Topic: Distance between a point and a line

Question: Find the distance between the point and the line.

Point $(1,-1,-1)$

Line $x=1-t$ $y=2t$ $z=-1$

Answer choices:

A $\frac{1}{\sqrt{5}}$

B $\frac{1}{\sqrt{25}}$

C 5

D $\sqrt{5}$

Solution: A

We have to start by converting the parametric equations to a vector equation. Since we have $x = 1 - t$, $y = 2t$, and $z = -1$, we get

$$r = (1 - t)\mathbf{i} + (2t)\mathbf{j} + (-1)\mathbf{k}$$

$$r = (1 - t)\mathbf{i} + 2t\mathbf{j} - \mathbf{k}$$

Now we'll rearrange the vector equation until it matches the format $r = r_0 + tv$.

$$r = \mathbf{i} - t\mathbf{i} + 2t\mathbf{j} - \mathbf{k}$$

$$r = (\mathbf{i} - \mathbf{k}) + (-t\mathbf{i} + 2t\mathbf{j})$$

$$r = (\mathbf{i} - \mathbf{k}) + t(-\mathbf{i} + 2\mathbf{j})$$

Matching this to $r = r_0 + tv$ gives us $r_0(1,0,-1)$ and $v\langle -1,2,0\rangle$. We'll rename the vector $v\langle -1,2,0\rangle$ to $a\langle -1,2,0\rangle$. We'll set a aside for a moment and work on the vector b, which connects the given point $(1,-1,-1)$ to the point on the line, $r_0(1,0,-1)$.

$$b\langle 1 - 1, -1 - 0, -1 - (-1)\rangle$$

$$b\langle 0,-1,0\rangle$$

Now we'll find the cross product of a and b.

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

$$a \times b = \left[(2)(0) - (0)(-1)\right]\mathbf{i} - \left[(-1)(0) - (0)(0)\right]\mathbf{j} + \left[(-1)(-1) - (2)(0)\right]\mathbf{k}$$

$$a \times b = (0 - 0)\mathbf{i} - (0 - 0)\mathbf{j} + (1 - 0)\mathbf{k}$$

$$a \times b = 0\mathbf{i} - 0\mathbf{j} + 1\mathbf{k}$$

$$a \times b = \langle 0,0,1 \rangle$$

Then we need the magnitude of the cross product of a and b.

$$\left| a \times b \right| = \sqrt{(0)^2 + (0)^2 + (1)^2}$$

$$\left| a \times b \right| = \sqrt{1}$$

$$\left| a \times b \right| = 1$$

We also need the magnitude of $a\langle -1,2,0 \rangle$.

$$|a| = \sqrt{(-1)^2 + (2)^2 + (0)^2}$$

$$|a| = \sqrt{1 + 4}$$

$$|a| = \sqrt{5}$$

Finally, we'll use the distance formula to find the distance from the point to the line.

$$d = \frac{|a \times b|}{|a|}$$

$$d = \frac{1}{\sqrt{5}}$$

Topic: Distance between a point and a line

Question: Find the distance between the point and the line.

Point $(1,1,1)$

Line $x = 2 + t$ $y = 1 - 2t$ $z = 3t$

Answer choices:

A $\sqrt{\frac{4}{3}}$

B $\sqrt{\frac{7}{12}}$

C $\sqrt{\frac{12}{7}}$

D $\sqrt{\frac{3}{4}}$

Solution: C

We have to start by converting the parametric equations to a vector equation. Since we have $x = 2 + t$, $y = 1 - 2t$, and $z = 3t$, we get

$$r = (2 + t)\mathbf{i} + (1 - 2t)\mathbf{j} + (3t)\mathbf{k}$$

$$r = (2 + t)\mathbf{i} + (1 - 2t)\mathbf{j} + 3t\mathbf{k}$$

Now we'll rearrange the vector equation until it matches the format $r = r_0 + tv$.

$$r = 2\mathbf{i} + t\mathbf{i} + \mathbf{j} - 2t\mathbf{j} + 3t\mathbf{k}$$

$$r = (2\mathbf{i} + \mathbf{j}) + (t\mathbf{i} - 2t\mathbf{j} + 3t\mathbf{k})$$

$$r = (2\mathbf{i} + \mathbf{j}) + t(\mathbf{i} - 2\mathbf{j} + 3\mathbf{k})$$

Matching this to $r = r_0 + tv$ gives us $r_0(2,1,0)$ and $v\langle 1, -2,3\rangle$. We'll rename the vector $v\langle 1, -2,3\rangle$ to $a\langle 1, -2,3\rangle$. We'll set a aside for a moment and work on the vector b, which connects the given point $(1,1,1)$ to the point on the line, $r_0(2,1,0)$.

$$b\langle 1 - 2, 1 - 1, 1 - 0\rangle$$

$$b\langle -1,0,1\rangle$$

Now we'll find the cross product of a and b.

