Differential Equations Quizzes

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MATH

Topic: Classifying differential equations

Question: Which equation is nonlinear?

Answer choices:

A $\quad yy' = x^2y - y^2x$

B $\quad y = e^xy' + 2x^3$

C $\quad y' - 2y = e^{x+y}$

D $\quad x^3(y'' - x) = y$

Solution: C

Answer choice A is a linear equation, which we can see if we rewrite it.

$$yy' = x^2y - y^2x$$

$$y' = x^2 - yx$$

$$y' + xy = x^2$$

Answer choice B is a linear equation, which we can see if we rewrite it.

$$y = e^xy' + 2x^3$$

$$e^xy' - y = -2x^3$$

Because the equation $y' - 2y = e^{x+y}$ contains e^y, it's non-linear.

Answer choice D is a linear equation, which we can see if we rewrite it.

$$x^3(y'' - x) = y$$

$$x^3y'' - x^4 = y$$

$$x^3y'' - y = x^4$$

Topic: Classifying differential equations

Question: Which equation is a third order linear equation?

Answer choices:

A $y'' - 2y + 3 = 0$

B $yy''' - 2x = y^2$

C $y''' + 3y = e^x$

D $yy' + ye^x = y^2$

Solution: C

The order of a differential equation is equivalent to the degree of the highest-degree derivative that appears in the equation.

Answer choices B and C contain a third derivative, so they're both third order differential equations.

But the equation $yy''' - 2x = y^2$ in answer choice B is nonlinear,

$$yy''' - 2x = y^2$$

$$yy''' - y^2 = 2x$$

$$y''' - y = \frac{2x}{y}$$

because $Q(x) = 2x/y$. Differential equations are linear when $Q(x)$ is x only, not in y.

Topic: Classifying differential equations

Question: Which linear equation is homogeneous?

Answer choices:

A $y'' + y = y' - 3$

B $y' = e^x y$

C $x^2 y''' + x = y + x^2 y$

D $xy'' - y' = y \sin x + x^2$

Solution: B

A linear equation is homogeneous when $Q(x) = 0$. Rewriting each of the four answer choices in standard form gives

$$y'' - y' + y = -3$$

$$y' - e^x y = 0$$

$$x^2 y''' - (1 + x^2)y = -x$$

$$xy'' - y' - (\sin x)y = x^2$$

Only the equation in answer choice B is homogeneous with $Q(x) = 0$.

Topic: Linear equations

Question: Find the solution to the linear differential equation.

$$\frac{dy}{dx} + 8x^3y = 12x^3$$

Answer choices:

A $\quad y = Ce^{-2x^4}$

B $\quad y = \frac{1}{2} + Ce^{-2x^4}$

C $\quad y = \frac{3}{2} + Ce^{-2x^4}$

D $\quad y = \frac{5}{2} + Ce^{-2x^4}$

Solution: C

The linear differential equation is already in standard form, so we can identify $P(x)$ and $Q(x)$.

$$P(x) = 8x^3$$

$$Q(x) = 12x^3$$

We use $P(x)$ to find the integrating factor.

$$\mu(x) = e^{\int P(x)\ dx}$$

$$\mu(x) = e^{\int 8x^3\ dx}$$

$$\mu(x) = e^{2x^4}$$

Multiply through the differential equation by the integrating factor.

$$e^{2x^4}\left(\frac{dy}{dx} + 8x^3y = 12x^3\right)$$

$$e^{2x^4}\frac{dy}{dx} + e^{2x^4}8x^3y = 12x^3e^{2x^4}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{2x^4}y) = 12x^3e^{2x^4}$$

then integrate, using a simple substitution with $u = 2x^4$ and $du/8 = x^3\ dx$ to integrate the right side.

$$\int \frac{d}{dx}(e^{2x^4}y) = \int 12x^3e^{2x^4}\ dx$$

$$e^{2x^4}y = 12\int e^u\left(\frac{du}{8}\right)$$

$$e^{2x^4}y = \frac{3}{2}e^u + C$$

$$e^{2x^4}y = \frac{3}{2}e^{2x^4} + C$$

$$y = \frac{3}{2} + Ce^{-2x^4}$$

Topic: Linear equations

Question: Find the solution to the linear differential equation.

$$xy' + x^2 - x^2y = 0$$

Answer choices:

A $y = 1 + Ce^{\frac{x^2}{2}}$

B $y = 3 + Ce^{\frac{x^2}{2}}$

C $y = 3 + Ce^{\frac{x^2}{4}}$

D $y = 1 + Ce^{\frac{x^2}{4}}$

Solution: A

Put the linear differential equation in standard form.

$$xy' + x^2 - x^2y = 0$$

$$y' + x - xy = 0$$

$$y' - xy = -x$$

Now we can identify $P(x)$ and $Q(x)$.

$$P(x) = -x$$

$$Q(x) = -x$$

We use $P(x)$ to find the integrating factor.

$$\mu(x) = e^{\int P(x)\ dx}$$

$$\mu(x) = e^{\int -x\ dx}$$

$$\mu(x) = e^{-\frac{x^2}{2}}$$

Multiply through the differential equation by the integrating factor.

$$e^{-\frac{x^2}{2}}(y' - xy = -x)$$

$$e^{-\frac{x^2}{2}}y' - e^{-\frac{x^2}{2}}xy = -xe^{-\frac{x^2}{2}}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{-\frac{x^2}{2}}y) = -xe^{-\frac{x^2}{2}}$$

then integrate, using a simple substitution with $u = -x^2/2$ and $du = -x\ dx$ to integrate the right side.

$$\int \frac{d}{dx}(e^{-\frac{x^2}{2}}y) = \int -xe^{-\frac{x^2}{2}}\ dx$$

$$e^{-\frac{x^2}{2}}y = \int e^u\ du$$

$$e^{-\frac{x^2}{2}}y = e^u + C$$

$$e^{-\frac{x^2}{2}}y = e^{-\frac{x^2}{2}} + C$$

Solve for y to find the solution to the linear differential equation.

$$y = 1 + Ce^{\frac{x^2}{2}}$$

Topic: Linear equations

Question: Find the solution to the linear differential equation.

$$xy' - 2y = x^2$$

Answer choices:

A $\quad y = x^2 \ln x + C$

B $\quad y = \dfrac{x^2}{\ln x}$

C $\quad y = Cx^2 \ln x$

D $\quad y = x^2(\ln x + C)$

Solution: D

Put the linear differential equation in standard form.

$$xy' - 2y = x^2$$

$$y' - \frac{2}{x}y = x$$

Now we can identify $P(x)$ and $Q(x)$.

$$P(x) = -\frac{2}{x}$$

$$Q(x) = x$$

We use $P(x)$ to find the integrating factor.

$$\mu(x) = e^{\int -\frac{2}{x}\ dx}$$

$$\mu(x) = e^{-2\ln x}$$

$$\mu(x) = e^{\ln x^{-2}}$$

$$\mu(x) = x^{-2}$$

Multiply through the differential equation by the integrating factor.

$$x^{-2}\left(y' - \frac{2}{x}y = x\right)$$

$$x^{-2}y' - 2x^{-3}y = x^{-1}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(x^{-2}y) = x^{-1}$$

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{1}{x}$$

then integrate.

$$\int \frac{d}{dx}\left(\frac{y}{x^2}\right) = \int \frac{1}{x}\ dx$$

$$\frac{y}{x^2} = \ln x + C$$

Solve for y to find the solution to the linear differential equation.

$$y = x^2(\ln x + C)$$

Topic: Initial value problems

Question: What can we interpret from an initial condition of $y(5) = 0$?

Answer choices:

A $C = 0$

B $x = 0$ and $y = 5$

C $C = 5$

D $x = 5$ and $y = 0$

Solution: D

An initial condition for a first order differential equation will take the form $y(x_0) = y_0$. To use it, we substitute $x = x_0$ and $y = y_0$ into the general solution to find the associated value of C. In our case, $x = 5$ and $y = 0$.

Topic: Initial value problems

Question: If $y(0) = -1$, solve the initial value problem.

$$y' + y = -x^2$$

Answer choices:

A $y = -x^2 + 2x - 2 + e^{-x}$

B $y = -x^2 + 2x - 2$

C $y = -x^2 + 2x - 2 - e^{-x}$

D $y = -x^2 - 2x + 2 - 3e^{-x}$

Solution: A

The linear differential equation is already in standard form, so we can identify $P(x)$ and $Q(x)$.

$$P(x) = 1$$

$$Q(x) = -x^2$$

We use $P(x)$ to find the integrating factor.

$$\mu(x) = e^{\int P(x)\ dx}$$

$$\mu(x) = e^{\int 1\ dx}$$

$$\mu(x) = e^x$$

Multiply through the differential equation by the integrating factor.

$$e^x\left(\frac{dy}{dx} + y = -x^2\right)$$

$$e^x\frac{dy}{dx} + e^x y = -x^2e^x$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^x y) = -x^2e^x$$

then integrate, using integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = e^x\ dx$, and $v = e^x$ to integrate the right side.

$$\int \frac{d}{dx}(e^x y) = -\int x^2e^x\ dx$$

$$e^x y = -\left(x^2e^x - \int 2xe^x\ dx\right)$$

$$e^x y = -x^2e^x + \int 2xe^x\ dx$$

Apply integration by parts a second time, this time with $u = 2x$, $du = 2\ dx$, $dv = e^x\ dx$, and $v = e^x$.

$$e^x y = -x^2e^x + \left(2xe^x - \int 2e^x\ dx\right)$$

$$e^x y = -x^2e^x + 2xe^x - 2e^x + C$$

$$y = -x^2 + 2x - 2 + \frac{C}{e^x}$$

Once we have this general solution, we recognize from the initial condition $y(0) = -1$ that $x = 0$ and $y = -1$, so we'll plug these values into the general solution,

$$-1 = -0^2 + 2(0) - 2 + \frac{C}{e^0}$$

and then simplify to solve for C.

$$-1 = -2 + C$$

$$C = 1$$

So the solution is

$$y = -x^2 + 2x - 2 + \frac{1}{e^x}$$

Topic: Initial value problems

Question: A function $y(x)$ is a solution of $y' + ky = 0$. Suppose that $y(0) = 1$ and $y(1) = e$. Find the value of the constant k.

Answer choices:

A -1

B 1

C 0

D e

Solution: A

The linear differential equation is already in standard form, so we can identify $P(x)$ and $Q(x)$.

$$P(x) = k$$

$$Q(x) = 0$$

We use $P(x)$ to find the integrating factor.

$$\mu(x) = e^{\int P(x)\ dx}$$

$$\mu(x) = e^{\int k\ dx}$$

$$\mu(x) = e^{kx}$$

Multiply through the differential equation by the integrating factor.

$$e^{kx}\left(\frac{dy}{dx} + ky = 0\right)$$

$$e^{kx}\frac{dy}{dx} + kye^{kx} = 0e^{kx}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(e^{kx}y) = 0$$

then integrate.

$$\int \frac{d}{dx}(e^{kx}y) = \int 0\ dx$$

$$e^{kx}y = C$$

$$y = Ce^{-kx}$$

Once we have this general solution, we recognize from the initial condition $y(0) = 1$ that $x = 0$ and $y = 1$ and from the initial condition $y(1) = e$ that $x = 1$ and $y = e$, so we'll plug these values into the general solution,

$$1 = Ce^{-k(0)}$$

$$e = Ce^{-k(1)}$$

and then simplify these to find C and k.

$$C = 1$$

$$e = e^{-k}$$

$$k = -1$$

We weren't asked for it, but now we also know that the solution to the differential equation is

$$y = 1e^{-(-1)x}$$

$$y = e^{x}$$

Topic: Separable equations

Question: Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{\sec^2 x}{\tan^2 y}$$

Answer choices:

A $\tan y - y = C \tan x$

B $\tan y - y = \tan x + C$

C $y = \tan x + C$

D $y(\tan y - 1) = \tan x + C$

Solution: B

Start by separating variables, putting y terms on the left and x terms on the right.

$$\frac{dy}{dx} = \frac{\sec^2 x}{\tan^2 y}$$

$$dy = \frac{\sec^2 x}{\tan^2 y}\ dx$$

$$\tan^2 y\ dy = \sec^2 x\ dx$$

Now integrate both sides, adding the constant of integration C to the right side.

$$\int \tan^2 y\ dy = \int \sec^2 x\ dx$$

$$\int \sec^2 y - 1\ dy = \int \sec^2 x\ dx$$

$$\tan y - y = \tan x + C$$

We'll let this be the implicit solution to the separable differential equation.

Topic: Separable equations

Question: Find the solution to the separable differential equation.

$$\frac{du}{dv} = \frac{3v\sqrt{1+u^2}}{u}$$

Answer choices:

A $\quad u = \sqrt{\frac{3}{2}v^2} + C$

B $\quad u = \sqrt{\frac{3}{2}v^2 + C}$

C $\quad u = \sqrt{\left(\frac{3}{2}v^2 + C\right)^2 - \frac{3}{2}}$

D $\quad u = \sqrt{\left(\frac{3}{2}v^2 + C\right)^2 - 1}$

Solution: D

Start by separating variables, putting u terms on the left and v terms on the right.

$$\frac{du}{dv}=\frac{3v\sqrt{1+u^2}}{u}$$

$$du=\frac{3v\sqrt{1+u^2}}{u}\ dv$$

$$u\ du=3v\sqrt{1+u^2}\ dv$$

$$\frac{u}{\sqrt{1+u^2}}\ du=3v\ dv$$

Now integrate both sides, using a substitution for the left side with $k=1+u^2$ and $du=dk/2u$, and adding the constant of integration C to the right side.

$$\int\frac{u}{\sqrt{k}}\left(\frac{dk}{2u}\right)=\int 3v\ dv$$

$$\frac{1}{2}\int k^{-\frac{1}{2}}\ dk=\int 3v\ dv$$

$$k^{\frac{1}{2}}=\frac{3}{2}v^2+C$$

Back-substitute for $k=1+u^2$.

$$(1+u^2)^{\frac{1}{2}}=\frac{3}{2}v^2+C$$

$$\sqrt{1+u^2}=\frac{3}{2}v^2+C$$

Solve for u to get the explicit the solution to the separable differential equation.

$$1+u^2=\left(\frac{3}{2}v^2+C\right)^2$$

$$u^2=\left(\frac{3}{2}v^2+C\right)^2-1$$

$$u=\sqrt{\left(\frac{3}{2}v^2+C\right)^2-1}$$

Topic: Separable equations

Question: Which first order differential equation is not separable?

Answer choices:

A $y' = e^{x-y}$

B $y' = 1 + \dfrac{x}{y}$

C $y' = y + \dfrac{y}{x}$

D $y' = \dfrac{x}{1+y}$

Solution: B

Answer choice A can be rewritten as

$$y' = e^{x-y}$$

$$\frac{dy}{dx} = e^x e^{-y}$$

$$\frac{dy}{dx} = \frac{e^x}{e^y}$$

$$e^y \ dy = e^x \ dx$$

The variables in answer choice B can't be separated, so the equation isn't separable.

Answer choice C can be rewritten as

$$y' = y + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{xy}{x} + \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y(x+1)}{x}$$

$$\frac{1}{y} \ dy = \frac{x+1}{x} \ dx$$

Answer choice D can be rewritten as

$$y' = \frac{x}{1+y}$$

$$dy = \frac{x}{1+y}\ dx$$

$$(1+y)\ dy = x\ dx$$

Topic: Substitutions

Question: Use a substitution to solve the separable differential equation.

$$y' = x + y$$

Answer choices:

A $y = Ce^x + x + 2$

B $y = Ce^x - x - 2$

C $y = Ce^x + x + 1$

D $y = Ce^x - x - 1$

Solution: D

If we choose the substitution $u = ax + by$, then we can set up the substitution

$$u = x + y$$

$$u' = 1 + y'$$

and then solve this second equation $u' = 1 + y'$ for y' to get $y' = u' - 1$. Then we'll substitute $y' = u' - 1$ into the left side of the original differential equation, and $u = x + y$ into the right side of the original differential equation, and we'll get

$$u' - 1 = u$$

$$u' = u + 1$$

$$\frac{du}{dx} = u + 1$$

Separate variables,

$$du = (u + 1)\ dx$$

$$\frac{1}{u + 1}\ du = dx$$

then integrate both sides.

$$\int \frac{1}{u + 1}\ du = \int dx$$

$$\ln|u + 1| = x + C$$

Now we'll solve for u.

$$e^{\ln|u+1|} = e^{x+C}$$

$$|u+1| = e^{x+C}$$

$$u+1 = Ce^x$$

$$u = Ce^x - 1$$

Back-substitute for $u = x + y$, then solve for y.

$$x + y = Ce^x - 1$$

$$y = Ce^x - x - 1$$

Topic: Substitutions

Question: Use a substitution to solve the separable differential equation.

$$y' = 3x + y$$

Answer choices:

A $y = Ce^x + 3x + 3$

B $y = Ce^x - 3x - 3$

C $y = Ce^x + x + 1$

D $y = Ce^x - x - 1$

Solution: B

If we choose the substitution $u = ax + by$, then we can set up the substitution

$$u = 3x + y$$

$$u' = 3 + y'$$

and then solve this second equation $u' = 3 + y'$ for y' to get $y' = u' - 3$. Then we'll substitute $y' = u' - 3$ into the left side of the original differential equation, and $u = 3x + y$ into the right side of the original differential equation, and we'll get

$$u' - 3 = u$$

$$u' = u + 3$$

$$\frac{du}{dx} = u + 3$$

Separate variables,

$$du = (u + 3)\ dx$$

$$\frac{1}{u + 3}\ du = dx$$

then integrate both sides.

$$\int \frac{1}{u + 3}\ du = \int dx$$

$$\ln|u + 3| = x + C$$

Now we'll solve for u.

$$e^{\ln|u+3|} = e^{x+C}$$

$$|u+3| = e^{x+C}$$

$$u + 3 = Ce^x$$

$$u = Ce^x - 3$$

Back-substitute for $u = 3x + y$, then solve for y.

$$3x + y = Ce^x - 3$$

$$y = Ce^x - 3x - 3$$

Topic: Substitutions

Question: Use a substitution to solve the separable differential equation.

$$y' = 2x + 2y$$

Answer choices:

A $\quad y = Ce^x - 2x - 2$

B $\quad y = Ce^x - x - 1$

C $\quad y = Ce^{2x} - x - \dfrac{1}{2}$

D $\quad y = \dfrac{1}{2}Ce^x - x - \dfrac{1}{4}$

Solution: C

If we choose the substitution $u = ax + by$, then we can set up the substitution

$$u = 2x + 2y$$

$$u' = 2 + 2y'$$

and then solve this second equation $u' = 2 + 2y'$ for y' to get $y' = (1/2)u' - 1$. Then we'll substitute $y' = (1/2)u' - 1$ into the left side of the original differential equation, and $u = 2x + 2y$ into the right side of the original differential equation, and we'll get

$$\frac{1}{2}u' - 1 = u$$

$$u' - 2 = 2u$$

$$u' = 2u + 2$$

$$\frac{du}{dx} = 2u + 2$$

Separate variables,

$$du = (2u + 2)\ dx$$

$$\frac{1}{2u + 2}\ du = dx$$

then integrate both sides.

$$\int \frac{1}{2u+2}\ du = \int dx$$

$$\frac{1}{2}\ln|u+1| = x + C$$

Now we'll solve for u,

$$\ln|u+1| = 2x + C$$

$$e^{\ln|u+1|} = e^{2x+C}$$

$$|u+1| = e^{2x+C}$$

$$u + 1 = Ce^{2x}$$

$$u = Ce^{2x} - 1$$

Back-substitute for $u = 2x + 2y$, then solve for y.

$$2x + 2y = Ce^{2x} - 1$$

$$2y = Ce^{2x} - 2x - 1$$

$$y = Ce^{2x} - x - \frac{1}{2}$$

Topic: Bernoulli equations

Question: Find the solution to the Bernoulli differential equation.

$$y' + \frac{y}{x} = xy^2$$

Answer choices:

A $\quad y = \dfrac{1}{-x^2 + Cx}$

B $\quad y = -x^2 + Cx$

C $\quad y = x^2 + C$

D $\quad y = \dfrac{1}{\sqrt{-x^2 + Cx}}$

Solution: A

With the equation in standard form, divide through by y^n. In this equation, that means we're dividing by y^2.

$$y' + \frac{y}{x} = xy^2$$

$$\frac{y'}{y^2} + \frac{y}{xy^2} = \frac{xy^2}{y^2}$$

$$y^{-2}y' + \frac{1}{x}y^{-1} = x$$

Our substitution is $v = y^{-1}$, so we'll differentiate to get

$$v' = -y^{-2}y'$$

and then solve this for $y^{-2}y'$.

$$y^{-2}y' = -v'$$

Now we can make substitutions into the Bernoulli equation.

$$y^{-2}y' + \frac{1}{x}y^{-1} = x$$

$$-v' + \frac{1}{x}v = x$$

Multiplying through by -1 puts the equation in standard form of a linear differential equation.

$$v' - \frac{1}{x}v = -x$$

To solve the linear equation, we'll use $P(x) = -1/x$ to find the integrating factor,

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int -\frac{1}{x}\ dx}$$

$$I(x) = e^{-\ln x}$$

$$I(x) = e^{\ln(x^{-1})}$$

$$I(x) = x^{-1}$$

and then multiply through the linear equation by $I(x)$.

$$v' - \frac{1}{x}v = -x$$

$$x^{-1}v' - \frac{1}{x}vx^{-1} = -x^{-1}x$$

$$\frac{1}{x}v' - \frac{1}{x^2}v = -1$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(vx^{-1}) = -1$$

then integrate both sides.

$$\int \frac{d}{dx}(vx^{-1}) = \int -1\ dx$$

$$vx^{-1} = -x + C$$

Solve for v.

$$v = -x^2 + Cx$$

Use $v = y^{-1}$ to back-substitute for v,

$$y^{-1} = -x^2 + Cx$$

then solve for y.

$$\frac{1}{y} = -x^2 + Cx$$

$$y = \frac{1}{-x^2 + Cx}$$

Topic: Bernoulli equations

Question: If $y(1) = 1$, find the solution to the Bernoulli differential equation.

$$xy' - y = y^3 \ln x$$

Answer choices:

A $\quad y = \dfrac{1}{\ln x + 1}$

B $\quad y = \dfrac{1}{\sqrt{-\ln x + 1}}$

C $\quad y = \dfrac{1}{2} - \ln x + \dfrac{1}{2}x^{-2}$

D $\quad y = \dfrac{1}{\sqrt{\frac{1}{2} - \ln x + \frac{1}{2}x^{-2}}}$

Solution: D

Start by rewriting the equation in standard form.

$$xy' - y = y^3 \ln x$$

$$y' - \frac{y}{x} = \frac{y^3 \ln x}{x}$$

With the equation in standard form, divide through by y^n. In this equation, that means we're dividing by y^3.

$$\frac{y'}{y^3} - \frac{y}{xy^3} = \frac{y^3 \ln x}{xy^3}$$

$$y'y^{-3} - \frac{1}{x}y^{-2} = \frac{\ln x}{x}$$

Our substitution is $v = y^{-2}$, so we'll differentiate to get

$$v' = -2y^{-3}y'$$

and then solve this for $y^{-3}y'$.

$$y^{-3}y' = -\frac{1}{2}v'$$

Now we can make substitutions into the Bernoulli equation.

$$y'y^{-3} - \frac{1}{x}y^{-2} = \frac{\ln x}{x}$$

$$-\frac{1}{2}v' - \frac{1}{x}v = \frac{\ln x}{x}$$

Multiplying through by -2 puts the equation in standard form of a linear differential equation.

$$v' + \frac{2}{x}v = -\frac{2\ln x}{x}$$

To solve the linear equation, we'll use $P(x) = 2/x$ to find the integrating factor,

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int \frac{2}{x}\ dx}$$

$$I(x) = e^{2\ln x}$$

$$I(x) = e^{\ln x^2}$$

$$I(x) = x^2$$

and then multiply through the linear equation by $I(x)$.

$$v' + \frac{2}{x}v = -\frac{2\ln x}{x}$$

$$x^2v' + \frac{2}{x}x^2v = -\frac{2\ln x}{x}x^2$$

$$x^2v' + 2xv = -2x\ln x$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(vx^2) = -2x\ln x$$

then integrate both sides. We'll use integration by parts on the right with $u = \ln x$, $du = (1/x)\ dx$, $dv = -2x\ dx$, and $v = -x^2$.

$$\int \frac{d}{dx}(vx^2) = \int -2x \ln x\ dx$$

$$vx^2 = -x^2 \ln x - \int -x^2 \left(\frac{1}{x}\right)\ dx$$

$$vx^2 = -x^2 \ln x + \int x\ dx$$

$$vx^2 = \frac{1}{2}x^2 - x^2 \ln x + C$$

Solve for v.

$$v = \frac{1}{2} - \ln x + Cx^{-2}$$

Use $v = y^{-2}$ to back-substitute for v,

$$y^{-2} = \frac{1}{2} - \ln x + Cx^{-2}$$

then solve for y.

$$\frac{1}{y^2} = \frac{1}{2} - \ln x + Cx^{-2}$$

$$y^2 = \frac{1}{\frac{1}{2} - \ln x + Cx^{-2}}$$

$$y = \frac{1}{\sqrt{\frac{1}{2} - \ln x + Cx^{-2}}}$$

Once we have this general solution, we recognize from the initial condition $y(1) = 1$ that $x = 1$ and $y = 1$, so we'll plug these values into the general solution to find C.

$$1 = \frac{1}{\sqrt{\frac{1}{2} - \ln 1 + 1^{-2}C}}$$

$$1 = \frac{1}{\sqrt{\frac{1}{2} + C}}$$

$$1 = \frac{1}{2} + C$$

$$C = \frac{1}{2}$$

Then the solution is

$$y = \frac{1}{\sqrt{\frac{1}{2} - \ln x + \frac{1}{2}x^{-2}}}$$

Topic: Bernoulli equations

Question: If $y(0) = 1$, find the solution to the Bernoulli differential equation.

$$y' + y = x^2y^2$$

Answer choices:

A $\quad y = \dfrac{1}{x^2 + 2x + 2 - e^{-x}}$

B $\quad y = \dfrac{1}{x^2 + 2x + 2 - e^{x}}$

C $\quad y = x^2 + 2x + 2 - e^{x}$

D $\quad y = \dfrac{1}{x^2 + 2x + 2}$

Solution: B

With the equation in standard form, divide through by y^n. In this equation, that means we're dividing by y^2.

$$y' + y = x^2y^2$$

$$\frac{y'}{y^2} + \frac{y}{y^2} = \frac{x^2y^2}{y^2}$$

$$y^{-2}y' + y^{-1} = x^2$$

Our substitution is $v = y^{-1}$, so we'll differentiate to get

$$v' = -y^{-2}y'$$

and then solve this for $y^{-2}y'$.

$$y^{-2}y' = -v'$$

Now we can make substitutions into the Bernoulli equation.

$$y^{-2}y' + y^{-1} = x^2$$

$$-v' + v = x^2$$

Multiplying through by -1 puts the equation in standard form of a linear differential equation.

$$v' - v = -x^2$$

To solve the linear equation, we'll use $P(x) = -1$ to find the integrating factor,

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int -1\ dx}$$

$$I(x) = e^{-x}$$

and then multiply through the linear equation by $I(x)$.

$$e^{-x}v' - e^{-x}v = -e^{-x}x^2$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(ve^{-x}) = -x^2e^{-x}$$

then integrate both sides. We'll use integration by parts on the right with $u = -x^2$, $du = -2x\ dx$, $dv = e^{-x}\ dx$, and $v = -e^{-x}$.

$$\int \frac{d}{dx}(ve^{-x})\ dx = \int -x^2e^{-x}\ dx$$

$$ve^{-x} = x^2e^{-x} - 2\int xe^{-x}\ dx$$

We'll apply integration by parts again on the right with $u = x$, $du = dx$, $dv = e^{-x}\ dx$, and $v = -e^{-x}$.

$$ve^{-x} = x^2e^{-x} - 2\left(-xe^{-x} + \int e^{-x}\ dx\right)$$

$$ve^{-x} = x^2e^{-x} + 2xe^{-x} + 2e^{-x} + C$$

Solve for v.

$$v = x^2 + 2x + 2 + Ce^x$$

Use $v = y^{-1}$ to back-substitute for v,

$$y^{-1} = x^2 + 2x + 2 + Ce^x$$

then solve for y.

$$\frac{1}{y} = x^2 + 2x + 2 + Ce^x$$

$$y = \frac{1}{x^2 + 2x + 2 + Ce^x}$$

Once we have this general solution, we recognize from the initial condition $y(0) = 1$ that $x = 0$ and $y = 1$, so we'll plug these values into the general solution to find C.

$$1 = \frac{1}{0^2 + 2(0) + 2 + Ce^0}$$

$$1 = \frac{1}{2 + C}$$

$$1 = 2 + C$$

$$C = -1$$

Then the solution is

$$y = \frac{1}{x^2 + 2x + 2 - e^x}$$

Topic: Homogeneous equations

Question: Find the solution to the differential equation.

$$(x+y)y' = x - y$$

Answer choices:

A $C = x^2 + 2xy + y^2$

B $C = \dfrac{1}{x^2 + 2xy + y^2}$

C $C = x^2 - 2xy - y^2$

D $C = 1 - 2xy - y^2$

Solution: C

We start by solving for y',

$$y' = \frac{x - y}{x + y}$$

then we'll multiply through the numerator and denominator on the right side by $1/x$.

$$y' = \frac{x - y}{x + y} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$y' = \frac{x\left(\frac{1}{x}\right) - y\left(\frac{1}{x}\right)}{x\left(\frac{1}{x}\right) + y\left(\frac{1}{x}\right)}$$

$$y' = \frac{1 - \frac{y}{x}}{1 + \frac{y}{x}}$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = \frac{1 - v}{1 + v}$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv' = \frac{1 - v}{1 + v} - v$$

$$x\frac{dv}{dx} = \frac{1 - 2v - v^2}{1 + v}$$

$$\frac{1 + v}{1 - 2v - v^2}\ dv = \frac{1}{x}\ dx$$

and then integrate both sides, using a substitution for the integral on the left, with $u = 1 - 2v - v^2$ and $du = (-2 - 2v)\ dv$, and $dv = (1/(-2 - 2v))\ du$.

$$\int \frac{1 + v}{1 - 2v - v^2}\ dv = \int \frac{1}{x}\ dx$$

$$\int \frac{1 + v}{u}\left(\frac{1}{-2 - 2v}\ du\right) = \int \frac{1}{x}\ dx$$

$$-\frac{1}{2}\int \frac{1 + v}{u}\left(\frac{1}{1 + v}\ du\right) = \int \frac{1}{x}\ dx$$

$$-\frac{1}{2}\int \frac{1}{u}\ du = \int \frac{1}{x}\ dx$$

$$-\frac{1}{2}\ln u = \ln x + C$$

$$\ln u^{-\frac{1}{2}} = \ln x + C$$

$$e^{\ln u^{-\frac{1}{2}}} = e^{\ln x + C}$$

$$u^{-\frac{1}{2}} = Cx$$

Back-substitute for u.

$$(1 - 2v - v^2)^{-\frac{1}{2}} = Cx$$

$$\frac{1}{1-2v-v^2}=Cx^2$$

$$1=Cx^2(1-2v-v^2)$$

Back-substitute for v.

$$1=Cx^2\left(1-2\frac{y}{x}-\frac{y^2}{x^2}\right)$$

$$1=C(x^2-2xy-y^2)$$

$$C=x^2-2xy-y^2$$

Topic: Homogeneous equations

Question: Find the solution to the differential equation.

$$x^3y' = x^2y + y^3$$

Answer choices:

A $\quad y = \pm \dfrac{x}{\sqrt{C - 2\ln x}}$

B $\quad y = \dfrac{1}{\sqrt{C - 2\ln x}}$

C $\quad y = \dfrac{x}{\sqrt{C - 2\ln x}}$

D $\quad y = \pm \dfrac{x}{\sqrt{C - \ln x}}$

Solution: A

To put the equation in standard form, we'll divide through by x^3.

$$\frac{x^3}{x^3}y' = \frac{x^2y}{x^3} + \frac{y^3}{x^3}$$

$$y' = \frac{y}{x} + \left(\frac{y}{x}\right)^3$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = v + v^3$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv' = v^3$$

$$x\frac{dv}{dx} = v^3$$

$$\frac{1}{v^3}\ dv = \frac{1}{x}\ dx$$

and then integrate both sides.

$$\int \frac{1}{v^3}\ dv = \int \frac{1}{x}\ dx$$

$$-\frac{1}{2v^2} = \ln x + C$$

$$\frac{1}{v^2} = -2\ln x + C$$

$$v^2 = \frac{1}{C - 2\ln x}$$

$$v = \pm\frac{1}{\sqrt{C - 2\ln x}}$$

Back-substitute for v.

$$\frac{y}{x} = \pm\frac{1}{\sqrt{C - 2\ln x}}$$

$$y = \pm\frac{x}{\sqrt{C - 2\ln x}}$$

Topic: Homogeneous equations

Question: Find the solution to the differential equation.

$$2xy' = 2y + 4\sqrt{xy}$$

Answer choices:

A $\quad y = x\sqrt{\ln x + C}$

B $\quad y = x(\ln x + C)^2$

C $\quad y = \sqrt{\ln x + C}$

D $\quad y = (\ln x + C)^2$

Solution: B

To put the equation in standard form, we'll divide through by $2x$.

$$\frac{2x}{2x}y' = \frac{2y}{2x} + \frac{4\sqrt{xy}}{2x}$$

$$y' = \frac{y}{x} + 2\sqrt{\frac{y}{x}}$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = v + 2\sqrt{v}$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv' = 2\sqrt{v}$$

$$x\frac{dv}{dx} = 2\sqrt{v}$$

$$\frac{1}{2\sqrt{v}}\ dv = \frac{1}{x}\ dx$$

and then integrate both sides.

$$\int \frac{1}{2\sqrt{v}}\ dv = \int \frac{1}{x}\ dx$$

$$\sqrt{v} = \ln x + C$$

$$v = (\ln x + C)^2$$

Back-substitute for v.

$$\frac{y}{x} = (\ln x + C)^2$$

$$y = x(\ln x + C)^2$$

Topic: Exact equations

Question: Which equation is an exact differential equation?

Answer choices:

A $(3x^2y - 3y^2)\ dx + (x^3 + 9xy^2)\ dy = 0$

B $(3x^2y - 3y^3)\ dx + (x^3 - 9xy^2)\ dy = 0$

C $(3x^2y - 5y^3)\ dy - (x^3 + 9xy^2)\ dx = 0$

D $(3x^2y - 3y^3)\ dy + (2x^2 + 9xy^2)\ dx = 0$

Solution: B

If we start with answer choice B, we can identify M and N as

$$M = 3x^2y - 3y^3$$

$$N = x^3 - 9xy^2$$

If we find $\partial M/\partial y$ and $\partial N/\partial x$,

$$\frac{\partial M}{\partial y} = 3x^2 - 9y^2$$

$$\frac{\partial N}{\partial x} = 3x^2 - 9y^2$$

Because these partial derivatives are equal,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

answer choice B is an exact differential equation.

We can use the same process to show that answer choice A is not an exact equation,

$$\frac{\partial}{\partial y}(3x^2y - 3y^2) \neq \frac{\partial}{\partial x}(x^3 + 9xy^2)$$

$$3x^2 - 6y \neq 3x^2 + 9y^2$$

that answer choice C is not an exact equation,

$$\frac{\partial}{\partial y}(-(x^3 + 9xy^2)) \neq \frac{\partial}{\partial x}(3x^2y - 5y^3)$$

$$\frac{\partial}{\partial y}(-x^3 - 9xy^2) \neq \frac{\partial}{\partial x}(3x^2y - 5y^3)$$

$$-18xy \neq 6xy$$

and that answer choice D is not an exact equation.

$$\frac{\partial}{\partial y}(2x^2 + 9xy^2) \neq \frac{\partial}{\partial x}(3x^2y - 3y^3)$$

$$18xy \neq 6xy$$

Topic: Exact equations

Question: Which values of a and b make the equation an exact differential equation?

$$(ay \sin x - \cos y)\ dx + (b \cos x + x \sin y)\ dy = 0$$

Answer choices:

A $(a, b) = (5,5)$

B $(a, b) = (-3, -5)$

C $(a, b) = (-5, -3)$

D $(a, b) = (5, -5)$

Solution: D

If the equation is exact, then its partial derivatives are equal.

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(ay\sin x-\cos y)=\frac{\partial}{\partial x}(b\cos x+x\sin y)$$

$$a\sin x+\sin y=-b\sin x+\sin y$$

These partial derivatives are only equal when $a=-b$, which is only true for answer choice D.

Alternatively, we could also substitute each answer choice into the equation, and then apply the partial derivatives test. If we use the values from answer choice D, we'll plug $a=5$ and $b=-5$ into the equation.

$$(5y\sin x-\cos y)\ dx+(-5\cos x+x\sin y)\ dy=0$$

Then we can use our partial derivatives test determine whether or not the equation is exact.

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(5y\sin x-\cos y)=\frac{\partial}{\partial x}(-5\cos x+x\sin y)$$

$$5\sin x+\sin y=5\sin x+\sin y$$

Since these partial derivatives are equal, the differential equation must be exact when $a = 5$ and $b = -5$.

We can use the same process to show that answer choice A is not an exact equation,

$$(5y\sin x - \cos y)\ dx + (5\cos x + x\sin y)\ dy = 0$$

$$\frac{\partial}{\partial y}(5y\sin x - \cos y) \neq \frac{\partial}{\partial x}(5\cos x + x\sin y)$$

$$5\sin x + \sin y \neq -5\sin x + \sin y$$

that answer choice B is not an exact equation,

$$(-3y\sin x - \cos y)\ dx + (-5\cos x + x\sin y)\ dy = 0$$

$$\frac{\partial}{\partial y}(-3y\sin x - \cos y) \neq \frac{\partial}{\partial x}(-5\cos x + x\sin y)$$

$$-3\sin x + \sin y \neq 5\sin x + \sin y$$

and that answer choice C is not an exact equation.

$$(-5y\sin x - \cos y)\ dx + (-3\cos x + x\sin y)\ dy = 0$$

$$\frac{\partial}{\partial y}(-5y\sin x - \cos y) \neq \frac{\partial}{\partial x}(-3\cos x + x\sin y)$$

$$-5\sin x + \sin y \neq 3\sin x + \sin y$$

Topic: Exact equations

Question: What is the solution to the exact differential equation?

$$(2xy + 2y)\ dx + (x^2 + 2x)\ dy = 0$$

Answer choices:

A $\quad c = x^2y - 2xy - y$

B $\quad c = x^2y + 2xy$

C $\quad c = 2x^2y + 2xy + y$

D $\quad c = x^2y + 4xy$

Solution: B

We'll check to see that the equation is exact.

$$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(2xy+2y)=\frac{\partial}{\partial x}(x^2+2x)$$

$$2x+2=2x+2$$

The functions $M(x,y)$ and $N(x,y)$ are equally easy to integrate, so we'll just use $M(x,y)$, and then Ψ can be given by

$$\Psi=\int M(x,y)\ dx+h(y)$$

$$\Psi=\int 2xy+2y\ dx+h(y)$$

$$\Psi=x^2y+2xy+h(y)$$

We'll differentiate both sides with respect to y,

$$\Psi_y=x^2+2x+h'(y)$$

and then substitute $\Psi_y=N(x,y)$ to solve for $h'(y)$.

$$x^2+2x=x^2+2x+h'(y)$$

$$0=h'(y)$$

To find $h(y)$, we'll integrate both sides of this equation with respect to y.

$$\int 0 \ dy = \int h'(y) \ dy$$

$$k_1 = h(y) + k_2$$

$$h(y) = k_2 - k_1$$

$$h(y) = k$$

Plugging this value for $h(y)$ into the equation for Ψ gives

$$\Psi = x^2y + 2xy + h(y)$$

$$\Psi = x^2y + 2xy + k$$

Setting $\Psi = c$ to find the solution to the exact differential equation, we get

$$c = x^2y + 2xy + k$$

$$c - k = x^2y + 2xy$$

$$c = x^2y + 2xy$$

Topic: Second order linear homogeneous equations

Question: Find the general solution to the second order equation.

$$y'' + 5y' + 4y = 0$$

Answer choices:

A $y(x) = c_1e^{4x} + c_2e^{x}$

B $y(x) = c_1e^{4x} + c_2e^{-x}$

C $y(x) = c_1e^{-4x} + c_2e^{-x}$

D $y(x) = c_1e^{-4x} + c_2e^{x}$

Solution: C

The characteristic equation associated with the differential equation is

$$r^2 + 5r + 4 = 0$$

This quadratic equation is easily factorable,

$$(r + 4)(r + 1) = 0$$

which means the roots will be

$$r_1 + 4 = 0 \qquad r_2 + 1 = 0$$

$$r_1 = -4 \qquad r_2 = -1$$

These are real-numbered distinct roots, so we call them distinct real roots. Therefore, the general solution to the second order equation is

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

$$y(x) = c_1e^{-4x} + c_2e^{-x}$$

Topic: Second order linear homogeneous equations

Question: Find the general solution to the second order equation.

$$y'' + 2y' + 11y = 0$$

Answer choices:

A $\quad y(x) = e^{-\sqrt{10}x}(c_1 \cos(-x) + c_2 \sin(-x))$

B $\quad y(x) = e^{\sqrt{10}x}(c_1 \cos(-x) + c_2 \sin(-x))$

C $\quad y(x) = e^{-x}(c_1 \cos(\sqrt{10}x) + c_2 \sin(\sqrt{10}x))$

D $\quad y(x) = e^{x}(c_1 \cos(\sqrt{10}x) + c_2 \sin(\sqrt{10}x))$

Solution: C

The characteristic equation associated with the differential equation is

$$r^2 + 2r + 11 = 0$$

The left side doesn't easily factor, so we'll apply the quadratic formula.

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(11)}}{2(1)}$$

$$r = \frac{-2 \pm \sqrt{-40}}{2}$$

Now we'll use the imaginary number to rewrite the root of the negative number.

$$r = \frac{-2 \pm \sqrt{(40)(-1)}}{2}$$

$$r = \frac{-2 \pm 2\sqrt{(10)(-1)}}{2}$$

$$r = -1 \pm \sqrt{10}i$$

These are complex-numbered roots, so we call them complex conjugate roots. If we match up $r = \alpha \pm \beta i$ with $r = -1 \pm \sqrt{10}i$, we identify $\alpha = -1$ and $\beta = \sqrt{10}$, and the general solution is therefore

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

$$y(x) = e^{-x}(c_1 \cos(\sqrt{10}x) + c_2 \sin(\sqrt{10}x))$$

Topic: Second order linear homogeneous equations

Question: If $y(0) = 2$ and $y'(0) = -5$, find the general solution to the second order equation.

$$25y'' + 30y' + 9y = 0$$

Answer choices:

A $\quad y(x) = 2e^{-\frac{3}{5}x} - \frac{19}{5}xe^{-\frac{3}{5}x}$

B $\quad y(x) = 2e^{-\frac{3}{5}x} - \frac{19}{5}e^{-\frac{3}{5}x}$

C $\quad y(x) = 2e^{-\frac{3}{5}x} - \frac{5}{7}xe^{-\frac{3}{5}x}$

D $\quad y(x) = 2e^{-\frac{3}{5}x} - \frac{5}{7}e^{-\frac{3}{5}x}$

Solution: A

The characteristic equation associated with this differential equation is

$$25r^2+30r+9=0$$

$$(5r+3)(5r+3)=0$$

The roots are therefore

$$5r_1+3=0 \qquad\qquad 5r_2+3=0$$

$$5r_1=-3 \qquad\qquad 5r_2=-3$$

$$r_1=-\frac{3}{5} \qquad\qquad r_2=-\frac{3}{5}$$

With these equal real roots, the general solution is

$$y(x)=c_1e^{-\frac{3}{5}x}+c_2xe^{-\frac{3}{5}x}$$

and its derivative is

$$y(x)=-\frac{3}{5}c_1e^{-\frac{3}{5}x}+c_2e^{-\frac{3}{5}x}-\frac{3}{5}c_2xe^{-\frac{3}{5}x}$$

We'll substitute the initial condition $y(0)=2$ into $y(x)$,

$$2=c_1e^{-\frac{3}{5}(0)}+c_2(0)e^{-\frac{3}{5}(0)}$$

$$2=c_1(1)+c_2(0)$$

$$c_1=2$$

and the condition $y'(0)=-5$ into the derivative.

$$-5 = -\frac{3}{5}c_1 e^{-\frac{3}{5}(0)} + c_2 e^{-\frac{3}{5}(0)} - \frac{3}{5}c_2(0)e^{-\frac{3}{5}(0)}$$

$$-5 = -\frac{3}{5}c_1(1) + c_2(1)$$

$$-5 = -\frac{3}{5}c_1 + c_2$$

$$-25 = -3c_1 + 5c_2$$

Substitute $c_1 = 2$ into this equation.

$$-25 = -3(2) + 5c_2$$

$$-19 = 5c_2$$

$$c_2 = -\frac{19}{5}$$

So the general solution is

$$y(x) = c_1 e^{-\frac{3}{5}x} + c_2 x e^{-\frac{3}{5}x}$$

$$y(x) = 2e^{-\frac{3}{5}x} - \frac{19}{5}xe^{-\frac{3}{5}x}$$

Topic: Reduction of order

Question: Use reduction of order to find the general solution to the differential equation, given $y_1 = e^x$.

$$y'' - 2y' + y = 0$$

Answer choices:

A $\quad y(x) = c_1e^x + c_2e^{-x}$

B $\quad y(x) = c_1e^x + c_2xe^x$

C $\quad y(x) = c_1e^x + c_2e^{2x}$

D $\quad y(x) = c_1e^x + c_2x$

Solution: B

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = ve^x$$

$$y_2' = v'e^x + ve^x$$

$$y_2'' = v''e^x + 2v'e^x + ve^x$$

Plug these values into the homogeneous differential equation.

$$v''e^x + 2v'e^x + ve^x - 2(v'e^x + ve^x) + ve^x = 0$$

$$v''e^x = 0$$

We'll make a substitution $w = v'$, and therefore $w' = v''$.

$$w'e^x = 0$$

This substitution changes the second order equation in v into a first order linear equation in w that we can easily solve.

$$e^x \frac{dw}{dx} = 0$$

$$\frac{dw}{dx} = 0$$

$$w = C$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w \ dx = \int C \ dx = Cx + k$$

We can choose any constants, so we'll choose $C = 1$ and $k = 0$ for simplicity. Then v is

$$v = x$$

and the second solution for the differential equation is

$$y_2 = ve^x$$

$$y_2 = xe^x$$

Because the solutions are $y_1 = e^x$ and $y_2 = xe^x$, the general solution to the differential equation is

$$y(x) = c_1e^x + c_2xe^x$$

Topic: Reduction of order

Question: Use reduction of order to find the general solution to the differential equation, given $y_1 = \cos x^2$.

$$y'' - \frac{1}{x}y' + 4x^2y = 0$$

Answer choices:

A $y = c_1 \cos x^2 + c_2 x \sin x^2$

B $y = c_1 \cos x^2 + c_2 \sin x$

C $y = e^x(c_1 \cos x^2 + c_2 \sin x^2)$

D $y = c_1 \cos x^2 + c_2 \sin x^2$

Solution: D

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = v\cos x^2$$

$$y_2' = v'\cos x^2 - 2xv\sin x^2$$

$$y_2'' = v''\cos x^2 - 4xv'\sin x^2 - 2v\sin x^2 - 4vx^2\cos x^2$$

Plug these values into the homogeneous differential equation.

$$v''\cos x^2 - 4xv'\sin x^2 - 2v\sin x^2 - 4vx^2\cos x^2$$

$$-\frac{1}{x}(v'\cos x^2 - 2xv\sin x^2) + 4x^2v\cos x^2 = 0$$

$$v''\cos x^2 - 4xv'\sin x^2 - 2v\sin x^2 - 4vx^2\cos x^2$$

$$-v'\frac{1}{x}\cos x^2 + 2v\sin x^2 + 4x^2v\cos x^2 = 0$$

$$v''\cos x^2 - v'\left(4x\sin x^2 + \frac{1}{x}\cos x^2\right) = 0$$

We'll make a substitution $w = v'$, and therefore $w' = v''$.

$$w'\cos x^2 - w\left(4x\sin x^2 + \frac{1}{x}\cos x^2\right) = 0$$

This substitution changes the second order equation in v into a first order linear equation in w that we can easily solve.

$$\cos x^2 \frac{dw}{dx} - w\left(4x \sin x^2 + \frac{1}{x}\cos x^2\right) = 0$$

$$\frac{dw}{dx} - w\left(\frac{4x \sin x^2}{\cos x^2} + \frac{\cos x^2}{x \cos x^2}\right) = 0$$

$$\frac{dw}{dx} - w\left(4x \tan x^2 + \frac{1}{x}\right) = 0$$

$$\frac{1}{w}\ dw = \left(4x \tan x^2 + \frac{1}{x}\right)\ dx$$

$$\int \frac{1}{w}\ dw = \int 4x \tan x^2 + \frac{1}{x}\ dx$$

$$\ln w = \ln x + C + \int 4x \tan x^2\ dx$$

Let $t = x^2$ and $dt = 2x\ dx$.

$$\ln w = \ln x + C + 2\int \tan t\ dt$$

$$\ln w = \ln x + C - 2\ln\cos t$$

$$\ln w = \ln x - 2\ln\cos x^2 + C$$

$$\ln w = \ln\frac{x}{\cos^2 x^2} + C$$

$$w = C\frac{x}{\cos^2 x^2}$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w\ dx = \int C \frac{x}{\cos^2 x^2}\ dx$$

Let $t = x^2$ and $dt = 2x\ dx$.

$$v = \int w\ dx = \frac{C}{2} \int \frac{1}{\cos^2 t}\ dt$$

$$v = \frac{C}{2} \tan t + k$$

$$v = \frac{C}{2} \tan x^2 + k$$

We can choose any constants, so we'll choose $C = 2$ and $k = 0$ for simplicity. Then v is

$$v = \tan x^2$$

and the second solution for the differential equation is

$$y_2 = v \cos x^2$$

$$y_2 = \tan x^2 \cos x^2$$

$$y_2 = \sin x^2$$

Because the solutions are $y_2 = \cos x^2$ and $y_2 = \sin x^2$, the general solution to the differential equation is

$$y(x) = c_1 \cos x^2 + c_2 \sin x^2$$

Topic: Reduction of order

Question: Use reduction of order to find the general solution to the differential equation, given $y_1 = \sin x$.

$$y'' + y = 0$$

Answer choices:

A $\quad y = e^x(c_1 \cos x + c_2 \sin x)$

B $\quad y = e^{-x}(c_1 \cos x + c_2 \sin x)$

C $\quad y = c_1 \sin x + c_2 \cos x$

D $\quad y = c_1 \sin x + c_2 x \cos x$

Solution: C

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = v \sin x$$

$$y_2' = v' \sin x + v \cos x$$

$$y_2'' = v'' \sin x + 2v' \cos x - v \sin x$$

Plug these values into the homogeneous differential equation.

$$v'' \sin x + 2v' \cos x - v \sin x + v \sin x = 0$$

$$v'' \sin x + 2v' \cos x = 0$$

We'll make a substitution $w = v'$, and therefore $w' = v''$.

$$w' \sin x + 2w \cos x = 0$$

This substitution changes the second order equation in v into a first order linear equation in w that we can easily solve.

$$\sin x \frac{dw}{dx} + 2w \cos x = 0$$

$$\frac{dw}{dx} + \frac{2 \cos x}{\sin x} w = 0$$

$$\frac{dw}{dx} + (2 \cot x)w = 0$$

With $P(x) = 2\cot x$ and $Q(x) = 0$, the integrating factor is

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int 2\cot x\ dx}$$

$$I(x) = e^{2\ln\sin x}$$

$$I(x) = \sin^2 x$$

Multiplying the first order differential equation in w by the integrating factor, and then simplifying, gives

$$\frac{d}{dx}(w\sin^2 x) = 0$$

Integrate both sides.

$$\int \frac{d}{dx}(w\sin^2 x)\ dx = \int 0\ dx$$

$$w\sin^2 x = C$$

$$w = \frac{C}{\sin^2 x}$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w\ dx = \int \frac{C}{\sin^2 x}\ dx = -C\cot x + k$$

We can choose any constants, so we'll choose $C = -1$ and $k = 0$ for simplicity. Then v is

$$v = \cot x$$

and the second solution for the differential equation is

$$y_2 = \cot x \sin x$$

$$y_2 = \cos x$$

Because the solutions are $y_1 = \sin x$ and $y_2 = \cos x$, the general solution to the differential equation is

$$y(x) = c_1 \sin x + c_2 \cos x$$

Topic: Undetermined coefficients for nonhomogeneous equations

Question: Find the general solution to the differential equation.

$$y'' + 3y' = 2x - 6e^{-3x}$$

Answer choices:

A $\quad y(x) = c_1 + c_2e^{-3x} + \frac{1}{3}x^2 + \frac{2}{9}x + 2xe^{-3x}$

B $\quad y(x) = c_1 + c_2e^{-3x} + \frac{1}{3}x^2 - \frac{2}{9}x + 2xe^{-3x}$

C $\quad y(x) = c_1 + c_2e^{-3x} + \frac{1}{3}x^2 + \frac{2}{9}x + 3xe^{-3x}$

D $\quad y(x) = c_1 + c_2e^{-3x} + \frac{1}{3}x^2 - \frac{2}{9}x + 3xe^{-3x}$

Solution: B

We'll replace $g(x) = 2x - 6e^{-3x}$ with $g(x) = 0$ to change the nonhomogeneous equation into a homogeneous equation

$$y'' + 3y' = 0$$

then we'll solve the associated characteristic equation.

$$r^2 + 3r = 0$$

$$r(r + 3) = 0$$

$$r = 0, \ -3$$

These are distinct real roots, so the complementary solution will be

$$y_c(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

$$y_c(x) = c_1e^{(0)x} + c_2e^{-3x}$$

$$y_c(x) = c_1(1) + c_2e^{-3x}$$

$$y_c(x) = c_1 + c_2e^{-3x}$$

For the particular solution, we'll use $Ax + B$ as our guess for the polynomial function, and we'll use Ce^{-3x} as our guess for the exponential function, so the guess will be

$$y_p(x) = Ax + B + Ce^{-3x}$$

Comparing this to the complementary solution, we can see that c_2e^{-3x} from the complementary solution and Ce^{-3x} from the particular solution are overlapping terms, so we'll multiply Ce^{-3x} from the particular solution by x, such that our guess becomes

$$y_p(x) = Ax + B + Cxe^{-3x}$$

If we go forward with this guess, we'll run into trouble when we take its derivatives. We'll lose the constant, and won't be able to equate coefficients. To fix this, we'll need to multiply $Ax + B$ by x as well.

$$y_p(x) = Ax^2 + Bx + Cxe^{-3x}$$

Taking the first and second derivatives of the guess gives

$$y_p'(x) = 2Ax + B + Ce^{-3x} - 3Cxe^{-3x}$$

$$y_p''(x) = 2A - 6Ce^{-3x} + 9Cxe^{-3x}$$

Plug the derivatives into the original differential equation.

$$2A - 6Ce^{-3x} + 9Cxe^{-3x} + 3(2Ax + B + Ce^{-3x} - 3Cxe^{-3x}) = 2x - 6e^{-3x}$$

$$2A - 6Ce^{-3x} + 9Cxe^{-3x} + 6Ax + 3B + 3Ce^{-3x} - 9Cxe^{-3x} = 2x - 6e^{-3x}$$

$$(2A + 3B) + 6Ax - 3Ce^{-3x} = 2x - 6e^{-3x}$$

$$(2A + 3B) + (6A)x + (-3C)e^{-3x} = (2)x + (-6)e^{-3x}$$

Equate coefficients.

$$2A + 3B = 0 \qquad 6A = 2 \qquad -3C = -6$$

$$2\left(\frac{1}{3}\right) + 3B = 0 \qquad A = \frac{1}{3} \qquad C = 2$$

$$3B = -\frac{2}{3}$$

$$B = -\frac{2}{9}$$

Then the particular solution is

$$y_p(x) = \frac{1}{3}x^2 - \frac{2}{9}x + 2xe^{-3x}$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 + c_2e^{-3x} + \frac{1}{3}x^2 - \frac{2}{9}x + 2xe^{-3x}$$

Topic: Undetermined coefficients for nonhomogeneous equations

Question: Find the general solution to the differential equation.

$$y'' + 6y' + 10y = 10x^2 - 8x - 10$$

Answer choices:

A $\quad y(x) = e^{x}(c_1 \sin(-3x) + c_2 \cos(-3x)) + x^2 - 2x$

B $\quad y(x) = e^{-3x}(c_1 \sin x + c_2 \cos x) + x^2 - 2x$

C $\quad y(x) = e^{x}(c_1 \sin(-3x) + c_2 \cos(-3x)) + x^2 + 2x$

D $\quad y(x) = e^{-3x}(c_1 \sin x + c_2 \cos x) - x^2 + 2x$

Solution: B

We'll replace $g(x) = 10x^2 - 8x - 10$ with $g(x) = 0$ to change the nonhomogeneous equation into a homogeneous equation

$$y'' + 6y' + 10y = 0$$

then we'll solve the associated characteristic equation.

$$r^2 + 6r + 10 = 0$$

$$r = \frac{-6 \pm 2i}{2}$$

$$r = -3 \pm i$$

These are complex roots, so the complementary solution will be

$$y_c(x) = e^{-3x}(c_1 \sin x + c_2 \cos x)$$

For the particular solution, we'll use $Ax^2 + Bx + C$ as our guess for the polynomial function.

$$y_p(x) = Ax^2 + Bx + C$$

Taking the first and second derivatives of the guess gives

$$y_p'(x) = 2Ax + B$$

$$y_p''(x) = 2A$$

Plug the derivatives into the original differential equation.

$$2A + 6(2Ax + B) + 10(Ax^2 + Bx + C) = 10x^2 - 8x - 10$$

$$2A + 12Ax + 6B + 10Ax^2 + 10Bx + 10C = 10x^2 - 8x - 10$$

$$10Ax^2 + x(12A + 10B) + 2A + 6B + +10C = 10x^2 - 8x - 10$$

Equate coefficients.

$10A = 10$ $\quad$ $12A + 10B = -8$ $\quad$ $2A + 6B + 10C = -10$

$A = 1$ $\quad$ $B = -2$ $\quad$ $C = 0$

Then the particular solution is

$$y_p(x) = x^2 - 2x$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = e^{-3x}(c_1 \sin x + c_2 \cos x) + x^2 - 2x$$

Topic: Undetermined coefficients for nonhomogeneous equations

Question: Find the general solution to the differential equation.

$$y'' - 4y = 13\cos(3x) - 39\sin(3x)$$

Answer choices:

A $\quad y(x) = c_1\sin(2x) + c_2\cos(2x) + 3\sin(3x) - \cos(3x)$

B $\quad y(x) = c_1e^{2x} + c_2e^{-2x} + 3\cos(3x) - \sin(3x)$

C $\quad y(x) = c_1\sin(2x) + c_2\cos(2x) + 3\cos(3x) - \sin(3x)$

D $\quad y(x) = c_1e^{2x} + c_2e^{-2x} + 3\sin(3x) - \cos(3x)$

Solution: D

We'll replace $g(x) = 13\cos(3x) - 39\sin(3x)$ with $g(x) = 0$ to change the nonhomogeneous equation into a homogeneous equation

$$y'' - 4y = 0$$

then we'll solve the associated characteristic equation.

$$r^2 - 4 = 0$$

$$r = -2, 2$$

These are distinct real roots, so the complementary solution will be

$$y_c(x) = c_1e^{2x} + c_2e^{-2x}$$

For the particular solution, we'll use $A\cos(3x) + B\sin(3x)$ as our guess for the particular solution.

$$y_p(x) = A\cos(3x) + B\sin(3x)$$

Taking the first and second derivatives of the guess gives

$$y_p'(x) = -3A\sin(3x) + 3B\cos(3x)$$

$$y_p''(x) = -9A\cos(3x) - 9B\sin(3x)$$

Plug the derivatives into the original differential equation.

$$-9A\cos(3x) - 9B\sin(3x) - 4(A\cos(3x) + B\sin(3x)) = 13\cos(3x) - 39\sin(3x)$$

$$-9A\cos(3x) - 9B\sin(3x) - 4A\cos(3x) - 4B\sin(3x) = 13\cos(3x) - 39\sin(3x)$$

$$-13A\cos(3x) - 13B\sin(3x) = 13\cos(3x) - 39\sin(3x)$$

Equate coefficients.

$$-13A = 13 \qquad -13B = -39$$

$$A = -1 \qquad B = 3$$

Then the particular solution is

$$y_p(x) = -\cos(3x) + 3\sin(3x)$$

and the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{2x} + c_2e^{-2x} + 3\sin(3x) - \cos(3x)$$

Topic: Variation of parameters for nonhomogeneous equations

Question: Find the general solution to the differential equation.

$$y'' - 2y' + y = \frac{3e^x}{2x}$$

Answer choices:

A $\quad y = c_1e^x + c_2xe^x + \frac{3}{2}e^x \ln|x|$

B $\quad y(x) = c_1e^x + c_2xe^x + \frac{3}{2}x \ln|x|$

C $\quad y(x) = c_1e^x + c_2xe^x + \frac{3}{2}xe^x - \frac{3}{2}xe^x \ln|x|$

D $\quad y(x) = c_1e^x + c_2xe^x - \frac{3}{2}xe^x + \frac{3}{2}xe^x \ln|x|$

Solution: D

The coefficient on y'' is already 1. We'll substitute $g(x)=0$ to rewrite the nonhomogeneous equation as a homogeneous equation.

$$y''-2y'+y=0$$

Now we'll use the associated characteristic equation to find roots.

$$r^2-2r+1=0$$

$$(r-1)(r-1)=0$$

$$r=1$$

We get equal real roots, so the complementary solution is

$$y_c(x)=c_1e^x+c_2xe^x$$

and the fundamental set of solutions and their derivatives are therefore

$$\{y_1,y_2\}=\{e^x,xe^x\}$$

$$\{y_1',y_2'\}=\{e^x,e^x+xe^x\}$$

Create a system of linear equations using the fundamental set and its derivatives.

$$u_1'y_1+u_2'y_2=0$$

$$u_1'e^x+u_2'xe^x=0$$

$$u_1'y_1'+u_2'y_2'=g(x)$$

$$u_1'e^x+u_2'(e^x+xe^x)=\frac{3e^x}{2x}$$

$$u_1'e^x + u_2'e^x + u_2'xe^x = \frac{3e^x}{2x}$$

Subtract one equation from the other.

$$(u_1'e^x + u_2'xe^x) - (u_1'e^x + u_2'e^x + u_2'xe^x) = 0 - \frac{3e^x}{2x}$$

$$-u_2'e^x = -\frac{3e^x}{2x}$$

$$u_2' = \frac{3}{2x}$$

Use this value for u_2' to find u_1'

$$u_1'e^x + \left(\frac{3}{2x}\right)xe^x = 0$$

$$u_1'e^x = -\frac{3}{2}e^x$$

$$u_1' = -\frac{3}{2}$$

Integrate u_1' and u_2' to find u_1 and u_2.

$$u_1 = \int -\frac{3}{2}\ dx \qquad u_2 = \int \frac{3}{2x}\ dx$$

$$u_1 = -\frac{3}{2}x \qquad u_2 = \frac{3}{2}\ln|x|$$

So the particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = -\frac{3}{2}x(e^x) + \frac{3}{2}\ln|x|(xe^x)$$

$$y_p(x) = -\frac{3}{2}xe^x + \frac{3}{2}xe^x\ln|x|$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1e^x + c_2xe^x - \frac{3}{2}xe^x + \frac{3}{2}xe^x\ln|x|$$

Topic: Variation of parameters for nonhomogeneous equations

Question: Find the general solution to the differential equation.

$$y'' - 3y' - 4y = e^{3x}$$

Answer choices:

A $\quad y(x) = c_1 e^{-x} + c_2 e^{4x} - \frac{e^{3x}}{4}$

B $\quad y(x) = c_1 e^{x} + c_2 e^{-4x} + \frac{e^{3x}}{4}$

C $\quad y(x) = c_1 e^{-x} + c_2 x e^{4x} - \frac{e^{3x}}{4}$

D $\quad y(x) = c_1 e^{x} + c_2 x e^{-4x} + \frac{e^{3x}}{4}$

Solution: A

The coefficient on y'' is already 1. We'll substitute $g(x)=0$ to rewrite the nonhomogeneous equation as a homogeneous equation.

$$y''-3y'-4y=0$$

Now we'll use the associated characteristic equation to find roots.

$$r^2-3r-4=0$$

$$(r-4)(r+1)=0$$

$$r=4,\ -1$$

We get distinct real roots, so the complementary solution is

$$y_c(x)=c_1e^{-x}+c_2e^{4x}$$

and the fundamental set of solutions and their derivatives are therefore

$$\{y_1,y_2\}=\{e^{-x},e^{4x}\}$$

$$\{y_1',y_2'\}=\{-e^{-x},4e^{4x}\}$$

Create a system of linear equations using the fundamental set and its derivatives.

$$u_1'y_1+u_2'y_2=0$$

$$u_1'e^{-x}+u_2'e^{4x}=0$$

$$u_1'y_1'+u_2'y_2'=g(x)$$

$$u_1'(-e^{-x})+u_2'(4e^{4x})=e^{3x}$$

$$-u_1'e^{-x}+4u_2'e^{4x}=e^{3x}$$

Add the equations.

$$(u_1'e^{-x} + u_2'e^{4x}) + (-u_1'e^{-x} + 4u_2'e^{4x}) = 0 + e^{3x}$$

$$5u_2'e^{4x} = e^{3x}$$

$$u_2' = \frac{e^{-x}}{5}$$

Use this value for u_2' to find u_1'

$$u_1'e^{-x} + \left(\frac{e^{-x}}{5}\right)e^{4x} = 0$$

$$u_1'e^{-x} = -\frac{e^{3x}}{5}$$

$$u_1' = -\frac{e^{4x}}{5}$$

Integrate u_1' and u_2' to find u_1 and u_2.

$$u_1 = \int -\frac{e^{4x}}{5}\ dx$$

$$u_1 = -\frac{e^{4x}}{20}$$

$$u_2 = \int \frac{e^{-x}}{5}\ dx$$

$$u_2 = -\frac{e^{-x}}{5}$$

So the particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = -\frac{e^{4x}}{20}(e^{-x}) - \frac{e^{-x}}{5}(e^{4x})$$

$$y_p(x) = -\frac{e^{3x}}{20} - \frac{e^{3x}}{5}$$

$$y_p(x) = -\frac{e^{3x}}{4}$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1e^{-x} + c_2e^{4x} - \frac{e^{3x}}{4}$$

Topic: Variation of parameters for nonhomogeneous equations

Question: Find the general solution to the differential equation.

$$y'' + 2y' + 2y = e^{-x}\cos^2 x$$

Answer choices:

A $\quad y(x) = e^{-x}\left(c_1\cos x + c_2\sin x + \frac{1}{3}\cos^2 x\right)$

B $\quad y(x) = e^{-x}\left(c_1\cos x + c_2\sin x + \frac{1}{3}\sin^2 x\right)$

C $\quad y(x) = e^{-x}\left(c_1\cos x + c_2\sin x + \frac{1}{3} + \frac{1}{3}\cos^2 x\right)$

D $\quad y(x) = e^{-x}\left(c_1\cos x + c_2\sin x + \frac{1}{3} + \frac{1}{3}\sin^2 x\right)$

Solution: D

The coefficient on y'' is already 1. We'll substitute $g(x) = 0$ to rewrite the nonhomogeneous equation as a homogeneous equation.

$$y'' + 2y' + 2y = 0$$

Now we'll use the associated characteristic equation to find roots.

$$r^2 + 2r + 2 = 0$$

$$(r + 1)^2 + 1 = 0$$

$$r = -1 \pm i$$

We get two complex roots, so the complementary solution is

$$y_c(x) = e^{-x}(c_1 \cos x + c_2 \sin x)$$

and the fundamental set of solutions and their derivatives are therefore

$$\{y_1, y_2\} = \{e^{-x} \cos x, e^{-x} \sin x\}$$

$$\{y_1', y_2'\} = \{-e^{-x} \cos x - e^{-x} \sin x, -e^{-x} \sin x + e^{-x} \cos x\}$$

Create a system of linear equations using the fundamental set and its derivatives. We find

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'e^{-x} \cos x + u_2'e^{-x} \sin x = 0$$

$$u_1' \cos x + u_2' \sin x = 0$$

and

$$u_1'y_1' + u_2'y_2' = g(x)$$

$$u_1'e^{-x}(-\cos x - \sin x) + u_2'e^{-x}(-\sin x + \cos x) = e^{-x}\cos^2 x$$

$$u_1'(-\cos x - \sin x) + u_2'(-\sin x + \cos x) = \cos^2 x$$

Add the equations.

$$-u_1'\sin x + u_2'\cos x = \cos^2 x$$

From first equation, we know

$$u_2' = -u_1'\frac{\cos x}{\sin x}$$

and we can substitute this into the sum of the equations to get

$$-u_1'\sin x + \left(-u_1'\frac{\cos x}{\sin x}\right)'\cos x = \cos^2 x$$

$$-u_1'\frac{\sin^2 x + \cos^2 x}{\sin x} = \cos^2 x$$

$$-u_1'\frac{1}{\sin x} = \cos^2 x$$

$$u_1' = -\sin x\cos^2 x$$

Use this value for u_1' to find u_2'.

$$u_2' = -(-\sin x\cos^2 x)\frac{\cos x}{\sin x}$$

$$u_2' = \cos^3 x$$

Integrate u_1' and u_2' to find u_1 and u_2. We get

$$u_1 = \int -\sin x \cos^2 x \ dx$$

$u = \cos x$ and $du = -\sin x \ dx$

$$u_1 = \int u^2 \ du$$

$$u_1 = \frac{1}{3}u^3 = \frac{1}{3}\cos^3 x$$

and

$$u_2 = \int \cos^3 x \ dx$$

$$u_2 = \int (1 - \sin^2 x)\cos x \ dx$$

$u = \sin x$ and $du = \cos x \ dx$

$$u_2 = \int 1 - u^2 \ du$$

$$u_2 = u - \frac{1}{3}u^3 = \sin x - \frac{1}{3}\sin^3 x$$

So the particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = \frac{\cos^3 x}{3}(e^{-x} \cos x) + \left(\sin x - \frac{\sin^3 x}{3} \right)(e^{-x} \sin x)$$

$$y_p(x) = \frac{\cos^4 x}{3} e^{-x} + \left(\sin^2 x - \frac{\sin^4 x}{3} \right) e^{-x}$$

$$y_p(x) = \left(\frac{\cos^4 x - \sin^4 x + 3 \sin^2 x}{3} \right) e^{-x}$$

$$y_p(x) = \left(\frac{(1 - \sin^2 x)^2 - \sin^4 x + 3 \sin^2 x}{3} \right) e^{-x}$$

$$y_p(x) = \left(\frac{1 - 2 \sin^2 x + \sin^4 x - \sin^4 x + 3 \sin^2 x}{3} \right) e^{-x}$$

$$y_p(x) = \frac{1 + \sin^2 x}{3} e^{-x}$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = e^{-x} \left(c_1 \cos x + c_2 \sin x + \frac{1}{3} + \frac{1}{3} \sin^2 x \right)$$

Topic: Fundamental solution sets and the Wronskian

Question: Find the Wronskian of the fundamental set of solutions.

$$\{e^{-2x}, e^{3x}\}$$

Answer choices:

A $W(e^{-2x}, e^{3x}) = -5e^{-x}$

B $W(e^{-2x}, e^{3x}) = -5e^{x}$

C $W(e^{-2x}, e^{3x}) = 5e^{x}$

D $W(e^{-2x}, e^{3x}) = 5e^{2x}$

Solution: C

Given the solution set $\{y_1, y_2\} = \{e^{-2x}, e^{3x}\}$, the Wronskian will be

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

$$W(e^{-2x}, e^{3x}) = \begin{vmatrix} e^{-2x} & e^{3x} \\ -2e^{-2x} & 3e^{3x} \end{vmatrix} = (e^{-2x})(3e^{3x}) - (e^{3x})(-2e^{-2x})$$

Simplifying the determinant, we get

$$W(e^{-2x}, e^{3x}) = 3e^x + 2e^x$$

$$W(e^{-2x}, e^{3x}) = 5e^x$$

Topic: Fundamental solution sets and the Wronskian

Question: Which solution set cannot be a fundamental set of solutions?

Answer choices:

A $\{e^{3x}, 7e^{-x}\}$

B $\{2e^{-x}, 7e^{x}\}$

C $\{e^{-x}, 7e^{-x}\}$

D $\{3e^{-x}, 7e^{x}\}$

Solution: C

A solution set will only be a fundamental set of solutions when the Wronskian of the set is non-zero. So the set that doesn't form a fundamental set will be the set for which $W(y_1, y_2) = 0$.

For answer choice A, the Wronskian is

$$W(e^{3x}, 7e^{-x}) = \begin{vmatrix} e^{3x} & 7e^{-x} \\ 3e^{3x} & -7e^{-x} \end{vmatrix}$$

$$W(e^{3x}, 7e^{-x}) = (e^{3x})(-7e^{-x}) - (7e^{-x})(3e^{3x})$$

$$W(e^{3x}, 7e^{-x}) = -7e^{2x} - 21e^{2x}$$

$$W(e^{3x}, 7e^{-x}) = -28e^{2x}$$

$$W(e^{3x}, 7e^{-x}) \neq 0$$

For answer choice B, the Wronskian is

$$W(2e^{-x}, 7e^{x}) = \begin{vmatrix} 2e^{-x} & 7e^{x} \\ -2e^{-x} & 7e^{x} \end{vmatrix}$$

$$W(2e^{-x}, 7e^{x}) = (2e^{-x})(7e^{x}) - (7e^{x})(-2e^{-x})$$

$$W(2e^{-x}, 7e^{x}) = 14 + 14$$

$$W(2e^{-x}, 7e^{x}) = 28$$

$$W(2e^{-x}, 7e^{x}) \neq 0$$

For answer choice C, the Wronskian is

$$W(e^{-x},7e^{-x}) = \begin{vmatrix} e^{-x} & 7e^{-x} \\ -e^{-x} & -7e^{-x} \end{vmatrix}$$

$$W(e^{-x},7e^{-x}) = (e^{-x})(-7e^{-x}) - (7e^{-x})(-e^{-x})$$

$$W(e^{-x},7e^{-x}) = -7e^{-2x} + 7e^{-2x}$$

$$W(e^{-x},7e^{-x}) = 0$$

For answer choice D, the Wronskian is

$$W(3e^{-x},7e^{x}) = \begin{vmatrix} 3e^{-x} & 7e^{x} \\ -3e^{-x} & 7e^{x} \end{vmatrix}$$

$$W(3e^{-x},7e^{x}) = (3e^{-x})(7e^{x}) - (7e^{x})(-3e^{-x})$$

$$W(3e^{-x},7e^{x}) = 21e^{-x}e^{x} + 21e^{x}e^{-x}$$

$$W(3e^{-x},7e^{x}) = 21 + 21$$

$$W(3e^{-x},7e^{x}) = 42$$

Topic: Fundamental solution sets and the Wronskian

Question: Find the Wronskian of the fundamental set of solutions of the differential equation, generated with $y(0) = 1$ and $y'(0) = 0$, and $y(0) = 0$ and $y'(0) = 1$.

$$y'' + y = 0$$

Answer choices:

A $\quad 1$

B $\quad e^x$

C $\quad \cos(2x)$

D $\quad \cos x - \sin x$

Solution: A

Normally, we would find the general solution to this second order equation by solving the associated characteristic equation,

$$r^2 + 1 = 0$$

$$r = \pm i$$

and then plugging these roots into the formula for the complementary solution with complex conjugate roots.

$$y_c(x) = c_1 \sin x + c_2 \cos x$$

The derivative of this complementary solution is

$$y_c'(x) = c_1 \cos x - c_2 \sin x$$

If we now plug in the initial conditions $y(x_0) = 1$ and $y'(x_0) = 0$, and $y(x_0) = 0$ and $y'(x_0) = 1$, we'll be able to generate another fundamental set of solutions. We'll use $x_0 = 0$, since we can choose any value of x_0 that we like, and using $x_0 = 0$ will be easiest.

For $y(0) = 1$, $\qquad 1 = c_1 \sin 0 + c_2 \cos 0$

For $y'(0) = 0$, $\qquad 0 = c_1 \cos 0 - c_2 \sin 0$

Then this system gives $c_1 = 0$ and $c_2 = 1$, so we could say that one solution in our fundamental set of solutions is

$$y_1(x) = \cos x$$

The second set of initial conditions gives

For $y(0) = 0$, $\quad 0 = c_1 \sin 0 + c_2 \cos 0$

For $y'(0) = 1$, $\quad 1 = c_1 \cos 0 - c_2 \sin 0$

Then this system gives $c_1 = 1$ and $c_2 = 0$, so we could say that the other solution in our fundamental set is

$$y_2(x) = \sin x$$

So one fundamental set of solutions is $\{y_1, y_2\} = \{\cos x, \sin x\}$, but another is

$$\{y_1', y_2'\} = \{-\sin x, \cos x\}$$

Now we can find the Wronskian.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W(y_1, y_2) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$$W(\cos x, \sin x) = \cos^2 x + \sin^2 x$$

$$W(\cos x, \sin x) = 1$$

Topic: Variation of parameters with the Wronskian

Question: Use variation of parameters and Wronskian integrals to find the general solution to the nonhomogeneous equation.

$$y'' - 5y' + 6y = \frac{1}{e^x}$$

Answer choices:

A $\quad y(x) = c_1e^{2x} + c_2e^{3x} - \frac{5}{12}e^{-x}$

B $\quad y(x) = c_1e^{2x} + c_2e^{3x} + \frac{1}{12}e^{-x}$

C $\quad y(x) = c_1e^{-2x} + c_2e^{3x} - \frac{5}{12}e^{-x}$

D $\quad y(x) = c_1e^{-2x} + c_2e^{3x} + \frac{1}{12}e^{-x}$

Solution: B

The associated homogeneous equation is

$$y'' - 5y' + 6y = 0$$

so the characteristic will be

$$r^2 - 5r + 6 = 0$$

$$(r - 2)(r - 3) = 0$$

$$r = 2, 3$$

With distinct real roots, the complementary solution is

$$y_c(x) = c_1e^{2x} + c_2e^{3x}$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{e^{2x}, e^{3x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(e^{2x}, e^{3x}) = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix}$$

$$W(e^{2x}, e^{3x}) = (e^{2x})(3e^{3x}) - (e^{3x})(2e^{2x})$$

$$W(e^{2x}, e^{3x}) = 3e^{5x} - 2e^{5x}$$

$$W(e^{2x}, e^{3x}) = e^{5x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)}\ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)}\ dx$$

$$y_p(x) = -e^{2x} \int \frac{e^{3x}\left(\frac{1}{e^x}\right)}{e^{5x}}\ dx + e^{3x} \int \frac{e^{2x}\left(\frac{1}{e^x}\right)}{e^{5x}}\ dx$$

$$y_p(x) = -e^{2x} \int e^{-3x}\ dx + e^{3x} \int e^{-4x}\ dx$$

$$y_p(x) = -e^{2x}\left(-\frac{1}{3}e^{-3x}\right) + e^{3x}\left(-\frac{1}{4}e^{-4x}\right)$$

$$y_p(x) = \frac{1}{3}e^{-x} - \frac{1}{4}e^{-x}$$

$$y_p(x) = \frac{1}{12}e^{-x}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{2x} + c_2e^{3x} + \frac{1}{12}e^{-x}$$

Topic: Variation of parameters with the Wronskian

Question: Use variation of parameters and Wronskian integrals to find the general solution to the nonhomogeneous equation.

$$y'' - 4y' + 4y = \frac{e^{2x}}{x^3}$$

Answer choices:

A $\quad y(x) = c_1e^{2x} + c_2xe^{2x} + \dfrac{e^{2x}}{2x}$

B $\quad y(x) = c_1e^{2x} + c_2xe^{2x} + \dfrac{e^{2x}}{x^3}$

C $\quad y(x) = c_1e^{2x} + c_2xe^{2x} + \dfrac{e^{2x}}{x^2}$

D $\quad y(x) = c_1e^{2x} + c_2xe^{2x} + \dfrac{e^{2x}}{2x^2}$

Solution: A

The associated homogeneous equation is

$$y'' - 4y' + 4y = 0$$

so the characteristic will be

$$r^2 - 4r + 4 = 0$$

$$(r-2)(r-2) = 0$$

$$r = 2,\ 2$$

With equal real roots, the complementary solution is

$$y_c(x) = c_1e^{2x} + c_2xe^{2x}$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{e^{2x}, xe^{2x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix}$$

$$W(e^{2x}, xe^{2x}) = (e^{2x})(e^{2x} + 2xe^{2x}) - (xe^{2x})(2e^{2x})$$

$$W(e^{2x}, xe^{2x}) = e^{4x} + 2xe^{4x} - 2xe^{4x}$$

$$W(e^{2x}, xe^{2x}) = e^{4x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} \ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \ dx$$

$$y_p(x) = -e^{2x} \int \frac{xe^{2x}\left(\frac{e^{2x}}{x^3}\right)}{e^{4x}} \ dx + xe^{2x} \int \frac{e^{2x}\left(\frac{e^{2x}}{x^3}\right)}{e^{4x}} \ dx$$

$$y_p(x) = -e^{2x} \int \frac{1}{x^2} \ dx + xe^{2x} \int \frac{1}{x^3} \ dx$$

$$y_p(x) = -e^{2x}\left(-\frac{1}{x}\right) - \frac{1}{2}xe^{2x}\left(\frac{1}{x^2}\right)$$

$$y_p(x) = \frac{2e^{2x}}{2x} - \frac{e^{2x}}{2x}$$

$$y_p(x) = \frac{e^{2x}}{2x}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 e^{2x} + c_2 xe^{2x} + \frac{e^{2x}}{2x}$$

Topic: Variation of parameters with the Wronskian

Question: Use variation of parameters and Wronskian integrals to find the general solution to the nonhomogeneous equation.

$$y'' - 6y' + 9y = \frac{e^{3x}}{x^5}$$

Answer choices:

A $\quad y(x) = c_1e^{3x} + c_2xe^{3x} + \dfrac{e^{3x}}{3x^4}$

B $\quad y(x) = c_1e^{3x} + c_2xe^{3x} + \dfrac{e^{3x}}{3x^3}$

C $\quad y(x) = c_1e^{3x} + c_2xe^{3x} + \dfrac{e^{3x}}{12x^3}$

D $\quad y(x) = c_1e^{3x} + c_2xe^{3x} + \dfrac{e^{3x}}{12x^4}$

Solution: C

The associated homogeneous equation is

$$y'' - 6y' + 9y = 0$$

so the characteristic will be

$$r^2 - 6r + 9 = 0$$

$$(r - 3)(r - 3) = 0$$

$$r = 3,\ 3$$

With equal real roots, the complementary solution is

$$y_c(x) = c_1e^{3x} + c_2xe^{3x}$$

and the fundamental set of solutions is

$$\{y_1, y_2\} = \{e^{3x}, xe^{3x}\}$$

Find the Wronskian for the fundamental solution set.

$$W(e^{3x}, xe^{3x}) = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & e^{3x} + 3xe^{3x} \end{vmatrix}$$

$$W(e^{3x}, xe^{3x}) = (e^{3x})(e^{3x} + 3xe^{3x}) - (xe^{3x})(3e^{3x})$$

$$W(e^{3x}, xe^{3x}) = e^{6x} + 3xe^{6x} - 3xe^{6x}$$

$$W(e^{3x}, xe^{3x}) = e^{6x}$$

Then the particular solution is

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)}\ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)}\ dx$$

$$y_p(x) = -e^{3x} \int \frac{xe^{3x}\left(\frac{e^{3x}}{x^5}\right)}{e^{6x}}\ dx + xe^{3x} \int \frac{e^{3x}\left(\frac{e^{3x}}{x^5}\right)}{e^{6x}}\ dx$$

$$y_p(x) = -e^{3x} \int \frac{1}{x^4}\ dx + xe^{3x} \int \frac{1}{x^5}\ dx$$

$$y_p(x) = -e^{3x}\left(-\frac{1}{3}x^{-3}\right) + xe^{3x}\left(-\frac{1}{4}x^{-4}\right)$$

$$y_p(x) = \frac{e^{3x}}{3x^3} - \frac{e^{3x}}{4x^3}$$

$$y_p(x) = \frac{e^{3x}}{12x^3}$$

Then the general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{3x} + c_2xe^{3x} + \frac{e^{3x}}{12x^3}$$

Topic: Initial value problems with nonhomogeneous equations

Question: Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' + 2y' + y = 4 \sin x$$

Answer choices:

A $\quad y(x) = 2xe^{-x} - 2 \sin x$

B $\quad y(x) = 2e^{-x} + 2xe^{-x} - 2 \cos x$

C $\quad y(x) = 2e^{-x} - 2 \cos x$

D $\quad y(x) = 2e^{-x} + 2xe^{-x} - 2 \sin x$

Solution: B

First we solve the associated homogeneous equation.

$$y'' + 2y' + y = 0$$

$$r^2 + 2r + 1 = 0$$

$$(r + 1)^2 = 0$$

$$r = -1, -1$$

So the complementary solution with equal real roots is

$$y_c(x) = c_1e^{-x} + c_2xe^{-x}$$

Our guess for the particular solution will be

$$y_p(x) = A\cos x + B\sin x$$

$$y_p'(x) = -A\sin x + B\cos x$$

$$y_p''(x) = -A\cos x - B\sin x$$

Substituting into the original differential equation gives

$$-A\cos x - B\sin x + 2(-A\sin x + B\cos x) + A\cos x + B\sin x = 4\sin x$$

$$-A\cos x - B\sin x - 2A\sin x + 2B\cos x + A\cos x + B\sin x = 4\sin x$$

$$2B\cos x - 2A\sin x = 4\sin x$$

Equating coefficients gives $A = -2$ and $B = 0$, and therefore

$$y_p(x) = -2\cos x$$

The general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{-x} + c_2xe^{-x} - 2\cos x$$

Differentiating the general solution gives

$$y'(x) = -c_1e^{-x} + c_2e^{-x} - c_2xe^{-x} + 2\sin x$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$ into the general solution and its derivative.

$$0 = c_1 + 0 - 2$$

$$0 = -c_1 + c_2 - 0 + 0$$

Simplifying these equations gives

$$c_1 = 2$$

$$c_2 = c_1 = 2$$

With $c_1 = 2$ and $c_2 = 2$, the general solution becomes

$$y(x) = 2e^{-x} + 2xe^{-x} - 2\cos x$$

Topic: Initial value problems with nonhomogeneous equations

Question: Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' - 6y' + 5y = 6e^{2x} + 4e^{3x}$$

Answer choices:

A $\quad y(x) = 2e^{x} - 2e^{2x} - e^{3x} + e^{5x}$

B $\quad y(x) = -2e^{x} + 2e^{2x} + e^{3x} - e^{5x}$

C $\quad y(x) = e^{x} - e^{2x} - 2e^{3x} + 2e^{5x}$

D $\quad y(x) = -e^{x} + e^{2x} + 2e^{3x} - 2e^{5x}$

Solution: A

First we solve the associated homogeneous equation.

$$y'' - 6y' + 5y = 0$$

$$r^2 - 6r + 5 = 0$$

$$(r-1)(r-5) = 0$$

$$r = 1,\ 5$$

So the complementary solution with distinct real roots is

$$y_c(x) = c_1e^x + c_2e^{5x}$$

Our guess for the particular solution will be

$$y_p(x) = Ae^{2x} + Be^{3x}$$

$$y_p'(x) = 2Ae^{2x} + 3Be^{3x}$$

$$y_p''(x) = 4Ae^{2x} + 9Be^{3x}$$

Substituting into the original differential equation gives

$$4Ae^{2x} + 9Be^{3x} - 6(2Ae^{2x} + 3Be^{3x}) + 5(Ae^{2x} + Be^{3x}) = 6e^{2x} + 4e^{3x}$$

$$4Ae^{2x} + 9Be^{3x} - 12Ae^{2x} - 18Be^{3x} + 5Ae^{2x} + 5Be^{3x} = 6e^{2x} + 4e^{3x}$$

$$-3Ae^{2x} - 4Be^{3x} = 6e^{2x} + 4e^{3x}$$

Equating coefficients gives $A = -2$ and $B = -1$, and therefore

$$y_p(x) = -2e^{2x} - e^{3x}$$

The general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^x + c_2e^{5x} - 2e^{2x} - e^{3x}$$

Differentiating the general solution gives

$$y'(x) = c_1e^x + 5c_2e^{5x} - 4e^{2x} - 3e^{3x}$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$ into the general solution and its derivative.

$$0 = c_1 + c_2 - 2 - 1$$

$$0 = c_1 + 5c_2 - 4 - 3$$

Simplifying these equations gives

$$c_1 + c_2 = 3$$

$$c_1 + 5c_2 = 7$$

With $c_1 = 2$ and $c_2 = 1$, the general solution becomes

$$y(x) = 2e^x + e^{5x} - 2e^{2x} - e^{3x}$$

$$y(x) = 2e^x - 2e^{2x} - e^{3x} + e^{5x}$$

Topic: Initial value problems with nonhomogeneous equations

Question: Solve the initial value problem for the second order nonhomogeneous equation, given $y(0)=5$ and $y'(0)=7$.

$$y''-2y'+10y=100x$$

Answer choices:

A $\quad y(x)=e^{x}\cos(3x)+4e^{x}\sin(3x)+20x+4$

B $\quad y(x)=2e^{x}\cos(3x)+e^{x}\sin(3x)+20x+4$

C $\quad y(x)=3e^{x}\cos(3x)-2e^{x}\sin(3x)+10x+2$

D $\quad y(x)=4e^{x}\cos(3x)-5e^{x}\sin(3x)+10x+2$

Solution: C

First we solve the associated homogeneous equation.

$$y'' - 2y' + 10y = 0$$

$$r^2 - 2r + 10 = 0$$

$$(r-1)^2 + 9 = 0$$

$$r = 1 \pm 3i$$

So the complementary solution with complex conjugate roots is

$$y_c(x) = c_1 e^x \cos(3x) + c_2 e^x \sin(3x)$$

Our guess for the particular solution will be

$$y_p(x) = Ax + B$$

$$y_p'(x) = A$$

$$y_p''(x) = 0$$

Substituting into the original differential equation gives

$$0 - 2A + 10(Ax + B) = 100x$$

$$-2A + 10Ax + 10B = 100x$$

Equating coefficients gives $A = 10$ and $B = 2$, and therefore

$$y_p(x) = 10x + 2$$

The general solution is

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 e^x \cos(3x) + c_2 e^x \sin(3x) + 10x + 2$$

Differentiating the general solution gives

$$y'(x) = e^x(c_1 \cos(3x) + c_2 \sin(3x)) + e^x(-3c_1 \sin(3x) + 3c_2 \cos(3x)) + 10$$

Substitute the initial conditions $y(0) = 5$ and $y'(0) = 7$ into the general solution and its derivative.

$$c_1 + 0 + 0 + 2 = 5$$

$$c_1 + 3c_2 + 10 = 7$$

Simplifying these equations gives

$$c_1 = 3$$

$$c_2 = -2$$

With $c_1 = 3$ and $c_2 = -2$, the general solution becomes

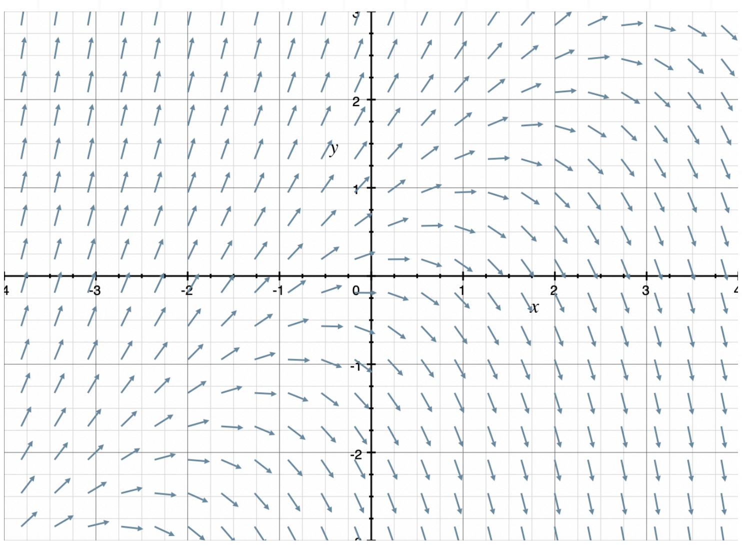
$$y(x) = 3e^x \cos(3x) - 2e^x \sin(3x) + 10x + 2$$

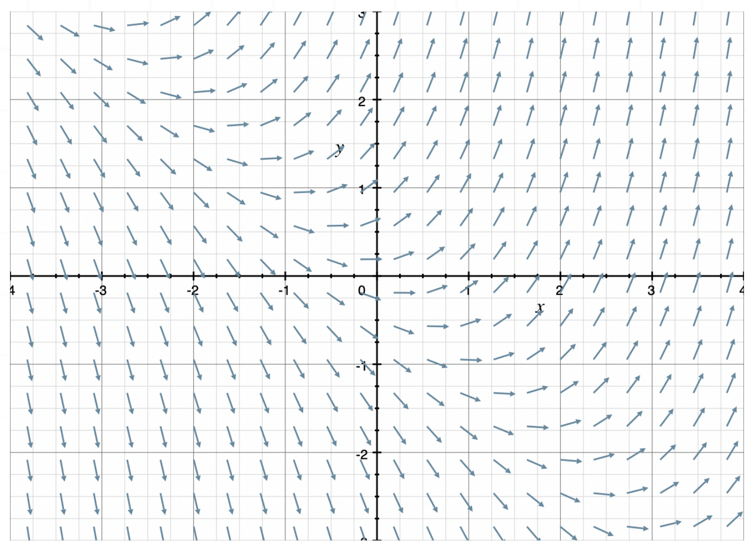
Topic: Direction fields and solution curves

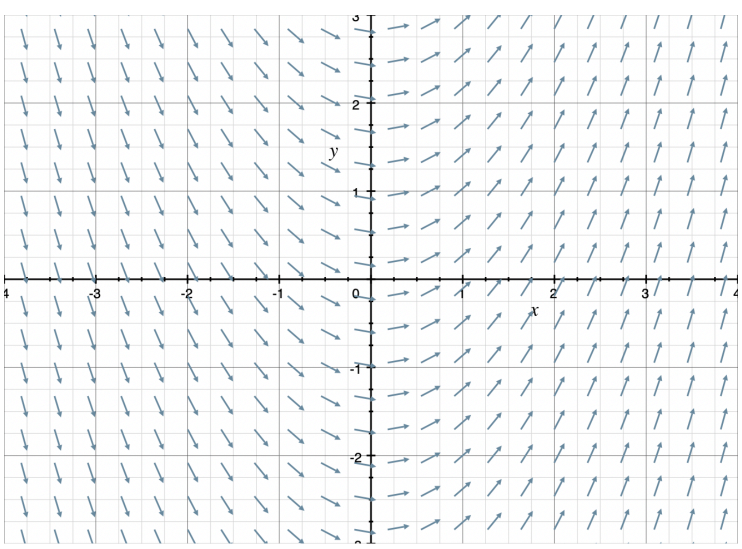
Question: Sketch the direction field of the differential equation.

