Calculus 3 Workbook Solutions

Triple integrals

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MATH

ITERATED INTEGRALS

1. Evaluate the iterated integral.

$$\int_{-2}^{3}\int_{0}^{\pi}\int_{-4}^{-2}\frac{2x^3}{x^2+1}\sin y(3z^2-4z+3)\ dz\ dy\ dx$$

Solution:

Since $f(x, y, z)$ can be factored as $a(x)b(y)c(z)$, we can rewrite the triple integral as a product of three single integrals.

$$\int_{-2}^{3}\frac{2x^3}{x^2+1}\ dx\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

Use substitution with $u=x^2$, $du=2x\ dx$, and u changing from 4 to 0 to evaluate the first integral.

$$\int_{4}^{0}\frac{u}{u+1}\ du\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

$$\int_{4}^{0}\frac{u+1-1}{u+1}\ du\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

$$\int_{4}^{0}1-\frac{1}{u+1}\ du\cdot\int_{0}^{\pi}\sin y\ dy\cdot\int_{-4}^{-2}3z^2-4z+3\ dz$$

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$$u - \ln|u+1| \Big|_4^0 \cdot \int_0^\pi \sin y\ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

$$[0 - \ln|0+1| - (4 - \ln|4+1|)] \cdot \int_0^\pi \sin y\ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

$$(-\ln 1 - 4 + \ln 5) \cdot \int_0^\pi \sin y\ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

$$(\ln 5 - 4) \cdot \int_0^\pi \sin y\ dy \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

Evaluate the second integral.

$$(\ln 5 - 4) \cdot \left(-\cos y \Big|_0^\pi\right) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

$$(\ln 5 - 4) \cdot (-\cos \pi - (-\cos 0)) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

$$(\ln 5 - 4) \cdot (-(-1) + 1) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

$$2(\ln 5 - 4) \cdot \int_{-4}^{-2} 3z^2 - 4z + 3\ dz$$

Evaluate the third integral.

$$2(\ln 5 - 4) \cdot \left(z^3 - 2z^2 + 3z \Big|_{-4}^{-2}\right)$$

$$2(\ln 5 - 4) \cdot [(-2)^3 - 2(-2)^2 + 3(-2) - ((-4)^3 - 2(-4)^2 + 3(-4))]$$

$$2(\ln 5 - 4) \cdot [-8 - 8 - 6 - (-64 - 32 - 12)]$$

$$2(\ln 5 - 4) \cdot (-8 - 8 - 6 + 64 + 32 + 12)$$

$$2(\ln 5 - 4) \cdot 86$$

$$172(\ln 5 - 4)$$

■ 2. Evaluate the iterated improper integral.

$$\int_0^\infty \int_0^\infty \int_1^\infty \frac{1}{(x+2y+z)^5}\ dz\ dy\ dx$$

Solution:

Integrate with respect to z, treating x and y as constants.

$$\int_0^\infty \int_0^\infty \int_1^\infty (x+2y+z)^{-5}\ dz\ dy\ dx$$

$$\int_0^\infty \int_0^\infty \frac{1}{-4}(x+2y+z)^{-4}\Big|_{z=1}^{z=\infty}\ dy\ dx$$

$$\int_0^\infty \int_0^\infty -\frac{1}{4(x+2y+z)^4}\Big|_{z=1}^{z=\infty}\ dy\ dx$$

$$\int_0^\infty \int_0^\infty \lim_{z\to\infty}\left[-\frac{1}{4(x+2y+z)^4}\right] - \left(-\frac{1}{4(x+2y+1)^4}\right)\ dy\ dx$$

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$$\int_0^\infty \int_0^\infty 0 + \frac{1}{4(x+2y+1)^4}\ dy\ dx$$

$$\int_0^\infty \int_0^\infty \frac{1}{4(x+2y+1)^4}\ dy\ dx$$

Integrate with respect to y, treating x as a constant.

$$\int_0^\infty \int_0^\infty \frac{1}{4}(x+2y+1)^{-4}\ dy\ dx$$

$$\int_0^\infty \frac{1}{-12}\left(\frac{1}{2}\right)(x+2y+1)^{-3}\Big|_{y=0}^{y=\infty}\ dx$$

$$\int_0^\infty -\frac{1}{24(x+2y+1)^3}\Big|_{y=0}^{y=\infty}\ dx$$

$$\int_0^\infty \lim_{y\to\infty}\left[-\frac{1}{24(x+2y+1)^3}\right] - \left(-\frac{1}{24(x+2(0)+1)^3}\right)\ dx$$

$$\int_0^\infty 0 + \frac{1}{24(x+1)^3}\ dx$$

$$\int_0^\infty \frac{1}{24(x+1)^3}\ dx$$

Integrate with respect to x.

$$\int_0^\infty \frac{1}{24}(x+1)^{-3}\ dx$$

$$\frac{1}{-48}(x+1)^{-2}\Big|_0^\infty$$

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$$-\frac{1}{48(x+1)^2}\Big|_0^{\infty}$$

$$\lim_{x\to\infty}\left[-\frac{1}{48(x+1)^2}\right]-\left(-\frac{1}{48(0+1)^2}\right)$$

$$0+\frac{1}{48}$$

$$\frac{1}{48}$$

■ 3. Evaluate the iterated integral.

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\int_{2y}^{x+\frac{\pi}{2}}\cos(x-2y+z)\ dz\ dy\ dx$$

Solution:

Integrate with respect to z, treating x any y as constants.

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\sin(x-2y+z)\Big|_{z=2y}^{z=x+\frac{\pi}{2}}\ dy\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\sin\left(x-2y+x+\frac{\pi}{2}\right)-\sin(x-2y+2y)\ dy\ dx$$

$$\int_0^{\frac{\pi}{2}}\int_x^{\frac{\pi}{4}}\sin\left(2x-2y+\frac{\pi}{2}\right)-\sin x\ dy\ dx$$

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Integrate with respect to y, treating x as a constant.

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos\left(2x-2y+\frac{\pi}{2}\right)-(\sin x)y\Big|_{y=x}^{y=\frac{\pi}{4}}\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos\left(2x-2\left(\frac{\pi}{4}\right)+\frac{\pi}{2}\right)-(\sin x)\frac{\pi}{4}-\left[-\frac{1}{2}\cos\left(2x-2x+\frac{\pi}{2}\right)-(\sin x)x\right]\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos(2x)-(\sin x)\frac{\pi}{4}-\left[-\frac{1}{2}\cos\left(\frac{\pi}{2}\right)-x\sin x\right]\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos(2x)-(\sin x)\frac{\pi}{4}+x\sin x\ dx$$

$$\int_0^{\frac{\pi}{2}} -\frac{1}{2}\cos(2x)+\left(x-\frac{\pi}{4}\right)\sin x\ dx$$

Integrate with respect to x.

$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}} x\sin x\ dx$$

Use integration by parts with $u=x$, $du=dx$, $dv=\sin x$, and $v=-\cos x$ to rewrite the remaining integral.

$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x\Big|_0^{\frac{\pi}{2}}+\left[-x\cos x\Big|_0^{\frac{\pi}{2}}-\int_0^{\frac{\pi}{2}}-\cos x\ dx\right]$$

$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x-x\cos x\Big|_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}}\cos x\ dx$$

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$$-\frac{1}{4}\sin(2x)+\frac{\pi}{4}\cos x - x\cos x + \sin x\Big|_0^{\frac{\pi}{2}}$$

Evaluate over the interval.

$$-\frac{1}{4}\sin\left(2\cdot\frac{\pi}{2}\right)+\frac{\pi}{4}\cos\frac{\pi}{2}-\frac{\pi}{2}\cos\frac{\pi}{2}+\sin\frac{\pi}{2}$$

$$-\left(-\frac{1}{4}\sin(2(0))+\frac{\pi}{4}\cos(0)-0\cos(0)+\sin(0)\right)$$

$$-\frac{1}{4}(0)+\frac{\pi}{4}(0)-\frac{\pi}{2}(0)+1-\left(-\frac{1}{4}(0)+\frac{\pi}{4}(1)-0+0\right)$$

$$1-\frac{\pi}{4}$$

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TRIPLE INTEGRALS

1. Evaluate the triple integral, where D is the box with opposite corners $(5,0,1)$ and $(14,2,10)$.

$$\iiint_D y \log \left(\frac{z^4}{(x-4)^2 \cdot 10^{y^2}} \right) \ dV$$

Solution:

Based on the coordinates of the opposite corners, x is defined on $[5,14]$, y is defined on $[0,2]$, and z is defined on $[1,10]$.

$$\int_1^{10} \int_0^2 \int_5^{14} y \log \left(\frac{z^4}{(x-4)^2 \cdot 10^{y^2}} \right) \ dx \ dy \ dz$$

Use laws of logs to simplify the integrand.

$$\int_1^{10} \int_0^2 \int_5^{14} y[-2\log(x-4) - \log 10^{y^2} + 4\log z] \ dx \ dy \ dz$$

$$\int_1^{10} \int_0^2 \int_5^{14} y[-2\log(x-4) - y^2 + 4\log z] \ dx \ dy \ dz$$

$$\int_1^{10} \int_0^2 \int_5^{14} -2y\log(x-4) - y^3 + 4y\log z \ dx \ dy \ dz$$

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$$\int_1^{10}\int_0^2\int_5^{14} -2y\log(x-4)\ dx\ dy\ dz - \int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz$$

$$+\int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

Work on the first integral.

$$\int_5^{14}\log(x-4)\ dx \cdot \int_0^2 -2y\ dy \cdot \int_1^{10}\ dz - \int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz$$

$$+\int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\left(\frac{(x-4)(\ln(x-4)-1)}{\ln 10}\Bigg|_5^{14}\right)\left(-y^2\Big|_0^2\right)\left(z\Big|_1^{10}\right) - \int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz$$

$$+\int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\left(\frac{(14-4)(\ln(14-4)-1)}{\ln 10} - \left(\frac{(5-4)(\ln(5-4)-1)}{\ln 10}\right)\right)(-2^2-(-0^2))(10-1)$$

$$-\int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz + \int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\left(\frac{10(\ln(10)-1)}{\ln 10} + \frac{1}{\ln 10}\right)(-4)(9)$$

$$-\int_1^{10}\int_0^2\int_5^{14} y^3\ dx\ dy\ dz + \int_1^{10}\int_0^2\int_5^{14} 4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2\int_5^{14}y^3\ dx\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

Work on the second integral.

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2 xy^3\Big|_{x=5}^{x=14}\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2 14y^3-5y^3\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\int_0^2 9y^3\ dy\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\frac{9}{4}y^4\Big|_{y=0}^{y=2}\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}\frac{9}{4}(2)^4-\frac{9}{4}(0)^4\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-\int_1^{10}36\ dz+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-36z\Big|_1^{10}+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-(36(10)-36(1))+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2\int_5^{14}4y\log z\ dx\ dy\ dz$$

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Work on the third integral.

