Calculus 3 Workbook Solutions

Sketching graphs and level curves

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SKETCHING GRAPHS OF MULTIVARIABLE FUNCTIONS

■ 1. Find the range of the function.

$$f(x, y) = x^2 + 2y^2 - 3$$

Solution:

Since $x^2 + 2y^2 \geq 0$, then $x^2 + 2y^2 = 0$ when $x = y = 0$.

$$f(0,0) = 0^2 + 2(0)^2 - 3$$

$$f(0,0) = -3$$

Since $x^2 + 2y^2$ can be arbitrarily large, the range of the $f(x, y)$ is $[-3,\infty)$.

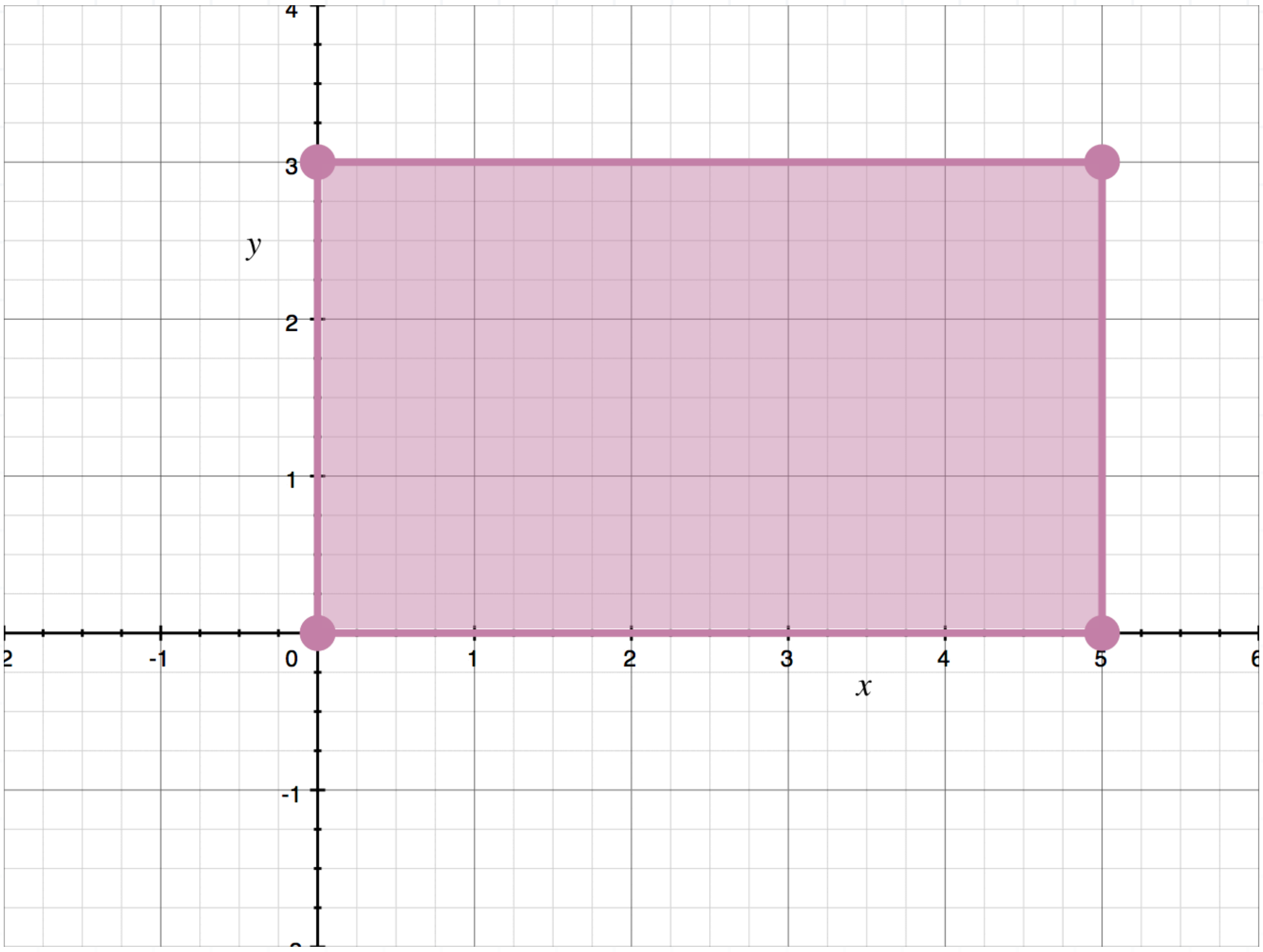
■ 2. Which function's domain is given by the graph, if the left and right sides of the rectangle are included in the domain, but the top and bottom sides are not?

