Parallel, perpendicular and angle between planes

Given two planes:

Plane	**Normal vector to the plane**
$a_1x + a_2y + a_3z = c$	$a\langle a_1, a_2, a_3\rangle$
$b_1x + b_2y + b_3z = d$	$b\langle b_1, b_2, b_3\rangle$

they will always be

parallel if the ratio equality is true.

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

perpendicular if the dot product of their normal vectors is 0.

$$a \cdot b = 0$$

set at a non-90° angle if the planes are neither parallel nor perpendicular, in which case the **angle** between the planes is given by

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where a and b are the normal vectors to the given planes, $a \cdot b$ is the dot product of the vectors, $|a|$ is the magnitude of the vector a (its length) and $|b|$ is the magnitude of the vector b

(its length). We can find the magnitude of both vectors using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

for a three-dimensional vector where the point (x_1, y_1, z_1) is the origin $(0,0,0)$.

Example

Say whether the planes are parallel, perpendicular, or neither. If the planes are neither parallel nor perpendicular, find the angle between the planes.

$$3x - y + 2z = 5$$

$$x + 4y + 3z = 1$$

First we'll find the normal vectors of the given planes.

Plane	Normal vector to the plane
$3x - y + 2z = 5$	$a\langle 3, -1, 2\rangle$
$x + 4y + 3z = 1$	$b\langle 1, 4, 3\rangle$

To say whether the planes are parallel, we'll set up our ratio inequality using the direction numbers from their normal vectors.

$$\frac{3}{1} = \frac{-1}{4} = \frac{2}{3}$$

Since the ratios are not equal, the planes are not parallel.

To say whether the planes are perpendicular, we'll take the dot product of their normal vectors.

$$a \cdot b = (3)(1) + (-1)(4) + (2)(3)$$

$$a \cdot b = 3 - 4 + 6$$

$$a \cdot b = 5$$

Since the dot product is not 0, the planes are not perpendicular.

Since the planes are not parallel or perpendicular, we know that they are set at a non-90° angle from one another, which is given by the formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

We need to find the dot product of the normal vectors, and the magnitude of each of them. We already know from our perpendicular test that their dot product is

$$a \cdot b = (3)(1) + (-1)(4) + (2)(3)$$

$$a \cdot b = 3 - 4 + 6$$

$$a \cdot b = 5$$

The magnitude of $a\langle 3, -1, 2\rangle$ is

$$|a| = \sqrt{(3-0)^2 + (-1-0)^2 + (2-0)^2}$$

$$|a| = \sqrt{9 + 1 + 4}$$

$$|a| = \sqrt{14}$$

The magnitude of $b\langle 1,4,3 \rangle$ is

$$|b| = \sqrt{(1-0)^2 + (4-0)^2 + (3-0)^2}$$

$$|b| = \sqrt{1 + 16 + 9}$$

$$|b| = \sqrt{26}$$

Plugging $a \cdot b = 5$, $|a| = \sqrt{14}$, and $|b| = \sqrt{26}$ into our cosine formula gives

$$\cos\theta = \frac{5}{\sqrt{14}\sqrt{26}}$$

$$\cos\theta = \frac{5}{\sqrt{364}}$$

$$\theta = \arccos\frac{5}{\sqrt{364}}$$

$$\theta = 74.8^\circ$$

The angle between the planes is 74.8°.
