Calculus 3 Workbook Solutions

Line integrals

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LINE INTEGRAL OF A CURVE

■ 1. Calculate the line integral over c, where c is the circle that lies in the plane $z=3$, with center on the z-axis and radius 4.

$$\int_c x^2+y^2+z^2\ ds$$

Solution:

The line integral over the curve for the function $f(x,y,z)$ is given by the formula

$$\int_c f(x,y,z)\ ds=\int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$$

Consider the parametrization of the circle.

$$x(t)=4\cos t$$

$$y(t)=4\sin t$$

$$z(t)=3$$

Over the whole circle, t changes from 0 to 2π, so $a=0$ and $b=2\pi$. Find the first-order derivatives.

$$x'(t)=-4\sin t$$

$$y'(t) = 4\cos t$$

$$z'(t) = 0$$

The function is

$$f(x(t), y(t), z(t)) = (4\cos t)^2 + (4\sin t)^2 + 3^2$$

$$f(x(t), y(t), z(t)) = 16(\cos^2 t + \sin^2 t) + 9$$

$$f(x(t), y(t), z(t)) = 16 + 9$$

$$f(x(t), y(t), z(t)) = 25$$

Therefore, the line integral over the curve is

$$\int_c f(x, y, z)\ ds = \int_0^{2\pi} 25\sqrt{(-4\sin t)^2 + (4\cos t)^2 + 0^2}\ dt$$

$$\int_c f(x, y, z)\ ds = \int_0^{2\pi} 25\sqrt{16\sin^2 t + 16\cos^2 t}\ dt$$

$$\int_c f(x, y, z)\ ds = \int_0^{2\pi} 25\sqrt{16}\ dt$$

$$\int_c f(x, y, z)\ ds = 100\int_0^{2\pi}\ dt$$

$$\int_c f(x, y, z)\ ds = 100\,(2\pi) = 200\pi$$

■ 2. Calculate the line integral P over c, where c is the part of the graph of the vector function $\vec{r}(t)$ between the points $(-2,6,-2)$ and $(4,9,1)$.

$$\vec{r}(t) = \langle 2t, t^2+5, t-1 \rangle$$

$$P = \int_c (y-z^2)\sqrt{5+x^2}\ ds$$

Solution:

The line integral over the curve for the function $f(x,y,z)$ is given by

$$\int_c f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\ dt$$

In order to find the value of $t=a$ that corresponds to $(-2,6,-2)$, solve the system of equations for t.

$$2t = -2$$

$$t^2+5=6$$

$$t-1=-2$$

By solving the system, we get $a=t=-1$. In order to find the value of $t=b$ that corresponds to $(4,9,1)$, solve the system of equations for t.

$$2t=4$$

$$t^2+5=9$$

$$t - 1 = 1$$

By solving the system, we get $b = t = 2$. Find the first-order derivatives.

$$x'(t) = 2$$

$$y'(t) = 2t$$

$$z'(t) = 1$$

The function is

$$f(x(t), y(t), z(t)) = (y(t) - z(t)^2)\sqrt{5 + x(t)^2}$$

$$f(x(t), y(t), z(t)) = (t^2 + 5 - (t - 1)^2)\sqrt{5 + (2t)^2}$$

$$f(x(t), y(t), z(t)) = (2t + 4)\sqrt{5 + 4t^2}$$

Therefore, the line integral over the curve is

$$\int_c f(x, y, z)\ ds = \int_{-1}^{2} (2t + 4)\sqrt{5 + 4t^2}\sqrt{2^2 + (2t)^2 + 1^2}\ dt$$

$$\int_c f(x, y, z)\ ds = \int_{-1}^{2} (2t + 4)\sqrt{5 + 4t^2}\sqrt{5 + 4t^2}\ dt$$

Since $5 + 4t^2 > 0$,

$$\int_c f(x, y, z)\ ds = \int_{-1}^{2} (2t + 4)(5 + 4t^2)\ dt$$

$$\int_c f(x,y,z)\ ds = \int_{-1}^{2} 8t^3 + 16t^2 + 10t + 20\ dt$$

$$\int_c f(x,y,z)\ ds = 2t^4 + \frac{16}{3}t^3 + 5t^2 + 20t\Big|_{-1}^{2}$$

$$\int_c f(x,y,z)\ ds = \left[2(2)^4 + \frac{16}{3}(2)^3 + 5(2)^2 + 20(2)\right] - \left[2(-1)^4 + \frac{16}{3}(-1)^3 + 5(-1)^2 + 20(-1)\right]$$

$$\int_c f(x,y,z)\ ds = \frac{404}{3} + \frac{55}{3} = 153$$

■ 3. Calculate the improper line integral over c, where c is the line of intersection of the surfaces $z - x^2 - y^2 + 2y + 1 = 0$ and $x - y - 1 = 0$.

$$\int_c \frac{1}{(1 + 8(x-1)y)^2}\ ds$$

Solution:

The line integral over the curve for the function $f(x,y,z)$ is given by

$$\int_c f(x,y,z)\ ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\ dt$$

Find the parametrization for $f(x(t),y(t),z(t))$. Let $y(t) = t$, then since $x - y - 1 = 0$,

$$x(t) = y(t) + 1 = t + 1$$

Substitute $x(t)$ and $y(t)$ into the equation $z - x^2 - y^2 + 2y + 1 = 0$.

$$z(t) - (t+1)^2 - t^2 + 2t + 1 = 0$$

$$z(t) = (t+1)^2 + t^2 - 2t - 1$$

$$z(t) = 2t^2$$

So the parametrization is

$$x(t) = t + 1$$

$$y(t) = t$$

$$z(t) = 2t^2$$

where t changes from $-\infty$ to ∞. Find the first-order derivatives.

$$x'(t) = 1$$

$$y'(t) = 1$$

$$z'(t) = 4t$$

The function is

$$f(x(t), y(t), z(t)) = \frac{1}{(1 + 8(x(t) - 1)y(t))^2}$$

$$f(x(t), y(t), z(t)) = \frac{1}{(1 + 8(t + 1 - 1)t)^2}$$

$$f(x(t), y(t), z(t)) = \frac{1}{(1 + 8t^2)^2}$$

Therefore, the line integral over the curve is

$$\int_c f(x,y,z)\ ds=\int_{-\infty}^{\infty}\frac{1}{(1+8t^2)^2}\sqrt{1^2+1^2+(4t)^2}\ dt$$

$$\int_c f(x,y,z)\ ds=\int_{-\infty}^{\infty}\frac{\sqrt{2+16t^2}}{(1+8t^2)^2}\ dt$$

$$\int_c f(x,y,z)\ ds=\int_{-\infty}^{\infty}\frac{\sqrt{2}\sqrt{1+8t^2}}{(1+8t^2)^2}\ dt$$

$$\int_c f(x,y,z)\ ds=\sqrt{2}\int_{-\infty}^{\infty}\frac{\sqrt{1+8t^2}}{(1+8t^2)^2}\ dt$$

Since the function under the integral is even,

$$\int_c f(x,y,z)\ ds=2\sqrt{2}\int_{0}^{\infty}\frac{\sqrt{1+8t^2}}{(1+8t^2)^2}\ dt$$

Make trigonometric substitution with

$$t=\frac{\tan u}{\sqrt{8}}\text{ and }dt=\frac{\sec^2 u}{\sqrt{8}}\ du$$

So $1+8t^2=1+\tan^2 u=\sec^2 u$, and u changes from 0 to $\pi/2$.

$$\int_c f(x,y,z)\ ds=2\sqrt{2}\int_{0}^{\pi/2}\frac{\sqrt{\sec^2 u}}{(\sec^2 u)^2}\left(\frac{\sec^2 u}{\sqrt{8}}\right)\ du$$

$$\int_c f(x,y,z)\ ds=\int_{0}^{\pi/2}\frac{\sec u}{\sec^4 u}\sec^2 u\ \ du$$

$$\int_c f(x, y, z)\ ds = \int_0^{\pi/2} \frac{1}{\sec u}\ du$$

$$\int_c f(x, y, z)\ ds = \int_0^{\pi/2} \cos u\ du$$

$$\int_c f(x, y, z)\ ds = (\sin u)\Big|_0^{\pi/2} = 1$$

LINE INTEGRAL OF A VECTOR FUNCTION

■ 1. Calculate the line integral of the vector function $\vec{F}(x,y) = \langle x + y, x - y \rangle$ over the curve $\vec{r}(t) = \langle t^2 - 1, t^2 + 1 \rangle$ for $-2 \le t \le 3$.

