Differential Equations Final Exam Solutions

krista king MATH

Differential Equations Final Exam Answer Key

1. (5 pts)	A	■	C	D	E
2. (5 pts)	A	B	C	D	■
3. (5 pts)	■	B	C	D	E
4. (5 pts)	A	B	■	D	E
5. (5 pts)	A	■	C	D	E
6. (5 pts)	■	B	C	D	E
7. (5 pts)	A	■	C	D	E
8. (5 pts)	A	B	C	D	■

9. (15 pts) $\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$+\frac{1}{6}\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} e^{-t} + \frac{3}{130}\begin{bmatrix} -3 \\ 7 \\ 27 \end{bmatrix} \sin 3t + \frac{1}{130}\begin{bmatrix} -7 \\ -27 \\ 63 \end{bmatrix} \cos 3t$$

10. (15 pts) $f(x) = 1 + \frac{L}{2} - \frac{L^2}{2} + \sum_{n=1}^{\infty} \frac{L((2-6L)(-1)^n - 2)}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right)$

$$+\sum_{n=1}^{\infty} \frac{(3L^2 - 6L + 2)(n\pi)^2(-1)^n + 6L^2(-1)^n - 6L^2 - 2(n\pi)^2}{(n\pi)^3} \sin\left(\frac{n\pi x}{L}\right)$$

11. (15 pts) $u(x, y) = 3e^{-36x} \sin\left(\frac{3y}{2}\right)$

12. (15 pts) $y(x) = c_0 + c_1 \sum_{k=1}^{\infty} \frac{1}{k!} x^k$

Differential Equations Final Exam Solutions

1. B. Change y' to dy/dx, and divide by x.

$$xy' + 4x^2y = 2x^2$$

$$x\frac{dy}{dx} + 4x^2y = 2x^2$$

$$\frac{dy}{dx} + 4xy = 2x$$

The linear differential equation is now in standard form, so we can identify $P(x) = 4x$ and $Q(x) = 2x$ and then use $P(x)$ to find the integrating factor.

$$\mu(x) = e^{\int P(x)\, dx}$$

$$\mu(x) = e^{\int 4x\, dx}$$

$$\mu(x) = e^{2x^2}$$

Multiply through the differential equation by the integrating factor.

$$e^{2x^2}\left(\frac{dy}{dx} + 4xy = 2x\right)$$

$$e^{2x^2}\frac{dy}{dx} + 4xye^{2x^2} = 2xe^{2x^2}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{2x^2}y) = 2xe^{2x^2}$$

then integrate, using $u = 2x^2$ and $du = 4x\ dx$ to integrate the right side.

$$\int \frac{d}{dx}(e^{2x^2}y) = \int 2xe^{2x^2}\ dx$$

$$e^{2x^2}y = \frac{1}{2}\int e^u\ du$$

$$e^{2x^2}y = \frac{1}{2}e^u + C$$

$$e^{2x^2}y = \frac{1}{2}e^{2x^2} + C$$

$$y = \frac{1}{2} + Ce^{-2x^2}$$

Once we have this general solution, we recognize from the initial condition $y(0) = 3/2$ that $x = 0$ and $y = 3/2$, so we'll plug these values into the general solution,

$$\frac{3}{2} = \frac{1}{2} + Ce^{-2(0)^2}$$

and then simplify to solve for C.

$$C = 1$$

So the solution is

$$y = \frac{1}{2} + e^{-2x^2}$$

2. E. Start by rewriting the Bernoulli equation in standard form.

$$xy' + y = y^2 \ln x$$

$$y' + \frac{y}{x} = \frac{y^2 \ln x}{x}$$

With the equation in standard form, divide through by y^n. In this equation, that means we're dividing by y^2.

$$\frac{y'}{y^2} + \frac{1}{xy} = \frac{\ln x}{x}$$

$$y'y^{-2} + \frac{1}{x}y^{-1} = \frac{\ln x}{x}$$

Our substitution is $v = y^{-1}$, and its derivative is $v' = -y^{-2}y'$. Substitute these into the Bernoulli equation.