A $\quad f(x, y) = 3y\sqrt{3x - x^2} + 4x\ln(5y - y^2)$

B $\quad f(x, y) = 3y\sqrt{5x - x^2} + 4x\ln(3y - y^2)$

C $\quad f(x, y) = 3x\sqrt{x^2 - 3x} + 4y\ln(y^2 - 5y)$

D $\quad f(x, y) = 3x\sqrt{x - 5x^2} + 4y\ln(y - 3y^2)$



Solution:

Because the left and right sides of the rectangle are included in the domain, but the top and bottom sides are not, the system of inequalities that describes the domain in the graph is $x \geq 0$, $x \leq 5$, $y > 0$, and $y < 3$.

The domain of the function in answer choice A is

$3x - x^2 \geq 0$ so $x(3 - x) \geq 0$, so $x \geq 0$ and $x \leq 3$

$5y - y^2 > 0$ so $y(5 - y) > 0$, so $y > 0$ and $y < 5$

The domain of the function in answer choice B is

$5x - x^2 \geq 0$ so $x(5 - x) \geq 0$, so $x \geq 0$ and $x \leq 5$

$3y - y^2 > 0$ so $y(3 - y) > 0$, so $y > 0$ and $y < 3$

The domain of the function in answer choice C is

$x^2 - 3x \geq 0$ so $x(x - 3) \geq 0$, so $x \geq 3$ and $x \leq 0$

$y^2 - 5y > 0$ so $y(y - 5) > 0$, so $y < 0$ and $y > 5$

The domain of the function in answer choice D is

$x - 5x^2 \geq 0$ so $x(1 - 5x) \geq 0$, so $x \geq 0$ and $x \leq 1/5$

$y - 3y^2 > 0$ so $y(1 - 3y) > 0$, so $y > 0$ and $y < 1/3$

The only matching domain comes from answer choice B.

■ 3. Find the value of the constant a for which $(2, -1, 0)$ lies on the graph of the function.

$$f(x, y) = x^2 + 2axy + y^2 - 1$$

Solution:

Substitute $x = 2$, $y = -1$, and $f(2, -1) = 0$ into the equation.

$$0 = 2^2 + 2a(2)(-1) + (-1)^2 - 1$$

$$0 = 4 - 4a + 1 - 1$$

$$4a = 4$$

$$a = 1$$

4. Find the intersection point of the function and the y-axis.

$$f(x, y) = \sqrt{x^2 - 5y + 15}$$

Solution:

At any intersection point with the y-axis, we know $x = z = 0$. Substitute $x = 0$ and $z = f(x, y) = 0$ into the function.

$$0 = \sqrt{0^2 - 5y + 15}$$

$$0 = \sqrt{-5y + 15}$$

$$-5y + 15 = 0$$

$$y = \frac{15}{5} = 3$$

So the function intersects the y-axis at $(0,3,0)$.

