Calculus 3 Workbook Solutions

Green's theorem

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GREEN'S THEOREM FOR ONE REGION

■ 1. Use Green's theorem to calculate the line integral of the vector field $\vec{F}(x, y)$ over the circle with the center at the origin and radius 4.

$$\vec{F}(x, y) = \left\langle \ln(x^2 + y^2 + 20) - 2y - 3x, \sqrt{x^2 + y^2 + 9} \right\rangle$$

Solution:

Let P and Q be the components of the vector field.

$$P(x, y) = \ln(x^2 + y^2 + 20) - 2y - 3x$$

$$Q(x, y) = \sqrt{x^2 + y^2 + 9}$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = \frac{2y}{x^2 + y^2 + 20} - 2$$

$$\frac{\partial Q}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + 9}}$$

In polar coordinates, parametrize the region bounded by the circle centered at the origin with radius 4.

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$r^2 = x^2 + y^2$$

$$dx\ dy = r\ dr\ d\phi$$

Then we get

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{x}{\sqrt{x^2 + y^2 + 9}} + 2 - \frac{2y}{x^2 + y^2 + 20}$$

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{r\cos\phi}{\sqrt{r^2 + 9}} + 2 - \frac{2r\sin\phi}{r^2 + 20}$$

The line integral is

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA = \int_0^4 \int_0^{2\pi} \left(\frac{r\cos\phi}{\sqrt{r^2 + 9}} + 2 - \frac{2r\sin\phi}{r^2 + 20}\right) r\ d\phi\ dr$$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA = \int_0^4 \int_0^{2\pi} \frac{r^2\cos\phi}{\sqrt{r^2 + 9}} + 2r - \frac{2r^2\sin\phi}{r^2 + 20}\ d\phi\ dr$$

Since the integral of sine and cosine functions over a 2π-period is 0, the integral simplifies to

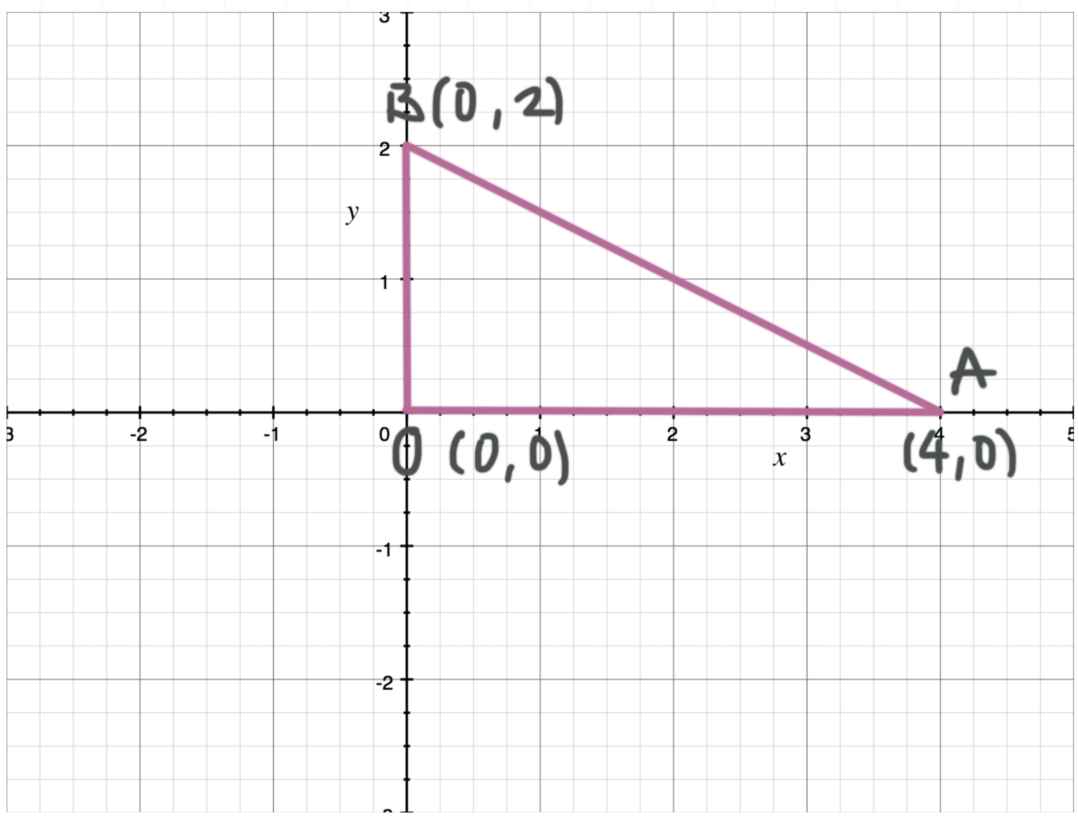
$$\int_0^4 \int_0^{2\pi} 2r\ d\phi\ dr$$

