Calculus 3 Workbook Solutions

Dot products

krista king MATH

DOT PRODUCT OF TWO VECTORS

■ 1. Find the dot product $\vec{a} \cdot \vec{b}$, where the vectors $\vec{a}$ and $\vec{b}$ have opposite directions, and $\vec{b}$ has a magnitude two times larger than $\vec{a} = \langle 2, -3, 5 \rangle$.

Solution:

Since $\vec{b}$ has opposite direction to $\vec{a}$, and a magnitude two times larger, $\vec{b} = -2\vec{a}$. So

$$\vec{b} = \langle -2(2), -2(-3), -2(5) \rangle$$

$$\vec{b} = \langle -4, 6, -10 \rangle$$

Then the dot product is

$$\vec{a} \cdot \vec{b} = x_a \cdot x_b + y_a \cdot y_b + z_a \cdot z_b$$

$$\vec{a} \cdot \vec{b} = (2)(-4) + (-3)(6) + (5)(-10)$$

$$\vec{a} \cdot \vec{b} = -76$$

■ 2. Find the value(s) of the parameter p such that the dot product of the vectors $\vec{a} = \langle p, 2p+1, 3 \rangle$ and $\vec{b} = \langle p-2, 5, -4 \rangle$ is 2.

Solution:

The dot product is

$$\vec{a}\cdot\vec{b}=x_a\cdot x_b+y_a\cdot y_b+z_a\cdot z_b$$

$$\vec{a}\cdot\vec{b}=p(p-2)+(2p+1)(5)+3(-4)$$

$$\vec{a}\cdot\vec{b}=p^2+8p-7$$

Since $\vec{a}\cdot\vec{b}=2$, we can write an equation for p.

$$p^2+8p-7=2$$

$$p^2+8p-9=0$$

$$(p+9)(p-1)=0$$

$p=-9$ and $p=1$

■ 3. Find the unit vector(s) $\vec{u}$ such that the dot product $\vec{a}\cdot\vec{u}$ reaches its maximum value, if $\vec{a}=\langle 2,2\rangle$.

Solution:

Since $\vec{u}$ is the unit vector, its magnitude is 1. Let ϕ be the angle between $\vec{u}$ and the positive direction of the x-axis. So $\vec{u}=\langle\cos\phi,\sin\phi\rangle$, where $0\le\phi<2\pi$. The dot product is

$$\vec{a}\cdot\vec{u}=x_a\cdot x_u+y_a\cdot y_u$$

$$\vec{a} \cdot \vec{u} = 2\cos\phi + 2\sin\phi$$

Let the function $f(\phi) = 2\cos\phi + 2\sin\phi$, then find the absolute maximum of $f(\phi)$ on the interval $[0,2\pi]$.

$$f'(\phi) = -2\sin\phi + 2\cos\phi = 0$$

This equation gives $2\sin\phi = 2\cos\phi$, and since $\cos\phi \neq 0$, $\tan\phi = 1$.

The critical points on $[0,2\pi]$ are $\pi/4$ and $5\pi/4$. Substitute these critical points and the bounds of the interval into $f(\phi)$.

$$f(0) = 2\cos(0) + 2\sin(0) = 2$$

$$f\left(\frac{\pi}{4}\right) = 2\cos\left(\frac{\pi}{4}\right) + 2\sin\left(\frac{\pi}{4}\right) = 2\sqrt{2}$$

$$f\left(\frac{5\pi}{4}\right) = 2\cos\left(\frac{5\pi}{4}\right) + 2\sin\left(\frac{5\pi}{4}\right) = -2\sqrt{2}$$

$$f(2\pi) = 2\cos(2\pi) + 2\sin(2\pi) = 2$$

So the function $f(\phi)$ has an absolute maximum at $\phi = \pi/4$. The unit vector is

$$\vec{u} = \left\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \right\rangle$$

$$\vec{u} = \left\langle \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right\rangle$$

The dot product is largest when $\vec{u}$ is the unit vector of $\vec{a}$ with the same direction. That's always true for any $\vec{a}$.

ANGLE BETWEEN TWO VECTORS

■ 1. Use dot products to find the angles between the vector $\vec{a}=\langle -2,4,-4\rangle$ and the positive direction of each major coordinate axis.

Solution:

The magnitude of $\vec{a}$ is

$$|\vec{a}|=\sqrt{(x_a)^2+(y_a)^2+(z_a)^2}$$

$$|\vec{a}|=\sqrt{(-2)^2+4^2+(-4)^2}$$

$$|\vec{a}|=6$$

The angle between $\vec{a}$ and $\vec{u_x}=\langle 1,0,0\rangle$ is

$$\cos\phi_x=\frac{\vec{a}\cdot\vec{u_x}}{|\vec{a}||\vec{u_x}|}$$

$$\cos\phi_x=\frac{(-2)(1)+(4)(0)+(-4)(0)}{(6)(1)}$$

$$\cos\phi_x=-\frac{1}{3}$$

$$\phi_x=\arccos-\frac{1}{3}\approx 1.911$$

$$\phi_x = \frac{1.911 \cdot 180^\circ}{\pi} \approx 109.5^\circ$$

The angle between $\vec{a}$ and $\vec{u}_y = \langle 0,1,0 \rangle$ is

$$\cos \phi_y = \frac{\vec{a} \cdot \vec{u}_y}{|\vec{a}||\vec{u}_y|}$$

$$\cos \phi_y = \frac{(-2)(0) + (4)(1) + (-4)(0)}{(6)(1)}$$

$$\cos \phi_y = \frac{2}{3}$$

$$\phi_y = \arccos \frac{2}{3} \approx 0.841$$

$$\phi_y = \frac{0.841 \cdot 180^\circ}{\pi} \approx 48^\circ$$

The angle between $\vec{a}$ and $\vec{u}_z = \langle 0,0,1 \rangle$ is

$$\cos \phi_z = \frac{\vec{a} \cdot \vec{u}_z}{|\vec{a}||\vec{u}_z|}$$

$$\cos \phi_z = \frac{(-2)(0) + (4)(0) + (-4)(1)}{(6)(1)}$$

$$\cos \phi_z = -\frac{2}{3}$$

$$\phi_z = \arccos\left(-\frac{2}{3}\right) \approx 2.3$$

$$\phi_z = \frac{2.3 \cdot 180^\circ}{\pi} \approx 132^\circ$$

■ 2. Find the angle between the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$, if $\vec{a}=\langle 3,-4,4\rangle$ and $\vec{b}=\langle -6,2,-1\rangle$.

