Differential Equations Workbook Solutions

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MATH

CLASSIFYING DIFFERENTIAL EQUATIONS

■ 1. Identify the order and linearity of the differential equation.

$$5y''' + 4xy = x^2$$

Solution:

The equation $5y''' + 4xy = x^2$ contains a third derivative, so it's a third order equation. We can see that the equation is linear, because the equation does not contain functions of y other than y itself and its derivatives.

■ 2. Say whether or not the linear equation is homogeneous.

$$y'' - x = 3y$$

Solution:

We can rewrite the equation as $y'' - 3y = x$, which allows us to see that the equation is linear and non-homogeneous.

■ 3. Identify the order and linearity of each differential equation.

$$x^3 + 3x \sin x = y'$$

$$x\frac{d^4y}{dx^4} - x\ln x = 0$$

$$y'' - 3y' + xy^2 = x$$

Solution:

The equation $x^3 + 3x\sin x = y'$ contains a first derivative, so it's a first order equation. We can rewrite it as $y' = x^3 + 3x\sin x$, which allows us to see that the equation is linear.

The equation

$$x\frac{d^4y}{dx^4} - x\ln x = 0$$

contains a fourth derivative, so it's a fourth order equation. We can rewrite it as $d^4y/dx^4 = \ln x$, which allows us to see that the equation is linear.

The equation $y'' - 3y' + xy^2 = x$ contains a second derivative, so it's a second order equation. Because the equation contains y^2, it's non-linear.

■ 4. Determine whether each linear equation is homogeneous or non-homogeneous.

$$y' - 3y = 0$$

$$(\sin x)y' = 0$$

$$\ln x - 3y'' = y$$

Solution:

Since $q(x) = 0$, the equation $y' - 3y = 0$ is homogeneous.

Since $q(x) = 0$, the equation $(\sin x)y' = 0$ is homogeneous.

We can rewrite $\ln x - 3y'' = y$ as $3y'' + y = \ln x$, which allows us to see that the equation is linear and non-homogeneous.

■ 5. Identify the order and linearity of each differential equation, then say whether or not each linear equation is homogeneous.

$$5y' - y''' - \ln y = x$$

$$y'' = 4y^2$$

$$y' = xy$$

Solution:

The equation $5y' - y''' - \ln y = x$ contains a third derivative, so it's a third order equation. We can rewrite it as $y''' - 5y' = -x - \ln y$, which allows us to see that the equation is non-linear, because the equation contains $\ln y$.

The equation $y'' = 4y^2$ contains a second derivative, so it's a second order equation. Because the equation contains y^2, it's non-linear.

The equation $y' = xy$ contains a first derivative, so it's a first order equation. We can rewrite it as $y' - xy = 0$, which allows us to see that the equation is linear and homogeneous.

6. Identify the order and linearity of the differential equation.

$$e^x y''' = e^y x$$

Solution:

The equation $e^x y''' = e^y x$ contains a third derivative, so it's a third order equation. Because the equation contains e^y, it's non-linear.

LINEAR EQUATIONS

■ 1. Find the solution to the linear differential equation.

$$y' - y\sin x = 0$$

Solution:

The linear differential equation is already in standard form, so we can use $P(x) = -\sin x$ to find the integrating factor.

$$\mu(x) = e^{\int -\sin x \, dx}$$

$$\mu(x) = e^{\cos x}$$

Multiply through the differential equation by the integrating factor.

$$e^{\cos x}\left(\frac{dy}{dx} - y\sin x = 0\right)$$

$$e^{\cos x}\frac{dy}{dx} - ye^{\cos x}\sin x = 0$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(ye^{\cos x}) = 0$$

then integrate and solve for y.

$$\int \frac{d}{dx}(ye^{\cos x}) = \int 0\ dx$$

$$ye^{\cos x} = C$$

$$y = Ce^{-\cos x}$$

■ 2. Solve the differential equation.

$$y' \cos x + y \sin x = 1$$

Solution:

Put the linear differential equation in standard form.

$$y' \cos x + y \sin x = 1$$

$$y' + y \tan x = \sec x$$

Use $P(x) = \tan x$ to find the integrating factor.

$$\mu(x) = e^{\int \tan x\ dx}$$

$$\mu(x) = e^{-\ln(\cos x)}$$

$$\mu(x) = \frac{1}{\cos x}$$

Multiply through the differential equation by the integrating factor.

$$\frac{1}{\cos x}(y' + y\tan x = \sec x)$$

$$\frac{1}{\cos x}y' + y\frac{\tan x}{\cos x} = \frac{\sec x}{\cos x}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}\left(\frac{1}{\cos x}y\right) = \frac{1}{\cos^2 x}$$

then integrate and solve for y.

$$\int \frac{d}{dx}\left(\frac{1}{\cos x}y\right) = \int \frac{1}{\cos^2 x}\ dx$$

$$\frac{1}{\cos x}y = \tan x + C$$

$$y = \cos x(\tan x + C)$$

$$y = \sin x + C\cos x$$

■ 3. Find the solution to the linear differential equation.

$$xy' + y = e^x$$

Solution:

Put the linear differential equation in standard form.

$$y' + \frac{1}{x}y = \frac{e^x}{x}$$

Use $P(x) = 1/x$ to find the integrating factor.

$$\mu(x) = e^{\int \frac{1}{x}\,dx}$$

$$\mu(x) = e^{\ln x}$$

$$\mu(x) = x$$

Multiply through the differential equation by the integrating factor.

$$x\left(y' + \frac{1}{x}y = \frac{e^x}{x}\right)$$

$$xy' + y = e^x$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(xy) = e^x$$

then integrate and solve for y.

$$xy = e^x + C$$

$$y = \frac{e^x + C}{x}$$

■ 4. Solve the differential equation.

$$xy' = x^3 - 3x^3y$$

Solution:

Put the linear differential equation in standard form.

$$xy' + 3x^3y = x^3$$

$$y' + 3x^2y = x^2$$

Use $P(x) = 3x^2$ to find the integrating factor.

$$\mu(x) = e^{\int 3x^2\ dx}$$

$$\mu(x) = e^{x^3}$$

Multiply through the differential equation by the integrating factor.

$$e^{x^3}(y' + 3x^2y = x^2)$$

$$e^{x^3}y' + 3x^2ye^{x^3} = x^2e^{x^3}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{x^3}y) = x^2e^{x^3}$$

then integrate, using a substitution with $u = x^3$ and $du = 3x^2\ dx$ to integrate the right side.

$$\int \frac{d}{dx}(e^{x^3}y) = \int x^2e^{x^3}\ dx$$

$$e^{x^3}y = \int \frac{e^u}{3}\ du$$

$$e^{x^3}y = \frac{1}{3}e^u + C$$

$$e^{x^3}y = \frac{1}{3}e^{x^3} + C$$

Solve for y to find the solution to the linear differential equation.

$$y = \frac{1}{3} + Ce^{-x^3}$$

■ 5. Find the solution to the linear differential equation.

$$xy' - 3y = x^2 - x + 1$$

Solution:

Put the linear differential equation in standard form.

$$xy' - 3y = x^2 - x + 1$$

$$y' - \frac{3}{x}y = x - 1 + \frac{1}{x}$$

Now we can use $P(x) = -3/x$ to find the integrating factor.

$$\mu(x) = e^{\int -\frac{3}{x}\ dx}$$

$$\mu(x) = e^{-3\ln x}$$

$$\mu(x) = x^{-3}$$

Multiply through the differential equation by the integrating factor.

$$x^{-3}\left(y'-\frac{3}{x}y=x-1+\frac{1}{x}\right)$$

$$x^{-3}y'-3x^{-4}y=x^{-2}-x^{-3}+x^{-4}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(x^{-3}y)=x^{-2}-x^{-3}+x^{-4}$$

then integrate and solve for y.

$$\int\frac{d}{dx}(x^{-3}y)=\int x^{-2}-x^{-3}+x^{-4}\ dx$$

$$x^{-3}y=-x^{-1}+\frac{1}{2}x^{-2}-\frac{1}{3}x^{-3}+C$$

$$x^{-3}y=-\frac{1}{x}+\frac{1}{2x^2}-\frac{1}{3x^3}+C$$

$$y=Cx^3-x^2+\frac{1}{2}x-\frac{1}{3}$$

■ 6. Solve the differential equation.

$$xy'-y=\sqrt{x}$$

Solution:

Put the linear differential equation in standard form.

$$y' - \frac{y}{x} = \frac{1}{\sqrt{x}}$$

Now we can use $P(x) = -1/x$ to find the integrating factor.

$$\mu(x) = e^{\int -\frac{1}{x}\ dx}$$

$$\mu(x) = e^{-\ln x}$$

$$\mu(x) = -\frac{1}{x}$$

Multiply through the differential equation by the integrating factor.

$$-\frac{1}{x}\left(y' - \frac{y}{x} = \frac{1}{\sqrt{x}}\right)$$

$$-\frac{1}{x}y' + \frac{y}{x^2} = -\frac{1}{x\sqrt{x}}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}\left(-\frac{y}{x}\right) = -\frac{1}{x\sqrt{x}}$$

then integrate and solve for y.

$$\int \frac{d}{dx}\left(-\frac{y}{x}\right) = \int -\frac{1}{x\sqrt{x}}\ dx$$

$$-\frac{y}{x}=\frac{2}{\sqrt{x}}+C$$

$$\frac{y}{x}=-\frac{2}{\sqrt{x}}+C$$

$$y=-2\sqrt{x}+Cx$$

INITIAL VALUE PROBLEMS

■ 1. Solve the initial value problem if $y(0)=0$.

$$5y'-10xy=25x$$

Solution:

Put the linear differential equation in standard form.

$$y'-2xy=5x$$

Now use $P(x)=-2x$ to find the integrating factor.

$$\mu(x)=e^{\int -2x\ dx}$$

$$\mu(x)=e^{-x^2}$$

Multiply through the differential equation by the integrating factor.

$$e^{-x^2}\left(\frac{dy}{dx}-2xy=5x\right)$$

$$e^{-x^2}\frac{dy}{dx}-2xye^{-x^2}=5xe^{-x^2}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{-x^2}y)=5xe^{-x^2}$$

then integrate, using a substitution with $u = -x^2$ and $du = -2x\ dx$ to integrate the right side.

$$\int \frac{d}{dx}(e^{-x^2}y) = \int 5xe^{-x^2}\ dx$$

$$e^{-x^2}y = \int -\frac{5}{2}e^u\ du$$

$$e^{-x^2}y = -\frac{5}{2}e^u + C$$

$$e^{-x^2}y = -\frac{5}{2}e^{-x^2} + C$$

$$y = -\frac{5}{2} + Ce^{x^2}$$

Once we have this general solution, we recognize from the initial condition $y(0) = 0$ that $x = 0$ and $y = 0$, so we'll plug these values into the general solution to solve for C.

$$0 = -\frac{5}{2} + Ce^{(0)^2}$$

$$0 = -\frac{5}{2} + C$$

$$C = \frac{5}{2}$$

So the solution to the differential equation is

$$y = -\frac{5}{2} + \frac{5}{2}e^{x^2}$$

2. Solve the initial value problem if $y(1) = -3$.

$$x\frac{dy}{dx} + 2y = 6x^2$$

Solution:

Put the linear differential equation in standard form.

$$x\frac{dy}{dx} + 2y = 6x^2$$

$$\frac{dy}{dx} + \frac{2}{x}y = 6x$$

Now we can use $P(x) = 2/x$ to find the integrating factor.

$$\mu(x) = e^{\int \frac{2}{x}\ dx}$$

$$\mu(x) = e^{2\ln x}$$

$$\mu(x) = x^2$$

Multiply through the differential equation by the integrating factor.

$$x^2\left(\frac{dy}{dx} + \frac{2}{x}y = 6x\right)$$

$$x^2\frac{dy}{dx} + 2xy = 6x^3$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(x^2y) = 6x^3$$

then integrate.

$$\int \frac{d}{dx}(x^2y) = \int 6x^3\ dx$$

$$x^2y = \frac{3}{2}x^4 + C$$

$$y = \frac{3}{2}x^2 + Cx^{-2}$$

Once we have this general solution, we recognize from the initial condition $y(1) = -3$ that $x = 1$ and $y = -3$, so we'll plug these values into the general solution to solve for C.

$$-3 = \frac{3}{2}(1)^2 + C(1)^{-2}$$

$$-3 = \frac{3}{2} + C$$

$$C = -\frac{9}{2}$$

So the solution to the differential equation is

$$y = \frac{3}{2}x^2 - \frac{9}{2}x^{-2}$$

■ 3. Solve the initial value problem if $y(0)=1$.

$$y'-y=2\sin(3x)$$

Solution:

The linear differential equation is already in standard form, so we can use $P(x)=-1$ to find the integrating factor.

$$\mu(x)=e^{\int -1\ dx}$$

$$\mu(x)=e^{-x}$$

Multiply through the differential equation by the integrating factor.

$$e^{-x}\left(\frac{dy}{dx}-y=2\sin(3x)\right)$$

$$e^{-x}\frac{dy}{dx}-ye^{-x}=2\sin(3x)e^{-x}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{-x}y)=2\sin(3x)e^{-x}$$

then integrate, using integration by parts to integrate the right side.

$$\int\frac{d}{dx}(e^{-x}y)=\int 2\sin(3x)e^{-x}\ dx$$

$$e^{-x}y=-\frac{1}{5}e^{-x}(\sin(3x)+3\cos(3x))+C$$

$$y = -\frac{1}{5}(\sin(3x) + 3\cos(3x)) + Ce^x$$

Once we have this general solution, we recognize from the initial condition $y(0) = 1$ that $x = 0$ and $y = 1$, so we'll plug these values into the general solution to solve for C.

$$1 = -\frac{1}{5}(\sin(3(0)) + 3\cos(3(0))) + Ce^0$$

$$1 = -\frac{3}{5} + C$$

$$C = \frac{8}{5}$$

So the solution to the differential equation is

$$y = -\frac{1}{5}(\sin(3x) + 3\cos(3x)) + \frac{8}{5}e^x$$

■ 4. A function $y(x)$ is a solution of the differential equation. Suppose that $y(1) = 1$ and $y(3) = 3$. Find the constant k and the solution $y(x)$.

$$y' - \frac{ky}{x} = 0$$

Solution:

The linear differential equation is already in standard form, so we can use $P(x) = -k/x$ to find the integrating factor.

$$\mu(x) = e^{\int -\frac{k}{x} \ dx}$$

$$\mu(x) = e^{-k \ln x}$$

$$\mu(x) = x^{-k}$$

Multiply through the differential equation by the integrating factor.

$$x^{-k}\left(\frac{dy}{dx} - \frac{ky}{x} = 0\right)$$

$$x^{-k}\frac{dy}{dx} - \frac{ky}{x}x^{-k} = 0$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(x^{-k}y) = 0$$

then integrate.

$$\int \frac{d}{dx}(x^{-k}y) = \int 0 \ dx$$

$$x^{-k}y = C$$

$$y = Cx^k$$

Once we have this general solution, we recognize from the initial condition $y(1) = 1$ that $x = 1$ and $y = 1$ and from the initial condition $y(3) = 3$ that $x = 3$ and $y = 3$, so we'll plug these values into the general solution,

$$1 = C(1)^k$$

$$3 = C(3)^k$$

and then simplify to find C and k.

$$C = 1$$

$$3 = 1(3)^k$$

$$k = 1$$

So the solution to the differential equation is

$$y = 1x^1$$

$$y = x$$

■ 5. Solve the initial value problem if $y(\ln 2) = 1$.

$$\frac{dy}{dx} - 6y = 2$$

Solution:

The linear differential equation is already in standard form, so we can use $P(x) = -6$ to find the integrating factor.

$$\mu(x) = e^{\int -6\, dx}$$

$$\mu(x) = e^{-6x}$$

Multiply through the differential equation by the integrating factor.

$$e^{-6x}\left(\frac{dy}{dx} - 6y = 2\right)$$

$$e^{-6x}\frac{dy}{dx} - 6ye^{-6x} = 2e^{-6x}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{-6x}y) = 2e^{-6x}$$

then integrate.

$$\int \frac{d}{dx}(e^{-6x}y) = \int 2e^{-6x}\ dx$$

$$e^{-6x}y = -\frac{1}{3}e^{-6x} + C$$

$$y = -\frac{1}{3} + Ce^{6x}$$

Once we have this general solution, we recognize from the initial condition $y(\ln 2) = 1$ that $x = \ln 2$ and $y = 1$, so we'll plug these values into the general solution to solve for C.

$$1 = -\frac{1}{3} + Ce^{6\ln 2}$$

$$1 = -\frac{1}{3} + 2^6C$$

$$C = \frac{1}{48}$$

So the particular solution to the differential equation is

$$y = -\frac{1}{3} + \frac{1}{48}e^{6x}$$

■ 6. A function $y(x)$ is a solution of $y' + x^k y = 0$. Suppose that $y(0) = 1$ and $y(1) = e^{-5}$. Find the constant k and the solution $y(x)$.

Solution:

The linear differential equation is already in standard form, so we can use $P(x) = x^k$ to find the integrating factor.

$$\mu(x) = e^{\int x^k \, dx}$$

$$\mu(x) = e^{\frac{x^{k+1}}{k+1}}$$

Multiply through the differential equation by the integrating factor.

$$e^{\frac{x^{k+1}}{k+1}}\left(\frac{dy}{dx} + x^k y = 0\right)$$

$$e^{\frac{x^{k+1}}{k+1}}\frac{dy}{dx} + x^k y e^{\frac{x^{k+1}}{k+1}} = 0$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{\frac{x^{k+1}}{k+1}}y) = 0$$

then integrate.

$$\int \frac{d}{dx}\left(e^{\frac{x^{k+1}}{k+1}}y\right) = \int 0\ dx$$

$$e^{\frac{x^{k+1}}{k+1}}y = C$$

$$y = Ce^{-\frac{x^{k+1}}{k+1}}$$

Once we have this general solution, we recognize from the initial condition $y(0) = 1$ that $x = 0$ and $y = 1$, and from the initial condition $y(1) = e^{-5}$ that $x = 1$ and $y = e^{-5}$, so we'll plug these values into the general solution,

$$1 = Ce^{-\frac{0^{k+1}}{k+1}}$$

$$e^{-5} = Ce^{-\frac{1^{k+1}}{k+1}}$$

and then simplify these to solve for C and k.

$$C = 1$$

$$e^{-5} = e^{-\frac{1}{k+1}}$$

$$5(k+1) = 1$$

$$k + 1 = \frac{1}{5}$$

$$k = -\frac{4}{5}$$

So the solution to the differential equation is

$$y = e^{-5x^{\frac{1}{5}}}$$

SEPARABLE EQUATIONS

■ 1. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

Solution:

Separate variables, moving y to the left and x to the right.

$$dy = \frac{3x^2}{y}\ dx$$

$$y\ dy = 3x^2\ dx$$

Integrate both sides, adding the constant of integration C to the right side.

$$\int y\ dy = \int 3x^2\ dx$$

$$\frac{1}{2}y^2 = x^3 + C$$

Solve for y to find the solution to the separable differential equation.

$$y^2 = 2(x^3 + C)$$

$$y = \pm\sqrt{2(x^3 + C)}$$

■ 2. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{x+2}{2y}$$

Solution:

Separate variables, moving y to the left and x to the right.

$$dy = \frac{x+2}{2y}\ dx$$

$$2y\ dy = (x+2)\ dx$$

Integrate both sides, adding the constant of integration C to the right side.

$$\int 2y\ dy = \int x+2\ dx$$

$$y^2 = \frac{1}{2}x^2 + 2x + C$$

Solve for y to find the solution to the separable differential equation.

$$y = \pm\sqrt{\frac{1}{2}x^2 + 2x + C}$$

■ 3. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{4x^3 - 1}{x(y^2 - 1)}$$

Solution:

Separate variables, moving y to the left and x to the right.

$$dy = \frac{4x^3 - 1}{x(y^2 - 1)}\ dx$$

$$(y^2 - 1)\ dy = \frac{4x^3 - 1}{x}\ dx$$

$$(y^2 - 1)\ dy = \left(\frac{4x^3}{x} - \frac{1}{x}\right)\ dx$$

$$(y^2 - 1)\ dy = \left(4x^2 - \frac{1}{x}\right)\ dx$$

Integrate both sides, adding the constant of integration C to the right side.

$$\int y^2 - 1\ dy = \int 4x^2 - \frac{1}{x}\ dx$$

$$\frac{1}{3}y^3 - y = \frac{4}{3}x^3 - \ln|x| + C$$

Solve for y to find the solution to the separable differential equation.

$$y^3 - 3y = 4x^3 - 3\ln|x| + 3C$$

$$y^3 - 3y = 4x^3 - 3\ln|x| + C$$

We can't solve this equation explicitly for y, so we'll leave this as the implicitly-defined general solution.

■ 4. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{x^2\sqrt{x^3 - 3}}{y^2}$$

Solution:

Start by separating variables, putting y terms on the left and x terms on the right.

$$dy = \frac{x^2\sqrt{x^3 - 3}}{y^2}\ dx$$

$$y^2\ dy = x^2\sqrt{x^3 - 3}\ dx$$

Now integrate both sides, using a substitution with $u = x^3 - 3$ and $du/3 = x^2\ dx$, and adding the constant of integration C to the right side.

$$\int y^2\ dy = \int \sqrt{u}\left(\frac{du}{3}\right)$$

$$\frac{1}{3}y^3 = \frac{2}{9}u^{\frac{3}{2}} + C$$

Back-substitute for $u = x^3 - 3$.

$$\frac{1}{3}y^3 = \frac{2}{9}(x^3 - 3)^{\frac{3}{2}} + C$$

Solve for y to get the explicit the solution to the separable differential equation.

$$y^3 = \frac{2}{3}(x^3 - 3)^{\frac{3}{2}} + C$$

$$y = \left(\frac{2}{3}(x^3 - 3)^{\frac{3}{2}} + C\right)^{\frac{1}{3}}$$

■ 5. Find the solution to the separable differential equation.

$$y' + \sin(x - y) = \sin(x + y)$$

Solution:

Separate variables, moving y to the left and x to the right.

$$y' = \sin(x + y) - \sin(x - y)$$

$$y' = 2\cos x \sin y$$

$$dy = 2\cos x \sin y \ dx$$

$$\frac{1}{\sin y} \ dy = 2\cos x \ dx$$

Integrate both sides, adding the constant of integration C to the right side.

$$\int \frac{1}{\sin y}\ dy = \int 2\cos x\ dx$$

$$\ln \left| \tan \frac{y}{2} \right| = 2\sin x + C$$

■ 6. Solve the initial value problem if $y(1) = -1$.

$$(xy^2 + x) + (x^2y - y)y' = 0$$

Solution:

Separate variables, moving y to the left and x to the right.

$$(xy^2 + x) + (x^2y - y)y' = 0$$

$$(x^2y - y)y' = -(xy^2 + x)$$

$$(x^2y - y)\ dy = -(xy^2 + x)\ dx$$

$$(x^2 - 1)y\ dy = -(y^2 + 1)x\ dx$$

$$\frac{y}{y^2 + 1}\ dy = -\frac{x}{x^2 - 1}\ dx$$

Integrate both sides, using substitutions on both sides, and adding the constant of integration C to the right side.

$$\int \frac{y}{y^2 + 1}\ dy = \int -\frac{x}{x^2 - 1}\ dx$$

$$\frac{1}{2}\ln|y^2+1| = -\frac{1}{2}\ln|x^2-1| + C$$

Once we have this general solution, we recognize from the initial condition $y(0) = -1$ that $x = 0$ and $y = -1$, so we'll plug these values into the general solution,

$$\frac{1}{2}\ln(2) = -\frac{1}{2}\ln(1) + C$$

$$\frac{1}{2}\ln(2) = C$$

So the solution to the differential equation is

$$\frac{1}{2}\ln(y^2+1) = -\frac{1}{2}\ln|x^2-1| + \frac{1}{2}\ln(2)$$

$$\ln(y^2+1) = -\ln|x^2-1| + \ln(2)$$

$$\ln(y^2+1) = \ln\frac{2}{|x^2-1|}$$

$$y^2+1 = \frac{2}{|x^2-1|}$$

$$y^2 = \frac{2}{|x^2-1|} - 1$$

$$y = \pm\sqrt{\frac{2}{|x^2-1|} - 1}$$

SUBSTITUTIONS

■ 1. Use a substitution to solve the separable differential equation.

$$y' = \sin(x + y)$$

Solution:

If we choose the substitution $u = ax + by$, then we can set $u = x + y$, $u' = 1 + y'$, and $y' = u' - 1$. Then we'll substitute $y' = u' - 1$ into the left side of the original differential equation, and $u = x + y$ into the right side of the original differential equation, and we'll get

$$u' - 1 = \sin u$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = \sin u + 1$$

$$du = (\sin u + 1)\ dx$$

$$\frac{1}{\sin u + 1}\ du = dx$$

and then integrate both sides.

$$\int \frac{1}{\sin u + 1}\ du = \int dx$$

$$\frac{2}{\cot\frac{u}{2}+1}=x+C$$

$$\frac{2}{x+C}=\cot\frac{u}{2}+1$$

$$\frac{2}{x+C}-1=\cot\frac{u}{2}$$

We'll back-substitute using $u=x+y$, leaving the equation solution implicitly defined in x and y.

$$\frac{2}{x+C}-1=\cot\left(\frac{x+y}{2}\right)$$

■ 2. Use a substitution to solve the separable differential equation.

$$y'=(x+y-5)^2$$

Solution:

If we choose the substitution $u=ax+by$, then we can set up $u=x+y$, $u'=1+y'$, and $y'=u'-1$. Then we'll substitute $y'=u'-1$ into the left side of the original differential equation, and $u=x+y$ into the right side of the original differential equation, and we'll get

$$u'-1=(u-5)^2$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = (u-5)^2 + 1$$

$$du = ((u-5)^2 + 1)\ dx$$

$$\frac{1}{(u-5)^2 + 1}\ du = dx$$

and then integrate both sides.

$$\int \frac{1}{(u-5)^2 + 1}\ du = \int dx$$

$$-\tan^{-1}(5-u) = x + C$$

$$\tan^{-1}(5-u) = -x + C$$

$$\tan(\tan^{-1}(5-u)) = \tan(-x + C)$$

$$5 - u = \tan(-x + C)$$

$$u = 5 - \tan(-x + C)$$

We'll back-substitute using $u = x + y$ and then solve for y to find the solution to the differential equation.

$$x + y = 5 - \tan(-x + C)$$

$$y = 5 - \tan(-x + C) - x$$

■ 3. Use a substitution to solve the initial value problem, if $y(0) = 0$.

$$y' = e^{5y-x}$$

Solution:

If we choose the substitution $u = ax + by$, then we can set $u = 5y - x$, $u' = 5y' - 1$, and $y' = (u' + 1)/5$. Then we'll substitute $y' = (u' + 1)/5$ into the left side of the original differential equation, and $u = 5y - x$ into the right side of the original differential equation, and we'll get

$$\frac{u' + 1}{5} = e^u$$

$$u' + 1 = 5e^u$$

$$u' = 5e^u - 1$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = 5e^u - 1$$

$$du = (5e^u - 1)\ dx$$

$$\frac{1}{5e^u - 1}\ du = dx$$

and then integrate both sides.

$$\int \frac{1}{5e^u - 1}\ du = \int dx$$

$$\int \frac{e^{-u}}{5 - e^{-u}}\ du = \int dx$$

$$\ln(5-e^{-u}) = x + C$$

$$e^{\ln(5-e^{-u})} = e^{x+C}$$

$$5 - e^{-u} = Ce^{x}$$

$$e^{-u} = 5 - Ce^{x}$$

$$-u = \ln(5 - Ce^{x})$$

$$u = -\ln(5 - Ce^{x})$$

We'll back-substitute using $u = 5y - x$ and then solve for y to find the solution to the differential equation.

$$5y - x = -\ln(5 - Ce^{x})$$

$$5y = x - \ln(5 - Ce^{x})$$

$$y = \frac{1}{5}(x - \ln(5 - Ce^{x}))$$

Once we have this general solution, we recognize from the initial condition $y(0) = 0$ that $x = 0$ and $y = 0$, so we'll plug these values into the general solution to find C,

$$0 = \frac{1}{5}(0 - \ln(5 - Ce^{0}))$$

$$0 = \ln(5 - C)$$

$$e^{0} = e^{\ln(5-C)^{-1}}$$

$$1 = \frac{1}{5 - C}$$

$$5 - C = 1$$

$$C = 4$$

and then the solution to the differential equation is

$$y = \frac{1}{5}(x - \ln(5 - 4e^x))$$

■ 4. Use a substitution to solve the separable differential equation.

$$y' = \frac{1}{6x - 3y}$$

Solution:

If we choose the substitution $u = ax + by$, then we can set $u = 6x - 3y$, $u' = 6 - 3y'$, and $y' = 2 - (1/3)u'$. Then we'll substitute $y' = 2 - (1/3)u'$ into the left side of the original differential equation, and $u = 6x - 3y$ into the right side of the original differential equation, and we'll get

$$2 - \frac{1}{3}u' = \frac{1}{u}$$

$$2 - \frac{1}{u} = \frac{1}{3}u'$$

$$\frac{6u - 3}{u} = u'$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = \frac{6u-3}{u}$$

$$du = \left(\frac{6u-3}{u}\right) dx$$

$$\frac{u}{6u-3} du = dx$$

and then integrate both sides.

$$\int \frac{u}{6u-3} du = \int dx$$

$$\frac{1}{3}\int \frac{u}{2u-1} du = \int dx$$

$$\frac{1}{6}\left(\int \frac{1}{2u-1} du + \int du\right) = \int dx$$

$$\frac{1}{6}(\ln(2u-1) + u) = x + C$$

Simplify to find the solution to the differential equation.

$$\ln(2u-1) + u = 6x + C$$

$$\ln(12x - 6y - 1) + 6x - 3y = 6x + C$$

$$\ln(12x - 6y - 1) - 3y = C$$

■ 5. Use a substitution to solve the initial value problem, if $y(0) = 8$.

$$y' + (9x + y - 1)^2 = 0$$

Solution:

If we choose the substitution $u = ax + by$, then we can set $u = 9x + y$, $u' = 9 + y'$, and $y' = u' - 9$. Then we'll substitute $y' = u' - 9$ and $u = 9x + y$ into the left side of the original differential equation, and we'll get

$$u' - 9 + (u - 1)^2 = 0$$

$$u' = 9 - (u - 1)^2$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = 9 - (u - 1)^2$$

$$du = (9 - (u - 1)^2)\ dx$$

$$\frac{1}{9 - (u - 1)^2}\ du = dx$$

and then integrate both sides.

$$\int \frac{1}{9 - (u - 1)^2}\ du = \int dx$$

$$\int \frac{du}{9 - (u^2 - 2u + 1)} = \int dx$$

$$\int \frac{du}{9 - u^2 + 2u - 1} = \int dx$$

$$\int \frac{du}{8+2u-u^2} = \int dx$$

$$-\int \frac{du}{u^2-2u-8} = \int dx$$

$$-\int \frac{du}{(u+2)(u-4)} = \int dx$$

Use partial fractions to rewrite the integral on the left.

$$\frac{1}{6}\int \frac{1}{u+2} - \frac{1}{u-4} = \int dx$$

$$\frac{1}{6}(\ln(u+2) - \ln(u-4)) = x + C$$

$$\ln\left(\frac{u+2}{u-4}\right) = 6x + C$$

Raise both sides to the base e, then solve for u.

$$\frac{u+2}{u-4} = Ce^{6x}$$

$$u + 2 = (u-4)Ce^{6x}$$

$$u + 2 = uCe^{6x} - 4Ce^{6x}$$

$$u - uCe^{6x} = -2 - 4Ce^{6x}$$

$$u(1 - Ce^{6x}) = -2 - 4Ce^{6x}$$

$$u = \frac{-2 - 4Ce^{6x}}{1 - Ce^{6x}}$$

We'll back-substitute using $u = 9x + y$ and then solve for y to find the solution to the differential equation.

$$9x + y = \frac{-2 - 4Ce^{6x}}{1 - Ce^{6x}}$$

$$y = \frac{-2 - 4Ce^{6x}}{1 - Ce^{6x}} - 9x$$

Once we have this general solution, we recognize from the initial condition $y(0) = 8$ that $x = 0$ and $y = 8$, so we'll plug these values into the general solution to find C.

$$8 = \frac{-2 - 4Ce^{6(0)}}{1 - Ce^{6(0)}} - 9(0)$$

$$8 = \frac{-2 - 4C}{1 - C}$$

$$8(1 - C) = -2 - 4C$$

$$8 - 8C = -2 - 4C$$

$$4C = 10$$

$$C = \frac{5}{2}$$

Then the solution is

$$y = \frac{-2 - 4\frac{5}{2}e^{6x}}{1 - \frac{5}{2}e^{6x}} - 9x$$

$$y = \frac{-2 - 10e^{6x}}{1 - \frac{5}{2}e^{6x}} - 9x$$

$$y = \frac{-4 - 20e^{6x}}{2 - 5e^{6x}} - 9x$$

$$y = \frac{20e^{6x} + 4}{5e^{6x} - 2} - 9x$$

■ 6. Use a substitution to solve the initial value problem, if $y(0) = 2$.

$$y' + 2x + 2y - 2xy = x^2 + y^2$$

Solution:

We can rewrite this equation as

$$y' + 2x + 2y - 2xy = x^2 + y^2$$

$$y' + 2(x + y) = x^2 + 2xy + y^2$$

$$y' + 2(x + y) = (x + y)^2$$

If we choose the substitution $u = ax + by$, then we can set $u = x + y$, $u' = 1 + y'$, and $y' = u' - 1$. Then we'll substitute $y' = u' - 1$ into the left side of the original differential equation, and $u = x + y$ into the right side of the original differential equation, and we'll get

$$u' - 1 + 2u = u^2$$

$$u' = u^2 - 2u + 1$$

$$u' = (u - 1)^2$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = (u - 1)^2$$

$$du = (u - 1)^2 \ dx$$

$$\frac{1}{(u - 1)^2} \ du = dx$$

and then integrate both sides.

$$\int \frac{1}{(u - 1)^2} \ du = \int dx$$

$$\frac{1}{1 - u} = x + C$$

$$\frac{1}{x + C} = 1 - u$$

$$u = 1 - \frac{1}{x + C}$$

We'll back-substitute using $u = x + y$ and then solve for y to find the solution to the differential equation.

$$x + y = 1 - \frac{1}{x + C}$$

$$y = 1 - \frac{1}{x + C} - x$$

Once we have this general solution, we recognize from the initial condition $y(0) = 2$ that $x = 0$ and $y = 2$, so we'll plug these values into the general solution to find C.

$$2 = 1 - \frac{1}{0 + C} - 0$$

$$2 = 1 - \frac{1}{C}$$

$$1 = -\frac{1}{C}$$

$$C = -1$$

Then the solution is

$$y = 1 - \frac{1}{x - 1} - x$$

BERNOULLI EQUATIONS

■ 1. Find the solution to the Bernoulli differential equation.

$$(y^4 + x^4y)\ dx - 3x^5\ dy = 0$$

Solution:

Rewrite the equation in standard form.

$$y^4 + x^4y = 3x^5\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^4 + x^4y}{3x^5}$$

Split the fraction so that we can separate terms.

$$\frac{dy}{dx} = \frac{y^4}{3x^5} + \frac{x^4y}{3x^5}$$

$$\frac{dy}{dx} - \frac{x^4y}{3x^5} = \frac{y^4}{3x^5}$$

$$\frac{dy}{dx} - \frac{x^4}{3x^5}y = \frac{1}{3x^5}y^4$$

$$y' - \frac{1}{3x}y = \frac{1}{3x^5}y^4$$

Divide through by $y^n = y^4$.

$$\frac{y'}{y^4} - \frac{y}{3xy^4} = \frac{y^4}{3x^5y^4}$$

$$y'y^{-4} - \frac{1}{3x}y^{-3} = \frac{1}{3x^5}$$

Our substitution is $v = y^{-3}$, so we'll differentiate to get

$$v' = -3y^{-4}y'$$

and then solve this for $y^{-4}y'$.

$$y^{-4}y' = -\frac{1}{3}v'$$

Now we can make substitutions into the Bernoulli equation.

$$-\frac{1}{3}v' - \frac{1}{3x}v = \frac{1}{3x^5}$$

$$v' + \frac{1}{x}v = -\frac{1}{x^5}$$

To find the solution to the linear equation, we'll find the integrating factor,

$$I(x) = e^{\int \frac{1}{x}\,dx}$$

$$I(x) = e^{\ln x}$$

$$I(x) = x$$

and then multiply through the linear equation by $I(x)$.

$$xv' + \frac{1}{x}xv = -\frac{1}{x^5}x$$

$$xv' + v = -\frac{1}{x^4}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(vx) = -\frac{1}{x^4}$$

then integrate both sides.

$$\int \frac{d}{dx}(vx) = \int -\frac{1}{x^4}\ dx$$

$$vx = \frac{1}{3x^3} + C$$

Solve for v.

$$v = \frac{1}{3x^4} + \frac{C}{x}$$

Use $v = y^{-3}$ to back-substitute for v,

$$y^{-3} = \frac{1 + 3x^3C}{3x^4}$$

then solve for y to find the solution.

$$\frac{1}{y^3} = \frac{1 + 3x^3C}{3x^4}$$

$$y^3 = \frac{3x^4}{1 + 3x^3C}$$

$$y = \sqrt[3]{\frac{3x^4}{1 + 3x^3C}}$$

■ 2. Find the solution to the Bernoulli differential equation.

$$xy' - 3y = xy^{\frac{5}{3}}$$

Solution:

Rewrite the equation in standard form.

$$y' - \frac{3y}{x} = y^{\frac{5}{3}}$$

Divide through by $y^n = y^{\frac{5}{3}}$.

$$\frac{y'}{y^{\frac{5}{3}}} - \frac{3y}{xy^{\frac{5}{3}}} = \frac{y^{\frac{5}{3}}}{y^{\frac{5}{3}}}$$

$$y'y^{-\frac{5}{3}} - \frac{3}{x}y^{-\frac{2}{3}} = 1$$

Our substitution is $v = y^{-\frac{2}{3}}$, so we'll differentiate to get

$$v' = -\frac{2}{3}y^{-\frac{5}{3}}y'$$

and then solve this for $y^{-\frac{5}{3}}y'$.

$$y^{-\frac{5}{3}}y' = -\frac{3}{2}v'$$

Now we can make substitutions into the Bernoulli equation.

$$-\frac{3}{2}v' - \frac{3}{x}v = 1$$

$$v' + \frac{2}{x}v = -\frac{2}{3}$$

To find the solution to the linear equation, we'll find the integrating factor,

$$I(x) = e^{\int \frac{2}{x}\,dx}$$

$$I(x) = e^{2\ln x}$$

$$I(x) = e^{\ln x^2}$$

$$I(x) = x^2$$

and then multiply through the linear equation by $I(x)$.

$$v'(x^2) + \frac{2}{x}v(x^2) = -\frac{2}{3}(x^2)$$

$$x^2v' + 2xv = -\frac{2}{3}x^2$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(x^2v) = -\frac{2}{3}x^2$$

then integrate both sides.

$$\int \frac{d}{dx}(x^2 v)\ dx = \int -\frac{2}{3}x^2\ dx$$

$$x^2 v = -\frac{2}{9}x^3 + C$$

Solve for v.

$$v = -\frac{2}{9}x + \frac{1}{x^2}C$$

Use $v = y^{-\frac{2}{3}}$ to back-substitute for v,

$$y^{-\frac{2}{3}} = -\frac{2}{9}x + \frac{1}{x^2}C$$

then solve for y.

$$\frac{1}{y^{\frac{2}{3}}} = -\frac{2}{9}x + \frac{1}{x^2}C$$

$$y^{\frac{2}{3}} = \frac{1}{-\frac{2}{9}x + \frac{1}{x^2}C}$$

$$y = \frac{1}{\sqrt{\left(-\frac{2}{9}x + \frac{1}{x^2}C\right)^3}}$$

Simplify and rewrite the solution.

$$y = \frac{1}{\sqrt{\left(\frac{1}{x^2}C - \frac{2}{9}x\right)^3}}$$

$$y = \frac{1}{\sqrt{\left(\frac{9C}{9x^2} - \frac{2x^3}{9x^2}\right)^3}}$$

$$y = \frac{1}{\left(\frac{9C - 2x^3}{9x^2}\right)^{\frac{3}{2}}}$$

$$y = \frac{1}{\frac{(9C - 2x^3)^{\frac{3}{2}}}{(9x^2)^{\frac{3}{2}}}}$$

$$y = \frac{1}{\frac{(9C - 2x^3)^{\frac{3}{2}}}{27x^3}}$$

$$y = \frac{27x^3}{(C - 2x^3)^{\frac{3}{2}}}$$

■ 3. Find the solution to the Bernoulli differential equation.

$$y' = y^2 \sin x + y \cot x$$

Solution:

Rewrite the equation in standard form.

$$y' - y \cot x = y^2 \sin x$$

Divide through by $y^n = y^2$.

$$\frac{y'}{y^2} - \frac{y \cot x}{y^2} = \frac{y^2 \sin x}{y^2}$$

$$y'y^{-2} - (\cot x)y^{-1} = \sin x$$

Our substitution is $v = y^{-1}$, so we'll differentiate to get

$$v' = -y^{-2}y'$$

and then solve this for $y^{-2}y'$.

$$y^{-2}y' = -v'$$

Now we can make substitutions into the Bernoulli equation.

$$-v' - (\cot x)v = \sin x$$

$$v' + (\cot x)v = -\sin x$$

To find the solution to the linear equation, we'll find the integrating factor,

$$I(x) = e^{\int \cot x \, dx}$$

$$I(x) = e^{\ln \sin x}$$

$$I(x) = \sin x$$

and then multiply through the linear equation by $I(x)$.

$$v' \sin x + (\cot x \sin x)v = -\sin x \sin x$$

$$v' \sin x + (\cot x \sin x)v = -\sin^2 x$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(v\sin x) = -\sin^2 x$$

then integrate both sides.

$$\int \frac{d}{dx}(v\sin x) = \int -\sin^2 x \ dx$$

$$v\sin x = -\frac{1}{2}x + \frac{1}{2}\sin x\cos x + C$$

Solve for v.

$$v = -\frac{1}{2}x\csc x + \frac{1}{2}\cos x + C\csc x$$

Use $v = y^{-1}$ to back-substitute for v,

$$y^{-1} = -\frac{1}{2}x\csc x + \frac{1}{2}\cos x + C\csc x$$

then solve for y.

$$\frac{1}{y} = -\frac{1}{2}x\csc x + \frac{1}{2}\cos x + C\csc x$$

$$y = \frac{1}{-\frac{1}{2}x\csc x + \frac{1}{2}\cos x + C\csc x}$$

■ 4. Find the solution to the Bernoulli differential equation, if $y(1) = 0$.

$$y' - \frac{y}{x} + \sqrt{y} = 0$$

Solution:

Rewrite the equation in standard form.

$$y' - \frac{y}{x} = -\sqrt{y}$$

Divide through by $y^n = y^{\frac{1}{2}}$.

$$\frac{y'}{y^{\frac{1}{2}}} - \frac{y}{xy^{\frac{1}{2}}} = -\frac{y^{\frac{1}{2}}}{y^{\frac{1}{2}}}$$

$$y'y^{-\frac{1}{2}} - \frac{1}{x}y^{\frac{1}{2}} = -1$$

Our substitution is $v = y^{\frac{1}{2}}$, so we'll differentiate to get

$$v' = \frac{1}{2}y^{-\frac{1}{2}}y'$$

and then solve this for $y^{-\frac{1}{2}}y'$.

$$y^{-\frac{1}{2}}y' = 2v'$$

Now we can make substitutions into the Bernoulli equation.

$$2v' - \frac{1}{x}v = -1$$

Multiplying through by $1/2$ puts the equation into standard form of a linear differential equation.

$$v' - \frac{1}{2x}v = -\frac{1}{2}$$

To find the solution to the linear equation, we'll find the integrating factor,

$$I(x) = e^{\int -\frac{1}{2x}\ dx}$$

$$I(x) = e^{-\frac{1}{2}\ln x}$$

$$I(x) = \frac{1}{x^{\frac{1}{2}}}$$

and then multiply through the linear equation by $I(x)$.

$$\frac{1}{x^{\frac{1}{2}}}v' - \frac{1}{x^{\frac{1}{2}}}\frac{1}{2x}v = -\frac{1}{2}\frac{1}{x^{\frac{1}{2}}}$$

$$\frac{1}{x^{\frac{1}{2}}}v' - \frac{1}{2x^{\frac{3}{2}}}v = -\frac{1}{2x^{\frac{1}{2}}}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}\left(\frac{1}{x^{\frac{1}{2}}}v\right) = -\frac{1}{2x^{\frac{1}{2}}}$$

then integrate both sides.

$$\int \frac{d}{dx}\left(\frac{1}{x^{\frac{1}{2}}}v\right) = \int -\frac{1}{2x^{\frac{1}{2}}}\ dx$$

$$\frac{1}{x^{\frac{1}{2}}}v = -x^{\frac{1}{2}} + C$$

Solve for v.

$$v = -x + Cx^{\frac{1}{2}}$$

Use $v = y^{\frac{1}{2}}$ to back-substitute for v,

$$y^{\frac{1}{2}} = -x + Cx^{\frac{1}{2}}$$

then solve for y.

$$y = (Cx^{\frac{1}{2}} - x)^2$$

Once we have this general solution, we recognize from the initial condition $y(1) = 0$ that $x = 1$ and $y = 0$, so we'll plug these values into the general solution to solve for C.

$$0 = (C(1)^{\frac{1}{2}} - 1)^2$$

$$0 = (C - 1)^2$$

$$0 = C - 1$$

$$C = 1$$

Then the solution is

$$y = (x^{\frac{1}{2}} - x)^2$$

■ 5. Find the solution to the Bernoulli differential equation, if $y(1) = 1$.

$$y' + \frac{2}{x}y = \frac{y^2}{x^2}$$

Solution:

Rewrite the equation in standard form.

$$\frac{y'}{y^2} + \frac{2y}{xy^2} = \frac{y^2}{x^2y^2}$$

$$y'y^{-2} + \frac{2}{x}y^{-1} = \frac{1}{x^2}$$

Our substitution is $v = y^{-1}$, so we'll differentiate to get

$$v' = -y^{-2}y'$$

and then solve this for $y^{-2}y'$.

$$y^{-2}y' = -v'$$

Now we can make substitutions into the Bernoulli equation.

$$y'y^{-2} + \frac{2}{x}y^{-1} = \frac{1}{x^2}$$

$$-v' + \frac{2}{x}v = \frac{1}{x^2}$$

$$v' - \frac{2}{x}v = -\frac{1}{x^2}$$

To find the solution to the linear equation, we'll find the integrating factor,

$$I(x) = e^{\int -\frac{2}{x}\ dx}$$

$$I(x) = e^{-2\ln x}$$

$$I(x) = e^{\ln x^{-2}}$$

$$I(x) = \frac{1}{x^2}$$

and then multiply through the linear equation by $I(x)$.

$$\frac{1}{x^2}v' - \frac{2}{x}\frac{1}{x^2}v = -\frac{1}{x^2}\frac{1}{x^2}$$

$$\frac{1}{x^2}v' - \frac{2}{x^3}v = -\frac{1}{x^4}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}\left(\frac{1}{x^2}v\right) = -\frac{1}{x^4}$$

then integrate both sides.

$$\int \frac{d}{dx}\left(\frac{1}{x^2}v\right) = \int -\frac{1}{x^4}\ dx$$

$$v\frac{1}{x^2} = \frac{1}{3x^3} + C$$

Solve for v.

$$v = \frac{1}{3x} + Cx^2$$

Use $v = y^{-1}$ to back-substitute for v,

$$y^{-1} = \frac{1}{3x} + Cx^2$$

then solve for y.

$$\frac{1}{y} = \frac{1}{3x} + Cx^2$$

$$y = \frac{1}{\frac{1}{3x} + Cx^2}$$

Once we have this general solution, we recognize from the initial condition $y(1) = 1$ that $x = 1$ and $y = 1$, so we'll plug these values into the general solution to solve for C.

$$1 = \frac{1}{\frac{1}{3(1)} + C(1)^2}$$

$$1 = \frac{1}{\frac{1}{3} + C}$$

$$1 = \frac{1}{3} + C$$

$$C = \frac{2}{3}$$

Then the solution is

$$y = \frac{1}{\frac{1}{3x} + \frac{2}{3}x^2}$$

■ 6. Find the solution to the Bernoulli differential equation, if $y(0) = 2$.

$$3y' + 2y = xy^5$$

Solution:

Rewrite the equation in standard form.

$$y' + \frac{2}{3}y = \frac{1}{3}xy^5$$

Divide through by $y^n = y^5$.

$$\frac{y'}{y^5} + \frac{2y}{3y^5} = \frac{xy^5}{3y^5}$$

$$y'y^{-5} + \frac{2}{3}y^{-4} = \frac{x}{3}$$

Our substitution is $v = y^{-4}$, so we'll differentiate to get

$$v' = -4y^{-5}y'$$

and then solve this for $y^{-5}y'$.

$$y^{-5}y' = -\frac{1}{4}v'$$

Now we can make substitutions into the Bernoulli equation.

$$-\frac{1}{4}v'+\frac{2}{3}v=\frac{x}{3}$$

$$v'-\frac{8}{3}v=-\frac{4x}{3}$$

To find the solution to the linear equation, we'll find the integrating factor,

$$I(x)=e^{\int -\frac{8}{3}\ dx}$$

$$I(x)=e^{-\frac{8}{3}x}$$

and then multiply through the linear equation by $I(x)$.

$$v'-\frac{8}{3}v=-\frac{4x}{3}$$

$$e^{-\frac{8}{3}x}v'-\frac{8}{3}e^{-\frac{8}{3}x}v=-\frac{4x}{3}e^{-\frac{8}{3}x}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(ve^{-\frac{8}{3}x})=-\frac{4}{3}xe^{-\frac{8}{3}x}$$

then integrate both sides.

$$\int \frac{d}{dx}(ve^{-\frac{8}{3}x})=\int -\frac{4}{3}xe^{-\frac{8}{3}x}\ dx$$

$$ve^{-\frac{8}{3}x}=\frac{1}{16}e^{-\frac{8}{3}x}(8x+3)+C$$

Solve for v.

$$v = \frac{1}{16}(8x + 3) + Ce^{\frac{8}{3}x}$$

Use $v = y^{-4}$ to back-substitute for v,

$$y^{-4} = \frac{1}{16}(8x + 3) + Ce^{\frac{8}{3}x}$$

then solve for y.

$$\frac{1}{y^4} = \frac{1}{16}(8x + 3) + Ce^{\frac{8}{3}x}$$

$$y^4 = \frac{1}{\frac{1}{16}(8x + 3) + Ce^{\frac{8}{3}x}}$$

$$y = \frac{1}{\sqrt[4]{\frac{1}{16}(8x + 3) + Ce^{\frac{8}{3}x}}}$$

Once we have this general solution, we recognize from the initial condition $y(0) = 2$ that $x = 0$ and $y = 2$, so we'll plug these values into the general solution to find C.

$$2 = \frac{1}{\sqrt[4]{\frac{1}{16}(8(0) + 3) + Ce^{\frac{8}{3}(0)}}}$$

$$2 = \frac{1}{\sqrt[4]{\frac{3}{16} + C}}$$

$$16 = \frac{1}{\frac{3}{16} + C}$$

$$\frac{3}{16} + C = \frac{1}{16}$$

$$C = -\frac{1}{8}$$

Then the solution is

$$y = \frac{1}{\sqrt[4]{\frac{1}{16}(8x+3) - \frac{1}{8}e^{\frac{8}{3}x}}}$$

HOMOGENEOUS EQUATIONS

1. Find the solution to the differential equation.

$$x^2yy' = x^3 + xy^2$$

Solution:

To put the equation in standard form, we'll divide through by x^2y.

$$\frac{x^2y}{x^2y}y' = \frac{x^3}{x^2y} + \frac{xy^2}{x^2y}$$

$$y' = \frac{x}{y} + \frac{y}{x}$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = v + \frac{1}{v}$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv' = \frac{1}{v}$$

$$x\frac{dv}{dx} = \frac{1}{v}$$

$$v \, dv = \frac{1}{x} \, dx$$

and then integrate both sides.

$$\int v \, dv = \int \frac{1}{x} \, dx$$

$$\frac{v^2}{2} = \ln x + C$$

$$v^2 = 2 \ln x + C$$

$$v = \pm \sqrt{2 \ln x + C}$$

Back-substitute for v.

$$\frac{y}{x} = \pm \sqrt{2 \ln x + C}$$

$$y = \pm x \sqrt{2 \ln x + C}$$

■ 2. Use a substitution to find a solution to the homogeneous equation.

$$x(x+y)y' - y(2x+y) = 0$$

Solution:

Rewrite the equation to solve for y'.

$$x(x+y)y' = y(2x+y)$$

$$x(x+y)y' = 2xy + y^2$$

$$y' = \frac{2xy + y^2}{x^2 + xy}$$

We'll multiply through the numerator and denominator on the right side by $1/x^2$.

$$y' = \frac{2xy + y^2}{x^2 + xy} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$y' = \frac{2xy\frac{1}{x^2} + y^2\frac{1}{x^2}}{x^2\frac{1}{x^2} + xy\frac{1}{x^2}}$$

$$y' = \frac{2\frac{y}{x} + \frac{y^2}{x^2}}{1 + \frac{y}{x}}$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = \frac{2v + v^2}{1 + v}$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv' = \frac{2v + v^2}{1 + v} - v$$

$$xv' = \frac{2v + v^2 - v - v^2}{1 + v}$$

$$xv' = \frac{v}{1+v}$$

$$x\frac{dv}{dx} = \frac{v}{1+v}$$

$$\frac{1+v}{v}\ dv = \frac{1}{x}\ dx$$

and then integrate both sides.

$$\int \frac{1+v}{v}\ dv = \int \frac{1}{x}\ dx$$

$$\ln v + v = \ln x + C$$

$$e^{\ln v + v} = e^{\ln x + C}$$

$$ve^v = Cx$$

Back-substitute for v.

$$\frac{y}{x}e^{\frac{y}{x}} = Cx$$

$$ye^{\frac{y}{x}} = Cx^2$$

■ 3. Rewrite the homogeneous equation in terms of y/x, but don't solve the separable equation.

$$yy' - x = \sqrt{x^2 + y^2}$$

Solution:

Solve for y' by dividing both sides by y.

$$yy' = \sqrt{x^2 + y^2} + x$$

$$y' = \frac{\sqrt{x^2 + y^2} + x}{y}$$

Break the fraction in two, then bring the entire first fraction under the root.

$$y' = \frac{\sqrt{x^2 + y^2}}{y} + \frac{x}{y}$$

$$y' = \sqrt{\frac{x^2 + y^2}{y^2}} + \frac{x}{y}$$

Split the fraction under the root in two, simplify, then rewrite the fractions to put them in terms of y/x.

$$y' = \sqrt{\frac{x^2}{y^2} + 1} + \frac{x}{y}$$

$$y' = \sqrt{\frac{1}{\left(\frac{y}{x}\right)^2} + 1} + \frac{1}{\frac{y}{x}}$$

■ 4. Use a substitution to find a solution to the homogeneous equation.

$$xyy' = x^2 e^{-\frac{y^2}{x^2}} + y^2$$

Solution:

To put the equation in standard form, we'll divide through by xy.

$$\frac{xyy'}{xy} = \frac{x^2 e^{-\frac{y^2}{x^2}}}{xy} + \frac{y^2}{xy}$$

$$y' = \frac{x}{y} e^{-\frac{y^2}{x^2}} + \frac{y}{x}$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = \frac{1}{v} e^{-v^2} + v$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv' = \frac{1}{v} e^{-v^2}$$

$$x \frac{dv}{dx} = \frac{1}{v} e^{-v^2}$$

$$ve^{v^2} \ dv = \frac{1}{x} \ dx$$

and then integrate both sides.

$$\int ve^{v^2}\ dv = \int \frac{1}{x}\ dx$$

$$\frac{1}{2}e^{v^2} = \ln x + C$$

$$e^{v^2} = 2\ln x + C$$

$$v^2 = \ln|2\ln x + C|$$

$$v = \pm\sqrt{\ln|2\ln x + C|}$$

Back-substitute for v.

$$\frac{y}{x} = \pm\sqrt{\ln|2\ln x + C|}$$

$$y = \pm x\sqrt{\ln|2\ln x + C|}$$

■ 5. Find the solution to the differential equation, if $y(1) = 1$.

$$x^2y' = 3xy + 2y^2$$

Solution:

To put the equation in standard form, we'll divide through by x^2.

$$\frac{x^2y'}{x^2} = \frac{3xy}{x^2} + \frac{2y^2}{x^2}$$

$$y' = \frac{3y}{x} + \frac{2y^2}{x^2}$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = 3v + 2v^2$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv' = 2v + 2v^2$$

$$x\frac{dv}{dx} = 2v + 2v^2$$

$$\frac{1}{2v + 2v^2}\ dv = \frac{1}{x}\ dx$$

and then integrate both sides and solve for v.

$$\int \frac{1}{2v + 2v^2}\ dv = \int \frac{1}{x}\ dx$$

$$\frac{1}{2}(\ln v - \ln(v + 1)) = \ln x + C$$

$$\ln v - \ln(v + 1) = 2\ln x + C$$

$$\ln \frac{v}{v + 1} = 2\ln x + C$$

$$\ln \frac{v}{v + 1} = \ln x^2 + C$$

$$\frac{v}{v + 1} = Cx^2$$

$$v = Cx^2v + Cx^2$$

$$v(1 - Cx^2) = Cx^2$$

$$v = \frac{Cx^2}{1 - Cx^2}$$

Back-substitute for v.

$$\frac{y}{x} = \frac{Cx^2}{1 - Cx^2}$$

$$y = \frac{Cx^3}{1 - Cx^2}$$

Once we have this general solution, we recognize from the initial condition $y(1) = 1$ that $x = 1$ and $y = 1$, so we'll plug these values into the general solution to solve for C.

$$1 = \frac{C}{1 - C}$$

$$1 - C = C$$

$$C = \frac{1}{2}$$

Then the solution to the differential equation is

$$y = \frac{\frac{1}{2}x^3}{1 - \frac{1}{2}x^2}$$

$$y = \frac{x^3}{2 - x^2}$$

■ 6. Use a substitution to find a solution to the homogeneous equation, if $y(1)=1$.

$$xy'+y\ln x-y=y\ln y$$

Solution:

To put the equation in standard form, we'll divide through by x.

$$xy'=y\ln y-y\ln x+y$$

$$y'=\frac{y}{x}\ln y-\frac{y}{x}\ln x+\frac{y}{x}$$

$$y'=\frac{y}{x}\ln\frac{y}{x}+\frac{y}{x}$$

Substitute $v=y/x$ and $y'=v+xv'$.

$$v+xv'=v\ln v+v$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv'=v\ln v$$

$$x\frac{dv}{dx}=v\ln v$$

$$\frac{1}{v\ln v}\ dv=\frac{1}{x}\ dx$$

and then integrate both sides.

$$\int \frac{1}{v \ln v}\ dv = \int \frac{1}{x}\ dx$$

$$\ln(\ln v) = \ln x + C$$

$$\ln v = Cx$$

$$v = Ce^x$$

Back-substitute for v.

$$\frac{y}{x} = Ce^x$$

$$y = Cxe^x$$

Once we have this general solution, we recognize from the initial condition $y(1) = 1$ that $x = 1$ and $y = 1$, so we'll plug these values into the general solution to solve for C.

$$1 = Ce$$

$$C = \frac{1}{e}$$

So the particular solution to the differential equation is

$$y = \frac{1}{e}xe^x$$

$$y = xe^{x-1}$$

EXACT EQUATIONS

■ 1. If the differential equation is exact, find its solution.

$$2xy + (x^2 - y^2)\frac{dy}{dx} = 0$$

Solution:

We'll check to see that the equation is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(2xy) = \frac{\partial}{\partial x}(x^2 - y^2)$$

$$2x = 2x$$

The functions $M(x, y)$ and $N(x, y)$ are equally easy to integrate, so we'll just use $M(x, y)$, and then Ψ can be given by

$$\Psi = \int M(x, y)\ dx + h(y)$$

$$\Psi = \int 2xy\ dx + h(y)$$

$$\Psi = x^2y + h(y)$$

We'll differentiate both sides with respect to y,

$$\Psi_y = x^2 + h'(y)$$

and then substitute $\Psi_y = N(x, y)$ to solve for $h'(y)$.

$$x^2 - y^2 = x^2 + h'(y)$$

$$-y^2 = h'(y)$$

To find $h(y)$, we'll integrate both sides of this equation with respect to y.

$$\int -y^2 \, dy = \int h'(y) \, dy$$

$$-\frac{y^3}{3} + k_1 = h(y) + k_2$$

$$h(y) = -\frac{y^3}{3} + k_1 - k_2$$

$$h(y) = k - \frac{y^3}{3}$$

Plugging this value for $h(y)$ into the equation for Ψ gives

$$\Psi = x^2y + h(y)$$

$$\Psi = x^2y + k - \frac{y^3}{3}$$

Finally, setting $\Psi = c_1$ to find the solution to the exact differential equation, we get

$$c_1 = x^2y + k - \frac{y^3}{3}$$

$$c_1 - k = x^2y - \frac{y^3}{3}$$

$$c = x^2y - \frac{y^3}{3}$$

■ 2. What is the solution to the exact differential equation?

$$(6y + xe^{-y})\ dy - e^{-y}\ dx = 0$$

Solution:

We'll check to see that the equation is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(-e^{-y}) = \frac{\partial}{\partial x}(6y + xe^{-y})$$

$$e^{-y} = e^{-y}$$

The functions $M(x, y)$ and $N(x, y)$ are equally easy to integrate, so we'll just use $M(x, y)$, and then Ψ can be given by

$$\Psi = \int M(x, y)\ dx + h(y)$$

$$\Psi = \int -e^{-y}\ dx + h(y)$$

$$\Psi = -xe^{-y} + h(y)$$

We'll differentiate both sides with respect to y,

$$\Psi_y = xe^{-y} + h'(y)$$

and then substitute $\Psi_y = N(x, y)$ to solve for $h'(y)$.

$$6y + xe^{-y} = xe^{-y} + h'(y)$$

$$6y = h'(y)$$

To find $h(y)$, we'll integrate both sides of this equation with respect to y.

$$\int 6y \ dy = \int h'(y) \ dy$$

$$3y^2 + k_1 = h(y) + k_2$$

$$h(y) = 3y^2 + k_1 - k_2$$

$$h(y) = 3y^2 + k$$

Plugging this value for $h(y)$ into the equation for Ψ gives

$$\Psi = -xe^{-y} + h(y)$$

$$\Psi = -xe^{-y} + 3y^2 + k$$

Finally, setting $\Psi = c_1$ to find the solution to the exact differential equation, we get

$$c_1 = -xe^{-y} + 3y^2 + k$$

$$c_1 - k = -xe^{-y} + 3y^2$$

$$c = -xe^{-y} + 3y^2$$

■ 3. Find the value of a that make the equation an exact differential equation.

$$(5\sin y - x^3)\ dx + ax\cos y\ dy = 0$$

Solution:

Since the equation is exact, we can use our partial derivatives test to find the value of a.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(5\sin y - x^3) = \frac{\partial}{\partial x}(ax\cos y)$$

$$5\cos y = a\cos y$$

These partial derivatives are equal when $a = 5$, so the equation is exact when $a = 5$.

■ 4. If the differential equation is exact, find its solution.

$$(x \sin y - 2x)\ dx + \left(\frac{x^2}{2}\cos y - 2y\right)\ dy = 0$$

Solution:

We'll check to see that the equation is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(x \sin y - 2x) = \frac{\partial}{\partial x}\left(\frac{x^2}{2}\cos y - 2y\right)$$

$$x \cos y = x \cos y$$

The functions $M(x, y)$ and $N(x, y)$ are equally easy to integrate, so we'll just use $M(x, y)$, and then Ψ can be given by

$$\Psi = \int M(x, y)\ dx + h(y)$$

$$\Psi = \int (x \sin y - 2x)\ dx + h(y)$$

$$\Psi = \frac{x^2}{2}\sin y - x^2 + h(y)$$

We'll differentiate both sides with respect to y,

$$\Psi_y = \frac{x^2}{2}\cos y + h'(y)$$

and then substitute $\Psi_y = N(x,y)$ to solve for $h'(y)$.

$$\frac{x^2}{2}\cos y - 2y = \frac{x^2}{2}\cos y + h'(y)$$

$$-2y = h'(y)$$

To find $h(y)$, we'll integrate both sides of this equation with respect to y.

$$\int -2y\ dy = \int h'(y)\ dy$$

$$-y^2 + k_1 = h(y) + k_2$$

$$h(y) = -y^2 + k_1 - k_2$$

$$h(y) = -y^2 + k$$

Plugging this value for $h(y)$ into the equation for Ψ gives

$$\Psi = \frac{x^2}{2}\sin y - x^2 + h(y)$$

$$\Psi = \frac{x^2}{2}\sin y - x^2 - y^2 + k$$

Finally, setting $\Psi = c_1$ to find the solution to the exact differential equation, we get

$$c_1 = \frac{x^2}{2}\sin y - x^2 - y^2 + k$$

$$c_1 - k = \frac{x^2}{2}\sin y - x^2 - y^2$$

$$c = \frac{x^2}{2}\sin y - x^2 - y^2$$

■ 5. Find the value of a that makes the equation an exact differential equation.

$$(2\cos x + 4\ln y)\ dx + \left(\frac{a^2x}{y} + e^y\right)\ dy = 0$$

Solution:

Since the equation is exact, we can use our partial derivatives test to find the value of a.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(2\cos x + 4\ln y) = \frac{\partial}{\partial x}\left(\frac{a^2x}{y} + e^y\right)$$

$$\frac{4}{y} = \frac{a^2}{y}$$

$$a^2 = 4$$

$$a = \pm 2$$

The partial derivatives are equal when $a = \pm 2$, which means the equation is exact when $a = \pm 2$.

■ 6. What is the solution to the exact differential equation?

$$(x\cos y + e^y)\ dy + (e^x + \sin y)\ dx = 0$$

Solution:

We'll check to see that the equation is exact.

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(e^x + \sin y) = \frac{\partial}{\partial x}(x\cos y + e^y)$$

$$\cos y = \cos y$$

The functions $M(x, y)$ and $N(x, y)$ are equally easy to integrate, so we'll just use $M(x, y)$, and then Ψ can be given by

$$\Psi = \int M(x, y)\ dx + h(y)$$

$$\Psi = \int (e^x + \sin y)\ dx + h(y)$$

$$\Psi = e^x + x\sin y + h(y)$$

We'll differentiate both sides with respect to y,

$$\Psi_y = x\cos y + h'(y)$$

and then substitute $\Psi_y = N(x, y)$ to solve for $h'(y)$.

$$x\cos y + e^y = x\cos y + h'(y)$$

$$e^y = h'(y)$$

To find $h(y)$, we'll integrate both sides of this equation with respect to y.

$$\int e^y\ dy = \int h'(y)\ dy$$

$$e^y + k_1 = h(y) + k_2$$

$$h(y) = e^y + k_1 - k_2$$

$$h(y) = e^y + k$$

Plugging this value for $h(y)$ into the equation for Ψ gives

$$\Psi = e^x + x\sin y + h(y)$$

$$\Psi = e^x + x\sin y + e^y + k$$

Finally, setting $\Psi = c_1$ to find the solution to the exact differential equation, we get

$$c_1 = e^x + x\sin y + e^y + k$$

$$c_1 - k = e^x + x\sin y + e^y$$

$$c = e^x + x \sin y + e^y$$

SECOND ORDER LINEAR HOMOGENEOUS EQUATIONS

■ 1. Find the general solution to the homogeneous equation.

$$y'' + 9y' + 8y = 0$$

Solution:

Factor the associated characteristic equation and solve for roots.

$$r^2 + 9r + 8 = 0$$

$$(r + 8)(r + 1) = 0$$

$$r = -8, \ -1$$

These are distinct real roots, so the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$y(x) = c_1 e^{-8x} + c_2 e^{-x}$$

■ 2. Find the solution to the homogeneous equation, if $y(0) = 5$ and $y'(0) = 2$.

$$y'' - 4y' + 5y = 0$$

Solution:

The characteristic equation associated with the differential equation is

$$r^2 - 4r + 5 = 0$$

Apply the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$r = \frac{4 \pm \sqrt{-4}}{2}$$

Use the imaginary number to rewrite the root of the negative number.

$$r = \frac{4 \pm \sqrt{(4)(-1)}}{2}$$

$$r = \frac{4 \pm 2\sqrt{-1}}{2}$$

$$r = 2 \pm i$$

These are complex conjugate roots. If we match up $r = \alpha \pm \beta i$ with $r = 2 \pm i$, we identify $\alpha = 2$ and $\beta = 1$, and the general solution is therefore

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

$$y(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$$

The derivative of the general solution is

$$y'(x) = 2e^{2x}(c_1 \cos x + c_2 \sin x) + e^{2x}(-c_1 \sin x + c_2 \cos x)$$

$$y'(x) = 2c_1e^{2x} \cos x + c_2e^{2x} \cos x + 2c_2e^{2x} \sin x - c_1e^{2x} \sin x$$

$$y'(x) = (2c_1 + c_2)e^{2x} \cos x + (2c_2 - c_1)e^{2x} \sin x$$

Substitute the initial conditions $y(0) = 5$ and $y'(0) = 2$ into the general solution and its derivative. We get

$$5 = e^{2(0)}(c_1 \cos(0) + c_2 \sin(0))$$

$$5 = 1(c_1(1) + 0)$$

$$c_1 = 5$$

and

$$2 = (2c_1 + c_2)e^{2(0)} \cos(0) + (2c_2 - c_1)e^{2(0)} \sin(0)$$

$$2 = (2c_1 + c_2)(1) + (2c_2 - c_1)(1)(0)$$

$$2 = 2c_1 + c_2$$

$$2 = 2(5) + c_2$$

$$c_2 = -8$$

After substituting the initial conditions, the general solution becomes

$$y(x) = e^{2x}(5 \cos x - 8 \sin x)$$

■ 3. Find the general solution to the homogeneous equation.

$$y'' - 10y' + 25y = 0$$

Solution:

Factor the associated characteristic equation and solve for roots.

$$r^2 - 10r + 25 = 0$$

$$(r - 5)(r - 5) = 0$$

$$r = 5, 5$$

These are equal real roots, so the general solution is

$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

$$y(x) = c_1 e^{5x} + c_2 x e^{5x}$$

■ 4. If $y(0) = 1$ and $y'(0) = 0$, find the solution to the homogeneous equation.

$$y'' + 4y' + 4y = 0$$

Solution:

Factor the associated characteristic equation and solve for roots.

$$r^2 + 4r + 4 = 0$$

$$(r + 2)(r + 2) = 0$$

$$r = -2, \ -2$$

These are equal real roots, so the general solution is and its derivative are

$$y(x) = c_1e^{-2x} + c_2xe^{-2x}$$

$$y'(x) = -2c_1e^{-2x} + c_2e^{-2x} - 2c_2xe^{-2x}$$

Substitute $y(0) = 1$ into $y(x)$.

$$y(0) = c_1e^{-2(0)} + c_2(0)e^{-2(0)}$$

$$1 = c_1(1) + 0$$

$$c_1 = 1$$

Substitute $y'(0) = 0$ into $y'(x)$.

$$y'(0) = -2c_1e^{-2(0)} + c_2e^{-2(0)} - 2c_2(0)e^{-2(0)}$$

$$0 = -2c_1(1) + c_2(1) - 0$$

$$c_2 = 2c_1$$

$$c_2 = 2(1)$$

$$c_2 = 2$$

After substituting the initial conditions, the general solution becomes

$$y(x) = e^{-2x} + 2xe^{-2x}$$

■ 5. If $y(0) = 2$ and $y'(0) = 3$, find the solution to the homogeneous equation.

$$y'' + 9y = 0$$

Solution:

The characteristic equation associated with the differential equation is

$$r^2 + 9 = 0$$

Apply the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{0 \pm \sqrt{0^2 - 4(1)(9)}}{2(1)}$$

$$r = \frac{\pm\sqrt{-36}}{2}$$

Use the imaginary number to rewrite the root of the negative number.

$$r = \frac{\pm\sqrt{(36)(-1)}}{2}$$

$$r = \frac{\pm 6\sqrt{-1}}{2}$$

$$r = \pm\, 3i$$

These are complex conjugate roots. If we match up $r = \alpha \pm \beta i$ with $r = \pm 3i$, we identify $\alpha = 0$ and $\beta = 3$, and the general solution is therefore

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

$$y(x) = e^{(0)x}(c_1 \cos(3x) + c_2 \sin(3x))$$

$$y(x) = c_1 \cos(3x) + c_2 \sin(3x)$$

The derivative of the general solution is

$$y'(x) = -3c_1 \sin(3x) + 3c_2 \cos(3x)$$

Substitute the initial conditions $y(0) = 2$ and $y'(0) = 3$ into the general solution and its derivative. We get

$$2 = c_1 \cos(3(0)) + c_2 \sin(3(0))$$

$$2 = c_1(1) + 0$$

$$c_1 = 2$$

and

$$3 = -3c_1 \sin(3(0)) + 3c_2 \cos(3(0))$$

$$3 = 0 + 3c_2(1)$$

$$c_2 = 1$$

After substituting the initial conditions, the general solution becomes

$$y(x) = 2\cos(3x) + \sin(3x)$$

■ 6. If $y(0)=1$ and $y'(0)=5$, find the solution to the homogeneous equation.

$$2y''+11y'+5y=0$$

Solution:

Factor the associated characteristic equation and solve for roots.

$$2r^2+11r+5=0$$

$$(2r+1)(r+5)=0$$

$$r=-\frac{1}{2},\ -5$$

These are distinct real roots, so the general solution is and its derivative are

$$y(x)=c_1e^{-\frac{1}{2}x}+c_2e^{-5x}$$

$$y'(x)=-\frac{1}{2}c_1e^{-\frac{1}{2}x}-5c_2e^{-5x}$$

Substitute $y(0)=1$ into $y(x)$.

$$y(0)=c_1e^{-\frac{1}{2}(0)}+c_2e^{-5(0)}$$

$$1=c_1(1)+c_2(1)$$

$$1=c_1+c_2$$

Substitute $y'(0) = 5$ into $y'(x)$.

$$y'(0) = -\frac{1}{2}c_1e^{-\frac{1}{2}(0)} - 5c_2e^{-5(0)}$$

$$5 = -\frac{1}{2}c_1(1) - 5c_2(1)$$

$$5 = -\frac{1}{2}c_1 - 5c_2$$

Multiply $1 = c_1 + c_2$ by 5 to get $5 = 5c_1 + 5c_2$, then add the equations.

$$5 + 5 = -\frac{1}{2}c_1 - 5c_2 + (5c_1 + 5c_2)$$

$$10 = -\frac{1}{2}c_1 + 5c_1$$

$$10 = \frac{9}{2}c_1$$

$$c_1 = \frac{20}{9}$$

Then c_2 is

$$1 = \frac{20}{9} + c_2$$

$$c_2 = -\frac{11}{9}$$

After substituting the initial conditions, the general solution becomes

$$y(x) = \frac{20}{9}e^{-\frac{1}{2}x} - \frac{11}{9}e^{-5x}$$

REDUCTION OF ORDER

■ 1. Use reduction of order to find the general solution to the differential equation, given $y_1 = e^{2x}$.

$$y'' + 3y' - 10y = 0$$

Solution:

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = ve^{2x}$$

$$y_2' = v'e^{2x} + 2ve^{2x}$$

$$y_2'' = v''e^{2x} + 4v'e^{2x} + 4ve^{2x}$$

Plug these values into the homogeneous differential equation.

$$v''e^{2x} + 4v'e^{2x} + 4ve^{2x} + 3(v'e^{2x} + 2ve^{2x}) - 10ve^{2x} = 0$$

$$v''e^{2x} + 7v'e^{2x} = 0$$

Make the substitution $w = v'$ and $w' = v''$.

$$w'e^{2x} + 7we^{2x} = 0$$

$$e^{2x}\frac{dw}{dv} + 7we^{2x} = 0$$

$$\frac{dw}{dv} + 7w = 0$$

With $P(x) = 7$ and $Q(x) = 0$, the integrating factor is

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int 7\ dx}$$

$$I(x) = e^{7x}$$

Multiplying the first order differential equation in w by the integrating factor, and then simplifying, gives

$$\frac{d}{dv}(we^{7x}) = 0$$

Integrate both sides.

$$\int \frac{d}{dv}(we^{7x})\ dv = \int 0\ dv$$

$$we^{7x} = C$$

$$w = Ce^{-7x}$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w = \int Ce^{-7x}\ dx = -\frac{1}{7}Ce^{-7x} + k$$

We can choose any constants, so we'll choose $C = -7$ and $k = 0$ for simplicity. Then v is

$$v = e^{-7x}$$

and the second solution for the differential equation is

$$y_2 = ve^{2x}$$

$$y_2 = e^{-7x}e^{2x}$$

$$y_2 = e^{-5x}$$

Because the solutions are $y_1 = e^{2x}$ and $y_2 = e^{-5x}$, the general solution to the differential equation is

$$y(x) = c_1e^{2x} + c_2e^{-5x}$$

■ 2. Use reduction of order to find the general solution to the differential equation, given $y_1 = x^2$.

$$x^2y'' - 7xy' + 12y = 0$$

Solution:

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = vx^2$$

$$y_2' = v'x^2 + 2vx$$

$$y_2'' = v''x^2 + 4v'x + 2v$$

Plug these values into the homogeneous differential equation.

$$x^2(v''x^2 + 4v'x + 2v) - 7x(v'x^2 + 2vx) + 12vx^2 = 0$$

$$v''x^4 + 4v'x^3 + 2vx^2 - 7v'x^3 - 14vx^2 + 12vx^2 = 0$$

$$v''x^4 - 3x^3v' = 0$$

Make the substitution $w = v'$ and $w' = v''$.

$$w'x^4 - 3x^3w = 0$$

$$x^4\frac{dw}{dv} - 3x^3w = 0$$

$$\frac{dw}{dv} - \frac{3}{x}w = 0$$

With $P(x) = -3/x$ and $Q(x) = 0$, the integrating factor is

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int -\frac{3}{x}\ dx}$$

$$I(x) = e^{-3\ln x}$$

$$I(x) = x^{-3}$$

Multiplying the first order differential equation in w by the integrating factor, and then simplifying, gives

$$\frac{d}{dv}(wx^{-3}) = 0$$

Integrate both sides.

$$\int \frac{d}{dv}(wx^{-3})\ dv = \int 0\ dv$$

$$wx^{-3} = C$$

$$w = Cx^3$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w = \int Cx^3\ dx = \frac{C}{4}x^4 + k$$

We can choose any constants, so we'll choose $C = 4$ and $k = 0$ for simplicity. Then v is

$$v = x^4$$

and the second solution for the differential equation is

$$y_2 = vx^2$$

$$y_2 = (x^4)x^2$$

$$y_2 = x^6$$

Because the solutions are $y_1 = x^2$ and $y_2 = x^6$, the general solution to the differential equation is

$$y(x) = c_1x^2 + c_2x^6$$

■ 3. Use reduction of order to find the general solution to the differential equation, given $y_1 = x^3$.

$$x^2y'' - 6y = x^3$$

Solution:

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = vx^3$$

$$y_2' = v'x^3 + 3vx^2$$

$$y_2'' = v''x^3 + 6v'x^2 + 6vx$$

Plug these values into the homogeneous differential equation.

$$x^2(v''x^3 + 6v'x^2 + 6vx) - 6vx^3 = x^3$$

$$v''x^5 + 6v'x^4 + 6vx^3 - 6vx^3 = = x^3$$

$$v''x^5 + 6v'x^4 = x^3$$

$$v''x^2 + 6v'x = 1$$

Make the substitution $w = v'$ and $w' = v''$.

$$w'x^2 + 6wx = 1$$

$$w' + \frac{6}{x}w = \frac{1}{x^2}$$

With $P(x) = 6/x$ and $Q(x) = 1/x^2$, the integrating factor is

$$\mu(x) = e^{\int P(x)\ dx}$$

$$\mu(x) = e^{\int \frac{6}{x}\ dx}$$

$$\mu(x) = e^{6 \ln x}$$

$$\mu(x) = x^6$$

Multiplying the first order differential equation in w by the integrating factor, and then simplifying, gives

$$\frac{d}{dv}(wx^6) = x^4$$

Integrate both sides.

$$\int \frac{d}{dv}(wx^6)\ dv = \int x^4\ dv$$

$$wx^6 = \frac{1}{5}x^5 + C$$

$$w = \frac{1}{5x} + \frac{C}{x^6}$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w = \int \frac{1}{5x} + \frac{C}{x^6}\ dx = \frac{1}{5}\ln x + \frac{C}{5}x^{-5} + k$$

We can choose any constants, so we'll choose $C = 5$ and $k = 0$ for simplicity. Then v is

$$v = \frac{1}{5}\ln x + \frac{1}{x^5}$$

and the second solution for the differential equation is

$$y_2 = vx^3$$

$$y_2 = \left(\frac{1}{5}\ln x + \frac{1}{x^5}\right)x^3$$

$$y_2 = \frac{1}{5}x^3\ln x + \frac{1}{x^2}$$

So the general solution to the differential equation is

$$y = c_1x^3 + c_2\left(\frac{1}{5}x^3\ln x + \frac{1}{x^2}\right)$$

■ 4. Use reduction of order to find the general solution to the differential equation, given $y_1 = \cos(2x)$.

$$y'' + 4y = 0$$

Solution:

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = v\cos(2x)$$

$$y_2' = v'\cos(2x) - 2v\sin(2x)$$

$$y_2'' = v''\cos(2x) - 4v'\sin(2x) - 4v\cos(2x)$$

Plug these values into the homogeneous differential equation.

$$v''\cos(2x) - 4v'\sin(2x) - 4v\cos(2x) + 4v\cos(2x) = 0$$

$$v''\cos(2x) - 4v'\sin(2x) = 0$$

$$v'' - 4v'\tan(2x) = 0$$

Make the substitution $w = v'$ and $w' = v''$.

$$w' - 4w\tan(2x) = 0$$

$$\frac{dw}{dv} = 4w\tan(2x)$$

$$\frac{1}{w}\ dw = 4\tan(2x)\ dx$$

Integrate both sides.

$$\int \frac{1}{w}\ dw = \int 4\tan(2x)\ dx$$

$$\ln w = -2\ln(\cos(2x)) + C$$

$$\ln w = \ln\left(\frac{1}{\cos^2(2x)}\right) + C$$

$$w = \frac{C}{\cos^2(2x)}$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w = \int \frac{C}{\cos^2(2x)}\ dx$$

$$v = \int w = \int \frac{C}{\cos^2(2x)} = \frac{C}{2}\tan(2x) + k$$

We can choose any constants, so we'll choose $C = 2$ and $k = 0$ for simplicity. Then v is

$$v = \tan(2x)$$

and the second solution for the differential equation is

$$y_2 = v\cos(2x)$$

$$y_2 = \tan(2x)\cos(2x)$$

$$y_2 = \sin(2x)$$

Because the solutions are $y_2 = \cos(2x)$ and $y_2 = \sin(2x)$, the general solution to the differential equation is

$$y(x) = c_1\cos(2x) + c_2\sin(2x)$$

■ 5. Use reduction of order to find the general solution to the differential equation, given $y_1 = x$.

$$(x+1)y'' + xy' - y = 0$$

Solution:

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = vx$$

$$y_2' = v'x + v$$

$$y_2'' = v''x + 2v'$$

Plug these values into the homogeneous differential equation.

$$(x+1)(v''x + 2v') + x(v'x + v) - vx = 0$$

$$v''(x^2 + x) + v'(x^2 + 2x + 2) = 0$$

Make the substitution $w = v'$ and $w' = v''$.

$$w'(x^2 + x) + w(x^2 + 2x + 2) = 0$$

$$\frac{dw}{w} = -\frac{x^2 + 2x + 2}{x^2 + x}\ dx$$

$$\frac{dw}{w} = \left(-1 - \frac{2}{x} + \frac{1}{1+x}\right)\ dx$$

$$\int \frac{dw}{w} = \int \left(-1 - \frac{2}{x} + \frac{1}{1+x}\right)\ dx$$

$$\ln w = -x - 2\ln x + \ln(x+1) + C$$

$$\ln w = -x + \ln\left(\frac{x+1}{x^2}\right) + C$$

$$w = Ce^{-x}\left(\frac{x+1}{x^2}\right)$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w = \int Ce^{-x}\left(\frac{x+1}{x^2}\right)\ dx = -\frac{C}{x}e^{-x} + k$$

We can choose any constants, so we'll choose $C = 1$ and $k = 0$ for simplicity. Then v is

$$v = \frac{1}{x}e^{-x}$$

and the second solution for the differential equation is

$$y_2 = vx$$

$$y_2 = \left(\frac{1}{x}e^{-x}\right)x$$

$$y_2 = e^{-x}$$

Because the solutions are $y_1 = x$ and $y_2 = e^{-x}$, the general solution to the differential equation is

$$y(x) = c_1 x + c_2 e^{-x}$$

■ 6. Use reduction of order to find the general solution to the differential equation, given $y_1 = e^{4x}$.

$$y'' - 5y' + 4y = 0$$

Solution:

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = ve^{4x}$$

$$y_2' = v'e^{4x} + 4ve^{4x}$$

$$y_2'' = v''e^{4x} + 8v'e^{4x} + 16ve^{4x}$$

Plug these values into the homogeneous differential equation.

$$v''e^{4x} + 8v'e^{4x} + 16ve^{4x} - 5(v'e^{4x} + 4ve^{4x}) + 4ve^{4x} = 0$$

$$v''e^{4x} + 3v'e^{4x} = 0$$

Make the substitution $w = v'$ and $w' = v''$.

$$w'e^{4x} + 3we^{4x} = 0$$

$$\frac{dw}{dv} = -3w$$

$$\frac{dw}{w} = -3\ dv$$

$$\ln|w| = -3x$$

$$w = Ce^{-3x}$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w = \int Ce^{-3x} = -\frac{1}{3}Ce^{-3x} + k$$

We can choose any constants, so we'll choose $C = -3$ and $k = 0$ for simplicity. Then v is

$$v = e^{-3x}$$

and the second solution for the differential equation is

$$y_2 = ve^{4x}$$

$$y_2 = e^{-3x}e^{4x}$$

$$y_2 = e^{x}$$

Because the solutions are $y_1 = e^{4x}$ and $y_2 = e^{x}$, the general solution to the differential equation is

$$y(x) = c_1e^{4x} + c_2e^{x}$$

UNDETERMINED COEFFICIENTS FOR NONHOMOGENEOUS EQUATIONS

■ 1. Find the general solution to the differential equation.

$$y'' - 2y' + y = x^3 - 6x^2 + 6x + 1$$

Solution:

Solve the characteristic equation associated with the homogeneous equation,

$$y'' - 2y' + y = 0$$

$$r^2 - 2r + 1 = 0$$

$$(r-1)^2 = 0$$

$$r = 1,\ 1$$

to find the complementary solution with equal real roots.

$$y_c(x) = c_1e^x + c_2xe^x$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = Ax^3 + Bx^2 + Cx + D$$

$$y_p'(x) = 3Ax^2 + 2Bx + C$$

$$y_p''(x) = 6Ax + 2B$$

Plug the derivatives into the original differential equation.

$$6Ax + 2B - 2(3Ax^2 + 2Bx + C) + Ax^3 + Bx^2 + Cx + D = x^3 - 6x^2 + 6x + 1$$

$$6Ax + 2B - 6Ax^2 - 4Bx - 2C + Ax^3 + Bx^2 + Cx + D = x^3 - 6x^2 + 6x + 1$$

$$Ax^3 + x^2(-6A + B) + x(6A - 4B + C) + 2B - 2C + D = x^3 - 6x^2 + 6x + 1$$

Equate coefficients.

$$A = 1$$

$$-6A + B = -6 \qquad B = 0$$

$$6A - 4B + C = 6 \qquad C = 0$$

$$2B - 2C + D = 1 \qquad D = 1$$

Then the particular solution is

$$y_p(x) = x^3 + 1$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^x + c_2xe^x + x^3 + 1$$

■ 2. Use the method of undetermined coefficient to solve the differential equation.

$$y'' - 3y' + 2y = e^{-x}(6x^2 - 10x + 20)$$

Solution:

Solve the characteristic equation associated with the homogeneous equation,

$$y'' - 3y' + 2y = 0$$

$$r^2 - 3r + 2 = 0$$

$$(r - 1)(r - 2) = 0$$

$$r = 1,\ 2$$

to find the complementary solution with distinct real roots.

$$y_c(x) = c_1e^x + c_2e^{2x}$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = e^{-x}(Ax^2 + Bx + C)$$

$$y_p'(x) = -e^{-x}(Ax^2 + Bx + C) + e^{-x}(2Ax + B) = e^{-x}(-Ax^2 + x(2A - B) + B - C)$$

$$y_p''(x) = e^{-x}(Ax^2 + x(-4A + B) + C - 2B + 2A)$$

Plug the derivatives into the original differential equation.

$$e^{-x}(Ax^2 + x(-4A + B) + C - 2B + 2A) - 3e^{-x}(-Ax^2 + x(2A - B) + B - C)$$

$$+2e^{-x}(Ax^2 + Bx + C) = e^{-x}(6x^2 - 10x + 20)$$

$$e^{-x}(6Ax^2 + x(-10A + 6B) + 2A - 5B + 6C) = e^{-x}(6x^2 - 10x + 20)$$

Equate coefficients.

$$6A = 6 \qquad A = 1$$

$$-10A + 6B = -10 \qquad B = 0$$

$$2A - 5B + 6C = 20 \qquad C = 3$$

Then the particular solution is

$$y_p(x) = e^{-x}(x^2 + 3)$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^x + c_2e^{2x} + e^{-x}(x^2 + 3)$$

■ 3. Find the general solution to the differential equation.

$$y'' - y' = e^{2x} + e^{3x} + e^{4x}$$

Solution:

Solve the characteristic equation associated with the homogeneous equation,

$$y'' - y' = 0$$

$$r^2 - r = 0$$

$$r(r-1)=0$$

$$r=0,\ 1$$

to find the complementary solution with distinct real roots.

$$y_c(x)=c_1+c_2e^x$$

Our guess for the particular solution and its derivatives will be

$$y_p(x)=Ae^{2x}+Be^{3x}+Ce^{4x}$$

$$y_p'(x)=2Ae^{2x}+3Be^{3x}+4Ce^{4x}$$

$$y_p''(x)=4Ae^{2x}+9Be^{3x}+16Ce^{4x}$$

Plug the derivatives into the original differential equation.

$$4Ae^{2x}+9Be^{3x}+16Ce^{4x}-2Ae^{2x}-3Be^{3x}-4Ce^{4x}=e^{2x}+e^{3x}+e^{4x}$$

$$2Ae^{2x}+6Be^{3x}+12Ce^{4x}=e^{2x}+e^{3x}+e^{4x}$$

Equate coefficients.

$$2A=1 \qquad A=\frac{1}{2}$$

$$6B=1 \qquad B=\frac{1}{6}$$

$$12C=1 \qquad C=\frac{1}{12}$$

Then the particular solution is

$$y_p(x) = \frac{1}{2}e^{2x} + \frac{1}{6}e^{3x} + \frac{1}{12}e^{4x}$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 + c_2e^x + \frac{1}{2}e^{2x} + \frac{1}{6}e^{3x} + \frac{1}{12}e^{4x}$$

■ 4. Solve the differential equation.

$$y'' + 9y = 12\sin(3x)$$

Solution:

Solve the characteristic equation associated with the homogeneous equation,

$$y'' + 9y = 0$$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

to find the complementary solution with complex conjugate roots.

$$y_c(x) = c_1\sin(3x) + c_2\cos(3x)$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = Ax\sin(3x) + Bx\cos(3x)$$

$$y_p'(x) = A\sin(3x) + 3Ax\cos(3x) + B\cos(3x) - 3Bx\sin(3x)$$

$$y_p''(x) = 6A\cos(3x) - 9Ax\sin(3x) - 6B\sin(3x) - 9Bx\cos(3x)$$

Plug the derivatives into the original differential equation.

$$6A\cos(3x) - 9Ax\sin(3x) - 6B\sin(3x) - 9Bx\cos(3x)$$

$$+9(Ax\sin(3x) + Bx\cos(3x)) = 12\sin(3x)$$

$$6A\cos(3x) - 9Ax\sin(3x) - 6B\sin(3x) - 9Bx\cos(3x)$$

$$+9Ax\sin(3x) + 9Bx\cos(3x) = 12\sin(3x)$$

$$6A\cos(3x) - 6B\sin(3x) = 12\sin(3x)$$

Equate coefficients.

$$6A = 0 \qquad A = 0$$

$$-6B = 12 \qquad B = -2$$

Then the particular solution is

$$y_p(x) = -2x\cos(3x)$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y_c(x) = c_1\sin(3x) + c_2\cos(3x) - 2x\cos(3x)$$

■ 5. Solve the differential equation.

$$y'' - 4y' + 4y = e^{2x}$$

Solution:

Solve the characteristic equation associated with the homogeneous equation,

$$y'' - 4y' + 4y = 0$$

$$r^2 - 4r + 4 = 0$$

$$(r - 2)^2 = 0$$

$$r = 2, 2$$

to find the complementary solution with equal real roots.

$$y_c(x) = c_1e^{2x} + c_2xe^{2x}$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = Ax^2e^{2x}$$

$$y_p'(x) = 2Ax^2e^{2x} + 2Axe^{2x} = 2Ae^{2x}(x^2 + x)$$

$$y_p''(x) = 4Ae^{2x}(x^2 + x) + 2Ae^{2x}(2x + 1) = 2Ae^{2x}(2x^2 + 4x + 1)$$

Plug the derivatives into the original differential equation.

$$2Ae^{2x}(2x^2+4x+1)-4(2Ae^{2x}(x^2+x))+4Ax^2e^{2x}=e^{2x}$$

$$2Ae^{2x}(2x^2+4x+1)-8Ae^{2x}(x^2+x))+4Ax^2e^{2x}=e^{2x}$$

Equate coefficients.

$$2Ae^{2x}=e^{2x} \qquad A=\frac{1}{2}$$

Then the particular solution is

$$y_p(x)=\frac{1}{2}x^2e^{2x}$$

and the general solution is

$$y(x)=c_1e^{2x}+c_2xe^{2x}+\frac{1}{2}x^2e^{2x}$$

■ 6. Solve the differential equation.

$$y''-4y'+3y=e^x+10\sin x+3x$$

Solution:

Solve the characteristic equation associated with the homogeneous equation,

$$y''-4y'+3y=0$$

$$r^2-4r+3=0$$

$$(r-1)(r-3)=0$$

$$r=1,\ 3$$

to find the complementary solution with distinct real roots.

$$y_c(x)=c_1e^x+c_2e^{3x}$$

Our guess for the particular solution and its derivatives will be

$$y_p(x)=Axe^x+B\sin x+C\cos x+Dx+E$$

$$y_p'(x)=Ae^x+Axe^x+B\cos x-C\sin x+D$$

$$y_p''(x)=2Ae^x+Axe^x-B\sin x-C\cos x$$

Plug the derivatives into the original differential equation.

$$2Ae^x+Axe^x-B\sin x-C\cos x-4(Ae^x+Axe^x+B\cos x-C\sin x+D)+$$

$$+3(Axe^x+B\sin x+C\cos x+Dx+E)=e^x+10\sin x+3x$$

$$-2Ae^x+\sin x(4C+2B)+\cos x(2C-4B)+3Dx+3E-4D=e^x+10\sin x+3x$$

Equate coefficients.

$$-2A=1 \qquad A=-1/2$$

$$4C-2B=10 \qquad B=1$$

$$2C-4B=0 \qquad C=2$$

$$3D=3 \qquad D=1$$

$$3E - 4D = 0 \qquad\qquad E = \frac{4}{3}$$

Then the particular solution is

$$y_p(x) = -\frac{1}{2}xe^x + \sin x + 2\cos x + x + \frac{4}{3}$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^x + c_2e^{3x} - \frac{1}{2}xe^x + \sin x + 2\cos x + x + \frac{4}{3}$$

VARIATION OF PARAMETERS FOR NONHOMOGENEOUS EQUATIONS

■ 1. Find the general solution to the differential equation.

$$y'' - 10y' + 25y = x^3e^{5x}$$

Solution:

Solve the associated homogeneous equation,

$$y'' - 10y' + 25y = 0$$

$$r^2 - 10r + 25 = 0$$

$$(r - 5)(r - 5) = 0$$

$$r = 5,\ 5$$

to find equal real roots, then build the complementary solution.

$$y_c(x) = c_1e^{r_1x} + c_2xe^{r_1x}$$

$$y_c(x) = c_1e^{5x} + c_2xe^{5x}$$

The fundamental set of solutions and its derivatives are therefore

$$\{y_1, y_2\} = \{e^{5x},\ xe^{5x}\}$$

$$\{y_1', y_2'\} = \{5e^{5x},\ e^{5x} + 5xe^{5x}\}$$

Create a system of linear equations,

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'e^{5x} + u_2'xe^{5x} = 0$$

$$u_1'y_1' + u_2'y_2' = g(x)$$

$$u_1'(5e^{5x}) + u_2'(e^{5x} + 5xe^{5x}) = x^3e^{5x}$$

$$5u_1'e^{5x} + u_2'e^{5x} + 5u_2'xe^{5x} = x^3e^{5x}$$

then solve it for u_2',

$$5u_1'e^{5x} + 5u_2'xe^{5x} - (5u_1'e^{5x} + u_2'e^{5x} + 5u_2'xe^{5x}) = 0 - x^3e^{5x}$$

$$5u_1'e^{5x} + 5u_2'xe^{5x} - 5u_1'e^{5x} - u_2'e^{5x} - 5u_2'xe^{5x} = -x^3e^{5x}$$

$$-u_2'e^{5x} = -x^3e^{5x}$$

$$u_2' = x^3$$

and then u_1'.

$$u_1'e^{5x} + (x^3)xe^{5x} = 0$$

$$u_1'e^{5x} = -x^4e^{5x}$$

$$u_1' = -x^4$$

Then u_1 and u_2 are

$$u_1 = \int u_1' = \int -x^4\ dx = -\frac{1}{5}x^5$$

$$u_2 = \int u_2' = \int x^3\ dx = \frac{1}{4}x^4$$

The particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = -\frac{1}{5}x^5(e^{5x}) + \frac{1}{4}x^4(xe^{5x})$$

$$y_p(x) = -\frac{1}{5}x^5e^{5x} + \frac{1}{4}x^5e^{5x}$$

$$y_p(x) = \frac{x^5e^{5x}}{20}$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1e^{5x} + c_2xe^{5x} + \frac{x^5e^{5x}}{20}$$

■ 2. Find the general solution to the differential equation.

$$y'' - 6y' + 8y = e^{3x} + 5$$

Solution:

Solve the associated homogeneous equation,

$$y'' - 6y' + 8y = 0$$

$$r^2 - 6r + 8 = 0$$

$$(r-2)(r-4)=0$$

$$r=2,\ 4$$

to find distinct real roots, then build the complementary solution.

$$y_c(x)=c_1e^{r_1x}+c_2e^{r_2x}$$

$$y_c(x)=c_1e^{2x}+c_2e^{4x}$$

The fundamental set of solutions and its derivatives are therefore

$$\{y_1,y_2\}=\{e^{2x},\ e^{4x}\}$$

$$\{y_1',y_2'\}=\{2e^{2x},\ 4e^{4x}\}$$

Create a system of linear equations,

$$u_1'y_1+u_2'y_2=0$$

$$u_1'e^{2x}+u_2'e^{4x}=0$$

$$u_1'y_1'+u_2'y_2'=g(x)$$

$$u_1'(2e^{2x})+u_2'(4e^{4x})=e^{3x}+5$$

$$2u_1'e^{2x}+4u_2'e^{4x}=e^{3x}+5$$

then solve it for u_2',

$$2u_1'e^{2x}+4u_2'e^{4x}-2(u_1'e^{2x}+u_2'e^{4x})=e^{3x}+5-0$$

$$2u_2'e^{4x}=e^{3x}+5$$

$$u_2'=\frac{e^{3x}+5}{2e^{4x}}$$

and then u_1'.

$$u_1' e^{2x} + u_2' e^{4x} = 0$$

$$u_1' e^{2x} + \frac{e^{3x} + 5}{2e^{4x}} \cdot e^{4x} = 0$$

$$u_1' e^{2x} + \frac{e^{3x} + 5}{2} = 0$$

$$u_1' = -\frac{e^{3x} + 5}{2e^{2x}}$$

Then u_1 and u_2 are

$$u_1 = \int u_1' = \int -\frac{e^{3x} + 5}{2e^{2x}} \ dx = -\frac{e^x}{2} + \frac{5}{4}e^{-2x}$$

$$u_2 = \int u_2' = \int \frac{e^{3x} + 5}{2e^{4x}} \ dx = -\frac{1}{2}e^{-x} - \frac{5}{8}e^{-4x}$$

The particular solution is

$$y_p(x) = u_1 y_1 + u_2 y_2$$

$$y_p(x) = \left(-\frac{e^x}{2} + \frac{5}{4}e^{-2x}\right)(e^{2x}) + \left(-\frac{1}{2}e^{-x} - \frac{5}{8}e^{-4x}\right)(e^{4x})$$

$$y_p(x) = -e^{3x} + \frac{5}{8}$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1 e^{2x} + c_2 e^{4x} - e^{3x} + \frac{5}{8}$$

■ 3. Find the general solution to the differential equation.

$$y'' - 7y' + 12y = \frac{e^{5x}}{e^{2x} + 1}$$

Solution:

Solve the associated homogeneous equation,

$$y'' - 7y' + 12y = 0$$

$$r^2 - 7r + 12 = 0$$

$$(r - 3)(r - 4) = 0$$

$$r = 3, 4$$

to find distinct real roots, then build the complementary solution.

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$y_c(x) = c_1 e^{3x} + c_2 e^{4x}$$

The fundamental set of solutions and its derivatives are therefore

$$\{y_1, y_2\} = \{e^{3x}, e^{4x}\}$$

$$\{y_1', y_2'\} = \{3e^{3x}, 4e^{4x}\}$$

Create a system of linear equations,

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'e^{3x} + u_2'e^{4x} = 0$$

$$u_1'y_1' + u_2'y_2' = g(x)$$

$$u_1'(3e^{3x}) + u_2'(4e^{4x}) = \frac{e^{5x}}{e^{2x} + 1}$$

$$3u_1'e^{3x} + 4u_2'e^{4x} = \frac{e^{5x}}{e^{2x} + 1}$$

then solve it for u_2',

$$u_1' = -u_2'e^x$$

$$u_2'e^{4x} = \frac{e^{5x}}{e^{2x} + 1}$$

$$u_2' = \frac{e^x}{e^{2x} + 1}$$

and then u_1'.

$$u_1' = -\frac{e^{2x}}{e^{2x} + 1}$$

Then using the substitution $t = e^{2x}$ and $dt = 2e^2x\ dx$ to find u_1, and the substitution $t = e^x$ and $dt = e^x\ dx$ to find u_2,

$$u_1 = \int u_1' = \int -\frac{e^{2x}}{e^{2x} + 1}\ dx = -\frac{1}{2}\ln(e^{2x} + 1)$$

$$u_2 = \int u_2' = \int \frac{e^x}{e^{2x} + 1}\ dx = \arctan e^x$$

the particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = -\frac{1}{2}\ln(e^{2x}+1)(e^{3x}) + \arctan e^x(e^{4x})$$

$$y_p(x) = -\frac{e^{3x}}{2}\ln(e^{2x}+1) + e^{4x}\arctan e^x$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1e^{3x} + c_2e^{4x} - \frac{e^{3x}}{2}\ln(e^{2x}+1) + e^{4x}\arctan e^x$$

■ 4. Find the general solution to the differential equation.

$$y'' - 6y' + 9y = e^{3x}\sin(2x+1)$$

Solution:

Solve the associated homogeneous equation,

$$y'' - 6y' + 9y = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$$r = 3,\ 3$$

to find equal real roots, then build the complementary solution.

$$y_c(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

$$y_c(x) = c_1 e^{3x} + c_2 x e^{3x}$$

The fundamental set of solutions and its derivatives are therefore

$$\{y_1, y_2\} = \{e^{3x},\ xe^{3x}\}$$

$$\{y_1', y_2'\} = \{3e^{3x},\ e^{3x} + 3xe^{3x}\}$$

Create a system of linear equations,

$$u_1' y_1 + u_2' y_2 = 0 \qquad u_1' y_1' + u_2' y_2' = g(x)$$

$$u_1' e^{3x} + u_2' x e^{3x} = 0 \qquad u_1'(3e^{3x}) + u_2'(e^{3x} + 3xe^{3x}) = e^{3x} \sin(2x+1)$$

$$3u_1' e^{3x} + u_2' e^{3x} + 3u_2' x e^{3x} = e^{3x} \sin(2x+1)$$

then solve it for u_2'

$$3u_1' e^{3x} + 3u_2' x e^{3x} - (3u_1' e^{3x} + u_2' e^{3x} + 3u_2' x e^{3x}) = 0 - e^{3x} \sin(2x+1)$$

$$-u_2' e^{3x} = -\, e^{3x} \sin(2x+1)$$

$$u_2' = \sin(2x+1)$$

and then hen u_1'.

$$u_1' e^{3x} + u_2' x e^{3x} = 0$$

$$u_1' = -\, x u_2'$$

$$u_1' = -x\sin(2x+1)$$

Then using integration by parts with $t = -x$, $dt = -dx$, $dv = \sin(2x+1)\ dx$, and $v = -(1/2)\cos(2x+1)$ to find u_1, and integrating directly to find u_2,

$$u_1 = \int u_1' = \int -x\sin(2x+1)\ dx = \frac{1}{2}x\cos(2x+1) - \frac{1}{4}\sin(2x+1)$$

$$u_2 = \int u_2' = \int \sin(2x+1)\ dx = -\frac{1}{2}\cos(2x+1)$$

the particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = \left(\frac{1}{2}x\cos(2x+1) - \frac{1}{4}\sin(2x+1)\right)(e^{3x}) - \frac{1}{2}\cos(2x+1)(xe^{3x})$$

$$y_p(x) = \frac{1}{2}xe^{3x}\cos(2x+1) - \frac{1}{4}e^{3x}\sin(2x+1) - \frac{1}{2}xe^{3x}\cos(2x+1)$$

$$y_p(x) = -\frac{1}{4}e^{3x}\sin(2x+1)$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1e^{3x} + c_2xe^{3x} - \frac{1}{4}e^{3x}\sin(2x+1)$$

■ 5. Find the general solution to the differential equation.

$$2y'' + 18y = \sin^4(3x)$$

Solution:

Rewrite the equation in standard form.

$$y'' + 9y = \frac{\sin^4(3x)}{2}$$

Solve the associated homogeneous equation,

$$y'' + 9y = 0$$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

to find complex conjugate roots, then build the complementary solution.

$$y_c(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

$$y_c(x) = c_1 \cos(3x) + c_2 \sin(3x)$$

The fundamental set of solutions and its derivatives are therefore

$$\{y_1, y_2\} = \{\cos(3x),\ \sin(3x)\}$$

$$\{y_1', y_2'\} = \{-3\sin(3x),\ 3\cos(3x)\}$$

Create a system of linear equations,

$$u_1'y_1 + u_2'y_2 = 0 \qquad\qquad u_1'y_1' + u_2'y_2' = g(x)$$

$$u_1' \cos(3x) + u_2' \sin(3x) = 0 \qquad u_1'(-3\sin(3x)) + u_2'(3\cos(3x)) = \frac{\sin^4(3x)}{2}$$

$$-3u_1' \sin(3x) + 3u_2' \cos(3x) = \frac{\sin^4(3x)}{2}$$

then solve it for u_2',

$$u_1' = -u_2' \frac{\sin(3x)}{\cos(3x)}$$

$$-3\left(-u_2' \frac{\sin(3x)}{\cos(3x)}\right)\sin(3x) + 3u_2' \cos(3x) = \frac{\sin^4(3x)}{2}$$

$$u_2' = \frac{\cos(3x)\sin^4(3x)}{6}$$

and then u_1'.

$$u_1' = -u_2' \frac{\sin(3x)}{\cos(3x)}$$

$$u_1' = -\frac{\cos(3x)\sin^4(3x)}{6} \cdot \frac{\sin(3x)}{\cos(3x)}$$

$$u_1' = -\frac{\sin^5(3x)}{6}$$

Then using the substitution $t = \cos(3x)$ and $dt = -(1/3)\sin(3x)\ dx$ to find u_1, and the substitution $t = \sin(3x)$ and $dt = (1/3)\cos(3x)\ dx$ to find u_2,

$$u_1 = \int u_1' = \int -\frac{\sin^5(3x)}{6}\ dx = -\frac{1}{6}\int (1 - \cos^2(3x))^2 \sin(3x)\ dx$$

$$= \frac{1}{18}\left(\cos(3x) - \frac{2}{3}\cos^3(3x) + \frac{1}{5}\cos^5(3x)\right)$$

$$u_2 = \int u_2' = \int \frac{\cos(3x)\sin^4(3x)}{6}\ dx = \frac{t^5}{90} = \frac{\sin^5(3x)}{90}$$

the particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = \frac{1}{18}\left(\cos(3x) - \frac{2}{3}\cos^3(3x) + \frac{1}{5}\cos^5(3x)\right)(\cos(3x)) + \frac{\sin^5(3x)}{90}(\sin(3x))$$

$$y_p(x) = \frac{1}{18}\left(\cos(3x) - \frac{2}{3}\cos^3(3x) + \frac{1}{5}\cos^5(3x)\right)(\cos(3x)) + \frac{\sin^5(3x)}{90}(\sin(3x))$$

$$y_p(x) = \frac{1}{18}\cos^2(3x) - \frac{1}{27}\cos^4(3x) + \frac{1}{90}$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1\cos(3x) + c_2\sin(3x) + \frac{1}{18}\cos^2(3x) - \frac{1}{27}\cos^4(3x) + \frac{1}{90}$$

■ 6. Find the general solution to the differential equation.

$$y'' - 4y' + 5y = e^{2x}\ln(1 + \sin x)$$

Solution:

Solve the associated homogeneous equation,

$$y'' - 4y' + 5y = 0$$

$$r^2 - 4r + 5 = 0$$

$$r = 2 \pm i$$

to find complex conjugate roots, then build the complementary solution.

$$y_c(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

$$y_c(x) = e^{2x}(c_1 \cos x + c_2 \sin x)$$

The fundamental set of solutions and its derivatives are therefore

$$\{y_1, y_2\} = \{e^{2x} \cos x, \ e^{2x} \sin x\}$$

$$\{y_1', y_2'\} = \{2e^{2x} \cos x - e^{2x} \sin x, \ 2e^{2x} \sin x + e^{2x} \cos x\}$$

Create a system of linear equations,

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' e^{2x} \cos x + u_2' e^{2x} \sin x = 0$$

and

$$u_1' y_1' + u_2' y_2' = g(x)$$

$$u_1'(2e^{2x} \cos x - e^{2x} \sin x) + u_2'(2e^{2x} \sin x + e^{2x} \cos x) = e^{2x} \ln(1 + \sin x)$$

then solve it for u_2' by rewriting the first equation for u_1',

$$u_1'e^{2x}\cos x = -u_2'e^{2x}\sin x$$

$$u_1' = -u_2'\tan x$$

and then plugging this value into the second equation in the system.

$$-u_2'\tan x(2e^{2x}\cos x - e^{2x}\sin x) + u_2'(2e^{2x}\sin x + e^{2x}\cos x) = e^{2x}\ln(1+\sin x)$$

$$u_2'(2e^{2x}\sin x + e^{2x}\cos x) - u_2'(2e^{2x}\sin x - e^{2x}\sin x\tan x) = e^{2x}\ln(1+\sin x)$$

$$u_2'(2e^{2x}\sin x + e^{2x}\cos x - 2e^{2x}\sin x + e^{2x}\sin x\tan x) = e^{2x}\ln(1+\sin x)$$

$$u_2'(e^{2x}\cos x + e^{2x}\sin x\tan x) = e^{2x}\ln(1+\sin x)$$

$$u_2'(\cos x + \sin x\tan x) = \ln(1+\sin x)$$

$$u_2' = \frac{\ln(1+\sin x)}{\cos x + \frac{\sin^2 x}{\cos x}}$$

$$u_2' = \frac{\ln(1+\sin x)}{\frac{\cos^2 x + \sin^2 x}{\cos x}}$$

$$u_2' = \frac{\ln(1+\sin x)\cos x}{\cos^2 x + \sin^2 x}$$

$$u_2' = \ln(1+\sin x)\cos x$$

Then find u_1'.

$$u_1' = -\ln(1+\sin x)\cos x\tan x$$

$$u_1' = -\ln(1+\sin x)\sin x$$

$$u_1' = -\sin x \ln(1 + \sin x)$$

Then using integration by parts with $u = \ln(1 + \sin x)$, $du = \cos x/(1 + \sin x)\ dx$, $dv = -\sin x\ dx$, and $v = \cos x$ to find u_1, and using a substitution with $t = \sin x$ and $dt = \cos x\ dx$, followed by integration by parts with $u = \ln(1 + t)\ du$, $du = 1/(t + 1)\ dt$, $dv = dt$, and $v = t$ to find u_2,

$$u_1 = \int u_1' = \int -\sin x \ln(1 + \sin x)\ dx = \cos x \ln(1 + \sin x) - x - \cos x$$

$$u_2 = \int u_2' = \int \cos x \ln(1 + \sin x)\ dx = (\sin x + 1)\ln(\sin x + 1) - \sin x$$

the particular solution is

$$y_p(x) = u_1 y_1 + u_2 y_2$$

$$y_p(x) = (\cos x \ln(1 + \sin x) - x - \cos x)(e^{-2x} \cos x)$$

$$+((\sin x + 1)\ln(\sin x + 1) - \sin x)(e^{-2x} \sin x)$$

$$y_p(x) = e^{-2x}(\ln(\sin x + 1)\ln(\sin x + 1) - x \sin x - 1)$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y = e^{-2x}(c_1 \cos x + c_2 \sin x + \ln(\sin x + 1)(\sin x + 1) - x \sin x - 1)$$

FUNDAMENTAL SOLUTION SETS AND THE WRONSKIAN

■ 1. Use the Wronskian to determine whether $\{y_1, y_2\} = \{e^x, e^{x+1}\}$ is the fundamental set of solutions for the differential equation.

$$y'' = y$$

Solution:

First we verify that y_1 and y_2 are solutions of the differential equation.

$$y_1'' = (e^x)'' = (e^x)' = e^x$$

$$y'' = y$$

$$e^x = e^x$$

$$y_2'' = (e^{x+1})'' = (e^{x+1})' = e^{x+1}$$

$$y'' = y$$

$$e^{x+1} = e^{x+1}$$

So $\{y_1, y_2\}$ is a set of solutions for differential equation. To verify whether the solution set is a fundamental set, we'll take the Wronskian.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(e^x, e^{x+1}) = \begin{vmatrix} e^x & e^{x+1} \\ e^x & e^{x+1} \end{vmatrix}$$

$$W(e^x, e^{x+1}) = (e^x)(e^{x+1}) - (e^{x+1})(e^x)$$

$$W(e^x, e^{x+1}) = 0$$

Since the Wronskian is zero, $\{y_1, y_2\}$ is not a fundamental set of solutions.

■ 2. Determine whether $\{y_1, y_2\} = \{\cos(2x),\ \cos(2x+1)\}$ is a fundamental set of solutions for the differential equation.

$$y'' + 4y = 0$$

Solution:

First we verify that y_1 and y_2 are solutions of the differential equation.

$$y_1'' = (\cos(2x))'' = (-2\sin(2x))' = -4\cos(2x)$$

$$y'' + 4y = 0$$

$$-4\cos(2x) + 4\cos(2x) = 0$$

$$0 = 0$$

$$y_2'' = (\cos(2x+1))'' = (-2\sin(2x+1))' = -4\cos(2x+1)$$

$$y'' + 4y = 0$$

$$-4\cos(2x+1)+4\cos(2x+1)=0$$

$$0=0$$

So $\{y_1, y_2\}$ is a set of solutions for differential equation. To verify whether the solution set is a fundamental set, we'll take the Wronskian.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(\cos(2x), \cos(2x+1)) = \begin{vmatrix} \cos(2x) & \cos(2x+1) \\ -2\sin(2x) & -2\sin(2x+1) \end{vmatrix}$$

$$W(\cos(2x), \cos(2x+1)) = (\cos(2x)(-2\sin(2x+1)) - (\cos(2x+1))(-2\sin(2x))$$

$$W(\cos(2x), \cos(2x+1)) = -2\sin 1$$

Since the Wronskian is non-zero, $\{y_1, y_2\}$ is a fundamental set of solutions.

■ 3. Find the Wronskian for the solution set.

$$\{e^{3x},\ xe^{3x}\}$$

Solution:

Find the derivatives of the solutions in the set.

$$(e^{3x})' = 3e^{3x}$$

$$(xe^{3x})' = e^{3x} + 3xe^{3x}$$

So the Wronskian for the set is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(e^{3x}, xe^{3x}) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix}$$

$$W(e^{3x}, xe^{3x}) = e^{3x}(e^{3x} + 3xe^{3x}) - xe^{3x}(3e^{3x})$$

$$W(e^{3x}, xe^{3x}) = e^{6x} + 3xe^{6x} - 3xe^{6x}$$

$$W(e^{3x}, xe^{3x}) = e^{6x}$$

■ 4. Find the Wronskian for the solution set.

$$\{e^{2x}\cos(2x),\ e^{2x}\sin(2x)\}$$

Solution:

Find the derivatives of the solutions in the set.

$$(e^{2x}\cos(2x))' = 2e^{2x}\cos(2x) - 2e^{2x}\sin(2x)$$

$$(e^{2x}\sin(2x))' = 2e^{2x}\sin(2x) + 2e^{2x}\cos(2x)$$

So the Wronskian for the set is

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$W(e^{2x}\cos(2x), e^{2x}\sin(2x)) = \begin{vmatrix} e^{2x}\cos(2x) & e^{2x}\sin(2x) \\ 2e^{2x}\cos(2x) - 2e^{2x}\sin(2x) & 2e^{2x}\sin(2x) + 2e^{2x}\cos(2x) \end{vmatrix}$$

$$W(e^{2x}\cos(2x), e^{2x}\sin(2x)) = e^{2x}\cos(2x)(2e^{2x}\cos(2x) - 2e^{2x}\sin(2x))$$

$$-e^{2x}\sin(2x)(2e^{2x}\sin(2x) + 2e^{2x}\cos(2x))$$

$$W(e^{2x}\cos(2x), e^{2x}\sin(2x)) = 2e^{4x}$$

■ 5. Find the fundamental set of solutions for the the second order differential equation.

$$y'' + 2y' + y = 0$$

Solution:

Normally, we would find the general solution to this second order equation by solving the associated characteristic equation,

$$r^2 + 2r + 1 = 0$$

$$(r + 1)^2 = 0$$

$$r = -1, \ -1$$

to get the complementary solution.

$$y_c(x) = c_1e^{-x} + c_2xe^{-x}$$

The derivative of the complementary solution is

$$y_c'(x) = -c_1e^{-x} + c_2e^{-x} - c_2xe^{-x}$$

If we now plug in the initial conditions $y(x_0) = 1$ and $y'(x_0) = 0$, and $y(x_0) = 0$ and $y'(x_0) = 1$, we'll be able to generate another fundamental set of solutions. We'll use $x_0 = 0$, since we can choose any value of x_0 that we like, and using $x_0 = 0$ will be the easiest.

For $y(0) = 1$, $\qquad 1 = c_1 \cdot 1 + c_2 \cdot 0$

For $y'(0) = 0$, $\qquad 0 = c_1 - c_2 + 0$

Then this system gives $c_1 = 1$ and $c_2 = 1$, so we could say that one solution in our fundamental set of solutions is

$$y_1(x) = e^{-x} + xe^{-x}$$

The second set of initial conditions gives

For $y(0) = 0$, $\qquad 0 = c_1 + 0$

For $y'(0) = 1$, $\qquad 1 = -c_1 + c_2$

Then this system gives $c_1 = 0$ and $c_2 = 1$, so we could say that the other solution in our fundamental set is

$$y_2(x) = xe^{-x}$$

So one fundamental set of solutions is $\{y_1, y_2\} = \{e^{-x} + xe^{-x}, xe^{-x}\}$.

6. Find the fundamental set of solutions for the the second order differential equation, generated by solutions of the initial value problems with $y(0) = 1$ and $y'(0) = 0$, and $y(0) = 0$ and $y'(0) = 1$.

$$y'' + 4y' = 0$$

Solution:

Normally, we would find the general solution to this second order equation by solving the associated characteristic equation,

$$r^2 + 4r = 0$$

$$r = 0, \ -4$$

to get the complementary solution.

$$y_c(x) = c_1 + c_2 e^{-4x}$$

The derivative of the complementary solution is

$$y_c'(x) = -4c_2 e^{-4x}$$

If we now plug in the initial conditions $y(x_0) = 1$ and $y'(x_0) = 0$, and $y(x_0) = 0$ and $y'(x_0) = 1$, we'll be able to generate another fundamental set of solutions. We'll use $x_0 = 0$, since we can choose any value of x_0 that we like, and using $x_0 = 0$ will be the easiest.

For $y(0) = 1$, $\qquad 1 = c_1 + c_2$

For $y'(0) = 0$, $\qquad 0 = -4c_2$

Then this system gives $c_1 = 1$ and $c_2 = 0$, so we could say that one solution in our fundamental set of solutions is

$$y_1(x) = 1$$

The second set of initial conditions gives

For $y(0) = 0$, $\qquad 0 = c_1 + c_2$

For $y'(0) = 1$, $\qquad 1 = -4c_2$

Then this system gives $c_1 = 1/4$ and $c_2 = -1/4$, so we could say that the other solution in our fundamental set is

$$y_2(x) = \frac{1}{4} - \frac{1}{4}e^{-4x}$$

So one fundamental set of solutions is

$$\{y_1, y_2\} = \left\{1, \frac{1}{4} - \frac{1}{4}e^{-4x}\right\}$$

We can verify that this set is a fundamental set by taking the Wronskian.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W\left(1, \frac{1}{4} - \frac{1}{4}e^{-4x}\right) = \begin{vmatrix} 1 & \frac{1}{4} - \frac{1}{4}e^{-4x} \\ 0 & e^{-4x} \end{vmatrix}$$

$$W\left(1, \frac{1}{4} - \frac{1}{4}e^{-4x}\right) = e^{-4x}$$

VARIATION OF PARAMETERS WITH THE WRONSKIAN

■ 1. Solve the differential equation.

$$y'' + 7y' + 12y = e^x + 7e^{3x}$$

Solution:

The associated homogeneous equation is

$$y'' + 7y' + 12y = 0$$

and solving the characteristic equation gives distinct real roots.

$$r^2 - 7r + 12 = 0$$

$$(r + 3)(r + 4) = 0$$

$$r = -3, \ -4$$

With distinct real roots, the complementary solution is

$$y_c(x) = c_1e^{-3x} + c_2e^{-4x}$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{e^{-3x}, \ e^{-4x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(e^{-3x}, e^{-4x}) = \begin{vmatrix} e^{-3x} & e^{-4x} \\ -3e^{-3x} & -4e^{-4x} \end{vmatrix}$$

$$W(e^{-3x}, e^{-4x}) = -4e^{-7x} + 3e^{-7x}$$

$$W(e^{-3x}, e^{-4x}) = -e^{-7x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)}\ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)}\ dx$$

$$y_p(x) = -e^{-3x} \int \frac{e^{-4x}(e^x + 7e^{3x})}{-e^{-7x}}\ dx + e^{-4x} \int \frac{e^{-3x}(e^x + 7e^{3x})}{-e^{-7x}}\ dx$$

$$y_p(x) = -e^{-3x} \int (-e^{4x} - 7e^{6x})\ dx + e^{-4x} \int (-e^{5x} - 7e^{7x})\ dx$$

$$y_p(x) = -e^{-3x}\left(-\frac{1}{4}e^{4x} - \frac{7}{6}e^{6x}\right) + e^{-4x}\left(-\frac{1}{5}e^{5x} - e^{7x}\right)$$

$$y_p(x) = \frac{1}{20}e^{x} - \frac{1}{6}e^{3x}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{-3x} + c_2e^{-4x} + \frac{1}{20}e^{x} - \frac{1}{6}e^{3x}$$

■ 2. Solve the differential equation.

$$y'' - 2y' = \frac{e^{3x}}{e^{2x} + 1}$$

Solution:

The associated homogeneous equation is

$$y'' - 2y' = 0$$

and solving the characteristic equation gives distinct real roots.

$$r^2 - 2r = 0$$

$$r(r - 2) = 0$$

$$r = 0,\ 2$$

With distinct real roots, the complementary solution is

$$y_c(x) = c_1 + c_2e^{2x}$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{1,\ e^{2x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(1, e^{2x}) = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix}$$

$$W(1, e^{2x}) = 2e^{2x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} \ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \ dx$$

$$y_p(x) = -\int \frac{e^{2x}(e^{3x})}{2e^{2x}(e^{2x}+1)} \ dx + e^{2x} \int \frac{e^{3x}}{2e^{2x}(e^{2x}+1)} \ dx$$

$$y_p(x) = -\int \frac{e^{3x}}{2(e^{2x}+1)} \ dx + e^{2x} \int \frac{e^{x}}{2(e^{2x}+1)} \ dx$$

$$y_p(x) = -\left(\frac{1}{2}e^x - \frac{1}{2}\arctan(e^x)\right) + \frac{1}{2}e^{2x}\arctan(e^x)$$

$$y_p(x) = -\frac{1}{2}e^x + \frac{1}{2}\arctan(e^x) + \frac{1}{2}e^{2x}\arctan(e^x)$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 + c_2 e^{2x} - \frac{1}{2}e^x + \frac{1}{2}\arctan(e^x) + \frac{1}{2}e^{2x}\arctan(e^x)$$

■ 3. Solve the differential equation.

$$y'' + 8y' + 16y = 21x^5 e^{-4x}$$

Solution:

The associated homogeneous equation is

$$y'' + 8y' + 16y = 0$$

and solving the characteristic equation gives equal real roots.

$$r^2 + 8r + 16 = 0$$

$$(r + 4)^2 = 0$$

$$r = -4, \ -4$$

With equal real roots, the complementary solution is

$$y_c(x) = c_1e^{-4x} + c_2xe^{-4x}$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{e^{-4x}, \ xe^{-4x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(e^{-4x}, xe^{-4x}) = \begin{vmatrix} e^{-4x} & xe^{-4x} \\ -4e^{-4x} & e^{-4x} - 4xe^{-4x} \end{vmatrix}$$

$$W(e^{-4x}, xe^{-4x}) = e^{-8x} - 4xe^{-8x} + 4xe^{-8x}$$

$$W(e^{-4x}, xe^{-4x}) = e^{-8x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} \ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \ dx$$

$$y_p(x) = -e^{-4x} \int \frac{xe^{-4x}(21x^5e^{-4x})}{e^{-8x}}\ dx + xe^{-4x} \int \frac{e^{-4x}(21x^5e^{-4x})}{e^{-8x}}\ dx$$

$$y_p(x) = -21e^{-4x} \int x^6\ dx + 21xe^{-4x} \int x^5\ dx$$

$$y_p(x) = -3e^{-4x}(x^7) + 7xe^{-4x} \left(\frac{1}{2}x^6 \right)$$

$$y_p(x) = \frac{1}{2}x^7e^{-4x}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{-4x} + c_2xe^{-4x} + \frac{1}{2}x^7e^{-4x}$$

4. Solve the differential equation.

$$y'' + 6y' + 9y = \frac{1}{e^{3x}\sqrt{1-x^2}}$$

Solution:

The associated homogeneous equation is

$$y'' + 6y' + 9y = 0$$

and solving the characteristic equation gives equal real roots.

$$r^2+6r+9=0$$

$$(r+3)^2=0$$

$$r=-3,\ -3$$

With equal real roots, the complementary solution is

$$y_c(x)=c_1e^{-3x}+c_2xe^{-3x}$$

and the fundamental set of solutions is

$$\{y_1,y_2\}=\{e^{-3x},\ xe^{-3x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(e^{-3x},xe^{-3x})=\begin{vmatrix} e^{-3x} & xe^{-3x} \\ -3e^{-3x} & e^{-3x}-3xe^{-3x} \end{vmatrix}$$

$$W(e^{-3x},xe^{-3x})=e^{-6x}-3xe^{-6x}+3xe^{-6x}$$

$$W(e^{-3x},xe^{-3x})=e^{-6x}$$

Then the particular solution is

$$y_p(x)=-y_1\int\frac{y_2g(x)}{W(y_1,y_2)}\ dx+y_2\int\frac{y_1g(x)}{W(y_1,y_2)}\ dx$$

$$y_p(x)=-e^{-3x}\int\frac{xe^{-3x}}{e^{-6x}\cdot e^{3x}\sqrt{1-x^2}}\ dx+xe^{-3x}\int\frac{e^{-3x}}{e^{-6x}\cdot e^{3x}\sqrt{1-x^2}}\ dx$$

$$\int \frac{xe^{-3x}}{e^{-6x} \cdot e^{3x}\sqrt{1-x^2}}\ dx = \int \frac{x}{\sqrt{1-x^2}}\ dx = -\sqrt{1-x^2}$$

$$\int \frac{e^{-3x}}{e^{-6x} \cdot e^{3x}\sqrt{1-x^2}}\ dx = \int \frac{1}{\sqrt{1-x^2}}\ dx = \arcsin x$$

$$y_p(x) = -e^{-3x}(-\sqrt{1-x^2}) + xe^{-3x} \arcsin x$$

$$y_p(x) = e^{-3x}\sqrt{1-x^2} + xe^{-3x} \arcsin x$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{-3x} + c_2xe^{-3x} + e^{-3x}\sqrt{1-x^2} + xe^{-3x} \arcsin x$$

■ 5. Solve the differential equation.

$$y'' + y = \frac{1}{\sin^3 x}$$

Solution:

The associated homogeneous equation is

$$y'' + y = 0$$

and solving the characteristic equation gives complex conjugate roots.

$$r^2 + 1 = 0$$

$$r = \pm i$$

With complex conjugate roots, the complementary solution is

$$y_c(x) = c_1 \cos x + c_2 \sin x$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{\cos x, \ \sin x\}$$

Find the Wronskian for the fundamental solution set.

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$W(\cos x, \sin x) = \cos^2 x + \sin^2 x$$

$$W(\cos x, \sin x) = 1$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} \ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \ dx$$

$$y_p(x) = -\cos x \int \frac{\sin x}{\sin^3 x} \ dx + \sin x \int \frac{\cos x}{\sin^3 x} \ dx$$

$$y_p(x) = -\cos x \int \frac{1}{\sin^2 x} \ dx + \sin x \int \frac{\cos x}{\sin^3 x} \ dx$$

$$\int \frac{1}{\sin^2 x} \ dx = -\cot x$$

$$\int \frac{\cos x}{\sin^3 x}\ dx = -\frac{1}{2\sin^2 x}$$

$$y_p(x) = -\cos x(-\cot x) + \sin x \left(-\frac{1}{2\sin^2 x}\right)$$

$$y_p(x) = \frac{\cos(2x)}{\sin x}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 \cos x + c_2 \sin x + \frac{\cos(2x)}{\sin x}$$

6. Solve the differential equation.

$$y'' + 4y = \cos^2(2x)$$

Solution:

The associated homogeneous equation is

$$y'' + 4y = 0$$

and solving the characteristic equation gives complex conjugate roots.

$$r^2 + 4 = 0$$

$$r = \pm 2i$$

With complex conjugate roots, the complementary solution is

$$y_c(x) = c_1 \cos(2x) + c_2 \sin(2x)$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{\cos(2x), \ \sin(2x)\}$$

Find the Wronskian for the fundamental solution set.

$$W(\cos(2x), \sin(2x)) = \begin{vmatrix} \cos(2x) & \sin(2x) \\ -2\sin(2x) & 2\cos(2x) \end{vmatrix}$$

$$W(\cos(2x), \sin(2x)) = 2\sin^2(2x) + 2\cos^2(2x)$$

$$W(\cos(2x), \sin(2x)) = 2$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} \ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \ dx$$

$$y_p(x) = -\cos(2x) \int \frac{\sin(2x)\cos^2(2x)}{2} \ dx + \sin(2x) \int \frac{\cos(2x)(1 - \sin^2(2x))}{2} \ dx$$

$$\int \frac{\sin(2x)\cos^2(2x)}{2} \ dx = -\frac{\cos^3(2x)}{12}$$

$$\int \frac{\cos(2x)(1 - \sin^2(2x))}{2} \ dx = \frac{\sin(2x)}{4} - \frac{\sin^3(2x)}{12}$$

$$y_p(x) = -\cos(2x)\left(-\frac{\cos^3(2x)}{12}\right) + \sin(2x)\left(\frac{\sin(2x)}{4} - \frac{\sin^3(2x)}{12}\right)$$

$$y_p(x) = \frac{\sin^2(2x)}{4} + \frac{\cos(4x)}{12}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 \cos(2x) + c_2 \sin(2x) + \frac{\sin^2(2x)}{4} + \frac{\cos(4x)}{12}$$

INITIAL VALUE PROBLEMS WITH NONHOMOGENEOUS EQUATIONS

■ 1. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0)=0$ and $y'(0)=2$.

$$y''-2y'-3y=10e^x\sin x$$

Solution:

Solving the characteristic equation for the associated homogeneous equation gives

$$y''-2y'-3y=0$$

$$r^2-2r-3=0$$

$$(r+1)(r-3)=0$$

$$r=-1,\ 3$$

so the complementary solution with distinct real roots is

$$y_c(x)=c_1e^{-x}+c_2e^{3x}$$

Our guess for the particular solution and its derivatives will be

$$y_p(x)=e^x(A\sin x+B\cos x)$$

$$y_p'(x)=e^x((A-B)\sin x+(A+B)\cos x)$$

$$y_p''(x) = e^x(-2B\sin x + 2A\cos x)$$

Plugging these into the original differential equation, we get

$$e^x(-2B\sin x + 2A\cos x) - 2e^x((A-B)\sin x + (A+B)\cos x)$$

$$-3e^x(A\sin x + B\cos x) = 10e^x\sin x$$

$$e^x(-5A\sin x - 5B\cos x) = 10e^x\sin x$$

Using undetermined coefficients, we find $A = -2$ and $B = 0$, so the particular solution is

$$y_p(x) = -2e^x\sin x$$

and the general solution and its derivative are

$$y(x) = c_1e^{-x} + c_2e^{3x} - 2e^x\sin x$$

$$y'(x) = -c_1e^{-x} + 3c_2e^{3x} - 2e^x\sin x - 2e^x\cos x$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 2$.

$$c_1 + c_2 - 0 = 0$$

$$-c_1 + 3c_2 - 0 - 2 = 2$$

We'll simplify these equations to get

$$c_1 = -c_2$$

$$c_2 + 3c_2 = 4$$

With $c_1 = -1$ and $c_2 = 1$, the solution to the initial value problems is

$$y(x) = -e^{-x} + e^{3x} - 2e^{x}\sin x$$

■ 2. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' - 4y = 8x^2 + 4$$

Solution:

Solving the characteristic equation for the associated homogeneous equation gives

$$y'' - 4y = 0$$

$$r^2 - 4 = 0$$

$$r = -2, 2$$

so the complementary solution with distinct real roots is

$$y_c(x) = c_1e^{2x} + c_2e^{-2x}$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = Ax^2 + Bx + C$$

$$y_p'(x) = 2Ax + B$$

$$y_p''(x) = 2A$$

Plugging these into the original differential equation, we get

$$2A - 4(Ax^2 + Bx + C) = 8x^2 + 4$$

$$2A - 4Ax^2 + 4Bx + 4C = 8x^2 + 4$$

Using undetermined coefficients, we find $A = -2$, $B = 0$, and $C = -2$, so the particular solution is

$$y_p(x) = -2x^2 - 2$$

and the general solution and its derivative are

$$y(x) = c_1e^{2x} + c_2e^{-2x} - 2x^2 - 2$$

$$y'(x) = 2c_1e^{2x} - 2c_2e^{-2x} - 4x$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$c_1 + c_2 - 0 - 2 = 0$$

$$2c_1 - 2c_2 - 0 = 0$$

We'll simplify these equations to get

$$c_1 + c_2 = 2$$

$$c_1 = c_2$$

With $c_1 = 1$ and $c_2 = 1$, the solution to the initial value problems is

$$y(x) = e^{2x} + e^{-2x} - 2x^2 - 2$$

■ 3. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' - 10y' + 25y = 28x^6e^{5x}$$

Solution:

Solving the characteristic equation for the associated homogeneous equation gives

$$y'' - 10y' + 25y = 0$$

$$r^2 - 10r + 25 = 0$$

$$(r - 5)^2 = 0$$

$$r = 5, 5$$

so the complementary solution with equal real roots is

$$y_c(x) = c_1e^{5x} + c_2xe^{5x}$$

The fundamental set of solutions is

$$\{y_1, y_2\} = \{e^{5x}, xe^{5x}\}$$

and the Wronskian for the fundamental set is

$$W(e^{5x}, xe^{5x}) = \begin{vmatrix} e^{5x} & xe^{5x} \\ 5e^{5x} & e^{5x} + 5xe^{5x} \end{vmatrix}$$

$$W(e^{5x}, xe^{5x}) = e^{10x} + 5xe^{10x} - 5xe^{10x}$$

$$W(e^{5x}, xe^{5x}) = e^{10x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} \ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \ dx$$

$$y_p(x) = -e^{5x} \int \frac{xe^{5x}(28x^6 e^{5x})}{e^{10x}} \ dx + xe^{5x} \int \frac{e^{5x}(28x^6 e^{5x})}{e^{10x}} \ dx$$

$$y_p(x) = -e^{5x} \int 28x^7 \ dx + xe^{5x} \int 28x^6 \ dx$$

$$y_p(x) = -e^{5x}\left(\frac{7}{2}x^8\right) + xe^{5x}(4x^7)$$

$$y_p(x) = \frac{1}{2}x^8 e^{5x}$$

The general solution and its derivative are

$$y(x) = c_1 e^{5x} + c_2 xe^{5x} + \frac{1}{2}x^8 e^{5x}$$

$$y'(x) = 5c_1 e^{5x} + c_2 e^{5x} + 5c_2 xe^{5x} + 4x^7 e^{5x} + \frac{5}{2}x^8 e^{5x}$$

Substitute the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$c_1 + 0 + 0 = 1$$

$$5c_1 + c_2 + 0 + 0 + 0 = 0$$

We'll simplify these equations to get

$$c_1 = 1$$

$$c_2 = -5c_1 = -5$$

With $c_1 = 1$ and $c_2 = -5$, the solution to the initial value problems is

$$y(x) = e^{5x} - 5xe^{5x} + \frac{1}{2}x^8e^{5x}$$

■ 4. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 1$ and $y'(0) = 0$.

$$4y'' + 4y' + y = 25\sin x + 4$$

Solution:

Solving the characteristic equation for the associated homogeneous equation gives

$$4y'' + 4y' + y = 0$$

$$4r^2 + 4r + 1 = 0$$

$$(2r + 1)^2 = 0$$

$$r = -\frac{1}{2}, \ -\frac{1}{2}$$

so the complementary solution with equal real roots is

$$y_c(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x}$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = A \sin x + B \cos x + C$$

$$y_p'(x) = A \cos x - B \sin x$$

$$y_p''(x) = -A \sin x - B \cos x$$

Plugging these into the original differential equation, we get

$$4(-A \sin x - B \cos x) + 4(A \cos x - B \sin x)$$

$$+A \sin x + B \cos x + C = 25 \sin x + 4$$

$$(-3A - 4B)\sin x + (4A - 3B)\cos x + C = 25 \sin x + 4$$

Using undetermined coefficients, we find $A = -3$, $B = -4$, and $C = 4$, so the particular solution is

$$y_p(x) = -3 \sin x - 4 \cos x + 4$$

and the general solution and its derivative are

$$y(x) = c_1 e^{-\frac{1}{2}x} + c_2 x e^{-\frac{1}{2}x} - 3 \sin x - 4 \cos x + 4$$

$$y'(x) = -\frac{1}{2}c_1 e^{-\frac{1}{2}x} + c_2 e^{-\frac{1}{2}x} - \frac{1}{2}c_2 x e^{-\frac{1}{2}x} - 3 \cos x + 4 \sin x$$

Substitute the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$c_1 + 0 - 4 + 4 = 1$$

$$-\frac{1}{2}c_1 + c_2 + 0 - 3 + 0 = 0$$

We'll simplify these equations to get

$$c_1 = 1$$

$$c_2 = \frac{7}{2}$$

With $c_1 = -1$ and $c_2 = 7/2$, the solution to the initial value problems is

$$y(x) = e^{-\frac{1}{2}x} + \frac{7}{2}xe^{-\frac{1}{2}x} - 3\sin x - 4\cos x + 4$$

■ 5. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 1$.

$$y'' + 4y' + 5y = 17e^{2x} + 40\sin(3x)$$

Solution:

Solving the characteristic equation for the associated homogeneous equation gives

$$y'' + 4y' + 5y = 0$$

$$r^2 + 4r + 5 = 0$$

$$(r + 2)^2 + 1 = 0$$

$$r = -2 \pm i$$

so the complementary solution with complex conjugate roots is

$$y_c(x) = c_1 e^{-2x} \cos x + c_2 e^{-2x} \sin x$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = Ae^{2x} + B\sin(3x) + C\cos(3x)$$

$$y_p'(x) = 2Ae^{2x} + 3B\cos(3x) - 3C\sin(3x)$$

$$y_p''(x) = 4Ae^{2x} - 9B\sin(3x) - 9C\cos(3x)$$

Plugging these into the original differential equation, we get

$$4Ae^{2x} - 9B\sin(3x) - 9C\cos(3x) + 4(2Ae^{2x} + 3B\cos(3x) - 3C\sin(3x))$$

$$+5(e^{2x} + B\sin(3x) + C\cos(3x)) = 17e^{2x} + 40\sin(3x)$$

$$17Ae^{2x} + (-4B - 12C)\sin(3x) + (12B - 4C)\cos(3x) = 17e^{2x} + 40\sin(3x)$$

Using undetermined coefficients, we find $A = 1$, $B = -1$, and $C = -3$, so the particular solution is

$$y_p(x) = e^{2x} - \sin(3x) - 3\cos(3x)$$

and the general solution and its derivative are

$$y(x) = c_1 e^{-2x}\cos(x) + c_2 e^{-2x}\sin(x) + e^{2x} - \sin(3x) - 3\cos(3x)$$

$$y'(x) = -2e^{-2x}(c_1\cos(x) + c_2\sin(x)) + e^{-2x}(-c_1\sin(x) + c_2\cos(x))$$

$$+2e^{2x} - 3\cos(3x) + 9\sin(3x)$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 1$.

$$c_1 + 0 + 1 + 0 - 3 = 0$$

$$-2c_1 + c_2 + 2 - 3 = 1$$

We'll simplify these equations to get

$$c_1 = 2$$

$$c_2 = 6$$

With $c_1 = 2$ and $c_2 = 6$, the solution to the initial value problems is

$$y(x) = 2e^{-2x}\cos(x) + 6e^{-2x}\sin(x) + e^{2x} - \sin(3x) - 3\cos(3x)$$

■ 6. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 2$ and $y'(0) = 2$.

$$y'' + 16y = 40\sin^2(3x)$$

Solution:

Solving the characteristic equation for the associated homogeneous equation gives

$$y'' + 16y = 0$$

$$r^2 + 16 = 0$$

$$r = \pm 4i$$

so the complementary solution with complex conjugate roots is

$$y_c(x) = c_1 \cos(4x) + c_2 \sin(4x)$$

Our guess for the particular solution and its derivatives will be

$$y_p(x) = A\cos(6x) + B\sin(6x) + C$$

$$y_p'(x) = -6A\sin(6x) + 6B\cos(6x)$$

$$y_p''(x) = -36A\cos(6x) - 36B\sin(6x)$$

Plugging these into the original differential equation, we get

$$-36A\cos(6x) - 36B\sin(6x) + 16(A\cos(6x) + B\sin(6x) + C) = 40\sin^2(3x)$$

Using undetermined coefficients, we find $A = 1$, $B = 0$, and $C = 5/4$, so the particular solution is

$$y_p(x) = \cos(6x) + \frac{5}{4}$$

and the general solution and its derivative are

$$y(x) = c_1 \cos(4x) + c_2 \sin(4x) + \cos(6x) + \frac{5}{4}$$

$$y'(x) = -4c_1 \sin(4x) + 4c_2 \cos(4x) - 6\sin(6x)$$

Substitute the initial conditions $y(0) = 2$ and $y'(0) = 2$.

$$c_1 + 0 + 1 + \frac{5}{4} = 2$$

$$0 + 4c_2 - 0 = 2$$

We'll simplify these equations to get

$$c_1 = -\frac{1}{4}$$

$$c_2 = \frac{1}{2}$$

With $c_1 = -1/4$ and $c_2 = 1/2$, the solution to the initial value problems is

$$y(x) = -\frac{1}{4}\cos(4x) + \frac{1}{2}\sin(4x) + \cos(6x) + \frac{5}{4}$$

DIRECTION FIELDS AND SOLUTION CURVES

■ 1. Sketch the direction field of the differential equation.

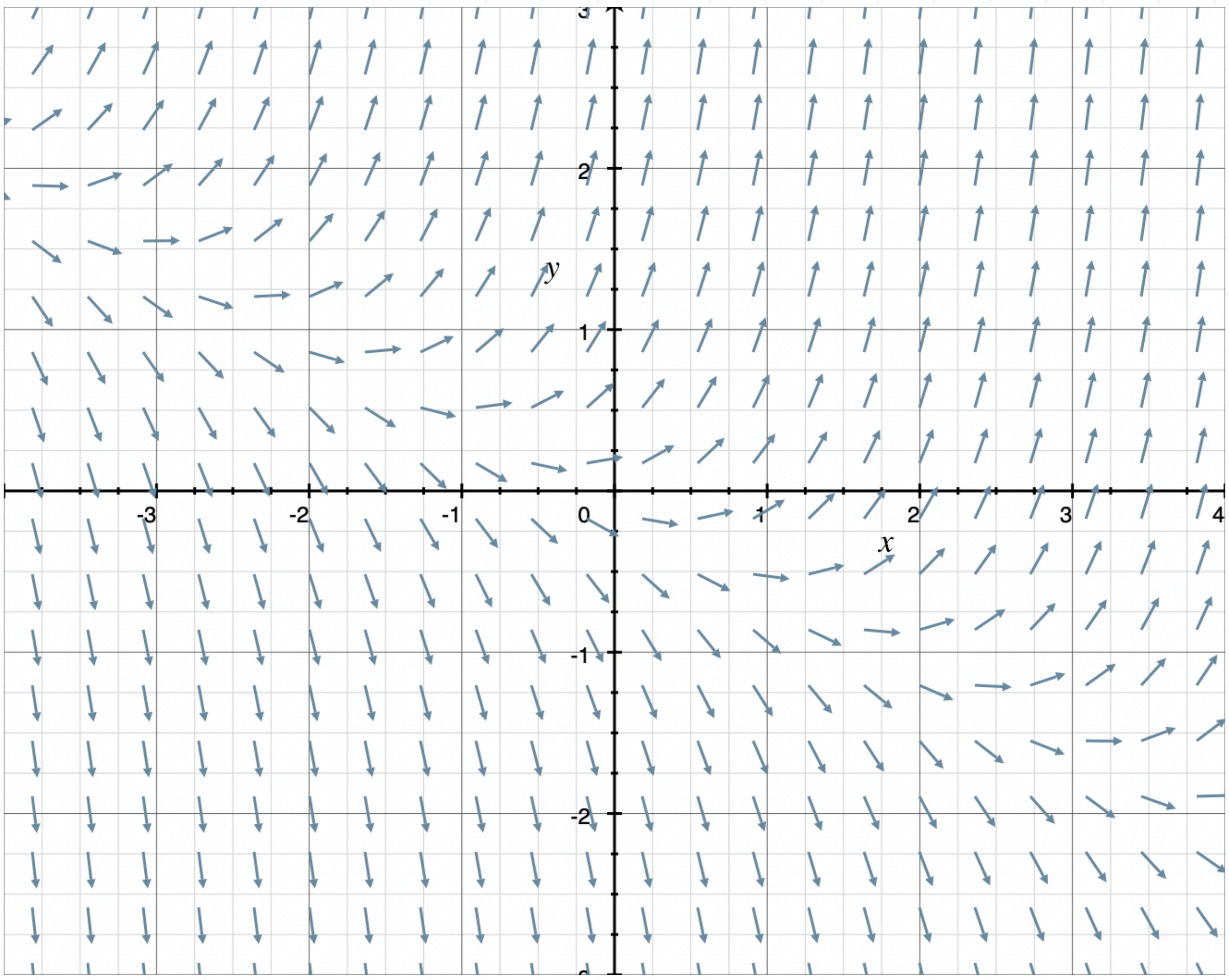
$$y' = 2y + x$$

Solution:

We'll make a table of values of y' for every combination of $x = \{-2, -1, 0, 1, 2\}$ and $y = \{-2, -1, 0, 1, 2\}$.

		x				
		-2	**-1**	**0**	**1**	**2**
y	**-2**	-6	-5	-4	-3	-2
	-1	-4	-3	-2	-1	0
	0	-2	-1	0	1	2
	1	0	1	2	3	4
	2	2	3	4	5	6

Plotting each of these values as a tangent, for instance plotting the slope -6 at the point $(-2, -2)$, gives us the beginning of a direction field. And if we keep adding more and more tangents to get finer detail, the direction field should look something like this:



■ 2. Sketch the direction field of the differential equation, and the solution curve passing through $(1,1)$.

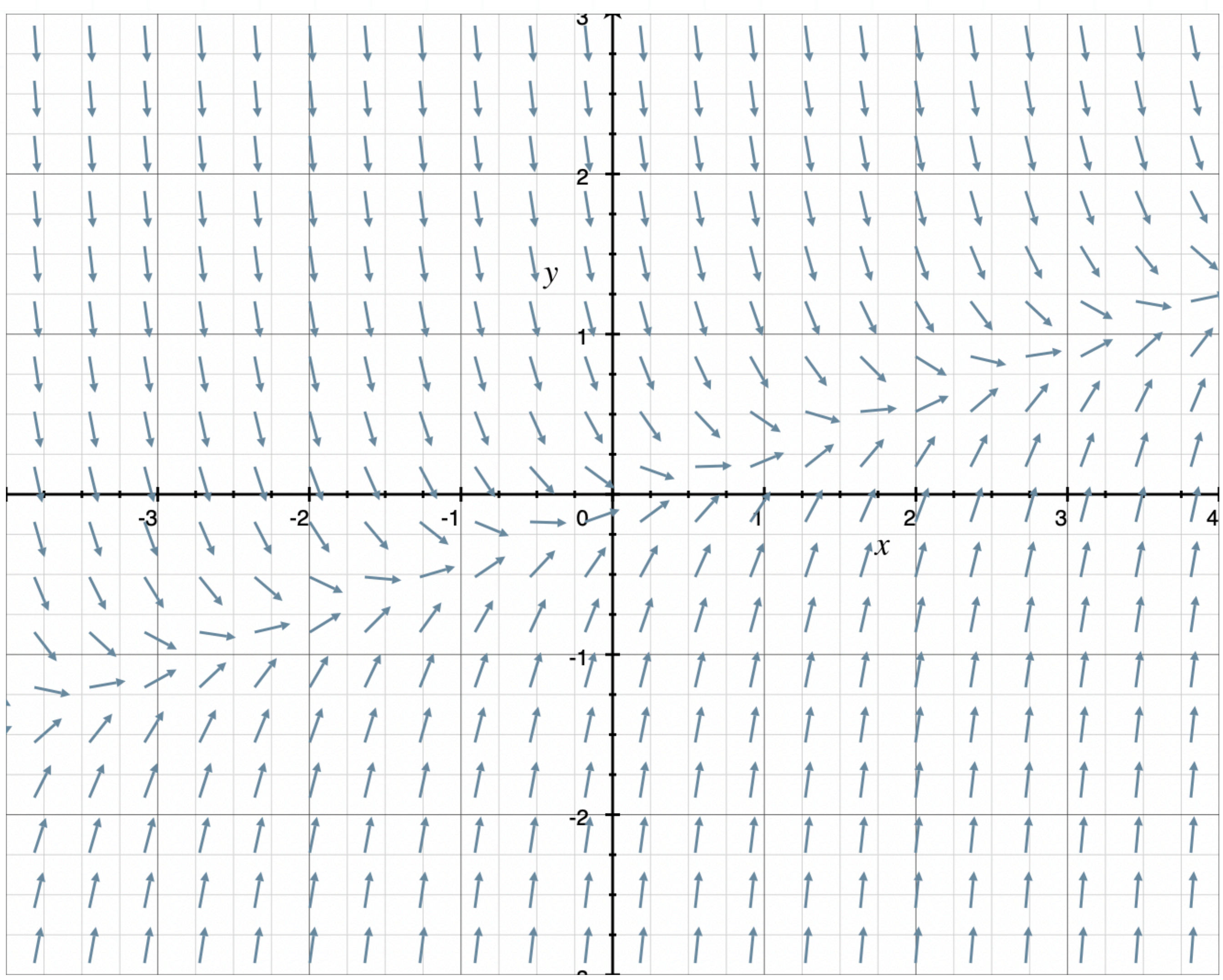
$$y' = -3y + x$$

Solution:

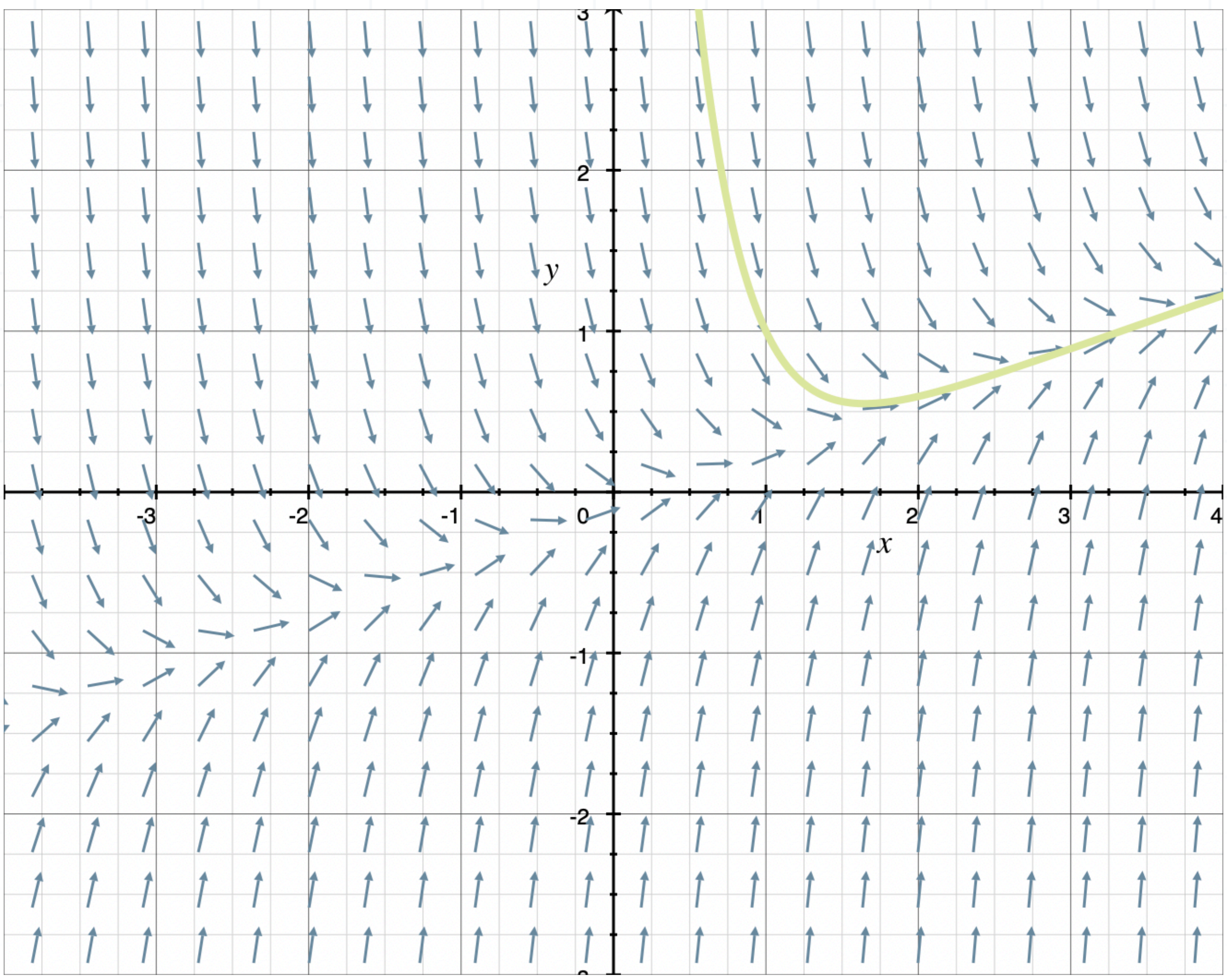
We'll make a table of values of y' for every combination of $x = \{-2, -1, 0, 1, 2\}$ and $y = \{-2, -1, 0, 1, 2\}$.

		x				
		-2	-1	0	1	2
y	-2	4	5	6	7	8
	-1	1	2	3	4	5
	0	-2	-1	0	1	2
	1	-5	-4	-3	-2	-1
	2	-8	-7	-6	-5	-4

Plotting each of these values as a tangent, for instance plotting the slope 4 at the point $(-2,-2)$, gives us the beginning of a direction field. And if we keep adding more and more tangents to get finer detail, the direction field should look something like this:



Following the tangents in the direction field around $(1,1)$, we can approximate the solution curve through that point.



