Calculus 3
Final Exam Solutions

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MATH

Calculus 3 Final Exam Answer Key

1. (5 pts)	A	■	C	D	E
2. (5 pts)	A	■	C	D	E
3. (5 pts)	A	B	■	D	E
4. (5 pts)	■	B	C	D	E
5. (5 pts)	A	B	■	D	E
6. (5 pts)	A	B	C	D	■
7. (5 pts)	A	B	C	■	E
8. (5 pts)	A	■	C	D	E

9. (15 pts) $x = 5 - 2t$, $y = 1 - 3t$, $z = -5t$

10. (15 pts) Local minimum at $(4,2)$

11. (15 pts) 72π

12. (15 pts) $\kappa(t) = \dfrac{4}{25}$

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1. B. Treat x- and y-terms as constants and take the derivative with respect to z. The function is

$$f(x, y, z) = 3x^2e^y \ln 2z$$

so the partial derivative with respect to z is

$$f_z = (3x^2e^y)\frac{2}{2z}$$

$$f_z = \frac{3x^2e^y}{z}$$

2. B. The derivative of $z = x - y^2$ with respect to t, where $x = 2t - 5t^3$ and $y = t^2$ can be found using chain rule.

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

We'll find each partial derivative.

$$\frac{\partial z}{\partial x} = 1$$

$$\frac{\partial x}{\partial t} = 2 - 15t^2$$

$$\frac{\partial z}{\partial y} = -2y$$

$$\frac{\partial y}{\partial t} = 2t$$

Then the derivative of $z = x - y^2$ with respect to t is

$$\frac{dz}{dt} = 1(2 - 15t^2) + (-2y)(2t)$$

$$\frac{dz}{dt} = 2 - 15t^2 - 4yt$$

Substitute t^2 in for y (given by the original equation $y = t^2$) so that dz/dt is in terms of t.

$$\frac{dz}{dt} = 2 - 15t^2 - 4(t^2)t$$

$$\frac{dz}{dt} = 2 - 15t^2 - 4t^3$$

$$\frac{dz}{dt} = -4t^3 - 15t^2 + 2$$

3. C. Find the gradient vector.

$$\nabla f(x, y) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

The partial derivatives of $f(x, y) = 3x^2 + 3xy$ are

$$\frac{\partial f}{\partial x} = 6x + 3y$$

$$\frac{\partial f}{\partial y} = 3x$$

so the gradient vector is

$$\nabla f(x,y) = \langle 6x + 3y, 3x \rangle$$

Now plug in the point $P(-1,2)$.

$$\langle 6(-1) + 3(2), 3(-1) \rangle$$

$$\langle 0, -3 \rangle$$

4. A. Use the formula

$$f_{\text{avg}} = \frac{1}{V(E)} \iiint_E f(x,y,z) \ dV$$

to find the average value of the function. To find the volume of region E, find the volume of the cube with side length 3.

$$V(E) = 3^3 = 27$$

Since all of the sides of the cube start at the origin $(0,0,0)$ and extend to 3 along the coordinate axes, the limits of integration for each variable will be $[0,3]$.

$$f_{\text{avg}} = \frac{1}{27} \int_0^3 \int_0^3 \int_0^3 3x^2yz \ dx \ dy \ dz$$

Always integrate from the inside out, so integrate with respect to x first, then evaluate over the interval.

$$f_{\text{avg}} = \frac{1}{27}\int_0^3\int_0^3 x^3yz\Big|_0^3 \ dy \ dz$$

$$f_{\text{avg}} = \frac{1}{27}\int_0^3\int_0^3 (3)^3yz - (0)^3yz \ dy \ dz$$

$$f_{\text{avg}} = \frac{1}{27}\int_0^3\int_0^3 27yz \ dy \ dz$$

$$f_{\text{avg}} = \int_0^3\int_0^3 yz \ dy \ dz$$

Now integrate with respect to y.

$$f_{\text{avg}} = \int_0^3 \frac{1}{2}y^2z\Big|_0^3 \ dz$$

$$f_{\text{avg}} = \int_0^3 \frac{1}{2}(3)^2z - \frac{1}{2}(0)^2z \ dz$$

$$f_{\text{avg}} = \int_0^3 \frac{9}{2}z \ dz$$

$$f_{\text{avg}} = \frac{9}{2} \cdot \frac{1}{2}z^2\Big|_0^3$$

$$f_{\text{avg}} = \frac{9}{2}\left[\frac{1}{2}(3)^2 - \frac{1}{2}(0)^2\right]$$

$$f_{\text{avg}} = \frac{9}{2}\left(\frac{9}{2}\right)$$

$$f_{\text{avg}} = \frac{81}{4}$$

5. C. Take the cross product of vectors $\vec{a} = \langle a_1, a_2, a_3 \rangle$ and $\vec{b} = \langle b_1, b_2, b_3 \rangle$ to find the vector orthogonal to the plane.

