Calculus 3 Workbook Solutions

Double integrals in polar coordinates

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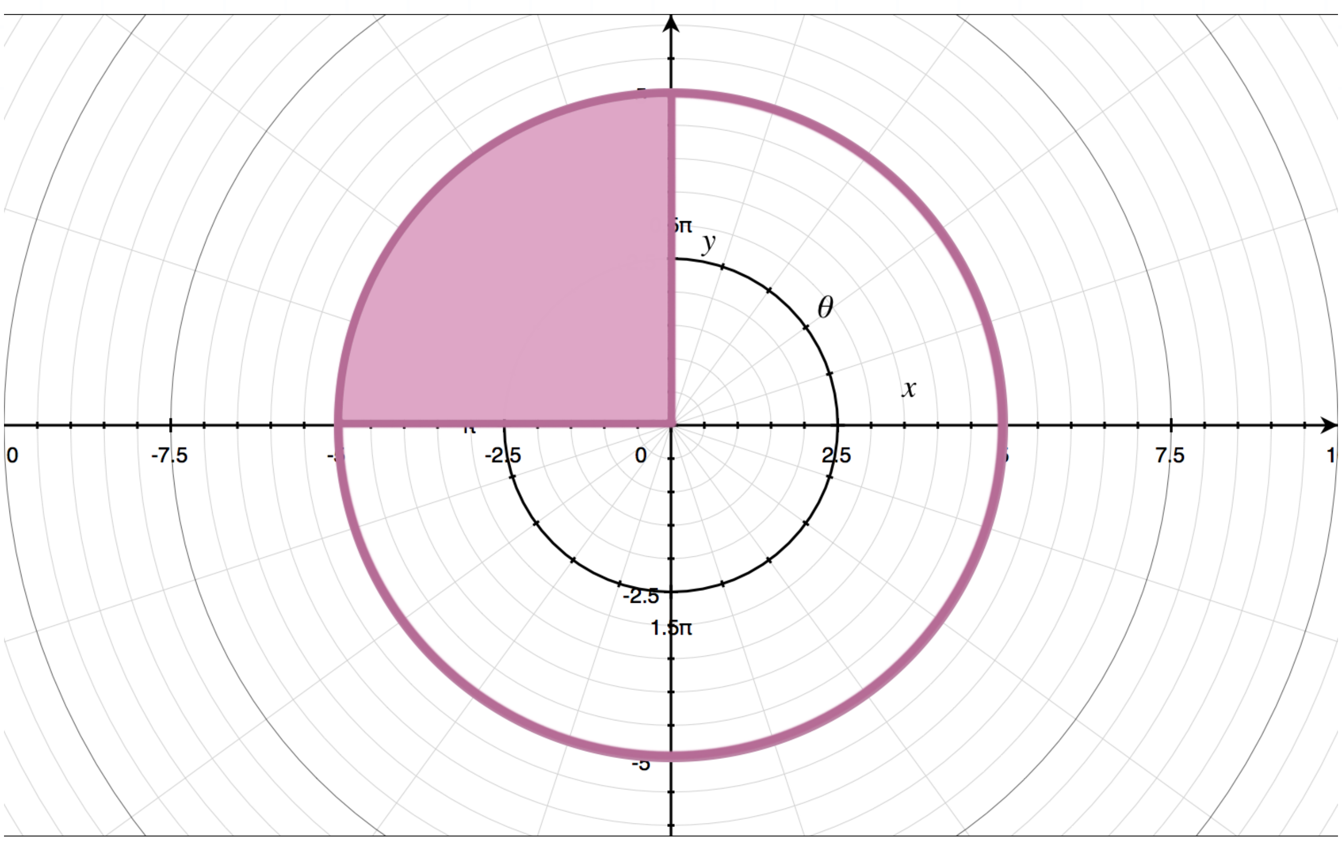
CHANGING ITERATED INTEGRALS TO POLAR COORDINATES

1. Convert the iterated integral to polar coordinates, then find its value.

$$\int_{-5}^{0}\int_{0}^{\sqrt{25-x^2}} xy\ dy\ dx$$

Solution:

The double integral shows that x is bounded on $x=[-5,0]$ while y is bounded on 0 to $\sqrt{25-x^2}$. So the area of integration is the part of the circle with center at the origin and radius 5, that lies in the second quadrant.



On this region, r is bounded on $r=[0,5]$ and θ is bounded on $\theta=[\pi/2,\pi]$. Converting the function to polar coordinates gives

$$xy$$

$$r\cos\theta\cdot r\sin\theta$$

$$r^2\cos\theta\sin\theta$$

So the integral in polar coordinates is

$$\int_{\pi/2}^{\pi}\int_0^5 r^2\cos\theta\sin\theta\ r\ dr\ d\theta$$

$$\int_{\pi/2}^{\pi}\int_0^5 r^3\cos\theta\sin\theta\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_{\pi/2}^{\pi}\frac{1}{4}r^4\cos\theta\sin\theta\Big|_{r=0}^{r=5}\ d\theta$$

$$\int_{\pi/2}^{\pi}\frac{1}{4}(5)^4\cos\theta\sin\theta-\frac{1}{4}(0)^4\cos\theta\sin\theta\ d\theta$$

$$\int_{\pi/2}^{\pi}\frac{625}{4}\cos\theta\sin\theta\ d\theta$$

Integrate with respect to θ, using a substitution with

$$u=\sin\theta$$

$\dfrac{du}{d\theta}=\cos\theta$, so $du=\cos\theta\ d\theta$ and $d\theta=\dfrac{du}{\cos\theta}$

The bounds $\theta=[\pi/2,\pi]$ become $u=[1,0]$, so the integral can be rewritten as

$$\int_1^0 \frac{625}{4}(\cos\theta)u\left(\frac{du}{\cos\theta}\right)$$

$$\int_1^0 \frac{625}{4}u\ du$$

Integrate and evaluate.

$$\frac{625}{8}u^2\Big|_1^0$$

$$\frac{625}{8}(0)^2 - \frac{625}{8}(1)^2$$

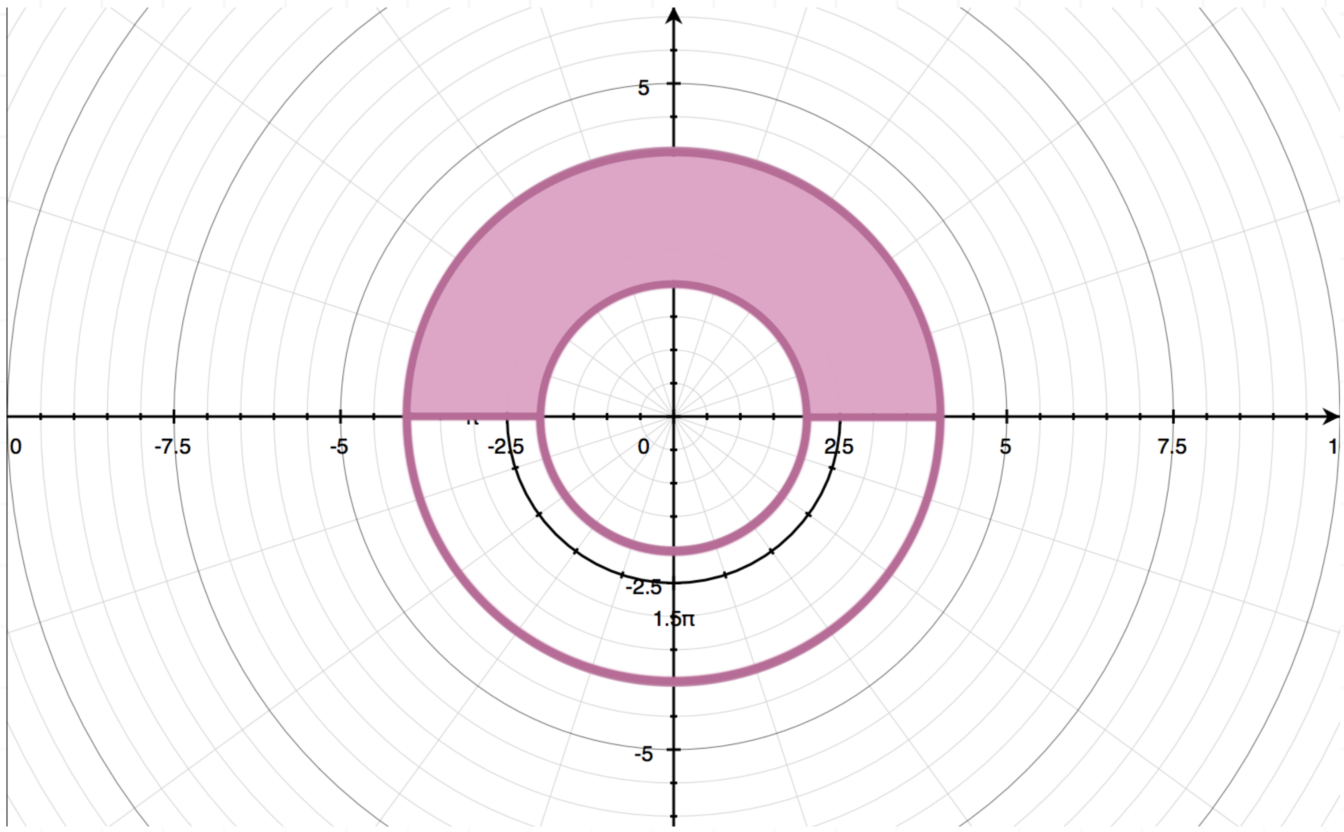
$$-\frac{625}{8}$$

■ 2. Convert the sum of iterated integrals to polar coordinates, then find its value.

$$\int_{-4}^{-2}\int_0^{\sqrt{16-x^2}} x-y\ dy\ dx + \int_{-2}^{2}\int_{\sqrt{4-x^2}}^{\sqrt{16-x^2}} x-y\ dy\ dx + \int_2^4\int_0^{\sqrt{16-x^2}} x-y\ dy\ dx$$

Solution:

If we consider the bounds on x and y from each integral, we can see that the region of integration is the area between the circles with centers at the origin and radii 2 and 4, in only the first and second quadrants.