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 4xy\log z\Big|_{x=5}^{x=14}\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 4(14)y\log z-4(5)y\log z\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 56y\log z-20y\log z\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10}\int_0^2 36y\log z\ dy\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10} 18y^2\log z\Big|_{y=0}^{y=2}\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10} 18(2)^2\log z-18(0)^2\log z\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\int_1^{10} 72\log z\ dz$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\left[\frac{72z(\ln z-1)}{\ln 10}\Big|_1^{10}\right]$$

$$\frac{324-360\ln(10)}{\ln 10}-324+\frac{72(10)(\ln 10-1)}{\ln 10}-\frac{72(1)(\ln 1-1)}{\ln 10}$$

$$\frac{324-360\ln 10}{\ln 10}-324+\frac{720\ln 10-720}{\ln 10}+\frac{72}{\ln 10}$$

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$$\frac{324 - 360 \ln 10 + 720 \ln 10 - 720 + 72}{\ln 10} - 324$$

$$\frac{360 \ln 10 - 324}{\ln 10} - 324$$

Find a common denominator.

$$\frac{360 \ln 10 - 324}{\ln 10} - \frac{324 \ln 10}{\ln 10}$$

$$\frac{360 \ln 10 - 324 \ln 10 - 324}{\ln 10}$$

$$\frac{36 \ln 10 - 324}{\ln 10}$$

■ 2. Evaluate the triple integral, where D is the right circular cylinder with radius 2, height 4, and a base that lies in the xy-plane with center at the origin.

$$\iiint_D e^{0.5z}\sqrt{x^2 + y^2}\ dV$$

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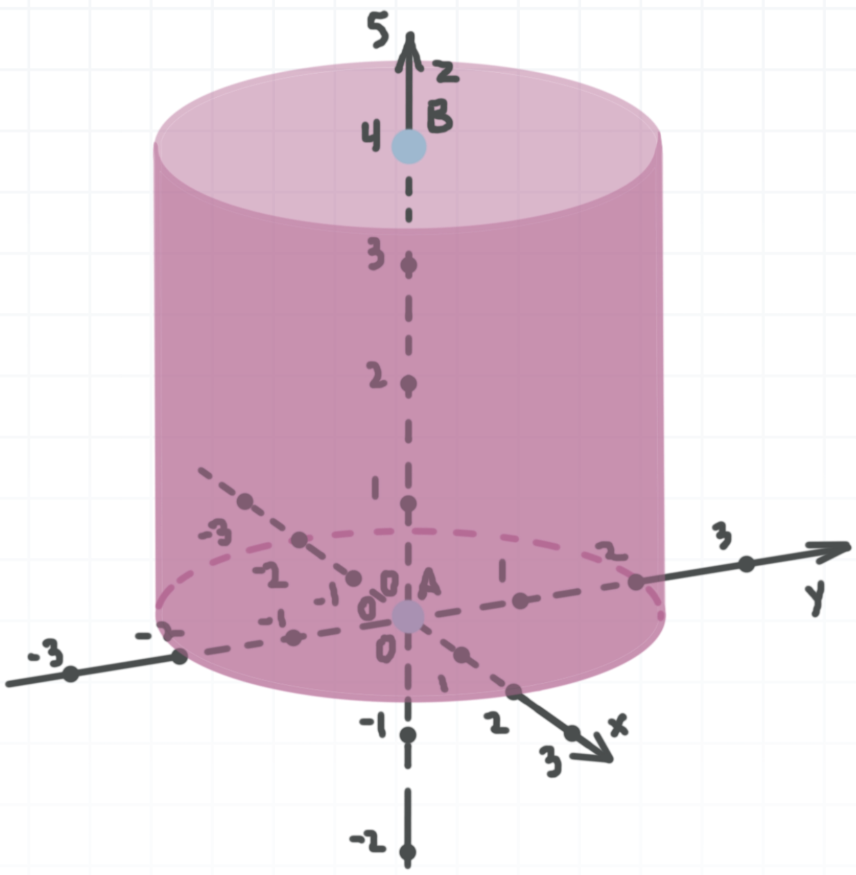

Solution:

The value of z is defined on $[0,4]$, and x and y change within the circle C with radius 2 that lies in the xy-plane with center at the origin. Rewrite the triple integral as a product of single and double integrals.

$$\iiint_D e^{0.5z}\sqrt{x^2+y^2}\ dV$$

$$\int_0^4 \iint_C e^{0.5z}\sqrt{x^2+y^2}\ dA\ dz$$

$$\int_0^4 e^{0.5z}\ dz \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

Work on the first integral.

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$$\frac{1}{0.5}e^{0.5z}\Big|_0^4 \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

$$(2e^{0.5(4)} - 2e^{0.5(0)}) \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

$$(2e^2 - 2) \cdot \iint_C \sqrt{x^2+y^2}\ dA$$

Convert the second integral to polar coordinates, then evaluate it.

$$(2e^2 - 2) \cdot \int_0^2 \int_0^{2\pi} r \cdot r\ d\theta\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 \int_0^{2\pi} r^2\ d\theta\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 r^2\theta\Big|_{\theta=0}^{\theta=2\pi}\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 r^2(2\pi) - r^2(0)\ dr$$

$$(2e^2 - 2) \cdot \int_0^2 2\pi r^2\ dr$$

$$(2e^2 - 2) \cdot \left[\frac{2}{3}\pi r^3\Big|_0^2\right]$$

$$(2e^2 - 2) \cdot \left[\frac{2}{3}\pi(2)^3 - \frac{2}{3}\pi(0)^3\right]$$

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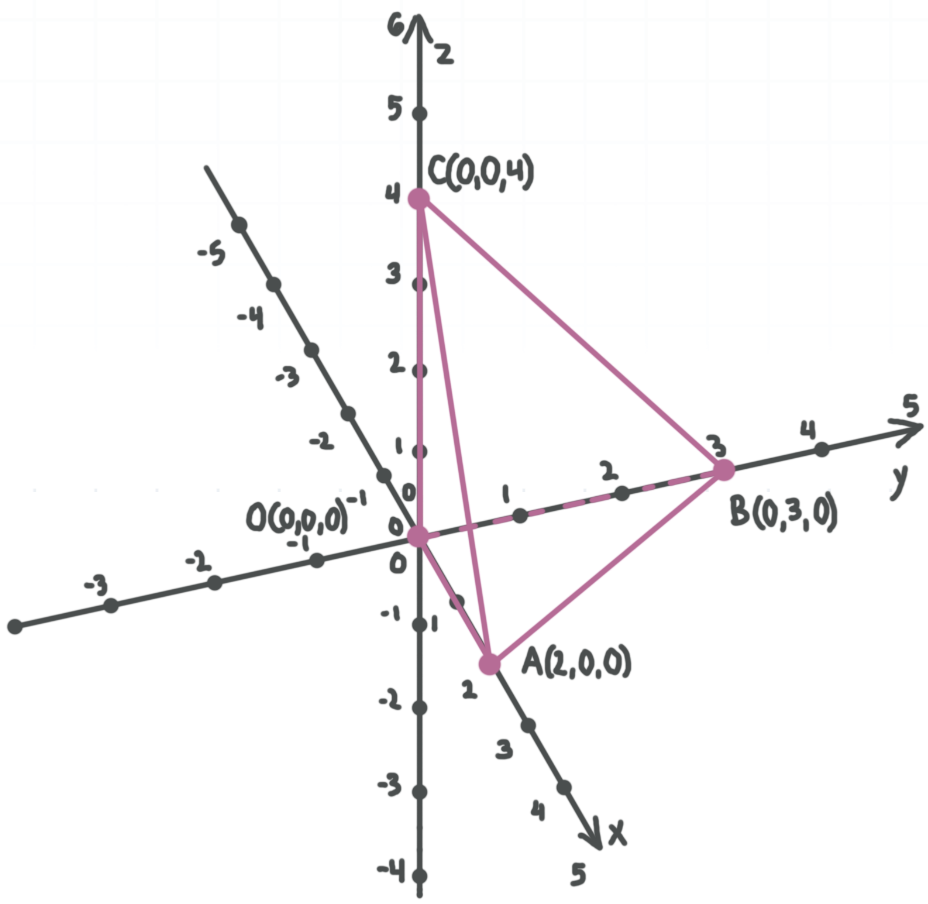
$$(2e^2 - 2) \cdot \frac{16}{3}\pi$$

$$\frac{32\pi(e^2 - 1)}{3}$$

■ 3. Evaluate the triple integral, where $ABCO$ is the irregular pyramid such that O is the origin and the vertices are $A(2,0,0)$, $B(0,3,0)$, and $C(0,0,4)$.

$$\iiint_{ABCO} 72xy \ dV$$



Solution:

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The equation of the line AB in the xy-plane is $y = -1.5x + 3$. So when x changes from 0 to 2, y changes from 0 to $-1.5x + 3$.

The equation of the plane ABC is $z = -2x - (4/3)y + 4$. So when x and y change within the triangle OAB, z changes from 0 to $-2x - (4/3)y + 4$.

Then we can write the triple integral as an iterated integral.

$$\int_0^2 \int_0^{-1.5x+3} \int_0^{-2x-\frac{4}{3}y+4} 72xy \ dz \ dy \ dx$$

Integrate with respect to z, treating x and y as constants.

$$\int_0^2 \int_0^{-1.5x+3} 72xyz \Big|_{z=0}^{z=-2x-\frac{4}{3}y+4} \ dy \ dx$$

$$\int_0^2 \int_0^{-1.5x+3} 72xy\left(-2x-\frac{4}{3}y+4\right) - 72xy(0) \ dy \ dx$$

$$\int_0^2 \int_0^{-1.5x+3} -144x^2y - 96xy^2 + 288xy \ dy \ dx$$

Integrate with respect to y, treating x as a constant.

$$\int_0^2 -72x^2y^2 - 32xy^3 + 144xy^2 \Big|_{y=0}^{y=-1.5x+3} \ dx$$

$$\int_0^2 -72x^2(-1.5x+3)^2 - 32x(-1.5x+3)^3 + 144x(-1.5x+3)^2$$

$$-(-72x^2(0)^2 - 32x(0)^3 + 144x(0)^2) \ dx$$

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$$\int_0^2 -72x^2(-1.5x+3)^2 - 32x(-1.5x+3)^3 + 144x(-1.5x+3)^2\ dx$$

$$\int_0^2 -72x^2\left(\frac{9}{4}x^2 - \frac{9}{2}x + 9\right) - 32x\left(-\frac{27}{8}x^3 + \frac{27}{2}x^2 - 27x + 27\right)$$

$$+144x\left(\frac{9}{4}x^2 - \frac{9}{2}x + 9\right)\ dx$$

$$\int_0^2 -162x^4 + 324x^3 - 648x^2 + 108x^4 - 432x^3 + 864x^2 - 864x$$

$$+324x^3 - 648x^2 + 1{,}296x\ dx$$

$$\int_0^2 -54x^4 + 216x^3 - 432x^2 + 432x\ dx$$

Integrate with respect to x.