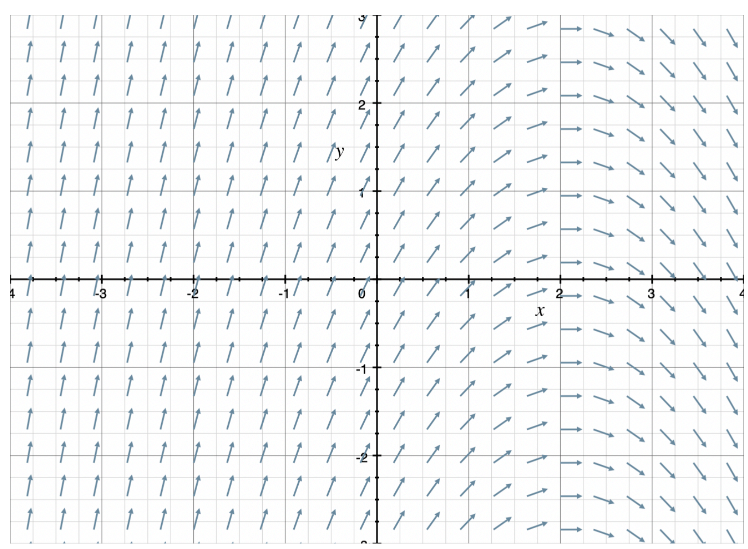
$$y' = y + x$$

Answer choices:

A 

B 

C 

D 

Solution: B

We'll pair up values $x = \{-2, -1,0,1,2\}$ with values $y = \{-2, -1,0,1,2\}$, calculating y' for each combination of values.

For instance, at $(x, y) = (-2, -2)$, we find

$$y' = y + x$$

$$y' = -2 - 2$$

$$y' = -4$$

Or at $(x, y) = (1,2)$, we find

$$y' = y + x$$

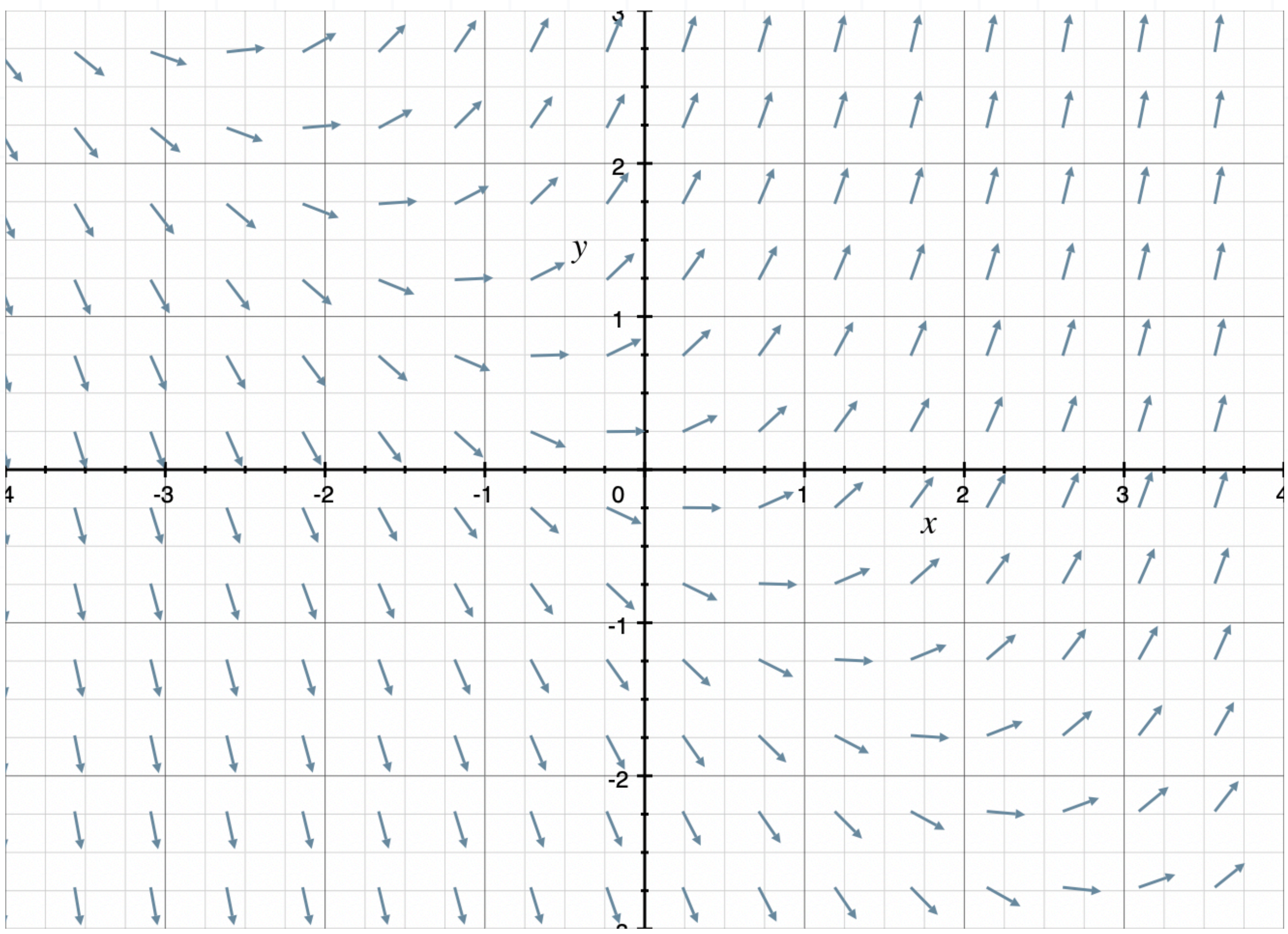
$$y' = 2 + 1$$

$$y' = 3$$

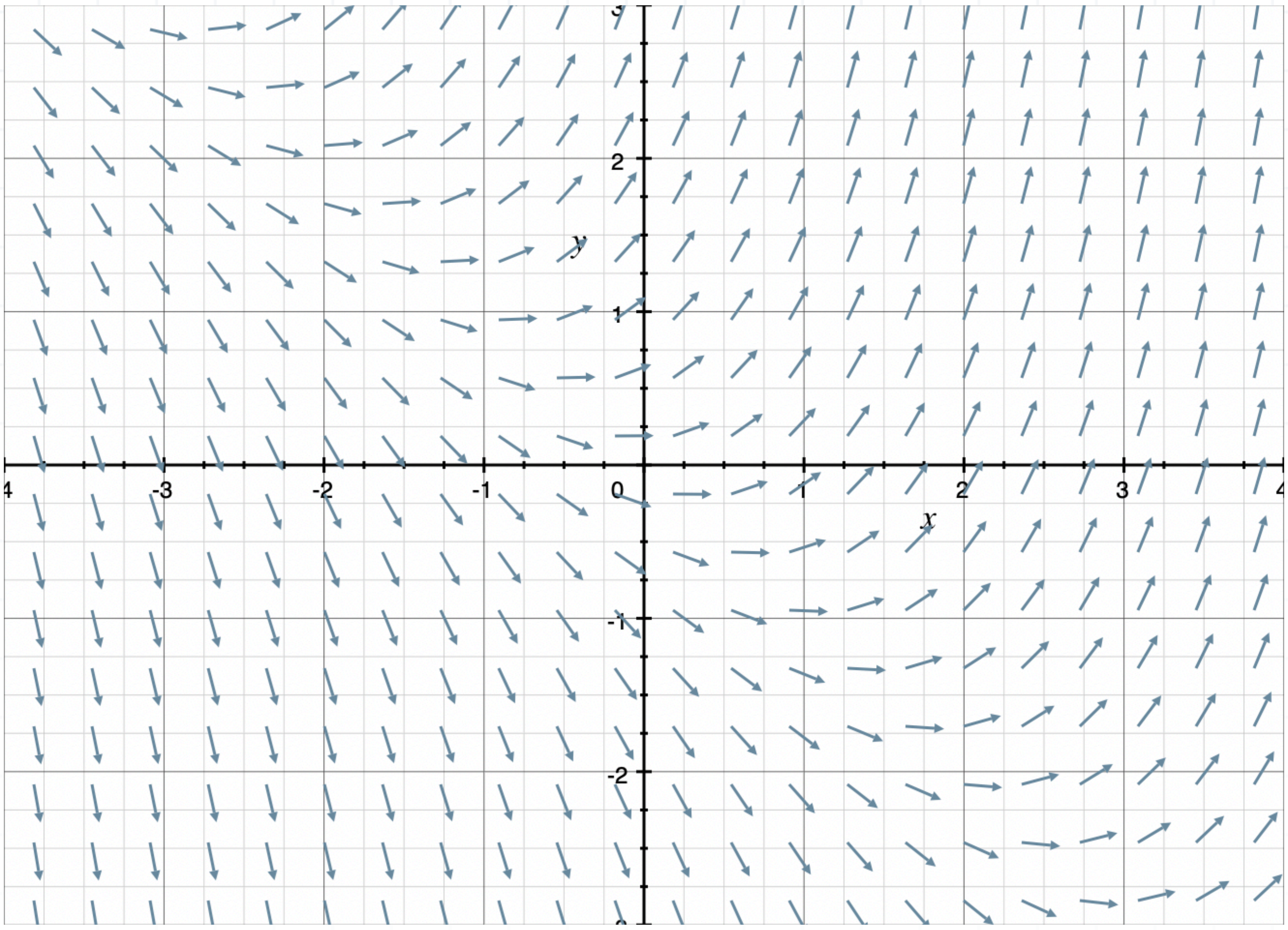
Let's make a table with each of these combinations.

		x				
		-2	**-1**	**0**	**1**	**2**
y	**-2**	-4	-3	-2	-1	0
	-1	-3	-2	-1	0	1
	0	-2	-1	0	1	2
	1	-1	0	1	2	3
	2	0	1	2	3	4

The values of y' that we found represent the slope of the function at the corresponding point (x, y). For example, we see the point $(2,-2)$, and the corresponding value $y'=0$. This means that the slope of the function at $(2,-2)$ is 0, so we'd draw a small, short horizontal line exactly at $(2,-2)$. Plotting all of the other point-slope pairs, the direction field starts to look something like this:



If we add more points, maybe ones that are half-way between those that we already found, a more complete direction field should look something like this:



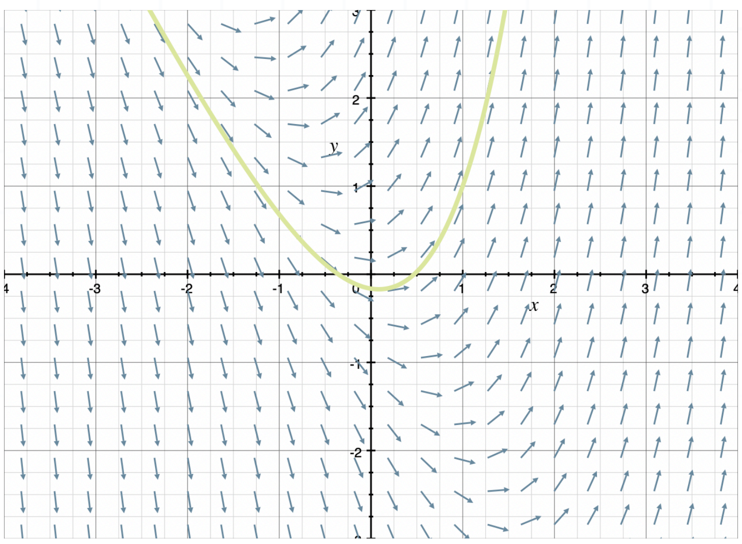
y
x
-4
-3
-2
-1
0
1
2
3
4
3
2
1
-1
-2
-3

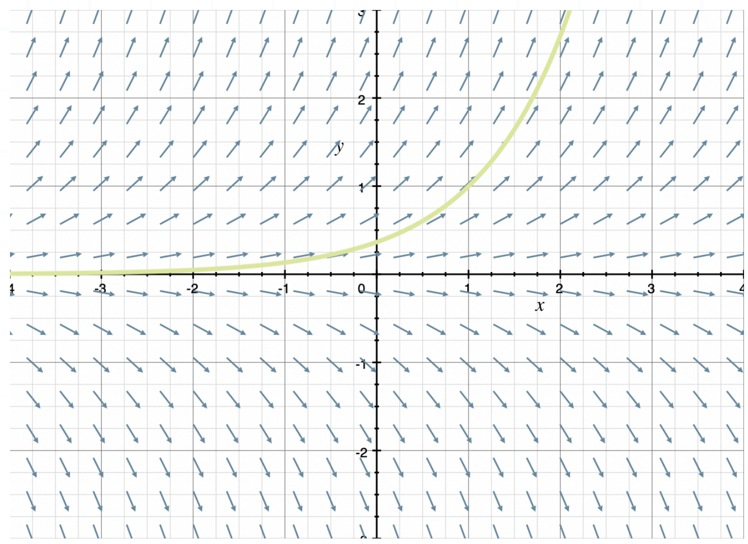
Topic: Direction fields and solution curves

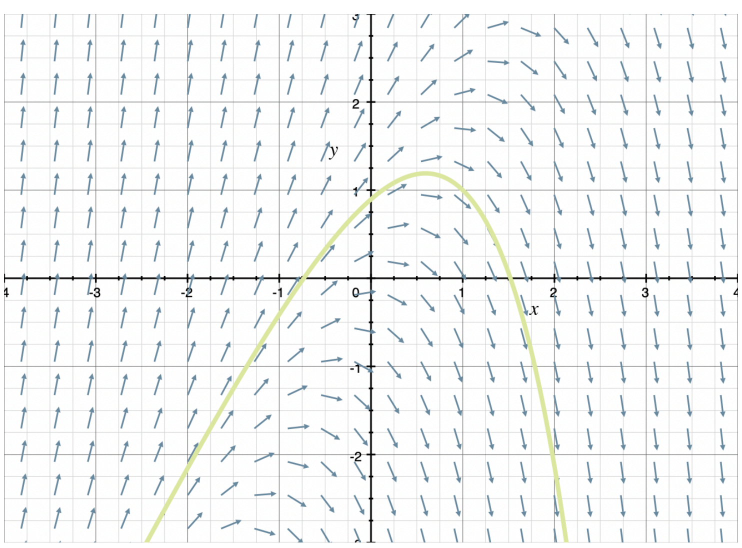
Question: Sketch the direction field for the differential equation, and the solution curve passing through $(1,1)$.

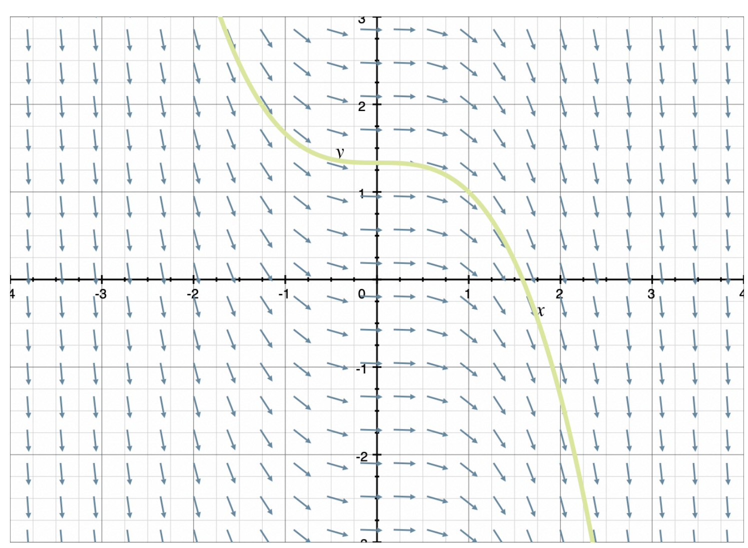
$$y' = y - 2x$$

Answer choices:

A 

B 

C 

D 

Solution: C

We'll pair up values $x = \{-2, -1,0,1,2\}$ with values $y = \{-2, -1,0,1,2\}$, calculating y' for each combination of values.

For instance, at $(x, y) = (-2, -2)$, we find

$$y' = y - 2x$$

$$y' = -2 - 2(-2)$$

$$y' = 2$$

Or at $(x, y) = (1,2)$, we find

$$y' = y - 2x$$

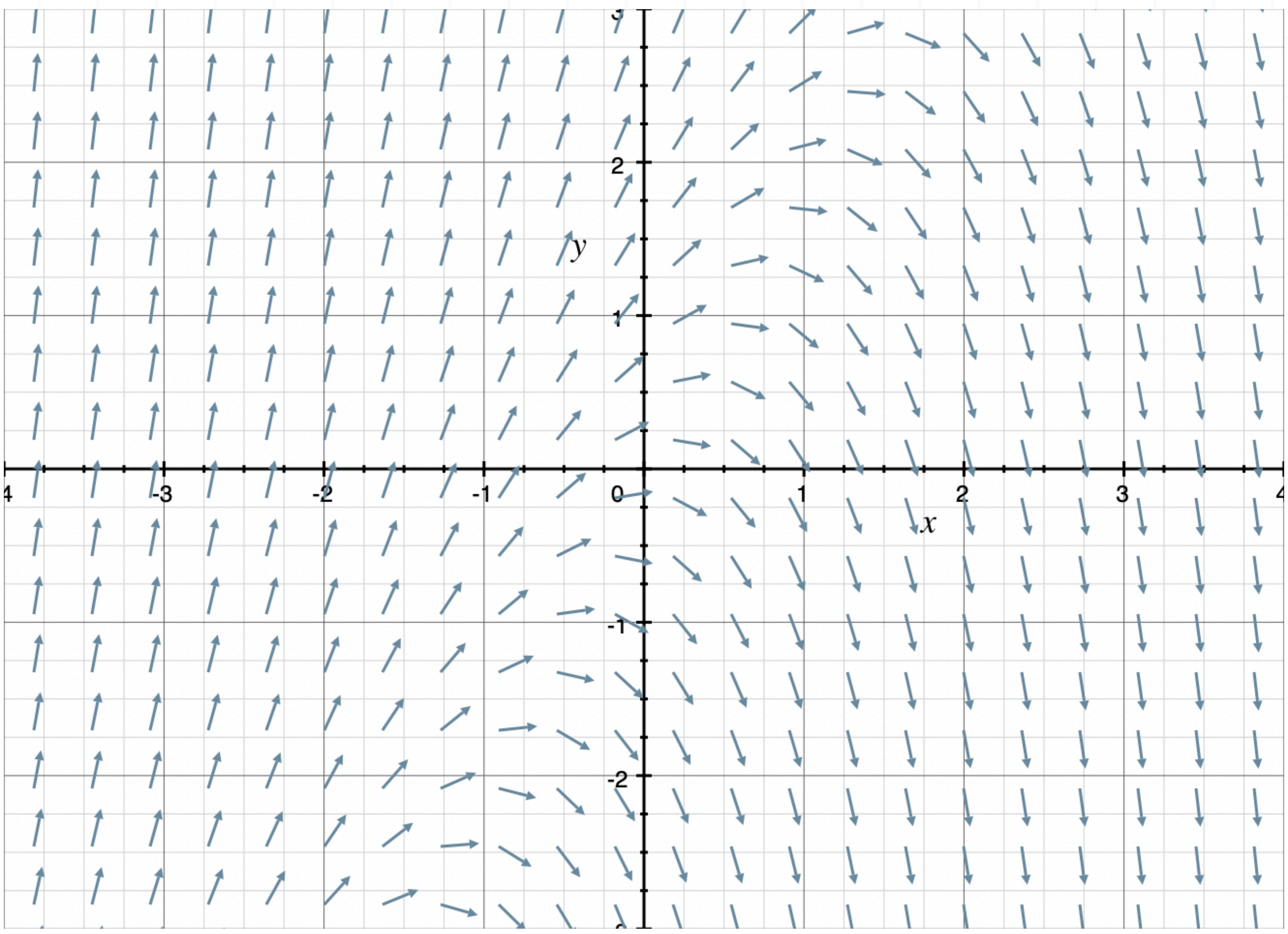
$$y' = 2 - 2(1)$$

$$y' = 0$$

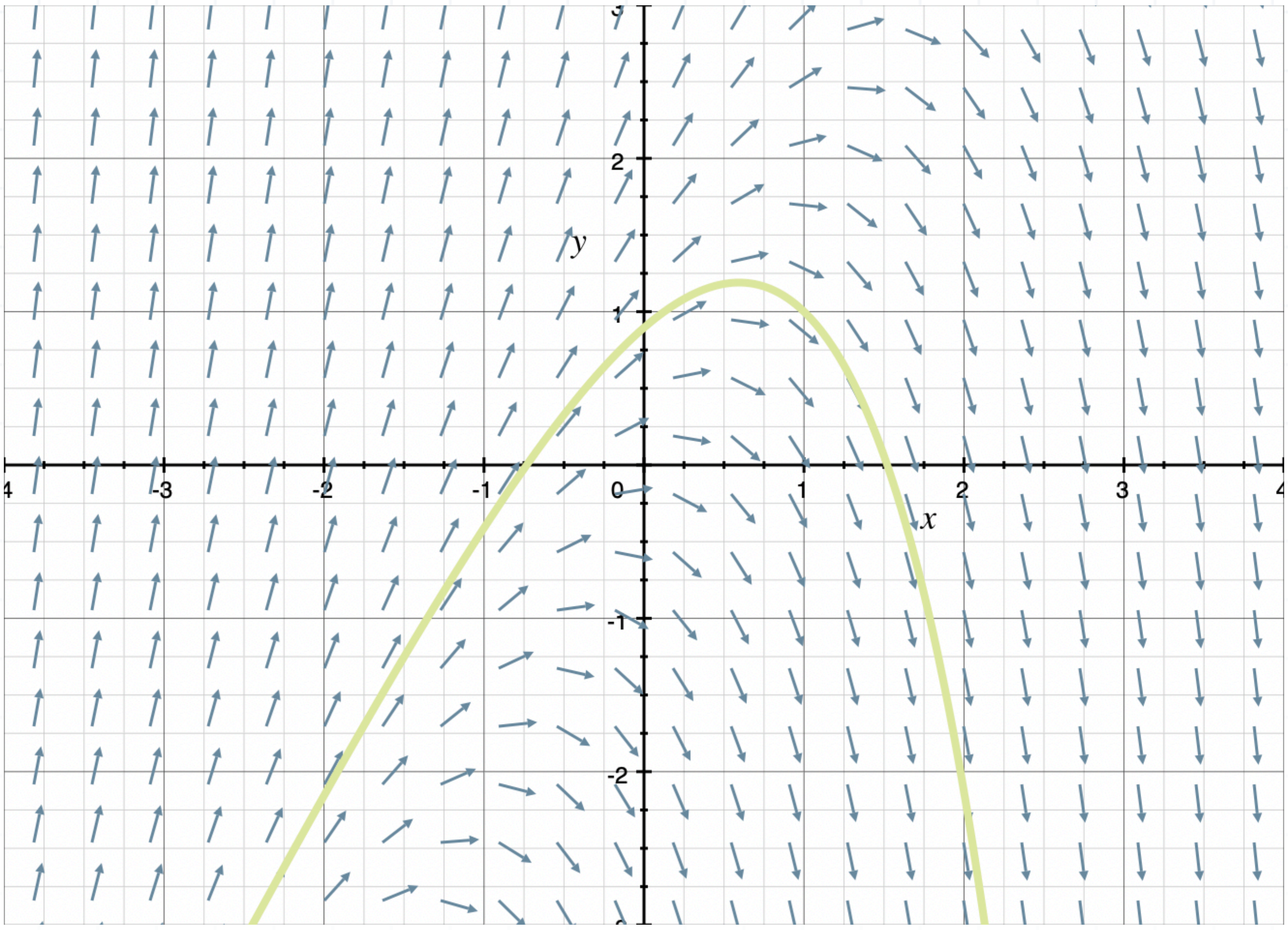
Let's make a table with each of these combinations.

		x				
		-2	**-1**	**0**	**1**	**2**
y	**-2**	2	0	-2	-4	-6
	-1	3	1	-1	-3	-5
	0	4	2	0	-2	-4
	1	5	3	1	-1	-3
	2	6	4	2	0	-2

The values of y' that we found represent the slope of the function at the corresponding point (x, y). For example, we see the point $(2, -2)$, and the corresponding value $y' = -6$. This means that the slope of the function at $(2, -2)$ is -6, so we'd draw a small, short line exactly at $(2, -2)$ with slope -6. Plotting all of the other point-slope pairs, the direction field starts to look something like this:



If we plot the point $(1,1)$, then follow the slopes in our direction field to both the left and right of $(1,1)$, our solution curve should look something like this:



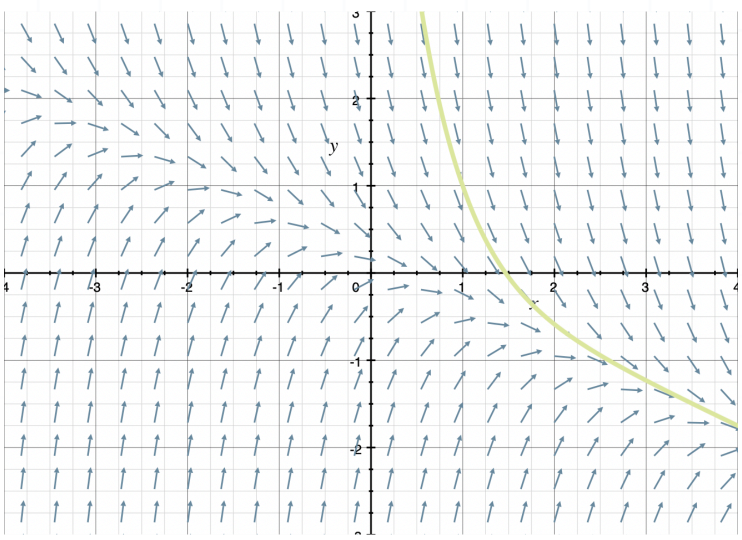
y
x

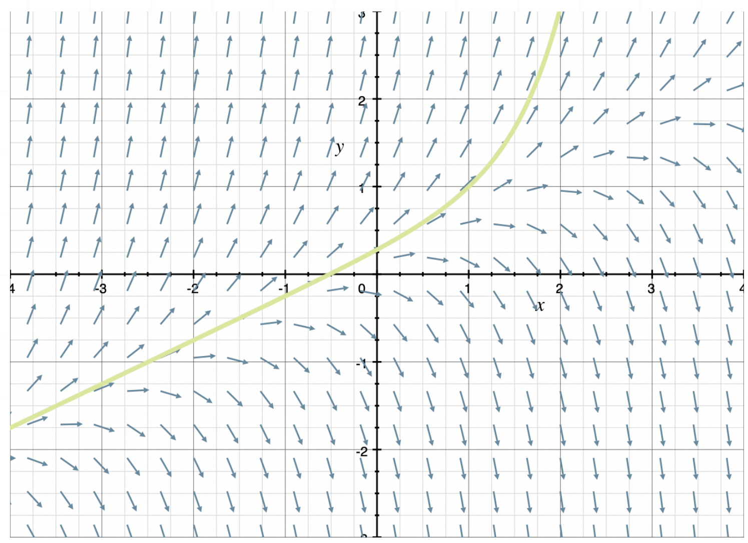
Topic: Direction fields and solution curves

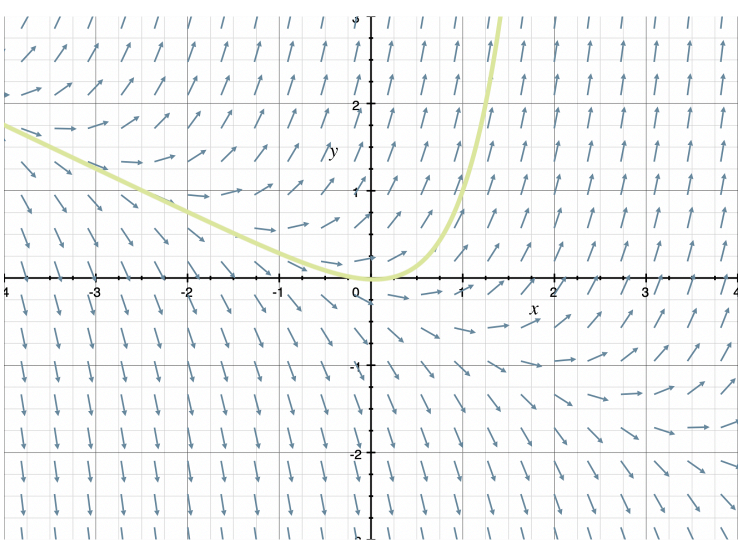
Question: Sketch the direction field for the differential equation and the solution curve passing through $(1,1)$.

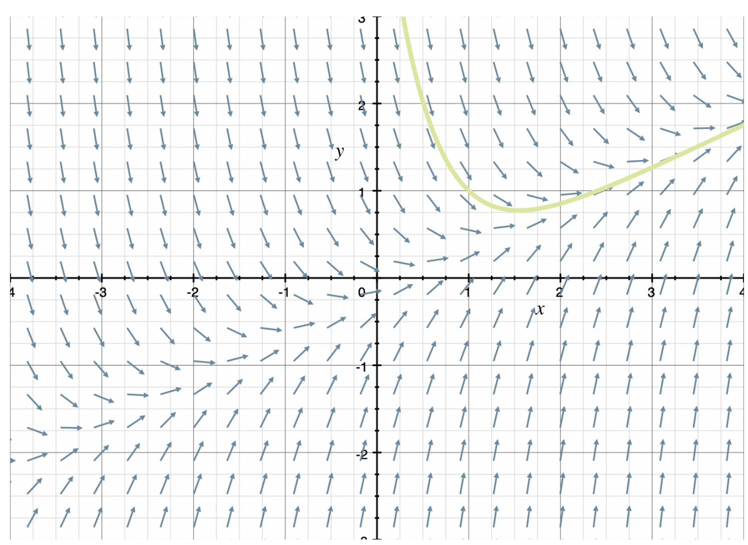
$$y' = -2y + x$$

Answer choices:

A 

B 

C 

D 

Solution: D

We'll pair up values $x = \{-2, -1, 0, 1, 2\}$ with values $y = \{-2, -1, 0, 1, 2\}$, calculating y' for each combination of values.

For instance, at $(x, y) = (-2, -2)$, we find

$$y' = -2y + x$$

$$y' = -2(-2) - 2$$

$$y' = 2$$

Or at $(x, y) = (1, 2)$, we find

$$y' = -2y + x$$

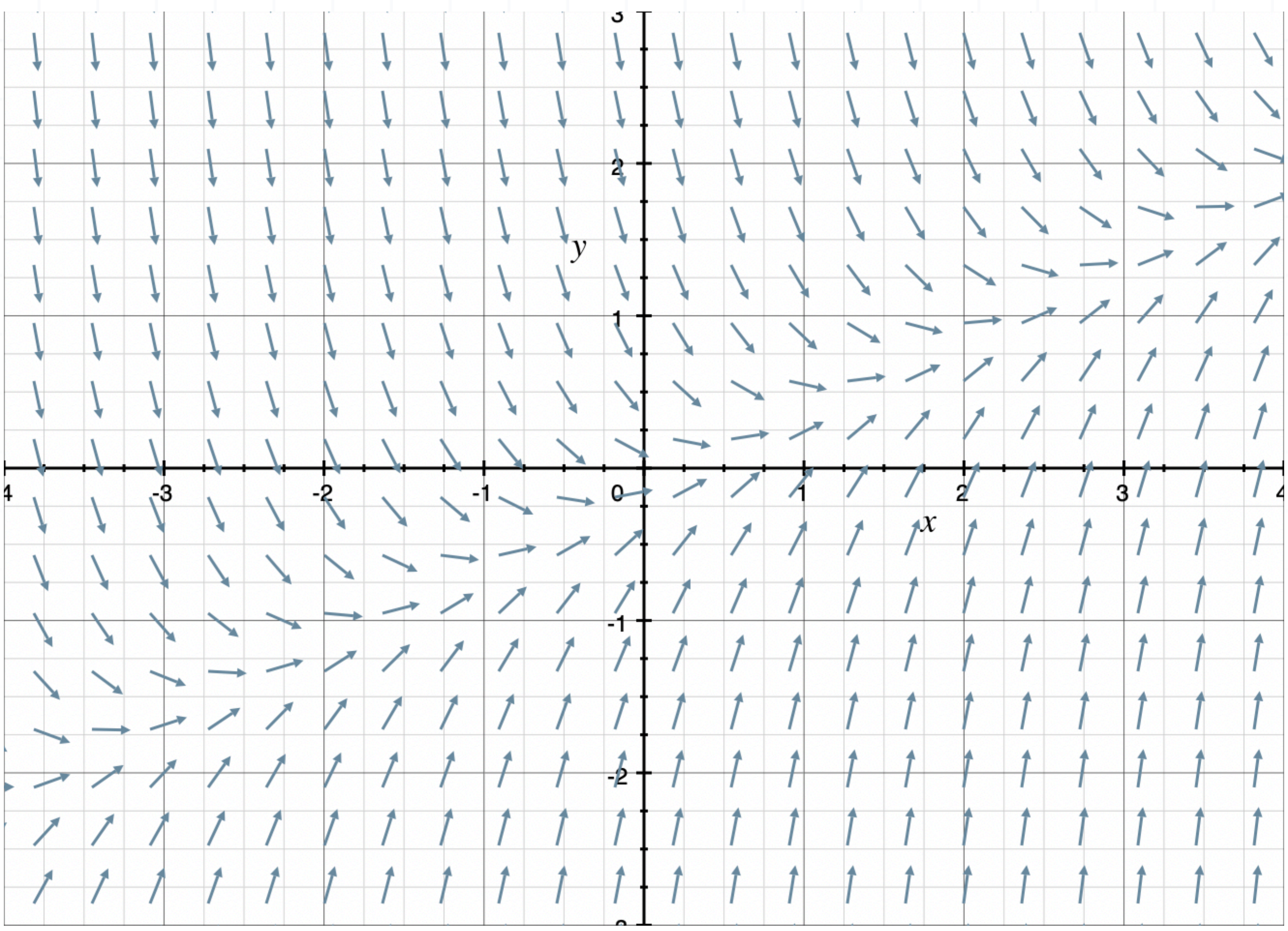
$$y' = -2(2) + 1$$

$$y' = -3$$

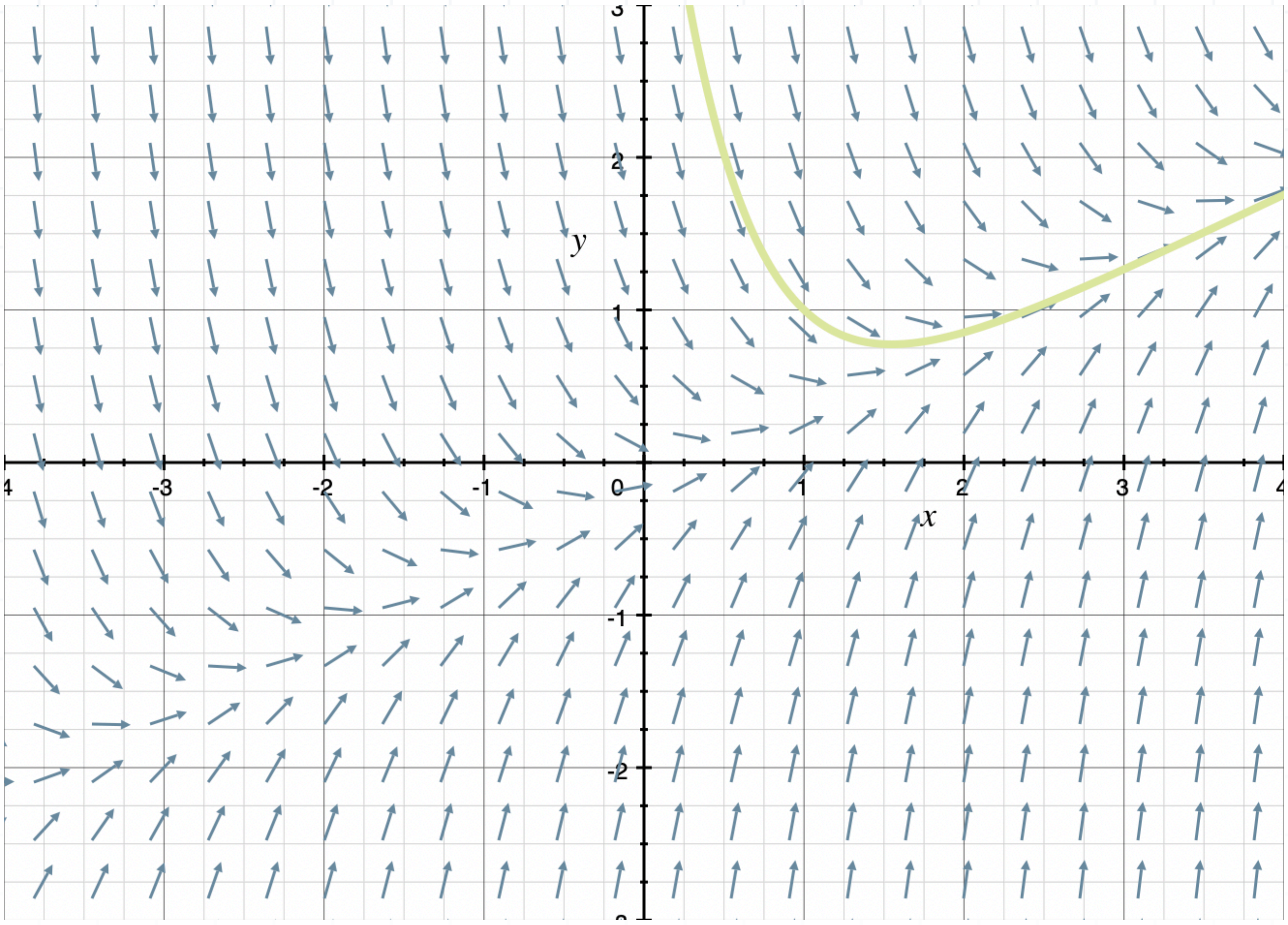
Let's make a table with each of these combinations.

		x				
		-2	-1	0	1	2
y	-2	2	3	4	5	6
	-1	0	1	2	3	4
	0	-2	-1	0	1	2
	1	-4	-3	-2	-1	0
	2	-6	-5	-4	-3	-2

The values of y' that we found represent the slope of the function at the corresponding point (x, y). For example, we see the point $(2, -2)$, and the corresponding value $y' = 6$. This means that the slope of the function at $(2, -2)$ is 6, so we'd draw a small, short line exactly at $(2, -2)$ with slope 6. Plotting all of the other point-slope pairs, the direction field starts to look something like this:



If we plot the point $(1,1)$, then follow the slopes in our direction field to both the left and right of $(1,1)$, our solution curve should look something like this:


y
x
-3
-2
-1
0
1
2
3
-2
-1
1
2

Topic: Intervals of validity

Question: Which differential equation has a solution with an interval of validity of $(-\infty, \infty)$?

Answer choices:

A $\quad e^{-x}\dfrac{dy}{dx} - yx^3 - x^2e^{-x} + 2e^x = 0$

B $\quad x\dfrac{dy}{dx} - (x^2 + 1)y = 3e^x$

C $\quad y' + e^{2x}(x - 2)y - \dfrac{1}{x} = x^2$

D $\quad x^2y' - xy\ln x = e^x$

Solution: A

Rewrite the first equation.

$$e^{-x}\frac{dy}{dx} - yx^3 - x^2e^{-x} + 2e^x = 0$$

$$e^{-x}\frac{dy}{dx} - yx^3 = x^2e^{-x} - 2e^x$$

$$\frac{dy}{dx} - yx^3e^x = x^2 - 2e^{2x}$$

When the equation is linear, we can find the interval of validity without calculating the equation's solution, because the interval of validity is determined by $P(x)$ and $Q(x)$ from the standard form of the linear equation,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

In our case,

$$P(x) = -x^3e^x$$

$$Q(x) = x^2 - 2e^{2x}$$

The functions $P(x)$ and $Q(x)$ are defined everywhere, $(-\infty, \infty)$, so the interval of validity is $(-\infty, \infty)$.

Topic: Intervals of validity

Question: Find the interval of validity for the differential equation if $y(1)=3$.

$$(x+1)y' = xy + x^2e^{-x^2}$$

Answer choices:

A $(-\infty,-1)$

B $(0,\infty)$

C $(-1,\infty)$

D $(-\infty,2)$

Solution: C

Put the equation into standard form of a linear equation.

$$(x+1)y' = xy + x^2e^{-x^2}$$

$$y' = \frac{xy}{x+1} + \frac{x^2e^{-x^2}}{x+1}$$

$$y' = \frac{x}{x+1}y + \frac{x^2}{x+1}e^{-x^2}$$

$$y' - \frac{x}{x+1}y = \frac{x^2}{x+1}e^{-x^2}$$

With the equation rewritten in standard form, we can identify

$$P(x) = -\frac{x}{x+1}$$

$$Q(x) = \frac{x^2}{x+1}e^{-x^2}$$

Both functions are undefined when the denominator is 0.

$$x+1=0$$

$$x=-1$$

This value separates $(-\infty,\infty)$ into two possible intervals of validity, $(-\infty,-1)$ and $(-1,\infty)$. Because we're given the initial condition $y(1)=3$, and the value $x_0=1$ is contained in $(-1,\infty)$, we can say that $(-1,\infty)$ is the interval of validity.

Topic: Intervals of validity

Question: Given the general solution to $y' = y^2xe^x$, and the initial condition $y(0) = 1$, find the actual solution and then determine its interval of validity.

$$y = \frac{1}{e^x(1-x) + C}$$

Answer choices:

A $(1,\infty)$

B $(-\infty,1)$

C $(-\infty,\infty)$

D $(-\infty,0)$

Solution: B

To find the actual solution, we'll substitute $x = 0$ and $y = 1$ from the initial condition,

$$1 = \frac{1}{e^0(1 - 0) + C}$$

$$1 = \frac{1}{1 + C}$$

$$1 + C = 1$$

$$C = 0$$

and then plug this back into the general solution.

$$y = \frac{1}{e^x(1 - x)}$$

The actual solution is undefined where the denominator is 0.

$$e^x(1 - x) = 0$$

Because $e^x > 0$ for every x, we can divide it out to get

$$1 - x = 0$$

$$x = 1$$

This value separates $(-\infty, \infty)$ into two possible intervals of validity, $(-\infty,1)$ and $(1,\infty)$. Because the initial condition is $y(0) = 1$, and the value $x_0 = 0$ is contained in $(-\infty,1)$, we can say that $(-\infty,1)$ is the interval of validity.

Topic: Euler's method

Question: If $y(1) = 0$, use Euler's method to approximate $y(3)$ with $n = 4$ steps.

$$y' = \frac{1}{2 + y}$$

Answer choices:

A $y(3) \approx 1$

B $y(3) \approx 1.69$

C $y(3) \approx 0.86$

D $y(3) \approx 0.82$

Solution: C

We're starting at $y(1)$ and we need to get to $y(3)$ in $n = 4$ steps, so

$$\Delta t = \frac{3-1}{4}$$

$$\Delta t = \frac{1}{2}$$

Starting with $(t_0, y_0) = (1,0)$, and with $\Delta t = 1/2$, we build our table and use it to approximate $y(3)$.

$$t_0 = 1 \qquad y_0 = 0 \qquad y_0 = 0$$

$$t_1 = \frac{3}{2} \qquad y_1 = 0 + \left(\frac{1}{2+0}\right)\left(\frac{1}{2}\right) \qquad y_1 = \frac{1}{4}$$

$$t_2 = 2 \qquad y_2 = \frac{1}{4} + \left(\frac{1}{2+\frac{1}{4}}\right)\left(\frac{1}{2}\right) \qquad y_2 = \frac{17}{36}$$

$$t_3 = \frac{5}{2} \qquad y_3 = \frac{17}{36} + \left(\frac{1}{2+\frac{17}{36}}\right)\left(\frac{1}{2}\right) \qquad y_3 = \frac{2{,}161}{3{,}204}$$

$$t_4 = 3 \qquad y_4 = \frac{2{,}161}{3{,}204} + \left(\frac{1}{2+\frac{2{,}161}{3{,}204}}\right)\left(\frac{1}{2}\right) \qquad y_4 \approx 0.86$$

Topic: Euler's method

Question: If $y(0) = 0$, use Euler's method to approximate $y(4)$ with $n = 5$ steps.

$$y' = 1 + y$$

Answer choices:

A $y(4) \approx 8.50$

B $y(4) \approx 17.90$

C $y(4) \approx 55.60$

D $y(4) \approx 9.50$

Solution: B

We're starting at $y(0)$ and we need to get to $y(4)$ in $n = 5$ steps, so

$$\Delta t = \frac{4-0}{5}$$

$$\Delta t = \frac{4}{5}$$

Starting with $(t_0, y_0) = (0,0)$, and with $\Delta t = 4/5$, we build our table and use it to approximate $y(4)$.

$$t_0 = 0 \qquad y_0 = 0 \qquad y_0 = 0$$

$$t_1 = \frac{4}{5} \qquad y_1 = 0 + (1+0)\left(\frac{4}{5}\right) \qquad y_1 = \frac{4}{5}$$

$$t_2 = \frac{8}{5} \qquad y_2 = \frac{4}{5} + \left(1 + \frac{4}{5}\right)\left(\frac{4}{5}\right) \qquad y_2 = \frac{56}{25}$$

$$t_3 = \frac{12}{5} \qquad y_3 = \frac{56}{25} + \left(1 + \frac{56}{25}\right)\left(\frac{4}{5}\right) \qquad y_3 = \frac{604}{125}$$

$$t_4 = \frac{16}{5} \qquad y_4 = \frac{604}{125} + \left(1 + \frac{604}{125}\right)\left(\frac{4}{5}\right) \qquad y_4 = \frac{5{,}936}{625}$$

$$t_5 = 4 \qquad y_5 = \frac{5{,}936}{625} + \left(1 + \frac{5{,}936}{625}\right)\left(\frac{4}{5}\right) \qquad y_5 = \frac{55{,}924}{3{,}125}$$

$$y_5 \approx 17.90$$

Topic: Euler's method

Question: Use Euler's Method with step size 0.5 to approximate the values y_1, y_2, y_3, and y_4 when $y(3) = 0$, then find $y(5)$.

$$y' = y - 2t$$

Answer choices:

A $y(5) \approx -28.5$

B $y(5) \approx -16$

C $y(5) \approx -47.75$

D $y(5) \approx -16.5$

Solution: A

Since we're given step-size directly, we already know that

$$\Delta t = 0.5$$

To start building our table, we identify $t_0 = 3$ and $y_0 = 0$ from our initial condition, $y(3) = 0$. Since we need to find y_4, we'll start adding $\Delta t = 0.5$ to t_0 until we reach t_4.

$t_0 = 3$	$y_0 = 0$	$y_0 = 0$
$t_1 = 3.5$		
$t_2 = 4$		
$t_3 = 4.5$		
$t_4 = 5$		

Now that we've built the outline for our table, we'll use Euler's formula to find y_1, y_2, y_3, and y_4.

$t_0 = 3$	$y_0 = 0$	$y_0 = 0$
$t_1 = 3.5$	$y_1 = 0 + (0 - 2(3))(0.5)$	$y_1 = -3$
$t_2 = 4$	$y_2 = -3 + (-3 - 2(3.5))(0.5)$	$y_2 = -8$
$t_3 = 4.5$	$y_3 = -8 + (-8 - 2(4))(0.5)$	$y_3 = -16$
$t_4 = 5$	$y_4 = -16 + (-16 - 2(4.5))(0.5)$	$y_4 = -28.5$

After filling out the table, we can say that the value of $y(5)$ is approximately

$$y(5) \approx -28.5$$

Topic: Autonomous equations and equilibrium solutions

Question: Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = 4y - 4y^2$$

Answer choices:

A $y = 0$ is a stable solution and $y = 1$ is an unstable solution

B $y = 0$ is an unstable solution and $y = 1$ is a stable solution

C $y = 0$ and $y = 1$ are unstable solutions

D $y = 0$ and $y = 1$ are stable solutions

Solution: B

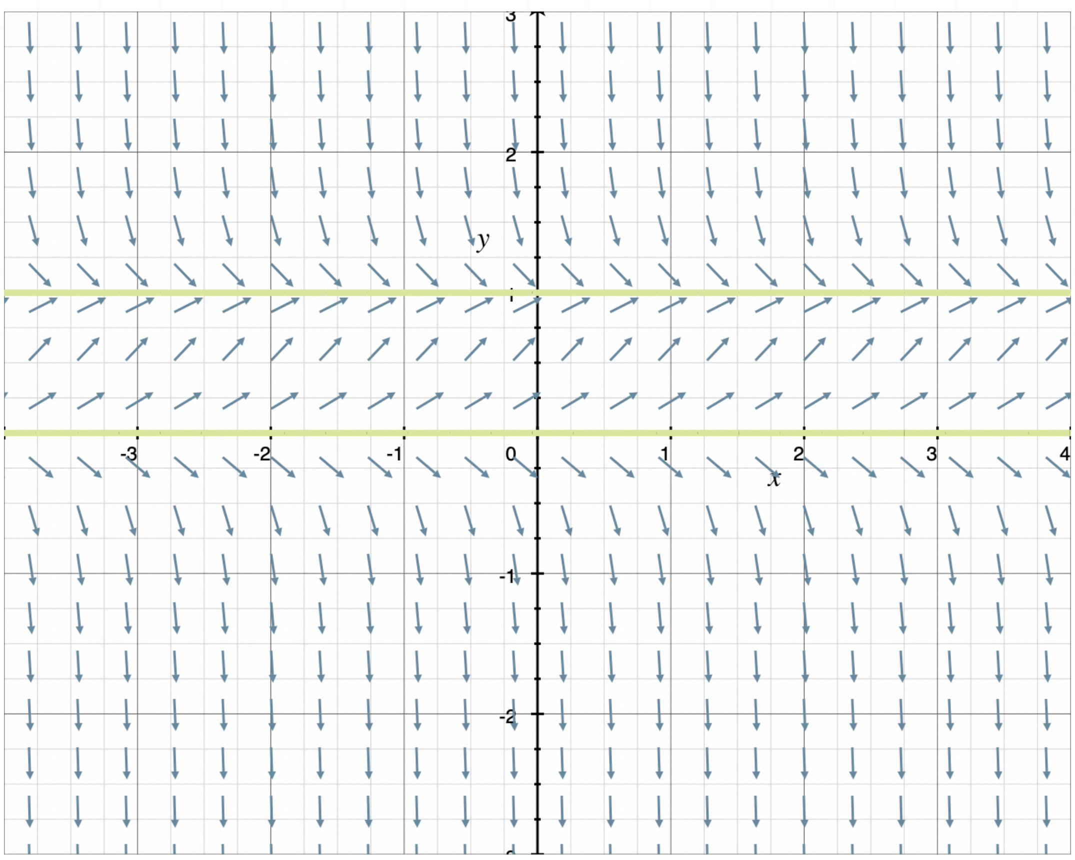
The autonomous differential equation has equilibrium solutions at

$$4y - 4y^2 = 0$$

$$y(1 - y) = 0$$

$$y = 0,\ 1$$

If we sketch the direction field and then overlay the equilibrium solutions, we get



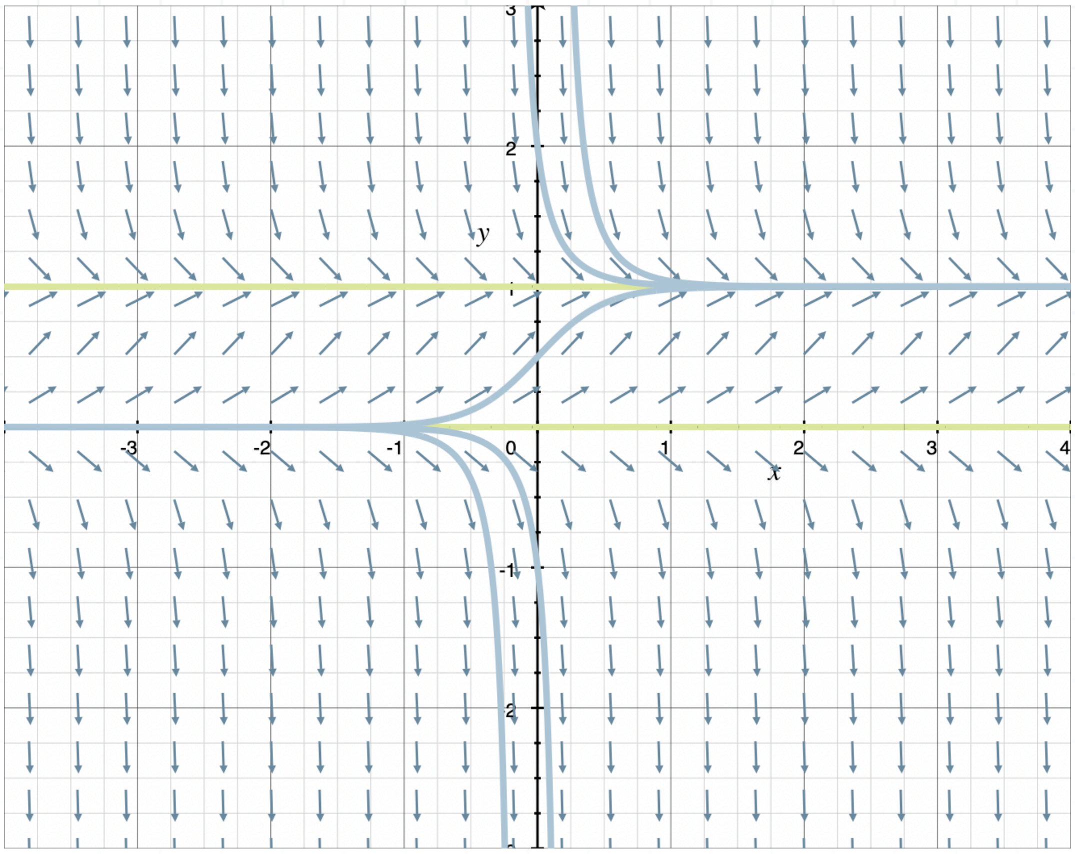
These two equilibrium solutions divide the vertical axis into three intervals,

$$y < 0$$

$$0 < y < 1$$

$$1 < y$$

We can use the direction field to sketch solution curves in each interval.



These solution curves approach $y = 1$ on both sides, but move away from $y = 0$ on both sides. Therefore, $y = 1$ is a stable equilibrium solution, while $y = 0$ is an unstable equilibrium solution.

Topic: Autonomous equations and equilibrium solutions

Question: Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = y(y - 2)$$

Answer choices:

A $\quad y = 0$ and $y = 2$ are unstable solutions

B $\quad y = 0$ and $y = 2$ are stable solutions

C $\quad y = 0$ is a stable solution and $y = 2$ is an unstable solution

D $\quad y = 0$ is an unstable solution and $y = 2$ is a stable solution

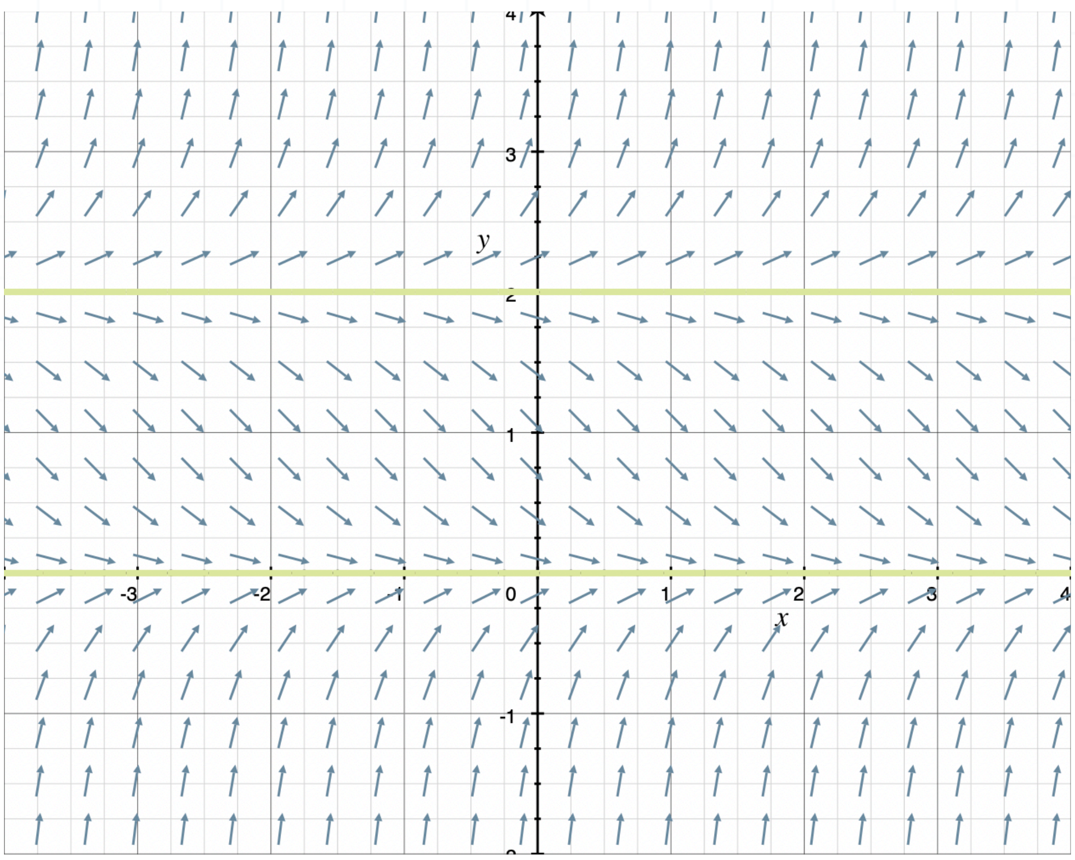
Solution: C

The autonomous differential equation has equilibrium solutions at

$$y(y-2)=0$$

$$y=0,\ 2$$

If we sketch the direction field and then overlay the equilibrium solutions, we get



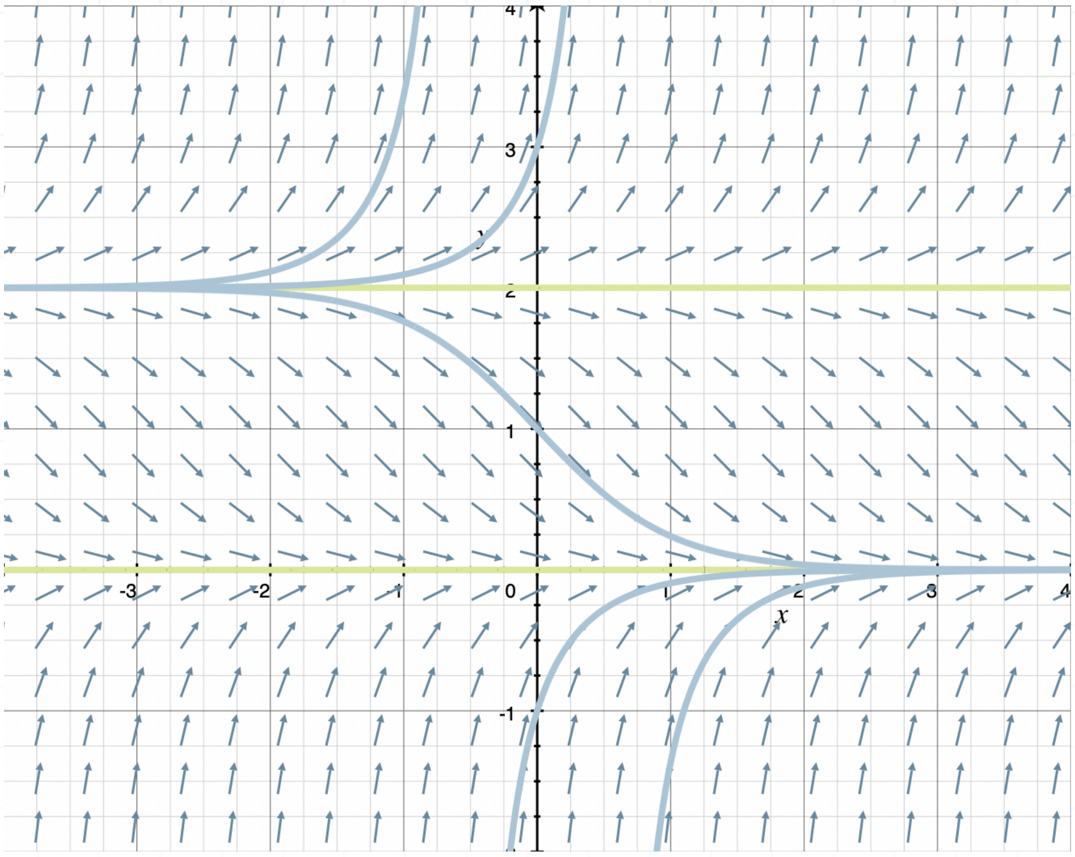
These two equilibrium solutions divide the vertical axis into three intervals,

$$y<0$$

$$0<y<2$$

$$2<y$$

We can use the direction field to sketch solution curves in each interval.



These solution curves approach $y = 0$ on both sides, but move away from $y = 2$ on both sides. Therefore, $y = 0$ is a stable equilibrium solution, while $y = 2$ is an unstable equilibrium solution.

Topic: Autonomous equations and equilibrium solutions

Question: Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$y' = y^3 - 5y^2 + 6y$$

Answer choices:

A $y = 0$, $y = 2$, and $y = 3$ are unstable solutions

B $y = 0$, $y = 2$, and $y = 3$ are stable solutions

C $y = 0$ and $y = 3$ are stable solutions and $y = 2$ is an unstable solution

D $y = 0$ and $y = 3$ are unstable solutions and $y = 2$ is stable solution

Solution: D

The autonomous differential equation has equilibrium solutions at

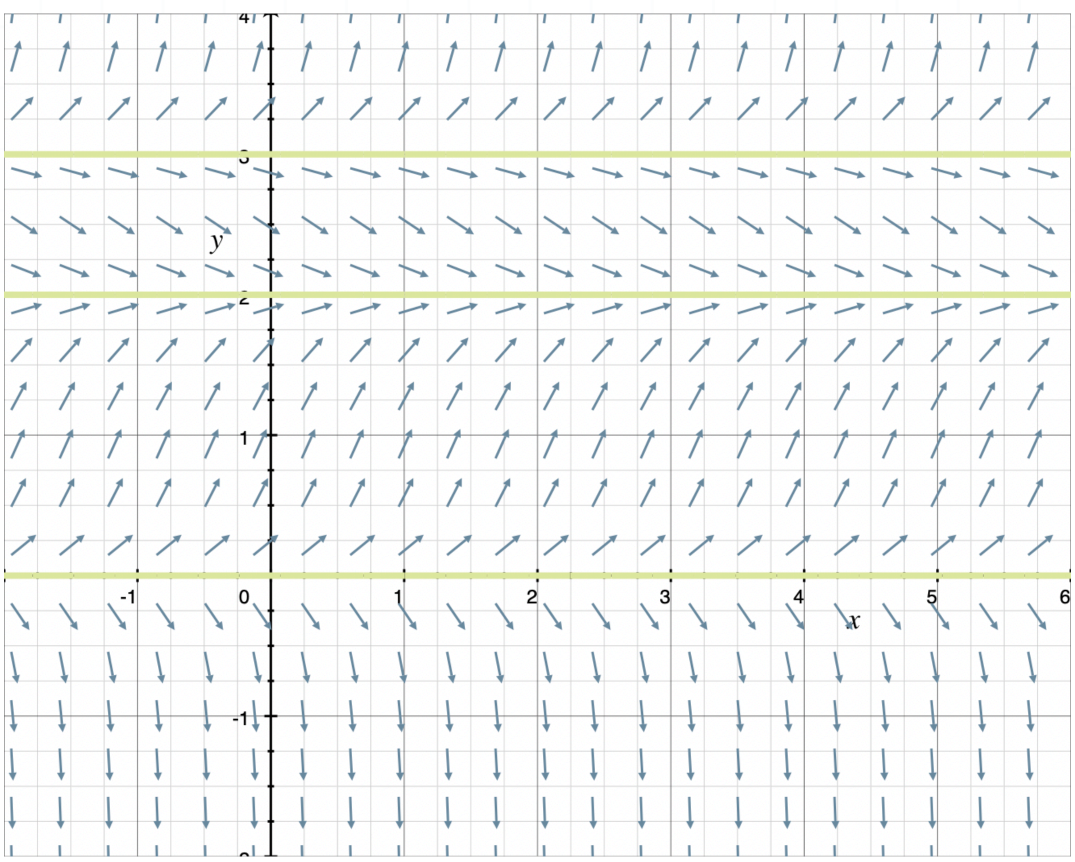
$$y^3 - 5y^2 + 6y = 0$$

$$y(y^2 - 5y + 6) = 0$$

$$y(y - 2)(y - 3) = 0$$

$$y = 0,\ 2,\ 3$$

If we sketch the direction field and then overlay the equilibrium solutions, we get



These three equilibrium solutions divide the vertical axis into four intervals,

$$y < 0$$

$$0 < y < 2$$

$$2 < y < 3$$

$$3 < y$$

and we can see from the direction field that solution curves will approach $y = 2$ on both sides, but move away from $y = 0$ and $y = 3$ on both sides. Therefore, $y = 2$ is a stable equilibrium solution, while $y = 0$ and $y = 3$ are unstable equilibrium solutions.

Topic: The logistic equation

Question: A population of insects grows exponentially with a growth constant $k=7$. How long will it take for the population to triple, assuming time t is measured in hours?

Answer choices:

A 0.649 hours

B 0.157 hours

C 2.333 hours

D 0.429 hours

Solution: B

If we say that P_0 is the original population, and $3P_0$ is triple the original population, then the exponential growth equation is

$$3P_0 = P_0 e^{7t}$$

$$\frac{3P_0}{P_0} = e^{7t}$$

$$3 = e^{7t}$$

Apply the natural log to both sides of the equation, canceling the $\ln e$, and then rearrange.

$$\ln 3 = \ln e^{7t}$$

$$\ln 3 = 7t$$

$$t = \frac{\ln 3}{7}$$

$$t \approx 0.157 \text{ hours}$$

Topic: The logistic equation

Question: A population of flies doubled in 5 hours. Assuming exponential growth, how long would it take for the population to increase by 10-fold?

Answer choices:

A 2.41 hours

B 3.32 hours

C 16.61 hours

D 1.51 hours

Solution: C

We've been told that the population doubles in 5 hours. If we call its original size P_0, then we can say that it grows to $2P_0$ in $t = 5$ hours. Substituting these values into the exponential growth equation gives

$$2P_0 = P_0 e^{k(5)}$$

$$\frac{2P_0}{P_0} = e^{k(5)}$$

$$2 = e^{5k}$$

To solve for k, we'll apply the natural log to both sides of the equation, cancelling the $\ln e$, and then rearrange.

$$\ln 2 = \ln e^{5k}$$

$$\ln 2 = 5k$$

$$k = \frac{\ln 2}{5}$$

Now that we have a value for the growth constant k, we can figure out how long it would take for the population to increase 10 fold. If we say that P_0 is the original population, and $10P_0$ is the resulting population that's 10 times larger, then the exponential equation is

$$10P_0 = P_0 e^{\frac{\ln 2}{5}t}$$

$$\frac{10P_0}{P_0} = e^{\frac{\ln 2}{5}t}$$

$$10 = e^{\frac{\ln 2}{5}t}$$

Apply the natural log to both sides of the equation, canceling the $\ln e$, and then rearrange.

$$\ln 10 = \ln e^{\frac{\ln 2}{5}t}$$

$$\ln 10 = \frac{\ln 2}{5}t$$

$$t = \frac{5 \ln 10}{\ln 2}$$

$t \approx 16.61$ hours

Topic: The logistic equation

Question: A population of rabbits is observed to be 300 rabbits at time $t = 0$, where t is measured in days. After 2 weeks, the population doubled. Assuming exponential growth, how long would it take for the population to grow to $1{,}200$ rabbits?

Answer choices:

A 7 days

B 4 weeks

C 11 weeks 4 days

D 7 weeks 4 days

Solution: B

After 2 weeks, the population doubled from 300 rabbits to 600 rabbits. Substituting these values into the exponential equation gives

$$600 = 300e^{k(14)}$$

$$\frac{600}{300} = e^{k(14)}$$

$$2 = e^{14k}$$

To solve for k, we'll apply the natural log to both sides of the equation, canceling the $\ln e$, and then rearrange.

$$\ln 2 = \ln e^{14k}$$

$$\ln 2 = 14k$$

$$k = \frac{\ln 2}{14}$$

Now that we have a value for the growth constant k, we can can figure out how long it'll take for the population to grow to $1,200$ rabbits.

$$1{,}200 = 300e^{\frac{\ln 2}{14}t}$$

$$4 = e^{\frac{\ln 2}{14}t}$$

Apply the natural log to both sides of the equation, canceling the $\ln e$, and then rearrange.

$$\ln 4 = \ln e^{\frac{\ln 2}{14}t}$$

$$\ln 4 = \frac{\ln 2}{14}t$$

$$t = \frac{14 \ln 4}{\ln 2}$$

$t \approx 28$ days

Topic: Predator-prey systems

Question: Is the system cooperative, competitive, or predator-prey?

$$\frac{dx}{dt} = 0.14x - 0.2xy$$

$$\frac{dy}{dt} = -0.09y + 0.10y^2 + 0.12xy$$

Answer choices:

A Cooperative

B Competitive

C Predator-prey

D None of these

Solution: C

The interaction term in the dx/dt equation is $-0.2xy$, and the interaction term in the dy/dt equation is $0.12xy$.

When these interaction terms have opposite signs, it means the system is predator-prey. In this case, we know the size of population x is decreasing because the sign of $-0.2xy$ is negative, while the size of population y is increasing because the sign of $0.12xy$ is positive.

Topic: Predator-prey systems

Question: Is the system cooperative, competitive, or predator-prey?

$$\frac{dx}{dt} = 0.8x + 0.15xy$$

$$\frac{dy}{dt} = 0.3y + 0.16xy$$

Answer choices:

A Cooperative

B Competitive

C Predator-prey

D None of these

Solution: A

The interaction term in the dx/dt equation is $0.15xy$, and the interaction term in the dy/dt equation is $0.16xy$.

When these interaction terms have positive signs, it means the system is cooperative. In this case, we know the size of population x is increasing because the sign of $0.15xy$ is positive, and the size of population y is also increasing because the sign of $0.16xy$ is positive.

Topic: Predator-prey systems

Question: Is the system cooperative, competitive, or predator-prey?

$$\frac{dx}{dt} = -0.3x - 0.21xy$$

$$\frac{dy}{dt} = 0.9y - 0.36xy$$

Answer choices:

A Cooperative

B Competitive

C Predator-prey

D None of these

Solution: B

The interaction term in the dx/dt equation is $-0.21xy$, and the interaction term in the dy/dt equation is $-0.36xy$.

When these interaction terms have negative signs, it means the system is competitive. In this case, we know the size of population x is decreasing because the sign of $-0.21xy$ is negative, and the size of population y is also decreasing because the sign of $-0.36xy$ is negative.

Topic: Exponential growth and decay

Question: If an amoeba population doubles in 4 hours, and assuming normal exponential growth, find the growth constant for the population.

Answer choices:

A $\quad k \approx 0.17$

B $\quad k \approx 0.50$

C $\quad k \approx 0.69$

D $\quad k \approx 5.77$

Solution: A

The size of the original population is P_0. Since the population doubles after 4 hours, we know $P(4) = 2P_0$. Plug this initial condition into the exponential growth equation.

$$P(t) = P_0 e^{kt}$$

$$2P_0 = P_0 e^{k(4)}$$

$$2 = e^{4k}$$

Take the natural log of both sides.

$$\ln 2 = \ln(e^{4k})$$

$$\ln 2 = 4k$$

$$k \approx 0.17$$

Topic: Exponential growth and decay

Question: If a bacteria population grows exponentially and increases fivefold in 6 hours, how long did it take for the population to double?

Answer choices:

A $t \approx 4.16$ hours

B $t \approx 13.93$ hours

C $t \approx 2.58$ hours

D $t \approx 0.43$ hours

Solution: C

The size of the original population is P_0. Since the population increases fivefold in 6 hours, we know $P(6)=5P_0$. Plug this initial condition into the exponential growth equation.

$$P(t)=P_0e^{kt}$$

$$5P_0=P_0e^{k(6)}$$

$$5=e^{6k}$$

Take the natural log of both sides.

$$\ln 5=\ln(e^{6k})$$

$$\ln 5=6k$$

$$k=\frac{\ln 5}{6}$$

This is the growth constant for the population, and we can plug it back in to figure out how long it takes for the population to double (reach a size of $2P_0$).

$$P(t)=P_0e^{kt}$$

$$2P_0=P_0e^{\frac{\ln 5}{6}t}$$

$$2=e^{\frac{\ln 5}{6}t}$$

Take the natural log of both sides.

$$\ln 2=\ln(e^{\frac{\ln 5}{6}t})$$

$$\ln 2 = \frac{\ln 5}{6} t$$

$$t = \frac{6 \ln 2}{\ln 5}$$

$$t \approx 2.58$$

Topic: Exponential growth and decay

Question: If a fruit fly population grows exponentially and increases sixfold in 7 days, how long did it take for the population to triple?

Answer choices:

A $t \approx 11.42$ days

B $t \approx 3.50$ days

C $t \approx 1.77$ days

D $t \approx 4.29$ days

Solution: D

The size of the original population is P_0. Since the population increases sixfold in 7 days, we know $P(7) = 6P_0$. Plug this initial condition into the exponential growth equation.

$$P(t) = P_0 e^{kt}$$

$$6P_0 = P_0 e^{k(7)}$$

$$6 = e^{7k}$$

Take the natural log of both sides.

$$\ln 6 = \ln(e^{7k})$$

$$\ln 6 = 7k$$

$$k = \frac{\ln 6}{7}$$

This is the growth constant for the population, and we can plug it back in to figure out how long it takes for the population to triple (reach a size of $3P_0$).

$$P(t) = P_0 e^{kt}$$

$$3P_0 = P_0 e^{\frac{\ln 6}{7}t}$$

$$3 = e^{\frac{\ln 6}{7}t}$$

Take the natural log of both sides.

$$\ln 3 = \ln(e^{\frac{\ln 6}{7}t})$$

$$\ln 3 = \frac{\ln 6}{7}t$$

$$t = \frac{7\ln 3}{\ln 6}$$

$$t \approx 4.29$$

Topic: Mixing problems

Question: A tank contains $1,000$ L of water and 10 kg of dissolved salt. Fresh water is filling the tank at 10 L/min, while the tank drains at 5 L/min. Assuming the solution stays perfectly mixed, how much salt is in the tank after t minutes?

Answer choices:

A $\quad y = 5e^{-\frac{1}{100}t}$

B $\quad y = 5e^{\frac{1}{100}t}$

C $\quad y = \dfrac{2,000}{200 + t}$

D $\quad y = \dfrac{10}{200 + t}$

Solution: C

The concentration of salt in the freshwater entering the tank is $C_1 = 0$ kg/min, while the fill rate is $r_1 = 10$ L/min.

With a fill rate of 10 L/min and a drain rate of 5 L/min, the initial $1{,}000$ L in the tank is gaining 5 L every minute, so the concentration of salt draining from the tank is $C_2 = y/(1{,}000 + 5t)$ kg/L, while the drain rate is $r_2 = 5$ L/min.

Plugging these values into the differential equation gives

$$\frac{dy}{dt} = C_1 r_1 - C_2 r_2$$

$$\frac{dy}{dt} = (0)(10) - \left(\frac{y}{1{,}000 + 5t}\right)(5)$$

$$\frac{dy}{dt} = -\frac{5y}{1{,}000 + 5t}$$

$$\frac{dy}{dt} = -\frac{y}{200 + t}$$

This is now a separable differential equation, so we'll separate variables and integrate both sides.

$$\int \frac{1}{y}\ dy = \int -\frac{1}{200 + t}\ dt$$

$$\ln|y| = -\ln|200 + t| + C$$

Solve for y.

$$e^{\ln|y|} = e^{-\ln|200+t|+C}$$

$$|y| = Ce^{-\ln|200+t|}$$

$$y = Ce^{-\ln|200+t|}$$

Because we're talking about a real-world problem in which time t will never be negative, $200 + t$ will always be positive, so we can take away the absolute value.

$$y = Ce^{-\ln(200+t)}$$

By laws of logarithms, the negative sign in front of the logarithm becomes the exponent on the argument.

$$y = Ce^{\ln(200+t)^{-1}}$$

$$y = C(200 + t)^{-1}$$

$$y = \frac{C}{200 + t}$$

The tank initially contained 10 kg of dissolved salt, so we'll plug $y(0) = 10$ into this general solution.

$$10 = \frac{C}{200 + 0}$$

$$C = 2{,}000$$

So the amount of salt in the tank at any time t can be modeled by

$$y = \frac{2{,}000}{200 + t}$$

Topic: Mixing problems

Question: A tank contains $1{,}000$ L of water and 15 kg of dissolved salt. Fresh water is filling the tank at 10 L/min, while the tank drains at 20 L/min. Assuming the solution stays perfectly mixed, how much salt is in the tank after 10 minutes?

Answer choices:

A 12.40 kg

B 12.15 kg

C 16.75 kg

D 13.60 kg

Solution: B

The concentration of salt in the fresh water entering the tank is $C_1 = 0$ kg/min, while the fill rate is $r_1 = 10$ L/min.

With a fill rate of 10 L/min and a drain rate of 20 L/min, the initial $1{,}000$ L of water in the tank is decreasing by 10 L every minute, so the concentration of salt draining from the tank is $C_2 = y/(1{,}000 - 10t)$ kg/L, while the drain rate is $r_2 = 20$ L/min.

Plugging these values into the differential equation gives

$$\frac{dy}{dt} = C_1 r_1 - C_2 r_2$$

$$\frac{dy}{dt} = (0)(10) - \left(\frac{y}{1{,}000 - 10t}\right)(20)$$

$$\frac{dy}{dt} = -\frac{20y}{1{,}000 - 10t}$$

$$\frac{dy}{dt} = -\frac{2y}{100 - t}$$

This is now a separable differential equation, so we'll separate variables and integrate both sides.

$$\int \frac{1}{y}\ dy = \int -\frac{2}{100 - t}\ dt$$

$$\ln|y| = 2\ln|100 - t| + C$$

Solve for y.

$$e^{\ln|y|} = e^{2\ln|100-t|+C}$$

$$|y| = Ce^{2\ln|100-t|}$$

$$y = Ce^{2\ln|100-t|}$$

By laws of logarithms,

$$y = Ce^{\ln|100-t|^2}$$

$$y = C|100 - t|^2$$

$$y = C(100 - t)^2$$

The tank initially contained 15 kg of dissolved salt, so we'll plug $y(0) = 15$ into this general solution.

$$15 = C(100 - 0)^2$$

$$15 = 10{,}000C$$

$$C = \frac{3}{2{,}000}$$

So the amount of salt in the tank at any time t can be modeled by

$$y = \frac{3}{2{,}000}(100 - t)^2$$

To find the amount of salt in the tank after 10 minutes, we'll evaluate at $t = 10$.

$$y(10) = \frac{3}{2{,}000}(100 - 10)^2$$

$$y(10) = \frac{3(8{,}100)}{2{,}000}$$

$$y(10) = \frac{243}{20}$$

$$y(10) = 12.15 \text{ kg}$$

Topic: Mixing problems

Question: A tank contains $2{,}000$ L of water and 25 kg of dissolved salt. Fresh water is filling the tank at 10 L/min, while the tank drains at 10 L/min. Assuming the solution stays perfectly mixed, how much salt is in the tank after 5 minutes?