In polar coordinates, that means the region is defined on $r = [2,4]$ and $\theta = [0,\pi]$. And the function converts to

$$x - y$$

$$r\cos\theta - r\sin\theta$$

$$r(\cos\theta - \sin\theta)$$

Then the integral in polar coordinates is

$$\int_0^\pi \int_2^4 r(\cos\theta - \sin\theta)r \ dr \ d\theta$$

$$\int_0^\pi \int_2^4 r^2(\cos\theta - \sin\theta) \ dr \ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{\pi} \frac{1}{3}r^3(\cos\theta - \sin\theta)\Big|_{r=2}^{r=4} \ d\theta$$

$$\int_0^{\pi} \frac{1}{3}(4)^3(\cos\theta - \sin\theta) - \frac{1}{3}(2)^3(\cos\theta - \sin\theta) \ d\theta$$

$$\int_0^{\pi} \frac{64}{3}(\cos\theta - \sin\theta) - \frac{8}{3}(\cos\theta - \sin\theta) \ d\theta$$

$$\frac{56}{3}\int_0^{\pi} \cos\theta - \sin\theta \ d\theta$$

Integrate with respect to θ.

$$\frac{56}{3}(\sin\theta + \cos\theta)\Big|_0^{\pi}$$

$$\frac{56}{3}(\sin\pi + \cos\pi) - \frac{56}{3}(\sin(0) + \cos(0))$$

$$\frac{56}{3}(0 + (-1)) - \frac{56}{3}(0 + 1)$$

$$-\frac{56}{3} - \frac{56}{3}$$

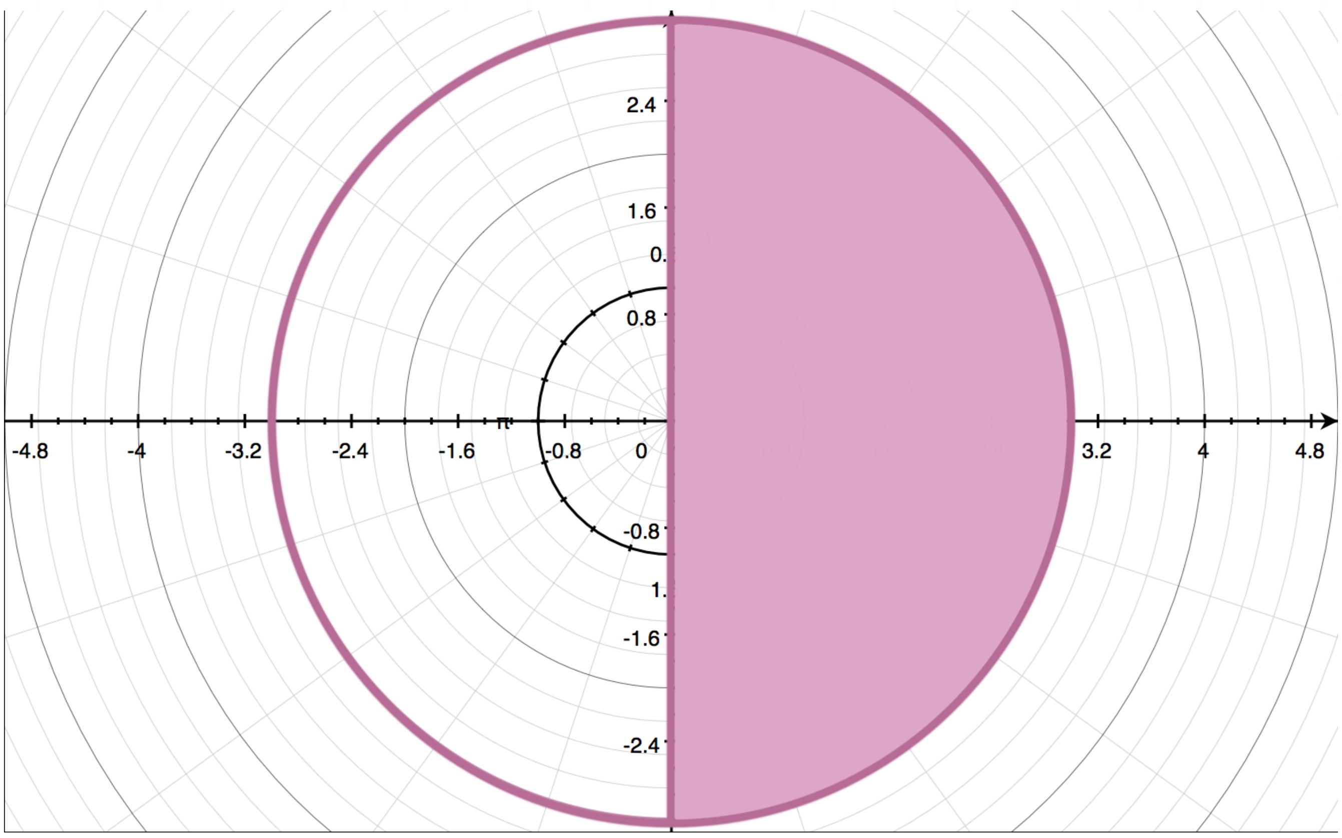
$$-\frac{112}{3}$$

■ 3. Convert the iterated integral to polar coordinates, then find its value.

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-y^2}} \ln(x^2+y^2)\ dx\ dy$$

Solution:

If we consider the bounds on x and y from the double integral, we can see that the region of integration is the part of the circle with center at the origin and radius 3, that lies in only the first and fourth quadrants.



In polar coordinates, that means the region is defined on $r=[0,3]$ and $\theta=[-\pi/2,\pi/2]$. And the function converts to

$$\ln(x^2+y^2)=\ln(r^2)=2\ln r$$

Then the integral in polar coordinates is

$$\int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\ln r)\ r\ d\theta\ dr$$

$$\int_0^3 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2r\ln r\ d\theta\ dr$$

Integrate with respect to θ, treating r as a constant.

$$\int_0^3 2r\theta\ln r\Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\ dr$$

$$\int_0^3 2r\left(\frac{\pi}{2}\right)\ln r - 2r\left(-\frac{\pi}{2}\right)\ln r\ dr$$

$$\int_0^3 r\pi\ln r + r\pi\ln r\ dr$$

$$\int_0^3 2\pi r\ln r\ dr$$

Integrate with respect to r, using integration by parts.

$u = \ln r$ with $du = \dfrac{1}{r}\ dr$

$dv = 2\pi r$ with $v = \pi r^2$

Then the integral becomes

$$\pi r^2\ln r\Big|_0^3 - \int_0^3 \pi r^2\left(\frac{1}{r}\ dr\right)$$

$$\pi r^2 \ln r \Big|_0^3 - \int_0^3 \pi r \ dr$$

$$\pi r^2 \ln r \Big|_0^3 - \left[\frac{1}{2}\pi r^2 \Big|_0^3 \right]$$

$$\pi r^2 \ln r - \frac{1}{2}\pi r^2 \Big|_0^3$$

$$\pi(3)^2 \ln 3 - \frac{1}{2}\pi(3)^2 - \left(\pi(0)^2 \ln 0 - \frac{1}{2}\pi(0)^2 \right)$$

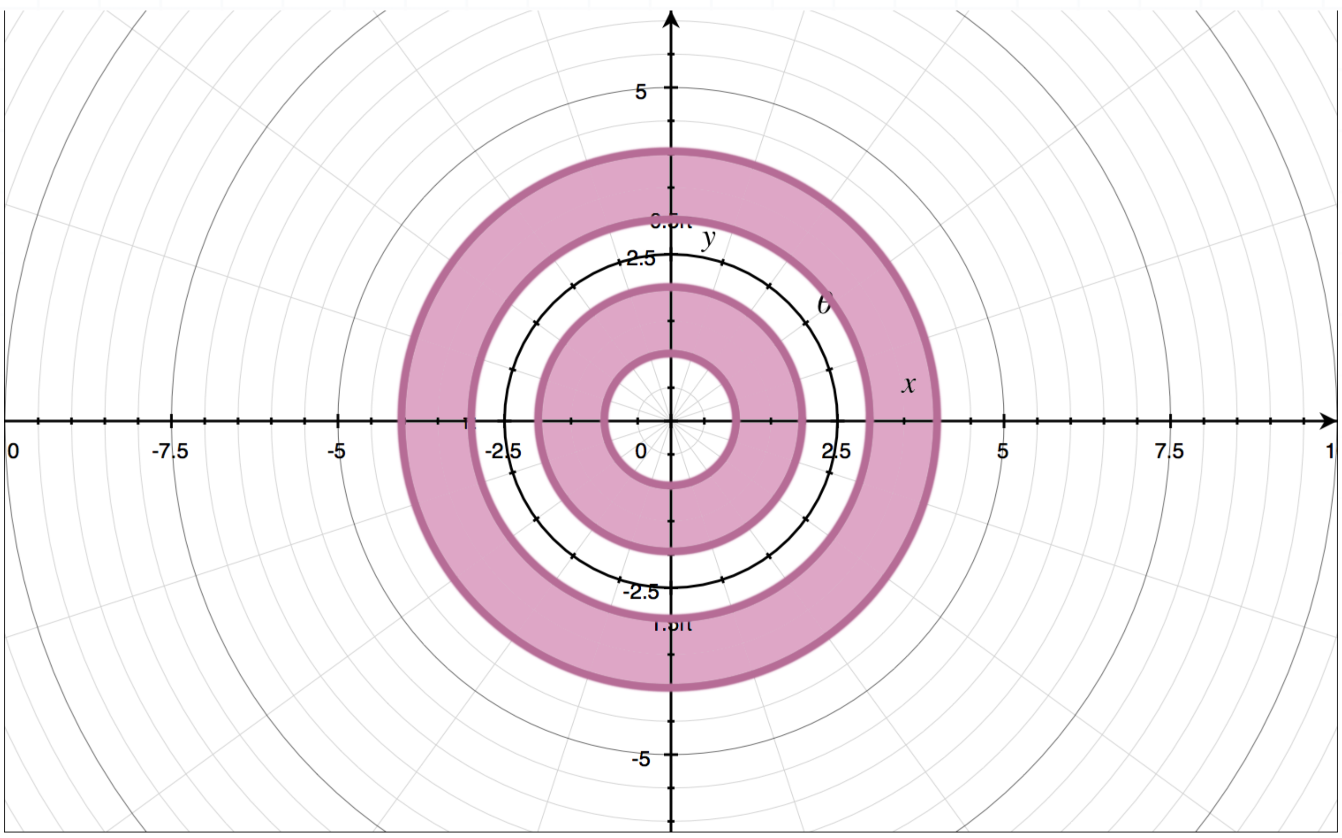
$$9\pi \ln 3 - \frac{9}{2}\pi - (0 - 0)$$

$$9\pi \ln 3 - \frac{9}{2}\pi$$

CHANGING DOUBLE INTEGRALS TO POLAR COORDINATES

■ 1. The region D consists of two rings centered at the origin, where the inner ring is defined on $r = [1,2]$ and the outer ring is defined on $r = [3,4]$. Convert the double integral to polar coordinates, then find its value.

$$\iint_D x + 2y \; dA$$



Solution:

We'll use separate integrals for each ring, but first, we'll convert the function.

