac{2}{3}\right),\left(\frac{\pi}{6},0\right),\left(\frac{\pi}{2},0\right)$$

$$\left(\frac{5\pi}{6},0\right),\left(\frac{\pi}{6},\frac{2}{3}\right),\left(\frac{\pi}{2},\frac{2}{3}\right),\left(\frac{5\pi}{6},\frac{2}{3}\right)$$

Evaluate $z=(y+3)\sin x$ at these nine midpoints.

$$z\left(\frac{\pi}{6},-\frac{2}{3}\right)=\left(-\frac{2}{3}+3\right)\sin\left(\frac{\pi}{6}\right)=\frac{7}{6}$$

$$z\left(\frac{\pi}{2},-\frac{2}{3}\right)=\left(-\frac{2}{3}+3\right)\sin\left(\frac{\pi}{2}\right)=\frac{7}{3}$$

$$z\left(\frac{5\pi}{6},-\frac{2}{3}\right)=\left(-\frac{2}{3}+3\right)\sin\left(\frac{5\pi}{6}\right)=\frac{7}{6}$$

$$z\left(\frac{\pi}{6},0\right)=(0+3)\sin\left(\frac{\pi}{6}\right)=\frac{3}{2}$$

$$z\left(\frac{\pi}{2},0\right)=(0+3)\sin\left(\frac{\pi}{2}\right)=3$$

$$z\left(\frac{5\pi}{6},0\right)=(0+3)\sin\left(\frac{5\pi}{6}\right)=\frac{3}{2}$$

$$z\left(\frac{\pi}{6},\frac{2}{3}\right)=\left(\frac{2}{3}+3\right)\sin\left(\frac{\pi}{6}\right)=\frac{11}{6}$$

$$z\left(\frac{\pi}{2},\frac{2}{3}\right)=\left(\frac{2}{3}+3\right)\sin\left(\frac{\pi}{2}\right)=\frac{11}{3}$$

$$z\left(\frac{5\pi}{6},\frac{2}{3}\right)=\left(\frac{2}{3}+3\right)\sin\left(\frac{5\pi}{6}\right)=\frac{11}{6}$$

The area of a $\pi/3\times 2/3$ rectangle is $2\pi/9$, so the Midpoint Rule volume estimate with rectangles of these dimensions is

$$V=\frac{2\pi}{9}\left[z\left(\frac{\pi}{6},-\frac{2}{3}\right)+z\left(\frac{\pi}{2},-\frac{2}{3}\right)+z\left(\frac{5\pi}{6},-\frac{2}{3}\right)\right.$$

$$\left.+z\left(\frac{\pi}{6},0\right)+z\left(\frac{\pi}{2},0\right)+z\left(\frac{5\pi}{6},0\right)\right.$$

$$+z\left(\frac{\pi}{6},\frac{2}{3}\right)+z\left(\frac{\pi}{2},\frac{2}{3}\right)+z\left(\frac{5\pi}{6},\frac{2}{3}\right)\Bigg]$$

$$V \approx \frac{2\pi}{9}\left(\frac{7}{6}+\frac{7}{3}+\frac{7}{6}+\frac{3}{2}+3+\frac{3}{2}+\frac{11}{6}+\frac{11}{3}+\frac{11}{6}\right)$$