Answer choices:

A 25.6 kg

B 25.1 kg

C 24.9 kg

D 24.4 kg

Solution: D

The concentration of salt in the fresh water entering the tank is $C_1 = 0$ kg/min, while the fill rate is $r_1 = 10$ L/min.

With a fill rate of 10 L/min and a drain rate 10 L/min, the total volume in the tank stays, which means the volume at any time t is $2{,}000$ L.

Plugging these values into the differential equation gives

$$\frac{dy}{dt} = C_1r_1 - C_2r_2$$

$$\frac{dy}{dt} = (0)(10) - \left(\frac{y}{2{,}000}\right)(10)$$

$$\frac{dy}{dt} = -\frac{10y}{2{,}000}$$

$$\frac{dy}{dt} = -\frac{y}{200}$$

This is now a separable differential equation, so we'll separate variables and integrate both sides.

$$\int \frac{1}{y}\ dy = \int -\frac{1}{200}\ dt$$

$$\ln|y| = -\frac{1}{200}t + C$$

Solve for y.

$$e^{\ln|y|} = e^{-\frac{1}{200}t+C}$$

$$|y| = Ce^{-\frac{1}{200}t}$$

$$y = Ce^{-\frac{1}{200}t}$$

The tank initially contained 25 kg of dissolved salt, so we'll plug $y(0) = 25$ into this general solution.

$$25 = Ce^{-\frac{1}{200}(0)}$$

$$C = 25$$

So the amount of salt in the tank at any time t can be modeled by

$$y = 25e^{-\frac{1}{200}t}$$

To find the amount of salt in the tank after 5 minutes, we'll evaluate at $t = 5$.

$$y = 25e^{-\frac{1}{200}(5)}$$

$$y = 25e^{-\frac{1}{40}}$$

$$y \approx 24.4 \text{ kg}$$

Topic: Newton's Law of Cooling

Question: You prepare a cup of coffee and measure its temperature at $100^\circ C$. After 2 minutes on the counter in your $25^\circ C$ kitchen, the coffee has cooled to $90^\circ C$. If you want to drink the coffee as soon as it cools to $50^\circ C$, how long will you need to wait?

Answer choices:

A 10.30 minutes

B 15.35 minutes

C 12.20 minutes

D 25 minutes

Solution: B

We'll list out what we know.

$T_0 = 100^\circ$ Initial temperature of the coffee

$T_a = 25^\circ$ Ambient temperature of the kitchen

$T(2) = 90^\circ$ At time $t = 2$ minutes, the coffee has cooled to 90°

If we plug everything we know into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 25 + (100 - 25)e^{-kt}$$

$$T(t) = 25 + 75e^{-kt}$$

Substitute the initial condition $T(2) = 90^\circ$,

$$T(2) = 25 + 75e^{-k(2)}$$

$$90 = 25 + 75e^{-2k}$$

in order to find a value for the decay constant k.

$$65 = 75e^{-2k}$$

$$\frac{13}{15} = e^{-2k}$$

$$\ln\frac{13}{15} = \ln(e^{-2k})$$

$$\ln\frac{13}{15} = -2k$$

$$k = -\frac{1}{2}\ln\frac{13}{15}$$

Substitute this value for k into the equation modeling temperature over time.

$$T(t) = 25 + 75e^{-\left(-\frac{1}{2}\ln\frac{13}{15}\right)t}$$

$$T(t) = 25 + 75e^{\left(\frac{1}{2}\ln\frac{13}{15}\right)t}$$

We want to find the time t at which the soup reaches 50°, so we'll substitute $T(t) = 50^\circ$.

$$50 = 25 + 75e^{\left(\frac{1}{2}\ln\frac{13}{15}\right)t}$$

$$25 = 75e^{\left(\frac{1}{2}\ln\frac{13}{15}\right)t}$$

$$\frac{1}{3} = e^{\left(\frac{1}{2}\ln\frac{13}{15}\right)t}$$

Apply the natural logarithm to both sides of the equation.

$$\ln\frac{1}{3} = \ln\left(e^{\left(\frac{1}{2}\ln\frac{13}{15}\right)t}\right)$$

$$\ln\frac{1}{3} = \left(\frac{1}{2}\ln\frac{13}{15}\right)t$$

$$2\ln\frac{1}{3} = \left(\ln\frac{13}{15}\right)t$$

$$t = \frac{2\ln\frac{1}{3}}{\ln\frac{13}{15}}$$

$t \approx 15.35$ minutes

Topic: Newton's Law of Cooling

Question: You light a scented candle in a glass container. When you blow out the flame, the temperature of the glass is $200^\circ C$. Room temperature is $20^\circ C$ and after 5 minutes the glass has cooled to $150^\circ C$. If it's safe to handle the glass at $40^\circ C$, how long should you wait before touching the glass?

Answer choices:

A 16.88 minutes

B 22.79 minutes

C 50.24 minutes

D 33.76 minutes

Solution: D

We'll list out what we know.

$T_0 = 200^\circ$	Initial temperature of the glass
$T_a = 20^\circ$	Ambient temperature in the room
$T(5) = 150^\circ$	At time $t = 5$ minutes, the glass has cooled to 150°

If we plug everything we know into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 20 + (200 - 20)e^{-kt}$$

$$T(t) = 20 + 180e^{-kt}$$

Substitute the initial condition $T(5) = 150^\circ$,

$$T(5) = 20 + 180e^{-k(5)}$$

$$150 = 20 + 180e^{-5k}$$

in order to find a value for the decay constant k.

$$130 = 180e^{-5k}$$

$$\frac{13}{18} = e^{-5k}$$

$$\ln\frac{13}{18} = \ln(e^{-5k})$$

$$\ln \frac{13}{18} = -5k$$

$$k = -\frac{1}{5} \ln \frac{13}{18}$$

Substitute this value for k into the equation modeling temperature over time.

$$T(t) = 20 + 180e^{-\left(-\frac{1}{5} \ln \frac{13}{18}\right)t}$$

$$T(t) = 20 + 180e^{\left(\frac{1}{5} \ln \frac{13}{18}\right)t}$$

We want to find the time t at which the glass reaches 40°, so we'll substitute $T(t) = 40^\circ$.

$$40 = 20 + 180e^{\left(\frac{1}{5} \ln \frac{13}{18}\right)t}$$

$$20 = 180e^{\left(\frac{1}{5} \ln \frac{13}{18}\right)t}$$

$$\frac{1}{9} = e^{\left(\frac{1}{5} \ln \frac{13}{18}\right)t}$$

Apply the natural logarithm to both sides of the equation.

$$\ln \frac{1}{9} = \ln \left(e^{\left(\frac{1}{5} \ln \frac{13}{18}\right)t}\right)$$

$$\ln \frac{1}{9} = \left(\frac{1}{5} \ln \frac{13}{18}\right) t$$

$$5 \ln \frac{1}{9} = \left(\ln \frac{13}{18}\right) t$$

$$t = \frac{5 \ln \frac{1}{9}}{\ln \frac{13}{18}}$$

$t \approx 33.76$ minutes

Topic: Newton's Law of Cooling

Question: At a local tea shop, tea that's boiling at $120^\circ C$ has just been removed from the heat and set out on the countertop, where the ambient temperature is $20^\circ C$. After 3 minutes, the tea has cooled to $114^\circ C$. If it's safe to drink the tea at $60^\circ C$, how long do we need to wait to drink the tea?

Answer choices:

A 40.04 minutes

B 42.44 minutes

C 44.42 minutes

D 48.24 minutes

Solution: C

We'll list out what we know.

$T_0 = 120^\circ C$	Initial temperature of the tea
$T_a = 20^\circ C$	Ambient temperature on the countertop
$T(3) = 114^\circ C$	At time $t = 3$ minutes, the tea has cooled to $114^\circ C$

If we plug everything we know into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 20 + (120 - 20)e^{-kt}$$

$$T(t) = 20 + 100e^{-kt}$$

Substitute the initial condition $T(3) = 114^\circ C$,

$$114 = 20 + 100e^{-3k}$$

in order to find a value for the decay constant k.

$$94 = 100e^{-3k}$$

$$0.94 = e^{-3k}$$

$$\ln 0.94 = \ln(e^{-3k})$$

$$\ln 0.94 = -3k$$

$$k = -\frac{1}{3}\ln 0.94$$

Substitute this value for k into the equation modeling temperature over time.

$$T(t) = 20 + 100e^{-\left(-\frac{1}{3}\ln 0.94\right)t}$$

$$T(t) = 20 + 100e^{\frac{1}{3}t\ln 0.94}$$

We want to find the time t at which the temperature of the tea reaches $60^\circ C$, so we'll substitute $T(t) = 60^\circ$.

$$60 = 20 + 100e^{\frac{1}{3}t\ln 0.94}$$

$$40 = 100e^{\frac{1}{3}t\ln 0.94}$$

$$\frac{2}{5} = e^{\frac{1}{3}t\ln 0.94}$$

Apply the natural logarithm to both sides of the equation.

$$\ln\frac{2}{5} = \ln e^{\frac{1}{3}t\ln 0.94}$$

$$\ln\frac{2}{5} = \frac{1}{3}t\ln 0.94$$

$$t = \frac{3\ln\frac{2}{5}}{\ln 0.94}$$

$t \approx 44.42$ minutes

Topic: Electrical series circuits

Question: If a battery supplies a constant voltage of 20 V, has an inductance of 2 h and a resistance of $10\ \Omega$, and assuming $i(0)=0$, find the function that models the current.

Answer choices:

A $\quad i(t)=1-e^{-5t}$

B $\quad i(t)=2-2e^{-5t}$

C $\quad i(t)=1-e^{-10t}$

D $\quad i(t)=2-2e^{-10t}$

Solution: B

This series circuit doesn't have a capacitor, which means we can remove that term from Kirchhoff's second law.

$$L\frac{di}{dt}+Ri+\frac{1}{C}q=E(t)$$

$$L\frac{di}{dt}+Ri=E(t)$$

Plugging in what we know, we get

$$2\frac{di}{dt}+10i=20$$

$$i'+5i=10$$

This is a linear differential equation with integrating factor

$$I(t)=e^{\int P\ dt}$$

$$I(t)=e^{\int 5\ dt}$$

$$I(t)=e^{5t}$$

Therefore, we can rewrite the linear equation as

$$e^{5t}\frac{di}{dt}+5e^{5t}i(t)=10e^{5t}$$

$$\frac{d}{dt}(e^{5t}i(t))=10e^{5t}$$

$$\int \frac{d}{dt}(e^{5t}i(t))\ dt = \int 10e^{5t}\ dt$$

$$e^{5t}i(t) = 2e^{5t} + C$$

$$i(t) = 2 + Ce^{-5t}$$

Substituting the initial condition $i(0) = 0$, we find

$$0 = 2 + Ce^{-5(0)}$$

$$0 = 2 + C(1)$$

$$C = -2$$

So we can write the equation modeling the current i over time t as

$$i(t) = 2 - 2e^{-5t}$$

Topic: Electrical series circuits

Question: If a battery supplies a constant voltage of 18 V, has an inductance of 3 h and a resistance of $12\ \Omega$, and assuming $i(0)=0$, find the current after 60 seconds.

Answer choices:

A $\quad 4.41$

B $\quad 4.50$

C $\quad 1.50$

D $\quad 1.47$

Solution: C

This series circuit doesn't have a capacitor, which means we can remove that term from Kirchhoff's second law.

$$L\frac{di}{dt}+Ri+\frac{1}{C}q=E(t)$$

$$L\frac{di}{dt}+Ri=E(t)$$

Plugging in what we know, we get

$$3\frac{di}{dt}+12i=18$$

$$i'+4i=6$$

This is a linear differential equation with integrating factor

$$I(t)=e^{\int P\ dt}$$

$$I(t)=e^{\int 4\ dt}$$

$$I(t)=e^{4t}$$

Therefore, we can rewrite the linear equation as

$$e^{4t}\frac{di}{dt}+4e^{4t}i(t)=6e^{4t}$$

$$\frac{d}{dt}(e^{4t}i(t))=6e^{4t}$$

$$\int \frac{d}{dt}(e^{4t}i(t))\ dt = \int 6e^{4t}\ dt$$

$$e^{4t}i(t) = \frac{3}{2}e^{4t} + C$$

$$i(t) = \frac{3}{2} + Ce^{-4t}$$

Substituting the initial condition $i(0) = 0$, we find

$$0 = \frac{3}{2} + Ce^{-4(0)}$$

$$C = -\frac{3}{2}$$

So we can write the equation modeling the current i over time t as

$$i(t) = \frac{3}{2} - \frac{3}{2}e^{-4t}$$

$$i(t) = \frac{3}{2}(1 - e^{-4t})$$

Assuming time is measured in seconds, the current after 60 seconds is

$$i(1) = \frac{3}{2}(1 - e^{-4(60)})$$

$$i(1) = \frac{3}{2}(1 - e^{-240})$$

$$i(1) = 1.5$$

Topic: Electrical series circuits

Question: In an electrical circuit, a battery supplies a constant voltage of 20 V; the circuit has an inductance of 4 h, a resistance of 3 Ω, and capacitance of 2 f. Given $q(0) = 2$ and $i(0) = 0$, find the charge $q(t)$ on the capacitor.

Answer choices:

A $\quad q(t) = 2e^{\frac{1}{2}t}(3 - e^{\frac{3}{4}t})$

B $\quad q(t) = 2e^{-\frac{1}{4}t}(2 - e^{-\frac{1}{4}t})$

C $\quad q(t) = 4e^{-\frac{1}{4}t}(1 - e^{-\frac{1}{4}t})$

D $\quad q(t) = 2e^{-\frac{1}{2}t}(1 - e^{-\frac{1}{4}t})$

Solution: B

This series circuit has a capacitor, which means we use Kirchhoff's second law.

$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=E(t)$$

Plugging in what we know, we get

$$4\frac{d^2q}{dt^2}+3\frac{dq}{dt}+\frac{1}{2}q=20$$

$$4q''+3q'+\frac{1}{2}q=20$$

$$8q''+6q'+q=40$$

The associated characteristic equation for this second order homogeneous equation is

$$8r^2+6r+1=0$$

$$(2r+1)(4r+1)=0$$

$$r=-\frac{1}{2},\ -\frac{1}{4}$$

Because we found distinct real roots, we know the circuit is overdamped. With distinct real roots, we can say that the general solution and its derivative are given by

$$q(t)=c_1e^{-\frac{1}{4}t}+c_2e^{-\frac{1}{2}t}$$

$$q'(t) = i(t) = -\frac{1}{4}c_1e^{-\frac{1}{4}t} - \frac{1}{2}c_2e^{-\frac{1}{2}t}$$

Substituting the initial conditions $q(0) = 2$ and $i(0) = 0$ into these equations, we get

$$2 = c_1e^{-\frac{1}{4}(0)} + c_2e^{-\frac{1}{2}(0)}$$

$$2 = c_1 + c_2$$

and

$$0 = -\frac{1}{4}c_1e^{-\frac{1}{4}(0)} - \frac{1}{2}c_2e^{-\frac{1}{2}(0)}$$

$$0 = -\frac{1}{4}c_1 - \frac{1}{2}c_2$$

$$0 = c_1 + 2c_2$$

$$c_1 = -2c_2$$

Substituting this value into the first equation gives

$$2 = -2c_2 + c_2$$

$$2 = -c_2$$

$$c_2 = -2$$

Plugging this back into the equation for c_1, we get

$$c_1 = -2(-2)$$

$$c_1 = 4$$

So the general solution modeling charge q on the capacitor over time t is

$$q(t) = c_1 e^{-\frac{1}{4}t} + c_2 e^{-\frac{1}{2}t}$$

$$q(t) = 4e^{-\frac{1}{4}t} - 2e^{-\frac{1}{2}t}$$

$$q(t) = 2e^{-\frac{1}{4}t}(2 - e^{-\frac{1}{4}t})$$

Topic: Spring and mass systems

Question: A mass of weight 50 pounds stretches a spring 10 feet. Find the equation that models the motion of the mass if we release the mass when $t=0$ from a position of 2 feet above equilibrium, with a downward velocity of 5 ft/s.

Answer choices:

A $\quad x(t)=-2\cos\left(\frac{4\sqrt{5}}{5}t\right)+\frac{5\sqrt{5}}{4}\sin\left(\frac{4\sqrt{5}}{5}t\right)$

B $\quad x(t)=2\cos\left(\frac{\sqrt{10}}{10}t\right)-\sin\left(\frac{\sqrt{10}}{10}t\right)$

C $\quad x(t)=\cos\left(\frac{\sqrt{10}}{10}t\right)-\sin\left(\frac{\sqrt{10}}{10}t\right)$

D $\quad x(t)=2\cos\left(\frac{4\sqrt{5}}{5}t\right)-\frac{5\sqrt{5}}{4}\sin\left(\frac{4\sqrt{5}}{5}t\right)$

Solution: A

First we'll use Hooke's Law to find the spring constant k.

$$F = ks$$

$$50 = k(10)$$

$$k = 5 \text{ lb/ft}$$

To convert weight into mass, we'll use $m = W/g = 50/32 = 25/16$ slug. Plugging everything we have into the second order equation, we get

$$\frac{25}{16}\left(\frac{d^2x}{dt^2}\right) + 5x = 0$$

$$\frac{d^2x}{dt^2} + \frac{16}{5}x = 0$$

From this equation, we see that $\omega = \sqrt{16/5} = 4/\sqrt{5}$, which means that the general solution is given by

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$x(t) = c_1 \cos\left(\frac{4}{\sqrt{5}}t\right) + c_2 \sin\left(\frac{4}{\sqrt{5}}t\right)$$

Its derivative is

$$x'(t) = -\frac{4}{\sqrt{5}}c_1 \sin\left(\frac{4}{\sqrt{5}}t\right) + \frac{4}{\sqrt{5}}c_2 \cos\left(\frac{4}{\sqrt{5}}t\right)$$

The question states that the initial position is $x(0) = -2$ and the initial velocity is $x'(0) = 5$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$-2 = c_1 \cos\left(\frac{4}{\sqrt{5}}(0)\right) + c_2 \sin\left(\frac{4}{\sqrt{5}}(0)\right)$$

$$-2 = c_1(1) + c_2(0)$$

$$c_1 = -2$$

and

$$5 = -\frac{4}{\sqrt{5}}c_1 \sin\left(\frac{4}{\sqrt{5}}(0)\right) + \frac{4}{\sqrt{5}}c_2 \cos\left(\frac{4}{\sqrt{5}}(0)\right)$$

$$5 = -\frac{4}{\sqrt{5}}c_1(0) + \frac{4}{\sqrt{5}}c_2(1)$$

$$c_2 = \frac{5\sqrt{5}}{4}$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = -2\cos\left(\frac{4}{\sqrt{5}}t\right) + \frac{5\sqrt{5}}{4}\sin\left(\frac{4}{\sqrt{5}}t\right)$$

$$x(t) = -2\cos\left(\frac{4\sqrt{5}}{5}t\right) + \frac{5\sqrt{5}}{4}\sin\left(\frac{4\sqrt{5}}{5}t\right)$$

Topic: Spring and mass systems

Question: A mass weighing 10 pounds stretches a spring 2 feet. Find the equation that models the motion of the mass if we release it when $t = 0$ from a position of 5 feet above equilibrium, with a downward velocity of 1 ft/s.

Answer choices:

A $\quad x(t) = 5\cos\left(\frac{\sqrt{2}}{2}t\right) - 2\sin\left(\frac{\sqrt{2}}{2}t\right)$

B $\quad x(t) = 5\cos\left(\frac{\sqrt{2}}{2}t\right) + 2\sin\left(\frac{\sqrt{2}}{2}t\right)$

C $\quad x(t) = -5\cos(4t) - \frac{1}{4}\sin(4t)$

D $\quad x(t) = -5\cos(4t) + \frac{1}{4}\sin(4t)$

Solution: C

First we'll use Hooke's Law to find the spring constant k.

$$F = ks$$

$$10 = k(2)$$

$$k = 5 \text{ lb/ft}$$

To convert weight into mass, we'll use $m = W/g = 10/32 = 5/16$ slug. Plugging everything we have into the second order equation, we get

$$\frac{5}{16}\left(\frac{d^2x}{dt^2}\right) + 5x = 0$$

$$\frac{d^2x}{dt^2} + 16x = 0$$

From this equation, we see that $\omega = \sqrt{16} = 4$, which means that the general solution is given by

$$x(t) = c_1\cos(\omega t) + c_2\sin(\omega t)$$

$$x(t) = c_1\cos(4t) + c_2\sin(4t)$$

Its derivative is

$$x'(t) = -4c_1\sin(4t) + 4c_2\cos(4t)$$

The question states that the initial position is $x(0) = -5$ and the initial velocity is $x'(0) = -1$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$-5 = c_1 \cos(4(0)) + c_2 \sin(4(0))$$

$$-5 = c_1(1) + c_2(0)$$

$$c_1 = -5$$

and

$$-1 = -4c_1 \sin(4(0)) + 4c_2 \cos(4(0))$$

$$-1 = -4c_1(0) + 4c_2(1)$$

$$c_2 = -\frac{1}{4}$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = -5\cos(4t) - \frac{1}{4}\sin(4t)$$

Topic: Spring and mass systems

Question: A mass weighing 16 pounds stretches a spring 8 feet, and a damping force that's triple the instantaneous velocity is acting on the spring/mass system. Find an equation that models the motion of the mass if it's initially released from 4 feet below equilibrium with an upward velocity of 6 ft/s.

Answer choices:

A $\quad x(t) = e^{3t}((3+\sqrt{5})e^{\sqrt{5}t} + (3-\sqrt{5})e^{-\sqrt{5}t})$

B $\quad x(t) = e^{-3t}((-3+\sqrt{5})e^{\sqrt{5}t} + (-3-\sqrt{5})e^{-\sqrt{5}t})$

C $\quad x(t) = e^{3t}\left(\left(1-\frac{\sqrt{5}}{2}\right)e^{\sqrt{5}t} + \left(-5-\frac{\sqrt{5}}{2}\right)e^{-\sqrt{5}t}\right)$

D $\quad x(t) = e^{-3t}\left(\left(2+\frac{9\sqrt{5}}{5}\right)e^{\sqrt{5}t} + \left(2-\frac{9\sqrt{5}}{5}\right)e^{-\sqrt{5}t}\right)$

Solution: D

Hooke's Law tells us that the spring constant is

$$F = ks$$

$$16 = k(8)$$

$$k = 2 \text{ lb/ft}$$

Now we'll use $W = mg$ to convert the weight into mass.

$$W = mg$$

$$16 = m(32)$$

$$m = \frac{1}{2} \text{ slug}$$

With the spring constant and the mass, and $\beta = 3$ since the damping force is triple the instantaneous velocity, we can plug everything into the differential equation.

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{3}{\frac{1}{2}}\left(\frac{dx}{dt}\right) + \frac{2}{\frac{1}{2}}x = 0$$

$$\frac{d^2x}{dt^2} + 6\left(\frac{dx}{dt}\right) + 4x = 0$$

The associated characteristic equation and its roots are

$$r^2 + 6r + 4 = 0$$

$$r = -3 \pm \sqrt{5}$$

Because we find roots where $\lambda^2 - \omega^2 > 0$, the spring and mass system is overdamped and the general solution is

$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega^2}t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}t})$$

$$x(t) = e^{-3t}(c_1 e^{\sqrt{5}t} + c_2 e^{-\sqrt{5}t})$$

Its derivative is

$$x'(t) = -3e^{-3t}(c_1 e^{\sqrt{5}t} + c_2 e^{-\sqrt{5}t}) + e^{-3t}(\sqrt{5}c_1 e^{\sqrt{5}t} - \sqrt{5}c_2 e^{-\sqrt{5}t})$$

$$x'(t) = e^{-3t}((-3 + \sqrt{5})c_1 e^{\sqrt{5}t} - (3 + \sqrt{5})c_2 e^{-\sqrt{5}t})$$

The question states that the initial position is $x(0) = 4$ and the initial velocity is $x'(0) = 6$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$4 = e^{-3(0)}(c_1 e^{\sqrt{5}(0)} + c_2 e^{-\sqrt{5}(0)})$$

$$4 = c_1 + c_2$$

$$c_1 = 4 - c_2$$

and

$$6 = e^{-3(0)}((-3 + \sqrt{5})c_1 e^{\sqrt{5}(0)} - (3 + \sqrt{5})c_2 e^{-\sqrt{5}(0)})$$

$$6 = (-3 + \sqrt{5})c_1 - (3 + \sqrt{5})c_2$$

$$6 = (-3 + \sqrt{5})(4 - c_2) - (3 + \sqrt{5})c_2$$

$$6 = 4(-3+\sqrt{5}) - (-3+\sqrt{5})c_2 - (3+\sqrt{5})c_2$$

$$6 = -12 + 4\sqrt{5} + (3 - \sqrt{5} - 3 - \sqrt{5})c_2$$

$$18 - 4\sqrt{5} = -2\sqrt{5}c_2$$

$$c_2 = 2 - \frac{9\sqrt{5}}{5}$$

Then the value of c_1 is

$$c_1 = 4 - c_2$$

$$c_1 = 4 - \left(2 - \frac{9\sqrt{5}}{5}\right)$$

$$c_1 = 2 + \frac{9\sqrt{5}}{5}$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = e^{-3t}\left(\left(2 + \frac{9\sqrt{5}}{5}\right)e^{\sqrt{5}t} + \left(2 - \frac{9\sqrt{5}}{5}\right)e^{-\sqrt{5}t}\right)$$

Topic: Power series basics

Question: Find the radius of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^2+n}$$

Answer choices:

A $\quad R = -\frac{1}{3}$

B $\quad R = \frac{1}{3}$

C $\quad R = -\frac{2}{3}$

D $\quad R = \frac{2}{3}$

Solution: B

First, let's rewrite the series.

$$\sum_{n=1}^{\infty} \frac{(3x-1)^n}{n^2+n}$$

Then applying the ratio test to the power series gives

$$L = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{\frac{(3x-1)^{n+1}}{(n+1)^2+n+1}}{\frac{(3x-1)^n}{n^2+n}} \right|$$

$$L = \lim_{n \to \infty} \left| \frac{(3x-1)^{n+1}}{(n+1)^2+n+1} \left(\frac{n^2+n}{(3x-1)^n} \right) \right|$$

$$L = \lim_{n \to \infty} \left| (3x-1) \left(\frac{n^2+n}{(n+1)^2+n+1} \right) \right|$$

$$L = \lim_{n \to \infty} \left| (3x-1) \left(\frac{n(n+1)}{(n+1)[(n+1)+1]} \right) \right|$$

$$L = \lim_{n \to \infty} \left| (3x-1) \left(\frac{n}{n+2} \right) \right|$$

$$L = |3x - 1| \lim_{n \to \infty} \left| \frac{1}{1 + \frac{2}{n}} \right|$$

$$L = |3x - 1| \left| \frac{1}{1 + 0} \right|$$

$$L = |3x - 1|$$

To determine where the series converges absolutely, we set this value $L < 1$.

$$|3x - 1| < 1$$

$$3 \left| x - \frac{1}{3} \right| < 1$$

$$\left| x - \frac{1}{3} \right| < \frac{1}{3}$$

$$-\frac{1}{3} < x - \frac{1}{3} < \frac{1}{3}$$

$$0 < x < \frac{2}{3}$$

So the interval of convergence is $(0,2/3)$, and the radius of convergence is $R = 1/3$.

Topic: Power series basics

Question: Find the Maclaurin series representation of the function.

$$f(x) = \frac{3}{1 + 2x^2}$$

Answer choices:

A $\sum_{n=0}^{\infty} (2x^2)^n$

B $\sum_{n=0}^{\infty} 3(2x^2)^n$

C $\sum_{n=0}^{\infty} (-1)^n(2x^2)^n$

D $\sum_{n=0}^{\infty} 3(-1)^n(2x^2)^n$

Solution: D

Because we already have the Maclaurin series representation of

$$f(x) = \frac{1}{1-x}$$

we can replace x with $-2x^2$.

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\frac{1}{1-(-2x^2)} = \sum_{n=0}^{\infty} (-2x^2)^n$$

$$\frac{1}{1+2x^2} = \sum_{n=0}^{\infty} (-1)^n(2x^2)^n$$

Then multiply the series by 3.

$$\frac{3}{1+2x^2} = \sum_{n=0}^{\infty} 3(-1)^n(2x^2)^n$$

Topic: Power series basics

Question: Find the first and second derivatives of the power series.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-3)^n$$

Answer choices:

A $\quad f'(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(x-3)^{n-1}$ and $f''(x) = \sum_{n=2}^{\infty} \frac{1}{(n-2)!}(x-3)^{n-2}$

B $\quad f'(x) = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}(x-3)^{n-1}$ and $f''(x) = \sum_{n=0}^{\infty} \frac{1}{(n-2)!}(x-3)^{n-2}$

C $\quad f'(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)}(x-3)^{n-1}$ and $f''(x) = \sum_{n=2}^{\infty} \frac{1}{(n-2)}(x-3)^{n-2}$

D $\quad f'(x) = \sum_{n=1}^{\infty} \frac{1}{(n+1)!}(x-3)^{n-1}$ and $f''(x) = \sum_{n=2}^{\infty} \frac{1}{(n+2)!}(x-3)^{n-2}$

Solution: A

To differentiate the power series representation, we can apply power rule for derivatives.

$$f(x) = \sum_{n=0}^{\infty} \frac{1}{n!}(x-3)^n$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{n}{n!}(x-3)^{n-1}$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{n}{n(n-1)(n-2)(n-3)\dots}(x-3)^{n-1}$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{1}{(n-1)(n-2)(n-3)\dots}(x-3)^{n-1}$$

$$f'(x) = \sum_{n=0}^{\infty} \frac{1}{(n-1)!}(x-3)^{n-1}$$

Alternatively, we could have expanded the original power series and differentiated term-by-term,

$$f(x) = 1 + (x-3) + \frac{1}{2!}(x-3)^2 + \frac{1}{3!}(x-3)^3 + \dots$$

$$f'(x) = 0 + 1 + \frac{1}{2!} \cdot 2(x-3) + \frac{1}{3!} \cdot 3(x-3)^2 + \dots$$

$$f'(x) = 1 + \frac{1}{1!}(x-3) + \frac{1}{2!}(x-3)^2 + \dots$$

and then we can see from the pattern of this expanded series that the power series representation of the derivative is

$$f'(x) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!}(x-3)^{n-1}$$

To find the second derivative, we can apply power rule again.

$$f''(x) = \sum_{n=0}^{\infty} \frac{n-1}{(n-1)!}(x-3)^{n-2}$$

$$f''(x) = \sum_{n=0}^{\infty} \frac{n-1}{(n-1)(n-2)(n-3)(n-4)\dots}(x-3)^{n-2}$$

$$f''(x) = \sum_{n=0}^{\infty} \frac{1}{(n-2)(n-3)(n-4)\dots}(x-3)^{n-2}$$

$$f''(x) = \sum_{n=0}^{\infty} \frac{1}{(n-2)!}(x-3)^{n-2}$$

Alternatively, we could have expanded the power series representation of the first derivative and differentiated term-by-term,

$$f'(x) = 1 + \frac{1}{1!}(x-3) + \frac{1}{2!}(x-3)^2 + \frac{1}{3!}(x-3)^3 + \dots$$

$$f''(x) = 0 + 1 + \frac{1}{2!} \cdot 2(x-3) + \frac{1}{3!} \cdot 3(x-3)^2 + \dots$$

$$f''(x) = 1 + \frac{1}{1!}(x-3) + \frac{1}{2!}(x-3)^2 + \dots$$

and then we can see from the pattern of this expanded series that the power series representation of the second derivative is

$$f(x) = \sum_{n=2}^{\infty} \frac{1}{(n-2)!}(x-3)^{n-2}$$

Topic: Adding power series

Question: Are the series in phase?

$$\sum_{n=1}^{\infty} nc_n x^{n-1}$$

$$3\sum_{n=0}^{\infty} c_n x^{n+2}$$

Answer choices:

A Yes, because the series begin at the same power of x

B Yes, because the series have matching indices

C No, because the series don't begin at the same power of x

D No, because the series don't have matching indices

Solution: C

Two series are in phase when they begin at the same power of x. The index of the first series

$$\sum_{n=1}^{\infty} nc_n x^{n-1}$$

is $n = 1$, which means the first term of the series will be

$$1c_1x^{1-1}$$

$$c_1x^0$$

The index of the second series

$$3\sum_{n=0}^{\infty} c_n x^{n+2}$$

is $n = 0$, which means the first term of the series will be

$$3c_0x^{0+2}$$

$$3c_0x^2$$

The first term of the first series is for x^0, while the first term of the second series is for x^2. Because each series begins at a different power of x, the series are not in phase.

Topic: Adding power series

Question: Find the difference of the power series.

$$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

Answer choices:

A $\sum_{k=0}^{\infty} ((k+1)c_{k+1} + c_k)x^k$

B $\sum_{k=0}^{\infty} ((k+1)c_{k+1} - c_k)x^k$

C $\sum_{k=0}^{\infty} (kc_k + c_{k+1})x^k$

D $\sum_{k=0}^{\infty} (kc_k - c_{k+1})x^k$

Solution: B

Both series begin with the x^0 term, so they're already in phase. To make their indices match, we'll make the substitution $k = n - 1$ and $n = k + 1$ into the first series, and make the substitution $k = n$ into the second series.

$$\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$\sum_{k=0}^{\infty} (k+1)c_{k+1}x^k - \sum_{k=0}^{\infty} c_k x^k$$

The series are now in phase with matching indices, so we'll add them and then factor out x^k.

$$\sum_{k=0}^{\infty} (k+1)c_{k+1}x^k - c_k x^k$$

$$\sum_{k=0}^{\infty} ((k+1)c_{k+1} - c_k)x^k$$

Topic: Adding power series

Question: Find the sum of the power series.

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - 2\sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n$$

Answer choices:

A $\quad \sum_{k=1}^{\infty} ((k+2)(k+1)c_{k+2} - 2kc_k + c_k)x^k$

B $\quad \sum_{k=1}^{\infty} ((k+2)c_k - 2kc_k + c_k)x^k$

C $\quad c_0 + 2c_2 + \sum_{k=1}^{\infty} ((k+2)(k+1)c_{k+2} - 2kc_k + c_k)x^k$

D $\quad c_0 + 2c_2 + \sum_{k=1}^{\infty} ((k+2)c_k - 2kc_k + c_k)x^k$

Solution: C

The first and third series begin with the x^0 term, but the second series begins with the x^1 term, so we'll pull out the x^0 term from both the first and the third series.

$$2(2-1)c_2x^{2-2} + \sum_{n=3}^{\infty} n(n-1)c_nx^{n-2} - 2\sum_{n=1}^{\infty} nc_nx^n + c_0x^0 + \sum_{n=1}^{\infty} c_nx^n$$

$$c_0 + 2c_2 + \sum_{n=3}^{\infty} n(n-1)c_nx^{n-2} - 2\sum_{n=1}^{\infty} nc_nx^n + \sum_{n=1}^{\infty} c_nx^n$$

Each series now begins with the x^1 term, so they're in phase. To make their indices match, we'll make the substitution $k = n - 2$ and $n = k + 2$ into the first series, and make the substitution $k = n$ into the second and third series.

$$c_0 + 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+2-1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_kx^k + \sum_{k=1}^{\infty} c_kx^k$$

$$c_0 + 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k - 2\sum_{k=1}^{\infty} kc_kx^k + \sum_{k=1}^{\infty} c_kx^k$$

The series are now in phase and the indices match, so we'll add them and then factor out x^k.

$$c_0 + 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1)c_{k+2}x^k - 2kc_kx^k + c_kx^k$$

$$c_0 + 2c_2 + \sum_{k=1}^{\infty} ((k+2)(k+1)c_{k+2} - 2kc_k + c_k)x^k$$

Topic: Power series solutions

Question: Find a power series solution to the differential equation.

$$y' - y = 0$$

Answer choices:

A $y = e^x$

B $y = c_0 e^x$

C $y = c_0 x e^x$

D $y = c_0 x e^{x^2}$

Solution: B

We'll use the standard form for the power series solution and its derivative,

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$$

to substitute into the differential equation.

$$y' - y = 0$$

$$\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

The first series will start with the $x^{1-1} = x^0$ term, while the second series will start with the x^0 term, so the series are already in phase. To make the indices match, we'll substitute $k = n - 1$ and $n = k + 1$ into the first series (the new index will be $k = 1 - 1 = 0$), and $k = n$ into the second series (the new index will be $k = 0$).

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

Now that the series are in phase with matching indices, we can combine them and then factor out x^k.

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)x^k - c_k x^k = 0$$

$$\sum_{k=0}^{\infty} (c_{k+1}(k+1) - c_k)x^k = 0$$

Equating coefficients and then solving for the coefficient with the largest subscript, c_{k+1}, gives the recurrence relation.

$$c_{k+1}(k+1) - c_k = 0$$

$$c_{k+1} = \frac{c_k}{k+1}$$

Plugging in $k = 0, 1, 2, 3, \ldots,$ we start to see a pattern.

$$k = 0 \qquad c_{0+1} = \frac{c_0}{0+1} \qquad c_1 = \frac{c_0}{1!}$$

$$k = 1 \qquad c_{1+1} = \frac{c_1}{1+1} \qquad c_2 = \frac{c_0}{2!}$$

$$k = 2 \qquad c_{2+1} = \frac{c_2}{2+1} \qquad c_3 = \frac{c_0}{3!}$$

$$k = 3 \qquad c_{3+1} = \frac{c_3}{3+1} \qquad c_4 = \frac{c_0}{4!}$$

Plugging these coefficients into the expansion of the power series solution, we get

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \ldots$$

$$y = c_0 + \frac{c_0}{1!}x + \frac{c_0}{2!}x^2 + \frac{c_0}{3!}x^3 + \frac{c_0}{4!}x^4 + \ldots$$

$$y = c_0\left(\frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots\right)$$

We can rewrite the series inside the parentheses.

$$y = c_0 \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

We recognize this series as the common Maclaurin series representation of e^x,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

so the general solution to the differential equation is

$$y = c_0 e^x$$

Topic: Power series solutions

Question: Find a power series solution to the differential equation.

$$y' = x^2y$$

Answer choices:

A $\quad y(x) = c_0e^x$

B $\quad y(x) = c_0e^{\frac{x^2}{2}}$

C $\quad y(x) = c_0e^{\frac{x^3}{3}}$

D $\quad y(x) = c_0e^{\frac{x^4}{4}}$

Solution: C

We'll use the standard form for the power series solution and its derivative,

$$y=\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+c_3x^3+\dots$$

$$y'=\sum_{n=1}^{\infty}c_nnx^{n-1}=c_1+2c_2x+3c_3x^2+4c_4x^3+\dots$$

to substitute into the differential equation.

$$y'=x^2y$$

$$y'-x^2y=0$$

$$\sum_{n=1}^{\infty}c_nnx^{n-1}-x^2\sum_{n=0}^{\infty}c_nx^n=0$$

$$\sum_{n=1}^{\infty}c_nnx^{n-1}-\sum_{n=0}^{\infty}c_nx^{n+2}=0$$

The first series will start with the $x^{1-1}=x^0$ term, while the second series will start with the $x^{0+2}=x^2$ term, so we'll pull out the x^0 and x^1 terms from the first series in order to put the series in phase.

$$c_1(1)x^{1-1}+c_2(2)x^{2-1}+\sum_{n=3}^{\infty}c_nnx^{n-1}-\sum_{n=0}^{\infty}c_nx^{n+2}=0$$

$$c_1x^0+2c_2x^1+\sum_{n=3}^{\infty}c_nnx^{n-1}-\sum_{n=0}^{\infty}c_nx^{n+2}=0$$

$$c_1 + 2c_2x + \sum_{n=3}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+2} = 0$$

Now the series are in phase (they both begin with the x^2 term), so we just need to make their indices match. We'll substitute $k = n - 1$ and $n = k + 1$ into the first series (the new index will be $k = 3 - 1 = 2$), and $k = n + 2$ and $n = k - 2$ into the second series (the new index will be $k = 2$).

$$c_1 + 2c_2x + \sum_{k=2}^{\infty} c_{k+1}(k+1)x^k - \sum_{k=2}^{\infty} c_{k-2}x^k = 0$$

Now that the series are in phase with matching indices, we can combine them and then factor out x^k.

$$c_1 + 2c_2x + \sum_{k=2}^{\infty} c_{k+1}(k+1)x^k - c_{k-2}x^k = 0$$

$$c_1 + 2c_2x + \sum_{k=2}^{\infty} (c_{k+1}(k+1) - c_{k-2})x^k = 0$$

Equating coefficients, we get

$c_1 = 0$ $\qquad\qquad k = 0$

$2c_2 = 0$ $\qquad\qquad k = 1$

$c_{k+1}(k+1) - c_{k-2} = 0$ $\qquad\qquad k = 2, 3, 4, 5, \ldots$

Solving the recurrence relation for the coefficient with the largest subscript, c_{k+1}, gives

$c_1 = 0$ $\qquad\qquad k = 0$

$$c_2 = 0 \qquad k = 1$$

$$c_{k+1} = \frac{c_{k-2}}{k+1} \qquad k = 2, 3, 4, 5, \ldots$$

Plugging in $k = 2, 3, 4, 5, \ldots$, we start to see a pattern.

		$k = 0$	$c_1 = 0$	$k = 1$	$c_2 = 0$
$k = 2$	$c_3 = \frac{c_0}{3}$	$k = 3$	$c_4 = 0$	$k = 4$	$c_5 = 0$
$k = 5$	$c_6 = \frac{c_0}{3 \cdot 6}$	$k = 6$	$c_7 = 0$	$k = 7$	$c_8 = 0$
$k = 8$	$c_9 = \frac{c_0}{3 \cdot 6 \cdot 9}$	$k = 9$	$c_{10} = 0$	$k = 10$	$c_{11} = 0$
$k = 11$	$c_{12} = \frac{c_0}{3 \cdot 6 \cdot 9 \cdot 12}$	...		...	
	...				

Plugging these coefficients into the expansion of the power series solution, we get

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \ldots$$

$$y = c_0 + (0)x + (0)x^2 + \frac{c_0}{3}x^3 + (0)x^4 + (0)x^5 + \frac{c_0}{3 \cdot 6}x^6$$

$$+(0)x^7 + (0)x^8 + \frac{c_0}{3 \cdot 6 \cdot 9}x^9 + \ldots$$

$$y = c_0 + \frac{c_0}{3}x^3 + \frac{c_0}{3 \cdot 6}x^6 + \frac{c_0}{3 \cdot 6 \cdot 9}x^9 + \frac{c_0}{3 \cdot 6 \cdot 9 \cdot 12}x^{12} + \ldots$$

$$y = c_0 \left(1 + \frac{1}{3}x^3 + \frac{1}{3 \cdot 6}x^6 + \frac{1}{3 \cdot 6 \cdot 9}x^9 + \frac{1}{3 \cdot 6 \cdot 9 \cdot 12}x^{12} + \dots \right)$$

$$y = c_0 \left(1 + \frac{1}{3^1}x^3 + \frac{1}{3^2(1 \cdot 2)}x^6 + \frac{1}{3^3(1 \cdot 2 \cdot 3)}x^9 + \frac{1}{3^4(1 \cdot 2 \cdot 3 \cdot 4)}x^{12} + \dots \right)$$

$$y = c_0 \left(\frac{1}{3^0(0!)}x^0 + \frac{1}{3^1(1!)}x^{3(1)} + \frac{1}{3^2(2!)}x^{3(2)} + \frac{1}{3^3(3!)}x^{3(3)} + \frac{1}{3^4(4!)}x^{3(4)} + \dots \right)$$

We can rewrite the series inside the parentheses.

$$y = c_0 \sum_{n=0}^{\infty} \frac{1}{3^n n!}x^{3n}$$

We could leave the series solution this way, but we can also rewrite the series as

$$y = c_0 \sum_{n=0}^{\infty} \frac{(x^3)^n}{3^n n!}$$

$$y = c_0 \sum_{n=0}^{\infty} \left(\frac{x^3}{3} \right)^n \frac{1}{n!}$$

$$y = c_0 \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3} \right)^n}{n!}$$

We recognize this series as the common Maclaurin series representation of e^x, except with $x^3/3$ instead of just x.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{\frac{x^3}{3}} = \sum_{n=0}^{\infty} \frac{\left(\frac{x^3}{3}\right)^n}{n!}$$

So the general solution to the differential equation is

$$y = c_0 e^{\frac{x^3}{3}}$$

Topic: Power series solutions

Question: Find a power series solution to the differential equation.

$$y'' + xy' + y = 0$$

Answer choices:

A $\quad y(x) = c_0 \sum_{n=0}^{\infty} \frac{(1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(2)^n n!}{(2n+1)!} x^{2n+1}$

B $\quad y(x) = c_0 \sum_{n=0}^{\infty} \frac{(1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=1}^{\infty} \frac{(2)^n n!}{(2n+1)!} x^{2n+1}$

C $\quad y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=1}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$

D $\quad y(x) = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$

Solution: D

We'll use the standard form for the power series solution and its derivatives,

$$y=\sum_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+c_3x^3+\dots$$

$$y'=\sum_{n=1}^{\infty}c_nnx^{n-1}=c_1+2c_2x+3c_3x^2+4c_4x^3+\dots$$

$$y''=\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}=2c_2+6c_3x+12c_4x^2+\dots$$

to substitute into the differential equation.

$$y''+xy'+y=0$$

$$\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}+x\sum_{n=1}^{\infty}c_nnx^{n-1}+\sum_{n=0}^{\infty}c_nx^n=0$$

$$\sum_{n=2}^{\infty}c_nn(n-1)x^{n-2}+\sum_{n=1}^{\infty}c_nnx^n+\sum_{n=0}^{\infty}c_nx^n=0$$

The first series will start with the $x^{2-2}=x^0$ term, the second series will start with the x^1 term, and the third series will start with the x^0 term. So we'll pull out the x^0 term from both the first and the third series in order to put the series in phase.

$$c_2(2)(2-1)x^{2-2}+\sum_{n=3}^{\infty}c_nn(n-1)x^{n-2}+\sum_{n=1}^{\infty}c_nnx^n+c_0x^0+\sum_{n=1}^{\infty}c_nx^n=0$$

$$c_0 + 2c_2 + \sum_{n=3}^{\infty} c_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=1}^{\infty} c_n x^n = 0$$

Now the series are in phase (they all begin with the x^1 term), so we just need to make their indices match. We'll substitute $k = n - 2$ and $n = k + 2$ into the first series (the new index will be $k = 3 - 2 = 1$), and we'll substitute $k = n$ into the second and third series (their new indices will be $k = 1$).

$$c_0 + 2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_k k x^k + \sum_{k=1}^{\infty} c_k x^k = 0$$

Now that the series are in phase with matching indices, we can combine them and then factor out x^k.

$$c_0 + 2c_2 + \sum_{k=1}^{\infty} c_{k+2}(k+2)(k+1)x^k + c_k k x^k + c_k x^k = 0$$

$$c_0 + 2c_2 + \sum_{k=1}^{\infty} (c_{k+2}(k+2)(k+1) + c_k k + c_k)x^k = 0$$

Equating coefficients, we get

$$c_0 + 2c_2 = 0 \qquad k = 0$$

$$c_{k+2}(k+2)(k+1) + c_k k + c_k = 0 \qquad k = 1, 2, 3, 4, \ldots$$

Solving the recurrence relation for the coefficient with the largest subscript, c_{k+2}, gives

$$c_0 + 2c_2 = 0 \qquad k = 0$$

$$c_{k+2} = -\frac{c_k}{k+2} \qquad k = 1,\ 2,\ 3,\ 4,\ \dots$$

Plugging in $k = 1,\ 2,\ 3,\ 4,\ \dots,$ we start to see a pattern.

$$k = 0 \qquad c_2 = -\frac{c_0}{2} \qquad k = 1 \qquad c_3 = -\frac{c_1}{3}$$

$$k = 2 \qquad c_4 = \frac{c_0}{2 \cdot 4} \qquad k = 3 \qquad c_5 = \frac{c_1}{3 \cdot 5}$$

$$k = 4 \qquad c_6 = -\frac{c_0}{2 \cdot 4 \cdot 6} \qquad k = 5 \qquad c_7 = -\frac{c_1}{3 \cdot 5 \cdot 7}$$

$$\dots \qquad \dots$$

Plugging these coefficients into the expansion of the power series solution, we get

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots$$

$$y = c_0 + c_1x - \frac{c_0}{2}x^2 - \frac{c_1}{3}x^3 + \frac{c_0}{2 \cdot 4}x^4 + \frac{c_1}{3 \cdot 5}x^5 - \frac{c_0}{2 \cdot 4 \cdot 6}x^6 - \frac{c_1}{3 \cdot 5 \cdot 7}x^7 + \dots$$

$$y = \left(c_0 - \frac{c_0}{2}x^2 + \frac{c_0}{2 \cdot 4}x^4 - \frac{c_0}{2 \cdot 4 \cdot 6}x^6 + \dots\right)$$

$$+\left(c_1x - \frac{c_1}{3}x^3 + \frac{c_1}{3 \cdot 5}x^5 - \frac{c_1}{3 \cdot 5 \cdot 7}x^7 + \dots\right)$$

$$y = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{2 \cdot 4}x^4 - \frac{1}{2 \cdot 4 \cdot 6}x^6 + \dots\right)$$

$$+c_1\left(x - \frac{1}{3}x^3 + \frac{1}{3 \cdot 5}x^5 - \frac{1}{3 \cdot 5 \cdot 7}x^7 + \dots\right)$$

We could leave the solution this way, or we could try to rewrite each series in order to simplify the general solution.

$$y=c_0\left(\frac{1}{2^0(0!)}x^{2(0)}-\frac{1}{2^1(1!)}x^{2(1)}+\frac{1}{2^2(2!)}x^{2(2)}-\frac{1}{2^3(3!)}x^{2(3)}+\dots\right)$$

$$+c_1\left(x^{2(0)+1}-\frac{1}{3}x^{2(1)+1}+\frac{1}{3\cdot 5}x^{2(2)+1}-\frac{1}{3\cdot 5\cdot 7}x^{2(3)+1}+\dots\right)$$

$$y=c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n}$$

$$+c_1\left(x^{2(0)+1}-\frac{1}{3}x^{2(1)+1}+\frac{1}{3\cdot 5}x^{2(2)+1}-\frac{1}{3\cdot 5\cdot 7}x^{2(3)+1}+\dots\right)$$

For the c_1 series, we want to try representing the denominator with factorial notation, but the denominators don't include even numbers. So we'll have to add those into the numerator to cancel them out when we switch to factorial notation.

$$y=c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n}$$

$$+c_1\left(x^{2(0)+1}-\frac{1(2)}{(2(1)+1)!}x^{2(1)+1}+\frac{1(2\cdot 4)}{(2(2)+1)!}x^{2(2)+1}-\frac{1(2\cdot 4\cdot 6)}{(2(3)+1)!}x^{2(3)+1}+\dots\right)$$

$$y=c_0\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n n!}x^{2n}$$

$$+c_1\left(x^{2(0)+1}-\frac{1(2^1)}{(2(1)+1)!}x^{2(1)+1}+\frac{1(2^2)(1\cdot 2)}{(2(2)+1)!}x^{2(2)+1}-\frac{1(2^3)(1\cdot 2\cdot 3)}{(2(3)+1)!}x^{2(3)+1}+\dots\right)$$

$$y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n}$$

$$+c_1 \left(\frac{2^0(0!)}{(2(0)+1)!} x^{2(0)+1} - \frac{2^1(1!)}{(2(1)+1)!} x^{2(1)+1} + \frac{2^2(2!)}{(2(2)+1)!} x^{2(2)+1} - \frac{2^3(3!)}{(2(3)+1)!} x^{2(3)+1} + \dots \right)$$

So the general solution to the differential equation is

$$y = c_0 \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n n!} x^{2n} + c_1 \sum_{n=0}^{\infty} \frac{(-2)^n n!}{(2n+1)!} x^{2n+1}$$

Topic: Nonpolynomial coefficients

Question: Find the first three terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y'' - (\cos x)y = 0$$

Answer choices:

A $\quad y = c_0\left(1 + \frac{1}{2}x^2 - \frac{1}{144}x^6 + \ldots\right) + c_1\left(x + \frac{1}{6}x^3 - \frac{1}{60}x^5 + \ldots\right)$

B $\quad y = c_0\left(1 + \frac{1}{6}x^2 + \frac{1}{24}x^6 + \ldots\right) + c_1\left(x + \frac{1}{2}x^3 + \frac{1}{6}x^5 + \ldots\right)$

C $\quad y = c_0\left(1 + \frac{1}{2}x^2 + \ldots\right) + c_1\left(x + \frac{1}{6}x^3 + \ldots\right)$

D $\quad y = c_0\left(1 + \frac{1}{6}x^2 + \ldots\right) + c_1\left(x + \frac{1}{2}x^3 + \ldots\right)$

Solution: A

We know $x_0 = 0$ is an ordinary point, because $\cos x$ is analytic there. And we know that the Maclaurin series representation of $\cos x$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots$$

So we can substitute into the differential equation to get

$$y'' - (\cos x)y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right) \sum_{n=0}^{\infty} c_n x^n = 0$$

Now we'll just expand each series through its first few terms,

$$(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots)$$

$$-\left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right)(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots) = 0$$

$$(2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots)$$

$$-(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots)$$

$$+\frac{1}{2!}x^2(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots)$$

$$-\frac{1}{4!}x^4(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots)$$

$$+\frac{1}{6!}x^6(c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + \dots) + \dots = 0$$

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$-(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots)$$

$$+\left(\frac{1}{2!}c_0x^2 + \frac{1}{2!}c_1x^3 + \frac{1}{2!}c_2x^4 + \frac{1}{2!}c_3x^5 + \frac{1}{2!}c_4x^6 + \dots\right)$$

$$-\left(\frac{1}{4!}c_0x^4 + \frac{1}{4!}c_1x^5 + \frac{1}{4!}c_2x^6 + \frac{1}{4!}c_3x^7 + \frac{1}{4!}c_4x^8 + \dots\right)$$

$$+\left(\frac{1}{6!}c_0x^6 + \frac{1}{6!}c_1x^7 + \frac{1}{6!}c_2x^8 + \frac{1}{6!}c_3x^9 + \frac{1}{6!}c_4x^{10} + \dots\right) + \dots = 0$$

and then collect equivalent powers of x.

$$(2c_2 - c_0) + (6c_3 - c_1)x + \left(12c_4 - c_2 + \frac{1}{2!}c_0\right)x^2$$

$$+\left(20c_5 - c_3 + \frac{1}{2!}c_1\right)x^3 + \left(30c_6 - c_4 + \frac{1}{2!}c_2 - \frac{1}{4!}c_0\right)x^4 + \dots$$

From this result, we get a system of equations.

$$2c_2 - c_0 = 0$$

$$6c_3 - c_1 = 0$$

$$12c_4 - c_2 + \frac{1}{2!}c_0 = 0$$

$$20c_5 - c_3 + \frac{1}{2!}c_1 = 0$$

$$30c_6 - c_4 + \frac{1}{2!}c_2 - \frac{1}{4!}c_0 = 0$$

which simplifies to

$$c_2 = \frac{1}{2}c_0$$

$$c_3 = \frac{1}{6}c_1$$

$$c_4 = 0$$

$$c_5 = -\frac{1}{60}c_1$$

$$c_6 = -\frac{1}{144}c_0$$

Pull these coefficients together into the power series solution.

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + \ldots$$

$$y = c_0 + c_1x + \frac{1}{2}c_0x^2 + \frac{1}{6}c_1x^3 + 0x^4 - \frac{1}{60}c_1x^5 + \ldots$$

$$y = c_0\left(1 + \frac{1}{2}x^2 - \frac{1}{144}x^6 + \ldots\right) + c_1\left(x + \frac{1}{6}x^3 - \frac{1}{60}x^5 + \ldots\right)$$

Topic: Nonpolynomial coefficients

Question: Find the first four terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y'' + e^x y' - y = 0$$

Answer choices:

A $\quad y = c_0\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right) + c_1\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \dots\right)$

B $\quad y = c_0\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \dots\right) + c_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right)$

C $\quad y = c_0\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right) - c_1\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots\right)$

D $\quad y = c_0\left(1 - \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots\right) - c_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right)$

Solution: B

We know $x_0 = 0$ is an ordinary point, because e^x is analytic there. And we know that the Maclaurin series representation of e^x is

$$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots$$

So we can substitute into the differential equation to get

$$y'' + e^x y' - y = 0$$

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right) \sum_{n=1}^{\infty} c_n n x^{n-1}$$

$$-\sum_{n=0}^{\infty} c_n x^n = 0$$

Now we'll just expand each series through its first few terms,

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$+\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots\right)(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$-(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots) = 0$$

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$+(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+x(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+\frac{1}{2}x^2(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+\frac{1}{6}x^3(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+\frac{1}{24}x^4(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$-(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots) = 0$$

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$+(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+(c_1x + 2c_2x^2 + 3c_3x^3 + 4c_4x^4 + 5c_5x^5 + \dots)$$

$$+\left(\frac{1}{2}c_1x^2 + c_2x^3 + \frac{3}{2}c_3x^4 + 2c_4x^5 + \frac{5}{2}c_5x^6 + \dots\right)$$

$$+\left(\frac{1}{6}c_1x^3 + \frac{1}{3}c_2x^4 + \frac{1}{2}c_3x^5 + \frac{2}{3}c_4x^6 + \frac{5}{6}c_5x^7 + \dots\right)$$

$$+\left(\frac{1}{24}c_1x^4 + \frac{1}{12}c_2x^5 + \frac{1}{8}c_3x^6 + \frac{1}{6}c_4x^7 + \frac{5}{24}c_5x^8 + \dots\right)$$

$$-(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots) = 0$$

and then collect equivalent powers of x.

$$(2c_2 + c_1 - c_0) + (6c_3 + 2c_2)x + \left(12c_4 + 3c_3 + c_2 + \frac{1}{2}c_1\right)x^2$$

$$+\left(20c_5+4c_4+2c_3+c_2+\frac{1}{6}c_1\right)x^3$$

$$+\left(30c_6+5c_5+3c_4+\frac{3}{2}c_3+\frac{1}{3}c_2+\frac{1}{24}c_1\right)x^4+\dots$$

From this result, we get a system of equations.

$$2c_2+c_1-c_0=0$$

$$6c_3+2c_2=0$$

$$12c_4+3c_3+c_2+\frac{1}{2}c_1=0$$

$$20c_5+4c_4+2c_3+c_2+\frac{1}{6}c_1=0$$

$$30c_6+5c_5+3c_4+\frac{3}{2}c_3+\frac{1}{3}c_2+\frac{1}{24}c_1=0$$

which simplifies to

$$c_2=-\frac{1}{2}c_1+\frac{1}{2}c_0$$

$$c_3=\frac{1}{6}c_1-\frac{1}{6}c_0$$

$$c_4=-\frac{1}{24}c_1$$

$$c_5=\frac{1}{120}c_1-\frac{1}{120}c_0$$

$$c_6=-\frac{1}{720}c_1+\frac{1}{240}c_0$$

Pull these coefficients together into the power series solution.

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+\dots$$

$$y = c_0 + c_1 x + \left(-\frac{1}{2}c_1 + \frac{1}{2}c_0\right)x^2 + \left(\frac{1}{6}c_1 - \frac{1}{6}c_0\right)x^3 + \left(-\frac{1}{24}c_1\right)x^4$$

$$+\left(\frac{1}{120}c_1 - \frac{1}{120}c_0\right)x^5 + \left(-\frac{1}{720}c_1 + \frac{1}{240}c_0\right)x^6 + \dots$$

$$y = c_0\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \frac{1}{240}x^6 - \dots\right)$$

$$+c_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{120}x^5 - \frac{1}{720}x^6 + \dots\right)$$

Simplify to just the first four terms.

$$y = c_0\left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{120}x^5 + \dots\right) + c_1\left(x - \frac{1}{2}x^2 + \frac{1}{6}x^3 - \frac{1}{24}x^4 + \dots\right)$$

Topic: Nonpolynomial coefficients

Question: Find the first three terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y'' - y' + \ln(1 + x)y = 0$$

Answer choices:

A $\quad y = c_0\left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \ldots\right) + c_1\left(1 - \frac{1}{6}x^3 - \frac{1}{60}x^5 + \ldots\right)$

B $\quad y = c_0\left(1 - \frac{1}{6}x^3 - \frac{1}{60}x^5 + \ldots\right) + c_1\left(x + \frac{1}{2}x^2 + \frac{1}{6}x^3 - \ldots\right)$

C $\quad y = c_0\left(x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \ldots\right) + c_1\left(1 + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \ldots\right)$

D $\quad y = c_0\left(1 + \frac{1}{6}x^3 + \frac{1}{60}x^5 + \ldots\right) + c_1\left(x - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \ldots\right)$

Solution: B

We know $x_0 = 0$ is an ordinary point, because $\ln(1+x)$ is analytic there. And we know that the Maclaurin series representation of $\ln(1+x)$ is

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots$$

So we can substitute into the differential equation to get

$$y'' - y' + \ln(1+x)y = 0$$

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} c_n n x^{n-1} + \ln(1+x)\sum_{n=0}^{\infty} c_n x^n = 0$$

Now we'll just expand each series through its first few terms,

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$-(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+\left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots\right)(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots) = 0$$

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots)$$

$$-(c_1 + 2c_2x + 3c_3x^2 + 4c_4x^3 + 5c_5x^4 + \dots)$$

$$+x(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots)$$

$$-\frac{1}{2}x^2(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots)$$

$$+\frac{1}{3}x^3(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots)$$

$$-\frac{1}{4}x^4(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots)$$

$$+\frac{1}{5}x^5(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + \dots) - \dots = 0$$

$$2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + 30c_6x^4 + \dots$$

$$-c_1 - 2c_2x - 3c_3x^2 - 4c_4x^3 - 5c_5x^4 - \dots$$

$$+c_0x + c_1x^2 + c_2x^3 + c_3x^4 + c_4x^5 + \dots$$

$$-\frac{1}{2}c_0x^2 - \frac{1}{2}c_1x^3 - \frac{1}{2}c_2x^4 - \frac{1}{2}c_3x^5 - \frac{1}{2}c_4x^6 - \dots$$

$$+\frac{1}{3}c_0x^3 + \frac{1}{3}c_1x^4 + \frac{1}{3}c_2x^5 + \frac{1}{3}c_3x^6 + \frac{1}{3}c_4x^7 + \dots$$

$$-\frac{1}{4}c_0x^4 - \frac{1}{4}c_1x^5 - \frac{1}{4}c_2x^6 - \frac{1}{4}c_3x^7 - \frac{1}{4}c_4x^8 - \dots$$

$$+\frac{1}{5}c_0x^5 + \frac{1}{5}c_1x^6 + \frac{1}{5}c_2x^7 + \frac{1}{5}c_3x^8 + \frac{1}{5}c_4x^9 + \dots = 0$$

and then collect equivalent powers of x.

$$(2c_2 - c_1) + (6c_3 - 2c_2 + c_0)x + \left(12c_4 - 3c_3 + c_1 - \frac{1}{2}c_0\right)x^2$$

$$+\left(20c_5 - 4c_4 + c_2 - \frac{1}{2}c_1 + \frac{1}{3}c_0\right)x^3$$

$$+\left(30c_6 - 5c_5 + c_3 - \frac{1}{2}c_2 + \frac{1}{3}c_1 - \frac{1}{4}c_0\right)x^4 + \ldots = 0$$

From this result, we get a system of equations.

$$2c_2 - c_1 = 0$$

$$6c_3 - 2c_2 + c_0 = 0$$

$$12c_4 - 3c_3 + c_1 - \frac{1}{2}c_0 = 0$$

$$20c_5 - 4c_4 + c_2 - \frac{1}{2}c_1 + \frac{1}{3}c_0 = 0$$

$$30c_6 - 5c_5 + c_3 - \frac{1}{2}c_2 + \frac{1}{3}c_1 - \frac{1}{4}c_0 = 0$$

which simplifies to

$$c_2 = \frac{1}{2}c_1 \qquad c_3 = \frac{1}{6}c_1 - \frac{1}{6}c_0$$

$$c_4 = -\frac{1}{24}c_1 \qquad c_5 = -\frac{1}{120}c_1 - \frac{1}{60}c_0$$

$$c_6 = -\frac{7}{720}c_1 + \frac{1}{90}c_0$$

Pull these coefficients together into the power series solution.

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + \ldots$$

$$y = c_0 + c_1x + \frac{1}{2}c_1x^2 + \left(\frac{1}{6}c_1 - \frac{1}{6}c_0\right)x^3 - \frac{1}{24}c_1x^4$$

$$+\left(-\frac{1}{120}c_1-\frac{1}{60}c_0\right)x^5+\left(-\frac{7}{720}c_1+\frac{1}{90}c_0\right)x^6+\dots$$

$$y=c_0\left(1-\frac{1}{6}x^3-\frac{1}{60}x^5+\frac{1}{90}x^6+\dots\right)$$

$$+c_1\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{24}x^4-\frac{1}{120}x^5-\frac{7}{720}x^6+\dots\right)$$

Simplify to just the first three terms.

$$y=c_0\left(1-\frac{1}{6}x^3-\frac{1}{60}x^5+\dots\right)+c_1\left(x+\frac{1}{2}x^2+\frac{1}{6}x^3-\dots\right)$$

Topic: Singular points and Frobenius' Theorem

Question: Find and classify any singular points of the differential equation.

$$(x^2 - 9)y'' + (x + 3)y' + 2y = 0$$

Answer choices:

A $x_0 = 3$ and $x_0 = -3$ are regular singular points

B $x_0 = 3$ and $x_0 = -3$ are irregular singular points

C $x_0 = -3$ is a regular singular point; $x_0 = 3$ is an irregular singular point

D $x_0 = 3$ is a regular singular point; $x_0 = -3$ is an irregular singular point

Solution: A

The differential equation

$$(x^2 - 9)y'' + (x + 3)y' + 2y = 0$$

has singular points at

$$x^2 - 9 = 0$$

$$x^2 = 9$$

$$x = \pm 3$$

We'll find $Q(x)$ and $R(x)$ and get

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)}$$

$$Q(x) = (x - x_0)\frac{x + 3}{x^2 - 9}$$

$$Q(x) = (x - x_0)\frac{x + 3}{(x + 3)(x - 3)}$$

$$Q(x) = (x - x_0)\frac{1}{x - 3}$$

and

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)}$$

$$R(x) = (x - x_0)^2\frac{2}{x^2 - 9}$$

$$R(x) = (x - x_0)^2 \frac{2}{(x+3)(x-3)}$$

At $x_0 = 3$, we find

$$Q(x) = (x - 3)\frac{1}{x - 3}$$

$$Q(x) = 1$$

and

$$R(x) = (x - 3)^2 \frac{2}{(x+3)(x-3)}$$

$$R(x) = \frac{2(x-3)}{x+3}$$

Both $Q(x)$ and $R(x)$ are analytic at $x = 3$ because

$$Q(3) = 1$$

$$R(3) = \frac{2(3-3)}{3+3} = 0$$

And at $x_0 = -3$, we find

$$Q(x) = (x - (-3))\frac{1}{x - 3}$$

$$Q(x) = \frac{x+3}{x-3}$$

and

$$R(x) = (x - (-3))^2 \frac{2}{(x+3)(x-3)}$$

$$R(x) = \frac{2(x+3)}{x-3}$$

Both $Q(x)$ and $R(x)$ are analytic at $x = -3$ because

$$Q(-3) = \frac{-3+3}{-3-3} = 0$$

$$R(-3) = \frac{2(-3+3)}{-3-3} = 0$$

So both $x_0 = 3$ and $x_0 = -3$ are regular singular points.

Topic: Singular points and Frobenius' Theorem

Question: Use the method of Frobenius to find the indicial equation and recurrence relation of the differential equation.

$$x^2y'' + \left(\frac{5}{3}x + x^2\right)y' - \frac{1}{3}y = 0$$

Answer choices:

A $\quad r^2 + \frac{2}{3}r - \frac{1}{3} = 0$ and $c_k = -\frac{(k+r-1)c_{k-1}}{3k^2 + 2k + 6kr + 2r + 3r^2 - 1}$

B $\quad r^2 - \frac{2}{3}r + \frac{1}{3} = 0$ and $c_k = \frac{(k+r-1)c_{k-1}}{3k^2 + 2k + 6kr + 2r + 3r^2 - 1}$

C $\quad r - \frac{1}{3} = 0$ and $c_k = -\frac{(k+r-1)c_{k-1}}{3k^2 + 2k + 6kr + 2r + 3r^2 - 1}$

D $\quad r + \frac{1}{3} = 0$ and $c_k = \frac{(k+r-1)c_{k-1}}{3k^2 + 2k + 6kr + 2r + 3r^2 - 1}$

Solution: A

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = x^2$$

$$q(x) = \frac{5}{3}x + x^2$$

$$r(x) = -\frac{1}{3}$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = x\frac{\frac{5}{3}x + x^2}{x^2} = \frac{5}{3} + x$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{\left(-\frac{1}{3}\right)}{x^2} = -\frac{1}{3}$$

We can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point. Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n + r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n + r)(n + r - 1)c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$x^2y''+\left(\frac{5}{3}x+x^2\right)y'-\frac{1}{3}y=0$$

$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r-2}$$

$$+\left(\frac{5}{3}x+x^2\right)\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}-\frac{1}{3}\sum_{n=0}^{\infty}c_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r}+\frac{5}{3}x\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}$$

$$+x^2\sum_{n=0}^{\infty}(n+r)c_nx^{n+r-1}-\sum_{n=0}^{\infty}\frac{1}{3}c_nx^{n+r}=0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)c_nx^{n+r}+\sum_{n=0}^{\infty}\frac{5}{3}(n+r)c_nx^{n+r}$$

$$+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r+1}-\sum_{n=0}^{\infty}\frac{1}{3}c_nx^{n+r}=0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty}\left[(n+r)(n+r-1)c_n+\frac{5}{3}(n+r)c_n-\frac{1}{3}c_n\right]x^{n+r}$$

$$+\sum_{n=0}^{\infty}(n+r)c_nx^{n+r+1}=0$$

Factor x^r out of the left side of the equation.

$$x^r\left[\sum_{n=0}^{\infty}\left((n+r)(n+r-1)c_n+\frac{5}{3}(n+r)c_n-\frac{1}{3}c_n\right)x^n+\sum_{n=0}^{\infty}(n+r)c_nx^{n+1}\right]=0$$

Now we need to make sure these two series are in phase. The first series starts with the x^0 term, while the second series starts with the x^1 term. So we'll pull the x^0 term out of the first series to put them in phase.

$$x^r\left[\left(r^2+\frac{2}{3}r-\frac{1}{3}\right)c_0+\sum_{n=1}^{\infty}\left((n+r)(n+r-1)c_n+\frac{5}{3}(n+r)c_n-\frac{1}{3}c_n\right)x^n\right.$$

$$\left.+\sum_{n=0}^{\infty}(n+r)c_nx^{n+1}\right]=0$$

Now that the series are in phase, we'll substitute $k=n$ into the first series, and $k=n+1$ and $n=k-1$ into the second series.

$$x^r\left[\left(r^2+\frac{2}{3}r-\frac{1}{3}\right)c_0+\sum_{k=1}^{\infty}\left((k+r)(k+r-1)c_k+\frac{5}{3}(k+r)c_k-\frac{1}{3}c_k\right)x^k\right.$$

$$\left.+\sum_{k=1}^{\infty}(k+r-1)c_{k-1}x^k\right]=0$$

Now that the series are in phase with matching indices, combine them.

$$x^r\left[\left(r^2+\frac{2}{3}r-\frac{1}{3}\right)c_0\right.$$

$$\left.+\sum_{k=1}^{\infty}\left((k+r)(k+r-1)c_k+\frac{5}{3}(k+r)c_k-\frac{1}{3}c_k+(k+r-1)c_{k-1}\right)x^k\right]=0$$

This equation gives

$$\left(r^2+\frac{2}{3}r-\frac{1}{3}\right)c_0=0 \qquad k=0$$

$$(k+r)(k+r-1)c_k+\frac{5}{3}(k+r)c_k-\frac{1}{3}c_k+(k+r-1)c_{k-1}=0 \qquad k=1,\ 2,\ 3,\ \dots$$

So the indicial equation and recurrence relation are

$$r^2+\frac{2}{3}r-\frac{1}{3}=0$$

$$c_k=-\frac{(k+r-1)c_{k-1}}{3k^2+2k+6kr+2r+3r^2-1}$$

Topic: Singular points and Frobenius' Theorem

Question: Use the method of Frobenius to find the general solution to the differential equation about $x_0 = 0$.

$$4xy'' + \frac{1}{2}y' + y = 0$$

Answer choices:

A $$y(x) = C_1\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

$$+C_2x^{\frac{7}{8}}\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

B $$y(x) = C_1x\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

$$+C_2x^{\frac{7}{8}}\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

C $$y(x) = C_1\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

$$+C_2x^{\frac{7}{8}}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

D $$y(x) = C_1x\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

$$+C_2x^{\frac{7}{8}}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

Solution: C

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = 4x$$

$$q(x) = \frac{1}{2}$$

$$r(x) = 1$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = x\frac{\frac{1}{2}}{4x} = \frac{1}{8}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{1}{4x} = \frac{1}{4}x$$

We can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point. Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$4xy'' + \frac{1}{2}y' + y = 0$$

$$4x\sum_{n=0}^{\infty} (n+r)(n+r-1)c_n x^{n+r-2} + \frac{1}{2}\sum_{n=0}^{\infty} (n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 4(n+r)(n+r-1)c_n x^{n+r-1} + \sum_{n=0}^{\infty} \frac{1}{2}(n+r)c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty} \left(4(n+r)(n+r-1)c_n + \frac{1}{2}(n+r)c_n\right) x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Factor x^r out of the left side of the equation.

$$x^r\left[\sum_{n=0}^{\infty} \left(4(n+r)(n+r-1)c_n + \frac{1}{2}(n+r)c_n\right) x^{n-1} + \sum_{n=0}^{\infty} c_n x^n\right] = 0$$

Now we need to make sure these two series are in phase. The first series starts with the x^{-1} term, while the second series starts with the x^0 term. So we'll pull the x^{-1} term out of the first series to put them in phase.

$$x^r\left[r\left(4r-\frac{7}{2}\right)c_0x^{-1}+\sum_{n=1}^{\infty}\left(4(n+r)(n+r-1)c_n+\frac{1}{2}(n+r)c_n\right)x^{n-1}\right.$$

$$\left.+\sum_{n=0}^{\infty}c_nx^n\right]=0$$

Now that the series are in phase, we'll substitute $k=n-1$ and $n=k+1$ into the first series, and $k=n$ into the second series.

$$x^r\left[r\left(4r-\frac{7}{2}\right)c_0x^{-1}+\sum_{k=0}^{\infty}\left(4(k+r+1)(k+r)c_{k+1}+\frac{1}{2}(k+r+1)c_{k+1}\right)x^k\right.$$

$$\left.+\sum_{k=0}^{\infty}c_kx^k\right]=0$$

Now that the series are in phase with matching indices, combine them.

$$x^r\left[r\left(4r-\frac{7}{2}\right)c_0x^{-1}\right.$$

$$\left.+\sum_{k=0}^{\infty}\left(4(k+r+1)(k+r)c_{k+1}+\frac{1}{2}(k+r+1)c_{k+1}+c_k\right)x^k\right]=0$$

$$x^r\left[r\left(4r-\frac{7}{2}\right)c_0x^{-1}\right.$$

$$+\sum_{k=0}^{\infty}\left(\left(4(k+r+1)(k+r)+\frac{1}{2}(k+r+1)\right)c_{k+1}+c_k\right)x^k\Bigg] = 0$$

This equation gives

$$r\left(4r-\frac{7}{2}\right)c_0 = 0$$

$$\left(4(k+r+1)(k+r)+\frac{1}{2}(k+r+1)\right)c_{k+1}+c_k = 0 \qquad k = 0, 1, 2, \dots$$

or

$$r\left(4r-\frac{7}{2}\right) = 0$$

$$c_{k+1} = -\frac{c_k}{(k+r+1)\left(4(k+r)+\frac{1}{2}\right)} \qquad k = 0, 1, 2, \dots$$

The indicial equation gives us $r_1 = 0$ and $r_2 = 7/8$. Substituting these indicial roots into the recurrence relation, we get

	For $r_1 = 0$	For $r_2 = 7/8$
	$c_{k+1} = -\dfrac{2c_k}{(k+1)(8k+1)}$	$c_{k+1} = -\dfrac{2c_k}{(8k+15)(k+1)}$
$k = 0$	$c_1 = -2c_0$	$c_1 = -\dfrac{2c_0}{15}$
$k = 1$	$c_2 = \dfrac{2c_0}{9}$	$c_2 = \dfrac{2c_0}{345}$

$k = 2 \qquad c_3 = -\frac{4c_0}{459} \qquad c_3 = -\frac{4c_0}{32{,}085}$

$\dots \qquad \dots \qquad \dots$

Forming these coefficients into series around $x_0 = 0$ gives,

$$c_0(x - x_0)^0 + c_1(x - x_0)^1 + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

$$c_0(x - x_0)^0 - 2c_0(x - x_0)^1 + \frac{2c_0}{9}(x - x_0)^2 - \frac{4c_0}{459}(x - x_0)^3 + \dots$$

$$c_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

and

$$c_0(x - x_0)^0 + c_1(x - x_0)^1 + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

$$c_0(x - x_0)^0 - \frac{2c_0}{15}(x - x_0)^1 + \frac{2c_0}{345}(x - x_0)^2 - \frac{4c_0}{32{,}085}(x - x_0)^3 + \dots$$

$$c_0\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

so the series solutions are

$$y_1(x) = c_0x^0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

$$y_2(x) = c_0x^{\frac{7}{8}}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

and the general solution on the interval $(0,\infty)$ is

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

$$y(x) = C_1c_0\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

$$+C_2c_0x^{\frac{7}{8}}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

But C_1c_0 and C_2c_0 are constants, so we can simplify them and write the general solution as

$$y(x) = C_1\left(1 - 2x + \frac{2}{9}x^2 - \frac{4}{459}x^3 + \dots\right)$$

$$+C_2x^{\frac{7}{8}}\left(1 - \frac{2}{15}x + \frac{2}{345}x^2 - \frac{4}{32{,}085}x^3 + \dots\right)$$

Topic: The Laplace transform

Question: Find the Laplace transform, given $s > -1$.

$$\mathscr{L}(e^{-t})$$

Answer choices:

A $\quad F(s) = \dfrac{1}{-s+1}$

B $\quad F(s) = \dfrac{1}{-s-1}$

C $\quad F(s) = \dfrac{1}{s+1}$

D $\quad F(s) = \dfrac{1}{s-1}$

Solution: C

We're transforming $f(t) = e^{-t}$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t) \ dt$$

$$F(s) = \int_0^\infty e^{-st} e^{-t} \ dt$$

$$F(s) = \int_0^\infty e^{-st-t} \ dt$$

$$F(s) = \int_0^\infty e^{(-s-1)t} \ dt$$

Now we can integrate and evaluate over the interval.

$$F(s) = \frac{1}{-s-1} e^{(-s-1)t} \Big|_0^\infty$$

$$F(s) = \lim_{t \to \infty} \left(\frac{1}{-s-1} e^{(-s-1)t} \right) - \frac{1}{-s-1} e^{(-s-1)(0)}$$

$$F(s) = \lim_{t \to \infty} \left(\frac{1}{-s-1} e^{(-s-1)t} \right) + \frac{1}{s+1}$$

If we evaluate the limit assuming $-s-1 < 0$, or $s > -1$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{1}{s+1}$$

Topic: The Laplace transform

Question: Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(3)$$

Answer choices:

A $F(s) = -\frac{3}{s}$

B $F(s) = \frac{3}{s}$

C $F(s) = -\frac{1}{3s}$

D $F(s) = \frac{1}{3s}$

Solution: B

We're transforming $f(t) = 3$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

$$F(s) = \int_0^\infty 3e^{-st}\ dt$$

Now we can integrate and evaluate over the interval.

$$F(s) = \frac{3}{-s} e^{-st} \Big|_0^\infty$$

$$F(s) = \lim_{t \to \infty} \left(\frac{3}{-s} e^{-st} \right) - \frac{3}{-s} e^{-s(0)}$$

$$F(s) = \lim_{t \to \infty} \left(\frac{3}{-s} e^{-st} \right) + \frac{3}{s}$$

If we evaluate the limit assuming $s > 0$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{3}{s}$$

Topic: The Laplace transform

Question: Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(t)$$

Answer choices:

A $\quad F(s) = \dfrac{1}{s^2}$

B $\quad F(s) = \dfrac{1}{s}$

C $\quad F(s) = \dfrac{1}{s} + \dfrac{1}{s^2}$

D $\quad F(s) = \dfrac{1}{s+1}$

Solution: A

We're transforming $f(t) = t$, so we'll plug this into the definition of the Laplace transform.

$$F(s) = \int_0^\infty e^{-st}f(t)\ dt$$

$$F(s) = \int_0^\infty e^{-st}t\ dt$$

$$F(s) = \int_0^\infty te^{-st}\ dt$$

Now we can integrate and evaluate over the interval.

$$F(s) = -\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\Bigg|_0^\infty$$

$$F(s) = \lim_{t\to\infty}\left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right) - \left(-\frac{(0)e^{-s\cdot 0}}{s} - \frac{e^{-s(0)}}{s^2}\right)$$

$$F(s) = \lim_{t\to\infty}\left(-\frac{te^{-st}}{s} - \frac{e^{-st}}{s^2}\right) + \frac{1}{s^2}$$

If we evaluate the limit assuming $s > 0$, the exponential tends toward 0 and the Laplace transform is

$$F(s) = \frac{1}{s^2}$$

Topic: Table of transforms

Question: Use a table of Laplace transforms to transform the function.

$$f(t) = 5t - 6t^2$$

Answer choices:

A $\quad F(s) = \frac{5}{s} - \frac{6}{s^2}$

B $\quad F(s) = \frac{5}{s^2} - \frac{6}{s^3}$

C $\quad F(s) = \frac{5}{s} - \frac{12}{s^2}$

D $\quad F(s) = \frac{5}{s^2} - \frac{12}{s^3}$

Solution: D

From the table of Laplace transforms, we know

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$$

With $n = 1$ and $n = 2$, we find

$$\mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(t^2) = \frac{2}{s^3}$$

Then the Laplace transform of $f(t) = 5t - 6t^2$ is

$$F(s) = 5\left(\frac{1}{s^2}\right) - 6\left(\frac{2}{s^3}\right)$$

$$F(s) = \frac{5}{s^2} - \frac{12}{s^3}$$

Topic: Table of transforms

Question: Use a table of Laplace transforms to transform the function.

$$f(t) = e^{3t} + 4$$

Answer choices:

A $F(s) = \dfrac{1}{s+3} - \dfrac{1}{4s}$

B $F(s) = \dfrac{1}{s-3} + \dfrac{1}{4s}$

C $F(s) = \dfrac{1}{s-3} + \dfrac{4}{s}$

D $F(s) = \dfrac{1}{s+3} - \dfrac{1}{s}$

Solution: C

From the table of Laplace transforms, we know

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathscr{L}(1) = \frac{1}{s}$$

With $a = 3$, we find

$$\mathscr{L}(e^{3t}) = \frac{1}{s-3}$$

Then the Laplace transform of $f(t) = e^{3t} + 4$ is

$$F(s) = \frac{1}{s-3} + 4\left(\frac{1}{s}\right)$$

$$F(s) = \frac{1}{s-3} + \frac{4}{s}$$

Topic: Table of transforms

Question: Use a table of Laplace transforms to transform the function.

$$f(t) = 5 + 3t + t^2$$

Answer choices:

A $\quad F(s) = 5 + \frac{3}{s} + \frac{2}{s^2}$

B $\quad F(s) = \frac{5}{s} + \frac{3}{s^2} + \frac{2}{s^3}$

C $\quad F(s) = \frac{5}{s} + \frac{3}{s^2} + \frac{1}{s^3}$

D $\quad F(s) = 5 + \frac{3}{s} + \frac{1}{s^2}$

Solution: B

From the table of Laplace transforms, we know

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}$$

$$\mathscr{L}(1) = \frac{1}{s}$$

With $n = 1$ and $n = 2$, we find

$$\mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(t^2) = \frac{2}{s^3}$$

Then the Laplace transform of $f(t) = 5 + 3t + t^2$ is

$$F(s) = 5\left(\frac{1}{s}\right) + 3\left(\frac{1}{s^2}\right) + \frac{2}{s^3}$$

$$F(s) = \frac{5}{s} + \frac{3}{s^2} + \frac{2}{s^3}$$

Topic: Exponential type

Question: Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = \sin t$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t \to \infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Answer choices:

A $\quad \alpha > -1$

B $\quad \alpha > 1$

C $\quad \alpha > e^{\pi}$

D $\quad \alpha > 0$

Solution: D

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{\sin t}{e^{\alpha t}}$$

The value of $\sin t$ is never greater than 1, but $e^{\alpha t}$ grows exponentially for any $\alpha > 0$. So the denominator will grow infinitely large, while the numerator remains small, so the limit will go to 0, and the Laplace transform $\mathscr{L}(f(t))$ is defined. Therefore $f(t) = \sin t$ is of exponential type for any $a > 0$.

Topic: Exponential type

Question: Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = t^n$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t \to \infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Answer choices:

A $\alpha > 1$

B $\alpha > 0$

C $\alpha > \ln n$

D $\alpha > -1$

Solution: B

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty} \frac{t^n}{e^{\alpha t}}$$

For the numerator, given any real number x, we know that

$$e^{\ln x} = x \quad \text{and} \quad \ln(e^x) = x$$

so we can write the limit as

$$\lim_{t\to\infty} \frac{t^n}{e^{\alpha t}} \to \lim_{t\to\infty} \frac{e^{\ln(t^n)}}{e^{\alpha t}} \to \lim_{t\to\infty} \frac{e^{n\ln t}}{e^{\alpha t}} \to \lim_{t\to\infty} e^{n\ln t-\alpha t}$$

We know that the only way for this limit to approach 0 is if the exponent tends to $-\infty$, in other words, if $n\ln t - \alpha t \to -\infty$ as $t \to \infty$.

This is always true, since logarithmic growth is always slower than linear growth. Therefore the function $f(t) = t^n$, for a fixed positive integer n, is of exponential type for any $\alpha > 0$.

Topic: Exponential type

Question: Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = b^t$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t \to \infty} \frac{f(t)}{e^{\alpha t}} = 0$$

Answer choices:

A $\alpha > 1$

B $\alpha > 0$

C $\alpha > \ln b$

D $\alpha > -1$

Solution: C

Substituting $f(t)$ into the limit gives

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{b^t}{e^{\alpha t}}$$

With this we can write

$$\lim_{t\to\infty}\frac{b^t}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{e^{\ln(b^t)}}{e^{\alpha t}} \to \lim_{t\to\infty}\frac{e^{t\ln b}}{e^{\alpha t}} \to \lim_{t\to\infty} e^{t(\ln b-\alpha)}$$

We know that the only way for this limit to approach 0 is if the coefficient of t in the exponent is negative. In other words, if $\ln b - \alpha < 0$. Therefore the function $f(t) = b^t$, for a fixed $b > 0$, is of exponential type for any $\alpha > \ln b$.