$$-v' + \frac{1}{x}v = \frac{\ln x}{x}$$

$$v' - \frac{1}{x}v = -\frac{\ln x}{x}$$

To solve the linear equation, we'll use $P(x) = -1/x$ to find the integrating factor,

$$I(x) = e^{\int P(x)\, dx}$$

$$I(x) = e^{\int -\frac{1}{x}\ dx}$$

$$I(x) = e^{-\ln x}$$

$$I(x) = e^{\ln(x^{-1})}$$

$$I(x) = \frac{1}{x}$$

and then multiply through the linear equation by $I(x)$.

$$\frac{1}{x}v' - \frac{1}{x^2}v = -\frac{\ln x}{x^2}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}\left(\frac{v}{x}\right) = -\frac{\ln x}{x^2}$$

then integrate both sides. We'll use integration by parts on the right with $u = \ln x$, $du = (1/x)\ dx$, $dv = -1/x^2\ dx$, and $v = 1/x$.

$$\int \frac{d}{dx}\left(\frac{v}{x}\right)\ dx = -\int \frac{\ln x}{x^2}\ dx$$

$$\frac{v}{x} = \frac{1}{x}\ln x - \int \frac{1}{x^2}\ dx$$

$$\frac{v}{x} = \frac{1}{x}\ln x + \frac{1}{x} + C$$

Solve for v.

$$v = \ln x + 1 + Cx$$

Use $v = y^{-1}$ to back-substitute for v,

$$y^{-1} = \ln x + 1 + Cx$$

then solve for y.

$$\frac{1}{y} = \ln x + 1 + Cx$$

$$y = \frac{1}{\ln x + 1 + Cx}$$

3. A. We'll use undetermined coefficients and set $g(t) = 0$ to find the associated homogeneous equation,

$$y'' - 2y' - 3y = 3t - 5\sin t$$

$$y'' - 2y' - 3y = 0$$

then solve the associated characteristic equation.

$$r^2 - 2r - 3 = 0$$

$$(r - 3)(r + 1) = 0$$

$$r = -1,\ 3$$

These are distinct real roots, so the complementary solution will be

$$y_c(t) = c_1e^{r_1t} + c_2e^{r_2t}$$

$$y_c(t) = c_1e^{-t} + c_2e^{3t}$$

Our guess for the particular solution will be

$$y_p(t) = At + B + C\cos t + D\sin t$$

Take the first and second derivatives of the guess.

$$y_p'(t) = A - C\sin t + D\cos t$$

$$y_p''(t) = -C\cos t - D\sin t$$

Plugging into the original differential equation, we get

$$y'' - 2y' - 3y = 3t - 5\sin t$$

$$-C\cos t - D\sin t - 2A + 2C\sin t - 2D\cos t - 3At - 3B - 3C\cos t - 3D\sin t$$

$$= 3t - 5\sin t$$

$$-3At - (2A + 3B) - \cos t(4C + 2D) - \sin t(4D - 2C) = 3t - 5\sin t$$

Equating coefficients gives $-3A = 3$, $2A + 3B = 0$, $4C + 2D = 0$, and $2C - 4D = -5$, and we can solve these equations as a system to get $A = -1$, $B = 2/3$, $C = -1/2$, and $D = 1$. So the particular solution is

$$y_p(t) = -t + \frac{2}{3} - \frac{1}{2}\cos t + \sin t$$

Putting this particular solution together with the complementary solution gives us the general solution to the nonhomogeneous differential equation.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{-t} + c_2e^{3t} - t + \frac{2}{3} - \frac{1}{2}\cos t + \sin t$$

We'll substitute the initial condition $y(0) = 1/6$ into $y(t)$,

$$\frac{1}{6} = c_1e^{-0} + c_2e^{3(0)} - 0 + \frac{2}{3} - \frac{1}{2}\cos(0) + \sin(0)$$

$$\frac{1}{6} = c_1 + c_2 + \frac{2}{3} - \frac{1}{2}$$

$$c_1 + c_2 = 0$$

and the condition $y'(0) = 0$ into the derivative.

