Equation of a plane

The equation of a plane is given by the formula

$$a(x-x_1)+b(y-y_1)+c(z-z_1)=0$$

where $\langle a,b,c\rangle$ are the direction numbers from the normal vector to the plane.

Given three points in the plane $P(P_1,P_2,P_3)$, $Q(Q_1,Q_2,Q_3)$ and $R(R_1,R_2,R_3)$, we can find the equation of the plane by

using the points to generate **two vectors**

$$\overrightarrow{PQ}=\left\langle (Q_1-P_1),(Q_2-P_2),(Q_3-P_3)\right\rangle$$

$$\overrightarrow{PR}=\left\langle (R_1-P_1),(R_2-P_2),(R_3-P_3)\right\rangle,$$

taking the **cross product** of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ to get the normal vector to the plane

$$\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ PQ_1 & PQ_2 & PQ_3\\ PR_1 & PR_2 & PR_3\end{vmatrix}=\mathbf{i}\begin{vmatrix}PQ_2 & PQ_3\\ PR_2 & PR_3\end{vmatrix}-\mathbf{j}\begin{vmatrix}PQ_1 & PQ_3\\ PR_1 & PR_3\end{vmatrix}+\mathbf{k}\begin{vmatrix}PQ_1 & PQ_2\\ PR_1 & PR_2\end{vmatrix}$$

$$=\mathbf{i}\left(PQ_2PR_3-PQ_3PR_2\right)-\mathbf{j}\left(PQ_1PR_3-PQ_3PR_1\right)$$

$$+\mathbf{k}\left(PQ_1PR_2-PQ_2PR_1\right)$$

and then plugging the given points and the normal vector into the **formula** for the equation of the plane.

Example

Find the equation of the plane that passes through the given points.

$$P(1,0,2)$$

$$Q(2,-1,3)$$

$$R(1,-1,2)$$

We'll start by using the given points P, Q and R to find two vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ that lie in the plane.

$$\overrightarrow{PQ} = \langle (2-1), (-1-0), (3-2) \rangle$$

$$\overrightarrow{PQ} = \langle 1, -1, 1 \rangle$$

and

$$\overrightarrow{PR} = \langle (1-1), (-1-0), (2-2) \rangle$$

$$\overrightarrow{PR} = \langle 0, -1, 0 \rangle$$

Taking the cross product of these two vectors, we get

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 0 & -1 & 0 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ -1 & 0 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ 0 & -1 \end{vmatrix}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \left[(-1)(0) - (1)(-1)\right] \mathbf{i} - \left[(1)(0) - (1)(0)\right] \mathbf{j} + \left[(1)(-1) - (-1)(0)\right] \mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = (0+1)\mathbf{i} - (0-0)\mathbf{j} + (-1-0)\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = 1\mathbf{i} - 0\mathbf{j} - 1\mathbf{k}$$

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \langle 1,0,-1 \rangle$$

Now we'll plug any of the given points, we'll use P, and the direction numbers from the cross product into the formula for the equation of the plane.

$$(1)(x-1) + (0)(y-0) + (-1)(z-2) = 0$$

$$x - 1 - z + 2 = 0$$

$$x - z = -1$$