$$x + 2y$$

$$r\cos\theta + 2r\sin\theta$$

$$r(\cos\theta + 2\sin\theta)$$

For each ring, θ is defined on $\theta = [0,2\pi]$, so the entire region can be integrated by

$$\int_0^{2\pi}\int_1^2 r(\cos\theta + 2\sin\theta)r\ dr\ d\theta + \int_0^{2\pi}\int_3^4 r(\cos\theta + 2\sin\theta)r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_1^2 r^2(\cos\theta + 2\sin\theta)\ dr\ d\theta + \int_0^{2\pi}\int_3^4 r^2(\cos\theta + 2\sin\theta)\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{2\pi} \frac{1}{3}r^3(\cos\theta + 2\sin\theta)\Big|_{r=1}^{r=2}\ d\theta + \int_0^{2\pi} \frac{1}{3}r^3(\cos\theta + 2\sin\theta)\Big|_{r=3}^{r=4}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{3}(2)^3(\cos\theta + 2\sin\theta) - \frac{1}{3}(1)^3(\cos\theta + 2\sin\theta)\ d\theta$$

$$+\int_0^{2\pi} \frac{1}{3}(4)^3(\cos\theta + 2\sin\theta) - \frac{1}{3}(3)^3(\cos\theta + 2\sin\theta)\ d\theta$$

$$\int_0^{2\pi} \frac{8}{3}(\cos\theta + 2\sin\theta) - \frac{1}{3}(\cos\theta + 2\sin\theta)\ d\theta$$

$$+\int_0^{2\pi} \frac{64}{3}(\cos\theta + 2\sin\theta) - \frac{27}{3}(\cos\theta + 2\sin\theta)\ d\theta$$

$$\int_0^{2\pi} \frac{7}{3}(\cos\theta + 2\sin\theta)\ d\theta + \int_0^{2\pi} \frac{37}{3}(\cos\theta + 2\sin\theta)\ d\theta$$

$$\int_0^{2\pi} \frac{7}{3}\cos\theta + \frac{14}{3}\sin\theta \ d\theta + \int_0^{2\pi} \frac{37}{3}\cos\theta + \frac{74}{3}\sin\theta \ d\theta$$

$$\int_0^{2\pi} \frac{7}{3}\cos\theta + \frac{14}{3}\sin\theta + \frac{37}{3}\cos\theta + \frac{74}{3}\sin\theta \ d\theta$$

$$\int_0^{2\pi} \frac{88}{3}\sin\theta + \frac{44}{3}\cos\theta \ d\theta$$

Integrate with respect to θ.

$$-\frac{88}{3}\cos\theta + \frac{44}{3}\sin\theta\Big|_0^{2\pi}$$

$$\frac{44}{3}\sin\theta - \frac{88}{3}\cos\theta\Big|_0^{2\pi}$$

$$\frac{44}{3}\sin(2\pi) - \frac{88}{3}\cos(2\pi) - \left(\frac{44}{3}\sin(0) - \frac{88}{3}\cos(0)\right)$$

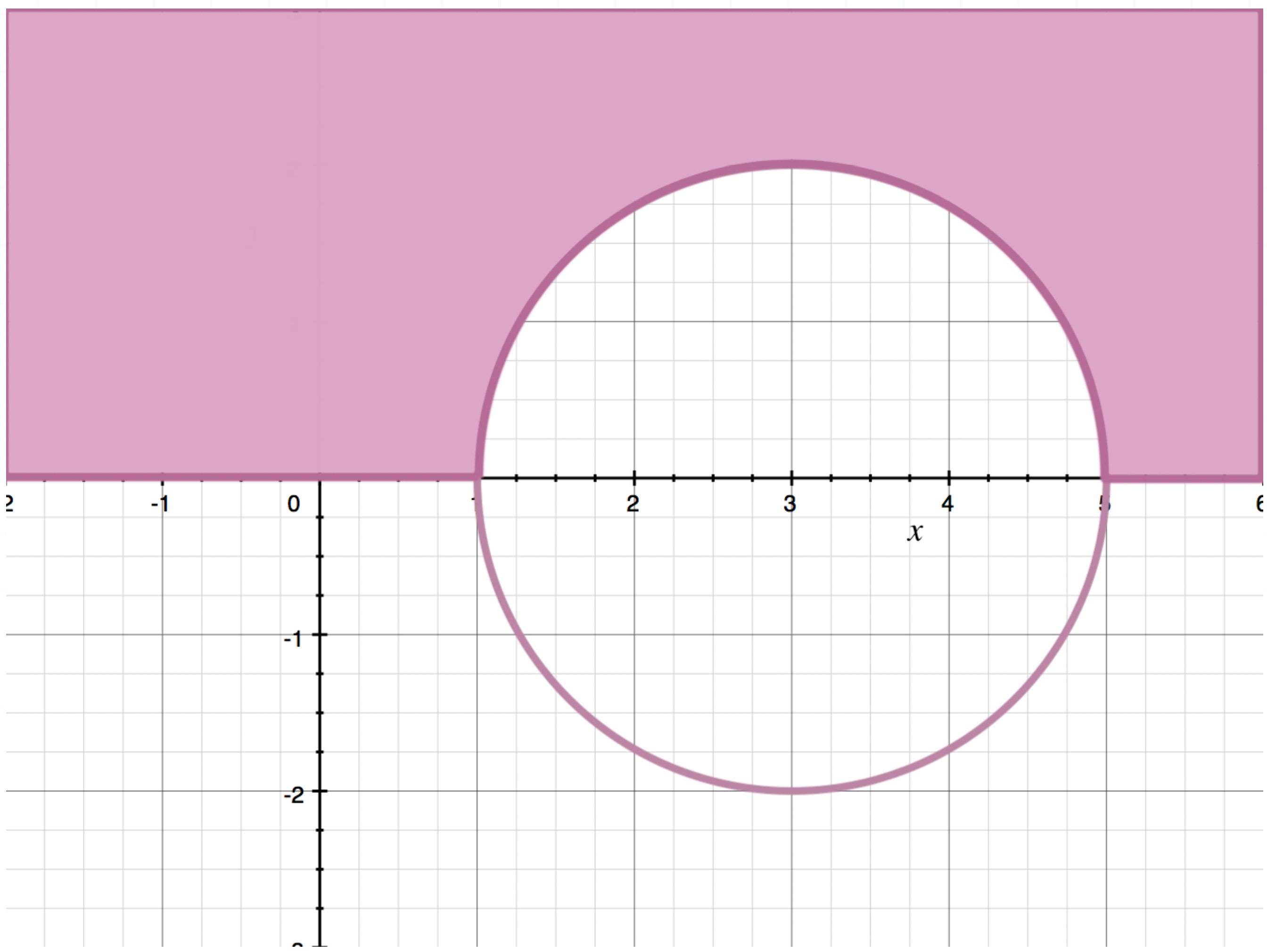
$$\frac{44}{3}(0) - \frac{88}{3}(1) - \left(\frac{44}{3}(0) - \frac{88}{3}(1)\right)$$

$$-\frac{88}{3} + \frac{88}{3}$$

$$0$$

■ 2. The region D consists of all the points in the first and second quadrants outside the circle centered at $(3,0)$ with radius $r = 2$. Convert the double integral to polar coordinates, using the conversion formulas $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$ for a circle shifted off the origin, then find its value.

$$\iint_D \frac{1}{((x-3)^2+y^2)^2}\ dA$$



Solution:

Applying $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$, the conversion formulas become $x = 3 + r\cos\theta$ and $y = r\sin\theta$. Using these, we'll convert the function to polar coordinates.

$$\frac{1}{((x-3)^2+y^2)^2}$$

$$\frac{1}{((3+r\cos\theta-3)^2+(r\sin\theta)^2)^2}$$

$$\frac{1}{((r\cos\theta)^2+(r\sin\theta)^2)^2}$$

$$\frac{1}{(r^2\cos^2\theta+r^2\sin^2\theta)^2}$$

$$\frac{1}{(r^2(\cos^2\theta+\sin^2\theta))^2}$$

$$\frac{1}{(r^2(1))^2}$$

$$\frac{1}{(r^2)^2}$$

$$\frac{1}{r^4}$$

The integral in polar coordinates is therefore

$$\int_0^{\pi}\int_2^{\infty}\frac{1}{r^4}r\ dr\ d\theta$$

$$\int_0^{\pi}\int_2^{\infty}\frac{1}{r^3}\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{\pi}-\frac{1}{2r^2}\Big|_{r=2}^{r=\infty}\ d\theta$$

$$\int_0^\pi -\frac{1}{2(\infty)^2} - \left(-\frac{1}{2(2)^2}\right) \ d\theta$$

$$\int_0^\pi 0 + \frac{1}{2(4)} \ d\theta$$

$$\int_0^\pi \frac{1}{8} \ d\theta$$

Integrate with respect to θ.

$$\frac{1}{8}\theta\Big|_0^\pi$$

$$\frac{1}{8}\pi - \frac{1}{8}(0)$$

$$\frac{\pi}{8}$$

■ 3. The region D consists of all the points inside the circle centered at $(-2,2)$ with radius $r = 1$. Convert the double integral to polar coordinates, using the conversion formulas $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$ for a circle shifted off the origin, then find its value.

$$\iint_D x^2 + y^2 \ dA$$

Solution:

Applying $x = x_0 + r\cos\theta$ and $y = y_0 + r\sin\theta$, the conversion formulas become $x = -2 + r\cos\theta$ and $y = 2 + r\sin\theta$. Using these, we'll convert the function to polar coordinates.

$$x^2 + y^2$$

$$(-2 + r\cos\theta)^2 + (2 + r\sin\theta)^2$$

$$4 - 4r\cos\theta + r^2\cos^2\theta + 4 + 4r\sin\theta + r^2\sin^2\theta$$

$$8 + 4r(\sin\theta - \cos\theta) + r^2(\cos^2\theta + \sin^2\theta)$$

$$8 + 4r(\sin\theta - \cos\theta) + r^2(1)$$

$$8 + 4r(\sin\theta - \cos\theta) + r^2$$

Therefore, the integral in polar coordinates is:

$$\int_0^{2\pi}\int_0^1 (8 + 4r(\sin\theta - \cos\theta) + r^2)r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 8r + 4r^2(\sin\theta - \cos\theta) + r^3\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{2\pi} 4r^2 + \frac{4}{3}r^3(\sin\theta - \cos\theta) + \frac{1}{4}r^4\Big|_{r=0}^{r=1}\ d\theta$$

$$\int_0^{2\pi} 4(1)^2 + \frac{4}{3}(1)^3(\sin\theta - \cos\theta) + \frac{1}{4}(1)^4$$

$$-\left(4(0)^2+\frac{4}{3}(0)^3(\sin\theta-\cos\theta)+\frac{1}{4}(0)^4\right)\ d\theta$$

$$\int_0^{2\pi} 4+\frac{4}{3}(\sin\theta-\cos\theta)+\frac{1}{4}\ d\theta$$

$$\int_0^{2\pi} \frac{17}{4}+\frac{4}{3}\sin\theta-\frac{4}{3}\cos\theta\ d\theta$$

Integrate with respect to θ.

$$\frac{17}{4}\theta-\frac{4}{3}\cos\theta-\frac{4}{3}\sin\theta\Big|_0^{2\pi}$$

$$\frac{17}{4}(2\pi)-\frac{4}{3}\cos(2\pi)-\frac{4}{3}\sin(2\pi)-\left(\frac{17}{4}(0)-\frac{4}{3}\cos(0)-\frac{4}{3}\sin(0)\right)$$

$$\frac{17\pi}{2}-\frac{4}{3}(1)-\frac{4}{3}(0)+\frac{4}{3}(1)+\frac{4}{3}(0)$$

$$\frac{17\pi}{2}-\frac{4}{3}+\frac{4}{3}$$

$$\frac{17\pi}{2}$$

SKETCHING AREA

1. Identify the region of integration given by the double integral.

$$\int_0^{2\pi}\int_0^{\frac{3}{\sqrt{1+1.25\cos^2\theta}}} f(r,\theta)\ dr\ d\theta$$

Solution:

Rewrite the upper bound on r.