■ 3. Sketch the direction field of the differential equation, and the solution curve passing through $(0,0)$.

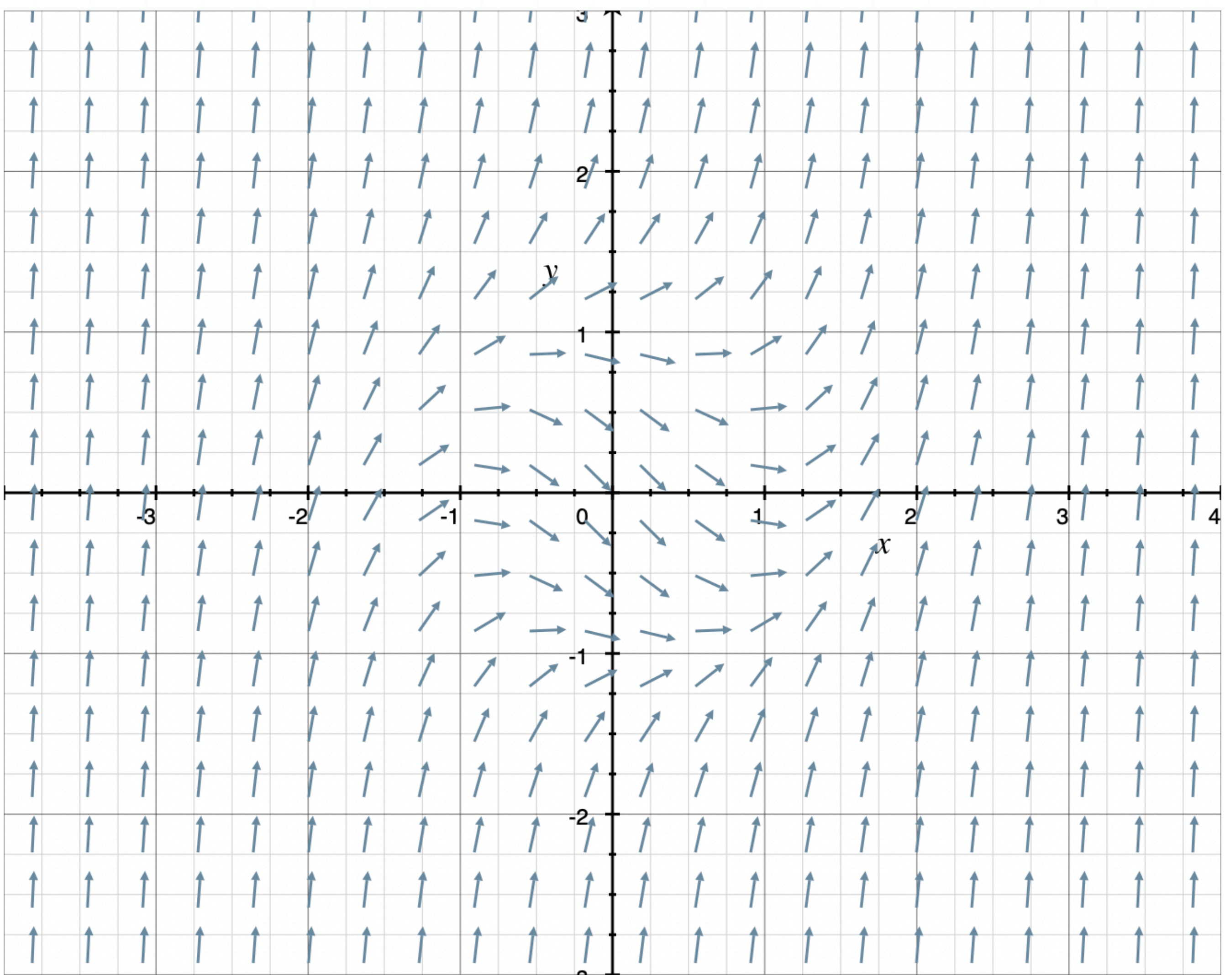
$$y' = y^2 + x^2 - 1$$

Solution:

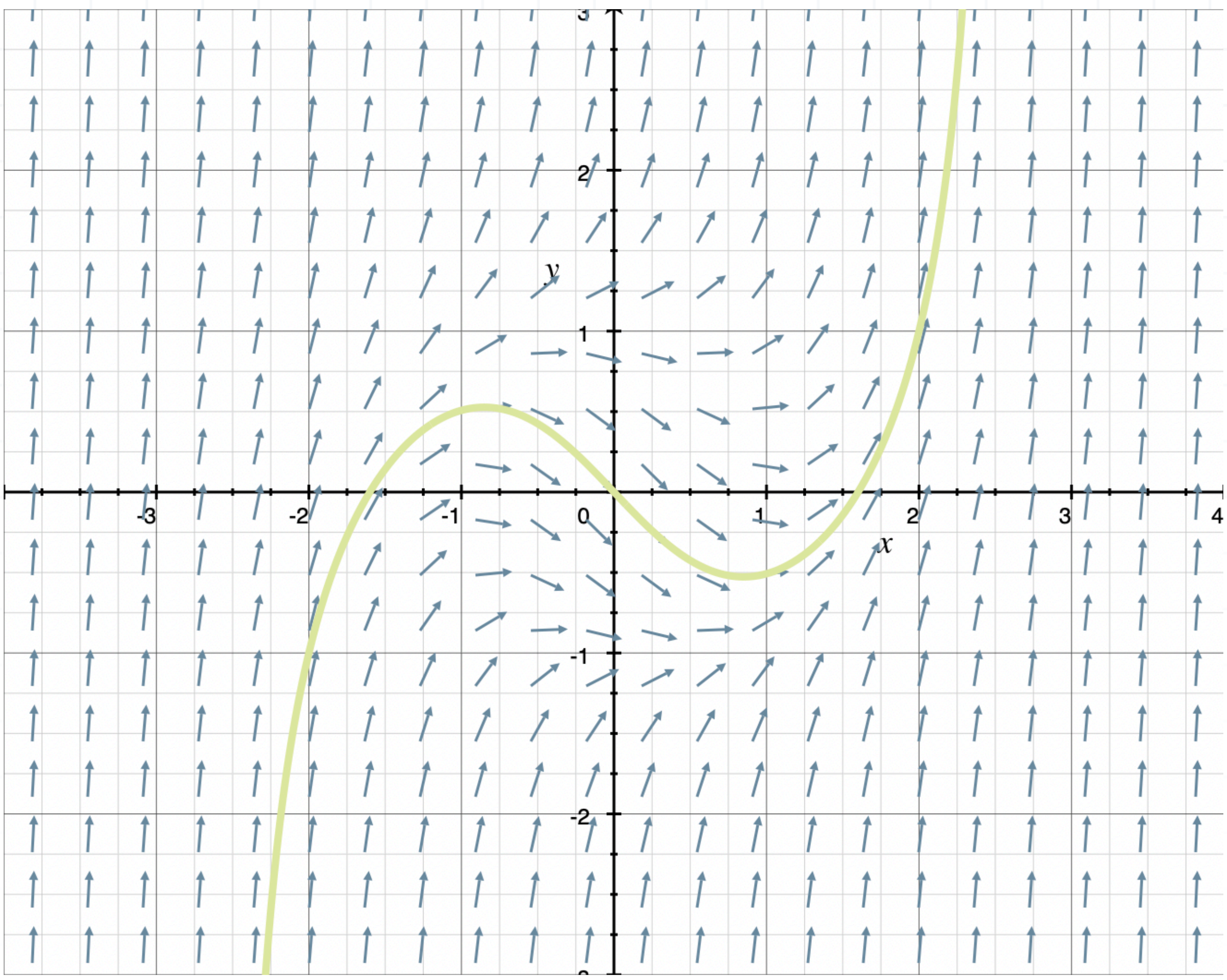
We'll make a table of values of y' for every combination of $x = \{-2, -1, 0, 1, 2\}$ and $y = \{-2, -1, 0, 1, 2\}$.

		x				
		-2	-1	0	1	2
y	-2	7	4	3	4	7
	-1	4	1	0	1	4
	0	3	0	-1	0	3
	1	4	1	0	1	4
	2	7	4	3	4	7

Plotting each of these values as a tangent, for instance plotting the slope 7 at the point $(-2,-2)$, gives us the beginning of a direction field. And if we keep adding more and more tangents to get finer detail, the direction field should look something like this:



Following the tangents in the direction field around $(0,0)$, we can approximate the solution curve through that point.



■ 4. Sketch the direction field for the differential equation, and the solution curve passing through $(0,1)$.

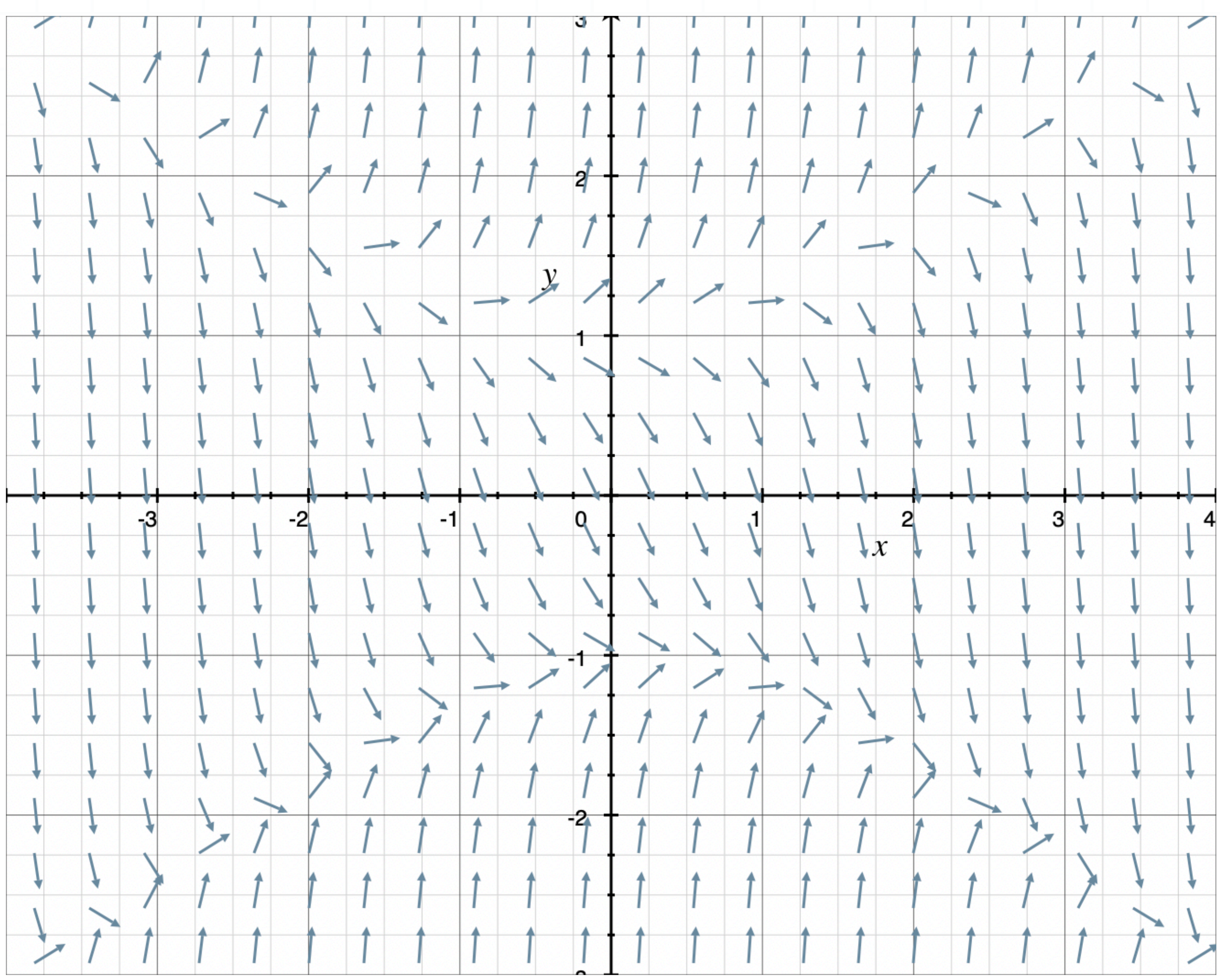
$$y' = 2y^2 - x^2 - 2$$

Solution:

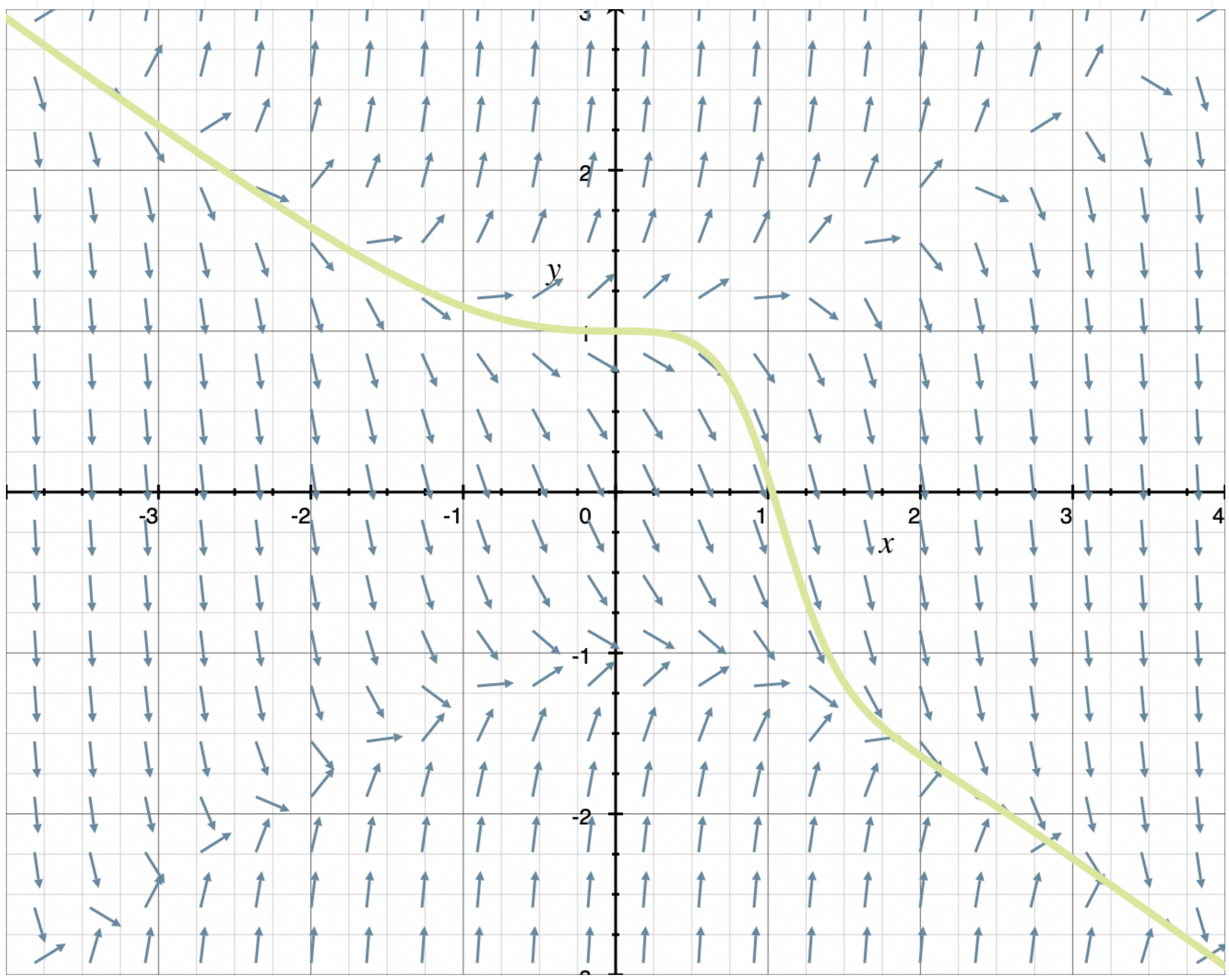
We'll make a table of values of y' for every combination of $x = \{-2, -1, 0, 1, 2\}$ and $y = \{-2, -1, 0, 1, 2\}$.

		x				
		-2	**-1**	**0**	**1**	**2**
	-2	2	5	6	5	2
	-1	-4	-1	0	-1	-4
y	**0**	-6	-3	-2	-3	-6
	1	-4	-1	0	-1	-4
	2	2	5	6	5	2

Plotting each of these values as a tangent, for instance plotting the slope 2 at the point $(-2,-2)$, gives us the beginning of a direction field. And if we keep adding more and more tangents to get finer detail, the direction field should look something like this:



Following the tangents in the direction field around $(0,1)$, we can approximate the solution curve through that point.



■ 5. Sketch the direction field of the differential equation.

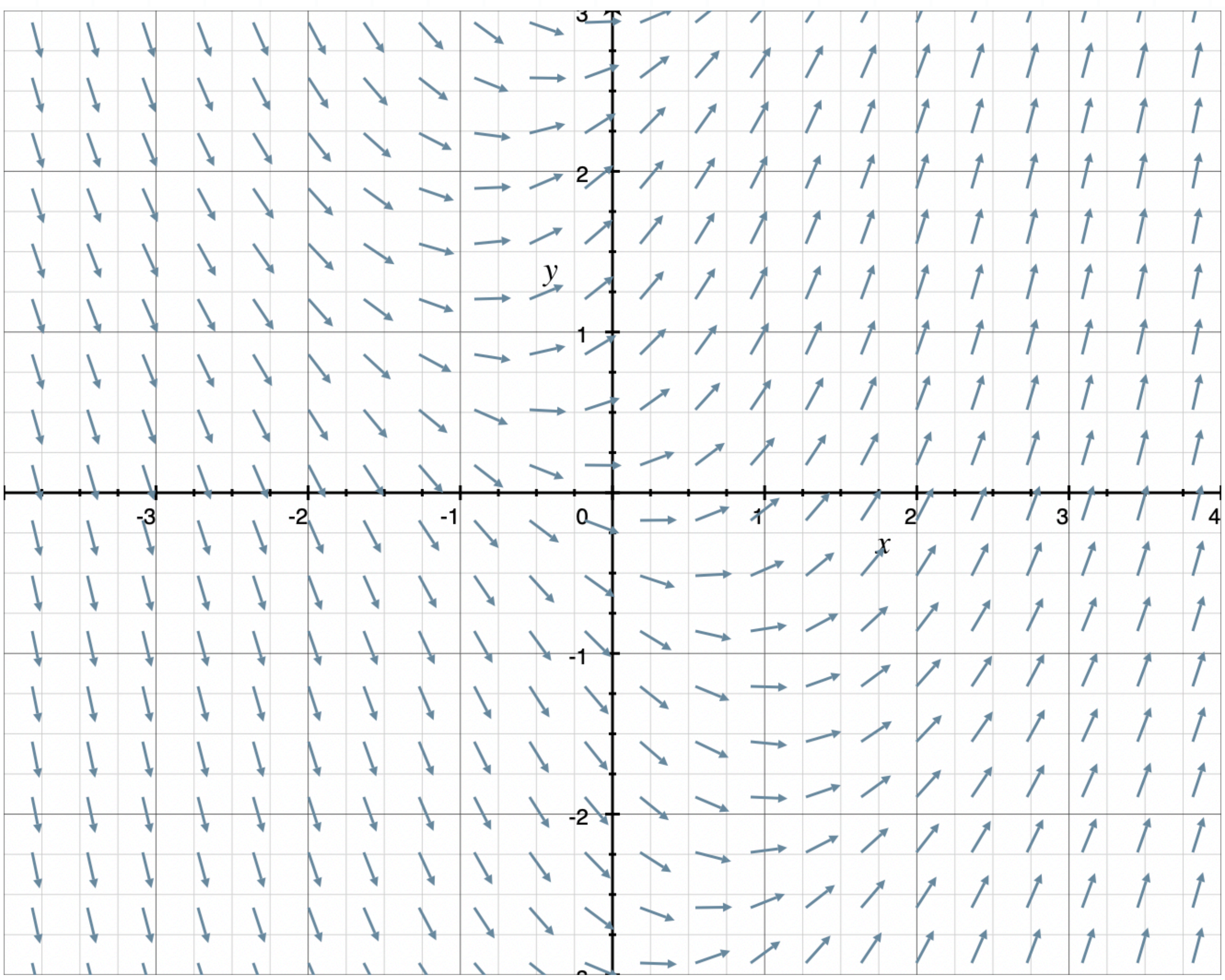
$$y' = \sin y + x$$

Solution:

We'll make a table of values of y' for every combination of $x = \{-2, -1, 0, 1, 2\}$ and $y = \{-\pi, -\pi/2, 0, \pi/2, \pi\}$.

		x				
		-2	-1	0	1	2
y	-2	-2	-1	0	1	2
	-1	-3	-2	-1	0	1
	0	-2	-1	0	1	2
	1	-1	0	1	2	3
	2	-2	-1	0	1	2

Plotting each of these values as a tangent, for instance plotting the slope -2 at the point $(-2,-2)$, gives us the beginning of a direction field. And if we keep adding more and more tangents to get finer detail, the direction field should look something like this:



■ 6. Sketch the direction field of the differential equation.

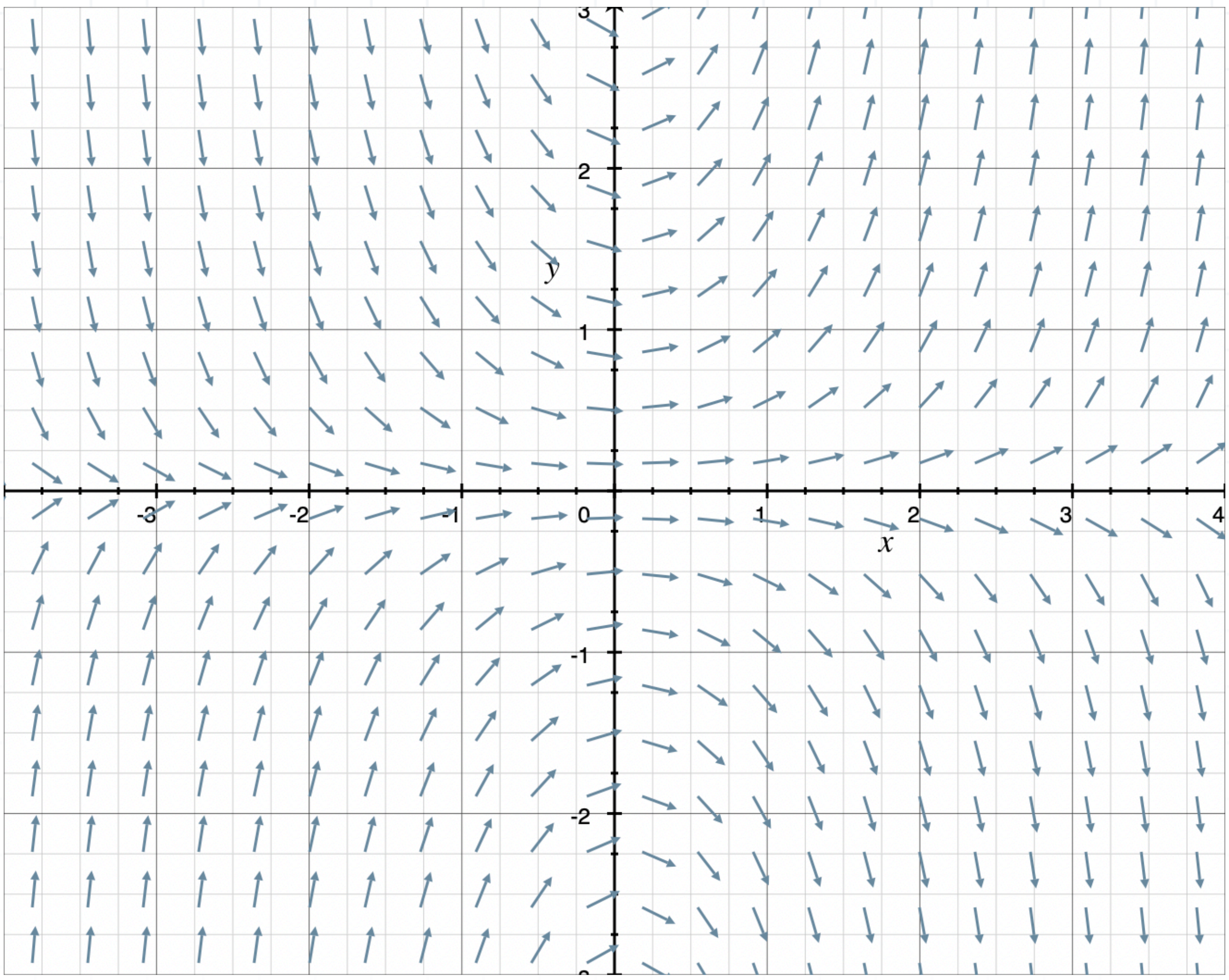
$$y' = xy$$

Solution:

We'll make a table of values of y' for every combination of $x = \{-2, -1, 0, 1, 2\}$ and $y = \{-2, -1, 0, 1, 2\}$.

		x				
		-2	**-1**	**0**	**1**	**2**
y	**-2**	4	2	0	-2	-4
	-1	2	1	0	-1	-2
	0	0	0	0	0	0
	1	-2	-1	0	1	2
	2	-4	-2	0	2	4

Plotting each of these values as a tangent, for instance plotting the slope 4 at the point $(-2, -2)$, gives us the beginning of a direction field. And if we keep adding more and more tangents to get finer detail, the direction field should look something like this:



y
x
3
2
1
0
-1
-2
-3
-3
-2
-1
1
2
3
4

INTERVALS OF VALIDITY

■ 1. Find the interval of validity for the solution to the differential equation, given $y(3) = -4$.

$$(x^2 - 25)y' - 7y - \ln(10 - x) = 0$$

Solution:

Put the equation into standard form for a linear equation.

$$(x^2 - 25)y' - 7y - \ln(10 - x) = 0$$

$$\frac{dy}{dx} - \frac{7y}{x^2 - 25} - \frac{\ln(10 - x)}{x^2 - 25} = 0$$

$$\frac{dy}{dx} - \frac{7}{x^2 - 25}y = \frac{\ln(10 - x)}{x^2 - 25}$$

From this new form, we can identify

$$P(x) = -\frac{7}{x^2 - 25}$$

$$Q(x) = \frac{\ln(10 - x)}{x^2 - 25}$$

both of which are undefined at

$$x^2 - 25 = 0$$

$$x^2 = 25$$

$$x = \pm 5$$

The functions $P(x)$ and $Q(x)$ are undefined for

$$10 - x < 0$$

$$x > 10$$

So we need to keep $x < 10$, which means we'll only consider the interval $(-\infty,10)$. The values $x = \pm 5$ divide this interval into three possible intervals of validity.

$$-\infty < x < -5 \qquad -5 < x < 5 \qquad 5 < x < 10$$

Because the initial condition is $y(3) = -4$, and the value $x_0 = 3$ is contained within $(-5,5)$, we can say that $(-5,5)$ is the interval of validity.

■ 2. Find the interval of validity for the solution to the differential equation, given $y(0) = 1$.

$$(x+2)y' + xye^{3x} - 5 = 0$$

Solution:

Put the equation into standard form for a linear equation.

$$(x+2)y' + xye^{3x} - 5 = 0$$

$$\frac{dy}{dx} + \frac{xye^{3x}}{x+2} - \frac{5}{x+2} = 0$$

$$\frac{dy}{dx} + \frac{xe^{3x}}{x+2}y = \frac{5}{x+2}$$

From this new form, we can identify

$$P(x) = \frac{xe^{3x}}{x+2}$$

$$Q(x) = \frac{5}{x+2}$$

both of which are undefined at

$$x + 2 = 0$$

$$x = -2$$

This value separates $(-\infty, \infty)$ into two possible intervals of validity.

$$-\infty < x < -2 \qquad\qquad -2 < x < \infty$$

Because the initial condition is $y(0) = 1$, and the value $x_0 = 0$ is contained within $(-2,\infty)$, we can say that $(-2,\infty)$ is the interval of validity.

■ 3. Find the interval of validity for the solution to the differential equation, given $y(0) = -1$, and solve the initial value problem to verify that the interval of validity is correct.

$$(x^2 - 1)y' + 2xy = x + 1$$

Solution:

Put the equation into standard form for a linear equation.

$$(x^2-1)y' + 2xy = x + 1$$

$$\frac{dy}{dx} + \frac{2xy}{x^2-1} = \frac{x+1}{x^2-1}$$

$$\frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{x-1}$$

From this new form, we can identify

$$P(x) = \frac{2x}{x^2-1}$$

$$Q(x) = \frac{1}{x-1}$$

which together are undefined at

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

These two values separate $(-\infty, \infty)$ into three possible intervals of validity.

$$-\infty < x < -1 \qquad -1 < x < 1 \qquad 1 < x < \infty$$

Because the initial condition is $y(0) = -1$, and the value $x_0 = 0$ is contained within $(-1,1)$, we can say that $(-1,1)$ is the interval of validity.

If we also wanted to find the particular solution by solving the initial value problem, we'd find the integrating factor,

$$\rho(x) = e^{\int P(x)\ dx}$$

$$\rho(x) = e^{\int \frac{2x}{x^2-1}\ dx}$$

$$\rho(x) = e^{\ln(x^2-1)}$$

$$\rho(x) = x^2 - 1$$

multiply through the linear equation,

$$\frac{dy}{dx}(x^2-1) + 2xy = \frac{1}{x-1}(x^2-1)$$

$$\frac{d}{dx}((x^2-1)y) = x + 1$$

integrate both sides,

$$\int \frac{d}{dx}(y(x^2-1))\ dx = \int x + 1\ dx$$

$$(x^2-1)y = \frac{x^2}{2} + x + C$$

and then solve for y to find the general solution.

$$y = \frac{x^2 + 2x + 2C}{2(x^2-1)}$$

Substituting the initial condition $y(0) = -1$ lets us solve for C,

$$-1 = \frac{0^2 + 2(0) + 2C}{2(0^2 - 1)}$$

$$-1 = \frac{2C}{-2}$$

$$2 = 2C$$

$$C = 1$$

which we can plug back into the general solution to find the particular solution.

$$y = \frac{x^2 + 2x + 2}{2(x^2 - 1)}$$

The solution is undefined where the denominator is 0.

$$x^2 - 1 = 0$$

$$x^2 = 1$$

$$x = \pm 1$$

These two values separate $(-\infty, \infty)$ into three possible intervals of validity.

$$-\infty < x < -1 \qquad -1 < x < 1 \qquad 1 < x < \infty$$

Because the initial condition is $y(0) = -1$, and the value $x_0 = 0$ is contained within $(-1,1)$, we can verify that $(-1,1)$ is the interval of validity.

■ 4. Find the interval of validity for the solution to the differential equation, given $y(0) = y_0$, for some positive y_0.

$$y' = 3x^2y^{\frac{4}{3}}$$

Solution:

This is a separable differential equation, which means it's non-linear, and therefore we'll need to find the solution to the equation before we can find the interval of validity. We'll first find the general solution by separating variables,

$$\frac{dy}{dx} = 3x^2y^{\frac{4}{3}}$$

$$dy = 3x^2y^{\frac{4}{3}}\ dx$$

$$\frac{1}{y^{\frac{4}{3}}}\ dy = 3x^2\ dx$$

integrating both sides, and then solving for y.

$$\int \frac{1}{y^{\frac{4}{3}}}\ dy = \int 3x^2\ dx$$

$$-\frac{3}{y^{\frac{1}{3}}} = x^3 + C$$

$$-3 = (x^3 + C)y^{\frac{1}{3}}$$

$$y = \left(-\frac{3}{x^3 + C}\right)^3$$

We'll substitute the initial condition $y(0) = y_0$ to find a value for C.

$$y_0 = \left(-\frac{3}{0^3 + C}\right)^3$$

$$C^3 = -\frac{27}{y_0}$$

$$C = -\frac{3}{\sqrt[3]{y_0}}$$

Now we can plug this value back into the general solution, which will give us the equation's actual solution.

$$y = \left(-\frac{3}{x^3 - \frac{3}{\sqrt[3]{y_0}}}\right)^3$$

$$y = \left(-\frac{3\sqrt[3]{y_0}}{x^3\sqrt[3]{y_0} - 3}\right)^3$$

$$y = -\frac{27y_0}{(x^3\sqrt[3]{y_0} - 3)^3}$$

This particular solution is undefined where the denominator is 0.

$$(x^3\sqrt[3]{y_0} - 3)^3 = 0$$

$$x^3\sqrt[3]{y_0} - 3 = 0$$

$$x = \frac{\sqrt[3]{3}}{\sqrt[9]{y_0}}$$

Because the solution is undefined at this value, it divides $(-\infty, \infty)$ into two potential intervals of validity.

$$-\infty < x < \frac{\sqrt[3]{3}}{\sqrt[9]{y_0}} \qquad \frac{\sqrt[3]{3}}{\sqrt[9]{y_0}} < x < \infty$$

Since we're using the initial condition $y(0) = y_0$, and $y_0 > 0$, the interval of validity is

$$\left(-\infty, \frac{\sqrt[3]{3}}{\sqrt[9]{y_0}}\right)$$

■ 5. Find the interval of validity for the solution to the differential equation.

$$\frac{dy}{dx} = y^2e^x$$

$$y(0) = \frac{1}{e^2 - 1}$$

Solution:

This is a separable differential equation, which means it's non-linear, and therefore we'll need to find the solution to the equation before we can find the interval of validity. We'll first find the general solution by separating variables,

$$\frac{dy}{dx} = y^2 e^x$$

$$dy = y^2 e^x \ dx$$

$$\frac{1}{y^2} \ dy = e^x \ dx$$

integrating both sides, and then solving for y.

$$\int \frac{1}{y^2} \ dy = \int e^x \ dx$$

$$-\frac{1}{y} = e^x + C$$

$$y = -\frac{1}{e^x + C}$$

We'll substitute the initial condition $y(0) = 1/(e^2 - 1)$ to find a value for C.

$$\frac{1}{e^2 - 1} = -\frac{1}{e^0 + C}$$

$$e^2 - 1 = -(e^0 + C)$$

$$e^2 - 1 = -1 - C$$

$$C = -e^2$$

Now we can plug this value back into the general solution, which will give us the equation's actual solution.

$$y = -\frac{1}{e^x - e^2}$$

The general solution is undefined when

$$e^x - e^2 = 0$$

$$e^x = e^2$$

$$x = 2$$

Because the solution is undefined at this value, it divides $(-\infty, \infty)$ into two potential intervals of validity.

$$-\infty < x < 2 \qquad\qquad 2 < x < \infty$$

Since we're using the initial condition $y(0) = 1/(e^2 - 1)$, and $x_0 = 0$ is contained within $(-\infty,2)$, we can say that $(-\infty,2)$ is the interval of validity.

■ 6. Find the interval of validity for the solution to the differential equation, given $y(1) = y_0$.

$$xy' = y^2$$

Solution:

This is a separable differential equation, which means it's non-linear, and therefore we'll need to find the solution to the equation before we can find the interval of validity. We'll first find the general solution by separating variables,

$$\frac{dy}{dx} = \frac{y^2}{x}$$

$$dy = \frac{y^2}{x}\ dx$$

$$\frac{1}{y^2}\ dy = \frac{1}{x}\ dx$$

integrating both sides, and then solving for y.

$$\int \frac{1}{y^2}\ dy = \int \frac{1}{x}\ dx$$

$$-\frac{1}{y} = \ln|x| + C$$

$$y = -\frac{1}{\ln|x| + C}$$

We'll substitute the initial condition $y(1) = y_0$ to find a value for C.

$$y_0 = -\frac{1}{\ln|1| + C}$$

$$y_0 = -\frac{1}{C}$$

$$C = -\frac{1}{y_0}$$

Now we can plug this value back into the general solution, which will give us the equation's actual solution.

$$y = -\frac{1}{\ln|x| - \frac{1}{y_0}}$$

$$y = \frac{-y_0}{\ln|x|y_0 - 1}$$

This particular solution is undefined where the denominator is 0.

$$\ln|x|y_0 - 1 = 0$$

$$\ln|x|y_0 = 1$$

$$\ln|x| = \frac{1}{y_0}$$

$$|x| = e^{\frac{1}{y_0}}$$

$$x = \pm e^{\frac{1}{y_0}}$$

Because we have $\ln|x|$ in the actual solution, we also have to have $x \neq 0$. Which means we've divided $(-\infty, \infty)$ into four potential intervals of validity.

$$-\infty < x < -e^{\frac{1}{y_0}} \qquad -e^{\frac{1}{y_0}} < x < 0 \qquad 0 < x < e^{\frac{1}{y_0}} \qquad e^{\frac{1}{y_0}} < x < \infty$$

Since we're using the initial condition $y(1) = y_0$, the interval of validity must contain $x = 1$. The third interval will only contain $x = 1$ when

$$e^{\frac{1}{y_0}} > 1$$

$$\ln(e^{\frac{1}{y_0}}) > \ln 1$$

$$\frac{1}{y_0} > 0$$

$$y_0 > 0$$

The fourth interval will only contain $x = 1$ when

$$e^{\frac{1}{y_0}} < 1$$

$$\ln(e^{\frac{1}{y_0}}) < \ln 1$$

$$\frac{1}{y_0} < 0$$

$$y_0 < 0$$

So we've found constraints for $y_0 < 0$ and $y_0 > 0$, but what about $y_0 = 0$? If we substitute $y_0 = 0$ into the actual solution, we get

$$y = \frac{-y_0}{\ln|x|y_0 - 1}$$

$$0 = \frac{-0}{\ln|x|(0) - 1}$$

This is true for all x, which means the interval of validity is $(-\infty, \infty)$ when $y_0 = 0$. So we can summarize our findings as

For $y_0 < 0$, the interval of validity is $e^{\frac{1}{y_0}} < x < \infty$

For $y_0 = 0$, the interval of validity is $(-\infty, \infty)$

For $y_0 > 0$, the interval of validity is $0 < x < e^{\frac{1}{y_0}}$

EULER'S METHOD

■ 1. If $y(0)=0$, use Euler's method to approximate $y(2)$ with $n=4$ steps.

$$y'=1+y$$

Solution:

We're starting at $y(0)$ and we need to get to $y(2)$ in $n=4$ steps, so

$$\Delta t=\frac{2-0}{4}$$

$$\Delta t=\frac{1}{2}$$

Starting with $(t_0,y_0)=(0,0)$, and with $\Delta t=1/2$, we build our table and use it to approximate $y(2)$.

$$t_0=0 \qquad y_0=0 \qquad y_0=0$$

$$t_1=\frac{1}{2} \qquad y_1=0+(1+0)\left(\frac{1}{2}\right) \qquad y_1=\frac{1}{2}$$

$$t_2=1 \qquad y_2=\frac{1}{2}+\left(1+\frac{1}{2}\right)\left(\frac{1}{2}\right) \qquad y_2=\frac{5}{4}$$

$$t_3=\frac{3}{2} \qquad y_3=\frac{5}{4}+\left(1+\frac{5}{4}\right)\left(\frac{1}{2}\right) \qquad y_3=\frac{19}{8}$$

$t_4 = 2 \qquad y_4 = \frac{19}{8} + \left(1 + \frac{19}{8}\right)\left(\frac{1}{2}\right) \qquad y_4 = \frac{65}{16}$

$y_4 \approx 4.06$

So, using $n = 4$ steps, Euler's method approximates $y(2) \approx 4.06$.

■ 2. If $y(2) = 0$, use Euler's method to approximate $y(4)$ with $n = 5$ steps.

$$y' = 3t + y$$

Solution:

We're starting at $y(2)$ and we need to get to $y(4)$ in $n = 5$ steps, so

$$\Delta t = \frac{4-2}{5}$$

$$\Delta t = \frac{2}{5}$$

Starting with $(t_0, y_0) = (2,0)$, and with $\Delta t = 2/5$, we can build our table and use it to approximate $y(4)$.

$t_0 = 2 \qquad y_0 = 0 \qquad y_0 = 0$

$t_1 = \frac{12}{5} \qquad y_1 = 0 + (3(2) + 0)\left(\frac{2}{5}\right) \qquad y_1 = \frac{12}{5}$

$$t_2 = \frac{14}{5} \qquad y_2 = \frac{12}{5} + \left(3\left(\frac{12}{5}\right) + \frac{12}{5}\right)\left(\frac{2}{5}\right) \qquad y_2 = \frac{156}{25}$$

$$t_3 = \frac{16}{5} \qquad y_3 = \frac{156}{25} + \left(3\left(\frac{14}{5}\right) + \frac{156}{25}\right)\left(\frac{2}{5}\right) \qquad y_3 = \frac{1{,}512}{125}$$

$$t_4 = \frac{18}{5} \qquad y_4 = \frac{1{,}512}{125} + \left(3\left(\frac{16}{5}\right) + \frac{1{,}512}{125}\right)\left(\frac{2}{5}\right) \qquad y_4 = \frac{12{,}984}{625}$$

$$t_5 = 4 \qquad y_5 = \frac{12{,}984}{625} + \left(3\left(\frac{18}{5}\right) + \frac{12{,}984}{625}\right)\left(\frac{2}{5}\right) \qquad y_5 = \frac{104{,}388}{3{,}125}$$

$$y_5 \approx 33.4$$

So, using $n = 5$ steps, Euler's method approximates $y(4) \approx 33.4$.

■ 3. If $y(0) = 1$, use Euler's method to approximate $y(1)$ with $n = 4$ steps.

$$y' = 1 - 2y$$

Solution:

We're starting at $y(0)$ and we need to get to $y(1)$ in $n = 4$ steps, so

$$\Delta t = \frac{1 - 0}{4}$$

$$\Delta t = \frac{1}{4}$$

Starting with $(t_0, y_0) = (0,1)$, and with $\Delta t = 1/4$, we build our table and use it to approximate $y(1)$.

$$t_0 = 0 \qquad y_0 = 1 \qquad y_0 = 1$$

$$t_1 = \frac{1}{4} \qquad y_1 = 1 + (1 - 2(1))\left(\frac{1}{4}\right) \qquad y_1 = \frac{3}{4}$$

$$t_2 = \frac{1}{2} \qquad y_2 = \frac{3}{4} + \left(1 - 2\left(\frac{3}{4}\right)\right)\left(\frac{1}{4}\right) \qquad y_2 = \frac{5}{8}$$

$$t_3 = \frac{3}{4} \qquad y_3 = \frac{5}{8} + \left(1 - 2\left(\frac{5}{8}\right)\right)\left(\frac{1}{4}\right) \qquad y_3 = \frac{9}{16}$$

$$t_4 = 1 \qquad y_4 = \frac{9}{16} + \left(1 - 2\left(\frac{9}{16}\right)\right)\left(\frac{1}{4}\right) \qquad y_4 = \frac{17}{32}$$

$$y_4 \approx 0.53$$

So, using $n = 4$ steps, Euler's method approximates $y(1) \approx 0.53$.

■ 4. Use Euler's method and five steps to approximate $y(3)$, given $y(2) = 1$.

$$y' = t^2 + y$$

Solution:

We're starting at $y(2)$ and we need to get to $y(3)$ in $n = 5$ steps, so

$$\Delta t = \frac{3-2}{5}$$

$$\Delta t = \frac{1}{5}$$

Starting with $(t_0, y_0) = (2,1)$, and with $\Delta t = 1/5$, we can build our table and use it to approximate $y(3)$.

$$t_0 = 2 \qquad y_0 = 1 \qquad y_0 = 1$$

$$t_1 = \frac{11}{5} \qquad y_1 = 1 + (2^2 + 1)\left(\frac{1}{5}\right) \qquad y_1 = 2$$

$$t_2 = \frac{12}{5} \qquad y_2 = 2 + \left(\left(\frac{11}{5}\right)^2 + 2\right)\left(\frac{1}{5}\right) \qquad y_2 = \frac{421}{125}$$

$$t_3 = \frac{13}{5} \qquad y_3 = \frac{421}{125} + \left(\left(\frac{12}{5}\right)^2 + \frac{421}{125}\right)\left(\frac{1}{5}\right) \qquad y_3 = \frac{3{,}246}{625}$$

$$t_4 = \frac{14}{5} \qquad y_4 = \frac{3{,}246}{625} + \left(\left(\frac{13}{5}\right)^2 + \frac{3{,}246}{625}\right)\left(\frac{1}{5}\right) \qquad y_4 = \frac{23{,}701}{3{,}125}$$

$$t_5 = 3 \qquad y_5 = \frac{23{,}701}{3{,}125} + \left(\left(\frac{14}{5}\right)^2 + \frac{23{,}701}{3{,}125}\right)\left(\frac{1}{5}\right) \qquad y_5 = \frac{166{,}706}{15{,}625}$$

$$y_5 \approx 10.67$$

So, using $n = 5$ steps, Euler's method approximates $y(3) \approx 10.67$.

■ 5. If $y(0) = 0$, use Euler's method to approximate $y(1)$ with $\Delta t = 0.2$.

$$y' + 3y = 1 - e^{-5t}$$

Solution:

Starting with $(t_0, y_0) = (0,0)$, and with $\Delta t = 0.2$, we can build our table and use it to approximate $y(1)$.

$t_0 = 0$	$y_0 = 0$	$y_0 = 0$
$t_1 = 0.2$	$y_1 = 0 + (-3(0) + 1 - e^{-5(0)})(0.2)$	$y_1 = 0$
$t_2 = 0.4$	$y_2 = 0 + (-3(0) + 1 - e^{-5(0.2)})(0.2)$	$y_2 \approx 0.1264$
$t_3 = 0.6$	$y_3 = 0.1264 + (-3(0.1264) + 1 - e^{-5(0.4)})(0.2)$	$y_3 \approx 0.2235$
$t_4 = 0.8$	$y_4 = 0.2235 + (-3(0.2235) + 1 - e^{-5(0.6)})(0.2)$	$y_4 \approx 0.2794$
$t_5 = 1$	$y_5 = 0.2794 + (-3(0.2794) + 1 - e^{-5(0.8)})(0.2)$	$y_5 \approx 0.3081$

So, using $n = 5$ steps, Euler's method approximates $y(1) \approx 0.3081$.

■ 6. If $y(0) = 3$, use Euler's method to approximate y_3 with $\Delta t = 0.1$, find the exact value and percentage error.

$$y' = 8e^{2t} - 3y$$

Solution:

Starting with $(t_0, y_0) = (0,3)$, and with $\Delta t = 0.1$, we can build our table and use it to approximate y_3.

$t_0 = 0$	$y_0 = 3$	$y_0 = 3$
$t_1 = 0.1$	$y_1 = 3 + (8e^{2(0)} - 3(3))(0.1)$	$y_1 = 2.9$
$t_2 = 0.2$	$y_2 = 2.9 + (8e^{2(0.1)} - 3(2.9))(0.1)$	$y_2 \approx 3.0071$
$t_3 = 0.3$	$y_2 = 3.0071 + (8e^{2(0.2)} - 3(3.0071))(0.1)$	$y_3 \approx 3.2984$

So Euler's method approximates $y(0.3) \approx 3.2984$.

AUTONOMOUS EQUATIONS AND EQUILIBRIUM SOLUTIONS

■ 1. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable without sketching a direction field.

$$\frac{dy}{dt} = -y^2 - y + 6$$

Solution:

The autonomous differential equation has equilibrium solutions at

$$-y^2 - y + 6 = 0$$

$$-(y+3)(y-2) = 0$$

$$y = -3,\ 2$$

These two equilibrium solutions divide the vertical axis into three intervals:

$$y < -3$$

$$-3 < y < 2$$

$$2 < y$$

We need a test-value in each of the three intervals, so we'll use $y = -4$, $y = 0$, and $y = 3$, then substitute these values into $f(y) = -y^2 - y + 6$.

$$f(-4) = -(-4)^2 - (-4) + 6 = -16 + 4 + 6 = -6 < 0$$

$$f(0) = -(0)^2 - (0) + 6 = 0 - 0 + 6 = 6 > 0$$

$$f(3) = -(3)^2 - (3) + 6 = -9 - 3 + 6 = -6 < 0$$

We find a negative value when $y = -4$, a positive value when $y = 0$, and a negative value when $y = 3$, so we can summarize these results in a table.

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
$(2,\infty)$	$-$	Decreasing/Falling
$(-3,2)$	$+$	Increasing/Rising
$(-\infty,-3)$	$-$	Decreasing/Falling

The solution curves approach $y = 2$ on both sides, but move away from $y = -3$ on both sides. Therefore, $y = 2$ is a stable equilibrium solution, while $y = -3$ is an unstable equilibrium solution.

■ 2. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable without sketching a direction field.

$$\frac{dy}{dt} = y^3 - 4y^2 + 4y$$

Solution:

The autonomous differential equation has equilibrium solutions at

$$y^3 - 4y^2 + 4y = 0$$

$$y(y-2)(y-2) = 0$$

$$y = 0,\ 2$$

These two equilibrium solutions divide the vertical axis into three intervals:

$$y < 0$$

$$0 < y < 2$$

$$2 < y$$

We need a test-value in each of the three intervals, so we'll use $y = -1$, $y = 1$, and $y = 3$, then substitute these values into $f(y) = y^3 - 4y^2 + 4y$.

$$f(-1) = (-1)^3 - 4(-1)^2 + 4(-1) = -1 - 4 - 4 = -9 < 0$$

$$f(1) = (1)^3 - 4(1)^2 + 4(1) = 1 - 4 + 4 = 1 > 0$$

$$f(3) = (3)^3 - 4(3)^2 + 4(3) = 27 - 36 + 12 = 3 > 0$$

We find a negative value when $y = -9$, and positive values when $y = 1$ and when $y = 3$, so we can summarize these results in a table.

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
$(2,\infty)$	$+$	Increasing/Rising
$(0,2)$	$+$	Increasing/Rising

$(-\infty,0)$	$-$	Decreasing/Falling

The solution curves move away from $y = 0$ on both sides, so $y = 0$ is an unstable equilibrium solution. The solution curves move toward $y = 2$ on one side and away from it on another side, so $y = 2$ is a semi-stable equilibrium solution.

■ 3. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = (y^2 - 9)(y - 1)^2$$

Solution:

The autonomous differential equation has equilibrium solutions at

$$(y^2 - 9)(y - 1)^2 = 0$$

$$(y - 3)(y + 3)(y - 1)^2 = 0$$

$$y = -3,\ 1,\ 3$$

These three equilibrium solutions divide the vertical axis into four intervals:

$$y < -3$$

$$-3 < y < 1$$

$$1 < y < 3$$

$$3 < y$$

We need a test-value in each of the four intervals, so we'll use $y = -4$, $y = 0$, $y = 2$, and $y = 4$, then substitute these values into $f(y) = (y^2 - 9)(y - 1)^2$.

$$f(-4) = ((-4)^2 - 9)(-4 - 1)^2 = (16 - 9)(-5)^2 = 175 > 0$$

$$f(0) = ((0)^2 - 9)(0 - 1)^2 = (0 - 9)(-1)^2 = -9 < 0$$

$$f(2) = (2^2 - 9)(2 - 1)^2 = (4 - 9)(1)^2 = -5 < 0$$

$$f(4) = (4^2 - 9)(4 - 1)^2 = (16 - 9)(3)^2 = 63 > 0$$

We find positive values when $y = -4$ and when $y = 4$, and negative values when $y = 0$ and when $y = 2$, so we can summarize these results in a table.

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
$(3,\infty)$	$+$	Increasing/Rising
$(1,3)$	$-$	Decreasing/Falling
$(-3,1)$	$-$	Decreasing/Falling
$(-\infty,-3)$	$+$	Increasing/Rising

The solution curves move toward $y = -3$ on both sides, so $y = -3$ is a stable equilibrium solution. The solution curves move away from $y = 3$ on both sides, so $y = 3$ is an unstable equilibrium solution. The solution curves are moving toward $y = 1$ on one side and away from it on another side, so $y = 1$ is a semi-stable equilibrium solution.

4. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = \sin y$$

Solution:

The autonomous differential equation has equilibrium solutions at

$$\sin y = 0$$

$y = \pi k$ for any integer k

These equilibrium solutions divide the vertical axis into an infinite number of intervals:

$$\dots$$

$$-2\pi < y < -\pi$$

$$-\pi < y < 0$$

$$0 < y < \pi$$

$$\pi < y < 2\pi$$

$$\dots$$

We need a test-value in each interval, so we'll use $y = -3\pi/2$, $y = -\pi/2$, $y = \pi/2$, and $y = 3\pi/2$, then substitute these values into $f(y) = \sin y$.

...

$$f\left(-\frac{3\pi}{2}\right) = \sin\left(-\frac{3\pi}{2}\right) = 1 > 0$$

$$f\left(-\frac{\pi}{2}\right) = \sin\left(-\frac{\pi}{2}\right) = -1 < 0$$

$$f\left(\frac{\pi}{2}\right) = \sin\left(\frac{\pi}{2}\right) = 1 > 0$$

$$f\left(\frac{3\pi}{2}\right) = \sin\left(\frac{3\pi}{2}\right) = -1 < 0$$

...

We find alternating positive and negative values as we move from one test interval to the next. We'll summarize these results in a table.

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
...	$+$	
$(\pi,2\pi)$	$-$	Decreasing/Falling
$(0,\pi)$	$+$	Increasing/Rising
$(-\pi,0)$	$-$	Decreasing/Falling
$(-2\pi,-\pi)$	$+$	Increasing/Rising

... −

Based on this pattern, the solution curves will move toward

$$y = \ldots -5\pi, \ -3\pi, \ -\pi, \pi, 3\pi, 5\pi, \ldots$$

$y = \pi + 2\pi k$ for any integer k

so these are all stable equilibrium solutions. On the other hand, the solution curves move away from

$$y = \ldots -4\pi, \ -2\pi, 0, 2\pi, 4\pi, \ldots$$

$y = 2\pi k$ for any integer k

so these are all unstable equilibrium solutions.

■ 5. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = |16 - y^2|$$

Solution:

The autonomous differential equation has equilibrium solutions at

$$|16 - y^2| = 0$$

$$16 - y^2 = 0$$

$$(4 - y)(4 + y) = 0$$

$$y = -4, 4$$

These two equilibrium solutions divide the vertical axis into three intervals:

$$y < -4$$

$$-4 < y < 4$$

$$4 < y$$

We need a test-value in each of the three intervals, so we'll use $y = -5$, $y = 0$, and $y = 5$, then substitute these values into $f(y) = |16 - y^2|$.

$$f(-5) = |16 - (-5)^2| = |16 - 25| = 9 > 0$$

$$f(0) = |16 - 0^2| = |16 - 0| = 16 > 0$$

$$f(5) = |16 - 5^2| = |16 - 25| = 9 > 0$$

We find positive values in each interval, which means both equilibrium solutions are semi-stable.

■ 6. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} + 3y + 6 = e^y(3y + 6)$$

Solution:

The autonomous differential equation has equilibrium solutions at

$$e^y(3y+6)-3y-6=0$$

$$e^y(3y+6)-(3y+6)=0$$

$$(3y+6)(e^y-1)=0$$

$$y=0,\ -2$$

These two equilibrium solutions divide the vertical axis into three intervals:

$$y<-2$$

$$-2<y<0$$

$$0<y$$

We need a test-value in each of the three intervals, so we'll use $y=-4$, $y=-1$, and $y=1$, then substitute these values into $f(y)=e^y(3y+6)-3y-6$.

$$f(-4)=e^{-4}(3(-4)+6)-3(-4)-6=\frac{1}{e^4}(-6)+6=-6\left(\frac{1-e^4}{e^4}\right)>0$$

$$f(-1)=e^{-1}(3(-1)+6)-3(-1)-6=\frac{1}{e}(3)-3=3\left(\frac{1-e}{e}\right)<0$$

$$f(1)=e^1(3(1)+6)-3(1)-6=e(9)-9=9e-9>0$$

We find positive values when $y = -4$ and when $y = 1$, and a negative value when $y = -1$, so we can summarize these results in a table.

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
$(0,\infty)$	$+$	Increasing/Rising
$(-2,0)$	$-$	Decreasing/Falling
$(-\infty,-2)$	$+$	Increasing/Rising

The solution curves move toward $y = -2$ on both sides, so $y = -2$ is a stable equilibrium solution. And the solution curves move away from $y = 0$ on both sides, so $y = 0$ is an unstable equilibrium solution.

THE LOGISTIC EQUATION

■ 1. Describe the growth pattern of a population of butterflies, assuming their population is modeled by the logistic growth equation, if the carrying capacity of their habitat is $3{,}700$ butterflies.

Solution:

Between $P = 0$ and $P = M = 3{,}700$, the butterfly population is always increasing. In other words, in the interval $0 < P < 3{,}700$, the butterfly population will grow slowly when P is close to 0, then growth will speed up, and then the growth rate will level off again as P gets close to $3{,}700$.

If the population starts at a size of exactly 0, of course it will neither increase nor decrease, as indicated by the equilibrium solution at $P = 0$.

If the population starts at a size of exactly $3{,}700$, then it will neither increase nor decrease, as indicated by the equilibrium solution at $P = 3{,}700$.

And if the butterfly population starts at a size greater than $3{,}700$, it will die off until it reaches stability at a size of $3{,}700$, as indicated by the stable equilibrium solution at $P = 3{,}700$.

■ 2. A bacteria population is observed to be $2{,}500$ at $t = 0$. Using the logistic growth model and a growth rate of 0.03 with a carrying capacity of $4{,}000$, write the logistic growth equation, then solve the initial value problem.

Solution:

Knowing that $k = 0.03$ and $M = 4{,}000$, we'll substitute into the logistic growth equation.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

$$\frac{dP}{dt} = 0.03P\left(1 - \frac{P}{4{,}000}\right)$$

$$\frac{dP}{dt} = 0.03P\left(\frac{4{,}000 - P}{4{,}000}\right)$$

$$\frac{dP}{dt} = \frac{0.03P(4{,}000 - P)}{4{,}000}$$

$$\frac{dP}{dt} = \frac{0.03}{4{,}000}(4{,}000P - P^2)$$

Separate variables, then integrate both sides of the equation.

$$\frac{1}{4{,}000P - P^2}\ dP = \frac{0.03}{4{,}000}\ dt$$

$$\int \frac{1}{4{,}000P - P^2}\ dP = \int \frac{0.03}{4{,}000}\ dt$$

$$\int \frac{1}{4{,}000P - P^2}\ dP = \frac{0.03}{4{,}000}t + C$$

Apply a partial fractions decomposition.

$$\frac{1}{P(4{,}000-P)}=\frac{A}{P}+\frac{B}{4{,}000-P}$$

$$1=A(4{,}000-P)+BP$$

$$1=4{,}000A-AP+BP$$

$$1=(-A+B)P+4{,}000A$$

Equating coefficients tells us that $A=1/4{,}000$, then that $-A+B=(-1/4{,}000)+B=0$, and therefore that $B=1/4{,}000$.

$$\int\frac{\frac{1}{4{,}000}}{P}+\frac{\frac{1}{4{,}000}}{4{,}000-P}\ dP=\frac{0.03}{4{,}000}t+C$$

$$\frac{1}{4{,}000}\int\frac{1}{P}+\frac{1}{4{,}000-P}\ dP=\frac{0.03}{4{,}000}t+C$$

$$\frac{1}{4{,}000}(\ln|P|-\ln|4{,}000-P|)=\frac{0.03}{4{,}000}t+C$$

$$\ln|P|-\ln|4{,}000-P|=0.03t+C$$

$$\ln\left|\frac{P}{4{,}000-P}\right|=0.03t+C$$

Solve for P to get the general solution.

$$\left|\frac{P}{4{,}000-P}\right|=e^{0.03t+C}$$

$$\frac{P}{4{,}000-P}=Ce^{0.03t}$$

Substitute the initial condition $P(0) = 2{,}500$.

$$\frac{2{,}500}{4{,}000 - 2{,}500} = Ce^{0.03(0)}$$

$$\frac{2{,}500}{1{,}500} = C$$

$$C = \frac{5}{3}$$

Then the actual solution is

$$\frac{P}{4{,}000 - P} = \frac{5}{3}e^{0.03t}$$

$$\frac{3P}{4{,}000 - P} = 5e^{0.03t}$$

■ 3. An insect population is observed to be 500 at time $t = 0$. Assuming the population follows a logistic growth model with a growth rate of 0.01 and a carrying capacity of $2{,}500$, solve the initial value problem.

Solution:

Knowing that $k = 0.01$ and $M = 2{,}500$, we'll substitute into the logistic growth equation.

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

$$\frac{dP}{dt} = 0.01P\left(1 - \frac{P}{2{,}500}\right)$$

$$\frac{dP}{dt} = 0.01P\left(\frac{2{,}500 - P}{2{,}500}\right)$$

$$\frac{dP}{dt} = \frac{0.01P(2{,}500 - P)}{2{,}500}$$

Separate variables and integrate both sides of the equation.

$$\frac{1}{P(2{,}500 - P)}\ dP = \frac{0.01}{2{,}500}\ dt$$

$$\int \frac{1}{P(2{,}500 - P)}\ dP = \int \frac{0.01}{2{,}500}\ dt$$

Apply a partial fractions decomposition.

$$\frac{1}{P(2{,}500 - P)} = \frac{A}{P} + \frac{B}{2{,}500 - P}$$

$$1 = A(2{,}500 - P) + BP$$

$$1 = 2{,}500A - AP + BP$$

$$1 = (-A + B)P + 2{,}500A$$

Equating coefficients tells us that $A = 1/2{,}500$, then that $-A + B = (-1/2{,}500) + B = 0$, and therefore that $B = 1/2{,}500$.

$$\int \frac{\frac{1}{2{,}500}}{P} + \frac{\frac{1}{2{,}500}}{2{,}500 - P}\ dP = \int \frac{0.01}{2{,}500}\ dt$$

$$\frac{1}{2{,}500}\int \frac{1}{P}+\frac{1}{2{,}500-P}\ dP=\frac{0.01}{2{,}500}t+C$$

$$\frac{1}{2{,}500}(\ln|P|+\ln|2{,}500-P|)=\frac{0.01}{2{,}500}t+C$$

$$\ln|P|+\ln|2{,}500-P|=0.01t+C$$

$$\ln|P(2{,}500-P)|=0.01t+C$$

$$e^{\ln|P(2{,}500-P)|}=e^{0.01t+C}$$

$$P(2{,}500-P)=Ce^{0.01t}$$

Substitute the initial condition $P(0)=500$ and solve for C.

$$500(2{,}500-500)=Ce^{0.01(0)}$$

$$C=1{,}000{,}000$$

So the actual solution is.

$$P(2{,}500-P)=1{,}000{,}000e^{0.01t}$$

■ 4. The population of Canada in $2{,}000$ and 2010 was 30.7 and 34 million people, respectively. Predict its population in 2050 and 2060 assuming an exponential model of population growth.

Solution:

Since our earliest observation is in 2000, we have $P_0 = P(0) = 30.7$, and after 10 years we find $P(10) = 34$. Substitute these values into exponential growth equation,

$$P(t) = P_0 e^{kt}$$

$$34 = 30.7e^{10k}$$

then solve for k.

$$1.11 = e^{10k}$$

$$\ln 1.11 = 10k$$

$$k \approx 0.01$$

To find the population in 2050, we'll use $t = 2050 - 2000 = 50$,

$$P(50) \approx 30.7e^{0.01(50)}$$

$$P(50) \approx 50.62$$

and to find the population in 2060, we'll use $t = 2060 - 2000 = 60$.

$$P(60) \approx 30.7e^{0.01(60)}$$

$$P(60) \approx 55.94$$

■ 5. A bacteria population increases 8-fold in 5 hours. Assuming exponential growth, how long did it take for the population to increase 5-fold?

Solution:

We've been told that the population grows to 8 times its original size. If we call its original size P_0, then we can say that it grows to $8P_0$ after 5 hours. Substituting these values into the exponential growth equation gives

$$8P_0 = P_0 e^{k(5)}$$

$$8 = e^{5k}$$

$$\ln 8 = 5k$$

$$k = \frac{\ln 8}{5}$$

Now that we have a value for the growth constant k, we can can figure out how long it took for the population to increase by 5-fold. If we say that P_0 is the original population, and $5P_0$ is five times the original population, then the exponential equation becomes

$$5P_0 = P_0 e^{\frac{\ln 8}{5}t}$$

$$5 = e^{\frac{\ln 8}{5}t}$$

$$\ln 5 = \ln e^{\frac{\ln 8}{5}t}$$

$$\ln 5 = \frac{\ln 8}{5}t$$

$$t = \frac{5 \ln 5}{\ln 8}$$

$$t \approx 3.87$$

This result tells us that the bacteria population increased 5-fold in about 3.87 hours, or about 3 hours and 52 minutes.

■ 6. The growth rate of a population of bears is modeled by the differential equation, where time t is measured in years. Solve the differential equation with the initial condition $P(0) = 30$.

$$\frac{dP}{dt} = 0.002P(400 - P)$$

Solution:

Separate variables,

$$\frac{dP}{P(400 - P)} = 0.002\ dt$$

$$\int \frac{dP}{P(400 - P)} = \int 0.002\ dt$$

then apply a partial fractions decomposition to integrate both sides.

$$\frac{1}{P(400 - P)} = \frac{A}{P} + \frac{B}{400 - P}$$

$$1 = A(400 - P) + BP$$

$$1 = 400A - AP + BP$$

$$1 = (-A + B)P + 400A$$

Equating coefficients tells us that $A = 1/400$, then that $-A + B = (-1/400) + B = 0$, and therefore that $B = 1/400$.

$$\int \frac{\frac{1}{400}}{P} + \frac{\frac{1}{400}}{400 - P}\ dP = \int 0.002\ dt$$

$$\frac{1}{400}(\ln|P| + \ln|400 - P|) = 0.002t + C$$

$$\ln|P(400 - P)| = 0.8t + C$$

$$e^{\ln|P(400-P)|} = e^{0.8t+C}$$

$$P(400 - P) = Ce^{0.8t}$$

Substitute the initial condition $P(0) = 30$ and solve for C.

$$30(400 - 30) = Ce^{0.8(0)}$$

$$C = 11{,}100$$

So the actual solution to the differential equation is

$$P(400 - P) = 11{,}100e^{0.8t}$$

PREDATOR-PREY SYSTEMS

■ 1. Is the system cooperative, competitive, or predator-prey? Find the equilibrium solutions of the system.

$$\frac{dx}{dt} = -0.16x + 0.08xy$$

$$\frac{dy}{dt} = 4.5y - 0.9xy$$

Solution:

The mixed xy term $0.08xy$ in the differential equation for dx/dt is positive, while the mixed xy term $-0.9xy$ in the differential equation for dy/dt is negative, which means this is a predator-prey system.

To find equilibrium solutions, we'll factor both equations to get

$$\frac{dx}{dt} = -0.16x + 0.08xy$$

$$\frac{dx}{dt} = 0.08x(-2 + y)$$

$$\frac{dx}{dt} = 0.08x(y - 2)$$

and

$$\frac{dy}{dt} = 4.5y - 0.9xy$$

$$\frac{dy}{dt} = 0.9y(5 - x)$$

Setting both equations equal to 0 gives

$$0.08x(y - 2) = 0$$

$0.08x = 0$ and $y - 2 = 0$

$x = 0$ and $y = 2$

and

$$0.9y(5 - x) = 0$$

$0.9y = 0$ and $5 - x = 0$

$y = 0$ and $x = 5$

We need to test both solutions from the first equation with both solutions from the second equation. So if we pair $x = 0$ with both $y = 0$ and $y = 2$, and then in addition we pair $x = 5$ with both $y = 0$ and $y = 2$, we get four combinations.

The solutions of the system are therefore

- $(x, y) = (0,0)$, where both populations are 0 and will remain there

- $(x, y) = (0,2)$, where the y population is stable at 2 while the x population is at 0

- $(x, y) = (5,0)$, where the x population is stable at 5 while the y population is at 0

- $(x, y) = (5,2)$, where the system is balanced

■ 2. Given the system modeling the interaction between populations of foxes and hares, find any equilibrium where the foxes and hares coexist, and give the size of each population.

$$\frac{dH}{dt} = 0.14H - 0.07FH$$

$$\frac{dF}{dt} = -0.65F + 0.05FH$$

Solution:

To find equilibrium solutions, we'll factor both equations to get

$$\frac{dH}{dt} = 0.14H - 0.07FH$$

$$\frac{dH}{dt} = 0.07H(2 - F)$$

and

$$\frac{dF}{dt} = -0.65F + 0.05FH$$

$$\frac{dF}{dt} = 0.05F(-13 + H)$$

$$\frac{dF}{dt} = 0.05F(H - 13)$$

Setting both equations equal to 0 gives

$$0.07H(2 - F) = 0$$

$0.07H = 0$ and $2 - F = 0$

$H = 0$ and $F = 2$

and

$$0.05F(H - 13) = 0$$

$0.05F = 0$ and $H - 13 = 0$

$F = 0$ and $H = 13$

We need to test both solutions from the first equation with both solutions from the second equation. So if we pair $H = 0$ with both $F = 0$ and $F = 2$, and then in addition we pair $H = 13$ with both $F = 0$ and $F = 2$, we get four combinations.

The only equilibrium solution where foxes and hares coexist is $(H, F) = (13, 2)$.

3. Determine whether the system is cooperative, competitive, or predator-prey, and find the maximum size of both populations when they are in equilibrium, assuming both populations are measured in millions.

$$\frac{dx}{dt} = 0.02x - 0.4x^2 + 0.005xy$$

$$\frac{dy}{dt} = 0.1y - 1.1y^2 + 0.05xy$$

Solution:

In both equations, the mixed xy has a positive sign, so the system is cooperative. To find the equilibrium solution, we'll start by factoring the equations.

$$\frac{dx}{dt} = 0.02x - 0.4x^2 + 0.005xy$$

$$\frac{dx}{dt} = 0.005x(4 - 80x + y)$$

$$\frac{dx}{dt} = 0.005x(y - 80x + 4)$$

and

$$\frac{dy}{dt} = 0.1y - 1.1y^2 + 0.05xy$$

$$\frac{dy}{dt} = 0.05y(2 - 22y + x)$$

$$\frac{dy}{dt} = 0.05y(x - 22y + 2)$$

Setting both equations equal to 0 gives

$0.005x(y - 80x + 4) = 0$

$0.005x = 0$ and $y - 80x + 4 = 0$

$x = 0$ and $80x - y = 4$

and

$0.05y(x - 22y + 2) = 0$

$0.05y = 0$ and $x - 22y + 2 = 0$

$y = 0$ and $22y - x = 2$

The equilibrium solution pairs are then

$x = 0$ and $y = 0$

$x = 0$ and $22y - x = 2$

$80x - y = 4$ and $y = 0$

$80x - y = 4$ and $22y - x = 2$

The first pair is simply $(x, y) = (0,0)$, and the second and third pairs can be given explicitly with a simple substitution as

$x = 0$ and $22y - x = 2$

$x = 0$ and $22y = 2$

$$x = 0 \text{ and } y = \frac{1}{11}$$

$$(x, y) = \left(0, \frac{1}{11}\right) \approx (0,\ 0.091)$$

and

$$80x - y = 4 \text{ and } y = 0$$

$$80x = 4 \text{ and } y = 0$$

$$x = \frac{1}{20} \text{ and } y = 0$$

$$(x, y) = \left(\frac{1}{20}, 0\right) \approx (0.05,\ 0)$$

For the last pair, we have to solve the system.

$$80x - y = 4 \text{ and } 22y - x = 2$$

$$(x, y) = \left(\frac{90}{1{,}759}, \frac{164}{1{,}759}\right) \approx (0.051,\ 0.093)$$

Therefore, we have the four combinations

$$(x, y) = (0,\ 0)$$

$$(x, y) \approx (0,\ 0.091)$$

$$(x, y) \approx (0.05,\ 0)$$

$$(x, y) \approx (0.051,\ 0.093)$$

The last pair is our equilibrium solution. Since the populations are measured in millions, the maximum size of population x is $51{,}000$ and the maximum size of population y is $93{,}000$.

■ 4. Determine whether the system is cooperative, competitive, or predator-prey, and find the equilibrium solutions, assuming both populations are measured in millions.

$$\frac{dx}{dt} = 0.004x - 0.2xy$$

$$\frac{dy}{dt} = 0.03y - 0.36y^2 - 0.6xy$$

Solution:

In both equations, the mixed xy has a negative sign, so the system is competitive. To find the equilibrium solution, we'll start by factoring the equations.

$$\frac{dx}{dt} = 0.004x - 0.2xy$$

$$\frac{dx}{dt} = 0.004x(1 - 50y)$$

and

$$\frac{dy}{dt} = 0.03y - 0.36y^2 - 0.6xy$$

$$\frac{dy}{dt} = 0.03y(1 - 12y - 20x)$$

Setting both equations equal to 0 gives

$$0.004x(1 - 50y) = 0$$

$$0.004x = 0 \text{ and } 50y = 1$$

$$x = 0 \text{ and } y = \frac{1}{50}$$

and

$$0.03y(1 - 12y - 20x) = 0$$

$$0.03y = 0 \text{ and } 1 - 12y - 20x = 0$$

$$y = 0 \text{ and } 12y + 20x = 1$$

The equilibrium solution pairs are then

$$x = 0 \text{ and } y = 0$$

$$x = 0 \text{ and } y = \frac{1}{50}$$

$$12y + 20x = 1 \text{ and } y = 0$$

$$12y + 20x = 1 \text{ and } y = \frac{1}{50}$$

The first and second pairs are simply $(x, y) = (0,0)$ and $(x, y) = (0,1/50)$, and the third and fourth pairs can be determined explicitly with a simple substitution.

$$12y + 20x = 1 \text{ and } y = 0$$

$$20x = 1 \text{ and } y = 0$$

$$x = \frac{1}{20} \text{ and } y = 0$$

$$(x, y) = \left(\frac{1}{20}, 0\right) = (0.05,\ 0)$$

and

$$12y + 20x = 1 \text{ and } y = \frac{1}{50}$$

$$20x = \frac{19}{25} \text{ and } y = \frac{1}{50}$$

$$x = \frac{19}{500} \text{ and } y = \frac{1}{50}$$

$$(x, y) = \left(\frac{19}{500}, \frac{1}{50}\right) = (0.038,\ 0.02)$$

Therefore, we have the four combinations

$$(x, y) = (0,\ 0)$$

$$(x, y) = (0,\ 0.02)$$

$$(x, y) = (0.05,\ 0)$$

$$(x, y) = (0.038,\ 0.02)$$

The last pair is our equilibrium solution. Since the populations are measured in millions, the maximum size of population x is $38,000$ and the maximum size of population y is $20,000$.

■ 5. Two species of bacteria are being cultivated in a petri dish. Every day, an amount of sugar is deposited in the petri dish and the species compete for this source of food. Determine whether the system is cooperative, competitive, or predator-prey and find the equilibrium solutions.

$$\frac{dx}{dt} = 0.001x - 0.03x^2 - 0.04xy$$

$$\frac{dy}{dt} = 0.002y - 0.05y^2 - 0.01xy$$

Solution:

In both equations, the mixed xy has a negative sign, so the system is competitive. To find the equilibrium solution, we'll start by factoring the equations.

$$\frac{dx}{dt} = 0.001x - 0.03x^2 - 0.04xy$$

$$\frac{dx}{dt} = 0.001x(1 - 30x - 4y)$$

and

$$\frac{dy}{dt} = 0.002y - 0.05y^2 - 0.01xy$$

$$\frac{dy}{dt} = 0.002y(1 - 25y - 5x)$$

Setting both equations equal to 0 gives

$0.001x(1 - 30x - 4y)$

$0.001x = 0$ and $1 - 30x - 4y = 0$

$x = 0$ and $30x + 4y = 1$

and

$0.002y(1 - 25y - 5x) = 0$

$0.002y = 0$ and $1 - 25y - 5x = 0$

$y = 0$ and $25y + 5x = 1$

The equilibrium solution pairs are then

$x = 0$ and $y = 0$

$x = 0$ and $30x + 4y = 1$

$25y + 5x = 1$ and $y = 0$

$25y + 5x = 1$ and $30x + 4y = 1$

The first pair is simply $(x, y) = (0,0)$, and the second and third pairs can be determined explicitly with a simple substitution.

$$x = 0 \text{ and } 30x + 4y = 1$$

$$x = 0 \text{ and } 4y = 1$$

$$x = 0 \text{ and } y = \frac{1}{4}$$

$$(x, y) = \left(0, \ \frac{1}{4}\right) = (0, \ 0.25)$$

and

$$25y + 5x = 1 \text{ and } y = 0$$

$$5x = 1 \text{ and } y = 0$$

$$x = \frac{1}{5} \text{ and } y = 0$$

$$(x, y) = \left(\frac{1}{5}, 0\right) = (0.2, \ 0)$$

For the last pair, we have to solve the system.

$$25y + 5x = 1 \text{ and } 30x + 4y = 1$$

$$(x, y) = \left(\frac{21}{730}, \frac{5}{146}\right) \approx (0.029, \ 0.034)$$

Therefore, we have the four combinations

$$(x, y) = (0, \ 0)$$

$$(x, y) = (0, \ 0.25)$$

$$(x, y) = (0.2,\ 0)$$

$$(x, y) \approx (0.029,\ 0.034)$$

■ 6. Is the system cooperative, competitive, or predator-prey? Find the equilibrium solutions of the system.

$$\frac{dx}{dt} = 0.48x - 0.16xy$$

$$\frac{dy}{dt} = 1.2y + 7.2xy$$

Solution:

The mixed xy term $-0.16xy$ in the differential equation for dx/dt is negative, while the mixed xy term $7.2xy$ in the differential equation for dy/dt is positive, which means this is a predator-prey system.

To find equilibrium solutions, we'll factor both equations to get

$$\frac{dx}{dt} = 0.48x - 0.16xy$$

$$\frac{dx}{dt} = 0.16x(3 - y)$$

and

$$\frac{dy}{dt} = 1.2y + 7.2xy$$

$$\frac{dy}{dt} = 1.2y(1 + 6x)$$

Setting both equations equal to 0 gives

$$0.16x(3 - y) = 0$$

$0.16x = 0$ and $3 - y = 0$

$x = 0$ and $y = 3$

and

$$1.2y(1 + 6x) = 0$$

$1.2y = 0$ and $1 + 6x = 0$

$y = 0$ and $x = -\frac{1}{6}$

If we pair $x = 0$ with $y = 0$, and we pair $x = -1/6$ with $y = 3$, we get two combinations.

- $(x, y) = (0,0)$, where both populations are 0 and will remain there

- $(x, y) = (-1/6,3)$, where the system is balanced

EXPONENTIAL GROWTH AND DECAY

■ 1. If world population was 2.56 million in 1950 and 3.04 million in 1960 and is growing exponentially, find a function modeling population in the second half of the 20th century, then use the function to estimate world population in 1984.

Solution:

The problem gives us the initial conditions $P(0) = 2{,}560$ and $P(10) = 3{,}040$. Substituting the first condition into the exponential growth equation, we get

$$P = Ce^{kt}$$

$$2{,}560 = Ce^{k(0)}$$

$$C = P_0 = 2{,}560$$

Now we'll substitute $C = P_0 = 2{,}560$ and $P(10) = 3{,}040$ into the exponential growth equation to find k, the constant of proportionality.

$$3{,}040 = 2{,}560e^{10k}$$

$$\frac{19}{16} = e^{10k}$$

Apply the natural log to both sides.

$$\ln \frac{19}{16} = \ln(e^{10k})$$

$$\ln \frac{19}{16} = 10k$$

$$k = \frac{1}{10} \ln \frac{19}{16}$$

Therefore, the function modeling population growth is

$$P(t) = 2{,}560e^{\frac{t}{10} \ln \frac{19}{16}}$$

To approximate world population in 1984, we evaluate this function at $t = 1984 - 1950 = 34$.

$$P(34) = 2{,}560e^{\frac{34}{10} \ln \frac{19}{16}}$$

$$P(34) \approx 4{,}591.92$$

Therefore, world population in 1984 was approximately 4.592 million.

■ 2. The size of a population of fleas increased by 4 fold in 2 months. How long did it take for the population to increase 5 fold?

Solution:

The problem gives us the conditions $P(0) = P_0$ and $P(2) = 4P_0$. Substituting the first condition into the exponential growth equation, we get

$$P = Ce^{kt}$$

$$P_0 = Ce^{k(0)}$$

$$P_0 = C$$

Now we'll substitute $C = P_0$ and $P(2) = 4P_0$ into the exponential growth equation to find k, the constant of proportionality.

$$P = P_0e^{kt}$$

$$4P_0 = P_0e^{k(2)}$$

$$4 = e^{2k}$$

Apply the natural log to both sides.

$$\ln 4 = \ln(e^{2k})$$

$$\ln 4 = 2k$$

$$k = \frac{\ln 4}{2}$$

$$k = \ln 2$$

Now we can use k to find the time it took for the population to increase 5 fold, or the time it took the population to reach a size of $5P_0$.

$$5P_0 = P_0e^{t\ln 2}$$

$$5 = e^{t\ln 2}$$

$$\ln 5 = \ln e^{t\ln 2}$$

$$\ln 5 = t \ln 2$$

$$t = \frac{\ln 5}{\ln 2}$$

$$t \approx 2.32$$

Therefore, the flea population increased 5 fold in about 2.32 months, or about 2 months and 10 days.

■ 3. The half-life of radium-226 is 1,590 years. If the substance decays exponentially, find a function that models the radioactive decay of a sample of 100 g of radium-226 over time t years. How much of the mass remains after 2,000 years?

Solution:

Since radium-226 decays exponentially, we can write

$$m(t) = m_0 e^{kt}$$

where $m(t)$ is the mass of radium- 226 after t years, m_0 is the initial amount of radium in the sample, and k is the decay constant. From the question we know

$$m(0) = m_0 = 100 \text{ g}$$

$$m(1{,}590) = \frac{m_0}{2} = 50 \text{ g}$$

We can substitute these values into the exponential decay equation in order to find the decay constant k.

$$m(1{,}590) = m_0 e^{1{,}590k}$$

$$50 = 100e^{1{,}590k}$$

$$\frac{1}{2} = e^{1{,}590k}$$

Apply the natural log to both sides.

$$\ln\frac{1}{2} = \ln(e^{1{,}590k})$$

$$\ln\frac{1}{2} = 1{,}590k$$

$$k = -\frac{1}{1{,}590}\ln 2$$

So the function modeling the decay of radium-226 is

$$m(t) = 100e^{-\frac{t}{1{,}590}\ln 2}$$

We'll substitute $t = 2{,}000$ to determine how much radium-226 remains after $2{,}000$ years.

$$m(2{,}000) = 100e^{-\frac{2{,}000}{1{,}590}\ln 2}$$

$$m(2{,}000) \approx 41.8 \text{ g}$$

■ 4. A population of $1,000$ bacteria, growing exponentially, takes 12 minutes to double in size. How long will it take for the population to reach 1 million?

Solution:

We know $P(0)=P_0=1,000$ and $P(12)=2P_0=2,000$. Substituting the first condition into the exponential growth equation, we get

$$P(t)=P_0e^{kt}$$

$$2,000=1,000e^{k(12)}$$

$$2=e^{12k}$$

Apply the natural log to both sides.

$$\ln 2=\ln(e^{12k})$$

$$\ln 2=12k$$

$$k=\frac{1}{12}\ln 2$$

So the growth of the bacteria population can be modeled by

$$P(t)=1,000e^{\frac{t}{12}\ln 2}$$

We'll substitute $P(t)=1,000,000$ to determine how long it will take for the population to reach that size.

$$1,000,000=1,000e^{\frac{t}{12}\ln 2}$$

$$1{,}000 = e^{\frac{t}{12}\ln 2}$$

Apply the natural log to both sides.

$$\ln 1{,}000 = \ln e^{\frac{t}{12}\ln 2}$$

$$\ln 1{,}000 = \frac{t}{12}\ln 2$$

$$t = 12\frac{\ln 1{,}000}{\ln 2}$$

$$t \approx 119.6$$

The size of the bacteria population will reach 1 million in approximately 119.6 minutes, or just under two hours.

■ 5. A sofa's price drops by 50% in 10 months. If the rate at which the price decreases is proportional to current price, how much will the price decrease in 2 years?

Solution:

We know $P(0) = P_0$ and $P(10) = (1/2)P_0$. Substituting the first condition into the exponential decay equation gives

$$P = Ce^{kt}$$

$$P_0 = Ce^{k(0)}$$

$$P_0 = C$$

Now we'll substitute $C = P_0$ and $P(10) = (1/2)P_0$ into the exponential equation to find k, the constant of proportionality.

$$P = P_0 e^{kt}$$

$$\frac{1}{2}P_0 = P_0 e^{k(10)}$$

$$\frac{1}{2} = e^{10k}$$

Apply the natural log to both sides.

$$\ln\frac{1}{2} = \ln(e^{10k})$$

$$\ln\frac{1}{2} = 10k$$

$$k = \frac{1}{10}\ln\frac{1}{2}$$

$$k = \frac{1}{10}(\ln 1 - \ln 2)$$

$$k = -\frac{1}{10}\ln 2$$

Now we can use k to find the price after 2 years, or 24 months.

$$P(24) = P_0 e^{\left(-\frac{1}{10}\ln 2\right)24}$$

$$P(24) = P_0 e^{-\frac{12}{5}\ln 2}$$

$$P(24) = P_0 e^{\ln 2^{-\frac{12}{5}}}$$

$$P(24) = P_0 2^{-\frac{12}{5}}$$

$$P(24) = 0.19P_0$$

The price will fall to 19% of the original amount, which means the price will drop by $100\% - 19\% = 81\%$ after 2 years.

■ 6. A radioactive element decays into nonradioactive substances. After 3 years, the radioactivity decreases by 5%. How many years would it take for the radioactivity to decrease by 90%?

Solution:

If we substitute $P(0) = P_0$ into the exponential decay equation, we get

$$P = Ce^{kt}$$

$$P_0 = Ce^{k(0)}$$

$$P_0 = C$$

Now we'll substitute $C = P_0$ and $P(3) = 0.95P_0$, since after 3 years the radioactivity decreases by 5%, into the decay equation to find k, the constant of proportionality.

$$P = P_0 e^{kt}$$

$$0.95P_0 = P_0 e^{k(3)}$$

$$0.95 = e^{3k}$$

Apply the natural log to both sides.

$$\ln 0.95 = \ln(e^{3k})$$

$$\ln 0.95 = 3k$$

$$k = \frac{1}{3} \ln 0.95$$

Now we can use k and $P(t) = 0.1P_0$ to find the time it takes for radioactivity to decrease by 90%.

$$0.1P_0 = P_0 e^{\frac{t}{3} \ln 0.95}$$

$$0.1 = e^{\frac{t}{3} \ln 0.95}$$

$$\ln 0.1 = \ln e^{\frac{t}{3} \ln 0.95}$$

$$\ln 0.1 = \frac{t}{3} \ln 0.95$$

$$t = \frac{3 \ln 0.1}{\ln 0.95}$$

$$t \approx 134.67$$

Therefore, it takes almost 135 years for radioactivity to decrease by 90%.

MIXING PROBLEMS

■ 1. A tank contains $5{,}000$ L of water and 100 kg of dissolved salt. Fresh water is entering the tank at 20 L/min, and the solution drains at a rate of 15 L/min. Assuming the solution in the tank remains perfectly mixed, what is the function that models the amount of salt $y(t)$ at any given minute t? How much salt remains in the tank after 15 days?

Solution:

From the question, we know

- $C_1 = 0$ kg/L because the freshwater being added to the tank contains no salt
- $r_1 = 20$ L/min
- $C_2 = \dfrac{y(t)\ \text{kg}}{(5{,}000 + 5t)\ \text{L}}$ because every minute 20 L enter the tank while 15 L exit, which means 5 L of solution is added to the initial $5{,}000$ L every minute
- $r_2 = 15$ L/min

So the differential equation modeling the change in salt over time is

$$\frac{dy}{dt} = \left(\frac{0\ \text{kg}}{\text{L}}\right)\left(\frac{20\ \text{L}}{\text{min}}\right) - \left(\frac{y(t)\ \text{kg}}{(5{,}000 + 5t)\ \text{L}}\right)\left(\frac{15\ \text{L}}{\text{min}}\right)$$

$$\frac{dy}{dt} = \frac{0\text{ kg}}{\text{min}} - \frac{15y(t)\text{ kg}}{(5{,}000 + 5t)\text{ min}}$$

$$\frac{dy}{dt} = -\frac{3y(t)\text{ kg}}{(1{,}000 + t)\text{ min}}$$

Separate variables and integrate both sides.

$$\int \frac{1}{y}\ dy = \int -\frac{3}{1{,}000 + t}\ dt$$

$$\ln|y| = -3\ln|1{,}000 + t| + C$$

Simplify the equation and solve it for y.

$$e^{\ln|y|} = e^{-3\ln|1{,}000+t|+C}$$

$$|y| = Ce^{-3\ln|1{,}000+t|}$$

$$y = Ce^{\ln|1{,}000+t|^{-3}}$$

$$y = C|1{,}000 + t|^{-3}$$

Then the general solution is

$$y(t) = \frac{C}{(1{,}000 + t)^3}$$

We were told that the tank initially contained 100 kg of dissolved salt, which means $y(0) = 100$ is an initial condition that we can substitute into the general solution to solve for the constant of integration,

$$100 = \frac{C}{(1{,}000 + 0)^3}$$

$$C = 10^{11}$$

and then we can put this value back into the general solution.

$$y(t) = \frac{10^{11}}{(10^3 + t)^3}$$

To find how much salt remains in the tank after 15 days we just plug in $t = 15(24)(60) = 21{,}600$ minutes to find the amount of salt in the tank after 15 days.

$$y(21{,}600) = \frac{10^{11}}{(1{,}000 + 21{,}600)^3}$$

$$y(21600) = \frac{10^{11}}{22{,}600^3}$$

$$y(21{,}600) \approx 0.008 \approx 8 \times 10^{-3} \text{ kg}$$

■ 2. A tank contains $2{,}000$ L of water and 500 kg of dissolved salt. Fresh water is entering the tank at 20 L/min, and the solution drains at a rate of 50 L/min. Assuming the solution in the tank remains perfectly mixed, what is the function $y(t)$ that models the amount of salt in the tank after t minutes?

Solution:

From the question, we know

- $C_1 = 0$ kg/L because the freshwater being added to the tank contains no salt

- $r_1 = 20$ L/min

- $C_2 = \frac{y(t)\text{ kg}}{(2{,}000 - 30t)\text{ L}}$ because every minute 20 L enter the tank while 50 L exit, which means 30 L of solution is taken away from the initial $2{,}000$ L every minute

- $r_2 = 50$ L/min

So the differential equation modeling the change in salt over time is

$$\frac{dy}{dt} = \left(\frac{0\text{ kg}}{\text{L}}\right)\left(\frac{20\text{ L}}{\text{min}}\right) - \left(\frac{y(t)\text{ kg}}{(2{,}000 - 30t)\text{ L}}\right)\left(\frac{50\text{ L}}{\text{min}}\right)$$

$$\frac{dy}{dt} = \frac{0\text{ kg}}{\text{min}} - \frac{50y(t)\text{ kg}}{(2{,}000 - 30t)\text{ min}}$$

$$\frac{dy}{dt} = -\frac{5y(t)\text{ kg}}{(200 - 3t)\text{ min}}$$

Separate variables and integrate both sides.

$$\int \frac{1}{y}\ dy = \int -\frac{5}{200 - 3t}\ dt$$

$$\ln|y| = \frac{5}{3}\ln|200 - 3t| + C$$

Simplify the equation and solve it for y.

$$e^{\ln|y|} = e^{\frac{5}{3}\ln|200 - 3t| + C}$$

$$|y| = Ce^{\frac{5}{3}\ln|200-3t|}$$

$$y = Ce^{\ln|200-3t|^{\frac{5}{3}}}$$

$$y = C|200 - 3t|^{\frac{5}{3}}$$

We were told that the tank initially contained 500 kg of dissolved salt, which means $y(0) = 500$ is an initial condition that we can substitute into the general solution to solve for the constant of integration,

$$500 = C|200 - 3(0)|^{\frac{5}{3}}$$

$$C \approx 0.073$$

and then we can put this value back into the general solution.

$$y(t) = 0.073|200 - 3t|^{\frac{5}{3}}$$

■ 3. A tank contains $5{,}000$ L of freshwater and is being filled with a brine mixture with a concentration of 150 g/L, at a rate of 20 L/min. Meanwhile, solution drains from the tank at a rate of 30 L/min. Assuming the solution in the tank remains perfectly mixed, what is the function $y(t)$ that models the amount of salt in the tank after t minutes?

Solution:

From the question, we know

- $C_1 = 0.15$ kg/L because the brine solution being added to the tank has a salt concentration of 150 g/L

- $r_1 = 20$ L/min

- $C_2 = \dfrac{y(t)\text{ kg}}{(5{,}000 - 10t)\text{ L}}$ because every minute 20 L enter the tank while 30 L exit, which means 10 L of solution is taken away from the initial $5{,}000$ L every minute

- $r_2 = 30$ L/min

So the differential equation modeling the change in salt over time is

$$\frac{dy}{dt} = \left(\frac{0.15\text{ kg}}{\text{L}}\right)\left(\frac{20\text{ L}}{\text{min}}\right) - \left(\frac{y(t)\text{ kg}}{(5{,}000 - 10t)\text{ L}}\right)\left(\frac{30\text{ L}}{\text{min}}\right)$$

$$\frac{dy}{dt} = \frac{3\text{ kg}}{\text{min}} - \frac{30y(t)\text{ kg}}{(5{,}000 - 10t)\text{ min}}$$

$$\frac{dy}{dt} = \frac{3\text{ kg}}{\text{min}} - \frac{3y(t)\text{ kg}}{(500 - t)\text{ min}}$$

Rewrite the equation in standard form of a linear equation.

$$y' + \left(\frac{3}{500 - t}\right)y = 3$$

Then the integrating factor is

$$I(t) = e^{\int P(t)\,dt}$$

$$I(t) = e^{\int \frac{3}{500 - t}\,dt}$$

$$I(t) = e^{-3\ln(500-t)}$$

$$I(t) = e^{\ln(500-t)^{-3}}$$

$$I(t) = \frac{1}{(500-t)^3}$$

Then we can rewrite the linear differential equation as

$$\frac{d}{dt}\left(\frac{y}{(500-t)^3}\right) = \frac{3}{(500-t)^3}$$

$$\int \frac{d}{dt}\left(\frac{y}{(500-t)^3}\right)\ dt = \int \frac{3}{(500-t)^3}\ dt$$

$$\frac{y}{(500-t)^3} = \frac{3}{2(500-t)^2} + C$$

So the general solution will be

$$y(t) = \frac{3}{2}(500-t) + C(500-t)^3$$

$$y(t) = 750 - \frac{3}{2}t + C(500-t)^3$$

The problem states that the tank was initially filled with freshwater, which means $y(0) = 0$.

$$0 = 750 - \frac{3}{2}(0) + C(500-0)^3$$

$$C = -\frac{750}{500^3}$$

$$C = -\frac{3}{500{,}000}$$

So the actual solution is

$$y(t) = 750 - \frac{3}{2}t - \frac{3(500 - t)^3}{500{,}000}$$

■ 4. A tank contains $1{,}000$ L of fresh water. A mix of water and salt is being added into the tank at a rate of 500 L/h, and the solution drains out at the same rate. The concentration of the solution that's being deposited varies periodically, according to the function $f(t) = 2\sin t$ kg/L. Assuming that the solution in the tank remains perfectly mixed, how much salt is in the tank after t hours?

Solution:

From the question, we know

- $C_1 = 2\sin t$ kg/L because the brine solution being added to the tank has a salt concentration modeled by $2\sin t$ kg/L

- $r_1 = 500$ L/h

- $C_2 = \frac{y(t)\text{ kg}}{1{,}000\text{ L}}$ because with equal input and output rates, the amount of solution in the tank stays constant over time

- $r_2 = 500$ L/h

So the differential equation modeling the change in salt over time is

$$\frac{dy}{dt} = \left(\frac{2\sin t \text{ kg}}{\text{L}}\right)\left(\frac{500 \text{ L}}{\text{h}}\right) - \left(\frac{y(t) \text{ kg}}{1000 \text{ L}}\right)\left(\frac{500 \text{ L}}{\text{h}}\right)$$

$$\frac{dy}{dt} = \frac{1{,}000\sin t \text{ kg}}{\text{h}} - \frac{y(t) \text{ kg}}{2 \text{ h}}$$

Rewrite the equation in standard form of a linear equation.

$$y' + \frac{1}{2}y = 1{,}000\sin t$$

Then the integrating factor is

$$I(t) = e^{\int P(t)\ dt}$$

$$I(t) = e^{\int \frac{1}{2}\ dt}$$

$$I(t) = e^{\frac{t}{2}}$$

Then we can rewrite the linear differential equation as

$$\frac{d}{dt}(ye^{\frac{t}{2}}) = 1{,}000e^{\frac{t}{2}}\sin t$$

$$\int \frac{d}{dt}(ye^{\frac{t}{2}})\ dt = \int 1{,}000e^{\frac{t}{2}}\sin t\ dt$$

$$ye^{\frac{t}{2}} = 1{,}000\int e^{\frac{t}{2}}\sin t\ dt$$

Use integration by parts with $u = \sin t$, $du = \cos t\ dt$, $dv = e^{\frac{t}{2}}\ dt$, and $v = 2e^{\frac{t}{2}}$ to integrate the right side.

$$\int e^{\frac{t}{2}} \sin t \ dt = 2e^{\frac{t}{2}} \sin t - 2 \int e^{\frac{t}{2}} \cos t \ dt$$

Use integration by parts again, this time with $u = \cos t$, $du = -\sin t \ dt$, $dv = e^{\frac{t}{2}} \ dt$, and $v = 2e^{\frac{t}{2}}$ to integrate the right side.

$$\int e^{\frac{t}{2}} \sin t \ dt = 2e^{\frac{t}{2}} \sin t - 2 \left[2e^{\frac{t}{2}} \cos t + 2 \int e^{\frac{t}{2}} \sin t \ dt \right]$$

$$\int e^{\frac{t}{2}} \sin t \ dt = 2e^{\frac{t}{2}} \sin t - 4e^{\frac{t}{2}} \cos t - 4 \int e^{\frac{t}{2}} \sin t \ dt$$

$$5 \int e^{\frac{t}{2}} \sin t \ dt = 2e^{\frac{t}{2}} (\sin t - 2 \cos t)$$

$$\int e^{\frac{t}{2}} \sin t \ dt = \frac{2}{5} e^{\frac{t}{2}} (\sin t - 2 \cos t)$$

Substituting back in our equation, we get

$$ye^{\frac{t}{2}} = 1{,}000 \left(\frac{2}{5} e^{\frac{t}{2}} (\sin t - 2 \cos t) \right) + C$$

$$ye^{\frac{t}{2}} = 400e^{\frac{t}{2}} (\sin t - 2 \cos t) + C$$

$$y(t) = 400(\sin t - 2 \cos t) + Ce^{-\frac{t}{2}}$$

Since the tank was filled with fresh water, the initial amount of salt is given by $y(0) = 0$, so we get

$$0 = 400(\sin 0 - 2 \cos 0) + Ce^{-\frac{0}{2}}$$

$$0 = 400(0 - 2) + C$$

$$0 = -800 + C$$

$$C = 800$$

Plug this value back into the general solution to get the actual solution.

$$y(t) = 400(\sin t - 2\cos t) + 800e^{-\frac{t}{2}}$$

$$y(t) = 400\sin t - 800\cos t + 800e^{-\frac{t}{2}}$$

■ 5. A tank holding $2{,}000$ L of fresh water is being filled at 400 L/h with a brine solution with a concentration of $f(t) = 5 + \cos(t/5)$ kg/L. Assuming that the solution in the tank remains perfectly mixed and drains at a rate equal to the fill rate, how much salt is in the tank after t hours?

Solution:

From the question, we know

- $C_1 = 5 + \cos\left(\frac{t}{5}\right)$ kg/L because the brine solution being added to the tank has a salt concentration modeled by $5 + \cos(t/5)$ kg/L
- $r_1 = 400$ L/h
- $C_2 = \frac{y(t)\text{ kg}}{2{,}000\text{ L}}$ because with equal input and output rates, the amount of solution in the tank stays constant over time

- $r_2 = 400$ L/h

So the differential equation modeling the change in salt over time is

$$\frac{dy}{dt} = \left(5 + \cos\left(\frac{t}{5}\right)\right)\frac{\text{kg}}{\text{L}}\left(\frac{400\text{ L}}{\text{h}}\right) - \left(\frac{y(t)\text{ kg}}{2{,}000\text{ L}}\right)\left(\frac{400\text{ L}}{\text{h}}\right)$$

$$\frac{dy}{dt} = \left(2{,}000 + 400\cos\left(\frac{t}{5}\right)\right)\frac{\text{kg}}{\text{h}} - \frac{y(t)\text{ kg}}{5\text{h}}$$

Rewrite the equation in standard form of a linear equation.

$$y' + \frac{1}{5}y = 2{,}000 + 400\cos\left(\frac{t}{5}\right)$$

Then the integrating factor is

$$I(t) = e^{\int P(t)\ dt}$$

$$I(t) = e^{\int \frac{1}{5}\ dt}$$

$$I(t) = e^{\frac{t}{5}}$$

Then we can rewrite the linear differential equation as

$$\frac{d}{dt}(ye^{\frac{t}{5}}) = 2{,}000e^{\frac{t}{5}} + 400e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)$$

$$\int \frac{d}{dt}(ye^{\frac{t}{5}})\ dt = \int 2{,}000e^{\frac{t}{5}} + 400e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt$$

$$ye^{\frac{t}{5}} = 2{,}000\int e^{\frac{t}{5}}\ dt + 400\int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt$$

$$ye^{\frac{t}{5}} = 10{,}000e^{\frac{t}{5}} + 400\int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt$$

Use integration by parts with $u = \cos(t/5)$, $du = -(1/5)\sin(t/5)\ dt$, $dv = e^{\frac{t}{5}}\ dt$, and $v = 5e^{\frac{t}{5}}$ to integrate the right side.

$$\int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt = 5e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right) + \int e^{\frac{t}{5}}\sin\left(\frac{t}{5}\right)\ dt$$

Use integration by parts again, this time with $u = \sin(t/5)$, $du = (1/5)\cos(t/5)\ dt$, $dv = e^{\frac{t}{5}}\ dt$, and $v = 5e^{\frac{t}{5}}$ to integrate the right side.

$$\int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt = 5e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right) + \left[5e^{\frac{t}{5}}\sin\left(\frac{t}{5}\right) - \int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt\right]$$

$$2\int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt = 5e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right) + 5e^{\frac{t}{5}}\sin\left(\frac{t}{5}\right)$$

$$\int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt = \frac{5}{2}e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right) + \frac{5}{2}e^{\frac{t}{5}}\sin\left(\frac{t}{5}\right)$$

$$\int e^{\frac{t}{5}}\cos\left(\frac{t}{5}\right)\ dt = \frac{5}{2}e^{\frac{t}{5}}\left(\cos\left(\frac{t}{5}\right) + \sin\left(\frac{t}{5}\right)\right)$$

Substituting back in our equation, we get

$$ye^{\frac{t}{5}} = 10{,}000e^{\frac{t}{5}} + 400\left[\frac{5}{2}e^{\frac{t}{5}}\left(\cos\left(\frac{t}{5}\right) + \sin\left(\frac{t}{5}\right)\right)\right] + C$$

$$ye^{\frac{t}{5}} = 10{,}000e^{\frac{t}{5}} + 1{,}000e^{\frac{t}{5}}\left(\cos\left(\frac{t}{5}\right) + \sin\left(\frac{t}{5}\right)\right) + C$$

$$y = 10{,}000 + 1{,}000 \left(\cos\left(\frac{t}{5}\right) + \sin\left(\frac{t}{5}\right) \right) + Ce^{-\frac{t}{5}}$$

Since the tank was filled with fresh water, the initial amount of salt is given by $y(0) = 0$, so we get

$$0 = 10{,}000 + 1{,}000 \left(\cos\left(\frac{0}{5}\right) + \sin\left(\frac{0}{5}\right) \right) + Ce^{-\frac{0}{5}}$$

$$0 = 10{,}000 + 1{,}000 + C$$

$$C = -11{,}000$$

Plug this value back into the general solution to get the actual solution.

$$y = 10{,}000 + 1{,}000 \left(\cos\left(\frac{t}{5}\right) + \sin\left(\frac{t}{5}\right) \right) - 11{,}000e^{-\frac{t}{5}}$$

$$y = 1{,}000 \left[\sin\left(\frac{t}{5}\right) + \cos\left(\frac{t}{5}\right) - 11e^{-\frac{t}{5}} + 10 \right]$$

■ 6. A tank contains $3{,}000$ L of water and 600 kg of dissolved salt. Fresh water is entering the tank at 10 L/min, and the solution drains out at the same rate. Assuming the solution in the tank remains perfectly mixed, what is the function that models the amount of salt $y(t)$ at any given minute t?

Solution:

From the question, we know

- $C_1 = 0$ kg/L because the freshwater being added to the tank contains no salt

- $r_1 = 10$ L/min

- $C_2 = \frac{y(t) \text{ kg}}{3{,}000 \text{ L}}$ because with equal input and output rates, the amount of solution in the tank stays constant over time

- $r_2 = 10$ L/min

So the differential equation modeling the change in salt over time is

$$\frac{dy}{dt} = \left(\frac{0 \text{ kg}}{\text{L}}\right)\left(\frac{10 \text{ L}}{\text{min}}\right) - \left(\frac{y(t) \text{ kg}}{3{,}000 \text{ L}}\right)\left(\frac{10 \text{ L}}{\text{min}}\right)$$

$$\frac{dy}{dt} = -\frac{y(t) \text{ kg}}{300 \text{ min}}$$

Separate variables and integrate both sides.

$$\int \frac{1}{y}\ dy = \int -\frac{1}{300}\ dt$$

$$\ln|y| = -\frac{1}{300}t + C$$

$$e^{\ln|y|} = e^{-\frac{1}{300}t+C}$$

$$|y| = Ce^{-\frac{1}{300}t}$$

Then the general solution is

$$y(t) = Ce^{-\frac{1}{300}t}$$

Since the tank initially contained 600 kg of dissolved salt, we know $y(0) = 600$ is an initial condition that we can substitute into the general solution to solve for the constant of integration,

$$600 = Ce^{-\frac{1}{300}(0)}$$

$$C = 600$$

and then we can put this value back into the general solution.

$$y(t) = 600e^{-\frac{1}{300}t}$$

NEWTON'S LAW OF COOLING

1. The temperature of a soup dropped from $100^\circ C$ to $95^\circ C$ in 5 minutes. How long would it take for the temperature to drop to $40^\circ C$ if ambient temperature is $25^\circ C$?

Solution:

If we plug everything we know,

$T_0 = 100^\circ$ Initial temperature of the soup

$T_a = 25^\circ$ Temperature of the environment

$T(5) = 95^\circ$ At $t = 5$ minutes, the soup has cooled to 95°

into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 25 + (100 - 25)e^{-kt}$$

$$T(t) = 25 + 75e^{-kt}$$

Substitute the initial condition $T(5) = 95^\circ$,

$$95 = 25 + 75e^{-k(5)}$$

$$95 = 25 + 75e^{-5k}$$

$$70 = 75e^{-5k}$$

$$\frac{70}{75} = e^{-5k}$$

$$\ln\frac{14}{15} = -5k$$

$$k = -\frac{1}{5}\ln\frac{14}{15}$$

then plug k into the equation modeling temperature over time.

$$T(t) = 25 + 75e^{-\left(-\frac{1}{5}\ln\frac{14}{15}\right)t}$$

$$T(t) = 25 + 75e^{\left(\frac{1}{5}\ln\frac{14}{15}\right)t}$$

We want to find the time t at which the soup reaches 40°, so we'll substitute $T(t) = 40^\circ$.

$$40 = 25 + 75e^{\left(\frac{1}{5}\ln\frac{14}{15}\right)t}$$

$$15 = 75e^{\left(\frac{1}{5}\ln\frac{14}{15}\right)t}$$

$$\frac{15}{75} = e^{\left(\frac{1}{5}\ln\frac{14}{15}\right)t}$$

$$\frac{1}{5} = e^{\left(\frac{1}{5}\ln\frac{14}{15}\right)t}$$

$$\ln\frac{1}{5} = \frac{t}{5}\ln\frac{14}{15}$$

$$5\ln\frac{1}{5} = t\ln\frac{14}{15}$$

$$t = \frac{5 \ln \frac{1}{5}}{\ln \frac{14}{15}}$$

$$t \approx 116.64$$

The soup will cool from $100^\circ C$ to $40^\circ C$ in about 116.64 minutes.

■ 2. The temperature of a liquid dropped from $350^\circ C$ to $300^\circ C$ in 30 minutes. What temperature would the liquid reach in 3 hours if the temperature of the environment is $30^\circ C$?

Solution:

If we plug everything we know,

$T_0 = 350^\circ$	Initial temperature of the liquid
$T_a = 30^\circ$	Temperature of the environment
$T(1/2) = 300^\circ$	At $t = 1/2$ hour, the liquid has cooled to 300°

into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 30 + (350 - 30)e^{-kt}$$

$$T(t) = 30 + 320e^{-kt}$$

Substitute the initial condition $T(1/2) = 300^\circ$,

$$300 = 30 + 320e^{-\frac{1}{2}k}$$

$$270 = 320e^{-\frac{1}{2}k}$$

$$\frac{270}{320} = e^{-\frac{1}{2}k}$$

$$\ln\frac{27}{32} = -\frac{1}{2}k$$

$$k = -2\ln\frac{27}{32}$$

then plug k into the equation modeling temperature over time.

$$T(t) = 30 + 320e^{-\left(-2\ln\frac{27}{32}\right)t}$$

$$T(t) = 30 + 320e^{2t\ln\frac{27}{32}}$$

We want to the temperature decrease over 3 hours, so we'll evaluate at $t = 3$ to find $T(3)$.

$$T(3) = 30 + 320e^{2(3)\ln\frac{27}{32}}$$

$$T(3) = 30 + 320e^{6\ln\frac{27}{32}}$$

$$T(3) = 30 + 320e^{\ln\left(\frac{27}{32}\right)^6}$$

$$T(3) = 30 + 320\left(\frac{27}{32}\right)^6$$

$$T(3) \approx 145.46$$

In 3 hours, the temperature of the liquid will fall by $350^\circ C - 145^\circ C = 205^\circ C$.

■ 3. The temperature of ice cream in a freezer is -20°. After two minutes outside the freezer, where the temperature is 35°, the temperature of the ice cream increased to -10°. What will be the temperature of the ice cream after 10 minutes?

Solution:

If we plug everything we know,

$T_0 = -20^\circ$ Initial temperature of the ice cream

$T_a = 35^\circ$ Temperature of the environment

$T(2) = -10^\circ$ At $t = 2$ minutes, the ice cream has warmed to -10°

into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 35 + (-20 - 35)e^{-kt}$$

$$T(t) = 35 - 55e^{-kt}$$

Substitute the initial condition $T(2) = -10^\circ$,

$$-10 = 35 - 55e^{-2k}$$

$$-45 = -55e^{-2k}$$

$$\frac{9}{11} = e^{-2k}$$

$$\ln\frac{9}{11} = -2k$$

$$k = -\frac{1}{2}\ln\frac{9}{11}$$

then plug k into the equation modeling temperature over time.

$$T(t) = 35 - 55e^{-\left(-\frac{1}{2}\ln\frac{9}{11}\right)t}$$

$$T(t) = 35 - 55e^{\frac{1}{2}t\ln\frac{9}{11}}$$

We want to find the temperature of the ice cream after 10 minutes, so we'll evaluate at $t = 10$ to find $T(10)$.

$$T(10) = 35 - 55e^{\frac{1}{2}(10)\ln\frac{9}{11}}$$

$$T(10) = 35 - 55e^{5\ln\frac{9}{11}}$$

$$T(10) = 35 - 55e^{\ln\left(\frac{9}{11}\right)^5}$$

$$T(10) = 35 - 55\left(\frac{9}{11}\right)^5$$

$$T(10) \approx 15$$

The temperature of the ice cream after 10 minutes will be about 15°.

■ 4. The temperature in a room is 25°, while outside temperature is -20°. After 5 minutes with the door open, the room temperature dropped to 20°. What will the temperature be in the room after 15 minutes, and when will the room temperature drop to 5°?

Solution:

If we plug everything we know,

$T_0 = 25^\circ$	Initial temperature of the room
$T_a = -20^\circ$	Temperature of the environment
$T(5) = 20^\circ$	At $t = 5$ minutes, the room has cooled to 20°

If we plug everything we know into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = -20 + (25 - (-20))e^{-kt}$$

$$T(t) = -20 + (25 + 20)e^{-kt}$$

$$T(t) = -20 + 45e^{-kt}$$

Substitute the initial condition $T(5) = 20^\circ$,

$$20 = -20 + 45e^{-5k}$$

$$40 = 45e^{-5k}$$

$$\frac{8}{9} = e^{-5k}$$

$$\ln\frac{8}{9} = -5k$$

$$k = -\frac{1}{5}\ln\frac{8}{9}$$

then plug k into the equation modeling temperature over time.

$$T(t) = -20 + 45e^{-\left(-\frac{1}{5}\ln\frac{8}{9}\right)t}$$

$$T(t) = -20 + 45e^{\frac{1}{5}t\ln\frac{8}{9}}$$

We want to find the temperature of the room after 15 minutes, so we'll evaluate at $t = 15$ to find $T(15)$.

$$T(15) = -20 + 45e^{\frac{1}{5}(15)\ln\frac{8}{9}}$$

$$T(15) = -20 + 45e^{3\ln\frac{8}{9}}$$

$$T(15) = -20 + 45e^{\ln\left(\frac{8}{9}\right)^3}$$

$$T(15) = -20 + 45\left(\frac{8}{9}\right)^3$$

$$T(10) \approx 11.6^\circ$$

The room temperature after 15 minutes will be $11.6^\circ C$. To find the time t at which the room temperature drops to 5°, we'll substitute $T(t) = 5^\circ$.

$$5 = -20 + 45e^{\frac{1}{5}t\ln\frac{8}{9}}$$

$$25 = 45e^{\frac{1}{5}t\ln\frac{8}{9}}$$

$$\frac{25}{45} = e^{\frac{1}{5}t\ln\frac{8}{9}}$$

$$\frac{5}{9} = e^{\frac{1}{5}t\ln\frac{8}{9}}$$

$$\ln\frac{5}{9} = \frac{1}{5}t\ln\frac{8}{9}$$

$$5\ln\frac{5}{9} = t\ln\frac{8}{9}$$

$$t = \frac{5\ln\frac{5}{9}}{\ln\frac{8}{9}}$$

$$t \approx 25$$

The room temperature drops to 5° after 25 minutes.

■ 5. A piece of metal is placed in a $20^\circ C$ room, and after 5 minutes the temperature of the metal is $120^\circ C$. After another 15 minutes the metal's temperature is $75^\circ C$. What was the initial temperature of the metal?

Solution:

If we plug everything we know,

$T_a = 20^\circ$ Temperature of the environment

$T(5) = 120^\circ$ At $t = 5$ minutes, the metal has cooled to 120°

$T(20) = 75^\circ$ At $t = 20$ minutes, the metal has cooled to 75°

into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 20 + (T_0 - 20)e^{-kt}$$

Substitute the conditions $T(5) = 120^\circ$ and $T(20) = 75^\circ$.

$$120 = 20 + (T_0 - 20)e^{-5k}$$

$$75 = 20 + (T_0 - 20)e^{-20k}$$

Simplify the system to

$$100 = (T_0 - 20)e^{-5k}$$

$$55 = (T_0 - 20)e^{-20k}$$

Divide the first equation by the second equation.

$$\frac{100}{55} = \frac{e^{-5k}}{e^{-20k}}$$

$$\frac{20}{11} = e^{15k}$$

$$\ln \frac{20}{11} = 15k$$

$$k = \frac{1}{15} \ln \frac{20}{11}$$

Substitute this value for k into the first equation of the system.

$$100 = (T_0 - 20)e^{-5\left(\frac{1}{15}\ln\frac{20}{11}\right)}$$

$$100 = (T_0 - 20)e^{-\frac{1}{3}\ln\frac{20}{11}}$$

$$100 = (T_0 - 20)e^{\ln\left(\frac{20}{11}\right)^{-\frac{1}{3}}}$$

$$100 = (T_0 - 20)\left(\frac{20}{11}\right)^{-\frac{1}{3}}$$

$$100\left(\frac{20}{11}\right)^{\frac{1}{3}} = (T_0 - 20)$$

$$T_0 = 100\left(\frac{20}{11}\right)^{\frac{1}{3}} + 20$$

$$T_0 \approx 142^\circ C$$

The initial temperature of the metal was approximately $142^\circ C$.

■ 6. Room temperature can only be measured at temperatures less than or equal to $25^\circ C$. We can't currently measure room temperature, which means the room must be warmer than $25^\circ C$. Meanwhile, outside temperature is $-5^\circ C$. After 15 minutes with the window open, room temperature has dropped to $20^\circ C$, and to $17^\circ C$ after another 5 minutes. What was the initial temperature of the room?

Solution:

If we plug everything we know,

$T_a = -5^\circ$	Temperature of the environment
$T(15) = 20^\circ$	At $t = 15$ minutes, the room has cooled to 20°
$T(20) = 17^\circ$	At $t = 20$ minutes, the room has cooled to 17°

into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = -5 + (T_0 + 5)e^{-kt}$$

Substitute the conditions $T(15) = 20^\circ$ and $T(20) = 17^\circ$.

$$20 = -5 + (T_0 + 5)e^{-15k}$$

$$17 = -5 + (T_0 + 5)e^{-20k}$$

Simplify the system to

$$25 = (T_0 + 5)e^{-15k}$$

$$22 = (T_0 + 5)e^{-20k}$$

Divide the first equation by the second equation.

$$\frac{25}{22} = \frac{e^{-15k}}{e^{-20k}}$$

$$\frac{25}{22} = e^{5k}$$

$$\ln\frac{25}{22}=5k$$

$$k=\frac{1}{5}\ln\frac{25}{22}$$

Substitute this value for k into the first equation of the system.

$$25=(T_0+5)e^{-15\left(\frac{1}{5}\ln\frac{25}{22}\right)}$$

$$25=(T_0+5)e^{-3\ln\frac{25}{22}}$$

$$25=(T_0+5)e^{\ln\left(\frac{25}{22}\right)^{-3}}$$

$$25=(T_0+5)\left(\frac{25}{22}\right)^{-3}$$

$$25\left(\frac{25}{22}\right)^{3}=T_0+5$$

$$T_0=25\left(\frac{25}{22}\right)^{3}-5$$

$$T_0\approx 31.7^\circ C$$

The initial temperature of the room was approximately $31.7^\circ C$.

ELECTRICAL SERIES CIRCUITS

■ 1. If a battery supplies a constant voltage of 9 V, has an inductance of 3 h and a resistance of $3\ \Omega$, and assuming $i(0)=0$, find the current after 120 seconds.

Solution:

This series circuit doesn't have a capacitor, which means we can remove that term from Kirchhoff's second law.

$$L\frac{di}{dt}+Ri+\frac{1}{C}q=E(t)$$

$$L\frac{di}{dt}+Ri=E(t)$$

Plugging in what we know, we get

$$3\frac{di}{dt}+3i=9$$

$$i'+i=3$$

This is a linear differential equation with integrating factor

$$I(t)=e^{\int P\ dt}$$

$$I(t)=e^{\int 1\ dt}$$

$$I(t) = e^t$$

Therefore, we can rewrite the linear equation as

$$e^t \frac{di}{dt} + e^t i(t) = 3e^t$$

$$\frac{d}{dt}(e^t i(t)) = 3e^t$$

$$\int \frac{d}{dt}(e^t i(t))\ dt = \int 3e^t\ dt$$

$$e^t i(t) = 3e^t + C$$

$$i(t) = 3 + Ce^{-t}$$

Substituting the initial condition $i(0) = 0$, we find

$$0 = 3 + Ce^0$$

$$C = -3$$

So we can write the equation modeling the current i over time t as

$$i(t) = 3 - 3e^{-t}$$

Assuming time is measured in minutes, the current after 120 seconds is

$$i(2) = 3 - 3e^{-120}$$

$$i(2) = 3$$

■ 2. Given $L = 10$ h, $R = 70\ \Omega$, $C = 0.01$ f, $E(t) = 0$, $q(0) = 0$, and $i(0) = 3$, find the charge $q(t)$ on the capacitor.

Solution:

Plugging what we know into the second order equation given by Kirchhoff's second law, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$10q'' + 70q' + \frac{1}{0.01}q = 0$$

$$q'' + 7q' + 10q = 0$$

The associated characteristic equation for this second order homogeneous equation is

$$r^2 + 7r + 10 = 0$$

$$(r+5)(r+2) = 0$$

$$r = -5,\ -2$$

Because we found distinct real roots, we know the circuit is overdamped. With distinct real roots, we can say that the general solution and its derivative are given by

$$q(t) = c_1e^{-5t} + c_2e^{-2t}$$

$$q'(t) = i(t) = -5c_1e^{-5t} - 2c_2e^{-2t}$$

Substituting the initial conditions $q(0) = 3$ and $i(0) = 0$ into these equations, we get

$$0 = c_1 e^{-5(0)} + c_2 e^{-2(0)}$$

$$0 = c_1 + c_2$$

$$c_2 = -c_1$$

and

$$3 = -5c_1 e^{-5(0)} - 2c_2 e^{-2(0)}$$

$$3 = -5c_1 - 2c_2$$

Substituting $c_2 = -c_1$ into the second equation gives

$$3 = -5c_1 + 2c_1$$

$$3 = -3c_1$$

$$c_1 = -1$$

Plugging this back into the equation for c_2, we get

$$c_2 = -(-1)$$

$$c_2 = 1$$

So the general solution can be written as

$$q(t) = -e^{-5t} + e^{-2t}$$

3. An inductance of 2 h is connected in series with a resistance of $10\ \Omega$ and a battery giving $E(t)=120$ V. Initially, the current is zero. Formulate and solve an initial value problem modeling the electrical circuit.

Solution:

This series circuit doesn't have a capacitor, which means we can remove that term from Kirchhoff's second law.

$$L\frac{di}{dt}+Ri+\frac{1}{C}q=E(t)$$

$$L\frac{di}{dt}+Ri=E(t)$$

Plugging in what we know, we get

$$2\frac{di}{dt}+10i=120$$

$$i'+5i=60$$

This is a linear differential equation with integrating factor

$$I(t)=e^{\int P\ dt}$$

$$I(t)=e^{\int 5\ dt}$$

$$I(t)=e^{5t}$$

Therefore, we can rewrite the linear equation as

$$e^{5t}\frac{di}{dt} + 5e^{5t}i(t) = 60e^{5t}$$

$$\frac{d}{dt}(e^{5t}i(t)) = 60e^{5t}$$

$$\int \frac{d}{dt}(e^{5t}i(t))\ dt = \int 60e^{5t}\ dt$$

$$e^{5t}i(t) = 12e^{5t} + C$$

$$i(t) = 12 + Ce^{-5t}$$

Substituting the initial condition $i(0) = 0$, we find

$$0 = 12 + Ce^0$$

$$C = -12$$

So we can write the equation modeling the current i over time t as

$$i(t) = 12 - 12e^{-5t}$$

■ 4. Suppose $L = 2$ h, $R = 6\ \Omega$, $E = 24e^{3t}$ V, and $i(0) = 0$. Formulate and solve an initial value problem that models the electrical circuit.

Solution:

This series circuit doesn't have a capacitor, which means we can remove that term from Kirchhoff's second law.

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$$

$$L\frac{di}{dt} + Ri = E(t)$$

Plugging in what we know, we get

$$2\frac{di}{dt} + 6i = 24e^{3t}$$

$$i' + 3i = 12e^{3t}$$

This is a linear differential equation with integrating factor

$$I(t) = e^{\int P\ dt}$$

$$I(t) = e^{\int 3\ dt}$$

$$I(t) = e^{3t}$$

Therefore, we can rewrite the linear equation as

$$e^{3t}\frac{di}{dt} + 3e^{3t}i(t) = 12e^{3t}e^{3t}$$

$$\frac{d}{dt}(e^{3t}i(t)) = 12e^{6t}$$

$$\int \frac{d}{dt}(e^{3t}i(t))\ dt = \int 12e^{6t}\ dt$$

$$e^{3t}i(t) = 2e^{6t} + C$$

$$i(t) = 2e^{3t} + Ce^{-3t}$$

Substituting the initial condition $i(0) = 0$, we find

$$0 = 2e^0 + Ce^0$$

$$C = -2$$

So we can write the equation modeling the current i over time t as

$$i(t) = 2e^{3t} - 2e^{-3t}$$

■ 5. Suppose $L = 2$ h, $R = 40\ \Omega$, $C = 0.005$ f, $E = 100$ V, $q(0) = 9$ C, and $q'(0) = i(0) = 0$. Formulate and solve an initial value problem that models this LRC circuit.

Solution:

Plugging what we know into the second order equation given by Kirchhoff's second law, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$2q'' + 40q' + \frac{1}{0.005}q = 100$$

$$q'' + 20q' + 100q = 50$$

The associated characteristic equation for this second order homogeneous equation is

$$r^2 + 20r + 100 = 0$$

$$(r + 10)(r + 10) = 0$$

$$r = -10, \ -10$$

Because we found equal real roots, we know the circuit is critically damped. With equal real roots, we can say that the complementary solution and its derivative are given by

$$q_c(t) = c_1 e^{-10t} + c_2 t e^{-10t}$$

$$q_c'(t) = i(t) = -10c_1 e^{-10t} + c_2 e^{-10t} - 10tc_2 e^{-10t}$$

Substituting the initial conditions $q(0) = 9$ and $i(0) = 0$ into these equations, we get

$$9 = c_1 e^{-10(0)} + c_2(0)e^{-10(0)}$$

$$c_1 = 9$$

and

$$0 = -10c_1 e^{-10(0)} + c_2 e^{-10(0)} - 10(0)c_2 e^{-10(0)}$$

$$0 = -10c_1 + c_2$$

$$c_2 = -10c_1$$

$$c_2 = -10(9)$$

$$c_2 = -90$$

So the complementary solution can be written as

$$q_c(t) = 9e^{-10t} - 90te^{-10t}$$

Our guess for the particular solution will be

$$y_p(x) = A$$

Taking the first and second derivatives of this guess, we get

$$y_p'(x) = 0$$

$$y_p''(x) = 0$$

Plugging the first two derivatives into the original differential equation, we get

$$0 + 20(0) + 100A = 50$$

$$A = \frac{1}{2}$$

$$y_p(x) = \frac{1}{2}$$

Putting this particular solution together with the complementary solution gives us the general solution to the differential equation.

$$q(t) = 9e^{-10t} - 90te^{-10t} + \frac{1}{2}$$

6. An LRC circuit is set up with an inductance of $1/2$ h, a resistance of $1\ \Omega$, and a capacitance of $2/5$ f. Assuming the initial charge is 2 C and the initial current is 6 A, find the solution function describing the charge on the capacitor at any time t.

Solution:

Plugging what we know into the second order equation given by Kirchhoff's second law, we get

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

$$\frac{1}{2}q''+q'+\frac{5}{2}q=0$$

$$q''+2q'+5q=0$$

The associated characteristic equation for this second order homogeneous equation is

$$r^2+2r+5=0$$

$$r=-1\pm 2i$$

Because we found complex conjugate roots, we know the circuit is underdamped. With complex roots, we can say that the general solution and its derivative are given by

$$q(t)=e^{-t}(c_1\cos(2t)+c_2\sin(2t))$$

$$q'(t) = i(t) = -e^{-t}(c_1 \cos(2t) + c_2 \sin(2t)) + e^{-t}(-2c_1 \sin(2t) + 2c_2 \cos(2t))$$

$$= e^{-t}((2c_2 - c_1)\cos(2t) - (c_2 + 2c_1)\sin(2t))$$

Substituting the initial conditions $q(0) = 2$ and $i(0) = 6$ into these equations, we get

$$2 = e^{-(0)}(c_1 \cos(2(0)) + c_2 \sin(2(0)))$$

$$2 = c_1$$

and

$$6 = e^{-0}((2c_2 - c_1)\cos(2(0)) - (c_2 + 2c_1)\sin(2(0)))$$

$$6 = 2c_2 - c_1$$

$$6 = 2c_2 - 2$$

$$c_2 = 4$$

So the general solution can be written as

$$q(t) = e^{-t}(2 \cos(2t) + 4 \sin(2t))$$

SPRING AND MASS SYSTEMS

■ 1. Find the movement of a 6 kg mass attached to a spring with spring constant $k = 3$ kg/s^2 and damping constant $\beta = 6$ kg/s, given $x(0) = 2$ and $x'(0) = 0$.

Solution:

The differential equation for this system is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

with $m = 6$, $k = 3$, and $\beta = 6$, where $2\lambda = \beta/m$ and $\omega^2 = k/m$. The value of λ is given by

$$2\lambda = \frac{\beta}{m}$$

$$\lambda = \frac{\beta}{2m}$$

$$\lambda = \frac{6}{2(6)} = \frac{1}{2}$$

and ω^2 is given by

$$\omega^2 = \frac{k}{m}$$

$$\omega^2 = \frac{3}{6} = \frac{1}{2}$$

The roots of the associated characteristic equation, $r^2 + 2\lambda r + \omega^2 = 0$, are

$$r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

$$r_{1,2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{1}{2}}$$

$$r_{1,2} = -\frac{1}{2} \pm i\frac{1}{2}$$

Because we find roots in which $\lambda^2 - \omega^2 < 0$, the spring and mass system is underdamped and the general solution and its derivative are

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}t + c_2 \sin\sqrt{\omega^2 - \lambda^2}t)$$

$$x(t) = e^{-\frac{1}{2}t}\left(c_1 \cos\frac{1}{2}t + c_2 \sin\frac{1}{2}t\right)$$

$$x'(t) = -\frac{1}{2}e^{-\frac{1}{2}t}\left(c_1 \cos\frac{t}{2} + c_2 \sin\frac{t}{2}\right) + e^{-\frac{1}{2}t}\left(-\frac{1}{2}c_1 \sin\frac{t}{2} + \frac{1}{2}c_2 \cos\frac{t}{2}\right)$$

$$= e^{-\frac{1}{2}t}\left(\left(-\frac{1}{2}c_1 + \frac{1}{2}c_2\right)\cos\frac{t}{2} + \left(-\frac{1}{2}c_1 - \frac{1}{2}c_2\right)\sin\frac{t}{2}\right)$$

The question states that the initial position is $x(0) = 2$ and the initial velocity is $x'(0) = 0$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$2 = e^{-\frac{1}{2}(0)}\left(c_1 \cos\frac{1}{2}(0) + c_2 \sin\frac{1}{2}(0)\right)$$

$$c_1 = 2$$

and

$$0 = e^{-\frac{1}{2}(0)}\left(\left(-\frac{1}{2}(2) + \frac{1}{2}c_2\right)\cos\frac{0}{2} + \left(-\frac{1}{2}(2) - \frac{1}{2}c_2\right)\sin\frac{0}{2}\right)$$

$$0 = -1 + \frac{1}{2}c_2$$

$$c_2 = 2$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = e^{-\frac{t}{2}}\left(2\cos\frac{t}{2} + 2\sin\frac{t}{2}\right)$$

$$x(t) = 2\sin\left(\frac{t}{2}\right) - 3\cos\left(\frac{t}{2}\right)$$

■ 2. Assume an object weighing 8 lbs stretches a spring 12 in. Find the equation of motion if the spring is released from equilibrium with an upward velocity of $\sqrt{2}$ ft/sec.

Solution:

First we'll use Hooke's Law to find the spring constant k.

$$F = ks$$

$$8 = k(1)$$

$$k = 8$$

Now we'll use $W = mg$ to convert the weight into mass.

$$W = mg$$

$$8 = m(32)$$

$$m = \frac{1}{4}$$

Plugging everything we have into the second order equation, we get

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{8}{\frac{1}{4}}x = 0$$

$$\frac{d^2x}{dt^2} + 32x = 0$$

From this equation, we see that $\omega = \sqrt{32} = 4\sqrt{2}$, which means that the general solution and its derivative are

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$x(t) = c_1 \cos(4\sqrt{2}t) + c_2 \sin(4\sqrt{2}t)$$

and

$$x'(t) = -4\sqrt{2}c_1\sin(4\sqrt{2}t) + 4\sqrt{2}c_2\cos(4\sqrt{2}t)$$

The question states that the initial position is $x(0) = 0$ and the initial velocity is $x'(0) = -\sqrt{2}$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$0 = c_1\cos(4\sqrt{2}(0)) + c_2\sin(4\sqrt{2}(0))$$

$$c_1 = 0$$

and

$$-\sqrt{2} = -4\sqrt{2}(0)\sin(4\sqrt{2}(0)) + 4\sqrt{2}c_2\cos(4\sqrt{2}(0))$$

$$c_2 = -\frac{1}{4}$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = (0)\cos(4\sqrt{2}t) - \frac{1}{4}\sin(4\sqrt{2}t)$$

$$x(t) = -\frac{1}{4}\sin(4\sqrt{2}t)$$

■ 3. An 8 lb weight is attached to an 11 ft spring. When the mass comes to rest in the equilibrium position, the spring measures 15 ft. The system is immersed in a medium that imparts a damping force equal to $3/2$ times the instantaneous velocity of the mass. Find the equation of motion if the mass is pushed upward from the equilibrium position with an initial upward velocity of 4 ft/sec. What is the position of the mass after 15 sec?

Solution:

Hooke's Law tells us that the spring constant is

$$F = ks$$

$$8 = k(15 - 11)$$

$$8 = k(4)$$

$$k = 2 \text{ lb/ft}$$

Now we'll use $W = mg$ to convert the weight into mass.

$$W = mg$$

$$8 = m(32)$$

$$m = \frac{1}{4} \text{ slug}$$

With the spring constant and the mass, and $\beta = 3/2$ since the damping force is $3/2$ times the instantaneous velocity, we can plug everything into the differential equation.

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{\frac{3}{2}}{\frac{1}{4}}\left(\frac{dx}{dt}\right) + \frac{2}{\frac{1}{4}}x = 0$$

$$\frac{d^2x}{dt^2} + 6\left(\frac{dx}{dt}\right) + 8x = 0$$

The associated characteristic equation and its roots are

$$r^2 + 6r + 8 = 0$$

$$r = -4, -2$$

Because we find roots in which $\lambda^2 - \omega^2 > 0$, the spring and mass system is overdamped and the general solution and its derivative are

$$x(t) = c_1e^{-4t} + c_2e^{-2t}$$

and

$$x'(t) = -4c_1e^{-4t} - 2c_2e^{-2t}$$

The question states that the initial position is $x(0) = 0$ and the initial velocity is $x'(0) = -4$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$0 = c_1e^{-4(0)} + c_2e^{-2(0)}$$

$$c_1 = -c_2$$

and

$$-4 = -4c_1e^{-4(0)} - 2c_2e^{-2(0)}$$

$$-4 = -4c_1 - 2c_2$$

$$-4 = 4c_2 - 2c_2$$

$$-4 = 2c_2$$

$$c_2 = -2$$

$$c_1 = 2$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = 2e^{-4t} - 2e^{-2t}$$

After 15 seconds the mass is at a position of

$$x(15) = 2e^{-4(15)} - 2e^{-2(15)} \approx -1.9 \times 10^{-13} \approx 0$$

so it's at equilibrium.

■ 4. A 2 kg mass stretches a spring with a length of 20 cm. The system is attached to a dashpot that creates a damping force equal to 28 times the instantaneous velocity of the mass. Find the motion equation if the mass is released from rest at a point 6 cm below equilibrium.

Solution:

Hooke's Law tells us that the spring constant is

$$F = ks$$

$$2(9.8) = k(0.2)$$

$$k = 98 \text{ lb/ft}$$

With the spring constant and the mass, and $\beta = 28$ since the damping force is 28 times the instantaneous velocity, we can plug everything into the differential equation.

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{28}{2}\left(\frac{dx}{dt}\right) + \frac{98}{2}x = 0$$

$$\frac{d^2x}{dt^2} + 14\left(\frac{dx}{dt}\right) + 49x = 0$$

The associated characteristic equation and its roots are

$$r^2 + 14r + 49 = 0$$

$$r = -7, -7$$

Because we find roots in which $\lambda^2 - \omega^2 = 0$, the spring and mass system is critically damped and the general solution and its derivative are

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

$$x(t) = e^{-7t}(c_1 + c_2 t)$$

and

$$x'(t) = -7e^{-7t}c_1 - 7te^{-7t}c_2 + e^{-7t}c_2$$

The question states that the initial position is $x(0) = 6$ and the initial velocity is $x'(0) = 0$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$6 = e^{-7(0)}(c_1 + c_2(0))$$

$$c_1 = 6$$

and

$$0 = -7e^{-7(0)}c_1 - 7(0)e^{-7(0)}c_2 + e^{-7(0)}c_2$$

$$0 = -7c_1 + c_2$$

$$0 = -7(6) + c_2$$

$$c_2 = 42$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = e^{-7t}(6 + 42t)$$

■ 5. A 16 lb weight stretches a spring 2 ft. Assume the damping force on the system is equal to the instantaneous velocity of the mass. Find the equation of motion of the mass.

Solution:

Hooke's Law tells us that the spring constant is

$$F = ks$$

$$16 = k(2)$$

$$k = 8 \text{ lb/ft}$$

Now we'll use $W = mg$ to convert the weight into mass.

$$W = mg$$

$$16 = m(32)$$

$$m = \frac{1}{2}$$

With the spring constant and the mass, and $\beta = 1$ since the damping force is equal to the instantaneous velocity, we can plug everything into the differential equation.

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{1}{\frac{1}{2}}\left(\frac{dx}{dt}\right) + \frac{8}{\frac{1}{2}}x = 0$$

$$\frac{d^2x}{dt^2} + 2\left(\frac{dx}{dt}\right) + 16x = 0$$

The associated characteristic equation and its roots are

$$r^2 + 2r + 16 = 0$$

$$r = -1 \pm i\sqrt{15}$$

Because we find roots in which $\lambda^2 - \omega^2 < 0$, the spring and mass system is underdamped and the general solution is

$$x(t) = e^{-\lambda t}(c_1 \cos \sqrt{\omega^2 - \lambda^2} t + c_2 \sin \sqrt{\omega^2 - \lambda^2} t)$$

$$x(t) = e^{-t}(c_1 \cos \sqrt{15} t + c_2 \sin \sqrt{15} t)$$

■ 6. A mass of 9.8 kg stretches a spring 2.45 m. Find the equation that models the motion of the mass if we release the mass when $t = 0$ from a position 5 m above equilibrium, with a downward velocity of 2 m/s.

Solution:

First we'll use Hooke's Law to find the spring constant k.

$$F = ks$$

$$9.8 = k(2.45)$$

$$k = 4$$

Now we'll use $W = mg$ to convert the weight into mass.

$$W = mg$$

$$9.8 = m(9.8)$$

$$m = 1$$

Plugging everything we have into the second order equation, we get

$$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{4}{1}x = 0$$

$$\frac{d^2x}{dt^2} + 4x = 0$$

From this equation, we see that $\omega = \sqrt{4} = 2$, which means that the general solution and its derivative are

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$x(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

and

$$x'(t) = -2c_1 \sin(2t) + 2c_2 \cos(2t)$$

The question states that the initial position is $x(0) = 5$ and the initial velocity is $x'(0) = -2$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$5 = c_1 \cos(2(0)) + c_2 \sin(2(0))$$

$$c_1 = 5$$

and

$$-2 = -2c_1 \sin(2(0)) + 2c_2 \cos(2(0))$$

$$c_2 = -1$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = 5\cos(2t) - \sin(2t)$$

POWER SERIES BASICS

■ 1. Where is the power series centered?

$$\sum_{n=0}^{\infty} c_n(x+3)^n = c_0 + c_1(x+3) + c_2(x+3)^2 + c_3(x+3)^3 + \dots$$

Solution:

A power series centered at x_0 is given by

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \dots$$

We also call this a power series in $x-x_0$. In the given series,

$$\sum_{n=0}^{\infty} c_n(x+3)^n = c_0 + c_1(x+3) + c_2(x+3)^2 + c_3(x+3)^3 + \dots$$

we can identify $x_0 = -3$, and say that the power series is centered at $x_0 = -3$.

■ 2. Find the radius of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{3^{2n}}{100^n}(10x)^n$$

Solution:

Let's rewrite the series.

$$\sum_{n=1}^{\infty}\left(\frac{3^2}{100}\right)^n 10^n x^n$$

$$\sum_{n=1}^{\infty}\left(\frac{3^2}{100}\cdot 10\right)^n x^n$$

$$\sum_{n=1}^{\infty}\left(\frac{9}{10}\right)^n x^n$$

Then applying the ratio test to the power series gives

$$L=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{\left(\frac{9}{10}\right)^{n+1}x^{n+1}}{\left(\frac{9}{10}\right)^n x^n}\right|$$

$$L=\lim_{n\to\infty}\left|\frac{9}{10}x\right|$$

$$L=\frac{9}{10}|x|$$

To determine where the series converges absolutely, we set up the inequality $L < 1$.

$$\frac{9}{10}|x| < 1$$

$$|x| < \frac{10}{9}$$

$$-\frac{10}{9} < x < \frac{10}{9}$$

So the radius of convergence is $R = 10/9$.

■ 3. Find the Maclaurin series representation of $f(x) = \cosh(9x^2)$.

Solution:

Because we already have the Maclaurin series representation of $f(x) = \cosh x$, we can simply replace x with $9x^2$.

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$$

$$\cosh(9x^2) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (9x^2)^{2n}$$

$$\cosh(9x^2) = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (9^{2n} x^{4n})$$

$$\cosh(9x^2) = \sum_{n=0}^{\infty} \frac{9^{2n}}{(2n)!} x^{4n}$$

■ 4. Find the Maclaurin series representation of the function.

$$f(x) = 4\ln(-2 - 3x^3)$$

Solution:

Because we already have the Maclaurin series representation of $f(x) = \ln(1 + x)$, we can start by rewriting the argument so that it starts with positive 1.

$$f(x) = 4\ln(-2 + 3 - 3 - 3x^3)$$

$$f(x) = 4\ln(1 - 3 - 3x^3)$$

$$f(x) = 4\ln(1 + (-3 - 3x^3))$$

Now we'll replace x with $-3 - 3x^3$ in the Maclaurin series representation of $f(x) = \ln(1 + x)$.

$$\ln(1 + x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$$

$$\ln(1 + (-3 - 3x^3)) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (-3 - 3x^3)^n$$

Now we'll multiply by 4 and simplify the final Maclaurin series representation.

$$4\ln(1+(-3-3x^3)) = 4\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(-3-3x^3)^n$$

$$4\ln(-2-3x^3) = \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n}((-1)(3+3x^3))^n$$

$$4\ln(-2-3x^3) = \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n}(-1)^n(3+3x^3)^n$$

$$4\ln(-2-3x^3) = \sum_{n=1}^{\infty}\frac{4(-1)^{2n+1}}{n}(3+3x^3)^n$$

The exponent $2n+1$ will create the a $-$, $+$, $-$, $+$, ... pattern, but the simpler exponent n accomplishes the same thing.

$$4\ln(-2-3x^3) = \sum_{n=1}^{\infty}\frac{4(-1)^{n}}{n}(3+3x^3)^n$$

■ 5. Find the Maclaurin series representation of the function.

$$f(x) = \frac{6}{\sqrt{2x-1}}$$

Solution:

Because we already have the Maclaurin series representation of $f(x) = 1/(1 - x)$, we can start by rewriting the function so that the denominator starts with positive 1.

$$f(x) = -\frac{6}{1 - \sqrt{2x}}$$

Now we'll replace x with $\sqrt{2x}$ in the Maclaurin series representation of $f(x) = 1/(1 - x)$.

$$\frac{1}{1 - x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1 - \sqrt{2x}} = \sum_{n=0}^{\infty} (\sqrt{2x})^n$$

Now we'll multiply by -6 and simplify the final Maclaurin series representation.

$$-\frac{6}{1 - \sqrt{2x}} = -6\sum_{n=0}^{\infty} (\sqrt{2x})^n$$

$$-\frac{6}{1 - \sqrt{2x}} = -\sum_{n=0}^{\infty} 6((2x)^{\frac{1}{2}})^n$$

$$-\frac{6}{1 - \sqrt{2x}} = -\sum_{n=0}^{\infty} 6(2x)^{\frac{n}{2}}$$

■ 6. Find the first three derivatives of the power series.

$$f(x) = \sum_{n=0}^{\infty} 3(-1)^n(2x^2)^n$$

Solution:

To differentiate the power series representation, we can apply power rule for derivatives.

$$f(x) = \sum_{n=0}^{\infty} 3(-1)^n(2x^2)^n$$

$$f'(x) = \sum_{n=0}^{\infty} 3n(-1)^n(2x^2)^{n-1}(4x)$$

$$f'(x) = \sum_{n=0}^{\infty} 12n(-1)^n x(2x^2)^{n-1}$$

To find the second derivative, we can apply power rule again.

$$f''(x) = \sum_{n=0}^{\infty} 12n(n-1)(-1)^n x(2x^2)^{n-2}(4x)$$

$$f''(x) = \sum_{n=0}^{\infty} 48n(n-1)(-1)^n x^2(2x^2)^{n-2}$$

ADDING POWER SERIES

■ 1. Are the power series in phase? Do the indices match?

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} x^{2n}$$

Solution:

The power series are in phase because both series begin with the x^0 term. But the indices don't match. Therefore, if we wanted to add the series, we'd need to make their indices match by substituting $k = n - 2$ and $n = k + 2$ into the first series, while substituting $k = 2n$ and $n = k/2$ into the second series.

■ 2. As written, can the power series be added?

$$\sum_{n=0}^{\infty} n!x^n \qquad 5 + \sum_{n=1}^{\infty} \frac{3^n}{n!} x^n$$

Solution:

As written, the series can't be added. They aren't in phase because the first series begins with the x^0 term, while the second series begins with the x^1 term. Furthermore, the indices of the series don't match.

To set up the series for addition, we'd pull the $n=0$ term out of the first series to put the series in phase. Then we'd make the substitution $n=k$ into both series to make the indices match at $k=1$.

■ 3. Find the sum.

$$\sum_{n=3}^{\infty}(2n-1)c_nx^{n-3}+\sum_{n=1}^{\infty}2nc_nx^{n-1}$$

Solution:

Both series begin with the x^0 term, so they're already in phase. To make their indices match, we'll make the substitution $k=n-3$ and $n=k+3$ into the first series, and make the substitution $k=n-1$ and $n=k+1$ into the second series.

$$\sum_{n=3}^{\infty}(2n-1)c_nx^{n-3}+\sum_{n=1}^{\infty}2nc_nx^{n-1}$$

$$\sum_{k=0}^{\infty}(2(k+3)-1)c_{k+3}x^k+\sum_{k=0}^{\infty}2(k+1)c_{k+1}x^k$$

The series are now in phase with matching indices, so we'll add them and then factor out x^k.

$$\sum_{k=0}^{\infty} (2(k+3)-1)c_{k+3}x^k + 2(k+1)c_{k+1}x^k$$

$$\sum_{k=0}^{\infty} [(2(k+3)-1)c_{k+3} + 2(k+1)c_{k+1}]x^k$$

4. Add the power series.

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \qquad x\sum_{n=0}^{\infty} c_n x^n$$

Solution:

Rewrite the second series by bringing the x into the series representation.

$$x\sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{n=0}^{\infty} c_n x^{n+1}$$

Now we can write the sum as

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

The first series begins with the x^0 term, while the second series begins with the x^1 term, so the series aren't in phase. To put them in phase, we'll pull the x^0 term out of the first series.

$$c_2 2(2-1)x^{2-2} + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$2c_2 + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

The series are now in phase, but their indices don't match. To make the indices match, we'll make the substitution $k = n - 2$ and $n = k + 2$ into the first series, and make the substitution $k = n + 1$ and $n = k - 1$ into the second series.

$$2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+2-1)x^k + \sum_{k=1}^{\infty} c_{k-1}x^k$$

The series are now in phase with matching indices, so we'll add them and then factor out x^k.

$$2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+2-1)x^k + c_{k-1}x^k$$

$$2c_2 + \sum_{k=1}^{\infty} [c_{k+2}(k+2)(k+2-1) + c_{k-1}]x^k$$

■ 5. Find the sum.

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + x\sum_{n=1}^{\infty} nc_nx^{n-1} - x\sum_{n=0}^{\infty} c_nx^n$$

Solution:

Rewrite the first series by splitting it into two series.

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + x\sum_{n=1}^{\infty} nc_nx^{n-1} - x\sum_{n=0}^{\infty} c_nx^n$$

$$x^2\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + x\sum_{n=1}^{\infty} nc_nx^{n-1} - x\sum_{n=0}^{\infty} c_nx^n$$

Now rewrite the first series by bringing the x^2 into the series representation, and rewrite the third and fourth series by bringing the x into the series representation.