$$V \approx \frac{2\pi}{9}\left(\frac{36}{6}+\frac{18}{3}+\frac{6}{2}+3\right)$$

$$V \approx \frac{2\pi}{9}(6+6+3+3)$$

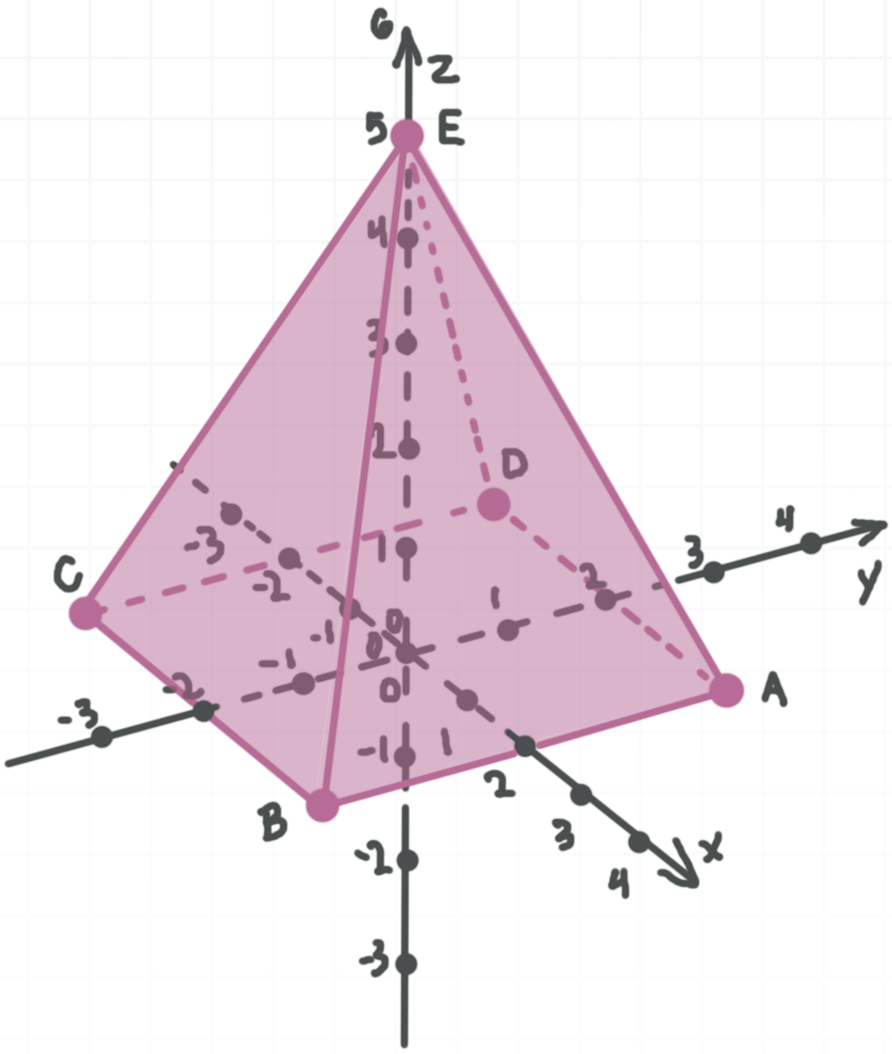
$$V \approx \frac{2\pi}{9}(18)$$

$$V \approx 4\pi$$

$$V \approx 12.57$$

If exact volume is 12, rectangles with dimensions $\pi/2 \times 1$ give an approximation of $V \approx 13.33$, and rectangles with dimensions $\pi/3 \times 2/3$ give an approximation of $V \approx 12.57$, we can say that the $\pi/3 \times 2/3$ rectangles give a better approximation.

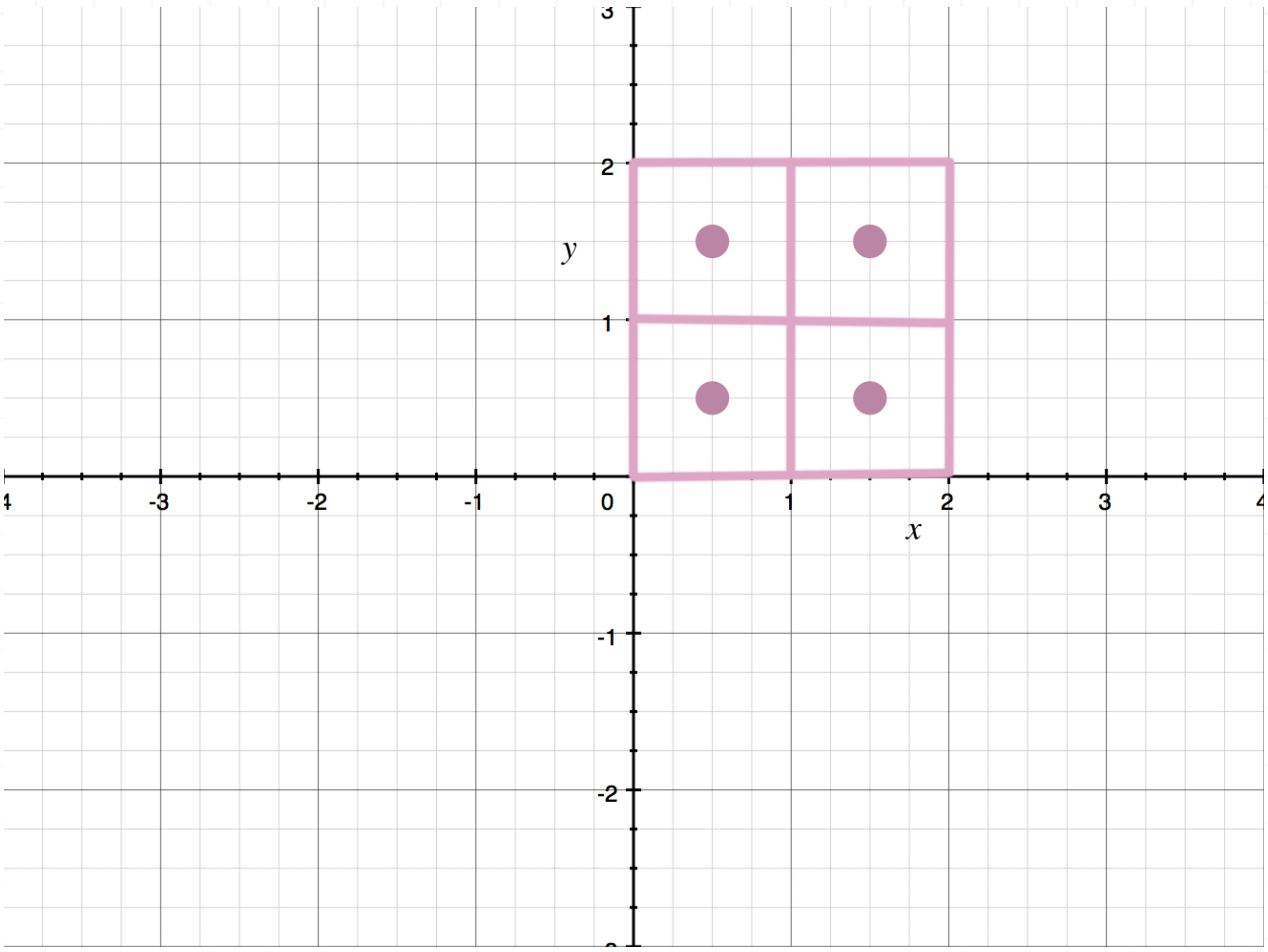
■ 2. Use Midpoint Rule and 1×1 squares to estimate the volume of the right square pyramid $ABCDE$ with base side length 4 and height 5, assuming its base lies in the xy-plane with its sides parallel to the major coordinate axes, and the vertex of the pyramid lies on z-axis. If the pyramid's exact volume is $V = 80/3$, find the percentage error of the approximation.