Topic: Partial fractions decompositions

Question: Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{5x+3}{(x+3)(x-3)}$$

Answer choices:

A $\quad f(x) = -\frac{3}{x-3} - \frac{2}{x+3}$

B $\quad f(x) = \frac{3}{x-3} - \frac{2}{x+3}$

C $\quad f(x) = \frac{2}{x+3} - \frac{3}{x-3}$

D $\quad f(x) = \frac{2}{x+3} + \frac{3}{x-3}$

Solution: D

These are distinct linear factors.

$$\frac{5x+3}{(x+3)(x-3)}=\frac{A}{x+3}+\frac{B}{x-3}$$

To solve for A, remove the $x+3$ factor and set $x=-3$ to find the value of the left side of the decomposition equation.

$$\frac{5x+3}{x-3}\to\frac{5(-3)+3}{-3-3}\to\frac{-12}{-6}\to 2$$

To solve for B, we'll remove the $x-3$ factor and set $x=3$.

$$\frac{5x+3}{x+3}\to\frac{5(3)+3}{3+3}\to\frac{18}{6}\to 3$$

Plugging $A=2$ and $B=3$ back into the partial fractions decomposition gives

$$f(x)=\frac{A}{x+3}+\frac{B}{x-3}$$

$$f(x)=\frac{2}{x+3}+\frac{3}{x-3}$$

Topic: Partial fractions decompositions

Question: Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{x^3-8x^2-1}{(x+3)(x-2)(x^2+1)}$$

Answer choices:

A $\quad f(x)=\frac{2}{x+3}+\frac{1}{x-2}+\frac{1}{x^2+1}$

B $\quad f(x)=\frac{2}{x+3}-\frac{1}{x-2}-\frac{1}{x^2+1}$

C $\quad f(x)=\frac{2}{x+3}-\frac{1}{x-2}+\frac{1}{x^2+1}$

D $\quad f(x)=\frac{2}{x+3}+\frac{1}{x-2}-\frac{1}{x^2+1}$

Solution: B

These are distinct linear and quadratic factors.

$$\frac{x^3 - 8x^2 - 1}{(x+3)(x-2)(x^2+1)} = \frac{A}{x+3} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1}$$

To solve for A, remove the $x+3$ factor and set $x = -3$ to find the value of the left side of the decomposition equation.

$$\frac{x^3 - 8x^2 - 1}{(x-2)(x^2+1)} \to \frac{(-3)^3 - 8(-3)^2 - 1}{(-3-2)((-3)^2+1)} \to \frac{-100}{-50} \to 2$$

To solve for B, we'll remove the $x-2$ factor and set $x = 2$.

$$\frac{x^3 - 8x^2 - 1}{(x+3)(x^2+1)} \to \frac{2^3 - 8(2)^2 - 1}{(2+3)(2^2+1)} \to \frac{-25}{25} \to -1$$

To solve for C, we'll multiply both sides of the equation by the denominator from the left side.

$$x^3 - 8x^2 - 1 = A(x-2)(x^2+1) + B(x+3)(x^2+1) + (Cx+D)(x+3)(x-2)$$

$$x^3 - 8x^2 - 1 = A(x^3 + x - 2x^2 - 2) + B(x^3 + x + 3x^2 + 3) + (Cx+D)(x^2 + x - 6)$$

$$x^3 - 8x^2 - 1 = (A+B+C)x^3 + (-2A+3B+C+D)x^2$$

$$+(A+B-6C+D)x + (-2A+3B-6D)$$

Equating coefficients gives us a system of equations.

$$A + B + C = 1$$

$$-2A + 3B + C + D = -8$$

$$A + B - 6C + D = 0$$

$$-2A + 3B - 6D = -1$$

Solving $A + B + C = 1$ for A gives $A = 1 - B - C$, so the system simplifies to

$$5B + 3C + D = -6$$

$$7C - D = 1$$

$$5B + 2C - 6D = 1$$

Solving $7C - D = 1$ for D gives $D = 7C - 1$, so the system simplifies to

$$B + 2C = -1$$

$$B - 8C = -1$$

Subtracting the second equation from the first gives

$$B + 2C - (B - 8C) = -1 - (-1)$$

$$B + 2C - B + 8C = -1 + 1$$

$$10C = 0$$

$$C = 0$$

Then $D = -1$, $B = -1$, and $A = 2$. Plugging $A = 2$, $B = -1$, $C = 0$, and $D = -1$ back into the partial fractions decomposition gives

$$f(x) = \frac{A}{x+3} + \frac{B}{x-2} + \frac{Cx+D}{x^2+1}$$

$$f(x) = \frac{2}{x+3} - \frac{1}{x-2} - \frac{1}{x^2+1}$$

Topic: Partial fractions decompositions

Question: Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{4x^2+10x+8}{(x+1)^3}$$

Answer choices:

A $\quad f(x)=\frac{4}{x+1}+\frac{2}{(x+1)^2}+\frac{2}{(x+1)^3}$

B $\quad f(x)=\frac{4}{x+1}-\frac{2}{(x+1)^2}+\frac{2}{(x+1)^3}$

C $\quad f(x)=\frac{4}{x+1}+\frac{2}{(x+1)^2}-\frac{2}{(x+1)^3}$

D $\quad f(x)=\frac{4}{x+1}-\frac{2}{(x+1)^2}-\frac{2}{(x+1)^3}$

Solution: A

These are repeated linear factors.

$$\frac{4x^2+10x+8}{(x+1)^3}=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

To solve for A, we'll multiply both sides of the equation by the denominator from the left side.

$$4x^2+10x+8=A(x+1)^2+B(x+1)+C$$

$$4x^2+10x+8=Ax^2+(2A+B)x+(A+B+C)$$

Equating coefficients gives us $A=4$ and then a system of equations.

$$2A+B=10$$

$$A+B+C=8$$

With $A=4$, we find $B=2$ and then $C=2$. Plugging $A=4$, $B=2$, and $C=2$ back into the partial fractions decomposition gives

$$f(x)=\frac{A}{x+1}+\frac{B}{(x+1)^2}+\frac{C}{(x+1)^3}$$

$$f(x)=\frac{4}{x+1}+\frac{2}{(x+1)^2}+\frac{2}{(x+1)^3}$$

Topic: Inverse Laplace transforms

Question: Find the inverse Laplace transform.

$$F(s) = \frac{s+5}{s^2+5s+6}$$

Answer choices:

A $f(t) = 3e^{2t} + 2e^{3t}$

B $f(t) = 3e^{-2t} + 2e^{-3t}$

C $f(t) = 3e^{2t} - 2e^{3t}$

D $f(t) = 3e^{-2t} - 2e^{-3t}$

Solution: D

We'll first factor the denominator of $F(s)$,

$$F(s) = \frac{s+5}{s^2+5s+6}$$

$$F(s) = \frac{s+5}{(s+2)(s+3)}$$

and then apply a partial fractions decomposition.

$$\frac{s+5}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$s+5 = A(s+3) + B(s+2)$$

To find A, we'll set $s = -2$ in order to eliminate B.

$$-2+5 = A((-2)+3) + B((-2)+2)$$

$$3 = A(1) + B(0)$$

$$A = 3$$

To find B, we'll set $s = -3$ in order to eliminate A.

$$-3+5 = A((-3)+3) + B((-3)+2)$$

$$2 = A(0) + B(-1)$$

$$2 = -B$$

$$B = -2$$

Substituting these values back into the decomposition gives

$$F(s) = \frac{3}{s+2} + \frac{-2}{s+3}$$

$$F(s) = 3\left(\frac{1}{s+2}\right) - 2\left(\frac{1}{s+3}\right)$$

$$F(s) = 3\left(\frac{1}{s-(-2)}\right) - 2\left(\frac{1}{s-(-3)}\right)$$

In this form, we can see that both values inside the parentheses resemble the Laplace transform

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

We can see that $a_1 = -2$ and $a_2 = -3$. We'll reverse the formula from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = 3e^{-2t} - 2e^{-3t}$$

Topic: Inverse Laplace transforms

Question: Find the inverse Laplace transform.

$$F(s) = \frac{s-2}{s^2+4s}$$

Answer choices:

A $\quad f(t) = -\frac{1}{2}t + \frac{3}{2}e^{-4t}$

B $\quad f(t) = -\frac{1}{2}t + \frac{3}{2}e^{4t}$

C $\quad f(t) = -\frac{1}{2} + \frac{3}{2}e^{-4t}$

D $\quad f(t) = -\frac{1}{2} + \frac{3}{2}e^{4t}$

Solution: C

We'll first factor the denominator of $F(s)$,

$$F(s) = \frac{s-2}{s^2+4s}$$

$$F(s) = \frac{s-2}{s(s+4)}$$

and then apply a partial fractions decomposition.

$$\frac{s-2}{s(s+4)} = \frac{A}{s} + \frac{B}{s+4}$$

$$s - 2 = A(s+4) + Bs$$

To find A, we'll set $s = 0$ in order to eliminate B.

$$0 - 2 = A(0+4) + B(0)$$

$$-2 = A(4) + B(0)$$

$$A = -\frac{1}{2}$$

To find B, we'll set $s = -4$ in order to eliminate A.

$$-4 - 2 = A(-4+4) + B(-4)$$

$$-6 = A(0) + B(-4)$$

$$-6 = -4B$$

$$B = \frac{3}{2}$$

Substituting these values back into the decomposition gives

$$F(s) = \frac{-\frac{1}{2}}{s} + \frac{\frac{3}{2}}{s+4}$$

$$F(s) = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{3}{2}\left(\frac{1}{s+4}\right)$$

$$F(s) = -\frac{1}{2}\left(\frac{1}{s}\right) + \frac{3}{2}\left(\frac{1}{s-(-4)}\right)$$

In this form, we can see that the values inside the parentheses resemble the Laplace transforms

$$\mathscr{L}(1) = \frac{1}{s} \qquad \mathscr{L}(e^{at}) = \frac{1}{s-a}$$

With $a = -4$, we'll reverse the formulas from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = -\frac{1}{2} + \frac{3}{2}e^{-4t}$$

Topic: Inverse Laplace transforms

Question: Find the inverse Laplace transform.

$$F(s) = \frac{3s+1}{2s^2+s-1}$$

Answer choices:

A $\quad f(t) = \frac{5}{3}e^{-\frac{1}{2}t} + \frac{2}{3}e^{-t}$

B $\quad f(t) = \frac{5}{6}e^{\frac{1}{2}t} + \frac{2}{3}e^{-t}$

C $\quad f(t) = \frac{5}{3}e^{-t} + \frac{2}{3}e^{-t}$

D $\quad f(t) = \frac{5}{6}e^{-t} + \frac{2}{3}e^{-t}$

Solution: B

We'll first factor the denominator of $F(s)$,

$$F(s)=\frac{3s+1}{2s^2+s-1}$$

$$F(s)=\frac{3s+1}{(2s-1)(s+1)}$$

and then apply a partial fractions decomposition.

$$\frac{3s+1}{(2s-1)(s+1)}=\frac{A}{2s-1}+\frac{B}{s+1}$$

$$3s+1=A(s+1)+B(2s-1)$$

To find A, we'll set $s=1/2$ in order to eliminate B.

$$3\left(\frac{1}{2}\right)+1=A\left(\frac{1}{2}+1\right)+B\left(2\left(\frac{1}{2}\right)-1\right)$$

$$\frac{5}{2}=A\left(\frac{3}{2}\right)+B(0)$$

$$A=\frac{5}{3}$$

To find B, we'll set $s=-1$ in order to eliminate A.

$$3(-1)+1=A((-1)+1)+B(2(-1)-1)$$

$$-2=A(0)+B(-3)$$

$$-2 = -3B$$

$$B = \frac{2}{3}$$

Substituting these values back into the decomposition gives

$$F(s) = \frac{\frac{5}{3}}{2s-1} + \frac{\frac{2}{3}}{s+1}$$

$$F(s) = \frac{5}{3}\left(\frac{1}{2s-1}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)$$

$$F(s) = \frac{5}{3}\left(\frac{\frac{1}{2}}{\frac{2s}{2}-\frac{1}{2}}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)$$

$$F(s) = \frac{5}{6}\left(\frac{1}{s-\frac{1}{2}}\right) + \frac{2}{3}\left(\frac{1}{s+1}\right)$$

$$F(s) = \frac{5}{6}\left(\frac{1}{s-\frac{1}{2}}\right) + \frac{2}{3}\left(\frac{1}{s-(-1)}\right)$$

In this form, we can see that both values inside the parentheses resemble the Laplace transform

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

We can see that $a_1 = 1/2$ and $a_2 = -1$. We'll reverse the formula from the table to rewrite the transform as

$$f(t)=\mathscr{L}^{-1}(F(s))=\frac{5}{6}e^{\frac{1}{2}t}+\frac{2}{3}e^{-t}$$

Topic: Transforming derivatives

Question: Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 0$ and $y'(0) = 0$.

Answer choices:

A	$\mathscr{L}(y'(t)) = s^2Y(s)$	$\mathscr{L}(y''(t)) = 0$
B	$\mathscr{L}(y'(t)) = sY(s)$	$\mathscr{L}(y''(t)) = 2sY(s)$
C	$\mathscr{L}(y'(t)) = sY(s)$	$\mathscr{L}(y''(t)) = s^2Y(s)$
D	$\mathscr{L}(y'(t)) = 0$	$\mathscr{L}(y''(t)) = sY(s)$

Solution: C

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - 0$$

$$\mathscr{L}(y'(t)) = sY(s)$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s(0) - 0$$

$$\mathscr{L}(y''(t)) = s^2Y(s)$$

So for any function $y(t)$ with the initial conditions $y(0) = 0$ and $y'(0) = 0$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s)$$

$$\mathscr{L}(y''(t)) = s^2Y(s)$$

Topic: Transforming derivatives

Question: Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0)=1/2$ and $y'(0)=3/4$.

Answer choices:

A $\mathscr{L}(y'(t))=sY(s)-e^{2t}$ $\mathscr{L}(y''(t))=sY(s)-e^{2t}-e^{\frac{4}{3}t}$

B $\mathscr{L}(y'(t))=sY(s)-\dfrac{1}{2}$ $\mathscr{L}(y''(t))=s^2Y(s)-\dfrac{s}{2}-\dfrac{3}{4}$

C $\mathscr{L}(y'(t))=2sY(s)$ $\mathscr{L}(y''(t))=\dfrac{3s}{4}Y(s)-\dfrac{1}{2}s$

D $\mathscr{L}(y'(t))=2sY(s)-\dfrac{1}{2}$ $\mathscr{L}(y''(t))=2sY(s)-\dfrac{3}{4}s$

Solution: B

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - \frac{1}{2}$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s\left(\frac{1}{2}\right) - \frac{3}{4}$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - \frac{s}{2} - \frac{3}{4}$$

So for any function $y(t)$ with the initial conditions $y(0) = 1/2$ and $y'(0) = 3/4$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s) - \frac{1}{2}$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - \frac{s}{2} - \frac{3}{4}$$

Topic: Transforming derivatives

Question: Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0)=\pi$ and $y'(0)=e$.

Answer choices:

A $\mathscr{L}(y'(t))=sY(s)-e^{\pi}$ $\mathscr{L}(y''(t))=s^2Y(s)-e^{\pi}s-(\pi+e)$

B $\mathscr{L}(y'(t))=sY(s)-e^{\pi t}$ $\mathscr{L}(y''(t))=s^2Y(s)-e^{\pi t}-e^{e}$

C $\mathscr{L}(y'(t))=sY(s)-\pi$ $\mathscr{L}(y''(t))=s^2Y(s)-\pi s-e$

D $\mathscr{L}(y'(t))=sY(s)-e$ $\mathscr{L}(y''(t))=s^2Y(s)-se-\pi$

Solution: C

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(t)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(t)) = sY(s) - \pi$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and t instead of f and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(t)) = s^2Y(s) - s\,(\pi) - e$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - \pi s - e$$

So for any function $y(t)$ with the initial conditions $y(0) = \pi$ and $y'(0) = e$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(t)) = sY(s) - \pi$$

$$\mathscr{L}(y''(t)) = s^2Y(s) - \pi s - e$$

Topic: Laplace transforms for initial value problems

Question: Find the solution to the second order equation, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' + 3y' + 2y = t$$

Answer choices:

A $\quad y(t) = \frac{1}{2}t - \frac{3}{4} + e^{t} - \frac{1}{4}e^{2t}$

B $\quad y(t) = \frac{1}{2}t - \frac{3}{4} + e^{2t} - \frac{1}{4}e^{-t}$

C $\quad y(t) = \frac{1}{2}t - \frac{3}{4} + e^{-t} - \frac{1}{4}e^{-2t}$

D $\quad y(t) = \frac{1}{2}t - \frac{3}{4} + e^{2t} - \frac{1}{4}e^{2t}$

Solution: C

From our table of Laplace transforms, we know

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(t) = \frac{1}{s^2}$$

Plugging these transforms into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s^2}$$

Now we'll plug in the initial conditions $y(0) = 0$ and $y'(0) = 0$ in order to simplify the transform.

$$s^2Y(s) - s(0) - 0 + 3(sY(s) - 0) + 2Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s^2}$$

Factor out $Y(s)$, then isolate it on the left side of the equation.

$$Y(s)(s^2 + 3s + 2) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2(s^2 + 3s + 2)}$$

$$Y(s) = \frac{1}{s^2(s+2)(s+1)}$$

We'll need to use a partial fractions decomposition.

$$\frac{1}{s^2(s+2)(s+1)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+2} + \frac{D}{s+1}$$

To solve for B, we'll remove the s^2 factor, set $s = 0$, and find the value of the left side.

$$\frac{1}{(s+2)(s+1)}$$

$$\frac{1}{(0+2)(0+1)} = \frac{1}{2(1)} = \frac{1}{2}$$

To solve for C, we'll remove the $(s+2)$ factor, set $s = -2$, and find the value of the left side.

$$\frac{1}{s^2(s+1)}$$

$$\frac{1}{(-2)^2(-2+1)} = \frac{1}{4(-1)} = -\frac{1}{4}$$

To solve for D, we'll remove the $(s+1)$ factor, set $s = -1$, and find the value of the left side.

$$\frac{1}{s^2(s+2)}$$

$$\frac{1}{(-1)^2(-1+2)} = \frac{1}{1(1)} = 1$$

Now that we have the values of B, C, and D, we can plug those into the decomposition,

$$\frac{1}{s^2(s+2)(s+1)}=\frac{A}{s}+\frac{\frac{1}{2}}{s^2}+\frac{-\frac{1}{4}}{s+2}+\frac{1}{s+1}$$

and then pick a value we haven't used for s yet ($s \neq -2,\ -1, 0$), like $s=1$, plug it in, and solve for A.

$$\frac{1}{1^2(1+2)(1+1)}=\frac{A}{1}+\frac{\frac{1}{2}}{1^2}+\frac{-\frac{1}{4}}{1+2}+\frac{1}{1+1}$$

$$\frac{1}{3(2)}=A+\frac{1}{2}-\frac{1}{12}+\frac{1}{2}$$

$$A=\frac{1}{6}+\frac{1}{12}-\frac{1}{2}-\frac{1}{2}$$

$$A=\frac{2}{12}+\frac{1}{12}-\frac{6}{12}-\frac{6}{12}$$

$$A=-\frac{9}{12}$$

$$A=-\frac{3}{4}$$

Plugging the values we found for A, B, C, and D back into the partial fractions decomposition gives

$$Y(s)=\frac{-\frac{3}{4}}{s}+\frac{\frac{1}{2}}{s^2}+\frac{-\frac{1}{4}}{s+2}+\frac{1}{s+1}$$

and then we can rearrange each term in the decomposition to make it easier to find a matching formula in the Laplace transform table.

$$Y(s) = -\frac{3}{4}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s^2}\right) - \frac{1}{4}\left(\frac{1}{s-(-2)}\right) + \frac{1}{s-(-1)}$$

The terms remaining inside the parentheses should remind us of the transforms

$$\mathscr{L}(1) = \frac{1}{s} \qquad \mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(e^{-2t}) = \frac{1}{s-(-2)} \qquad \mathscr{L}(e^{-t}) = \frac{1}{s-(-1)}$$

So we'll make these substitutions to put the equation back in terms of the original variable t, instead of the transform variable s, which will give us the solution to the second order nonhomogeneous differential equation.

$$y(t) = -\frac{3}{4}(1) + \frac{1}{2}(t) - \frac{1}{4}e^{-2t} + e^{-t}$$

$$y(t) = -\frac{3}{4} + \frac{1}{2}t - \frac{1}{4}e^{-2t} + e^{-t}$$

$$y(t) = \frac{1}{2}t - \frac{3}{4} + e^{-t} - \frac{1}{4}e^{-2t}$$

Topic: Laplace transforms for initial value problems

Question: Find the solution to the second order equation, given $y(0) = 4$ and $y'(0) = 9$.

$$y''(t) - 3y'(t) + 2y(t) = 4e^{3t}$$

Answer choices:

A $\quad y(t) = 2e^{3t} + e^{2t} + e^{t}$

B $\quad y(t) = e^{3t} + 2e^{2t} + e^{t}$

C $\quad y(t) = e^{4t} + 2e^{3t} + e^{2t}$

D $\quad y(t) = 2e^{3t} + 2e^{2t} + e^{t}$

Solution: A

From our table of Laplace transforms, we know

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

Plugging these transforms into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = 4\left(\frac{1}{s-3}\right)$$

Now we'll plug in the initial conditions $y(0) = 4$ and $y'(0) = 9$ in order to simplify the transform.

$$s^2Y(s) - s(4) - 9 - 3(sY(s) - 4) + 2Y(s) = 4\left(\frac{1}{s-3}\right)$$

$$s^2Y(s) - 4s - 9 - 3sY(s) + 12 + 2Y(s) = \frac{4}{s-3}$$

Solve for $Y(s)$ by collecting all the $Y(s)$ terms on one side, and moving all other terms to the other side.

$$s^2Y(s) - 3sY(s) + 2Y(s) = \frac{4}{s-3} + 4s - 3$$

Factor out $Y(s)$, then isolate it on the left side of the equation.

$$Y(s)(s^2 - 3s + 2) = \frac{4}{s-3} + 4s - 3$$

$$Y(s)(s^2 - 3s + 2) = \frac{4}{s-3} + \frac{(4s-3)(s-3)}{s-3}$$

$$Y(s)(s^2 - 3s + 2) = \frac{4s^2 - 15s + 13}{s-3}$$

$$Y(s) = \frac{4s^2 - 15s + 13}{(s-3)(s-2)(s-1)}$$

We'll need to use a partial fractions decomposition.

$$\frac{4s^2 - 15s + 13}{(s-3)(s-2)(s-1)} = \frac{A}{s-3} + \frac{B}{s-2} + \frac{C}{s-1}$$

To solve for A, we'll remove the $(s-3)$ factor, set $s = 3$, and find the value of the left side.

$$\frac{4s^2 - 15s + 13}{(s-2)(s-1)}$$

$$\frac{4(3)^2 - 15(3) + 13}{(3-2)(3-1)} = \frac{4(9) - 45 + 13}{1(2)} = \frac{4}{2} = 2$$

To solve for B, we'll remove the $(s-2)$ factor, set $s = 2$, and find the value of the left side.

$$\frac{4s^2 - 15s + 13}{(s-3)(s-1)}$$

$$\frac{4(2)^2 - 15(2) + 13}{(2-3)(2-1)} = \frac{4(4) - 30 + 13}{-1(1)} = \frac{-1}{-1} = 1$$

To solve for C, we'll remove the $(s-1)$ factor, set $s=1$, and find the value of the left side.

$$\frac{4s^2-15s+13}{(s-3)(s-2)}$$

$$\frac{4(1)^2-15(1)+13}{(1-3)(1-2)}=\frac{4(1)-15+13}{-2(-1)}=\frac{2}{2}=1$$

Plugging the values we found for A, B, and C back into the partial fractions decomposition gives

$$Y(s)=\frac{2}{s-3}+\frac{1}{s-2}+\frac{1}{s-1}$$

and then we can rearrange each term in the decomposition to make it easier to find a matching formula in the Laplace transform table.

$$Y(s)=2\left(\frac{1}{s-3}\right)+\frac{1}{s-2}+\frac{1}{s-1}$$

The terms remaining should remind us of the transforms

$$\mathscr{L}(e^{3t})=\frac{1}{s-3} \qquad \mathscr{L}(e^{2t})=\frac{1}{s-2} \qquad \mathscr{L}(e^{t})=\frac{1}{s-1}$$

So we'll make these substitutions to put the equation back in terms of the original variable t, instead of the transform variable s, which will give us the solution to the second order nonhomogeneous differential equation.

$$y(t)=2e^{3t}+e^{2t}+e^{t}$$

Topic: Laplace transforms for initial value problems

Question: Find the solution to the second order equation, given $y(0) = 1$ and $y'(0) = 2$.

$$y'' - 5y' + 6y = 3t$$

Answer choices:

A $\quad y(t) = \frac{5}{12} + \frac{1}{2}t + \frac{1}{4}e^{2t} - \frac{1}{3}e^{3t}$

B $\quad y(t) = \frac{5}{12} + \frac{1}{2}t + \frac{1}{4}e^{2t} + \frac{1}{3}e^{3t}$

C $\quad y(t) = \frac{5}{12} + \frac{1}{2}t - \frac{1}{12}e^{t}$

D $\quad y(t) = \frac{5}{12} + \frac{1}{2}t + \frac{7}{12}e^{5t}$

Solution: B

From our table of Laplace transforms, we know

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(t) = \frac{1}{s^2}$$

Plugging these transforms into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) - 5(sY(s) - y(0)) + 6Y(s) = 3\left(\frac{1}{s^2}\right)$$

Now we'll plug in the initial conditions $y(0) = 1$ and $y'(0) = 2$ in order to simplify the transform.

$$s^2Y(s) - s(1) - 2 - 5(sY(s) - 1) + 6Y(s) = 3\left(\frac{1}{s^2}\right)$$

$$s^2Y(s) - s - 2 - 5sY(s) + 5 + 6Y(s) = \frac{3}{s^2}$$

Solve for $Y(s)$ by collecting all the $Y(s)$ terms on one side, and moving all other terms to the other side.

$$s^2Y(s) - 5sY(s) + 6Y(s) = \frac{3}{s^2} + s - 3$$

Factor out $Y(s)$, then isolate it on the left side of the equation.

$$Y(s)(s^2 - 5s + 6) = \frac{3}{s^2} + s - 3$$

$$Y(s)(s^2 - 5s + 6) = \frac{3}{s^2} + \frac{s^2(s-3)}{s^2}$$

$$Y(s)(s^2 - 5s + 6) = \frac{s^3 - 3s^2 + 3}{s^2}$$

$$Y(s) = \frac{s^3 - 3s^2 + 3}{s^2(s-3)(s-2)}$$

We'll need to use a partial fractions decomposition.

$$\frac{s^3 - 3s^2 + 3}{s^2(s-3)(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-3} + \frac{D}{s-2}$$

To solve for B, we'll remove the s^2 factor, set $s = 0$, and find the value of the left side.

$$\frac{s^3 - 3s^2 + 3}{(s-3)(s-2)}$$

$$\frac{0^3 - 3(0)^2 + 3}{(0-3)(0-2)} = \frac{3}{-3(-2)} = \frac{3}{6} = \frac{1}{2}$$

To solve for C, we'll remove the $(s-3)$ factor, set $s = 3$, and find the value of the left side.

$$\frac{s^3 - 3s^2 + 3}{s^2(s-2)}$$

$$\frac{3^3 - 3(3)^2 + 3}{3^2(3-2)} = \frac{27 - 3(9) + 3}{9(1)} = \frac{3}{9} = \frac{1}{3}$$

To solve for D, we'll remove the $(s-2)$ factor, set $s=2$, and find the value of the left side.

$$\frac{s^3-3s^2+3}{s^2(s-3)}$$

$$\frac{2^3-3(2)^2+3}{2^2(2-3)}=\frac{8-3(4)+3}{4(-1)}=\frac{-1}{-4}=\frac{1}{4}$$

Now that we have the values of B, C, and D, we can plug those into the decomposition,

$$\frac{s^3-3s^2+3}{s^2(s-3)(s-2)}=\frac{A}{s}+\frac{\frac{1}{2}}{s^2}+\frac{\frac{1}{3}}{s-3}+\frac{\frac{1}{4}}{s-2}$$

and then pick a value we haven't used for s yet ($s\neq 0, 2, 3$), like $s=1$, plug it in, and solve for A.

$$\frac{1^3-3(1)^2+3}{1^2(1-3)(1-2)}=\frac{A}{1}+\frac{\frac{1}{2}}{1^2}+\frac{\frac{1}{3}}{1-3}+\frac{\frac{1}{4}}{1-2}$$

$$\frac{1-3+3}{-2(-1)}=A+\frac{1}{2}+\frac{\frac{1}{3}}{-2}+\frac{\frac{1}{4}}{-1}$$

$$\frac{1}{2}=A+\frac{1}{2}-\frac{1}{6}-\frac{1}{4}$$

$$A=\frac{1}{2}-\frac{1}{2}+\frac{1}{6}+\frac{1}{4}$$

$$A=\frac{4}{24}+\frac{6}{24}$$

$$A = \frac{5}{12}$$

Plugging the values we found for A, B, C, and D back into the partial fractions decomposition gives

$$Y(s) = \frac{\frac{5}{12}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{3}}{s-3} + \frac{\frac{1}{4}}{s-2}$$

and then we can rearrange each term in the decomposition to make it easier to find a matching formula in the Laplace transform table.

$$Y(s) = \frac{5}{12}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{3}\left(\frac{1}{s-3}\right) + \frac{1}{4}\left(\frac{1}{s-2}\right)$$

The terms remaining inside the parentheses should remind us of the transforms

$$\mathscr{L}(1) = \frac{1}{s} \qquad \mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(e^{3t}) = \frac{1}{s-3} \qquad \mathscr{L}(e^{2t}) = \frac{1}{s-2}$$

So we'll make these substitutions to put the equation back in terms of the original variable t, instead of the transform variable s, which will give us the solution to the second order nonhomogeneous differential equation.

$$y(t) = \frac{5}{12}(1) + \frac{1}{2}(t) + \frac{1}{3}e^{3t} + \frac{1}{4}e^{2t}$$

$$y(t) = \frac{5}{12} + \frac{1}{2}t + \frac{1}{3}e^{3t} + \frac{1}{4}e^{2t}$$

$$y(t) = \frac{5}{12} + \frac{1}{2}t + \frac{1}{4}e^{2t} + \frac{1}{3}e^{3t}$$

Topic: Step functions

Question: Represent $f(t)$ in terms of unit step functions.

$$f(t) = \begin{cases} 5 & 0 \le t < 2 \\ 2 & 2 \le t < 3 \\ 7 & 3 \le t < 5 \\ 0 & t \ge 5 \end{cases}$$

Answer choices:

A $f(t) = 5 - 3u_2(t) + 5u_3(t) - 7u_5(t)$

B $f(t) = 5 + 3u_2(t) - 5u_3(t) + 7u_5(t)$

C $f(t) = 5 - 3u_2(t) + 5u_3(t)$

D $f(t) = 5u_2(t) + 2u_3(t) + 7u_5(t)$

Solution: A

Remember that a Heaviside function represents a switch, so we need one Heaviside function every time we make a switch in the value of $f(t)$. There are three switches, one from 5 to 2, one from 2 to 7, and one from 7 to 0, so we'll need three Heaviside functions in order to express $f(t)$.

We write a Heaviside function as $u_c(t)$, where the switch occurs at $t = c$. The switches in this function $f(t)$ are at $t = 2$, $t = 3$, and $t = 5$, which means the three Heaviside functions we'll need are $u_2(t)$, $u_3(t)$, and $u_5(t)$.

To express $f(t)$, we start with the idea that $f(t)$ has a value of 5 when it's "off." When the first switch at $t = 2$ turns on, the value of the function immediately shifts from 5 to 2 (down 3 units), so we can write

$$f(t) \approx 5 - 3u_2(t)$$

We use -3 as a coefficient on $u_2(t)$, since $5 - 2 = 3$, and 2 is the value the function takes on after the first switch. Then moving from 2 to 7 is an increase of 5, which means we need to multiply $u_3(t)$ by 5,

$$f(t) \approx 5 - 3u_2(t) + 5u_3(t)$$

and moving from 7 to 0 is a decrease of 7, which means we need to multiply $u_5(t)$ by -7, and this will give us the full expression of $f(t)$ in terms of step functions.

$$f(t) \approx 5 - 3u_2(t) + 5u_3(t) - 7u_5(t)$$

Topic: Step functions

Question: Which function has an (on at 7)-to-off switch at $t = 10$?

Answer choices:

A $f(t) = 10 - 10u_7(t)$

B $f(t) = 10u_7(t)$

C $f(t) = 7u_{10}(t)$

D $f(t) = 7 - 7u_{10}(t)$

Solution: D

An (on at n)-to-off switch can be modeled by

$$f(t) = n(1 - u_c(t)) = \begin{cases} n & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

If we want the switch to be "on" at a value of 7, and then turn "off" to a value of 0 when we arrive at the switching point $t = 10$, then we'd write the function as

$$f(t) = 7(1 - u_{10}(t)) = \begin{cases} 7 & 0 \le t < 10 \\ 0 & t \ge 10 \end{cases}$$

$$f(t) = 7 - 7u_{10}(t) = \begin{cases} 7 & 0 \le t < 10 \\ 0 & t \ge 10 \end{cases}$$

Topic: Step functions

Question: How many switches does the function have?

$$f(t) = 9 + 2u_2(t) - 3u_5(t)$$

Answer choices:

A $\quad 1$

B $\quad 2$

C $\quad 3$

D $\quad 5$

Solution: B

The unit step functions show us that we have switches at $t = 2$ and $t = 5$. The first term, 9, just gives us the "starting point" of the function.

Topic: Second Shifting Theorem

Question: For $x \geq 0$, rewrite the function with a 2-unit shift to the right.

$$f(x) = e^{3x} - 5x^2 + 7x\sin(4x)$$

Answer choices:

A $\quad (e^{3x} - 5x^2 + 7x\sin(4x))u(x-2) \qquad x \geq 0$

B $\quad (e^{3(x-2)} - 5(x-2)^2 + 7(x-2)\sin(4(x-2)))u(x-2) \qquad x \geq 0$

C $\quad (e^{3(x-2)} - 5(x-2)^2 + 7(x-2)\sin(4(x-2)))u(x-2) \qquad x \geq 2$

D $\quad e^{3(x-2)} - 5(x-2)^2 + 7(x-2)\sin(4(x-2)) \qquad x \geq 2$

Solution: B

Because we want to shift the function 2 units to the right, we can start by substituting $c = 2$ into our formula.

$$f(x-c)u(x-c) = \begin{cases} 0 & 0 \le x < c \\ f(x-c) & x \ge c \end{cases}$$

$$f(x-2)u(x-2) = \begin{cases} 0 & 0 \le x < 2 \\ f(x-2) & x \ge 2 \end{cases}$$

Now we just need to replace $f(x-2)$, which is

$$f(x) = e^{3x} - 5x^2 + 7x\sin(4x)$$

$$f(x-2) = e^{3(x-2)} - 5(x-2)^2 + 7(x-2)\sin(4(x-2))$$

So the shifted function, for $x \ge 0$, is represented by

$$(e^{3(x-2)} - 5(x-2)^2 + 7(x-2)\sin(4(x-2)))u(x-2)$$

Topic: Second Shifting Theorem

Question: What is the Laplace transform of the function?

$$f(x) = \begin{cases} 0 & 0 \le t < 5 \\ (x-5)\sin(x-5) & t \ge 5 \end{cases}$$

Answer choices:

A $\mathscr{L}(f(x)) = e^{5s}\left(\dfrac{2s}{(s^2+1)^2}\right)$

B $\mathscr{L}(f(x)) = \dfrac{2s}{(s^2+1)^2}$

C $\mathscr{L}(f(x)) = e^{-5s}\left(\dfrac{2s}{(s^2+1)^2}\right)$

D $\mathscr{L}(f(x)) = e^{s}\left(\dfrac{2s}{(s^2+1)^2}\right)$

Solution: C

The function $f(x)$ is a shifted version of the function $g(x) = x \sin x$ for $x \geq 0$ by 5 units to the right. In other words $f(x) = g(x-5)u(x-5)$,

$$\mathscr{L}(g(t-c)u(t-c)) = e^{-cs}G(s)$$

$$\mathscr{L}(f(x)) = \mathscr{L}(g(x-5)u(x-5)) = e^{-5s}G(s)$$

where $G(x)$ is the Laplace transform of the function $g(x)$.

$$G(s) = \mathscr{L}(g(x)) = \mathscr{L}(x \sin x)$$

Using the transform formula,

$$\mathscr{L}(t \sin(at)) = \frac{2as}{(s^2 + a^2)^2}$$

we get

$$G(s) = \mathscr{L}(g(x)) = \frac{2s}{(s^2+1)^2}$$

Then the transform of $f(x)$ is

$$\mathscr{L}(f(x)) = e^{-5s}\left(\frac{2s}{(s^2+1)^2}\right)$$

Topic: Second Shifting Theorem

Question: Which function represents $f(t)$, shifted 7 units to the right, for $t \geq 0$?

Answer choices:

A $\quad f(t-7)u(t-7)$

B $\quad f(t)u(t-7)$

C $\quad f(t-7)u(t)$

D $\quad f(t)u(t)$

Solution: A

Because we want to shift the function 7 units to the right, we can start by substituting $c=7$ into the Second Shifting Theorem formula.

$$f(x-c)u(x-c)=\begin{cases} 0 & 0 \le x < c \\ f(x-c) & x \ge c \end{cases}$$

$$f(x-7)u(x-7)=\begin{cases} 0 & 0 \le x < 7 \\ f(x-7) & x \ge 7 \end{cases}$$

Topic: Laplace transforms of step functions

Question: Find the Laplace transform of the function $g(t)$.

$$g(t) = 5u_1(t) - 9u_3(t)$$

Answer choices:

A $\mathscr{L}(g(t)) = \frac{5e^{-s}}{s} - \frac{9e^{-3s}}{s}$

B $\mathscr{L}(g(t)) = \frac{e^{-s}}{s} - \frac{e^{-3s}}{s}$

C $\mathscr{L}(g(t)) = \frac{5e^{-s}}{s} + \frac{9e^{-3s}}{s}$

D $\mathscr{L}(g(t)) = \frac{e^{-s}}{s} + \frac{e^{-3s}}{s}$

Solution: A

We know that the Laplace transform of $u_c(t)$ is given by

$$\mathscr{L}(u_c(t)) = \frac{e^{-sc}}{s}$$

Therefore, the Laplace transform of $5u_1(t)$ must be

$$\frac{5e^{-s}}{s}$$

and the Laplace transform of $-9u_3(t)$ must be

$$-\frac{9e^{-3s}}{s}$$

Putting these together, the Laplace transform is

$$\mathscr{L}(g(t)) = \frac{5e^{-s}}{s} - \frac{9e^{-3s}}{s}$$

Topic: Laplace transforms of step functions

Question: Find the Laplace transform of the function $f(t)$.

$$f(t) = \begin{cases} 2 & 0 \le x < 1 \\ 5 & 1 \le x < 3 \\ -1 & x \ge 3 \end{cases}$$

Answer choices:

A $\mathscr{L}(f(t)) = 2 + 5e^{-s} - e^{-3s}$

B $\mathscr{L}(f(t)) = 2 + \dfrac{3e^{-s}}{s} - \dfrac{6e^{-3s}}{s}$

C $\mathscr{L}(f(t)) = \dfrac{2}{s} + \dfrac{5e^{-s}}{s} - \dfrac{e^{-3s}}{s}$

D $\mathscr{L}(f(t)) = \dfrac{2}{s} + \dfrac{3e^{-s}}{s} - \dfrac{6e^{-3s}}{s}$

Solution: D

Remember that a Heaviside function represents a switch, so we need one Heaviside function every time we make a switch in the value of $f(t)$. There are two switches, one from 2 to 5, and one from 5 to -1, so we'll need two Heaviside functions in order to express $f(t)$.

We write a Heaviside function as $u_c(t)$, where the switch occurs at $t = c$. The switches in this function $f(t)$ are at $t = 1$ and $t = 3$, which means the two Heaviside functions we'll need are $u_1(t)$, and $u_3(t)$.

When the first switch turns on at $t = 1$, the value of the function immediately shifts from 2 to 5 ($5 - 2 = 3$), so we can write

$$f(t) \approx 2 + 3u_1(t)$$

When the second switch turns on at $t = 5$, the value of the function immediately shifts from 5 to -1 ($-1 - 5 = -6$), so we can write

$$f(t) \approx 2 + 3u_1(t) - 6u_3(t)$$

We know that the Laplace transform of $u_c(t)$ is given by

$$\mathscr{L}(u_c(t)) = \frac{e^{-sc}}{s}$$

Therefore, the Laplace transform of $3u_1(t)$ must be

$$\frac{3e^{-s}}{s}$$

and the Laplace transform of $-6u_3(t)$ must be

$$-\frac{6e^{-3s}}{s}$$

Putting these together, the Laplace transform is

$$\mathscr{L}(f(t)) = \frac{2}{s} + \frac{3e^{-s}}{s} - \frac{6e^{-3s}}{s}$$

Topic: Laplace transforms of step functions

Question: Find the inverse Laplace transform.

$$G(s) = \frac{24e^{-5s}}{s^5}$$

Answer choices:

A $\mathscr{L}^{-1}(G(s)) = t^4u(t-5)$

B $\mathscr{L}^{-1}(G(s)) = (t-5)^4u(t-5)$

C $\mathscr{L}^{-1}(G(s)) = (t-5)^4$

D $\mathscr{L}^{-1}(G(s)) = (t-5)^4u(t)$

Solution: B

We'll start by pulling e^{-5s} out of the fraction, so that the function is then in the form $G(s) = e^{-5s}F(s)$.

$$G(s) = \frac{24e^{-5s}}{s^5}$$

$$G(s) = e^{-5s}\left(\frac{24}{s^5}\right)$$

$$F(s) = \frac{24}{s^5}$$

The inverse transform of $F(s)$ is

$$\mathscr{L}^{-1}(F(s)) = \mathscr{L}^{-1}\left(\frac{4!}{s^{4+1}}\right) = t^4$$

Which means the inverse transform of $G(s)$ is

$$\mathscr{L}^{-1}(G(s)) = u_5(t)f(t-5)$$

$$\mathscr{L}^{-1}(G(s)) = (t-5)^4u(t-5)$$

Topic: Step functions with initial value problems

Question: Solve the initial value problem, given $y(0) = 0$ and $y'(0) = 2$.

$$y'' - 3y' + 2y = e^{3t} - \sin(t-2)u(t-2)$$

Answer choices:

A $\quad Y(s) = \dfrac{1}{(s-3)(s-2)(s-1)} - \dfrac{e^{-2s}}{(s^2+1)(s-2)(s-1)} + \dfrac{2}{(s-2)(s-1)}$

B $\quad Y(s) = \dfrac{1}{(s-3)(s-2)(s-1)} - \dfrac{1}{(s^2+1)(s-2)(s-1)} + \dfrac{2}{(s-2)(s-1)}$

C $\quad Y(s) = \dfrac{1}{(s-3)} - \dfrac{e^{-2s}}{(s^2+1)} + \dfrac{2}{(s-2)(s-1)}$

D $\quad Y(s) = \dfrac{1}{(s-3)(s-2)(s-1)} - \dfrac{e^{-2s}}{(s^2+1)(s-2)(s-1)}$

Solution: A

Using the Laplace transform formula,

$$\mathscr{L}(u_c(t)g(t-c)) = e^{-cs}\mathscr{L}(g(t))$$

the transformed equation will be

$$\mathscr{L}(y'') - 3\mathscr{L}(y') + 2\mathscr{L}(y) = \mathscr{L}(e^{3t}) - \mathscr{L}(\sin(t-2)u(t-2))$$

$$s^2Y(s) - sy(0) - y'(0) - 3(sY(s) - y(0)) + 2Y(s) = \frac{1}{s-3} - e^{-2s}\left(\frac{1}{s^2+1}\right)$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 2$, then solve for $Y(s)$.

$$s^2Y(s) - s(0) - 2 - 3(sY(s) - 0) + 2Y(s) = \frac{1}{s-3} - e^{-2s}\left(\frac{1}{s^2+1}\right)$$

$$s^2Y(s) - 2 - 3sY(s) + 2Y(s) = \frac{1}{s-3} - e^{-2s}\left(\frac{1}{s^2+1}\right)$$

$$(s^2 - 3s + 2)Y(s) = \frac{1}{s-3} - e^{-2s}\left(\frac{1}{s^2+1}\right) + 2$$

$$Y(s) = \frac{1}{(s-3)(s^2-3s+2)} - \frac{e^{-2s}}{(s^2+1)(s^2-3s+2)} + \frac{2}{(s^2-3s+2)}$$

$$Y(s) = \frac{1}{(s-3)(s-2)(s-1)} - \frac{e^{-2s}}{(s^2+1)(s-2)(s-1)} + \frac{2}{(s-2)(s-1)}$$

Topic: Step functions with initial value problems

Question: Solve the initial value problem, given $y(0) = 0$ and $y'(0) = 1$.

$$y'' = 2u(t - 3)$$

Answer choices:

A $\quad y(t) = t^2 + t$

B $\quad y(t) = t^2 u(t - 3) + t$

C $\quad y(t) = 2(t - 3)^2 u(t - 3) + t$

D $\quad y(t) = (t - 3)^2 u(t - 3) + t$

Solution: D

Using the Laplace transform formula,

$$\mathscr{L}(u_c(t)) = \frac{e^{-cs}}{s}$$

the transformed equation will be

$$\mathscr{L}(y'') = 2\mathscr{L}(u(t-3))$$

$$s^2Y(s) - sy(0) - y'(0) = e^{-3s}\left(\frac{2}{s}\right)$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 1$, then solve for $Y(s)$.

$$s^2Y(s) - s(0) - 1 = e^{-3s}\left(\frac{2}{s}\right)$$

$$s^2Y(s) = e^{-3s}\left(\frac{2}{s}\right) + 1$$

$$Y(s) = e^{-3s}\left(\frac{2}{s^3}\right) + \frac{1}{s^2}$$

$$Y(s) = e^{-3s}\left(\frac{2}{s^3}\right) + \frac{1}{s^2}$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$, the inverse transform is

$$y(t) = (t-3)^2u(t-3) + t$$

Topic: Step functions with initial value problems

Question: Find $Y(s)$, given $y(0)=-1$ and $y'(0)=1$.

$$y''+6y=e^{2t}u(t-7)$$

Answer choices:

A $\quad Y(s)=e^{-7s}\left(\frac{1}{(s-2)(s^2+6)}\right)+\frac{1}{s^2+6}-\frac{s}{s^2+6}$

B $\quad Y(s)=e^{14-7s}\left(\frac{1}{(s-2)(s^2+6)}\right)+\frac{1}{s^2+6}-\frac{s}{s^2+6}$

C $\quad Y(s)=e^{14-7s}\left(\frac{1}{(s-2)(s^2+6)}\right)$

D $\quad Y(s)=e^{-7s}\left(\frac{1}{(s-2)(s^2+6)}\right)$

Solution: B

Remember that each function must be shifted by a proper amount. So, getting things set up for the proper shifts gives us,

$$g(t) = e^{2t}u(t-7)$$

$$g(t+c) = e^{2(t+7)}$$

$$g(t+c) = e^{14}e^{2t}$$

and

$$\mathscr{L}(u_c(t)g(t)) = e^{-cs}\mathscr{L}(g(t+c))$$

$$\mathscr{L}(u_7(t)g(t)) = e^{-7s}\mathscr{L}(e^{14}e^{2t})$$

$$\mathscr{L}(u_7(t)g(t)) = e^{-7s}e^{14}\mathscr{L}(e^{2t})$$

$$\mathscr{L}(u_7(t)g(t)) = e^{14-7s}\mathscr{L}(e^{2t})$$

Then the transformed equation will be

$$\mathscr{L}(y'') + 6\mathscr{L}(y) = \mathscr{L}(e^{2t}u(t-7))$$

$$s^2Y(s) - sy(0) - y'(0) + 6Y(s) = e^{14-7s}\left(\frac{1}{s-2}\right)$$

Substitute the initial conditions $y(0) = -1$ and $y'(0) = 1$, then solve for $Y(s)$.

$$s^2Y(s) - s(-1) - 1 + 6Y(s) = e^{14-7s}\left(\frac{1}{s-2}\right)$$

$$s^2Y(s) + s - 1 + 6Y(s) = e^{14-7s}\left(\frac{1}{s-2}\right)$$

$$(s^2 + 6)Y(s) = e^{14-7s}\left(\frac{1}{s-2}\right) + 1 - s$$

$$Y(s) = e^{14-7s}\left(\frac{1}{(s-2)(s^2+6)}\right) + \frac{1}{s^2+6} - \frac{s}{s^2+6}$$

Topic: The Dirac delta function

Question: Solve the initial value problem and find the general solution, given $y(0) = 0$ and $y'(0) = -2$.

$$y'' + 3y' + 2y = 7\delta(t - 9)$$

Answer choices:

A $y(t) = 7e^{9-t}u(t - 9) + 2e^{-3t}$

B $y(t) = 7e^{9-t}u(t - 9) - 2e^{-3t}$

C $y(t) = 7e^{9-t}u(t - 9) + 7e^{18-2t}u(t - 9) + 2e^{-t} - 2e^{-2t}$

D $y(t) = 7e^{9-t}u(t - 9) - 7e^{18-2t}u(t - 9) - 2e^{-t} + 2e^{-2t}$

Solution: D

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'') + 3\mathscr{L}(y') + 2\mathscr{L}(y) = 7\mathscr{L}(\delta(t-9))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

$$\mathscr{L}(y(t)) = Y(s)$$

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

we get

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 2Y(s) = 7e^{-9s}$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = -2$.

$$s^2Y(s) - s(0) - (-2) + 3sY(s) - 3(0) + 2Y(s) = 7e^{-9s}$$

$$s^2Y(s) + 2 + 3sY(s) + 2Y(s) = 7e^{-9s}$$

Solve for $Y(s)$.

$$(s^2 + 3s + 2)Y(s) = 7e^{-9s} - 2$$

$$Y(s) = \frac{7}{s^2 + 3s + 2}e^{-9s} - \frac{2}{s^2 + 3s + 2}$$

$$Y(s) = 7\frac{1}{(s+2)(s+1)}e^{-9s} - 2\frac{1}{(s+2)(s+1)}$$

Apply a partial fractions decomposition.

$$\frac{1}{(s+2)(s+1)} = \frac{A}{s+2} + \frac{B}{s+1}$$

To solve for A, remove the $s+2$ factor from the denominator of the left side, then substitute $s = -2$ into the left side.

$$\frac{1}{s+1} \to \frac{1}{-2+1} \to \frac{1}{-1} \to -1$$

To solve for B, remove the $s+1$ factor from the denominator of the left side, then substitute $s = -1$ into the left side.

$$\frac{1}{s+2} \to \frac{1}{-1+2} \to \frac{1}{1} \to 1$$

So the decomposition is

$$\frac{1}{(s+2)(s+1)} = \frac{-1}{s+2} + \frac{1}{s+1}$$

$$\frac{1}{(s+2)(s+1)} = \frac{1}{s+1} - \frac{1}{s+2}$$

Then the Laplace transform becomes

$$Y(s) = 7\left(\frac{1}{s+1} - \frac{1}{s+2}\right)e^{-9s} - 2\left(\frac{1}{s+1} - \frac{1}{s+2}\right)$$

$$Y(s) = 7\left(\frac{1}{s+1}\right)e^{-9s} - 7\left(\frac{1}{s+2}\right)e^{-9s} - 2\left(\frac{1}{s+1}\right) + 2\left(\frac{1}{s+2}\right)$$

Now that we've broken down the transform this way, we can apply the inverse transform to find the general solution to the second order nonhomogeneous equation.

$$y(t) = 7(e^{-(t-9)})u(t-9) - 7(e^{-2(t-9)})u(t-9) - 2e^{-t} + 2e^{-2t}$$

$$y(t) = 7e^{9-t}u(t-9) - 7e^{18-2t}u(t-9) - 2e^{-t} + 2e^{-2t}$$

Topic: The Dirac delta function

Question: Solve the initial value problem and find the general solution, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' + 5y = 2\delta(t-2) + u(t-2)$$

Answer choices:

A $\quad y(t) = \frac{1}{5}u(t-2)(\sin(\sqrt{5}(t-2)) - \cos(\sqrt{5}(t-2)))$

B $\quad y(t) = \frac{1}{5}u(t-2)(2\sqrt{5}\sin(\sqrt{5}(t-2)) - \cos(\sqrt{5}(t-2)) + 1)$

C $\quad y(t) = \frac{1}{5}u(t-2)(\sin(\sqrt{5}(t-2)) + \cos(\sqrt{5}(t-2)))$

D $\quad y(t) = \frac{1}{5}u(t-2)(2\sqrt{5}\sin(\sqrt{5}(t-2)) + \cos(\sqrt{5}(t-2)) - 1)$

Solution: B

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'') + 5\mathscr{L}(y) = 2\mathscr{L}(\delta(t-2)) + \mathscr{L}(u(t-2))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y(t)) = Y(s)$$

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

$$\mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

we get

$$s^2Y(s) - sy(0) - y'(0) + 5Y(s) = 2e^{-2s} + \frac{e^{-2s}}{s}$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$s^2Y(s) - s(0) - 0 + 5Y(s) = 2e^{-2s} + \frac{e^{-2s}}{s}$$

$$s^2Y(s) + 5Y(s) = 2e^{-2s} + \frac{e^{-2s}}{s}$$

Solve for $Y(s)$.

$$(s^2 + 5)Y(s) = 2e^{-2s} + \frac{e^{-2s}}{s}$$

$$Y(s) = \frac{2e^{-2s}}{s^2+5} + \frac{e^{-2s}}{s(s^2+5)}$$

$$Y(s) = 2\left(\frac{1}{s^2+5}\right)e^{-2s} + \frac{1}{s(s^2+5)}e^{-2s}$$

Apply a partial fractions decomposition.

$$\frac{1}{s(s^2+5)} = \frac{A}{s} + \frac{Bs+C}{s^2+5}$$

Expand the decomposition,

$$1 = A(s^2+5) + (Bs+C)s$$

$$1 = As^2 + 5A + Bs^2 + Cs$$

$$1 = (A+B)s^2 + (C)s + (5A)$$

then equate coefficients.

$$A + B = 0$$

$$C = 0$$

$$5A = 1$$

From these equations, we see $A = 1/5$, and then $B = -1/5$. Therefore, the decomposition is

$$\frac{1}{s(s^2+5)} = \frac{\frac{1}{5}}{s} + \frac{-\frac{1}{5}s+0}{s^2+5}$$

$$\frac{1}{s(s^2+5)} = \frac{1}{5}\left(\frac{1}{s}\right) - \frac{1}{5}\left(\frac{s}{s^2+5}\right)$$

Then the Laplace transform can be rewritten as

$$Y(s) = 2\left(\frac{1}{s^2+5}\right)e^{-2s} + \left(\frac{1}{5}\left(\frac{1}{s}\right) - \frac{1}{5}\left(\frac{s}{s^2+5}\right)\right)e^{-2s}$$

$$Y(s) = \frac{2\sqrt{5}}{5}\left(\frac{\sqrt{5}}{s^2+5}\right)e^{-2s} + \frac{1}{5}\left(\frac{1}{s}\right)e^{-2s} - \frac{1}{5}\left(\frac{s}{s^2+5}\right)e^{-2s}$$

Now that we've broken down the transform this way, we can apply the inverse transform to find the general solution to the second order nonhomogeneous equation.

$$y(t) = \frac{2\sqrt{5}}{5}\sin(\sqrt{5}(t-2))u(t-2) + \frac{1}{5}u(t-2) - \frac{1}{5}\cos(\sqrt{5}(t-2))u(t-2)$$

$$y(t) = \frac{1}{5}u(t-2)(2\sqrt{5}\sin(\sqrt{5}(t-2)) - \cos(\sqrt{5}(t-2)) + 1)$$

Topic: The Dirac delta function

Question: Solve the initial value problem and find the general solution, given $y(0) = 0$ and $y'(0) = 3$.

$$y'' + y' = \delta(t - 3)$$

Answer choices:

A $y(t) = u(t - 3) - e^{3-t}u(t - 3) - e^{-3t}$

B $y(t) = u(t - 3) + e^{-t}u(t - 3) + 3e^{-t} + 3$

C $y(t) = u(t - 3) - u(t - 3)e^{-(t-3)} + 3 - 3e^{-t}$

D $y(t) = u(t - 3) - e^{-t}u(t - 3) + e^{-3t}$

Solution: C

Apply the Laplace transform to both sides of the second order equation.

$$\mathscr{L}(y'') + \mathscr{L}(y') = \mathscr{L}(\delta(t-3))$$

Using formulas from our table of transforms,

$$\mathscr{L}(y''(t)) = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y'(t)) = sY(s) - y(0)$$

$$\mathscr{L}(y(t)) = Y(s)$$

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

we get

$$s^2Y(s) - sy(0) - y'(0) + (sY(s) - y(0)) = e^{-3s}$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 3$.

$$s^2Y(s) - s(0) - 3 + sY(s) - 0 = e^{-3s}$$

$$s^2Y(s) - 3 + sY(s) = e^{-3s}$$

Solve for $Y(s)$.

$$(s^2 + s)Y(s) = e^{-3s} + 3$$

$$Y(s) = e^{-3s}\frac{1}{s^2 + s} + \frac{3}{s^2 + s}$$

$$Y(s) = e^{-3s}\left(\frac{1}{s(s+1)}\right) + 3\left(\frac{1}{s(s+1)}\right)$$

Apply a partial fractions decomposition.

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

To solve for A, remove the s factor from the denominator of the left side, then substitute $s = 0$ into the left side.

$$\frac{1}{s+1} \to \frac{1}{0+1} \to \frac{1}{1} \to 1$$

To solve for B, remove the $s+1$ factor from the denominator of the left side, then substitute $s = -1$ into the left side.

$$\frac{1}{s} \to \frac{1}{-1} \to -1$$

So the decomposition is

$$\frac{1}{s(s+1)} = \frac{1}{s} + \frac{-1}{s+1}$$

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}$$

Then the Laplace transform becomes

$$Y(s) = e^{-3s}\left(\frac{1}{s} - \frac{1}{s+1}\right) + 3\left(\frac{1}{s} - \frac{1}{s+1}\right)$$

$$Y(s) = e^{-3s}\left(\frac{1}{s}\right) - e^{-3s}\left(\frac{1}{s+1}\right) + 3\left(\frac{1}{s}\right) - 3\left(\frac{1}{s+1}\right)$$

Now that we've broken down the transform this way, we can apply the inverse transform to find the general solution to the second order nonhomogeneous equation.

$$y(t) = u(t-3) - u(t-3)e^{-(t-3)} + 3 - 3e^{-t}$$

Topic: Convolution integrals

Question: Find the convolution of $f(t) = \sin t$ and $g(t) = e^{-t}$.

Answer choices:

A $\quad f(t) * g(t) = \frac{1}{2}(e^{-t} + \sin t - \cos t)$

B $\quad f(t) * g(t) = \frac{1}{2}(e^{-t} - \sin t - \cos t)$

C $\quad f(t) * g(t) = e^{-t} - 2\sin t$

D $\quad f(t) * g(t) = e^{-t} - 2\cos t$

Solution: A

To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions that we need for our convolution integral.

$$f(\tau)=\sin\tau$$

$$g(t-\tau)=e^{-(t-\tau)}=e^{\tau-t}$$

So the convolution integral gives

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t)*g(t)=\int_0^t \sin\tau(e^{\tau-t})\ d\tau$$

$$f(t)*g(t)=\int_0^t \sin\tau(e^{-t}e^{\tau})\ d\tau$$

$$f(t)*g(t)=e^{-t}\int_0^t e^{\tau}\sin\tau\ d\tau$$

We'll use integration by parts to evaluate the integral, letting

$$u=\sin\tau \qquad dv=e^{\tau}\ d\tau$$

$$du=\cos\tau\ d\tau \qquad v=e^{\tau}$$

Then the convolution is given by

$$f(t)*g(t)=e^{-t}\left[uv\Big|_0^t-\int_0^t v\ du\right]$$

$$f(t) * g(t) = e^{-t}\left[e^{\tau}\sin\tau\Big|_0^t - \int_0^t e^{\tau}\cos\tau\ d\tau\right]$$

We'll use integration by parts to evaluate the integral, letting

$u = \cos\tau$ $\quad$ $dv = e^{\tau}\ d\tau$

$du = -\sin\tau\ d\tau$ $\quad$ $v = e^{\tau}$

Then the convolution is given by

$$f(t) * g(t) = e^{-t}\left[e^{\tau}\sin\tau\Big|_0^t - \left(e^{\tau}\cos\tau\Big|_0^t + \int_0^t e^{\tau}\sin\tau\ d\tau\right)\right]$$

$$f(t) * g(t) = e^{-t}\left(e^{\tau}\sin\tau\Big|_0^t - e^{\tau}\cos\tau\Big|_0^t - \int_0^t e^{\tau}\sin\tau\ d\tau\right)$$

Integrate the remaining function, and evaluate over the interval.

$$f(t) * g(t) = e^{-t}\left(e^{t}\sin t - e^{t}\cos t + 1 - \int_0^t e^{\tau}\sin\tau\ d\tau\right)$$

$$f(t) * g(t) = \sin t - \cos t + e^{-t} - e^{-t}\int_0^t e^{\tau}\sin\tau\ d\tau$$

$$e^{-t}\int_0^t e^{\tau}\sin\tau\ d\tau = \sin t - \cos t + e^{-t} - e^{-t}\int_0^t e^{\tau}\sin\tau\ d\tau$$

$$2e^{-t}\int_0^t e^{\tau}\sin\tau\ d\tau = \sin t - \cos t + e^{-t}$$

$$e^{-t}\int_0^t e^{\tau}\sin\tau\ d\tau = \frac{\sin t - \cos t + e^{-t}}{2}$$

$$f(t) * g(t) = \sin t - \cos t + e^{-t} - \frac{\sin t - \cos t + e^{-t}}{2}$$

$$f(t) * g(t) = \frac{\sin t - \cos t + e^{-t}}{2}$$

Topic: Convolution integrals

Question: Find the convolution of $f(t) = t^3$ and $g(t) = e^t$.

Answer choices:

A $f(t) * g(t) = -t^3 + t + e^t$

B $f(t) * g(t) = -t^3 - 3t^2 - 6t - 6 + 6e^t$

C $f(t) * g(t) = -t^3 + t^2 - t + e^t$

D $f(t) * g(t) = -3t^3 - 3t^2 - t - 1 + 6e^t$

Solution: B

To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions that we need for our convolution integral.

$$f(\tau)=\tau^3$$

$$g(t-\tau)=e^{t-\tau}$$

So the convolution integral gives

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t)*g(t)=\int_0^t \tau^3e^{t-\tau}\ d\tau$$

$$f(t)*g(t)=\int_0^t \tau^3e^te^{-\tau}\ d\tau$$

$$f(t)*g(t)=e^t\int_0^t \tau^3e^{-\tau}\ d\tau$$

We'll use integration by parts to evaluate the integral, letting

$$u=\tau^3 \qquad dv=e^{-\tau}\ d\tau$$

$$du=3\tau^2\ d\tau \qquad v=-e^{-\tau}$$

Then the convolution is given by

$$f(t)*g(t)=e^t\left[uv\Big|_0^t-\int_0^t v\ du\right]$$

$$f(t) * g(t) = e^t \left[-\tau^3 e^{-\tau} \Big|_0^t + 3 \int_0^t e^{-\tau} \tau^2 \ d\tau \right]$$

We'll use integration by parts to evaluate the integral, letting

$$u = \tau^2 \qquad dv = e^{-\tau} \ d\tau$$

$$du = 2\tau \ d\tau \qquad v = -e^{-\tau}$$

Then the convolution is given by

$$f(t) * g(t) = e^t \left[-\tau^3 e^{-\tau} \Big|_0^t + 3 \left(-\tau^2 e^{-\tau} \Big|_0^t + 2 \int_0^t \tau e^{-\tau} \ d\tau \right) \right]$$

$$f(t) * g(t) = e^t \left[-\tau^3 e^{-\tau} \Big|_0^t - 3\tau^2 e^{-\tau} \Big|_0^t + 6 \int_0^t \tau e^{-\tau} \ d\tau \right]$$

We'll use integration by parts to evaluate the integral, letting

$$u = \tau \qquad dv = e^{-\tau} \ d\tau$$

$$du = d\tau \qquad v = -e^{-\tau}$$

Then the convolution is given by

$$f(t) * g(t) = e^t \left[-\tau^3 e^{-\tau} \Big|_0^t - 3\tau^2 e^{-\tau} \Big|_0^t + 6 \left(-\tau e^{-\tau} \Big|_0^t + \int_0^t e^{-\tau} \ d\tau \right) \right]$$

$$f(t) * g(t) = e^t \left(-\tau^3 e^{-\tau} \Big|_0^t - 3\tau^2 e^{-\tau} \Big|_0^t - 6\tau e^{-\tau} \Big|_0^t + 6 \int_0^t e^{-\tau} \ d\tau \right)$$

$$f(t) * g(t) = e^t \left(-\tau^3 e^{-\tau} - 3\tau^2 e^{-\tau} - 6\tau e^{-\tau} - 6e^{-\tau} \Big|_0^t \right)$$

Integrate the remaining function, and evaluate over the interval.

$$f(t) * g(t) = e^t(-t^3 e^{-t} - 3t^2 e^{-t} - 6te^{-t} - 6e^{-t} + 6)$$

$$f(t) * g(t) = -t^3 - 3t^2 - 6t - 6 + 6e^t$$

Topic: Convolution integrals

Question: Use a convolution integral to find the inverse transform.

$$H(s) = \left(\frac{s}{s^2+1}\right)\left(\frac{1}{(s-1)^2}\right)$$

Answer choices:

A $\quad f(t) * g(t) = te^t - \cos t$

B $\quad f(t) * g(t) = e^{2t} + t\sin t$

C $\quad f(t) * g(t) = \frac{1}{2}(te^t - \sin t)$

D $\quad f(t) * g(t) = e^t - \cos t$

Solution: C

In the function $H(s)$, we have

$$F(s) = \frac{s}{s^2 + 1}$$

$$G(s) = \frac{1}{(s-1)^2}$$

Using the Laplace transforms

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

$$\mathscr{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

we get

$$f(t) = \cos t$$

$$g(t) = te^t$$

To find the convolution, we'll first find $f(\tau)$ and $g(t-\tau)$, the two functions that we need for our convolution integral.

$$f(\tau) = \cos \tau$$

$$g(t-\tau) = (t-\tau)e^{t-\tau}$$

So the convolution integral gives

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t (t - \tau)(\cos \tau)e^{t-\tau} \ d\tau$$

$$f(t) * g(t) = \int_0^t (t \cos \tau)e^{t-\tau} \ d\tau - \int_0^t (\tau \cos \tau)e^{t-\tau} \ d\tau$$

$$f(t) * g(t) = te^t \int_0^t e^{-\tau} \cos \tau \ d\tau - e^t \int_0^t \tau e^{-\tau} \cos \tau \ d\tau$$

We'll use integration by parts to evaluate the second integral, letting

$$u = \tau \qquad dv = e^{-\tau} \cos \tau \ d\tau$$

$$du = d\tau \qquad v = \frac{1}{2}e^{-\tau}(\sin \tau - \cos \tau)$$

Then the convolution is given by

$$\int_0^t \tau e^{-\tau} \cos \tau \ d\tau = uv\Big|_0^t - \int_0^t v \ du$$

$$\int_0^t \tau e^{-\tau} \cos \tau \ d\tau = \frac{1}{2}\tau e^{-\tau}(\sin \tau - \cos \tau)\Big|_0^t - \int_0^t \frac{1}{2}e^{-\tau}(\sin \tau - \cos \tau) \ d\tau$$

$$\int_0^t \tau e^{-\tau} \cos \tau \ d\tau = \frac{1}{2}\tau e^{-\tau}(\sin \tau - \cos \tau)\Big|_0^t - \frac{1}{2}\int_0^t \frac{d}{d\tau}(-e^{-\tau} \sin \tau) \ d\tau$$

$$\int_0^t \tau e^{-\tau} \cos \tau \ d\tau = \frac{1}{2}te^{-t}(\sin t - \cos t) + \frac{1}{2}e^{-\tau} \sin \tau\Big|_0^t$$

$$\int_0^t \tau e^{-\tau} \cos \tau \ d\tau = \frac{1}{2}te^{-t}(\sin t - \cos t) + \frac{1}{2}e^{-t} \sin t$$

$$\int_0^t \tau e^{-\tau} \cos \tau \ d\tau = \frac{e^{-t}}{2}(t \sin t - t \cos t + \sin t)$$

We'll use integration by parts to evaluate the integral first integral, letting

$$u = \cos \tau \qquad dv = e^{-\tau} \ d\tau$$

$$du = -\sin \tau \ d\tau \qquad v = -e^{-\tau}$$

Then the convolution is given by

$$\int_0^t e^{-\tau} \cos \tau \ d\tau = uv\Big|_0^t - \int_0^t v \ du$$

$$\int_0^t e^{-\tau} \cos \tau \ d\tau = -e^{-\tau} \cos \tau\Big|_0^t - \int_0^t e^{-\tau} \sin \tau \ d\tau$$

We'll use integration by parts to evaluate the integral this remaining integral, letting

$$u = \sin \tau \qquad dv = e^{-\tau} \ d\tau$$

$$du = \cos \tau \ d\tau \qquad v = -e^{-\tau}$$

Then the value of the integral is

$$\int_0^t e^{-\tau} \cos \tau \ d\tau = -e^{-\tau} \cos \tau\Big|_0^t - \left(-e^{-\tau} \sin \tau + \int_0^t e^{-\tau} \cos \tau \ d\tau\right)$$

$$\int_0^t e^{-\tau} \cos \tau \ d\tau = -e^{-\tau} \cos \tau\Big|_0^t + e^{-\tau} \sin \tau\Big|_0^t - \int_0^t e^{-\tau} \cos \tau \ d\tau$$

$$2\int_0^t e^{-\tau}\cos\tau\ d\tau = -e^{-\tau}\cos\tau\Big|_0^t + e^{-\tau}\sin\tau\Big|_0^t$$

$$2\int_0^t e^{-\tau}\cos\tau\ d\tau = -e^{-t}\cos t + 1 + e^{-t}\sin t$$

$$\int_0^t e^{-\tau}\cos\tau\ d\tau = \frac{e^{-t}\sin t - e^{-t}\cos t + 1}{2}$$

Integrate the remaining function, and evaluate over the interval.

$$f(t) * g(t) = te^t\left(\frac{e^{-t}\sin t - e^{-t}\cos t + 1}{2}\right) - e^t\left(\frac{e^{-t}}{2}(t\sin t - t\cos t + \sin t)\right)$$

$$f(t) * g(t) = \frac{t\sin t - t\cos t + te^t}{2} - \frac{t\sin t - t\cos t + \sin t}{2}$$

$$f(t) * g(t) = \frac{te^t - \sin t}{2}$$

Topic: Convolution integrals for initial value problems

Question: Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=1$ and $y'(0)=1$.

$$y''+2y=g(t)$$

Answer choices:

A $\quad y(t)=\frac{\sqrt{2}}{2}\left(\sqrt{2}\cos(\sqrt{2}t)+\sin(\sqrt{2}t)+\int_0^t \sin(\sqrt{2}\tau)g(t+\tau)\ d\tau\right)$

B $\quad y(t)=\frac{\sqrt{2}}{2}\left(\sqrt{2}\cos(\sqrt{2}t)+\sin(\sqrt{2}t)+\int_0^t \sin(\sqrt{2}\tau)g(t-\tau)\ d\tau\right)$

C $\quad y(t)=\frac{\sqrt{2}}{2}\left(\sqrt{2}\cos(\sqrt{2}t)+\sin(\sqrt{2}t)+\int_0^t \cos(\sqrt{2}\tau)g(t+\tau)\ d\tau\right)$

D $\quad y(t)=\frac{\sqrt{2}}{2}\left(\sqrt{2}\cos(\sqrt{2}t)+\sin(\sqrt{2}t)+\int_0^t \cos(\sqrt{2}\tau)g(t-\tau)\ d\tau\right)$

Solution: B

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) + 2Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = 1$ and $y'(0) = 1$ in order to simplify the transform.

$$s^2Y(s) - s(1) - 1 + 2Y(s) = G(s)$$

$$s^2Y(s) - s - 1 + 2Y(s) = G(s)$$

We'll solve for $Y(s)$ by gathering all the $Y(s)$ terms on the left, and moving all other terms to the right, then factoring out a $Y(s)$.

$$s^2Y(s) + 2Y(s) = G(s) + s + 1$$

$$Y(s)(s^2 + 2) = G(s) + s + 1$$

$$Y(s) = \frac{s}{s^2 + 2} + \frac{1}{s^2 + 2} + G(s)\left(\frac{1}{s^2 + 2}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s. If we start with the first term, we can see its similarity to the formula for the transform of $\cos(at)$.

$$\mathcal{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

If we say $a = \sqrt{2}$, then the inverse transform of that first term is

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 + 2}\right) = \cos(\sqrt{2}t)$$

The second term from $Y(s)$ should remind us of the formula for the transform of $\sin(at)$.

$$\mathcal{L}(\sin(at)) = \frac{a}{s^2 + a^2}$$

Let's rewrite the second term so that it better matches the transform formula for $\sin(at)$.

$$\frac{1}{s^2 + 2}$$

$$\frac{\frac{\sqrt{2}}{\sqrt{2}}}{s^2 + (\sqrt{2})^2}$$

$$\frac{\frac{1}{\sqrt{2}}\sqrt{2}}{s^2 + (\sqrt{2})^2}$$

$$\frac{1}{\sqrt{2}}\left(\frac{\sqrt{2}}{s^2 + (\sqrt{2})^2}\right)$$

Now with $a = \sqrt{2}$, the inverse transform of that second term is

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+2}\right)=\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$$

Finding the inverse transform of the last term needs the convolution integral. We already know

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+2}\right)=\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$$

and we can say that the inverse transform of $G(s)$ is $g(t)$, so for our convolution integral, we'll use the functions

$$f(t)=\frac{1}{\sqrt{2}}\sin(\sqrt{2}t)$$

$$g(t)=g(t)$$

Plugging these into the convolution integral, we get

$$f(t)*g(t)=\int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t)*g(t)=\frac{1}{\sqrt{2}}\int_0^t \sin(\sqrt{2}\tau)g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$,

$$Y(s)=\frac{s}{s^2+2}+\frac{1}{s^2+2}+G(s)\left(\frac{1}{s^2+2}\right)$$

gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = \cos(\sqrt{2}t) + \frac{1}{\sqrt{2}}\sin(\sqrt{2}t) + \frac{1}{\sqrt{2}}\int_0^t \sin(\sqrt{2}\tau)g(t-\tau)\ d\tau$$

$$y(t) = \frac{1}{\sqrt{2}}\left(\sqrt{2}\cos(\sqrt{2}t) + \sin(\sqrt{2}t) + \int_0^t \sin(\sqrt{2}\tau)g(t-\tau)\ d\tau\right)$$

$$y(t) = \frac{\sqrt{2}}{2}\left(\sqrt{2}\cos(\sqrt{2}t) + \sin(\sqrt{2}t) + \int_0^t \sin(\sqrt{2}\tau)g(t-\tau)\ d\tau\right)$$

Topic: Convolution integrals for initial value problems

Question: Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=1$ and $y'(0)=1$.

$$3y''+y=g(t)$$

Answer choices:

A $\quad y(t)=\cos\left(\frac{\sqrt{3}}{3}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right)+\int_0^t\cos\left(\frac{\sqrt{3}}{3}\tau\right)g(t+\tau)\ d\tau$

B $\quad y(t)=\cos\left(\frac{\sqrt{3}}{3}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right)+\int_0^t\sin\left(\frac{\sqrt{3}}{3}\tau\right)g(t-\tau)\ d\tau$

C $\quad y(t)=\cos\left(\frac{\sqrt{3}}{3}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right)+\sqrt{3}\int_0^t\sin\left(\frac{\sqrt{3}}{3}\tau\right)g(t+\tau)\ d\tau$

D $\quad y(t)=\cos\left(\frac{\sqrt{3}}{3}t\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right)+\frac{\sqrt{3}}{3}\int_0^t\sin\left(\frac{\sqrt{3}}{3}\tau\right)g(t-\tau)\ d\tau$

Solution: D

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$3[s^2Y(s) - sy(0) - y'(0)] + Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = 1$ and $y'(0) = 1$ in order to simplify the transform.

$$3[s^2Y(s) - s(1) - 1] + Y(s) = G(s)$$

$$3s^2Y(s) - 3s - 3 + Y(s) = G(s)$$

We'll solve for $Y(s)$ by gathering all the $Y(s)$ terms on the left, and moving all other terms to the right, then factoring out a $Y(s)$.

$$3s^2Y(s) + Y(s) = G(s) + 3s + 3$$

$$Y(s)(3s^2 + 1) = G(s) + 3s + 3$$

$$Y(s) = 3\left(\frac{s}{3s^2+1}\right) + 3\left(\frac{1}{3s^2+1}\right) + G(s)\left(\frac{1}{3s^2+1}\right)$$

$$Y(s) = 3\left(\frac{\frac{1}{3}s}{s^2+\frac{1}{3}}\right) + 3\left(\frac{\frac{1}{3}}{s^2+\frac{1}{3}}\right) + G(s)\left(\frac{\frac{1}{3}}{s^2+\frac{1}{3}}\right)$$

$$Y(s) = \frac{s}{s^2 + \frac{1}{3}} + \frac{1}{s^2 + \frac{1}{3}} + \frac{1}{3}G(s)\left(\frac{1}{s^2 + \frac{1}{3}}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s. If we start with the first term, we can see its similarity to the formula for the transform of $\cos(at)$.

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2 + a^2}$$

If we say $a = \sqrt{1/3} = 1/\sqrt{3} = \sqrt{3}/3$, then the inverse transform of that first term is

$$\mathscr{L}^{-1}\left(\frac{s}{s^2 + \frac{1}{3}}\right) = \cos\left(\frac{\sqrt{3}}{3}t\right)$$

The second term from $Y(s)$ should remind us of the formula for the transform of $\sin(at)$.

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2 + a^2}$$

Let's rewrite the second term so that it better matches the transform formula for $\sin(at)$.

$$\frac{1}{s^2 + \frac{1}{3}}$$

$$\frac{\frac{\frac{\sqrt{3}}{3}}{\frac{\sqrt{3}}{3}}}{s^2+\left(\frac{\sqrt{3}}{3}\right)^2}$$

$$\frac{\frac{1}{\frac{\sqrt{3}}{3}}\left(\frac{\sqrt{3}}{3}\right)}{s^2+\left(\frac{\sqrt{3}}{3}\right)^2}$$

$$\sqrt{3}\left(\frac{\frac{\sqrt{3}}{3}}{s^2+\left(\frac{\sqrt{3}}{3}\right)^2}\right)$$

Now with $a=\sqrt{3}/3$, the inverse transform of that second term is

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+\frac{1}{3}}\right)=\sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right)$$

Finding the inverse transform of the last term needs the convolution integral. We already know

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+\frac{1}{3}}\right)=\sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right)$$

and we can say that the inverse transform of $G(s)$ is $g(t)$, so for our convolution integral, we'll use the functions

$$f(t) = \sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right)$$

$$g(t) = g(t)$$

Plugging these into the convolution integral, we get

$$f(t) * g(t) = \int_0^t f(t)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \sqrt{3}\int_0^t \sin\left(\frac{\sqrt{3}}{3}\tau\right)g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$,

$$Y(s) = \frac{s}{s^2+\frac{1}{3}} + \frac{1}{s^2+\frac{1}{3}} + \frac{1}{3}G(s)\left(\frac{1}{s^2+\frac{1}{3}}\right)$$

gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = \cos\left(\frac{\sqrt{3}}{3}t\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}}{3}t\right) + \frac{\sqrt{3}}{3}\int_0^t \sin\left(\frac{\sqrt{3}}{3}\tau\right)g(t-\tau)\ d\tau$$

Topic: Convolution integrals for initial value problems

Question: Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=1$ and $y'(0)=-2$.

$$2y''+5y=g(t)$$

Answer choices:

A $\quad y(t)=\cos\left(\frac{\sqrt{10}}{2}t\right)-\frac{2\sqrt{10}}{5}\sin\left(\frac{\sqrt{10}}{2}t\right)+\frac{\sqrt{10}}{5}\int_0^t \sin\left(\frac{\sqrt{10}}{2}\tau\right)g(t-\tau)\ d\tau$

B $\quad y(t)=\cos\left(\frac{\sqrt{10}}{2}t\right)-\frac{4\sqrt{10}}{5}\sin\left(\frac{\sqrt{10}}{2}t\right)+\frac{\sqrt{10}}{5}\int_0^t \sin\left(\frac{\sqrt{10}}{2}\tau\right)g(t-\tau)\ d\tau$

C $\quad y(t)=\cos\left(\frac{\sqrt{10}}{2}t\right)-\frac{2\sqrt{10}}{5}\sin\left(\frac{\sqrt{10}}{2}t\right)+\frac{\sqrt{10}}{10}\int_0^t \sin\left(\frac{\sqrt{10}}{2}\tau\right)g(t-\tau)\ d\tau$

D $\quad y(t)=\cos\left(\frac{\sqrt{10}}{2}t\right)-\frac{4\sqrt{10}}{5}\sin\left(\frac{\sqrt{10}}{2}t\right)+\frac{\sqrt{10}}{20}\int_0^t \sin\left(\frac{\sqrt{10}}{2}\tau\right)g(t-\tau)\ d\tau$

Solution: C

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$2[s^2Y(s) - sy(0) - y'(0)] + 5Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = 1$ and $y'(0) = -2$ in order to simplify the transform.

$$2[s^2Y(s) - s(1) - (-2)] + 5Y(s) = G(s)$$

$$2s^2Y(s) - 2s + 4 + 5Y(s) = G(s)$$

We'll solve for $Y(s)$ by gathering all the $Y(s)$ terms on the left, and moving all other terms to the right, then factoring out a $Y(s)$.

$$2s^2Y(s) + 5Y(s) = G(s) + 2s - 4$$

$$Y(s)(2s^2 + 5) = G(s) + 2s - 4$$

$$Y(s) = 2\left(\frac{s}{2s^2+5}\right) - 4\left(\frac{1}{2s^2+5}\right) + G(s)\left(\frac{1}{2s^2+5}\right)$$

$$Y(s) = 2\left(\frac{\frac{1}{2}s}{s^2+\frac{5}{2}}\right) - 4\left(\frac{\frac{1}{2}}{s^2+\frac{5}{2}}\right) + G(s)\left(\frac{\frac{1}{2}}{s^2+\frac{5}{2}}\right)$$

$$Y(s)=\frac{s}{s^2+\frac{5}{2}}-2\left(\frac{1}{s^2+\frac{5}{2}}\right)+\frac{1}{2}G(s)\left(\frac{1}{s^2+\frac{5}{2}}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s. If we start with the first term, we can see its similarity to the formula for the transform of $\cos(at)$.

$$\mathscr{L}(\cos(at))=\frac{s}{s^2+a^2}$$

If we say $a=\sqrt{5/2}$, then the inverse transform of that first term is

$$\mathscr{L}^{-1}\left(\frac{s}{s^2+\frac{5}{2}}\right)=\cos\left(\sqrt{\frac{5}{2}}t\right)$$

The second term from $Y(s)$ should remind us of the formula for the transform of $\sin(at)$.

$$\mathscr{L}(\sin(at))=\frac{a}{s^2+a^2}$$

Let's rewrite the second term so that it better matches the transform formula for $\sin(at)$.

$$\frac{1}{s^2+\frac{5}{2}}$$

$$\frac{\frac{\sqrt{\frac{5}{2}}}{\sqrt{\frac{5}{2}}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2}$$

$$\frac{\frac{1}{\sqrt{\frac{5}{2}}}\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2}$$

$$\sqrt{\frac{2}{5}}\left(\frac{\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2}\right)$$

Plug this rewritten term back into the equation for $Y(s)$.

$$Y(s) = \frac{s}{s^2 + \frac{5}{2}} - 2 \cdot \sqrt{\frac{2}{5}}\left(\frac{\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2}\right) + \frac{1}{2}G(s) \cdot \sqrt{\frac{2}{5}}\left(\frac{\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2}\right)$$

$$Y(s) = \frac{s}{s^2 + \frac{5}{2}} - \frac{2\sqrt{2}}{\sqrt{5}}\left(\frac{\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2}\right) + \frac{\sqrt{2}}{2\sqrt{5}}G(s)\left(\frac{\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2}\right)$$

Now with $a = \sqrt{5/2}$, the inverse transform of everything inside the parentheses of the second term is

$$\mathscr{L}^{-1}\left(\frac{\sqrt{\frac{5}{2}}}{s^2+\left(\sqrt{\frac{5}{2}}\right)^2}\right) = \sin\left(\sqrt{\frac{5}{2}}t\right)$$

Finding the inverse transform of the last term needs the convolution integral. We already know

$$\mathscr{L}^{-1}\left(\frac{\sqrt{\frac{5}{2}}}{s^2+\left(\sqrt{\frac{5}{2}}\right)^2}\right) = \sin\left(\sqrt{\frac{5}{2}}t\right)$$

and we can say that the inverse transform of $G(s)$ is $g(t)$, so for our convolution integral, we'll use the functions

$$f(t) = \sin\left(\sqrt{\frac{5}{2}}t\right)$$

$$g(t) = g(t)$$

Plugging these into the convolution integral, we get

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t \sin\left(\sqrt{\frac{5}{2}}\tau\right) g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$,

$$Y(s) = \frac{s}{s^2 + \frac{5}{2}} - \frac{2\sqrt{2}}{\sqrt{5}} \left(\frac{\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2} \right) + \frac{\sqrt{2}}{2\sqrt{5}} G(s) \left(\frac{\sqrt{\frac{5}{2}}}{s^2 + \left(\sqrt{\frac{5}{2}}\right)^2} \right)$$

gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = \cos\left(\sqrt{\frac{5}{2}}t\right) - \frac{2\sqrt{2}}{\sqrt{5}} \sin\left(\sqrt{\frac{5}{2}}t\right) + \frac{\sqrt{2}}{2\sqrt{5}} \int_0^t \sin\left(\sqrt{\frac{5}{2}}\tau\right) g(t-\tau)\ d\tau$$

$$y(t) = \cos\left(\frac{\sqrt{10}}{2}t\right) - \frac{2\sqrt{10}}{5} \sin\left(\frac{\sqrt{10}}{2}t\right) + \frac{\sqrt{10}}{10} \int_0^t \sin\left(\frac{\sqrt{10}}{2}\tau\right) g(t-\tau)\ d\tau$$

Topic: Matrix basics

Question: Find the determinant of the matrix.

$$A = \begin{bmatrix} 3 & 6 & 0 \\ 2 & 6 & 4 \\ 0 & 5 & 7 \end{bmatrix}$$

Answer choices:

A $|A| = 0$

B $|A| = -6$

C $|A| = -12$

D $|A| = -18$

Solution: D

To find the determinant of matrix A, we'll break the 3×3 determinant into 2×2 determinants. We'll use the first row as coefficients, remembering to alternate signs, $+$, $-$, $+$.

$$\begin{vmatrix} 3 & 6 & 0 \\ 2 & 6 & 4 \\ 0 & 5 & 7 \end{vmatrix}$$

$$3\begin{vmatrix} 6 & 4 \\ 5 & 7 \end{vmatrix} - 6\begin{vmatrix} 2 & 4 \\ 0 & 7 \end{vmatrix} + 0\begin{vmatrix} 2 & 6 \\ 0 & 5 \end{vmatrix}$$

$$3\begin{vmatrix} 6 & 4 \\ 5 & 7 \end{vmatrix} - 6\begin{vmatrix} 2 & 4 \\ 0 & 7 \end{vmatrix}$$

Simplify the 2×2 determinants.

$$3[(6)(7) - (4)(5)] - 6[(2)(7) - (4)(0)]$$

$$3(42 - 20) - 6(14)$$

$$66 - 84$$

$$-18$$

Topic: Matrix basics

Question: Find any Eigenvalues of the matrix.

$$B = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Answer choices:

A $\lambda = 4$

B $\lambda = -4, 4$

C $\lambda = 0, 4$

D $\lambda = -4, 0, 4$

Solution: A

To find the Eigenvalues of matrix B, we'll start by finding $B - \lambda I$.

$$B - \lambda I = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$B - \lambda I = \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$B - \lambda I = \begin{bmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{bmatrix}$$

Then we'll find the determinant of this matrix, using the top row as coefficients, and remembering to alternate signs, $+$, $-$, $+$.

$$|B - \lambda I| = \begin{vmatrix} 4-\lambda & 1 & 0 \\ 0 & 4-\lambda & 1 \\ 0 & 0 & 4-\lambda \end{vmatrix}$$

$$|B - \lambda I| = (4-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} - 1\begin{vmatrix} 0 & 1 \\ 0 & 4-\lambda \end{vmatrix} + 0\begin{vmatrix} 0 & 4-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|B - \lambda I| = (4-\lambda)\begin{vmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{vmatrix} - \begin{vmatrix} 0 & 1 \\ 0 & 4-\lambda \end{vmatrix}$$

$$|B - \lambda I| = (4-\lambda)[(4-\lambda)(4-\lambda) - (1)(0)] - [(0)(4-\lambda) - (1)(0)]$$

$$|B - \lambda I| = (4-\lambda)(4-\lambda)(4-\lambda)$$

Eigenvalues exist where $|B - \lambda I| = 0$.

$$(4-\lambda)(4-\lambda)(4-\lambda)=0$$

Because the equation is true for $\lambda = 4$, this is an Eigenvalue of the matrix.

