Calculus 3 Workbook Solutions

Applications of double integrals

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DOUBLE INTEGRALS TO FIND MASS AND CENTER OF MASS

■ 1. The circular disk with radius 12 has density $\delta = 1/(r+4)$, where r is the distance to the center of disk. Find the mass and center of mass of the disk.

Solution:

The mass of the disk is given by the double integral

$$\iint_D \delta(x, y)\ dA$$

but we'll need to convert it to polar coordinates using

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dx\ dy = r\ dr\ d\theta$$

The area of integration is the circle with center at the origin and radius 12. So the polar coordinate r changes from 0 to 12, and θ changes from 0 to 2π. So the integral in polar coordinates is

$$\int_0^{2\pi}\int_0^{12} \frac{1}{r+4} r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^{12}\frac{r}{r+4}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^{12}\frac{r+4-4}{r+4}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^{12}1-\frac{4}{r+4}\ dr\ d\theta$$

Integrate with respect to r.

$$\int_0^{2\pi} r-4\ln(r+4)\Big|_0^{12}\ d\theta$$

$$\int_0^{2\pi} 12-4\ln(12+4)-[0-4\ln(0+4)]\ d\theta$$

$$\int_0^{2\pi} 12-4\ln(12+4)-0+4\ln(0+4)\ d\theta$$

$$\int_0^{2\pi} 12-4\ln(16)+4\ln(4)\ d\theta$$

Integrate with respect to θ.

$$12\theta-4\ln(16)\theta+4\ln(4)\theta\Big|_0^{2\pi}$$

$$12(2\pi)-4\ln(16)(2\pi)+4\ln(4)(2\pi)-[12(0)-4\ln(16)(0)+4\ln(4)(0)]$$

$$24\pi-8\pi\ln(16)+8\pi\ln(4)$$

Use laws of logs to simplify.

$$24\pi - 8\pi \ln(4^2) + 8\pi \ln(4)$$

$$24\pi - 8(2)\pi \ln(4) + 8\pi \ln(4)$$

$$24\pi - 16\pi \ln(4) + 8\pi \ln(4)$$

$$24\pi - 8\pi \ln(4)$$

Since the disk is symmetric and has symmetric density about x-axis and y-axis, its center of mass is the center of the disk.

■ 2. The rectangular plate with length 4 m and width 2 m has density $\delta = 2d$ kg/m^2, where d is the distance from its left 2 m side. Find the mass and center of mass of the plate.

Solution:

The mass of the plate is given by the double integral:

$$\int\int_D \delta(x, y)\ dA$$

Let's place the origin at the bottom left corner of the plate, and put the edges of the plate along the x- and y-axes. So the plate is defined on $x = [0,4]$ and $y = [0,2]$, and its density is $2x$. So the double integral is

$$\int_0^2 \int_0^4 2x\ dx\ dy$$

Integrate with respect to x.

$$\int_0^2 x^2 \Big|_0^4\ dy$$

$$\int_0^2 4^2 - 0^2\ dy$$

$$\int_0^2 16\ dy$$

Integrate with respect to y.

$$16y \Big|_0^2$$

$$16(2) - 16(0)$$

32 kg

Find the x-coordinate of the center of mass.

$$\bar{x} = \frac{1}{M} \int \int_D x\delta(x, y)\ dA$$

$$\bar{x} = \frac{1}{32} \int_0^2 \int_0^4 2x^2\ dx\ dy$$

$$\bar{x} = \frac{1}{32} \int_0^2 \frac{2}{3}x^3 \Big|_0^4 \ dy$$

$$\bar{x} = \frac{1}{32} \int_0^2 \frac{2}{3}(4)^3 - \frac{2}{3}(0)^3 \ dy$$

$$\bar{x} = \frac{1}{32} \int_0^2 \frac{128}{3} \ dy$$

$$\bar{x} = \frac{4}{3} \int_0^2 \ dy$$

$$\bar{x} = \frac{4}{3}y \Big|_0^2$$

$$\bar{x} = \frac{4}{3}(2) - \frac{4}{3}(0)$$

$$\bar{x} = \frac{8}{3}$$

Find the y-coordinate of the center of mass.

$$\bar{y} = \frac{1}{M} \iint_D y\delta(x, y) \ dA$$

$$\bar{y} = \frac{1}{32} \int_0^2 \int_0^4 2xy \ dx \ dy$$

$$\bar{y} = \frac{1}{32} \int_0^2 x^2 y \Big|_{x=0}^{x=4} \ dy$$

$$\bar{y} = \frac{1}{32}\int_0^2 4^2y - 0^2y\ dy$$

$$\bar{y} = \frac{1}{32}\int_0^2 16y\ dy$$

$$\bar{y} = \frac{1}{2}\int_0^2 y\ dy$$

$$\bar{y} = \frac{1}{2}\left(\frac{1}{2}y^2\right)\Big|_0^2$$

$$\bar{y} = \frac{1}{4}y^2\Big|_0^2$$

$$\bar{y} = \frac{1}{4}(2)^2 - \frac{1}{4}(0)^2$$

$$\bar{y} = 1$$

Therefore, the mass is $M = 32$ kg, and the center of mass is at $(\bar{x}, \bar{y}) = (8/3,1)$.

