Calculus 1 Quizzes

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MATH

Topic: Idea of the limit

Question: What statement is being made by the limit equation?

$$\lim_{x \to 3} (x^2 - 1) = 8$$

Answer choices:

A The limit as x approaches 8 of the function $f(x) = x^2 - 1$ is 3.

B The limit as x approaches 3 of the function $f(x) = x^2 - 1$ is not 8.

C The limit as x approaches 8 of the function $f(x) = x^2 - 1$ is not 3.

D The limit as x approaches 3 of the function $f(x) = x^2 - 1$ is 8.

Solution: D

Break down the limit

$$\lim_{x \to 3}(x^2 - 1) = 8$$

into its component parts:

- x approaches 3
- the function is $f(x) = x^2 - 1$
- the value of the limit is 8

Putting these pieces together gives a full statement of the limit:

"The limit as x approaches 3 of the function $f(x) = x^2 - 1$ is equal to 8."

Topic: Idea of the limit

Question: Use limit notation to write the limit of the function $f(x)$ as x approaches 3.

$$f(x) = \frac{x-6}{x}$$

Answer choices:

A $\lim_{x \to -3} f(x) = \frac{x-6}{x}$

B $\lim_{x \to 3} f(x) = \frac{x-6}{x}$

C $\lim_{x \to 3} \frac{x-6}{x}$

D $\lim_{x \to -3} \frac{x-6}{x}$

Solution: C

When a is the value that x approaches, and $f(x)$ is the given function, the limit is written as

$$\lim_{x \to a} f(x)$$

In this case, x approaches 3, so $a = 3$, and the function is

$$f(x) = \frac{x - 6}{x}$$

So we'd write the limit as

$$\lim_{x \to 3} \frac{x - 6}{x}$$

Topic: Idea of the limit

Question: Evaluate the limit.

$$\lim_{x \to 3} \frac{x-6}{x}$$

Answer choices:

A -3

B 3

C -1

D 1

Solution: C

To evaluate the limit,

$$\lim_{x \to 3} \frac{x - 6}{x}$$

plug the value that's being approached into the function, then simplify the answer.

$$\frac{3 - 6}{3}$$

$$\frac{-3}{3}$$

$$-1$$

Topic: One-sided limits

Question: Find the left-hand limit.

$$\lim_{x \to 2^-} \frac{|x-2|}{x-2}$$

Answer choices:

A -1

B 1

C -2

D 2

Solution: A

If we try substitution to evaluate the limit, we get the undefined value $0/0$. Instead, let's try substituting a value to the left of $x = 2$ that's very close to $x = 2$, like $x = 1.9999$.

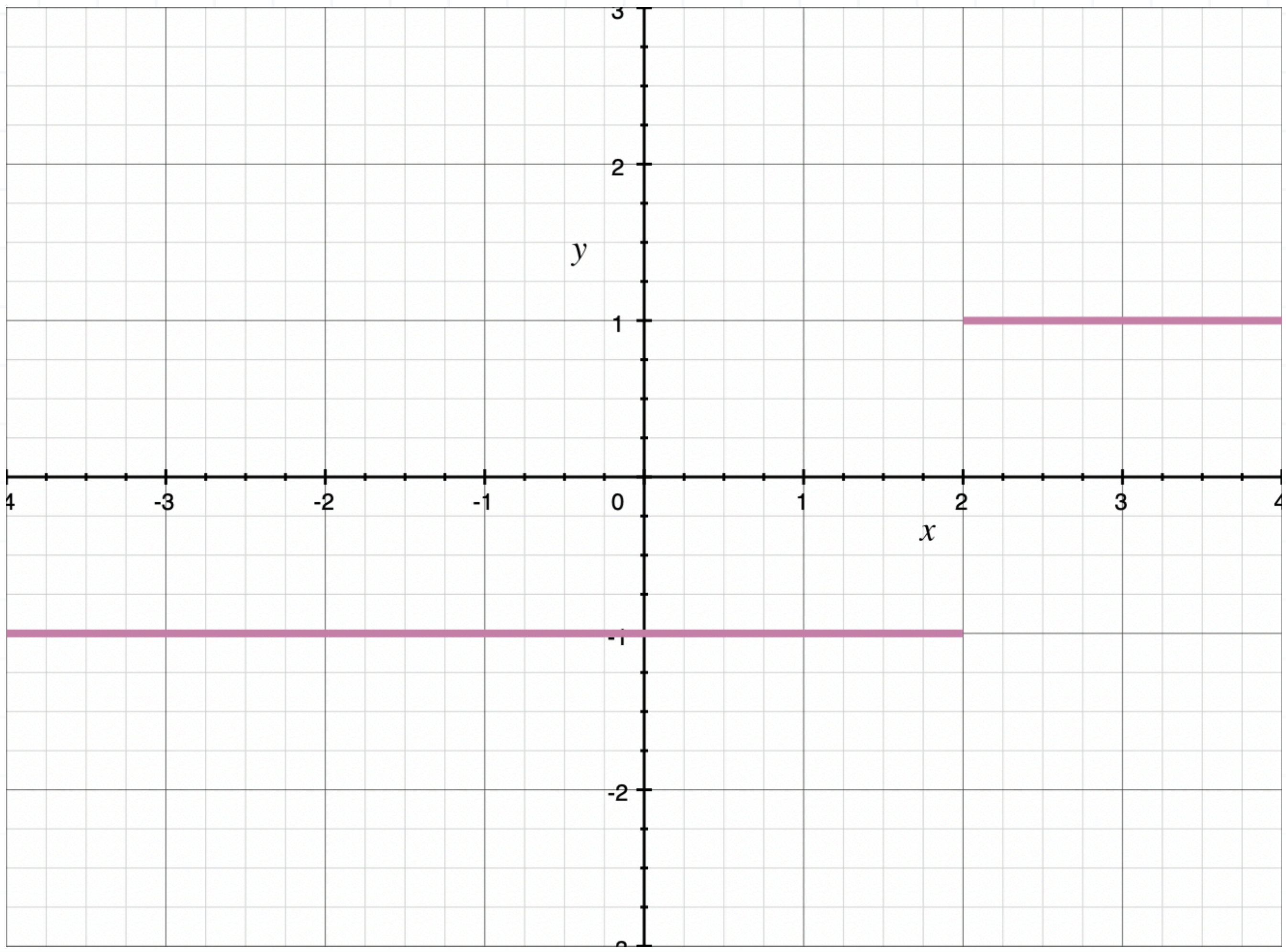
$$\frac{|1.9999 - 2|}{1.9999 - 2}$$

$$\frac{|-0.0001|}{-0.0001}$$

$$\frac{0.0001}{-0.0001}$$

$$-1$$

As we approach $x = 2$ from the left, the function is a constant -1 (the numerator is always positive and the denominator is always negative). The graph of the function confirms this value for the left-hand limit.


y
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3
2
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0
-1
-2
-3
-4
-3
-2
-1
1
2
3
4

Topic: One-sided limits

Question: Find the right-hand limit.

$$\lim_{x \to 2^+} \frac{|x-2|}{x-2}$$

Answer choices:

A -1

B 1

C -2

D 2

Solution: B

If we try substitution to evaluate the limit, we get the undefined value $0/0$. Instead, let's try substituting a value to the right of $x = 2$ that's very close to $x = 2$, like $x = 2.0001$.

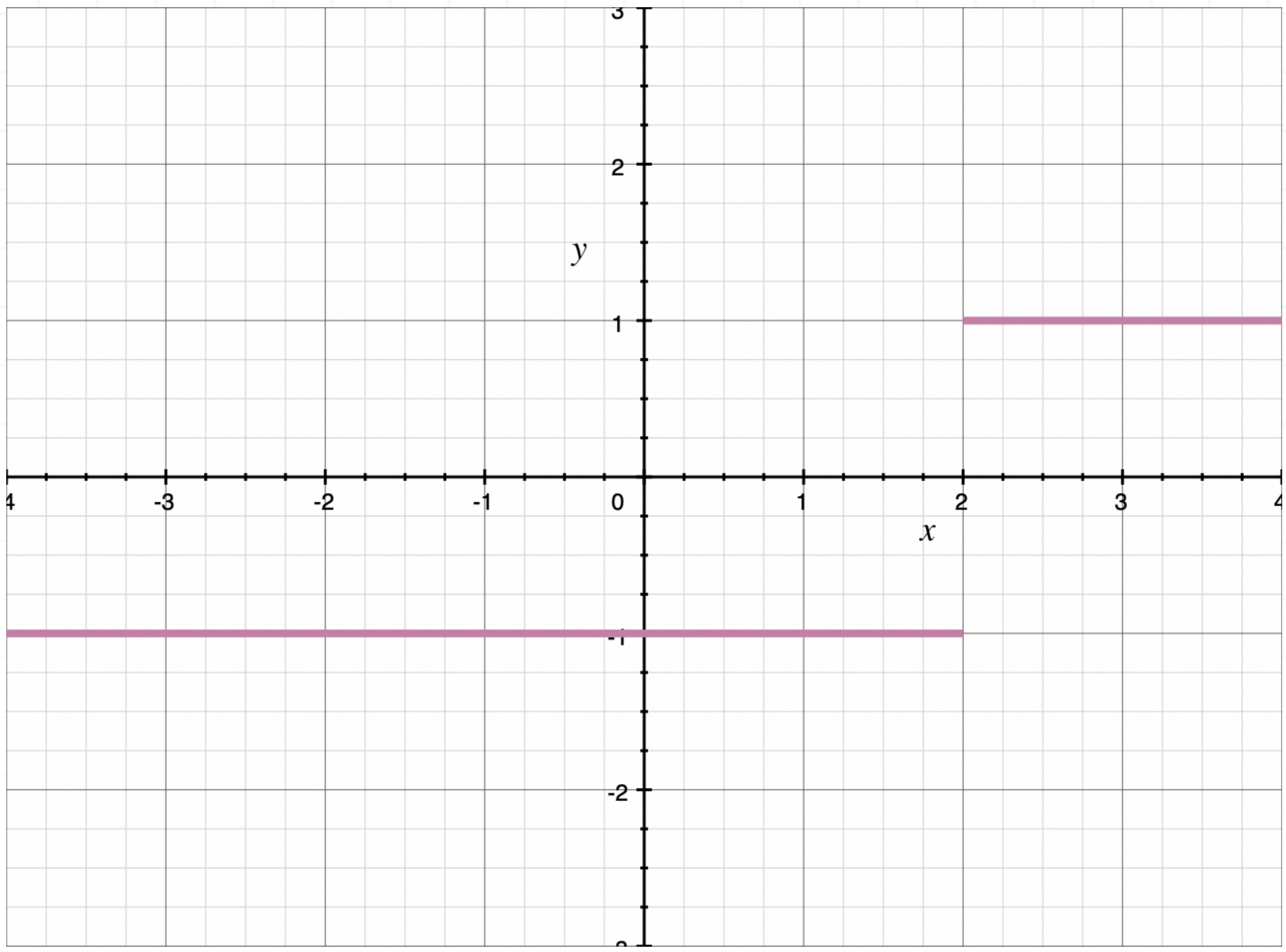
$$\frac{|2.0001 - 2|}{2.0001 - 2}$$

$$\frac{|0.0001|}{0.0001}$$

$$\frac{0.0001}{0.0001}$$

$$1$$

As we approach $x = 2$ from the right, the function is a constant 1 (the numerator is always positive and the denominator is always positive). The graph of the function confirms this value for the right-hand limit.


3
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0
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2
3
4
x
-1
-2
-3

Topic: One-sided limits

Question: Find the limit.

$$\lim_{x \to 2} \frac{|x-2|}{x-2}$$

Answer choices:

A -1

B 1

C -2

D Does not exist (DNE)

Solution: D

We can see the left-hand limit of the function at $x = 2$ if we try substituting $x = 1.9999$.

$$\frac{|1.9999 - 2|}{1.9999 - 2}$$

$$\frac{|-0.0001|}{-0.0001}$$

$$\frac{0.0001}{-0.0001}$$

$$-1$$

We can see the right-hand limit of the function at $x = 2$ if we try substituting $x = 2.0001$.

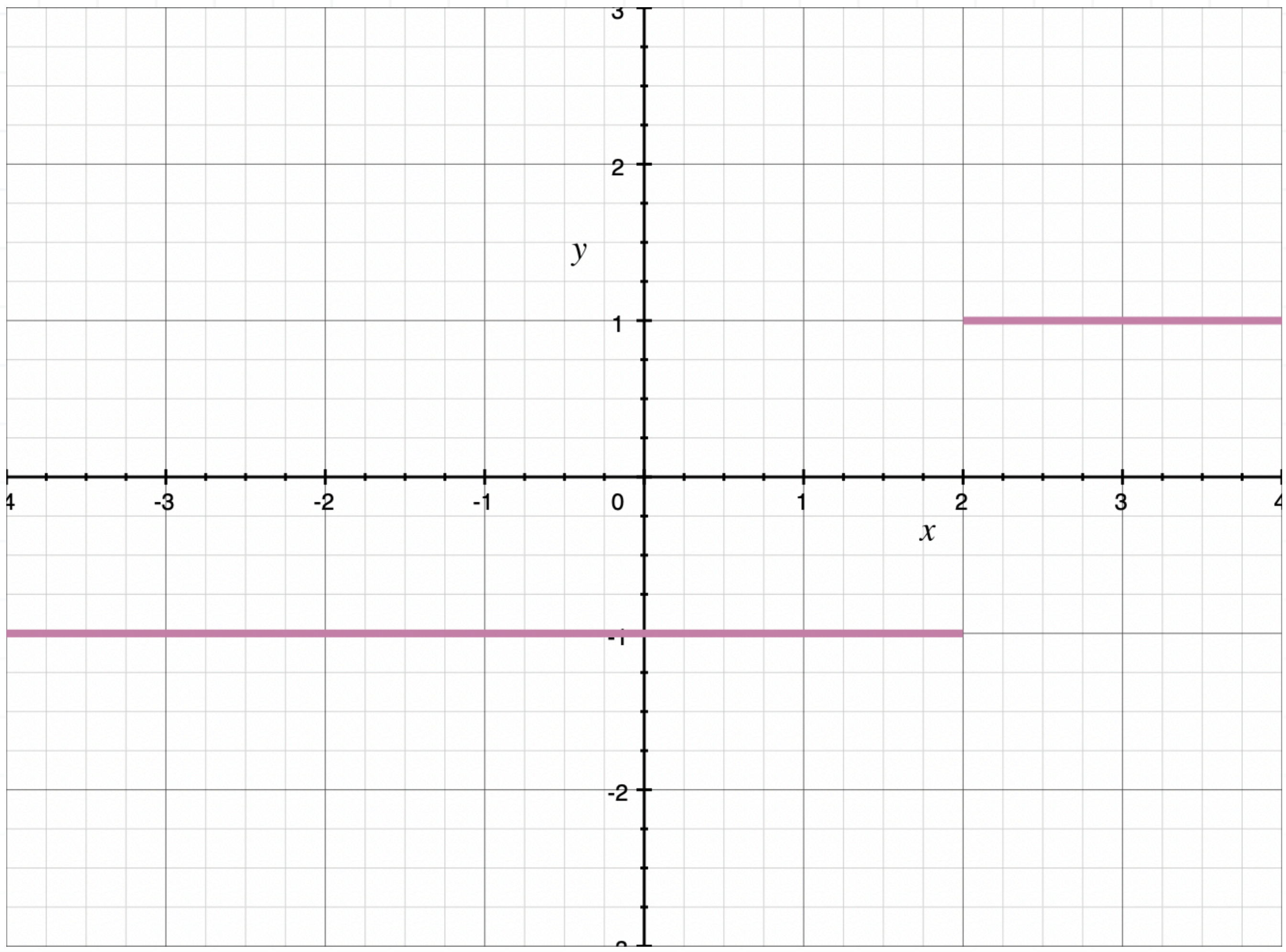
$$\frac{|2.0001 - 2|}{2.0001 - 2}$$

$$\frac{|0.0001|}{0.0001}$$

$$\frac{0.0001}{0.0001}$$

$$1$$

The graph of the function confirms these one-sided limits.



Because the one-sided limits aren't equivalent, the general limit of the function doesn't exist at $x = 2$.

Topic: Proving that the limit does not exist

Question: How do we know that the limit does not exist?

$$\lim_{x \to 0} \frac{1}{x}$$

Answer choices:

A The left-hand limit is 0.

B The right-hand limit does not equal the left-hand limit.

C The right-hand limit is equal to the left-hand limit.

D The right-hand limit exists.

Solution: B

We know that the general limit only exists if both the left- and right-hand limits exist. We can't find the limit using substitution, so we'll pick values close to $x = 0$, but on either side of it, to get an idea of what the one-sided limits are doing.

$$f(-0.0001) = \frac{1}{-0.0001} = -10{,}000$$

$$f(0.0001) = \frac{1}{0.0001} = 10{,}000$$

These values tell us that the left-hand limit is $-\infty$, and the right-hand limit is ∞. Because the one-sided limits aren't equal, the general limit does not exist.

Topic: Proving that the limit does not exist

Question: How do we know that the limit does not exist?

$$\lim_{x \to 0} \frac{\sqrt{x^2}}{x}$$

Answer choices:

A The right-hand limit is 1.

B The right-hand limit is equal to the left-hand limit.

C The right-hand limit does not equal the left-hand limit.

D The left-hand limit does exist.

Solution: C

We know that the general limit only exists if both the left- and right-hand limits exist. We can't find the limit using substitution, so we'll pick values close to $x = 0$, but on either side of it, to get an idea of what the one-sided limits are doing.

$$f(-0.0001) = \frac{\sqrt{(-0.0001)^2}}{-0.0001} = \frac{0.0001}{-0.0001} = -1$$

$$f(0.0001) = \frac{\sqrt{(0.0001)^2}}{0.0001} = \frac{0.0001}{0.0001} = 1$$

These values tell us that the left-hand limit is -1, and the right-hand limit is 1. Because the one-sided limits aren't equal, the general limit does not exist.

Topic: Proving that the limit does not exist

Question: How do we know that the limit does not exist?

$$\lim_{x\to 1} \ln(x-1)$$

Answer choices:

A The right-hand limit does not exist.

B The right-hand limit is equal to the left-hand limit.

C The left-hand limit is approaching $-\infty$.

D The left-hand limit does not exist.

Solution: D

We know that the general limit only exists if both the left- and right-hand limits exist. We can't find the limit using substitution (because the natural log function is undefined when the argument is 0, and substituting $x = 1$ makes the argument 0), so we'll pick values close to $x = 1$, but on either side of it, to get an idea of what the one-sided limits are doing.

$$f(0.9999) = \ln(0.9999 - 1) = \ln(-0.0001) = \text{DNE}$$

$$f(1.0001) = \ln(1.0001 - 1) = \ln(0.0001) = -\infty$$

The natural log function isn't defined in real numbers for negative arguments. So, because we end up with $\ln(-0.0001)$ when we investigate the left-hand limit, we can say that the left-hand limit does not exist (DNE).

This fact alone tells us that the general limit does not exist, since both of the one-sided limits must exist in order for the general limit to be defined.

Topic: Precise definition of the limit

Question: Which of these is the precise definition of the limit?

Answer choices:

A Let f be a function defined on a closed interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < x - c < \delta$, then $f(x) - L < \epsilon$.

B Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

C Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $|f(x) - L| < \epsilon$, then $0 < |x - c| < \delta$.

Solution: B

The correct statement of the precise definition of the limit is:

Let f be a function defined on an open interval containing c (except possibly at c itself) and let L be a real number. The statement $\lim_{x \to c} f(x) = L$ means that for each $\epsilon > 0$ there exists a $\delta > 0$ such that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.

Topic: Precise definition of the limit

Question: Use the the precise definition of the limit to prove the value of the limit by finding a relationship between ϵ and δ that guarantees the limit exists.

$$\lim_{x \to 2}(x - 1) = 1$$

Answer choices:

A $\quad \delta = \epsilon^2$

B $\quad \delta = \sqrt{\epsilon}$

C $\quad \delta = \epsilon$

D $\quad \delta = \frac{\epsilon}{2}$

Solution: C

To prove the limit equation,

$$\lim_{x \to 2}(x - 1) = 1$$

we need to show that, on some open interval surrounding $x = 2$, for every $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|(x - 1) - 1| < \epsilon \text{ whenever } 0 < |x - 2| < \delta$$

Let $\epsilon > 0$ and $0 < |x - 2| < \delta$. We need to find a δ (which will be in terms of ϵ) that will give $|(x - 1) - 1| < \epsilon$. So,

$$|(x - 1) - 1| < \epsilon$$

$$|x - 2| < \epsilon$$

Now if $|x - 2| < \epsilon$ and $0 < |x - 2| < \delta$, then if $\epsilon > 0$, then $\delta = \epsilon$. Therefore, the limit equation is true.

Topic: Precise definition of the limit

Question: True or false? The precise definition of the limit implies that picking a value of x inside the δ interval will return a resulting value in the ϵ interval.

Answer choices:

A True

B False

Solution: A

According to the epsilon-delta definition of the limit, choosing a value for x between $x - \delta$ and $x + \delta$ will return a function value between $L - \epsilon$ and $L + \epsilon$.

Topic: Limits of combinations

Question: If $f(x) = x - 5$ and $g(x) = 3$, evaluate the limit.

$$\lim_{x\to 2}[f(x) - g(x)]$$

Answer choices:

A $\lim_{x\to 2}[f(x) - g(x)] = 6$

B $\lim_{x\to 2}[f(x) - g(x)] = -6$

C $\lim_{x\to 2}[f(x) - g(x)] = 0$

D $\lim_{x\to 2}[f(x) - g(x)] = \infty$

Solution: B

We'll start by distributing the limit across the combination.

$$\lim_{x \to 2} [f(x) - g(x)]$$

$$\lim_{x \to 2} f(x) - \lim_{x \to 2} g(x)$$

$$\lim_{x \to 2} (x - 5) - \lim_{x \to 2} (3)$$

Now we'll substitute the value we're approaching into each function.

$$(2 - 5) - (3)$$

$$-3 - 3$$

$$-6$$

Topic: Limits of combinations

Question: If $f(x) = x^3$ and $g(x) = 2 - x^2$, evaluate the limit.

$$\lim_{x\to 3} 2f(x)g(x)$$

Answer choices:

A $\lim_{x\to 3} 2f(x)g(x) = 189$

B $\lim_{x\to 3} 2f(x)g(x) = -189$

C $\lim_{x\to 3} 2f(x)g(x) = 378$

D $\lim_{x\to 3} 2f(x)g(x) = -378$

Solution: D

We'll start by distributing the limit across the combination.

$$\lim_{x\to 3} 2f(x)g(x)$$

$$\lim_{x\to 3} 2f(x) \lim_{x\to 3} g(x)$$

$$2\lim_{x\to 3} f(x) \lim_{x\to 3} g(x)$$

$$2\lim_{x\to 3}(x^3) \lim_{x\to 3}(2-x^2)$$

Now we'll substitute the value we're approaching into each function.

$$2(3^3)(2-3^2)$$

$$2(27)(-7)$$

$$-378$$

Topic: Limits of combinations

Question: If $f(x) = x^2 + 2x + 1$ and $g(x) = x - 1$, evaluate the limit.

$$\lim_{x \to -1} \frac{f(x)}{4g(x)}$$

Answer choices:

A $\lim_{x \to -1} \frac{f(x)}{4g(x)} = -\infty$

B $\lim_{x \to -1} \frac{f(x)}{4g(x)} = \infty$

C $\lim_{x \to -1} \frac{f(x)}{4g(x)} = 0$

D The limit does not exist (DNE)

Solution: C

We'll start by plugging $f(x)$ and $g(x)$ into the limit.

$$\lim_{x \to -1} \frac{f(x)}{4g(x)}$$

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{4(x - 1)}$$

Now we'll substitute the value we're approaching into the function.

$$\frac{(-1)^2 + 2(-1) + 1}{4(-1 - 1)}$$

$$\frac{1 - 2 + 1}{4(-2)}$$

$$\frac{0}{-8}$$

$$0$$

Topic: Limits of composites

Question: If $f(x) = x^3$ and $g(x) = x^2 + 3$, evaluate the limit.

$$\lim_{x\to 5} f(g(x))$$

Answer choices:

A $\lim_{x\to 5} f(g(x)) = 21{,}952$

B $\lim_{x\to 5} f(g(x)) = 81$

C $\lim_{x\to 5} f(g(x)) = 15{,}628$

D $\lim_{x\to 5} f(g(x)) = 253$

Solution: A

First find the composite $f(g(x))$, when $f(x) = x^3$ and $g(x) = x^2 + 3$.

$$f(x) = x^3$$

$$f(g(x)) = (x^2 + 3)^3$$

Then find the limit of the composite function.

$$\lim_{x \to 5} f(g(x))$$

$$\lim_{x \to 5} (x^2 + 3)^3$$

$$(5^2 + 3)^3$$

$$28^3$$

$$21{,}952$$

Topic: Limits of composites

Question: If $f(x) = \cos x$ and $g(x) = x + 4$, evaluate the limit.

$$\lim_{x \to -4} f(g(x))$$

Answer choices:

A $\lim_{x \to -4} f(g(x)) = -1$

B $\lim_{x \to -4} f(g(x)) = 0$

C $\lim_{x \to -4} f(g(x)) = 1$

D The limits does not exist (DNE)

Solution: C

First find the composite $f(g(x))$, when $f(x) = \cos x$ and $g(x) = x + 4$.

$$f(x) = \cos x$$

$$f(g(x)) = \cos(x + 4)$$

Then find the limit of the composite function.

$$\lim_{x \to -4} f(g(x))$$

$$\lim_{x \to -4} \cos(x + 4)$$

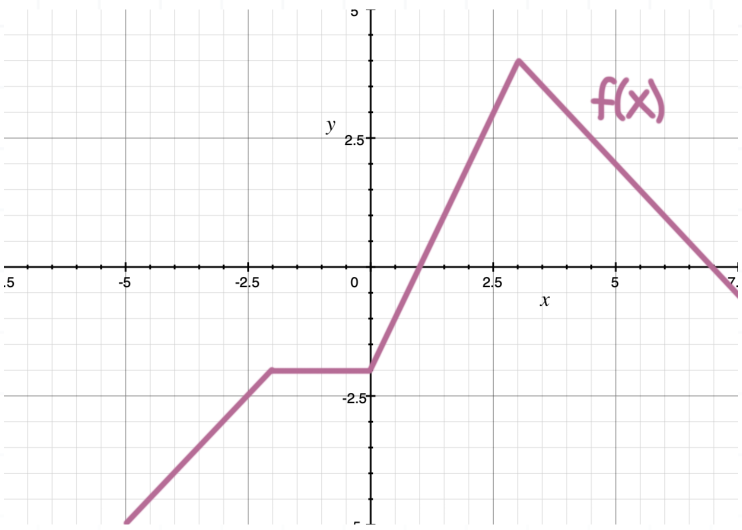
$$\cos(-4 + 4)$$

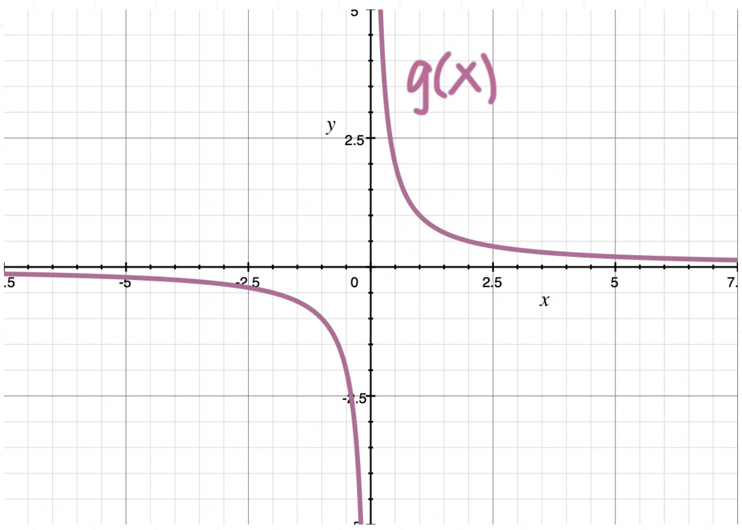
$$\cos(0)$$

$$1$$

Topic: Limits of composites

Question: Given the graphs of $f(x)$ and $g(x)$, find $\lim_{x\to 0} f(g(x))$.





Answer choices:

A $\lim_{x\to 0} f(g(x)) = 0$

B $\lim_{x\to 0} f(g(x)) = \infty$

C $\lim_{x\to 0} f(g(x)) = -2$

D $\lim_{x\to 0} f(g(x)) = \text{DNE}$

Solution: D

Use the theorem for limits of composite functions.

$$\lim_{x \to 0} f(g(x)) = f(\lim_{x \to 0} g(x))$$

From the graph of $g(x)$, we can see that

$$\lim_{x \to 0} g(x) = \text{DNE}$$

Therefore, $\lim_{x \to 0} f(g(x))$ does not exist.

Topic: Point discontinuities

Question: Find the removable discontinuities of the function.

$$f(x) = \frac{x-5}{x^2-25}$$

Answer choices:

A $x = 0$

B $x = -5$

C $x = 5$

D $x = 25$

Solution: C

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{x - 5}{x^2 - 25}$$

$$f(x) = \frac{x - 5}{(x + 5)(x - 5)}$$

The factor $x - 5$ can be canceled from the numerator and denominator.

$$f(x) = \frac{1}{x + 5}$$

Because $x = 5$ is a value that *would have* made the denominator 0, but we canceled it out when we canceled $x - 5$, we know that the function has a removable discontinuity at $x = 5$.

Topic: Point discontinuities

Question: Find the removable discontinuities of the function.

$$f(x) = \frac{x-1}{x^2+x-2}$$

Answer choices:

A $x = 1$

B $x = -1$

C $x = 2$

D $x = -2$

Solution: A

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{x-1}{x^2+x-2}$$

$$f(x) = \frac{x-1}{(x+2)(x-1)}$$

The factor $x-1$ can be canceled from the numerator and denominator.

$$f(x) = \frac{1}{x+2}$$

Because $x=1$ is a value that *would have* made the denominator 0, but we canceled it out when we canceled $x-1$, we know that the function has a removable discontinuity at $x=1$.

Topic: Point discontinuities

Question: Find the removable discontinuities of the function.

$$f(x) = \frac{x^2 + 2x - 3}{x^2 + x - 2}$$

Answer choices:

A $x = -3$

B $x = -2$

C $x = -1$

D $x = 1$

Solution: D

Factor the numerator and denominator as completely as possible.

$$f(x) = \frac{x^2 + 2x - 3}{x^2 + x - 2}$$

$$f(x) = \frac{(x+3)(x-1)}{(x+2)(x-1)}$$

The factor $x - 1$ can be canceled from the numerator and denominator.

$$f(x) = \frac{x+3}{x+2}$$

Because $x = 1$ is a value that *would have* made the denominator 0, but we canceled it out when we canceled $x - 1$, we know that the function has a removable discontinuity at $x = 1$.

Topic: Jump discontinuities

Question: Which of the following statements is true?

Answer choices:

A A jump discontinuity occurs when the right- and left-hand limits both exist, but aren't equal.

B A jump discontinuity occurs when the right- and left-hand limits are not equal, and only one exists.

C A jump discontinuity occurs when the right- and left-hand limits do not exist.

D A jump discontinuity occurs when the right- and left-hand limits both exist, and they're equal.

Solution: A

For a jump discontinuity to occur, the left- and right-hand limits must both exist, but they must not be equal. In other words, the left-hand limit will exist, and the right-hand limit will exist, but the left- and right-hand limits will have different values.

Topic: Jump discontinuities

Question: Choose the correct description of the jump discontinuity that exists in the function.

$$f(x) = \frac{x}{|x|}$$

Answer choices:

A The function has a jump discontinuity at $x = 1$.

B The function has a jump discontinuity at $x = -1$.

C The function has a jump discontinuity at $x = \infty$.

D The function has a jump discontinuity at $x = 0$.

Solution: D

To answer this question, we could investigate the limit at each of the values given in the answer choices.

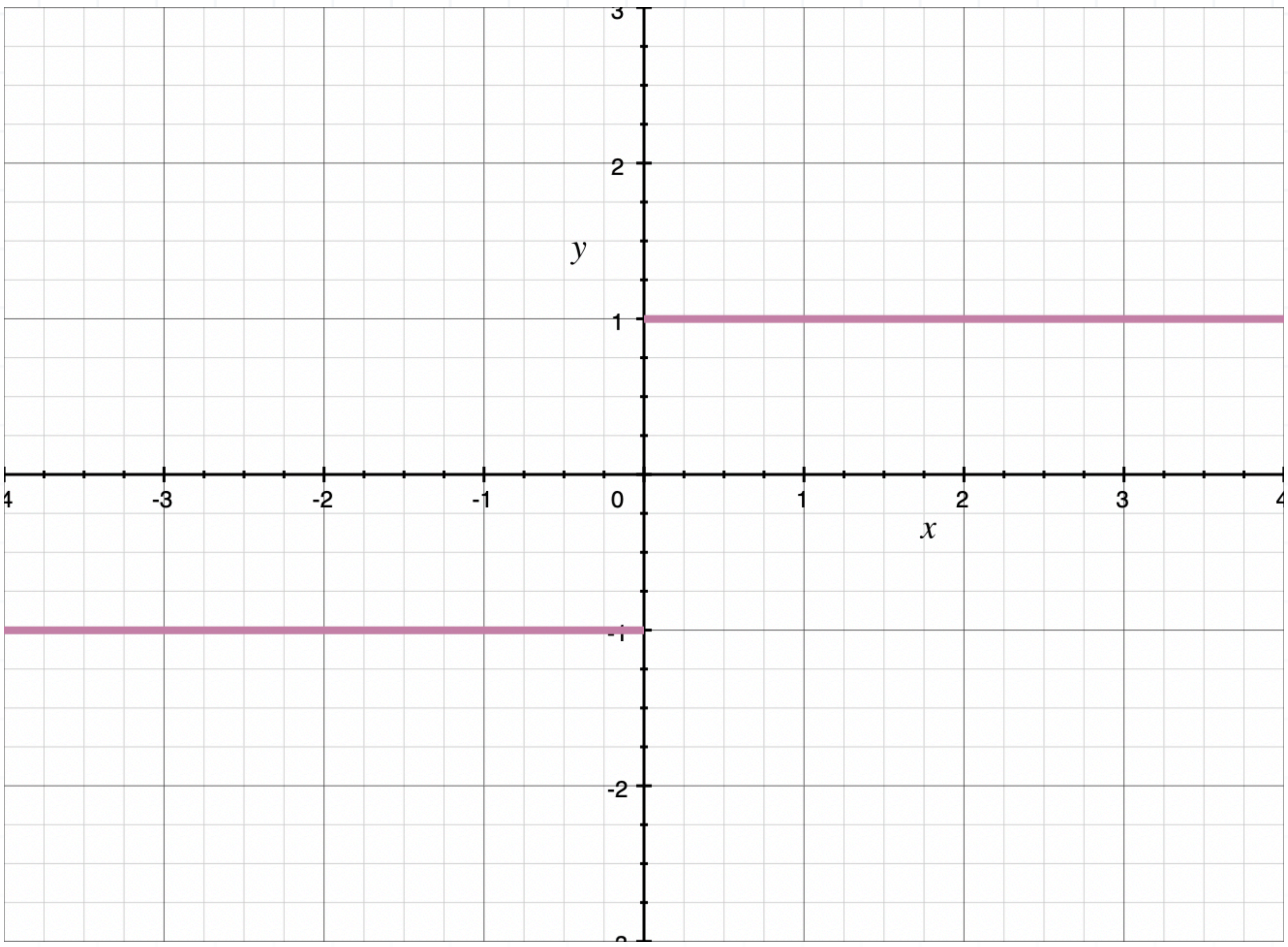
However, if we consider the answer choices briefly, or if we investigate answer choice D, we see that the function is undefined at $x = 0$. So $x = 0$ is an interesting place to start.

We'll look at what the function is doing on either side of $x = 0$.

$$f(-0.0001) = \frac{-0.0001}{|-0.0001|} = \frac{-0.0001}{0.0001} = -1$$

$$f(0.0001) = \frac{0.0001}{|0.0001|} = \frac{0.0001}{0.0001} = 1$$

What we see when we substitute values on either side of $x = 0$ is that, no matter which values we pick, any value to the left of $x = 0$ will return a value of -1, and any value to the right of $x = 0$ will return a value of 1.



Therefore, the jump discontinuity occurs at $x = 0$.

Topic: Jump discontinuities

Question: Choose the correct description of the jump discontinuity.

$$f(x) = \frac{x-1}{|x-1|}$$

Answer choices:

A The function has a jump discontinuity at $x = -1$.

B The function has a jump discontinuity at $x = 1$.

C The function has a jump discontinuity at $x = \infty$.

D The function has a jump discontinuity at $x = 0$.

Solution: B

To answer this question, we could investigate the limit at each of the values given in the answer choices.

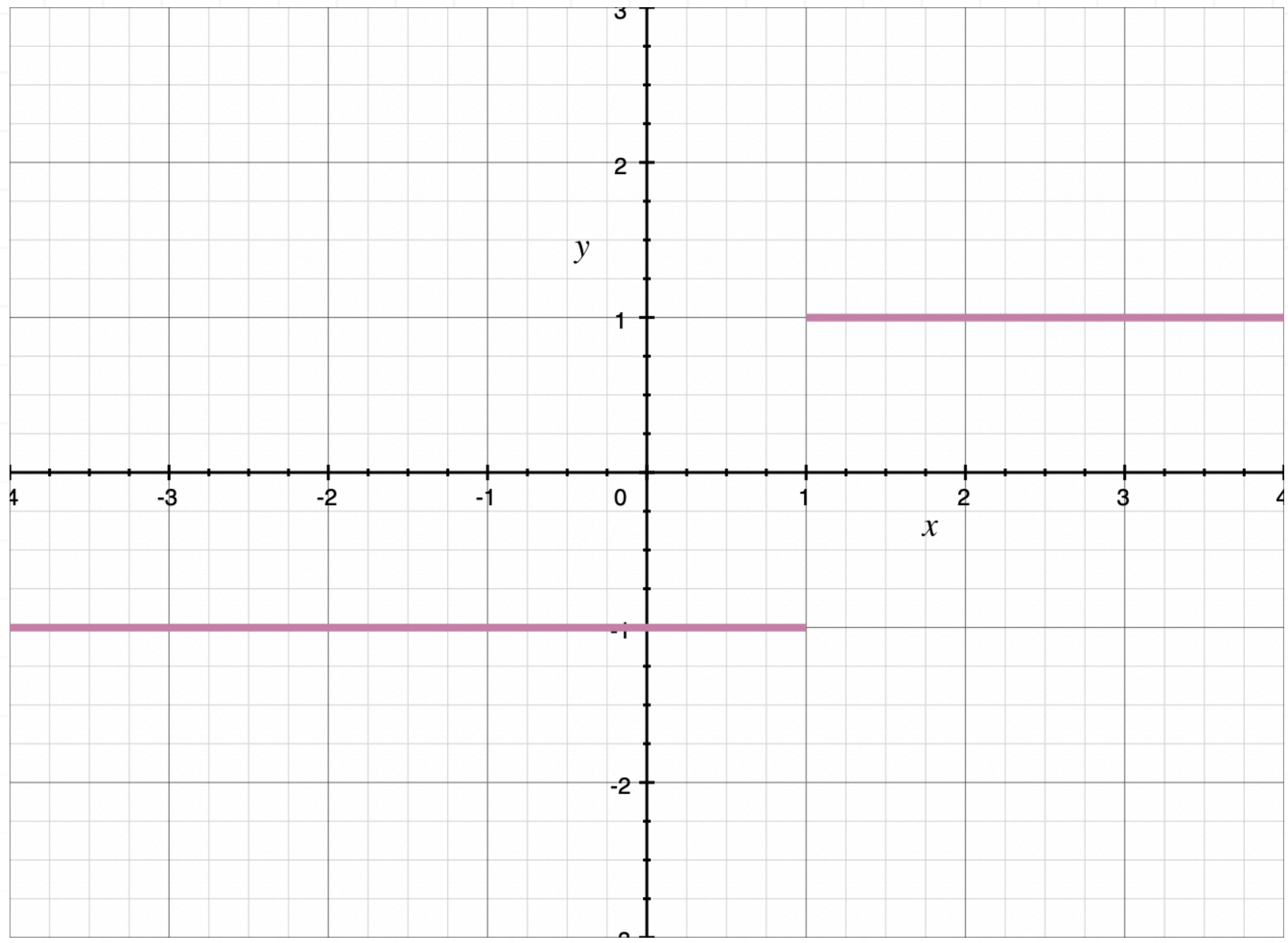
However, if we consider the answer choices briefly, or if we investigate answer choice B, we see that the function is undefined at $x=1$. So $x=1$ is an interesting place to start.

We'll look at what the function is doing on either side of $x=1$.

$$f(0.9999)=\frac{0.9999-1}{|0.9999-1|}=\frac{-0.0001}{|-0.0001|}=\frac{-0.0001}{0.0001}=-1$$

$$f(1.0001)=\frac{1.0001-1}{|1.0001-1|}=\frac{0.0001}{|0.0001|}=\frac{0.0001}{0.0001}=1$$

What we see when we substitute values on either side of $x=1$ is that, no matter which values we pick, any value to the left of $x=1$ will return a value of -1, and any value to the right of $x=1$ will return a value of 1.



Therefore, the jump discontinuity occurs at $x = 1$.

Topic: Infinite discontinuities

Question: Choose the correct statement.

Answer choices:

A An infinite discontinuity exists where the right- and left-hand limits both approach ∞, or both approach $-\infty$.