Solution:

The sum of the vectors is

$$\vec{a}+\vec{b}=\langle x_a+x_b, y_a+y_b, z_a+z_b\rangle$$

$$\vec{a}+\vec{b}=\langle 3-6,-4+2,4-1\rangle$$

$$\vec{a}+\vec{b}=\langle -3,-2,3\rangle$$

The difference of the vectors is

$$\vec{a}-\vec{b}=\langle x_a-x_b, y_a-y_b, z_a-z_b\rangle$$

$$\vec{a}-\vec{b}=\langle 3-(-6),-4-2,4-(-1)\rangle$$

$$\vec{a}-\vec{b}=\langle 9,-6,5\rangle$$

The angle between the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is given by

$$\cos\phi = \frac{(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})}{|\vec{a}+\vec{b}||\vec{a}-\vec{b}|}$$

The dot product is

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=\langle -3,-2,3\rangle\cdot\langle 9,-6,5\rangle$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=(-3)(9)+(-2)(-6)+(3)(5)$$

$$(\vec{a}+\vec{b})\cdot(\vec{a}-\vec{b})=0$$

Since the dot product is 0, $\cos\phi=0$ and $\phi=90^\circ$. So the vector $\vec{a}+\vec{b}$ is perpendicular to the vector $\vec{a}-\vec{b}$. The angle between the vectors $\vec{a}+\vec{b}$ and $\vec{a}-\vec{b}$ is equal to 90° not for any pair of vectors, but only for the vectors which have equal magnitude, i.e. only if $|\vec{a}|=|\vec{b}|$.

■ 3. Find the two vectors $\vec{b_1}$ and $\vec{b_2}$ with magnitude 5 that each have an angle of 30° with $\vec{a}=\langle -2,1\rangle$.

Solution:

The angle between $\vec{a}$ and the positive direction of the x-axis, which we'll represent with $\vec{u_x}=\langle 1,0,0\rangle$, is

$$\cos\phi=\frac{\vec{a}\cdot\vec{u_x}}{|\vec{a}||\vec{u_x}|}$$

$$\cos\phi=\frac{\langle -2,1\rangle\cdot\langle 1,0\rangle}{\sqrt{(-2)^2+1^2}\cdot 1}$$

$$\cos\phi = \frac{(-2)(1)+(1)(0)}{\sqrt{5}}$$

$$\cos\phi = -\frac{2\sqrt{5}}{5}$$

$$\phi = \arccos\left(-\frac{2\sqrt{5}}{5}\right) \approx 2.678$$

$$\phi = \frac{2.678 \cdot 180^\circ}{\pi} \approx 153.4^\circ$$

Since $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ have an angle of 30° with $\overrightarrow{a}$,

$$\phi_1 = 153.4^\circ + 30^\circ = 183.4^\circ$$

$$\phi_2 = 153.4^\circ - 30^\circ = 123.4^\circ$$

The formula for the vector $\overrightarrow{c}$ with magnitude M and angle α is given by

$$\overrightarrow{c} = \langle M\cos\alpha, M\sin\alpha\rangle$$

Therefore,

$$\overrightarrow{b_1} = \langle 5\cos 183.4^\circ, 5\sin 183.4^\circ\rangle$$

$$\overrightarrow{b_1} = \langle -5, -0.3\rangle$$

and

$$\overrightarrow{b_2} = \langle 5\cos 123.4^\circ, 5\sin 123.4^\circ\rangle$$

$$\overrightarrow{b_2} = \langle -2.8, 4.2\rangle$$

ORTHOGONAL, PARALLEL, OR NEITHER

■ 1. Find the terminal point B of the vector $\overrightarrow{AB}$ that has initial point $A(2,0,-1)$, magnitude 24, and is parallel to the vector $\vec{c}=\langle -2,4,4\rangle$.

Solution:

Parallel vectors have the same direction. Find the unit vector $\vec{u}$ in the direction of $\vec{c}$ (which is in the same direction as $\overrightarrow{AB}$).

$$\vec{u}=\frac{\vec{c}}{|\vec{c}|}$$

$$\vec{u}=\frac{\langle -2,4,4\rangle}{\sqrt{(-2)^2+4^2+4^2}}$$

$$\vec{u}=\frac{\langle -2,4,4\rangle}{6}$$

$$\vec{u}=\left\langle -\frac{1}{3},\frac{2}{3},\frac{2}{3}\right\rangle$$

Since $\overrightarrow{AB}$ has the same direction as $\vec{u}$ and a magnitude of 24,

$$\overrightarrow{AB}=24\vec{u}$$

$$\overrightarrow{AB}=24\left\langle -\frac{1}{3},\frac{2}{3},\frac{2}{3}\right\rangle$$

$$\overrightarrow{AB} = \langle -8,16,16 \rangle$$

Since A is the initial point of $\overrightarrow{AB}$, and B is the terminal point,

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

We know

$x_B - 2 = -8$ so $x_B = -6$

$y_B - 0 = 16$ so $y_B = 16$

$z_B - (-1) = 16$ so $z_B = 15$

■ 2. Find two vectors $\overrightarrow{b_1}$ and $\overrightarrow{b_2}$ with magnitude 2, that are orthogonal to $\overrightarrow{a} = \langle 3, -1 \rangle$.