$$-\frac{54}{5}x^5 + 54x^4 - \frac{432}{3}x^3 + 216x^2\Big|_0^2$$

$$-\frac{54}{5}(2)^5 + 54(2)^4 - \frac{432}{3}(2)^3 + 216(2)^2$$

$$-\left(-\frac{54}{5}(0)^5 + 54(0)^4 - \frac{432}{3}(0)^3 + 216(0)^2\right)$$

$$-\frac{54}{5}(32) + 54(16) - \frac{432}{3}(8) + 216(4)$$

$$-\frac{1{,}728}{5} + 864 - \frac{3{,}456}{3} + 864$$

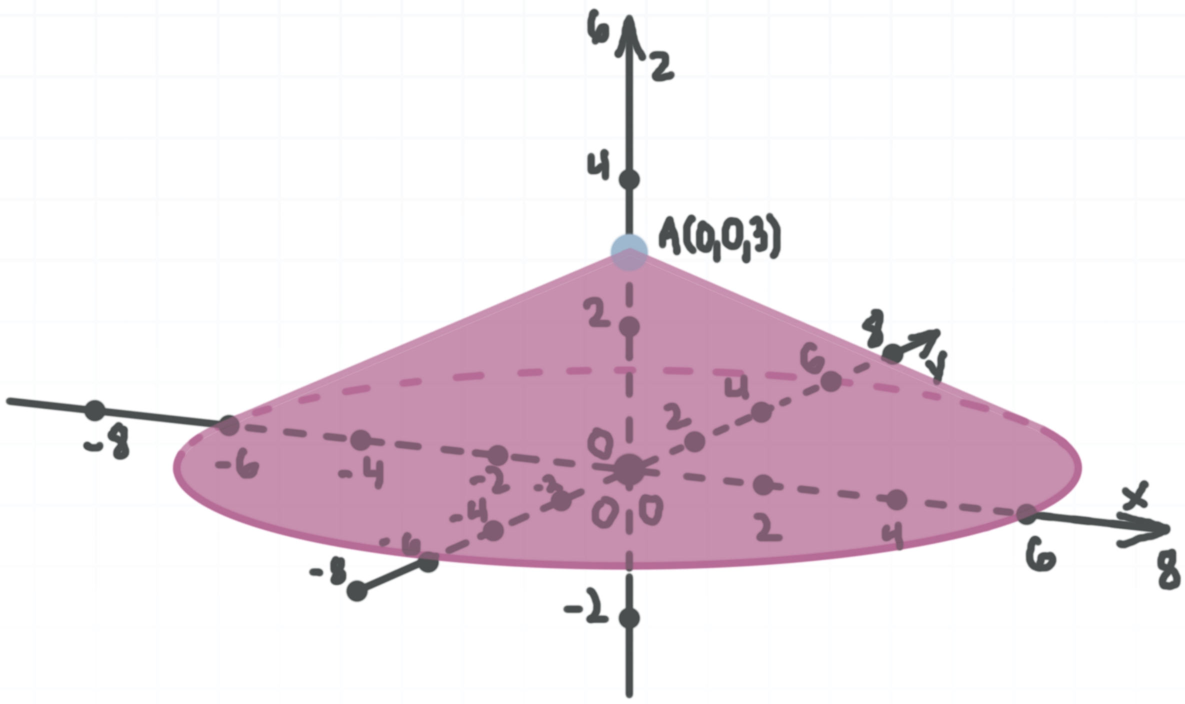
$$-\frac{5,184}{15}-\frac{17,280}{15}+\frac{25,920}{15}$$

$$\frac{3,456}{15}$$

$$\frac{1,152}{5}$$

AVERAGE VALUE

■ 1. Use triple integrals to find the average value of the function $f(x, y, z) = x^2 + y^2 + z^2$ over a right circular cone with radius $R = 6$, height $h = 3$, and a base that lies in the xy-plane with center at the origin.



Solution:

The volume of the right circular cone is

$$V(E) = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi \cdot (6)^2 \cdot 3 = 36\pi$$

The value of z in the cone is defined from 0 to 3, and x and y change within the circle C with radius r and center $(0,0,z)$ that lies in the plane parallel to xy-plane. Find r using the section through OA and the x-axis.

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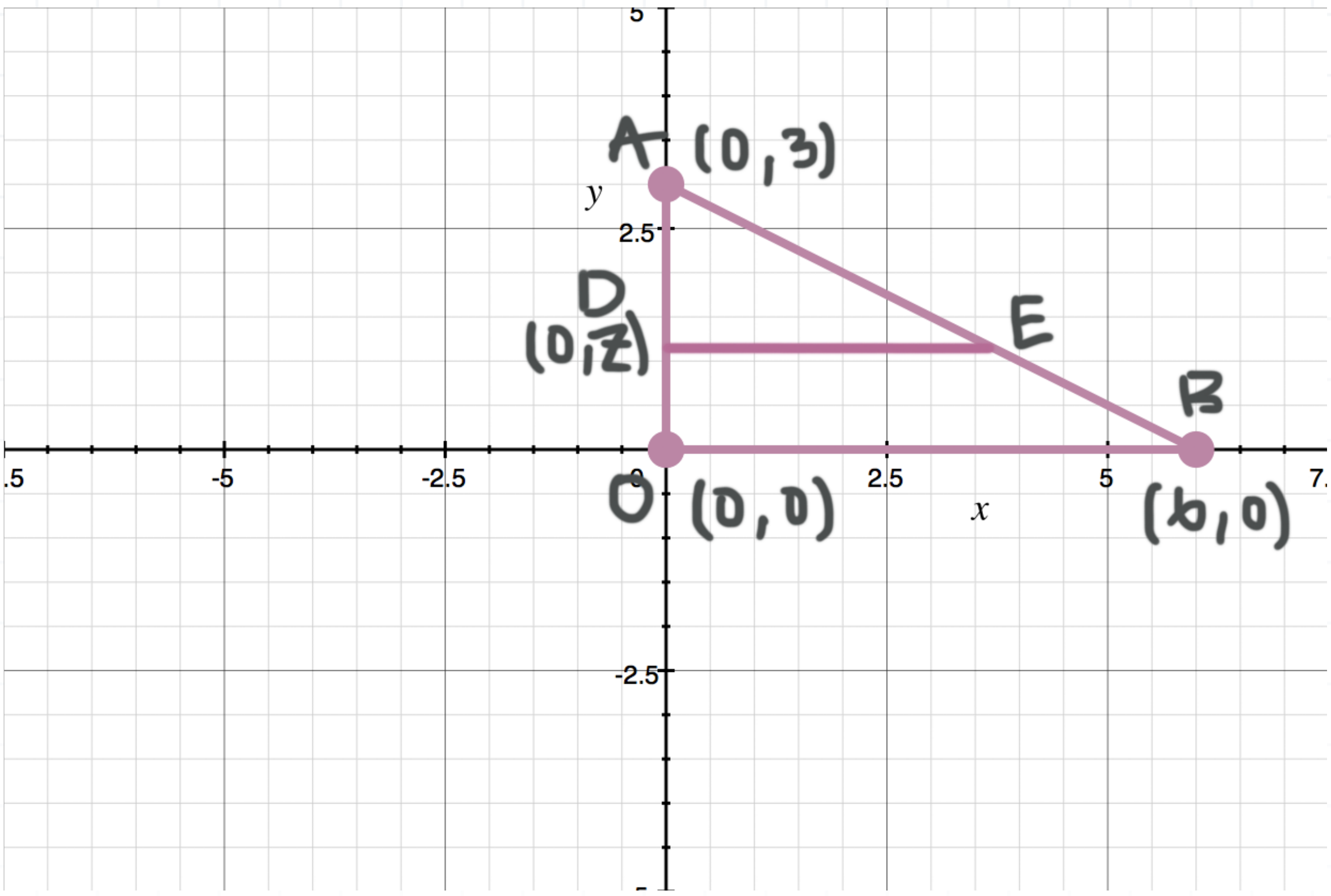


In the triangle OAB, $OB = R = 6$, $OA = h = 3$, $OD = z$, and $AD = h - z = 3 - z$. Since DE is parallel to OB, the triangles OAB and DAE are similar, so

$$\frac{DE}{OB} = \frac{DA}{OA}$$

$$\frac{DE}{6} = \frac{3 - z}{3}$$

$$DE = 2(3 - z) = 6 - 2z$$

Then the triple integral will be

$$\iiint_E x^2 + y^2 + z^2 \ dV$$

$$\int_0^3 \iint_C x^2 + y^2 + z^2 \ dA \ dz$$

Convert the inner integral to polar coordinates.

$$\int_0^3 \int_0^{6-2z} \int_0^{2\pi} (r^2 + z^2) \cdot r \ d\theta \ dr \ dz$$

$$\int_0^3 \int_0^{6-2z} \int_0^{2\pi} r^3 + rz^2 \ d\theta \ dr \ dz$$

$$\int_0^3 \left[\int_0^{6-2z} r^3 + rz^2 \ dr \cdot \int_0^{2\pi} \ d\theta \right] \ dz$$

Integrate with respect to θ.

$$\int_0^3 \left[\int_0^{6-2z} r^3 + rz^2 \ dr \cdot \theta \Big|_0^{2\pi} \right] \ dz$$

$$\int_0^3 \left[\int_0^{6-2z} r^3 + rz^2 \ dr \cdot (2\pi - 0) \right] \ dz$$

$$\int_0^3 2\pi \int_0^{6-2z} r^3 + rz^2 \ dr \ dz$$

Integrate with respect to r.

$$\int_0^3 2\pi \left(\frac{1}{4}r^4 + \frac{1}{2}r^2z^2 \right) \Bigg|_{r=0}^{r=6-2z} \ dz$$

$$\int_0^3 2\pi \left(\frac{1}{4}(6-2z)^4 + \frac{1}{2}(6-2z)^2z^2 \right) - 2\pi \left(\frac{1}{4}(0)^4 + \frac{1}{2}(0)^2z^2 \right) \ dz$$

$$\int_0^3 \frac{\pi}{2}(6-2z)^4+\pi(6-2z)^2z^2\ dz$$

$$\int_0^3 \frac{\pi}{2}(36-24z+4z^2)^2+\pi(36-24z+4z^2)z^2\ dz$$

$$\int_0^3 \frac{\pi}{2}(16z^4-192z^3+288z^2-1{,}152z+1{,}296)+36\pi z^2-24\pi z^3+4\pi z^4\ dz$$

$$\int_0^3 8\pi z^4-96\pi z^3+144\pi z^2-576\pi z+648\pi+36\pi z^2-24\pi z^3+4\pi z^4\ dz$$

$$\int_0^3 12\pi z^4-120\pi z^3+180\pi z^2-576\pi z+648\pi\ dz$$

Integrate with respect to z.