Topic: Matrix basics

Question: Put the matrix into reduced row-echelon form.

$$L = \begin{bmatrix} 2 & 4 & 8 \\ 5 & 3 & 2 \\ 3 & 0 & 5 \end{bmatrix}$$

Answer choices:

A $\quad L = \begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & 18/7 \\ 0 & 0 & 59/7 \end{bmatrix}$

B $\quad L = \begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 0 & 59/7 \\ 0 & 1 & 18/7 \end{bmatrix}$

C $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

D $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Solution: D

Following the Gauss-Jordan elimination algorithm, we'll multiply through the first row by $1/2$ to change the leading entry in that row to 1.

$$\begin{bmatrix} 2(1/2) & 4(1/2) & 8(1/2) \\ 5 & 3 & 2 \\ 3 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 5 & 3 & 2 \\ 3 & 0 & 5 \end{bmatrix}$$

To change the 5 in the first column to a 0, we'll subtract 5 multiples of the first row from the second row, and use the result to replace the second row, $R_2 - 5R_1 \rightarrow R_2$.

$$\begin{bmatrix} 1 & 2 & 4 \\ 5-5(1) & 3-5(2) & 2-5(4) \\ 3 & 0 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & -7 & -18 \\ 3 & 0 & 5 \end{bmatrix}$$

To change the 3 in the first column to a 0, we'll subtract 3 multiples of the first row from the third row, and use the result to replace the third row, $R_3 - 3R_1 \rightarrow R_3$.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & -7 & -18 \\ 3-3(1) & 0-3(2) & 5-3(4) \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & -7 & -18 \\ 0 & -6 & -7 \end{bmatrix}$$

Multiply through the second row by $-1/7$ to change the leading entry in that row to 1.

$$\begin{bmatrix} 1 & 2 & 4 \\ 0(-1/7) & -7(-1/7) & -18(-1/7) \\ 0 & -6 & -7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 18/7 \\ 0 & -6 & -7 \end{bmatrix}$$

To change the 2 in the second column to a 0, we'll subtract 2 multiples of the second row from the first row, and use the result to replace the first row, $R_1 - 2R_2 \to R_1$.

$$\begin{bmatrix} 1-2(0) & 2-2(1) & 4-2(18/7) \\ 0 & 1 & 18/7 \\ 0 & -6 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & 18/7 \\ 0 & -6 & -7 \end{bmatrix}$$

To change the -6 in the second column to a 0, we'll add 6 multiples of the second row to the third row, and use the result to replace the third row, $R_3 + 6R_2 \to R_3$.

$$\begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & 18/7 \\ 0+6(0) & -6+6(1) & -7+6(18/7) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & 18/7 \\ 0 & 0 & 59/7 \end{bmatrix}$$

Multiply through the third row by $7/59$ to change the leading entry in that row to 1.

$$\begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & 18/7 \\ 0(7/59) & 0(7/59) & 59/7(7/59) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8/7 \\ 0 & 1 & 18/7 \\ 0 & 0 & 1 \end{bmatrix}$$

To change the $-8/7$ in the third column to a 0, we'll add $8/7$ multiples of the third row to the first row, and use the result to replace the first row, $R_1 + (8/7)R_3 \to R_1$.

$$\begin{bmatrix} 1+(8/7)(0) & 0+(8/7)(0) & -8/7+(8/7)(1) \\ 0 & 1 & 18/7 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 18/7 \\ 0 & 0 & 1 \end{bmatrix}$$

To change the $18/7$ in the third column to a 0, we'll subtract $88/7$ multiples of the third row from the second row, and use the result to replace the second row, $R_2 - (18/7)R_3 \to R_2$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 - (18/7)(0) & 1 - (18/7)(0) & 18/7 - (18/7)(1) \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topic: Building systems

Question: Rewrite the system of differential equations in matrix form.

$$x_1' = x_2 - 3x_3$$

$$x_2' = -2x_1 + x_3$$

$$x_3' = 5x_2 - x_1 + 4x_3$$

Answer choices:

A $\quad \vec{x}' = \begin{bmatrix} 0 & 1 & -3 \\ -2 & 0 & 1 \\ -1 & 5 & 4 \end{bmatrix} \vec{x}$

B $\quad \vec{x}' = \begin{bmatrix} 0 & 1 & -3 \\ 2 & 0 & 1 \\ 5 & -1 & 4 \end{bmatrix} \vec{x}$

C $\quad \vec{x}' = \begin{bmatrix} 0 & 1 & 3 \\ 2 & 0 & 1 \\ 1 & 5 & 4 \end{bmatrix} \vec{x}$

D $\quad \vec{x}' = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & 5 \\ -3 & 1 & 4 \end{bmatrix} \vec{x}$

Solution: A

To write the system in matrix form $\vec{x}' = A\vec{x}$, we define the vector $\vec{x}'$, the matrix A, and the vector $\vec{x}$ as

$$\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 & -3 \\ -2 & 0 & 1 \\ -1 & 5 & 4 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Then we can write the system in matrix form.

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & -3 \\ -2 & 0 & 1 \\ -1 & 5 & 4 \end{bmatrix} \vec{x}$$

Topic: Building systems

Question: Convert the fourth order linear differential equation into a system of differential equations.

$$y^{(4)} - y''' + y'' - y' + 3e^t = 0$$

Answer choices:

A $$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3e^t \end{bmatrix}$$

B $$\vec{x}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 3e^t \end{bmatrix}$$

C $$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3e^t \end{bmatrix}$$

D $$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3e^t \end{bmatrix}$$

Solution: D

We'll start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

$$x_3(t) = y''(t)$$

$$x_4(t) = y'''(t)$$

Now we need to solve the original differential equation for the fourth derivative, $y^{(4)}(t)$.

$$y^{(4)} - y''' + y'' - y' + 3e^t = 0$$

$$y^{(4)} = y''' - y'' + y' - 3e^t$$

Then if we take the derivatives of the equations for $x_1(t)$, $x_2(t)$, $x_3(t)$, and $x_4(t)$, we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = x_3(t)$$

$$x_3'(t) = y'''(t) = x_4(t)$$

$$x_4'(t) = y^{(4)}(t) = y''' - y'' + y' - 3e^t = x_4(t) - x_3(t) + x_2(t) - 3e^t$$

Simplifying these equations gives us a system of equations that's equivalent to the original fourth order differential equation.

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = x_4$$

$$x_4' = x_2 - x_3 + x_4 - 3e^t$$

And if we wanted to write this nonhomogeneous system as a matrix equation, we'd get

$$\vec{x}' = A\vec{x} + F$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3e^t \end{bmatrix}$$

Topic: Building systems

Question: Convert the third order linear differential equation into a system of differential equations in the form $\vec{x}' = A\vec{x} + F$, given $\vec{x}'(0) = B$.

$$y''' - 2y'' + (\cos t)y' - 3y - \sin t = 0$$

$$y(0) = 0,\ y'(0) = 1,\ y''(0) = 3$$

Answer choices:

A $\quad \vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & \cos t & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix},\ \vec{x}'(0) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$

B $\quad \vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -\cos t & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix},\ \vec{x}'(0) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

C $\quad \vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & -\cos t & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix},\ \vec{x}'(0) = \begin{bmatrix} 0 \\ -1 \\ -3 \end{bmatrix}$

D $\quad \vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & \cos t & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ -\sin t \end{bmatrix},\ \vec{x}'(0) = \begin{bmatrix} -3 \\ 0 \\ -1 \end{bmatrix}$

Solution: B

We'll start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

$$x_3(t) = y''(t)$$

Now we need to solve the original differential equation for $y'''(t)$.

$$y''' - 2y'' + (\cos t)y' - 3y - \sin t = 0$$

$$y''' = 2y'' - (\cos t)y' + 3y + \sin t$$

Then if we take the derivatives of the equations for $x_1(t)$, $x_2(t)$, and $x_3(t)$, we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = x_3(t)$$

$$x_3'(t) = y'''(t) = 2y'' - (\cos t)y' + 3y + \sin t = 2x_3(t) - (\cos t)x_2(t) + 3x_1(t) + \sin t$$

Simplifying these equations gives us a system of equations that's equivalent to the original third order differential equation.

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3x_1 - (\cos t)x_2 + 2x_3 + \sin t$$

Rewrite the initial conditions.

$$y(0) = 0 \qquad x_1(0) = 0$$

$$y'(0) = 1 \qquad x_2(0) = 1$$

$$y''(0) = 3 \qquad x_3(0) = 3$$

And if we wanted to write this nonhomogeneous system as a matrix equation, we'd get

$$\vec{x}' = A\vec{x} + F \qquad \vec{x}'(0) = B$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -\cos t & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix} \qquad \vec{x}'(0) = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$$

Topic: Solving systems

Question: Which system of differential equations has a coefficient matrix that gives distinct real eigenvalues?

Answer choices:

A $x_1' = 2x_2$ and $x_2' = 4x_2 - 2x_1$

B $x_1' = x_1 + 3x_2$ and $x_2' = x_2 - 3x_1$

C $x_1' = x_1 + 4x_2$ and $x_2' = 2x_1 + 3x_2$

D $x_1' = 6x_1 - 3x_2$ and $x_2' = 3x_1$

Solution: C

The Eigenvalues of the system in answer choice A are given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} -\lambda & 2 \\ -2 & 4-\lambda \end{vmatrix} = 0$$

$$-\lambda(4 - \lambda) - 2(-2) = 0$$

$$\lambda^2 - 4\lambda + 4 = 0$$

$$(\lambda - 2)^2 = 0$$

$$\lambda_1 = \lambda_2 = 2$$

The Eigenvalues of the system in answer choice B are given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 3 \\ -3 & 1-\lambda \end{vmatrix} = 0$$

$$(1 - \lambda)(1 - \lambda) - 3(-3) = 0$$

$$(1 - \lambda)^2 + 9 = 0$$

$$\lambda_{1,2} = 1 \pm 3i$$

The Eigenvalues of the system in answer choice C are given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(3-\lambda) - 8 = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

$$(\lambda + 1)(\lambda - 5) = 0$$

$$\lambda_1 = -1, 5$$

The Eigenvalues of the system in answer choice D are given by

$$|A - \lambda I| = 0$$

$$\begin{vmatrix} 6-\lambda & -3 \\ 3 & 0-\lambda \end{vmatrix} = 0$$

$$(6-\lambda)(0-\lambda) - (-3)(3) = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)^2 = 0$$

$$\lambda_1 = \lambda_2 = 3$$

The only coefficient matrix that gives distinct real roots is the coefficient matrix for the system in answer choice C.

Topic: Solving systems

Question: Determine which vector satisfies the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \vec{x}$$

Answer choices:

A $\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$

B $\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-2t}$

C $\begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$

D $\begin{bmatrix} 2 \\ -1 \end{bmatrix} e^{t}$

Solution: A

Plugging the vector from answer choice A and its derivative into the matrix equation that represents the system shows that the vector satisfies the system.

$$\begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} + 2e^{2t} \\ -2e^{2t} + 4e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} 4e^{2t} \\ 2e^{2t} \end{bmatrix}$$

Plugging the vector from answer choice B and its derivative into the matrix equation that represents the system shows that the vector is not a solution to the system.

$$\begin{bmatrix} 2e^{-2t} \\ -4e^{-2t} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} -e^{-2t} \\ 2e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{-2t} \\ -4e^{-2t} \end{bmatrix} = \begin{bmatrix} -e^{-2t} + 4e^{-2t} \\ e^{-2t} + 8e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{-2t} \\ -4e^{-2t} \end{bmatrix} \neq \begin{bmatrix} 3e^{-2t} \\ 9e^{-2t} \end{bmatrix}$$

Plugging the vector from answer choice C and its derivative into the matrix equation that represents the system shows that the vector is not a solution to the system.

$$\begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} e^{2t} + 4e^{2t} \\ -e^{2t} + 8e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{2t} \\ 4e^{2t} \end{bmatrix} \neq \begin{bmatrix} 5e^{2t} \\ 7e^{2t} \end{bmatrix}$$

Plugging the vector from answer choice D and its derivative into the matrix equation that represents the system shows that the vector is not a solution to the system.

$$\begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix} = \begin{bmatrix} 2e^{t} - 2e^{t} \\ -2e^{t} - 4e^{t} \end{bmatrix}$$

$$\begin{bmatrix} 2e^{t} \\ -e^{t} \end{bmatrix} \neq \begin{bmatrix} 0 \\ -6e^{t} \end{bmatrix}$$

Topic: Solving systems

Question: Which vector set could be a fundamental set of solutions to a system of differential equations?

Answer choices:

A $\begin{bmatrix}-2\\3\end{bmatrix}e^{-5t}$ and $\begin{bmatrix}2\\3\end{bmatrix}e^{7t}$

B $\begin{bmatrix}1\\2\end{bmatrix}e^{-5t}$ and $\begin{bmatrix}-2\\-4\end{bmatrix}e^{7t}$

C $\begin{bmatrix}-1\\2\end{bmatrix}e^{t}$ and $\begin{bmatrix}3\\-6\end{bmatrix}e^{2t}$

D $\begin{bmatrix}e^{3t}\\-4e^{3t}\end{bmatrix}$ and $\begin{bmatrix}-2e^{t}\\8e^{t}\end{bmatrix}$

Solution: A

If the Wronskian of a vector set is non-zero, then the vectors are linearly independent and could therefore be a fundamental set of solutions to some system of differential equations.

The Wronskian of the vector set in answer choice A is non-zero.

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} -2e^{-5t} & 2e^{7t} \\ 3e^{-5t} & 3e^{7t} \end{vmatrix}$$

$$W(\vec{x}_1, \vec{x}_2) = -2e^{-5t}(3e^{7t}) - 2e^{7t}(3e^{-5t})$$

$$W(\vec{x}_1, \vec{x}_2) = -6e^{2t} - 6e^{2t}$$

$$W(\vec{x}_1, \vec{x}_2) = -12e^{2t}$$

$$W(\vec{x}_1, \vec{x}_2) \neq 0$$

The Wronskian of the vector set in answer choice B is zero.

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} e^{-5t} & -2e^{7t} \\ 2e^{-5t} & -4e^{7t} \end{vmatrix}$$

$$W(\vec{x}_1, \vec{x}_2) = -4e^{2t} - (-2e^{7t})(2e^{-5t})$$

$$W(\vec{x}_1, \vec{x}_2) = 0$$

The Wronskian of the vector set in answer choice C is zero.

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} -e^{t} & 3e^{2t} \\ 2e^{t} & -6e^{2t} \end{vmatrix}$$

$$W(\vec{x}_1, \vec{x}_2) = 6e^{3t} - 6e^{3t}$$

$$W(\vec{x}_1, \vec{x}_2) = 0$$

The Wronskian of the vector set in answer choice D is zero.

$$W(\vec{x}_1, \vec{x}_2) = \begin{vmatrix} e^{3t} & -2e^t \\ -4e^{3t} & 8e^t \end{vmatrix}$$

$$W(\vec{x}_1, \vec{x}_2) = 8e^{4t} - 8e^{4t}$$

$$W(\vec{x}_1, \vec{x}_2) = 0$$

Topic: Distinct real Eigenvalues

Question: Find the solution to the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{4t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t}$

Solution: B

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

and the determinant $|A - \lambda I|$ is

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)^2 - 9$$

$$|A - \lambda I| = \lambda^2 - 2\lambda - 8$$

Solve the characteristic equation to find the Eigenvalues.

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda + 2)(\lambda - 4) = 0$$

$$\lambda = -2, 4$$

Then for these Eigenvalues, $\lambda_1 = -2$ and $\lambda_2 = 4$, we find

$$A-(-2)I=\begin{bmatrix}1-(-2) & 3\\ 3 & 1-(-2)\end{bmatrix}$$

$$A-(-2)I=\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}$$

and

$$A-4I=\begin{bmatrix}1-4 & 3\\ 3 & 1-4\end{bmatrix}$$

$$A-4I=\begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix} \qquad \begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix}$$

$$\begin{bmatrix}3 & 3\\ 0 & 0\end{bmatrix} \qquad \begin{bmatrix}-3 & 3\\ 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix} \qquad \begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$$

If we turn these back into systems of equations (in both cases we only need to consider the equation that we get from the first row of each matrix), we get

$$k_1+k_2=0 \qquad k_1-k_2=0$$

$$k_1=-k_2 \qquad k_1=k_2$$

From the first system, we'll choose $k_2 = -1$, which results in $k_1 = 1$. And from the second system, we'll choose $k_2 = 1$, which results in $k_1 = 1$.

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then the solutions to the system are

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$

Therefore, the general solution to the homogeneous system will be

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$

Topic: Distinct real Eigenvalues

Question: Find the solution to the system of differential equations.

$$x_1' = 3x_1 + 2x_2$$

$$x_2' = 7x_1 - 2x_2$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} e^{5t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-4t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} e^{-4t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$

Solution: D

The coefficient matrix is

$$A = \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix}$$

and the determinant $|A - \lambda I|$ is

$$|A - \lambda I| = \left| \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 3 & 2 \\ 7 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & 2 \\ 7 & -2 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (3 - \lambda)(-2 - \lambda) - 7(2)$$

$$|A - \lambda I| = \lambda^2 - \lambda - 20$$

Solve the characteristic equation to find the Eigenvalues.

$$\lambda^2 - \lambda - 20 = 0$$

$$(\lambda - 5)(\lambda + 4) = 0$$

$$\lambda = -4, 5$$

Then for these Eigenvalues, $\lambda_1 = -4$ and $\lambda_2 = 5$, we find

$$A-(-4)I=\begin{bmatrix}3-(-4) & 2 \\ 7 & -2-(-4)\end{bmatrix}$$

$$A-(-4)I=\begin{bmatrix}7 & 2 \\ 7 & 2\end{bmatrix}$$

and

$$A-5I=\begin{bmatrix}3-5 & 2 \\ 7 & -2-5\end{bmatrix}$$

$$A-5I=\begin{bmatrix}-2 & 2 \\ 7 & -7\end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix}7 & 2 \\ 7 & 2\end{bmatrix} \qquad \begin{bmatrix}-2 & 2 \\ 7 & -7\end{bmatrix}$$

$$\begin{bmatrix}7 & 2 \\ 0 & 0\end{bmatrix} \qquad \begin{bmatrix}1 & -1 \\ 7 & -7\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{2}{7} \\ 0 & 0\end{bmatrix} \qquad \begin{bmatrix}1 & -1 \\ 0 & 0\end{bmatrix}$$

If we turn these back into systems of equations (in both cases we only need to consider the equation that we get from the first row of each matrix), we get

$$k_1+\frac{2}{7}k_2=0 \qquad k_1-k_2=0$$

$$k_1 = -\frac{2}{7}k_2 \qquad\qquad k_1 = k_2$$

From the first system, we'll choose $k_2 = -7$, which results in $k_1 = 2$. And from the second system, we'll choose $k_2 = 1$, which results in $k_1 = 1$.

$$\vec{k_1} = \begin{bmatrix} 2 \\ -7 \end{bmatrix} \qquad\qquad \vec{k_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} \qquad\qquad \vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{-4t} \qquad\qquad \vec{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

Therefore, the general solution to the homogeneous system will be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ -7 \end{bmatrix} e^{-4t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{5t}$$

Topic: Distinct real Eigenvalues

Question: Find the solution to the system of differential equations.

$$x_1' = x_1 + 6x_2$$

$$x_2' = 4x_1 - x_2$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{5t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{-5t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{5t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{5t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} te^{5t}$

Solution: A

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 6 \\ 4 & -1 \end{bmatrix}$$

and the determinant $|A - \lambda I|$ is

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 6 \\ 4 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 6 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 6 \\ 4 & -1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)(-1-\lambda) - (6)(4)$$

$$|A - \lambda I| = \lambda^2 - 25$$

Solve the characteristic equation to find the Eigenvalues.

$$\lambda^2 - 25 = 0$$

$$(\lambda - 5)(\lambda + 5) = 0$$

$$\lambda = -5, 5$$

Then for these Eigenvalues, $\lambda_1 = -5$ and $\lambda_2 = 5$, we find

$$A - (-5)I = \begin{bmatrix} 1-(-5) & 6 \\ 4 & -1-(-5) \end{bmatrix}$$

$$A - (-5)I = \begin{bmatrix} 6 & 6 \\ 4 & 4 \end{bmatrix}$$

and

$$A - 5I = \begin{bmatrix} 1-5 & 6 \\ 4 & -1-5 \end{bmatrix}$$

$$A - 5I = \begin{bmatrix} -4 & 6 \\ 4 & -6 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} 6 & 6 \\ 4 & 4 \end{bmatrix} \qquad \begin{bmatrix} -4 & 6 \\ 4 & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 4 & 4 \end{bmatrix} \qquad \begin{bmatrix} -4 & 6 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations (in both cases we only need to consider the equation that we get from the first row of each matrix), we get

$$k_1 + k_2 = 0 \qquad k_1 - \frac{3}{2}k_2 = 0$$

$$k_1 = -k_2 \qquad\qquad k_1 = \frac{3}{2}k_2$$

From the first system, we'll choose $k_2 = -1$, which results in $k_1 = 1$. And from the second system, we'll choose $k_2 = 2$, which results in $k_1 = 3$.

$$\vec{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad\qquad \vec{k_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} \qquad\qquad \vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t} \qquad\qquad \vec{x_2} = \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{5t}$$

Therefore, the general solution to the homogeneous system will be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 3 \\ 2 \end{bmatrix} e^{5t}$$

Topic: Equal real Eigenvalues with multiplicity two

Question: Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} \vec{x}$$

Answer choices:

A $\vec{x} = c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3 \left(\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} te^{5t} + \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} e^{5t} \right)$

B $\vec{x} = c_1 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} te^{5t}$

C $\vec{x} = c_1 \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^{5t} + c_3 \left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} te^{5t} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{5t} \right)$

D $\vec{x} = c_1 \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^{5t} + c_3 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} te^{5t}$

Solution: C

We'll need to find the determinant $|A - \lambda I|$,

$$|A - \lambda I| = \left| \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 5 & -4 & 0 \\ 1 & 0 & 2 \\ 0 & 2 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 5-\lambda & -4 & 0 \\ 1 & -\lambda & 2 \\ 0 & 2 & 5-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (5-\lambda) \begin{vmatrix} -\lambda & 2 \\ 2 & 5-\lambda \end{vmatrix} - (-4) \begin{vmatrix} 1 & 2 \\ 0 & 5-\lambda \end{vmatrix} + 0 \begin{vmatrix} 1 & -\lambda \\ 0 & 2 \end{vmatrix}$$

$$|A - \lambda I| = (5-\lambda)[(-\lambda)(5-\lambda) - (2)(2)] + 4[(1)(5-\lambda) - (2)(0)]$$

$$|A - \lambda I| = (5-\lambda)(-5\lambda + \lambda^2 - 4) + 4(5-\lambda)$$

$$|A - \lambda I| = -25\lambda + 5\lambda^2 - 20 + 5\lambda^2 - \lambda^3 + 4\lambda + 20 - 4\lambda$$

$$|A - \lambda I| = -\lambda^3 + 10\lambda^2 - 25\lambda$$

Solve the characteristic equation to find the Eigenvalues.

$$-\lambda^3 + 10\lambda^2 - 25\lambda = 0$$

$$\lambda(\lambda^2 - 10\lambda + 25) = 0$$

$$\lambda(\lambda-5)(\lambda-5)=0$$

$$\lambda=0,\ 5,\ 5$$

Let's deal with $\lambda_1=0$ first. We find

$$A-0I=\begin{bmatrix}5-0 & -4 & 0\\ 1 & -0 & 2\\ 0 & 2 & 5-0\end{bmatrix}$$

$$A-0I=\begin{bmatrix}5 & -4 & 0\\ 1 & 0 & 2\\ 0 & 2 & 5\end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix}1 & -\frac{4}{5} & 0\\ 1 & 0 & 2\\ 0 & 2 & 5\end{bmatrix}\to\begin{bmatrix}1 & -\frac{4}{5} & 0\\ 0 & \frac{4}{5} & 2\\ 0 & 2 & 5\end{bmatrix}\to\begin{bmatrix}1 & -\frac{4}{5} & 0\\ 0 & 1 & \frac{5}{2}\\ 0 & 2 & 5\end{bmatrix}$$

$$\to\begin{bmatrix}1 & -\frac{4}{5} & 0\\ 0 & 1 & \frac{5}{2}\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & \frac{5}{2}\\ 0 & 0 & 0\end{bmatrix}$$

Turning these first two rows into equations gives

$$k_1+2k_3=0$$

$$k_2+\frac{5}{2}k_3=0$$

or

$$k_1 = -2k_3$$

$$k_2 = -\frac{5}{2}k_3$$

Let's choose $k_3 = -2$, which then gives $k_1 = 4$ and $k_2 = 5$.

$$\vec{k_1} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

Using this vector, one solution is

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix} e^{0t}$$

$$\vec{x_1} = \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix}$$

For the two remaining Eigenvalues, $\lambda_2 = \lambda_3 = 5$, we'll find

$$A - 5I = \begin{bmatrix} 5-5 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 5-5 \end{bmatrix}$$

$$A - 5I = \begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -5 & 2 \\ 0 & -4 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Turning these first two rows into equations gives

$$k_1 + 2k_3 = 0$$

$$k_2 = 0$$

or

$$k_1 = -2k_3$$

$$k_2 = 0$$

Let's choose $k_3 = -1$, which then gives $k_1 = 2$.

$$\overrightarrow{k_2} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Other than the trivial solution $\overrightarrow{k_2} = (0,0,0)$, any other (k_1, k_2, k_3) set we find from $k_1 = -2k_3$ and $k_2 = 0$ will result in a vector that's linearly dependent with the $\overrightarrow{k_2} = (2,0,-1)$ vector we already found. So we can say that the Eigenvalue $\lambda_2 = \lambda_3 = 5$ produces only one Eigenvector.

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^{5t}$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_2 = \lambda_3 = 5$, we have to use $\overrightarrow{k_2} = (2,0,-1)$ to find a second solution.

$$(A - \lambda_1 I)\overrightarrow{p_1} = \overrightarrow{k_2}$$

$$\begin{bmatrix} 0 & -4 & 0 \\ 1 & -5 & 2 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$-4p_2 = 2$$

$$p_1 - 5p_2 + 2p_3 = 0$$

$$2p_2 = -1$$

The first and third equations both give $p_2 = -1/2$, so the system simplifies to just the second equation,

$$p_1 - 5\left(-\frac{1}{2}\right) + 2p_3 = 0$$

$$p_1 = -\frac{5}{2} - 2p_3$$

There are an infinite number of value pairs we can choose for $\overrightarrow{p_1} = (p_1, -1/2, p_3)$. Let's pick simple values, like $\overrightarrow{p_1} = (-1/2, -1/2, -1)$. But ideally, we'd like to simplify the solution by eliminating fractions and not leading with a negative value, so we'll multiply through $\overrightarrow{p_1}$ by -2 to rewrite it as

$$\vec{p_1} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

Then our second solution will be

$$\vec{x_3} = \vec{k_1} t e^{\lambda_1 t} + \vec{p_1} e^{\lambda_1 t}$$

$$\vec{x_3} = \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} t e^{5t} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{5t}$$

and the general solution to the homogeneous system is

$$\vec{x} = c_1 \vec{x_1} + c_2 \vec{x_2} + c_3 \vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 4 \\ 5 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} e^{5t} + c_3 \left(\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} t e^{5t} + \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} e^{5t} \right)$$

Topic: Equal real Eigenvalues with multiplicity two

Question: Find the general solution to the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{t} + c_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} e^{t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} te^{-t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} e^{-t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} te^{-t} + c_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} e^{-t}$

Solution: C

We'll need to find the determinant $|A-\lambda I|$,

$$|A-\lambda I| = \left|\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right|$$

$$|A-\lambda I| = \left|\begin{bmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ -1 & -2 & -2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right|$$

$$|A-\lambda I| = \begin{vmatrix} -\lambda & 2 & 1 \\ 1 & 1-\lambda & 1 \\ -1 & -2 & -2-\lambda \end{vmatrix}$$

$$|A-\lambda I| = -\lambda \begin{vmatrix} 1-\lambda & 1 \\ -2 & -2-\lambda \end{vmatrix} - 2 \begin{vmatrix} 1 & 1 \\ -1 & -2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 1-\lambda \\ -1 & -2 \end{vmatrix}$$

$$|A-\lambda I| = -\lambda[(1-\lambda)(-2-\lambda)-(1)(-2)] - 2[(1)(-2-\lambda)-(1)(-1)]$$

$$+1[(1)(-2)-(1-\lambda)(-1)]$$

$$|A-\lambda I| = -\lambda(\lambda+\lambda^2) - 2(-1-\lambda) + (-1-\lambda)$$

$$|A-\lambda I| = -\lambda^2-\lambda^3+2+2\lambda-1-\lambda$$

$$|A-\lambda I| = -\lambda^3-\lambda^2+\lambda+1$$

Solve the characteristic equation to find the Eigenvalues.

$$-\lambda^3-\lambda^2+\lambda+1=0$$

$$-(\lambda-1)(\lambda+1)^2=0$$

$$\lambda = -1, \ -1, 1$$

Let's deal with $\lambda_1 = 1$ first. We find

$$A - (1)I = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 1-1 & 1 \\ -1 & -2 & -2-1 \end{bmatrix}$$

$$A - (1)I = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 1 & 0 & 1 \\ -1 & -2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 2 & 2 \\ 0 & -4 & -4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & -4 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Turning these first two rows into equations gives

$$k_1 + k_3 = 0$$

$$k_2 + k_3 = 0$$

or

$$k_1 = -k_3$$

$$k_2 = -k_3$$

Let's choose $k_3 = -1$, which then gives $k_1 = 1$ and $k_2 = 1$.

$$\vec{k_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

Using this vector, one solution is

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^t$$

For the two remaining Eigenvalues, $\lambda_2 = \lambda_3 = -1$, we'll find

$$A - (-1)I = \begin{bmatrix} -(-1) & 2 & 1 \\ 1 & 1-(-1) & 1 \\ -1 & -2 & -2-(-1) \end{bmatrix}$$

$$A - (-1)I = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ -1 & -2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Turning this first row into an equation gives

$$k_1 + 2k_2 + k_3 = 0$$

or

$$k_1 = -2k_2 - k_3$$

Let's choose $k_2 = 0$ and $k_3 = -1$, which then gives $k_1 = 1$.

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

We could also choose $k_2 = -1$ and $k_3 = 0$, which then gives $k_1 = 2$.

$$\overrightarrow{k_3} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$$

So we can say that the Eigenvalue $\lambda_2 = \lambda_3 = -1$ produces two linearly independent Eigenvectors.

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t} \qquad \overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} \qquad \overrightarrow{x_3} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} e^{-t}$$

Therefore, the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2} + c_3 \overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} e^{-t}$$

Topic: Equal real Eigenvalues with multiplicity two

Question: Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{3t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{3t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{3t} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{3t} \right)$

D $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{3t} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^{3t} \right)$

Solution: C

We'll need to find the determinant $|A - \lambda I|$,

$$|A - \lambda I| = \left| \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 3 & 1 & 0 \\ 0 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & 1 & 0 \\ 0 & 4-\lambda & -1 \\ 0 & 0 & 3-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (3-\lambda) \begin{vmatrix} 4-\lambda & -1 \\ 0 & 3-\lambda \end{vmatrix} - 1 \begin{vmatrix} 0 & -1 \\ 0 & 3-\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & 4-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (3-\lambda)[(4-\lambda)(3-\lambda) - (0)(-1)]$$

$$|A - \lambda I| = (3-\lambda)(4-\lambda)(3-\lambda)$$

Solve the characteristic equation to find the Eigenvalues.

$$(3-\lambda)(4-\lambda)(3-\lambda) = 0$$

$$\lambda = 4,\ 3,\ 3$$

Let's deal with $\lambda_1 = 4$ first. We find

$$A - 4I = \begin{bmatrix} 3-4 & 1 & 0 \\ 0 & 4-4 & -1 \\ 0 & 0 & 3-4 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Turning these first two rows into equations gives

$$k_1 - k_2 = 0$$

$$k_3 = 0$$

or

$$k_1 = k_2$$

$$k_3 = 0$$

Let's choose $k_2 = 1$, which then gives $k_1 = 1$ and $k_3 = 0$.

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Using this vector, one solution is

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{4t}$$

For the two remaining Eigenvalues, $\lambda_2 = \lambda_3 = 3$, we'll find

$$A - 3I = \begin{bmatrix} 3-3 & 1 & 0 \\ 0 & 4-3 & -1 \\ 0 & 0 & 3-3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Turning these first two rows into equations gives

$$k_2 = 0$$

$$k_3 = 0$$

Let's choose $k_1 = 1$, which then gives

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

So we can say that the Eigenvalue $\lambda_2 = \lambda_3 = 3$ produces only one Eigenvector.

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t}$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_2 = \lambda_3 = 3$, we have to use $\overrightarrow{k_2} = (1,0,0)$ to find a second solution.

$$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_2}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$p_2 = 1$$

$$p_2 - p_3 = 0$$

or

$$p_2 = 1$$

$$p_3 = 1$$

Let's choose $p_1 = 0$, which then gives

$$\overrightarrow{p_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Then our second solution will be

$$\overrightarrow{x_3} = \overrightarrow{k_2} t e^{\lambda_2 t} + \overrightarrow{p_1} e^{\lambda_3 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t e^{3t} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{3t}$$

and the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{3t} + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{3t} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{3t} \right)$$

Topic: Equal real Eigenvalues with multiplicity three

Question: Find the general solution to the system of differential equations.

$$x_1'(t) = x_1$$

$$x_2'(t) = 2x_1 + 2x_2 - x_3$$

$$x_3'(t) = x_2$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$

B $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t \right) + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t \right)$

C $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^t + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$

D $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t \right) + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t \right)$

Solution: D

We can represent this system in matrix form as

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$

We'll need to find the determinant $|A - \lambda I|$,

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 2-\lambda & -1 \\ 0 & 1 & -\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 1 & -\lambda \end{vmatrix} - 0\begin{vmatrix} 2 & -1 \\ 0 & -\lambda \end{vmatrix} + 0\begin{vmatrix} 2 & 2-\lambda \\ 0 & 1 \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)[(2-\lambda)(-\lambda) - (-1)(1)]$$

$$|A - \lambda I| = (1-\lambda)(-2\lambda + \lambda^2 + 1)$$

$$|A - \lambda I| = -2\lambda + \lambda^2 + 1 + 2\lambda^2 - \lambda^3 - \lambda$$

$$|A - \lambda I| = 1 - 3\lambda + 3\lambda^2 - \lambda^3$$

Solve the characteristic equation to find the Eigenvalues.

$$1-3\lambda+3\lambda^2-\lambda^3=0$$

$$\lambda^3-3\lambda^2+3\lambda-1=0$$

$$(\lambda-1)(\lambda-1)(\lambda-1)=0$$

Then for these Eigenvalues, $\lambda_1=\lambda_2=\lambda_3=1$, we find

$$A-1I=\begin{bmatrix}1-1 & 0 & 0\\ 2 & 2-1 & -1\\ 0 & 1 & -1\end{bmatrix}$$

$$A-1I=\begin{bmatrix}0 & 0 & 0\\ 2 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix}2 & 1 & -1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{1}{2} & -\frac{1}{2}\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

If we turn these first two rows into equations, we get

$$k_1=0$$

$$k_2-k_3=0$$

So $k_2=k_3$, and we can choose $k_2=k_3=1$.

$$\overrightarrow{k_1}=\begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

Other than the trivial solution $\vec{k_1} = (0,0,0)$, any other (k_1, k_2, k_3) set we find from $(k_1, k_2, k_3) = (0, k_2, k_3)$ will result in a vector that's linearly dependent with the $\vec{k_1} = (0,1,1)$ vector we already found. So we can say that the Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = 1$ produces only one Eigenvector.

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t$$

Because we only find one Eigenvector for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have to use $\vec{k_1} = (0,1,1)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$2p_1 + p_2 - p_3 = 1$$

$$p_2 - p_3 = 1$$

Substituting $p_2 - p_3 = 1$ into the first equation gives

$$2p_1 + 1 = 1$$

$$2p_1 = 0$$

$$p_1 = 0$$

We can choose any values for p_2 and p_3 that satisfy $p_2 - p_3 = 1$, so we'll pick $(p_2, p_3) = (1,0)$ to get

$$\vec{p_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then our second solution will be

$$\vec{x_2} = \vec{k_1} te^{\lambda_1 t} + \vec{p_1} e^{\lambda_1 t}$$

$$\vec{x_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

Now we'll use $\vec{k_1} = (0,1,1)$ and $\vec{p_1} = (0,1,0)$ to find a third solution.

$$(A - \lambda_1 I)\vec{q_1} = \vec{p_1}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$2q_1 + q_2 - q_3 = 1$$

$$q_2 - q_3 = 0$$

Substituting $q_2 - q_3 = 0$ into the first equation gives

$$2q_1 + 0 = 1$$

$$q_1 = \frac{1}{2}$$

Plugging this back into the first equation gives

$$2\left(\frac{1}{2}\right)+q_2-q_3=1$$

$$q_2-q_3=0$$

$$q_2=q_3$$

We can choose any values for q_2 and q_3, so we'll pick $(q_2,q_3)=(0,0)$ to get

$$\vec{q_1}=\begin{bmatrix}\frac{1}{2}\\0\\0\end{bmatrix}$$

Ideally, we'd like to simplify the solution by eliminating fractions, so we'll multiply through $\vec{q_1}$ by 2 to rewrite it as

$$\vec{q_1}=\begin{bmatrix}1\\0\\0\end{bmatrix}$$

Then our third solution will be

$$\vec{x_3}=\vec{k_1}\frac{t^2}{2}e^{\lambda_1 t}+\vec{p_1}te^{\lambda_1 t}+\vec{q_1}e^{\lambda_1 t}$$

$$\vec{x_3}=\begin{bmatrix}0\\1\\1\end{bmatrix}\frac{t^2}{2}e^t+\begin{bmatrix}0\\1\\0\end{bmatrix}te^t+\begin{bmatrix}1\\0\\0\end{bmatrix}e^t$$

and the general solution to the homogeneous system is

$$\vec{x}=c_1\vec{x_1}+c_2\vec{x_2}+c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t \right) + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t \right)$$

Topic: Equal real Eigenvalues with multiplicity three

Question: Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right)$

$\quad +c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right)$

B $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right)$

C $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right) + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^{2t} \right)$

Solution: A

We'll need to find the determinant $|A - \lambda I|$,

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 0 \\ 1 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 1 & 0 \\ -1 & 3-\lambda & 0 \\ 1 & 0 & 2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda) \begin{vmatrix} 3-\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - 1 \begin{vmatrix} -1 & 0 \\ 1 & 2-\lambda \end{vmatrix} + 0 \begin{vmatrix} -1 & 3-\lambda \\ 1 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)[(3-\lambda)(2-\lambda) - (0)(0)] - [(-1)(2-\lambda) - (1)(0)]$$

$$|A - \lambda I| = (1-\lambda)(3-\lambda)(2-\lambda) + (2-\lambda)$$

$$|A - \lambda I| = (2-\lambda)^3$$

Solve the characteristic equation to find the Eigenvalues.

$$(2-\lambda)^3 = 0$$

$$\lambda = 2, \ 2, \ 2$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} 1-2 & 1 & 0 \\ -1 & 3-2 & 0 \\ 1 & 0 & 2-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn these first two rows into equations, we get

$$k_1 = 0$$

$$k_2 = 0$$

So we can choose $k_3 = 1$ to get

$$\overrightarrow{k_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Other than the trivial solution $\overrightarrow{k_1} = (0,0,0)$, any other (k_1, k_2, k_3) set we find from $(k_1, k_2, k_3) = (0,0,k_3)$ will result in a vector that's linearly dependent with the $\overrightarrow{k_1} = (0,0,1)$ vector we already found. So we can say that the Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = 2$ produces only one Eigenvector.

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Because we only find one Eigenvector for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$, we have to use $\vec{k_1} = (0,0,1)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$-p_1 + p_2 = 0$$

$$p_1 = 1$$

Substituting $p_1 = 1$ into the first equation gives

$$p_2 = 1$$

We can choose any values for p_3, so we'll pick $p_3 = 0$ to get

$$\vec{p_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Then our second solution will be

$$\vec{x_2} = \vec{k_1} t e^{\lambda_1 t} + \vec{p_1} e^{\lambda_1 t}$$

$$\vec{x_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Now we'll use $\vec{p_1} = (1,1,0)$ to find a third solution.

$$(A - \lambda_1 I)\vec{q_1} = \vec{p_1}$$

$$\begin{bmatrix} -1 & 1 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$-q_1 + q_2 = 1$$

$$q_1 = 0$$

Substituting $q_1 = 0$ into the first equation gives

$$0 + q_2 = 1$$

$$q_2 = 1$$

We can choose any value for q_3, so we'll pick $q_3 = 0$ to get

$$\vec{q_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then our third solution will be

$$\vec{x_3} = \vec{k_1}\frac{t^2}{2}e^{\lambda_1 t} + \vec{p_1}te^{\lambda_1 t} + \vec{q_1}e^{\lambda_1 t}$$

$$\vec{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

and the general solution to the homogeneous system is

$$\vec{x} = c_1 \vec{x}_1 + c_2 \vec{x}_2 + c_3 \vec{x}_3$$

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right) + c_3 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \frac{t^2}{2} e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right)$$

Topic: Equal real Eigenvalues with multiplicity three

Question: Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} -1 & 5 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} te^{-t} \right)$

C $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^{-t} + c_3 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{t^2}{2} e^{-t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{-t} \right)$

Solution: B

We'll need to find the determinant $|A-\lambda I|$,

$$|A-\lambda I| = \left| \begin{bmatrix} -1 & 5 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$|A-\lambda I| = \left| \begin{bmatrix} -1 & 5 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 5 & 0 \\ 0 & -1-\lambda & 0 \\ 0 & 2 & -1-\lambda \end{vmatrix}$$

$$|A-\lambda I| = (-1-\lambda) \begin{vmatrix} -1-\lambda & 0 \\ 2 & -1-\lambda \end{vmatrix} - 5 \begin{vmatrix} 0 & 0 \\ 0 & -1-\lambda \end{vmatrix} + 0 \begin{vmatrix} 0 & -1-\lambda \\ 0 & 2 \end{vmatrix}$$

$$|A-\lambda I| = (-1-\lambda)[(-1-\lambda)(-1-\lambda)-(0)(2)] - 5[(0)(-1-\lambda)-(0)(0)]$$

$$|A-\lambda I| = (-1-\lambda)(-1-\lambda)(-1-\lambda)$$

Solve the characteristic equation to find the Eigenvalues.

$$(-1-\lambda)(-1-\lambda)(-1-\lambda) = 0$$

$$\lambda = -1, \ -1, \ -1$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = -1$, we find

$$A-(-1)I=\begin{bmatrix} -1-(-1) & 5 & 0 \\ 0 & -1-(-1) & 0 \\ 0 & 2 & -1-(-1) \end{bmatrix}$$

$$A-(-1)I=\begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this first row into an equation, we get

$$k_2 = 0$$

So we can choose $k_1 = 1$ and $k_3 = 0$ to get

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We could also choose $k_1 = 0$ and $k_3 = 1$ to get

$$\overrightarrow{k_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

So we can say that the Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = -1$ produces only two Eigenvectors.

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} \qquad \vec{x_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

Because we find two Eigenvectors for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = -1$, we have to find a third solution.

$$(A - \lambda_1 I)\vec{p_1} = a_1\vec{k_1} + a_2\vec{k_2}$$

$$\begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ a_2 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$5p_2 = a_1$$

$$2p_2 = a_2$$

We can choose any value for p_2, so we'll pick $p_2 = 1$ with $p_1 = 0$ and $p_3 = 0$.

$$\vec{p_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Then our third solution will be

$$\vec{x_3} = \vec{p_1}e^{\lambda_1 t} + (a_1\vec{k_1} + a_2\vec{k_2})te^{\lambda_1 t}$$

$$\vec{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + \left(5\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + 2\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right) te^{-t}$$

$$\vec{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} te^{-t}$$

and the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{-t} + \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix} te^{-t} + \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} te^{-t} \right)$$

Topic: Complex Eigenvalues

Question: Solve the system of differential equations.

$$\vec{x}' = \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ 1 + 3i \end{bmatrix} e^{3it} + c_2 \begin{bmatrix} 2 \\ 1 - 3i \end{bmatrix} e^{-3it}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ 1 + 3i \end{bmatrix} e^{-3it} + c_2 \begin{bmatrix} 2 \\ 1 - 3i \end{bmatrix} e^{3it}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ 3i \end{bmatrix} e^{3it} + c_2 \begin{bmatrix} 2 \\ -3i \end{bmatrix} e^{-3it}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ 3i \end{bmatrix} e^{-3it} + c_2 \begin{bmatrix} 2 \\ -3i \end{bmatrix} e^{3it}$

Solution: A

We'll need to find the determinant $|A-\lambda I|$,

$$|A-\lambda I| = \left| \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A-\lambda I| = \left| \begin{bmatrix} -1 & 2 \\ -5 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A-\lambda I| = \begin{vmatrix} -1-\lambda & 2 \\ -5 & 1-\lambda \end{vmatrix}$$

$$|A-\lambda I| = (-1-\lambda)(1-\lambda)-(2)(-5)$$

$$|A-\lambda I| = \lambda^2+9$$

Solve the characteristic equation to find the Eigenvalues.

$$\lambda^2+9=0$$

$$\lambda = \pm 3i$$

Then for these complex conjugate Eigenvalues, $\lambda_1 = 3i$ and $\lambda_2 = -3i$, we find

$$A-(3i)I = \begin{bmatrix} -1-3i & 2 \\ -5 & 1-3i \end{bmatrix}$$

and

$$A-(-3i)I = \begin{bmatrix} -1+3i & 2 \\ -5 & 1+3i \end{bmatrix}$$

If we turn these matrices back into systems of equations, we get

$$(-1-3i)k_1 + 2k_2 = 0 \qquad (-1+3i)k_1 + 2k_2 = 0$$

$$-5k_1 + (1-3i)k_2 = 0 \qquad -5k_1 + (1+3i)k_2 = 0$$

From the first equation in the first system for $\lambda_1 = 3i$, we find

$$k_2 = \left(\frac{1}{2} + \frac{3}{2}i\right) k_1$$

Choosing $k_1 = 2$ gives $k_2 = 1 + 3i$, which leads us to the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 2 \\ 1+3i \end{bmatrix}$$

and the corresponding solution vector

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1+3i \end{bmatrix} e^{3it}$$

In the same way, from the first equation in the second system for $\lambda_2 = -3i$, we find

$$k_2 = \left(\frac{1}{2} - \frac{3}{2}i\right) k_1$$

Choosing $k_1 = 2$ gives $k_2 = 1 - 3i$, which leads us to the Eigenvector

$$\overrightarrow{k_2} = \begin{bmatrix} 2 \\ 1-3i \end{bmatrix}$$

and the corresponding solution vector

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 2 \\ 1 - 3i \end{bmatrix} e^{-3it}$$

Therefore, the general solution will be

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 2 \\ 1 + 3i \end{bmatrix} e^{3it} + c_2 \begin{bmatrix} 2 \\ 1 - 3i \end{bmatrix} e^{-3it}$$

Topic: Complex Eigenvalues

Question: Solve the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2-i)t} + c_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(2+i)t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1+i \\ 1 \end{bmatrix} e^{(2+i)t} + c_2 \begin{bmatrix} 1-i \\ 1 \end{bmatrix} e^{(2-i)t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(2-i)t} + c_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(2+i)t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 1+i \end{bmatrix} e^{(2+i)t} + c_2 \begin{bmatrix} 1 \\ 1-i \end{bmatrix} e^{(2-i)t}$

Solution: B

We'll need to find the determinant $|A - \lambda I|$,

$$|A - \lambda I| = \left| \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & -2 \\ 1 & 1 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (3 - \lambda)(1 - \lambda) - (-2)(1)$$

$$|A - \lambda I| = \lambda^2 - 4\lambda + 5$$

Solve the characteristic equation to find the Eigenvalues.

$$\lambda^2 - 4\lambda + 5 = 0$$

$$(\lambda - 2)^2 = -1$$

$$\lambda = 2 \pm i$$

Then for these complex conjugate Eigenvalues, $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$, we find

$$A - (2 + i)I = \begin{bmatrix} 3 - (2 + i) & -2 \\ 1 & 1 - (2 + i) \end{bmatrix}$$

$$A - (2 + i)I = \begin{bmatrix} 1 - i & -2 \\ 1 & -1 - i \end{bmatrix}$$

and

$$A - (2 - i)I = \begin{bmatrix} 3 - (2 - i) & -2 \\ 1 & 1 - (2 - i) \end{bmatrix}$$

$$A - (2 - i)I = \begin{bmatrix} 1 + i & -2 \\ 1 & -1 + i \end{bmatrix}$$

If we turn these matrices back into systems of equations, we get

$$(1 - i)k_1 - 2k_2 = 0 \qquad (1 + i)k_1 - 2k_2 = 0$$

$$k_1 + (-1 - i)k_2 = 0 \qquad k_1 + (-1 + i)k_2 = 0$$

From the second equation in the first system for $\lambda_1 = 2 + i$, we find

$$k_1 = (1 + i)k_2$$

Choosing $k_2 = 1$ gives $k_1 = 1 + i$, which leads us to the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix}$$

and the corresponding solution vector

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} e^{2+it}$$

In the same way, from the second equation in the second system for $\lambda_2 = 2 - i$, we find

$$k_1 = (1 - i)k_2$$

Choosing $k_2 = 1$ gives $k_1 = 1 - i$, which leads us to the Eigenvector

$$\overrightarrow{k_2} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix}$$

and the corresponding solution vector

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} e^{2 - it}$$

Therefore, the general solution will be

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 + i \\ 1 \end{bmatrix} e^{(2+i)t} + c_2 \begin{bmatrix} 1 - i \\ 1 \end{bmatrix} e^{(2-i)t}$$

Topic: Complex Eigenvalues

Question: Solve the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 2 & 4 \\ -5 & 6 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1+2i \\ 2 \end{bmatrix} e^{(4-4i)t} + c_2 \begin{bmatrix} 1-2i \\ 2 \end{bmatrix} e^{(4+4i)t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1+2i \\ 2 \end{bmatrix} e^{(4+4i)t} + c_2 \begin{bmatrix} 1-2i \\ 2 \end{bmatrix} e^{(4-4i)t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ 1+2i \end{bmatrix} e^{(4-4i)t} + c_2 \begin{bmatrix} 2 \\ 1-2i \end{bmatrix} e^{(4+4i)t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 2 \\ 1+2i \end{bmatrix} e^{(4+4i)t} + c_2 \begin{bmatrix} 2 \\ 1-2i \end{bmatrix} e^{(4-4i)t}$

Solution: D

We'll need to find the determinant $|A - \lambda I|$,

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 4 \\ -5 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 4 \\ -5 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & 4 \\ -5 & 6 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (2 - \lambda)(6 - \lambda) - (4)(-5)$$

$$|A - \lambda I| = \lambda^2 - 8\lambda + 32$$

Solve the characteristic equation to find the Eigenvalues.

$$\lambda^2 - 8\lambda + 32 = 0$$

$$(\lambda - 4)^2 = -16$$

$$\lambda = 4 \pm 4i$$

Then for these complex conjugate Eigenvalues, $\lambda_1 = 4 + 4i$ and $\lambda_2 = 4 - 4i$, we find

$$A - (4 + 4i)I = \begin{bmatrix} 2 - (4 + 4i) & 4 \\ -5 & 6 - (4 + 4i) \end{bmatrix}$$

$$A - (4 + 4i)I = \begin{bmatrix} -2 - 4i & 4 \\ -5 & 2 - 4i \end{bmatrix}$$

and

$$A - (4-4i)I = \begin{bmatrix} 2-(4-4i) & 4 \\ -5 & 6-(4-4i) \end{bmatrix}$$

$$A - (4-4i)I = \begin{bmatrix} -2+4i & 4 \\ -5 & 2+4i \end{bmatrix}$$

If we turn these matrices back into systems of equations, we get

$$(-2-4i)k_1 + 4k_2 = 0 \qquad (-2+4i)k_1 + 4k_2 = 0$$

$$-5k_1 + (2-4i)k_2 = 0 \qquad -5k_1 + (2+4i)k_2 = 0$$

From the first equation in the first system for $\lambda_1 = 4 + 4i$, we find

$$k_2 = \left(\frac{1}{2} + i\right) k_1$$

Choosing $k_1 = 2$ gives $k_2 = 1 + 2i$, which leads us to the Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 2 \\ 1+2i \end{bmatrix}$$

and the corresponding solution vector

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1+2i \end{bmatrix} e^{(4+4i)t}$$

In the same way, from the first equation in the second system for $\lambda_2 = 4 - 4i$, we find

$$k_2 = \left(\frac{1}{2} - i\right) k_1$$

Choosing $k_1 = 2$ gives $k_2 = 1 - 2i$, which leads us to the Eigenvector

$$\overrightarrow{k_2} = \begin{bmatrix} 2 \\ 1 - 2i \end{bmatrix}$$

and the corresponding solution vector

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 2 \\ 1 - 2i \end{bmatrix} e^{(4-4i)t}$$

Therefore, the general solution will be

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 2 \\ 1 + 2i \end{bmatrix} e^{(4+4i)t} + c_2 \begin{bmatrix} 2 \\ 1 - 2i \end{bmatrix} e^{(4-4i)t}$$

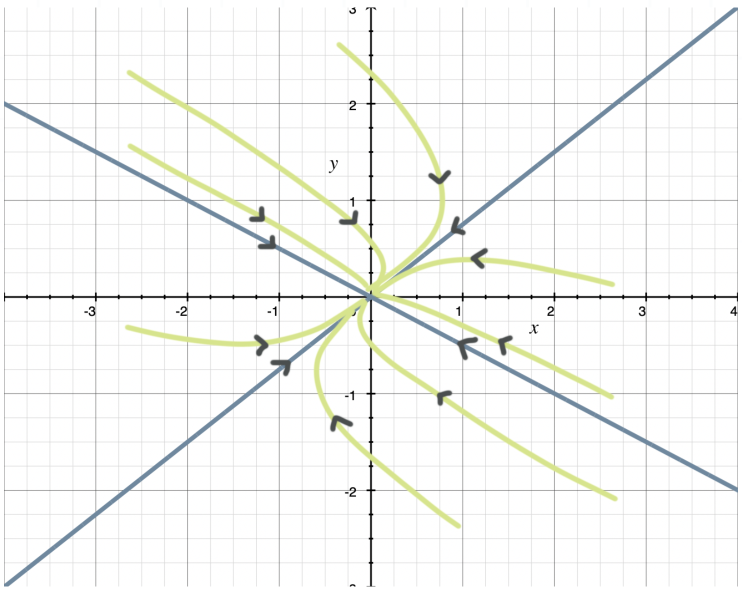
Topic: Phase portraits for distinct real Eigenvalues

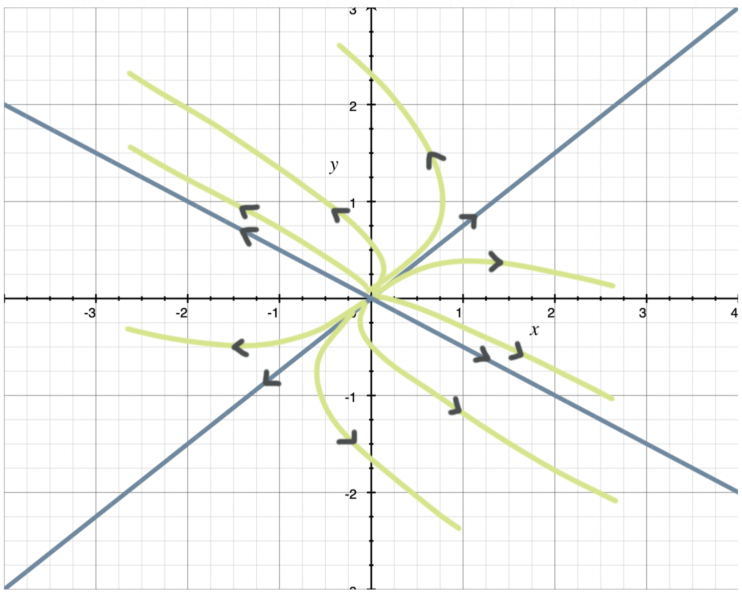
Question: Sketch the phase portrait of the system.

$$x_1' = -\frac{5}{2}x_1 + 2x_2$$

$$x_2' = \frac{3}{4}x_1 - 2x_2$$

Answer choices:

A


B


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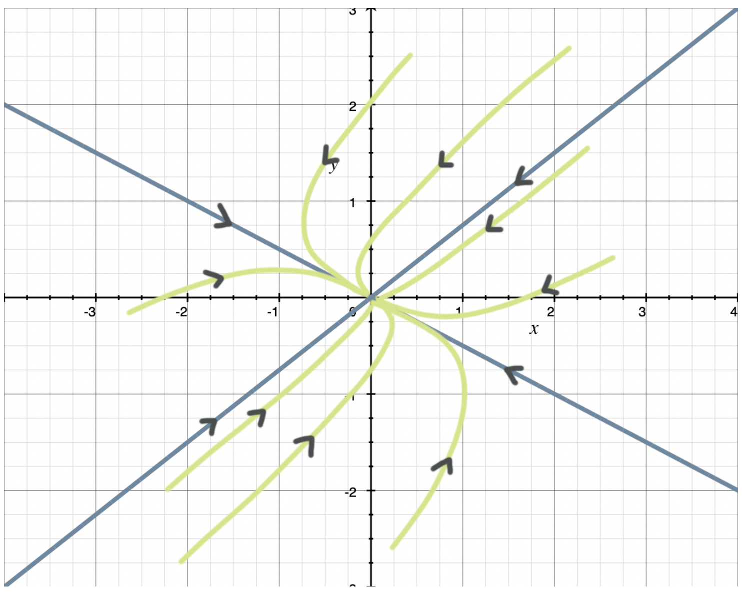
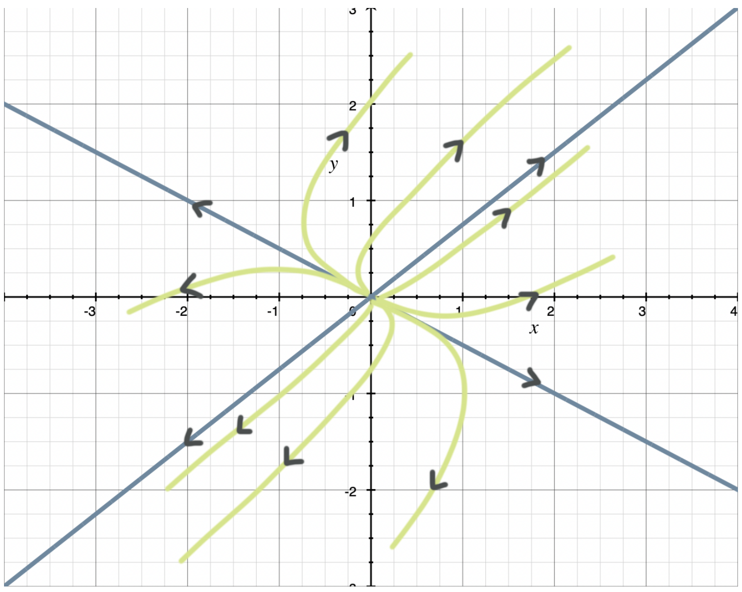


D


Solution: A

The coefficient matrix and $A-\lambda I$ are,

$$A=\begin{bmatrix}-\frac{5}{2} & 2\\ \frac{3}{4} & -2\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}-\frac{5}{2}-\lambda & 2\\ \frac{3}{4} & -2-\lambda\end{bmatrix}$$

and the characteristic equation

$$\left(-\frac{5}{2}-\lambda\right)(-2-\lambda)-(2)\left(\frac{3}{4}\right)=0$$

$$\lambda^2+\frac{9}{2}\lambda+\frac{7}{2}=0$$

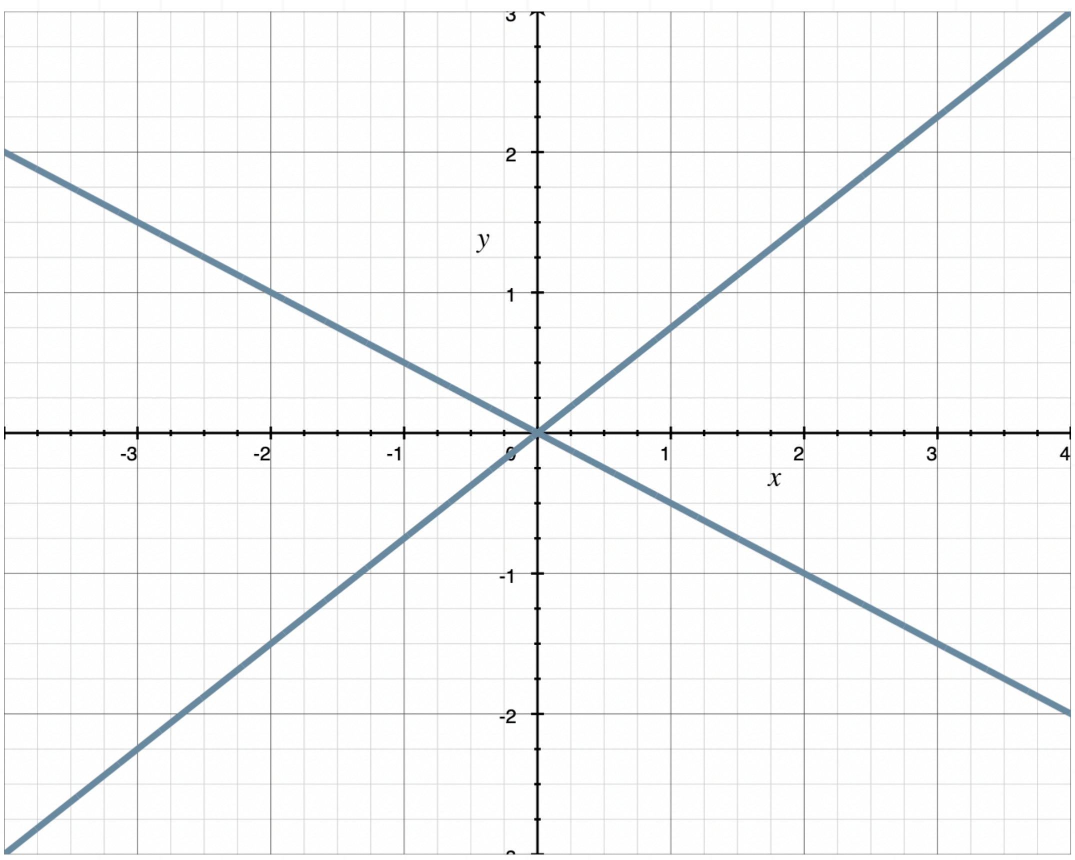
$$\left(\lambda+\frac{9}{4}\right)^2=\frac{25}{16}$$

$$\lambda=-1,\ -\frac{7}{2}$$

gives the Eigenvalues $\lambda_1=-1$ and $\lambda_2=-7/2$, and their associated Eigenvectors

$$\overrightarrow{k_1}=\begin{bmatrix}1\\ \frac{3}{4}\end{bmatrix}\qquad\qquad \overrightarrow{k_2}=\begin{bmatrix}1\\ -\frac{1}{2}\end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (1,3/4)$ lies along the line $y = (3/4)x$, and the Eigenvector $\overrightarrow{k_2} = (1, -1/2)$ lies along the line $y = (-1/2)x$, so we'll sketch these lines.



The Eigenvalue associated with $\overrightarrow{k_1} = (1,3/4)$ is $\lambda = -1$, which means the direction along that trajectory is toward the origin. The Eigenvalue associated with $\overrightarrow{k_2} = (1, -1/2)$ is $\lambda = -7/2$, which means the direction along that trajectory is also toward the origin.

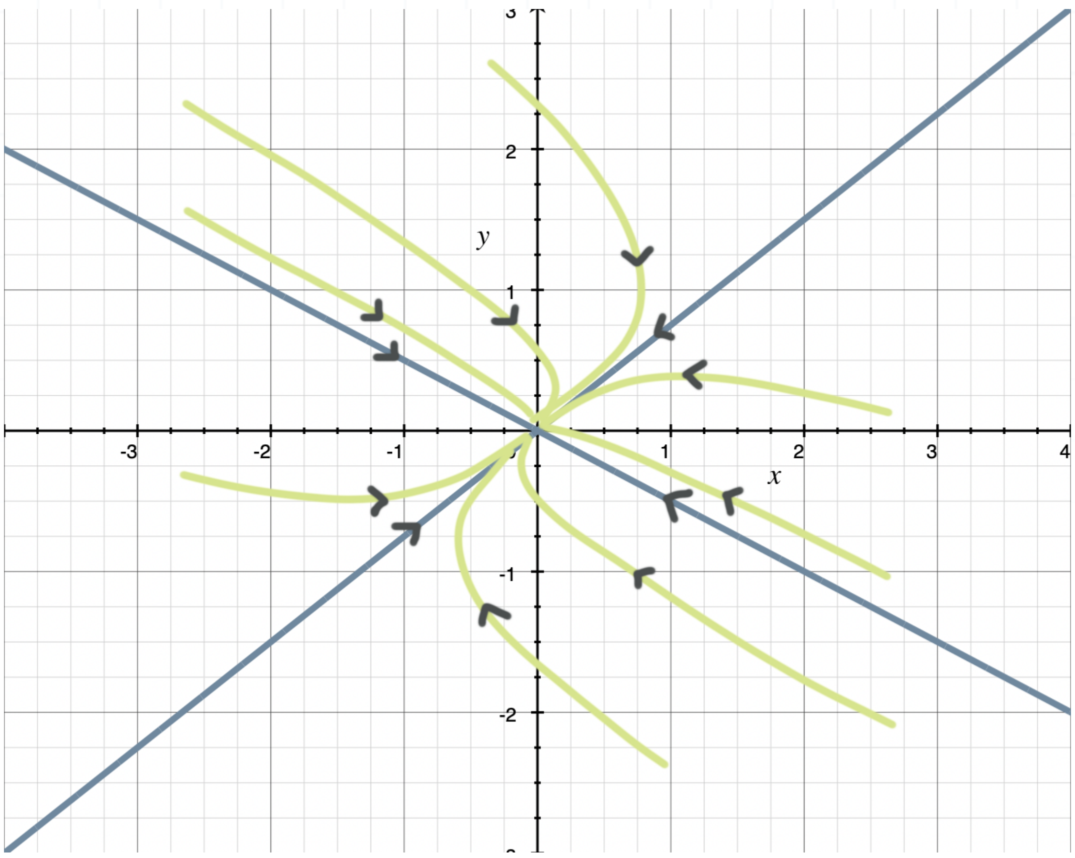
Because both Eigenvalues are negative, we're dealing with an asymptotically stable attractor node that attracts all trajectories.

To apply the $t \to \pm\infty$ test, we'll use the general solution to the system,

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ \frac{3}{4} \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ -\frac{1}{2} \end{bmatrix} e^{-\frac{7}{2}t}$$

Because $e^{-\frac{7}{2}t}$ dominates e^{-t} as $t \to -\infty$, the $\overrightarrow{k_1} = (1,3/4)$ vector drops away first, meaning that our trajectories are going to “start” parallel to $\overrightarrow{k_2} = (1, -1/2)$. On the other end, $e^{-\frac{7}{2}t}$ goes to 0 faster than e^{-t} as $t \to \infty$, so the $\overrightarrow{k_2} = (1, -1/2)$ vector will drop away first, meaning that our trajectories are going to “end” parallel to $\overrightarrow{k_1} = (1,3/4)$.

Therefore, starting the trajectories parallel to $\overrightarrow{k_2} = (1, -1/2)$ and ending them parallel to $\overrightarrow{k_1} = (1,3/4)$ means that the phase portrait must look something like



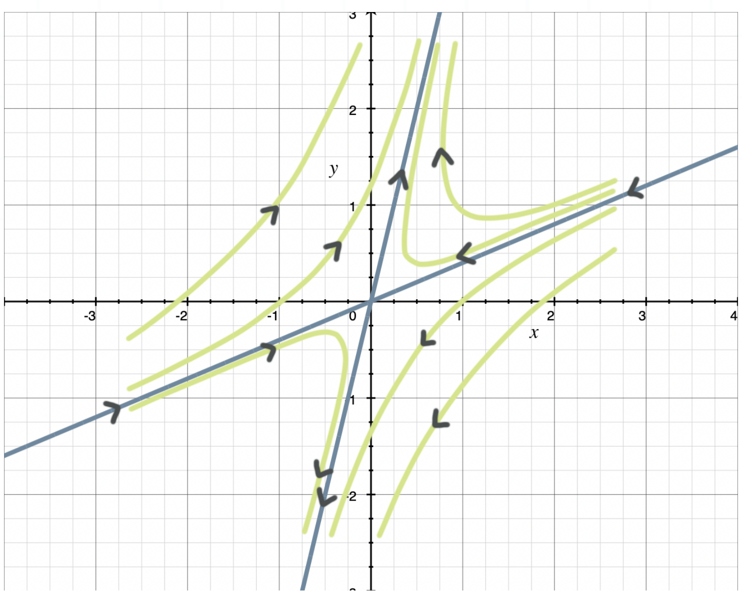
Topic: Phase portraits for distinct real Eigenvalues

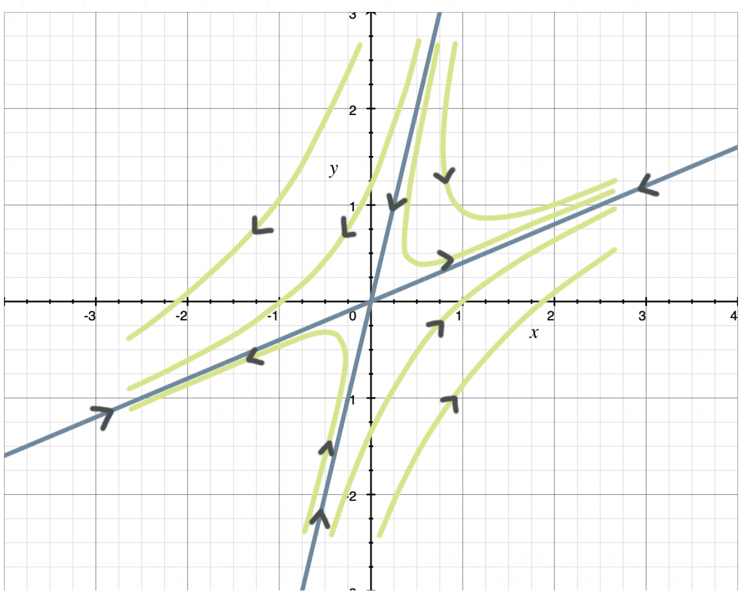
Question: Sketch the phase portrait of the system.

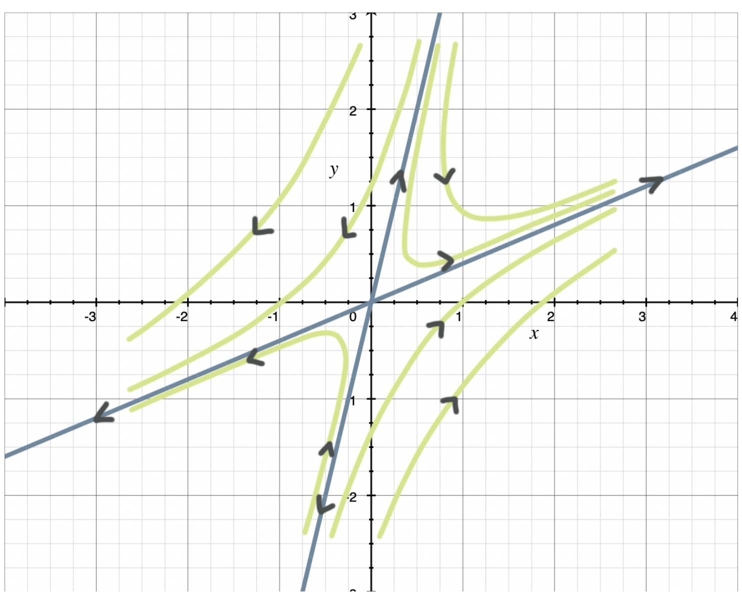
$$x_1' = 10x_1 - 5x_2$$

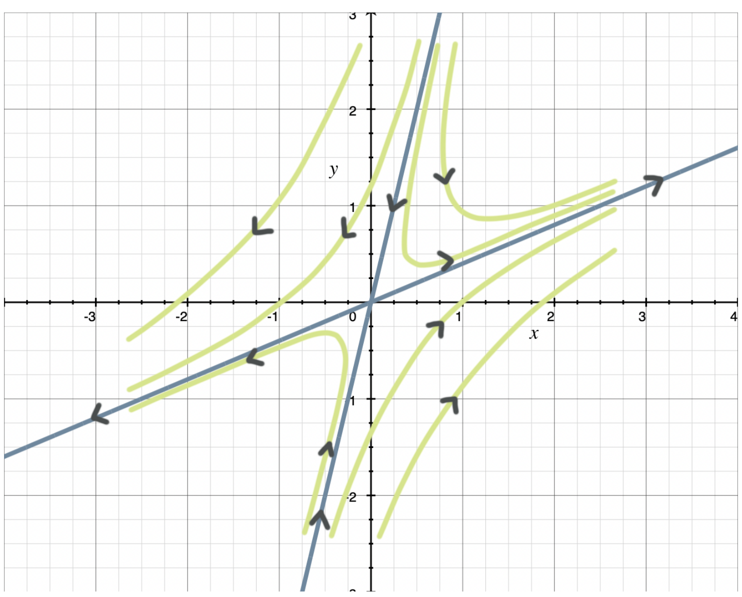
$$x_2' = 8x_1 - 12x_2$$

Answer choices:

A 

B 

C 

D 

Solution: D

The coefficient matrix and $A - \lambda I$ are,

$$A = \begin{bmatrix} 10 & -5 \\ 8 & -12 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 10 - \lambda & -5 \\ 8 & -12 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(10 - \lambda)(-12 - \lambda) - (-5)(8) = 0$$

$$\lambda^2 + 2\lambda - 80 = 0$$

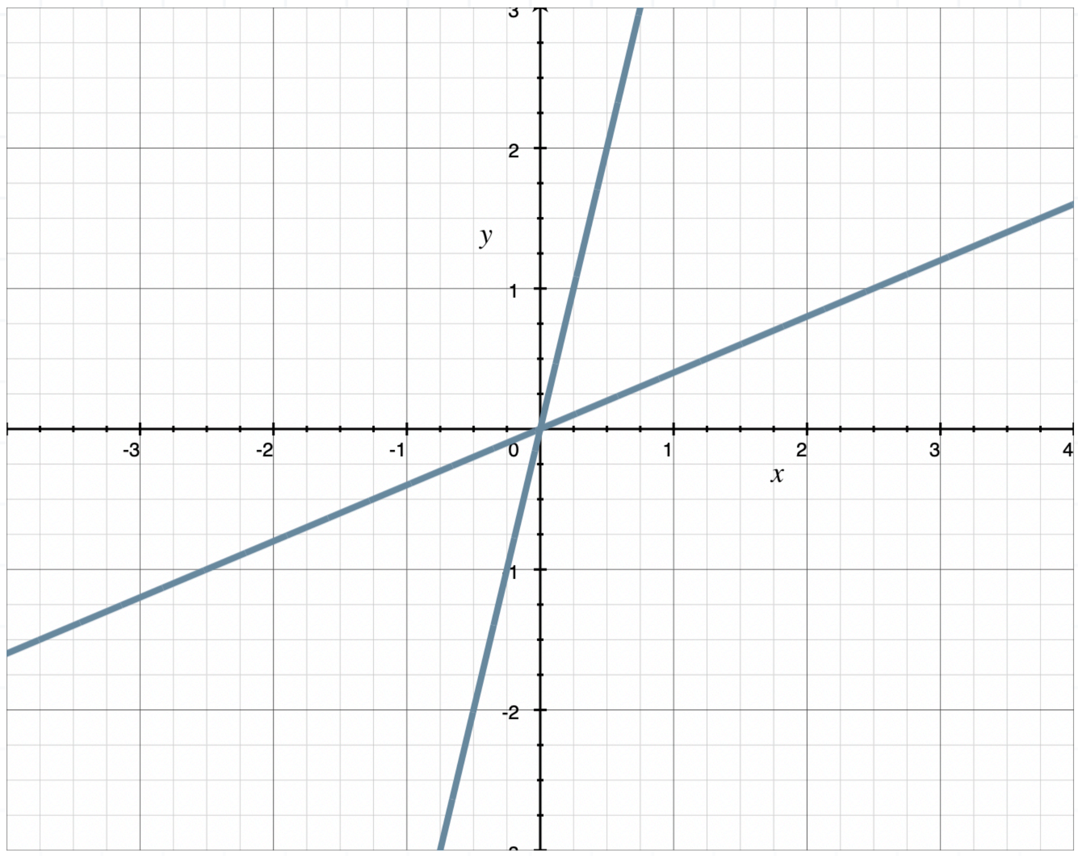
$$(\lambda + 10)(\lambda - 8) = 0$$

$$\lambda = -10,\ 8$$

gives the Eigenvalues $\lambda_1 = -10$ and $\lambda_2 = 8$, and their associated Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (1,4)$ lies along the line $y = 4x$, and the Eigenvector $\overrightarrow{k_2} = (5,2)$ lies along the line $y = (2/5)x$, so we'll sketch these lines.



The Eigenvalue associated with $\overrightarrow{k_1} = (1,4)$ is $\lambda = -10$, which means the direction along that trajectory is toward the origin. The Eigenvalue associated with $\overrightarrow{k_2} = (5,2)$ is $\lambda = 8$, which means the direction along that trajectory is away from the origin.

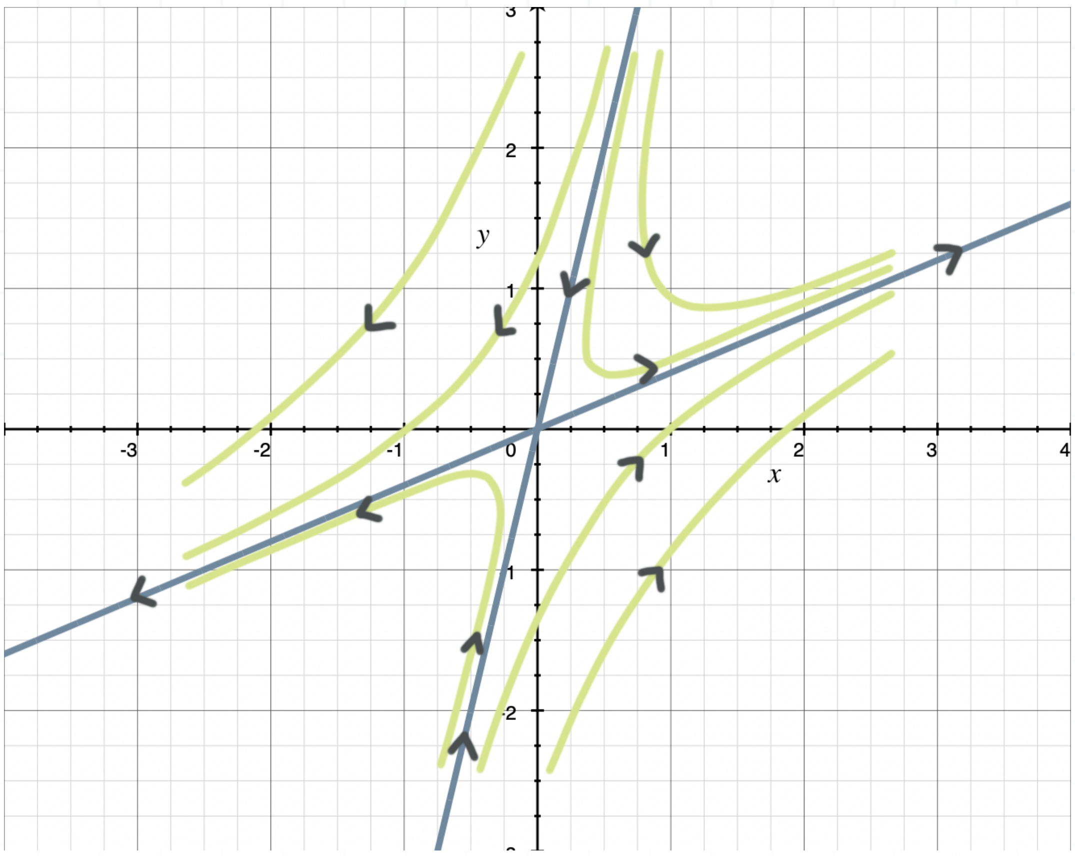
Because the Eigenvalues have opposite signs, we're dealing with an unstable saddle point.

To apply the $t \to \pm\infty$ test, we'll use the general solution to the system,

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 4 \end{bmatrix} e^{-10t} + c_2 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{8t}$$

Because e^{-10t} dominates e^{8t} as $t \to -\infty$, the $\overrightarrow{k_2} = (5,2)$ vector drops away first, meaning that our trajectories are going to "start" parallel to $\overrightarrow{k_1} = (1,4)$. On the other end, e^{8t} dominates e^{-10t} as $t \to \infty$, so the $\overrightarrow{k_1} = (1,4)$ vector will drop away first, meaning that our trajectories are going to "end" parallel to $\overrightarrow{k_2} = (5,2)$.

