Calculus 3 Workbook Solutions

Partial derivatives

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PARTIAL DERIVATIVES

■ 1. Find $f_x + f_y$.

$$f(x, y) = \sqrt{\sin(x + y)}$$

Solution:

The partial derivatives f_x and f_y are

$$f_x = \frac{\frac{\partial}{\partial x}\sin(x + y)}{2\sqrt{\sin(x + y)}} = \frac{\cos(x + y)}{2\sqrt{\sin(x + y)}}$$

$$f_y = \frac{\frac{\partial}{\partial y}\sin(x + y)}{2\sqrt{\sin(x + y)}} = \frac{\cos(x + y)}{2\sqrt{\sin(x + y)}}$$

Then the sum is

$$f_x + f_y = 2\frac{\cos(x + y)}{2\sqrt{\sin(x + y)}} = \frac{\cos(x + y)}{\sqrt{\sin(x + y)}}$$

■ 2. Find f_r and f_θ.

$$f(r, \theta) = r^2(\sin 2\theta - \cos 2\theta)$$

Solution:

The partial derivatives are

$$f_r = 2r(\sin 2\theta - \cos 2\theta)$$

$$f_\theta = r^2(2\cos 2\theta + 2\sin 2\theta) = 2r^2(\cos 2\theta + \sin 2\theta)$$

■ 3. Find u_s and u_t.

$$u(t, s) = 2^{\frac{t}{s}}$$

Solution:

The partial derivatives are

$$u_s = \ln(2)2^{\frac{t}{s}}\frac{\partial}{\partial s}\left(\frac{t}{s}\right)$$

$$u_s = \ln(2)2^{\frac{t}{s}}\left(-\frac{t}{s^2}\right)$$

$$u_s = \frac{-t\ln(2)}{s^2}2^{\frac{t}{s}}$$

and

$$u_t = \ln(2)2^{\frac{t}{s}}\frac{\partial}{\partial t}\left(\frac{t}{s}\right)$$

$$u_t = \ln(2)2^{\frac{t}{s}}\left(\frac{1}{s}\right)$$

$$u_t = \frac{\ln(2)}{s}2^{\frac{t}{s}}$$

■ 4. Find the point (x, y) where $f_x = f_y = 0$.

$$f(x, y) = 3x^2 - 2xy + 3y^2 - 4x + 2y - 1$$

Solution:

The partial derivatives are

$$f_x = 6x - 2y - 4 = 0$$

$$f_x = 3x - y - 2 = 0$$

and

$$f_y = -2x + 6y + 2 = 0$$

$$f_y = -x + 3y + 1 = 0$$

That makes the system of linear equations

$$3x - y - 2 = 0$$

$$-x + 3y + 1 = 0$$

The first equation solves for y as $y = 3x - 2$. Then substituting this into $-x + 3y + 1 = 0$ gives

$$-x + 3(3x - 2) + 1 = 0$$

$$8x - 5 = 0$$

$$x = \frac{5}{8}$$

Which means

$$y = 3 \cdot \frac{5}{8} - 2$$

$$y = -\frac{1}{8}$$

Then the point where $f_x = f_y = 0$ is

$$\left(\frac{5}{8}, -\frac{1}{8}\right)$$

PARTIAL DERIVATIVES IN THREE OR MORE VARIABLES

■ 1. Find $f_x^2 + f_y^2 + f_z^2$.

$$f(x, y, z) = \tan(x^2 + y^2 + z^2)$$

Solution:

The partial derivatives of f are

$$f_x = \sec^2(x^2 + y^2 + z^2)\frac{\partial}{\partial x}(x^2 + y^2 + z^2)$$

$$f_x = 2x \sec^2(x^2 + y^2 + z^2)$$

and

$$f_y = \sec^2(x^2 + y^2 + z^2)\frac{\partial}{\partial y}(x^2 + y^2 + z^2)$$

$$f_y = 2y \sec^2(x^2 + y^2 + z^2)$$

and

$$f_z = \sec^2(x^2 + y^2 + z^2)\frac{\partial}{\partial z}(x^2 + y^2 + z^2)$$

$$f_z = 2z \sec^2(x^2 + y^2 + z^2)$$

Then the sum is

$$f_x^2 + f_y^2 + f_z^2 = (2x\sec^2(x^2+y^2+z^2))^2 + (2y\sec^2(x^2+y^2+z^2))^2$$

$$+(2z\sec^2(x^2+y^2+z^2))^2$$

$$f_x^2 + f_y^2 + f_z^2 = 4(x^2+y^2+z^2)\sec^4(x^2+y^2+z^2)$$

■ 2. Find f_u, f_v, and f_w.

