Calculus 3 Workbook Solutions

Lines and planes

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MATH

VECTOR, PARAMETRIC, AND SYMMETRIC EQUATIONS OF A LINE

■ 1. Find the vector equation of the line that passes through $A(1,0,-2)$ and is parallel to the symmetric equation.

$$\frac{x-3}{2}=-\frac{y}{2}=z+1$$

Solution:

Rewrite the symmetric equation in standard form.

$$\frac{x-3}{2}=\frac{y+0}{-2}=\frac{z+1}{1}$$

So the parallel vector is $\langle 2,-2,1\rangle$. Since we also know the line passes through $A(1,0,-2)$, the equation is

$$r=(1\mathbf{i}+0\mathbf{j}-2\mathbf{k})+t(2\mathbf{i}-2\mathbf{j}+\mathbf{k})$$

$$r=\mathbf{i}+2t\mathbf{i}-2t\mathbf{j}-2\mathbf{k}+t\mathbf{k}$$

$$r=(1+2t)\mathbf{i}+(-2t)\mathbf{j}+(-2+t)\mathbf{k}$$

■ 2. Find the parametric equation of the line that passes through $A(2,3,-2)$ and $B(0,-1,5)$.

Solution:

Vector AB is

$$AB = \langle 0-2, -1-3, 5-(-2) \rangle$$

$$AB = \langle -2, -4, 7 \rangle$$

Since the line should be parallel to $AB = \langle -2, -4, 7 \rangle$ and passes through $A(2,3,-2)$, the parametric equation is

$$x = 2 - 2t$$

$$y = 3 - 4t$$

$$z = -2 + 7t$$

■ 3. Which line passes through $A(1,0,-4)$?

A $x = 1 - 2t,\ y = -5t,\ z = 3 - 4t$

B $x = 2 - 5t,\ y = 5t,\ z = 2 - 4t$

C $x = 6 - t,\ y = 5 - t,\ z = 6 - 2t$

D $x = 6 + t,\ y = 5 + t,\ z = 6 + t$

Solution:

If the line passes though $A(1,0,-4)$, then there's a value t_0 where

$$x(t_0) = 1$$

$$y(t_0) = 0$$

$$z(t_0) = -4$$

We can use $(1,0,-4)$ to check each of the answer choices.

For answer choice A, start with $y = -5t$. Substituting $y(t_0) = 0$ gives $0 = -5t_0$, and then $t_0 = 0$. Plugging into $x = 1 - 2t$ and $z = 3 - 4t$ gives

$$x(t_0) = 1 - 2t_0 = 1 - 2(0) = 1$$

$$z(t_0) = 3 - 4t_0 = 3$$

Putting these values together gives $(1,0,3)$, which doesn't match $(1,0,-4)$, so let's try answer choice B. Start with $y = 5t$. Substituting $y(t_0) = 0$ gives $0 = 5t_0$, and then $t_0 = 0$. Plugging into $x = 2 - 5t$ and $z = 2 - 4t$ gives

$$x(t_0) = 2 - 5t_0 = 2 - 5(0) = 2$$

$$z(t_0) = 2 - 4t_0 = 2 - 4(0) = 2$$

Putting these values together gives $(2,0,2)$, which doesn't match $(1,0,-4)$, so let's try answer choice C. Start with $y = 5 - t$. Substituting $y(t_0) = 0$ gives $0 = 5 - t_0$, and then $t_0 = 5$. Plugging into $x = 6 - t$ and $z = 6 - 2t$ gives

$$x(t_0) = 6 - t_0 = 6 - 5 = 1$$

$$z(t_0) = 6 - 2t_0 = 6 - 2(5) = -4$$

Putting these together gives $(1,0,-4)$, so answer choice C is correct.

PARALLEL, INTERSECTING, SKEW AND PERPENDICULAR LINES

■ 1. For $A(1,0,-1)$, $B(1,3,0)$, $C(0,0,2)$, and $D(-1,-2,3)$, are lines AB and CD parallel, intersecting, skew, or perpendicular?

Solution:

Find vectors AB and CD and the equations of the corresponding lines. The vector AB is

$$AB = \langle 1-1, 3-0, 0-(-1) \rangle = \langle 0,3,1 \rangle$$

Its parametric equation is

$$x = 1 + 0t = 1$$

$$y = 0 + 3t = 3t$$

$$z = -1 + 1t = t - 1$$

The vector CD is

$$CD = \langle -1-0, -2-0, 3-2 \rangle = \langle -1,-2,1 \rangle$$

Its parametric equation is

$$x = 0 - 1t = -t$$

$$y = 0 - 2t = -2t$$

$$z = 2 + 1t = 2 + t$$

Since $0/(-1)$ isn't equal to $3/(-2)$, AB isn't parallel to CD. To check if the lines intersect, solve the system of equations.

$$1 = -s$$

$$3t = -2s$$

$$t - 1 = 2 + s$$

So

$$s = -1$$

$$3t = 2$$

$$t = 2$$

Since the system has no solutions, the lines don't intersect, so the lines are skew.