Solution:

The line integral of the vector function $\vec{F}(x,y)$ over the curve $\vec{r}(t)$ is given by

$$\int_c \vec{F}(x,y)\ ds = \int_a^b \vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t)\ dt$$

Plug $x(t) = t^2 - 1$ and $y(t) = t^2 + 1$ into the expression for $F(x,y)$.

$$\vec{F}(x(t),y(t)) = \langle t^2 - 1 + t^2 + 1, t^2 - 1 - (t^2 + 1) \rangle$$

$$\vec{F}(x(t),y(t)) = \langle 2t^2, -2 \rangle$$

Find the first-order derivative.

$$\vec{r}\,'(t) = \langle 2t, 2t \rangle$$

The dot product is

$$\vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t) = \langle 2t^2, -2 \rangle \cdot \langle 2t, 2t \rangle$$

$$\vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t) = 2t^2\,(2t) - 2\,(2t)$$

$$\vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t) = 4t^3 - 4t$$

Therefore, the line integral over the curve is

$$\int_c F(x,y)\,ds = \int_{-2}^{3} 4t^3 - 4t\ dt$$

$$\int_c F(x,y)\,ds = t^4 - 2t^2 \Big|_{-2}^{3}$$

$$\int_c F(x,y)\,ds = [3^4 - 2(3)^2] - [(-2)^4 - 2(-2)^2] = 55$$

■ 2. Calculate the line integral of the vector function $\overrightarrow{F}(x,y,z) = \langle xyz, -z, y \rangle$ over c, where c is the ellipse that lies in the plane $x = -4$ with the center on the x-axis, a semi-axis of 2 in the y-direction, and a semi-axis of 5 in the z-direction.

Solution:

The line integral of the vector function $\overrightarrow{F}(x,y,z)$ over the curve $\overrightarrow{r}(t)$ is given by the formula

$$\int_c \overrightarrow{F}(x,y,z)\ ds = \int_a^b \overrightarrow{F}(x(t),y(t),z(t)) \cdot \overrightarrow{r}'(t)\ dt$$

where $\overrightarrow{F}(x(t),y(t),z(t)) \cdot \overrightarrow{r}'(t)$ is the dot product of the two vectors.

Parametrize the ellipse.

$$x(t) = -4$$

$$y(t) = 2\cos t$$

$$z(t) = 5\sin t$$

Over the whole ellipse, t changes from 0 to 2π, so $a = 0$ and $b = 2\pi$.

Plug $x(t) = -4$, $y(t) = 2\cos t$, and $z(t) = 5\sin t$ into the expression for $F(x, y, z)$.

$$\vec{F}(x(t), y(t), z(t)) = \langle -4 \cdot 2\cos t \cdot 5\sin t, -5\sin t, 2\cos t\rangle$$

$$\vec{F}(x(t), y(t), z(t)) = \langle -40\cos t\sin t, -5\sin t, 2\cos t\rangle$$

$$\vec{F}(x(t), y(t), z(t)) = \langle -20\sin 2t, -5\sin t, 2\cos t\rangle$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}\,'(t) = \langle 0, -2\sin t, 5\cos t\rangle$$

The dot product is

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t) = \langle -20\sin 2t, -5\sin t, 2\cos t\rangle \cdot \langle 0, -2\sin t, 5\cos t\rangle$$

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t) = 10\sin^2 t + 10\cos^2 t$$

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}\,'(t) = 10$$

Therefore, the line integral over the curve is

$$\int_C F(x, y, z)\ ds = \int_0^{2\pi} 10\ dt = 10(2\pi) = 20\pi$$

■ 3. Calculate the improper line integral of the vector function $\vec{F}(x,y,z)$ over the curve $\vec{r}(t)=\langle e^t,-e^{-t},2t\rangle$ for $t\geq 0$.

$$\vec{F}(x,y,z)=\left\langle y^2,\frac{3}{x^2},2xy^2z\right\rangle$$

Solution:

The line integral of the vector function $\vec{F}(x,y,z)$ over the curve $\vec{r}(t)$ is given by

$$\int_C \vec{F}(x,y,z)\ ds=\int_a^b \vec{F}(x(t),y(t),z(t))\cdot\vec{r}\,'(t)\ dt$$

where $\vec{F}(x(t),y(t),z(t))\cdot\vec{r}\,'(t)$ is the dot product of the two vectors.

Plug $x(t)=e^t$, $y(t)=-e^{-t}$, and $z(t)=2t$ into the expression for $F(x,y,z)$.

$$\vec{F}(x(t),y(t),z(t))=\left\langle(-e^{-t})^2,\frac{3}{(e^t)^2},\ 2e^t\cdot(-e^{-t})^2\cdot 2t\right\rangle$$

$$\vec{F}(x(t),y(t),z(t))=\left\langle e^{-2t},\frac{3}{e^{2t}},4te^{-t}\right\rangle$$

$$\vec{F}(x(t),y(t),z(t))=\langle e^{-2t},3e^{-2t},4te^{-t}\rangle$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}\,'(t)=\langle e^t,e^{-t},2\rangle$$

The dot product is

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t) = \langle e^{-2t}, 3e^{-2t}, 4te^{-t} \rangle \cdot \langle e^t, e^{-t}, 2 \rangle$$

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t) = e^{-2t}e^t + 3e^{-2t}e^{-t} + 8te^{-t}$$

$$\vec{F}(x(t), y(t), z(t)) \cdot \vec{r}'(t) = e^{-t} + 3e^{-3t} + 8te^{-t}$$

Therefore, the line integral over the curve is

$$\int_c F(x, y, z)\ ds = \int_0^\infty e^{-t} + 3e^{-3t} + 8te^{-t}\ dt$$

$$\int_c F(x, y, z)\ ds = \int_0^\infty e^{-t}\ dt + 3\int_0^\infty e^{-3t}\ dt + 8\int_0^\infty te^{-t}\ dt$$

Take the integral of te^{-t} using integration by parts with $u = t$, $du = dt$, $dv = e^{-t}\ dt$, and $v = -e^{-t}$.

$$\int te^{-t}\ dt = -te^{-t} + \int e^{-t}\ dt$$

So

$$\int_0^\infty e^{-t}\ dt + 3\int_0^\infty e^{-3t}\ dt + 8\int_0^\infty te^{-t}\ dt$$

$$-e^{-t}\Big|_0^\infty + (-e^{-3t})\Big|_0^\infty + 8(-te^{-t})\Big|_0^\infty + 8(-e^{-t})\Big|_0^\infty$$

$$9(-e^{-t})\Big|_0^\infty + (-e^{-3t})\Big|_0^\infty + 8(-te^{-t})\Big|_0^\infty$$

$$9\lim_{t\to\infty}(-e^{-t}) - 9(-e^0) + \lim_{t\to\infty}(-e^{-3t}) - (-e^0) + 8\lim_{t\to\infty}(-te^{-t}) - 8(-0 \cdot e^0)$$

$$9 \cdot 0 + 9 + 0 + 1 + 8 \cdot 0 - 8 \cdot 0 = 10$$

POTENTIAL FUNCTION OF A CONSERVATIVE VECTOR FIELD

■ 1. Determine whether or not the vector field is conservative.

$$\vec{F}(x, y, z) = \left\langle \ln(2y+z), \frac{2x}{2y+z}, \frac{x}{2y+z} \right\rangle$$

Solution:

If a vector field $F: R^3 \to R^3$ is continuously differentiable in a simply-connected domain $W \in R^3$ and its curl is zero, then F is conservative within the domain W.

$$\vec{F}(x, y, z) = \langle F_x, F_y, F_z \rangle$$

Recall that the curl of a vector field in three dimensions is given by

$$\text{curl } F = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

In other words, curl $F = \vec{0}$ if

$$\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z}$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

The domain of the vector field is the half-space $2y + z > 0$, which is a simply-connected space.

Check if the curl is zero. We get

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x}{2y+z}\right) = -\frac{2x}{(2y+z)^2}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}\left(\frac{2x}{2y+z}\right) = -\frac{2x}{(2y+z)^2}$$

and

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(\ln(2y+z)) = \frac{1}{2y+z}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x}{2y+z}\right) = \frac{1}{2y+z}$$

and

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2x}{2y+z}\right) = \frac{2}{2y+z}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(\ln(2y+z)) = \frac{2}{2y+z}$$

Because each set of partial derivatives is equivalent, the curl is 0.

