Vector, parametric and symmetric equations of the line

Vector, parametric, and symmetric equations are different types of equations that can be used to represent the same line. We use different equations at different times to tell us information about the line, so we need to know how to find all three types of equations.

The **vector** equation of a line is given by

$$r = r_0 + tv$$

where r_0 is a point on the line and v is a parallel vector

The **parametric** equations of a line are given by

$$x = a$$

$$y = b$$

$$z = c$$

where a, b and c are the coefficients from the vector equation $r = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$

The **symmetric** equations of a line are given by

$$\frac{x - a_1}{v_1} = \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$

where $a(a_1, a_2, a_3)$ are the coordinates from a point on the line and v_1, v_2 and v_3 are the coordinates from a parallel vector.

Example

Find the vector, parametric and symmetric equations of the line that passes through the point $a(2, -1, 3)$ and is parallel to the vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

Before we get started, we can say that the given point $a(2, -1, 3)$ can also be represented by $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$. Additionally, we know that the given vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$ can be represented by $\langle 2, -1, 4 \rangle$, or $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$. To summarize what we know, we have

$$(2, -1, 3) \qquad\qquad 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$$

$$\langle 2, -1, 4 \rangle \qquad\qquad 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$$

To find the vector equation of the line, we'll use $r = r_0 + tv$, where r_0 is the point on the line $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and v is the parallel vector $2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$.

$$r = (2\mathbf{i} - \mathbf{j} + 3\mathbf{k}) + t(2\mathbf{i} - \mathbf{j} + 4\mathbf{k})$$

$$r = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k} + 2\mathbf{i}t - \mathbf{j}t + 4\mathbf{k}t$$

$$r = (2\mathbf{i} + 2\mathbf{i}t) + (-\mathbf{j} - \mathbf{j}t) + (3\mathbf{k} + 4\mathbf{k}t)$$

$$r = (2 + 2t)\mathbf{i} + (-1 - t)\mathbf{j} + (3 + 4t)\mathbf{k}$$

With the vector equation of this line in hand, it'll be very easy for us to find the parametric equations of the line, because all we have to do is take the coefficients from the vector equation, and the parametric equations are

$$x = 2 + 2t$$

$$y = -1 - t$$

$$z = 3 + 4t$$

To find the symmetric equations, we'll just plug the given coordinate point in for a_1, a_2 and a_3, plus the coefficients from the perpendicular vector in for v_1, v_2 and v_3.

$$\frac{x-2}{2} = \frac{y-(-1)}{-1} = \frac{z-3}{4}$$

$$\frac{x-2}{2} = -y-1 = \frac{z-3}{4}$$

In conclusion, we've found the following three equations for the same line:

Vector $\qquad\qquad\qquad\qquad r = (2+2t)\mathbf{i} + (-1-t)\mathbf{j} + (3+4t)\mathbf{k}$

Parametric $\qquad\qquad\qquad x = 2+2t,\ y = -1-t,\ z = 3+4t$

Symmetric $\qquad\qquad\qquad \dfrac{x-2}{2} = -y-1 = \dfrac{z-3}{4}$

There's one other thing you need to be aware of when you're finding symmetric equations. Sometimes v_1, v_2 or v_3 will be equal to 0. In this case, you pull that particular fraction out of the symmetric equation, put it by itself, and don't divide by 0. So, if the formula for symmetric equations is

$$\frac{x-a_1}{v_1} = \frac{y-a_2}{v_2} = \frac{z-a_3}{v_3}$$

and $v_1 = 0$, then the symmetric equations become

$$x - a_1, \ \frac{y - a_2}{v_2} = \frac{z - a_3}{v_3}$$