$$r=\frac{3}{\sqrt{1+1.25\cos^2\theta}}$$

$$r^2=\frac{9}{1+1.25\cos^2\theta}$$

$$r^2(1+1.25\cos^2\theta)=9$$

$$r^2+1.25r^2\cos^2\theta=9$$

$$r^2+1.25(r\cos\theta)^2=9$$

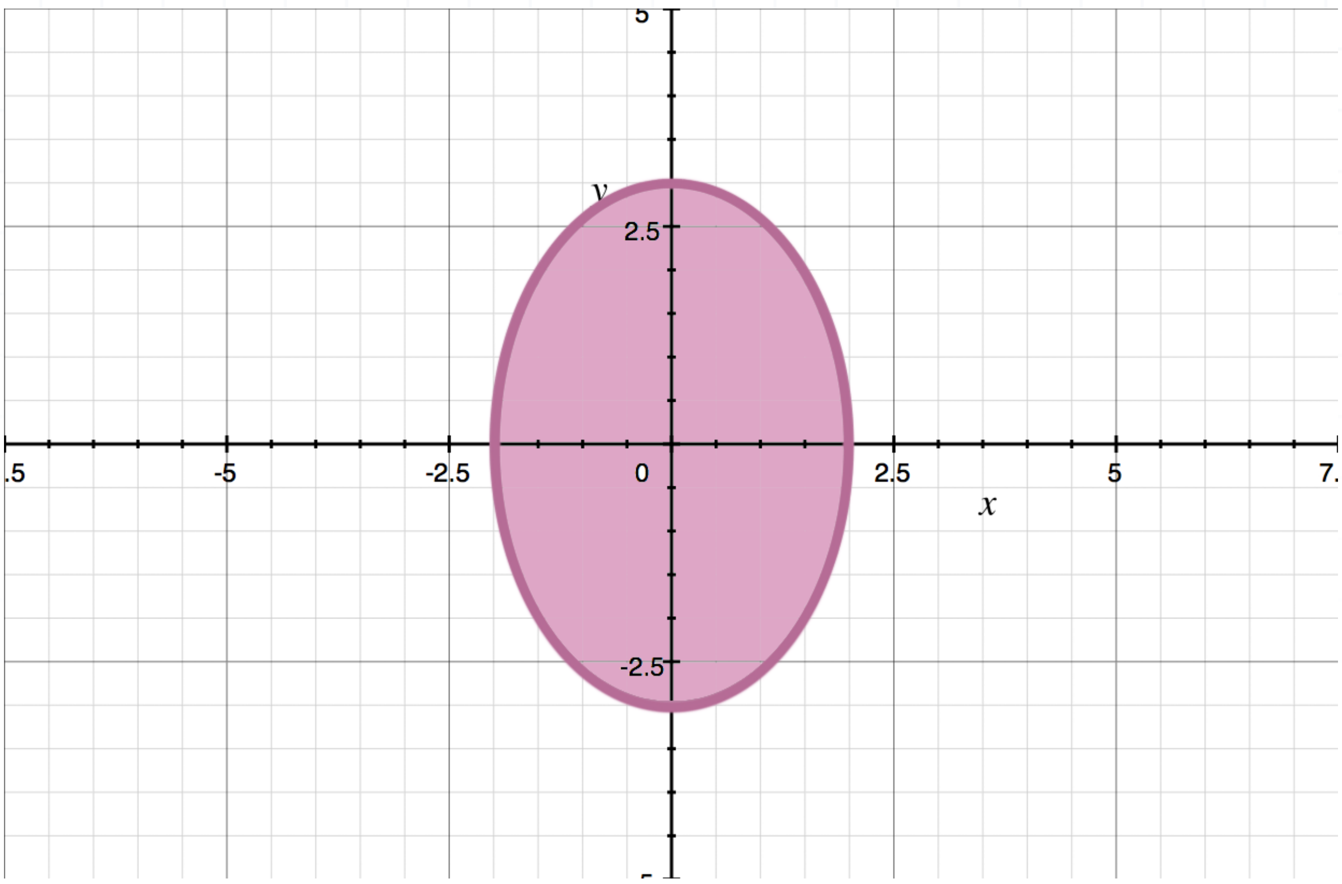
Substitute $r^2=x^2+y^2$ and $x=r\cos\theta$ to convert this upper bound into rectangular coordinates.

$$x^2+y^2+\frac{5}{4}x^2=9$$

$$\frac{9}{4}x^2 + y^2 = 9$$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

We see now that the upper bound on r is the ellipse centered at the origin with horizontal radius 2 and vertical radius 3.



Going back to the original integral, r is bounded below at 0 and above by this ellipse, and θ is bounded on $[0,2\pi]$, which means the region of integration is this entire ellipse.

■ 2. Identify the region of integration given by the double integral.

$$\int_0^5 \int_0^{\frac{1}{3}\cos^{-1}\left(\frac{r}{5}\right)} f(r,\theta)\ d\theta\ dr$$

Solution:

Rewrite the upper bound on θ into rectangular coordinates.

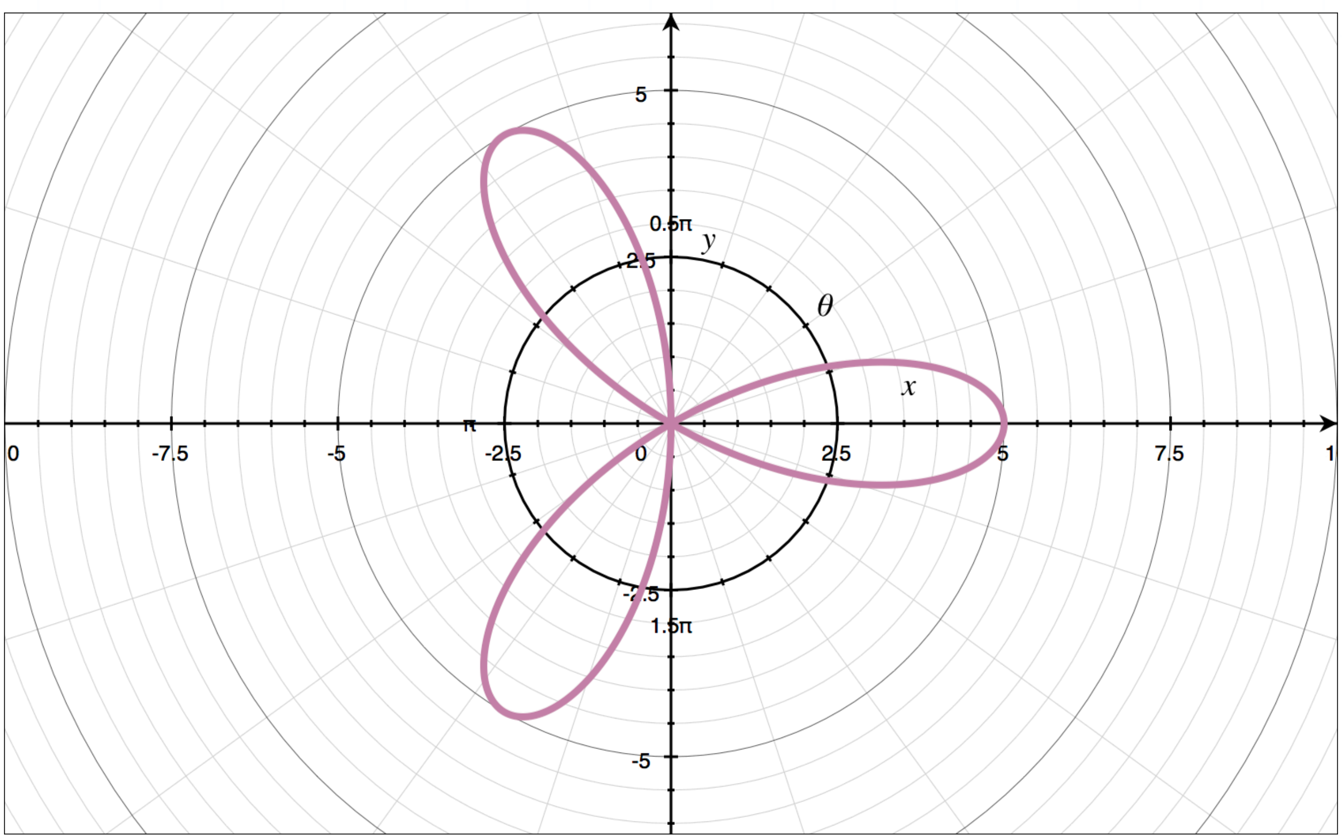
$$\theta = \frac{1}{3}\cos^{-1}\left(\frac{r}{5}\right)$$

$$3\theta = \cos^{-1}\left(\frac{r}{5}\right)$$

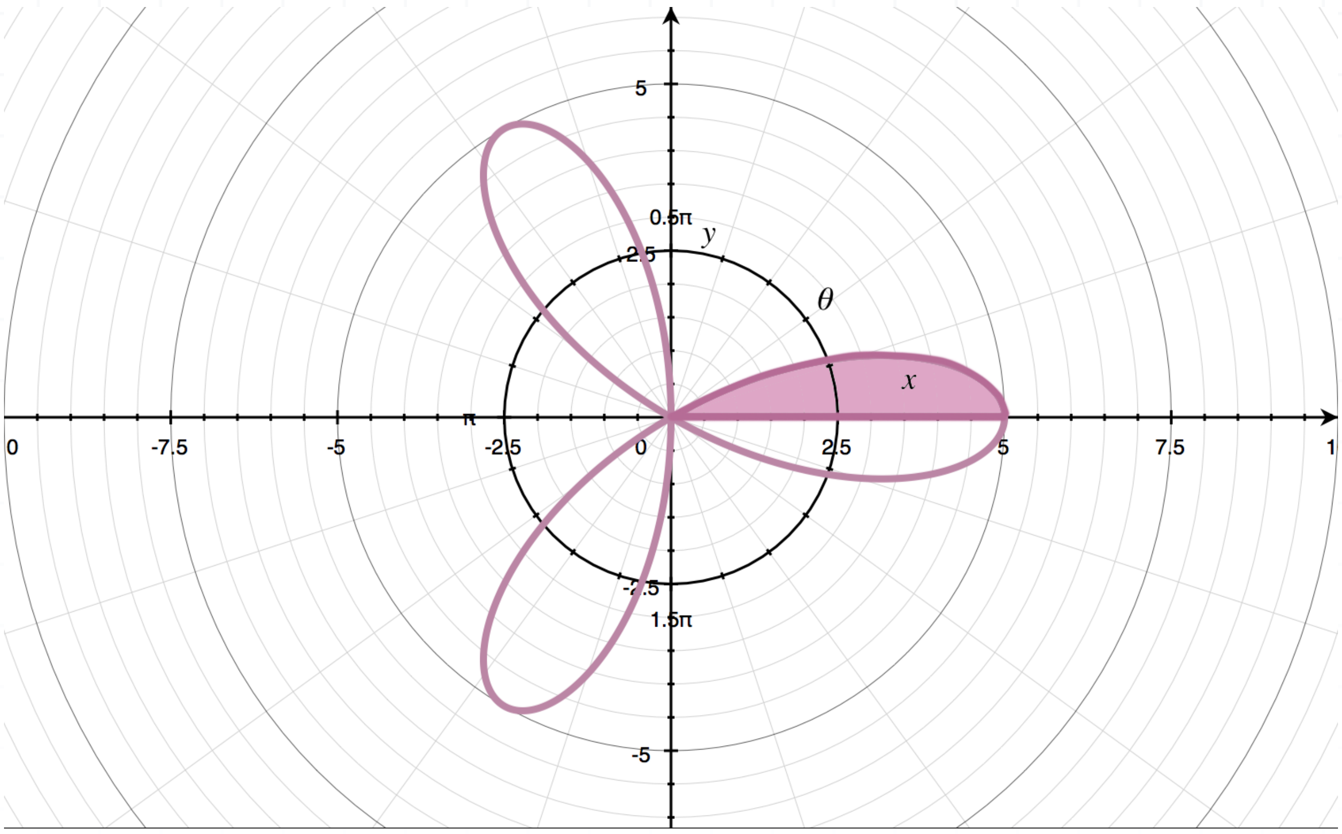
$$\cos(3\theta) = \frac{r}{5}$$

$$r = 5\cos(3\theta)$$

This polar curve is a three-petaled rose whose petals extend to $r = 5$.