Therefore, starting the trajectories parallel to $\vec{k_1} = (1,4)$ and ending them parallel to $\vec{k_2} = (5,2)$ means that the phase portrait must look something like



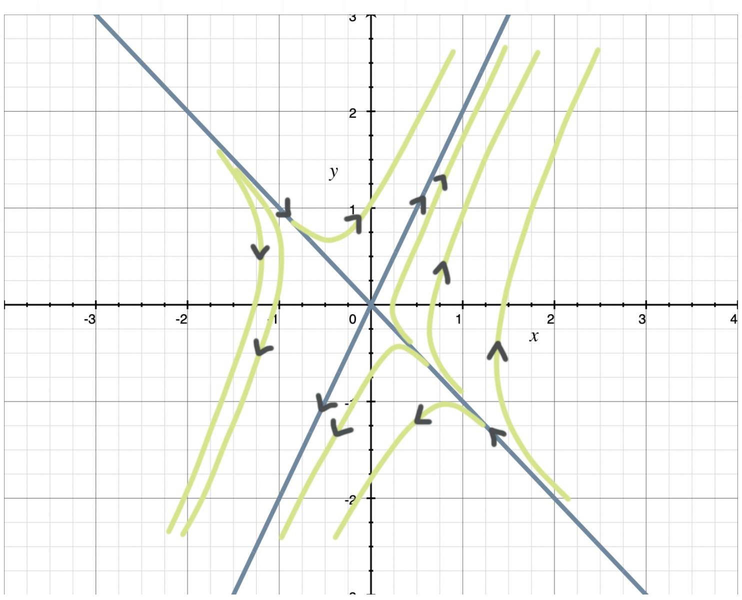
Topic: Phase portraits for distinct real Eigenvalues

Question: Sketch the phase portrait of the system.

$$x_1' = x_1 + 2x_2$$

$$x_2' = 4x_1 + 3x_2$$

Answer choices:

A


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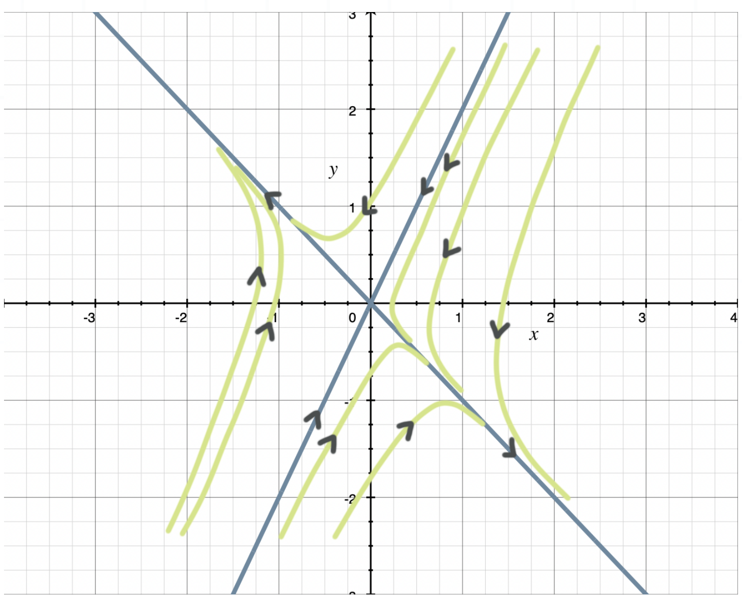
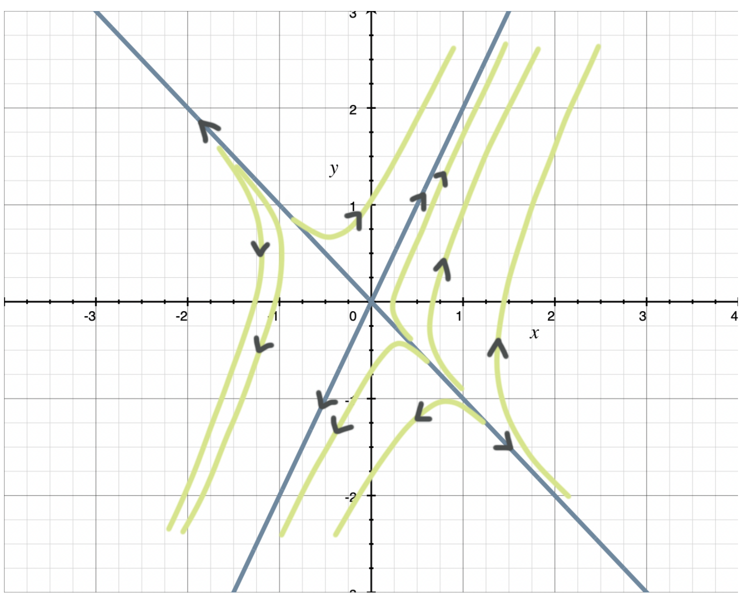
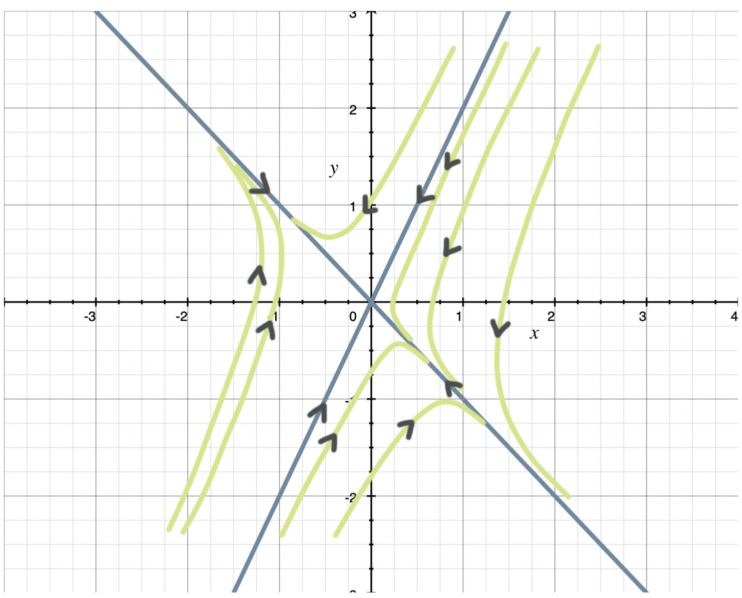


C


D


Solution: A

The coefficient matrix and $A - \lambda I$ are,

$$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 \\ 4 & 3 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(1 - \lambda)(3 - \lambda) - (2)(4) = 0$$

$$\lambda^2 - 4\lambda - 5 = 0$$

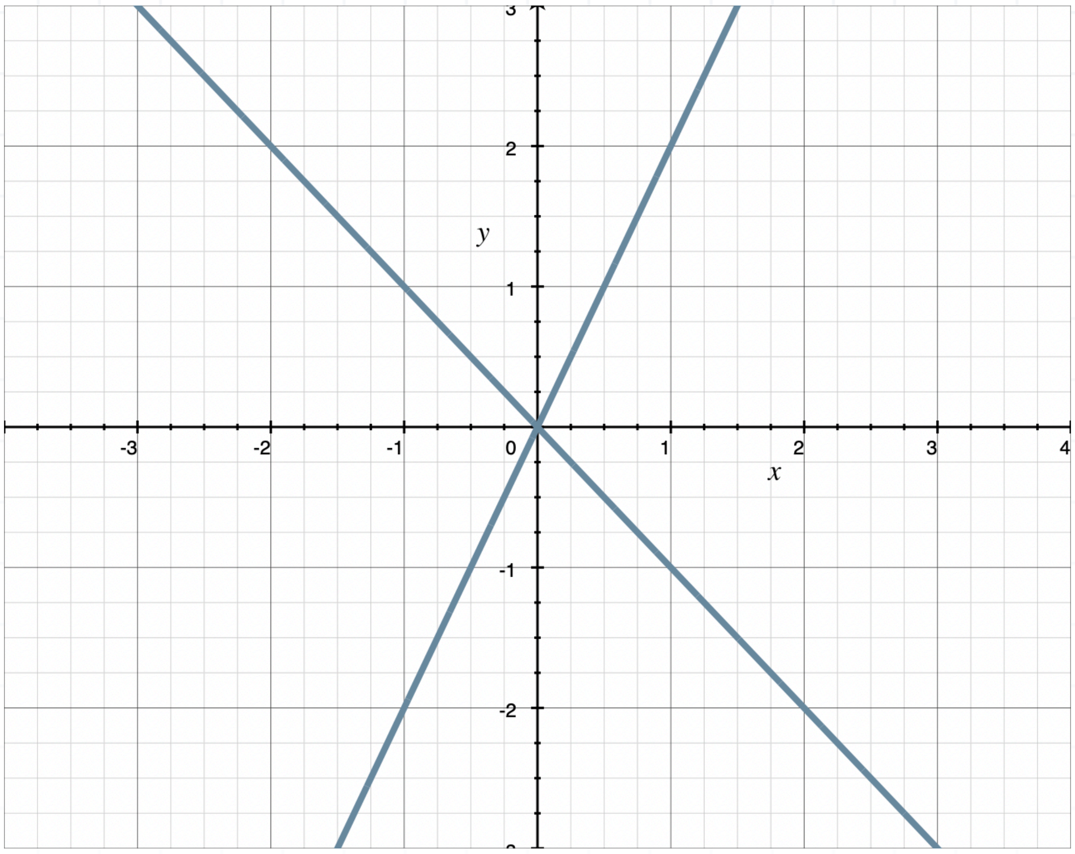
$$(\lambda + 1)(\lambda - 5) = 0$$

$$\lambda = -1, 5$$

gives the Eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 5$, and their associated Eigenvectors

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (1, -1)$ lies along the line $y = -x$, and the Eigenvector $\overrightarrow{k_2} = (1,2)$ lies along the line $y = 2x$, so we'll sketch these lines.



The Eigenvalue associated with $\vec{k_1} = (1, -1)$ is $\lambda = -1$, which means the direction along that trajectory is toward the origin. The Eigenvalue associated with $\vec{k_2} = (1,2)$ is $\lambda = 5$, which means the direction along that trajectory is away from the origin.

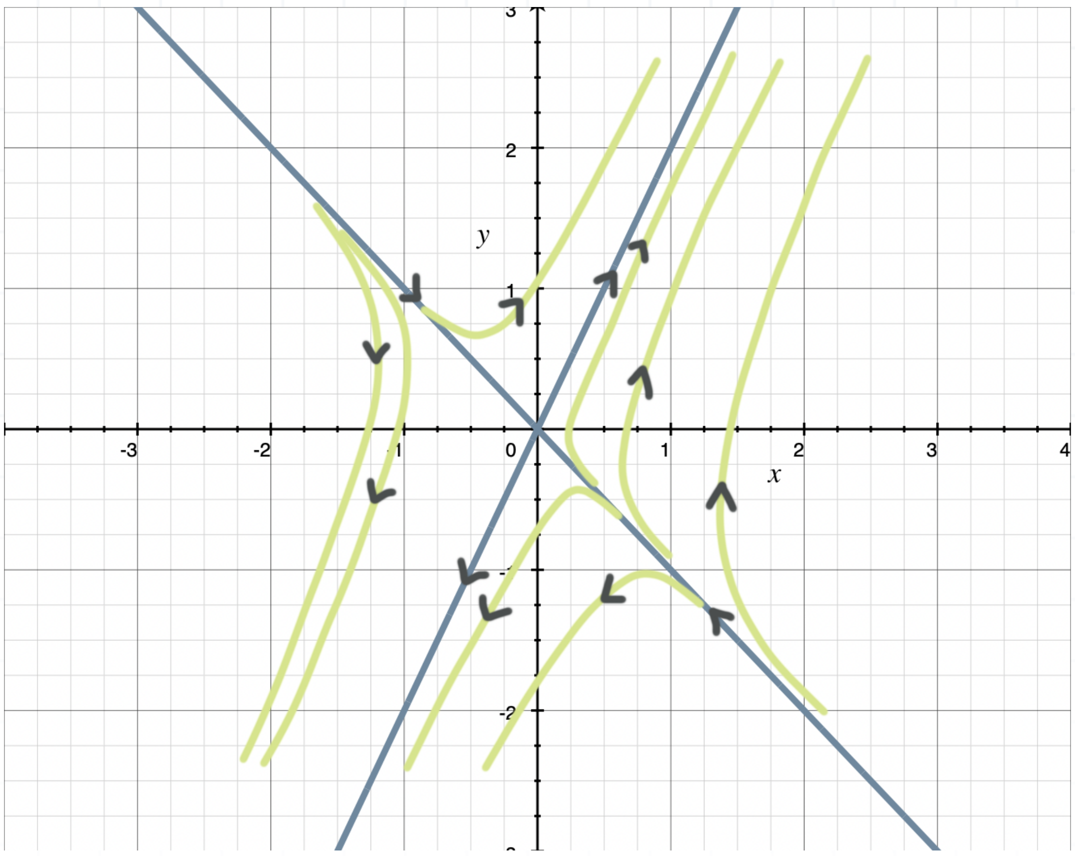
Because the Eigenvalues have opposite signs, we're dealing with an unstable saddle point.

To apply the $t \to \pm \infty$ test, we'll use the general solution to the system,

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{5t}$$

Because e^{-t} dominates e^{5t} as $t \to -\infty$, the $\vec{k_2} = (1,2)$ vector drops away first, meaning that our trajectories are going to "start" parallel to $\vec{k_1} = (1, -1)$. On the other end, e^{5t} dominates e^{-t} as $t \to \infty$, so the $\vec{k_1} = (1, -1)$ vector will drop away first, meaning that our trajectories are going to "end" parallel to $\vec{k_2} = (1,2)$.

Therefore, starting the trajectories parallel to $\overrightarrow{k_1} = (1, -1)$ and ending them parallel to $\overrightarrow{k_2} = (1,2)$ means that the phase portrait must look something like



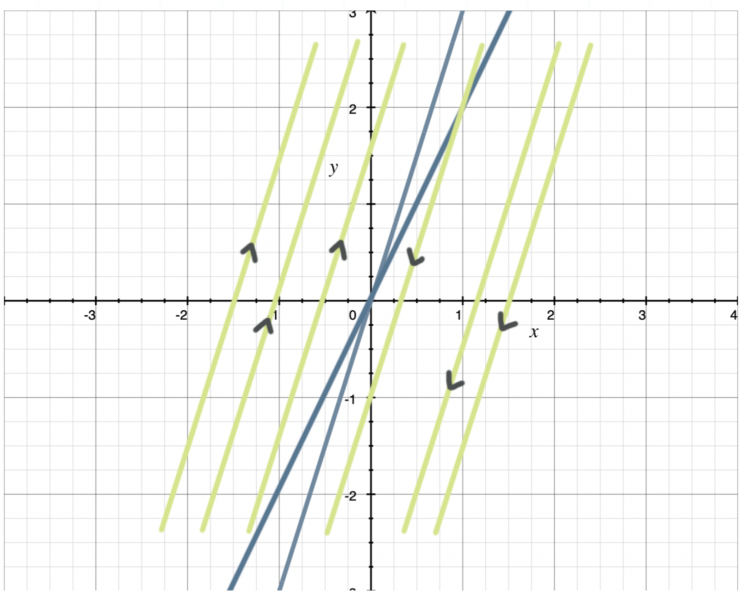
Topic: Phase portraits for equal real Eigenvalues

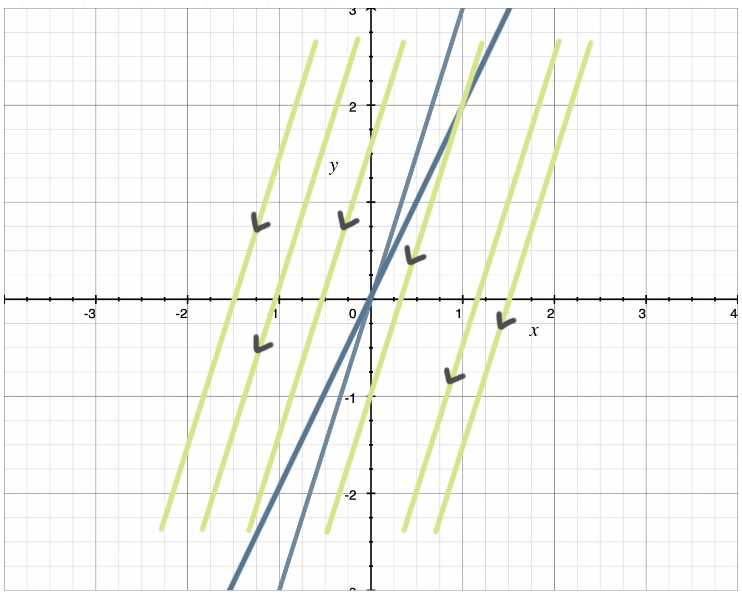
Question: Sketch the phase portrait of the system.

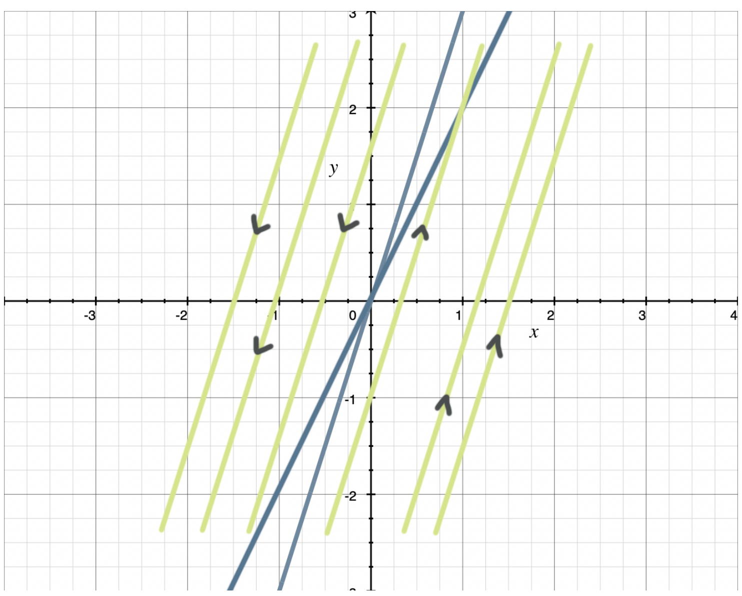
$$x_1' = 3x_1 - x_2$$

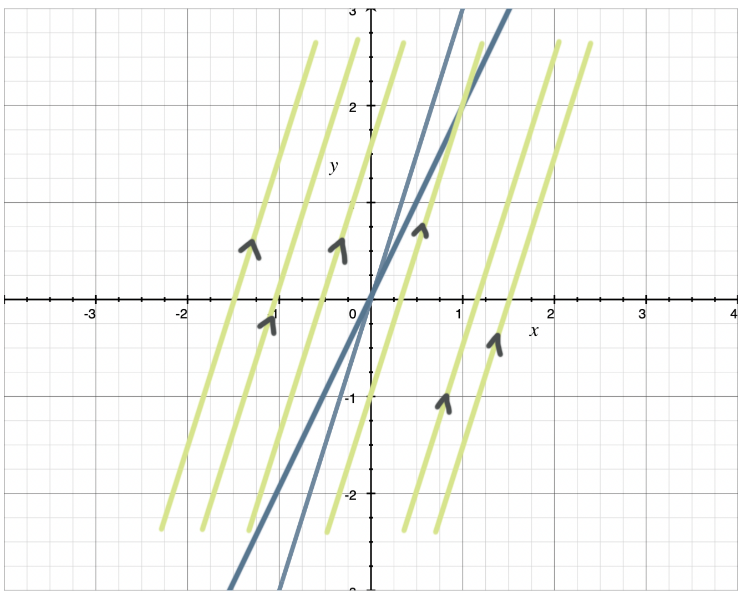
$$x_2' = 9x_1 - 3x_2$$

Answer choices:

A


B


C


D


Solution: C

The coefficient matrix and $A-\lambda I$ are,

$$A=\begin{bmatrix}3&-1\\9&-3\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}3-\lambda&-1\\9&-3-\lambda\end{bmatrix}$$

and the characteristic equation

$$(3-\lambda)(-3-\lambda)-(-1)(9)=0$$

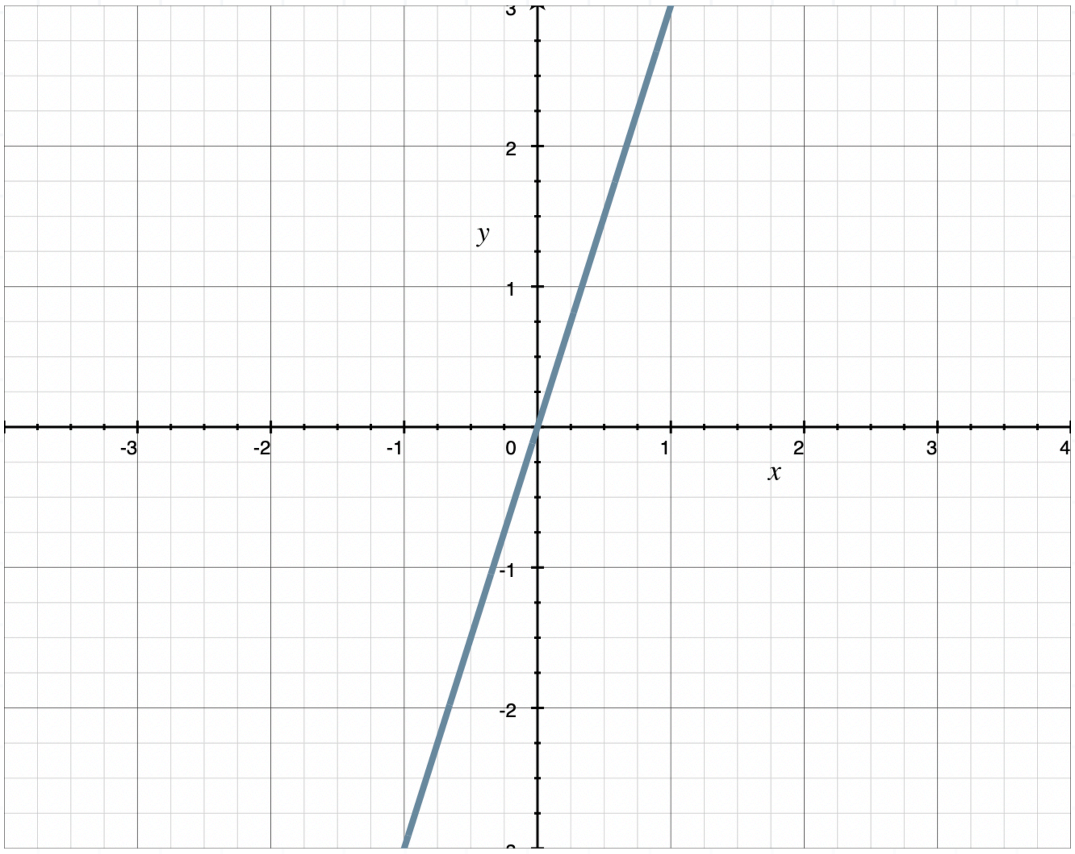
$$\lambda^2=0$$

$$\lambda=0$$

gives the Eigenvalue $\lambda_1=0$ and the associated Eigenvector

$$\overrightarrow{k_1}=\begin{bmatrix}1\\3\end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1}=(1,3)$ lies along the line $y=3x$, so we'll sketch this line.



The Eigenvalue associated with $\overrightarrow{k_1} = (1,3)$ is $\lambda = 0$, and we'll use $\overrightarrow{k_1} = (1,3)$ to find $\overrightarrow{p_1}$.

$$(A - (0)I)\overrightarrow{p_1} = \overrightarrow{k_1}$$

$$\begin{bmatrix} 3 & -1 \\ 9 & -3 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 3p_1 - p_2 \\ 9p_1 - 3p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Turning this matrix equation into a system of equations gives

$$3p_1 - p_2 = 1$$

$$9p_1 - 3p_2 = 3$$

and then

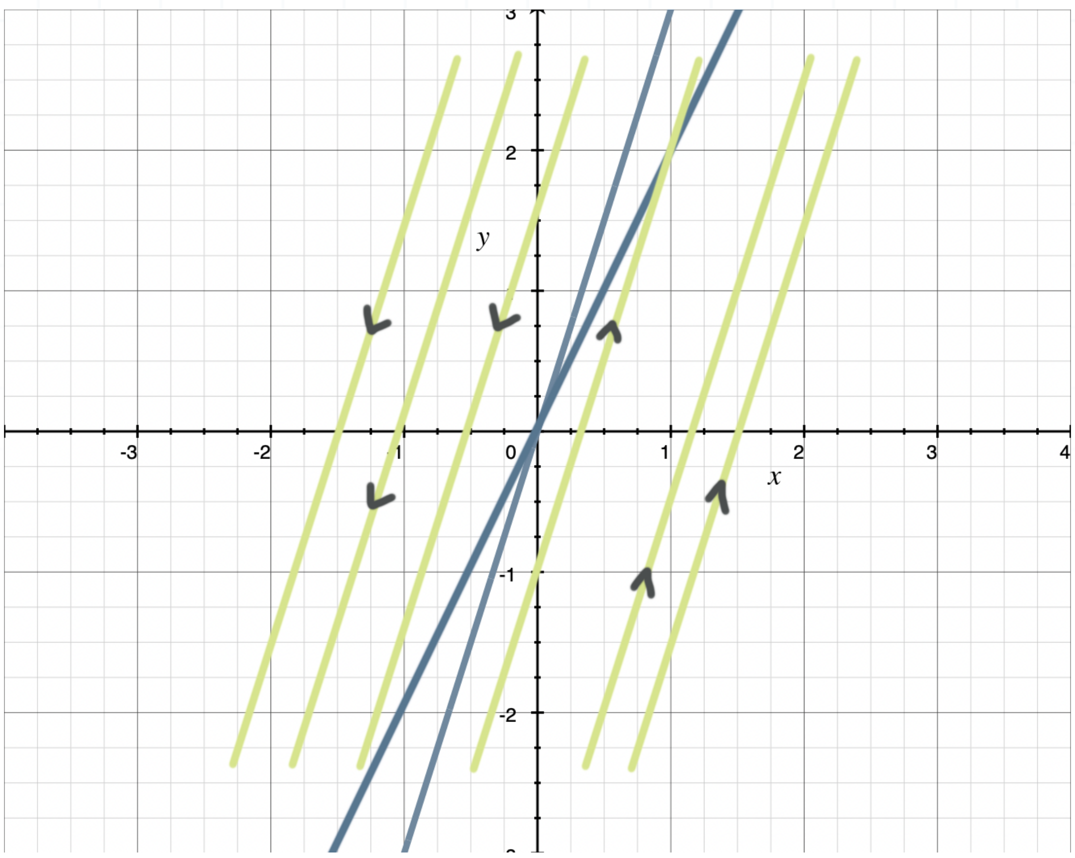
$$3p_1 - p_2 = 1$$

$$3p_1 = p_2 + 1$$

So we'll use

$$\vec{p_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

The vector $\vec{p_1} = (1,2)$ lies along the line $y = 2x$. All trajectories are parallel to $\vec{k_1} = (1,3)$ and every point along $\vec{p_1} = (1,2)$ is an equilibrium point.



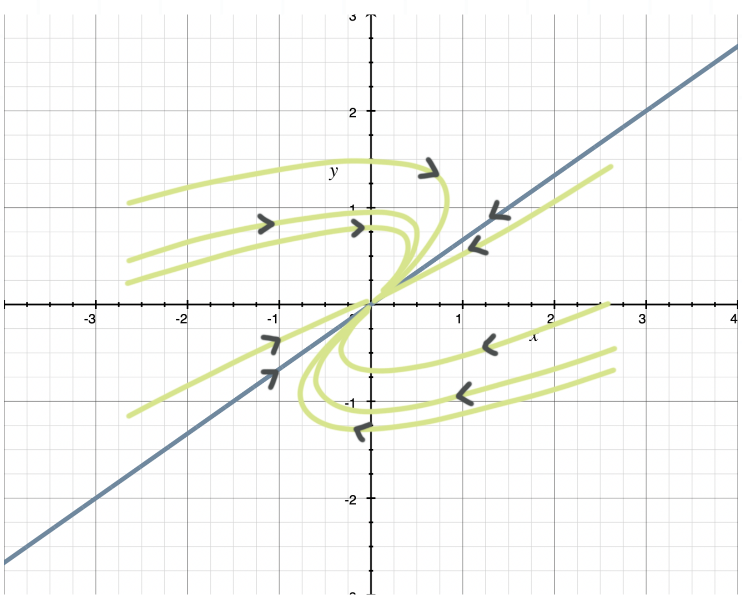
Topic: Phase portraits for equal real Eigenvalues

Question: Sketch the phase portrait of the system.

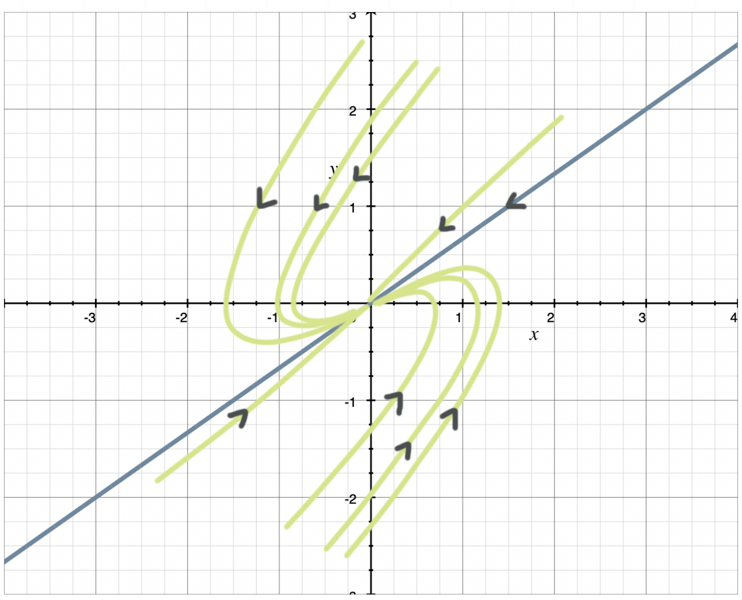
$$x_1' = 12x_1 - 9x_2$$

$$x_2' = 4x_1$$

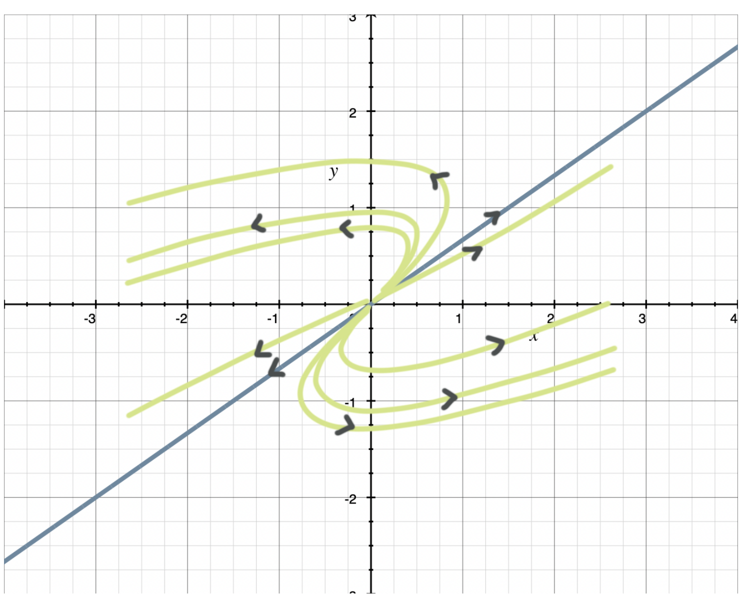
Answer choices:



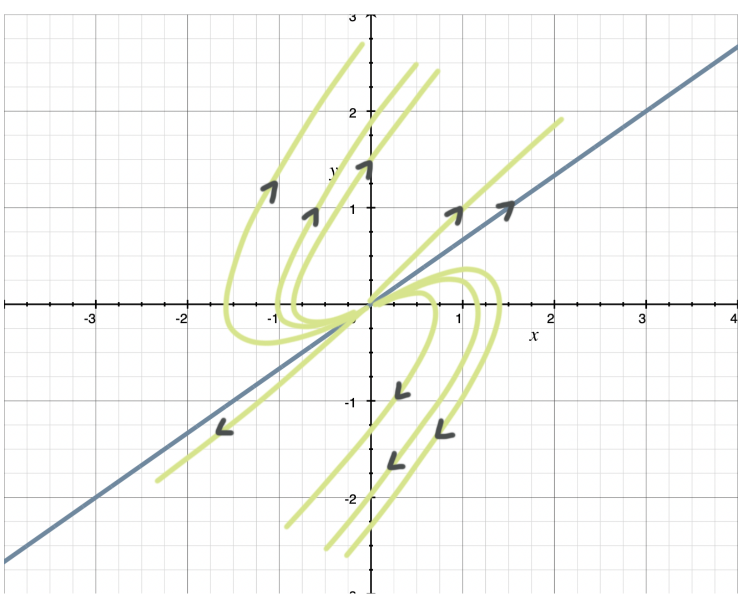
A



B



C



D

Solution: C

The coefficient matrix and $A - \lambda I$ are,

$$A = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 12 - \lambda & -9 \\ 4 & -\lambda \end{bmatrix}$$

and the characteristic equation

$$(12 - \lambda)(-\lambda) - (-9)(4) = 0$$

$$\lambda^2 - 12\lambda + 36 = 0$$

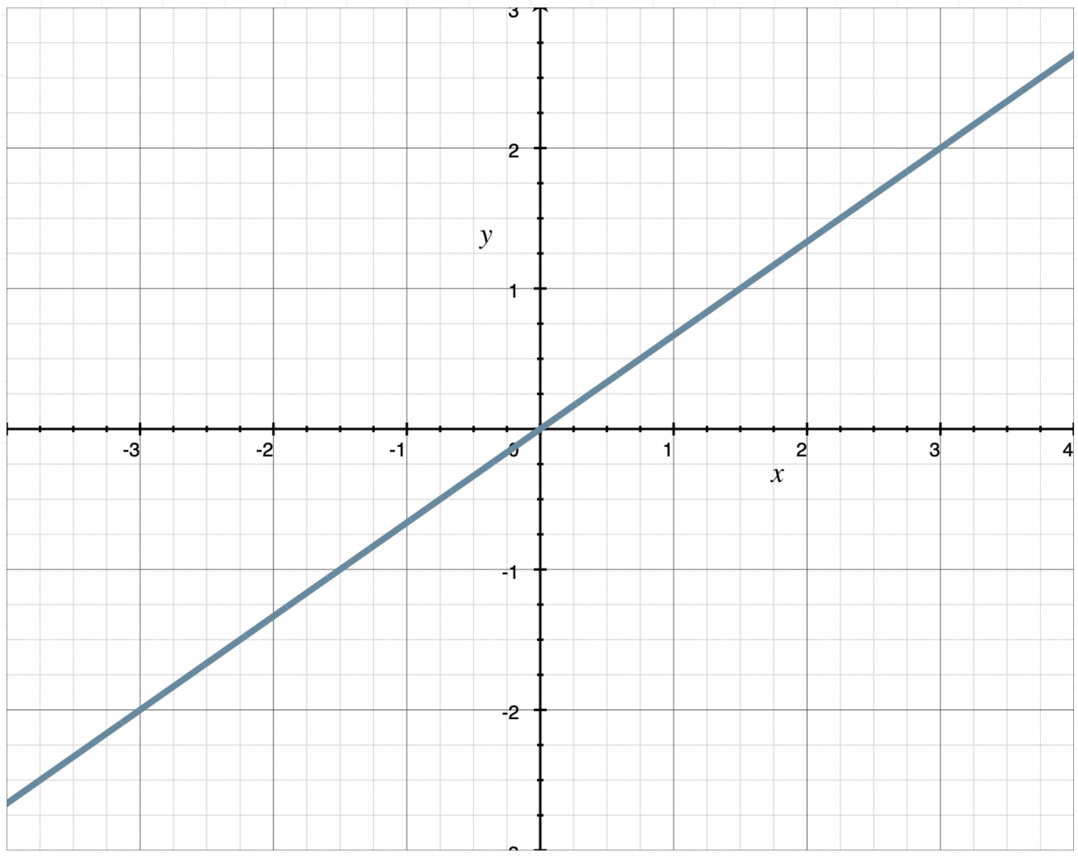
$$(\lambda - 6)(\lambda - 6) = 0$$

$$\lambda = 6, 6$$

gives the Eigenvalues $\lambda_1 = \lambda_2 = 6$, and the associated Eigenvector

$$\overrightarrow{k_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (3,2)$ lies along the line $y = (2/3)x$, so we'll sketch this line.



The Eigenvalue associated with $\overrightarrow{k_1} = (3,2)$ is $\lambda = 6$, which means the direction along that trajectory is away from the origin.

Because the Eigenvalue is positive, we're dealing with an unstable repeller node that repels all trajectories.

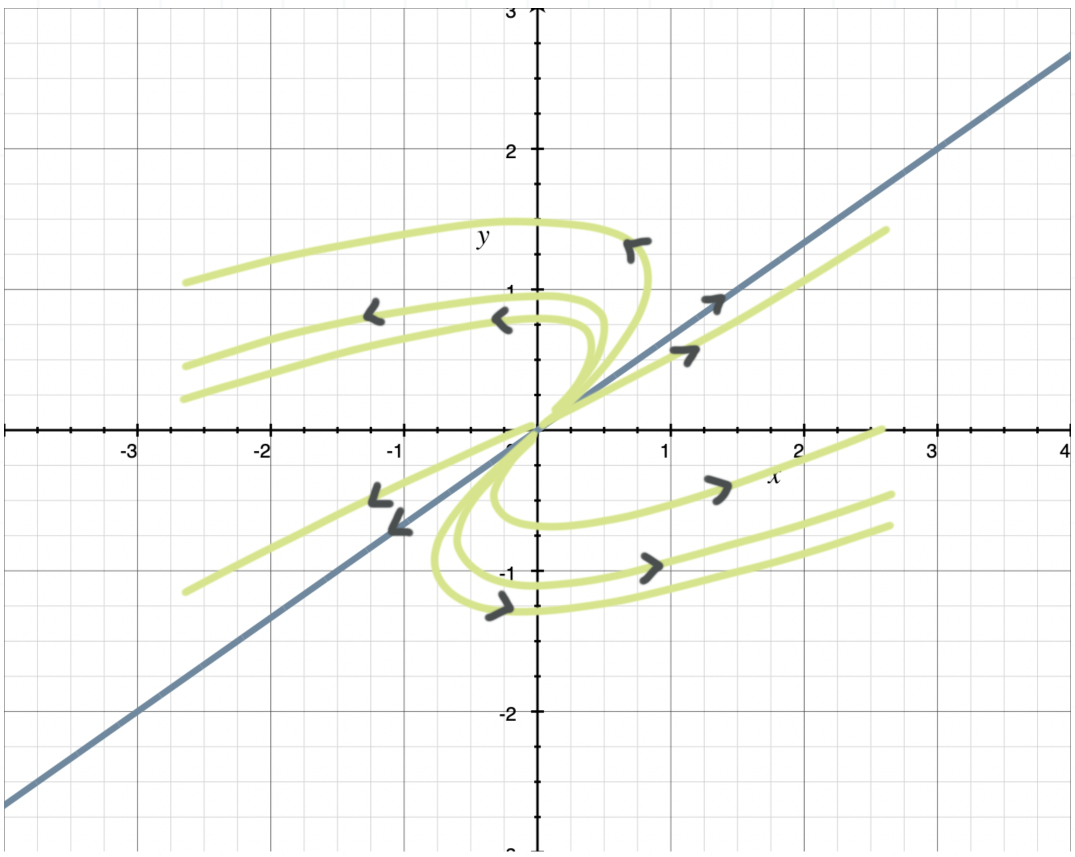
If we test the vector $\overrightarrow{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\overrightarrow{x}' = \begin{bmatrix} 12 & -9 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{x}' = \begin{bmatrix} 12(1) - 9(0) \\ 4(1) + 0(0) \end{bmatrix}$$

$$\overrightarrow{x}' = \begin{bmatrix} 12 \\ 4 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(12,4)$ (toward the first quadrant), which means that the phase portrait must look something like



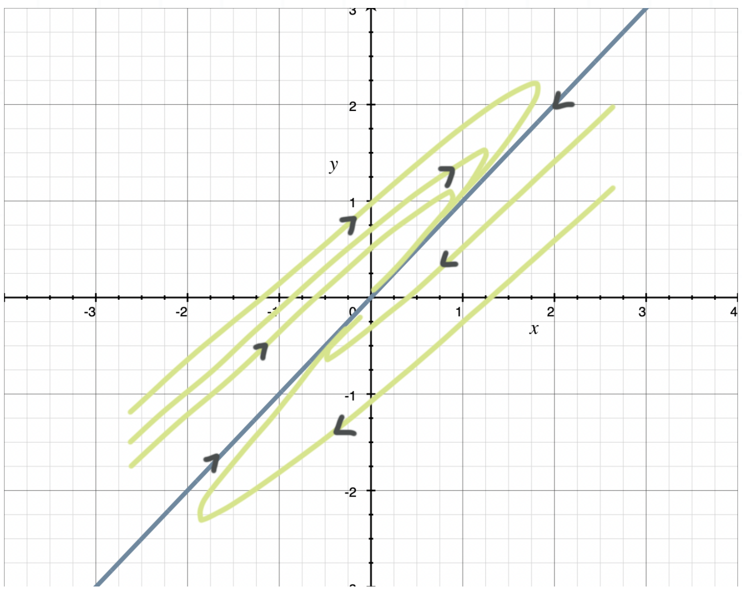
Topic: Phase portraits for equal real Eigenvalues

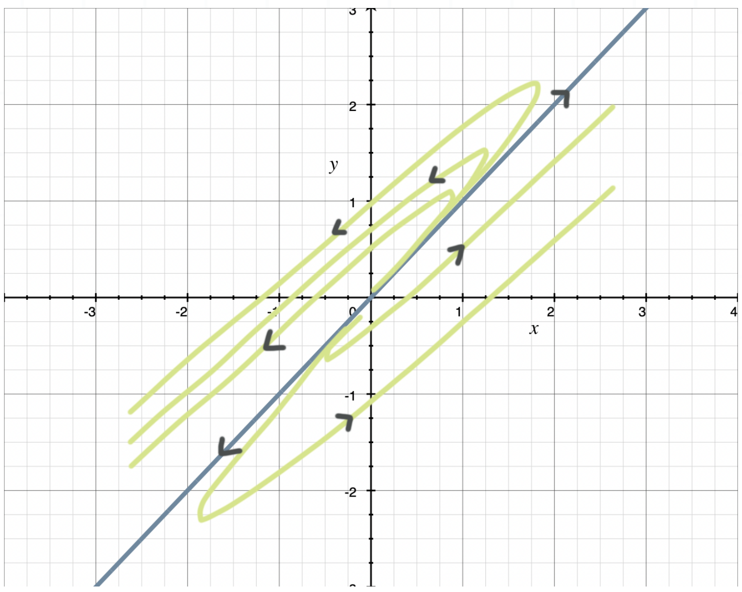
Question: Sketch the phase portrait of the system.

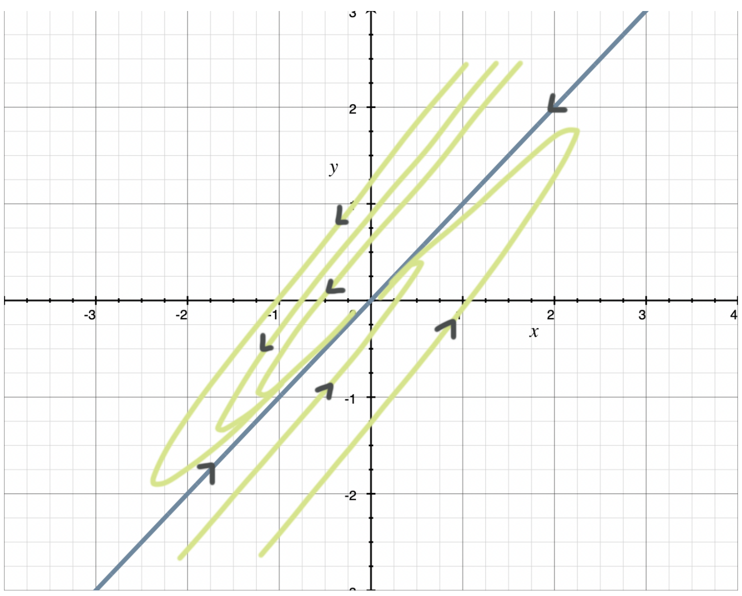
$$x_1' = -6x_1 + 5x_2$$

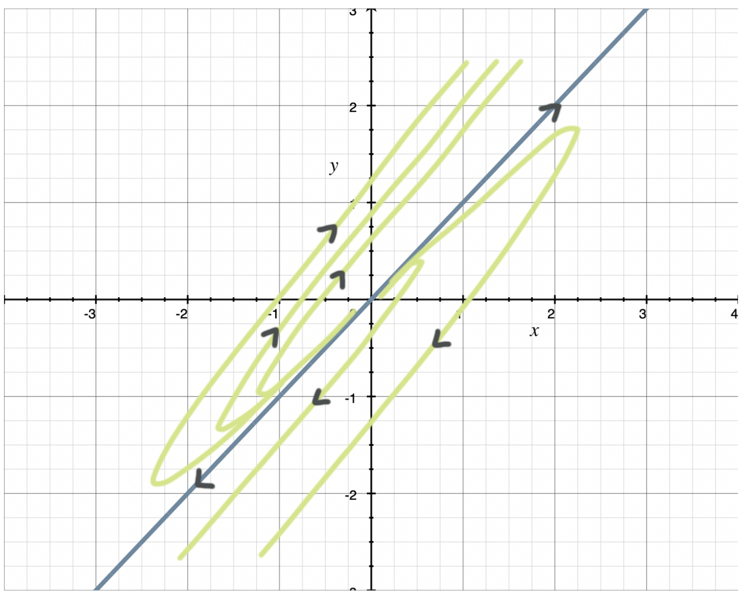
$$x_2' = -5x_1 + 4x_2$$

Answer choices:

A 

B 

C 

D 

Solution: A

The coefficient matrix and $A-\lambda I$ are,

$$A=\begin{bmatrix}-6&5\\-5&4\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}-6-\lambda&5\\-5&4-\lambda\end{bmatrix}$$

and the characteristic equation

$$(-6-\lambda)(4-\lambda)-(5)(-5)=0$$

$$\lambda^2+2\lambda+1=0$$

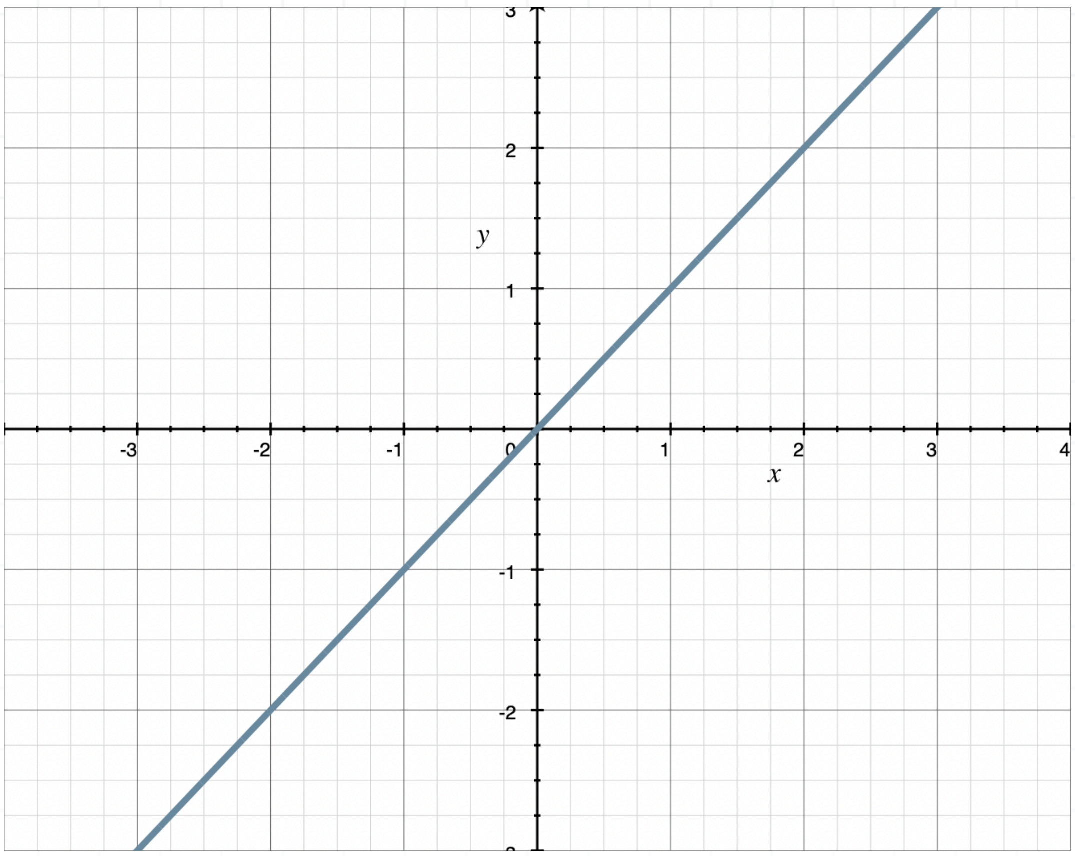
$$(\lambda+1)(\lambda+1)=0$$

$$\lambda=-1,\ -1$$

gives the Eigenvalues $\lambda_1=\lambda_2=-1$, and the associated Eigenvector

$$\vec{k_1}=\begin{bmatrix}1\\1\end{bmatrix}$$

The Eigenvector $\vec{k_1}=(1,1)$ lies along the line $y=x$, so we'll sketch this line.



The Eigenvalue associated with $\overrightarrow{k_1} = (1,1)$ is $\lambda = -1$, which means the direction along that trajectory is toward the origin.

Because the Eigenvalue is negative, we're dealing with an asymptotically stable attractor node that attracts all trajectories.

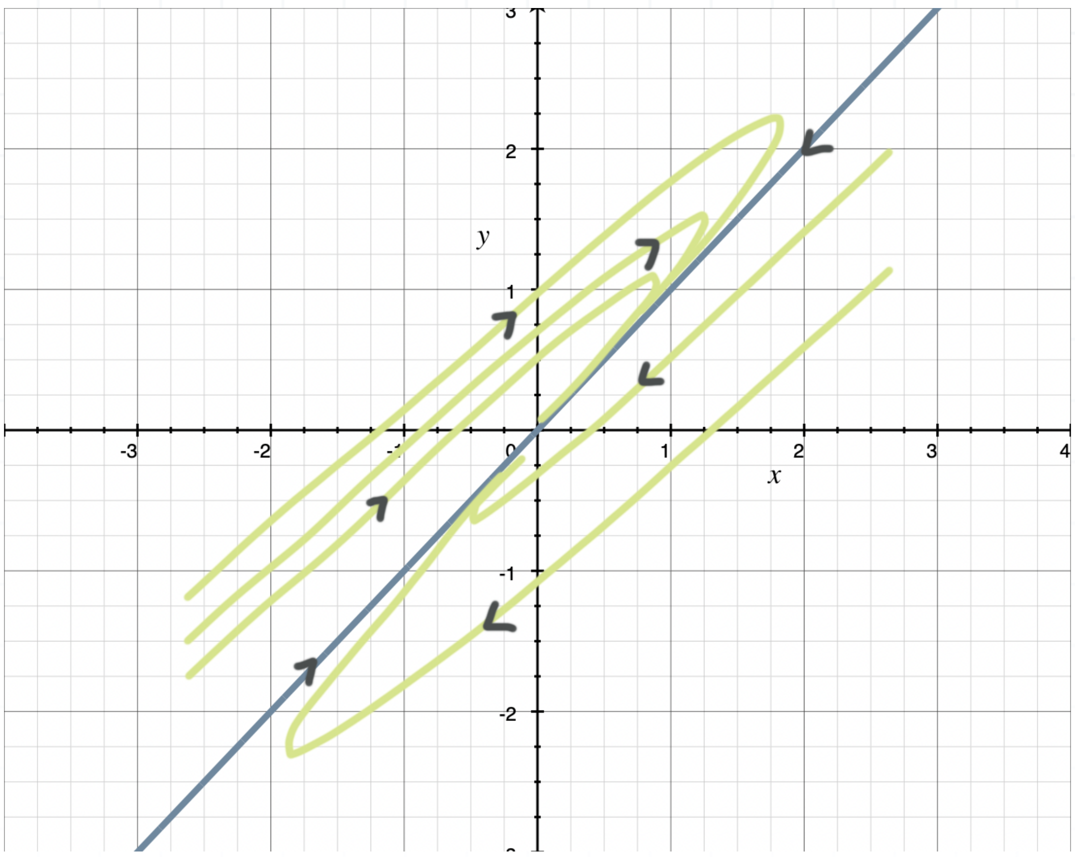
If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} -6 & 5 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} -6(1) + 5(0) \\ -5(1) + 4(0) \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} -6 \\ -5 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(-6,-5)$ (toward the third quadrant), which means that the phase portrait must look something like



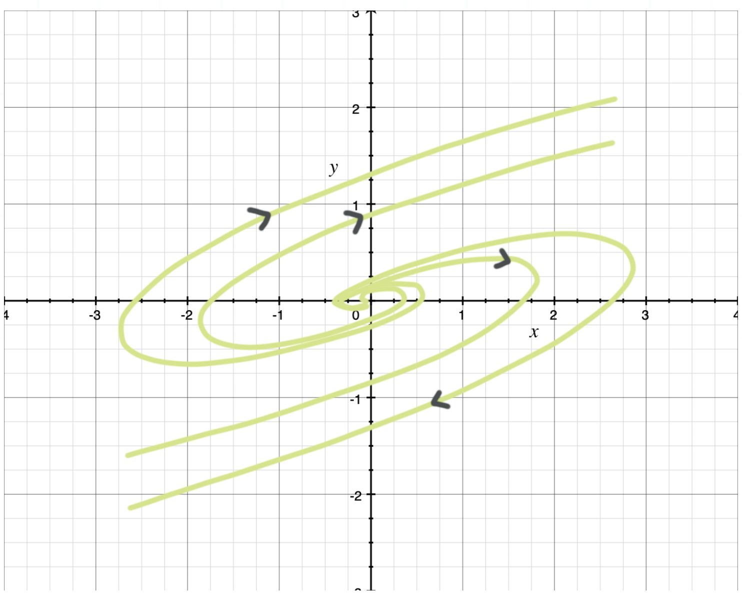
Topic: Phase portraits for complex Eigenvalues

Question: Sketch the phase portrait of the system.

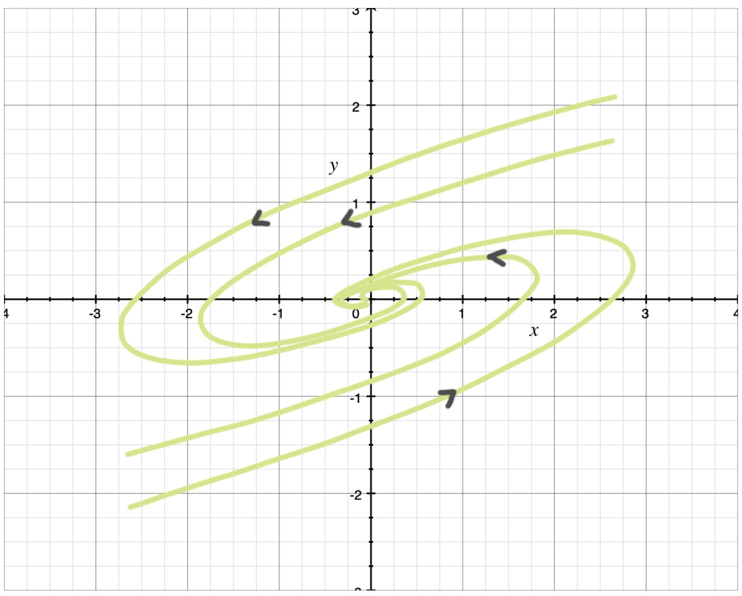
$$x_1' = x_1 - 8x_2$$

$$x_2' = x_1 - 3x_2$$

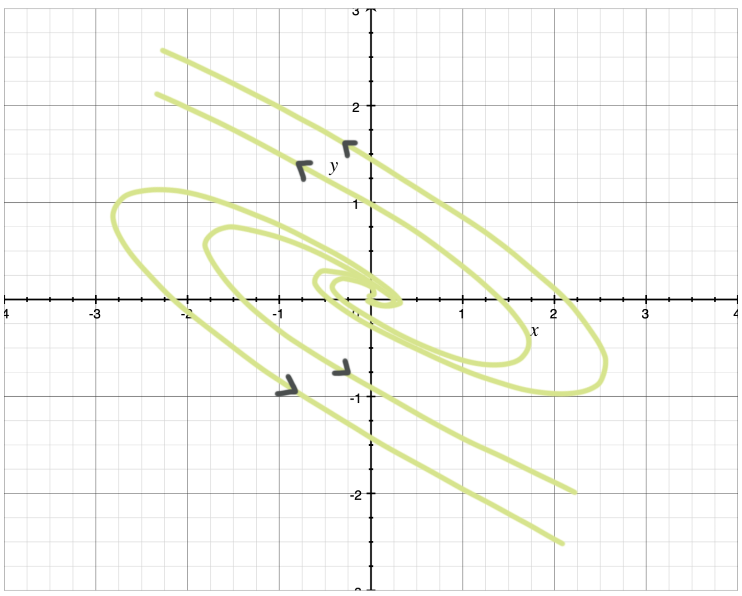
Answer choices:



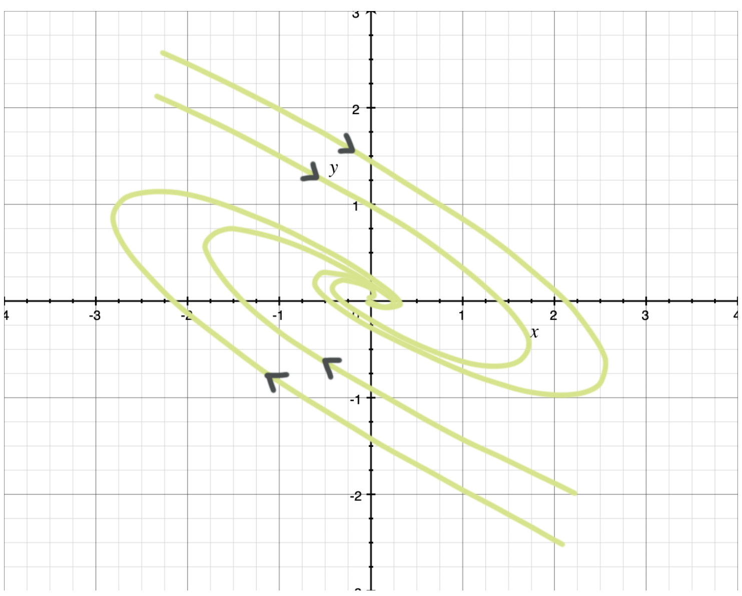
A



B



C



D

Solution: B

The coefficient matrix and $A - \lambda I$ are,

$$A = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & -8 \\ 1 & -3 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(1 - \lambda)(-3 - \lambda) - (-8)(1) = 0$$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$(\lambda + 1)^2 = -4$$

$$\lambda = -1 \pm 2i$$

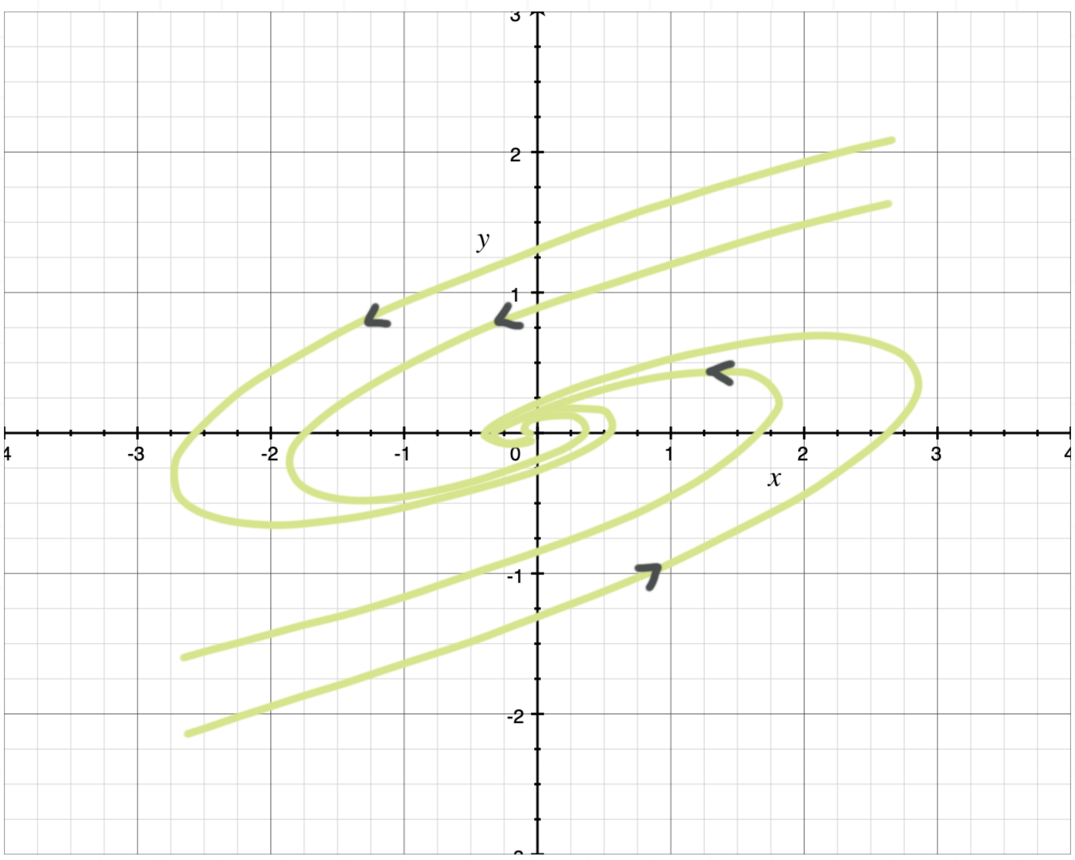
These are Eigenvalues with a negative real part, $\alpha = -1 < 0$, which means we're dealing with an asymptotically stable attractor spiral that attracts all trajectories.

If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 1 & -8 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(1,1)$ (into the first quadrant), which means that the phase portrait must look something like



Topic: Phase portraits for complex Eigenvalues

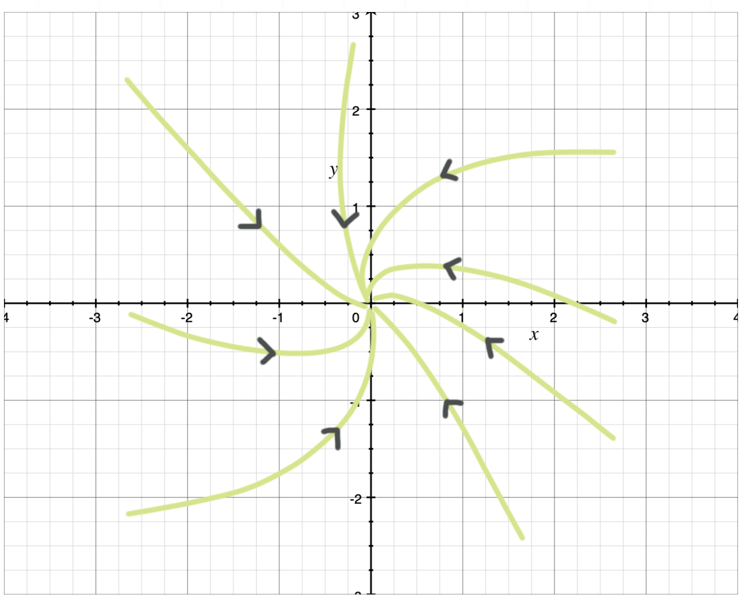
Question: Sketch the phase portrait of the system.

$$x_1' = 5x_1 + x_2$$

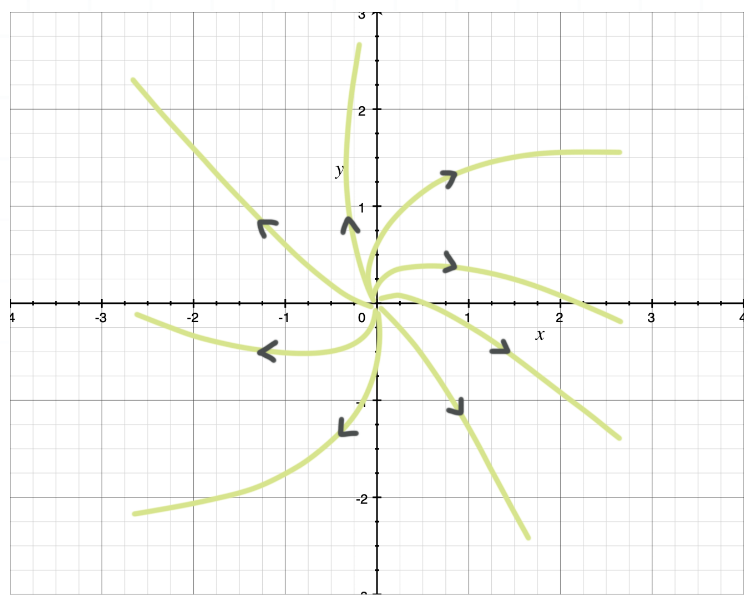
$$x_2' = -2x_1 + 3x_2$$

Answer choices:

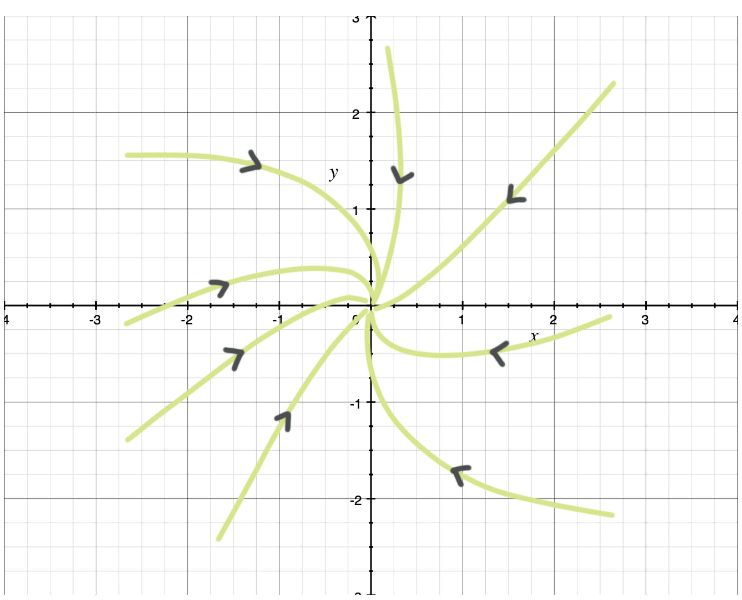
A



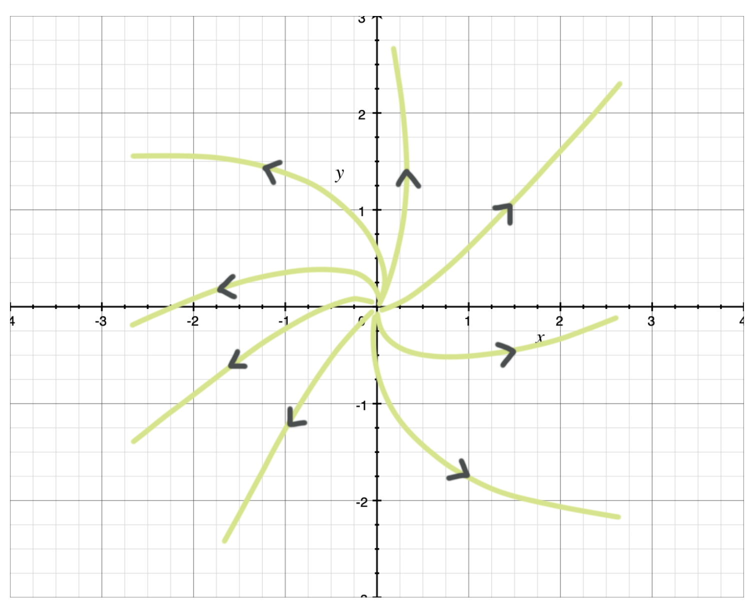
B



C



D



Solution: B

The coefficient matrix and $A - \lambda I$ are,

$$A = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 5 - \lambda & 1 \\ -2 & 3 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(5 - \lambda)(3 - \lambda) - (1)(-2) = 0$$

$$\lambda^2 - 8\lambda + 17 = 0$$

$$(\lambda - 4)^2 = -1$$

$$\lambda = 4 \pm 1i$$

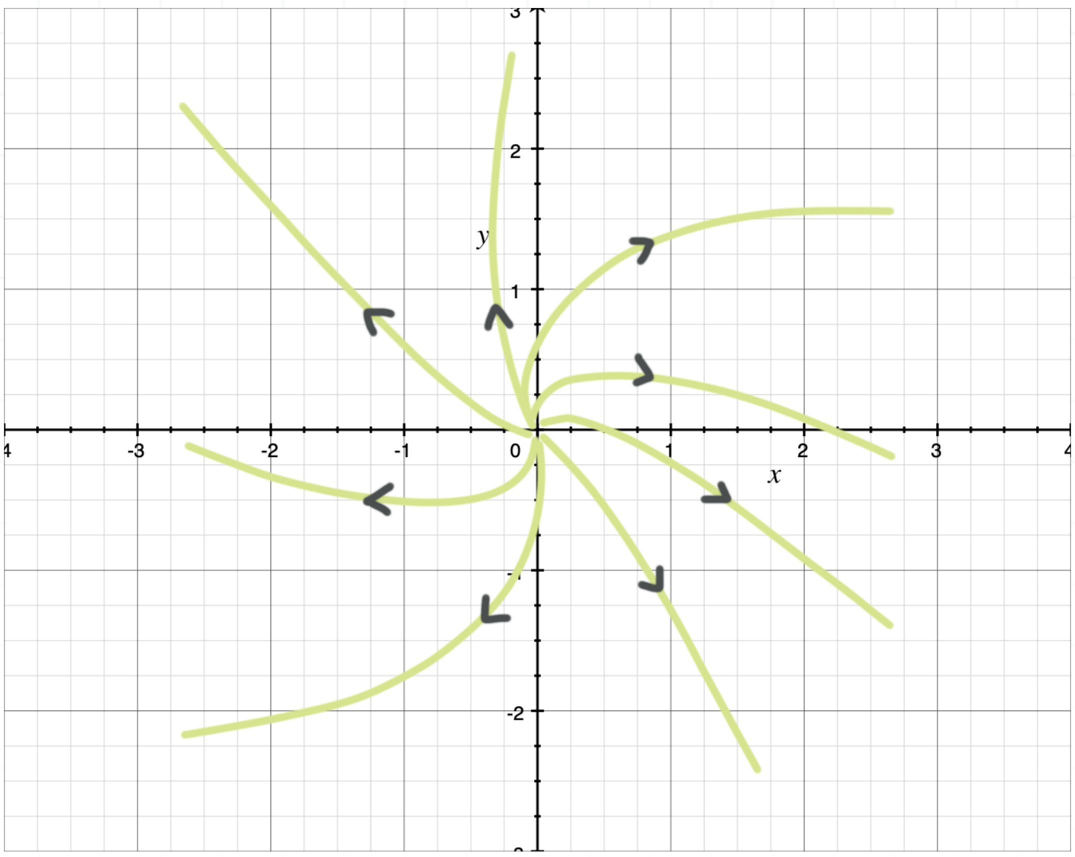
These are Eigenvalues with a positive real part, $\alpha = 4 > 0$, which means we're dealing with an unstable repeller spiral that repels all trajectories.

If we test the vector $\overrightarrow{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\overrightarrow{x}' = \begin{bmatrix} 5 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\overrightarrow{x}' = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(5,-2)$ (into the fourth quadrant), which means that the phase portrait must look something like



Topic: Phase portraits for complex Eigenvalues

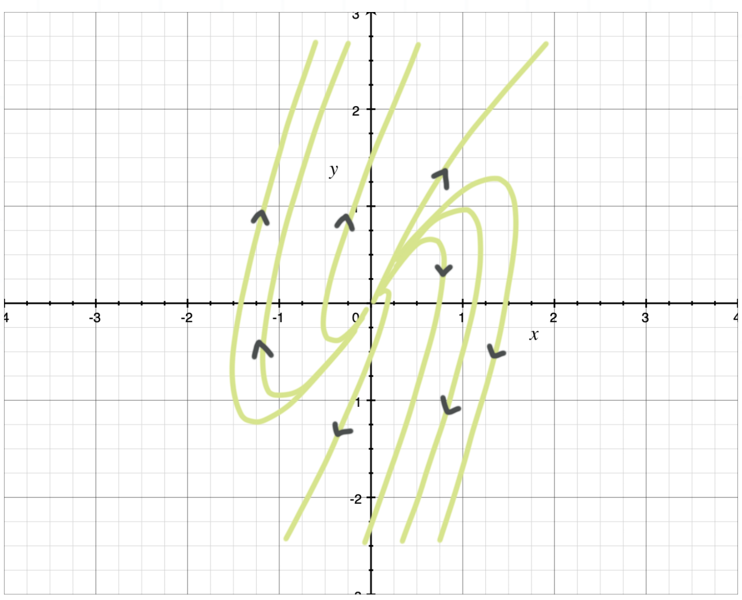
Question: Sketch the phase portrait of the system.

$$x_1' = 4x_1 - 5x_2$$

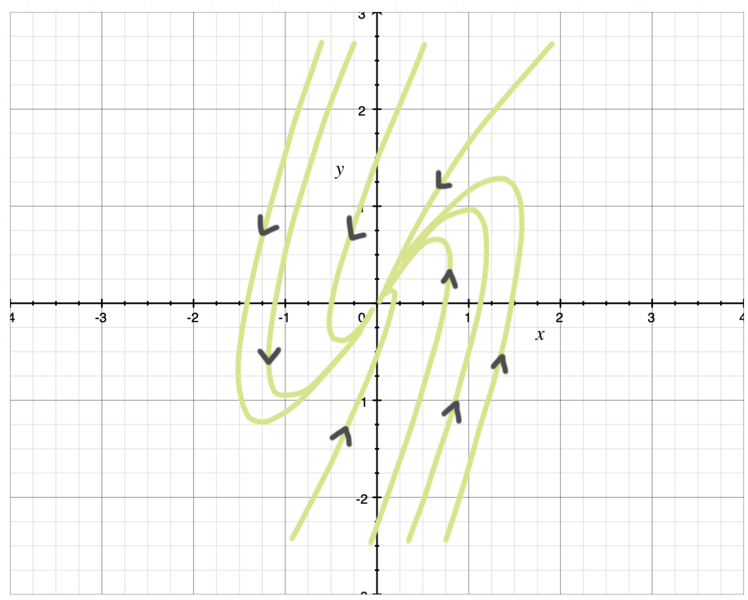
$$x_2' = 5x_1 - 4x_2$$

Answer choices:

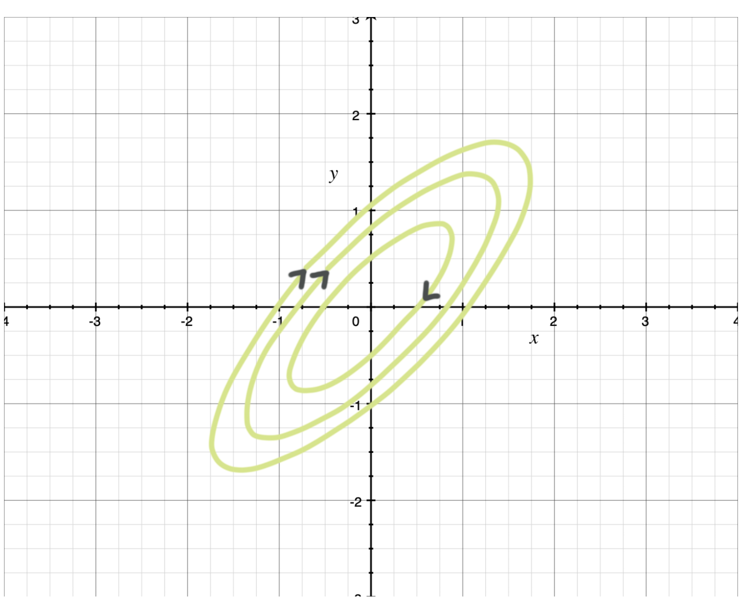
A



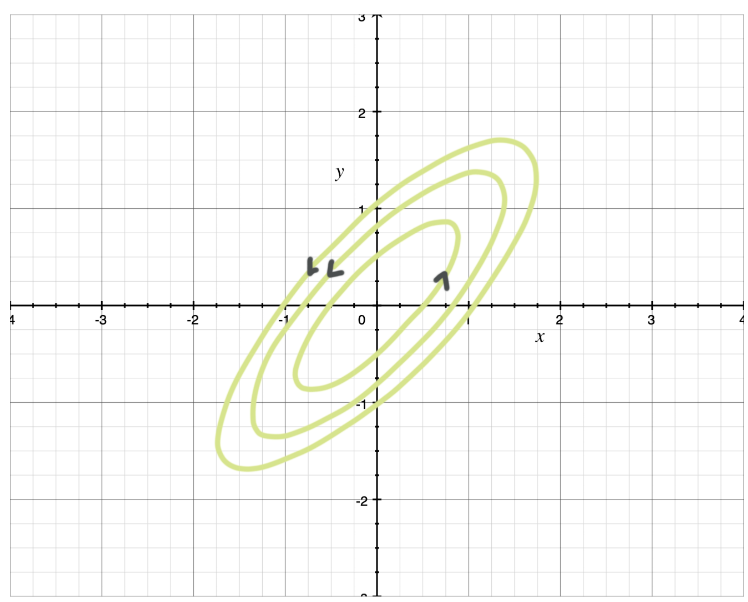
B



C



D



Solution: D

The coefficient matrix and $A - \lambda I$ are,

$$A = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & -5 \\ 5 & -4 - \lambda \end{bmatrix}$$

and the characteristic equation

$$(4 - \lambda)(-4 - \lambda) - (-5)(5) = 0$$

$$\lambda^2 = -9$$

$$\lambda = \pm\sqrt{3}i$$

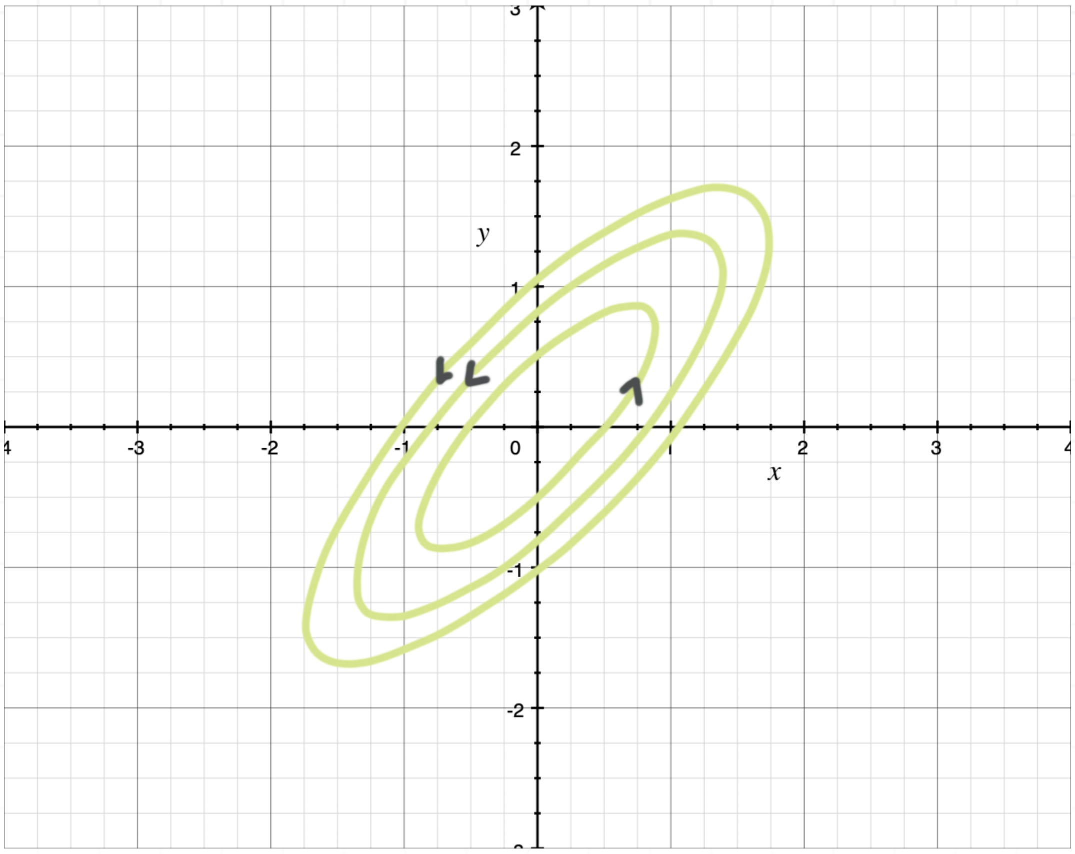
These are Eigenvalues with no real part, $\alpha = 0$, which means we're dealing with a stable center that neither repels nor attracts trajectories.