Solution:

Let $\overrightarrow{u}$ be the unit vector orthogonal to $\overrightarrow{a}$. Let ϕ be the angle between $\overrightarrow{u}$ and the positive direction of the x-axis. So $\overrightarrow{u} = \langle \cos\phi, \sin\phi \rangle$. Since $\overrightarrow{u}$ is orthogonal to $\overrightarrow{a}$, the dot product is 0.

$$\overrightarrow{u} \cdot \overrightarrow{a} = 0$$

$$\langle \cos\phi, \sin\phi \rangle \cdot \langle 3, -1 \rangle = 0$$

$$3\cos\phi - \sin\phi = 0$$

$$\sin\phi = 3\cos\phi$$

$$\tan\phi = 3$$

$$\phi_1 = \arctan 3 \approx 1.249$$

$$\phi_2 = \pi + \arctan 3 \approx 4.39$$

The formula for the vector $\overrightarrow{c}$ with magnitude M and angle α is given by

$$\overrightarrow{c} = \langle M\cos\alpha, M\sin\alpha\rangle$$

Plug in $M = 2$ and $\alpha = \phi_1, \phi_2$.

$$\overrightarrow{b_1} = \langle 2\cos(1.249), 2\sin(1.249)\rangle$$

$$\overrightarrow{b_1} = \langle 0.6, 1.9\rangle$$

Similarly,

$$\overrightarrow{b_2} = \langle 2\cos(4.39), 2\sin(4.39)\rangle$$

$$\overrightarrow{b_2} = \langle -0.6, -1.9\rangle$$

■ 3. Find value(s) of the parameter p, such that the vectors $\overrightarrow{a} = \langle p, p+3, 6-p\rangle$ and $\overrightarrow{b} = \langle p-1, 4, 2\rangle$ are (a) parallel, and (b) orthogonal.

Solution:

(a) The vectors are parallel if their respective coordinates are proportional.

$$\frac{p}{p-1}=\frac{p+3}{4}=\frac{6-p}{2}$$

Solve the second equation for p.

$$\frac{p+3}{4}=\frac{6-p}{2}$$

$$p+3=2(6-p)$$

$$3p=9$$

$$p=3$$

Check if the first equation holds.

$$\frac{3}{3-1}=\frac{3+3}{4}=\frac{3}{2}$$

So the vectors $\vec{a}$ and $\vec{b}$ are parallel if $p=3$.

(b) The vectors are orthogonal if their dot product is 0.

$$\vec{a}\cdot\vec{b}=0$$

$$\langle p,p+3,6-p\rangle\cdot\langle p-1,4,2\rangle=0$$

$$(p)(p-1)+(p+3)(4)+(6-p)(2)=0$$

$$p^2+p+24=0$$

Since this equation has no real solutions, the vectors $\vec{a}$ and $\vec{b}$ can't be orthogonal for any p.

ACUTE ANGLE BETWEEN THE LINES

■ 1. Find the acute angle between the lines.

Line 1: $x = 2t + 1$, $y = t - 4$, $z = 6$

Line 2: $\dfrac{x-1}{4} = \dfrac{y+1}{5} = z$

Solution:

The angle between the lines is the same as the angle between their direction vectors.

The direction vector of the first line given in parametric form is $\vec{a} = \langle 2,1,0 \rangle$, and the direction vector of the second line given in symmetric form is $\vec{b} = \langle 4,5,1 \rangle$. Then the angle ϕ between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\phi = \frac{\vec{a} \cdot \vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\phi = \frac{(2)(4) + (1)(5) + (0)(1)}{\sqrt{2^2 + 1^2 + 0^2} \cdot \sqrt{4^2 + 5^2 + 1^2}}$$

$$\cos\phi = \frac{13}{\sqrt{5} \cdot \sqrt{42}}$$

$$\phi = \arccos\frac{13}{\sqrt{210}} \approx 0.46$$

$$\phi = \frac{0.46 \cdot 180^\circ}{\pi} \approx 26^\circ$$

■ 2. Find the acute angle between the line and the plane.

Line: $x = t + 7, \ y = -2t - 5, \ z = 3t + 6$

Plane: $3x - y - 4z + 15 = 0$

Solution:

Let ϕ be the acute angle between the line and the vector $\vec{n}$, which is the normal (orthogonal) vector to the given plane. Then the angle between the line and the plane is $90^\circ - \phi$.

The direction vector of the line given in parametric form is $\vec{a} = \langle 1, -2, 3 \rangle$, and the normal vector to the given plane is $\vec{n} = \langle 3, -1, -4 \rangle$.

The angle ϕ between $\vec{a}$ and $\vec{n}$ is given by

$$\cos\phi = \frac{\vec{a} \cdot \vec{n}}{|\vec{a}||\vec{n}|}$$

$$\cos\phi = \frac{(1)(3) + (-2)(-1) + (3)(-4)}{\sqrt{1^2 + (-2)^2 + 3^2} \cdot \sqrt{3^2 + (-1)^2 + (-4)^2}}$$

$$\cos\phi = \frac{-7}{\sqrt{14}\cdot\sqrt{26}}$$

$$\phi = \arccos\frac{-7}{\sqrt{364}} \approx 1.9465$$

$$\phi = \frac{1.9465 \cdot 180^\circ}{\pi} \approx 111.5^\circ$$

Since $\phi > 90^\circ$, the acute angle between the line and the vector $\overrightarrow{n}$ is

$$180^\circ - 111.5^\circ = 68.5^\circ$$

The acute angle between the line and the plane is

$$90^\circ - 68.5^\circ = 21.5^\circ$$

■ 3. Find the acute angle between the planes.

Plane 1: $x - 2y + 1 = 0$

Plane 2: $x + y + 2z + 4 = 0$

Solution:

The angle between the planes is equal to the angle between their normal vectors. The normal vector to the first plane is $\overrightarrow{n_1} = \langle 1, -2, 0\rangle$ and the normal vector to the second plane is $\overrightarrow{n_2} = \langle 1,1,2\rangle$.