■ 2. Find the line L_2 that passes through $A(1,0,1)$ and is perpendicular to L_1.

$$L_1: \quad x = 2 - 2t,\ y = t,\ z = 3 + t$$

Solution:

The vector for L_1 is $a = \langle -2,1,1 \rangle$.

Let B be the point on L_1 where L_1 intersects L_2. Then B has coordinates $(2-2s, s, 3+s)$. So vector AB is

$$AB = \langle 2-2s-1, s-0, 3+s-1 \rangle$$

$$AB = \langle 1-2s, s, 2+s \rangle$$

Since AB is perpendicular to a, we know that $AB \cdot a = 0$, so

$$-2(1-2s) + 1(s) + 1(2+s) = 0$$

$$-2 + 4s + s + 2 + s = 0$$

$$6s = 0$$

$$s = 0$$

So $AB = \langle 1,0,2 \rangle$. Since L_2 passes through $A(1,0,1)$ and is parallel to the vector $AB = \langle 1,0,2 \rangle$, its equation is

$$x = 1 + t$$

$$y = 0$$

$$z = 1 + 2t$$

■ 3. Which line is perpendicular to L_1?

L_1: $\quad x = 2t,\ y = 21 - t,\ z = 6 - t$

A $\quad x = 2 - 3t,\ y = 7 + 5t,\ z = 2t$

B $x = 2 + 3t,\ y = 7 + 5t,\ z = t$

C $x = 2 + 3t,\ y = 7 - 5t,\ z = -t$

D $x = 2 - 3t,\ y = 5 - 5t,\ z = -t$

Solution:

The vector for L_1 is $a = \langle 2, -1, -1 \rangle$, so check each answer choice to see if its vector is perpendicular to a. If we call the vector for each answer choice b, then we can say $b \cdot a = 0$.

For answer choice A, $b = \langle -3,5,2 \rangle$ and $b \cdot a = 2(-3) - 1(5) - 1(2) = -13 \neq 0$.

For answer choice B, $b = \langle 3,5,1 \rangle$ and $b \cdot a = 2(3) - 1(5) - 1(1) = 0$. Let's check if this line intersects L_1.

$$2 + 3t = 2s$$

$$7 + 5t = 21 - s$$

$$t = 6 - s$$

Substitute $6 - s$ for t into the first equation.

$$2 + 3(6 - s) = 2s$$

$$20 - 3s = 2s$$

$$s = 4$$

Then $t = 6 - 4 = 2$ and $7 + 5(2) = 21 - 4$.

For answer choice C, $b = \langle 3, -5, -1 \rangle$ and $b \cdot a = 2(3) - 1(-5) - 1(-1) = 12 \neq 0$.

For answer choice D, $b = \langle -3, -5, -1 \rangle$ and $b \cdot a = 2(-3) - 1(-5) - 1(-1) = 0$. Let's check if this line intersects L_1.

$$2 - 3t = 2s$$

$$5 - 5t = 21 - s$$

$$-t = 6 - s$$

$$t = s - 6$$

Substitute $s - 6$ for t to the first equation:

$$2 - 3(s - 6) = 2s$$

$$20 - 3s = 2s$$

$$s = 4$$

Then $t = 4 - 6 = -2$ and $5 - 5(-2) = 21 - 4$.

From the given choices, only the line in answer choice B is perpendicular and intersects L_1.

EQUATION OF A PLANE

■ 1. Find the equation of a plane that passes through $A(1,4,-2)$ and is perpendicular to the line.

$$r=\langle 1,3,3\rangle+t\langle -2,3,1\rangle$$

Solution:

Since the plane is perpendicular to the line with vector $\langle -2,3,1\rangle$, it has the same normal vector, $\langle -2,3,1\rangle$.

The equation of the plane that passes through $A(1,4,-2)$ and has the normal vector $\langle -2,3,1\rangle$ is

$$-2(x-1)+3(y-4)+1(z+2)=0$$

$$-2x+2+3y-12+z+2=0$$

$$-2x+3y+z=8$$

■ 2. Find the equation of a plane that passes through $A(1,4,-2)$ and the line given by the parametric equation.

$$x=2-4t,\ y=3t,\ z=1+t$$

Solution:

Evaluating the parametric equation at $t = 0$ gives $(2,0,1)$. The vector of the parametric equation is $a = \langle -4,3,1 \rangle$.

