Integral of a vector function

To find the integral of a vector function $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$, we simply replace each coefficient with its integral. In other words, the integral of the vector function is

$$\int r(t)\ dt = \mathbf{i}\int r(t)_1\ dt + \mathbf{j}\int r(t)_2\ dt + \mathbf{k}\int r(t)_3\ dt$$

If the vector function is given as $r(t) = \left\langle r(t)_1, r(t)_2, r(t)_3 \right\rangle$, then its integral is

$$\int r(t) = \left\langle \int r(t)_1\ dt, \int r(t)_2\ dt, \int r(t)_3\ dt \right\rangle$$

Example

Find the integral of the vector function over the interval $[0,\pi]$.

$$r(t) = \sin(2t)\mathbf{i} + 2e^{2t}\mathbf{j} + 4t^3\mathbf{k}$$

Remember that we're only taking the integrals of the coefficients, which means $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ will be left alone.

$$\int_0^\pi r(t)\ dt = \frac{-\cos(2t)}{2}\Big|_0^\pi \mathbf{i} + \frac{2e^{2t}}{2}\Big|_0^\pi \mathbf{j} + \frac{4t^4}{4}\Big|_0^\pi \mathbf{k}$$

$$\int_0^\pi r(t)\ dt = \frac{-\cos(2t)}{2}\Big|_0^\pi \mathbf{i} + e^{2t}\Big|_0^\pi \mathbf{j} + t^4\Big|_0^\pi \mathbf{k}$$

Evaluating over the interval $[0,\pi]$, we get

$$\int_0^{\pi} r(t)\ dt = \left[\frac{-\cos(2\pi)}{2} - \frac{-\cos(2(0))}{2}\right]\mathbf{i} + \left[e^{2\pi} - e^{2(0)}\right]\mathbf{j} + \left[\pi^4 - 0^4\right]\mathbf{k}$$

$$\int_0^{\pi} r(t)\ dt = \left[\frac{-\cos(2\pi)}{2} + \frac{\cos 0}{2}\right]\mathbf{i} + \left(e^{2\pi} - 1\right)\mathbf{j} + \left(\pi^4 - 0\right)\mathbf{k}$$

$$\int_0^{\pi} r(t)\ dt = \left(\frac{-1}{2} + \frac{1}{2}\right)\mathbf{i} + \left(e^{2\pi} - 1\right)\mathbf{j} + \pi^4\mathbf{k}$$

$$\int_0^{\pi} r(t)\ dt = 0\mathbf{i} + \left(e^{2\pi} - 1\right)\mathbf{j} + \pi^4\mathbf{k}$$

$$\int_0^{\pi} r(t)\ dt = \left(e^{2\pi} - 1\right)\mathbf{j} + \pi^4\mathbf{k}$$

This is the integral of the vector function. We could also write it in the form

$$\int_0^{\pi} r(t)\ dt = \left\langle 0, e^{2\pi} - 1, \pi^4\right\rangle$$