Calculus 3 Workbook Solutions

Differentials

krista king MATH

DIFFERENTIAL OF A MULTIVARIABLE FUNCTION

■ 1. Find the differential of the multivariable function.

$$f(r,\theta)=\frac{r^2}{\sin 2\theta+\cos 2\theta}$$

Solution:

Find partial derivatives of f with respect to r and θ.

$$f_r=\frac{2r}{\sin 2\theta+\cos 2\theta}$$

and

$$f_\theta=\frac{(0)(\sin 2\theta+\cos 2\theta)-(r^2)(2\cos 2\theta-2\sin 2\theta)}{(\sin 2\theta+\cos 2\theta)^2}$$

$$f_\theta=-\frac{2r^2(\cos 2\theta-\sin 2\theta)}{(\sin 2\theta+\cos 2\theta)^2}$$

So the differential is

$$df=\frac{2r}{\sin 2\theta+\cos 2\theta}\ dr-\frac{2r^2(\cos 2\theta-\sin 2\theta)}{(\sin 2\theta+\cos 2\theta)^2}\ d\theta$$

■ 2. Find the differential of the multivariable function.

$$U(u, v, w) = \frac{(2u+1)^2(3v+4)}{\sqrt{w-2}}$$

Solution:

Rewrite the function by bringing the denominator up into the numerator.

$$U(u, v, w) = (2u+1)^2(3v+4)(w-2)^{-\frac{1}{2}}$$

Find partial derivatives of U with respect to u, v, and w.

$$U_u = 2(2)(2u+1)(3v+4)(w-2)^{-\frac{1}{2}}$$

$$U_u = 4(2u+1)(3v+4)(w-2)^{-\frac{1}{2}}$$

and

$$U_v = (2u+1)^2(3)(w-2)^{-\frac{1}{2}}$$

$$U_v = 3(2u+1)^2(w-2)^{-\frac{1}{2}}$$

and

$$U_w = -\frac{1}{2}(2u+1)^2(3v+4)(w-2)^{-\frac{1}{2}}$$

Then the differential is

$$dU = 4(2u+1)(3v+4)(w-2)^{-\frac{1}{2}}\ du + 3(2u+1)^2(w-2)^{-\frac{1}{2}}\ dv$$

$$-\frac{1}{2}(2u+1)^2(3v+4)(w-2)^{-\frac{3}{2}}\ dw$$

$$dU = \frac{4(2u+1)(3v+4)}{\sqrt{w-2}}\ du + \frac{3(2u+1)^2}{\sqrt{w-2}}\ dv$$

$$-\frac{(2u+1)^2(3v+4)}{2\sqrt{(w-2)^3}}\ dw$$

■ 3. Find the differential of the multivariable function at $(-6,2)$.

$$f(x,y) = 4\log_2(x^2 - 2xy + y^2)$$

Solution:

Rewrite the function.

$$f(x,y) = 4\log_2(x-y)^2$$

Since for the point $(-6,2)$ the expression $(x-y)$ is negative,

$$\log_2(x-y)^2 = 2\log_2(y-x)$$

So

$$f(x,y) = 8\log_2(y-x)$$

Find partial derivatives.

$$f_x = \frac{8}{\ln(2)(y-x)} \cdot \frac{\partial}{\partial x}(y-x) = -\frac{8}{\ln(2)(y-x)}$$

$$f_x(-6,2) = -\frac{8}{\ln(2)(2-(-6))} = -\frac{1}{\ln(2)}$$

and

$$f_y = \frac{8}{\ln(2)(y-x)} \cdot \frac{\partial}{\partial y}(y-x) = \frac{8}{\ln(2)(y-x)}$$

$$f_y(-6,2) = \frac{8}{\ln(2)(2-(-6))} = \frac{1}{\ln(2)}$$

So the differential is

$$df = -\frac{dx}{\ln 2} + \frac{dy}{\ln 2}$$

■ 4. Find the point(s) where the differential of the multivariable function is equal to 0 (i.e., find the critical points of the function).

$$f(s,t) = 3t^4 - 2t^2s - s^2 + 16t + 5$$

Solution:

Find the partial derivatives.

$$f_s = -2t^2 - 2s = -2(t^2 + s)$$

$$f_t = 3(4t^3) - 2(2t)s + 16 = 12t^3 - 4ts + 16 = 4(3t^3 - ts + 4)$$

Since $df = 0$, we get a system of nonlinear equations.

$$3t^3 - ts + 4 = 0$$

$$t^2 + s = 0$$

The second equation solves for s as $s = -t^2$. Then by by substitution, we get

$$3t^3 - t(-t^2) + 4 = 0$$

$$3t^3 + t^3 + 4 = 0$$

$$4t^3 + 4 = 0$$

$$t^3 = -1$$

$$t = -1$$

Then $s = -(-1)^2 = -1$, and the differential is 0 at $(-1, -1)$.