The normal vector to the plane is given by the cross product $n = AB \times a$ where $AB = \langle 1, -4,3 \rangle$.

$$AB \times a = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 3 \\ -4 & 3 & 1 \end{vmatrix}$$

$$AB \times a = \mathbf{i}((-4)(1) - (3)(3)) - \mathbf{j}((1)(1) - (3)(-4)) + \mathbf{k}((1)(3) - (-4)(-4))$$

$$AB \times a = -13\mathbf{i} - 13\mathbf{j} - 13\mathbf{k}$$

The equation of the plane that passes through $B(2,0,1)$ and has the normal vector $\langle -13, -13, -13 \rangle$ is

$$-13(x-2) - 13(y-0) - 13(z-1) = 0$$

$$(x-2) + (y-0) + (z-1) = 0$$

$$x + y + z - 3 = 0$$

$$x + y + z = 3$$

■ 3. Which of the lines lie in the plane $2x - y + 3z = 1$? Choose as many of the answer choices as are correct.

A $\quad x = 1 + 2t,\ y = 1 - 3t,\ z = -5t$

B $x = 1 - 2t$, $y = 1 - 5t$, $z = 4t$

C $x = 1 + 4t$, $y = 1 + t$, $z = -3t$

D $x = 1 + 2t$, $y = 1 + t$, $z = -t$

Solution:

A line lies in the plane if any two points on the line lie in that plane. So we can check two points on the line of each answer choice to see if they satisfy the plane equation.

All of the lines pass through the point $(1,1,0)$ so let's check that in the plane equation.

$$2(1) - (1) + 3(0) = 1$$

$$2 - 1 + 0 = 1$$

$$1 = 1$$

So the first point on each line lies in the plane. Let's take the second point on each line for $t = 1$ and substitute them into the plane equation.

For answer choice A, $t = 1$ gives $(3, -2, -5)$, and then

$$2(3) - (-2) + 3(-5) = 1$$

$$6 + 2 - 15 = 1$$

$$-7 = 1$$

For answer choice B, $t=1$ gives $(-1,-4,4)$, and then

$$2(-1)-(-4)+3(4)=1$$

$$-2+4+12=1$$

$$14=1$$

For answer choice C, $t=1$ gives $(5,2,-3)$, and then

$$2(5)-(2)+3(-3)=1$$

$$10-2-9=1$$

$$-1=1$$

For answer choice D, $t=1$ gives $(3,2,-1)$, and then

$$2(3)-(2)+3(-1)=1$$

$$6-2-3=1$$

$$1=1$$

Because none of the equations held true for the lines in A, B, and C, none of those lines lie in the plane. But the line in D gave us $1=1$, a true equation, which means the line in D is the only line that lies in the plane.

■ 4. Find the equation of a plane that passes through the intersecting lines L_1 and L_2.

$$L_1: \quad \frac{x-2}{2} = \frac{y+3}{3} = \frac{z}{2}$$

$$L_2: \quad \frac{x-2}{-1} = \frac{y+3}{2} = \frac{z}{5}$$

Solution:

The vectors for L_1 and L_2 are $n_1 = \langle 2,3,2 \rangle$ and $n_2 = \langle -1,2,5 \rangle$ respectively. The normal vector to the plane is the cross-product $n = n_1 \times n_2$.

$$n_1 \times n_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ -1 & 2 & 5 \end{vmatrix}$$

$$n_1 \times n_2 = \mathbf{i}((3)(5) - (2)(2)) - \mathbf{j}((2)(5) - (2)(-1)) + \mathbf{k}((2)(2) - (3)(-1))$$

$$n_1 \times n_2 = 11\mathbf{i} - 12\mathbf{j} + 7\mathbf{k}$$

The lines intersect at $(2,-3,0)$. Since the equation of the plane passes through $(2,-3,0)$ and has the normal vector $\langle 11,-12,7 \rangle$, its equation is

$$11(x-2) - 12(y+3) + 7(z-0) = 0$$

$$11x - 22 - 12y - 36 + 7z = 0$$

$$11x - 12y + 7z = 58$$

■ 5. Find the equation of a plane that passes through the parallel lines L_1 and L_2.

L_1: $\quad r = \langle 1,2,-4 \rangle + t\langle 0,1,-1 \rangle$

L_2: $\quad r = \langle 2,-3,0 \rangle + t\langle 0,1,-1 \rangle$

Solution:

$A(1,2,-4)$ is a point on L_1 and $B(2,-3,0)$ is a point on L_2. The vector for both lines is $a = \langle 0,1,-1 \rangle$. So the normal vector to the plane is the cross-product $n = AB \times a$, where $AB = \langle 1,-5,4 \rangle$.

$$AB \times a = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -5 & 4 \\ 0 & 1 & -1 \end{vmatrix}$$

$$AB \times a = \mathbf{i}((-5)(-1) - (4)(1)) - \mathbf{j}((1)(-1) - (4)(0)) + \mathbf{k}((1)(1) - (-5)(0))$$

$$AB \times a = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

The equation of the plane that passes through $A(1,2,-4)$ and has the normal vector $\langle 1,1,1 \rangle$ is

$$1(x-1) + 1(y-2) + 1(z+4) = 0$$

$$x - 1 + y - 2 + z + 4 = 0$$

$$x + y + z = -1$$

INTERSECTION OF A LINE AND A PLANE

■ 1. Find the x-, y-, and z-intercepts of the plane.

