Topic: Three dimensions, two constraints

Question: Find the extrema of the function, subject to the given constraints.

$$f(x, y, z) = x^2 + y^2 + z^2$$

when $2x + y - z = 4$ and $3x + 5y + 9z = 12$

Answer choices:

A Local maximum at $\left(\frac{76}{49}, -\frac{54}{49}, \frac{10}{49}\right)$

B Local minimum at $\left(\frac{76}{49}, -\frac{54}{49}, \frac{10}{49}\right)$

C Local maximum at $\left(\frac{76}{49}, \frac{54}{49}, \frac{10}{49}\right)$

D Local minimum at $\left(\frac{76}{49}, \frac{54}{49}, \frac{10}{49}\right)$

Solution: D

We'll start by moving all terms in the constraint equation to one side, until the equation is equal to 0. Then we'll replace the 0 with $g(x, y, z)$ and $h(x, y, z)$.

$$2x + y - z = 4$$

$$2x + y - z - 4 = 0$$

$$g(x, y, z) = 2x + y - z - 4$$

and

$$3x + 5y + 9z = 12$$

$$3x + 5y + 9z - 12 = 0$$

$$h(x, y, z) = 3x + 5y + 9z - 12$$

Next we'll find the first-order partial derivatives of $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial f}{\partial z} = 2z$$

and

$$\frac{\partial g}{\partial x} = 2$$

$$\frac{\partial g}{\partial y} = 1$$

$$\frac{\partial g}{\partial z} = -1$$

and

$$\frac{\partial h}{\partial x} = 3$$

$$\frac{\partial h}{\partial y} = 5$$

$$\frac{\partial h}{\partial z} = 9$$

We'll multiply the partial derivatives of g by the Lagrange multiplier λ, and the partial derivatives of h by the Lagrange multiplier μ

$$\frac{\partial g}{\partial x} = 2\lambda$$

$$\frac{\partial g}{\partial y} = 1\lambda = \lambda$$

$$\frac{\partial g}{\partial z} = -1\lambda = -\lambda$$

and

$$\frac{\partial h}{\partial x} = 3\mu$$

$$\frac{\partial h}{\partial y} = 5\mu$$

$$\frac{\partial h}{\partial z} = 9\mu$$

Then we'll set the partial derivatives of f equal to the sum of the corresponding partial derivatives from g and h, making sure to use the equations that include the Lagrange multiplier.

For the partial derivatives with respect to x we get

$$2x = 2\lambda + 3\mu$$

$$x = \lambda + \frac{3}{2}\mu$$

For the partial derivatives with respect to y we get

$$2y = \lambda + 5\mu$$

$$y = \frac{1}{2}\lambda + \frac{5}{2}\mu$$

For the partial derivatives with respect to z we get

$$2z = -\lambda + 9\mu$$

$$z = -\frac{1}{2}\lambda + \frac{9}{2}\mu$$

Now we'll take these values for x, y and z and plug them into both constraint equations.

$$2x + y - z = 4$$

$$2\left(\lambda + \frac{3}{2}\mu\right) + \left(\frac{1}{2}\lambda + \frac{5}{2}\mu\right) - \left(-\frac{1}{2}\lambda + \frac{9}{2}\mu\right) = 4$$

$$2\lambda + 3\mu + \frac{1}{2}\lambda + \frac{5}{2}\mu + \frac{1}{2}\lambda - \frac{9}{2}\mu = 4$$

$$4\lambda + 6\mu + \lambda + 5\mu + \lambda - 9\mu = 8$$

$$6\lambda + 2\mu = 8$$

$$3\lambda + \mu = 4$$

and

$$3x + 5y + 9z = 12$$

$$3\left(\lambda + \frac{3}{2}\mu\right) + 5\left(\frac{1}{2}\lambda + \frac{5}{2}\mu\right) + 9\left(-\frac{1}{2}\lambda + \frac{9}{2}\mu\right) = 12$$

$$3\lambda + \frac{9}{2}\mu + \frac{5}{2}\lambda + \frac{25}{2}\mu - \frac{9}{2}\lambda + \frac{81}{2}\mu = 12$$

$$6\lambda + 9\mu + 5\lambda + 25\mu - 9\lambda + 81\mu = 24$$

$$2\lambda + 115\mu = 24$$

Now we'll solve these remaining equations as a system of equations.

[1] $3\lambda + \mu = 4$

[2] $2\lambda + 115\mu = 24$

We'll multiply [1] by 2 and [2] by 3 to get 6λ in both equations.