$$f(u, v, w) = u^{v^w}$$

Solution:

The partial derivatives are

$$f_u = v^w \cdot u^{v^w - 1}$$

and

$$f_v = u^{v^w} \cdot \ln u \cdot \frac{\partial}{\partial v} v^w$$

$$f_v = u^{v^w} \cdot \ln u \cdot w \cdot v^{w-1}$$

and

$$f_w = u^{v^w} \cdot \ln u \cdot \frac{\partial}{\partial w} v^w$$

$$f_w = u^{v^w} \cdot \ln u \cdot \ln v \cdot v^w$$

3. Find the point (a, b, c, d) where $f_a = f_b = f_c = f_d = 0$.

$$f(a, b, c, d) = a^2 + b^2 - c^2 - d^2 + 4ab - 4cd - 6a + 6c + 8 = 0$$

Solution:

The partial derivatives are

$$f_a = 2a + 4b - 6 = 0$$

$$f_b = 2b + 4a = 0$$

$$f_c = -2c - 4d + 6 = 0$$

$$f_d = -2d - 4c = 0$$

These make a system of linear equations.

$$a + 2b - 3 = 0$$

$$b + 2a = 0$$

$$c + 2d - 3 = 0$$

$$d + 2c = 0$$

The solution to the system is $a = -1$, $b = 2$, $c = -1$, and $d = 2$, or $(a, b, c, d) = (-1,2, -1,2)$.

HIGHER ORDER PARTIAL DERIVATIVES

1. Find f_{uvw}.

$$f(u,v,w)=\sqrt{u^2+v^2+w^2}$$

Solution:

Find f_u.

$$f_u=\frac{1}{2\sqrt{u^2+v^2+w^2}}\frac{\partial}{\partial u}(u^2+v^2+w^2)$$

$$f_u=\frac{2u}{2\sqrt{u^2+v^2+w^2}}$$

$$f_u=\frac{u}{\sqrt{u^2+v^2+w^2}}$$

Then f_{uv} is

$$f_{uv}=-\frac{u}{2(u^2+v^2+w^2)^{3/2}}\frac{\partial}{\partial v}(u^2+v^2+w^2)$$

$$f_{uv}=-\frac{2uv}{2(u^2+v^2+w^2)^{3/2}}$$

$$f_{uv}=-\frac{uv}{(u^2+v^2+w^2)^{3/2}}$$

Then f_{uvw} is

$$f_{uvw} = \frac{3uv}{2(u^2+v^2+w^2)^{5/2}} \frac{\partial}{\partial w}(u^2+v^2+w^2)$$

$$f_{uvw} = \frac{6uvw}{2(u^2+v^2+w^2)^{5/2}}$$

$$f_{uvw} = \frac{3uvw}{(u^2+v^2+w^2)^{5/2}}$$

■ 2. Find and identify the curve for the set of the points (x, y) where $f_{xx} = f_{yy}$.

$$f(x, y) = 3x^3 - 4x^2y + y^3 - x^2 + 5y + 7$$

Solution:

Find f_{xx} and f_{yy}.

$$f_x = 9x^2 - 8xy - 2x$$

$$f_{xx} = 18x - 8y - 2$$

and

$$f_y = -4x^2 + 3y^2 + 5$$

$$f_{yy} = 6y$$

Since $f_{xx} = f_{yy}$, we get the equation

$$18x - 8y - 2 = 6y$$

$$9x - 7y - 1 = 0$$

This is the equation of a line in the xy-plane with the equation $9x - 7y - 1 = 0$.