2. Find the potential function of the vector field.

$$\vec{F}(x,y) = \langle \cos(x-3y)+5, -3\cos(x-3y)-8 \rangle$$

Solution:

A potential function $f(x,y)$ of a vector field $\vec{F}(x,y)$ satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x,y) = F_x(x,y) \text{ and } \frac{\partial f}{\partial y}(x,y) = F_y(x,y)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x,y) = \cos(x-3y)+5$$

Integrate both sides with respect to x, treating y as a constant.

$$f(x,y) = \int \cos(x-3y)+5\ dx$$

$$f(x,y) = \sin(x-3y)+5x+C(y)$$

Differentiate $f(x,y)$ with respect to y, treating x as a constant.

$$\frac{\partial f}{\partial y}(x,y) = \frac{\partial}{\partial y}\left(\sin(x-3y)+5x+C(y)\right) = -3\cos(x-3y)+C'(y)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x,y) = -3\cos(x-3y)-8$$

$$-3\cos(x-3y)+C'(y)=-3\cos(x-3y)-8$$

$$C'(y)=-8$$

Integrate both sides with respect to y.

$$C(y)=\int -8\ dy=-8y+c$$

Therefore,

$$f(x,y)=\sin(x-3y)+5x-8y+c$$

For any conservative vector field, there exist an infinite number of possible potential functions, which each vary by an additive constant c.

3. Find the potential function of the vector field.

$$\vec{F}(x,y,z)=\langle z^2\,2^{x+4y}\ln 2,\ z^2\,2^{x+4y+2}\ln 2,\ z\,2^{x+4y+1}-6z^2\rangle$$

Solution:

A potential function $f(x,y,z)$ of a vector field $\vec{F}(x,y,z)$ satisfies the equality $\nabla f=\vec{F}$, or

$$\frac{\partial f}{\partial x}(x,y,z)=F_x(x,y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z)=F_y(x,y,z)$$

$$\frac{\partial f}{\partial z}(x,y,z) = F_z(x,y,z)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x,y,z) = z^2 2^{x+4y} \ln 2$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x,y,z) = \int z^2 2^{x+4y} \ln 2 \ dx$$

$$f(x,y,z) = z^2 2^{4y} \int 2^x \ln 2 \ dx$$

$$f(x,y,z) = z^2 2^{4y} \cdot 2^x + C(y,z)$$

So

$$f(x,y,z) = z^2 2^{x+4y} + C(y,z)$$

Differentiate $f(x,y,z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x,y,z) = \frac{\partial}{\partial y}(z^2 2^{x+4y} + C(y,z))$$

$$\frac{\partial f}{\partial y}(x,y,z) = 4z^2 2^{x+4y} \ln 2 + \frac{\partial C}{\partial y}(y,z)$$

$$\frac{\partial f}{\partial y}(x,y,z) = z^2 2^{x+4y+2} \ln 2 + \frac{\partial C}{\partial y}(y,z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x,y,z) = z^2\,2^{x+4y+2}\ln 2$$

$$z^2\,2^{x+4y+2}\ln 2 + \frac{\partial C}{\partial y}(y,z) = z^2\,2^{x+4y+2}\ln 2$$

$$\frac{\partial C}{\partial y}(y,z) = 0$$

Therefore, $C(y,z)$ is a constant in terms of y, or

$$C(y,z) = C(z)$$

So

$$f(x,y,z) = z^2\,2^{x+4y} + C(z)$$

Similarly, differentiate $f(x,y,z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x,y,z) = \frac{\partial}{\partial z}(z^2\,2^{x+4y} + C(z))$$

$$\frac{\partial f}{\partial z}(x,y,z) = 2z\,2^{x+4y} + C'(z)$$

$$\frac{\partial f}{\partial z}(x,y,z) = z\,2^{x+4y+1} + C'(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x,y,z) = z\,2^{x+4y+1} - 6z^2$$

So

$$z\,2^{x+4y+1} + C'(z) = z\,2^{x+4y+1} - 6z^2$$

$$C'(z) = -\,6z^2$$

Integrate both sides with respect to z.

$$C(z) = \int -\,6z^2\ dz$$

$$C(z) = -\,2z^3 + c$$

Therefore,

$$f(x, y, z) = z^2\,2^{x+4y} - 2z^3 + c$$

For any conservative vector field, there exist an infinite number of possible potential functions, which vary by an additive constant c.

POTENTIAL FUNCTION OF A CONSERVATIVE VECTOR FIELD TO EVALUATE A LINE INTEGRAL

■ 1. Calculate the line integral of the conservative vector field $\vec{F}(x,y)$ over the curve $\vec{r}(t)=\langle 9\arctan^2 t, t^4-2t^2+2\rangle$ between $(0,2)$ and $(\pi^2,5)$.

$$\vec{F}(x,y)=\left\langle \frac{y}{\sqrt{x}}, 2(y+\sqrt{x})\right\rangle$$

Solution:

The line integral of the conservative vector field is independent of the curve, and can be calculated as

$$\int_c \vec{F}\cdot d\vec{r}=\int_c \nabla f\cdot d\vec{r}=f(\vec{r}_2)-f(\vec{r}_1)$$

Find the potential function $f(x,y)$ of a vector field $\vec{F}(x,y)$ that satisfies the equality $\nabla f=\vec{F}$, or

$$\frac{\partial f}{\partial x}(x,y)=F_x(x,y) \text{ and } \frac{\partial f}{\partial y}(x,y)=F_y(x,y)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x,y)=\frac{y}{\sqrt{x}}$$

Integrate both sides with respect to x, treating y as a constant.

$$f(x, y) = \int \frac{y}{\sqrt{x}}\ dx$$

$$f(x, y) = 2y\sqrt{x} + C(y)$$

Differentiate $f(x, y)$ with respect to y, treating x as a constant.

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}(2y\sqrt{x} + C(y)) = 2\sqrt{x} + C'(y)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y) = 2(y + \sqrt{x})$$

So

$$2\sqrt{x} + C'(y) = 2(y + \sqrt{x})$$

$$C'(y) = 2y$$

Integrate both sides with respect to y.

$$C(y) = \int 2y\ dy = y^2 + c$$

Therefore,

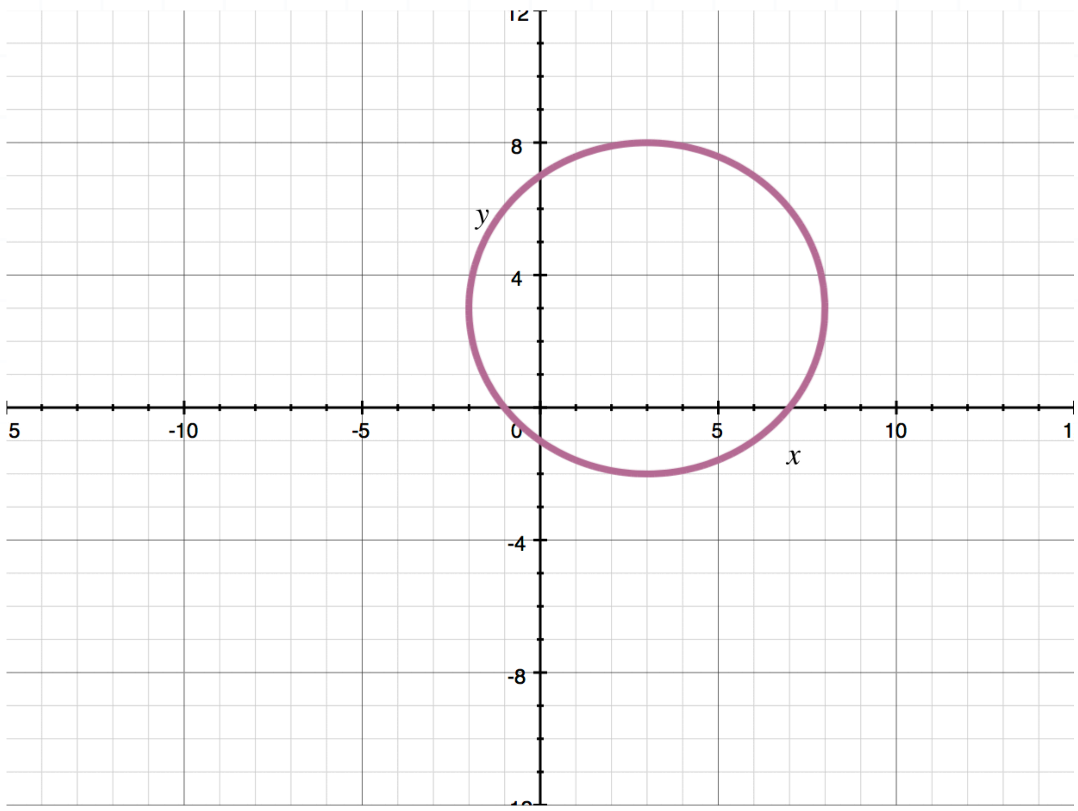
$$f(x, y) = 2y\sqrt{x} + y^2 + c$$

So the line integral is

$$\int_C \vec{F} \cdot d\vec{r} = f(\pi^2,5) - f(0,2)$$

$$\int_C \vec{F} \cdot d\vec{r} = 2 \cdot 5\sqrt{\pi^2} + 5^2 + c - (2 \cdot 2\sqrt{0} + 2^2 + c) = 10\pi + 21$$