$$y'(t) = -c_1e^{-t} + 3c_2e^{3t} - 1 + \frac{1}{2}\sin t + \cos t$$

$$0 = -c_1e^{-0} + 3c_2e^{3(0)} - 1 + \frac{1}{2}\sin(0) + \cos(0)$$

$$0 = -c_1 + 3c_2 - 1 + 1$$

$$0 = -c_1 + 3c_2$$

$$c_1 - 3c_2 = 0$$

Solve the system of equations

$$c_1 - 3c_2 = 0$$

$$c_1 + c_2 = 0$$

to get $c_1 = 0$ and $c_2 = 0$. Then the general solution is

$$y(t) = -t + \frac{2}{3} - \frac{1}{2}\cos t + \sin t$$

4. C. The homogeneous equation associated with

$$y'' - y' - 6y = \frac{1}{e^{2x}}$$

is

$$y'' - y' - 6y = 0$$

so the characteristic equation will be

$$r^2 - r - 6 = 0$$

$$(r+2)(r-3) = 0$$

$$r = -2,\ 3$$

These are distinct real roots, so the complementary solution is

$$y_c(x) = c_1e^{-2x} + c_2e^{3x}$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{e^{-2x}, e^{3x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(e^{-2x}, e^{3x}) = \begin{vmatrix} e^{-2x} & e^{3x} \\ -2e^{-2x} & 3e^{3x} \end{vmatrix}$$

$$W(e^{-2x}, e^{3x}) = (e^{-2x})(3e^{3x}) - (e^{3x})(-2e^{-2x})$$

$$W(e^{-2x}, e^{3x}) = 3e^{x} + 2e^{x}$$

$$W(e^{-2x}, e^{3x}) = 5e^{x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)}\ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)}\ dx$$

$$y_p(x) = -e^{-2x} \int \frac{e^{3x}\left(\frac{1}{e^{2x}}\right)}{5e^{x}}\ dx + e^{3x} \int \frac{e^{-2x}\left(\frac{1}{e^{2x}}\right)}{5e^{x}}\ dx$$

$$y_p(x) = -e^{-2x} \int \frac{1}{5}\ dx + e^{3x} \int \frac{1}{5}e^{-5x}\ dx$$

$$y_p(x) = -e^{-2x}\left(\frac{1}{5}x\right) + e^{3x}\left(-\frac{1}{25}e^{-5x}\right)$$

$$y_p(x) = -\frac{1}{5}xe^{-2x} - \frac{1}{25}e^{-2x}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{-2x} + c_2e^{3x} - \frac{1}{5}xe^{-2x} - \frac{1}{25}e^{-2x}$$

5. B. Since we're given step-size directly, we already know that

$$\Delta t = 0.2$$

To start building our table, we identify $t_0 = 0$ and $y_0 = 1$ from the initial condition $y(0) = 1$.

$$t_0 = 0 \qquad y_0 = 1 \qquad y_0 = 1$$

$$t_1 = 0.2 \qquad y_1 = 1 + (2(1) + 0)(0.2) \qquad y_1 = 1.4$$

$$t_2 = 0.4 \qquad y_2 = 1.4 + (2(1.4) + 0.2)(0.2) \qquad y_2 = 2$$

$$t_3 = 0.6 \qquad y_3 = 2 + (2(2) + 0.4)(0.2) \qquad y_3 = 2.88$$

$$t_4 = 0.8 \qquad y_4 = 2.88 + (2(2.88) + 0.6)(0.2) \qquad y_4 = 4.152$$

$$t_5 = 1 \qquad y_5 = 4.152 + (2(4.152) + 0.8)(0.2) \qquad y_5 = 5.9728$$

After filling out the table, we can say that the value of $y(1)$ is approximately

$$y(1) \approx 5.97$$

6. A. After 2 hours, the population doubled from 20 to 40. Substituting these values into the exponential equation gives

$$40 = 20e^{k(120)}$$

$$2 = e^{120k}$$

To solve for k, we'll apply the natural log to both sides of the equation, canceling the $\ln e$, and then rearrange.

