Calculus 3 Workbook Solutions

Derivatives and integrals of vector functions

krista king MATH

DERIVATIVE OF A VECTOR FUNCTION

■ 1. Find the second order derivative of the vector function.

$$\vec{r}(t) = \left\langle \sqrt{t}, \frac{2}{t}, e^{t+3} \right\rangle$$

Solution:

Differentiate each component individually with respect to t.

$$r_1'(t) = (t^{1/2})' = \frac{1}{2}t^{-1/2}$$

$$r_1''(t) = \left(\frac{1}{2}t^{-1/2}\right)' = -\frac{1}{4}t^{-3/2} = -\frac{1}{4t^{3/2}}$$

$$r_2'(t) = (2t^{-1})' = -2t^{-2}$$

$$r_2''(t) = (-2t^{-2})' = 4t^{-3} = \frac{4}{t^3}$$

$$r_3'(t) = (e^{t+3})' = e^{t+3}$$

$$r_3''(t) = (e^{t+3})' = e^{t+3}$$

■ 2. Find the Jacobian matrix of the vector function at $(u, v) = (1,2)$.

$$\vec{r}(u,v) = \langle 2uv + 1, u^2 + v^2 \rangle$$

Solution:

The Jacobian is given by

$$\frac{\partial \vec{r}(u,v)}{\partial(u,v)} = \begin{bmatrix} \frac{\partial r_1}{\partial u} & \frac{\partial r_1}{\partial v} \\ \frac{\partial r_2}{\partial u} & \frac{\partial r_2}{\partial v} \end{bmatrix}$$

$$\frac{\partial \vec{r}(u,v)}{\partial(u,v)} = \begin{bmatrix} 2v & 2u \\ 2u & 2v \end{bmatrix}$$

Evaluate at $u = 1$ and $v = 2$.

$$\frac{\partial \vec{r}(1,2)}{\partial(u,v)} = \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}$$

■ 3. Find the Jacobian matrix for the vector function.

$$\vec{r}(t,s) = \langle \ln(ts), 3t + 2s - 1, \sin(t + s) \rangle$$

Solution:

The Jacobian is given by

$$\frac{\partial \vec{r}(t,s)}{\partial(t,s)} = \begin{bmatrix} \frac{\partial r_1}{\partial t} & \frac{\partial r_1}{\partial s} \\ \frac{\partial r_2}{\partial t} & \frac{\partial r_2}{\partial s} \\ \frac{\partial r_3}{\partial t} & \frac{\partial r_3}{\partial s} \end{bmatrix}$$

$$\frac{\partial \vec{r}(t,s)}{\partial(t,s)} = \begin{bmatrix} \frac{1}{t} & \frac{1}{s} \\ 3 & 2 \\ \cos(t+s) & \cos(t+s) \end{bmatrix}$$

UNIT TANGENT VECTOR

■ 1. Find the unit tangent vector to the function that that sits at a 30° angle.

$$\vec{r}(t) = \langle t^2 + 4, 2t^3 - 3 \rangle$$

Solution:

If the vector $\langle u, v \rangle$ has an angle of $\phi = 30^\circ$, then

$$\tan \phi = \tan 30^\circ$$

$$\frac{v}{u} = \frac{1}{\sqrt{3}}$$

To find the components of the tangent vector, differentiate $\vec{r}(t)$ with respect to t.

$$r_1'(t) = 2t$$

$$r_2'(t) = 6t^2$$

Since

$$\frac{r_2'(t)}{r_1'(t)} = \frac{1}{\sqrt{3}}$$

for some $t = t_0$,

$$\frac{6t_0^2}{2t_0} = \frac{1}{\sqrt{3}}$$

$$3t_0 = \frac{1}{\sqrt{3}}$$

$$t_0 = \frac{1}{3\sqrt{3}}$$

To find the tangent vector, plug $t_0 = 1/3\sqrt{3}$ into $\vec{r}\,'(t)$.

$$r'_1\left(\frac{1}{3\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$$

$$r'_2\left(\frac{1}{3\sqrt{3}}\right) = 6 \cdot \frac{1}{3^2 \cdot 3} = \frac{2}{9}$$

So the tangent vector is

$$\vec{r}\,'(t_0) = \left\langle \frac{2}{3\sqrt{3}}, \frac{2}{9} \right\rangle$$

The magnitude of the tangent vector is

$$|\vec{r}\,'(t_0)| = \sqrt{\left(\frac{2}{3\sqrt{3}}\right)^2 + \left(\frac{2}{9}\right)^2} = \sqrt{\frac{16}{81}} = \frac{4}{9}$$