The angle ϕ between $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ is given by

$$\cos\phi = \frac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{|\overrightarrow{n_1}||\overrightarrow{n_2}|}$$

$$\cos\phi = \frac{(1)(1) + (-2)(1) + (0)(2)}{\sqrt{1^2 + (-2)^2 + 0^2} \cdot \sqrt{1^2 + 1^2 + 2^2}}$$

$$\cos\phi = \frac{-1}{\sqrt{5} \cdot \sqrt{6}}$$

$$\phi = \arccos\frac{-1}{\sqrt{30}} \approx 1.7544$$

$$\phi = \frac{1.7544 \cdot 180^\circ}{\pi} \approx 100.5^\circ$$

Therefore, the acute angle between the planes is

$$180^\circ - 100.5 \circ = 79.5^\circ$$

ACUTE ANGLES BETWEEN THE CURVES

■ 1. Find the acute angle(s) between the curves.

$$x^2 + y^2 = 4$$

$$x^2 + 4y^2 = 4$$

Solution:

Set the curves equal to each other to find the points where they intersect.

$$x^2 + y^2 - 4 = x^2 + 4y^2 - 4$$

$$3y^2 = 0$$

$$y = 0$$

Substitute $y = 0$ into the first equation in order to solve for x.

$$x^2 + (0)^2 = 4$$

$$x^2 = 4$$

$x_1 = -2$ and $x_2 = 2$

So we have two intersection points, $(-2,0)$ and $(2,0)$. Now differentiate the first equation with respect to x.

$$2x + 2yy' = 0$$

$$x + yy' = 0$$

Substitute $(-2,0)$ for (x, y).

$$(-2) + (0)y' = 0$$

$$-2 = 0$$

So the derivative does not exist. Since the implicit function is differentiable at this point, but the derivative does not exist, the tangent line is vertical.

Similarly, for the point $(2,0)$ the derivative does not exist, and so the tangent line is vertical.

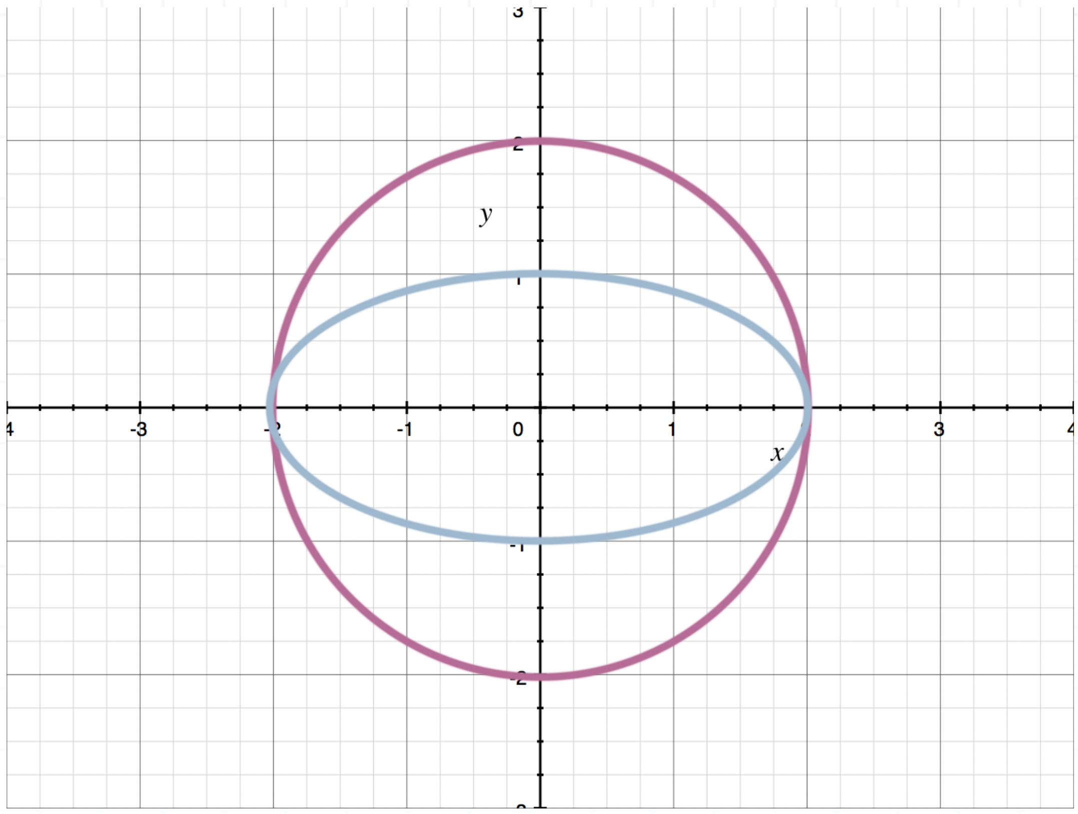
Differentiate the second equation with respect to x.

$$2x + 8yy' = 0$$

$$x + 4yy' = 0$$

The derivative does not exist at both intersection points, so the tangent lines are vertical.

Therefore, the angles between the curves are equal to 0 for both intersection points, $(-2,0)$ and $(2,0)$. We can confirm the result by sketching both curves. The circle $x^2 + y^2 = 4$ is centered at the origin with radius 2, and the ellipse $x^2 + 4y^2 = 4$ is centered at the origin with a horizontal semi-axis of 2 and a vertical semi-axis of 1.



2. Find the acute angle(s) between the curves given in parametric form.

$$x = t^2 + 1,\ y = 2t^2 + t - 3,\ z = t - 1$$

$$x = 2s^2 - 7,\ y = s - 5,\ z = s - 3$$

Solution:

Set the curves equal to one another to find the points where they intersect.

$$t^2 + 1 = 2s^2 - 7$$

$$2t^2 + t - 3 = s - 5$$

$$t - 1 = s - 3$$

Solve the system of equations for t and s. In the third equation, isolate t and substitute it into the first equation.

$$t = s - 2$$

$$(s-2)^2 + 1 = 2s^2 - 7$$

$$s^2 - 4s + 4 + 1 = 2s^2 - 7$$

$$s^2 + 4s - 12 = 0$$

$$(s-2)(s+6) = 0$$

$s = 2$, and then $t = 2 - 2 = 0$

or

$s = -6$, and then $t = -6 - 2 = -8$

Substitute each solution into the second equation

$$2(0)^2 + (0) - 3 = (2) - 5$$

$$2(-8)^2 + (-8) - 3 = (-6) - 5$$

The first equation is true and the second is false, so we have only one solution, which is $t = 0$ and $s = 2$. The point of intersection is

$$x(0) = (0)^2 + 1 = 1$$

$$y(0) = 2(0)^2 + (0) - 3 = -3$$

$$z(0) = (0) - 1 = -1$$

Therefore, the curves intersect at the point $(1,-3,-1)$.