$$\ln 2 = \ln e^{120k}$$

$$\ln 2 = 120k$$

$$k = \frac{\ln 2}{120}$$

Now that we have a value for the growth constant k, we can can figure out how long it'll take for the population to grow to 100.

$$100 = 20e^{\frac{\ln 2}{120}t}$$

$$5 = e^{\frac{\ln 2}{120}t}$$

Apply the natural log to both sides of the equation, canceling the $\ln e$, and then rearrange.

$$\ln 5 = \ln e^{\frac{\ln 2}{120}t}$$

$$\ln 5 = \frac{\ln 2}{120}t$$

$$t = \frac{120 \ln 5}{\ln 2}$$

$t \approx 279$ minutes

$t \approx 4$ hours 39 minutes

7. B. The coefficient matrix of the system $x_1' = x_1 + 3x_2$ and $x_2' = 2x_2$ is

$$A = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$$

and the determinant $|A-\lambda I|$ is

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 3\\ 0 & 2-\lambda\end{vmatrix}$$

$$|A-\lambda I|=(1-\lambda)(2-\lambda)-3(0)$$

$$|A-\lambda I|=(1-\lambda)(2-\lambda)$$

Solve the characteristic equation to find the Eigenvalues.

$$(1-\lambda)(2-\lambda)=0$$

$$\lambda=1,\ 2$$

Then for these Eigenvalues, $\lambda_1=1$ and $\lambda_2=2$, we find

$$A-1I=\begin{bmatrix}0 & 3\\ 0 & 1\end{bmatrix} \qquad A-2I=\begin{bmatrix}-1 & 3\\ 0 & 0\end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix}0 & 3\\ 0 & 1\end{bmatrix} \qquad \begin{bmatrix}-1 & 3\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix} \qquad \begin{bmatrix}1 & -3\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$$

If we turn these back into systems of equations (in both cases we only need to consider the equation that we get from the first row of each matrix), we get

$$k_2 = 0 \qquad\qquad k_1 = 3k_2$$

From the first system, we'll choose $k_1 = 1$. And from the second system, we'll choose $k_2 = 1$, which results in $k_1 = 3$.

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad \overrightarrow{k_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Then the solutions to the system are

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t} \qquad\qquad \overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \qquad\qquad \overrightarrow{x_2} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$$

Therefore, the general solution to the homogeneous system will be

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t}$$

8. E. Substituting $u(x,0) = 0$ into the product solution gives

$$u(x, y) = v(x)w(y)$$

$$0 = v(x)w(0)$$

When $v(x) = 0$ we get the trivial solution, so we'll only use $w(0) = 0$.

Substituting $u(0,y) = 0$ into the product solution gives

$$u(x, y) = v(x)w(y)$$

$$u(0,y) = v(0)w(y) = 0$$

When $w(y) = 0$ we get the trivial solution, so we'll only use $v(0) = 0$.

And if we differentiate the product solution and then substitute the partial derivative condition, we get

$$\frac{\partial u}{\partial x} = v'(x)w(y)$$

$$0 = v(x)w'(0)$$

When $v(x) = 0$ we get the trivial solution, so we'll only use $w'(0) = 0$. So the boundary conditions become $v(0) = 0$, $w(0) = 0$, and $w'(0) = 0$.

9. Our standard procedure here will be to start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

$$x_3(t) = y''(t)$$

Now we need to solve the original differential equation for $y'''(t)$.

$$y''' - 3y'' + 2y' = e^{-t} - 3\sin 3t$$

$$y''' = 3y'' - 2y' + e^{-t} - 3\sin 3t$$

Then if we take the derivatives of the equations for $x_1(t)$, $x_2(t)$, and $x_3(t)$, we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = x_3(t)$$

$$x_3'(t) = y'''(t) = 3y'' - 2y' + e^{-t} - 3\sin 3t = 3x_3(t) - 2x_2(t) + e^{-t} - 3\sin 3t$$

Simplifying these equations gives us a system of equations that's equivalent to the original third order differential equation.

