Calculus 3 Workbook Solutions

Applications of triple integrals

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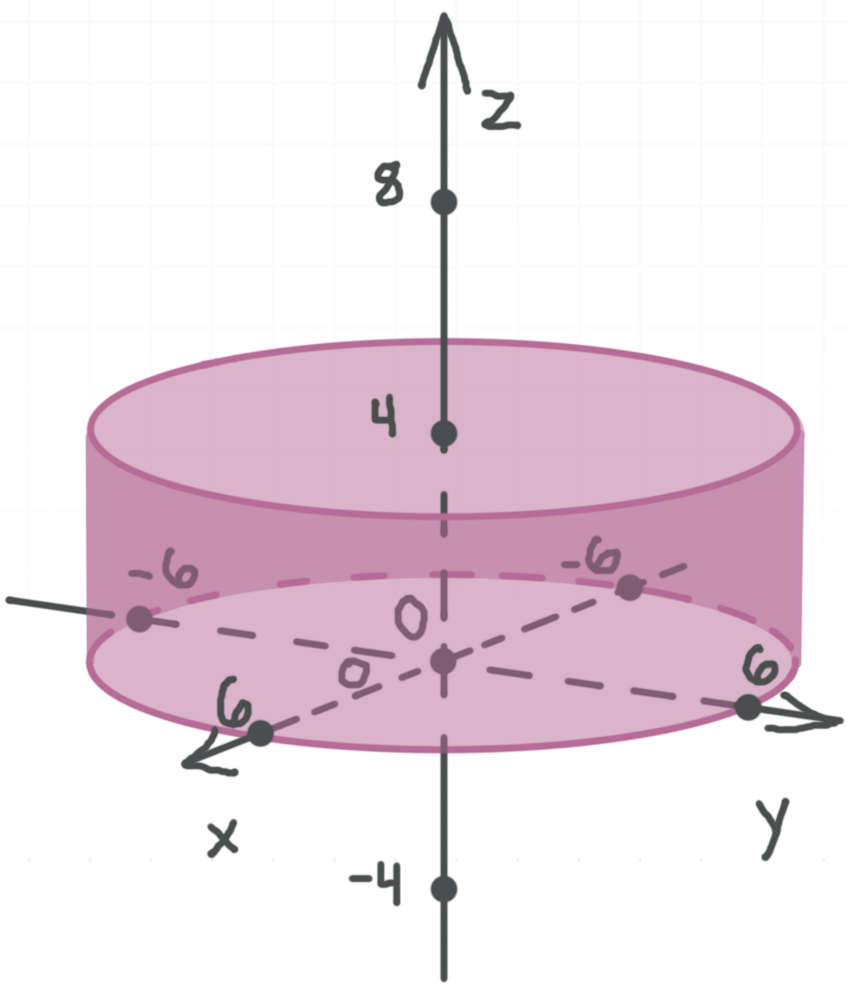
MATH

TRIPLE INTEGRALS TO FIND MASS AND CENTER OF MASS

■ 1. The disk with radius 6 and height 4 has density $\delta = 1/(d+2)$, where d is the distance to the central axis of the disk. Find the mass and center of mass of the disk.



Solution:

Consider the cylinder with radius 6 and height 4, whose base lies in the xy-plane with its center at the origin. Within the cylinder, r changes from 0 to 6, θ changes from 0 to 2π, and z changes from 0 to 4. The density function is

$$\delta = \frac{1}{r+2}$$

Therefore, the integral representing mass in cylindrical coordinates is

$$\int_0^4 \int_0^{2\pi} \int_0^6 \frac{1}{r+2} \ r \ dr \ d\theta \ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^6 \frac{r+2-2}{r+2} \ dr \ d\theta \ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^6 \frac{r+2}{r+2} - \frac{2}{r+2} \ dr \ d\theta \ dz$$

$$\int_0^4 \int_0^{2\pi} \int_0^6 1 - \frac{2}{r+2} \ dr \ d\theta \ dz$$

Integrate with respect to r.