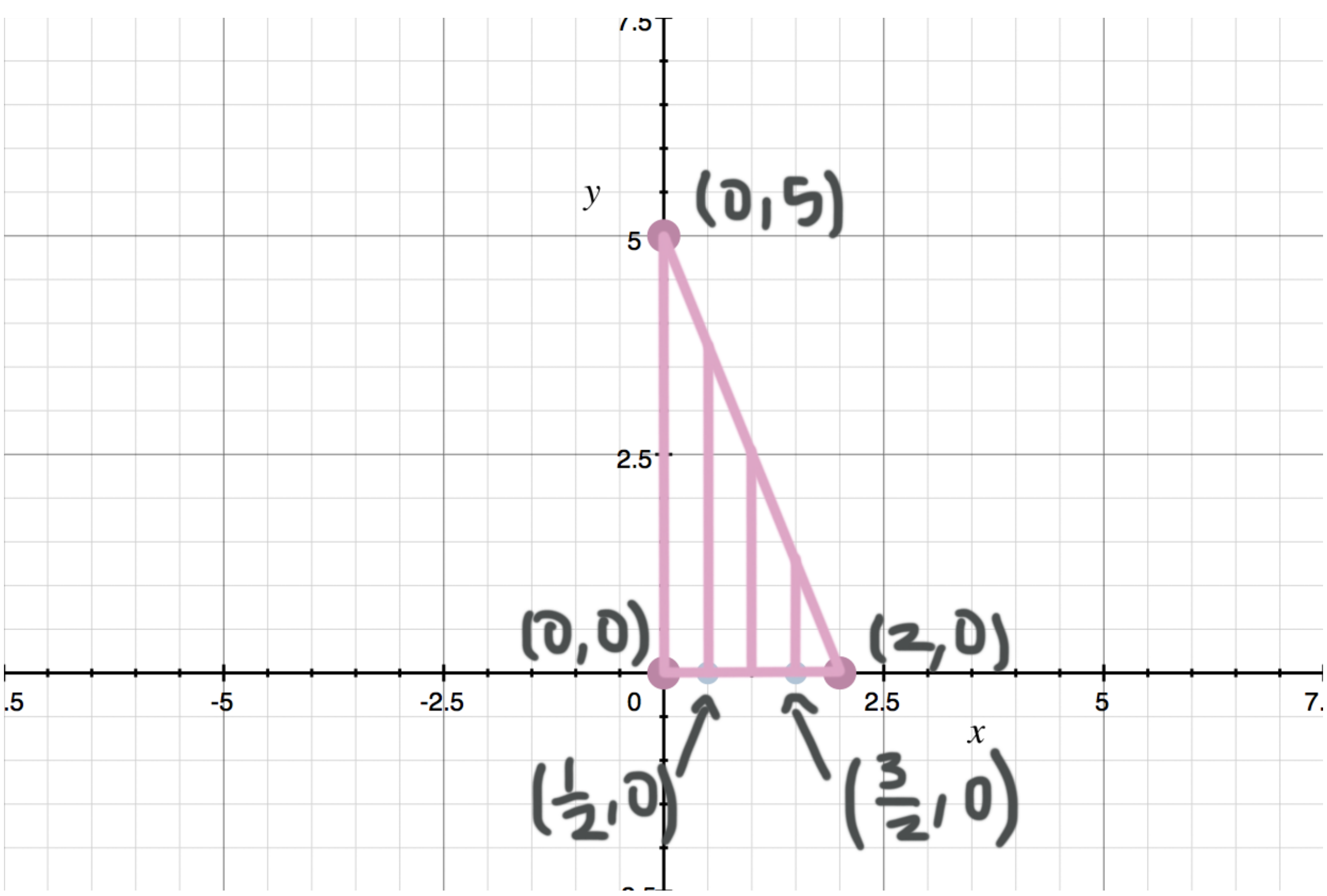
Solution:

Since a right square pyramid is symmetric and can be split into four equal parts, we can approximate the volume of the pyramid lying in the first quadrant, then multiply that volume by 4 to get an approximation of total volume.

Sketch the region of the base in the first quadrant only, the 1×1 squares, and their midpoints.



The midpoints are $(0.5,0.5), (1.5,0.5), (0.5,1.5),$ and $(1.5,1.5)$. To find z for each of these midpoints, we look at the triangle OAE.



Because $(0.5,0.5)$ and $(1.5,1.5)$ lie on the segment OA, the value of z at both points can be given by the equation of AE. Since $(2,0)$ and $(0,5)$ define the endpoints of AE, the equation of AE is

$$y = -\frac{5}{2}x + 5$$

So at $(0.5,0.5)$, z is

$$z = y = -\frac{5}{2}(0.5) + 5$$

$$z = y = 3.75$$

And at $(1.5,1.5)$, z is

$$z = y = -\frac{5}{2}(1.5) + 5$$

$$z = y = 1.25$$

This isn't one of the midpoints, but the point $(1,1)$ is also along OA, which means the value at that point can be given by the equation of AE.

$$z = y = -\frac{5}{2}(1) + 5$$

$$z = y = 2.5$$

We need this value in order to calculate the z-values for the midpoints $(1.5,0.5)$ and $(0.5,1.5)$. We can get the z-value of $(1.5,0.5)$ by recognizing that $(1.5,0.5)$ is the midpoint of $(2,0,0)$ and $(1,1,2.5)$.

$$z = \frac{0 + 2.5}{2} = 1.25$$

Similarly, we can get the z-value of $(0.5,1.5)$ by recognizing that $(0.5,1.5)$ is the midpoint of $(0,2,0)$ and $(1,1,2.5)$.

$$z = \frac{0 + 2.5}{2} = 1.25$$

Since the area of a 1×1 square is $A = 1$, the volume approximation is

$$1 \cdot |3.75 + 1.25 + 1.25 + 1.25| = 7.5$$

This is the approximation of the pyramid's volume in the first quadrant only, so the approximation of its total volume is $4 \cdot 7.5 = 30$.