At $t=0$, the first curve has values

$$x'(t)=2t,\ x'(0)=0$$

$$y'(t)=4t+1,\ y'(0)=1$$

$$z'(t)=1,\ z'(0)=1$$

So the tangent vector for the first curve is $\vec{a}=\langle 0,1,1\rangle$.

At $s=2$, the second curve has values

$$x'(s)=4s,\ x'(2)=8$$

$$y'(s)=1,\ y'(2)=1$$

$$z'(s)=1,\ z'(2)=1$$

So the tangent vector for the second curve is $\vec{b}=\langle 8,1,1\rangle$.

The angle ϕ between $\vec{a}$ and $\vec{b}$ is given by

$$\cos\phi=\frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

$$\cos\phi=\frac{(0)(8)+(1)(1)+(1)(1)}{\sqrt{0^2+1^2+1^2}\cdot\sqrt{8^2+1^2+1^2}}$$

$$\cos\phi=\frac{2}{\sqrt{2}\cdot\sqrt{66}}$$

$$\cos\phi = \frac{1}{\sqrt{33}}$$

$$\phi = \arccos\frac{1}{\sqrt{33}} \approx 1.4$$

$$\phi = \frac{1.4 \cdot 180^\circ}{\pi} \approx 80^\circ$$

■ 3. Find the value of the parameter p such that $f(x) = e^x$ and $g(x) = e^{-x} + 2p$ are orthogonal at the point(s) of intersection.

Solution:

Set the curves equal to each other to find the points where they intersect.

$$e^x = e^{-x} + 2p$$

Make a substitution of $u = e^x$.

$$u = \frac{1}{u} + 2p$$

$$u^2 - 2pu - 1 = 0$$

Use the quadratic formula to solve the equation.

$$u = p \pm \sqrt{p^2 + 1}$$

$$e^x = p \pm \sqrt{p^2 + 1}$$

Since $p - \sqrt{p^2 + 1} < 0$, we have only one solution, which is $e^x = p + \sqrt{p^2 + 1}$.

Then the intersection point is given by

$$x = \ln(p + \sqrt{p^2 + 1}) \text{ and } y = p + \sqrt{p^2 + 1}$$

Find the slope for $f(x)$ and $g(x)$ at this intersection point.

$$f'(x) = e^x = p + \sqrt{p^2 + 1}$$

$$g'(x) = -e^{-x} = -\frac{1}{p + \sqrt{p^2 + 1}}$$

So the tangent vectors for the functions $f(x)$ and $g(x)$ are

$$\overrightarrow{a_f} = \left\langle 1, p + \sqrt{p^2 + 1} \right\rangle$$

$$\overrightarrow{a_g} = \left\langle 1, -\frac{1}{p + \sqrt{p^2 + 1}} \right\rangle$$

The curves are orthogonal if the dot product of their tangent vectors at the intersection point is 0.

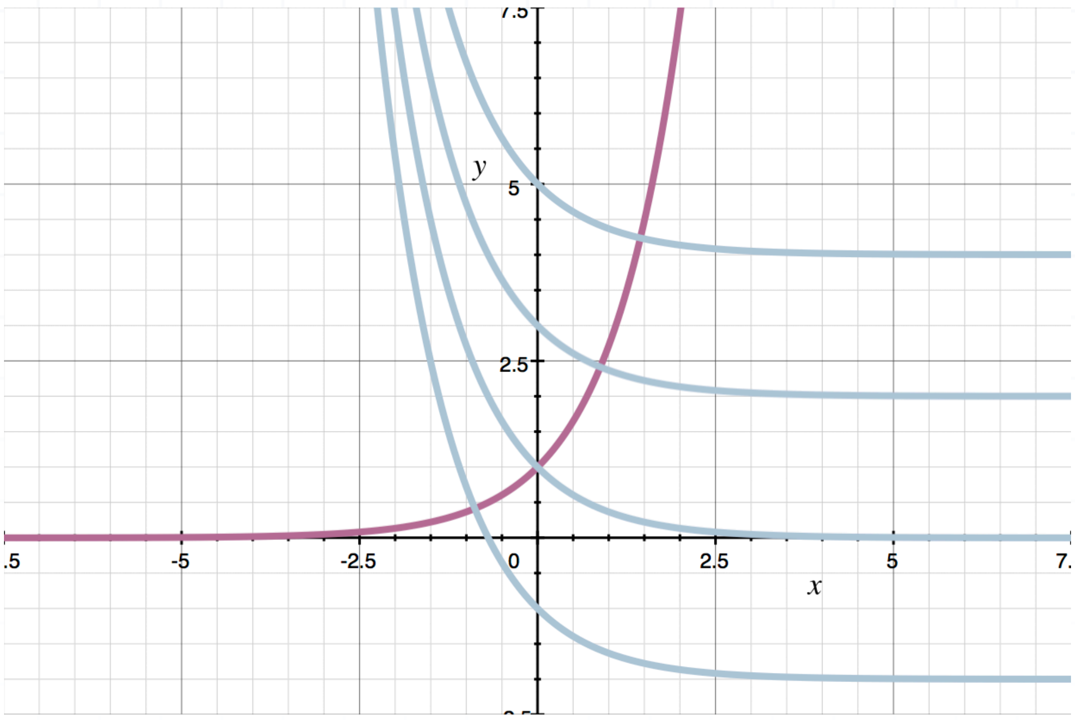
$$\overrightarrow{a_f} \cdot \overrightarrow{a_g} = 0$$

$$\left\langle 1, p + \sqrt{p^2 + 1} \right\rangle \cdot \left\langle 1, -\frac{1}{p + \sqrt{p^2 + 1}} \right\rangle = 0$$

$$(1)(1)+(p+\sqrt{p^2+1})\left(-\frac{1}{p+\sqrt{p^2+1}}\right)=0$$

$$1-1=0$$

So these curves are orthogonal for any real value of p.



