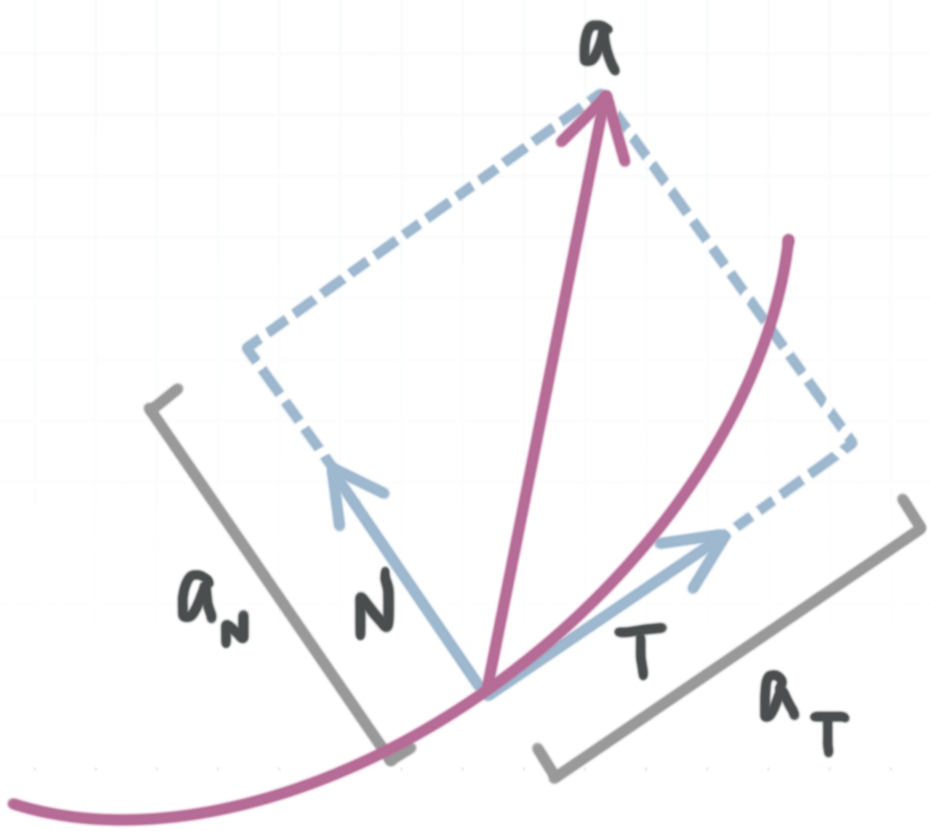
Tangential and normal components of the acceleration vector

At any given point along a curve, we can find the acceleration vector a that represents acceleration at that point. If we find the unit tangent vector T and the unit normal vector N at the same point, then the tangential component of acceleration a_T and the normal component of acceleration a_N are shown in the diagram below.



Tangential component of acceleration $$a_T = \frac{r'(t) \cdot r''(t)}{\left| r'(t) \right|}$$

Normal component of acceleration $$a_N = \frac{\left| r'(t) \times r''(t) \right|}{\left| r'(t) \right|}$$

In these formulas for the tangential and normal components,

$r(t)$ is the position vector, $r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$

$r'(t)$ is its first derivative, $r'(t) = r'(t)_1\mathbf{i} + r'(t)_2\mathbf{j} + r'(t)_3\mathbf{k}$

$r''(t)$ is its second derivative, $r''(t) = r''(t)_1\mathbf{i} + r''(t)_2\mathbf{j} + r''(t)_3\mathbf{k}$

$r'(t) \cdot r''(t)$ is the dot product of the first and second derivatives, $r'(t) \cdot r''(t) = r'(t)_1 r''(t)_1 + r'(t)_2 r''(t)_2 + r'(t)_3 r''(t)_3$

$\left|r'(t)\right|$ is the magnitude of the first derivative,

$\left|r'(t)\right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$

$\left|r'(t) \times r''(t)\right|$ is the magnitude of the cross product of the first and second derivatives, where the cross product is

$$r'(t) \times r''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r'(t)_1 & r'(t)_2 & r'(t)_3 \\ r''(t)_1 & r''(t)_2 & r''(t)_3 \end{vmatrix}$$

We'll start by finding each of the pieces in the list above, and then we'll plug them into the formulas for the tangential and normal components of the acceleration vector.