Because exact volume is given as $V = 80/3$, the percentage error is

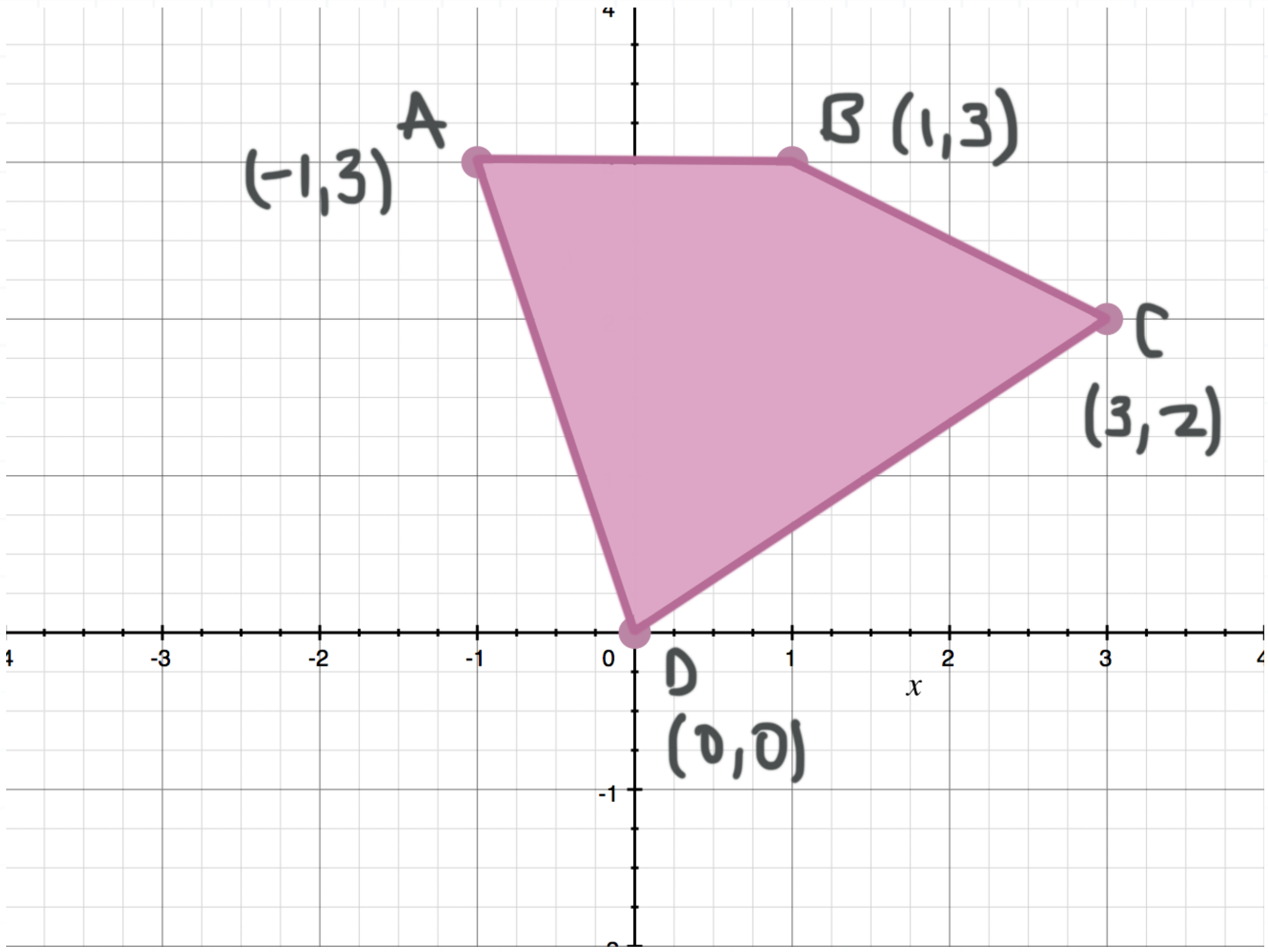
$$E_P = \frac{|\text{Exact volume} - \text{Approximate volume}|}{\text{Exact volume}} \cdot 100\,\%$$