$$\int_0^4 \int_0^{2\pi} r - 2\ln|r+2| \Big|_0^6 \ d\theta \ dz$$

$$\int_0^4 \int_0^{2\pi} 6 - 2\ln|6+2| - (0 - 2\ln|0+2|) \ d\theta \ dz$$

$$\int_0^4 \int_0^{2\pi} 6 - 2\ln 8 + 2\ln 2 \ d\theta \ dz$$

Integrate with respect to θ.

$$\int_0^4 6\theta - 2\theta\ln 8 + 2\theta\ln 2 \Big|_0^{2\pi} \ dz$$

$$\int_0^4 6(2\pi) - 2(2\pi)\ln 8 + 2(2\pi)\ln 2 \ dz$$

$$\int_0^4 12\pi - 4\pi \ln 8 + 4\pi \ln 2 \ dz$$

Integrate with respect to z.

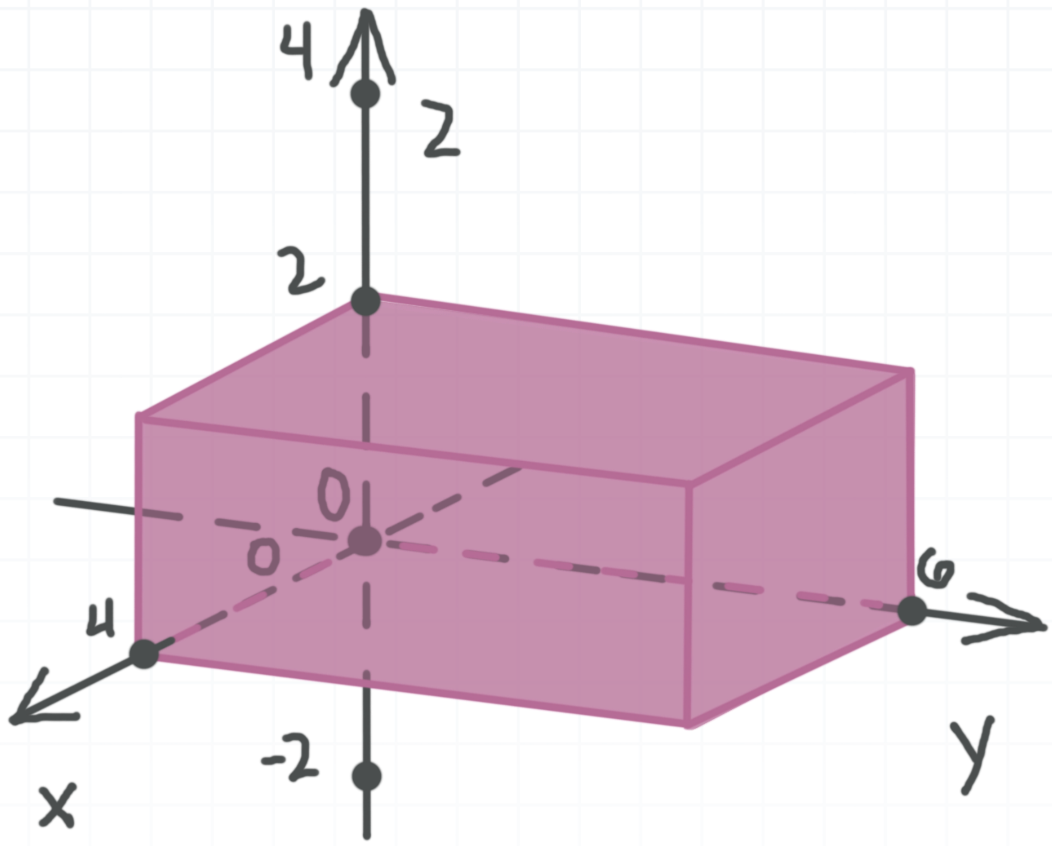
$$12\pi z - 4\pi z \ln 8 + 4\pi z \ln 2 \Big|_0^4$$

$$12\pi(4) - 4\pi(4)\ln 8 + 4\pi(4)\ln 2$$

$$48\pi - 16\pi \ln 8 + 16\pi \ln 2$$

Since the disk is symmetric and has symmetric density about the x- and y-axes, its center of mass lies on its axis. Also, since the mass is equally distributed along the z-axis, its center of mass lies on the half of its axis. Therefore, the center of mass has coordinates $(0,0,2)$.

