Iterated and triple integrals

Iterated integrals are double or triple integrals whose limits of integration are already specified. An iterated triple integral might look like

$$\int_0^1 \int_1^3 \int_0^2 x^2y^3 \ dx \ dy \ dz$$

In this case, because the integral ends in $dx \ dy \ dz$ and we always integrate "inside out", we'd integrate first with respect to x, then with respect to y, and lastly with respect to z, evaluating over the associated interval after each integration.

You'll also see triple integrals in which the limits of integration have not yet been specified, like

$$\iiint_E 7xy^2 \ dV$$

Instead of a *triple iterated integral* (where the word *iterated* indicates the presence of the limits of integration), we just call this a *triple integral.* Since we can't solve the triple integral without finding limits of integration, we'll calculate limits of integration for each variable, then add them into the triple integral to turn it into a triple iterated integral.

Let's try solving a triple iterated integral.

Example

Evaluate the iterated integral.

$$\int_0^1 \int_1^3 \int_0^2 x^2y^3 \ dx \ dy \ dz$$

We always work our way "inside out" in order to evaluate iterated integrals. Since dx is listed closest to the "inside", we know we have to integrate with respect to x first, and that the limits of integration $[2,0]$ on the innermost integral are associated with x.

Integrating with respect to x (keeping y and z constant), and then evaluating over the interval $[2,0]$ gives

$$\int_0^1 \int_1^3 \left(\frac{1}{3}x^3y^3 \Big|_{x=0}^{x=2} \right) \ dy \ dz$$

$$\int_0^1 \int_1^3 \left[\frac{1}{3}(2)^3y^3 - \frac{1}{3}(0)^3y^3 \right] \ dy \ dz$$

$$\int_0^1 \int_1^3 \frac{8}{3}y^3 \ dy \ dz$$

Since dy is listed next, we know we have to integrate with respect to y (keeping z constant), and then evaluate over the interval $[1,3]$.

$$\int_0^1 \left(\frac{2}{3}y^4 \Big|_{y=1}^{y=3} \right) \ dz$$

$$\int_0^1 \left[\frac{2}{3}(3)^4 - \frac{2}{3}(1)^4 \right] \ dz$$

$$\int_0^1 \left(\frac{162}{3} - \frac{2}{3}\right)\ dz$$

$$\int_0^1 \frac{160}{3}\ dz$$

Since dz is listed last, we know we have to integrate with respect to z, and then evaluate over the interval $[0,1]$.

$$\frac{160}{3}z\Big|_0^1$$

$$\frac{160}{3}(1) - \frac{160}{3}(0)$$

$$\frac{160}{3}$$

This is the value of the triple iterated integral.

Now let's do a triple integral without limits of integration to see how it's different.