$$\frac{12\pi}{5}z^5-30\pi z^4+60\pi z^3-288\pi z^2+648\pi z\Big|_0^3$$

$$\frac{12\pi}{5}(3)^5-30\pi(3)^4+60\pi(3)^3-288\pi(3)^2+648\pi(3)$$

$$-\left(\frac{12\pi}{5}(0)^5-30\pi(0)^4+60\pi(0)^3-288\pi(0)^2+648\pi(0)\right)$$

$$\frac{2{,}916\pi}{5}-2{,}430\pi+1{,}620\pi-2{,}592\pi+1{,}944\pi$$

$$-\frac{4{,}374\pi}{5}$$

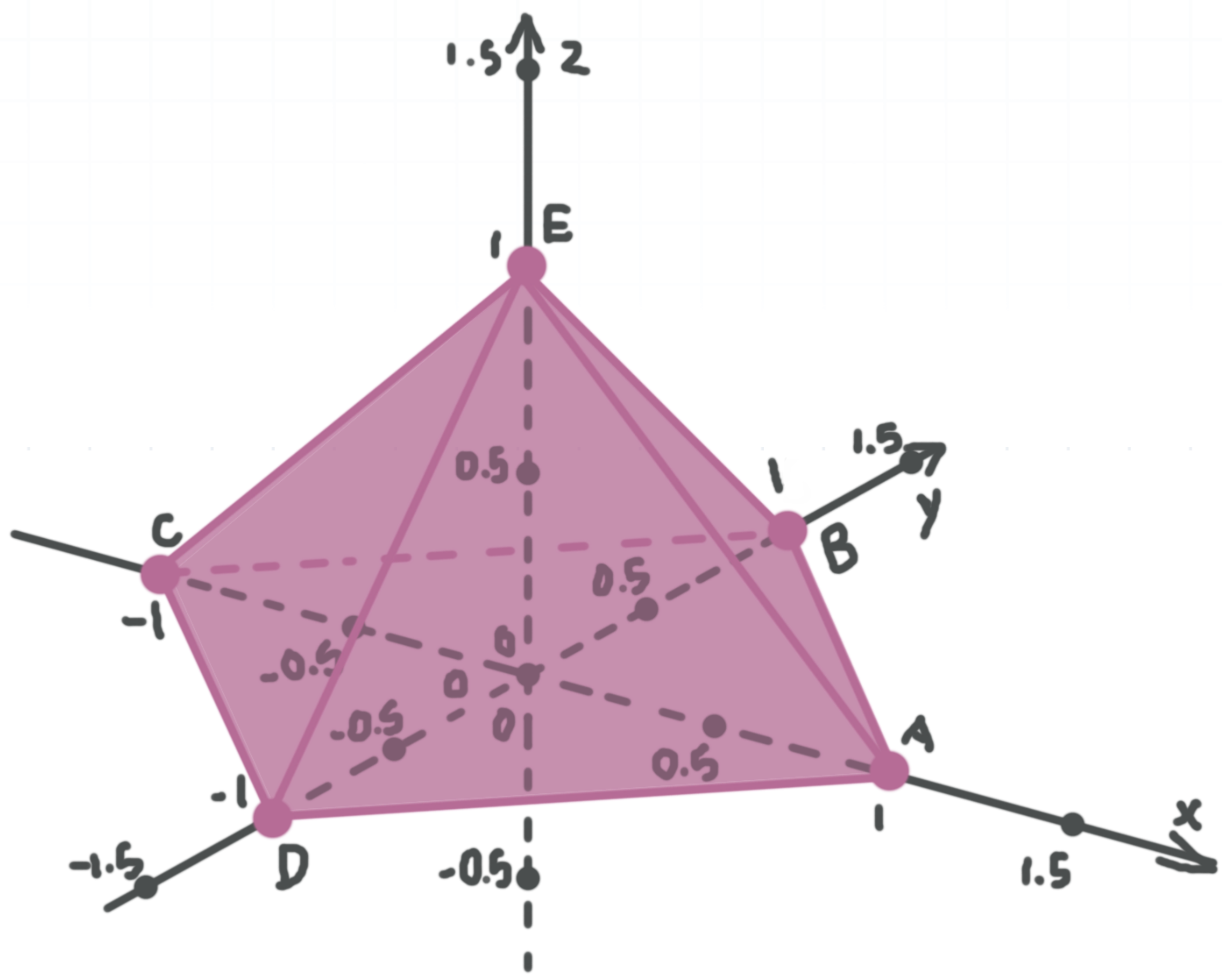
Then the average value is

$$f_{avg}(x, y, z) = \frac{1}{V(E)} \iiint_E f(x, y, z)\ dV$$

$$f_{avg}(x, y, z) = \frac{1}{36\pi}\left(-\frac{4{,}374\pi}{5}\right)$$

$$f_{avg}(x, y, z) = -24.3$$

■ 2. Use triple integrals to find the average value of the function $f(x, y, z) = |xyz|$ over a regular pyramid $ABCDE$, where $A(1,0,0)$, $B(0,1,0)$, $C(-1,0,0)$, $D(0,-1,0)$, and $E(0,0,1)$.



Solution:

The volume of the regular pyramid is

$$V(G) = \frac{1}{3}a^2h = \frac{1}{3}(\sqrt{2})^2 \cdot 1 = \frac{2}{3}$$

Because the pyramid is regular and centered at the origin, it has the same volume in each octant. Therefore, we can calculate the volume in just the first octant, and multiply the result by 4 to get total volume.

The equation of the line AB in the xy-plane is $y = 1 - x$. So when x changes from 0 to 1, y changes from 0 to $1 - x$. The equation of the plane ABE is $z = -x - y + 1$. So when x and y change within the triangle OAB, z changes from 0 to $-x - y + 1$. Therefore, the triple integral can be rewritten as an iterated integral.

$$4\iiint_{G_1} xyz\ dV$$

$$4\int_0^1 \int_0^{1-x} \int_0^{-x-y+1} xyz\ dz\ dy\ dx$$

Integrate with respect to z.

$$4\int_0^1 \int_0^{1-x} \frac{1}{2}xyz^2\Big|_{z=0}^{z=-x-y+1}\ dy\ dx$$

$$4\int_0^1 \int_0^{1-x} \frac{1}{2}xy(-x-y+1)^2 - \frac{1}{2}xy(0)^2\ dy\ dx$$

$$4\int_0^1 \int_0^{1-x} \frac{1}{2}xy(x^2 - xy - x + xy + y^2 - y - x - y + 1)\ dy\ dx$$

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$$4\int_0^1\int_0^{1-x}\frac{1}{2}x^3y-x^2y+\frac{1}{2}xy-xy^2+\frac{1}{2}xy^3\ dy\ dx$$

$$\int_0^1\int_0^{1-x}2x^3y-4x^2y+2xy-4xy^2+2xy^3\ dy\ dx$$

Integrate with respect to y.

$$\int_0^1 x^3y^2-2x^2y^2+xy^2-\frac{4}{3}xy^3+\frac{1}{2}xy^4\Big|_{y=0}^{y=1-x}\ dx$$

$$\int_0^1 x^3(1-x)^2-2x^2(1-x)^2+x(1-x)^2-\frac{4}{3}x(1-x)^3+\frac{1}{2}x(1-x)^4$$

$$-\left(x^3(0)^2-2x^2(0)^2+x(0)^2-\frac{4}{3}x(0)^3+\frac{1}{2}x(0)^4\right)\ dx$$

$$\int_0^1 (x^3-2x^2+x)(1-x)^2+x(1-x)^3\left(-\frac{4}{3}+\frac{1}{2}(1-x)\right)\ dx$$

$$\int_0^1 (x^3-2x^2+x)(1-x)^2+\left(-\frac{4}{3}x+\frac{1}{2}x-\frac{1}{2}x^2\right)(1-x)^3\ dx$$

$$\int_0^1 (x^3-2x^2+x)(1-x)^2-\left(\frac{1}{2}x^2+\frac{5}{6}x\right)(1-x)^3\ dx$$

$$\int_0^1 x^5-4x^4+6x^3-4x^2+x-\left(\frac{1}{2}x^2+\frac{5}{6}x\right)(1-3x+3x^2-x^3)\ dx$$

$$\int_0^1 x^5-4x^4+6x^3-4x^2+x-\left(-\frac{1}{2}x^5+\frac{2}{3}x^4+x^3-2x^2+\frac{5}{6}x\right)\ dx$$

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$$\int_0^1 \frac{3}{2}x^5 - \frac{14}{3}x^4 + 5x^3 - 2x^2 + \frac{1}{6}x\ dx$$

Integrate with respect to x.

$$\frac{1}{4}x^6 - \frac{14}{15}x^5 + \frac{5}{4}x^4 - \frac{2}{3}x^3 + \frac{1}{12}x^2\Big|_0^1$$

$$\frac{1}{4}(1)^6 - \frac{14}{15}(1)^5 + \frac{5}{4}(1)^4 - \frac{2}{3}(1)^3 + \frac{1}{12}(1)^2$$

$$-\left(\frac{1}{4}(0)^6 - \frac{14}{15}(0)^5 + \frac{5}{4}(0)^4 - \frac{2}{3}(0)^3 + \frac{1}{12}(0)^2\right)$$

$$\frac{1}{4} - \frac{14}{15} + \frac{5}{4} - \frac{2}{3} + \frac{1}{12}$$

$$\frac{15}{60} - \frac{56}{60} + \frac{75}{60} - \frac{40}{60} + \frac{5}{60}$$

$$-\frac{1}{60}$$

Then the average value is

$$f_{avg}(x, y, z) = \frac{1}{V(G)}\iiint_G f(x, y, z)\ dV$$

$$f_{avg}(x, y, z) = \frac{1}{\frac{2}{3}}\left(-\frac{1}{60}\right)$$

$$f_{avg}(x, y, z) = -\frac{1}{40}$$

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■ 3. Use triple integrals to find the average value of the function $f(x, y, z) = 2x - 3y + z$ over a layer bounded by the planes $z = 2$ and $z = 4$.

