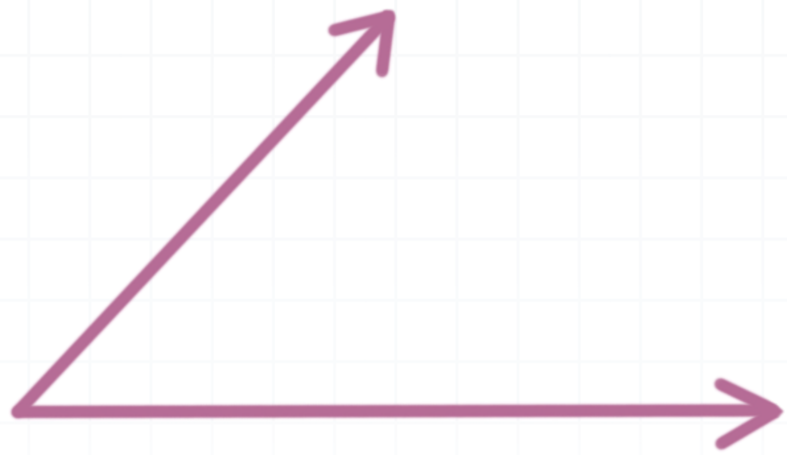
Acute angle between the curves

An acute angle is angle that's less than 90°, like this:



If we want to find the acute angle between two curves, we'll find the tangent lines to both curves at their point(s) of intersection, convert the tangent lines to standard vector form and then use the formula

$$\cos\theta = \frac{a \cdot b}{|a||b|}$$

where a and b are the given vectors, $a \cdot b$ is the dot product of the vectors, $|a|$ is the magnitude of the vector a (its length) and $|b|$ is the magnitude of the vector b (its length). We can find the magnitude of both vectors using the distance formula

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

for a two-dimensional vector where the point (x_1, y_1) is the origin $(0,0)$.

If the formula above gives a result that's greater than 90°, then we've found the obtuse angle between the lines. To find the acute angle, we just subtract the obtuse angle from 180°, and we'll get the acute angle.

Example

Find the acute angle between the curves.

$$y = x^2$$

$$y = 2x^2 - 1$$

We'll start by setting the curves equal to each other and solving for x, in order to find the point(s) where the curves intersect each other.

$$x^2 = 2x^2 - 1$$

$$-x^2 = -1$$

$$x^2 = 1$$

$$x = \pm 1$$

Since we have two points of intersection, we'll need to find two acute angles, one for each of the points of intersection.

We'll plug both values of x into $y = x^2$ to find the corresponding y-values. We can use either curve; they should both return the same y-values.

For $x = 1$:

$$y = x^2$$

$$y = (1)^2$$

$$y = 1$$

$$(1,1)$$

For $x = -1$:

$$y = x^2$$

$$y = (-1)^2$$

$$y = 1$$

$$(-1,1)$$

We need to find the tangent lines for both curves at each of the points of intersection. Remember that to find a tangent line, we'll take the derivative of the function, then evaluate the derivative at the point of intersection to find the slope of the tangent line there. Then we'll plug the slope and the tangent point into the point-slope formula to find the equation of the tangent line.

At $(1,1)$,

for $y = x^2$:

$$y' = 2x$$

$$y'(1,1) = 2(1)$$

$$y'(1,1) = 2$$

for $y = 2x^2 - 1$:

$$y' = 4x$$

$$y' = 4(1)$$

$$y'(1,1) = 4$$

At $(-1,1)$,

for $y = x^2$:

for $y = 2x^2 - 1$:

$$y' = 2x$$

$$y'(-1,1) = 2(-1)$$

$$y'(-1,1) = -2$$

$$y' = 4x$$

$$y' = 4(-1)$$

$$y'(-1,1) = -4$$

Plugging the slopes and the intersection points into the point-slope formula for the equation of a line, we get

At $(1,1)$,

for $y = x^2$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 2(x - 1)$$

$$y - 1 = 2x - 2$$

$$y = 2x - 1$$

for $y = 2x^2 - 1$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 4(x - 1)$$

$$y - 1 = 4x - 4$$

$$y = 4x - 3$$

At $(-1,1)$,

for $y = x^2$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -2\left[x - (-1)\right]$$

$$y - 1 = -2(x + 1)$$

$$y - 1 = -2x - 2$$

$$y = -2x - 1$$

for $y = 2x^2 - 1$:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = -4\left[x - (-1)\right]$$

$$y - 1 = -4(x + 1)$$

$$y - 1 = -4x - 4$$

$$y = -4x - 3$$

We need to convert our tangent line equations to standard vector form.