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3x_3 - 2x_2 + e^{-t} - 3\sin 3t$$

And if we wanted to write this nonhomogeneous system as a matrix equation, we would get

$$\vec{x}' = A\vec{x} + F$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ e^{-t} - 3\sin 3t \end{bmatrix}$$

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3 - \lambda \end{bmatrix}$$

and then find its determinant $|A-\lambda I|$.

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 0 & -2 & 3-\lambda \end{vmatrix}$$

$$|A-\lambda I| = -\lambda \begin{vmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{vmatrix}$$

$$|A-\lambda I| = -\lambda[(-\lambda)(-3-\lambda)-(1)(-2)]$$

$$|A-\lambda I| = -\lambda(3\lambda+\lambda^2+2)$$

$$|A-\lambda I| = -\lambda(\lambda^2+3\lambda+2)$$

$$|A-\lambda I| = -\lambda(\lambda+1)(\lambda+2)$$

Solve the characteristic equation for the Eigenvalues.

$$-\lambda(\lambda+1)(\lambda+2)=0$$

$$\lambda = 0, \ -1, \ -2$$

We'll handle $\lambda_1 = 0$ first, starting by finding $A-(0)I$.

$$A-(0)I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix}$$

We get the system of equations

$$k_2 = 0$$

$$k_3 = 0$$

If we choose $k_1 = 1$, then the Eigenvector is

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and therefore the solution vector is

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

For $\lambda_2 = -1$, we start by finding $A - (-1)I$.

$$A - (-1)I = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From this matrix, we find $k_1 = k_2 = k_3 = 0$, so the Eigenvector is

$$\vec{k_2} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

and therefore the solution vector is

$$\vec{x_2} = \vec{k_2} e^{\lambda_2 t} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} e^{-t}$$

For $\lambda_3 = -2$, we start by finding $A - (-2)I$.

$$A-(-2)I=\begin{bmatrix}2&1&0\\0&2&1\\0&-2&5\end{bmatrix}\to\begin{bmatrix}1&\frac{1}{2}&0\\0&2&1\\0&-2&5\end{bmatrix}\to\begin{bmatrix}1&0&-\frac{1}{4}\\0&1&\frac{1}{2}\\0&0&6\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

From this matrix, we find $k_1=k_2=k_3=0$, so the Eigenvector is

$$\overrightarrow{k_3}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

and therefore the solution vector is

$$\overrightarrow{x_3}=\overrightarrow{k_3}e^{\lambda_3 t}=\begin{bmatrix}0\\0\\0\end{bmatrix}e^{-2t}$$

So the complementary solution to the homogeneous system is

$$\overrightarrow{x_c}=c_1\overrightarrow{x_1}+c_2\overrightarrow{x_2}+c_3\overrightarrow{x_3}$$

$$\overrightarrow{x_c}=c_1\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Now we can turn to finding the particular solution. We’ll rewrite the forcing function vector as

$$F=\begin{bmatrix}0\\0\\e^{-t}-3\sin 3t\end{bmatrix}$$

$$F=\begin{bmatrix}0\\0\\1\end{bmatrix}e^{-t}+\begin{bmatrix}0\\0\\-3\end{bmatrix}\sin 3t$$

We want a particular solution in the same form, so our guess will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^{-t} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \sin 3t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \cos 3t$$

$$\vec{x_p'} = \begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} e^{-t} + \begin{bmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \end{bmatrix} \cos 3t + \begin{bmatrix} -3c_1 \\ -3c_2 \\ -3c_3 \end{bmatrix} \sin 3t$$

Then we'll plug these into the matrix equation representing the system of differential equations, $\vec{x}' = A\vec{x} + F$.