$$\vec{a} \times \vec{b} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$\vec{a} \times \vec{b} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$\vec{a} \times \vec{b} = (2(1) - (3)(2))\mathbf{i} - (1(1) - (3)(-1))\mathbf{j} + (1(2) - (2)(-1))\mathbf{k}$$

$$\vec{a} \times \vec{b} = (2 - 6)\mathbf{i} - (1 + 3)\mathbf{j} + (2 + 2)\mathbf{k}$$

$$\vec{a} \times \vec{b} = -4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$$

$$\vec{a} \times \vec{b} = 4(-\mathbf{i} - \mathbf{j} + \mathbf{k})$$

$$\vec{a} \times \vec{b} = -\mathbf{i} - \mathbf{j} + \mathbf{k}$$

$$\langle -1, -1, 1 \rangle$$

6. E. To find the unit vector we'll need to use the formula

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

We have to first find $r'(t)$. If $r(t) = (2 + t^2)\mathbf{i} + e^{-t}\mathbf{j} + \cos 2t\mathbf{k}$, then

$$r'(t) = 2t\mathbf{i} - e^{-t}\mathbf{j} - 2\sin 2t\mathbf{k}$$

Now plug $t = 0$ into $r'(t)$.

$$r'(0) = 2(0)\mathbf{i} - e^{-(0)}\mathbf{j} - 2\sin(2(0))\mathbf{k}$$

$$r'(0) = -\mathbf{j}$$

Find $|r'(0)|$.

$$|r'(0)| = \sqrt{(-1)^2}$$

$$|r'(0)| = 1$$

Plug everything into the formula.

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$T(t) = \frac{-\mathbf{j}}{1}$$

$$T(t) = -\mathbf{j}$$

7. D. To find the arc length, we'll need to use the formula

$$\int_a^b |r'(t)|\ dt$$

Find $r'(t)$. If $r(t) = 2\cos t\mathbf{i} - 2\sin t\mathbf{j} + t\mathbf{k}$, then

$$r'(t) = -2\sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{k}$$

Find $|r'(t)|$.

$$|r'(t)| = \sqrt{(-2\sin t)^2 + (-2\cos t)^2 + 1^2}$$

$$|r'(t)| = \sqrt{4\sin^2 t + 4\cos^2 t + 1}$$

$$|r'(t)| = \sqrt{4(\sin^2 t + \cos^2 t) + 1}$$

Remember the trig identity $\sin^2 t + \cos^2 t = 1$ and substitute.

$$|r'(t)| = \sqrt{4(1) + 1}$$

$$|r'(t)| = \sqrt{5}$$

Plug $|r'(t)|$ into the formula and integrate from $[0,2\pi]$.

$$\int_0^{2\pi} \sqrt{5}\ dt$$

$$\sqrt{5}t\Big|_0^{2\pi}$$

$$\sqrt{5}(2\pi) - \sqrt{5}(0)$$

$$2\sqrt{5}\pi$$

8. B. First find the velocity of the ball using the formula

$$v(t) = v(t_0) + \int a(t)\ dt$$

$$v(t) = 6\mathbf{i} + 16\mathbf{k} + \int 16\mathbf{j} - 32\mathbf{k}\ dt$$

$$v(t) = 6\mathbf{i} + 16\mathbf{k} + 16t\mathbf{j} - 32t\mathbf{k}$$

$$v(t) = 6\mathbf{i} + 16t\mathbf{j} + (16 - 32t)\mathbf{k}$$

Find the position function using the formula

$$r(t) = r(t_0) + \int v(t)\ dt$$

Since we're finding how far away the ball is from its initial position, use $(0,0,0)$ as the initial position.

$$r(t) = 0 + \int 6\mathbf{i} + 16t\mathbf{j} + (16 - 32t)\mathbf{k}\ dt$$

$$r(t) = 6t\mathbf{i} + 8t^2\mathbf{j} + (16t - 16t^2)\mathbf{k}$$

Since we are asked to find how far the ball has traveled at $t = 1$, find $r(1)$.