■ 2. The rectangular plate with base dimensions 4×5 m and height 2 m has density $\delta = 4d$ kg/m^2, where d is the distance from its 4×2 m left face. Find the mass and center of mass of the plate.



Solution:

Place the origin at the bottom left corner of the plate, and the x- and y-axes along the dimensions 4 and 5 respectively. So the value of x changes from 0 to 4, y changes from 0 to 5, and z changes from 0 to 2. The density is equal to $4y$, so the triple integral will be

$$\int_0^2 \int_0^5 \int_0^4 4y \ dx \ dy \ dz$$

$$\int_0^2 \int_0^5 4xy \Big|_0^4 \ dy \ dz$$

$$\int_0^2 \int_0^5 16y \ dy \ dz$$

$$\int_0^2 8y^2\Big|_0^5 \ dz$$

$$\int_0^2 200 \ dz$$

$$200z\Big|_0^2$$

$$400$$

So the mass M of the plate is 400 kg. Find the center of mass. We get

$$\bar{x} = \frac{1}{M}\iiint_V x\delta(x, y, z) \ dV$$

$$\bar{x} = \frac{1}{400}\int_0^2\int_0^5\int_0^4 4xy \ dx \ dy \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2\int_0^5 2x^2y\Big|_{x=0}^{x=4} \ dy \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2\int_0^5 32y \ dy \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2 16y^2\Big|_0^5 \ dz$$

$$\bar{x} = \frac{1}{400}\int_0^2 400 \ dz$$

$$\bar{x} = \frac{400z}{400}\Big|_0^2$$

$$\bar{x} = z\Big|_0^2$$

$$\bar{x} = 2$$

and

$$\bar{y} = \frac{1}{M}\iiint_V y\delta(x, y, z)\ dV$$

$$\bar{y} = \frac{1}{400}\int_0^2\int_0^5\int_0^4 4y^2\ dx\ dy\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2\int_0^5 4xy^2\Big|_{x=0}^{x=4}\ dy\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2\int_0^5 16y^2\ dy\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2 \frac{16}{3}y^3\Big|_0^5\ dz$$

