Calculus 3 Workbook Solutions

Limits and continuity

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DOMAIN OF A MULTIVARIABLE FUNCTION

■ 1. Find the domain of the multivariable function.

$$f(x, y) = \sqrt{\sin(2x + y)}$$

Solution:

An expression under the square root should be nonnegative, so

$$\sin(2x + y) \geq 0$$

The function $\sin t \geq 0$ if $2\pi k \leq t \leq \pi + 2\pi k$ for any integer k. So

$$2\pi k \leq 2x + y \leq \pi + 2\pi k \text{ for any integer } k$$

■ 2. Find the domain of the multivariable function.

$$f(x, y) = (x^2 - y^2)\tan(2x)\cot(y + \pi)$$

Solution:

The argument of a tangent function can't be equal to

$$\frac{\pi}{2} + \pi k \text{ for any integer } k$$

so

$$2x \neq \frac{\pi}{2} + \pi k$$

$$x \neq \frac{\pi}{4} + \frac{\pi k}{2} \text{ for any integer } k$$

The argument of a cotangent function can't be equal to

$$\pi m \text{ for any integer } m$$

so

$$y + \pi \neq \pi m$$

$$y \neq \pi(m-1) \text{ for any integer } m$$

Let $n = m - 1$ where n is also any integer, then

$$y \neq \pi n \text{ for any integer } n$$

So the domain of the function is

$$x \neq \frac{\pi}{4} + \frac{\pi k}{2} \text{ for any integer } k$$

$$y \neq \pi n \text{ for any integer } n$$

■ 3. Find the domain of the multivariable function.

$$f(x, y) = \sin(3x + y)\log_{x-y}(x^2)$$

Solution:

The domain of the logarithmic function $\log_a b$ is $a > 0$, $a \neq 1$, and $b > 0$. So

$$x - y > 0$$

$$x - y \neq 1$$

$$x^2 > 0$$

Since x^2 is always greater than 0 except $x = 0$, we can say $x^2 > 0$ if $x \neq 0$. The domain of the function is

$$x - y > 0$$

$$x - y \neq 1$$

$$x \neq 0$$

■ 4. Find the set of points that lie within the domain of the multivariable function.

$$f(x, y) = 3\sqrt{x^2 + 2x + y^2 - 4y - 4}$$

Solution:

An expression under the square root should be nonnegative, so

$$x^2 + 2x + y^2 - 4y - 4 \geq 0$$

Complete the square with respect to each variable.

$$(x^2 + 2x + 1 - 1) + (y^2 - 4y + 4 - 4) - 4 \geq 0$$

$$(x + 1)^2 - 1 + (y - 2)^2 - 4 - 4 \geq 0$$

$$(x + 1)^2 + (y - 2)^2 - 9 \geq 0$$

$$(x + 1)^2 + (y - 2)^2 \geq 3^2$$

The domain is all points except the inner points of the circle with center at $(-1,2)$ and radius 3.

■ 5. Find the set of points that lie within the domain of the multivariable function.

$$f(x, y) = (2xy)^{-\frac{3}{4}}$$

Solution:

The function can be rewritten as

$$f(x, y) = \frac{1}{(2xy)^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{(2xy)^3}}$$

An expression under the square root should be positive.

$$(2xy)^3 > 0$$

$$2xy > 0$$

$$xy > 0$$

So x and y must be both positive, or both be negative. Which means the domain will be all points in quadrants I and III in the xy-plane.

LIMIT OF A MULTIVARIABLE FUNCTION

■ 1. If the limit exists, find its value.

$$\lim_{(x,y)\to(0,0)} \ln(2x+3ey+e^2)$$

Solution:

Since the function is continuous at $(0,0)$, just substitute $(0,0)$ for (x,y).

$$\lim_{(x,y)\to(0,0)} \ln(2(0)+3e(0)+e^2)$$

$$\lim_{(x,y)\to(0,0)} \ln(e^2)$$

$$2$$

■ 2. If the limit exists, find its value.

$$\lim_{(x,y)\to(\pi,\frac{\pi}{2})} \frac{\sin(3x+y)}{\cos(x-2y)}$$

Solution:

Since the function is continuous at $(\pi, \pi/2)$, just substitute the respective values for (x, y).