$$\begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} e^{-t} + \begin{bmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \end{bmatrix} \cos 3t + \begin{bmatrix} -3c_1 \\ -3c_2 \\ -3c_3 \end{bmatrix} \sin 3t$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 3 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^{-t} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \sin 3t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \cos 3t \right]$$

$$+ \begin{bmatrix} 0 \\ 0 \\ e^{-t} - 3 \sin 3t \end{bmatrix}$$

$$\begin{bmatrix} -a_1 \\ -a_2 \\ -a_3 \end{bmatrix} e^{-t} + \begin{bmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \end{bmatrix} \cos 3t + \begin{bmatrix} -3c_1 \\ -3c_2 \\ -3c_3 \end{bmatrix} \sin 3t$$

$$= \begin{bmatrix} a_2 \\ a_3 \\ -2a_2 + 3a_3 + 1 \end{bmatrix} e^{-t} + \begin{bmatrix} b_2 \\ b_3 \\ -2b_2 + 3b_3 - 3 \end{bmatrix} \sin 3t + \begin{bmatrix} c_2 \\ c_3 \\ -2c_2 + 3c_3 \end{bmatrix} \cos 3t$$

Breaking this equation into a system of equations gives

$$-a_1 = a_2 \qquad 3b_1 = c_2 \qquad -3c_1 = b_2$$

$$-a_2 = a_3 \qquad 3b_2 = c_3 \qquad -3c_2 = b_3$$

$$-a_3 = -2a_2 + 3a_3 + 1 \qquad 3b_3 = -2c_2 + 3c_3 \qquad -3c_3 = -2b_2 + 3b_3 - 3$$

Putting these results together gives $\overrightarrow{a} = (a_1, a_2, a_3) = (-1/6, 1/6, -1/6)$, $\overrightarrow{b} = (b_1, b_2, b_3) = (-9/130, 21/130, 81/130)$, and $\overrightarrow{c} = (c_1, c_2, c_3) = (-7/130, -27/130, 63/130)$.

Therefore, the particular solution is

$$\overrightarrow{x_p} = \frac{1}{6} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} e^{-t} + \frac{3}{130} \begin{bmatrix} -3 \\ 7 \\ 27 \end{bmatrix} \sin 3t + \frac{1}{130} \begin{bmatrix} -7 \\ -27 \\ 63 \end{bmatrix} \cos 3t$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{6} \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix} e^{-t} + \frac{3}{130} \begin{bmatrix} -3 \\ 7 \\ 27 \end{bmatrix} \sin 3t + \frac{1}{130} \begin{bmatrix} -7 \\ -27 \\ 63 \end{bmatrix} \cos 3t$$

10. For A_0 we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{0} 2x+2\ dx + \frac{1}{2L}\int_{0}^{L} 4x-3x^2\ dx$$

$$A_0 = \frac{1}{2L}(x^2+2x)\Big|_{-L}^{0} + \frac{1}{2L}(2x^2-x^3)\Big|_{0}^{L}$$

$$A_0 = \frac{1}{2L}(-L^2+2L+2L^2-L^3)$$

$$A_0 = \frac{1}{2L}(2L+L^2-L^3)$$

$$A_0 = 1+\frac{L}{2}-\frac{L^2}{2}$$

And for A_n we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{0} (2x+2)\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} (4x-3x^2)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=2x+2$, $du=2\ dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$ to evaluate the first integral.

$$A_n = \frac{1}{L}\left[(2x+2)\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{2L}{n\pi}\int \sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_{-L}^{0}$$

$$+\frac{1}{L}\int_0^L (4x-3x^2)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{1}{L}\left[\frac{2L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{0}+\frac{1}{L}\int_0^L (4x-3x^2)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{1}{L}\left[\frac{2L^2}{(n\pi)^2}(1-(-1)^n)\right]+\frac{1}{L}\int_0^L (4x-3x^2)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{2L}{(n\pi)^2}(1-(-1)^n)+\frac{1}{L}\int_0^L (4x-3x^2)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=4x-3x^2$, $du=(4-6x)\ dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$ to evaluate the second integral.

$$A_n=\frac{2L}{(n\pi)^2}(1-(-1)^n)+\frac{1}{L}\int_0^L (4x-3x^2)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{2L}{(n\pi)^2}(1-(-1)^n)$$

$$+\frac{1}{L}\left[(4x-3x^2)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)-\frac{L}{n\pi}\int (4-6x)\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$A_n=\frac{2L}{(n\pi)^2}(1-(-1)^n)-\frac{1}{n\pi}\int_0^L (4-6x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=4-6x$, $du=-6\ dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$ to evaluate the second integral.