$$\int_0^4 2r\ dr \cdot \int_0^{2\pi}\ d\phi$$

$$r^2\Big|_0^4 \cdot 2\pi$$

$$(4^2 - 0^2)(2\pi) = 32\pi$$

■ 2. Use Green's theorem to calculate the line integral of the vector field $\overrightarrow{F}(x, y) = \langle y(y^2 + \sin x), y^2 - \cos x \rangle$ over the triangle OAB, where $O(0,0)$, $A(4,0)$, and $B(0,2)$.



Solution:

Let P and Q be the components of the vector field.

$$P(x, y) = y(y^2 + \sin x)$$

$$Q(x, y) = y^2 - \cos x$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = 3y^2 + \sin x$$

$$\frac{\partial Q}{\partial x} = \sin x$$

Then the difference of partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \sin x - 3y^2 - \sin x = -3y^2$$

In the triangle OAB, the value of x changes from 0 to 4 and the value of y changes from 0 to the line AB. Find the equation of the line through the points $(4,0)$ and $(0,2)$. The equation of the line with slope k which passes through (x_0, y_0) is

$$y - y_0 = k(x - x_0)$$

The line AB has a slope of -0.5, and passes through the point $(0,2)$. So its equation is

$$y - 2 = -0.5\,(x - 0)$$

$$y = -0.5x + 2$$

Therefore, in the triangle OAB, y changes from 0 to $-0.5x + 2$, and the line integral is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_0^4 \int_0^{-0.5x+2} -3y^2\ dy\ dx$$

Then the integral on the right simplifies to

$$\int_0^4 -y^3 \Big|_0^{-0.5x+2} \ dx$$

$$\int_0^4 -(-0.5x+2)^3 - (-0^3) \ dx$$

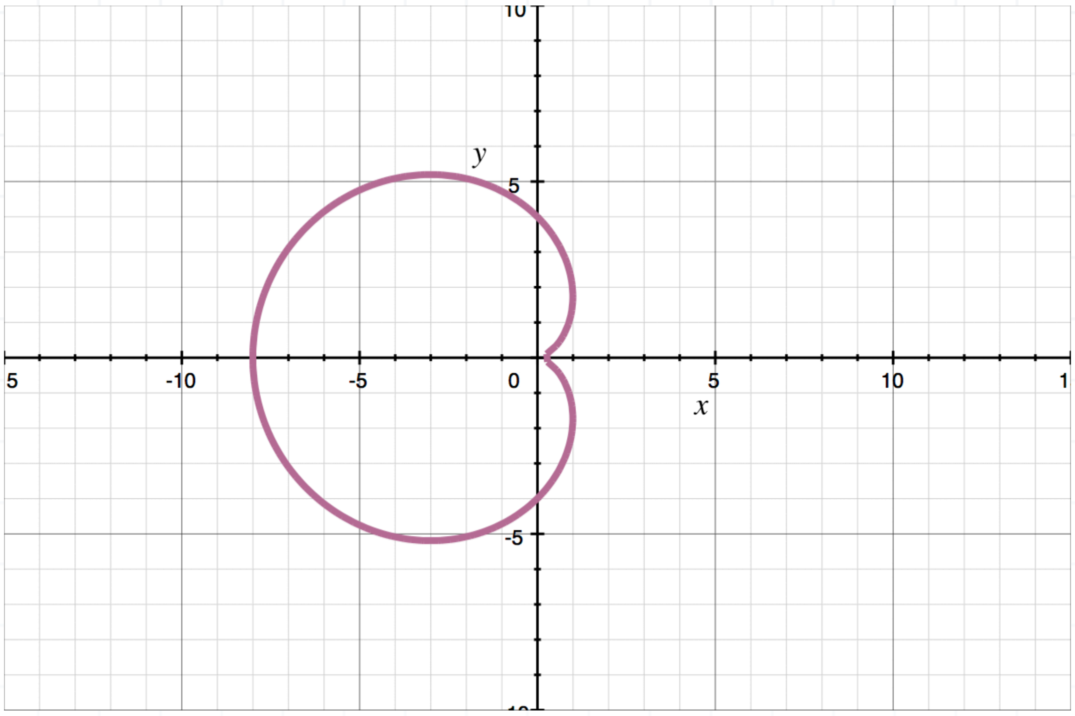
$$\int_0^4 (0.5x-2)^3 \ dx$$

$$\frac{1}{2}(0.5x-2)^4 \Big|_0^4$$

$$\frac{1}{2}(0.5 \cdot 4 - 2)^4 - \frac{1}{2}(0.5 \cdot 0 - 2)^4$$

$$\frac{0^4}{2} - \frac{2^4}{2} = -8$$

■ 3. Use Green's theorem to calculate the line integral of the vector field $\overrightarrow{F}(x,y) = \langle x^3 - y^3, x^3 + y^3 \rangle$ over the cardioid $(x^2+y^2)^2 + 8x(x^2+y^2) - 16y^2 = 0$.