$$\lim_{(x,y)\to(\pi,\frac{\pi}{2})} \frac{\sin(3\pi + \frac{\pi}{2})}{\cos(\pi - 2\frac{\pi}{2})}$$

$$\lim_{(x,y)\to(\pi,\frac{\pi}{2})} \frac{\sin\left(\frac{7\pi}{2}\right)}{\cos(0)}$$

$$\frac{-1}{1}$$

$$-1$$

■ 3. If the limit exists, find its value.

$$\lim_{(x,y)\to(-\infty,-\infty)} (x^3 + 4y)(\sin(x^2 + 2y) + 3)$$

Solution:

Since $-1 \le \sin t \le 1$, then

$$-1 \le \sin(x^2 + 2y) \le 1$$

$$-1 + 3 \le \sin(x^2 + 2y) + 3 \le 1 + 3$$

$$2 \le \sin(x^2 + 2y) + 3 \le 4$$

If $x \to -\infty$ and $y \to -\infty$, then $x^3 + 4y \to -\infty$. So

$$\lim_{(x,y)\to(-\infty,-\infty)} (x^3 + 4y)(\sin(x^2 + 2y) + 3) \le \lim_{(x,y)\to(-\infty,-\infty)} 4(x^3 + 4y) = -\infty$$

$$\lim_{(x,y)\to(-\infty,-\infty)} (x^3 + 4y)(\sin(x^2 + 2y) + 3) = -\infty$$

■ 4. If the limit exists, find its value.

$$\lim_{(x,y)\to 0,0)} \frac{4x^4 - y^4}{2x^2 + y^2}$$

Solution:

Rewrite the function as

$$\frac{(2x^2 - y^2)(2x^2 + y^2)}{2x^2 + y^2}$$

$$2x^2 - y^2$$

This function is continuous at all real values of x and y.

$$\lim_{(x,y)\to(0,0)} \frac{4x^4 - y^4}{2x^2 + y^2} = \lim_{(x,y)\to(0,0)} (2x^2 - y^2) = 0$$

■ 5. If the limit exists, find its value.

$$\lim_{(x,y)\to(\infty,\infty)} 2^y - x^2$$

Solution:

In order for this limit to exist, the function must be approaching the same value regardless of the path that we take as we move in towards (∞, ∞).

Consider the path $y = \log_2(x^2)$.

$$\lim_{(x,\log_2(x^2))\to(\infty,\infty)} (2^{\log_2(x^2)} - x^2)$$

$$\lim_{(x,\log_2(x^2))\to(\infty,\infty)} (x^2 - x^2) = 0$$

Then consider the path $y = x$.

$$\lim_{(x,x)\to(\infty,\infty)} (2^x - x^2) = \infty$$

Since the limits from two different paths are not equal, the limit does not exist.

■ 6. If the limit exists, find its value.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 + 2x^2y^2 - xy}{2x^3 + y^2}$$

Solution:

In order for this limit to exist, the function must be approaching the same value regardless of the path that we take as we approach $(0,0)$.

Consider the path $y = x$.

$$\lim_{(x,x)\to(0,0)} \frac{x^4 + 2x^2(x)^2 - x(x)}{2x^3 + x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^4 - x^2}{2x^3 + x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^2 - 1}{2x + 1}$$

$$\frac{3(0)^2 - 1}{2(0) + 1} = -1$$

Consider the path $y = -x$.

$$\lim_{(x,x)\to(0,0)} \frac{x^4 + 2x^2(-x)^2 - x(-x)}{2x^3 + (-x)^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^4 + x^2}{2x^3 + x^2}$$

$$\lim_{(x,x)\to(0,0)} \frac{3x^2 + 1}{2x + 1}$$

$$\frac{3(0)^2 + 1}{2(0) + 1} = 1$$

Since the limits from two different paths are not equal, the limit does not exist.

PRECISE DEFINITION OF THE LIMIT FOR MULTIVARIABLE FUNCTIONS

1. Which value of δ can be used to apply the precise definition of the limit to $f(x, y)$ with $\epsilon = 0.002$ at the point $(0,0)$?

$$f(x, y) = (x^2 + y^2)(3 - xy)$$

Solution:

We need to find a δ such that $|f(x, y) - f(0,0)| < \epsilon$ whenever $0 < \sqrt{(x-0)^2 + (y-0)^2} < \delta$.