DIRECTION COSINES AND DIRECTION ANGLES

■ 1. Find the direction angles of the linear combination $\overrightarrow{c} = 2\overrightarrow{a} - 3\overrightarrow{b}$, where $\overrightarrow{a} = \langle 3,1, -3\rangle$ and $\overrightarrow{b} = \langle 0, -2, -2\rangle$.

Solution:

Since $\overrightarrow{c}$ is the linear combination of two vectors, we can compute its coordinates as

$$\overrightarrow{c} = \langle 2x_a - 3x_b, 2y_a - 3y_b, 2z_a - 3z_b\rangle$$

$$\overrightarrow{c} = \langle 2(3) - 3(0), 2(1) - 3(-2), 2(-3) - 3(-2)\rangle$$

$$\overrightarrow{c} = \langle 6,8,0\rangle$$

The magnitude of the vector $\overrightarrow{c}$ is given by

$$|\overrightarrow{c}| = \sqrt{x_c^2 + y_c^2 + z_c^2}$$

$$|\overrightarrow{c}| = \sqrt{6^2 + 8^2 + 0^2}$$

$$|\overrightarrow{c}| = \sqrt{100} = 10$$

The direction angle α of the vector $\overrightarrow{c}$ with respect to the x-axis is

$$\alpha = \arccos \frac{x_c}{|\overrightarrow{c}|}$$

$$\alpha = \arccos\frac{6}{10} = \arccos\frac{3}{5} \approx 0.927$$

$$\alpha = \frac{0.927 \cdot 180^\circ}{\pi} \approx 53^\circ$$

The direction angle with respect to the y-axis is

$$\beta = \arccos\frac{y_c}{|\vec{c}\,|}$$

$$\beta = \arccos\frac{8}{10} = \arccos\frac{4}{5} \approx 0.64$$

$$\beta = \frac{0.64 \cdot 180^\circ}{\pi} \approx 37^\circ$$

The direction angle with respect to the z-axis is

$$\gamma = \arccos\frac{z_c}{|\vec{c}\,|}$$

$$\gamma = \arccos\frac{0}{10} = \arccos 0 = \frac{\pi}{2}$$

$$\gamma = 90^\circ$$

■ 2. Find the vector $\vec{a}$ with magnitude 6 that has direction angles 120°, 45°, and 135° with respect to x, y, and z-axes, respectively.

Solution:

The cosine functions of the direction angles of the vector $\vec{a}$ are given by

$$\cos\alpha = \frac{x_a}{|\vec{a}|}$$

$$\cos\beta = \frac{y_a}{|\vec{a}|}$$

$$\cos\gamma = \frac{z_a}{|\vec{a}|}$$

Solve these equations for x_a, y_a, and z_a.

$$x_a = |\vec{a}|\cos\alpha$$

$$y_a = |\vec{a}|\cos\beta$$

$$z_a = |\vec{a}|\cos\gamma$$

Plug in $|\vec{a}| = 6$, $\alpha = 120^\circ$, $\beta = 45^\circ$, and $\gamma = 135^\circ$.

$$x_a = 6\cos 120^\circ = 6\left(-\frac{1}{2}\right) = -3$$

$$y_a = 6\cos 45^\circ = 6\left(\frac{\sqrt{2}}{2}\right) = 3\sqrt{2}$$

$$z_a = 6\cos 135^\circ = 6\left(-\frac{\sqrt{2}}{2}\right) = -3\sqrt{2}$$

■ 3. Find the vector $\vec{a}$ that has an x-coordinate of 2, y-coordinate of -1, and direction angle with respect to the z-axis of $\pi/3$.

Solution:

Let z be the unknown coordinate of the vector $\vec{a}$, so that $\vec{a} = \langle 2, -1, z \rangle$, then consider the direction angle γ of the vector $\vec{a}$ with respect to the z-axis.

$$\cos\gamma = \frac{z}{|\vec{a}|}$$

The magnitude of the vector $\vec{a}$ is

$$|\vec{a}| = \sqrt{x_a^2 + y_a^2 + z_a^2}$$

$$|\vec{a}| = \sqrt{2^2 + (-1)^2 + z^2}$$

$$|\vec{a}| = \sqrt{5 + z^2}$$

Plug $|\vec{a}|$ and $\gamma = \pi/3$ into the expression for $\cos\gamma$, then solve the equation for z.

$$\cos\left(\frac{\pi}{3}\right) = \frac{z}{\sqrt{5 + z^2}}$$

$$\frac{1}{2} = \frac{z}{\sqrt{5 + z^2}}$$

Square both sides, then multiply through by $4(5 + z^2)$.

$$\frac{1}{4} = \frac{z^2}{5 + z^2}$$

$$5 + z^2 = 4z^2$$

$$3z^2 = 5$$

$$z^2 = \frac{5}{3}$$

$$z_1 = -\frac{\sqrt{15}}{3} \text{ and } z_2 = \frac{\sqrt{15}}{3}$$

Since the vector $\vec{a}$ has the positive direction angle with respect to the z-axis, it also has a positive z-coordinate. So

$$z = \frac{\sqrt{15}}{3} \text{ and } \vec{a} = \left\langle 2, -1, \frac{\sqrt{15}}{3} \right\rangle$$

SCALAR EQUATION OF A LINE

■ 1. Find the parametric scalar equations of the line that pass through the points $A(5,4,-3)$ and $B(1,0,3)$.

Solution:

To find the parametric equations of a line, use the formulas

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Take $A(5,4,-3)$ as a point on the line, and $\overrightarrow{AB}$ as a direction vector. Since A is the initial point of the vector $\overrightarrow{AB}$, and B is the terminal point,

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

$$\overrightarrow{AB} = \langle 1-5, 0-4, 3-(-3) \rangle$$

$$\overrightarrow{AB} = \langle -4, -4, 6 \rangle$$

Plug these values into the equations for the line.

$$x = 5 - 4t$$

$$y = 4 - 4t$$

$$z = -3 + 6t$$

■ 2. Find the parametric scalar equations of the line that passes through the point $A(4,-1,0)$ and is orthogonal to the plane $x + 2y - z = 7$.