Because the interval over which θ is defined is $\theta = [0{,}5\cos(3\theta)]$, we'll start at the angle $\theta = 0$ and rotate counterclockwise until we run into the curve. Therefore, we can say that the region of integration is defined as just the upper-half of the first petal.



■ 3. Identify the region of integration given by the double integral.

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{2\cos\theta}^{4\cos\theta} f(r,\theta)\ dr\ d\theta$$

Solution:

The region is defined on $r = 2\cos\theta$ to $r = 4\cos\theta$, so we'll convert these bounds using $x^2 + y^2 = r^2$ and $x = r\cos\theta$.

$$r = 2\cos\theta$$

$$\sqrt{x^2+y^2} = 2\frac{x}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 2x$$

$$x^2-2x+y^2 = 0$$

Complete the square.

$$x^2-2x+1-1+y^2 = 0$$

$$(x-1)^2-1+y^2 = 0$$

$$(x-1)^2+y^2 = 1$$

So the lower bound on r is the circle centered at $(1,0)$ with radius 1. Now convert the upper bound.

$$r = 4\cos\theta$$

$$\sqrt{x^2+y^2} = 4\frac{x}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 4x$$

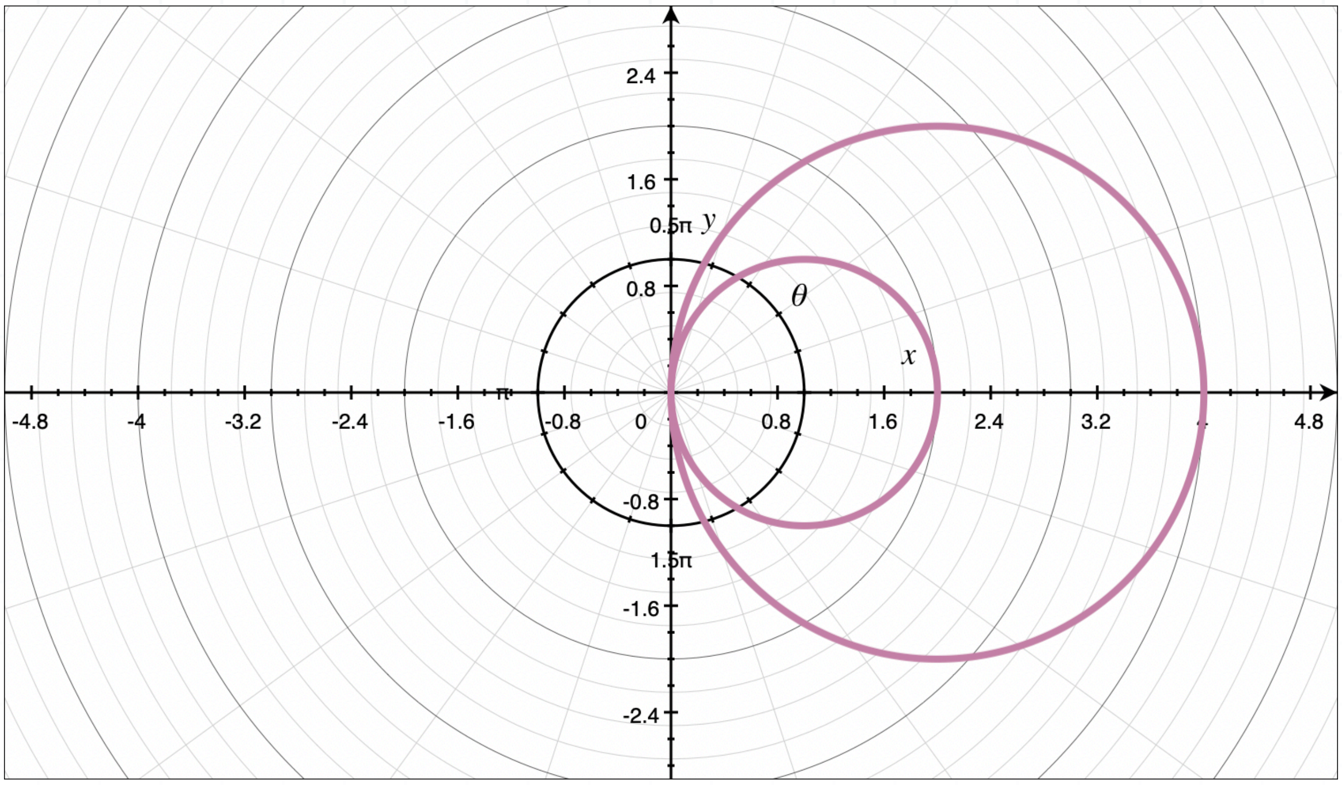
$$x^2-4x+y^2 = 0$$

$$x^2-4x+4-4+y^2 = 0$$

$$(x-2)^2-4+y^2 = 0$$

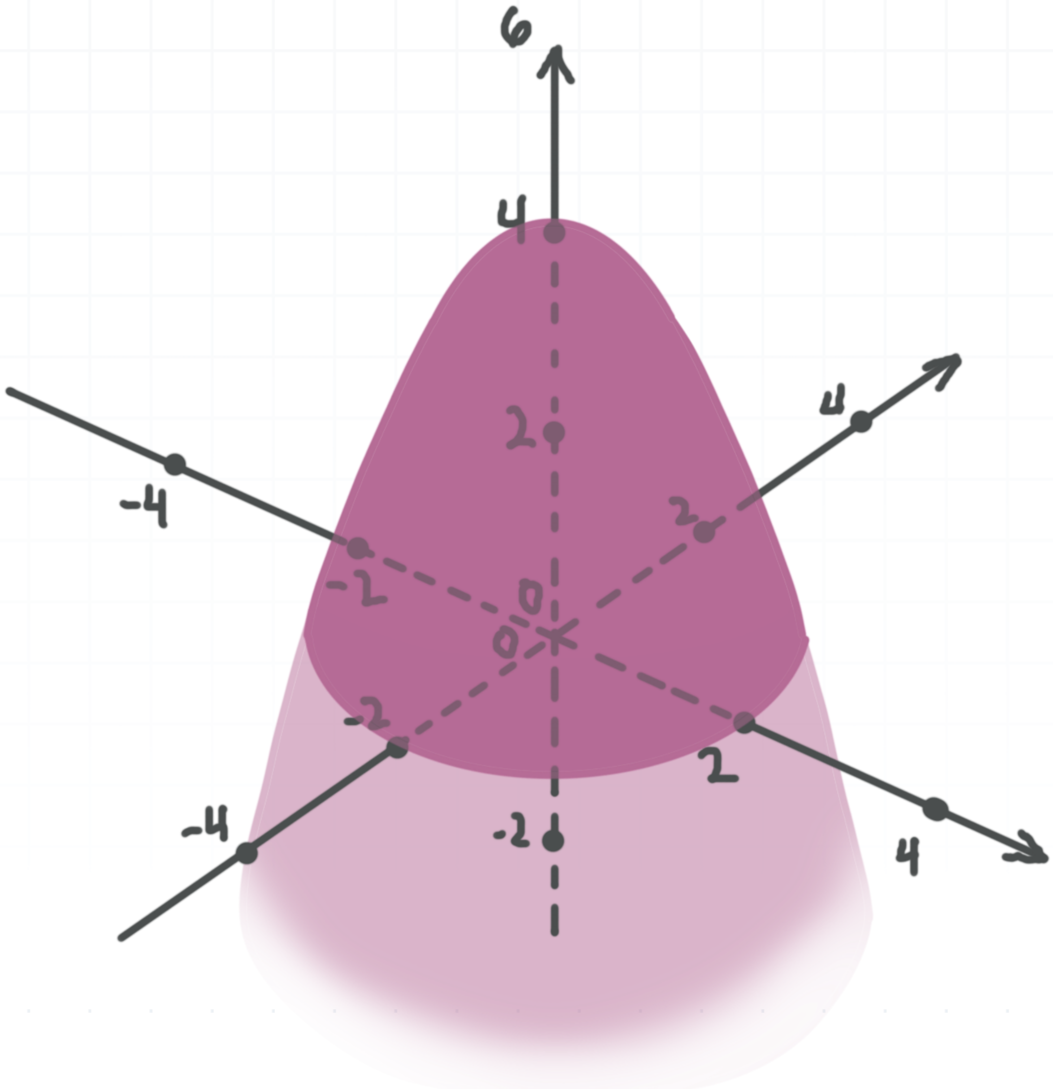
$$(x-2)^2+y^2 = 4$$

So the upper bound on r is the circle centered at $(2,0)$ with radius 2, and the region of integration is the area between $(x-1)^2+y^2=1$ and $(x-2)^2+y^2=4$.



FINDING AREA

■ 1. Find area of the surface $x^2 + y^2 + z - 4 = 0$ above the xy-plane.