$$E_P = \frac{|80/3 - 30|}{80/3} \cdot 100\,\%$$

$$E_P = 12.5\,\%$$

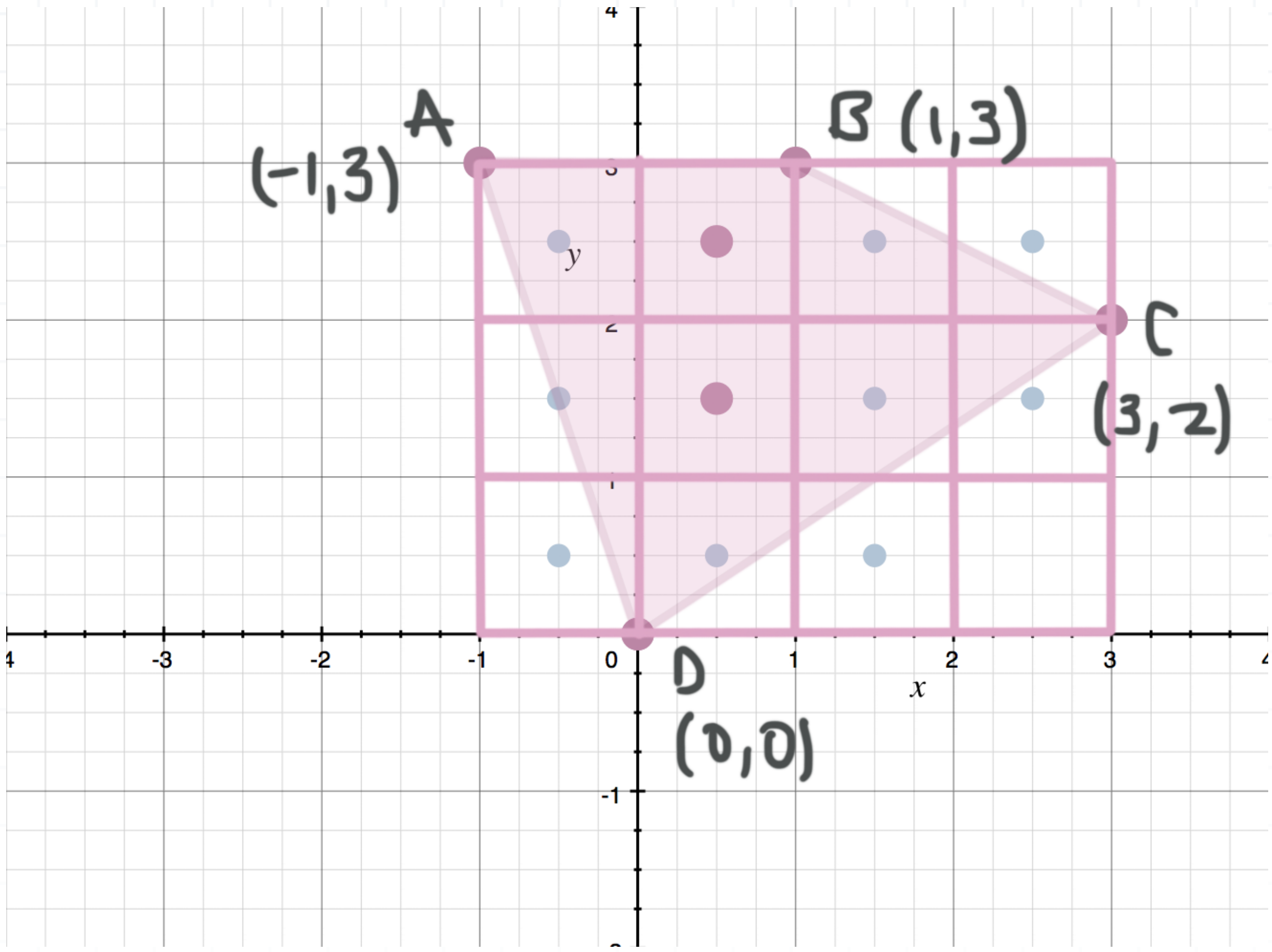
■ 3. Use Midpoint Rule with 1×1 squares to estimate the value of the given integral, where $ABCD$ is the quadrilateral shown in the xy-plane below. For any square that lie only partially inside the region, divide the area in half, then round the final approximation to the nearest tenth.

$$\iint_{ABCD} x + \sqrt{y+3}\ dy\ dx$$



Solution:

Sketch the region, the 1×1 squares, and each of their midpoints. We'll use purple dots for midpoints of squares that lie completely within the region, and blue dots for midpoints of squares that lie only partially within the region.



Evaluate $z = x + \sqrt{y+3}$ at each midpoint, rounding each value to the nearest hundredth.

$$z(0.5,1.5) = 0.5 + \sqrt{1.5+3} = 2.62$$

$$z(0.5,2.5) = 0.5 + \sqrt{2.5+3} = 2.85$$

$$z(-0.5,0.5) = -0.5 + \sqrt{0.5+3} = 1.37$$

$$z(0.5,0.5) = 0.5 + \sqrt{0.5+3} = 2.37$$

$$z(1.5,0.5) = 1.5 + \sqrt{0.5+3} = 3.37$$

$$z(-0.5,1.5) = -0.5 + \sqrt{1.5+3} = 1.62$$

$$z(1.5,1.5) = 1.5 + \sqrt{1.5+3} = 3.62$$

$$z(2.5,1.5) = 2.5 + \sqrt{1.5 + 3} = 4.62$$

$$z(-0.5,2.5) = -0.5 + \sqrt{2.5 + 3} = 1.85$$

$$z(1.5,2.5) = 1.5 + \sqrt{2.5 + 3} = 3.85$$

$$z(2.5,2.5) = 2.5 + \sqrt{2.5 + 3} = 4.85$$

A 1×1 square has area $A = 1$, and if we divide area of partially contained squares by 2, then the integral approximation is