Solution:

Since the plane has the equation $x + 2y - z = 7$, its normal vector is $\langle 1,2,-1\rangle$. Also, since the line is orthogonal to the plane, we can use the normal vector of the plane as the direction vector of the line.

To find the parametric equations of a line, use the formulas

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Take $A(4,-1,0)$ as a point on the line, and $\langle 1,2,-1\rangle$ as a direction vector.

$$x = 4 + (1)t = 4 + t$$

$$y = -1 + (2)t = -1 + 2t$$

$$z = 0 + (-1)t = -t$$

■ 3. Find the parametric scalar equations of the line that forms the intersection of the planes $2x + 3y - z = 1$ and $x - y + 4z = -4$.

Solution:

There are an infinite number of correct answers for this problem, because we can choose any point on the line, and also any direction vector. Let's choose the points at $x = 0$ and $x = 1$.

Substitute $x = 0$ and solve the system for y and z to get

$$3y - z = 1$$

$$-y + 4z = -4$$

and then

$$z = 3y - 1$$

$$-y + 4(3y - 1) = -4$$

So

$$11y - 4 = -4$$

$y = 0$ and $z = -1$.

Substitute $x = 1$ to get

$$2(1) + 3y - z = 1$$

$$(1) - y + 4z = -4$$

and then

$$z = 3y + 1$$

$$1 - y + 4(3y + 1) = -4$$

So

$$11y = -9$$

$$y = -\frac{9}{11} \text{ and } z = -\frac{16}{11}$$

So we have two points on the line, $A(0,0,-1)$ and $B(1,-9/11,-16/11)$. To find the parametric equations of a line, use the formulas

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Take $A(0,0,-1)$ as a point on the line and $\overrightarrow{AB}$ as a direction vector. Since A is the initial point of the vector $\overrightarrow{AB}$, and B is the terminal point,

$$\overrightarrow{AB} = \langle x_B - x_A, y_B - y_A, z_B - z_A \rangle$$

$$\overrightarrow{AB} = \left\langle 1 - 0, -\frac{9}{11} - 0, -\frac{16}{11} - (-1) \right\rangle$$

$$\overrightarrow{AB} = \left\langle 1, -\frac{9}{11}, -\frac{5}{11} \right\rangle$$

Plug these values into the equations for the line.

$$x = 0 + (1)t = t$$

$$y = 0 - \frac{9}{11}t = -\frac{9}{11}t$$

$$z = -1 - \frac{5}{11}t$$

SCALAR EQUATION OF A PLANE

1. Find the scalar equations of the plane, given its vector equation.

$$\langle 1,2,-1\rangle \cdot (\vec{r}-\langle 0,5,-4\rangle)=0$$

Solution:

The scalar equation of the plane in general form is

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

From the vector equation of the plane we can get a normal vector to the plane $\vec{n}=\langle 1,2,-1\rangle$, and a point that lies in the plane $(0,5,-4)$. Plug these values into the scalar equation of the plane.

$$1(x-0)+2(y-5)+(-1)(z-(-4))=0$$

$$x+2y-z-14=0$$

2. Find the scalar equations of the plane that passes through the points $A(2,0,1)$, $B(-1,3,2)$, and $C(1,1,-4)$.

Solution:

The scalar equation of the plane in general form is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Let $\vec{n} = \langle a, b, c \rangle$ be the unknown normal vector of the plane, then $\vec{n}$ is orthogonal to any vector in the plane. Since $\vec{n}$ is orthogonal to $\overrightarrow{AB}$ and $\overrightarrow{AC}$,

$$\vec{n} \cdot \overrightarrow{AB} = 0$$

$$\vec{n} \cdot \overrightarrow{AC} = 0$$

Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \langle -1 - 2, 3 - 0, 2 - 1 \rangle = \langle -3, 3, 1 \rangle$$

$$\overrightarrow{AC} = \langle 1 - 2, 1 - 0, -4 - 1 \rangle = \langle -1, 1, -5 \rangle$$

So

$$\langle a, b, c \rangle \cdot \langle -3, 3, 1 \rangle = 0$$

$$\langle a, b, c \rangle \cdot \langle -1, 1, -5 \rangle = 0$$

Therefore, we have a system of equations in terms of a, b, and c.

$$-3a + 3b + c = 0$$

$$-a + b - 5c = 0$$

The system has an infinite number of solutions since there are an infinite number of normal vectors to the plane. Let's choose the vector with $a = 1$, and solve the system for the associated values of b and c. We get

$$-3 + 3b + c = 0$$

$$-1 + b - 5c = 0$$

and then

$$c = 3 - 3b$$

$$-1 + b - 5(3 - 3b) = 0$$

Therefore, $b = 1$ and $c = 0$. So the normal vector is $\vec{n} = \langle 1,1,0 \rangle$. Plug $\vec{n}$ and $A(2,0,1)$ into the scalar equation of the plane.

$$1(x - 2) + 1(y - 0) + 0(z - 1) = 0$$

$$x + y - 2 = 0$$

■ 3. Find the scalar equation of a plane(s) that's 6 units from, and parallel to, the plane $x - 2y + 2z - 2 = 0$.

Solution:

Since the planes are parallel, they have the same normal vector $\vec{n} = \langle 1, -2,2 \rangle$. Let's take any point in the given plane, then find the points that are at a distance of 6 from it in the direction of $\pm\vec{n}$.

Let $x = 0$ and $y = 0$. Then $2z - 2 = 0$ and $z = 1$. So the point $A(0,0,1)$ is in the given plane. The magnitude of $\vec{n}$ is

$$|\vec{n}| = \sqrt{1^2 + (-2)^2 + 2^2} = \sqrt{9} = 3$$

Let O be the origin and B be the point at a distance of 6 from A in the direction of $\vec{n}$. Since $|\vec{n}|=3$ and $AB=6$, we can find the coordinates of B using a vector equation.

$$\overrightarrow{OB}=\overrightarrow{OA}+2\vec{n}$$

$$\overrightarrow{OB}=\langle 0,0,1\rangle+2\langle 1,-2,2\rangle$$

$$\overrightarrow{OB}=\langle 2,-4,5\rangle$$

So the point B has coordinates $(2,-4,5)$. Similarly, let C be the point at a distance of 6 from A in the direction of $-\vec{n}$.

$$\overrightarrow{OC}=\langle 0,0,1\rangle-2\langle 1,-2,2\rangle$$

$$\overrightarrow{OC}=\langle -2,4,-3\rangle$$

So the point C has coordinates $(-2,4,-3)$. Plug $\vec{n}$ and $B(2,-4,5)$ into the scalar equation of the plane.