B An infinite discontinuity exists where the right-hand limit approaches $-\infty$ while the left-hand limit approaches ∞.

C An infinite discontinuity exists where the right-hand limit approaches ∞ while the left-hand limit approaches $-\infty$.

D All of the above are true.

Solution: D

In answer choices A, B, and C, both the left-hand limit and right-hand limit are tending toward an infinite value, whether that value is $-\infty$ or ∞.

Wherever both the left- and right-hand limit are approaching an infinite value, the function has a vertical asymptote, and therefore an infinite discontinuity.

Topic: Infinite discontinuities

Question: Choose the correct description of the infinite discontinuity.

$$f(x) = \frac{1}{x}$$

Answer choices:

A The function has an infinite discontinuity at $x = 1$.

B The function has an infinite discontinuity at $x = -\infty$.

C The function has an infinite discontinuity at $x = 0$.

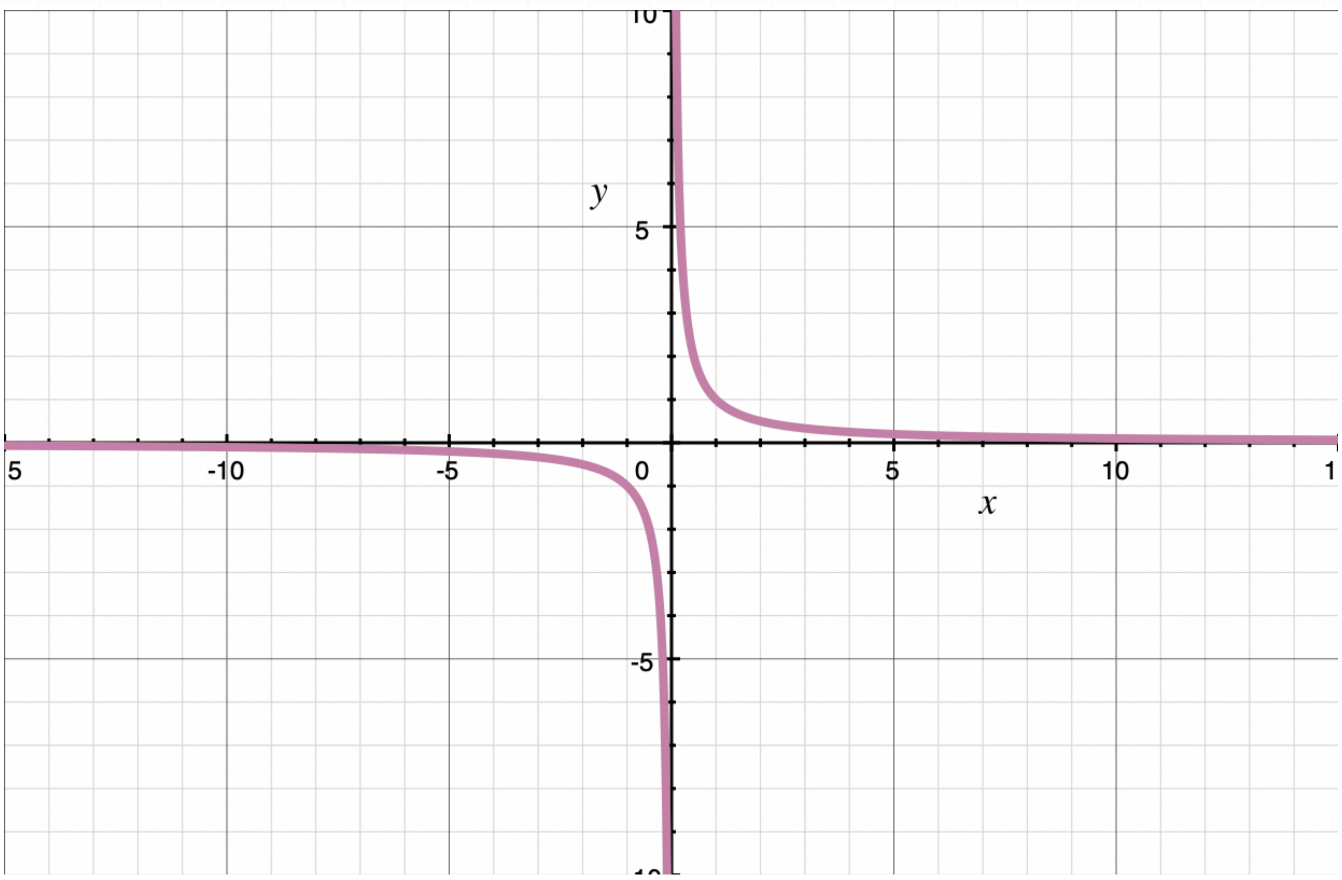
D The function has an infinite discontinuity at $x = \infty$.

Solution: C

The denominator of the function will be 0 when $x = 0$. That discontinuity can't be removed by canceling factors from the function, so $x = 0$ doesn't represent a point discontinuity.

Which means the function has a vertical asymptote, and therefore an infinite discontinuity, at $x = 0$.

The graph of the function confirms the discontinuity.



Topic: Infinite discontinuities

Question: Choose the correct description of the infinite discontinuity.

$$f(x) = \frac{x}{x-4}$$

Answer choices:

A The function has an infinite discontinuity at $x = 4$.

B The function has an infinite discontinuity at $x = -\infty$.

C The function has an infinite discontinuity at $x = -4$.

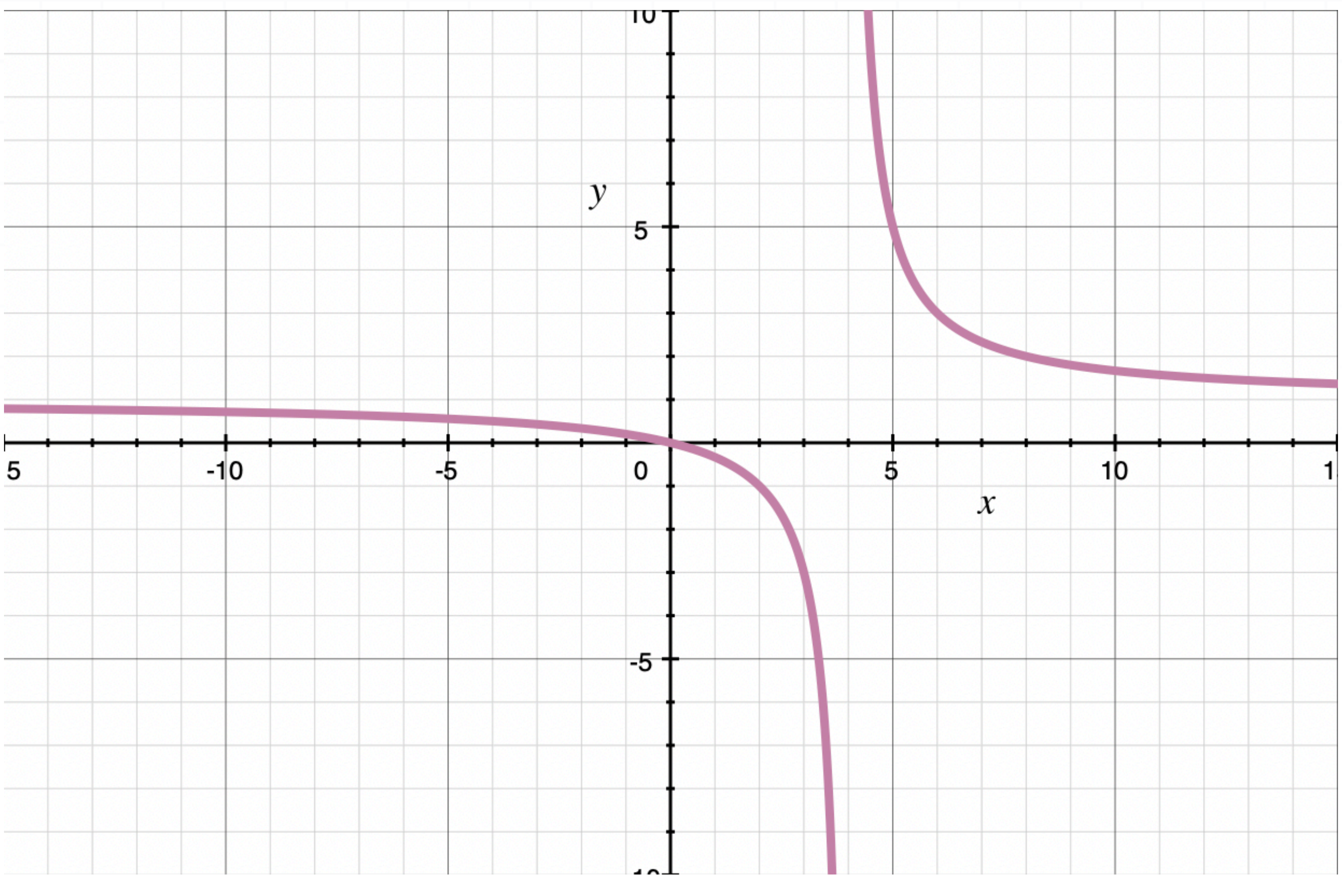
D The function has an infinite discontinuity at $x = \infty$.

Solution: A

The denominator of the function will be 0 when $x = 4$. That discontinuity can't be removed by canceling factors from the function, so $x = 4$ doesn't represent a point discontinuity.

Which means the function has a vertical asymptote, and therefore an infinite discontinuity, at $x = 4$.

The graph of the function confirms the discontinuity.



Topic: Endpoint discontinuities

Question: Which of the following statements is true?

Answer choices:

A The endpoint of an interval is discontinuous because one of the one-sided limits will be 0.

B The endpoint of an interval is discontinuous because one of the one-sided limits will be ∞.

C The endpoint of an interval is discontinuous because one of the one-sided limits will not exist.

D The endpoint of an interval is discontinuous because both of the one-sided limits will not exist.

Solution: C

The endpoint of an interval is discontinuous because one of the one-sided limits does not exist.

Because the function stops at an endpoint, either the left-hand limit will exist while the right-hand limit does not, or the right-hand limit will exist while the left-hand limit does not.

Topic: Endpoint discontinuities

Question: If the function $f(x) = x^2$ is only defined on $[1,4]$, and does not extend beyond that interval, what are the discontinuities of the function?

Answer choices:

A Endpoint discontinuities at $x = 0,\ 4$.

B A jump discontinuity at $x = 0$.

C Endpoint discontinuities at $x = 1,\ 4$ and a jump discontinuity at $x = 0$.

D Endpoint discontinuities at $x = 1,\ 4$.

Solution: D

The endpoints of an interval are discontinuous for a function because one of the one-sided limits will not exist at each endpoint.

The function $f(x) = x^2$ is a continuous function, but the interval $[1,4]$ means that there will be endpoint discontinuities at $x = 1$ and $x = 4$.

At $x = 1$, only the right-hand limit exists. The left-hand limit would be outside the function's domain. By the definition of continuity (that the left-hand limit exists, the right-hand limit exists, and the left- and right-hand limits are equal), that means the function isn't continuous at $x = 1$, so there's an endpoint discontinuity there.

At $x = 4$, only the left-hand limit exists. The right-hand limit is outside the function's domain. By the definition of continuity, that means the function isn't continuous at $x = 4$, so there's an endpoint discontinuity there.

Topic: Endpoint discontinuities

Question: What are the discontinuities of the function on the interval $[2,5]$?

$$f(x) = \sqrt{x}$$

Answer choices:

A Endpoint discontinuities at $x = 2$ and $x = 5$ and when $x \geq 0$.

B Endpoint discontinuities at $x = 2$ and $x = 5$.

C Endpoint discontinuities at $x = 2$ and $x = 5$ and when $x \leq 0$.

D Endpoint discontinuities at $x = 0$ and $x = 5$.

Solution: B

The function $f(x) = \sqrt{x}$ is a continuous function when $x \geq 0$ but the interval $[2,5]$ means that there will be endpoint discontinuities at the points $x = 2$ and $x = 5$.

An endpoint discontinuity exists at $x = 2$ because the left-hand limit doesn't exist there, and an endpoint discontinuity exists at $x = 5$ because the right-hand limit doesn't exist there.

Topic: Intermediate Value Theorem with an interval

Question: Which statement is true?

Answer choices:

A The IVT only applies to discontinuous functions.

B The IVT only applies when there's no interval.

C The IVT only applies to open intervals.

D The IVT only applies to closed intervals.

Solution: D

The Intermediate Value Theorem states that for a function on a closed interval $[a, b]$ where the function is continuous on the interval, a point c exists on the interval where $f(c) = k$.

$f(a) < k < f(b)$ and $a < c < b$

Topic: Intermediate Value Theorem with an interval

Question: Use the Intermediate Value Theorem to choose an interval over which $f(x) = x^2 + 2x - 35$ is guaranteed to have a root.

Answer choices:

A $[0,2]$

B $[0,10]$

C $[8,10]$

D $[-2,0]$

Solution: B

This function is quadratic function, so we know that it's continuous. Evaluate the function at both endpoints of the interval $[0,10]$.

$$f(0) = 0^2 + 2(0) - 35$$

$$f(0) = -35$$

and

$$f(10) = 10^2 + 2(10) - 35$$

$$f(10) = 85$$

Because the function is below the x-axis at the left edge of the interval, and above the x-axis at the right edge of the interval, we can say $f(a) < f(c) < f(b)$, or more specifically, $-35 < f(c) < 85$, where $f(c) = 0$.

Therefore, by the intermediate value theorem, it must be true that the function has a root on the interval $[0,10]$.

Topic: Intermediate Value Theorem with an interval

Question: Is there a root for the function $f(x) = x^2 - 4$ on the interval $[1,6]$?

Answer choices:

A Yes, there's a root at $(0,4)$.

B Yes, there's a root at $(0, -4)$.

C Yes, there's a root at $(2,0)$.

D Yes, there's a root at $(-2,0)$.

Solution: C

This function is quadratic function, so we know that it's continuous. Evaluate the function at both endpoints of the interval $[1,6]$.

$$f(1) = 1^2 - 4$$

$$f(1) = 1 - 4$$

$$f(1) = -3$$

and

$$f(6) = 6^2 - 4$$

$$f(6) = 36 - 4$$

$$f(6) = 32$$

The IVT confirms that the function has a root on the interval, because the function's value crosses from below the x-axis to above the x-axis at some point within that interval.

To find the root, which is the point where the graph of the function crosses the x-axis, we'll set the function equal to 0.

$$x^2 - 4 = 0$$

$$x^2 = 4$$

$$x = \pm 2$$

Therefore, the root in the interval $[1,6]$ is at $x = 2$, and that coordinate point is $(2,0)$.

Topic: Intermediate Value Theorem without an interval

Question: If we're trying to use the Intermediate Value Theorem to prove the existence of a root for the function, but no interval is given, what should we do?

Answer choices:

A We should give up, because there's no way to use the IVT if no interval is given.

B We should give up, because by definition, the function has no root if no interval is given.

C We should try to find our own interval, and the only way to do this is to guess random intervals.

D We should try to find our own interval, and to do this, we can try to consider what we might already know about the function's values, we can look at the graph of the function, or otherwise try to be strategic about how to narrow down a useful interval.

Solution: D

When no interval is given to us in which we should look for the root, it still may be possible to use the Intermediate Value Theorem to prove the existence of a root.

But we have to find an interval first. In order to do that, we can employ different techniques, and get creative in order to narrow down what interval we might be able to use.

For instance, graphing the function might show us the approximate location of the root, and we can pick values for the interval that are on either side of the root.

Or, if we know something about the shape of the graph of the function, and we can use that information to narrow down an interval, we can take that approach as well.

Topic: Intermediate Value Theorem without an interval

Question: There are no real roots for the function $f(x) = \sin x$.

Answer choices:

A True

B False

Solution: B

No interval is given, but the sine function oscillates up and down, reaching its lowest value of -1 and highest value of 1.

If it oscillates back and forth from -1 below the x-axis, to 1 above the x-axis, and because $f(x) = \sin x$ is a continuous function, it must cross the x-axis at some point, which means it has at least one real root somewhere in its domain.

Topic: Intermediate Value Theorem without an interval

Question: Find an interval for the function $f(x) = \cos x$ on which the function has a real root.

Answer choices:

A $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$

B $\left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$

C $\left(\frac{\pi}{4}, \frac{3\pi}{4}\right)$

D $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$

Solution: D

We could answer this question by evaluating the function $f(x) = \cos x$ at the endpoints of each interval given in the answer choices.

If we did that, only answer choices C and D give a negative value at one edge of the interval and a positive value at the other edge of the interval. From there, only answer choice D is a closed interval, so answer choice D can be the only correct answer.

Alternatively, we could have looked at the graph of $f(x) = \cos x$ and investigated its value at the endpoints of each of the intervals. We'd see that the graph was above the x-axis everywhere from $-\pi/4$ to $\pi/4$, thereby eliminating answer choices A and B, allowing us to conclude that D must be the correct choice, since C is not a closed interval.

Topic: Solving with substitution

Question: Use substitution to evaluate the limit.

$$\lim_{x \to 5} \frac{x^2 + 2x + 1}{x + 5}$$

Answer choices:

A 12

B 6

C 1.6

D 3.6

Solution: D

Substitute $x = 5$ into the function to evaluate the limit.

$$f(x) = \frac{x^2 + 2x + 1}{x + 5}$$

$$f(5) = \frac{5^2 + 2(5) + 1}{5 + 5}$$

$$f(5) = \frac{36}{10}$$

$$f(5) = 3.6$$

Topic: Solving with substitution

Question: Use substitution to evaluate the limit.

$$\lim_{x\to 6}(x^3 + 6 - 3x)$$

Answer choices:

A $\quad 204$

B $\quad 198$

C $\quad 240$

D $\quad 234$

Solution: A

Substitute $x = 6$ into the function to evaluate the limit.

$$f(x) = x^3 + 6 - 3x$$

$$f(6) = 6^3 + 6 - 3(6)$$

$$f(6) = 216 + 6 - 18$$

$$f(6) = 204$$

Topic: Solving with substitution

Question: Use substitution to evaluate the limit.

$$\lim_{x \to 0} \frac{-1}{3(x+3)}$$

Answer choices:

A $-\frac{1}{9}$

B $\frac{1}{9}$

C $-\frac{1}{6}$

D $\frac{1}{6}$

Solution: A

Substitute $x=0$ into the function to evaluate the limit.

$$f(x)=\frac{-1}{3(x+3)}$$

$$f(0)=\frac{-1}{3(0+3)}$$

$$f(0)=\frac{-1}{3(3)}$$

$$f(0)=-\frac{1}{9}$$

Topic: Solving with factoring

Question: Use factoring to find the limit.

$$\lim_{t \to -1} \frac{(t+1)(t^2 - t + 1)}{t + 1}$$

Answer choices:

A 0

B 3

C -1

D ∞

Solution: B

The numerator and denominator share a common factor of $t+1$, which can be canceled from the function.

$$\lim_{t \to -1} \frac{(t+1)(t^2-t+1)}{t+1}$$

$$\lim_{t \to -1} (t^2-t+1)$$

Now use substitution to evaluate the limit.

$$(-1)^2-(-1)+1$$

$$1+1+1$$

$$3$$

Topic: Solving with factoring

Question: Use factoring to find the limit.

$$\lim_{x \to 2} \frac{x^2 - 2x}{x - 2}$$

Answer choices:

A $\quad 4$

B $\quad -4$

C $\quad 2$

D $\quad -2$

Solution: C

Factor the numerator and denominator as completely as possible.

$$\lim_{x \to 2} \frac{x^2 - 2x}{x - 2}$$

$$\lim_{x \to 2} \frac{x(x - 2)}{x - 2}$$

Cancel the common factor of $x - 2$.

$$\lim_{x \to 2} x$$

Then use direct substitution to evaluate the limit.

$$2$$

Topic: Solving with factoring

Question: Use factoring to find the limit.

$$\lim_{x \to 3} \frac{x^2 - 7x + 12}{x^2 - 9}$$

Answer choices:

A $\frac{1}{3}$

B $-\frac{1}{3}$

C $\frac{1}{6}$

D $-\frac{1}{6}$

Solution: D

Factor the numerator and denominator as completely as possible.

$$\lim_{x \to 3} \frac{x^2 - 7x + 12}{x^2 - 9}$$

$$\lim_{x \to 3} \frac{(x - 4)(x - 3)}{(x + 3)(x - 3)}$$

Cancel the common factor of $x - 3$.

$$\lim_{x \to 3} \frac{x - 4}{x + 3}$$

Then use direct substitution to evaluate the limit.

$$\frac{3 - 4}{3 + 3}$$

$$-\frac{1}{6}$$

Topic: Solving with conjugate method

Question: Use conjugate method to find the limit.

$$\lim_{m \to 0} \frac{\sqrt{m+4}-2}{m}$$

Answer choices:

A 0

B 4

C $\frac{1}{4}$

D ∞

Solution: C

The conjugate of $\sqrt{m+4}-2$ is $\sqrt{m+4}+2$. Multiply both the numerator and denominator by this conjugate.

$$\lim_{m\to 0}\frac{\sqrt{m+4}-2}{m}\left(\frac{\sqrt{m+4}+2}{\sqrt{m+4}+2}\right)$$

$$\lim_{m\to 0}\frac{m+4+2\sqrt{m+4}-2\sqrt{m+4}-4}{m(\sqrt{m+4}+2)}$$

$$\lim_{m\to 0}\frac{m+4-4}{m(\sqrt{m+4}+2)}$$

$$\lim_{m\to 0}\frac{m}{m(\sqrt{m+4}+2)}$$

Cancel the common factor of m from both the numerator and denominator.

$$\lim_{m\to 0}\frac{1}{\sqrt{m+4}+2}$$

Now use substitution to evaluate the limit.

$$\frac{1}{\sqrt{0+4}+2}$$

$$\frac{1}{2+2}$$

$$\frac{1}{4}$$

Topic: Solving with conjugate method

Question: Use conjugate method to find the limit.

$$\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}$$

Answer choices:

A 6

B 3

C 9

D 0

Solution: A

The conjugate of $\sqrt{x}-3$ is $\sqrt{x}+3$. Multiply both the numerator and denominator by this conjugate.

$$\lim_{x \to 9} \frac{x-9}{\sqrt{x}-3}\left(\frac{\sqrt{x}+3}{\sqrt{x}+3}\right)$$

$$\lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$$

$$\lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{x+3\sqrt{x}-3\sqrt{x}-9}$$

$$\lim_{x \to 9} \frac{(x-9)(\sqrt{x}+3)}{x-9}$$

Cancel the common factor of $x-9$ from both the numerator and denominator.

$$\lim_{x \to 9}(\sqrt{x}+3)$$

Now use substitution to evaluate the limit.

$$\sqrt{9}+3$$

$$3+3$$

$$6$$

Topic: Solving with conjugate method

Question: Use conjugate method to find the limit.

$$\lim_{x \to 16} \frac{16 - x}{4 - \sqrt{x}}$$

Answer choices:

A 4

B 8

C 0

D 16

Solution: B

The conjugate of $4-\sqrt{x}$ is $4+\sqrt{x}$. Multiply both the numerator and denominator by this conjugate.

$$\lim_{x\to 16}\frac{16-x}{4-\sqrt{x}}\left(\frac{4+\sqrt{x}}{4+\sqrt{x}}\right)$$

$$\lim_{x\to 16}\frac{(16-x)(4+\sqrt{x})}{(4-\sqrt{x})(4+\sqrt{x})}$$

$$\lim_{x\to 16}\frac{(16-x)(4+\sqrt{x})}{16+4\sqrt{x}-4\sqrt{x}-x}$$

$$\lim_{x\to 16}\frac{(16-x)(4+\sqrt{x})}{16-x}$$

Cancel the common factor of $16-x$ from both the numerator and denominator.

$$\lim_{x\to 16}(4+\sqrt{x})$$

Now use substitution to evaluate the limit.

$$4+\sqrt{16}$$

$$4+4$$

$$8$$

Topic: Infinite limits and vertical asymptotes

Question: Evaluate the limit.

$$\lim_{x \to 2} \frac{\sqrt{x^2+1}}{(x-2)^2}$$

Answer choices:

A 0

B 1

C ∞

D $-\infty$

Solution: C

We can't use substitution to evaluate the limit, because we get a 0 in the denominator and therefore an undefined value.

$$\frac{\sqrt{2^2+1}}{(2-2)^2}$$

$$\frac{\sqrt{5}}{0^2}$$

$$\frac{\sqrt{5}}{0}$$

So we'll test values close to, and on either side of, $x=2$ to see how the function is behaving close to that point.

$$f(1.9999)=\frac{\sqrt{1.9999^2+1}}{(1.9999-2)^2}\approx\frac{\sqrt{4.9999}}{0.00000001}\approx 225{,}000{,}000$$

$$f(2.0001)=\frac{\sqrt{2.0001^2+1}}{(2.0001-1)^2}\approx\frac{\sqrt{5.0004}}{0.00000001}\approx 225{,}000{,}000$$

The specific values we find aren't important. All that matters is that we realize that these extremely large values tell us that the function is approaching ∞ on both sides of $x=2$, so we can say that the value of the limit is ∞.

$$\lim_{x\to 2}\frac{\sqrt{x^2+1}}{(x-2)^2}=\infty$$

Topic: Infinite limits and vertical asymptotes

Question: Evaluate the limit.

$$\lim_{x\to\frac{\pi}{2}} \tan x$$

Answer choices:

A $\lim_{x\to\frac{\pi}{2}^-} \tan x = -\infty$ $\lim_{x\to\frac{\pi}{2}^+} \tan x = \infty$

B $\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$ $\lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty$

C $\lim_{x\to\frac{\pi}{2}^-} \tan x = -\infty$ $\lim_{x\to\frac{\pi}{2}^+} \tan x = -\infty$

D $\lim_{x\to\frac{\pi}{2}^-} \tan x = \infty$ $\lim_{x\to\frac{\pi}{2}^+} \tan x = \infty$

Solution: B

We can't use substitution to evaluate the limit, because we get a 0 in the denominator and therefore an undefined value.

$$\tan\frac{\pi}{2}$$

$$\frac{\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}$$

$$\frac{1}{0}$$

So we'll test values close to, and on either side of, $x=\pi/2$ to see how the function is behaving close to that point.

$$f\left(\frac{49\pi}{100}\right)=\tan\frac{49\pi}{100}\approx 31.82$$

$$f\left(\frac{51\pi}{100}\right)=\tan\frac{51\pi}{100}\approx -31.82$$

The function is approaching ∞ to the left of $x=\pi/2$ and $-\infty$ to the right of $x=\pi/2$, so the general limit does not exist. But the one-sided limits are

$$\lim_{x\to\frac{\pi}{2}^-}\tan x=\infty$$

$$\lim_{x\to\frac{\pi}{2}^+}\tan x=-\infty$$

Topic: Infinite limits and vertical asymptotes

Question: Evaluate the limit.

$$\lim_{x \to \pi} \cot x$$

Answer choices:

A 0

B 1

C ∞

D Does not exist (DNE)

Solution: D

We can't use substitution to evaluate the limit, because we get a 0 in the denominator and therefore an undefined value.

$$\cot \pi$$

$$\frac{\cos \pi}{\sin \pi}$$

$$\frac{-1}{0}$$

So we'll test values close to, and on either side of, $x = \pi$ to see how the function is behaving close to that point.

$$f\left(\frac{99\pi}{100}\right) = \cot \frac{99\pi}{100} \approx -31.82$$

$$f\left(\frac{101\pi}{100}\right) = \cot \frac{101\pi}{100} \approx 31.82$$

The function is approaching $-\infty$ to the left of $x = \pi$ and ∞ to the right of $x = \pi$, so the general limit does not exist.

Topic: Limits at infinity and horizontal asymptotes

Question: Evaluate the limit.

$$\lim_{x\to\infty} \frac{6}{4x^2}$$

Answer choices:

A $\quad \frac{3}{2}$

B $\quad 0$

C $\quad \infty$

D $\quad 1$

Solution: B

To find the limit as $x \to \infty$, we'll look at the highest-degree terms in both the numerator and denominator.

The highest-degree term in the numerator is 6, which has a degree of 0. The highest-degree term in the denominator is $4x^2$, which has a degree of 2. Therefore,

N < D: $0 < 2$

When the degree of the numerator is less that the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

So, as $x \to \infty$, the limit is 0.

Topic: Limits at infinity and horizontal asymptotes

Question: Evaluate the limit.

$$\lim_{x \to \infty} \frac{6x^3 + 2x^2 - x + 1}{8x^3 - 1}$$

Answer choices:

A -1

B ∞

C $\frac{3}{4}$

D 1

Solution: C

To find the limit as $x \to \infty$, we'll look at the highest-degree terms in both the numerator and denominator.

The highest-degree term in the numerator is $6x^3$, which has a degree of 3. The highest-degree term in the denominator is $8x^3$, which has a degree of 3. Therefore,

N = D: $3 = 3$

When the degree of the numerator is equal to the degree of the denominator, the function has a horizontal asymptote given by the ratio of the coefficients on those highest-degree terms.

$$\frac{6x^3 + 2x^2 - x + 1}{8x^3 - 1}$$

$$\frac{6x^3}{8x^3}$$

$$\frac{6}{8}$$

$$\frac{3}{4}$$

So, as $x \to \infty$, the limit is $3/4$.

Topic: Limits at infinity and horizontal asymptotes

Question: Evaluate the limit.

$$\lim_{x \to \infty} \frac{x^2 + 1}{5x^4 + 2x^2}$$

Answer choices:

A $\sqrt{5}$

B $\frac{\sqrt{5}}{5}$

C ∞

D 0

Solution: D

To find the limit as $x \to \infty$, we'll look at the highest-degree terms in both the numerator and denominator.

The highest-degree term in the numerator is x^2, which has a degree of 2. The highest-degree term in the denominator is $5x^4$, which has a degree of 4. Therefore,

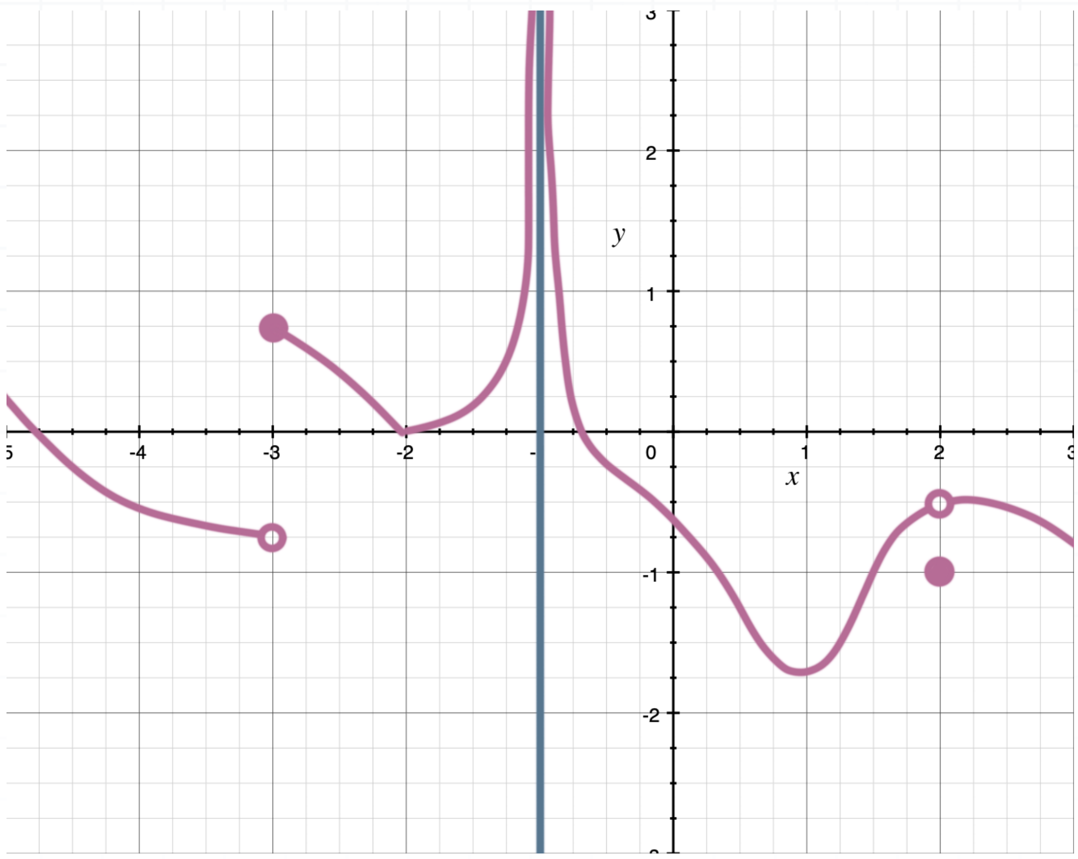
N < D: $2 < 4$

When the degree of the numerator is less that the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

So, as $x \to \infty$, the limit is 0.

Topic: Crazy graphs

Question: Use the graph to find the function's limit as $x \to -1$.



Answer choices:

A $\lim_{x \to -1} f(x) = 0$

B $\lim_{x \to -1} f(x) = \text{DNE}$

C $\lim_{x \to -1} f(x) = \infty$

D $\lim_{x \to -1} f(x) = -\infty$

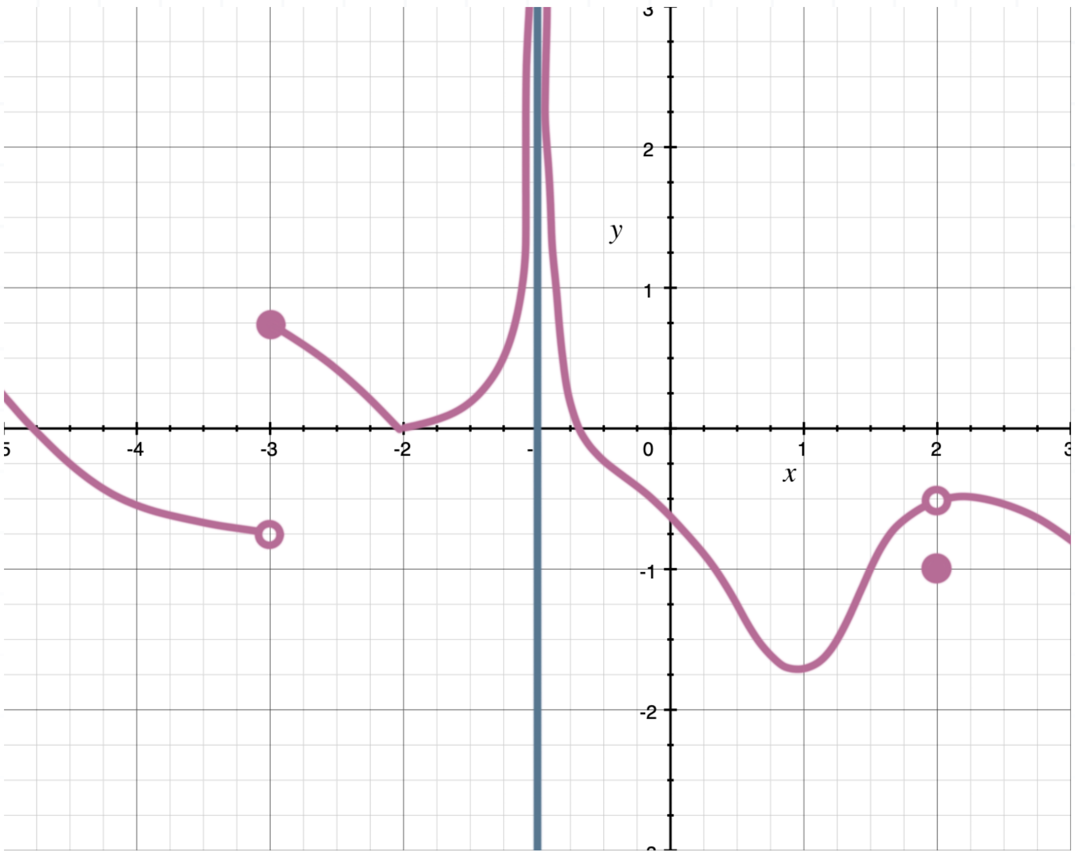
Solution: C

Using the graph, at $x = -1$, the function is approaching ∞ from the left side and ∞ from the right side, so the one-sided limits are equal and the general limit is also ∞.

$$\lim_{x \to -1} f(x) = \infty$$

Topic: Crazy graphs

Question: Use the graph to find the function's limit as $x \to -3^-$ and $x \to -3^+$.



Answer choices:

A $\lim_{x \to -3^-} f(x) = 0.75$ $\lim_{x \to -3^+} f(x) = -0.75$

B $\lim_{x \to -3^-} f(x) = -0.75$ $\lim_{x \to -3^+} f(x) = -0.75$

C $\lim_{x \to -3^-} f(x) = 0.75$ $\lim_{x \to -3^+} f(x) = 0.75$

D $\lim_{x \to -3^-} f(x) = -0.75$ $\lim_{x \to -3^+} f(x) = 0.75$

Solution: D

Using the graph, we'll look at the limit as x gets close to -3 from the left side. We can see that

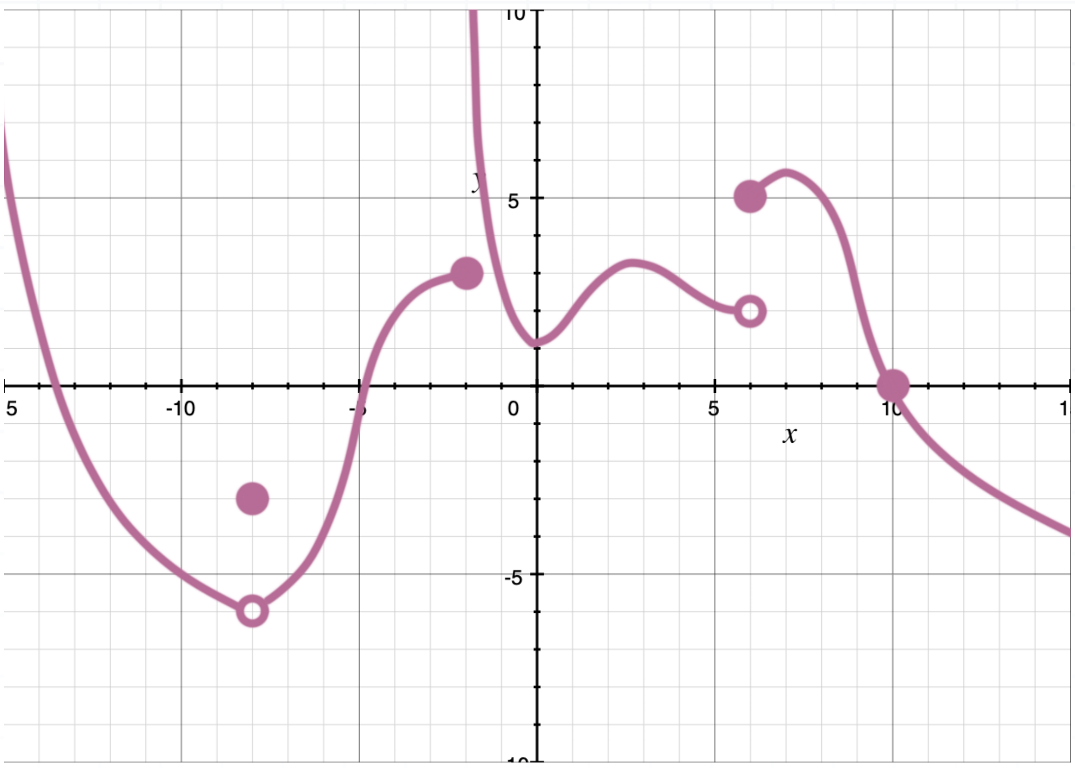
$$\lim_{x\to -3^-} f(x) = -0.75$$

And as x gets close to -3 from the right side, we can see that

$$\lim_{x\to -3^+} f(x) = 0.75$$

Topic: Crazy graphs

Question: Use the graph to find the function's limit as $x \to -2$.



Answer choices:

A $\lim_{x \to -2} f(x) = \text{DNE}$

B $\lim_{x \to -2} f(x) = \infty$

C $\lim_{x \to -2} f(x) = 3$

D $\lim_{x \to -2} f(x) = 0$

Solution: A

Using the graph, we'll look at the limit as x gets close to -2 from the left side. We can see that

$$\lim_{x \to -2^-} f(x) = 3$$

And as x gets close to -2 from the right side, we can see that

$$\lim_{x \to -2^+} f(x) = \infty$$

Because the left- and right-hand limits aren't equal, we've proven that the general limit does not exist at $x = -2$.

