Implicit differentiation for multivariable functions

When we want to use implicit differentiation to find partial derivatives of multivariable functions, we'll use the following formulas.

For multivariable functions **in two variables**:

$$\frac{dy}{dx}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

For multivariable functions **in three variables**:

$$\frac{\partial z}{\partial x}=-\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

and

$$\frac{\partial z}{\partial y}=-\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Before we can use these formulas to find derivatives or partial derivatives of the original function, we'll need to rearrange it so that it's equal to 0, and then rename it $F(x,y)$ (for two variable functions) or $F(x,y,z)$ (for three variable functions).

Let's try an example with a multivariable function in two variables.

Example

Use implicit differentiation to find the partial derivative of the multivariable function.

$$xe^{2y} = x^2 - y^3$$

We'll start by rearranging the function so that it's equal to 0, then we'll call it $F(x, y)$.

$$0 = x^2 - y^3 - xe^{2y}$$

$$F(x, y) = x^2 - y^3 - xe^{2y}$$

Now we'll take partial derivatives of F with respect to x and y.

$$\frac{\partial F}{\partial x} = 2x - e^{2y}$$

and

$$\frac{\partial F}{\partial y} = -3y^2 - xe^{2y}(2)$$

$$\frac{\partial F}{\partial y} = -3y^2 - 2xe^{2y}$$

We'll plug the partial derivatives into the formula for the derivative of a multivariable function with two variables.

$$\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

$$\frac{dy}{dx} = -\frac{2x - e^{2y}}{-3y^2 - 2xe^{2y}}$$

$$\frac{dy}{dx} = \frac{2x - e^{2y}}{3y^2 + 2xe^{2y}}$$

This is the partial derivative of $xe^{2y} = x^2 - y^3$.

Now we'll try an example with a multivariable function in three variables.

Example

Use implicit differentiation to find the partial derivatives of the multivariable function.

$$x^2 \sin z = 3yz + 2x^3$$

We'll start by rearranging the function so that it's equal to 0, then we'll call it $F(x, y, z)$.

$$0 = 3yz + 2x^3 - x^2 \sin z$$

$$F(x, y, z) = 3yz + 2x^3 - x^2 \sin z$$

Now we'll take partial derivatives of F with respect to x, y, and z.

$$\frac{\partial F}{\partial x} = 6x^2 - 2x \sin z$$

$$\frac{\partial F}{\partial y} = 3z$$

$$\frac{\partial F}{\partial z} = 3y - x^2 \cos z$$

We'll plug the partial derivatives into the formula for the derivative of a multivariable function with three variables.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial x} = -\frac{6x^2 - 2x \sin z}{3y - x^2 \cos z}$$

and

$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

$$\frac{\partial z}{\partial y} = -\frac{3z}{3y - x^2 \cos z}$$

These are the partial derivatives of the $x^2 \sin z = 3yz + 2x^3$.
