Expressing the integral six ways

There are six ways to express an iterated triple integral. While the function $f(x,y,z)$ inside the integral always stays the same, the order of integration will change, and the limits of integration will change to match the order.

With order of integration x, y, z $\quad \iiint_E f(x,y,z)\ dx\ dy\ dz$

With order of integration x, z, y $\quad \iiint_E f(x,y,z)\ dx\ dz\ dy$

With order of integration y, x, z $\quad \iiint_E f(x,y,z)\ dy\ dx\ dz$

With order of integration y, z, x $\quad \iiint_E f(x,y,z)\ dy\ dz\ dx$

With order of integration z, x, y $\quad \iiint_E f(x,y,z)\ dz\ dx\ dy$

With order of integration z, y, x $\quad \iiint_E f(x,y,z)\ dz\ dy\ dx$

The only hard part of these problems is finding the limits of integration for each of the three individual integrals in each of the six triple iterated integrals.

Remember that, with all iterated integrals, you work your way from the inside toward the outside. So, if the integral ends in $dx\ dy\ dz$, it means you integrate from the inside out, from left to right, first with respect to x, then with respect to y, and then with respect to z.

$$\iiint_E f(x,y,z)\ dx\ dy\ dz$$

Since you're integrating with respect to x first, and you'll need to integrate with respect to y and z later, you need to have y and z variables left over after you integrate with respect to x and evaluate over the associated limits of integration. This means that the inner integral needs to have limits of integration in terms of y and z.

$$\int\int\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$$

Once you've integrated with respect to x, you'll have only y and z variables remaining. You'll integrate with respect to y, plugging in the limits of integration associated with y. Since you need to leave z variables in the function in order to later integrate with respect to z, that means your limits of integration for y need to be in terms of z.

$$\int\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$$

Finally, once you've eliminated y and have only z variables remaining, the limits of integration associated with z should be constants, so that your final answer is a constant.

$$\int_{z}^{z}\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$$

Notice in the chart below how all of the innermost integrals have limits of integration in terms of two variables, the second integral has limits of integration in terms of one variable, and the outermost integral has constant limits of integration.

With order of integration x, y, z $\quad \int_{z}^{z}\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz$

With order of integration x, z, y $\quad \int_{y}^{y}\int_{z(y)}^{z(y)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dz\ dy$

With order of integration y, x, z $\quad \int_{z}^{z}\int_{x(z)}^{x(z)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dx\ dz$

With order of integration y, z, x $\quad \int_{x}^{x}\int_{z(x)}^{z(x)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dz\ dx$

With order of integration z, x, y $\quad \int_{y}^{y}\int_{x(y)}^{x(y)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dx\ dy$

With order of integration z, y, x $\quad \int_{x}^{x}\int_{y(x)}^{y(x)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dy\ dx$

It’s best to find all of the limits of integration you’ll need before you start writing down all of the integrals. The easiest way to keep the limits of integration organized is with the chart below.

	x	y	z	Multivariable
x	**Constants** Set $y=0$, $z=0$ Solve for x	$x(y)$ Set $z=0$ Solve for x	$x(z)$ Set $y=0$ Solve for x	$x(y,z)$ Solve for x
y	$y(x)$ Set $z=0$ Solve for y	**Constants** Set $x=0$, $z=0$ Solve for y	$y(z)$ Set $x=0$ Solve for y	$y(x,z)$ Solve for y
z	$z(x)$ Set $y=0$ Solve for z	$z(y)$ Set $x=0$ Solve for z	**Constants** Set $x=0$, $y=0$ Solve for z	$z(x,y)$ Solve for z

Once you've got the chart completely filled in, you'll be able to go straight to the limits of integration you need for each integral.

Example

Express the triple iterated integral

$$\iiint_E f(x,y,z)\ dV$$

six ways for the volume bounded by the given curves.

$$y = x^2 + 15z^2 - 9$$

$$y = 0$$

We'll start by creating the chart for the limits of integration.

	x	y	z	Multivariable
x	$0=x^2+15(0)^2-9$ $x^2=9$ $x=\pm 3$	$y=x^2+15(0)^2-9$ $x^2=y+9$ $x=\pm\sqrt{y+9}$	$0=x^2+15z^2-9$ $x^2=9-15z^2$ $x=\pm\sqrt{9-15z^2}$	$y=x^2+15z^2-9$ $x^2=y-15z^2+9$ $x=\pm\sqrt{y-15z^2+9}$
y	$y=x^2+15(0)^2-9$ $y=x^2-9$	$y=(0)^2+15(0)^2-9$ $y=-9$	$y=(0)^2+15z^2-9$ $y=15z^2-9$	$y=x^2+15z^2-9$
z	$0=x^2+15z^2-9$ $15z^2=9-x^2$ $z=\pm\sqrt{(9-x^2)/15}$	$y=(0)^2+15z^2-9$ $15z^2=y+9$ $z=\pm\sqrt{(y+9)/15}$	$0=(0)^2+15z^2-9$ $15z^2=9$ $z=\pm\sqrt{3/5}$	$15z^2=y-x^2+9$ $z=\pm\sqrt{\frac{y-x^2+9}{15}}$