Solution:

Since the region of integration is an infinite layer, the average value of the function $f(x, y, z)$ over the region E could be calculated using a limit.

$$f_{avg}(x, y, z) = \lim_{t \to \infty} \frac{1}{V(E(t))} \iiint_{E(t)} f(x, y, z)\ dV$$

The box $E(t)$ has dimensions $x \in [-t, t]$, $y \in [-t, t]$, and $z \in [2,4]$, so the volume of $E(t)$ is

$$V(E(t)) = (2)(2t)(2t) = 8t^2$$

Then the triple iterated integral is

$$\int_{-t}^{t} \int_{-t}^{t} \int_{2}^{4} 2x - 3y + z\ dz\ dy\ dx$$

Integrate with respect to z.

$$\int_{-t}^{t} \int_{-t}^{t} 2xz - 3yz + \frac{1}{2}z^2 \Big|_{z=2}^{z=4}\ dy\ dx$$

$$\int_{-t}^{t} \int_{-t}^{t} 2x(4) - 3y(4) + \frac{1}{2}(4)^2 - \left(2x(2) - 3y(2) + \frac{1}{2}(2)^2\right)\ dy\ dx$$

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$$\int_{-t}^{t}\int_{-t}^{t} 8x - 12y + 8 - 4x + 6y - 2\ dy\ dx$$

$$\int_{-t}^{t}\int_{-t}^{t} 4x - 6y + 6\ dy\ dx$$

Integrate with respect to y.

$$\int_{-t}^{t} 4xy - 3y^2 + 6y\Big|_{y=-t}^{y=t}\ dx$$

$$\int_{-t}^{t} 4x(t) - 3(t)^2 + 6(t) - (4x(-t) - 3(-t)^2 + 6(-t))\ dx$$

$$\int_{-t}^{t} 4xt - 3t^2 + 6t + 4xt + 3t^2 + 6t\ dx$$

$$\int_{-t}^{t} 8xt + 12t\ dx$$

Integrate with respect to x.

$$4x^2t + 12xt\Big|_{x=-t}^{x=t}$$

$$4t^2t + 12(t)t - (4(-t)^2t + 12(-t)t)$$

$$4t^3 + 12t^2 - 4t^3 + 12t^2$$

$$24t^2$$

Then the average value is

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$$f_{avg}(x,y,z) = \lim_{t \to \infty} \frac{1}{8t^2}(24t^2)$$

$$f_{avg}(x,y,z) = \lim_{t \to \infty} 3$$

$$f_{avg}(x,y,z) = 3$$

FINDING VOLUME

■ 1. Find the volume given by the triple integral.

$$\int_{-4}^{6}\int_{3-2x^2}^{10}\int_{2x-y}^{12-y} dz\ dy\ dx$$

Solution:

Integrate with respect to z.

$$\int_{-4}^{6}\int_{3-2x^2}^{10} z\Big|_{z=2x-y}^{z=12-y} dy\ dx$$

$$\int_{-4}^{6}\int_{3-2x^2}^{10} 12-y-(2x-y)\ dy\ dx$$

$$\int_{-4}^{6}\int_{3-2x^2}^{10} 12-2x\ dy\ dx$$

Integrate with respect to y.

$$\int_{-4}^{6} 12y-2xy\Big|_{y=3-2x^2}^{y=10} dx$$

$$\int_{-4}^{6} 12(10)-2x(10)-(12(3-2x^2)-2x(3-2x^2))\ dx$$

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$$\int_{-4}^{6} 120 - 20x - (36 - 24x^2 - 6x + 4x^3)\ dx$$

$$\int_{-4}^{6} 120 - 20x - 36 + 24x^2 + 6x - 4x^3\ dx$$

$$\int_{-4}^{6} 84 - 14x + 24x^2 - 4x^3\ dx$$

Integrate with respect to x.

$$84x - 7x^2 + 8x^3 - x^4 \Big|_{-4}^{6}$$

$$84(6) - 7(6)^2 + 8(6)^3 - 6^4 - (84(-4) - 7(-4)^2 + 8(-4)^3 - (-4)^4)$$

$$504 - 252 + 1{,}728 - 1{,}296 - (-336 - 112 - 512 - 256)$$

$$504 - 252 + 1{,}728 - 1{,}296 + 336 + 112 + 512 + 256$$

$$1{,}900$$

■ 2. Use a triple integral to find the volume of the solid bounded by the circular paraboloid $4 - x^2 - y^2 - z = 0$ and the xy-plane.

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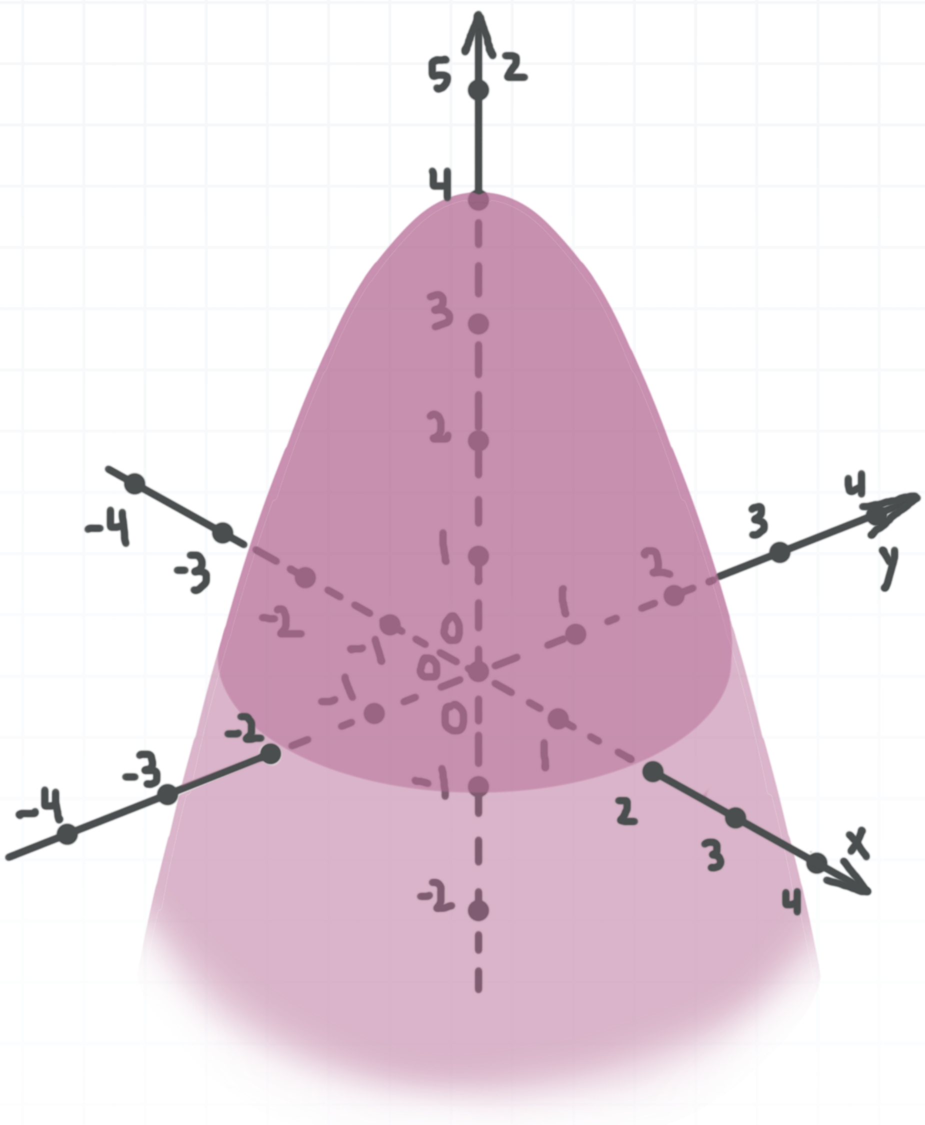

Solution:

The value of z is defined from 0 to 4, x and y change within the circle C with radius $4 - z$ and center $(0,0,z)$ that lies in the plane parallel to the xy-plane. So the volume is given by

$$V = \int_0^4 \iint_C dA\ dz$$

$$V = \int_0^4 \pi(4 - z)^2\ dz$$

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$$V=\int_0^4 \pi z^2-8\pi z+16\pi\ dz$$

Integrate with respect to z.

$$V=\frac{\pi}{3}z^3-4\pi z^2+16\pi z\Big|_0^4$$

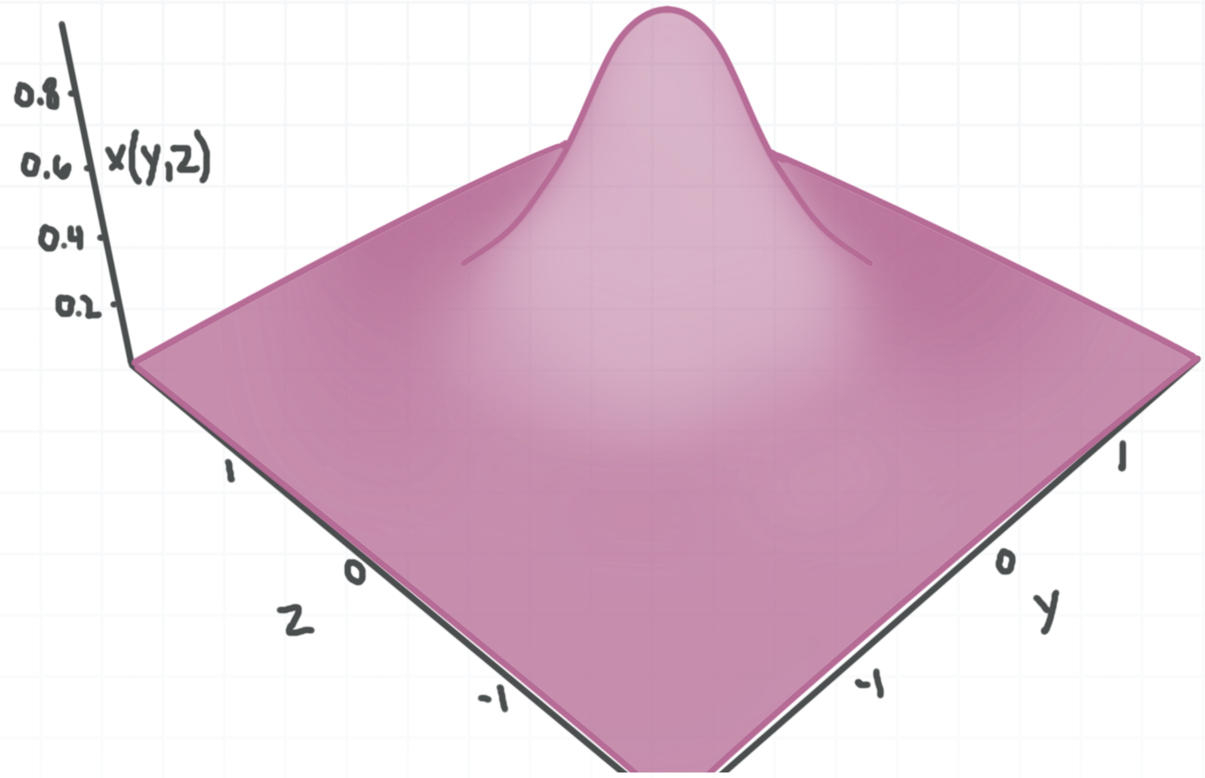
$$V=\frac{\pi}{3}(4)^3-4\pi(4)^2+16\pi(4)-\left(\frac{\pi}{3}(0)^3-4\pi(0)^2+16\pi(0)\right)$$

$$V=\frac{64\pi}{3}-64\pi+64\pi$$

$$V=\frac{64\pi}{3}$$

■ 3. Use a triple integral to find the volume of the solid bounded by the surface $x=g(y,z)$ and the yz-plane.

$$x=\frac{1}{(y^2+z^2+1)^2}$$



Solution:

The volume of a region E is equal to the triple integral with $f(x, y, z) = 1$, i.e.

$$V = \iiint_E dV$$

Since $g(y, z) \geq 0$, $g(y, z) \to 0$ when $y, z \to \infty$, $g(y, z) \leq 1$, and $g(y, z) = 1$ when $y = z = 0$, the value of x changes from 0 to 1, y and z change within the given region.