$$\bar{y} = \frac{1}{400}\int_0^2 \frac{2{,}000}{3}\ dz$$

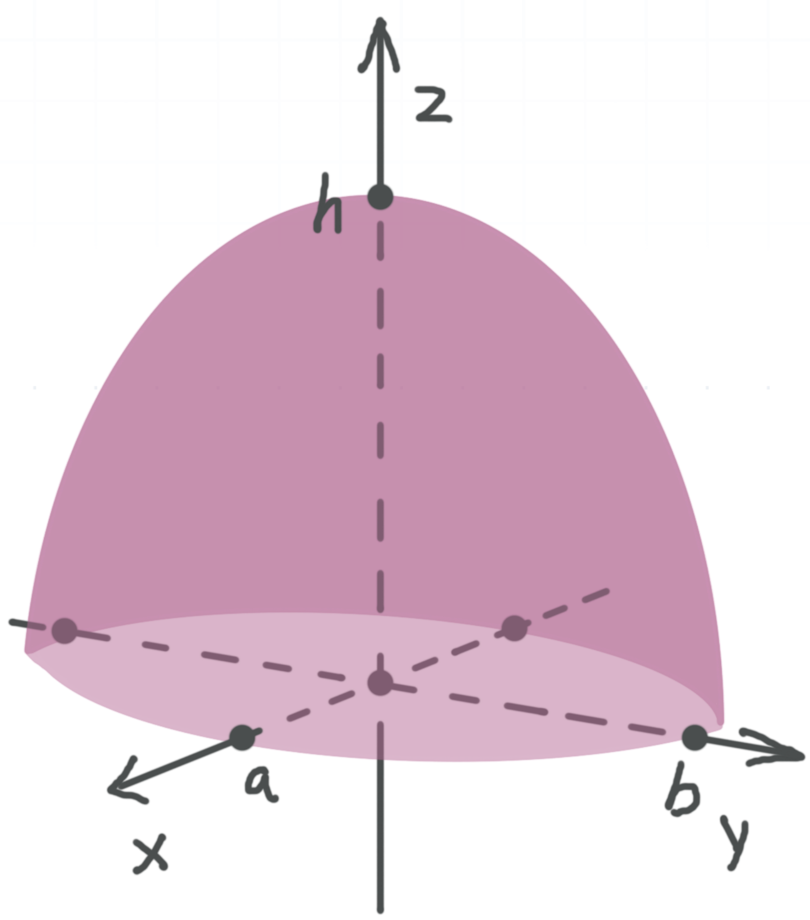
$$\bar{y} = \int_0^2 \frac{5}{3}\ dz$$

$$\bar{y} = \frac{5}{3}z\Big|_0^2$$

$$\bar{y} = \frac{10}{3}$$

Since the mass is equally distributed along the z-axis, the center of mass lies at half-plate height, so $\bar{z} = 2/2 = 1$, and the center of mass is at $(\bar{x}, \bar{y}, \bar{z}) = (2,10/3,1)$.

■ 3. The half ellipsoid has a base with semi-axes a and b, height h, and constant density δ. Find its mass and center of mass.



Solution:

Position the base in the xy-plane with the semi-axes along the x and y axes. Then the equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{h^2}=1$$

Within the half ellipsoid, the value of ρ changes from 0 to 1, θ changes from 0 to 2π, and ϕ changes from 0 to $\pi/2$. The partial derivatives are

$$\frac{\partial x}{\partial \rho}=\frac{\partial}{\partial \rho}(a\rho\sin\phi\cos\theta)=a\sin\phi\cos\theta$$

$$\frac{\partial x}{\partial \theta}=\frac{\partial}{\partial \theta}(a\rho\sin\phi\cos\theta)=-a\rho\sin\phi\sin\theta$$

$$\frac{\partial x}{\partial \phi}=\frac{\partial}{\partial \phi}(a\rho\sin\phi\cos\theta)=a\rho\cos\phi\cos\theta$$

$$\frac{\partial y}{\partial \rho}=\frac{\partial}{\partial \rho}(b\rho\sin\phi\sin\theta)=b\sin\phi\sin\theta$$

$$\frac{\partial y}{\partial \theta}=\frac{\partial}{\partial \theta}(b\rho\sin\phi\sin\theta)=b\rho\sin\phi\cos\theta$$

$$\frac{\partial y}{\partial \phi}=\frac{\partial}{\partial \phi}(b\rho\sin\phi\sin\theta)=b\rho\cos\phi\sin\theta$$

$$\frac{\partial z}{\partial \rho}=\frac{\partial}{\partial \rho}(h\rho\cos\phi)=h\cos\phi$$

$$\frac{\partial z}{\partial \theta}=\frac{\partial}{\partial \theta}(h\rho\cos\phi)=0$$

$$\frac{\partial z}{\partial \phi}=\frac{\partial}{\partial \phi}(h\rho\cos\phi)=-h\rho\sin\phi$$