$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2+2} + \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + \sum_{n=1}^{\infty} nc_nx^{n-1+1} - \sum_{n=0}^{\infty} c_nx^{n+1}$$

$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n} + \sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} + \sum_{n=1}^{\infty} nc_nx^{n} - \sum_{n=0}^{\infty} c_nx^{n+1}$$

The first series begins with the x^2 term, the second series begins with the x^0 term, and the third and fourth series begin with the x^1 term, so the series aren't in phase. To put them in phase, we'll pull the x^0 and x^1 terms out of the second series, and we'll pull the x^1 term out of the third and fourth series.

$$\sum_{n=2}^{\infty} n(n-1)c_nx^n + 2(2-1)c_2x^{2-2} + 3(3-1)c_3x^{3-2} + \sum_{n=4}^{\infty} n(n-1)c_nx^{n-2}$$

$$+1c_1x^1 + \sum_{n=2}^{\infty} nc_nx^n - c_0x^{0+1} - \sum_{n=1}^{\infty} c_nx^{n+1}$$

$$2c_2 + (6c_3 + c_1 - c_0)x + \sum_{n=2}^{\infty} n(n-1)c_nx^n + \sum_{n=4}^{\infty} n(n-1)c_nx^{n-2}$$

$$+\sum_{n=2}^{\infty} nc_nx^n - \sum_{n=1}^{\infty} c_nx^{n+1}$$

The series are now in phase, but their indices don't match. To make the indices match, we'll make the substitution $k = n$ into the first and third series, make the substitution $k = n - 2$ and $n = k + 2$ into the second series, and make the substitution $k = n + 1$ and $n = k - 1$ into the fourth series.

$$2c_2 + (6c_3 + c_1 - c_0)x + \sum_{k=2}^{\infty} k(k-1)c_kx^k + \sum_{k=2}^{\infty} (k+2)(k+1)c_{k+2}x^k$$

$$+\sum_{k=2}^{\infty} kc_kx^k - \sum_{k=2}^{\infty} c_{k-1}x^k$$

The series are now in phase with matching indices, so we'll add them and then factor out x^k.

$$2c_2 + (6c_3 + c_1 - c_0)x + \sum_{k=2}^{\infty} k(k-1)c_kx^k + (k+2)(k+1)c_{k+2}x^k + kc_kx^k - c_{k-1}x^k$$

$$2c_2 + (6c_3 + c_1 - c_0)x + \sum_{k=2}^{\infty} [k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_{k-1}]x^k$$

$$2c_2 + (6c_3 + c_1 - c_0)x + \sum_{k=2}^{\infty} [(k+2)(k+1)c_{k+2} + k^2 c_k - c_{k-1}]x^k$$

■ 6. Find the sum.

$$3x^2 \sum_{n=0}^{\infty} (n-2)(n-1)nc_n x^n - (x^2 - 2) \sum_{n=1}^{\infty} c_n x^{n+1} + (x+1) \sum_{n=2}^{\infty} n(n+1)c_n x^{n-1}$$

Solution:

Rewrite the second series by splitting it into two series.

$$3x^2 \sum_{n=0}^{\infty} (n-2)(n-1)nc_n x^n - x^2 \sum_{n=1}^{\infty} c_n x^{n+1} + 2 \sum_{n=1}^{\infty} c_n x^{n+1}$$

$$+(x+1) \sum_{n=2}^{\infty} n(n+1)c_n x^{n-1}$$

Rewrite the last series by splitting it into two series.

$$3x^2 \sum_{n=0}^{\infty} (n-2)(n-1)nc_n x^n - x^2 \sum_{n=1}^{\infty} c_n x^{n+1} + 2 \sum_{n=1}^{\infty} c_n x^{n+1}$$

$$+x \sum_{n=2}^{\infty} n(n+1)c_n x^{n-1} + \sum_{n=2}^{\infty} n(n+1)c_n x^{n-1}$$

Bring the coefficients into each series.

$$\sum_{n=0}^{\infty} 3(n-2)(n-1)nc_nx^{n+2} - \sum_{n=1}^{\infty} c_nx^{n+3} + \sum_{n=1}^{\infty} 2c_nx^{n+1}$$

$$+\sum_{n=2}^{\infty} n(n+1)c_nx^n + \sum_{n=2}^{\infty} n(n+1)c_nx^{n-1}$$

The first, third, and fourth series begin with the x^2 term, the second series begins with the x^4 term, and the fifth series begins with the x^1 term, so the series aren't in phase. To put them in phase, we'll pull all the x^0, x^1, x^2, and x^3 terms from each series.

$$3(0-2)(0-1)(0)c_0x^{0+2} + 3(1-2)(1-1)(1)c_1x^{1+2} + \sum_{n=2}^{\infty} 3(n-2)(n-1)nc_nx^{n+2}$$

$$-\sum_{n=1}^{\infty} c_nx^{n+3}$$

$$+2c_1x^{1+1} + 2c_2x^{2+1} + \sum_{n=3}^{\infty} 2c_nx^{n+1}$$

$$+2(2+1)c_2x^2 + 3(3+1)c_3x^3 + \sum_{n=4}^{\infty} n(n+1)c_nx^n$$

$$+2(2+1)c_2x^{2-1} + 3(3+1)c_3x^{3-1} + 4(4+1)c_4x^{4-1} + \sum_{n=5}^{\infty} n(n+1)c_nx^{n-1}$$

$$\sum_{n=2}^{\infty} 3(n-2)(n-1)nc_nx^{n+2} - \sum_{n=1}^{\infty} c_nx^{n+3} + 2c_1x^2 + 2c_2x^3 + \sum_{n=3}^{\infty} 2c_nx^{n+1}$$

$$+6c_2x^2 + 12c_3x^3 + \sum_{n=4}^{\infty} n(n+1)c_nx^n$$

$$+6c_2x + 12c_3x^2 + 20c_4x^3 + \sum_{n=5}^{\infty} n(n+1)c_nx^{n-1}$$

$$6c_2x + (2c_1 + 6c_2 + 12c_3)x^2 + (2c_2 + 12c_3 + 20c_4)x^3$$

$$+\sum_{n=2}^{\infty} 3(n-2)(n-1)nc_nx^{n+2} - \sum_{n=1}^{\infty} c_nx^{n+3} + \sum_{n=3}^{\infty} 2c_nx^{n+1}$$

$$+\sum_{n=4}^{\infty} n(n+1)c_nx^n + \sum_{n=5}^{\infty} n(n+1)c_nx^{n-1}$$

The series are now in phase, but their indices don't match. To make the indices match, we'll make the substitution $k = n + 2$ and $n = k - 2$ into the first series, make the substitution $k = n + 3$ and $n = k - 3$ into the second series, make the substitution $k = n + 1$ and $n = k - 1$ into the third series, make the substitution $k = n$ into the fourth series, and make the substitution $k = n - 1$ and $n = k + 1$ into the fifth series.

$$6c_2x + (2c_1 + 6c_2 + 12c_3)x^2 + (2c_2 + 12c_3 + 20c_4)x^3$$

$$+\sum_{k=4}^{\infty} 3(k-4)(k-3)(k-2)c_{k-2}x^k - \sum_{k=4}^{\infty} c_{k-3}x^k + \sum_{k=4}^{\infty} 2c_{k-1}x^k$$

$$+\sum_{k=4}^{\infty} k(k+1)c_kx^k + \sum_{k=4}^{\infty} (k+1)(k+2)c_{k+1}x^k$$

The series are now in phase with matching indices, so we'll add them and then factor out x^k.

$$6c_2x + (2c_1 + 6c_2 + 12c_3)x^2 + (2c_2 + 12c_3 + 20c_4)x^3$$

$$+\sum_{k=4}^{\infty}[3(k-4)(k-3)(k-2)c_{k-2}-c_{k-3}+2c_{k-1}+k(k+1)c_k$$

$$+(k+1)(k+2)c_{k+1}]x^k$$

$$6c_2x+(2c_1+6c_2+12c_3)x^2+(2c_2+12c_3+20c_4)x^3$$

$$+\sum_{k=4}^{\infty}[(k+1)(k+2)c_{k+1}+k(k+1)c_k+2c_{k-1}$$

$$+3(k-4)(k-3)(k-2)c_{k-2}-c_{k-3}]x^k$$

POWER SERIES SOLUTIONS

■ 1. Is $x_0 = 4$ an ordinary point of the differential equation?

$$\frac{1}{x-4}y'' + (x^2+2)y' + 3y = 0$$

Solution:

No, $x_0 = 4$ is a singular point of the differential equation, because the coefficient on y'' is undefined there. In other words, $1/(x-4)$ is undefined at $x_0 = 4$.

■ 2. Can we find a power series solution to the differential equation around $x_0 = -1$?

$$(x-1)y'' + xy' + (x^2+2)y = \sqrt{x-1}$$

Solution:

Even though all of the polynomial coefficients on the left side of the equation are all defined at $x_0 = -1$, the function on the right side of the equation, $g(x) = \sqrt{x-1}$, is undefined there, which means we won't be able to find a power series solution to the differential equation at that point.

■ 3. Find the minimum radius of convergence of a power series solution of the differential equation about the ordinary point $x_0 = -2$.

$$x^3y'' + (2x + 1)y' - y = 0$$

Solution:

The singular points of the differential equation occur where $x^3 = 0$, or at $x = 0$. In the plane, we consider the ordinary point $x_0 = -2$ to occur at $(-2,0)$, and the singular point $x = 0$ to occur at $(0,0)$. The distance from the ordinary point $(-2,0)$ to the singular point $(0,0)$ is 2. So the minimum radius of convergence is

$$|x - (-2)| < 2$$

$$|x + 2| < 2$$

■ 4. Find a power series solution to the differential equation, given $y(0) = 0$ and $y'(0) = 1$.

$$(x^2 + 1)y'' + 2xy' = 0$$

Solution:

We'll use the standard form for the first two derivatives of the power series solution,

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \dots$$

to substitute into the differential equation.

$$(x^2+1)y'' + 2xy' = 0$$

$$(x^2+1)\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 2x\sum_{n=1}^{\infty} c_n n x^{n-1} = 0$$

$$x^2\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + 2x\sum_{n=1}^{\infty} c_n n x^{n-1} = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n} + \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} 2c_n n x^{n} = 0$$

The first series will start with the x^2 term, the second series will start with the x^0 term, and the third series will start with the x^1 term. We'll pull out all the x^0 and x^1 terms to put the series in phase.

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n} + c_2 2(2-1)x^{2-2} + c_3 3(3-1)x^{3-2} + \sum_{n=4}^{\infty} c_n n(n-1)x^{n-2}$$

$$+2c_1 1x^1 + \sum_{n=2}^{\infty} 2c_n n x^{n} = 0$$

$$2c_2 + (6c_3 + 2c_1)x + \sum_{n=2}^{\infty} c_n n(n-1)x^n + \sum_{n=4}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=2}^{\infty} 2c_n n x^n = 0$$

Now the series are in phase (they all begin with the x^2 term), so we just need to make their indices match. We'll substitute $k = n$ into the first and third series, and substitute $k = n - 2$ and $n = k + 2$ into the second series.

$$2c_2 + (6c_3 + 2c_1)x + \sum_{k=2}^{\infty} c_k k(k-1)x^k + \sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k$$

$$+ \sum_{k=2}^{\infty} 2c_k k x^k = 0$$

Now that the series are in phase with matching indices, we can combine them and then factor out x^k.

$$2c_2 + (6c_3 + 2c_1)x + \sum_{k=2}^{\infty} c_k k(k-1)x^k + c_{k+2}(k+2)(k+1)x^k + 2c_k k x^k = 0$$

$$2c_2 + (6c_3 + 2c_1)x + \sum_{k=2}^{\infty} [c_k k(k-1) + c_{k+2}(k+2)(k+1) + 2c_k k]x^k = 0$$

$$2c_2 + (6c_3 + 2c_1)x + \sum_{k=2}^{\infty} [c_{k+2}(k+2)(k+1) + c_k k(k+1)]x^k = 0$$

Equating coefficients, we get

$2c_2 = 0$	$k = 0$
$6c_3 + 2c_1 = 0$	$k = 1$
$c_{k+2}(k+2)(k+1) + c_k k(k+1) = 0$	$k = 2, 3, 4, \dots$

Solving the recurrence relation for the coefficient with the largest subscript, c_{k+2}, gives

$$c_2 = 0 \qquad k = 0$$

$$c_3 = -\frac{c_1}{3} \qquad k = 1$$

$$c_{k+2} = -\frac{c_k k}{k+2} \qquad k = 2, 3, 4, \dots$$

Plugging in $k = 2, 3, 4, \dots$, we start to see a pattern.

$$k = 0 \qquad c_2 = 0 \qquad k = 1 \qquad c_3 = -\frac{c_1}{3}$$

$$k = 2 \qquad c_4 = 0 \qquad k = 3 \qquad c_5 = \frac{c_1}{5}$$

$$k = 4 \qquad c_6 = 0 \qquad k = 5 \qquad c_7 = -\frac{c_1}{7}$$

$$\dots \qquad \dots$$

Plugging these coefficients into the expansion of the power series solution, we get

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$y = c_0 + c_1x + 0x^2 - \frac{c_1}{3}x^3 + 0x^4 + \frac{c_1}{5}x^5 + 0x^6 - \frac{c_1}{7}x^7 + 0x^8 + \dots$$

$$y = c_0 + c_1\left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots\right)$$

So the general solution to the differential equation is

$$y = c_0 + c_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

We've been told that $y(0) = 0$ and $y'(0) = 1$, so we can plug these into the formula for the series solution and its derivative to get

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$0 = c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 + \dots$$

$$0 = c_0$$

and

$$y' = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

$$1 = c_1 + 2c_2(0) + 3c_3(0)^2 + \dots$$

$$1 = c_1$$

If we plug $c_0 = 0$ and $c_1 = 1$ back into the general solution, we get

$$y = 0 + 1 \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

We know that this is the power series representation of $\tan^{-1} x$,

$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$$

which means we can write the general solution to the differential equation as

$$y = \tan^{-1} x$$

■ 5. Find a power series solution to the differential equation, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' - xy' - y = 0$$

Solution:

We'll use the standard form for the power series solution and its derivatives,

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \ldots$$

to substitute into the differential equation.

$$y'' - xy' - y = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=0}^{\infty} c_n x^n = 0$$

The first series will start with the $x^{2-2} = x^0$ term, the second series will start with the x^1 term, and the third series will start with the x^0 term. So we'll pull out the x^0 term from both the first and the third series in order to put the series in phase.

$$c_2(2)(2-1)x^{2-2} + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - c_0 x^0 - \sum_{n=1}^{\infty} c_n x^n = 0$$

$$-c_0 + 2c_2 + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=1}^{\infty} c_n x^n = 0$$

Now the series are in phase (they all begin with the x^1 term), so we just need to make their indices match. We'll substitute $k = n - 2$ and $n = k + 2$ into the first series (the new index will be $k = 3 - 2 = 1$), and we'll substitute $k = n$ into the second and third series (their new indices will be $k = 1$).

$$-c_0 + 2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_k k x^k - \sum_{k=1}^{\infty} c_k x^k = 0$$

Now that the series are in phase with matching indices, we can combine them and then factor out x^k.

$$-c_0 + 2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k - c_k k x^k - c_k x^k = 0$$

$$-c_0 + 2c_2 + \sum_{k=1}^{\infty} (c_{k+2}(k+2)(k+1) - c_k k - c_k)x^k = 0$$

Equating coefficients, we get

$$-c_0 + 2c_2 = 0 \qquad k = 0$$

$$c_{k+2}(k+2)(k+1) - c_k k - c_k = 0 \qquad k = 1, 2, 3, 4, \ldots$$

Solving the recurrence relation for the coefficient with the largest subscript, c_{k+2}, gives

$$-c_0 + 2c_2 = 0 \qquad k = 0$$

$$c_{k+2} = \frac{c_k}{k+2} \qquad k = 1, 2, 3, 4, \ldots$$

Plugging in $k = 1, 2, 3, 4, \ldots$, we start to see a pattern.

$$k = 0 \qquad c_2 = \frac{c_0}{2} \qquad\qquad k = 1 \qquad c_3 = \frac{c_1}{3}$$

$$k = 2 \qquad c_4 = \frac{c_0}{2 \cdot 4} \qquad\qquad k = 3 \qquad c_5 = \frac{c_1}{3 \cdot 5}$$

$$k = 4 \qquad c_6 = \frac{c_0}{2 \cdot 4 \cdot 6} \qquad\qquad k = 5 \qquad c_7 = \frac{c_1}{3 \cdot 5 \cdot 7}$$

$$\ldots \qquad\qquad \ldots$$

Plugging these coefficients into the expansion of the power series solution, we get

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

$$y = c_0 + c_1x + \frac{c_0}{2}x^2 + \frac{c_1}{3}x^3 + \frac{c_0}{2 \cdot 4}x^4 + \frac{c_1}{3 \cdot 5}x^5 + \frac{c_0}{2 \cdot 4 \cdot 6}x^6 + \frac{c_1}{3 \cdot 5 \cdot 7}x^7 + \dots$$

$$y = \left(c_0 + \frac{c_0}{2}x^2 + \frac{c_0}{2 \cdot 4}x^4 + \frac{c_0}{2 \cdot 4 \cdot 6}x^6 + \dots\right)$$

$$+\left(c_1x + \frac{c_1}{3}x^3 + \frac{c_1}{3 \cdot 5}x^5 + \frac{c_1}{3 \cdot 5 \cdot 7}x^7 + \dots\right)$$

$$y = c_0\left(1 + \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 + \frac{1}{2 \cdot 4 \cdot 6}x^6 + \dots\right)$$

$$+c_1\left(x + \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 + \frac{1}{3 \cdot 5 \cdot 7}x^7 + \dots\right)$$

We could leave the solution this way, or we could try to rewrite each series in order to simplify the general solution.

$$y = c_0\left(\frac{1}{2^0(0!)}x^{2(0)} + \frac{1}{2^1(1!)}x^{2(1)} + \frac{1}{2^2(2!)}x^{2(2)} + \frac{1}{2^3(3!)}x^{2(3)} + \dots\right)$$

$$+c_1\left(\frac{2^0(0!)}{(2(0)+1)!}x^{2(0)+1} + \frac{2^1(1!)}{(2(1)+1)!}x^{2(1)+1} + \frac{2^2(2!)}{(2(2)+1)!}x^{2(2)+1}\right.$$

$$\left.+\frac{2^3(3!)}{(2(3)+1)!}x^{2(3)+1} + \dots\right)$$

So the general solution to the differential equation is

$$y = c_0\sum_{n=0}^{\infty}\frac{1}{2^n n!}x^{2n} + c_1\sum_{n=0}^{\infty}\frac{2^n n!}{(2n+1)!}x^{2n+1}$$

We've been told that $y(0)=1$ and $y'(0)=0$, so we can plug these into the formula for the series solution and its derivative to get

$$y=c_0+c_1x+c_2x^2+c_3x^3+\dots$$

$$1=c_0+c_1(0)+c_2(0)^2+c_3(0)^3+\dots$$

$$1=c_0$$

and

$$y'=c_1+2c_2x+3c_3x^2+\dots$$

$$0=c_1+2c_2(0)+3c_3(0)^2+\dots$$

$$0=c_1$$

If we plug $c_0=1$ and $c_1=0$ back into the general solution, we get

$$y=(1)\sum_{n=0}^{\infty}\frac{1}{2^nn!}x^{2n}+(0)\sum_{n=0}^{\infty}\frac{2^nn!}{(2n+1)!}x^{2n+1}$$

$$y=\sum_{n=0}^{\infty}\frac{1}{2^nn!}x^{2n}$$

We can leave the solution this way, or we can rewrite it as

$$y=\sum_{n=0}^{\infty}\left(\frac{x^2}{2}\right)^n\frac{1}{n!}$$

$$y=\sum_{n=0}^{\infty}\frac{\left(\frac{x^2}{2}\right)^n}{n!}$$

If we think back to our table of common power series, we remember that the series representation of e^x is given as

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{\frac{x^2}{2}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^2}{2}\right)^n}{n!}$$

Therefore, we can write the general solution to the differential equation as

$$y = e^{\frac{x^2}{2}}$$

■ 6. Find a power series solution to the differential equation, given $y(0) = 0$ and $y'(0) = 1$.

$$y'' + x^2y' + xy = 0$$

Solution:

We'll use the standard form for the power series solution and its derivatives,

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = 2c_2 + 6c_3x + 12c_4x^2 + \dots$$

to substitute into the differential equation.

$$y'' + x^2y' + xy = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + x^2 \sum_{n=1}^{\infty} c_n n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

The first series will start with the $x^{2-2} = x^0$ term, the second series will start with the $x^{1+1} = x^2$ term, and the third series will start with the $x^{0+1} = x^1$ term. So we'll pull out the x^0 and x^1 terms from the first series, and we'll pull out the x^1 term from the third series. Doing so will put the series in phase.

$$c_2(2)(2-1)x^{2-2} + c_3(3)(3-1)x^{3-2} + \sum_{n=4}^{\infty} c_n n(n-1)x^{n-2}$$

$$+ \sum_{n=1}^{\infty} c_n n x^{n+1} + c_0x^{0+1} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$2c_2 + (c_0 + 6c_3)x + \sum_{n=4}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

Now the series are in phase (they all begin with the x^2 term), so we just need to make their indices match. We'll substitute $k = n - 2$ and $n = k + 2$ into the first series (the new index will be $k = 4 - 2 = 2$), and we'll substitute

$k = n + 1$ and $n = k - 1$ into the second and third series (their new indices will be $k = 2$).

$$2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=2}^{\infty} c_{k-1}(k-1)x^k + \sum_{k=2}^{\infty} c_{k-1}x^k = 0$$

Now that the series are in phase with matching indices, we can combine them and then factor out x^k.

$$2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} c_{k+2}(k+2)(k+1)x^k + c_{k-1}(k-1)x^k + c_{k-1}x^k = 0$$

$$2c_2 + (c_0 + 6c_3)x + \sum_{k=2}^{\infty} (c_{k+2}(k+2)(k+1) + c_{k-1}(k-1) + c_{k-1})x^k = 0$$

Equating coefficients, we get

$$2c_2 = 0 \qquad k = 0$$

$$c_0 + 6c_3 = 0 \qquad k = 1$$

$$c_{k+2}(k+2)(k+1) + c_{k-1}(k-1) + c_{k-1} = 0 \qquad k = 2, 3, 4, 5, \ldots$$

Solving the recurrence relation for the coefficient with the largest subscript, c_{k+2}, gives

$$c_2 = 0 \qquad k = 0$$

$$c_3 = -\frac{c_0}{6} \qquad k = 1$$

$$c_{k+2} = -\frac{kc_{k-1}}{(k+2)(k+1)} \qquad k = 2, 3, 4, 5, \ldots$$

Plugging in $k = 2, 3, 4, 5, \ldots$, we start to see a pattern.

For $k = 0, 3, 6, 9, \ldots$

$$k = 0 \qquad c_2 = 0$$

$$k = 3 \qquad c_5 = 0$$

$$k = 6 \qquad c_8 = 0$$

$$k = 9 \qquad c_{11} = 0$$

For $k = 1, 4, 7, 10, \ldots$

$$k = 1 \qquad c_3 = -\frac{c_0}{6}$$

$$k = 4 \qquad c_6 = \frac{4c_0}{(6)(6)(5)}$$

$$k = 7 \qquad c_9 = -\frac{7(4)c_0}{(9)(8)(6)(6)(5)}$$

$$k = 10 \qquad c_{12} = \frac{10(7)(4)c_0}{(12)(11)(9)(8)(6)(6)(5)}$$

For $k = 2, 5, 8, 11, \ldots$

$$k = 2 \qquad c_4 = -\frac{2c_1}{(4)(3)}$$

$$k = 5 \qquad c_7 = \frac{5(2)c_1}{(7)(6)(4)(3)}$$

$$k = 8 \qquad c_{10} = -\frac{8(5)(2)c_1}{(10)(9)(7)(6)(4)(3)}$$

$$k = 11 \qquad c_{13} = \frac{11(8)(5)(2)c_1}{(13)(12)(10)(9)(7)(6)(4)(3)}$$

We’ve been told that $y(0) = 0$ and $y'(0) = 1$, so we can plug these into the formula for the series solution and its derivative to get

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

$$0 = c_0 + c_1(0) + c_2(0)^2 + c_3(0)^3 + \dots$$

$$0 = c_0$$

and

$$y' = c_1 + 2c_2x + 3c_3x^2 + \dots$$

$$1 = c_1 + 2c_2(0) + 3c_3(0)^2 + \dots$$

$$1 = c_1$$

Which means our coefficients become

$$k = 0 \quad c_2 = 0 \qquad k = 1 \quad c_3 = 0 \qquad k = 2 \quad c_4 = -\frac{2}{(4)(3)}$$

$$k = 3 \quad c_5 = 0 \qquad k = 4 \quad c_6 = 0 \qquad k = 5 \quad c_7 = \frac{5(2)}{(7)(6)(4)(3)}$$

$$k = 6 \quad c_8 = 0 \qquad k = 7 \quad c_9 = 0 \qquad k = 8 \quad c_{10} = -\frac{8(5)(2)}{(10)(9)(7)(6)(4)(3)}$$

$$k=9 \quad c_{11}=0 \qquad k=10 \quad c_{12}=0 \qquad k=11 \quad c_{13}=\frac{11(8)(5)(2)}{(13)(12)(10)(9)(7)(6)(4)(3)}$$

Plugging these coefficients into the expansion of the power series solution, we get

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+c_8x^8+\dots$$

$$y=0+x+0x^2-0x^3-\frac{2}{(4)(3)}x^4+0x^5+0x^6+\frac{5(2)}{(7)(6)(4)(3)}x^7$$

$$+0x^8-0x^9-\frac{8(5)(2)}{(10)(9)(7)(6)(4)(3)}x^{10}+\dots$$

$$y=x-\frac{2}{(4)(3)}x^4+\frac{5(2)}{(7)(6)(4)(3)}x^7-\frac{8(5)(2)}{(10)(9)(7)(6)(4)(3)}x^{10}$$

$$+\frac{11(8)(5)(2)}{(13)(12)(10)(9)(7)(6)(4)(3)}x^{13}-\dots$$

We could leave the solution this way, or we could try to rewrite each series in order to simplify the general solution.

$$y=x-\frac{2(2)}{4!}x^4+\frac{5(5)(2)(2)}{7!}x^7-\frac{8(8)(5)(5)(2)(2)}{10!}x^{10}$$

$$+\frac{11(11)(8)(8)(5)(5)(2)(2)}{13!}x^{13}-\dots$$

$$y=x-\frac{(2)^2}{4!}x^4+\frac{(2\cdot 5)^2}{7!}x^7-\frac{(2\cdot 5\cdot 8)^2}{10!}x^{10}+\frac{(2\cdot 5\cdot 8\cdot 11)^2}{13!}x^{13}-\dots$$

So the general solution to the differential equation is

$$y = x + \sum_{n=1}^{\infty} (-1)^n \frac{2^2 \cdot 5^2 \cdot 8^2 \cdot 11^2 \cdot \ldots \cdot (3n-1)^2 x^{3n+1}}{(3n+1)!}$$

NONPOLYNOMIAL COEFFICIENTS

■ 1. Solve the differential equation around $x_0 = 0$.

$$y'' + \frac{1}{1-x}y' = 0$$

Solution:

We know that the power series representations of the first two derivatives y' and y'' are

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots$$

and that the power series representation of the coefficient function on y' is

$$\frac{1}{1-x} = \sum_{n=0} x^n = 1 + x + x^2 + x^3 + x^4 + \dots$$

Plugging all of these values into the original differential equation gives

$$\sum_{n=2} n(n-1)c_nx^{n-2} + (1 + x + x^2 + x^3 + x^4 + \dots)\sum_{n=1} nc_nx^{n-1} = 0$$

Replace each power series with its expanded version.

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$+(1 + x + x^2 + x^3 + x^4 + \dots)(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots) = 0$$

$$2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots$$

$$+c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots$$

$$+c_1x + 2c_2x^2 + 3c_3x^3 + 4c_4x^4 + 5c_5x^5 + \dots$$

$$+c_1x^2 + 2c_2x^3 + 3c_3x^4 + 4c_4x^5 + 5c_5x^6 + \dots$$

$$+c_1x^3 + 2c_2x^4 + 3c_3x^5 + 4c_4x^6 + 5c_5x^7 + \dots$$

$$+c_1x^4 + 2c_2x^5 + 3c_3x^6 + 4c_4x^7 + 5c_5x^8 + \dots = 0$$

$$(2c_2 + c_1) + (6c_3 + 2c_2 + c_1)x + (12c_4 + 3c_3 + 2c_2 + c_1)x^2$$

$$+(20c_5 + 4c_4 + 3c_3 + 2c_2 + c_1)x^3 + \dots = 0$$

This lets us build a system of equations.

$$2c_2 + c_1 = 0$$

$$6c_3 + 2c_2 + c_1 = 0$$

$$12c_4 + 3c_3 + 2c_2 + c_1 = 0$$

$$20c_5 + 4c_4 + 3c_3 + 2c_2 + c_1 = 0$$

Solving this system, we get $c_2 = -c_1/2$, $c_3 = 0$, $c_4 = 0$, and $c_5 = 0$. Therefore, the power series representation of the solution to the differential equation will be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

$$y = c_0 + c_1x - \frac{c_1}{2}x^2$$

■ 2. Find the first six terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y'' - e^xy = 0$$

Solution:

We know $x_0 = 0$ is an ordinary point because e^x is analytic there. And we know that the Maclaurin series representation of e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$$

Plugging this and the power series representations of y and y'',

$$y = \sum_{n=0}^{\infty} c_nx^n = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_nn(n-1)x^{n-2} = 2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots$$

into the original differential equation, we get

$$\sum_{n=2}^{\infty} n(n-1)c_nx^{n-2} - \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right)\sum_{n=0}^{\infty} c_nx^n = 0$$

Now we'll just expand each series through its first few terms.

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$-\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots\right)(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots) = 0$$

$$2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots$$

$$-c_0 - c_1x - c_2x^2 - c_3x^3 - c_4x^4 - \dots$$

$$-c_0x - c_1x^2 - c_2x^3 - c_3x^4 - c_4x^5 - \dots$$

$$-c_0\frac{x^2}{2!} - c_1\frac{x^3}{2!} - c_2\frac{x^4}{2!} - c_3\frac{x^5}{2!} - c_4\frac{x^6}{2!} - \dots$$

$$-c_0\frac{x^3}{3!} - c_1\frac{x^4}{3!} - c_2\frac{x^5}{3!} - c_3\frac{x^6}{3!} - c_4\frac{x^7}{3!} - \dots$$

$$-c_0\frac{x^4}{4!} - c_1\frac{x^5}{4!} - c_2\frac{x^6}{4!} - c_3\frac{x^7}{4!} - c_4\frac{x^8}{4!} - \dots$$

$$-c_0\frac{x^5}{5!} - c_1\frac{x^6}{5!} - c_2\frac{x^7}{5!} - c_3\frac{x^8}{5!} - c_4\frac{x^9}{5!} - \dots = 0$$

$$(2c_2 - c_0) + (6c_3 - c_1 - c_0)x + \left(12c_4 - c_2 - c_1 - \frac{c_0}{2!}\right)x^2$$

$$+\left(20c_5 - c_3 - c_2 - \frac{c_1}{2!} - \frac{c_0}{3!}\right)x^3 + \dots = 0$$

This lets us build a system of equations.

$$2c_2 - c_0 = 0$$

$$6c_3 - c_1 - c_0 = 0$$

$$12c_4 - c_2 - c_1 - \frac{c_0}{2!} = 0$$

$$20c_5 - c_3 - c_2 - \frac{c_1}{2!} - \frac{c_0}{3!} = 0$$

This system simplifies to

$$c_2 = \frac{c_0}{2}$$

$$c_3 = \frac{c_0 + c_1}{6}$$

$$c_4 = \frac{c_0 + c_1}{12}$$

$$c_5 = \frac{5c_0 + 4c_1}{120}$$

Therefore, the power series representation of the solution to the differential equation will be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

$$y = c_0 + c_1x + \frac{c_0}{2}x^2 + \frac{c_0 + c_1}{6}x^3 + \frac{c_0 + c_1}{12}x^4 + \frac{5c_0 + 4c_1}{120}x^5 + \dots$$

$$y = c_0\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \dots\right) + c_1\left(x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{30}x^5 + \dots\right)$$

■ 3. Find the first five terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y' + e^{3x}y = 0$$

Solution:

We know $x_0 = 0$ is an ordinary point because e^{3x} is analytic there. And we know that the Maclaurin series representation of e^{3x} is

$$e^{3x} = \sum_{n=0}^{\infty} \frac{3^n x^n}{n!} = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \frac{243x^5}{5!} + \ldots$$

Plugging this and the power series representations of y and y',

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \ldots$$

into the original differential equation, we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \frac{243x^5}{5!} + \ldots\right)\sum_{n=0}^{\infty} c_n x^n = 0$$

Now we'll just expand each series through its first few terms.

$$(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+\left(1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \frac{81x^4}{4!} + \frac{243x^5}{5!} + \dots\right)$$

$$(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots) = 0$$

$$c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots$$

$$+c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$+3c_0x + 3c_1x^2 + 3c_2x^3 + 3c_3x^4 + 3c_4x^5 + \dots$$

$$+\frac{9c_0}{2!}x^2 + \frac{9c_1}{2!}x^3 + \frac{9c_2}{2!}x^4 + \frac{9c_3}{2!}x^5 + \frac{9c_4}{2!}x^6 + \dots$$

$$+\frac{27c_0}{3!}x^3 + \frac{27c_1}{3!}x^4 + \frac{27c_2}{3!}x^5 + \frac{27c_3}{3!}x^6 + \frac{27c_4}{3!}x^7 + \dots$$

$$+\frac{81c_0}{4!}x^4 + \frac{81c_1}{4!}x^5 + \frac{81c_2}{4!}x^6 + \frac{81c_3}{4!}x^7 + \frac{81c_4}{4!}x^8 + \dots$$

$$+\frac{243c_0}{5!}x^5 + \frac{243c_1}{5!}x^6 + \frac{243c_2}{5!}x^7 + \frac{243c_3}{5!}x^8 + \frac{243c_4}{5!}x^9 + \dots = 0$$

$$(c_1 + c_0) + (2c_2 + c_1 + 3c_0)x + \left(3c_3 + c_2 + 3c_1 + \frac{9c_0}{2!}\right)x^2$$

$$+\left(4c_4 + c_3 + 3c_2 + \frac{9c_1}{2!} + \frac{27c_0}{3!}\right)x^3 + \dots = 0$$

This lets us build a system of equations.

$$c_1 + c_0 = 0$$

$$2c_2 + c_1 + 3c_0 = 0$$

$$3c_3 + c_2 + 3c_1 + \frac{9c_0}{2!} = 0$$

$$4c_4 + c_3 + 3c_2 + \frac{9c_1}{2!} + \frac{27c_0}{3!} = 0$$

This system simplifies to

$$c_1 = -c_0$$

$$c_2 = -c_0$$

$$c_3 = -\frac{c_0}{6}$$

$$c_4 = \frac{19c_0}{24}$$

Therefore, the power series representation of the solution to the differential equation will be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

$$y(x) = c_0 - c_0x - c_0x^2 - \frac{c_0}{6}x^3 + \frac{19c_0}{24}x^4 + \dots$$

■ 4. Find the first eight terms of the power series solution of the differential equation around the ordinary point $x_0 = 0$.

$$y'' + y\cos x = 0$$

Solution:

We know $x_0 = 0$ is an ordinary point because $\cos x$ is analytic there. And we know that the Maclaurin series representation of $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Plugging this and the power series representations of y and y'',

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots$$

into the original differential equation, we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right) \sum_{n=0}^{\infty} c_n x^n = 0$$

Now we'll just expand each series through its first few terms.

$$(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots)$$

$$+\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots)$$

$$+(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots)$$

$$-\left(\frac{c_0}{2!}x^2 + \frac{c_1}{2!}x^3 + \frac{c_2}{2!}x^4 + \frac{c_3}{2!}x^5 + \dots\right)$$

$$+\left(\frac{c_0}{4!}x^4 + \frac{c_1}{4!}x^5 + \frac{c_2}{4!}x^6 + \frac{c_3}{4!}x^7 + \dots\right)$$

$$-\left(\frac{c_0}{6!}x^6 + \frac{c_1}{6!}x^7 + \frac{c_2}{6!}x^8 + \frac{c_3}{6!}x^9 + \dots\right) - \dots = 0$$

$$(2c_2 + c_0) + (6c_3 + c_1)x + \left(12c_4 + c_2 - \frac{c_0}{2!}\right)x^2$$

$$+\left(20c_5 + c_3 - \frac{c_1}{2!}\right)x^3 + \left(30c_6 + c_4 - \frac{c_2}{2!} + \frac{c_0}{4!}\right)x^4$$

$$+\left(42c_7 + c_5 - \frac{c_3}{2!} + \frac{c_1}{4!}\right)x^5 + \dots = 0$$

This lets us build a system of equations.

$$2c_2 + c_0 = 0$$

$$6c_3 + c_1 = 0$$

$$12c_4 + c_2 - \frac{c_0}{2!} = 0$$

$$20c_5 + c_3 - \frac{c_1}{2!} = 0$$

$$30c_6 + c_4 - \frac{c_2}{2!} + \frac{c_0}{4!} = 0$$

$$42c_7 + c_5 - \frac{c_3}{2!} + \frac{c_1}{4!} = 0$$

This system simplifies to

$$c_2 = -\frac{c_0}{2} \qquad c_3 = -\frac{c_1}{6}$$

$$c_4 = \frac{c_0}{12} \qquad c_5 = \frac{c_1}{30}$$

$$c_6 = -\frac{c_0}{80} \qquad c_7 = -\frac{19c_1}{5{,}040}$$

Therefore, the power series representation of the solution to the differential equation will be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

$$y(x) = c_0 + c_1x - \frac{c_0}{2}x^2 - \frac{c_1}{6}x^3 + \frac{c_0}{12}x^4 + \frac{c_1}{30}x^5 - \frac{c_0}{80}x^6 + \frac{19c_1}{5{,}040}x^7 + \dots$$

■ 5. Find the first six terms of the power series solution of the differential equation around the ordinary point $x_0 = 0$.

$$y'' - \ln(1 + 5x)y = 0$$

Solution:

We know $x_0 = 0$ is an ordinary point because $\ln(1 + 5x)$ is analytic there. And we know that the Maclaurin series representation of $\ln(1 + 5x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \dots$$

$$\ln(1+5x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} (5x)^n = 5x - \frac{1}{2}(5x)^2 + \frac{1}{3}(5x)^3 - \frac{1}{4}(5x)^4 + \dots$$

$$\ln(1+5x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}5^n}{n} x^n = 5x - \frac{25}{2}x^2 + \frac{125}{3}x^3 - \frac{625}{4}x^4 + \dots$$

Plugging this and the power series representations of y and y'',

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots$$

into the original differential equation, we get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \left(5x - \frac{25}{2}x^2 + \frac{125}{3}x^3 - \frac{625}{4}x^4 + \dots\right) \sum_{n=0}^{\infty} c_n x^n = 0$$

Now we'll just expand each series through its first few terms.

$$(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots)$$

$$-\left(5x - \frac{25}{2}x^2 + \frac{125}{3}x^3 - \frac{625}{4}x^4 + \dots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots) = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + \dots)$$

$$-5x(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots)$$

$$+\frac{25}{2}x^2(c_0+c_1x+c_2x^2+c_3x^3+\dots)$$

$$-\frac{125}{3}x^3(c_0+c_1x+c_2x^2+c_3x^3+\dots)-\dots=0$$

$$(2c_2+6c_3x+12c_4x^2+20c_5x^3+\dots)$$

$$-(5c_0x+5c_1x^2+5c_2x^3+5c_3x^4+\dots)$$

$$+\left(\frac{25c_0}{2}x^2+\frac{25c_1}{2}x^3+\frac{25c_2}{2}x^4+\frac{25c_3}{2}x^5+\dots\right)$$

$$-\left(\frac{125c_0}{3}x^3+\frac{125c_1}{3}x^4+\frac{125c_2}{3}x^5+\frac{125c_3}{3}x^6+\dots\right)-\dots=0$$

$$2c_2+(6c_3-5c_0)x+\left(12c_4-5c_1+\frac{25c_0}{2}\right)x^2$$

$$+\left(20c_5-5c_2+\frac{25c_1}{2}-\frac{125c_0}{3}\right)x^3+\dots=0$$

This lets us build a system of equations.

$$2c_2=0$$

$$6c_3-5c_0=0$$

$$12c_4-5c_1+\frac{25c_0}{2}=0$$

$$20c_5-5c_2+\frac{25c_1}{2}-\frac{125c_0}{3}=0$$

This system simplifies to

$$c_2 = 0$$

$$c_3 = \frac{5c_0}{6}$$

$$c_4 = \frac{10c_1 - 25c_0}{24}$$

$$c_5 = \frac{50c_0 - 15c_1}{24}$$

Therefore, the power series representation of the solution to the differential equation will be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \ldots$$

$$y = c_0 + c_1x + \frac{5c_0}{6}x^3 + \frac{10c_1 - 25c_0}{24}x^4 + \frac{50c_0 - 15c_1}{24}x^5 + \ldots$$

■ 6. Find the first four terms of the power series solution of the differential equation around the ordinary point $x_0 = 2$.

$$y'' + \cosh(x-2)y = 0$$

Solution:

We know $x_0 = 0$ is an ordinary point because $\cosh(x-2)$ is analytic there. And we know that the Maclaurin series representation of $\cosh(x-2)$ is

$$\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!} + \dots$$

$$\cosh(x-2) = \sum_{n=0}^{\infty} \frac{(x-2)^{2n}}{(2n)!} = 1 + \frac{(x-2)^2}{2!} + \frac{(x-2)^4}{4!} + \frac{(x-2)^6}{6!} + \frac{(x-2)^8}{8!} + \dots$$

Plugging this and the power series representations of $y(x-2)$ and $y''(x-2)$,

$$y(x-2) = \sum_{n=0}^{\infty} c_n(x-2)^n$$

$$= c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + c_4(x-2)^4 + \dots$$

$$y''(x-2) = \sum_{n=2}^{\infty} c_n n(n-1)(x-2)^{n-2}$$

$$= 2c_2 + 6c_3(x-2) + 12c_4(x-2)^2 + 20c_5(x-2)^3 + 30c_6(x-2)^4 + \dots$$

into the original differential equation, we get

$$\sum_{n=2}^{\infty} n(n-1)c_n(x-2)^{n-2} + \sum_{n=0}^{\infty} \frac{(x-2)^{2n}}{(2n)!} \sum_{n=0}^{\infty} c_n(x-2)^n = 0$$

Now we'll just expand each series through its first few terms.

$$(2c_2 + 6c_3(x-2) + 12c_4(x-2)^2 + 20c_5(x-2)^3 + \dots)$$

$$+\left(1 + \frac{(x-2)^2}{2!} + \frac{(x-2)^4}{4!} + \frac{(x-2)^6}{6!} + \dots\right)$$

$$(c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \dots) + \dots = 0$$

$$(2c_2 + 6c_3(x-2) + 12c_4(x-2)^2 + 20c_5(x-2)^3 + \dots)$$

$$+(c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \dots)$$

$$+\frac{(x-2)^2}{2!}(c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \dots)$$

$$+\frac{(x-2)^4}{4!}(c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \dots) + \dots = 0$$

$$(2c_2 + 6c_3(x-2) + 12c_4(x-2)^2 + 20c_5(x-2)^3 + \dots)$$

$$+(c_0 + c_1(x-2) + c_2(x-2)^2 + c_3(x-2)^3 + \dots)$$

$$+\left(\frac{c_0}{2!}(x-2)^2 + \frac{c_1}{2!}(x-2)^3 + \frac{c_2}{2!}(x-2)^4 + \frac{c_3}{2!}(x-2)^5 + \dots\right)$$

$$+\left(\frac{c_0}{4!}(x-2)^4 + \frac{c_1}{4!}(x-2)^5 + \frac{c_2}{4!}(x-2)^6 + \frac{c_3}{4!}(x-2)^7 + \dots\right) + \dots = 0$$

$$(2c_2 + c_0) + (6c_3 + c_1)(x-2) + \left(12c_4 + c_2 + \frac{c_0}{2!}\right)(x-2)^2$$

$$+\left(20c_5 + c_3 + \frac{c_1}{2!}\right)(x-2)^3 + \dots = 0$$

This lets us build a system of equations.

$$2c_2 + c_0 = 0$$

$$6c_3 + c_1 = 0$$

$$12c_4 + c_2 + \frac{c_0}{2!} = 0$$

$$20c_5 + c_3 + \frac{c_1}{2!} = 0$$

This system simplifies to

$$c_2 = -\frac{c_0}{2}$$

$$c_3 = -\frac{c_1}{6}$$

$$c_4 = 0$$

Therefore, the power series representation of the solution to the differential equation will be

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots$$

$$y = c_0 + c_1(x-2) - \frac{c_0}{2}(x-2)^2 - \frac{c_1}{6}(x-2)^3 + \dots$$

SINGULAR POINTS AND FROBENIUS' THEOREM

■ 1. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$2x^2y'' + x(1-2x)y' + 4xy = 0$$

Solution:

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = 2x^2$$

$$q(x) = x(1-2x)$$

$$r(x) = 4x$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = x\frac{x(1-2x)}{2x^2} = \frac{1-2x}{2}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{4x}{2x^2} = 2x$$

Because both denominators simplify to constants, we can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$2x^2y'' + x(1-2x)y' + 4xy = 0$$

$$2x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + x(1-2x) \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$+4x \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r+1}$$

$$+\sum_{n=0}^{\infty} 4c_n x^{n+r+1} = 0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty} [4-2(n+r)]c_n x^{n+r+1} + \sum_{n=0}^{\infty} [2(n+r)(n+r-1)+(n+r)]c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}[4-2(n+r)]c_nx^{n+r+1}+\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_nx^{n+r}=0$$

$$x^r\left[\sum_{n=0}^{\infty}[4-2(n+r)]c_nx^{n+1}+\sum_{n=0}^{\infty}(n+r)(2n+2r-1)c_nx^n\right]=0$$

The first series starts with the x^1 term, while the second series starts with the x^0 term, so we'll pull the x^0 term out of the second series.

$$x^r\left[\sum_{n=0}^{\infty}[4-2(n+r)]c_nx^{n+1}+r(2r-1)c_0+\sum_{n=1}^{\infty}(n+r)(2n+2r-1)c_nx^n\right]=0$$

Now that the series are in phase, we'll substitute $k=n+1$ and $n=k-1$ into the first series, and $k=n$ into the second series.

$$x^r\left[r(2r-1)c_0+\sum_{k=1}^{\infty}[4-2(k+r-1)]c_{k-1}x^k+\sum_{k=1}^{\infty}(k+r)(2k+2r-1)c_kx^k\right]=0$$

Now that the series are in phase with matching indices, combine them.

$$x^r\left[r(2r-1)c_0+\sum_{k=1}^{\infty}[(6-2k-2r)c_{k-1}+(k+r)(2k+2r-1)c_k]x^k\right]=0$$

This equation gives

$$r(2r-1)c_0=0 \qquad k=0$$

$$(6-2k-2r)c_{k-1}+(k+r)(2k+2r-1)c_k=0 \qquad k=1,2,3,\ldots$$

or

$$r(2r-1)=0 \qquad k=0$$

$$c_k=-\frac{(6-2k-2r)c_{k-1}}{(k+r)(2k+2r-1)} \qquad k=1, 2, 3, \dots$$

The indicial equation gives us $r_1=1/2$ and $r_2=0$. Substituting these indicial roots into the recurrence relation, we get

	For $r_1=1/2$	For $r_2=0$
	$c_k=-\frac{(5-2k)c_{k-1}}{k(2k+1)}$	$c_k=-\frac{(6-2k)c_{k-1}}{k(2k-1)}$
$k=1$	$c_1=-c_0$	$c_1=-4c_0$
$k=2$	$c_2=\frac{1}{10}c_0$	$c_2=\frac{4}{3}c_0$
$k=3$	$c_3=\frac{1}{210}c_0$	$c_3=0$
...	...	...

Forming these coefficients into series around $x_0=0$ gives,

$$c_0(x-x_0)^0+c_1(x-x_0)^1+c_2(x-x_0)^2+c_3(x-x_0)^3+\dots$$

$$c_0x^0-c_0x^1+\frac{1}{10}c_0x^2+\frac{1}{210}c_0x^3+\dots$$

$$c_0\left(1-x+\frac{1}{10}x^2+\frac{1}{210}x^3+\dots\right)$$

and

$$c_0(x-x_0)^0+c_1(x-x_0)^1+c_2(x-x_0)^2+c_3(x-x_0)^3+\dots$$

$$c_0x^0-4c_0x^1+\frac{4}{3}c_0x^2+\dots$$

$$c_0\left(1-4x+\frac{4}{3}x^2+\dots\right)$$

So the series solutions are

$$y_1(x)=c_0x^{\frac{1}{2}}\left(1-x+\frac{1}{10}x^2+\frac{1}{210}x^3+\dots\right)$$

$$y_2(x)=c_0x^0\left(1-4x+\frac{4}{3}x^2+\dots\right)$$

and the general solution is

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$y(x)=C_1c_0x^{\frac{1}{2}}\left(1-x+\frac{1}{10}x^2+\frac{1}{210}x^3+\dots\right)+C_2c_0\left(1-4x+\frac{4}{3}x^2+\dots\right)$$

$$y(x)=C_1x^{\frac{1}{2}}\left(1-x+\frac{1}{10}x^2+\dots\right)+C_2\left(1-4x+\frac{4}{3}x^2+\dots\right)$$

■ 2. Determine the regularity of the singular point $x_0=0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$4xy''+2y'+y=0$$

Solution:

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = 4x$$

$$q(x) = 2$$

$$r(x) = 1$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = x\frac{2}{4x} = \frac{1}{2}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{1}{4x} = \frac{x}{4}$$

Because both denominators simplify to constants, we can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$4xy'' + 2y' + y = 0$$

$$4x \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + 2\sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} 2(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty} 2(n+r)(2n+2r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$x^r \left[\sum_{n=0}^{\infty} 2(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \right] = 0$$

The first series starts with the x^{-1} term, while the second series starts with the x^0 term, so we'll pull the x^{-1} term out of the first series.

$$x^r \left[2r(2r-1)c_0 x^{-1} + \sum_{n=1}^{\infty} 2(n+r)(2n+2r-1)c_n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n \right] = 0$$

Now that the series are in phase, we'll substitute $k = n-1$ and $n = k+1$ into the first series, and $k = n$ into the second series.

$$x^r \left[2r(2r-1)c_0x^{-1} + \sum_{k=0}^{\infty} 2(k+r+1)(2k+2r+1)c_{k+1}x^k + \sum_{k=0}^{\infty} c_kx^k \right] = 0$$

Now that the series are in phase with matching indices, combine them.

$$x^r \left[2r(2r-1)c_0x^{-1} + \sum_{k=0}^{\infty} [2(k+r+1)(2k+2r+1)c_{k+1} + c_k]x^k \right] = 0$$

This equation gives

$$2r(2r-1)c_0 = 0$$

$$2(k+r+1)(2k+2r+1)c_{k+1} + c_k = 0 \qquad k = 0, 1, 2, \dots$$

or

$$r(2r-1) = 0$$

$$c_{k+1} = -\frac{c_k}{2(k+r+1)(2k+2r+1)} \qquad k = 0, 1, 2, \dots$$

The indicial equation gives us $r_1 = 0$ and $r_2 = 1/2$. Substituting these indicial roots into the recurrence relation, we get

	For $r_1 = 1/2$	For $r_2 = 0$
	$c_{k+1} = -\frac{c_k}{2(k+\frac{3}{2})(2k+2)}$	$c_{k+1} = -\frac{c_k}{2(k+1)(2k+1)}$
$k = 0$	$c_1 = -\frac{c_0}{3!}$	$c_1 = -\frac{c_0}{2!}$

$k = 1 \qquad c_2 = \frac{c_0}{5!} \qquad c_2 = \frac{c_0}{4!}$

$k = 2 \qquad c_3 = -\frac{c_0}{7!} \qquad c_3 = -\frac{c_0}{6!}$

$\dots \qquad \dots \qquad \dots$

Forming these coefficients into series around $x_0 = 0$ gives,

$$c_0(x - x_0)^0 + c_1(x - x_0)^1 + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

$$c_0(x - 0)^0 - \frac{1}{3!}c_0(x - 0)^1 + \frac{1}{5!}c_0(x - 0)^2 + \frac{1}{7!}c_0(x - 0)^3 - \dots$$

$$c_0\left(1 - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \dots\right)$$

and

$$c_0(x - x_0)^0 + c_1(x - x_0)^1 + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

$$c_0(x - 0)^0 - \frac{1}{2!}c_0(x - 0)^1 + \frac{1}{4!}c_0(x - 0)^2 + \frac{1}{6!}c_0(x - 0)^3 + \dots$$

$$c_0\left(1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \dots\right)$$

So the series solutions are

$$y_1(x) = c_0x^0\left(1 - \frac{1}{2!}x + \frac{1}{4!}x^2 - \frac{1}{6!}x^3 + \dots\right) = c_0\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^k$$

$$y_2(x) = c_0x^{\frac{1}{2}}\left(1 - \frac{1}{3!}x + \frac{1}{5!}x^2 - \frac{1}{7!}x^3 + \dots\right) = c_0x^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^k$$

and the general solution is

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

$$y(x) = C_1c_0\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^k + C_2c_0x^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^k$$

$$y(x) = C_1\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^k + C_2x^{\frac{1}{2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^k$$

■ 3. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$x^2y'' + x^2y' - 6y = 0$$

Solution:

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = x^2$$

$$q(x) = x^2$$

$$r(x) = -6$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = x\frac{x^2}{x^2} = x$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{-6}{x^2} = -6$$

Because both denominators simplify to constants, we can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$x^2y'' + x^2y' - 6y = 0$$

$$x^2\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + x^2\sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} - 6\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r+1} - \sum_{n=0}^{\infty} 6c_n x^{n+r} = 0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)c_n - 6c_n]x^{n+r} + \sum_{n=0}^{\infty}(n+r)c_n x^{n+r+1} = 0$$

$$x^r\left[\sum_{n=0}^{\infty}[(n+r)(n+r-1)-6]c_n x^n + \sum_{n=0}^{\infty}(n+r)c_n x^{n+1}\right] = 0$$

The first series starts with the x^0 term, while the second series starts with the x^1 term, so we'll pull the x^0 term out of the first series.

$$x^r\left[(r^2-r-6)c_0 + \sum_{n=1}^{\infty}[(n+r)(n+r-1)-6]c_n x^n + \sum_{n=0}^{\infty}(n+r)c_n x^{n+1}\right] = 0$$

Now that the series are in phase, we'll substitute $k=n+1$ and $n=k-1$ into the second series, and $k=n$ into the first series.

$$x^r\left[(r^2-r-6)c_0 + \sum_{k=1}^{\infty}[(k+r)(k+r-1)-6]c_k x^k + \sum_{k=1}^{\infty}(k+r-1)c_{k-1}x^k\right] = 0$$

Now that the series are in phase with matching indices, combine them.

$$x^r\left[(r^2-r-6)c_0 + \sum_{k=1}^{\infty}[[(k+r)(k+r-1)-6]c_k + (k+r-1)c_{k-1}]x^k\right] = 0$$

This equation gives

$$(r^2-r-6)c_0 = 0 \qquad k=0$$

$$[(k+r)(k+r-1)-6]c_k + (k+r-1)c_{k-1} = 0 \qquad k=1, 2, 3, \ldots$$

or

$$(r-3)(r+2)=0 \qquad k=0$$

$$c_k=-\frac{(k+r-1)c_{k-1}}{(k+r)(k+r-1)-6} \qquad k=1,2,3,\ldots$$

The indicial equation gives us $r_1=3$ and $r_2=-2$. Substituting these indicial roots into the recurrence relation, we get

	For $r_1=3$	For $r_2=-2$
	$c_k=-\frac{(k+2)c_{k-1}}{(k+3)(k+2)-6}$	$c_k=-\frac{(k-3)c_{k-1}}{(k-2)(k-3)-6}$
$k=1$	$c_1=-\frac{c_0}{2}$	$c_1=-\frac{c_0}{2}$
$k=2$	$c_2=\frac{c_0}{7}$	$c_2=\frac{c_0}{12}$
$k=3$	$c_3=-\frac{5c_0}{168}$	$c_3=0$
...	...	...

Forming these coefficients into series around $x_0=0$ gives,

$$c_0(x-x_0)^0+c_1(x-x_0)^1+c_2(x-x_0)^2+c_3(x-x_0)^3+\ldots$$

$$c_0x^0-\frac{1}{2}c_0x^1+\frac{1}{7}c_0x^2-\frac{5}{168}c_0x^3+\ldots$$

$$c_0\left(1-\frac{1}{2}x+\frac{1}{7}x^2-\frac{5}{168}x^3+\ldots\right)$$

and

$$c_0(x-x_0)^0+c_1(x-x_0)^1+c_2(x-x_0)^2+c_3(x-x_0)^3+\dots$$

$$c_0x^0-\frac{1}{2}c_0x^1+\frac{1}{12}c_0x^2+\dots$$

$$c_0\left(1-\frac{1}{2}x+\frac{1}{12}x^2+\dots\right)$$

So the series solutions are

$$y_1(x)=c_0x^3\left(1-\frac{1}{2}x+\frac{1}{7}x^2-\frac{5}{168}x^3+\dots\right)$$

$$y_2(x)=c_0x^{-2}\left(1-\frac{1}{2}x+\frac{1}{12}x^2+\dots\right)$$

and the general solution is

$$y(x)=C_1y_1(x)+C_2y_2(x)$$

$$y(x)=C_1c_0x^3\left(1-\frac{1}{2}x+\frac{1}{7}x^2-\dots\right)$$

$$+C_2c_0x^{-2}\left(1-\frac{1}{2}x+\frac{1}{12}x^2+\dots\right)$$

$$y(x)=C_1x^3\left(1-\frac{1}{2}x+\frac{1}{7}x^2-\dots\right)+C_2x^{-2}\left(1-\frac{1}{2}x+\frac{1}{12}x^2+\dots\right)$$

■ 4. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$8x^2y'' + x^2y' + (2 - x)y = 0$$

Solution:

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = 8x^2$$

$$q(x) = x^2$$

$$r(x) = 2 - x$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = x\frac{x^2}{8x^2} = \frac{x}{8}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{2 - x}{8x^2} = \frac{2 - x}{8}$$

Because both denominators simplify to constants, we can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$8x^2y'' + x^2y' + (2-x)y = 0$$

$$8x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + x^2 \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$+(2-x)\sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 8(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r)c_n x^{n+r+1}$$

$$+\sum_{n=0}^{\infty} 2c_n x^{n+r} - \sum_{n=0}^{\infty} c_n x^{n+r+1} = 0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty} (8(n+r)(n+r-1)+2)c_n x^{n+r} + \sum_{n=0}^{\infty} (n+r-1)c_n x^{n+r+1} = 0$$

$$x^r \left[\sum_{n=0}^{\infty} (8(n+r)(n+r-1)+2)c_n x^n + \sum_{n=0}^{\infty} (n+r-1)c_n x^{n+1} \right] = 0$$

The first series starts with the x^0 term, while the second series starts with the x^1 term, so we'll pull the x^0 term out of the first series.

$$x^r \left[(8r(r-1)+2)c_0 + \sum_{n=1}^{\infty} (8(n+r)(n+r-1)+2)c_n x^n + \sum_{n=0}^{\infty} (n+r-1)c_n x^{n+1} \right] = 0$$

Now that the series are in phase, we'll substitute $k = n + 1$ and $n = k - 1$ into the second series, and $k = n$ into the first series.

$$x^r \left[2(4r^2 - 4r + 1)c_0 + \sum_{k=1}^{\infty} (8(k+r)(k+r-1)+2)c_k x^k + \sum_{k=1}^{\infty} (k+r-2)c_{k-1} x^k \right] = 0$$

Now that the series are in phase with matching indices, combine them.

$$x^r \left[2(2r-1)^2 c_0 + \sum_{k=1}^{\infty} [(8(k+r)(k+r-1)+2)c_k + (k+r-2)c_{k-1}]x^k \right] = 0$$

This equation gives

$$2(2r-1)^2 c_0 = 0 \qquad k = 0$$

$$(8(k+r)(k+r-1)+2)c_k + (k+r-2)c_{k-1} = 0 \qquad k = 1, 2, 3, \dots$$

or

$$(2r-1)^2 = 0 \qquad k = 0$$

$$c_k = -\frac{(k+r-2)c_{k-1}}{8(k+r)(k+r-1)+2} \qquad k = 1, 2, 3, \dots$$

The indicial equation gives us $r_1 = r_2 = 1/2$. Substituting this indicial root into the recurrence relation, we get

For $r_1 = r_2 = 1/2$

$$c_k = -\frac{(k-\frac{3}{2})c_{k-1}}{8(k+\frac{1}{2})(k-\frac{1}{2})+2}$$

$$c_k = -\frac{(2k-3)c_{k-1}}{16k^2}$$

$k = 1$ $\quad c_1 = \frac{1}{16}c_0$

$k = 2$ $\quad c_2 = -\frac{1}{1{,}024}c_0$

$k = 3$ $\quad c_3 = \frac{1}{49{,}152}c_0$

... ...

Forming these coefficients into series around $x_0 = 0$ gives,

$$c_0(x-x_0)^0 + c_1(x-x_0)^1 + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \ldots$$

$$c_0x^0 + \frac{1}{16}c_0x^1 - \frac{1}{1{,}024}c_0x^2 + \frac{1}{49{,}152}c_0x^3 + \ldots$$

$$c_0\left(1 + \frac{1}{16}x - \frac{1}{1{,}024}x^2 + \frac{1}{49{,}152}x^3 + \ldots\right)$$

So the series solutions are

$$y_1(x) = c_0 x^{\frac{1}{2}} \left(1 + \frac{1}{16}x - \frac{1}{1{,}024}x^2 + \frac{1}{49{,}152}x^3 + \dots \right)$$

and

$$y_2(x) = y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n+r_1}$$

$$y_2(x) = c_0 x^{\frac{1}{2}} \ln x \left(1 + \frac{1}{16}x - \frac{1}{1{,}024}x^2 + \frac{1}{49{,}152}x^3 + \dots \right) + \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

and the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$y(x) = C_1 c_0 x^{\frac{1}{2}} \left(1 + \frac{1}{16}x - \frac{1}{1{,}024}x^2 + \frac{1}{49{,}152}x^3 + \dots \right)$$

$$+ C_2 \left(c_0 x^{\frac{1}{2}} \ln x \left(1 + \frac{1}{16}x - \frac{1}{1{,}024}x^2 + \frac{1}{49{,}152}x^3 + \dots \right) + \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}} \right)$$

$$y(x) = C_1 x^{\frac{1}{2}} \left(1 + \frac{1}{16}x - \frac{1}{1{,}024}x^2 + \frac{1}{49{,}152}x^3 + \dots \right)$$

$$+ C_2 x^{\frac{1}{2}} \ln x \left(1 + \frac{1}{16}x - \frac{1}{1{,}024}x^2 + \frac{1}{49{,}152}x^3 + \dots \right) + C_2 \sum_{n=0}^{\infty} b_n x^{n+\frac{1}{2}}$$

■ 5. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$3x^2y'' + 3xy' - 2xy = 0$$

Solution:

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = 3x^2$$

$$q(x) = 3x$$

$$r(x) = -2x$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = x\frac{3x}{3x^2} = 1$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{-2x}{3x^2} = -\frac{2x}{3}$$

Because both denominators simplify to constants, we can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$3x^2y'' + 3xy' - 2xy = 0$$

$$3x^2 \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + 3x \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$-2x \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)(n+r-1)c_n x^{n+r} + \sum_{n=0}^{\infty} 3(n+r)c_n x^{n+r} - \sum_{n=0}^{\infty} 2c_n x^{n+r+1} = 0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty} (3(n+r)(n+r-1) + 3(n+r))c_n x^{n+r} - \sum_{n=0}^{\infty} 2c_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)((n+r-1)+1)c_n x^{n+r} - \sum_{n=0}^{\infty} 2c_n x^{n+r+1} = 0$$

$$\sum_{n=0}^{\infty} 3(n+r)^2 c_n x^{n+r} - \sum_{n=0}^{\infty} 2c_n x^{n+r+1} = 0$$

$$x^r \left[\sum_{n=0}^{\infty} 3(n+r)^2 c_n x^n - \sum_{n=0}^{\infty} 2c_n x^{n+1} \right] = 0$$

The first series starts with the x^0 term, while the second series starts with the x^1 term, so we'll pull the x^0 term out of the first series.

$$x^r \left[3r^2c_0 + \sum_{n=1}^{\infty} 3(n+r)^2 c_n x^n - \sum_{n=0}^{\infty} 2c_n x^{n+1}\right] = 0$$

Now that the series are in phase, we'll substitute $k = n + 1$ and $n = k - 1$ into the second series, and $k = n$ into the first series.

$$x^r \left[3r^2c_0 + \sum_{k=1}^{\infty} 3(k+r)^2 c_k x^k - \sum_{k=1}^{\infty} 2c_{k-1} x^k\right] = 0$$

Now that the series are in phase with matching indices, combine them.

$$x^r \left[3r^2c_0 + \sum_{k=1}^{\infty} [3(k+r)^2 c_k - 2c_{k-1}]x^k\right] = 0$$

This equation gives

$$3r^2c_0 = 0 \qquad k = 0$$

$$3(k+r)^2c_k - 2c_{k-1} = 0 \qquad k = 1, 2, 3, \ldots$$

or

$$r_1 = r_2 = 0 \qquad k = 0$$

$$c_k = \frac{2c_{k-1}}{3(k+r)^2} \qquad k = 1, 2, 3, \ldots$$

The indicial equation gives us $r_1 = r_2 = 0$. Substituting this indicial root into the recurrence relation, we get

For $r_1 = r_2 = 0$

$$c_{k+1} = \frac{2c_k}{3(k+1)^2}$$

$k = 1$ $$c_2 = \frac{2^2}{3^2 \cdot 2^2}c_0$$

$k = 2$ $$c_3 = \frac{2^3}{3^3 \cdot 3^2 \cdot 2^2}c_0$$

... ...

Forming these coefficients into series around $x_0 = 0$ gives,

$$c_0(x-0)^0 + \frac{2}{3}c_0(x-0)^1 + \frac{2^2}{3^2(2!)^2}c_0(x-0)^2 + \dots$$

So the series solutions are

$$y_1(x) = \sum_{k=0}^{\infty} \frac{2^k}{3^k((k+1)!)^2}c_0 \cdot x^k = c_0 \sum_{k=0}^{\infty} \frac{2^k}{3^k((k+1)!)^2}x^k$$

$$y_2(x) = c_0 \ln x \sum_{k=0}^{\infty} \frac{2^k}{3^k((k+1)!)^2}x^k + b_0 \sum_{k=0}^{\infty} \frac{2^k}{3^k((k+1)!)^2}x^k$$

and the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$y(x) = C_1 c_0 \sum_{k=0}^{\infty} \frac{2^k}{3^k((k+1)!)^2}x^k + C_2 c_0 \ln x \sum_{k=0}^{\infty} \frac{2^k}{3^k((k+1)!)^2}x^k$$

$$+C_2b_0\sum_{k=0}^{\infty}\frac{2^k}{3^k((k+1)!)^2}x^k$$

$$y(x)=C_1\sum_{k=0}^{\infty}\frac{2^k}{3^k((k+1)!)^2}x^k+C_2\ln x\sum_{k=0}^{\infty}\frac{2^k}{3^k((k+1)!)^2}x^k$$

$$+C_3\sum_{k=0}^{\infty}\frac{2^k}{3^k((k+1)!)^2}x^k$$

■ 6. Determine the regularity of the singular point $x_0=-1$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$x(x+1)y''+3(x+1)y'+y=0$$

Solution:

Matching this differential equation to the standard form $p(x)y''+q(x)y'+r(x)y=0$, we can identify

$$p(x)=x(x+1)$$

$$q(x)=3(x+1)$$

$$r(x)=1$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = (x+1)\frac{3(x+1)}{x(x+1)} = \frac{3(x+1)}{x}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = (x+1)^2\frac{1}{x(x+1)} = \frac{x+1}{x}$$

$$\lim_{x \to -1} Q(x) = \lim_{x \to -1} R(x) = 0$$

Because the denominators of $Q(x)$ and $R(x)$ are constants, these functions are analytic about $x_0 = -1$, so $x_0 = -1$ is a regular singular point.

Now we'll use

$$y = \sum_{n=0}^{\infty} c_n(x+1)^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n(x+1)^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n(x+1)^{n+r-2}$$

to make substitutions into the differential equation.

$$x(x+1)y'' + 3(x+1)y' + y = 0$$

$$x(x+1)\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n(x+1)^{n+r-2} + 3(x+1)\sum_{n=0}^{\infty} (n+r)c_n(x+1)^{n+r-1}$$

$$+\sum_{n=0}^{\infty} c_n(x+1)^{n+r} = 0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(x+1)^{n+r-2}+x\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(x+1)^{n+r-2}$$

$$+3x\sum_{n=0}^{\infty}(n+r)c_n(x+1)^{n+r-1}+3\sum_{n=0}^{\infty}(n+r)c_n(x+1)^{n+r-1}$$

$$+\sum_{n=0}^{\infty}c_n(x+1)^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(x+1)^{n+r}+\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n(x+1)^{n+r-1}$$

$$+\sum_{n=0}^{\infty}3(n+r)c_n(x+1)^{n+r}+\sum_{n=0}^{\infty}3(n+r)c_n(x+1)^{n+r-1}$$

$$+\sum_{n=0}^{\infty}c_n(x+1)^{n+r}=0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty}[(n+r)(n+r-1)+3(n+r)+1]c_n(x+1)^{n+r}$$

$$+\sum_{n=0}^{\infty}[(n+r)(n+r-1)+3(n+r)]c_n(x+1)^{n+r-1}=0$$

$$\sum_{n=0}^{\infty}[(n+r)(n+r+2)+1]c_n(x+1)^{n+r}$$

$$+\sum_{n=0}^{\infty}(n+r)(n+r+2)c_n(x+1)^{n+r-1}=0$$

$$(x+1)^r \left[\sum_{n=0}^{\infty} [(n+r)(n+r+2)+1]c_n(x+1)^n \right.$$

$$\left. + \sum_{n=0}^{\infty} (n+r)(n+r+2)c_n(x+1)^{n-1} \right] = 0$$

The first series starts with the $(x+1)^0$ term, while the second series starts with the $(x+1)^{-1}$ term, so we'll pull the $(x+1)^{-1}$ term out of the second series.

$$(x+1)^r \left[\sum_{n=0}^{\infty} [(n+r)(n+r+2)+1]c_n(x+1)^n \right.$$

$$\left. + r(r+2)c_0(x+1)^{-1} + \sum_{n=1}^{\infty} (n+r)(n+r+2)c_n(x+1)^{n-1} \right] = 0$$

Now that the series are in phase, we'll substitute $k=n-1$ and $n=k+1$ into the second series, and $k=n$ into the first series.

$$(x+1)^r \left[\sum_{k=0}^{\infty} [(k+r)(k+r+2)+1]c_k(x+1)^k \right.$$

$$\left. + r(r+2)c_0(x+1)^{-1} + \sum_{k=0}^{\infty} (k+r+1)(k+r+3)c_{k+1}(x+1)^k \right] = 0$$

Now that the series are in phase with matching indices, combine them.

$$(x+1)^r \left[r(r+2)c_0(x+1)^{-1} \right.$$

$$+\sum_{k=0}^{\infty}[[(k+r)(k+r+2)+1]c_k+(k+r+1)(k+r+3)c_{k+1}](x+1)^k\Bigg]=0$$

This equation gives

$$r(r+2)c_0=0$$

$$[(k+r)(k+r+2)+1]c_k+(k+r+1)(k+r+3)c_{k+1}=0 \quad k=0,\ 1,\ 2,\ \dots$$

or

$$r(r+2)=0$$

$$c_{k+1}=-\frac{[(k+r)(k+r+2)+1]c_k}{(k+r+1)(k+r+3)} \qquad k=0,\ 1,\ 2,\ \dots$$

The indicial equation gives us $r_1=0$ and $r_2=-2$. Substituting these indicial roots into the recurrence relation, we get

	For $r_1=0$	For $r_2=-2$
	$c_{k+1}=-\dfrac{(k+1)c_k}{k+3}$	$c_{k+1}=-\dfrac{(k-1)c_k}{k+1}$
$k=0$	$c_1=-\dfrac{c_0}{3}$	$c_1=c_0$
$k=1$	$c_2=\dfrac{c_0}{6}$	$c_2=0$
$k=2$	$c_3=-\dfrac{c_0}{10}$	$c_3=0$
...	...	...

Forming these coefficients into series around $x_0 = -1$ gives,

$$c_0(x - x_0)^0 + c_1(x - x_0)^1 + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

$$c_0(x + 1)^0 - \frac{c_0}{3}(x + 1)^1 + \frac{c_0}{6}(x + 1)^2 - \frac{c_0}{10}(x + 1)^3 + \dots$$

$$c_0 \left(-\frac{1}{3}(x + 1) + \frac{1}{6}(x + 1)^2 - \frac{1}{10}(x + 1)^3 + \dots \right)$$

and

$$c_0(x - x_0)^0 + c_1(x - x_0)^1 + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

$$c_0(x + 1)^0 + c_0(x + 1)^1 + 0(x + 1)^2 + 0(x + 1)^3 + \dots$$

$$c_0(1 + (x + 1))$$

$$c_0(x + 2)$$

So the series solutions are

$$y_1(x) = c_0 \left(-\frac{1}{3}(x + 1) + \frac{1}{6}(x + 1)^2 - \frac{1}{10}(x + 1)^3 + \dots \right)$$

$$y_2(x) = c_0 x^{-2}(x + 2)$$

and the general solution is

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

$$y(x) = C_1 c_0 \left(-\frac{1}{3}(x + 1) + \frac{1}{6}(x + 1)^2 - \frac{1}{10}(x + 1)^3 + \dots \right) + C_2 c_0 x^{-2}(x + 2)$$

$$y(x) = C_1\left(-\frac{1}{3}(x+1) + \frac{1}{6}(x+1)^2 - \frac{1}{10}(x+1)^3 + \dots\right) + C_2x^{-2}(x+2)$$

THE LAPLACE TRANSFORM

■ 1. Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(t+5)$$

Solution:

We're transforming $f(t) = t + 5$, so we'll plug the first term, $f(t) = t$, into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

$$F(s) = \int_0^\infty te^{-st}\ dt$$

Now we can integrate and evaluate over the interval.

$$F(s) = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\Bigg|_0^\infty$$

$$F(s) = \lim_{t \to \infty}\left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right) - \left(-\frac{(0)e^{-s(0)}}{s} - \frac{e^{-s(0)}}{s^2}\right)$$

$$F(s) = \lim_{t \to \infty}\left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right) + \frac{1}{s^2}$$

If we evaluate the limit assuming $s > 0$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{1}{s^2}$$

Now we're transforming $f(t) = 5$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t) \ dt$$

$$F(s) = \int_0^\infty 5e^{-st} \ dt$$

Now we can integrate and evaluate over the interval.

$$F(s) = \frac{5}{-s} e^{-st} \Big|_0^\infty$$

$$F(s) = \lim_{t \to \infty} \left(\frac{5}{-s} e^{-st} \right) - \frac{5}{-s} e^{-s(0)}$$

$$F(s) = \lim_{t \to \infty} \left(\frac{5}{-s} e^{-st} \right) + \frac{5}{s}$$

If we evaluate the limit assuming $s > 0$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{5}{s}$$

So the Laplace transform of $\mathscr{L}(t + 5)$ is

$$F(s) = \frac{1}{s^2} + \frac{5}{s}$$

■ 2. Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(\sin t)$$

Solution:

We're transforming $f(t) = \sin t$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

$$F(s) = \int_0^\infty e^{-st} \sin t\ dt$$

Now we can integrate with integration by parts using $u = e^{-st}$, $du = -se^{-st}\ dt$, $dv = \sin t\ dt$, and $v = -\cos t$.

$$\int e^{-st} \sin t\ dt = (e^{-st})(-\cos t) - \int (-\cos t)(-se^{-st})\ dt$$

$$\int e^{-st} \sin t\ dt = -e^{-st} \cos t - s \int e^{-st} \cos t\ dt$$

Use integration by parts using $u = e^{-st}$, $du = -se^{-st}\ dt$, $dv = \cos t\ dt$, and $v = \sin t$.

$$\int e^{-st}\sin t\ dt = -e^{-st}\cos t - s\left(e^{-st}\sin t - \int (-se^{-st})\sin t\ dt\right)$$

$$\int e^{-st}\sin t\ dt = -e^{-st}\cos t - se^{-st}\sin t - s^2\int e^{-st}\sin t\ dt$$

Move the integral to the left side to consolidate like integrals.

$$\int e^{-st}\sin t\ dt + s^2\int e^{-st}\sin t\ dt = -e^{-st}\cos t - se^{-st}\sin t$$

$$(1+s^2)\int e^{-st}\sin t\ dt = -e^{-st}\cos t - se^{-st}\sin t$$

$$\int e^{-st}\sin t\ dt = -\frac{1}{s^2+1}e^{-st}\cos t - \frac{s}{s^2+1}e^{-st}\sin t$$

$$\int e^{-st}\sin t\ dt = -\frac{e^{-st}}{s^2+1}(\cos t + s\sin t)$$

$$\int_0^\infty e^{-st}\sin t\ dt = -\frac{e^{-st}}{s^2+1}(\cos t + s\sin t)\Big|_0^\infty$$

Evaluate over the interval.

$$F(s) = \lim_{t\to\infty}\left(-\frac{e^{-st}}{s^2+1}(\cos t + s\sin t)\right) - \left(-\frac{e^{-s(0)}}{s^2+1}(\cos(0) + s\sin(0))\right)$$

$$F(s) = \lim_{t\to\infty}\left(-\frac{e^{-st}}{s^2+1}(\cos t + s\sin t)\right) + \frac{1}{s^2+1}$$

If we evaluate the limit assuming $s > 0$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{1}{s^2 + 1}$$

■ 3. Find the Laplace transform, given $s > 1$.

$$\mathscr{L}(e^t)$$

Solution:

We're transforming $f(t) = e^t$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t) \ dt$$

$$F(s) = \int_0^\infty e^{-st} e^t \ dt$$

$$F(s) = \int_0^\infty e^{-st+t} \ dt$$

$$F(s) = \int_0^\infty e^{(-s+1)t} \ dt$$

Now we can integrate and evaluate over the interval.

$$F(s) = \frac{1}{-s+1} e^{(-s+1)t} \Big|_0^\infty$$

$$F(s) = \lim_{t \to \infty} \left(\frac{1}{-s+1} e^{(-s+1)t} \right) - \frac{1}{-s+1} e^{(-s+1)(0)}$$

$$F(s) = \lim_{t \to \infty} \left(\frac{1}{-s+1} e^{(-s+1)t} \right) + \frac{1}{s-1}$$

If we evaluate the limit assuming $-s+1<0$, or $s>1$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{1}{s-1}$$

■ 4. Find the Laplace transform, given $s>0$.

$$\mathscr{L}(5t)$$

Solution:

We're transforming $f(t)=5t$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

$$F(s) = 5\int_0^\infty te^{-st}\ dt$$

Now we can integrate with integration by parts using $u=t$, $du=dt$, $dv=e^{-st}\ dt$, and $v=(-1/s)e^{-st}$.

$$\int te^{-st}\ dt = t\left(-\frac{1}{s}\right)e^{-st} - \int\left(-\frac{1}{s}\right)e^{-st}\ dt$$

$$\int te^{-st}\ dt = -\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st}\ dt$$

$$\int te^{-st}\ dt = -\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}$$

Evaluate over the interval.

$$F(s) = 5\left(-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right)\Bigg|_0^\infty$$

$$F(s) = \frac{-5te^{-st}}{s} - \frac{5e^{-st}}{s^2}\Bigg|_0^\infty$$

$$F(s) = \lim_{t\to\infty}\left(\frac{-5te^{-st}}{s} - \frac{5e^{-st}}{s^2}\right) - \left(\frac{-5(0)e^{-s(0)}}{s} - \frac{5e^{-s(0)}}{s^2}\right)$$

$$F(s) = \lim_{t\to\infty}\left(\frac{-5te^{-st}}{s} - \frac{5e^{-st}}{s^2}\right) + \frac{5}{s^2}$$

If we evaluate the limit assuming $s > 0$, the exponentials tend toward 0 and the Laplace transform is

$$F(s) = \frac{5}{s^2}$$

■ 5. Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(7)$$

Solution:

We're transforming $f(t) = 7$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

$$F(s) = \int_0^\infty 7e^{-st}\ dt$$

Now we can integrate and evaluate over the interval.

$$F(s) = \frac{7}{-s} e^{-st} \Big|_0^\infty$$

$$F(s) = \lim_{t \to \infty} \left(\frac{7}{-s} e^{-st} \right) - \frac{7}{-s} e^{-s(0)}$$

$$F(s) = \lim_{t \to \infty} \left(\frac{7}{-s} e^{-st} \right) + \frac{7}{s}$$

If we evaluate the limit assuming $s > 0$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{7}{s}$$

■ 6. Find the Laplace transform, given $s>0$.

$$\mathscr{L}(\cos t)$$

Solution:

We're transforming $f(t)=\cos t$, so we'll plug this into the definition of the Laplace transform.

$$F(s)=\int_0^\infty e^{-st}f(t)\ dt$$

$$F(s)=\int_0^\infty e^{-st}\cos t\ dt$$

Now we can integrate with integration by parts using $u=e^{-st}$, $du=-se^{-st}\ dt$, $dv=\cos t\ dt$, and $v=\sin t$.

$$\int e^{-st}\cos t\ dt=e^{-st}\sin t-\int \sin t(-se^{-st}\ dt)$$

$$\int e^{-st}\cos t\ dt=e^{-st}\sin t+s\int e^{-st}\sin t\ dt$$

Use integration by parts using $u=e^{-st}$, $du=-se^{-st}\ dt$, $dv=\sin t\ dt$, and $v=-\cos t$.

$$\int e^{-st}\cos t\ dt=e^{-st}\sin t+s\left(e^{-st}(-\cos t)-\int(-\cos t)(-se^{-st}\ dt)\right)$$

$$\int e^{-st}\cos t\ dt=e^{-st}\sin t-se^{-st}\cos t-s^2\int e^{-st}\cos t\ dt$$

Move the integral to the left side to consolidate like integrals.

$$\int e^{-st}\cos t\ dt + s^2\int e^{-st}\cos t\ dt = e^{-st}\sin t - se^{-st}\cos t$$

$$(1+s^2)\int e^{-st}\cos t\ dt = e^{-st}\sin t - se^{-st}\cos t$$

$$\int e^{-st}\cos t\ dt = \frac{1}{s^2+1}e^{-st}\sin t - \frac{s}{s^2+1}e^{-st}\cos t$$

Evaluate over the interval.

$$F(s) = \frac{1}{s^2+1}e^{-st}\sin t - \frac{s}{s^2+1}e^{-st}\cos t\Big|_0^\infty$$

$$F(s) = \lim_{t\to\infty}\left(\frac{1}{s^2+1}e^{-st}\sin t - \frac{s}{s^2+1}e^{-st}\cos t\right)$$

$$-\left(\frac{1}{s^2+1}e^{-s(0)}\sin(0) - \frac{s}{s^2+1}e^{-s(0)}\cos(0)\right)$$

$$F(s) = \lim_{t\to\infty}\left(\frac{1}{s^2+1}e^{-st}\sin t - \frac{s}{s^2+1}e^{-st}\cos t\right) + \frac{s}{s^2+1}$$

If we evaluate the limit assuming $s > 0$, the exponentials tend toward 0 and the Laplace transform is

$$F(s) = \frac{s}{s^2+1}$$

TABLE OF TRANSFORMS

■ 1. Use a table of Laplace transforms to transform the function.

$$f(t) = \cos(3t) + 5e^{-7t}$$

Solution:

From the table of Laplace transforms, we know

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

$$\mathscr{L}(e^{at}) = \frac{1}{s - a}$$

When we apply these transform formulas to the terms in our function, we get

$$\mathscr{L}(\cos(3t)) = \frac{s}{s^2 + 3^2} = \frac{s}{s^2 + 9}$$

$$\mathscr{L}(e^{-7t}) = \frac{1}{s - (-7)} = \frac{1}{s + 7}$$

Then the Laplace transform of $f(t) = 5 + 3t + t^2$ is

$$F(s) = \frac{s}{s^2 + 9} + 5\left(\frac{1}{s + 7}\right)$$

$$F(s) = \frac{s}{s^2+9} + \frac{5}{s+7}$$

2. Use a table of Laplace transforms to transform the function.

$$f(t) = t^2 + 3t - e^{3t}$$

Solution:

From the table of Laplace transforms, we know

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

With $n = 1$ in $3t$ and $n = 2$ in t^2, we find

$$\mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(t^2) = \frac{2}{s^3}$$

$$\mathscr{L}(e^{3t}) = \frac{1}{s-3}$$

Then the Laplace transform of $f(t) = t^2 + 3t - e^{3t}$ is

$$F(s) = \frac{2}{s^3} + 3\left(\frac{1}{s}\right) - \frac{1}{s-3}$$

$$F(s) = \frac{2}{s^3} + \frac{3}{s} - \frac{1}{s-3}$$

■ 3. Use a table of Laplace transforms to transform the function.

$$f(t) = 2t^4 - 3\sin(2t)$$

Solution:

From the table of Laplace transforms, we know

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2 + a^2}$$

When we apply these transform formulas to the terms in our function, we get

$$\mathscr{L}(t^4) = \frac{4!}{s^5} = \frac{24}{s^5}$$

$$\mathscr{L}(\sin(2t)) = \frac{2}{s^2 + 2^2} = \frac{2}{s^2 + 4}$$

Then the Laplace transform of $f(t) = 2t^4 - 3\sin(2t)$ is

$$F(s) = 2\left(\frac{24}{s^5}\right) - 3\left(\frac{2}{s^2 + 4}\right)$$

$$F(s)=\frac{48}{s^5}-\frac{6}{s^2+4}$$

■ 4. Use a table of Laplace transforms to transform the function.

$$f(t)=e^{-5t}-t\sin(3t)$$

Solution:

From the table of Laplace transforms, we know

$$\mathscr{L}(e^{at})=\frac{1}{s-a}$$

$$\mathscr{L}(t\sin(at))=\frac{2as}{(s^2+a^2)^2}$$

When we apply these transform formulas to the terms in our function, we get

$$\mathscr{L}(e^{-5t})=\frac{1}{s-(-5)}=\frac{1}{s+5}$$

$$\mathscr{L}(t\sin(3t))=\frac{2(3)s}{(s^2+3^2)^2}=\frac{6s}{(s^2+9)^2}$$

Then the Laplace transform of $f(t)=e^{-5t}-t\sin(3t)$ is

$$F(s)=\frac{1}{s+5}-\frac{6s}{(s^2+9)^2}$$

■ 5. Use a table of Laplace transforms to transform the function.

$$f(t) = e^{6t}\cos t + t^4$$

Solution:

From the table of Laplace transforms, we know

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathscr{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2}$$

When we apply these transform formulas to the terms in our function, we get

$$\mathscr{L}(t^4) = \frac{4!}{s^{4+1}} = \frac{24}{s^5}$$

$$\mathscr{L}(e^{6t}\cos t) = \frac{s-6}{(s-6)^2+1^2} = \frac{s-6}{(s-6)^2+1}$$

Then the Laplace transform of $f(t) = e^{6t}\cos t + t^4$ is

$$F(s) = \frac{s-6}{(s-6)^2+1} + \frac{24}{s^5}$$

■ 6. Use a table of Laplace transforms to transform the function.

$$f(t) = \cos(5t) + 3t^3 - e^{-3t}$$

Solution:

From the table of Laplace transforms, we know

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathscr{L}(e^{at}) = \frac{1}{s - a}$$

When we apply these transform formulas to the terms in our function, we get

$$\mathscr{L}(\cos(5t)) = \frac{s}{s^2 + 5^2} = \frac{s}{s^2 + 25}$$

$$\mathscr{L}(t^3) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

$$\mathscr{L}(e^{-3t}) = \frac{1}{s - (-3)} = \frac{1}{s + 3}$$

Then the Laplace transform of $f(t) = \cos(5t) + 3t^3 - e^{-3t}$ is

$$F(s) = \frac{s}{s^2 + 25} + 3\left(\frac{6}{s^4}\right) - \left(\frac{1}{s + 3}\right)$$

$$F(s) = \frac{s}{s^2 + 25} + \frac{18}{s^4} - \frac{1}{s + 3}$$

EXPONENTIAL TYPE

■ 1. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = c$, is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t \to \infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Solution:

Substituting $f(t)$ into the limit gives

$$\lim_{t \to \infty} \frac{f(t)}{e^{\alpha t}} \to \lim_{t \to \infty} \frac{c}{e^{\alpha t}}$$

The value of the numerator will remain constant while the value of the denominator grows exponentially for any $\alpha > 0$. So the limit will go to 0. Therefore $f(t) = c$ is of exponential type for any $\alpha > 0$.

■ 2. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = t$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t \to \infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Solution:

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{t}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{e^{\ln(t)}}{e^{\alpha t}} \to \lim_{t\to\infty} e^{\ln(t)-\alpha t}$$

We know that the only way for this limit to approach 0 is if the exponent tends to $-\infty$, i.e., $\ln(t) - \alpha t \to -\infty$ as $t \to \infty$. This is always true, since logarithmic growth is always slower than linear growth. Therefore $f(t) = t$ is of exponential type for any $\alpha > 0$.

■ 3. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = t^2 + t$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}} = 0$$

Solution:

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{t^2+t}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{t^2}{e^{\alpha t}} + \lim_{t\to\infty}\frac{t}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{e^{2\ln(t)}}{e^{\alpha t}} + \lim_{t\to\infty}\frac{e^{\ln(t)}}{e^{\alpha t}}$$

We just need to compare the growth of the exponents in both limits, and we see that it's a comparison between logarithmic growth and linear growth. So for any $\alpha > 0$, the denominator will grow faster than the numerator in both limits. Therefore $f(t) = t^2 + t$ is of exponential type for any $\alpha > 0$.

■ 4. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = t\cos(bt)$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Solution:

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty} \frac{t\cos(bt)}{e^{\alpha t}}$$

We know the cosine function satisfies $-1 \leq \cos t \leq 1$, so

$$\lim_{t\to\infty} \frac{-t}{e^{\alpha t}} \leq \lim_{t\to\infty} \frac{t\cos(bt)}{e^{\alpha t}} \leq \lim_{t\to\infty} \frac{t}{e^{\alpha t}}$$

$$-\lim_{t\to\infty} \frac{t}{e^{\alpha t}} \leq \lim_{t\to\infty} \frac{t\cos(bt)}{e^{\alpha t}} \leq \lim_{t\to\infty} \frac{t}{e^{\alpha t}}$$

Since t is of exponential type for any $\alpha > 0$, it follows that

$$0 \leq \lim_{t\to\infty} \frac{t\cos(bt)}{e^{\alpha t}} \leq 0$$

Therefore $f(t) = t\cos(bt)$ is of exponential type for any $\alpha > 0$.

■ 5. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = \ln(t)$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Solution:

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty} \frac{\ln(t)}{e^{\alpha t}}$$

We have a ratio between two expressions depending on t, but we know the denominator grows faster than the numerator, such that for any $\alpha > 0$ this limit will go to 0. So $f(t) = \ln(t)$ is of exponential type for any $\alpha > 0$.

■ 6. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = te^{bt}$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Solution:

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty} \frac{te^{bt}}{e^{\alpha t}} \to \lim_{t\to\infty} te^{(b-\alpha)t}$$

We know that the only way for this limit to approach 0 is if the exponent is negative, i.e., $b-\alpha<0$. Then, for any $\alpha > b$,

$$\lim_{t\to\infty} te^{(b-\alpha)t} = 0$$

Therefore, $f(t) = te^{bt}$ is of exponential type for any $\alpha > b$.

PARTIAL FRACTIONS DECOMPOSITIONS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4}$$

Solution:

The decomposition with repeated linear factors is

$$\frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4} = \frac{A}{x-2} + \frac{B}{(x-2)^2} + \frac{C}{(x-2)^3} + \frac{D}{(x-2)^4}$$

Multiply through the equation by the denominator from the left side.

$$x^3 - 2x^2 + 2x - 3 = A(x-2)^3 + B(x-2)^2 + C(x-2) + D$$

$$x^3 - 2x^2 + 2x - 3 = Ax^3 + (-6A + B)x^2$$

$$+(12A - 4B + C)x + (-8A + 4B - 2C + D)$$

Build a system of equations.

$$A = 1$$

$$-6A + B = -2$$

$$12A - 4B + C = 2$$

$$-8A + 4B - 2C + D = -3$$

Then we can say

$$-6A + B = -2$$

$$-6 + B = -2$$

$$B = 4$$

and

$$12A - 4B + C = 2$$

$$12 - 16 + C = 2$$

$$C = 6$$

and finally

$$-8A + 4B - 2C + D = -3$$

$$-8 + 16 - 12 + D = -3$$

$$D = 1$$

Plug $A = 1$, $B = 4$, $C = 6$, and $D = 1$ into the decomposition.

$$f(x) = \frac{1}{x-2} + \frac{4}{(x-2)^2} + \frac{6}{(x-2)^3} + \frac{1}{(x-2)^4}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 4x^2 - 10}{x^2(x+1)(x-1)}$$

Solution:

The decomposition with distinct linear factors is

$$\frac{x^3 + 4x^2 - 10}{x^2(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{x-1}$$

To solve for B, remove the x^2 factor, then substitute $x = 0$ into the left side.

$$\frac{x^3 + 4x^2 - 10}{(x+1)(x-1)} \to \frac{0^3 + 4(0)^2 - 10}{(0+1)(0-1)} \to \frac{-10}{-1} \to 10$$

To solve for C, remove the $x + 1$ factor, then substitute $x = -1$ into the left side.

$$\frac{x^3 + 4x^2 - 10}{x^2(x-1)} \to \frac{(-1)^3 + 4(-1)^2 - 10}{(-1)^2(-1-1)} \to \frac{-7}{-2} \to \frac{7}{2}$$

To solve for D, remove the $x - 1$ factor, then substitute $x = 1$ into the left side.

$$\frac{x^3 + 4x^2 - 10}{x^2(x+1)} \to \frac{1^3 + 4(1)^2 - 10}{1^2(1+1)} \to \frac{-5}{2} \to -\frac{5}{2}$$

To solve for A, substitute $B = 10$, $C = 7/2$, and $D = -5/2$, plus a value for x that we haven't used yet ($x \neq -1, 0, 1$), like $x = 2$.

$$\frac{2^3+4(2)^2-10}{2^2(2+1)(2-1)}=\frac{A}{2}+\frac{10}{2^2}+\frac{\frac{7}{2}}{2+1}+\frac{-\frac{5}{2}}{2-1}$$

$$\frac{7}{6}=\frac{A}{2}+\frac{5}{2}+\frac{7}{6}-\frac{5}{2}$$

$$\frac{A}{2}=0$$

$$A=0$$

Plug $A=0$, $B=10$, $C=7/2$, and $D=-5/2$ into the decomposition.

$$f(x)=\frac{0}{x}+\frac{10}{x^2}+\frac{\frac{7}{2}}{x+1}+\frac{-\frac{5}{2}}{x-1}$$

$$f(x)=10\left(\frac{1}{x^2}\right)+\frac{7}{2}\left(\frac{1}{x+1}\right)-\frac{5}{2}\left(\frac{1}{x-1}\right)$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{2}{(x-1)(x+1)}$$

Solution:

The decomposition with distinct linear factors is

$$\frac{2}{(x-1)(x+1)}=\frac{A}{x-1}+\frac{B}{x+1}$$

To solve for A, remove the $x-1$ factor, then substitute $x=1$ into the left side.

$$\frac{2}{x+1} \to \frac{2}{1+1} \to 1$$

To solve for B, remove the $x+1$ factor, then substitute $x=-1$ into the left side.

$$\frac{2}{x-1} \to \frac{2}{-1-1} \to -1$$

Plug $A=1$ and $B=-1$ into the decomposition.

$$f(x)=\frac{1}{x-1}-\frac{1}{x+1}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{2x^4+16}{x(x^2+2)^2}$$

Solution:

The decomposition with distinct linear and repeated quadratic factors is

$$\frac{2x^4+16}{x(x^2+2)^2}=\frac{A}{x}+\frac{Bx+C}{x^2+2}+\frac{Dx+E}{(x^2+2)^2}$$

Multiply through the equation by the denominator from the left side.

$$2x^4 + 16 = A(x^2 + 2)^2 + (Bx + C)x(x^2 + 2) + (Dx + E)x$$

$$2x^4 + 16 = (A + B)x^4 + Cx^3 + (4A + 2B + D)x^2 + (2C + E)x + 4A$$

Build a system of equations.

$$A + B = 2$$

$$C = 0$$

$$4A + 2B + D = 0$$

$$2C + E = 0$$

$$4A = 16$$

We have $A = 4$ and $C = 0$, so $B = -2$ and $E = 0$. Then $D = -12$. Plug these values into the decomposition.

$$f(x) = \frac{4}{x} + \frac{-2x + 0}{x^2 + 2} + \frac{-12x + 0}{(x^2 + 2)^2}$$

$$f(x) = \frac{4}{x} - \frac{2x}{x^2 + 2} - \frac{12x}{(x^2 + 2)^2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x + 2}{(x - 1)^2}$$

Solution:

The decomposition with repeated linear factors is

$$\frac{x+2}{(x-1)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2}$$

Multiply through the equation by the denominator from the left side.

$$x + 2 = Ax + (-A + B)$$

Then $A = 1$, and $-A + B = 2$, so $B = 3$. Plug $A = 1$ and $B = 3$ into the decomposition.

$$f(x) = \frac{1}{x-1} + \frac{3}{(x-1)^2}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 2x^2 - 3x + 1}{x^3(x^2+1)^2}$$

Solution:

The decomposition with repeated linear and repeated quadratic factors is

$$\frac{x^3 + 2x^2 - 3x + 1}{x^3(x^2+1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

Multiply through the equation by the denominator from the left side.

$$x^3+2x^2-3x+1=Ax^2(x^2+1)^2+Bx(x^2+1)^2$$

$$+C(x^2+1)^2+(Dx+E)x^3(x^2+1)+(Fx+G)x^3$$

$$x^3+2x^2-3x+1=Ax^6+2Ax^4+Ax^2+Bx^5+2Bx^3+Bx$$

$$+Cx^4+2Cx^2+C+Dx^6+Dx^4+Ex^5+Ex^3+Fx^4+Gx^3$$

$$x^3+2x^2-3x+1=(A+D)x^6+(B+E)x^5+(2A+C+D+F)x^4$$

$$+(2B+E+G)x^3+(A+2C)x^2+Bx+C$$

Build a system of equations.

$$A+D=0$$

$$B+E=0$$

$$2A+C+D+F=0$$

$$2B+E+G=1$$

$$A+2C=2$$

$$B=-3$$

$$C=1$$

We have $B=-3$ and $C=1$, so $A=0$ and $E=3$. Then $D=0$, $G=4$, and $F=-1$. Plug these values into the decomposition.

$$f(x)=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x^3}+\frac{Dx+E}{x^2+1}+\frac{Fx+G}{(x^2+1)^2}$$

$$f(x) = \frac{0}{x} + \frac{-3}{x^2} + \frac{1}{x^3} + \frac{0x+3}{x^2+1} + \frac{-x+4}{(x^2+1)^2}$$

$$f(x) = -\frac{3}{x^2} + \frac{1}{x^3} + \frac{3}{x^2+1} - \frac{x-4}{(x^2+1)^2}$$

INVERSE LAPLACE TRANSFORMS

■ 1. Find the inverse Laplace transform.

$$F(s) = \frac{s+3}{s^2 - 3s + 2}$$

Solution:

We'll first factor the denominator of $F(s)$,

$$F(s) = \frac{s+3}{s^2 - 3s + 2}$$

$$F(s) = \frac{s+3}{(s-1)(s-2)}$$

and then apply a partial fractions decomposition.

$$\frac{s+3}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$s + 3 = A(s-2) + B(s-1)$$

To find A, we'll set $s = 1$ in order to eliminate B.

$$1 + 3 = A(1-2) + B(1-1)$$

$$4 = A(-1) + B(0)$$

$$A = -4$$

To find B, we'll set $s = 2$ in order to eliminate A.

$$2 + 3 = A(2 - 2) + B(2 - 1)$$

$$5 = A(0) + B(1)$$

$$B = 5$$

Substituting these values back into the decomposition gives

$$F(s) = \frac{-4}{s-1} + \frac{5}{s-2}$$

$$F(s) = -4\left(\frac{1}{s-1}\right) + 5\left(\frac{1}{s-2}\right)$$

In this form, we can see that the values inside the parentheses resemble the Laplace transform

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

With $a_1 = 1$ and $a_2 = 2$, we'll reverse the formulas from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = -4e^{t} + 5e^{2t}$$

■ 2. Use an inverse Laplace transform to find $f(t)$.

$$F(s) = \frac{2s - 5}{s^3 + s^2 - 12s}$$

Solution:

We'll first factor the denominator of $F(s)$,

$$F(s) = \frac{2s-5}{s^3+s^2-12s}$$

$$F(s) = \frac{2s-5}{s(s-3)(s+4)}$$

and then apply a partial fractions decomposition.

$$\frac{2s-5}{s(s-3)(s+4)} = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s+4}$$

$$2s-5 = A(s-3)(s+4) + Bs(s+4) + Cs(s-3)$$

To find A, we'll set $s=0$ in order to eliminate B and C.

$$2(0)-5 = A(0-3)(0+4) + B(0)(0+4) + C(0)(0-3)$$

$$-5 = A(-12) + B(0) + C(0)$$

$$A = \frac{5}{12}$$

To find B, we'll set $s=3$ in order to eliminate A and C.

$$2(3)-5 = A(3-3)(3+4) + B(3)(3+4) + C(3)(3-3)$$

$$1 = A(0) + B(21) + C(0)$$

$$B = \frac{1}{21}$$

To find C, we'll set $s = -4$ in order to eliminate A and B.

$$2(-4) - 5 = A(-4 - 3)(-4 + 4) + B(-4)(-4 + 4) + C(-4)(-4 - 3)$$

$$-13 = A(0) + B(0) + C(28)$$

$$C = -\frac{13}{28}$$

Substituting these values back into the decomposition gives

$$F(s) = \frac{\frac{5}{12}}{s} + \frac{\frac{1}{21}}{s - 3} + \frac{-\frac{13}{28}}{s + 4}$$

$$F(s) = \frac{5}{12}\left(\frac{1}{s}\right) + \frac{1}{21}\left(\frac{1}{s - 3}\right) - \frac{13}{28}\left(\frac{1}{s + 4}\right)$$

$$F(s) = \frac{5}{12}\left(\frac{1}{s}\right) + \frac{1}{21}\left(\frac{1}{s - 3}\right) - \frac{13}{28}\left(\frac{1}{s - (-4)}\right)$$

In this form, we can see that the values inside the parentheses resemble the Laplace transforms

$$\mathscr{L}(1) = \frac{1}{s} \qquad\qquad \mathscr{L}(e^{at}) = \frac{1}{s - a}$$

With $a = -4$, we'll reverse the formulas from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = \frac{5}{12} + \frac{1}{21}e^{3t} - \frac{13}{28}e^{-4t}$$

3. Find the inverse Laplace transform.

$$F(s) = \frac{s-3}{s^2+4s+4}$$

Solution:

We'll first factor the denominator of $F(s)$,

$$F(s) = \frac{s-3}{s^2+4s+4}$$

$$F(s) = \frac{s-3}{(s+2)^2}$$

and then apply a partial fractions decomposition.

$$\frac{s-3}{(s+2)^2} = \frac{A}{s+2} + \frac{B}{(s+2)^2}$$

$$s-3 = A(s+2) + B$$

To find B, we'll set $s = -2$ in order to eliminate A.

$$-2-3 = A(-2+2) + B$$

$$-5 = A(0) + B$$

$$B = -5$$

Find A.

$$s - 3 = A(s + 2) - 5$$

$$s - 3 = (A)s + (2A - 5)$$

Equating coefficients gives $A = 1$. Substituting these values back into the decomposition gives

$$F(s) = \frac{1}{s+2} + \frac{-5}{(s+2)^2}$$

$$F(s) = \frac{1}{s+2} - 5\left(\frac{1}{(s+2)^2}\right)$$

$$F(s) = \frac{1}{s-(-2)} - 5\left(\frac{1}{(s-(-2))^2}\right)$$

In this form, we can see that the values inside the parentheses resemble the Laplace transforms

$$\mathscr{L}(e^{at}) = \frac{1}{s-a} \qquad \mathscr{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

With $a = -4$, we'll reverse the formulas from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = e^{-2t} - 5te^{-2t}$$

■ 4. Use an inverse Laplace transform to find $f(t)$.

$$F(s) = \frac{3s^3 + 6s^2 + 27s + 24}{s^4 + 13s + 36}$$

Solution:

We'll first factor the denominator of $F(s)$,

$$F(s)=\frac{3s^3+6s^2+27s+24}{s^4+13s+36}$$

$$F(s)=\frac{3s^3+6s^2+27s+24}{(s^2+4)(s^2+9)}$$

and then apply a partial fractions decomposition.

$$\frac{3s^3+6s^2+27s+24}{(s^2+4)(s^2+9)}=\frac{3s(s^2+9)+6(s^2+4)}{(s^2+4)(s^2+9)}$$

$$\frac{3s^3+6s^2+27s+24}{(s^2+4)(s^2+9)}=\frac{3s}{s^2+4}+\frac{6}{s^2+9}$$

Substituting these values back into the decomposition gives

$$F(s)=\frac{3s}{s^2+4}+\frac{6}{s^2+9}$$

$$F(s)=3\left(\frac{s}{s^2+4}\right)+6\left(\frac{1}{s^2+9}\right)$$

$$F(s)=3\left(\frac{s}{s^2+2^2}\right)+6\left(\frac{1}{s^2+3^2}\right)$$

In this form, we can see that the values inside the parentheses resemble the Laplace transforms

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2 + a^2} \qquad \mathscr{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

With $a_1 = 2$ and $a_2 = 3$, we'll reverse the formulas from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = 3\cos(2t) + 2\sin(3t)$$

■ 5. Use an inverse Laplace transform to find $f(t)$.

$$F(s) = \frac{s-2}{s^2 - 4s + 5}$$

Solution:

We'll first factor the denominator of $F(s)$,

$$F(s) = \frac{s-2}{s^2 - 4s + 5}$$

$$F(s) = \frac{s-2}{(s-2)^2 + 1}$$

Now the function resembles the Laplace transform

$$\mathscr{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2 + b^2}$$

With $a = 2$ and $b = 1$, we'll reverse the formula from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = e^{2t}\cos t$$

■ 6. Find the inverse Laplace transform.

$$F(s) = \frac{s-2}{2s^2 + 2s + 2}$$

Solution:

We'll first factor the denominator of $F(s)$,

$$F(s) = \frac{s-2}{2s^2 + 2s + 2}$$

$$F(s) = \frac{s-2}{2(s^2 + s + 1)}$$

$$F(s) = \frac{s-2}{2\left(\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}\right)}$$

To make the numerator more closely match the denominator, we'll rewrite it as

$$F(s) = \frac{s + \frac{1}{2} - \frac{5}{2}}{2\left(\left(s+\frac{1}{2}\right)^2 + \frac{3}{4}\right)}$$

$$F(s) = \frac{s + \frac{1}{2}}{2\left(\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}\right)} - \frac{\frac{5}{2}}{2\left(\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}\right)}$$

$$F(s) = \frac{1}{2}\left(\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right) - \frac{5}{4}\left(\frac{1}{\left(s + \frac{1}{2}\right)^2 + \frac{3}{4}}\right)$$

$$F(s) = \frac{1}{2}\left(\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right) - \frac{5}{4}\left(\frac{1}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

$$F(s) = \frac{1}{2}\left(\frac{s + \frac{1}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right) - \frac{5}{4}\left(\frac{2}{\sqrt{3}}\right)\left(\frac{\frac{\sqrt{3}}{2}}{\left(s + \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}\right)$$

In this form, we can see that the values inside the parentheses resemble the Laplace transforms

$$\mathscr{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2 + b^2} \qquad \mathscr{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2 + b^2}$$

With $a = -1/2$ and $b = \sqrt{3}/2$, we'll reverse the formulas from the table to rewrite the transform as

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2}e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{5}{2\sqrt{3}}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{1}{2}e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{5\sqrt{3}}{6}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = \frac{e^{-\frac{t}{2}}}{2}\left[\cos\left(\frac{\sqrt{3}}{2}t\right) - \frac{5\sqrt{3}}{3}\sin\left(\frac{\sqrt{3}}{2}t\right)\right]$$

TRANSFORMING DERIVATIVES

■ 1. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 1$ and $y'(0) = 1$.

Solution:

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - 1$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s(1) - 1$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - s - 1$$

So for any function $y(t)$ with the initial conditions $y(0) = 1$ and $y'(0) = 1$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s) - 1$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - s - 1$$

■ 2. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 0$ and $y'(0) = 2$.

Solution:

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - 0$$

$$\mathscr{L}(y'(t)) = sY(s)$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s(0) - 2$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 2$$

So for any function $y(t)$ with the initial conditions $y(0) = 0$ and $y'(0) = 2$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 2$$

■ 3. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 1$ and $y'(0) = -2$.

Solution:

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - 1$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s(1) - (-2)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - s + 2$$

So for any function $y(t)$ with the initial conditions $y(0) = 1$ and $y'(0) = -2$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s) - 1$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - s + 2$$

■ 4. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 3$ and $y'(0) = -1/4$.

Solution:

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - 3$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s(3) - \left(-\frac{1}{4}\right)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 3s + \frac{1}{4}$$

So for any function $y(t)$ with the initial conditions $y(0) = 3$ and $y'(0) = -1/4$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s) - 3$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 3s + \frac{1}{4}$$

■ 5. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 4$ and $y'(0) = 2/3$.

Solution:

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - 4$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s(4) - \frac{2}{3}$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 4s - \frac{2}{3}$$

So for any function $y(t)$ with the initial conditions $y(0) = 4$ and $y'(0) = 2/3$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s) - 4$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 4s - \frac{2}{3}$$

■ 6. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 7$ and $y'(0) = 1/7$.

Solution:

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - 7$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s(7) - \frac{1}{7}$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 7s - \frac{1}{7}$$

So for any function $y(t)$ with the initial conditions $y(0) = 7$ and $y'(0) = 1/7$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s) - 7$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - 7s - \frac{1}{7}$$

LAPLACE TRANSFORMS FOR INITIAL VALUE PROBLEMS

■ 1. Find the solution to the second order equation, given $y(0)=1$ and $y'(0)=-1$.

$$y''+3y'+2y=6t$$

Solution:

Substitute values from the table of Laplace transforms,

$$\mathscr{L}(y'')=s^2Y(s)-sy(0)-y'(0)$$

$$\mathscr{L}(y')=sY(s)-y(0)$$

$$\mathscr{L}(y)=Y(s)$$

$$\mathscr{L}(t)=\frac{1}{s^2}$$

into the differential equation,

$$s^2Y(s)-sy(0)-y'(0)+3[sY(s)-y(0)]+2Y(s)=6\left(\frac{1}{s^2}\right)$$

then plug in the initial conditions $y(0)=1$ and $y'(0)=-1$.

$$s^2Y(s)-s(1)-(-1)+3[sY(s)-1]+2Y(s)=6\left(\frac{1}{s^2}\right)$$

$$s^2Y(s) - s + 1 + 3sY(s) - 3 + 2Y(s) = \frac{6}{s^2}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{6}{s^2} + s + 2$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{6}{s^2} + \frac{s^2(s+2)}{s^2}$$

$$Y(s)(s^2 + 3s + 2) = \frac{s^3 + 2s^2 + 6}{s^2}$$

$$Y(s) = \frac{s^3 + 2s^2 + 6}{s^2(s+1)(s+2)}$$

Use a partial fractions decomposition.

$$\frac{s^3 + 2s^2 + 6}{s^2(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{s+2}$$

To solve for B, remove the s^2 factor and set $s = 0$.

$$\frac{s^3 + 2s^2 + 6}{(s+1)(s+2)}$$

$$\frac{0^3 + 2(0)^2 + 6}{(0+1)(0+2)} = \frac{6}{1(2)} = 3$$

To solve for C, remove the $(s+1)$ factor and set $s = -1$.

$$\frac{s^3 + 2s^2 + 6}{s^2(s+2)}$$

$$\frac{(-1)^3+2(-1)^2+6}{(-1)^2(-1+2)}=\frac{-1+2+6}{1(1)}=7$$

To solve for D, remove the $(s+2)$ factor and set $s=-2$.

$$\frac{s^3+2s^2+6}{s^2(s+1)}$$

$$\frac{(-2)^3+2(-2)^2+6}{(-2)^2(-2+1)}=\frac{-8+2(4)+6}{4(-1)}=\frac{6}{-4}=-\frac{3}{2}$$

Use the values of B, C, and D and a value we haven't used for s yet ($s \neq -2, -1, 0$), like $s=1$, to solve for A.

$$\frac{1^3+2(1)^2+6}{1^2(1+1)(1+2)}=\frac{A}{1}+\frac{3}{1^2}+\frac{7}{1+1}+\frac{-\frac{3}{2}}{1+2}$$

$$\frac{1+2+6}{2(3)}=A+3+\frac{7}{2}-\frac{\frac{3}{2}}{3}$$

$$\frac{3}{2}=A+3+\frac{7}{2}-\frac{1}{2}$$

$$A=\frac{3}{2}-3-\frac{7}{2}+\frac{1}{2}$$

$$A=-\frac{9}{2}$$

Then the partial fractions decomposition is

$$Y(s)=\frac{-\frac{9}{2}}{s}+\frac{3}{s^2}+\frac{7}{s+1}+\frac{-\frac{3}{2}}{s+2}$$

$$Y(s) = -\frac{9}{2}\left(\frac{1}{s}\right) + 3\left(\frac{1}{s^2}\right) + 7\left(\frac{1}{s-(-1)}\right) - \frac{3}{2}\left(\frac{1}{s-(-2)}\right)$$

and using the transforms

$$\mathscr{L}(1) = \frac{1}{s} \qquad \mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(e^{-t}) = \frac{1}{s-(-1)} \qquad \mathscr{L}(e^{-2t}) = \frac{1}{s-(-2)}$$

the solution to the second order differential equation is

$$y(t) = -\frac{9}{2}(1) + 3(t) + 7e^{-t} - \frac{3}{2}e^{-2t}$$

$$y(t) = -\frac{9}{2} + 3t + 7e^{-t} - \frac{3}{2}e^{-2t}$$

■ 2. Find the solution to the second order equation, given $y(0) = 1$ and $y'(0) = -1$.

$$3y'' + 4y' + y = -t$$

Solution:

Substitute values from the table of Laplace transforms,

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(t) = \frac{1}{s^2}$$

into the differential equation,

$$3[s^2Y(s) - sy(0) - y'(0)] + 4[sY(s) - y(0)] + Y(s) = -\left(\frac{1}{s^2}\right)$$

then plug in the initial conditions $y(0) = 1$ and $y'(0) = -1$.

$$3[s^2Y(s) - s(1) - (-1)] + 4[sY(s) - 1] + Y(s) = -\left(\frac{1}{s^2}\right)$$

$$3s^2Y(s) - 3s + 3 + 4sY(s) - 4 + Y(s) = -\frac{1}{s^2}$$

$$3s^2Y(s) + 4sY(s) + Y(s) = -\frac{1}{s^2} + 3s + 1$$

$$3s^2Y(s) + 4sY(s) + Y(s) = -\frac{1}{s^2} + \frac{s^2(3s+1)}{s^2}$$

$$Y(s)(3s^2 + 4s + 1) = \frac{3s^3 + s^2 - 1}{s^2}$$

$$Y(s) = \frac{3s^3 + s^2 - 1}{s^2(3s+1)(s+1)}$$

Use a partial fractions decomposition.

$$\frac{3s^3 + s^2 - 1}{s^2(3s+1)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{3s+1} + \frac{D}{s+1}$$

To solve for B, remove the s^2 factor and set $s=0$.

$$\frac{3s^3+s^2-1}{(3s+1)(s+1)}$$

$$\frac{3(0)^3+0^2-1}{(3(0)+1)(0+1)}=\frac{-1}{1(1)}=-1$$

To solve for C, remove the $(3s+1)$ factor and set $s=-1/3$.

$$\frac{3s^3+s^2-1}{s^2(s+1)}$$

$$\frac{3\left(-\frac{1}{3}\right)^3+\left(-\frac{1}{3}\right)^2-1}{\left(-\frac{1}{3}\right)^2\left(-\frac{1}{3}+1\right)}=\frac{3\left(-\frac{1}{27}\right)+\frac{1}{9}-1}{\frac{1}{9}\left(-\frac{1}{3}+1\right)}=\frac{-\frac{1}{9}+\frac{1}{9}-1}{\frac{2}{27}}=-\frac{27}{2}$$

To solve for D, remove the $(s+1)$ factor and set $s=-1$.

$$\frac{3s^3+s^2-1}{s^2(3s+1)}$$

$$\frac{3(-1)^3+(-1)^2-1}{(-1)^2(3(-1)+1)}=\frac{3(-1)+1-1}{1(-3+1)}=\frac{-3}{-2}=\frac{3}{2}$$

Use the values of B, C, and D and a value we haven't used for s yet ($s\neq-1,\ -1/3,\ 0$), like $s=1$, to solve for A.

$$\frac{3(1)^3+1^2-1}{1^2(3(1)+1)(1+1)}=\frac{A}{1}+\frac{-1}{1^2}+\frac{-\frac{27}{2}}{3(1)+1}+\frac{\frac{3}{2}}{1+1}$$

$$\frac{3+1-1}{4(2)} = A - 1 + \frac{-\frac{27}{2}}{4} + \frac{\frac{3}{2}}{2}$$

$$\frac{3}{8} = A - 1 - \frac{27}{8} + \frac{3}{4}$$

$$A = \frac{3}{8} + 1 + \frac{27}{8} - \frac{3}{4}$$

$$A = \frac{3}{8} + \frac{8}{8} + \frac{27}{8} - \frac{6}{8}$$

$$A = \frac{32}{8}$$

$$A = 4$$

Then the partial fractions decomposition is

$$Y(s) = \frac{4}{s} + \frac{-1}{s^2} + \frac{-\frac{27}{2}}{3s+1} + \frac{\frac{3}{2}}{s+1}$$

$$Y(s) = 4\left(\frac{1}{s}\right) - \frac{1}{s^2} - \frac{27}{2}\left(\frac{1}{3s+1}\right) + \frac{3}{2}\left(\frac{1}{s+1}\right)$$

$$Y(s) = 4\left(\frac{1}{s}\right) - \frac{1}{s^2} - \frac{27}{2}\left(\frac{\frac{1}{3}}{s+\frac{1}{3}}\right) + \frac{3}{2}\left(\frac{1}{s-(-1)}\right)$$

$$Y(s) = 4\left(\frac{1}{s}\right) - \frac{1}{s^2} - \frac{9}{2}\left(\frac{1}{s+\frac{1}{3}}\right) + \frac{3}{2}\left(\frac{1}{s-(-1)}\right)$$

$$Y(s) = 4\left(\frac{1}{s}\right) - \frac{1}{s^2} - \frac{9}{2}\left(\frac{1}{s - \left(-\frac{1}{3}\right)}\right) + \frac{3}{2}\left(\frac{1}{s - (-1)}\right)$$

and using the transforms

$$\mathscr{L}(1) = \frac{1}{s} \qquad \mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(e^{-\frac{1}{3}t}) = \frac{1}{s - \left(-\frac{1}{3}\right)} \qquad \mathscr{L}(e^{-t}) = \frac{1}{s - (-1)}$$

the solution to the second order differential equation is

$$y(t) = 4(1) - t - \frac{9}{2}e^{-\frac{1}{3}t} + \frac{3}{2}e^{-t}$$

$$y(t) = 4 - t - \frac{9}{2}e^{-\frac{1}{3}t} + \frac{3}{2}e^{-t}$$

■ 3. Find the solution to the second order equation, given $y(0) = 1$ and $y'(0) = 4$.

$$y'' - 5y' = 10t - 2$$

Solution:

Substitute values from the table of Laplace transforms,

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(1) = \frac{1}{s}$$

into the differential equation,

$$s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) = 10\left(\frac{1}{s^2}\right) - 2\left(\frac{1}{s}\right)$$

then plug in the initial conditions $y(0) = 1$ and $y'(0) = 4$.

$$s^2Y(s) - s(1) - 4 - 5(sY(s) - 1) = 10\left(\frac{1}{s^2}\right) - 2\left(\frac{1}{s}\right)$$

$$s^2Y(s) - s - 4 - 5sY(s) + 5 = \frac{10}{s^2} - \frac{2}{s}$$

$$Y(s)(s^2 - 5s) = \frac{10 - 2s}{s^2} + s - 1$$

$$Y(s)s(s - 5) = \frac{10 - 2s + s^3 - s^2}{s^2}$$

$$Y(s) = \frac{s^3 - s^2 - 2s + 10}{s^3(s - 5)}$$

Use a partial fractions decomposition.

$$\frac{s^3 - s^2 - 2s + 10}{s^3(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-5}$$

Multiplying both sides by the denominator from the left side, we get

$$s^3 - s^2 - 2s + 10 = As^2(s-5) + Bs(s-5) + C(s-5) + Ds^3$$

$$s^3 - s^2 - 2s + 10 = As^3 - 5As^2 + Bs^2 - 5Bs + Cs - 5C + Ds^3$$

$$s^3 - s^2 - 2s + 10 = (A+D)s^3 + (B-5A)s^2 + (C-5B)s - 5C$$

This is an equality between two polynomials of the variable s, so the corresponding coefficients must be equal, which gives us

$$A + D = 1$$

$$-5A + B = -1$$

$$-5B + C = -2$$

$$-5C = 10$$

Solving this system gives $A = 1/5$, $B = 0$, $C = -2$, and $D = 4/5$. Then the partial fractions decomposition is

$$Y(s) = \frac{\frac{1}{5}}{s} + \frac{0}{s^2} + \frac{-2}{s^3} + \frac{\frac{4}{5}}{s-5}$$

$$Y(s) = \frac{1}{5}\left(\frac{1}{s}\right) + 0 - \left(\frac{2}{s^3}\right) + \frac{4}{5}\left(\frac{1}{s-5}\right)$$

and using the transforms

$$\mathcal{L}(1) = \frac{1}{s} \qquad \mathcal{L}(t^2) = \frac{2}{s^3} \qquad \mathcal{L}(e^{5t}) = \frac{1}{s-5}$$

the solution to the second order differential equation is

$$y(t) = \frac{4}{5}e^{5t} - t^2 + \frac{1}{5}$$

■ 4. Find the solution to the second order equation, given $y(0) = 2$ and $y'(0) = 2$.

$$y'' - 4y = -13\cos(3t)$$

Solution:

Substitute values from the table of Laplace transforms,

$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}(y') = sY(s) - y(0)$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(\cos(3t)) = \frac{s}{s^2+9}$$

into the differential equation,

$$s^2Y(s) - sy(0) - y'(0) - 4Y(s) = -13\left(\frac{s}{s^2+9}\right)$$

then plug in the initial conditions $y(0) = 2$ and $y'(0) = 2$.

$$s^2Y(s) - s(2) - 2 - 4Y(s) = -13\left(\frac{s}{s^2+9}\right)$$

$$s^2Y(s) - 4Y(s) - 2s - 2 = -\frac{13s}{s^2+9}$$

$$(s^2-4)Y(s) - 2s - 2 = -\frac{13s}{s^2+9}$$

$$(s^2-4)Y(s) = -\frac{13s}{s^2+9} + 2s + 2$$

$$(s^2-4)Y(s) = \frac{-13s + (2s+2)(s^2+9)}{s^2+9}$$

$$Y(s) = \frac{-13s + 2s^3 + 2s^2 + 18s + 18}{(s-2)(s+2)(s^2+9)}$$

$$Y(s) = \frac{2s^3 + 2s^2 + 5s + 18}{(s-2)(s+2)(s^2+9)}$$

Use a partial fractions decomposition.

$$\frac{2s^3 + 2s^2 + 5s + 18}{(s-2)(s+2)(s^2+9)} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{Cs+D}{s^2+9}$$

Multiplying both sides by the denominator from the left side, we get

$$2s^3 + 2s^2 + 5s + 18 = A(s+2)(s^2+9) + B(s-2)(s^2+9)$$

$$+(Cs+D)(s-2)(s+2)$$

$$2s^3 + 2s^2 + 5s + 18 = A(s^3 + 2s^2 + 9s + 18) + B(s^3 - 2s^2 + 9s - 18)$$

$$+C(s^3 - 4s) + D(s^2 - 4)$$

$$2s^3 + 2s^2 + 5s + 18 = s^3(A + B + C) + s^2(2A - 2B + D)$$

$$+s(9A + 9B - 4C) + 18A - 18B - 4D$$

This is an equality between two polynomials of the variable s, so the corresponding coefficients must be equal, which gives us

$$A + B + C = 2$$

$$2A - 2B + D = 2$$

$$9A + 9B - 4C = 5$$

$$18A - 18B - 4D = 18$$

Solving this system gives $A = 1$, $B = 0$, $C = 1$, and $D = 0$. Then the partial fractions decomposition is

$$Y(s) = \frac{1}{s-2} + \frac{0}{s+2} + \frac{1s+0}{s^2+9}$$

$$Y(s) = \frac{1}{s-2} + \frac{s}{s^2+9}$$

and using the transforms

$$\mathscr{L}(e^{2t}) = \frac{1}{s-2} \qquad \mathscr{L}(\cos(3t)) = \frac{s}{s^2+9}$$

the solution to the second order differential equation is

$$y(t) = e^{2t} + \cos(3t)$$

■ 5. Find the solution to the second order equation, given $y(2) = 1$ and $y'(2) = -2$.

$$y'' + 2y' - 3y = 0$$

Solution:

Before we do anything else, we need to shift the initial conditions from $t = 2$ to $t = 0$. We'll define a new variable η, and we want η to be equal to 0 so that we can set the initial conditions at $y(\eta = 0)$ and $y'(\eta = 0)$. So η and t are related by

$$\eta = t - 2$$

$$t = \eta + 2$$

Then we want to make substitutions for t into the differential equation.

$$y''(\eta + 2) + 2y'(\eta + 2) - 3y(\eta + 2) = 0$$

Next, we'll define a new function $u(\eta)$ with

$$u(\eta) = y(\eta + 2)$$

$$u'(\eta) = y'(\eta + 2)$$

$$u''(\eta) = y''(\eta + 2)$$

Then the initial conditions are

$$u(0) = y(0+2) = y(2) = 1$$

$$u'(0) = y'(0+2) = y'(2) = -2$$

As a result, the new initial value problem is

$$u'' + 2u' - 3u = 0 \text{ with } u(0) = 1 \text{ and } u'(0) = -2$$

From our table of Laplace transforms, we know

$$\mathscr{L}(u'') = s^2U(s) - su(0) - u'(0)$$

$$\mathscr{L}(u') = sU(s) - u(0)$$

Plugging these transforms into the differential equation gives

$$s^2U(s) - su(0) - u'(0) + 2(sU(s) - u(0)) - 3U(s) = 0$$

$$s^2U(s) - su(0) - u'(0) + 2sU(s) - 2u(0) - 3U(s) = 0$$

Now we'll plug in the initial conditions $u(0) = 1$ and $u'(0) = -2$ in order to simplify the transform.

$$s^2U(s) - s(1) - (-2) + 2sU(s) - 2(1) - 3U(s) = 0$$

$$s^2U(s) - s + 2sU(s) - 3U(s) = 0$$

Solve for $U(s)$ by collecting all the $U(s)$ terms on one side, and moving all other terms to the other side.

$$s^2U(s) + 2sU(s) - 3U(s) = s$$

Factor out $U(s)$, then isolate it on the left side of the equation.

$$U(s)(s^2 + 2s - 3) = s$$

$$U(s) = \frac{s}{s^2 + 2s - 3}$$

$$U(s) = \frac{s}{(s-1)(s+3)}$$

We'll need to use a partial fractions decomposition.

$$\frac{s}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

To solve for A, we'll remove the $(s-1)$ factor, set $s = 1$, and find the value of the left side.

$$\frac{s}{s+3} = \frac{1}{1+3} = \frac{1}{4}$$

$$A = \frac{1}{4}$$

To solve for B, we'll remove the $(s+3)$ factor, set $s = -3$, and find the value of the left side.

$$\frac{s}{s-1} = \frac{-3}{-3-1} = \frac{3}{4}$$

$$B = \frac{3}{4}$$

Plugging the values we found for A and B back into the partial fractions decomposition gives

$$U(s)=\frac{\frac{1}{4}}{s-1}+\frac{\frac{3}{4}}{s+3}$$

and then we can rearrange each term in the decomposition to make it easier to find a matching formula in the Laplace transform table.

$$U(s)=\frac{1}{4}\left(\frac{1}{s-1}\right)+\frac{3}{4}\left(\frac{1}{s+3}\right)$$

The values remaining inside the parentheses should remind us of the transforms

$$\mathscr{L}(e^{\eta})=\frac{1}{s-1} \qquad \mathscr{L}(e^{-3\eta})=\frac{1}{s+3}$$

or equivalently, the inverse transforms

$$\mathscr{L}^{-1}\left(\frac{1}{s-1}\right)=e^{\eta} \qquad \mathscr{L}^{-1}\left(\frac{1}{s+3}\right)=e^{-3\eta}$$

So we'll make these substitutions to put the equation back in terms of η, instead of the transform variable s,

$$u(\eta)=\frac{1}{4}e^{\eta}+\frac{3}{4}e^{-3\eta}$$

Because $y(t)=u(\eta)=u(t-2)$ and $\eta=t-2$, the solution to the original differential equation can be written as

$$u(\eta)=\frac{1}{4}e^{t-2}+\frac{3}{4}e^{-3(t-2)}$$

$$u(\eta)=\frac{1}{4}e^{t-2}+\frac{3}{4}e^{-3t+6}$$

$$u(\eta) = \frac{1}{4e^2}e^t + \frac{3e^6}{4}e^{-3t}$$

■ 6. Find the solution to the second order equation, given $y(\pi) = 2$ and $y'(\pi) = 0$.

$$y'' + 4y = 0$$

Solution:

Before we do anything else, we need to shift the initial conditions from $t = \pi$ to $t = 0$. We'll define a new variable η, and we want η to be equal to 0 so that we can set the initial conditions at $y(\eta = 0)$ and $y'(\eta = 0)$. So η and t are related by

$$\eta = t - \pi$$

$$t = \eta + \pi$$

Then we want to make substitutions for t into the differential equation.

$$y''(\eta + \pi) + 4y(\eta + \pi) = 0$$

Next, we'll define a new function $u(\eta)$ with

$$u(\eta) = y(\eta + \pi)$$

$$u'(\eta) = y'(\eta + \pi)$$

$$u''(\eta) = y''(\eta + \pi)$$

Then the initial conditions are

$$u(0) = y(0 + \pi) = y(\pi) = 2$$

$$u'(0) = y'(0 + \pi) = y'(\pi) = 0$$

As a result, the new initial value problem is

$u'' + 4u = 0$ with $u(0) = 2$ and $u'(0) = 0$

From our table of Laplace transforms, we know

$$\mathscr{L}(u'') = s^2U(s) - su(0) - u'(0)$$

Plugging this transform into the differential equation gives

$$s^2U(s) - su(0) - u'(0) + 4U(s) = 0$$

Now we'll plug in the initial conditions $u(0) = 2$ and $u'(0) = 0$ in order to simplify the transform.

$$s^2U(s) - s(2) - 0 + 4U(s) = 0$$

$$s^2U(s) - 2s + 4U(s) = 0$$

Solve for $U(s)$ by collecting all the $U(s)$ terms on one side, and moving all other terms to the other side.

$$s^2U(s) + 4U(s) = 2s$$

Factor out $U(s)$, then isolate it on the left side of the equation.

$$U(s)(s^2 + 4) = 2s$$

$$U(s) = \frac{2s}{s^2 + 4}$$

$$U(s) = 2\left(\frac{s}{s^2 + 4}\right)$$

The value remaining inside the parentheses should remind us of the transform

$$\mathscr{L}(\cos(2t)) = \frac{s}{s^2 + 4}$$

or equivalently, the inverse transform

$$\mathscr{L}^{-1}\left(\frac{s}{s^2 + 4}\right) = \cos(2t)$$

So we'll make this substitution to put the equation back in terms of η, instead of the transform variable s,

$$u(\eta) = 2\cos(2\eta)$$

Because $y(t) = u(\eta) = u(t - \pi)$ and $\eta = t - \pi$, the solution to the original differential equation can be written as

$$u(\eta) = 2\cos(2(t - \pi))$$

$$u(\eta) = 2\cos(2t - 2\pi)$$

$$u(\eta) = 2(\cos(2t)\cos(2\pi) + \sin(2t)\sin(2\pi))$$

$$u(\eta) = 2(\cos(2t)(1) + \sin(2t)(0))$$

$$u(\eta) = 2\cos(2t)$$

STEP FUNCTIONS

■ 1. Write $f(t)$ in terms of step functions, if $f(t)$ has a value of -3 at $t=0$, jumps down 2 units at $t=4$, up 7 units at $t=5$, and if $f(t)=9$ for $t \geq 6$.

Solution:

At the starting point, $f(t)=-3$, so we can start the piecewise function as

$$f(t)=\begin{cases} -3 & 0 \leq t < 4 \\ & 4 \leq t < 5 \\ & 5 \leq t < 6 \\ & t \geq 6 \end{cases}$$

At $t=4$, the function drops down 2 units, which means that it takes on the value $-3-2=-5$.

$$f(t)=\begin{cases} -3 & 0 \leq t < 4 \\ -5 & 4 \leq t < 5 \\ & 5 \leq t < 6 \\ & t \geq 6 \end{cases}$$

In the same way, the function jumps up 7 units at $t=5$. So $-5+7=2$, and

$$f(t)=\begin{cases} -3 & 0 \leq t < 4 \\ -5 & 4 \leq t < 5 \\ 2 & 5 \leq t < 6 \\ & t \geq 6 \end{cases}$$

We're told that the function has a value of 9 for $t \geq 6$, which means that the function has jumps up 7 units at $t = 6$, from $f(t) = 2$ to $f(t) = 9$.

$$f(t) = \begin{cases} -3 & 0 \leq t < 4 \\ -5 & 4 \leq t < 5 \\ 2 & 5 \leq t < 6 \\ 9 & t \geq 6 \end{cases}$$

This piecewise function $f(t)$ can be represented in terms of unit step function as

$$f(t) = -3 - 2u_4(t) + 7u_5(t) + 7u_6(t)$$

■ 2. Write an (on at 5)-to-off function that has a switch at $t = 8$.

Solution:

Since we want the switch to be "on" at a value of 5, and then turn "off" to a value of 0 when we arrive at the switching point $t = 8$, we'd write the equation of the function as

$$5 - 5u_8(t) = \begin{cases} 5 - 5(0) & 0 \leq t < 8 \\ 5 - 5(1) & t \geq 8 \end{cases}$$

$$5 - 5u_8(t) = \begin{cases} 5 & 0 \leq t < 8 \\ 0 & t \geq 8 \end{cases}$$

■ 3. Express the piecewise function in terms of unit step functions.

$$f(t) = \begin{cases} \cos t & 0 \le t < 2\pi \\ 0 & t \ge 2\pi \end{cases}$$

Solution:

The first thing we notice about $f(t)$ is that it follows an "on-to-off" pattern instead of the default "off-to-on" pattern. This "(on at n)-to-off" pattern is modeled by

$$n(1 - u_c(t)) = \begin{cases} n & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

which means we can rewrite $f(t)$ as

$$f(t) = \cos t(1 - u_{2\pi}(t))$$

$$f(t) = \cos t - u_{2\pi}(t)\cos t$$

■ 4. Rewrite the function $g(t)$ in terms of Heaviside functions.

$$g(t) = \begin{cases} 0 & 0 \le t < 1 \\ t^2 - 3 & 1 \le t < 2 \\ 5 & 2 \le t < 3 \\ t^3 + 2 & t \ge 3 \end{cases}$$

Solution:

There are three switches, one from 0 to $t^2 - 3$, one from $t^2 - 3$ to 5, and one from 5 to $t^3 + 2$, so we'll need three Heaviside functions in order to express $f(t)$.

We write a Heaviside function as $u_c(t)$, where the switch occurs at $t = c$. The switches in this function $f(t)$ are at $t = 1$, $t = 2$, and $t = 3$, which means the three Heaviside functions we'll need are $u_1(t)$, $u_2(t)$, and $u_3(t)$.

To express $f(t)$, we start with the idea that $f(t)$ is and "off-to-on" non-constant function. For the first switch at $t = 1$, we can write

$$g(t) \approx (t^2 - 3)u_1(t)$$

Then moving from $t^2 - 3$ to 5 is a change of $5 - (t^2 - 3) = 8 - t^2$, which means we need to multiply $u_2(t)$ by $8 - t^2$.

$$g(t) \approx (t^2 - 3)u_1(t) + (8 - t^2)u_2(t)$$

And moving from 5 to $t^3 + 2$ is a change of $t^3 + 2 - 5 = t^3 - 3$, which means we need to multiply $u_3(t)$ by $t^3 - 3$, and this will give us the full expression of $g(t)$ in terms of step functions.

$$g(t) \approx (t^2 - 3)u_1(t) + (8 - t^2)u_2(t) + (t^3 - 3)u_3(t)$$

■ 5. Describe all switches of the function $f(t) = (-5 + 6u_2(t) - 3u_5(t))(1 - u_7(t))$ and represent it as a piecewise function.

Solution:

The function $f(t)$ is a product of the function $-5+6u_2(t)-3u_5(t)$ and the function $1-u_7(t)$. The $1-u_7(t)$ function tells us that $f(t)$ has an "on-to-off" switch at $t=7$.

The function $-5+6u_2(t)-3u_5(t)$ has switches at $t=2$ and $t=5$, and the -5 gives us the "starting point." So the $1-u_7(t)$ tells us that $f(t)$ is "on" for the interval $0\le t<7$, then turns off to a zero value for $t\ge 7$. So, we can write

$$f(t)=\begin{cases} -5 & 0\le t<2 \\ & 2\le t<5 \\ & 5\le t<7 \\ 0 & t\ge 7 \end{cases}$$

Then we consider the coefficients on the unit step functions, 6 and -3. We'll add 6 to -5 to get $-5+6=1$ for the second piece,

$$f(t)=\begin{cases} -5 & 0\le t<2 \\ 1 & 2\le t<5 \\ & 5\le t<7 \\ 0 & t\ge 7 \end{cases}$$

then add -3 to get $1-3=-2$ for the third piece.

$$f(t)=\begin{cases} -5 & 0\le t<2 \\ 1 & 2\le t<5 \\ -2 & 5\le t<7 \\ 0 & t\ge 7 \end{cases}$$

■ 6. Describe all switches of the function $f(t) = 4 - u_1(t) + 5u_3(t) + 3u_4(t) - 6u_8(t)$ and represent it as a piecewise function.

Solution:

The unit step functions show us that we have switches at $t = 1$, 3, 4, and 8, and the 4 gives us the "starting point." So, we can write the following

$$f(t) = \begin{cases} 4 & 0 \le t < 1 \\ & 1 \le t < 3 \\ & 3 \le t < 4 \\ & 4 \le t < 8 \\ & t \ge 8 \end{cases}$$

Then we consider the coefficients on the unit step functions, -1, 5, 3, and -6. We'll add -1 to 4 to get $4 - 1 = 3$ for the second piece,

$$f(t) = \begin{cases} 4 & 0 \le t < 1 \\ 3 & 1 \le t < 3 \\ & 3 \le t < 4 \\ & 4 \le t < 8 \\ & t \ge 8 \end{cases}$$

then add 5 to get $3 + 5 = 8$ for the third piece,

$$f(t) = \begin{cases} 4 & 0 \le t < 1 \\ 3 & 1 \le t < 3 \\ 8 & 3 \le t < 4 \\ & 4 \le t < 8 \\ & t \ge 8 \end{cases}$$

then add 3 to get $8+3=11$ for the fourth piece,

$$f(t)=\begin{cases} 4 & 0\le t<1 \\ 3 & 1\le t<3 \\ 8 & 3\le t<4 \\ 11 & 4\le t<8 \\ & t\ge 8 \end{cases}$$

and then finally add -6 to get $11-6=5$ for the last piece.

$$f(t)=\begin{cases} 4 & 0\le t<1 \\ 3 & 1\le t<3 \\ 8 & 3\le t<4 \\ 11 & 4\le t<8 \\ 5 & t\ge 8 \end{cases}$$

SECOND SHIFTING THEOREM

■ 1. Use a step function to represent shifting the portion of $f(x)$ for $x \geq 0$ to the right 2 units, while turning off the function on $0 \leq x < 2$. Then take its Laplace transform.

$$f(x) = 5 - x^2 + \cos(3x) - 4e^{5x}$$

Solution:

Because we want to shift the curve 2 units to the right, we can start by substituting $c = 2$ into our formula for this kind of shift.

$$f(x-c)u(x-c) = \begin{cases} 0 & 0 \leq x < c \\ f(x-c) & x \geq c \end{cases}$$

$$f(x-2)u(x-2) = \begin{cases} 0 & 0 \leq x < 2 \\ f(x-2) & x \geq 2 \end{cases}$$

Now we just need to replace $f(x-2)$.

$$f(x) = 5 - x^2 + \cos(3x) - 4e^{5x}$$

$$f(x-2) = 5 - (x-2)^2 + \cos(3(x-2)) - 4e^{5(x-2)}$$

So the shifted function is represented by

$$(5 - (x-2)^2 + \cos(3(x-2)) - 4e^{5(x-2)})u(x-2)$$

$$= \begin{cases} 0 & 0 \le x < 2 \\ 5 - (x-2)^2 + \cos(3(x-2)) - 4e^{5(x-2)} & x \ge 2 \end{cases}$$

The Laplace transform of $f(x)$ is

$$F(s) = \mathscr{L}(f(x)) = \mathscr{L}(5) - \mathscr{L}(x^2) + \mathscr{L}(\cos(3x)) - 4\mathscr{L}(e^{5x})$$

$$F(s) = \mathscr{L}(f(x)) = \frac{5}{s} - \frac{2!}{s^3} + \frac{s}{s^2 + 3^2} - 4\frac{1}{s-5}$$

$$F(s) = \mathscr{L}(f(x)) = \frac{5}{s} - \frac{2}{s^3} + \frac{s}{s^2 + 9} - \frac{4}{s-5}$$

Then by the Second Shifting Theorem, the Laplace transform of the shifted function is

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(f(t-2)u(t-2)) = e^{-2s}\left(\frac{5}{s} - \frac{2}{s^3} + \frac{s}{s^2+9} - \frac{4}{s-5}\right)$$

■ 2. Use a step function to represent shifting the portion of $f(x)$ for $x \ge 0$ to the right 3 units, while turning off the function on $0 \le x < 3$. Then take its Laplace transform.

$$f(x) = 7x \sin x - 4\cos(3x + 9)$$

Solution:

Because we want to shift the curve 3 units to the right, we can start by substituting $c = 3$ into our formula for this kind of shift.

$$f(x-c)u(x-c) = \begin{cases} 0 & 0 \le x < c \\ f(x-c) & x \ge c \end{cases}$$

$$f(x-3)u(x-3) = \begin{cases} 0 & 0 \le x < 3 \\ f(x-3) & x \ge 3 \end{cases}$$

Now we just need to replace $f(x-3)$.

$$f(x) = 7x \sin x - 4\cos(3x+9)$$

$$f(x-3) = 7(x-3)\sin(x-3) - 4\cos(3(x-3)+9)$$

$$f(x-3) = 7(x-3)\sin(x-3) - 4\cos(3x)$$

So the shifted function is represented by

$$(7(x-3)\sin(x-3) - 4\cos(3x))u(x-3)$$

$$= \begin{cases} 0 & 0 \le x < 3 \\ 7(x-3)\sin(x-3) - 4\cos(3x) & x \ge 3 \end{cases}$$

The Laplace transform of $f(x)$ is

$$F(s) = \mathscr{L}(f(x)) = 7\mathscr{L}(x \sin x) - 4\mathscr{L}(\cos(3x+9))$$

$$\cos(3x+9) = \cos(3x)\cos 9 - \sin(3x)\sin 9$$

$$F(s) = \mathscr{L}(f(x)) = 7\mathscr{L}(x \sin x) - 4\mathscr{L}(\cos(3x)\cos 9 - \sin(3x)\sin 9)$$

$$F(s) = \mathscr{L}(f(x)) = 7\left(\frac{2s}{(s^2+1^2)^2}\right) - 4\left(\cos 9\left(\frac{s}{s^2+9}\right) - \sin 9\left(\frac{3}{s^2+9}\right)\right)$$

$$F(s) = \mathscr{L}(f(x)) = \frac{14s}{(s^2+1)^2} - \frac{4s\cos 9}{s^2+9} + \frac{12\sin 9}{s^2+9}$$

Then by the Second Shifting Theorem, the Laplace transform of the shifted function is

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(f(t-3)u(t-3)) = e^{-3s}\left(\frac{14s}{(s^2+1)^2} - \frac{4s\cos 9}{s^2+9} + \frac{12\sin 9}{s^2+9}\right)$$

■ 3. Find the Laplace transform of the function.

$$g(t) = \begin{cases} 0 & 0 \le x < 7 \\ e^{3t-21} & x \ge 7 \end{cases}$$

Solution:

Given $f(t) = e^{3t}$ for $t \ge 0$, we can rewrite the function as

$$g(t) = f(t-7)u(t-7)$$

By the Second Shifting Theorem we have

$$\mathscr{L}(f(t-7)u(t-7)) = e^{-7s}F(s)$$

To find the Laplace transform of this function, we'll need the Laplace transform of $f(t)$.

$$F(s) = \mathscr{L}(f(t)) = \mathscr{L}(e^{3t})$$

$$F(s) = \mathscr{L}(f(t)) = \frac{1}{s-3}$$

Then by the Second Shifting Theorem, the Laplace transform of the shifted function is

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-7s}\left(\frac{1}{s-3}\right)$$

and therefore

$$\mathscr{L}(g(t)) = e^{-7s}\left(\frac{1}{s-3}\right)$$

■ 4. Find the Laplace transform of the function.

$$g(t) = \begin{cases} 0 & 0 \le x < 4 \\ e^{t-4}\sin(3t-12) - (t-4)^3 e^{4t-16} & x \ge 4 \end{cases}$$

Solution:

Given $f(t) = e^t \sin(3t) - t^3 e^{4t}$ for $t \ge 0$, rewrite the function as

$$g(t) = f(t-4)u(t-4)$$

By the Second Shifting Theorem we have

$$\mathscr{L}(f(t-4)u(t-4)) = e^{-4s}F(s)$$

To find the Laplace transform of this function, we'll need the Laplace transform of $f(t)$.

$$F(s) = \mathscr{L}(f(t)) = \mathscr{L}(e^t \sin(3t)) - \mathscr{L}(t^3 e^{4t})$$

$$F(s) = \mathscr{L}(f(t)) = \frac{3}{(s-1)^2+9} - \frac{3!}{(s-4)^4}$$

$$F(s) = \mathscr{L}(f(t)) = \frac{3}{(s-1)^2+9} - \frac{6}{(s-4)^4}$$

Then by the Second Shifting Theorem, the Laplace transform of the shifted function is

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-4s}\left(\frac{3}{(s-1)^2+9} - \frac{6}{(s-4)^4}\right)$$

and therefore

$$\mathscr{L}(g(t)) = e^{-4s}\left(\frac{3}{(s-1)^2+9} - \frac{6}{(s-4)^4}\right)$$

■ 5. Calculate the Laplace transform.

$$\mathscr{L}(e^{3t-6}\cos(4-2t)u(t-2))$$

Solution:

Given $f(t)=e^{3t}\cos(-2t)$, or $f(t)=e^{3t}\cos(2t)$, rewrite the function as

$$e^{3t-6}\cos(4-2t)u(t-2)=e^{3(t-2)}\cos(-2(t-2)u(t-2))=f(t-2)$$

By the Second Shifting Theorem we have

$$\mathscr{L}(f(t-2)u(t-2))=e^{-2s}F(s)$$

To find the Laplace transform of this function, we'll need the Laplace transform of $f(t)$.

$$F(s)=\mathscr{L}(f(t))=\mathscr{L}(e^{3t}\cos(2t))$$

$$F(s)=\mathscr{L}(f(t))=\frac{s-3}{(s-3)^2+4}$$

Then by the Second Shifting Theorem, the Laplace transform of the shifted function is

$$\mathscr{L}(f(t-c)u(t-c))=e^{-cs}F(s)$$

$$\mathscr{L}(f(t-c)u(t-c))=e^{-2s}\left(\frac{s-3}{(s-3)^2+4}\right)$$

■ 6. Find the Laplace transform of the function.

$$f(t) = \begin{cases} 1 - t & 0 \le x < 2 \\ -1 & 2 \le x < 4 \\ (5t - 20)^3 e^{3t-12} & x \ge 4 \end{cases}$$

Solution:

Rewrite the function as

$$f(t) = 1 - t + (-1 - (1 - t))u(t - 2) + ((5t - 20)^3 e^{3t-12} - (-1))u(t - 4)$$

$$f(t) = 1 - t + (t - 2)u(t - 2) + (125(t - 4)^3 e^{3(t-4)} + 1)u(t - 4)$$

The Laplace transform of the first term is

$$\mathscr{L}(1 - t) = \frac{1}{s} - \frac{1}{s^2}$$

By the Second Shifting Theorem, the Laplace transform of the second term is

$$\mathscr{L}(f(t - c)u(t - c)) = e^{-cs}F(s)$$

$$\mathscr{L}((t - 2)u(t - 2)) = e^{-2s}\frac{1}{s^2}$$

The Laplace transform of the third term is

$$\mathscr{L}(125(t - 4)^3 e^{3(t-4)} + 1)u(t - 4)) = e^{-4s}\left(125\frac{3!}{(s - 3)^4} + \frac{1}{s}\right)$$

$$\mathscr{L}(125(t - 4)^3 e^{3(t-4)} + 1)u(t - 4)) = e^{-4s}\left(\frac{750}{(s - 3)^4} + \frac{1}{s}\right)$$

Then the Laplace transform of the function is

$$\mathscr{L}(f(t)) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-2s} + \left(\frac{750}{(s-3)^4} + \frac{1}{s}\right)e^{-4s}$$

LAPLACE TRANSFORMS OF STEP FUNCTIONS

■ 1. Find the inverse Laplace transform.

$$F(s) = \frac{3se^{-7s}}{s^2 - 5s + 6}$$

Solution:

We'll start by pulling e^{-7s} out of the fraction, so that the function is then in the form $F(s) = e^{-7s}G(s)$, and

$$F(s) = e^{-7s}\frac{3s}{s^2 - 5s + 6}$$

$$G(s) = \frac{3s}{s^2 - 5s + 6} = \frac{3s}{(s-2)(s-3)}$$

Use partial fractions to decompose $G(s)$.

$$\frac{3s}{(s-2)(s-3)} = \frac{A}{s-2} + \frac{B}{s-3}$$

To find the value of A, we'll remove the $s - 2$ factor from the denominator of the left side, then set $s = 2$.

$$\frac{3s}{s-3} \to \frac{3(2)}{2-3} \to \frac{6}{-1} \to -6$$

To find the value of B, we'll remove the $s-3$ factor from the denominator of the left side, then set $s=3$.

$$\frac{3s}{s-2} \to \frac{3(3)}{3-2} \to \frac{9}{1} \to 9$$

So the partial fractions decomposition is

$$G(s) = \frac{-6}{s-2} + \frac{9}{s-3}$$

$$G(s) = -6\left(\frac{1}{s-2}\right) + 9\left(\frac{1}{s-3}\right)$$

The inverse transform of $G(s)$ is therefore

$$g(t) = -6e^{2t} + 9e^{3t}$$

and the inverse transform of $F(s)$ is

$$\mathscr{L}^{-1}(F(s)) = u_7(t)g(t-7) = (-6e^{2(t-7)} + 9e^{3(t-7)})u(t-7)$$

■ 2. Find the inverse Laplace transform.

$$G(s) = \frac{(2s^2+1)e^{-3s}}{s^3-4s^2+3s}$$

Solution:

We'll start by pulling e^{-3s} out of the fraction, so that the function is then in the form $G(s) = e^{-3s}F(s)$.

$$G(s) = e^{-3s}\frac{2s^2+1}{s^3-4s^2+3s}$$

$$F(s) = \frac{2s^2+1}{s^3-4s^2+3s} = \frac{2s^2+1}{s(s-1)(s-3)}$$

Use partial fractions to decompose $F(s)$.

$$\frac{2s^2+1}{s(s-1)(s-3)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-3}$$

To find the value of A, we'll remove the s factor from the denominator of the left side, then set $s = 0$.

$$\frac{2s^2+1}{(s-1)(s-3)} \to \frac{2(0)^2+1}{(0-1)(0-3)} \to \frac{1}{(-1)(-3)} \to \frac{1}{3}$$

To find the value of B, we'll remove the $s-1$ factor from the denominator of the left side, then set $s = 1$.

$$\frac{2s^2+1}{s(s-3)} \to \frac{2(1)^2+1}{1(1-3)} \to \frac{3}{-2} \to -\frac{3}{2}$$

To find the value of C, we'll remove the $s-3$ factor from the denominator of the left side, then set $s = 3$.

$$\frac{2s^2+1}{s(s-1)} \to \frac{2(3)^2+1}{3(3-1)} \to \frac{19}{6}$$

So the partial fractions decomposition is

$$F(s)=\frac{\frac{1}{3}}{s}+\frac{-\frac{3}{2}}{s-1}+\frac{\frac{19}{6}}{s-3}$$

$$F(s)=\frac{1}{3}\left(\frac{1}{s}\right)-\frac{3}{2}\left(\frac{1}{s-1}\right)+\frac{19}{6}\left(\frac{1}{s-3}\right)$$

Then the inverse transform of $F(s)$ is

$$f(t)=\frac{1}{3}-\frac{3}{2}e^{t}+\frac{19}{6}e^{3t}$$

Which means the inverse transform of $G(s)$ is

$$\mathscr{L}^{-1}(G(s))=u_3(t)f(t-3)=\left(\frac{1}{3}-\frac{3}{2}e^{t-3}+\frac{19}{6}e^{3(t-3)}\right)u(t-3)$$

■ 3. Find the Laplace transform of the function $g(t)$.

$$g(t)=5u_1(t)+6(t-3)^3u_3(t)+7e^4u_4(t)$$

Solution:

We know that the Laplace transform of $u_c(t)$ is given by

$$\mathscr{L}(u_c(t))=\frac{e^{-sc}}{s}$$

Therefore, the Laplace transform of $5u_1(t)$ must be

$$\frac{5e^{-s}}{s}$$

To find the transform of $6(t-3)^3u_3(t)$, we identify that $c=3$, and then we can make a substitution of $t+3$ into $f(t-c)$ to determine the function before it was shifted.

$$6(t-3)^3$$

$$6(t+3-3)^3$$

$$6t^3$$

So the Laplace transform of the second term will be given by

$$\mathscr{L}(u_c(t)f(t-c))=e^{-sc}F(s)$$

$$\mathscr{L}(6(t-3)^3u_3(t))=e^{-3s}\left(6\frac{3!}{s^{3+1}}\right)$$

$$\mathscr{L}(6(t-3)^3u_3(t))=e^{-3s}\left(\frac{36}{s^4}\right)$$

$$\mathscr{L}(6(t-3)^3u_3(t))=\frac{36e^{-3s}}{s^4}$$

Similarly, the Laplace transform of the third term will be given by

$$\mathscr{L}(u_c(t)f(t-c))=e^{-sc}F(s)$$

$$\mathscr{L}(7e^4u_4(t))=e^{-4s}\left(7e^4\frac{1}{s}\right)$$

$$\mathscr{L}(7e^4u_4(t))=\frac{e^{4-4s}}{s}$$

Putting these three transforms together, the Laplace transform is

$$\mathscr{L}(g(t)) = \frac{5e^{-s}}{s} + \frac{36e^{-3s}}{s^4} + \frac{e^{4-4s}}{s}$$

■ 4. Find the Laplace transform of the function $g(t)$.

$$f(t) = \begin{cases} 3 & 0 \le t < 2 \\ t+1 & 2 \le t < 5 \\ 4t - 14 & t \ge 5 \end{cases}$$

Solution:

There are two switches in the function, one from 3 to $t+1$, and one from $t+1$ to $4t-14$, so we'll need two Heaviside functions to express $f(t)$. We'll use $u_2(t)$ and $u_5(t)$, since the switches occur at $t=2$ and $t=5$.