■ 3. Find and identify the curve(s) for the set of the points (x, y) where $f_{xx} = f_{yy}$.

$$f(x, y) = \sin(x^2 + y^2)$$

Solution:

Find f_{xx}.

$$f_x = \cos(x^2 + y^2)\frac{\partial}{\partial x}(x^2 + y^2)$$

$$f_x = 2x\cos(x^2 + y^2)$$

so

$$f_{xx} = 2\cos(x^2 + y^2) + 2x\frac{\partial}{\partial x}\cos(x^2 + y^2)$$

$$f_{xx} = 2\cos(x^2 + y^2) + 2x \cdot (-2x\sin(x^2 + y^2))$$

$$f_{xx} = 2\cos(x^2 + y^2) - 4x^2\sin(x^2 + y^2)$$

Because x and y are symmetric,

$$f_{yy} = 2\cos(x^2 + y^2) - 4y^2 \sin(x^2 + y^2)$$

So $f_{xx} = f_{yy}$ gives

$$2\cos(x^2 + y^2) - 4x^2 \sin(x^2 + y^2) = 2\cos(x^2 + y^2) - 4y^2 \sin(x^2 + y^2)$$

$$(x^2 - y^2)\sin(x^2 + y^2) = 0$$

The solutions are

$$x^2 - y^2 = 0$$

$$(x - y)(x + y) = 0$$

$x = y$ or $x = -y$

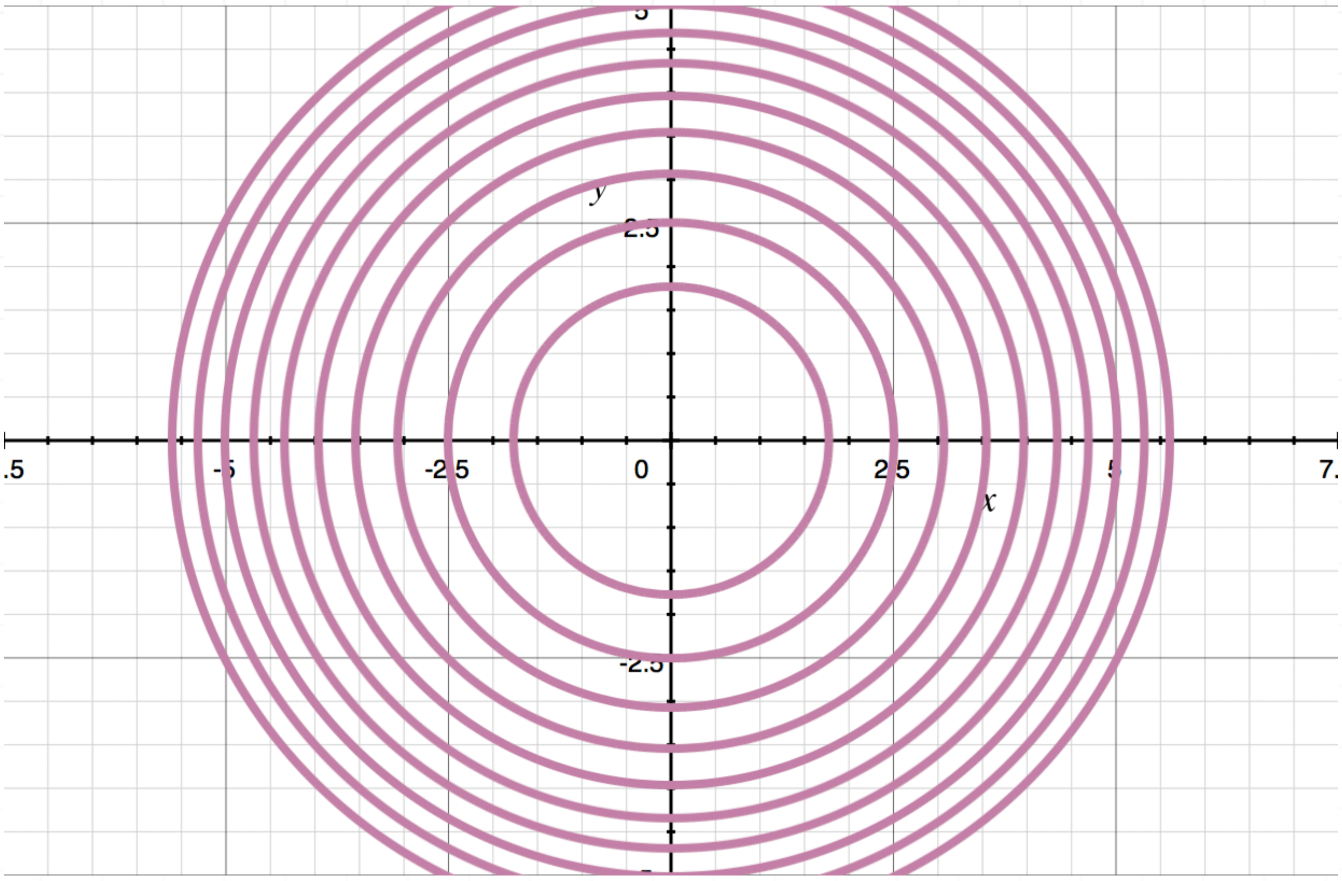
and

$$\sin(x^2 + y^2) = 0$$

$x^2 + y^2 = \pi k$ for any integer k

Since $x^2 + y^2 \geq 0$, $x^2 + y^2 = \pi k$ for any nonnegative integer k. We have a sequence of concentrated circles with radii $\sqrt{\pi k}$.

So the set of points is the pair of intersecting lines, $x = y$ and $x = -y$, and a sequence of concentrated circles with radii $\sqrt{\pi k}$ for any nonnegative integer k.



■ 4. Find all four second-order partial derivatives for the function. Is $f_{ts} = f_{st}$?

$$f(t, s) = e^{ts}$$

Solution:

Find the partial derivatives

$$f_t = se^{ts}$$

$$f_{tt} = s^2e^{ts}$$

$$f_{ts} = tse^{ts} + e^{ts}$$

and

$$f_s = te^{ts}$$

$$f_{ss} = t^2 e^{ts}$$

$$f_{st} = tse^{ts} + e^{ts}$$

The four partial derivatives are