$$2x - 3y + z = 6$$

Solution:

The x-intercept will exist where $y = 0$ and $z = 0$.

$$2x - 3(0) + (0) = 6$$

$$2x = 6$$

$$x = 3$$

The y-intercept will exist where $x = 0$ and $z = 0$.

$$2(0) - 3y + (0) = 6$$

$$-3y = 6$$

$$y = -2$$

The z-intercept will exist where $x = 0$ and $y = 0$.

$$2(0) - 3(0) + z = 6$$

$$z = 6$$

The x-, y-, and z-intercepts are $(3,0,0)$, $(0,-2,0)$, and $(0,0,6)$ respectively.

■ 2. Find the intersection of AB and the plane $x + 2y + 4z = 12$, where A and B are the points $A(0,1,-2)$ and $B(-1,2,0)$, or determine that the line segment and the plane do no intersect.

Solution:

The line segment AB intersects the plane if the line that passes through the points A and B intersects the plane at some point C, and this point C lies between the points A and B.

The vector for AB is

$$AB = \langle -1-0, 2-1, 0-(-2) \rangle$$

$$AB = \langle -1, 1, 2 \rangle$$

The parametric equation of the line AB that passes through A is

$$x = -t$$

$$y = 1 + t$$

$$z = -2 + 2t$$

Substitute x, y, and z into the plane equation.

$$(-t) + 2(1 + t) + 4(-2 + 2t) = 12$$

$$9t - 6 = 12$$

$$t = 2$$

Plugging $t = 2$ back into the parametric equation gives the intersection point $(-2,3,2)$.

This point on the line AB lies between A and B if its coordinates lie between corresponding coordinates of the points A and B. But looking at the x-coordinates of each point, -2 doesn't lie between 0 and -1, so $(-2,3,2)$ doesn't lie on the interval AB, so AB and the plane do no intersect.

■ 3. Find the value of the constant p for which the line doesn't intersect the plane.

The line $\dfrac{x+1}{2} = \dfrac{y}{3} = z - 1$

The plane $px + 2y + z = 4$

Solution:

Rewrite the line equation in parametric form as

$$x = 2t - 1$$

$$y = 3t$$

$$z = t + 1$$

Substitute x, y, and z into the plane equation.

$$p(2t-1)+2(3t)+(t+1)=4$$

$$2tp+6t+t-p+1-4=0$$

$$t(2p+7)-p-3=0$$

$$t(2p+7)=p+3$$

$$t=\frac{p+3}{2p+7}$$

This equation for t is valid only if

$$2p+7\neq 0$$

$$2p\neq -7$$

$$p\neq -\frac{7}{2}$$

The line doesn't intersect the plane if $p=-7/2$.

PARALLEL, PERPENDICULAR, AND ANGLE BETWEEN PLANES

■ 1. Find the equation of the plane that passes through the points $A(1,0,-1)$ and $B(0,1,-1)$ and is perpendicular to the plane $x+2y+3z=6$.

Solution:

Let $n=\langle n_1,n_2,n_3\rangle$ be the normal vector to the plane we need to find. Since there are an infinite number of such vectors with different length, we can take a vector with an x-coordinate 1, like $n=\langle 1,n_2,n_3\rangle$. Let's find other components of n.

Since n is perpendicular to the plane $x+2y+3z=6$,

$$\langle 1,n_2,n_3\rangle\cdot\langle 1,2,3\rangle=0$$

$$1+2n_2+3n_3=0$$

Since n is the normal vector to the plane which includes the vector AB, n is also perpendicular to AB.

$$n\cdot AB=0$$

$$\langle 1,n_2,n_3\rangle\cdot\langle -1,1,0\rangle=0$$

$$-1+n_2=0$$

We have the system of linear equations in terms of n_2 and n_3.

From the second equation, we get $n_2 = 1$. Then substitute into the first equation.

$$1 + 2(1) + 3n_3 = 0$$

$$3 + 3n_3 = 0$$

$$n_3 = -1$$

So $n = \langle 1,1,-1\rangle$. Let's find the equation of the plane with normal vector $\langle 1,1,-1\rangle$ which passes through $A(1,0,-1)$.

$$1(x-1) + 1(y-0) - 1(z+1) = 0$$

$$x + y - z - 2 = 0$$

■ 2. Find the equation of the plane that passes through $A(3,2,-4)$ and is parallel to the plane $-x + 3y - 2z = 4$.

Solution:

Since the plane is parallel to $-x + 3y - 2z = 4$, it has the same normal vector, $\langle -1,3,-2\rangle$.