■ 5. Find and identify the set of point(s) where the differential of the multivariable function $f(x, y)$ doesn't depend on dy.

$$f(x, y) = \cos(e^{x^2+y})$$

Solution:

Since $df = f_x \ dx$ for these points, $f_y = 0$.

$$f_y = -\sin(e^{x^2+y})\frac{\partial}{\partial y}e^{x^2+y}$$

$$f_y = -\sin(e^{x^2+y})e^{x^2+y}\frac{\partial}{\partial y}(x^2+y)$$

$$f_y = -\sin(e^{x^2+y})e^{x^2+y} = 0$$

Since $e^{x^2+y} > 0$ for any (x, y), $\sin(e^{x^2+y}) = 0$ and $e^{x^2+y} = \pi n$ for any integer n.

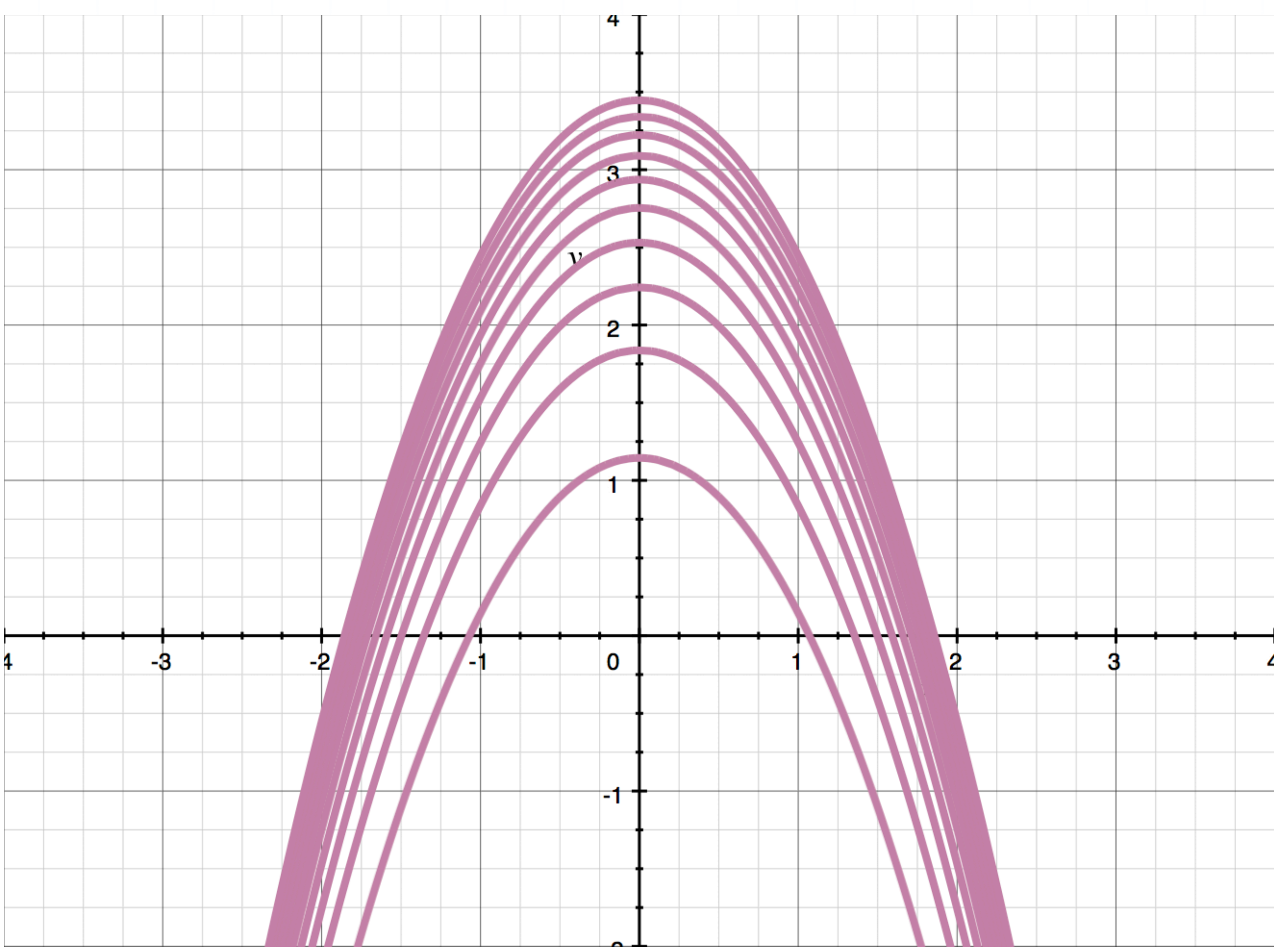
Since $e^{x^2+y} > 0$ for any (x, y), πn should be positive, i.e. $e^{x^2+y} = \pi n$ for any positive integer n.

Solve the equation for y.

$$x^2 + y = \ln(\pi n)$$

$$y = -x^2 + \ln(\pi n) \text{ for integer } n \geq 1$$

This is the sequence of parabolas with vertices at the points $(0, \ln(\pi n))$ for integers $n \geq 1$.



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