Lagrange multipliers

We already know how to find critical points of a multivariable function and use the second derivative test to classify those critical points.

But sometimes we're asked to find and classify the critical points of a multivariable function that's subject to a secondary constraint equation.

Multivariable function	$z = f(x, y)$
Constraint equation	$g(x, y) = c$

To find the critical points of a function like this one, we can use the Lagrange Multiplier λ (lambda) to develop a system of simultaneous equations that will allow us to solve for critical points. We'll follow these steps:

1. Bring the constant c in $g(x, y)$ over to the left-hand side, so that you end up with two functions in the same form, $f(x, y) = \ldots$ and $g(x, y) = \ldots$.

2. Take the partial derivatives of $f(x, y)$ and $g(x, y)$.

3. Multiply the partial derivatives of $g(x, y)$ by λ.

4. Create the system of simultaneous equations $\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}\lambda$ and $\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}\lambda$

5. Solve both equations for λ, then set them equal to each other.

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6. Solve for one variable in terms of the other, then plug into the original constraint equation to find values for both x and y.

7. Use the second derivative test to classify the critical point.

$$D(x,y,\lambda) = \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

If $D(x,y) < 0$, then f has a **saddle point** at (x,y)

If $D(x,y) = 0$, then the test is **inconclusive**

If $D(x,y) > 0$ and

$\frac{\partial^2 f}{\partial x^2}(x,y) > 0$, then f has a **local minimum** at (x,y)

$\frac{\partial^2 f}{\partial x^2}(x,y) < 0$, then f has a **local maximum** at (x,y)

Example

Find the maximum and minimum of the function with the given constraint.

$$f(x,y) = x^2 + 2y^2 - 120$$

if $2y - 4x = 24$

$f(x,y)$ is the function, and it's subject to the constraint $2y - 4x = 24$.

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We need to set the constraint equation equal to 0 so that we can get it into the same form as our function, setting it equal to $g(x, y)$.

$$2y - 4x - 24 = 0$$

$$g(x, y) = 2y - 4x - 24$$

Now we can find the partial derivatives of the function $f(x, y)$ and the constraint equation $g(x, y)$.

For $f(x, y)$:

$$\frac{\partial f}{\partial x} = 2x$$

and

$$\frac{\partial f}{\partial y} = 4y$$

For $g(x, y)$:

$$\frac{\partial g}{\partial x} = -4$$

and

$$\frac{\partial g}{\partial y} = 2$$

We'll multiply the partial derivatives of the constraint equation by λ, and get

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$$\frac{\partial g}{\partial x}\lambda = -4\lambda$$

and

$$\frac{\partial g}{\partial y}\lambda = 2\lambda$$

Next, we create this system of simultaneous equations:

$$\frac{\partial f}{\partial x} = \frac{\partial g}{\partial x}\lambda$$

$$2x = -4\lambda$$

$$\lambda = -\frac{x}{2}$$

and

$$\frac{\partial f}{\partial y} = \frac{\partial g}{\partial y}\lambda$$

$$4y = 2\lambda$$

$$\lambda = 2y$$

With two equations for λ, we can set them equal to one another, and then solve for y in terms of x.

$$2y = -\frac{x}{2}$$

$$y = -\frac{x}{4}$$

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Plugging this y-value into the constraint equation and solving for x, we get

$$2y - 4x = 24$$

$$2\left(-\frac{x}{4}\right) - 4x = 24$$

$$-\frac{x}{2} - 4x = 24$$

$$-x - 8x = 48$$

$$-9x = 48$$

$$x = -\frac{48}{9}$$

Now we can plug this real-number value of x into the equation to find a real-number value of y.

$$2y - 4\left(-\frac{48}{9}\right) = 24$$

$$2y + \frac{192}{9} = 24$$

$$2y + \frac{64}{3} = 24$$

$$6y + 64 = 72$$

$$6y = 8$$

$$y = \frac{4}{3}$$

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This tells us that our single critical point is

$$\left(-\frac{48}{9},\frac{4}{3}\right)$$

Now we need to use the second derivative test to classify this critical point. To do so, we'll find the second-order partial derivatives of f.

$$\frac{\partial^2 f}{\partial x^2}=2$$

$$\frac{\partial^2 f}{\partial y^2}=4$$

$$\frac{\partial^2 f}{\partial x \partial y}=0$$

Plug the second-order partial derivatives into the second derivative test formula.

$$D(x,y,\lambda)=\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}-\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x,y,\lambda)=(2)(4)-(0)^2$$

$$D(x,y,\lambda)=8$$

Since we have no variables remaining in $D(x,y,\lambda)$, this means we'll have the same result for all possible points. Because $D>0$, we have to look at $\partial^2 f/\partial x^2$.

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$$\frac{\partial^2 f}{\partial x^2}\left(-\frac{48}{9},\frac{4}{3}\right)=2$$

Since $2 > 0$, the function has a local minimum at this critical point.

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