Let's find the equation of the plane with normal vector $\langle -1,3,-2\rangle$ which passes through the point $A(3,2,-4)$.

$$-1(x-3) + 3(y-2) - 2(z+4) = 0$$

$$-x + 3 + 3y - 6 - 2z - 8 = 0$$

$$-x + 3y - 2z - 11 = 0$$

■ 3. Find the equation of the plane a that passes through the point $A(1,-2,-3)$ and form equal angles with all of the coordinate planes, xy, yz, and xz.

Solution:

Let $n = \langle n_1, n_2, n_3 \rangle$ be the normal vector to the plane a. The xy-plane has normal vector $\langle 0,0,1 \rangle$, and cosine of the angle between a and the xy-plane is

$$\frac{n \cdot \langle 0,0,1 \rangle}{|n|} = \frac{n_3}{|n|}$$

Similarly, cosine of the angle between a and the yz-plane is

$$\frac{n \cdot \langle 1,0,0 \rangle}{|n|} = \frac{n_1}{|n|}$$

Cosine of the angle between a and the xz-plane is

$$\frac{n \cdot \langle 0,1,0 \rangle}{|n|} = \frac{n_2}{|n|}$$

Since all three angles are equal,

$$\frac{n_1}{|n|} = \frac{n_2}{|n|} = \frac{n_3}{|n|}$$

$$n_1 = n_2 = n_3$$

We can use a normal vector of any length, so we can choose the vector $n = \langle 1,1,1 \rangle$. Let's find the equation of the plane with normal vector $\langle 1,1,1 \rangle$ which passes through $A(1, -2, -3)$.

$$1(x-1) + 1(y+2) + 1(z+3) = 0$$

$$x - 1 + y + 2 + z + 3 = 0$$

$$x + y + z + 4 = 0$$

PARAMETRIC EQUATIONS FOR THE LINE OF INTERSECTION OF TWO PLANES

■ 1. Find the parametric equations for the line of intersection of the planes with normal vectors $a = \langle 2,0,-1\rangle$ and $b = \langle 1,2,-3\rangle$, and that have the common point $A(1,2,2)$.

Solution:

To find the vector of a line, we need to take the cross product of the normal vectors of the planes.

$$a \times b = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 1 & 2 & -3 \end{vmatrix}$$

$$a \times b = \mathbf{i}((0)(-3) - (-1)(2)) - \mathbf{j}((2)(-3) - (-1)(1)) + \mathbf{k}((2)(2) - (0)(1))$$

$$a \times b = 2\mathbf{i} + 5\mathbf{j} + 4\mathbf{k}$$

Since point A is common to both planes, it lies on the line of intersection of the planes. Let's find the equation of the line with vector $\langle 2,5,4\rangle$ which passes through the point $A(1,2,2)$.

$$x = 1 + 2t$$

$$y = 2 + 5t$$

$$z = 2 + 4t$$

■ 2. Find the parametric equations for the line of intersection of the plane $2x - 3y - 4z = 2$ with xz-plane.

Solution:

Since the intersection line lies on the xz-plane, $y = 0$. Substitute $y = 0$ into the plane equation.

$$2x - 4z = 2$$

$$x - 2z = 1$$

So we have the equation of a line on the xz-plane. To get a parametric equation, introduce $z = t$, and find x from the equation $x = 2t + 1$. So the parametric equations for the line of intersection is

$$x = 2t + 1$$

$$y = 0$$

$$z = t$$

■ 3. Find the equations of a plane a that's perpendicular to the plane b, which is $x - 3y + z = 2$, and intersects b along the line given by the parametric equation.

$$x = 2t$$

$$y = 1 + t$$

$$z = 2 - t$$

Solution:

The normal vector of the plane b is $\langle 1, -3, 1 \rangle$. The vector of the line of intersection is $\langle 2, 1, -1 \rangle$.

Since the plane a is perpendicular to the plane b, their normal vectors are perpendicular.

Also, since the line of intersection lies in the plane a, it's perpendicular to the normal vector of a.

So we can find the normal vector of a as a cross product.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$\mathbf{i}((-3)(-1) - (1)(1)) - \mathbf{j}((1)(-1) - (1)(2)) + \mathbf{k}((1)(1) - (-3)(2))$$

$$2\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$$

Let's take the point on the plane a by plugging $t = 0$ into the equation for the line of intersection.

$$x = 2(0) = 0$$

$$y = 1 + 0 = 1$$

$$z = 2 - 0 = 2$$

Let's find the equation of the plane with normal vector $\langle 2,3,7 \rangle$ which passes through $(0,1,2)$.

$$2(x - 0) + 3(y - 1) + 7(z - 2) = 0$$

$$2x + 3y - 3 + 7z - 14 = 0$$

$$2x + 3y + 7z - 17 = 0$$

SYMMETRIC EQUATIONS FOR THE LINE OF INTERSECTION OF TWO PLANES

■ 1. Find the symmetric equations for the line of intersection of the plane a, which is $2x - y + 3z = 12$, and the plane a' that's symmetric to the plane a with respect to the xz-plane.