When the first switch at $t=2$ turns on, the value of the function shifts from 3 to $t+1$ ($t+1-3=t-2$), so we can write

$$f(t) \approx 3 + (t-2)u_2(t)$$

When the second switch at $t=5$ turns on, the value of the function shifts from $t+1$ to $4t-14$ ($4t-14-(t+1)=3t-15$), so we can write

$$f(t) \approx 3 + (t-2)u_2(t) + (3t-15)u_5(t)$$

We know that the Laplace transform of $u_c(t)$ is given by

$$\mathscr{L}(u_c(t)) = \frac{e^{-sc}}{s}$$

Therefore, the Laplace transform of $(t-2)u_2(t)$ must be,

$$\frac{e^{-2s}}{s^2}$$

the Laplace transform of $(3t-15)u_5(t)$ must be,

$$\frac{3e^{-5s}}{s^2}$$

and putting these transforms together, the Laplace transform of $f(t)$ is

$$\mathscr{L}(f(t))=\frac{3}{s}+\frac{e^{-2s}}{s^2}+\frac{3e^{-5s}}{s^2}$$

■ 5. Find the Laplace transform of the function $g(t)$.

$$f(t)=\begin{cases}0 & 0\le t<\frac{\pi}{2} \\ 3\sin t & \frac{\pi}{2}\le t<\pi \\ \cos t & t\ge\pi\end{cases}$$

Solution:

There are two switches in the function, one from 0 to $3\sin t$, and one from $3\sin t$ to $\cos t$, so we'll need two Heaviside functions to express $f(t)$. We'll use $u_{\frac{\pi}{2}}(t)$ and $u_\pi(t)$, since the switches occur at $t=\pi/2$ and $t=\pi$.

When the first switch at $t = \pi/2$ turns on, the value of the function shifts from 0 to $3 \sin t$ ($3 \sin t - 0 = 3 \sin t$), so we can write

$$f(t) \approx 0 + 3u_{\frac{\pi}{2}}(t)\sin t$$

When the second switch at $t = \pi$ turns on, the value of the function shifts from $3 \sin t$ to $\cos t$ ($\cos t - 3 \sin t$), so we can write

$$f(t) \approx 0 + 3u_{\frac{\pi}{2}}(t)\sin t + (\cos t - 3 \sin t)u_{\pi}(t)$$

$$f(t) \approx 3u_{\frac{\pi}{2}}(t)\sin t + (\cos t - 3 \sin t)u_{\pi}(t)$$

Using identities from Trigonometry, we can rewrite values from $f(t)$ as

$$\sin t = \cos\left(t - \frac{\pi}{2}\right)$$

$$\cos t = -\cos(t - \pi)$$

$$\sin t = -\sin(t - \pi)$$

so,

$$\cos t - 3 \sin t = -\cos(t - \pi) + 3 \sin(t - \pi)$$

and then

$$f(t) \approx 3u_{\frac{\pi}{2}}(t)\cos\left(t - \frac{\pi}{2}\right) + (-\cos(t - \pi) + 3 \sin(t - \pi))u_{\pi}(t)$$

$$f(t) \approx 3u_{\frac{\pi}{2}}(t)\cos\left(t - \frac{\pi}{2}\right) - \cos(t - \pi)u_{\pi}(t) + 3 \sin(t - \pi)u_{\pi}(t)$$

The Laplace transform of of the first term must be

$$3\mathscr{L}\left(u\left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right)\right)=3e^{-\frac{\pi}{2}s}\frac{s}{s^2+1^2}$$

$$3\mathscr{L}\left(u\left(t-\frac{\pi}{2}\right)\cos\left(t-\frac{\pi}{2}\right)\right)=\frac{3se^{-\frac{\pi}{2}s}}{s^2+1}$$

The Laplace transform of $\cos(t-\pi)u_\pi(t)$ must be

$$\mathscr{L}(u(t-\pi)\cos(t-\pi))=e^{-\pi s}\frac{s}{s^2+1^2}$$

$$\mathscr{L}(u(t-\pi)\cos(t-\pi))=\frac{se^{-\pi s}}{s^2+1}$$

The Laplace transform of $3\sin(t-\pi)u_\pi(t)$ must be

$$3\mathscr{L}(u(t-\pi)\sin(t-\pi))=\frac{3(1)e^{-\pi s}}{s^2+1}$$

$$3\mathscr{L}(u(t-\pi)\sin(t-\pi))=\frac{3e^{-\pi s}}{s^2+1}$$

Putting these three transforms together, the Laplace transform is

$$\mathscr{L}(f(t))=\frac{3se^{-\frac{\pi}{2}s}}{s^2+1}-\frac{se^{-\pi s}}{s^2+1}+\frac{3e^{-\pi s}}{s^2+1}$$

■ 6. Find the inverse Laplace transform.

$$G(s)=\frac{3}{s}+e^{-s}\left(\frac{1}{s}-\frac{2}{s-4}\right)+e^{-3s}\frac{1}{s-9}$$

Solution:

We can think of the second term of the function as $G_1(s) = e^{-s}F_1(s)$.

$$G_1(s) = e^{-s}\left(\frac{1}{s} - \frac{2}{s-4}\right)$$

The inverse transform of $F_1(s)$ is

$$f_1(t) = 1 - 2e^{4t}$$

which means the inverse transform of $G_1(s)$ is

$$\mathcal{L}^{-1}(G_1(s)) = u_1(t)f(t-1)$$

$$\mathcal{L}^{-1}(G_1(s)) = (1 - 2e^{4(t-1)})u(t-1)$$

$$\mathcal{L}^{-1}(G_1(s)) = (1 - 2e^{4t-4})u(t-1)$$

We can think of the third term of the function as $G_2(s) = e^{-3s}F_2(s)$.

$$G_2(s) = e^{-3s}\frac{1}{s-9}$$

The inverse transform of $F_2(s)$ is

$$f_2(t) = e^{9t}$$

which means the inverse transform of $G_2(s)$ is

$$\mathcal{L}^{-1}(G_2(s)) = u_3(t)f(t-3)$$

$$\mathscr{L}^{-1}(G_2(s)) = e^{9(t-3)}u(t-3)$$

$$\mathscr{L}^{-1}(G_2(s)) = e^{9t-27}u(t-3)$$

Putting these together, the inverse Laplace transform is

$$\mathscr{L}^{-1}(G(s)) = 3 + (1 - 2e^{4t-4})u(t-1) + e^{9t-27}u(t-3)$$

STEP FUNCTIONS WITH INITIAL VALUE PROBLEMS

■ 1. Solve the initial value problem, given $y(0)=0$ and $y'(0)=0$.

$$y''=\begin{cases} 2 & 0\le t<4 \\ e^{2t} & t\ge 4 \end{cases}$$

Solution:

The forcing function could also be written as

$$g(t)=\begin{cases} 2 & 0\le t<4 \\ e^{2t} & t\ge 4 \end{cases}$$

$$g(t)=2+(e^{2t}-2)u(t-4)$$

Because the step function shows $c=4$, we need $g(t)=e^{2t}-2$ to include a shift of 4 as well. We can fix the shift using

$$\mathscr{L}(u_c(t)g(t))=e^{-cs}\mathscr{L}(g(t+c))$$

So we'll find

$$g(t+4)$$

$$e^{2(t+4)}-2$$

$$\mathscr{L}(u_4(t)g(t))=e^{-4s}\mathscr{L}(e^{2(t+4)}-2)$$

Then using the Laplace transforms of $\sin t$ and the step function on the right side, the transformed equation becomes

$$\mathscr{L}(y'') = \mathscr{L}(2) + e^{-4s}\mathscr{L}(e^{2(t+4)} - 2)$$

$$s^2Y(s) - sy(0) - y'(0) = \frac{2}{s} + e^{-4s}\mathscr{L}(e^{2t}e^8 - 2)$$

$$s^2Y(s) - sy(0) - y'(0) = \frac{2}{s} + e^{-4s}\left(e^8\frac{1}{s-2} - \frac{2}{s}\right)$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$, then solve for $Y(s)$.

$$s^2Y(s) = \frac{2}{s} + e^{-4s}\left(e^8\frac{1}{s-2} - \frac{2}{s}\right)$$

$$Y(s) = \frac{2}{s^3} + e^{-4s}\left(e^8\frac{1}{s^2(s-2)} - \frac{2}{s^3}\right)$$

Let's apply a partial fractions decomposition.

$$\frac{1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2}$$

Multiply through both sides of the decomposition equation by the denominator from the left side.

$$1 = As(s-2) + B(s-2) + Cs^2$$

$$1 = As^2 - 2As + Bs - 2B + Cs^2$$

$$1 = s^2(A + C) + s(B - 2A) - 2B$$

Equating coefficients gives $A = -1/4$, $B = -1/2$, and $C = 1/4$, and plugging these into the decomposition gives

$$\frac{1}{s^2(s-2)} = \frac{-\frac{1}{4}}{s} + \frac{-\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s-2}$$

$$\frac{1}{s^2(s-2)} = -\frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s-2}\right)$$

So $Y(s)$ is

$$Y(s) = \frac{2}{s^3} + e^{-4s}\left(e^8\left(-\frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s-2}\right)\right) - \frac{2}{s^3}\right)$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$, the general solution to the second order equation is

$$y(t) = t^2 + u(t-4)\left(e^8\left(-\frac{1}{4} - \frac{1}{2}(t-4) + \frac{1}{4}e^{2(t-4)}\right) - (t-4)^2\right)$$

■ 2. Solve the initial value problem, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' - 9y = e^{2t-10}u(t-5)$$

Solution:

Rewrite the function as

$$y'' - 9y = e^{2(t-5)}u(t-5)$$

Both the exponential and the step function show a shift of $c = 5$, which means that the transformed equation becomes

$$\mathscr{L}(y'') - 9\mathscr{L}(y) = e^{-5s}\mathscr{L}(e^{2t})$$

$$s^2Y(s) - sy(0) - y'(0) - 9Y(s) = e^{-5s}\frac{1}{s-2}$$

Substitute the initial conditions $y(0) = 1$ and $y'(0) = 0$, then solve for $Y(s)$.

$$s^2Y(s) - s - 9Y(s) = e^{-5s}\frac{1}{s-2}$$

$$(s^2 - 9)Y(s) = e^{-5s}\frac{1}{s-2} + s$$

$$Y(s) = e^{-5s}\frac{1}{(s-2)(s+3)(s-3)} + \frac{s}{(s+3)(s-3)}$$

Apply a partial fractions decomposition to the first term.

$$\frac{1}{(s-2)(s+3)(s-3)} = \frac{A}{s-2} + \frac{B}{s+3} + \frac{C}{s-3}$$

To solve for A, we'll remove the factor of $s - 2$ from the denominator on the left side, then set $s = 2$.

$$\frac{1}{(s+3)(s-3)} = \frac{1}{(2+3)(2-3)} = -\frac{1}{5} = A$$

To solve for B, we'll remove the factor of $s + 3$ from the denominator on the left side, then set $s = -3$.

$$\frac{1}{(s-2)(s-3)}=\frac{1}{(-3-2)(-3-3)}=\frac{1}{30}=B$$

To solve for C, we'll remove the factor of $s-3$ from the denominator on the left side, then set $s=3$.

$$\frac{1}{(s-2)(s+3)}=\frac{1}{(3-2)(3+3)}=\frac{1}{6}=C$$

Plugging $A=-1/5$, $B=1/30$, and $C=1/6$ into the decomposition gives

$$\frac{1}{(s-2)(s+3)(s-3)}=\frac{-\frac{1}{5}}{s-2}+\frac{\frac{1}{30}}{s+3}+\frac{\frac{1}{6}}{s-3}$$

$$\frac{1}{(s-2)(s+3)(s-3)}=-\frac{1}{5}\left(\frac{1}{s-2}\right)+\frac{1}{30}\left(\frac{1}{s+3}\right)+\frac{1}{6}\left(\frac{1}{s-3}\right)$$

Now apply a partial fractions decomposition to the second term.

$$\frac{s}{(s+3)(s-3)}=\frac{A}{s+3}+\frac{B}{s-3}$$

To solve for A, we'll remove the factor of $s+3$ from the denominator on the left side, then set $s=-3$.

$$\frac{s}{s-3}=\frac{-3}{-3-3}=\frac{1}{2}=A$$

To solve for B, we'll remove the factor of $s-3$ from the denominator on the left side, then set $s=3$.

$$\frac{s}{s+3}=\frac{3}{3+3}=\frac{1}{2}=B$$

Plugging $A = 1/2$, and $B = 1/2$ into the decomposition gives

$$\frac{s}{(s+3)(s-3)} = \frac{\frac{1}{2}}{s+3} + \frac{\frac{1}{2}}{s-3}$$

$$\frac{s}{(s+3)(s-3)} = \frac{1}{2}\left(\frac{1}{s+3}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right)$$

So $Y(s)$ is

$$Y(s) = e^{-5s}\left(-\frac{1}{5}\left(\frac{1}{s-2}\right) + \frac{1}{30}\left(\frac{1}{s+3}\right) + \frac{1}{6}\left(\frac{1}{s-3}\right)\right)$$

$$+\frac{1}{2}\left(\frac{1}{s+3}\right) + \frac{1}{2}\left(\frac{1}{s-3}\right)$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$, the general solution to the second order equation is

$$y(t) = u(t-5)\left(-\frac{1}{5}e^{2(t-5)} + \frac{1}{30}e^{-3(t-5)} + \frac{1}{6}e^{3(t-5)}\right) + \frac{1}{2}e^{-3t} + \frac{1}{2}e^{3t}$$

■ 3. Solve the initial value problem, given $y(0) = -1$ and $y'(0) = 1$.

$$y'' - 4y' = g(t)$$

$$g(t) = \begin{cases} 0 & 0 \le t < 5 \\ -3 & t \ge 5 \end{cases}$$

Solution:

The forcing function could also be written as

$$g(t) = -3u(t-5)$$

so

$$y'' - 4y' = -3u(t-5)$$

Applying a Laplace transform to the second order equation, we get

$$\mathscr{L}(y'') - 4\mathscr{L}(y') = -3\mathscr{L}(u(t-5))$$

$$s^2Y(s) - sy(0) - y'(0) - 4(sY(s) - y(0)) = -3e^{-5s}\frac{1}{s}$$

Substitute the initial conditions $y(0) = -1$ and $y'(0) = 1$, then solve for $Y(s)$.

$$s^2Y(s) + s - 1 - 4(sY(s) + 1) = -3e^{-5s}\frac{1}{s}$$

$$s^2Y(s) + s - 1 - 4sY(s) - 4 = -3e^{-5s}\frac{1}{s}$$

$$(s^2 - 4s)Y(s) + s - 5 = -3e^{-5s}\frac{1}{s}$$

$$(s^2 - 4s)Y(s) = -3e^{-5s}\frac{1}{s} + 5 - s$$

$$Y(s) = -3e^{-5s}\frac{1}{s(s^2 - 4s)} + \frac{5 - s}{s^2 - 4s}$$

$$Y(s) = e^{-5s}\frac{-3}{s^2(s-4)} + \frac{5 - s}{s(s-4)}$$

Apply a partial fractions decomposition to the first term.

$$-\frac{3}{s^2(s-4)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-4}$$

To solve for C, we'll remove the factor of $s-4$ from the denominator on the left side, then set $s=4$.

$$-\frac{3}{s^2}=-\frac{3}{4^2}=-\frac{3}{16}=C$$

To find A and B, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$-3=As(s-4)+B(s-4)+Cs^2$$

$$-3=As^2-4As+Bs-4B+Cs^2$$

$$-3=s^2(A+C)+s(B-4A)-4B$$

Equating coefficients gives $A=3/16$ and $B=3/4$, so we'll plug these values into the decomposition.

$$-\frac{3}{s^2(s-4)}=\frac{3}{16}\left(\frac{1}{s}\right)+\frac{3}{4}\left(\frac{1}{s^2}\right)-\frac{3}{16}\left(\frac{1}{s-4}\right)$$

Apply a partial fractions decomposition to the second term.

$$\frac{5-s}{s(s-4)}=\frac{D}{s}+\frac{E}{s-4}$$

To solve for E, we'll remove the factor of $s-4$ from the denominator on the left side, then set $s=4$.

$$\frac{5-s}{s}=\frac{5-4}{4}=\frac{1}{4}=E$$

To solve for D, we'll remove the factor of s from the denominator on the left side, then set $s=0$.

$$\frac{5-s}{s-4}=\frac{5-0}{0-4}=-\frac{5}{4}=D$$

Plugging $D=-5/4$ and $E=1/4$ into the decomposition gives

$$\frac{5-s}{s(s-4)}=-\frac{5}{4}\left(\frac{1}{s}\right)+\frac{1}{4}\left(\frac{1}{s-4}\right)$$

So $Y(s)$ is

$$Y(s)=e^{-5s}\left(\frac{3}{16}\left(\frac{1}{s}\right)+\frac{3}{4}\left(\frac{1}{s^2}\right)-\frac{3}{16}\left(\frac{1}{s-4}\right)\right)-\frac{5}{4}\left(\frac{1}{s}\right)+\frac{1}{4}\left(\frac{1}{s-4}\right)$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s))=f(t-c)u(t-c)$, the general solution to the second order equation is

$$y(t)=u(t-5)\left(\frac{3}{16}+\frac{3}{4}(t-5)-\frac{3}{16}e^{4(t-5)}\right)-\frac{5}{4}+\frac{1}{4}e^{4t}$$

■ 4. Solve the initial value problem, given $y(0)=0$ and $y'(0)=0$.

$$y''-5y'+6y=tu(t-4)$$

Solution:

The forcing function can also be written as

$$g(t) = \begin{cases} 0 & 0 \le t < 4 \\ t & t \ge 4 \end{cases}$$

Because the step function shows $c = 4$, we need $f(t) = t$ to include a shift of 4 as well. We can fix the shift using

$$\mathscr{L}(u_c(t)g(t)) = e^{-cs}\mathscr{L}(g(t + c))$$

So we'll find

$$g(t + 4)$$

$$t + 4$$

Then the Laplace transform of the differential equation becomes

$$\mathscr{L}(y'') - 5\mathscr{L}(y') + 6\mathscr{L}(y) = \mathscr{L}(tu(t - 4))$$

$$\mathscr{L}(y'') - 5\mathscr{L}(y') + 6\mathscr{L}(y) = e^{-4s}\mathscr{L}(t + 4)$$

$$s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 6Y(s) = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$, then solve for $Y(s)$.

$$s^2Y(s) - 5sY(s) + 6Y(s) = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

$$(s^2 - 5s + 6)Y(s) = e^{-4s}\left(\frac{1}{s^2} + \frac{4}{s}\right)$$

$$(s^2 - 5s + 6)Y(s) = e^{-4s}\left(\frac{1 + 4s}{s^2}\right)$$

$$Y(s) = e^{-4s}\left(\frac{1 + 4s}{s^2(s^2 - 5s + 6)}\right)$$

$$Y(s) = e^{-4s}\left(\frac{1 + 4s}{s^2(s - 2)(s - 3)}\right)$$

Let's apply a partial fractions decomposition.

$$\frac{1 + 4s}{s^2(s - 2)(s - 3)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 2} + \frac{D}{s - 3}$$

To solve for C, we'll remove the factor of $s - 2$ from the denominator on the left side, then set $s = 2$.

$$\frac{1 + 4s}{s^2(s - 3)} \to \frac{1 + 4(2)}{2^2(2 - 3)} \to \frac{9}{-4} \to -\frac{9}{4}$$

To solve for D, we'll remove the factor of $s - 3$ from the denominator on the left side, then set $s = 3$.

$$\frac{1 + 4s}{s^2(s - 2)} \to \frac{1 + 4(3)}{3^2(3 - 2)} \to \frac{13}{9}$$

To find A and B, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$1 + 4s = As(s - 2)(s - 3) + B(s - 2)(s - 3) + Cs^2(s - 3) + Ds^2(s - 2)$$

$$1 + 4s = As^3 - 5As^2 + 6As + Bs^2 - 5Bs + 6B + Cs^3 - 3Cs^2 + Ds^3 - 2Ds^2$$

$$1 + 4s = s^3(A + C + D) + s^2(-5A + B - 3C - 2D) + s(6A - 5B) + 6B$$

Equating coefficients gives a system of equations,

$$A + C + D = 0$$

$$-5A + B - 3C - 2D = 0$$

$$6A - 5B = 4$$

and if we solve this system we find $A = 29/36$ and $B = 1/6$. Plugging these values, $C = -9/4$, and $D = 13/9$ into the decomposition gives

$$\frac{1+4s}{s^2(s-2)(s-3)} = \frac{29}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right) - \frac{9}{4}\left(\frac{1}{s-2}\right) + \frac{13}{9}\left(\frac{1}{s-3}\right)$$

So $Y(s)$ is

$$Y(s) = e^{-4s}\left(\frac{29}{36}\left(\frac{1}{s}\right) + \frac{1}{6}\left(\frac{1}{s^2}\right) - \frac{9}{4}\left(\frac{1}{s-2}\right) + \frac{13}{9}\left(\frac{1}{s-3}\right)\right)$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$, the inverse transform is

$$y(t) = u(t-4)\left(\frac{29}{36} + \frac{1}{6}(t-4) - \frac{9}{4}e^{2(t-4)} + \frac{13}{9}e^{3(t-4)}\right)$$

■ 5. Solve the initial value problem, given $y(0) = -2$ and $y'(0) = 1$.

$$y'' - y' = \sin(3t-9)u(t-3)$$

Solution:

We know from our table to Laplace transforms that

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2}$$

Using these formulas, the transform of the differential equation becomes

$$\mathscr{L}(y'') - \mathscr{L}(y') = \mathscr{L}(\sin(3(t-3))u(t-3))$$

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) = e^{-3s}\frac{3}{s^2+9}$$

Substitute the initial conditions $y(0) = -2$ and $y'(0) = 1$, then solve for $Y(s)$.

$$s^2Y(s) + 2s - 1 - (sY(s) + 2) = e^{-3s}\frac{3}{s^2+9}$$

$$s^2Y(s) + 2s - 1 - sY(s) - 2 = e^{-3s}\frac{3}{s^2+9}$$

$$(s^2 - s)Y(s) + 2s - 3 = e^{-3s}\frac{3}{s^2+9}$$

$$(s^2 - s)Y(s) = e^{-3s}\frac{3}{s^2+9} + 3 - 2s$$

$$Y(s) = e^{-3s}\frac{3}{s(s^2+9)(s-1)} + \frac{3-2s}{s(s-1)}$$

Let's apply a partial fractions decomposition for the first term.

$$\frac{3}{s(s^2+9)(s-1)} = \frac{A}{s} + \frac{B}{s-1} + \frac{Cs+D}{s^2+9}$$

To solve for A, we'll remove the factor of s from the denominator on the left side, then set $s = 0$.

$$\frac{3}{(s^2+9)(s-1)} \to \frac{3}{(0^2+9)(0-1)} \to \frac{3}{9(-1)} \to -\frac{3}{9} \to -\frac{1}{3}$$

To solve for B, we'll remove the factor of $s-1$ from the denominator on the left side, then set $s = 1$.

$$\frac{3}{s(s^2+9)} \to \frac{3}{1(1^2+9)} \to \frac{3}{10}$$

To find C and D, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$3 = A(s^2+9)(s-1) + Bs(s^2+9) + (Cs+D)s(s-1)$$

$$3 = As^3 - As^2 + 9As - 9A + Bs^3 + 9Bs + Cs^3 - Cs^2 + Ds^2 - Ds$$

$$3 = s^3(A+B+C) + s^2(-A-C+D) + s(9A+9B-D) - 9A$$

Equate coefficients, then substitute the values $A = -1/3$ and $B = 3/10$ that we already found in order to find C.

$$A + B + C = 0$$

$$-\frac{1}{3} + \frac{3}{10} + C = 0$$

$$C = \frac{1}{30}$$

Then equate coefficients and substitute these values in order to find D.

$$-A - C + D = 0$$

$$\frac{1}{3} - \frac{1}{30} + D = 0$$

$$D = -\frac{3}{10}$$

Plugging $A = -1/3$, $B = 3/10$, $C = 1/30$, and $D = -3/10$ into the decomposition equation gives

$$\frac{3}{s(s^2+9)(s-1)} = \frac{-\frac{1}{3}}{s} + \frac{\frac{3}{10}}{s-1} + \frac{\frac{1}{30}s - \frac{3}{10}}{s^2+9}$$

$$\frac{1}{s(s^2+9)(s-1)} = -\frac{1}{3}\left(\frac{1}{s}\right) + \frac{3}{10}\left(\frac{1}{s-1}\right) + \frac{1}{30}\left(\frac{s-9}{s^2+9}\right)$$

$$\frac{1}{s(s^2+9)(s-1)} = -\frac{1}{3}\left(\frac{1}{s}\right) + \frac{3}{10}\left(\frac{1}{s-1}\right) + \frac{1}{30}\left(\frac{s}{s^2+9} - \frac{9}{s^2+9}\right)$$

Now let's apply a partial fractions decomposition for the second term.

$$\frac{3-2s}{s(s-1)} = \frac{E}{s} + \frac{F}{s-1}$$

To solve for E, we'll remove the factor of s from the denominator on the left side, then set $s = 0$.

$$\frac{3-2s}{s-1} \to \frac{3-2(0)}{0-1} \to \frac{3}{-1} \to -3$$

To solve for F, we'll remove the factor of $s-1$ from the denominator on the left side, then set $s=1$.

$$\frac{3-2s}{s} \to \frac{3-2(1)}{1} \to 1$$

Plugging $E=-3$, and $F=1$ into the decomposition equation gives

$$\frac{3-2s}{s(s-1)} = -\frac{3}{s} + \frac{1}{s-1}$$

So $Y(s)$ is

$$Y(s) = e^{-3s}\left(-\frac{1}{3}\left(\frac{1}{s}\right) + \frac{3}{10}\left(\frac{1}{s-1}\right) + \frac{1}{30}\left(\frac{s}{s^2+9} - \frac{9}{s^2+9}\right)\right) - \frac{3}{s} + \frac{1}{s-1}$$

$$Y(s) = -3\left(\frac{1}{s}\right) + \frac{1}{s-1} - \frac{1}{3}\left(\frac{1}{s}\right)e^{-3s} + \frac{3}{10}\left(\frac{1}{s-1}\right)e^{-3s}$$

$$+\frac{1}{30}\left(\frac{s}{s^2+9}\right)e^{-3s} - \frac{1}{30}\left(\frac{9}{s^2+9}\right)e^{-3s}$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$, the inverse transform is

$$y(t) = -3 + e^t - \frac{1}{3}(1)u(t-3) + \frac{3}{10}e^{t-3}u(t-3)$$

$$+\frac{1}{30}\cos(3(t-3))u(t-3) - \frac{3}{30}\sin(3(t-3))u(t-3)$$

$$y(t) = -3 + e^t - \frac{1}{3}u(t-3) + \frac{3}{10}e^{t-3}u(t-3)$$

$$+\frac{1}{30}\cos(3(t-3))u(t-3)-\frac{1}{10}\sin(3(t-3))u(t-3)$$

■ 6. Solve the initial value problem, given $y(0) = 0$ and $y'(0) = 2$.

$$y'' - 6y' = \begin{cases} 1 & 0 \le t < 2 \\ 3t + 1 & t \ge 2 \end{cases}$$

Solution:

The forcing function can also be written as

$$g(t) = 1 + 3tu(t-2)$$

Because the step function shows $c = 2$, we need $f(t) = 3t$ to include a shift of 2 as well. We can fix the shift using

$$\mathscr{L}(u_c(t)g(t)) = e^{-cs}\mathscr{L}(g(t+c))$$

So we'll find

$$g(t+2)$$

$$3(t+2)$$

Then using the Laplace transforms of t and the step function on the right side, the Laplace transform of the differential equation will be

$$\mathscr{L}(y'') - 6\mathscr{L}(y') = \mathscr{L}(1) + \mathscr{L}(3tu(t-2))$$

$$s^2Y(s) - sy(0) - y'(0) - 6sY(s) + 6y(0) = \frac{1}{s} + 3e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 2$, then solve for $Y(s)$.

$$s^2Y(s) - 2 - 6sY(s) = \frac{1}{s} + 3e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

$$(s^2 - 6s)Y(s) = \frac{1}{s} + 3e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + 2$$

$$Y(s) = \frac{1}{s(s^2 - 6s)} + 3e^{-2s}\left(\frac{1 + 2s}{s^2(s^2 - 6s)}\right) + \frac{2}{s^2 - 6s}$$

$$Y(s) = \frac{1 + 2s}{s(s^2 - 6s)} + 3e^{-2s}\left(\frac{1 + 2s}{s^2(s^2 - 6s)}\right)$$

$$Y(s) = \frac{1 + 2s}{s^2(s - 6)} + e^{-2s}\left(\frac{3 + 6s}{s^3(s - 6)}\right)$$

Let's apply a partial fractions decomposition to the first term.

$$\frac{1 + 2s}{s^2(s - 6)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 6}$$

To solve for C, we'll remove the factor of $s - 6$ from the denominator on the left side, then set $s = 6$.

$$\frac{1 + 2s}{s^2} \to \frac{1 + 2(6)}{6^2} \to \frac{13}{36}$$

To find B and C, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$1 + 2s = As(s - 6) + B(s - 6) + Cs^2$$

$$1 + 2s = As^2 - 6As + Bs - 6B + Cs^2$$

$$1 + 2s = s^2(A + C) + s(-6A + B) - 6B$$

Then equating coefficients gives $A = -13/36$ and $B = -1/6$. Plugging these values and $C = 13/36$ into the decomposition gives

$$\frac{1 + 2s}{s^2(s - 6)} = -\frac{13}{36}\left(\frac{1}{s}\right) - \frac{1}{6}\left(\frac{1}{s^2}\right) + \frac{13}{36}\left(\frac{1}{s - 6}\right)$$

Let's apply a partial fractions decomposition to the second term.

$$\frac{3 + 6s}{s^3(s - 6)} = \frac{D}{s} + \frac{E}{s^2} + \frac{F}{s^3} + \frac{G}{s - 6}$$

To solve for G, we'll remove the factor of $s - 6$ from the denominator on the left side, then set $s = 6$.

$$\frac{3 + 6s}{s^3} \to \frac{3 + 6(6)}{6^3} \to \frac{39}{216} \to \frac{13}{72}$$

To find D, E, and F, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$3 + 6s = Ds^2(s - 6) + Es(s - 6) + F(s - 6) + Gs^3$$

$$3 + 6s = Ds^3 - 6Ds^2 + Es^2 - 6Es + Fs - 6F + Gs^3$$

$$3 + 6s = s^3(D + G) + s^2(-6D + E) + s(-6E + F) - 6F$$

Then equating coefficients lets us find $D - 13/72$, $E = -13/12$, and $F = -1/2$. Plugging these values and $G = 13/72$ into the decomposition equation gives

$$\frac{3+6s}{s^3(s-6)} = -\frac{13}{72}\left(\frac{1}{s}\right) - \frac{13}{12}\left(\frac{1}{s^2}\right) - \frac{1}{2}\left(\frac{1}{s^3}\right) + \frac{13}{72}\left(\frac{1}{s-6}\right)$$

So $Y(s)$ is

$$Y(s) = -\frac{13}{36}\left(\frac{1}{s}\right) - \frac{1}{6}\left(\frac{1}{s^2}\right) + \frac{13}{36}\left(\frac{1}{s-6}\right)$$

$$+e^{-2s}\left(-\frac{13}{72}\left(\frac{1}{s}\right) - \frac{13}{12}\left(\frac{1}{s^2}\right) - \frac{1}{2}\left(\frac{1}{s^3}\right) + \frac{13}{72}\left(\frac{1}{s-6}\right)\right)$$

$$Y(s) = -\frac{13}{36}\left(\frac{1}{s}\right) - \frac{1}{6}\left(\frac{1}{s^2}\right) + \frac{13}{36}\left(\frac{1}{s-6}\right)$$

$$-\frac{13}{72}\left(\frac{1}{s}\right)e^{-2s} - \frac{13}{12}\left(\frac{1}{s^2}\right)e^{-2s} - \frac{1}{2}\left(\frac{1}{s^3}\right)e^{-2s} + \frac{13}{72}\left(\frac{1}{s-6}\right)e^{-2s}$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$, the inverse transform is

$$y(t) = -\frac{13}{36} - \frac{1}{6}t + \frac{13}{36}e^{6t} - \frac{13}{72}u(t-2) - \frac{13}{12}(t-2)u(t-2)$$

$$-\frac{1}{4}(t-2)^2u(t-2) + \frac{13}{72}e^{6(t-2)}u(t-2)$$

THE DIRAC DELTA FUNCTION

■ 1. Solve the initial value problem, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' + 4y = 4\delta(t-2)$$

Solution:

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'') + 4\mathscr{L}(y) = 4\mathscr{L}(\delta(t-2))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y(t)) = Y(s)$$

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

we get

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 4e^{-2s}$$

Substitute the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$s^2Y(s) - s(1) - 0 + 4Y(s) = 4e^{-2s}$$

$$s^2Y(s) - s + 4Y(s) = 4e^{-2s}$$

Solve for $Y(s)$.

$$s^2Y(s) + 4Y(s) = 4e^{-2s} + s$$

$$(s^2 + 4)Y(s) = 4e^{-2s} + s$$

$$Y(s) = 4e^{-2s}\frac{1}{s^2 + 4} + \frac{s}{s^2 + 4}$$

$$Y(s) = 2e^{-2s}\frac{2}{s^2 + 4} + \frac{s}{s^2 + 4}$$

From our table of Laplace transforms, we know

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2 + a^2}$$

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

So the general solution to the second order equation is

$$y(t) = 2u(t-2)\sin(2(t-2)) + \cos(2t)$$

■ 2. Solve the initial value problem, given $y(0) = 0$ and $y'(0) = 1$.

$$y'' + 2y' + 3y = \delta(t-5)$$

Solution:

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'') + 2\mathscr{L}(y') + 3\mathscr{L}(y) = \mathscr{L}(\delta(t-5))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

$$\mathscr{L}(y(t)) = Y(s)$$

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

we get

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + 3Y(s) = e^{-5s}$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 1$.

$$s^2Y(s) - 1 + 2sY(s) + 3Y(s) = e^{-5s}$$

Solve for $Y(s)$.

$$(s^2 + 2s + 3)Y(s) = e^{-5s} + 1$$

$$Y(s) = e^{-5s}\frac{1}{s^2 + 2s + 3} + \frac{1}{s^2 + 2s + 3}$$

$$Y(s) = e^{-5s}\frac{1}{(s+1)^2 + 2} + \frac{1}{(s+1)^2 + 2}$$

$$Y(s) = e^{-5s}\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{(s^2+1) + (\sqrt{2})^2}\right) + \frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{(s^2+1) + (\sqrt{2})^2}\right)$$

From our table of Laplace transforms, we know

$$\mathscr{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

So the general solution to the second order equation is

$$y(t) = \frac{1}{\sqrt{2}}u(t-5)e^{-(t-5)}\sin(\sqrt{2}(t-5)) + \frac{1}{\sqrt{2}}e^{-t}\sin(\sqrt{2}t)$$

$$y(t) = \frac{\sqrt{2}}{2}e^{-t}[u(t-5)e^{5}\sin(\sqrt{2}(t-5)) + \sin(\sqrt{2}t)]$$

■ 3. Solve the initial value problem, given $y(0)=2$ and $y'(0)=0$.

$$y''-y'=3u(t-1)$$

Solution:

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'')-\mathscr{L}(y')=3\mathscr{L}(u(t-1))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s)-sy(0)-y'(0)$$

$$\mathscr{L}(y'(t)) = sY(s)-y(0)$$

$$\mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

we get

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) = \frac{3e^{-s}}{s}$$

Substitute the initial conditions $y(0) = 2$ and $y'(0) = 0$.

$$s^2Y(s) - sy(0) - y'(0) - sY(s) + y(0) = \frac{3e^{-s}}{s}$$

$$s^2Y(s) - 2s - sY(s) + 2 = \frac{3e^{-s}}{s}$$

Solve for $Y(s)$.

$$(s^2 - s)Y(s) = \frac{3e^{-s}}{s} + 2s - 2$$

$$Y(s) = \frac{3e^{-s}}{s(s^2 - s)} + \frac{2s - 2}{s^2 - s}$$

$$Y(s) = \frac{3e^{-s}}{s^2(s - 1)} + \frac{2(s - 1)}{s(s - 1)}$$

$$Y(s) = e^{-s}\frac{3}{s^2(s - 1)} + \frac{2}{s}$$

Apply a partial fractions decomposition.

$$\frac{1}{s^2(s - 1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s - 1}$$

To solve for C, remove the $s - 1$ factor from the denominator of the left side, then substitute $s = 1$ into the left side.

$$\frac{1}{s^2} \to \frac{1}{1^2} \to 1$$

To find A and B, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$1 = As(s-1) + B(s-1) + Cs^2$$

$$1 = As^2 - As + Bs - B + Cs^2$$

$$1 = s^2(A + C) + s(B - A) - B$$

Then equating coefficients gives

$-B = 1$ so $B = -1$

$B - A = 0$ so $B = A = -1$

So the decomposition is

$$\frac{1}{s^2(s-1)} = -\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1}$$

Then the Laplace transform becomes

$$Y(s) = 3e^{-s}\left(-\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s-1}\right) + \frac{2}{s}$$

$$Y(s) = -3e^{-s}\left(\frac{1}{s}\right) - 3e^{-s}\left(\frac{1}{s^2}\right) + 3e^{-s}\left(\frac{1}{s-1}\right) + 2\left(\frac{1}{s}\right)$$

Now that we've broken down the transform this way, we can apply the inverse transform to find the general solution to the second order nonhomogeneous equation.

$$y(t) = -3u(t-1) - 3(t-1)u(t-1) + 3e^{t-1}u(t-1) + 2$$

$$y(t) = 3u(t-1)(-1-(t-1)+e^{t-1}) + 2$$

$$y(t) = 3u(t-1)(-1-t+1+e^{t-1}) + 2$$

$$y(t) = 3u(t-1)(e^{t-1}-t) + 2$$

■ 4. Solve the initial value problem, given $y(0) = 1$ and $y'(0) = 1$.

$$y'' + y' - 2y = 3u(t-4)$$

Solution:

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'') + \mathscr{L}(y') - 2\mathscr{L}(y) = 3\mathscr{L}(u(t-4))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

$$\mathscr{L}(y(t)) = Y(s)$$

$$\mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

we get

$$s^2Y(s) - sy(0) - y'(0) + (sY(s) - y(0)) - 2Y(s) = \frac{3e^{-4s}}{s}$$

Substitute the initial conditions $y(0) = 1$ and $y'(0) = 1$.

$$s^2Y(s) - s - 1 + (sY(s) - 1) - 2Y(s) = \frac{3e^{-4s}}{s}$$

$$s^2Y(s) - s - 1 + sY(s) - 1 - 2Y(s) = \frac{3e^{-4s}}{s}$$

$$s^2Y(s) - s - 2 + sY(s) - 2Y(s) = \frac{3e^{-4s}}{s}$$

$$s^2Y(s) + sY(s) - 2Y(s) = \frac{3e^{-4s}}{s} + s + 2$$

Solve for $Y(s)$.

$$(s^2 + s - 2)Y(s) = \frac{3e^{-4s}}{s} + s + 2$$

$$Y(s) = \frac{3e^{-4s}}{s(s^2 + s - 2)} + \frac{s + 2}{s^2 + s - 2}$$

$$Y(s) = \frac{3e^{-4s}}{s(s - 1)(s + 2)} + \frac{s + 2}{(s - 1)(s + 2)}$$

$$Y(s) = \frac{3e^{-4s}}{s(s - 1)(s + 2)} + \frac{1}{s - 1}$$

Apply a partial fractions decomposition.

$$\frac{1}{s(s - 1)(s + 2)} = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s + 2}$$

To solve for A, remove the s factor from the denominator of the left side, then substitute $s = 0$ into the left side.

$$\frac{1}{(s-1)(s+2)} \to \frac{1}{(0-1)(0+2)} \to -\frac{1}{2}$$

To solve for B, remove the $s-1$ factor from the denominator of the left side, then substitute $s = 1$ into the left side.

$$\frac{1}{s(s+2)} \to \frac{1}{1(1+2)} \to \frac{1}{3}$$

To solve for C, remove the $s+2$ factor from the denominator of the left side, then substitute $s = -2$ into the left side.

$$\frac{1}{s(s-1)} \to \frac{1}{-2(-2-1)} \to \frac{1}{6}$$

So the decomposition is

$$\frac{1}{s(s-1)(s-2)} = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{3}\left(\frac{1}{s-1}\right) + \frac{1}{6}\left(\frac{1}{s+2}\right)$$

Then the Laplace transform becomes

$$Y(s) = 3e^{-4s}\left(-\frac{1}{2}\left(\frac{1}{s}\right) + \frac{1}{3}\left(\frac{1}{s-1}\right) + \frac{1}{6}\left(\frac{1}{s+2}\right)\right) + \frac{1}{s-1}$$

$$Y(s) = -\frac{3}{2}e^{-4s}\left(\frac{1}{s}\right) + e^{-4s}\left(\frac{1}{s-1}\right) + \frac{1}{2}e^{-4s}\left(\frac{1}{s+2}\right) + \frac{1}{s-1}$$

Now that we've broken down the transform this way, we can apply the inverse transform to find the general solution to the second order nonhomogeneous equation.

$$y(t) = -\frac{3}{2}u(t-4) + u(t-4)e^{t-4} + \frac{1}{2}u(t-4)e^{-2(t-4)} + e^t$$

■ 5. Solve the initial value problem, given $y(0) = -2$ and $y'(0) = 1$.

$$y'' - 2y = u(t-7) - 2\delta(t-7)$$

Solution:

Apply the Laplace transform to both sides of the second order equation.

$$\mathcal{L}(y'') - 2\mathcal{L}(y) = \mathcal{L}(u(t-7)) - 2\mathcal{L}(\delta(t-7))$$

Using formulas from our table of transforms,

$$\mathcal{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}(y(t)) = Y(s)$$

$$\mathcal{L}(\delta(t-c)) = e^{-cs}$$

$$\mathcal{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

we get

$$s^2Y(s) - sy(0) - y'(0) - 2Y(s) = \frac{e^{-7s}}{s} - 2e^{-7s}$$

Substitute the initial conditions $y(0) = -2$ and $y'(0) = 1$.

$$s^2Y(s) + 2s - 1 - 2Y(s) = \frac{e^{-7s}}{s} - 2e^{-7s}$$

$$s^2Y(s) - 2Y(s) = \frac{e^{-7s}}{s} - 2e^{-7s} - 2s + 1$$

Solve for $Y(s)$.

$$(s^2 - 2)Y(s) = \frac{e^{-7s}}{s} - 2e^{-7s} - 2s + 1$$

$$(s^2 - 2)Y(s) = e^{-7s}\frac{1 - 2s}{s} - 2s + 1$$

$$Y(s) = e^{-7s}\frac{1 - 2s}{s(s^2 - 2)} - \frac{2s}{s^2 - 2} + \frac{1}{s^2 - 2}$$

Apply a partial fractions decomposition.

$$\frac{1 - 2s}{s(s - \sqrt{2})(s + \sqrt{2})} = \frac{A}{s} + \frac{B}{s - \sqrt{2}} + \frac{C}{s + \sqrt{2}}$$

To solve for A, remove the s factor from the denominator of the left side, then substitute $s = 0$ into the left side.

$$\frac{1 - 2s}{(s - \sqrt{2})(s + \sqrt{2})} \to \frac{1 - 2(0)}{(0 - \sqrt{2})(0 + \sqrt{2})} \to -\frac{1}{2}$$

To solve for B, remove the $s-\sqrt{2}$ factor from the denominator of the left side, then substitute $s=\sqrt{2}$ into the left side.

$$\frac{1-2s}{s(s+\sqrt{2})} \to \frac{1-2\sqrt{2}}{\sqrt{2}(\sqrt{2}+\sqrt{2})} \to \frac{1-2\sqrt{2}}{4}$$

To solve for C, remove the $s+\sqrt{2}$ factor from the denominator of the left side, then substitute $s=-\sqrt{2}$ into the left side.

$$\frac{1-2s}{s(s-\sqrt{2})} \to \frac{1-2(-\sqrt{2})}{-\sqrt{2}(-\sqrt{2}-\sqrt{2})} \to \frac{1+2\sqrt{2}}{4}$$

So the decomposition is

$$\frac{1-2s}{s(s^2-2)} = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{1-2\sqrt{2}}{4}\left(\frac{1}{s-\sqrt{2}}\right) + \frac{1+2\sqrt{2}}{4}\left(\frac{1}{s+\sqrt{2}}\right)$$

Then the Laplace transform becomes

$$Y(s) = e^{-7s}\left(-\frac{1}{2}\left(\frac{1}{s}\right) + \frac{1-2\sqrt{2}}{4}\left(\frac{1}{s-\sqrt{2}}\right) + \frac{1+2\sqrt{2}}{4}\left(\frac{1}{s+\sqrt{2}}\right)\right)$$

$$-\frac{2s}{s^2-2} + \frac{1}{s^2-2}$$

$$Y(s) = -\frac{1}{2}e^{-7s}\left(\frac{1}{s}\right) + \frac{1-2\sqrt{2}}{4}e^{-7s}\left(\frac{1}{s-\sqrt{2}}\right) + \frac{1+2\sqrt{2}}{4}e^{-7s}\left(\frac{1}{s+\sqrt{2}}\right)$$

$$-\frac{2s}{s^2-2} + \frac{1}{s^2-2}$$

From our table of Laplace transforms, we know

$$\mathscr{L}(\sinh(at)) = \frac{a}{s^2 - a^2} \qquad \mathscr{L}(\cosh(at)) = \frac{s}{s^2 - a^2}$$

so the general solution to the second order equation is

$$y(t) = -\frac{1}{2}u(t-7) + \frac{1-2\sqrt{2}}{4}e^{\sqrt{2}(t-7)}u(t-7) + \frac{1+2\sqrt{2}}{4}e^{-\sqrt{2}(t-7)}u(t-7)$$

$$-2\cosh(\sqrt{2}t) + \frac{\sqrt{2}}{2}\sinh(\sqrt{2}t)$$

■ 6. Solve the initial value problem, given $y(0) = -1$ and $y'(0) = -2$.

$$y'' + 2y' + y = 7\delta(t-5) + 5u(t-3)$$

Solution:

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'') + 2\mathscr{L}(y') + \mathscr{L}(y) = 7\mathscr{L}(\delta(t-5)) + 5\mathscr{L}(u(t-3))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

$$\mathscr{L}(y(t)) = Y(s)$$

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

$$\mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

we get

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = 7e^{-5s} + \frac{5e^{-3s}}{s}$$

Substitute the initial conditions $y(0) = -1$ and $y'(0) = -2$.

$$s^2Y(s) + s + 2 + 2(sY(s) + 1) + Y(s) = 7e^{-5s} + \frac{5e^{-3s}}{s}$$

$$s^2Y(s) + s + 2 + 2sY(s) + 2 + Y(s) = 7e^{-5s} + \frac{5e^{-3s}}{s}$$

$$s^2Y(s) + s + 4 + 2sY(s) + Y(s) = 7e^{-5s} + \frac{5e^{-3s}}{s}$$

Solve for $Y(s)$.

$$(s^2 + 2s + 1)Y(s) = 7e^{-5s} + \frac{5e^{-3s}}{s} - s - 4$$

$$Y(s) = \frac{7e^{-5s}}{s^2 + 2s + 1} + \frac{5e^{-3s}}{s(s^2 + 2s + 1)} - \frac{s}{s^2 + 2s + 1} - \frac{4}{s^2 + 2s + 1}$$

$$Y(s) = \frac{7e^{-5s}}{(s+1)^2} + \frac{5e^{-3s}}{s(s+1)^2} - \frac{s}{(s+1)^2} - \frac{4}{(s+1)^2}$$

Apply a partial fractions decomposition.

$$\frac{1}{s(s+1)^2}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}$$

To solve for A, remove the s factor from the denominator of the left side, then substitute $s=0$ into the left side.

$$\frac{1}{(s+1)^2}\to\frac{1}{(0+1)^2}\to\frac{1}{1}\to 1$$

To find B and C, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$1=A(s+1)^2+Bs(s+1)+Cs$$

$$1=As^2+2As+A+Bs^2+Bs+Cs$$

$$1=(A+B)s^2+(2A+B+C)s+A$$

Then equating coefficients gives

$$A+B=0$$

$$B=-A=-1$$

and

$$2A+B+C=0$$

$$2-1+C=0$$

$$C=-1$$

So the decomposition is

$$\frac{1}{s(s+1)^2} = \frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Apply a partial fractions decomposition.

$$\frac{s}{(s+1)^2} = \frac{D}{s+1} + \frac{E}{(s+1)^2}$$

To find D and E, we'll multiply through both sides of the decomposition equation by the denominator from the left side.

$$s = D(s+1) + E$$

$$s = Ds + D + E$$

Then equating coefficients gives

$$D = 1$$

and

$$D + E = 0$$

$$E = -D = -1$$

So the decomposition is

$$\frac{s}{(s+1)^2} = \frac{1}{s+1} - \frac{1}{(s+1)^2}$$

Then the Laplace transform becomes

$$Y(s) = 7e^{-5s}\frac{1}{(s+1)^2} + 5e^{-3s}\left(\frac{1}{s} - \frac{1}{s+1} - \frac{1}{(s+1)^2}\right)$$

$$-\left(\frac{1}{s+1}-\frac{1}{(s+1)^2}\right)-\frac{4}{(s+1)^2}$$

$$Y(s)=7e^{-5s}\frac{1}{(s+1)^2}+5e^{-3s}\left(\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right)$$

$$-\frac{1}{s+1}+\frac{1}{(s+1)^2}-\frac{4}{(s+1)^2}$$

$$Y(s)=7e^{-5s}\frac{1}{(s+1)^2}+5e^{-3s}\left(\frac{1}{s}-\frac{1}{s+1}-\frac{1}{(s+1)^2}\right)-\frac{1}{s+1}-\frac{3}{(s+1)^2}$$

$$Y(s)=7e^{-5s}\frac{1}{(s+1)^2}+5e^{-3s}\left(\frac{1}{s}\right)-5e^{-3s}\left(\frac{1}{s+1}\right)-5e^{-3s}\left(\frac{1}{(s+1)^2}\right)$$

$$-\frac{1}{s+1}-3\left(\frac{1}{(s+1)^2}\right)$$

From our table of Laplace transforms, we know

$$\mathscr{L}(t^n e^{at})=\frac{n!}{(s-a)^{n+1}}$$

so the general solution of the second order equation is

$$y(t)=7(t-5)e^{-(t-5)}u(t-5)+5u(t-3)-5u(t-3)e^{-(t-3)}$$

$$-5u(t-3)(t-3)e^{-(t-3)}-e^{-t}-3te^{-t}$$

$$y(t)=5u(t-3)(1-e^{3-t}-te^{3-t}+3e^{3-t})+7(t-5)e^{-(t-5)}u(t-5)-e^{-t}-3te^{-t}$$

$$y(t)=5u(t-3)(1+2e^{3-t}-te^{3-t})+7(t-5)e^{5-t}u(t-5)-e^{-t}-3te^{-t}$$

CONVOLUTION INTEGRALS

1. Find the convolution of $f(t) = e^{2t}$ and $g(t) = 2t$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

Solution:

From the table of Laplace transforms, we know that the transforms of $f(t) = e^{2t}$ and $g(t) = 2t$ are

$$\mathscr{L}(f(t)) = F(s) = \frac{1}{s-2}$$

$$\mathscr{L}(g(t)) = G(s) = \frac{2}{s^2}$$

The product of these transforms is

$$F(s)G(s) = \frac{1}{s-2}\left(\frac{2}{s^2}\right)$$

$$F(s)G(s) = \frac{2}{s^2(s-2)}$$

So, our expectation is that the Laplace transform of the convolution $f(t) * g(t)$ will give us this same result. To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions we need for our convolution integral.

$$f(\tau) = e^{2\tau}$$

$$g(t-\tau)=2(t-\tau)$$

So, the convolution integral gives

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t)*g(t)=\int_0^t 2(t-\tau)e^{2\tau}\ d\tau$$

$$f(t)*g(t)=2t\int_0^t e^{2\tau}\ d\tau-2\int_0^t \tau e^{2\tau}\ d\tau$$

$$f(t)*g(t)=2t\left(\frac{e^{2\tau}}{2}\Bigg|_0^t\right)-2\int_0^t \tau e^{2\tau}\ d\tau$$

$$f(t)*g(t)=2t\left(\frac{e^{2t}}{2}-\frac{1}{2}\right)-2\int_0^t \tau e^{2\tau}\ d\tau$$

$$f(t)*g(t)=te^{2t}-t-2\int_0^t \tau e^{2\tau}\ d\tau$$

Use integration by parts with $u=\tau$, $du=d\tau$, $dv=e^{2\tau}\ d\tau$, and $v=(1/2)e^{2\tau}$.

$$f(t)*g(t)=te^{2t}-t-2\left(\frac{1}{2}\tau e^{2\tau}-\int\frac{1}{2}e^{2\tau}\ d\tau\right)\Bigg|_0^t$$

$$f(t)*g(t)=te^{2t}-t-\left(\tau e^{2\tau}-\frac{1}{2}e^{2\tau}\right)\Bigg|_0^t$$

$$f(t) * g(t) = te^{2t} - t - \left[te^{2t} - \frac{1}{2}e^{2t} - \left((0)e^{2(0)} - \frac{1}{2}e^{2(0)}\right)\right]$$

$$f(t) * g(t) = te^{2t} - t - te^{2t} + \frac{1}{2}e^{2t} - \frac{1}{2}$$

$$f(t) * g(t) = \frac{1}{2}e^{2t} - t - \frac{1}{2}$$

In theory, if we take the Laplace transform of this new function, we should get back the product of the transforms that we found earlier.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

$$F(s)G(s) = \mathscr{L}\left(\frac{1}{2}e^{2t} - t - \frac{1}{2}\right)$$

$$F(s)G(s) = \mathscr{L}\left(\frac{1}{2}e^{2t}\right) - \mathscr{L}(t) - \mathscr{L}\left(\frac{1}{2}\right)$$

$$F(s)G(s) = \frac{1}{2}\mathscr{L}(e^{2t}) - \mathscr{L}(t) - \frac{1}{2}\mathscr{L}(1)$$

$$F(s)G(s) = \frac{1}{2}\left(\frac{1}{s-2}\right) - \frac{1}{s^2} - \frac{1}{2}\left(\frac{1}{s}\right)$$

$$F(s)G(s) = \frac{1}{2(s-2)} - \frac{1}{s^2} - \frac{1}{2s}$$

$$F(s)G(s) = \frac{s^2 - 2(s-2) - s(s-2)}{2s^2(s-2)}$$

$$F(s)G(s) = \frac{s^2 - 2s + 4 - s^2 + 2s}{2s^2(s-2)}$$

$$F(s)G(s) = \frac{4}{2s^2(s-2)}$$

$$F(s)G(s) = \frac{2}{s^2(s-2)}$$

We found a value that matches the one we found earlier, so we've shown that the product of the transforms is equivalent to the Laplace transform of the convolution.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

■ 2. Use a convolution integral to find the inverse transform.

$$H(s) = \frac{s^2}{(s^2+1)^2}$$

Solution:

We could rewrite the transform as

$$H(s) = \left(\frac{s}{s^2+1}\right)\left(\frac{s}{s^2+1}\right)$$

So in this case we have,

$$F(s) = \frac{s}{s^2+1}$$

$$G(s) = \frac{s}{s^2+1}$$

Use the Laplace transform

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

to define

$$f(t) = \cos t$$

$$g(t) = \cos t$$

To find the convolution, we'll first find $f(\tau)$ and $g(t - \tau)$, the two functions that we need for our convolution integral.

$$f(\tau) = \cos \tau$$

$$g(t - \tau) = \cos(t - \tau)$$

So the convolution integral gives

$$f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t \cos \tau \cos(t - \tau)\ d\tau$$

Using the trigonometric identity,

$$\cos a \cos b = \frac{1}{2}[\cos(a - b) + \cos(a + b)]$$

$$\cos \tau \cos(t - \tau) = \frac{1}{2}[\cos(\tau - (t - \tau)) + \cos(\tau + t - \tau)]$$

$$\cos\tau\cos(t-\tau)=\frac{1}{2}[\cos(2\tau-t)+\cos t]$$

we get

$$f(t)*g(t)=\int_0^t\frac{1}{2}[\cos(2\tau-t)+\cos t]\ d\tau$$

$$f(t)*g(t)=\frac{1}{2}\int_0^t\cos(2\tau-t)+\cos t\ d\tau$$

$$f(t)*g(t)=\frac{1}{2}\int_0^t\cos(2\tau-t)\ d\tau+\frac{1}{2}\int_0^t\cos t\ d\tau$$

$$f(t)*g(t)=\frac{1}{2}\left(\frac{1}{2}\sin(2\tau-t)\right)+\frac{1}{2}\tau\cos t\Big|_0^t$$

$$f(t)*g(t)=\frac{1}{4}\sin(2\tau-t)+\frac{1}{2}\tau\cos t\Big|_0^t$$

Evaluate over the interval.

$$f(t)*g(t)=\frac{1}{4}\sin(2t-t)+\frac{1}{2}t\cos t-\left(\frac{1}{4}\sin(2(0)-t)+\frac{1}{2}(0)\cos t\right)$$

$$f(t)*g(t)=\frac{1}{4}\sin t+\frac{1}{2}t\cos t-\frac{1}{4}\sin(-t)$$

$$f(t)*g(t)=\frac{1}{4}\sin t+\frac{1}{2}t\cos t+\frac{1}{4}\sin t$$

$$f(t)*g(t)=\frac{1}{2}\sin t+\frac{1}{2}t\cos t$$

■ 3. Find the convolution of $f(t) = \cos t$ and $g(t) = t^2$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

Solution:

From the table of Laplace transforms, we know that the transforms of $f(t) = \cos t$ and $g(t) = t^2$ are

$$\mathscr{L}(f(t)) = F(s) = \frac{s}{s^2+1}$$

$$\mathscr{L}(g(t)) = G(s) = \frac{2}{s^3}$$

The product of these transforms is

$$F(s)G(s) = \frac{s}{s^2+1}\left(\frac{2}{s^3}\right)$$

$$F(s)G(s) = \frac{2}{s^2(s^2+1)}$$

So our expectation is that the Laplace transform of the convolution $f(t) * g(t)$ will give us this same result. To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions we need for our convolution integral.

$$f(\tau) = \cos\tau$$

$$g(t-\tau) = (t-\tau)^2$$

So, the convolution integral gives

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t (t-\tau)^2 \cos\tau\ d\tau$$

$$f(t) * g(t) = \int_0^t (t^2 - 2t\tau + \tau^2)\cos\tau\ d\tau$$

$$f(t) * g(t) = t^2 \int_0^t \cos\tau\ d\tau - 2t\int_0^t \tau\cos\tau\ d\tau + \int_0^t \tau^2\cos\tau\ d\tau$$

$$f(t) * g(t) = t^2 \sin\tau \Big|_0^t - 2t\int_0^t \tau\cos\tau\ d\tau + \int_0^t \tau^2\cos\tau\ d\tau$$

$$f(t) * g(t) = t^2 \sin t - 2t\int_0^t \tau\cos\tau\ d\tau + \int_0^t \tau^2\cos\tau\ d\tau$$

Use integration by parts with $u = \tau^2$, $du = 2\tau\ d\tau$, $dv = \cos\tau\ d\tau$, and $v = \sin\tau$.

$$\int_0^t \tau^2\cos\tau\ d\tau = \tau^2\sin\tau\Big|_0^t - 2\int_0^t \tau\sin\tau\ d\tau$$

$$\int_0^t \tau^2\cos\tau\ d\tau = t^2\sin t - 2\int_0^t \tau\sin\tau\ d\tau$$

Use integration by parts again with $u = \tau$, $du = d\tau$, $dv = \sin\tau\ d\tau$, and $v = -\cos\tau$.

$$\int_0^t \tau^2 \cos \tau \ d\tau = t^2 \sin t - 2\left(-\tau \cos \tau \Big|_0^t - \int_0^t (-\cos \tau) \ d\tau\right)$$

$$\int_0^t \tau^2 \cos \tau \ d\tau = t^2 \sin t - 2\left(-t \cos t + \int_0^t \cos \tau \ d\tau\right)$$

$$\int_0^t \tau^2 \cos \tau \ d\tau = t^2 \sin t - 2\left(-t \cos t + (\sin \tau)\Big|_0^t\right)$$

$$\int_0^t \tau^2 \cos \tau \ d\tau = t^2 \sin t - 2(-t \cos t + \sin t)$$

$$\int_0^t \tau^2 \cos \tau \ d\tau = t^2 \sin t + 2t \cos t - 2 \sin t$$

Then the convolution is given by

$$f(t) * g(t) = t^2 \sin t - 2t \int_0^t \tau \cos \tau \ d\tau + t^2 \sin t + 2t \cos t - 2 \sin t$$

$$f(t) * g(t) = 2t^2 \sin t + 2t \cos t - 2 \sin t - 2t \int_0^t \tau \cos \tau \ d\tau$$

Use integration by parts with $u = \tau$, $du = d\tau$, $dv = \cos \tau \ d\tau$, and $v = \sin \tau$.

$$\int_0^t \tau \cos \tau \ d\tau = \tau \sin \tau \Big|_0^t - \int_0^t \sin \tau \ d\tau$$

$$\int_0^t \tau \cos \tau \ d\tau = t \sin t - (-\cos \tau)\Big|_0^t$$

$$\int_0^t \tau \cos \tau \, d\tau = t \sin t - (-\cos t + 1)$$

$$\int_0^t \tau \cos \tau \, d\tau = t \sin t + \cos t - 1$$

Then the convolution is given by

$$f(t) * g(t) = 2t^2 \sin t + 2t \cos t - 2 \sin t - 2t(t \sin t + \cos t - 1)$$

$$f(t) * g(t) = 2t^2 \sin t + 2t \cos t - 2 \sin t - 2t^2 \sin t - 2t \cos t + 2t$$

$$f(t) * g(t) = 2t - 2 \sin t$$

In theory, if we take the Laplace transform of this new function, we should get back the product of the transforms that we found earlier.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

$$F(s)G(s) = \mathscr{L}(2t - 2 \sin t)$$

$$F(s)G(s) = \mathscr{L}(2t) - \mathscr{L}(2 \sin t)$$

$$F(s)G(s) = 2\mathscr{L}(t) - 2\mathscr{L}(\sin t)$$

$$F(s)G(s) = 2\left(\frac{1}{s^2}\right) - 2\left(\frac{1}{s^2 + 1}\right)$$

$$F(s)G(s) = \frac{2}{s^2} - \frac{2}{s^2 + 1}$$

$$F(s)G(s) = \frac{2s^2 + 2 - 2s^2}{s^2(s^2 + 1)}$$

$$F(s)G(s) = \frac{2}{s^2(s^2+1)}$$

We found a value that matches the one we found earlier, so we've shown that the product of the transforms is equivalent to the Laplace transform of the convolution.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

■ 4. Find the convolution of $f(t) = e^{2t}$ and $g(t) = e^{-3t}$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

Solution:

From the table of Laplace transforms, we know that the transforms of $f(t) = e^{2t}$ and $g(t) = e^{-3t}$ are

$$\mathscr{L}(f(t)) = F(s) = \frac{1}{s-2}$$

$$\mathscr{L}(g(t)) = G(s) = \frac{1}{s+3}$$

The product of these transforms is

$$F(s)G(s) = \frac{1}{s-2}\left(\frac{1}{s+3}\right)$$

$$F(s)G(s) = \frac{1}{(s-2)(s+3)}$$

So our expectation is that the Laplace transform of the convolution $f(t) * g(t)$ will give us this same result. To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions we need for our convolution integral.

$$f(\tau) = e^{2\tau}$$

$$g(t-\tau) = e^{-3(t-\tau)} = e^{3\tau-3t}$$

So, the convolution integral gives

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{2\tau}e^{3\tau-3t}\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{2\tau+3\tau-3t}\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{5\tau-3t}\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{5\tau}e^{-3t}\ d\tau$$

$$f(t) * g(t) = e^{-3t}\int_0^t e^{5\tau}\ d\tau$$

$$f(t) * g(t) = e^{-3t} \left(\frac{e^{5\tau}}{5} \right) \Bigg|_0^t$$

$$f(t) * g(t) = e^{-3t} \left(\frac{e^{5t}}{5} - \frac{1}{5} \right)$$

$$f(t) * g(t) = \frac{e^{5t}e^{-3t}}{5} - \frac{e^{-3t}}{5}$$

$$f(t) * g(t) = \frac{e^{2t}}{5} - \frac{e^{-3t}}{5}$$

In theory, if we take the Laplace transform of this new function, we should get back the product of the transforms that we found earlier.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

$$F(s)G(s) = \mathscr{L} \left(\frac{e^{2t}}{5} - \frac{e^{-3t}}{5} \right)$$

$$F(s)G(s) = \mathscr{L} \left(\frac{e^{2t}}{5} \right) - \mathscr{L} \left(\frac{e^{-3t}}{5} \right)$$

$$F(s)G(s) = \frac{1}{5}\mathscr{L}(e^{2t}) - \frac{1}{5}\mathscr{L}(e^{-3t})$$

$$F(s)G(s) = \frac{1}{5} \left(\frac{1}{s-2} \right) - \frac{1}{5} \left(\frac{1}{s+3} \right)$$

$$F(s)G(s) = \frac{1}{5} \left(\frac{1}{s-2} - \frac{1}{s+3} \right)$$

$$F(s)G(s) = \frac{1}{5}\left(\frac{s+3-(s-2)}{(s-2)(s+3)}\right)$$

$$F(s)G(s) = \frac{1}{5}\left(\frac{5}{(s-2)(s+3)}\right)$$

$$F(s)G(s) = \frac{1}{(s-2)(s+3)}$$

We found a value that matches the one we found earlier, so we've shown that the product of the transforms is equivalent to the Laplace transform of the convolution.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

■ 5. Use a convolution integral to find the inverse transform of the following transform.

$$H(s) = \frac{1}{(s-5)(s-6)}$$

Solution:

We could rewrite the transform as

$$H(s) = \left(\frac{1}{s-5}\right)\left(\frac{1}{s-6}\right)$$

So, in this case we have,

$$\mathscr{L}(f(t)) = F(s) = \frac{1}{s-5}$$

$$\mathscr{L}(g(t)) = G(s) = \frac{1}{s-6}$$

Use the Laplace transform

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

to define

$$f(t) = e^{5t}$$

$$g(t) = e^{6t}$$

To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions that we need for our convolution integral.

$$f(\tau) = e^{5\tau}$$

$$g(t-\tau) = e^{6(t-\tau)} = e^{6t-6\tau}$$

So the convolution integral gives

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{5\tau}e^{6t-6\tau}\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{5\tau+6t-6\tau}\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{6t-\tau} \ d\tau$$

$$f(t) * g(t) = \int_0^t e^{-\tau} e^{6t} \ d\tau$$

$$f(t) * g(t) = e^{6t} \int_0^t e^{-\tau} \ d\tau$$

$$f(t) * g(t) = e^{6t} (-e^{-\tau}) \Big|_0^t$$

$$f(t) * g(t) = e^{6t}(-e^{-t} + 1)$$

$$f(t) * g(t) = -e^{5t} + e^{6t}$$

In theory, if we take the Laplace transform of this new function, we should get back the product of the transforms that we found earlier.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

$$F(s)G(s) = \mathscr{L}(-e^{5t} + e^{6t})$$

$$F(s)G(s) = \mathscr{L}(e^{6t}) - \mathscr{L}(e^{5t})$$

$$F(s)G(s) = \frac{1}{s-6} - \frac{1}{s-5}$$

$$F(s)G(s) = \frac{s-5-(s-6)}{(s-5)(s-6)}$$

$$F(s)G(s) = \frac{1}{(s-5)(s-6)}$$

We found a value that matches the one we found earlier, so we've shown that the product of the transforms is equivalent to the Laplace transform of the convolution.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

■ 6. Find the convolution of $f(t) = t^3$ and $g(t) = 8$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

Solution:

From the table of Laplace transforms, we know that the transforms of $f(t) = t^3$ and $g(t) = 8$ are

$$\mathscr{L}(f(t)) = F(s) = \frac{6}{s^4}$$

$$\mathscr{L}(g(t)) = G(s) = \frac{8}{s}$$

The product of these transforms is

$$F(s)G(s) = \frac{6}{s^4}\left(\frac{8}{s}\right)$$

$$F(s)G(s) = \frac{48}{s^5}$$

So our expectation is that the Laplace transform of the convolution $f(t) * g(t)$ will give us this same result. To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions we need for our convolution integral.

$$f(\tau) = \tau^3$$

$$g(t-\tau) = 8$$

So, the convolution integral gives

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t 8\tau^3\ d\tau$$

$$f(t) * g(t) = 2\tau^4 \Big|_0^t$$

$$f(t) * g(t) = 2t^4$$

In theory, if we take the Laplace transform of this new function, we should get back the product of the transforms that we found earlier.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

$$F(s)G(s) = \mathscr{L}(2t^4)$$

$$F(s)G(s) = 2\mathscr{L}(t^4)$$

$$F(s)G(s) = 2\left(\frac{4!}{s^5}\right)$$

$$F(s)G(s) = 2\left(\frac{24}{s^5}\right)$$

$$F(s)G(s) = \frac{48}{s^5}$$

We found a value that matches the one we found earlier, so we've shown that the product of the transforms is equivalent to the Laplace transform of the convolution.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

CONVOLUTION INTEGRALS FOR INITIAL VALUE PROBLEMS

■ 1. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=0$ and $y'(0)=0$.

$$y''+4y=g(t)$$

Solution:

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'')=s^2Y(s)-sy(0)-y'(0)$$

$$\mathscr{L}(y)=Y(s)$$

$$\mathscr{L}(g(t))=G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s)-sy(0)-y'(0)+4Y(s)=G(s)$$

Now we'll plug in the initial conditions $y(0)=0$ and $y'(0)=0$ in order to simplify the transform.

$$s^2Y(s)-s(0)-0+4Y(s)=G(s)$$

$$s^2Y(s)+4Y(s)=G(s)$$

$$Y(s)(s^2+4)=G(s)$$

$$Y(s) = G(s)\left(\frac{1}{s^2+4}\right)$$

Rewrite the transform function.

$$Y(s) = G(s)\left(\frac{\frac{2}{2}}{s^2+2^2}\right)$$

$$Y(s) = G(s)\left(\frac{1}{2}\left(\frac{2}{s^2+2^2}\right)\right)$$

Using the Laplace transform formula,

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2}$$

with $a = 2$ we get

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin(2t)$$

and we can say that the inverse transform of $G(s)$ is $g(t)$, so for our convolution integral we'll use the functions

$$f(t) = \frac{1}{2}\sin(2t)$$

$$g(t) = g(t)$$

Plugging these into the convolution integral, we get

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \frac{1}{2}\int_0^t \sin(2\tau)g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$,

$$Y(s) = G(s)\left(\frac{1}{s^2+4}\right)$$

gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = \frac{1}{2}\int_0^t \sin(2\tau)g(t-\tau)\ d\tau$$

■ 2. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = 0$ and $y'(0) = -2$.

$$y'' + 2y' - 3y = g(t)$$

Solution:

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) - 3Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = 0$ and $y'(0) = -2$ in order to simplify the transform.

$$s^2Y(s) + 2 + 2sY(s) - 3Y(s) = G(s)$$

$$s^2Y(s) + 2sY(s) - 3Y(s) = G(s) - 2$$

$$(s^2 + 2s - 3)Y(s) = G(s) - 2$$

$$(s - 1)(s + 3)Y(s) = G(s) - 2$$

$$Y(s) = (G(s) - 2)\left(\frac{1}{(s-1)(s+3)}\right)$$

Applying a partial fraction decomposition we get

$$\frac{1}{(s-1)(s+3)} = \frac{A}{s-1} + \frac{B}{s+3}$$

To solve for A, we'll remove the factor of $s - 1$ from the denominator on the left side, then set $s = 1$.

$$\frac{1}{s+3} \to \frac{1}{1+3} \to \frac{1}{4}$$

To solve for B, we'll remove the factor of $s+3$ from the denominator on the left side, then set $s=-3$.

$$\frac{1}{s-1} \to \frac{1}{-3-1} \to -\frac{1}{4}$$

Plugging $A=1/4$, and $B=-1/4$ into the decomposition gives

$$\frac{1}{(s-1)(s+3)} = \frac{1}{4}\left(\frac{1}{s-1}\right) - \frac{1}{4}\left(\frac{1}{s+3}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s.

$$\frac{1}{(s-1)(s+3)} = \frac{1}{4}e^t - \frac{1}{4}e^{-3t}$$

Plugging all of these values back into the equation for $Y(s)$, we get

$$Y(s) = (G(s)-2)\left(\frac{1}{(s-1)(s+3)}\right)$$

$$Y(s) = (G(s)-2)\left(\frac{1}{4}\left(\frac{1}{s-1}\right) - \frac{1}{4}\left(\frac{1}{s+3}\right)\right)$$

$$Y(s) = \frac{1}{4}G(s)\left(\frac{1}{s-1}\right) - \frac{1}{4}G(s)\left(\frac{1}{s+3}\right) - \frac{1}{2}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s+3}\right)$$

The inverse transform of $G(s)$ is $g(t)$, so for our convolution integral for the first term we'll use the functions

$f_1(t) = e^t$ and $g(t) = g(t)$

Plugging these into the convolution integral, we get

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f_1(t) * g(t) = \int_0^t e^{\tau}g(t-\tau)\ d\tau$$

And for our convolution integral for the second term we'll use the functions

$f_2(t) = e^{-3t}$ and $g(t) = g(t)$

Plugging these into the convolution integral, we get

$$f_2(t) * g(t) = \int_0^t e^{-3\tau}g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$, gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = \frac{1}{4}\int_0^t e^{\tau}g(t-\tau)\ d\tau - \frac{1}{4}\int_0^t e^{-3\tau}g(t-\tau)\ d\tau - \frac{1}{2}e^t + \frac{1}{2}e^{-3t}$$

■ 3. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' - y' - 2y = e^{5t}$$

Solution:

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) - (sY(s) - y(0)) - 2Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = 1$ and $y'(0) = 0$ in order to simplify the transform.

$$s^2Y(s) - s(1) - (sY(s) - 1) - 2Y(s) = G(s)$$

$$s^2Y(s) - s - sY(s) + 1 - 2Y(s) = G(s)$$

$$s^2Y(s) - sY(s) - 2Y(s) = G(s) + s - 1$$

$$(s^2 - s - 2)Y(s) = G(s) + s - 1$$

$$(s - 2)(s + 1)Y(s) = G(s) + s - 1$$

$$Y(s) = G(s)\left(\frac{1}{(s-2)(s+1)}\right) + (s-1)\left(\frac{1}{(s-2)(s+1)}\right)$$

Applying a partial fractions decomposition for the first term, we get

$$\frac{1}{(s-2)(s+1)}=\frac{A}{s-2}+\frac{B}{s+1}$$

To solve for A, we'll remove the factor of $s-2$ from the denominator on the left side, then set $s=2$.

$$\frac{1}{s+1}\to\frac{1}{2+1}\to\frac{1}{3}$$

To solve for B, we'll remove the factor of $s+1$ from the denominator on the left side, then set $s=-1$.

$$\frac{1}{s-2}\to\frac{1}{-1-2}\to-\frac{1}{3}$$

Plugging $A=1/3$, and $B=-1/3$ into the decomposition gives

$$\frac{1}{(s-2)(s+1)}=\frac{1}{3}\left(\frac{1}{s-2}\right)-\frac{1}{3}\left(\frac{1}{s+1}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s.

$$\frac{1}{(s-2)(s+1)}=\frac{1}{3}e^{2t}-\frac{1}{3}e^{-t}$$

Applying a partial fractions decomposition for the second term, we get

$$\frac{s-1}{(s-2)(s+1)}=\frac{C}{s-2}+\frac{D}{s+1}$$

To solve for C, we'll remove the factor of $s-2$ from the denominator on the left side, then set $s=2$.

$$\frac{s-1}{s+1} \to \frac{2-1}{2+1} \to \frac{1}{3}$$

To solve for D, we'll remove the factor of $s+1$ from the denominator on the left side, then set $s = -1$.

$$\frac{s-1}{s-2} \to \frac{-1-1}{-1-2} \to \frac{2}{3}$$

Plugging $C = 1/3$, and $D = 2/3$ into the decomposition gives

$$\frac{s-1}{(s-2)(s+1)} = \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s.

$$\frac{s-1}{(s-2)(s+1)} = \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Plugging all of these values back into the equation for $Y(s)$, we get

$$Y(s) = G(s)\left(\frac{1}{3}\left(\frac{1}{s-2}\right) - \frac{1}{3}\left(\frac{1}{s+1}\right)\right) + \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)$$

$$Y(s) = \frac{1}{3}G(s)\left(\frac{1}{s-2}\right) - \frac{1}{3}G(s)\left(\frac{1}{s+1}\right) + \frac{1}{3}\left(\frac{1}{s-2}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)$$

The inverse transform of $G(s)$ is $g(t)$, so for our convolution integral, we'll use the functions

$f_1(t) = e^{2t}$ and $g(t) = g(t)$

for the first term, and

$$f_2(t) = e^{-t} \text{ and } g(t) = g(t)$$

for the second term. Plugging these into the convolution integral, we get

$$f_1(t) * g(t) = \int_0^t e^{2\tau} g(t-\tau)\ d\tau$$

and

$$f_2(t) * g(t) = \int_0^t e^{-\tau} g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$, we get the general solution of the second order differential equation.

$$y(t) = \frac{1}{3}\int_0^t e^{2\tau} g(t-\tau)\ d\tau - \frac{1}{3}\int_0^t e^{-\tau} g(t-\tau)\ d\tau + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Since $g(t) = e^{5t}$, then $g(t-\tau) = e^{5(t-\tau)} = e^{5t-5\tau}$ and

$$y(t) = \frac{1}{3}\int_0^t e^{2\tau} e^{5t-5\tau}\ d\tau - \frac{1}{3}\int_0^t e^{-\tau} e^{5t-5\tau}\ d\tau + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

$$y(t) = \frac{1}{3}e^{5t}\int_0^t e^{-3\tau}\ d\tau - \frac{1}{3}e^{5t}\int_0^t e^{-6\tau}\ d\tau + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

$$y(t) = -\frac{1}{9}e^{5t}e^{-3\tau}\Big|_0^t + \frac{1}{18}e^{5t}e^{-6\tau}\Big|_0^t + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

Evaluate over the interval.

$$y(t) = -\frac{1}{9}e^{5t}e^{-3t} - \left(-\frac{1}{9}e^{5t}e^{-3(0)}\right) + \frac{1}{18}e^{5t}e^{-6t} - \left(\frac{1}{18}e^{5t}e^{-6(0)}\right) + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

$$y(t) = -\frac{1}{9}e^{5t}e^{-3t} + \frac{1}{9}e^{5t} + \frac{1}{18}e^{5t}e^{-6t} - \frac{1}{18}e^{5t} + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

$$y(t) = -\frac{1}{9}e^{2t} + \frac{1}{9}e^{5t} + \frac{1}{18}e^{-t} - \frac{1}{18}e^{5t} + \frac{1}{3}e^{2t} + \frac{2}{3}e^{-t}$$

$$y(t) = \frac{13}{18}e^{-t} + \frac{2}{9}e^{2t} + \frac{1}{18}e^{5t}$$

■ 4. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = 2$ and $y'(0) = -1$.

$$y'' + 3y' = g(t)$$

Solution:

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) = G(s)$$

Now we'll plug in the initial conditions $y(0) = 2$ and $y'(0) = -1$ in order to simplify the transform.

$$s^2Y(s) - s(2) + 1 + 3(sY(s) - 2) = G(s)$$

$$s^2Y(s) - 2s + 1 + 3sY(s) - 6 = G(s)$$

$$s^2Y(s) - 2s + 3sY(s) - 5 = G(s)$$

$$s^2Y(s) + 3sY(s) = G(s) + 2s + 5$$

$$(s^2 + 3s)Y(s) = G(s) + 2s + 5$$

Solve for the transform $Y(s)$.

$$Y(s) = \frac{2s}{s^2 + 3s} + \frac{5}{s^2 + 3s} + G(s)\left(\frac{1}{s^2 + 3s}\right)$$

$$Y(s) = \frac{2s}{s(s + 3)} + \frac{5}{s(s + 3)} + G(s)\left(\frac{1}{s(s + 3)}\right)$$

$$Y(s) = \frac{2}{s + 3} + \frac{5}{s(s + 3)} + G(s)\left(\frac{1}{s(s + 3)}\right)$$

Applying a partial fractions decomposition we get

$$\frac{1}{s(s + 3)} = \frac{A}{s} + \frac{B}{s + 3}$$

To solve for A, we'll remove the factor of s from the denominator on the left side, then set $s = 0$.

$$\frac{1}{s + 3} \to \frac{1}{0 + 3} \to \frac{1}{3}$$

To solve for B, we'll remove the factor of $s+3$ from the denominator on the left side, then set $s=-3$.

$$\frac{1}{s} \to \frac{1}{-3} \to -\frac{1}{3}$$

Plugging $A=1/3$, and $B=-1/3$ into the decomposition gives

$$\frac{1}{s(s+3)} = \frac{1}{3}\left(\frac{1}{s}\right) - \frac{1}{3}\left(\frac{1}{s+3}\right)$$

Plugging this back into the transform gives

$$Y(s) = \frac{2}{s+3} + 5\left(\frac{1}{3}\left(\frac{1}{s}\right) - \frac{1}{3}\left(\frac{1}{s+3}\right)\right) + G(s)\left(\frac{1}{3}\left(\frac{1}{s}\right) - \frac{1}{3}\left(\frac{1}{s+3}\right)\right)$$

$$Y(s) = \frac{2}{s+3} + \frac{5}{3}\left(\frac{1}{s}\right) - \frac{5}{3}\left(\frac{1}{s+3}\right) + \frac{1}{3}G(s)\left(\frac{1}{s}\right) - \frac{1}{3}G(s)\left(\frac{1}{s+3}\right)$$

The inverse transform of $G(s)$ is $g(t)$, so for the convolution integral for the third term, we'll use

$f_1(t)=1$ and $g(t)=g(t)$

and for the convolution integral for the third term, we'll use

$f_2(t)=e^{-3t}$ and $g(t)=g(t)$

So the convolution integrals are

$$f_1(t) * g(t) = \int_0^t 1 \cdot g(t-\tau)\ d\tau = \int_0^t g(t-\tau)\ d\tau$$

and

$$f_2(t) * g(t) = \int_0^t e^{-3\tau} g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$ gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = 2e^{-3t} + \frac{5}{3} - \frac{5}{3}e^{-3t} + \frac{1}{3}\int_0^t g(t-\tau)\ d\tau - \frac{1}{3}\int_0^t e^{-3\tau} g(t-\tau)\ d\tau$$

$$y(t) = \frac{1}{3}e^{-3t} + \frac{5}{3} + \frac{1}{3}\int_0^t g(t-\tau)\ d\tau - \frac{1}{3}\int_0^t e^{-3\tau} g(t-\tau)\ d\tau$$

$$y(t) = \frac{1}{3}\left(e^{-3t} + 5 + \int_0^t g(t-\tau)\ d\tau - \int_0^t e^{-3\tau} g(t-\tau)\ d\tau\right)$$

■ 5. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = 1$ and $y'(0) = 1$.

$$y'' - 5y' + 4y = g(t)$$

Solution:

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2 Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}(y') = sY(s) - y(0)$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 4Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = 1$ and $y'(0) = 1$ in order to simplify the transform.

$$s^2Y(s) - s(1) - 1 - 5(sY(s) - 1) + 4Y(s) = G(s)$$

$$s^2Y(s) - s - 1 - 5sY(s) + 5 + 4Y(s) = G(s)$$

$$s^2Y(s) - s + 4 - 5sY(s) + 4Y(s) = G(s)$$

$$s^2Y(s) - 5sY(s) + 4Y(s) = G(s) + s - 4$$

$$(s^2 - 5s + 4)Y(s) = G(s) + s - 4$$

$$(s - 4)(s - 1)Y(s) = G(s) + s - 4$$

$$Y(s) = \frac{s-4}{(s-4)(s-1)} + G(s)\left(\frac{1}{(s-4)(s-1)}\right)$$

$$Y(s) = \frac{1}{s-1} + G(s)\left(\frac{1}{(s-4)(s-1)}\right)$$

Applying a partial fractions decomposition we get

$$\frac{1}{(s-4)(s-1)}=\frac{A}{s-4}+\frac{B}{s-1}$$

To solve for A, we'll remove the factor of $s-4$ from the denominator on the left side, then set $s=4$.

$$\frac{1}{s-1}\to\frac{1}{4-1}\to\frac{1}{3}$$

To solve for B, we'll remove the factor of $s-1$ from the denominator on the left side, then set $s=1$.

$$\frac{1}{s-4}\to\frac{1}{1-4}\to-\frac{1}{3}$$

Plugging $A=1/3$ and $B=-1/3$ into the decomposition gives

$$\frac{1}{(s-4)(s-1)}=\frac{1}{3}\left(\frac{1}{s-4}\right)-\frac{1}{3}\left(\frac{1}{s-1}\right)$$

Plugging all of these values back into the equation for $Y(s)$, we get

$$Y(s)=\frac{1}{s-1}+G(s)\left(\frac{1}{3}\left(\frac{1}{s-4}\right)-\frac{1}{3}\left(\frac{1}{s-1}\right)\right)$$

$$Y(s)=\frac{1}{s-1}+\frac{1}{3}G(s)\left(\frac{1}{s-4}\right)-\frac{1}{3}G(s)\left(\frac{1}{s-1}\right)$$

The inverse transform of $G(s)$ is $g(t)$, so for the convolution of the second term we'll use

$f_1(t)=e^{4t}$ and $g(t)=g(t)$

and for the convolution of the third term we'll use

$f_2(t) = e^t$ and $g(t) = g(t)$

Plugging these into the convolution integral, we get

$$f_1(t) * g(t) = \int_0^t e^{4\tau} g(t - \tau)\ d\tau$$

and

$$f_2(t) * g(t) = \int_0^t e^{\tau} g(t - \tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$ gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = e^t + \frac{1}{3}\int_0^t e^{4\tau} g(t - \tau)\ d\tau - \frac{1}{3}\int_0^t e^{\tau} g(t - \tau)\ d\tau$$

■ 6. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = -1$ and $y'(0) = 0$.

$$y'' - 2y' - 8y = g(t)$$

Solution:

From a table of Laplace transforms, we know that

$$\mathcal{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathcal{L}(y') = sY(s) - y(0)$$

$$\mathcal{L}(y) = Y(s)$$

$$\mathcal{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) - 8Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = -1$ and $y'(0) = 0$ in order to simplify the transform.

$$s^2Y(s) - s(-1) - 0 - 2(sY(s) - (-1)) - 8Y(s) = G(s)$$

$$s^2Y(s) + s - 2sY(s) - 2 - 8Y(s) = G(s)$$

$$s^2Y(s) - 2sY(s) - 8Y(s) = G(s) - s + 2$$

$$(s^2 - 2s - 8)Y(s) = G(s) - s + 2$$

$$(s - 4)(s + 2)Y(s) = G(s) - s + 2$$

$$Y(s) = G(s)\left(\frac{1}{(s-4)(s+2)}\right) + \frac{2-s}{(s-4)(s+2)}$$

Applying a partial fractions decomposition for the first term, we get

$$\frac{1}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

To solve for A, we'll remove the factor of $s-4$ from the denominator on the left side, then set $s=4$.

$$\frac{1}{s+2} \to \frac{1}{4+2} \to \frac{1}{6}$$

To solve for B, we'll remove the factor of $s+2$ from the denominator on the left side, then set $s=-2$.

$$\frac{1}{s-4} \to \frac{1}{-2-4} \to -\frac{1}{6}$$

Plugging $A=1/6$ and $B=-1/6$ into the decomposition gives

$$\frac{1}{(s-4)(s+2)} = \frac{1}{6}\left(\frac{1}{s-4}\right) - \frac{1}{6}\left(\frac{1}{s+2}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s.

$$\frac{1}{(s-4)(s+2)} = \frac{1}{6}e^{4t} - \frac{1}{6}e^{-2t}$$

Applying a partial fractions decomposition for the second term, we get

$$\frac{2-s}{(s-4)(s+2)} = \frac{C}{s-4} + \frac{D}{s+2}$$

To solve for C, we'll remove the factor of $s-4$ from the denominator on the left side, then set $s=4$.

$$\frac{2-s}{s+2} \to \frac{2-4}{4+2} \to -\frac{2}{6} \to -\frac{1}{3}$$

To solve for D, we'll remove the factor of $s+2$ from the denominator on the left side, then set $s=-2$.

$$\frac{2-s}{s-4} \to \frac{2-(-2)}{-2-4} \to -\frac{4}{6} \to -\frac{2}{3}$$

Plugging $C=-1/3$ and $D=-2/3$ into the decomposition gives

$$\frac{2-s}{(s-4)(s+2)} = -\frac{1}{3}\left(\frac{1}{s-4}\right) - \frac{2}{3}\left(\frac{1}{s+2}\right)$$

Plugging all of these values back into the equation for $Y(s)$, we get

$$Y(s) = G(s)\left(\frac{1}{6}\left(\frac{1}{s-4}\right) - \frac{1}{6}\left(\frac{1}{s+2}\right)\right) - \frac{1}{3}\left(\frac{1}{s-4}\right) - \frac{2}{3}\left(\frac{1}{s+2}\right)$$

$$Y(s) = \frac{1}{6}G(s)\left(\frac{1}{s-4}\right) - \frac{1}{6}G(s)\left(\frac{1}{s+2}\right) - \frac{1}{3}\left(\frac{1}{s-4}\right) - \frac{2}{3}\left(\frac{1}{s+2}\right)$$

The inverse transform of $G(s)$ is $g(t)$, so for the convolution integral of the first term we'll use

$f_1(t) = e^{4t}$ and $g(t) = g(t)$

and for the convolution integral of the second term we'll use

$f_2(t) = e^{-2t}$ and $g(t) = g(t)$

Plugging these into the convolution integral, we get

$$f_1(t) * g(t) = \int_0^t e^{4\tau} g(t-\tau) \ d\tau$$

and

$$f_2(t) * g(t) = \int_0^t e^{-2\tau} g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$ gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = \frac{1}{6}\int_0^t e^{4\tau} g(t-\tau)\ d\tau - \frac{1}{6}\int_0^t e^{-2\tau} g(t-\tau)\ d\tau - \frac{1}{3}e^{4t} - \frac{2}{3}e^{-2t}$$

$$y(t) = \frac{1}{6}\left(\int_0^t e^{4\tau} g(t-\tau)\ d\tau - \int_0^t e^{-2\tau} g(t-\tau)\ d\tau - 2e^{4t} - 4e^{-2t}\right)$$

MATRIX BASICS

■ 1. Find $3A-(1/2)B+2C$.

$$A=\begin{bmatrix} 0 & 1 & -\frac{1}{3} \\ 2 & \frac{1}{3} & 4 \\ \frac{2}{3} & -1 & 2 \end{bmatrix} \qquad B=\begin{bmatrix} 2 & 0 & -4 \\ -2 & 6 & 4 \\ 10 & 0 & 0 \end{bmatrix} \qquad C=\begin{bmatrix} 1 & 0 & 3 \\ -2 & 10 & 1 \\ -4 & 3 & 0 \end{bmatrix}$$

Solution:

Because all three matrices have identical dimensions (they are all 3×3 matrices), we can find the sum.

$$3A-\frac{1}{2}B+2C=3\begin{bmatrix} 0 & 1 & -\frac{1}{3} \\ 2 & \frac{1}{3} & 4 \\ \frac{2}{3} & -1 & 2 \end{bmatrix}-\frac{1}{2}\begin{bmatrix} 2 & 0 & -4 \\ -2 & 6 & 4 \\ 10 & 0 & 0 \end{bmatrix}+2\begin{bmatrix} 1 & 0 & 3 \\ -2 & 10 & 1 \\ -4 & 3 & 0 \end{bmatrix}$$

$$3A-\frac{1}{2}B+2C=\begin{bmatrix} 0 & 3 & -1 \\ 6 & 1 & 12 \\ 2 & -3 & 6 \end{bmatrix}-\begin{bmatrix} 1 & 0 & -2 \\ -1 & 3 & 2 \\ 5 & 0 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 0 & 6 \\ -4 & 20 & 2 \\ -8 & 6 & 0 \end{bmatrix}$$

$$3A-\frac{1}{2}B+2C=\begin{bmatrix} 0-1+2 & 3-0+0 & -1-(-2)+6 \\ 6-(-1)+(-4) & 1-3+20 & 12-2+2 \\ 2-5+(-8) & -3-0+6 & 6-0+0 \end{bmatrix}$$

$$3A - \frac{1}{2}B + 2C = \begin{bmatrix} 1 & 3 & 7 \\ 3 & 18 & 12 \\ -11 & 3 & 6 \end{bmatrix}$$

■ 2. For the 2×2 matrix, compute the determinant.

$$\begin{bmatrix} \frac{3}{2} & 1 \\ 2 & -6 \end{bmatrix}$$

Solution:

The determinant of the 2×2 matrix is given by

$$\begin{vmatrix} \frac{3}{2} & 1 \\ 2 & -6 \end{vmatrix} = \frac{3}{2}(-6) - 1(2)$$

$$\begin{vmatrix} \frac{3}{2} & 1 \\ 2 & -6 \end{vmatrix} = -11$$

■ 3. For the 3×3 matrix, compute the determinant.

$$\begin{bmatrix} -1 & 5 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 4 & 0 & 2 \end{bmatrix}$$

Solution:

The determinant of the 3×3 matrix is given by

$$\begin{vmatrix} -1 & 5 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 4 & 0 & 2 \end{vmatrix} = -\begin{vmatrix} -\frac{3}{2} & 1 \\ 0 & 2 \end{vmatrix} - 5\begin{vmatrix} \frac{1}{2} & 1 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} \frac{1}{2} & -\frac{3}{2} \\ 4 & 0 \end{vmatrix}$$

$$\begin{vmatrix} -1 & 5 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 4 & 0 & 2 \end{vmatrix} = -\left(-\frac{3}{2}(2) - 1(0)\right) - 5\left(\frac{1}{2}(2) - 1(4)\right)$$

$$\begin{vmatrix} -1 & 5 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 4 & 0 & 2 \end{vmatrix} = -(-3) - 5(1-4)$$

$$\begin{vmatrix} -1 & 5 & 0 \\ \frac{1}{2} & -\frac{3}{2} & 1 \\ 4 & 0 & 2 \end{vmatrix} = 18$$

■ 4. Find the Eigenvalues and Eigenvectors of the matrix.

$$A = \begin{bmatrix} 1 & 3 \\ -1 & 5 \end{bmatrix}$$

Solution:

Since the dimension of the matrix is 2×2, we'll use I_2 to find $A-\lambda I$.

$$A-\lambda I=\begin{bmatrix}1 & 3\\ -1 & 5\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}1 & 3\\ -1 & 5\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}1-\lambda & 3\\ -1 & 5-\lambda\end{bmatrix}$$

Take the determinant,

$$|A-\lambda I|=\begin{vmatrix}1-\lambda & 3\\ -1 & 5-\lambda\end{vmatrix}$$

$$|A-\lambda I|=(1-\lambda)(5-\lambda)-3(-1)$$

$$|A-\lambda I|=5-6\lambda+\lambda^2+3$$

$$|A-\lambda I|=\lambda^2-6\lambda+8$$

then the characteristic equation is

$$\lambda^2-6\lambda+8=0$$

$$(\lambda-2)(\lambda-4)=0$$

$$\lambda=2,\ 4$$

Therefore, the Eigenvalues are $\lambda_1=2$ and $\lambda_2=4$.

For $\lambda_1 = 2$, we find

$$A - 2I = \begin{bmatrix} 1-2 & 3 \\ -1 & 5-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 3 \\ -1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 \\ 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - 3k_2 = 0$$

$$k_1 = 3k_2$$

If we choose $k_2 = 1$, we get $k_1 = 3$, and so the Eigenvector for $\lambda_1 = 2$ is

$$\vec{k_1} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

For $\lambda_2 = 4$, we find

$$A - 4I = \begin{bmatrix} 1-4 & 3 \\ -1 & 5-4 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -3 & 3 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - k_2 = 0$$

$$k_1 = k_2$$

If we choose $k_2 = 1$, we get $k_1 = 1$, and so the Eigenvector for $\lambda_2 = 4$ is

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

■ 5. Find the Eigenvalues and Eigenvectors of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -4 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$

Solution:

Since the dimension of the matrix is 3×3, we'll use I_3 to find $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -4 & 0 \\ -2 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -4 & 0 \\ -2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ -4 & -4-\lambda & 0 \\ -2 & 1 & 2-\lambda \end{bmatrix}$$

Take the determinant,

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ -4 & -4-\lambda & 0 \\ -2 & 1 & 2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (-\lambda)\begin{vmatrix} -4-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} -4 & 0 \\ -2 & 2-\lambda \end{vmatrix} + 0\begin{vmatrix} -4 & -4-\lambda \\ -2 & 1 \end{vmatrix}$$

$$|A - \lambda I| = -\lambda((-4-\lambda)(2-\lambda) - 0(1)) - 1((-4)(2-\lambda) - 0(-2))$$

$$|A - \lambda I| = \lambda(4+\lambda)(2-\lambda) + 4(2-\lambda)$$

$$|A - \lambda I| = (2-\lambda)(\lambda^2 + 4\lambda + 4)$$

then the characteristic equation is

$$(2-\lambda)(\lambda+2)(\lambda+2) = 0$$

$$\lambda = 2, \ -2, \ -2$$

Therefore, the Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = \lambda_3 = -2$.

For $\lambda_1 = 2$, we find

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ -4 & -4-2 & 0 \\ -2 & 1 & 2-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ -4 & -6 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & 0 \\ -4 & -6 & 0 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ -4 & -6 & 0 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -8 & 0 \\ 2 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -8 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 = 0$$

$$k_2 = 0$$

If we choose $k_3 = 1$, the Eigenvector for $\lambda_1 = 2$ is

$$\vec{k_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

For $\lambda_2 = \lambda_3 = -2$, we find

$$A - (-2)I = \begin{bmatrix} -(-2) & 1 & 0 \\ -4 & -4-(-2) & 0 \\ -2 & 1 & 2-(-2) \end{bmatrix}$$

$$A-(-2)I=\begin{bmatrix}2 & 1 & 0\\ -4 & -2 & 0\\ -2 & 1 & 4\end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix}2 & 1 & 0\\ -4 & -2 & 0\\ -2 & 1 & 4\end{bmatrix}\to\begin{bmatrix}1 & \frac{1}{2} & 0\\ -4 & -2 & 0\\ -2 & 1 & 4\end{bmatrix}\to\begin{bmatrix}1 & \frac{1}{2} & 0\\ 0 & 0 & 0\\ -2 & 1 & 4\end{bmatrix}$$

$$\to\begin{bmatrix}1 & \frac{1}{2} & 0\\ 0 & 0 & 0\\ 0 & 2 & 4\end{bmatrix}\to\begin{bmatrix}1 & \frac{1}{2} & 0\\ 0 & 2 & 4\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{1}{2} & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 2\\ 0 & 0 & 0\end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1=0$$

and

$$k_2+2k_3=0$$

$$k_2=-2k_3$$

If we choose $k_3=-1$, we get $k_2=2$, and so the Eigenvector for $\lambda_2=\lambda_3=-2$ is

$$\overrightarrow{k_2}=\begin{bmatrix}0\\ 2\\ -1\end{bmatrix}$$

■ 6. Find the Eigenvalues and Eigenvectors of the matrix.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & -9 \\ 4 & 1 & 0 \end{bmatrix}$$

Solution:

Since the dimension of the matrix is 3×3, we'll use I_3 to find $A - \lambda I$.

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & -9 \\ 4 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & -9 \\ 4 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ -3 & -\lambda & -9 \\ 4 & 1 & -\lambda \end{bmatrix}$$

Take the determinant,

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ -3 & -\lambda & -9 \\ 4 & 1 & -\lambda \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)\begin{vmatrix} -\lambda & -9 \\ 1 & -\lambda \end{vmatrix} - 0\begin{vmatrix} -3 & -9 \\ 4 & -\lambda \end{vmatrix} + 0\begin{vmatrix} -3 & -\lambda \\ 4 & 1 \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)((-\lambda)(-\lambda) - (-9)(1))$$

$$|A-\lambda I| = (2-\lambda)(\lambda^2+9)$$

then the characteristic equation is

$$(2-\lambda)(\lambda^2+9) = 0$$

$$\lambda = 2, \ \pm 3i$$

Therefore, the Eigenvalues are $\lambda_1 = 2$ and $\lambda_{2,3} = \pm 3i$.

For $\lambda_1 = 2$, we find

$$A - 2I = \begin{bmatrix} 2-2 & 0 & 0 \\ -3 & -2 & -9 \\ 4 & 1 & -2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ -3 & -2 & -9 \\ 4 & 1 & -2 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & 0 \\ -3 & -2 & -9 \\ 4 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} -3 & -2 & -9 \\ 4 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & 3 \\ 4 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & \frac{2}{3} & 3 \\ 0 & -\frac{5}{3} & -14 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{3} & 3 \\ 0 & 1 & \frac{42}{5} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{13}{5} \\ 0 & 1 & \frac{42}{5} \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - \frac{13}{5}k_3 = 0$$

$$k_1 = \frac{13}{5}k_3$$

and

$$k_2 + \frac{42}{5}k_3 = 0$$

$$k_2 = -\frac{42}{5}k_3$$

If we choose $k_3 = 5$, we get $k_1 = 13$ and $k_2 = -42$, and so the Eigenvector for $\lambda_1 = 2$ is

$$\vec{k_1} = \begin{bmatrix} 13 \\ -42 \\ 5 \end{bmatrix}$$

For $\lambda_2 = 3i$, we find

$$A - (3i)I = \begin{bmatrix} 2-3i & 0 & 0 \\ -3 & -3i & -9 \\ 4 & 1 & -3i \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 2-3i & 0 & 0 \\ -3 & -3i & -9 \\ 4 & 1 & -3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -3 & -3i & -9 \\ 4 & 1 & -3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3i & -9 \\ 4 & 1 & -3i \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3i & -9 \\ 0 & 1 & -3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3i \\ 0 & 1 & -3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3i \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 = 0$$

and

$$k_2 - (3i)k_3 = 0$$

$$k_2 = 3ik_3$$

If we choose $k_3 = 1$, we get $k_2 = 3i$, and so the Eigenvector for $\lambda_2 = 3i$ is

$$\overrightarrow{k_2} = \begin{bmatrix} 0 \\ 3i \\ 1 \end{bmatrix}$$

For $\lambda_3 = -3i$, we find

$$A - (-3i)I = \begin{bmatrix} 2+3i & 0 & 0 \\ -3 & 3i & -9 \\ 4 & 1 & 3i \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 2+3i & 0 & 0 \\ -3 & 3i & -9 \\ 4 & 1 & 3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -3 & 3i & -9 \\ 4 & 1 & 3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3i & -9 \\ 4 & 1 & 3i \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3i & -9 \\ 0 & 1 & 3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3i \\ 0 & 1 & 3i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 3i \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 = 0$$

and

$$k_2 + (3i)k_3 = 0$$

$$k_2 = -3ik_3$$

If we choose $k_3 = 1$, we get $k_2 = -3i$, and so the Eigenvector for $\lambda_3 = -3i$ is

$$\overrightarrow{k_2} = \begin{bmatrix} 0 \\ -3i \\ 1 \end{bmatrix}$$

BUILDING SYSTEMS

■ 1. Rewrite the system of differential equations in matrix form.

$$x_1' = 3x_1 - 4x_3 + e^t$$

$$x_2' = 5x_2 + x_1 - 13x_3$$

$$x_3' = \sin(t^2) - 3x_1 - 2x_2$$

Solution:

Rewrite the system.

$$x_1' = 3x_1 + 0x_2 - 4x_3 + e^t$$

$$x_2' = 1x_1 + 5x_2 - 13x_3$$

$$x_3' = -3x_1 - 2x_2 + 0x_3 + \sin(t^2)$$

Since we have a nonhomogeneous system, we'll get a matrix equation in the form

$$\vec{x}' = A\vec{x} + F$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 5 & -13 \\ -3 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \\ \sin(t^2) \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 3 & 0 & -4 \\ 1 & 5 & -13 \\ -3 & -2 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} e^t \\ 0 \\ \sin(t^2) \end{bmatrix}$$

■ 2. Rewrite the system of differential equations with the initial conditions in matrix form.

$$x_1' = -x_1 + 2x_3 - t^2 \qquad x_1(0) = 2$$

$$x_2' = 9x_2 - x_3 \qquad x_2(0) = -3$$

$$x_3' = 2x_2 - 4x_3 + 5\cos t \qquad x_3(0) = 0$$

Solution:

Rewrite the system.

$$x_1' = -1x_1 + 0x_2 + 2x_3 - t^2$$

$$x_2' = 0x_1 + 9x_2 - 1x_3$$

$$x_3' = 0x_1 + 2x_2 - 4x_3 + 5\cos t$$

Since we have a nonhomogeneous system, we'll get a matrix equation in the form

$$\vec{x}' = A\vec{x} + F$$

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 9 & -1 \\ 0 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -t^2 \\ 0 \\ 5\cos t \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 9 & -1 \\ 0 & 2 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} -t^2 \\ 0 \\ 5\cos t \end{bmatrix}$$

The given set of initial conditions can be rewritten together as

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

and so the initial value problem can also be written as

$$\vec{x}' = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 9 & -1 \\ 0 & 2 & -4 \end{bmatrix} \vec{x} + \begin{bmatrix} -t^2 \\ 0 \\ 5\cos t \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$$

■ 3. Convert the fourth order linear differential equation into a system of differential equations in matrix form.

$$3y^{(4)} + 6y''' - 18y'' - 20t\cos(t^2) = e^t$$

Solution:

Solve the differential equation for the highest order derivative, $y^{(4)}$.

$$3y^{(4)} + 6y''' - 18y'' - 20t\cos(t^2) = e^t$$

$$3y^{(4)} = e^t + 20t\cos(t^2) - 6y''' + 18y''$$

$$y^{(4)} = \frac{1}{3}e^t + \frac{20}{3}t\cos(t^2) - 2y''' + 6y''$$

Then define

$$x_1(t) = y(t) \qquad x_1'(t) = y'(t) = x_2(t)$$

$$x_2(t) = y'(t) \qquad x_2'(t) = y''(t) = x_3(t)$$

$$x_3(t) = y''(t) \qquad x_3'(t) = y'''(t) = x_4(t)$$

$$x_4(t) = y'''(t) \qquad x_4'(t) = y^{(4)}(t) = \frac{1}{3}e^t + \frac{20}{3}t\cos(t^2) - 2y''' + 6y''$$

$$= 6x_3 - 2x_4 + \frac{20}{3}t\cos(t^2) + \frac{1}{3}e^t$$

Rewriting the nonhomogeneous system as a matrix equation gives

$$\vec{x}' = A\vec{x} + F$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 6 & -2 \end{bmatrix}\vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \frac{20}{3}t\cos(t^2) + \frac{1}{3}e^t \end{bmatrix}$$

■ 4. Convert the given initial value problem into a system of differential equations and rewrite it as a matrix equation.

$$y''' - 4ty' + 7y - t^2y'' = 2t^3 - 4$$

$$y(0) = -2, \ y'(0) = 1, \ y''(0) = 0$$

Solution:

Solve the differential equation for the highest order derivative, y'''.

$$y''' - t^2y'' - 4ty' + 7y = 2t^3 - 4$$

$$y''' = -7y + 4ty' + t^2y'' + 2t^3 - 4$$

Then define

$$x_1(t) = y(t) \qquad x_1'(t) = y'(t) = x_2(t)$$

$$x_2(t) = y'(t) \qquad x_2'(t) = y''(t) = x_3(t)$$

$$x_3(t) = y''(t) \qquad x_3'(t) = y'''(t) = -7y + 4ty' + t^2y'' + 2t^3 - 4$$

$$= -7x_1 + 4tx_2 + t^2x_3 + 2t^3 - 4$$

Rewriting the nonhomogeneous system as a matrix equation gives

$$\overrightarrow{x}' = A\overrightarrow{x} + F$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 4t & t^2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 2t^3 - 4 \end{bmatrix}$$

Rewrite the given set of initial conditions,

$$x_1(0) = y(0) = -2$$

$$x_2(0) = y'(0) = 1$$

$$x_3(0) = y''(0) = 0$$

and then the initial value problem can be written as

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -7 & 4t & t^2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 2t^3 - 4 \end{bmatrix} \qquad \vec{x}(0) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

■ 5. Convert the third order linear differential equation into a system of differential equations in matrix form.

$$e^t y''' - 5y'' + y' - t^2 y + e^{2t} = 0$$

Solution:

Solve the differential equation for the highest order derivative, y'''.

$$e^t y''' - 5y'' + y' - t^2 y + e^{2t} = 0$$

$$e^t y''' = t^2 y - y' + 5y'' - e^{2t}$$

$$y''' = t^2e^{-t}y - e^{-t}y' + 5e^{-t}y'' - e^t$$

Then define

$$x_1(t) = y(t) \qquad x_1'(t) = y'(t) = x_2(t)$$

$$x_2(t) = y'(t) \qquad x_2'(t) = y''(t) = x_3(t)$$

$$x_3(t) = y''(t) \qquad x_3'(t) = y'''(t) = t^2e^{-t}y - e^{-t}y' + 5e^{-t}y'' - e^t$$

$$= t^2e^{-t}x_1 - e^{-t}x_2 + 5e^{-t}x_3 - e^t$$

Rewriting the nonhomogeneous system as a matrix equation gives

$$\vec{x}' = A\vec{x} + F$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t^2e^{-t} & -e^{-t} & 5e^{-t} \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ -e^t \end{bmatrix}$$

■ 6. Convert the fourth order linear differential equation with the given initial conditions into a system of differential equations in matrix form.

$$y^{(4)} - (\cos t)y''' + 3e^ty'' - 2y = 0$$

$$y(0) = 3, \ y'(0) = 1, \ y''(0) = -2, \ y'''(0) = 0$$

Solution:

Solve the differential equation for the highest order derivative, $y^{(4)}$.

$$y^{(4)} - (\cos t)y''' + 3e^t y'' - 2y = 0$$

$$y^{(4)} = 2y - 3e^t y'' + (\cos t)y'''$$

Then define

$$x_1(t) = y(t) \qquad x_1'(t) = y'(t) = x_2(t)$$

$$x_2(t) = y'(t) \qquad x_2'(t) = y''(t) = x_3(t)$$

$$x_3(t) = y''(t) \qquad x_3'(t) = y'''(t) = x_4(t)$$

$$x_4(t) = y'''(t) \qquad x_4'(t) = y^{(4)}(t) = 2y - 3e^t y'' + (\cos t)y'''$$

$$= 2x_1 - 3e^t x_3 + (\cos t)x_4$$

Rewriting the nonhomogeneous system as a matrix equation gives

$$\vec{x}' = A\vec{x} + F$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -3e^t & \cos t \end{bmatrix} \vec{x}$$

SOLVING SYSTEMS

■ 1. Show that the vector is a solution to the system.

$$x_1' = x_1 + 3x_2 - e^{2t}$$

$$x_2' = 3x_1 + x_2$$

$$\vec{x_1} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{1}{8} \\ \frac{3}{8} \end{bmatrix} e^{2t}$$

Solution:

Rewrite the system in matrix form.

$$\vec{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

Rewrite the solution vector as

$$\vec{x_1} = \begin{bmatrix} -3e^{-2t} \\ 3e^{-2t} \end{bmatrix} + \begin{bmatrix} \frac{1}{8}e^{2t} \\ \frac{3}{8}e^{2t} \end{bmatrix}$$

$$\vec{x_1} = \begin{bmatrix} -3e^{-2t} + \frac{1}{8}e^{2t} \\ 3e^{-2t} + \frac{3}{8}e^{2t} \end{bmatrix}$$

The derivative of the solution vector is

$$\vec{x_1}' = \begin{bmatrix} 6e^{-2t} + \frac{1}{4}e^{2t} \\ -6e^{-2t} + \frac{3}{4}e^{2t} \end{bmatrix}$$

Plug the solution and its derivative into the matrix equation.

$$\vec{x} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6e^{-2t} + \frac{1}{4}e^{2t} \\ -6e^{-2t} + \frac{3}{4}e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} -3e^{-2t} + \frac{1}{8}e^{2t} \\ 3e^{-2t} + \frac{3}{8}e^{2t} \end{bmatrix} + \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6e^{-2t} + \frac{1}{4}e^{2t} \\ -6e^{-2t} + \frac{3}{4}e^{2t} \end{bmatrix} = \begin{bmatrix} 1\left(-3e^{-2t} + \frac{1}{8}e^{2t}\right) + 3\left(3e^{-2t} + \frac{3}{8}e^{2t}\right) \\ 3\left(-3e^{-2t} + \frac{1}{8}e^{2t}\right) + 1\left(3e^{-2t} + \frac{3}{8}e^{2t}\right) \end{bmatrix} + \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6e^{-2t} + \frac{1}{4}e^{2t} \\ -6e^{-2t} + \frac{3}{4}e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{-2t} + \frac{5}{4}e^{2t} \\ -6e^{-2t} + \frac{3}{4}e^{2t} \end{bmatrix} + \begin{bmatrix} -e^{2t} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6e^{-2t} + \frac{1}{4}e^{2t} \\ -6e^{-2t} + \frac{3}{4}e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{-2t} + \frac{1}{4}e^{2t} \\ -6e^{-2t} + \frac{3}{4}e^{2t} \end{bmatrix}$$

Because we found equivalent vectors on both sides of the equation, we've shown that the vector $\vec{x_1}$ is a solution to the system of differential equations.

■ 2. Show that the vector is a solution to the system.

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \vec{x}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} e^{2t}$$

Solution:

Rewrite the solution vector as

$$\vec{x_1} = \begin{bmatrix} e^{2t} \\ -e^t - 2e^{2t} \\ e^t + 2e^{2t} \end{bmatrix}$$

The derivative of the solution vector is

$$\vec{x_1}' = \begin{bmatrix} 2e^{2t} \\ -e^t - 4e^{2t} \\ e^t + 4e^{2t} \end{bmatrix}$$

Plug the solution and its derivative into the matrix equation.

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \vec{x}$$

$$\begin{bmatrix} 2e^{2t} \\ -e^t - 4e^{2t} \\ e^t + 4e^{2t} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \begin{bmatrix} e^{2t} \\ -e^t - 2e^{2t} \\ e^t + 2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{2t} \\ -e^t - 4e^{2t} \\ e^t + 4e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} - e^t - 2e^{2t} + e^t + 2e^{2t} \\ -2e^{2t} - e^t - 2e^{2t} \\ 2e^{2t} - e^t - 2e^{2t} + 2e^t + 4e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{2t} \\ -e^t - 4e^{2t} \\ e^t + 4e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} \\ -e^t - 4e^{2t} \\ e^t + 4e^{2t} \end{bmatrix}$$

Because we found equivalent vectors on both sides of the equation, we've shown that the vector $\vec{x_1}$ is a solution to the system of differential equations.

■ 3. Confirm that the vectors are linearly independent.

$$\vec{x_1} = \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix} e^{3t} \qquad \vec{x_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{6t} \qquad \vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{7t}$$

Solution:

Find the Wronskian of the vector set.

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = \begin{vmatrix} 3e^{3t} & 0 & e^{7t} \\ -3e^{3t} & 0 & e^{7t} \\ 7e^{3t} & e^{6t} & e^{7t} \end{vmatrix}$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 3e^{3t} \begin{vmatrix} 0 & e^{7t} \\ e^{6t} & e^{7t} \end{vmatrix} - 0 \begin{vmatrix} -3e^{3t} & e^{7t} \\ 7e^{3t} & e^{7t} \end{vmatrix} + e^{7t} \begin{vmatrix} -3e^{3t} & 0 \\ 7e^{3t} & e^{6t} \end{vmatrix}$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 3e^{3t}(0 - e^{13t}) + e^{7t}(-3e^{9t} - 0)$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = -3e^{16t} - 3e^{16t}$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = -6e^{16t}$$

Because $e^{16t} \neq 0$ for any t, the Wronskian is non-zero, which means that the vector set is linearly independent.

■ 4. Verify that the solution vectors satisfy the system of differential equations. Calculate the Wronskian of the solution vectors and write the general solution of the homogeneous system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$$

$$\{\vec{x_1}, \vec{x_2}, \vec{x_3}\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{t}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{-t}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \right\}$$

Solution:

We can verify that each of the vectors $\{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$ are solutions by substituting them into the matrix equation. First though, we'll need the derivative of each solution vector.

$$\vec{x_1} = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} \qquad \vec{x_1}' = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

$$\vec{x_2} = \begin{bmatrix} -e^{-t} \\ -2e^{-t} \\ e^{-t} \end{bmatrix} \qquad \vec{x_2}' = \begin{bmatrix} e^{-t} \\ 2e^{-t} \\ -e^{-t} \end{bmatrix}$$

$$\vec{x_3} = \begin{bmatrix} -e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix} \qquad \vec{x_3}' = \begin{bmatrix} -2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix}$$

Substitute $\vec{x_1}$ and its derivative $\vec{x_1}'$ into the matrix equation.

$$\begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

$$\begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} = \begin{bmatrix} e^t + 0 + 0 \\ -e^t + 0 + e^t \\ 0 + 0 + e^t \end{bmatrix}$$

$$\begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

Substitute $\vec{x_2}$ and its derivative $\vec{x_2}'$ into the matrix equation.

$$\begin{bmatrix} e^{-t} \\ 2e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -e^{-t} \\ -2e^{-t} \\ e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} \\ 2e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{-t} + 2e^{-t} + 0 \\ e^{-t} + 0 + e^{-t} \\ 0 - 2e^{-t} + e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} \\ 2e^{-t} \\ -e^{-t} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ 2e^{-t} \\ -e^{-t} \end{bmatrix}$$

Substitute $\vec{x_3}$ and its derivative $\vec{x_3}'$ into the matrix equation.

$$\begin{bmatrix} -2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -e^{2t} \\ e^{2t} \\ e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} -2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} -e^{2t} - e^{2t} + 0 \\ e^{2t} + 0 + e^{2t} \\ 0 + e^{2t} + e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} -2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} -2e^{2t} \\ 2e^{2t} \\ 2e^{2t} \end{bmatrix}$$

Because we found equivalent vectors on both sides of the equation in each of these three cases, we've shown that all three vectors are solutions to the system of differential equations.

Now let's take the Wronskian of the solution set.

$$W(\vec{x_1},\vec{x_2},\vec{x_3})=\begin{vmatrix} e^t & -e^{-t} & -e^{2t} \\ 0 & -2e^{-t} & e^{2t} \\ e^t & e^{-t} & e^{2t} \end{vmatrix}$$

$$W(\vec{x_1},\vec{x_2},\vec{x_3})=e^t\begin{vmatrix} -2e^{-t} & e^{2t} \\ e^{-t} & e^{2t} \end{vmatrix}-(-e^{-t})\begin{vmatrix} 0 & e^{2t} \\ e^t & e^{2t} \end{vmatrix}+(-e^{2t})\begin{vmatrix} 0 & -2e^{-t} \\ e^t & e^{-t} \end{vmatrix}$$

$$W(\vec{x_1},\vec{x_2},\vec{x_3})=e^t(-2e^{-t}e^{2t}-e^{2t}e^{-t})+e^{-t}(0-e^{2t}e^t)-e^{2t}(0+2e^{-t}e^t)$$

$$W(\vec{x_1},\vec{x_2},\vec{x_3})=-2e^te^{-t}e^{2t}-e^te^{2t}e^{-t}-e^{-t}e^{2t}e^t-2e^{2t}e^{-t}e^t$$

$$W(\vec{x_1},\vec{x_2},\vec{x_3})=-6e^{2t}$$

Because the Wronskian is non-zero, we can confirm that the vectors in the solution set are linearly independent, which means that the vector set represents a fundamental set of solutions.

Given that this is a fundamental set of solutions, we can write the general solution of the system as

$$\vec{x}=c_1\vec{x_1}+c_2\vec{x_2}+c_3\vec{x_3}$$

$$\vec{x}=c_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}e^t+c_2\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}e^{-t}+c_3\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}e^{2t}$$

■ 5. Verify that the solution vectors satisfy the system of differential equations. Calculate the Wronskian of the solution vectors and write the general solution of the homogeneous system of differential equations.

$$\vec{x}' = \begin{bmatrix} 6 & 5 \\ -5 & 0 \end{bmatrix} \vec{x}$$

$$\{\vec{x}_1, \vec{x}_2\} = \left\{ \begin{bmatrix} -3\cos(4t) + 4\sin(4t) \\ 5\cos(4t) \end{bmatrix} e^{3t}, \begin{bmatrix} -4\cos(4t) - 3\sin(4t) \\ 5\sin(4t) \end{bmatrix} e^{3t} \right\}$$

Solution:

We can verify that each of the vectors $\{\vec{x}_1, \vec{x}_2\}$ are solutions by substituting them into the matrix equation. First though, we'll need the derivative of each solution vector.

$$\vec{x}_1 = \begin{bmatrix} e^{3t}(-3\cos(4t) + 4\sin(4t)) \\ 5e^{3t}\cos(4t) \end{bmatrix} \qquad \vec{x}_1{}' = \begin{bmatrix} e^{3t}(7\cos(4t) + 24\sin(4t)) \\ 5e^{3t}(3\cos(4t) - 4\sin(4t)) \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} e^{3t}(-4\cos(4t) - 3\sin(4t)) \\ 5e^{3t}\sin(4t) \end{bmatrix} \qquad \vec{x}_2{}' = \begin{bmatrix} e^{3t}(-24\cos(4t) + 7\sin(4t)) \\ 5e^{3t}(3\sin(4t) + 4\cos(4t)) \end{bmatrix}$$

Substitute $\vec{x}_1$ and its derivative $\vec{x}_1{}'$ into the matrix equation.

$$\begin{bmatrix} e^{3t}(7\cos(4t) + 24\sin(4t)) \\ 5e^{3t}(3\cos(4t) - 4\sin(4t)) \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} e^{3t}(-3\cos(4t) + 4\sin(4t)) \\ 5e^{3t}\cos(4t) \end{bmatrix}$$

$$\begin{bmatrix} e^{3t}(7\cos(4t) + 24\sin(4t)) \\ 5e^{3t}(3\cos(4t) - 4\sin(4t)) \end{bmatrix} = \begin{bmatrix} e^{3t}(7\cos(4t) + 24\sin(4t)) \\ 5e^{3t}(3\cos(4t) - 4\sin(4t)) \end{bmatrix}$$

Substitute $\vec{x}_2$ and its derivative $\vec{x}_2{}'$ into the matrix equation.

$$\begin{bmatrix} e^{3t}(-24\cos(4t)+7\sin(4t)) \\ 5e^{3t}(3\sin(4t)+4\cos(4t)) \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -5 & 0 \end{bmatrix} \begin{bmatrix} e^{3t}(-4\cos(4t)-3\sin(4t)) \\ 5e^{3t}\sin(4t) \end{bmatrix}$$

$$\begin{bmatrix} e^{3t}(-24\cos(4t)+7\sin(4t)) \\ 5e^{3t}(3\sin(4t)+4\cos(4t)) \end{bmatrix} = \begin{bmatrix} e^{3t}(-24\cos(4t)+7\sin(4t)) \\ 5e^{3t}(4\cos(4t)+3\sin(4t)) \end{bmatrix}$$

Because we found equivalent vectors on both sides of the equation in each of these two cases, we've shown that both vectors are solutions to the system of differential equations.

Now let's take the Wronskian of the solution set.

$$W(\vec{x_1},\vec{x_2}) = \begin{vmatrix} e^{3t}(-3\cos(4t)+4\sin(4t)) & e^{3t}(-4\cos(4t)-3\sin(4t)) \\ 5e^{3t}\cos(4t) & 5e^{3t}\sin(4t) \end{vmatrix}$$

$$W(\vec{x_1},\vec{x_2}) = 5e^{6t}\sin(4t)(-3\cos(4t)+4\sin(4t)) - 5e^{6t}\cos(4t)(-4\cos(4t)-3\sin(4t))$$

$$W(\vec{x_1},\vec{x_2}) = 5e^{6t}(-3\cos(4t)\sin(4t)+4\sin^2(4t)+4\cos^2(4t)+3\cos(4t)\sin(4t))$$

$$W(\vec{x_1},\vec{x_2}) = 20e^{6t}(\sin^2(4t)+\cos^2(4t))$$

Since $\sin^2(4t)+\cos^2(4t)=1$, we get

$$W(\vec{x_1},\vec{x_2}) = 20e^{6t}$$

Because the Wronskian is non-zero, we can confirm that the vectors in the solution set are linearly independent, which means that the vector set represents a fundamental set of solutions.

Given that this is a fundamental set of solutions, we can write the general solution of the system as

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} -3\cos(4t) + 4\sin(4t) \\ 5\cos(4t) \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -4\cos(4t) - 3\sin(4t) \\ 5\sin(4t) \end{bmatrix} e^{3t}$$

■ 6. Verify that the solution vectors satisfy the system of differential equations. Calculate the Wronskian of the solution vectors and write the general solution of the homogeneous system of differential equations.

$$\vec{x}' = \begin{bmatrix} 5 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \vec{x}$$

$$\{\vec{x_1}, \vec{x_2}, \vec{x_3}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{5t}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{5t} \right\}$$

Solution:

We can verify that each of the vectors $\{\vec{x_1}, \vec{x_2}, \vec{x_3}\}$ are solutions by substituting them into the matrix equation. First though, we'll need the derivative of each solution vector.

$$\vec{x_1} = \begin{bmatrix} e^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix} \qquad \vec{x_1}' = \begin{bmatrix} 3e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix}$$

$$\vec{x_2} = \begin{bmatrix} e^{5t} \\ 0 \\ 0 \end{bmatrix} \qquad \vec{x_2}' = \begin{bmatrix} 5e^{5t} \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{x_3} = \begin{bmatrix} 0 \\ -e^{5t} \\ e^{5t} \end{bmatrix} \qquad \vec{x_3}' = \begin{bmatrix} 0 \\ -5e^{5t} \\ 5e^{5t} \end{bmatrix}$$

Substitute $\vec{x_1}$ and its derivative $\vec{x_1}'$ into the matrix equation.

$$\begin{bmatrix} 3e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 5 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} e^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} 3e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 5e^{3t} - e^{3t} - e^{3t} \\ 4e^{3t} - e^{3t} \\ -e^{3t} + 4e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} 3e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix}$$

Substitute $\vec{x_2}$ and its derivative $\vec{x_2}'$ into the matrix equation.

$$\begin{bmatrix} 5e^{5t} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} e^{5t} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5e^{5t} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5e^{5t} - 0 - 0 \\ 0 + 0 - 0 \\ 0 - 0 + 0 \end{bmatrix}$$

$$\begin{bmatrix} 5e^{5t} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5e^{5t} \\ 0 \\ 0 \end{bmatrix}$$

Substitute $\vec{x_3}$ and its derivative $\vec{x_3}'$ into the matrix equation.

$$\begin{bmatrix} 0 \\ -5e^{5t} \\ 5e^{5t} \end{bmatrix} = \begin{bmatrix} 5 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \begin{bmatrix} 0 \\ -e^{5t} \\ e^{5t} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -5e^{5t} \\ 5e^{5t} \end{bmatrix} = \begin{bmatrix} e^{5t} - e^{5t} \\ -4e^{5t} - e^{5t} \\ e^{5t} + 4e^{5t} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -5e^{5t} \\ 5e^{5t} \end{bmatrix} = \begin{bmatrix} 0 \\ -5e^{5t} \\ 5e^{5t} \end{bmatrix}$$

Because we found equivalent vectors on both sides of the equation in each of these three cases, we've shown that all three vectors are solutions to the system of differential equations.

Now let's take the Wronskian of the solution set.

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = \begin{vmatrix} e^{3t} & e^{5t} & 0 \\ e^{3t} & 0 & -e^{5t} \\ e^{3t} & 0 & e^{5t} \end{vmatrix}$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = e^{3t} \begin{vmatrix} 0 & -e^{5t} \\ 0 & e^{5t} \end{vmatrix} - e^{5t} \begin{vmatrix} e^{3t} & -e^{5t} \\ e^{3t} & e^{5t} \end{vmatrix} + 0 \begin{vmatrix} e^{3t} & 0 \\ e^{3t} & 0 \end{vmatrix}$$

$$W(\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}) = e^{3t}(0 - 0) - e^{5t}(e^{3t}e^{5t} + e^{5t}e^{3t})$$

$$W(\overrightarrow{x_1}, \overrightarrow{x_2}, \overrightarrow{x_3}) = -2e^{13t}$$

Because the Wronskian is non-zero, we can confirm that the vectors in the solution set are linearly independent, which means that the vector set represents a fundamental set of solutions.

Given that this is a fundamental set of solutions, we can write the general solution of the system as

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{5t}$$

DISTINCT REAL EIGENVALUES

■ 1. Solve the system of differential equations.

$$x_1' = 4x_1 + 8x_2$$

$$x_2' = x_1 + 2x_2$$

Solution:

The coefficient matrix is

$$A = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & 8 \\ 1 & 2 - \lambda \end{bmatrix}$$

Then the determinant is

$$|A - \lambda I| = \begin{vmatrix} 4 - \lambda & 8 \\ 1 & 2 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (4 - \lambda)(2 - \lambda) - (8)(1)$$

$$|A - \lambda I| = 8 - 6\lambda + \lambda^2 - 8$$

$$|A - \lambda I| = \lambda^2 - 6\lambda$$

and the characteristic equation is

$$\lambda^2 - 6\lambda = 0$$

$$\lambda(\lambda - 6) = 0$$

$$\lambda = 0, 6$$

Then for these Eigenvalues, $\lambda_1 = 0$ and $\lambda_2 = 6$, we find

$$A - (0)I = \begin{bmatrix} 4-0 & 8 \\ 1 & 2-0 \end{bmatrix}$$

$$A - (0)I = \begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix}$$

and

$$A - 6I = \begin{bmatrix} 4-6 & 8 \\ 1 & 2-6 \end{bmatrix}$$

$$A - 6I = \begin{bmatrix} -2 & 8 \\ 1 & -4 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} 4 & 8 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} -2 & 8 \\ 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \qquad \begin{bmatrix} 1 & -4 \\ 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -4 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations (in both cases we only need to consider the equation that we get from the first row of each matrix), we get

$$k_1 + 2k_2 = 0 \qquad k_1 - 4k_2 = 0$$

$$k_1 = -2k_2 \qquad k_1 = 4k_2$$

From the first system, we'll choose $k_2 = -1$, which results in $k_1 = 2$. And from the second system, we'll choose $k_2 = 1$, which results in $k_1 = 4$.

$$\vec{k_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{0t} \qquad \vec{x_2} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{6t}$$

$$\vec{x_1} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Therefore, the general solution to the homogeneous system must be

$$\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{6t}$$

■ 2. Solve the system of differential equations.

$$x_1' = 7x_1 + 6x_2 + 3x_3$$

$$x_2' = 2x_2 - x_1 + x_3$$

$$x_3' = x_1 + 2x_2 + x_3$$

Solution:

The coefficient matrix is

$$A = \begin{bmatrix} 7 & 6 & 3 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 7 & 6 & 3 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 7 & 6 & 3 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 7-\lambda & 6 & 3 \\ -1 & 2-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{bmatrix}$$

Then the determinant is

$$|A - \lambda I| = \begin{vmatrix} 7-\lambda & 6 & 3 \\ -1 & 2-\lambda & 1 \\ 1 & 2 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (7-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} - 6\begin{vmatrix} -1 & 1 \\ 1 & 1-\lambda \end{vmatrix} + 3\begin{vmatrix} -1 & 2-\lambda \\ 1 & 2 \end{vmatrix}$$

$$|A - \lambda I| = (7-\lambda)((2-\lambda)(1-\lambda) - 1(2)) - 6((-1)(1-\lambda) - 1(1))$$

$$+3((-1)(2) - (2-\lambda)(1))$$

$$|A - \lambda I| = (7-\lambda)(2 - 3\lambda + \lambda^2 - 2) - 6(\lambda - 1 - 1) + 3(-2 - 2 + \lambda)$$

$$|A - \lambda I| = (7-\lambda)\lambda(\lambda - 3) - 6(\lambda - 2) + 3(\lambda - 4)$$

$$|A - \lambda I| = \lambda(7-\lambda)(\lambda - 3) - 6\lambda + 12 + 3\lambda - 12$$

$$|A - \lambda I| = \lambda((7-\lambda)(\lambda - 3) - 3)$$

$$|A - \lambda I| = \lambda(7\lambda - 21 - \lambda^2 + 3\lambda - 3)$$

$$|A - \lambda I| = \lambda(-\lambda^2 + 10\lambda - 24)$$

and the characteristic equation is

$$\lambda(-\lambda^2 + 10\lambda - 24) = 0$$

$$\lambda_1 = 0 \qquad\qquad -\lambda^2 + 10\lambda - 24 = 0$$

$$\lambda^2 - 10\lambda + 24 = 0$$

$$(\lambda - 6)(\lambda - 4) = 0$$

$$\lambda_{2,3} = 6, 4$$

For the Eigenvalue $\lambda_1 = 0$ we find

$$A - (0)I = \begin{bmatrix} 7-0 & 6 & 3 \\ -1 & 2-0 & 1 \\ 1 & 2 & 1-0 \end{bmatrix}$$

$$A - (0)I = \begin{bmatrix} 7 & 6 & 3 \\ -1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 2 & 1 \\ 7 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \\ 7 & 6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 4 & 2 \\ 0 & -8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & -8 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & -8 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this back into a system of equations, we get

$$k_1 = 0$$

and

$$k_2 + \frac{1}{2}k_3 = 0$$

$$k_2 = -\frac{1}{2}k_3$$

We'll choose $k_3 = 2$, which results in $k_2 = -1$, and so we get the Eigenvalue

$$\vec{k_1} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

and therefore the solution vector

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} e^{0t}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

For the Eigenvalue $\lambda_2 = 6$ we find

$$A - 6I = \begin{bmatrix} 7-6 & 6 & 3 \\ -1 & 2-6 & 1 \\ 1 & 2 & 1-6 \end{bmatrix}$$

$$A - 6I = \begin{bmatrix} 1 & 6 & 3 \\ -1 & -4 & 1 \\ 1 & 2 & -5 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 6 & 3 \\ 0 & 2 & 4 \\ 1 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 3 \\ 0 & 2 & 4 \\ 0 & -4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & -4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -9 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this back into a system of equations, we get

$$k_1 - 9k_3 = 0$$

$$k_1 = 9k_3$$

and

$$k_2 + 2k_3 = 0$$

$$k_2 = -2k_3$$

We'll choose $k_3 = 1$, which results in $k_1 = 9$ and $k_2 = -2$, and so we get the Eigenvector

$$\overrightarrow{k_2} = \begin{bmatrix} 9 \\ -2 \\ 1 \end{bmatrix}$$

and therefore the solution vector

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 9 \\ -2 \\ 1 \end{bmatrix} e^{6t}$$

For the Eigenvalue $\lambda_3 = 4$ we find

$$A - 4I = \begin{bmatrix} 7-4 & 6 & 3 \\ -1 & 2-4 & 1 \\ 1 & 2 & 1-4 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} 3 & 6 & 3 \\ -1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this back into a system of equations, we get

$$k_1 + 2k_2 = 0$$

$$k_1 = -2k_2$$

and

$$k_3 = 0$$

We'll choose $k_2 = -1$, which results in $k_1 = 2$, and so we get the Eigenvector

$$\overrightarrow{k_3} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

and therefore the solution vector

$$\overrightarrow{x_3} = \overrightarrow{k_3}e^{\lambda_3 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} e^{4t}$$

Therefore, the general solution to the homogeneous system must be

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 9 \\ -2 \\ 1 \end{bmatrix} e^{6t} + c_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} e^{4t}$$

■ 3. Solve the system of differential equations, given $x_1(0) = -2$ and $x_2(0) = 19$.

$$x_1' = x_1 + 2x_2$$

$$x_2' = 5x_1 - 2x_2$$

Solution:

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 5 & -2 - \lambda \end{bmatrix}$$

Then the determinant is

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 2 \\ 5 & -2 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (1 - \lambda)(-2 - \lambda) - (2)(5)$$

$$|A - \lambda I| = -2 + \lambda + \lambda^2 - 10$$

$$|A - \lambda I| = \lambda^2 + \lambda - 12$$

and the characteristic equation is

$$\lambda^2 + \lambda - 12 = 0$$

$$(\lambda - 3)(\lambda + 4) = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = -4$$

For the Eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -4$, we find

$$A - 3I = \begin{bmatrix} 1 - 3 & 2 \\ 5 & -2 - 3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix}$$

and

$$A - (-4)I = \begin{bmatrix} 1+4 & 2 \\ 5 & -2+4 \end{bmatrix}$$

$$A - (-4)I = \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} -2 & 2 \\ 5 & -5 \end{bmatrix} \qquad \begin{bmatrix} 5 & 2 \\ 5 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix} \qquad \begin{bmatrix} 5 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{2}{5} \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 - k_2 = 0 \qquad k_1 + \frac{2}{5}k_2 = 0$$

$$k_1 = k_2 \qquad k_1 = -\frac{2}{5}k_2$$

From the first system, we'll choose $k_2 = 1$, which results in $k_1 = 1$. And from the second system, we'll choose $k_2 = -5$, which results in $k_1 = 2$.

$$\vec{k_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$$

Then the solutions to the system are

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 2 \\ -5 \end{bmatrix} e^{-4t}$$

Therefore, the general solution to the homogeneous system must be

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 2 \\ -5 \end{bmatrix} e^{-4t}$$

Rewrite the general solution,

$$\overrightarrow{x} = \begin{bmatrix} c_1e^{3t} & 2c_2e^{-4t} \\ c_1e^{3t} & -5c_2e^{-4t} \end{bmatrix}$$

then substitute the initial condition,

$$\overrightarrow{x}(0) = \begin{bmatrix} -2 \\ 19 \end{bmatrix}$$

to get

$$\begin{bmatrix} -2 \\ 19 \end{bmatrix} = \begin{bmatrix} c_1e^{3(0)} & 2c_2e^{-4(0)} \\ c_1e^{3(0)} & -5c_2e^{-4(0)} \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 19 \end{bmatrix} = \begin{bmatrix} c_1 & 2c_2 \\ c_1 & -5c_2 \end{bmatrix}$$

This gives the system of equations

$$c_1 + 2c_2 = -2$$

$$c_1 - 5c_2 = 19$$

We can rewrite the system as

$$c_1 = -2 - 2c_2$$

$$c_1 = 19 + 5c_2$$

to get

$$-2 - 2c_2 = 19 + 5c_2$$

$$7c_2 = -21$$

$$c_2 = -3$$

and

$$c_1 = -2 - 2(-3)$$

$$c_1 = -2 + 6$$

$$c_1 = 4$$

The solution of the system of differential equations with the given initial conditions is

$$\vec{x} = 4\begin{bmatrix}1\\1\end{bmatrix}e^{3t} - 3\begin{bmatrix}2\\-5\end{bmatrix}e^{-4t}$$

$$\vec{x} = \begin{bmatrix}4\\4\end{bmatrix}e^{3t} + \begin{bmatrix}-6\\15\end{bmatrix}e^{-4t}$$

■ 4. Convert the differential equation into a system of equations, then use the system to solve the initial value problem, given $y(0) = 2$ and $y'(0) = 6$.

$$y'' - 6y' - 7y = 0$$

Solution:

Solve the original differential equation for $y''(t)$.

$$y'' - 6y' - 7y = 0$$

$$y'' = 6y' + 7y$$

We can start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

Then if we take the derivatives of $x_1(t)$ and $x_2(t)$, we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = 6y'(t) + 7y(t) = 6x_2(t) + 7x_1(t)$$

Therefore, we have the system of differential equations

$$x_1' = x_2$$

$$x_2' = 7x_1 + 6x_2$$

The coefficient matrix is

$$A = \begin{bmatrix} 0 & 1 \\ 7 & 6 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ 7 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ 7 & 6 - \lambda \end{bmatrix}$$

Then the determinant is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 7 & 6 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = -\lambda(6 - \lambda) - 1(7)$$

$$|A - \lambda I| = -6\lambda + \lambda^2 - 7$$

$$|A - \lambda I| = \lambda^2 - 6\lambda - 7$$

So the characteristic equation is

$$\lambda^2 - 6\lambda - 7 = 0$$

$$(\lambda - 7)(\lambda + 1) = 0$$

$$\lambda_1 = 7 \qquad \lambda_2 = -1$$

Then for these Eigenvalues, $\lambda_1 = 7$ and $\lambda_2 = -1$, we find

$$A - 7I = \begin{bmatrix} -7 & 1 \\ 7 & 6 - 7 \end{bmatrix}$$

$$A - 7I = \begin{bmatrix} -7 & 1 \\ 7 & -1 \end{bmatrix}$$

and

$$A - (-1)I = \begin{bmatrix} 1 & 1 \\ 7 & 6+1 \end{bmatrix}$$

$$A - (-1)I = \begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} -7 & 1 \\ 7 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{7} \\ 7 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{7} \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 7 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 - \frac{1}{7}k_2 = 0$$

$$k_1 = \frac{1}{7}k_2$$

$$k_1 + k_2 = 0$$

$$k_1 = -k_2$$

From the first system, we'll choose $k_2 = 7$, which results in $k_1 = 1$. And from the second system, we'll choose $k_2 = -1$, which results in $k_1 = 1$. So the Eigenvectors are

$$\vec{k_1} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} e^{7t} \qquad \vec{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Therefore, the general solution to the homogeneous system must be

$$\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 7 \end{bmatrix} e^{7t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Rewrite the initial conditions $y(0) = 2$ and $y'(0) = 6$ as

$$x_1(0) = y(0) = 2$$

$$x_2(0) = y'(0) = 6$$

With these two values, we can say

$$\vec{x}(0) = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$$

and then substitute this into the general solution.

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 7 \end{bmatrix} e^{7(0)} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-(0)}$$

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 6 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 7c_1 - c_2 \end{bmatrix}$$

We get the system of equations

$$c_1 + c_2 = 2$$

$$7c_1 - c_2 = 6$$

Solving this system, we'll get

$$c_1 = 1$$

$$c_2 = 1$$

Therefore, the solution to the system is

$$\vec{x} = \begin{bmatrix} 1 \\ 7 \end{bmatrix} e^{7t} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

And the solution $y(t)$ to the initial value problem is

$$y(t) = x_1(t)$$

$$y(t) = e^{7t} + e^{-t}$$

■ 5. Convert the differential equation into a system of equations, then solve the system.

$$y''' - 2y'' - 9y' + 18y = 0$$

Solution:

Solve the original differential equation for $y'''(t)$.

$$y''' = 2y'' + 9y' - 18y$$

We can start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

$$x_3(t) = y''(t)$$

Then if we take the derivatives of $x_1(t)$, $x_2(t)$, and $x_3(t)$ we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = x_3(t)$$

$$x_3'(t) = y'''(t) = 2y'' + 9y' - 18y = 2x_3 + 9x_2 - 18x_1$$

Therefore, we have the system of differential equations

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = -18x_1 + 9x_2 + 2x_3$$

The coefficient matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -18 & 9 & 2 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -18 & 9 & 2-\lambda \end{bmatrix}$$

Then the determinant is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ -18 & 9 & 2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = -\lambda \begin{vmatrix} -\lambda & 1 \\ 9 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ -18 & 2-\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & -\lambda \\ -18 & 9 \end{vmatrix}$$

$$|A - \lambda I| = -\lambda((-\lambda)(2-\lambda) - 1(9)) - 1(0(2-\lambda) - 1(-18)) + 0((0(9) - (-\lambda)(-18))$$

$$|A - \lambda I| = -\lambda(\lambda^2 - 2\lambda - 9) - 18$$

$$|A - \lambda I| = -\lambda^3 + 2\lambda^2 + 9\lambda - 18$$

$$|A - \lambda I| = -\lambda^2(\lambda - 2) + 9(\lambda - 2)$$

$$|A - \lambda I| = (\lambda - 2)(9 - \lambda^2)$$

So the characteristic equation is

$$(\lambda - 2)(9 - \lambda^2) = 0$$

$$(\lambda - 2)(3 - \lambda)(3 + \lambda) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = 3 \qquad \lambda_3 = -3$$

For the Eigenvalue $\lambda_1 = 2$ we find

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -18 & 9 & 2-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ -18 & 9 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -2 & 1 \\ -18 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Turning this back into a system of equations, we get

$$k_1 - \frac{1}{4}k_3 = 0$$

$$k_1 = \frac{1}{4}k_3$$

and

$$k_2 - \frac{1}{2}k_3 = 0$$

$$k_2 = \frac{1}{2}k_3$$

We'll choose $k_3 = 4$, which results in $k_1 = 1$ and $k_2 = 2$, and therefore we get the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2t}$$

For the Eigenvalue $\lambda_2 = 3$ we find

$$A - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ -18 & 9 & 2-3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ -18 & 9 & -1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & -3 & 1 \\ -18 & 9 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & -3 & 1 \\ 0 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 3 & -1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Turning this back into a system of equations, we get

$$k_1 - \frac{1}{9}k_3 = 0$$

$$k_1 = \frac{1}{9}k_3$$

and

$$k_2 - \frac{1}{3}k_3 = 0$$

$$k_2 = \frac{1}{3}k_3$$

We'll choose $k_3 = 9$, which results in $k_1 = 1$ and $k_2 = 3$, and therefore we get the Eigenvector

$$\vec{k_2} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

and the solution vector

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t}$$

For the Eigenvalue $\lambda_3 = -3$ we find

$$A - (-3)I = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -18 & 9 & 2+3 \end{bmatrix}$$

$$A - (-3)I = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ -18 & 9 & 5 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 3 & 1 \\ -18 & 9 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 3 & 1 \\ 0 & 15 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} \\ 0 & 15 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & \frac{1}{3} \\ 0 & 15 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Turning this back into a system of equations, we get

$$k_1 - \frac{1}{9}k_3 = 0$$

$$k_1 = \frac{1}{9}k_3$$

and

$$k_2 + \frac{1}{3}k_3 = 0$$

$$k_2 = -\frac{1}{3}k_3$$

We'll choose $k_3 = 9$, which results in $k_1 = 1$ and $k_2 = -3$, and therefore we get the Eigenvector

$$\vec{k_3} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix}$$

and the solution vector

$$\vec{x_3} = \vec{k_3} e^{\lambda_3 t}$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix} e^{-3t}$$

So the general solution is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 9 \end{bmatrix} e^{-3t}$$

And the solution $y(t)$ to the original differential equation is

$$y(t) = c_1e^{2t} + c_2e^{3t} + c_3e^{-3t}$$

■ 6. Solve the initial value problem, given $y(0) = 2$, $y'(0) = 6$, and $y''(0) = 16$.

$$y''' - 6y'' + 11y' - 6y = 0$$

Solution:

Solve the original differential equation for y'''.

$$y''' = 6y'' - 11y' + 6y$$

We can start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

$$x_3(t) = y''(t)$$

Then if we take the derivatives of $x_1(t)$, $x_2(t)$, and $x_3(t)$ we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = x_3(t)$$

$$x_3'(t) = y'''(t) = 6y'' - 11y' + 6y = 6x_3 - 11x_2 + 6x_1$$

Therefore, we have the system of differential equations

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 6x_1 - 11x_2 + 6x_3$$

The coefficient matrix is

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -11 & 6-\lambda \end{bmatrix}$$

Then the determinant is

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 6 & -11 & 6-\lambda \end{vmatrix}$$

$$|A - \lambda I| = -\lambda \begin{vmatrix} -\lambda & 1 \\ -11 & 6-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & 1 \\ 6 & 6-\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & -\lambda \\ 6 & -11 \end{vmatrix}$$

$$|A - \lambda I| = -\lambda(-\lambda(6-\lambda) - 1(-11)) - 1(0(6-\lambda) - 1(6)) + 0(0(-11) - (-\lambda)(6))$$

$$|A - \lambda I| = -\lambda(\lambda^2 - 6\lambda + 11) + 6$$

$$|A - \lambda I| = -\lambda^3 + 6\lambda^2 - 11\lambda + 6$$

So the characteristic equation is

$$-\lambda^3 + 6\lambda^2 - 11\lambda + 6 = 0$$

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0$$

$$(\lambda - 1)(\lambda - 2)(3 - \lambda) = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = 2 \qquad \lambda_3 = 3$$

For the Eigenvalue $\lambda_1 = 1$ we find

$$A - (1)I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 6-1 \end{bmatrix}$$

$$A - (1)I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 6 & -11 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -5 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Turning this back into a system of equations, we get

$$k_1 - k_3 = 0$$

$$k_1 = k_3$$

and

$$k_2 - k_3 = 0$$

$$k_2 = k_3$$

We'll choose $k_3 = 1$, which results in $k_1 = 1$ and $k_2 = 1$, and therefore we get the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t$$

For the Eigenvalue $\lambda_2 = 2$ we find

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 6-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -2 & 1 \\ 6 & -11 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & -2 & 1 \\ 0 & -8 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & -8 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & -8 & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Turning this back into a system of equations, we get

$$k_1 - \frac{1}{4}k_3 = 0$$

$$k_1 = \frac{1}{4}k_3$$

and

$$k_2 - \frac{1}{2}k_3 = 0$$

$$k_2 = \frac{1}{2}k_3$$

We'll choose $k_3 = 4$, which results in $k_1 = 1$ and $k_2 = 2$, and therefore we get the Eigenvector

$$\vec{k_2} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$$

and the solution vector

$$\vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2t}$$

For the Eigenvalue $\lambda_3 = 3$ we find

$$A - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 6-3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -3 & 1 & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & -3 & 1 \\ 6 & -11 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & -3 & 1 \\ 0 & -9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} & 0 \\ 0 & 1 & -\frac{1}{3} \\ 0 & -9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & -9 & 3 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{9} \\ 0 & 1 & -\frac{1}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

Turning this back into a system of equations, we get

$$k_1 - \frac{1}{9}k_3 = 0$$

$$k_1 = \frac{1}{9}k_3$$

and

$$k_2 - \frac{1}{3}k_3 = 0$$

$$k_2 = \frac{1}{3}k_3$$

We'll choose $k_3 = 9$, which results in $k_1 = 1$ and $k_2 = 3$, and therefore we get the Eigenvector

$$\vec{k_3} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

and the solution vector

$$\vec{x_3} = \vec{k_3}e^{\lambda_3 t}$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t}$$

So the general solution is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t}$$

The initial conditions can be rewritten as

$$\vec{x}(0) = \begin{bmatrix} 2 \\ 6 \\ 16 \end{bmatrix}$$

so we'll plug this into the general solution to get

$$\begin{bmatrix} 2 \\ 6 \\ 16 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{(0)} + c_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} e^{2(0)} + c_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3(0)}$$

$$\begin{bmatrix} 2 \\ 6 \\ 16 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix}$$

So we get the system of equations

$$c_1 + c_2 + c_3 = 2$$

$$c_1 + 2c_2 + 3c_3 = 6$$

$$c_1 + 4c_2 + 9c_3 = 16$$

After solving the system, we get

$$c_1 = -1$$

$$c_2 = 2$$

$$c_3 = 1$$

The solution is therefore

$$\vec{x} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 4 \\ 8 \end{bmatrix} e^{2t} + \begin{bmatrix} 1 \\ 3 \\ 9 \end{bmatrix} e^{3t}$$

Therefore, the solution to the initial value problem is

$$y(t) = -e^t + 2e^{2t} + e^{3t}$$

EQUAL REAL EIGENVALUES WITH MULTIPLICITY TWO

■ 1. Solve the system of the differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

Solution:

We'll find $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

and then we'll find its determinant.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$|A-\lambda I| = (1-\lambda)\begin{vmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 2 \\ 0 & 2-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 1-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A-\lambda I| = (1-\lambda)[(1-\lambda)(2-\lambda) - 2(0)]$$

$$-[(0)(2-\lambda) - 2(0)] + [(0)(0) - (1-\lambda)(0)]$$

$$|A-\lambda I| = (1-\lambda)(1-\lambda)(2-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(1-\lambda)(1-\lambda)(2-\lambda) = 0$$

$$\lambda_1 = \lambda_2 = 1 \qquad\qquad \lambda_3 = 2$$

We'll handle $\lambda_3 = 2$ first, starting with finding $A - \lambda_3 I$.

$$A - 2I = \begin{bmatrix} 1-2 & 1 & 1 \\ 0 & 1-2 & 2 \\ 0 & 0 & 2-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - 3k_3 = 0$$

$$k_1 = 3k_3$$

and

$$k_2 - 2k_3 = 0$$

$$k_2 = 2k_3$$

If we choose $k_3 = 1$, we find $k_1 = 3$ and $k_2 = 2$ and therefore we get the Eigenvector

$$\vec{k_3} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

and the solution vector

$$\vec{x_3} = \vec{k_3} e^{\lambda_3 t} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

Then for the repeated Eigenvalues $\lambda_1 = \lambda_2 = 1$, we find

$$A - (1)I = \begin{bmatrix} 1-1 & 1 & 1 \\ 0 & 1-1 & 2 \\ 0 & 0 & 2-1 \end{bmatrix}$$

$$A - (1)I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_2 = 0$$

$$k_3 = 0$$

Then there's only one linearly independent Eigenvector for $\lambda_1 = \lambda_2 = 1$, and we'll choose

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_1 = \lambda_2 = 1$, we have to use $\vec{k_1} = (1,0,0)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This matrix equation gives $p_2 + p_3 = 1$ and $p_3 = 0$, and therefore $p_2 = 1$. We'll choose $p_1 = 0$, so

$$\vec{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then the third solution is

$$\overrightarrow{x_2} = \overrightarrow{k_1} t e^{\lambda_1 t} + \overrightarrow{p} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

Then the general solution is

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2} + c_3 \overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t \right) + c_3 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

■ 2. Solve the system of the differential equations.

$$\overrightarrow{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \overrightarrow{x}$$

Solution:

We'll find $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 1 & 1 \\ 2 & 2-\lambda & 2 \\ 3 & 3 & 3-\lambda \end{bmatrix}$$

and then we'll find its determinant.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 1 \\ 2 & 2-\lambda & 2 \\ 3 & 3 & 3-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} 2-\lambda & 2 \\ 3 & 3-\lambda \end{vmatrix} - 1\begin{vmatrix} 2 & 2 \\ 3 & 3-\lambda \end{vmatrix} + 1\begin{vmatrix} 2 & 2-\lambda \\ 3 & 3 \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)[(2-\lambda)(3-\lambda)-(2)(3)]$$

$$-[(2)(3-\lambda)-(2)(3)]+[(2)(3)-(2-\lambda)(3)]$$

$$|A - \lambda I| = (1-\lambda)(6-5\lambda+\lambda^2-6)-(6-2\lambda-6)+(6-6+3\lambda)$$

$$|A - \lambda I| = (1-\lambda)(-5\lambda+\lambda^2)+2\lambda+3\lambda$$

$$|A - \lambda I| = -5\lambda+\lambda^2+5\lambda^2-\lambda^3+5\lambda$$

$$|A - \lambda I| = 6\lambda^2-\lambda^3$$

$$|A - \lambda I| = \lambda^2(6-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$\lambda^2(6-\lambda)=0$$

$$\lambda_1 = 6 \qquad \lambda_2 = \lambda_3 = 0$$

We'll handle $\lambda_1 = 6$ first, starting with finding $A - \lambda_1 I$.

$$A - 6I = \begin{bmatrix} 1-6 & 1 & 1 \\ 2 & 2-6 & 2 \\ 3 & 3 & 3-6 \end{bmatrix}$$

$$A - 6I = \begin{bmatrix} -5 & 1 & 1 \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{5} \\ 2 & -4 & 2 \\ 3 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & -\frac{18}{5} & \frac{12}{5} \\ 3 & 3 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & -\frac{18}{5} & \frac{12}{5} \\ 0 & \frac{18}{5} & -\frac{12}{5} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{2}{3} \\ 0 & \frac{18}{5} & -\frac{12}{5} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & \frac{18}{5} & -\frac{12}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - \frac{1}{3}k_3 = 0$$

$$k_1 = \frac{1}{3}k_3$$

and

$$k_2 - \frac{2}{3}k_3 = 0$$

$$k_2 = \frac{2}{3}k_3$$

If we choose $k_3 = 3$, we find $k_1 = 1$ and $k_2 = 2$ and therefore we get the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

and the solution vector

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} e^{6t}$$

Then for the repeated Eigenvalues $\lambda_2 = \lambda_3 = 0$, we find

$$A - (0)I = \begin{bmatrix} 1-0 & 1 & 1 \\ 2 & 2-0 & 2 \\ 3 & 3 & 3-0 \end{bmatrix}$$

$$A - (0)I = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 3 & 3 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 + k_2 + k_3 = 0$$

$$k_1 = -k_2 - k_3$$

We can choose $k_2 = 0$ and $k_3 = -1$ to get $k_1 = 1$ and therefore

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

But we can also choose $k_2 = 1$ and $k_3 = -1$ to get $k_1 = 0$ and therefore

$$\overrightarrow{k_3} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

These linearly independent Eigenvectors give the solution vectors

$$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\overrightarrow{x_3} = \overrightarrow{k_3}e^{\lambda_3 t} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Then the general solution is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

■ 3. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ -1 & 1 & 0 \end{bmatrix} \vec{x}$$

Solution:

We'll find $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ -1 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ -1 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 & 2 \\ 2 & -\lambda & 4 \\ -1 & 1 & -\lambda \end{bmatrix}$$

and then we'll find its determinant.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & 2 \\ 2 & -\lambda & 4 \\ -1 & 1 & -\lambda \end{vmatrix}$$

$$|A - \lambda I| = (3-\lambda)\begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} 2 & 4 \\ -1 & -\lambda \end{vmatrix} + 2\begin{vmatrix} 2 & -\lambda \\ -1 & 1 \end{vmatrix}$$

$$|A - \lambda I| = (3-\lambda)[(-\lambda)(-\lambda) - (4)(1)]$$

$$+[(2)(-\lambda) - (4)(-1)] + 2[(2)(1) - (-\lambda)(-1)]$$

$$|A - \lambda I| = (3 - \lambda)(\lambda^2 - 4) + (-2\lambda + 4) + 2(2 - \lambda)$$

$$|A - \lambda I| = (3 - \lambda)(\lambda^2 - 4) - 2(\lambda - 2) - 2(\lambda - 2)$$

$$|A - \lambda I| = (\lambda - 2)[(3 - \lambda)(\lambda + 2) - 4]$$

$$|A - \lambda I| = (\lambda - 2)(-\lambda^2 + \lambda + 2)$$

$$|A - \lambda I| = -(\lambda - 2)^2(\lambda + 1)$$

Solve the characteristic equation for the Eigenvalues.

$$-(\lambda - 2)^2(\lambda + 1) = 0$$

$$\lambda_1 = -1 \qquad\qquad \lambda_2 = \lambda_3 = 2$$

We'll handle $\lambda_1 = -1$ first, starting with finding $A - \lambda_1 I$.

$$A - (-1)I = \begin{bmatrix} 3+1 & -1 & 2 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A - (-1)I = \begin{bmatrix} 4 & -1 & 2 \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{2} \\ 2 & 1 & 4 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{3}{2} & 3 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{2} \\ 0 & \frac{3}{2} & 3 \\ 0 & \frac{3}{4} & \frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & \frac{1}{2} \\ 0 & 1 & 2 \\ 0 & \frac{3}{4} & \frac{3}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & \frac{3}{4} & \frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 + k_3 = 0$$

$$k_1 = -k_3$$

and

$$k_2 + 2k_3 = 0$$

$$k_2 = -2k_3$$

If we choose $k_3 = -1$, we find $k_1 = 1$ and $k_2 = 2$ and therefore we get the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{-t}$$

Then for the repeated Eigenvalues $\lambda_2 = \lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} 3-2 & -1 & 2 \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -2 & 4 \\ -1 & 1 & -2 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ -1 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - k_2 + 2k_3 = 0$$

$$k_1 = k_2 - 2k_3$$

We can choose $k_2 = 1$ and $k_3 = 0$ to get $k_1 = 1$ and therefore

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

But we can also choose $k_2 = 0$ and $k_3 = -1$ to get $k_1 = 2$ and therefore

$$\overrightarrow{k_3} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

These linearly independent Eigenvectors give the solution vectors

$$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

$$\vec{x}_3 = \vec{k}_3 e^{\lambda_3 t} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^{2t}$$

Then the general solution is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^{2t}$$

■ 4. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

Solution:

We'll find $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

and then we'll find its determinant.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 3 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)\begin{vmatrix} 2-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & 3 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)[(2-\lambda)(1-\lambda) - 3(0)]$$

$$+[(-1)(1-\lambda) - 3(0)] + 0[(-1)(0) - (2-\lambda)(0)]$$

$$|A - \lambda I| = (2-\lambda)(2-\lambda)(1-\lambda) - (1-\lambda)$$

$$|A - \lambda I| = (1-\lambda)[(2-\lambda)(2-\lambda) - 1]$$

$$|A - \lambda I| = (1-\lambda)(4 - 4\lambda + \lambda^2 - 1)$$

$$|A - \lambda I| = (1-\lambda)(\lambda^2 - 4\lambda + 3)$$

$$|A - \lambda I| = (1-\lambda)(\lambda - 3)(\lambda - 1)$$

Solve the characteristic equation for the Eigenvalues.

$$-(\lambda - 1)(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda_1 = \lambda_2 = 1 \qquad \lambda_3 = 3$$

We'll handle $\lambda_3 = 3$ first, starting with finding $A - \lambda_3 I$.

$$A - 3I = \begin{bmatrix} 2-3 & -1 & 0 \\ -1 & 2-3 & 3 \\ 0 & 0 & 1-3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & 3 \\ 0 & 0 & -2 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 3 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 + k_2 = 0$$

$$k_1 = -k_2$$

and

$$k_3 = 0$$

If we choose $k_2 = -1$, we find $k_1 = 1$ and therefore we get the Eigenvector

$$\overrightarrow{k_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and the solution vector

$$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-t}$$

Then for the repeated Eigenvalues $\lambda_1 = \lambda_2 = 1$, we find

$$A - (1)I = \begin{bmatrix} 2-1 & -1 & 0 \\ -1 & 2-1 & 3 \\ 0 & 0 & 1-1 \end{bmatrix}$$

$$A - (1)I = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - k_2 = 0$$

$$k_1 = k_2$$

and

$$k_3 = 0$$

We can choose $k_2 = 1$ to get $k_1 = 1$ and therefore the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} = \begin{bmatrix}1\\1\\0\end{bmatrix}e^t$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_1 = \lambda_2 = 1$, we have to use $\vec{k_1} = (1,1,0)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

$$\begin{bmatrix}1 & -1 & 0\\-1 & 1 & 3\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}p_1\\p_2\\p_3\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix}$$

This matrix equation gives $p_1 - p_2 = 1$ and $-p_1 + p_2 + 3p_3 = 1$. If we solve this system, we find

$$\vec{p} = \begin{bmatrix}1\\0\\\frac{2}{3}\end{bmatrix}$$

Then the third solution is

$$\vec{x_2} = \vec{k_1}te^{\lambda_1 t} + \vec{p}e^{\lambda_1 t} = \begin{bmatrix}1\\1\\0\end{bmatrix}te^t + \begin{bmatrix}1\\0\\\frac{2}{3}\end{bmatrix}e^t$$

Then the general solution is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \\ \frac{2}{3} \end{bmatrix} e^t \right) + c_3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{3t}$$

■ 5. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \vec{x}$$

Solution:

We'll find $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{bmatrix}$$

and then we'll find its determinant.