Before we go on, notice that we've found two limits of integration in every section of the chart above, except in the row for y. It'll often be the case with these types of problems, that you have one variable for which you're only able to find one limit of integration.

Usually you can find the other one by going back to the question we were asked at the beginning. In this case, we were told that $y=0$, so we can use $y=0$ as the second limit of integration. We'll revise the chart by adding this in.

	x	y	z	Multivariable
x	$0=x^2+15(0)^2-9$ $x^2=9$ $x=\pm 3$	$y=x^2+15(0)^2-9$ $x^2=y+9$ $x=\pm\sqrt{y+9}$	$0=x^2+15z^2-9$ $x^2=9-15z^2$ $x=\pm\sqrt{9-15z^2}$	$y=x^2+15z^2-9$ $x^2=y-15z^2+9$ $x=\pm\sqrt{y-15z^2+9}$
y	$y=x^2+15(0)^2-9$ $y=x^2-9$ $y=0$	$y=(0)^2+15(0)^2-9$ $y=-9$ $y=0$	$y=(0)^2+15z^2-9$ $y=15z^2-9$ $y=0$	$y=x^2+15z^2-9$ $y=0$
z	$0=x^2+15z^2-9$ $15z^2=9-x^2$ $z=\pm\sqrt{(9-x^2)/15}$	$y=(0)^2+15z^2-9$ $15z^2=y+9$ $z=\pm\sqrt{(y+9)/15}$	$0=(0)^2+15z^2-9$ $15z^2=9$ $z=\pm\sqrt{3/5}$	$15z^2=y-x^2+9$ $z=\pm\sqrt{\frac{y-x^2+9}{15}}$

Now we can pull the limits of integration we've found into each of the six triple iterated integrals.

$$\int_z^z\int_{y(z)}^{y(z)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dy\ dz \qquad \int_{-\sqrt{3/5}}^{\sqrt{3/5}}\int_{15z^2-9}^{0}\int_{-\sqrt{y-15z^2+9}}^{\sqrt{y-15z^2+9}} f(x,y,z)\ dx\ dy\ dz$$

$$\int_y^y\int_{z(y)}^{z(y)}\int_{x(y,z)}^{x(y,z)} f(x,y,z)\ dx\ dz\ dy \qquad \int_{-9}^{0}\int_{-\sqrt{(y+9)/15}}^{\sqrt{(y+9)/15}}\int_{-\sqrt{y-15z^2+9}}^{\sqrt{y-15z^2+9}} f(x,y,z)\ dx\ dz\ dy$$

$$\int_z^z\int_{x(z)}^{x(z)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dx\ dz \qquad \int_{-\sqrt{3/5}}^{\sqrt{3/5}}\int_{-\sqrt{9-15z^2}}^{\sqrt{9-15z^2}}\int_{x^2+15z^2-9}^{0} f(x,y,z)\ dy\ dx\ dz$$

$$\int_x^x\int_{z(x)}^{z(x)}\int_{y(x,z)}^{y(x,z)} f(x,y,z)\ dy\ dz\ dx \qquad \int_{-3}^{3}\int_{-\sqrt{(9-x^2)/15}}^{\sqrt{(9-x^2)/15}}\int_{x^2+15z^2-9}^{0} f(x,y,z)\ dy\ dz\ dx$$

$$\int_{y}^{y}\int_{x(y)}^{x(y)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dx\ dy \qquad \int_{-9}^{0}\int_{-\sqrt{y+9}}^{\sqrt{y+9}}\int_{-\sqrt{(y-x^2+9)/15}}^{\sqrt{(y-x^2+9)/15}} f(x,y,z)\ dz\ dx\ dy$$

$$\int_{x}^{x}\int_{y(x)}^{y(x)}\int_{z(x,y)}^{z(x,y)} f(x,y,z)\ dz\ dy\ dx \qquad \int_{-3}^{3}\int_{x^2-9}^{0}\int_{-\sqrt{(y-x^2+9)/15}}^{\sqrt{(y-x^2+9)/15}} f(x,y,z)\ dz\ dy\ dx$$