Volume of the parallelepiped from vectors

If we need to find the volume of a parallelepiped and we're given three vectors, all we have to do is find the scalar triple product of the three vectors:

$$\left|a \cdot (b \times c)\right|$$

where the given vectors are $a\langle a_1, a_2, a_3\rangle$, $b\langle b_1, b_2, b_3\rangle$ and $c\langle c_1, c_2, c_3\rangle$. $b \times c$ is the cross product of b and c, and we'll find it using the 3×3 matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

We'll convert the result of the cross product into standard vector form, and then take the dot product of $a\langle a_1, a_2, a_3\rangle$ and the vector result of $b \times c$. The final answer is the value of the scalar triple product, which is the volume of the parallelepiped.

Example

Find the volume of the parallelepiped given by the vectors.

$$a\langle 2, -1, 3\rangle$$

$$b\langle 3, 2, -4\rangle$$

$$c\langle -2, 0, 1\rangle$$

We'll start by taking the cross product of b and c.

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & -4 \\ -2 & 0 & 1 \end{vmatrix}$$

$$b \times c = \mathbf{i} \begin{vmatrix} 2 & -4 \\ 0 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 3 & -4 \\ -2 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 3 & 2 \\ -2 & 0 \end{vmatrix}$$

$$b \times c = \left[(2)(1) - (-4)(0)\right] \mathbf{i} - \left[(3)(1) - (-4)(-2)\right] \mathbf{j} + \left[(3)(0) - (2)(-2)\right] \mathbf{k}$$

$$b \times c = (2 + 0)\mathbf{i} - (3 - 8)\mathbf{j} + (0 + 4)\mathbf{k}$$

$$b \times c = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$$

$$b \times c = \langle 2,5,4 \rangle$$

Now we'll take the dot product of $a\langle 2, -1,3 \rangle$ and $b \times c = \langle 2,5,4 \rangle$.

$$\left| a \cdot (b \times c) \right| = (2)(2) + (-1)(5) + (3)(4)$$

$$\left| a \cdot (b \times c) \right| = 4 - 5 + 12$$

$$\left| a \cdot (b \times c) \right| = 11$$

The volume of the parallelepiped is 11.