Topic: Trigonometric limits

Question: Evaluate the limit.

$$\lim_{x \to 0} \frac{\cos x \sin x}{x}$$

Answer choices:

A $\quad 0$

B $\quad -1$

C $\quad 1$

D $\quad$ Does not exist (DNE)

Solution: C

If we use direct substitution to evaluate the limit, we get the undefined value $0/0$.

$$\frac{\cos(0)\sin(0)}{0}$$

$$\frac{1(0)}{0}$$

$$\frac{0}{0}$$

But if we rewrite the limit as

$$\lim_{x\to 0} \cos x \cdot \frac{\sin x}{x}$$

$$\lim_{x\to 0} \cos x \cdot \lim_{x\to 0} \frac{\sin x}{x}$$

then we see that we have the product of two of the three key trig limit formulas,

$\lim_{x\to 0} \cos x = 1$ and $\lim_{x\to 0} \frac{\sin x}{x} = 1$

So we can evaluate the limit using these formulas.

$$\lim_{x\to 0} \cos x \cdot \lim_{x\to 0} \frac{\sin x}{x}$$

$$1 \cdot 1$$

1

Topic: Trigonometric limits

Question: Use a reciprocal identity to move the function toward one of the key trig limits, and then evaluate the limit.

$$\lim_{x \to 0} \frac{7}{x \csc x}$$

Answer choices:

A $\quad 0$

B $\quad 7$

C $\quad -7$

D $\quad \infty$

Solution: B

Rewrite the function as using the reciprocal identity that relates $\sin x$ and $\csc x$.

$$\lim_{x \to 0} \frac{7}{x \csc x}$$

$$\lim_{x \to 0} \frac{7}{\frac{x}{\sin x}}$$

$$\lim_{x \to 0} \frac{7 \sin x}{x}$$

$$7 \lim_{x \to 0} \frac{\sin x}{x}$$

We know the value of the trig limit

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

Therefore,

$$7(1)$$

$$7$$

Topic: Trigonometric limits

Question: Use the conjugate method, and then the Pythagorean identity $\sin^2 x + \cos^2 x = 1$, to evaluate the limit.

$$\lim_{h \to 0} \frac{\cos h - 1}{h}$$

Answer choices:

A 0

B 1

C -1

D ∞

Solution: A

If we use direct substitution to evaluate the limit, we get the undefined value $0/0$.

$$\frac{\cos(0)-1}{0}$$

$$\frac{1-1}{0}$$

$$\frac{0}{0}$$

But we've been asked to start with conjugate method, anyway. We'll multiply both the numerator and denominator of the function by the conjugate of $\cos h - 1$.

$$\lim_{h\to 0}\frac{\cos h-1}{h}\left(\frac{\cos h+1}{\cos h+1}\right)$$

$$\lim_{h\to 0}\frac{\cos^2 h+\cos h-\cosh h-1}{h(\cos h+1)}$$

$$\lim_{h\to 0}\frac{\cos^2 h-1}{h(\cos h+1)}$$

If we factor out a negative sign, we can rewrite the limit as

$$\lim_{h\to 0}-\frac{1-\cos^2 h}{h(\cos h+1)}$$

We were told in the question to use the Pythagorean identity $\sin^2 x+\cos^2 x=1$, which we can rewrite.

$$\sin^2 x + \cos^2 x = 1$$

$$\sin^2 x = 1 - \cos^2 x$$

Now that the right side of this trigonometric identity matches the numerator of the function, we can make a substitution.

$$\lim_{h \to 0} -\frac{\sin^2 h}{h(\cos h + 1)}$$

Now we'll rewrite the limit

$$\lim_{h \to 0} \left(-\frac{\sin h}{h} \cdot \frac{\sin h}{\cos h + 1} \right)$$

One of the three key trig limits is

$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

which means we can simplify the limit to

$$\lim_{h \to 0} \left(-\frac{\sin h}{\cos h + 1} \right)$$

Now we can use substitution to evaluate the limit.

$$-\frac{\sin(0)}{\cos(0) + 1}$$

$$-\frac{0}{1 + 1}$$

$$-\frac{0}{2}$$

0

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 1/2$. If it's discontinuous, redefine the function to make it continuous.

$$f(x) = \begin{cases} |2x - 1| & x < \frac{1}{2} \\ \frac{3}{4} & x = \frac{1}{2} \\ \frac{2x - 1}{2} & x > \frac{1}{2} \end{cases}$$

Answer choices:

A The function is continuous at $x = 1/2$.

B The function is discontinuous at $x = 1/2$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 1/2$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x - 1| & x < \frac{1}{2} \\ \frac{1}{2} & x = \frac{1}{2} \\ \frac{2x - 1}{2} & x > \frac{1}{2} \end{cases}$$

D The function is discontinuous at $x = 1/2$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} |2x-1| & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \\ \frac{2x-1}{2} & x > \frac{1}{2} \end{cases}$$

Solution: D

In order for the function to be continuous at $x = 1/2$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 1/2$.

We already know that the value of the function at $x = 1/2$ is $3/4$, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to $3/4$. If they exist but aren't equal to $3/4$, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 1/2$.

To look at the left-hand limit, we'll use the first "piece" of our piecewise-defined function, because it defines the function to the left of $x = 1/2$ (the domain of that piece is $x < 1/2$).

$$\lim_{x \to (1/2)^-} |2x - 1|$$

Since the domain of $|2x - 1|$ is $x < 1/2$, we know that no matter what value in the domain we plug in for, we're always going to get a negative value for $2x - 1$. That means we can take away the absolute value bars as long as we put a negative sign in front of $2x - 1$.

$$\lim_{x \to (1/2)^-} -(2x - 1)$$

$$\lim_{x \to (1/2)^-} 1 - 2x$$

$$1 - 2\left(\frac{1}{2}\right)$$

$$1 - 1$$

$$0$$

We know now that the left-hand limit exists, and that it's equal to 0. Let's look at the right-hand limit by using the third "piece" of the piecewise-defined function, since it defines the function to the right of $x = 1/2$ (the domain of that piece is $x > 1/2$).

$$\lim_{x \to (1/2)^+} \frac{2x - 1}{2}$$

$$\frac{2\left(\frac{1}{2}\right)-1}{2}$$

$$\frac{1-1}{2}$$

$$0$$

We know now that the right-hand limit exists, and that it's equal to 0.

Since the left-hand limit exists at $x=1/2$, the right-hand limit exists at $x=1/2$, and the left- and right-hand limits are equal and therefore the general limit exists at $x=1/2$, but the general limit at $x=1/2$ isn't equal to the value of the function at $x=1/2$, the function is discontinuous and we need to redefine the function in order to make it continuous at $x=1/2$.

So we just redefine the value of the function at $x=1/2$ to be equal to the general limit at $x=1/2$ that we found earlier by taking the left- and right-hand limits at $x=1/2$.

$$f(x)=\begin{cases} |2x-1| & x<\frac{1}{2} \\ 0 & x=\frac{1}{2} \\ \frac{2x-1}{2} & x>\frac{1}{2} \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x=0$. If it's discontinuous, redefine the function to make it continuous.

$$f(x)=\begin{cases}\frac{\sqrt{4x+4}-2}{2x} & x>0\\ -2 & x=0\\ \frac{x}{x^2+2x} & x<0\end{cases}$$

Answer choices:

A The function is continuous at $x=0$.

B The function is discontinuous at $x=0$ and the discontinuity is non-removable.

C The function is discontinuous at $x=0$. The discontinuity can be removed by redefining the function as

$$f(x)=\begin{cases}\frac{\sqrt{4x+4}-2}{2x} & x>0\\ \frac{1}{2} & x=0\\ \frac{x}{x^2+2x} & x<0\end{cases}$$

D The function is discontinuous at $x=0$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \frac{\sqrt{4x+4}-2}{2x} & x > 0 \\ 0 & x = 0 \\ \frac{x}{x^2+2x} & x < 0 \end{cases}$$

Solution: C

In order for the function to be continuous at $x = 0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 0$.

We already know that the value of the function at $x = 0$ is -2, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to -2. If they exist but aren't equal to -2, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 0$.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x\to 0^-} \frac{x}{x^2+2x}$$

$$\lim_{x\to 0^-} \frac{x}{x(x+2)}$$

$$\lim_{x\to 0^-} \frac{1}{x+2}$$

$$\frac{1}{0+2}$$

$$\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $1/2$. Let's look at the right-hand limit by using the first "piece" of our piecewise-defined function, since it defines the function to the right of $x=0$ (the domain of that piece is $x>0$).

$$\lim_{x\to 0^+} \frac{\sqrt{4x+4}-2}{2x}$$

$$\lim_{x\to 0^+} \frac{\sqrt{4x+4}-2}{2x}\left(\frac{\sqrt{4x+4}+2}{\sqrt{4x+4}+2}\right)$$

$$\lim_{x\to 0^+} \frac{4x+4-4}{2x(\sqrt{4x+4}+2)}$$

$$\lim_{x\to 0^+} \frac{2}{\sqrt{4x+4}+2}$$

$$\lim_{x\to 0^+} \frac{2}{\sqrt{4(x+1)}+2}$$

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x+1}+1}$$

$$\frac{1}{\sqrt{0+1}+1}$$

$$\frac{1}{1+1}$$

$$\frac{1}{2}$$

We know now that the right-hand limit exists, and that it's equal to $1/2$.

Since the left-hand limit exists at $x=0$, the right-hand limit exists at $x=0$, and the left- and right-hand limits are equal and therefore the general limit exists at $x=0$, but the general limit at $x=0$ isn't equal to the value of the function at $x=0$, the function is discontinuous and we need to redefine the function in order to make it continuous at $x=0$.

So we just redefine the value of the function at $x=0$ to be equal to the general limit at $x=0$ that we found earlier by taking the left- and right-hand limits at $x=0$.

$$f(x)=\begin{cases} \dfrac{\sqrt{4x+4}-2}{2x} & x>0 \\ \dfrac{1}{2} & x=0 \\ \dfrac{x}{x^2+2x} & x<0 \end{cases}$$

Topic: Making the function continuous

Question: Determine whether the function is continuous at $x = 0$. If it's discontinuous, redefine the function to make it continuous.

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ 0 & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Answer choices:

A The function is continuous at $x = 0$.

B The function is discontinuous at $x = 0$ and the discontinuity is non-removable.

C The function is discontinuous at $x = 0$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ \dfrac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

D The function is discontinuous at $x = 0$. The discontinuity can be removed by redefining the function as

$$f(x) = \begin{cases} \dfrac{\sqrt{\cos x} - 1}{\sin^2 x} & x > 0 \\ -\dfrac{1}{2} & x = 0 \\ \dfrac{4 - x}{x^2 - 2x - 8} & x < 0 \end{cases}$$

Solution: B

In order for the function to be continuous at $x = 0$,

- the left-hand limit must exist,
- the right-hand limit must exist,
- the left- and right-hand limits must be equal to one another, and
- the general limit must be equal to the value of the function at $x = 0$.

We already know that the value of the function at $x = 0$ is 0, because that's the second "piece" of the piecewise-defined function we were given.

Now we just need to show that the left- and right-hand limits both exist and that they're both equal to 0. If they exist but aren't equal to 0, then we'll have to "plug the hole" and remove the discontinuity by redefining the function at $x = 0$.

To look at the left-hand limit, we'll use the third "piece" of the piecewise-defined function, because it defines the function to the left of $x = 0$ (the domain of that piece is $x < 0$).

$$\lim_{x\to 0^-} \frac{4-x}{x^2-2x-8}$$

$$\lim_{x\to 0^-} \frac{4-x}{(x-4)(x+2)}$$

$$\lim_{x\to 0^-} -\frac{x-4}{(x-4)(x+2)}$$

$$\lim_{x\to 0^-} -\frac{1}{x+2}$$

$$-\frac{1}{0+2}$$

$$-\frac{1}{2}$$

We know now that the left-hand limit exists, and that it's equal to $-1/2$. Let's look at the right-hand limit by using the first "piece" of the piecewise-defined function, since it defines the function to the right of $x=0$ (the domain of that piece is $x>0$). Substitute using a Pythagorean identity.

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x}-1}{\sin^2 x}$$

$$\lim_{x\to 0^+} \frac{\sqrt{\cos x}-1}{1-\cos^2 x}$$

Factor the denominator in order to simplify the fraction and then evaluate the limit.

$$\lim_{x\to0^+}\frac{\sqrt{\cos x}-1}{(1+\cos x)(1-\cos x)}$$

$$\lim_{x\to0^+}\frac{\sqrt{\cos x}-1}{(1+\cos x)\left[(1+\sqrt{\cos x})(1-\sqrt{\cos x})\right]}$$

$$\lim_{x\to0^+}-\frac{1-\sqrt{\cos x}}{(1+\cos x)(1+\sqrt{\cos x})(1-\sqrt{\cos x})}$$

$$\lim_{x\to0^+}-\frac{1}{(1+\cos x)(1+\sqrt{\cos x})}$$

$$-\frac{1}{(1+\cos(0))(1+\sqrt{\cos(0)})}$$

$$-\frac{1}{(1+1)(1+1)}$$

$$-\frac{1}{4}$$

We know now that the right-hand limit exists, and that it's equal to $-1/4$.

Since the left-hand limit exists at $x=0$, the right-hand limit exists at $x=0$, but the left- and right-hand limits are not equal to another, that means the general limit does not exist at $x=0$. Furthermore, because the one-sided limits are unequal, it means the discontinuity is non-removable.

Topic: Squeeze Theorem

Question: Use Squeeze Theorem to evaluate the limit.

$$\lim_{x \to 0} x^2 \cos x$$

Answer choices:

A ∞

B -1

C 0

D 1

Solution: C

We know the value of the cosine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \le \cos x \le 1$$

Multiply through the inequality by x^2 to get the function at the center of the inequality to match the one we were given.

$$-x^2 \le x^2 \cos x \le x^2$$

Apply the limit throughout the inequality.

$$\lim_{x \to 0} -x^2 \le \lim_{x \to 0} x^2 \cos x \le \lim_{x \to 0} x^2$$

$$-0^2 \le \lim_{x \to 0} x^2 \cos x \le 0^2$$

$$0 \le \lim_{x \to 0} x^2 \cos x \le 0$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x \to 0} x^2 \cos x = 0$$

Topic: Squeeze Theorem

Question: Use Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{\sin(6x)}{x^2}$$

Answer choices:

A -1

B 0

C ∞

D 1

Solution: B

We know the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \le \sin(6x) \le 1$$

Divide through the inequality by x^2 to get the function at the center of the inequality to match the one we were given.

$$-\frac{1}{x^2} \le \frac{\sin(6x)}{x^2} \le \frac{1}{x^2}$$

Apply the limit throughout the inequality.

$$\lim_{x\to\infty} -\frac{1}{x^2} \le \lim_{x\to\infty} \frac{\sin(6x)}{x^2} \le \lim_{x\to\infty} \frac{1}{x^2}$$

$$-\frac{1}{\infty^2} \le \lim_{x\to\infty} \frac{\sin(6x)}{x^2} \le \frac{1}{\infty^2}$$

$$0 \le \lim_{x\to\infty} \frac{\sin(6x)}{x^2} \le 0$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x\to\infty} \frac{\sin(6x)}{x^2} = 0$$

Topic: Squeeze Theorem

Question: Use Squeeze Theorem to evaluate the limit.

$$\lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5}$$

Answer choices:

A $\quad \frac{1}{3}$

B $\quad \infty$

C $\quad 0$

D $\quad 3$

Solution: A

We know the value of the sine function oscillates back and forth between -1 and 1, so we'll start with

$$-1 \le \sin(4x) \le 1$$

Add $2x^3$ to each part of the inequality.

$$2x^3 - 1 \le 2x^3 + \sin(4x) \le 2x^3 + 1$$

Divide through the inequality by $6x^3 + 5$ to get the function at the center of the inequality to match the one we were given.

$$\frac{2x^3 - 1}{6x^3 + 5} \le \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \frac{2x^3 + 1}{6x^3 + 5}$$

Apply the limit throughout the inequality.

$$\lim_{x \to \infty} \frac{2x^3 - 1}{6x^3 + 5} \le \lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \lim_{x \to \infty} \frac{2x^3 + 1}{6x^3 + 5}$$

$$\frac{2}{6} \le \lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \frac{2}{6}$$

$$\frac{1}{3} \le \lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} \le \frac{1}{3}$$

We've squeezed the limit we wanted to find in the inequality, so we can say the value of the limit is

$$\lim_{x \to \infty} \frac{2x^3 + \sin(4x)}{6x^3 + 5} = \frac{1}{3}$$

Topic: Definition of the derivative

Question: Use the definition of the derivative to find the simplified form of the limit.

$$f(x) = x^3 - 2x$$

Answer choices:

A $\quad f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - 2(x + \Delta x)}{\Delta x}$

B $\quad f'(x) = \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + \Delta x^2 - 2)$

C $\quad f'(x) = \lim_{\Delta x \to 0} (\Delta x^2 - 2)$

D $\quad f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - (x^3 - 2x)}{\Delta x}$

Solution: B

After replacing x with $(x+\Delta x)$ in $f(x)$,

$$f(x)=(x+\Delta x)^3-2(x+\Delta x)$$

we'll substitute for $f(x+\Delta x)$.

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-2(x+\Delta x)-f(x)}{\Delta x}$$

Then plug $f(x)$ into the definition.

$$f'(x)=\lim_{\Delta x\to 0}\frac{(x+\Delta x)^3-2(x+\Delta x)-(x^3-2x)}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{x^3+x^2\Delta x+2x^2\Delta x+2x\Delta x^2+x\Delta x^2+\Delta x^3-2x-2\Delta x-x^3+2x}{\Delta x}$$

Collect like terms,

$$f'(x)=\lim_{\Delta x\to 0}\frac{x^2\Delta x+2x^2\Delta x+2x\Delta x^2+x\Delta x^2+\Delta x^3-2x-2\Delta x+2x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{x^2\Delta x+2x^2\Delta x+2x\Delta x^2+x\Delta x^2+\Delta x^3-2\Delta x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{3x^2\Delta x+3x\Delta x^2+\Delta x^3-2\Delta x}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x)=\lim_{\Delta x\to 0}\frac{\Delta x(3x^2+3x\Delta x+\Delta x^2-2)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (3x^2 + 3x\Delta x + \Delta x^2 - 2)$$

Topic: Definition of the derivative

Question: Use the definition of the derivative to find the derivative of the function.

$$f(x) = x^2$$

Answer choices:

A $f'(x) = 0$

B $f'(x) = 2$

C $f'(x) = 2x$

D $f'(x) = x^2 + 2x$

Solution: C

After replacing x with $(x + \Delta x)$ in $f(x)$,

$$f(x) = (x + \Delta x)^2$$

we'll substitute for $f(x + \Delta x)$.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - f(x)}{\Delta x}$$

Then plug $f(x)$ into the definition.

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x + \Delta x)^2 - x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{x^2 + x\Delta x + x\Delta x + \Delta x^2 - x^2}{\Delta x}$$

Collect like terms,

$$f'(x) = \lim_{\Delta x \to 0} \frac{x\Delta x + x\Delta x + \Delta x^2}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (2x + \Delta x)$$

Now we evaluate the limit using substitution, which means we'll substitute $\Delta x = 0$.

$$f'(x) = 2x + 0$$

$$f'(x) = 2x$$

Topic: Definition of the derivative

Question: Use the definition of the derivative to find the derivative of the function.

$$f(x) = 2 - x^2 + x$$

Answer choices:

A $\quad f'(x) = 2$

B $\quad f'(x) = 2x$

C $\quad f'(x) = -2x$

D $\quad f'(x) = -2x + 1$

Solution: D

After replacing x with $(x+\Delta x)$ in $f(x)$,

$$f(x)=2-(x+\Delta x)^2+(x+\Delta x)$$

we'll substitute for $f(x+\Delta x)$.

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-(x+\Delta x)^2+(x+\Delta x)-f(x)}{\Delta x}$$

Then plug $f(x)$ into the definition.

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-(x+\Delta x)^2+(x+\Delta x)-(2-x^2+x)}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-(x^2+x\Delta x+x\Delta x+\Delta x^2)+x+\Delta x-2+x^2-x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-x^2-x\Delta x-x\Delta x-\Delta x^2+x+\Delta x-2+x^2-x}{\Delta x}$$

Collect like terms,

$$f'(x)=\lim_{\Delta x\to 0}\frac{2-x\Delta x-x\Delta x-\Delta x^2+x+\Delta x-2-x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{-x\Delta x-x\Delta x-\Delta x^2+x+\Delta x-x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{-x\Delta x-x\Delta x-\Delta x^2+\Delta x}{\Delta x}$$

$$f'(x)=\lim_{\Delta x\to 0}\frac{-2x\Delta x-\Delta x^2+\Delta x}{\Delta x}$$

then factor Δx out of the numerator and cancel out that common factor from the numerator and denominator.

$$f'(x) = \lim_{\Delta x \to 0} \frac{\Delta x(-2x - \Delta x + 1)}{\Delta x}$$

$$f'(x) = \lim_{\Delta x \to 0} (-2x - \Delta x + 1)$$

Now we evaluate the limit using substitution, which means we'll substitute $\Delta x = 0$.

$$f'(x) = -2x - 0 + 1$$

$$f'(x) = -2x + 1$$

Topic: Power rule

Question: Find the derivative.

$$y = -3x^3$$

Answer choices:

A $y' = -9x^4$

B $y' = 9x^2$

C $y' = -9x^2$

D $y' = -6x^2$

Solution: C

Apply power rule to differentiate the equation.

$$y' = -3(3)x^{3-1}$$

$$y' = -9x^2$$

Topic: Power rule

Question: Find the derivative.

$$y = 5x^5 - 4x^2$$

Answer choices:

A $y' = 5x^4 - 4x$

B $y' = 25x^4 - 8x$

C $y' = 25x^5 - 8x^2$

D $y' = 25x^3 - 8$

Solution: B

Apply power rule to differentiate the equation, one term at a time.

$$y' = 5(5)x^{5-1} - 4(2)x^{2-1}$$

$$y' = 25x^4 - 8x^1$$

$$y' = 25x^4 - 8x$$

Topic: Power rule

Question: Find the derivative.

$$y = 3x^7 - 9x^2 + 21$$

Answer choices:

A $y' = 21x^{-6} - 18x$

B $y' = 3x(7x^5 - 6x)$

C $y' = 21x^8 - 18x^2$

D $y' = 21x^6 - 18x$

Solution: D

Apply power rule to differentiate the equation, one term at a time.

$$y' = 3(7)x^{7-1} - 9(2)x^{2-1} + 0$$

$$y' = 21x^6 - 18x^1$$

$$y' = 21x^6 - 18x$$

Topic: Power rule for negative powers

Question: Differentiate the function.

$$f(x) = x^{-2}$$

Answer choices:

A $f'(x) = -3x^{-2}$

B $f'(x) = -2x^{-3}$

C $f'(x) = -2x^{-2}$

D $f'(x) = -2x^{-1}$

Solution: B

Apply power rule to differentiate the function.

$$f'(x) = -2x^{-2-1}$$

$$f'(x) = -2x^{-3}$$

Topic: Power rule for negative powers

Question: Differentiate the function.

$$f(x) = 3x^{-5} - 4$$

Answer choices:

A $\quad f'(x) = -12x^{-6}$

B $\quad f'(x) = -15x^{-4}$

C $\quad f'(x) = -5x^{-6}$

D $\quad f'(x) = -15x^{-6}$

Solution: D

Apply power rule to differentiate the equation, one term at a time.

$$f'(x) = 3(-5)x^{-5-1} - 0$$

$$f'(x) = -15x^{-6}$$

Topic: Power rule for negative powers

Question: Differentiate the function.

$$f(x) = \frac{4}{x^2} - \frac{6}{x} + 2$$

Answer choices:

A $f'(x) = -8x^{-3} + 6x^{-2}$

B $f'(x) = -8x^{-4} + 6x^{-3}$

C $f'(x) = -2x^{-3} + x^{-2}$

D $f'(x) = -8x^{-2} + 6x^{-1}$

Solution: A

First, rewrite the fractions by changing the signs of the exponents.

$$f(x) = 4x^{-2} - 6x^{-1} + 2$$

Apply power rule to differentiate the equation, one term at a time.

$$f'(x) = 4(-2)x^{-2-1} - 6(-1)x^{-1-1} + 0$$

$$f'(x) = -8x^{-3} + 6x^{-2}$$

Topic: Power rule for fractional powers

Question: Differentiate the function.

$$f(x) = \sqrt{x}$$

Answer choices:

A $f'(x) = -\frac{1}{2}\sqrt{x}$

B $f'(x) = -\frac{1}{2\sqrt{x}}$

C $f'(x) = \frac{1}{2\sqrt{x}}$

D $f'(x) = \frac{1}{2}\sqrt{x}$

Solution: C

Rewrite the function by converting the radical into a fractional exponent.

$$f(x) = x^{\frac{1}{2}}$$

Apply power rule to differentiate the function.

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1}$$

$$f'(x) = \frac{1}{2}x^{-\frac{1}{2}}$$

Because the original function was given in terms of a root, rewrite this answer with a root instead of a fractional exponent.

$$f'(x) = \frac{1}{2x^{\frac{1}{2}}}$$

$$f'(x) = \frac{1}{2\sqrt{x}}$$

Topic: Power rule for fractional powers

Question: Differentiate the function.

$$f(x) = \frac{4}{\sqrt{x}}$$

Answer choices:

A $\quad f'(x) = -\frac{2}{\sqrt{x^3}}$

B $\quad f'(x) = -\frac{2}{\sqrt{x}}$

C $\quad f'(x) = -2\sqrt{x^3}$

D $\quad f'(x) = -2\sqrt{x}$

Solution: A

Rewrite the function by converting the radical into a fractional exponent.

$$f(x) = \frac{4}{x^{\frac{1}{2}}}$$

$$f(x) = 4x^{-\frac{1}{2}}$$

Apply power rule to differentiate the function.

$$f'(x) = 4\left(-\frac{1}{2}\right)x^{-\frac{1}{2}-1}$$

$$f'(x) = -\frac{4}{2}x^{-\frac{1}{2}-\frac{2}{2}}$$

$$f'(x) = -2x^{-\frac{3}{2}}$$

Because the original function was given in terms of a root, rewrite this answer with a root instead of a fractional exponent.

$$f'(x) = -\frac{2}{x^{\frac{3}{2}}}$$

$$f'(x) = -\frac{2}{\sqrt{x^3}}$$

Topic: Power rule for fractional powers

Question: Differentiate the function.

$$f(x) = \frac{5}{\sqrt[3]{x^2}}$$

Answer choices:

A $\quad f'(x) = -\frac{10}{3}\sqrt[3]{x^5}$

B $\quad f'(x) = -\frac{10}{3\sqrt[3]{x^5}}$

C $\quad f'(x) = -\frac{10}{3\sqrt[3]{x}}$

D $\quad f'(x) = -\frac{10}{3}\sqrt[3]{x}$

Solution: B

Rewrite the function by converting the radical into a fractional exponent.

$$f(x) = \frac{5}{x^{\frac{2}{3}}}$$

$$f(x) = 5x^{-\frac{2}{3}}$$

Apply power rule to differentiate the function.

$$f'(x) = 5\left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1}$$

$$f'(x) = -\frac{10}{3}x^{-\frac{2}{3}-\frac{3}{3}}$$

$$f'(x) = -\frac{10}{3}x^{-\frac{5}{3}}$$

Because the original function was given in terms of a root, rewrite this answer with a root instead of a fractional exponent.

$$f'(x) = -\frac{10}{3x^{\frac{5}{3}}}$$

$$f'(x) = -\frac{10}{3\sqrt[3]{x^5}}$$

Topic: Product rule with two functions

Question: Find the derivative.

$$y = (x^2 + 2)(x^3 + 1)$$

Answer choices:

A $\quad y' = 5x^3 + 6x + 2$

B $\quad y' = x^4 + 12x^2 + 2x$

C $\quad y' = 5x^4 + 6x^2 + 2x$

D $\quad y' = 5x^4 - 6x^2 + 2x$

Solution: C

Let $f(x) = x^2 + 2$ and $g(x) = x^3 + 1$, and then apply product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (x^2 + 2)(3x^2) + (2x)(x^3 + 1)$$

Expand the derivative, then collect like terms.

$$y' = 3x^2(x^2) + 3x^2(2) + 2x(x^3) + 2x(1)$$

$$y' = 3x^4 + 6x^2 + 2x^4 + 2x$$

$$y' = 5x^4 + 6x^2 + 2x$$

Topic: Product rule with two functions

Question: Find the derivative.

$$y = (3x^2 + 2x)(x^4 - 3x + 1)$$

Answer choices:

A $y' = 18x^4 + 10x^3 - 27x^2 - 6x + 2$

B $y' = 10x^5 + 27x^4 - 27x^2 - 6x + 2$

C $y' = 18x^5 + 10x^4 + 27x^2 - 6x + 2$

D $y' = 18x^5 + 10x^4 - 27x^2 - 6x + 2$

Solution: D

Let $f(x) = 3x^2 + 2x$ and $g(x) = x^4 - 3x + 1$, and then apply product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (3x^2 + 2x)(4x^3 - 3) + (6x + 2)(x^4 - 3x + 1)$$

Expand the derivative, then collect like terms.

$$y' = 3x^2(4x^3 - 3) + 2x(4x^3 - 3) + 6x(x^4 - 3x + 1) + 2(x^4 - 3x + 1)$$

$$y' = 3x^2(4x^3) - 3x^2(3) + 2x(4x^3) - 2x(3)$$

$$+6x(x^4) - 6x(3x) + 6x(1) + 2(x^4) - 2(3x) + 2(1)$$

$$y' = 12x^5 - 9x^2 + 8x^4 - 6x + 6x^5 - 18x^2 + 6x + 2x^4 - 6x + 2$$

$$y' = 12x^5 + 6x^5 + 8x^4 + 2x^4 - 9x^2 - 18x^2 - 6x + 6x - 6x + 2$$

$$y' = 18x^5 + 10x^4 - 27x^2 - 6x + 2$$

Topic: Product rule with two functions

Question: Find the derivative.

$$h(x) = (3x^2 - 7)(x^2 - 4x + 3)$$

Answer choices:

A $h'(x) = 6x^3 - 36x^2 + 4x + 28$

B $h'(x) = 12x^3 - 36x^2 + 32x + 28$

C $h'(x) = 12x^3 - 12x^2 + 4x + 28$

D $h'(x) = 12x^3 - 36x^2 + 4x + 28$

Solution: D

Let $f(x) = 3x^2 - 7$ and $g(x) = x^2 - 4x + 3$, and then apply product rule.

$$h'(x) = f(x)g'(x) + f'(x)g(x)$$

$$h'(x) = (3x^2 - 7)(2x - 4) + (6x)(x^2 - 4x + 3)$$

Expand the derivative, then collect like terms.

$$h'(x) = (3x^2 - 7)(2x) - (3x^2 - 7)(4) + (6x)(x^2) - (6x)(4x) + (6x)(3)$$

$$h'(x) = 2x(3x^2) - 2x(7) - 4(3x^2) - 4(-7) + (6x)(x^2) - (6x)(4x) + (6x)(3)$$

$$h'(x) = 6x^3 - 14x - 12x^2 + 28 + 6x^3 - 24x^2 + 18x$$

$$h'(x) = 6x^3 + 6x^3 - 12x^2 - 24x^2 - 14x + 18x + 28$$

$$h'(x) = 12x^3 - 36x^2 + 4x + 28$$

Topic: Product rule with three or more functions

Question: Use the product rule to find the derivative of the function.

$$f(x) = (3x^2)(x)(-2x^4)$$

Answer choices:

A $f'(x) = 6x^7$

B $f'(x) = 42x^6$

C $f'(x) = -6x^7$

D $f'(x) = -42x^6$

Solution: D

Let $r(x) = 3x^2$, $s(x) = x$, and $t(x) = -2x^4$. Find each of their derivatives.

$$r(x) = 3x^2$$

$$r'(x) = 6x$$

and

$$s(x) = x$$

$$s'(x) = 1$$

and

$$t(x) = -2x^4$$

$$t'(x) = -8x^3$$

Apply product rule.

$$f'(x) = r'(x)s(x)t(x) + r(x)s'(x)t(x) + r(x)s(x)t'(x)$$

$$f'(x) = (6x)(x)(-2x^4) + (3x^2)(1)(-2x^4) + (3x^2)(x)(-8x^3)$$

Expand the derivative, then collect like terms.

$$f'(x) = (6x^2)(-2x^4) + (3x^2)(-2x^4) + (3x^3)(-8x^3)$$

$$f'(x) = -12x^6 - 6x^6 - 24x^6$$

$$f'(x) = -42x^6$$

Topic: Product rule with three or more functions

Question: Use the product rule to find the derivative of the function.

$$f(x) = (x + 3)(2x^2 - 5)(-x^3 + 2)$$

Answer choices:

A $f'(x) = -12x^3$

B $f'(x) = -12x^5 - 30x^4 + 20x^3 + 57x^2 + 24x - 10$

C $f'(x) = -22x^4 - 36x^3 + 15x^2 + 8x$

D $f'(x) = -2x^5 + 2x^3 - x^2 + 12x - 10$

Solution: B

Let $r(x) = x + 3$, $s(x) = 2x^2 - 5$, and $t(x) = -x^3 + 2$. Find each of their derivatives.

$$r(x) = x + 3$$

$$r'(x) = 1$$

and

$$s(x) = 2x^2 - 5$$

$$s'(x) = 4x$$

and

$$t(x) = -x^3 + 2$$

$$t'(x) = -3x^2$$

Apply product rule.

$$f'(x) = r'(x)s(x)t(x) + r(x)s'(x)t(x) + r(x)s(x)t'(x)$$

$$f'(x) = (1)(2x^2 - 5)(-x^3 + 2) + (x + 3)(4x)(-x^3 + 2) + (x + 3)(2x^2 - 5)(-3x^2)$$

Expand the derivative, then collect like terms.

$$f'(x) = -2x^5 + 4x^2 + 5x^3 - 10$$

$$+(4x^2 + 12x)(-x^3 + 2) + (2x^3 - 5x + 6x^2 - 15)(-3x^2)$$

$$f'(x) = -2x^5 + 5x^3 + 4x^2 - 10$$

$$+(-4x^5+8x^2-12x^4+24x)+(-6x^5+15x^3-18x^4+45x^2)$$

$$f'(x)=-2x^5+5x^3+4x^2-10-4x^5+8x^2-12x^4+24x-6x^5+15x^3-18x^4+45x^2$$

$$f'(x)=-2x^5-4x^5-6x^5-12x^4-18x^4+5x^3+15x^3+4x^2+8x^2+45x^2+24x-10$$

$$f'(x)=-12x^5-30x^4+20x^3+57x^2+24x-10$$

Topic: Product rule with three or more functions

Question: Use the product rule to find the derivative of the function.

$$y = (3x^5 - 4)^3$$

Answer choices:

A $y' = 45x^4(3x^5 - 4)^2$

B $y' = (15x^4)^3$

C $y' = 3(15x^4)^2(3x^5 - 4)$

D $y' = 3(15x^4)^3$

Solution: A

Let $r(x) = 3x^5 - 4$, $s(x) = 3x^5 - 4$, and $t(x) = 3x^5 - 4$. Find each of their derivatives.

$$r(x) = 3x^5 - 4$$

$$r'(x) = 15x^4$$

and

$$s(x) = 3x^5 - 4$$

$$s'(x) = 15x^4$$

and

$$t(x) = 3x^5 - 4$$

$$t'(x) = 15x^4$$

Apply product rule.

$$f'(x) = (15x^4)(3x^5 - 4)(3x^5 - 4) + (3x^5 - 4)(15x^4)(3x^5 - 4) + (3x^5 - 4)(3x^5 - 4)(15x^4)$$

$$f'(x) = 3(15x^4)(3x^5 - 4)(3x^5 - 4)$$

$$f'(x) = 45x^4(3x^5 - 4)(3x^5 - 4)$$

$$f'(x) = 45x^4(3x^5 - 4)^2$$

Topic: Quotient rule

Question: Find the derivative.

$$y = \frac{2x^2 - 1}{3x + 5}$$

Answer choices:

A $\quad y' = \dfrac{6x^2 - 20x + 3}{(3x + 5)^2}$

B $\quad y' = \dfrac{6x^2 + 20x + 3}{3x + 5}$

C $\quad y' = \dfrac{6x^2 + 20x + 3}{(3x + 5)^2}$

D $\quad y' = \dfrac{6x^2 + 10x + 3}{(3x + 5)^2}$

Solution: C

Apply the quotient rule to find the derivative.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(4x)(3x+5) - (2x^2-1)(3)}{(3x+5)^2}$$

Simplify the derivative.

$$y' = \frac{(12x^2+20x) - (6x^2-3)}{(3x+5)^2}$$

$$y' = \frac{12x^2+20x-6x^2+3}{(3x+5)^2}$$

$$y' = \frac{6x^2+20x+3}{(3x+5)^2}$$

Topic: Quotient rule

Question: Find the derivative.

$$y = \frac{x^2 - x + 1}{x^2 + 1}$$

Answer choices:

A $\quad y' = \dfrac{x^2 - 1}{(x^2 + 1)^2}$

B $\quad y' = \dfrac{x - 1}{(x^2 + 1)^2}$

C $\quad y' = \dfrac{x^2 - 1}{x^2 + 1}$

D $\quad y' = \dfrac{x^2}{(x^2 + 1)^2}$

Solution: A

Apply the quotient rule to find the derivative.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(2x-1)(x^2+1) - (x^2-x+1)(2x)}{(x^2+1)^2}$$

Simplify the derivative.

$$y' = \frac{(2x^3+2x-x^2-1) - (2x^3-2x^2+2x)}{(x^2+1)^2}$$

$$y' = \frac{2x^3+2x-x^2-1-2x^3+2x^2-2x}{(x^2+1)^2}$$

$$y' = \frac{x^2-1}{(x^2+1)^2}$$

Topic: Quotient rule

Question: Find the derivative.

$$y = \frac{1}{3x^2 + 1}$$

Answer choices:

A $\quad y' = -\dfrac{6x}{(3x^2 + 1)^2}$

B $\quad y' = \dfrac{6x}{(3x^2 + 1)^2}$

C $\quad y' = -\dfrac{6x}{3x^2 + 1}$

D $\quad y' = -\dfrac{6}{(3x^2 + 1)^2}$

Solution: A

Apply the quotient rule to find the derivative.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(0)(3x^2 + 1) - (1)(6x)}{(3x^2 + 1)^2}$$

Simplify the derivative.

$$y' = \frac{-6x}{(3x^2 + 1)^2}$$

$$y' = -\frac{6x}{(3x^2 + 1)^2}$$

Topic: Trigonometric derivatives

Question: Find the derivative of the trigonometric function.

$$y = 4\sec x - 3x^2 \tan x$$

Answer choices:

A $\quad y' = 4\sec x \tan x - 3x^2 \sec^2 x$

B $\quad y' = 4\sec x \tan x - 6x \sec^2 x$

C $\quad y' = 4\sec x \tan x - 3x^2 \sec^2 x - 6x \tan x$

D $\quad y' = (4 - 6x)\tan x - 3x^2 \sec^2 x$

Solution: C

Let's look at one term at a time. The derivative of $4 \sec x$ is

$$4 \sec x \tan x$$

To find the derivative of $-3x^2 \tan x$, we'll need to use product rule. If $f(x) = -3x^2$ and $f'(x) = -6x$, and $g(x) = \tan x$ and $g'(x) = \sec^2 x$, then we can plug directly into the product rule formula.

$$f(x)g'(x) + f'(x)g(x)$$

$$(-3x^2)(\sec^2 x) + (-6x)(\tan x)$$

$$-3x^2 \sec^2 x - 6x \tan x$$

Putting these derivatives together, we get

$$y' = 4 \sec x \tan x - 3x^2 \sec^2 x - 6x \tan x$$

Topic: Trigonometric derivatives

Question: Find the derivative of the trigonometric function.

$$y = 2\sin x \csc x$$

Answer choices:

A $y' = 0$

B $y' = -4\cot x$

C $y' = 4\cot x$

D $y' = -2\cot x + 2\tan x$

Solution: A

Use the product rule with

$$f(x) = 2\sin x$$

$$f'(x) = 2\cos x$$

and

$$g(x) = \csc x$$

$$g'(x) = -\csc x \cot x$$

Then the derivative is

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (2\sin x)(-\csc x \cot x) + (2\cos x)(\csc x)$$

$$y' = -2\sin x \csc x \cot x + 2\cos x \csc x$$

$$y' = -2\cot x + 2\cot x$$

$$y' = 0$$

Topic: Trigonometric derivatives

Question: Find the derivative of the trigonometric function.

$$y = 7\cot x$$

Answer choices:

A $y' = -7\csc^2 x$

B $y' = 7\csc^2 x$

C $y' = 7\csc x$

D $y' = -7\csc x$

Solution: A

The derivative of $y = \cot x$ is $y' = -\csc^2 x$. So the derivative will be

$$y' = 7(-\csc^2 x)$$

$$y' = -7\csc^2 x$$

Topic: Exponential derivatives

Question: Find the derivative of the exponential function.

$$y = 3e^x + 10x$$

Answer choices:

A $y' = 3e^x$

B $y' = 3e^x + 10$

C $y' = e^x + 10$

D $y' = e^x$

Solution: B

The base is e and the exponent is x, so the derivative of $3e^x$ is $3e^x$ and the derivative of the function is

$$y' = 3e^x + 10$$

Topic: Exponential derivatives

Question: Find the derivative of the exponential function.

$$y = 7^x - 3x^{-4}$$

Answer choices:

A $\quad y' = 7^x \ln(7) + 12x^{-5}$

B $\quad y' = 7^x \ln(7) + 12x^{-3}$

C $\quad y' = 7^x \ln(7)$

D $\quad y' = 7^x \ln(7) - 3x^{-5}$

Solution: A

In the function 7^x, $a = 7$ and the exponent is x. We'll differentiate by applying the formula for exponential derivatives.

$$a^x(\ln a)$$

$$7^x(\ln(7))$$

$$7^x \ln(7)$$

Then the derivative is

$$y' = 7^x \ln(7) + 12x^{-5}$$

Topic: Exponential derivatives

Question: Find the derivative of the exponential function.

$$y = 4xe^x - x^4 2^x$$

Answer choices:

A $y' = 4xe^x + 2^x x^3(x \ln 2 + 4)$

B $y' = 4xe^x + 2^x x^4 \ln 2$

C $y' = 4e^x(x + 1) - 2^x x^3(x \ln 2 + 4)$

D $y' = 4e^x(x + 1) - 2^x x^3(x \ln 2 - 4)$

Solution: C

We'll apply product rule with

$$f(x) = 4x$$

$$f'(x) = 4$$

and

$$g(x) = e^x$$

$$g'(x) = e^x$$

Then the derivative is

$$f(x)g'(x) + f'(x)g(x)$$

$$(4x)(e^x) + (4)(e^x)$$

$$4xe^x + 4e^x$$

We'll apply product rule with

$$f(x) = x^4$$

$$f'(x) = 4x^3$$

and

$$g(x) = 2^x$$

$$g'(x) = 2^x \ln 2$$

Then the derivative is

$$f(x)g'(x) + f'(x)g(x)$$

$$(x^4)(2^x \ln 2) + (4x^3)(2^x)$$

$$x^4 2^x \ln 2 + 4x^3 2^x$$

Then the derivative of the function is

$$y' = 4xe^x + 4e^x - (2^x x^4 \ln 2 + 4 \cdot 2^x x^3)$$

$$y' = 4e^x(x+1) - 2^x x^3(x \ln 2 + 4)$$

Topic: Logarithmic derivatives

Question: Find the derivative of the logarithmic function.

$$y = 9x^2 + 2e^x \ln x$$

Answer choices:

A $\quad y' = 18x + 2e^x \ln x + \dfrac{2}{x}$

B $\quad y' = 18x + \dfrac{2e^x}{x}$

C $\quad y' = 18x + 2e^x \ln x$

D $\quad y' = 18x + 2e^x \ln x + \dfrac{2e^x}{x}$

Solution: D

We need to take the derivative one term at a time, applying the derivative formulas for the natural log. We'll also need to apply product rule to the second term.

$$y' = 18x + 2e^x \ln x + 2e^x \left(\frac{1}{x} \right)$$

$$y' = 18x + 2e^x \ln x + \frac{2e^x}{x}$$

Topic: Logarithmic derivatives

Question: Find the derivative of the logarithmic function.

$$y = 2^x - 3\log_2 x$$

Answer choices:

A $\quad y' = 2^x \ln 2 - \dfrac{3}{x}$

B $\quad y' = 2^x - \dfrac{3}{x}$

C $\quad y' = \dfrac{2^x}{\ln 2} - \dfrac{3}{x \ln 2}$

D $\quad y' = 2^x \ln 2 - \dfrac{3}{x \ln 2}$

Solution: D

We need to take the derivative one term at a time, applying the derivative formulas for the natural log.

$$y' = 2^x \ln 2 - 3\left(\frac{1}{x \ln 2}\right)$$

$$y' = 2^x \ln 2 - \frac{3}{x \ln 2}$$

Topic: Logarithmic derivatives

Question: Find the derivative of the logarithmic function.

$$f(x) = 4 \ln x$$

Answer choices:

A $f'(x) = \dfrac{4}{x}$

B $f'(x) = 4x$

C $f'(x) = 4$

D $f'(x) = x$

Solution: A

The derivative is

$$f'(x) = 4\left(\frac{1}{x}\right)$$

$$f'(x) = \frac{4}{x}$$

Topic: Chain rule with power rule

Question: Apply power rule and chain rule to find the derivative.

$$y = (5x^2 + 2x - 8)^5$$

Answer choices:

A $\quad y' = (5x + 2)(5x^2 + 2x - 8)^4$

B $\quad y' = (50x + 10)(5x^2 + 2x - 8)^5$

C $\quad y' = (50x + 10)(5x^2 + 2x - 8)^4$

D $\quad y' = (50x - 10)(5x^2 + 2x - 8)^4$

Solution: C

Use substitution with $u = 5x^2 + 2x - 8$ and $u' = 10x + 2$, and rewrite the function with the substitution.

$$y = u^5$$

Then the derivative is

$$y' = 5u^4u'$$

Back-substitute.

$$y' = 5(5x^2 + 2x - 8)^4(10x + 2)$$

$$y' = (50x + 10)(5x^2 + 2x - 8)^4$$

Topic: Chain rule with power rule

Question: Apply power rule and chain rule to find the derivative.

$$f(x) = 8(6x^2 + 2)^4$$

Answer choices:

A $f'(x) = 384x(6x^2 + 2)^3$

B $f'(x) = 384(6x^2 + 2)^3$

C $f'(x) = 32x(6x^2 + 2)^3$

D $f'(x) = 32(6x^2 + 2)^3$

Solution: A

Use substitution with $u = 6x^2 + 2$ and $u' = 12x$, and rewrite the function with the substitution.

$$f(x) = 8u^4$$

Then the derivative is

$$f'(x) = 32u^3u'$$

Back-substitute.

$$f'(x) = 32(6x^2 + 2)^3(12x)$$

$$f'(x) = 384x(6x^2 + 2)^3$$

Topic: Chain rule with power rule

Question: Apply power rule and chain rule to find the derivative.

$$f(y) = (y^3 + 1)^{25}$$

Answer choices:

A $f'(y) = (3y^2)^{25}$

B $f'(y) = 25(y^3 + 1)^{24}$

C $f'(y) = 75y^2(y^3 + 1)^{24}$

D $f'(y) = 25(3y^2)^{24}$

Solution: C

Use substitution with $u = y^3 + 1$ and $u' = 3y^2$, and rewrite the function with the substitution.

$$f(y) = u^{25}$$

Then the derivative is

$$f'(y) = 25u^{24}u'$$

Back-substitute.

$$f'(y) = 25(y^3 + 1)^{24}(3y^2)$$

$$f'(y) = 75y^2(y^3 + 1)^{24}$$

Topic: Chain rule with trig, log, and exponential functions

Question: Find the derivative of the trigonometric function.

$$y = \sin(3x^2 + 11x)$$

Answer choices:

A $\quad y' = -(6x + 11)\cos(3x^2 + 11x)$

B $\quad y' = -(6x + 11)\sin(3x^2 + 11x)$

C $\quad y' = (6x + 11)\cos(3x^2 + 11x)$

D $\quad y' = (6x + 11)\sin(3x^2 + 11x)$

Solution: C

Set $u=3x^2+11x$ and $u'=6x+11$. Then $y=\sin u$, and the derivative is

$$y'=\cos u\cdot u'$$

$$y'=\cos(3x^2+11x)\cdot(6x+11)$$

$$y'=(6x+11)\cos(3x^2+11x)$$

Topic: Chain rule with trig, log, and exponential functions

Question: Find the derivative of the exponential function.

$$y = e^{\sqrt{x+1}}$$

Answer choices:

A $\quad y' = \dfrac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$

B $\quad y' = \dfrac{e^{\sqrt{x}}}{2\sqrt{x+1}}$

C $\quad y' = \dfrac{e^{\sqrt{x+1}}}{\sqrt{x+1}}$

D $\quad y' = e^{\sqrt{x+1}}$

Solution: A

Make a substitution, letting $u=\sqrt{x+1}$ and

$$u'=\frac{1}{2\sqrt{x+1}}$$

Then the function is

$$y=e^u$$

and the derivative is

$$y'=e^u\cdot u'$$

$$y'=e^{\sqrt{x+1}}\cdot\frac{1}{2\sqrt{x+1}}$$

$$y'=\frac{e^{\sqrt{x+1}}}{2\sqrt{x+1}}$$

Topic: Chain rule with trig, log, and exponential functions

Question: Find the derivative of the logarithmic function.

$$y = \ln(x^2 - 5x)$$

Answer choices:

A $\quad y' = \dfrac{2x + 5}{x^2 + 5x}$

B $\quad y' = \dfrac{2x - 5}{x^2 - 5}$

C $\quad y' = \dfrac{5 - 2x}{x^2 - 5x}$

D $\quad y' = \dfrac{2x - 5}{x^2 - 5x}$

Solution: D

Let $u = x^2 - 5x$ and $u' = 2x - 5$. Then the function is

$$y = \ln u$$

and the derivative is

$$y' = \frac{1}{u} \cdot u'$$

$$y' = \frac{1}{x^2 - 5x} \cdot (2x - 5)$$

$$y' = \frac{2x - 5}{x^2 - 5x}$$

Topic: Chain rule with product rule

Question: Apply product rule and chain rule to find the derivative.

$$y = (4x - 7)^2(2x + 3)$$

Answer choices:

A $\quad y' = (4x - 7)(12x + 5)$

B $\quad y' = 2(4x - 7)(2x + 3)$

C $\quad y' = 2(4x - 7)(12x + 5)$

D $\quad y' = 2(4x - 7)^3(12x + 5)$

Solution: C

Set $f(x) = (4x - 7)^2$ and $g(x) = 2x + 3$. Then

$$f(x) = (4x - 7)^2$$

$$f'(x) = 2(4x - 7)(4)$$

$$f'(x) = 8(4x - 7)$$

and

$$g(x) = 2x + 3$$

$$g'(x) = 2$$

Now we can apply product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = ((4x - 7)^2)(2) + (8(4x - 7))(2x + 3)$$

The two terms $2(4x - 7)^2$ and $8(4x - 7)(2x + 3)$ share a common factor of $2(4x - 7)$, so factor that out.

$$y' = 2(4x - 7)\left[(4x - 7) + 4(2x + 3)\right]$$

$$y' = 2(4x - 7)(4x - 7 + 8x + 12)$$

$$y' = 2(4x - 7)(12x + 5)$$

Topic: Chain rule with product rule

Question: Apply product rule and chain rule to find the derivative.

$$y = 2\sin x^2 \sec(2x^3 + 3)$$

Answer choices:

A $y' = 12x^2 \sin x^2 \sec(2x^3 + 3)\tan(2x^3 + 3) + 4x \cos x^2 \sec(2x^3 + 3)$

B $y' = 2\sin x^2 \sec(2x^3 + 3)\tan(2x^3 + 3) + 2x \cos x \sec(2x^3 + 3)$

C $y' = 12x^2 \sin x^2 \sec(2x^3 + 3)\tan(2x^3 + 3) + 4x \cos x \sec(2x^3 + 3)$

D $y' = 12x^2 \sin x^2 \sec(2x^3 + 3) + 4x \sin x^2 \sec(2x^3 + 3)$

Solution: A

Use the product rule with

$$f(x) = 2\sin x^2$$

$$f'(x) = 2\cos x^2(2x)$$

$$f'(x) = 4x\cos x^2$$

and

$$g(x) = \sec(2x^3 + 3)$$

$$g'(x) = \sec(2x^3 + 3)\tan(2x^3 + 3)(6x^2)$$

$$g'(x) = 6x^2\sec(2x^3 + 3)\tan(2x^3 + 3)$$

Then the derivative is

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (2\sin x^2)(6x^2\sec(2x^3 + 3)\tan(2x^3 + 3)) + (4x\cos x^2)(\sec(2x^3 + 3))$$

$$y' = 12x^2\sin x^2\sec(2x^3 + 3)\tan(2x^3 + 3) + 4x\cos x^2\sec(2x^3 + 3)$$

Topic: Chain rule with product rule

Question: Find the derivative of the exponential function.

$$y = 4xe^{5x^2-2}$$

Answer choices:

A $\quad y' = 4e^{5x^2-2}(5x + 1)$

B $\quad y' = 4e^{5x^2-2}(5x^2 + 1)$

C $\quad y' = 4e^{5x^2-2}(10x^2 + 1)$

D $\quad y' = 4e^{5x^2-2}(10x + 1)$

Solution: C

We'll apply product rule with

$$f(x) = 4x$$

$$f'(x) = 4$$

and

$$g(x) = e^{5x^2-2}$$

$$g'(x) = 10xe^{5x^2-2}$$

Then the derivative is

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (4x)(10xe^{5x^2-2}) + (4)(e^{5x^2-2})$$

$$y' = 40x^2e^{5x^2-2} + 4e^{5x^2-2}$$

The terms share a common factor of $4e^{5x^2-2}$, so factor that out.

$$y' = 4e^{5x^2-2}(10x^2 + 1)$$

Topic: Chain rule with quotient rule

Question: Apply quotient rule and chain rule to find the derivative.

$$y = \frac{(2x^2+1)^3}{x}$$

Answer choices:

A $\quad y' = 12x(2x^2+1)^2$

B $\quad y' = \dfrac{(2x^2+1)^2(10x^2-1)}{x^2}$

C $\quad y' = \dfrac{1}{12x(2x^2+1)^2}$

D $\quad y' = \dfrac{1-10x^2}{(2x^2+1)^4}$

Solution: B

List out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = (2x^2 + 1)^3$$

$$f'(x) = 3(2x^2 + 1)^2(4x)$$

$$f'(x) = 12x(2x^2 + 1)^2$$

and

$$g(x) = x$$

$$g'(x) = 1$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(12x(2x^2 + 1)^2)(x) - ((2x^2 + 1)^3)(1)}{x^2}$$

$$y' = \frac{12x^2(2x^2 + 1)^2 - (2x^2 + 1)^3}{x^2}$$

Within the numerator, we have a common factor of $(2x^2 + 1)^2$, so factor that out.

$$y' = \frac{(2x^2 + 1)^2(12x^2 - (2x^2 + 1))}{x^2}$$

$$y' = \frac{(2x^2 + 1)^2(12x^2 - 2x^2 - 1)}{x^2}$$

$$y' = \frac{(2x^2+1)^2(10x^2-1)}{x^2}$$

Topic: Chain rule with quotient rule

Question: Apply quotient rule and chain rule to find the derivative.

$$y = \frac{4}{(x^2 - 1)^3}$$

Answer choices:

A $\quad y' = -\frac{24x^2}{(x^2 - 1)^4}$

B $\quad y' = \frac{4}{(x^2 - 1)^3}$

C $\quad y' = -\frac{24x}{(x^2 - 1)^4}$

D $\quad y' = -\frac{12}{(x^2 - 1)^4}$

Solution: C

List out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = 4$$

$$f'(x) = 0$$

and

$$g(x) = (x^2 - 1)^3$$

$$g'(x) = 3(x^2 - 1)^2(2x) = 6x(x^2 - 1)^2$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(0)((x^2 - 1)^3) - (4)(6x(x^2 - 1)^2)}{[(x^2 - 1)^3]^2}$$

$$y' = \frac{0 - 24x(x^2 - 1)^2}{(x^2 - 1)^6}$$

$$y' = -\frac{24x(x^2 - 1)^2}{(x^2 - 1)^6}$$

$$y' = -\frac{24x}{(x^2 - 1)^4}$$

Topic: Chain rule with quotient rule

Question: Apply quotient rule and chain rule to find the derivative.

$$y = \frac{(3x^2+4)^2}{(4-2x)^4}$$

Answer choices:

A $\quad y' = -\frac{16(3x^2+4)}{(4-2x)^4}$

B $\quad y' = -\frac{16(3x^2+4)(3x+2)}{(4-2x)^5}$

C $\quad y' = \frac{16(3x^2+4)}{(4-2x)^4}$

D $\quad y' = \frac{16(3x^2+4)(3x+2)}{(4-2x)^5}$

Solution: D

List out $f(x)$ and $g(x)$ and their derivatives.

$$f(x) = (3x^2 + 4)^2$$

$$f'(x) = 2(3x^2 + 4)(6x)$$

$$f'(x) = 12x(3x^2 + 4)$$

and

$$g(x) = (4 - 2x)^4$$

$$g'(x) = 4(4 - 2x)^3(-2)$$

$$g'(x) = -8(4 - 2x)^3$$

Now we can plug these values directly into the quotient rule formula.

$$y' = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$$

$$y' = \frac{(12x(3x^2 + 4))((4 - 2x)^4) - ((3x^2 + 4)^2)(-8(4 - 2x)^3)}{[(4 - 2x)^4]^2}$$

$$y' = \frac{12x(3x^2 + 4)(4 - 2x)^4 + 8(3x^2 + 4)^2(4 - 2x)^3}{(4 - 2x)^8}$$

Within the fraction, we have a common factor of $(4 - 2x)^3$, so cancel that out.

$$y' = \frac{12x(3x^2 + 4)(4 - 2x) + 8(3x^2 + 4)^2}{(4 - 2x)^5}$$

Within the numerator, we have a common factor of $4(3x^2+4)$, so factor that out.

$$y' = \frac{4(3x^2+4)\left[3x(4-2x)+2(3x^2+4)\right]}{(4-2x)^5}$$

$$y' = \frac{4(3x^2+4)\left[(12x-6x^2)+(6x^2+8)\right]}{(4-2x)^5}$$

$$y' = \frac{4(3x^2+4)(12x-6x^2+6x^2+8)}{(4-2x)^5}$$

$$y' = \frac{4(3x^2+4)(12x+8)}{(4-2x)^5}$$

Factor out another 4 from the $12x+8$.

$$y' = \frac{16(3x^2+4)(3x+2)}{(4-2x)^5}$$

Topic: Inverse trigonometric derivatives

Question: Find the derivative of the inverse trig function.

$$f(x) = \tan^{-1}(x^2 - 1)$$

Answer choices:

A $\quad f'(x) = \dfrac{2x}{x^4 - 2x^2 + 2}$

B $\quad f'(x) = \dfrac{1}{x^4 - 2x^2 + 2}$

C $\quad f'(x) = \dfrac{2x}{1 + 4x^2}$

D $\quad f'(x) = \dfrac{1}{1 + 4x^2}$

Solution: A

Apply the formula for the derivative of inverse tangent, with $g(x) = x^2 - 1$, in order to differentiate the function.

$$y' = \frac{g'(x)}{1 + [g(x)]^2}$$

$$y' = \frac{2x}{1 + (x^2 - 1)^2}$$

Simplify the derivative.

$$y' = \frac{2x}{1 + (x^4 - 2x^2 + 1)}$$

$$y' = \frac{2x}{x^4 - 2x^2 + 2}$$

Topic: Inverse trigonometric derivatives

Question: Find the derivative of the inverse trig function.

$$y = \frac{1}{\cos^{-1} x}$$

Answer choices:

A $\quad y' = \dfrac{1}{1 - x^2}$

B $\quad y' = \dfrac{1}{(\cos^{-1} x)^2 \sqrt{1 - x^2}}$

C $\quad y' = -\sqrt{1 - x^2}$

D $\quad y' = \dfrac{\sqrt{1 - x^2}}{(\cos^{-1} x)^2}$

Solution: B

Rewrite the function.

$$y = \frac{1}{\arccos x}$$

$$y = (\arccos x)^{-1}$$

Use substitution with $u = \arccos x$ and

$$u' = -\frac{1}{\sqrt{1 - x^2}}$$

Then the function is

$$y = u^{-1}$$

The derivative is

$$y' = -u^{-2}u'$$

$$y' = -(\arccos x)^{-2}\left(-\frac{1}{\sqrt{1 - x^2}}\right)$$

$$y' = \frac{1}{(\arccos x)^2}\left(\frac{1}{\sqrt{1 - x^2}}\right)$$

$$y' = \frac{1}{(\cos^{-1} x)^2\sqrt{1 - x^2}}$$

Topic: Inverse trigonometric derivatives

Question: Find the derivative of the inverse trig function.

$$y = x\sin^{-1}\sqrt{x}$$

Answer choices:

A $\quad y' = x\sin^{-1}\sqrt{x} + \dfrac{1}{\sqrt{1-x}}$

B $\quad y' = \sin^{-1}\sqrt{x} + \dfrac{x}{\sqrt{1-x}}$

C $\quad y' = x\sin^{-1}\sqrt{x} + \dfrac{1}{2\sqrt{x(1-x)}}$

D $\quad y' = \dfrac{\sqrt{x}}{2\sqrt{1-x}} + \sin^{-1}\sqrt{x}$

Solution: D

We need to apply product rule, with

$$f(x) = x$$

$$f'(x) = 1$$

and

$$g(x) = \sin^{-1}\sqrt{x}$$

$$g'(x) = \frac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

Take the derivative using product rule.

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (x)\left[\frac{1}{\sqrt{1-(\sqrt{x})^2}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)\right] + (1)(\sin^{-1}\sqrt{x})$$

$$y' = \frac{1}{2}x\left(\frac{1}{x^{\frac{1}{2}}\sqrt{1-x}}\right) + \sin^{-1}\sqrt{x}$$

$$y' = \frac{x}{2\sqrt{x}\sqrt{1-x}} + \sin^{-1}\sqrt{x}$$

$$y' = \frac{\sqrt{x}\sqrt{x}}{2\sqrt{x}\sqrt{1-x}} + \sin^{-1}\sqrt{x}$$

$$y' = \frac{\sqrt{x}}{2\sqrt{1-x}} + \sin^{-1}\sqrt{x}$$

Topic: Hyperbolic derivatives

Question: Find the derivative of the hyperbolic function.

$$y = \cosh(x^2 - 1)$$

Answer choices:

A $y' = -2x\sinh(x^2 - 1)$

B $y' = -2x\cosh(x^2 - 1)$

C $y' = 2x\sinh(x^2 - 1)$

D $y' = 2x\cosh(x^2 - 1)$

Solution: C

Make a substitution, letting $u = x^2 - 1$ and $u' = 2x$. Then $y = \cosh u$, and the derivative is

$$y' = \sinh u \cdot u'$$

$$y' = \sinh(x^2 - 1) \cdot (2x)$$

$$y' = 2x \sinh(x^2 - 1)$$

Topic: Hyperbolic derivatives

Question: Find the derivative of the hyperbolic function.

$$y = x^2 \sinh x$$

Answer choices:

A $\quad y' = x \cosh x + 2 \sinh x$

B $\quad y' = x^2 \cosh x + 2x \sinh x$

C $\quad y' = x \cosh x - 2 \sinh x$

D $\quad y' = x^2 \cosh x - 2x \sinh x$

Solution: B

Use the product rule with

$$f(x) = x^2$$

$$f'(x) = 2x$$

and

$$g(x) = \sinh x$$

$$g'(x) = \cosh x$$

Then the derivative is

$$y' = f(x)g'(x) + f'(x)g(x)$$

$$y' = (x^2)(\cosh x) + (2x)(\sinh x)$$

$$y' = x^2 \cosh x + 2x \sinh x$$

Topic: Hyperbolic derivatives

Question: Find the derivative of the hyperbolic function.

$$f(x) = \sinh(\cosh(2x))$$

Answer choices:

A $f'(x) = 2\sinh(2x)\sinh(\sinh(2x))$

B $f'(x) = 2\sinh(2x)\cosh(\sinh(2x))$

C $f'(x) = 2\sinh(2x)\cosh(\cosh(2x))$

D $f'(x) = 2\cosh(2x)\cosh(\cosh(2x))$

Solution: C

Use a substitution with $u = \cosh(2x)$ and $u' = 2\sinh(2x)$. Then the function can be rewritten as

$$f(x) = \sinh u$$

and the derivative is

$$f'(x) = \cosh u \cdot u'$$

$$f'(x) = \cosh(\cosh(2x)) \cdot (2\sinh(2x))$$

$$f'(x) = 2\sinh(2x)\cosh(\cosh(2x))$$

Topic: Inverse hyperbolic derivatives

Question: Find the derivative of the inverse hyperbolic function.

$$y = \cosh^{-1}(x^3)$$

Answer choices:

A $\quad y' = \dfrac{3x^2}{\sqrt{x^6 - 1}}$ $\quad$ with $x^3 < 1$

B $\quad y' = \dfrac{3x^2}{\sqrt{x^6 + 1}}$ $\quad$ with $x^3 < 1$

C $\quad y' = \dfrac{3x^2}{\sqrt{x^6 + 1}}$ $\quad$ with $x^3 > 1$

D $\quad y' = \dfrac{3x^2}{\sqrt{x^6 - 1}}$ $\quad$ with $x^3 > 1$

Solution: D

Apply the formula for the derivative of inverse hyperbolic cosine, with $g(x) = x^3$ and $g'(x) = 3x^2$.

$$y' = \frac{g'(x)}{\sqrt{[g(x)]^2 - 1}} \qquad \text{with } x^3 > 1$$

$$y' = \frac{3x^2}{\sqrt{(x^3)^2 - 1}} \qquad \text{with } x^3 > 1$$

$$y' = \frac{3x^2}{\sqrt{x^6 - 1}} \qquad \text{with } x^3 > 1$$

Topic: Inverse hyperbolic derivatives

Question: Find the derivative of the inverse hyperbolic function.

$$y = \tanh^{-1}(2x^5 - 1)$$

Answer choices:

A $\quad y' = \dfrac{5}{2x + 2x^6}$ $\quad$ with $|2x^5 - 1| < 1$

B $\quad y' = \dfrac{1}{2x - 2x^6}$ $\quad$ with $|2x^5 - 1| < 1$

C $\quad y' = \dfrac{5}{x - x^6}$ $\quad$ with $|2x^5 - 1| < 1$

D $\quad y' = \dfrac{5}{2x - 2x^6}$ $\quad$ with $|2x^5 - 1| < 1$

Solution: D

Apply the formula for the derivative of inverse hyperbolic tangent, with $g(x) = 2x^5 - 1$ and $g'(x) = 10x^4$.

$$y' = \frac{g'(x)}{1 - [g(x)]^2} \quad \text{with } |g(x)| < 1$$

$$y' = \frac{10x^4}{1 - (2x^5 - 1)^2} \quad \text{with } |2x^5 - 1| < 1$$

Simplify the derivative.

$$y' = \frac{10x^4}{1 - (4x^{10} - 4x^5 + 1)} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{10x^4}{1 - 4x^{10} + 4x^5 - 1} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{10x^4}{-4x^{10} + 4x^5} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{5}{-2x^6 + 2x} \quad \text{with } |2x^5 - 1| < 1$$

$$y' = \frac{5}{2x - 2x^6} \quad \text{with } |2x^5 - 1| < 1$$

Topic: Inverse hyperbolic derivatives

Question: Find the derivative of the inverse hyperbolic function.

$$f(x) = (\sinh^{-1}(2x^3))^4$$

Answer choices:

A $\quad f'(x) = \dfrac{24x^2(\sinh^{-1}(2x^3))^3}{\sqrt{x^2+1}}$

B $\quad f'(x) = \dfrac{24x^2(\sinh^{-1}(2x^3))^3}{\sqrt{4x^6+1}}$

C $\quad f'(x) = \dfrac{6x^2(\sinh^{-1}(2x^3))^3}{\sqrt{x^2+1}}$

D $\quad f'(x) = \dfrac{6x^2(\sinh^{-1}(2x^3))^3}{\sqrt{4x^6+1}}$

Solution: B

Use a substitution with $u = \sinh^{-1}(2x^3)$ and

$$u' = \frac{6x^2}{\sqrt{(2x^3)^2 + 1}}$$

$$u' = \frac{6x^2}{\sqrt{4x^6 + 1}}$$

Then the function is

$$f(x) = u^4$$

and the derivative is

$$f'(x) = 4u^3 \cdot u'$$

$$f'(x) = 4(\sinh^{-1}(2x^3))^3 \cdot \frac{6x^2}{\sqrt{4x^6 + 1}}$$

$$f'(x) = \frac{24x^2(\sinh^{-1}(2x^3))^3}{\sqrt{4x^6 + 1}}$$

Topic: Logarithmic differentiation

Question: Use logarithmic differentiation to find the derivative.

$$y = 3^{5x}$$

Answer choices:

A $y' = 15^{5x}(\ln 3)$

B $y' = 3^{4x}(5 \ln 3)$

C $y' = 3^{5x}(5 \ln 3)$

D $y' = 3^{5x}(\ln 15)$

Solution: C

Apply the natural log to both sides of the equation.

$$y = 3^{5x}$$

$$\ln y = \ln(3^{5x})$$

Use laws of logs to rewrite the equation.

$$\ln y = 5x \ln 3$$

Take the derivative of both sides of the equation and remembering to multiply by y' when we take the derivative of y.

$$\frac{1}{y}y' = 5 \ln 3$$

Solve for y' in terms of x by isolating y' and substituting for y.

$$y' = 5y \ln 3$$

$$y' = 5(3^{5x})\ln 3$$

$$y' = 3^{5x}(5 \ln 3)$$

Topic: Logarithmic differentiation

Question: Use logarithmic differentiation to find the derivative.

$$y = 3\ln(5x)$$

Answer choices:

A $\quad y' = \dfrac{5}{x}$

B $\quad y' = \dfrac{3}{x}$

C $\quad y' = \dfrac{3}{5x}$

D $\quad y' = \dfrac{15}{x}$

Solution: B

Apply the natural log to both sides of the equation.

$$y = 3\ln(5x)$$

$$\ln y = \ln(3\ln(5x))$$

Use laws of logs to rewrite the equation.

$$\ln y = \ln 3 + \ln(\ln(5x))$$

Take the derivative of both sides of the equation, remembering to multiply by y' when we take the derivative of y.

$$\frac{1}{y}y' = 0 + \frac{1}{\ln(5x)}\left(\frac{1}{5x}(5)\right)$$

$$\frac{1}{y}y' = \frac{1}{x\ln(5x)}$$

Solve for y' in terms of x by isolating y' and substituting for y.

$$y' = \frac{y}{x\ln(5x)}$$

$$y' = \frac{3\ln(5x)}{x\ln(5x)}$$

$$y' = \frac{3}{x}$$

Topic: Logarithmic differentiation

Question: Use logarithmic differentiation to find the derivative.

$$y = \ln(6x^2 - 3x + 9)^{4x}$$

Answer choices:

A $\quad y' = 4\ln(6x^2 - 3x + 9)$

B $\quad y' = \dfrac{4x(4x-1)}{2x^2 - x + 3}$

C $\quad y' = 4\ln(6x^2 - 3x + 9) + \dfrac{4x(4x-1)}{2x^2 - x + 3}$

D $\quad y' = 4\ln(6x^2 - 3x + 9) + \dfrac{1}{6x^2 - 3x + 9}$

Solution: C

Apply the natural log to both sides of the equation.

$$y = \ln(6x^2 - 3x + 9)^{4x}$$

$$\ln y = \ln(\ln(6x^2 - 3x + 9)^{4x})$$

Use laws of logs to rewrite the equation.

$$\ln y = \ln(4x \ln(6x^2 - 3x + 9))$$

$$\ln y = \ln(4x) + \ln(\ln(6x^2 - 3x + 9))$$

Take the derivative of both sides of the equation, remembering to multiply by y' when we take the derivative of y.

$$\frac{1}{y}y' = \frac{1}{4x}(4) + \frac{1}{\ln(6x^2 - 3x + 9)}\left(\frac{1}{6x^2 - 3x + 9}(12x - 3)\right)$$

$$\frac{1}{y}y' = \frac{1}{x} + \frac{1}{\ln(6x^2 - 3x + 9)}\left(\frac{12x - 3}{6x^2 - 3x + 9}\right)$$

$$\frac{1}{y}y' = \frac{1}{x} + \frac{12x - 3}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}$$

Solve for y' in terms of x by isolating y' and substituting for y.

$$y' = y\left[\frac{1}{x} + \frac{12x - 3}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}\right]$$

$$y' = \ln(6x^2 - 3x + 9)^{4x}\left[\frac{1}{x} + \frac{12x - 3}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}\right]$$

$$y' = \frac{\ln(6x^2 - 3x + 9)^{4x}}{x} + \frac{(12x - 3)\ln(6x^2 - 3x + 9)^{4x}}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}$$

Use laws of logarithms to simplify the derivative.

$$y' = \frac{4x\ln(6x^2 - 3x + 9)}{x} + \frac{4x(12x - 3)\ln(6x^2 - 3x + 9)}{(6x^2 - 3x + 9)\ln(6x^2 - 3x + 9)}$$

$$y' = 4\ln(6x^2 - 3x + 9) + \frac{4x(12x - 3)}{6x^2 - 3x + 9}$$

$$y' = 4\ln(6x^2 - 3x + 9) + \frac{12x(4x - 1)}{3(2x^2 - x + 3)}$$

$$y' = 4\ln(6x^2 - 3x + 9) + \frac{4x(4x - 1)}{2x^2 - x + 3}$$

Topic: Tangent lines

Question: Find the equation of the tangent line to the function at $(1, -2)$.

$$g(x) = 3x^2 - 6x + 1$$

Answer choices:

A $y = -2$

B $x + y = -2$

C $y = 2$

D $x - y = 2$

Solution: A

If we know that the tangent line intersects the curve at $(1, -2)$, we don't need to find $g(1)$, because it's the y-value where the tangent line intersects the curve, so $g(1) = -2$.

Take the derivative of the function.

$$g'(x) = 6x - 6$$

Find the slope of the tangent line at $(1, -2)$ by evaluating the derivative at that point.

$$g'(1) = 6(1) - 6$$

$$g'(1) = 0$$

The slope of the tangent line at $(1, -2)$ is $g'(1) = 0$.

Finally, substitute both $g(1)$ and $g'(1)$ into the tangent line formula, along with $a = 1$, since this is the value at which we're finding the equation of the tangent line.

$$y = g(a) + g'(a)(x - a)$$

$$y = -2 + 0(x - 1)$$

$$y = -2$$

Topic: Tangent lines

Question: Find the equation of the tangent line to the function at $(1,1/2)$.

$$f(x) = \frac{1}{x^2 + 1}$$

Answer choices:

A $y = -\frac{1}{2}x + 1$

B $y = x - 1$

C $y = -2x + 2$

D $y = \frac{1}{2}x - 1$

Solution: A

Use quotient rule to take the derivative of the function.

$$f'(x) = \frac{(0)(x^2+1) - (1)(2x)}{(x^2+1)^2}$$

$$f'(x) = \frac{0 - 2x}{(x^2+1)^2}$$

$$f'(x) = -\frac{2x}{(x^2+1)^2}$$

Find the slope of the tangent line at $(1,1/2)$ by evaluating the derivative at that point.

$$f'(1) = -\frac{2(1)}{(1^2+1)^2}$$

$$f'(1) = -\frac{1}{2}$$

Now we can find the equation of the tangent line by plugging the slope $f'(1) = 1/2$ and the point $(1,1/2)$ into the formula for the equation of the tangent line.

$$y = f(a) + f'(a)(x - a)$$

$$y = f(1) - \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2} - \frac{1}{2}(x - 1)$$

$$y = \frac{1}{2} - \frac{1}{2}x + \frac{1}{2}$$

$$y = -\frac{1}{2}x + 1$$

Topic: Tangent lines

Question: Where on the interval $-1 \leq x \leq 1$ does the function have horizontal tangent lines?

$$f(x) = x^3 - x - 3$$

Answer choices:

A At $x = 0$

B At $x = \pm \frac{\sqrt{3}}{3}$

C At $x = \pm \sqrt{3}$

D At $x = \pm 3$

Solution: B

Take the derivative of the function.

$$f'(x) = 3x^2 - 1$$

Horizontal tangent lines exist when $f'(x) = 0$, so we'll set the derivative equal to 0.

$$3x^2 - 1 = 0$$

$$3x^2 = 1$$

$$x^2 = \frac{1}{3}$$

$$x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{1}}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}} = \pm\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \pm\frac{\sqrt{3}}{3}$$

On the interval $-1 \le x \le 1$, the function has two horizontal tangent lines, located at

$$x = \pm\frac{\sqrt{3}}{3}$$

Topic: Value that makes two tangent lines parallel

Question: What is the value of a such that the tangent lines of $f(x)$ at $x = a$ and $x = a + 1$ are parallel?

$$f(x) = x^3 + x^2 + x - 1$$

Answer choices:

A $\quad a = -\frac{1}{2}$

B $\quad a = \frac{1}{2}$

C $\quad a = -\frac{5}{6}$

D $\quad a = \frac{5}{6}$

Solution: C

Start by finding the derivative of $f(x)$.

$$f'(x) = 3x^2 + 2x + 1$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 3a^2 + 2a + 1$$

$$f'(a+1) = 3(a+1)^2 + 2(a+1) + 1$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 2a + 1 = 3(a+1)^2 + 2(a+1) + 1$$

$$3a^2 + 2a + 1 = 3(a^2 + 2a + 1) + 2a + 2 + 1$$

$$3a^2 + 2a + 1 = 3a^2 + 6a + 3 + 2a + 2 + 1$$

Collect like terms and solve for a.

$$2a + 1 = 6a + 3 + 2a + 2 + 1$$

$$1 = 6a + 3 + 2 + 1$$

$$0 = 6a + 3 + 2$$

$$0 = 6a + 5$$

$$6a = -5$$

$$a = -\frac{5}{6}$$

Topic: Value that makes two tangent lines parallel

Question: What is the value of a such that the tangent lines of $f(x)$ at $x=a$ and $x=a+1$ are parallel?

$$f(x)=2x^3-x^2+x+12$$

Answer choices:

A $\quad a=\frac{1}{3}$

B $\quad a=-\frac{1}{3}$

C $\quad a=\frac{1}{2}$

D $\quad a=-\frac{1}{2}$

Solution: B

Start by finding the derivative of $f(x)$.

$$f'(x) = 6x^2 - 2x + 1$$

Now we'll plug both $x = a$ and $x = a + 1$ into the derivative.

$$f'(a) = 6a^2 - 2a + 1$$

$$f'(a+1) = 6(a+1)^2 - 2(a+1) + 1$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$6a^2 - 2a + 1 = 6(a+1)^2 - 2(a+1) + 1$$

$$6a^2 - 2a + 1 = 6(a^2 + 2a + 1) - 2a - 2 + 1$$

$$6a^2 - 2a + 1 = 6a^2 + 12a + 6 - 2a - 2 + 1$$

Collect like terms and solve for a.

$$-2a + 1 = 12a + 6 - 2a - 2 + 1$$

$$1 = 12a + 6 - 2 + 1$$

$$0 = 12a + 6 - 2$$

$$0 = 12a + 4$$

$$12a = -4$$

$$a = -\frac{1}{3}$$

Topic: Value that makes two tangent lines parallel

Question: What is the value of a such that the tangent lines of $f(x)$ at $x = a$ and $x = a + 2$ are parallel?

$$f(x) = x^3 + 3x^2 - x - 5$$

Answer choices:

A $\quad a = \frac{1}{2}$

B $\quad a = -\frac{1}{2}$

C $\quad a = 2$

D $\quad a = -2$

Solution: D

Start by finding the derivative of $f(x)$.

$$f'(x) = 3x^2 + 6x - 1$$

Now we'll plug both $x = a$ and $x = a + 2$ into the derivative.

$$f'(a) = 3a^2 + 6a - 1$$

$$f'(a + 2) = 3(a + 2)^2 + 6(a + 2) - 1$$

These represent the slope of each tangent line, so we'll set them equal to one another.

$$3a^2 + 6a - 1 = 3(a + 2)^2 + 6(a + 2) - 1$$

$$3a^2 + 6a - 1 = 3(a^2 + 4a + 4) + 6a + 12 - 1$$

$$3a^2 + 6a - 1 = 3a^2 + 12a + 12 + 6a + 12 - 1$$

Collect like terms and solve for a.

$$6a - 1 = 12a + 12 + 6a + 12 - 1$$

$$-1 = 12a + 12 + 12 - 1$$

$$0 = 12a + 12 + 12$$

$$0 = 12a + 24$$

$$12a = -24$$

$$a = -2$$

Topic: Values that make the function differentiable

Question: Which values of a and b would make the function differentiable?

$$f(x) = \begin{cases} 3x^2 & x > 1 \\ bx^2 - a & x \leq 1 \end{cases}$$

Answer choices:

A	$a = 3$	and	$b = 0$
B	$a = 0$	and	$b = 0$
C	$a = 0$	and	$b = 3$
D	$a = 3$	and	$b = 3$

Solution: C

The break point of the function is at $x = 1$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x = 1$ equal to one another.

$$\lim_{x \to 1^+} 3x^2 = \lim_{x \to 1^-} (bx^2 - a)$$

$$3(1)^2 = b(1)^2 - a$$

$$3 = b - a$$

$$b - a = 3$$

Now we'll work on smoothness by setting the one-sided limits of the derivatives of each piece at the break point $x = 1$ equal to one another.

$$\lim_{x \to 1^+} 6x = \lim_{x \to 1^-} 2bx$$

$$6(1) = 2b(1)$$

$$6 = 2b$$

$$b = 3$$

Pull together these two equations into a system of equations.

$$b = 3$$

$$b - a = 3$$

We need to solve the system, which we can do by substituting the first equation $b = 3$ into the second equation.

$$3 - a = 3$$

$$-a = 0$$

$$a = 0$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = 0$ and $b = 3$.

Topic: Values that make the function differentiable

Question: Which values of a and b would make the function differentiable?

$$f(x) = \begin{cases} x^2 - 5 & x > 3 \\ 4x^2 - 2ax + b & x \leq 3 \end{cases}$$

Answer choices:

A	$a = 22$	and	$b = 9$
B	$a = 22$	and	$b = 22$
C	$a = 9$	and	$b = 9$
D	$a = 9$	and	$b = 22$

Solution: D

The break point of the function is at $x = 3$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x = 3$ equal to one another.

$$\lim_{x \to 3^+} (x^2 - 5) = \lim_{x \to 3^-} (4x^2 - 2ax + b)$$

$$3^2 - 5 = 4(3)^2 - 2a(3) + b$$

$$9 - 5 = 4(9) - 6a + b$$

$$4 = 36 - 6a + b$$

$$-6a + b = -32$$

$$6a - b = 32$$

Now we'll work on smoothness by setting the one-sided limits of the derivatives of each piece at the break point $x = 3$ equal to one another.

$$\lim_{x \to 3^+} 2x = \lim_{x \to 3^-} (8x - 2a)$$

$$2(3) = 8(3) - 2a$$

$$6 = 24 - 2a$$

$$-18 = -2a$$

$$a = 9$$

Pull together these two equations into a system of equations.

$$a = 9$$

$$6a - b = 32$$

We need to solve the system, which we can do by substituting the first equation $a = 9$ into the second equation.

$$6(9) - b = 32$$

$$54 - b = 32$$

$$-b = -22$$

$$b = 22$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = 9$ and $b = 22$.