5. Write the equation of the function $f(x, y)$ shifted in a positive direction along the x-axis by 2 units.

$$f(x, y) = x^2y^2 - 2xy - 4y^2 - 4y + 4x$$

Solution:

Shifting a function in a positive direction along the x-axis by a is equivalent to replacing x with $(x-a)$, where a is a constant. So substitute $(x-2)$ for x into the equation, calling it a new function $g(x,y)$.

$$g(x,y)=(x-2)^2y^2-2(x-2)y-4y^2-4y+4(x-2)$$

$$g(x,y)=(x^2-4x+4)y^2-2(x-2)y-4y^2-4y+4(x-2)$$

$$g(x,y)=x^2y^2-4xy^2+4y^2-4y^2-2xy+4y-4y+4x-8$$

$$g(x,y)=x^2y^2-4xy^2-2xy+4x-8$$

■ 6. Which function A, B, C, or D is a reflection of $f(x,y)$ over the xz-plane? Hint: Use the even identity $\cos(-t)=\cos t$ to simplify.

$$f(x,y)=\cos(x^2-y^2+2xy)$$

$$A(x,y)=\cos(-x^2+y^2+2xy)$$

$$B(x,y)=\cos(x^2-y^2+2xy)$$

$$C(x,y)=\cos(x^2+y^2-2xy)$$

$$D(x,y)=\cos(-x^2-y^2-2xy)$$

Solution:

The reflection of a function over the xz-plane is equivalent to replacing y with $-y$. So substitute $-y$ for y into $f(x, y)$.

$$f(x,-y)=\cos(x^2-(-y)^2+2x(-y))$$

$$f(x,-y)=\cos(x^2-y^2-2xy)$$

$$f(x,-y)=\cos(-(y^2-x^2+2xy))$$

Since $\cos t$ is an even function, $\cos(-t)=\cos t$, so

$$f(x,-y)=\cos(y^2-x^2+2xy)$$

This matches

$$A(x,y)=\cos(-x^2+y^2+2xy)$$

■ 7. Find the absolute maximum of the function.

$$f(x,y)=5-2x^2-7y^2$$

Solution:

The absolute maximum is the highest point over the entire domain of a function. If (x_0, y_0) is the point on the xy-plane where the absolute maximum occurs, then

$$f(x_0,y_0)\geq f(x,y)$$

for any (x, y) in the domain. Since $-2x^2 - 7y^2 \leq 0$, then $-2x^2 - 7y^2 = 0$ when $x = y = 0$.

$$f(0,0) = 5 - 2(0)^2 - 7(0)^2$$

$$f(0,0) = 5$$

So the absolute maximum occurs at $(0,0)$ and has a value of 5.

SKETCHING LEVEL CURVES OF MULTIVARIABLE FUNCTIONS

■ 1. Find the level curve of $f(x, y)$ when $z = 5$.

$$f(x, y) = x^2 - 2xy + 6y - 4$$

Solution:

Substitute $f(x, y) = z = 5$ into the equation and then solve for x.

$$x^2 - 2xy + 6y - 4 = 5$$

$$x^2 - 2xy + 6y - 9 = 0$$

$$-2y(x - 3) + (x - 3)(x + 3) = 0$$

$$(x - 3)(-2y + x + 3) = 0$$

$x = 3$ or $x = 2y - 3$

So for $z = 5$, the level curve includes two lines:

$y = \dfrac{3 + x}{2}$ and $x = 3$

■ 2. Find the level curve of $f(x, y)$ which passes through $(0,1,z)$.

$$f(x, y) = 2x^2 - y + 2$$

Solution:

Substitute $f(x, y) = z$ into the equation, then solve the equation for y.

$$2x^2 - y + 2 = z$$

$$y = 2x^2 + 2 - z$$

Since this is a parabola that passes through $(0,1,z)$, substitute 0 for x and 1 for y, then solve for z.