■ 2. Calculate the line integral of the conservative vector field $\vec{F}(x,y) = \langle x^2 + y^2, 2xy + 1 \rangle$ over the part of the circle with center at $(3,3)$ and radius 5, that lies in the first quadrant, with clockwise rotation.



Solution:

The line integral of the conservative vector field is independent of the curve, and can be calculated as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f(\vec{r_2}) - f(\vec{r_1})$$

Let's find the potential function $f(x, y)$ of a vector field $\vec{F}(x, y)$ that satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y) = F_x(x, y) \text{ and } \frac{\partial f}{\partial y}(x, y) = F_y(x, y)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x, y) = x^2 + y^2$$

Integrate both sides with respect to x, treating y as a constant.

$$f(x, y) = \int x^2 + y^2 \ dx$$

$$f(x, y) = \frac{x^3}{3} + xy^2 + C(y)$$

Differentiate $f(x, y)$ with respect to y, treating x as a constant.

$$\frac{\partial f}{\partial y}(x, y) = \frac{\partial}{\partial y}\left(\frac{x^3}{3} + xy^2 + C(y)\right) = 2xy + C'(y)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y) = 2xy + 1$$

So

$$2xy + C'(y) = 2xy + 1$$

$$C'(y) = 1$$

Integrate both sides with respect to y.

$$C(y) = \int 1\ dy = y + c$$

Therefore,

$$f(x, y) = \frac{x^3}{3} + xy^2 + y + c$$

The initial point of the curve is $(0,7)$, and the terminal point is $(7,0)$, so the line integral is

$$\int_c \vec{F} \cdot d\vec{r} = f(7,0) - f(0,7)$$

$$\int_c \vec{F} \cdot d\vec{r} = \frac{7^3}{3} + 7 \cdot 0^2 + 0 + c - \left(\frac{0^3}{3} + 0 \cdot 7^2 + 7 + c\right) = \frac{322}{3}$$

■ 3. Calculate the line integral of the conservative vector field $\vec{F}(x, y, z) = \langle y^2, 2xy, (1 + z)^{-1} \rangle$ over the curve $\vec{r}(t) = \langle \sin(\pi t^2), t^3 e^{t-1}, (t - 2)^2 \rangle$ for $1 \le t \le 2$.

Solution:

The line integral of the conservative vector field is independent of the curve, and can be calculated as

$$\int_c \vec{F} \cdot d\vec{r} = \int_c \nabla f \cdot d\vec{r} = f(\vec{r_2}) - f(\vec{r_1})$$

Find the potential function $f(x, y, z)$ of a vector field $\vec{F}(x, y, z)$ that satisfies the equality $\nabla f = \vec{F}$, or

$$\frac{\partial f}{\partial x}(x, y, z) = F_x(x, y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

From the first equation,

$$\frac{\partial f}{\partial x}(x, y, z) = y^2$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x, y, z) = \int y^2 \ dx = xy^2 + C(y, z)$$

Differentiate $f(x, y, z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}(xy^2 + C(y, z))$$

$$\frac{\partial f}{\partial y}(x, y, z) = 2xy + \frac{\partial C}{\partial y}(y, z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y, z) = 2xy$$

So

$$2xy + \frac{\partial C}{\partial y}(y, z) = 2xy$$

$$\frac{\partial C}{\partial y}(y, z) = 0$$

Therefore, $C(y, z)$ is a constant in terms of y, or

$$C(y, z) = C(z)$$

So

$$f(x, y, z) = xy^2 + C(z)$$

Similarly, differentiate $f(x, y, z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{\partial}{\partial z}(xy^2 + C(z)) = C'(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x, y, z) = (1 + z)^{-1}$$

So

$$C'(z) = (1 + z)^{-1}$$

Integrate both sides with respect to z.

$$C(z) = \int (1+z)^{-1} \ dz$$

$$C(z) = \ln(1+z) + c$$

Therefore,

$$f(x, y, z) = xy^2 + \ln(1+z) + c$$

Calculate the endpoints of the curve.

$$\vec{r}(1) = \langle \sin(\pi \cdot 1^2), 1^3 e^{1-1}, (1-2)^2 \rangle = \langle 0,1,1 \rangle$$

$$\vec{r}(2) = \langle \sin(\pi \cdot 2^2), 2^3 e^{2-1}, (2-2)^2 \rangle = \langle 0,8e,0 \rangle$$

So the line integral is

$$\int_c \vec{F} \cdot d\vec{r} = f(0,8e,0) - f(0,1,1)$$

$$\int_c \vec{F} \cdot d\vec{r} = 0 \cdot (8e)^2 + \ln(1+0) + c - (0 \cdot 1^2 + \ln(1+1) + c)$$

$$\int_c \vec{F} \cdot d\vec{r} = -\ln 2$$

INDEPENDENCE OF PATH

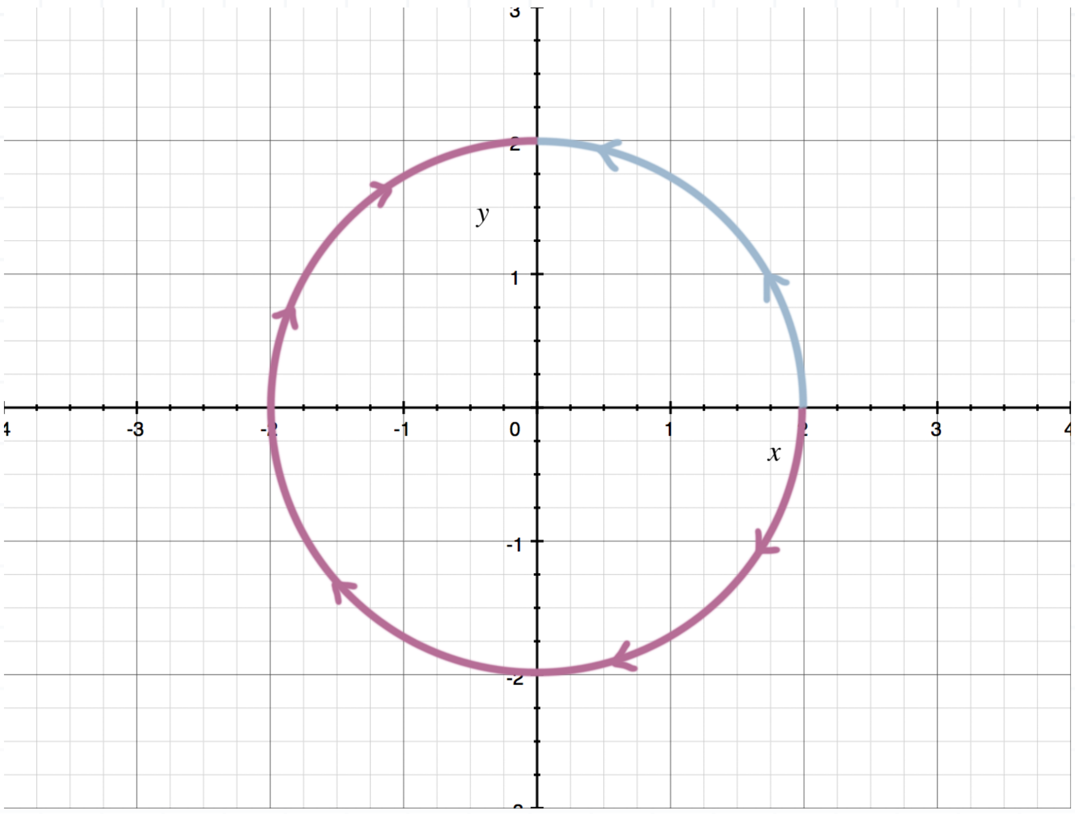
■ 1. Check if the line integral of the vector field $\vec{F}(x,y)$ is independent of path for any curve connecting the points $(2,0)$ and $(0,2)$. If it *is* independent of path, then prove it. If not, give a counterexample.

$$\vec{F}(x,y) = \left\langle \frac{4y}{x^2+y^2}, \frac{-4x}{x^2+y^2} \right\rangle$$

Solution:

It's easy to check that curl $F = 0$, but since the domain of the vector field has a hole at $(0,0)$, this vector field more probably is not conservative.