Solution:

To get the equation of the plane a', just substitute $(-y)$ for y into the equation of the plane a.

$$2x + y + 3z = 12$$

The line of intersection of two symmetric planes with respect to the xz-plane lies in the xz-plane, so $y = 0$.

$$2x + 3z = 12$$

To obtain the symmetric equations for the line of intersection, isolate x and divide each side of the equation by 6 (least common multiple of 2 and 3).

$$2x = -(3z - 12)$$

$$\frac{2x}{6} = -\frac{3z - 12}{6}$$

$$\frac{x}{3} = -\frac{z - 4}{2}$$

So the symmetric equation for the line of intersection of the planes is

$$y = 0, \ \frac{x}{3} = -\frac{z-4}{2}$$

■ 2. Find the symmetric equations for the line of intersection of the plane $6x - 5y + z = 10$ with yz-plane.

Solution:

Since the intersection line lies in the yz-plane, that means $x = 0$. So substitute $x = 0$ into the plane equation.

$$-5y + z = 10$$

So we have the equation of a line in the yz-plane. To get the symmetric equation, isolate z and divide each side of the equation by 5.

$$z = 5y + 10$$

$$\frac{z}{5} = \frac{5y + 10}{5}$$

$$\frac{z}{5} = y + 2$$

So the symmetric equation for the line of intersection of the planes is

$$x = 0, \ \frac{z}{5} = y + 2$$

■ 3. Find the equations of a plane a that's perpendicular to the plane b, which is $2x-y-z=3$, and intersects b along the line

$$\frac{x-1}{3}=\frac{y+2}{2}=\frac{z}{2}$$

Solution:

The normal vector of the plane b is $\langle 2,-1,-1\rangle$. The vector of the line of intersection is $\langle 3,2,2\rangle$. Since the plane a is perpendicular to the plane b, their normal vectors are perpendicular.

And since the line of intersection lies in the plane a, it's perpendicular to the normal vector of a.

So we can find the normal vector of a as the cross product.

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & -1 \\ 3 & 2 & 2 \end{vmatrix}$$

$$\mathbf{i}((-1)(2)-(-1)(2))-\mathbf{j}((2)(2)-(-1)(3))+\mathbf{k}((2)(2)-(-1)(3))$$

$$0\mathbf{i}-7\mathbf{j}+7\mathbf{k}$$

Let's take the point on the plane a by plugging $z=0$ into the intersection line equation.

$x-1=0$ so $x=1$

$y + 2 = 0$ so $y = -2$

Let's find the equation of the plane with normal vector $\langle 0, -7, 7 \rangle$ which passes through $(1, -2, 0)$.

$$0(x - 1) - 7(y + 2) + 7(z - 0) = 0$$

$$-7y - 14 + 7z = 0$$

$$-y - 2 + z = 0$$

$$-y + z = 2$$

Note that the plane a appears to be parallel to x-axis. The equation of the plane is

$$-y + z = 2$$

DISTANCE BETWEEN A POINT AND A LINE

■ 1. Determine the length of the height of triangle ABC, that's perpendicular to BC, if $A(2,0,-1)$, $B(4,5,2)$, and $C(4,3,0)$.

Solution:

The height of triangle ABC is equal to the distance between the point A and the line passing through B and C. By the distance formula between the line BC and the point A,

$$d=\frac{|AB\times AC|}{|BC|}$$

The numerator is twice the area of the triangle ABC, and the denominator is the length of the base of the triangle, BC. The vectors of the triangle are

$$AB=\langle 2,5,3\rangle$$

$$AC=\langle 2,3,1\rangle$$

$$BC=\langle 0,-2,-2\rangle$$

Then

$$|BC|=\sqrt{(0)^2+(-2)^2+(-2)^2}=2\sqrt{2}$$

and the cross product $AB\times AC$ is

$$AB \times AC = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 5 & 3 \\ 2 & 3 & 1 \end{vmatrix}$$

$$AB \times AC = \mathbf{i}((5)(1) - (3)(3)) - \mathbf{j}((2)(1) - (3)(2)) + \mathbf{k}((2)(3) - (5)(2))$$

$$AB \times AC = -4\mathbf{i} + 4\mathbf{j} - 4\mathbf{k}$$

Then

$$|AB \times AC| = \sqrt{(-4)^2 + 4^2 + (-4)^2} = 4\sqrt{3}$$

By the distance formula,

$$d = \frac{4\sqrt{3}}{2\sqrt{2}} = \sqrt{6}$$

■ 2. Find the sum of distances from $A(3,3,-1)$ to all of the coordinate axes, x, y, and z.