Solution:

Consider the standard equation of the cardioid.

$$(x^2+y^2)^2+4ax(x^2+y^2)-4a^2y^2=0$$

The equation of the cardioid in polar coordinates is

$$r(\phi)=2a(1-\cos\phi) \text{ with } 0\le\phi\le 2\pi$$

In this case, $a=2$, so

$$r(\phi)=4(1-\cos\phi) \text{ with } 0\le\phi\le 2\pi$$

Consider the conversion to polar coordinates.

$$x=r\cos\phi$$

$$y=r\sin\phi$$

$$0 \le \phi \le 2\pi$$

$$r^2 = x^2 + y^2$$

$$dx\ dy = r\ dr\ d\phi$$

Within the region enclosed by the cardioid, ϕ changes from 0 to 2π, and r changes from 0 to $4(1 - \cos\phi)$. Now let P and Q be the components of the vector field.

$$P(x, y) = x^3 - y^3$$

$$Q(x, y) = x^3 + y^3$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = -3y^2$$

$$\frac{\partial Q}{\partial x} = 3x^2$$

Then the difference of the partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2 - (-3y^2) = 3(y^2 + x^2) = 3r^2$$

Then the line integral is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_0^{2\pi} \int_0^{4(1-\cos\phi)} (3r^2)r\ dr\ d\phi$$

and the right side of this equation simplifies to

$$\int_0^{2\pi} \int_0^{4(1-\cos\phi)} 3r^3 \ dr \ d\phi$$

$$\int_0^{2\pi} \frac{3}{4}r^4 \Big|_0^{4(1-\cos\phi)} \ d\phi$$

$$\int_0^{2\pi} \frac{3}{4} \cdot 4^4(1-\cos\phi)^4 - \frac{3}{4} \cdot 0^4 \ d\phi$$

$$\int_0^{2\pi} 192(1-\cos\phi)^4 \ d\phi$$

Expand the integrand.

$$192 \int_0^{2\pi} \cos^4\phi - 4\cos^3\phi + 6\cos^2\phi - 4\cos\phi + 1 \ d\phi$$

$$192 \int_0^{2\pi} \frac{1}{2}\cos 2\phi + \frac{1}{8}\cos 4\phi + \frac{3}{8} - (3\cos\phi + \cos 3\phi)$$

$$+3\cos 2\phi + 3 - 4\cos\phi + 1 \ d\phi$$

Since the integral of cosine functions over a 2π-period is 0, the integral becomes

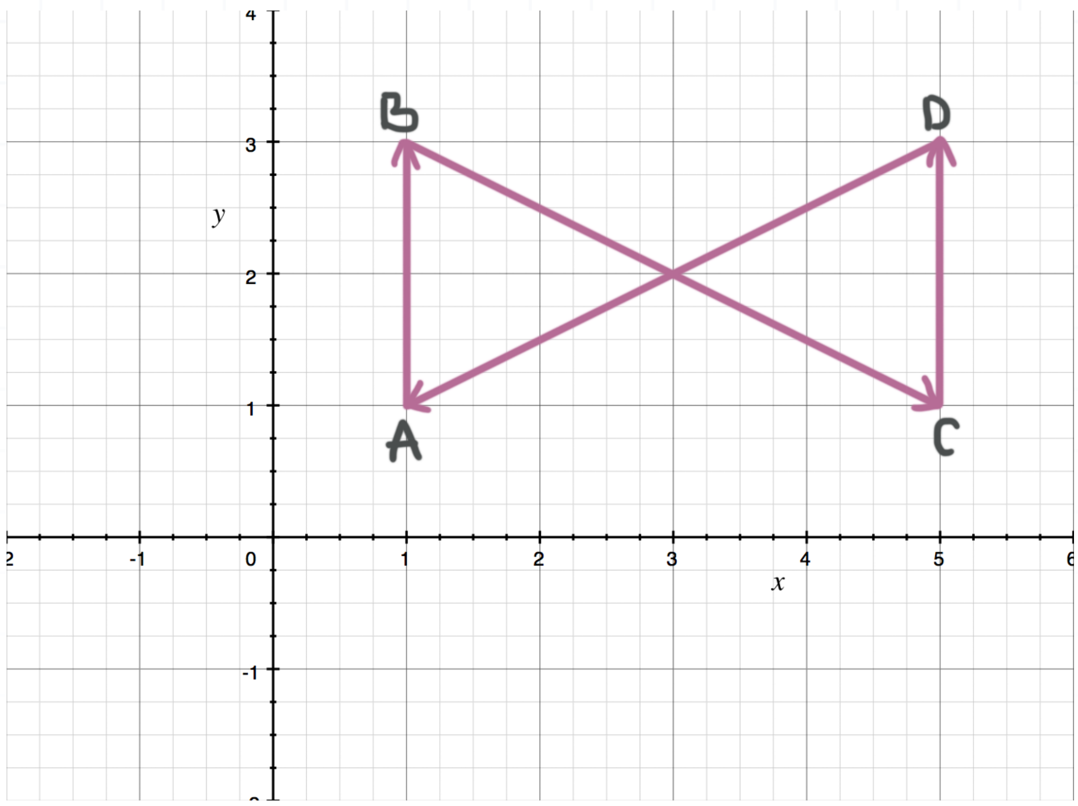
$$192 \int_0^{2\pi} \frac{3}{8} + 3 + 1 \ d\phi$$