$$1 = 2(0)^2 + 2 - z$$

$$1 = 0 + 2 - z$$

$$z = 2 - 1$$

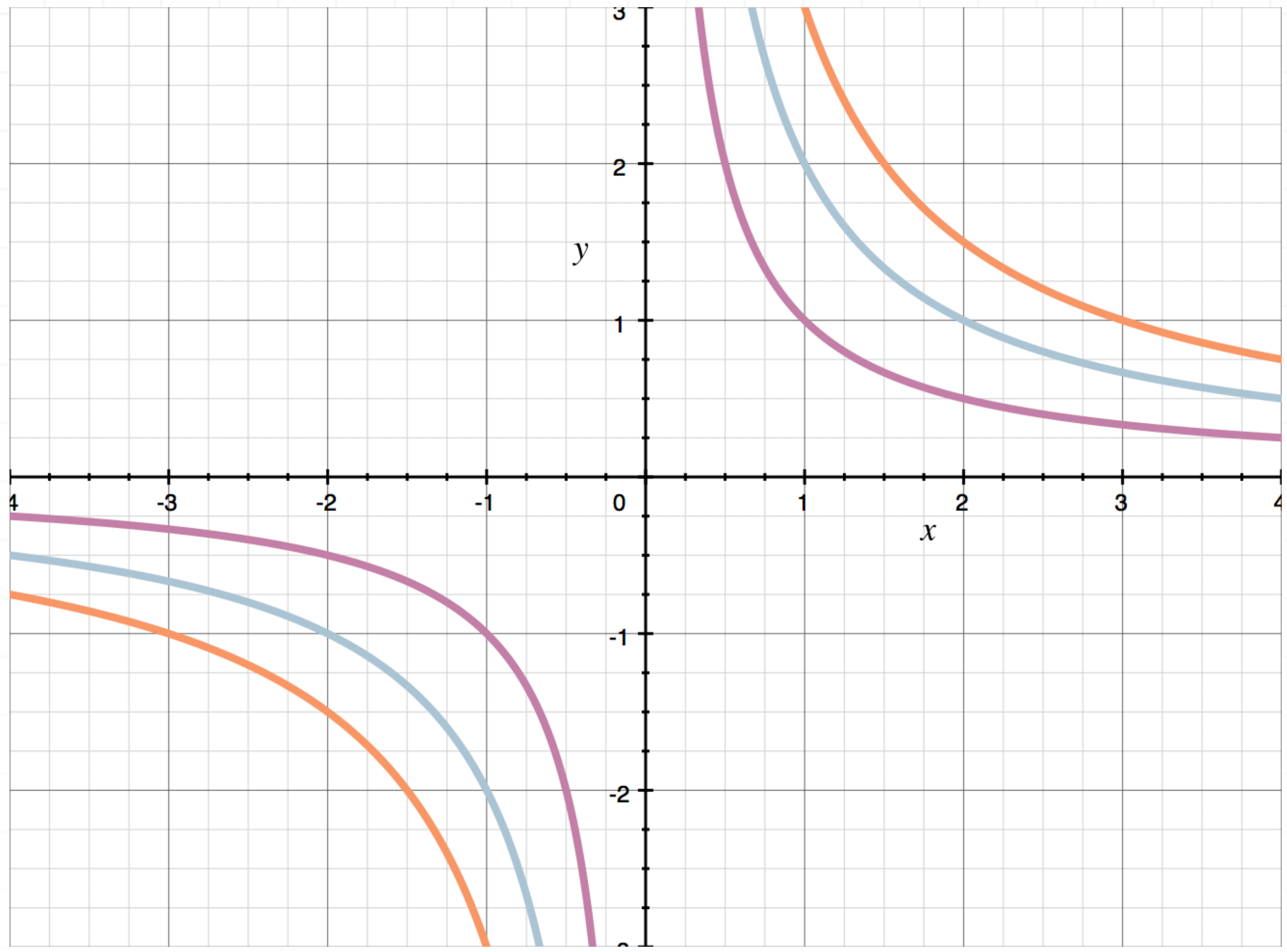
$$z = 1$$

Substitute $z = 1$ back into the level curve equation to get the equation of the level curve at $z = 1$.

$$y = 2x^2 + 2 - 1$$

$$y = 2x^2 + 1$$

■ 3. The graph shows level curves of $f(x, y) = 4xy$. Find the value of z that corresponds to the light blue curve.