$$r(1) = 6(1)\mathbf{i} + 8(1)^2\mathbf{j} + (16(1) - 16(1)^2)\mathbf{k}$$

$$r(1) = 6\mathbf{i} + 8\mathbf{j} + (16 - 16)\mathbf{k}$$

$$r(1) = 6\mathbf{i} + 8\mathbf{j}$$

The magnitude of $r(1)$ is how far the ball has traveled.

$$|r(1)| = \sqrt{6^2 + 8^2}$$

$$|r(1)| = \sqrt{36 + 64}$$

$$|r(1)| = \sqrt{100}$$

$$|r(1)| = 10$$

9. Start by finding the vector equation for the line where the planes intersect each other. The formula is $r = r_0 + tv$. Find v by taking the cross product of the normal vectors $a = \langle 1,1,-1 \rangle$ and $b = \langle 2,-3,1 \rangle$ of the planes.

$$v = a \times b = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{k}$$

$$v = a \times b = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

$$v = (1(1) - (-1)(-3))\mathbf{i} - (1(1) - (-1)(2))\mathbf{j} + (1(-3) - (1)(2))\mathbf{k}$$

$$v = (1 - 3)\mathbf{i} - (1 + 2)\mathbf{j} + (-3 - 2)\mathbf{k}$$

$$v = -2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$$

To find a point on the line of intersection, or r_0, set $z = 0$ and solve for x and y in both equations.

$$x + y = 6$$

$$2x - 3y = 7$$

Multiply the first equation by -2 to get $-2x - 2y = -12$, then add this to the second equation to eliminate x.

$$-2x + 2x - 2y - 3y = -12 + 7$$

$$-5y = -5$$

$$y = 1$$

Plug $y = 1$ into one of the original equations and solve for x.

$$x + 1 = 6$$

$$x = 5$$

Then we can say $r_0 = \langle 5,1,0 \rangle$, which means we get

$$r = r_0 + tv$$

$$r = 5\mathbf{i} + 1\mathbf{j} + 0\mathbf{k} + t(-2\mathbf{i} - 3\mathbf{j} - 5\mathbf{k})$$

$$r = (5 - 2t)\mathbf{i} + (1 - 3t)\mathbf{j} - 5t\mathbf{k}$$

Therefore, the parametric equations for the line of intersection are

$$x = 5 - 2t$$

$$y = 1 - 3t$$

$$z = -5t$$

10. Start by finding $g(x, y)$ by moving all terms in the constraint equation to one side.

$$2x + y = 10$$

$$g(x, y) = 2x + y - 10$$

Find the first-order partial derivatives of $f(x, y)$ and $g(x, y)$.

$$\frac{\partial f}{\partial x} = 2x$$

$$\frac{\partial f}{\partial y} = 2y$$

$$\frac{\partial g}{\partial x} = 2$$

$$\frac{\partial g}{\partial y} = 1$$

Multiply the partial derivatives of g by the Lagrange multiplier λ and set equal to the respective partial derivatives of f.

$2x = 2\lambda$, or $x = \lambda$

$2y = 1\lambda$, or $2y = \lambda$

Since $x = \lambda$ and $2y = \lambda$, we can say

$$2y = x$$

$$y = \frac{1}{2}x$$

Plug $y = (1/2)x$ back into the constraint equation and solve for x.

$$2x + y = 10$$

$$2x + \frac{1}{2}x = 10$$

$$\frac{5}{2}x = 10$$

$$x = 10 \cdot \frac{2}{5}$$

$$x = 4$$

Plug $x = 4$ back into the constraint equation and solve for y.

$$2x + y = 10$$

$$2(4) + y = 10$$

$$8 + y = 10$$

$$y = 2$$

The critical point is $(4,2)$, but now we need to determine if the critical point is a maximum, minimum, or saddle point. Find the second-order partial derivatives of $f(x, y)$.

$$\frac{\partial^2 f}{\partial x^2} = 2$$

$$\frac{\partial^2 f}{\partial y^2} = 2$$

$$\frac{\partial^2 f}{\partial x \partial y} = 0$$

Plug these into the second derivative test formula.

$$D(x, y, \lambda) = \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right) - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$$

$$D(x, y, \lambda) = (2)(2) - (0)^2$$

$$D(x, y, \lambda) = 4$$

Since $D > 0$ and $\partial^2 f/\partial x^2 = 2 > 0$, the critical point is a local minimum.