Finally, the unit tangent vector is

$$\frac{\vec{r}\,'(t_0)}{|\vec{r}\,'(t_0)|} = \left\langle \frac{2}{3\sqrt{3}} \cdot \frac{9}{4}, \frac{2}{9} \cdot \frac{9}{4} \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2} \right\rangle$$

■ 2. Find the tangent vector at the point $(-1,0,1)$.

$$\vec{r}(t) = \langle 2t^3 - 3t^2 + 5t - 5,\ \sin(\pi t), e^{t-1} \rangle$$

Solution:

Identify the value of t that corresponds to $(-1,0,1)$. We could use $r_1(t) = -1$, $r_2(t) = 0$, or $r_3(t) = 1$. The first and the second equations will probably give us several solutions, so let's use the third one.

$$e^{t-1} = 1$$

$$t - 1 = 0$$

$$t = 1$$

Check that the other equations hold, $r_1(1) = -1$ and $r_2(1) = 0$.

$$2(1)^3 - 3(1)^2 + 5(1) - 5 = -1$$

$$\sin(\pi(1)) = \sin 0 = 0$$

In order to find the tangent vector, differentiate each term individually with respect to t.

$$r_1'(t) = 6t^2 - 6t + 5$$

$$r_2'(t) = \pi\cos(\pi t)$$

$$r_3'(t) = e^{t-1}$$

Plug in $t = 1$.

$$r_1'(1) = 6(1)^2 - 6(1) + 5 = 5$$

$$r_2'(1) = \pi\cos(\pi \cdot 1) = -\pi$$

$$r_3'(1) = e^{t-1} = 1$$

■ 3. Find the point(s) where the unit tangent vector to the curve is orthogonal to the xz-plane

$$\vec{r}(t) = \langle t^3 + 2, 5t^2 - 3t + 8, t^2 + 5\rangle$$

Solution:

There are only two unit vectors orthogonal to the xz-plane, which are $\langle 0,1,0\rangle$ and $\langle 0,-1,0\rangle$. Identify the value of t that corresponds to these tangent vectors by differentiating each term of the vector function with respect to t.

$$r_1'(t) = 3t^2$$

$$r_2'(t) = 10t - 3$$

$$r_3'(t) = 2t$$

Since $r_1'(t) = 0$ and $r_3'(t) = 0$, we can conclude that $t = 0$, so $r_2'(0) = 10 \cdot 0 - 3 = -3$. Since the tangent vector is $\langle 0, -3, 0 \rangle$, the unit tangent vector is $\langle 0, -1, 0 \rangle$. Find the point for $t = 0$.

$$\overrightarrow{r}(0) = \langle 0^3 + 2, 5 \cdot 0^2 - 3 \cdot 0 + 8, 0^2 + 5 \rangle = \langle 2, 8, 5 \rangle$$

PARAMETRIC EQUATIONS OF THE TANGENT LINE

■ 1. Find the parametric equation of the tangent line to $\vec{r}(u)$ at $u=-2$.

$$\vec{r}(u)=\langle e^{u+3}, \ln(1-u)\rangle$$

Solution:

Plug in $u=-2$ to find the coordinates of the point.

$$\vec{r}(-2)=\langle e^{-2+3}, \ln[1-(-2)]\rangle$$

$$\vec{r}(-2)=\langle e, \ln 3\rangle$$

Find the tangent vector at $u=-2$.

$$\vec{r}\,'(u)=\left\langle e^{u+3}, \frac{1}{u-1}\right\rangle$$

$$\vec{r}\,'(-2)=\left\langle e^{-2+3}, \frac{1}{-2-1}\right\rangle=\left\langle e, -\frac{1}{3}\right\rangle$$

The vector equation of the line with this direction vector, which passes through the point $(e, \ln 3)$, is

$$\vec{L}(u)=\langle e, \ln 3\rangle+t\left\langle e, -\frac{1}{3}\right\rangle$$

So the parametric equation is

$$x = e + te$$

$$y = \ln 3 - \frac{t}{3}$$

■ 2. Find the parametric equation(s) of the tangent line to the function $\vec{r}(t)$ that passes through the origin.

$$\vec{r}(t) = \langle 2t^2, 3t + 3, t + 1 \rangle$$

Solution:

The origin doesn't lie on the given curve, so we don't know the point where the tangent line touches the curve. Let T be the value of parameter t such that the tangent line touches the curve at $t = T$, then find the equation of the tangent line at this point. The coordinates of the point are

$$\vec{r}(T) = \langle 2T^2, 3T + 3, T + 1 \rangle$$

The direction vector is $\vec{r}\,'(t) = \langle 4t, 3, 1 \rangle$, so at the point $t = T$, $\vec{r}\,'(T) = \langle 4T, 3, 1 \rangle$.