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 1 & 0 & 5-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (5-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & 5-\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & 0 \\ 1 & 5-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 2-\lambda \\ 1 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (5-\lambda)[(2-\lambda)(5-\lambda) - 0(0)] - 0 + 0$$

$$|A - \lambda I| = (5-\lambda)(2-\lambda)(5-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(5-\lambda)(2-\lambda)(5-\lambda) = 0$$

$$\lambda_1 = \lambda_2 = 5 \qquad \lambda_3 = 2$$

We'll handle $\lambda_3 = 2$ first, starting with finding $A - \lambda_3 I$.

$$A - 2I = \begin{bmatrix} 5-2 & 0 & 0 \\ 1 & 2-2 & 0 \\ 1 & 0 & 5-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 3 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 = 0$$

$$k_3 = 0$$

We can choose $k_2 = 1$ to get the Eigenvector

$$\vec{k_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and the solution vector

$$\vec{x_3} = \vec{k_3} e^{\lambda_3 t} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Then for the repeated Eigenvalues $\lambda_1 = \lambda_2 = 5$, we find

$$A - 5I = \begin{bmatrix} 5-5 & 0 & 0 \\ 1 & 2-5 & 0 \\ 1 & 0 & 5-5 \end{bmatrix}$$

$$A - 5I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 = 0$$

$$k_2 = 0$$

We can choose $k_3 = 1$ to get the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the solution vector

$$\overrightarrow{x_1} = \overrightarrow{k}_1 e^{\lambda_1 t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{5t}$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_1 = \lambda_2 = 5$, we have to use $\overrightarrow{k_1} = (0,0,1)$ to find a second solution.

$$(A - \lambda_1 I)\overrightarrow{p_1} = \overrightarrow{k_1}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & -3 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

This matrix equation gives $p_1 - 3p_2 = 0$ and $p_1 = 1$. If we solve this system, we find

$$\overrightarrow{p} = \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

Then the third solution is

$$\overrightarrow{x_2} = \overrightarrow{k_1} te^{\lambda_1 t} + \overrightarrow{p} e^{\lambda_1 t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{5t} + \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix} e^{5t}$$

Then the general solution is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{5t} + \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix} e^{5t} \right) + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

■ 6. Find the general solution to the system of differential equations.

$$\overrightarrow{x}' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 0 \\ 4 & 0 & 1 \end{bmatrix} \overrightarrow{x}$$

Solution:

We'll find $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 0 \\ 4 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 0 \\ 4 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 4 \\ 0 & -3-\lambda & 0 \\ 4 & 0 & 1-\lambda \end{bmatrix}$$

and then we'll find its determinant.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 4 \\ 0 & -3-\lambda & 0 \\ 4 & 0 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} -3-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0\begin{vmatrix} 0 & 0 \\ 4 & 1-\lambda \end{vmatrix} + 4\begin{vmatrix} 0 & -3-\lambda \\ 4 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)[(-3-\lambda)(1-\lambda) - 0(0)] - 0 + 4[0(0) - (-3-\lambda)(4)]$$

$$|A - \lambda I| = -(1-\lambda)(3+\lambda)(1-\lambda) + 16(3+\lambda)$$

$$|A - \lambda I| = -(3+\lambda)[(1-\lambda)^2 - 16]$$

$$|A - \lambda I| = -(3+\lambda)(1 - 2\lambda + \lambda^2 - 16)$$

$$|A - \lambda I| = -(3+\lambda)(\lambda^2 - 2\lambda - 15)$$

$$|A - \lambda I| = -(\lambda+3)(\lambda+3)(\lambda-5)$$

Solve the characteristic equation for the Eigenvalues.

$$-(\lambda+3)(\lambda+3)(\lambda-5) = 0$$

$$\lambda_1 = 5 \qquad \lambda_2 = \lambda_3 = -3$$

We'll handle $\lambda_1 = 5$ first, starting with finding $A - \lambda_1 I$.

$$A - 5I = \begin{bmatrix} 1-5 & 0 & 4 \\ 0 & -3-5 & 0 \\ 4 & 0 & 1-5 \end{bmatrix}$$

$$A - 5I = \begin{bmatrix} -4 & 0 & 4 \\ 0 & -8 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & -8 & 0 \\ 4 & 0 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - k_3 = 0$$

$$k_1 = k_3$$

and

$$k_2 = 0$$

We can choose $k_3 = 1$ to get $k_1 = 1$ and therefore the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{5t}$$

Then for the repeated Eigenvalues $\lambda_2 = \lambda_3 = -3$, we find

$$A + 3I = \begin{bmatrix} 1+3 & 0 & 4 \\ 0 & -3+3 & 0 \\ 4 & 0 & 1+3 \end{bmatrix}$$

$$A + 3I = \begin{bmatrix} 4 & 0 & 4 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 4 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - k_3 = 0$$

$$k_1 = k_3$$

We can choose $k_2 = 0$ and $k_3 = 1$ to get the Eigenvector

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and the solution vector

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t}$$

But we could also choose $k_2 = 1$ and $k_3 = 1$ to get the Eigenvector

$$\overrightarrow{k_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and the solution vector

$$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-3t}$$

Then the general solution is

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2} + c_3 \overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-3t}$$

EQUAL REAL EIGENVALUES WITH MULTIPLICITY THREE

■ 1. Find the general solution to the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

Solution:

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & 3 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1 - \lambda) \begin{vmatrix} 1 - \lambda & 2 \\ 0 & 1 - \lambda \end{vmatrix} - 2 \begin{vmatrix} 0 & 2 \\ 0 & 1 - \lambda \end{vmatrix} + 3 \begin{vmatrix} 0 & 1 - \lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (1 - \lambda)((1 - \lambda)(1 - \lambda) - (2)(0))$$

$$-2((0)(1 - \lambda) - (2)(0)) + 3((0)(0) - (1 - \lambda)(0))$$

$$|A - \lambda I| = (1 - \lambda)(1 - \lambda)(1 - \lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(1 - \lambda)(1 - \lambda)(1 - \lambda) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we find

$$A - (1)I = \begin{bmatrix} 1 - 1 & 2 & 3 \\ 0 & 1 - 1 & 2 \\ 0 & 0 & 1 - 1 \end{bmatrix}$$

$$A - (1)I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 & \frac{3}{2} \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & \frac{3}{2} \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get $k_2 = 0$ and $k_3 = 0$. We can choose any value for k_1. If we choose $k_1 = 1$, we find

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Other than the trivial solution $\vec{k_1} = (0,0,0)$, any other (k_1, k_2, k_3) set we find from $(k_1, k_2, k_3) = (k_1,0,0)$ will result in a vector that's linearly dependent with the $\vec{k_1} = (1,0,0)$ vector we already found. So we can say that the Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = 1$ produces only one Eigenvector.

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$$

Because we only find one Eigenvector for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have to use $\vec{k_1} = (1,0,0)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p} = \vec{k_1}$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives $2p_2 + 3p_3 = 1$ and $2p_3 = 0$, which means we find $p_2 = 1/2$ and $p_3 = 0$, and we can pick any value for p_1 so we'll choose $p_1 = 0$.

$$\vec{p} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Then our second solution will be

$$\vec{x_2} = \vec{k_1} t e^{\lambda_1 t} + \vec{p} e^{\lambda_1 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t e^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} e^t$$

Now we'll use $\vec{k_1} = (1,0,0)$ and $\vec{p} = (0,1/2,0)$ to find a third solution.

$$(A - \lambda_1 I)\vec{q} = \vec{p}$$

$$\begin{bmatrix} 0 & 2 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives $2q_2 + 3q_3 = 0$ and $2q_3 = 1/2$, which means we find $q_2 = -3/8$ and $q_3 = 1/4$, and we can pick any value for q_1, so we'll pick $q_1 = 0$.

$$\vec{q} = \begin{bmatrix} 0 \\ -\frac{3}{8} \\ \frac{1}{4} \end{bmatrix}$$

Then our third solution will be

$$\vec{x_3} = \vec{k_1} \frac{t^2}{2} e^{\lambda_1 t} + \vec{p} t e^{\lambda_1 t} + \vec{q} e^{\lambda_1 t}$$

$$\vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -\frac{3}{8} \\ \frac{1}{4} \end{bmatrix} e^t$$

So the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} e^t \right)$$

$$+c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -\frac{3}{8} \\ \frac{1}{4} \end{bmatrix} e^t \right)$$

■ 2. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

Solution:

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 2 \\ 0 & 0 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} - 0\begin{vmatrix} 0 & 2 \\ 0 & 1-\lambda \end{vmatrix} + 1\begin{vmatrix} 0 & 1-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)((1-\lambda)(1-\lambda) - (0)(0))$$

$$-0((0)(1-\lambda) - (2)(0)) + ((0)(0) - (1-\lambda)(0))$$

$$|A - \lambda I| = (1-\lambda)(1-\lambda)(1-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(1-\lambda)(1-\lambda)(1-\lambda) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 1$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we find

$$A - (1)I = \begin{bmatrix} 1-1 & 0 & 1 \\ 0 & 1-1 & 2 \\ 0 & 0 & 1-1 \end{bmatrix}$$

$$A - (1)I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix gives us the equation $k_3 = 0$, and we can pick an values for k_1 and k_2, so we could choose $k_1 = 1$ and $k_2 = 0$.

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We could also choose $k_1 = 0$ and $k_2 = 1$.

$$\vec{k_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

The solution vectors associated with these Eigenvectors are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$$

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

Because we only find two linearly independent Eigenvectors for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have to use $\overrightarrow{k_1} = (1,0,0)$ and $\overrightarrow{k_2} = (0,1,0)$ to find a third solution.

$$(A - \lambda_1 I)\overrightarrow{p} = a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$p_3 = a_1$$

$$2p_3 = a_2$$

Choosing $p_3 = 1$ gives $a_1 = 1$ and $a_2 = 2$. We can choose any value for p_1 and p_2, so we'll pick $p_1 = 0$ and $p_2 = 0$ to get

$$\overrightarrow{p} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and

$$a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2} = 1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

Then our third solution will be

$$\overrightarrow{x_3} = (a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2})te^{\lambda_3 t} + \overrightarrow{p}e^{\lambda_3 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t$$

and the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t \right)$$

■ 3. Find the general solution to the system of differential equations.

$$\overrightarrow{x}' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \overrightarrow{x}$$

Solution:

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 3-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (3-\lambda)(3-\lambda)(3-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(3-\lambda)(3-\lambda)(3-\lambda) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 3$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = 3$, we find

$$A - 3I = \begin{bmatrix} 3-3 & 0 & 0 \\ 0 & 3-3 & 0 \\ 0 & 0 & 3-3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can choose any values for k_1, k_2, and k_3, so we'll use 1, 0, and 0 in all three combinations to get the three linearly independent Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{k_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \vec{k_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and then the solution vectors

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t}$$

$$\vec{x_2} = \vec{k_2} e^{\lambda_2 t} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t}$$

$$\vec{x_3} = \vec{k_3} e^{\lambda_3 t} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

Then the general solution to the homogeneous system is

$$\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2} + c_3 \vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

■ 4. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 0 & 2 & 6 \\ -2 & -4 & -5 \\ 0 & 0 & -2 \end{bmatrix} \vec{x}$$

Solution:

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 6 \\ -2 & -4 & -5 \\ 0 & 0 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 2 & 6 \\ -2 & -4 & -5 \\ 0 & 0 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 2 & 6 \\ -2 & -4-\lambda & -5 \\ 0 & 0 & -2-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 2 & 6 \\ -2 & -4-\lambda & -5 \\ 0 & 0 & -2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (-\lambda)\begin{vmatrix} -4-\lambda & -5 \\ 0 & -2-\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & -5 \\ 0 & -2-\lambda \end{vmatrix} + 6\begin{vmatrix} -2 & -4-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (-\lambda)((-4-\lambda)(-2-\lambda) - (-5)(0))$$

$$-2((-2)(-2-\lambda) - (-5)(0)) + 6((-2)(0) - (-4-\lambda)(0))$$

$$|A - \lambda I| = (-\lambda)(-4-\lambda)(-2-\lambda) + 4(-2-\lambda)$$

$$|A - \lambda I| = (-2-\lambda)[(-\lambda)(-4-\lambda) + 4]$$

$$|A - \lambda I| = (-2 - \lambda)(\lambda^2 + 4\lambda + 4)$$

$$|A - \lambda I| = (-2 - \lambda)(\lambda + 2)^2$$

$$|A - \lambda I| = -(2 + \lambda)^3$$

Solve the characteristic equation for the Eigenvalues.

$$-(2 + \lambda)^3 = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = -2$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = -2$, we find

$$A - (-2)I = \begin{bmatrix} 2 & 2 & 6 \\ -2 & -4+2 & -5 \\ 0 & 0 & -2+2 \end{bmatrix}$$

$$A - (-2)I = \begin{bmatrix} 2 & 2 & 6 \\ -2 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 3 \\ -2 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Turning the matrix into a system of equations gives

$$k_1 + k_2 = 0$$

$$k_1 = -k_2$$

and

$$k_3 = 0$$

We can choose $k_2 = -1$, which gives $k_1 = 1$ and therefore the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-2t}$$

Because we only find one linearly independent Eigenvector for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -2$, we have to use $\vec{k_1} = (1, -1, 0)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p} = \vec{k_1}$$

$$\begin{bmatrix} 2 & 2 & 6 \\ -2 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$2p_1 + 2p_2 + 6p_3 = 1$$

$$-2p_1 - 2p_2 - 5p_3 = -1$$

We can choose any value for p_3, so we'll choose $p_3 = 0$ to simplify the system to just $2p_1 + 2p_2 = 1$. We'll choose $p_1 = 1$ and $p_2 = -1/2$ to get

$$\vec{p} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Then our second solution will be

$$\vec{x_2} = \vec{k_1} te^{\lambda_1 t} + \vec{p} e^{\lambda_1 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} e^{-2t}$$

Now we'll use $\vec{k_1} = (1, -1, 0)$ and $\vec{p} = (1, -1/2, 0)$ to find a third solution.

$$(A - \lambda_1 I)\vec{q} = \vec{p}$$

$$\begin{bmatrix} 2 & 2 & 6 \\ -2 & -2 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$2q_1 + 2q_2 + 6q_3 = 1$$

$$-2q_1 - 2q_2 - 5q_3 = -\frac{1}{2}$$

We can choose any value for q_3, so we'll choose $q_3 = 1/2$ to simplify the system to just $q_1 + q_2 = -1$. Then we'll choose $q_1 = -1$ and $q_2 = 0$ to get the Eigenvector

$$\vec{q} = \begin{bmatrix} -1 \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

Then the solution vector will be

$$\vec{x_3} = \vec{k_1}\frac{t^2}{2}e^{\lambda_1 t} + \vec{p}te^{\lambda_1 t} + \vec{q}e^{\lambda_1 t}$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \frac{t^2}{2}e^{-2t} + \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} -1 \\ 0 \\ \frac{1}{2} \end{bmatrix} e^{-2t}$$

and the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{-2t} + c_2\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} e^{-2t} \right)$$

$$+c_3\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \frac{t^2}{2}e^{-2t} + \begin{bmatrix} 1 \\ -\frac{1}{2} \\ 0 \end{bmatrix} te^{-2t} + \begin{bmatrix} -1 \\ 0 \\ \frac{1}{2} \end{bmatrix} e^{-2t} \right)$$

■ 5. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} -3 & 2 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & -1 \end{bmatrix} \vec{x}$$

Solution:

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} -3 & 2 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -3 & 2 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -3-\lambda & 2 & 0 \\ -2 & 1-\lambda & 0 \\ -2 & 2 & -1-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} -3-\lambda & 2 & 0 \\ -2 & 1-\lambda & 0 \\ -2 & 2 & -1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (-3-\lambda)\begin{vmatrix} 1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} - 2\begin{vmatrix} -2 & 0 \\ -2 & -1-\lambda \end{vmatrix} + 0\begin{vmatrix} -2 & 1-\lambda \\ -2 & 2 \end{vmatrix}$$

$$|A - \lambda I| = (-3-\lambda)((1-\lambda)(-1-\lambda) - (2)(0))$$

$$-2((-2)(-1-\lambda) - (2)(0)) + 0((-2)(2) - (1-\lambda)(-2))$$

$$|A - \lambda I| = (-3 - \lambda)(1 - \lambda)(-1 - \lambda) + 4(-1 - \lambda)$$

$$|A - \lambda I| = -(1 + \lambda)^3$$

Solve the characteristic equation for the Eigenvalues.

$$-(1 + \lambda)(1 + \lambda)(1 + \lambda) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = -1$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = -1$, we find

$$A - (-1)I = \begin{bmatrix} -3+1 & 2 & 0 \\ -2 & 1+1 & 0 \\ -2 & 2 & -1+1 \end{bmatrix}$$

$$A - (-1)I = \begin{bmatrix} -2 & 2 & 0 \\ -2 & 2 & 0 \\ -2 & 2 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 2 & 0 \\ -2 & 2 & 0 \\ -2 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can rewrite the matrix as the equation

$$k_1 - k_2 = 0$$

$$k_1 = k_2$$

We can choose any value for k_3, so we'll pick $k_3 = 0$ and $k_2 = 1$ to get $k_1 = 1$ and the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

But we could also choose $k_3 = 1$ and $k_2 = 1$ to get $k_1 = 1$ and the Eigenvector

$$\vec{k_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

The Eigenvectors give us two solution vectors,

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

$$\vec{x_2} = \vec{k_2} e^{\lambda_2 t} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t}$$

Because we only find two linearly independent Eigenvectors for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -1$, we have to use $\vec{k_1} = (1,1,0)$ and $\vec{k_2} = (1,1,1)$ to find a third solution.

$$(A - \lambda_1 I)\vec{p_1} = a_1 \vec{k_1} + a_2 \vec{k_2}$$

$$\begin{bmatrix} -2 & 2 & 0 \\ -2 & 2 & 0 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 2 & 0 \\ -2 & 2 & 0 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 \\ a_1 + a_2 \\ a_2 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$-2p_1 + 2p_2 = a_1 + a_2$$

$$-2p_1 + 2p_2 = a_2$$

So, $a_1 + a_2 = a_2$, or $a_1 = 0$. The system simplifies to just $-2p_1 + 2p_2 = a_2$, and we can choose $p_1 = 1$, $p_2 = 0$, and $p_3 = 0$, and we get $a_2 = -2$.

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and

$$a_1\vec{k_1} + a_2\vec{k_2} = 0\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$a_1\vec{k_1} + a_2\vec{k_2} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix}$$

Then the solution vector will be

$$\vec{x_3} = (a_1\vec{k_1} + a_2\vec{k_2})te^{\lambda_3 t} + \vec{p}e^{\lambda_3 t}$$

$$\vec{x_3} = \begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}$$

and the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{-t} + c_3\left(\begin{bmatrix} -2 \\ -2 \\ -2 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}\right)$$

■ 6. Find the general solution to the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

Solution:

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 2-\lambda \end{vmatrix} + 0\begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (2 - \lambda)((2 - \lambda)(2 - \lambda) - (1)(0))$$

$$-((0)(2 - \lambda) - (1)(0)) + 0((0)(0) - (2 - \lambda)(0))$$

$$|A - \lambda I| = (2 - \lambda)(2 - \lambda)(2 - \lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(2 - \lambda)(2 - \lambda)(2 - \lambda) = 0$$

$$\lambda_1 = \lambda_2 = \lambda_3 = 2$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} 2-2 & 1 & 0 \\ 0 & 2-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We get the system of equations

$$k_2 = 0$$

$$k_3 = 0$$

We can choose any value for k_1, so we'll pick $k_1 = 1$ and we'll get the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$$

Because we only find one linearly independent Eigenvector for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$, we have to use $\vec{k_1} = (1,0,0)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p} = \vec{k_1}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$p_2 = 1$$

$$p_3 = 0$$

We can choose any value for p_1, so we'll pick $p_1 = 0$ to get the Eigenvector

$$\vec{p} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and the solution vector

$$\vec{x_2} = \vec{k_1}te^{\lambda_1 t} + \vec{p}e^{\lambda_1 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Now we'll use $\overrightarrow{k_1} = (1,0,0)$ and $\overrightarrow{p} = (0,1,0)$ to find a third solution.

$$(A - \lambda_1 I)\overrightarrow{q} = \overrightarrow{p}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$q_2 = 0$$

$$q_3 = 1$$

We can choose any value for q_1, so we'll pick $q_1 = 0$ to get the Eigenvector

$$\overrightarrow{q} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and the solution vector

$$\overrightarrow{x_3} = \overrightarrow{k_1}\frac{t^2}{2}e^{\lambda_1 t} + \overrightarrow{p}te^{\lambda_1 t} + \overrightarrow{q}e^{\lambda_1 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2}e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Then the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right)$$

$$+c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} \right)$$

COMPLEX EIGENVALUES

■ 1. Find the general solution to the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} \vec{x}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} 2-\lambda & 4 \\ -2 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (4)(-2)$$

$$\begin{vmatrix} 2-\lambda & 4 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 - 4 + 8$$

$$\begin{vmatrix} 2-\lambda & 4 \\ -2 & -2-\lambda \end{vmatrix} = \lambda^2 + 4$$

Solve the characteristic equation for the Eigenvalues.

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

Then for these complex conjugate Eigenvalues, $\lambda_1 = 2i$ and $\lambda_2 = -2i$, we find

$$A - (2i)I = \begin{bmatrix} 2-2i & 4 \\ -2 & -2-2i \end{bmatrix} \qquad A - (-2i)I = \begin{bmatrix} 2+2i & 4 \\ -2 & -2+2i \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} 2-2i & 4 \\ -2 & -2-2i \end{bmatrix} \qquad \begin{bmatrix} 2+2i & 4 \\ -2 & -2+2i \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{1-i} \\ -2 & -2-2i \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{2}{1+i} \\ -2 & -2+2i \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{1-i} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{2}{1+i} \\ 0 & 0 \end{bmatrix}$$

If we turn these matrices back into systems of equations, we get

$$k_1 + \left(\frac{2}{1-i}\right) k_2 = 0 \qquad k_1 + \left(\frac{2}{1+i}\right) k_2 = 0$$

$$k_1 = \left(-\frac{2}{1-i}\right) k_2 \qquad k_1 = \left(-\frac{2}{1+i}\right) k_2$$

Choosing $k_2 = i - 1$ gives $k_1 = 2$ in the first system, and choosing $k_2 = -(1+i)$ gives $k_1 = 2$ in the second system, which gives the Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 2 \\ i-1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 2 \\ -i-1 \end{bmatrix}$$

and the solution vectors

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 2 \\ i-1 \end{bmatrix} e^{(2i)t} \qquad \vec{x_2} = \begin{bmatrix} 2 \\ -i-1 \end{bmatrix} e^{(-2i)t}$$

Then the general solution will be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ i-1 \end{bmatrix} e^{(2i)t} + c_2 \begin{bmatrix} 2 \\ -i-1 \end{bmatrix} e^{(-2i)t}$$

■ 2. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} -6 & 4 \\ -8 & 2 \end{bmatrix} \vec{x}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} -6-\lambda & 4 \\ -8 & 2-\lambda \end{vmatrix} = (-6-\lambda)(2-\lambda) - (4)(-8)$$

$$\begin{vmatrix} -6-\lambda & 4 \\ -8 & 2-\lambda \end{vmatrix} = -12 + 6\lambda - 2\lambda + \lambda^2 + 32$$

$$\begin{vmatrix} -6-\lambda & 4 \\ -8 & 2-\lambda \end{vmatrix} = \lambda^2 + 4\lambda + 20$$

Solve the characteristic equation for the Eigenvalues.

$$\lambda^2 + 4\lambda + 20 = 0$$

$$(\lambda + 2)^2 + 16 = 0$$

$$\lambda + 2 = \pm 4i$$

$$\lambda = -2 \pm 4i$$

Then for these complex conjugate Eigenvalues, $\lambda_1 = -2 + 4i$ and $\lambda_2 = -2 - 4i$, we find

$$A - (-2+4i)I = \begin{bmatrix} -6-(-2+4i) & 4 \\ -8 & 2-(-2+4i) \end{bmatrix}$$

$$A - (-2+4i)I = \begin{bmatrix} -4-4i & 4 \\ -8 & 4-4i \end{bmatrix}$$

and

$$A - (-2-4i)I = \begin{bmatrix} -6-(-2-4i) & 4 \\ -8 & 2-(-2-4i) \end{bmatrix}$$

$$A - (-2-4i)I = \begin{bmatrix} -4+4i & 4 \\ -8 & 4+4i \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} -4-4i & 4 \\ -8 & 4-4i \end{bmatrix} \qquad \begin{bmatrix} -4+4i & 4 \\ -8 & 4+4i \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{-1-i} \\ -8 & 4-4i \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{-1+i} \\ -8 & 4+4i \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{-1-i} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{-1+i} \\ 0 & 0 \end{bmatrix}$$

If we turn these matrices back into systems of equations, we get

$$k_1 + \left(\frac{1}{-1-i}\right) k_2 = 0 \qquad\qquad k_1 + \left(\frac{1}{-1+i}\right) k_2 = 0$$

$$k_1 = \left(\frac{1}{1+i}\right) k_2 \qquad\qquad k_1 = \left(\frac{1}{1-i}\right) k_2$$

Choosing $k_2 = 1 + i$ gives $k_1 = 1$ in the first system, and choosing $k_2 = 1 - i$ gives $k_1 = 1$ in the second system, which gives the Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} \qquad\qquad \vec{k_2} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$

and the solution vectors

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad\qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(2+4i)t} \qquad\qquad \vec{x_2} = \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(-2-4i)t}$$

Then the general solution will be

$$\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(-2+4i)t} + c_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(-2-4i)t}$$

■ 3. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 5 & 5 \end{bmatrix} \vec{x}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} 1-\lambda & -1 \\ 5 & 5-\lambda \end{vmatrix} = (1-\lambda)(5-\lambda) - (-1)(5)$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 5 & 5-\lambda \end{vmatrix} = 5 - \lambda - 5\lambda + \lambda^2 + 5$$

$$\begin{vmatrix} 1-\lambda & -1 \\ 5 & 5-\lambda \end{vmatrix} = \lambda^2 - 6\lambda + 10$$

Solve the characteristic equation for the Eigenvalues.

$$\lambda^2 - 6\lambda + 10 = 0$$

$$(\lambda - 3)^2 + 1 = 0$$

$$\lambda = 3 \pm i$$

Then for the complex conjugate Eigenvalues, $\lambda = 3 \pm i$, we find

$$A - (3+i)I = \begin{bmatrix} 1-(3+i) & -1 \\ 5 & 5-(3+i) \end{bmatrix}$$

$$A - (3+i)I = \begin{bmatrix} -2-i & -1 \\ 5 & 2-i \end{bmatrix}$$

and

$$A - (3 - i)I = \begin{bmatrix} 1 - (3 - i) & -1 \\ 5 & 5 - (3 - i) \end{bmatrix}$$

$$A - (3 - i)I = \begin{bmatrix} -2 + i & -1 \\ 5 & 2 + i \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} -2 - i & -1 \\ 5 & 2 - i \end{bmatrix} \qquad \begin{bmatrix} -2 + i & -1 \\ 5 & 2 + i \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2 + i} \\ 5 & 2 - i \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2 - i} \\ 5 & 2 + i \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2 + i} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & \frac{1}{2 - i} \\ 0 & 0 \end{bmatrix}$$

If we turn these matrices back into systems of equations, we get

$$k_1 + \left(\frac{1}{2 + i}\right) k_2 = 0 \qquad k_1 + \left(\frac{1}{2 - i}\right) k_2 = 0$$

$$k_1 = -\left(\frac{1}{2 + i}\right) k_2 \qquad k_1 = -\left(\frac{1}{2 - i}\right) k_2$$

Choosing $k_2 = -(2 + i)$ gives $k_1 = 1$ in the first system, and choosing $k_2 = -(2 - i)$ gives $k_1 = 1$ in the second system, which gives the Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 1 \\ -(2 + i) \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 1 \\ -(2 - i) \end{bmatrix}$$

and the solution vectors

$$\vec{x}_1 = \vec{k}_1 e^{\lambda_1 t} \qquad \vec{x}_2 = \vec{k}_2 e^{\lambda_2 t}$$

$$\vec{x}_1 = \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) e^{3t} \qquad \vec{x}_2 = \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \sin t \right) e^{3t}$$

Therefore, the general solution will be

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2$$

$$\vec{x} = c_1 \left(\begin{bmatrix} 1 \\ -2 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t \right) e^{3t}$$

$$+c_2 \left(\begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ -2 \end{bmatrix} \sin t \right) e^{3t}$$

■ 4. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 6 & -10 \\ 5 & -4 \end{bmatrix} \vec{x}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} 6 - \lambda & -10 \\ 5 & -4 - \lambda \end{vmatrix} = (6 - \lambda)(-4 - \lambda) - (5)(-10)$$

$$\begin{vmatrix} 6-\lambda & -10 \\ 5 & -4-\lambda \end{vmatrix} = -24 - 2\lambda + \lambda^2 + 50$$

$$\begin{vmatrix} 6-\lambda & -10 \\ 5 & -4-\lambda \end{vmatrix} = (\lambda - 1)^2 + 25$$

Solve the characteristic equation for the Eigenvalues.

$$(\lambda - 1)^2 + 25 = 0$$

$$\lambda = 1 \pm 5i$$

Then for these complex conjugate Eigenvalues, $\lambda_1 = 1 + 5i$ and $\lambda_2 = 1 - 5i$, we find

$$A - (1+5i)I = \begin{bmatrix} 6-(1+5i) & -10 \\ 5 & -4-(1+5i) \end{bmatrix}$$

$$A - (1+5i)I = \begin{bmatrix} 5-5i & -10 \\ 5 & -5-5i \end{bmatrix}$$

and

$$A - (1-5i)I = \begin{bmatrix} 6-(1-5i) & -10 \\ 5 & -4-(1-5i) \end{bmatrix}$$

$$A - (1-5i)I = \begin{bmatrix} 5+5i & -10 \\ 5 & -5+5i \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} 5-5i & -10 \\ 5 & -5-5i \end{bmatrix} \qquad \begin{bmatrix} 5+5i & -10 \\ 5 & -5+5i \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{2}{1-i} \\ 5 & -5-5i \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{2}{1+i} \\ 5 & -5+5i \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{2}{1-i} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{2}{1+i} \\ 0 & 0 \end{bmatrix}$$

If we turn matrices back into systems of equations, we get

$$k_1 + \left(-\frac{2}{1-i}\right)k_2 = 0 \qquad k_1 + \left(-\frac{2}{1+i}\right)k_2 = 0$$

$$k_1 = \left(\frac{2}{1-i}\right)k_2 \qquad k_1 = \left(\frac{2}{1+i}\right)k_2$$

Choosing $k_2 = 1 - i$ gives $k_1 = 2$ in the first system, and choosing $k_2 = 1 + i$ gives $k_1 = 2$ in the second system, which gives the Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix}$$

and the solution vectors

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1-i \end{bmatrix} e^{(1+5i)t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 2 \\ 1+i \end{bmatrix} e^{(1-5i)t}$$

Then the general solution will be

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 1-i \end{bmatrix} e^{(1+5i)t} + c_2 \begin{bmatrix} 2 \\ 1+i \end{bmatrix} e^{(1-5i)t}$$

■ 5. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 5 \\ 2 & -1 & 4 \end{bmatrix} \vec{x}$$

Solution:

The determinant $|A - \lambda I|$ is

$$|A - \lambda I| = \begin{vmatrix} -2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 5 \\ 2 & -1 & 4-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (-2-\lambda)\begin{vmatrix} 2-\lambda & 5 \\ -1 & 4-\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & 5 \\ 2 & 4-\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 2-\lambda \\ 2 & -1 \end{vmatrix}$$

$$|A - \lambda I| = (-2-\lambda)((2-\lambda)(4-\lambda)+5)$$

$$|A - \lambda I| = (-2-\lambda)((\lambda-3)^2+4)$$

Solve the characteristic equation for the Eigenvalues.

$$(-2-\lambda)((\lambda-3)^2+4) = 0$$

$$\lambda_1 = -2 \qquad\qquad \lambda_{2,3} = 3 \pm 2i$$

Then for $\lambda_1 = -2$, we find

$$A - (-2)I = \begin{bmatrix} -2+2 & 0 & 0 \\ 1 & 2+2 & 5 \\ 2 & -1 & 4+2 \end{bmatrix}$$

$$A - (-2)I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 4 & 5 \\ 2 & -1 & 6 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 4 & 5 \\ 2 & -1 & 6 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 \\ 0 & -9 & -4 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 \\ 0 & 1 & \frac{4}{9} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{29}{9} \\ 0 & 1 & \frac{4}{9} \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix back into a system of equations, we get

$$k_1 + \frac{29}{9}k_3 = 0$$

$$k_1 = -\frac{29}{9}k_3$$

and

$$k_2 + \frac{4}{9}k_3 = 0$$

$$k_2 = -\frac{4}{9}k_3$$

If we choose $k_3 = -9$, we find $k_1 = 29$ and $k_2 = 4$, and therefore we find the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 29 \\ 4 \\ -9 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 29 \\ 4 \\ -9 \end{bmatrix} e^{-2t}$$

Then for the complex Eigenvalue $\lambda_2 = 3 + 2i$, we find

$$A - (3+2i)I = \begin{bmatrix} -2-(3+2i) & 0 & 0 \\ 1 & 2-(3+2i) & 5 \\ 2 & -1 & 4-(3+2i) \end{bmatrix}$$

$$A - (3+2i)I = \begin{bmatrix} -5-2i & 0 & 0 \\ 1 & -1-2i & 5 \\ 2 & -1 & 1-2i \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & -1-2i & 5 \\ 2 & -1 & 1-2i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1-2i & 5 \\ 2 & -1 & 1-2i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1-2i & 5 \\ 0 & -1 & 1-2i \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1-2i \\ 0 & -1-2i & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1+2i \\ 0 & -1-2i & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1+2i \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix back into a system of equations, we get

$$k_1 = 0$$

and

$$k_2 + (-1 + 2i)k_3 = 0$$

$$k_2 = (1 - 2i)k_3$$

Choosing $k_3 = 1$ gives $k_2 = 1 - 2i$, which gives the Eigenvector

$$\vec{k_2} = \begin{bmatrix} 0 \\ 1 - 2i \\ 1 \end{bmatrix}$$

and the solution vectors

$$\vec{x_2} = \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \sin(2t) \right) e^{3t}$$

$$\vec{x_3} = \left(\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \cos(2t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \sin(2t) \right) e^{3t}$$

Therefore, the general solution will be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 29 \\ 4 \\ -9 \end{bmatrix} e^{-2t} + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \cos(2t) - \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \sin(2t) \right) e^{3t}$$

$$+c_3\left(\begin{bmatrix}0\\-2\\0\end{bmatrix}\cos(2t)+\begin{bmatrix}0\\1\\1\end{bmatrix}\sin(2t)\right)e^{3t}$$

■ 6. Find the general solution to the system of differential equations.

$$\vec{x}\,'=\begin{bmatrix}1&2&3\\0&-1&2\\0&-5&1\end{bmatrix}\vec{x}$$

Solution:

The determinant $|A-\lambda I|$ is

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&2&3\\0&-1-\lambda&2\\0&-5&1-\lambda\end{vmatrix}$$

$$|A-\lambda I|=(1-\lambda)\begin{vmatrix}-1-\lambda&2\\-5&1-\lambda\end{vmatrix}-2\begin{vmatrix}0&2\\0&1-\lambda\end{vmatrix}+3\begin{vmatrix}0&-1-\lambda\\0&-5\end{vmatrix}$$

$$|A-\lambda I|=(1-\lambda)((-1-\lambda)(1-\lambda)+10)$$

$$|A-\lambda I|=(1-\lambda)(\lambda^2+9)$$

Solve the characteristic equation for the Eigenvalues.

$$\lambda_1=1 \qquad\qquad \lambda_{2,3}=\pm 3i$$

Then for $\lambda_1=1$, we find

$$A - (1)I = \begin{bmatrix} 1-1 & 2 & 3 \\ 0 & -1-1 & 2 \\ 0 & -5 & 1-1 \end{bmatrix}$$

$$A - (1)I = \begin{bmatrix} 0 & 2 & 3 \\ 0 & -2 & 2 \\ 0 & -5 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix back into a systems of equations, we get

$$k_2 = 0$$

$$k_3 = 0$$

If we choose $k_1 = 1$, we get the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$$

Then for the complex Eigenvalue $\lambda_2 = 3i$, we find

$$A-(3i)I=\begin{bmatrix}1-3i & 2 & 3\\ 0 & -1-3i & 2\\ 0 & -5 & 1-3i\end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix}1 & \frac{2}{1-3i} & \frac{3}{1-3i}\\ 0 & -1-3i & 2\\ 0 & -5 & 1-3i\end{bmatrix}\to\begin{bmatrix}1 & \frac{1+3i}{5} & \frac{3(1+3i)}{10}\\ 0 & -1-3i & 2\\ 0 & -5 & 1-3i\end{bmatrix}\to\begin{bmatrix}1 & \frac{1+3i}{5} & \frac{3(1+3i)}{10}\\ 0 & -5 & 1-3i\\ 0 & -1-3i & 2\end{bmatrix}$$

$$\to\begin{bmatrix}1 & \frac{1+3i}{5} & \frac{3(1+3i)}{10}\\ 0 & 1 & \frac{-1+3i}{5}\\ 0 & -1-3i & 2\end{bmatrix}\to\begin{bmatrix}1 & \frac{1+3i}{5} & \frac{3(1+3i)}{10}\\ 0 & 1 & \frac{-1+3i}{5}\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & \frac{7+9i}{10}\\ 0 & 1 & \frac{-1+3i}{5}\\ 0 & 0 & 0\end{bmatrix}$$

If we turn this matrix back into a system of equations, we get

$$k_1+\left(\frac{7+9i}{10}\right)k_3=0$$

$$k_1=-\left(\frac{7+9i}{10}\right)k_3$$

and

$$k_2+\left(\frac{-1+3i}{5}\right)k_3=0$$

$$k_2=\left(\frac{1-3i}{5}\right)k_3$$

Choosing $k_3 = 10$ gives $k_1 = -7 - 9i$ and $k_2 = 2 - 6i$, which gives us the Eigenvector

$$\vec{k_2} = \begin{bmatrix} -7 - 9i \\ 2 - 6i \\ 10 \end{bmatrix}$$

and the solution vector

$$\vec{x_2} = \begin{bmatrix} -7 \\ 2 \\ 10 \end{bmatrix} \cos(3t) - \begin{bmatrix} -9 \\ -6 \\ 0 \end{bmatrix} \sin(3t)$$

$$\vec{x_3} = \begin{bmatrix} -9 \\ -6 \\ 0 \end{bmatrix} \cos(3t) + \begin{bmatrix} -7 \\ 2 \\ 10 \end{bmatrix} \sin(3t)$$

Therefore, the general solution will be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} -7 \\ 2 \\ 10 \end{bmatrix} \cos(3t) - \begin{bmatrix} -9 \\ -6 \\ 0 \end{bmatrix} \sin(3t) \right)$$

$$+c_3 \left(\begin{bmatrix} -9 \\ -6 \\ 0 \end{bmatrix} \cos(3t) + \begin{bmatrix} -7 \\ 2 \\ 10 \end{bmatrix} \sin(3t) \right)$$

PHASE PORTRAITS FOR DISTINCT REAL EIGENVALUES

■ 1. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -4 & 4 \\ -2 & 5 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -4-\lambda & 4 \\ -2 & 5-\lambda \end{bmatrix}$$

and the characteristic equation

$$(-4-\lambda)(5-\lambda) - (4)(-2) = 0$$

$$\lambda^2 - \lambda - 12 = 0$$

$$(\lambda - 4)(\lambda + 3) = 0$$

$$\lambda = -3, 4$$

gives the Eigenvalues $\lambda_1 = -3$ and $\lambda_2 = 4$, and their associated Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (4,1)$ lies along the line $y = (1/4)x$, and the Eigenvector $\overrightarrow{k_2} = (1,2)$ lies along the line $y = 2x$, so we'll sketch these lines to represent the Eigenvectors in our phase portrait.

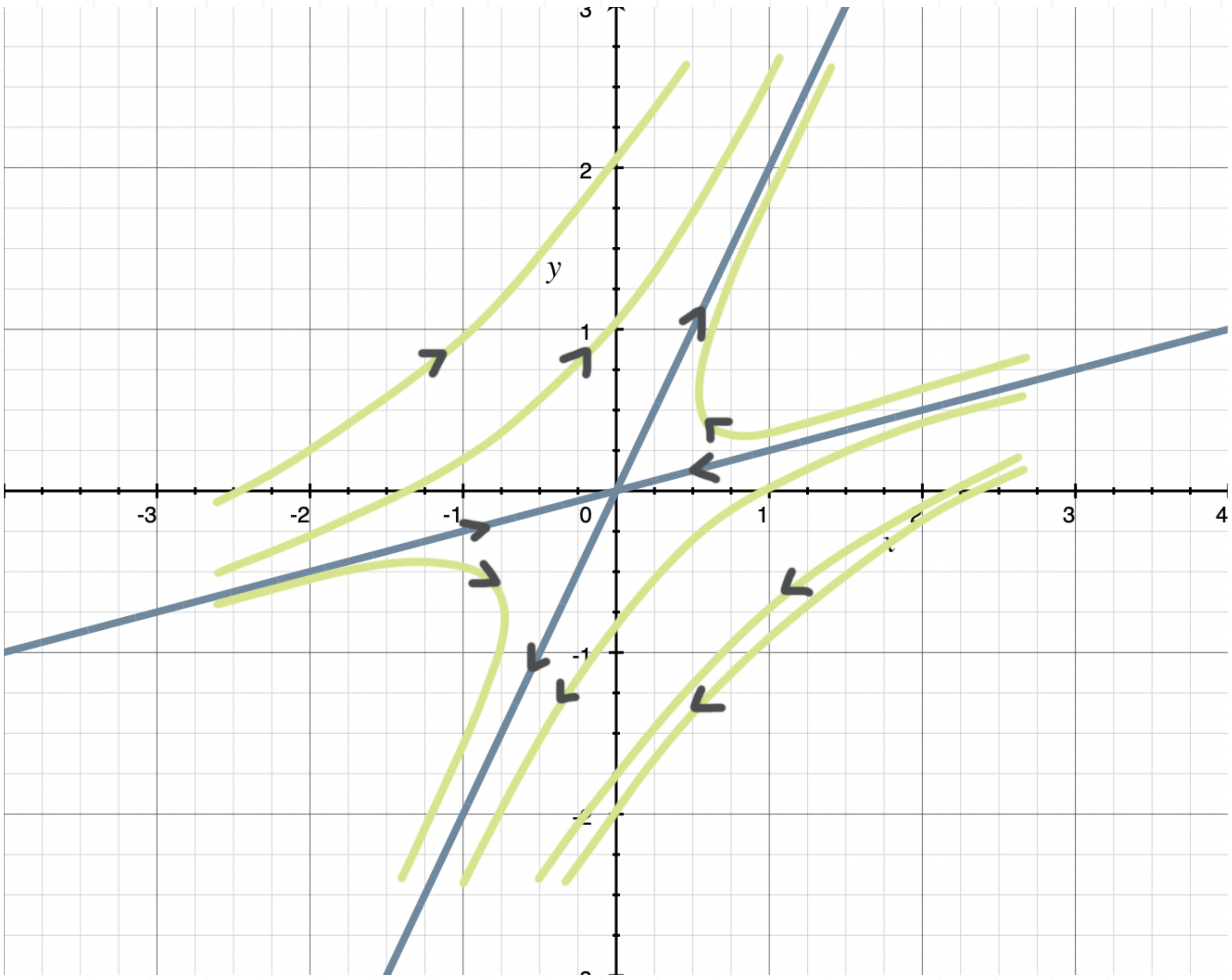
The Eigenvalue associated with $\overrightarrow{k_1} = (4,1)$ is $\lambda = -3$, which means the direction along that trajectory is toward the origin. The Eigenvalue associated with $\overrightarrow{k_2} = (1,2)$ is $\lambda_2 = 4$, which means the direction along that trajectory is away from the origin.

Because the Eigenvalues have opposite signs, we're dealing with an unstable saddle point that attracts trajectories in some places and repels them in others.

To apply the $t \to \pm\infty$ test, we'll use the general solution to the system,

$$\vec{x} = c_1 \begin{bmatrix} 4 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{4t}$$

Because e^{-3t} dominates e^{4t} as $t \to -\infty$, the $\overrightarrow{k_2} = (1,2)$ vector drops away first, meaning that our trajectories are going to "start" parallel to $\overrightarrow{k_1} = (4,1)$. On the other end, e^{-3t} goes to 0 faster than e^{4t} as $t \to \infty$, so the $\overrightarrow{k_1} = (4,1)$ vector will drop away first, meaning that our trajectories are going to "end" parallel to $\overrightarrow{k_2} = (1,2)$.



■ 2. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 \\ 4 & -1 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(2-\lambda)(-1-\lambda)-(1)(4)=0$$

$$\lambda^2-\lambda-6=0$$

$$(\lambda-3)(\lambda+2)=0$$

$$\lambda=3,\ -2$$

gives the Eigenvalues $\lambda_1=3$ and $\lambda_2=-2$, and their associated Eigenvectors

$$\overrightarrow{k_1}=\begin{bmatrix}1\\1\end{bmatrix} \qquad \overrightarrow{k_2}=\begin{bmatrix}1\\-4\end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1}=(1,1)$ lies along the line $y=x$, and the Eigenvector $\overrightarrow{k_2}=(1,-4)$ lies along the line $y=-4x$, so we'll sketch these lines to represent the Eigenvectors in our phase portrait.

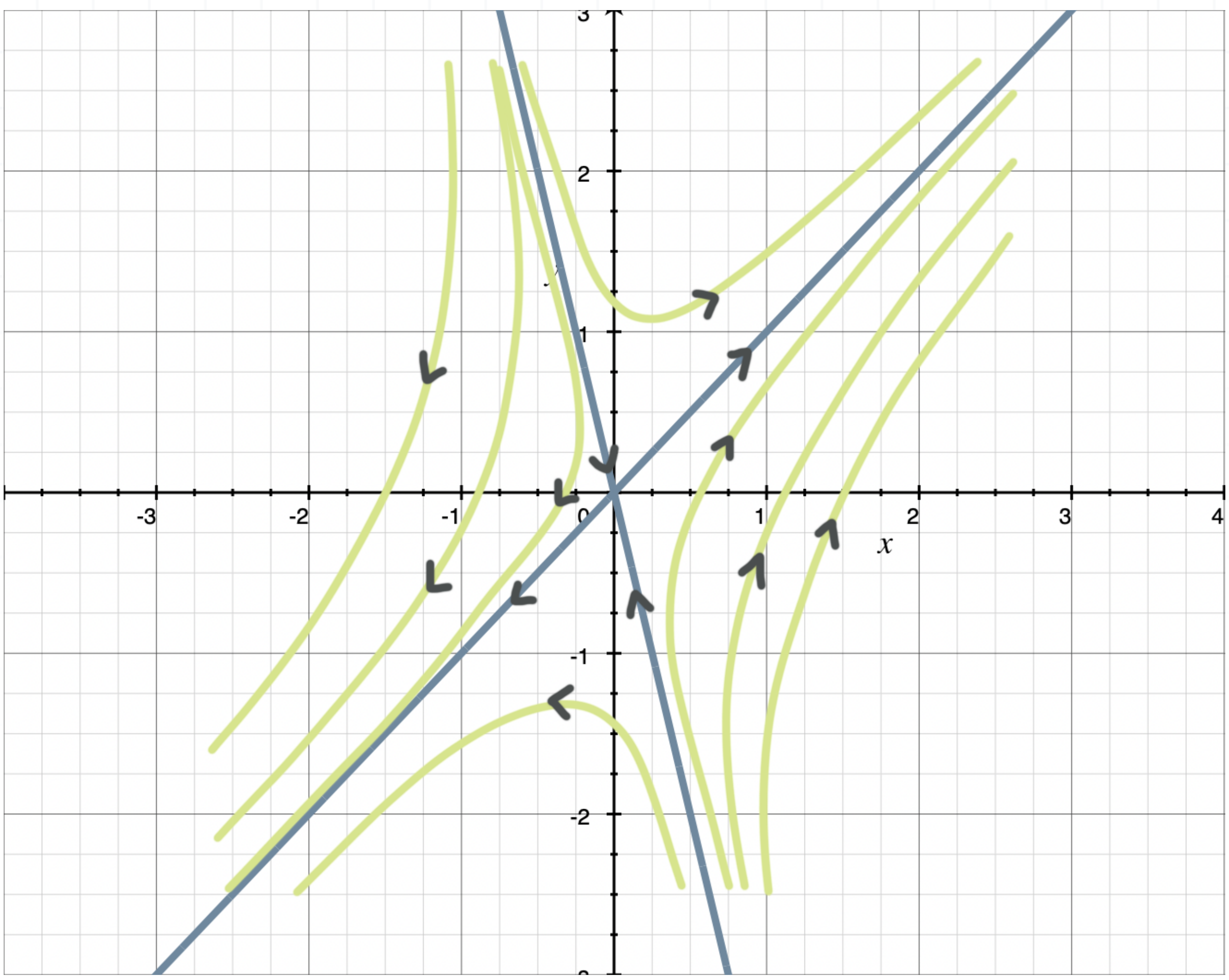
The Eigenvalue associated with $\overrightarrow{k_1}=(1,1)$ is $\lambda=3$, which means the direction along that trajectory is away from the origin. The Eigenvalue associated with $\overrightarrow{k_2}=(1,-4)$ is $\lambda_2=-2$, which means the direction along that trajectory is toward the origin.

Because the Eigenvalues have opposite signs, we're dealing with an unstable saddle point that attracts trajectories in some places and repels them in others.

To apply the $t\to\pm\infty$ test, we'll use the general solution to the system,

$$\overrightarrow{x}=c_1\begin{bmatrix}1\\1\end{bmatrix}e^{3t}+c_2\begin{bmatrix}1\\-4\end{bmatrix}e^{-2t}$$

Because e^{-2t} dominates e^{3t} as $t \to -\infty$, the $\overrightarrow{k_1} = (1,1)$ vector drops away first, meaning that our trajectories are going to "start" parallel to $\overrightarrow{k_2} = (1,-4)$. On the other end, e^{-2t} goes to 0 faster than e^{3t} as $t \to \infty$, so the $\overrightarrow{k_2} = (1,-4)$ vector will drop away first, meaning that our trajectories are going to "end" parallel to $\overrightarrow{k_1} = (1,1)$.



■ 3. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & 1 \\ 4 & 2 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(2 - \lambda)(2 - \lambda) - (1)(4) = 0$$

$$\lambda^2 - 4\lambda = 0$$

$$\lambda(\lambda - 4) = 0$$

$$\lambda = 0, 4$$

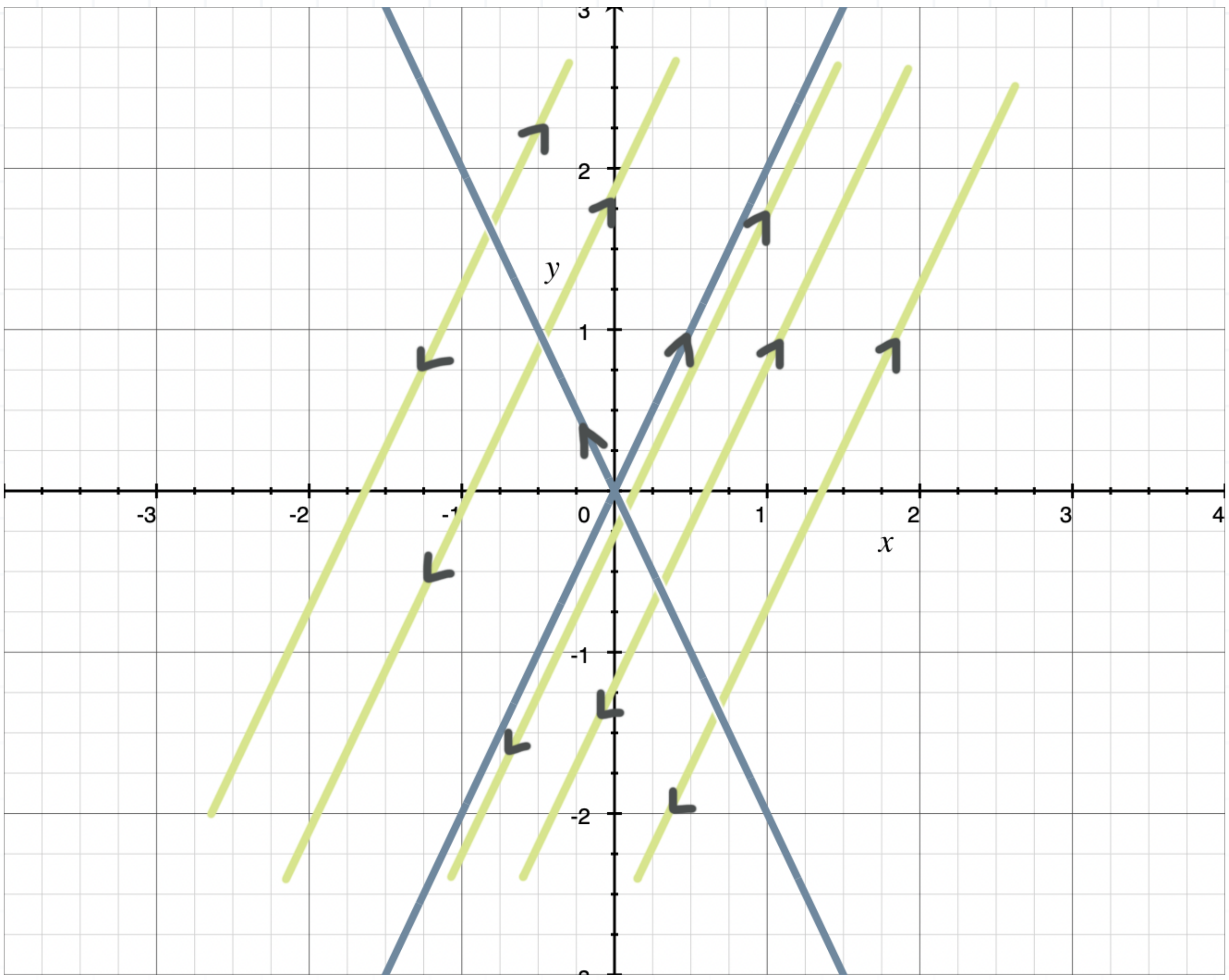
gives the Eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 4$, and their associated Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (1, -2)$ lies along the line $y = -2x$, and the Eigenvector $\overrightarrow{k_2} = (1,2)$ lies along the line $y = 2x$, so we'll sketch these lines to represent the Eigenvectors in our phase portrait.

The Eigenvalue associated with $\overrightarrow{k_1} = (1, -2)$ is $\lambda = 0$, which means equilibrium exists along the entire line $\overrightarrow{k_1} = (1, -2)$. The Eigenvalue associated with $\overrightarrow{k_2} = (1,2)$ is $\lambda_2 = 4$, which means the direction along that trajectory is away from the origin.

Because the non-zero Eigenvalue is positive, we're dealing with an unstable line of equilibrium along $\overrightarrow{k_1} = (1, -2)$ that repels all trajectories.



■ 4. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 5 - \lambda & 1 \\ 4 & 2 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(5-\lambda)(2-\lambda)-(1)(4)=0$$

$$\lambda^2-7\lambda+6=0$$

$$(\lambda-6)(\lambda-1)=0$$

$$\lambda=6,\ 1$$

gives the Eigenvalues $\lambda_1=6$ and $\lambda_2=1$, and their associated Eigenvectors

$$\vec{k_1}=\begin{bmatrix}1\\1\end{bmatrix} \qquad \vec{k_2}=\begin{bmatrix}1\\-4\end{bmatrix}$$

The Eigenvector $\vec{k_1}=(1,1)$ lies along the line $y=x$, and the Eigenvector $\vec{k_2}=(1,-4)$ lies along the line $y=-4x$, so we'll sketch these lines to represent the Eigenvectors in our phase portrait.

The Eigenvalue associated with $\vec{k_1}=(1,1)$ is $\lambda=6$, which means the direction along that trajectory is away from the origin. The Eigenvalue associated with $\vec{k_2}=(1,-4)$ is $\lambda_2=1$, which means the direction along that trajectory is also away from the origin.

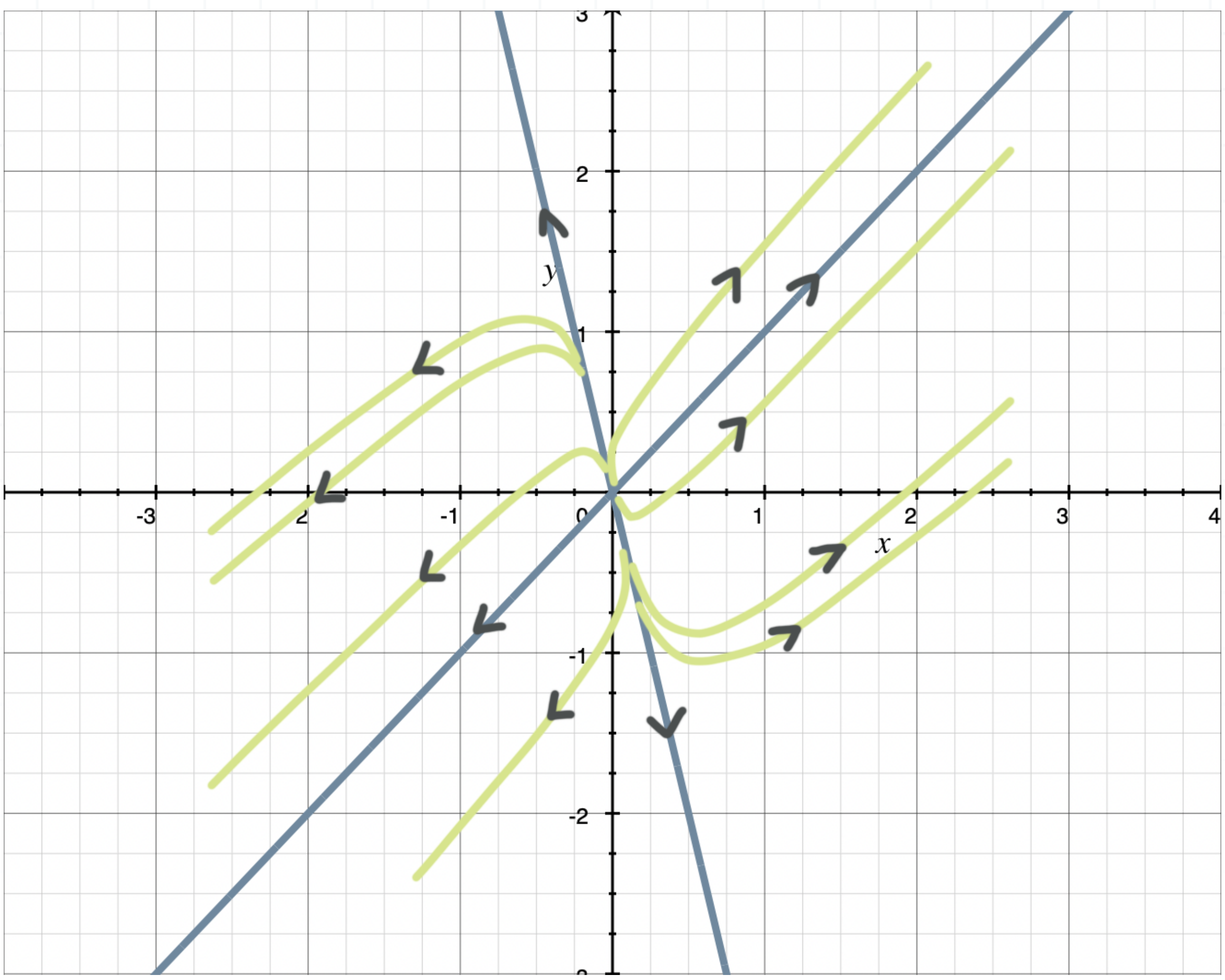
Because both Eigenvalues are positive, we're dealing with an unstable repeller node that repels all trajectories.

To apply the $t\to\pm\infty$ test, we'll use the general solution to the system,

$$\vec{x}=c_1\begin{bmatrix}1\\1\end{bmatrix}e^{6t}+c_2\begin{bmatrix}1\\-4\end{bmatrix}e^{t}$$

Because e^t dominates e^{6t} as $t\to-\infty$, the $\vec{k_1}=(1,1)$ vector drops away first, meaning that our trajectories are going to "start" parallel to $\vec{k_2}=(1,-4)$.

On the other end, e^{6t} dominates e^{t} as $t \to \infty$, so the $\overrightarrow{k_2} = (1, -4)$ vector will drop away first, meaning that our trajectories are going to "end" parallel to $\overrightarrow{k_1} = (1,1)$.



■ 5. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -2-\lambda & 4 \\ 0 & -\lambda \end{bmatrix}$$

and the characteristic equation

$$(-2-\lambda)(-\lambda) - (4)(0) = 0$$

$$\lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda + 2) = 0$$

$$\lambda = 0, \ -2$$

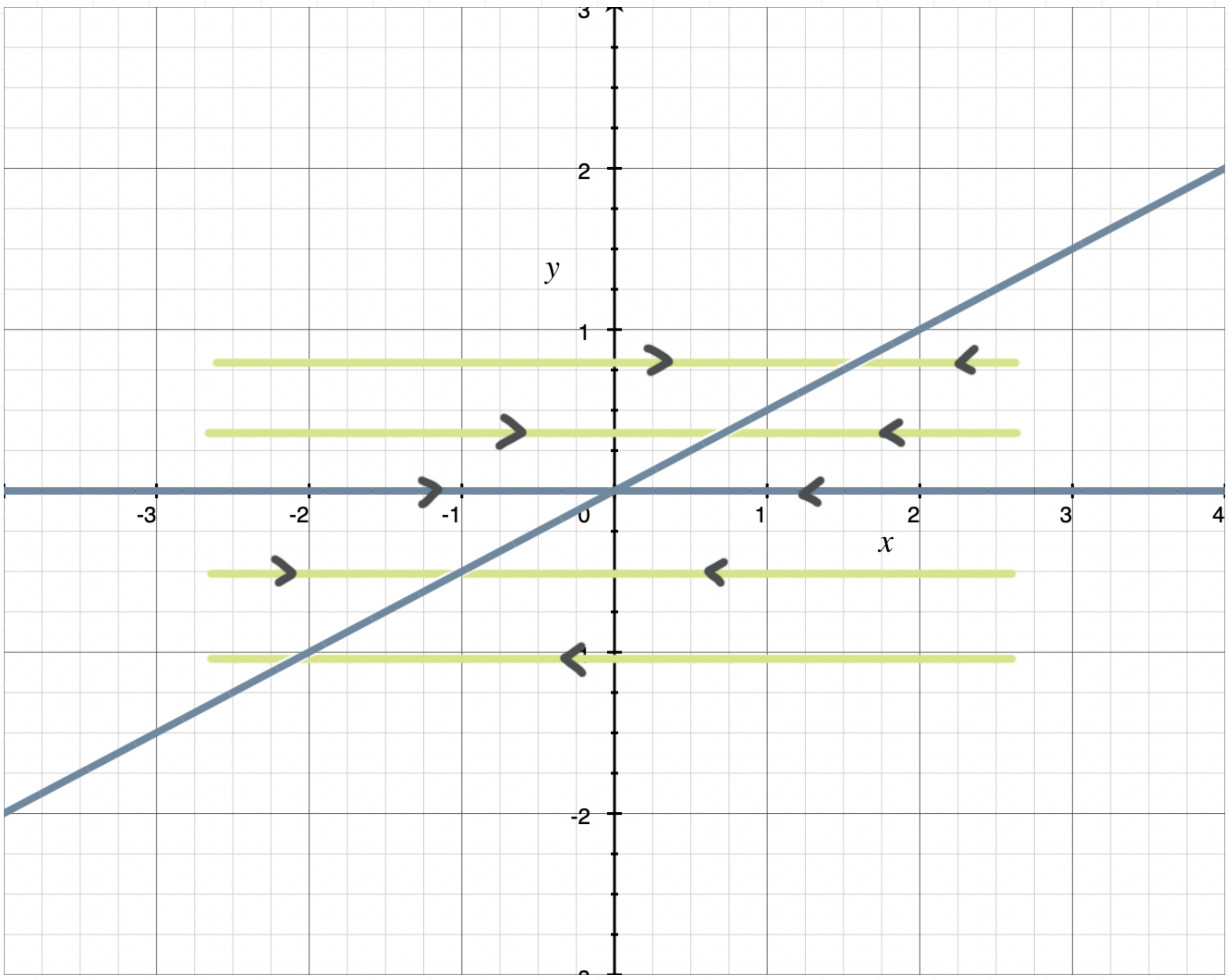
gives the Eigenvalues $\lambda_1 = 0$ and $\lambda_2 = -2$, and their associated Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (2,1)$ lies along the line $y = (1/2)x$, and the Eigenvector $\overrightarrow{k_2} = (1,0)$ lies along the line $y = 0$, so we'll sketch these lines to represent the Eigenvectors in our phase portrait.

The Eigenvalue associated with $\overrightarrow{k_1} = (2,1)$ is $\lambda = 0$, which means equilibrium will exist everywhere along the line $y = (1/2)x$. The Eigenvalue associated with $\overrightarrow{k_2} = (1,0)$ is $\lambda_2 = -2$, which means the direction along that trajectory is away from the origin.

Because the non-zero Eigenvalue is negative, we're dealing with a stable line of equilibrium that attracts all trajectories.



■ 6. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -3 & -1 \\ 0 & -4 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -3 - \lambda & -1 \\ 0 & -4 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(-3-\lambda)(-4-\lambda)-(-1)(0)=0$$

$$\lambda^2+7\lambda+12=0$$

$$(\lambda+4)(\lambda+3)=0$$

$$\lambda=-3,\ -4$$

gives the Eigenvalues $\lambda_1=-3$ and $\lambda_2=-4$, and their associated Eigenvectors

$$\vec{k_1}=\begin{bmatrix}1\\0\end{bmatrix} \qquad \vec{k_2}=\begin{bmatrix}1\\1\end{bmatrix}$$

The Eigenvector $\vec{k_1}=(1,0)$ lies along the line $y=0$, and the Eigenvector $\vec{k_2}=(1,1)$ lies along the line $y=x$, so we'll sketch these lines to represent the Eigenvectors in our phase portrait.

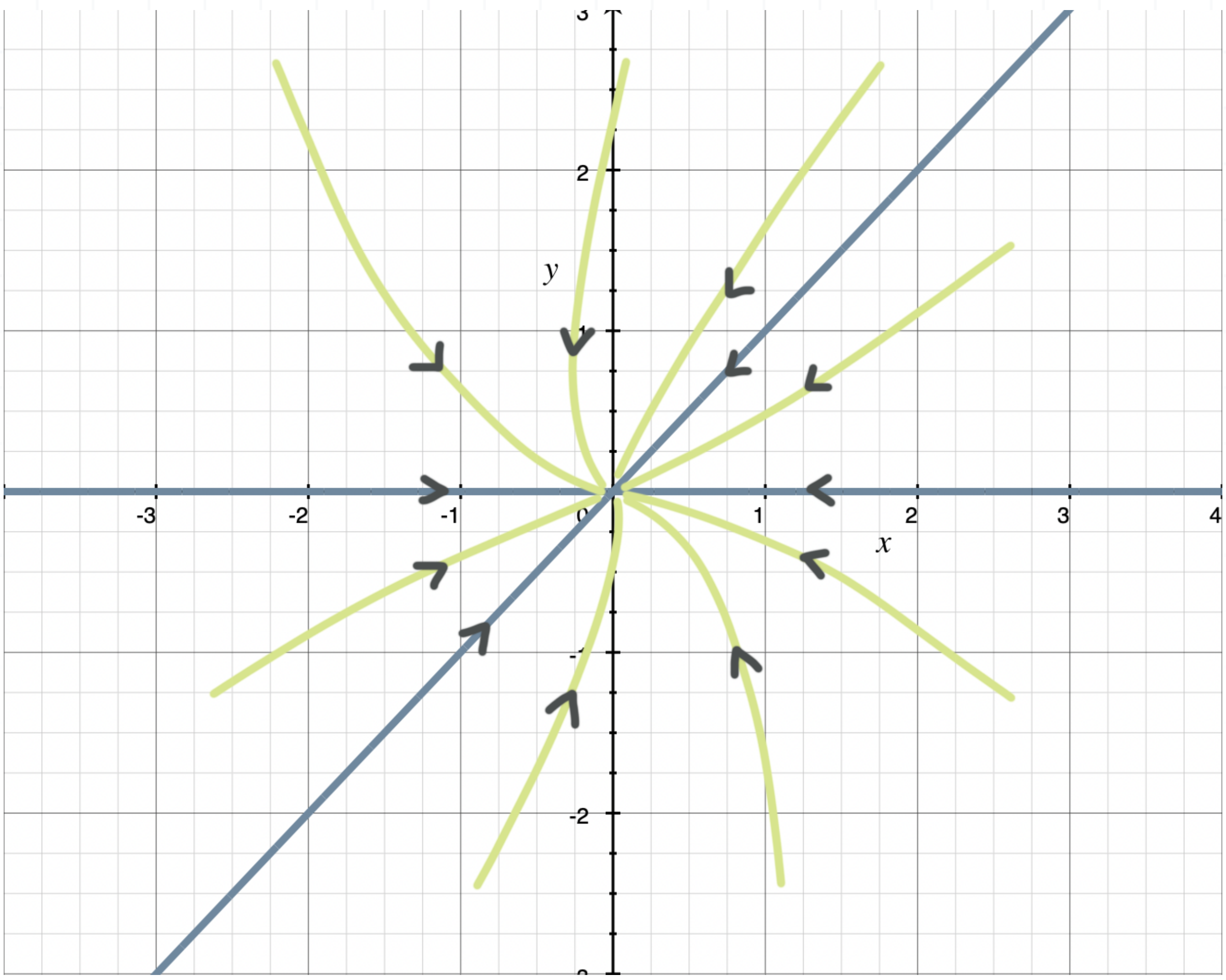
The Eigenvalue associated with $\vec{k_1}=(1,0)$ is $\lambda=-3$, which means the direction along that trajectory is toward the origin. The Eigenvalue associated with $\vec{k_2}=(1,1)$ is $\lambda_2=-4$, which means the direction along that trajectory is also toward the origin.

Because both the Eigenvalues are negative, we're dealing with an asymptotically stable node that attracts all trajectories.

To apply the $t\to\pm\infty$ test, we'll use the general solution to the system,

$$\vec{x}=c_1\begin{bmatrix}1\\0\end{bmatrix}e^{-3t}+c_2\begin{bmatrix}1\\1\end{bmatrix}e^{-4t}$$

Because e^{-4t} dominates e^{-3t} as $t \to -\infty$, the $\overrightarrow{k_1} = (1,0)$ vector drops away first, meaning that our trajectories are going to "start" parallel to $\overrightarrow{k_2} = (1,1)$. On the other end, e^{-4t} goes to 0 faster than e^{-3t} as $t \to \infty$, so the $\overrightarrow{k_2} = (1,1)$ vector will drop away first, meaning that our trajectories are going to "end" parallel to $\overrightarrow{k_1} = (1,0)$.



PHASE PORTRAITS FOR EQUAL REAL EIGENVALUES

■ 1. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 6 & 2 \\ -8 & -2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 6 - \lambda & 2 \\ -8 & -2 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(6 - \lambda)(-2 - \lambda) - (2)(-8) = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 2, 2$$

gives the Eigenvalues $\lambda_1 = \lambda_2 = 2$, and the associated Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

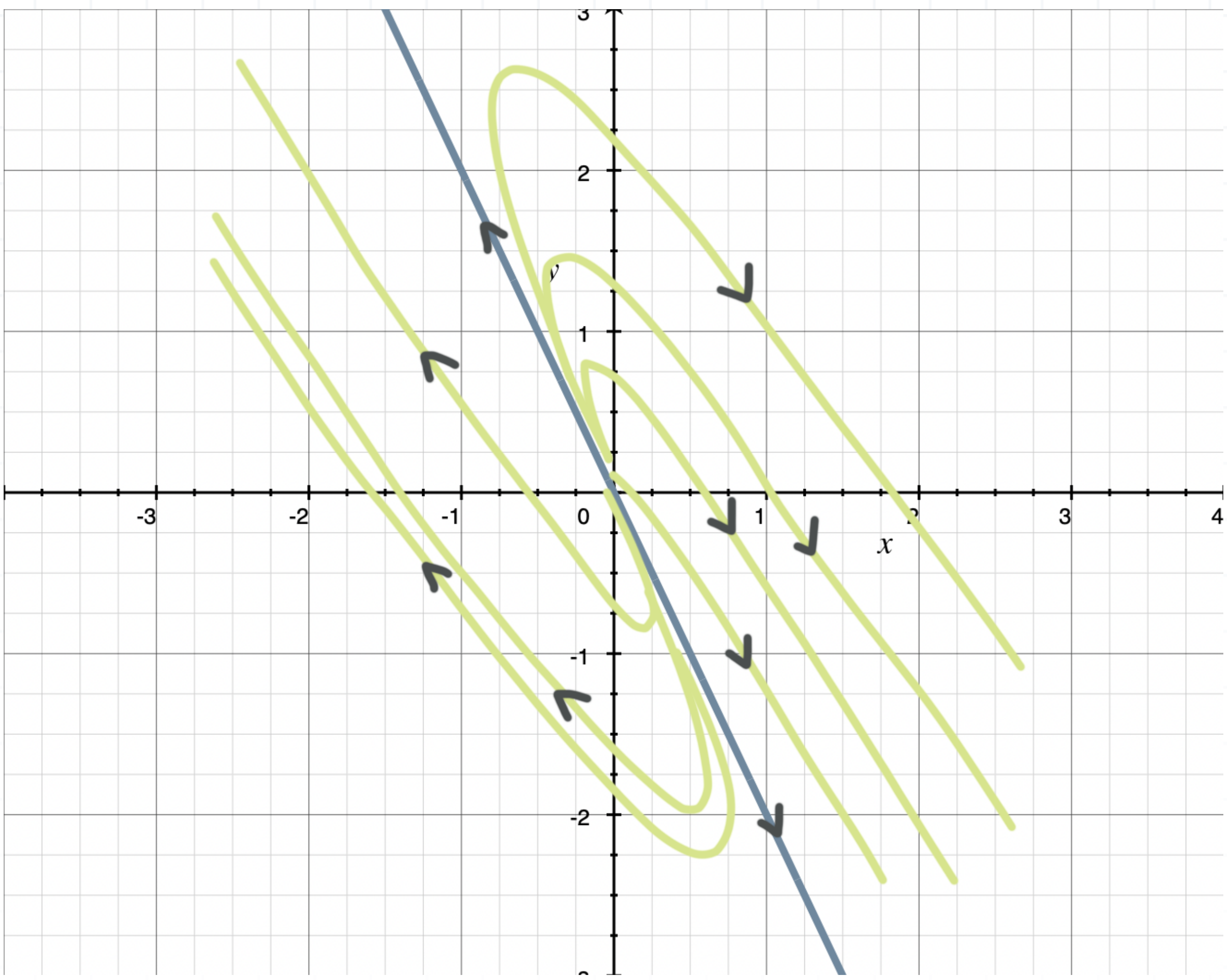
The Eigenvector $\vec{k_1} = (1, -2)$ lies along the line $y = -2x$, so we'll sketch this line in our phase portrait. The Eigenvalue associated with $\vec{k_1} = (1, -2)$ is $\lambda = 2$, which means the direction along that trajectory is away from the origin. Because the Eigenvalue is positive, we're dealing with an unstable repeller node that repels all trajectories.

If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 6 & 2 \\ -8 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(6, -8)$ (toward the third quadrant), which means that the phase portrait must look something like



■ 2. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -3-\lambda & -4 \\ 4 & 5-\lambda \end{bmatrix}$$

and the characteristic equation

$$(-3-\lambda)(5-\lambda)-(-4)(4)=0$$

$$\lambda^2-2\lambda+1=0$$

$$(\lambda-1)(\lambda-1)=0$$

$$\lambda=1,\ 1$$

gives the Eigenvalues $\lambda_1=\lambda_2=1$, and the associated Eigenvector

$$\overrightarrow{k_1}=\begin{bmatrix}1\\-1\end{bmatrix}$$

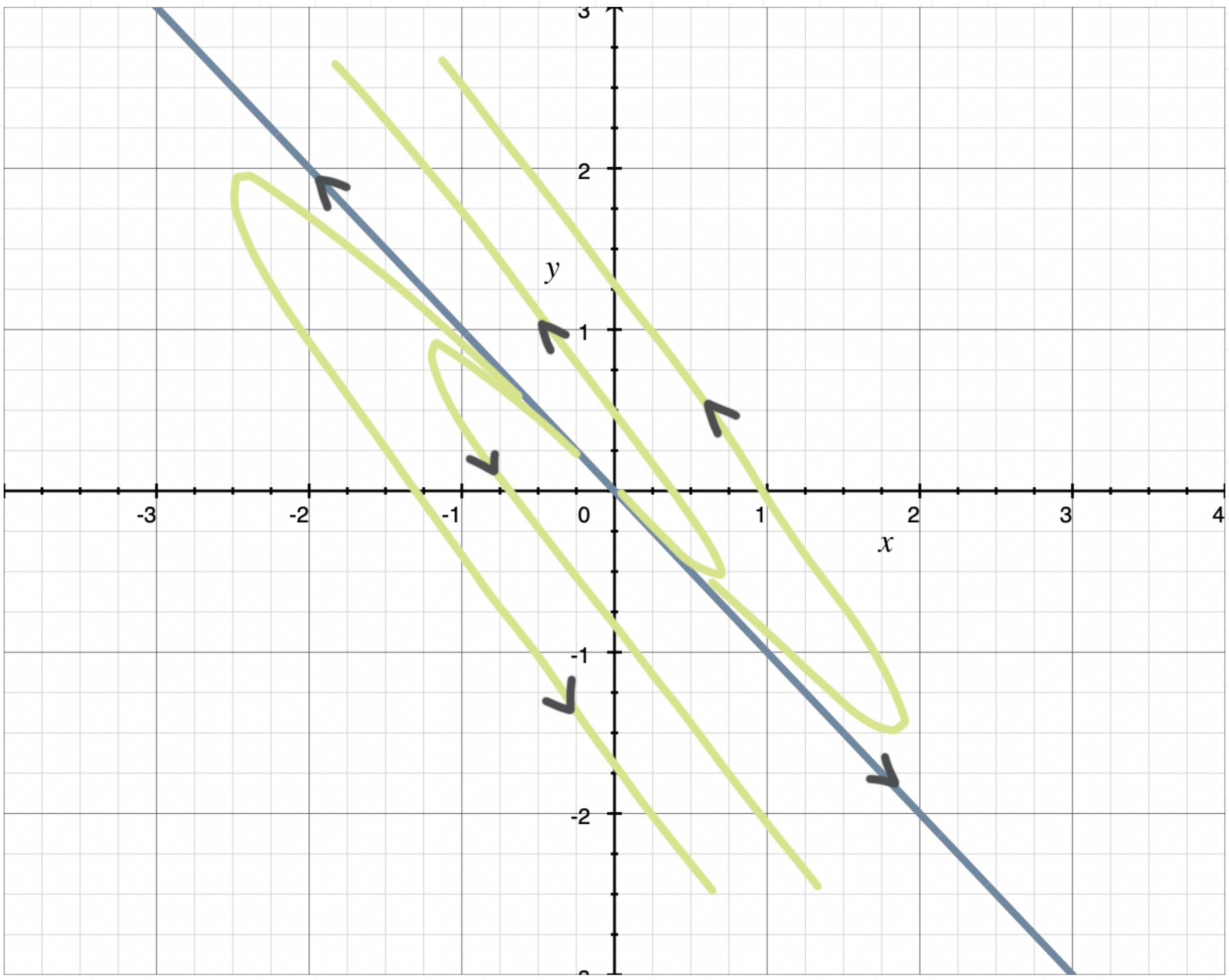
The Eigenvector $\overrightarrow{k_1}=(1,-1)$ lies along the line $y=-x$, so we'll sketch this line in our phase portrait. The Eigenvalue associated with $\overrightarrow{k_1}=(1,-1)$ is $\lambda=1$, which means the direction along that trajectory is away from the origin. Because the Eigenvalue is positive, we're dealing with an unstable repeller node that repels all trajectories.

If we test the vector $\overrightarrow{x}=(1,0)$ in the matrix equation associated with the system, we get

$$\overrightarrow{x}'=\begin{bmatrix}-3&-4\\4&5\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\overrightarrow{x}'=\begin{bmatrix}-3\\4\end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(-3,4)$ (toward the second quadrant), which means that the phase portrait must look something like



■ 3. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -1 & 9 \\ -1 & -7 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -1 - \lambda & 9 \\ -1 & -7 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(-1-\lambda)(-7-\lambda)-(9)(-1)=0$$

$$\lambda^2+8\lambda+16=0$$

$$(\lambda+4)(\lambda+4)=0$$

$$\lambda=-4,\ -4$$

gives the Eigenvalues $\lambda_1=\lambda_2=-4$, and the associated Eigenvector

$$\vec{k_1}=\begin{bmatrix}3\\-1\end{bmatrix}$$

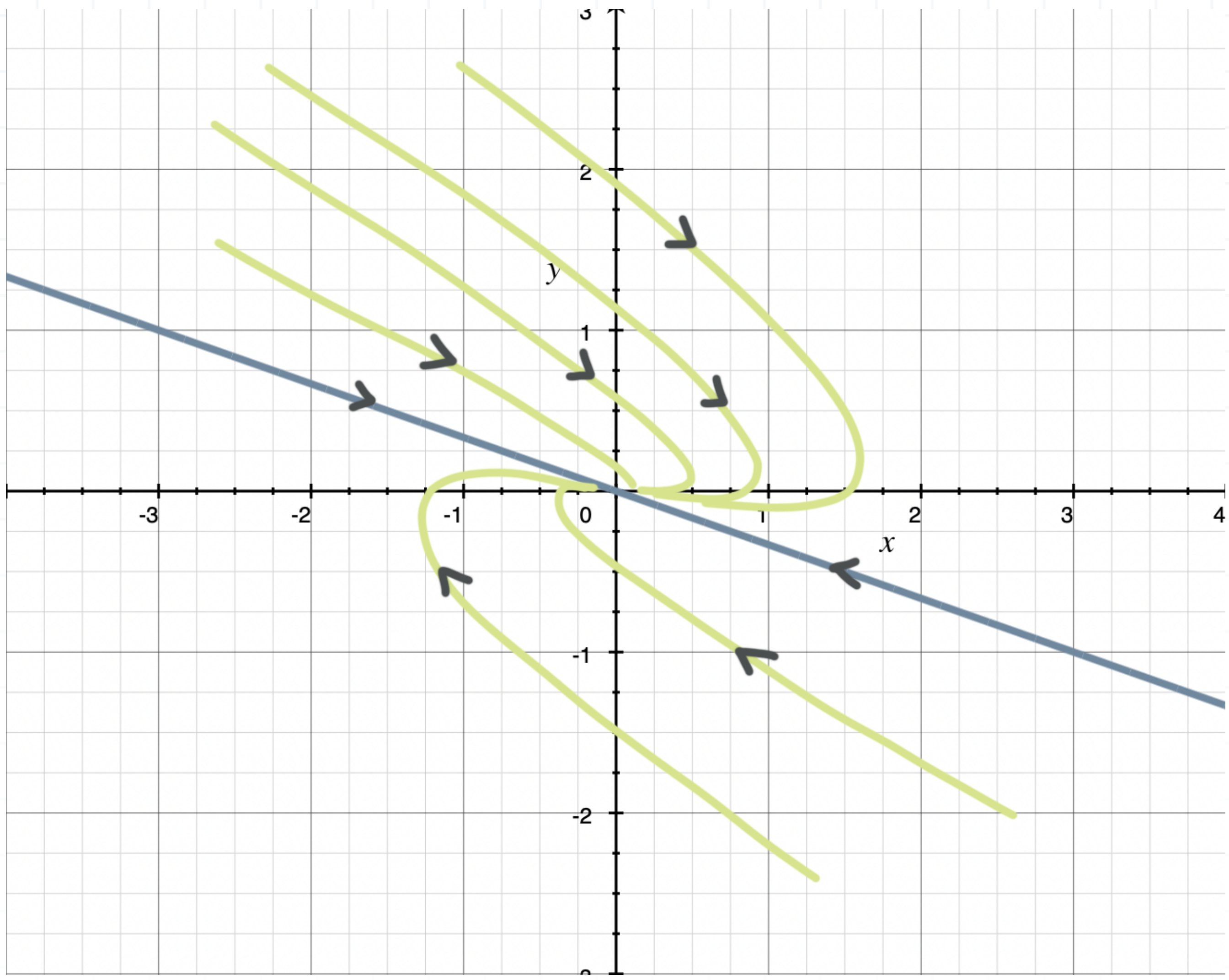
The Eigenvector $\vec{k_1}=(3,-1)$ lies along the line $y=(-1/3)x$, so we'll sketch this line in our phase portrait. The Eigenvalue associated with $\vec{k_1}=(3,-1)$ is $\lambda=-4$, which means the direction along that trajectory is toward the origin. Because the Eigenvalue is negative, we're dealing with an asymptotically stable attractor node that attracts all trajectories.

If we test the vector $\vec{x}=(1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}'=\begin{bmatrix}-1&9\\-1&-7\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\vec{x}'=\begin{bmatrix}-1\\-1\end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(-1,-1)$ (toward the third quadrant), which means that the phase portrait must look something like



■ 4. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 10 & 4 \\ -4 & 2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 10 - \lambda & 4 \\ -4 & 2 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(10 - \lambda)(2 - \lambda) - (4)(-4) = 0$$

$$\lambda^2 - 12\lambda + 36 = 0$$

$$(\lambda - 6)(\lambda - 6) = 0$$

$$\lambda = 6,\ 6$$

gives the Eigenvalues $\lambda_1 = \lambda_2 = 6$, and the associated Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

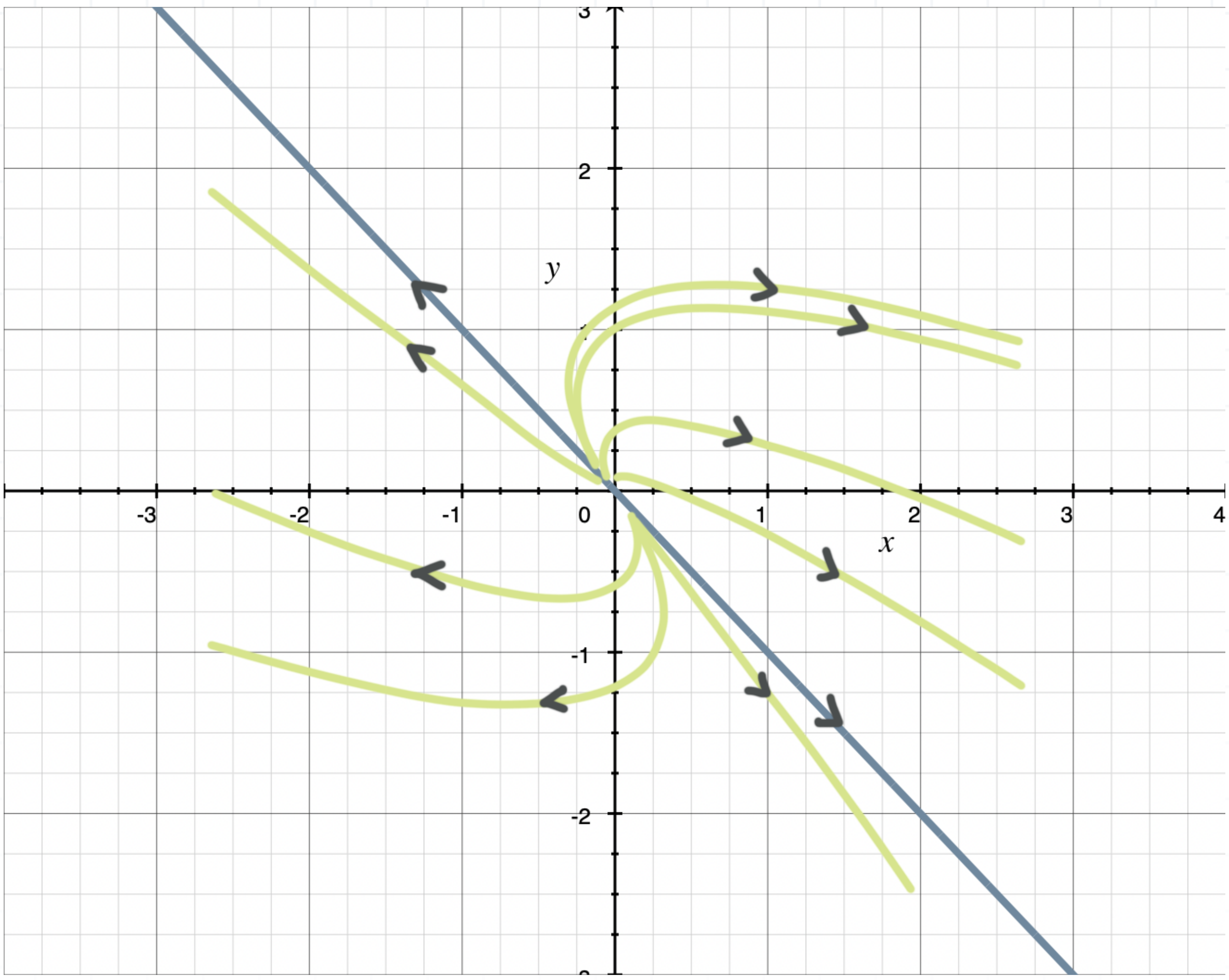
The Eigenvector $\vec{k_1} = (1, -1)$ lies along the line $y = -x$, so we'll sketch this line in our phase portrait. The Eigenvalue associated with $\vec{k_1} = (1, -1)$ is $\lambda = 6$, which means the direction along that trajectory is away from the origin. Because the Eigenvalue is positive, we're dealing with an unstable repeller node that repels all trajectories.

If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 10 & 4 \\ -4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 10 \\ -4 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(10, -4)$ (toward the fourth quadrant), which means that the phase portrait must look something like



■ 5. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 16 & -5 \\ 5 & 6 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 16 - \lambda & -5 \\ 5 & 6 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(16-\lambda)(6-\lambda)-(-5)(5)=0$$

$$\lambda^2-22\lambda+121=0$$

$$(\lambda-11)(\lambda-11)=0$$

$$\lambda=11,\ 11$$

gives the Eigenvalues $\lambda_1=\lambda_2=11$, and the associated Eigenvector

$$\overrightarrow{k_1}=\begin{bmatrix}1\\1\end{bmatrix}$$

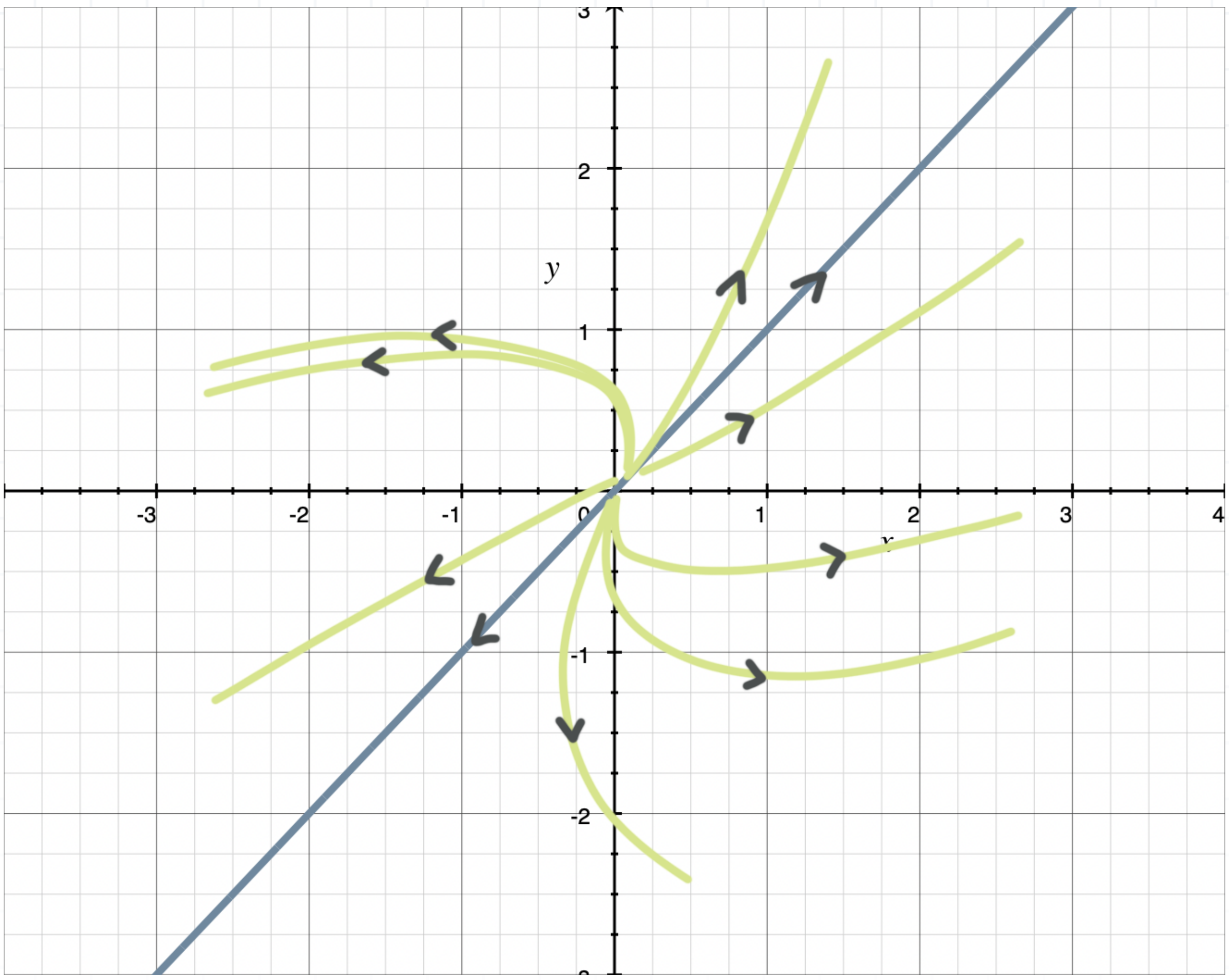
The Eigenvector $\overrightarrow{k_1}=(1,1)$ lies along the line $y=x$, so we'll sketch this line in our phase portrait. The Eigenvalue associated with $\overrightarrow{k_1}=(1,1)$ is $\lambda=11$, which means the direction along that trajectory is away from the origin. Because the Eigenvalue is positive, we're dealing with an unstable repeller node that repels all trajectories.

If we test the vector $\overrightarrow{x}=(1,0)$ in the matrix equation associated with the system, we get

$$\overrightarrow{x}'=\begin{bmatrix}16&-5\\5&6\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\overrightarrow{x}'=\begin{bmatrix}16\\5\end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(16,5)$ (toward the first quadrant), which means that the phase portrait must look something like



■ 6. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & 4 \\ -1 & -2 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(2-\lambda)(-2-\lambda)-(4)(-1)=0$$

$$\lambda^2=0$$

$$\lambda=0,\ 0$$

gives the Eigenvalues $\lambda_1=\lambda_2=0$, and the associated Eigenvector

$$\overrightarrow{k_1}=\begin{bmatrix}2\\-1\end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1}=(2,-1)$ lies along the line $y=(-1/2)x$, so we'll sketch this line in our phase portrait. The Eigenvalue associated with $\overrightarrow{k_1}=(2,-1)$ is $\lambda=0$, which means we need to find a second Eigenvector $\overrightarrow{p_1}$.

$$(A-\lambda I)\overrightarrow{p_1}=\overrightarrow{k_1}$$

$$\begin{bmatrix}2&4\\-1&-2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

Turn this matrix back into a system of equations.

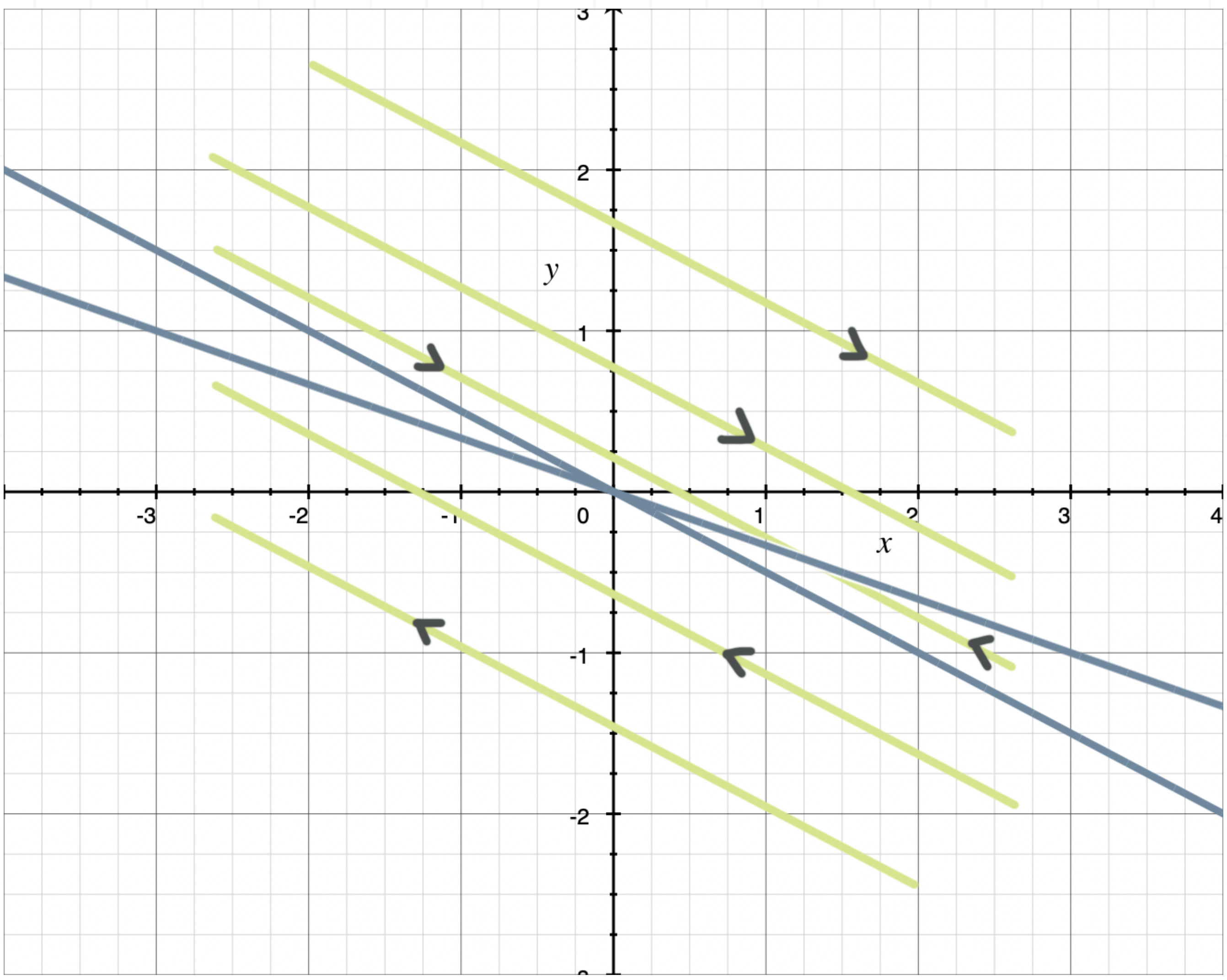
$$2p_1+4p_2=2$$

$$-p_1-2p_2=-1$$

The system simplifies to just $p_1+2p_2=1$. If we choose $p_1=3$ and $p_2=-1$, we get

$$\overrightarrow{p_1}=\begin{bmatrix}3\\-1\end{bmatrix}$$

The trajectories are parallel to $\overrightarrow{k_1} = (2, -1)$, and equilibrium exists at every point along $\overrightarrow{p_1} = (3, -1)$.



PHASE PORTRAITS FOR COMPLEX EIGENVALUES

■ 1. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 5 & 1 \\ -5 & 2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 5-\lambda & 1 \\ -5 & 2-\lambda \end{bmatrix}$$

so the characteristic equation gives the Eigenvalues for the system.

$$(5-\lambda)(2-\lambda) - (1)(-5) = 0$$

$$\lambda^2 - 7\lambda + 15 = 0$$

$$\lambda = \frac{7 \pm \sqrt{11}i}{2}$$

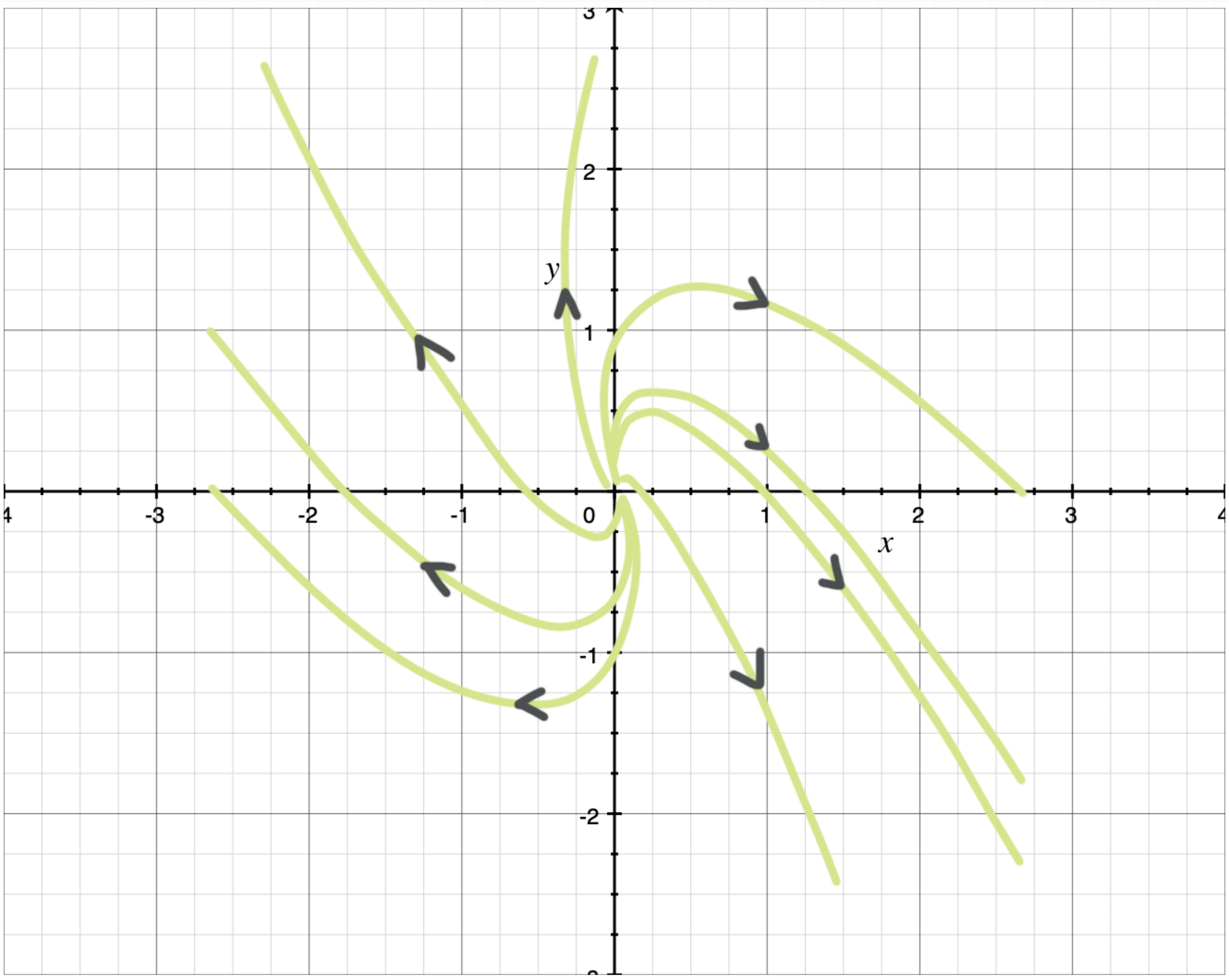
These are Eigenvalues with a positive real part, $\alpha = 7/2 > 0$, which means we're dealing with an unstable spiral that repels all trajectories.

If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 5 & 1 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 5 \\ -5 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(5, -5)$ (into the fourth quadrant), which means that the phase portrait must look something like



■ 2. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -3 \\ 2 & 2-\lambda \end{bmatrix}$$

so the characteristic equation gives the Eigenvalues for the system.

$$(1-\lambda)(2-\lambda) - (-3)(2) = 0$$

$$\lambda^2 - 3\lambda + 8 = 0$$

$$\lambda = \frac{3 \pm \sqrt{23}i}{2}$$

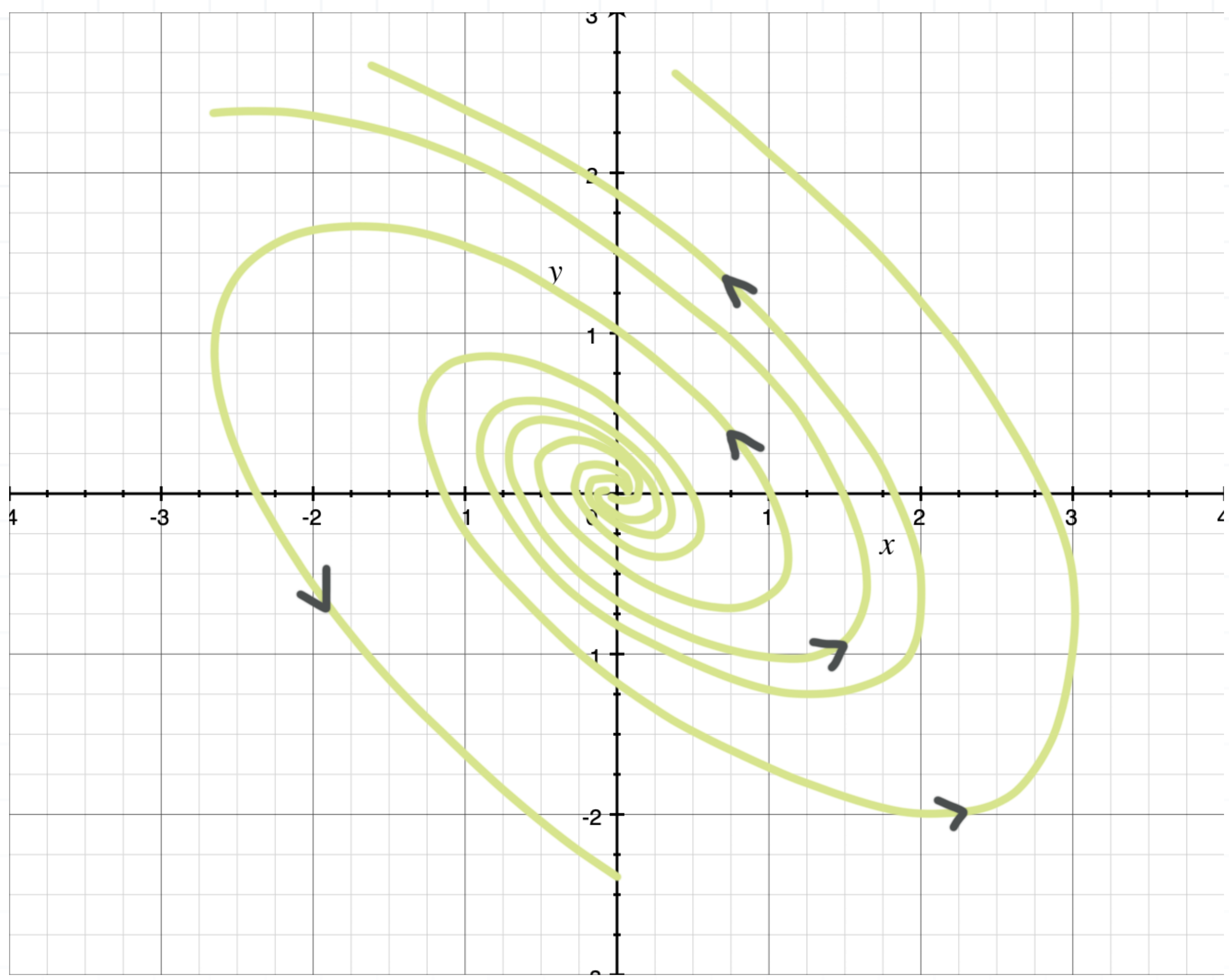
These are Eigenvalues with a positive real part, $\alpha = 3/2 > 0$, which means we're dealing with an unstable spiral that repels all trajectories.

If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(1,2)$ (into the first quadrant), which means that the phase portrait must look something like



■ 3. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & -1 \\ 10 & -4 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & -1 \\ 10 & -4 - \lambda \end{bmatrix}$$

so the characteristic equation gives the Eigenvalues for the system.

$$(2-\lambda)(-4-\lambda)-(-1)(10)=0$$

$$\lambda^2+2\lambda+2=0$$

$$\lambda=-1\pm i$$

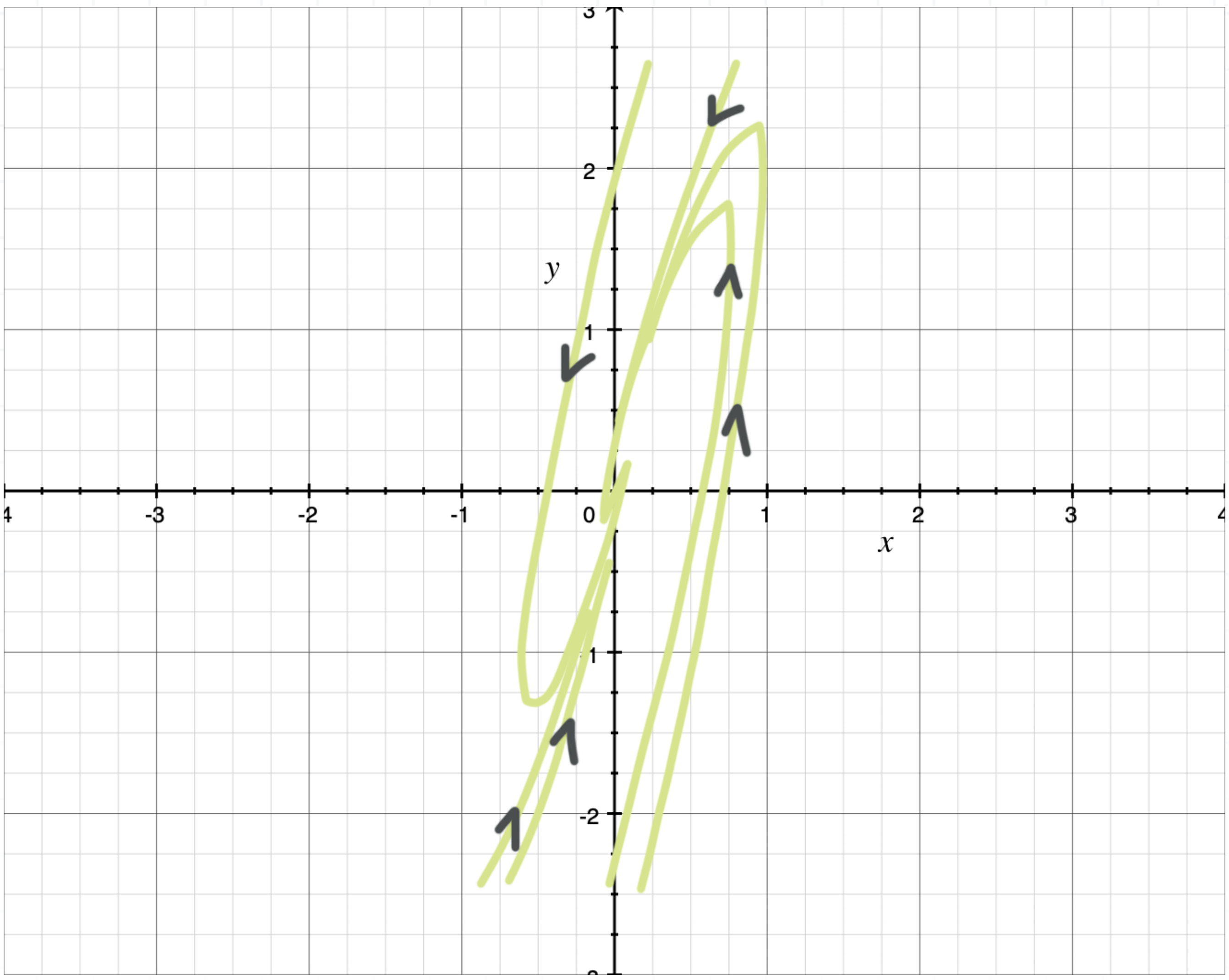
These are Eigenvalues with a negative real part, $\alpha=-1<0$, which means we're dealing with an asymptotically stable spiral that attracts all trajectories.

If we test the vector $\overrightarrow{x}=(1,0)$ in the matrix equation associated with the system, we get

$$\overrightarrow{x}'=\begin{bmatrix}2 & -1\\ 10 & -4\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\overrightarrow{x}'=\begin{bmatrix}2\\ 10\end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(2,10)$ (into the first quadrant), which means that the phase portrait must look something like



■ 4. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -2 - \lambda & 2 \\ -1 & -\lambda \end{bmatrix}$$

so the characteristic equation gives the Eigenvalues for the system.

$$(-2-\lambda)(-\lambda)-(2)(-1)=0$$

$$\lambda^2+2\lambda+2=0$$

$$\lambda=-1\pm i$$

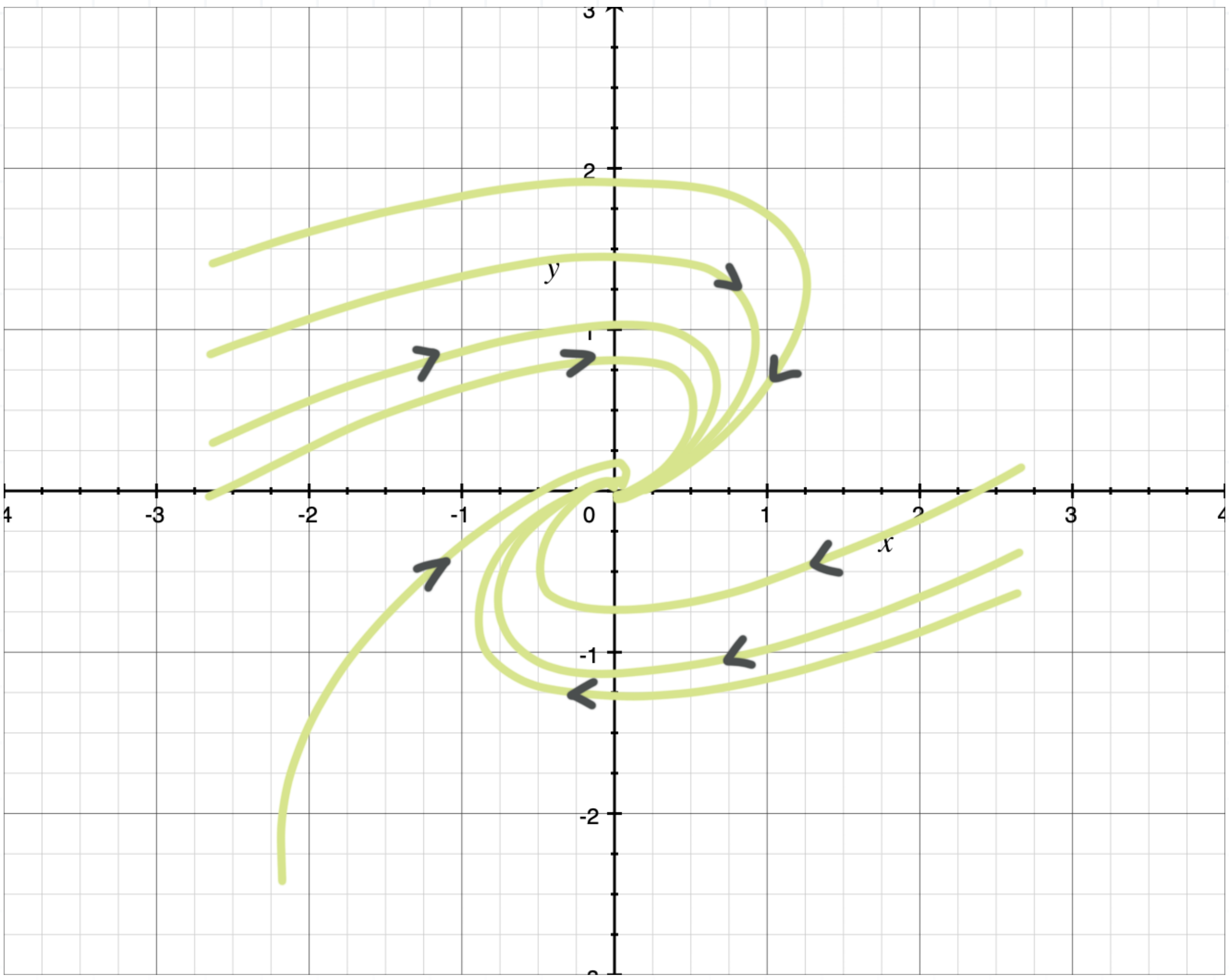
These are Eigenvalues with a negative real part, $\alpha=-1<0$, which means we're dealing with an asymptotically stable spiral that attracts all trajectories.

If we test the vector $\overrightarrow{x}=(1,0)$ in the matrix equation associated with the system, we get

$$\overrightarrow{x}'=\begin{bmatrix}-2 & 2\\ -1 & 0\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\overrightarrow{x}'=\begin{bmatrix}-2\\ -1\end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(-2,-1)$ (into the third quadrant), which means that the phase portrait must look something like



■ 5. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -2 & -4 \\ 2 & 2 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -2 - \lambda & -4 \\ 2 & 2 - \lambda \end{bmatrix}$$

so the characteristic equation gives the Eigenvalues for the system.

$$(-2-\lambda)(2-\lambda)-(-4)(2)=0$$

$$\lambda^2+4=0$$

$$\lambda=\pm 2i$$

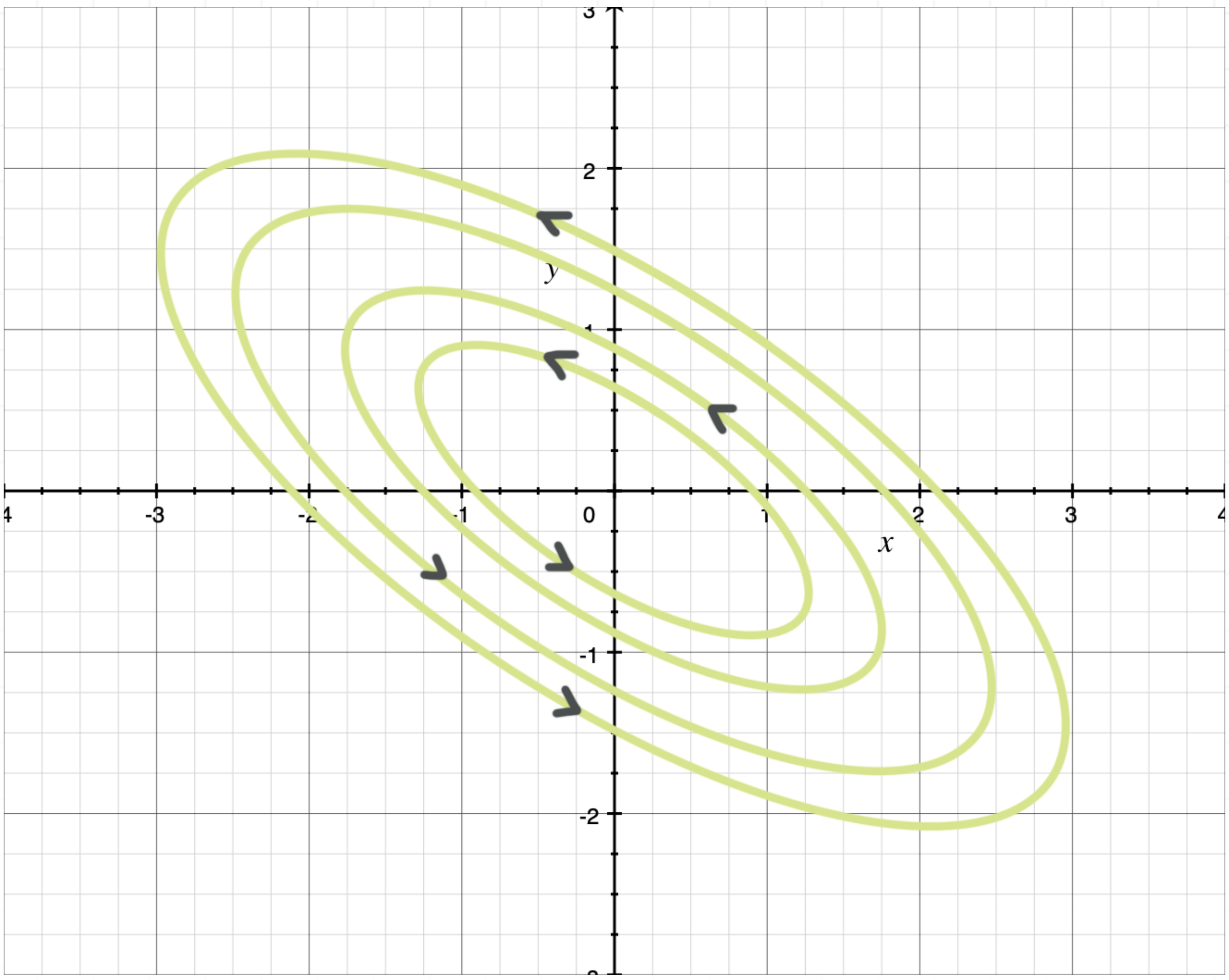
These are Eigenvalues with no real part, $\alpha=0$, which means we're dealing with a stable center that neither repels nor attracts trajectories.

If we test the vector $\vec{x}=(1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}'=\begin{bmatrix}-2 & -4\\ 2 & 2\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}$$

$$\vec{x}'=\begin{bmatrix}-2\\ 2\end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(-2,2)$ (into the second quadrant), which means that the phase portrait must look something like



■ 6. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 1 & 2 \\ -13 & -1 \end{bmatrix}$$

Solution:

The matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ -13 & -1-\lambda \end{bmatrix}$$

so the characteristic equation gives the Eigenvalues for the system.

$$(1-\lambda)(-1-\lambda)-(2)(-13)=0$$

$$\lambda^2+25=0$$

$$\lambda=\pm 5i$$

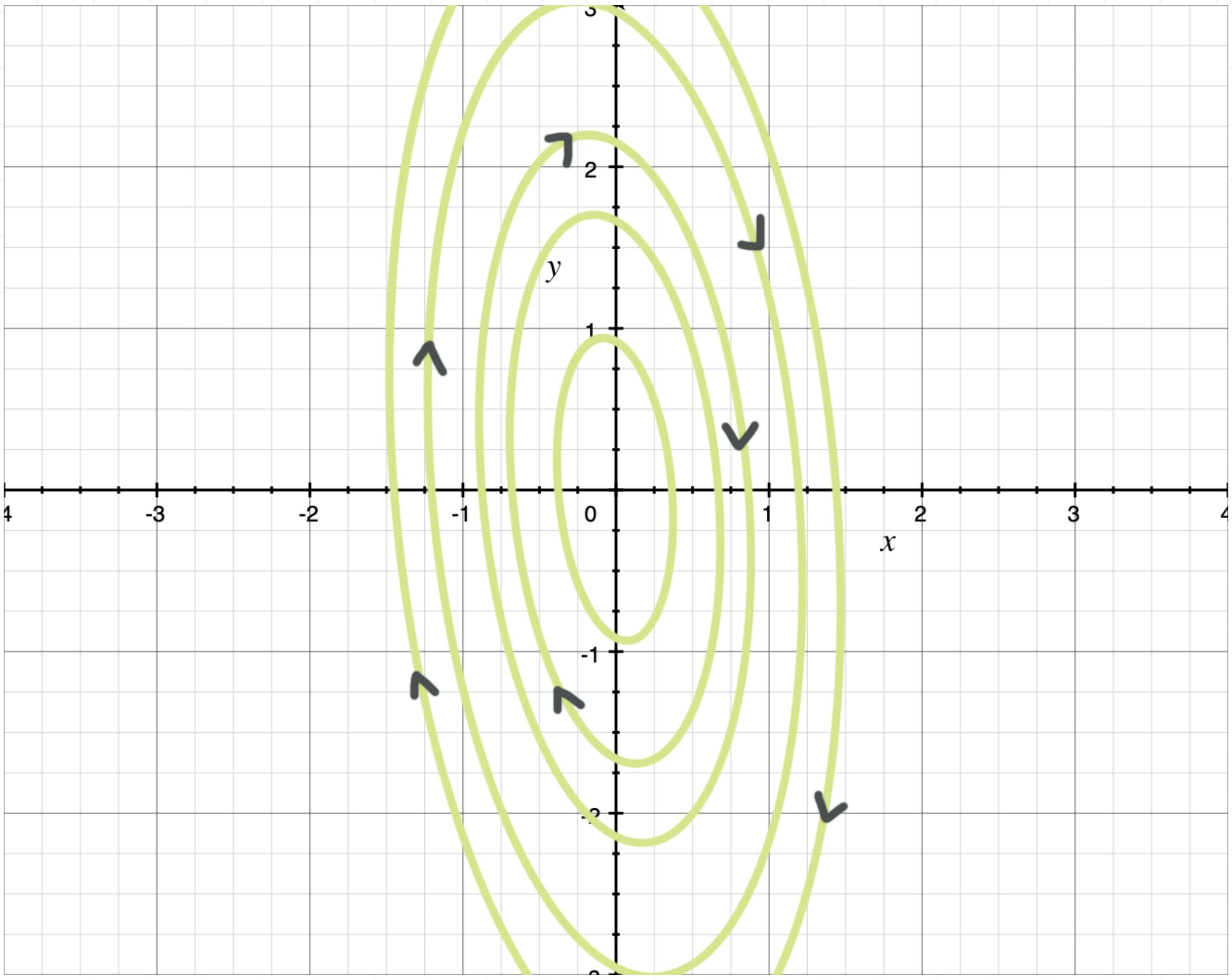
These are Eigenvalues with no real part, $\alpha=0$, which means we're dealing with a stable center that neither repels nor attracts trajectories.

If we test the vector $\overrightarrow{x}=(1,0)$ in the matrix equation associated with the system, we get

$$\overrightarrow{x}'=\begin{bmatrix}1 & 2\\-13 & -1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}$$

$$\overrightarrow{x}'=\begin{bmatrix}1\\-13\end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(1,-13)$ (into the fourth quadrant), which means that the phase portrait must look something like


y
x
-3
-2
-1
0
1
2
3
1
2
-1

UNDETERMINED COEFFICIENTS FOR NONHOMOGENEOUS SYSTEMS

■ 1. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos t \\ 2 \sin t \end{bmatrix}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$(2-\lambda)(-\lambda) - (1)(-1)$$

$$\lambda^2 - 2\lambda + 1$$

Solve the characteristic equation.

$$\lambda^2 - 2\lambda + 1 = 0$$

$$(\lambda - 1)^2 = 0$$

$$\lambda_{1,2} = 1,\ 1$$

Then for the Eigenvalues $\lambda_1 = \lambda_2 = 1$, we find

$$A - 1I = \begin{bmatrix} 2-1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$A - I = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

Put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

If we turn this matrix back into a system of equations, we get

$$k_1 + k_2 = 0$$

$$k_1 = -k_2$$

Choose $k_2 = -1$ to get $k_1 = 1$, which gives the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the solution vector

$$\vec{x_1} = \vec{k_1}e^{\lambda t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_1 = \lambda_2 = 1$, we have to use $\vec{k_1} = (1, -1)$ to find a second solution.

$$(A - \lambda I)\vec{p_1} = \vec{k_1}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$p_1 + p_2 = 1$$

$$-p_1 - p_2 = -1$$

If we choose $p_2 = 0$, we find $p_1 = 1$, and we get the Eigenvector

$$\overrightarrow{p_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

and the solution vector

$$\overrightarrow{x_2} = \overrightarrow{k_1} te^{\lambda_1 t} + \overrightarrow{p_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \right)$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix}$$

$$F = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin t$$

We want a particular solution in the same form, so our guess will be

$$\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin t$$

$$\vec{x}_p{}' = \begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} \sin t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos t$$

Plugging this into the matrix equation representing the system, we get

$$\begin{bmatrix} -a_1 \\ -a_2 \end{bmatrix} \sin t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos t = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \cos t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \sin t \right] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \sin t$$

$$\begin{bmatrix} -a_1 \sin t + b_1 \cos t \\ -a_2 \sin t + b_2 \cos t \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} a_1 \cos t + b_1 \sin t \\ a_2 \cos t + b_2 \sin t \end{bmatrix} + \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix}$$

$$\begin{bmatrix} -a_1 \sin t + b_1 \cos t \\ -a_2 \sin t + b_2 \cos t \end{bmatrix} = \begin{bmatrix} 2a_1 \cos t + 2b_1 \sin t + a_2 \cos t + b_2 \sin t \\ -a_1 \cos t - b_1 \sin t \end{bmatrix} + \begin{bmatrix} \cos t \\ 2\sin t \end{bmatrix}$$

$$\begin{bmatrix} -a_1 \sin t + b_1 \cos t \\ -a_2 \sin t + b_2 \cos t \end{bmatrix} = \begin{bmatrix} (2a_1 + a_2 + 1)\cos t + (2b_1 + b_2)\sin t \\ -a_1 \cos t + (2 - b_1)\sin t \end{bmatrix}$$

Breaking this equation into a system of equations gives

$$-a_1 \sin t + b_1 \cos t = (2a_1 + a_2 + 1)\cos t + (2b_1 + b_2)\sin t$$

$$-a_2 \sin t + b_2 \cos t = -a_1 \cos t + (2 - b_1)\sin t$$

These equations can each be broken into its own system.

$$-a_1 = 2b_1 + b_2 \qquad\qquad -a_2 = 2 - b_1$$

$$b_1 = 2a_1 + a_2 + 1 \qquad\qquad b_2 = -a_1$$

Solving these systems together gives $\vec{a} = (a_1, a_2) = (1/2, -2)$ and $\vec{b} = (b_1, b_2) = (0, -1/2)$. Therefore, the particular solution is

$$\vec{x}_p = \begin{bmatrix} \frac{1}{2} \\ -2 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \sin t$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \right) + \begin{bmatrix} \frac{1}{2} \\ -2 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} \sin t$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t \right) + \begin{bmatrix} \frac{1}{2} \cos t \\ -2 \cos t - \frac{1}{2} \sin t \end{bmatrix}$$

■ 2. Use undetermined coefficients to find the general solution to the initial value problem.

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^{3t} \\ t^2 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix}$$

$$(-\lambda)(-\lambda) - 1(1)$$

$$\lambda^2 - 1$$

Solve the characteristic equation.

$$\lambda^2 - 1 = 0$$

$$\lambda_1 = 1 \qquad \lambda_2 = -1$$

Then for these Eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = -1$, we find

$$A - 1I = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad A + 1I = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 - k_2 = 0 \qquad k_1 + k_2 = 0$$

$$k_1 = k_2 \qquad k_1 = -k_2$$

If we choose $k_2 = 1$ in the first system as $k_2 = -1$ in the second system, we get the Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

and the solution vectors

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t \qquad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} 2e^{3t} \\ t^2 \end{bmatrix}$$

$$F = \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^2$$

We want a particular solution in the same form, so our guess will be

$$\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\vec{x}_p{}' = \begin{bmatrix} 3a_1 \\ 3a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 2b_1 \\ 2b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Plugging this into the matrix equation representing the system, we get

$$\begin{bmatrix} 3a_1 \\ 3a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 2b_1 \\ 2b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right]$$

$$+ \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^2$$

$$\begin{bmatrix} 3a_1e^{3t} + 2b_1t + c_1 \\ 3a_2e^{3t} + 2b_2t + c_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_1e^{3t} + b_1t^2 + c_1t + d_1 \\ a_2e^{3t} + b_2t^2 + c_2t + d_2 \end{bmatrix} + \begin{bmatrix} 2e^{3t} \\ t^2 \end{bmatrix}$$

$$\begin{bmatrix} 3a_1e^{3t} + 2b_1t + c_1 \\ 3a_2e^{3t} + 2b_2t + c_2 \end{bmatrix} = \begin{bmatrix} a_2e^{3t} + b_2t^2 + c_2t + d_2 + 2e^{3t} \\ a_1e^{3t} + b_1t^2 + c_1t + d_1 + t^2 \end{bmatrix}$$

Breaking this equation into a system of two equations gives

$$3a_1e^{3t} + 2b_1t + c_1 = (a_2 + 2)e^{3t} + b_2t^2 + c_2t + d_2$$

$$3a_2e^{3t} + 2b_2t + c_2 = a_1e^{3t} + (b_1 + 1)t^2 + c_1t + d_1$$

These equations can each be broken into its own system.

$$3a_1 = a_2 + 2 \qquad\qquad 3a_2 = a_1$$

$$b_2 = 0 \qquad\qquad b_1 + 1 = 0$$

$$2b_1 = c_2 \qquad\qquad 2b_2 = c_1$$

$$c_1 = d_2 \qquad\qquad c_2 = d_1$$

Putting these results together gives $\vec{a} = (a_1, a_2) = (3/4, 1/4)$, $\vec{b} = (b_1, b_2) = (-1, 0)$, $\vec{c} = (c_1, c_2) = (0, -2)$, and $\vec{d} = (d_1, d_2) = (-2, 0)$. Therefore, the particular solution is

$$\vec{x}_p = \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix} e^{3t} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ -2 \end{bmatrix} t + \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} \frac{3}{4}e^{3t} - t^2 - 2 \\ \frac{1}{4}e^{3t} - 2t \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{3}{4}e^{3t} - t^2 - 2 \\ \frac{1}{4}e^{3t} - 2t \end{bmatrix}$$

Plug in the initial condition,

$$\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{3}{4} - 2 \\ \frac{1}{4} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + \frac{3}{4} - 2 \\ c_1 - c_2 + \frac{1}{4} \end{bmatrix}$$

then break the matrix equation into a system of equations.

$$c_1 + c_2 + \frac{3}{4} - 2 = \frac{1}{4}$$

$$c_1 - c_2 + \frac{1}{4} = \frac{3}{4}$$

Solving the system gives $c_1 = 1$ and $c_2 = 1/2$, which means the solution to the system is

$$\vec{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{3}{4}e^{3t} - t^2 - 2 \\ \frac{1}{4}e^{3t} - 2t \end{bmatrix}$$

■ 3. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 6e^{4t} \\ \cos(3t) \\ 1 \end{bmatrix}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} 1-\lambda & -1 & 1 \\ 1 & 1-\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix}$$

$$(1-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & -\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 2 & -\lambda \end{vmatrix} + 1\begin{vmatrix} 1 & 1-\lambda \\ 2 & -1 \end{vmatrix}$$

$$(1-\lambda)((1-\lambda)(-\lambda) - (-1)(-1)) + (1(-\lambda) - (-1)(2)) + (1(-1) - (1-\lambda)(2))$$

$$(1-\lambda)(\lambda^2 - \lambda - 1) - \lambda + 2 + 2\lambda - 3$$

$$-\lambda^3 + 2\lambda^2 + \lambda - 2$$

$$-\lambda^2(\lambda - 2) + (\lambda - 2)$$

Solve the characteristic equation.

$$(\lambda - 2)(-\lambda^2 + 1) = 0$$

$$\lambda_1 = 2 \qquad \lambda_2 = -1 \qquad \lambda_3 = 1$$

Then for these Eigenvalues, $\lambda_1 = 2$, $\lambda_2 = -1$, and $\lambda_3 = 1$, we find

$$\begin{bmatrix} -1 & -1 & 1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ 2 & -1 & -2 \end{bmatrix} \qquad \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & -1 \\ 2 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 0 \\ 0 & -3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \\ 0 & -5 & 3 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & \frac{1}{5} \\ 0 & 1 & -\frac{3}{5} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 - k_3 = 0 \qquad k_1 + \frac{1}{5}k_3 = 0 \qquad k_1 - k_3 = 0$$

$$k_2 = 0 \qquad k_2 - \frac{3}{5}k_3 = 0 \qquad k_2 - k_3 = 0$$

These systems simplify to

$$k_1 = k_3 \qquad k_1 = -\frac{1}{5}k_3 \qquad k_1 = k_3$$

$$k_2 = 0 \qquad k_2 = \frac{3}{5}k_3 \qquad k_2 = k_3$$

From these systems, we find the Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} \qquad \vec{k_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and then the solution vectors

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t} \qquad \vec{x_3} = \vec{k_3} e^{\lambda_3 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} \qquad \vec{x_2} = \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} e^{-t} \qquad \vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1 \vec{x_1} + c_2 \vec{x_2} + c_3 \vec{x_3}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} 6e^{4t} \\ \cos(3t) \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} e^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos(3t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

We want a particular solution in the same form, so our guess will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^{4t} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cos(3t) + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \sin(3t) + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$\vec{x}_p{}' = \begin{bmatrix} 4a_1 \\ 4a_2 \\ 4a_3 \end{bmatrix} e^{4t} + \begin{bmatrix} -3b_1 \\ -3b_2 \\ -3b_3 \end{bmatrix} \sin(3t) + \begin{bmatrix} 3c_1 \\ 3c_2 \\ 3c_3 \end{bmatrix} \cos(3t)$$

Plugging these into the matrix equation representing the system, we get

$$\begin{bmatrix} 4a_1 \\ 4a_2 \\ 4a_3 \end{bmatrix} e^{4t} + \begin{bmatrix} -3b_1 \\ -3b_2 \\ -3b_3 \end{bmatrix} \sin(3t) + \begin{bmatrix} 3c_1 \\ 3c_2 \\ 3c_3 \end{bmatrix} \cos(3t)$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^{4t} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cos(3t) + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \sin(3t) + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \right]$$

$$+ \begin{bmatrix} 6 \\ 0 \\ 0 \end{bmatrix} e^{4t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4a_1e^{4t} - 3b_1 \sin(3t) + 3c_1 \cos(3t) \\ 4a_2e^{4t} - 3b_2 \sin(3t) + 3c_2 \cos(3t) \\ 4a_3e^{4t} - 3b_3 \sin(3t) + 3c_3 \cos(3t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} a_1e^{4t} + b_1 \cos(3t) + c_1 \sin(3t) + d_1 \\ a_2e^{4t} + b_2 \cos(3t) + c_2 \sin(3t) + d_2 \\ a_3e^{4t} + b_3 \cos(3t) + c_3 \sin(3t) + d_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 6e^{4t} \\ \cos(3t) \\ 1 \end{bmatrix}$$

Breaking this equation into a system of three equations gives

$$4a_1e^{4t} - 3b_1 \sin(3t) + 3c_1 \cos(3t) = (a_1 - a_2 + a_3 + 6)e^{4t} + (b_1 - b_2 + b_3)\cos(3t)$$

$$+(c_1-c_2+c_3)\sin(3t)+(d_1-d_2+d_3)$$

$$4a_2e^{4t}-3b_2\sin(3t)+3c_2\cos(3t)=(a_1+a_2-a_3)e^{4t}+(b_1+b_2-b_3+1)\cos(3t)$$

$$+(c_1+c_2-c_3)\sin(3t)+(d_1+d_2-d_3)$$

$$4a_3e^{4t}-3b_3\sin(3t)+3c_3\cos(3t)=(2a_1-a_2)e^{4t}+(2b_1-b_2)\cos(3t)$$

$$+(2c_1-c_2)\sin(3t)+(2d_1-d_2+1)$$

These equations can each be broken into its own system.

$$4a_1=a_1-a_2+a_3+6 \qquad 4a_2=a_1+a_2-a_3 \qquad 4a_3=2a_1-a_2$$

$$-3b_1=c_1-c_2+c_3 \qquad -3b_2=c_1+c_2-c_3 \qquad -3b_3=2c_1-c_2$$

$$3c_1=b_1-b_2+b_3 \qquad 3c_2=b_1+b_2-b_3+1 \qquad 3c_3=2b_1-b_2$$

$$0=d_1-d_2+d_3 \qquad 0=d_1+d_2-d_3 \qquad 0=2d_1-d_2+1$$

Putting these results together gives $\overrightarrow{a}=(a_1,a_2,a_3)=(11/5,2/5,1)$, $\overrightarrow{b}=(b_1,b_2,b_3)=(7/130,-1/10,7/130)$, $\overrightarrow{c}=(c_1,c_2,c_3)=(9/130,3/10,9/130)$, and $\overrightarrow{d}=(d_1,d_2,d_3)=(0,1,1)$. Therefore, the particular solution is

$$\overrightarrow{x_p}=\begin{bmatrix}\frac{11}{5}\\ \frac{2}{5}\\ 1\end{bmatrix}e^{4t}+\begin{bmatrix}\frac{7}{130}\\ -\frac{1}{10}\\ \frac{7}{130}\end{bmatrix}\cos(3t)+\begin{bmatrix}\frac{9}{130}\\ \frac{3}{10}\\ \frac{9}{130}\end{bmatrix}\sin(3t)+\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -1 \\ 3 \\ 5 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t}$$

$$+ \begin{bmatrix} \frac{11}{5} \\ \frac{2}{5} \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} \frac{7}{130} \\ -\frac{1}{10} \\ \frac{7}{130} \end{bmatrix} \cos(3t) + \begin{bmatrix} \frac{9}{130} \\ \frac{3}{10} \\ \frac{9}{130} \end{bmatrix} \sin(3t) + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

■ 4. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} -e^t \\ 3t^2 \\ 2e^{-t} \end{bmatrix}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} 3-\lambda & -1 & 1 \\ -1 & 5-\lambda & -1 \\ 1 & -1 & 3-\lambda \end{vmatrix}$$

$$(3-\lambda) \begin{vmatrix} 5-\lambda & -1 \\ -1 & 3-\lambda \end{vmatrix} - (-1) \begin{vmatrix} -1 & -1 \\ 1 & 3-\lambda \end{vmatrix} + 1 \begin{vmatrix} -1 & 5-\lambda \\ 1 & -1 \end{vmatrix}$$

$$(3-\lambda)((5-\lambda)(3-\lambda)-(-1)(-1))+((-1)(3-\lambda)-(-1)(1))+((-1)(-1)-(5-\lambda)(1))$$

$$(3-\lambda)(\lambda^2-8\lambda+14)+\lambda-2+\lambda-4$$

$$(3-\lambda)(\lambda^2-8\lambda+14)-2(3-\lambda)$$

$$(3-\lambda)(\lambda^2-8\lambda+12)$$

Solve the characteristic equation.

$$(3-\lambda)(\lambda-6)(\lambda-2)=0$$

$$\lambda_1=2 \qquad \lambda_2=3 \qquad \lambda_3=6$$

Then for these Eigenvalues, $\lambda_1=2$, $\lambda_2=3$, and $\lambda_3=6$, we find

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

Put these three matrices into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 1 \\ -1 & 3 & -1 \\ 1 & -1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 0 \end{bmatrix} \qquad \begin{bmatrix} -3 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & -1 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ -1 & 2 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & -3 \\ -3 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & -3 \\ 0 & -4 & -8 \\ 0 & -2 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn these matrices back into systems of equations, we get

$$k_1 + k_3 = 0 \qquad k_1 - k_3 = 0 \qquad k_1 - k_3 = 0$$

$$k_2 = 0 \qquad k_2 - k_3 = 0 \qquad k_2 + 2k_3 = 0$$

Solving these systems gives the Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \overrightarrow{k_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

and the solution vectors

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t} \qquad \overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} \qquad \overrightarrow{x_3} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{6t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\vec{x_c} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{6t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} -e^t \\ 3t^2 \\ 2e^{-t} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} e^{-t}$$

We want a particular solution in the same form, so our guess will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} e^{-t}$$

$$\vec{x_p}' = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} 2b_1 \\ 2b_2 \\ 2b_3 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} -e_1 \\ -e_2 \\ -e_3 \end{bmatrix} e^{-t}$$

Plugging these into the matrix equation representing the system, we get

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} 2b_1 \\ 2b_2 \\ 2b_3 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} -e_1 \\ -e_2 \\ -e_3 \end{bmatrix} e^{-t}$$

$$= \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} e^{-t} \right]$$

$$+\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} e^{-t}$$

$$\begin{bmatrix} a_1e^t + 2b_1t + c_1 - e_1e^{-t} \\ a_2e^t + 2b_2t + c_2 - e_2e^{-t} \\ a_3e^t + 2b_3t + c_3 - e_3e^{-t} \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \begin{bmatrix} a_1e^t + b_1t^2 + c_1t + d_1 + e_1e^{-t} \\ a_2e^t + b_2t^2 + c_2t + d_2 + e_2e^{-t} \\ a_3e^t + b_3t^2 + c_3t + d_3 + e_3e^{-t} \end{bmatrix}$$

$$+\begin{bmatrix} -e^t \\ 3t^2 \\ 2e^{-t} \end{bmatrix}$$

Breaking this equation into a system of three equations gives

$$a_1e^t + 2b_1t + c_1 - e_1e^{-t} = (3a_1 - a_2 + a_3 - 1)e^t + (3b_1 - b_2 + b_3)t^2$$

$$+(3c_1 - c_2 + c_3)t + (3d_1 - d_2 + d_3) + (3e_1 - e_2 + e_3)e^{-t}$$

$$a_2e^t + 2b_2t + c_2 - e_2e^{-t} = (-a_1 + 5a_2 - a_3)e^t + (-b_1 + 5b_2 - b_3 + 3)t^2$$

$$+(-c_1 + 5c_2 - c_3)t + (-d_1 + 5d_2 - d_3) + (-e_1 + 5e_2 - e_3)e^{-t}$$

$$a_3e^t + 2b_3t + c_3 - e_3e^{-t} = (a_1 - a_2 + 3a_3)e^t + (b_1 - b_2 + 3b_3)t^2$$

$$+(c_1 - c_2 + 3c_3)t + (d_1 - d_2 + 3d_3) + (e_1 - e_2 + 3e_3 + 2)e^{-t}$$

These equations can each be broken into its own system.

$$a_1 = 3a_1 - a_2 + a_3 - 1 \qquad a_2 = -a_1 + 5a_2 - a_3 \qquad a_3 = a_1 - a_2 + 3a_3$$

$$0 = 3b_1 - b_2 + b_3 \qquad 0 = -b_1 + 5b_2 - b_3 + 3 \qquad 0 = b_1 - b_2 + 3b_3$$

$$2b_1 = 3c_1 - c_2 + c_3 \qquad 2b_2 = -c_1 + 5c_2 - c_3 \qquad 2b_3 = c_1 - c_2 + 3c_3$$

$$c_1 = 3d_1 - d_2 + d_3 \qquad c_2 = -d_1 + 5d_2 - d_3 \qquad c_3 = d_1 - d_2 + 3d_3$$

$$-e_1 = 3e_1 - e_2 + e_3 \qquad -e_2 = -e_1 + 5e_2 - e_3 \qquad -e_3 = e_1 - e_2 + 3e_3 + 2$$

Putting these results together gives $\overrightarrow{a} = (a_1, a_2, a_3) = (7/10, 1/10, -3/10)$, $\overrightarrow{b} = (b_1, b_2, b_3) = (-1/6, -2/3, -1/6)$, $\overrightarrow{c} = (c_1, c_2, c_3) = (-1/6, -1/3, -1/6)$, $\overrightarrow{d} = (d_1, d_2, d_3) = (-7/108, -5/54, -7/108)$, and $\overrightarrow{e} = (e_1, e_2, e_3) = (5/42, -1/14, -23/42)$. Therefore, the particular solution is

$$\overrightarrow{x_p} = \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \\ -\frac{3}{10} \end{bmatrix} e^t + \begin{bmatrix} -\frac{1}{6} \\ -\frac{2}{3} \\ -\frac{1}{6} \end{bmatrix} t^2 + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} t + \begin{bmatrix} -\frac{7}{108} \\ -\frac{5}{54} \\ -\frac{7}{108} \end{bmatrix} + \begin{bmatrix} \frac{5}{42} \\ -\frac{1}{14} \\ -\frac{23}{42} \end{bmatrix} e^{-t}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} e^{6t}$$

$$+ \begin{bmatrix} \frac{7}{10} \\ \frac{1}{10} \\ -\frac{3}{10} \end{bmatrix} e^t + \begin{bmatrix} -\frac{1}{6} \\ -\frac{2}{3} \\ -\frac{1}{6} \end{bmatrix} t^2 + \begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} t + \begin{bmatrix} -\frac{7}{108} \\ -\frac{5}{54} \\ -\frac{7}{108} \end{bmatrix} + \begin{bmatrix} \frac{5}{42} \\ -\frac{1}{14} \\ -\frac{23}{42} \end{bmatrix} e^{-t}$$

■ 5. Use the method of undetermined coefficients to find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} \vec{x} + \begin{bmatrix} t + e^t \\ 2t^2 + 1 \end{bmatrix}$$

Solution:

The determinant $|A - \lambda I|$ is

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & 0 \\ 1 & 4 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (3 - \lambda)(4 - \lambda) - (0)(1)$$

$$|A - \lambda I| = (3 - \lambda)(4 - \lambda)$$

Solve the characteristic equation.

$$(3 - \lambda)(4 - \lambda) = 0$$

$$\lambda_1 = 3 \qquad \lambda_2 = 4$$

Then for these Eigenvalues, $\lambda_1 = 3$ and $\lambda_2 = 4$, we find

$$A - 3I = \begin{bmatrix} 3 - 3 & 0 \\ 1 & 4 - 3 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} 3 - 4 & 0 \\ 1 & 4 - 4 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \qquad A - 4I = \begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 + k_2 = 0$$

$$k_1 = -k_2$$

$$k_1 = 0$$

For the first system, we choose $(k_1, k_2) = (1, -1)$. And for the second system, we choose $(k_1, k_2) = (0,1)$. We find the Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\vec{k_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and the solution vectors

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$$

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} t + e^t \\ 2t^2 + 1 \end{bmatrix}$$

$$F = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t^2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We want a particular solution in the same form, so our guess will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\vec{x_p}' = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} 2b_1 \\ 2b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Plugging these into the matrix equation representing the system, we get

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} 2b_1 \\ 2b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right]$$

$$+ \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 2 \end{bmatrix} t^2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a_1e^t + 2b_1t + c_1 \\ a_2e^t + 2b_2t + c_2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a_1e^t + b_1t^2 + c_1t + d_1 \\ a_2e^t + b_2t^2 + c_2t + d_2 \end{bmatrix} + \begin{bmatrix} t + e^t \\ 2t^2 + 1 \end{bmatrix}$$

Breaking this equation into a system of two equations gives

$$a_1e^t + 2b_1t + c_1 = 3(a_1e^t + b_1t^2 + c_1t + d_1) + 0(a_2e^t + b_2t^2 + c_2t + d_2) + t + e^t$$

$$a_2e^t + 2b_2t + c_2 = 1(a_1e^t + b_1t^2 + c_1t + d_1) + 4(a_2e^{3t} + b_2t^2 + c_2t + d_2) + 2t^2 + 1$$

which simplifies to

$$a_1e^t + 2b_1t + c_1 = (3a_1 + 1)e^t + 3b_1t^2 + (3c_1 + 1)t + 3d_1$$

$$a_2e^t + 2b_2t + c_2 = (a_1 + 4a_2)e^t + (b_1 + 4b_2 + 2)t^2 + (c_1 + 4c_2)t + (d_1 + 4d_2 + 1)$$

These equations can each be broken into its own system.

$$a_1 = 3a_1 + 1 \qquad a_2 = a_1 + 4a_2$$

$$0 = 3b_1 \qquad 0 = b_1 + 4b_2 + 2$$

$$2b_1 = 3c_1 + 1 \qquad 2b_2 = c_1 + 4c_2$$

$$c_1 = 3d_1 \qquad c_2 = d_1 + 4d_2 + 1$$

Solving these eight equations as a system gives $(a_1, a_2) = (-1/2, 1/6)$, $(b_1, b_2) = (0, -1/2)$, $(c_1, c_2) = (-1/3, -1/6)$, and $(d_1, d_2) = (-1/9, -19/72)$. Therefore, the particular solution is

$$\vec{x}_p = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{6} \end{bmatrix} e^t + \begin{bmatrix} 0 \\ -\frac{1}{2} \end{bmatrix} t^2 + \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} t + \begin{bmatrix} -\frac{1}{9} \\ -\frac{19}{72} \end{bmatrix}$$

Now we can rewrite the particular solution as one vector.

$$\vec{x}_p = \begin{bmatrix} -\frac{1}{2}e^t - \frac{1}{3}t - \frac{1}{9} \\ \frac{1}{6}e^t - \frac{1}{2}t^2 - \frac{1}{6}t - \frac{19}{72} \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{4t} + \begin{bmatrix} -\frac{1}{2}e^t - \frac{1}{3}t - \frac{1}{9} \\ \frac{1}{6}e^t - \frac{1}{2}t^2 - \frac{1}{6}t - \frac{19}{72} \end{bmatrix}$$

■ 6. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 3e^t \\ 3t + 2e^{3t} \\ t^3 - 4e^t \end{bmatrix}$$

Solution:

The determinant $|A - \lambda I|$ is

$$\begin{vmatrix} -\lambda & 1 & 0 \\ -4 & 4-\lambda & 0 \\ -2 & 1 & 2-\lambda \end{vmatrix}$$

$$(-\lambda) \begin{vmatrix} 4-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} -4 & 0 \\ -2 & 2-\lambda \end{vmatrix} + 0 \begin{vmatrix} -4 & 4-\lambda \\ -2 & 1 \end{vmatrix}$$

$$(-\lambda)((4-\lambda)(2-\lambda) - (0)(1)) - ((-4)(2-\lambda) - (0)(-2)) + 0$$

$$(-\lambda)(4-\lambda)(2-\lambda)+4(2-\lambda)$$

$$(2-\lambda)(\lambda^2-4\lambda+4)$$

Solve the characteristic equation.

$$(2-\lambda)(\lambda^2-4\lambda+4)=0$$

$$(2-\lambda)(\lambda-2)(\lambda-2)=0$$

$$\lambda_1=\lambda_2=\lambda_3=2$$

Then for these Eigenvalues, $\lambda_1=\lambda_2=\lambda_3=2$, we find

$$A-2I=\begin{bmatrix}-2 & 1 & 0\\ -4 & 4-2 & 0\\ -2 & 1 & 2-2\end{bmatrix}$$

$$A-2I=\begin{bmatrix}-2 & 1 & 0\\ -4 & 2 & 0\\ -2 & 1 & 0\end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix}-2 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & -\frac{1}{2} & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

If we turn this matrix back into a system of equations, we get

$$k_1-\frac{1}{2}k_2=0$$

$$k_1=\frac{1}{2}k_2$$

So we can choose $k_2 = 2$ and $k_3 = 0$ to get

$$\vec{k_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

We could also choose $k_2 = 0$ and $k_3 = 1$ to get

$$\vec{k_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So we can say that the Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = 2$ produces only two linearly independent Eigenvectors.