$$192 \cdot \frac{35}{8} \int_0^{2\pi} \ d\phi$$

$$840 \cdot 2\pi = 1{,}680\pi$$

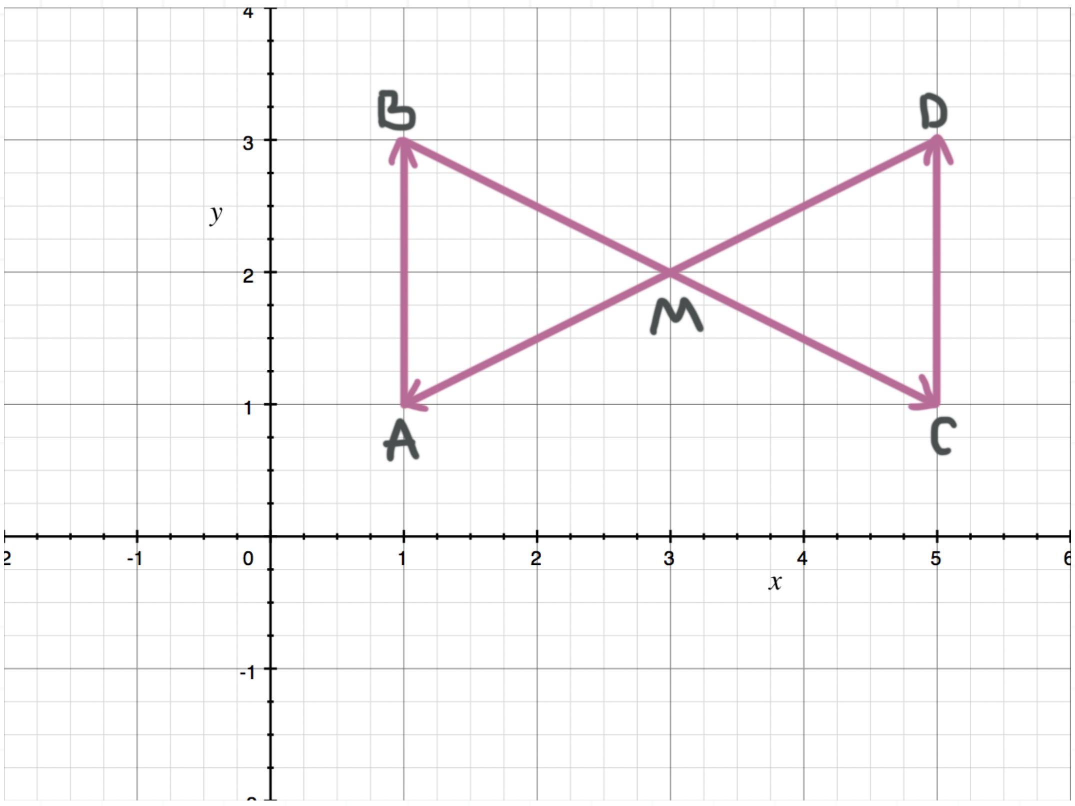
GREEN'S THEOREM FOR TWO REGIONS

1. Use Green's theorem to calculate the line integral of the vector field $\vec{F}(x,y) = \langle x^2, x^3y \rangle$ over the piecewise linear closed curve $ABCDA$, where $A(1,1)$, $B(1,3)$, $C(5,1)$, and $D(5,3)$.