Topic: Values that make the function differentiable

Question: Which values of a and b would make the function differentiable?

$$f(x) = \begin{cases} ax^2 + 10 & x \leq 2 \\ x^2 - 6x + b & x > 2 \end{cases}$$

Answer choices:

A $\quad a = \frac{1}{2} \quad$ and $\quad b = 16$

B $\quad a = -\frac{1}{2} \quad$ and $\quad b = -16$

C $\quad a = \frac{1}{2} \quad$ and $\quad b = -16$

D $\quad a = -\frac{1}{2} \quad$ and $\quad b = 16$

Solution: D

The break point of the function is at $x = 2$, because that's where the first piece of the function ends and the second piece takes over.

We'll work on continuity first by setting the one-sided limits at the break point $x = 2$ equal to one another.

$$\lim_{x\to 2^-} (ax^2 + 10) = \lim_{x\to 2^+} (x^2 - 6x + b)$$

$$a(2)^2 + 10 = 2^2 - 6(2) + b$$

$$4a + 10 = 4 - 12 + b$$

$$4a - b = -18$$

Now we'll work on smoothness by setting the one-sided limits of the derivatives of each piece at the break point $x = 2$ equal to one another.

$$\lim_{x\to 2^-} 2ax = \lim_{x\to 2^+} (2x - 6)$$

$$2a(2) = 2(2) - 6$$

$$4a = 4 - 6$$

$$4a = -2$$

$$a = -\frac{1}{2}$$

Pull together these two equations into a system of equations.

$$a = -\frac{1}{2}$$

$$4a - b = -18$$

We need to solve the system, which we can do by substituting the first equation $a = -1/2$ into the second equation.

$$4\left(-\frac{1}{2}\right) - b = -18$$

$$-2 - b = -18$$

$$-b = -16$$

$$b = 16$$

Therefore, the values of the constants a and b that make $f(x)$ differentiable are $a = -1/2$ and $b = 16$.

Topic: Normal lines

Question: Find the equation of the normal line to the function at $(1,2)$.

$$f(x) = 2x^4$$

Answer choices:

A $\quad y = 8x - 6$

B $\quad y = -\frac{1}{8}x - \frac{17}{8}$

C $\quad y = -\frac{1}{8}x + \frac{17}{8}$

D $\quad y = 8x - 10$

Solution: C

Take the derivative of the function,

$$f'(x) = 8x^3$$

and then evaluate it at $(1,2)$.

$$f'(1) = 8(1)^3$$

$$f'(1) = 8$$

This is the slope of the tangent line at $(1,2)$. Since $m = 8$, we'll take the negative reciprocal to find n, the slope of the normal line.

$$n = -\frac{1}{8}$$

We'll plug $n = -1/8$ and the point $(1,2)$ into the point-slope formula for the equation of the line. Once we simplify, we'll have the equation of the normal line to the function at $(1,2)$.

$$y - y_1 = n(x - x_1)$$

$$y - 2 = -\frac{1}{8}(x - 1)$$

$$y - 2 = -\frac{1}{8}x + \frac{1}{8}$$

$$y = -\frac{1}{8}x + \frac{1}{8} + \frac{16}{8}$$

$$y = -\frac{1}{8}x + \frac{17}{8}$$

Topic: Normal lines

Question: Find the equation of the normal line to the function at $(3,6)$.

$$f(x) = x\sqrt{x+1}$$

Answer choices:

A $\quad y = -\frac{4}{11}x + \frac{78}{11}$

B $\quad y = \frac{11}{4}x - \frac{57}{4}$

C $\quad y = \frac{11}{4}x - \frac{9}{4}$

D $\quad y = -\frac{4}{11}x - \frac{54}{11}$

Solution: A

Take the derivative of the function,

$$f'(x) = (1)(\sqrt{x+1}) + (x)\left(\frac{1}{2}(x+1)^{-\frac{1}{2}}\right)$$

$$f'(x) = \sqrt{x+1} + \frac{x}{2\sqrt{x+1}}$$

and then evaluate it at $(3,6)$.

$$f'(3) = \sqrt{3+1} + \frac{3}{2\sqrt{3+1}}$$

$$f'(3) = 2 + \frac{3}{2(2)}$$

$$f'(3) = \frac{8}{4} + \frac{3}{4}$$

$$f'(3) = \frac{11}{4}$$

This is the slope of the tangent line at $(3,6)$. Since $m = 11/4$, we'll take the negative reciprocal to find n, the slope of the normal line.

$$n = -\frac{4}{11}$$

We'll plug $n = -4/11$ and the point $(3,6)$ into the point-slope formula for the equation of the line. Once we simplify, we'll have the equation of the normal line to the function at $(3,6)$.

$$y - y_1 = n(x - x_1)$$

$$y - 6 = -\frac{4}{11}(x - 3)$$

$$y - 6 = -\frac{4}{11}x + \frac{12}{11}$$

$$y = -\frac{4}{11}x + \frac{12}{11} + \frac{66}{11}$$

$$y = -\frac{4}{11}x + \frac{78}{11}$$

Topic: Normal lines

Question: Find the equation of the normal line to the function at $(2,2)$.

$$f(x) = \frac{2x^2}{x+2}$$

Answer choices:

A $\quad y = -\frac{2}{3}x - \frac{2}{3}$

B $\quad y = \frac{3}{2}x - 1$

C $\quad y = \frac{3}{2}x - 5$

D $\quad y = -\frac{2}{3}x + \frac{10}{3}$

Solution: D

Take the derivative of the function,

$$f'(x) = \frac{(4x)(x+2) - (2x^2)(1)}{(x+2)^2}$$

$$f'(x) = \frac{4x^2 + 8x - 2x^2}{(x+2)^2}$$

$$f'(x) = \frac{2x^2 + 8x}{(x+2)^2}$$

and then evaluate it at $(2,2)$.

$$f'(2) = \frac{2(2)^2 + 8(2)}{(2+2)^2}$$

$$f'(2) = \frac{2(4) + 16}{4^2}$$

$$f'(2) = \frac{8 + 16}{16}$$

$$f'(2) = \frac{3}{2}$$

This is the slope of the tangent line at $(2,2)$. Since $m = 3/2$, we'll take the negative reciprocal to find n, the slope of the normal line.

$$n = -\frac{2}{3}$$

We'll plug $n = -2/3$ and the point $(2,2)$ into the point-slope formula for the equation of the line. Once we simplify, we'll have the equation of the normal line to the function at $(2,2)$.

$$y - y_1 = n(x - x_1)$$

$$y - 2 = -\frac{2}{3}(x - 2)$$

$$y - 2 = -\frac{2}{3}x + \frac{4}{3}$$

$$y = -\frac{2}{3}x + \frac{4}{3} + \frac{6}{3}$$

$$y = -\frac{2}{3}x + \frac{10}{3}$$

Topic: Average rate of change

Question: Find the average rate of change of the function on $[0,2]$.

$$f(x) = x^2$$

Answer choices:

A $\frac{\Delta f}{\Delta x} = 0$

B $\frac{\Delta f}{\Delta x} = \frac{1}{2}$

C $\frac{\Delta f}{\Delta x} = 1$

D $\frac{\Delta f}{\Delta x} = 2$

Solution: D

From the interval, we know $x_1 = 0$ and $x_2 = 2$. We'll find $f(x_1)$ and $f(x_2)$ by plugging these values into the function. We get

$$f(0) = 0^2$$

$$f(0) = 0$$

and

$$f(2) = 2^2$$

$$f(2) = 4$$

Now we can plug the values we've found into the formula for average rate fo change.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(2) - f(0)}{2 - 0}$$

$$\frac{\Delta f}{\Delta x} = \frac{4 - 0}{2}$$

$$\frac{\Delta f}{\Delta x} = \frac{4}{2}$$

$$\frac{\Delta f}{\Delta x} = 2$$

Topic: Average rate of change

Question: Find the average rate of change of the function on $[1,4]$.

$$f(x) = \frac{2x}{3x^2 - 1}$$

Answer choices:

A $\quad \frac{\Delta f}{\Delta x} = -\frac{47}{13}$

B $\quad \frac{\Delta f}{\Delta x} = -\frac{13}{47}$

C $\quad \frac{\Delta f}{\Delta x} = \frac{47}{13}$

D $\quad \frac{\Delta f}{\Delta x} = \frac{13}{47}$

Solution: B

From the interval, we know $x_1 = 1$ and $x_2 = 4$. We'll find $f(x_1)$ and $f(x_2)$ by plugging these values into the function. We get

$$f(1) = \frac{2(1)}{3(1)^2 - 1}$$

$$f(1) = \frac{2}{2}$$

$$f(1) = 1$$

and

$$f(4) = \frac{2(4)}{3(4)^2 - 1}$$

$$f(4) = \frac{8}{48 - 1}$$

$$f(4) = \frac{8}{47}$$

Now we can plug the values we've found into the formula for average rate fo change.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(4) - f(1)}{4 - 1}$$

$$\frac{\Delta f}{\Delta x} = \frac{\frac{8}{47} - 1}{3}$$

$$\frac{\Delta f}{\Delta x} = \frac{\frac{8}{47} - \frac{47}{47}}{3}$$

$$\frac{\Delta f}{\Delta x} = \frac{-\frac{39}{47}}{3}$$

$$\frac{\Delta f}{\Delta x} = -\frac{39}{47} \cdot \frac{1}{3}$$

$$\frac{\Delta f}{\Delta x} = -\frac{13}{47}$$

Topic: Average rate of change

Question: Find the average rate of change of the function on $[2,3]$.

$$f(x) = 6e^x - 4\sqrt{x^3}$$

Answer choices:

A $\dfrac{\Delta f}{\Delta x} = 6e^2(e-1) - 4(3\sqrt{3} - 2\sqrt{2})$

B $\dfrac{\Delta f}{\Delta x} = e^2(6e+1) + 3\sqrt{3} + 8\sqrt{2}$

C $\dfrac{\Delta f}{\Delta x} = 6e^2(e+1) + 4(3\sqrt{3} + 2\sqrt{2})$

D $\dfrac{\Delta f}{\Delta x} = e^2(6e-1) - 3\sqrt{3} + 8\sqrt{2}$

Solution: A

From the interval, we know $x_1 = 2$ and $x_2 = 3$. We'll find $f(x_1)$ and $f(x_2)$ by plugging these values into the function. We get

$$f(3) = 6e^3 - 4\sqrt{3^3}$$

$$f(3) = 6e^3 - 4\sqrt{27}$$

$$f(3) = 6e^3 - 12\sqrt{3}$$

and

$$f(2) = 6e^2 - 4\sqrt{2^3}$$

$$f(2) = 6e^2 - 4\sqrt{8}$$

$$f(2) = 6e^2 - 8\sqrt{2}$$

Now we can plug the values we've found into the formula for average rate fo change.

$$\frac{\Delta f}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\frac{\Delta f}{\Delta x} = \frac{f(3) - f(2)}{3 - 2}$$

$$\frac{\Delta f}{\Delta x} = \frac{6e^3 - 12\sqrt{3} - (6e^2 - 8\sqrt{2})}{3 - 2}$$

$$\frac{\Delta f}{\Delta x} = \frac{6e^3 - 12\sqrt{3} - 6e^2 + 8\sqrt{2}}{1}$$

$$\frac{\Delta f}{\Delta x} = 6e^3 - 12\sqrt{3} - 6e^2 + 8\sqrt{2}$$

$$\frac{\Delta f}{\Delta x} = 6e^2(e - 1) - 4(3\sqrt{3} - 2\sqrt{2})$$

Topic: Implicit differentiation

Question: Using implicit differentiation, we...

Answer choices:

A treat x as a variable and y as a function

B treat x as a variable and y as a variable

C treat x as a function and y as a function

D treat x as a function and y as a variable

Solution: A

When we use implicit differentiation, it’s important to remember that we can’t treat y as a variable, the same way we would if we were differentiating “normally.”

In contrast, we have to treat y as a function of y in terms of x, and therefore apply chain rule whenever we take the derivative of y, which means we multiply by y'.

Topic: Implicit differentiation

Question: Every time we take the derivative of y, implicit differentiation requires us to multiply by which of the following?

Answer choices:

A 1

B y

C 0

D y'

Solution: D

We have to treat y as a function of y in terms of x, and therefore apply chain rule whenever we take the derivative of y, which means we multiply by y' every time we differentiate y.

Topic: Implicit differentiation

Question: Use implicit differentiation to find the derivative.

$$x^2 - y^2 = 9$$

Answer choices:

A $y' = -\frac{x}{y}$

B $y' = -\frac{y}{x}$

C $y' = \frac{x}{y}$

D $y' = \frac{y}{x}$

Solution: C

Using implicit differentiation to take the derivative of both sides of the equation gives

$$2x - 2yy' = 0$$

$$-2yy' = -2x$$

$$yy' = x$$

$$y' = \frac{x}{y}$$

Topic: Equation of the tangent line with implicit differentiation

Question: Find the equation of the tangent line to the curve at $(3,2)$.

$$\frac{x^2}{9} + \frac{y^2}{4} = 2$$

Answer choices:

A $\quad y = -\frac{2}{3}x + 4$

B $\quad y = -\frac{2}{3}x - 4$

C $\quad y = \frac{2}{3}x + 4$

D $\quad y = \frac{2}{3}x - 4$

Solution: A

Using implicit differentiation to find the derivative of the curve, we get

$$\frac{2x}{9}+\frac{2y}{4}y'=0$$

Simplify and solve for y'.

$$\frac{2y}{4}y'=-\frac{2x}{9}$$

$$y'=-\frac{2x(4)}{9(2y)}$$

$$y'=-\frac{8x}{18y}$$

$$y'=-\frac{4x}{9y}$$

Evaluate the derivative at $(3,2)$ to find the slope of the tangent line.

$$m=-\frac{4(3)}{9(2)}$$

$$m=-\frac{12}{18}$$

$$m=-\frac{2}{3}$$

Plug the slope $m=-2/3$ and the point of tangency $(3,2)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 2 = -\frac{2}{3}(x - 3)$$

$$y - 2 = -\frac{2}{3}x + 2$$

$$y = -\frac{2}{3}x + 4$$

Topic: Equation of the tangent line with implicit differentiation

Question: Find the equation of the tangent line to the curve at $(0,1)$.

$$5x^2 + y^2 + 4xy = 1$$

Answer choices:

A $y = -2x - 1$

B $y = -2x + 1$

C $y = 2x - 1$

D $y = 2x + 1$

Solution: B

Using implicit differentiation to find the derivative of the curve, we get

$$10x + 2yy' + [(4)(y) + (4x)(1)(y')] = 0$$

$$10x + 2yy' + 4y + 4xy' = 0$$

Simplify and solve for y'.

$$2yy' + 4xy' = -10x - 4y$$

$$y'(2y + 4x) = -10x - 4y$$

$$y' = -\frac{10x + 4y}{2y + 4x}$$

$$y' = -\frac{5x + 2y}{y + 2x}$$

Evaluate the derivative at $(0,1)$ to find the slope of the tangent line.

$$m = -\frac{5(0) + 2(1)}{(1) + 2(0)}$$

$$m = -\frac{2}{1}$$

$$m = -2$$

Plug the slope $m = -2$ and the point of tangency $(0,1)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2(x - 0)$$

$$y - 1 = -2x$$

$$y = -2x + 1$$

Topic: Equation of the tangent line with implicit differentiation

Question: Find the equation of the tangent line to the curve at $(1,4)$.

$$8 + x^2y^2 - 5xy = 4$$

Answer choices:

A $y = 4x - 8$

B $y = 4x + 8$

C $y = -4x - 8$

D $y = -4x + 8$

Solution: D

Using implicit differentiation to find the derivative of the curve, we get

$$0 + [(2x)(y^2) + (x^2)(2yy')] - [(5)(y) + (5x)(1)(y')] = 0$$

$$2xy^2 + 2x^2yy' - 5y - 5xy' = 0$$

Simplify and solve for y'.

$$2x^2yy' - 5xy' = 5y - 2xy^2$$

$$y'(2x^2y - 5x) = 5y - 2xy^2$$

$$y' = \frac{5y - 2xy^2}{2x^2y - 5x}$$

Evaluate the derivative at $(1,4)$ to find the slope of the tangent line.

$$m = \frac{5(4) - 2(1)(4)^2}{2(1)^2(4) - 5(1)}$$

$$m = \frac{20 - 32}{8 - 5}$$

$$m = \frac{-12}{3}$$

$$m = -4$$

Plug the slope $m = -4$ and the point of tangency $(1,4)$ into the point-slope formula for the equation of a line.

$$y - y_1 = m(x - x_1)$$

$$y - 4 = -4(x - 1)$$

$$y - 4 = -4x + 4$$

$$y = -4x + 8$$

Topic: Higher-order derivatives

Question: Find the second derivative of the function.

$$y = e^{-4x^2}$$

Answer choices:

A $\quad y'' = 8e^{-4x^2}(2x^2 - 1)$

B $\quad y' = 16xe^{-4x^2}$

C $\quad y'' = 64xe^{-4x^2}$

D $\quad y'' = 8e^{-4x^2}(8x^2 - 1)$

Solution: D

First we need to find the first derivative using the chain rule

$$y = e^{-4x^2}$$

$$y' = e^{-4x^2}(-8x)$$

$$y' = -8xe^{-4x^2}$$

Now we can find the second derivative by taking the derivative of the first derivative using the product rule.

$$y'' = (y')' = (-8xe^{-4x^2})'$$

$$y'' = -8e^{-4x^2} + (-8xe^{-4x^2}(-8x))$$

$$y'' = -8e^{-4x^2} + 64x^2e^{-4x^2}$$

$$y'' = 8e^{-4x^2}(8x^2 - 1)$$

Topic: Higher-order derivatives

Question: Find the second derivative of the function.

$$y = (5x + 7)^8$$

Answer choices:

A $\quad y'' = 1{,}400(5x + 7)^6$

B $\quad y'' = 140(5x + 7)^6$

C $\quad y'' = 280(5x + 7)^6$

D $\quad y'' = 200(5x + 7)^6$

Solution: A

First we need to find the first derivative using the chain rule

$$y = (5x + 7)^8$$

$$y' = 8(5x + 7)^7(5)$$

$$y' = 40(5x + 7)^7$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (40(5x + 7)^7)'$$

$$y'' = 40 \cdot 7(5x + 7)^6(5)$$

$$y'' = 1{,}400(5x + 7)^6$$

Topic: Higher-order derivatives

Question: Find the third derivative of the function.

$$y = 4x^4 - 5x^3 + 2x$$

Answer choices:

A $y''' = 96$

B $y''' = 96x - 30$

C $y''' = 66x$

D $y''' = 48x - 30$

Solution: B

First we need to find the first derivative using the chain rule

$$y = 4x^4 - 5x^3 + 2x$$

$$y' = 16x^3 - 15x^2 + 2$$

Now we can find the second derivative by taking the derivative of the first derivative.

$$y'' = (y')' = (16x^3 - 15x^2 + 2)'$$

$$y'' = 48x^2 - 30x$$

Find the third derivative by taking the derivative of the second derivative.

$$y''' = (y'')' = (48x^2 - 30x)'$$

$$y''' = 96x - 30$$

Topic: Second derivatives with implicit differentiation

Question: Use implicit differentiation to find the second derivative.

$$x^3 + y^3 = 9$$

Answer choices:

A $\quad y'' = -\frac{2x(y^3 - x^3)}{y^6}$

B $\quad y'' = -\frac{2x(x^3 + y^3)}{y^5}$

C $\quad y'' = -\frac{2x(y^3 - x^3)}{y^5}$

D $\quad y'' = -\frac{2x(x^3 + y^3)}{y^6}$

Solution: B

The first derivative is

$$3x^2 + 3y^2y' = 0$$

Solve for y'.

$$3y^2y' = -3x^2$$

$$y' = \frac{-3x^2}{3y^2}$$

$$y' = \frac{-x^2}{y^2}$$

Use quotient rule to find the second derivative.

$$y'' = \frac{(-2x)(y^2) - (-x^2)(2yy')}{(y^2)^2}$$

$$y'' = \frac{-2xy^2 + 2x^2yy'}{y^4}$$

$$y'' = \frac{2xy(-y + xy')}{y^4}$$

$$y'' = \frac{2x(-y + xy')}{y^3}$$

Substitute for the first derivative.

$$y'' = \frac{2x\left[-y + x\left(\frac{-x^2}{y^2}\right)\right]}{y^3}$$

$$y'' = \frac{2x\left(-y - \frac{x^3}{y^2}\right)}{y^3}$$

Find a common denominator within the numerator, then combine the fractions in the numerator into one fraction.

$$y'' = \frac{2x\left(\frac{-y^3 - x^3}{y^2}\right)}{y^3}$$

$$y'' = 2x\left(\frac{-y^3 - x^3}{y^5}\right)$$

$$y'' = -\frac{2x(x^3 + y^3)}{y^5}$$

Topic: Second derivatives with implicit differentiation

Question: Use implicit differentiation to find the second derivative.

$$x^2 + 2xy + y^3 = 25$$

Answer choices:

A $\quad y'' = \dfrac{8x^2 - 24x^2y + 16xy - 48xy^2 - 18y^4}{(2x + 3y^2)^3}$

B $\quad y'' = -\dfrac{8x^2 - 24x^2y + 16xy - 48xy^2 - 18y^4}{(2x + 3y^2)^3}$

C $\quad y'' = -\dfrac{24x^2 + 18y^4}{(2x + 3y^2)^3}$

D $\quad y'' = \dfrac{24x^2 + 18y^4}{(2x + 3y^2)^3}$

Solution: A

The first derivative is

$$2x + [(2)(y) + (2x)(1)(y')] + 3y^2y' = 0$$

$$2x + 2y + 2xy' + 3y^2y' = 0$$

Solve for y'.

$$2xy' + 3y^2y' = -2x - 2y$$

$$y'(2x + 3y^2) = -2x - 2y$$

$$y' = \frac{-2x - 2y}{2x + 3y^2}$$

Use quotient rule to find the second derivative.

$$y'' = \frac{(-2 - 2(1)(y'))(2x + 3y^2) - (-2x - 2y)(2 + 6yy')}{(2x + 3y^2)^2}$$

$$y'' = \frac{(-2 - 2y')(2x + 3y^2) + (2x + 2y)(2 + 6yy')}{(2x + 3y^2)^2}$$

$$y'' = \frac{-4x - 6y^2 - 4xy' - 6y^2y' + 4x + 12xyy' + 4y + 12y^2y'}{(2x + 3y^2)^2}$$

Collect like terms, then factor out y'.

$$y'' = \frac{-4xy' + 12xyy' + 6y^2y' + 4y - 6y^2}{(2x + 3y^2)^2}$$

$$y'' = \frac{y'(-4x + 12xy + 6y^2) + 4y - 6y^2}{(2x + 3y^2)^2}$$

Substitute for the first derivative.

$$y'' = \frac{\frac{-2x - 2y}{2x + 3y^2}(-4x + 12xy + 6y^2) + 4y - 6y^2}{(2x + 3y^2)^2}$$

$$y'' = \frac{\frac{(-2x - 2y)(-4x + 12xy + 6y^2)}{2x + 3y^2} + \frac{(4y - 6y^2)(2x + 3y^2)}{2x + 3y^2}}{(2x + 3y^2)^2}$$

$$y'' = \frac{\frac{(-2x - 2y)(-4x + 12xy + 6y^2) + (4y - 6y^2)(2x + 3y^2)}{2x + 3y^2}}{(2x + 3y^2)^2}$$

$$y'' = \frac{(-2x - 2y)(-4x + 12xy + 6y^2) + (4y - 6y^2)(2x + 3y^2)}{(2x + 3y^2)^3}$$

Expand the numerator, then simplify.

$$y'' = \frac{(8x^2 - 24x^2y - 12xy^2 + 8xy - 24xy^2 - 12y^3) + (8xy + 12y^3 - 12xy^2 - 18y^4)}{(2x + 3y^2)^3}$$

$$y'' = \frac{8x^2 - 24x^2y - 12xy^2 + 8xy - 24xy^2 - 12y^3 + 8xy + 12y^3 - 12xy^2 - 18y^4}{(2x + 3y^2)^3}$$

$$y'' = \frac{8x^2 - 24x^2y + 16xy - 48xy^2 - 18y^4}{(2x + 3y^2)^3}$$

Topic: Second derivatives with implicit differentiation

Question: Use implicit differentiation to find the second derivative.

$$x^2y^2 + 3xy = 100$$

Answer choices:

A $\quad y'' = -\dfrac{y}{x^2}$

B $\quad y'' = \dfrac{y}{x^2}$

C $\quad y'' = -\dfrac{2y}{x^2}$

D $\quad y'' = \dfrac{2y}{x^2}$

Solution: D

The first derivative is

$$[(2x)(y^2) + (x^2)(2y)(y')] + [(3)(y) + (3x)(1)(y')] = 0$$

$$2xy^2 + 2x^2yy' + 3y + 3xy' = 0$$

Solve for y'.

$$2x^2yy' + 3xy' = -2xy^2 - 3y$$

$$y'(2x^2y + 3x) = -2xy^2 - 3y$$

$$y' = \frac{-2xy^2 - 3y}{2x^2y + 3x}$$

Factor and simplify the first derivative.

$$y' = \frac{-y(2xy + 3)}{x(2xy + 3)}$$

$$y' = -\frac{y}{x}$$

Use the quotient rule to find the second derivative.

$$y'' = -\frac{(y')(x) - (y)(1)}{x^2}$$

$$y'' = -\frac{xy' - y}{x^2}$$

Substitute for the first derivative.

$$y'' = -\frac{x\left(-\frac{y}{x}\right) - y}{x^2}$$

$$y'' = -\frac{-y - y}{x^2}$$

$$y'' = \frac{y + y}{x^2}$$

$$y'' = \frac{2y}{x^2}$$

Topic: Critical points and the first derivative test

Question: Find the critical point of the function.

$$f(x) = x^2 - 10x + 2$$

Answer choices:

A $x = \frac{1}{5}$

B $x = 5$

C $x = -5$

D $x = -\frac{1}{5}$

Solution: B

Take the derivative of the function.

$$f(x) = x^2 - 10x + 2$$

$$f'(x) = 2x - 10$$

This derivative exists everywhere. Set the derivative equal to 0 and solve for x.

$$2x - 10 = 0$$

$$2x = 10$$

$$x = 5$$

The function has one potential critical point at $x = 5$.

Topic: Critical points and the first derivative test

Question: Where is the function increasing and decreasing?

$$f(x) = x^2$$

Answer choices:

A Increasing on $x < 1$ and decreasing on $x > 1$

B Increasing on $x < 0$ and decreasing on $x > 0$

C Increasing on $x > 0$ and decreasing on $x < 0$

D Increasing on $x > 1$ and decreasing on $x < 1$

Solution: C

Find the derivative.

$$f(x) = x^2$$

$$f'(x) = 2x$$

This derivative exists everywhere. Set the derivative equal to 0 and solve for x.

$$0 = 2x$$

$$x = 0$$

Investigate the critical point $x = 0$ by testing $x = -1$ and $x = 1$ in the first derivative.

$$f'(-1) = 2(-1)$$

$$f'(-1) = -2$$

and

$$f'(1) = 2(1)$$

$$f'(1) = 2$$

On the left side of $x = 0$ the derivative is negative so the function is decreasing. On the right side of $x = 0$ the derivative is positive so the function is increasing.

Topic: Critical points and the first derivative test

Question: Where is the function increasing and decreasing?

$$f(x) = x^4 - 4x^3 + 4x^2 - 7$$

Answer choices:

A Decreasing on $x < 0$ and $1/2 < x < 3/2$, increasing on $0 < x < 1/2$ and $x > 3/2$

B Decreasing on $0 < x < 1/2$ and $x > 3/2$, increasing on $x < 0$ and $1/2 < x < 3/2$

C Decreasing on $x < 0$ and $1 < x < 2$, increasing on $0 < x < 1$ and $x > 2$

D Decreasing on $0 < x < 1$ and $x > 2$, increasing on $x < 0$ and $1 < x < 2$

Solution: C

Take the first derivative of the function.

$$f(x) = x^4 - 4x^3 + 4x^2 - 7$$

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$f'(x) = 4x(x^2 - 3x + 2)$$

$$f'(x) = 4x(x - 2)(x - 1)$$

This derivative exists everywhere. Set the derivative equal to 0 and solve for x.

$$4x(x - 2)(x - 1) = 0$$

$$x = 0,\ 1,\ 2$$

Investigate each interval by evaluating the first derivative at $x = -1$, $x = 1/2$, $x = 3/2$, and $x = 3$.

$$f'(-1) = 4(-1)^3 - 12(-1)^2 + 8(-1)$$

$$f'(-1) = -4 - 12 - 8$$

$$f'(-1) = -24$$

and

$$f'\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^3 - 12\left(\frac{1}{2}\right)^2 + 8\left(\frac{1}{2}\right)$$

$$f'\left(\frac{1}{2}\right) = 4\left(\frac{1}{8}\right) - 12\left(\frac{1}{4}\right) + 4$$

$$f'\left(\frac{1}{2}\right) = \frac{1}{2} - 3 + 4$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{2}$$

and

$$f'\left(\frac{3}{2}\right) = 4\left(\frac{3}{2}\right)^3 - 12\left(\frac{3}{2}\right)^2 + 8\left(\frac{3}{2}\right)$$

$$f'\left(\frac{3}{2}\right) = 4\left(\frac{27}{8}\right) - 12\left(\frac{9}{4}\right) + 12$$

$$f'\left(\frac{3}{2}\right) = \frac{27}{2} - 27 + 12$$

$$f'\left(\frac{3}{2}\right) = -\frac{3}{2}$$

and

$$f'(3) = 4(3)^3 - 12(3)^2 + 8(3)$$

$$f'(3) = 4(27) - 12(9) + 24$$

$$f'(3) = 108 - 108 + 24$$

$$f'(3) = 24$$

To the left of $x = 0$ the derivative is negative so the function is decreasing. Between $x = 0$ and $x = 1$, the derivative is positive so the function is increasing. Between $x = 1$ and $x = 2$, the derivative is negative so the function is decreasing. To the right of $x = 2$ the derivative is positive so the function is increasing.

The function $f(x) = x^4 - 4x^3 + 4x^2 - 7$ is decreasing when $x < 0$, increasing between $x = 0$ and $x = 1$, decreasing between $x = 1$ and $x = 2$, and increasing when $x > 2$.

Topic: Inflection points and the second derivative test

Question: Find the function's inflection points.

$$f(x) = x^4 - 6x^2 - 3x + 2$$

Answer choices:

A $(1,0)$ and $(-1,-6)$

B $(-1,0)$ and $(1,6)$

C $(-1,0)$ and $(1,-6)$

D $(-1,-6)$ and $(0,1)$

Solution: C

Find the second derivative of the function.

$$f(x) = x^4 - 6x^2 - 3x + 2$$

$$f'(x) = 4x^3 - 12x - 3$$

$$f''(x) = 12x^2 - 12$$

Set the second derivative equal to 0 and solve for x.

$$12x^2 - 12 = 0$$

$$12x^2 = 12$$

$$x^2 = 1$$

$$x = \pm 1$$

There are two possible inflection points at $x = -1$ and $x = 1$. Investigate $x = -1$ by testing $x = -2$ and $x = 0$ in the second derivative.

$$f''(-2) = 12(-2)^2 - 12$$

$$f''(-2) = 36$$

and

$$f''(0) = 12(0)^2 - 12$$

$$f''(0) = -12$$

Since $f''(-2) = 36 > 0$, the function is concave up to the left of $x = -1$, and since $f''(0) = -12 < 0$, the function is concave down to the right of $x = -1$.

Because the function changes concavity at $x = -1$ and $f''(x)$ is continuous, there's an inflection point there. We'll get the y-coordinate of the inflection point by substituting $x = -1$ into $f(x)$.

$$f(-1) = (-1)^4 - 6(-1)^2 - 3(-1) + 2$$

$$f(-1) = 0$$

The function has an inflection point at $(-1,0)$.

Investigate $x = 1$ by testing $x = 0$ and $x = 2$ into the second derivative.

$$f''(0) = 12(0)^2 - 12$$

$$f''(0) = -12$$

and

$$f''(2) = 12(2)^2 - 12$$

$$f''(2) = 36$$

Since $f''(0) = -12 < 0$, the function is concave down to the left of $x = 1$, and since $f''(2) = 36 > 0$, the function is concave up to the right of $x = 1$.

Because the function changes concavity at $x = 1$ and $f''(x)$ is continuous, there's an inflection point there. We'll get the y-coordinate of the inflection point by substituting $x = 1$ into $f(x)$.

$$f(1) = (1)^4 - 6(1)^2 - 3(1) + 2$$

$$f(1) = -6$$

The function has a second inflection point at $(1, -6)$.

Topic: Inflection points and the second derivative test

Question: Use the Second Derivative Test to classify the critical points at $x = 0$ and $x = 2$.

$$f''(x) = -6x + 6$$

Answer choices:

A Relative minimum at $x = 0$; Relative maximum at $x = 2$

B Relative minimum at $x = 2$; Relative maximum at $x = 0$

C Relative minima at $x = 0$ and $x = 2$

D Relative maxima at $x = 0$ and $x = 2$

Solution: A

The second derivative is positive at $x = 0$,

$$f''(0) = -6(0) + 6 = 6 > 0$$

so the function is concave up at that critical point, which means there's a relative minimum there.

The second derivative is negative at $x = 2$,

$$f''(2) = -6(2) + 6 = -6 < 0$$

so the function is concave down at that critical point, which means there's a relative maximum there.

Topic: Inflection points and the second derivative test

Question: Use the second derivative test to find the function's extrema?

$$f(x) = x^2 + x + 4$$

Answer choices:

A The function has a local minimum at $x = -1/2$.

B The function has a local maximum at $x = -1/2$.

C The function has a local minimum at $x = 1/2$.

D The function has a local maximum at $x = 1/2$.

Solution: A

Take the first derivative.

$$f(x) = x^2 + x + 4$$

$$f'(x) = 2x + 1$$

Set the derivative equal to 0 and solve for x.

$$2x + 1 = 0$$

$$2x = -1$$

$$x = -\frac{1}{2}$$

There's one critical point at $x = -1/2$. Take the second derivative.

$$f'(x) = 2x + 1$$

$$f''(x) = 2$$

Substitute the critical point $x = -1/2$ into the second derivative.

$$f''\left(-\frac{1}{2}\right) = 2$$

Because the second derivative is positive at the critical point, it means there's a local minimum at $x = -1/2$.

Topic: Intercepts and vertical asymptotes

Question: Find the function's vertical asymptote.

$$f(x) = \frac{1}{x^2}$$

Answer choices:

A The function has a vertical asymptote at $x = 1$

B The function has a vertical asymptote at $x = 0$

C The function has a vertical asymptote at $x = \infty$

D The function has a vertical asymptote at $x = -1$

Solution: B

Set the function's denominator equal to 0.

$$x^2 = 0$$

$$x = 0$$

This is the value that makes the denominator 0, so the function has a vertical asymptote at $x = 0$.

Topic: Intercepts and vertical asymptotes

Question: Find the function's vertical asymptotes.

$$f(x) = \frac{x-2}{x^2-3}$$

Answer choices:

A The function has vertical asymptotes at $x = -2$ and $x = 2$

B The function has vertical asymptotes at $x = -3$ and $x = 3$

C The function has vertical asymptotes at $x = -\sqrt{3}$ and $x = \sqrt{3}$

D The function has vertical asymptotes at $x = -\sqrt{2}$ and $x = \sqrt{2}$

Solution: C

Set the function's denominator equal to 0.

$$x^2 - 3 = 0$$

$$x^2 = 3$$

$$x = \pm\sqrt{3}$$

These are the values that make the denominator 0, so the function has vertical asymptotes at $x = \pm\sqrt{3}$.

Topic: Intercepts and vertical asymptotes

Question: Find the function's vertical asymptotes.

$$f(x) = \frac{2x}{x^2 - 4x + 3}$$

Answer choices:

A The function has vertical asymptotes at $x = 1$ and $x = -3$

B The function has vertical asymptotes at $x = -1$ and $x = -3$

C The function has vertical asymptotes at $x = -1$ and $x = 3$

D The function has vertical asymptotes at $x = 1$ and $x = 3$

Solution: D

Set the function's denominator equal to 0.

$$x^2 - 4x + 3 = 0$$

$$(x - 1)(x - 3) = 0$$

$x = 3$ or $x = 1$

These are the values that make the denominator 0, so the function has vertical asymptotes at $x = 1$ and $x = 3$.

Topic: Horizontal and slant asymptotes

Question: Find the function's horizontal asymptote(s).

$$f(x) = \frac{4x^3 - 2x^2 + 1}{2x^3 - 3x}$$

Answer choices:

A $y = 0$

B $y = -3$

C $y = 2$

D $y = \pm 2$

Solution: C

The degree of the numerator is 3 and the degree of the denominator is 3, so the degree of the numerator is equal to the degree of the denominator

Therefore, the equation of the horizontal asymptote is given as the ratio of the coefficients on the highest-degree terms, and the function has the horizontal asymptote at $y = 2$.

$$y = \frac{4}{2} = 2$$

Topic: Horizontal and slant asymptotes

Question: Find the function's horizontal asymptote(s).

$$y = \frac{x^5 - x + 6}{x^7 - x^4 + 3x^2 - 1}$$

Answer choices:

A The function has a horizontal asymptote at $y = 1$

B The function has a horizontal asymptote at $y = 5/7$

C The function has a horizontal asymptote at $y = 0$

D The function has no horizontal asymptote

Solution: C

The x^5 term is the highest-degree term in the numerator, and the x^7 term is the highest-degree term in the denominator.

Because the degree of the numerator is less than the degree of the denominator, the function has a horizontal asymptote at $y = 0$.

Topic: Horizontal and slant asymptotes

Question: Find the function's slant asymptote(s).

$$f(x) = \frac{x^2 - x + 3}{x + 1}$$

Answer choices:

A The function has a slant asymptote at $y = x + 2 + \dfrac{5}{x+1}$

B The function has a slant asymptote at $y = x - 2 + \dfrac{5}{x+1}$

C The function has a slant asymptote at $y = x - 2$

D The function has a slant asymptote at $y = x + 2$

Solution: C

The degree of the numerator is exactly one greater than the degree of the denominator, $2 > 1$, so the function has a slant asymptote.

We want to do polynomial long division with the function, which we set up as

$$x+1 \enclose{longdiv}{x^2-x+3}$$

If we work through this division, we end up with

$$f(x) = x - 2 + \frac{5}{x+1}$$

The slant asymptote is what we get when we remove the remainder from this rewritten function. If we remove the remainder, we get

$$f(x) = x - 2$$

So the equation of the slant asymptote is

$$y = x - 2$$

Topic: Sketching graphs

Question: If the first derivative of the function is positive, then the function is...

Answer choices:

A ... concave down

B ... concave up

C ... decreasing

D ... increasing

Solution: D

Where the first derivative of a function is positive, the function itself is increasing.

Topic: Sketching graphs

Question: If $g'(-2) = 0$ and $g''(-2) > 0$, which of the following must be true?

Answer choices:

A The function has an inflection point at $x = -2$.

B The function has a local minimum at $x = -2$.

C The function has a local maximum at $x = -2$.

D The function has an x-intercept at $x = -2$.

Solution: B

If $g'(-2) = 0$, then $x = -2$ is critical point of the function. If $g''(x) > 0$ at a critical point, there's a local minimum there.

Topic: Sketching graphs

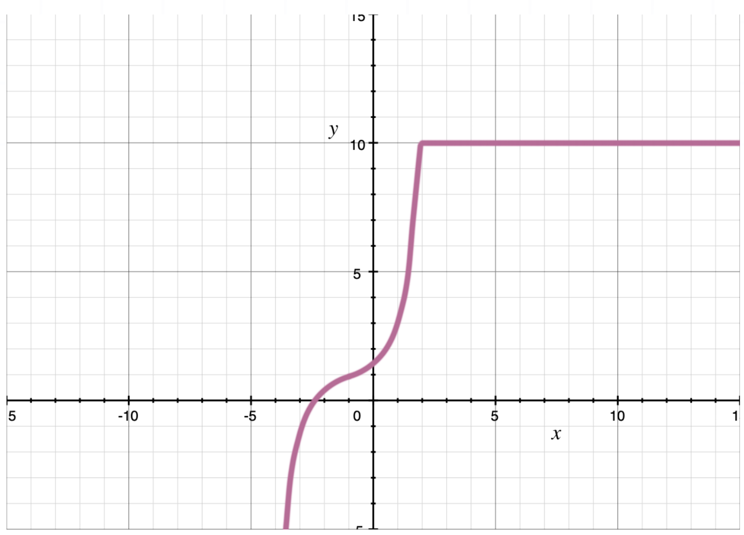
Question: Which of the following is a graph of the function with the given properties?