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{2t} \qquad \vec{x_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Because we find two Eigenvectors for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$, we have to find a third solution.

$$(A - \lambda_1 I)\vec{p_1} = a_1 \vec{k_1} + a_2 \vec{k_2}$$

$$\begin{bmatrix} -2 & 1 & 0 \\ -4 & 2 & 0 \\ -2 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 2a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_2 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$-2p_1 + p_2 = a_1$$

$$-4p_1 + 2p_2 = 2a_1$$

$$-2p_1 + p_2 = a_2$$

We can choose $p_1 = -1$ with $p_2 = 0$ and $p_3 = 0$ to get the Eigenvector

$$\overrightarrow{p_1} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}$$

Then the solution vector will be

$$\overrightarrow{x_3} = \overrightarrow{p_1} e^{\lambda_1 t} + (a_1 \overrightarrow{k_1} + a_2 \overrightarrow{k_2}) t e^{\lambda_1 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \left(2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right) t e^{2t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} t e^{2t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2} + c_3 \overrightarrow{x_3}$$

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} t e^{2t} \right)$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} 3e^t \\ 3t + 2e^{3t} \\ t^3 - 4e^t \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} t$$

We want a particular solution in the same form, so our guess will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} e^{3t} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} t^3 + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} t^2 + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} t + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

$$\vec{x_p}' = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \end{bmatrix} e^{3t} + \begin{bmatrix} 3c_1 \\ 3c_2 \\ 3c_3 \end{bmatrix} t^2 + \begin{bmatrix} 2d_1 \\ 2d_2 \\ 2d_3 \end{bmatrix} t + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

Plugging this into the matrix equation representing the system, we get

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} 3b_1 \\ 3b_2 \\ 3b_3 \end{bmatrix} e^{3t} + \begin{bmatrix} 3c_1 \\ 3c_2 \\ 3c_3 \end{bmatrix} t^2 + \begin{bmatrix} 2d_1 \\ 2d_2 \\ 2d_3 \end{bmatrix} t + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} e^{3t} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} t^3 + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} t^2 + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} t + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} \right]$$

$$+ \begin{bmatrix} 3 \\ 0 \\ -4 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} t$$

$$\begin{bmatrix} a_1e^t + 3b_1e^{3t} + 3c_1t^2 + 2d_1t + e_1 \\ a_2e^t + 3b_2e^{3t} + 3c_2t^2 + 2d_2t + e_2 \\ a_3e^t + 3b_3e^{3t} + 3c_3t^2 + 2d_3t + e_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1e^t + b_1e^{3t} + c_1t^3 + d_1t^2 + e_1t + f_1 \\ a_2e^t + b_2e^{3t} + c_2t^3 + d_2t^2 + e_2t + f_2 \\ a_3e^t + b_3e^{3t} + c_3t^3 + d_3t^2 + e_3t + f_3 \end{bmatrix}$$

$$+ \begin{bmatrix} 3e^t \\ 3t + 2e^{3t} \\ t^3 - 4e^t \end{bmatrix}$$

Breaking this equation into a system of three equations gives

$$a_1e^t + 3b_1e^{3t} + 3c_1t^2 + 2d_1t + e_1 = (a_2 + 3)e^t + b_2e^{3t} + c_2t^3 + d_2t^2 + e_2t + f_2$$

$$a_2e^t + 3b_2e^{3t} + 3c_2t^2 + 2d_2t + e_2 = (-4a_1 - 4a_2)e^t + (-4b_1 - 4b_2 + 2)e^{3t}$$

$$+(-4c_1 - 4c_2)t^3 + (-4d_1 - 4d_2)t^2 + (-4e_1 - 4e_2 + 3)t + (-4f_1 - 4f_2)$$

$$a_3e^t + 3b_3e^{3t} + 3c_3t^2 + 2d_3t + e_3 = (-2a_1 + a_2 + 2a_3 - 4)e^t + (-2b_1 + b_2 + 2b_3)e^{3t}$$

$$+(-2c_1 + c_2 - 2c_3 + 1)t^3 + (-2d_1 + d_2 - 2d_3)t^2$$

$$+(-2e_1 + e_2 - 2e_3)t + (-2f_1 + f_2 - 2f_3)$$

These equations can each be broken into its own system.

$$a_1 = a_2 + 3 \qquad a_2 = -4a_1 - 4a_2 \qquad a_3 = -2a_1 + a_2 + 2a_3 - 4$$

$$3b_1 = b_2 \qquad 3b_2 = -4b_1 - 4b_2 + 2 \qquad 3b_3 = -2b_1 + b_2 + 2b_3$$

$$0 = c_2 \qquad 0 = -4c_1 - 4c_2 \qquad 0 = -2c_1 + c_2 - 2c_3 + 1$$

$$3c_1 = d_2 \qquad 3c_2 = -4d_1 - 4d_2 \qquad 3c_3 = -2d_1 + d_2 - 2d_3$$

$$2d_1 = e_2 \qquad 2d_2 = -4e_1 - 4e_2 + 3 \qquad 2d_3 = -2e_1 + e_2 - 2e_3$$

$$e_1 = f_2 \qquad\qquad e_2 = -4f_1 - 4f_2 \qquad\qquad e_3 = -2f_1 + f_2 - 2f_3$$

Putting these results together gives $\vec{a} = (a_1, a_2, a_3) = (-3, -4,2)$, $\vec{b} = (b_1, b_2, b_3) = (2,6,2)$, $\vec{c} = (c_1, c_2, c_3) = (0,0, -1/2)$, $\vec{d} = (d_1, d_2, d_3) = (0,0, -3/4)$, $\vec{e} = (e_1, e_2, e_3) = (3/4,0,0)$, and $\vec{f} = (f_1, f_2, f_3) = (3/4,3/4,3/8)$. Therefore, the particular solution becomes

$$\vec{x}_p = \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ -\frac{3}{4} \end{bmatrix} t^2 + \begin{bmatrix} \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{8} \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 2 \\ 4 \\ 2 \end{bmatrix} te^{2t} \right)$$

$$+ \begin{bmatrix} -3 \\ -4 \\ 2 \end{bmatrix} e^t + \begin{bmatrix} 2 \\ 6 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} t^3 + \begin{bmatrix} 0 \\ 0 \\ -\frac{3}{4} \end{bmatrix} t^2 + \begin{bmatrix} \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} \frac{3}{4} \\ \frac{3}{4} \\ \frac{3}{8} \end{bmatrix}$$

VARIATION OF PARAMETERS FOR NONHOMOGENEOUS SYSTEMS

■ 1. Use the method of variation of parameters to find the general solution to the system, given the complementary solution.

$$\vec{x}' = A\vec{x} + \begin{bmatrix} e^{-3t} \\ 2e^t + 1 \end{bmatrix}$$

$$\vec{x}_c = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} e^{5t}$$

Solution:

We already have the complementary solution, so we'll use the solution vectors $\vec{x}_1$ and $\vec{x}_2$ to form $\Phi(t)$.

$$\Phi(t) = \begin{bmatrix} e^{2t} & 2e^{5t} \\ 3e^{2t} & 7e^{5t} \end{bmatrix}$$

Find the inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} e^{2t} & 2e^{5t} & | & 1 & 0 \\ 3e^{2t} & 7e^{5t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2e^{3t} & | & e^{-2t} & 0 \\ 3e^{2t} & 7e^{5t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2e^{3t} & | & e^{-2t} & 0 \\ 0 & e^{5t} & | & -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2e^{3t} & | & e^{-2t} & 0 \\ 0 & 1 & | & -3e^{-5t} & e^{-5t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 7e^{-2t} & -2e^{-2t} \\ 0 & 1 & | & -3e^{-5t} & e^{-5t} \end{bmatrix}$$

The inverse matrix is

$$\Phi^{-1}(t) = \begin{bmatrix} 7e^{-2t} & -2e^{-2t} \\ -3e^{-5t} & e^{-5t} \end{bmatrix}$$

so we'll integrate the product of the inverse matrix and $F(t)$ to get

$$\int \Phi^{-1}(t)F(t)\ dt$$

$$\int \begin{bmatrix} 7e^{-2t} & -2e^{-2t} \\ -3e^{-5t} & e^{-5t} \end{bmatrix} \begin{bmatrix} e^{-3t} \\ 2e^{t} + 1 \end{bmatrix}\ dt$$

$$\int \begin{bmatrix} 7e^{-5t} - 4e^{-t} - 2e^{-2t} \\ -3e^{-8t} + 2e^{-4t} + e^{-5t} \end{bmatrix}\ dt$$

$$\begin{bmatrix} -\frac{7}{5}e^{-5t} + 4e^{-t} + e^{-2t} \\ \frac{3}{8}e^{-8t} - \frac{1}{2}e^{-4t} - \frac{1}{5}e^{-5t} \end{bmatrix}$$

Then the particular solution is given by

$$\vec{x}_p = \Phi(t)\int \Phi^{-1}(t)F(t)\ dt$$

$$\vec{x}_p = \begin{bmatrix} e^{2t} & 2e^{5t} \\ 3e^{2t} & 7e^{5t} \end{bmatrix} \begin{bmatrix} -\frac{7}{5}e^{-5t} + 4e^{-t} + e^{-2t} \\ \frac{3}{8}e^{-8t} - \frac{1}{2}e^{-4t} - \frac{1}{5}e^{-5t} \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} -\frac{7}{5}e^{-3t} + 4e^t + 1 + \frac{3}{4}e^{-3t} - e^t - \frac{2}{5} \\ -\frac{21}{5}e^{-3t} + 12e^t + 3 + \frac{21}{8}e^{-3t} - \frac{7}{2}e^t - \frac{7}{5} \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} -\frac{13}{20}e^{-3t} + 3e^t + \frac{3}{5} \\ -\frac{63}{40}e^{-3t} + \frac{17}{2}e^t + \frac{8}{5} \end{bmatrix}$$

$$\vec{x}_p = -\frac{1}{20}\begin{bmatrix} 13 \\ \frac{63}{2} \end{bmatrix} e^{-3t} + \begin{bmatrix} 3 \\ \frac{17}{2} \end{bmatrix} e^t + \frac{1}{5}\begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Then the general solution is

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1\begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2\begin{bmatrix} 2 \\ 7 \end{bmatrix} e^{5t} - \frac{1}{20}\begin{bmatrix} 13 \\ \frac{63}{2} \end{bmatrix} e^{-3t} + \begin{bmatrix} 3 \\ \frac{17}{2} \end{bmatrix} e^t + \frac{1}{5}\begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

■ 2. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix} \vec{x} + \begin{bmatrix} 2t + 1 \\ e^t \end{bmatrix}$$

Solution:

We need to start by solving the corresponding homogeneous system,

$$\vec{x}' = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix} \vec{x}$$

so we'll find $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 6 - \lambda & -8 \\ 4 & -6 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = \lambda^2 - 36 + 32 = \lambda^2 - 4$$

$$|A - \lambda I| = (\lambda + 2)(\lambda - 2)$$

Solve the characteristic equation for the Eigenvalues of the system.

$$(\lambda + 2)(\lambda - 2) = 0$$

$$\lambda = 2, \ -2$$

Then for these Eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = -2$, we find

$$A - 2I = \begin{bmatrix} 4 & -8 \\ 4 & -8 \end{bmatrix} \qquad A + 2I = \begin{bmatrix} 8 & -8 \\ 4 & -4 \end{bmatrix}$$

Put these matrices into reduced row-echelon form.

$$\begin{bmatrix} 4 & -8 \\ 4 & -8 \end{bmatrix} \qquad \begin{bmatrix} 8 & -8 \\ 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 4 & -8 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ 4 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

If we turn each matrix back into a system of equations, we get

$$k_1 - 2k_2 = 0 \qquad k_1 - k_2 = 0$$

$$k_1 = 2k_2 \qquad k_1 = k_2$$

For the first system, we choose $(k_1, k_2) = (2,1)$, and for the second system, we choose $(k_1, k_2) = (1,1)$, and we get the Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then the solution vectors are

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} \qquad \vec{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t}$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\vec{x_1}$ and $\vec{x_2}$ to form $\Phi(t)$,

$$\Phi(t) = \begin{bmatrix} 2e^{2t} & e^{-2t} \\ e^{2t} & e^{-2t} \end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} 2e^{2t} & e^{-2t} & | & 1 & 0 \\ e^{2t} & e^{-2t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2}e^{-4t} & | & \frac{1}{2}e^{-2t} & 0 \\ e^{2t} & e^{-2t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2}e^{-4t} & | & \frac{1}{2}e^{-2t} & 0 \\ 0 & \frac{1}{2}e^{-2t} & | & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2}e^{-4t} & | & \frac{1}{2}e^{-2t} & 0 \\ 0 & 1 & | & -e^{2t} & 2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & e^{-2t} & -e^{-2t} \\ 0 & 1 & | & -e^{2t} & 2e^{2t} \end{bmatrix}$$

The inverse matrix is

$$\Phi^{-1}(t) = \begin{bmatrix} e^{-2t} & -e^{-2t} \\ -e^{2t} & 2e^{2t} \end{bmatrix}$$

so we'll integrate the product of the inverse matrix and $F(t)$ to get

$$\int \Phi^{-1}(t)F(t)\ dt$$

$$\int \begin{bmatrix} e^{-2t} & -e^{-2t} \\ -e^{2t} & 2e^{2t} \end{bmatrix} \begin{bmatrix} 2t+1 \\ e^{t} \end{bmatrix}\ dt$$

$$\int \begin{bmatrix} 2te^{-2t} + e^{-2t} - e^{-t} \\ -2te^{2t} - e^{2t} + 2e^{3t} \end{bmatrix}\ dt$$

$$\begin{bmatrix} -te^{-2t} - e^{-2t} + e^{-t} \\ -te^{2t} + \frac{2}{3}e^{3t} \end{bmatrix}$$

Then the particular solution is given by

$$\vec{x_p} = \Phi(t) \int \Phi^{-1}(t)F(t)\ dt$$

$$\vec{x_p} = \begin{bmatrix} 2e^{2t} & e^{-2t} \\ e^{2t} & e^{-2t} \end{bmatrix} \begin{bmatrix} -te^{-2t} - e^{-2t} + e^{-t} \\ -te^{2t} + \frac{2}{3}e^{3t} \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} -2t - 2 + 2e^t - t + \frac{2}{3}e^t \\ -t - 1 + e^t - t + \frac{2}{3}e^t \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} -3 \\ -2 \end{bmatrix} t + \begin{bmatrix} -2 \\ -1 \end{bmatrix} + \begin{bmatrix} \frac{8}{3} \\ \frac{5}{3} \end{bmatrix} e^t$$

Then the general solution is

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} - \begin{bmatrix} 3 \\ 2 \end{bmatrix} t - \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \frac{1}{3}e^t \begin{bmatrix} 8 \\ 5 \end{bmatrix}$$

■ 3. Solve the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} \sin t + \cos t \\ \sin t \end{bmatrix}$$

Solution:

We need to start by solving the corresponding homogeneous system,

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x}$$

so we'll find $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix}$$

$$|A - \lambda I| = \lambda^2 + 1$$

Solve the characteristic equation for the Eigenvalues of the system.

$$\lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

Then for these Eigenvalues, $\lambda_1 = i$ and $\lambda_2 = -i$, we find

$$A - iI = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

Put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & i \\ -1 & -i \end{bmatrix}$$

$$\begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

Rewriting the matrix as a system of equations gives

$$k_1 + ik_2 = 0$$

$$k_1 = -ik_2$$

We'll choose $k_1 = i$ and $k_2 = -1$ to get the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} i \\ -1 \end{bmatrix}$$

$$\overrightarrow{k_1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} i$$

Then the solutions to the system are

$$\overrightarrow{x_1} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} \cos t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t = \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \sin t = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ to form $\Phi(t)$,

$$\Phi(t) = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} -\sin t & \cos t & | & 1 & 0 \\ -\cos t & -\sin t & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\cot t & | & -\frac{1}{\sin t} & 0 \\ -\cos t & -\sin t & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\cot t & | & -\frac{1}{\sin t} & 0 \\ 0 & -\frac{1}{\sin t} & | & -\frac{\cos t}{\sin t} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\cot t & | & -\frac{1}{\sin t} & 0 \\ 0 & 1 & | & \cos t & -\sin t \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -\sin t & -\cos t \\ 0 & 1 & | & \cos t & -\sin t \end{bmatrix}$$

The inverse matrix is

$$\Phi^{-1}(t) = \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix}$$

so we'll integrate the product of the inverse matrix and $F(t)$ to get

$$\int \Phi^{-1}(t)F(t)\ dt$$

$$\int \begin{bmatrix} -\sin t & -\cos t \\ \cos t & -\sin t \end{bmatrix} \begin{bmatrix} \sin t + \cos t \\ \sin t \end{bmatrix}\ dt$$

$$\int \begin{bmatrix} -\sin^2 t - 2\sin t\cos t \\ \sin t\cos t + \cos^2 t - \sin^2 t \end{bmatrix}\ dt$$

$$\int \begin{bmatrix} -\frac{1}{2} + \frac{1}{2}\cos(2t) - \sin(2t) \\ \frac{1}{2}\sin(2t) + \cos(2t) \end{bmatrix} \ dt$$

$$\begin{bmatrix} -\frac{t}{2} + \frac{1}{4}\sin(2t) + \frac{1}{2}\cos(2t) \\ -\frac{1}{4}\cos(2t) + \frac{1}{2}\sin(2t) \end{bmatrix}$$

Then the particular solution is given by

$$\vec{x_p} = \Phi(t) \int \Phi^{-1}(t) F(t) \ dt$$

$$\vec{x_p} = \begin{bmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{bmatrix} \begin{bmatrix} -\frac{t}{2} + \frac{1}{4}\sin(2t) + \frac{1}{2}\cos(2t) \\ -\frac{1}{4}\cos(2t) + \frac{1}{2}\sin(2t) \end{bmatrix}$$

$$\vec{x_p} = \begin{bmatrix} \frac{t}{2}\sin t - \frac{1}{4}\sin(2t)\sin t - \frac{1}{2}\cos(2t)\sin t - \frac{1}{4}\cos t\cos(2t) + \frac{1}{2}\cos t\sin(2t) \\ \frac{t}{2}\cos t - \frac{1}{4}\sin(2t)\cos t - \frac{1}{2}\cos(2t)\cos t + \frac{1}{4}\cos(2t)\sin t - \frac{1}{2}\sin(2t)\sin t \end{bmatrix}$$

$$\vec{x_p} = \frac{1}{4}\begin{bmatrix} 2t - \sin(2t) - 2\cos(2t) \\ \cos(2t) - 2\sin(2t) \end{bmatrix} \sin t + \frac{1}{4}\begin{bmatrix} -\cos(2t) + 2\sin(2t) \\ 2t - \sin(2t) - 2\cos(2t) \end{bmatrix} \cos t$$

Then the general solution is

$$\vec{x} = \vec{x_c} + \vec{x_p}$$

$$\vec{x} = c_1 \begin{bmatrix} -\sin t \\ -\cos t \end{bmatrix} + c_2 \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 2t - \sin(2t) - 2\cos(2t) \\ \cos(2t) - 2\sin(2t) \end{bmatrix} \sin t$$

$$+\frac{1}{4}\begin{bmatrix} -\cos(2t) + 2\sin(2t) \\ 2t - \sin(2t) - 2\cos(2t) \end{bmatrix} \cos t$$

■ 4. Solve the system of differential equations.

$$x_1' = -5x_1 + 12x_2 + e^t$$

$$x_2' = -4x_1 + 9x_2 + 3e^{-t}$$

Solution:

We need to start by solving the corresponding homogeneous system,

$$\vec{x}' = \begin{bmatrix} -5 & 12 \\ -4 & 9 \end{bmatrix} \vec{x}$$

so we'll find $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} -5 & 12 \\ -4 & 9 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} -5 & 12 \\ -4 & 9 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} -5 - \lambda & 12 \\ -4 & 9 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = \lambda^2 - 4\lambda - 45 + 48 = (\lambda - 1)(\lambda - 3)$$

Solve the characteristic equation for the Eigenvalues of the system.

$$(\lambda - 1)(\lambda - 3) = 0$$

$$\lambda = 1, 3$$

Then for these Eigenvalues, $\lambda_1 = 1$ and $\lambda_2 = 3$, we find

$$A - I = \begin{bmatrix} -6 & 12 \\ -4 & 8 \end{bmatrix} \qquad A - 3I = \begin{bmatrix} -8 & 12 \\ -4 & 6 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} -6 & 12 \\ -4 & 8 \end{bmatrix} \qquad \begin{bmatrix} -8 & 12 \\ -4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -4 & 8 \end{bmatrix} \qquad \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 - 2k_2 = 0 \qquad 2k_1 - 3k_2 = 0$$

$$k_1 = 2k_2 \qquad 2k_1 = 3k_2$$

For the first system, we choose $(k_1, k_2) = (2,1)$ and from the second system, we choose $(k_1, k_2) = (3,2)$, which gives the Eigenvectors

$$\vec{k_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and the solution vectors

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t \qquad \vec{x}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x}_c = c_1\vec{x}_1 + c_2\vec{x}_2$$

$$\vec{x}_c = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t}$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\vec{x}_1$ and $\vec{x}_2$ to form $\Phi(t)$,

$$\Phi(t) = \begin{bmatrix} 2e^t & 3e^{3t} \\ e^t & 2e^{3t} \end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} 2e^t & 3e^{3t} & | & 1 & 0 \\ e^t & 2e^{3t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2}e^{2t} & | & \frac{1}{2}e^{-t} & 0 \\ e^t & 2e^{3t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2}e^{2t} & | & \frac{1}{2}e^{-t} & 0 \\ 0 & \frac{1}{2}e^{3t} & | & -\frac{1}{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2}e^{2t} & | & \frac{1}{2}e^{-t} & 0 \\ 0 & 1 & | & -e^{-3t} & 2e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 2e^{-t} & -3e^{-t} \\ 0 & 1 & | & -e^{-3t} & 2e^{-3t} \end{bmatrix}$$

The inverse matrix is

$$\Phi^{-1}(t) = \begin{bmatrix} 2e^{-t} & -3e^{-t} \\ -e^{-3t} & 2e^{-3t} \end{bmatrix}$$

so we'll integrate the product of the inverse matrix and $F(t)$ to get

$$\int \Phi^{-1}(t)F(t)\ dt$$

$$\int \begin{bmatrix} 2e^{-t} & -3e^{-t} \\ -e^{-3t} & 2e^{-3t} \end{bmatrix} \begin{bmatrix} e^{t} \\ 3e^{-t} \end{bmatrix}\ dt$$

$$\int \begin{bmatrix} 2 - 9e^{-2t} \\ -e^{-2t} + 6e^{-4t} \end{bmatrix}\ dt$$

$$\begin{bmatrix} 2t + \frac{9}{2}e^{-2t} \\ \frac{1}{2}e^{-2t} - \frac{3}{2}e^{-4t} \end{bmatrix}$$

Then the particular solution is given by

$$\vec{x_p} = \Phi(t) \int \Phi^{-1}(t)F(t)\ dt$$

$$\vec{x}_p = \begin{bmatrix} 2e^t & 3e^{3t} \\ e^t & 2e^{3t} \end{bmatrix} \begin{bmatrix} 2t + \frac{9}{2}e^{-2t} \\ \frac{1}{2}e^{-2t} - \frac{3}{2}e^{-4t} \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} 4te^t + 9e^{-t} + \frac{3}{2}e^t - \frac{9}{2}e^{-t} \\ 2te^t + \frac{9}{2}e^{-t} + e^t - 3e^{-t} \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^t + \begin{bmatrix} \frac{9}{2} \\ \frac{3}{2} \end{bmatrix} e^{-t}$$

$$\vec{x}_p = \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^t + \frac{3}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t}$$

Then the general solution is

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{3t} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} \frac{3}{2} \\ 1 \end{bmatrix} e^t + \frac{3}{2} \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-t}$$

■ 5. Solve the system of differential equations.

$$x_1' = x_1 + 3x_2 + 1 + e^{2t}$$

$$x_2' = x_2 + e^t$$

Solution:

We need to start by solving the corresponding homogeneous system,

$$\vec{x}' = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \vec{x}$$

so we'll find $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 3 \\ 0 & 1 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (1 - \lambda)^2 - 0 = (1 - \lambda)^2$$

Solve the characteristic equation for the Eigenvalues of the system.

$$(1 - \lambda)^2 = 0$$

$$\lambda = 1$$

Then for the Eigenvalue $\lambda = 1$, we find

$$A - I = \begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix}$$

Put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

If we turn this matrix back into a system of equations, we get just $k_2 = 0$, so we'll choose $k_1 = 1$ with $k_2 = 0$ to find the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then we'll use this Eigenvector to find a second solution vector.

$$\begin{bmatrix} 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This matrix equation gives the system of equations $3p_2 = 1$ and $p_1 = 0$, and then the solution vector

$$\vec{p} = \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix} e^t \right)$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\vec{x_1}$ and $\vec{x_2}$ to form $\Phi(t)$,

$$\Phi(t) = \begin{bmatrix} e^t & te^t \\ 0 & \frac{1}{3}e^t \end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} e^t & te^t & | & 1 & 0 \\ 0 & \frac{1}{3}e^t & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t & | & e^{-t} & 0 \\ 0 & \frac{1}{3}e^t & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t & | & e^{-t} & 0 \\ 0 & 1 & | & 0 & 3e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & e^{-t} & -3te^{-t} \\ 0 & 1 & | & 0 & 3e^{-t} \end{bmatrix}$$

The inverse matrix is

$$\Phi^{-1}(t) = \begin{bmatrix} e^{-t} & -3te^{-t} \\ 0 & 3e^{-t} \end{bmatrix}$$

so we'll integrate the product of the inverse matrix and $F(t)$ to get

$$\int \Phi^{-1}(t)F(t)\ dt$$

$$\int \begin{bmatrix} e^{-t} & -3te^{-t} \\ 0 & 3e^{-t} \end{bmatrix} \begin{bmatrix} e^{2t}+1 \\ e^t \end{bmatrix}\ dt$$

$$\int \begin{bmatrix} e^t + e^{-t} - 3t \\ 3 \end{bmatrix}\ dt$$

$$\begin{bmatrix} e^t - e^{-t} - \frac{3t^2}{2} \\ 3t \end{bmatrix}$$

Then the particular solution is given by

$$\overrightarrow{x_p} = \Phi(t)\int \Phi^{-1}(t)F(t)\ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^t & te^t \\ 0 & \frac{1}{3}e^t \end{bmatrix}\begin{bmatrix} e^t - e^{-t} - \frac{3t^2}{2} \\ 3t \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t} - 1 - \frac{3t^2}{2}e^t + 3t^2e^t \\ te^t \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t} - 1 + \frac{3t^2}{2}e^t \\ te^t \end{bmatrix}$$

Then the general solution is

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}e^t + c_2\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ \frac{1}{3} \end{bmatrix}e^t\right) + \begin{bmatrix} e^{2t} - 1 + \frac{3t^2}{2}e^t \\ te^t \end{bmatrix}$$

■ 6. Solve the system of differential equations.

$$\overrightarrow{x}' = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}\overrightarrow{x} + \begin{bmatrix} t^5 \\ t^3 \end{bmatrix}$$

Solution:

We need to start by solving the corresponding homogeneous system,

$$\vec{x}' = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \vec{x}$$

so we'll find $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 2 \\ 0 & -\lambda \end{vmatrix}$$

$$|A - \lambda I| = \lambda^2$$

Solve the characteristic equation for the Eigenvalues of the system.

$$\lambda^2 = 0$$

$$\lambda = 0$$

Then for the Eigenvalue $\lambda = 0$, we find

$$A - (0)I = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix}$$

Put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

If we rewrite this matrix as a system of equations, we get just $k_2 = 0$, so we'll choose $k_1 = 1$ with $k_2 = 0$ to get the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Then we'll use this Eigenvector to find a second solution vector.

$$\begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This matrix equation gives the system of equations $2p_2 = 1$, so we find $p_2 = 1/2$ and choose $p_1 = 0$ to get the second solution vector

$$\overrightarrow{p} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \right)$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ to form $\Phi(t)$,

$$\Phi(t) = \begin{bmatrix} 1 & t \\ 0 & \frac{1}{2} \end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} 1 & t & | & 1 & 0 \\ 0 & \frac{1}{2} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t & | & 1 & 0 \\ 0 & 1 & | & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 1 & -2t \\ 0 & 1 & | & 0 & 2 \end{bmatrix}$$

The inverse matrix is

$$\Phi^{-1}(t) = \begin{bmatrix} 1 & -2t \\ 0 & 2 \end{bmatrix}$$

so we'll integrate the product of the inverse matrix and $F(t)$ to get

$$\int \Phi^{-1}(t)F(t)\ dt$$

$$\int \begin{bmatrix} 1 & -2t \\ 0 & 2 \end{bmatrix} \begin{bmatrix} t^5 \\ t^3 \end{bmatrix}\ dt$$

$$\int \begin{bmatrix} t^5 - 2t^4 \\ 2t^3 \end{bmatrix}\ dt$$

$$\begin{bmatrix} \frac{1}{6}t^6 - \frac{2}{5}t^5 \\ \frac{1}{2}t^4 \end{bmatrix}$$

Then the particular solution is given by

$$\overrightarrow{x_p} = \Phi(t)\int \Phi^{-1}(t)F(t)\ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} 1 & t \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{6}t^6 - \frac{2}{5}t^5 \\ \frac{1}{2}t^4 \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} \frac{1}{6}t^6 + \frac{1}{10}t^5 \\ \frac{1}{4}t^4 \end{bmatrix}$$

Then the general solution is

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \left(\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} t \right) + \begin{bmatrix} \frac{1}{6}t^6 + \frac{1}{10}t^5 \\ \frac{1}{4}t^4 \end{bmatrix}$$

THE MATRIX EXPONENTIAL

■ 1. Use an inverse Laplace transform to calculate the matrix exponential.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$$

Solution:

First, we'll find $sI - A$.

$$sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s-1 & 2 \\ -3 & s+1 \end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \,|\, I]$ into $[I \,|\, (sI - A)^{-1}]$.

$$\begin{bmatrix} s-1 & 2 & | & 1 & 0 \\ -3 & s+1 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{s-1} & | & \frac{1}{s-1} & 0 \\ -3 & s+1 & | & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & \frac{2}{s-1} & | & \frac{1}{s-1} & 0 \\ 0 & \frac{5+s^2}{s-1} & | & \frac{3}{s-1} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{2}{s-1} & | & \frac{1}{s-1} & 0 \\ 0 & 1 & | & \frac{3}{5+s^2} & \frac{s-1}{5+s^2} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & | & \frac{s+1}{5+s^2} & \frac{-2}{5+s^2} \\ 0 & 1 & | & \frac{3}{5+s^2} & \frac{s-1}{5+s^2} \end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s+1}{5+s^2} & \frac{-2}{5+s^2} \\ \frac{3}{5+s^2} & \frac{s-1}{5+s^2} \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+5} + \frac{1}{s^2+5} & \frac{-2}{s^2+5} \\ \frac{3}{s^2+5} & \frac{s}{s^2+5} - \frac{1}{s^2+5} \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} \cos(\sqrt{5}t) + \frac{1}{\sqrt{5}}\sin(\sqrt{5}t) & -\frac{2}{\sqrt{5}}\sin(\sqrt{5}t) \\ \frac{3}{\sqrt{5}}\sin(\sqrt{5}t) & \cos(\sqrt{5}t) - \frac{1}{\sqrt{5}}\sin(\sqrt{5}t) \end{bmatrix}$$

and then this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} \cos(\sqrt{5}t) + \frac{1}{\sqrt{5}}\sin(\sqrt{5}t) & -\frac{2}{\sqrt{5}}\sin(\sqrt{5}t) \\ \frac{3}{\sqrt{5}}\sin(\sqrt{5}t) & \cos(\sqrt{5}t) - \frac{1}{\sqrt{5}}\sin(\sqrt{5}t) \end{bmatrix}$$

■ 2. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \vec{x}$$

Solution:

First, we'll find $sI - A$.

$$sI - A = s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}1 & -1\\-3 & 3\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}1 & -1\\-3 & 3\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s-1 & 1\\3 & s-3\end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A\,|\,I]$ into $[I\,|\,(sI - A)^{-1}]$.

$$\begin{bmatrix}s-1 & 1 & | & 1 & 0\\3 & s-3 & | & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & \frac{1}{s-1} & | & \frac{1}{s-1} & 0\\3 & s-3 & | & 0 & 1\end{bmatrix}$$

$$\to \begin{bmatrix}1 & \frac{1}{s-1} & | & \frac{1}{s-1} & 0\\0 & \frac{s^2-4s}{s-1} & | & -\frac{3}{s-1} & 1\end{bmatrix} \to \begin{bmatrix}1 & \frac{1}{s-1} & | & \frac{1}{s-1} & 0\\0 & 1 & | & \frac{-3}{s(s-4)} & \frac{s-1}{s(s-4)}\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 0 & | & \frac{3}{4s} + \frac{1}{4(s-4)} & \frac{-1}{s(s-4)}\\0 & 1 & | & \frac{-3}{s(s-4)} & \frac{s-1}{s(s-4)}\end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix}\frac{3}{4s} + \frac{1}{4(s-4)} & \frac{-1}{s(s-4)}\\\frac{-3}{s(s-4)} & \frac{s-1}{s(s-4)}\end{bmatrix} = \begin{bmatrix}\frac{3}{4s} + \frac{1}{4(s-4)} & \frac{1}{4s} - \frac{1}{4(s-4)}\\\frac{3}{4s} - \frac{3}{4(s-4)} & \frac{1}{4s} + \frac{3}{4(s-4)}\end{bmatrix}$$

$$\mathscr{L}^{-1}((sI-A)^{-1}) = \begin{bmatrix} \frac{3}{4}+\frac{1}{4}e^{4t} & \frac{1}{4}-\frac{1}{4}e^{4t} \\ \frac{3}{4}-\frac{3}{4}e^{4t} & \frac{1}{4}+\frac{3}{4}e^{4t} \end{bmatrix}$$

and then this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} \frac{3}{4}+\frac{1}{4}e^{4t} & \frac{1}{4}-\frac{1}{4}e^{4t} \\ \frac{3}{4}-\frac{3}{4}e^{4t} & \frac{1}{4}+\frac{3}{4}e^{4t} \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the general solution to the homogeneous system is

$$\vec{x} = e^{At}C$$

$$\vec{x} = \begin{bmatrix} \frac{3}{4}+\frac{1}{4}e^{4t} & \frac{1}{4}-\frac{1}{4}e^{4t} \\ \frac{3}{4}-\frac{3}{4}e^{4t} & \frac{1}{4}+\frac{3}{4}e^{4t} \end{bmatrix} \begin{bmatrix} 4c_1 \\ 4c_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} (3c_1+c_2)+(c_1-c_2)e^{4t} \\ (3c_1+c_2)-3(c_1-c_2)e^{4t} \end{bmatrix}$$

$$\vec{x} = (3c_1+c_2)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + (c_1-c_2)\begin{bmatrix} e^{4t} \\ -3e^{4t} \end{bmatrix}$$

Simplifying the constants, we get

$$\vec{x} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ -3 \end{bmatrix}e^{4t}$$

■ 3. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} \vec{x}$$

Solution:

First, we'll find $sI - A$.

$$sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 1 \\ 3 & s-2 \end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \,|\, I]$ into $[I \,|\, (sI - A)^{-1}]$.

$$\begin{bmatrix} s & 1 & | & 1 & 0 \\ 3 & s-2 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{s} & | & \frac{1}{s} & 0 \\ 3 & s-2 & | & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & \frac{1}{s} & | & \frac{1}{s} & 0 \\ 0 & \frac{s^2 - 2s - 3}{s} & | & -\frac{3}{s} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{s} & | & \frac{1}{s} & 0 \\ 0 & 1 & | & \frac{-3}{(s+1)(s-3)} & \frac{s}{(s+1)(s-3)} \end{bmatrix}$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & \frac{1}{s}+\frac{3}{s(s+1)(s-3)} & \frac{-1}{(s+1)(s-3)} \\ 0 & 1 & \frac{-3}{(s+1)(s-3)} & \frac{s}{(s+1)(s-3)} \end{array}\right]$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI-A)^{-1} = \begin{bmatrix} \frac{1}{s}+\frac{3}{s(s+1)(s-3)} & \frac{-1}{(s+1)(s-3)} \\ \frac{-3}{(s+1)(s-3)} & \frac{s}{(s+1)(s-3)} \end{bmatrix} = \begin{bmatrix} \frac{\frac{3}{4}}{s+1}+\frac{\frac{1}{4}}{s-3} & \frac{\frac{1}{4}}{s+1}+\frac{-\frac{1}{4}}{s-3} \\ \frac{\frac{3}{4}}{s+1}+\frac{-\frac{3}{4}}{s-3} & \frac{\frac{1}{4}}{s+1}+\frac{\frac{3}{4}}{s-3} \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI-A)^{-1}) = \begin{bmatrix} \frac{3}{4}e^{-t}+\frac{1}{4}e^{3t} & \frac{1}{4}e^{-t}-\frac{1}{4}e^{3t} \\ \frac{3}{4}e^{-t}-\frac{3}{4}e^{3t} & \frac{1}{4}e^{-t}+\frac{3}{4}e^{3t} \end{bmatrix}$$

and then this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} \frac{3}{4}e^{-t}+\frac{1}{4}e^{3t} & \frac{1}{4}e^{-t}-\frac{1}{4}e^{3t} \\ \frac{3}{4}e^{-t}-\frac{3}{4}e^{3t} & \frac{1}{4}e^{-t}+\frac{3}{4}e^{3t} \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the general solution to the homogeneous system is

$$\vec{x} = e^{At}C$$

$$\vec{x} = \begin{bmatrix} \frac{3}{4}e^{-t}+\frac{1}{4}e^{3t} & \frac{1}{4}e^{-t}-\frac{1}{4}e^{3t} \\ \frac{3}{4}e^{-t}-\frac{3}{4}e^{3t} & \frac{1}{4}e^{-t}+\frac{3}{4}e^{3t} \end{bmatrix} \begin{bmatrix} 4c_1 \\ 4c_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} (3c_1+c_2)e^{-t}+(c_1-c_2)e^{3t} \\ (3c_1+c_2)e^{-t}-3(c_1-c_2)e^{3t} \end{bmatrix}$$

$$\vec{x} = (3c_1 + c_2)\begin{bmatrix} e^{-t} \\ e^{-t} \end{bmatrix} + (c_1 - c_2)\begin{bmatrix} e^{3t} \\ -3e^{3t} \end{bmatrix}$$

Simplifying the constants, we get

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{3t}$$

■ 4. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 6t^2 - te^{-3t} \\ \sin t \end{bmatrix}$$

Solution:

First, we'll find $sI - A$.

$$sI - A = s\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s + 2 & 0 \\ 0 & s - 1 \end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \mid I]$ into $[I \mid (sI - A)^{-1}]$.

$$\begin{bmatrix} s+2 & 0 & | & 1 & 0 \\ 0 & s-1 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{1}{s+2} & 0 \\ 0 & 1 & | & 0 & \frac{1}{s-1} \end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

and then this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the complementary solution will be

$$\vec{x_c} = e^{At}C$$

$$\vec{x_c} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x_c} = \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^{t} \end{bmatrix}$$

To find the particular solution, we'll need e^{-As}, which we find by making the substitution $t = s$ into e^{At},

$$e^{As} = \begin{bmatrix} e^{-2s} & 0 \\ 0 & e^{s} \end{bmatrix}$$

and then calculating the inverse of this resulting e^{As}.

$$\left[\begin{array}{cc|cc} e^{-2s} & 0 & 1 & 0 \\ 0 & e^{s} & 0 & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & e^{2s} & 0 \\ 0 & 1 & 0 & e^{-s} \end{array}\right]$$

So e^{-As} is given by this resulting matrix.

$$e^{-As} = \begin{bmatrix} e^{2s} & 0 \\ 0 & e^{-s} \end{bmatrix}$$

We'll find $F(s)$ by substituting $t = s$ into F.

$$F(s) = \begin{bmatrix} 6s^2 - se^{-3s} \\ \sin s \end{bmatrix}$$

Therefore, the particular solution will be

$$\vec{x_p} = e^{At} \int_{t_0}^{t} e^{-As} F(s)\ ds$$

$$\vec{x_p} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{2s} & 0 \\ 0 & e^{-s} \end{bmatrix} \begin{bmatrix} 6s^2 - se^{-3s} \\ \sin s \end{bmatrix}\ ds$$

$$\vec{x_p} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix} \int_0^t \begin{bmatrix} 6s^2e^{2s} - se^{-s} \\ e^{-s} \sin s \end{bmatrix}\ ds$$

Now we'll integrate and then evaluate on $[0,t]$.

$$\vec{x_p} = \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} e^{2t}\left(3t^2 - 3t + \frac{3}{2}\right) + e^{-t}(t+1) - \frac{5}{2} \\ -\frac{1}{2}e^{-t}(\sin t + \cos t) + \frac{1}{2} \end{bmatrix}$$

$$\vec{x_p} = \begin{bmatrix} 3t^2 - 3t + \frac{3}{2} + e^{-3t}(t+1) - \frac{5}{2}e^{-2t} \\ -\frac{1}{2}(\sin t + \cos t) + \frac{1}{2}e^t \end{bmatrix}$$

Then the general solution is the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x_c} + \vec{x_p}$$

$$\vec{x} = \begin{bmatrix} c_1 e^{-2t} \\ c_2 e^t \end{bmatrix} + \begin{bmatrix} 3t^2 - 3t + \frac{3}{2} + e^{-3t}(t+1) - \frac{5}{2}e^{-2t} \\ -\frac{1}{2}(\sin t + \cos t) + \frac{1}{2}e^t \end{bmatrix}$$

■ 5. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 3 \\ 3 & -1 & -4 \end{bmatrix} \vec{x}$$

Solution:

First, we'll find $sI - A$.

$$sI - A = s\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}-1 & 0 & 0\\-2 & 0 & 3\\3 & -1 & -4\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s & 0 & 0\\0 & s & 0\\0 & 0 & s\end{bmatrix} - \begin{bmatrix}-1 & 0 & 0\\-2 & 0 & 3\\3 & -1 & -4\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s+1 & 0 & 0\\2 & s & -3\\-3 & 1 & s+4\end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \,|\, I]$ into $[I \,|\, (sI - A)^{-1}]$.

$$\begin{bmatrix}s+1 & 0 & 0 & | & 1 & 0 & 0\\2 & s & -3 & | & 0 & 1 & 0\\-3 & 1 & s+4 & | & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & | & \frac{1}{s+1} & 0 & 0\\2 & s & -3 & | & 0 & 1 & 0\\-3 & 1 & s+4 & | & 0 & 0 & 1\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 0 & 0 & | & \frac{1}{s+1} & 0 & 0\\0 & s & -3 & | & -\frac{2}{s+1} & 1 & 0\\0 & 1 & s+4 & | & \frac{3}{s+1} & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & | & \frac{1}{s+1} & 0 & 0\\0 & 1 & s+4 & | & \frac{3}{s+1} & 0 & 1\\0 & s & -3 & | & -\frac{2}{s+1} & 1 & 0\end{bmatrix}$$

$$\to \begin{bmatrix}1 & 0 & 0 & | & \frac{1}{s+1} & 0 & 0\\0 & 1 & s+4 & | & \frac{3}{s+1} & 0 & 1\\0 & 0 & -(s+1)(s+3) & | & -\frac{3s+2}{s+1} & 1 & -s\end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s+1} & 0 & 0 \\ 0 & 1 & s+4 & | & \frac{3}{s+1} & 0 & 1 \\ 0 & 0 & 1 & | & \frac{3s+2}{(s+1)^2(s+3)} & \frac{-1}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s+1} & 0 & 0 \\ 0 & 1 & 0 & | & \frac{1-2s}{(s+1)^2(s+3)} & \frac{s+4}{(s+1)(s+3)} & \frac{3}{(s+1)(s+3)} \\ 0 & 0 & 1 & | & \frac{3s+2}{(s+1)^2(s+3)} & \frac{-1}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ \frac{-2s+1}{(s+1)^2(s+3)} & \frac{s+4}{(s+1)(s+3)} & \frac{3}{(s+1)(s+3)} \\ \frac{3s+2}{(s+1)^2(s+3)} & \frac{-1}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}$$

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & 0 \\ \frac{3}{2(s+1)^2} - \frac{7}{4(s+1)} + \frac{7}{4(s+3)} & \frac{3}{2(s+1)} - \frac{1}{2(s+3)} & \frac{3}{2(s+1)} - \frac{3}{2(s+3)} \\ -\frac{1}{2(s+1)^2} + \frac{7}{4(s+1)} - \frac{7}{4(s+3)} & -\frac{1}{2(s+1)} + \frac{1}{2(s+3)} & -\frac{1}{2(s+1)} + \frac{3}{2(s+3)} \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} e^{-t} & 0 & 0 \\ \frac{3}{2}te^{-t} - \frac{7}{4}e^{-t} + \frac{7}{4}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{3}{2}e^{-t} - \frac{3}{2}e^{-3t} \\ -\frac{1}{2}te^{-t} + \frac{7}{4}e^{-t} - \frac{7}{4}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}$$

and then this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} e^{-t} & 0 & 0 \\ \frac{3}{2}te^{-t} - \frac{7}{4}e^{-t} + \frac{7}{4}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{3}{2}e^{-t} - \frac{3}{2}e^{-3t} \\ -\frac{1}{2}te^{-t} + \frac{7}{4}e^{-t} - \frac{7}{4}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the general solution will be

$$\vec{x} = \begin{bmatrix} e^{-t} & 0 & 0 \\ \frac{3}{2}te^{-t} - \frac{7}{4}e^{-t} + \frac{7}{4}e^{-3t} & \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{3}{2}e^{-t} - \frac{3}{2}e^{-3t} \\ -\frac{1}{2}te^{-t} + \frac{7}{4}e^{-t} - \frac{7}{4}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\vec{x}_c = \begin{bmatrix} c_1e^{-t} \\ \frac{3}{2}c_1te^{-t} + \left(-\frac{7}{4}c_1 + \frac{3}{2}c_2 + \frac{3}{2}c_3\right)e^{-t} + \left(\frac{7}{4}c_1 - \frac{1}{2}c_2 - \frac{3}{2}c_3\right)e^{-3t} \\ -\frac{1}{2}c_1te^{-t} + \left(\frac{7}{4}c_1 - \frac{1}{2}c_2 - \frac{1}{2}c_3\right)e^{-t} + \left(-\frac{7}{4}c_{+}\frac{1}{2}c_2 + \frac{3}{2}c_3\right)e^{-3t} \end{bmatrix}$$

■ 6. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^{3t} \\ 6t^2 \end{bmatrix}$$

Solution:

First, we'll find $sI - A$.

$$sI - A = s\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1 & 4\\0 & 0 & -2\\0 & 0 & 0\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s & 0 & 0\\0 & s & 0\\0 & 0 & s\end{bmatrix} - \begin{bmatrix}0 & 1 & 4\\0 & 0 & -2\\0 & 0 & 0\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s & -1 & -4\\0 & s & 2\\0 & 0 & s\end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \,|\, I]$ into $[I \,|\, (sI - A)^{-1}]$.

$$\begin{bmatrix}s & -1 & -4 & | & 1 & 0 & 0\\0 & s & 2 & | & 0 & 1 & 0\\0 & 0 & s & | & 0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & -\frac{1}{s} & -\frac{4}{s} & | & \frac{1}{s} & 0 & 0\\0 & 1 & \frac{2}{s} & | & 0 & \frac{1}{s} & 0\\0 & 0 & 1 & | & 0 & 0 & \frac{1}{s}\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -\frac{1}{s} & 0 & | & \frac{1}{s} & 0 & \frac{4}{s^2}\\0 & 1 & 0 & | & 0 & \frac{1}{s} & -\frac{2}{s^2}\\0 & 0 & 1 & | & 0 & 0 & \frac{1}{s}\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & | & \frac{1}{s} & \frac{1}{s^2} & \frac{-2+4s}{s^3}\\0 & 1 & 0 & | & 0 & \frac{1}{s} & -\frac{2}{s^2}\\0 & 0 & 1 & | & 0 & 0 & \frac{1}{s}\end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix}\frac{1}{s} & \frac{1}{s^2} & -\frac{2}{s^3} + \frac{4}{s^2}\\0 & \frac{1}{s} & -\frac{2}{s^2}\\0 & 0 & \frac{1}{s}\end{bmatrix}$$

$$\mathscr{L}^{-1}((sI-A)^{-1}) = \begin{bmatrix} 1 & t & -t^2+4t \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix}$$

and then this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} 1 & t & -t^2+4t \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the complementary solution will be

$$\vec{x_c} = e^{At}C$$

$$\vec{x_c} = \begin{bmatrix} 1 & t & -t^2+4t \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\vec{x_c} = \begin{bmatrix} c_1 + c_2t - c_3t^2 + 4c_3t \\ c_2 - 2c_3t \\ c_3 \end{bmatrix}$$

To find the particular solution, we'll need e^{-As}, which we find by making the substitution $t = s$ into e^{At},

$$e^{As} = \begin{bmatrix} 1 & s & -s^2+4s \\ 0 & 1 & -2s \\ 0 & 0 & 1 \end{bmatrix}$$

and then calculating the inverse of this resulting e^{As}.

$$\begin{bmatrix} 1 & s & -s^2+4s & | & 1 & 0 & 0 \\ 0 & 1 & -2s & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & s & 0 & | & 1 & 0 & s^2-4s \\ 0 & 1 & 0 & | & 0 & 1 & 2s \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & 1 & -s & -s^2-4s \\ 0 & 1 & 0 & | & 0 & 1 & 2s \\ 0 & 0 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

So e^{-As} is given by this resulting matrix.

$$e^{-As} = \begin{bmatrix} 1 & -s & -s^2-4s \\ 0 & 1 & 2s \\ 0 & 0 & 1 \end{bmatrix}$$

We'll find $F(s)$ by substituting $t = s$ into F.

$$F(s) = \begin{bmatrix} 0 \\ e^{3s} \\ 6s^2 \end{bmatrix}$$

Therefore, the particular solution will be

$$\vec{x_p} = e^{At} \int_{t_0}^{t} e^{-As} F(s) \ ds$$

$$\vec{x_p} = \begin{bmatrix} 1 & t & -t^2+4t \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix} \int_0^t \begin{bmatrix} 1 & -s & -s^2-4s \\ 0 & 1 & 2s \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ e^{3s} \\ 6s^2 \end{bmatrix} \ ds$$

$$\vec{x}_p = \begin{bmatrix} 1 & t & -t^2+4t \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix} \int_0^t \begin{bmatrix} -se^{3s} - 6s^4 - 24s^3 \\ e^{3s} + 12s^3 \\ 6s^2 \end{bmatrix} \ ds$$

Now we'll integrate and then evaluate on $[0,t]$.

$$\vec{x}_p = \begin{bmatrix} 1 & t & -t^2+4t \\ 0 & 1 & -2t \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{6t^5}{5} - 6t^4 + \frac{1}{9}e^{3t}(1-3t) - \frac{1}{9} \\ \frac{1}{3}(9t^4 + e^{3t} - 1) \\ 2t^3 \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} -\frac{t^5}{5} + 2t^4 - \frac{1}{3}t + \frac{1}{9}e^{3t} - \frac{1}{9} \\ -t^4 + \frac{1}{3}e^{3t} - \frac{1}{3} \\ 2t^3 \end{bmatrix}$$

Then the general solution is the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} c_2 + 4c_3 \\ -2c_3 \\ 0 \end{bmatrix} t + \begin{bmatrix} -c_3 \\ 0 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} -\frac{t^5}{5} + 2t^4 - \frac{1}{3}t + \frac{1}{9}e^{3t} - \frac{1}{9} \\ -t^4 + \frac{1}{3}e^{3t} - \frac{1}{3} \\ 2t^3 \end{bmatrix}$$

HOMOGENEOUS HIGHER ORDER EQUATIONS

■ 1. Find the general solution to the third order homogeneous differential equation.

$$y''' - 3y'' - y' + 3y = 0$$

Solution:

We'll rewrite the differential equation as its associated characteristic equation, and then use a calculator to factor the polynomial.

$$r^3 - 3r^2 - r + 3 = 0$$

$$(r + 1)(r - 1)(r - 3) = 0$$

From the factored form of the polynomial, we can see that the equation has distinct real roots $r_1 = -1$, $r_2 = 1$, and $r_3 = 3$. And the general solution is therefore

$$y(t) = c_1 e^{-t} + c_2 e^t + c_3 e^{3t}$$

■ 2. Find the general solution to the third order homogeneous differential equation.

$$y''' + 6y'' + 8y' = 0$$

Solution:

We'll rewrite the differential equation as its associated characteristic equation, and then use a calculator to factor the polynomial.

$$r^3 + 6r^2 + 8r = 0$$

$$r(r + 2)(r + 4) = 0$$

From the factored form of the polynomial, we can see that the equation has distinct real roots $r_1 = 0$, $r_2 = -2$, and $r_3 = -4$. And the general solution is therefore

$$y(t) = c_1 + c_2 e^{-2t} + c_3 e^{-4t}$$

■ 3. Find the general solution to the third order homogeneous differential equation.

$$y''' + 15y'' + 75y' + 125y = 0$$

Solution:

We'll rewrite the differential equation as its associated characteristic equation, and then use a calculator to factor the polynomial.

$$r^3 + 15r^2 + 75r + 125 = 0$$

$$(r+5)^3 = 0$$

From the factored form of the polynomial, we can see that the equation has three equal real roots $r_1 = r_2 = r_3 = -5$. And the general solution is therefore

$$y(t) = c_1e^{-5t} + c_2te^{-5t} + c_3t^2e^{-5t}$$

■ 4. Find the general solution to the third order homogeneous differential equation.

$$y''' - 5y'' + 12y' - 8y = 0$$

Solution:

We'll rewrite the differential equation as its associated characteristic equation, and then use a calculator to factor the polynomial.

$$r^3 - 5r^2 + 12r - 8 = 0$$

$$(r-1)(r^2 - 4r + 8) = 0$$

From the factored form of the polynomial, we can see that the equation has one real root $r_1 = 1$, and two complex conjugate roots $r_{2,3} = 2 \pm 2i$. And the general solution is therefore

$$y(t) = c_1e^t + c_2e^{2t}\cos(2t) + c_3e^{2t}\sin(2t)$$

■ 5. Find the general solution to the fifth order homogeneous differential equation.

$$y^{(5)} - 2y^{(4)} + 2y''' = 0$$

Solution:

We'll rewrite the differential equation as its associated characteristic equation, and then use a calculator to factor the polynomial.

$$r^5 - 2r^4 + 2r^3 = 0$$

$$r^3(r^2 - 2r + 2) = 0$$

From the factored form of the polynomial, we can see that the equation has three equal real roots $r_1 = r_2 = r_3 = 0$, and two complex conjugate roots $r_{4,5} = 1 \pm i$. And the general solution is therefore

$$y(t) = c_1 + c_2t + c_3t^2 + c_4e^t \cos t + c_5e^t \sin t$$

■ 6. Find the general solution to the seventh order homogeneous differential equation.

$$y^{(7)} + 15y^{(6)} + 125y^{(5)} + 595y^{(4)} + 1{,}795y''' + 2{,}861y'' + 2{,}175y' + 625y = 0$$

Solution:

We'll rewrite the differential equation as its associated characteristic equation, and then use a calculator to factor the polynomial.

$$r^7 + 15r^6 + 125r^5 + 595r^4 + 1{,}795r^3 + 2{,}861r^2 + 2{,}175r + 625 = 0$$

$$(r+1)^3(r^2 + 6r + 25)^2 = 0$$

From the factored form of the polynomial, we can see that the equation has three equal real roots $r_1 = r_2 = r_3 = -1$, and a pair of complex conjugate roots with multiplicity two, $r_{4,5} = -3 \pm 4i$ and $r_{6,7} = -3 \pm 4i$. And the general solution is therefore

$$y(t) = c_1e^{-t} + c_2te^{-t} + c_3t^2e^{-t} + c_4e^{-3t}\cos(4t) + c_5e^{-3t}\sin(4t)$$

$$+c_6te^{-3t}\cos(4t) + c_7te^{-3t}\sin(4t)$$

UNDETERMINED COEFFICIENTS FOR HIGHER ORDER EQUATIONS

■ 1. Find the general solution to the third order differential equation.

$$y''' + y'' - 4y' - 4y = 8t^2$$

Solution:

Find the roots of the characteristic equation.

$$r^3 + r^2 - 4r - 4 = 0$$

$$(r + 1)(r^2 - 4) = 0$$

$$(r + 1)(r + 2)(r - 2) = 0$$

$$r = -1, 2, \ -2$$

With three distinct roots, the complementary solution is

$$y_c(t) = c_1e^{-t} + c_2e^{2t} + c_3e^{-2t}$$

Our guess for the particular solution and its derivatives will be

$$y_p(t) = At^2 + Bt + C$$

$$y_p'(t) = 2At + B$$

$$y_p''(t) = 2A$$

$$y_p'''(t) = 0$$

Plugging these into the differential equation gives

$$0 + 2A - 8At - 4B - 4At^2 - 4Bt - 4C = 8t^2$$

$$-4At^2 - (8A + 4B)t + 2A - 4B - 4C = 8t^2$$

Equating coefficients gives us the system of equations

$$-4A = 8$$

$$-8A - 4B = 0$$

$$2A - 4B - 4C = 0$$

From this system, we find $A = -2$, $B = 4$, and $C = -5$, so the particular solution will be

$$y_p(t) = -2t^2 + 4t - 5$$

The general solution will be the sum of the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{-t} + c_2e^{2t} + c_3e^{-2t} - 2t^2 + 4t - 5$$

■ 2. Find the general solution to the third order differential equation.

$$y''' - 6y'' + 9y' - 4y = 1 + e^{2t} + 9e^{4t}$$

Solution:

Find the roots of the characteristic equation.

$$r^3 - 6r^2 + 9r - 4 = 0$$

$$(r-1)^2(r-4) = 0$$

$$r = 1,\ 1,\ 4$$

With a pair of repeated roots and one distinct root, the complementary solution is

$$y_c(t) = c_1e^t + c_2te^t + c_3e^{4t}$$

Our guess for the particular solution would be

$$y_p = A + Be^{2t} + Ce^{4t}$$

but Ce^{4t} overlaps with c_3e^{4t} from the complementary solution. So we multiply that term by t to get Cte^{4t}. So the particular solution and its derivatives will be

$$y_p(t) = A + Be^{2t} + Cte^{4t}$$

$$y_p'(t) = 2Be^{2t} + Ce^{4t} + 4Cte^{4t}$$

$$y_p''(t) = 4Be^{2t} + 8Ce^{4t} + 16Cte^{4t}$$

$$y_p'''(t) = 8Be^{2t} + 48Ce^{4t} + 64Cte^{4t}$$

Plugging these into the differential equation gives

$$8Be^{2t}+64Cte^{4t}+48Ce^{4t}-24Be^{2t}-48Ce^{4t}-96Cte^{4t}$$

$$+18Be^{2t}+9Ce^{4t}+36Cte^{4t}-4A-4Be^{2t}-4Cte^{4t}$$

$$=1+e^{2t}+9e^{4t}$$

$$-2Be^{2t}+9Ce^{4t}-4A=1+e^{2t}+9e^{4t}$$

Equating coefficients gives us the system of equations

$$-4A=1$$

$$-2B=1$$

$$9C=9$$

From this system, we find $A=-1/4$, $B=-1/2$, and $C=1$, so the particular solution will be

$$y_p(t)=-\frac{1}{4}-\frac{1}{2}e^{2t}+te^{4t}$$

The general solution will be the sum of the complementary and particular solutions.

$$y(t)=y_c(t)+y_p(t)$$

$$y(t)=c_1e^t+c_2te^t+c_3e^{4t}-\frac{1}{4}-\frac{1}{2}e^{2t}+te^{4t}$$

■ 3. Find the general solution to the differential equation. Hint: $x^4 + 4x^3 + 7x^2 + 6x + 2 = (x+1)^2(x^2 + 2x + 2)$.

$$y'''' + 4y''' + 7y'' + 6y' + 2y = e^{-t}(\cos t + 2\sin t + 4)$$

Solution:

Find the roots of the characteristic equation.

$$r^4 + 4r^3 + 7r^2 + 6r + 2 = 0$$

$$(r+1)^2(r^2 + 2r + 2) = 0$$

$$(r+1)^2(r + i + 1)(r - i + 1) = 0$$

$$r = -1, \ -1, \ -1 \pm i$$

With a pair of repeated roots and a pair of complex conjugate roots, the complementary solution is

$$y_c(t) = c_1e^{-t} + c_2te^{-t} + c_3e^{-t}\cos t + c_4e^{-t}\sin t$$

Our guess for the particular solution would be

$$y_p = Ae^{-t}\cos t + Be^{-t}\sin t + Ce^{-t}$$

but $Ae^{-t}\cos t$ and $Be^{-t}\sin t$ overlap with $c_3e^{-t}\cos t$ and $c_4e^{-t}\sin t$ from the complementary solution, so we multiply those terms by t. And Ce^{-t} overlaps with c_1e^{-t}, so we'll multiply it by t^2 (if we only multiplied by t, then we'd still have an overlap with c_2te^{-t}). So the particular solution and its derivatives will be

$$y_p(t) = Ate^{-t}\cos t + Bte^{-t}\sin t + Ct^2e^{-t}$$

$$y_p'(t) = e^{-t}[\cos t(A - At + Bt) + \sin t(B - Bt - At) + 2Ct - Ct^2]$$

$$y_p''(t) = e^{-t}[(2At - 2A - 2B)\sin t + (-2Bt - 2A + 2B)\cos t + C(t^2 - 4t + 2)]$$

$$y_p'''(t) = e^{-t}[(-2At + 2Bt + 6A)\sin t + (2At + 2Bt - 6B)\cos t + C(-t^2 + 6t - 6)]$$

$$y_p''''(t) = e^{-t}[(-4Bt - 8A + 8B)\sin t + (-4At + 8A + 8B)\cos t + C(t^2 - 8t + 12)]$$

Plugging these into the differential equation gives

$$e^{-t}[(-4Bt - 8A + 8B)\sin t + (-4At + 8A + 8B)\cos t + C(t^2 - 8t + 12)]$$

$$+4e^{-t}[(-2At + 2Bt + 6A)\sin t + (2At + 2Bt - 6B)\cos t + C(-t^2 + 6t - 6)]$$

$$+7e^{-t}[(2At - 2A - 2B)\sin t + (-2Bt - 2A + 2B)\cos t + C(t^2 - 4t + 2)]$$

$$+6e^{-t}[\cos t(A - At + Bt) + \sin t(B - Bt - At) + 2Ct - Ct^2]$$

$$+2[Ate^{-t}\cos t + Bte^{-t}\sin t + Ct^2e^{-t}]$$

$$= e^{-t}(\cos t + 2\sin t + 4)$$

$$e^{-t}[(-4Bt - 8A + 8B)\sin t + (-4At + 8A + 8B)\cos t + C(t^2 - 8t + 12)]$$

$$+e^{-t}[(-8At + 8Bt + 24A)\sin t + (8At + 8Bt - 24B)\cos t + C(-4t^2 + 24t - 24)]$$

$$+e^{-t}[(14At - 14A - 14B)\sin t + (-14Bt - 14A + 14B)\cos t + C(7t^2 - 28t + 14)]$$

$$+e^{-t}[(6A - 6At + 6Bt)\cos t + (6B - 6Bt - 6At)\sin t + C(12t - 6t^2)]$$

$$+2Ate^{-t}\cos t + 2Bte^{-t}\sin t + 2Ct^2e^{-t}$$

$$= e^{-t}(\cos t + 2\sin t + 4)$$

$$e^{-t}[(-4Bt - 8A + 8B - 8At + 8Bt + 24A + 14At - 14A - 14B + 6B - 6Bt - 6At + 2Bt)\sin t$$

$$+(-4At + 8A + 8B + 8At + 8Bt - 24B - 14Bt - 14A + 14B + 6A - 6At + 6Bt + 2At)\cos t$$

$$+C(t^2 - 8t + 12 - 4t^2 + 24t - 24 + 7t^2 - 28t + 14 + 12t - 6t^2 + 2t^2)]$$

$$= e^{-t}(\cos t + 2\sin t + 4)$$

$$e^{-t}(2A\sin t - 2B\cos t + 2C) = e^{-t}(\cos t + 2\sin t + 4)$$

Equating coefficients gives us the system of equations

$$2A = 2$$

$$-2B = 1$$

$$2C = 4$$

From this system, we find $A = 1$, $B = -1/2$, and $C = 2$, so the particular solution will be

$$y_p(t) = te^{-t}\cos t - \frac{1}{2}te^{-t}\sin t + 2t^2e^{-t}$$

The general solution will be the sum of the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{-t} + c_2te^{-t} + c_3e^{-t}\cos t + c_4e^{-t}\sin t$$

$$+te^{-t}\cos t - \frac{1}{2}te^{-t}\sin t + 2t^2e^{-t}$$

■ 4. Find the general solution to the differential equation.

$$y''' + 9y'' + 27y' + 27y = 20t^2e^{-3t}$$

Solution:

Find the roots of the characteristic equation.

$$r^3 + 9r^2 + 27r + 27 = 0$$

$$(r+3)^3 = 0$$

$$r = -3,\ -3,\ -3$$

With a repeated root with multiplicity three, the complementary solution is

$$y_c(t) = c_1e^{-3t} + c_2te^{-3t} + c_3t^2e^{-3t}$$

Our guess for the particular solution would be

$$y_p(t) = (At^2 + Bt + C)e^{-3t}$$

$$y_p(t) = At^2e^{-3t} + Bte^{-3t} + Ce^{-3t}$$

but all three of the terms in this guess overlap with a term in the complementary solution. Multiplying the particular solution by t would eliminate the overlap in one term, multiplying by t^2 would eliminate the overlap in two terms, but multiplying by t^3 is the only way to eliminate the overlap in all three terms. So the particular solution and its derivatives will be

$$y_p(t) = (At^5 + Bt^4 + Ct^3)e^{-3t}$$

$$y_p'(t) = (-3At^5 + (5A - 3B)t^4 + (4B - 3C)t^3 + 3Ct^2)e^{-3t}$$

$$y_p''(t) = (9At^5 + (-30A + 9B)t^4 + (20A - 24B + 9C)t^3 + (12B - 18C)t^2 + 6Ct)e^{-3t}$$

$$y_p'''(t) = [-27At^5 + (135A - 27B)t^4 + (-180A + 108B - 27C)t^3$$

$$+(60A - 108B + 81C)t^2 + (24B - 54C)t + 6C]e^{-3t}$$

Plugging these into the differential equation gives

$$[-27At^5 + (135A - 27B)t^4 + (-180A + 108B - 27C)t^3$$

$$+(60A - 108B + 81C)t^2 + (24B - 54C)t + 6C]e^{-3t}$$

$$+9(9At^5 + (-30A + 9B)t^4 + (20A - 24B + 9C)t^3 + (12B - 18C)t^2 + 6Ct)e^{-3t}$$

$$+27(-3At^5 + (5A - 3B)t^4 + (4B - 3C)t^3 + 3Ct^2)e^{-3t}$$

$$+27(At^5 + Bt^4 + Ct^3)e^{-3t} = 20t^2e^{-3t}$$

$$(60At^2 + 24Bt + 6C)e^{-3t} = 20t^2e^{-3t}$$

Equating coefficients gives us the system of equations

$$60A = 20$$

$$24B = 0$$

$$6C = 0$$

From this system, we find $A = 1/3$, $B = 0$, and $C = 0$, so the particular solution will be

$$y_p(t) = \frac{1}{3}t^5e^{-3t}$$

The general solution will be the sum of the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{-3t} + c_2te^{-3t} + c_3t^2e^{-3t} + \frac{1}{3}t^5e^{-3t}$$

■ 5. Find the general solution to the differential equation.

$$y''' + 4y' = \cos(2t) - 1$$

Solution:

Find the roots of the characteristic equation.

$$r^3 + 4r = 0$$

$$r(r^2 + 4) = 0$$

$$r = 0, \ \pm 2i$$

With one distinct root and a pair of complex conjugate roots, the complementary solution is

$$y_c(t) = c_1 + c_2 \cos(2t) + c_3 \sin(2t)$$

Our guess for the particular solution would be

$$y_p(t) = A\cos(2t) + B\sin(2t) + C$$

but all three of the terms in this guess overlap with a term in the complementary solution. Multiplying the particular solution by t eliminates the overlap in all three terms, so the particular solution and its derivatives will be

$$y_p(t) = t(A\cos(2t) + B\sin(2t) + C)$$

$$y_p'(t) = (A + 2Bt)\cos(2t) + (B - 2At)\sin(2t) + C$$

$$y_p''(t) = (-4A - 4Bt)\sin(2t) + (4B - 4At)\cos(2t)$$

$$y_p'''(t) = (-12A - 8Bt)\cos(2t) + (-12B + 8At)\sin(2t)$$

Plugging these into the differential equation gives

$$(-12A - 8Bt)\cos(2t) + (-12B + 8At)\sin(2t)$$

$$+4(A + 2Bt)\cos(2t) + 4(B - 2At)\sin(2t) + 4C$$

$$= \cos(2t) - 1$$

$$-8A\cos(2t) - 8B\sin(2t) + 4C = \cos(2t) - 1$$

Equating coefficients gives us the system of equations

$$-8A = 1$$

$$-8B = 0$$

$$4C = -1$$

From this system, we find $A = -1/8$, $B = 0$, and $C = -1/4$, so the particular solution will be

$$y_p(t) = -\frac{1}{8}t\cos(2t) - \frac{t}{4}$$

The general solution will be the sum of the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1 + c_2\cos(2t) + c_3\sin(2t) - \frac{1}{8}t\cos(2t) - \frac{t}{4}$$

■ 6. Find the general solution to the differential equation.

$$y'''' + 2y'' + y = t^2 + 8\sin t$$

Solution:

Find the roots of the characteristic equation.

$$r^4 + 2r^2 + 1 = 0$$

$$(r^2+1)^2=0$$

$$r^2+1=0$$

$$r^2=-1$$

$$r=\pm i$$

With one repeated pair of complex conjugate roots, the complementary solution is

$$y_c(t)=c_1\cos t+c_2\sin t+c_3t\cos t+c_4t\sin t$$

Our guess for the particular solution would be

$$y_p(t)=At^2+Bt+C+D\sin t+E\cos t$$

but $D\sin t$ and $E\cos t$ overlap with terms in the complementary solution, so we'll multiply them by t^2, and our guess for the particular solution and its derivatives will be

$$y_p(t)=At^2+Bt+C+Dt^2\sin t+Et^2\cos t$$

$$y_p'(t)=2At+B+2t(D\sin t+E\cos t)+t^2(D\cos t-E\sin t)$$

$$y_p''(t)=2A+2(D\sin t+E\cos t)+4t(D\cos t-E\sin t)+t^2(-D\sin t-E\cos t)$$

$$y_p'''(t)=-6E\sin t+6D\cos t+6t(-D\sin t-E\cos t)+t^2(E\sin t-D\cos t)$$

$$y_p''''(t)=-12E\cos t-12D\sin t+8t(-D\cos t+E\sin t)+t^2(E\cos t+D\sin t)$$

Plugging these into the differential equation gives

$$-12E\cos t - 12D\sin t + 8t(-D\cos t + E\sin t) + t^2(E\cos t + D\sin t)$$

$$+2[2A + 2(D\sin t + E\cos t) + 4t(D\cos t - E\sin t) + t^2(-D\sin t - E\cos t)]$$

$$+At^2 + Bt + C + Dt^2\sin t + Et^2\cos t = t^2 + 8\sin t$$

$$(-12E - 8Dt + Et^2 + 4E + 8Dt - 2Et^2 + Et^2)\cos t$$

$$+(-12D + 8Et + Dt^2 + 4D - 8Et - 2Dt^2 + Dt^2)\sin t$$

$$+At^2 + Bt + 4A + C = t^2 + 8\sin t$$

$$-8E\cos t - 8D\sin t + At^2 + Bt + 4A + C = t^2 + 8\sin t$$

Equating coefficients gives us the system of equations

$$A = 1$$

$$B = 0$$

$$4A + C = 0$$

$$-8D = 8$$

$$-8E = 0$$

From this system, we find $A = 1$, $B = 0$, $C = -4$, $D = -1$, and $E = 0$, so the particular solution will be

$$y_p(t) = t^2 - 4 - t^2\sin t$$

The general solution will be the sum of the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t + t^2 - 4 - t^2 \sin t$$

VARIATION OF PARAMETERS FOR HIGHER ORDER EQUATIONS

■ 1. Use variation of parameters to find the general solution to the differential equation.

$$y''' - y'' = te^t + 2t + 1 + 3\sin t$$

Solution:

Find the roots of the characteristic equation.

$$y''' - y'' = 0$$

$$r^3 - r^2 = 0$$

$$r^2(r-1) = 0$$

$$r = 0,\ 0,\ 1$$

With a pair of repeated roots and one distinct root, the complementary solution is

$$y_c(t) = c_1e^{r_1t} + c_2te^{r_1t} + c_3e^{r_2t}$$

$$y_c(t) = c_1e^{0t} + c_2te^{0t} + c_3e^{1t}$$

$$y_c(t) = c_1 + c_2t + c_3e^t$$

From the complementary solution, we can say that the fundamental set of solutions is $\{1, t, e^t\}$. With three solutions in the fundamental set, W, W_1, W_2, and W_3 are

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & t & e^t \\ 0 & 1 & e^t \\ 0 & 0 & e^t \end{vmatrix} = e^t$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & t & e^t \\ 0 & 1 & e^t \\ 1 & 0 & e^t \end{vmatrix} = 0\begin{vmatrix} 1 & e^t \\ 0 & e^t \end{vmatrix} - t\begin{vmatrix} 0 & e^t \\ 1 & e^t \end{vmatrix} + e^t\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= te^t - e^t = e^t(t-1)$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & e^t \\ 0 & 0 & e^t \\ 0 & 1 & e^t \end{vmatrix} = 1\begin{vmatrix} 0 & e^t \\ 1 & e^t \end{vmatrix} - 0\begin{vmatrix} 0 & e^t \\ 0 & e^t \end{vmatrix} + e^t\begin{vmatrix} 0 & 0 \\ 0 & 1 \end{vmatrix} = -e^t$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

Now we'll use these values and $g(t)$ from the nonhomogeneous equation to find u_1', u_2', and u_3'. We'll get

$$u_1' = \frac{g(t)W_1}{W}$$

$$u_1' = \frac{(te^t + 2t + 1 + 3\sin t)(e^t(t-1))}{e^t}$$

$$u_1' = (te^t + 2t + 1 + 3\sin t)(t - 1)$$

$$u_1' = t^2e^t + 2t^2 + t + 3t\sin t - te^t - 2t - 1 - 3\sin t$$

$$u_1' = t^2e^t + 2t^2 - t + 3t\sin t - te^t - 1 - 3\sin t$$

and

$$u_2' = \frac{g(t)W_2}{W}$$

$$u_2' = \frac{(te^t + 2t + 1 + 3\sin t)(-e^t)}{e^t}$$

$$u_2' = -\,te^t - 2t - 1 - 3\sin t$$

and

$$u_3' = \frac{g(t)W_3}{W}$$

$$u_3' = \frac{(te^t + 2t + 1 + 3\sin t)(1)}{e^t}$$

$$u_3' = t + 2te^{-t} + e^{-t} + 3e^{-t}\sin t$$

Use integration by parts to find u_1 from u_1'.

$$u_1 = \int t^2e^t + 2t^2 - t + 3t\sin t - te^t - 1 - 3\sin t\ dt$$

$$u_1 = t^2e^t - \int 2te^t\ dt + \frac{2}{3}t^3 - \frac{1}{2}t^2 - 3t\cos t + 3\int \cos t\ dt - \int te^t\ dt - t + 3\cos t$$

$$u_1 = t^2e^t - 3te^t + 3\int e^t\ dt + \frac{2}{3}t^3 - \frac{1}{2}t^2 - 3t\cos t + 3\sin t - t + 3\cos t$$

$$u_1 = t^2e^t - 3te^t + 3e^t + \frac{2}{3}t^3 - \frac{1}{2}t^2 - t - 3t\cos t + 3\sin t + 3\cos t$$

Integrate u_2' to find u_2.

$$u_2 = \int -te^t - 2t - 1 - 3\sin t\ dt$$

$$u_2 = -te^t + \int e^t\ dt - t^2 - t + 3\cos t$$

$$u_2 = -te^t + e^t - t^2 - t + 3\cos t$$

Integrate u_3' to find u_3.

$$u_3 = \int t + 2te^{-t} + e^{-t} + 3e^{-t}\sin t\ dt$$

$$u_3 = \frac{1}{2}t^2 - e^{-t} + \int 2te^{-t}\ dt + \int 3e^{-t}\sin t\ dt$$

$$u_3 = \frac{1}{2}t^2 - e^{-t} + \left(-2te^{-t} - \int -2e^{-t}\ dt\right) + \int 3e^{-t}\sin t\ dt$$

$$u_3 = \frac{1}{2}t^2 - e^{-t} - 2te^{-t} + \int 2e^{-t}\ dt + \int 3e^{-t}\sin t\ dt$$

$$u_3 = \frac{1}{2}t^2 - e^{-t} - 2te^{-t} - 2e^{-t} + \int 3e^{-t}\sin t\ dt$$

$$u_3 = \frac{1}{2}t^2 - 3e^{-t} - 2te^{-t} + \int 3e^{-t}\sin t\ dt$$

$$u_3 = \frac{1}{2}t^2 - 3e^{-t} - 2te^{-t} - \frac{3}{2}e^{-t}\cos t - \frac{3}{2}e^{-t}\sin t$$

$$u_3 = \frac{t^2}{2} - \frac{3}{e^t} - \frac{2t}{e^t} - \frac{3\cos t}{2e^t} - \frac{3\sin t}{2e^t}$$

With u_1, u_2, and u_3, we can say that the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \left(t^2e^t - 3te^t + 3e^t + \frac{2}{3}t^3 - \frac{1}{2}t^2 - t - 3t\cos t + 3\sin t + 3\cos t\right)(1)$$

$$+(-te^t + e^t - t^2 - t + 3\cos t)(t) + \left(\frac{t^2}{2} - \frac{3}{e^t} - \frac{2t}{e^t} - \frac{3\cos t}{2e^t} - \frac{3\sin t}{2e^t}\right)(e^t)$$

$$y_p(t) = t^2e^t - 3te^t + 3e^t + \frac{2}{3}t^3 - \frac{1}{2}t^2 - t - 3t\cos t + 3\sin t + 3\cos t$$

$$-t^2e^t + te^t - t^3 - t^2 + 3t\cos t + \frac{1}{2}t^2e^t - 3 - 2t - \frac{3\cos t}{2} - \frac{3\sin t}{2}$$

$$y_p(t) = -2te^t + 3e^t + \frac{1}{2}t^2e^t - \frac{1}{3}t^3 - \frac{3}{2}t^2 - 3t - 3 + \frac{3}{2}\sin t + \frac{3}{2}\cos t$$

Adding this particular solution to the complementary solution gives us the general solution $y(t)$.

$$y(t) = c_1 + c_2t + c_3e^t - 2te^t + 3e^t + \frac{t^2}{2}e^{-t} - \frac{1}{3}t^3 - \frac{3}{2}t^2 - 3t - 3 + \frac{3}{2}\sin t + \frac{3}{2}\cos t$$

■ 2. Use variation of parameters to find the general solution to the differential equation.

$$y'' - 5y' + 6y = te^{2t} - e^{3t}$$

Solution:

Find the roots of the characteristic equation.

$$y'' - 5y' + 6y = 0$$

$$r^2 - 5r + 6 = 0$$

$$(r-2)(r-3) = 0$$

$$r = 2,\ 3$$

With two distinct roots, the complementary solution is

$$y_c(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

$$y_c(t) = c_1 e^{2t} + c_2 e^{3t}$$

From the complementary solution, we can say that the fundamental set of solutions is $\{e^{2t}, e^{3t}\}$. Then W, W_1, and W_2 are

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = (e^{2t})(3e^{3t}) - (e^{3t})(2e^{2t}) = e^{5t}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = \begin{vmatrix} 0 & e^{3t} \\ 1 & 3e^{3t} \end{vmatrix} = (0)(3e^{3t}) - (1)(e^{3t}) = -e^{3t}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = \begin{vmatrix} e^{2t} & 0 \\ 2e^{2t} & 1 \end{vmatrix} = (e^{2t})(1) - (0)(2e^{2t}) = e^{2t}$$

Now we'll use these values and $g(t)$ from the nonhomogeneous equation to find u_1' and u_2'.

$$u_1' = \frac{g(t)W_1}{W} = \frac{(te^{2t} - e^{3t})(-e^{3t})}{e^{5t}} = -t + e^t$$

$$u_2' = \frac{g(t)W_2}{W} = \frac{(te^{2t} - e^{3t})(e^{2t})}{e^{5t}} = te^{-t} - 1$$

Integrate u_1' to find u_1.

$$u_1 = \int (-t + e^t)\ dt$$

$$u_1 = -\frac{1}{2}t^2 + e^t$$

Integrate u_2' to find u_2.

$$u_2 = \int (te^{-t} - 1)\ dt$$

$$u_2 = -te^{-t} + \int e^{-t}\ dt - t$$

$$u_2 = -te^{-t} - e^{-t} - t$$

With u_1 and u_2, we can say that the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2$$

$$y_p(t) = \left(-\frac{1}{2}t^2 + e^t\right)(e^{2t}) + (-te^{-t} - e^{-t} - t)(e^{3t})$$

$$y_p(t) = -\frac{1}{2}t^2e^{2t} + e^{3t} - te^{2t} - e^{2t} - te^{3t}$$

Adding this particular solution to the complementary solution gives us the general solution $y(t)$.

$$y(t) = c_1e^{2t} + c_2e^{3t} - \frac{1}{2}t^2e^{2t} + e^{3t} - te^{2t} - e^{2t} - te^{3t}$$

■ 3. Use variation of parameters to find the general solution to the differential equation.

$$y''' - y'' - 6y' = t^2e^{3t} - t + 1$$

Solution:

Find the roots of the characteristic equation.

$$y''' - y'' - 6y' = 0$$

$$r^3 - r^2 - 6r = 0$$

$$r(r+2)(r-3) = 0$$

$$r = 0, \ -2, \ 3$$

With three distinct roots, the complementary solution is

$$y_c(t) = c_1 e^{r_1 t} + c_2 e^{r_1 t} + c_3 e^{r_2 t}$$

$$y_c(t) = c_1 e^{0t} + c_2 e^{-2t} + c_3 e^{3t}$$

$$y_c(t) = c_1 + c_2 e^{-2t} + c_3 e^{3t}$$

From the complementary solution, we can say that the fundamental set of solutions is $\{1, e^{-2t}, e^{3t}\}$. Then W, W_1, W_2, and W_3 are

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & e^{-2t} & e^{3t} \\ 0 & -2e^{-2t} & 3e^{3t} \\ 0 & 4e^{-2t} & 9e^{3t} \end{vmatrix} = -30e^t$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & e^{-2t} & e^{3t} \\ 0 & -2e^{-2t} & 3e^{3t} \\ 1 & 4e^{-2t} & 9e^{3t} \end{vmatrix} = 5e^t$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & e^{3t} \\ 0 & 0 & 3e^{3t} \\ 0 & 1 & 9e^{3t} \end{vmatrix} = -3e^{3t}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} 1 & e^{-2t} & 0 \\ 0 & -2e^{-2t} & 0 \\ 0 & 4e^{-2t} & 1 \end{vmatrix} = -2e^{-2t}$$

Now we'll use these values and $g(t)$ from the nonhomogeneous equation to find u_1', u_2', and u_3'.

$$u_1' = \frac{g(t)W_1}{W}$$

$$u_1' = \frac{(t^2e^{3t} - t + 1)(5e^t)}{-30e^t}$$

$$u_1' = -\frac{1}{6}t^2e^{3t} + \frac{1}{6}t - \frac{1}{6}$$

and

$$u_2' = \frac{g(t)W_2}{W}$$

$$u_2' = \frac{(t^2e^{3t} - t + 1)(-3e^{3t})}{-30e^t}$$

$$u_2' = \frac{1}{10}t^2e^{5t} - \frac{1}{10}te^{2t} + \frac{1}{10}e^{2t}$$

and

$$u_3' = \frac{g(t)W_3}{W}$$

$$u_3' = \frac{(t^2e^{3t} - t + 1)(-2e^{-2t})}{-30e^t}$$

$$u_3' = \frac{1}{15}t^2 - \frac{1}{15}te^{-3t} + \frac{1}{15}e^{-3t}$$

Use integration by parts to find u_1 from u_1'.

$$u_1 = \int -\frac{1}{6}t^2e^{3t} + \frac{1}{6}t - \frac{1}{6}\ dt$$

$$u_1 = -\frac{1}{18}t^2e^{3t} + \frac{1}{9}\int te^{3t}\ dt + \frac{1}{12}t^2 - \frac{1}{6}t$$

$$u_1=-\frac{1}{18}t^2e^{3t}+\frac{1}{27}te^{3t}-\frac{1}{27}\int e^{3t}\ dt+\frac{1}{12}t^2-\frac{1}{6}t$$

$$u_1=-\frac{1}{18}t^2e^{3t}+\frac{1}{27}te^{3t}-\frac{1}{81}e^{3t}+\frac{1}{12}t^2-\frac{1}{6}t$$

Integrate u_2' to find u_2.

$$u_2=\int\frac{1}{10}t^2e^{5t}-\frac{1}{10}te^{2t}+\frac{1}{10}e^{2t}\ dt$$

$$u_2=\frac{1}{50}t^2e^{5t}-\frac{1}{25}\int te^{5t}\ dt-\frac{1}{20}te^{2t}+\frac{1}{20}\int e^{2t}\ dt+\frac{1}{20}e^{2t}$$

$$u_2=\frac{1}{50}t^2e^{5t}-\frac{1}{125}te^{5t}+\frac{1}{125}\int e^{5t}\ dt-\frac{1}{20}te^{2t}+\frac{1}{40}e^{2t}+\frac{1}{20}e^{2t}$$

$$u_2=\frac{1}{50}t^2e^{5t}-\frac{1}{125}te^{5t}+\frac{1}{625}e^{5t}-\frac{1}{20}te^{2t}+\frac{3}{40}e^{2t}$$

Integrate u_3' to find u_3.

$$u_3=\int\frac{1}{15}t^2-\frac{1}{15}te^{-3t}+\frac{1}{15}e^{-3t}\ dt$$

$$u_3=\frac{1}{45}t^3+\frac{1}{45}te^{-3t}-\frac{1}{45}\int e^{-3t}\ dt-\frac{1}{45}e^{-3t}$$

$$u_3=\frac{1}{45}t^3+\frac{1}{45}te^{-3t}+\frac{1}{135}e^{-3t}-\frac{1}{45}e^{-3t}$$

$$u_3=\frac{1}{45}t^3+\frac{1}{45}te^{-3t}-\frac{2}{135}e^{-3t}$$

With u_1, u_2, and u_3, we can say that the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \left(-\frac{1}{18}t^2e^{3t} + \frac{1}{27}te^{3t} - \frac{1}{81}e^{3t} + \frac{1}{12}t^2 - \frac{1}{6}t\right)(1)$$

$$+\left(\frac{1}{50}t^2e^{5t} - \frac{1}{125}te^{5t} + \frac{1}{625}e^{5t} - \frac{1}{20}te^{2t} + \frac{3}{40}e^{2t}\right)e^{-2t}$$

$$+\left(\frac{1}{45}t^3 + \frac{1}{45}te^{-3t} - \frac{2}{135}e^{-3t}\right)e^{3t}$$

$$y_p(t) = -\frac{1}{18}t^2e^{3t} + \frac{1}{27}te^{3t} - \frac{1}{81}e^{3t} + \frac{1}{12}t^2 - \frac{1}{6}t$$

$$+\frac{1}{50}t^2e^{3t} - \frac{1}{125}te^{3t} + \frac{1}{625}e^{3t} - \frac{1}{20}t + \frac{3}{40} + \frac{1}{45}t^3e^{3t} + \frac{1}{45}t - \frac{2}{135}$$

$$y_p(t) = \frac{1}{45}t^3e^{3t} - \frac{8}{225}t^2e^{3t} + \frac{98}{3{,}375}te^{3t} - \frac{544}{50{,}625}e^{3t} + \frac{1}{12}t^2 - \frac{7}{36}t + \frac{13}{216}$$

Adding this particular solution to the complementary solution gives us the general solution $y(t)$.

$$y(t) = c_1 + c_2e^{-2t} + c_3e^{3t}$$

$$+\frac{1}{45}t^3e^{3t} - \frac{8}{225}t^2e^{3t} + \frac{98}{3{,}375}te^{3t} - \frac{544}{50{,}625}e^{3t} + \frac{1}{12}t^2 - \frac{7}{36}t + \frac{13}{216}$$

■ 4. Use variation of parameters to find the general solution to the differential equation.

$$y''' - 6y'' + 5y' + 12y = 10te^{3t} - 4\cos(2t)$$

Solution:

Find the roots of the characteristic equation.

$$y''' - 6y'' + 5y' + 12y = 0$$

$$r^3 - 6r^2 + 5r + 12 = 0$$

$$(r+1)(r-3)(r-4) = 0$$

$$r = -1,\ 3,\ 4$$

With three distinct roots, the complementary solution is

$$y_c(t) = c_1e^{r_1t} + c_2e^{r_1t} + c_3e^{r_2t}$$

$$y_c(t) = c_1e^{-t} + c_2e^{3t} + c_3e^{4t}$$

From the complementary solution, we can say that the fundamental set of solutions is $\{e^{-t}, e^{3t}, e^{4t}\}$. Then W, W_1, W_2, and W_3 are

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{3t} & e^{4t} \\ -e^{-t} & 3e^{3t} & 4e^{4t} \\ e^{-t} & 9e^{3t} & 16e^{4t} \end{vmatrix}$$

$$= e^{-t}\begin{vmatrix} 3e^{3t} & 4e^{4t} \\ 9e^{3t} & 16e^{4t} \end{vmatrix} - e^{3t}\begin{vmatrix} -e^{-t} & 4e^{4t} \\ e^{-t} & 16e^{4t} \end{vmatrix} + e^{4t}\begin{vmatrix} -e^{-t} & 3e^{3t} \\ e^{-t} & 9e^{3t} \end{vmatrix}$$

$$= e^{-t}(48e^{7t} - 36e^{7t}) - e^{3t}(-16e^{3t} - 4e^{3t}) + e^{4t}(-9e^{2t} - 3e^{2t})$$

$$= e^{-t}(12e^{7t}) - e^{3t}(-20e^{3t}) + e^{4t}(-12e^{2t})$$

$$= 12e^{6t} + 20e^{6t} - 12e^{6t} = 20e^{6t}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & e^{3t} & e^{4t} \\ 0 & 3e^{3t} & 4e^{4t} \\ 1 & 9e^{3t} & 16e^{4t} \end{vmatrix}$$

$$= 0 \begin{vmatrix} 3e^{3t} & 4e^{4t} \\ 9e^{3t} & 16e^{4t} \end{vmatrix} - e^{3t} \begin{vmatrix} 0 & 4e^{4t} \\ 1 & 16e^{4t} \end{vmatrix} + e^{4t} \begin{vmatrix} 0 & 3e^{3t} \\ 1 & 9e^{3t} \end{vmatrix}$$

$$= 0 - e^{3t}(0 - 4e^{4t}) + e^{4t}(0 - 3e^{3t})$$

$$= 4e^{7t} - 3e^{7t} = e^{7t}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-t} & 0 & e^{4t} \\ -e^{-t} & 0 & 4e^{4t} \\ e^{-t} & 1 & 16e^{4t} \end{vmatrix}$$

$$= e^{-t} \begin{vmatrix} 0 & 4e^{4t} \\ 1 & 16e^{4t} \end{vmatrix} - 0 \begin{vmatrix} -e^{-t} & 4e^{4t} \\ e^{-t} & 16e^{4t} \end{vmatrix} + e^{4t} \begin{vmatrix} -e^{-t} & 0 \\ e^{-t} & 1 \end{vmatrix}$$

$$= e^{-t}(0 - 4e^{4t}) - 0 + e^{4t}(-e^{-t} - 0)$$

$$= -4e^{3t} - e^{3t} = -5e^{3t}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} e^{-t} & e^{3t} & 0 \\ -e^{-t} & 3e^{3t} & 0 \\ e^{-t} & 9e^{3t} & 1 \end{vmatrix}$$

$$= e^{-t} \begin{vmatrix} 3e^{3t} & 0 \\ 9e^{3t} & 1 \end{vmatrix} - e^{3t} \begin{vmatrix} -e^{-t} & 0 \\ e^{-t} & 1 \end{vmatrix} + 0 \begin{vmatrix} -e^{-t} & 3e^{3t} \\ e^{-t} & 9e^{3t} \end{vmatrix}$$

$$= e^{-t}(3e^{3t} - 0) - e^{3t}(-e^{-t} - 0) + 0$$

$$= 3e^{2t} + e^{2t} = 4e^{2t}$$

Now we'll use these values and $g(t)$ from the nonhomogeneous equation to find u_1', u_2', and u_3'.

$$u_1' = \frac{g(t)W_1}{W}$$

$$u_1' = \frac{(10te^{3t} - 4\cos(2t))(e^{7t})}{20e^{6t}}$$

$$u_1' = \frac{1}{2}te^{4t} - \frac{1}{5}e^{t}\cos(2t)$$

and

$$u_2' = \frac{g(t)W_2}{W}$$

$$u_2' = \frac{(10te^{3t} - 4\cos(2t))(-5e^{3t})}{20e^{6t}}$$

$$u_2' = -\frac{5}{2}t + e^{-3t}\cos(2t)$$

and

$$u_3' = \frac{g(t)W_3}{W}$$

$$u_3' = \frac{(10te^{3t} - 4\cos(2t))(4e^{2t})}{20e^{6t}}$$

$$u_3' = 2te^{-t} - \frac{4}{5}e^{-4t}\cos(2t)$$

Use integration by parts to find u_1 from u_1'.

$$u_1 = \int \frac{1}{2}te^{4t} - \frac{1}{5}e^{t}\cos(2t)\ dt$$

$$u_1 = \frac{1}{8}te^{4t} - \frac{1}{8}\int e^{4t}\ dt - \frac{1}{5}\int e^{t}\cos(2t)\ dt$$

$$u_1 = \frac{1}{8}te^{4t} - \frac{1}{32}e^{4t} - \frac{1}{5}\int e^{t}\cos(2t)\ dt$$

$$\int e^{t}\cos(2t)\ dt = e^{t}\cos(2t) + 2\int e^{t}\sin(2t)\ dt$$

$$\int e^{t}\cos(2t)\ dt = e^{t}\cos(2t) + 2e^{t}\sin(2t) - 4\int e^{t}\cos(2t)\ dt$$

$$5\int e^{t}\cos(2t)\ dt = e^{t}\cos(2t) + 2e^{t}\sin(2t)$$

$$\int e^{t}\cos(2t)\ dt = \frac{1}{5}e^{t}\cos(2t) + \frac{2}{5}e^{t}\sin(2t)$$

$$u_1 = \frac{1}{8}te^{4t} - \frac{1}{32}e^{4t} - \frac{1}{25}e^{t}\cos(2t) - \frac{2}{25}e^{t}\sin(2t)$$

Use integration by parts to find u_2 from u_2'.

$$u_2 = \int -\frac{5}{2}t + e^{-3t}\cos(2t)\ dt$$

$$u_2 = -\frac{5}{4}t^2 + \int e^{-3t}\cos(2t)\ dt$$

$$\int e^{-3t}\cos(2t)\ dt = -\frac{1}{3}e^{-3t}\cos(2t) - \frac{2}{3}\int e^{-3t}\sin(2t)\ dt$$

$$\int e^{-3t}\cos(2t)\ dt = -\frac{1}{3}e^{-3t}\cos(2t) + \frac{2}{9}e^{-3t}\sin(2t) - \frac{4}{9}\int e^{-3t}\cos(2t)\ dt$$

$$\frac{13}{9}\int e^{-3t}\cos(2t)\ dt = -\frac{1}{3}e^{-3t}\cos(2t) + \frac{2}{9}e^{-3t}\sin(2t)$$

$$\int e^{-3t}\cos(2t)\ dt = -\frac{3}{13}e^{-3t}\cos(2t) + \frac{2}{13}e^{-3t}\sin(2t)$$

$$u_2 = -\frac{5}{4}t^2 - \frac{3}{13}e^{-3t}\cos(2t) + \frac{2}{13}e^{-3t}\sin(2t)$$

Use integration by parts to find u_3 from u_3'.

$$u_3 = \int 2te^{-t} - \frac{4}{5}e^{-4t}\cos(2t)\ dt$$

$$u_3 = -2te^{-t} + 2\int e^{-t}\ dt - \frac{4}{5}\int e^{-4t}\cos(2t)\ dt$$

$$u_3 = -2te^{-t} - 2e^{-t} - \frac{4}{5}\int e^{-4t}\cos(2t)\ dt$$

$$\int e^{-4t}\cos(2t)\ dt = -\frac{1}{4}e^{-4t}\cos(2t) - \frac{1}{2}\int e^{-4t}\sin(2t)\ dt$$

$$\int e^{-4t}\cos(2t)\ dt = -\frac{1}{4}e^{-4t}\cos(2t) + \frac{1}{8}e^{-4t}\sin(2t) - \frac{1}{4}\int e^{-4t}\cos(2t)\ dt$$

$$\frac{5}{4}\int e^{-4t}\cos(2t)\ dt = -\frac{1}{4}e^{-4t}\cos(2t) + \frac{1}{8}e^{-4t}\sin(2t)$$

$$\int e^{-4t}\cos(2t)\ dt = -\frac{1}{5}e^{-4t}\cos(2t) + \frac{1}{10}e^{-4t}\sin(2t)$$

$$u_3 = -2te^{-t} - 2e^{-t} + \frac{4}{25}e^{-4t}\cos(2t) - \frac{2}{25}e^{-4t}\sin(2t)$$

With u_1, u_2, and u_3, we can say that the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \left(\frac{1}{8}te^{4t} - \frac{1}{32}e^{4t} - \frac{1}{25}e^{t}\cos(2t) - \frac{2}{25}e^{t}\sin(2t)\right)(e^{-t})$$

$$+\left(-\frac{5}{4}t^2 - \frac{3}{13}e^{-3t}\cos(2t) + \frac{2}{13}e^{-3t}\sin(2t)\right)e^{3t}$$

$$+\left(-2te^{-t} - 2e^{-t} + \frac{4}{25}e^{-4t}\cos(2t) - \frac{2}{25}e^{-4t}\sin(2t)\right)e^{4t}$$

$$y_p(t) = \frac{1}{8}te^{3t} - \frac{1}{32}e^{3t} - \frac{1}{25}\cos(2t) - \frac{2}{25}\sin(2t)$$

$$-\frac{5}{4}t^2e^{3t} - \frac{3}{13}\cos(2t) + \frac{2}{13}\sin(2t) - 2te^{3t} - 2e^{3t} + \frac{4}{25}\cos(2t) - \frac{2}{25}\sin(2t)$$

$$y_p(t) = -\frac{5}{4}t^2e^{3t} - \frac{15}{8}te^{3t} - \frac{65}{32}e^{3t} - \frac{36}{325}\cos(2t) - \frac{2}{325}\sin(2t)$$

Adding this particular solution to the complementary solution gives us the general solution $y(t)$.

$$y(t) = c_1e^{-t} + c_2e3t + c_3e^{4t} - \frac{5}{4}t^2e^{3t} - \frac{15}{8}te^{3t} - \frac{65}{32}e^{3t}$$

$$-\frac{36}{325}\cos(2t) - \frac{2}{325}\sin(2t)$$

■ 5. Use variation of parameters to find the general solution to the differential equation.

$$y''' - y'' + 9y' - 9y = 60\cos(3t)$$

Solution:

Find the roots of the characteristic equation.

$$y''' - y'' + 9y' - 9y = 0$$

$$r^3 - r^2 + 9r - 9 = 0$$

$$(r-1)(r^2+9) = 0$$

$$r = 1, \ \pm 3i$$

With a pair of complex conjugate roots and one distinct root, the complementary solution is

$$y_c(t) = c_1e^t + c_2\cos(3t) + c_3\sin(3t)$$

From the complementary solution, we can say that the fundamental set of solutions is $\{e^t, \cos(3t), \sin(3t)\}$. Then W, W_1, W_2, and W_3 are

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^t & \cos(3t) & \sin(3t) \\ e^t & -3\sin(3t) & 3\cos(3t) \\ e^t & -9\cos(3t) & -9\sin(3t) \end{vmatrix}$$

$$= e^t \begin{vmatrix} -3\sin(3t) & 3\cos(3t) \\ -9\cos(3t) & -9\sin(3t) \end{vmatrix} - \cos(3t) \begin{vmatrix} e^t & 3\cos(3t) \\ e^t & -9\sin(3t) \end{vmatrix}$$

$$+\sin(3t) \begin{vmatrix} e^t & -3\sin(3t) \\ e^t & -9\cos(3t) \end{vmatrix}$$

$$= e^t(27\sin^2(3t) + 27\cos^2(3t)) - \cos(3t)(-9e^t\sin(3t) - 3e^t\cos(3t))$$

$$+\sin(3t)(-9e^t\cos(3t) + 3e^t\sin(3t))$$

$$= 27e^t + 9e^t\sin(3t)\cos(3t) + 3e^t\cos^2(3t) - 9e^t\sin(3t)\cos(3t) + 3e^t\sin^2(3t)$$

$$= 27e^t + 3e^t$$

$$= 30e^t$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & \cos(3t) & \sin(3t) \\ 0 & -3\sin(3t) & 3\cos(3t) \\ 1 & -9\cos(3t) & -9\sin(3t) \end{vmatrix}$$

$$= 0 \begin{vmatrix} -3\sin(3t) & 3\cos(3t) \\ -9\cos(3t) & -9\sin(3t) \end{vmatrix} - \cos(3t) \begin{vmatrix} 0 & 3\cos(3t) \\ 1 & -9\sin(3t) \end{vmatrix}$$

$$+\sin(3t) \begin{vmatrix} 0 & -3\sin(3t) \\ 1 & -9\cos(3t) \end{vmatrix}$$

$$= 0 - \cos(3t)(-3\cos(3t)) + \sin(3t)(3\sin 3(3t)t)$$

$$= 3\cos^2(3t) + 3\sin^2(3t)$$

$$= 3$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} e^t & 0 & \sin(3t) \\ e^t & 0 & 3\cos(3t) \\ e^t & 1 & -9\sin(3t) \end{vmatrix}$$

$$= e^t \begin{vmatrix} 0 & 3\cos(3t) \\ 1 & -9\sin(3t) \end{vmatrix} - 0 \begin{vmatrix} e^t & 3\cos(3t) \\ e^t & -9\sin(3t) \end{vmatrix} + \sin(3t) \begin{vmatrix} e^t & 0 \\ e^t & 1 \end{vmatrix}$$

$$= e^t(-3\cos(3t)) - 0 + \sin(3t)(e^t)$$

$$= -3e^t\cos(3t) + e^t\sin(3t)$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} e^t & \cos(3t) & 0 \\ e^t & -3\sin(3t) & 0 \\ e^t & -9\cos(3t) & 1 \end{vmatrix}$$

$$= e^t \begin{vmatrix} -3\sin(3t) & 0 \\ -9\cos(3t) & 1 \end{vmatrix} - \cos(3t) \begin{vmatrix} e^t & 0 \\ e^t & 1 \end{vmatrix} + 0 \begin{vmatrix} e^t & -3\sin(3t) \\ e^t & -9\cos(3t) \end{vmatrix}$$

$$= e^t(-3\sin(3t)) - \cos(3t)(e^t) + 0$$

$$= -3e^t\sin(3t) - e^t\cos(3t)$$

Now we'll use these values and $g(t)$ from the nonhomogeneous equation to find u_1', u_2', and u_3'.

$$u_1' = \frac{g(t)W_1}{W}$$

$$u_1' = \frac{3(60\cos(3t))}{30e^t}$$

$$u_1' = 6e^{-t}\cos(3t)$$

and

$$u_2' = \frac{g(t)W_2}{W}$$

$$u_2' = \frac{(60\cos(3t))(-3e^t\cos(3t) + e^t\sin(3t))}{30e^t}$$

$$u_2' = -6\cos^2(3t) + 2\sin(3t)\cos(3t)$$

$$u_2' = -3 - 3\cos(6t) + \sin(6t)$$

and

$$u_3' = \frac{g(t)W_3}{W}$$

$$u_3' = \frac{(60\cos(3t))(-3e^t\sin(3t) - e^t\cos(3t))}{30e^t}$$

$$u_3' = -6\sin(3t)\cos(3t) - 2\cos^2(3t)$$

$$u_3' = -3\sin(6t) - 1 - \cos(6t)$$

Use integration by parts to find u_1 from u_1'.

$$u_1 = \int 6e^{-t}\cos(3t)\ dt = 6\int e^{-t}\cos(3t)\ dt$$

$$\int e^{-t}\cos(3t)\ dt = -e^{-t}\cos(3t) - 3\int e^{-t}\sin(3t)\ dt$$

$$\int e^{-t}\cos(3t)\ dt = -e^{-t}\cos(3t) + 3e^{-t}\sin(3t) - 9\int e^{-t}\cos(3t)\ dt$$

$$10\int e^{-t}\cos(3t)\ dt = -e^{-t}\cos(3t) + 3e^{-t}\sin(3t)$$

$$\int e^{-t}\cos(3t)\ dt = -\frac{1}{10}e^{-t}\cos(3t) + \frac{3}{10}e^{-t}\sin(3t)$$

$$u_1 = 6\left(-\frac{1}{10}e^{-t}\cos(3t) + \frac{3}{10}e^{-t}\sin(3t)\right)$$

$$u_1 = -\frac{3}{5}e^{-t}\cos(3t) + \frac{9}{5}e^{-t}\sin(3t)$$

Integrate u_2' to find u_2.

$$u_2 = \int -3 - 3\cos(6t) + \sin(6t)\ dt$$

$$u_2 = -3t - \frac{1}{2}\sin(6t) - \frac{1}{6}\cos(6t)$$

Integrate u_3' to find u_3.

$$u_3 = \int -3\sin(6t) - 1 - \cos(6t)\ dt$$

$$u_3 = \frac{1}{2}\cos(6t) - t - \frac{1}{6}\sin(6t)$$

With u_1, u_2, and u_3, we can say that the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \left(-\frac{3}{5}e^{-t}\cos(3t) + \frac{9}{5}e^{-t}\sin(3t)\right)e^t + \left(-3t - \frac{1}{2}\sin(6t) - \frac{1}{6}\cos(6t)\right)\cos(3t)$$

$$+\left(\frac{1}{2}\cos(6t) - t - \frac{1}{6}\sin(6t)\right)\sin(3t)$$

$$y_p(t) = -\frac{3}{5}\cos(3t) + \frac{9}{5}\sin(3t) + \left(-3t - \frac{1}{2}\sin(6t) - \frac{1}{6}\cos(6t)\right)\cos(3t)$$

$$+\left(\frac{1}{2}\cos(6t) - t - \frac{1}{6}\sin(6t)\right)\sin(3t)$$

Adding this particular solution to the complementary solution gives us the general solution $y(t)$.

$$y(t) = c_1e^t + c_2\cos(3t) + c_3\sin(3t) - \frac{3}{5}\cos(3t) + \frac{9}{5}\sin(3t)$$

$$+\left(-3t - \frac{1}{2}\sin(6t) - \frac{1}{6}\cos(6t)\right)\cos(3t)$$

$$+\left(\frac{1}{2}\cos(6t) - t - \frac{1}{6}\sin(6t)\right)\sin(3t)$$

■ 6. Use variation of parameters to find the general solution to the differential equation, given $y(0) = 0$, $y'(0) = -1$, $y''(0) = 10$.

$$y''' + y'' - y' - y = 8e^{-t}$$

Solution:

Find the roots of the characteristic equation.

$$y''' + y'' - y' - y = 0$$

$$r^3 + r^2 - r - 1 = 0$$

$$(r + 1)(r^2 - 1) = 0$$

$$r = -1, \ -1, \ 1$$

With a pair of equal roots and one distinct root, the complementary solution is

$$y_c(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t} + c_3 e^{r_2 t}$$

$$y_c(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 e^{t}$$

From the complementary solution, we can say that the fundamental set of solutions is $\{e^{-t}, te^{-t}, e^t\}$. Then W, W_1, W_2, and W_3 are

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} & e^t \\ -e^{-t} & e^{-t}(1 - t) & e^t \\ e^{-t} & e^{-t}(t - 2) & e^t \end{vmatrix} = 4e^{-t}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & te^{-t} & e^t \\ 0 & e^{-t}(1 - t) & e^t \\ 1 & e^{-t}(t - 2) & e^t \end{vmatrix} = 2t - 1$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-t} & 0 & e^t \\ -e^{-t} & 0 & e^t \\ e^{-t} & 1 & e^t \end{vmatrix} = -2$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} e^{-t} & te^{-t} & 0 \\ -e^{-t} & e^{-t}(1-t) & 0 \\ e^{-t} & e^{-t}(t-2) & 1 \end{vmatrix} = e^{-2t}$$

Now we'll use these values and $g(t)$ from the nonhomogeneous equation to find u_1', u_2', and u_3'.

$$u_1' = \frac{g(t)W_1}{W} = \frac{8e^{-t}(2t-1)}{4e^{-t}} = 4t - 2$$

$$u_2' = \frac{g(t)W_2}{W} = \frac{8e^{-t}(-2)}{4e^{-t}} = -4$$

$$u_3' = \frac{g(t)W_3}{W} = \frac{8e^{-t}(e^{-2t})}{4e^{-t}} = 2e^{-2t}$$

Use integration by parts to find u_1 from u_1'.

$$u_1 = \int 4t - 2 \ dt$$

$$u_1 = -2t^2 - 2t$$

Integrate u_2' to find u_2.

$$u_2 = \int -4 \ dt$$

$$u_2 = -4t$$

Integrate u_3' to find u_3.

$$u_3 = \int 2e^{-2t}\ dt$$

$$u_3 = -e^{-2t}$$

With u_1, u_2, and u_3, we can say that the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = (2t^2 - 2t)e^{-t} - 4t(te^{-t}) - e^{-2t}e^t$$

$$y_p(t) = 2t^2e^{-t} - 2te^{-t} - 4t^2e^{-t} - e^{-t}$$

$$y_p(t) = -2t^2e^{-t} - 2te^{-t} - e^{-t}$$

Adding this particular solution to the complementary solution gives us the general solution $y(t)$.

$$y(t) = c_1e^{-t} + c_2te^{-t} + c_3e^t - 2t^2e^{-t} - 2te^{-t} - e^{-t}$$

The derivatives of the general solution are

$$y'(t) = -c_1e^{-t} + c_2e^{-t} - c_2te^{-t} + c_3e^t - 2te^{-t} + 2t^2e^{-t} - e^{-t}$$

$$y''(t) = c_1e^{-t} - 2c_2e^{-t} + c_2te^{-t} + c_3e^t - e^{-t} + 6te^{-t} - 2t^2e^{-t}$$

Substituting the initial conditions,

$$y(0) = c_1 + c_3 - 1 = 0$$

$$y'(0) = -c_1 + c_2 + c_3 - 1 = -1$$

$$y''(0) = c_1 - 2c_2 + c_3 - 1 = 10$$

and then solving these as a system of equations gives $c_1 = 3$, $c_2 = 5$, and $c_3 = -2$. So the actual solution is

$$y(t) = 3e^{-t} + 5te^{-t} - 2e^{t} - 2t^2e^{-t} - 2te^{-t} - e^{-t}$$

$$y(t) = 2e^{-t} + 3te^{-t} - 2e^{t} - 2t^2e^{-t}$$

LAPLACE TRANSFORMS FOR HIGHER ORDER EQUATIONS

■ 1. Use the Laplace transform to solve the differential equation, given $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$.

$$y''' + 6y'' + 8y' = 8e^{-2t}$$

Solution:

Apply the Laplace transform to both sides of the differential equation.

$$\mathcal{L}(y''') + 6\mathcal{L}(y'') + 8\mathcal{L}(y') = 8\mathcal{L}(e^{-2t})$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 6(s^2Y(s) - sy(0) - y'(0))$$

$$+8(sY(s) - y(0)) = \frac{8}{s+2}$$

Substitute the initial conditions $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$,

$$Y(s)(s^3 + 6s^2 + 8s) = \frac{8}{s+2}$$

then solve for $Y(s)$.

$$Y(s) = \frac{8}{(s+2)(s^3 + 6s^2 + 8s)}$$

$$Y(s) = \frac{8}{s(s+2)^2(s+4)}$$

Apply a partial fractions decomposition.

$$\frac{8}{s(s+2)^2(s+4)} = \frac{A}{s} + \frac{B}{(s+2)^2} + \frac{C}{s+2} + \frac{D}{s+4}$$

We can quickly find $A = 1/2$, $B = -2$, and $D = -1/2$, then we'll plug these values into the decomposition, along with $s = 1$, to solve for C.

$$\frac{8}{(1+2)^2(1+4)} = \frac{1}{2(1)} - \frac{2}{(1+2)^2} + \frac{C}{1+2} - \frac{1}{2(1+4)}$$

$$\frac{C}{3} = \frac{8}{45} - \frac{1}{2} + \frac{2}{9} + \frac{1}{10}$$

$$C = 0$$

Then the Laplace transform is

$$Y(s) = \frac{1}{2s} - \frac{2}{(s+2)^2} - \frac{1}{2(s+4)}$$

Applying the inverse Laplace transform to each term gives the general solution to the differential equation gives

$$y(t) = \frac{1}{2} - 2te^{-2t} - \frac{1}{2}e^{-4t}$$

■ 2. Use the Laplace transform to solve the differential equation, given $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 1$.

$$y''' - 3y' + 2y = 12e^t + 24e^{2t}$$

Solution:

Apply the Laplace transform to both sides of the differential equation.

$$\mathscr{L}(y''') - 3\mathscr{L}(y') + 2\mathscr{L}(y) = 12\mathscr{L}(e^t) + 24\mathscr{L}(e^{2t})$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{12}{s-1} + \frac{24}{s-2}$$

Substitute the initial conditions $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 1$,

$$Y(s)(s^3 - 3s + 2) = \frac{12}{s-1} + \frac{24}{s-2} + 1 = \frac{s^2 + 33s - 46}{(s-1)(s-2)}$$

then solve for $Y(s)$.

$$Y(s) = \frac{s^2 + 33s - 46}{(s-1)(s-2)(s-1)^2(s+2)} = \frac{s^2 + 33s - 46}{(s-1)^3(s-2)(s+2)}$$

Apply a partial fractions decomposition.

$$\frac{s^2 + 33s - 46}{(s-1)^3(s-2)(s+2)} = \frac{A}{(s-1)^3} + \frac{B}{(s-1)^2} + \frac{C}{s-1} + \frac{D}{s-2} + \frac{E}{s+2}$$

We can quickly find $A = 4$, $D = 6$, and $E = -1$, then we'll plug these values into the decomposition, along with $s = -1$,

$$\frac{(-1)^2 + 33(-1) - 46}{(-1-1)^3(-1-2)(-1+2)} = \frac{4}{(-1-1)^3} + \frac{B}{(-1-1)^2} + \frac{C}{-1-1} + \frac{6}{-1-2} - \frac{1}{-1+2}$$

$$B - 2C = 1$$

and then separately $s = 0$,

$$\frac{0^2 + 33(0) - 46}{(0-1)^3(0-2)(0+2)} = \frac{4}{(0-1)^3} + \frac{B}{(0-1)^2} + \frac{C}{0-1} + \frac{6}{0-2} - \frac{1}{0+2}$$

$$B - C = -4$$

to get the system of equations

$$B - 2C = 1$$

$$B - C = -4$$

Solving this system gives $B = -9$ and $C = -5$. Then the Laplace transform is

$$Y(s) = \frac{4}{(s-1)^3} - \frac{9}{(s-1)^2} - \frac{5}{s-1} + \frac{6}{s-2} - \frac{1}{s+2}$$

Applying the inverse Laplace transform to each term gives the general solution to the differential equation gives

$$y(t) = \frac{4}{2!}t^2e^t - 9te^t - 5e^t + 6e^{2t} - e^{-2t}$$

$$y(t) = 2t^2e^t - 9te^t - 5e^t + 6e^{2t} - e^{-2t}$$

■ 3. Use the Laplace transform to solve the differential equation, given $y(0) = 0$, $y'(0) = 1$, and $y''(0) = 2$.

$$y''' - y'' - 5y' - 3y = 2e^{3t} + 10\sin t$$

Solution:

Apply the Laplace transform to both sides of the differential equation.

$$\mathscr{L}(y''') - \mathscr{L}(y'') - 5\mathscr{L}(y') - 3\mathscr{L}(y) = 2\mathscr{L}(e^{3t}) + 10\mathscr{L}(\sin t)$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - s^2Y(s) + sy(0) + y'(0)$$

$$-5sY(s) + 5y(0) - 3Y(s) = \frac{2}{s-3} + \frac{10}{s^2+1}$$

Substitute the initial conditions $y(0) = 0$, $y'(0) = 1$, and $y''(0) = 2$,

$$Y(s)(s^3 - s^2 - 5s - 3) = \frac{2}{s-3} + \frac{10}{s^2+1} + s + 1$$

$$Y(s)(s^3 - s^2 - 5s - 3) = \frac{s^4 - 2s^3 + 8s - 31}{(s-3)(s^2+1)}$$

then solve for $Y(s)$.

$$Y(s) = \frac{s^4 - 2s^3 + 8s - 31}{(s-3)^2(s^2+1)(s+1)^2}$$

Apply a partial fractions decomposition.

$$\frac{s^4 - 2s^3 + 8s - 31}{(s^2+1)(s+1)^2(s-3)^2} = \frac{As+B}{s^2+1} + \frac{C}{(s+1)^2} + \frac{D}{s+1} + \frac{E}{(s-3)^2} + \frac{F}{s-3}$$

We can quickly find $C = -9/8$ and $E = 1/8$, then we'll plug these values into the decomposition, along with $s = -2$,

$$\frac{(-2)^4 - 2(-2)^3 + 8(-2) - 31}{((-2)^2 + 1)(-2 + 1)^2(-2 - 3)^2}$$

$$= \frac{A(-2) + B}{(-2)^2 + 1} - \frac{9}{8(-2 + 1)^2} + \frac{D}{-2 + 1} + \frac{1}{8(-2 - 3)^2} + \frac{F}{-2 - 3}$$

$$2A - B + 5D + F = -5$$

and then separately $s = 0$,

$$\frac{0^4 - 2(0)^3 + 8(0) - 31}{(0^2 + 1)(0 + 1)^2(0 - 3)^2} = \frac{A(0) + B}{0^2 + 1} - \frac{9}{8(0 + 1)^2} + \frac{D}{0 + 1} + \frac{1}{8(0 - 3)^2} + \frac{F}{0 - 3}$$

$$3B + 3D - F = -7$$

and then separately $s = 1$,

$$\frac{1^4 - 2(1)^3 + 8(1) - 31}{(1^2 + 1)(1 + 1)^2(1 - 3)^2} = \frac{A(1) + B}{1^2 + 1} - \frac{9}{8(1 + 1)^2} + \frac{D}{1 + 1} + \frac{1}{8(1 - 3)^2} + \frac{F}{1 - 3}$$

$$A + B + D - F = -1$$

and then separately $s = 2$,

$$\frac{2^4 - 2(2)^3 + 8(2) - 31}{(2^2 + 1)(2 + 1)^2(2 - 3)^2} = \frac{A(2) + B}{2^2 + 1} - \frac{9}{8(2 + 1)^2} + \frac{D}{2 + 1} + \frac{1}{8(2 - 3)^2} + \frac{F}{2 - 3}$$

$$6A + 3B + 5D - 15F = -5$$

to get the system of equations

$$2A - B + 5D + F = -5$$

$$3B + 3D - F = -7$$

$$A+B+D-F=-1$$

$$6A+3B+5D-15F=-5$$

We can use a matrix to solve the system for the values of A, B, D, and F.

$$\begin{bmatrix} 2 & -1 & 5 & 1 & | & -5 \\ 0 & 3 & 3 & -1 & | & -7 \\ 1 & 1 & 1 & -1 & | & -1 \\ 6 & 3 & 5 & -15 & | & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 & | & -1 \\ 2 & -1 & 5 & 1 & | & -5 \\ 0 & 3 & 3 & -1 & | & -7 \\ 6 & 3 & 5 & -15 & | & -5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & -1 & | & -1 \\ 0 & -3 & 3 & 3 & | & -3 \\ 0 & 3 & 3 & -1 & | & -7 \\ 6 & 3 & 5 & -15 & | & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & -1 & | & -1 \\ 0 & -3 & 3 & 3 & | & -3 \\ 0 & 3 & 3 & -1 & | & -7 \\ 0 & -3 & -1 & -9 & | & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & -1 & | & -1 \\ 0 & 1 & -1 & -1 & | & 1 \\ 0 & 3 & 3 & -1 & | & -7 \\ 0 & -3 & -1 & -9 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & | & -2 \\ 0 & 1 & -1 & -1 & | & 1 \\ 0 & 3 & 3 & -1 & | & -7 \\ 0 & -3 & -1 & -9 & | & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 0 & | & -2 \\ 0 & 1 & -1 & -1 & | & 1 \\ 0 & 0 & 6 & 2 & | & -10 \\ 0 & -3 & -1 & -9 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & 0 & | & -2 \\ 0 & 1 & -1 & -1 & | & 1 \\ 0 & 0 & 6 & 2 & | & -10 \\ 0 & 0 & -4 & -12 & | & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 2 & 0 & | & -2 \\ 0 & 1 & -1 & -1 & | & 1 \\ 0 & 0 & 1 & \frac{1}{3} & | & -\frac{5}{3} \\ 0 & 0 & -4 & -12 & | & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & | & \frac{4}{3} \\ 0 & 1 & -1 & -1 & | & 1 \\ 0 & 0 & 1 & \frac{1}{3} & | & -\frac{5}{3} \\ 0 & 0 & -4 & -12 & | & 4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & | & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & | & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & | & -\frac{5}{3} \\ 0 & 0 & -4 & -12 & | & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & | & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & | & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & | & -\frac{5}{3} \\ 0 & 0 & 0 & -\frac{32}{3} & | & -\frac{8}{3} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -\frac{2}{3} & | & \frac{4}{3} \\ 0 & 1 & 0 & -\frac{2}{3} & | & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & | & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & | & \frac{1}{4} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & | & \frac{3}{2} \\ 0 & 1 & 0 & -\frac{2}{3} & | & -\frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{3} & | & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & | & \frac{1}{4} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & | & \frac{3}{2} \\ 0 & 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & | & -\frac{5}{3} \\ 0 & 0 & 0 & 1 & | & \frac{1}{4} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & | & \frac{3}{2} \\ 0 & 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & | & -\frac{7}{4} \\ 0 & 0 & 0 & 1 & | & \frac{1}{4} \end{bmatrix}$$

Solving this system gives $A = 3/2$, $B = -1/2$, $D = -7/4$, and $F = 1/4$. Then the Laplace transform is

$$Y(s) = \frac{\frac{3}{2}s - \frac{1}{2}}{s^2 + 1} + \frac{-\frac{9}{8}}{(s+1)^2} + \frac{-\frac{7}{4}}{s+1} + \frac{\frac{1}{8}}{(s-3)^2} + \frac{\frac{1}{4}}{s-3}$$

$$Y(s) = \frac{1}{2}\left(\frac{3s-1}{s^2+1}\right) - \frac{9}{8}\left(\frac{1}{(s+1)^2}\right) - \frac{7}{4}\left(\frac{1}{s+1}\right)$$

$$+\frac{1}{8}\left(\frac{1}{(s-3)^2}\right)+\frac{1}{4}\left(\frac{1}{s-3}\right)$$

$$Y(s)=\frac{3}{2}\left(\frac{s}{s^2+1^2}\right)-\frac{1}{2}\left(\frac{1}{s^2+1^2}\right)-\frac{9}{8}\left(\frac{1}{(s-(-1))^{1+1}}\right)-\frac{7}{4}\left(\frac{1}{s-(-1)}\right)$$

$$+\frac{1}{8}\left(\frac{1}{(s-3)^{1+1}}\right)+\frac{1}{4}\left(\frac{1}{s-3}\right)$$

Applying the inverse Laplace transform to each term gives the general solution to the differential equation gives

$$y(t)=\frac{3}{2}\cos t-\frac{1}{2}\sin t-\frac{9}{8}te^{-t}-\frac{7}{4}e^{-t}+\frac{1}{8}te^{3t}+\frac{1}{4}e^{3t}$$

■ 4. Use the Laplace transform to solve the differential equation, given $y(0)=0$, $y'(0)=0$, and $y''(0)=0$.

$$y'''-2y''+y'-2y=5+25\sin t$$

Solution:

Apply the Laplace transform to both sides of the differential equation.

$$\mathscr{L}(y''')-2\mathscr{L}(y'')+\mathscr{L}(y')-2\mathscr{L}(y)=\mathscr{L}(5)+25\mathscr{L}(\sin t)$$

$$s^3Y(s)-s^2y(0)-sy'(0)-y''(0)-2s^2Y(s)+2sy(0)+2y'(0)$$

$$+sY(s)-y(0)-2Y(s)=\frac{5}{s}+\frac{25}{s^2+1}$$

Substitute the initial conditions $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$,

$$Y(s)(s^3 - 2s^2 + s - 2) = \frac{5}{s} + \frac{25}{s^2+1} = \frac{5s^2 + 25s + 5}{s(s^2+1)}$$

then solve for $Y(s)$.

$$Y(s) = \frac{5(s^2 + 5s + 1)}{s(s^2+1)^2(s-2)}$$

Apply a partial fractions decomposition.

$$\frac{5(s^2+5s+1)}{s(s-2)(s^2+1)^2} = \frac{A}{s} + \frac{B}{s-2} + \frac{Cs+D}{s^2+1} + \frac{Es+F}{(s^2+1)^2}$$

We can quickly find $A = -5/2$ and $B = 3/2$, then we'll plug these values into the decomposition, along with $s = 1$,

$$\frac{5(1^2+5(1)+1)}{1(1-2)(1^2+1)^2} = -\frac{5}{2(1)} + \frac{3}{2(1-2)} + \frac{C(1)+D}{1^2+1} + \frac{E(1)+F}{(1^2+1)^2}$$

$$2C + 2D + E + F = -19$$

and then separately $s = -1$,

$$\frac{5((-1)^2+5(-1)+1)}{(-1)((-1)-2)((-1)^2+1)^2} = -\frac{5}{2(-1)} + \frac{3}{2((-1)-2)} + \frac{C(-1)+D}{(-1)^2+1} + \frac{E(-1)+F}{((-1)^2+1)^2}$$

$$-2C + 2D - E + F = -13$$

and then separately $s = -2$,

$$\frac{5((-2)^2+5(-2)+1)}{-2(-2-2)((-2)^2+1)^2} = -\frac{5}{2(-2)} + \frac{3}{2(-2-2)} + \frac{C(-2)+D}{(-2)^2+1} + \frac{E(-2)+F}{((-2)^2+1)^2}$$

$$-10C + 5D - 2E + F = -25$$

and then separately $s = 3$,

$$\frac{5(3^2 + 5(3) + 1)}{3(3-2)(3^2+1)^2} = -\frac{5}{2(3)} + \frac{3}{2(3-2)} + \frac{C(3)+D}{3^2+1} + \frac{E(3)+F}{(3^2+1)^2}$$

$$30C + 10D + 3E + F = -25$$

to get the system of equations

$$2C + 2D + E + F = -19$$

$$-2C + 2D - E + F = -13$$

$$-10C + 5D - 2E + F = -25$$

$$30C + 10D + 3E + F = -25$$

We can use a matrix to solve the system for the values of C, D, E, and F.

$$\begin{bmatrix} 2 & 2 & 1 & 1 & | & -19 \\ -2 & 2 & -1 & 1 & | & -13 \\ -10 & 5 & -2 & 1 & | & -25 \\ 30 & 10 & 3 & 1 & | & -25 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & | & -\frac{19}{2} \\ -2 & 2 & -1 & 1 & | & -13 \\ -10 & 5 & -2 & 1 & | & -25 \\ 30 & 10 & 3 & 1 & | & -25 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & | & -\frac{19}{2} \\ 0 & 4 & 0 & 2 & | & -32 \\ -10 & 5 & -2 & 1 & | & -25 \\ 30 & 10 & 3 & 1 & | & -25 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & | & -\frac{19}{2} \\ 0 & 4 & 0 & 2 & | & -32 \\ 0 & 15 & 3 & 6 & | & -120 \\ 30 & 10 & 3 & 1 & | & -25 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & | & -\frac{19}{2} \\ 0 & 4 & 0 & 2 & | & -32 \\ 0 & 15 & 3 & 6 & | & -120 \\ 0 & -20 & -12 & -14 & | & 260 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & \frac{1}{2} & \frac{1}{2} & | & -\frac{19}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 15 & 3 & 6 & | & -120 \\ 0 & -20 & -12 & -14 & | & 260 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & | & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 15 & 3 & 6 & | & -120 \\ 0 & -20 & -12 & -14 & | & 260 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & | & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 0 & 3 & -\frac{3}{2} & | & 0 \\ 0 & -20 & -12 & -14 & | & 260 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & | & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 0 & 3 & -\frac{3}{2} & | & 0 \\ 0 & 0 & -12 & -4 & | & 100 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & 0 & | & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & -12 & -4 & | & 100 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & | & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & -12 & -4 & | & 100 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & | & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & -10 & | & 100 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{4} & | & -\frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & 1 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & \frac{1}{2} & | & -8 \\ 0 & 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & 1 & | & -10 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & -3 \\ 0 & 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & 1 & | & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & | & -3 \\ 0 & 0 & 1 & 0 & | & -5 \\ 0 & 0 & 0 & 1 & | & -10 \end{bmatrix}$$

Solving this system gives $C = 1$, $D = -3$, $E = -5$, and $F = -10$. Then the Laplace transform is

$$Y(s) = -\frac{5}{2s} + \frac{3}{2(s-2)} + \frac{s-3}{s^2+1} - \frac{5s+10}{(s^2+1)^2}$$

$$Y(s) = -\frac{5}{2}\left(\frac{1}{s}\right) + \frac{3}{2}\left(\frac{1}{s-2}\right) + \frac{s}{s^2+1}$$

$$-3\left(\frac{1}{s^2+1}\right) - \frac{5}{2}\left(\frac{2s}{(s^2+1)^2}\right) - 5\left(\frac{2}{(s^2+1)^2}\right)$$

Applying the inverse Laplace transform to each term gives the general solution to the differential equation gives

$$y(t) = -\frac{5}{2} + \frac{3}{2}e^{2t} + \cos t - 3\sin t - \frac{5}{2}t\sin t - 5(\sin t - t\cos t)$$

$$y(t) = -\frac{5}{2} + \frac{3}{2}e^{2t} - 8\sin t + \cos t - \frac{5}{2}t\sin t + 5t\cos t$$

■ 5. Use the Laplace transform to solve the differential equation, given $y(0) = -2$, $y'(0) = 3$, and $y''(0) = 6$.

$$y''' - 6y'' + 12y' - 8y = 27e^{-t} - 2e^{t}$$

Solution:

Apply the Laplace transform to both sides of the differential equation.

$$\mathscr{L}(y''') - 6\mathscr{L}(y'') + 12\mathscr{L}(y') - 8\mathscr{L}(y) = 27\mathscr{L}(e^{-t}) - 2\mathscr{L}(e^{t})$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 6(s^2Y(s) - sy(0) - y'(0))$$

$$+12(sY(s) - y(0)) - 8Y(s) = \frac{27}{s+1} - \frac{2}{s-1}$$

Substitute the initial conditions $y(0) = -2$, $y'(0) = 3$, and $y''(0) = 6$,

$$Y(s)(s^3 - 6s^2 + 12s - 8) = \frac{27}{s+1} - \frac{2}{s-1} - 2s^2 + 15s - 36$$

$$Y(s)(s^3 - 6s^2 + 12s - 8) = \frac{-2s^4 + 15s^3 - 34s^2 + 10s + 7}{(s+1)(s-1)}$$

then solve for $Y(s)$.

$$Y(s) = \frac{-2s^4 + 15s^3 - 34s^2 + 10s + 7}{(s+1)(s-1)(s-2)^3}$$

Apply a partial fractions decomposition.

$$\frac{-2s^4 + 15s^3 - 34s^2 + 10s + 7}{(s+1)(s-1)(s-2)^3}$$

$$= \frac{A}{s+1} + \frac{B}{s-1} + \frac{C}{(s-2)^3} + \frac{D}{(s-2)^2} + \frac{E}{s-2}$$

We can quickly find $A = -1$, $B = 2$, and $C = -7$, then we'll plug these values into the decomposition, along with $s = 0$,

$$\frac{-2(0)^4 + 15(0)^3 - 34(0)^2 + 10(0) + 7}{(0+1)(0-1)(0-2)^3}$$

$$= -\frac{1}{0+1} + \frac{2}{0-1} - \frac{7}{(0-2)^3} + \frac{D}{(0-2)^2} + \frac{E}{0-2}$$

$$D - 2E = 12$$

and then separately $s = -2$,

$$\frac{-2(-2)^4 + 15(-2)^3 - 34(-2)^2 + 10(-2) + 7}{(-2+1)(-2-1)(-2-2)^3}$$

$$= -\frac{1}{-2+1} + \frac{2}{-2-1} - \frac{7}{(-2-2)^3} + \frac{D}{(-2-2)^2} + \frac{E}{-2-2}$$

$$D - 4E = 18$$

to get the system of equations

$$D - 2E = 12$$

$$D - 4E = 18$$

Solving this system gives $D = 6$ and $E = -3$. Then the Laplace transform is

$$Y(s) = -\frac{1}{s+1} + \frac{2}{s-1} - \frac{7}{(s-2)^3} + \frac{6}{(s-2)^2} - \frac{3}{s-2}$$

Applying the inverse Laplace transform to each term gives the general solution to the differential equation gives

$$y(t) = -e^{-t} + 2e^{t} - \frac{7}{2}t^2e^{2t} + 6te^{2t} - 3e^{2t}$$

■ 6. Use the Laplace transform to solve the differential equation, given $y(0) = 3$, $y'(0) = 0$, and $y''(0) = -9$.

$$y''' + 3y'' + 4y' + 2y = -8\sin t - 6\cos t$$

Solution:

Apply the Laplace transform to both sides of the differential equation.

$$\mathscr{L}(y''') + 3\mathscr{L}(y'') + 4\mathscr{L}(y') + 2\mathscr{L}(y) = -8\mathscr{L}(\sin t) - 6\mathscr{L}(\cos t)$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 3s^2Y(s) - 3sy(0) - 3y'(0)$$

$$+4sY(s) - 4y(0) + 2Y(s) = -\frac{8}{s^2+1} - \frac{6s}{s^2+1}$$

Substitute the initial conditions $y(0) = 3$, $y'(0) = 0$, and $y''(0) = -9$,

$$Y(s)(s^3 + 3s^2 + 4s + 2) = \frac{-8-6s}{s^2+1} + 3s^2 + 9s + 3$$

$$Y(s)(s^3 + 3s^2 + 4s + 2) = \frac{3s^4 + 9s^3 + 6s^2 + 3s - 5}{s^2+1}$$

then solve for $Y(s)$.

$$Y(s) = \frac{3s^4 + 9s^3 + 6s^2 + 3s - 5}{(s^2+1)(s^2+2s+2)(s+1)}$$

Apply a partial fractions decomposition.

$$\frac{3s^4+9s^3+6s^2+3s-5}{(s^2+1)(s^2+2s+2)(s+1)}=\frac{A}{s+1}+\frac{Bs+C}{s^2+1}+\frac{Ds+E}{s^2+2s+2}$$

We can quickly find $A=-4$, then we'll plug this value into the decomposition, along with $s=-2$,

$$\frac{3(-2)^4+9(-2)^3+6(-2)^2+3(-2)-5}{((-2)^2+1)((-2)^2+2(-2)+2)(-2+1)}$$

$$=-\frac{4}{-2+1}+\frac{B(-2)+C}{(-2)^2+1}+\frac{D(-2)+E}{(-2)^2+2(-2)+2}$$

$$4B-2C+10D-5E=29$$

and then separately $s=0$,

$$\frac{3(0)^4+9(0)^3+6(0)^2+3(0)-5}{(0^2+1)(0^2+2(0)+2)(0+1)}=-\frac{4}{0+1}+\frac{B(0)+C}{0^2+1}+\frac{D(0)+E}{0^2+2(0)+2}$$

$$2C+E=3$$

and then separately $s=1$,

$$\frac{3(1)^4+9(1)^3+6(1)^2+3(1)-5}{(1^2+1)(1^2+2(1)+2)(1+1)}=-\frac{4}{1+1}+\frac{B(1)+C}{1^2+1}+\frac{D(1)+E}{1^2+2(1)+2}$$

$$5B+5C+2D+2E=28$$

and then separately $s=2$,

$$\frac{3(2)^4+9(2)^3+6(2)^2+3(2)-5}{(2^2+1)(2^2+2(2)+2)(2+1)}=-\frac{4}{2+1}+\frac{B(2)+C}{2^2+1}+\frac{D(2)+E}{2^2+2(2)+2}$$

$$4B + 2C + 2D + E = 23$$

to get the system of equations

$$4B - 2C + 10D - 5E = 29$$

$$2C + E = 3$$

$$5B + 5C + 2D + 2E = 28$$

$$4B + 2C + 2D + E = 23$$

We can use a matrix to solve the system for the values of B, C, D, and E.

$$\begin{bmatrix} 4 & -2 & 10 & -5 & | & 29 \\ 0 & 2 & 0 & 1 & | & 3 \\ 5 & 5 & 2 & 2 & | & 28 \\ 4 & 2 & 2 & 1 & | & 23 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} & -\frac{5}{4} & | & \frac{29}{4} \\ 0 & 2 & 0 & 1 & | & 3 \\ 5 & 5 & 2 & 2 & | & 28 \\ 4 & 2 & 2 & 1 & | & 23 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} & -\frac{5}{4} & | & \frac{29}{4} \\ 0 & 2 & 0 & 1 & | & 3 \\ 0 & \frac{15}{2} & -\frac{21}{2} & \frac{33}{4} & | & -\frac{33}{4} \\ 4 & 2 & 2 & 1 & | & 23 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} & -\frac{5}{4} & | & \frac{29}{4} \\ 0 & 2 & 0 & 1 & | & 3 \\ 0 & \frac{15}{2} & -\frac{21}{2} & \frac{33}{4} & | & -\frac{33}{4} \\ 0 & 4 & -8 & 6 & | & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} & -\frac{5}{4} & | & \frac{29}{4} \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & \frac{15}{2} & -\frac{21}{2} & \frac{33}{4} & | & -\frac{33}{4} \\ 0 & 4 & -8 & 6 & | & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{5}{2} & -1 & | & 8 \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & \frac{15}{2} & -\frac{21}{2} & \frac{33}{4} & | & -\frac{33}{4} \\ 0 & 4 & -8 & 6 & | & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{5}{2} & -1 & | & 8 \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & 0 & -\frac{21}{2} & \frac{9}{2} & | & -\frac{39}{2} \\ 0 & 4 & -8 & 6 & | & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{5}{2} & -1 & | & 8 \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{7} & | & \frac{13}{7} \\ 0 & 4 & -8 & 6 & | & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{5}{2} & -1 & | & 8 \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{7} & | & \frac{13}{7} \\ 0 & 0 & -8 & 4 & | & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{14} & | & \frac{47}{14} \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{7} & | & \frac{13}{7} \\ 0 & 0 & -8 & 4 & | & -12 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{14} & | & \frac{47}{14} \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{7} & | & \frac{13}{7} \\ 0 & 0 & 0 & \frac{4}{7} & | & \frac{20}{7} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{14} & | & \frac{47}{14} \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{7} & | & \frac{13}{7} \\ 0 & 0 & 0 & 1 & | & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & \frac{1}{2} & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{7} & | & \frac{13}{7} \\ 0 & 0 & 0 & 1 & | & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & -\frac{3}{7} & | & \frac{13}{7} \\ 0 & 0 & 0 & 1 & | & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & | & 3 \\ 0 & 1 & 0 & 0 & | & -1 \\ 0 & 0 & 1 & 0 & | & 4 \\ 0 & 0 & 0 & 1 & | & 5 \end{bmatrix}$$

Solving this system gives $B = 9/4$, $C = -5/2$, $D = 11/2$, and $E = 8$. Then the Laplace transform is

$$Y(s) = -\frac{4}{s+1} + \frac{3s-1}{s^2+1} + \frac{4s+5}{s^2+2s+2}$$

$$Y(s) = -4\left(\frac{1}{s-(-1)}\right) + 3\left(\frac{s}{s^2+1^2}\right) - \frac{1}{s^2+1^2}$$

$$+4\left(\frac{s-(-1)}{(s-(-1))^2+1^2}\right) - 4\left(\frac{1}{(s-(-1))^2+1^2}\right)$$

$$+5\left(\frac{1}{(s-(-1))^2+1^2}\right)$$

Applying the inverse Laplace transform to each term gives the general solution to the differential equation gives

$$y(t) = -4e^{-t} + 3\cos t - \sin t + 4e^{-t}\cos t - 4e^{-t}\sin t + 5e^{-t}\sin t$$

$$y(t) = -4e^{-t} + 3\cos t - \sin t + 4e^{-t}\cos t + e^{-t}\sin t$$

SYSTEMS OF HIGHER ORDER EQUATIONS

■ 1. Solve the system of differential equations.

$$x_1' = 2x_1$$

$$x_2' = 4x_3 - x_2$$

$$x_3' = -3x_1$$

Solution:

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 4 \\ -3 & 0 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 4 \\ -3 & 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & -1-\lambda & 4 \\ -3 & 0 & -\lambda \end{bmatrix}$$

and then finding its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 0 \\ 0 & -1-\lambda & 4 \\ -3 & 0 & -\lambda \end{vmatrix}$$

$$|A-\lambda I| = (2-\lambda)\begin{vmatrix} -1-\lambda & 4 \\ 0 & -\lambda \end{vmatrix} - (0)\begin{vmatrix} 0 & 4 \\ -3 & -\lambda \end{vmatrix} + (0)\begin{vmatrix} 0 & -1-\lambda \\ -3 & 0 \end{vmatrix}$$

$$|A-\lambda I| = (2-\lambda)((-1-\lambda)(-\lambda)-(4)(0)) - 0 + 0$$

$$|A-\lambda I| = (2-\lambda)(-1-\lambda)(-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(2-\lambda)(-1-\lambda)(-\lambda) = 0$$

$$\lambda = -1, 0, 2$$

For the Eigenvalue $\lambda_1 = -1$, we find

$$A + 1I = \begin{bmatrix} 2+1 & 0 & 0 \\ 0 & -1+1 & 4 \\ -3 & 0 & 1 \end{bmatrix}$$

$$A + 1I = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ -3 & 0 & 1 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$3k_1 = 0$$

$$4k_3 = 0$$

$$-3k_1 + k_3 = 0$$

we get $k_1 = 0$, $k_3 = 0$, and we'll choose $k_2 = 1$. Then the solution vector is

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

For the Eigenvalue $\lambda_2 = 0$, we find

$$A - (0)I = \begin{bmatrix} 2-0 & 0 & 0 \\ 0 & -1-0 & 4 \\ -3 & 0 & 0 \end{bmatrix}$$

$$A - (0)I = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 4 \\ -3 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$2k_1 = 0$$

$$-k_2 + 4k_3 = 0$$

$$-3k_1 = 0$$

we get $k_1 = 0$, and we'll choose $k_3 = 1$ to get $k_2 = 4$. The the solution vector is

$$\vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} e^{0t} = \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix}$$

For the Eigenvalue $\lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} 2-2 & 0 & 0 \\ 0 & -1-2 & 4 \\ -3 & 0 & -2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -3 & 4 \\ -3 & 0 & -2 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$-3k_2 + 4k_3 = 0$$

$$-3k_1 - 2k_3 = 0$$

we'll choose $k_3 = 3$ to get $k_1 = -2$ and $k_2 = 4$. Then the solution vector is

$$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} e^{2t}$$

So the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2} + c_3 \overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 4 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 4 \\ 3 \end{bmatrix} e^{2t}$$

■ 2. Solve the system of differential equations.