If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 4 & -5 \\ 5 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(4,5)$ (into the first quadrant), which means that the phase portrait must look something like


y
x
-3
-2
-1
0
1
2
3
2
1
-1
-2

Topic: Undetermined coefficients for nonhomogeneous systems

Question: Given the vector F from a system $\vec{x}' = A\vec{x} + F$, and without knowing the complementary solution of the system, what should be the initial guess for the particular solution?

$$F = \begin{bmatrix} \cos(2t) \\ -\sin t \\ t^2 - 3 \end{bmatrix}$$

Answer choices:

A $\quad \vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cos(2t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \sin t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} t^2$

B $\quad \vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cos(2t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \sin t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} t^2 + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

C $\quad \vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \sin(2t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cos(2t) + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} t^2 + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$

D $\quad \vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \sin(2t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cos(2t) + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t$

$$+\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} t^2 + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} t + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Solution: D

Start by rewriting F as the sum of separate vectors.

$$F = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cos(2t) + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix}$$

The guess for the particular solution for the $\cos(2t)$ portion will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \sin(2t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cos(2t)$$

The guess for the particular solution for the $\sin t$ portion will be

$$\vec{x_p} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t$$

The guess for the particular solution for the $t^2 - 3$ portion will be

$$\vec{x}_p = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} t^2 + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} t + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

So the guess for the complete particular solution, without accounting for any overlap with the complementary solution, will be

$$\vec{x}_p = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \sin(2t) + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \cos(2t) + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t$$

$$+ \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} t^2 + \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix} t + \begin{bmatrix} g_1 \\ g_2 \\ g_3 \end{bmatrix}$$

Topic: Undetermined coefficients for nonhomogeneous systems

Question: Use the method of undetermined coefficients to find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \vec{x} + \begin{bmatrix} -e^{3t} \\ t^2 + 5 \end{bmatrix}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} -e^{3t} \\ \frac{1}{2}e^{3t} - \frac{1}{5}t^2 - \frac{2}{25}t - \frac{127}{125} \end{bmatrix}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} -e^{3t} \\ \frac{1}{2}e^{3t} - \frac{1}{5}t^2 - \frac{2}{25}t - \frac{127}{125} \end{bmatrix}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{5t} + \begin{bmatrix} e^{3t} \\ 2e^{3t} + \frac{1}{5}t^2 - 2t + 1 \end{bmatrix}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} e^{3t} \\ 2e^{3t} + \frac{1}{5}t^2 - 2t + 1 \end{bmatrix}$

Solution: B

We have to start by finding the complementary solution, which means we'll find $|A-\lambda I|$.

$$|A-\lambda I| = \left|\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right|$$

$$|A-\lambda I| = \left|\begin{bmatrix}2 & 0\\1 & 5\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right|$$

$$|A-\lambda I| = \begin{vmatrix}2-\lambda & 0\\1 & 5-\lambda\end{vmatrix}$$

$$|A-\lambda I| = (2-\lambda)(5-\lambda)-(0)(1)$$

$$|A-\lambda I| = (2-\lambda)(5-\lambda)$$

So the characteristic equation is

$$(2-\lambda)(5-\lambda)=0$$

$$\lambda = 2, 5$$

Then for these Eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 5$, we find

$$A-2I = \begin{bmatrix}2-2 & 0\\1 & 5-2\end{bmatrix}$$

$$A-2I = \begin{bmatrix}0 & 0\\1 & 3\end{bmatrix}$$

and

$$A - 5I = \begin{bmatrix} 2-5 & 0 \\ 1 & 5-5 \end{bmatrix}$$

$$A - 5I = \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix} \qquad \begin{bmatrix} -3 & 0 \\ 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 + 3k_2 = 0 \qquad k_1 = 0$$

$$k_1 = -3k_2$$

For the first system, we choose $(k_1, k_2) = (3, -1)$. And from the second system, we choose $(k_1, k_2) = (0,1)$.

$$\vec{k_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{2t} \qquad \vec{x_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} -e^{3t} \\ t^2 + 5 \end{bmatrix}$$

$$F = \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

We want a particular solution in the same form, so our guess will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

$$\vec{x_p}' = \begin{bmatrix} 3a_1 \\ 3a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 2b_1 \\ 2b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Plugging this into the matrix equation representing the system of differential equations $\vec{x}' = A\vec{x} + F$, we get

$$\begin{bmatrix} 3a_1 \\ 3a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} 2b_1 \\ 2b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3t} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} t + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \right]$$

$$+ \begin{bmatrix} -1 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 3a_1e^{3t} + 2b_1t + c_1 \\ 3a_2e^{3t} + 2b_2t + c_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} a_1e^{3t} + b_1t^2 + c_1t + d_1 \\ a_2e^{3t} + b_2t^2 + c_2t + d_2 \end{bmatrix} + \begin{bmatrix} -e^{3t} \\ t^2 + 5 \end{bmatrix}$$

Breaking this equation into a system of two equations gives

$$3a_1e^{3t} + 2b_1t + c_1 = 2(a_1e^{3t} + b_1t^2 + c_1t + d_1) + 0(a_2e^{3t} + b_2t^2 + c_2t + d_2) - e^{3t}$$

$$3a_2e^{3t} + 2b_2t + c_2 = 1(a_1e^{3t} + b_1t^2 + c_1t + d_1) + 5(a_2e^{3t} + b_2t^2 + c_2t + d_2) + t^2 + 5$$

which simplifies to

$$3a_1e^{3t} + 2b_1t + c_1 = (2a_1 - 1)e^{3t} + 2b_1t^2 + 2c_1t + 2d_1$$

$$3a_2e^{3t} + 2b_2t + c_2 = (a_1 + 5a_2)e^{3t} + (b_1 + 5b_2 + 1)t^2 + (c_1 + 5c_2)t + (d_1 + 5d_2 + 5)$$

These equations can each be broken into its own system.

$$3a_1 = 2a_1 - 1$$

$$0 = 2b_1$$

$$2b_1 = 2c_1$$

$$c_1 = 2d_1$$

$$3a_2 = a_1 + 5a_2$$

$$0 = b_1 + 5b_2 + 1$$

$$2b_2 = c_1 + 5c_2$$

$$c_2 = d_1 + 5d_2 + 5$$

Solving these eight equations as a system gives $(a_1, a_2) = (-1, 1/2)$, $(b_1, b_2) = (0, -1/5)$, $(c_1, c_2) = (0, -2/25)$, and $(d_1, d_2) = (0, -127/125)$. Therefore, the particular solution becomes

$$\overrightarrow{x_p} = \begin{bmatrix} -1 \\ \frac{1}{2} \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ -\frac{1}{5} \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ -\frac{2}{25} \end{bmatrix} t + \begin{bmatrix} 0 \\ -\frac{127}{125} \end{bmatrix}$$

Now we can rewrite the particular solution as one vector.

$$\overrightarrow{x_p} = \begin{bmatrix} -e^{3t} \\ \frac{1}{2}e^{3t} - \frac{1}{5}t^2 - \frac{2}{25}t - \frac{127}{125} \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{5t} + \begin{bmatrix} -e^{3t} \\ \frac{1}{2}e^{3t} - \frac{1}{5}t^2 - \frac{2}{25}t - \frac{127}{125} \end{bmatrix}$$

Topic: Undetermined coefficients for nonhomogeneous systems

Question: Use the method of undetermined coefficients to find the general solution to the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} t+2 \\ -3e^{3t} \\ \sin t \end{bmatrix}$$

Answer choices:

A $$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} e^{(2+\sqrt{2})t} + c_3 \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix} e^{(2-\sqrt{2})t}$$

$$+ \begin{bmatrix} -\frac{3}{4}t - \frac{19}{8} + 3e^{3t} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ -\frac{1}{4}t - \frac{9}{8} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ \frac{1}{2}t + 2 + 3e^{3t} - \frac{6}{17}\sin t - \frac{7}{17}\cos t \end{bmatrix}$$

B $$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix} te^{-2t}$$

$$+\begin{bmatrix} -\frac{3}{4}t - \frac{19}{8} + 3e^{3t} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ -\frac{1}{4}t - \frac{9}{8} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ \frac{1}{2}t + 2 + 3e^{3t} - \frac{6}{17}\sin t - \frac{7}{17}\cos t \end{bmatrix}$$

C $$\vec{x} = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}e^{(2+\sqrt{2})t} + c_3\begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}e^{(2-\sqrt{2})t}$$

$$+\begin{bmatrix} -6t - 19 + 3e^{3t} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ -2t - 9 + 3e^{3t} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ 4t + 16 + 3e^{3t} - \frac{6}{17}\sin t - \frac{7}{17}\cos t \end{bmatrix}$$

D $$\vec{x} = c_1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}e^{2t} + c_2\begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix}e^{-2t} + c_3\begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}te^{-2t}$$

$$+\begin{bmatrix} -6t - 19 + 3e^{3t} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ -2t - 9 + 3e^{3t} + \frac{1}{17}\sin t + \frac{4}{17}\cos t \\ 4t + 16 + 3e^{3t} - \frac{6}{17}\sin t - \frac{7}{17}\cos t \end{bmatrix}$$

Solution: A

We have to start by finding the complementary solution, which means we'll find $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 2-\lambda & 0 & 1 \\ 0 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda) \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} - 0 \begin{vmatrix} 0 & 1 \\ 1 & 2-\lambda \end{vmatrix} + 1 \begin{vmatrix} 0 & 2-\lambda \\ 1 & 1 \end{vmatrix}$$

$$|A - \lambda I| = (2-\lambda)[(2-\lambda)(2-\lambda) - (1)(1)] + [(0)(1) - (2-\lambda)(1)]$$

$$|A - \lambda I| = (2-\lambda)(3 - 4\lambda + \lambda^2) - (2-\lambda)$$

$$|A - \lambda I| = (2-\lambda)(3 - 4\lambda + \lambda^2 - 1)$$

$$|A - \lambda I| = (2-\lambda)(\lambda^2 - 4\lambda + 2)$$

So the characteristic equation is

$$(2-\lambda)(\lambda^2 - 4\lambda + 2) = 0$$

$$\lambda = 2,\ 2 \pm \sqrt{2}$$

Then for these Eigenvalues, $\lambda_1 = 2$, $\lambda_2 = 2 + \sqrt{2}$, and $\lambda_3 = 2 - \sqrt{2}$, we find

$$A - 2I = \begin{bmatrix} 2-2 & 0 & 1 \\ 0 & 2-2 & 1 \\ 1 & 1 & 2-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

and

$$A - (2+\sqrt{2})I = \begin{bmatrix} 2-(2+\sqrt{2}) & 0 & 1 \\ 0 & 2-(2+\sqrt{2}) & 1 \\ 1 & 1 & 2-(2+\sqrt{2}) \end{bmatrix}$$

$$A - (2+\sqrt{2})I = \begin{bmatrix} -\sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & 1 \\ 1 & 1 & -\sqrt{2} \end{bmatrix}$$

and

$$A - (2-\sqrt{2})I = \begin{bmatrix} 2-(2-\sqrt{2}) & 0 & 1 \\ 0 & 2-(2-\sqrt{2}) & 1 \\ 1 & 1 & 2-(2-\sqrt{2}) \end{bmatrix}$$

$$A - (2-\sqrt{2})I = \begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & \sqrt{2} & 1 \\ 1 & 1 & \sqrt{2} \end{bmatrix}$$

Put these three matrices into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -\sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & 1 \\ 1 & 1 & -\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -\sqrt{2} \\ -\sqrt{2} & 0 & 1 \\ 0 & -\sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -\sqrt{2} \\ 0 & \sqrt{2} & -1 \\ 0 & -\sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -\sqrt{2} \\ 0 & 1 & -\frac{\sqrt{2}}{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & -\frac{\sqrt{2}}{2} \\ 0 & -\sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} \sqrt{2} & 0 & 1 \\ 0 & \sqrt{2} & 1 \\ 1 & 1 & \sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & \sqrt{2} \\ \sqrt{2} & 0 & 1 \\ 0 & \sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & \sqrt{2} \\ 0 & -\sqrt{2} & -1 \\ 0 & \sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & \sqrt{2} \\ 0 & 1 & \frac{\sqrt{2}}{2} \\ 0 & \sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & \frac{\sqrt{2}}{2} \\ 0 & \sqrt{2} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & -\frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & \frac{\sqrt{2}}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 + k_2 = 0 \qquad k_1 - \frac{\sqrt{2}}{2}k_3 = 0 \qquad k_1 - \frac{\sqrt{2}}{2}k_3 = 0$$

$$k_3 = 0 \qquad k_2 - \frac{\sqrt{2}}{2}k_3 = 0 \qquad k_2 + \frac{\sqrt{2}}{2}k_3 = 0$$

Rewriting these systems gives

$$k_1 = -k_2 \qquad k_1 = \frac{\sqrt{2}}{2}k_3 \qquad k_1 = \frac{\sqrt{2}}{2}k_3$$

$$k_3 = 0 \qquad k_2 = \frac{\sqrt{2}}{2}k_3 \qquad k_2 = -\frac{\sqrt{2}}{2}k_3$$

For the first system, we choose $(k_1, k_2, k_3) = (1, -1, 0)$, from the second system, we choose $(k_1, k_2, k_3) = (1, 1, \sqrt{2})$, and from the third system, we choose $(k_1, k_2, k_3) = (1, -1, \sqrt{2})$.

$$\vec{k_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} \qquad \vec{k_3} = \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2}e^{\lambda_2 t} \qquad \vec{x_3} = \vec{k_3}e^{\lambda_3 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} \qquad \vec{x_2} = \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} e^{(2+\sqrt{2})t} \qquad \vec{x_3} = \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix} e^{(2-\sqrt{2})t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \\ \sqrt{2} \end{bmatrix} e^{(2+\sqrt{2})t} + c_3 \begin{bmatrix} 1 \\ -1 \\ \sqrt{2} \end{bmatrix} e^{(2-\sqrt{2})t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} t+2 \\ -3e^{3t} \\ \sin t \end{bmatrix}$$

$$F = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t$$

We want a particular solution in the same form, so our guess will be

$$\vec{x_p} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{3t} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \sin t + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \cos t$$

$$\vec{x}_p' = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 3c_1 \\ 3c_2 \\ 3c_3 \end{bmatrix} e^{3t} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} -e_1 \\ -e_2 \\ -e_3 \end{bmatrix} \sin t$$

Plugging this into the matrix equation representing the system of differential equations $\vec{x}' = A\vec{x} + F$, we get

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} + \begin{bmatrix} 3c_1 \\ 3c_2 \\ 3c_3 \end{bmatrix} e^{3t} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} -e_1 \\ -e_2 \\ -e_3 \end{bmatrix} \sin t$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{3t} + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \sin t + \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \cos t \right]$$

$$+ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} e^{3t} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t$$

$$\begin{bmatrix} a_1 + 3c_1e^{3t} + d_1 \cos t - e_1 \sin t \\ a_2 + 3c_2e^{3t} + d_2 \cos t - e_2 \sin t \\ a_3 + 3c_3e^{3t} + d_3 \cos t - e_3 \sin t \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} a_1t + b_1 + c_1e^{3t} + d_1 \sin t + e_1 \cos t \\ a_2t + b_2 + c_2e^{3t} + d_2 \sin t + e_2 \cos t \\ a_3t + b_3 + c_3e^{3t} + d_3 \sin t + e_3 \cos t \end{bmatrix} + \begin{bmatrix} t+2 \\ -3e^{3t} \\ \sin t \end{bmatrix}$$

Breaking this equation into a system of three equations gives

$$a_1 + 3c_1e^{3t} + d_1\cos t - e_1\sin t = 2(a_1t + b_1 + c_1e^{3t} + d_1\sin t + e_1\cos t)$$

$$+0(a_2t + b_2 + c_2e^{3t} + d_2\sin t + e_2\cos t)$$

$$+1(a_3t + b_3 + c_3e^{3t} + d_3\sin t + e_3\cos t) + t + 2$$

$$a_2 + 3c_2e^{3t} + d_2\cos t - e_2\sin t = 0(a_1t + b_1 + c_1e^{3t} + d_1\sin t + e_1\cos t)$$

$$+2(a_2t + b_2 + c_2e^{3t} + d_2\sin t + e_2\cos t)$$

$$+1(a_3t + b_3 + c_3e^{3t} + d_3\sin t + e_3\cos t) - 3e^{3t}$$

$$a_3 + 3c_3e^{3t} + d_3\cos t - e_3\sin t = 1(a_1t + b_1 + c_1e^{3t} + d_1\sin t + e_1\cos t)$$

$$+1(a_2t + b_2 + c_2e^{3t} + d_2\sin t + e_2\cos t)$$

$$+2(a_3t + b_3 + c_3e^{3t} + d_3\sin t + e_3\cos t) + \sin t$$

which simplifies to

$$a_1 + 3c_1e^{3t} + d_1\cos t - e_1\sin t = 2a_1t + 2b_1 + 2c_1e^{3t} + 2d_1\sin t + 2e_1\cos t$$

$$+a_3t + b_3 + c_3e^{3t} + d_3\sin t + e_3\cos t + t + 2$$

$$a_2 + 3c_2e^{3t} + d_2\cos t - e_2\sin t = 2a_2t + 2b_2 + 2c_2e^{3t} + 2d_2\sin t + 2e_2\cos t$$

$$+a_3t + b_3 + c_3e^{3t} + d_3\sin t + e_3\cos t - 3e^{3t}$$

$$a_3 + 3c_3e^{3t} + d_3\cos t - e_3\sin t = a_1t + b_1 + c_1e^{3t} + d_1\sin t + e_1\cos t$$

$$+a_2t + b_2 + c_2e^{3t} + d_2\sin t + e_2\cos t$$

$$+2a_3t + 2b_3 + 2c_3e^{3t} + 2d_3\sin t + 2e_3\cos t + \sin t$$

and then to

$$(a_1) + (3c_1)e^{3t} + (d_1)\cos t + (-e_1)\sin t = (2a_1 + a_3 + 1)t + (2b_1 + b_3 + 2)$$

$$+(2c_1 + c_3)e^{3t} + (2e_1 + e_3)\cos t + (2d_1 + d_3)\sin t$$

$$(a_2) + (3c_2)e^{3t} + (d_2)\cos t + (-e_2)\sin t = (2a_2 + a_3)t + (2b_2 + b_3)$$

$$+(2c_2 + c_3 - 3)e^{3t} + (2e_2 + e_3)\cos t + (2d_2 + d_3)\sin t$$

$$(a_3) + (3c_3)e^{3t} + (d_3)\cos t + (-e_3)\sin t = (a_1 + a_2 + 2a_3)t + (b_1 + b_2 + 2b_3)$$

$$+(c_1 + c_2 + 2c_3)e^{3t} + (e_1 + e_2 + 2e_3)\cos t + (d_1 + d_2 + 2d_3 + 1)\sin t$$

These equations can each be broken into its own system.

$$0 = 2a_1 + a_3 + 1 \qquad 0 = 2a_2 + a_3 \qquad 0 = a_1 + a_2 + 2a_3$$

$$a_1 = 2b_1 + b_3 + 2 \qquad a_2 = 2b_2 + b_3 \qquad a_3 = b_1 + b_2 + 2b_3$$

$$3c_1 = 2c_1 + c_3 \qquad 3c_2 = 2c_2 + c_3 - 3 \qquad 3c_3 = c_1 + c_2 + 2c_3$$

$$d_1 = 2e_1 + e_3 \qquad d_2 = 2e_2 + e_3 \qquad d_3 = e_1 + e_2 + 2e_3$$

$$-e_1 = 2d_1 + d_3 \qquad -e_2 = 2d_2 + d_3 \qquad -e_3 = d_1 + d_2 + 2d_3 + 1$$

Solving these 15 equations as a system gives $(a_1, a_2, a_3) = (-3/4, -1/4, 1/2)$, $(b_1, b_2, b_3) = (-19/8, -9/8, 2)$, $(c_1, c_2, c_3) = (3, 0, 3)$, $(d_1, d_2, d_3) = (1/17, 1/17, -6/17)$, and $(e_1, e_2, e_3) = (4/17, 4/17, -7/17)$. Therefore, the particular solution becomes

$$\vec{x}_p=\begin{bmatrix}-\frac{3}{4}\\-\frac{1}{4}\\\frac{1}{2}\end{bmatrix}t+\begin{bmatrix}-\frac{19}{8}\\-\frac{9}{8}\\2\end{bmatrix}+\begin{bmatrix}3\\0\\3\end{bmatrix}e^{3t}+\begin{bmatrix}\frac{1}{17}\\\frac{1}{17}\\-\frac{6}{17}\end{bmatrix}\sin t+\begin{bmatrix}\frac{4}{17}\\\frac{4}{17}\\-\frac{7}{17}\end{bmatrix}\cos t$$

Now we can rewrite the particular solution as one vector.

$$\vec{x}_p=\begin{bmatrix}-\frac{3}{4}t-\frac{19}{8}+3e^{3t}+\frac{1}{17}\sin t+\frac{4}{17}\cos t\\-\frac{1}{4}t-\frac{9}{8}+\frac{1}{17}\sin t+\frac{4}{17}\cos t\\\frac{1}{2}t+2+3e^{3t}-\frac{6}{17}\sin t-\frac{7}{17}\cos t\end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x}=\vec{x}_c+\vec{x}_p$$

$$\vec{x}=c_1\begin{bmatrix}1\\-1\\0\end{bmatrix}e^{2t}+c_2\begin{bmatrix}1\\1\\\sqrt{2}\end{bmatrix}e^{(2+\sqrt{2})t}+c_3\begin{bmatrix}1\\-1\\\sqrt{2}\end{bmatrix}e^{(2-\sqrt{2})t}$$

$$+\begin{bmatrix}-\frac{3}{4}t-\frac{19}{8}+3e^{3t}+\frac{1}{17}\sin t+\frac{4}{17}\cos t\\-\frac{1}{4}t-\frac{9}{8}+\frac{1}{17}\sin t+\frac{4}{17}\cos t\\\frac{1}{2}t+2+3e^{3t}-\frac{6}{17}\sin t-\frac{7}{17}\cos t\end{bmatrix}$$

Topic: Variation of parameters for nonhomogeneous systems

Question: Given the vector F and the complementary solution for a system of differential equations $\overrightarrow{x}' = A\overrightarrow{x} + F$, use variation of parameters to find the system's particular solution.

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 3 \\ 5 \end{bmatrix} e^{-3t}$$

$$F(t) = \begin{bmatrix} e^t \\ -\sin t \end{bmatrix}$$

Answer choices:

A $\quad \overrightarrow{x_p} = \begin{bmatrix} 7 \\ -25 \end{bmatrix} e^t + \frac{17}{47} \begin{bmatrix} \frac{41}{5} \\ 87 \end{bmatrix} \cos t - \frac{17}{47} \begin{bmatrix} \frac{623}{5} \\ 61 \end{bmatrix} \sin t$

B $\quad \overrightarrow{x_p} = \begin{bmatrix} 7 \\ -25 \end{bmatrix} e^t - \frac{17}{47} \begin{bmatrix} \frac{623}{5} \\ 61 \end{bmatrix} \sin t$

C $\quad \overrightarrow{x_p} = \frac{1}{22} \begin{bmatrix} -7 \\ 25 \end{bmatrix} e^t + \frac{1}{110} \begin{bmatrix} -3 \\ 17 \end{bmatrix} \cos t + \frac{1}{110} \begin{bmatrix} -15 \\ 9 \end{bmatrix} \sin t$

D $\quad \overrightarrow{x_p} = -\frac{1}{22} \begin{bmatrix} 7 \\ -25 \end{bmatrix} e^t - \frac{11}{940} \begin{bmatrix} \frac{623}{5} \\ 61 \end{bmatrix} \sin t$

Solution: C

We already have the complementary solution, so we'll use the solution vectors $\vec{x_1}$ and $\vec{x_2}$ to form $\Phi(t)$.

$$\Phi(t)=\begin{bmatrix} e^{2t} & 3e^{-3t} \\ -2e^{2t} & 5e^{-3t} \end{bmatrix}$$

Find the inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} e^{2t} & 3e^{-3t} & | & 1 & 0 \\ -2e^{2t} & 5e^{-3t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3e^{-5t} & | & e^{-2t} & 0 \\ -2e^{2t} & 5e^{-3t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3e^{-5t} & | & e^{-2t} & 0 \\ 0 & 11e^{-3t} & | & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3e^{-5t} & | & e^{-2t} & 0 \\ 0 & 1 & | & \frac{2}{11}e^{3t} & \frac{1}{11}e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{5}{11}e^{-2t} & -\frac{3}{11}e^{-2t} \\ 0 & 1 & | & \frac{2}{11}e^{3t} & \frac{1}{11}e^{3t} \end{bmatrix}$$

Now with

$$\Phi(t)=\begin{bmatrix} e^{2t} & 3e^{-3t} \\ -2e^{2t} & 5e^{-3t} \end{bmatrix} \quad \Phi^{-1}(t)=\begin{bmatrix} \frac{5}{11}e^{-2t} & -\frac{3}{11}e^{-2t} \\ \frac{2}{11}e^{3t} & \frac{1}{11}e^{3t} \end{bmatrix} \quad F(t)=\begin{bmatrix} e^{t} \\ -\sin t \end{bmatrix}$$

the particular solution from the method of variation of parameters is given by

$$\vec{x_p} = \Phi(t) \int \Phi^{-1}(t) F(t)\ dt$$

$$\vec{x_p} = \begin{bmatrix} e^{2t} & 3e^{-3t} \\ -2e^{2t} & 5e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{5}{11}e^{-2t} & -\frac{3}{11}e^{-2t} \\ \frac{2}{11}e^{3t} & \frac{1}{11}e^{3t} \end{bmatrix} \begin{bmatrix} e^t \\ -\sin t \end{bmatrix}\ dt$$

$$\vec{x_p} = \begin{bmatrix} e^{2t} & 3e^{-3t} \\ -2e^{2t} & 5e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{5}{11}e^{-2t}(e^t) - \frac{3}{11}e^{-2t}(-\sin t) \\ \frac{2}{11}e^{3t}(e^t) + \frac{1}{11}e^{3t}(-\sin t) \end{bmatrix}\ dt$$

$$\vec{x_p} = \begin{bmatrix} e^{2t} & 3e^{-3t} \\ -2e^{2t} & 5e^{-3t} \end{bmatrix} \int \begin{bmatrix} \frac{5}{11}e^{-t} + \frac{3}{11}e^{-2t}\sin t \\ \frac{2}{11}e^{4t} - \frac{1}{11}e^{3t}\sin t \end{bmatrix}\ dt$$

Integrate, then simplify the result.

$$\vec{x_p} = \begin{bmatrix} e^{2t} & 3e^{-3t} \\ -2e^{2t} & 5e^{-3t} \end{bmatrix} \begin{bmatrix} -\frac{5}{11}e^{-t} - \frac{3}{55}e^{-2t}(\cos t + 2\sin t) \\ \frac{1}{22}e^{4t} + \frac{1}{110}e^{3t}(\cos t - 3\sin t) \end{bmatrix}$$

$$\vec{x_p} = \begin{bmatrix} e^{2t}(-\frac{5}{11}e^{-t} - \frac{3}{55}e^{-2t}(\cos t + 2\sin t)) + 3e^{-3t}(\frac{1}{22}e^{4t} + \frac{1}{110}e^{3t}(\cos t - \sin t)) \\ -2e^{2t}(-\frac{5}{11}e^{-t} - \frac{3}{55}e^{-2t}(\cos t + 2\sin t)) + 5e^{-3t}(\frac{1}{22}e^{4t} + \frac{1}{110}e^{3t}(\cos t - 3\sin t)) \end{bmatrix}$$

$$\vec{x_p} = \begin{bmatrix} -\frac{5}{11}e^t - \frac{3}{55}(\cos t + 2\sin t) + \frac{3}{22}e^t + \frac{3}{110}(\cos t - \sin t) \\ \frac{10}{11}e^t + \frac{6}{55}(\cos t + 2\sin t) + \frac{5}{22}e^t + \frac{5}{110}(\cos t - 3\sin t) \end{bmatrix}$$

$$\vec{x_p}=\begin{bmatrix}-\frac{5}{11}e^t+\frac{3}{22}e^t-\frac{3}{55}\cos t+\frac{3}{110}\cos t-\frac{6}{55}\sin t-\frac{3}{110}\sin t\\ \frac{10}{11}e^t+\frac{5}{22}e^t+\frac{6}{55}\cos t+\frac{5}{110}\cos t+\frac{12}{55}\sin t-\frac{15}{110}\sin t\end{bmatrix}$$

$$\vec{x_p}=\begin{bmatrix}-\frac{7}{22}e^t-\frac{3}{110}\cos t-\frac{15}{110}\sin t\\ \frac{25}{22}e^t+\frac{17}{110}\cos t+\frac{9}{110}\sin t\end{bmatrix}$$

Breaking apart the particular solution gives

$$\vec{x_p}=\begin{bmatrix}-\frac{7}{22}e^t\\ \frac{25}{22}e^t\end{bmatrix}+\begin{bmatrix}-\frac{3}{110}\cos t\\ \frac{17}{110}\cos t\end{bmatrix}+\begin{bmatrix}-\frac{15}{110}\sin t\\ \frac{9}{110}\sin t\end{bmatrix}$$

$$\vec{x_p}=\frac{1}{22}\begin{bmatrix}-7\\25\end{bmatrix}e^t+\frac{1}{110}\begin{bmatrix}-3\\17\end{bmatrix}\cos t+\frac{1}{110}\begin{bmatrix}-15\\9\end{bmatrix}\sin t$$

Topic: Variation of parameters for nonhomogeneous systems

Question: Use the method of variation of parameters to find the general solution to the system.

$$x_1'(t) = 2x_1 + x_2$$

$$x_2'(t) = 3x_2 + te^t$$

Answer choices:

A $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} te^t$

B $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 2 \\ -1 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^t$

C $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t$

D $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \frac{1}{4} \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^t$

Solution: D

Start by writing the nonhomogeneous system in matrix form.

$$\vec{x}' = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} te^t$$

Now we'll find $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & 1 \\ 0 & 3 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (2 - \lambda)(3 - \lambda) - (1)(0)$$

$$|A - \lambda I| = (2 - \lambda)(3 - \lambda)$$

So the characteristic equation is

$$(2 - \lambda)(3 - \lambda) = 0$$

$$\lambda = 2, 3$$

Then for these Eigenvalues, $\lambda_1 = 2$ and $\lambda_2 = 3$, we find

$$A - 2I = \begin{bmatrix} 2 - 2 & 1 \\ 0 & 3 - 2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

and

$$A - 3I = \begin{bmatrix} 2-3 & 1 \\ 0 & 3-3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_2 = 0$$

$$-k_1 + k_2 = 0$$

$$k_1 = k_2$$

For the first system, we choose $(k_1, k_2) = (1,0)$. And from the second system, we choose $(k_1, k_2) = (1,1)$.

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{k_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} \qquad \vec{x_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1 \vec{x_1} + c_2 \vec{x_2}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\vec{x_1}$ and $\vec{x_2}$ to form $\Phi(t)$,

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix} e^{2t} & e^{3t} & | & 1 & 0 \\ 0 & e^{3t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & e^{t} & | & e^{-2t} & 0 \\ 0 & e^{3t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & e^{t} & | & e^{-2t} & 0 \\ 0 & 1 & | & 0 & e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & e^{-2t} & -e^{-2t} \\ 0 & 1 & | & 0 & e^{-3t} \end{bmatrix}$$

Now with

$$\Phi(t) = \begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix} \qquad \Phi^{-1}(t) = \begin{bmatrix} e^{-2t} & -e^{-2t} \\ 0 & e^{-3t} \end{bmatrix} \qquad F(t) = \begin{bmatrix} 0 \\ te^t \end{bmatrix}$$

the particular solution from the method of variation of parameters is given by

$$\overrightarrow{x_p} = \Phi(t) \int \Phi^{-1}(t) F(t) \ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix} \int \begin{bmatrix} e^{-2t} & -e^{-2t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 0 \\ te^t \end{bmatrix} \ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix} \int \begin{bmatrix} e^{-2t}(0) - e^{-2t}(te^t) \\ 0(0) + e^{-3t}(te^t) \end{bmatrix} \ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix} \int \begin{bmatrix} -te^{-t} \\ te^{-2t} \end{bmatrix} \ dt$$

Integrate, then simplify the result.

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t} & e^{3t} \\ 0 & e^{3t} \end{bmatrix} \begin{bmatrix} te^{-t} + e^{-t} \\ -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t}(te^{-t} + e^{-t}) + e^{3t}\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) \\ 0(te^{-t} + e^{-t}) + e^{3t}\left(-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t}\right) \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} \frac{1}{2}te^t + \frac{3}{4}e^t \\ -\frac{1}{2}te^t - \frac{1}{4}e^t \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} \frac{1}{2}te^t + \frac{3}{4}e^t \\ -\frac{1}{2}te^t - \frac{1}{4}e^t \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^t + \frac{1}{4} \begin{bmatrix} 3 \\ -1 \end{bmatrix} e^t$$

Topic: Variation of parameters for nonhomogeneous systems

Question: Use the method of variation of parameters to find the general solution to the system.

$$\vec{x}\,' = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} \vec{x} + \begin{bmatrix} e^{2t} \\ e^{t} + 1 \end{bmatrix}$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{5}{6} \\ 1 \end{bmatrix} e^{t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{5}{6} \\ 1 \end{bmatrix} e^{t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{5}{6} \\ 1 \end{bmatrix} e^{2t} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{5}{6} \\ 1 \end{bmatrix} e^{2t} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{t} - \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

Solution: D

Start by finding $|A - \lambda I|$.

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right|$$

$$|A - \lambda I| = \left| \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right|$$

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & -1 \\ 6 & -3 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (2 - \lambda)(-3 - \lambda) - (-1)(6)$$

$$|A - \lambda I| = -6 + \lambda + \lambda^2 + 6$$

So the characteristic equation is

$$\lambda^2 + \lambda = 0$$

$$\lambda(\lambda + 1) = 0$$

$$\lambda = 0, \ -1$$

Then for these Eigenvalues we find

$$A - 0I = \begin{bmatrix} 2 - 0 & -1 \\ 6 & -3 - 0 \end{bmatrix}$$

$$A - 0I = \begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix}$$

and

$$A-(-1)I=\begin{bmatrix} 2-(-1) & -1 \\ 6 & -3-(-1) \end{bmatrix}$$

$$A-(-1)I=\begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} 2 & -1 \\ 6 & -3 \end{bmatrix} \qquad \begin{bmatrix} 3 & -1 \\ 6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 6 & -3 \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{1}{3} \\ 6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1-\frac{1}{2}k_2=0 \qquad k_1-\frac{1}{3}k_2=0$$

$$k_1=\frac{1}{2}k_2 \qquad k_1=\frac{1}{3}k_2$$

For the first system, we choose $(k_1, k_2)=(1,2)$. And from the second system, we choose $(k_1, k_2)=(1,3)$.

$$\vec{k_1}=\begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{k_2}=\begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \vec{x_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t}$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\vec{x_1}$ and $\vec{x_2}$ to form $\Phi(t)$,

$$\Phi(t) = \begin{bmatrix} 1 & e^{-t} \\ 2 & 3e^{-t} \end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t) \,|\, I]$ into $[I \,|\, \Phi^{-1}(t)]$.

$$\begin{bmatrix} 1 & e^{-t} & | & 1 & 0 \\ 2 & 3e^{-t} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & e^{-t} & | & 1 & 0 \\ 0 & e^{-t} & | & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & e^{-t} & | & 1 & 0 \\ 0 & 1 & | & -2e^{t} & e^{t} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 3 & -1 \\ 0 & 1 & | & -2e^t & e^t \end{bmatrix}$$

Now with

$$\Phi(t) = \begin{bmatrix} 1 & e^{-t} \\ 2 & 3e^{-t} \end{bmatrix} \qquad \Phi^{-1}(t) = \begin{bmatrix} 3 & -1 \\ -2e^t & e^t \end{bmatrix} \qquad F(t) = \begin{bmatrix} e^{2t} \\ e^t + 1 \end{bmatrix}$$

the particular solution from the method of variation of parameters is given by

$$\overrightarrow{x_p} = \Phi(t) \int \Phi^{-1}(t) F(t) \ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} 1 & e^{-t} \\ 2 & 3e^{-t} \end{bmatrix} \int \begin{bmatrix} 3 & -1 \\ -2e^t & e^t \end{bmatrix} \begin{bmatrix} e^{2t} \\ e^t + 1 \end{bmatrix} \ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} 1 & e^{-t} \\ 2 & 3e^{-t} \end{bmatrix} \int \begin{bmatrix} 3(e^{2t}) - 1(e^t + 1) \\ -2e^t(e^{2t}) + e^t(e^t + 1) \end{bmatrix} \ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} 1 & e^{-t} \\ 2 & 3e^{-t} \end{bmatrix} \int \begin{bmatrix} 3e^{2t} - e^t - 1 \\ -2e^{3t} + e^{2t} + e^t \end{bmatrix} \ dt$$

Integrate, then simplify the result.

$$\overrightarrow{x_p} = \begin{bmatrix} 1 & e^{-t} \\ 2 & 3e^{-t} \end{bmatrix} \begin{bmatrix} \frac{3}{2}e^{2t} - e^t - t \\ -\frac{2}{3}e^{3t} + \frac{1}{2}e^{2t} + e^t \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} 1\left(\frac{3}{2}e^{2t} - e^t - t\right) + e^{-t}\left(-\frac{2}{3}e^{3t} + \frac{1}{2}e^{2t} + e^t\right) \\ 2\left(\frac{3}{2}e^{2t} - e^t - t\right) + 3e^{-t}\left(-\frac{2}{3}e^{3t} + \frac{1}{2}e^{2t} + e^t\right) \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} \frac{5}{6}e^{2t} - \frac{1}{2}e^t - t + 1 \\ e^{2t} - \frac{1}{2}e^t - 2t + 3 \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{5}{6}e^{2t} - \frac{1}{2}e^t - t + 1 \\ e^{2t} - \frac{1}{2}e^t - 2t + 3 \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{-t} + \begin{bmatrix} \frac{5}{6} \\ 1 \end{bmatrix} e^{2t} - \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t - \begin{bmatrix} 1 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Topic: The matrix exponential

Question: Use the matrix exponential to find the general solution to the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} \cosh t \\ \sinh t \end{bmatrix}$$

Answer choices:

A $\quad \vec{x} = \begin{bmatrix} c_1 + t \\ c_2 \end{bmatrix} \sinh t + \begin{bmatrix} c_2 \\ c_1 + t \end{bmatrix} \cosh t$

B $\quad \vec{x} = \begin{bmatrix} c_2 \\ c_1 + t \end{bmatrix} \sinh t + \begin{bmatrix} c_1 + t \\ c_2 \end{bmatrix} \cosh t$

C $\quad \vec{x} = \begin{bmatrix} c_1 t \\ c_2 \end{bmatrix} \sinh t + \begin{bmatrix} c_2 \\ c_1 t \end{bmatrix} \cosh t$

D $\quad \vec{x} = \begin{bmatrix} c_2 \\ c_1 t \end{bmatrix} \sinh t + \begin{bmatrix} c_1 t \\ c_2 \end{bmatrix} \cosh t$

Solution: B

First, we'll find $sI - A$.

$$sI - A = s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s-0 & 0-1\\0-1 & s-0\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s & -1\\-1 & s\end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A\,|\,I]$ into $[I\,|\,(sI - A)^{-1}]$.

$$\begin{bmatrix}s & -1 & | & 1 & 0\\-1 & s & | & 0 & 1\end{bmatrix} \to \begin{bmatrix}-1 & s & | & 0 & 1\\s & -1 & | & 1 & 0\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -s & | & 0 & -1\\s & -1 & | & 1 & 0\end{bmatrix} \to \begin{bmatrix}1 & -s & | & 0 & -1\\0 & s^2-1 & | & 1 & s\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -s & | & 0 & -1\\0 & 1 & | & \frac{1}{s^2-1} & \frac{s}{s^2-1}\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & \frac{s}{s^2-1} & \frac{1}{s^2-1}\\0 & 1 & | & \frac{1}{s^2-1} & \frac{s}{s^2-1}\end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s}{s^2 - 1} & \frac{1}{s^2 - 1} \\ \frac{1}{s^2 - 1} & \frac{s}{s^2 - 1} \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

and then say that this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the complementary solution will be

$$\vec{x_c} = e^{At}C$$

$$\vec{x_c} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x_c} = \begin{bmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{bmatrix}$$

To find the particular solution, we'll need e^{-As}, which we find by making the substitution $t = s$ into e^{At},

$$e^{As} = \begin{bmatrix} \cosh s & \sinh s \\ \sinh s & \cosh s \end{bmatrix}$$

and then calculating the inverse of this resulting e^{As}.

$$\begin{bmatrix} \cosh s & \sinh s & | & 1 & 0 \\ \sinh s & \cosh s & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{\sinh s}{\cosh s} & | & \frac{1}{\cosh s} & 0 \\ \sinh s & \cosh s & | & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & \frac{\sinh s}{\cosh s} & | & \frac{1}{\cosh s} & 0 \\ 0 & \frac{1}{\cosh s} & | & -\frac{\sinh s}{\cosh s} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{\sinh s}{\cosh s} & | & \frac{1}{\cosh s} & 0 \\ 0 & 1 & | & -\sinh s & \cosh s \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & | & \cosh s & -\sinh s \\ 0 & 1 & | & -\sinh s & \cosh s \end{bmatrix}$$

So e^{-As} is given by this resulting matrix.

$$e^{-As} = \begin{bmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{bmatrix}$$

We'll find $F(s)$ by substituting $t = s$ into F.

$$F(s) = \begin{bmatrix} \cosh s \\ \sinh s \end{bmatrix}$$

Therefore, the particular solution will be

$$\vec{x_p} = e^{At} \int_{t_0}^{t} e^{-As} F(s) \ ds$$

$$\vec{x_p} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \int_0^t \begin{bmatrix} \cosh s & -\sinh s \\ -\sinh s & \cosh s \end{bmatrix} \begin{bmatrix} \cosh s \\ \sinh s \end{bmatrix} \ ds$$

$$\vec{x_p} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \int_0^t \begin{bmatrix} \cosh^2 s - \sinh^2 s \\ -\sinh s \cosh s + \sinh s \cosh s \end{bmatrix} \ ds$$

$$\vec{x_p} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \int_0^t \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ ds$$

Now we'll integrate and then evaluate on $[0,t]$.

$$\vec{x_p} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \begin{bmatrix} s \\ 0 \end{bmatrix} \Bigg|_0^t$$

$$\vec{x_p} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \left(\begin{bmatrix} t \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right)$$

$$\vec{x_p} = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix} \begin{bmatrix} t \\ 0 \end{bmatrix}$$

Finally, we can do the last matrix multiplication to get $\vec{x_p}$.

$$\vec{x_p} = \begin{bmatrix} \cosh t(t) + \sinh t(0) \\ \sinh t(t) + \cosh t(0) \end{bmatrix}$$

$$\vec{x_p} = \begin{bmatrix} t \cosh t \\ t \sinh t \end{bmatrix}$$

Then the general solution is the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x_c} + \vec{x_p}$$

$$\vec{x} = \begin{bmatrix} c_1 \cosh t + c_2 \sinh t \\ c_1 \sinh t + c_2 \cosh t \end{bmatrix} + \begin{bmatrix} t \cosh t \\ t \sinh t \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} c_2 \\ c_1 + t \end{bmatrix} \sinh t + \begin{bmatrix} c_1 + t \\ c_2 \end{bmatrix} \cosh t$$

Topic: The matrix exponential

Question: Use the matrix exponential to find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad \vec{x} = \begin{bmatrix} c_1 \\ -2c_1t + c_2 \\ 3c_1t^2 + (5c_1 + c_2)t + c_3 \end{bmatrix}$

B $\quad \vec{x} = \begin{bmatrix} c_1 \\ -2c_1t + c_2 \\ 3c_1t^2 + 5c_2t + c_3 \end{bmatrix}$

C $\quad \vec{x} = \begin{bmatrix} c_1 \\ 3c_1t + c_2 \\ \frac{3}{2}c_1t^2 + (5c_1 + c_2)t + c_3 \end{bmatrix}$

D $\quad \vec{x} = \begin{bmatrix} c_1 \\ 3c_1t + c_2 \\ \frac{3}{2}c_1t^2 + 5c_2t + c_3 \end{bmatrix}$

Solution: C

First, we'll find $sI - A$.

$$sI - A = s\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \\ 5 & 1 & 0 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s-0 & 0-0 & 0-0 \\ 0-3 & s-0 & 0-0 \\ 0-5 & 0-1 & s-0 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 & 0 \\ -3 & s & 0 \\ -5 & -1 & s \end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \mid I]$ into $[I \mid (sI - A)^{-1}]$.

$$\begin{bmatrix} s & 0 & 0 & | & 1 & 0 & 0 \\ -3 & s & 0 & | & 0 & 1 & 0 \\ -5 & -1 & s & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s} & 0 & 0 \\ -3 & s & 0 & | & 0 & 1 & 0 \\ -5 & -1 & s & | & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s} & 0 & 0 \\ 0 & s & 0 & | & \frac{3}{s} & 1 & 0 \\ -5 & -1 & s & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s} & 0 & 0 \\ 0 & s & 0 & | & \frac{3}{s} & 1 & 0 \\ 0 & -1 & s & | & \frac{5}{s} & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s} & 0 & 0 \\ 0 & 1 & 0 & | & \frac{3}{s^2} & \frac{1}{s} & 0 \\ 0 & -1 & s & | & \frac{5}{s} & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s} & 0 & 0 \\ 0 & 1 & 0 & | & \frac{3}{s^2} & \frac{1}{s} & 0 \\ 0 & 0 & s & | & \frac{5s+3}{s^2} & \frac{1}{s} & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{s} & 0 & 0 \\ 0 & 1 & 0 & | & \frac{3}{s^2} & \frac{1}{s} & 0 \\ 0 & 0 & 1 & | & \frac{5s+3}{s^3} & \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s} & 0 & 0 \\ \frac{3}{s^2} & \frac{1}{s} & 0 \\ \frac{5s+3}{s^3} & \frac{1}{s^2} & \frac{1}{s} \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{bmatrix}$$

and then say that this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the general solution to the homogeneous system is

$$\vec{x} = e^{At}C$$

$$\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 3t & 1 & 0 \\ \frac{3}{2}t^2 + 5t & t & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} c_1 \\ 3c_1t + c_2 \\ \frac{3}{2}c_1t^2 + (5c_1 + c_2)t + c_3 \end{bmatrix}$$

Topic: The matrix exponential

Question: Use an inverse Laplace transform to calculate the matrix exponential.

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}$$

Answer choices:

A $\quad \dfrac{\sqrt{5}}{5}e^{\frac{3}{2}t}\begin{bmatrix} \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right) - 3\sinh\left(\frac{\sqrt{5}}{2}t\right) & 2\sinh\left(\frac{\sqrt{5}}{2}t\right) \\ -2\sinh\left(\frac{\sqrt{5}}{2}t\right) & \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right) + 3\sinh\left(\frac{\sqrt{5}}{2}t\right) \end{bmatrix}$

B $\quad \dfrac{\sqrt{5}}{5}e^{\frac{3}{2}t}\begin{bmatrix} \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right) + 3\sinh\left(\frac{\sqrt{5}}{2}t\right) & -2\sinh\left(\frac{\sqrt{5}}{2}t\right) \\ 2\sinh\left(\frac{\sqrt{5}}{2}t\right) & \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right) - 3\sinh\left(\frac{\sqrt{5}}{2}t\right) \end{bmatrix}$

C $\quad \dfrac{\sqrt{5}}{5}e^{\frac{3}{2}t}\begin{bmatrix} \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right) & 2\sinh\left(\frac{\sqrt{5}}{2}t\right) \\ -2\sinh\left(\frac{\sqrt{5}}{2}t\right) & \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right) \end{bmatrix}$

D $\quad \frac{\sqrt{5}}{5}e^{\frac{3}{2}t}\begin{bmatrix}\sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right) & -2\sinh\left(\frac{\sqrt{5}}{2}t\right)\\ 2\sinh\left(\frac{\sqrt{5}}{2}t\right) & \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right)\end{bmatrix}$

Solution: B

First, we'll find $sI-A$.

$$sI-A=s\begin{bmatrix}1&0\\0&1\end{bmatrix}-\begin{bmatrix}3&-1\\1&0\end{bmatrix}$$

$$sI-A=\begin{bmatrix}s&0\\0&s\end{bmatrix}-\begin{bmatrix}3&-1\\1&0\end{bmatrix}$$

$$sI-A=\begin{bmatrix}s-3&0-(-1)\\0-1&s-0\end{bmatrix}$$

$$sI-A=\begin{bmatrix}s-3&1\\-1&s\end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI-A\,|\,I]$ into $[I\,|\,(sI-A)^{-1}]$.

$$\begin{bmatrix}s-3&1&|&1&0\\-1&s&|&0&1\end{bmatrix}\to\begin{bmatrix}1&\frac{1}{s-3}&|&\frac{1}{s-3}&0\\-1&s&|&0&1\end{bmatrix}$$

$$\to \left[\begin{array}{cc|cc} 1 & \frac{1}{s-3} & \frac{1}{s-3} & 0 \\ 0 & \frac{s(s-3)+1}{s-3} & \frac{1}{s-3} & 1 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & \frac{1}{s-3} & \frac{1}{s-3} & 0 \\ 0 & 1 & \frac{1}{s(s-3)+1} & \frac{s-3}{s(s-3)+1} \end{array}\right]$$

$$\to \left[\begin{array}{cc|cc} 1 & 0 & \frac{s}{s^2-3s+1} & -\frac{1}{s^2-3s+1} \\ 0 & 1 & \frac{1}{s^2-3s+1} & \frac{s-3}{s^2-3s+1} \end{array}\right]$$

Before we can apply an inverse transform to the inverse matrix, we need to rewrite the entries to prepare them for the inverse Laplace transform.

$$(sI-A)^{-1}=$$

$$\begin{bmatrix} \frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}+\frac{3\sqrt{5}}{5}\left(\frac{\frac{\sqrt{5}}{2}}{\left(s-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right) & -\frac{2\sqrt{5}}{5}\left(\frac{\frac{\sqrt{5}}{2}}{\left(s-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right) \\ \frac{2\sqrt{5}}{5}\left(\frac{\frac{\sqrt{5}}{2}}{\left(s-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right) & \frac{s-\frac{3}{2}}{\left(s-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}-\frac{3\sqrt{5}}{5}\left(\frac{\frac{\sqrt{5}}{2}}{\left(s-\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2}\right) \end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$\mathscr{L}^{-1}((sI-A)^{-1})=$$

$$\begin{bmatrix} e^{\frac{3}{2}t}\cosh\left(\frac{\sqrt{5}}{2}t\right)+\frac{3\sqrt{5}}{5}e^{\frac{3}{2}t}\sinh\left(\frac{\sqrt{5}}{2}t\right) & -\frac{2\sqrt{5}}{5}e^{\frac{3}{2}t}\sinh\left(\frac{\sqrt{5}}{2}t\right) \\ \frac{2\sqrt{5}}{5}e^{\frac{3}{2}t}\sinh\left(\frac{\sqrt{5}}{2}t\right) & e^{\frac{3}{2}t}\cosh\left(\frac{\sqrt{5}}{2}t\right)-\frac{3\sqrt{5}}{5}e^{\frac{3}{2}t}\sinh\left(\frac{\sqrt{5}}{2}t\right) \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI-A)^{-1})=$$

$$\frac{\sqrt{5}}{5}e^{\frac{3}{2}t}\begin{bmatrix} \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right)+3\sinh\left(\frac{\sqrt{5}}{2}t\right) & -2\sinh\left(\frac{\sqrt{5}}{2}t\right) \\ 2\sinh\left(\frac{\sqrt{5}}{2}t\right) & \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right)-3\sinh\left(\frac{\sqrt{5}}{2}t\right) \end{bmatrix}$$

and then say that this result is the matrix exponential.

$$e^{At}=$$

$$\frac{\sqrt{5}}{5}e^{\frac{3}{2}t}\begin{bmatrix} \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right)+3\sinh\left(\frac{\sqrt{5}}{2}t\right) & -2\sinh\left(\frac{\sqrt{5}}{2}t\right) \\ 2\sinh\left(\frac{\sqrt{5}}{2}t\right) & \sqrt{5}\cosh\left(\frac{\sqrt{5}}{2}t\right)-3\sinh\left(\frac{\sqrt{5}}{2}t\right) \end{bmatrix}$$

Topic: Homogeneous higher order equations

Question: Find the general solution to the third order homogeneous differential equation.

$$y''' - 6y'' + 11y' - 6 = 0$$

Answer choices:

A $\quad y(t) = c_1t^2e^t + c_2e^{6t} + c_3e^{11t}$

B $\quad y(t) = c_1e^t + c_2e^{2t} + c_3e^{3t}$

C $\quad y(t) = c_1e^{-6t} + c_2e^{6t} + c_3e^{11t}$

D $\quad y(t) = c_1 + c_2te^{2t} + c_3t^2e^{3t}$

Solution: B

We'll write the characteristic polynomial associated with the differential equation, then factor it.

$$r^3 - 6r^2 + 11r - 6 = 0$$

$$(r - 1)(r - 2)(r - 3) = 0$$

The equation has distinct real roots $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$. The root $r_1 = 1$ will contribute c_1e^t to the solution, $r_2 = 2$ will contribute c_2e^{2t}, and $r_3 = 3$ will contribute c_3e^{3t}, so the general solution is the sum of all three,

$$y(t) = c_1e^t + c_2e^{2t} + c_3e^{3t}$$

Topic: Homogeneous higher order equations

Question: A fifth order homogeneous differential equation has roots $r_1 = -4$, $r_2 = -4$, $r_3 = -4$, and r_4 and r_5 are roots that are not equal to -4. What will the portion of the solution that accounts for r_1, r_2, and r_3 look like?

Answer choices:

A $c_1e^{-4t} + c_2te^{-4t} + c_3e^{5t}$

B $c_1e^{-4t} + c_2e^{-4t} + c_3e^{-4t}$

C $c_1e^{-4t} + c_2t^{-4}e^t + c_3t^4e^t$

D $c_1e^{-4t} + c_2te^{-4t} + c_3t^2e^{-4t}$

Solution: D

When the differential equation has k equal real roots (call these roots r_1), the portion of the general solution that accounts for those roots will be

$$c_1e^{r_1t} + c_2te^{r_1t} + \dots + c_kt^{k-1}e^{r_1t}$$

In this problem, the root -4 appears three times. In other words, we have three equal roots, $r_1 = r_2 = r_3 = -4$. Plugging $k = 3$ and $r_1 = -4$ into the formula above, the portion of the general solution that accounts for the equal real roots $r_1 = r_2 = r_3 = -4$ will be

$$c_1e^{-4t} + c_2te^{-4t} + c_3t^2e^{-4t}$$

Topic: Homogeneous higher order equations

Question: Given the sixth order homogeneous differential equation and its associated characteristic polynomial, find the general solution to the differential equation.

$$y^{(6)} - 10y^{(5)} + 48y^{(4)} - 132y''' + 221y'' - 210y' + 98y = 0$$

$$(r^2 - 2r + 2)(r^2 - 4r + 7)^2 = 0$$

Answer choices:

A $\quad y(t) = c_1e^t \cos t + c_2e^t \sin t$

$$+c_3e^{2t}\cos(\sqrt{3}t) + c_4e^{2t}\sin(\sqrt{3}t) + c_5te^{2t}\cos(\sqrt{3}t) + c_6te^{2t}\sin(\sqrt{3}t)$$

B $\quad y(t) = c_1 \cos t + c_2 \sin t$

$$+c_3e^{4t}\cos(\sqrt{7}t) + c_4e^{4t}\sin(\sqrt{7}t) + c_5t^2e^{4t}\cos(\sqrt{7}t) + c_6t^2e^{4t}\sin(\sqrt{7}t)$$

C $\quad y(t) = c_1e^t + c_2e^{-2t} + c_3e^{2t} + c_4te^t + c_5e^{-4t} + c_6e^{7t}$

D $\quad y(t) = c_1e^t \cos t + c_2e^t \sin t$

$$+c_3e^{3t}\cos(2t) + c_4e^{3t}\sin(2t) + c_5te^{3t}\cos(2t) + c_6te^{3t}\sin(2t)$$

Solution: A

Solving the characteristic polynomial gives

$$r^2 - 2r + 2 = 0$$

$$r^2 - 2r + 1 - 1 + 2 = 0$$

$$(r-1)^2 = -1$$

$$r = 1 \pm i$$

and

$$r^2 - 4r + 7 = 0$$

$$r^2 - 4r + 4 - 4 + 7 = 0$$

$$(r-2)^2 = -3$$

$$r = 2 \pm \sqrt{3}i$$

The roots $r = 1 \pm i$ have multiplicity 1, and the roots $r = 2 \pm \sqrt{3}i$ have multiplicity 2. Which means the roots $r_{1,2} = 1 \pm i$ will contribute

$$c_1 e^{\alpha t} \cos(\beta t) + c_2 e^{\alpha t} \sin(\beta t)$$

$$c_1 e^t \cos t + c_2 e^t \sin t$$

to the general solution, while the roots $r_{3,4} = 2 \pm \sqrt{3}i$ and $r_{5,6} = 2 \pm \sqrt{3}i$ will contribute

$$c_3 e^{\alpha t} \cos(\beta t) + c_4 e^{\alpha t} \sin(\beta t) + c_5 t e^{\alpha t} \cos(\beta t) + c_6 t e^{\alpha t} \sin(\beta t)$$

$$c_3e^{2t}\cos(\sqrt{3}t)+c_4e^{2t}\sin(\sqrt{3}t)+c_5te^{2t}\cos(\sqrt{3}t)+c_6te^{2t}\sin(\sqrt{3}t)$$

Therefore, the general solution to the sixth order homogeneous differential equation is

$$y(t)=c_1e^t\cos t+c_2e^t\sin t$$

$$+c_3e^{2t}\cos(\sqrt{3}t)+c_4e^{2t}\sin(\sqrt{3}t)+c_5te^{2t}\cos(\sqrt{3}t)+c_6te^{2t}\sin(\sqrt{3}t)$$

Topic: Undetermined coefficients for higher order equations

Question: Find the general solution to the second order differential equation.

$$y'' + 4y' + 3y = 9t$$

Answer choices:

A $\quad y(t) = c_1e^{4t} + c_2e^{3t} + t - 1$

B $\quad y(t) = c_1e^{-t} + c_2e^{-3t} + 3t - 4$

C $\quad y(t) = c_1te^{t} + c_2t^2e^{3t} - 3t$

D $\quad y(t) = c_1 + c_2e^{-t} + 6$

Solution: B

Before we find the general solution, we need to find the complementary solution and the particular solution.

To find the complementary solution $y_c(t)$, we solve the homogeneous form of the differential equation by factoring and finding the roots of the characteristic polynomial.

$$r^2 + 4r + 3 = 0$$

$$(r + 1)(r + 3) = 0$$

The roots are $r_1 = -1$ and $r_2 = -3$, each with multiplicity 1, so the complementary solution is

$$y_c(t) = c_1 e^{-t} + c_2 e^{-3t}$$

To find the particular solution $y_p(t)$, our guess and its derivatives will be

$$y_p(t) = At + B$$

$$y_p'(t) = A$$

$$y_p''(t) = 0$$

We plug these into the differential equation and get

$$0 + 4(A) + 3(At + B) = 9t$$

$$4A + 3At + 3B = 9t$$

$$(3A)t + (4A + 3B) = 9t$$

Equating coefficients gives us the system of equations

$$3A = 9$$

$$4A + 3B = 0$$

When we solve this system of equations, we find $A = 3$ and $B = -4$, so the particular solution will be

$$y_p(t) = 3t - 4$$

To find the general solution, we'll sum the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{-t} + c_2e^{-3t} + 3t - 4$$

Topic: Undetermined coefficients for higher order equations

Question: Given the complementary solution, find the general solution to the second order differential equation.

$$y'' - 2y' = 3te^{-t} + 2e^{-t}$$

$$y_c(t) = c_1 + c_2 e^{2t}$$

Answer choices:

A $\quad y(t) = c_1 + c_2 e^{2t} + (2t + 3)e^{-t}$

B $\quad y(t) = c_1 + c_2 e^{2t} + (3t + 2)e^{-t}$

C $\quad y(t) = c_1 + c_2 e^{2t} + (t + 2)e^{-t}$

D $\quad y(t) = c_1 + c_2 t^2 e^{2t} + 3e^{-t}$

Solution: C

Since $g(t)$ is the product of a polynomial and an exponential,

$$g(t) = 3te^{-t} + 2e^{-t}$$

$$g(t) = (3t + 2)e^{-t}$$

we guess that the particular solution has the form

$$y_p(t) = (At + B)e^{-t}$$

$$y_p(t) = Ate^{-t} + Be^{-t}$$

The first and second derivatives of $y_p(t)$ are

$$y_p'(t) = A(e^{-t} - te^{-t}) - Be^{-t}$$

$$y_p'(t) = -Ate^{-t} + (A - B)e^{-t}$$

and

$$y_p''(t) = -A(e^{-t} - te^{-t}) - (A - B)e^{-t}$$

$$y_p''(t) = Ate^{-t} + (-2A + B)e^{-t}$$

We plug these into the differential equation and simplify.

$$Ate^{-t} + (-2A + B)e^{-t} - 2(-Ate^{-t} + (A - B)e^{-t}) = 3te^{-t} + 2e^{-t}$$

$$Ate^{-t} - 2Ae^{-t} + Be^{-t} + 2Ate^{-t} - 2Ae^{-t} + 2Be^{-t} = 3te^{-t} + 2e^{-t}$$

$$3Ate^{-t} + (-4A + 3B)e^{-t} = 3te^{-t} + 2e^{-t}$$

Equating coefficients gives a system of equations.

$$3A = 3$$

$$-4A + 3B = 2$$

Solving the system gives $A = 1$ and $B = 2$, so the particular solution is

$$y_p(t) = te^{-t} + 2e^{-t}$$

$$y_p(t) = (t + 2)e^{-t}$$

To find the general solution, we'll sum the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t$$

$$y(t) = c_1 + c_2e^{2t} + (t + 2)e^{-t}$$

Topic: Undetermined coefficients for higher order equations

Question: Find the best guess for the form for the particular solution of the fourth order differential equation.

$$y^{(4)} - 8y''' + 42y'' - 104y' + 169y = e^{2t}\sin(3t)$$

Answer choices:

A $\quad y_p(t) = Ae^{2t}\cos(3t) + Be^{2t}\sin(3t)$

B $\quad y_p(t) = Ae^{2t}\sin(3t)$

C $\quad y_p(t) = Ate^{2t}\cos(3t) + Bte^{2t}\sin(3t)$

D $\quad y_p(t) = At^2e^{2t}\cos(3t) + Bt^2e^{2t}\sin(3t)$

Solution: D

The first step is to find the complementary solution. We need to factor the characteristic polynomial of the differential equation and find its roots.

$$r^4 - 8r^3 + 42r^2 - 104r + 169 = 0$$

$$(r^2 - 4r + 13)^2 = 0$$

The roots are $r_{1,2} = 2 \pm 3i$ with multiplicity 2, so the complementary solution is

$$y_c(t) = c_1 e^{2t} \cos(3t) + c_2 e^{2t} \sin(3t) + c_3 t e^{2t} \cos(3t) + c_4 t e^{2t} \sin(3t)$$

For the particular solution, since $g(t)$ is the product of a sine function and an exponential function, our guess should be

$$y_p(t) = Ae^{2t} \cos(3t) + Be^{2t} \sin(3t)$$

This guess overlaps with the first two terms of the complementary solution. If we multiply through by t,

$$y_p(t) = Ate^{2t} \cos(3t) + Bte^{2t} \sin(3t)$$

then the guess will overlap will the last two terms of the complementary solution. So we'll multiply through by t again to get the best version of our guess for the particular solution.

$$y_p(t) = At^2 e^{2t} \cos(3t) + Bt^2 e^{2t} \sin(3t)$$

Topic: Variation of parameters for higher order equations

Question: Use variation of parameters to find the general solution to the differential equation.

$$y''' + y' = \sec t$$

Answer choices:

A $\quad y(t) = c_1 + c_2 \cos t + c_3 \sin t + \ln|\sec t + \tan t| - t \cos t + (\sin t)\ln|\cos t|$

B $\quad y(t) = c_1 e^{-t} + c_2 t e^{-t} + c_3 \cos t + \sec^2 t + \csc t - t \sin t$

C $\quad y(t) = c_1 e^{-4t} + c_2 \cos t + c_3 \sin t + \cos t + \sin t + 1$

D $\quad y(t) = c_1 + c_2 \cos t + c_3 \sin t + \ln|\sec^2 t| + t^2 \csc t + 3t^3$

Solution: A

Our first step is to get the characteristic polynomial and calculate its roots so we can find the complementary solution.

$$r^3 + r = 0$$

$$r(r^2 + 1) = 0$$

The roots are $r_1 = 0$, $r_2 = i$, and $r_3 = -i$, so the complementary solution is

$$y_c(t) = c_1 + c_2 \cos t + c_3 \sin t$$

We now have the fundamental set of solutions $\{1, \cos t, \sin t\}$, and from here we can build the Wronskian determinants.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{vmatrix}$$

$$= 1 \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} - 0 \begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix} + 0 \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \sin^2 t + \cos^2 t$$

$$= 1$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{vmatrix}$$

$$= 0 \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix} - 0 \begin{vmatrix} \cos t & \sin t \\ -\cos t & -\sin t \end{vmatrix} + 1 \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix}$$

$$= \cos^2 t + \sin^2 t$$

$$= 1$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{vmatrix}$$

$$= 1 \begin{vmatrix} 0 & \cos t \\ 1 & -\sin t \end{vmatrix} - 0 \begin{vmatrix} 0 & \sin t \\ 1 & -\sin t \end{vmatrix} + 0 \begin{vmatrix} 0 & \sin t \\ 0 & \cos t \end{vmatrix}$$

$$= 0 - \cos t$$

$$= -\cos t$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -\sin t & 0 \\ -\cos t & 1 \end{vmatrix} - 0 \begin{vmatrix} \cos t & 0 \\ -\cos t & 1 \end{vmatrix} + 0 \begin{vmatrix} \cos t & 0 \\ -\sin t & 0 \end{vmatrix}$$

$$= -\sin t - 0$$

$$= -\sin t$$

Then the unknowns are

$$u_1' = \frac{g(t)W_1}{W} = \frac{\sec t}{1} = \sec t$$

$$u_2' = \frac{g(t)W_2}{W} = \frac{(\sec t)(-\cos t)}{1} = -1$$

$$u_3' = \frac{g(t)W_3}{W} = \frac{(\sec t)(-\sin t)}{1} = -\tan t$$

Next, we integrate each of these to get u_1, u_2, and u_3.

$$u_1 = \int \frac{g(t)W_1}{W}\ dt = \int \sec t\ dt = \ln|\sec t + \tan t|$$

$$u_2 = \int \frac{g(t)W_2}{W}\ dt = \int -\ dt = -t$$

$$u_3 = \int \frac{g(t)W_3}{W}\ dt = \int -\tan t\ dt = \ln|\cos t|$$

Then the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \ln|\sec t + \tan t| - t\cos t + (\sin t)\ln|\cos t|$$

The general solution is the sum of the complementary solution and the particular solution,

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1 + c_2\cos t + c_3\sin t + \ln|\sec t + \tan t| - t\cos t + (\sin t)\ln|\cos t|$$

Topic: Variation of parameters for higher order equations

Question: Use variation of parameters to find the general solution to the differential equation.

$$y''' - 3y'' - 4y' + 12y = t$$

Answer choices:

A $\quad y(t) = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^{3t} + \frac{1}{10}t + \frac{1}{30}$

B $\quad y(t) = c_1 e^{-2t} + c_2 e^{2t} + c_3 e^{3t} + \frac{1}{12}t + \frac{1}{36}$

C $\quad y(t) = c_1 e^{3t} + c_2 e^{4t} + c_3 e^{5t} + 2t + \frac{1}{18}$

D $\quad y(t) = c_1 e^{3t} + c_2 e^{4t} + c_3 e^{5t} + \frac{1}{10}t + \frac{1}{30}$

Solution: B

Our first step is to get the characteristic polynomial and calculate its roots so we can find the complementary solution.

$$r^3 - 3r^2 - 4r + 12 = 0$$

$$(r + 2)(r - 2)(r - 3) = 0$$

The roots are $r_1 = -2$, $r_2 = 2$, and $r_3 = 3$, so the complementary solution is

$$y_c(t) = c_1e^{-2t} + c_2e^{2t} + c_3e^{3t}$$

We now have the fundamental set of solutions $\{e^{-2t}, e^{2t}, e^{3t}\}$, and from here we can build the Wronskian determinants.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-2t} & e^{2t} & e^{3t} \\ -2e^{-2t} & 2e^{2t} & 3e^{3t} \\ 4e^{-2t} & 4e^{2t} & 9e^{3t} \end{vmatrix}$$

$$= e^{-2t}\begin{vmatrix} 2e^{2t} & 3e^{3t} \\ 4e^{2t} & 9e^{3t} \end{vmatrix} - e^{2t}\begin{vmatrix} -2e^{-2t} & 3e^{3t} \\ 4e^{-2t} & 9e^{3t} \end{vmatrix} + e^{3t}\begin{vmatrix} -2e^{-2t} & 2e^{2t} \\ 4e^{-2t} & 4e^{2t} \end{vmatrix}$$

$$= e^{-2t}(18e^{5t} - 12e^{5t}) - e^{2t}(-18e^{t} - 12e^{t}) + e^{3t}(-8 - 8)$$

$$= e^{-2t}(6e^{5t}) - e^{2t}(-30e^{t}) + e^{3t}(-16)$$

$$= 6e^{3t} + 30e^{3t} - 16e^{3t}$$

$$= 20e^{3t}$$

$$W_1=\begin{vmatrix}0 & y_2 & y_3\\ 0 & y_2' & y_3'\\ 1 & y_2'' & y_3''\end{vmatrix}=\begin{vmatrix}0 & e^{2t} & e^{3t}\\ 0 & 2e^{2t} & 3e^{3t}\\ 1 & 4e^{2t} & 9e^{3t}\end{vmatrix}$$

$$=0\begin{vmatrix}2e^{2t} & 3e^{3t}\\ 4e^{2t} & 9e^{3t}\end{vmatrix}-0\begin{vmatrix}e^{2t} & e^{3t}\\ 4e^{2t} & 9e^{3t}\end{vmatrix}+1\begin{vmatrix}e^{2t} & e^{3t}\\ 2e^{2t} & 3e^{3t}\end{vmatrix}$$

$$=3e^{5t}-2e^{5t}$$

$$=e^{5t}$$

$$W_2=\begin{vmatrix}y_1 & 0 & y_3\\ y_1' & 0 & y_3'\\ y_1'' & 1 & y_3''\end{vmatrix}=\begin{vmatrix}e^{-2t} & 0 & e^{3t}\\ -2e^{-2t} & 0 & 3e^{3t}\\ 4e^{-2t} & 1 & 9e^{3t}\end{vmatrix}$$

$$=-0\begin{vmatrix}-2e^{-2t} & 3e^{3t}\\ 4e^{-2t} & 9e^{3t}\end{vmatrix}+0\begin{vmatrix}e^{-2t} & e^{3t}\\ 4e^{-2t} & 9e^{3t}\end{vmatrix}-1\begin{vmatrix}e^{-2t} & e^{3t}\\ -2e^{-2t} & 3e^{3t}\end{vmatrix}$$

$$=-(3e^{t}+2e^{t})$$

$$=-5e^{t}$$

$$W_3=\begin{vmatrix}y_1 & y_2 & 0\\ y_1' & y_2' & 0\\ y_1'' & y_2'' & 1\end{vmatrix}=\begin{vmatrix}e^{-2t} & e^{2t} & 0\\ -2e^{-2t} & 2e^{2t} & 0\\ 4e^{-2t} & 4e^{2t} & 1\end{vmatrix}$$

$$=0\begin{vmatrix}-2e^{-2t} & 2e^{2t}\\ 4e^{-2t} & 4e^{2t}\end{vmatrix}-0\begin{vmatrix}e^{-2t} & e^{2t}\\ 4e^{-2t} & 4e^{2t}\end{vmatrix}+1\begin{vmatrix}e^{-2t} & e^{2t}\\ -2e^{-2t} & 2e^{2t}\end{vmatrix}$$

$$=2+2$$

$$= 4$$

Then the unknowns are

$$u_1' = \frac{g(t)W_1}{W} = \frac{te^{5t}}{20e^{3t}} = \frac{1}{20}te^{2t}$$

$$u_2' = \frac{g(t)W_2}{W} = \frac{-5te^{t}}{20e^{3t}} = -\frac{1}{4}te^{-2t}$$

$$u_3' = \frac{g(t)W_3}{W} = \frac{4t}{20e^{3t}} = \frac{1}{5}te^{-3t}$$

Next, we integrate each of these to get u_1, u_2, and u_3.

$$u_1 = \int \frac{g(t)W_1}{W}\ dt = \int \frac{1}{20}te^{2t}\ dt = \frac{1}{40}te^{2t} - \frac{1}{80}e^{2t}$$

$$u_2 = \int \frac{g(t)W_2}{W}\ dt = \int -\frac{1}{4}te^{-2t}\ dt = \frac{1}{8}te^{-2t} + \frac{1}{16}e^{-2t}$$

$$u_3 = \int \frac{g(t)W_3}{W}\ dt = \int \frac{1}{5}te^{-3t}\ dt = -\frac{1}{15}te^{-3t} - \frac{1}{45}e^{-3t}$$

Then the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \left(\frac{1}{40}te^{2t} - \frac{1}{80}e^{2t}\right)e^{-2t} + \left(\frac{1}{8}te^{-2t} + \frac{1}{16}e^{-2t}\right)e^{2t} + \left(-\frac{1}{15}te^{-3t} - \frac{1}{45}e^{-3t}\right)e^{3t}$$

$$y_p(t) = \frac{1}{40}t - \frac{1}{80} + \frac{1}{8}t + \frac{1}{16} - \frac{1}{15}t - \frac{1}{45}$$

$$y_p(t) = \frac{1}{12}t + \frac{1}{36}$$

The general solution is the sum of the complementary solution and the particular solution,

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{-2t} + c_2e^{2t} + c_3e^{3t} + \frac{1}{12}t + \frac{1}{36}$$

Topic: Variation of parameters for higher order equations

Question: Use variation of parameters to find the general solution to the differential equation.

$$y''' + y'' - y' + 15y = \frac{1}{2}e^{3t}$$

Answer choices:

A $\quad y(t) = c_1e^{4t} + c_2e^{-4t}\cos(4t) + c_3e^{-4t}\sin(4t) + \frac{1}{72}e^{-3t}$

B $\quad y(t) = c_1e^{-2t} + c_2e^{t}\cos t + c_3e^{t}\sin t + \frac{1}{128}e^{2t}\sin(2t)$

C $\quad y(t) = c_1e^{-3t} + c_2e^{t}\cos(2t) + c_3e^{t}\sin(2t) + \frac{1}{96}e^{3t}$

D $\quad y(t) = c_1 + c_2e^{2t}\cos t + c_3e^{2t}\sin t + \frac{1}{56}e^{-2t}$

Solution: C

Our first step is to get the characteristic polynomial, factor it, and calculate its roots so we can find the complementary solution.

$$r^3 + r^2 - r + 15 = 0$$

$$(r+3)(r^2 - 2r + 5) = 0$$

The roots are $r_1 = -3$, $r_2 = 1 + 2i$, and $r_3 = 1 - 2i$, so the complementary solution is

$$y_c(t) = c_1 e^{-3t} + c_2 e^t \cos(2t) + c_3 e^t \sin(2t)$$

We now have the fundamental set of solutions $\{e^{-3t}, e^t \cos(2t), e^t \sin(2t)\}$, and from here we can build the Wronskian determinants.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-3t} & e^t \cos(2t) & e^t \sin(2t) \\ -3e^{-3t} & e^t \cos(2t) - 2e^t \sin(2t) & e^t \sin(2t) + 2e^t \cos(2t) \\ 9e^{-3t} & -3e^t \cos(2t) - 4e^t \sin(2t) & -3e^t \sin(2t) + 4e^t \cos(2t) \end{vmatrix}$$

$$= e^{-3t} \begin{vmatrix} e^t \cos 2t - 2e^t \sin(2t) & e^t \sin(2t) + 2e^t \cos(2t) \\ -3e^t \cos(2t) - 4e^t \sin(2t) & -3e^t \sin(2t) + 4e^t \cos(2t) \end{vmatrix}$$

$$+3e^{-3t} \begin{vmatrix} e^t \cos(2t) & e^t \sin(2t) \\ -3e^t \cos(2t) - 4e^t \sin(2t) & -3e^t \sin(2t) + 4e^t \cos(2t) \end{vmatrix}$$

$$+9e^{-3t} \begin{vmatrix} e^t \cos(2t) & e^t \sin(2t) \\ e^t \cos(2t) - 2e^t \sin(2t) & e^t \sin(2t) + 2e^t \cos(2t) \end{vmatrix}$$

$$= e^{-3t}(10e^{2t}) + 3e^{-3t}(4e^{2t}) + 9e^{-3t}(2e^{2t})$$

$$= 10e^{-t} + 12e^{-t} + 18e^{-t}$$

$$= 40e^{-t}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & e^t\cos(2t) & e^t\sin(2t) \\ 0 & e^t\cos(2t) - 2e^t\sin(2t) & e^t\sin(2t) + 2e^t\cos(2t) \\ 1 & -3e^t\cos(2t) - 4e^t\sin(2t) & -3e^t\sin(2t) + 4e^t\cos(2t) \end{vmatrix}$$

$$= 1 \begin{vmatrix} e^t\cos(2t) & e^t\sin(2t) \\ e^t\cos(2t) - 2e^t\sin(2t) & e^t\sin(2t) + 2e^t\cos(2t) \end{vmatrix}$$

$$= 1(2e^{2t})$$

$$= 2e^{2t}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} e^{-3t} & 0 & e^t\sin(2t) \\ -3e^{-3t} & 0 & e^t\sin(2t) + 2e^t\cos(2t) \\ 9e^{-3t} & 1 & -3e^t\sin(2t) + 4e^t\cos(2t) \end{vmatrix}$$

$$= -1 \begin{vmatrix} e^{-3t} & e^t\sin(2t) \\ -3e^{-3t} & e^t\sin(2t) + 2e^t\cos(2t) \end{vmatrix}$$

$$= -4e^{-2t}\sin(2t) - 2e^{-2t}\cos(2t)$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} e^{-3t} & e^t\cos(2t) & 0 \\ -3e^{-3t} & e^t\cos(2t) - 2e^t\sin(2t) & 0 \\ 9e^{-3t} & -3e^t\cos(2t) - 4e^t\sin(2t) & 1 \end{vmatrix}$$

$$= 1 \begin{vmatrix} e^{-3t} & e^t\cos(2t) \\ -3e^{-3t} & e^t\cos(2t) - 2e^t\sin(2t) \end{vmatrix}$$

$$= 4e^{-2t}\cos(2t) - 2e^{-2t}\sin(2t)$$

Then the unknowns are

$$u_1' = \frac{g(t)W_1}{W} = \frac{\left(\frac{1}{2}e^{3t}\right)(2e^{2t})}{40e^{-t}} = \frac{1}{40}e^{6t}$$

$$u_2' = \frac{g(t)W_2}{W} = \frac{\left(\frac{1}{2}e^{3t}\right)(-4e^{-2t}\sin(2t) - 2e^{-2t}\cos(2t))}{40e^{-t}}$$

$$= -\frac{1}{20}e^{2t}\sin(2t) - \frac{1}{40}e^{2t}\cos(2t)$$

$$u_3' = \frac{g(t)W_3}{W} = \frac{\left(\frac{1}{2}e^{3t}\right)(4e^{-2t}\cos(2t) - 2e^{-2t}\sin(2t))}{40e^{-t}}$$

$$= \frac{1}{20}e^{2t}\cos(2t) - \frac{1}{40}e^{2t}\sin(2t)$$

Next, we integrate each of these to get u_1, u_2, and u_3.

$$u_1 = \int \frac{g(t)W_1}{W}\ dt = \int \frac{1}{40}e^{6t} = \frac{1}{240}e^{6t}$$

$$u_2 = \int \frac{g(t)W_2}{W}\ dt = \int -\frac{1}{20}e^{2t}\sin(2t) - \frac{1}{40}e^{2t}\cos(2t)\ dt$$

$$= -\frac{3}{160}e^{2t}\sin(2t) + \frac{1}{160}e^{2t}\cos(2t)$$

$$u_3 = \int \frac{g(t)W_3}{W}\ dt = \int \frac{1}{20}e^{2t}\cos(2t) - \frac{1}{40}e^{2t}\sin(2t)\ dt$$

$$= \frac{1}{160}e^{2t}\sin(2t) + \frac{3}{160}e^{2t}\cos(2t)$$

Then the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \left(\frac{1}{240}e^{6t}\right)(e^{-3t}) + \left(-\frac{3}{160}e^{2t}\sin(2t) + \frac{1}{160}e^{2t}\cos(2t)\right)(e^t\cos(2t))$$

$$+\left(\frac{1}{160}e^{2t}\sin(2t) + \frac{3}{160}e^{2t}\cos(2t)\right)(e^t\sin(2t))$$

$$y_p(t) = \frac{1}{240}e^{3t} - \frac{3}{160}e^{3t}\sin(2t)\cos(2t) + \frac{1}{160}e^{3t}\cos^2(2t)$$

$$+\frac{1}{160}e^{3t}\sin^2(2t) + \frac{3}{160}e^{3t}\sin(2t)\cos(2t)$$

$$y_p(t) = \frac{1}{240}e^{3t} + \frac{1}{160}e^{3t}(\sin^2(2t) + \cos^2(2t))$$

$$y_p(t) = \frac{1}{240}e^{3t} + \frac{1}{160}e^{3t}$$

$$y_p(t) = \frac{1}{96}e^{3t}$$

And the general solution is the sum of the complementary solution and the particular solution,

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{-3t} + c_2e^t\cos(2t) + c_3e^t\sin(2t) + \frac{1}{96}e^{3t}$$

Topic: Laplace transforms for higher order equations

Question: Use a Laplace transform to find the solution to the third order differential equation, given $y(0)=0$, $y'(0)=0$, and $y''(0)=0$.

$$y'''-6y''+11y'-6y=e^{4t}$$

Answer choices:

A $\quad y(t)=-\frac{1}{6}e^{t}+\frac{1}{2}e^{2t}-\frac{1}{2}e^{3t}+\frac{1}{6}e^{4t}$

B $\quad y(t)=\frac{1}{8}e^{t}+\frac{1}{4}e^{2t}+\frac{1}{16}e^{3t}+\frac{1}{2}e^{4t}$

C $\quad y(t)=1+2e^{t}+3e^{2t}+4e^{8t}$

D $\quad y(t)=-\frac{1}{3}e^{-3t}+\frac{1}{9}e^{-t}-\frac{1}{9}e^{t}+\frac{1}{3}e^{3t}$

Solution: A

Apply the transform to both sides of the differential equation.

$$\mathscr{L}(y''') - 6\mathscr{L}(y'') + 11\mathscr{L}(y') - 6\mathscr{L}(y) = \mathscr{L}(e^{4t})$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 6(s^2Y(s) - sy(0) - y'(0))$$

$$+11(sY(s) - y(0)) - 6Y(s) = \frac{1}{s-4}$$

Substitute the initial conditions $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$,

$$s^3Y(s) - 6s^2Y(s) + 11sY(s) - 6Y(s) = \frac{1}{s-4}$$

then solve for $Y(s)$.

$$(s^3 - 6s^2 + 11s - 6)Y(s) = \frac{1}{s-4}$$

$$Y(s) = \frac{1}{(s-4)(s^3 - 6s^2 + 11s - 6)}$$

$$Y(s) = \frac{1}{(s-1)(s-2)(s-3)(s-4)}$$

Apply a partial fractions decompositions.

$$\frac{1}{(s-1)(s-2)(s-3)(s-4)} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{s-3} + \frac{D}{s-4}$$

Solving the decomposition equation gives $A = -1/6$, $B = 1/2$, $C = -1/2$, and $D = 1/6$, so the Laplace transform becomes

$$Y(s) = \frac{-\frac{1}{6}}{s-1} + \frac{\frac{1}{2}}{s-2} + \frac{-\frac{1}{2}}{s-3} + \frac{\frac{1}{6}}{s-4}$$

$$Y(s) = -\frac{1}{6}\left(\frac{1}{s-1}\right) + \frac{1}{2}\left(\frac{1}{s-2}\right) - \frac{1}{2}\left(\frac{1}{s-3}\right) + \frac{1}{6}\left(\frac{1}{s-4}\right)$$

Applying the inverse transform to each term gives the general solution to the differential equation.

$$y(t) = -\frac{1}{6}e^{t} + \frac{1}{2}e^{2t} - \frac{1}{2}e^{3t} + \frac{1}{6}e^{4t}$$

Topic: Laplace transforms for higher order equations

Question: Use a Laplace transform to find the solution to the third order differential equation, given $y(0) = 2$, $y'(0) = -1$, and $y''(0) = -3$.

$$y''' + y'' + 4y' + 4y = 0$$

Answer choices:

A $\quad y(t) = 2e^t + 4e^{6t} + \cos(2t) - \sin(2t)$

B $\quad y(t) = e^{2t} - \sin(2t)$

C $\quad y(t) = e^{-t} + e^{2it}$

D $\quad y(t) = e^{-t} + \cos(2t)$

Solution: D

Apply the transform to both sides of the differential equation.

$$\mathscr{L}(y''') + \mathscr{L}(y'') + 4\mathscr{L}(y') + 4\mathscr{L}(y) = \mathscr{L}(0)$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + s^2Y(s) - sy(0) - y'(0)$$

$$+4(sY(s) - y(0)) + 4Y(s) = 0$$

$$s^3Y(s) - 2s^2 + s + 3 + s^2Y(s) - 2s + 1 + 4(sY(s) - 2) + 4Y(s) = 0$$

$$(s^3 + s^2 + 4s + 4)Y(s) - 2s^2 - s - 4 = 0$$

$$(s^3 + s^2 + 4s + 4)Y(s) = 2s^2 + s + 4$$

Solve for $Y(s)$.

$$Y(s) = \frac{2s^2 + s + 4}{s^3 + s^2 + 4s + 4}$$

$$Y(s) = \frac{2s^2 + s + 4}{(s+1)(s^2+4)}$$

Apply a partial fractions decomposition.

$$\frac{2s^2 + s + 4}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4}$$

Solving the decomposition equation gives $A = 1$, $B = 1$, and $C = 0$, so the Laplace transform becomes

$$Y(s) = \frac{1}{s+1} + \frac{1}{s^2+4}$$

Applying the inverse transform to each term gives the general solution to the differential equation.

$$y(t) = e^{-t} + \cos(2t)$$

Topic: Laplace transforms for higher order equations

Question: Use a Laplace transform to find the solution to the third order differential equation, given $y(0)=0$, $y'(0)=0$, and $y''(0)=1$.

$$y'''+2y''+4y'+8y=te^{2t}$$

Answer choices:

A $\quad y(t)=-\frac{3}{64}e^{2t}+\frac{3}{32}te^{2t}+\frac{23}{64}e^{-2t}+\frac{7}{32}\cos(2t)-\frac{9}{128}\sin(2t)$

B $\quad y(t)=\frac{7}{128}e^{t}+\frac{1}{12}te^{3t}+\frac{8}{19}e^{-t}+\frac{7}{32}\cos(4t)+\frac{7}{64}\sin(4t)$

C $\quad y(t)=-\frac{3}{128}e^{2t}+\frac{1}{32}te^{2t}+\frac{17}{128}e^{-2t}-\frac{7}{64}\cos(2t)+\frac{9}{64}\sin(2t)$

D $\quad y(t)=-\frac{7}{64}e^{2t}+\frac{1}{64}te^{2t}+\frac{34}{128}e^{-2t}-\frac{14}{128}\cos(2t)+\frac{11}{64}\sin(2t)$

Solution: C

Apply the transform to both sides of the differential equation.

$$\mathscr{L}(y''') + 2\mathscr{L}(y'') + 4\mathscr{L}(y') + 8\mathscr{L}(y) = \mathscr{L}(te^{2t})$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) + 2(s^2Y(s) - sy(0) - y'(0))$$

$$+4(sY(s) - y(0)) + 8Y(s) = \frac{1}{(s-2)^2}$$

$$s^3Y(s) - 1 + 2s^2Y(s) + 4sY(s) + 8Y(s) = \frac{1}{(s-2)^2}$$

$$(s^3 + 2s^2 + 4s + 8)Y(s) = \frac{1}{(s-2)^2} + 1$$

$$(s^3 + 2s^2 + 4s + 8)Y(s) = \frac{s^2 - 4s + 5}{(s-2)^2}$$

Solve for $Y(s)$.

$$Y(s) = \frac{s^2 - 4s + 5}{(s-2)^2(s^3 + 2s^2 + 4s + 8)}$$

$$Y(s) = \frac{s^2 - 4s + 5}{(s-2)^2(s+2)(s^2+4)}$$

Apply a partial fractions decomposition.

$$\frac{s^2 - 4s + 5}{(s-2)^2(s+2)(s^2+4)} = \frac{A}{s-2} + \frac{B}{(s-2)^2} + \frac{C}{s+2} + \frac{Ds + E}{s^2 + 4}$$

Solving the decomposition equation gives $A = -3/128$, $B = 1/32$, $C = 17/128$, $D = -7/64$, and $E = 9/32$, so the Laplace transform becomes

$$Y(s) = \frac{-\frac{3}{128}}{s-2} + \frac{\frac{1}{32}}{(s-2)^2} + \frac{\frac{17}{128}}{s+2} + \frac{-\frac{7}{64}s + \frac{9}{32}}{s^2+4}$$

$$Y(s) = \frac{-\frac{3}{128}}{s-2} + \frac{\frac{1}{32}}{(s-2)^2} + \frac{\frac{17}{128}}{s+2} + \frac{-\frac{7}{64}s}{s^2+4} + \frac{\frac{9}{32}}{s^2+4}$$

Applying the inverse transform to each term gives the general solution to the differential equation.

$$y(t) = -\frac{3}{128}e^{2t} + \frac{1}{32}te^{2t} + \frac{17}{128}e^{-2t} - \frac{7}{64}\cos(2t) + \frac{9}{64}\sin(2t)$$

Topic: Systems of higher order equations

Question: Solve the system of differential equations.

$$x_1' = x_1 + 4x_3$$

$$x_2' = 2x_2$$

$$x_3' = 3x_1 + x_2 - 3x_3$$

Answer choices:

A $\quad \vec{x} = c_1 \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{4t} + c_3 \begin{bmatrix} 4 \\ -1 \\ 6 \end{bmatrix} e^{5t}$

B $\quad \vec{x} = c_1 \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix} e^{-5t}$

C $\quad \vec{x} = c_1 \begin{bmatrix} 8 \\ 3 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix} e^{-5t}$

D $\quad \vec{x} = c_1 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} e^{-5t}$

Solution: D

We'll need to start by finding the matrix $A-\lambda I$,

$$A-\lambda I=\begin{bmatrix}1&0&4\\0&2&0\\3&1&-3\end{bmatrix}-\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}1&0&4\\0&2&0\\3&1&-3\end{bmatrix}-\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}1-\lambda&0&4\\0&2-\lambda&0\\3&1&-3-\lambda\end{bmatrix}$$

and then find its determinant $|A-\lambda I|$.

$$|A-\lambda I|=\begin{vmatrix}1-\lambda&0&4\\0&2-\lambda&0\\3&1&-3-\lambda\end{vmatrix}$$

$$|A-\lambda I|=(1-\lambda)\begin{vmatrix}2-\lambda&0\\1&-3-\lambda\end{vmatrix}+4\begin{vmatrix}0&2-\lambda\\3&1\end{vmatrix}$$

$$|A-\lambda I|=(1-\lambda)(2-\lambda)(-3-\lambda)+4(2-\lambda)(-3)$$

$$|A-\lambda I|=-6+7\lambda-\lambda^3-24+12\lambda$$

$$|A-\lambda I|=-\lambda^3+19\lambda-30$$

$$|A-\lambda I|=(2-\lambda)(3-\lambda)(-5-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(2-\lambda)(3-\lambda)(-5-\lambda)=0$$

$$\lambda = 2,\ 3,\ -5$$

We'll handle $\lambda_1 = 2$ first, starting by finding $A - 2I$.

$$A - 2I = \begin{bmatrix} 1-2 & 0 & 4 \\ 0 & 2-2 & 0 \\ 3 & 1 & -3-2 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 4 \\ 0 & 0 & 0 \\ 3 & 1 & -5 \end{bmatrix}$$

We get the system of equations

$$-k_1 + 4k_3 = 0$$

$$3k_1 + k_2 - 5k_3 = 0$$

Solving the first equation gives $k_1 = 4k_3$. Plugging this into the second equation gives

$$3(4k_3) + k_2 - 5k_3 = 0$$

$$k_2 + 7k_3 = 0$$

$$k_2 = -7k_3$$

If we choose $k_3 = 1$, then the Eigenvector is

$$\vec{k_1} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix}$$

and therefore the solution vector is

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} e^{2t}$$

For $\lambda_2 = 3$, we start by finding $A - 3I$.

$$A - 3I = \begin{bmatrix} 1-3 & 0 & 4 \\ 0 & 2-3 & 0 \\ 3 & 1 & -3-3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & 4 \\ 0 & -1 & 0 \\ 3 & 1 & -6 \end{bmatrix}$$

We get the system of equations

$$-2k_1 + 4k_3 = 0$$

$$-k_2 = 0$$

$$3k_1 + k_2 - 6k_3 = 0$$

We can see from the second equation that $k_2 = 0$, and from the first and third equations that $k_1 = 2k_3$. If we choose $k_3 = 1$, then the Eigenvector is

$$\vec{k_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

and therefore the solution vector is

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{3t}$$

For $\lambda_3 = -5$, we start by finding $A + 5I$.

$$A + 5I = \begin{bmatrix} 1+5 & 0 & 4 \\ 0 & 2+5 & 0 \\ 3 & 1 & -3+5 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 4 \\ 0 & 7 & 0 \\ 3 & 1 & 2 \end{bmatrix}$$

We get the system of equations

$$6k_1 + 4k_3 = 0$$

$$7k_2 = 0$$

$$3k_1 + k_2 + 2k_3 = 0$$

We can see from the second equation that $k_2 = 0$, and from the first and third equations that $k_1 = -(2/3)k_3$. If we choose $k_3 = -3$, then the Eigenvector is

$$\overrightarrow{k_3} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

and therefore the solution vector is

$$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t} = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} e^{-5t}$$

So the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} e^{-5t}$$

Topic: Systems of higher order equations

Question: Solve the system of differential equations.

$$x_1' = -x_2$$

$$x_2' = x_1$$

$$x_3' = x_3$$

Answer choices:

A $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} i \\ 1 \\ -i \end{bmatrix} e^{it} + c_3 \begin{bmatrix} -i \\ 0 \\ 1 \end{bmatrix} e^{-it}$

B $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e^{it} + c_3 \begin{bmatrix} -i \\ 1 \\ 0 \end{bmatrix} e^{-it}$

C $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 0 \\ 1 \\ i \end{bmatrix} e^{it} + c_3 \begin{bmatrix} 0 \\ -i \\ 1 \end{bmatrix} e^{-it}$

D $\quad \overrightarrow{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix} e^{it} + c_3 \begin{bmatrix} 1 \\ 1 \\ -i \end{bmatrix} e^{-it}$

Solution: B

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1 - \lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & -1 & 0 \\ 1 & -\lambda & 0 \\ 0 & 0 & 1 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = -\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & 1 - \lambda \end{vmatrix} + 1 \begin{vmatrix} 1 & 0 \\ 0 & 1 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = -\lambda(-\lambda(1 - \lambda)) + 1 - \lambda$$

$$|A - \lambda I| = -\lambda^3 + \lambda^2 - \lambda + 1$$

$$|A - \lambda I| = (\lambda^2 + 1)(1 - \lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(\lambda^2 + 1)(1 - \lambda) = 0$$

$$\lambda = 1, \ i, \ \ -i$$

We'll handle $\lambda_1 = 1$ first, starting by finding $A - 1I$.

$$A - 1I = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} -1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We get the system of equations

$$-k_1 - k_2 = 0$$

$$k_1 - k_2 = 0$$

Solving this system gives us $k_1 = k_2 = 0$. If we choose $k_3 = 1$, then the Eigenvector is

$$\overrightarrow{k_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore the solution vector is

$$\overrightarrow{x_1} = c_1 \overrightarrow{k_1} e^{\lambda_1 t} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t$$

For $\lambda_2 = i$, we start by finding $A - iI$.

$$A - iI = \begin{bmatrix} -i & -1 & 0 \\ 1 & -i & 0 \\ 0 & 0 & 1-i \end{bmatrix}$$

We get the system of equations

$$-ik_1 - k_2 = 0$$

$$k_1 - ik_2 = 0$$

$$(1 - i)k_3 = 0$$

Solving this system gives us $k_1 = ik_2$ and $k_3 = 0$. If we choose $k_2 = 1$, then the Eigenvector is

$$\overrightarrow{k_2} = \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix}$$

So the solution vector is

$$\overrightarrow{x_2} = c_2\overrightarrow{k_2}e^{\lambda_2 t} = c_2 \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e^{it}$$

For $\lambda_2 = -i$, we start by finding $A + iI$.

$$A - I = \begin{bmatrix} i & -1 & 0 \\ 1 & i & 0 \\ 0 & 0 & 1+i \end{bmatrix}$$

We get the system of equations

$$ik_1 - k_2 = 0$$

$$k_1 + ik_2 = 0$$

$$(1 + i)k_3 = 0$$

Solving this system gives us $k_1 = -ik_2$ and $k_3 = 0$. If we choose $k_2 = -1$, then the Eigenvector is

$$\overrightarrow{k_3} = \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix}$$

So the solution vector is

$$\overrightarrow{x_3} = c_3\overrightarrow{k_3}e^{\lambda_3 t} = c_3 \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix} e^{-it}$$

So the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} i \\ 1 \\ 0 \end{bmatrix} e^{it} + c_3 \begin{bmatrix} i \\ -1 \\ 0 \end{bmatrix} e^{-it}$$

Topic: Systems of higher order equations

Question: Solve the system of differential equations.

$$x_1' = x_1 + 2x_2 + 2x_3$$

$$x_2' = x_2 + x_3$$

$$x_3' = x_3$$

Answer choices:

A $$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} e^t \right) + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} e^t \right)$$

B $$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ \frac{2}{3} \\ 0 \end{bmatrix} e^{4t} \right) + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} \frac{2}{3} \\ 0 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} \frac{4}{7} \\ -\frac{3}{8} \\ \frac{1}{2} \end{bmatrix} e^t \right)$$

C $$\vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix} e^t \right) + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ \frac{3}{4} \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} e^t \right)$$

D $$\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} e^t \right) + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{bmatrix} e^{3t} \right)$$

Solution: A

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 2 & 2 \\ 0 & 1 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 1 - \lambda & 2 & 2 \\ 0 & 1 - \lambda & 1 \\ 0 & 0 & 1 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (1 - \lambda) \begin{vmatrix} 1 - \lambda & 1 \\ 0 & 1 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (1 - \lambda)^3$$

Solve the characteristic equation for the Eigenvalues.

$$(1 - \lambda)^3 = 0$$

$$\lambda = 1,\ 1,\ 1$$

We'll start by finding $A - 1I$.

$$A - 1I = \begin{bmatrix} 1-1 & 2 & 2 \\ 0 & 1-1 & 1 \\ 0 & 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We get the system of equations

$$2k_2 + 2k_3 = 0$$

$$k_3 = 0$$

So $k_2 = k_3 = 0$, and if we choose $k_1 = 1$, we find only one Eigenvector.

