Green's theorem for two regions

Green's theorem gives us a way to change a line integral into a double integral. If a line integral is particularly difficult to evaluate, then using Green's theorem to change it to a double integral might be a good way to approach the problem.

If we want to find the area of a region which is the union of two simple regions, and the original line integral has the form

$$\oint_c P\ dx + Q\ dy$$

then we can apply Green's theorem to change the line integral into a double integral in the form

$$\iint_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA + \iint_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\ dA$$

where

$\frac{\partial Q}{\partial x}$ is the partial derivative of Q with respect to x

$\frac{\partial P}{\partial y}$ is the partial derivative of P with respect to y

If we choose to use Green's theorem and change the line integral to a double integral, we'll need to find limits of integration for both x and y so that we can evaluate the double integral as an iterated integral. Often the limits for x and y will be given to us in the problem.

Example

Solve the line integral for the triangular region with vertices at $(0,0)$, $(1,1)$ and $(2,0)$.

$$\oint_c \left(5\sin x + 5y\right)\ dx + \left(5x^2 - 3y^2\right)\ dy$$

Since the integral we were given matches the form

$$\oint_c P\ dx + Q\ dy$$

we know we can use Green's theorem to change it to

$$\int\int_{R_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA + \int\int_{R_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\ dA$$

We'll start by finding partial derivatives.

