Unit tangent vector

To find the unit tangent vector for a vector function, we use the formula

$$T(t) = \frac{r'(t)}{||r'(t)||}$$

where $r'(t)$ is the derivative of the vector function $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$ and t is given.

Remember that $||r'(t)||$ is the magnitude of the derivative of the vector function at time t. We can find $\left|r'(t)\right|$ using the formula

$$||r'(t)|| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

Example

Find the unit tangent vector of the vector function at $t = 1$.

$$r(t) = 4t^3\mathbf{i} + 6t\mathbf{j} + 4t\ln(t)\mathbf{k}$$

We'll start by finding the derivative of the vector function $r(t) = 4t^3\mathbf{i} + 6t\mathbf{j} + 4t\ln(t)\mathbf{k}$ at time $t = 1$ so that we can plug it into the formula for the unit tangent vector. To find the derivative, we'll just replace each of the coefficients with their derivatives. The derivative of $4t^3$ is $12t^2$; the derivative of $6t$ is 6; the derivative of $4t\ln(t)$ using product rule is $(4)(\ln(t)) + (4t)(1/t)$.

$$r'(t) = 12t^2\mathbf{i} + 6\mathbf{j} + \left[(4)(\ln(t)) + (4t)\left(\frac{1}{t}\right)\right]\mathbf{k}$$

$$r'(t) = 12t^2\mathbf{i} + 6\mathbf{j} + \left[4\ln(t) + 4\right]\mathbf{k}$$

Now we'll find the value of the derivative at $t = 1$.

$$r'(1) = 12(1)^2\mathbf{i} + 6\mathbf{j} + \left[4\ln(1) + 4\right]\mathbf{k}$$

$$r'(1) = 12\mathbf{i} + 6\mathbf{j} + \left[4(0) + 4\right]\mathbf{k}$$

$$r'(1) = 12\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}$$

Now we'll use the values from the derivative to find the magnitude of the vector function at $t = 1$ so that we can plug it into the formula for the unit tangent vector.

$$||r'(t)|| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$||r'(1)|| = \sqrt{12^2 + 6^2 + 4^2}$$

$$||r'(1)|| = \sqrt{144 + 36 + 16}$$

$$||r'(1)|| = \sqrt{196}$$

$$||r'(1)|| = 14$$

Plugging everything into the formula for the unit tangent vector, we get

$$T(1) = \frac{12\mathbf{i} + 6\mathbf{j} + 4\mathbf{k}}{14}$$

$$T(1) = \frac{12}{14}\mathbf{i} + \frac{6}{14}\mathbf{j} + \frac{4}{14}\mathbf{k}$$

$$T(1) = \frac{6}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{2}{7}\mathbf{k}$$

which is the equation of the unit tangent vector for $r(t) = 4t^3\mathbf{i} + 6t\mathbf{j} + 4t\ln(t)\mathbf{k}$.