Solution:

Since the given curve crosses itself, we can't apply Green's theorem to it directly. Let $M(3,2)$ be the point of intersection of DA and BC.



So the line integral over the curve $ABCDA$ is

$$\int_{ABCDA} \vec{F}(x,y)\ ds = \int_{ABM} \vec{F}(x,y)\ ds + \int_{MCDM} \vec{F}(x,y)\ ds + \int_{MA} \vec{F}(x,y)\ ds$$

$$\int_{ABCDA} \vec{F}(x,y)\ ds = \int_{ABMA} \vec{F}(x,y)\ ds + \int_{MCDM} \vec{F}(x,y)\ ds$$

So the line integral over the self-intersecting curve $ABCD$ is equal to the sum of two line integrals, over two closed curves $ABMA$ and $MCDM$, and we can apply Green's theorem to each curve separately.

Consider the triangle ABM. In this triangle x changes from 1 to 3, and y changes from the line DA to the line BC.

Let's find the equation of the line DA that passes through the points $(5,3)$ and $(1,1)$. The equation of the line with slope k which passes through the point (x_0, y_0) is

$$y - y_0 = k(x - x_0)$$

The line AD has a slope of 0.5, and passes through $(1,1)$. So its equation is

$$y - 1 = 0.5(x - 1)$$

$$y = 0.5x + 0.5$$

Similarly, let's find the equation of the line BC which passes through the points $(1,3)$ and $(5,1)$. The line BC has a slope of -0.5, and passes through $(1,3)$. So its equation is

$$y - 3 = -0.5(x - 1)$$

$$y = -0.5x + 3.5$$

So in the triangle ABM, y changes from $0.5x + 0.5$ to $-0.5x + 3.5$. For the triangle ABM, the components of the vector field will be

$$P(x, y) = x^2$$

$$Q(x, y) = x^3y$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = 0$$

$$\frac{\partial Q}{\partial x} = 3x^2y$$

Then the difference of the partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2y - 0 = 3x^2y$$

Therefore, the line integral over the triangle ABM is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_1^3 \int_{0.5x+0.5}^{-0.5x+3.5} 3x^2y\ dy\ dx$$

$$\int_1^3 3x^2 \left(\int_{0.5x+0.5}^{-0.5x+3.5} y\ dy \right) dx$$

$$3\int_1^3 x^2 \left(\frac{y^2}{2} \Big|_{0.5x+0.5}^{-0.5x+3.5} \right) dx$$

$$3\int_1^3 x^2 \left(\frac{(-0.5x+3.5)^2}{2} - \frac{(0.5x+0.5)^2}{2} \right) dx$$

$$3\int_1^3 x^2(-2)(x-3)\ dx$$

$$6\int_1^3 3x^2 - x^3\ dx$$

$$6\left(x^3 - \frac{x^4}{4} \right)\Big|_1^3$$

$$6\left(3^3 - \frac{3^4}{4} \right) - 6\left(1^3 - \frac{1^4}{4} \right) = 36$$

Consider the triangle MCD. In this triangle, x changes from 3 to 5, and y changes from the line BC to the line DA, from $-0.5x+3.5$ to $0.5x+0.5$. Therefore, the line integral over the triangle MCD is

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_3^5 \int_{-0.5x+3.5}^{0.5x+0.5} 3x^2y\ dy\ dx$$

Then the integral on the right simplifies to

$$\int_3^5 3x^2 \left(\int_{-0.5x+3.5}^{0.5x+0.5} y\ dy \right)\ dx$$

$$3\int_3^5 x^2 \left(\frac{y^2}{2} \Big|_{-0.5x+3.5}^{0.5x+0.5} \right)\ dx$$

$$3\int_3^5 x^2 \left(\frac{(0.5x+0.5)^2}{2} - \frac{(-0.5x+3.5)^2}{2} \right)\ dx$$