■ 3. Some gas is distributed above the line with density $\delta = e^{-ad^2}$, where d is the distance to point A on the line, and a is a constant. Find the total mass of the gas and its center of mass.

Solution:

The mass of the gas is given by the double integral

$$\iint_D \delta(x,y)\ dA$$

Let's place the origin at the point A on the line, and the x-axis on this line. Then x is defined from $-\infty$ to ∞ and y is defined from 0 to ∞. The density is $\delta = e^{-a(x^2+y^2)}$. Then the mass is

$$\int_0^\infty \int_{-\infty}^\infty e^{-a(x^2+y^2)}\ dx\ dy$$

$$\int_0^\infty \int_{-\infty}^\infty e^{-ax^2}e^{-ay^2}\ dx\ dy$$

$$\int_0^\infty e^{-ay^2}\ dy \cdot \int_{-\infty}^\infty e^{-ax^2}\ dx$$

These integrals can't be calculated directly, but we know their values.

$$\int_{-\infty}^\infty e^{-ax^2}\ dx = \sqrt{\frac{\pi}{a}}$$

Since the function is symmetric,

$$\int_0^\infty e^{-ay^2}\ dy = \frac{1}{2}\sqrt{\frac{\pi}{a}}$$

So the expression becomes

$$\frac{1}{2}\sqrt{\frac{\pi}{a}} \cdot \sqrt{\frac{\pi}{a}}$$

$$\frac{1}{2}\left(\frac{\pi}{a}\right)$$

$$\frac{\pi}{2a}$$

Since the region and density are symmetric about y-axis, $\bar{x} = 0$. Calculate $\bar{y}$.

$$\bar{y} = \frac{1}{M} \iint_D y\delta(x, y) \ dA$$

$$\bar{y} = \frac{2a}{\pi} \int_0^{\infty} \int_{-\infty}^{\infty} ye^{-a(x^2+y^2)} \ dx \ dy$$

$$\bar{y} = \frac{2a}{\pi} \int_0^{\infty} ye^{-ay^2} dy \cdot \int_{-\infty}^{\infty} e^{-ax^2} dx$$

$$\bar{y} = \frac{2a}{\pi} \int_0^{\infty} ye^{-ay^2} dy \cdot \sqrt{\frac{\pi}{a}}$$

$$\bar{y} = \frac{\sqrt{a}\sqrt{a}\sqrt{\pi}}{\sqrt{a}\sqrt{\pi}\sqrt{\pi}} \int_0^{\infty} 2ye^{-ay^2} dy$$

$$\bar{y} = \frac{\sqrt{a}}{\sqrt{\pi}} \int_0^{\infty} 2ye^{-ay^2} dy$$

$$\bar{y} = \sqrt{\frac{a}{\pi}} \int_0^{\infty} 2ye^{-ay^2} dy$$

Use substitution with $u = y^2$, $du = 2y \ dy$, and where u changes from 0 to ∞.

$$\bar{y} = \sqrt{\frac{a}{\pi}} \int_0^{\infty} 2ye^{-ay^2} dy$$

$$\bar{y} = \sqrt{\frac{a}{\pi}} \int_0^{\infty} e^{-au} \ du$$

$$\bar{y} = \sqrt{\frac{a}{\pi}} \left(\frac{1}{-a} e^{-au} \right) \Big|_0^{\infty}$$

$$\bar{y} = \left[\lim_{u \to \infty} \sqrt{\frac{a}{\pi}} \left(\frac{1}{-a} e^{-au} \right) \right] - \sqrt{\frac{a}{\pi}} \left(\frac{1}{-a} e^{-a(0)} \right)$$

$$\bar{y} = \sqrt{\frac{a}{\pi}} \left(\frac{1}{-a} (0) \right) + \sqrt{\frac{a}{\pi}} \left(\frac{1}{a} \right)$$

$$\bar{y} = \frac{1}{\sqrt{a\pi}}$$

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