$$x = \frac{1}{(y^2 + z^2 + 1)^2}$$

$$(y^2 + z^2 + 1)^2 = \frac{1}{x}$$

$$y^2 + z^2 + 1 = \frac{1}{\sqrt{x}}$$

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$$y^2 + z^2 = \frac{1}{\sqrt{x}} - 1$$

$$y^2 + z^2 = x^{-\frac{1}{2}} - 1$$

So y and z change within the circle C with radius $\sqrt{x^{-\frac{1}{2}} - 1}$ and center $(x,0,0)$ that lies in the plane parallel to the yz-plane. So the volume is given by

$$V = \int_0^1 \iint_C \ dA \ dx$$

$$V = \int_0^1 \pi(x^{-\frac{1}{2}} - 1) \ dx$$

$$V = \int_0^1 \pi x^{-\frac{1}{2}} - \pi \ dx$$

Integrate with respect to x.

$$V = 2\pi x^{\frac{1}{2}} - \pi x \Big|_0^1$$

$$V = 2\pi(1)^{\frac{1}{2}} - \pi(1) - (2\pi(0)^{\frac{1}{2}} - \pi(0))$$

$$V = 2\pi - \pi$$

$$V = \pi$$

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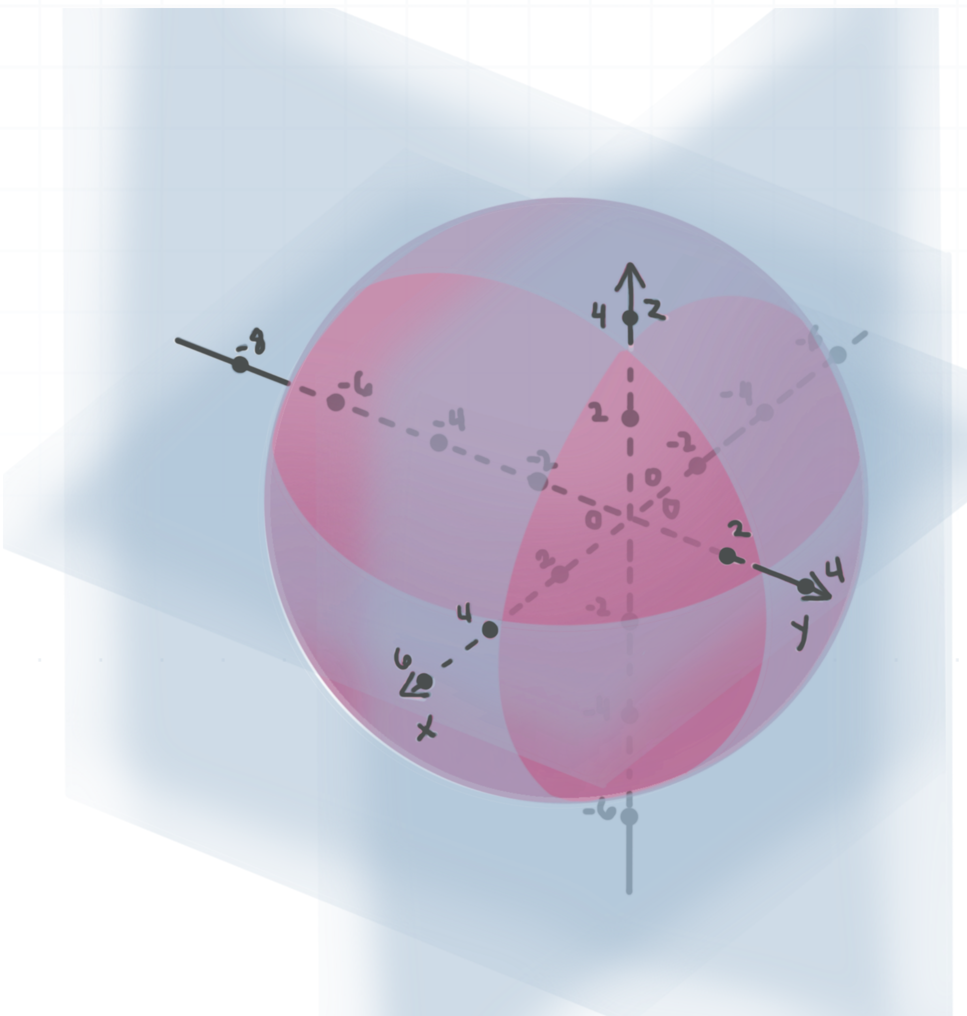
EXPRESSING THE INTEGRAL SIX WAYS

■ 1. Represent the triple integral as an iterated integral in which the order of integration is $dx\ dz\ dy$, where E is the part of the sphere with center at $(-1,-2,-1)$ and radius 25, lying in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$).

$$\iiint_E f(x,y,z)\ dV$$



Solution:

The equation of the sphere is

$$(x+1)^2+(y+2)^2+(z+1)^2=25$$

To find the interval over which y is defined, plug in $x=z=0$.

$$(0+1)^2+(y+2)^2+(0+1)^2=25$$

$$(y+2)^2=23$$

$$y=-2\pm\sqrt{23}$$

In the first octant, y is defined from 0 to $-2+\sqrt{23}$.

The innermost integral is in terms of x, so we can find the limits of integration on x by solving the equation of the sphere.

$$(x+1)^2+(y+2)^2+(z+1)^2=25$$

$$(x+1)^2=25-(y+2)^2-(z+1)^2$$

$$x=-1\pm\sqrt{25-(y+2)^2-(z+1)^2}$$

So in the first octant x is defined from 0 to $-1+\sqrt{25-(y+2)^2-(z+1)^2}$.

Next we'll try to find bounds for the middle integral. We'll set $x=0$ in the sphere equation, since by this point in the integral we would have evaluated for z. We'll solve this equation for z.

$$(0+1)^2+(y+2)^2+(z+1)^2=25$$

$$1+(y+2)^2+(z+1)^2=25$$

$$(z+1)^2=24-(y+2)^2$$

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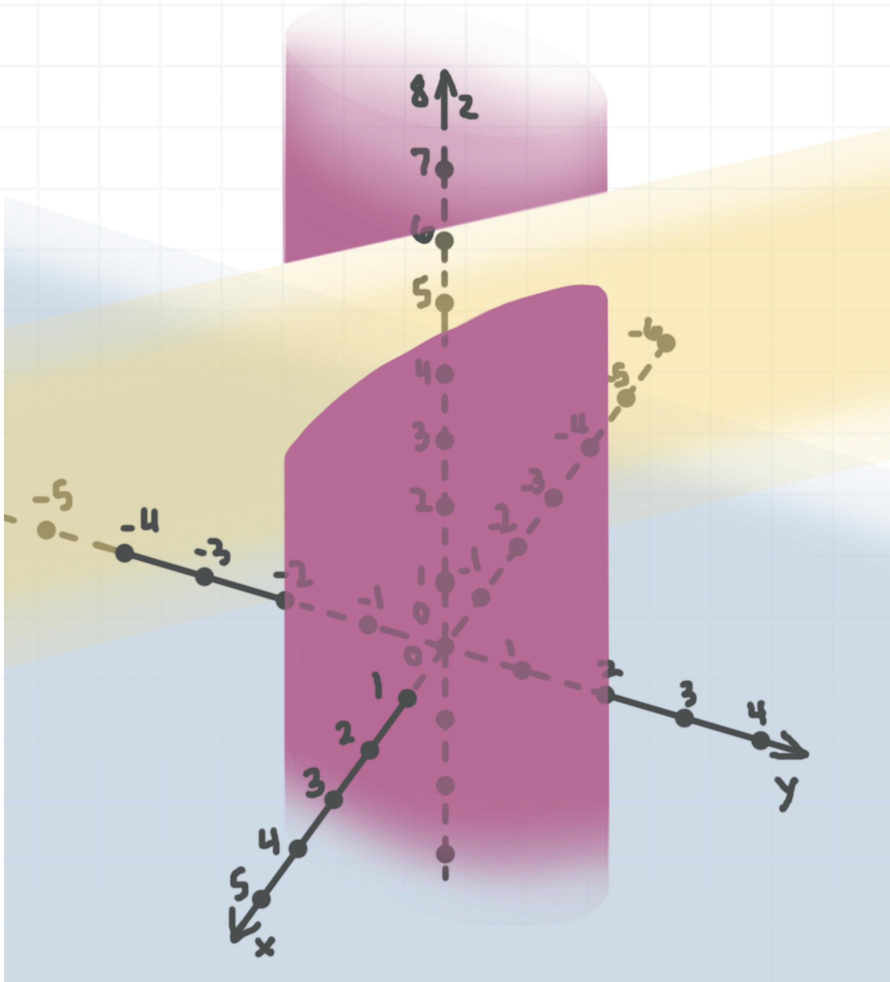
$$z = -1 \pm \sqrt{24 - (y+2)^2}$$

So in the first octant z is defined from 0 to $-1 + \sqrt{24 - (y+2)^2}$. Therefore, the iterated integral is given by

$$\int_0^{-2+\sqrt{23}} \int_0^{-1+\sqrt{24-(y+2)^2}} \int_0^{-1+\sqrt{25-(y+2)^2-(z+1)^2}} f(x, y, z)\ dx\ dz\ dy$$

■ 2. Represent the triple integral as an iterated integral using the order of integration $dz\ dy\ dx$, where E is the part of the cylinder $4x^2 + y^2 = 4$, between the planes $z = -3$ and $x + y - z + 4 = 0$.

$$\iiint_E f(x, y, z)\ dV$$



Solution:

To find an upper bound for z, solve the plane equation for z.

$$x + y - z + 4 = 0$$

$$z = x + y + 4$$

So z is defined from -3 to $x + y + 4$. The values of x and y for each z change within the ellipse $4x^2 + y^2 = 4$.