Solution:

The distance between A and the x-axis is the distance between $(3,3,-1)$ and its projection on this axis, $(3,0,0)$.

$$\sqrt{(3-3)^2 + (3-0)^2 + (-1-0)^2} = \sqrt{10}$$

The distance between A and the y-axis is the distance between $(3,3,-1)$ and its projection on this axis, $(0,3,0)$.

$$\sqrt{(3-0)^2+(3-3)^2+(-1-0)^2}=\sqrt{10}$$

The distance between A and the z-axis is the distance between $(3,3,-1)$ and its projection on this axis, $(0,0,-1)$.

$$\sqrt{(3-0)^2+(3-0)^2+(-1-(-1))^2}=3\sqrt{2}$$

The total distance is therefore

$$\sqrt{10}+\sqrt{10}+3\sqrt{2}$$

$$2\sqrt{10}+3\sqrt{2}$$

■ 3. Find the distance between $A(0,-2,1)$, and the line.

$$\frac{x+1}{2}=\frac{y-2}{-1}=\frac{z}{2}$$

Solution:

The point on the line $B(-1,2,0)$, and the vector of the line is $a=\langle 2,-1,2\rangle$. By the distance formula between the line and the point A,

$$d=\frac{|AB\times a|}{|a|}$$

where

$$AB = \langle -1,4,-1\rangle$$

$$|a| = \sqrt{(2)^2 + (-1)^2 + (2)^2} = 3$$

The cross product $AB \times a$ is

$$AB \times a = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 4 & -1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$AB \times a = \mathbf{i}((4)(2) - (-1)(-1)) - \mathbf{j}((-1)(2) - (-1)(2)) + \mathbf{k}((-1)(-1) - (4)(2))$$

$$AB \times a = 7\mathbf{i} + 0\mathbf{j} - 7\mathbf{k}$$

Then

$$|AB \times a| = \sqrt{(7)^2 + 0^2 + (-7)^2} = 7\sqrt{2}$$

By the distance formula,

$$d = \frac{7\sqrt{2}}{3}$$

DISTANCE BETWEEN A POINT AND A PLANE

■ 1. Determine the length of the height of tetrahedron $ABCD$, that's perpendicular to the plane BCD, if $A(1,0,-1)$, $B(0,1,0)$, $C(2,3,4)$, and $D(2,2,2)$.

Solution:

The height of tetrahedron $ABCD$ is equal to the distance between the point A and the plane passing through B, C, and D. The normal vector to BCD is the cross product $n = BC \times BD$, where $BC = \langle 2,2,4 \rangle$ and $BD = \langle 2,1,2 \rangle$.

$$BC \times BD = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 4 \\ 2 & 1 & 2 \end{vmatrix}$$

$$BC \times BD = \mathbf{i}((2)(2) - (4)(1)) - \mathbf{j}((2)(2) - (4)(2)) + \mathbf{k}((2)(1) - (2)(2))$$

$$BC \times BD = 0\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$$

Then

$$|n| = \sqrt{(0)^2 + 4^2 + (-2)^2} = 2\sqrt{5}$$

By the distance formula between the plane BCD and the point A,

$$d = \frac{|n \cdot AB|}{|n|}$$

with $AB = \langle -1,1,1 \rangle$. So

$$d = \frac{|\langle 0,4,-2\rangle \cdot \langle -1,1,1\rangle|}{2\sqrt{5}}$$

$$d = \frac{0(-1)+4(1)-2(1)}{2\sqrt{5}}$$

$$d = \frac{2}{2\sqrt{5}}$$

$$d = \frac{1}{\sqrt{5}}$$

■ 2. Find the sum of distances from the point $A(2,3,-5)$ to all of the coordinate planes, xy, yz, and xz.

Solution:

The distance between the point A and the xy-plane is the distance between $(2,3,-5)$ and its projection on this plane, $(2,3,0)$. In other words, the distance between the point A and the xy-plane is equal to the absolute value of the z-coordinate of the point A, -5.

$$\sqrt{(2-2)^2+(3-3)^2+(-5-0)^2} = 5$$

Similarly, the distance between the point A and yz-plane is equal to the absolute value of the x-coordinate of the point A, 2.