$$f(0,0) = (0^2 + 0^2)(3 - (0)(0)) = 0$$

$$|3 - xy| \le |3| + |xy| = 3 + |x||y|$$

$$|x| = \sqrt{x^2} \le \sqrt{x^2 + y^2} = \delta$$

Similarly,

$$|y| = \sqrt{y^2} \le \sqrt{x^2 + y^2} = \delta$$

So

$$|3 - xy| \le 3 + |x||y| \le 3 + \delta^2$$

Finally,

$$|f(x, y) - f(0,0)| = |(x^2 + y^2)(3 - xy)| \leq \delta^2(3 + \delta^2)$$

Since δ is relatively small, $3 + \delta^2 \leq 4$. So

$$|f(x, y) - f(0,0)| \leq 4\delta^2$$

Let $\epsilon = 4\delta^2$. Then

$$\delta = \frac{\sqrt{\epsilon}}{2}$$

■ 2. Which value of δ can be used to apply the precise definition of the limit to $f(x, y)$ with $\epsilon = 0.001$ at the point $(0,0)$? Hint: Use the polar form of the function.

$$f(x, y) = \frac{5x^2y}{x^2 + y^2}$$

Solution:

We need to find a δ such that $|f(x, y) - \lim_{(x,y)\to(0,0)} f(x, y)| < \epsilon$ whenever $0 < \sqrt{(x - 0)^2 + (y - 0)^2} < \delta$.

Since $f(x, y)$ is not continuous at $(0,0)$, we can switch to polar coordinates to investigate it. Substituting $x^2 + y^2 = r^2$, $x = r\cos\theta$, and $y = r\sin\theta$, we rewrite the function in polar coordinates.

$$f(r,\theta)=\frac{5(r\cos\theta)^2(r\sin\theta)}{r^2}$$

$$f(r,\theta)=\frac{5r^3\cos^2\theta\sin\theta}{r^2}$$

$$f(r,\theta)=5r\cos^2\theta\sin\theta$$

Since $0\le\sqrt{(x-0)^2+(y-0)^2}\le\delta$, then $0\le r\le\delta$. And if $x=0$ and $y=0$, then $r=0$.

$$\lim_{(x,y)\to(0,0)}f(x,y)=\lim_{r\to 0}f(r,\theta)=\lim_{r\to 0}5r\cos^2\theta\sin\theta=0$$

$$|f(r,\theta)-0|=|5r\cos^2\theta\sin\theta|=5r|\cos^2\theta\sin\theta|$$

Since $|\sin\theta|\le 1$ and $|\cos\theta|\le 1$,

$$|f(r,\theta)-0|\le 5r\le 5\delta$$

Let $\epsilon=5\delta$. Then

$$\delta=\frac{1}{5}\epsilon$$

■ 3. We know that $f(x,y)$ is a continuous function, and that for any real $\epsilon>0$, there exists a $\delta>0$ such that $\sqrt{(x-4)^2+(y+3)^2}<\delta$ implies $|f(x,y)-7|<\epsilon$. If the limit exists, find its value.

$$\lim_{(x,y)\to(4,-3)}(f(x,y))^2$$

Solution:

From the given statement, by the precise definition of the limit there exists

$$\lim_{(x,y)\to(4,-3)} f(x,y)$$

$$\lim_{(x,y)\to(4,-3)} f(x,y) = f(4,-3) = 7$$

By properties of limits,

$$\lim_{(x,y)\to(4,-3)} (f(x,y))^2 = \left(\lim_{(x,y)\to(4,-3)} f(x,y)\right)^2 = (7)^2 = 49$$

■ 4. We know that $f(x,y)$ and $g(x,y)$ are continuous functions, and that for any real $\epsilon > 0$, there exists a $\delta > 0$ such that $\sqrt{(x-2)^2+y^2} < \delta$ implies $|f(x,y)+3| + |g(x,y)-5| < \epsilon$. If the limit exists, find its value

$$\lim_{(x,y)\to(2,0)} (3f(x,y) - 2g(x,y))$$

Solution:

From the given statement,

$$|f(x,y)+3| \le |f(x,y)+3| + |g(x,y)-5| < \epsilon$$

So by the precise definition of the limit there exists

$$\lim_{(x,y)\to(2,0)} f(x,y)$$

$$\lim_{(x,y)\to(2,0)} f(x,y) = f(2,0) = -3$$

Similarly, for the function $g(x,y)$,

$$|g(x,y)-5| \le |f(x,y)+3| + |g(x,y)-5| < \epsilon$$

$$\lim_{(x,y)\to(2,0)} g(x,y) = g(2,0) = 5$$

By properties of limits,

$$\lim_{(x,y)\to(2,0)} (3f(x,y) - 2g(x,y))$$

$$3\lim_{(x,y)\to(2,0)} f(x,y) - 2\lim_{(x,y)\to(2,0)} g(x,y)$$

$$3(-3) - 2(5) = -19$$

■ 5. We know that for any real $\epsilon > 0$, there exists a $\delta > 0$ such that

for $x > 0$, $\sqrt{x^2+y^2} < \delta$ implies $|f(x,y)-4| < \epsilon$

for $x \le 0$, $\sqrt{x^2+y^2} < \delta$ implies $|f(x,y)+4| < \epsilon$

If the limit exists, find its value.

$$\lim_{(x,y)\to(0,0)} 3^{f(x,y)}$$

Solution:

From the given statement, by the precise definition of the limit, if (x, y) approaches $(0,0)$ along the path $y = x$ for $x > 0$, then

$$\lim_{(x,x)\to(0,0)} f(x, x) = 4$$

But if (x, y) approaches $(0,0)$ along the path $y = x$ for $x < 0$, then

$$\lim_{(x,x)\to(0,0)} f(x, x) = -4$$

So the limit

$$\lim_{(x,y)\to(0,0)} f(x, y)$$

does not exist, and by the properties of limits,

$$\lim_{(x,y)\to(0,0)} 3^{f(x,y)}$$

also does not exist.

■ 6. We know that for any real $\epsilon > 0$, there exists a $\delta > 0$ such that $\sqrt{(x+1)^2 + (y-12)^2} < \delta$ implies $f(x, y) > \epsilon$. If the limit exists, find its value.

$$\lim_{(x,y)\to(-1,12)} (f(x,y) - 13)$$

Solution:

From the given statement, by the precise definition of the limit, there exists

$$\lim_{(x,y)\to(-1,12)} f(x,y)$$

$$\lim_{(x,y)\to(-1,12)} f(x,y) = \infty$$

By the properties of limits,

$$\lim_{(x,y)\to(-1,12)} (f(x,y) - 12)$$

$$\lim_{(x,y)\to(-1,12)} f(x,y) - \lim_{(x,y)\to(-1,12)} 12$$

$$\infty - 12$$

$$\infty$$

DISCONTINUITIES OF MULTIVARIABLE FUNCTIONS

■ 1. Find any discontinuities of the function.

$$f(x, y) = 3^{x^2-2y^2+\sqrt{x^2+5y^2-x+1}}$$

Solution:

The power function 3^t is continuous for every real number t. An expression under the square root should be nonnegative, so

$$x^2 + 5y^2 - x + 1 \geq 0$$

$$x^2 - 2(0.5)x + 0.25 - 0.25 + 5y^2 + 1 \geq 0$$

$$(x - 0.5)^2 - 0.25 + 5y^2 + 1 \geq 0$$

$$(x - 0.5)^2 + 5y^2 + 0.75 \geq 0$$

Since $(x - 0.5)^2 \geq 0$ and $5y^2 \geq 0$ and $0.75 > 0$, the sum of these terms is always positive, so

$$(x - 0.5)^2 + 5y^2 + 0.75 > 0$$

So the given function is continuous for all real numbers x and y.

■ 2. Find any discontinuities of the function.

$$f(x, y) = \sqrt{\sin x \cos y + \sin y \cos x}$$

Solution:

An expression under the square root should be nonnegative, so

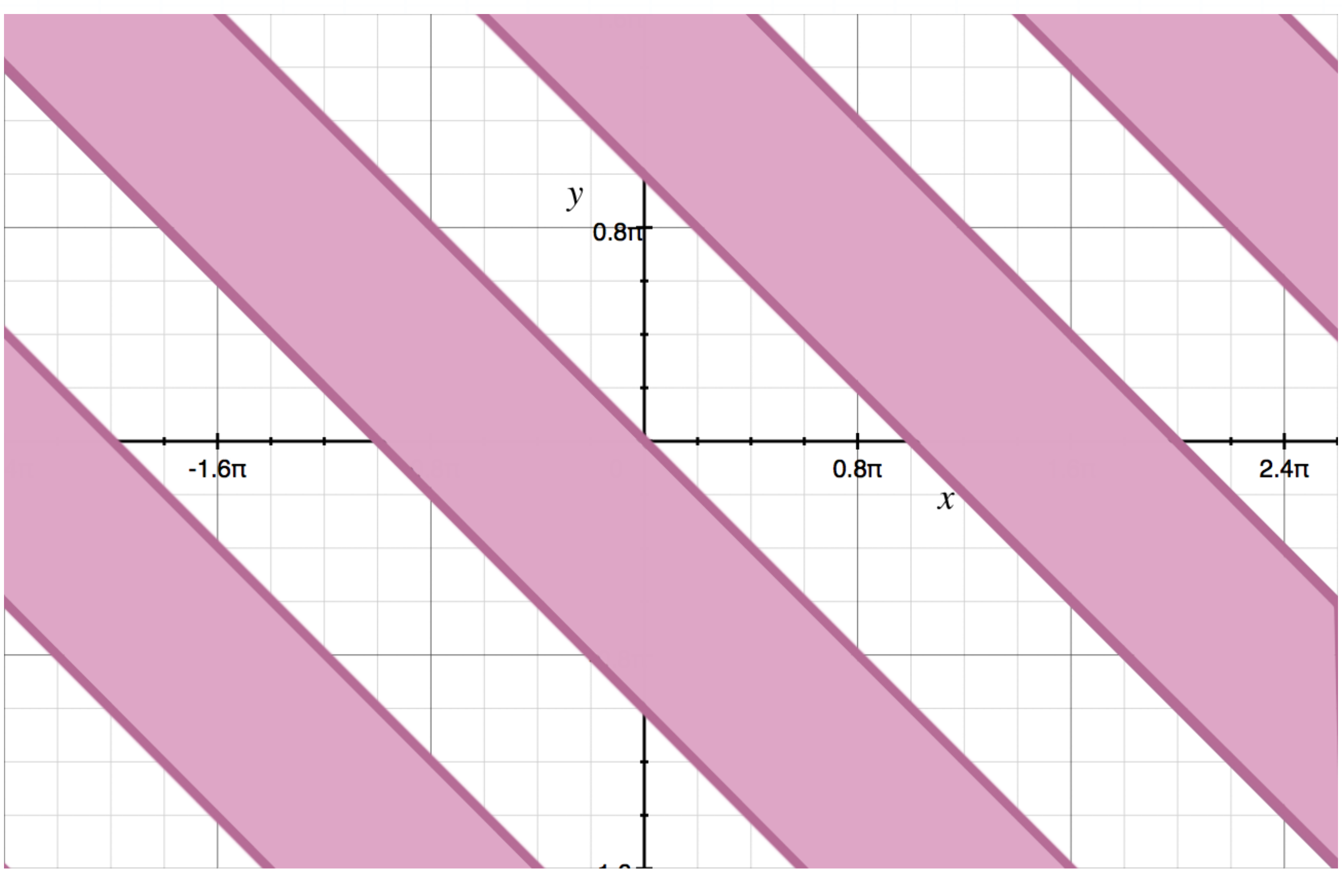
$$\sin x \cos y + \sin y \cos x \geq 0$$

$$\sin(x + y) \geq 0$$

The function is discontinuous if $\sin(x + y) < 0$. Solve the trigonometric inequality

$$2\pi k - \pi < x + y < 2\pi k$$

$$-x - \pi + 2\pi k < y < -x + 2\pi k$$



The function is discontinuous when $-x-\pi+2\pi k<y<-x+2\pi k$ for any integer k.

■ 3. Find any discontinuities of the function.

$$f(x,y)=\begin{cases}\frac{4x^2-y^2}{2x-y} & y\neq 2x\\ 0 & y=2x\end{cases}$$

Solution:

Simplify the function for $y\neq 2x$.

$$\frac{4x^2-y^2}{2x-y}=\frac{(2x-y)(2x+y)}{2x-y}=2x+y$$

So for all of the points $y\neq 2x$ the function $f(x,y)$ is continuous.

For the points $y=2x$ the function is continuous only at the points (x_0,y_0) where

$$\lim_{(x,y)\to(x_0,y_0)} 2x+y=0$$

$$2x_0+y_0=0$$

Since $y_0=2x_0$, we have

$$2x_0+2x_0=0$$

$$4x_0 = 0$$

$$x_0 = 0$$

Which gives $y_0 = 0$. Therefore, for the points $y = 2x$, the function is continuous only at $(0,0)$. So the function is continuous for all real numbers x and y, excluding the points $y = 2x$, but including the point $(0,0)$.