The vector equation of the line with the direction vector $\langle 4T, 3, 1 \rangle$, which passes through the point $(2T^2, 3T + 3, T + 1)$, is

$$\vec{L}(t) = \langle 2T^2, 3T + 3, T + 1 \rangle + t\langle 4T, 3, 1 \rangle$$

So the parametric equations are

$$x = 2T^2 + 4Tt$$

$$y = 3T + 3 + 3t$$

$$z = T + 1 + t$$

Since this line passes through the origin, there exist values of t and T such that $x(t) = 0$, $y(t) = 0$, and $z(t) = 0$, so we need to solve the system of equations for t and T.

$$2T^2 + 4Tt = 0$$

$$3T + 3 + 3t = 0$$

$$T + 1 + t = 0$$

The second and third equations are equivalent, so solve the third equation for t and substitute the result into the first equation.

$$2T^2 + 4T(-T - 1) = 0$$

$$t = -T - 1$$

This gives

$$2T^2 - 4T^2 - 4T = 0$$

$$t = -T - 1$$

and then

$$T(T + 2) = 0$$

$$t = -T - 1$$

So we have two solutions, which are $T=0$ with $t=-1$, and $T=-2$ with $t=1$. So there are two tangent lines at different points on the curve which pass through the origin. Plug the values of T into the parametric equation of the line.

At the first tangent line for $T=0$,

$$x=0$$

$$y=3+3t$$

$$z=1+t$$

At the second tangent line for $T=-2$,

$$x=2(-2)^2+4(-2)t=8-8t$$

$$y=3(-2)+3+3t=-3+3t$$

$$z=-2+1+t=-1+t$$

■ 3. Find the equation of the tangent plane to the surface $\vec{r}(t,s)$ at the point $t=1$ and $s=4$.

$$\vec{r}(t,s)=\langle t^2+s^2,-3t+5,2s+1\rangle$$

Solution:

First, we need to find any two tangent vectors to the plane $\vec{a}$ and $\vec{b}$ at the given point, then we can find the normal vector to the plane as the cross product $\vec{a} \times \vec{b}$. The simplest way to find two tangent vectors is

(a) keep $s = 4$, consider $\vec{r}(t,4)$ as a function of one variable t, and find the tangent vector at $t = 1$, or

(b) vise versa, keeping $t = 1$, considering $\vec{r}(1,s)$ as a function of one variable s, and find the tangent vector at $s = 4$.

(a) Set $s = 4$:

$$\vec{r}(t,4) = \langle t^2 + 16, -3t + 5, 9 \rangle$$

$$\vec{r}\,'(t,4) = \langle 2t, -3, 0 \rangle$$

Plug in $t = 1$ to get the tangent vector.

$$\vec{r}\,'(1,4) = \langle 2 \cdot 1, -3, 0 \rangle$$

$$\vec{a} = \langle 2, -3, 0 \rangle$$

(b) Set $t = 1$:

$$\vec{r}(1,s) = \langle s^2 + 1, 2, 2s + 1 \rangle$$

$$\vec{r}\,'(1,s) = \langle 2s, 0, 2 \rangle$$

Plug in $s = 4$ to get the tangent vector.

$$\vec{r}\,'(1,4) = \langle 2 \cdot 4, 0, 2 \rangle$$

$$\vec{b} = \langle 8, 0, 2 \rangle$$

Next, find the normal vector $\vec{n}=\vec{a}\times\vec{b}$ to the plane. The cross product of two vectors $\vec{a}$ and $\vec{b}$ is given by

$$\vec{a}\times\vec{b}=\mathbf{i}(a_2b_3-a_3b_2)-\mathbf{j}(a_1b_3-a_3b_1)+\mathbf{k}(a_1b_2-a_2b_1)$$

Plug in $\vec{a}=\langle 2,-3,0\rangle$ and $\vec{b}=\langle 8,0,2\rangle$.

$$\vec{a}\times\vec{b}=\mathbf{i}(-3\cdot 2-0\cdot 0)-\mathbf{j}(2\cdot 2-0\cdot 8)+\mathbf{k}(2\cdot 0-(-3)\cdot 8)$$