Solution:

The light blue curve corresponds to the hyperbola $y = 2/x$, or $xy = 2$. So substitute the 2 for xy into the $f(x, y)$.

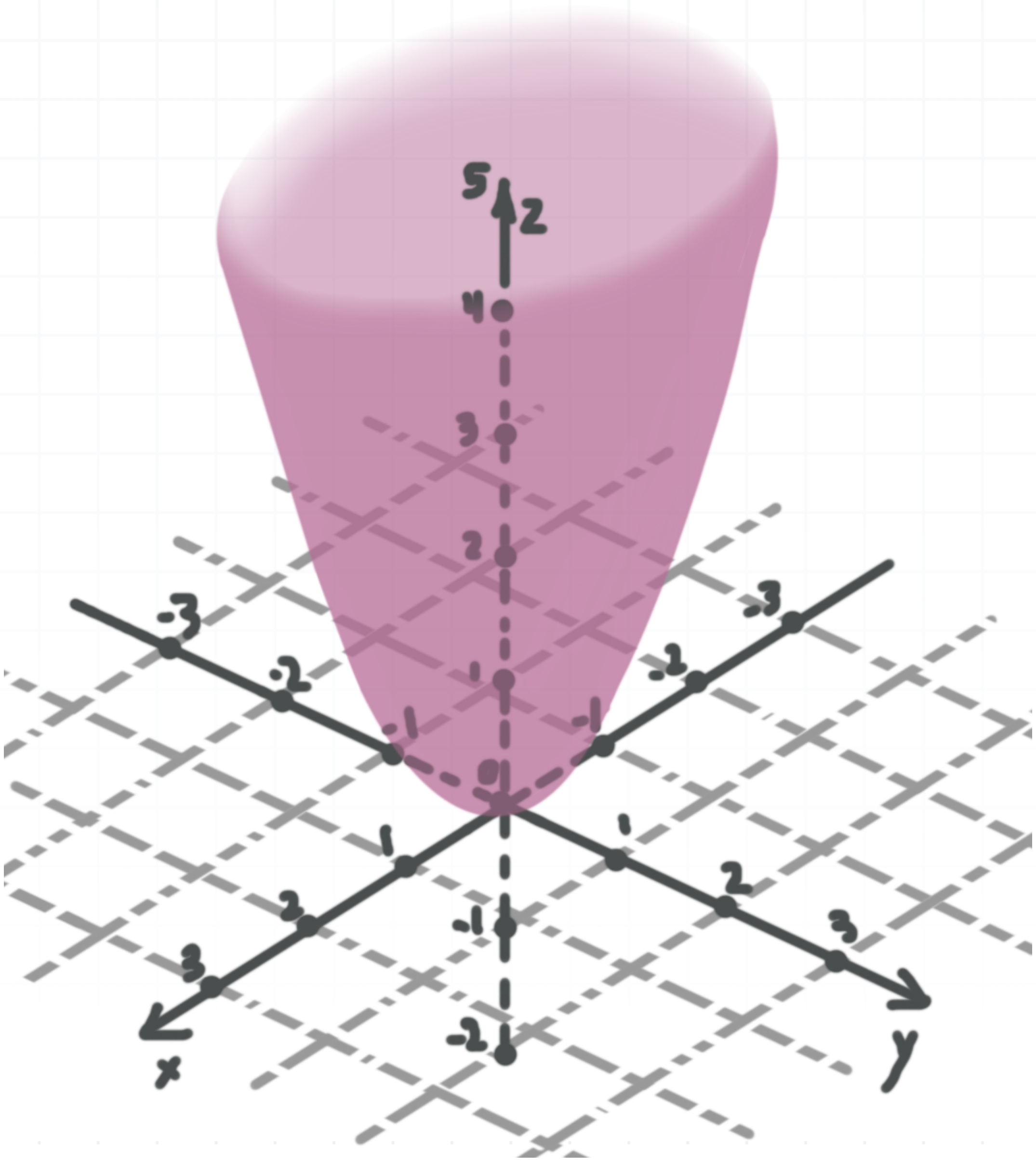
$$f(x, y) = 4(xy)$$

$$f(x, y) = 4(2)$$

$$f(x, y) = 8$$

So the light blue level curve corresponds to $z = 8$.