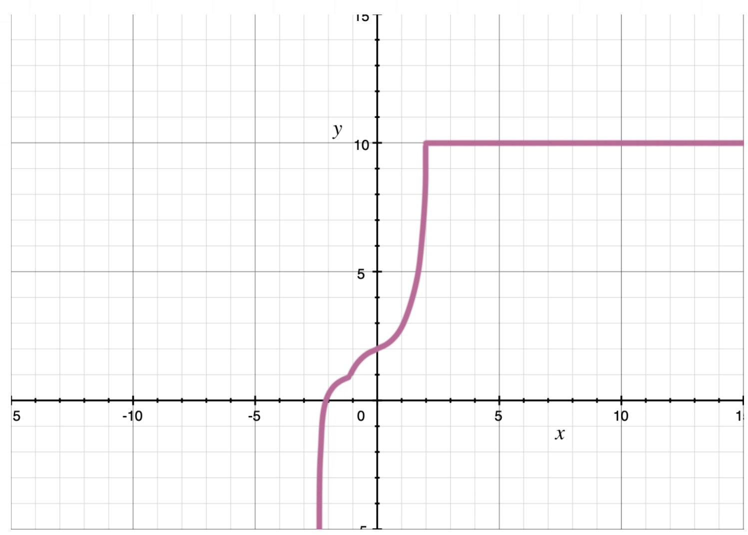
$$f'(x) < 0 \text{ and } f''(x) > 0 \text{ for } x \le -1$$

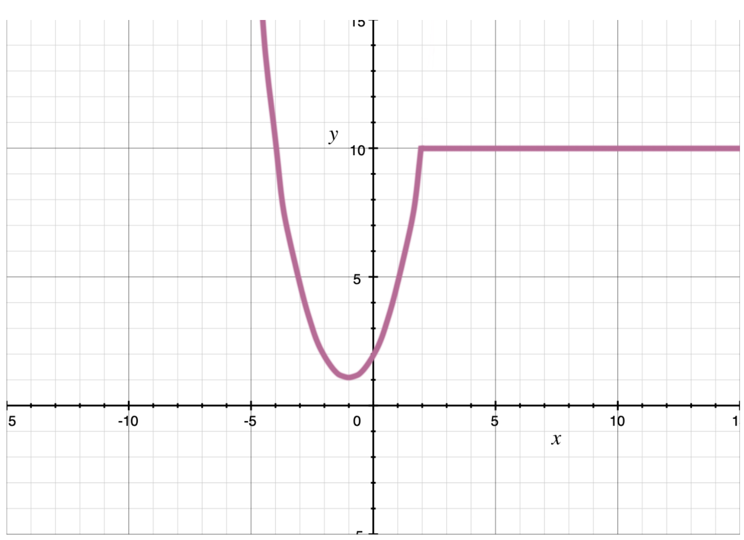
$$f'(x) > 0 \text{ and } f''(x) > 0 \text{ for } -1 < x < 2$$

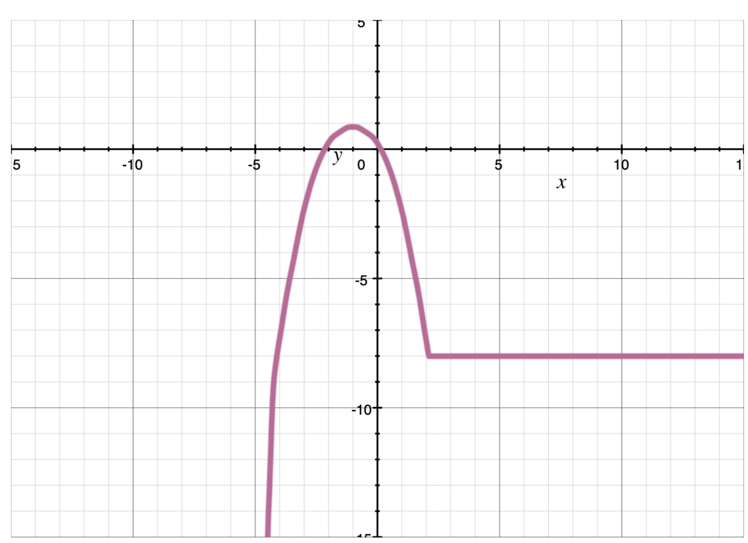
$$f(x) = 10 \text{ for } x \ge 2$$

Answer choices:

A 

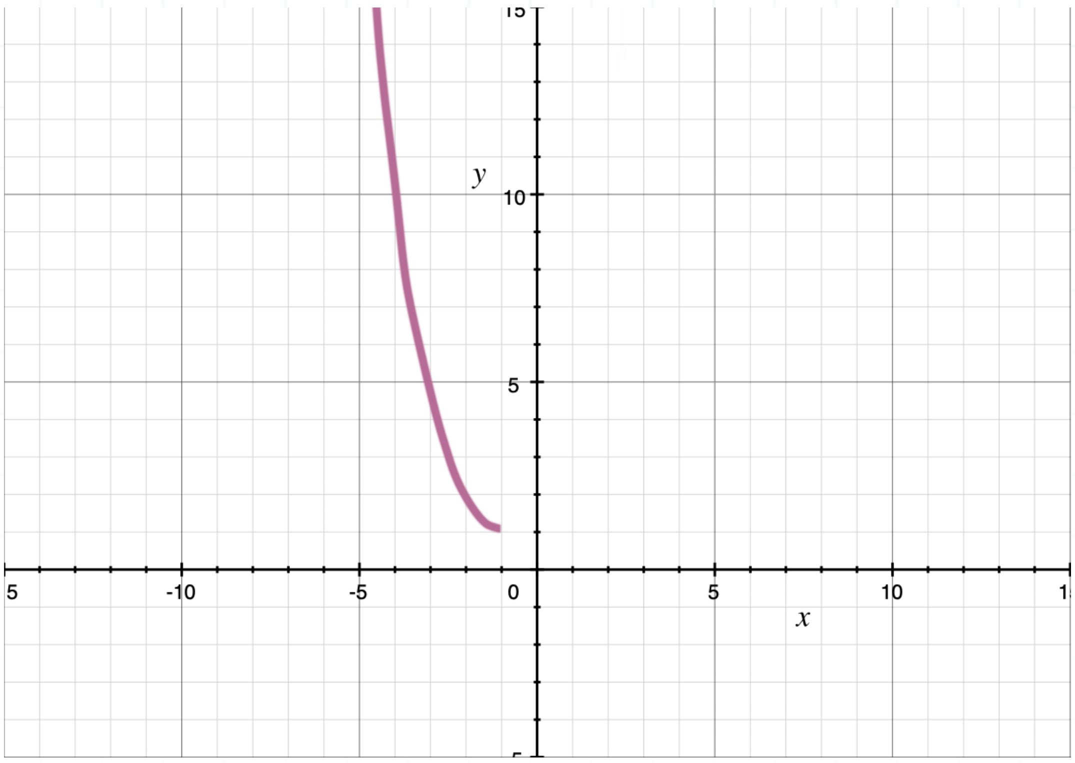
B 

C 

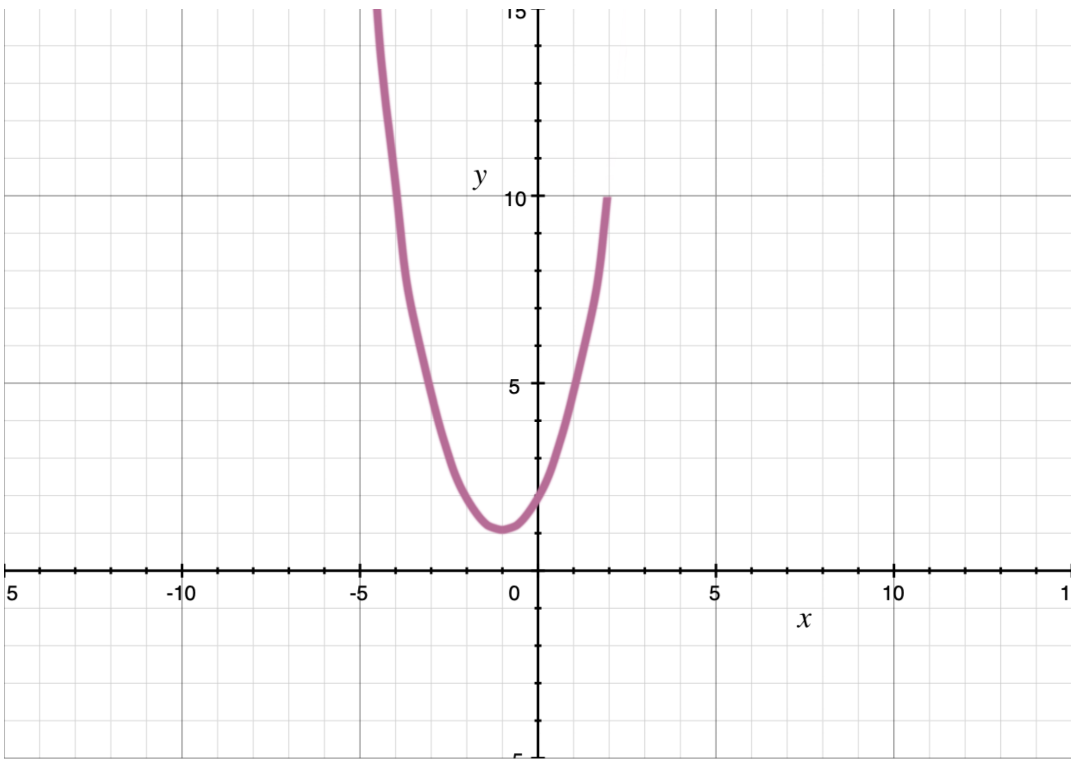
D 

Solution: C

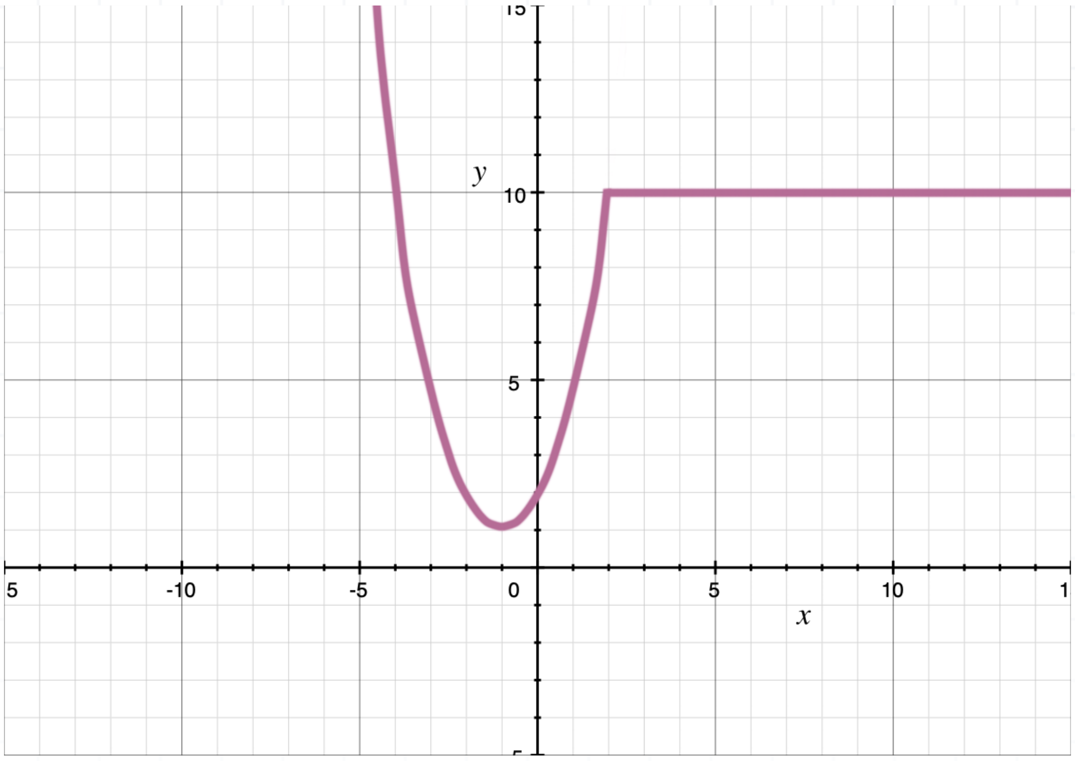
For $x \leq -1$, the function is decreasing since $f'(x) < 0$, and concave up since $f''(x) > 0$.



For $-1 < x < 2$, the function is increasing since $f'(x) > 0$, and concave up since $f''(x) > 0$.



For $x \geq 2$, the function's value is $f(x) = 10$.



Topic: Extrema on a closed interval

Question: Find the absolute extrema of the function on the interval $[1,2]$.

$$f(x) = -\frac{1}{x^2}$$

Answer choices:

A	Minimum at $(1,1)$	Maximum at $(2,4)$
B	Minimum at $(1,-1)$	Maximum at $\left(2,-\frac{1}{4}\right)$
C	Minimum at $\left(2,\frac{1}{4}\right)$	Maximum at $(1,1)$
D	No Minimum	No Maximum

Solution: B

Find the first derivative, then set it equal to 0 and solve for x in order to find critical points.

$$f'(x) = \frac{2}{x^3}$$

$$0 = \frac{2}{x^3}$$

The function has no critical points, so we only need to check the function's value at the endpoints of the interval.

At $x = 1$,

$$f(1) = -\frac{1}{1^2}$$

$$f(1) = -1$$

At $x = 2$,

$$f(2) = -\frac{1}{2^2}$$

$$f(2) = -\frac{1}{4}$$

If we order these points from least to greatest in terms of the function's value, we get

$$(1, -1)$$

$$\left(2, -\frac{1}{4}\right)$$

So on the interval $[1,2]$, the function has an absolute minimum at $(1, -1)$ and an absolute maximum at $(2, -1/4)$.

Topic: Extrema on a closed interval

Question: Find the absolute extrema of the function on the interval $[0,3]$.

$$f(x) = x^2 - 4x$$

Answer choices:

A Global minimum at $(3,-3)$; Global maximum at $(2,-4)$

B Global maximum at $(2,-4)$; Global maximum at $(3,-3)$

C Global minimum at $(0,0)$; Global maximum at $(2,-4)$

D Global minimum at $(2,-4)$; Global maximum at $(0,0)$

Solution: D

Find the first derivative,

$$f'(x) = 2x - 4$$

$$f'(x) = 2(x - 2)$$

then set it equal to 0 and solve for x.

$$2(x - 2) = 0$$

$$x - 2 = 0$$

$$x = 2$$

Absolute extrema could occur at this critical point and/or at the endpoints of the interval. So we'll find the value of $f(x)$ at each of these points.

At $x = 0$,

$$f(0) = 0^2 - 4(0)$$

$$f(0) = 0$$

At $x = 2$,

$$f(2) = 2^2 - 4(2)$$

$$f(2) = 4 - 8$$

$$f(2) = -4$$

At $x = 3$,

$$f(3) = 3^2 - 4(3)$$

$$f(3) = 9 - 12$$

$$f(3) = -3$$

If we rank these points from least to greatest in terms of the function's value, we get

$$(2, -4)$$

$$(3, -3)$$

$$(0,0)$$

So on the interval $[0,3]$, the function has an absolute minimum at $(2, -4)$ and an absolute maximum at $(0,0)$.

Topic: Extrema on a closed interval

Question: Find the absolute extrema of the function on the interval $[0,2]$.

$$f(x) = x^3 - 3x$$

Answer choices:

A Global minimum at $(1, -2)$; Global maximum at $(2,2)$

B Global minimum at $(2,2)$; Global maximum at $(1, -2)$

C Global minimum at $(-1,2)$; Global maximum at $(2,2)$

D Global minimum at $(1, -2)$; Global maxima at $(-1,2)$ and $(2,2)$

Solution: A

Find the first derivative,

$$f'(x) = 3x^2 - 3$$

$$f'(x) = 3(x^2 - 1)$$

$$f'(x) = 3(x + 1)(x - 1)$$

then set it equal to 0 and solve for x.

$$3(x + 1)(x - 1) = 0$$

$$x = -1, 1$$

The critical point $x = -1$ is outside the interval $[0,2]$, so we'll ignore it. Then we can say that absolute extrema could occur at just $x = 1$ and/or at the endpoints of the interval. So we'll find the value of $f(x)$ at each of these points.

At $x = 0$,

$$f(0) = 0^3 - 3(0)$$

$$f(0) = 0 - 0$$

$$f(0) = 0$$

At $x = 1$,

$$f(1) = 1^3 - 3(1)$$

$$f(1) = 1 - 3$$

$$f(1) = -2$$

At $x = 2$,

$$f(2) = 2^3 - 3(2)$$

$$f(2) = 8 - 6$$

$$f(2) = 2$$

If we rank these points from least to greatest in terms of the function's value, we get

$$(1, -2)$$

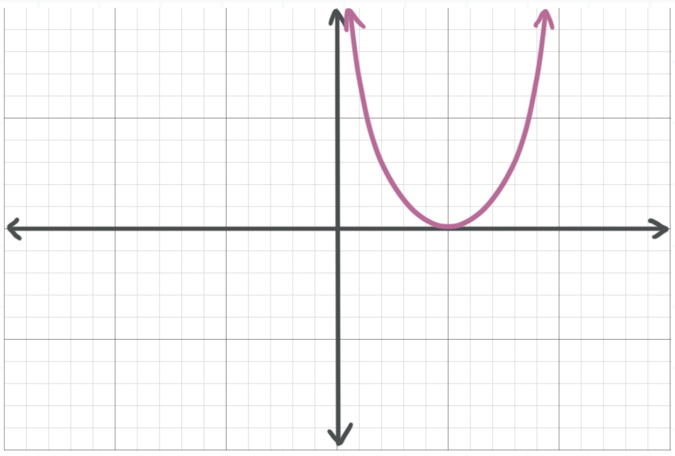
$$(0,0)$$

$$(2,2)$$

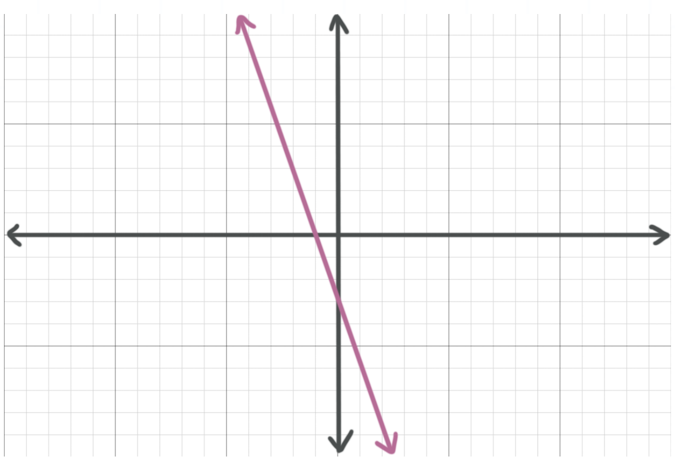
So on the interval $[0,2]$, the function has an absolute minimum at $(1, -2)$ and an absolute maximum at $(2,2)$.

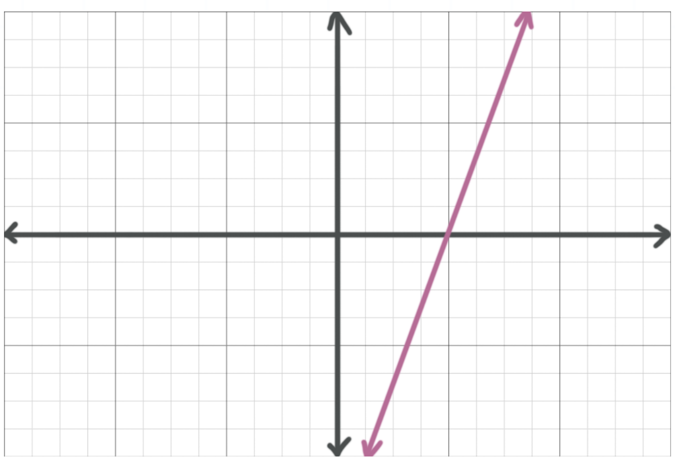
Topic: Sketching f(x) from f'(x)

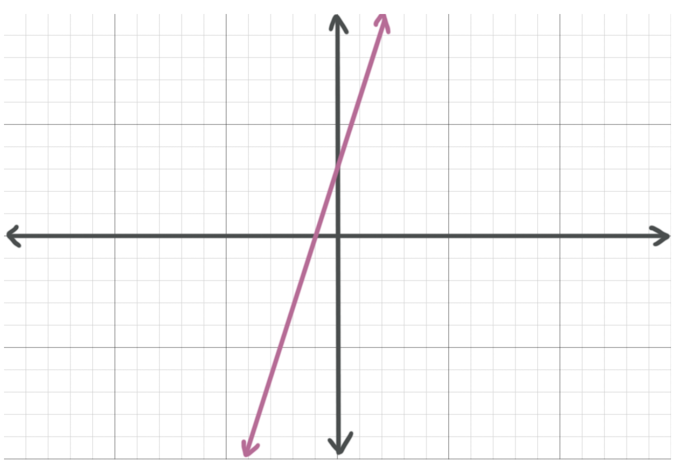
Question: Given the graph of $f(x)$, which is a possible graph of $f'(x)$?

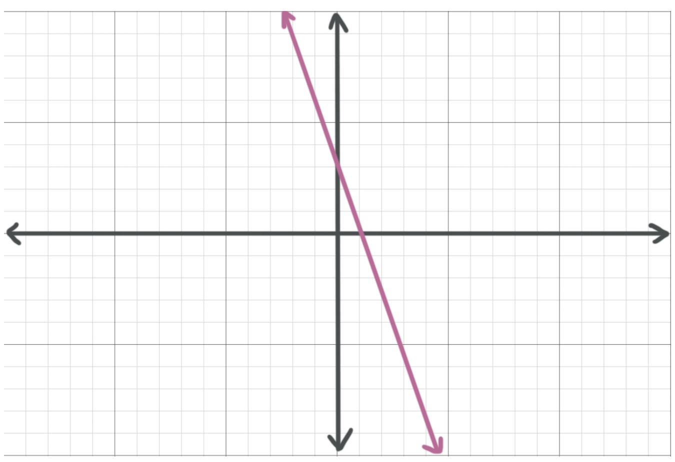


Answer choices:

A 

B 

C 

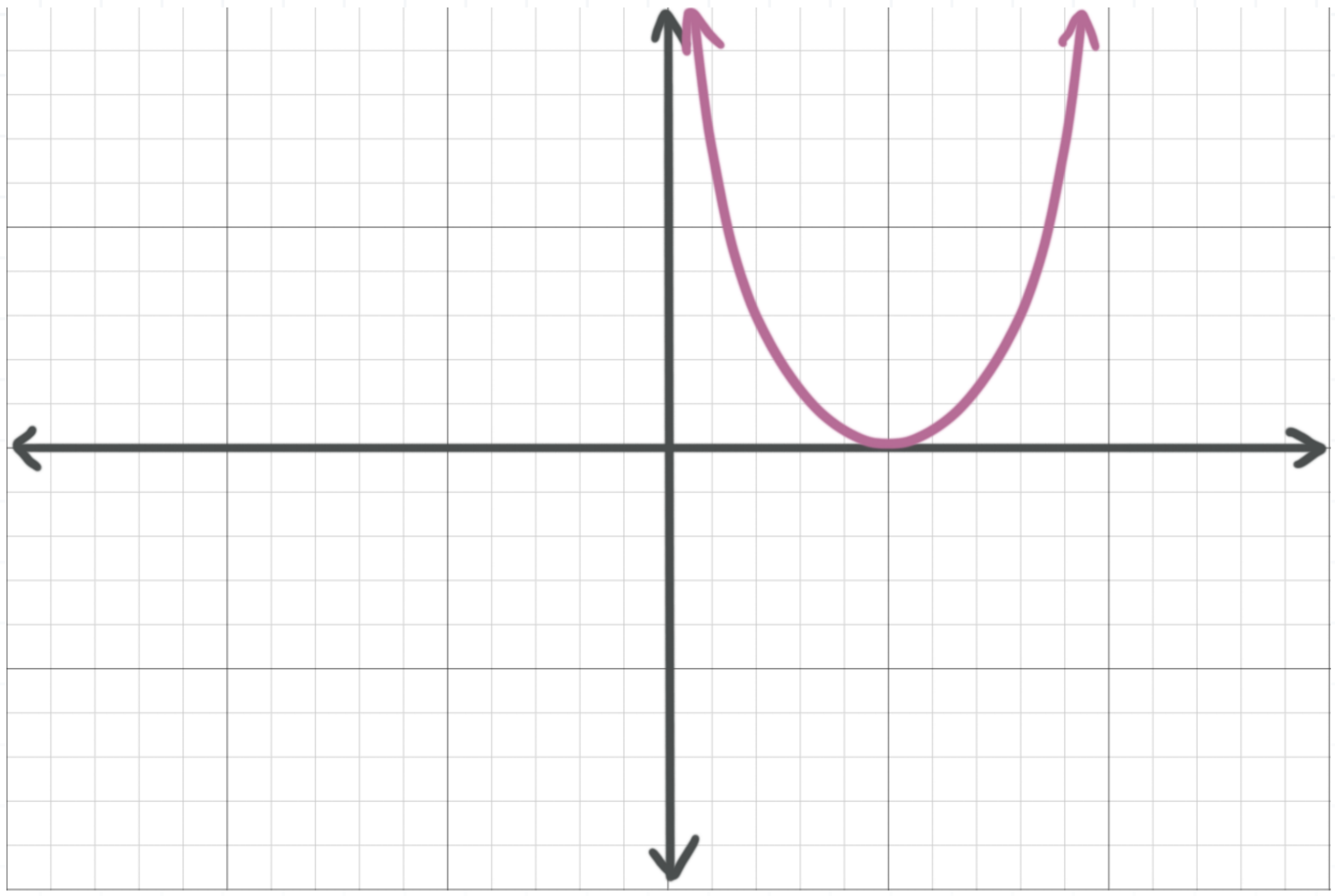
D 

Solution: B

To take a sketch of $f(x)$ and translate it into a sketch of a possible $f'(x)$, we'll use the chart that compares the graphs of those two functions.

$f(x)$	$f'(x)$
Critical point	0 **(x-intercept)**
Increasing	**Positive (above the x-axis)**
Decreasing	**Negative (below the x-axis)**
Inflection point	**Critical point**
Concave up	**Increasing**
Concave down	**Decreasing**

If we consider the graph of $f(x)$ we've been given,



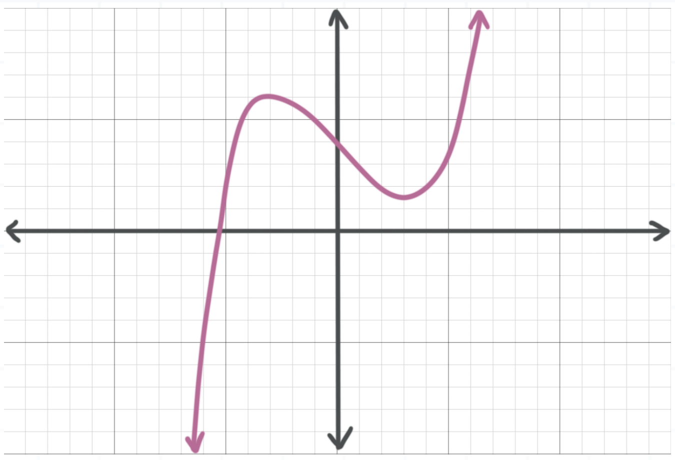
we can see right away that it's concave up everywhere, which means the graph of $f'(x)$ will be increasing everywhere, which eliminates answer choices A and D.

We can also see that the graph of $f(x)$ is decreasing to the left of what looks like $x = 5$, and then increasing to the right of that point. If that's the case, then the graph of $f'(x)$ should be negative (below the x-axis) to the left of $x = 5$ and positive (above the x-axis) to the right of $x = 5$.

That matches answer choice B, and not answer choice C.

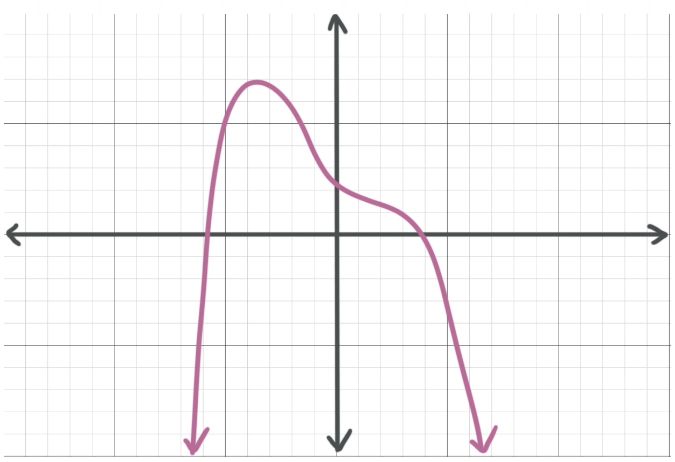
Topic: Sketching f(x) from f'(x)

Question: Given the graph of $f'(x)$, which is a possible graph of $f(x)$?

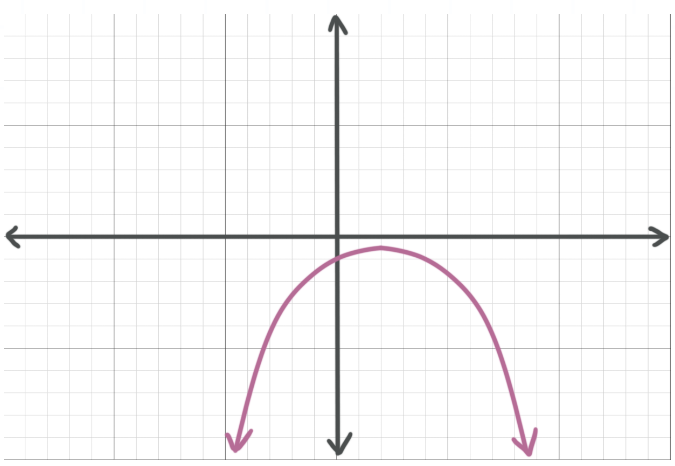


Answer choices:

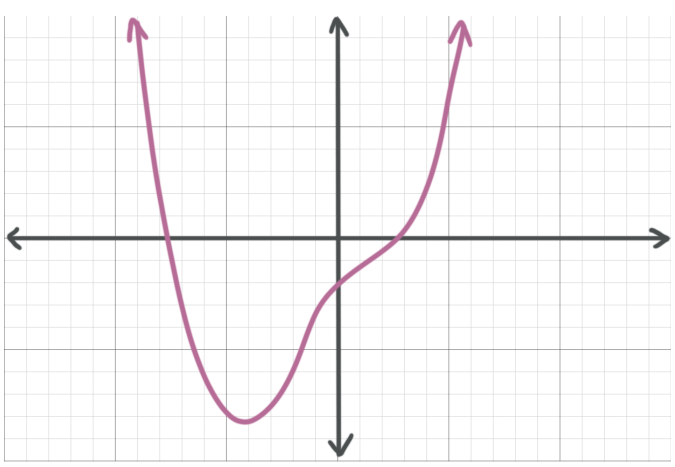
A



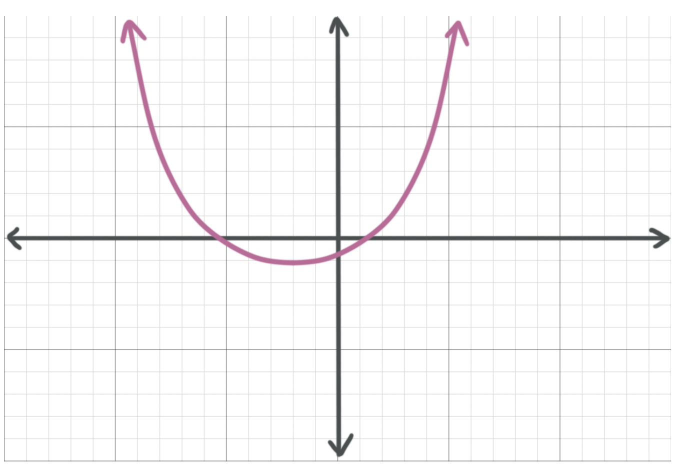
B



C



D

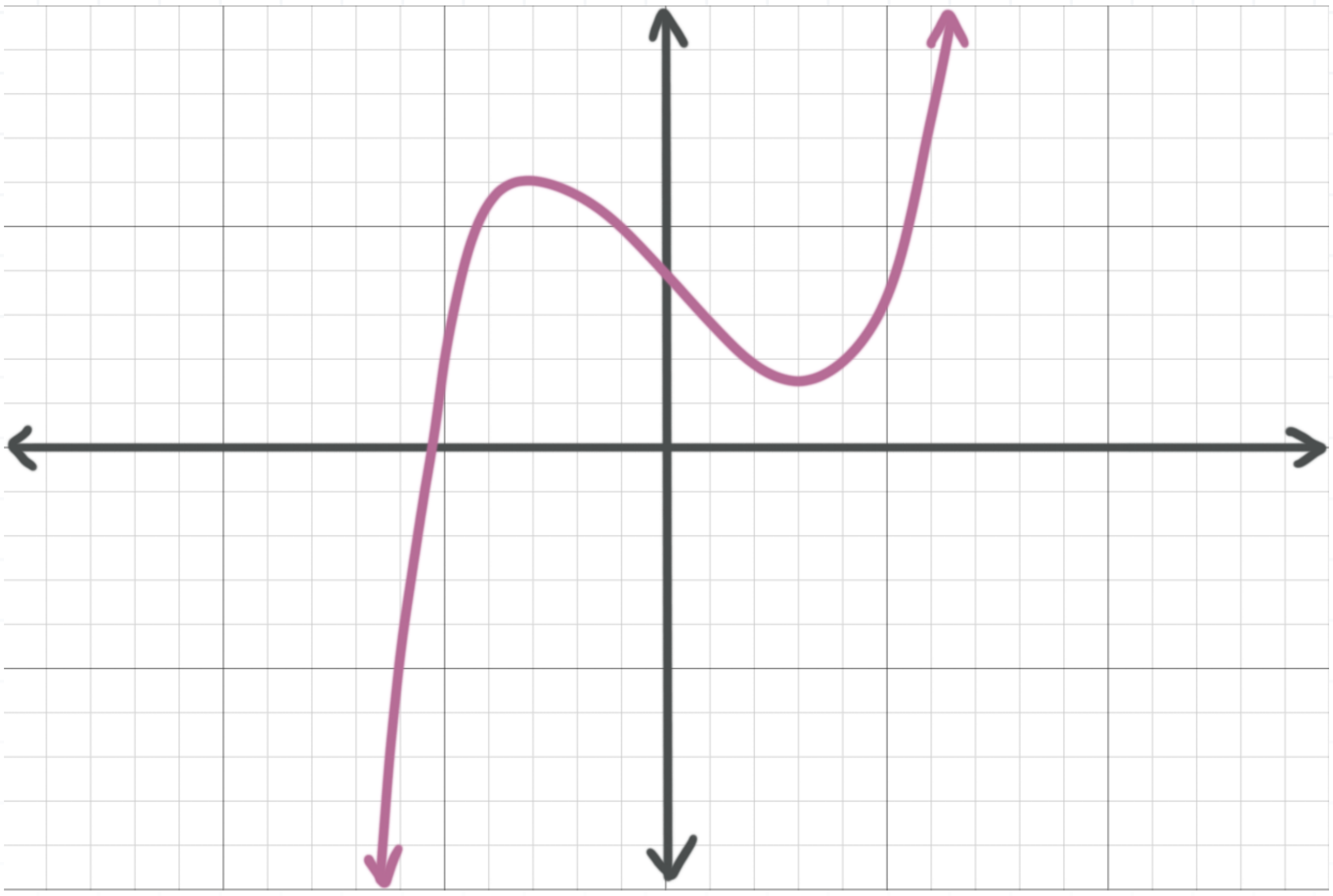


Solution: C

To take a sketch of $f'(x)$ and translate it into a sketch of a possible $f(x)$, we'll use the chart that compares the graphs of those two functions.

$f(x)$	$f'(x)$
Critical point	0 (x-intercept)
Increasing	Positive (above the x-axis)
Decreasing	Negative (below the x-axis)
Inflection point	Critical point
Concave up	Increasing
Concave down	Decreasing

If we consider the graph of $f'(x)$ we've been given,

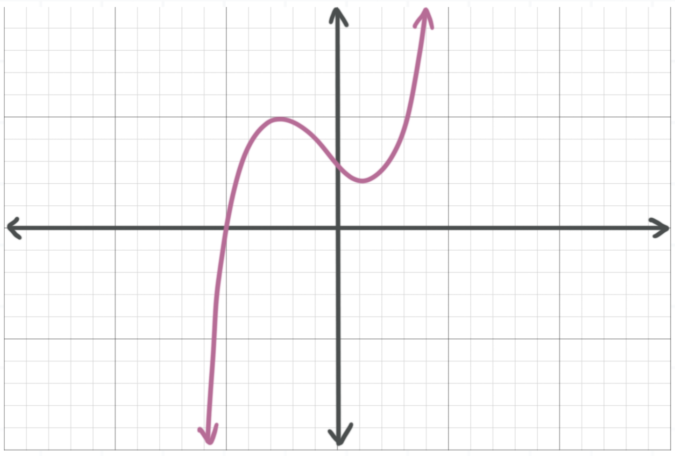


we can see right away that, as we move from left to right, it's increasing, then decreasing, then increasing again. Which means that the graph of $f(x)$, as we move from left to right, must be concave up, then concave down, then concave up again.

The only graph matching that concavity pattern is the graph in answer choice C.

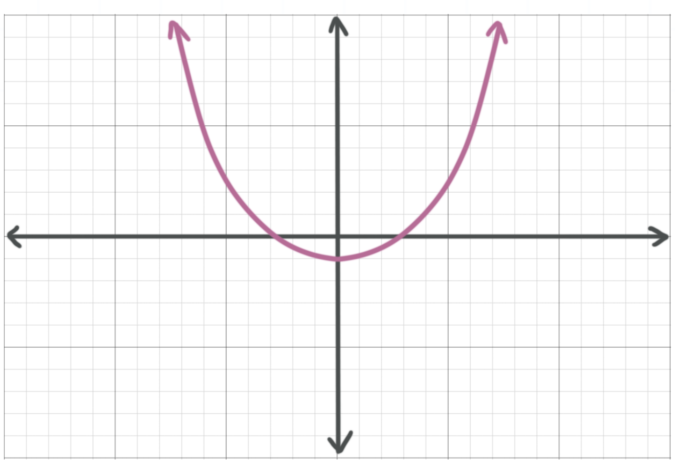
Topic: Sketching f(x) from f'(x)

Question: Given the graph of $f(x)$, which is a possible graph of $f'(x)$?

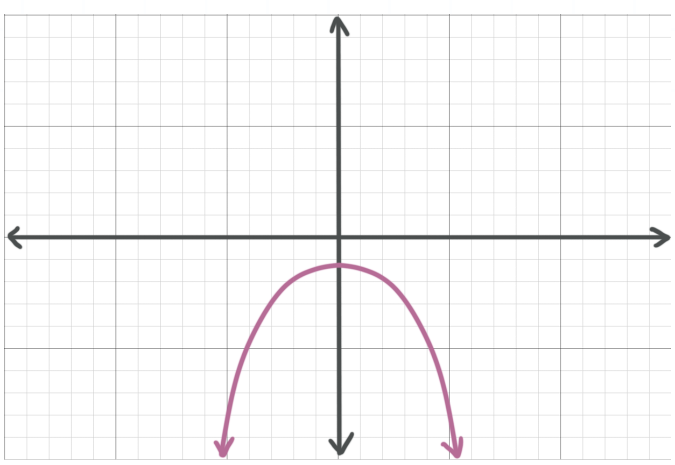


Answer choices:

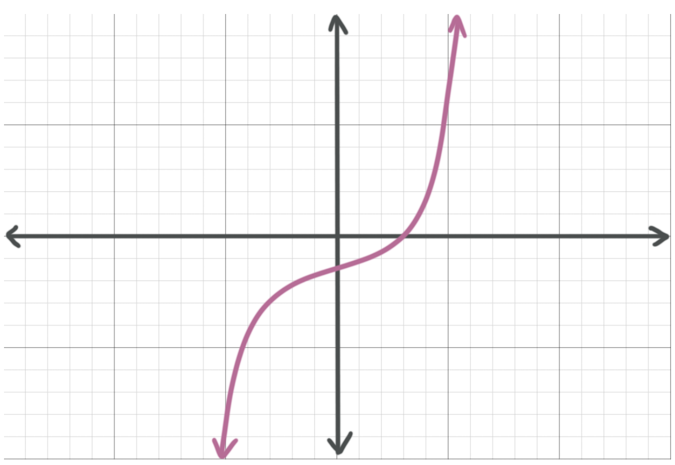
A



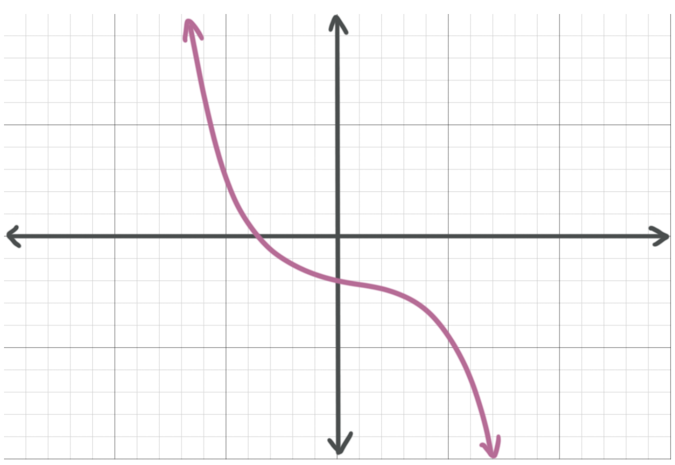
B



C



D

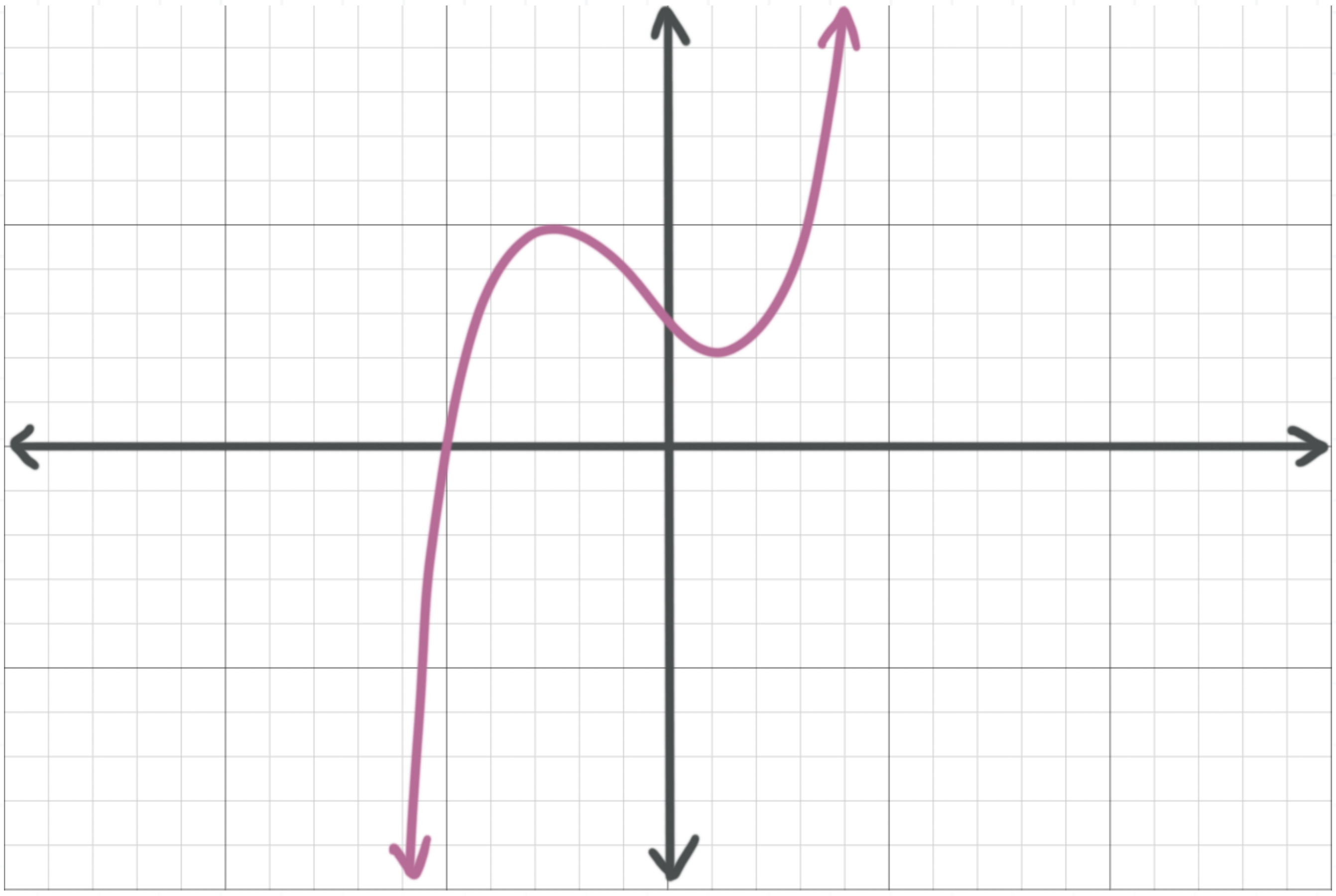


Solution: A

To take a sketch of $f(x)$ and translate it into a sketch of a possible $f'(x)$, we'll use the chart that compares the graphs of those two functions.