At $(1,1)$,

for $y = x^2$:

$$y = 2x - 1$$

$$-2x + y = -1$$

$$\langle -2,1 \rangle$$

for $y = 2x^2 - 1$:

$$y = 4x - 3$$

$$-4x + y = -3$$

$$\langle -4,1 \rangle$$

At $(-1,1)$,

for $y = x^2$:

$$y = -2x - 1$$

$$2x + y = -1$$

$$\langle 2,1 \rangle$$

for $y = 2x^2 - 1$:

$$y = -4x - 3$$

$$4x + y = -3$$

$$\langle 4,1 \rangle$$

To summarize our findings so far, we can say that we need to find the acute angle

between the vectors $a = \langle -2,1 \rangle$ and $b = \langle -4,1 \rangle$ at the point $(1,1)$

between the vectors $c = \langle 2,1 \rangle$ and $d = \langle 4,1 \rangle$ at the point $(-1,1)$

Before we can use the cosine formula to find the acute angle, we need to find the dot products $a \cdot b$ and $c \cdot d$ and the magnitude of each vector.

For $a = \langle -2,1 \rangle$ and $b = \langle -4,1 \rangle$ at the point $(1,1)$:

For $a \cdot b$:

$$a \cdot b = (-2)(-4) + (1)(1)$$

$$a \cdot b = 8 + 1$$

$$a \cdot b = 9$$

For $|a|$:

$$|a| = \sqrt{(-2-0)^2 + (1-0)^2}$$

$$|a| = \sqrt{4+1}$$

$$|a| = \sqrt{5}$$

For $|b|$:

$$|b| = \sqrt{(-4-0)^2 + (1-0)^2}$$

$$|b| = \sqrt{16+1}$$

$$|b| = \sqrt{17}$$

For $c = \langle 2,1 \rangle$ and $d = \langle 4,1 \rangle$ at the point $(-1,1)$:

For $c \cdot d$:

$$c \cdot d = (2)(4) + (1)(1)$$

$$c \cdot d = 8 + 1$$

$$c \cdot d = 9$$

For $|c|$:

$$|c| = \sqrt{(2-0)^2 + (1-0)^2}$$

$$|c| = \sqrt{4+1}$$

$$|c| = \sqrt{5}$$

For $|d|$:

$$|d| = \sqrt{(4-0)^2 + (1-0)^2}$$

$$|d| = \sqrt{16+1}$$

$$|d| = \sqrt{17}$$

Finally, plug the dot products and magnitudes we've found into our formula.

For $a = \langle -2,1 \rangle$ and $b = \langle -4,1 \rangle$ at the point $(1,1)$:

$$\cos\theta = \frac{9}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{9}{\sqrt{85}}$$

$$\theta = \arccos\frac{9}{\sqrt{85}}$$

$$\theta = 12.5^\circ$$

For $c = \langle 2,1 \rangle$ and $d = \langle 4,1 \rangle$ at the point $(-1,1)$:

$$\cos\theta = \frac{9}{\sqrt{5}\sqrt{17}}$$

$$\cos\theta = \frac{9}{\sqrt{85}}$$

$$\theta = \arccos\frac{9}{\sqrt{85}}$$

$$\theta = 12.5^\circ$$

In conclusion, we can say that

the acute angle between the tangent lines $y = 2x - 1$ and $y = 4x - 3$ at the tangent point $(1,1)$ is 12.5°

the acute angle between the tangent lines $y = -2x - 1$ and $y = -4x - 3$ at the tangent point $(-1,1)$ is 12.5°

And therefore, we can say that

the acute angle between the curves $y = x^2$ and $y = 2x^2 - 1$ at the intersection point $(1,1)$ is 12.5°

the acute angle between the curves $y = x^2$ and $y = 2x^2 - 1$ at the intersection point $(-1,1)$ is 12.5°