$$x_1' = 2x_2 - 2x_3$$

$$x_2' = 2x_1 + 4x_2 + 4x_3$$

$$x_3' = -2x_1 + 4x_2 - 3x_3$$

Solution:

We'll need to start by finding the matrix $A-\lambda I$,

$$A-\lambda I=\begin{bmatrix}0&2&-2\\2&4&4\\-2&4&-3\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}0&2&-2\\2&4&4\\-2&4&-3\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}-\lambda&2&-2\\2&4-\lambda&4\\-2&4&-3-\lambda\end{bmatrix}$$

and then finding its determinant $|A-\lambda I|$.

$$|A-\lambda I|=\begin{vmatrix}-\lambda&2&-2\\2&4-\lambda&4\\-2&4&-3-\lambda\end{vmatrix}$$

$$|A-\lambda I|=(-\lambda)\begin{vmatrix}4-\lambda&4\\4&-3-\lambda\end{vmatrix}-2\begin{vmatrix}2&4\\-2&-3-\lambda\end{vmatrix}+(-2)\begin{vmatrix}2&4-\lambda\\-2&4\end{vmatrix}$$

$$|A-\lambda I|=(-\lambda)((4-\lambda)(-3-\lambda)-(4)(4))-2((2)(-3-\lambda)-(-2)(4))$$

$$-2((2)(4)-(4-\lambda)(-2))$$

$$|A-\lambda I|=(-\lambda)(\lambda^2-\lambda-28)-2(-2\lambda+2)-2(16-2\lambda)$$

$$|A-\lambda I|=-\lambda^3+\lambda^2+36\lambda-36$$

$$|A - \lambda I| = (\lambda^2 - 36)(1 - \lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(\lambda - 6)(\lambda + 6)(1 - \lambda) = 0$$

$$\lambda = -6, 1, 6$$

For the Eigenvalue $\lambda_1 = -6$, we find

$$A + 6I = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 4+6 & 4 \\ -2 & 4 & -3+6 \end{bmatrix}$$

$$A + 6I = \begin{bmatrix} 6 & 2 & -2 \\ 2 & 10 & 4 \\ -2 & 4 & 3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 10 & 4 \\ 6 & 2 & -2 \\ -2 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 2 \\ 6 & 2 & -2 \\ -2 & 4 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 2 \\ 0 & -28 & -14 \\ 0 & 14 & 7 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 5 & 2 \\ 0 & 14 & 7 \\ 0 & -28 & -14 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 2 \\ 0 & 14 & 7 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$k_1 - \frac{1}{2}k_3 = 0$$

$$k_2 + \frac{1}{2}k_3 = 0$$

we'll choose $k_3 = 2$ to get $k_1 = 1$ and $k_2 = -1$. Then the solution vector is

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} e^{-6t}$$

For the Eigenvalue $\lambda_2 = 1$, we find

$$A - 1I = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 4-1 & 4 \\ -2 & 4 & -3-1 \end{bmatrix}$$

$$A - 1I = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 3 & 4 \\ -2 & 4 & -4 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 2 \\ 2 & 3 & 4 \\ -2 & 4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$k_1 + 2k_3 = 0$$

$$k_2 = 0$$

we get $k_2 = 0$ and we'll choose $k_3 = 1$ to get $k_1 = -2$. Then the solution vector is

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{t}$$

For the Eigenvalue $\lambda_3 = 6$, we find

$$A - 6I = \begin{bmatrix} -6 & 2 & -2 \\ 2 & 4-6 & 4 \\ -2 & 4 & -3-6 \end{bmatrix}$$

$$A - 6I = \begin{bmatrix} -6 & 2 & -2 \\ 2 & -2 & 4 \\ -2 & 4 & -9 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & -2 & 4 \\ -6 & 2 & -2 \\ -2 & 4 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 10 \\ 0 & 2 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & -4 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$k_1 - \frac{1}{2}k_3 = 0$$

$$k_2 - \frac{5}{2}k_3 = 0$$

we'll choose $k_3 = 2$ to get $k_1 = 1$ and $k_2 = 5$. Then the solution vector is

$$\vec{x_3} = \vec{k_3}e^{\lambda_3 t}$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} e^{6t}$$

So the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} e^{-6t} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 1 \\ 5 \\ 2 \end{bmatrix} e^{6t}$$

■ 3. Solve the system of differential equations.

$$x_1' = 2x_1 + 4x_3$$

$$x_2' = -x_1 + 2x_2$$

$$x_3' = -x_3$$

Solution:

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 0 & 4 \\ -1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 0 & 4 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

and then finding its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 4 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)\begin{vmatrix} 2-\lambda & 0 \\ 0 & -1-\lambda \end{vmatrix} - (0)\begin{vmatrix} -1 & 0 \\ 0 & -1-\lambda \end{vmatrix} + 4\begin{vmatrix} -1 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)((2-\lambda)(-1-\lambda) - (0)(0)) - 0 + 4((-1)(0) - (2-\lambda)(0))$$

$$|A - \lambda I| = (2-\lambda)(2-\lambda)(-1-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(2-\lambda)(2-\lambda)(-1-\lambda) = 0$$

$$\lambda = -1, 2, 2$$

For the Eigenvalue $\lambda_1 = -1$, we find

$$A + 1I = \begin{bmatrix} 2+1 & 0 & 4 \\ -1 & 2+1 & 0 \\ 0 & 0 & -1+1 \end{bmatrix}$$

$$A + 1I = \begin{bmatrix} 3 & 0 & 4 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$3k_1 + 4k_3 = 0$$

$$-k_1 + 3k_2 = 0$$

we'll choose $k_3 = 9$ to get $k_1 = -12$ and $k_2 = -4$. Then the solution vector is

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} -12 \\ -4 \\ 9 \end{bmatrix} e^{-t}$$

For the Eigenvalue $\lambda_2 = \lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} 2-2 & 0 & 4 \\ -1 & 2-2 & 0 \\ 0 & 0 & -1-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 0 & 0 & 4 \\ -1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$4k_3 = 0$$

$$-k_1 = 0$$

$$-3k_3 = 0$$

we get $k_1 = 0$ and $k_3 = 0$, and we'll choose $k_2 = 1$. Then the solution vector is

$$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_2 = \lambda_3 = 2$, we have to use $\overrightarrow{k_2} = (0,1,0)$ to find a second solution.

$$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_2}$$

$$\begin{bmatrix} 0 & 0 & 4 \\ -1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$4p_3 = 0$$

$$-p_1 = 1$$

$$-3p_3 = 0$$

we get $p_1 = -1$ and $p_3 = 0$, and we'll choose $p_2 = 0$. Then the second solution will be

$$\overrightarrow{x_3} = \overrightarrow{k_2}te^{\lambda_2 t} + \overrightarrow{p_1}e^{\lambda_2 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$$

So the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} -12 \\ -4 \\ 9 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{2t} \right)$$

■ 4. Solve the system of differential equations.

$$x_2' = -x_1 + 3x_2 + 5x_3$$

$$x_3' = 2x_1 - 2x_2 + x_3$$

Solution:

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & 5 \\ 2 & -2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & 5 \\ 2 & -2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 0 & 0 \\ -1 & 3-\lambda & 5 \\ 2 & -2 & 1-\lambda \end{bmatrix}$$

and then finding its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 0 & 0 \\ -1 & 3-\lambda & 5 \\ 2 & -2 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (-\lambda)\begin{vmatrix} 3-\lambda & 5 \\ -2 & 1-\lambda \end{vmatrix} - (0)\begin{vmatrix} -1 & 5 \\ 2 & 1-\lambda \end{vmatrix} + (0)\begin{vmatrix} -1 & 3-\lambda \\ 2 & -2 \end{vmatrix}$$

$$|A - \lambda I| = (-\lambda)((3-\lambda)(1-\lambda) - (5)(-2)) - 0 + 0$$

$$|A - \lambda I| = (-\lambda)(\lambda^2 - 4\lambda + 13)$$

Solve the characteristic equation for the Eigenvalues.

$$(-\lambda)(\lambda^2 - 4\lambda + 13) = 0$$

$$\lambda = 0,\ 2 \pm 3i$$

For the Eigenvalue $\lambda_1 = 0$, we find

$$A - (0)I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3-0 & 5 \\ 2 & -2 & 1-0 \end{bmatrix}$$

$$A - (0)I = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 3 & 5 \\ 2 & -2 & 1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -3 & -5 \\ 2 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 \\ 0 & 4 & 11 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 \\ 0 & 1 & \frac{11}{4} \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{13}{4} \\ 0 & 1 & \frac{11}{4} \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$k_1 + \frac{13}{4}k_3 = 0$$

$$k_2 + \frac{11}{4}k_3 = 0$$

we'll choose $k_3 = 4$ to get $k_1 = -13$ and $k_2 = -11$. Then the solution vector is

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} -13 \\ -11 \\ 4 \end{bmatrix}$$

For the Eigenvalue $\lambda_2 = 2 + 3i$, we find

$$A - (2+3i)I = \begin{bmatrix} -2-3i & 0 & 0 \\ -1 & 3-2-3i & 5 \\ 2 & -2 & 1-2-3i \end{bmatrix}$$

$$A - (2+3i)I = \begin{bmatrix} -2-3i & 0 & 0 \\ -1 & 1-3i & 5 \\ 2 & -2 & -1-3i \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$(-2-3i)k_1 = 0$$

$$-k_1 + (1-3i)k_2 + 5k_3 = 0$$

$$2k_1 - 2k_2 + (-1-3i)k_3 = 0$$

we get $k_1 = 0$ and we'll choose $k_3 = -2$ to get $k_2 = 1 + 3i$. Then the solution vector is

$$\vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 0 \\ 1+3i \\ -2 \end{bmatrix} e^{(2+3i)t}$$

For the Eigenvalue $\lambda_3 = 2 - 3i$, we find

$$A - (2-3i)I = \begin{bmatrix} -2+3i & 0 & 0 \\ -1 & 3-2+3i & 5 \\ 2 & -2 & 1-2+3i \end{bmatrix}$$

$$A - (2-3i)I = \begin{bmatrix} -2+3i & 0 & 0 \\ -1 & 1+3i & 5 \\ 2 & -2 & -1+3i \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$(-2+3i)k_1 = 0$$

$$-k_1 + (1+3i)k_2 + 5k_3 = 0$$

$$2k_1 - 2k_2 + (-1+3i)k_3 = 0$$

we get $k_1 = 0$ and we'll choose $k_3 = -2$ to get $k_2 = 1 - 3i$. Then the solution vector is

$$\vec{x_3} = \vec{k_3} e^{\lambda_3 t}$$

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 1 - 3i \\ -2 \end{bmatrix} e^{(2-3i)t}$$

So the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3$$

$$\vec{x} = c_1 \begin{bmatrix} -13 \\ -11 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 + 3i \\ -2 \end{bmatrix} e^{(2+3i)t} + c_3 \begin{bmatrix} 0 \\ 1 - 3i \\ -2 \end{bmatrix} e^{(2-3i)t}$$

■ 5. Solve the system of differential equations.

$$x_1' = x_1 - x_2$$

$$x_2' = 5x_1 - x_2$$

$$x_3' = -3x_1 + x_2 + 4x_3$$

Solution:

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 0 \\ 5 & -1 & 0 \\ -3 & 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & -1 & 0 \\ 5 & -1 & 0 \\ -3 & 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -1 & 0 \\ 5 & -1-\lambda & 0 \\ -3 & 1 & 4-\lambda \end{bmatrix}$$

and then finding its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 & 0 \\ 5 & -1-\lambda & 0 \\ -3 & 1 & 4-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} -1-\lambda & 0 \\ 1 & 4-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 5 & 0 \\ -3 & 4-\lambda \end{vmatrix} + (0)\begin{vmatrix} 5 & -1-\lambda \\ -3 & 1 \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)((-1-\lambda)(4-\lambda) - (1)(0)) + ((5)(4-\lambda) - (-3)(0)) + 0$$

$$|A - \lambda I| = (1-\lambda)(-1-\lambda)(4-\lambda) + (5)(4-\lambda)$$

$$|A - \lambda I| = (4-\lambda)(\lambda^2 + 4)$$

Solve the characteristic equation for the Eigenvalues.

$$(4-\lambda)(\lambda^2+4) = 0$$

$$\lambda = 4, \ \pm 2i$$

For the Eigenvalue $\lambda_1 = 4$, we find

$$A - 4I = \begin{bmatrix} 1-4 & -1 & 0 \\ 5 & -1-4 & 0 \\ -3 & 1 & 4-4 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -3 & -1 & 0 \\ 5 & -5 & 0 \\ -3 & 1 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 5 & -5 & 0 \\ -3 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ -3 & -1 & 0 \\ -3 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & -4 & 0 \\ 0 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$k_1 = 0$$

$$k_2 = 0$$

we get $k_1 = 0$ and $k_2 = 0$ and we'll choose $k_3 = 1$. Then the solution vector is

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{4t}$$

For the Eigenvalue $\lambda_2 = 2i$, we find

$$A - 2iI = \begin{bmatrix} 1 - 2i & -1 & 0 \\ 5 & -1 - 2i & 0 \\ -3 & 1 & 4 - 2i \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$(1 - 2i)k_1 - k_2 = 0$$

$$5k_1 + (-1 - 2i)k_2 = 0$$

$$-3k_1 + k_2 + (4 - 2i)k_3 = 0$$

we'll choose $k_3 = 2$ to get $k_1 = 1 - 3i$ and $k_2 = -5 - 5i$. Then the solution vector is

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 - 3i \\ -5 - 5i \\ 2 \end{bmatrix} e^{2it}$$

For the Eigenvalue $\lambda_3 = -2i$, we find

$$A + 2iI = \begin{bmatrix} 1 + 2i & -1 & 0 \\ 5 & -1 + 2i & 0 \\ -3 & 1 & 4 + 2i \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$(1 + 2i)k_1 - k_2 = 0$$

$$5k_1 + (-1 + 2i)k_2 = 0$$

$$-3k_1 + k_2 + (4 + 2i)k_3 = 0$$

we'll choose $k_3 = 2$ to get $k_1 = 1 + 3i$ and $k_2 = -5 + 5i$. Then the solution vector is

$$\vec{x_3} = \vec{k_3}e^{\lambda_3 t}$$

$$\vec{x_3} = \begin{bmatrix} 1+3i \\ -5+5i \\ 2 \end{bmatrix} e^{-2it}$$

So the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1-3i \\ -5-5i \\ 2 \end{bmatrix} e^{2it} + c_3 \begin{bmatrix} 1+3i \\ -5+5i \\ 2 \end{bmatrix} e^{-2it}$$

■ 6. Solve the system of differential equations.

$$x_1' = -x_1 + 3x_2 - x_3$$

$$x_2' = -x_2 + 2x_3$$

$$x_3' = -x_3$$

Solution:

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} -1 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -1 & 3 & -1 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -1-\lambda & 3 & -1 \\ 0 & -1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

and then finding its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 3 & -1 \\ 0 & -1-\lambda & 2 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (-1-\lambda)\begin{vmatrix} -1-\lambda & 2 \\ 0 & -1-\lambda \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 0 & -1-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 0 & -1-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (-1-\lambda)((-1-\lambda)(-1-\lambda) - (2)(0)) - 0 + 0$$

$$|A - \lambda I| = (-1-\lambda)(-1-\lambda)(-1-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(-1-\lambda)(-1-\lambda)(-1-\lambda) = 0$$

$$\lambda = -1, \ -1, \ -1$$

For the Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = -1$, we find

$$A + 1I = \begin{bmatrix} -1+1 & 3 & -1 \\ 0 & -1+1 & 2 \\ 0 & 0 & -1+1 \end{bmatrix}$$

$$A + 1I = \begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$3k_2 - k_3 = 0$$

$$2k_3 = 0$$

we get $k_2 = 0$ and $k_3 = 0$, and we'll choose $k_1 = 1$. Then the solution vector is

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t}$$

Because we only find one Eigenvector for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -1$, we have to use $\overrightarrow{k_1} = (1,0,0)$ to find a second solution.

$$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_1}$$

$$\begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$3p_2 - p_3 = 1$$

$$2p_3 = 0$$

we get $p_3 = 0$ and $p_2 = 1/3$, and we'll choose $p_1 = 1$. Then the second solution will be

$$\overrightarrow{x_2} = \overrightarrow{k_1} te^{\lambda_2 t} + \overrightarrow{p_1} e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix} e^{-t}$$

Now we'll use $\overrightarrow{p_1} = (1,1/3,0)$ to find a third solution.

$$(A - \lambda_3 I)\overrightarrow{q_1} = \overrightarrow{p_1}$$

$$\begin{bmatrix} 0 & 3 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

Rewriting this matrix as a system of equations,

$$3q_2 - q_3 = 1$$

$$2q_3 = \frac{1}{3}$$

we get $q_2 = 7/18$ and $q_3 = 1/6$, and we'll choose $q_1 = 1$. Then the second solution will be

$$\overrightarrow{x_3} = \overrightarrow{k_1} \frac{t^2}{2} e^{\lambda_3 t} + \overrightarrow{p_1} te^{\lambda_3 t} + \overrightarrow{q_1} e^{\lambda_3 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{-t} + \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ \frac{7}{18} \\ \frac{1}{6} \end{bmatrix} e^{-t}$$

So the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix} e^{-t} \right) + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{-t} + \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} 1 \\ \frac{7}{18} \\ \frac{1}{6} \end{bmatrix} e^{-t} \right)$$

SERIES SOLUTIONS OF HIGHER ORDER EQUATIONS

■ 1. Find a power series solution in x to the differential equation.

$$y''' - xy' = 0$$

Solution:

Substitute y' and y''' into the differential equation.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} - x\sum_{n=1}^{\infty} c_n n x^{n-1} = 0$$

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} - \sum_{n=1}^{\infty} c_n n x^{n} = 0$$

To put these series in phase, we'll pull the x^0 term out of the first series.

$$c_3 \cdot 3(3-1)(3-2)x^{3-3} + \sum_{n=4}^{\infty} c_n n(n-1)(n-2)x^{n-3} - \sum_{n=1}^{\infty} c_n n x^{n} = 0$$

$$6c_3 + \sum_{n=4}^{\infty} c_n n(n-1)(n-2)x^{n-3} - \sum_{n=1}^{\infty} c_n n x^{n} = 0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n - 3$ and $n = k + 3$ into the first series, and $k = n$ into the second series.

$$6c_3 + \sum_{k=1}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k - \sum_{k=1}^{\infty} c_k k x^k = 0$$

With the series in phase and matching indices, we can finally add them.

$$6c_3 + \sum_{k=1}^{\infty} [c_{k+3}(k+1)(k+2)(k+3) - c_k k]x^k = 0$$

We get

$$6c_3 = 0$$

$$c_3 = 0$$

and

$$c_{k+3}(k+1)(k+2)(k+3) - c_k k = 0 \qquad k = 1, 2, 3, \ldots$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+3}.

$$c_{k+3} = \frac{k}{(k+1)(k+2)(k+3)}c_k \qquad k = 1, 2, 3, \ldots$$

Now we'll start plugging in values $k = 1, 2, 3, \ldots$.

$$k = 1 \quad c_4 = \frac{c_1}{4!} \qquad k = 2 \quad c_5 = \frac{4c_2}{5!} \qquad k = 3 \quad c_6 = 0$$

$$k = 4 \quad c_7 = \frac{4c_1}{7!} \qquad k = 5 \quad c_8 = \frac{20c_2}{8!} \qquad k = 6 \quad c_9 = 0$$

$$\vdots \qquad \vdots \qquad \vdots$$

Then the general solution is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

$$y = c_0 + c_1x + c_2x^2 + \frac{1}{4!}c_1x^4 + \frac{4}{5!}c_2x^5 + \frac{4}{7!}c_1x^7 + \frac{20}{8!}c_2x^8 + \dots$$

$$y = c_0 + c_1\left(x + \frac{1}{4!}x^4 + \frac{4}{7!}x^7 + \dots\right) + c_2\left(x^2 + \frac{4}{5!}x^5 + \frac{20}{8!}x^8\right) + \dots$$

■ 2. Find a power series solution in x to the differential equation.

$$y''' - y'' = 0$$

Solution:

Substitute y'' and y''' into the differential equation.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} - \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} = 0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n - 3$ and $n = k + 3$ into the first series, and $k = n - 2$ and $n = k + 2$ into the second series.

$$\sum_{k=0}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k - \sum_{k=0}^{\infty} c_{k+2}(k+2)(k+1)x^k = 0$$

With the series in phase and matching indices, we can finally add them.

$$\sum_{k=0}^{\infty}[c_{k+3}(k+3)(k+2)(k+1)-c_{k+2}(k+2)(k+1)]x^k=0$$

We get

$$c_{k+3}(k+3)(k+2)(k+1)-c_{k+2}(k+2)(k+1)=0 \qquad k=0,\ 1,\ 2,\ \dots$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+3}.

$$c_{k+3}=\frac{c_{k+2}}{k+3} \qquad k=0,\ 1,\ 2,\ \dots$$

Now we'll start plugging in values $k=0,\ 1,\ 2,....$

$$k=0 \qquad c_3=\frac{c_2}{3}$$

$$k=1 \qquad c_4=\frac{c_2}{3\cdot 4}$$

$$k=2 \qquad c_5=\frac{c_2}{3\cdot 4\cdot 5}$$

Then the general solution is

$$y=c_0+c_1x+c_2x^2+c_3x^3+\dots$$

$$y=c_0+c_1x+c_2x^2+\frac{c_2}{3}x^3+\frac{c_2}{3\cdot 4}x^4+\frac{c_2}{3\cdot 4\cdot 5}x^5+\dots$$

$$y=c_0+c_1x+c_2x^2+\frac{2c_2}{3!}x^3+\frac{2c_2}{4!}x^4+\frac{2c_2}{5!}x^5+\dots$$

$$y=c_0+c_1x+c_2\left(x^2+\frac{2}{3!}x^3+\frac{2}{4!}x^4+\frac{2}{5!}x^5+\dots\right)$$

■ 3. Find a power series solution in x to the differential equation.

$$xy''' - y' = 0$$

Solution:

Substitute y' and y''' into the differential equation.

$$x\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} - \sum_{n=1}^{\infty} c_n n x^{n-1} = 0$$

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^{n-1} = 0$$

To put these series in phase, we'll pull the x^0 term out of the second series.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-2} - c_1 x^{1-1} - \sum_{n=2}^{\infty} c_n n x^{n-1} = 0$$

$$-c_1 + \sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-2} - \sum_{n=2}^{\infty} c_n n x^{n-1} = 0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n - 2$ and $n = k + 2$ into the first series, and $k = n - 1$ and $n = k + 1$ into the second series.

$$-c_1 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)k \cdot x^k - \sum_{k=1}^{\infty} c_{k+1}(k+1)x^k = 0$$

With the series in phase and matching indices, we can finally add them.

$$-c_1 + \sum_{k=1}^{\infty} [c_{k+2}(k+2)(k+1)k - c_{k+1}(k+1)]x^k = 0$$

We get

$$c_1 = 0$$

and

$$c_{k+2}(k+2)(k+1)k - c_{k+1}(k+1) = 0 \qquad k = 1, 2, 3, ...$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+2}.

$$c_{k+2} = \frac{c_{k+1}}{k(k+2)} \qquad k = 1, 2, 3, ...$$

Now we'll start plugging in values $k = 1, 2, 3, ...$

$$k = 1 \qquad c_3 = \frac{2c_2}{3!}$$

$$k = 2 \qquad c_4 = \frac{c_2}{4!}$$

$$k = 3 \qquad c_5 = \frac{2c_2}{3!5!}$$

$$k = 4 \qquad c_6 = \frac{2c_2}{4!6!}$$

$$k = 5 \qquad c_7 = \frac{2c_2}{5!7!}$$

$$k = 6 \qquad c_8 = \frac{2c_2}{6!8!}$$

$$\vdots$$

$$c_m = \frac{2c_2}{(m-2)!m!} \qquad m \geq 3$$

Then the general solution is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + \dots$$

$$y = c_0 + c_1x + c_2x^2 + \frac{2c_2}{1!3!}x^3 + \frac{2c_2}{2!4!}x^4 + \frac{2c_2}{3!5!}x^5 + \dots$$

$$y = c_0 + c_1x + \sum_{m=2}^{\infty} \frac{2c_2}{(m-2)!m!}x^m$$

■ 4. Find a power series solution in x to the differential equation.

$$x^2y''' + y'' - xy = 0$$

Solution:

Substitute y, y'', and y''' into the differential equation.

$$x^2\sum_{n=3}^{\infty} c_nn(n-1)(n-2)x^{n-3} + \sum_{n=2}^{\infty} c_nn(n-1)x^{n-2} - x\sum_{n=0}^{\infty} c_nx^n = 0$$

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-1} + \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

To put these series in phase, we'll pull the x^0 and x^1 terms out of the second series and pull the x^1 term out of the third series.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-1} + c_2 \cdot 2(2-1)x^{2-2} + c_3 \cdot 3(3-1)x^{3-2}$$

$$+\sum_{n=4}^{\infty} c_n n(n-1)x^{n-2} - c_0 x - \sum_{n=1}^{\infty} c_n x^{x+1} = 0$$

$$2c_2 + (6c_3 - c_0)x + \sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-1} + \sum_{n=4}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n - 3$ and $n = k + 3$ into the first series, $k = n - 4$ and $n = k + 4$ into the second series, and $k = n - 1$ and $n = k + 1$ into the third series.

$$2c_2 + (6c_3 - c_0)x + \sum_{k=0}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^{k+2}$$

$$+\sum_{k=0}^{\infty} c_{k+4}(k+4)(k+3)x^{k+2} - \sum_{k=0}^{\infty} c_{k+1}x^{k+2} = 0$$

With the series in phase and matching indices, we can finally add them.

$$2c_2 + (6c_3 - c_0)x$$

$$+\sum_{k=0}^{\infty} [c_{k+3}(k+1)(k+2)(k+3) + c_{k+4}(k+3)(k+4) - c_{k+1}]x^{k+2} = 0$$

We get

$$2c_2 = 0$$

$$c_2 = 0$$

and

$$6c_3 - c_0 = 0$$

$$c_3 = \frac{c_0}{6}$$

and

$$c_{k+3}(k+1)(k+2)(k+3) + c_{k+4}(k+3)(k+4) - c_{k+1} = 0 \qquad k = 0, 1, 2, ...$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+4}.

$$c_{k+4} = \frac{c_{k+1} - c_{k+3}(k+1)(k+2)(k+3)}{(k+3)(k+4)} \qquad k = 0, 1, 2, ...$$

Now we'll start plugging in values $k = 0, 1, 2, \dots$.

$$k = 0 \qquad c_4 = \frac{c_1 - c_0}{12}$$

$$k = 1 \qquad c_5 = \frac{-c_1 + c_0}{10}$$

$$k = 2 \qquad c_6 = \frac{c_1}{5} - \frac{7}{36}c_0$$

$$k = 3 \qquad c_7 = \frac{31}{56}c_0 - \frac{41}{72}c_1$$

$$\vdots$$

Then the general solution is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$$

$$y = c_0 + c_1 x + \frac{c_0}{6}x^3 + \frac{c_1 - c_0}{12}x^4 + \frac{-c_1 + c_0}{10}x^5$$

$$+\left(\frac{c_1}{5} - \frac{7}{36}c_0\right)x^6 + \left(\frac{31}{56}c_0 - \frac{41}{72}c_1\right)x^7 + \dots$$

$$y = c_0\left(1 + \frac{1}{6}x^3 - \frac{1}{12}x^4 + \frac{1}{10}x^5 - \frac{7}{36}x^6 + \frac{31}{56}x^7 - \dots\right)$$

$$+c_1\left(x + \frac{1}{12}x^4 - \frac{1}{10}x^5 + \frac{1}{5}x^6 - \frac{41}{72}x^7 + \dots\right)$$

■ 5. Find a power series solution in x to the differential equation.

$$y''' + xy' + y = 0$$

Solution:

Substitute y, y', and y''' into the differential equation.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^n = 0$$

To put these series in phase, we'll pull the x^0 term out of the first and third series.

$$c_3 \cdot 3(3-1)(3-2)x^{3-3} + \sum_{n=4}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^n$$

$$+c_0 x^0 + \sum_{n=1}^{\infty} c_n x^n = 0$$

$$6c_3 + c_0 + \sum_{n=4}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=1}^{\infty} c_n x^n = 0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n-3$ and $n = k+3$ into the first series, and $k = n$ into the second and third series.

$$6c_3 + c_0 + \sum_{k=1}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_k k x^k + \sum_{k=1}^{\infty} c_k x^k = 0$$

With the series in phase and matching indices, we can finally add them.

$$6c_3 + c_0 + \sum_{k=1}^{\infty} [c_{k+3}(k+3)(k+2)(k+1) + c_k(k+1)]x^k = 0$$

We get

$$6c_3 + c_0 = 0$$

$$c_3 = -\frac{c_0}{6}$$

and

$$c_{k+3}(k+3)(k+2)(k+1) + c_k(k+1) = 0 \qquad k = 1, 2, 3...$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+3}.

$$c_{k+3} = -\frac{c_k}{(k+3)(k+2)} \qquad k = 1, 2, 3...$$

Now we'll start plugging in values $k = 1, 2, 3....$

$$k = 0 \quad c_3 = -\frac{c_0}{6} \qquad k = 1 \quad c_4 = -\frac{c_1}{3 \cdot 4} \qquad k = 2 \quad c_5 = -\frac{c_2}{4 \cdot 5}$$

$$k = 3 \quad c_6 = \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6} \qquad k = 4 \quad c_7 = \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7} \qquad k = 5 \quad c_8 = \frac{c_2}{4 \cdot 5 \cdot 7 \cdot 8}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

Then the general solution is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \dots$$

$$y = c_0 + c_1x + c_2x^2 - \frac{c_0}{6}x^3 - \frac{c_1}{3 \cdot 4}x^4 - \frac{c_2}{4 \cdot 5}x^5 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6}x^6 + \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7}x^7 + \dots$$

$$y = c_0\left(1 - \frac{1}{6}x^3 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6}x^6 - \dots\right) + c_1\left(x - \frac{c_1}{3 \cdot 4}x^4 + \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7}x^7 - \dots\right)$$

$$+c_2\left(x^2-\frac{c_2}{4\cdot 5}x^5+\dots\right)$$

■ 6. Find a power series solution in x to the differential equation.

$$y'''+xy''-y=0$$

Solution:

Substitute y, y'', and y''' into the differential equation.

$$\sum_{n=3}^{\infty}c_nn(n-1)(n-2)x^{n-3}+x\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}-\sum_{n=0}^{\infty}c_nx^n=0$$

$$\sum_{n=3}^{\infty}c_nn(n-1)(n-2)x^{n-3}+\sum_{n=2}^{\infty}c_nn(n-1)x^{n-1}-\sum_{n=0}^{\infty}c_nx^n=0$$

To put these series in phase, we'll pull the x^0 terms out of the first and third series.

$$c_3\cdot 3(3-1)(3-2)x^{3-3}+\sum_{n=4}^{\infty}c_nn(n-1)(n-2)x^{n-3}+\sum_{n=2}^{\infty}c_nn(n-1)x^{n-1}$$

$$-c_0x^0-\sum_{n=1}^{\infty}c_nx^n=0$$

$$6c_3-c_0+\sum_{n=4}^{\infty}c_nn(n-1)(n-2)x^{n-3}+\sum_{n=2}^{\infty}c_nn(n-1)x^{n-1}-\sum_{n=1}^{\infty}c_nx^n=0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n - 3$ and $n = k + 3$ into the first series, $k = n - 1$ and $n = k + 1$ into the second series, and $k = n$ into the third series.

$$6c_3 - c_0 + \sum_{k=1}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_{k+1}(k+1)kx^k - \sum_{k=1}^{\infty} c_k x^k = 0$$

With the series in phase and matching indices, we can finally add them.

$$6c_3 - c_0 + \sum_{k=1}^{\infty} [c_{k+3}(k+3)(k+2)(k+1) + c_{k+1}(k+1)k - c_k]x^k = 0$$

We get

$$6c_3 - c_0 = 0$$

$$c_3 = \frac{c_0}{6}$$

and

$$c_{k+3}(k+3)(k+2)(k+1) + c_{k+1}(k+1)k - c_k = 0 \qquad k = 1, 2, 3...$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+3}.

$$c_{k+3} = \frac{c_k - c_{k+1}(k+1)k}{(k+3)(k+2)(k+1)} \qquad k = 1, 2, 3...$$

Now we'll start plugging in values $k = 1, 2, 3....$

$$k = 1 \qquad c_4 = \frac{c_1 - 2c_2}{24}$$

$$k = 2 \qquad c_5 = \frac{c_2 - c_0}{60}$$

$$k = 3 \qquad c_6 = \frac{c_0 - 3c_1 + 6c_2}{720}$$

$$k = 4 \qquad c_7 = \frac{8c_0 + c_1 - 10c_2}{5{,}040}$$

$$\vdots$$

Then the general solution is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots$$

$$y = c_0 + c_1x + c_2x^2 + \frac{c_0}{6}x^3 + \frac{c_1 - 2c_2}{24}x^4 + \frac{c_2 - c_0}{60}x^5$$

$$+\frac{c_0 - 3c_1 + 6c_2}{720}x^6 + \frac{8c_0 + c_1 - 10c_2}{5{,}040}x^7 + \ldots$$

$$y = c_0\left(1 + \frac{1}{6}x^3 - \frac{1}{60}x^5 + \frac{1}{720}x^6 + \frac{1}{630}x^7 + \ldots\right)$$

$$+c_1\left(x + \frac{1}{24}x^4 - \frac{1}{240}x^6 + \frac{1}{5{,}040}x^7 + \ldots\right)$$

$$+c_2\left(x^2 - \frac{1}{12}x^4 + \frac{1}{60}x^5 + \frac{1}{120}x^6 - \frac{1}{504}x^7 + \ldots\right)$$

FOURIER SERIES REPRESENTATIONS

■ 1. Find the Fourier series representation of $f(x) = (2x+1)^2$ on $-L \le x \le L$.

Solution:

For A_0, we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} (2x+1)^2\ dx$$

$$A_0 = \frac{1}{6L}(2x+1)^3 \cdot \frac{1}{2}\Big|_{-L}^{L}$$

$$A_0 = \frac{1}{12L}((1+2L)^3 - (1-2L)^3)$$

$$A_0 = \frac{1}{12L}(12L + 16L^3)$$

$$A_0 = 1 + \frac{4}{3}L^2$$

For A_n, we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} (2x+1)^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = (2x+1)^2$, $du = 4(2x+1)\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{1}{L}\left[(2x+1)^2\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} - \frac{4L}{n\pi}\int_{-L}^{L}(2x+1)\sin\left(\frac{n\pi x}{L}\right)\ dx\right]$$

$$A_n = -\frac{4}{n\pi}\int_{-L}^{L}(2x+1)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 2x+1$, $du = 2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = -\frac{4}{n\pi}\left[-(2x+1)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{2L}{n\pi}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\ dx\right]$$

$$A_n = -\frac{4}{n\pi}\left[-(2L+1)\frac{L}{n\pi}(-1)^n + (1-2L)\frac{L}{n\pi}(-1)^n + \frac{2L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}\right]$$

$$A_n = \frac{16L^2}{n^2\pi^2}(-1)^n$$

For B_n, we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} (2x+1)^2\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=(2x+1)^2$, $du=4(2x+1)\ dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$.

$$B_n=\frac{1}{L}\left[-(2x+1)^2\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}+\frac{4L}{n\pi}\int_{-L}^{L}(2x+1)\cos\left(\frac{n\pi x}{L}\right)\ dx\right]$$

Use integration by parts with $u=2x+1$, $du=2\ dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$.

$$\int_{-L}^{L}(2x+1)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$=(2x+1)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}-\frac{2L}{n\pi}\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$\int_{-L}^{L}(2x+1)\cos\left(\frac{n\pi x}{L}\right)\ dx=\frac{2L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}=0$$

Therefore,

$$B_n=\frac{1}{L}\left[-(2x+1)^2\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}+0\right]$$

$$B_n=\frac{1}{L}\left(-(2L+1)^2\frac{L}{n\pi}(-1)^n+(1-2L)^2\frac{L}{n\pi}(-1)^n\right)$$

$$B_n=\frac{(-1)^n}{n\pi}(-8L)$$

$$B_n=-\frac{8L(-1)^n}{n\pi}$$

Then the Fourier series representation of $f(x)=(2x+1)^2$ on $-L \le x \le L$ is

$$f(x)=1+\frac{4}{3}L^2+\sum_{n=1}^{\infty}\frac{16L^2}{n^2\pi^2}(-1)^n\cos\left(\frac{n\pi x}{L}\right)-\sum_{n=1}^{\infty}\frac{8L}{n\pi}(-1)^n\sin\left(\frac{n\pi x}{L}\right)$$

■ 2. Find the Fourier series representation of $f(x)=e^x$ on $-\pi \le x \le \pi$.

Solution:

For A_0, we get

$$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^x\ dx$$

$$A_0=\frac{1}{2\pi}e^x\Big|_{-\pi}^{\pi}$$

$$A_0=\frac{1}{2\pi}(e^{\pi}-e^{-\pi})$$

For A_n, we get

$$A_n=\frac{1}{\pi}\int_{-\pi}^{\pi}e^x\cos(nx)\ dx$$

Use integration by parts with $u=e^x$, $du=e^x\ dx$, $dv=\cos(nx)\ dx$, and $v=(1/n)\sin(nx)$.

$$A_n = \frac{1}{\pi}\left[e^x \frac{1}{n}\sin(nx)\Big|_{-\pi}^{\pi} - \frac{1}{n}\int_{-\pi}^{\pi} e^x \sin(nx)\ dx\right]$$

$$A_n = -\frac{1}{n\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\ dx$$

Use integration by parts with $u = e^x$, $du = e^x\ dx$, $dv = \sin(nx)\ dx$, and $v = -(1/n)\cos(nx)$.

$$A_n = -\frac{1}{n\pi}\left[-\frac{1}{n}e^x \cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} e^x \cos(nx)\ dx\right]$$

$$A_n = -\frac{1}{n\pi}\left(-\frac{1}{n}(e^{\pi} - e^{-\pi})(-1)^n + \frac{\pi}{n}A_n\right)$$

$$A_n\left(1 + \frac{1}{n^2}\right) = \frac{(e^{\pi} - e^{-\pi})(-1)^n}{n^2\pi}$$

$$A_n = \frac{(e^{\pi} - e^{-\pi})(-1)^n}{\pi(n^2 + 1)}$$

For B_n, we get

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^x \sin(nx)\ dx$$

Use integration by parts with $u = e^x$, $du = e^x\ dx$, $dv = \sin(nx)\ dx$, and $v = -(1/n)\cos(nx)$.

$$B_n = \frac{1}{\pi}\left[-\frac{1}{n}e^x \cos(nx)\Big|_{-\pi}^{\pi} + \frac{1}{n}\int_{-\pi}^{\pi} e^x \cos(nx)\ dx\right]$$

$$B_n = \frac{1}{\pi}\left(-\frac{1}{n}(-1)^n(e^{\pi} - e^{-\pi}) + \frac{\pi}{n}A_n\right)$$

$$B_n = -\frac{n}{\pi(n^2+1)}(e^{\pi} - e^{-\pi})(-1)^n$$

Then the Fourier series representation of $f(x) = e^x$ on $-\pi \le x \le \pi$ is

$$f(x) = \frac{1}{2\pi}(e^{\pi} - e^{-\pi}) + \sum_{n=1}^{\infty} \frac{(e^{\pi} - e^{-\pi})(-1)^n}{\pi(n^2+1)}\cos(nx) + \sum_{n=1}^{\infty} -\frac{n(-1)^n(e^{\pi} - e^{-\pi})}{\pi(n^2+1)}\sin(nx)$$

■ 3. Find the Fourier series representation of the function on $-\pi \le x \le \pi$.

$$f(x) = \sin\left(\frac{x}{2}\right)$$

Solution:

Since $f(x)$ is odd, we know $A_0 = A_n = 0$. For B_n, we get

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right)\sin\left(\frac{n\pi x}{\pi}\right)\ dx$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \sin\left(\frac{x}{2}\right)\sin(nx)\ dx$$

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \frac{1}{2}\cos\left(\left(n - \frac{1}{2}\right)x\right) - \cos\left(\left(n + \frac{1}{2}\right)x\right)\ dx$$

$$B_n = \frac{1}{2\pi} \left(\frac{1}{n - \frac{1}{2}} \sin \left(n - \frac{1}{2} \right) x - \frac{1}{n + \frac{1}{2}} \sin \left(n + \frac{1}{2} \right) x \right) \Bigg|_{-\pi}^{\pi}$$

$$B_n = \frac{1}{2\pi} \left(\frac{1}{n - \frac{1}{2}} \sin \left(n - \frac{1}{2} \right) \pi - \frac{1}{n + \frac{1}{2}} \sin \left(n + \frac{1}{2} \right) \pi \right.$$

$$\left. + \frac{1}{n - \frac{1}{2}} \sin \left(n - \frac{1}{2} \right) \pi - \frac{1}{n + \frac{1}{2}} \sin \left(n + \frac{1}{2} \right) \pi \right)$$

$$B_n = \frac{1}{2\pi} \left(\frac{2}{n - \frac{1}{2}} \sin \left(n\pi - \frac{\pi}{2} \right) - \frac{2}{n + \frac{1}{2}} \sin \left(n\pi + \frac{\pi}{2} \right) \right)$$

$$B_n = \frac{1}{2\pi} \left(\frac{2}{n - \frac{1}{2}} \left(\sin(n\pi) \cos \left(\frac{\pi}{2} \right) - \cos(n\pi) \sin \left(\frac{\pi}{2} \right) \right) \right.$$

$$\left. - \frac{2}{n + \frac{1}{2}} \left(\sin(n\pi) \cos \left(\frac{\pi}{2} \right) + \cos(n\pi) \sin \left(\frac{\pi}{2} \right) \right) \right)$$

$$B_n = \frac{1}{2\pi} \left(\frac{2}{n - \frac{1}{2}} (0 - \cos(n\pi)) - \frac{2}{n + \frac{1}{2}} (0 + \cos(n\pi)) \right)$$

$$B_n = \frac{1}{2\pi} \left(-\frac{4}{2n - 1} \cos(n\pi) - \frac{4}{2n + 1} \cos(n\pi) \right)$$

$$B_n = -\frac{8n(-1)^n}{\pi(4n^2 - 1)}$$

Then the Fourier series representation of $f(x)$ on $-\pi \leq x \leq \pi$ is

$$f(x) = \sum_{n=1}^{\infty} -\frac{8n(-1)^n}{\pi(4n^2-1)} \sin(nx)$$

■ 4. Find the Fourier series representation of $f(x) = x \sin x$ on $-\pi \leq x \leq \pi$.

Solution:

Since $f(x)$ is even, we know $B_n = 0$. For A_0, we get

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x \sin x \ dx$$

$$A_0 = \frac{1}{\pi} \int_{0}^{\pi} x \sin x \ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin x \ dx$, and $v = -\cos x$.

$$A_0 = \frac{1}{\pi} \left(-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \ dx \right)$$

$$A_0 = \frac{1}{\pi} \left(-x \cos x \Big|_0^{\pi} + \int_0^{\pi} \cos x \ dx \right)$$

$$A_0 = \frac{1}{\pi} \left(\pi + \sin x \Big|_0^{\pi} \right) = 1$$

For A_n, we get

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\ dx$$

$$A_n = \frac{2}{\pi}\int_{0}^{\pi} f(x)\ \cos(nx)\ dx$$

$$A_n = \frac{2}{\pi}\int_{0}^{\pi} x\sin x\cos(nx)\ dx$$

$$A_n = \frac{2}{\pi}\int_{0}^{\pi} \frac{x}{2}(\sin(1+n)x + \sin(1-n)x)\ dx$$

$$A_n = \frac{1}{\pi}\int_{0}^{\pi} x(\sin(1+n)x + \sin(1-n)x)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(1+n)x + \sin(1-n)x\ dx$, and

$$v = -\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x \text{ with } n \neq 1$$

Then we get

$$A_n = \frac{1}{\pi}\left(x\left(-\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right)\Bigg|_0^{\pi}\right.$$

$$\left. + \int_0^{\pi}\frac{1}{n+1}\cos(n+1)x\ dx - \int_0^{\pi}\frac{1}{n-1}\cos(n-1)x\ dx\right)$$

$$A_n = \frac{1}{\pi}\left(\pi\left(-\frac{1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n+1}\right)\right.$$

$$\left. + \frac{1}{(n+1)^2}\sin(n+1)x\Big|_0^\pi - \frac{1}{(n-1)^2}\sin(n-1)x\Big|_0^\pi\right)$$

$$A_n = \frac{2}{n^2-1}(-1)^{n+1} \text{ with } n \neq 1$$

Because the value we found for A_n wasn't valid for $n = 1$, we'll also find A_1.

$$A_1 = \frac{2}{\pi}\int_0^\pi x \sin x \cos x \ dx$$

$$A_1 = \frac{1}{\pi}\int_0^\pi x \sin(2x) \ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(2x) \ dx$, and $v = -(1/2)\cos(2x)$.

$$A_1 = \frac{1}{\pi}\left(-\frac{1}{2}x\cos(2x)\Big|_0^\pi + \frac{1}{2}\int_0^\pi \cos(2x) \ dx\right)$$

$$A_1 = \frac{1}{\pi}\left(-\frac{\pi}{2} + \frac{1}{4}\sin(2x)\Big|_0^\pi\right)$$

$$A_1 = -\frac{1}{2}$$

Then the Fourier series representation of $f(x) = x\sin x$ on $-\pi \leq x \leq \pi$ is

$$f(x) = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2 - 1}\cos(nx)$$

■ 5. Find the Fourier series representation of $f(x) = x^2 + \sin x$ on $-L \le x \le L$.

Solution:

The function x^2 is even, so $B_n = 0$ in its Fourier representation. For A_0, we get

$$A_0 = \frac{1}{L}\int_0^L x^2\ dx$$

$$A_0 = \frac{1}{L} \cdot \frac{x^3}{3}\Big|_0^L = \frac{L^2}{3}$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{L}\left(x^2\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2L}{n\pi}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{2}{L}\left(0 - \frac{2L}{n\pi}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = -\frac{4}{n\pi}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = -\frac{4}{n\pi}\left(-x\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = -\frac{4}{n\pi}\left(-\frac{L^2}{n\pi}\cos(n\pi) + \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{4L^2}{n^2\pi^2}(-1)^n + \frac{4}{n\pi}\cdot\frac{L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$A_n = \frac{4L^2}{n^2\pi^2}(-1)^n$$

The function $\sin(x)$ is odd, so $A_0 = A_n = 0$ in its Fourier representation. For B_n, we get

$$B_n = \frac{2}{L}\int_0^L \sin x \sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L \left(\cos\left(\left(\frac{n\pi}{L} - 1\right)x\right) - \cos\left(\left(\frac{n\pi}{L} + 1\right)x\right)\right)\ dx$$

$$B_n = \frac{1}{L}\left(\frac{L}{n\pi - L}\sin\left(\frac{n\pi}{L} - 1\right)x - \frac{L}{n\pi + L}\sin\left(\frac{n\pi}{L} + 1\right)x\right)\Bigg|_0^L$$

$$B_n = \frac{1}{L}\Bigg(\frac{L}{n\pi - L}\left(\sin\left(\frac{n\pi x}{L}\right)\cos x - \cos\left(\frac{n\pi x}{L}\right)\sin x\right)$$

$$-\frac{L}{n\pi + L}\left(\sin\left(\frac{n\pi x}{L}\right)\cos x + \cos\left(\frac{n\pi x}{L}\right)\sin x\right)\Bigg)\Bigg|_0^L$$

$$B_n = \frac{1}{L}\left(\frac{L}{n\pi - L}(-\cos(n\pi)\sin L) - \frac{L}{n\pi + L}(\cos(n\pi)\sin L)\right)$$

$$B_n = \frac{-\sin L\cos(n\pi)}{n\pi - L} - \frac{\sin L\cos(n\pi)}{n\pi + L}$$

$$B_n = -\frac{2n\pi(-1)^n\sin L}{n^2\pi^2 - L^2}$$

Therefore, the Fourier series representation of $f(x) = x^2 + \sin x$ on $-L \le x \le L$ is

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty}\frac{4L^2}{n^2\pi^2}(-1)^n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} -\frac{2n\pi\sin L(-1)^n}{n^2\pi^2 - L^2}\sin\left(\frac{n\pi x}{L}\right)$$

■ 6. Find the Fourier series representation of $f(x) = \cosh x$ on $-\pi \le x \le \pi$.

Solution:

Since $f(x)$ is even, we know $B_n = 0$. For A_0, we get

$$A_0 = \frac{1}{\pi}\int_0^{\pi} \cosh x \ dx$$

$$A_0 = \frac{1}{\pi}\sinh x\Big|_0^{\pi} = \frac{\sinh \pi}{\pi}$$

For A_n, we get

$$A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx) \ dx$$

$$A_n = \frac{1}{\pi}\int_0^{\pi} (e^x + e^{-x})\cos(nx) \ dx$$

Use integration by parts with $u = e^x + e^{-x}$, $du = (e^x - e^{-x}) \ dx$, $dv = \cos(nx) \ dx$, and $v = (1/n)\sin(nx)$.

$$A_n = \frac{1}{\pi}\left((e^x + e^{-x})\frac{1}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi} (e^x - e^{-x})\sin(nx) \ dx \right)$$

$$A_n = \frac{1}{\pi}\left(0 - \frac{1}{n}\int_0^{\pi} (e^x - e^{-x})\sin(nx) \ dx \right)$$

$$A_n = -\frac{1}{n\pi}\int_0^{\pi} (e^x - e^{-x})\sin(nx) \ dx$$

Use integration by parts with $u = e^x - e^{-x}$, $du = (e^x + e^{-x}) \ dx$, $dv = \sin(nx) \ dx$, and $v = -(1/n)\cos(nx)$.

$$A_n = -\frac{1}{n\pi}\left(-\frac{1}{n}(e^x - e^{-x})\cos(nx)\ dx\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}(e^x + e^{-x})\cos(nx)\ dx\right)$$

$$A_n = -\frac{1}{n\pi}\left(-\frac{2}{n}(-1)^n\sinh\pi + \frac{\pi}{n}A_n\right)$$

$$A_n = \frac{2}{n^2\pi}(-1)^n\sinh\pi - \frac{1}{n^2}A_n$$

$$A_n\left(1 + \frac{1}{n^2}\right) = \frac{2(-1)^n\sinh\pi}{n^2\pi}$$

$$A_n = \frac{2(-1)^n\sinh\pi}{\pi(n^2+1)}$$

Then the Fourier series representation of $f(x) = \cosh x$ on $-\pi \le x \le \pi$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n\cos(nx)$$

$$f(x) = \frac{\sinh\pi}{\pi} + \sum_{n=1}^{\infty}\frac{2(-1)^n\sinh\pi}{\pi(n^2+1)}\cos(nx)$$

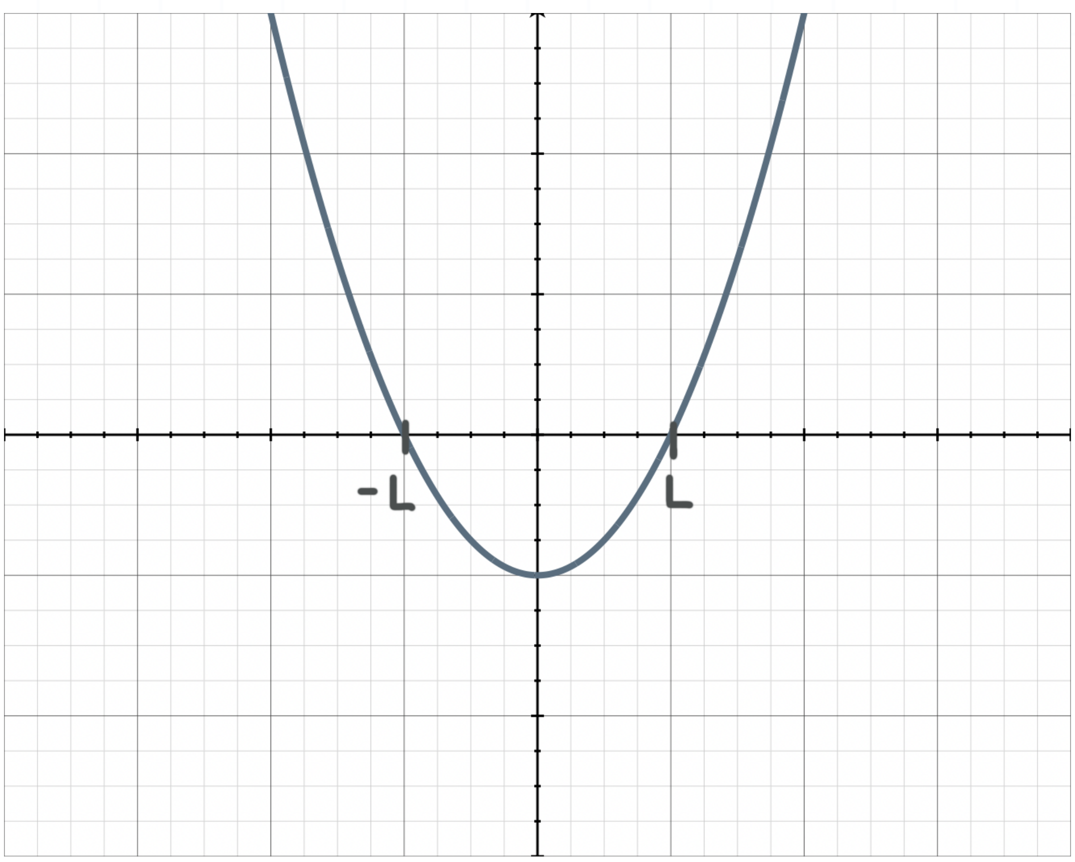
PERIODIC FUNCTIONS AND PERIODIC EXTENSIONS

■ 1. Sketch the function's periodic extension on the interval $-L \leq x \leq L$, given some positive value of L.

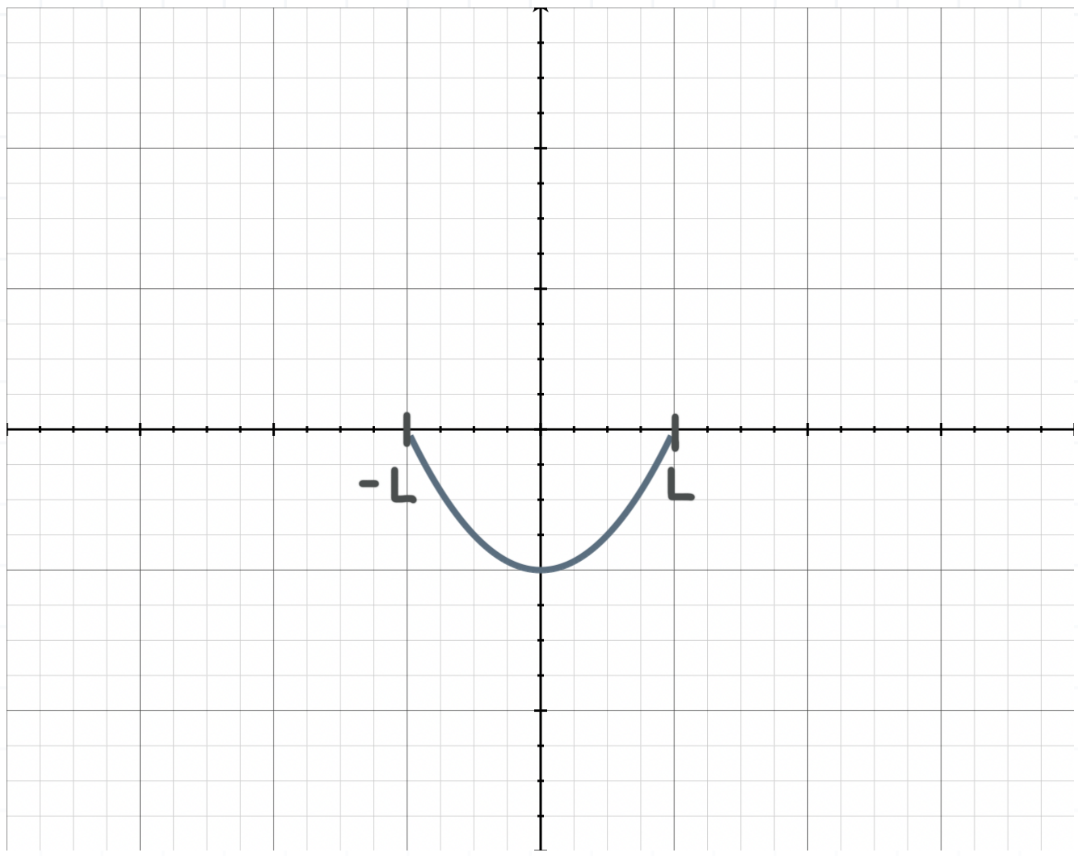
$$f(x) = \frac{1}{L}x^2 - L$$

Solution:

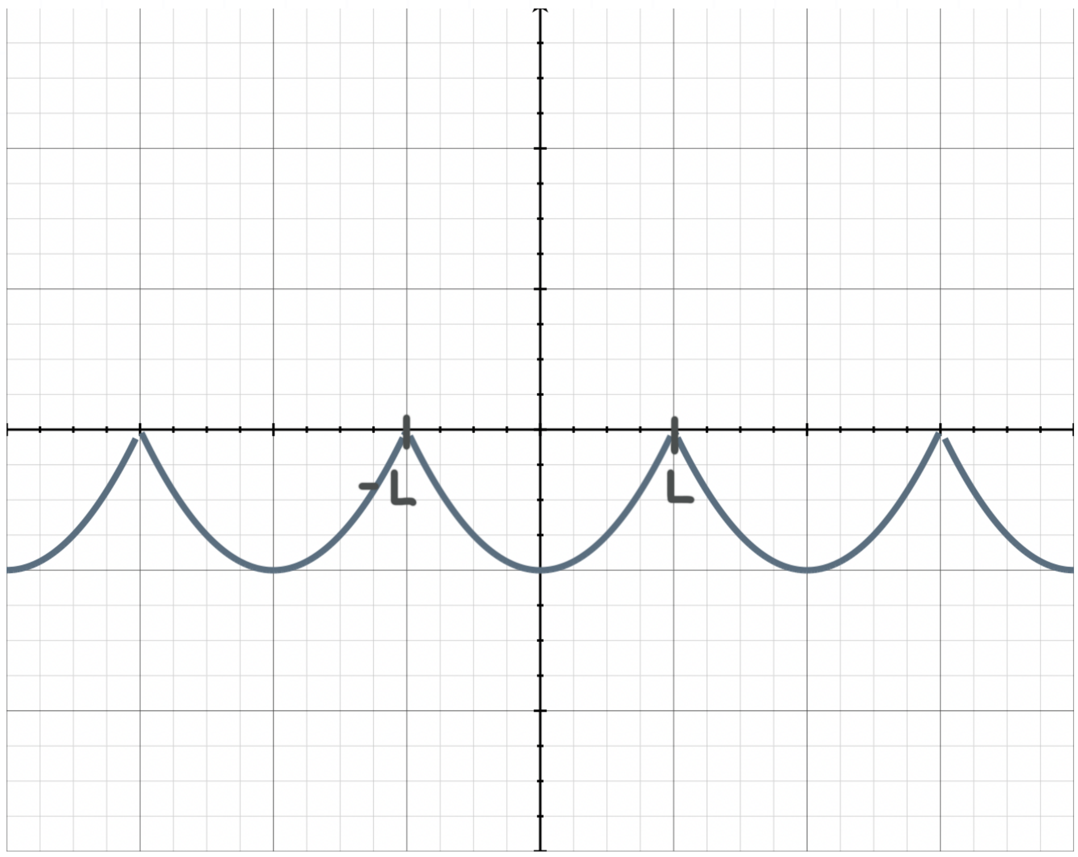
The sketch of $f(x)$ is



If we limit the graph to the interval $-L \leq x \leq L$, then the section of the graph on $-L \leq x \leq L$ is



If we repeat this section over and over again on both sides, we get a sketch of the periodic extension of $f(x)$.

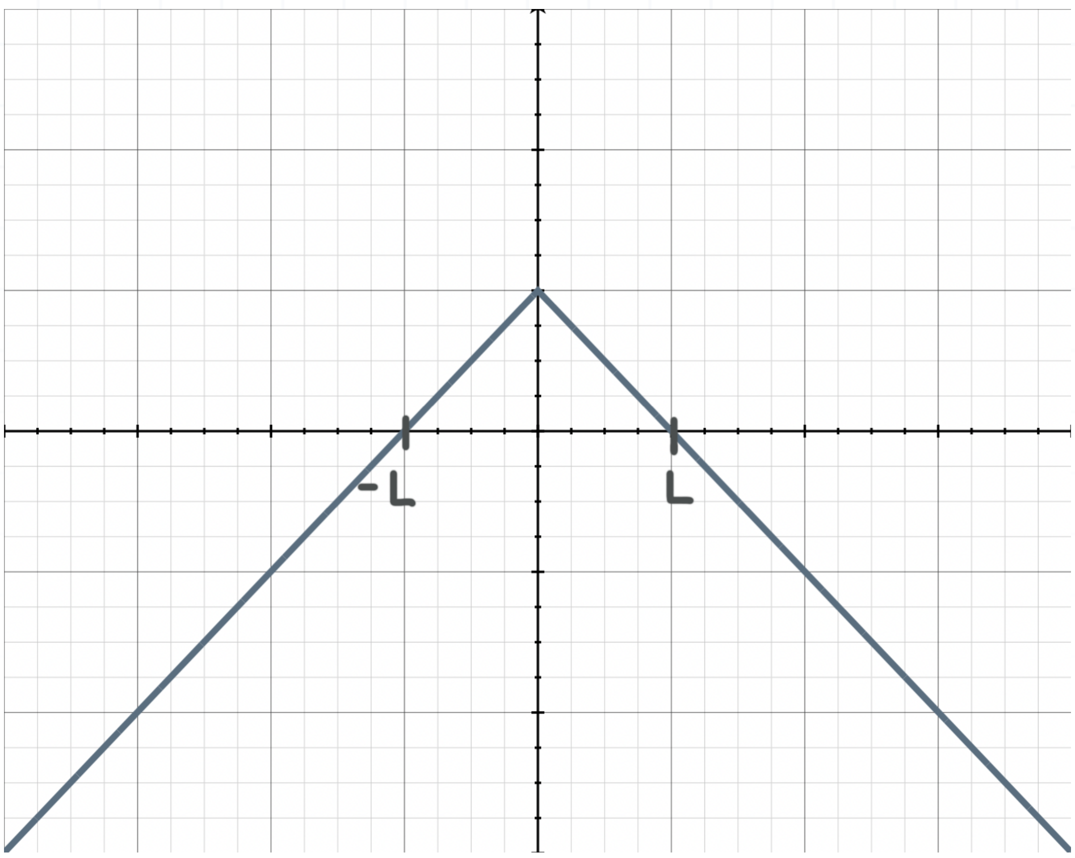


■ 2. Sketch the function's periodic extension on the interval $-L \leq x \leq L$.

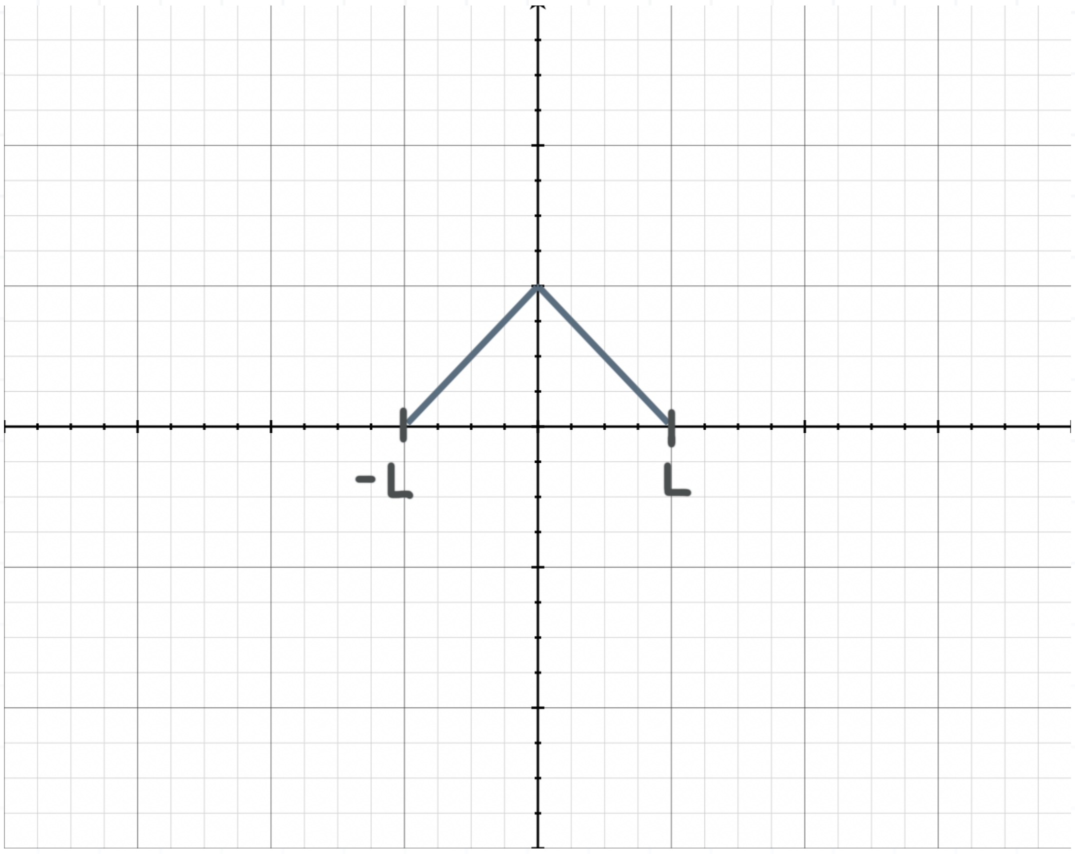
$$f(x) = \begin{cases} L + x & x < 0 \\ L - x & x \geq 0 \end{cases}$$

Solution:

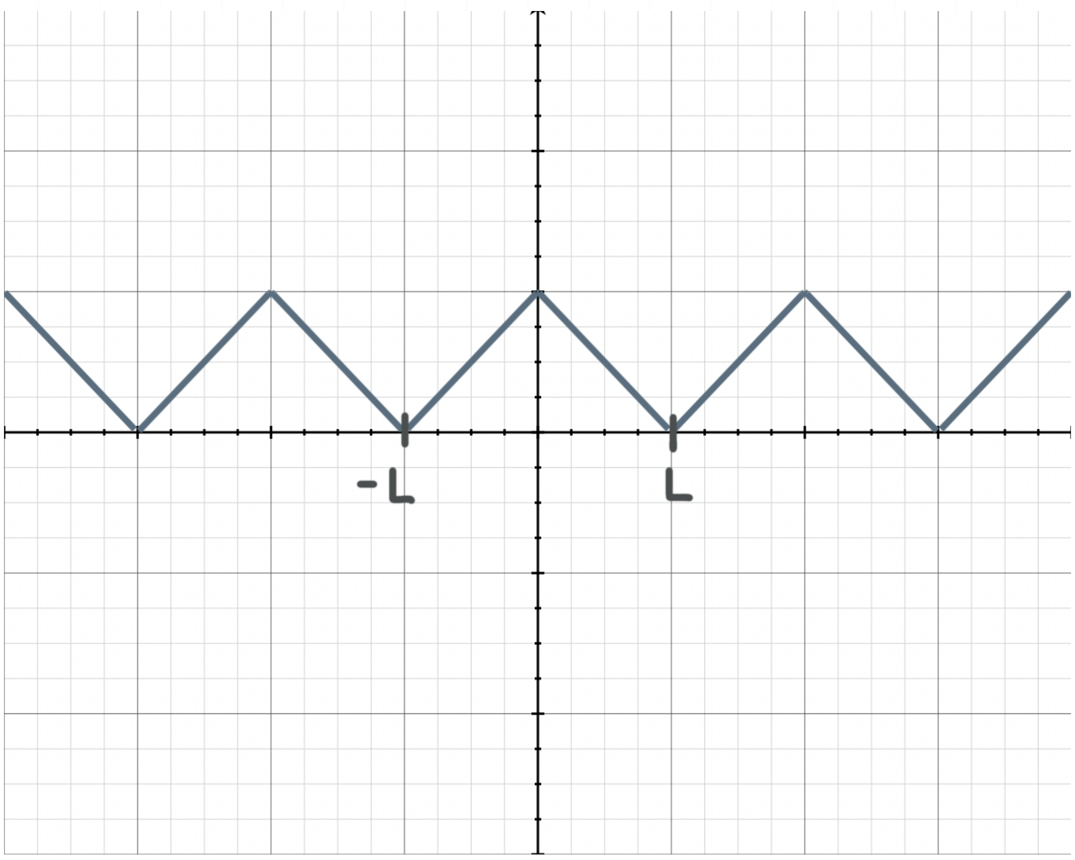
The sketch of $f(x)$ is



If we limit the graph to the interval $-L \leq x \leq L$, then the section of the graph on $-L \leq x \leq L$ is



If we repeat this section over and over again on both sides, we get a sketch of the periodic extension of $f(x)$.



3. Find the even and odd extensions of the function, given some positive value of L. Then sketch both extensions.

$$f(x) = \frac{1}{L^2}x^3 - \frac{1}{L}x^2$$

Solution:

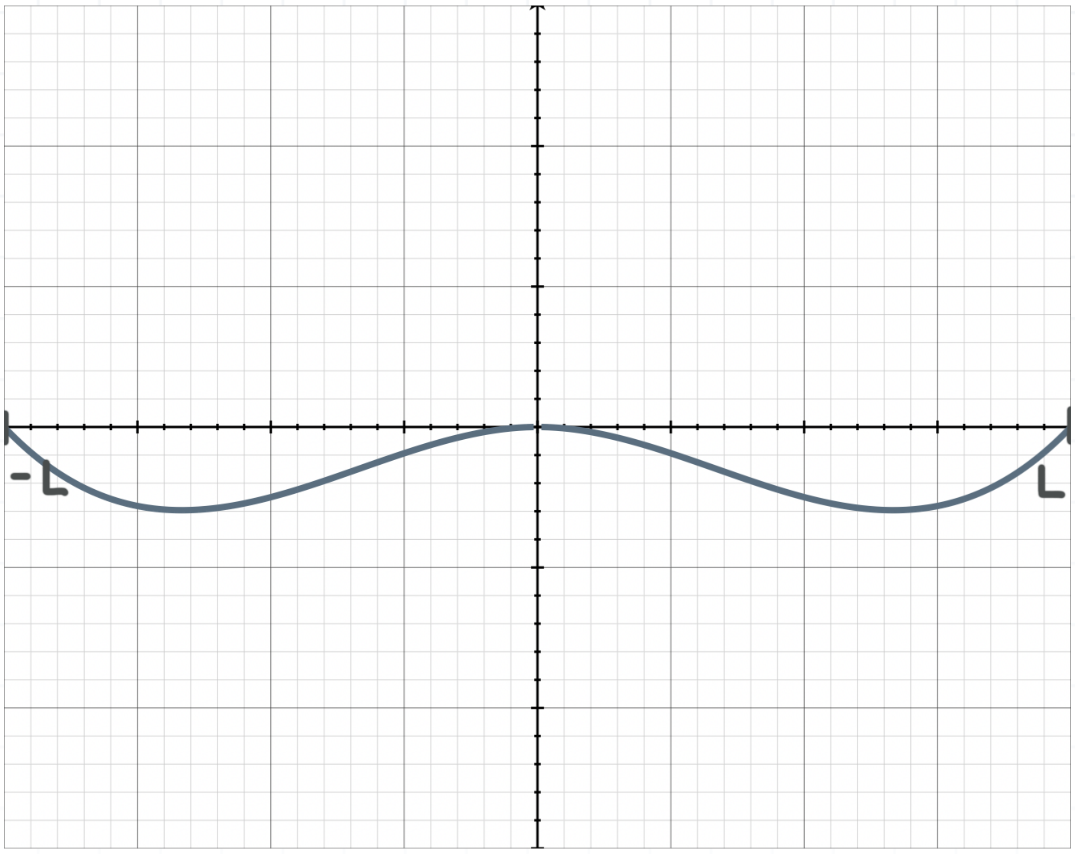
The even extension of $f(x)$ is

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{L^2}x^3 - \frac{1}{L}x^2 & 0 \le x \le L \\ \frac{1}{L^2}(-x)^3 - \frac{1}{L}(-x)^2 & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{L^2}x^3 - \frac{1}{L}x^2 & 0 \le x \le L \\ -\frac{1}{L^2}x^3 - \frac{1}{L}x^2 & -L \le x < 0 \end{cases}$$

The sketch of the even extension is



The odd extension of $f(x)$ is

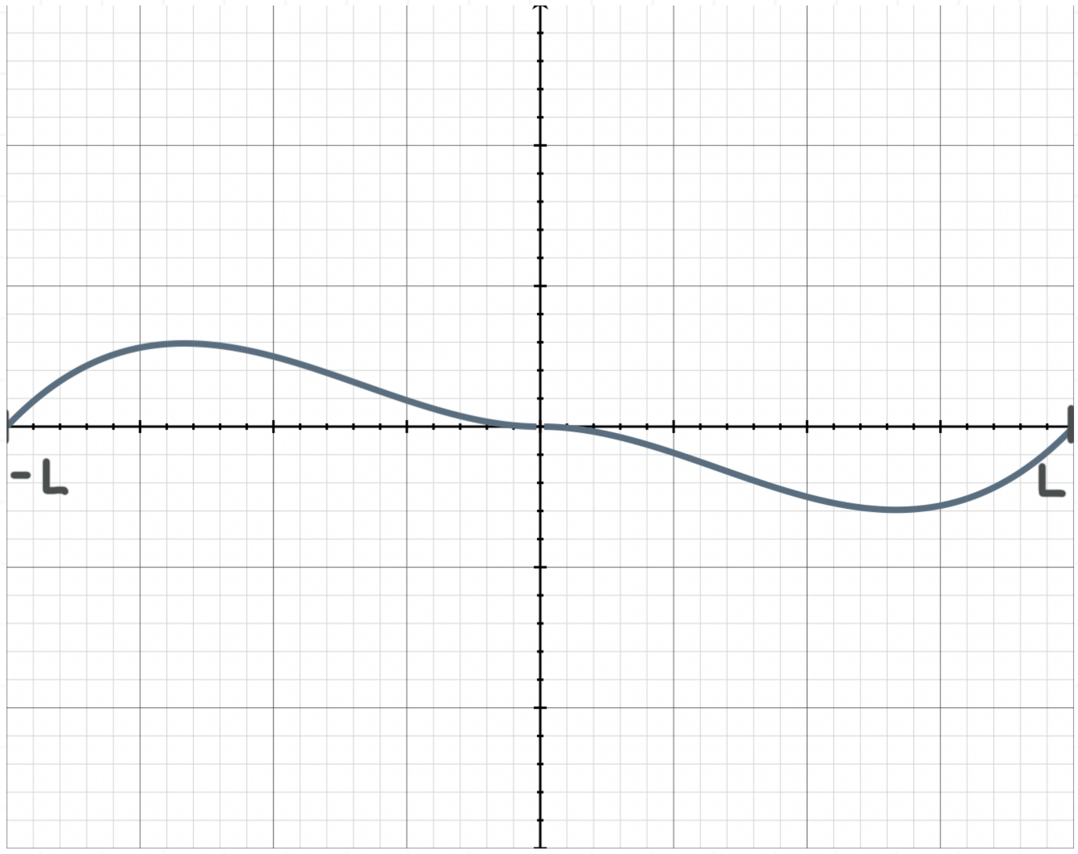
$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{L^2}x^3 - \frac{1}{L}x^2 & 0 \le x \le L \\ -\left(\frac{1}{L^2}(-x)^3 - \frac{1}{L}(-x)^2\right) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{L^2}x^3 - \frac{1}{L}x^2 & 0 \le x \le L \\ -\left(-\frac{1}{L^2}x^3 - \frac{1}{L}x^2\right) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{1}{L^2}x^3 - \frac{1}{L}x^2 & 0 \le x \le L \\ \frac{1}{L^2}x^3 + \frac{1}{L}x^2 & -L \le x < 0 \end{cases}$$

The sketch of the odd extension is

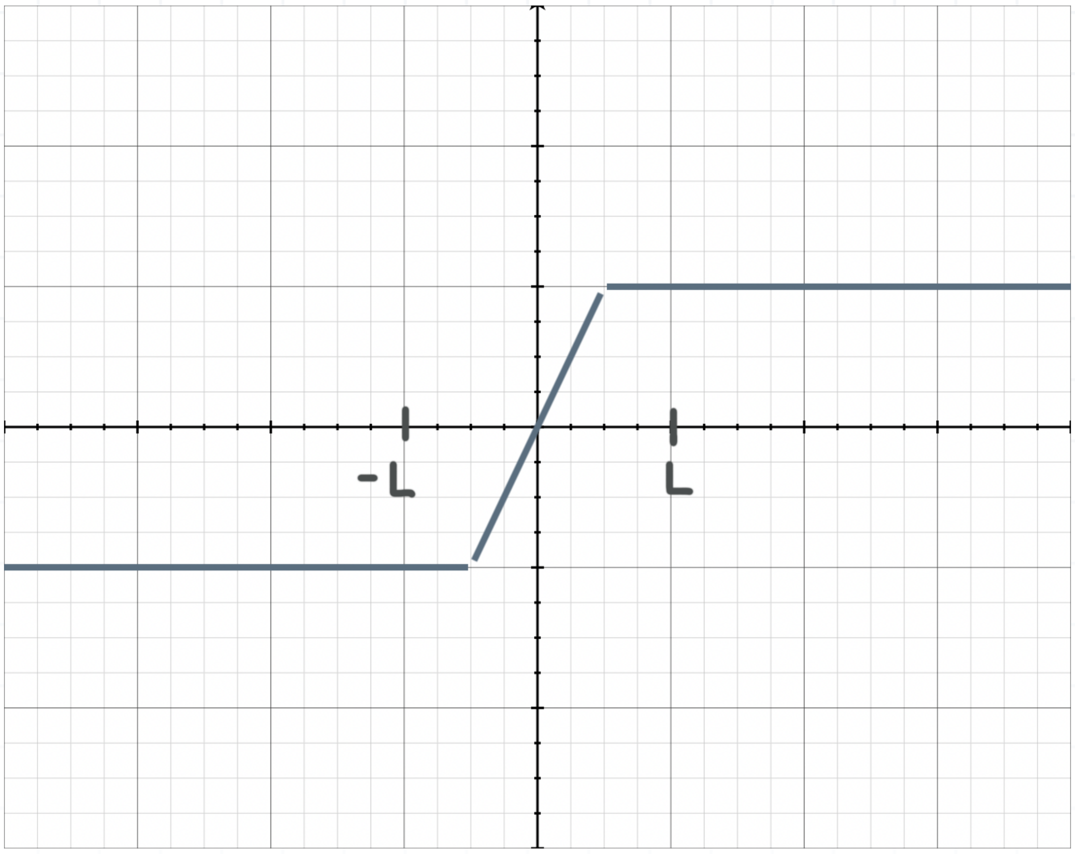


■ 4. Find the even and odd extensions of the function, given some positive value of L. Compare both sketches.

$$f(x) = \begin{cases} L^2 & \frac{L}{2} < x \\ 2Lx & -\frac{L}{2} \le x \le \frac{L}{2} \\ -L^2 & x < -\frac{L}{2} \end{cases}$$

Solution:

The sketch of $f(x)$ is

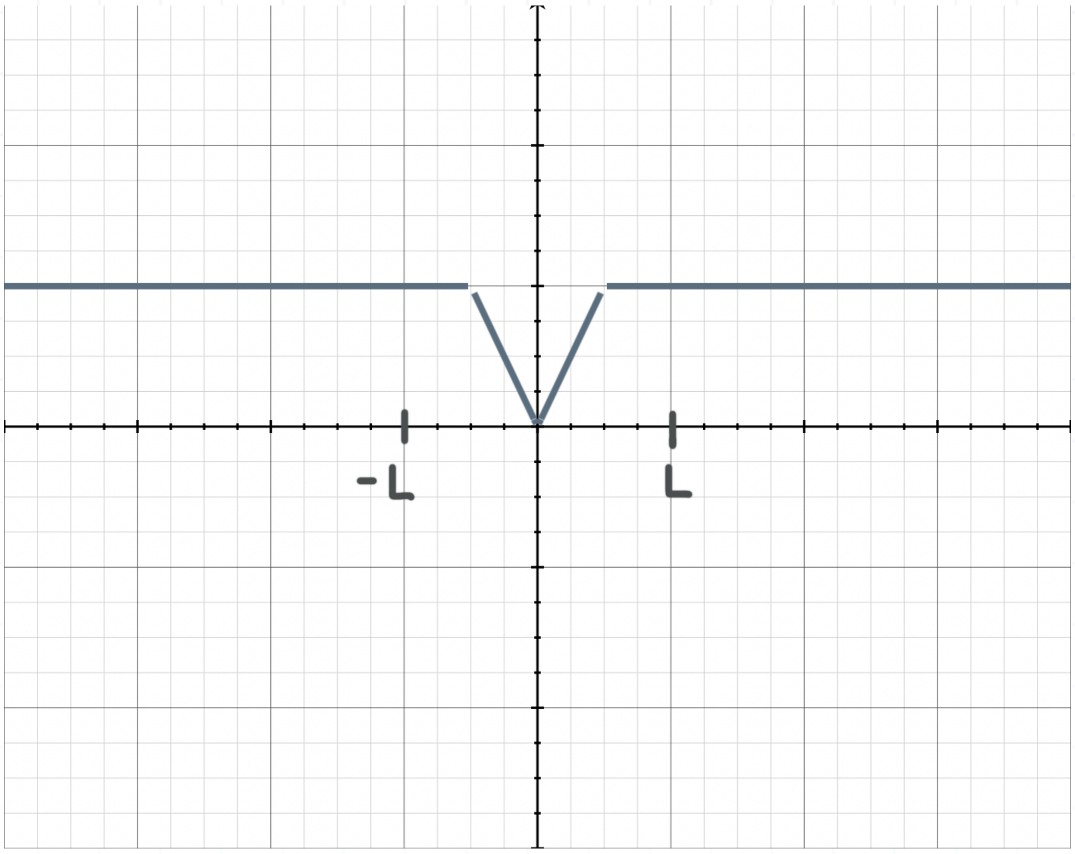


The even extension of $f(x)$ is

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} L^2 & \frac{L}{2} < x \le L \\ 2Lx & 0 \le x \le \frac{L}{2} \\ 2L(-x) & -\frac{L}{2} \le x < 0 \\ -L^2 & -L \le x < -\frac{L}{2} \end{cases} = \begin{cases} L^2 & \frac{L}{2} < x \le L \\ 2Lx & 0 \le x \le \frac{L}{2} \\ -2Lx & -\frac{L}{2} \le x < 0 \\ -L^2 & -L \le x < -\frac{L}{2} \end{cases}$$

The sketch of the even extension is



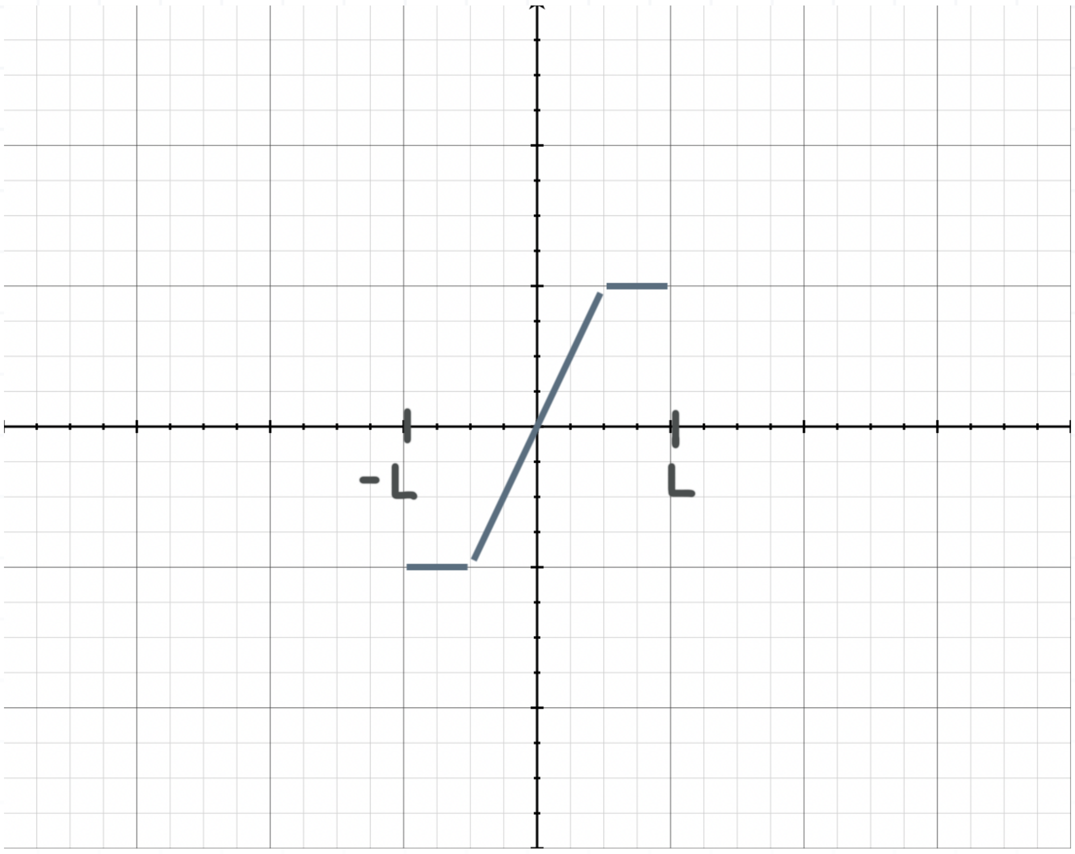
The odd extension of $f(x)$ is

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} L^2 & \frac{L}{2} < x \le L \\ 2Lx & 0 \le x \le \frac{L}{2} \\ -(2L(-x)) & -\frac{L}{2} \le x < 0 \\ -L^2 & -L \le x < -\frac{L}{2} \end{cases}$$

$$g(x) = \begin{cases} L^2 & \frac{L}{2} < x \le L \\ 2Lx & 0 \le x \le \frac{L}{2} \\ 2Lx & -\frac{L}{2} \le x < 0 \\ -L^2 & -L \le x < -\frac{L}{2} \end{cases} = \begin{cases} L^2 & \frac{L}{2} < x \le L \\ 2Lx & -\frac{L}{2} \le x \le \frac{L}{2} \\ -L^2 & -L \le x < -\frac{L}{2} \end{cases}$$

The sketch of the odd extension is



Since the sketches of the function and its odd extension are equivalent on $-L \leq x \leq L$, the function is odd.

■ 5. Find the even and odd extensions of the function, given some positive value of L.

$$f(x) = e^{L-2x}$$

Solution:

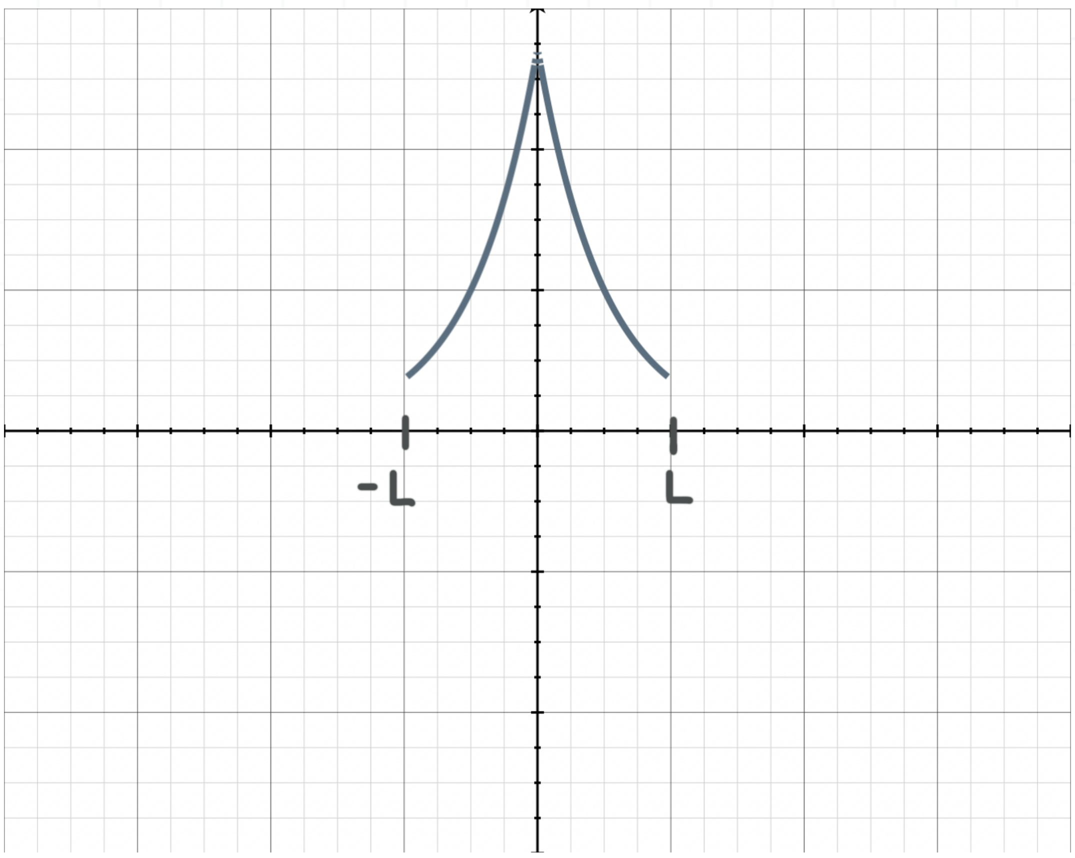
The even extension of $f(x)$ is

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} e^{L-2x} & 0 \le x \le L \\ e^{L-2(-x)} & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} e^{L-2x} & 0 \le x \le L \\ e^{L+2x} & -L \le x < 0 \end{cases}$$

The sketch of the even extension is



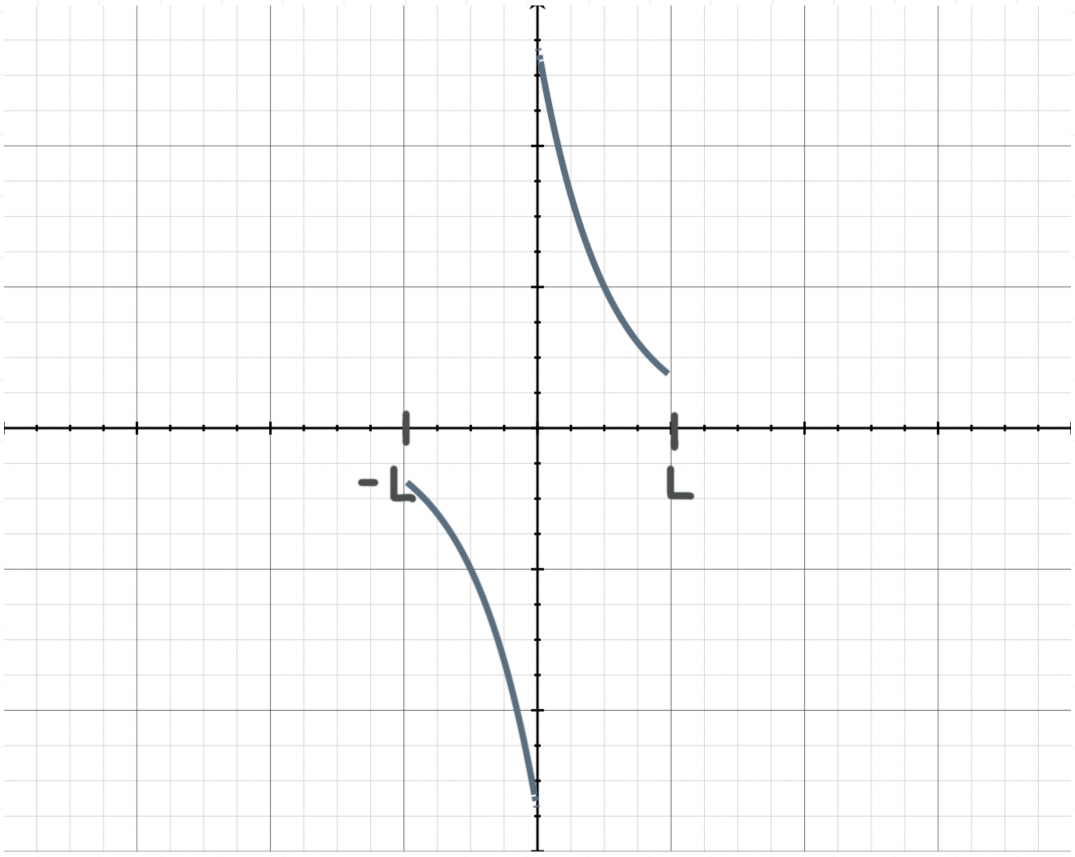
The odd extension of $f(x)$ is

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} e^{L-2x} & 0 \le x \le L \\ -e^{L-2(-x)} & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} e^{L-2x} & 0 \le x \le L \\ -e^{L+2x} & -L \le x < 0 \end{cases}$$

The sketch of the odd extension is



■ 6. Sketch the periodic extensions of the even and odd extensions of the function $f(x) = x$ on $-L \leq x \leq L$.

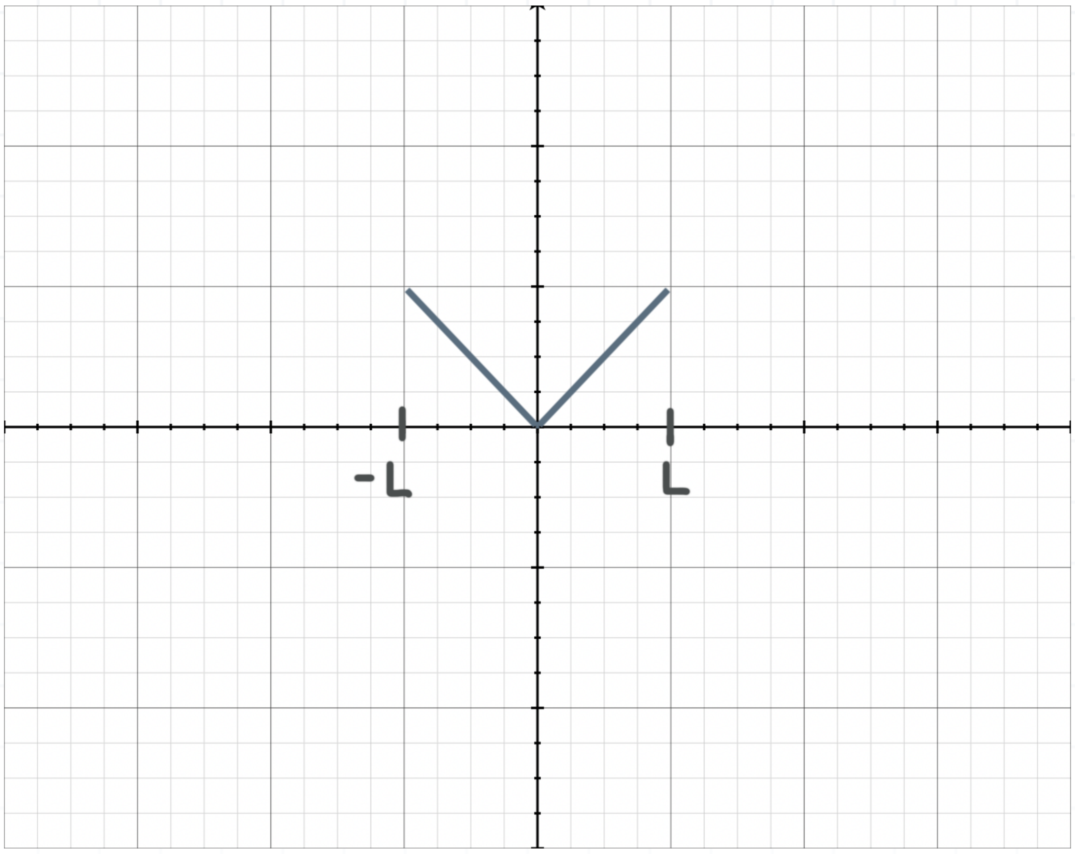
Solution:

The even extension of $f(x)$ is

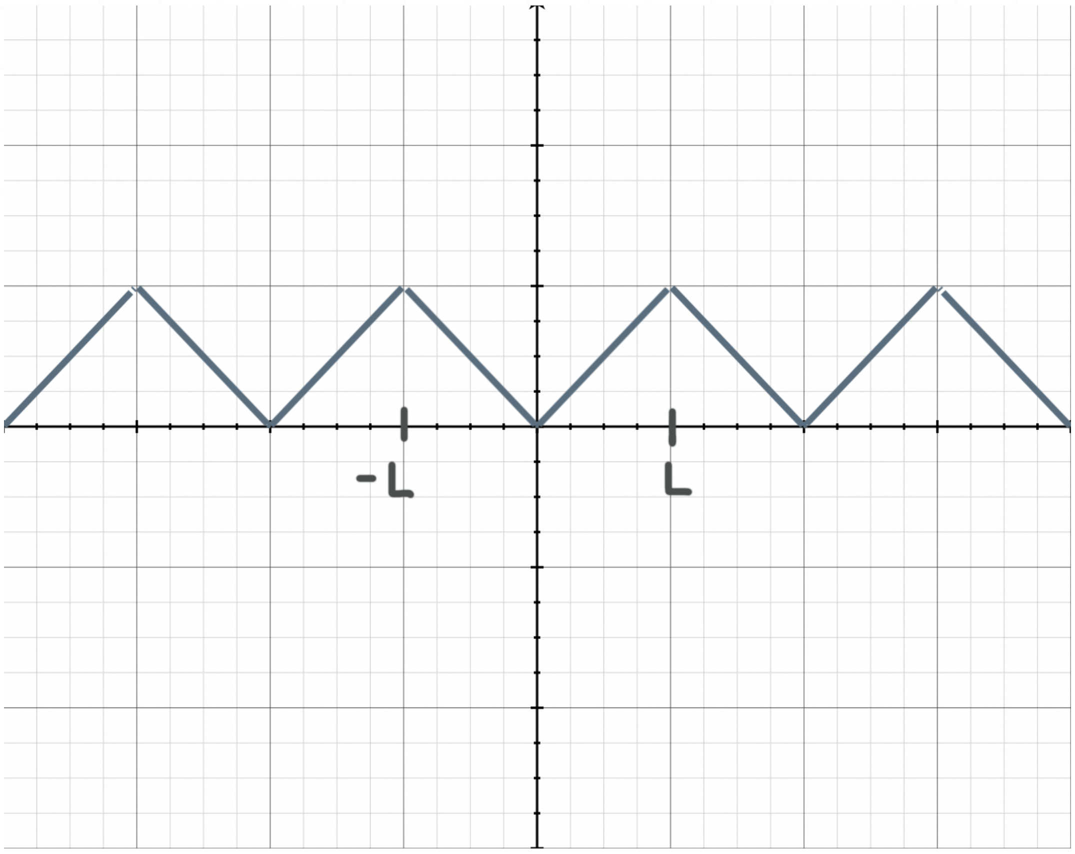
$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} x & 0 \leq x \leq L \\ -x & -L \leq x < 0 \end{cases}$$

The sketch of the even extension is



Then the sketch of the periodic extension of the even extension is



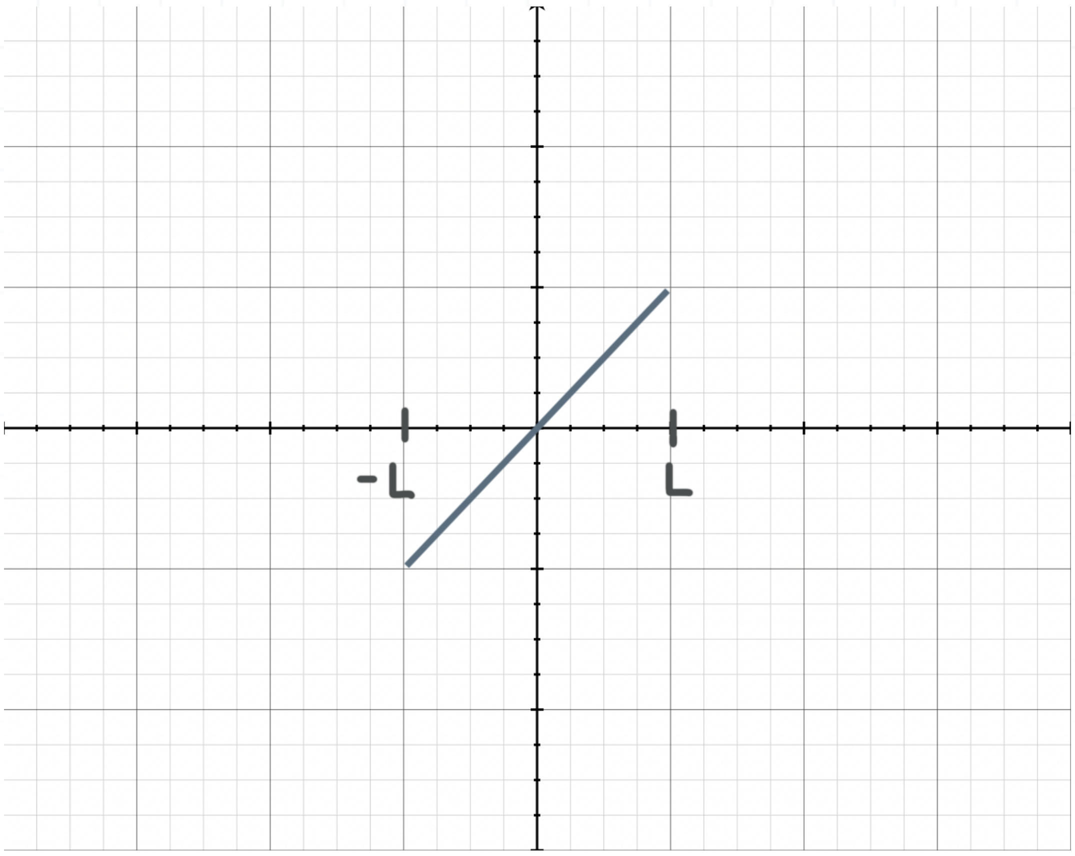
The odd extension of $f(x)$ is

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

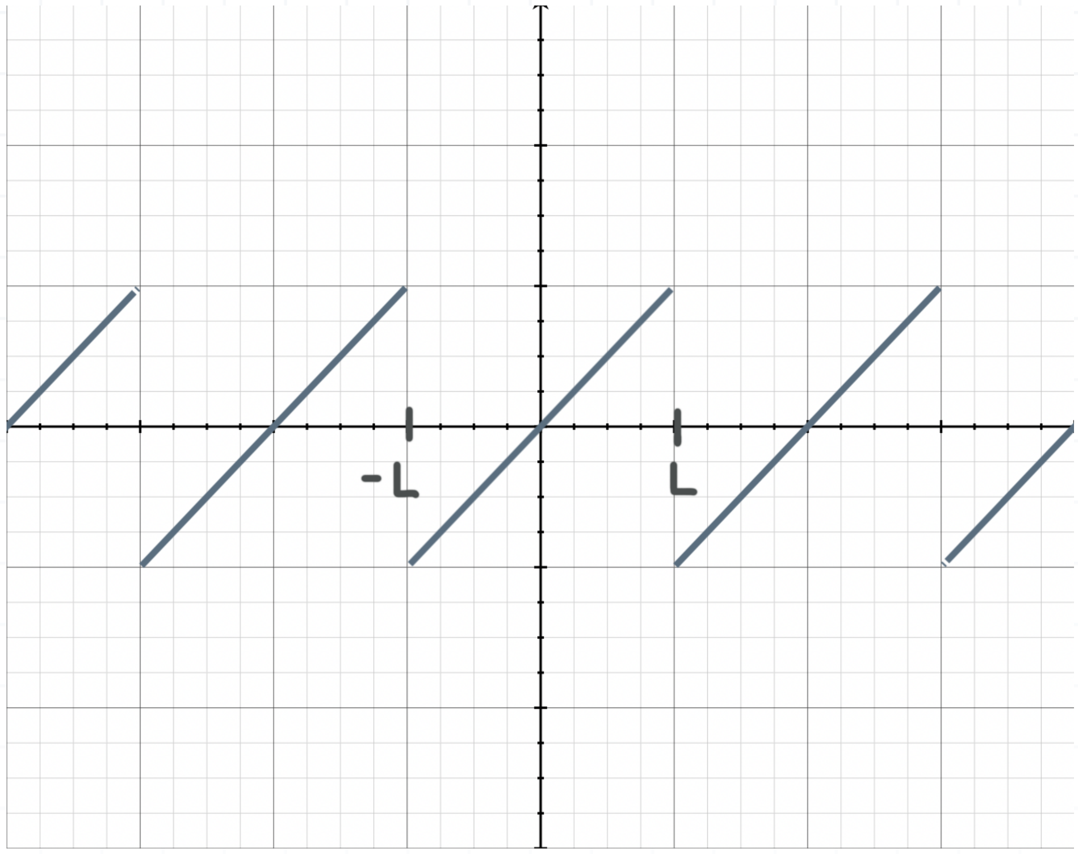
$$g(x) = \begin{cases} x & 0 \le x \le L \\ -(-x) & -L \le x < 0 \end{cases}$$

$g(x) = x$ on $-L \le x \le L$

The sketch of the odd extension is



Then the sketch of the periodic extension of the odd extension is


-L
L

REPRESENTING PIECEWISE FUNCTIONS

■ 1. Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} -2x & -L \le x < 0 \\ 3x & 0 \le x \le L \end{cases}$$

Solution:

For A_0 we get

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \ dx$$

$$A_0 = \frac{1}{2L} \left(\int_{-L}^{0} -2x \ dx + \int_{0}^{L} 3x \ dx \right)$$

$$A_0 = \frac{1}{2L} \left(-x^2 \Big|_{-L}^{0} + \frac{3}{2}x^2 \Big|_{0}^{L} \right)$$

$$A_0 = \frac{5}{4}L$$

For A_n we get

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) \ dx$$

$$A_n = \frac{1}{L}\left(\int_{-L}^{0} -2x\cos\left(\frac{n\pi x}{L}\right)\ dx + \int_{0}^{L} 3x\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = -\frac{2}{L}\int_{-L}^{0} x\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{3}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = -\frac{2}{L}\left(\frac{xL}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} - \frac{L}{n\pi}\int_{-L}^{0}\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$+\frac{3}{L}\left(\frac{xL}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L} - \frac{L}{n\pi}\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = -\frac{2}{L}\left(\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0}\right) + \frac{3}{L}\left(\left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right)$$

$$A_n = -\frac{2}{L}\left(\left(\frac{L}{n\pi}\right)^2 - \left(\frac{L}{n\pi}\right)^2(-1)^n\right) + \frac{3}{L}\left(\left(\frac{L}{n\pi}\right)^2(-1)^n - \left(\frac{L}{n\pi}\right)^2\right)$$

$$A_n = \frac{5L}{n^2\pi^2}((-1)^n - 1)$$

For B_n we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\left(\int_{-L}^{0} -2x \sin\left(\frac{n\pi x}{L}\right)\ dx + \int_{0}^{L} 3x \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$B_n = -\frac{2}{L}\left(\int_{-L}^{0} x \sin\left(\frac{n\pi x}{L}\right)\ dx\right) + \frac{3}{L}\left(\int_{0}^{L} x \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{2}{L}\left(-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$+\frac{3}{L}\left(-\frac{xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \int_0^L \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$B_n = -\frac{2}{L}\left(-\frac{L^2}{n\pi}(-1)^n + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$$

$$+\frac{3}{L}\left(-\frac{L^2}{n\pi}(-1)^n + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$$

$$B_n = -\frac{2}{L}\left(-\frac{L^2}{n\pi}(-1)^n\right) + \frac{3}{L}\left(-\frac{L^2}{n\pi}(-1)^n\right)$$

$$B_n = -\frac{L}{n\pi}(-1)^n$$

Then the Fourier series representation of the piecewise function on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{5}{4}L + \sum_{n=1}^{\infty} \frac{5L}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi x}{L}\right) - \sum_{n=1}^{\infty} \frac{L}{n\pi}(-1)^n \sin\left(\frac{n\pi x}{L}\right)$$

■ 2. Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} 2x + 4 & -L \le x < 0 \\ 4 - 2x & 0 \le x \le L \end{cases}$$

Solution:

Since $f(x)$ is even,

$$f(-x) = \begin{cases} -2x + 4 & 0 \le x \le L \\ 4 + 2x & -L \le x < 0 \end{cases}$$

$$f(-x) = \begin{cases} 2x + 4 & -L \le x < 0 \\ 4 - 2x & 0 \le x \le L \end{cases}$$

$$f(-x) = f(x)$$

we know $B_n = 0$. For A_0 we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\left(\int_{-L}^{0} -2x + 4\ dx + \int_{0}^{L} 4 + 2x\ dx\right)$$

$$A_0 = \frac{1}{2L}\left((-x^2 + 4x)\Big|_{-L}^{0} + (4x + x^2)\Big|_{0}^{L}\right)$$

$$A_0 = \frac{1}{2L}(L^2 + 4L + 4L + L^2)$$

$$A_0 = \frac{1}{2L}(8L + 2L^2)$$

$$A_0 = 4 + L$$

For A_n, we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{0} (4 - 2x)\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} (4 + 2x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 4 - 2x$, $du = -2\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ in the first integral and with $u = 4 + 2x$, $du = 2\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ in the second integral.

$$A_n = \frac{1}{L}\left((4 - 2x)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} + \frac{2L}{n\pi}\int_{-L}^{0}\sin\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left. + (4 + 2x)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L} - \frac{2L}{n\pi}\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{1}{L}\left(\frac{2L}{n\pi}\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} - \frac{2L}{n\pi}\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right)$$

$$A_n = \frac{1}{L}\left(-\frac{2L^2}{(n\pi)^2}(1-(-1)^n) + \frac{2L^2}{(n\pi)^2}((-1)^n - 1)\right)$$

$$A_n = \frac{4L}{n^2\pi^2}((-1)^n - 1)$$

Then the Fourier series representation of the piecewise function on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 4 + L + \sum_{n=1}^{\infty} \frac{4L}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi x}{L}\right)$$

■ 3. Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} 1 & -L \le x < 0 \\ 3\left(x + \frac{L}{2}\right) + 1 & 0 \le x \le L \end{cases}$$

Solution:

For A_0, we get

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L} \left(\int_{-L}^{0} 1\ dx + \int_{0}^{L} \left(3\left(x + \frac{L}{2} \right) + 1 \right)\ dx \right)$$

$$A_0 = \frac{1}{2L} \left(x \Big|_{-L}^{0} + \left(\frac{3}{2}x^2 + \frac{3L}{2}x + x \right) \Big|_{0}^{L} \right)$$

$$A_0 = \frac{1}{2L} \left(L + \left(\frac{3L^2}{2} + \frac{3L^2}{2} + L \right) \right)$$

$$A_0 = \frac{1}{2L}(2L + 3L^2)$$

$$A_0 = 1 + \frac{3L}{2}$$

For A_n, we get

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L} \right)\ dx$$

$$A_n = \frac{1}{L} \left(\int_{-L}^{0} \cos\left(\frac{n\pi x}{L} \right)\ dx + \int_{0}^{L} \left(3\left(x + \frac{L}{2} \right) + 1 \right) \cos\left(\frac{n\pi x}{L} \right)\ dx \right)$$

$$A_n = \frac{1}{L} \left(\frac{L}{n\pi} \sin\left(\frac{n\pi x}{L} \right) \Big|_{-L}^{0} + \int_{0}^{L} \left(3\left(x + \frac{L}{2} \right) + 1 \right) \cos\left(\frac{n\pi x}{L} \right)\ dx \right)$$

Use integration by parts with $u = 3(x + L/2) + 1$, $du = 3\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{1}{L}\left(\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} + \left(3\left(x+\frac{L}{2}\right)+1\right)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right.$$

$$\left. -\frac{3L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{1}{L}\left(\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} + \left(3\left(x+\frac{L}{2}\right)+1\right)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right.$$

$$\left. +\frac{3L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right)$$

$$A_n = \frac{1}{L}\left(-\frac{L}{n\pi}\sin(-n\pi) + \left(3\left(L+\frac{L}{2}\right)+1\right)\frac{L}{n\pi}\sin(n\pi)\right.$$

$$\left. +\frac{3L^2}{(n\pi)^2}\cos(n\pi) - \frac{3L^2}{(n\pi)^2}\right)$$

$$A_n = \frac{1}{L}\left(\frac{L}{n\pi}\sin(n\pi) + \frac{3L^2}{(n\pi)^2}(-1)^n - \frac{3L^2}{(n\pi)^2}\right)$$

$$A_n = \frac{3L}{(n\pi)^2}((-1)^n - 1)$$

For B_n, we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\left(\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right)\ dx + \int_{0}^{L}\left(3\left(x+\frac{L}{2}\right)+1\right)\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u = 3(x + L/2) + 1$, $du = 3\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{1}{L}\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} - \left(3\left(x+\frac{L}{2}\right)+1\right)\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right.$$

$$\left.+\frac{3L}{n\pi}\int_{0}^{L}\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$B_n = \frac{1}{L}\left(-\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} - \left(3\left(x+\frac{L}{2}\right)+1\right)\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right.$$

$$\left.+\frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}\right)$$

$$B_n = \frac{1}{L}\left(-\frac{L}{n\pi} + \frac{L}{n\pi}\cos(-n\pi) - \left(3\left(L+\frac{L}{2}\right)+1\right)\left(\frac{L}{n\pi}\right)\cos(n\pi)\right.$$

$$\left.+\left(3\left(\frac{L}{2}\right)+1\right)\left(\frac{L}{n\pi}\right) + \frac{3L^2}{(n\pi)^2}\sin(n\pi)\right)$$

$$B_n = \frac{1}{L}\left(-\frac{L}{n\pi} + \frac{L}{n\pi}(-1)^n - \left(\frac{9L}{2}+1\right)\left(\frac{L}{n\pi}\right)(-1)^n + \left(\frac{3L}{2}+1\right)\left(\frac{L}{n\pi}\right)\right)$$

$$B_n = \left(-\frac{1}{n\pi} + \frac{1}{n\pi}(-1)^n - \frac{1}{n\pi}\left(\frac{9L}{2} + 1\right)(-1)^n + \frac{1}{n\pi}\left(\frac{3L}{2} + 1\right)\right)$$

$$B_n = \frac{1}{n\pi}\left((-1)^n - \left(\frac{9L}{2} + 1\right)(-1)^n + \frac{3L}{2}\right)$$

$$B_n = \frac{3L}{2n\pi}(1 - 3(-1)^n)$$

Then the Fourier series representation of the piecewise function on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 1 + \frac{3L}{2} + \sum_{n=1}^{\infty} \frac{3L}{(n\pi)^2}((-1)^n - 1)\cos\left(\frac{n\pi x}{L}\right)$$

$$+\sum_{n=1}^{\infty} \frac{3L}{2n\pi}(1 - 3(-1)^n)\sin\left(\frac{n\pi x}{L}\right)$$

■ 4. Find the Fourier series representation of the piecewise function on $-\pi \le x \le \pi$.

$$f(x) = \begin{cases} (x+1)^2 & -\pi \le x < 0 \\ 2 - (x-1)^2 & 0 \le x \le \pi \end{cases}$$

Solution:

For A_0, we get

$$A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\ dx$$

$$A_0 = \frac{1}{2\pi} \left(\int_{-\pi}^{0} (x+1)^2\ dx + \int_{0}^{\pi} (2 - (x-1)^2)\ dx \right)$$

$$A_0 = \frac{1}{2\pi} \left(\left(\frac{x^3}{3} + x^2 + x \right) \Bigg|_{-\pi}^{0} + \left(2x - \left(\frac{x^3}{3} - x^2 + x \right) \right) \Bigg|_{0}^{\pi} \right)$$

$$A_0 = \frac{1}{2\pi} \left(\left(\frac{x^3}{3} + x^2 + x \right) \Bigg|_{-\pi}^{0} + \left(-\frac{x^3}{3} + x^2 + x \right) \Bigg|_{0}^{\pi} \right)$$

$$A_0 = \frac{1}{2\pi} \left(\frac{\pi^3}{3} - \pi^2 + \pi - \frac{\pi^3}{3} + \pi^2 + \pi \right)$$

$$A_0 = 1$$

For A_n, we get

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\cos(nx)\ dx$$

$$A_n = \frac{1}{\pi} \left(\int_{-\pi}^{0} (x+1)^2\cos(nx)\ dx + \int_{0}^{\pi} (2 - (x-1)^2)\cos(nx)\ dx \right)$$

Since

$$\int_{-\pi}^{0} (x+1)^2\cos(nx)\ dx = \int_{0}^{\pi} (1-x)^2\cos(nx)\ dx$$

we get

$$A_n = \frac{1}{\pi}\left(\int_0^{\pi} (1-x)^2\cos(nx)\ dx\right.$$

$$\left. + \int_0^{\pi} 2\cos(nx)\ dx\ - \int_0^{\pi} (x-1)^2\cos(nx)\ dx\ \right)$$

$$A_n = \frac{1}{\pi}\int_0^{\pi} 2\cos(nx)\ dx$$

$$A_n = \frac{1}{\pi} \cdot \frac{2}{n}\sin(nx)\Big|_0^{\pi} = 0$$

For B_n, we get

$$B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\ dx$$

$$B_n = \frac{1}{\pi}\left(\int_{-\pi}^{0} (x+1)^2\sin(nx)\ dx + \int_0^{\pi} (2-(x-1)^2)\sin(nx)\ dx\right)$$

Since

$$\int_{-\pi}^{0} (x+1)^2\sin(nx)\ dx = \int_0^{\pi} -(x-1)^2\sin(nx)\ dx$$

we get

$$B_n = \frac{1}{\pi}\left(\int_0^{\pi} -(x-1)^2\sin(nx)\ dx + \int_0^{\pi}(2-(x-1)^2)\sin(nx)\ dx\right)$$

$$B_n = \frac{1}{\pi}\int_0^{\pi}(2-2(x-1)^2)\sin(nx)\ dx$$

$$B_n = -\frac{1}{\pi}\int_0^{\pi}(2x^2-4x)\sin(nx)\ dx$$

Use integration by parts with $u = 2x^2 - 4x$, $du = (4x-4)\ dx$, $dv = \sin(nx)\ dx$, and $v = -(1/n)\cos(nx)$.

$$B_n = -\frac{1}{\pi}\left(-\frac{1}{n}(2x^2-4x)\cos(nx)\Big|_0^{\pi} + \frac{4}{n}\int_0^{\pi}(x-1)\cos(nx)\ dx\right)$$

$$B_n = -\frac{1}{\pi}\left(-\frac{1}{n}(2\pi^2-4\pi)(-1)^n + \frac{4}{n}\int_0^{\pi}(x-1)\cos(nx)\ dx\right)$$

Use integration by parts with $u = x - 1$, $du = \ dx$, $dv = \cos(nx)\ dx$, and $v = (1/n)\sin(nx)$.

$$B_n = -\frac{1}{\pi}\left(-\frac{2\pi}{n}(\pi-2)(-1)^n + \frac{4}{n}\left(\frac{(x-1)}{n}\sin(nx)\Big|_0^{\pi} - \frac{1}{n}\int_0^{\pi}\sin(nx)\ dx\right)\right)$$

$$B_n = -\frac{1}{\pi}\left(-\frac{2\pi}{n}(\pi-2)(-1)^n + \frac{4}{n}\left(0 + \frac{1}{n^2}\cos(nx)\Big|_0^{\pi}\right)\right)$$

$$B_n = \frac{2(\pi-2)}{n}(-1^n) - \frac{4}{\pi n^3}\cos(nx)\Big|_0^{\pi}$$

$$B_n = \frac{2(\pi - 2)}{n}(-1^n) - \frac{4}{\pi n^3}((-1)^n - 1)$$

Then the Fourier series representation of the piecewise function on $-\pi \leq x \leq \pi$ is

$$f(x) = 1 + \sum_{n=1}^{\infty} \left(\left(\frac{2(\pi - 2)}{n} - \frac{4}{\pi n^3} \right) (-1)^n + \frac{4}{\pi n^3} \right) \sin(nx)$$

■ 5. Find the Fourier series representation of the piecewise function.

$$f(x) = \begin{cases} 3x + 6 & -L \leq x < -1 \\ -3x & -1 \leq x \leq 1 \\ 3x - 6 & 1 < x \leq L \end{cases}$$

Solution:

The function $f(x)$ is odd,

$$f(-x) = \begin{cases} 3(-x) + 6 & -L \leq -x < -1 \\ -3(-x) & -1 \leq -x \leq 1 \\ 3(-x) - 6 & 1 < -x \leq L \end{cases}$$

$$f(-x) = \begin{cases} -3x - 6 & -L \leq x < -1 \\ 3x & -1 \leq x \leq 1 \\ -3x + 6 & 1 < x \leq L \end{cases}$$

$$f(-x) = \begin{cases} -(3x+6) & -L \le x < -1 \\ -(-3x) & -1 \le x \le 1 \\ -(3x-6) & 1 < x \le L \end{cases}$$

$$f(-x) = -f(x)$$

so we know $A_0 = A_n = 0$. For B_n, we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\left(\int_{-L}^{-1} (3x+6)\sin\left(\frac{n\pi x}{L}\right)\ dx + \int_{-1}^{1} -3x\sin\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left.+\int_{1}^{L} (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u = 3x + 6$, $du = 3\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ for the first integral.

$$B_n = \frac{1}{L}\left((3x+6)\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{-1} + \frac{3L}{n\pi}\int_{-L}^{-1}\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left.+\int_{-1}^{1} -3x\sin\left(\frac{n\pi x}{L}\right)\ dx + \int_{1}^{L} (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$B_n = \frac{1}{L}\left(-\frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right)\frac{L(-3L+6)}{n\pi}(-1)^n + \frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{-1}\right.$$

$$+\int_{-1}^{1} -3x \sin\left(\frac{n\pi x}{L}\right)\ dx + \int_{1}^{L} (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

$$B_n = \frac{1}{L}\Bigg(-\frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) + \frac{L(-3L+6)}{n\pi}(-1)^n - \frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right)$$

$$+\int_{-1}^{1} -3x \sin\left(\frac{n\pi x}{L}\right)\ dx + \int_{1}^{L} (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

Use integration by parts with $u = -3x$, $du = -3\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ for the second integral.

$$B_n = \frac{1}{L}\Bigg(-\frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) + \frac{L(-3L+6)}{n\pi}(-1)^n - \frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right)$$

$$+\frac{3xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-1}^{1} - \frac{3L}{n\pi}\int_{-1}^{1}\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\int_{1}^{L} (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

$$B_n = \frac{1}{L}\Bigg(-\frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) + \frac{L(-3L+6)}{n\pi}(-1)^n - \frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right)$$

$$+\frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) + \frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) - \frac{3L}{n\pi}\int_{-1}^{1}\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\int_{1}^{L} (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

$$B_n = \frac{1}{L}\Bigg(\frac{L(-3L+6)}{n\pi}(-1)^n - \frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right)$$

$$+\frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) - \frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-1}^{1} + \int_1^L (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

$$B_n = \frac{1}{L}\Bigg(\frac{L(-3L+6)}{n\pi}(-1)^n - \frac{3L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right)$$

$$+\frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) - \frac{6L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right) + \int_1^L (3x-6)\sin\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

Use integration by parts with $u = 3x - 6$, $du = 3\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ for the second integral.

$$B_n = \frac{1}{L}\Bigg(\frac{L(-3L+6)}{n\pi}(-1)^n - \frac{9L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right) + \frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right)$$

$$+(3x-6)\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_1^L + \frac{3L}{n\pi}\int_1^L \cos\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

$$B_n = \frac{1}{L}\Bigg(\frac{L(-3L+6)}{n\pi}(-1)^n - \frac{9L^2}{(n\pi)^2}\sin\left(\frac{n\pi}{L}\right) + \frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right)$$

$$-(3L-6)\frac{L}{n\pi}(-1)^n - \frac{3L}{n\pi}\cos\left(\frac{n\pi}{L}\right) + \frac{3L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_1^L\Bigg)$$

$$B_n = -\frac{12L}{n^2\pi^2}\sin\left(\frac{n\pi}{L}\right) - \frac{6(L-2)}{n\pi}(-1)^n$$

Then the Fourier series representation is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \left(-\frac{12L}{n^2\pi^2} \sin\left(\frac{n\pi}{L}\right) - \frac{6(L-2)}{n\pi}(-1)^n\right) \sin\left(\frac{n\pi x}{L}\right)$$

6. Find the Fourier series representation of the piecewise function.

$$f(x) = \begin{cases} 2 - 2x & -L \le x < 0 \\ 4x + 3 & 0 \le x \le L \end{cases}$$

Solution:

For A_0, we get

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x) \ dx$$

$$A_0 = \frac{1}{2L} \left(\int_{-L}^{0} (2 - 2x) \ dx + \int_{0}^{L} (4x + 3) \ dx \right)$$

$$A_0 = \frac{1}{2L} \left(2x - x^2 \Big|_{-L}^{0} + (2x^2 + 3x) \Big|_{0}^{L} \right)$$

$$A_0 = \frac{1}{2L}(2L + L^2 + 2L^2 + 3L)$$

$$A_0 = \frac{1}{2}(3L + 5)$$

For A_n, we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\left(\int_{-L}^{0} (2 - 2x)\cos\left(\frac{n\pi x}{L}\right)\ dx + \int_{0}^{L} (4x + 3)\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Since

$$\int_{-L}^{0} (2 - 2x)\cos\left(\frac{n\pi x}{L}\right)\ dx = \int_{0}^{L} (2 + 2x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

we get

$$A_n = \frac{1}{L}\int_{0}^{L} (2 + 2x + 4x + 3)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{0}^{L} (6x + 5)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 6x + 5$, $du = 6\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{1}{L}\left((6x + 5)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{6L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{1}{L}\left(0 - \frac{6L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{1}{L}\left(\frac{6L}{n\pi} \cdot \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$$

$$A_n = \frac{6L}{n^2\pi^2}((-1)^n - 1)$$

For B_n, we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\left(\int_{-L}^{0} (2 - 2x)\sin\left(\frac{n\pi x}{L}\right)\ dx + \int_0^L (4x + 3)\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Since

$$\int_{-L}^{0} (2 - 2x)\sin\left(\frac{n\pi x}{L}\right)\ dx = \int_0^L (-2 - 2x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

we get

$$B_n = \frac{1}{L}\int_{-L}^{0} (-2 - 2x + 4x + 3)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{0} (2x + 1)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 2x + 1$, $du = 2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{1}{L}\left(-(2x+1)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{2L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$B_n = \frac{1}{L}\left(-\frac{L}{n\pi}((2L+1)(-1)^n - 1) + \frac{2L^2}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$$

$$B_n = \frac{1}{L}\left(-\frac{L}{n\pi}((2L+1)(-1)^n - 1) + 0\right)$$

$$B_n = \frac{1-(2L+1)(-1)^n}{n\pi}$$

Then the Fourier series representation of the piecewise function is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{1}{2}(3L+5) + \sum_{n=1}^{\infty}\frac{6L}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi x}{L}\right)$$

$$+\sum_{n=1}^{\infty}\frac{1-(2L+1)(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

CONVERGENCE OF A FOURIER SERIES

■ 1. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

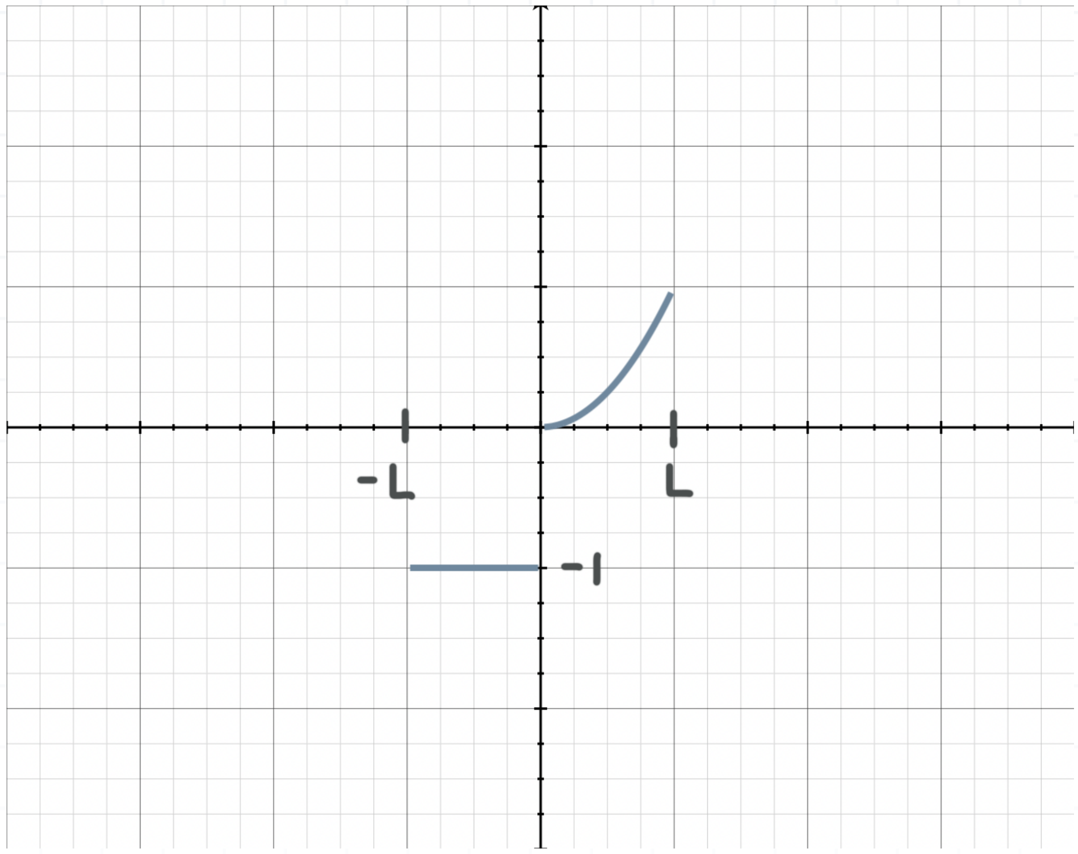
$$f(x) = \begin{cases} -1 & -L \leq x < 0 \\ x^2 & 0 \leq x \leq L \end{cases}$$

$$f(x) = -\frac{1}{2} + \frac{L^2}{6} + \frac{2L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

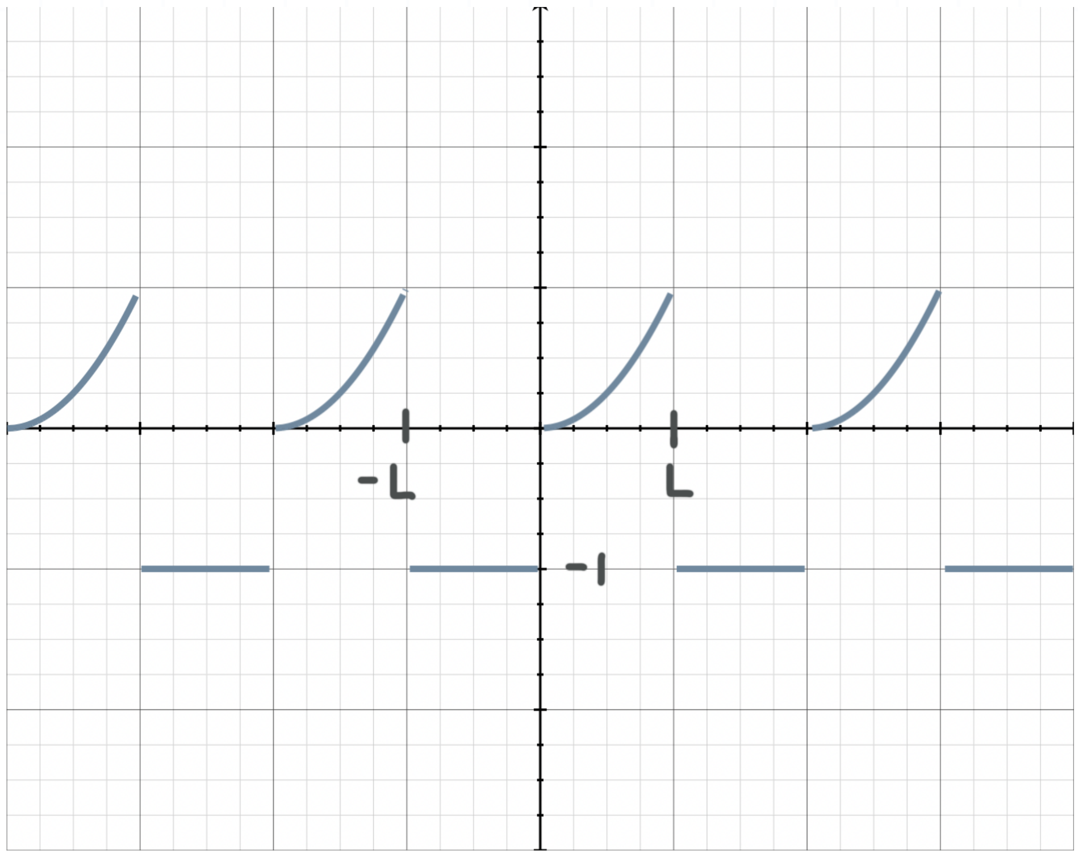
$$+ \sum_{n=1}^{\infty} \frac{1}{n^3\pi^3} \left[(2L^2 - L^2n^2\pi^2 - n^2\pi^2)(-1)^n + n^2\pi^2 - 2L^2\right] \sin\left(\frac{n\pi x}{L}\right)$$

Solution:

A sketch of the function $f(x)$ is



The sketch of its periodic extension, $g(x)$, is



From the graph of the periodic extension, we can see that $g(x)$ has a jump discontinuity at $x = 0$, within the interval $-L \leq x \leq L$, as well as jump discontinuities at $-L$ and L.

Because the function is piecewise smooth on the individual intervals $-L \le x \le 0$ and $0 \le x \le L$, the function $f(x)$ and the periodic extension $g(x)$ are both continuous there, so the Fourier series representation will converge to the periodic extension $g(x)$, and therefore also to the original function $f(x)$. Because of the jump discontinuity at $x = 0$, the Fourier series representation at that point will converge to

$$\frac{\lim_{x\to 0^-} g(x) + \lim_{x\to 0^+} g(x)}{2}$$

$$\frac{-1+0}{2}$$

$$-\frac{1}{2}$$

At $x = -L$, the Fourier series representation will converge to

$$\frac{\lim_{x\to -L^-} g(x) + \lim_{x\to -L^+} g(x)}{2}$$

$$\frac{L^2+(-1)}{2}$$

$$\frac{L^2-1}{2}$$

and at $x = L$ the Fourier series representation will converge to

$$\frac{\lim_{x\to L^-} g(x) + \lim_{x\to L^+} g(x)}{2}$$

$$\frac{L^2-1}{2}$$

The Fourier series representation converges to $f(x)$, since $f(x)$ is piecewise smooth and has a finite number of jump discontinuities on any interval with finite endpoints.

■ 2. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

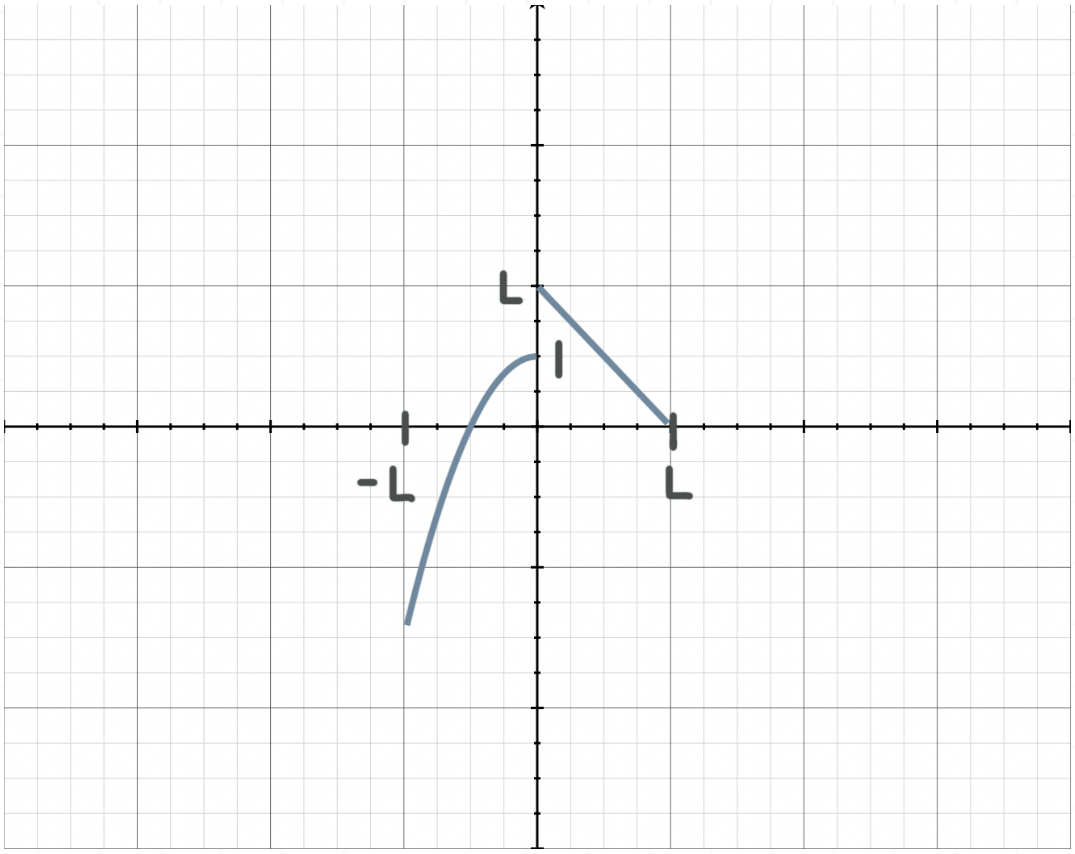
$$f(x) = \begin{cases} -x^2 + 1 & -L \le x < 0 \\ L - x & 0 \le x \le L \end{cases}$$

$$f(x) = \frac{6 + 3L - 2L^2}{12} + \frac{L}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (2L + 1)(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

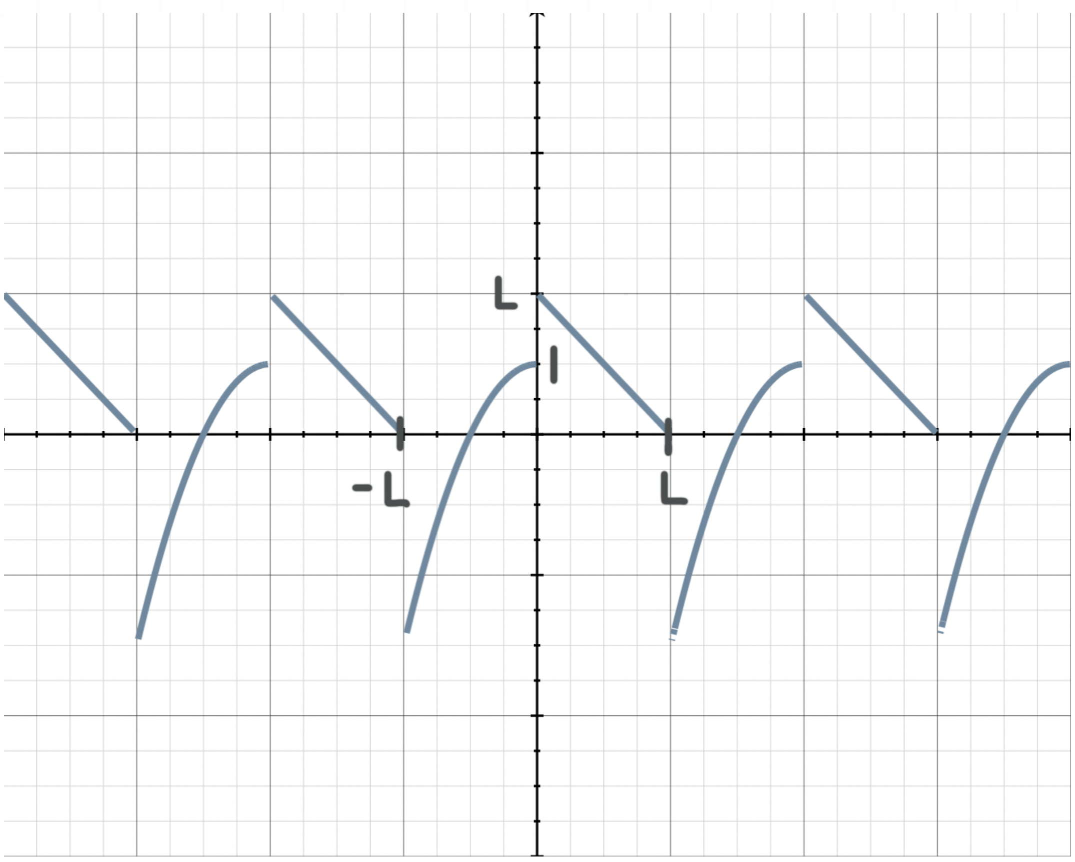
$$+ \frac{1}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[(n^2\pi^2(1 - L^2) + 2L^2)(-1)^n + n^2\pi^2(L - 1) - 2L^2\right] \sin\left(\frac{n\pi x}{L}\right)$$

Solution:

A sketch of the function $f(x)$ is



The sketch of its periodic extension, $g(x)$, is



From the graph of the periodic extension, we can see that $g(x)$ has a jump discontinuity at $x = 0$, within the interval $-L \leq x \leq L$, as well as jump discontinuities at $-L$ and L, assuming $L \neq 1$.

Because the function is piecewise smooth on the individual intervals $-L \leq x \leq 0$ and $0 \leq x \leq L$, the function $f(x)$ and the periodic extension $g(x)$ are both continuous there, so the Fourier series representation will converge to the periodic extension $g(x)$, and therefore also to the original function $f(x)$. Because of the jump discontinuity at $x = 0$, the Fourier series representation at that point will converge to

$$\frac{\lim_{x \to 0^-} g(x) + \lim_{x \to 0^+} g(x)}{2}$$

$$\frac{1 + L}{2}$$

At $x = -L$, the Fourier series representation will converge to

$$\frac{\lim_{x \to -L^-} g(x) + \lim_{x \to -L^+} g(x)}{2}$$

$$\frac{0 + (-L^2 + 1)}{2}$$

$$\frac{1 - L^2}{2}$$

and at $x = L$ the Fourier series representation will converge to

$$\frac{\lim_{x \to L^-} g(x) + \lim_{x \to L^+} g(x)}{2}$$

$$\frac{0 + (-L^2 + 1)}{2}$$

$$\frac{1 - L^2}{2}$$

The Fourier series representation converges to $f(x)$, since $f(x)$ is piecewise smooth and has a finite number of jump discontinuities on any interval with finite endpoints.

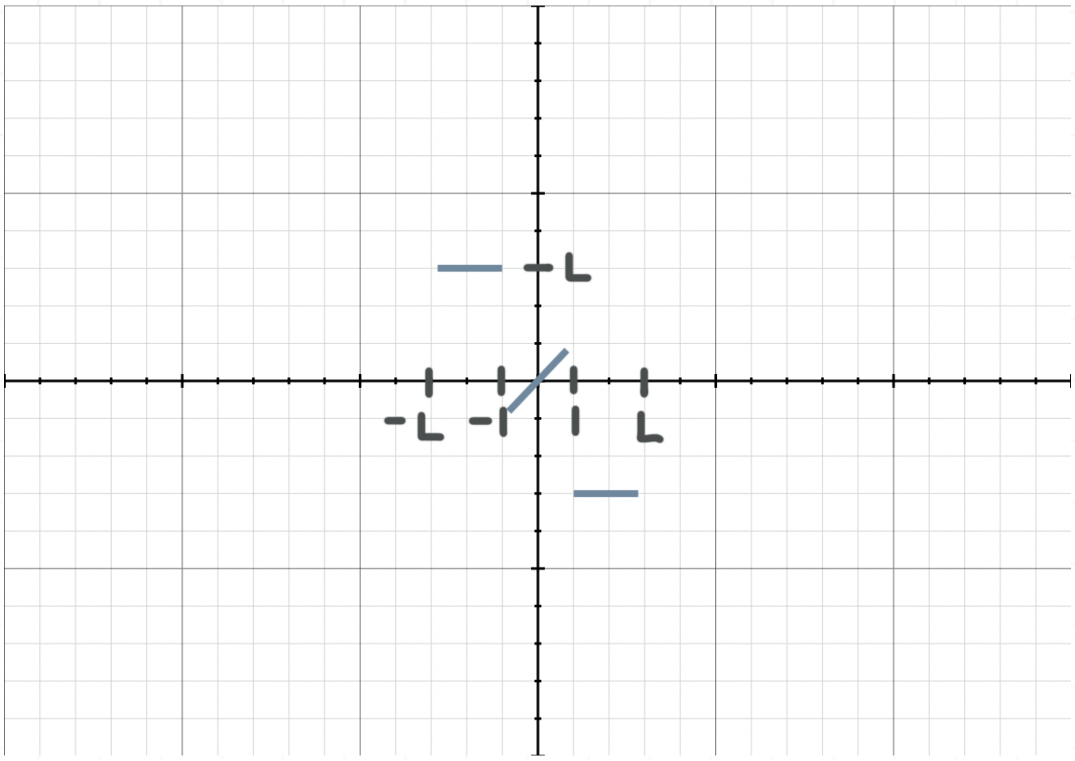
■ 3. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$, assuming $L > 1$.

$$f(x) = \begin{cases} L & -L \le x < -1 \\ x & -1 \le x \le 1 \\ -L & 1 < x \le L \end{cases}$$

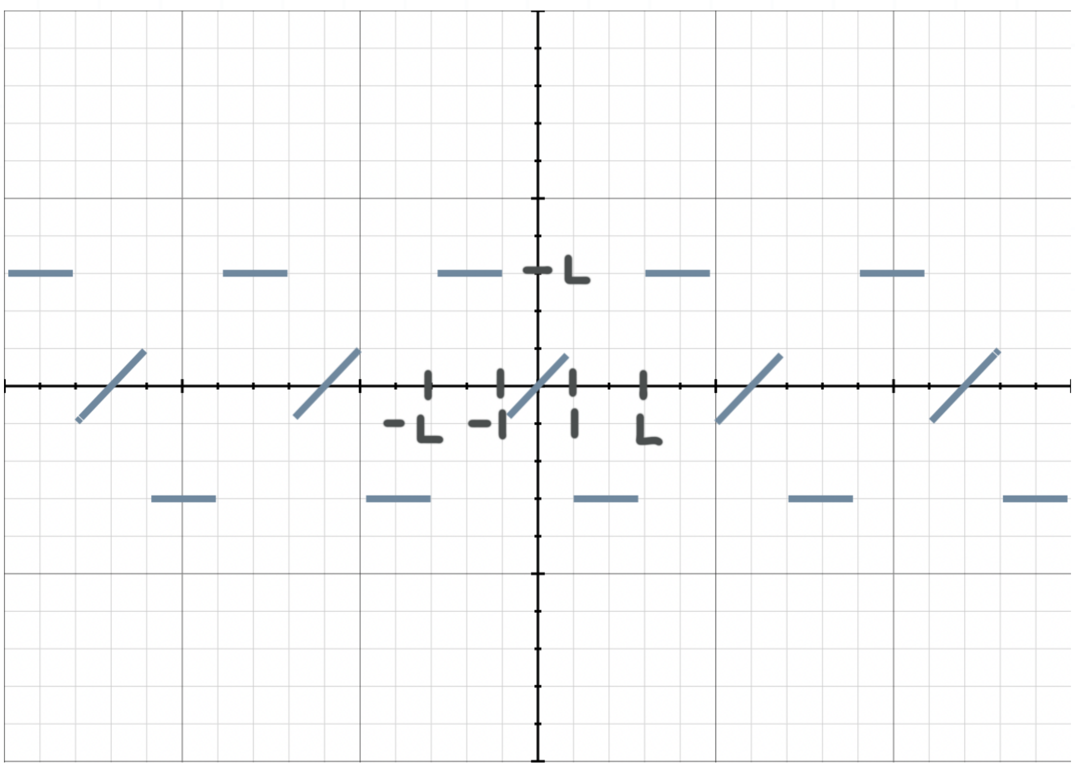
$$f(x) = \sum_{n=1}^{\infty} \left[\frac{2L}{n\pi}(-1)^n - \frac{2(L+1)}{n\pi} \cos\left(\frac{n\pi}{L}\right) + \frac{2L}{n^2\pi^2} \sin\left(\frac{n\pi}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

Solution:

A sketch of the function $f(x)$ is



The sketch of its periodic extension, $g(x)$, is



From the graph of the periodic extension, we can see that $g(x)$ has a jump discontinuity at $x = \pm 1$, within the interval $-L \leq x \leq L$, as well as jump discontinuities at $-L$ and L.

Because the function is piecewise smooth on the individual intervals $-L \leq x \leq -1$, $-1 \leq x \leq 1$, and $1 \leq x \leq L$, the function $f(x)$ and the periodic extension $g(x)$ are both continuous there, so the Fourier series representation will converge to the periodic extension $g(x)$, and therefore also to the original function $f(x)$. Because of the jump discontinuities at $x = \pm 1$, the Fourier series representation at $x = 1$ will converge to

$$\frac{\lim_{x\to 1^-} g(x) + \lim_{x\to 1^+} g(x)}{2}$$

$$\frac{1 + (-L)}{2}$$

$$\frac{1 - L}{2}$$

and the Fourier series representation at $x = -1$ will converge to

$$\frac{\lim_{x\to -1^-} g(x) + \lim_{x\to -1^+} g(x)}{2}$$

$$\frac{L + (-1)}{2}$$

$$\frac{L - 1}{2}$$

At $x = -L$, the Fourier series representation will converge to

$$\frac{\lim_{x\to -L^-} g(x) + \lim_{x\to -L^+} g(x)}{2}$$

$$\frac{-L + L}{2} = 0$$

and at $x = L$ the Fourier series representation will converge to

$$\frac{\lim_{x \to L^-} g(x) + \lim_{x \to L^+} g(x)}{2}$$

$$\frac{-L + L}{2} = 0$$

The Fourier series representation converges to $f(x)$, since $f(x)$ is piecewise smooth and has a finite number of jump discontinuities on any interval with finite endpoints.

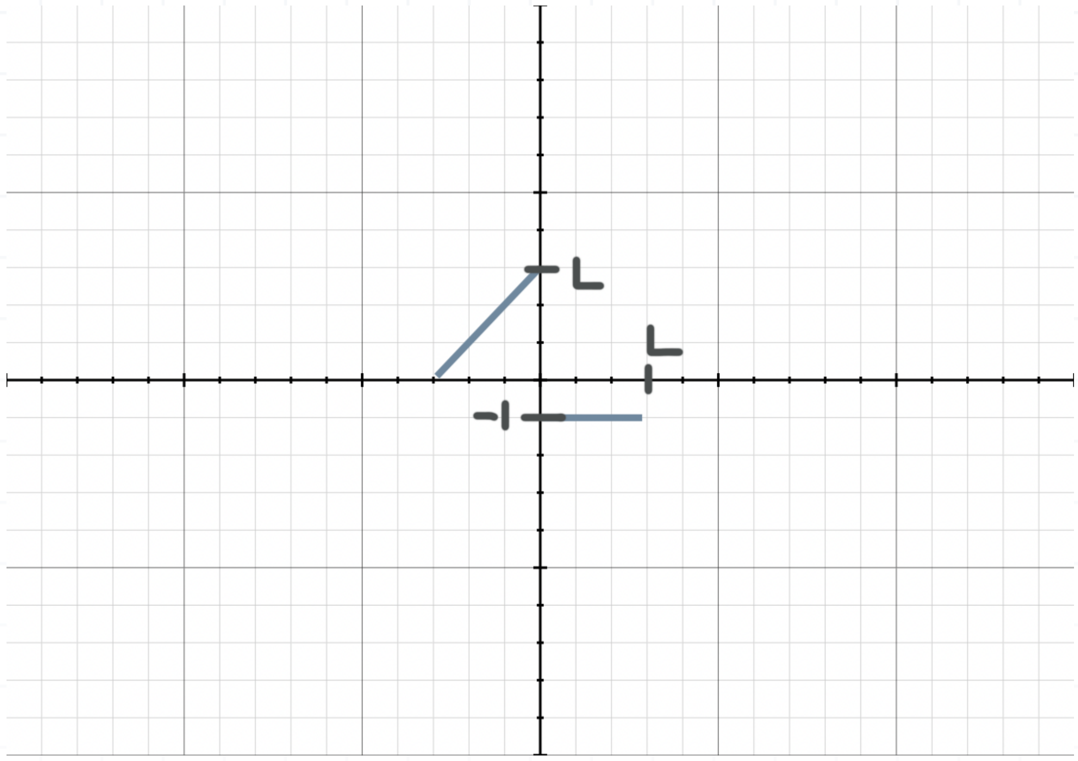
■ 4. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

$$f(x) = \begin{cases} x + L & -L \le x < 0 \\ -1 & 0 \le x \le L \end{cases}$$

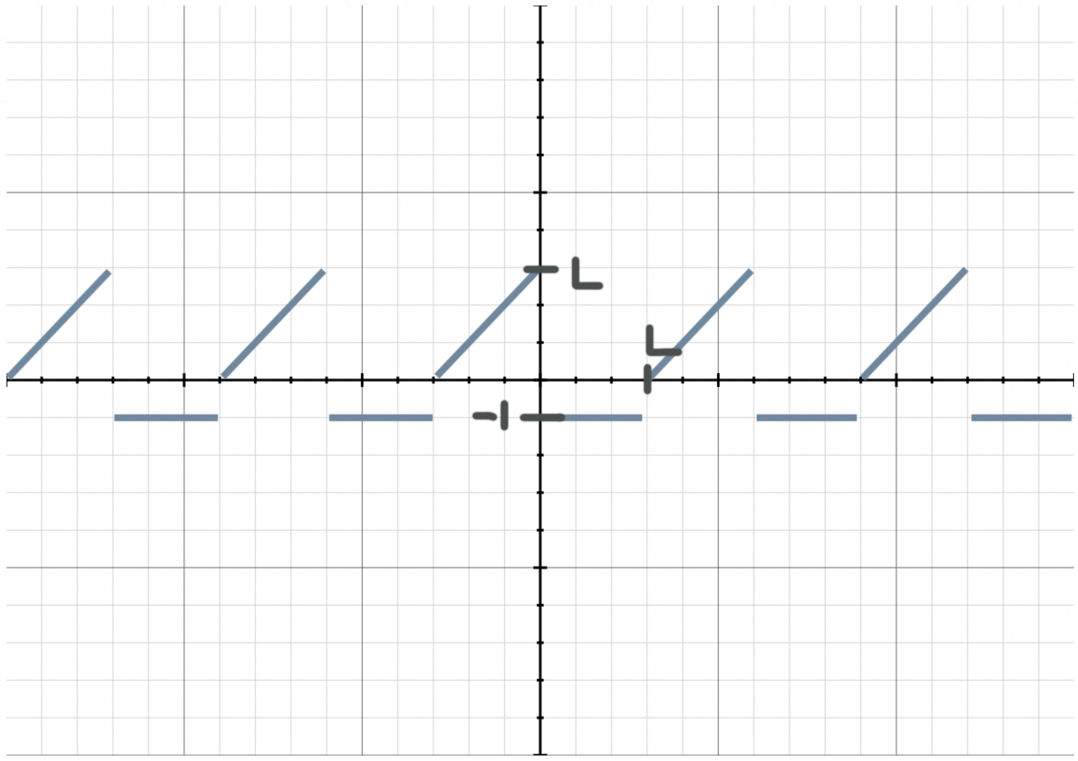
$$f(x) = \frac{L-2}{4} + \frac{L}{\pi^2} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1 - L}{n} \sin\left(\frac{n\pi x}{L}\right)$$

Solution:

A sketch of the function $f(x)$ is



The sketch of its periodic extension, $g(x)$, is



From the graph of the periodic extension, we can see that $g(x)$ has a jump discontinuity at $x = 0$, within the interval $-L \leq x \leq L$, as well as jump discontinuities at $-L$ and L.