11. Convert the rectangular coordinates to cylindrical coordinates using the following conversion formulas:

$$r^2 = x^2 + y^2$$

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

Convert the cylinder to cylindrical coordinates.

$$x^2 + y^2 = 9$$

$$r^2 = 9$$

$$r = 3$$

This represents the distance from the origin. Since we need to stay within the cylinder, the limits of integration for r will be $[0,3]$. Since $z = z$ in cylindrical coordinates, $z = 0$ will remain the same. Now we'll convert the cone to cylindrical coordinates.

$$z^2 = 4x^2 + 4y^2$$

$$z^2 = 4(x^2 + y^2)$$

$$z^2 = 4r^2$$

$$z = 2r$$

Since we need to stay below the cone, the limits of integration for z will be $[0,2r]$. Since there's no restriction on where inside the cylinder, the limits of integration for θ in cylindrical coordinates will be the full $[0,2\pi]$.

Convert the function to cylindrical coordinates and remember that you must always multiply by r when converting the function.

$$f(x, y, z) = 2$$

$$f(r, \theta, z) = 2r$$

The triple integral therefore becomes

$$\iiint_E 2\ dV = \int_0^{2\pi} \int_0^3 \int_0^{2r} 2r\ dz\ dr\ d\theta$$

Integrate working from the inside out, so first integrate with respect to z.

$$\int_0^{2\pi}\int_0^3\int_0^{2r} 2r\ dz\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 2rz\Big|_0^{2r}\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 2r(2r)-2r(0)\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^3 4r^2\ dr\ d\theta$$

Integrate with respect to r.

$$\int_0^{2\pi} \frac{4}{3}r^3\Big|_0^3\ d\theta$$

$$\int_0^{2\pi} \frac{4}{3}(3)^3-\frac{4}{3}(0)^3\ d\theta$$

$$\int_0^{2\pi} 36\ d\theta$$

Integrate with respect to θ.

$$36\theta\Big|_0^{2\pi}$$

$$36(2\pi)-36(0)$$

$$72\pi$$

12. We can find curvature using the formula

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|}$$

But to find the unit tangent vector, we have to use

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

And in order to get the unit tangent vector, we need the derivative of the vector function. If $r(t) = 3t\mathbf{i} + 4\sin t\mathbf{j} - 4\cos t\mathbf{k}$, then

$$r'(t) = 3\mathbf{i} + 4\cos t\mathbf{j} + 4\sin t\mathbf{k}$$

Then we can say

$$|r(t)| = \sqrt{3^2 + (4\sin t)^2 + (4\cos t)^2}$$

$$|r(t)| = \sqrt{9 + 16\sin^2 t + 16\cos^2 t}$$

$$|r(t)| = \sqrt{9 + 16(\sin^2 t + \cos^2 t)}$$

Use the trig identity $\sin^2 t + \cos^2 t = 1$ to substitute.

$$|r(t)| = \sqrt{9 + 16(1)}$$

$$|r(t)| = \sqrt{25}$$

$$|r(t)| = 5$$

Then the unit tangent vector is

$$T(t) = \frac{r'(t)}{|r'(t)|}$$

$$T(t) = \frac{3\mathbf{i} + 4\cos t\mathbf{j} + 4\sin t\mathbf{k}}{5}$$

$$T(t) = \frac{3}{5}\mathbf{i} + \frac{4}{5}\cos t\mathbf{j} + \frac{4}{5}\sin t\mathbf{k}$$

Find the magnitude of the derivative of the unit tangent vector.

$$T'(t) = -\frac{4}{5}\sin t\mathbf{j} + \frac{4}{5}\cos t\mathbf{k}$$

$$|T'(t)| = \sqrt{\left(-\frac{4}{5}\sin t\right)^2 + \left(\frac{4}{5}\cos t\right)^2}$$

$$|T'(t)| = \sqrt{\frac{16}{25}\sin^2 t + \frac{16}{25}\cos^2 t}$$

$$|T'(t)| = \sqrt{\frac{16}{25}(\sin^2 t + \cos^2 t)}$$

$$|T'(t)| = \sqrt{\frac{16}{25}(1)}$$

$$|T'(t)| = \frac{4}{5}$$

Then we can say that the curvature is

$$\kappa(t) = \frac{|T'(t)|}{|r'(t)|}$$

$$\kappa(t) = \frac{\frac{4}{5}}{5}$$

$$\kappa(t) = \frac{4}{25}$$

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