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = \mathbf{i}\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

$$a \times b = \left[(-2)(1) - (3)(0)\right]\mathbf{i} - \left[(1)(1) - (3)(-1)\right]\mathbf{j} + \left[(1)(0) - (-2)(-1)\right]\mathbf{k}$$

$$a \times b = (-2 - 0)\mathbf{i} - (1 + 3)\mathbf{j} + (0 - 2)\mathbf{k}$$

$$a \times b = -2\mathbf{i} - 4\mathbf{j} - 2\mathbf{k}$$

$$a \times b = \langle -2, -4, -2 \rangle$$

Then we need the magnitude of the cross product of a and b.

$$\left|a \times b\right| = \sqrt{(-2)^2 + (-4)^2 + (-2)^2}$$

$$\left|a \times b\right| = \sqrt{4 + 16 + 4}$$

$$\left|a \times b\right| = \sqrt{24}$$

We also need the magnitude of $a\langle 1, -2, 3 \rangle$.

$$|a| = \sqrt{(1)^2 + (-2)^2 + (3)^2}$$

$$|a| = \sqrt{1 + 4 + 9}$$

$$|a| = \sqrt{14}$$

Finally, we'll use the distance formula to find the distance from the point to the line.

$$d = \frac{\left| a \times b \right|}{|a|}$$

$$d = \frac{\sqrt{24}}{\sqrt{14}}$$

$$d = \sqrt{\frac{24}{14}}$$

$$d = \sqrt{\frac{12}{7}}$$

Topic: Distance between a point and a line

Question: Find the distance between the point and the line.

Point $(2,-4,5)$

Line $x=-3+2t$ $y=3+t$ $z=2-5t$

Answer choices:

A $\sqrt{\frac{19}{2}}$

B $\sqrt{\frac{391}{5}}$

C $\sqrt{\frac{2}{19}}$

D $\sqrt{\frac{5}{391}}$

Solution: B

We have to start by converting the parametric equations to a vector equation. Since we have $x = -3 + 2t$, $y = 3 + t$, and $z = 2 - 5t$, we get

$$r = (-3 + 2t)\mathbf{i} + (3 + t)\mathbf{j} + (2 - 5t)\mathbf{k}$$

Now we'll rearrange the vector equation until it matches the format $r = r_0 + tv$.

$$r = -3\mathbf{i} + 2t\mathbf{i} + 3\mathbf{j} + t\mathbf{j} + 2\mathbf{k} - 5t\mathbf{k}$$

$$r = (-3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + (2t\mathbf{i} + t\mathbf{j} - 5t\mathbf{k})$$

$$r = (-3\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) + t(2\mathbf{i} + \mathbf{j} - 5\mathbf{k})$$

Matching this to $r = r_0 + tv$ gives us $r_0(-3,3,2)$ and $v\langle 2,1,-5\rangle$. We'll rename the vector $v\langle 2,1,-5\rangle$ to $a\langle 2,1,-5\rangle$. We'll set a aside for a moment and work on the vector b, which connects the given point $(2,-4,5)$ to the point on the line, $r_0(-3,3,2)$.

$$b\langle 2-(-3), -4-3, 5-2\rangle$$

$$b\langle 5,-7,3\rangle$$

Now we'll find the cross product of a and b.

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$a \times b = \mathbf{i} \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} - \mathbf{j} \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} + \mathbf{k} \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}$$

$$a \times b = \left(a_2b_3 - a_3b_2\right)\mathbf{i} - \left(a_1b_3 - a_3b_1\right)\mathbf{j} + \left(a_1b_2 - a_2b_1\right)\mathbf{k}$$

$$a \times b = \left[(1)(3) - (-5)(-7)\right]\mathbf{i} - \left[(2)(3) - (-5)(5)\right]\mathbf{j} + \left[(2)(-7) - (1)(5)\right]\mathbf{k}$$

$$a \times b = (3 - 35)\mathbf{i} - (6 + 25)\mathbf{j} + (-14 - 5)\mathbf{k}$$

$$a \times b = -32\mathbf{i} - 31\mathbf{j} - 19\mathbf{k}$$

$$a \times b = \langle -32, -31, -19 \rangle$$

Then we need the magnitude of the cross product of a and b.

$$\left| a \times b \right| = \sqrt{(-32)^2 + (-31)^2 + (-19)^2}$$

$$\left| a \times b \right| = \sqrt{1{,}024 + 961 + 361}$$

$$\left| a \times b \right| = \sqrt{2{,}346}$$

We also need the magnitude of $a\langle 2,1,-5 \rangle$.

$$|a| = \sqrt{(2)^2 + (1)^2 + (-5)^2}$$

$$|a| = \sqrt{4 + 1 + 25}$$

$$|a| = \sqrt{30}$$

Finally, we'll use the distance formula to find the distance from the point to the line.

$$d = \frac{|a \times b|}{|a|}$$

$$d = \frac{\sqrt{2{,}346}}{\sqrt{30}}$$

$$d = \sqrt{\frac{2{,}346}{30}}$$

$$d = \sqrt{\frac{391}{5}}$$