$$3\int_3^5 x^2 \cdot 2(x-3)\ dx$$

$$6\int_3^5 x^3 - 3x^2\ dx$$

$$6\left(\frac{x^4}{4} - x^3 \right)\Big|_3^5$$

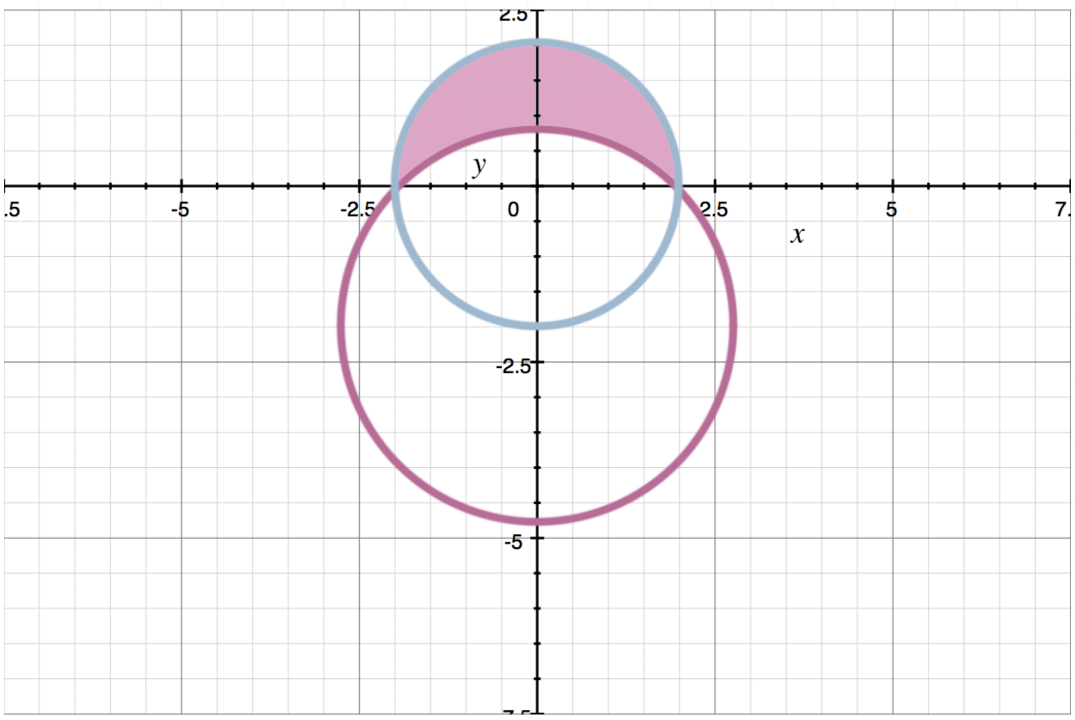
$$6\left(\frac{5^4}{4} - 5^3 \right) - 6\left(\frac{3^4}{4} - 3^3 \right) = 228$$

In total, the integral over the path $ABCDA$ is

$$36 + 228 = 264$$

2. Use Green's theorem (in reverse order) to calculate the double integral over the region D inside the circle $C_1: x^2 + y^2 = 4$, but outside the circle $C_2: x^2 + (y+2)^2 = 8$.

$$\iint_D 3x^2 \, dA$$



Solution:

In order to apply Green's theorem, we need to find any vector field $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$, such that

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 3x^2$$

For simplicity, let

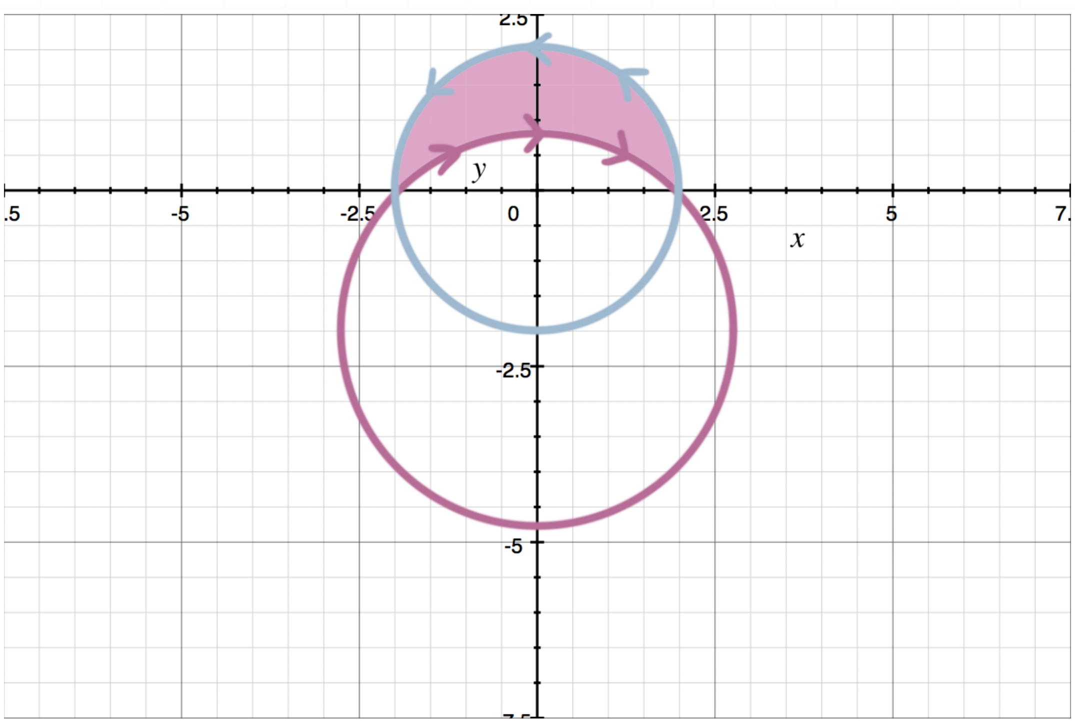
$$\frac{\partial Q}{\partial x} = 3x^2$$

$$\frac{\partial P}{\partial y} = 0$$

After integration, we have $\vec{F}(x, y) = \langle 0, x^3 \rangle$. Apply Green's theorem in reverse order.