To give a counterexample, we can compute the line integrals over two curves that lie on different sides of the hole at $(0,0)$. Consider the circle with center at $(0,0)$ and radius 2, which passes through $(2,0)$ and $(0,2)$. Then, consider two paths between these points along the circle, clockwise and counterclockwise.



Consider the parametrization of the circle.

$$\vec{r}(t) = \langle 2\cos t, 2\sin t \rangle$$

For the clockwise path, t changes from 0 to $-3\pi/2$, and for the counterclockwise path, t changes from 0 to $\pi/2$.

The line integral of the vector function $\vec{F}(x, y)$ over the curve $\vec{r}(t)$ is given by

$$\int_c \vec{F}(x, y)\ ds = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{r}'(t)\ dt$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle$$

The function is

$$F(x(t),y(t))=\left\langle \frac{4(2\sin t)}{(2\cos t)^2+(2\sin t)^2},-\frac{4(2\cos t)}{(2\cos t)^2+(2\sin t)^2}\right\rangle$$

$$F(x(t),y(t))=\left\langle \frac{8\sin t}{4(\cos^2 t+\sin^2 t)},-\frac{8\cos t}{4(\cos^2 t+\sin^2 t)}\right\rangle$$

$$F(x(t),y(t))=\left\langle \frac{8\sin t}{4},-\frac{8\cos t}{4}\right\rangle$$

$$F(x(t),y(t))=\langle 2\sin t,-2\cos t\rangle$$

The dot product is

$$\vec{F}(x(t),y(t))\cdot\vec{r}\,'(t)=\langle 2\sin t,-2\cos t\rangle\cdot\langle -2\sin t,2\cos t\rangle$$

$$\vec{F}(x(t),y(t))\cdot\vec{r}\,'(t)=-4\sin^2 t-4\cos^2 t$$

$$\vec{F}(x(t),y(t))\cdot\vec{r}\,'(t)=-4$$

So the line integral over the clockwise circle is

$$\int_{cw} F(x,y)\ ds=\int_0^{-3\pi/2}-4\ dt=-4\left(-\frac{3\pi}{2}\right)=6\pi$$

and the line integral over the counterclockwise circle is

$$\int_{ccw} F(x,y)\ ds=\int_0^{\pi/2}-4\ dt=-4\left(\frac{\pi}{2}\right)=-2\pi$$

Since these integrals over different paths aren't equal, the line integral of the vector field is dependent of path.

2. Check if the line integral of the vector field $\vec{F}(x,y)$ is independent of path for any curve that lies within the rectangle given by $1 < x < 5$ and $1 < y < 5$, and that connects the points $(2,4)$ and $(4,2)$.

$$\vec{F}(x,y) = \left\langle \frac{2(x-1)}{(x^2-2x+y^2+1)^2}, \frac{2y}{(x^2-2x+y^2+1)^2} \right\rangle$$

Solution:

Find the domain of the vector field.

$$(x^2-2x+y^2+1)^2 \neq 0$$

$$x^2-2x+y^2+1 \neq 0$$

$$(x-1)^2+y^2 \neq 0$$

So the domain of the vector field is all points except the point $(1,0)$. Since the domain is not simply-connected, the vector field is not conservative over the domain, and therefore the line integral of the vector field is dependent of path. But since the point $(1,0)$ lies outside the rectangle given by $1 < x < 5$ and $1 < y < 5$, which is an open simply-connected set, the vector field is likely conservative within the rectangle. Let's prove that.

We know that if a vector field $F: R^2 \to R^2$ is continuously differentiable in a simply-connected domain $W \in R^2$ (or differentiable in an open simply-connected domain) and its curl is 0, then F is conservative within the domain W.

The vector field is differentiable on the open rectangle, so we just need to check its curl. Recall that curl of a vector field in two dimensions is given by

$$\text{curl}\, F = \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial t}$$

Find partial derivatives.

$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}\left(\frac{2y}{(x^2 - 2x + y^2 + 1)^2}\right) = -\frac{8(x-1)y}{(x^2 - 2x + y^2 + 1)^3}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}\left(\frac{2(x-1)}{(x^2 - 2x + y^2 + 1)^2}\right) = -\frac{8(x-1)y}{(x^2 - 2x + y^2 + 1)^3}$$

So the vector field is conservative in the given rectangle and therefore its line integral is independent of path.

■ 3. Determine whether the line integral of the vector field $\overrightarrow{F}(x, y, z)$ is independent of path for any curve that connects any two points within the vector field's domain.

$$\overrightarrow{F}(x, y, z) = \langle x\ln(x^2 + y^2 + z^2 - 9), y\ln(x^2 + y^2 + z^2 - 9), z\ln(x^2 + y^2 + z^2 - 9)\rangle$$

Solution:

Find the domain of the vector field.

$$x^2 + y^2 + z^2 - 9 > 0$$

$$x^2 + y^2 + z^2 > 3^2$$

So the domain of the vector field $\overrightarrow{F}$ is the set of all points outside the sphere centered at the origin with radius 3.

We know that if a vector field $F : R^3 \to R^3$ is continuously differentiable in a simply-connected domain $W \in R^3$ (or differentiable in an open simply-connected domain) and its curl is 0, then F is conservative within the domain W.

The set is called simply-connected if every path between any two points can be continuously transformed staying within the set into any other such path between the points. So the domain of the vector field $\overrightarrow{F}$ is simply-connected.

The set R^2 other than the circle, *is not* simply connected, but for any $n > 2$, the set R^n other than the sphere, *is* simply connected.

The domain is an open set, and the vector field is differentiable over the entire domain. Let's determine whether or not the curl of $\overrightarrow{F}$ is 0.

Remember that the curl of a vector field in three dimensions is given by

$$\text{curl}\, F = \left\langle \frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right\rangle$$

In other words, curl $F = \overrightarrow{0}$ if

$$\frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z}$$

$$\frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}$$

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

If we verify that each of these partial derivative equations are true, we find

$$\frac{\partial F_z}{\partial y} = \frac{\partial}{\partial y}(z\ln(x^2+y^2+z^2-9)) = \frac{2yz}{x^2+y^2+z^2-9}$$

$$\frac{\partial F_y}{\partial z} = \frac{\partial}{\partial z}(y\ln(x^2+y^2+z^2-9)) = \frac{2yz}{x^2+y^2+z^2-9}$$

and

$$\frac{\partial F_x}{\partial z} = \frac{\partial}{\partial z}(x\ln(x^2+y^2+z^2-9)) = \frac{2xz}{x^2+y^2+z^2-9}$$

$$\frac{\partial F_z}{\partial x} = \frac{\partial}{\partial x}(z\ln(x^2+y^2+z^2-9)) = \frac{2xz}{x^2+y^2+z^2-9}$$

