Parametric equations for the line of intersection of two planes

If two planes intersect each other, the intersection will always be a line.

The vector equation for the line of intersection is given by

$$r = r_0 + tv$$

where r_0 is a point on the line and v is the vector result of the cross product of the normal vectors of the two planes.

The parametric equations for the line of intersection are given by

$x = a$, $y = b$, and $z = c$

where a, b and c are the coefficients from the vector equation $r = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$.

Example

Find the parametric equations for the line of intersection of the planes.

$$2x + y - z = 3$$

$$x - y + z = 3$$

We need to find the vector equation of the line of intersection. In order to get it, we'll need to first find v, the cross product of the normal vectors of the given planes.

The normal vectors for the planes are

Plane	Normal vector to the plane
$2x + y - z = 3$	$a\langle 2,1,-1\rangle$
$x - y + z = 3$	$b\langle 1,-1,1\rangle$

The cross product of the normal vectors is

$$v = |a \times b| = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{vmatrix}$$

$$v = |a \times b| = \mathbf{i}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} - \mathbf{j}\begin{vmatrix} 2 & -1 \\ 1 & 1 \end{vmatrix} + \mathbf{k}\begin{vmatrix} 2 & 1 \\ 1 & -1 \end{vmatrix}$$

$$v = |a \times b| = \left[(1)(1) - (-1)(-1)\right]\mathbf{i} - \left[(2)(1) - (-1)(1)\right]\mathbf{j} + \left[(2)(-1) - (1)(1)\right]\mathbf{k}$$

$$v = |a \times b| = (1 - 1)\mathbf{i} - (2 + 1)\mathbf{j} + (-2 - 1)\mathbf{k}$$

$$v = |a \times b| = 0\mathbf{i} - 3\mathbf{j} - 3\mathbf{k}$$

$$v = |a \times b| = \langle 0, -3, -3\rangle$$

We also need a point on the line of intersection. To get it, we'll use the equations of the given planes as a system of linear equations. If we set $z = 0$ in both equations, we get

$$2x + y - z = 3$$

$$2x + y - 0 = 3$$

$$2x + y = 3$$

and

$$x - y + z = 3$$

$$x - y + 0 = 3$$

$$x - y = 3$$

Now we'll add the equations together.

$$(2x + x) + (y - y) = 3 + 3$$

$$3x + 0 = 6$$

$$x = 2$$

Plugging $x = 2$ back into $x - y = 3$, we get

$$2 - y = 3$$

$$-y = 1$$

$$y = -1$$

Putting these values together, the point on the line of intersection is

$$(2, -1, 0)$$

$$r_0 = 2\mathbf{i} - \mathbf{j} + 0\mathbf{k}$$

$$r_0 = \langle 2, -1, 0 \rangle$$

Now we'll plug v and r_0 into the vector equation.

$$r = r_0 + tv$$

$$r = (2\mathbf{i} - \mathbf{j} + 0\mathbf{k}) + t(0\mathbf{i} - 3\mathbf{j} - 3\mathbf{k})$$

$$r = 2\mathbf{i} - \mathbf{j} + 0\mathbf{k} + 0\mathbf{i}t - 3\mathbf{j}t - 3\mathbf{k}t$$

$$r = 2\mathbf{i} - \mathbf{j} - 3\mathbf{j}t - 3\mathbf{k}t$$

$$r = (2)\mathbf{i} + (-1 - 3t)\mathbf{j} + (-3t)\mathbf{k}$$

With the vector equation for the line of intersection in hand, we can find the parametric equations for the same line. Matching up $r = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ with our vector equation $r = (2)\mathbf{i} + (-1 - 3t)\mathbf{j} + (-3t)\mathbf{k}$, we can say that

$$a = 2$$

$$b = -1 - 3t$$

$$c = -3t$$

Therefore, the parametric equations for the line of intersection are

$$x = 2$$

$$y = -1 - 3t$$

$$z = -3t$$