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$1(x-2)+(-2)(y+4)+2(z-5)=0$$

$$x-2y+2z-20=0$$

Plug $\vec{n}$ and $C(-2,4,-3)$ into the scalar equation of the plane.

$$1(x+2)+(-2)(y-4)+2(z+3)=0$$

$$x-2y+2z+16=0$$

SCALAR AND VECTOR PROJECTIONS

■ 1. Find the vector sum of projections of the vector $\vec{a} = \langle 13, -8, 9 \rangle$ onto the three coordinate axes.

Solution:

The vector projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is given by

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}$$

While finding the vector projection onto a line, we can choose any direction vector of the line. So let's find the vector projection of $\vec{a}$ onto the three unit vectors, $\vec{x} = \langle 1, 0, 0 \rangle$, $\vec{y} = \langle 0, 1, 0 \rangle$, and $\vec{z} = \langle 0, 0, 1 \rangle$.

Since the magnitude of each unit vector is 1,

$$\text{proj}_{\vec{x}}(\vec{a}) = (\vec{a} \cdot \vec{x})\vec{x} = (13(1) - 8(0) + 9(0))\langle 1, 0, 0 \rangle = \langle 13, 0, 0 \rangle$$

$$\text{proj}_{\vec{y}}(\vec{a}) = (\vec{a} \cdot \vec{y})\vec{y} = (13(0) - 8(1) + 9(0))\langle 0, 1, 0 \rangle = \langle 0, -8, 0 \rangle$$

$$\text{proj}_{\vec{z}}(\vec{a}) = (\vec{a} \cdot \vec{z})\vec{z} = (13(0) - 8(0) + 9(1))\langle 0, 0, 1 \rangle = \langle 0, 0, 9 \rangle$$

The sum of the vector projections is

$$\text{proj}_{\vec{x}}(\vec{a}) + \text{proj}_{\vec{y}}(\vec{a}) + \text{proj}_{\vec{z}}(\vec{a}) = \langle 13, 0, 0 \rangle + \langle 0, -8, 0 \rangle + \langle 0, 0, 9 \rangle = \langle 13, -8, 9 \rangle$$

In fact, the vector sum of the projections of any vector $\vec{a}$ onto the coordinate axes is always equal to the vector $\vec{a}$ itself.

■ 2. Find the projection of the vector $\vec{a} = \langle 4,3,-1 \rangle$ onto the plane Q, which is given by $2x - y + 2z - 7 = 0$.

Solution:

The vector projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is given by

$$\text{proj}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|^2}\vec{b}$$

The projection of $\vec{a}$ onto a plane can be calculated by subtracting the component of $\vec{a}$ that's orthogonal to the plane, from $\vec{a}$. So

$$\text{proj}_Q(\vec{a}) = \vec{a} - \text{proj}_{\vec{n}}(\vec{a}) = \vec{a} - \frac{\vec{a} \cdot \vec{n}}{|\vec{n}|^2}\vec{n}$$

Since the plane has equation $2x - y + 2z - 7 = 0$, its normal vector is $\vec{n} = \langle 2,-1,2 \rangle$. The magnitude of $\vec{n}$ is

$$|\vec{n}| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$$

The dot product of $\vec{a}$ and $\vec{n}$ is

$$\vec{a} \cdot \vec{n} = \langle 4,3,-1 \rangle \cdot \langle 2,-1,2 \rangle$$

$$\vec{a} \cdot \vec{n} = (4)(2) + (3)(-1) + (-1)(2)$$

$$\vec{a} \cdot \vec{n} = 3$$

Plug these values into the formula for a vector projection onto a plane.

$$\text{proj}_Q(\vec{a}) = \langle 4,3,-1 \rangle - \frac{3}{3^2} \langle 2,-1,2 \rangle$$

$$\text{proj}_Q(\vec{a}) = \left\langle 4 - \frac{2}{3}, 3 + \frac{1}{3}, -1 - \frac{2}{3} \right\rangle$$

$$\text{proj}_Q(\vec{a}) = \left\langle \frac{10}{3}, \frac{10}{3}, -\frac{5}{3} \right\rangle$$

■ 3. Find the vector $\vec{a}$ if its scalar projections onto the vectors $\vec{b} = \langle 4,-3 \rangle$ and $\vec{c} = \langle 0,2 \rangle$ are both 3.

Solution:

The scalar projection of a vector $\vec{a}$ onto another vector $\vec{b}$ is given by

$$\text{comp}_{\vec{b}}(\vec{a}) = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|}$$

Let x and y be the coordinates of the vector $\vec{a}$. The magnitudes of $\vec{b}$ and $\vec{c}$ are

$$|\vec{b}| = \sqrt{4^2 + (-3)^2} = \sqrt{25} = 5$$

$$|\vec{c}| = \sqrt{0^2 + 2^2} = \sqrt{4} = 2$$

Since the scalar projection of $\vec{a}$ onto $\vec{b}$ is 3,

$$\frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = 3$$

$$\frac{4x - 3y}{5} = 3$$

$$4x - 3y = 15$$

Similarly, since the scalar projection of $\vec{a}$ onto $\vec{c}$ is 3,

$$\frac{\vec{a} \cdot \vec{c}}{|\vec{c}|} = 3$$

$$\frac{0 \cdot x + 2y}{2} = 3$$

$$2y = 6$$

$$y = 3$$

Substitute $y = 3$ into $4x - 3y = 15$ in order to solve for x.

$$4x - 3(3) = 15$$

$$4x = 24$$

$$x = 6$$

Then the vector $\vec{a}$ is given by $\vec{a} = \langle 6,3 \rangle$.

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