$f(x)$	$f'(x)$
Critical point	0 **(x-intercept)**
Increasing	**Positive (above the x-axis)**
Decreasing	**Negative (below the x-axis)**
Inflection point	**Critical point**
Concave up	**Increasing**
Concave down	**Decreasing**

If we consider the graph of $f(x)$ we've been given,



we can see right away that, as we move from left to right, it’s concave down and then concave up, which means that, as we move from left to right, the graph of $f'(x)$ will be decreasing and then increasing.

The only graph matching that direction pattern is the graph in answer choice A.

Topic: Linear approximation

Question: Find the linear approximation of the function at $a = 0$.

$$f(x) = \cos x$$

Answer choices:

A $L(x) = 1$

B $L(x) = x$

C $L(x) = x - 1$

D $L(x) = -x$

Solution: A

Take the derivative.

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

Evaluate the original function at $a = 0$.

$$f(0) = \cos 0$$

$$f(0) = 1$$

Evaluate the derivative at $a = 0$.

$$f'(0) = -\sin 0$$

$$f'(0) = 0$$

Substitute all of these pieces into the linear approximation formula.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + 0(x - 0)$$

$$L(x) = 1$$

Topic: Linear approximation

Question: Use linear approximation to estimate $e^{-0.1}$.

Answer choices:

A 0.1

B 0

C 0.9

D 1.1

Solution: C

Given the value of the function we're asked to estimate, it's clear that the function should be e^x. Instead of trying to find $f(-0.1)$, let's use a linear approximation equation and $a = 0$ to get an approximation for $f(-0.1)$.

Take the derivative.

$$f(x) = e^x$$

$$f'(x) = e^x$$

Evaluate the original function at $a = 0$.

$$f(0) = e^0$$

$$f(0) = 1$$

Evaluate the derivative at $a = 0$.

$$f'(0) = e^0$$

$$f'(0) = 1$$

Substitute all of these pieces into the linear approximation formula.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 1 + 1(x - 0)$$

$$L(x) = 1 + x$$

Now that we've built the linear approximation equation, we can substitute $x = -0.1$.

$$L(-0.1) = 1 - 0.1$$

$$L(-0.1) = 0.9$$

Topic: Linear approximation

Question: Find the linear approximation of the function at $a = 2$.

$$f(x) = (x + 4)^2$$

Answer choices:

A $L(x) = 1 + x$

B $L(x) = 12 + 12x$

C $L(x) = -12 - 12x$

D $L(x) = 1 - x$

Solution: B

Take the derivative.

$$f(x) = (x+4)^2$$

$$f'(x) = 2(x+4)(1)$$

$$f'(x) = 2x + 8$$

Evaluate the original function at $a = 2$.

$$f(2) = (2+4)^2$$

$$f(2) = 36$$

Evaluate the derivative at $a = 2$.

$$f'(2) = 2(2) + 8$$

$$f'(2) = 12$$

Substitute all of these pieces into the linear approximation formula.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 36 + 12(x - 2)$$

$$L(x) = 36 + 12x - 24$$

$$L(x) = 12 + 12x$$

Topic: Estimating a root

Question: Use linear approximation to estimate $\sqrt[3]{9}$.

Answer choices:

A $\frac{41}{12}$

B $\frac{23}{12}$

C $\frac{7}{12}$

D $\frac{25}{12}$

Solution: D

We don't know the value of $\sqrt[3]{9}$, but we know that $\sqrt[3]{8} = 2$. So instead of trying to calculate $\sqrt[3]{9}$ directly, let's use the function $f(x) = \sqrt[3]{x}$.

Differentiate the function

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

and then evaluate it at $a = 8$.

$$f'(8) = \frac{1}{3(8)^{\frac{2}{3}}}$$

$$f'(8) = \frac{1}{3(8^{\frac{1}{3}})^2}$$

$$f'(8) = \frac{1}{3(2)^2}$$

$$f'(8) = \frac{1}{3(4)}$$

$$f'(8) = \frac{1}{12}$$

So along the function $f(x) = \sqrt[3]{x}$, we have the point of tangency $(8,2)$ and the slope $m = 1/12$. Substitute these into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 2 + \frac{1}{12}(x - 8)$$

$$L(x) = 2 + \frac{1}{12}x - \frac{8}{12}$$

$$L(x) = \frac{1}{12}x - \frac{8}{12} + \frac{24}{12}$$

$$L(x) = \frac{1}{12}x + \frac{16}{12}$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[3]{9}$. Substitute $x = 9$.

$$L(9) = \frac{1}{12}(9) + \frac{16}{12}$$

$$L(9) = \frac{9}{12} + \frac{16}{12}$$

$$L(9) = \frac{25}{12}$$

Topic: Estimating a root

Question: Use linear approximation to estimate $\sqrt[3]{29}$.

Answer choices:

A $\frac{83}{27}$

B $\frac{25}{27}$

C $\frac{137}{27}$

D $\frac{79}{27}$

Solution: A

We don't know the value of $\sqrt[3]{29}$, but we know that $\sqrt[3]{27} = 3$. So instead of trying to calculate $\sqrt[3]{29}$ directly, let's use the function $f(x) = \sqrt[3]{x}$.

Differentiate the function

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}}$$

$$f'(x) = \frac{1}{3x^{\frac{2}{3}}}$$

and then evaluate it at $a = 27$.

$$f'(27) = \frac{1}{3(27)^{\frac{2}{3}}}$$

$$f'(27) = \frac{1}{3(27^{\frac{1}{3}})^2}$$

$$f'(27) = \frac{1}{3(3)^2}$$

$$f'(27) = \frac{1}{3(9)}$$

$$f'(27) = \frac{1}{27}$$

So along the function $f(x) = \sqrt[3]{x}$, we have the point of tangency $(27,3)$ and the slope $m = 1/27$. Substitute these into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 3 + \frac{1}{27}(x - 27)$$

$$L(x) = 3 + \frac{1}{27}x - 1$$

$$L(x) = \frac{1}{27}x + 2$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[3]{29}$. Substitute $x = 29$.

$$L(29) = \frac{1}{27}(29) + 2$$

$$L(29) = \frac{29}{27} + \frac{54}{27}$$

$$L(29) = \frac{83}{27}$$

Topic: Estimating a root

Question: Use linear approximation to estimate $\sqrt[4]{79}$.

Answer choices:

A $\frac{322}{54}$

B $\frac{164}{54}$

C $\frac{161}{54}$

D $\frac{82}{54}$

Solution: C

We don't know the value of $\sqrt[4]{79}$, but we know that $\sqrt[4]{81} = 3$. So instead of trying to calculate $\sqrt[4]{79}$ directly, let's use the function $f(x) = \sqrt[4]{x}$.

Differentiate the function

$$f'(x) = \frac{1}{4}x^{-\frac{3}{4}}$$

$$f'(x) = \frac{1}{4x^{\frac{3}{4}}}$$

and then evaluate it at $a = 81$.

$$f'(81) = \frac{1}{4(81)^{\frac{3}{4}}}$$

$$f'(81) = \frac{1}{4(81^{\frac{1}{4}})^3}$$

$$f'(81) = \frac{1}{4(3)^3}$$

$$f'(81) = \frac{1}{4(27)}$$

$$f'(81) = \frac{1}{108}$$

So along the function $f(x) = \sqrt[4]{x}$, we have the point of tangency $(81,3)$ and the slope $m = 1/108$. Substitute these into the linear approximation equation.

$$L(x) = f(a) + f'(a)(x - a)$$

$$L(x) = 3 + \frac{1}{108}(x - 81)$$

$$L(x) = 3 + \frac{1}{108}x - \frac{81}{108}$$

$$L(x) = \frac{1}{108}x - \frac{81}{108} + \frac{324}{108}$$

$$L(x) = \frac{1}{108}x + \frac{243}{108}$$

Now that we have the linear approximation equation, we can use it to estimate $\sqrt[4]{79}$. Substitute $x = 79$.

$$L(79) = \frac{1}{108}(79) + \frac{243}{108}$$

$$L(79) = \frac{79}{108} + \frac{243}{108}$$

$$L(79) = \frac{322}{108}$$

$$L(79) = \frac{161}{54}$$

Topic: Absolute, relative, and percentage error

Question: Use a linear approximation to estimate the value of $\sqrt{99}$, then find the absolute error of the estimate.

Answer choices:

A $E_A(100) \approx 0.0050$

B $E_A(100) \approx 0.0001$

C $E_A(99) \approx 0.0050$

D $E_A(99) \approx 0.0001$

Solution: D

The root $\sqrt{99}$ is very close to $\sqrt{100}$, which we already know is 10. So instead of thinking specifically about $\sqrt{99}$, let's think about $\sqrt{x}$ and use the function $f(x)=\sqrt{x}$.

Differentiate the function,

$$f'(x)=\frac{1}{2}x^{-\frac{1}{2}}$$

$$f'(x)=\frac{1}{2x^{\frac{1}{2}}}$$

$$f'(x)=\frac{1}{2\sqrt{x}}$$

then evaluate the derivative at $x=100$.

$$f'(100)=\frac{1}{2\sqrt{100}}$$

$$f'(100)=\frac{1}{2(10)}$$

$$f'(100)=\frac{1}{20}$$

So the linear approximation line intersects $f(x)=\sqrt{x}$ at the point of tangency $(100,10)$, and has a slope of $m=1/20$. Substitute these values into the linear approximation equation.

$$L(x)=f(a)+f'(a)(x-a)$$

$$L(x) = 10 + \frac{1}{20}(x - 100)$$

$$L(x) = 10 + \frac{1}{20}x - \frac{100}{20}$$

$$L(x) = \frac{1}{20}x - \frac{100}{20} + \frac{200}{20}$$

$$L(x) = \frac{1}{20}x + \frac{100}{20}$$

Then the linear approximation at $x = 99$ is

$$L(99) = \frac{1}{20}(99) + \frac{100}{20}$$

$$L(99) = \frac{99}{20} + \frac{100}{20}$$

$$L(99) = \frac{199}{20}$$

$$L(99) = 9.95$$

In comparison, the actual value of $\sqrt{99}$ is

$$f(x) = \sqrt{x}$$

$$f(99) = \sqrt{99}$$

$$f(99) \approx 9.9499$$

Therefore, the absolute error of the approximation is

$$E_A(a) = |f(a) - L(a)|$$

$$E_A(99) = |f(99) - L(99)|$$

$$E_A(99) \approx |9.9499 - 9.95|$$

$$E_A(99) \approx |-0.0001|$$

$$E_A(99) \approx 0.0001$$

Topic: Absolute, relative, and percentage error

Question: If the absolute error of a linear approximation estimate of $\sqrt{99}$ is $E_A(99) = 0.0001$, then find the relative error of the estimate.

Answer choices:

A $E_R(99) \approx 0.00005001$

B $E_R(99) \approx 0.00001005$

C $E_R(100) \approx 0.00005001$

D $E_R(100) \approx 0.00001005$

Solution: B

The actual value of $\sqrt{99}$ is

$$f(x) = \sqrt{x}$$

$$f(99) = \sqrt{99}$$

$$f(99) \approx 9.9499$$

and we've been told that the absolute error of the estimate is

$$E_A(99) = 0.0001$$

The relative error is

$$E_R(a) = \frac{E_A(a)}{f(a)}$$

$$E_R(99) = \frac{E_A(99)}{f(99)}$$

$$E_R(99) \approx \frac{0.0001}{9.9499}$$

$$E_R(99) \approx 0.00001005$$

Topic: Absolute, relative, and percentage error

Question: If the absolute error of a linear approximation estimate of $\sqrt{99}$ is $E_A(99) = 0.0001$ and the relative error is $E_R(99) \approx 0.00001005$, then find the percentage error of the estimate.

Answer choices:

A $E_P(100) \approx 0.001005$

B $E_P(100) \approx 0.00001005$

C $E_P(99) \approx 0.001005$

D $E_P(99) \approx 0.00001005$

Solution: C

The actual value of $\sqrt{99}$ is

$$f(x) = \sqrt{x}$$

$$f(99) = \sqrt{99}$$

$$f(99) \approx 9.9499$$

and we've been told that the absolute error of the estimate is $E_A(99) = 0.0001$ and that the relative error of the estimate is $E_R(99) \approx 0.00001005$.

The percentage error is

$$E_P(a) = 100\% \cdot E_R(a)$$

$$E_P(99) = 100\% \cdot E_R(99)$$

$$E_P(99) \approx 100\% \cdot 0.00001005$$

$$E_P(99) \approx 0.001005$$

Topic: Radius of the balloon

Question: A thin sheet of ice is in the shape of a circle. If the ice is melting in such a way that the area of the sheet is decreasing at a rate of 0.25 m^2/s, at what rate is the radius decreasing when the area of the sheet is 25 m^2?

Answer choices:

A $-\frac{0.025}{\sqrt{\pi}}$ m/s

B -400π cm/s

C -112π cm/s

D $-\frac{1}{400\sqrt{\pi}}$ m/s

Solution: A

The formula for the area of a circle is

$$A = \pi r^2$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dA}{dt} = \pi(2r)\frac{dr}{dt}$$

$$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

From the question, we know that $dA/dt = -0.25$ and that $A = 25$, so first we need to find the radius.

$$A = \pi r^2$$

$$25 = \pi r^2$$

$$r^2 = \frac{25}{\pi}$$

$$r = \frac{5}{\sqrt{\pi}}$$

Now we'll plug in everything we know.

$$-0.25 = 2\pi\left(\frac{5}{\sqrt{\pi}}\right)\frac{dr}{dt}$$

$$-0.25 = 10\sqrt{\pi}\frac{dr}{dt}$$

Solve for dr/dt, which is the rate we were asked to find.

$$\frac{dr}{dt} = -\frac{0.25}{10\sqrt{\pi}}$$

$$\frac{dr}{dt} = -\frac{0.025}{\sqrt{\pi}}$$

Topic: Radius of the balloon

Question: Air is being pumped into a spherical balloon at a rate of 192π m^3/hr. How fast is the balloon's surface area changing when $r = 4$ m?

Answer choices:

A $\quad \dfrac{1}{96\pi}$ m^2/hr

B $\quad 3$ m^2/hr

C $\quad 96\pi$ m^2/hr

D $\quad \dfrac{1}{3\pi}$ m^2/hr

Solution: C

The formula for the volume of a sphere is

$$V=\frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt}=\frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dV/dt=192\pi$ and that $r=4$, so we'll plug those in.

$$192\pi=4\pi(4)^2\frac{dr}{dt}$$

$$192\pi=64\pi\frac{dr}{dt}$$

Solve for dr/dt,

$$\frac{dr}{dt}=\frac{192\pi}{64\pi}$$

$$\frac{dr}{dt}=3$$

The formula for the surface area of a sphere is

$$S=4\pi r^2$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dS}{dt} = 4\pi(2r)\frac{dr}{dt}$$

$$\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$$

From the question, we know that $dr/dt = 3$ and that $r = 4$, so we'll plug those in.

$$\frac{dS}{dt} = 8\pi(4)(3)$$

$$\frac{dS}{dt} = 96\pi$$

Topic: Radius of the balloon

Question: Air is being sucked out of a spherical balloon so that its volume is decreasing by 250 cm^3/s. How fast is the radius decreasing when the radius is 5 cm?

Answer choices:

A $\frac{5}{2\pi}$ cm/s

B $-\frac{5}{2\pi}$ cm/s

C $\frac{2}{5\pi}$ cm/s

D $-\frac{2}{5\pi}$ cm/s

Solution: B

The formula for the volume of a sphere is

$$V = \frac{4}{3}\pi r^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{4}{3}\pi(3r^2)\frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2\frac{dr}{dt}$$

From the question, we know that $dV/dt = -250$ and that $r = 5$, so we'll plug those in.

$$-250 = 4\pi(5)^2\frac{dr}{dt}$$

$$-250 = 100\pi\frac{dr}{dt}$$

Solve for dr/dt, which is the rate we were asked to find.

$$\frac{dr}{dt} = -\frac{250}{100\pi}$$

$$\frac{dr}{dt} = -\frac{5}{2\pi}$$

Topic: Price of the product

Question: An item is currently selling for $\$50$/unit. The quantity supplied is decreasing by 10 units/week. At what rate is the price of the item changing?

$$q = 4{,}000e^{-0.01p}$$

Answer choices:

A Increasing by $\$2.43$ per week

B Decreasing by $\$1.86$ per week

C Decreasing by $\$6.59$ per week

D Increasing by $\$0.41$ per week

Solution: D

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q = 4{,}000e^{-0.01p}$$

$$\frac{dq}{dt} = -40e^{-0.01p}\frac{dp}{dt}$$

From the question, we know that $p = 50$ and $dq/dt = -10$, so we'll plug those in.

$$-10 = -40e^{-0.01(50)}\frac{dp}{dt}$$

$$-10 = -40e^{-0.50}\frac{dp}{dt}$$

Solve for dp/dt, which is the rate we were asked to find.

$$\frac{dp}{dt} = \frac{-10}{-40e^{-0.50}}$$

$$\frac{dp}{dt} = \frac{1}{4e^{-0.50}}$$

$$\frac{dp}{dt} = \frac{e^{0.50}}{4}$$

$$\frac{dp}{dt} \approx \$0.41$$

Topic: Price of the product

Question: A company has determined that the demand curve for their product is given by q, where p is the price in dollars, and q is the quantity in millions. If the company is increasing the price of the product by $\$1.25$ per week, find the rate at which demand is changing when the price is $\$25$.

$$q = \sqrt{2{,}000 - p^2}$$

Answer choices:

A Increasing by 0.85 million items per week

B Decreasing by 1.85 million items per week

C Decreasing by 0.85 million items per week

D Increasing by 1.85 million items per week

Solution: C

Use implicit differentiation to take the derivative of both sides of the quantity equation.

$$q = \sqrt{2{,}000 - p^2}$$

$$\frac{dq}{dt} = \frac{1}{2}(2{,}000 - p^2)^{-\frac{1}{2}}\left(-2p\frac{dp}{dt}\right)$$

$$\frac{dq}{dt} = -p(2{,}000 - p^2)^{-\frac{1}{2}}\frac{dp}{dt}$$

From the question, we know that $p = 25$ and $dp/dt = 1.25$, so we'll plug those in.

$$\frac{dq}{dt} = -25(2{,}000 - 25^2)^{-\frac{1}{2}}(1.25)$$

$$\frac{dq}{dt} \approx -0.85$$

Demand is decreasing by 0.85 million items per week.

Topic: Price of the product

Question: Suppose that the price p of a product is given by the demand function, where q represents the demand quantity. If the daily demand is decreasing at a rate of 8 units per day, at what rate is the price changing when the demand is 30 units?

$$p = \frac{900 - 10q}{250 - 2q}$$

Answer choices:

A The price is changing at a rate of $\$6.45$ per day

B The price is changing at a rate of $\$0.155$ per day

C The price is changing at a rate of $\$6.45$ per week

D The price is changing at a rate of $\$0.155$ per week

Solution: B

Use implicit differentiation to take the derivative of both sides of the price equation.

$$p = \frac{900 - 10q}{250 - 2q}$$

$$\frac{dp}{dt} = \frac{-10\frac{dq}{dt}(250 - 2q) - \left(-2\frac{dq}{dt}\right)(900 - 10q)}{(250 - 2q)^2}$$

$$\frac{dp}{dt} = \frac{\frac{dq}{dt}(-2{,}500 + 20q) + \frac{dq}{dt}(1{,}800 - 20q)}{(250 - 2q)^2}$$

$$\frac{dp}{dt} = \frac{\frac{dq}{dt}(-2{,}500 + 20q + 1{,}800 - 20q)}{(250 - 2q)^2}$$

$$\frac{dp}{dt} = \frac{-700\frac{dq}{dt}}{(250 - 2q)^2}$$

From the question, we know that $q = 30$ and $dq/dt = -8$, so we'll plug those in.

$$\frac{dp}{dt} = \frac{-700(-8)}{(250 - 2(30))^2}$$

$$\frac{dp}{dt} = \frac{5{,}600}{36{,}100}$$

$$\frac{dp}{dt} = 0.155$$

Topic: Water level in the tank

Question: Water is being pumped into a rectangular tank at a rate of 0.8 cubic feet per minute. How fast is the water level rising if the base of the tank is a rectangle with dimensions 4×5 feet?

Answer choices:

A 25 ft/m

B 0.4 ft/m

C 0.04 ft/m

D 0.8 ft/m

Solution: C

The formula for the volume of a rectangular prism is

$$V = lwh$$

From the question, we know that $w = 4$ and $l = 5$, so we'll plug that in.

$$V = 5(4)h$$

$$V = 20h$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = 20(1)\frac{dh}{dt}$$

$$\frac{dV}{dt} = 20\frac{dh}{dt}$$

From the question, we know that $dV/dt = 0.8$, so make that substitution.

$$0.8 = 20\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{0.8}{20}$$

$$\frac{dh}{dt} = 0.04$$

Topic: Water level in the tank

Question: Water is being pumped from a cylindrical tank with a radius of 3 ft at a rate of 18 cubic feet per minute. How fast is the water level falling when the water is 2 ft deep?

Answer choices:

A $-\frac{2}{\pi}$ ft/min

B -2 ft/min

C $-\pi$ ft/min

D $-\frac{\pi}{2}$ ft/min

Solution: A

The formula for the volume of a cylinder is

$$V = \pi r^2 h$$

From the question, we know that $r = 3$, so plug that in.

$$V = \pi(3)^2 h$$

$$V = 9\pi h$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = 9\pi(1)\frac{dh}{dt}$$

$$\frac{dV}{dt} = 9\pi\frac{dh}{dt}$$

From the question, we know that $dV/dt = -18$, so make that substitution.

$$-18 = 9\pi\frac{dh}{dt}$$

$$\frac{dh}{dt} = -\frac{18}{9\pi}$$

$$\frac{dh}{dt} = -\frac{2}{\pi}$$

Topic: Water level in the tank

Question: An inverted pyramid is standing on its tip. The base of the pyramid is 4×4 meters, and the depth is 9 meters. If oil is flowing into the vat at 8 m^3/min, how fast is the oil level rising when the depth of the oil is 7 meters?

Answer choices:

A $\frac{3}{2}$ m/min

B $\frac{81}{98}$ m/min

C $\frac{16}{81}$ m/min

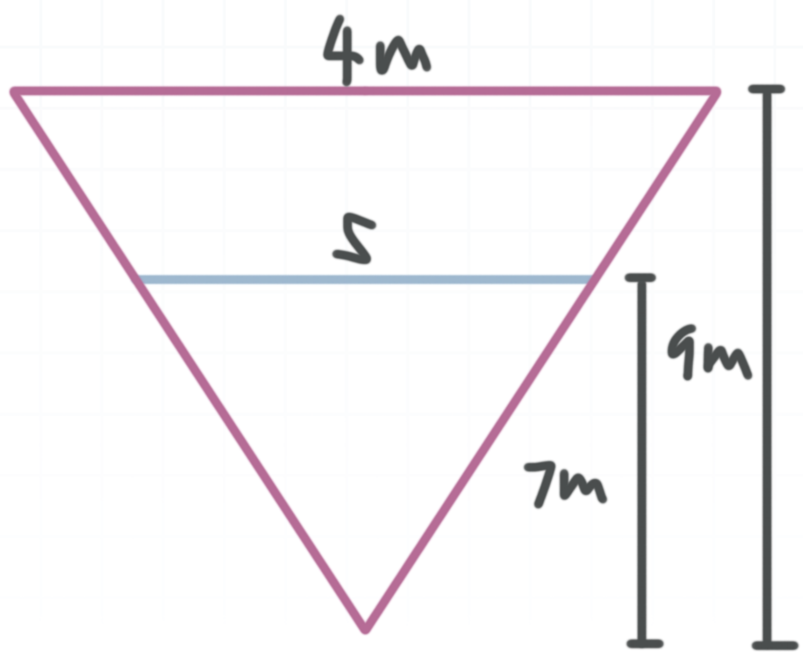
D $\frac{81}{16}$ m/min

Solution: B

The formula for the volume of a pyramid is

$$V = \frac{1}{3}s^2h$$

We want to express volume as a function of h only. Using the diagram of a cross-section of the pyramid,



and similar triangles, we see that

$$\frac{h}{s} = \frac{9}{4}$$

$$s = \frac{4}{9}h$$

Then the volume of the water is given by

$$V = \frac{1}{3}\left(\frac{4}{9}h\right)^2 h$$

$$V = \frac{1}{3}\left(\frac{16}{81}\right)h^3$$

$$V = \frac{16}{243}h^3$$

Use implicit differentiation to take the derivative of both sides.

$$(1)\frac{dV}{dt} = \frac{16}{243}(3h^2)\frac{dh}{dt}$$

$$\frac{dV}{dt} = \frac{16}{81}h^2\frac{dh}{dt}$$

From the question, we know that $dV/dt = 8$ and $h = 7$, so make those substitutions.

$$8 = \frac{16}{81}(7)^2\frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{8(81)}{16(49)}$$

$$\frac{dh}{dt} = \frac{81}{98}$$

Topic: Observer and the airplane

Question: An airplane is flying horizontally at 720 miles/hr, 3 miles above the ground, when it passes an observer on the ground. How fast is the distance between the person and the plane increasing 10 seconds later?

Answer choices:

A About 400 miles/hr

B About 500 miles/hr

C About 600 miles/hr

D About 700 miles/hr

Solution: A

The horizontal path of the plane, the vertical distance between the observer and that path, and the diagonal connecting the observer to the plane's location, together form a right triangle.

So we'll use the Pythagorean theorem, which relates the three side lengths of a right triangle.

$$a^2+b^2=c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt}+2b\frac{db}{dt}=2c\frac{dc}{dt}$$

We'll call a the vertical distance, b the horizontal distance, and c the diagonal distance. We know from the question that $a=3$. And because a stays constant, $da/dt=0$.

$$2(3)(0)+2b\frac{db}{dt}=2c\frac{dc}{dt}$$

$$2b\frac{db}{dt}=2c\frac{dc}{dt}$$

$$b\frac{db}{dt}=c\frac{dc}{dt}$$

Convert $t=10$ seconds to hours,

$$x\text{ hours}=10\text{ seconds}\times\frac{1\text{ minute}}{60\text{ seconds}}\times\frac{1\text{ hour}}{60\text{ minutes}}$$

$$x \text{ hours} = \frac{1}{360} \text{ hours}$$

then use it to find the horizontal distance b.

$$b \text{ miles} = \frac{1}{360} \text{ hours} \times \frac{720 \text{ miles}}{\text{hour}}$$

$$b \text{ miles} = 2 \text{ miles}$$

With $a = 3$ and $b = 2$, we can find c.

$$a^2 + b^2 = c^2$$

$$3^2 + 2^2 = c^2$$

$$9 + 4 = c^2$$

$$13 = c^2$$

$$c \approx 3.61$$

Substitute $b = 2$ and $c = 3.61$, along with $db/dt = 720$, into the derivative. This will give us the rate at which the distance between the observer and the airplane is increasing.

$$b\frac{db}{dt} = c\frac{dc}{dt}$$

$$2(720) = (3.61)\ \frac{dc}{dt}$$

$$\frac{dc}{dt} \approx 399$$

Topic: Observer and the airplane

Question: Car A starts 50 miles directly west of a car B and begins moving east at 75 mph. At the same moment, car B begins moving south at 90 mph. At what rate is the distance between the cars changing after 20 minutes?

Answer choices:

A About 39 miles/hr

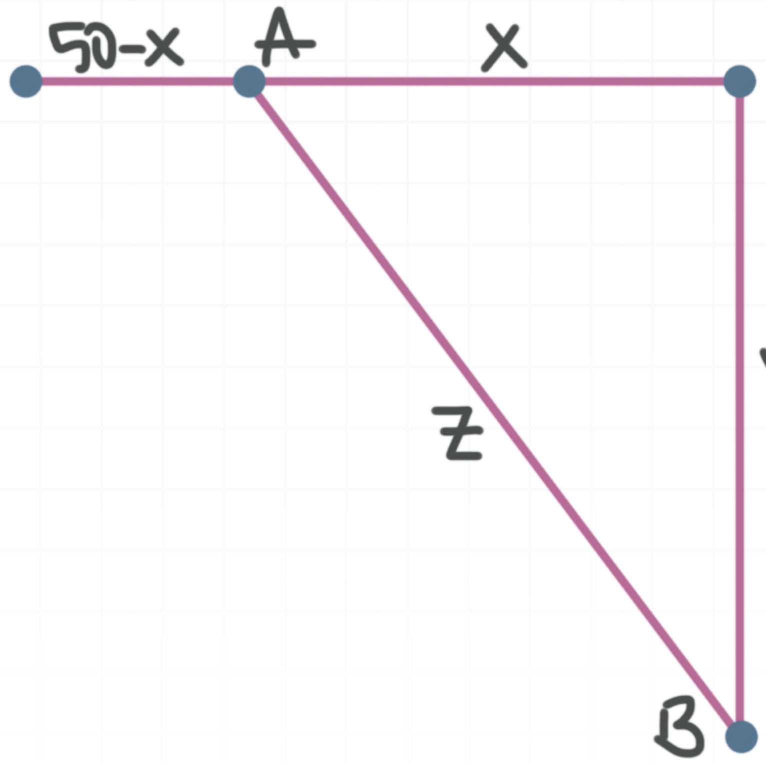
B About 21 miles/hr

C About 18 miles/hr

D About 58 miles/hr

Solution: B

Draw a diagram.



We'll use the Pythagorean Theorem, which relates the three side lengths of a right triangle.

$$x^2 + y^2 = z^2$$

Use implicit differentiation to take the derivative of both sides.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

Convert $t = 20$ minutes to hours.

$$x \text{ hours} = 20 \text{ minutes} \times \frac{1 \text{ minute}}{60 \text{ seconds}}$$

$$x \text{ hours} = \frac{1}{3} \text{ hours}$$

If $t = 1/3$ hr, then car A has traveled east,

$$\frac{1}{3} \text{ hours} \times \frac{75 \text{ miles}}{\text{hour}} = 25 \text{ miles}$$

$$x = 50 - 25 = 25 \text{ miles}$$

and car B has traveled south

$$\frac{1}{3} \text{ hours} \times \frac{90 \text{ miles}}{\text{hour}} = 30 \text{ miles}$$

With $x = 25$ and $y = 30$, we can find z.

$$x^2 + y^2 = z^2$$

$$30^2 + 25^2 = z^2$$

$$1{,}525 = z^2$$

$$z \approx 39$$

Substitute what we know into the derivative, then solve for dz/dt.

$$25(-75) + 30(90) = 39\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{25(-75) + 30(90)}{39}$$

$$\frac{dz}{dt} \approx 21$$

Topic: Observer and the airplane

Question: A truck is 40 miles north of an intersection, traveling toward the intersection at 35 mph. At the same time, another car is 30 miles west of the intersection, traveling away from the intersection at 45 mph. Is the distance between the vehicles increasing or decreasing at that moment and at what rate?

Answer choices:

A Decreasing at a rate of 55 mph

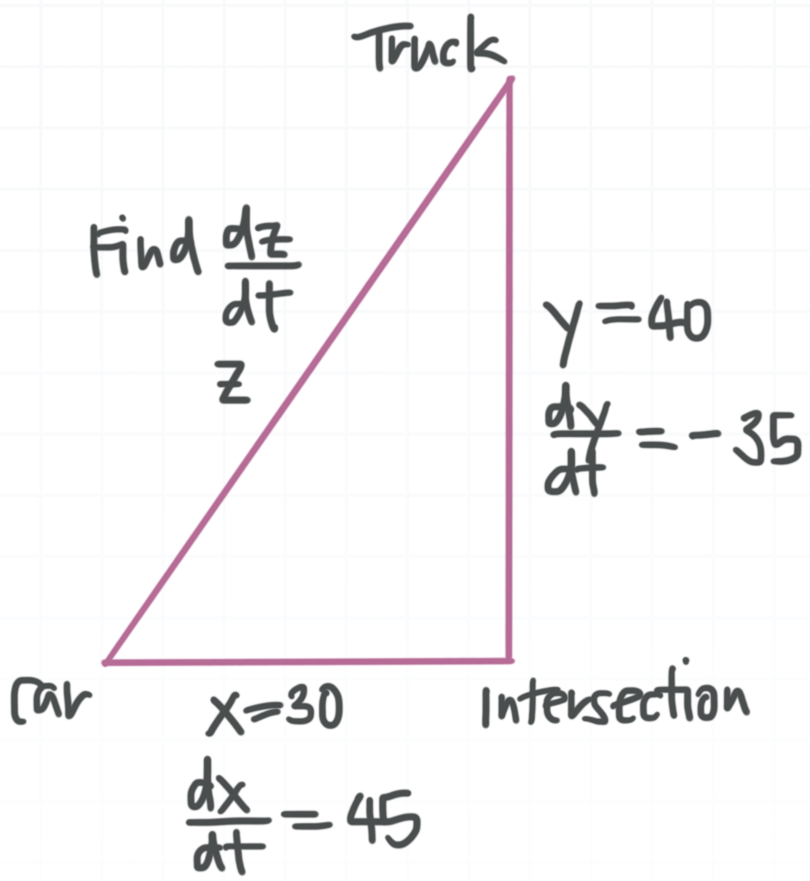
B Decreasing at a rate of 1 mph

C Increasing at a rate of 55 mph

D Increasing at a rate of 1 mph

Solution: B

Draw a diagram.



Use the Pythagorean Theorem $x^2 + y^2 = z^2$, then differentiate with respect to time.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 2z\frac{dz}{dt}$$

$$x\frac{dx}{dt} + y\frac{dy}{dt} = z\frac{dz}{dt}$$

We can use the Pythagorean Theorem to find the distance between the truck and the car, z, when $x = 30$ and $y = 40$.

$$x^2 + y^2 = z^2$$

$$30^2 + 40^2 = z^2$$

$$900 + 1{,}600 = z^2$$

$$2{,}500 = z^2$$

$$z = 50$$

Substitute what we know into the derivative, then solve for dz/dt.

$$30(45) + 40(-35) = 50\frac{dz}{dt}$$

$$\frac{dz}{dt} = \frac{30(45) + 40(-35)}{50}$$

$$\frac{dz}{dt} = -\frac{50}{50} = -1$$

The distance between the two vehicles is decreasing at a rate of 1 mph.

Topic: Ladder sliding down the wall

Question: A gardener's shovel is 1 m long and leaning against a fence, sliding down the fence at a rate of 0.25 m/s. When the top of the shovel is 0.5 m off the ground, at what rate is the bottom of the shovel sliding along the ground away from the fence?

Answer choices:

A $\frac{3\sqrt{3}}{4}$ m/s

B $\frac{4\sqrt{3}}{3}$ m/s

C $\frac{4}{3}$ m/s

D $\frac{\sqrt{3}}{12}$ m/s

Solution: D

The ground, the fence, and the shovel form a right triangle, so we'll use the Pythagorean Theorem as the equation that relates the side lengths.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

From the question we know that the length of the shovel is $c = 1$, and that the length of the shovel doesn't change, so $dc/dt = 0$.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2(1)(0)$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$a\frac{da}{dt} + b\frac{db}{dt} = 0$$

If we say that the vertical fence is side b, and that the horizontal ground is side a, then the question tells us that $b = 1/2$ and that $db/dt = -1/4$.

$$a\frac{da}{dt} + \frac{1}{2}\left(-\frac{1}{4}\right) = 0$$

$$a\frac{da}{dt} - \frac{1}{8} = 0$$

Find the value of a when $b = 1/2$ and $c = 1$.

$$a^2+b^2=c^2$$

$$a^2+\left(\frac{1}{2}\right)^2=1^2$$

$$a^2+\frac{1}{4}=1$$

$$a^2=\frac{3}{4}$$

$$a=\frac{\sqrt{3}}{2}$$

We're asked to solve for da/dt, so we'll plug in this value of a that we've found and then solve the equation for da/dt.

$$\left(\frac{\sqrt{3}}{2}\right)\frac{da}{dt}-\frac{1}{8}=0$$

$$\left(\frac{\sqrt{3}}{2}\right)\frac{da}{dt}=\frac{1}{8}$$

$$\frac{da}{dt}=\frac{2}{8\sqrt{3}}$$

$$\frac{da}{dt}=\frac{1}{4\sqrt{3}}$$

Rationalize the denominator.

$$\frac{1}{4\sqrt{3}}\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$$

$$\frac{\sqrt{3}}{4(3)}$$

$$\frac{\sqrt{3}}{12}$$

Topic: Ladder sliding down the wall

Question: A 5 foot ladder is sliding down a vertical wall while its bottom slides away from the wall at 3 ft/s. How fast is the top moving when the top is 4 feet off the ground?

Answer choices:

A $\quad -\frac{9}{4}$ ft/s

B $\quad -\frac{4}{9}$ ft/s

C $\quad -\frac{3}{2}$ ft/s

D $\quad -\frac{2}{3}$ ft/s

Solution: A

The ground, the wall, and the ladder form a right triangle, so we'll use the Pythagorean Theorem as the equation that relates the side lengths.

$$a^2 + b^2 = c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2c\frac{dc}{dt}$$

From the question we know that the length of the ladder is $c = 5$, and that the length of the ladder doesn't change, so $dc/dt = 0$.

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 2(5)(0)$$

$$2a\frac{da}{dt} + 2b\frac{db}{dt} = 0$$

$$a\frac{da}{dt} + b\frac{db}{dt} = 0$$

If we say that the vertical wall is side b, and that the horizontal ground is side a, then the question tells us that $b = 4$ and that $da/dt = 3$.

$$a(3) + 4\frac{db}{dt} = 0$$

$$3a + 4\frac{db}{dt} = 0$$

Find the value of a when $b = 4$ and $c = 5$.

$$a^2 + b^2 = c^2$$

$$a^2 + 4^2 = 5^2$$

$$a^2 + 16 = 25$$

$$a^2 = 9$$

$$a = 3$$

We're asked to solve for db/dt, so we'll plug in this value of a that we've found and then solve the equation for db/dt.

$$3(3) + 4\frac{db}{dt} = 0$$

$$9 + 4\frac{db}{dt} = 0$$

$$4\frac{db}{dt} = -9$$

$$\frac{db}{dt} = -\frac{9}{4}$$

Topic: Ladder sliding down the wall

Question: A 13-foot ladder is leaning against a wall. The base of the ladder is pushed toward the wall at the rate of 5 ft/s. When the base of the ladder is 12 feet from the wall, at what rate is the top of the ladder moving up the wall?