$$A = 1 \cdot (2.62 + 2.85)$$

$$+0.5(1.37 + 2.37 + 3.37 + 1.62 + 3.62 + 4.62 + 1.85 + 3.85 + 4.85)$$

$$A = 5.47 + 0.5(27.52)$$

$$A \approx 19.2$$

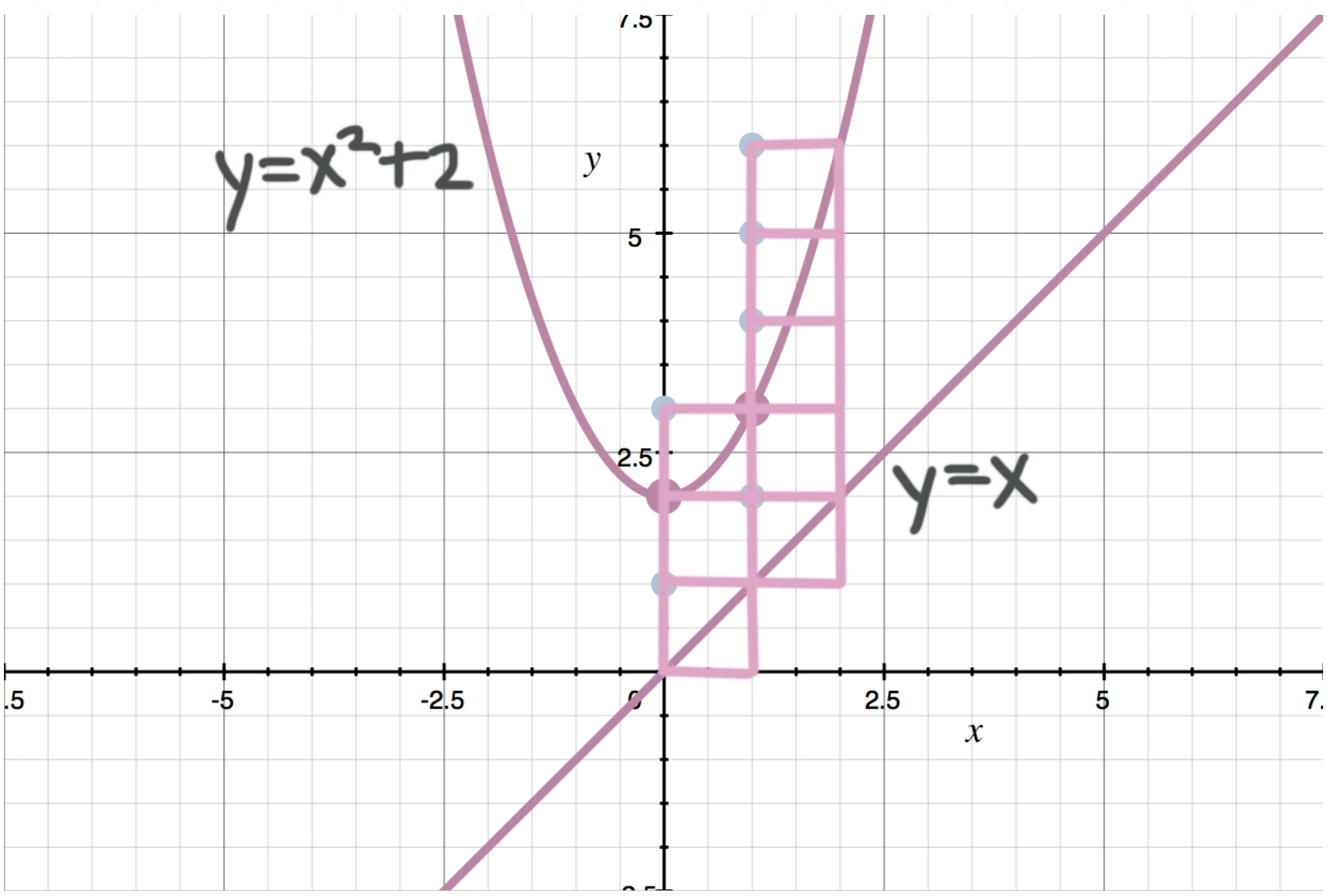
RIEMANN SUMS FOR DOUBLE INTEGRALS

■ 1. Use Riemann sums to approximate the integral, using squares with sides of length 1, and the upper-left corner of the squares. If the square partially lies within the domain, then divide its area in half.

$$\int_{0}^{2}\int_{x}^{x^2+2} x^2 + 2x - 3y + 1 \ dy \ dx$$

Solution:

Sketch in the upper-left corner of each square.



Find the value of z for each point. The purple points are for squares that lie fully in the domain,

$$z(0,2) = 0^2 + 2(0) - 3(2) + 1 = -5$$

$$z(1,3) = 1^2 + 2(1) - 3(3) + 1 = -5$$

and the blue points are for squares partially in the domain.