[3] $6\lambda + 2\mu = 8$

[4] $6\lambda + 345\mu = 72$

Now we'll subtract [3] from [4], which will cancel λ and allow us to solve for μ.

$$6\lambda + 345\mu - (6\lambda + 2\mu) = 72 - (8)$$

$$343\mu = 64$$

$$\mu = \frac{64}{343}$$

Now plug this back into [1] to solve for λ.

$$3\lambda + \mu = 4$$

$$3\lambda + \frac{64}{343} = 4$$

$$1{,}029\lambda + 64 = 1{,}372$$

$$1{,}029\lambda = 1{,}308$$

$$\lambda = \frac{1{,}308}{1{,}029}$$

$$\lambda = \frac{436}{343}$$

Putting these values for λ and μ back into our equations for x, y, and z, we get

$$x = \lambda + \frac{3}{2}\mu$$

$$x = \frac{436}{343} + \frac{3}{2}\left(\frac{64}{343}\right)$$

$$x = \frac{436}{343} + \frac{192}{686}$$

$$x = \frac{872}{686} + \frac{192}{686}$$

$$x = \frac{1{,}064}{686}$$

$$x = \frac{76}{49}$$

and

$$y = \frac{1}{2}\lambda + \frac{5}{2}\mu$$

$$y = \frac{1}{2}\left(\frac{436}{343}\right) + \frac{5}{2}\left(\frac{64}{343}\right)$$

$$y = \frac{436}{686} + \frac{320}{686}$$

$$y = \frac{756}{686}$$

$$y = \frac{54}{49}$$

and

$$z = -\frac{1}{2}\lambda + \frac{9}{2}\mu$$

$$z = -\frac{1}{2}\left(\frac{436}{343}\right) + \frac{9}{2}\left(\frac{64}{343}\right)$$

$$z = -\frac{436}{686} + \frac{576}{686}$$

$$z = \frac{140}{686}$$

$$z = \frac{10}{49}$$

The critical point for the function is therefore given by

$$\left(\frac{76}{49}, \frac{54}{49}, \frac{10}{49}\right)$$

$f(x, y, z) = x^2 + y^2 + z^2$ is the standard equation of a paraboloid, and we know that figure will only have one critical point. Since the paraboloid opens up, the critical point must be a minimum.

Topic: Three dimensions, two constraints

Question: What is the maximum temperature on the intersection of the sphere and the plane?

The function $F(x, y, z) = 5x + 5y + z^2 + 12$ models the temperature where the sphere

$$x^2 + y^2 + z^2 = \frac{21}{4}$$

intersects the plane

$$x + y - z = \frac{3}{2}$$

Answer choices:

A $\quad 27.18$

B $\quad 27.87$

C $\quad 28.68$

D $\quad 29.66$

Solution: A

The constraints are

$$g(x) = x^2 + y^2 + z^2 - \frac{21}{4}$$

$$h(x) = x + y - z - \frac{3}{2}$$

Then applying ∇F, ∇g, and ∇h gives us

$$F_x(x, y, z) = \lambda g_x(x, y, z) + \rho_x(x, y, z)$$

$$5 = 2\lambda x + \rho$$

and

$$F_y(x, y, z) = \lambda g_y(x, y, z) + \rho g_y(x, y, z)$$

$$5 = 2\lambda y + \rho$$

and

$$F_z(x, y, z) = \lambda g_z(x, y, z) + \rho g_z(x, y, z)$$

$$2z = 2\lambda z - \rho$$

Set up the following system of equations, and then solve.

$$5 = 2\lambda x + \rho$$

$$5 = 2\lambda y + \rho$$

$$2z = 2\lambda z - \rho$$

$$x^2 + y^2 + z^2 = \frac{21}{4}$$

$$x + y - z = \frac{3}{2}$$

Subtract the first two equations, and simplify.

$$\lambda(x - y) = 0$$

$$2z = 2\lambda z - \rho$$

$$x^2 + y^2 + z^2 = \frac{21}{4}$$

$$x + y - z = \frac{3}{2}$$

The first equation of the system above indicates that either $x = y$ or $\lambda = 0$. Replacing $x = y$ in the fourth and fifth equations results in

$$2x^2 + z^2 = \frac{21}{4}$$

$$2x - z = \frac{3}{2}$$

The solutions of this system are

$$(x, z) = \left(\frac{1 + \sqrt{3}}{2}, \frac{-1 + 2\sqrt{3}}{2}\right)$$

$$(x, z) = \left(\frac{1 - \sqrt{3}}{2}, \frac{-1 - 2\sqrt{3}}{2}\right)$$

Combining these solutions with their associated y-values gives the critical points

$$(x, z) = \left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\right)$$

$$(x, z) = \left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{-1-2\sqrt{3}}{2}\right)$$

Replacing $\lambda = 0$ in $5 = 2\lambda y + \rho$ results in $\rho = 5$. Replacing these values in $2z = 2\lambda z - \rho$ and solving the system of the last three equations yields the critical points in the imaginary plane which can't be considered.

Therefore we have only two critical points for which the measures of temperature must be calculated:

$$F\left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\right) = 5\left(\frac{1+\sqrt{3}}{2}\right) + 5\left(\frac{1+\sqrt{3}}{2}\right) + \left(\frac{-1+2\sqrt{3}}{2}\right)^2 + 12$$

$$F\left(\frac{1+\sqrt{3}}{2}, \frac{1+\sqrt{3}}{2}, \frac{-1+2\sqrt{3}}{2}\right) \approx 27.18$$

$$F\left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{-1-2\sqrt{3}}{2}\right) = 5\left(\frac{1-\sqrt{3}}{2}\right) + 5\left(\frac{1-\sqrt{3}}{2}\right) + \left(\frac{-1-2\sqrt{3}}{2}\right)^2 + 12$$

$$F\left(\frac{1-\sqrt{3}}{2}, \frac{1-\sqrt{3}}{2}, \frac{-1-2\sqrt{3}}{2}\right) \approx 13.32$$

Comparing the values of the temperature function at two critical points shows that the maximum temperature is 27.18.

