Topic: Double integrals to find mass and center of mass

Question: The vertices of a triangle-shaped lamina are $(4,2)$, $(0,2)$, and $(0,0)$. The density of the lamina is defined by $\delta = 4x + 2y$. What is the mass of the lamina?

Answer choices:

A $\quad m = \frac{1,024}{3}$

B $\quad m = \frac{1,024}{5}$

C $\quad m = 512$

D $\quad m = 32$

Solution: D

Of the given vertices $(4,2)$, $(0,2)$, and $(0,0)$, two of them are on the horizontal line $y = 2$. And two of the vertices are on the line $x = 0$. So $y = 2$ and $x = 0$ are two boundaries of the triangle.

The third boundary is a line that connects $(4,2)$ to $(0,0)$. The equation of this line is $y = (1/2)x$ or $x = 2y$. Now we can plug the bounds and the density equation $\delta = 4x + 2y$ into the double integral.

$$m = \int_0^2 \int_0^{2y} 4x + 2y \ dx \ dy$$

Integrate with respect to x, then evaluate over the interval.

$$m = \int_0^2 2x^2 + 2xy \Big|_{x=0}^{x=2y} \ dy$$

$$m = \int_0^2 2(2y)^2 + 2(2y)y - \left(2(0)^2 + 2(0)y\right) \ dy$$

$$m = \int_0^2 8y^2 + 4y^2 \ dy$$

$$m = \int_0^2 12y^2 \ dy$$

Integrate with respect to y, then evaluate over the interval.

$$m = 4y^3 \Big|_0^2$$

$$m = 4(2)^3 - 4(0)^3$$

$$m = 32$$

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Question: A region is bounded by the parabolas $y = 3x - x^2$ and $y = 2x^2 - 6x$. What is the center of the mass of the region?

Answer choices:

A $\left(\frac{3}{2}, -\frac{9}{10}\right)$

B $\left(\frac{3}{2}, \frac{9}{20}\right)$

C $\left(\frac{9}{7}, -\frac{9}{10}\right)$

D $\left(\frac{9}{2}, \frac{9}{2}\right)$

Solution: A

The points of intersection of the parabolas

$$y = 3x - x^2$$

$$y = 2x^2 - 6x$$

are $(0,0)$, and $(3,0)$. Putting this information into a double integral gives

$$A = \int_R dA$$

$$A = \int_0^3 \int_{2x^2-6x}^{3x-x^2} dy\ dx$$

Integrate with respect to y, then evaluate over the interval.

$$A = \int_0^3 y\Big|_{y=2x^2-6x}^{y=3x-x^2} dx$$

$$A = \int_0^3 3x - x^2 - \left(2x^2 - 6x\right)\ dx$$

$$A = \int_0^3 3x - x^2 - 2x^2 + 6x\ dx$$

$$A = \int_0^3 9x - 3x^2\ dx$$

Integrate with respect to x, then evaluate over the interval.

$$A=\frac{9}{2}x^2-x^3\Big|_0^3$$

$$A=\frac{9}{2}(3)^2-(3)^3-\left(\frac{9}{2}(0)^2-(0)^3\right)$$

$$A=\frac{81}{2}-27$$

$$A=\frac{81}{2}-\frac{54}{2}$$

$$A=\frac{27}{2}$$

Now that we have the area, we need to find M_x and M_y.

$$M_x=\int_0^3\int_{2x^2-6x}^{3x-x^2}y\ dy\ dx$$

$$M_x=\int_0^3\frac{1}{2}y^2\Big|_{y=2x^2-6x}^{y=3x-x^2}\ dx$$

$$M_x=\int_0^3\frac{1}{2}\left(3x-x^2\right)^2-\frac{1}{2}\left(2x^2-6x\right)^2\ dx$$

$$M_x=\int_0^3\frac{1}{2}\left(9x^2-6x^3+x^4\right)-\frac{1}{2}\left(4x^4-24x^3+36x^2\right)\ dx$$

$$M_x=\int_0^3\frac{9}{2}x^2-3x^3+\frac{1}{2}x^4-2x^4+12x^3-18x^2\ dx$$

$$M_x = \int_0^3 -\frac{3}{2}x^4 + 9x^3 - \frac{27}{2}x^2\ dx$$

$$M_x = -\frac{3}{10}x^5 + \frac{9}{4}x^4 - \frac{9}{2}x^3 \Big|_0^3$$

$$M_x = -\frac{3}{10}(3)^5 + \frac{9}{4}(3)^4 - \frac{9}{2}(3)^3 - \left(-\frac{3}{10}(0)^5 + \frac{9}{4}(0)^4 - \frac{9}{2}(0)^3\right)$$

$$M_x = -\frac{729}{10} + \frac{729}{4} - \frac{243}{2}$$

$$M_x = -\frac{1,458}{20} + \frac{3,645}{20} - \frac{2,430}{20}$$

$$M_x = -\frac{243}{20}$$

And for M_y we get

$$M_y = \int_0^3 \int_{2x^2-6x}^{3x-x^2} x\ dy\ dx$$

$$M_y = \int_0^3 x\left(3x - x^2\right) - x\left(2x^2 - 6x\right)\ dx$$

$$M_y = \int_0^3 3x^2 - x^3 - 2x^3 + 6x^2\ dx$$

$$M_y = \int_0^3 9x^2 - 3x^3\ dx$$

$$M_y = 3x^3 - \frac{3}{4}x^4 \Big|_0^3$$

$$M_y = 3(3)^3 - \frac{3}{4}(3)^4 - \left(3(0)^3 - \frac{3}{4}(0)^4\right)$$

$$M_y = 81 - \frac{243}{4}$$

$$M_y = \frac{81}{4}$$

Now that we have area, plus M_x and M_y, we can find $\bar{x}$ and $\bar{y}$.

$$\bar{x} = \frac{M_y}{A} = \frac{\frac{81}{4}}{\frac{27}{2}} = \frac{3}{2}$$

$$\bar{y} = \frac{M_x}{A} = \frac{-\frac{243}{20}}{\frac{27}{2}} = -\frac{9}{10}$$

Therefore, the center of mass is at

$$\left(\frac{3}{2}, -\frac{9}{10}\right)$$

Topic: Double integrals to find mass and center of mass

Question: The mass of a circular plate is given, where a is the length of a radius. What is the relationship between the radius of the given circular object and its density?

$$M = \frac{4\pi a^3}{3}$$

Answer choices:

A The density of the circular object is equal to its radius.

B The radius of the circular object is three times its density.

C The density of the circular object is twice its radius.

D The radius of the circular object is twice its density.

Solution: C

Start with a circle with radius a. Then for the the mass of the plate we get

$$M = \iint_R r \ dA$$

$$M = \int_0^{2\pi} \int_0^a (2r)(r) \ dr \ d\theta$$

$$M = \int_0^{2\pi} \frac{2}{3} r^3 \Big|_0^a \ d\theta$$

$$M = \int_0^{2\pi} \frac{2}{3} a^3 \ d\theta$$

$$M = \frac{2}{3} a^3 \theta \Big|_0^{2\pi}$$

$$M = \frac{4\pi a^3}{3}$$

Because we got the equation of the mass we were given, we know that answer choice C is correct.