Finding volume

We already know that we can use double integrals to find the volume below a function over some region $R = [a, b] \times [c, d]$.

We use the double integral formula

$$V = \int\int_D f(x, y)\ dA$$

to find volume, where D represents the region over which we're integrating, and $f(x, y)$ is the curve below which we want to find volume. We need to turn the double integral into an iterated integral by finding limits of integration for x and y.

Example

Find volume below the function over the region D.

$$z = 2xy$$

where D is the triangle bounded by the lines $y = 1$, $x = 1$, and $y = 3 - x$

The first thing we'll do is sketch the region D. It'll be easy if we solve for the intersection points of the three lines.

We'll find the intersection of $y = 1$ and $x = 1$.

Pairing $x = 1$ with $y = 1$, the intersection point is $(1,1)$.

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We'll find the intersection of $y = 1$ and $y = 3 - x$.

$$3 - x = 1$$

$$-x = -2$$

$$x = 2$$

Pairing $x = 2$ with $y = 1$, the intersection point is $(2,1)$.

We'll find the intersection of $x = 1$ and $y = 3 - x$.

$$y = 3 - x$$

$$y = 3 - 1$$

$$y = 2$$

Pairing $x = 1$ with $y = 2$, the intersection point is $(1,2)$.

If we plot these points and sketch the lines that connect them, we see the triangular region D.

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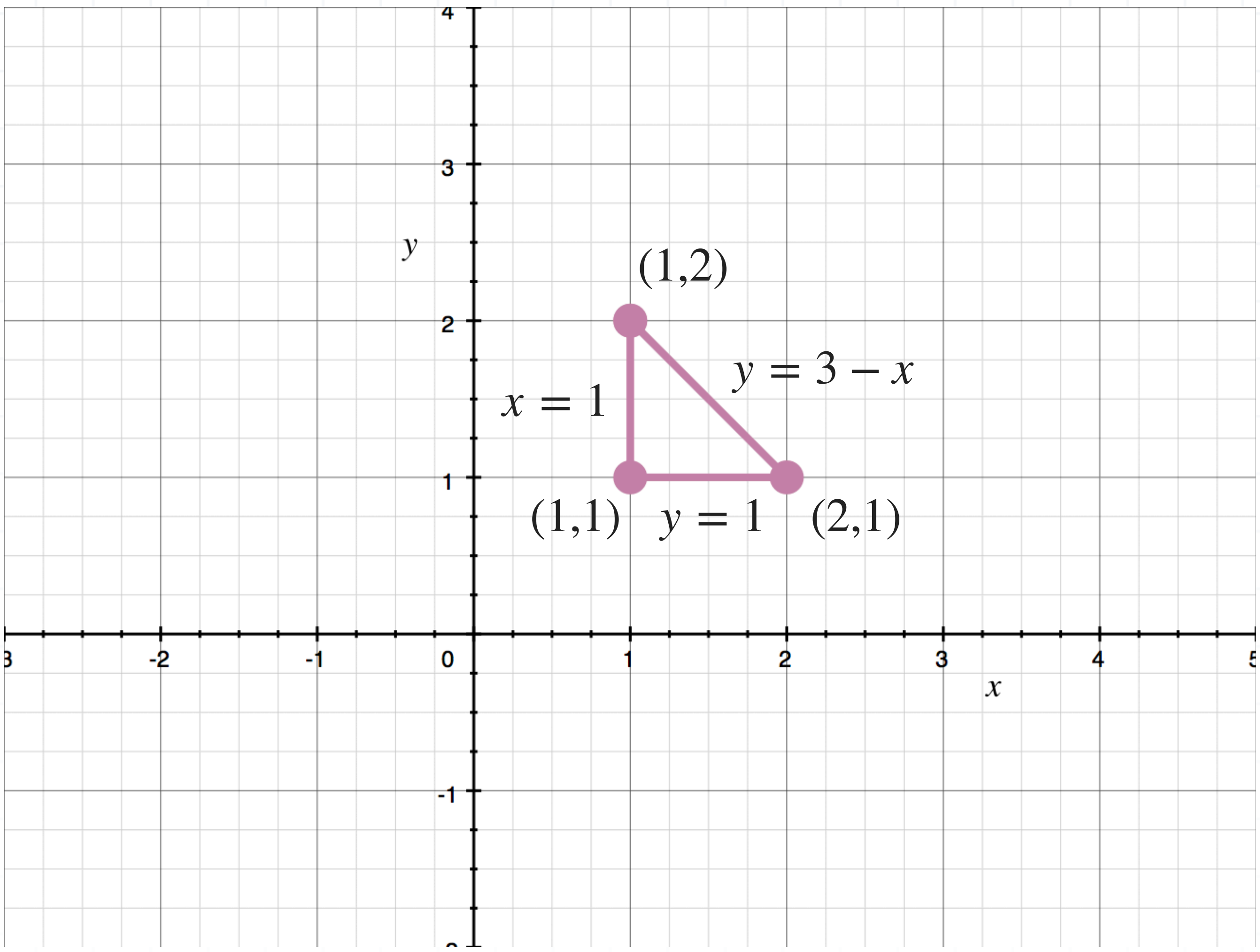

Since we only have one complex equation, $y = 3 - x$ and it's solved for y, we'll integrate with respect to y first, which means we'll treat this as a Type I integral, and so the inner integral will have limits of integration for y.

If we divide the triangular region D into vertical slices, the tops of those slices are defined by the line $y = 3 - x$, and the bottoms are defined by $y = 1$. Looking at the sketch of region D, we can see that x is defined on $[1,2]$. Therefore, we'll get

$$V = \int\int_D f(x, y)\ dA$$

$$V = \int_1^2 \int_1^{3-x} 2xy\ dy\ dx$$

We'll integrate first with respect to y.

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$$V=\int_{1}^{2} xy^2\Big|_{y=1}^{y=3-x}\ dx$$

$$V=\int_{1}^{2} x(3-x)^2-x(1)^2\ dx$$

$$V=\int_{1}^{2} x\left(9-6x+x^2\right)-x\ dx$$

$$V=\int_{1}^{2} 9x-6x^2+x^3-x\ dx$$

$$V=\int_{1}^{2} 8x-6x^2+x^3\ dx$$

Then we'll integrate with respect to x.

$$V=4x^2-2x^3+\frac{1}{4}x^4\Big|_{1}^{2}$$

$$V=4(2)^2-2(2)^3+\frac{1}{4}(2)^4-\left[4(1)^2-2(1)^3+\frac{1}{4}(1)^4\right]$$

$$V=16-16+4-\left(4-2+\frac{1}{4}\right)$$

$$V=2-\frac{1}{4}$$

$$V=\frac{7}{4}$$

We can say that the volume under the curve $z=2xy$ over the region D is 7/4.

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