Finally, the distance between the point A and the xz-plane is equal to the absolute value of the y-coordinate of the point A, 3. The total distance is

$$5 + 2 + 3 = 10$$

3. Find the points on the line L_1 that lie at a distance of 6 from plane a.

$$L_1: \qquad x = 1 + t,\ y = 2t,\ z = 3 - 2t$$

$$a: \qquad x + 2y - 2z = 4$$

Solution:

By the distance formula between an arbitrary point on the line L_1 and the plane a,

$$d = \frac{|x + 2y - 2z - 4|}{\sqrt{1 + 2^2 + (-2)^2}}$$

$$d = \frac{|x + 2y - 2z - 4|}{3}$$

Substitute $x = 1 + t$, $y = 2t$, and $z = 3 - 2t$.

$$d = \frac{|(1 + t) + 2(2t) - 2(3 - 2t) - 4|}{3}$$

$$d = \frac{|1 + t + 4t - 6 + 4t - 4|}{3}$$

$$d = \frac{|9t - 9|}{3}$$

$$d = |3t - 3|$$

Since the distance is equal to 6, we get an equation for t.

$$d = |3t - 3| = 6$$

$3t - 3 = 6$ or $3t - 3 = -6$

$3t = 9$ or $3t = -3$

$t = 3$ or $t = -1$

Coordinates of the point for $t = 3$ are $(1 + 3, 2(3), 3 - 2(3)) = (4, 6, -3)$, and the coordinates of the point for $t = -1$ are $(1 - 1, 2(-1), 3 - 2(-1)) = (0, -2, 5)$.

So the points on the line are

$$(4, 6, -3)$$

$$(0, -2, 5)$$

DISTANCE BETWEEN PARALLEL PLANES

■ 1. Determine the length of the height of triangular prism $ABCA_1B_1C_1$ that's perpendicular to the plane ABC if $A(2,0,1)$, $B(1,0,0)$, $C(2,2,3)$, $A_1(3,2,5)$, $B_1(2,2,4)$, and $C_1(3,4,7)$.

Solution:

The height of triangular prism $ABCA_1B_1C_1$ is equal to the distance between the prism bases, ABC and $A_1B_1C_1$, and equal to the distance between parallel planes ABC and $A_1B_1C_1$.

Find the distance between the parallel planes as the distance between $A_1(3,2,5)$ and plane ABC. The normal vector to ABC is the cross-product $n = BA \times BC$, where $BA = \langle 1,0,1 \rangle$ and $BC = \langle 1,2,3 \rangle$.

$$BA \times BC = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix}$$

$$BA \times BC = \mathbf{i}((0)(3) - (1)(2)) - \mathbf{j}((1)(3) - (1)(1)) + \mathbf{k}((1)(2) - (0)(1))$$

$$BA \times BC = -2\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$$

Then

$$|n| = \sqrt{(-2)^2 + (-2)^2 + (2)^2} = 2\sqrt{3}$$

By the distance formula between the plane ABC and the point A_1,

$$d = \frac{|n \cdot AA_1|}{|n|}$$

$$AA_1 = \langle 1,2,4 \rangle$$

So

$$d = \frac{|\langle -2,-2,2 \rangle \cdot \langle 1,2,4 \rangle|}{2\sqrt{3}}$$

$$d = \frac{-2(1) - 2(2) + 2(4)}{2\sqrt{3}}$$

$$d = \frac{2}{2\sqrt{3}}$$

$$d = \frac{1}{\sqrt{3}}$$

■ 2. Find the equations of the two planes that are parallel to the given plane and that lie at a distance of 2 from it.

$$2x - 2y - z = 3$$

Solution:

Since the parallel planes have the same normal vector, they have the equation

$$2x - 2y - z = k$$

where k is some arbitrary real number. Taking a point on this plane with $x = y = 0$, then $z = -k$. So by the distance between $(0,0,-k)$ and the given plane,

$$d = \frac{|2(0) - 2(0) - (-k) - 3|}{\sqrt{2^2 + (-2)^2 + (-1)^2}}$$

$$d = \frac{|k - 3|}{3}$$

Since the distance is equal to 2, we can get a value for k.

$$d = \frac{|k - 3|}{3} = 2$$

$$|k - 3| = 6$$

$$k = -3 \text{ or } k = 9$$

So the parallel planes are

$$2x - 2y - z = -3$$

$$2x - 2y - z = 9$$

3. Find the equations of the two parallel planes a and b that pass through $A(2,-6,3)$ and $B(7,10,11)$ respectively, and have the largest possible distance between them.

Solution:

We'll get the largest possible distance between the planes a and b if the interval AB is perpendicular to each plane. So the vector AB is a normal vector to both planes, which gives $AB = \langle 5,16,8\rangle$.

The equation of the plane that passes through $A(2,-6,3)$ and has the normal vector $AB = \langle 5,16,8\rangle$ is

$$5(x-2)+16(y+6)+8(z-3)=0$$

$$5x-10+16y+96+8z-24=0$$

$$5x+16y+8z=-62$$

The equation of the plane that passes through $B(7,10,11)$ and has the normal vector $AB = \langle 5,16,8\rangle$ is

$$5(x-7)+16(y-10)+8(z-11)=0$$

$$5x-35+16y-160+8z-88=0$$

$$5x+16y+8z=283$$

So the parallel planes are

$$5x+16y+8z=-62$$

$$5x + 16y + 8z = 283$$

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