Solution:

Determinate the curve of intersection of the surface $x^2 + y^2 + z - 4 = 0$ and the xy-plane $z = 0$.

$$x^2 + y^2 + z - 4 = 0$$

$$x^2 + y^2 + 0 = 4$$

$$x^2 + y^2 = 2^2$$

The curve of intersection is the circle with the center at the origin and radius 2, which means x is defined on $x = [-2,2]$ and y is defined on $y = [-\sqrt{4 - x^2}, \sqrt{4 - x^2}]$. The function is $z = 4 - x^2 - y^2$ and its partial derivatives are

$$z_x = -2x$$

$$z_y = -2y$$

So the area of the surface is

$$\iint_D \sqrt{1 + z_x^2 + z_y^2}\ dA$$

$$\int_{-2}^{2} \int_{-\sqrt{4 - x^2}}^{\sqrt{4 - x^2}} \sqrt{1 + 4x^2 + 4y^2}\ dy\ dx$$

$$\int_{-2}^{2} \int_{-\sqrt{4 - x^2}}^{\sqrt{4 - x^2}} \sqrt{1 + 4(x^2 + y^2)}\ dy\ dx$$

Convert to polar coordinates, remembering that the bounds define the circle $x^2 + y^2 = 2^2$, which means r is defined on $r = [0,2]$ and θ is defined on $\theta = [0,2\pi]$.

$$\int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2}\ r\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant, by using a substitution.

$$u = 4r^2$$

$$\frac{du}{dr} = 8r \text{, so } du = 8r\ dr \text{ and } dr = \frac{du}{8r}$$

The bounds $r = [0,2]$ convert to $u = [0,16]$.

$$\int_0^{2\pi} \int_0^{16} \sqrt{1+u}\ r \left(\frac{du}{8r}\right)\ d\theta$$

$$\int_0^{2\pi} \int_0^{16} \frac{1}{8}\sqrt{1+u}\ du\ d\theta$$

$$\int_0^{2\pi} \int_0^{16} \frac{1}{8}(1+u)^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u.

$$\int_0^{2\pi} \frac{1}{8} \cdot \frac{2}{3}(1+u)^{\frac{3}{2}}\Big|_0^{16}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{12}(1+u)^{\frac{3}{2}}\Big|_0^{16}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{12}(1+16)^{\frac{3}{2}} - \frac{1}{12}(1+0)^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} \frac{1}{12}(17)^{\frac{3}{2}} - \frac{1}{12}(1)^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} \frac{17^{\frac{3}{2}}}{12} - \frac{1}{12}\ d\theta$$

$$\int_0^{2\pi} \frac{17^{\frac{3}{2}} - 1}{12} \ d\theta$$

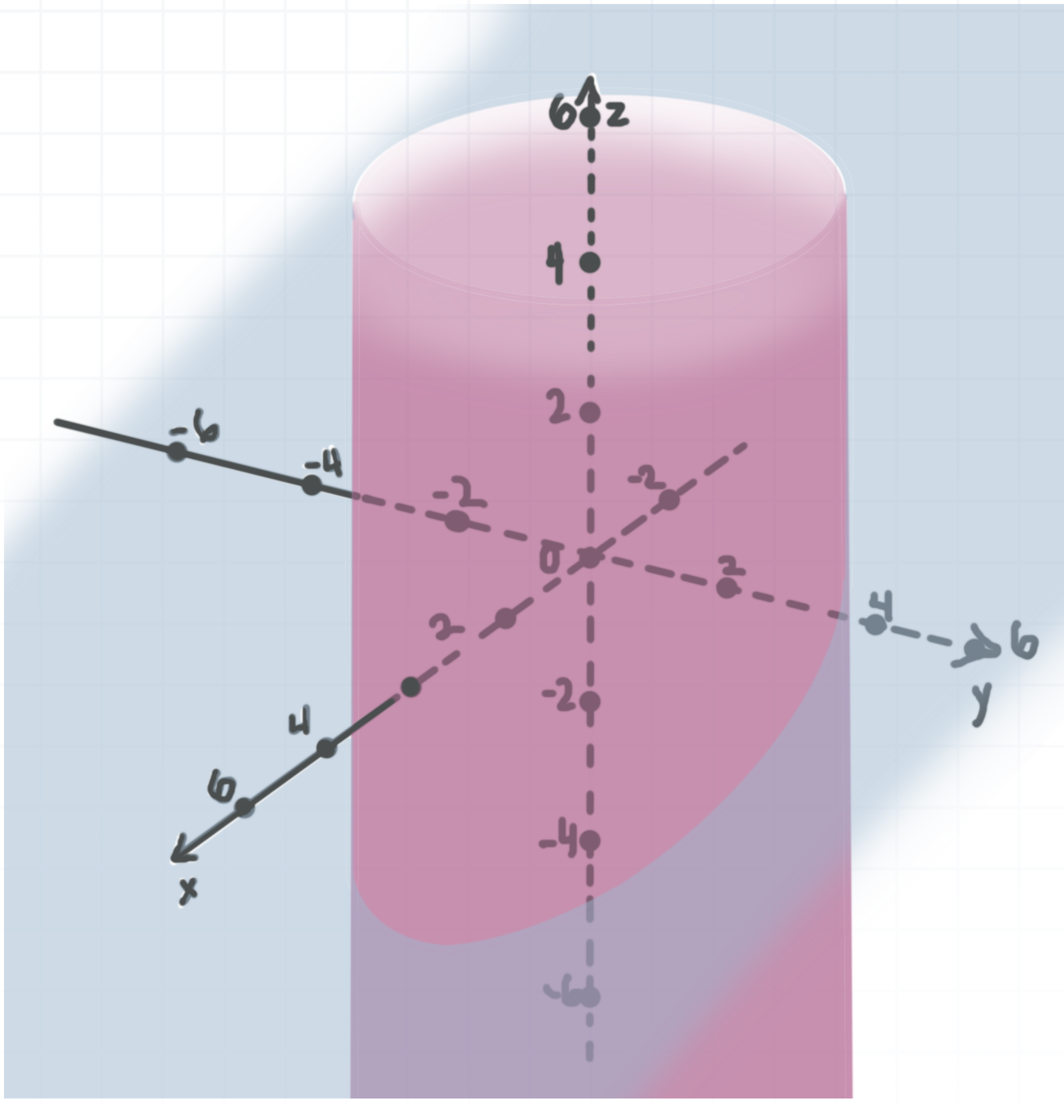
Integrate with respect to θ.

$$\frac{17^{\frac{3}{2}} - 1}{12}\theta \Big|_0^{2\pi}$$

$$\frac{17^{\frac{3}{2}} - 1}{12}(2\pi) - \frac{17^{\frac{3}{2}} - 1}{12}(0)$$

$$\frac{(17^{\frac{3}{2}} - 1)\pi}{6}$$

■ 2. Find area of the part of the plane $2x - y + 3z - 3 = 0$ that lies within the cylinder $(x - 3)^2 + (y - 2)^2 = 3^2$.



Solution:

The plane is

$$2x - y + 3z - 3 = 0$$

$$z = -\frac{2}{3}x + \frac{1}{3}y + 1$$

and its partial derivatives are

$$z_x = -\frac{2}{3}$$

$$z_y = \frac{1}{3}$$

So the area of the surface is

$$\iint_D \sqrt{1 + z_x^2 + z_y^2}\ dA$$

$$\iint_D \sqrt{1 + \frac{4}{9} + \frac{1}{9}}\ dA$$

$$\iint_D \sqrt{\frac{14}{9}}\ dA$$

$$\iint_D \frac{\sqrt{14}}{3}\ dA$$

The region of integration is the circle centered at $(3,2)$ with radius 3, so r is defined on $r = [0,3]$ and θ is defined on $[0,2\pi]$. Then the integral in polar coordinates is

$$\frac{\sqrt{14}}{3} \int_0^{2\pi} \int_0^3 r\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\frac{\sqrt{14}}{3} \int_0^{2\pi} \frac{1}{2}r^2 \Big|_{r=0}^{r=3}\ d\theta$$

$$\frac{\sqrt{14}}{3} \int_0^{2\pi} \frac{1}{2}(3)^2 - \frac{1}{2}(0)^2\ d\theta$$

$$\frac{\sqrt{14}}{3}\int_0^{2\pi}\frac{9}{2}\ d\theta$$

Integrate with respect to θ.

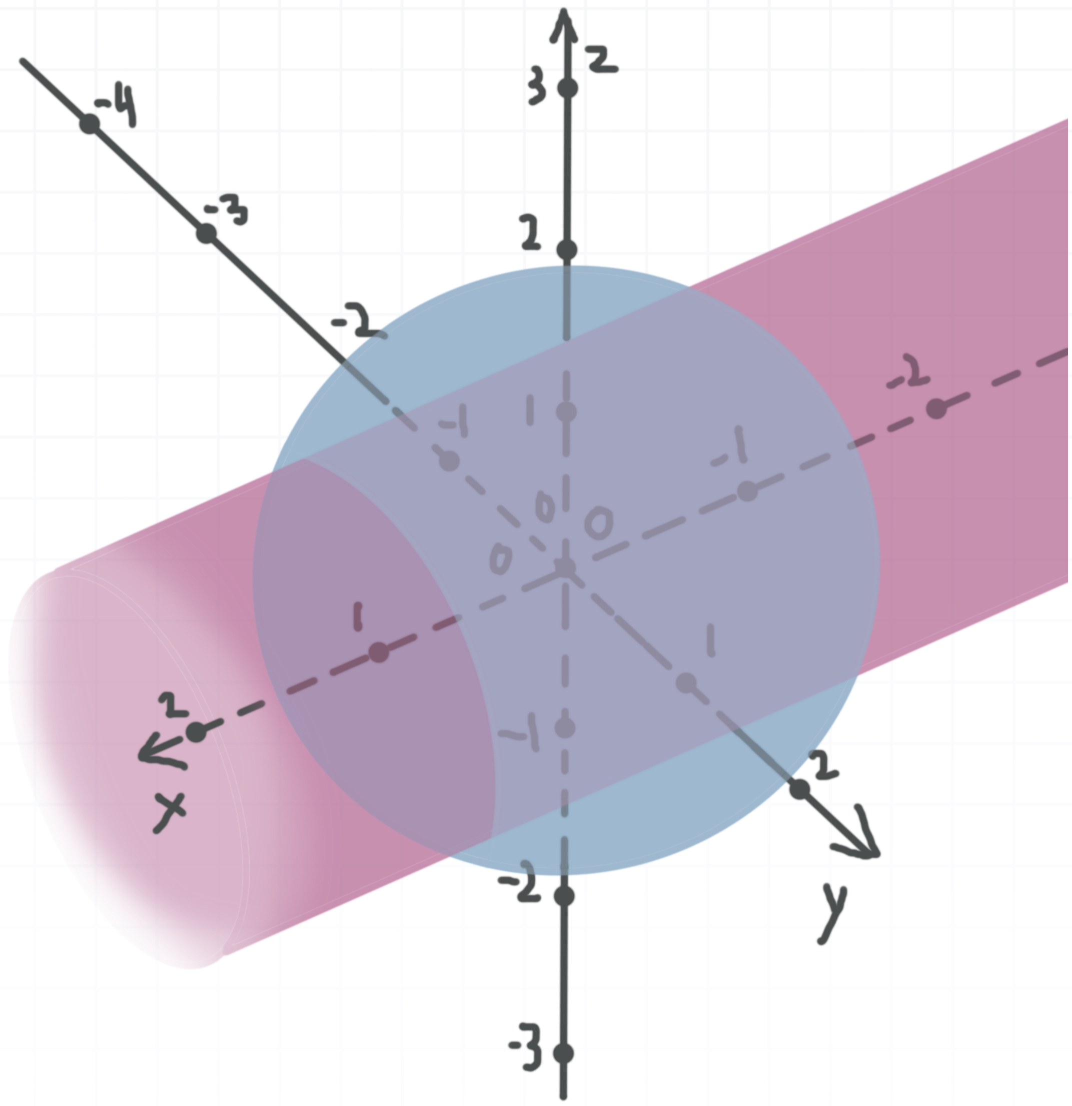
$$\frac{\sqrt{14}}{3}\left(\frac{9}{2}\theta\right)\Big|_0^{2\pi}$$

$$\frac{3\sqrt{14}}{2}\theta\Big|_0^{2\pi}$$

$$\frac{3\sqrt{14}}{2}(2\pi)-\frac{3\sqrt{14}}{2}(0)$$

$$3\sqrt{14}\pi$$

■ 3. Find area of the sphere $x^2+y^2+z^2-2=0$ that lies within the cylinder $y^2+z^2=1$.



