Symmetric equations for the line of intersection of two planes

If two planes intersect, their intersection will always be a line. The only exception is when the planes are identical, in which case they intersect everywhere.

The vector equation for the line of intersection of the planes is given by

$$r = r_0 + tv$$

where r_0 is a point on the line and v is the cross product of the normal vectors of the two planes. If we break this vector equation into parametric equations for the plane, we get

$$x = x_0 + at$$

$$y = y_0 + bt$$

$$z = z_0 + ct$$

Solving each of the parametric equations for t gives

$$t = \frac{x - x_0}{a}$$

$$t = \frac{y - y_0}{b}$$

$$t = \frac{z - z_0}{c}$$

Then, because we have three values all equal to t, we know those values must be equal to each other, and we find

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

where (x_0, y_0, z_0) is a point on the line, and a, b, and c come from the cross product $v = \langle a, b, c \rangle$. These are the symmetric equations for the line of intersection of two planes.

However, realize that any zero value in the cross product vector $v = \langle 0, b, c \rangle$ or $v = \langle a, 0, c \rangle$ or $v = \langle a, b, 0 \rangle$ will result in division by 0 in the symmetric equations. In that case, we'll substitute the corresponding parametric equation in place of the symmetric equation. For instance, given $v = \langle 0, b, c \rangle$, the parametric equation for x will be

$$x = x_0 + at$$

$$x = x_0 + 0t$$

$$x = x_0$$

Then if we wanted to express the equation of the line in terms of its parametric equations, we'd give

$$x = x_0, \ \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Let's do an example so that we can work through a step-by-step process for finding the equation of the line of intersection of two planes.

Example

Find the symmetric equations for the line of intersection of the planes.

$$2x + y - z = 3$$

$$x - y + z = 3$$

The normal vectors for the planes are $a\langle 2,1,-1\rangle$ for the plane $2x + y - z = 3$ and $b\langle 1,-1,1\rangle$ for the plane $x - y + z = 3$. So the cross product of the normal vectors is

$$v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$v = \mathbf{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$v = [(1)(1) - (-1)(-1)]\mathbf{i} - [(2)(1) - (-1)(1)]\mathbf{j} + [(2)(-1) - (1)(1)]\mathbf{k}$$

$$v = (1 - 1)\mathbf{i} - (2 + 1)\mathbf{j} + (-2 - 1)\mathbf{k}$$

$$v = 0\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$$

$$v = \langle 0, -3, -3\rangle$$

To find a point on the line of intersection, we'll set $z = 0$ in both equations.

$$2x + y - z = 3$$

$$2x + y - 0 = 3$$

$$2x + y = 3$$

and

$$x - y + z = 3$$

$$x - y + 0 = 3$$

$$x - y = 3$$

We'll add the equations to find x.

$$(2x + x) + (y - y) = 3 + 3$$

$$3x + 0 = 6$$

$$x = 2$$

Plugging $x = 2$ back into $x - y = 3$, we get

$$2 - y = 3$$

$$-y = 1$$

$$y = -1$$

Putting these values together, the point on the line of intersection is $(2, -1, 0)$. Then with the cross product $v = \langle 0, -3, -3 \rangle$ and $(2, -1, 0)$, we'll build the parametric equations.

$$x = 2 + 0t \qquad y = -1 - 3t \qquad z = 0 - 3t$$

$$x = 2 \qquad z = -3t$$

Solve these equations for the parameter t.

$$x = 2 \qquad t = -\frac{y + 1}{3} \qquad t = -\frac{z}{3}$$

The equation $x = 2$ can't be solved for t, so we'll separate it from the other two equations for t, listing the symmetric equations for the line of intersection of the planes as

$$x = 2, \ -\frac{y+1}{3} = -\frac{z}{3}$$
