**Topic**: Midpoint rule for double integrals

**Question**: The value of the integral is to be estimated on the rectangle $R = [0,2] \times [0,8]$, where the rectangle is divided into $2 \times 2$ subrectangles. Which expression can be used as Midpoint Rule to estimate the value of the integral?

$$\iint_R x^2 + y^2 \ dA$$

**Answer choices**:

A $\displaystyle \iint_R x^2 + y^2 \ dA \approx 4\left(\frac{17}{4} + \frac{25}{4} + \frac{145}{4} + \frac{153}{4}\right)$

B $\displaystyle \iint_R x^2 + y^2 \ dA \approx 4\left(\frac{9}{4} + \frac{25}{4} + \frac{136}{4} + \frac{153}{4}\right)$

C $\displaystyle \iint_R x^2 + y^2 \ dA \approx 2\left(\frac{17}{4} + \frac{25}{4} + \frac{145}{4} + \frac{153}{4}\right)$

D $\displaystyle \iint_R x^2 + y^2 \ dA \approx 2\left(\frac{9}{4} + \frac{25}{4} + \frac{136}{4} + \frac{153}{4}\right)$

**Solution**: A

The rectangle $R$ is bounded by the lines $x=0$, $x=2$, $y=0$, and $y=8$. Because we want $2$ subrectangles across and $2$ subrectangles down, that means that we'll have $2\times 2=4$ total subrectangles, each with dimensions $x\times y=1\times 4$.

Which means, using midpoints, the Riemann sum estimate is given by

$$\iint_R x^2+y^2\ dA\approx\sum_{i=1}^{2}\sum_{j=1}^{2}f(x_i,y_j)\Delta A=\Delta A\left[f\left(\frac{1}{2},2\right)+f\left(\frac{3}{2},2\right)+f\left(\frac{1}{2},6\right)+f\left(\frac{3}{2},6\right)\right]$$

Plugging each of the midpoints into the given integrand gives

$$\iint_R x^2+y^2\ dA\approx 4\left[\left(\frac{1}{4}+4\right)+\left(\frac{9}{4}+4\right)+\left(\frac{1}{4}+36\right)+\left(\frac{9}{4}+36\right)\right]$$

$$\iint_R x^2+y^2\ dA\approx 4\left[\left(\frac{1}{4}+\frac{16}{4}\right)+\left(\frac{9}{4}+\frac{16}{4}\right)+\left(\frac{1}{4}+\frac{144}{4}\right)+\left(\frac{9}{4}+\frac{144}{4}\right)\right]$$

$$\iint_R x^2+y^2\ dA\approx 4\left(\frac{17}{4}+\frac{25}{4}+\frac{145}{4}+\frac{153}{4}\right)$$

**Topic**: Midpoint rule for double integrals

**Question**: Using midpoint rule to estimate the double integral gave $3{,}584$. Which dimensions describe the rectangle $R$ beneath the volume if $\Delta A = 16$, and the rectangle is divided into $m = 2$ sub-squares across by $n = 4$ sub-squares down?

$$\iint_R xy + x - y \ dA$$

**Answer choices**:

A $\quad R = [-8{,}8] \times [0{,}16]$

B $\quad R = [0{,}8] \times [-8{,}16]$

C $\quad R = [0{,}8] \times [0{,}16]$

D $\quad R = [0{,}16] \times [0{,}8]$

**Solution**: C

Because the problem says that we are dividing the underlying rectangle $R$ into sub-squares, we know that $\Delta A$ must be given by

$$x^2 = \Delta A$$

$$x^2 = 16$$

$$x = 4$$

So the dimensions of the sub-squares are $4 \times 4$. Then, because we know that we have $2$ rectangles across by $4$ rectangles down, using the Midpoint Rule with the information we've been given, plus the dimensions from answer choice C, we get

$$\iint_R xy + x - y \ dA \approx \sum_{i=1}^{2} \sum_{j=1}^{4} f(x_i, y_j) \Delta A$$

$$= \Delta A \left[ f(2,2) + f(2,6) + f(2,10) + f(2,14) + f(6,2) + f(6,6) + f(6,10) + f(6,14) \right]$$