Solution:

Since the area consists of two equal parts, lying on either side of the yz-plane, we'll calculate the area for $x > 0$, then double the result. Because the cylinder is parallel to the x-axis, we should use the function $x = f(y, z)$.

$$x^2 + y^2 + z^2 - 2 = 0$$

$$x^2 = 2 - y^2 - z^2$$

$$x = \sqrt{2 - y^2 - z^2}$$

Then the partial derivatives are

$$x_y = \frac{-y}{\sqrt{2 - y^2 - z^2}}$$

$$x_z = \frac{-z}{\sqrt{2 - y^2 - z^2}}$$

So the area of the surface is

$$\iint_D \sqrt{1 + z_x^2 + z_y^2}\ dA$$

$$\iint_D \sqrt{1 + \frac{y^2}{2 - y^2 - z^2} + \frac{z^2}{2 - y^2 - z^2}}\ dA$$

$$\iint_D \frac{\sqrt{(2 - y^2 - z^2) + y^2 + z^2}}{\sqrt{2 - y^2 - z^2}}\ dA$$

$$\iint_D \frac{\sqrt{2}}{\sqrt{2 - y^2 - z^2}}\ dA$$

$$\sqrt{2}\iint_D \frac{1}{\sqrt{2 - y^2 - z^2}}\ dA$$

If we use $r^2 = y^2 + z^2$ and $dy\ dz = r\ dr\ d\theta$ to convert the function, we get

$$\sqrt{2}\iint_D \frac{1}{\sqrt{2 - r^2}}\ dA$$

The region of integration is the circle centered at the origin with radius 1, so r is defined on $r=[0,1]$ and θ is defined on $[0,2\pi]$. Then the integral in polar coordinates is

$$\int_0^{2\pi}\int_0^1 \frac{1}{\sqrt{2-r^2}}r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{2-r^2}}\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant, using a substitution with

$$u=r^2$$

$$\frac{du}{dr}=2r\text{, so }du=2r\ dr\text{ and }dr=\frac{du}{2r}$$

If r is defined on $r=[0,1]$, then u is defined on $u=[0,1]$.

$$\int_0^{2\pi}\int_0^1 \frac{r}{\sqrt{2-u}}\left(\frac{du}{2r}\right)\ d\theta$$

$$\int_0^{2\pi}\int_0^1 \frac{1}{2\sqrt{2-u}}\ du\ d\theta$$

$$\int_0^{2\pi}\int_0^1 \frac{1}{2}(2-u)^{-\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u, treating θ as a constant.

$$\int_0^{2\pi} \frac{1}{2} \cdot \frac{2}{-1}(2-u)^{\frac{1}{2}} \Big|_{u=0}^{u=1} \ d\theta$$

$$\int_0^{2\pi} -\sqrt{2-u} \Big|_{u=0}^{u=1} \ d\theta$$

$$\int_0^{2\pi} -\sqrt{2-1} - (-\sqrt{2-0}) \ d\theta$$

$$\int_0^{2\pi} -\sqrt{1} + \sqrt{2} \ d\theta$$

$$\int_0^{2\pi} \sqrt{2} - 1 \ d\theta$$

Integrate with respect to θ.

$$(\sqrt{2}-1)\theta \Big|_0^{2\pi}$$

$$(\sqrt{2}-1)(2\pi) - (\sqrt{2}-1)(0)$$

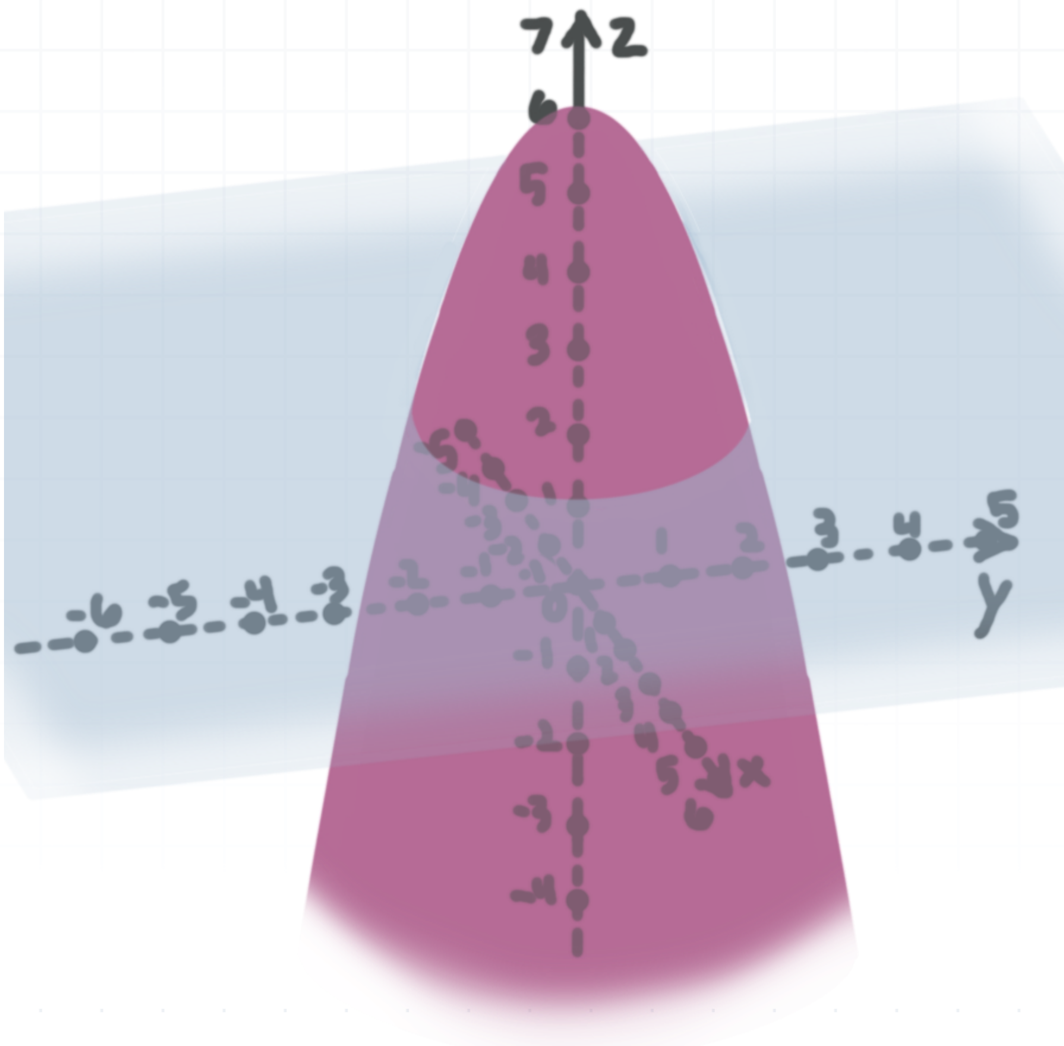
$$2\pi(\sqrt{2}-1)$$

This is exactly half the area, so we'll double this to find total area.

$$4\pi(\sqrt{2}-1)$$

FINDING VOLUME

■ 1. Find the volume of the region bounded $x^2 + y^2 + z - 6 = 0$ and $z = 2$.



Solution:

Finding the volume bounded by $x^2 + y^2 + z - 6 = 0$ and $z = 2$ is equivalent to finding the volume bounded by $x^2 + y^2 + z - 4 = 0$ and $z = 0$. Find the curve of intersection of these last two surfaces.

$$x^2 + y^2 + 0 - 4 = 0$$

$$x^2+y^2=2^2$$

So the curve of intersection is the circle centered at the origin with radius 2. And we can rewrite $x^2+y^2+z-4=0$ as the function $z=4-x^2-y^2$, which means the volume can be defined by

$$\iint_D 4-x^2-y^2\ dA$$

Since the region of integration is the circle centered at the origin with radius 2, the value of r is defined on $r=[0,2]$, and the value of θ is defined on $[0,2\pi]$. Using the conversion formula $r^2=x^2+y^2$, the integral in the polar coordinates is

$$\int_0^{2\pi}\int_0^2 (4-r^2)r\ dr\ d\theta$$

$$\int_0^{2\pi}\int_0^2 4r-r^3\ dr\ d\theta$$

Integrate with respect to r, treating θ as a constant.

$$\int_0^{2\pi} 2r^2-\frac{1}{4}r^4\Big|_{r=0}^{r=2}\ d\theta$$

$$\int_0^{2\pi} 2(2)^2-\frac{1}{4}(2)^4-\left(2(0)^2-\frac{1}{4}(0)^4\right)\ d\theta$$

$$\int_0^{2\pi} 2(4)-\frac{1}{4}(16)\ d\theta$$

$$\int_0^{2\pi} 8 - 4 \ d\theta$$

$$\int_0^{2\pi} 4 \ d\theta$$

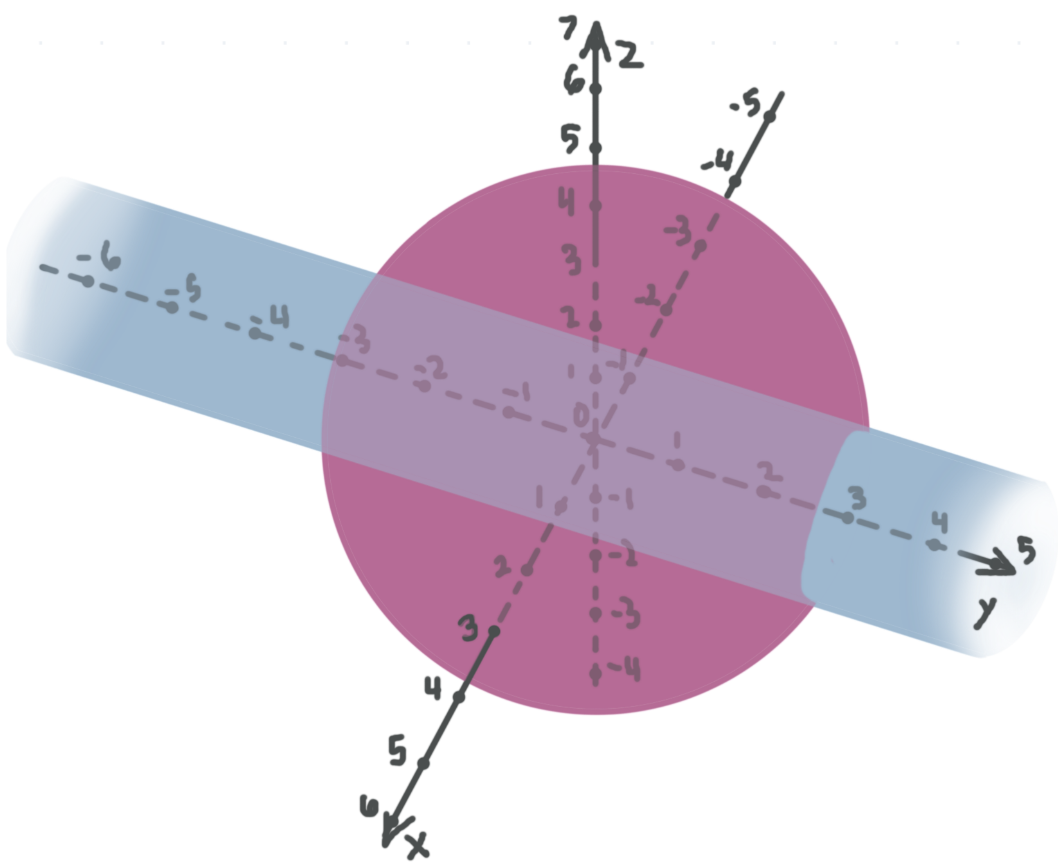
Integrate with respect to θ.

$$4\theta \Big|_0^{2\pi}$$

$$4(2\pi) - 4(0)$$

$$8\pi$$

■ 2. Find volume of the sphere $x^2 + y^2 + z^2 - 9 = 0$ that lies within the cylinder $x^2 + z^2 = 1$.