In the outermost integral, x changes from -1 to 1, and to find the bounds for y in the middle integral we'll solve the ellipse equation for y.

$$4x^2 + y^2 = 4$$

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$$y^2 = 4 - 4x^2$$

$$y = \pm 2\sqrt{1 - x^2}$$

Therefore, y changes from $-2\sqrt{1-x^2}$ to $2\sqrt{1-x^2}$, and the iterated integral is

$$\int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \int_{-3}^{x+y+4} f(x, y, z)\ dz\ dy\ dx$$

■ 3. Represent the triple integral as an improper iterated integral using the order $dx\ dy\ dz$, where E is interior of the circular paraboloid $x^2 - 4x + y^2 + 6y - z + 12 = 0$.

$$\iiint_E f(x, y, z)\ dV$$

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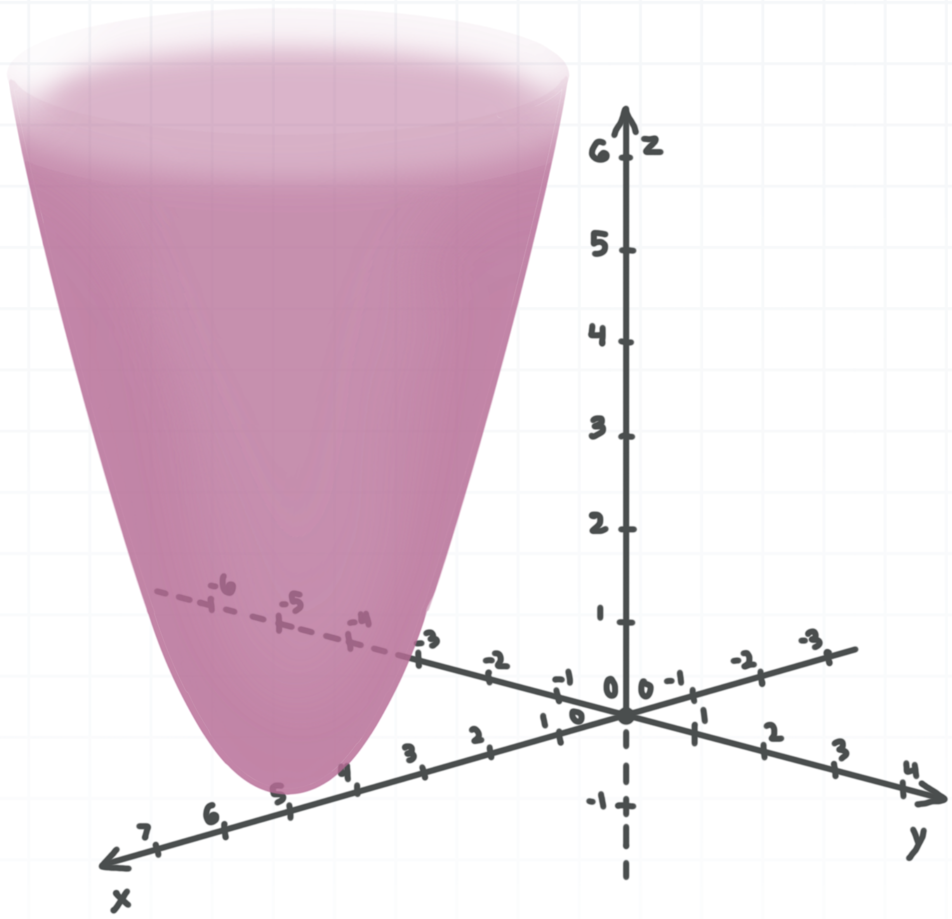

Solution:

Rewrite the paraboloid equation in standard form.

$$x^2 - 4x + y^2 + 6y - z + 12 = 0$$

$$(x^2 - 4x + 4 - 4) + (y^2 + 6y + 9 - 9) - z + 12 = 0$$

$$(x - 2)^2 - 4 + (y + 3)^2 - 9 - z + 12 = 0$$

$$(x - 2)^2 + (y + 3)^2 - z - 1 = 0$$

$$(x - 2)^2 + (y + 3)^2 = z + 1$$

Since the vertex is at $(2, -3, -1)$, the value of z changes from -1 to ∞.

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The values of x and y change for each z within the circle $(x-2)^2+(y+3)^2=z+1$ with center at $(2,-3,z)$ and radius $r=\sqrt{z+1}$ that lies in the plane parallel to the xy-plane.

So y changes from $-3-r$ to $-3+r$, or from $-3-\sqrt{z+1}$ to $-3+\sqrt{z+1}$. To find the bounds on x, solve the equation of the circle equation for x.

$$(x-2)^2+(y+3)^2=z+1$$

$$(x-2)^2=z+1-(y+3)^2$$

$$x-2=\pm\sqrt{z+1-(y+3)^2}$$

$$x=2\pm\sqrt{z+1-(y+3)^2}$$

Then x changes from $2-\sqrt{z+1-(y+3)^2}$ to $2+\sqrt{z+1-(y+3)^2}$, and the iterated integral is

$$\int_{-1}^{\infty}\int_{-3-\sqrt{z+1}}^{-3+\sqrt{z+1}}\int_{2-\sqrt{z+1-(y+3)^2}}^{2+\sqrt{z+1-(y+3)^2}} f(x,y,z)\ dx\ dy\ dz$$

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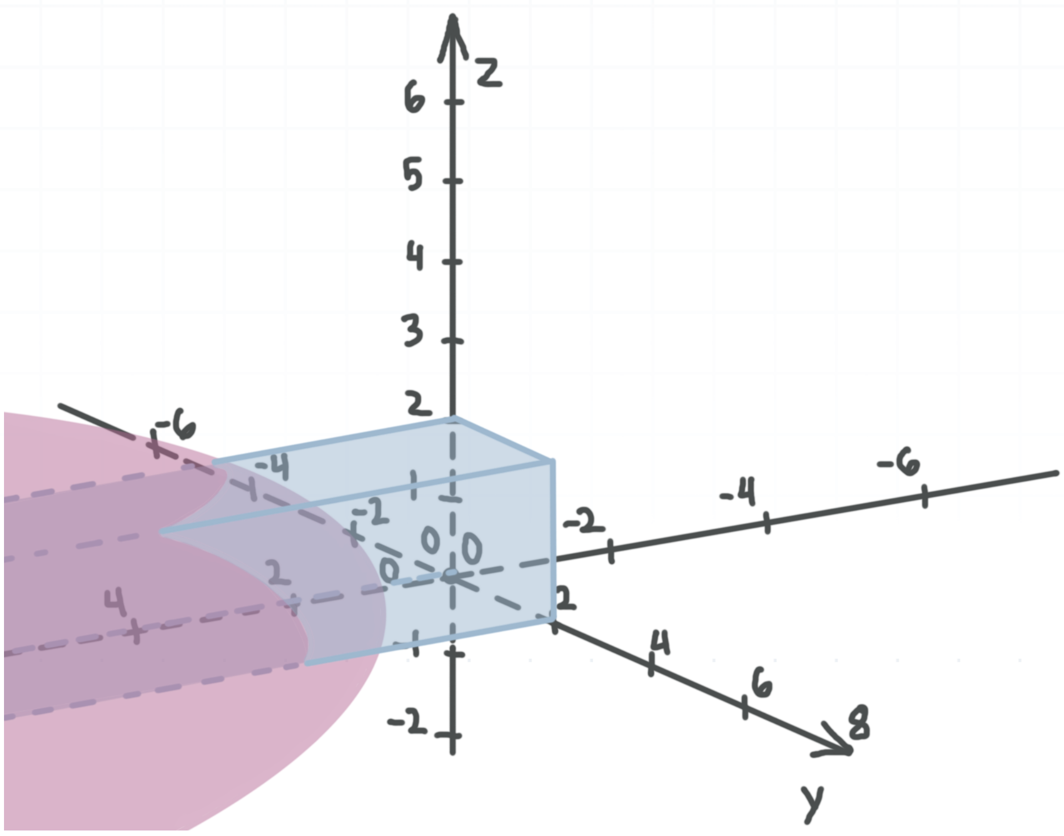
TYPE I, II, AND III REGIONS

■ 1. Evaluate the triple integral, where E is the region that lies in the first octant ($x \geq 0$, $y \geq 0$, $z \geq 0$), and is bounded by the surfaces $y = 2$, $z = 2$, and $x - 0.5y^2 - 0.5z^2 - 1 = 0$.

$$\iiint_E 4x + 2y - 2z \, dV$$



Solution:

The value of y changes from 0 to 2, z changes from 0 to 2, and x changes from 0 to $x = 0.5y^2 + 0.5z^2 + 1$. Since the function needs to be integrated with respect to x first, we can treat E as a type I region. So we can rewrite the triple integral as an iterated integral.

$$\int_0^2 \int_0^2 \int_0^{0.5y^2+0.5z^2+1} 4x + 2y - 2z \ dx \ dy \ dz$$

Integrate with respect to x.

$$\int_0^2 \int_0^2 2x^2 + 2xy - 2xz \Big|_{x=0}^{x=0.5y^2+0.5z^2+1} \ dy \ dz$$

$$\int_0^2 \int_0^2 2(0.5y^2 + 0.5z^2 + 1)^2 + 2(0.5y^2 + 0.5z^2 + 1)y - 2(0.5y^2 + 0.5z^2 + 1)z$$

$$-(2(0)^2 + 2(0)y - 2(0)z) \ dy \ dz$$

$$\int_0^2 \int_0^2 \frac{1}{2}y^4 + 2y^2 + 2 + y^2z^2 + 2z^2 + \frac{1}{2}z^4 + y^3 + yz^2 + 2y - y^2z - z^3 - 2z \ dy \ dz$$

Integrate with respect to y.

$$\int_0^2 \frac{1}{10}y^5 + \frac{2}{3}y^3 + 2y + \frac{1}{3}y^3z^2 + 2yz^2 + \frac{1}{2}yz^4$$

$$+\frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + y^2 - \frac{1}{3}y^3z - yz^3 - 2yz \Big|_{y=0}^{y=2} \ dz$$

$$\int_0^2 \frac{1}{10}y^5 + \frac{2}{3}y^3 + 2y + \frac{1}{3}y^3z^2 + 2yz^2 + \frac{1}{2}yz^4$$

$$+\frac{1}{4}y^4 + \frac{1}{2}y^2z^2 + y^2 - \frac{1}{3}y^3z - yz^3 - 2yz \Big|_{y=0}^{y=2} \ dz$$

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$$\int_0^2 \frac{1}{10}(2)^5 + \frac{2}{3}(2)^3 + 2(2) + \frac{1}{3}(2)^3z^2$$

$$+2(2)z^2 + \frac{1}{2}(2)z^4 + \frac{1}{4}(2)^4 + \frac{1}{2}(2)^2z^2 + (2)^2 - \frac{1}{3}(2)^3z - (2)z^3 - 2(2)z$$

$$-\Bigg(\frac{1}{10}(0)^5 + \frac{2}{3}(0)^3 + 2(0) + \frac{1}{3}(0)^3z^2$$

$$+2(0)z^2 + \frac{1}{2}(0)z^4 + \frac{1}{4}(0)^4 + \frac{1}{2}(0)^2z^2 + (0)^2 - \frac{1}{3}(0)^3z - (0)z^3 - 2(0)z\Bigg)\ dz$$

$$\int_0^2 \frac{16}{5} + \frac{16}{3} + 4 + \frac{8}{3}z^2 + 4z^2 + z^4 + 4 + 2z^2 + 4 - \frac{8}{3}z - 2z^3 - 4z\ dz$$

$$\int_0^2 z^4 - 2z^3 + \frac{26}{3}z^2 - \frac{20}{3}z + \frac{308}{15}\ dz$$

Integrate with respect to z.