$$\iint_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \ dA = \oint_c P \ dx + Q \ dy$$

where D is the region enclosed by curve c. In other words, the double integral over D is equal to the line integral over the closed curve.



Calculate the line integrals over the top and bottom parts of the closed curve individually. In order to find the line integral over the part of the circle C_1, consider the polar parametrization for the circle centered at the origin with radius 2.

$$x = 2\cos t$$

$$y = 2\sin t$$

where t changes from 0 to π. So

$$\vec{F}(x, y) = \langle 0,(2\cos t)^3\rangle = \langle 0,8\cos^3 t\rangle$$

$$dy = 2\cos t$$

Therefore, the line integral over the bound of D which lies on C_1 is

$$\int_0^\pi (8\cos^3 t)(2\cos t)\ dt$$

$$\int_0^\pi 16\cos^4 t\ dt$$

$$\int_0^\pi 16 \cdot \frac{1}{8}(4\cos 2t + \cos 4t + 3)\ dt$$

$$2\int_0^\pi 4\cos 2t + \cos 4t + 3\ dt$$

Since the integral of cosine functions over a 2π-period is 0, we can ignore the first and second terms of the integral.

$$2\int_0^\pi 3\ dt$$

$$6\int_0^\pi dt = 6\pi$$

In order to find the line integral over the part of the circle C_2, consider the polar parametrization for the circle centered at $(0, -2)$ with radius $\sqrt{8}$.

$$x = \sqrt{8}\cos t$$

$$y = -2 + \sqrt{8}\ \sin t$$

where t changes from $3\pi/4$ to $\pi/4$. So

$$\vec{F}(x, y) = \langle 0, (\sqrt{8}\cos t)^3 \rangle = \langle 0, 8\sqrt{8}\cos^3 t \rangle$$

$$dy = \sqrt{8}\cos t$$

Therefore, the line integral over the bound of D which lies on C_2 is

$$\int_{3\pi/4}^{\pi/4} (8\sqrt{8}\cos^3 t)(\sqrt{8}\cos t)\ dt$$

$$\int_{3\pi/4}^{\pi/4} 64\cos^4 t\ dt$$

$$\int_{3\pi/4}^{\pi/4} 64 \cdot \frac{1}{8}(4\cos 2t + \cos 4t + 3)\ dt$$

$$8\int_{3\pi/4}^{\pi/4} 4\cos 2t + \cos 4t + 3\ dt$$

Since the integral of cosine functions over a 2π-period is

$$8\int_{3\pi/4}^{\pi/4} 4\cos 2t + 3\ dt$$

$$8(2\sin(2t) + 3t)\Big|_{3\pi/4}^{\pi/4}$$

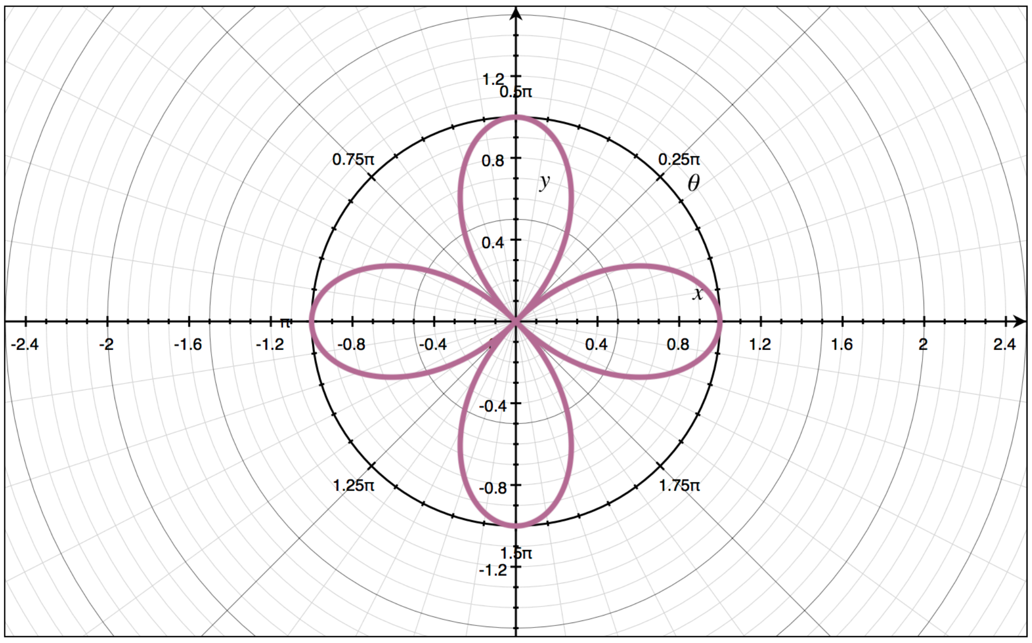
$$8\left[2\sin\left(2\cdot\frac{\pi}{4}\right)+3\cdot\frac{\pi}{4}\right]-8\left[2\sin\left(2\cdot\frac{3\pi}{4}\right)+3\cdot\frac{3\pi}{4}\right]$$