$$z(0,1) = 0^2 + 2(0) - 3(1) + 1 = -2$$

$$z(0,3) = 0^2 + 2(0) - 3(3) + 1 = -8$$

$$z(1,2) = 1^2 + 2(1) - 3(2) + 1 = -2$$

$$z(1,4) = 1^2 + 2(1) - 3(4) + 1 = -8$$

$$z(1,5) = 1^2 + 2(1) - 3(5) + 1 = -11$$

$$z(1,6) = 1^2 + 2(1) - 3(6) + 1 = -14$$

The area of a square with side 1 is 1. So the integral approximation is

$$1\left(-5 - 5 + \frac{1}{2}(-2 - 8 - 2 - 8 - 11 - 14)\right)$$

$$-10 + \frac{1}{2}(-45)$$

$$-\frac{20}{2} - \frac{45}{2}$$

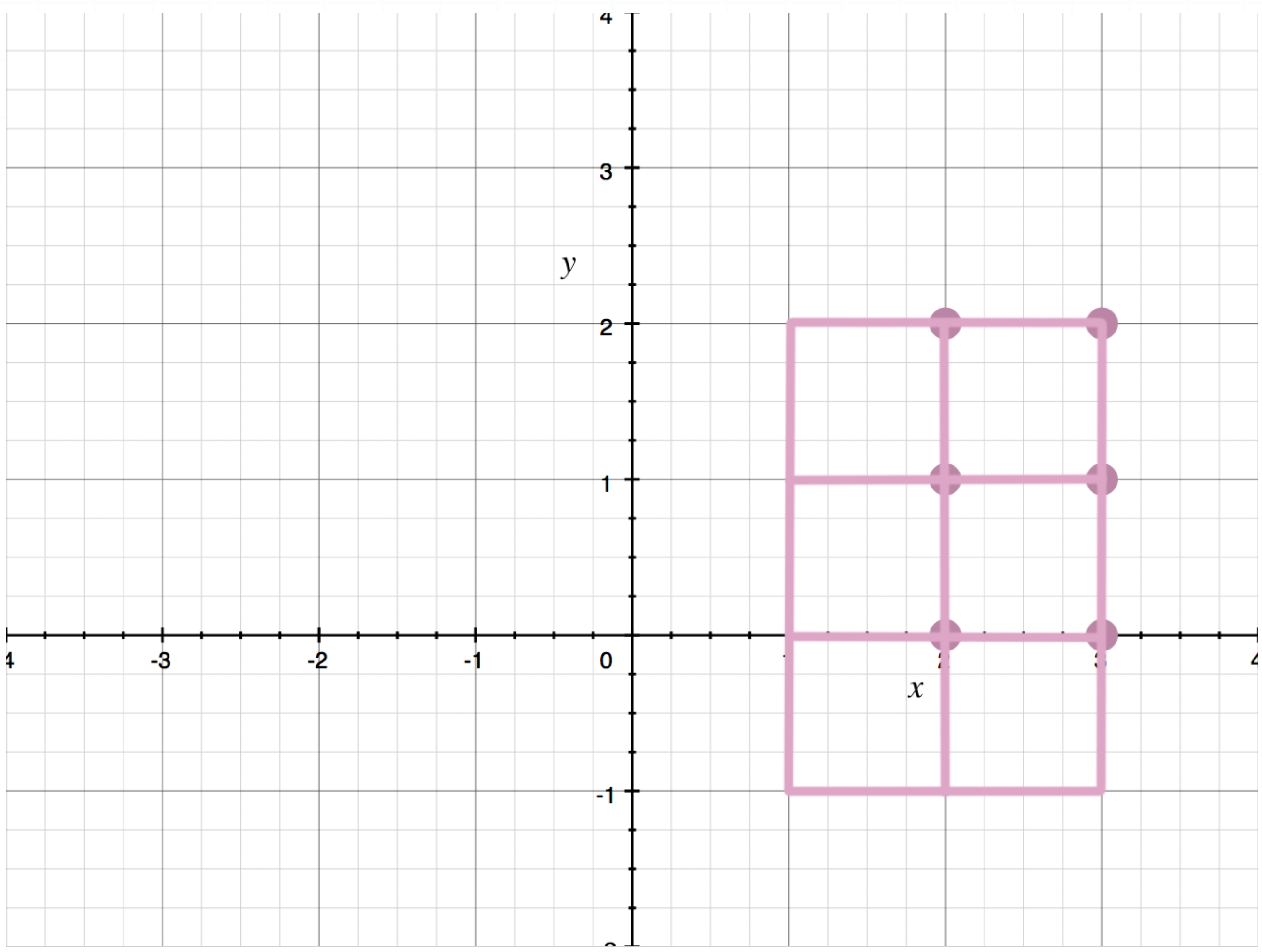
$$-\frac{65}{2}$$

■ 2. Use Riemann sums to approximate the integral over the rectangle $R = [1,3] \times [-1,2]$, using squares with sides of length 1, and the upper-right corners of the squares. Round your answer to the nearest tenth.

$$\iint_R \ln(x^2 + y + 2) \ dy \ dx$$

Solution:

Sketch in the upper-right corner of each square.



Find the value of z for each point, rounding the values to the nearest hundredth.

$$z(2,0) = \ln(2^2 + 0 + 2) \approx 1.79$$

$$z(3,0) = \ln(3^2 + 0 + 2) \approx 2.40$$

$$z(2,1) = \ln(2^2 + 1 + 2) \approx 1.95$$

$$z(3,1) = \ln(3^2 + 1 + 2) \approx 2.48$$

$$z(2,2) = \ln(2^2 + 2 + 2) \approx 2.08$$

$$z(3,2) = \ln(3^2 + 2 + 2) \approx 2.56$$

The area of a square with side 1 is 1. So the integral approximation, rounded to the nearest tenth, is