Therefore,

$$\vec{n}=-6\mathbf{i}-4\mathbf{j}+24\mathbf{k}$$

Plug in $t=1$ and $s=4$ to find the coordinates.

$$\vec{r}(1,4)=\langle 1^2+4^2,-3\cdot 1+5,2\cdot 4+1\rangle$$

$$\vec{r}(1,4)=\langle 17,2,9\rangle$$

The plane with normal vector $\vec{n}=\langle -6,-4,24\rangle$ which passes through the point $(17,2,9)$ has the equation

$$-6(x-17)-4(y-2)+24(z-9)=0$$

$$3x+2y-12z+53=0$$

INTEGRAL OF A VECTOR FUNCTION

■ 1. Find the integral of the vector function.

$$\int \langle e^{3u-2}, e^{5-u}, \sin^2(u-\pi) \rangle \ du$$

Solution:

Integrate each component individually.

$$\int e^{3u-2} \ du = \frac{e^{3u-2}}{3} + C_1$$

$$\int e^{5-u} \ du = -e^{5-u} + C_2$$

$$\int \sin^2(u-\pi) \ du = \int \frac{1}{2} - \frac{1}{2}\cos(2u-2\pi) \ du$$

$$= \int \frac{1}{2} \ du - \int \frac{1}{2}\cos(2u) \ du$$

$$= \frac{u}{2} - \frac{\sin(2u)}{4} + C_3$$

■ 2. Find the improper integral of the vector function.

$$\int_2^\infty \left\langle \frac{t-2}{t^3-8}, 2^{-t+1} \right\rangle \ dt$$

Solution:

Integrate each component individually, starting with the first component.

$$\int_2^\infty \frac{t-2}{t^3-8} \ dt$$

$$= \int_2^\infty \frac{t-2}{(t-2)(t^2+2t+4)} \ dt$$

$$= \int_2^\infty \frac{1}{t^2+2t+4} \ dt$$

$$= \int_2^\infty \frac{1}{(t+1)^2+3} \ dt$$

Substitute $u = t + 1$, $du = dt$, and u changing from 3 to ∞.

$$\int_3^\infty \frac{1}{u^2+3} \ du$$

$$\left. \frac{\arctan \frac{u}{\sqrt{3}}}{\sqrt{3}} \right|_3^\infty$$

$$\lim_{u\to\infty} \frac{\arctan \frac{u}{\sqrt{3}}}{\sqrt{3}} - \frac{\arctan \frac{3}{\sqrt{3}}}{\sqrt{3}}$$

$$\frac{\pi}{2\sqrt{3}} - \frac{\pi}{3\sqrt{3}} = \frac{\pi}{6\sqrt{3}}$$

The integral of the second component is

$$\int_2^\infty 2^{-t+1}\ dt = \left[-\frac{2^{-t+1}}{\ln 2}\right]\Bigg|_2^\infty$$

$$= \lim_{t\to\infty} -\frac{2^{-t+1}}{\ln 2} + \frac{2^{-2+1}}{\ln 2}$$

$$= 0 + \frac{1}{2\ln 2} = \frac{1}{\ln 4}$$

■ 3. Find the double integral of the vector function, where R is the square $[0,\pi] \times [0,\pi]$.

$$\int\int_R \langle ts,\ \sin(t-s)\rangle\ dA$$

Solution:

Integrate the first component.

$$\int\int_R ts\ dA$$

$$\int_0^\pi t\ dt \cdot \int_0^\pi s\ ds$$

$$\frac{t^2}{2}\bigg|_0^{\pi} \cdot \frac{s^2}{2}\bigg|_0^{\pi}$$

$$\left(\frac{\pi^2}{2}-\frac{0^2}{2}\right)\left(\frac{\pi^2}{2}-\frac{0^2}{2}\right)$$

$$\frac{\pi^4}{4}$$

Integrate the second component.

$$\iint_R \sin(t-s)\ dA$$

$$\int_0^{\pi}\int_0^{\pi} \sin(t-s)\ dt\ ds$$

Integrate with respect to t, treating s as a constant.

$$\int_0^{\pi} \sin(t-s)\ dt$$

$$-\cos(t-s)\bigg|_0^{\pi}$$

$$-\cos(\pi-s)+\cos(0-s)=2\cos s$$

Integrate with respect to s.

$$\int_0^{\pi} 2\cos s\ ds$$

$$2\sin s \Big|_0^{\pi}$$

$$2\sin\pi - 2\sin 0 = 0$$

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