The Jacobian of the transformation is

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=\begin{vmatrix}\frac{\partial x}{\partial\rho} & \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi}\\ \frac{\partial y}{\partial\rho} & \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi}\\ \frac{\partial z}{\partial\rho} & \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi}\end{vmatrix}$$

$$=\frac{\partial x}{\partial\rho}\left(\frac{\partial y}{\partial\theta}\cdot\frac{\partial z}{\partial\phi}-\frac{\partial y}{\partial\phi}\cdot\frac{\partial z}{\partial\theta}\right)-\frac{\partial x}{\partial\theta}\left(\frac{\partial y}{\partial\rho}\cdot\frac{\partial z}{\partial\phi}-\frac{\partial y}{\partial\phi}\cdot\frac{\partial z}{\partial\rho}\right)$$

$$+\frac{\partial x}{\partial\phi}\left(\frac{\partial y}{\partial\rho}\cdot\frac{\partial z}{\partial\theta}-\frac{\partial y}{\partial\theta}\cdot\frac{\partial z}{\partial\rho}\right)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=a\sin\phi\cos\theta(-b\rho\sin\phi\cos\theta\cdot h\rho\sin\phi-b\rho\cos\phi\sin\theta\cdot 0)$$

$$+a\rho\sin\phi\sin\theta(-b\sin\phi\sin\theta\cdot h\rho\sin\phi-b\rho\cos\phi\sin\theta\cdot h\cos\phi)$$

$$+a\rho\cos\phi\cos\theta(b\sin\phi\sin\theta\cdot 0-b\rho\sin\phi\cos\theta\cdot h\cos\phi)$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-abh\rho^2\sin^3\phi\cos^2\theta-abh\rho^2\sin\phi\sin^2\theta-abh\rho^2\sin\phi\cos^2\phi\cos^2\theta$$

$$\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)}=-abh\rho^2\sin\phi$$

Then the mass in spherical coordinates is given by

$$\delta\int_0^{\frac{\pi}{2}}\int_0^{2\pi}\int_0^1 abh\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$abh\delta \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$abh\delta \int_0^{\frac{\pi}{2}} \sin\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^2 \ d\rho$$

Evaluate each integral.

$$abh\delta \left(-\cos\phi \Big|_0^{\frac{\pi}{2}} \right) \left(\theta \Big|_0^{2\pi} \right) \left(\frac{1}{3}\rho^3 \Big|_0^1 \right)$$

$$abh\delta(1)(2\pi)\left(\frac{1}{3}\right)$$

$$\frac{2\pi abh\delta}{3}$$

Since the half ellipsoid is symmetric about the x- and y-axis, its center of mass lies on the z-axis, and

$$\bar{z} = \frac{1}{M} \iiint_V z\delta(x, y, z) \ dV$$

$$\bar{z} = \frac{3}{2\pi abh\delta} \cdot \delta \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 h\rho\cos\phi \ abh\rho^2 \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\bar{z} = \frac{3h}{2\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^3 \cos\phi \sin\phi \ d\rho \ d\theta \ d\phi$$

$$\bar{z} = \frac{3h}{4\pi} \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_0^1 \rho^3 \sin 2\phi \ d\rho \ d\theta \ d\phi$$

$$\bar{z} = \frac{3h}{4\pi} \int_0^{\frac{\pi}{2}} \sin 2\phi \ d\phi \cdot \int_0^{2\pi} d\theta \cdot \int_0^1 \rho^3 \ d\rho$$

Evaluate each integral.

$$\bar{z} = \frac{3h}{4\pi} \left(-\frac{1}{2} \cos 2\phi \Big|_0^{\frac{\pi}{2}} \right) \left(\theta \Big|_0^{2\pi} \right) \left(\frac{1}{4} \rho^4 \Big|_0^1 \right)$$

$$\bar{z} = \frac{3h}{4\pi} (1)(2\pi) \left(\frac{1}{4} \right)$$

$$\bar{z} = \frac{3h}{8}$$

So the center of mass is at $(0,0,3h/8)$.

MOMENTS OF INERTIA

1. The spherical solid object with radius 5 has density $\delta = d^2$, where d is the distance to the center of the sphere. Find the moment of inertia of the object about any line that passes through its center.

Solution:

Since the sphere is symmetric about any line that passes through its center, and has symmetrical density, it has the same moment of inertia about any such line. Find, for example, the moment of inertia about the z-axis.