and

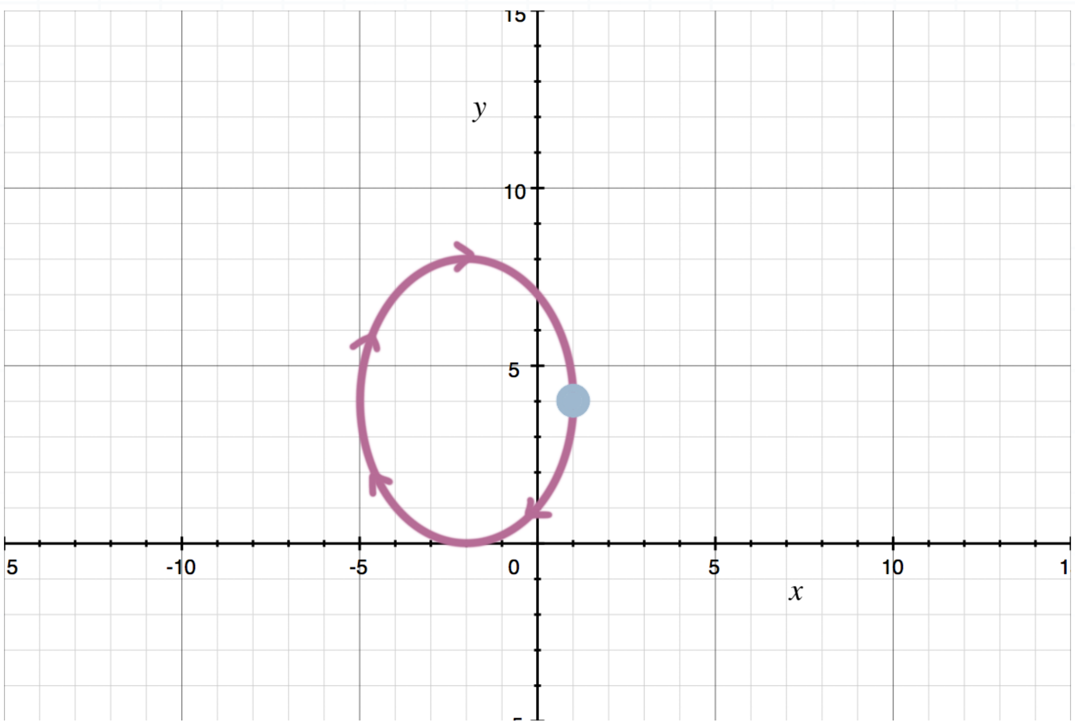
$$\frac{\partial F_y}{\partial x} = \frac{\partial}{\partial x}(y\ln(x^2+y^2+z^2-9)) = \frac{2xy}{x^2+y^2+z^2-9}$$

$$\frac{\partial F_x}{\partial y} = \frac{\partial}{\partial y}(x\ln(x^2+y^2+z^2-9)) = \frac{2xy}{x^2+y^2+z^2-9}$$

Therefore, since the vector field is differentiable in the open simply-connected domain and its curl is 0, it's conservative and therefore not dependent of path.

WORK DONE BY A FORCE FIELD

■ 1. Calculate the work done by the force field $\vec{F}(x,y) = \langle 25x^2 + 9y^2 + 1, x - y - 3 \rangle$ to move an object clockwise along the ellipse centered at $(-2,4)$ with semi-axis of 3 in the x-direction and semi-axis of 5 in the y-direction.



Solution:

The work done by the force field $\vec{F}(x,y)$ to move an object along the curve $\vec{r}(t)$ is equal to the line integral of the vector function $\vec{F}(x,y)$ over the curve $\vec{r}(t)$.

$$\int_C \vec{F}(x,y)\ ds = \int_a^b \vec{F}(x(t),y(t)) \cdot \vec{r}'(t)\ dt$$

The work done by the conservative force field along the closed curve is always 0. Since the given force field $\overrightarrow{F}(x, y)$ is not conservative, we need to calculate the work done directly by definition.

The standard parametrization of the ellipse with the center at (x_0, y_0), semi-axis of a in the x-direction, and semi-axis of b in the y-direction, is

$$\overrightarrow{r}(t) = \langle x_0 + a\cos t, y_0 + b\sin t\rangle$$

Plug in the values $(x_0, y_0) = (-2,4)$, $a = 3$, and $b = 5$.

$$\overrightarrow{r}(t) = \langle -2 + 3\cos t, 4 + 5\sin t\rangle$$

Since the work done by the force field to move an object along the closed curve is independent of the initial point, let's choose the point $(1,4)$, which corresponds to the value $t = 0$. Since the object is moved clockwise, t changes from 0 to -2π.

Plug $x(t) = -2 + 3\cos t$ and $y(t) = 4 + 5\sin t$ into the expression for $F(x, y)$.

$$\overrightarrow{F}(x(t), y(t)) = \langle 25(-2 + 3\cos t)^2 + 9(4 + 5\sin t)^2 + 1, -2 + 3\cos t - (4 + 5\sin t) - 3\rangle$$

$$\overrightarrow{F}(x(t), y(t)) = \langle -300\cos t + 360\sin t + 470, 3\cos t - 5\sin t - 9\rangle$$

Find the first-order derivative of $\overrightarrow{r}(t)$.

$$\overrightarrow{r}'(t) = \langle -3\sin t, 5\cos t\rangle$$

The dot product is

$$\overrightarrow{F}(x(t), y(t)) \cdot \overrightarrow{r}'(t) = \langle -300\cos t + 360\sin t + 470, 3\cos t - 5\sin t - 9\rangle$$

$$\cdot \langle -3\sin t, 5\cos t\rangle$$

$$\vec{F}(x(t),y(t))\cdot\vec{r}\,'(t)=-3\sin t(-300\cos t+360\sin t+470)$$

$$+5\cos t(3\cos t-5\sin t-9)$$

$$\vec{F}(x(t),y(t))\cdot\vec{r}\,'(t)=-1{,}080\sin^2 t+15\cos^2 t-1{,}410\sin t-45\cos t+875\cos t\sin t$$

Therefore, the line integral over the curve is

$$\int_0^{-2\pi}-1{,}080\sin^2 t+15\cos^2 t-1{,}410\sin t-45\cos t+875\cos t\sin t\ dt$$

$$\int_0^{-2\pi}-540(1-\cos 2t)+\frac{15}{2}(\cos 2t+1)-1{,}410\sin t-45\cos t+\frac{875}{2}\sin 2t\ dt$$

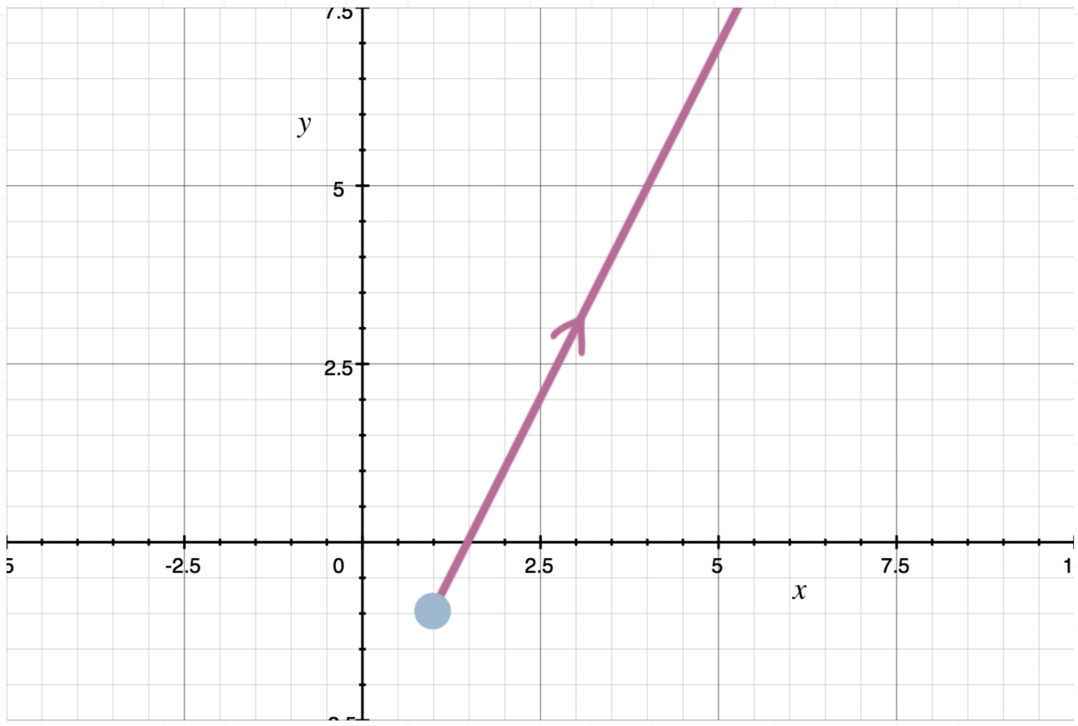
$$\int_0^{-2\pi}-1{,}410\sin t+\frac{875}{2}\sin 2t-45\cos t+\frac{1{,}095}{2}\cos 2t-\frac{1{,}065}{2}\ dt$$

Since the integral of sine and cosine functions over a 2π period is always 0,

$$\int_0^{-2\pi}-\frac{1{,}065}{2}\ dt=-\frac{1{,}065}{2}(-2\pi)=1{,}065\pi$$

■ 2. Find the work done by the force field $\vec{F}(x,y)$ to move an object infinitely along the line $y=2x-3$, starting from $(1,-1)$, in the positive direction of x.

$$\vec{F}(x,y)=\left\langle xe^{-y},\frac{y+2}{x^3}\right\rangle$$



Solution:

The work done by the force field $\vec{F}(x,y)$ to move an object along the curve $\vec{r}(t)$ is equal to the line integral of the vector function $\vec{F}(x,y)$ over the curve $\vec{r}(t)$.

$$\int_C \vec{F}(x,y)\ ds = \int_a^b \vec{F}(x(t),y(t)) \cdot \vec{r}\,'(t)\ dt$$

Consider the parametrization of the line $y = 2x - 3$ as $\vec{r}(t) = \langle t, 2t - 3 \rangle$.

