Average value

To find the average value of a function over some object E, we'll use the formula

$$f_{avg} = \frac{1}{V(E)} \iiint_E f(x, y, z)\ dV$$

where $V(E)$ is the volume of the object E.

In order to use the formula, we'll have find the volume of the object, plus the domain of x, y, and z so that we can set limits of integration, turn the triple integral into an iterated integral, and replace dV with $dz\ dy\ dx$.

Example

Find the average value of the function over a cube with side length 2, lying in the first octant with one corner at the origin $(0,0,0)$ and three sides lying in the coordinate planes.

$$f(x, y, z) = 3xyz^2$$

We'll start by finding the volume of the cube. Since we're dealing with a cube with side length 2, the volume will be

$$V(E) = (2)(2)(2)$$

$$V(E) = 8$$

To find the limits of integration, we have to look at the object we've been given. In this case, it's a cube whose corner is sitting at $(0,0,0)$ on the origin. Since the cube has side length 2, the limits of integration are $x=[0,2]$, $y=[0,2]$ and $z=[0,2]$.

Plugging everything we've found into the triple integral formula for average value, including the function itself, we get

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2\int_0^2 3xyz^2\ dz\ dy\ dx$$

$$f_{avg}=\frac{3}{8}\int_0^2\int_0^2\int_0^2 xyz^2\ dz\ dy\ dx$$

Integrating with respect to z, we get

$$f_{avg}=\frac{3}{8}\int_0^2\int_0^2 \frac{1}{3}xyz^3\Big|_{z=0}^{z=2}\ dy\ dx$$

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 xyz^3\Big|_{z=0}^{z=2}\ dy\ dx$$

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 xy(2)^3-xy(0)^3\ dy\ dx$$

$$f_{avg}=\frac{1}{8}\int_0^2\int_0^2 8xy\ dy\ dx$$

Now we'll integrate with respect to y.

$$f_{avg}=\frac{1}{8}\int_0^2 8\left(\frac{1}{2}\right)xy^2\Big|_{y=0}^{y=2}\ dx$$

$$f_{avg}=\frac{1}{2}\int_0^2 xy^2\Big|_{y=0}^{y=2}\ dx$$

$$f_{avg}=\frac{1}{2}\int_0^2 x(2)^2-x(0)^2\ dx$$

$$f_{avg}=\frac{1}{2}\int_0^2 4x\ dx$$

Finally we'll integrate with respect to x.

$$f_{avg}=\frac{1}{2}\left(\frac{4}{2}x^2\right)\Big|_0^2$$

$$f_{avg}=x^2\Big|_0^2$$

$$f_{avg}=(2)^2-(0)^2$$

$$f_{avg}=4$$

The average value of the function $f(x,y,z)=3xyz^2$ over the cube E is 4.