The moment of inertia about the z-axis is given by the triple integral.

$$\iiint_E (x^2 + y^2)\delta(x, y, z)\ dV$$

The function is

$$(x^2 + y^2)\delta(x, y, z)$$

$$\left[(\rho \sin\phi \cos\theta)^2 + (\rho \sin\phi \sin\theta)^2\right] \cdot \rho^2$$

$$\rho^4 \sin^2\phi$$

Therefore, the integral in spherical coordinates representing the moment of inertia about the z-axis is

$$\int_0^{\pi}\int_0^{2\pi}\int_0^5 \rho^4\sin^2\phi\cdot\rho^2\sin\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^5 \rho^6\sin^3\phi\ d\rho\ d\theta\ d\phi$$

$$\int_0^{\pi}\sin^3\phi\ d\phi\cdot\int_0^{2\pi}d\theta\cdot\int_0^5\rho^6\ d\rho$$

Use the reduction formula

$$\int\sin^m x\ dx=-\frac{\cos x\sin^{m-1}x}{m}+\frac{m-1}{m}\int\sin^{-2+m}x\ dx, \text{ where } m=3$$

to simplify the first integral.

$$\left(-\frac{1}{3}\sin^2\phi\cos\phi\Big|_0^{\pi}+\frac{2}{3}\int_0^{\pi}\sin\phi\ d\phi\right)\cdot\int_0^{2\pi}d\theta\cdot\int_0^5\rho^6\ d\rho$$

Evaluate each integral.

$$\left(-\frac{1}{3}\sin^2\phi\cos\phi-\frac{2}{3}\cos\phi\Big|_0^{\pi}\right)(2\pi-0)\left(\frac{1}{7}(5)^7-\frac{1}{7}(0)^7\right)$$

$$\left(-\frac{1}{3}\sin^2\pi\cos\pi-\frac{2}{3}\cos\pi+\frac{1}{3}\sin^2(0)\cos(0)+\frac{2}{3}\cos(0)\right)(2\pi)\left(\frac{1}{7}(5)^7\right)$$

$$\left(-\frac{1}{3}(0)(-1)-\frac{2}{3}(-1)+\frac{1}{3}(0)(1)+\frac{2}{3}(1)\right)(2\pi)\left(\frac{1}{7}(5)^7\right)$$

$$\left(\frac{2}{3}+\frac{2}{3}\right)(2\pi)\left(\frac{78{,}125}{7}\right)$$

$$\frac{8\pi}{3}\left(\frac{78{,}125}{7}\right)$$

$$\frac{625{,}000\pi}{21}$$

■ 2. A box (rectangular cuboid) has length 6, width 4, height 2, and constant density δ. Find the moment of inertia of the box about all of its edges.

Solution:

Although the box has 12 edges, it has only three different moments of inertia, because the moments about the edges of length 6 are equal, the moments about the edges of length 4 are equal, and the moments about the edges of length 2 are equal.

Place one corner of the box at the origin, with one edge each along the three major axes. Then the moment of inertia about the x-axis is given by

$$\iiint_E (y^2+z^2)\delta(x,y,z)\ dV$$

$$\delta\iiint_E y^2+z^2\ dV$$

$$\delta\int_0^2\int_0^4\int_0^6 y^2+z^2\ dx\ dy\ dz$$

Integrate with respect to x.

$$\delta\int_0^2\int_0^4 xy^2+xz^2\Big|_{x=0}^{x=6}\ dy\ dz$$

$$\delta\int_0^2\int_0^4 6y^2+6z^2\ dy\ dz$$

Integrate with respect to y.

$$\delta\int_0^2 2y^3+6yz^2\Big|_{y=0}^{y=4}\ dz$$

$$\delta\int_0^2 2(4)^3+6(4)z^2-(2(0)^3+6(0)z^2)\ dz$$

$$\delta\int_0^2 128+24z^2\ dz$$

Integrate with respect to z.