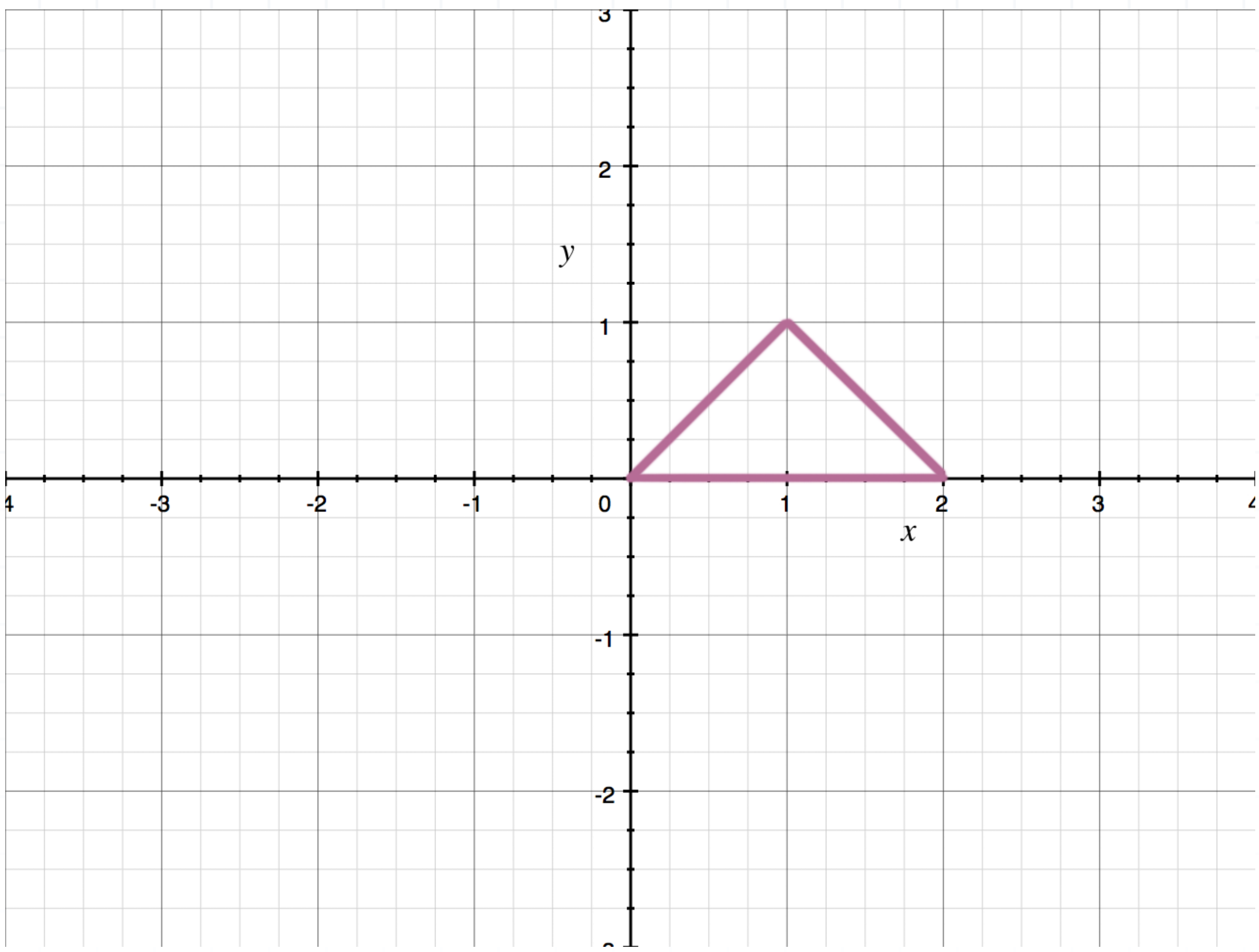
Since $Q(x, y) = 5x^2 - 3y^2$,

$$\frac{\partial Q}{\partial x} = 10x$$

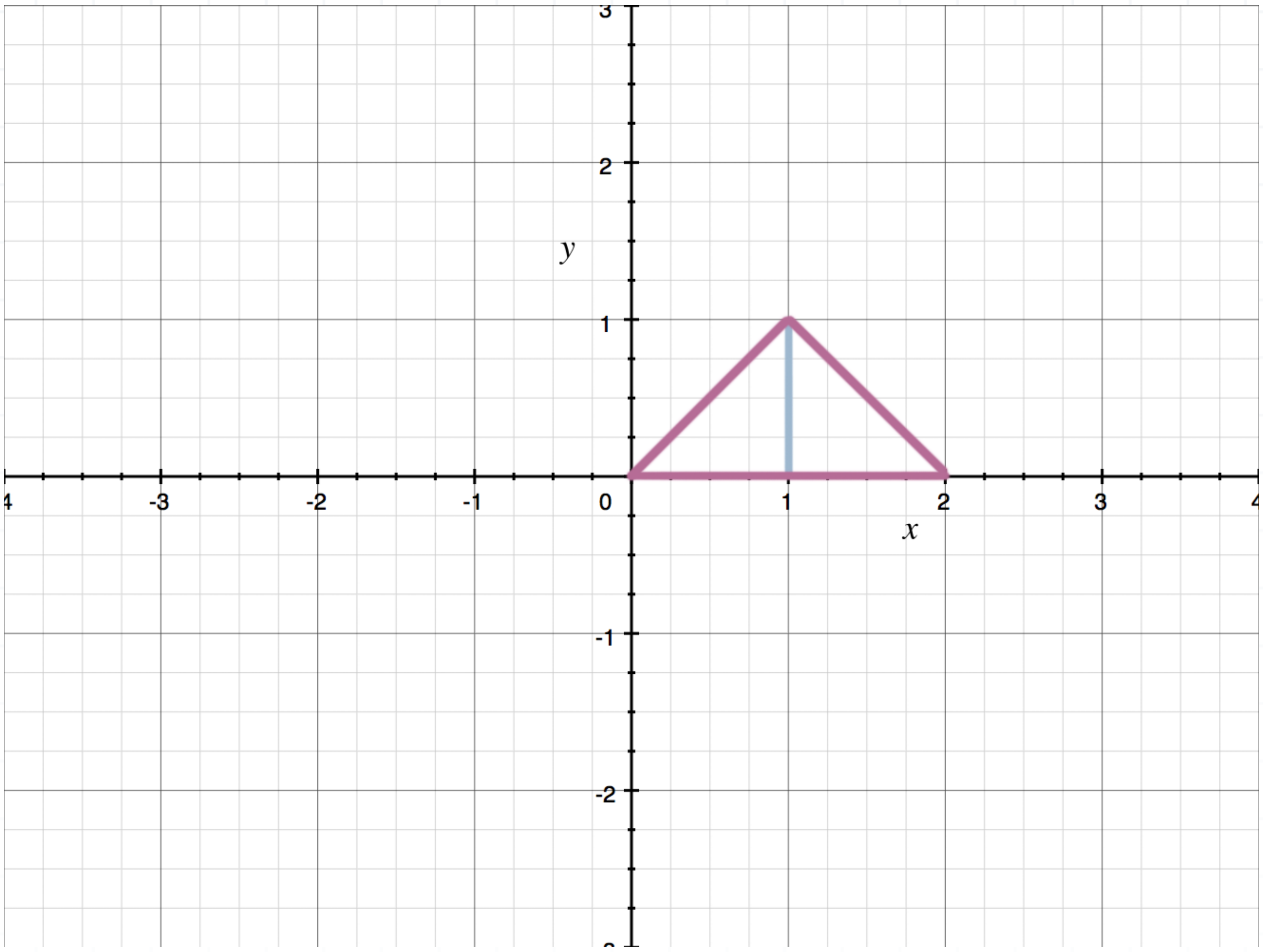
Since $P(x, y) = 5\sin x + 5y$,

$$\frac{\partial P}{\partial y} = 5$$

Now we just need to sketch the region so that we can find limits of integration.



Since the line connecting $(0,0)$ and $(1,1)$ is a different function than the line connecting $(1,1)$ and $(2,0)$, we'll need to divide the region into two parts, separated by the line $x = 1$.



Looking at the sketch of the region, we can say that the region on the left is defined for x on $[0,1]$ and the region on the right is defined for x on $[1,2]$. To find the interval for y for each region, we'll have to find the equation of the lines connecting the points. The equation of the line connecting $(0,0)$ and $(1,1)$ is $y=x$. The equation of the line connecting $(1,1)$ and $(2,0)$ is $y=-x+2$. Therefore, the equation for area is

$$\int_0^1 \int_0^x 10x-5\ dy\ dx+\int_1^2 \int_0^{-x+2} 10x-5\ dy\ dx$$