$$f_{tt} = s^2 e^{ts}$$

$$f_{ss} = t^2 e^{ts}$$

$$f_{ts} = f_{st} = tse^{ts} + e^{ts}$$

■ 5. Find the nth-order partial derivatives $\partial^n/\partial x^n$ and $\partial^n/\partial y^n$ by looking for patterns in the partial derivatives with respect to x and y.

$$f(x, y) = 2^{2x+4y}$$

Solution:

Look for a pattern in the partial derivatives with respect to x.

$$f_x = \ln(2) 2^{2x+4y} \frac{\partial}{\partial x}(2x + 4y)$$

$$f_x = 2 \ln(2) 2^{2x+4y} = \ln(2) \cdot 2^{2x+4y+1}$$

$$f_{xx} = \ln^2(2) 2^{2x+4y+1} \frac{\partial}{\partial x}(2x + 4y + 1)$$

$$f_{xx} = 2 \ln^2(2) 2^{2x+4y+1} = \ln^2(2) \cdot 2^{2x+4y+2}$$

$$f_{xxx} = \ln^3(2)2^{2x+4y+2}\frac{\partial}{\partial x}(2x+4y+2)$$

$$f_{xxx} = 2\ln^3(2)2^{2x+4y+2} = \ln^3(2)\cdot 2^{2x+4y+3}$$

...

$$\frac{\partial^n}{\partial x^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+n}$$

Look for a pattern in the partial derivatives with respect to y.

$$f_y = \ln(2)2^{2x+4y}\frac{\partial}{\partial y}(2x+4y)$$

$$f_y = 4\ln(2)2^{2x+4y} = \ln(2)\cdot 2^{2x+4y+2}$$

$$f_{yy} = \ln^2(2)2^{2x+4y+2}\frac{\partial}{\partial y}(2x+4y+2)$$

$$f_{yy} = 4\ln^2(2)2^{2x+4y+2} = \ln^2(2)\cdot 2^{2x+4y+4}$$

...

$$\frac{\partial^n}{\partial y^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+2n}$$

So the nth-order partial derivatives are

$$\frac{\partial^n}{\partial x^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+n}$$

$$\frac{\partial^n}{\partial y^n}f(x,y) = \ln^n(2)\cdot 2^{2x+4y+2n}$$

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