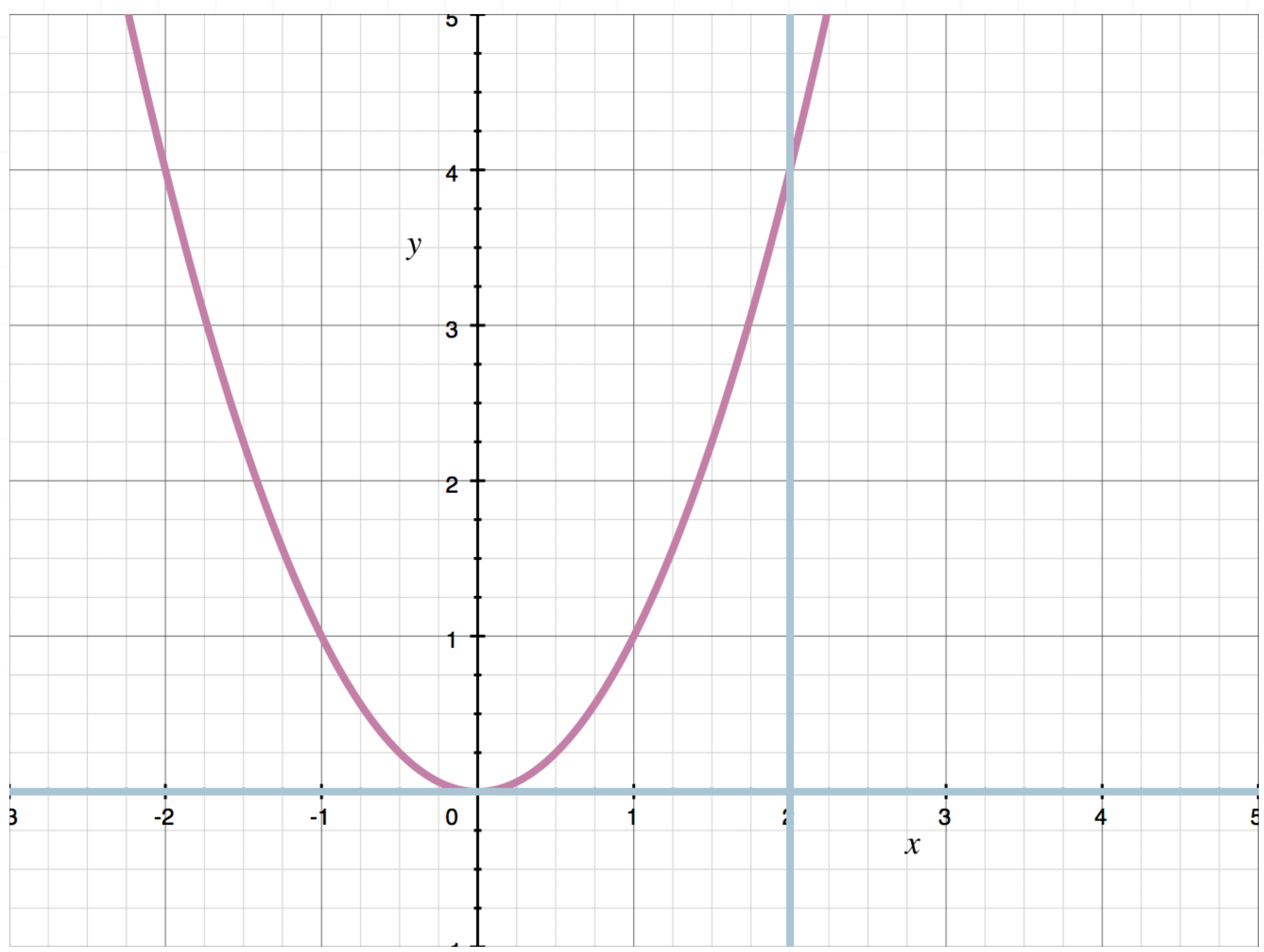
Example

Evaluate the triple integral if E is the region below $z = x + y - 1$ but above the region bounded by $y = x^2$, $y = 0$ and $x = 2$.

$$\iiint_E 7xy^2\ dV$$

We know that the region E lies below $z = x + y - 1$ but above the region bounded by $y = x^2$, $y = 0$ and $x = 2$.

If we graph the region bounded by $y = x^2$, $y = 0$ and $x = 2$, we can see that it lies in the xy-plane.



Since we're looking for the volume directly above the that planar region and not outside it, we can use the planar region to define the limits of integration for x and y.

Based on the graph, we can see that x is defined on the interval $[0,2]$, and that y is defined on the interval $\left[0,x^2\right]$.

Since the volume we're solving for sits on top of the region graphed above in the xy-plane, we know that the lower limit of integration for z is 0. Since the volume lies under $z = x + y - 1$, the upper limit of integration will be $x + y - 1$, which means that z is defined on the interval $[0,x + y - 1]$.

Generally speaking, we put the most complicated limits of integration on the innermost integral, and the simplest limits of integration on the outermost integral. Since the limits of integration for z are defined in terms of two variables, we'll put those on the innermost integral. The limits of integration for y are defined in terms of one variable, so those will come next. Since the limits of integration for x are constants, those will come last on the outermost integral.

$$\iiint_E 7xy^2\ dV = \int_0^2 \int_0^{x^2} \int_0^{x+y-1} 7xy^2\ dz\ dy\ dx$$

Now we're dealing with a triple iterated integral, which we already know how to solve. We'll start by integrating with respect to z (holding x and y constant), and then we'll evaluate over the interval for z.

$$\int_0^2 \int_0^{x^2} \left[7xy^2z \Big|_{z=0}^{z=x+y-1} \right]\ dy\ dx$$

$$\int_0^2 \int_0^{x^2} 7xy^2(x + y - 1) - 7xy^2(0)\ dy\ dx$$

$$\int_0^2 \int_0^{x^2} 7x^2y^2 + 7xy^3 - 7xy^2\ dy\ dx$$

Integrating with respect to y (holding x constant), and then evaluating over the interval for y, we get

$$\int_0^2 \left[\frac{7}{3}x^2y^3 + \frac{7}{4}xy^4 - \frac{7}{3}xy^3 \Big|_{y=0}^{y=x^2}\right] \ dx$$

$$\int_0^2 \frac{7}{3}x^2\left(x^2\right)^3 + \frac{7}{4}x\left(x^2\right)^4 - \frac{7}{3}x\left(x^2\right)^3 - \left[\frac{7}{3}x^2(0)^3 + \frac{7}{4}x(0)^4 - \frac{7}{3}x(0)^3\right] \ dx$$

$$\int_0^2 \frac{7}{3}x^2\left(x^6\right) + \frac{7}{4}x\left(x^8\right) - \frac{7}{3}x\left(x^6\right) \ dx$$

$$\int_0^2 \frac{7}{3}x^8 + \frac{7}{4}x^9 - \frac{7}{3}x^7 \ dx$$

Finally, integrating with respect to x and then evaluating over the interval for x, we get

$$\frac{7}{27}x^9 + \frac{7}{40}x^{10} - \frac{7}{24}x^8 \Big|_0^2$$

$$\frac{7}{27}(2)^9 + \frac{7}{40}(2)^{10} - \frac{7}{24}(2)^8 - \left[\frac{7}{27}(0)^9 + \frac{7}{40}(0)^{10} - \frac{7}{24}(0)^8\right]$$

$$\frac{7}{27}(512) + \frac{7}{40}(1{,}024) - \frac{7}{24}(256)$$

$$7(256)\left[\frac{1}{27}(2) + \frac{1}{40}(4) - \frac{1}{24}\right]$$

$$1{,}792\left(\frac{2}{27} + \frac{1}{10} - \frac{1}{24}\right)$$

$$1{,}792\left[\frac{2}{27}\left(\frac{80}{80}\right)+\frac{1}{10}\left(\frac{216}{216}\right)-\frac{1}{24}\left(\frac{90}{90}\right)\right]$$

$$1{,}792\left(\frac{160}{2{,}160}+\frac{216}{2{,}160}-\frac{90}{2{,}160}\right)$$

$$1{,}792\left(\frac{286}{2{,}160}\right)$$

$$1{,}792\left(\frac{143}{1{,}080}\right)$$

$$896\left(\frac{143}{540}\right)$$

$$\frac{128{,}128}{540}$$

$$\frac{32{,}032}{135}$$

This is the value of the triple integral.