Example

Find the tangential and normal components of the acceleration vector.

$$r(t) = 2t^2\mathbf{i} + 4t\mathbf{j} + 3t^3\mathbf{k}$$

We'll start by finding $r'(t)$, the derivative of the position function. To find the derivative, we'll just replace the coefficients on $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ with their derivatives.

$$r'(t) = 4t\mathbf{i} + 4\mathbf{j} + 9t^2\mathbf{k}$$

can also be written as $r'(t) = \left\langle 4t, 4, 9t^2 \right\rangle$

We'll repeat the process to find the second derivative.

$$r''(t) = 4\mathbf{i} + 0\mathbf{j} + 18t\mathbf{k}$$

$$r''(t) = 4\mathbf{i} + 18t\mathbf{k}$$

can also be written as $r''(t) = \langle 4, 0, 18t \rangle$

Now we'll find the dot product of the first and second derivatives.

$$r'(t) \cdot r''(t) = (4t)(4) + (4)(0) + (9t^2)(18t)$$

$$r'(t) \cdot r''(t) = 16t + 0 + 162t^3$$

$$r'(t) \cdot r''(t) = 16t + 162t^3$$

$$r'(t) \cdot r''(t) = 162t^3 + 16t$$

Now we'll find the magnitude of the first derivative.

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

$$\left| r'(t) \right| = \sqrt{(4t)^2 + (4)^2 + \left(9t^2\right)^2}$$

$$\left| r'(t) \right| = \sqrt{16t^2 + 16 + 81t^4}$$

$$\left|r'(t)\right| = \sqrt{81t^4 + 16t^2 + 16}$$

Finally, we'll get the cross product of the first and second derivatives, then find its magnitude.

$$r'(t) \times r''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r'(t)_1 & r'(t)_2 & r'(t)_3 \\ r''(t)_1 & r''(t)_2 & r''(t)_3 \end{vmatrix}$$

$$r'(t) \times r''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4t & 4 & 9t^2 \\ 4 & 0 & 18t \end{vmatrix}$$

$$r'(t) \times r''(t) = \begin{vmatrix} 4 & 9t^2 \\ 0 & 18t \end{vmatrix} \mathbf{i} - \begin{vmatrix} 4t & 9t^2 \\ 4 & 18t \end{vmatrix} \mathbf{j} + \begin{vmatrix} 4t & 4 \\ 4 & 0 \end{vmatrix} \mathbf{k}$$

$$r'(t) \times r''(t) = \left[(4)(18t) - (0)\left(9t^2\right)\right] \mathbf{i} - \left[(4t)(18t) - (4)\left(9t^2\right)\right] \mathbf{j} + \left[(4t)(0) - (4)(4)\right] \mathbf{k}$$

$$r'(t) \times r''(t) = (72t - 0)\,\mathbf{i} - \left(72t^2 - 36t^2\right)\mathbf{j} + \left(0 - 16\right)\mathbf{k}$$

$$r'(t) \times r''(t) = 72t\mathbf{i} - 36t^2\mathbf{j} - 16\mathbf{k}$$

$$r'(t) \times r''(t) = 4\left(18t\mathbf{i} - 9t^2\mathbf{j} - 4\mathbf{k}\right)$$

can also be written as $r'(t) \times r''(t) = 4\left\langle 18t, -9t^2, -4\right\rangle$

Now we just need the magnitude of the cross product.

$$\left|r'(t) \times r''(t)\right| = 4\sqrt{(18t)^2 + \left(-9t^2\right)^2 + (-4)^2}$$

$$\left|r'(t) \times r''(t)\right| = 4\sqrt{324t^2 + 81t^4 + 16}$$

$$\left|r'(t)\times r''(t)\right| = 4\sqrt{81t^4+324t^2+16}$$

We've finally found everything we need to solve for the tangential and normal components of acceleration. Plugging in what we know, we get

The tangential component of acceleration

$$a_T = \frac{r'(t)\cdot r''(t)}{\left|r'(t)\right|}$$

$$a_T = \frac{162t^3+16t}{\sqrt{81t^4+16t^2+16}}$$

The normal component of acceleration

$$a_N = \frac{\left|r'(t)\times r''(t)\right|}{\left|r'(t)\right|}$$

$$a_N = \frac{4\sqrt{81t^4+324t^2+16}}{\sqrt{81t^4+16t^2+16}}$$

$$a_N = 4\sqrt{\frac{81t^4+324t^2+16}{81t^4+16t^2+16}}$$

These are the tangential and normal components of the acceleration vector.