$$\delta(128z+8z^3)\Big|_0^2$$

$$\delta(128(2)+8(2)^3)-\delta(128(0)+8(0)^3)$$

$$\delta(256+64)$$

$$320\delta$$

In the same way, find the moment of inertia about the y-axis.

$$\delta \int_0^2 \int_0^4 \int_0^6 x^2 + z^2 \ dx \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 \frac{1}{3}x^3 + xz^2 \Big|_{x=0}^{x=6} \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 72 + 6z^2 \ dy \ dz$$

$$\delta \int_0^2 72y + 6yz^2 \Big|_{y=0}^{y=4} \ dz$$

$$\delta \int_0^2 288 + 24z^2 \ dz$$

$$\delta(288z + 8z^3) \Big|_0^2$$

$$\delta(288(2) + 8(2)^3)$$

$$\delta(576 + 64)$$

$$640\delta$$

In the same way, find the moment of inertia about the z-axis.

$$\delta \int_0^2 \int_0^4 \int_0^6 x^2 + y^2 \ dx \ dy \ dz$$

$$\delta \int_0^2 \int_0^4 \frac{1}{3}x^3 + xy^2 \Big|_{x=0}^{x=6} \ dy \ dz$$

$$\delta\int_0^2\int_0^4\frac{1}{3}(6)^3+6y^2\ dy\ dz$$

$$\delta\int_0^2\int_0^4 72+6y^2\ dy\ dz$$

$$\delta\int_0^2 72y+2y^3\Big|_0^4\ dz$$

$$\delta\int_0^2 288+128\ dz$$

$$\delta\int_0^2 416\ dz$$

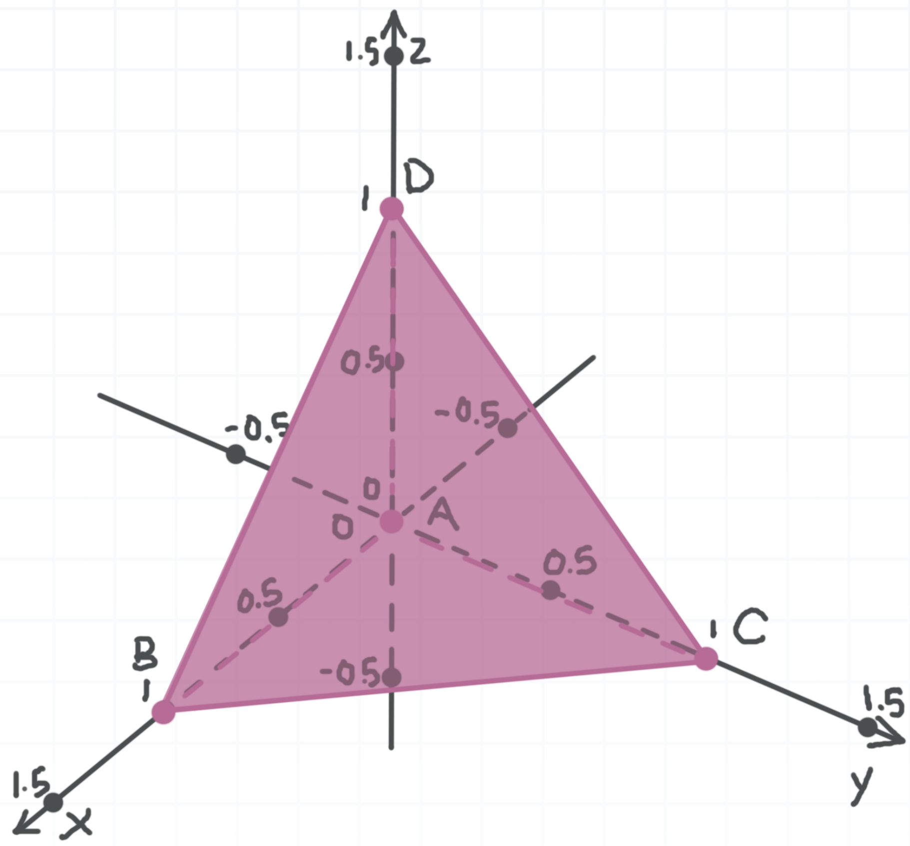
$$\delta(416z)\Big|_0^2$$

$$\delta(416(2))$$

$$832\delta$$

Therefore, the moments of inertia are 320δ, 640δ, and 832δ.