Plugging each of the midpoints into the integrand gives

$$V \approx 16 [((2)(2) + 2 - 2) + ((2)(6) + 2 - 6) + ((2)(10) + 2 - 10) + ((2)(14) + 2 - 14)$$

$$+((6)(2) + 6 - 2) + ((6)(6) + 6 - 6) + ((6)(10) + 6 - 10) + ((6)(14) + 6 - 14)]$$

$$V \approx 16 [(4) + (12 - 4) + (20 - 8) + (28 - 12) + (12 + 4) + (36) + (60 - 4) + (84 - 8)]$$

$$V \approx 16(4 + 8 + 12 + 16 + 16 + 36 + 56 + 76)$$

$$V \approx 3{,}584$$

**Topic**: Midpoint rule for double integrals

**Question**: The double integral is defined on both the rectangle $K$ and the square $L$. Using midpoint rule, which is the approximation of $V_K - V_L$?

$$\int\int x^2 - y \ dA$$

| | | |
|---|---|---|
| Rectangle $K$: | $K = [0,6] \times [0,4]$ | $m = 3, \ n = 2, \ \Delta A = 4$ |
| Square $L$: | $L = [0,4] \times [0,4]$ | $m = 2, \ n = 1, \ \Delta A = 8$ |

**Answer choices**:

A $\quad$ 166

B $\quad$ 184

C $\quad$ 128

D $\quad$ 126

**Solution**: B

For rectangle $K$, given $K = [0,6] \times [0,4]$ and that $m = 3$, $n = 2$, and $\Delta A = 4$, we must be dividing $K$ into sub-squares with dimensions $2 \times 2$, such that we have $m = 3$ squares across between $x = 0$ and $x = 6$, and $n = 2$ squares down between $y = 0$ and $y = 4$.

Therefore, we can set up the Riemann sum with midpoints as

$$\iint_K x^2 - y \ dA \approx \sum_{i=1}^{3} \sum_{j=1}^{2} f(x_i, y_j) \Delta A$$

$$= \Delta A \left[ f(1,1) + f(1,3) + f(3,1) + f(3,3) + f(5,1) + f(5,3) \right]$$

If we plug these points into the integrand, and add in $\Delta A = 4$, we get

$$V_K \approx 4 \left[ \left(1^2 - 1\right) + \left(1^2 - 3\right) + \left(3^2 - 1\right) + \left(3^2 - 3\right) + \left(5^2 - 1\right) + \left(5^2 - 3\right) \right]$$

$$V_K \approx 4(0 - 2 + 8 + 6 + 24 + 22)$$

$$V_K \approx 232$$

For square $L$, given $L = [0,4] \times [0,4]$ and that $m = 2$, $n = 1$, and $\Delta A = 8$, we must be dividing $L$ into sub-rectangles with dimensions $2 \times 4$, such that we have $m = 2$ rectangles across between $x = 0$ and $x = 4$, and $n = 1$ rectangle down between $y = 0$ and $y = 4$.

Therefore, we can set up the Riemann sum with midpoints as

$$\iint_L x^2 - y \ dA \approx \sum_{i=1}^{2} \sum_{j=1}^{1} f(x_i, y_j) \Delta A$$

$$= \Delta A\left[f(1,2) + f(3,2)\right]$$

If we plug these points into the integrand, and add in $\Delta A = 8$, we get

$$V_L \approx 8\left[\left(1^2 - 2\right) + \left(3^2 - 2\right)\right]$$

$$V_L \approx 8(-1 + 7)$$

$$V_L \approx 48$$

Therefore,

$$V_K - V_L = 232 - 48$$

$$V_K - V_L = 184$$