Plug $x(t) = t$ and $y(t) = 2t - 3$ into the expression for $F(x,y)$.

$$\vec{F}(x(t),y(t)) = \left\langle te^{-(2t-3)}, \frac{2t - 3 + 2}{t^3} \right\rangle$$

$$\vec{F}(x(t),y(t)) = \left\langle te^{-2t+3}, \frac{2t - 1}{t^3} \right\rangle$$

Find the first-order derivative of $\vec{r}(t)$.

$$\vec{r}\,'(t) = \langle 1, 2 \rangle$$

The dot product is

$$\vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t) = \left\langle te^{-2t+3}, \frac{2t-1}{t^3} \right\rangle \cdot \langle 1, 2 \rangle$$

$$\vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t) = 1 \cdot te^{-2t+3} + 2 \cdot \frac{2t-1}{t^3}$$

$$\vec{F}(x(t), y(t)) \cdot \vec{r}\,'(t) = te^{-2t+3} + 4t^{-2} - 2t^{-3}$$

Therefore, the line integral over the half-line is

$$\int_c F(x, y)\ ds = \int_1^\infty te^{-2t+3} + 4t^{-2} - 2t^{-3}\ dt$$

$$\int_c F(x, y)\ ds = \int_1^\infty te^{-2t+3}\ dt + \int_1^\infty 4t^{-2} - 2t^{-3}\ dt$$

Take the integral of te^{-2t+3} using integration by parts with $u = t$, $du = dt$, $dv = e^{-2t+3}\ dt$, and $v = -\frac{1}{2}e^{-2t+3}$.

$$\int te^{-2t+3}\ dt = -\frac{t}{2}e^{-2t+3} + \int \frac{1}{2}e^{-2t+3}\ dt$$

So

$$\int_1^\infty te^{-2t+3}\ dt + \int_1^\infty 4t^{-2} - 2t^{-3}\ dt$$

$$-\frac{t}{2}e^{-2t+3}\Big|_1^{\infty}+\int_1^{\infty}\frac{1}{2}e^{-2t+3}\ dt+\int_1^{\infty}4t^{-2}-2t^{-3}\ dt$$

$$-\frac{t}{2}e^{-2t+3}\Big|_1^{\infty}+\left(-\frac{1}{4}e^{-2t+3}\right)\Big|_1^{\infty}+\left(-\frac{4}{t}+\frac{1}{t^2}\right)\Big|_1^{\infty}$$

$$\lim_{t\to\infty}\left(-\frac{t}{2}e^{-2t+3}\right)-\left(-\frac{1}{2}e^{-2+3}\right)+\lim_{t\to\infty}\left(-\frac{1}{4}e^{-2t+3}\right)-\left(-\frac{1}{4}e^{-2+3}\right)$$

$$+\lim_{t\to\infty}\left(-\frac{4}{t}+\frac{1}{t^2}\right)-\left(-\frac{4}{1}+\frac{1}{1^2}\right)$$

$$0+\frac{e}{2}+0+\frac{e}{4}+0+3=3+\frac{3e}{4}$$

3. Find the work done by the conservative force field $\vec{F}(x,y,z)$ to move an object between the four points $A(0,-1,2)$, $B(1,1,3)$, $C(2,3,0)$, and $D(0,2,1)$ (starting from A to B, then to C, and finally to D).

$$\vec{F}(x,y,z)=\langle 1+4x+yz+3z^2, xz-1, x(y+6z)\rangle$$

Solution:

The work done by the force field $\vec{F}$ to move an object along the path is equal to the line integral of the vector function $\vec{F}$ along this path. Since the given vector field is conservative, the line integral is independent of path, and therefore the work done to move the object between the points A, B,

C, and D is equal to the work done to move the object from point A to point D.

The line integral of the conservative vector field over the segment AD can be calculated as

$$\int_c \overrightarrow{F} \cdot d\overrightarrow{r} = \int_c \nabla f \cdot d\overrightarrow{r} = f_D - f_A = f(0,2,1) - f(0,-1,2)$$

where f is the potential function of the vector field, $\nabla f = \overrightarrow{F}$. In other words,

$$\frac{\partial f}{\partial x}(x, y, z) = F_x(x, y, z)$$

$$\frac{\partial f}{\partial y}(x, y, z) = F_y(x, y, z)$$

$$\frac{\partial f}{\partial z}(x, y, z) = F_z(x, y, z)$$

Find the potential function of given vector field. From the first equation,

$$\frac{\partial f}{\partial x}(x, y, z) = 1 + 4x + yz + 3z^2$$

Integrate both sides with respect to x, treating y and z as constants.

$$f(x, y, z) = \int 1 + 4x + yz + 3z^2 \ dx = x + 2x^2 + xyz + 3xz^2 + C(y, z)$$

Differentiate $f(x, y, z)$ with respect to y, treating x and z as constants.

$$\frac{\partial f}{\partial y}(x, y, z) = \frac{\partial}{\partial y}(x + 2x^2 + xyz + 3xz^2 + C(y, z))$$

$$\frac{\partial f}{\partial y}(x, y, z) = xz + \frac{\partial C}{\partial y}(y, z)$$

From the second equation,

$$\frac{\partial f}{\partial y}(x, y, z) = xz - 1$$

So

$$xz + \frac{\partial C}{\partial y}(y, z) = xz - 1$$

$$\frac{\partial C}{\partial y}(y, z) = -1$$

Integrate both sides with respect to y, treating z as a constant.

$$C(y, z) = \int -1 \ dy$$

$$C(y, z) = -y + C(z)$$

So

$$f(x, y, z) = x + 2x^2 + xyz + 3xz^2 - y + C(z)$$

Similarly, differentiate $f(x, y, z)$ with respect to z, treating x and y as constants.

$$\frac{\partial f}{\partial z}(x, y, z) = \frac{\partial}{\partial z}x + 2x^2 + xyz + 3xz^2 - y + C(z) = xy + 6xz + C'(z)$$