$$\frac{1}{5}z^5 - \frac{1}{2}z^4 + \frac{26}{9}z^3 - \frac{20}{6}z^2 + \frac{308}{15}z\Big|_0^2$$

$$\frac{1}{5}(2)^5 - \frac{1}{2}(2)^4 + \frac{26}{9}(2)^3 - \frac{20}{6}(2)^2 + \frac{308}{15}(2)$$

$$-\left(\frac{1}{5}(0)^5 - \frac{1}{2}(0)^4 + \frac{26}{9}(0)^3 - \frac{20}{6}(0)^2 + \frac{308}{15}(0)\right)$$

$$\frac{32}{5} - 8 + \frac{208}{9} - \frac{40}{3} + \frac{616}{15}$$

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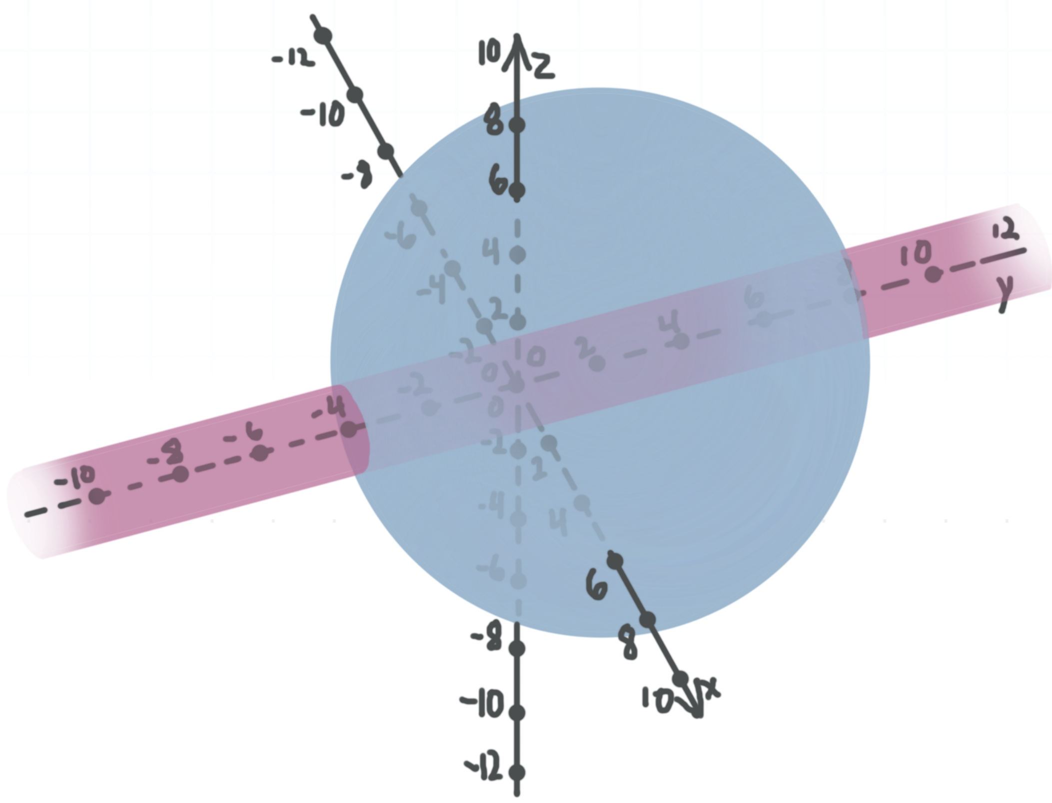
$$\frac{288}{45} - \frac{360}{45} + \frac{1{,}040}{45} - \frac{600}{45} + \frac{1{,}848}{45}$$

$$\frac{288}{45} - \frac{360}{45} + \frac{1{,}040}{45} - \frac{600}{45} + \frac{1{,}848}{45}$$

$$\frac{2{,}216}{45}$$

■ 2. Use a triple integral to find the volume of the region E that's bounded by the cylinder $x^2 + z^2 = 1$ and the sphere $x^2 + (y - 2)^2 + z^2 = 36$.



Solution:

The values of x and z change within the circle with the canter at the origin and the radius 1. To find the bounds for y, let's solve the equation of the sphere for y.

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$$x^2 + (y-2)^2 + z^2 = 36$$

$$(y-2)^2 = 36 - x^2 - z^2$$

$$y - 2 = \pm\sqrt{36 - x^2 - z^2}$$

$$y = 2 \pm \sqrt{36 - x^2 - z^2}$$

So y changes from $2 - \sqrt{36 - x^2 - z^2}$ to $2 + \sqrt{36 - x^2 - z^2}$. Since the function need to be integrated with respect to y first, we can treat E as a type II region. So we can rewrite the triple integral as an iterated integral.

$$\iint_C \int_{2-\sqrt{36-x^2-z^2}}^{2+\sqrt{36-x^2-z^2}} dy\ dA$$

Integrate with respect to y.

$$\iint_C y\Big|_{2-\sqrt{36-x^2-z^2}}^{2+\sqrt{36-x^2-z^2}} dA$$

$$\iint_C 2 + \sqrt{36 - x^2 - z^2} - (2 - \sqrt{36 - x^2 - z^2})\ dA$$

$$\iint_C 2 + \sqrt{36 - x^2 - z^2} - 2 + \sqrt{36 - x^2 - z^2}\ dA$$

$$\iint_C \sqrt{36 - x^2 - z^2} + \sqrt{36 - x^2 - z^2}\ dA$$

$$\iint_C 2\sqrt{36 - x^2 - z^2}\ dA$$

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Convert the integral into polar coordinates.

$$\int_0^1 \int_0^{2\pi} 2\sqrt{36 - r^2} \cdot r \ d\theta \ dr$$

$$\int_0^1 2r\sqrt{36 - r^2} \ dr \cdot \int_0^{2\pi} \ d\theta$$

$$2\pi \int_0^1 2r\sqrt{36 - r^2} \ dr$$

Integrate with respect to r, using a substitution with $r^2 = u$, $du = 2r \ dr$, and where u changes from 0 to 1.

$$2\pi \int_0^1 \sqrt{36 - u} \ du$$

$$2\pi \left(-\frac{2(36 - u)^{\frac{3}{2}}}{3} \right) \Bigg|_0^1$$

$$-\frac{4\pi(36 - u)^{\frac{3}{2}}}{3} \Bigg|_0^1$$

$$-\frac{4\pi(36 - 1)^{\frac{3}{2}}}{3} - \left(-\frac{4\pi(36 - 0)^{\frac{3}{2}}}{3} \right)$$

$$-\frac{4\pi(35)^{\frac{3}{2}}}{3} + \frac{4\pi(36)^{\frac{3}{2}}}{3}$$

$$\frac{4\pi(6^3) - 4\pi(35)^{\frac{3}{2}}}{3}$$

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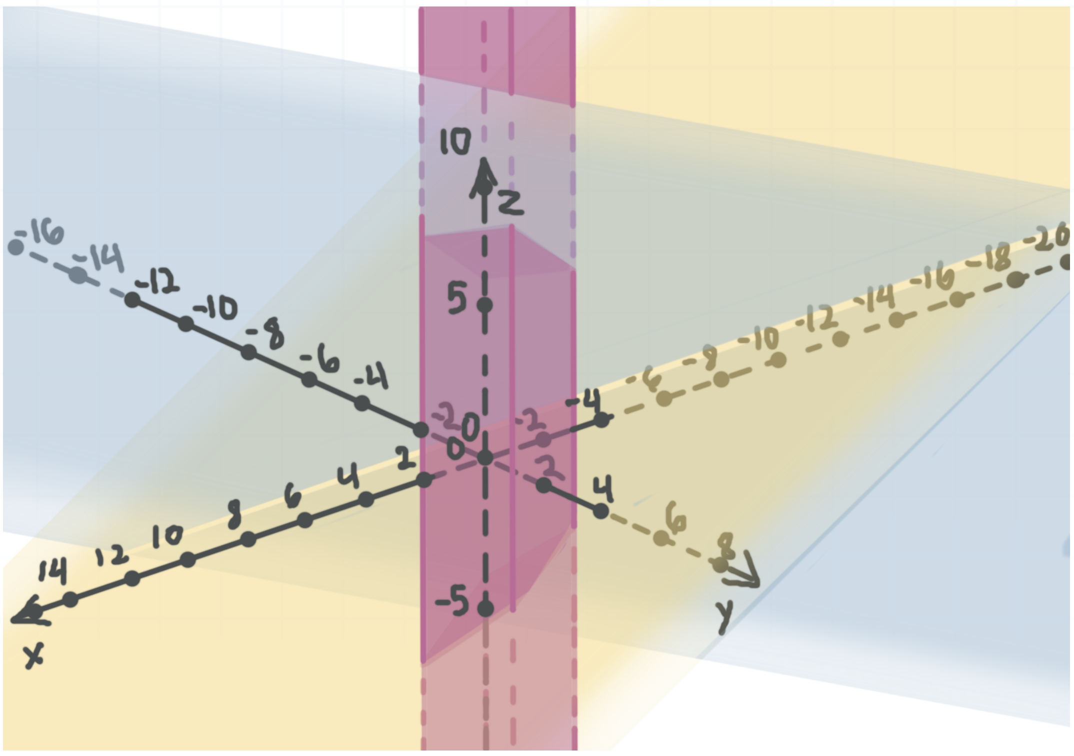
$$\frac{864\pi - 140\pi\sqrt{35}}{3}$$

■ 3. Use a triple integral to find the volume of the region E that lies in the first and fifth octants ($x \geq 0$, $y \geq 0$), and is bounded by the planes $x = 2$, $y = 3$, $2x + y - 2z + 12 = 0$, and $x - y + z + 4 = 0$.



Solution:

The value of x changes from 0 to 2, y changes from 0 to 3. To find the bounds for z, we'll solve the plane equations for z.

$$z = -x + y - 4$$

$$z = x + 0.5y + 6$$

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So z changes from $-x+y-4$ to $x+0.5y+6$. Since the function needs to be integrated with respect to z first, we can treat E as a type III region. So we can rewrite the triple integral as an iterated integral.

$$\int_0^2 \int_0^3 \int_{-x+y-4}^{x+0.5y+6} \ dz \ dy \ dx$$

Integrate with respect to z.

$$\int_0^2 \int_0^3 z \Big|_{z=-x+y-4}^{z=x+0.5y+6} \ dy \ dx$$

$$\int_0^2 \int_0^3 x+0.5y+6-(-x+y-4) \ dy \ dx$$

$$\int_0^2 \int_0^3 2x-0.5y+10 \ dy \ dx$$

Integrate with respect to y.

$$\int_0^2 2xy-0.25y^2+10y \Big|_{y=0}^{y=3} \ dx$$

$$\int_0^2 2x(3)-0.25(3)^2+10(3)-(2x(0)-0.25(0)^2+10(0)) \ dx$$

$$\int_0^2 6x-2.25+30 \ dx$$

$$\int_0^2 6x+27.75 \ dx$$

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Integrate with respect to x.

$$3x^2 + 27.75x \Big|_0^2$$

$$3(2)^2 + 27.75(2) - (3(0)^2 + 27.75(0))$$

$$12 + 55.5$$

$$67.5$$

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