$$1(1.79 + 2.40 + 1.95 + 2.48 + 2.08 + 2.56)$$

$$1.79 + 2.40 + 1.95 + 2.48 + 2.08 + 2.56$$

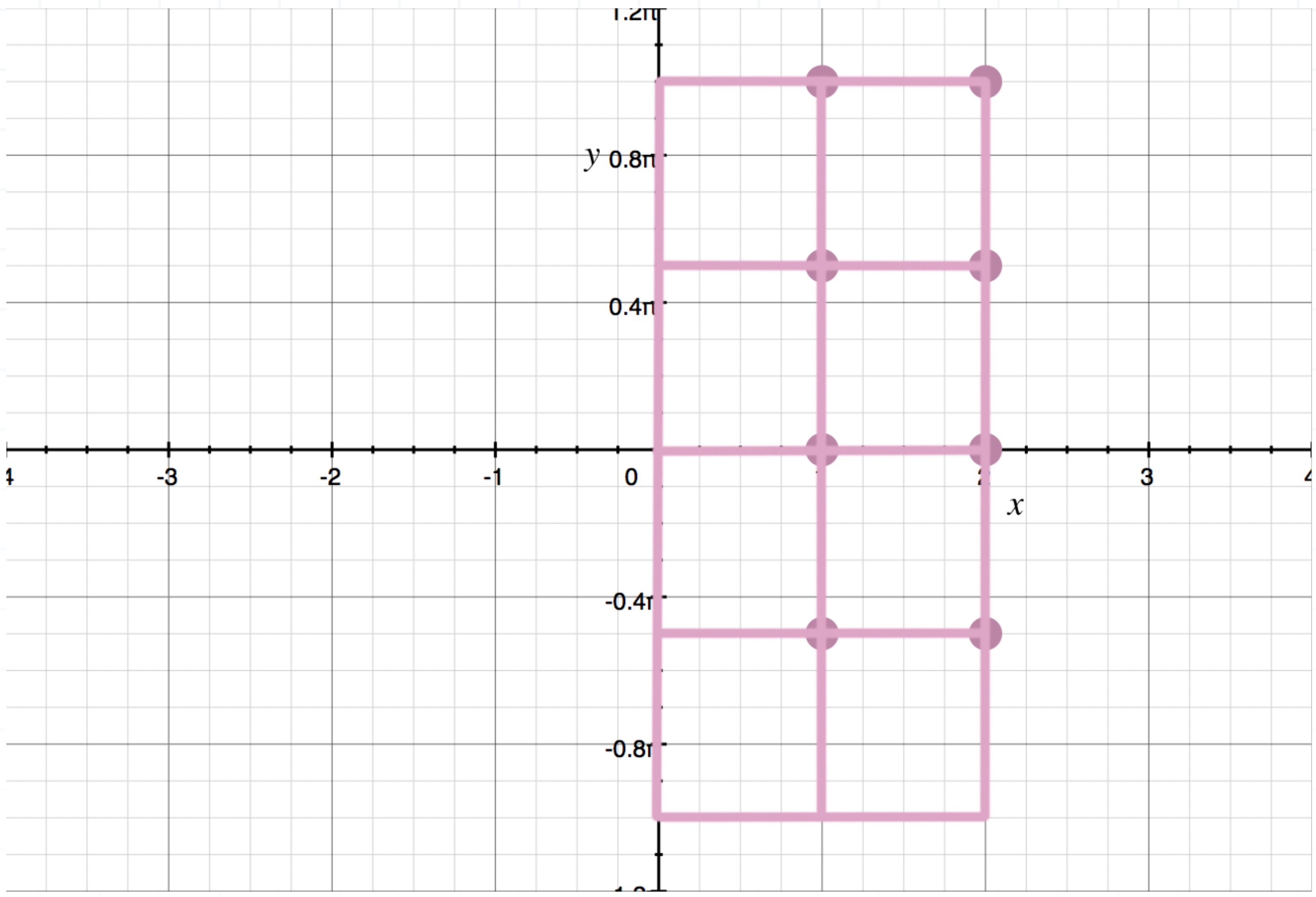
$$13.3$$

■ 3. Use Riemann sums to approximate the integral, using rectangles with sides $1 \times \pi/2$, and the upper-right corners of the rectangles. If the exact value is 14π, find the percentage error of the approximation.

$$\int_{-\pi}^{\pi} \int_{0}^{2} 2x + \sin^2 y + 1 \ dx \ dy$$

Solution:

Sketch in the upper-right corner of each rectangle.



Find the value of z for each point.

$$z(1,-\pi/2)=2(1)+\sin^2(-\pi/2)+1=4$$

$$z(2,-\pi/2)=2(2)+\sin^2(-\pi/2)+1=6$$

$$z(1,0)=2(1)+\sin^2(0)+1=3$$

$$z(2,0)=2(2)+\sin^2(0)+1=5$$

$$z(1,\pi/2)=2(1)+\sin^2(\pi/2)+1=4$$

$$z(2,\pi/2)=2(2)+\sin^2(\pi/2)+1=6$$

$$z(1,\pi)=2(1)+\sin^2(\pi)+1=3$$

$$z(2,\pi)=2(2)+\sin^2(\pi)+1=5$$

The area of a rectangle with dimensions $1 \times \pi/2$ is $\pi/2$. So the integral approximation is

$$\frac{\pi}{2} \cdot (4 + 6 + 3 + 5 + 4 + 6 + 3 + 5)$$

$$\frac{\pi}{2}(36)$$

$$18\pi$$

The percentage error is given by

$$\%\ \text{error} = \frac{|\text{Exact value} - \text{Approximate value}|}{\text{Exact value}} \cdot 100\,\%$$

$$\%\ \text{error} = \frac{|14\pi - 18\pi|}{14\pi} \cdot 100\,\%$$

$$\%\ \text{error} = \frac{|-4\pi|}{14\pi} \cdot 100\,\%$$

$$\%\ \text{error} = \frac{4\pi}{14\pi} \cdot 100\,\%$$

$$\%\ \text{error} = \frac{2}{7} \cdot 100\,\%$$

$$\%\ \text{error} = 28.6\,\%$$

W W W . K R I S T A K I N G M A T H . C O M