$$A_n = \frac{2L}{(n\pi)^2}(1-(-1)^n)$$

$$-\frac{1}{n\pi}\left[-(4-6x)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{6L}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$A_n = \frac{2L}{(n\pi)^2}(1-(-1)^n)$$

$$-\frac{1}{n\pi}\left[-(4-6x)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{6L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$A_n = \frac{2L}{(n\pi)^2}(1-(-1)^n) - \frac{1}{n\pi}\left[-(4-6x)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$A_n = \frac{2L}{(n\pi)^2}(1-(-1)^n) - \frac{1}{n\pi}\left[(6L-4)\frac{L}{n\pi}(-1)^n + \frac{4L}{n\pi}\right]$$

$$A_n = \frac{2L}{(n\pi)^2}(1-(-1)^n) - \frac{L}{(n\pi)^2}[(6L-4)(-1)^n + 4]$$

$$A_n = \frac{L}{(n\pi)^2}[2 - 2(-1)^n - (6L-4)(-1)^n - 4]$$

$$A_n = \frac{L}{(n\pi)^2}[(2-6L)(-1)^n - 2]$$

For B_n we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{0} (2x+2)\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} (4x-3x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 2x + 2$, $du = 2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ to evaluate the first integral.

$$B_n = \left[-(2x+2)\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_{-L}^{0}$$

$$+\frac{1}{L}\int_{0}^{L} (4x-3x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \left[-(2x+2)\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{0}$$

$$+\frac{1}{L}\int_{0}^{L} (4x-3x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \left[-(2x+2)\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{0} + \frac{1}{L}\int_{0}^{L} (4x-3x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \left[-\frac{2}{n\pi} + (-2L+2)\frac{1}{n\pi}(-1)^n\right] + \frac{1}{L}\int_{0}^{L} (4x-3x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{n\pi}[(-2L+2)(-1)^n - 2] + \frac{1}{L}\int_{0}^{L} (4x-3x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 4x - 3x^2$, $du = (4 - 6x)\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ to evaluate the first integral.

$$B_n = \frac{1}{n\pi}[(-2L+2)(-1)^n - 2]$$

$$+\left[-(4x-3x^2)\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{n\pi}\int (4-6x)\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \frac{1}{n\pi}[(-2L+2)(-1)^n - 2] + \frac{(3L^2-4L)}{n\pi}(-1)^n$$

$$+\left[\frac{1}{n\pi}\int (4-6x)\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = 4 - 6x$, $du = -6\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ to evaluate the first integral.

$$B_n = \frac{1}{n\pi}[(3L^2-6L+2)(-1)^n - 2]$$

$$+\frac{1}{n\pi}\left[(4-6x)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{6L}{n\pi}\int \sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \frac{1}{n\pi}[(3L^2-6L+2)(-1)^n - 2] + \frac{1}{n\pi}\left[\frac{6L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \frac{1}{n\pi}[(3L^2-6L+2)(-1)^n - 2] + \frac{1}{n\pi}\left[\frac{6L^2}{(n\pi)^2}((-1)^n - 1)\right]$$

$$B_n = \frac{1}{n\pi}[(3L^2-6L+2)(-1)^n - 2] + \frac{6L^2}{(n\pi)^3}((-1)^n - 1)$$

$$B_n = \frac{(3L^2 - 6L + 2)(n\pi)^2(-1)^n + 6L^2(-1)^n - 6L^2 - 2(n\pi)^2}{(n\pi)^3}$$

Then the Fourier series representation of the piecewise function on $-L \leq x \leq L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 1 + \frac{L}{2} - \frac{L^2}{2} + \sum_{n=1}^{\infty} \frac{L((2 - 6L)(-1)^n - 2)}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right)$$

$$+ \sum_{n=1}^{\infty} \frac{(3L^2 - 6L + 2)(n\pi)^2(-1)^n + 6L^2(-1)^n - 6L^2 - 2(n\pi)^2}{(n\pi)^3} \sin\left(\frac{n\pi x}{L}\right)$$

