Calculus 3 Workbook Solutions

Linear approximation and linearization

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LINEAR APPROXIMATION IN TWO VARIABLES

■ 1. Find the linear approximation of the function at $(1,1)$ and use it to approximate $f(0.99,0.99)$. Compare it to the exact value of $f(0.99,0.99)$.

$$f(t,s)=\sqrt{3t^2+s^2}$$

Solution:

The linearization of f at (a,b) is

$$L(t,s)=f(a,b)+f_t(a,b)(t-a)+f_s(a,b)(s-b)$$

Since $(a,b)=(1,1)$,

$$f(1,1)=\sqrt{3(1)^2+1^2}=\sqrt{4}=2$$

The partial derivatives at $(1,1)$ are

$$f_s(t,s)=\frac{2s}{2\sqrt{3t^2+s^2}}=\frac{s}{\sqrt{3t^2+s^2}}$$

$$f_s(1,1)=\frac{1}{\sqrt{3(1)^2+1^2}}=\frac{1}{2}$$

and

$$f_t(t,s)=\frac{6t}{2\sqrt{3t^2+s^2}}=\frac{3t}{\sqrt{3t^2+s^2}}$$

$$f_t(1,1) = \frac{3}{\sqrt{3(1)^2 + 1^2}} = \frac{3}{2}$$

The linear approximation of f at $(1,1)$ is

$$L(t, s) = 2 + \frac{3}{2}(t - 1) + \frac{1}{2}(s - 1)$$

$$L(t, s) = 1 + \frac{3}{2}t - \frac{3}{2} + \frac{1}{2}s - \frac{1}{2}$$

$$L(t, s) = \frac{3}{2}t + \frac{1}{2}s$$

The linear approximation at $(0.99,0.99)$ is

$$L(0.99,0.99) = \frac{3}{2} \cdot 0.99 + \frac{1}{2} \cdot 0.99 = 1.98$$

and $f(0.99,0.99)$ is

$$f(0.99,0.99) = \sqrt{3(0.99)^2 + (0.99)^2} = 1.98$$

So at $(0.99,0.99)$,

$$f(0.99,0.99) = L(0.99,0.99) = 1.98$$

■ 2. Calculate the percentage error of the linear approximation of the function at $f(0.9e,0.81)$. Use the initial point $(e,1)$.

$$f(x, y) = \ln\left(\frac{x^2}{y}\right)$$

Solution:

The linearization of a function f at (a, b) is

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

The function f can be rewritten as

$$f(x, y) = \ln(x^2) - \ln y$$

$$f(x, y) = 2\ln x - \ln y$$

Since $(a, b) = (e,1)$,

$$f(e,1) = 2\ln(e) - \ln(1) = 2(1) - 0 = 2$$

The partial derivatives of f are

$$f_x(x, y) = 2 \cdot \frac{1}{x} - 0 = \frac{2}{x}$$

$$f_x(e,1) = \frac{2}{e}$$

and

$$f_y(x, y) = 0 - \frac{1}{y} = -\frac{1}{y}$$

$$f_y(e,1) = -\frac{1}{1} = -1$$

The linear approximation of f at $(e,1)$ is

$$L(x,y)=2+\frac{2}{e}(x-e)-1(y-1)$$

$$L(x,y)=2+\frac{2}{e}\cdot x-2-y+1$$

$$L(x,y)=\frac{2}{e}x-y+1$$

The linear approximation at $(0.9e,0.81)$ is

$$f(0.9e,0.81)\approx L(0.9e,0.81)=\frac{2}{e}\cdot 0.9e-0.81+1=1.99$$

and $f(0.9e,0.81)$ is

$$f(0.9e,0.81)=\ln\left(\frac{(0.9e)^2}{0.81}\right)=\ln\left(\frac{0.81e^2}{0.81}\right)=\ln(e^2)=2$$

The percentage error is

$$\frac{|1.99-2|}{2}\cdot 100\,\%=0.5\,\%$$

■ 3. Find the linear approximation of the function at $(0,0)$ and use it to approximate $f(0.2,0.01)$. Round to two decimal places.

$$f(u,v)=3e^{2u-7v}$$

Solution:

The linearization of a function f at (a, b) is

$$L(u, v) = f(a, b) + f_u(a, b)(u - a) + f_v(a, b)(v - b)$$

Since $(a, b) = (0,0)$,

$$f(0,0) = 3e^0 = 3$$

The partial derivatives of f are

$$f_u(u, v) = 3 \cdot 2e^{2u-7v} = 6e^{2u-7v}$$

$$f_u(0,0) = 6e^0 = 6$$

and

$$f_v(u, v) = 3 \cdot (-7)e^{2u-7v} = -21e^{2u-7v}$$

$$f_v(u, v) = -21e^0 = -21$$

The linear approximation of at $(0,0)$ is

$$L(u, v) = 3 + 6(u - 0) - 21(v - 0)$$

$$L(u, v) = 3 + 6u - 21v$$

So

$$f(0.2,0.01) \approx L(0.2,0.01)$$

$$f(0.2,0.01) \approx 3 + 6 \cdot 0.2 - 21 \cdot 0.01$$

$$f(0.2,0.01) \approx 3.99$$

4. Find the values of a and b and write down the linear approximation of the function at (a,b), given that $f_x(a,b)=-5$ and $f_y(a,b)=11$.