Because the function is piecewise smooth on the individual intervals $-L \leq x \leq 0$ and $0 \leq x \leq L$, the function $f(x)$ and the periodic extension $g(x)$

are both continuous there, so the Fourier series representation will converge to the periodic extension $g(x)$, and therefore also to the original function $f(x)$. Because of the jump discontinuity at $x = 0$, the Fourier series representation at that point will converge to

$$\frac{\lim_{x \to 0^-} g(x) + \lim_{x \to 0^+} g(x)}{2}$$

$$\frac{L + (-1)}{2}$$

$$\frac{L - 1}{2}$$

At $x = -L$, the Fourier series representation will converge to

$$\frac{\lim_{x \to -L^-} g(x) + \lim_{x \to -L^+} g(x)}{2}$$

$$\frac{-1 + 0}{2}$$

$$-\frac{1}{2}$$

and at $x = L$ the Fourier series representation will converge to

$$\frac{\lim_{x \to L^-} g(x) + \lim_{x \to L^+} g(x)}{2}$$

$$\frac{-1 + 0}{2}$$

$$-\frac{1}{2}$$

The Fourier series representation converges to $f(x)$, since $f(x)$ is piecewise smooth and has a finite number of jump discontinuities on any interval with finite endpoints.

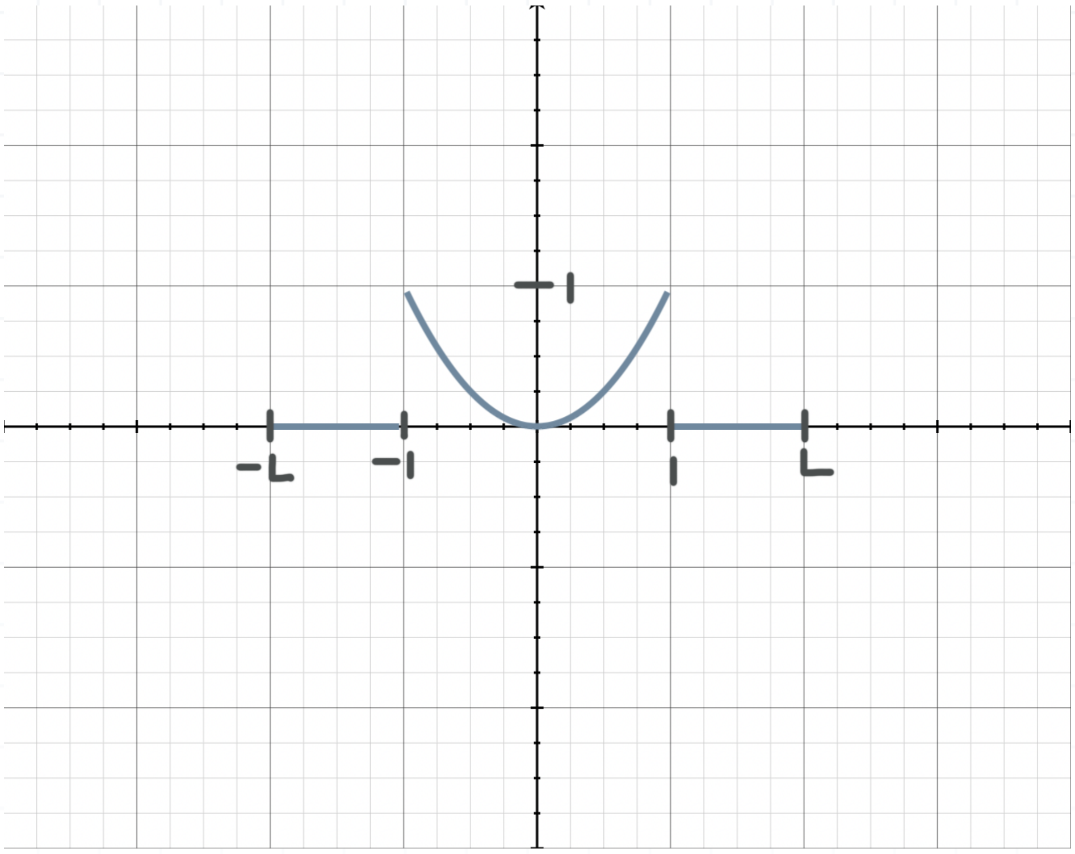
5. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$, assuming $L > 1$.

$$f(x) = \begin{cases} x^2 & |x| < 1 \\ 0 & 1 \le |x| \le L \end{cases}$$

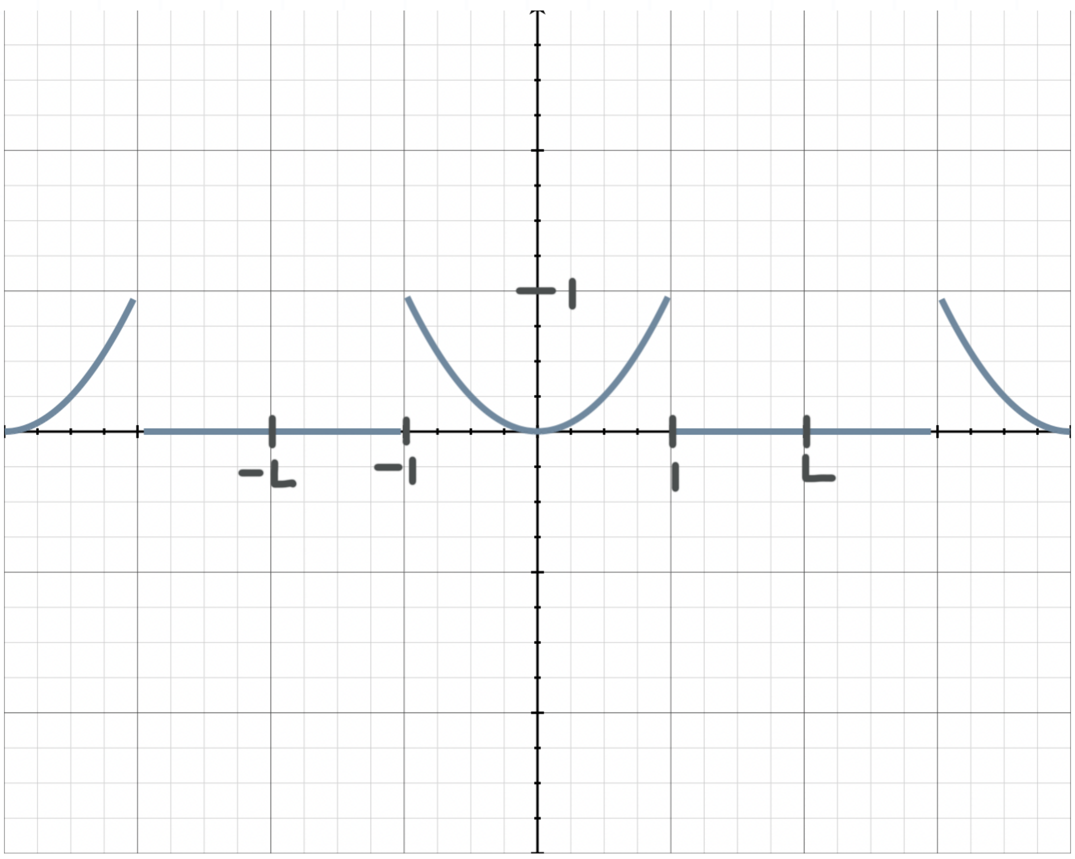
$$f(x) = \frac{1}{3L} + \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[(n^2\pi^2 - 2L^2)\sin\left(\frac{n\pi}{L}\right) + n\pi L\cos\left(\frac{n\pi}{L}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

Solution:

A sketch of the function $f(x)$ is



The sketch of its periodic extension, $g(x)$, is



From the graph of the periodic extension, we can see that $g(x)$ has a jump discontinuity at $x = \pm 1$, within the interval $-L \leq x \leq L$.

Because the function is piecewise smooth on the individual intervals $-L \le x \le -1$, $-1 \le x \le 1$, and $1 \le x \le L$, the function $f(x)$ and the periodic extension $g(x)$ are both continuous there, so the Fourier series representation will converge to the periodic extension $g(x)$, and therefore also to the original function $f(x)$. Because of the jump discontinuities at $x = \pm 1$, the Fourier series representation at $x = 1$ will converge to

$$\frac{\lim_{x\to 1^-} g(x) + \lim_{x\to 1^+} g(x)}{2}$$

$$\frac{1+0}{2}$$

$$\frac{1}{2}$$

and the Fourier series representation at $x = -1$ will converge to

$$\frac{\lim_{x\to -1^-} g(x) + \lim_{x\to -1^+} g(x)}{2}$$

$$\frac{0+1}{2}$$

$$\frac{1}{2}$$

The Fourier series representation converges to $f(x)$, since $f(x)$ is piecewise smooth and has a finite number of jump discontinuities on any interval with finite endpoints.

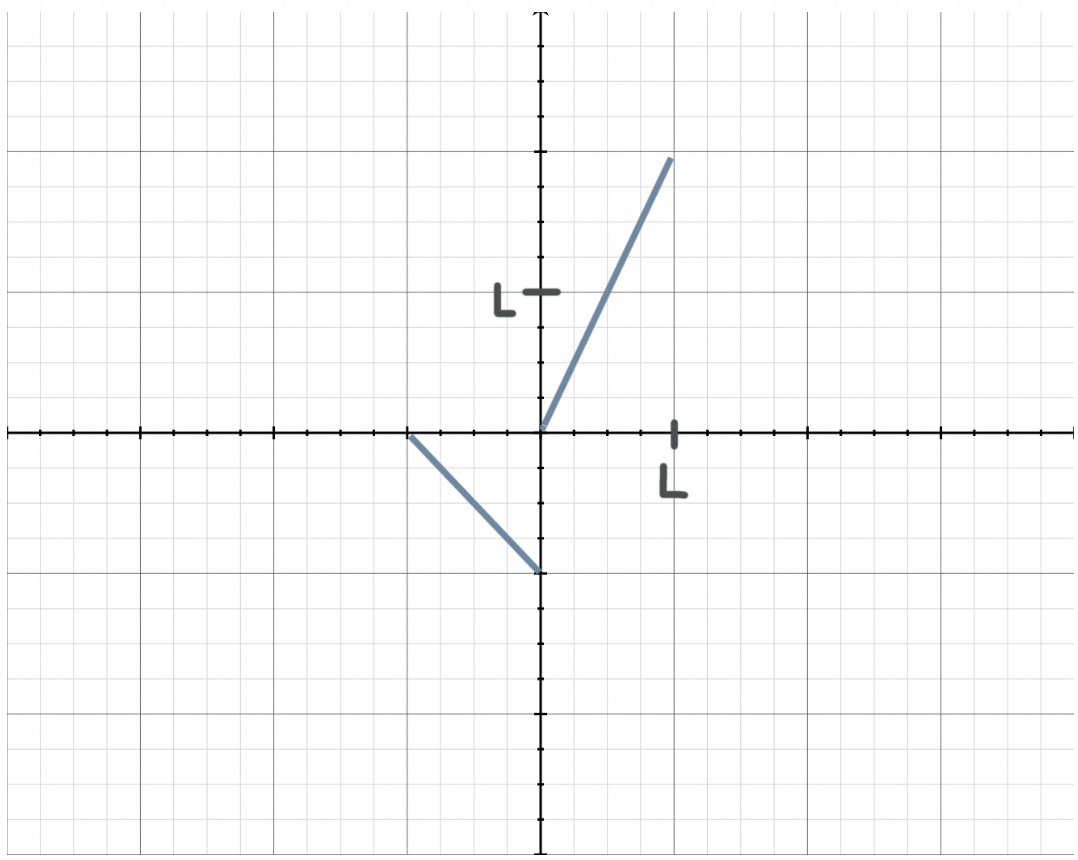
■ 6. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

$$f(x) = \begin{cases} Lx & 0 \le x \le L \\ -x - L & -L \le x < 0 \end{cases}$$

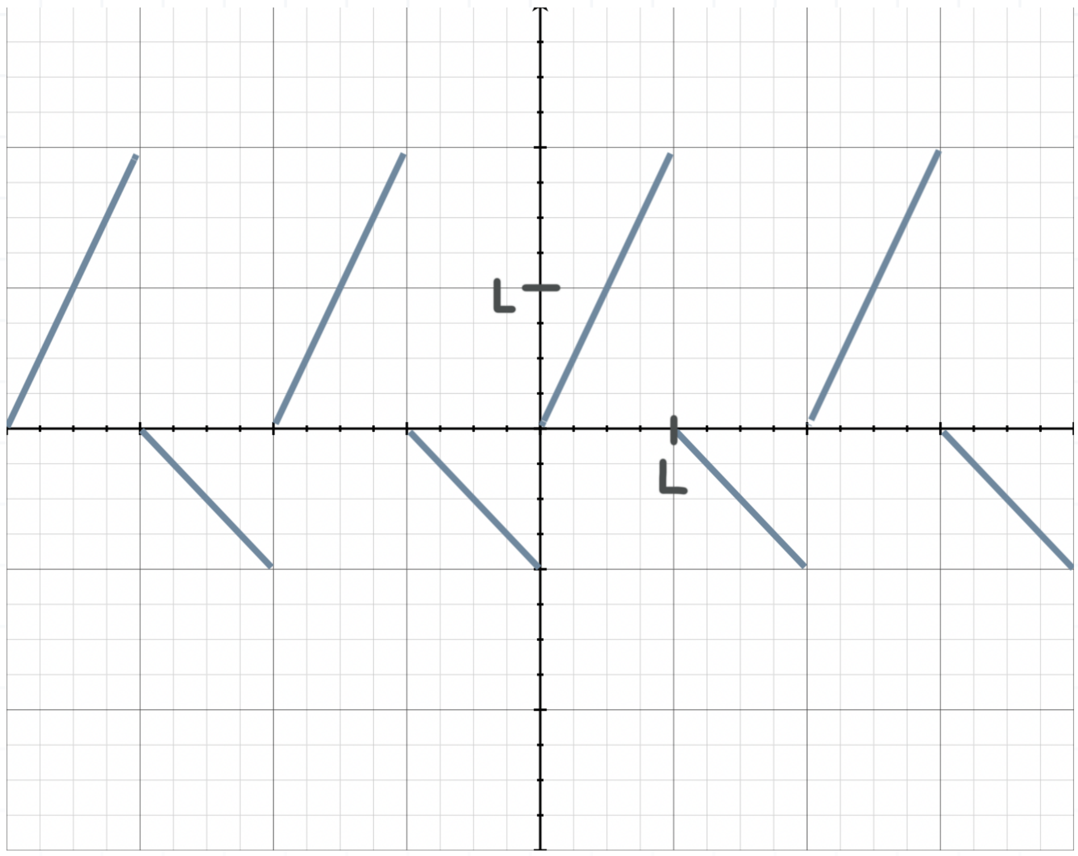
$$f(x) = \frac{L^2 - L}{4} + \frac{L}{\pi}\sum_{n=1}^{\infty} \frac{1 - L(-1)^n}{n} \sin\left(\frac{n\pi x}{L}\right) + \frac{L(L+1)}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

Solution:

A sketch of the function $f(x)$ is



The sketch of its periodic extension, $g(x)$, is



From the graph of the periodic extension, we can see that $g(x)$ has a jump discontinuity at $x = 0$, within the interval $-L \leq x \leq L$, as well as jump discontinuities at $-L$ and L.

Because the function is piecewise smooth on the individual intervals $-L \leq x \leq 0$ and $0 \leq x \leq L$, the function $f(x)$ and the periodic extension $g(x)$ are both continuous there, so the Fourier series representation will converge to the periodic extension $g(x)$, and therefore also to the original function $f(x)$. Because of the jump discontinuity at $x = 0$, the Fourier series representation at that point will converge to

$$\frac{\lim_{x \to 0^-} g(x) + \lim_{x \to 0^+} g(x)}{2}$$

$$\frac{-L + 0}{2}$$

$$-\frac{L}{2}$$

At $x = -L$, the Fourier series representation will converge to

$$\frac{\lim_{x\to -L^-} g(x) + \lim_{x\to -L^+} g(x)}{2}$$

$$\frac{L^2+0}{2}$$

$$\frac{L^2}{2}$$

and at $x = L$ the Fourier series representation will converge to

$$\frac{\lim_{x\to L^-} g(x) + \lim_{x\to L^+} g(x)}{2}$$

$$\frac{L^2+0}{2}$$

$$\frac{L^2}{2}$$

The Fourier series representation converges to $f(x)$, since $f(x)$ is piecewise smooth and has a finite number of jump discontinuities on any interval with finite endpoints.

FOURIER COSINE SERIES

1. Find the Fourier cosine series representation of the function $f(x) = 3x + 3$ on $0 \leq x \leq L$.

Solution:

For A_0, we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\int_0^L 3x + 3\ dx$$

$$A_0 = \frac{1}{L}\left(\frac{3}{2}x^2 + 3x\Big|_0^L\right)$$

$$A_0 = \frac{3}{2}L + 3$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L (3x + 3)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 3x + 3$, $du = 3\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{L}\left((3x+3)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{3L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{2}{L}\left(0 + \frac{3L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$$

$$A_n = \frac{6L}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$A_n = \frac{6L}{n^2\pi^2}((-1)^n - 1)$$

Then the Fourier cosine series representation is

$$f(x) = \frac{3}{2}L + 3 + \sum_{n=1}^{\infty}\frac{6L}{n^2\pi^2}((-1)^n - 1)\cos\left(\frac{n\pi x}{L}\right)$$

■ 2. Find the Fourier cosine series representation of the function $f(x) = 3x^2 + 2x$ on $0 \le x \le L$.

Solution:

For A_0, we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\int_0^L 3x^2 + 2x\ dx$$

$$A_0 = \frac{1}{L}\left(x^3 + x^2\Big|_0^L\right)$$

$$A_0 = L^2 + L$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L (3x^2 + 2x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 3x^2 + 2x$, $du = 6x + 2\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{L}\left((3x^2 + 2x)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{L}{n\pi}\int_0^L (6x + 2)\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u = 6x + 2$, $du = 6\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = \frac{2}{L}\left(0 - \frac{L}{n\pi}\left(-(6x + 2)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{6L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\ dx\right)\right)$$

$$A_n = -\frac{2}{n\pi}\left(-(6x+2)\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{12}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = -\frac{2}{n\pi}\left(-\frac{6L^2+2L}{n\pi}(-1)^n + \frac{2L}{n\pi} + \frac{12L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$$

$$A_n = -\frac{2}{n\pi}\left(-\frac{6L^2+2L}{n\pi}(-1)^n + \frac{2L}{n\pi} + 0\right)$$

$$A_n = \frac{12L^2+4L}{(n\pi)^2}(-1)^n - \frac{4L}{(n\pi)^2}$$

Then the Fourier cosine series representation is

$$f(x) = L^2 + L + \sum_{n=1}^{\infty}\left(\frac{12L^2+4L}{(n\pi)^2}(-1)^n - \frac{4L}{(n\pi)^2}\right)\cos\left(\frac{n\pi x}{L}\right)$$

■ 3. Find the Fourier cosine series representation of the function $f(x) = x^3$ function on $0 \le x \le \pi$.

Solution:

For A_0, we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{\pi} \int_0^\pi f(x)\ dx$$

$$A_0 = \frac{1}{\pi} \int_0^\pi x^3\ dx$$

$$A_0 = \frac{1}{\pi} \cdot \frac{x^4}{4} \Big|_0^\pi$$

$$A_0 = \frac{\pi^3}{4}$$

For A_n, we get

$$A_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx)\ dx$$

$$A_n = \frac{2}{\pi} \int_0^\pi x^3 \cos(nx)\ dx$$

Use integration by parts with $u = x^3$, $du = 3x^2\ dx$, $dv = \cos(nx)\ dx$, and $v = (1/n)\sin(nx)$.

$$A_n = \frac{2}{\pi} \left(\frac{x^3}{n} \sin(nx) \Big|_0^\pi - \frac{3}{n} \int_0^\pi x^2 \sin(nx)\ dx \right)$$

$$A_n = \frac{2}{\pi} \left(0 - \frac{3}{n} \int_0^\pi x^2 \sin(nx)\ dx \right)$$

Use integration by parts with $u=x^2$, $du=2x\ dx$, $dv=\sin(nx)\ dx$, and $v=-(1/n)\cos(nx)$.

$$A_n=-\frac{6}{n\pi}\left(-\frac{x^2}{n}\cos(nx)\Big|_0^\pi+\frac{2}{n}\int_0^\pi x\cos(nx)\ dx\right)$$

$$A_n=-\frac{6}{n\pi}\left(-\frac{\pi^2}{n}(-1)^n+\frac{2}{n}\int_0^\pi x\cos(nx)\ dx\right)$$

Use integration by parts with $u=x$, $du=dx$, $dv=\cos(nx)\ dx$, and $v=(1/n)\sin(nx)$.

$$A_n=\frac{6\pi^2(-1)^n}{n^2\pi}-\frac{12}{n^2\pi}\left(\frac{x}{n}\sin(nx)\Big|_0^\pi-\frac{1}{n}\int_0^\pi \sin(nx)\ dx\right)$$

$$A_n=\frac{6\pi(-1)^n}{n^2}-\frac{12}{n^2\pi}\left(0-\frac{1}{n}\int_0^\pi \sin(nx)\ dx\right)$$

$$A_n=\frac{6\pi(-1)^n}{n^2}+\frac{12}{n^3\pi}\left(-\frac{1}{n}\cos(nx)\right)\Big|_0^\pi$$

$$A_n=\frac{6\pi}{n^2}(-1)^n-\frac{12}{n^4\pi}((-1)^n-1)$$

Then the Fourier cosine series representation is

$$f(x)=\frac{\pi^3}{4}+\sum_{n=1}^{\infty}\left(\frac{6\pi}{n^2}(-1)^n+\frac{12}{n^4\pi}(1-(-1)^n)\right)\cos(nx)$$

■ 4. Find the Fourier cosine series representation of the function $f(x) = e^x$ on $0 \le x \le L$.

Solution:

For A_0, we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\int_0^L e^x\ dx$$

$$A_0 = \frac{1}{L}e^x\Big|_0^L$$

$$A_0 = \frac{1}{L}(e^L - 1)$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L e^x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = e^x$, $du = e^x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n=\frac{2}{L}\left(\frac{e^xL}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L-\frac{L}{n\pi}\int_0^L e^x\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n=\frac{2}{L}\left(0-\frac{L}{n\pi}\int_0^L e^x\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u=e^x$, $du=e^x\ dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$.

$$A_n=-\frac{2}{n\pi}\left(-\frac{e^xL}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L+\frac{L}{n\pi}\int_0^L e^x\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n=-\frac{2}{n\pi}\left(-\frac{e^LL}{n\pi}(-1)^n+\frac{L}{n\pi}+\frac{L}{n\pi}\cdot\frac{LA_n}{2}\right)$$

$$A_n=\frac{2Le^L(-1)^n}{n^2\pi^2}-\frac{2L}{n^2\pi^2}-\frac{L^2A_n}{n^2\pi^2}$$

$$A_n\left(1+\frac{L^2}{n^2\pi^2}\right)=\frac{2L}{n^2\pi^2}(e^L(-1)^n-1)$$

$$A_n=\frac{2L(e^L(-1)^n-1)}{n^2\pi^2+L^2}$$

Then the Fourier cosine series representation is

$$f(x)=\frac{1}{L}(e^L-1)+\sum_{n=1}^{\infty}\frac{2L(e^L(-1)^n-1)}{n^2\pi^2+L^2}\cos\left(\frac{n\pi x}{L}\right)$$

■ 5. Find the Fourier cosine series representation of the function $f(x) = x \cos x$ on $0 \le x \le 2\pi$.

Solution:

For A_0, we get

$$A_0 = \frac{1}{L} \int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\ dx$$

$$A_0 = \frac{1}{2\pi} \int_0^{2\pi} x \cos x\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos x\ dx$, and $v = \sin x$.

$$A_0 = \frac{1}{2\pi} \left(x \sin x \Big|_0^{2\pi} - \int_0^{2\pi} \sin x\ dx \right)$$

$$A_0 = \frac{1}{2\pi} \left(0 - \int_0^{2\pi} \sin x\ dx \right)$$

$$A_0 = \frac{1}{2\pi} \cos x \Big|_0^{2\pi} = 0$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{2\pi}\int_0^{2\pi} x\cos x\cos\left(\frac{nx}{2}\right)\ dx$$

Use the trig identity

$$\cos A\cos B = \frac{1}{2}[\cos(A-B)+\cos(A+B)]$$

to get

$$A_n = \frac{1}{2\pi}\int_0^{2\pi} x\left(\cos\left(1+\frac{n}{2}\right)x+\cos\left(1-\frac{n}{2}\right)x\right)\ dx$$

Use integration by parts with $u=x$, $du=dx$, and

$$dv = \left(\cos\left(1+\frac{n}{2}\right)x+\cos\left(1-\frac{n}{2}\right)x\right)\ dx$$

$$v = \frac{2}{2+n}\sin\left(1+\frac{n}{2}\right)x+\frac{2}{2-n}\sin\left(1-\frac{n}{2}\right)x$$

Then we get

$$A_n = \frac{1}{2\pi}\left[x\left(\frac{2}{2+n}\sin\left(1+\frac{n}{2}\right)x+\frac{2}{2-n}\sin\left(1-\frac{n}{2}\right)x\right)\Bigg|_0^{2\pi}\right.$$

$$\left.-\int_0^{2\pi}\frac{2}{2+n}\sin\left(1+\frac{n}{2}\right)x+\frac{2}{2-n}\sin\left(1-\frac{n}{2}\right)x\ dx\right]$$

$$A_n = \frac{1}{2\pi}\left[0 + \left(\left(\frac{2}{2+n}\right)^2 \cos\left(1+\frac{n}{2}\right)x + \left(\frac{2}{2-n}\right)^2 \cos\left(1-\frac{n}{2}\right)x\right)\Big|_0^{2\pi}\right]$$

$$A_n = \frac{4}{2\pi}\left(\frac{1}{(2+n)^2}(\cos(2\pi+\pi n)-1) + \frac{1}{(2-n)^2}(\cos(2\pi-\pi n)-1)\right)$$

$$A_n = \frac{2}{\pi}\left(\frac{1}{(2+n)^2}(\cos \pi n - 1) + \frac{1}{(2-n)^2}(\cos \pi n - 1)\right)$$

$$A_n = \frac{2}{\pi}\cdot\frac{2n^2+8}{(2+n)^2(2-n)^2}((-1)^n-1)$$

$$A_n = \frac{4(n^2+4)((-1)^n-1)}{\pi(n^2-4)^2} \text{ with } n \neq 2$$

We need to find A_2, so

$$A_2 = \frac{1}{\pi}\int_0^{2\pi} x\cos^2 x \ dx$$

Use integration by parts with $u = x$, $du = dx$, and

$$dv = \cos^2 x \ dx = \frac{1+\cos(2x)}{2} \ dx$$

$$v = \frac{x}{2} + \frac{1}{4}\sin(2x)$$

Then we get

$$A_2 = \frac{1}{\pi}\left(\left(\frac{x^2}{2} + \frac{1}{4}x\sin(2x)\right)\Big|_0^{2\pi} - \int_0^{2\pi}\left(\frac{x}{2}+\frac{1}{4}\sin(2x)\right) \ dx\right)$$

$$A_2 = \frac{1}{\pi}\left(2\pi^2 + 0 - \left(\frac{x^2}{4} + \frac{1}{8}\cos(2x)\right)\Big|_0^{2\pi}\right)$$

$$A_2 = \frac{1}{\pi}(2\pi^2 - \pi^2)$$

$$A_2 = \pi$$

Then A_1 is

$$A_1 = \frac{4(1^2 + 4)((-1)^1 - 1)}{\pi(1^2 - 4)^2}$$

$$A_1 = -\frac{40}{9\pi}$$

Then the Fourier cosine series representation is

$$f(x) = -\frac{40}{9\pi}\cos\left(\frac{x}{2}\right) + \pi\cos x + \sum_{n=3}^{\infty}\frac{4(n^2 + 4)((-1)^n - 1)}{\pi(n^2 - 4)^2}\cos\left(\frac{nx}{2}\right)$$

■ 6. Find the Fourier cosine series representation of the function $f(x) = \sin(3x + 5)$ on $0 \le x \le \pi$.

Solution:

For A_0, we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \sin(3x+5)\ dx$$

$$A_0 = -\frac{1}{3\pi}\cos(3x+5)\Big|_0^\pi$$

$$A_0 = -\frac{1}{3\pi}(\cos(3\pi+5) - \cos 5)$$

$$A_0 = -\frac{1}{3\pi}((\cos 3\pi \cos 5 - \sin 3\pi \sin 5) - \cos 5)$$

$$A_0 = -\frac{1}{3\pi}(-\cos 5 - \cos 5)$$

$$A_0 = \frac{2\cos 5}{3\pi}$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)\ dx$$

$$A_n = \frac{2}{\pi}\int_0^\pi \sin(3x+5)\cos(nx)\ dx$$

Use the trigonometric identity

$$\sin A \cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

we get

$$A_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}[\sin((n+3)x+5) + \sin((3-n)x+5)]\ dx$$

$$A_n = -\frac{1}{\pi}\left[\frac{1}{n+3}\cos((n+3)x+5) + \frac{1}{3-n}\cos((3-n)x+5)\right]\Bigg|_0^{\pi}$$

$$A_n = -\frac{1}{\pi}\left[\frac{1}{n+3}(\cos((n+3)\pi+5) - \cos 5) + \frac{1}{3-n}(\cos((3-n)\pi+5) - \cos 5)\right]$$

Use the trigonometric identity

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

we get

$$A_n = -\frac{1}{\pi}\Bigg[\frac{1}{n+3}(\cos(\pi(n+3))\cos 5 - \sin(\pi(n+3))\sin 5 - \cos 5)$$

$$+\frac{1}{3-n}(\cos(\pi(3-n))\cos 5 - \sin(\pi(3-n))\sin 5 - \cos 5)\Bigg]$$

$$A_n = -\frac{1}{\pi}\Bigg[\frac{1}{n+3}(\cos(\pi(n+3))\cos 5 + 0 - \cos 5)$$

$$+\frac{1}{3-n}(\cos(\pi(3-n))\cos 5 + 0 - \cos 5)\Bigg]$$

$$A_n = -\frac{1}{\pi}\left[\frac{1}{n+3}(-\cos(n\pi)\cos 5 - \cos 5) + \frac{1}{3-n}(-\cos(n\pi)\cos 5 - \cos 5)\right]$$

$$A_n = \frac{1}{\pi}\left(\cos 5((-1)^n + 1)\left(\frac{1}{n+3} + \frac{1}{3-n}\right)\right)$$

$$A_n = \frac{6\cos 5((-1)^n + 1)}{\pi(9 - n^2)} \text{ with } n \neq 3$$

We need to find A_1, A_2, and A_3. We get

$$A_1 = \frac{6\cos 5((-1)^1 + 1)}{\pi(9 - 1^2)} = 0$$

$$A_2 = \frac{6\cos 5((-1)^2 + 1)}{\pi(9 - 2^2)} = \frac{12\cos 5}{5\pi}$$

For A_3, we find

$$A_3 = \frac{2}{\pi}\int_0^{\pi} \sin(3x + 5)\cos(3x)\ dx$$

Use the trigonometric identity

$$\sin A \cos B = \frac{1}{2}[\sin(A + B) + \sin(A - B)]$$

to get

$$A_3 = \frac{1}{\pi}\int_0^{\pi} \sin(6x + 5) + \sin 5\ dx$$

$$A_3 = \frac{1}{\pi}\left(\frac{1}{6}\cos(6x+5)\Big|_0^{\pi} + (\sin 5)x\Big|_0^{\pi}\right)$$

$$A_3 = \frac{1}{\pi}\left(\frac{1}{6}\cos(6\pi+5) - \frac{1}{6}\cos 5 + \pi\sin 5\right)$$

$$A_3 = \frac{1}{\pi}\left(\frac{1}{6}\cos 5 - \frac{1}{6}\cos 5 + \pi\sin 5\right)$$

$$A_3 = \frac{1}{\pi}\sin 5 \cdot \pi = \sin 5$$

Then the Fourier cosine series representation is

$$f(x) = \frac{12\cos 5}{5\pi}\cos(2x) + \sin 5\cos(3x) + \sum_{n=4}^{\infty}\frac{6\cos 5((-1)^n+1)}{\pi(9-n^2)}\cos(nx)$$

FOURIER SINE SERIES

■ 1. Find the Fourier sine series representation of the function $f(x) = 2x^3 - 3x$ on $-L \le x \le L$.

Solution:

Since $f(-x) = -f(x)$, the function is odd.

$$f(-x) = 2(-x)^3 - 3(-x)$$

$$f(-x) = -(2x^3 - 3x)$$

For B_n, we get

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L (2x^3 - 3x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{4}{L}\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{6}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^3$, $du = 3x^2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the first integral and $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the second integral.

$$B_n = \left[\frac{4}{L} \cdot \frac{-L}{n\pi} x^3 \cos\left(\frac{n\pi x}{L}\right) - \frac{4}{L} \cdot \frac{-L}{n\pi} \int 3x^2 \cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left. - \frac{6}{L} \cdot \frac{-L}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \frac{6}{L} \cdot \frac{-L}{n\pi} \int \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-4}{n\pi} x^3 \cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \frac{12}{n\pi} \int x^2 \cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left. - \frac{6}{n\pi} \int \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = \left[\frac{-4}{n\pi} x^3 \cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi} x \cos\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. + \frac{12}{n\pi}\left(\frac{L}{n\pi} x^2 \sin\left(\frac{n\pi x}{L}\right) - \frac{2L}{n\pi} \int x \sin\left(\frac{n\pi x}{L}\right)\ dx\right) - \frac{6L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-4}{n\pi} x^3 \cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \frac{12L}{(n\pi)^2} x^2 \sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. - \frac{24L}{(n\pi)^2} \int x \sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{6L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the second integral.

$$B_n = \left[\frac{-4}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) + \frac{12L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. -\frac{24L}{(n\pi)^2}\left(\frac{-L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) - \frac{-L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right) - \frac{6L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-4}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) + \frac{12L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. +\frac{24L^2}{(n\pi)^3}x\cos\left(\frac{n\pi x}{L}\right) - \frac{24L^2}{(n\pi)^3}\int\cos\left(\frac{n\pi x}{L}\right)\ dx - \frac{6L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-4}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) + \frac{12L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. +\frac{24L^2}{(n\pi)^3}x\cos\left(\frac{n\pi x}{L}\right) - \frac{24L^3}{(n\pi)^4}\sin\left(\frac{n\pi x}{L}\right) - \frac{6L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \frac{-4}{n\pi}L^3(-1)^n + \frac{6}{n\pi}L(-1)^n + \frac{24L^3}{(n\pi)^3}L(-1)^n$$

$$B_n = \frac{(-4L^3 + 6L)}{n\pi}(-1)^n + \frac{24L^3}{(n\pi)^3}(-1)^n$$

Then the Fourier sine series representation of $f(x) = 2x^3 - 3x$ on $-L \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \left(\frac{-4L^3 + 6L}{n\pi}(-1)^n + \frac{24L^3}{(n\pi)^3}(-1)^n\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{2L}{\pi^3} \sum_{n=1}^{\infty} ((-2L^2 + 3)\pi^2 n^2 + 12L^2)\frac{(-1)^n}{n^3} \sin\left(\frac{n\pi x}{L}\right)$$

■ 2. Find the Fourier sine series representation of the function $f(x) = -x^3$ on $-L \le x \le L$.

Solution:

Since $f(-x) = -f(x)$, the function is odd.

$$f(-x) = -(-x)^3$$

$$f(-x) = -(-x^3)$$

For B_n, we get

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

$$B_n = \frac{2}{L}\int_0^L (-x^3)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

$$B_n = -\frac{2}{L}\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^3$, $du = 3x^2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \left[\frac{-2}{L}\cdot\frac{-L}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) - \frac{-2}{L}\cdot\frac{-L}{n\pi}\int 3x^2\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \left[\frac{2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) - \frac{6}{n\pi}\int x^2\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = \left[\frac{2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) - \frac{6}{n\pi}\cdot\frac{L}{n\pi}x^2\sin\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi}\cdot\frac{L}{n\pi}\cdot\int 2x\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \left[\frac{2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) - \frac{6L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right) + \frac{12L}{(n\pi)^2}\int x\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \Bigg[\frac{2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) - \frac{6L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right)$$

$$+\frac{12L}{(n\pi)^2}\cdot\frac{-L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) - \frac{12L}{(n\pi)^2}\cdot\frac{-L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\Bigg]\Bigg|_0^L$$

$$B_n = \left[\frac{2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) - \frac{6L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left.-\frac{12L^2}{(n\pi)^3}x\cos\left(\frac{n\pi x}{L}\right) + \frac{12L^3}{(n\pi)^4}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

Substitute $\cos(n\pi) = (-1)^n$.

$$B_n = \frac{2}{n\pi}L^3(-1)^n - \frac{12L^3}{(n\pi)^3}(-1)^n$$

Then the Fourier sine series representation of $f(x) = -x^3$ on $-L \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty}\left(\frac{2}{n\pi}L^3 - \frac{12L^3}{(n\pi)^3}\right)(-1)^n\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{2L^3}{\pi^3}\sum_{n=1}^{\infty}(\pi^2n^2 - 6)\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right)$$

■ 3. Find the Fourier sine series representation of the function $f(x) = x^3 - 2x$ on $-L \le x \le L$.

Solution:

Since $f(-x) = -f(x)$, the function is odd.

$$f(-x) = (-x)^3 - 2(-x)$$

$$f(-x) = -(x^3 - 2x)$$

For B_n, we get

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L (x^3 - 2x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L x^3\sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{4}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^3$, $du = 3x^2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the first integral and $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the second integral.

$$B_n = \left[\frac{2}{L}\cdot\frac{-L}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) - \frac{2}{L}\cdot\frac{-L}{n\pi}\int 3x^2\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left.-\frac{4}{L}\cdot\frac{-L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) + \frac{4}{L}\cdot\frac{-L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi}\int x^2\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left.+\frac{4}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) - \frac{4}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = \left[\frac{-2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6}{n\pi}\cdot\frac{L}{n\pi}x^2\sin\left(\frac{n\pi x}{L}\right) - \frac{6}{n\pi}\cdot\frac{L}{n\pi}\int 2x\sin\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left. + \frac{4}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) - \frac{4}{n\pi}\cdot\frac{-L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right) - \frac{6L}{(n\pi)^2}\int x\sin\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left. + \frac{4}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) + \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \left[\frac{-2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right) - \frac{12L}{(n\pi)^2}\cdot\frac{-L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. + \frac{12L}{(n\pi)^2}\cdot\frac{-L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{4}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) + \frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-2}{n\pi}x^3\cos\left(\frac{n\pi x}{L}\right) + \frac{6L}{(n\pi)^2}x^2\sin\left(\frac{n\pi x}{L}\right) + \frac{12L^2}{(n\pi)^3}x\cos\left(\frac{n\pi x}{L}\right)\right.$$

$$-\frac{12L^3}{(n\pi)^4}\sin\left(\frac{n\pi x}{L}\right)+\frac{4}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)+\frac{4L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg]\Bigg|_0^L$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$ and $\cos(\pm n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = \frac{-2L^3}{n\pi}(-1)^n + \frac{12L^3}{(n\pi)^3}(-1)^n + \frac{4L}{n\pi}(-1)^n$$

$$B_n = \frac{2L(-1)^n}{\pi^3 n^3}(6L^2 - L^2\pi^2 n^2 + 2\pi^2 n^2)$$

Then the Fourier series representation of $f(x) = x^3 - 2x$ on $-L \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} \frac{2L(-1)^n}{\pi^3 n^3}(6L^2 - L^2\pi^2 n^2 + 2\pi^2 n^2)\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{2L}{\pi^3}\sum_{n=1}^{\infty}(6L^2 - L^2\pi^2 n^2 + 2\pi^2 n^2)\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right)$$

■ 4. Find the Fourier sine series representation of the function $f(x) = x - x^2$ on $0 \le x \le L$.

Solution:

The function isn't odd, so we'll find its odd extension on $-L \le x \le L$.

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x - x^2 & 0 \le x \le L \\ -((-x) - (-x)^2) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x - x^2 & 0 \le x \le L \\ x + x^2 & -L \le x < 0 \end{cases}$$

Because the odd extension is an odd function, we can find its Fourier sine series representation, starting with calculating B_n.

$$B_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L (x - x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{1}{L}\int_0^L x^2\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the first integral and $u = x^2$, $du = 2x\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the second integral.

$$B_n = \left[\frac{1}{L}\cdot\frac{-L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) - \frac{1}{L}\cdot\frac{-L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left. -\frac{1}{L}\cdot\frac{-L}{n\pi}x^2\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{L}\cdot\frac{-L}{n\pi}\int 2x\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \left[\frac{-1}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) \right.$$

$$\left. + \frac{1}{n\pi} x^2 \cos\left(\frac{n\pi x}{L}\right) - \frac{2}{n\pi} \int x \cos\left(\frac{n\pi x}{L}\right)\ dx \right] \Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = \left[\frac{-1}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{n\pi} x^2 \cos\left(\frac{n\pi x}{L}\right) \right.$$

$$\left. - \frac{2L}{(n\pi)^2} x \sin\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2} \int \sin\left(\frac{n\pi x}{L}\right)\ dx \right] \Bigg|_0^L$$

$$B_n = \left[\frac{-1}{n\pi} x \cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) + \frac{1}{n\pi} x^2 \cos\left(\frac{n\pi x}{L}\right) \right.$$

$$\left. - \frac{2L}{(n\pi)^2} x \sin\left(\frac{n\pi x}{L}\right) - \frac{2L^2}{(n\pi)^3} \cos\left(\frac{n\pi x}{L}\right) \right] \Bigg|_0^L$$

Evaluate the interval using $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$.

$$B_n = \frac{-L}{n\pi}(-1)^n + \frac{L^2}{n\pi}(-1)^n - \frac{2L^2}{(n\pi)^3}(-1)^n + \frac{2L^2}{(n\pi)^3}$$

$$B_n = \frac{L}{\pi^3 n^3}((L\pi^2 n^2 - \pi^2 n^2 - 2L)(-1)^n + 2L)$$

Then the Fourier sine series representation of $f(x) = x - x^2$ on $0 \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L}{\pi^3} \sum_{n=1}^{\infty} ((L\pi^2 n^2 - \pi^2 n^2 - 2L)(-1)^n + 2L)\frac{1}{n^3} \sin\left(\frac{n\pi x}{L}\right)$$

■ 5. Find the Fourier sine series representation of $f(x) = x^2 - Lx$ on $0 \le x \le L$.

Solution:

The function isn't odd, so we'll find its odd extension on $-L \le x \le L$.

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 - Lx & 0 \le x \le L \\ -((-x)^2 - (-x)L) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 - Lx & 0 \le x \le L \\ -x^2 - Lx & -L \le x < 0 \end{cases}$$

Because the odd extension is an odd function, we can find its Fourier sine series representation, starting with calculating B_n.

$$B_n = \frac{1}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \, dx$$

$$B_n = \frac{1}{L}\int_0^L (x^2 - Lx)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{1}{L}\int_0^L Lx \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the second integral and $u = x^2$, $du = 2x\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ in the first integral.

$$B_n = \left[\frac{1}{L}\cdot\frac{-L}{n\pi}x^2\cos\left(\frac{n\pi x}{L}\right) - \frac{1}{L}\cdot\frac{-L}{n\pi}\int 2x\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left.-\frac{-L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) + \frac{-L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \left[-\frac{1}{n\pi}x^2\cos\left(\frac{n\pi x}{L}\right) + \frac{2}{n\pi}\int x\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left.+\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right) - \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = \left[-\frac{1}{n\pi}x^2\cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2}x\sin\left(\frac{n\pi x}{L}\right) - \frac{2L}{(n\pi)^2}\int\sin\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$+\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)-\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg]\Bigg|_0^L$$

$$B_n=\Bigg[-\frac{1}{n\pi}x^2\cos\left(\frac{n\pi x}{L}\right)+\frac{2L}{(n\pi)^2}x\sin\left(\frac{n\pi x}{L}\right)+\frac{2L^2}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right)$$

$$+\frac{L}{n\pi}x\cos\left(\frac{n\pi x}{L}\right)-\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg]\Bigg|_0^L$$

Evaluate the interval using $\sin(n\pi)=0$ and $\cos(n\pi)=(-1)^n$.

$$B_n=\frac{-L^2}{n\pi}(-1)^n+\frac{2L^2}{(n\pi)^3}(-1)^n+\frac{L^2}{n\pi}(-1)^n$$

$$B_n=\frac{2L^2}{(n\pi)^3}(-1)^n$$

Then the Fourier sine series representation of $f(x)=x^2-Lx$ on $0\le x\le L$ is

$$f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\frac{2L^2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin\left(\frac{n\pi x}{L}\right)$$

■ 6. Find the Fourier sine series representation of $f(x)=e^{L-x}+5$ on $0\le x\le L$.

Solution:

The function isn't odd, so we'll find its odd extension on $-L \le x \le L$.

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} e^{L-x} + 5 & 0 \le x \le L \\ -(e^{L-(-x)} + 5) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} e^{L-x} + 5 & 0 \le x \le L \\ -e^{L+x} - 5 & -L \le x < 0 \end{cases}$$

Because the odd extension is an odd function, we can find its Fourier sine series representation, starting with calculating B_n.

$$B_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L (e^{L-x} + 5)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_0^L 5\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = e^{L-x}$, $du = -e^{L-x}\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \left[\frac{1}{L}\cdot\frac{-L}{n\pi}e^{L-x}\cos\left(\frac{n\pi x}{L}\right) - \frac{1}{L}\cdot\frac{-L}{n\pi}\int(-e^{L-x})\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$+\frac{5}{L}\cdot\frac{-L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg]\Bigg|_0^L$$

$$B_n=\Bigg[-\frac{1}{n\pi}e^{L-x}\cos\left(\frac{n\pi x}{L}\right)-\frac{1}{n\pi}\int e^{L-x}\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$-\frac{5}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg]\Bigg|_0^L$$

Use integration by parts with $u=e^{L-x}$, $du=-e^{L-x}\ dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$.

$$B_n=\Bigg[-\frac{1}{n\pi}e^{L-x}\cos\left(\frac{n\pi x}{L}\right)-\frac{1}{n\pi}\cdot\frac{L}{n\pi}e^{L-x}\sin\left(\frac{n\pi x}{L}\right)$$

$$+\frac{1}{n\pi}\cdot\frac{L}{n\pi}\int(-e^{L-x})\sin\left(\frac{n\pi x}{L}\right)\ dx-\frac{5}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg]\Bigg|_0^L$$

$$B_n=\Bigg[-\frac{1}{n\pi}e^{L-x}\cos\left(\frac{n\pi x}{L}\right)-\frac{L}{(n\pi)^2}e^{L-x}\sin\left(\frac{n\pi x}{L}\right)$$

$$-\frac{L}{(n\pi)^2}\int e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx-\frac{5}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg]\Bigg|_0^L$$

So we get

$$\frac{1}{L}\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx+\frac{1}{L}\int_0^L 5\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$= \left[-\frac{1}{n\pi}e^{L-x}\cos\left(\frac{n\pi x}{L}\right) - \frac{L}{(n\pi)^2}e^{L-x}\sin\left(\frac{n\pi x}{L}\right) \right.$$

$$\left. -\frac{L}{(n\pi)^2}\int e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{5}{n\pi}\cos\left(\frac{n\pi x}{L}\right) \right] \Bigg|_0^L$$

$$\frac{1}{L}\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx + \left[\frac{5}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$= \left[-\frac{1}{n\pi}e^{L-x}\cos\left(\frac{n\pi x}{L}\right) - \frac{L}{(n\pi)^2}e^{L-x}\sin\left(\frac{n\pi x}{L}\right) \right.$$

$$\left. -\frac{L}{(n\pi)^2}\int e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{5}{n\pi}\cos\left(\frac{n\pi x}{L}\right) \right] \Bigg|_0^L$$

Evaluate the interval using $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$.

$$\frac{1}{L}\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{5}{n\pi}(-1)^n - \frac{5}{n\pi}$$

$$= -\frac{1}{n\pi}(-1)^n + \frac{1}{n\pi}e^L - \frac{L}{(n\pi)^2}\int e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{5}{n\pi}(-1)^n + \frac{5}{n\pi}$$

$$\left(\frac{1}{L} + \frac{L}{(n\pi)^2}\right)\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$= -\frac{1}{n\pi}(-1)^n + \frac{1}{n\pi}e^L - \frac{5}{n\pi}(-1)^n + \frac{5}{n\pi} - \frac{5}{n\pi}(-1)^n + \frac{5}{n\pi}$$

$$\left(\frac{(n\pi)^2 + L^2}{L(n\pi)^2}\right)\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx = -\frac{1}{n\pi}(-1)^n + \frac{1}{n\pi}e^L - \frac{10}{n\pi}(-1)^n + \frac{10}{n\pi}$$

$$\left(\frac{(n\pi)^2+L^2}{L(n\pi)^2}\right)\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx=-\frac{11}{n\pi}(-1)^n+\frac{1}{n\pi}e^L+\frac{10}{n\pi}$$

$$\int_0^L e^{L-x}\sin\left(\frac{n\pi x}{L}\right)\ dx=\left(\frac{Ln\pi}{(n\pi)^2+L^2}\right)(-11(-1)^n+e^L+10)$$

Then

$$B_n=-\frac{1}{n\pi}(-1)^n+\frac{1}{n\pi}e^L-\frac{L}{(n\pi)^2}\left[\left(\frac{Ln\pi}{(n\pi)^2+L^2}\right)(-11(-1)^n+e^L+10)\right]$$

$$-\frac{5}{n\pi}(-1)^n+\frac{5}{n\pi}$$

$$B_n=-\frac{1}{n\pi}(-1)^n+\frac{1}{n\pi}e^L-\left(\frac{L^2}{(n\pi)^3+L^2n\pi}\right)(-11(-1)^n+e^L+10)$$

$$-\frac{5}{n\pi}(-1)^n+\frac{5}{n\pi}$$

$$B_n=-\frac{6}{n\pi}(-1)^n+\frac{1}{n\pi}e^L+\frac{5}{n\pi}-\left(\frac{L^2}{(n\pi)^3+L^2n\pi}\right)(-11(-1)^n+e^L+10)$$

Then the Fourier sine series representation of $f(x)=e^{L-x}+5$ on $0\le x\le L$ is

$$f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\sum_{n=1}^{\infty}\left(-\frac{6}{n\pi}(-1)^n+\frac{1}{n\pi}e^L+\frac{5}{n\pi}\right.$$

$$-\left(\frac{L^2}{(n\pi)^3 + L^2 n\pi}\right)[-11(-1)^n + e^L + 10]\bigg)\sin\left(\frac{n\pi x}{L}\right)$$

COSINE AND SINE SERIES OF PIECEWISE FUNCTIONS

■ 1. Find the Fourier cosine series representation of the piecewise function on $0 \le x \le L$.

$$f(x) = \begin{cases} 0 & 0 \le x \le \frac{L}{2} \\ x^2 & \frac{L}{2} < x \le L \end{cases}$$

Solution:

The function is even so $B_n = 0$. For A_0, we get

$$A_0 = \frac{1}{L} \int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L} \left(\int_0^{\frac{L}{2}} 0\ dx + \int_{\frac{L}{2}}^L x^2\ dx \right)$$

$$A_0 = \frac{1}{L} \cdot \frac{x^3}{3} \Big|_{\frac{L}{2}}^L$$

$$A_0 = \frac{1}{L} \left(\frac{L^3}{3} - \frac{L^3}{24} \right)$$

$$A_0 = \frac{7L^2}{24}$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\left(\int_0^{\frac{L}{2}} 0\cos\left(\frac{n\pi x}{L}\right)\ dx + \int_{\frac{L}{2}}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{L}\left(x^2\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L} - \frac{2L}{n\pi}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = \frac{2}{L}\left(0 - \frac{L^3}{4n\pi}\sin\left(\frac{n\pi}{2}\right)\right) - \frac{4}{n\pi}\left(-x\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L} + \frac{L}{n\pi}\int_{\frac{L}{2}}^{L}\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = -\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{n^2\pi^2}\left((-1)^n - \frac{1}{2}\cos\left(\frac{n\pi}{2}\right)\right) - \frac{4L^2}{n^3\pi^3}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L}$$

$$A_n = -\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{n^2\pi^2}\left((-1)^n - \frac{1}{2}\cos\left(\frac{n\pi}{2}\right)\right) + \frac{4L^2}{n^3\pi^3}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \left(\frac{4L^2}{n^3\pi^3} - \frac{L^2}{2n\pi}\right)\sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{n^2\pi^2}\left((-1)^n - \frac{1}{2}\cos\left(\frac{n\pi}{2}\right)\right)$$

Then the Fourier cosine series representation of the piecewise function on $0 \le x \le L$ is

$$f(x) = \frac{7L^2}{24}$$

$$+\sum_{n=1}^{\infty}\left(\left(\frac{4L^2}{n^3\pi^3}-\frac{L^2}{2n\pi}\right)\sin\left(\frac{n\pi}{2}\right)+\frac{4L^2}{n^2\pi^2}\left((-1)^n-\frac{1}{2}\cos\left(\frac{n\pi}{2}\right)\right)\right)\cos\left(\frac{n\pi x}{L}\right)$$

■ 2. Find the Fourier sine series representation of the piecewise function on $0 \le x \le 2\pi$.

$$f(x) = \begin{cases} 2x & 0 \le x \le \pi \\ 4\pi - 2x & \pi < x \le 2\pi \end{cases}$$

Solution:

The odd extension $g(x)$ of the piecewise function $f(x)$ will be

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} 2x & 0 \le x \le \pi \\ 4\pi - 2x & \pi < x \le 2\pi \\ 2x & -\pi \le x < 0 \\ -4\pi - 2x & -2\pi < x < -\pi \end{cases}$$

To find the Fourier sine series, we'll calculate B_n.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{\pi}\left(\int_0^\pi 2x\sin\left(\frac{nx}{2}\right)\ dx + \int_\pi^{2\pi}(4\pi - 2x)\sin\left(\frac{nx}{2}\right)\ dx\right)$$

Use integration by parts with $u = 2x$, $du = 2\ dx$, $dv = \sin(nx/2)\ dx$, and $v = -(2/n)\cos(nx/2)$ in the first integral and with $u = 4\pi - 2x$, $du = -2\ dx$, $dv = \sin(nx/2)\ dx$, and $v = -(2/n)\cos(nx/2)$ in the second integral.

$$B_n = \frac{1}{\pi}\Bigg(-\frac{4x}{n}\cos\left(\frac{nx}{2}\right)\Big|_0^\pi + \frac{4}{n}\int_0^\pi \cos\left(\frac{nx}{2}\right)\ dx$$

$$+(4\pi - 2x)\left(-\frac{2}{n}\cos\left(\frac{nx}{2}\right)\right)\Big|_\pi^{2\pi} - \frac{4}{n}\int_\pi^{2\pi}\cos\left(\frac{nx}{2}\right)\ dx\Bigg)$$

$$B_n = \frac{1}{\pi}\Bigg(-\frac{4\pi}{n}\cos\left(\frac{n\pi}{2}\right) + 0 + \frac{8}{n^2}\sin\left(\frac{nx}{2}\right)\Big|_0^\pi$$

$$+0 + \frac{4\pi}{n}\cos\left(\frac{n\pi}{2}\right) - \frac{8}{n^2}\sin\left(\frac{nx}{2}\right)\Big|_\pi^{2\pi}\Bigg)$$

$$B_n = \frac{1}{\pi}\left(\frac{8}{n^2}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n^2}\sin\left(\frac{n\pi}{2}\right)\right)$$

$$B_n = \frac{16}{n^2\pi}\sin\left(\frac{n\pi}{2}\right)$$

Then the Fourier cosine series representation of the piecewise function on $0 \le x \le 2\pi$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{16}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{L}\right)$$

■ 3. Find the Fourier cosine series representation of the piecewise function on $0 \le x \le \pi$.

$$f(x) = \begin{cases} x^2 & 0 \le x \le \frac{\pi}{2} \\ x & \frac{\pi}{2} < x \le \pi \end{cases}$$

Solution:

The even extension $g(x)$ of the piecewise function $f(x)$ will be

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x & \frac{\pi}{2} < x \le \pi \\ x^2 & 0 \le x \le \frac{\pi}{2} \\ x^2 & -\frac{\pi}{2} \le x < 0 \\ -x & -\pi \le x < -\frac{\pi}{2} \end{cases}$$

To find the Fourier cosine series, we'll calculate A_0 and A_n. For A_0, we get

$$A_0 = \frac{1}{\pi} \int_0^{\pi} f(x)\ dx$$

$$A_0 = \frac{1}{\pi} \left(\int_0^{\frac{\pi}{2}} x^2\ dx + \int_{\frac{\pi}{2}}^{\pi} x\ dx \right)$$

$$A_0 = \frac{1}{\pi} \left(\frac{x^3}{3} \Bigg|_0^{\frac{\pi}{2}} + \frac{x^2}{2} \Bigg|_{\frac{\pi}{2}}^{\pi} \right)$$

$$A_0 = \frac{1}{\pi} \left(\frac{\pi^3}{24} + \frac{\pi^2}{2} - \frac{\pi^2}{8} \right)$$

$$A_0 = \frac{\pi^2}{24} + \frac{3\pi}{8}$$

For A_n, we get

$$A_n = \frac{2}{L} \int_0^{L} f(x) \cos \left(\frac{n\pi x}{L} \right)\ dx$$

$$A_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos(nx)\ dx$$

$$A_n = \frac{2}{\pi} \left(\int_0^{\frac{\pi}{2}} x^2 \cos(nx)\ dx + \int_{\frac{\pi}{2}}^{\pi} x \cos(nx)\ dx \right)$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(nx)\ dx$, and $v = (1/n)\sin(nx)$ in the first integral and with $u = x$, $du = dx$, $dv = \cos(nx)\ dx$, and $v = (1/n)\sin(nx)$ in the second integral.

$$A_n = \frac{2}{\pi}\left(\frac{x^2}{n}\sin(nx)\Big|_0^{\frac{\pi}{2}} - \frac{2}{n}\int_0^{\frac{\pi}{2}} x\sin(nx)\ dx + \frac{x}{n}\sin(nx)\Big|_{\frac{\pi}{2}}^{\pi} - \frac{1}{n}\int_{\frac{\pi}{2}}^{\pi} \sin(nx)\ dx\right)$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(nx)\ dx$, and $v = -(1/n)\cos(nx)$.

$$A_n = \frac{2}{\pi}\left(\frac{\pi^2}{4n}\sin\left(\frac{n\pi}{2}\right) - 0 - \frac{2}{n}\left(-\frac{x}{n}\cos(nx) + \frac{1}{n^2}\sin(nx)\right)\Big|_0^{\frac{\pi}{2}}\right.$$

$$\left. +0 - \frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}\cos(nx)\Big|_{\frac{\pi}{2}}^{\pi}\right)$$

$$A_n = \frac{2}{\pi}\left[\frac{\pi^2}{4n}\sin\left(\frac{n\pi}{2}\right) + \frac{\pi}{n^2}\cos\left(\frac{n\pi}{2}\right) - \frac{2}{n^3}\sin\left(\frac{n\pi}{2}\right)\right.$$

$$\left. -\frac{\pi}{2n}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}(-1)^n - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right]$$

$$A_n = \frac{2}{\pi}\left[\left(\frac{\pi^2}{4n} - \frac{2}{n^3} - \frac{\pi}{2n}\right)\sin\left(\frac{n\pi}{2}\right) + \frac{\pi - 1}{n^2}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n^2}(-1)^n\right]$$

$$A_n = \left(\frac{\pi}{2n} - \frac{4}{n^3\pi} - \frac{1}{n}\right)\sin\left(\frac{n\pi}{2}\right) + \frac{2(\pi - 1)}{n^2\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{n^2\pi}(-1)^n$$

Then the Fourier cosine series representation of the piecewise function on $0 \le x \le \pi$ is

$$f(x) = \frac{\pi^2}{24} + \frac{3\pi}{8}$$

$$+\sum_{n=1}^{\infty}\left(\left(\frac{\pi}{2n}-\frac{4}{n^3\pi}-\frac{1}{n}\right)\sin\left(\frac{n\pi}{2}\right)+\frac{2(\pi-1)}{n^2\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{2}{n^2\pi}(-1)^n\right)\cos(nx)$$

■ 4. Find the Fourier sine series representation of the piecewise function on $0 \le x \le \pi$.

$$f(x)=\begin{cases}\sin x & 0 \le x \le \frac{\pi}{2}\\ \sin 2x & \frac{\pi}{2} < x \le \pi\end{cases}$$

Solution:

The function is odd so $A_0 = A_n = 0$. For B_n, we get

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n=\frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\ dx$$

$$B_n=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\sin x\sin(nx)\ dx+\int_{\frac{\pi}{2}}^{\pi}\sin(2x)\sin(nx)\ dx\right)$$

Use the trigonometric identity

$$\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))$$

to get

$$B_n = \frac{1}{\pi}\left(\int_0^{\frac{\pi}{2}} \cos(n-1)x - \cos(n+1)x \ dx\right.$$

$$\left. + \int_{\frac{\pi}{2}}^{\pi} \cos(n-2)x - \cos(n+2)x \ dx\right)$$

$$B_n = \frac{1}{\pi}\left(\left(\frac{1}{n-1}\sin(n-1)x - \frac{1}{n+1}\sin(n+1)x\right)\Bigg|_0^{\frac{\pi}{2}}\right.$$

$$\left. + \left(\frac{1}{n-2}\sin(n-2)x - \frac{1}{n+2}\sin(n+2)x\right)\Bigg|_{\frac{\pi}{2}}^{\pi}\right)$$

$$B_n = \frac{1}{\pi}\left(\frac{1}{n-1}\sin\left(\frac{n\pi}{2} - \frac{\pi}{2}\right) - \frac{1}{n+1}\sin\left(\frac{n\pi}{2} + \frac{\pi}{2}\right)\right.$$

$$\left. +0 - \frac{1}{n-2}\sin\left(\frac{n\pi}{2} - \pi\right) + \frac{1}{n+2}\sin\left(\frac{n\pi}{2} + \pi\right)\right)$$

Use the trigonometric identities

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

to get

$$B_n = \frac{1}{\pi}\left(-\frac{1}{n-1}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n+1}\cos\left(\frac{n\pi}{2}\right)\right.$$

$$+\frac{1}{n-2}\sin\left(\frac{n\pi}{2}\right)-\frac{1}{n+2}\sin\left(\frac{n\pi}{2}\right)\Bigg)$$

$$B_n=\frac{1}{\pi}\left(-\frac{2n}{n^2-1}\cos\left(\frac{n\pi}{2}\right)+\frac{4}{n^2-4}\sin\left(\frac{n\pi}{2}\right)\right)$$ with $n\neq 1,2$

$$B_1=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\sin x\sin x\ dx+\int_{\frac{\pi}{2}}^{\pi}\sin(2x)\sin x\ dx\right)$$

Use the trigonometric identities

$$\sin^2 x=\frac{1-\cos(2x)}{2}$$

$$\sin A\sin B=\frac{1}{2}(\cos(A-B)-\cos(A+B))$$

to get

$$B_1=\frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}}\frac{1-\cos(2x)}{2}\ dx+\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi}(\cos x-\cos(3x))\ dx\right)$$

$$B_1=\frac{2}{\pi}\left(\left(\frac{x}{2}-\frac{\sin(2x)}{4}\right)\Bigg|_0^{\frac{\pi}{2}}+\frac{1}{2}\left(\sin x-\frac{1}{3}\sin(3x)\right)\Bigg|_{\frac{\pi}{2}}^{\pi}\right)$$

$$B_1=\frac{2}{\pi}\left(\frac{\pi}{4}-0+\frac{1}{2}\left(0-1-0-\frac{1}{3}\right)\right)$$

$$B_1=\frac{2}{\pi}\left(\frac{\pi}{4}-\frac{2}{3}\right)$$

$$B_1 = \frac{1}{2} - \frac{4}{3\pi}$$

and

$$B_2 = \frac{2}{\pi}\left(\int_0^{\frac{\pi}{2}} \sin x \sin(2x)\ dx + \int_{\frac{\pi}{2}}^{\pi} \sin^2(2x)\ dx\right)$$

Use the trigonometric identities

$$\sin^2 x = \frac{1 - \cos(2x)}{2}$$

$$\sin A \sin B = \frac{1}{2}(\cos(A - B) - \cos(A + B))$$

to get

$$B_2 = \frac{2}{\pi}\left(\frac{1}{2}\int_0^{\frac{\pi}{2}} \cos x - \cos(3x)\ dx + \int_{\frac{\pi}{2}}^{\pi} \frac{1 - \cos(4x)}{2}\ dx\right)$$

$$B_2 = \frac{2}{\pi}\left[\frac{1}{2}\left(\sin x - \frac{1}{3}\sin(3x)\right)\Bigg|_0^{\frac{\pi}{2}} + \left(\frac{x}{2} - \frac{\sin(4x)}{8}\right)\Bigg|_{\frac{\pi}{2}}^{\pi}\right]$$

$$B_2 = \frac{2}{\pi}\left(\frac{1}{2}\left(1 - 0 + \frac{1}{3} - 0\right) + \frac{\pi}{2} - 0 - \frac{\pi}{4} + 0\right)$$

$$B_2 = \frac{2}{\pi}\left(\frac{2}{3} + \frac{\pi}{4}\right)$$

$$B_2 = \frac{4}{3\pi} + \frac{1}{2}$$

Then the Fourier sine series representation of the piecewise function on $0 \le x \le \pi$ is

$$f(x) = \left(\frac{1}{2} - \frac{4}{3\pi}\right)\sin x + \left(\frac{1}{2} + \frac{4}{3\pi}\right)\sin(2x)$$

$$+\sum_{n=1}^{\infty} \frac{1}{\pi}\left(-\frac{2n}{n^2-1}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n^2-4}\sin\left(\frac{n\pi}{2}\right)\right)\sin(nx)$$

■ 5. Find the Fourier cosine series representation of the piecewise function on $0 \le x \le L$.

$$f(x) = \begin{cases} 3x & 0 \le x \le \frac{L}{3} \\ 3x - L & \frac{L}{3} < x \le \frac{2L}{3} \\ 3x - 2L & \frac{2L}{3} < x < L \end{cases}$$

Solution:

The even extension $g(x)$ of the piecewise function $f(x)$ will be

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} 3x - 2L & \frac{2L}{3} < x < L \\ 3x - L & \frac{L}{3} < x \le \frac{2L}{3} \\ 3x & 0 \le x \le \frac{L}{3} \\ -3x & -\frac{L}{3} \le x < 0 \\ -3x - L & -\frac{2L}{3} \le x < -\frac{L}{3} \\ -3x - 2L & -L < x < -\frac{2L}{3} \end{cases}$$

To find the Fourier cosine series, we'll calculate A_0 and A_n. For A_0, we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\left(\int_0^{\frac{L}{3}} 3x\ dx + \int_{\frac{L}{3}}^{\frac{2L}{3}} (3x - L)\ dx + \int_{\frac{2L}{3}}^{L} (3x - 2L)\ dx\right)$$

$$A_0 = \frac{1}{L}\left(\frac{3x^2}{2}\Bigg|_0^{\frac{L}{3}} + \left(\frac{3x^2}{2} - Lx\right)\Bigg|_{\frac{L}{3}}^{\frac{2L}{3}} + \left(\frac{3x^2}{2} - 2Lx\right)\Bigg|_{\frac{2L}{3}}^{L}\right)$$

$$A_0 = \frac{1}{L}\left(\frac{L^2}{6} + \frac{2L^2}{3} - \frac{2L^2}{3} - \frac{L^2}{6} + \frac{L^2}{3} + \frac{3L^2}{2} - 2L^2 - \frac{2L^2}{3} + \frac{4L^2}{3}\right)$$

$$A_0 = \frac{1}{L}\left(\frac{L^2}{2}\right)$$

$$A_0 = \frac{L}{2}$$

For A_n, we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\left(\int_0^{\frac{L}{3}} 3x\cos\left(\frac{n\pi x}{L}\right)\ dx + \int_{\frac{L}{3}}^{\frac{2L}{3}} (3x-L)\cos\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$\left.+\int_{\frac{2L}{3}}^{L} (3x-2L)\cos\left(\frac{n\pi x}{L}\right)\ dx\right)$$

Use integration by parts with $u = 3x$, $du = 3\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ in the first integral, with $u = 3x - L$, $du = 3\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ in the second integral, and with $u = 3x - 2L$, $du = 3\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ in the third integral.

$$A_n = \frac{2}{L}\left(3x\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{3}} - \int_0^{\frac{L}{3}} \frac{3L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\ dx\right.$$

$$+(3x-L)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{3}}^{\frac{2L}{3}} - \int_{\frac{L}{3}}^{\frac{2L}{3}} \frac{3L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$\left.+(3x-2L)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{2L}{3}}^{L} - \int_{\frac{2L}{3}}^{L} \frac{3L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{2}{L}\left(3x\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{3}} + (3x-L)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{3}}^{\frac{2L}{3}}\right.$$

$$+(3x-2L)\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{2L}{3}}^{L}-\int_0^L \frac{3L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\ dx\Bigg)$$

$$A_n=\frac{2}{L}\left(\frac{L^2}{n\pi}\sin\left(\frac{n\pi}{3}\right)-0+\frac{L^2}{n\pi}\sin\left(\frac{2n\pi}{3}\right)-0+\frac{L^2}{n\pi}\sin(n\pi)\right.$$

$$\left.-0+\frac{3L^2}{n^2\pi^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L\right)$$

$$A_n=\frac{2L}{n\pi}\left(\sin\left(\frac{n\pi}{3}\right)+\sin\left(\frac{2n\pi}{3}\right)\right)+\frac{6L}{n^2\pi^2}((-1)^n-1)$$

Then the Fourier cosine series representation of the piecewise function on $0\le x\le L$ is

$$f(x)=\frac{L}{2}$$

$$+\sum_{n=1}^{\infty}\left(\frac{2L}{n\pi}\left(\sin\left(\frac{n\pi}{3}\right)+\sin\left(\frac{2n\pi}{3}\right)\right)+\frac{6L}{n^2\pi^2}((-1)^n-1)\right)\cos\left(\frac{n\pi x}{L}\right)$$

■ 6. Find the Fourier sine series representation of the piecewise function on $0\le x\le 2$.

$$f(x)=\begin{cases}x^2 & 0\le x\le 1\\ x^2-1 & 1<x\le 2\end{cases}$$

Solution:

The odd extension $g(x)$ of the piecewise function $f(x)$ will be

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 - 1 & 1 < x \le 2 \\ x^2 & 0 \le x \le 1 \\ -x^2 & -1 \le x < 0 \\ -x^2 + 1 & -2 \le x < -1 \end{cases}$$

To find the Fourier sine series, we'll calculate B_n.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{2}\int_0^2 f(x)\sin\left(\frac{n\pi x}{2}\right)\ dx$$

$$B_n = \int_0^1 x^2 \sin\left(\frac{n\pi x}{2}\right)\ dx + \int_1^2 (x^2 - 1)\sin\left(\frac{n\pi x}{2}\right)\ dx$$

$$B_n = \int_0^1 x^2 \sin\left(\frac{n\pi x}{2}\right)\ dx + \int_1^2 x^2 \sin\left(\frac{n\pi x}{2}\right)\ dx - \int_1^2 \sin\left(\frac{n\pi x}{2}\right)\ dx$$

$$B_n = \int_0^2 x^2 \sin\left(\frac{n\pi x}{2}\right)\ dx - \int_1^2 \sin\left(\frac{n\pi x}{2}\right)\ dx$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(2/n\pi)\cos(n\pi x/2)$.

$$B_n = -\frac{2x^2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 + \frac{4}{n\pi}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)\ dx + \frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_1^2$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/2)\ dx$, and $v = (2/n\pi)\sin(n\pi x/2)$.

$$B_n = -\frac{8}{n\pi}(-1)^n + 0 + \frac{4}{n\pi}\left(\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg|_0^2 - \frac{2}{n\pi}\int_0^2 \sin\left(\frac{n\pi x}{2}\right)\ dx\right)$$

$$+\frac{2}{n\pi}(-1)^n - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$B_n = -\frac{6}{n\pi}(-1)^n - \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{n\pi}\left(0 + \frac{4}{(n\pi)^2}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2\right)$$

$$B_n = -\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) + 3(-1)^n\right) + \frac{16}{n^3\pi^3}\cos\left(\frac{n\pi x}{2}\right)\Bigg|_0^2$$

$$B_n = -\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) + 3(-1)^n\right) + \frac{16}{n^3\pi^3}((-1)^n - 1)$$

Then the Fourier sine series representation of the piecewise function on $0 \le x \le 2$ is

$$f(x) = \sum_{n=1}^{\infty}\left(-\frac{2}{n\pi}\left(\cos\left(\frac{n\pi}{2}\right) + 3(-1)^n\right) + \frac{16}{n^3\pi^3}((-1)^n - 1)\right)\sin\left(\frac{n\pi x}{2}\right)$$

SEPARATION OF VARIABLES

■ 1. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$\frac{1}{k^2}\left(\frac{\partial^2 u}{\partial x^2}\right)+\frac{\partial u}{\partial t}=0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x,t)=v(x)w(t)$$

$$\frac{\partial u}{\partial t}=v(x)w'(t)$$

$$\frac{\partial u}{\partial x}=v'(x)w(t) \qquad \frac{\partial^2 u}{\partial x^2}=v''(x)w(t)$$

Then we'll plug the derivatives into the partial differential equation.

$$\frac{1}{k^2}v''(x)w(t)+v(x)w'(t)=0$$

$$\frac{1}{k^2}v''(x)w(t)=-v(x)w'(t)$$

$$\frac{1}{k^2}\left(\frac{d^2 v}{dx^2}\right)w(t)=-v(x)\frac{dw}{dt}$$

$$\left(\frac{1}{k^2 v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(t)}\right)\frac{dw}{dt}$$

The equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$\left(\frac{1}{k^2 v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(t)}\right)\frac{dw}{dt} = -\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{1}{k^2 v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} + \lambda k^2 v = 0$$

$$r^2 + \lambda k^2 = 0$$

$$r = \pm k\sqrt{-\lambda}$$

$$v(x) = c_1\cos(k\sqrt{\lambda}x) + c_2\sin(k\sqrt{\lambda}x)$$

and

$$-\left(\frac{1}{w(t)}\right)\frac{dw}{dt} = -\lambda$$

$$\frac{dw}{dt} = \lambda w$$

$$\ln w = \lambda t + C$$

$$w = Ce^{\lambda t}$$

Then

$$u(x,t) = c_1 e^{\lambda t}\cos(k\sqrt{\lambda}x) + c_2 e^{\lambda t}\sin(k\sqrt{\lambda}x)$$

■ 2. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$16u_{yy} + u_{xx} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x,y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = v'(x)w(y) \qquad \frac{\partial u}{\partial y} = v(x)w'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)w(y) \qquad \frac{\partial^2 u}{\partial y^2} = v(x)w''(y)$$

Then we'll plug the derivatives into the partial differential equation.

$$16v(x)w''(y) + v''(x)w(y) = 0$$

$$16v(x)w''(y) = -v''(x)w(y)$$

$$\frac{16w''(y)}{w(y)} = -\frac{v''(x)}{v(x)}$$

The equation is telling us that a function in terms of x will always be equivalent to a function in terms of y, regardless of the values of x and y that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$\frac{16w''(y)}{w(y)} = -\frac{v''(x)}{v(x)} = -\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{16}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$16\left(\frac{d^2w}{dy^2}\right) + \lambda w = 0$$

$$16r^2 + \lambda = 0$$

$$r = \pm\frac{\sqrt{-\lambda}}{4}$$

$$w(y) = c_1\cos\left(\frac{\sqrt{\lambda}}{4}y\right) + c_2\sin\left(\frac{\sqrt{\lambda}}{4}y\right)$$

and

$$-\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$v(x) = c_3 e^{\sqrt{\lambda}x} + c_4 e^{-\sqrt{\lambda}x}$$

Then

$$u(x, y) = \left(c_1 \cos\left(\frac{\sqrt{\lambda}}{4}y\right) + c_2 \sin\left(\frac{\sqrt{\lambda}}{4}y\right)\right)(c_3 e^{\sqrt{\lambda}x} + c_4 e^{-\sqrt{\lambda}x})$$

■ 3. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$9\frac{\partial^2 u}{\partial x^2} - 4\frac{\partial^2 u}{\partial y^2} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = v'(x)w(y) \qquad \frac{\partial u}{\partial y} = v(x)w'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)w(y) \qquad \frac{\partial^2 u}{\partial y^2} = v(x)w''(y)$$

Then we'll plug the derivatives into the partial differential equation.

$$9v''(x)w(y) - 4v(x)w''(y) = 0$$

$$9v''(x)w(y) = 4v(x)w''(y)$$

$$\frac{9v''(x)}{v(x)} = \frac{4w''(y)}{w(y)}$$

The equation is telling us that a function in terms of x will always be equivalent to a function in terms of y, regardless of the values of x and y that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$\frac{4w''(y)}{w(y)} = \frac{9v''(x)}{v(x)}$$

$$\left(\frac{9}{v(x)}\right)\frac{d^2v}{dx^2} = \left(\frac{4}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{9}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$9\frac{d^2v}{dx^2} + \lambda v = 0$$

$$9r^2 + \lambda = 0$$

$$r = \pm \frac{\sqrt{-\lambda}}{3}$$

$$v(x) = c_1 \cos\left(\frac{\sqrt{\lambda}}{3}x\right) + c_2 \sin\left(\frac{\sqrt{\lambda}}{3}x\right)$$

and

$$\left(\frac{4}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$4\frac{d^2w}{dy^2} + \lambda w = 0$$

$$4r^2 + \lambda = 0$$

$$r = \pm \frac{\sqrt{-\lambda}}{2}$$

$$w(y) = c_3 \cos\left(\frac{\sqrt{\lambda}}{2}y\right) + c_4 \sin\left(\frac{\sqrt{\lambda}}{2}y\right)$$

Then

$$u(x, y) = \left(c_1 \cos\left(\frac{\sqrt{\lambda}}{3}x\right) + c_2 \sin\left(\frac{\sqrt{\lambda}}{3}x\right)\right)\left(c_3 \cos\left(\frac{\sqrt{\lambda}}{2}y\right) + c_4 \sin\left(\frac{\sqrt{\lambda}}{2}y\right)\right)$$

■ 4. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$-\frac{\partial^2 u}{\partial x^2} + 25\frac{\partial^2 u}{\partial t^2} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x, t) = v(x)w(t)$$

$$\frac{\partial u}{\partial x} = v'(x)w(t) \qquad \frac{\partial u}{\partial t} = v(x)w'(t)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)w(t) \qquad \frac{\partial^2 u}{\partial t^2} = v(x)w''(t)$$

Then we'll plug the derivatives into the partial differential equation.

$$-v''(x)w(t) + 25v(x)w''(t) = 0$$

$$-v''(x)w(t) = -25v(x)w''(t)$$

$$-\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{25}{w(t)}\right)\frac{d^2w}{dt^2}$$

The equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$-\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{25}{w(t)}\right)\frac{d^2w}{dt^2} = -\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$-\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$v(x) = c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x}$$

and

$$-\left(\frac{25}{w(t)}\right)\frac{d^2w}{dt^2} = -\lambda$$

$$25\frac{d^2w}{dt^2} - \lambda w = 0$$

$$25r^2 - \lambda = 0$$

$$r = \pm\frac{\sqrt{\lambda}}{5}$$

$$w(t) = c_3e^{\frac{\sqrt{\lambda}}{5}t} + c_4e^{-\frac{\sqrt{\lambda}}{5}t}$$

Then

$$u(x,y) = (c_1e^{\sqrt{\lambda}x} + c_2e^{-\sqrt{\lambda}x})(c_3e^{\frac{\sqrt{\lambda}}{5}t} + c_4e^{-\frac{\sqrt{\lambda}}{5}t})$$

■ 5. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$-\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t}=0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x,t)=v(x)w(t)$$

$$\frac{\partial u}{\partial t}=v(x)w'(t)$$

$$\frac{\partial u}{\partial x}=v'(x)w(t) \qquad \frac{\partial^2 u}{\partial x^2}=v''(x)w(t)$$

Then we'll plug the derivatives into the partial differential equation.

$$-v''(x)w(t)+v(x)w'(t)=0$$

$$-v''(x)w(t)=-\,v(x)w'(t)$$

$$-\frac{d^2v}{dx^2}w(t)=-\,v(x)\frac{dw}{dt}$$

$$-\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2}=-\left(\frac{1}{w(t)}\right)\frac{dw}{dt}$$

The equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$-\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(t)}\right)\frac{dw}{dt} = -\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$-\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$v(x) = c_1e^{\sqrt{\lambda}x} + c_2e^{\sqrt{\lambda}x}$$

and

$$-\left(\frac{1}{w(t)}\right)\frac{dw}{dt} = -\lambda$$

$$\frac{dw}{dt} = \lambda w$$

$$\ln w = \lambda t + C$$

$$w = Ce^{\lambda t}$$

Then

$$u(x, t) = Ce^{\lambda t}(c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x})$$

$$u(x, t) = c_1 e^{\sqrt{\lambda}x+\lambda t} + c_2 e^{-\sqrt{\lambda}x+\lambda t}$$

■ 6. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial y^2} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = v'(x)w(y) \qquad \frac{\partial u}{\partial y} = v(x)w'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)w(y) \qquad \frac{\partial^2 u}{\partial y^2} = v(x)w''(y)$$

Then we'll plug the derivatives into the partial differential equation.

$$v'(x)w(y) + v(x)w''(y) = 0$$

$$v'(x)w(y) = -v(x)w''(y)$$

$$\frac{dv}{dx}w(y) = -v(x)\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v(x)}\right)\frac{dv}{dx} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2}$$

The equation is telling us that a function in terms of x will always be equivalent to a function in terms of y, regardless of the values of x and y that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$\left(\frac{1}{v(x)}\right)\frac{dv}{dx} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{1}{v(x)}\right)\frac{dv}{dx} = -\lambda$$

$$\frac{dv}{dx} = -\lambda v$$

$$\ln v = -\lambda x + C$$

$$v = Ce^{-\lambda x}$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

$$r^2 - \lambda = 0$$

$$r = \pm\sqrt{\lambda}$$

$$w(y) = c_1e^{\sqrt{\lambda}y} + c_2e^{-\sqrt{\lambda}y}$$

Then

$$u(x, y) = Ce^{\lambda x}(c_1e^{\sqrt{\lambda}y} + c_2e^{-\sqrt{\lambda}y})$$

$$u(x, y) = c_1e^{\sqrt{\lambda}y-\lambda x} + c_2e^{-\sqrt{\lambda}y-\lambda x}$$

BOUNDARY VALUE PROBLEMS

■ 1. Solve the boundary value problem, if $y(0) = 3$ and $y'(0) = -4$.

$$y'' + 4y = 0$$

Solution:

Solving the associated characteristic equation gives

$$r^2 + 4 = 0$$

$$r^2 = -4$$

$$r = \pm 2i$$

which means the general solution to the homogeneous equation is

$$y(t) = c_1 \cos(2t) + c_2 \sin(2t)$$

Substituting $y(0) = 3$ into the general solution gives

$$3 = c_1 \cos(2 \cdot 0) + c_2 \sin(2 \cdot 0)$$

$$3 = c_1(1) + c_2(0)$$

$$c_1 = 3$$

Then we'll have

$$y(t) = 3\cos(2t) + c_2\sin(2t)$$

Now find $y'(t)$ and substitute $y'(0) = -4$.

$$y'(t) = -6\sin(2t) + 2c_2\cos(2t)$$

$$-4 = -6\sin(2 \cdot 0) + 2c_2\cos(2 \cdot 0)$$

$$-4 = 2c_2$$

$$c_2 = -2$$

Then the solution to the boundary value problem is

$$y(t) = 3\cos(2t) - 2\sin(2t)$$

■ 2. Solve the boundary value problem, if $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 5$.

$$y''' - y'' + 9y' - 9y = 0$$

Solution:

Solving the associated characteristic equation gives

$$r^3 - r^2 + 9r - 9 = 0$$

$$r^2(r - 1) + 9(r - 1) = 0$$

$$(r - 1)(r^2 + 9) = 0$$

$$r = 1, \pm 3i$$

which means the general solution to the homogeneous equation is

$$y(t) = c_1 e^t + c_2 \cos(3t) + c_3 \sin(3t)$$

$$y'(t) = c_1 e^t - 3c_2 \sin(3t) + 3c_3 \cos(3t)$$

$$y''(t) = c_1 e^t - 9c_2 \cos(3t) - 9c_3 \sin(3t)$$

Substituting $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 5$ into the general solution gives

$$c_1 + c_2 = 1$$

$$c_1 + 3c_3 = 0$$

$$c_1 - 9c_2 = 5$$

Solving this system gives $c_1 = 7/5$, $c_2 = -2/5$, and $c_3 = -7/15$. Then the solution to the boundary value problem is

$$y(t) = \frac{7}{5}e^t - \frac{2}{5}\cos(3t) - \frac{7}{15}\sin(3t)$$

■ 3. Solve the boundary value problem, if $u(\pi/3,t) = 2e^{3t}$, $(\partial u/\partial x)(\pi/3,t) = -3e^{3t}$, and $-\lambda = -9$.

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial u}{\partial t} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x,t) = v(x)w(t)$$

$$\frac{\partial u}{\partial t} = v(x)w'(t)$$

$$\frac{\partial u}{\partial x} = v'(x)w(t) \qquad \frac{\partial^2 u}{\partial x^2} = v''(x)w(t)$$

Then we'll plug the derivatives into the partial differential equation.

$$v''(x)w(t) + 3v(x)w'(t) = 0$$

$$v''(x)w(t) = -3v(x)w'(t)$$

$$\frac{d^2v}{dx^2}w(t) = -3v(x)\frac{dw}{dt}$$

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{3}{w(t)}\right)\frac{dw}{dt}$$

The equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{3}{w(t)}\right)\frac{dw}{dt} = -9$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -9$$

$$\frac{d^2v}{dx^2} + 9v = 0$$

$$r^2 + 9 = 0$$

$$r = \pm 3i$$

$$v(x) = c_1\cos(3x) + c_2\sin 93x)$$

and

$$-\left(\frac{3}{w(t)}\right)\frac{dw}{dt} = -9$$

$$\left(\frac{1}{w(t)}\right)\frac{dw}{dt} = 3$$

$$\ln w = 3t + C$$

$$w = Ce^{3t}$$

Then

$$u(x,t) = c_1e^{3t}\cos(3x) + c_2e^{3t}\sin(3x)$$

$$\frac{\partial u}{\partial x}(x,t) = -3c_1e^{3t}\sin(3x) + 3c_2e^{3t}\cos(3x)$$

Substituting $u = (\pi/3,t) = 2e^{3t}$ and $(\partial u/\partial x)(\pi/3,t) = -3e^{3t}$ into the general solution gives

$$-c_1e^{3t} = 2e^{3t}$$

$$-3c_2e^{3t} = -3e^{3t}$$

Then we'll have

$$c_1 = -2$$

$$c_2 = 1$$

Then the solution to the boundary value problem is

$$u(x,t) = -2e^{3t}\cos(3x) + e^{3t}\sin(3x)$$

■ 4. Solve the boundary value problem, if $u(\pi,t) = 0$, $u(x,0) = -2\cos(x/2)$, $(\partial u/\partial x)(\pi,t) = 2e^{-t} - e^{t}$, $(\partial u/\partial t)(x,0) = 6\cos(x/2)$, and $-\lambda = -1$.

$$4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x,t) = v(x)w(t)$$

$$\frac{\partial u}{\partial x} = v'(x)w(t) \qquad \frac{\partial^2 u}{\partial x^2} = v''(x)w(t)$$

$$\frac{\partial u}{\partial t} = v(x)w'(t) \qquad \frac{\partial^2 u}{\partial t^2} = v(x)w''(t)$$

Then we'll plug the derivatives into the partial differential equation.

$$4v''(x)w(t) + v(x)w''(t) = 0$$

$$4v''(x)w(t) = -v(x)w''(t)$$

$$4\frac{d^2v}{dx^2}w(t) = -v(x)\frac{d^2w}{dt^2}$$

$$\left(\frac{4}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(t)}\right)\frac{d^2w}{dt^2}$$

The equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$\left(\frac{4}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(t)}\right)\frac{d^2w}{dt^2} = -1$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{4}{v(x)}\right)\frac{d^2v}{dx^2} = -1$$

$$4\frac{d^2v}{dx^2} + v = 0$$

$$4r^2 + 1 = 0$$

$$r = \pm \frac{1}{2}i$$

$$v(x) = c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)$$

and

$$-\left(\frac{1}{w(t)}\right)\frac{d^2w}{dt^2} = -1$$

$$\frac{d^2w}{dt^2} - w = 0$$

$$r^2 - 1 = 0$$

$$r = \pm 1$$

$$w(t) = c_3e^t + c_4e^{-t}$$

Then

$$u(x,t) = \left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)(c_3e^t + c_4e^{-t})$$

$$\frac{\partial u}{\partial x}(x,t) = \left(-\frac{1}{2}c_1 \sin\left(\frac{x}{2}\right) + \frac{1}{2}c_2 \cos\left(\frac{x}{2}\right)\right)(c_3e^t + c_4e^{-t})$$

$$\frac{\partial u}{\partial t}(x,t) = \left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)(c_3e^t - c_4e^{-t})$$

Substituting $u(\pi,t) = 0$, $u(x,0) = -2\cos(x/2)$, $(\partial u/\partial x)(\pi,t) = 2e^{-t} - e^t$, and $(\partial u/\partial t)(x,0) = 6\cos(x/2)$ into the general solution gives

$$c_2(c_3e^t + c_4e^{-t}) = 0$$

$$\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)(c_3 + c_4) = -2\cos\left(\frac{x}{2}\right)$$

$$-\frac{1}{2}c_1(c_3e^t + c_4e^{-t}) = 2e^{-t} - e^t$$

$$\left(c_1 \cos\left(\frac{x}{2}\right) + c_2 \sin\left(\frac{x}{2}\right)\right)(c_3e - c_4) = 6\cos\left(\frac{x}{2}\right)$$

If we choose $c_1 = 2$, we find $c_1 = 2$, $c_2 = 0$, $c_3 = 1$, and $c_4 = -2$. Then the solution to the boundary value problem is

$$u(x,t) = 2\cos\left(\frac{x}{2}\right)(e^t - 2e^{-t})$$

■ 5. Solve the boundary value problem, if $u(0,y) = e^{8y} + 3e^{-8y}$, $u(x,0) = 8e^{2x} - 4e^{-2x}$, $(\partial u/\partial x)(0,y) = 6e^{8y} + 18e^{-8y}$, $(\partial u/\partial y)(x,0) = 16e^{-2x} - 32e^{2x}$, and $-\lambda = -64$.

$$\frac{\partial^2 u}{\partial y^2} - 16\frac{\partial^2 u}{\partial x^2} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x,y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = v'(x)w(y) \qquad \frac{\partial^2 u}{\partial x^2} = v''(x)w(y)$$

$$\frac{\partial u}{\partial y} = v(x)w'(y) \qquad \frac{\partial^2 u}{\partial y^2} = v(x)w''(y)$$

Then we'll plug the derivatives into the partial differential equation.

$$-16v''(x)w(y) + v(x)w''(y) = 0$$

$$-16v''(x)w(y) = -v(x)w''(y)$$

$$-16\frac{d^2v}{dx^2}w(y) = -v(x)\frac{d^2w}{dy^2}$$

$$-\left(\frac{16}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2}$$

The equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda = -64$.

$$-\left(\frac{16}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -64$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$-\left(\frac{16}{v(x)}\right)\frac{d^2v}{dx^2} = -64$$

$$\frac{d^2v}{dx^2} - 4v = 0$$

$$r^2 - 4 = 0$$

$$r = \pm 2$$

$$v(x) = c_1e^{2x} + c_2e^{-2x}$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -64$$

$$\frac{d^2w}{dy^2} - 64w = 0$$

$$r^2 - 64 = 0$$

$$r = \pm 8$$

$$w(y) = c_3e^{8y} + c_4e^{-8y}$$

Then

$$u(x, y) = (c_1e^{2x} + c_2e^{-2x})(c_3e^{8y} + c_4e^{-8y})$$

$$\frac{\partial u}{\partial x}(x, y) = (2c_1e^{2x} - 2c_2e^{-2x})(c_3e^{8y} + c_4e^{-8y})$$

$$\frac{\partial u}{\partial y}(x, y) = (c_1e^{2x} + c_2e^{-2x})(8c_3e^{8y} - 8c_4e^{-8y})$$

Substituting $u(0,y) = e^{8y} + 3e^{-8y}$, $u(x,0) = 8e^{2x} - 4e^{-2x}$, $(\partial u/\partial x)(0,y) = 6e^{8y} + 18e^{-8y}$, and $(\partial u/\partial y)(x,0) = 16e^{-2x} - 32e^{2x}$ into the general solution gives

$$(c_1 + c_2)(c_3e^{8y} + c_4e^{-8y}) = e^{8y} + 3e^{-8y}$$

$$(c_1e^{2x} + c_2e^{-2x})(c_3 + c_4) = 8e^{2x} - 4e^{-2x}$$

$$(2c_1 - 2c_2)(c_3e^{8y} + c_4e^{-8y}) = 6e^{8y} + 18e^{-8y}$$

$$(c_1e^{2x} + c_2e^{-2x})(8c_3 - 8c_4) = 16e^{-2x} - 32e^{2x}$$

Then we'll have

$$(c_1 + c_2)c_3 = 1$$

$$(c_1 + c_2)c_4 = 3$$

$$2(c_1 - c_2)c_3 = 6$$

$$2(c_1 - c_2)c_4 = 18$$

If we solve this system, we find $c_1 = 2$, $c_2 = -1$, $c_3 = 1$, and $c_4 = 3$. Then the solution to the boundary value problem is

$$u(x, y) = (2e^{2x} - e^{-2x})(e^{8y} + 3e^{-8y})$$

■ 6. Solve the boundary value problem, if $u = (0,y) = 4\sin(\sqrt{\lambda}y)$.

$$3\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0$$

Solution:

We'll find the first and second derivatives of the product solution.

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = v'(x)w(y)$$

$$\frac{\partial u}{\partial y} = v(x)w'(y) \qquad \frac{\partial^2 u}{\partial y^2} = v(x)w''(y)$$

Then we'll plug the derivatives into the partial differential equation.

$$3v'(x)w(y) - v(x)w''(y) = 0$$

$$3v'(x)w(y) = v(x)w''(y)$$

$$3\frac{dv}{dx}w(y) = v(x)\frac{d^2w}{dy^2}$$

$$\left(\frac{3}{v(x)}\right)\frac{dv}{dx} = \left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2}$$

The equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. The only way this can possibly be true is if both functions are constant, so we'll set the equation equal to $-\lambda$.

$$\left(\frac{3}{v(x)}\right)\frac{dv}{dx} = \left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{3}{v(x)}\right)\frac{dv}{dx} = -\lambda$$

$$3\ln v = -\lambda x + C$$

$$v = Ce^{-\frac{\lambda}{3}x}$$

and

$$\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} + \lambda w = 0$$

$$r^2 + \lambda = 0$$

$$r = \pm\sqrt{-\lambda}$$

$$w(y) = c_1\cos(\sqrt{\lambda}y) + c_2\sin(\sqrt{\lambda}y)$$

Then

$$u(x,y) = Ce^{-\frac{\lambda}{3}x}(c_1\cos(\sqrt{\lambda}y) + c_2\sin(\sqrt{\lambda}y))$$

$$u(x,y) = c_1e^{-\frac{\lambda}{3}x}\cos(\sqrt{\lambda}y) + c_2e^{-\frac{\lambda}{3}x}\sin(\sqrt{\lambda}y)$$

Substituting $u = (0,y) = 4\sin(\sqrt{\lambda}y)$ into the general solution gives

$$c_1\cos(\sqrt{\lambda}y) + c_2\sin(\sqrt{\lambda}y) = 4\sin(\sqrt{\lambda}y)$$

We'll have $c_1 = 0$ and $c_2 = 4$, which means the solution to the boundary value problem is

$$u(x, y) = 4e^{-\frac{\lambda}{3}x}\sin(\sqrt{\lambda}y)$$

THE HEAT EQUATION

■ 1. Find a solution to the partial differential equation if $u(x,0) = \sin(\pi x) + 3\sin(\pi x/2)$ and $u(0,t) = u(6,t) = 0$.

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le 6$$

Solution:

If we start with the product solution $u(x,t) = v(x)w(t)$, then we know

$$v(x)w'(t) = 9v''(x)w(t)$$

$$\frac{w'(t)}{9w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Break the equation into two ordinary differential equations.

$$w' = -9\lambda w$$

$$v'' = -\lambda v$$

The solution to the first equation, which is the first order differential equation, is

$$w(t) = Ce^{-9\lambda t}$$

To solve the second equation, we find the associated characteristic equation

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$ and the solution is

$$v(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions $v(0) = 0$ and $v(6) = 0$ gives

$$c_1 + c_2 = 0$$

$$c_1 e^{-6\sqrt{-\lambda}} + c_2 e^{6\sqrt{-\lambda}} = 0$$

which allows use to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$, and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v(6) = c_1 + 6c_2 = 0$$

which allows us to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$, and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives us

$$v(0) = c_1 = 0$$

$$v(6) = c_1 \cos(6\sqrt{\lambda}) + c_2 \sin(6\sqrt{\lambda}) = 0$$

$$c_2 \sin(6\sqrt{\lambda}) = 0$$

We'll find a non-trivial solution which satisfies

$$\sin(6\sqrt{\lambda}) = 0$$

$$6\sqrt{\lambda} = n\pi \qquad n = 1, 2, 3, \dots$$

$$\lambda_n = \left(\frac{\pi n}{6}\right)^2$$

So the general solution for $\lambda > 0$ becomes

$$v(x) = c_2 \sin\left(\sqrt{\left(\frac{\pi n}{6}\right)^2} x\right)$$

$$v(x) = c_2 \sin\left(\frac{n\pi x}{6}\right) \qquad n = 1, 2, 3...$$

$$w(t) = Ce^{-9\lambda t}$$

$$w(t) = Ce^{-9\left(\frac{\pi n}{6}\right)^2 t}$$

Putting the results together from both equations, we get the product solution to the heat equation.

$$u(x,t) = v(x)w(t)$$

$$u(x,t) = c_2 \sin\left(\frac{n\pi x}{6}\right) Ce^{-9\left(\frac{n\pi}{6}\right)^2 t} \qquad n = 1, 2, 3...$$

While c_2 will be different for each n, the constant $C \times c_2$ will also depend on n, so let's rename it as B_n.

$$u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{\pi n x}{6} e^{-9\left(\frac{\pi n}{6}\right)^2 t}$$

Matching up the boundary conditions gives us

$$u(x,0) = \sin(\pi x) + 3\sin(\pi x/2)$$

and then

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{6}\right)(1) = \sin(\pi x) + 3\sin\left(\frac{\pi x}{2}\right)$$

We get $B_6 = 1$, $B_3 = 3$, with all other values of $B_n = 0$, so the solution is

$$u(x,t) = e^{-9\pi^2 t}\sin(\pi x) + 3e^{-\frac{9}{4}\pi^2 t}\sin\left(\frac{\pi x}{2}\right)$$

■ 2. Find a solution to the partial differential equation if $u(x,0) = 2\cos(\pi x/2L)$, $u_x(0,t) = 0$, and $u(L,t) = 0$.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le L$$

Solution:

We start with the product solution $u(x, t) = v(x)w(t)$, then

$$v(x)w'(t) = 4v''(x)w(t)$$

$$\frac{w'(t)}{4w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Break the equation into two ordinary differential equations.

$$w' = -4\lambda w$$

$$v'' = -\lambda v$$

The solution to the first equation, which is the first order differential equation, is

$$w(t) = Ce^{-4\lambda t}$$

To solve the second equation, we find the associated characteristic equation

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$, and the solution is

$$v(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives us

$$v'(0) = -c_1\sqrt{-\lambda} + c_2\sqrt{-\lambda} = 0 \qquad c_1 = c_2$$

$$v(L) = c_1 e^{-L\sqrt{-\lambda}} + c_2 e^{L\sqrt{-\lambda}} = 0$$

which allows use to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$, and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives us

$$v'(0) = c_2 = 0$$

$$v(L) = c_1 + c_2 L = 0$$

which allows us to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$, and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives us

$$v'(0) = \sqrt{\lambda}c_2 = 0 \qquad c_2 = 0$$

$$v(L) = c_1 \cos(\sqrt{\lambda}L) + c_2 \sin(\sqrt{\lambda}L) = c_1 \cos(\sqrt{\lambda}L) = 0$$

We'll find a non-trivial solution which satisfies

$$\cos(\sqrt{\lambda}L) = 0$$

$$\sqrt{\lambda}L = \frac{\pi}{2} + n\pi$$

$$\lambda_n = \left(\frac{\pi + 2n\pi}{2L}\right)^2$$

So the general solution for $\lambda > 0$ becomes

$$v(x) = c_1 \cos\frac{(\pi + 2n\pi)x}{2L}$$

$$w(t) = Ce^{-4\lambda t}$$

$$w(t) = Ce^{-4\left(\frac{\pi + 2n\pi}{2L}\right)^2 t} = Ce^{-\left(\frac{\pi(2n+1)}{L}\right)^2 t} \qquad n = 1, 2, 3...$$

Putting the results together from both equations, we get the product solution to the heat equation.

$$u(x, t) = v(x)w(t)$$

$$u(x, t) = c_1 \cos\left(\frac{(\pi + 2n\pi)x}{2L}\right)Ce^{-\left(\frac{\pi(2n+1)}{L}\right)^2 t}$$

While c_1 will be different for each n, the constant $C \times c_1$ will also depend on n, so let's rename it as A_n.

$$u_n(x, t) = A_n \cos\left(\frac{(\pi + 2n\pi)x}{2L}\right)e^{-\left(\frac{\pi(2n+1)}{L}\right)^2 t} \qquad n = 1, 2, 3...$$

The solution is

$$u(x,t)=\sum_{n=1}^{\infty}A_n\cos\left(\frac{(\pi+2n\pi)x}{2L}\right)e^{-\left(\frac{\pi(2n+1)}{L}\right)^2 t}$$

Matching up the boundary conditions gives us

$$u(x,0)=2\cos\left(\frac{\pi x}{2L}\right)$$

and then

$$\sum_{n=1}^{\infty}A_n\cos\left(\frac{(\pi+2n\pi)x}{2L}\right)=2\cos\left(\frac{\pi x}{2L}\right)$$

We get $A_0=2$, but then $A_1=A_2=A_3=\ldots=0$, so the solution is

$$u(x,t)=2\cos\left(\frac{\pi x}{2L}\right)e^{-\frac{\pi^2}{L^2}t}$$

■ 3. Find a solution to the partial differential equation if $u(x,0)=4\sin(\pi x/6)\cos^2(\pi x/6)$, $u(0,t)=0$, and $u_x(3,t)=0$.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0\le x\le 3$$

Solution:

We start with the product solution $u(x,t)=v(x)w(t)$, then

$$v(x)w'(t) = v''(x)w(t)$$

$$\frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Break the equation into two ordinary differential equations

$$w' = -\lambda w$$

$$v'' = -\lambda v$$

The solution to the first equation, which is the first order differential equation, is

$$w(t) = Ce^{-\lambda t}$$

To solve the second equation, we find the associated characteristic equation

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$, and the solution is

$$v(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives

$$v(0) = c_1 + c_2 = 0 \qquad\qquad c_1 = c_2$$

$$v'(3) = -3\sqrt{-\lambda}c_1 e^{-3\sqrt{-\lambda}} + 3\sqrt{-\lambda}c_2 e^{3\sqrt{-\lambda}} = 0$$

which allows us to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$, and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v(3) = c_2 = 0$$

which allows us to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$, and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v'(3) = -c_1 \sin(3\sqrt{\lambda}) + c_2 \cos(3\sqrt{\lambda}) = c_2 \cos(3\sqrt{\lambda}) = 0$$

We'll find a non-trivial solution which satisfies

$$\cos(3\sqrt{\lambda}) = 0$$

$$3\sqrt{\lambda} = \frac{\pi}{2} + n\pi$$

$$\lambda_n = \left(\frac{\pi + 2n\pi}{6}\right)^2$$

So the general solution for $\lambda > 0$ becomes

$$v(x) = c_2 \sin \frac{(\pi + 2n\pi)x}{6}$$

$$w(t) = Ce^{-\lambda t}$$

$$w(t) = Ce^{-\left(\frac{\pi + 2n\pi}{6}\right)^2 t} \qquad n = 1, 2, 3...$$

Putting the results together from both equations, we get the product solution to the heat equation.

$$u(x, t) = v(x)w(t)$$

$$u(x, t) = c_2 \sin \frac{(\pi + 2n\pi)x}{6} Ce^{-\left(\frac{\pi(2n+1)}{6}\right)^2 t}$$

While c_2 will be different for each n, the constant $C \times c_2$ will also depend on n, so let's rename it as B_n.

$$u_n(x, t) = B_n \sin\left(\frac{(\pi + 2n\pi)x}{6}\right) e^{-\left(\frac{\pi(2n+1)}{6}\right)^2 t}, \quad n = 1, 2, 3...$$

The solution is

$$u(x, t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{(\pi + 2n\pi)x}{6}\right) e^{-\left(\frac{\pi(2n+1)}{6}\right)^2 t}$$

Matching up the boundary conditions gives

$$u(x,0) = 4\sin\left(\frac{\pi x}{6}\right)\cos^2\left(\frac{\pi x}{6}\right)$$

Then we get

$$\sum_{n=0}^{\infty} B_n \cos\left(\frac{(\pi + 2n\pi)x}{6}\right) = 4\sin\left(\frac{\pi x}{6}\right)\cos^2\left(\frac{\pi x}{6}\right)$$

$$\sum_{n=0}^{\infty} B_n \cos\left(\frac{(\pi + 2n\pi)x}{6}\right) = 2\sin\left(\frac{\pi x}{3}\right)\cos\left(\frac{\pi x}{6}\right)$$

$$\sum_{n=0}^{\infty} B_n \cos\left(\frac{(\pi + 2n\pi)x}{6}\right) = \sin\left(\frac{\pi x}{6}\right) + \sin\left(\frac{\pi x}{2}\right)$$

We find $B_0 = 1$, $B_1 = 1$, and $B_2 = B_3 = \ldots = 0$, so the solution is

$$u(x,t) = \sin\left(\frac{\pi x}{6}\right)e^{-\frac{\pi^2}{36}t} + \sin\left(\frac{\pi x}{2}\right)e^{-\frac{\pi^2}{4}t}$$

■ 4. Find a solution to the partial differential equation if $u(x,0) = 2x + 5$, $u_x(0,t) = 0$, and $u_x(L,t) = 0$.

$$\frac{\partial u}{\partial t} = 25\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le L$$

Solution:

We start with the product solution $u(x,t) = v(x)w(t)$, then

$$v(x)w'(t) = 25v''(x)w(t)$$

$$\frac{w'(t)}{25w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Break the equation into two ordinary differential equations.

$$w' = -25\lambda w$$

$$v'' = -\lambda v$$

The solution to the first equation, which is the first order differential equation, is

$$w(t) = Ce^{-25\lambda t}$$

To solve the second equation, we find the associated characteristic equation

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$, and the solution is

$$v(x) = c_1e^{-\sqrt{-\lambda}x} + c_2e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives

$$v'(0) = -c_1\sqrt{-\lambda} + c_2\sqrt{-\lambda} = 0$$

$$v'(L) = -c_1\sqrt{-\lambda}e^{-L\sqrt{-\lambda}} + c_2\sqrt{-\lambda}e^{-L\sqrt{-\lambda}} = 0$$

which allows us to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$, and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives

$$v'(0) = c_2 = 0$$

$$v'(L) = c_2 = 0$$

so there's non-trivial solution $v(x) = C_1$ and $w(t) = C_2$.

If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$, and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives us

$$v'(0) = \sqrt{\lambda}c_2 = 0 \qquad\qquad c_2 = 0$$

$$v'(L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) + c_2\sqrt{\lambda}\cos(\sqrt{\lambda}L) = -c_1\sqrt{\lambda}\sin(\sqrt{\lambda}L) = 0$$

We'll find a non-trivial solution which satisfies

$$\sin(\sqrt{\lambda}L) = 0$$

$$\sqrt{\lambda}L = n\pi$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2$$

So the general solution for $\lambda > 0$ becomes

$$v(x) = c_1 \cos\left(\frac{n\pi x}{L}\right)$$

$$w(t) = Ce^{-25\lambda t}$$

$$w(t) = Ce^{-25\left(\frac{\pi n}{L}\right)^2 t} \qquad n = 1, 2, 3...$$

Putting the results together from both equations, we get the product solution to the heat equation.

$$u(x, t) = v(x)w(t)$$

$$u_n(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)e^{-25\left(\frac{n\pi}{L}\right)^2 t}$$

Matching up the boundary conditions gives

$$u(x,0) = 2x + 5$$

Then we get

$$A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)e^{-25\left(\frac{n\pi}{L}\right)^2 t} = 2x + 5$$

We use the formulas for A_0 and A_n from the Fourier series representation to get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\int_0^L 2x + 5\ dx = \frac{1}{L}(x^2 + 5x)\Big|_0^L = L + 5$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L (2x + 5)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts, letting $u = 2x + 5$, $du = 2\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{n\pi}(2x + 5)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{4}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = -\frac{4}{n\pi}\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$A_n = \frac{4L}{(n\pi)^2}((-1)^n - 1)$$

So the solution is

$$u(x,t) = 5 + L + \sum_{n=1}^{\infty}\frac{4L}{(n\pi)^2}((-1)^n - 1)\cos\left(\frac{n\pi x}{L}\right)e^{-25\left(\frac{n\pi}{L}\right)^2 t}$$

■ 5. Find a solution to the partial differential equation if $u(x,0) = x\sin x$, $u(0,t) = 0$, and $u(\pi, t) = 0$.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le \pi$$

Solution:

We start with the product solution $u(x, t) = v(x)w(t)$, then

$$v(x)w'(t) = 4v''(x)w(t)$$

$$\frac{w'(t)}{4w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Break the equation into two ordinary differential equations.

$$w' = -4\lambda w$$

$$v'' = -\lambda v$$

The solution to the first equation, which is the first order differential equation, is

$$w(t) = Ce^{-4\lambda t}$$

To solve the second equation, we find the associated characteristic equation.

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$, and the solution is

$$v(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives

$$v(0) = c_1 + c_2 = 0 \qquad\qquad c_1 = c_2$$

$$v(\pi) = c_1 e^{-\pi\sqrt{-\lambda}} + c_2 e^{\pi\sqrt{-\lambda}} = 0$$

which allows us to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$, and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives us

$$v(0) = c_1 = 0$$

$$v(\pi) = c_1 + c_2\pi = 0$$

which allows us to determine that $c_1 = c_2 = 0$, from which we can get the trivial solution only $u(x, t) = 0$.

If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$, and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v(\pi) = c_1 \cos(\pi\sqrt{\lambda}) + c_2 \sin(\pi\sqrt{\lambda}) = c_2 \sin(\pi\sqrt{\lambda}) = 0$$

We'll find a non-trivial solution which satisfies

$$\sin(\pi\sqrt{\lambda}) = 0$$

$$\pi\sqrt{\lambda} = n\pi$$

$$\lambda_n = n^2$$

So the general solution for $\lambda > 0$ becomes

$$v(x) = c_2 \sin(nx)$$

$$w(t) = Ce^{-4\lambda t}$$

$$w(t) = Ce^{-4n^2t} \qquad n = 1, 2, 3...$$

Putting the results together from both equations, we get the product solution to the heat equation.

$$u(x, t) = v(x)w(t)$$

$$u(x, t) = c_2 \sin(nx)Ce^{-4n^2t}$$

While c_2 will be different for each n, the constant $C \times c_2$ will also depend on n, so let's rename it as B_n.

$$u_n(x, t) = B_n \sin(nx)e^{-4n^2t} \qquad n = 1, 2, 3...$$

The solution is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(nx)e^{-4n^2t}$$

Matching up the boundary conditions gives

$$u(x,0) = x \sin x$$

Then we get

$$\sum_{n=1}^{\infty} B_n \sin(nx)e^{-4n^2t} = x \sin x$$

We use the formula for B_n from the Fourier series representation to get

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{\pi}\int_0^\pi x \sin x \sin(nx)\ dx$$

$$B_n = \frac{1}{\pi}\int_0^\pi x(\cos(n-1)x - \cos(n+1)x)\ dx$$

Use integration by parts, letting $u = x$, $du = dx$, $dv = \cos(n-1)x - \cos(n+1)x\ dx$, and

$$v = -\frac{1}{n-1}\sin(n-1)x + \frac{1}{n+1}\sin(n+1)x$$

and we'll get

$$B_n = \frac{1}{\pi}\left(x \left(-\frac{1}{n-1}\sin(n-1)x + \frac{1}{n+1}\sin(n+1)x \right) \Big|_0^\pi \right.$$

$$\left. - \int_0^\pi \left(-\frac{1}{n-1}\sin(n-1)x + \frac{1}{n+1}\sin(n+1)x \right) dx \right)$$

$$B_n = \frac{1}{\pi}\left(-\frac{1}{(n-1)^2}\cos(n-1)x + \frac{1}{(n+1)^2}\cos(n+1)x \right) \Big|_0^\pi$$

$$B_n = \frac{1}{\pi}\left(\frac{1}{(n+1)^2}((-1)^{n+1}-1) - \frac{1}{(n-1)^2}((-1)^{n-1}-1) \right)$$

$$B_n = -\frac{4n((-1)^n+1)}{\pi(n^2-1)^2}, \ n \neq 1$$

For $n = 1$ specifically, we find

$$B_1 = \frac{2}{\pi}\int_0^\pi x\sin^2 x \ dx$$

$$B_1 = \frac{1}{\pi}\int_0^\pi x(1-\cos 2x) \ dx$$

$$B_1 = -\frac{1}{\pi}\left(\frac{x^2}{2}\right)\Big|_0^\pi - \frac{1}{\pi}\int_0^\pi x\cos(2x) \ dx$$

Use integration by parts, letting $u = x$, $du = dx$, $dv = \cos(2x) \ dx$, and $v = (1/2)\sin(2x)$.

$$B_1 = \frac{\pi}{2} - \frac{1}{\pi}\left(\frac{x}{2}\sin(2x)\Big|_0^\pi - \frac{1}{2}\int_0^\pi \sin(2x) \ dx \right)$$

$$B_1 = \frac{\pi}{2} + \frac{1}{2\pi}\left(-\frac{1}{2}\cos(2x)\Big|_0^{\pi}\right)$$

$$B_1 = \frac{\pi}{2}$$

Then the solution is

$$u(x,t) = \frac{\pi}{2}e^{-4t}\sin x - \sum_{n=2}^{\infty}\frac{4n((-1)^n+1)}{\pi(n^2-1)^2}\sin(nx)e^{-4n^2t}$$

■ 6. Find a solution to the partial differential equation if $u(x,0) = \sin(\pi x)$, $u_x(0,t) = 0$, and $u_x(5,t) = 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le 5$$

Solution:

We start with the product solution $u(x,\ t) = v(x)w(t)$, then

$$v(x)w'(t) = v''(x)w(t)$$

$$\frac{w'(t)}{w(t)} = \frac{v''(x)}{v(x)} = -\lambda$$

Break the equation into two ordinary differential equations.

$$w' = -\lambda w$$

$$v'' = -\lambda v$$

The solution to the first equation, which is the first order differential equation, is

$$w(t) = Ce^{-\lambda t}$$

To solve the second equation, we find the associated characteristic equation.

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$, and the solution is

$$v(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives

$$v'(0) = -c_1\sqrt{-\lambda} + c_2\sqrt{-\lambda} = 0 \qquad\qquad c_1 = c_2$$

$$v'(5) = -c_1\sqrt{-\lambda}e^{-5\sqrt{-\lambda}} + c_2\sqrt{-\lambda}e^{5\sqrt{-\lambda}} = 0$$

which allows use to determine that $c_1 = c_2 = 0$, from which we get only the trivial solution $u(x, t) = 0$.

If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$, and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives

$$v'(0) = c_2 = 0$$

$$v'(5) = c_2 = 0$$

so the solution is $v_0(x) = c_1$ and $w_0(t) = C$.

If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$, and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives

$$v'(0) = \sqrt{\lambda}c_2 = 0 \qquad\qquad c_2 = 0$$

$$v'(5) = -c_1\sqrt{\lambda}\sin(5\sqrt{\lambda}) + c_2\cos\sqrt{\lambda}(5\sqrt{\lambda}) = -c_1\sqrt{\lambda}\sin(5\sqrt{\lambda}) = 0$$

We'll find a non-trivial solution which satisfies

$$\sin(5\sqrt{\lambda}) = 0$$

$$5\sqrt{\lambda} = n\pi$$

$$\lambda_n = \left(\frac{n\pi}{5}\right)^2$$

So the general solution for $\lambda > 0$ becomes

$$v(x) = c_1 \cos\left(\frac{n\pi x}{5}\right)$$

$$w(t) = Ce^{-\lambda t}$$

$$w(t) = Ce^{-\left(\frac{n\pi}{5}\right)^2 t} \qquad n = 1, 2, 3...$$

Putting the results together from both equations, we get the product solution to the heat equation.

$$u(x, t) = v(x)w(t)$$

$$u(x, t) = c_1 C + c_1 \cos\left(\frac{n\pi x}{5}\right) Ce^{-\left(\frac{n\pi}{5}\right)^2 t}$$

While c_1 will be different for each n, the constant $C \times c_1$ will also depend on n, so let's rename it as A_n.

$$u_n(x, t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{5}\right) e^{-\left(\frac{n\pi}{5}\right)^2 t} \qquad n = 1, 2, 3...$$

Matching up the boundary conditions gives

$$u(x,0) = \sin(\pi x)$$

We use the formulas for A_0 and A_n from the Fourier series representation to get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{5}\int_0^5 \sin(\pi x)\ dx = \frac{1}{5}\left(-\frac{1}{\pi}\right)\cos(\pi x)\Big|_0^5 = \frac{2}{5\pi}$$

and

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{5}\int_0^5 \sin(\pi x)\cos\left(\frac{n\pi x}{5}\right)\ dx$$

$$A_n = \frac{1}{5}\int_0^5 \left(\sin \pi x\left(1-\frac{n}{5}\right) + \sin \pi x\left(1+\frac{n}{5}\right)\right) dx$$

$$A_n = -\frac{1}{5}\left(\frac{1}{\pi\left(1-\frac{n}{5}\right)}\cos\left(\pi+\frac{n\pi}{5}\right)x + \frac{1}{\pi\left(1+\frac{n}{5}\right)}\cos\left(\pi+\frac{n\pi}{5}\right)x\right)\Bigg|_0^5$$

$$A_n = -\frac{10(1+(-1)^n)}{\pi(n^2-25)},\ n \neq 5$$

For $n=5$ specifically, we find

$$A_5 = \frac{2}{5}\int_0^5 \sin(\pi x)\cos(\pi x)\ dx$$

$$A_5 = \frac{1}{5}\int_0^5 \sin(2\pi x)\ dx = \frac{1}{10\pi}\cos(2\pi x)\Big|_0^5 = 0$$

so the solution is

$$u(x,t) = \frac{2}{5\pi} + \sum_{n=1, n\neq 5}^{\infty} -\frac{10(1+(-1)^n)}{\pi(n^2-25)}\cos\left(\frac{n\pi x}{5}\right)e^{-\left(\frac{n\pi}{5}\right)^2 t}$$

CHANGING THE TEMPERATURE BOUNDARIES

■ 1. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \pi$$

$$u(x,0) = 12x + \pi$$

$$u(0,t) = \pi \qquad u(\pi, t) = 3\pi$$

Solution:

Find the function that models equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E = \pi + \frac{3\pi - \pi}{\pi}x$$

$$u_E = 2x + \pi$$

Find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{\pi}\int_0^\pi (12x + \pi - 2x - \pi)\sin(nx)\ dx$$

$$B_n = \frac{2}{\pi}\int_0^{\pi} 10x\sin(nx)\ dx$$

Integrate, using integration by parts for the second integral with $u = x$, $du = dx$, $dv = \sin(nx)\ dx$, and $v = -(1/n)\cos(nx)$.

$$B_n = \frac{20}{\pi}\left(x\left(-\frac{1}{n}\cos(nx)\right)\Bigg|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\ dx\right)$$

$$B_n = \frac{20}{\pi}\left(-\frac{\pi}{n}(-1)^n + \frac{1}{n^2}\sin(nx)\Bigg|_0^{\pi}\right) = \frac{20}{n}(-1)^{n+1}$$

Then the solution of the equation is

$$u(x,t) = u_E + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 2x + \pi + \sum_{n=1}^{\infty}\frac{20(-1)^{n+1}}{n}\sin(nx)\ e^{-2n^2t}$$

■ 2. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 4$$

$$u(x,0) = x^2 + 1$$

$$u(0,t) = 1 \qquad u(4,t) = 17$$

Solution:

Find the function that models equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E = 1 + \frac{17 - 1}{4}x$$

$$u_E = 1 + 4x$$

Find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{2}\int_0^4 (x^2 + 1 - 1 - 4x)\sin\left(\frac{n\pi x}{4}\right)\ dx$$

$$B_n = \frac{1}{2}\int_0^4 (x^2 - 4x)\sin\left(\frac{n\pi x}{4}\right)\ dx$$

Integrate, using integration by parts for the second integral with $u = x^2 - 4x$, $du = (2x - 4)\ dx$, $dv = \sin(n\pi x/4)\ dx$, and $v = -(4/n\pi)\cos(n\pi x/4)$.

$$B_n = \frac{1}{2}\left[(x^2 - 4x)\left(-\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)\right)\Bigg|_0^4 + \frac{4}{n\pi}\int_0^4 (2x - 4)\cos\left(\frac{n\pi x}{4}\right)\ dx\right]$$

$$B_n = \frac{2}{n\pi}\int_0^4 (2x - 4)\cos\left(\frac{n\pi x}{4}\right)\ dx$$

Integrate, using integration by parts for the second integral with $u = 2x - 4$, $du = 2\ dx$, $dv = \cos(n\pi x/4)\ dx$, and $v = (4/n\pi)\sin(n\pi x/4)$.

$$B_n = \frac{2}{n\pi}\left[(2x-4)\frac{4}{n\pi}\sin\left(\frac{n\pi x}{4}\right)\Big|_0^4 - \frac{8}{n\pi}\int_0^4 \sin\left(\frac{n\pi x}{4}\right)\ dx\right]$$

$$B_n = \frac{16}{n^2\pi^2}\cdot\frac{4}{n\pi}\cos\left(\frac{n\pi x}{4}\right)\Big|_0^4 = \frac{64}{n^3\pi^3}((-1)^n - 1)$$

Then the solution of the equation is

$$u(x,t) = u_E + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 1 + 4x + \sum_{n=1}^{\infty}\frac{64}{n^3\pi^3}((-1)^n - 1)\sin\left(\frac{n\pi x}{4}\right)e^{-\frac{n^2\pi^2}{4}t}$$

■ 3. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1$$

$$u(x,0) = x + \sin(\pi x)$$

$$u(0,t) = 4 \qquad u(1,t) = 5$$

Solution:

Find the function that models equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E = 4 + \frac{5-4}{1}x$$

$$u_E = x + 4$$

Find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = 2\int_0^1 (x + \sin(\pi x) - x - 4)\sin(n\pi x)\ dx$$

$$B_n = 2\int_0^1 \sin(\pi x)\sin(n\pi x) - 4\sin(n\pi x)\ dx$$

$$B_n = \int_0^1 \cos(n-1)\pi x - \cos(n+1)\pi x - 8\sin(n\pi x)\ dx$$

$$B_n = \frac{1}{\pi(n-1)}\sin(n-1)\pi x - \frac{1}{\pi(n+1)}\sin(n+1)\pi x + \frac{8}{n\pi}\cos(n\pi x)\Big|_0^1$$

$$B_n = \frac{8}{n\pi}((-1)^n - 1),\ n \neq 1$$

For $n = 1$ specifically, we get

$$B_1 = 2\int_0^1 (\sin \pi x - 4)\sin(\pi x)\ dx$$

$$B_1 = \int_0^1 1 - \cos(2\pi x) - 8\sin(\pi x)\ dx$$

$$B_1 = \left(x - \frac{1}{2\pi}\sin(2\pi x) + \frac{8}{\pi}\cos(\pi x) \right) \Bigg|_0^1 = 1 - \frac{16}{\pi}$$

Then the solution of the equation is

$$u(x,t) = u_E + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = x + 4 + \left(1 - \frac{16}{\pi}\right)\sin(\pi x)e^{-3\pi^2 t} + \sum_{n=2}^{\infty} \frac{8((-1)^n - 1)}{\pi n}\sin(n\pi x)e^{-3\pi^2 n^2 t}$$

■ 4. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 10$$

$$u(x,0) = 6 - 2x$$

$$u(0,t) = 10 \qquad u(10,t) = 50$$

Solution:

Find the function that models equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E = 10 + \frac{50 - 10}{10}x$$

$$u_E = 10 + 4x$$

Find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{5}\int_0^{10} (6 - 2x - 10 - 4x)\sin\left(\frac{n\pi x}{10}\right)\ dx$$

$$B_n = -\frac{1}{5}\int_0^{10} (4 + 6x)\sin\left(\frac{n\pi x}{10}\right)\ dx$$

Integrate, using integration by parts for the second integral with $u = 6x + 4$, $du = 6\ dx$, $dv = \sin(n\pi x/10)\ dx$, and $v = -(10/n\pi)\cos(n\pi x/10)$.

$$B_n = -\frac{1}{5}\left(-\frac{10}{n\pi}(6x + 4)\cos\left(\frac{n\pi x}{10}\right)\Big|_0^{10} + \frac{60}{n\pi}\int_0^{10}\cos\left(\frac{n\pi x}{10}\right)\ dx\right)$$

$$B_n = \frac{2}{n\pi}(64(-1)^n - 4) - \frac{12}{n\pi}\left(\frac{10}{n\pi}\right)\sin\left(\frac{n\pi x}{10}\right)\Big|_0^{10}$$

$$B_n = -\frac{8}{n\pi} + \frac{128(-1)^n}{n\pi} = \frac{8}{n\pi}(16(-1)^n - 1)$$

Then the solution of the equation is

$$u(x,t) = u_E + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 10 + 4x + \sum_{n=1}^{\infty} \frac{8}{n\pi}(16(-1)^n - 1)\sin\left(\frac{n\pi x}{10}\right)e^{-5\left(\frac{n\pi}{10}\right)^2 t}$$

■ 5. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \frac{\pi}{2}$$

$$u(x,0) = \cos x$$

$$u(0,t) = 1 \qquad u\left(\frac{\pi}{2}, t\right) = 0$$

Solution:

Find the function that models equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E = 1 + \frac{0 - 1}{\frac{\pi}{2}}x$$

$$u_E = 1 - \frac{2}{\pi}x$$

Find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{4}{\pi}\int_0^{\frac{\pi}{2}}\left(\cos x + \frac{2}{\pi}x - 1\right)\sin(2nx)\ dx$$

$$B_n = \frac{4}{\pi}\left[\int_0^{\frac{\pi}{2}}\frac{2}{\pi}x\sin(2nx)\ dx\right.$$

$$\left.+\int_0^{\frac{\pi}{2}}\frac{1}{2}\sin(2n-1)x + \frac{1}{2}\sin(2n+1)x - \sin(2nx)\ dx\right]$$

Integrate, using integration by parts for the first integral with $u = x$, $du = \ dx$, $dv = \sin(2nx)\ dx$, and $v = -(1/2n)\cos(2nx)$.

$$B_n = \frac{8}{\pi^2}\left(-\frac{1}{2n}x\cos(2nx)\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\frac{1}{2n}\cos(2nx)\ dx\right)$$

$$-\frac{2}{\pi(2n-1)}\cos(2n-1)x\Big|_0^{\frac{\pi}{2}} - \frac{2}{\pi(2n+1)}\cos(2n+1)x\Big|_0^{\frac{\pi}{2}} + \frac{2}{n\pi}\cos(2nx)\Big|_0^{\frac{\pi}{2}}$$

$$B_n = -\frac{4}{n\pi^2}\left((-1)^n\frac{\pi}{2}\right) + \frac{2}{n^2\pi^2}\sin(2nx)\Big|_0^{\frac{\pi}{2}}$$

$$+\frac{2}{\pi(2n-1)} + \frac{2}{\pi(2n+1)} + \frac{2}{n\pi}((-1)^n - 1)$$

$$B_n = \frac{2(n(2n+1) + n(2n-1) - 4n^2 + 1)}{n\pi(4n^2-1)} = \frac{2}{n\pi(4n^2-1)}$$

Then the solution of the equation is

$$u(x,t) = u_E + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 1 - \frac{2}{\pi}x + \sum_{n=1}^{\infty} \frac{2}{n\pi(4n^2 - 1)} \sin(2nx)\ e^{-4n^2 t}$$

■ 6. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \pi$$

$$u(x,0) = e^x + x + 1$$

$$u(0,t) = 1 \qquad u(\pi, t) = 1 + \pi$$

Solution:

Find the function that models equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E = 1 + \frac{1 + \pi - 1}{\pi}x$$

$$u_E = x + 1$$

Find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} (e^x + x + 1 - x - 1)\sin(nx)\ dx$$

$$B_n = \frac{2}{\pi} \int_0^{\pi} e^x \sin(nx)\ dx$$

Integrate, using integration by parts for the second integral with $u = e^x$, $du = e^x\ dx$, $dv = \sin(nx)\ dx$, and $v = -(1/n)\cos(nx)$.

$$B_n = \frac{2}{\pi} \left(-\frac{1}{n} e^x \cos(nx) \Big|_0^{\pi} + \frac{1}{n} \int_0^{\pi} e^x \cos(nx)\ dx \right)$$

Integrate, using integration by parts for the second integral with $u = e^x$, $du = e^x\ dx$, $dv = \cos(nx)\ dx$, and $v = (1/n)\sin(nx)$.

$$B_n = \frac{2}{\pi} \left[-\frac{1}{n}(e^{\pi}(-1)^n - 1) + \frac{1}{n} \left(\frac{1}{n} e^x \sin(nx) \Big|_0^{\pi} - \frac{1}{n} \int_0^{\pi} e^x \sin(nx)\ dx \right) \right]$$

$$B_n = -\frac{2}{n\pi}(e^{\pi}(-1)^n - 1) - \frac{2}{\pi n^2} \left(\frac{\pi}{2} \right) B_n$$

$$B_n \left(1 + \frac{1}{n^2} \right) = \frac{2}{n\pi}(1 - e^{\pi}(-1)^n)$$

$$B_n = \frac{2n(1 - e^{\pi}(-1)^n)}{\pi(n^2 + 1)}$$

Then the solution of the equation is

$$u(x,t) = u_E + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L} \right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 1 + x + \sum_{n=1}^{\infty} \frac{2n(1 - e^{\pi}(-1)^n)}{\pi(n^2 + 1)} \sin(nx)\ e^{-2n^2 t}$$

LAPLACE'S EQUATION

■ 1. Use the product solution to find $u_1(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \leq x \leq \pi$ and $0 \leq y \leq 1$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = 3\sin(2x) \qquad u(0,y) = 0$$

$$u(x,1) = 0 \qquad u(\pi, y) = 0$$

Solution:

If we find the first and second derivatives of the product solution,

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2 v}{dx^2}w + v\frac{d^2 w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

Our boundary conditions are $w(1) = 0$, $v(0) = 0$, and $v(\pi) = 0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2 \qquad v_n(x) = C\sin(nx) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} - n^2w = 0$$

$$w(1) = 0$$

$$w(y) = c_1e^{-ny} + c_2e^{ny}$$

Equivalently, this can be rewritten as

$$w(y) = c_1 \sinh n(y-1) + c_2 \cosh n(y-1)$$

which allows us to apply $w(1) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$w(y) = c_1 \sinh n(y-1)$$

Then the product solution is

$$u(x, y) = v(x)w(y)$$

$$u(x, y) = C\sin(nx)c_1 \sinh n(y-1)$$

$$u_n(x, y) = B_n \sin(nx) \sinh n(y-1) \qquad n = 1, 2, 3, \ldots$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \sin(nx) \sinh n(y-1)$$

Substitute $u(x,0) = 3\sin(2x)$.

$$\sum_{n=1}^{\infty} B_n \sin(nx) \sinh n(y-1) = 3\sin(2x)$$

$$B_2 = -\frac{3}{\sinh 2}$$

Other than B_2, we get $B_1 = B_3 = \ldots = B_n = 0$. Then the solution is

$$u(x,y) = -\frac{3}{\sinh 2} \sin(2x) \sinh(2y-2)$$

■ 2. Use the product solution to find $u(x,y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \le x \le 3$ and $0 \le y \le 3$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = 0 \qquad\qquad u(0,y) = 0$$

$$u(x,3) = 2\sin(\pi x) + \sin(2\pi x) \qquad\qquad u(3,y) = 0$$

Solution:

If we find the first and second derivatives of the product solution,

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2 v}{dx^2}w + v\frac{d^2 w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2 v}{dx^2}w = -v\frac{d^2 w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2 v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2 w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2 v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2 w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2 v}{dx^2} = -\lambda$$

$$\frac{d^2 v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

Our boundary conditions are $w(0) = 0$, $v(0) = 0$, and $v(3) = 0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{3}\right)^2 \qquad v_n(x) = C\sin(\sqrt{\lambda}x) = C\sin\left(\frac{n\pi x}{3}\right) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

$$w(0) = 0$$

$$w(y) = c_1e^{-\sqrt{\lambda}y} + c_2e^{\sqrt{\lambda}y}$$

Equivalently, this can be rewritten as

$$w(y) = c_1\sinh(\sqrt{\lambda}y) + c_2\cosh(\sqrt{\lambda}y)$$

which allows us to apply $w(0) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$w(y) = c_1 \sinh(\sqrt{\lambda} y)$$

$$w(y) = c_1 \sinh\left(\frac{n\pi y}{3}\right)$$

Then the product solution is

$$u(x, y) = v(x)w(y)$$

$$u(x, y) = C \sin\left(\frac{n\pi x}{3}\right) c_1 \sinh\left(\frac{n\pi y}{3}\right)$$

$$u_n(x, y) = B_n \sin\left(\frac{n\pi x}{3}\right) \sinh\left(\frac{n\pi y}{3}\right) \qquad n = 1, 2, 3, \ldots$$

$$u_n(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{3}\right) \sinh\left(\frac{n\pi y}{3}\right)$$

Substitute $u(x,3) = 2\sin(\pi x) + \sin(2\pi x)$.

$$\sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{3}\right) \sinh(n\pi) = 2\sin(\pi x) + \sin(2\pi x)$$

$$B_3 = \frac{2}{\sinh(3\pi)}, \ B_6 = \frac{1}{\sinh(6\pi)}$$

Then the solution is

$$u(x, y) = \frac{2}{\sinh(3\pi)} \sin(\pi x)\sinh(\pi y) + \frac{1}{\sinh(6\pi)} \sin(2\pi x)\sinh(2\pi y)$$

■ 3. Use the product solution to find $u(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \le x \le \pi$ and $0 \le y \le 2\pi$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u_y(x,0) = 0 \qquad u(0,y) = 2 + \cos(2y)$$

$$u_y(x,2\pi) = 0 \qquad u(\pi, y) = 0$$

Solution:

If we find the first and second derivatives of the product solution,

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2 v}{dx^2}w + v\frac{d^2 w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = \lambda$$

$$\frac{d^2v}{dx^2} = \lambda v(x)$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

$$\frac{d^2w}{dy^2} = -\lambda w(y)$$

$$\frac{d^2w}{dy^2} + \lambda w = 0$$

Our boundary conditions are $w(0) = 0$, $w(2\pi) = 0$, and $v(\pi) = 0$, and the solution to the second ordinary differential equation boundary value problem is

$$\lambda_0 = 0 \qquad\qquad w_0(y) = C$$

$$\lambda_n = \left(\frac{n\pi}{2\pi}\right)^2 = \frac{n^2}{4} \qquad\qquad w_n(x) = C\cos(\sqrt{\lambda}y) = C\cos\left(\frac{ny}{2}\right) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for $\lambda_0 = 0$ into the first equation is

$$v(x) = c_1 + c_2 x$$

and from $v(\pi) = 0$ we have

$$c_1 + c_2\pi = 0$$

$$v_0(x) = -c_2\pi + c_2 x = c_2(x - \pi)$$

When we plug this value for $\lambda_n > 0$ into the first equation, we get

$$v(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}$$

Equivalently, this can be rewritten as

$$v(x) = c_1 \sinh\sqrt{\lambda}(x - \pi) + c_2 \cosh\sqrt{\lambda}(x - \pi)$$

which allows us to apply $v(\pi) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$v(x) = c_1 \sinh\frac{n(x - \pi)}{2}$$

Then the product solution is

$$u(x,y)=v(x)w(y)$$

$$u(x,y)=c_2(x-\pi)C+c_1\sinh\left(\frac{n(x-\pi)}{2}\right)C\cos\left(\frac{ny}{2}\right)$$

$$u_n(x,y)=A_0(x-\pi)+A_n\sinh\left(\frac{n(x-\pi)}{2}\right)\cos\left(\frac{ny}{2}\right) \qquad n=1,\ 2,\ 3,\ \dots$$

$$u_n(x,y)=A_0(x-\pi)+\sum_{n=1}^{\infty}A_n\sinh\left(\frac{n(x-\pi)}{2}\right)\cos\left(\frac{ny}{2}\right)$$

Substitute $u(0,y)=2+\cos(2y)$.

$$A_0(-\pi)+\sum_{n=1}^{\infty}A_n\sinh\left(-\frac{n\pi}{2}\right)\cos\left(\frac{ny}{2}\right)=2+\cos(2y)$$

$$A_0=-\frac{2}{\pi}, \qquad A_4=-\frac{1}{\sinh(2\pi)}$$

Then the solution is

$$u(x,y)=-\frac{2}{\pi}(x-\pi)-\frac{1}{\sinh 2\pi}\sinh(2x-2\pi)\cos(2y)$$

■ 4. Use the product solution to find $u(x,y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0\le x\le 4$ and $0\le y\le 6$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = 0 \qquad u_x(0,y) = 0$$

$$u_y(x,6) = 0 \qquad u(4,y) = \sin\left(\frac{\pi y}{4}\right) + 5\sin\left(\frac{\pi y}{12}\right)$$

Solution:

If we find the first and second derivatives of the product solution,

$$u(x,y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2 v}{dx^2}w + v\frac{d^2 w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2 v}{dx^2}w = -v\frac{d^2 w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2 v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2 w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = \lambda$$

$$\frac{d^2v}{dx^2} = \lambda v(x)$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

$$\frac{d^2w}{dy^2} = -\lambda w(y)$$

$$\frac{d^2w}{dy^2} + \lambda w = 0$$

Our boundary conditions are $w(0) = 0$, $w(6) = 0$, and $v(0) = 0$, and the solution to the second ordinary differential equation boundary value problem is

$$w(y) = c_1 \cos(\sqrt{\lambda}y) + c_2 \sin(\sqrt{\lambda}y)$$

$$w(0) = 0, \text{ so } c_1 = 0$$

$$w'(6) = -c_1\sqrt{\lambda}\sin(6\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(6\sqrt{\lambda}) = 0$$

and the solution is

$$\cos 6\sqrt{\lambda} = 0$$

$$\lambda_n = \left(\frac{\frac{\pi}{2} + n\pi}{6}\right)^2 = \left(\frac{\pi + 2n\pi}{12}\right)^2$$

$$w(y) = c_2 \sin\frac{(\pi + 2n\pi)y}{12} \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the first equation, we get

$$v(x) = c_3 e^{-\sqrt{\lambda}x} + c_4 e^{\sqrt{\lambda}x}$$

Equivalently, this can be rewritten as

$$v(x) = c_3 \sinh(\sqrt{\lambda}x) + c_4 \cosh(\sqrt{\lambda}x)$$

which allows us to apply $v'(0) = 0$. When we do, we find $c_3 = 0$, and the solution is

$$v(x) = c_4 \cosh(\sqrt{\lambda}x)$$

$$v(x) = c_4 \cosh\left(\frac{(\pi + 2n\pi)x}{12}\right)$$

Then the product solution is

$$u(x, y) = v(x)w(y)$$

$$u(x,y) = c_4 \cosh\left(\frac{(\pi + 2n\pi)x}{12}\right) c_2 \sin\left(\frac{(\pi + 2n\pi)y}{12}\right)$$

$$u_n(x,y) = B_n \cosh\left(\frac{(\pi + 2n\pi)x}{12}\right) \sin\left(\frac{(\pi + 2n\pi)y}{12}\right) \qquad n = 1, 2, 3, \ldots$$

$$u(x,y) = \sum_{n=1}^{\infty} B_n \cosh\left(\frac{(\pi + 2n\pi)x}{12}\right) \sin\left(\frac{(\pi + 2n\pi)y}{12}\right)$$

Substitute

$$u(4,y) = \sin\left(\frac{\pi y}{4}\right) + 5\sin\left(\frac{\pi y}{12}\right)$$

to get

$$\sum_{n=1}^{\infty} B_n \cosh\left(\frac{(\pi + 2n\pi)}{3}\right) \sin\left(\frac{(\pi + 2n\pi)y}{12}\right) = \sin\left(\frac{\pi y}{4}\right) + 5\sin\left(\frac{\pi y}{12}\right)$$

$$B_0 = \frac{5}{\cosh\left(\frac{\pi}{3}\right)},\ B_1 = \frac{1}{\cosh \pi}$$

Other than B_0 and B_1, we know $B_2 = B_3 = \ldots = B_n = 0$. Then the solution is

$$u(x,y) = \frac{5}{\cosh\left(\frac{\pi}{3}\right)} \cosh\left(\frac{\pi x}{12}\right) \sin\left(\frac{\pi y}{12}\right) + \frac{1}{\cosh \pi} \cosh\left(\frac{\pi x}{4}\right) \sin\left(\frac{\pi y}{4}\right)$$

■ 5. Use the product solution to find $u_1(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \le x \le 1$ and $0 \le y \le 1$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = x \qquad u(0,y) = 0$$

$$u(x,1) = 0 \qquad u(1,y) = \sin(3\pi y)$$

Solution:

We can say that $u = u_1 + u_2$, where u_1 is the solution of

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = x \qquad u(0,y) = 0$$

$$u(x,1) = 0 \qquad u(1,y) = 0$$

and u_2 is the solution of

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = 0 \qquad u(0,y) = 0$$

$$u(x,1) = 0 \qquad u(1,y) = \sin(3\pi y)$$

Let's start with u_1. If we find the first and second derivatives of the product solution,

$$u_1(x,y) = v(x)w(y)$$

$$\frac{\partial u_1}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u_1}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{d^2v}{dx^2}w \qquad \frac{\partial^2 u_1}{\partial y^2} = v\frac{d^2w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2v}{dx^2}w + v\frac{d^2w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

Our boundary conditions are $w(1) = 0$, $v(0) = 0$, and $v(1) = 0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_n = (n\pi)^2 \qquad v_n(x) = C\sin(n\pi x) \qquad n = 1, 2, 3, \dots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

$$w(1) = 0$$

$$w(y) = c_1 e^{-\sqrt{\lambda}y} + c_2 e^{\sqrt{\lambda}y}$$

Equivalently, this can be rewritten as

$$w(y) = c_1 \sinh\sqrt{\lambda}(y-1) + c_2 \cosh\sqrt{\lambda}(y-1)$$

which allows us to apply $w(1) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$w(y) = c_1 \sinh \sqrt{\lambda}(y - 1)$$

$$w(y) = c_1 \sinh n\pi(y - 1)$$

Then the product solution is

$$u_1(x, y) = v(x)w(y)$$

$$u_1(x, y) = C \sin(n\pi x)c_1 \sinh n\pi(y - 1)$$

$$u_1(x, y) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh n\pi(y - 1)$$

Substitute $u(x,0) = x$.

$$\sum_{n=1}^{\infty} B_n \sin(n\pi x)\sinh(-n\pi) = x$$

Find B_n.

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = 2\int_0^1 x \sin(n\pi x)\ dx$$

Integrate, using integration by parts with $u = x$, $du = dx$, $dv = \sin n\pi x\ dx$, and $v = -(1/n\pi)\cos(n\pi x)$.

$$B_n = -2\left(-\frac{1}{n\pi}x\cos(n\pi x)\Big|_0^1 + \frac{1}{n\pi}\int_0^1 \cos(n\pi x)\ dx\right)$$

$$B_n = -2\left(-\frac{1}{n\pi}(-1)^n + \frac{1}{(n\pi)^2}\sin(n\pi x)\Big|_0^1\right) = -2\left(-\frac{1}{n\pi}(-1)^n + 0\right)$$

$$B_n = \frac{2(-1)^n}{n\pi}$$

Then the solution is

$$u_1(x,y) = \sum_{n=1}^{\infty}\frac{2(-1)^n}{n\pi}\sin(n\pi x)\sinh n\pi(y-1)$$

Solve for u_2,

$$\frac{\partial u_2}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u_2}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{d^2v}{dx^2}w \qquad \frac{\partial^2 u_2}{\partial y^2} = v\frac{d^2w}{dy^2}$$

then plug the second derivatives into Laplace's equation,

$$\frac{d^2v}{dx^2}w + v\frac{d^2w}{dy^2} = 0$$

so that we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = \lambda$$

$$\frac{d^2v}{dx^2} = \lambda v(x)$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

$$\frac{d^2w}{dy^2} = -\lambda w(y)$$

$$\frac{d^2w}{dy^2} + \lambda w = 0$$

Our boundary conditions are $w(1) = 0$, $v(0) = 0$, and $v(1) = 0$, and the solution to the second ordinary differential equation boundary value problem is

$$\lambda_n = (n\pi)^2 \qquad w_n(y) = C\sin(n\pi y) \qquad n = 1, 2, 3, \dots$$

When we plug this value for λ into the first equation, we get

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

$$v(x) = c_1 e^{-\sqrt{\lambda}x} + c_2 e^{\sqrt{\lambda}x}$$

Equivalently, this can be rewritten as

$$v(x) = c_1 \sinh(\sqrt{\lambda}x) + c_2 \cosh(\sqrt{\lambda}x)$$

which allows us to apply $v(0) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$v(x) = c_1 \sinh(\sqrt{\lambda}x)$$

$$v(x) = c_1 \sinh(n\pi x)$$

Then the product solution is

$$u_2(x, y) = v(x)w(y)$$

$$u_2(x, y) = c_1 \sinh(n\pi x) C \sin(n\pi y)$$

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi x)\sin(n\pi y)$$

Substitute $u(1,y) = \sin(3\pi y)$.

$$\sum_{n=1}^{\infty} B_n \sinh(n\pi)\sin(n\pi y) = \sin(3\pi y)$$

$$B_3 = \frac{1}{\sin(3\pi)}$$

Other than B_3, we know $B_1 = B_2 = B_4 = \ldots = B_n = 0$. Then the solution is

$$u_2(x, y) = \frac{1}{\sin(3\pi)} \sinh(3\pi x)\sin(3\pi y)$$

And therefore,

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2(-1)^n}{n\pi} \sin(n\pi x)\sinh \pi n(y-1) + \frac{1}{\sin(3\pi)} \sinh(3\pi x)\sin(3\pi y)$$

■ 6. Use the product solution to find $u_1(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \le x \le 2\pi$ and $0 \le y \le 2\pi$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = \cos x \qquad u_x(0,y) = 0$$

$$u(x,2\pi) = \cos x \qquad u_x(2\pi, y) = 0$$

Solution:

We can say that $u = u_1 + u_2$, where $u_1$is the solution of

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = \cos x \qquad u_x(0,y) = 0$$

$$u(x,2\pi) = 0 \qquad u_x(2\pi, y) = 0$$

and u_2 is the solution of

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = 0 \qquad u_x(0,y) = 0$$

$$u(x,2\pi) = \cos x \qquad u_x(2\pi, y) = 0$$

Lets start with u_1. If we find the first and second derivatives of the product solution,

$$u_1(x, y) = v(x)w(y)$$

$$\frac{\partial u_1}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u_1}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u_1}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u_1}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2 v}{dx^2}w + v\frac{d^2 w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

The boundary conditions are $w(2\pi)=0$, $v'(0)=0$, and $v'(2\pi)=0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_0 = 0 \qquad\qquad v_0 = C$$

$$\lambda_n = \left(\frac{n\pi}{2\pi}\right)^2 = \frac{n^2}{4} \qquad v_n(x) = C\cos\left(\frac{nx}{2}\right) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

For $\lambda_0 = 0$, the solution is

$$w_0(y) = c_1 + c_2(y - 2\pi)$$

$w(2\pi) = 0$, so $c_1 = 0$ and $w_0(y) = c_2(y - 2\pi)$

For $\lambda_n = n^2/4$, the solution is

$$w(y) = c_1e^{-\sqrt{\lambda}y} + c_2e^{\sqrt{\lambda}y}$$

Equivalently, this can be rewritten as

$$w(y) = c_1\sinh\sqrt{\lambda}(y - 2\pi) + c_2\cosh\sqrt{\lambda}(y - 2\pi)$$

which allows us to apply $w(2\pi) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$w(y) = c_1\sinh\sqrt{\lambda}(y - 2\pi)$$

$$w(y) = c_1 \sinh\left(\frac{n(y - 2\pi)}{2}\right)$$

Then the product solution is

$$u_1(x, y) = v(x)w(y)$$

$$u_1(x, y) = A_0(y - 2\pi) + \sum_{n=1}^{\infty} A_n \cos\left(\frac{nx}{2}\right) \sinh\left(\frac{n(y - 2\pi)}{2}\right)$$

Substitute $u_1(x,0) = \cos x$.

$$A_0(-2\pi) + \sum_{n=1}^{\infty} A_n \cos\left(\frac{nx}{2}\right) \sinh(-n\pi) = \cos x$$

$$A_2 = -\frac{1}{\sinh(2\pi)}$$

Other than A_2, we know $A_0 = A_1 = A_3 = \ldots = A_n = 0$. Then the solution is

$$u_1(x, y) = -\frac{1}{\sinh(2\pi)} \cos x \sinh(y - 2\pi)$$

Solve for u_2,

$$\frac{\partial u_2}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u_2}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u_2}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u_2}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2v}{dx^2}w + v\frac{d^2w}{dy^2} = 0$$

so that we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

The boundary conditions are $w(2\pi) = 0$, $v'(0) = 0$, and $v'(2\pi) = 0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_0 = 0 \qquad\qquad v_0 = C$$

$$\lambda_n = \left(\frac{n\pi}{2\pi}\right)^2 = \frac{n^2}{4} \qquad v_n(x) = C\cos\left(\frac{nx}{2}\right) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

For $\lambda_0 = 0$, the solution is

$$w_0(y) = c_1 + c_2 y$$

$w(0) = 0$, so $c_1 = 0$ and $w_0(y) = c_2 y$

For $\lambda_n = n^2/4$, the solution is

$$w(y) = c_1 e^{-\sqrt{\lambda}y} + c_2 e^{\sqrt{\lambda}y}$$

Equivalently, this can be rewritten as

$$w(y) = c_1 \sinh(\sqrt{\lambda}y) + c_2 \cosh(\sqrt{\lambda}y)$$

which allows us to apply $w(0) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$w(y) = c_1 \sinh(\sqrt{\lambda}y)$$

$$w(y) = c_1 \sinh\left(\frac{ny}{2}\right)$$

Then the product solution is

$$u_2(x, y) = v(x)w(y)$$

$$u_2(x, y) = A_0 y + \sum_{n=1}^{\infty} A_n \cos\left(\frac{nx}{2}\right) \sinh\left(\frac{ny}{2}\right)$$

Substitute $u_2(x,2\pi) = \cos x$.

$$A_0(2\pi) + \sum_{n=1}^{\infty} A_n \cos\left(\frac{nx}{2}\right) \sinh(n\pi) = \cos x$$

$$A_2 = \frac{1}{\sinh(2\pi)}$$

Other than A_2, we know $A_0 = A_1 = A_3 = \ldots = A_n = 0$. Then the solution is

$$u_2(x, y) = \frac{1}{\sinh(2\pi)} \cos x \sinh y$$

And therefore,

$$u(x, y) = -\frac{1}{\sinh(2\pi)} \cos x \sinh(y - 2\pi) + \frac{1}{\sinh(2\pi)} \cos x \sinh y$$

$$u(x, y) = \frac{\cos x}{\sinh(2\pi)}(\sinh y - \sinh(y - 2\pi))$$

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