Unit tangent and unit normal vectors

The unit tangent vector $T(t)$ of a vector function

$$r(t) = r(t)_1\mathbf{i} + r(t)_2\mathbf{j} + r(t)_3\mathbf{k}$$

is the vector that is 1 unit long and tangent to the vector function at the point t.

$$T(t) = \frac{r'(t)}{\left| r'(t) \right|}$$

Remember that $\left| r'(t) \right|$ is the magnitude of the derivative of the vector function at time t, and we can find it using the formula

$$\left| r'(t) \right| = \sqrt{\left[r'(t)_1\right]^2 + \left[r'(t)_2\right]^2 + \left[r'(t)_3\right]^2}$$

The unit normal vector $N(t)$ of the same vector function is the vector that is 1 unit long and perpendicular to the unit tangent vector at the same point t.

$$N(t) = \frac{T'(t)}{\left| T'(t) \right|}$$

Remember that $\left| T'(t) \right|$ is the magnitude of the derivative of the unit tangent vector at time t, and we can find it using the formula

$$\left| T'(t) \right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

Example

Find the unit normal vector of the vector function at $t=1$.

$$r(t)=t\mathbf{i}+t^2\mathbf{j}+2\mathbf{k}$$

In order to find the unit normal vector, we'll have to start by finding the unit tangent vector, given by

$$T(t)=\frac{r'(t)}{\left|r'(t)\right|}$$

We'll take the derivative of the vector function to get $r'(t)$. Remember, we only have to take the derivatives of the coefficients, leaving $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ alone.

$$r(t)=t\mathbf{i}+t^2\mathbf{j}+2\mathbf{k}$$

$$r'(t)=1\mathbf{i}+2t\mathbf{j}+0\mathbf{k}$$

$$r'(t)=\mathbf{i}+2t\mathbf{j}$$

Now we can use $r'(t)$ to find $\left|r'(t)\right|$.

$$\left|r'(t)\right|=\sqrt{\left[r'(t)_1\right]^2+\left[r'(t)_2\right]^2+\left[r'(t)_3\right]^2}$$

$$\left|r'(t)\right|=\sqrt{(1)^2+(2t)^2+(0)^2}$$

$$\left|r'(t)\right|=\sqrt{1+4t^2}$$

Plugging these into the formula for the unit tangent vector, we get

$$T(t)=\frac{r'(t)}{\left|r'(t)\right|}$$

$$T(t)=\frac{\mathbf{i}+2t\mathbf{j}}{\sqrt{1+4t^2}}$$

Now that we have an equation for the unit tangent vector, we can use it to find the unit normal vector, given by

$$N(t)=\frac{T'(t)}{\left|T'(t)\right|}$$

We'll have to start by taking the derivative of the unit tangent vector to get $T'(t)$. We'll need quotient rule to do this.

$$T(t)=\frac{\mathbf{i}+2t\mathbf{j}}{\sqrt{1+4t^2}}$$

$$T'(t)=\frac{(0\mathbf{i}+2\mathbf{j})\sqrt{1+4t^2}-(\mathbf{i}+2t\mathbf{j})\frac{1}{2}(1+4t^2)^{-\frac{1}{2}}(8t)}{\left(\sqrt{1+4t^2}\right)^2}$$

$$T'(t)=\frac{2\mathbf{j}\sqrt{1+4t^2}-4t(\mathbf{i}+2t\mathbf{j})(1+4t^2)^{-\frac{1}{2}}}{1+4t^2}$$

$$T'(t)=\frac{2\mathbf{j}\sqrt{1+4t^2}-\frac{4t\left(\mathbf{i}+2t\mathbf{j}\right)}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t)=\frac{\frac{2\mathbf{j}\sqrt{1+4t^2}\sqrt{1+4t^2}}{\sqrt{1+4t^2}}-\frac{4t\left(\mathbf{i}+2t\mathbf{j}\right)}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t) = \frac{\frac{2\mathbf{j}(1+4t^2)}{\sqrt{1+4t^2}} - \frac{4t(\mathbf{i}+2t\mathbf{j})}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t) = \frac{\frac{2\mathbf{j}(1+4t^2) - 4t(\mathbf{i}+2t\mathbf{j})}{\sqrt{1+4t^2}}}{1+4t^2}$$

$$T'(t) = \frac{2\mathbf{j}\left(1+4t^2\right) - 4t\left(\mathbf{i}+2t\mathbf{j}\right)}{\left(1+4t^2\right)^{\frac{3}{2}}}$$

$$T'(t) = \frac{2\mathbf{j} + 8t^2\mathbf{j} - 4t\mathbf{i} - 8t^2\mathbf{j}}{\left(1+4t^2\right)^{\frac{3}{2}}}$$

$$T'(t) = \frac{-4t\mathbf{i} + 2\mathbf{j}}{\left(1+4t^2\right)^{\frac{3}{2}}}$$

Next, we'll solve for $T'(1)$ by plugging $t = 1$ into the derivative we just found.

$$T'(1) = \frac{-4(1)\mathbf{i} + 2\mathbf{j}}{\left(1+4(1)^2\right)^{\frac{3}{2}}}$$

$$T'(1) = \frac{-4\mathbf{i} + 2\mathbf{j}}{5^{\frac{3}{2}}}$$

$$T'(1) = -\frac{4}{5\sqrt{5}}\mathbf{i} + \frac{2}{5\sqrt{5}}\mathbf{j}$$

Now we'll find the magnitude $\left|T'(t)\right|$.

$$\left|T'(t)\right| = \sqrt{\left[T'(t)_1\right]^2 + \left[T'(t)_2\right]^2 + \left[T'(t)_3\right]^2}$$

$$\left|T'(1)\right| = \sqrt{\left(-\frac{4}{5\sqrt{5}}\right)^2 + \left(\frac{2}{5\sqrt{5}}\right)^2 + (0)^2}$$

$$\left|T'(1)\right| = \sqrt{\frac{16}{25(5)} + \frac{4}{25(5)}}$$

$$\left|T'(1)\right| = \sqrt{\frac{20}{125}}$$

$$\left|T'(1)\right| = \sqrt{\frac{4}{25}}$$

$$\left|T'(1)\right| = \frac{2}{5}$$

Finally, we'll plug everything we've found into the formula for the unit normal vector, and we'll get

$$N(t) = \frac{T'(t)}{\left|T'(t)\right|}$$

$$N(1) = \frac{-\frac{4}{5\sqrt{5}}\mathbf{i} + \frac{2}{5\sqrt{5}}\mathbf{j}}{\frac{2}{5}}$$

$$N(1) = -\frac{4(5)}{5(2)\sqrt{5}}\mathbf{i} + \frac{2(5)}{5(2)\sqrt{5}}\mathbf{j}$$

$$N(1) = -\frac{2}{\sqrt{5}}\mathbf{i} + \frac{1}{\sqrt{5}}\mathbf{j}$$

Since multiplying a vector by a constant scalar doesn't change its value, we can multiply by $\sqrt{5}$, and simplify the equation of the unit normal vector to

$$N(1) = -2\mathbf{i} + \mathbf{j}$$

This is the unit normal vector of the vector function $r(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}$ at the point $t = 1$.