Topic: Three dimensions, two constraints

Question: Find the extrema of the function $F(x, y, z) = x^2 + y^2 + z^2 + 8$ subject to the constraints $x + y - z = 12$ and $x - y - z = 6$.

Answer choices:

A 53.5

B 55.5

C 57.5

D 59.5

Solution: C

The constraints are

$$g(x) = x + y - z - 12$$

$$h(x) = x - y - z - 6$$

Then applying ∇F, ∇g, and ∇h results in:

$$F_x(x, y, z) = \lambda g_x(x, y, z) + \rho_x(x, y, z)$$

$$2x = \lambda + \rho$$

and

$$F_y(x, y, z) = \lambda g_y(x, y, z) + \rho g_y(x, y, z)$$

$$2y = \lambda - \rho$$

and

$$F_z(x, y, z) = \lambda g_z(x, y, z) + \rho g_z(x, y, z)$$

$$2z = -\lambda - \rho$$

Set up the following system of equations.

$$2x = \lambda + \rho$$

$$2y = \lambda - \rho$$

$$2z = -\lambda - \rho$$

Solving for x, y, and z, we get

$$x=\frac{\lambda+\rho}{2}$$

$$y=\frac{\lambda-\rho}{2}$$

$$z=-\frac{\lambda+\rho}{2}$$

Now we'll plug these values into $g(x)$ and $h(x)$ to get

$$x+y-z=12$$

$$\frac{\lambda+\rho}{2}+\frac{\lambda-\rho}{2}-\left(-\frac{\lambda+\rho}{2}\right)=12$$

$$\frac{\lambda+\rho}{2}+\frac{\lambda-\rho}{2}+\frac{\lambda+\rho}{2}=12$$

$$\lambda+\rho+\lambda-\rho+\lambda+\rho=24$$

$$3\lambda+\rho=24$$

and

$$x-y-z=6$$

$$\frac{\lambda+\rho}{2}-\frac{\lambda-\rho}{2}-\left(-\frac{\lambda+\rho}{2}\right)=6$$

$$\frac{\lambda+\rho}{2}-\frac{\lambda-\rho}{2}+\frac{\lambda+\rho}{2}=6$$

$$\lambda + \rho - (\lambda - \rho) + \lambda + \rho = 12$$

$$\lambda + \rho - \lambda + \rho + \lambda + \rho = 12$$

$$\lambda + 3\rho = 12$$

If we multiply $\lambda + 3\rho = 12$ by 3, we get $3\lambda + 9\rho = 36$. Now we can subtract $3\lambda + 9\rho = 36$ from $3\lambda + \rho = 24$ in order to solve for ρ.

$$3\lambda + \rho - (3\lambda + 9\rho) = 24 - 36$$

$$3\lambda + \rho - 3\lambda - 9\rho = -12$$

$$\rho - 9\rho = -12$$

$$-8\rho = -12$$

$$\rho = \frac{-12}{-8} = \frac{3}{2}$$

We'll use $\rho = 3/2$ to find the value for λ.

$$3\lambda + \rho = 24$$

$$3\lambda + \frac{3}{2} = 24$$

$$6\lambda + 3 = 48$$

$$6\lambda = 45$$

$$\lambda = \frac{45}{6} = \frac{15}{2}$$

With values of ρ and λ, we can find values for x, y, and z.

$$2x = \frac{15}{2} + \frac{3}{2}$$

$$2x = \frac{18}{2}$$

$$x = \frac{18}{4} = \frac{9}{2}$$

and

$$2y = \frac{15}{2} - \frac{3}{2}$$

$$2y = \frac{12}{2}$$

$$2y = 6$$

$$y = \frac{6}{2} = 3$$

and

$$2z = -\frac{15}{2} - \frac{3}{2}$$

$$2z = -\frac{18}{2}$$

$$z = -\frac{18}{4} = -\frac{9}{2}$$

The solutions of this system of equations are

$$\left(\frac{9}{2},3,-\frac{9}{2}\right)$$

The value of the given function at the critical point is

$$F(x,y,z)=x^2+y^2+z^2+8$$

$$F(x,y,z)=\left(\frac{9}{2}\right)^2+3^2+\left(-\frac{9}{2}\right)^2+8$$

$$F(x,y,z)=\frac{81}{4}+9+\frac{81}{4}+8$$

$$F(x,y,z)=\frac{162}{4}+17$$

$$F(x,y,z)=\frac{162}{4}+\frac{68}{4}$$

$$F(x,y,z)=\frac{230}{4}$$

$$F(x,y,z)=57.5$$