$$f(x,y)=x^2-3y^2-7x-y$$

Solution:

The linearization of f at (a,b) is

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

Since $f_x(a,b)=-5$ and $f_y(a,b)=11$,

$$L(x,y)=f(a,b)-5(x-a)+11(y-b)$$

Because $f_x(x,y)=2x-7$ and $f_x(a,b)=-5$,

$$2a-7=-5$$

$$a=1$$

And because $f_y(x,y)=-6y-1$ and $f_y(a,b)=11$,

$$-6b-1=11$$

$$b=-2$$

So

$$f(a,b)=1^2-3(-2)^2-7-(-2)=-16$$

The $a=1$, $b=-2$, and the linear approximation of f at $(1,-2)$ is

$$L(x,y)=-16-5(x-1)+11(y+2)$$

$$L(x,y)=-16-5x+5+11y+22$$

$$L(x,y)=11-5x+11y$$

■ 5. Find $f_x(1,2)$ and $f_y(1,2)$, given that $f(1,2)=5$, $L(1,2.1)=5.5$, and $L(1.1,1.95)=5.4$, where $L(x,y)$ is the linear approximation of $f(x,y)$ at $(1,2)$.

Solution:

The linearization of f at (a,b) is

$$L(x,y)=f(a,b)+f_x(a,b)(x-a)+f_y(a,b)(y-b)$$

Since $(a,b)=(1,2)$ and $f(1,2)=5$, the linear approximation of f at $(1,2)$ is

$$L(x,y)=5+f_x(1,2)(x-1)+f_y(1,2)(y-2)$$

Since $L(1,2.1)=5.5$, substitute $x=1$ and $y=2.1$ into this linear approximation equation.

$$5+f_x(1-1)+f_y(2.1-2)=5.5$$

$$5+0.1f_y=5.5$$

$$f_y=5$$

So $f_y(1,2)=5$. In the same way, since $L(1.1,1.95)=5.4$, substitute $x=1.1$ and $y=1.95$ into the linear approximation equation.

$$5+f_x(1.1-1)+f_y(1.95-2)=5.4$$

$$5+0.1f_x-0.05f_y=5.4$$

Substitute $f_y=5$ and solve for f_x.

$$0.1f_x-5\cdot 0.05=0.4$$

$$0.1f_x=0.65$$

$$f_x=6.5$$

So $f_x(1,2)=6.5$, and the partial derivatives at $(1,2)$ are

$$f_x(1,2)=6.5$$

$$f_y(1,2)=5$$

LINEARIZATION OF A MULTIVARIABLE FUNCTION

■ 1. Find the percentage error of the linear approximation of the function at $(3.2, -1.1, 0.8)$, if the initial point is $(3, -1, 1)$.

$$f(x, y, z) = 2xy^2z^3$$

Solution:

The linearization of f at (a, b, c) is

$$L(x, y, z) = f(a, b, c) + f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c)$$

Since $(a, b, c) = (3, -1, 1)$,

$$f(3, -1, 1) = 2(3)(-1)^2(1)^3$$

$$f(3, -1, 1) = 6$$

The partial derivatives of f at $(3, -1, 1)$ are

$$f_x(x, y, z) = 2y^2z^3$$

$$f_x(3, -1, 1) = 2(-1)^2(1)^3 = 2$$

and

$$f_y(x, y, z) = 2x(2y)z^3 = 4xyz^3$$

$$f_y(3, -1, 1) = 4(3)(-1)(1)^3 = -12$$

and

$$f_z(x, y, z) = 2xy^2(3z^2) = 6xy^2z^2$$

$$f_z(3, -1, 1) = 6(3)(-1)^2(1)^2 = 18$$

So the linear approximation of f at $(3, -1, 1)$ is

$$L(x, y, z) = 6 + 2(x - 3) - 12(y + 1) + 18(z - 1)$$

$$L(x, y, z) = 6 + 2x - 6 - 12y - 12 + 18z - 18$$

$$L(x, y, z) = 2x - 12y + 18z - 30$$

Then the approximation of $(3.2, -1.1, 0.8)$ is

$$L(3.2, -1.1, 0.8) = 2(3.2) - 12(-1.1) + 18(0.8) - 30$$

$$L(3.2, -1.1, 0.8) = 4$$

The exact value of f at $(3.2, -1.1, 0.8)$ is

$$f(3.2, -1.1, 0.8) = 2(3.2)(-1.1)^2(0.8)^3 = 3.964928$$

So the percentage error is

$$\frac{|4 - 3.964928|}{3.964928} \cdot 100\% \approx 0.9\%$$

■ 2. Find the linear approximation of the function at $(2, \pi/6, -\pi/6)$ and use it to approximate $R(2, 0.5, -0.5)$. Round to two decimal places.

$$R(r, \phi, \theta) = r^2 \sin(2\phi)\cos(\theta + \pi)$$

Solution:

The linearization of R at (a, b, c) is

$$L(r, \phi, \theta) = R(a, b, c) + R_r(a, b, c)(r - a) + R_\phi(a, b, c)(\phi - b) + R_\theta(a, b, c)(\theta - c)$$

Since $(a, b, c) = (2, \pi/6, -\pi/6)$,

$$R\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = (2)^2 \sin\left(2 \cdot \frac{\pi}{6}\right)\cos\left(-\frac{\pi}{6} + \pi\right)$$

$$R\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = -3$$

The partial derivatives of R at $(2, \pi/6, -\pi/6)$ are

$$R_r(r, \phi, \theta) = 2r\sin(2\phi)\cos(\theta + \pi)$$

$$R_r\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = 2(2)\sin\left(2 \cdot \frac{\pi}{6}\right)\cos\left(-\frac{\pi}{6} + \pi\right) = -3$$

and

$$R_\phi(r, \phi, \theta) = 2r^2\cos(2\phi)\cos(\theta + \pi)$$

$$R_\phi\left(2, \frac{\pi}{6}, -\frac{\pi}{6}\right) = 2(2)^2\cos\left(2 \cdot \frac{\pi}{6}\right)\cos\left(-\frac{\pi}{6} + \pi\right) = -2\sqrt{3}$$

and

$$R_\theta(r, \phi, \theta) = -r^2\sin(2\phi)\sin(\theta + \pi)$$

$$R_\theta\left(2,\frac{\pi}{6},-\frac{\pi}{6}\right)=-(2)^2\sin\left(2\cdot\frac{\pi}{6}\right)\sin\left(-\frac{\pi}{6}+\pi\right)=-\sqrt{3}$$

So the linear approximation of R at $(2,\pi/6,-\pi/6)$ is

$$L(r,\phi,\theta)=-3-3(r-2)-2\sqrt{3}\left(\phi-\frac{\pi}{6}\right)-\sqrt{3}\left(\theta+\frac{\pi}{6}\right)$$

$$L(r,\phi,\theta)=-3r-2\sqrt{3}\phi-\sqrt{3}\theta+3+\frac{\sqrt{3}\pi}{6}$$

Then the approximation of $(2,0.5,-0.5)$ is

$$L(2,0.5,-0.5)=-3(2)-2\sqrt{3}(0.5)-\sqrt{3}(-0.5)+3+\frac{\sqrt{3}\pi}{6}$$

$$L(2,0.5,-0.5)=-3-\frac{\sqrt{3}}{2}+\frac{\sqrt{3}\pi}{6}$$

■ 3. Find the values of the first order partial derivatives of $f(x,y,z)$ at $(3,4,-8)$, where $L(x,y,z)$ is the linear approximation of the function $f(x,y,z)$ at $(3,4,-8)$, and $f(3,4,-8)=3$.

$$L(3.1,4.2,-8.1)=3$$

$$L(3.2,3.9,-7.8)=3.4$$

$$L(2.9,4.3,-8.1)=2.8$$

Solution:

The linearization of f at (a,b,c) is

$$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$$

Since $(a,b,c) = (3,4,-8)$ and $f(3,4,-8) = 3$, the linear approximation of f at $(3,4,-8)$ is

$$L(x,y,z) = 3 + f_x(x-3) + f_y(y-4) + f_z(z+8)$$

Since $L(3.1,4.2,-8.1) = 3$, substitute 3.1 for x, 4.2 for y, and -8.1 for z into the equation.

$$3 + f_x(3.1-3) + f_y(4.2-4) + f_z(-8.1+8) = 3$$

$$3 + 0.1f_x + 0.2f_y - 0.1f_z = 3$$

In the same way, using $L(3.2,3.9,-7.8) = 3.4$, we get

$$3 + f_x(3.2-3) + f_y(3.9-4) + f_z(-7.8+8) = 3.4$$

$$3 + 0.2f_x - 0.1f_y + 0.2f_z = 3.4$$

Using $L(2.9,4.3,-8.1) = 2.8$, we get

$$3 + f_x(2.9-3) + f_y(4.3-4) + f_z(-8.1+8) = 2.8$$

$$3 - 0.1f_x + 0.3f_y - 0.1f_z = 2.8$$

So we have three linear equations in terms of f_x, f_y, and f_z.

$$3 + 0.1f_x + 0.2f_y - 0.1f_z = 3$$

$$3 + 0.2f_x - 0.1f_y + 0.2f_z = 3.4$$

$$3 - 0.1f_x + 0.3f_y - 0.1f_z = 2.8$$

Simplify, and multiply each equation by 10.

$$f_x + 2f_y - f_z = 0$$

$$2f_x - f_y + 2f_z = 4$$

$$-f_x + 3f_y - f_z = -2$$

Solve the system of equations for f_x, f_y, and f_z.

$$f_x = 1$$

$$f_y = 0$$

$$f_z = 1$$

So the partial derivatives are

$$f_x(3,4,-8) = 1$$

$$f_y(3,4,-8) = 0$$

$$f_z(3,4,-8) = 1$$

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