Scalar triple product to prove vectors are coplanar

The scalar triple product $\left|a \cdot (b \times c)\right|$ of three vectors $a\langle a_1, a_2, a_3\rangle$, $b\langle b_1, b_2, b_3\rangle$ and $c\langle c_1, c_2, c_3\rangle$ will be equal to 0 when the vectors are coplanar, which means that the vectors all lie in the same plane.

a, b, and c are coplanar if $\left|a \cdot (b \times c)\right| = 0$

$b \times c$ is the cross product of b and c, and we'll find it using the 3×3 matrix

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \mathbf{i}\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - \mathbf{j}\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + \mathbf{k}\begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

$$= \mathbf{i}\left(b_2c_3 - b_3c_2\right) - \mathbf{j}\left(b_1c_3 - b_3c_1\right) + \mathbf{k}\left(b_1c_2 - b_2c_1\right)$$

We'll convert the result of the cross product into standard vector form, and then take the dot product of $a\langle a_1, a_2, a_3\rangle$ and the vector result of $b \times c$.

$$\left|a \cdot (b \times c)\right|$$

The final answer is the scalar triple product. If it's equal to 0, then we've proven that the vectors are coplanar.

Example

Prove that the vectors are coplanar.

$a\langle 3,3,-3\rangle$

$b\langle 1,0,-2\rangle$

$$c\langle 2,3,-1\rangle$$

We'll use the scalar triple product, and we'll start by calculating the cross product of b and c, $b \times c$.

$$b \times c = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 2 & 3 & -1 \end{vmatrix}$$

$$b \times c = \mathbf{i} \begin{vmatrix} 0 & -2 \\ 3 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & -2 \\ 2 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix}$$

$$b \times c = \mathbf{i}\left[(0)(-1)-(-2)(3)\right] - \mathbf{j}\left[(1)(-1)-(-2)(2)\right] + \mathbf{k}\left[(1)(3)-(0)(2)\right]$$

$$b \times c = \mathbf{i}(0+6) - \mathbf{j}(-1+4) + \mathbf{k}(3-0)$$

$$b \times c = 6\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

$$b \times c = \langle 6,-3,3\rangle$$

Next we'll take the dot product of $a\langle 3,3,-3\rangle$ and $b \times c = \langle 6,-3,3\rangle$.

$$\left|a \cdot (b \times c)\right| = (3)(6) + (3)(-3) + (-3)(3)$$

$$\left|a \cdot (b \times c)\right| = 18 - 9 - 9$$

$$\left|a \cdot (b \times c)\right| = 0$$

Since the scalar triple product of the vectors $a\langle 3,3,-3\rangle$, $b\langle 1,0,-2\rangle$ and $c\langle 2,3,-1\rangle$ is equal to 0,

$$\left|a \cdot (b \times c)\right| = 0$$

the vectors a, b, and c are coplanar.