Now we'll integrate both double integrals with respect to y and evaluate over the associated intervals.

$$\int_0^1 10xy-5y\Big|_{y=0}^{y=x}\ dx+\int_1^2 10xy-5y\Big|_{y=0}^{y=-x+2}\ dx$$

$$\int_0^1 10x^2 - 5x - \left[10x(0) - 5(0)\right] \ dx$$

$$+\int_1^2 10x(-x+2) - 5(-x+2) - \left[10x(0) - 5(0)\right] \ dx$$

$$\int_0^1 10x^2 - 5x \ dx + \int_1^2 -10x^2 + 20x + 5x - 10 \ dx$$

$$\int_0^1 10x^2 - 5x \ dx + \int_1^2 -10x^2 + 25x - 10 \ dx$$

Now we'll integrate with respect to x and evaluate over each interval.

$$\frac{10}{3}x^3 - \frac{5}{2}x^2\Big|_0^1 - \frac{10}{3}x^3 + \frac{25}{2}x^2 - 10x\Big|_1^2$$

$$\frac{10}{3}(1)^3 - \frac{5}{2}(1)^2 - \left[\frac{10}{3}(0)^3 - \frac{5}{2}(0)^2\right]$$

$$-\frac{10}{3}(2)^3 + \frac{25}{2}(2)^2 - 10(2) - \left[-\frac{10}{3}(1)^3 + \frac{25}{2}(1)^2 - 10(1)\right]$$

$$\frac{10}{3} - \frac{5}{2} - \frac{80}{3} + \frac{100}{2} - 20 + \frac{10}{3} - \frac{25}{2} + 10$$

$$\frac{10}{3} - \frac{5}{2} - \frac{80}{3} + 50 - 20 + \frac{10}{3} - \frac{25}{2} + 10$$

$$-\frac{60}{3} - \frac{30}{2} + 40$$

$$-20 - 15 + 40$$

$$5$$

This is the area of the region.