$$8\left(2+\frac{3\pi}{4}\right)-8\left[2\cdot(-1)+\frac{9\pi}{4}\right]=32-12\pi$$

In total, the integral over the region D is

$$6\pi+32-12\pi=32-6\pi$$

■ 3. Use Green's theorem to calculate the line integral of the vector field $\vec{F}(x,y)=\langle e^{x^2}-2y,y^2+2x\rangle$ over the four-petaled rose $r=\cos 2\phi$.



Solution:

The polar equation of the polar four-petaled rose is $r = \cos 2\phi$, where $-\pi/4 \le \phi \le \pi/4$ for the right petal, $\pi/4 \le \phi \le 3\pi/4$ for the bottom petal, $3\pi/4 \le \phi \le 5\pi/4$ for the left petal, and $5\pi/4 \le \phi \le 7\pi/4$ for the top petal.

Consider the standard conversion to polar coordinates.

$$x = r\cos\phi$$

$$y = r\sin\phi$$

$$0 \le \phi \le 2\pi$$

$$r^2 = x^2 + y^2$$

$$dx\ dy = r\ dr\ d\phi$$

Apply Green's theorem for each petal individually. Let P and Q be the components of the vector field.

$$P(x, y) = e^{x^2} - 2y$$

$$Q(x, y) = y^2 + 2x$$

Take partial derivatives.

$$\frac{\partial P}{\partial y} = -2$$

$$\frac{\partial Q}{\partial x} = 2$$

Then the difference between the partial derivatives is

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2 - (-2) = 4$$

Within the region enclosed by the right rose petal, ϕ changes from $-\pi/4$ to $\pi/4$, and r changes from 0 to $\cos 2\phi$. Therefore, the line integral is

$$\iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Within the region enclosed by the bottom rose petal, ϕ changes from $\pi/4$ to $3\pi/4$, so the line integral is

$$\iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{\pi/4}^{3\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Within the region enclosed by the left rose petal, ϕ changes from $3\pi/4$ to $5\pi/4$, so the line integral is

$$\iint_{D_3} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{3\pi/4}^{5\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Within the region enclosed by the top rose petal, ϕ changes from $5\pi/4$ to $7\pi/4$, so the line integral is

$$\iint_{D_4} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\ dA = \int_{5\pi/4}^{7\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Sum the four integrals.

$$\int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi + \int_{\pi/4}^{3\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi + \int_{3\pi/4}^{5\pi/4} \int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

$$+\int_{5\pi/4}^{7\pi/4}\int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

$$\int_{-\pi/4}^{7\pi/4}\int_0^{\cos 2\phi} 4r\ dr\ d\phi$$

Calculate the inner integral by treating ϕ as a constant.

$$\int_0^{\cos 2\phi} 4r\ dr$$

$$2r^2\Big|_0^{\cos 2\phi}$$

$$2\cos^2 2\phi - (2\cdot 0^2)$$

$$2\cos^2 2\phi$$

Calculate the outer integral.

$$\int_{-\pi/4}^{7\pi/4} 2\cos^2 2\phi\ d\phi$$

$$\int_{-\pi/4}^{7\pi/4} \cos 4\phi + 1\ d\phi$$

Since the integral of cosine functions over a 2π-period is

$$\int_{-\pi/4}^{7\pi/4} 1\ d\phi$$

$$\phi\Big|_{-\pi/4}^{7\pi/4}$$

$$\frac{7\pi}{4}-\left(-\frac{\pi}{4}\right)=2\pi$$

W W W . K R I S T A K I N G M A T H . C O M