4. Think about the shape of the level curves of the graph of the elliptic paraboloid. Are they lines, ellipses, parabolas, or hyperbolas?



Solution:

A level curve is a set of points where the function takes a constant value of z. For the graph shown of the elliptic paraboloid, the level curves are ellipses.

MATCHING THE FUNCTION WITH THE GRAPH AND LEVEL CURVES

■ 1. Which statement is true for the graph of the function?

$$x^2 - 2y^2 + z^2 - 8y - 6z = 0$$

A The graph is the hyperboloid centered at $(0,2,-3)$.

B The graph is the hyperboloid centered at $(0,-2,3)$.

C The graph is the ellipsoid centered at $(0,2,-3)$.

D The graph is the ellipsoid centered at $(0,-2,3)$.

Solution:

Transform the equation into standard form by completing the square with respect to y and z.

$$x^2 - 2(y^2 + 4y + 4 - 4) + (z^2 - 6z + 9 - 9) = 0$$

$$x^2 - 2((y+2)^2 - 4) + ((z-3)^2 - 9) = 0$$

$$x^2 - 2(y+2)^2 + 8 + (z-3)^2 - 9 = 0$$

$$x^2 - 2(y+2)^2 + (z-3)^2 = 1$$

$$x^2 - \frac{(y+2)^2}{1/2} + (z-3)^2 = 1$$

The quadric is a hyperboloid in standard form, with its center at $(0,-2,3)$, which is answer choice B.

■ 2. Find the equation of ellipsoid centered at $(2,0,2)$ that has the level curve $(x-2)^2+4y^2=0.75$ for $z=1.5$.

Solution:

An ellipsoid is given in standard form by

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}+\frac{(z-l)^2}{c^2}=1$$

Substitute $(2,0,2)$ for (h,k,l).

$$\frac{(x-2)^2}{a^2}+\frac{y^2}{b^2}+\frac{(z-2)^2}{c^2}=1$$

To find the level curve for $z=1.5$, substitute into the equation.

$$\frac{(x-2)^2}{a^2}+\frac{y^2}{b^2}+\frac{(1.5-2)^2}{c^2}=1$$

$$\frac{(x-2)^2}{a^2}+\frac{y^2}{b^2}=1-\frac{0.25}{c^2}$$

Since the level curve is $(x-2)^2+4y^2=0.75$, we can create a system of equations.

$$a^2=1$$

$$b^2 = 1/4$$

$1 - \dfrac{0.25}{c^2} = 0.75$, so $0.25c^2 = 0.25$

so

$$a = 1$$

$$b = 1/2$$

$$c = 1$$

Substitute these values into the ellipsoid's equation.

$$\frac{(x-2)^2}{1^2} + \frac{y^2}{\left(\frac{1}{2}\right)^2} + \frac{(z-2)^2}{1^2} = 1$$

$$(x-2)^2 + 4y^2 + (z-2)^2 = 1$$

■ 3. Which of the surfaces has the same level curves for any z?

A The plane $2x + 3y + z = 1$

B The ellipsoid $x^2 + 2y^2 + 4z^2 - 4x - 2y = 1$

C The cylinder $2x^2 + y^2 - 5x + 7y = 1$

D The elliptic cone $x^2 + 3y^2 - z^2 = 0$

Solution:

Only the cylinder $2x^2 + y^2 - 5x + 7y = 1$ has the same level curves for any z. Since z-coordinate is missing in the cylinder equation, when solved for y it'll give the same solution for any z.

We also know that a cylinder is parallel to the z-axis and therefore has to have the same level curves.

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