■ 3. The tetrahedron $ABCD$ has constant density δ. Find the moment of inertia of the solid about the line AB, where $A(0,0,0)$, $B(1,0,0)$, $C(0,1,0)$, and $D(0,0,1)$.



Solution:

The moment of inertia about the line AB (x-axis) is given by the triple integral.

$$\iiint_E (y^2 + z^2)\delta(x, y, z)\ dV$$

$$\delta \iiint_E y^2 + z^2\ dV$$

The equation of the line BC in the xy-plane is $y = 1 - x$. So when x changes from 0 to 1, y changes from 0 to $1 - x$. The equation of the plane BCD is $z = 1 - x - y$. So when x and y change within the triangle ABC, z changes from 0 to $1 - x - y$. Therefore, the integral representing the moment of inertia about the x-axis is given by

$$\delta \int_0^1 \int_0^{1-x} \int_0^{1-x-y} y^2 + z^2 \ dz \ dy \ dx$$

Integrate with respect to z.

$$\delta \int_0^1 \int_0^{1-x} y^2 z + \frac{1}{3}z^3 \Big|_{z=0}^{z=1-x-y} \ dy \ dx$$

$$\delta \int_0^1 \int_0^{1-x} y^2(1-x-y) + \frac{1}{3}(1-x-y)^3 \ dy \ dx$$

$$\delta \int_0^1 \int_0^{1-x} y^2 - xy^2 - y^3 + \frac{1}{3} - \frac{1}{3}x^3 + x^2 - x - x^2y + 2xy - xy^2 - y + y^2 - \frac{1}{3}y^3 \ dy \ dx$$

$$\delta \int_0^1 \int_0^{1-x} \frac{1}{3} - \frac{1}{3}x^3 + x^2 - x - x^2y + 2xy - 2xy^2 - y + 2y^2 - \frac{4}{3}y^3 \ dy \ dx$$

Integrate with respect to y.

$$\delta \int_0^1 \frac{1}{3}y - \frac{1}{3}x^3y + x^2y - xy - \frac{1}{2}x^2y^2 + xy^2 - \frac{2}{3}xy^3 - \frac{1}{2}y^2 + \frac{2}{3}y^3 - \frac{1}{3}y^4 \Big|_{y=0}^{y=1-x} \ dx$$

$$\delta \int_0^1 \frac{1}{3}(1-x) - \frac{1}{3}x^3(1-x) + x^2(1-x) - x(1-x) - \frac{1}{2}x^2(1-x)^2$$

$$+x(1-x)^2 - \frac{2}{3}x(1-x)^3 - \frac{1}{2}(1-x)^2 + \frac{2}{3}(1-x)^3 - \frac{1}{3}(1-x)^4 \ dx$$

$$\delta \int_0^1 \frac{1}{6} - \frac{2}{3}x + x^2 - \frac{2}{3}x^3 + \frac{1}{6}x^4 \ dx$$

Integrate with respect to x.

$$\delta\left(\frac{1}{6}x-\frac{1}{3}x^2+\frac{1}{3}x^3-\frac{1}{6}x^4+\frac{1}{30}x^5\right)\Big|_0^1$$

$$\delta\left(\frac{1}{6}(1)-\frac{1}{3}(1)^2+\frac{1}{3}(1)^3-\frac{1}{6}(1)^4+\frac{1}{30}(1)^5\right)$$

$$\delta\left(\frac{1}{6}-\frac{1}{3}+\frac{1}{3}-\frac{1}{6}+\frac{1}{30}\right)$$

$$\frac{1}{30}\delta$$

Therefore, the moment of inertia about the x-axis is $\delta/30$.

W W W . K R I S T A K I N G M A T H . C O M