From the third equation,

$$\frac{\partial f}{\partial z}(x, y, z) = x(y + 6z)$$

So

$$xy + 6xz + C'(z) = xy + 6xz$$

$$C'(z) = 0$$

Therefore, $C(z)$ is a constant,

$$C(y, z) = c$$

So

$$f(x, y, z) = xyz + 2x^2 + 3xz^2 + x - y + c$$

So the line integral is equal to

$$\int_c \vec{F} \cdot d\vec{r} = f(0,2,1) - f(0,-1,2)$$

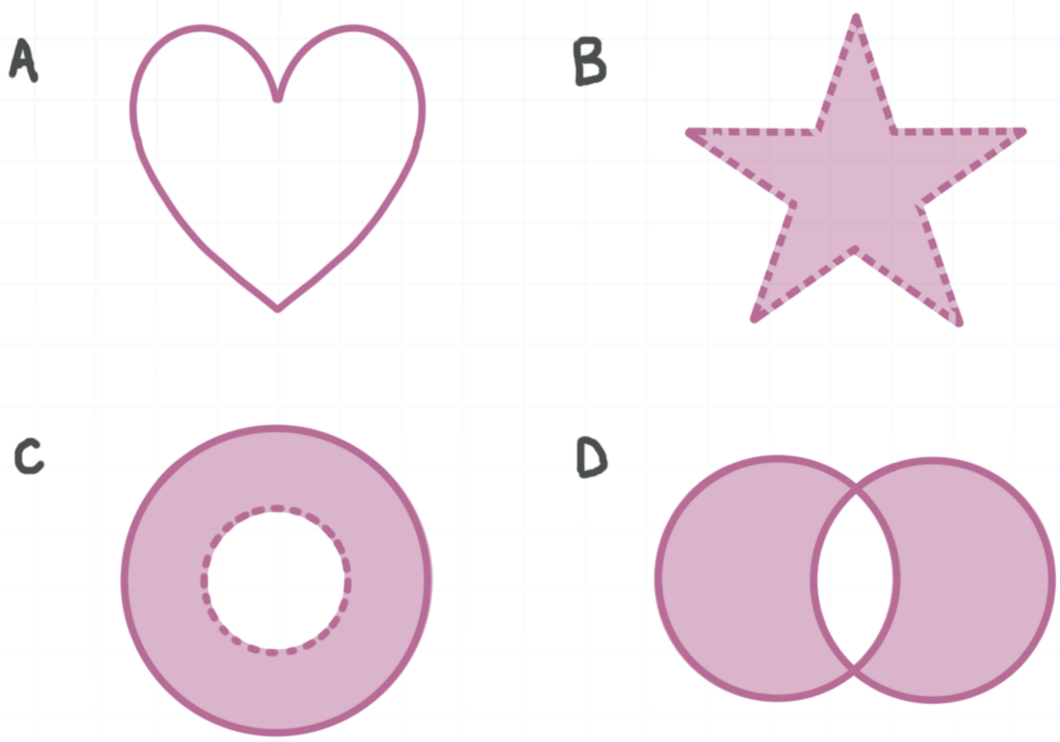
$$\int_c \vec{F} \cdot d\vec{r} = 0 \cdot 2 \cdot 1 + 2 \cdot 0^2 + 3 \cdot 0 \cdot 1^2 + 0 - 2 + c$$

$$-(0 \cdot (-1) \cdot 2 + 2 \cdot 0^2 + 3 \cdot 0 \cdot 2^2 + 0 - (-1) + c)$$

$$\int_c \vec{F} \cdot d\vec{r} = -3$$

OPEN, CONNECTED, AND SIMPLY CONNECTED

■ 1. Determine whether each set is open, closed, connected, or simply-connected.



Solution:

The set A is not open since it includes boundary points along heart's border, it's closed since it contains all of its boundary points, it's connected since it's possible to connect any two points by a path, but it's not simply-connected.

The set B is open since it doesn't contain any boundary points, it's not closed since it doesn't contain any boundary points, it's connected since it's possible to connect any two points by a path, and it's simply connected.

The set C is not open since it contains its boundary points along the outer circle, it's not closed since it doesn't contain its boundary points along the inner circle, it's connected since it's possible to connect any two points by a path, but it's not simply-connected because it has a hole.

The set D is not open since it contains all of its boundary points, it's closed since it contains all of its boundary points, it's connected since it's possible to connect any two points by a path, but it's not simply-connected because it has a hole.

■ 2. Find the domain D of the vector field $\vec{F}$, then determine whether it's open, closed, connected, or simply-connected.

$$\vec{F}(u,v) = \left\langle \sqrt{36 - 9u^2 - 4v^2}, \log_2(uv - v) \right\rangle$$

Solution:

Let's find the domains D_1 and D_2 of each component of the vector field individually, then find their intersection. Find the domain of the first component.

$$36 - 9u^2 - 4v^2 \geq 0$$

$$9u^2 + 4v^2 \leq 36$$

$$\frac{u^2}{2^2} + \frac{v^2}{3^2} \leq 1$$

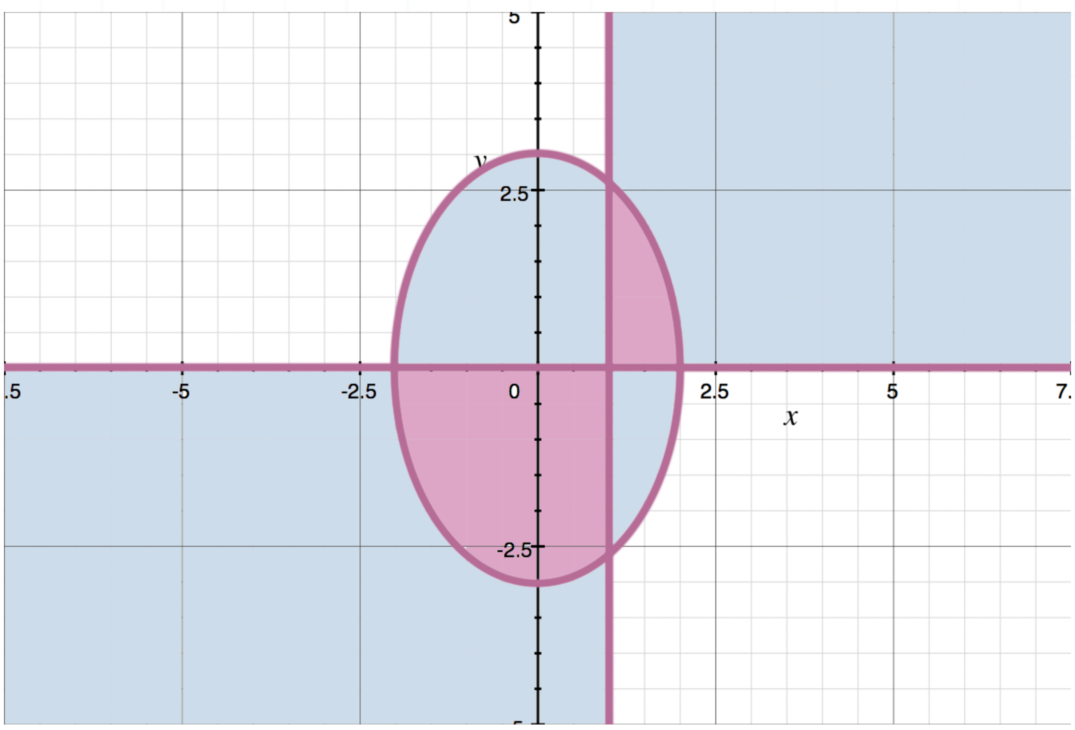
So the domain D_1 is the set of inner points of the ellipse centered at $(0,0)$ with u-semi-axis 2, and v-semi-axis 3.

Find the domain of the second component.

$$(u-1)v > 0$$

So D_2 consists of two sets, $u > 1$, $v > 0$ (above the line $v = 0$, right of the line $u = 1$) and $u < 1$, $v < 0$ (below the line $v = 0$ and left of the line $u = 1$).

Then the intersection of D_1 and D_2 is shown looks like this:



D is not open since it contains boundary points along ellipse border, it's not closed since its borders along the lines $u = 1$ and $v = 0$ are open, it's not connected since it's impossible to connect any two points from the two parts of D_2, and it's not simply-connected since it's not connected.

■ 3. Find the domain D of the vector field $\vec{F}$, then determine whether it's open, closed, connected, or simply-connected.

$$\vec{F}(x, y, z) = \left\langle \ln(4x - x^2 - y^2 - z^2), \frac{3x}{y^2 + z^2}, \frac{y}{x + 8} \right\rangle$$

Solution:

Let's find the domains D_1, D_2, and D_3 of each component of the vector field individually, then find their intersection.

The domain of the first component is

$$4x - x^2 - y^2 - z^2 > 0$$

$$x^2 - 4x + y^2 + z^2 < 0$$

$$x^2 - 4x + 4 - 4 + y^2 + z^2 < 0$$

$$(x - 2)^2 + y^2 + z^2 < 2^2$$

So the domain D_1 is the set of inner points of the sphere centered at $(2,0,0)$ with radius 2.

The domain of the second component is

$$y^2 + z^2 \neq 0$$

so y and z can't be 0 simultaneously, which means the domain D_2 is the set of all points except the x-axis.

The domain of the third component is

$$x + 8 \neq 0$$

So the domain D_3 is the set of all points except the plane $x = -8$.

Since the plane $x = -8$ has no common points with the sphere, the intersection of D_1, D_2, and D_3 is the set of inner points of the sphere centered at $(2,0,0)$ with radius 2, except the points that lie on the x-axis.

D is open since it doesn't contain any of its boundary points, and so D is not closed. It's connected since it's possible to connect any two points in D with a path that lies completely in D, and D isn't simply-connected.

W W W . K R I S T A K I N G M A T H . C O M