■ 4. Find and classify any discontinuities of the function.

$$f(x, y) = \frac{7x - y}{4x^2 + y^2 - 4x + 1}$$

Solution:

The denominator should be nonzero.

$$4x^2 + y^2 - 4x + 1 \neq 0$$

$$(2x - 1)^2 + y^2 \neq 0$$

The denominator equals 0 only at the point where $2x - 1 = 0$ and $y = 0$, or $(1/2,0)$. To classify the discontinuity, investigate the limit.

$$\lim_{(x,y)\to(1/2,0)} \frac{7x - y}{4x^2 + y^2 - 4x + 1}$$

Since the numerator at $(1/2,0)$ is positive, $7(1/2)+0=7/2>0$, and the denominator is positive, the function tends to infinity as (x,y) approaches $(1/2,0)$. So at $(1/2,0)$, the function has an infinite discontinuity.

So the single discontinuity at $(1/2,0)$ is an infinite discontinuity.

■ 5. Find and classify any discontinuities of the function.

$$f(x,y)=\frac{x^2-9y^2-2x+1}{|x-1|+|3y|}$$

Solution:

Simplify the function.

$$f(x,y)=\frac{(x-1)^2-9y^2}{|x-1|+3|y|}$$

$$f(x,y)=\frac{|x-1|^2-9|y|^2}{|x-1|+3|y|}$$

$$f(x,y)=\frac{(|x-1|-3|y|)(|x-1|+3|y|)}{|x-1|+3|y|}$$

$f(x,y)=|x-1|-3|y|$, assuming $|x-1|+3|y|\neq 0$

This function is continuous for all real numbers x and y.

If $|x-1|+3|y|=0$, then $x-1=0$ and $y=0$. So the function is discontinuous at $(1,0)$. Since the function $|x-1|-3|y|$ is continuous and finite at $(1,0)$, the function has a removable discontinuity at this point.

So the single discontinuity at $(1,0)$ is removable.

COMPOSITIONS OF MULTIVARIABLE FUNCTIONS

■ 1. Find $f(g(x, y))$.

$$f(t) = \ln(3t)$$

$$g(x, y) = \frac{x+1}{y+2}$$

Solution:

Substitute $g(x, y)$ for t into $f(t)$.

$$f(x, y) = \ln\left(3\frac{x+1}{y+2}\right)$$

$$f(x, y) = \ln 3 + \ln(x+1) - \ln(y+2)$$

■ 2. Find $f(x(t), y(t))$.

$$f(x, y) = x^2 - y^2 + 3$$

$$x(t) = \sqrt{t-5}$$

$$y(t) = 2^{t+2}$$

Solution:

Substitute $x(t)$ for x and 2^{t+2} for y into $f(x, y)$.

$$f(t) = (\sqrt{t-5})^2 - (2^{t+2})^2 + 3$$

$$f(t) = t - 5 - (2^{t+2})^2 + 3$$

$$f(t) = t - 5 - 2^{2t+4} + 3$$

$$f(t) = t - 2 - 2^{2t+4}$$

■ 3. Find $f(u(x, y), v(x, y))$.

$$f(u, v) = u^2 + v^2 + \frac{u - v}{\sqrt{2}}$$

$$u(x, y) = \sin(x + y)$$

$$v(x, y) = \cos(x + y)$$

Solution:

Substitute u and v into f.

$$f(u, v) = u^2 + v^2 + \frac{u - v}{\sqrt{2}}$$

$$f(x, y) = (\sin(x + y))^2 + (\cos(x + y))^2 + \frac{\sin(x + y) - \cos(x + y)}{\sqrt{2}}$$

Using the trig identity $\sin^2(a) + \cos^2(a) = 1$ simplifies the equation to

$$f(x, y) = 1 + \frac{\sin(x+y) - \cos(x+y)}{\sqrt{2}}$$

$$f(x, y) = 1 + \frac{\sin(x+y)}{\sqrt{2}} - \frac{\cos(x+y)}{\sqrt{2}}$$

Because

$$\cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$$

the function $f(x, y)$ can be rewritten as

$$f(x, y) = 1 + \cos\frac{\pi}{4}\sin(x+y) - \sin\frac{\pi}{4}\cos(x+y)$$

By the trigonometric identity $\sin(a-b) = \sin a \cos b - \cos a \sin b$, the equation becomes

$$f(x, y) = 1 + \sin\left(x + y - \frac{\pi}{4}\right)$$

W W W . K R I S T A K I N G M A T H . C O M