Directional derivatives

The directional derivative of a multivariable function takes into account the direction (given by the unit vector $\vec{u}$) as well as the partial derivatives of the function with respect to each of the variables.

In a **two variable function**, the formula for the directional derivative is

$$D_u f(x,y) = a\left(\frac{\partial f}{\partial x}\right) + b\left(\frac{\partial f}{\partial y}\right)$$

where

a and b come from the unit vector $\vec{u} = \langle a, b \rangle$

If asked to find the directional derivative in the direction of $\vec{v} = \langle c, d \rangle$, we'll need to convert $\vec{v} = \langle c, d \rangle$ to the unit vector using

$$\vec{u} = \left\langle \frac{c}{\sqrt{c^2+d^2}}, \frac{d}{\sqrt{c^2+d^2}} \right\rangle$$

$\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x

$\frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y

In a **three variable function**, the formula for the directional derivative is

$$D_u f(x,y,z) = a\left(\frac{\partial f}{\partial x}\right) + b\left(\frac{\partial f}{\partial y}\right) + c\left(\frac{\partial f}{\partial z}\right)$$

where

a, b and c come from the unit vector $\vec{u} = \langle a, b, c \rangle$

If asked to find the directional derivative in the direction of $\vec{v} = \langle d, e, f \rangle$, we'll need to convert $\vec{v} = \langle d, e, f \rangle$ to the unit vector using

$$\vec{u} = \left\langle \frac{d}{\sqrt{d^2 + e^2 + f^2}}, \frac{e}{\sqrt{d^2 + e^2 + f^2}}, \frac{f}{\sqrt{d^2 + e^2 + f^2}} \right\rangle$$

$\frac{\partial f}{\partial x}$ is the partial derivative of f with respect to x

$\frac{\partial f}{\partial y}$ is the partial derivative of f with respect to y

$\frac{\partial f}{\partial z}$ is the partial derivative of f with respect to z

Let's try an example with a two variable function.

Example

Find the directional derivative of the function.

$$f(x, y) = 2x^3 + 3x^2y + y^2$$

in the direction $\vec{v} = \langle 1,2 \rangle$

at the point $P(1, -2)$

We'll start by converting the given vector to its unit vector form.

$$\vec{u}=\left\langle \frac{c}{\sqrt{c^2+d^2}},\frac{d}{\sqrt{c^2+d^2}} \right\rangle$$

$$\vec{u}=\left\langle \frac{1}{\sqrt{(1)^2+(2)^2}},\frac{2}{\sqrt{(1)^2+(2)^2}} \right\rangle$$

$$\vec{u}=\left\langle \frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}} \right\rangle$$

Now we'll find the partial derivatives of f with respect to x and y.

$$\frac{\partial f}{\partial x}=6x^2+6xy$$

and

$$\frac{\partial f}{\partial y}=3x^2+2y$$

With the unit vector and the partial derivatives, we have everything we need to plug into our formula for the directional derivative.

$$D_uf(x,y)=\frac{1}{\sqrt{5}}\left(6x^2+6xy\right)+\frac{2}{\sqrt{5}}\left(3x^2+2y\right)$$

We want to find the directional derivative at the point $P(1,-2)$, so we'll plug this into the equation we just found for the directional derivative, and we'll get

$$D_u f(1, -2) = \frac{1}{\sqrt{5}}\left[6(1)^2 + 6(1)(-2)\right] + \frac{2}{\sqrt{5}}\left[3(1)^2 + 2(-2)\right]$$

$$D_u f(1, -2) = \frac{-6}{\sqrt{5}} + \frac{-2}{\sqrt{5}}$$

$$D_u f(1, -2) = \frac{-8}{\sqrt{5}}$$

This is the directional derivative of the function $f(x, y) = 2x^3 + 3x^2y + y^2$ in the direction $\vec{v} = \langle 1,2 \rangle$ at the point $P(1, -2)$.