11. If we start with the product solution $u(x, y) = v(x)w(y)$, then we know

$$v'(x)w(y) - 16v(x)w''(y) = 0$$

$$\frac{v'}{v}(x) = \frac{16w''}{w}(y) = -\lambda$$

Break this equation into two ordinary differential equations.

$$v' = -\lambda v \qquad\qquad 16w'' = -\lambda w$$

The solution to the first equation, which is a linear differential equation, is $v(x) = Ce^{-\lambda x}$. To solve the second equation, we find the associated characteristic equation.

$$16r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm (1/4)\sqrt{-\lambda}$ and the solution is

$$w(y) = c_1 \cos \frac{\sqrt{\lambda}}{4} y + c_2 \sin \frac{\sqrt{\lambda}}{4} y$$

Applying the boundary conditions gives

$$w(0) = c_1 = 0$$

$$w'(\pi) = \frac{c_2\sqrt{\lambda}}{4} \sin \frac{\sqrt{\lambda}}{4} \pi = 0$$

$$\frac{\sqrt{\lambda}}{4} \pi = \frac{\pi}{2} + n\pi$$

$$\lambda = (2 + 4n)^2$$

So,

$$w(y) = c_2 \sin \left(\frac{1}{2} + n \right) y$$

Plugging the value of λ we found into the solution equation for v, we get

$$v(x) = Ce^{-\lambda x}$$

$$v(x) = Ce^{-(2+4n)^2 x}$$

Putting our results together from the first and second order equations, we get the product solution to the heat equation.

$$u(x, y) = v(x)w(y)$$

$$u_n(x, y) = \sum_{n=0}^{\infty} B_n \sin\left(\left(\frac{1}{2} + n\right) y\right) e^{-(2+4n)^2 x} \text{ with } n = 1,\ 2,\ 3,\ \dots$$

Matching this up to the boundary condition

$$u(0,y) = 3 \sin\left(\frac{3y}{2}\right)$$

gives

$$\sum_{n=0}^{\infty} B_n \sin\left(\left(\frac{1}{2} + n\right) y\right) = 3 \sin\left(\frac{3y}{2}\right)$$

So we have $B_1 = 3$ and $B_0 = B_2 = \dots = 0$.

$$u(x, y) = 3e^{-36x} \sin\left(\frac{3y}{2}\right)$$

12. We'll substitute y' and y'' into the differential equation.

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^{n-1} = 0$$

The series are in phase, but the indices don't match. We can substitute $k = n - 2$ and $n = k + 2$ into the first series, and $k = n - 1$ and $n = k + 1$ into the second series.

$$\sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=0}^{\infty} c_{k+1}(k+1)x^k = 0$$

With the series in phase and matching indices, we can now add them.

$$\sum_{k=0}^{\infty} [c_{k+2}(k+2)(k+1) - c_{k+1}(k+1)]x^k = 0$$

$$k = 0,\ 1,\ 2,\ 3,\ \dots \quad c_{k+2}(k+2)(k+1) - c_{k+1}(k+1) = 0$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+2}.

$$c_{k+2} = \frac{c_{k+1}}{(k+2)}$$

Now we'll start plugging in values $k = 0,\ 1,\ 2,\ 3,\ \dots$.

$$k = 0 \qquad c_2 = \frac{c_1}{2!} \qquad\qquad k = 1 \qquad c_3 = \frac{c_2}{3} = \frac{c_1}{3!}$$

$$k = 2 \qquad c_4 = \frac{c_3}{4} = \frac{c_1}{4!} \qquad\qquad k = 3 \qquad c_5 = \frac{c_4}{5} = \frac{c_1}{5!}$$

$$c_k = \frac{c_1}{k!}$$

Now we can write the general solution to the differential equation as

$$y(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \dots$$

$$y(x) = c_0 + c_1x + \frac{c_1}{2!}x^2 + \frac{c_1}{3!}x^3 + \frac{c_1}{4!}x^4 + \cdots$$

$$y(x) = c_0 + c_1 \sum_{k=1}^{\infty} \frac{1}{k!} x^k$$

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