Answer choices:

A $\quad -\frac{25}{12}$ ft/s

B $\quad 12$ ft/s

C $\quad \frac{25}{12}$ ft/s

D $\quad -12$ ft/s

Solution: B

The ground, the wall, and the ladder form a right triangle, so we'll use the Pythagorean Theorem as the equation that relates the side lengths.

$$a^2+b^2=c^2$$

Use implicit differentiation to take the derivative of both sides.

$$2a\frac{da}{dt}+2b\frac{db}{dt}=2c\frac{dc}{dt}$$

From the question we know that the length of the ladder is $c=13$, and that the length of the ladder doesn't change, so $dc/dt=0$.

$$2a\frac{da}{dt}+2b\frac{db}{dt}=2(13)(0)$$

$$2a\frac{da}{dt}+2b\frac{db}{dt}=0$$

$$a\frac{da}{dt}+b\frac{db}{dt}=0$$

If we say that the vertical wall is side b, and that the horizontal ground is side a, then the question tells us that $a=12$ and that $da/dt=-5$.

$$12(-5)+b\frac{db}{dt}=0$$

$$-60+b\frac{db}{dt}=0$$

Find the value of b when $a=12$ and $c=13$.

$$a^2 + b^2 = c^2$$

$$12^2 + b^2 = 13^2$$

$$144 + b^2 = 169$$

$$b^2 = 25$$

$$b = 5$$

We're asked to solve for db/dt, so we'll plug in this value of a that we've found and then solve the equation for db/dt.

$$-60 + 5\frac{db}{dt} = 0$$

$$5\frac{db}{dt} = 60$$

$$\frac{db}{dt} = 12$$

Topic: Applied optimization

Question: A rancher plans to build a fenced rectangular area adjacent to an existing stone wall. He wants the fence to enclose $160{,}000$ square meters for his horses, but he's low on fencing. Which dimensions require the least amount of fencing?

Answer choices:

A $\quad 400 \times 400$

B $\quad 283 \times 565$

C $\quad 4{,}000 \times 8{,}000$

D $\quad 100 \times 1{,}600$

Solution: B

In this problem, we want to minimize the perimeter of a field and we know that we will require the fence to enclose $160{,}000$ ft square meters for the horses.

Since one side of the enclosure doesn't need any fencing, the amount of fencing needed to enclose the area is

$$P = 2x + y$$

where x is the length of a side adjacent to the wall and y is the length of the side opposite the wall. We can also write an equation for area in terms of x and y, and then solve it for y.

$$A = xy$$

$$160{,}000 = xy$$

$$y = \frac{160{,}000}{x}$$

Substitute this value into the perimeter equation.

$$P = 2x + \frac{160{,}000}{x}$$

Take the derivative,

$$P' = 2 - \frac{160{,}000}{x^2}$$

then set it equal to 0 to find critical points.

$$2 - \frac{160,000}{x^2} = 0$$

$$2 = \frac{160,000}{x^2}$$

$$2x^2 = 160,000$$

$$x^2 = 80,000$$

$$x = \sqrt{80,000}$$

$$x \approx 283$$

Use the first derivative test with test values of 280 and 290 to confirm that $x \approx 283$ represents a minimum.

$$P'(280) = 2 - \frac{160,000}{280^2}$$

$$P'(280) \approx -0.04$$

and

$$P'(290) = 2 - \frac{160,000}{290^2}$$

$$P'(290) \approx 0.10$$

Because we get a negative value to the left of the critical point and a positive value to the right, $x \approx 283$ represents a minimum. The associated value for y is

$$y = \frac{160,000}{283}$$

$$y \approx 565$$

Topic: Applied optimization

Question: With $1{,}000$ m of new fencing material, you need to enclose a rectangular yard and maximize its area. What dimensions should you use?

Answer choices:

A 250×250

B 250×750

C 500×500

D 125×375

Solution: A

If we use l for length and w for width, we can say the perimeter of the yard is

$$P = 2l + 2w$$

$$1{,}000 = 2l + 2w$$

and that the area is

$$A = lw$$

We want to maximize area, so we'll solve the perimeter equation for width.

$$2w = 1{,}000 - 2l$$

$$w = 500 - l$$

Substitute this value into the area equation.

$$A = l(500 - l)$$

$$A = 500l - l^2$$

Take the derivative, then set it equal to 0 to find critical points.

$$A' = 500 - 2l$$

$$500 - 2l = 0$$

$$2l = 500$$

$$l = 250$$

Use the first derivative test with test values of $l = 240$ and $l = 260$ to make sure that $l = 250$ represents a maximum.

$$A'(240) = 500 - 2(240)$$

$$A'(240) = 500 - 480$$

$$A'(240) = 20$$

and

$$A'(260) = 500 - 2(260)$$

$$A'(260) = 500 - 520$$

$$A'(260) = -20$$

Since we got a positive value to the left of the critical point and a negative value to the right of it, we know $l = 250$ represents a maximum. The corresponding width is

$$w = 500 - 250$$

$$w = 250$$

Topic: Applied optimization

Question: You want to construct a box with a square bottom and you only have 36 m^2 of material. Assuming you use all of the material, what is the maximum volume of the box?

Answer choices:

A $12\sqrt{6}$ m^3

B 36 m^3

C 18 m^3

D $6\sqrt{6}$ m^3

Solution: D

The volume of a box is always given by $V = lwh$, but since we've been told that the box has a square base, we know that $l = w$, so we can simplify the volume equation to $V = l^2h$.

The surface area of a box is always given by $A = 2lw + 2lh + 2wh$. But since $l = w$, we can simplify this as

$$A = 2ll + 2lh + 2lh$$

$$A = 2l^2 + 4lh$$

We know that total surface area is 36, so

$$36 = 2l^2 + 4lh$$

$$18 = l^2 + 2lh$$

Solve this area equation for height h.

$$2lh = 18 - l^2$$

$$h = \frac{18 - l^2}{2l}$$

Substitute this value into the volume equation.

$$V = l^2h$$

$$V = l^2\left(\frac{18 - l^2}{2l}\right)$$

$$V = \frac{18l - l^3}{2}$$

$$V = 9l - \frac{1}{2}l^3$$

Take the derivative,

$$V' = 9 - \frac{3}{2}l^2$$

then set it equal to 0 to find critical points.

$$9 - \frac{3}{2}l^2 = 0$$

$$\frac{3}{2}l^2 = 9$$

$$l^2 = 6$$

$$l = \pm\sqrt{6}$$

It's nonsensical to have a negative length, so the only critical point is $l = \sqrt{6}$. Use the first derivative test with test values of $l = 2$ and $l = 3$ to confirm that the critical point represents a maximum.

$$V'(2) = 9 - \frac{3}{2}(2)^2$$

$$V'(2) = 9 - 6$$

$$V'(2) = 3$$

and

$$V'(3) = 9 - \frac{3}{2}(3)^2$$

$$V'(3) = \frac{18}{2} - \frac{27}{2}$$

$$V'(3) = -\frac{9}{2}$$

Since we get a positive value to the left of the critical point and a negative value to the right of it, the function has a maximum at the critical point.

We found the length l associated with the critical point, but we were asked for the maximum volume, so now we just need to find the volume that corresponds with this length, which we'll do by plugging $l = \sqrt{6}$ into $V = 9l - (1/2)l^3$.

$$V = 9\sqrt{6} - \frac{1}{2}(\sqrt{6})^3$$

$$V = 9\sqrt{6} - \frac{6}{2}\sqrt{6}$$

$$V = 9\sqrt{6} - 3\sqrt{6}$$

$$V = 6\sqrt{6}$$

The maximum volume of the box is $6\sqrt{6}$ m^3.

Topic: Mean Value Theorem

Question: Which is the correct statement of the Mean Value Theorem?

Answer choices:

A If f is continuous and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

B If f is continuous on the closed interval $[a, b]$, then there exists a number c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

C If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

D If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Solution: D

The Mean Value Theorem states:

If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

Topic: Mean Value Theorem

Question: Which statement is true?

Answer choices:

A The Mean Value Theorem applies to functions that are continuous and differentiable on a given interval (a, b) and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(a - b)$.

B The Mean Value Theorem applies to functions that are discontinuous and differentiable on a given interval $[a, b]$ and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(b - a)$.

C The Mean Value Theorem applies to functions that are continuous on a given interval $[a, b]$ and differentiable on a given interval (a, b) and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(b - a)$.

D The Mean Value Theorem applies to functions that are discontinuous and differentiable on a given interval $[a, b]$ and states that there will be a point c in the interval such that $f(b) - f(a) = f'(c)(a - b)$.

Solution: C

The Mean Value Theorem applies to functions that are continuous on a given interval $[a, b]$ and differentiable on a given interval (a, b) and states that there will be a point c in the interval such that

$$f(b) - f(a) = f'(c)(b - a)$$

Topic: Mean Value Theorem

Question: Two police officers are sitting separately along a highway, 3 miles apart. A car passes the first officer at 60 mph, and passes the second officer two minutes later at 58 mph. Can the officers prove that the car was speeding (going faster than 65 mph) at some point between them?

Answer choices:

A Yes, they can prove the car was speeding

B No, they can't prove the car was speeding

C It's impossible to say one way or the other

Solution: A

The officers can use the Mean Value Theorem to prove that the car was traveling faster than the 65 mph speed limit at some point between them.

Let time $t = 0$ and position $s = 0$ when the car passes the first officer, and let $t = 2/60 = 1/30$ hr (after converting minutes to hours) and $s(1/30) = 3$ miles. Then the average speed of the car over the 3 miles (or 2 minutes) is

$$v_{avg} = \frac{s\left(\frac{1}{30}\right) - s(0)}{\frac{1}{30} - 0}$$

$$v_{avg} = \frac{3 - 0}{\frac{1}{30}}$$

$$v_{avg} = 3 \cdot \frac{30}{1}$$

$$v_{avg} = 90 \text{ mph}$$

By the Mean Value Theorem, the car must have been traveling at 90 mph at some point along the 3-mile stretch.

Topic: Rolle's Theorem

Question: Which of these is not part of Rolle's Theorem?

Answer choices:

A The function $f(x)$ must be continuous on the closed interval $[a, b]$.

B That the function $f(x)$ meets the condition $f(a) = f(b)$ for the closed interval $[a, b]$.

C The function $f(x)$ can be integrated over the open interval (a, b).

D The function $f(x)$ is differentiable on the open interval (a, b).

Solution: C

Rolle's Theorem requires three conditions be met in order for its conclusion to be true:

- The function $f(x)$ must be continuous on the closed interval $[a, b]$
- The function $f(x)$ must be differentiable on the open interval (a, b)
- The function $f(x)$ meets the condition $f(a) = f(b)$ for the interval $[a, b]$

If these three conditions are met, Rolle's Theorem states that there must exist a point c within the interval (a, b) where $f'(c) = 0$.

Topic: Rolle's Theorem

Question: Does the function meet the criteria of Rolle's Theorem on the interval $[0,1]$?

$$f(x) = x^2 - x + 6$$

Answer choices:

A Yes, it's continuous and differentiable over the interval, and $f(0) = f(1)$.

B Yes, it's continuous and $f(0) \neq f(1)$.

C No, it's not differentiable over the interval.

D No, it's discontinuous, and $f(0) \neq f(1)$.

Solution: A

Since this is a polynomial function, and we know that polynomial functions are continuous for all real numbers, we know that the function is continuous and differentiable on the interval $[0,1]$.

Confirm that $f(0) = f(1)$.

$$f(0) = 0^2 - 0 + 6$$

$$f(0) = 6$$

and

$$f(1) = 1^2 - 1 + 6$$

$$f(1) = 6$$

Since $f(0) = 6 = f(1)$, we've confirmed that this function over the given interval meets all three conditions of Rolle's Theorem.

Topic: Rolle's Theorem

Question: Use Rolle's Theorem to find the point in the interval $[0,4]$ where the function has a horizontal tangent line.

$$f(x) = -x^2 + 4x + 16$$

Answer choices:

A $(0,4)$

B $(-2,20)$

C $(0,-4)$

D $(2,20)$

Solution: D

Since this is a polynomial function, and we know that polynomial functions are continuous for all real numbers, we know that the function is continuous and differentiable on the interval $[0,4]$. Evaluating the function at the endpoints of the interval, we get

$$f(0) = -0^2 + 4(0) + 16$$

$$f(0) = 16$$

and

$$f(4) = -4^2 + 4(4) + 16$$

$$f(4) = 16$$

Since $f(0) = 16 = f(4)$, we've confirmed that the function over the given interval meets all three conditions of Rolle's Theorem.

Now we can find the point c by solving the equation $f'(c) = 0$.

$$f'(x) = -2x + 4$$

$$-2c + 4 = 0$$

$$2c = 4$$

$$c = 2$$

To find the coordinate point associated with $c = 2$, we'll plug it back into the original function.

$$f(2) = -2^2 + 4(2) + 16$$

$$f(2) = -4 + 8 + 16$$

$$f(2) = 20$$

The conclusion of Rolle's Theorem tells us that the function has a horizontal tangent line at $(2,20)$ inside the interval $[0,4]$.

Topic: Newton's Method

Question: Use Newton's Method to approximate to three decimal places the root of the function on the interval $[1/2,2]$.

$$f(x) = x^2 - 1$$

Answer choices:

A $\quad x = 1.500$

B $\quad x = 1.000$

C $\quad x = 2.000$

D $\quad x = 0.500$

Solution: B

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^2 - 1 = 0$$

Take the derivative of the function.

$$f(x_n) = x_n^2 - 1$$

$$f'(x_n) = 2x_n$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 1}{2x_n}$$

Since we know the interval where the function has a solution, then we can use the midpoint of the interval as x_0, $x_0 = (2 + 1/2)/2 = 5/4$, and work our problem with the number of decimals we were asked for.

$$x_0 = 1.25$$

$$x_1 = 1.250 - \frac{1.250^2 - 1}{2(1.250)} = 1.025$$

$$x_2 = 1.025 - \frac{1.025^2 - 1}{2(1.025)} = 1.000$$

$$x_3 = 1.000 - \frac{1.000^2 - 1}{2(1.000)} = 1.000$$

Since these last two approximations were identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 1.000$.

Topic: Newton's Method

Question: Use Newton’s Method to approximate to three decimal places the root of the function on the interval $[1,2]$.

$$f(x) = 2x^2 - 3$$

Answer choices:

A $x = 1.525$

B $x = 1.255$

C $x = 1.522$

D $x = 1.225$

Solution: D

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$2x^2 - 3 = 0$$

Take the derivative of the function.

$$f(x_n) = 2x_n^2 - 3$$

$$f'(x_n) = 4x_n$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{2x_n^2 - 3}{4x_n}$$

Let's start with the left endpoint of the interval, $x_n = 1$, and work our problem with the number of decimals we were asked for.

$$x_0 = 1.000$$

$$x_1 = 1.000 - \frac{2(1.000)^2 - 3}{4(1.000)} = 1.250$$

$$x_2 = 1.250 - \frac{2(1.250)^2 - 3}{4(1.250)} = 1.225$$

$$x_3 = 1.225 - \frac{2(1.225)^2 - 3}{4(1.225)} = 1.225$$

Since these last two approximations were identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 1.225$.

Topic: Newton's Method

Question: Use Newton’s Method to approximate to three decimal places the root of the function on the interval $[3,4]$.

$$f(x) = x^2 - 3x - 1$$

Answer choices:

A $x = 3.303$

B $x = 3.322$

C $x = 3.032$

D $x = 3.332$

Solution: A

When we use Newton's Method, the function must be in the form $f(x) = 0$.

$$x^2 - 3x - 1 = 0$$

Take the derivative of the function.

$$f(x_n) = x_n^2 - 3x_n - 1$$

$$f'(x_n) = 2x_n - 3$$

Then the Newton's Method formula will be

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 3x_n - 1}{2x_n - 3}$$

Let's start with the left endpoint of the interval, $x_n = 3$, and work our problem with the number of decimals we were asked for.

$$x_0 = 3.000$$

$$x_1 = 3.000 - \frac{(3.000)^2 - 3(3.000) - 1}{2(3.000) - 3} = 3.333$$

$$x_2 = 3.333 - \frac{(3.333)^2 - 3(3.333) - 1}{2(3.333) - 3} = 3.303$$

$$x_3 = 3.303 - \frac{(3.303)^2 - 3(3.303) - 1}{2(3.303) - 3} = 3.303$$

Since these last two approximations were identical to three decimal places, we can stop and conclude that an approximation of the root of the function in the given interval is $x = 3.303$.

Topic: L'Hospital's Rule

Question: Use L'Hospital's Rule to evaluate the limit.

$$\lim_{x \to \infty} \frac{e^{2x}}{x^3}$$

Answer choices:

A 0

B 1

C ∞

D $-\infty$

Solution: C

Since we get the indeterminate form ∞/∞ with direct substitution,

$$\lim_{x\to\infty}\frac{e^{2x}}{x^3}$$

$$\frac{\infty}{\infty}$$

we apply L'Hospital's Rule.

$$\lim_{x\to\infty}\frac{2e^{2x}}{3x^2}$$

$$\lim_{x\to\infty}\frac{4e^{2x}}{6x}$$

$$\lim_{x\to\infty}\frac{8e^{2x}}{6}$$

Now when we evaluate, we get ∞.

$$\frac{\infty}{6}$$

$$\infty$$

Since the last application of the rule allowed us to evaluate the limit by direct substitution without giving us an indeterminate form, we've found that the limit is ∞.

Topic: L'Hospital's Rule

Question: Use L’Hospital’s Rule to evaluate the limit.

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

Answer choices:

A 0

B 1

C ∞

D $-\infty$

Solution: B

Since we get the indeterminate form $0/0$ with direct substitution, but we can't eliminate the zero in the denominator by factoring,

$$\lim_{x \to 1} \frac{\ln x}{x - 1}$$

$$\lim_{x \to 1} \frac{\ln 0}{-1}$$

we apply L'Hospital's Rule.

$$\lim_{x \to 1} \frac{\frac{1}{x}}{1}$$

$$\lim_{x \to 1} \frac{1}{x}$$

We get $1/1 = 1$ when we evaluate the limit. Since the last application of the rule allowed us to evaluate the limit by direct substitution without giving us an indeterminate form, we've found that the limit is 1.

Topic: L'Hospital's Rule

Question: Use L’Hospital’s rule to evaluate the limit.

$$\lim_{x \to 4} (5 - x)^{\frac{1}{4 - x}}$$

Answer choices:

A 0

B 1

C ∞

D e

Solution: D

If we try substitution to evaluate at $x = 4$, we get an indeterminate form.

$$(5-4)^{\frac{1}{4-4}}$$

$$1^{\infty}$$

Because we get an indeterminate form, we want to use L'Hospital's Rule. But before we do, we need to get the fraction by itself. So we'll set the limit equal to y,

$$y = \lim_{x \to 4}(5-x)^{\frac{1}{4-x}}$$

and then take the natural log of both sides.

$$\ln y = \lim_{x \to 4} \ln\left((5-x)^{\frac{1}{4-x}}\right)$$

$$\ln y = \lim_{x \to 4} \frac{1}{4-x} \ln(5-x)$$

$$\ln y = \lim_{x \to 4} \frac{\ln(5-x)}{4-x}$$

With the limit rewritten, we'll apply L'Hospital's rule to the fraction.

$$\ln y = \lim_{x \to 4} \frac{\frac{1}{5-x}(-1)}{-1}$$

$$\ln y = \lim_{x \to 4} \frac{-\frac{1}{5-x}}{-1}$$

$$\ln y = \lim_{x \to 4} \frac{1}{5 - x}$$

Evaluate the limit,

$$\ln y = \frac{1}{5 - 4}$$

$$\ln y = \frac{1}{1}$$

$$\ln y = 1$$

then raise both sides to the base e to solve for y.

$$e^{\ln y} = e^1$$

$$y = e$$

Remember earlier that we set the limit equal to y,

$$y = \lim_{x \to 4} (5 - x)^{\frac{1}{4 - x}}$$

so because we now have two values both equal to y, we can set those values equal to each other.

$$\lim_{x \to 4} (5 - x)^{\frac{1}{4 - x}} = e$$

Topic: Position, velocity, and acceleration

Question: Find the velocity function.

$$x(t) = 4t^2 - 6t + 2$$

Answer choices:

A $v(t) = 8t - 4$

B $v(t) = 8t + 6$

C $v(t) = 4t - 6$

D $v(t) = 8t - 6$

Solution: D

Take the derivative of the position function.

$$x(t) = 4t^2 - 6t + 2$$

$$x'(t) = 8t - 6$$

Velocity is the derivative of position.

$$v(t) = 8t - 6$$

Topic: Position, velocity, and acceleration

Question: Find the position of a car when its velocity is zero.

$$x(t) = 4t^2 - 8t + 10$$

Answer choices:

A $x = 6$

B $x = 7$

C $x = 10$

D $x = 0$

Solution: A

Take the derivative of the position function.

$$x(t) = 4t^2 - 8t + 10$$

$$x'(t) = 8t - 8$$

Velocity is the derivative of position.

$$v(t) = 8t - 8$$

We need to find time when velocity is 0.

$$8t - 8 = 0$$

$$8t = 8$$

$$t = 1$$

Velocity is 0 when $t = 1$. To find position at the same time, substitute $t = 1$ into the position function.

$$x(1) = 4(1)^2 - 8(1) + 10$$

$$x(1) = 4 - 8 + 10$$

$$x(1) = 6$$

Topic: Position, velocity, and acceleration

Question: Use the position function to find the velocity of a rocket at $t = 4$.

$$x(t) = 6t^3 - t^2 + 3t - 9$$

Answer choices:

A $v(4) = 238$

B $v(4) = 371$

C $v(4) = 283$

D $v(4) = 317$

Solution: C

Take the derivative of the position function.

$$x(t) = 6t^3 - t^2 + 3t - 9$$

$$x'(t) = 18t^2 - 2t + 3$$

Velocity is the derivative of position.

$$v(t) = 18t^2 - 2t + 3$$

We need to find velocity when $t = 4$, so we'll plug $t = 4$ into the velocity function we just found.

$$v(4) = 18(4)^2 - 2(4) + 3$$

$$v(4) = 288 - 8 + 3$$

$$v(4) = 283$$

Topic: Ball thrown up from the ground

Question: A ball's thrown straight up from the ground with initial velocity $v_0 = 64$ ft/s. What is the ball's maximum height, and what is its velocity when it hits the ground?

Answer choices:

A Maximum height is 32 ft; Velocity is -32 ft/s

B Maximum height is 32 ft; Velocity is -64 ft/s

C Maximum height is 64 ft; Velocity is -64 ft/s

D Maximum height is 64 ft; Velocity is -32 ft/s

Solution: C

Substitute $g = 32$ ft/s², $v_0 = 64$ ft/s, and $y_0 = 0$ into the vertical motion formula.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(32)t^2 + (64)t + 0$$

$$y(t) = -16t^2 + 64t$$

To find velocity when the ball hits the ground, set the position function equal to 0, since height is 0 when the ball hits the ground.

$$-16t^2 + 64t = 0$$

$$-16t(t - 4) = 0$$

$$t = 0,\ 4$$

We know that the height is 0 when the ball is initially thrown up from the ground at $t = 0$, which means it hits the ground again when $t = 4$.

To find velocity when the ball hits the ground at $t = 4$, we need to find the velocity function by taking the derivative of the position function.

$$y'(t) = -32t + 64$$

$$v(t) = -32t + 64$$

Substitute $t = 4$ to find velocity when the ball hits the ground.

$$v(4) = -32(4) + 64$$

$$v(4) = -128 + 64$$

$$v(4) = -64$$

The ball's velocity when it hits the ground is -64 ft/s.

The ball reaches its maximum height when $v(t) = 0$, so set the velocity function equal to 0.

$$-32t + 64 = 0$$

$$32t = 64$$

$$t = 2$$

The ball reaches maximum height at $t = 2$, so substitute $t = 2$ into the position function.

$$y(2) = -16(2)^2 + 64(2)$$

$$y(2) = -64 + 128$$

$$y(2) = 64$$

The ball's maximum height is 64 ft.

Topic: Ball thrown up from the ground

Question: A ball's thrown straight up from the top of a 240 foot tall building with initial velocity $v_0 = 32$ ft/s. When will the ball reach a height of 256 feet?

Answer choices:

A After $t = 1$ s

B After $t = 2$ s

C After $t = 4$ s

D The ball never reaches a height of 256 feet

Solution: A

Substitute $g=32$ ft/s², $v_0=32$ ft/s, and $y_0=240$ into the vertical motion formula.

$$y(t)=-\frac{1}{2}gt^2+v_0t+y_0$$

$$y(t)=-\frac{1}{2}(32)t^2+(32)t+240$$

$$y(t)=-16t^2+32t+240$$

To find the time at which the ball reaches a height of 256 feet, set the position function equal to 256.

$$-16t^2+32t+240=256$$

$$-16t^2+32t-16=0$$

$$-16(t^2-2t+1)=0$$

$$-16(t-1)^2=0$$

$$t=1$$

Topic: Ball thrown up from the ground

Question: An apple is thrown straight up from the ground with an initial velocity of 100 m/s. Assuming constant gravity, find the apple's maximum height.

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

Answer choices:

A $\quad 520.1$ m

B $\quad 512.0$ m

C $\quad 51.02$ m

D $\quad 510.2$ m

Solution: D

Substitute $g = 9.8$ m/s², $v_0 = 100$ m/s, and $y_0 = 0$ into the vertical motion formula.

$$s(t) = -\frac{1}{2}gt^2 + v_0 t + y_0$$

$$s(t) = -\frac{1}{2}(9.8)t^2 + 100t + 0$$

$$s(t) = -4.9t^2 + 100t$$

Take the derivative of the position function.

$$s'(t) = -9.8t + 100$$

$$v(t) = -9.8t + 100$$

The apple reaches its maximum height when $v(t) = 0$, so set the velocity function equal to 0.

$$-9.8t + 100 = 0$$

$$9.8t = 100$$

$$t = 10.2$$

The apple reaches maximum height at $t = 10.2$, so substitute $t = 10.2$ into the position function.

$$s(10.2) = -4.9(10.2)^2 + 100(10.2)$$

$$s(10.2) \approx -509.8 + 1{,}020$$

$$s(10.2) \approx 510.2$$

The apple's maximum height is about 510.2 m.

Topic: Coin dropped from the roof

Question: A pumpkin is dropped from the top of a building and falls 5 m to the ground. Find instantaneous velocity at $t = 0.5$ seconds.

Answer choices:

A -29.4 m/s

B -39.1 m/s

C -4.9 m/s

D -9.8 m/s

Solution: C

Plugging everything we know into the formula for standard projectile motion, we get

$$s(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$s(t) = -\frac{1}{2}(9.8)t^2 + 0t + 5$$

$$s(t) = -4.9t^2 + 5$$

Take the derivative of the position function to get the velocity function.

$$s'(t) = v(t) = -9.8t$$

Substitute $t = 0.5$ to find instantaneous velocity at that time.

$$v(0.5) = -9.8(0.5)$$

$$v(0.5) = -4.9$$

The instantaneous velocity at $t = 0.5$ is -4.9 m/s. Because the velocity is negative, it means that the pumpkin is falling toward the ground.

Topic: Coin dropped from the roof

Question: A baseball is dropped from the top of a bridge that's 8 m high. Find its average velocity during the first second of its fall.

Answer choices:

A $\quad -4.9$ m/s

B $\quad -39.2$ m/s

C $\quad -15.6$ m/s

D $\quad -31.2$ m/s

Solution: A

Plugging everything we know into the formula for standard projectile motion, we get

$$s(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$s(t) = -\frac{1}{2}(9.8)t^2 + 0t + 8$$

$$s(t) = -4.9t^2 + 8$$

Substitute $t_1 = 0$ and $t_2 = 1$ into the formula for average velocity.

$$v_{avg} = \frac{s(t_2) - s(t_1)}{t_2 - t_1}$$

$$v_{avg} = \frac{s(1) - s(0)}{1 - 0}$$

$$v_{avg} = s(1) - s(0)$$

Find $s(0)$ and $s(1)$.

$$s(0) = -4.9(0)^2 + 8$$

$$s(0) = 8$$

and

$$s(1) = -4.9(1)^2 + 8$$

$$s(1) = 3.1$$

Substitute these values into the average velocity equation.

$$v_{avg} = \frac{3.1 - 8}{1 - 0}$$

$$v_{avg} = -4.9$$

$$v_{avg} = -4.9 \text{ m/s}$$

Topic: Coin dropped from the roof

Question: A coin is dropped from the roof of a 400 ft building with an initial velocity of -64 ft/s. When does it hit the ground and what is the velocity at that time?

Answer choices:

A The coin hits the ground after 7.78 s at -108.48 ft/s

B The coin hits the ground after 3.39 s at -108.48 ft/s

C The coin hits the ground after 3.39 s at -172.48 ft/s

D The coin hits the ground after 7.78 s at -172.48 ft/s

Solution: C

Substitute $g = 32$ ft/s², $v_0 = -64$ ft/s, and $y_0 = 400$ into the vertical motion formula.

$$y(t) = -\frac{1}{2}gt^2 + v_0t + y_0$$

$$y(t) = -\frac{1}{2}(32)t^2 + (-64)t + 400$$

$$y(t) = -16t^2 - 64t + 400$$

$$y(t) = -16(t^2 + 4t - 25)$$

To find velocity when the coin hits the ground, set the position function equal to 0, since height is 0 when the coin hits the ground.

$$-16(t^2 + 4t - 25) = 0$$

$$t^2 + 4t - 25 = 0$$

Use the quadratic formula to find the roots of the function.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-4 \pm \sqrt{4^2 - 4(1)(-25)}}{2(1)}$$

$$t = \frac{-4 \pm \sqrt{16 + 100}}{2}$$

$$t = \frac{-4 \pm 2\sqrt{29}}{2}$$

$$t = -2 \pm \sqrt{29}$$

$$t \approx -7.39, 3.39$$

A negative value for time is nonsensical, which means the coin hits the ground when $t \approx 3.39$.

To find velocity when the coin hits the ground at $t \approx 3.39$, we need to find the velocity function by taking the derivative of the position function.

$$y'(t) = -32t - 64$$

$$v(t) = -32t - 64$$

Substitute $t \approx 3.39$ to find velocity when the ball hits the ground.

$$v(3.39) = -32(3.39) - 64$$

$$v(3.39) = -108.48 - 64$$

$$v(3.39) = -172.48$$

The coin's velocity when it hits the ground is -172.48 ft/s.

Topic: Marginal cost, revenue, and profit

Question: The cost function C models the weekly expenses of a balloon company. What is the company's weekly marginal cost?

$$C(x) = 1.5x + 300$$

Answer choices:

A $\$1.00$

B $\$1.50$

C $\$1.05$

D $\$300$

Solution: B

We can find marginal cost by taking the derivative of the cost formula.

$$C(x) = 1.5x + 300$$

$$C'(x) = 1.5$$

The balloon company's weekly marginal cost is $\$1.50$.

Topic: Marginal cost, revenue, and profit

Question: If a candy company's weekly revenue is modeled by R, how many units should they sell in order to maximize weekly revenue?

$$R(x) = -0.52x^2 + 12x$$

Answer choices:

A 12

B 44

C 23

D 32

Solution: A

To find the marginal revenue function R', take the derivative of R.

$$R(x) = -0.52x^2 + 12x$$

$$R'(x) = -1.04x + 12$$

Set marginal revenue equal to 0, then solve for x.

$$-1.04x + 12 = 0$$

$$1.04x = 12$$

$$x = 11.5$$

Since we can't sell a partial unit, we'll round to $x = 12$. The candy company needs to sell 12 units in order to maximize weekly revenue.

Topic: Marginal cost, revenue, and profit

Question: The cell phone store has monthly costs described by $C(x) = 22.5x + 675$ and monthly revenue described by $R(x) = 0.89x^2 - 22x$. What's their marginal profit if they sell $1,000$ units this month?

Answer choices:

A $\$1,753.50$

B $\$844,825.00$

C $\$1,735.50$

D $\$846,175.00$

Solution: C

Create a profit equation by subtracting costs from revenue.

$$P(x) = 0.89x^2 - 22x - (22.5x + 675)$$

$$P(x) = 0.89x^2 - 22x - 22.5x - 675$$

$$P(x) = 0.89x^2 - 44.5x - 675$$

To find the marginal profit function, take the derivative of the profit function.

$$P'(x) = 1.78x - 44.5$$

The marginal profit when the stores sells $1{,}000$ units is therefore

$$P'(1{,}000) = 1.78(1{,}000) - 44.5$$

$$P'(1{,}000) = 1{,}735.50$$

Topic: Half life

Question: Americium-241 has a half-life of 432 years. Find the decay constant.

Answer choices:

A 0.160

B 0.0160

C 0.00160

D 0.000160

Solution: C

The half life equation is

$$\frac{1}{2} = e^{-kt}$$

Solve this for the decay constant k.

$$\ln\frac{1}{2} = \ln e^{-kt}$$

$$\ln\frac{1}{2} = -kt$$

$$k = -\frac{\ln\frac{1}{2}}{t}$$

Use laws of logarithms to rewrite the log.

$$k = -\frac{\ln 1 - \ln 2}{t}$$

$$k = -\frac{0 - \ln 2}{t}$$

$$k = \frac{\ln 2}{t}$$

Substitute $t = 432$.

$$k = \frac{\ln 2}{432}$$

$$k \approx 0.00160$$

Topic: Half life

Question: Carbon-19 has a decay constant of 0.000121. Find its half life.

Answer choices:

A $5,782$ years

B $5,728$ years

C $5,278$ years

D $5,827$ years

Solution: B

The half life equation is

$$\frac{1}{2} = e^{-kt}$$

Solve this for time t.

$$\ln\frac{1}{2} = \ln e^{-kt}$$

$$\ln\frac{1}{2} = -kt$$

$$t = -\frac{1}{k}\ln\frac{1}{2}$$

Use laws of logarithms to rewrite the log.

$$t = -\frac{1}{k}(\ln 1 - \ln 2)$$

$$t = -\frac{1}{k}(0 - \ln 2)$$

$$t = \frac{\ln 2}{k}$$

Substitute the decay constant $k = 0.000121$.

$$t = \frac{\ln 2}{0.000121}$$

$$t \approx 5{,}728$$

Topic: Half life

Question: Plutonium-239 has a half-life of $24{,}110$ years. Find the decay constant.

Answer choices:

A 0.0000287

B 0.000287

C 0.000000287

D 0.00000287

Solution: A

The half life equation is

$$\frac{1}{2} = e^{-kt}$$

Solve this for the decay constant k.

$$\ln\frac{1}{2} = \ln e^{-kt}$$

$$\ln\frac{1}{2} = -kt$$

$$k = -\frac{\ln\frac{1}{2}}{t}$$

Use laws of logarithms to rewrite the log.

$$k = -\frac{\ln 1 - \ln 2}{t}$$

$$k = -\frac{0 - \ln 2}{t}$$

$$k = \frac{\ln 2}{t}$$

Substitute $t = 24{,}110$.

$$k = \frac{\ln 2}{24{,}110}$$

$$k \approx 0.0000287$$

Topic: Newton's Law of Cooling

Question: The function T models the temperature (in Celsius) of a cooling object. What is the starting temperature of the object?

$$T(t) = 14e^{-5t}$$

Answer choices:

A $5^\circ\ C$

B $14^\circ\ C$

C $8^\circ\ C$

D $7^\circ\ C$

Solution: B

The starting temperature of the object is given by T when $t = 0$. So substitute $t = 0$ into the temperature function.

$$T(t) = 14e^{-5t}$$

$$T(0) = 14e^{-5(0)}$$

$$T(0) = 14e^{0}$$

$$T(0) = 14(1)$$

$$T(0) = 14$$

Topic: Newton's Law of Cooling

Question: The function T models the temperature (in Celsius) of a cooling object. What is the approximate temperature of the object after 1 hour?

$$T(t) = 8e^{-t}$$

Answer choices:

A $0^\circ\ C$

B $1^\circ\ C$

C $8^\circ\ C$

D $3^\circ\ C$

Solution: D

The temperature of the object after 1 hour is given by T when $t = 1$. So substitute $t = 1$ into the temperature function.

$$T(t) = 8e^{-t}$$

$$T(1) = 8e^{-1}$$

$$T(1) \approx 8(0.37)$$

$$T(1) \approx 3$$

Topic: Newton's Law of Cooling

Question: The function T models the temperature (in Celsius) of a cooling object. How many hours does it take to cool the object to 100°?

$$T(t) = 124e^{-0.6t}$$

Answer choices:

A 0.35 hours

B 0.70 hours

C 3.5 hours

D 7.0 hours

Solution: A

The time it takes to cool the object to 100° is given by T when $T=100$. So substitute $T=100$ into the temperature function.

$$T(t)=124e^{-0.6t}$$

$$100=124e^{-0.6t}$$

$$0.81=e^{-0.6t}$$

Apply the natural logarithm to both sides.

$$\ln 0.81=\ln e^{-0.6t}$$

$$-0.21=-0.6t$$

$$t=\frac{-0.21}{-0.6}$$

$$t=0.35$$

Topic: Sales decline

Question: A t-shirt company noticed that their inventory of blue shirts was declining exponentially at a rate of 23% per year. They currently have 300 blue shirts in stock and they don't plan to purchase any more. How many blue shirts will they have in stock in 3 years?

Answer choices:

A 250

B 200

C 150

D 100

Solution: C

Both the decline and the time have units in years, so with matching units we can plug directly into the sales decline formula to find the number of blue shirts the company will have in stock in 3 years.

$$F = Pe^{-rt}$$

$$F = 300e^{-0.23(3)}$$

$$F = 300e^{-0.69}$$

$$F \approx 150$$

Topic: Sales decline

Question: A pet store noticed that sales of generic cat food was declining at an exponential rate of 8% per year. If they currently sell 600 bags of generic cat food in one year, how many years will it take before they're only selling 100 bags annually?

Answer choices:

A 22.4 years

B 24.4 years

C 24.2 years

D 42.4 years

Solution: A

Both the decline and the time have units in years, so with matching units we can plug directly into the sales decline formula to find the number of years until sales reach the level of 100 bags per year.

$$F = Pe^{-rt}$$

$$100 = 600e^{(-0.08)t}$$

$$\frac{1}{6} = e^{-0.08t}$$

Apply the natural logarithm to both sides.

$$\ln\frac{1}{6} = \ln(e^{-0.08t})$$

$$\ln\frac{1}{6} = -0.08t$$

$$t = \frac{\ln\frac{1}{6}}{-0.08}$$

$$t \approx 22.4$$

Topic: Sales decline

Question: Pixie stick sales are declining at a candy store. Two years ago, the store sold 450 pixie sticks, but this year they're only selling 150. Assuming that sales have declined exponentially, what's been the annual rate of decline?

Answer choices:

A $5\ \%$

B $5.50\ \%$

C $0.55\ \%$

D $55\ \%$

Solution: D

Both the decline and the time have units in years, so with matching units we can plug directly into the sales decline formula to find the rate of decline.

$$F = Pe^{-rt}$$

$$150 = 450e^{-r(2)}$$

$$\frac{1}{3} = e^{-2r}$$

Apply the natural logarithm to both sides.

$$\ln\frac{1}{3} = \ln(e^{-2r})$$

$$\ln\frac{1}{3} = -2r$$

$$r = \frac{\ln\frac{1}{3}}{-2}$$

$$r = 0.55$$

Topic: Compounding interest

Question: Four years ago, you owed $\$10$ on your credit card. Since then, you haven't made any payments. If the card carries an annual interest rate of $22\,\%$, compounded continuously, how much do you owe on the credit card today?

Answer choices:

A $\$21.14$

B $\$24.11$

C $\$21.41$

D $\$14.11$

Solution: B

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential growth formula to find the amount you currently owe on the credit card.

$$A = Pe^{rt}$$

$$A = 10e^{(0.22)(4)}$$

$$A = 24.11$$

Topic: Compounding interest

Question: You made an initial investment of $\$1{,}000$ at an annual interest rate of $4.5\,\%$, compounded continuously. If the investment is currently worth $\$5{,}632$, how many years have you had the investment?

Answer choices:

A 384 years

B $3{,}840$ years

C 38.4 years

D 3.84 years

Solution: C

Both the interest rate and time have units in years, so with matching units we can plug directly into the exponential growth formula to find the number of years we've held the investment.

$$A = Pe^{rt}$$

$$5{,}632 = 1{,}000e^{(0.045)t}$$

$$5.632 = e^{0.045t}$$

Apply the natural logarithm to both sides.

$$\ln 5.632 = \ln(e^{0.045t})$$

$$\ln 5.632 = 0.045t$$

$$t = \frac{\ln 5.632}{0.045}$$

$$t \approx 38.4$$

Topic: Compounding interest

Question: Twenty years ago, you purchased $\$5,000$ worth of stock. This stock paid an interest compounded monthly, and today your shares are worth $\$45,000$. What was the interest rate?

Answer choices:

A $111\,\%$

B $0.11\,\%$

C $1.10\,\%$

D $11.04\,\%$

Solution: D

Because interest is compounded monthly, $n=12$. Using the compound interest formula, plug in what we know.

$$A=P\left(1+\frac{r}{n}\right)^{nt}$$

$$45{,}000=5{,}000\left(1+\frac{r}{12}\right)^{12(20)}$$

$$9=\left(1+\frac{r}{12}\right)^{240}$$

$$\sqrt[240]{9}=\left(1+\frac{r}{12}\right)$$

$$1.0092=1+\frac{r}{12}$$

$$r=12(1.0092-1)$$

$$r=0.1104$$

The interest rate was 11.04%.

W W W . K R I S T A K I N G M A T H . C O M