Solution:

Since the region consists of two equal parts on either side of the xz-plane, we'll calculate the volume for $y > 0$, then double the result.

The cylinder is parallel to the y-axis, so we'll use the function $y = f(x, z)$.

$$x^2 + y^2 + z^2 - 9 = 0$$

$$y^2 = 9 - x^2 - z^2$$

$$y = \sqrt{9 - x^2 - z^2}$$

The volume of the region is

$$\iint_D \sqrt{9 - x^2 - z^2} \ dA$$

Since the region of integration is the circle centered at the origin with radius 1, the value of r is defined on $r = [0,1]$ and the value of θ is defined on $\theta = [0,2\pi]$. Then using the conversion formulas $r^2 = x^2 + z^2$ and $dx \ dz = r \ dr \ d\theta$, the integral in polar coordinates is

$$\int_0^{2\pi} \int_0^1 \sqrt{9 - r^2} \ r \ dr \ d\theta$$

Integrate with respect to r, treating θ as a constant, using a substitution.

$$u = r^2$$

$$\frac{du}{dr}=2r\text{, so } du=2r\ dr \text{ and } dr=\frac{du}{2r}$$

If r is defined on $r=[0,1]$, then u is defined on $u=[0,1]$.

$$\int_0^{2\pi}\int_0^1 \sqrt{9-u}\ r\left(\frac{du}{2r}\right)\ d\theta$$

$$\int_0^{2\pi}\int_0^1 \frac{1}{2}(9-u)^{\frac{1}{2}}\ du\ d\theta$$

Integrate with respect to u, treating θ as a constant.

$$\int_0^{2\pi} -\frac{1}{2}\cdot\frac{2}{3}(9-u)^{\frac{3}{2}}\Big|_{u=0}^{u=1}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(9-u)^{\frac{3}{2}}\Big|_{u=0}^{u=1}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(9-1)^{\frac{3}{2}}+\frac{1}{3}(9-0)^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}8^{\frac{3}{2}}+\frac{1}{3}9^{\frac{3}{2}}\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(8^{\frac{1}{2}})^3+\frac{1}{3}(9^{\frac{1}{2}})^3\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}(2\sqrt{2})^3+\frac{1}{3}(3)^3\ d\theta$$

$$\int_0^{2\pi} -\frac{1}{3}8(2)\sqrt{2}+9\ d\theta$$

$$\int_0^{2\pi} 9-\frac{16\sqrt{2}}{3}\ d\theta$$

Integrate with respect to θ.

$$\left(9-\frac{16\sqrt{2}}{3}\right)\theta\Big|_0^{2\pi}$$

$$\left(9-\frac{16\sqrt{2}}{3}\right)(2\pi)-\left(9-\frac{16\sqrt{2}}{3}\right)(0)$$

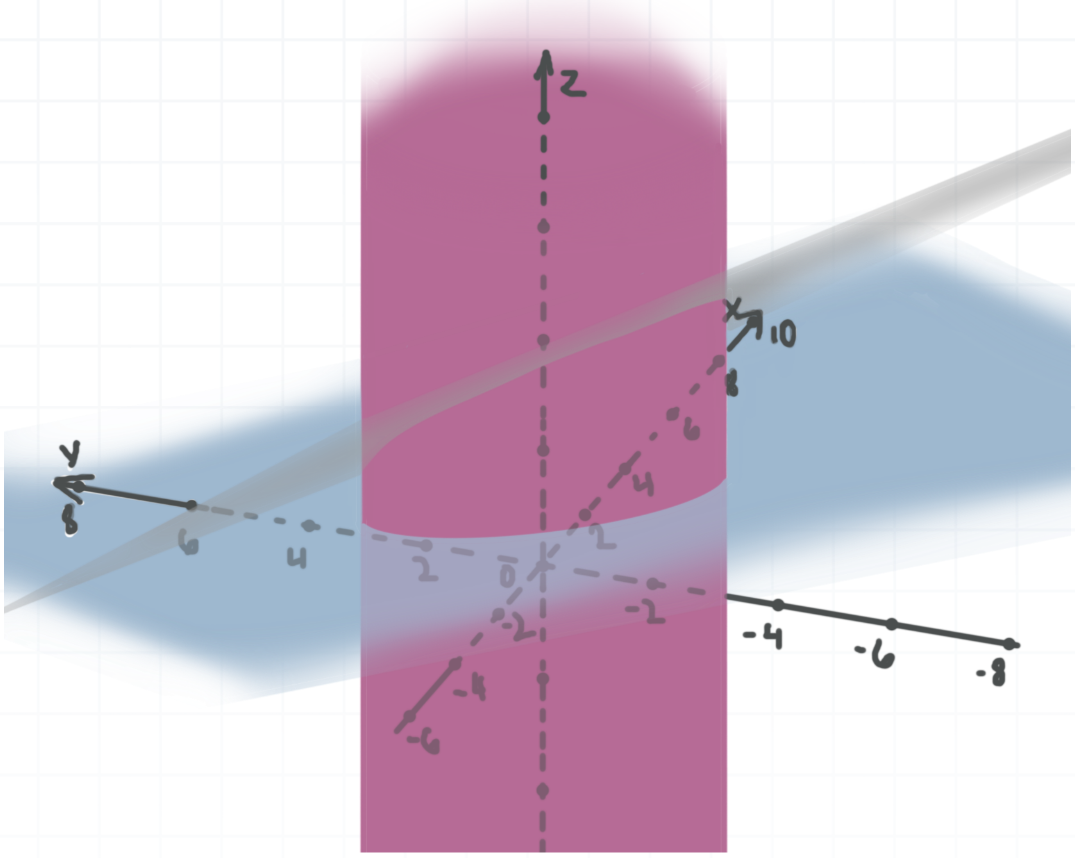
$$2\pi\left(9-\frac{16\sqrt{2}}{3}\right)$$

$$18\pi-\frac{32\sqrt{2}\pi}{3}$$

This is half of the total area, so we'll double it to get total area.

$$36\pi-\frac{64\sqrt{2}\pi}{3}$$

■ 3. Find the volume bounded by the cylinder $x^2+y^2=9$, the plane $x+y+5z-6=0$, and the plane $x+2y+3z-10=0$.



Solution:

We need to find the volume below the plane $x + y + 5z - 6 = 0$, then find the volume below the plane $x + 2y + 3z - 10 = 0$, and then calculate the difference.

The equation of the first plane is

$$x + y + 5z - 6 = 0$$

$$z = \frac{1}{5}(6 - x - y)$$

Then the volume is

$$\iint_D \frac{1}{5}(6 - x - y)\ dA$$

Since the region of integration is the circle centered at the origin with radius 3, we know that r is defined on $r=[0,3]$ and θ is defined on $\theta=[0,2\pi]$. Then the integral in polar coordinates is

$$\frac{1}{5}\int_0^{2\pi}\int_0^3 (6-r\cos\theta-r\sin\theta)r\ dr\ d\theta$$

$$\frac{1}{5}\int_0^{2\pi}\int_0^3 6r-r^2\cos\theta-r^2\sin\theta\ dr\ d\theta$$

Let's change the order of integration to simplify the integration.

$$\frac{1}{5}\int_0^{3}\int_0^{2\pi} 6r-r^2\cos\theta-r^2\sin\theta\ d\theta\ dr$$

Integrate with respect to θ, treating r as a constant.

$$\frac{1}{5}\int_0^3 6r\theta-r^2\sin\theta+r^2\cos\theta\Big|_{\theta=0}^{\theta=2\pi}\ dr$$

$$\frac{1}{5}\int_0^3 6r(2\pi)-r^2\sin(2\pi)+r^2\cos(2\pi)-(6r(0)-r^2\sin(0)+r^2\cos(0))\ dr$$

$$\frac{1}{5}\int_0^3 12\pi r-r^2(0)+r^2(1)-(-r^2(0)+r^2(1))\ dr$$

$$\frac{1}{5}\int_0^3 12\pi r+r^2-r^2\ dr$$

$$\frac{1}{5}\int_0^3 12\pi r\ dr$$

Integrate with respect to r.

$$\frac{1}{5}\cdot 6\pi r^2\Big|_0^3$$

$$\frac{6}{5}\pi r^2\Big|_0^3$$

$$\frac{6}{5}\pi(3)^2-\frac{6}{5}\pi(0)^2$$

$$\frac{54}{5}\pi$$

The equation of the second plane is

$$x+2y+3z-10=0$$

$$z=\frac{1}{3}(10-x-2y)$$

Then the volume is

$$\int\int_D \frac{1}{3}(10-x-2y)\ dA$$

$$\frac{1}{3}\int_0^{2\pi}\int_0^3 (10-r\cos\theta-2r\sin\theta)r\ dr\ d\theta$$

$$\frac{1}{3}\int_0^{2\pi}\int_0^3 10r-r^2\cos\theta-2r^2\sin\theta\ dr\ d\theta$$

Let's change the order of integration to simplify the integration.

$$\frac{1}{3}\int_0^3 \int_0^{2\pi} 10r - r^2\cos\theta - 2r^2\sin\theta\ d\theta\ dr$$

Integrate with respect to θ, treating r as a constant.

$$\frac{1}{3}\int_0^3 10r\theta - r^2\sin\theta + 2r^2\cos\theta\Big|_{\theta=0}^{\theta=2\pi}\ dr$$

$$\frac{1}{3}\int_0^3 10r(2\pi) - r^2\sin(2\pi) + 2r^2\cos(2\pi) - (10r(0) - r^2\sin(0) + 2r^2\cos(0))\ dr$$

$$\frac{1}{3}\int_0^3 20\pi r - r^2(0) + 2r^2(1) - (-r^2(0) + 2r^2(1))\ dr$$

$$\frac{1}{3}\int_0^3 20\pi r + 2r^2 - 2r^2\ dr$$

$$\frac{1}{3}\int_0^3 20\pi r\ dr$$

Integrate with respect to r.

$$\frac{1}{3}\cdot 10\pi r^2\Big|_0^3$$

$$\frac{10}{3}\pi r^2\Big|_0^3$$

$$\frac{10}{3}\pi(3)^2 - \frac{10}{3}\pi(0)^2$$

$$\frac{90}{3}\pi$$

$$30\pi$$

Then the difference is

$$30\pi - \frac{54\pi}{5}$$

$$\frac{150\pi}{5} - \frac{54\pi}{5}$$

$$\frac{96\pi}{5}$$

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