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and therefore the solution vector is

$$\overrightarrow{x_1} = c_1\overrightarrow{k_1}e^{\lambda_1 t} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$$

Because we only find one linearly independent Eigenvector for the one Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = 1$, we have to use $\overrightarrow{k_1} = (1,0,0)$ to find a second solution.

$$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_1}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$2p_2 + 2p_3 = 1$$

$$p_3 = 0$$

Solving this system gives $p_2 = 1/2$ and $p_3 = 0$. If we choose $p_1 = 0$, then the Eigenvector is

$$\overrightarrow{p_1} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

and therefore the solution vector is

$$\overrightarrow{x_2} = \overrightarrow{k_1} te^{\lambda_1 t} + \overrightarrow{p_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} e^t$$

But we still need another solution. We'll use $\overrightarrow{p_1} = (0,1/2,0)$ to find a third solution.

$$(A - \lambda_1 I)\overrightarrow{q_1} = \overrightarrow{p_1}$$

$$\begin{bmatrix} 0 & 2 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$2q_2 + 2q_3 = 0$$

$$q_3 = \frac{1}{2}$$

Solving this system gives $q_2 = -1/2$ and $q_3 = 1/2$. If we choose $q_1 = 0$, then the Eigenvector is

$$\overrightarrow{q_1} = \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

and therefore the solution vector is

$$\overrightarrow{x_3} = \overrightarrow{k_1}\frac{t^2}{2}e^{\lambda_1 t} + \overrightarrow{p_1}te^{\lambda_1 t} + \overrightarrow{q_1}e^{\lambda_1 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\frac{t^2}{2}e^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix}te^t + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}e^t$$

So the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} e^t \right) + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2} e^t + \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix} e^t \right)$$

Topic: Series solutions of higher order equations

Question: Find a power series solution in x to the differential equation.

$$y''' + y = 0$$

Answer choices:

A $\quad y = c_0 + c_1x + c_2x^2 + \dfrac{(-1)^{m+1}c_0}{(2m-2)!}x^{2m} + \dfrac{(-1)^m c_1}{(2m+4)!}x^{2m+1} + \dfrac{2(-1)^m c_2}{(2m+2)!}x^{2m+2} + \cdots$

B $\quad y = c_0 + c_1x + c_2x^2 + \dfrac{(-1)^m c_0}{(3m+3)!}x^{3m+1} + \dfrac{(-1)^m c_1}{(3m+4)!}x^{3m+2} + \dfrac{3(-1)^m c_2}{(3m+5)!}x^{3m+3} + \cdots$

C $\quad y = c_0 + c_1x + c_2x^2 + \dfrac{(-1)^m c_0}{(3m)!}x^{3m} + \dfrac{(-1)^m c_1}{(3m+1)!}x^{3m+1} + \dfrac{2(-1)^m c_2}{(3m+2)!}x^{3m+2} + \cdots$

D $\quad y = c_0 + c_1x + c_2x^2 + \dfrac{(-1)^m c_0}{(4m)!}x^{4m} + \dfrac{(-1)^m c_1}{(4m+1)!}x^{4m+1} + \dfrac{3(-1)^m c_2}{(4m+2)!}x^{4m+2} + \cdots$

Solution: C

We'll substitute y and y''' into the differential equation.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=0}^{\infty} c_n x^n = 0$$

The series are in phase, but the indices don't match. We can substitute $k = n - 3$ and $n = k + 3$ into the first series, and $k = n$ into the second series.

$$\sum_{k=0}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k = 0$$

With the series in phase and matching indices, we can now add them.

$$\sum_{k=0}^{\infty} [c_{k+3}(k+3)(k+2)(k+1) + c_k]x^k = 0$$

$$k = 0, 1, 2, 3, \ldots \qquad c_{k+3}(k+3)(k+2)(k+1) + c_k = 0$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+3}.

$$c_{k+3} = \frac{-c_k}{(k+3)(k+2)(k+1)}$$

Now we'll start plugging in values $k = 0, 1, 2, 3, \ldots$.

$$k = 0 \quad c_3 = \frac{-c_0}{(2)(3)} \qquad\qquad k = 1 \quad c_4 = \frac{-c_1}{(2)(3)(4)}$$

$$k = 2 \quad c_5 = \frac{-c_2}{(3)(4)(5)} \qquad\qquad k = 3 \quad c_6 = \frac{c_0}{(2)(3)(4)(5)(6)}$$

$$k = 4 \quad c_7 = \frac{c_1}{(2)(3)(4)(5)(6)(7)} \qquad k = 5 \quad c_8 = \frac{c_2}{(3)(4)(5)(6)(7)(8)}$$

$$k = 6 \quad c_9 = \frac{c_0}{(2)(3)(4)(5)(6)(7)(8)(9)} \qquad k = 7 \quad c_{10} = \frac{-c_1}{(2)(3)(4)(5)(6)(7)(8)(9)(10)}$$

$$\vdots \qquad\qquad \vdots$$

We can start to see a pattern if we look at every third term. We can break the coefficients into three cases, given by c_{3m}, c_{3m+1}, and c_{3m+2} for $m = 1, 2, 3, \ldots.$

$$c_{3m} = \frac{(-1)^m c_0}{(3m)!} \qquad m = 1, 2, 3, \ldots$$

$$c_{3m+1} = \frac{(-1)^m c_1}{(3m+1)!} \qquad m = 1, 2, 3, \ldots$$

$$c_{3m+2} = \frac{2(-1)^m c_2}{(3m+2)!} \qquad m = 1, 2, 3, \ldots$$

Now we can write the general solution to the differential equation as

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \ldots$$

$$y = c_0 + c_1x + c_2x^2 + c_{3m}x^{3m} + c_{3m+1}x^{3m+1} + c_{3m+2}x^{3m+2} + \cdots$$

$$y = c_0 + c_1x + c_2x^2 + \frac{(-1)^m c_0}{(3m)!}x^{3m} + \frac{(-1)^m c_1}{(3m+1)!}x^{3m+1} + \frac{2(-1)^m c_2}{(3m+2)!}x^{3m+2} + \cdots$$

Topic: Series solutions of higher order equations

Question: Find a power series solution in x to the differential equation.

$$y''' + xy' + xy = 0$$

Answer choices:

A $$y = c_0\left(1 - \frac{1}{30}x^4 + \frac{3}{334}x^7 + \frac{1}{2{,}048}x^8 + \dots\right)$$

$$+c_1\left(x - \frac{1}{24}x^3 - \frac{1}{60}x^4 + \frac{1}{1{,}260}x^6 + \frac{1}{2{,}688}x^7 + \frac{1}{30{,}240}x^8 + \dots\right)$$

$$+c_2\left(x^2 - \frac{1}{15}x^5 - \frac{1}{10}x^6 + \frac{1}{1{,}008}x^8 + \frac{1}{10{,}080}x^9 + \dots\right)$$

B $$y = c_0\left(1 - \frac{3}{8}x^5 + \frac{4}{7}x^8 + \frac{5}{6}x^{10} + \dots\right)$$

$$+c_1\left(x - \frac{1}{3}x^4 - \frac{2}{3}x^5 + \frac{4}{3}x^6 + \frac{10}{7}x^7 + \frac{1}{3}x^9 + \dots\right)$$

$$+c_2\left(x^2 - \frac{2}{11}x^7 - \frac{4}{9}x^8 + \frac{4}{13}x^{10} + \frac{5}{16}x^{11} + \dots\right)$$

C
$$y = c_0\left(1 - \frac{1}{12}x^4 + \frac{1}{630}x^7 + \frac{1}{4,032}x^8 + \dots\right)$$

$$+c_1\left(x - \frac{1}{48}x^4 - \frac{1}{120}x^5 + \frac{1}{2,320}x^7 + \frac{1}{5,376}x^8 + \frac{1}{60,480}x^9 + \dots\right)$$

$$+c_2\left(x^2 - \frac{1}{30}x^5 - \frac{1}{120}x^6 + \frac{1}{2,016}x^8 + \frac{1}{6,048}x^9 + \dots\right)$$

D
$$y = c_0\left(1 - \frac{1}{24}x^4 + \frac{1}{1,260}x^7 + \frac{1}{8,064}x^8 + \dots\right)$$

$$+c_1\left(x - \frac{1}{24}x^4 - \frac{1}{60}x^5 + \frac{1}{1,260}x^7 + \frac{1}{2,688}x^8 + \frac{1}{30,240}x^9 + \dots\right)$$

$$+c_2\left(x^2 - \frac{1}{30}x^5 - \frac{1}{120}x^6 + \frac{1}{2,016}x^8 + \frac{1}{6,048}x^9 + \dots\right)$$

Solution: D

We'll substitute y, y', and y''' into the differential equation.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + x\sum_{n=1}^{\infty} c_n n x^{n-1} + x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

The first series begins with the x^0 term while the second and third series both begin with the x^1 term. So to put these series in phase, we'll pull the x^0 term out of the first series.

$$c_3 3(3-1)(3-2)x^{3-3} + \sum_{n=4}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n - 3$ and $n = k + 3$ into the first series, $k = n$ into the second series, and $k = n + 1$ and $n = k - 1$ into the third series.

$$6c_3 + \sum_{k=1}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k + \sum_{k=1}^{\infty} c_k k x^k + \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

With the series in phase and matching indices, we can now add them.

$$6c_3 + \sum_{k=1}^{\infty} [c_{k+3}(k+3)(k+2)(k+1) + c_k k + c_{k-1}]x^k = 0$$

The $6c_3$ value in front of the series is associated with the $k = 0$ term, while the remaining series still represents the $k = 1, 2, 3, \ldots$ terms.

$k = 0$ $\qquad\qquad$ $6c_3 = 0$

$\qquad\qquad\qquad\quad$ $c_3 = 0$

$k = 1, 2, 3, \ldots$ $\qquad\qquad$ $c_{k+3}(k+3)(k+2)(k+1) + c_k k + c_{k-1} = 0$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+3}.

$$c_{k+3} = \frac{-c_{k-1} - c_k k}{(k+3)(k+2)(k+1)}$$

Now we'll start plugging in values $k = 1, 2, 3, \dots$.

$$k = 0 \qquad c_3 = 0$$

$$k = 1 \qquad c_4 = \frac{-c_0 - c_1}{24} \qquad k = 2 \qquad c_5 = \frac{-c_1 - 2c_2}{60}$$

$$k = 3 \qquad c_6 = \frac{-c_2}{120} \qquad k = 4 \qquad c_7 = \frac{c_0 + c_1}{1{,}260}$$

$$k = 5 \qquad c_8 = \frac{c_0 + 3c_1 + 4c_2}{8{,}064} \qquad k = 6 \qquad c_9 = \frac{c_1 + 5c_2}{30{,}240}$$

$$\vdots \qquad \vdots$$

There's not an obvious pattern here. As we've seen, the higher the order of the differential equation, the less likely it becomes that we'll be able to find clean series representations.

So without worrying about the pattern, we'll just use these values to say that the general solution is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$$

$$y = c_0 + c_1 x + c_2 x^2 + \frac{-c_0 - c_1}{24} x^4 + \frac{-c_1 - 2c_2}{60} x^5 + \frac{-c_2}{120} x^6 + \dots$$

$$y = c_0 \left(1 - \frac{1}{24} x^4 + \frac{1}{1{,}260} x^7 + \frac{1}{8{,}064} x^8 + \dots \right)$$

$$+c_1\left(x-\frac{1}{24}x^4-\frac{1}{60}x^5+\frac{1}{1{,}260}x^7+\frac{1}{2{,}688}x^8+\frac{1}{30{,}240}x^9+\ldots\right)$$

$$+c_2\left(x^2-\frac{1}{30}x^5-\frac{1}{120}x^6+\frac{1}{2{,}016}x^8+\frac{1}{6{,}048}x^9+\ldots\right)$$

Topic: Series solutions of higher order equations

Question: Find a power series solution in x to the differential equation.

$$y''' + x^2y' + xy = 0$$

Answer choices:

A
$$y = c_0 + c_1x + c_2x^2 + \frac{(-1)^m 1^2 5^2 \cdots (2m-3)^2 c_0}{(2m)!}x^{2m}$$

$$+\frac{(-1)^m 2^2 6^2 \cdots (2m-2)^2 c_1}{(2m+1)!}x^{2m+1}$$

$$+\frac{2(-1)^m 3^2 7^2 \cdots (2m-1)^2 c_2}{(2m+2)!}x^{2m+2} + \cdots$$

B
$$y = c_0 + c_1x + c_2x^2 + \frac{(-1)^m 2^2 8^2 \cdots (4m-3)^2 c_0}{(4m-2)!}x^{4m}$$

$$+\frac{(-1)^m 3^2 9^2 \cdots (4m-4)^2 c_1}{(4m-3)!}x^{4m+1}$$

$$+\frac{2(-1)^m 4^2 11^2 \cdots (4m-5)^2 c_2}{(4m-6)!}x^{4m+2} + \cdots$$

C $$y = c_0 + c_1x + c_2x^2 + \frac{(-1)^m 2^2 5^2 \cdots (3m-2)^2 c_0}{(3m)!}x^{3m}$$

$$+\frac{(-1)^m 3^2 6^2 \cdots (3m-2)^2 c_1}{(3m+1)!}x^{3m+1}$$

$$+\frac{2(-1)^m 4^2 7^2 \cdots (3m-1)^2 c_2}{(3m+2)!}x^{3m+2} + \cdots$$

D $$y = c_0 + c_1x + c_2x^2 + \frac{(-1)^m 1^2 5^2 \cdots (4m-3)^2 c_0}{(4m)!}x^{4m}$$

$$+\frac{(-1)^m 2^2 6^2 \cdots (4m-2)^2 c_1}{(4m+1)!}x^{4m+1}$$

$$+\frac{2(-1)^m 3^2 7^2 \cdots (4m-1)^2 c_2}{(4m+2)!}x^{4m+2} + \cdots$$

Solution: D

We'll substitute y, y', and y''' into the differential equation.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + x^2 \sum_{n=1}^{\infty} c_n n x^{n-1} + x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

The first series begins with the x^0 term, the second series begins with the x^2 term, and the third series begins with the x^1 term. So to put these series in phase, we'll pull the x^0 and x^1 terms out of the first series and pull the x^1 term out of the third series.

$$c_3 3(3-1)(3-2)x^{3-3} + c_4 4(4-1)(4-2)x^{4-3} + \sum_{n=5}^{\infty} c_n n(n-1)(n-2)x^{n-3}$$

$$+\sum_{n=1}^{\infty} c_n n x^{n+1} + c_0 x^{0+1} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

$$6c_3 + (c_0 + 24c_4)x + \sum_{n=5}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^{n+1} + \sum_{n=1}^{\infty} c_n x^{n+1} = 0$$

Now the series are in phase, but the indices don't match. We can substitute $k = n-3$ and $n = k+3$ into the first series, and $k = n+1$ and $n = k-1$ into the second and third series.

$$6c_3 + (c_0 + 24c_4)x + \sum_{k=2}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k$$

$$+\sum_{k=2}^{\infty} c_{k-1}(k-1)x^k + \sum_{k=2}^{\infty} c_{k-1}x^k = 0$$

With the series in phase and matching indices, we can finally add them.

$$6c_3 + (c_0 + 24c_4)x + \sum_{k=2}^{\infty} c_{k+3}(k+3)(k+2)(k+1)x^k + c_{k-1}(k-1)x^k + c_{k-1}x^k = 0$$

$$6c_3 + (c_0 + 24c_4)x + \sum_{k=2}^{\infty} [c_{k+3}(k+3)(k+2)(k+1) + c_{k-1}(k-1) + c_{k-1}]x^k = 0$$

$$6c_3 + (c_0 + 24c_4)x + \sum_{k=2}^{\infty} [c_{k+3}(k+3)(k+2)(k+1) + c_{k-1}(k-1+1)]x^k = 0$$

$$6c_3 + (c_0 + 24c_4)x + \sum_{k=2}^{\infty} [c_{k+3}(k+3)(k+2)(k+1) + c_{k-1}k]x^k = 0$$

This equation gives us

$$k = 0 \qquad 6c_3 = 0$$

$$c_3 = 0$$

$$k = 1 \qquad c_0 + 24c_4 = 0$$

$$c_4 = \frac{-c_0}{(2)(3)(4)}$$

$$k = 2, 3, 4, \ldots \qquad c_{k+3}(k+3)(k+2)(k+1) + c_{k-1}k = 0$$

We'll solve the recurrence relation for the coefficient with the largest subscript, c_{k+3}.

$$c_{k+3} = -\frac{c_{k-1}k}{(k+3)(k+2)(k+1)}$$

Now we'll start plugging in values $k = 2, 3, 4, \ldots$. In this step, it's best to keep the multiplications written out rather than calculating them so we can try to find identify a pattern.

$$k = 0 \quad c_3 = 0$$

$$k = 1 \quad c_4 = \frac{-c_0}{(2)(3)(4)} \qquad k = 2 \quad c_5 = \frac{-2c_1}{(3)(4)(5)}$$

$$k=3 \quad c_6=\frac{-3c_2}{(4)(5)(6)} \qquad k=4 \quad c_7=0$$

$$k=5 \quad c_8=\frac{5c_0}{(2)(3)(4)(6)(7)(8)} \qquad k=6 \quad c_9=\frac{(2)(6)c_1}{(3)(4)(5)(7)(8)(9)}$$

$$k=7 \quad c_{10}=\frac{(3)(7)c_2}{(4)(5)(6)(8)(9)(10)} \qquad k=8 \quad c_{11}=0$$

$$k=9 \quad c_{12}=\frac{-(5)(9)c_0}{(2)(3)(4)(6)(7)(8)(10)(11)(12)}$$

$$k=10 \quad c_{13}=\frac{-(2)(6)(10)c_1}{(3)(4)(5)(7)(8)(9)(11)(12)(13)}$$

$$k=11 \quad c_{14}=\frac{-(3)(7)(11)c_2}{(4)(5)(6)(8)(9)(10)(12)(13)(14)}$$

$$\vdots$$

We can start to see a pattern if we look at every fourth term. We can break the coefficients into four cases, given by c_{4m}, c_{4m+1}, c_{4m+2}, and c_{4m+3} for $m=1,\ 2,\ 3,\ \ldots$.

$$c_{4m}=\frac{(-1)^m 1^2 5^2 \cdots (4m-3)^2 c_0}{(4m)!} \qquad m=1,\ 2,\ 3,\ \ldots$$

$$c_{4m+1}=\frac{(-1)^m 2^2 6^2 \cdots (4m-2)^2 c_1}{(4m+1)!} \qquad m=1,\ 2,\ 3,\ \ldots$$

$$c_{4m+2}=\frac{2(-1)^m 3^2 7^2 \cdots (4m-1)^2 c_2}{(4m+2)!} \qquad m=1,\ 2,\ 3,\ \ldots$$

$$c_{4m+3} = 0$$

Now we can write the general solution to the differential equation as

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \dots$$

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_{4m}x^{4m} + c_{4m+1}x^{4m+1} + c_{4m+2}x^{4m+2} + c_{4m+3}x^{4m+3} + \cdots$$

$$y = c_0 + c_1x + c_2x^2 + \frac{(-1)^m 1^2 5^2 \cdots (4m-3)^2 c_0}{(4m)!}x^{4m}$$

$$+\frac{(-1)^m 2^2 6^2 \cdots (4m-2)^2 c_1}{(4m+1)!}x^{4m+1}$$

$$+\frac{2(-1)^m 3^2 7^2 \cdots (4m-1)^2 c_2}{(4m+2)!}x^{4m+2} + \cdots$$

Topic: Fourier series representations

Question: Find the Fourier series representation of the function on $-L \leq x \leq L$.

$$f(x) = 1 + x + x^2$$

Answer choices:

A $\quad f(x) = 1 + \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = 1 + \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right)$

Solution: A

The function isn't even because $f(-x) \neq f(x)$ and it's not odd because $f(-x) \neq -f(x)$.

$$f(-x) = 1 - x + (-x)^2$$

$$f(-x) = 1 - x + x^2$$

So we'll start by finding A_0 and the coefficients A_n and B_n. For A_0, we get

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L} \int_{-L}^{L} 1 + x + x^2\ dx$$

$$A_0 = \frac{1}{2L} \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 \right) \Big|_{-L}^{L}$$

$$A_0 = \frac{1}{2L} \left(L + \frac{1}{2}L^2 + \frac{1}{3}L^3 \right) - \frac{1}{2L} \left(-L + \frac{1}{2}(-L)^2 + \frac{1}{3}(-L)^3 \right)$$

$$A_0 = \frac{1}{2} \left(1 + \frac{1}{2}L + \frac{1}{3}L^2 \right) - \frac{1}{2} \left(-1 + \frac{1}{2}L - \frac{1}{3}L^2 \right)$$

$$A_0 = \frac{1}{2} \left(1 + \frac{1}{2}L + \frac{1}{3}L^2 + 1 - \frac{1}{2}L + \frac{1}{3}L^2 \right)$$

$$A_0 = 1 + \frac{L^2}{3}$$

For A_n, we get

$$A_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{1}{L}\int_{-L}^{L}(1+x+x^2)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{1}{L}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\ dx+\frac{1}{L}\int_{-L}^{L}x\cos\left(\frac{n\pi x}{L}\right)\ dx+\frac{1}{L}\int_{-L}^{L}x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}+\frac{1}{L}\int_{-L}^{L}x\cos\left(\frac{n\pi x}{L}\right)\ dx+\frac{1}{L}\int_{-L}^{L}x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=x$, $du=dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$.

$$A_n=\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}-\frac{1}{n\pi}\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{1}{L}\int_{-L}^{L}x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}$$

$$+\frac{1}{L}\int_{-L}^{L}x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=x^2$, $du=2x\ dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$.

$$A_n=\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)$$

$$+\frac{x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L}-\frac{2}{n\pi}\int_{-L}^{L}x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=x$, $du=dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$.

$$A_n=\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)+\frac{x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

$$+\frac{2Lx}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L}-\frac{2L}{(n\pi)^2}\int_{-L}^{L}\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)+\frac{x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

$$+\frac{2Lx}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)-\frac{2L^2}{(n\pi)^3}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L}$$

$$A_n=\left(\frac{1+x+x^2}{n\pi}-\frac{2L^2}{(n\pi)^3}\right)\sin\left(\frac{n\pi x}{L}\right)+\left(\frac{L+2Lx}{(n\pi)^2}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L}$$

$$A_n=\left(\frac{1+L+L^2}{n\pi}-\frac{2L^2}{(n\pi)^3}\right)\sin(n\pi)+\left(\frac{L+2L^2}{(n\pi)^2}\right)\cos(n\pi)$$

$$-\left(\frac{1-L+L^2}{n\pi}-\frac{2L^2}{(n\pi)^3}\right)\sin(-n\pi)-\left(\frac{L-2L^2}{(n\pi)^2}\right)\cos(-n\pi)$$

Using the even-odd trigonometric identities $\sin(-x)=-\sin x$ and $\cos(-x)=\cos x$, we can simplify A_n to

$$A_n=\left(\frac{1+L+L^2}{n\pi}-\frac{2L^2}{(n\pi)^3}\right)\sin(n\pi)+\left(\frac{L+2L^2}{(n\pi)^2}\right)\cos(n\pi)$$

$$+\left(\frac{1-L+L^2}{n\pi}-\frac{2L^2}{(n\pi)^3}\right)\sin(n\pi)-\left(\frac{L-2L^2}{(n\pi)^2}\right)\cos(n\pi)$$

For $n=1,2,3,\ldots$, $\sin(n\pi)=0$, and $\cos(n\pi)=(-1)^n$, so the expression for A_n simplifies to

$$A_n=\left(\frac{L+2L^2}{(n\pi)^2}\right)(-1)^n-\left(\frac{L-2L^2}{(n\pi)^2}\right)(-1)^n$$

$$A_n=\frac{L(-1)^n+2L^2(-1)^n}{(n\pi)^2}-\frac{L(-1)^n-2L^2(-1)^n}{(n\pi)^2}$$

$$A_n=\frac{L(-1)^n+2L^2(-1)^n-L(-1)^n+2L^2(-1)^n}{(n\pi)^2}$$

$$A_n=\frac{4L^2(-1)^n}{(n\pi)^2}$$

$$A_n=\left(\frac{2L}{n\pi}\right)^2(-1)^n$$

And for B_n, we get

$$B_n=\frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n=\frac{1}{L}\int_{-L}^{L} (1+x+x^2)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} \sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{1}{n\pi}\int_{-L}^{L} \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L}$$

$$+\frac{1}{L}\int_{-L}^{L} x^2\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$-\frac{x^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{L} + \frac{2}{n\pi}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$-\frac{x^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2Lx}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L} - \frac{2L}{(n\pi)^2}\int_{-L}^{L}\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$-\frac{x^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2Lx}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^2}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L}$$

$$B_n = \left(-\frac{1+x+x^2}{n\pi} + \frac{2L^2}{(n\pi)^3}\right)\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L+2Lx}{(n\pi)^2}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L}$$

$$B_n = \left(-\frac{1+L+L^2}{n\pi} + \frac{2L^2}{(n\pi)^3}\right)\cos(n\pi) + \left(\frac{L+2L^2}{(n\pi)^2}\right)\sin(n\pi)$$

$$-\left(-\frac{1-L+L^2}{n\pi} + \frac{2L^2}{(n\pi)^3}\right)\cos(-n\pi) - \left(\frac{L-2L^2}{(n\pi)^2}\right)\sin(-n\pi)$$

Using the even-odd trigonometric identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$, we can simplify B_n to

$$B_n = \left(-\frac{1+L+L^2}{n\pi} + \frac{2L^2}{(n\pi)^3}\right)\cos(n\pi) + \left(\frac{L+2L^2}{(n\pi)^2}\right)\sin(n\pi)$$

$$-\left(-\frac{1-L+L^2}{n\pi} + \frac{2L^2}{(n\pi)^3}\right)\cos(n\pi) + \left(\frac{L-2L^2}{(n\pi)^2}\right)\sin(n\pi)$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$, and $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = \left(-\frac{1+L+L^2}{n\pi} + \frac{2L^2}{(n\pi)^3}\right)(-1)^n - \left(-\frac{1-L+L^2}{n\pi} + \frac{2L^2}{(n\pi)^3}\right)(-1)^n$$

$$B_n = -\frac{1+L+L^2}{n\pi}(-1)^n + \frac{2L^2}{(n\pi)^3}(-1)^n + \frac{1-L+L^2}{n\pi}(-1)^n - \frac{2L^2}{(n\pi)^3}(-1)^n$$

$$B_n = \frac{1-L+L^2-1-L-L^2}{n\pi}(-1)^n$$

$$B_n = \frac{-2L}{n\pi}(-1)^n$$

$$B_n = \frac{2L}{n\pi}(-1)^{n+1}$$

Then the function's Fourier series representation on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 1 + \frac{L^2}{3} + \sum_{n=1}^{\infty}\left(\frac{2L}{n\pi}\right)^2(-1)^n\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty}\frac{2L}{n\pi}(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 1 + \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

Topic: Fourier series representations

Question: Find the Fourier series representation of the function on $-L \leq x \leq L$.

$$f(x) = x - x^3$$

Answer choices:

A $\quad f(x) = \frac{L}{\pi} + \frac{2L}{\pi}\sum_{n=1}^{\infty}\left(\frac{L^2 - 1}{n} - \frac{6L^2}{(n\pi)^2}\right)(-1)^n \cos\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = \frac{L}{\pi} + \sum_{n=1}^{\infty}\left(\frac{L^2 - 1}{n} - \frac{6L^2}{(n\pi)^2}\right)\cos\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = \frac{2L}{\pi}\sum_{n=1}^{\infty}\left(\frac{L^2 - 1}{n} - \frac{6L^2}{(n\pi)^2}\right)(-1)^n \sin\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = \sum_{n=1}^{\infty}\left(\frac{L^2 - 1}{n} - \frac{6L^2}{(n\pi)^2}\right)\sin\left(\frac{n\pi x}{L}\right)$

Solution: C

The function is odd because $f(-x)=-f(x)$,

$$f(-x)=-x-(-x)^3$$

$$f(-x)=-x+x^3$$

$$f(-x)=-(x-x^3)$$

$$f(-x)=-f(x)$$

Because the function is odd, $A_0=0$ and $A_n=0$, and we only need to calculate B_n. For B_n, we get

$$B_n=\frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n=\frac{2}{L}\int_0^L (x-x^3)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n=\frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx-\frac{2}{L}\int_0^L x^3\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=x^3$, $du=3x^2\ dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$.

$$B_n=\frac{2x^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L+\frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx-\frac{6}{n\pi}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=x^2$, $du=2x\ dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$.

$$B_n = \frac{2x^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{6Lx^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L + \left(\frac{2}{L} + \frac{12L}{(n\pi)^2}\right)\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{2x^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{6Lx^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$-\left(\frac{2}{L} + \frac{12L}{(n\pi)^2}\right)\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \left(\frac{2}{L} + \frac{12L}{(n\pi)^2}\right)\frac{L}{n\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2x^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{6Lx^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$-\left(\frac{2}{L} + \frac{12L}{(n\pi)^2}\right)\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{2}{L} + \frac{12L}{(n\pi)^2}\right)\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$B_n = \left(\frac{2x^3 - 2x}{n\pi} - \frac{12L^2x}{(n\pi)^3}\right)\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{2L - 6Lx^2}{(n\pi)^2} + \frac{12L^3}{(n\pi)^4}\right)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$B_n = \left(\frac{2L^3 - 2L}{n\pi} - \frac{12L^3}{(n\pi)^3}\right)\cos(n\pi) + \left(\frac{2L - 6L^3}{(n\pi)^2} + \frac{12L^3}{(n\pi)^4}\right)\sin(n\pi) - \frac{12L^3}{(n\pi)^4}\sin(0)$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$, and $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = \left(\frac{2L^3 - 2L}{n\pi} - \frac{12L^3}{(n\pi)^3}\right)(-1)^n$$

Then the function's Fourier series representation on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \left(\frac{L^2 - 1}{n} - \frac{6L^2}{(n\pi)^2}\right)(-1)^n \sin\left(\frac{n\pi x}{L}\right)$$

Topic: Fourier series representations

Question: Find the Fourier series representation of the function on $-L \leq x \leq L$.

$$f(x) = x^2 + x^4$$

Answer choices:

A $$f(x) = \sum_{n=1}^{\infty} \left(\frac{2L^2 + 1}{n^2} - \frac{12L^2}{n^4\pi^2} \right) (-1)^n \sin \left(\frac{n\pi x}{L} \right)$$

B $$f(x) = \left(\frac{2L}{\pi} \right)^2 \sum_{n=1}^{\infty} \left(\frac{2L^2 + 1}{n^2} - \frac{12L^2}{n^4\pi^2} \right) (-1)^n \sin \left(\frac{n\pi x}{L} \right)$$

C $$f(x) = \frac{L^2 + L^4}{5} + \sum_{n=1}^{\infty} \left(\frac{2L^2 + 1}{n^2} - \frac{12L^2}{n^4\pi^2} \right) (-1)^n \cos \left(\frac{n\pi x}{L} \right)$$

D $$f(x) = \frac{L^2}{3} + \frac{L^4}{5} + \left(\frac{2L}{\pi} \right)^2 \sum_{n=1}^{\infty} \left(\frac{2L^2 + 1}{n^2} - \frac{12L^2}{n^4\pi^2} \right) (-1)^n \cos \left(\frac{n\pi x}{L} \right)$$

Solution: D

The function is even because $f(-x)=f(x)$,

$$f(-x)=(-x)^2+(-x)^4$$

$$f(-x)=x^2+x^4$$

$$f(-x)=f(x)$$

Because the function is even, $B_n=0$ and we only need to calculate A_0 and A_n. For A_0, we get

$$A_0=\frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0=\frac{1}{L}\int_0^L x^2+x^4\ dx$$

$$A_0=\frac{1}{L}\left(\frac{1}{3}x^3+\frac{1}{5}x^5\right)\Big|_0^L$$

$$A_0=\frac{1}{L}\left(\frac{1}{3}L^3+\frac{1}{5}L^5\right)$$

$$A_0=\frac{L^2}{3}+\frac{L^4}{5}$$

For A_n, we get

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L (x^2 + x^4)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{2}{L}\int_0^L x^4\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^4$, $du = 4x^3\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2x^4}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\ dx - \frac{8}{n\pi}\int_0^L x^3\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^3$, $du = 3x^2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = \frac{2x^4}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{8Lx^3}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$+\frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\ dx - \frac{24L}{(n\pi)^2}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2x^4}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{8Lx^3}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \left(\frac{2}{L} - \frac{24L}{(n\pi)^2}\right)\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2x^4}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{8Lx^3}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{2x^2}{n\pi} - \frac{24L^2x^2}{(n\pi)^3}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$-\left(\frac{4}{n\pi}-\frac{48L^2}{(n\pi)^3}\right)\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=x$, $du=dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$.

$$A_n=\frac{2x^4}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{8Lx^3}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)+\left(\frac{2x^2}{n\pi}-\frac{24L^2x^2}{(n\pi)^3}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$+\left(\frac{4Lx}{(n\pi)^2}-\frac{48L^3x}{(n\pi)^4}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L-\left(\frac{4L}{(n\pi)^2}-\frac{48L^3}{(n\pi)^4}\right)\int_0^L\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{2x^4}{n\pi}\sin\left(\frac{n\pi x}{L}\right)+\frac{8Lx^3}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)+\left(\frac{2x^2}{n\pi}-\frac{24L^2x^2}{(n\pi)^3}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$+\left(\frac{4Lx}{(n\pi)^2}-\frac{48L^3x}{(n\pi)^4}\right)\cos\left(\frac{n\pi x}{L}\right)-\left(\frac{4L^2}{(n\pi)^3}-\frac{48L^4}{(n\pi)^5}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$A_n=\left(\frac{2x^4+2x^2}{n\pi}-\frac{24L^2x^2+4L^2}{(n\pi)^3}+\frac{48L^4}{(n\pi)^5}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$+\left(\frac{8Lx^3+4Lx}{(n\pi)^2}-\frac{48L^3x}{(n\pi)^4}\right)\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

For $n=1, 2, 3, \ldots$, $\sin(n\pi)=0$, and $\cos(n\pi)=(-1)^n$, so the expression for A_n simplifies to

$$A_n=\left(\frac{8Lx^3+4Lx}{(n\pi)^2}-\frac{48L^3x}{(n\pi)^4}\right)(-1)^n\Bigg|_0^L$$

$$A_n = \left(\frac{8L^4 + 4L^2}{(n\pi)^2} - \frac{48L^4}{(n\pi)^4} \right) (-1)^n$$

$$A_n = \left(\frac{2L}{\pi} \right)^2 \left(\frac{2L^2 + 1}{n^2} - \frac{12L^2}{n^4\pi^2} \right) (-1)^n$$

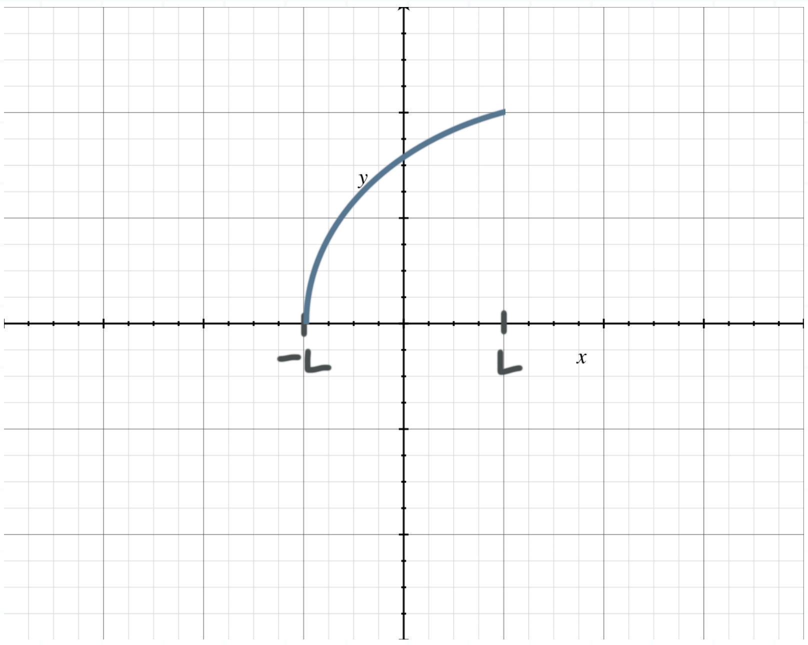
Then the function's Fourier series representation on $-L \leq x \leq L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos \left(\frac{n\pi x}{L} \right) + \sum_{n=1}^{\infty} B_n \sin \left(\frac{n\pi x}{L} \right)$$

$$f(x) = \frac{L^2}{3} + \frac{L^4}{5} + \left(\frac{2L}{\pi} \right)^2 \sum_{n=1}^{\infty} \left(\frac{2L^2 + 1}{n^2} - \frac{12L^2}{n^4\pi^2} \right) (-1)^n \cos \left(\frac{n\pi x}{L} \right)$$

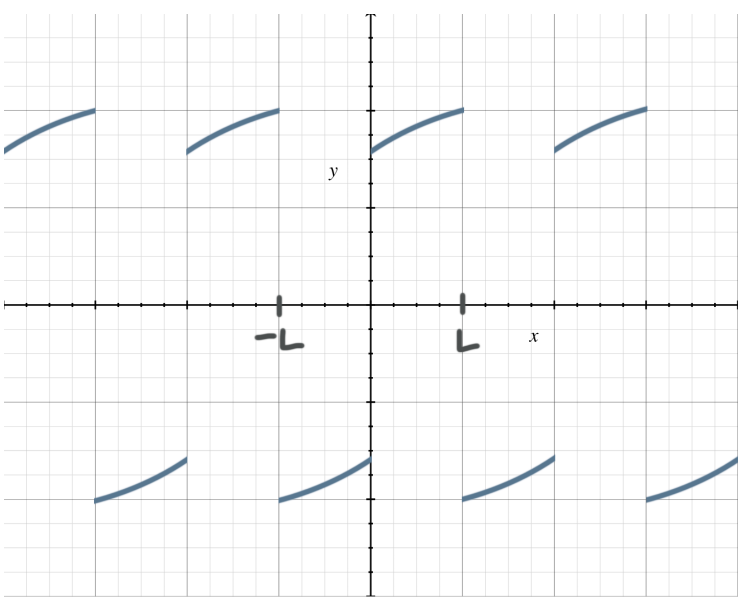
Topic: Periodic functions and periodic extensions

Question: Sketch the function’s periodic extension.

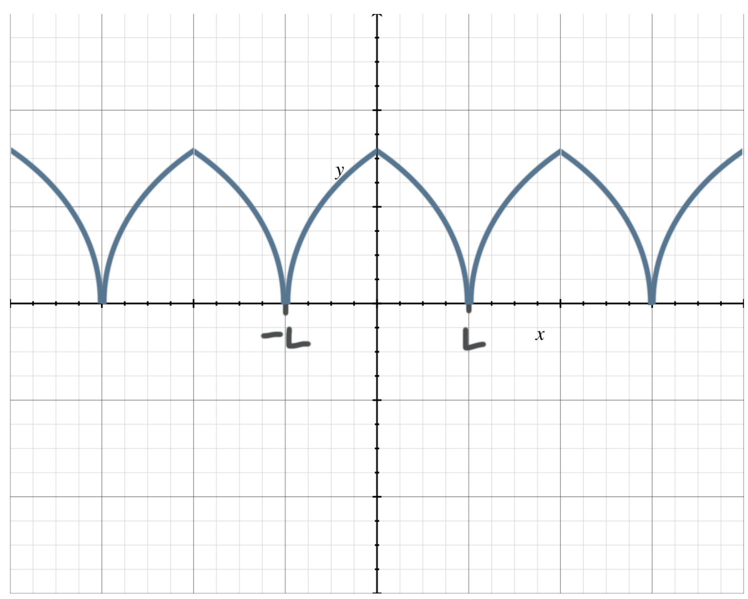


Answer choices:

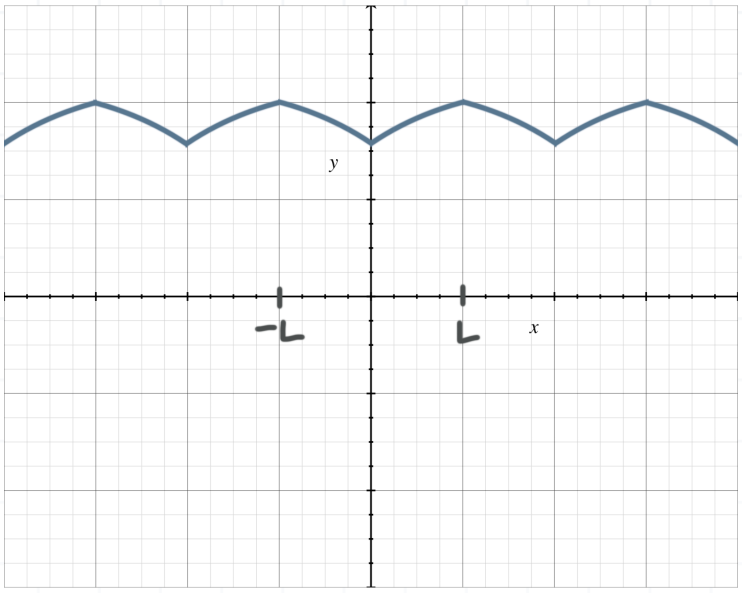
A



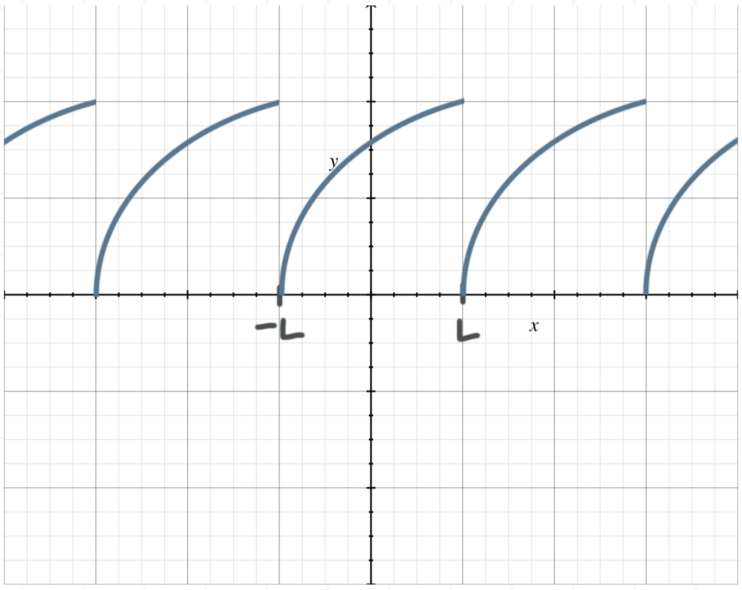
B



C



D



Solution: D

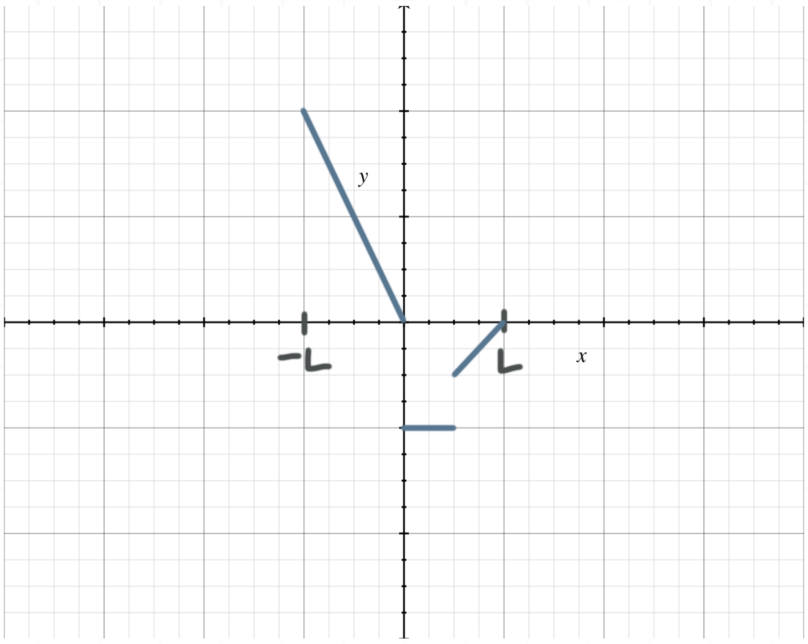
The periodic extension of a function is the function we get when we repeat the portion of the graph on $-L \leq x \leq L$ over and over to both the left and right, on $(-\infty, -L)$ and on (L, ∞). Answer choice D is the graph of the periodic extension.

Answer choice A is the odd extension of the portion of the graph on $0 \leq x \leq L$, and answer choice C is the even extension of the portion of the graph on $0 \leq x \leq L$.

We could essentially say that answer choice B is the even extension of the portion of the graph on $-L \leq x \leq 0$, but that's not something we typically do.

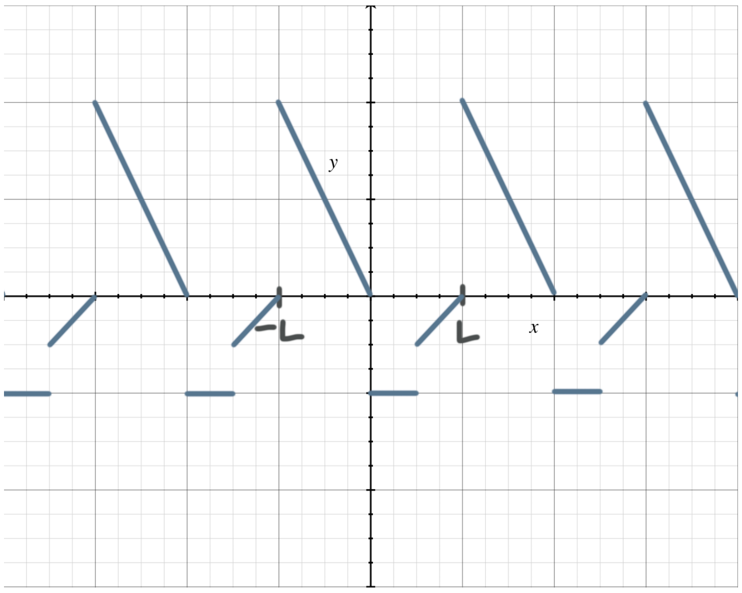
Topic: Periodic functions and periodic extensions

Question: Sketch the function's even extension.

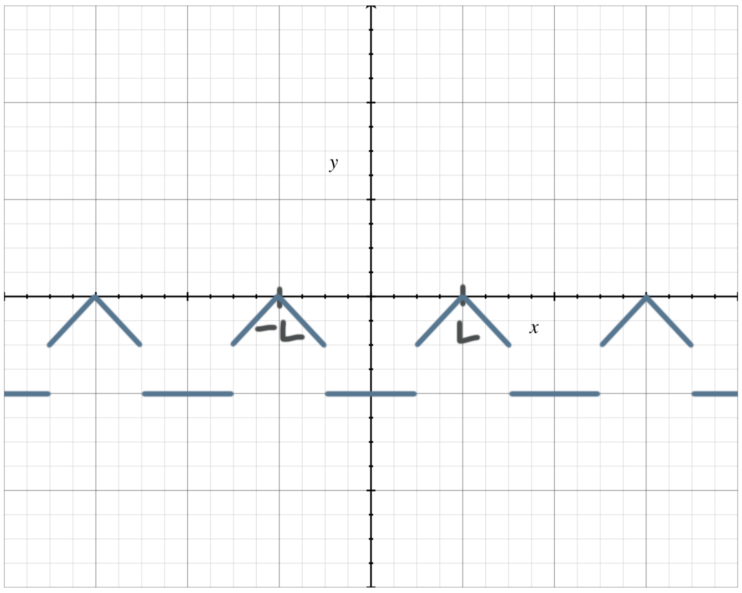


Answer choices:

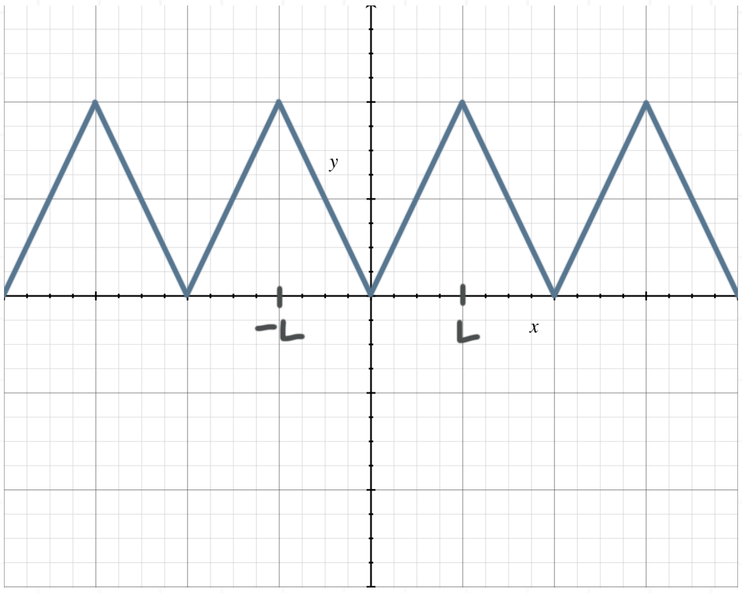
A



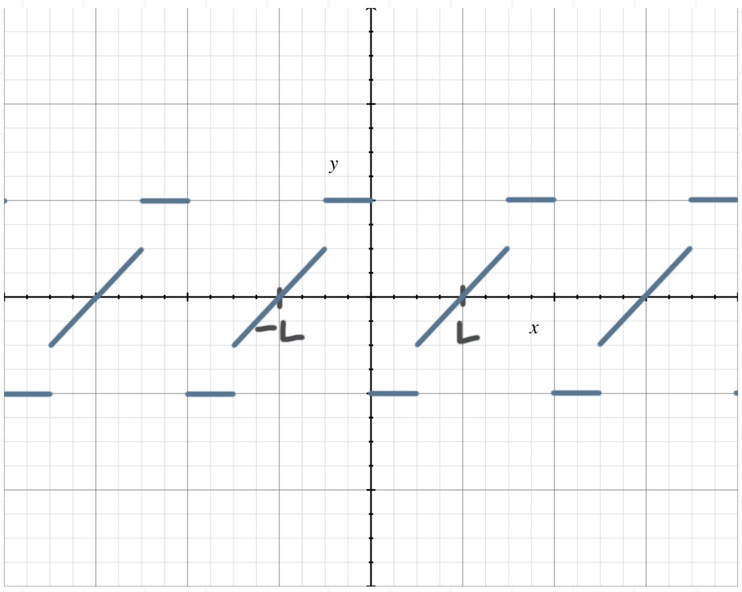
B



C



D



Solution: B

The even extension of a function is the function we get when we reflect the portion of the graph on $0 \leq x \leq L$ into the interval $-L \leq x \leq 0$, and then repeat that symmetrical graph over and over to both the left and right, on $(-\infty, -L)$ and on (L, ∞). Answer choice B is the graph of the even extension.

Answer choice A is the periodic extension of the portion of the graph on $-L \leq x \leq L$, and answer choice D is the odd extension of the portion of the graph on $0 \leq x \leq L$.

We could essentially say that answer choice C is the even extension of the portion of the graph on $-L \leq x \leq 0$, but that's not something we typically do.

Topic: Periodic functions and periodic extensions

Question: Find the even and odd extensions of the function.

$$f(x) = 1 + x + x^2$$

Answer choices:

A $\quad E(x) = \begin{cases} 1 - x + x^2 & 0 \leq x \leq L \\ 1 + x + x^2 & -L \leq x < 0 \end{cases}$ and $O(x) = \begin{cases} -1 + x - x^2 & 0 \leq x \leq L \\ 1 + x + x^2 & -L \leq x < 0 \end{cases}$

B $\quad E(x) = \begin{cases} 1 + x + x^2 & 0 \leq x \leq L \\ 1 - x + x^2 & -L \leq x < 0 \end{cases}$ and $O(x) = \begin{cases} 1 + x + x^2 & 0 \leq x \leq L \\ -1 + x - x^2 & -L \leq x < 0 \end{cases}$

C $\quad E(x) = \begin{cases} -1 + x - x^2 & 0 \leq x \leq L \\ 1 + x + x^2 & -L \leq x < 0 \end{cases}$ and $O(x) = \begin{cases} 1 - x + x^2 & 0 \leq x \leq L \\ 1 + x + x^2 & -L \leq x < 0 \end{cases}$

D $\quad E(x) = \begin{cases} 1 + x + x^2 & 0 \leq x \leq L \\ -1 + x - x^2 & -L \leq x < 0 \end{cases}$ and $O(x) = \begin{cases} 1 + x + x^2 & 0 \leq x \leq L \\ 1 - x + x^2 & -L \leq x < 0 \end{cases}$

Solution: B

The even extension of $f(x) = 1 + x + x^2$ is

$$E(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

$$E(x) = \begin{cases} 1 + x + x^2 & 0 \le x \le L \\ 1 - x + (-x)^2 & -L \le x < 0 \end{cases}$$

$$E(x) = \begin{cases} 1 + x + x^2 & 0 \le x \le L \\ 1 - x + x^2 & -L \le x < 0 \end{cases}$$

The odd extension of $f(x) = 1 + x + x^2$ is

$$O(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$O(x) = \begin{cases} 1 + x + x^2 & 0 \le x \le L \\ -(1 - x + (-x)^2) & -L \le x < 0 \end{cases}$$

$$O(x) = \begin{cases} 1 + x + x^2 & 0 \le x \le L \\ -1 + x - x^2 & -L \le x < 0 \end{cases}$$

Topic: Representing piecewise functions

Question: Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} -1 & -L \le x < 0 \\ 1 & 0 \le x \le L \end{cases}$$

Answer choices:

A $\quad f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = \frac{L^2}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{n\pi x}{L}\right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = \frac{L^2}{2} - \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$

Solution: A

We'll need to calculate A_0, A_n, and B_n to find the Fourier series representation.

For A_0 we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{0} -1\ dx + \frac{1}{2L}\int_{0}^{L} 1\ dx$$

$$A_0 = -\frac{1}{2L}x\Big|_{-L}^{0} + \frac{1}{2L}x\Big|_{0}^{L}$$

$$A_0 = -\frac{1}{2L}(0) - \left(-\frac{1}{2L}(-L)\right) + \frac{1}{2L}L - \frac{1}{2L}(0)$$

$$A_0 = -\frac{1}{2} + \frac{1}{2}$$

$$A_0 = 0$$

And for A_n we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{0} -1\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} 1\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = -\frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} + \frac{1}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}$$

$$A_n = -\frac{1}{n\pi}\sin\left(\frac{n\pi(0)}{L}\right) - \left(-\frac{1}{n\pi}\sin\left(\frac{n\pi(-L)}{L}\right)\right) + \frac{1}{n\pi}\sin\left(\frac{n\pi L}{L}\right) - \left(\frac{1}{n\pi}\sin\left(\frac{n\pi(0)}{L}\right)\right)$$

$$A_n = \frac{1}{n\pi}\sin(-n\pi) + \frac{1}{n\pi}\sin(n\pi)$$

$$A_n = -\frac{1}{n\pi}\sin(n\pi) + \frac{1}{n\pi}\sin(n\pi)$$

$$A_n = 0$$

For B_n we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{1}{L}\int_{-L}^{0}\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L}\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{-L}^{0} - \frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{0}^{L}$$

$$B_n = \frac{1}{n\pi}\cos\left(\frac{n\pi(0)}{L}\right) - \frac{1}{n\pi}\cos\left(-\frac{n\pi L}{L}\right) - \left[\frac{1}{n\pi}\cos\left(\frac{n\pi L}{L}\right) - \frac{1}{n\pi}\cos\left(\frac{n\pi(0)}{L}\right)\right]$$

$$B_n = \frac{1}{n\pi} - \frac{1}{n\pi}\cos(-n\pi) - \frac{1}{n\pi}\cos(n\pi) + \frac{1}{n\pi}$$

$$B_n = \frac{2}{n\pi} - \frac{1}{n\pi}\cos(n\pi) - \frac{1}{n\pi}\cos(n\pi)$$

$$B_n = \frac{2}{n\pi} - \frac{2}{n\pi}\cos(n\pi)$$

$$B_n = \frac{2}{n\pi}(1 + (-1)^{n+1})$$

Then the Fourier series representation of the piecewise function on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

Topic: Representing piecewise functions

Question: Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} -2x & -L \le x < 0 \\ 2x & 0 \le x \le L \end{cases}$$

Answer choices:

A $\quad f(x) = 2L + \frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = 2L + \frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi x}{L}\right) + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n(L+4)}{n} \sin\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = L + \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = L + \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi x}{L}\right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n(L+2)}{n} \sin\left(\frac{n\pi x}{L}\right)$

Solution: C

We'll need to calculate A_0, A_n, and B_n to find the Fourier series representation.

For A_0 we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{0} -2x\ dx + \frac{1}{2L}\int_{0}^{L} 2x\ dx$$

$$A_0 = -\frac{1}{2L}x^2\Big|_{-L}^{0} + \frac{1}{2L}x^2\Big|_{0}^{L}$$

$$A_0 = -\frac{1}{2L}(0)^2 - \left(-\frac{1}{2L}(-L)^2\right) + \frac{1}{2L}L^2 - \frac{1}{2L}(0)^2$$

$$A_0 = \frac{1}{2L}L^2 + \frac{1}{2L}L^2$$

$$A_0 = L$$

And for A_n we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{0} -2x\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} 2x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = -\frac{2}{L}\int_{-L}^{0} x\cos\left(\frac{n\pi x}{L}\right)\,dx + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\,dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ to evaluate the first integral.

$$A_n = -\frac{2}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{L}{n\pi}\int \sin\left(\frac{n\pi x}{L}\right)\,dx\right]\Bigg|_{-L}^{0} + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$A_n = -\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0} + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$A_n = -\frac{2(0)}{n\pi}\sin\left(\frac{n\pi(0)}{L}\right) - \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi(0)}{L}\right)$$

$$-\left[-\frac{2(-L)}{n\pi}\sin\left(-\frac{n\pi L}{L}\right) - \frac{2L}{(n\pi)^2}\cos\left(-\frac{n\pi L}{L}\right)\right] + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$A_n = -\frac{2L}{(n\pi)^2} - \left(\frac{2L}{n\pi}\sin(-n\pi) - \frac{2L}{(n\pi)^2}\cos(-n\pi)\right) + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$A_n = -\frac{2L}{(n\pi)^2} + \frac{2L}{n\pi}\sin(n\pi) + \frac{2L}{(n\pi)^2}\cos(n\pi) + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$A_n = \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2} + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

$$A_n = \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2} + \frac{2}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\,dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ to evaluate the second integral.

$$A_n = \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2} + \frac{2}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{L}{n\pi}\int \sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$A_n = \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2} + \left[\frac{2x}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$A_n = \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2} + \left[\frac{2L}{n\pi}\sin\left(\frac{n\pi L}{L}\right) + \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi L}{L}\right)\right]$$

$$-\left[\frac{2(0)}{n\pi}\sin\left(\frac{n\pi(0)}{L}\right) + \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi(0)}{L}\right)\right]$$

$$A_n = \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2} + \frac{2L}{n\pi}\sin(n\pi) + \frac{2L}{(n\pi)^2}\cos(n\pi) - \frac{2L}{(n\pi)^2}$$

$$A_n = \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2} + \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2}$$

$$A_n = \frac{4L}{(n\pi)^2}(-1)^n - \frac{4L}{(n\pi)^2}$$

$$A_n = \frac{4L}{(n\pi)^2}((-1)^n - 1)$$

For B_n we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{0} -2x\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} 2x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{2}{L}\int_{-L}^{0} x\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$ to evaluate the first integral.

$$B_n = \left[\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{2}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_{-L}^{0} + \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \left[\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{0} + \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \left[\frac{2(0)}{n\pi}\cos\left(\frac{n\pi(0)}{L}\right) - \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi(0)}{L}\right)\right]$$

$$-\left[\frac{2(-L)}{n\pi}\cos\left(-\frac{n\pi L}{L}\right) - \frac{2L}{(n\pi)^2}\sin\left(-\frac{n\pi L}{L}\right)\right] + \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2L}{n\pi}\cos(-n\pi) + \frac{2L}{(n\pi)^2}\sin(-n\pi) + \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2L}{n\pi}\cos(n\pi) - \frac{2L}{(n\pi)^2}\sin(n\pi) + \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2L}{n\pi}(-1)^n + \frac{2}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -\ (L/n\pi)\cos(n\pi x/L)$ to evaluate the first integral.

$$B_n = \frac{2L}{n\pi}(-1)^n + \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \frac{2L}{n\pi}(-1)^n + \left[-\frac{2x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \frac{2L}{n\pi}(-1)^n + \left[-\frac{2L}{n\pi}\cos\left(\frac{n\pi L}{L}\right) + \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi L}{L}\right)\right]$$

$$-\left[-\frac{2(0)}{n\pi}\cos\left(\frac{n\pi(0)}{L}\right) + \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi(0)}{L}\right)\right]$$

$$B_n = \frac{2L}{n\pi}(-1)^n - \frac{2L}{n\pi}\cos(n\pi) + \frac{2}{(n\pi)^2}\sin(n\pi)$$

$$B_n = \frac{2L}{n\pi}(-1)^n - \frac{2L}{n\pi}(-1)^n$$

$$B_n = 0$$

Then the Fourier series representation of the piecewise function on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = L + \frac{4L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

Topic: Representing piecewise functions

Question: Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} -\frac{L}{2} - x & -L \le x < -\frac{L}{2} \\ 0 & -\frac{L}{2} \le x \le \frac{L}{2} \\ -\frac{L}{2} + x & \frac{L}{2} < x \le L \end{cases}$$

Answer choices:

A $\quad f(x) = \frac{L}{4} + \sum_{n=1}^{\infty} \frac{4L}{(n\pi)^2} \left[(-1)^n - \cos\left(\frac{n\pi}{2}\right) \right] \cos\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = \frac{L^2}{4} + \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \cos\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = \frac{L^2}{4} - \frac{L}{\pi} \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} + \frac{2}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$

$$+\frac{2L}{\pi^2} \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n^2} - \frac{1}{n^2} \cos\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{L}\right)$$

D $\quad f(x) = \frac{L^2}{4} - \frac{L}{\pi} \sum_{n=1}^{\infty} \left(\frac{(-1)^n}{n} + \frac{2}{n^2\pi} \sin\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{L}\right)$

$$+\frac{2L}{\pi^2}\sum_{n=1}^{\infty}\left(\frac{(-1)^n}{n^2}-\frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right)\cos\left(\frac{n\pi x}{L}\right)$$

Solution: A

We see that $f(-x)=f(x)$,

$$f(-x)=\begin{cases}-\frac{L}{2}-x & -L<x\leq-\frac{L}{2}\\ 0 & -\frac{L}{2}\leq x\leq\frac{L}{2}\\ -\frac{L}{2}+x & \frac{L}{2}\leq x<L\end{cases}=f(x)$$

so the function $f(x)$ is even, which means $B_n=0$.

For A_0 we get

$$A_0=\frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0=\frac{1}{L}\int_{-L}^{-\frac{L}{2}}\left(-\frac{L}{2}-x\right)\ dx+\frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}}0\ dx+\frac{1}{L}\int_{\frac{L}{2}}^{L}\left(-\frac{L}{2}+x\right)\ dx$$

$$A_0=-\frac{1}{L}\int_{-L}^{-\frac{L}{2}}\frac{L}{2}+x\ dx-\frac{1}{L}\int_{\frac{L}{2}}^{L}\frac{L}{2}-x\ dx$$

$$A_0=-\frac{1}{L}\left(\frac{L}{2}x+\frac{1}{2}x^2\right)\Bigg|_{-L}^{-\frac{L}{2}}-\frac{1}{L}\left(\frac{L}{2}x-\frac{1}{2}x^2\right)\Bigg|_{\frac{L}{2}}^{L}$$

$$A_0 = -\frac{x}{2} - \frac{x^2}{2L}\Bigg|_{-L}^{-\frac{L}{2}} + \frac{x^2}{2L} - \frac{x}{2}\Bigg|_{\frac{L}{2}}^{L}$$

$$A_0 = -\frac{-\frac{L}{2}}{2} - \frac{\left(-\frac{L}{2}\right)^2}{2L} - \left(-\frac{-L}{2} - \frac{(-L)^2}{2L}\right) + \frac{L^2}{2L} - \frac{L}{2} - \left(\frac{\left(\frac{L}{2}\right)^2}{2L} - \frac{\frac{L}{2}}{2}\right)$$

$$A_0 = \frac{L}{4} - \frac{L}{8} - \frac{L}{8} + \frac{L}{4}$$

$$A_0 = \frac{L}{4}$$

Then for A_n we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_{-L}^{-\frac{L}{2}} \left(-\frac{L}{2} - x\right)\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} 0\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{2}{L}\int_{\frac{L}{2}}^{L} \left(-\frac{L}{2} + x\right)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = -\int_{-L}^{-\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right)\ dx - \frac{2}{L}\int_{-L}^{-\frac{L}{2}} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$-\int_{\frac{L}{2}}^{L} \cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{2}{L}\int_{\frac{L}{2}}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = -\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{-\frac{L}{2}} - \frac{2}{L}\int_{-L}^{-\frac{L}{2}} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$-\frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L} + \frac{2}{L}\int_{\frac{L}{2}}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ to evaluate the remaining two integrals.

$$A_n = -\frac{L+2x}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{-\frac{L}{2}} + \frac{2}{n\pi}\int_{-L}^{-\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{2x-L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L} - \frac{2}{n\pi}\int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = -\frac{L+2x}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{-\frac{L}{2}}$$

$$+\frac{2x-L}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L}$$

$$A_n = -\frac{L+2\left(-\frac{L}{2}\right)}{n\pi}\sin\left(\frac{n\pi\left(-\frac{L}{2}\right)}{L}\right) - \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi\left(-\frac{L}{2}\right)}{L}\right)$$

$$-\left[-\frac{L+2(-L)}{n\pi}\sin\left(\frac{n\pi(-L)}{L}\right) - \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi(-L)}{L}\right)\right]$$

$$+\frac{2L-L}{n\pi}\sin\left(\frac{n\pi L}{L}\right) + \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi L}{L}\right)$$

$$-\left[\frac{2\left(\frac{L}{2}\right)-L}{n\pi}\sin\left(\frac{n\pi\left(\frac{L}{2}\right)}{L}\right)+\frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi\left(\frac{L}{2}\right)}{L}\right)\right]$$

$$A_n=-\frac{2L}{(n\pi)^2}\cos\left(-\frac{n\pi}{2}\right)-\frac{L}{n\pi}\sin(-n\pi)+\frac{2L}{(n\pi)^2}\cos(-n\pi)$$

$$+\frac{L}{n\pi}\sin(n\pi)+\frac{2L}{(n\pi)^2}\cos(n\pi)-\frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)$$

$$A_n=-\frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}\sin(n\pi)+\frac{2L}{(n\pi)^2}\cos(n\pi)$$

$$+\frac{L}{n\pi}\sin(n\pi)+\frac{2L}{(n\pi)^2}\cos(n\pi)-\frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)$$

$$A_n=-\frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}(0)+\frac{2L}{(n\pi)^2}(-1)^n$$

$$+\frac{L}{n\pi}(0)+\frac{2L}{(n\pi)^2}(-1)^n-\frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)$$

$$A_n=\frac{4L}{(n\pi)^2}\left[(-1)^n-\cos\left(\frac{n\pi}{2}\right)\right]$$

Then the Fourier series representation of the piecewise function is

$$f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\frac{L}{4}+\sum_{n=1}^{\infty}\frac{4L}{(n\pi)^2}\left[(-1)^n-\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

Topic: Convergence of a Fourier series

Question: Find any points where the Fourier series representation doesn't converge to the periodic extension of $f(x)$.

$$f(x) = \begin{cases} (x+1)^2 & -L \leq x < 0 \\ (x-1)^2 & 0 \leq x \leq L \end{cases}$$

Answer choices:

A The function converges at every point

B $x = 2Ln$

C $x = L + 2Ln$

D $x = Ln$

Solution: A

The function $f(x)$ is continuous and smooth on intervals $(-L,0)$ and $(0,L)$, so its Fourier series representation converges to $f(x)$ on these intervals, and we only need to check convergence at $x = 0$ and $x = L$.

At $x = 0$,

$$f(x) = (0 - 1)^2 = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (x + 1)^2 = 1$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (x - 1)^2 = 1$$

Because these three values are equivalent, $f(x)$ is continuous at $x = 0$ and its Fourier representation converges to $f(x)$ at $x = 0$.

At $x = L$,

$$f(L) = (L - 1)^2$$

$$f(-L) = (-L + 1)^2 = (L - 1)^2$$

Therefore, $f(L) = f(-L)$, and the periodic extension of $f(x)$ is continuous at $x = L$. Then the Fourier series representation of $f(x)$ converges to $f(x)$ at $x = L$ and $x = -L$. Therefore, the Fourier series converges to the periodic extension of $f(x)$ at every point.

Topic: Convergence of a Fourier series

Question: Find any points where the Fourier series representation doesn't converge to the periodic extension of $f(x)$.

$$f(x) = \begin{cases} \arctan\left(\frac{1}{x}\right) & x \in [-L,0) \cup (0,L] \\ \frac{\pi}{2} & x = 0 \end{cases}$$

Answer choices:

A The function converges at every point

B $x = 2Ln$

C $x = L + 2Ln$

D $x = Ln$

Solution: D

The function $f(x)$ is continuous and smooth on the intervals $(-L,0)$ and $(L,0)$, so its Fourier series representation converges to $f(x)$ in these intervals, and we only need to check convergence at $x=0$ and $x=\pm L$.

At $x=0$,

$$f(x)=\frac{\pi}{2}$$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0^-}\arctan\left(\frac{1}{x}\right)=-\frac{\pi}{2}$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0^+}\arctan\left(\frac{1}{x}\right)=\frac{\pi}{2}$$

Because the one-sided limits aren't equivalent, the function has a jump discontinuity at $x=0$, and the Fourier series representation will converge to the average of the one-sided limits,

$$\frac{-\frac{\pi}{2}+\frac{\pi}{2}}{2}=0$$

At $x=\pm L$,

$$\lim_{x\to -L} f(x)=f(-L)=\arctan\left(\frac{1}{-L}\right)=-\arctan\left(\frac{1}{L}\right)$$

$$\lim_{x\to L} f(x)=f(L)=\arctan\left(\frac{1}{L}\right)$$

Because the one-sided limits aren't equivalent, the function has a jump discontinuity at $x = L$, and the Fourier series representation will converge to the average of the one-sided limits,

$$\frac{-\arctan\left(\frac{1}{L}\right) + \arctan\left(\frac{1}{L}\right)}{2} = 0$$

Therefore, the Fourier series representation will not converge to the periodic extension of $f(x)$ at every $x = 0 + 2Ln$ or at every $x = L + 2Ln$. We can express these together as every $x = Ln$.

Topic: Convergence of a Fourier series

Question: Find any points where the Fourier series representation doesn't converge to the periodic extension of $f(x)$.

$$f(x) = \begin{cases} 2 + x^2 & x \in [-L,0) \\ 1 & x = 0 \\ x^2 & x \in (0,L] \end{cases}$$

Answer choices:

A The function converges at every point

B $x = 2Ln$

C $x = (2n + 1)L$

D $x = Ln$

Solution: C

The function $f(x)$ is continuous and smooth on the intervals $(-L,0)$ and $(L,0)$, so its Fourier series representation converges to $f(x)$ at in these intervals, and we only need to check convergence at $x = 0$ and $x = L$.

At $x = 0$,

$$f(x) = 1$$

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2 + x^2) = 2$$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$$

Because the one-sided limits aren't equivalent, the function has a jump discontinuity at $x = 0$, and the Fourier series representation will converge to the average of the one-sided limits,

$$\frac{2 + 0}{2} = 1 = f(x)$$

So the Fourier series actually converges to the value of $f(x)$ at $x = 0$.

At $x = L$,

$$\lim_{x \to -L} f(x) = f(-L) = 2 + L^2$$

$$\lim_{x \to L} f(x) = f(L) = L^2$$

Because the one-sided limits aren't equivalent, the function has a jump discontinuity at $x = L$, and the Fourier series representation will converge to the average of the one-sided limits,

$$\frac{2 + L^2 + L^2}{2} = 1 + L^2 = f(L)$$

Therefore, the Fourier series representation will not converge to the periodic extension of $f(x)$ at every

$$x = L + 2Ln$$

$$x = (2n + 1)L$$

Topic: Fourier cosine series

Question: Find the Fourier cosine series of $f(x)=\sin x$ on $0\le x\le \pi$.

Answer choices:

A $\quad f(x)=\frac{1}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n+1}{1+n^2}\cos(nx)$

B $\quad f(x)=\frac{1}{\pi}+\frac{1}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n+1}{1-n^2}\cos(nx)$

C $\quad f(x)=\frac{2}{\pi}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n+1}{1+n^2}\cos(nx)$

D $\quad f(x)=\frac{2}{\pi}+\frac{2}{\pi}\sum_{n=2}^{\infty}\frac{(-1)^n+1}{1-n^2}\cos(nx)$

Solution: D

For A_0 we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{\pi}\int_0^\pi \sin x\ dx$$

$$A_0 = -\frac{1}{\pi}\cos x\Big|_0^\pi$$

$$A_0 - \frac{1}{\pi}(-1-1)$$

$$A_0 = \frac{2}{\pi}$$

And for A_n we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{\pi}\int_0^\pi \sin x\cos(nx)\ dx$$

$$A_n = \frac{1}{\pi}\int_0^\pi \sin((n+1)x) - \sin((n-1)x)\ dx$$

$$A_n = -\frac{1}{\pi}\left(\frac{1}{n+1}\cos((n+1)x) - \frac{1}{n-1}\cos((n-1)x)\right)\Bigg|_0^\pi$$

$$A_n = \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1} - \frac{\cos((n+1)\pi)}{n+1} + \frac{\cos((n-1)\pi)}{n-1}\right)$$

$$A_n = \frac{1}{\pi}\left(\frac{1-\cos((n+1)\pi)}{n+1} - \frac{1-\cos((n-1)\pi)}{n-1}\right)$$

$$A_n = \frac{1}{\pi}\left(\frac{1-(-1)^{n+1}}{n+1} - \frac{1-(-1)^{n+1}}{n-1}\right)$$

$$A_n = -\frac{1}{\pi}\left(\frac{((-1)^{n+1}-1)(n-1) - ((-1)^{n+1}-1)(n+1)}{n^2-1}\right)$$

$$A_n = -\frac{1}{\pi}\left(\frac{((-1)^{n+1}-1)(n-1-n-1)}{n^2-1}\right)$$

$$A_n = \frac{2}{\pi}\left(\frac{(-1)^n+1}{1-n^2}\right) \text{ for } n \neq 1$$

And for $n = 1$, we have

$$A_1 = \frac{2}{\pi}\int_0^\pi \sin x \cos x \ dx = \frac{1}{\pi}\int_0^\pi \sin 2x \ dx = \frac{1}{2\pi}\cos 2x\Big|_0^\pi = \frac{1}{2\pi}(1-1) = 0$$

Therefore, the Fourier cosine series representation of $\sin x$ on $0 \leq x \leq \pi$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{2}{\pi} + \frac{2}{\pi}\sum_{n=2}^{\infty} \frac{(-1)^n+1}{1-n^2}\cos(nx)$$

Topic: Fourier cosine series

Question: Find the Fourier cosine series of $f(x) = 2x + 5$ on $0 \le x \le L$.

Answer choices:

A $\quad f(x) = \frac{5}{2} + \frac{L}{2} - \frac{2L}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = 5 + L - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n^2} \cos\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = \frac{5 + L}{2} - \frac{2L + 5}{\pi} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \cos\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = 5 + L - \frac{4L + 10}{\pi} \sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n} \cos\left(\frac{n\pi x}{L}\right)$

Solution: B

For A_0 we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\int_0^L 2x + 5\ dx$$

$$A_0 = \frac{1}{L}(x^2 + 5x)\Big|_0^L$$

$$A_0 = L + 5$$

And for A_n we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L (2x + 5)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 2x + 5$, $du = 2\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$ to evaluate the integral.

$$A_n = \frac{2}{L}\left(\frac{L}{n\pi}(2x + 5)\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{2L}{n\pi}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx\right)$$

$$A_n = \frac{2}{L}\left(\frac{L}{n\pi}(2L + 5)\sin(n\pi) - \frac{5L}{n\pi}\sin 0 + \frac{2L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right)\Big|_0^L$$

$$A_n = \frac{2}{L}\left(\frac{2L^2}{(n\pi)^2}(\cos(n\pi) - 1)\right)$$

$$A_n = -\frac{4L}{(n\pi)^2}(1 - (-1)^n)$$

$$A_n = -\frac{4L}{(n\pi)^2}(1 + (-1)^{n+1})$$

Therefore, the Fourier cosine series representation of $f(x) = 2x + 5$ on $0 \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 5 + L - \frac{4L}{\pi^2}\sum_{n=1}^{\infty} \frac{1 + (-1)^{n+1}}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

Topic: Fourier cosine series

Question: Find the Fourier cosine series of $f(x) = (x-3)^2$ on $0 \le x \le \pi$.

Answer choices:

A $\quad f(x) = \dfrac{(\pi-3)^3}{9\pi} + \dfrac{3}{\pi} + \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{(\pi-3)(-1)^n+3}{3n^2}\cos(nx)$

B $\quad f(x) = \dfrac{(\pi-3)^3}{9} + 3 + \dfrac{2}{\pi^2}\sum_{n=1}^{\infty} \dfrac{(\pi-3)(-1)^n+3}{3n^2}\cos(nx)$

C $\quad f(x) = \dfrac{(\pi-3)^3}{3\pi} + \dfrac{9}{\pi} + \dfrac{4}{\pi}\sum_{n=1}^{\infty} \dfrac{(\pi-3)(-1)^n+3}{n^2}\cos(nx)$

D $\quad f(x) = \dfrac{\pi-3}{3} + 9 + \dfrac{2}{\pi^2}\sum_{n=1}^{\infty} \dfrac{(\pi-3)(-1)^n+3}{n^2}\cos(nx)$

Solution: C

For A_0 we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{\pi}\int_0^\pi (x-3)^2\ dx$$

$$A_0 = \frac{(x-3)^3}{3\pi}\Big|_0^\pi$$

$$A_0 = \frac{(\pi-3)^3}{3\pi} + \frac{9}{\pi}$$

And for A_n we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{\pi}\int_0^\pi (x-3)^2\cos(nx)\ dx$$

Use integration by parts with $u=(x-3)^2$, $du=2(x-3)\ dx$, $dv=\cos(nx)\ dx$, and $v=(1/n)\sin(nx)$ to evaluate the integral.

$$A_n = \frac{2}{\pi}\left(\frac{1}{n}(x-3)^2\sin(nx)\Big|_0^\pi - \frac{2}{n}\int_0^\pi (x-3)\sin(nx)\ dx\right)$$

$$A_n = -\frac{4}{n\pi}\int_0^\pi (x-3)\sin(nx)\ dx$$

Use integration by parts with $u = x - 3$, $du = dx$, $dv = \sin(nx)\ dx$, and $v = -(1/n)\cos(nx)$ to evaluate the integral.

$$A_n = -\frac{4}{n\pi}\left(-\frac{1}{n}(x-3)\cos(nx)\Big|_0^{\pi} + \frac{1}{n}\int_0^{\pi}\cos(nx)\ dx\right)$$

$$A_n = -\frac{4}{n\pi}\left(-\frac{\pi-3}{n}(-1)^n - \frac{3}{n} + \frac{1}{n^2}\sin(nx)\right)\Big|_0^{\pi}$$

$$A_n = \frac{4}{n^2\pi}((\pi-3)(-1)^n + 3)$$

Therefore, the Fourier cosine series representation of $f(x) = (x-3)^2$ on $0 \le x \le \pi$ is

$$f(x) = \frac{(\pi-3)^3}{3\pi} + \frac{9}{\pi} + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(\pi-3)(-1)^n + 3}{n^2}\cos(nx)$$

Topic: Fourier sine series

Question: Find the Fourier sine series representation of $f(x) = x + x^3$ on $-L \leq x \leq L$.

Answer choices:

A $$f(x) = \frac{2L}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}(6L^2 - (n\pi L)^2 - (n\pi)^2)\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{\pi}$$

B $$f(x) = \frac{2L}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}(6L^2 - (n\pi L)^2 - (n\pi)^2)\sin\left(\frac{n\pi x}{L}\right)$$

C $$f(x) = \frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\left(\frac{n\pi x}{L}\right)$$

D $$f(x) = 1 + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\left(\sin\left(\frac{n\pi x}{L}\right) + n\cos\left(\frac{n\pi x}{L}\right)\right)$$

Solution: B

Since $f(-x) = -f(x)$, the function is odd.

$$f(-x) = (-x) + (-x)^3$$

$$f(-x) = -x - x^3$$

$$-f(x) = -(x + x^3)$$

$$-f(x) = -x - x^3$$

For B_n, we get

$$B_n = \frac{2}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = \frac{2}{L} \int_0^L (x + x^3) \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right) \ dx + \frac{2}{L} \int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right) \ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L) \ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{2Lx}{Ln\pi} \cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{2L}{n\pi L} \int_0^L \cos\left(\frac{n\pi x}{L}\right) \ dx + \frac{2}{L} \int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = -\frac{2x}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{2}{L} \int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = -\frac{2L}{n\pi}\cos(n\pi) + \frac{2L}{(n\pi)^2}\sin(n\pi) - \left(0 + \frac{2L}{(n\pi)^2}\sin 0\right) + \frac{2}{L}\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{2L}{n\pi}\cos(n\pi) + \frac{2}{L}\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Now use integration by parts with $u = x^3$, $du = 3x^2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx = -\frac{Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{L}{n\pi}\int_0^L 3x^2 \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx = -\frac{Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L + \frac{3L}{n\pi}\int_0^L x^2 \cos\left(\frac{n\pi x}{L}\right)\ dx$$

Now use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx = \left(-\frac{Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{3L^2x^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right)\Big|_0^L$$

$$-\frac{6L^2}{(n\pi)^2}\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Now use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$\int_0^L x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx = -\frac{Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{3L^2x^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$+\frac{6L^3x}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L+\frac{6L^3}{(n\pi)^3}\int_0^L\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$\int_0^L x^3\sin\left(\frac{n\pi x}{L}\right)\ dx=-\frac{Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{3L^2x^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

$$+\frac{6L^3x}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right)+\frac{6L^4}{(n\pi)^4}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$\int_0^L x^3\sin\left(\frac{n\pi x}{L}\right)\ dx=-\frac{L^4}{n\pi}\cos(n\pi)+\frac{3L^4}{(n\pi)^2}\sin(n\pi)+\frac{6L^4}{(n\pi)^3}\cos(n\pi)$$

$$+\frac{6L^4}{(n\pi)^4}\sin(n\pi)-\frac{6L^4}{(n\pi)^4}\sin(0)$$

$$\int_0^L x^3\sin\left(\frac{n\pi x}{L}\right)\ dx=-\frac{L^4}{n\pi}(-1)^n+\frac{6L^4}{(n\pi)^3}(-1)^n$$

Plugging this back into the equation for B_n, we get

$$B_n=-\frac{2L}{n\pi}\cos(n\pi)+\frac{2}{L}\left(-\frac{L^4}{n\pi}(-1)^n+\frac{6L^4}{(n\pi)^3}(-1)^n\right)$$

$$B_n=-\frac{2L}{n\pi}(-1)^n-\frac{2L^3}{n\pi}(-1)^n+\frac{12L^3}{(n\pi)^3}(-1)^n$$

$$B_n=\frac{2L}{(n\pi)^3}(-1)^n(6L^2-L^2(n\pi)^2-(n\pi)^2)$$

Then the Fourier sine series representation of $f(x)=x+x^3$ on $-L\leq x\leq L$ is

$$f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\frac{2L}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}(6L^2-(n\pi L)^2-(n\pi)^2)\sin\left(\frac{n\pi x}{L}\right)$$

Topic: Fourier sine series

Question: Find the Fourier sine series representation of $f(x) = x - L$ on $0 \leq x \leq L$.

Answer choices:

A $\quad f(x) = -\frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = \frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n - 1}{n^2}\sin\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = \frac{L}{\pi^2} + \sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\left(\frac{n\pi x}{L}\right) + \frac{(-1)^n}{n}\cos\left(\frac{n\pi x}{L}\right)\right)$

D $\quad f(x) = 1 - \frac{\pi}{L} + \sum_{n=1}^{\infty}\frac{1}{n}\left(\sin\left(\frac{n\pi x}{L}\right) + \cos\left(\frac{n\pi x}{L}\right)\right)$

Solution: A

Since the function is not odd, let's find the function's odd extension on $-L \le x \le L$.

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x - L & 0 \le x \le L \\ x + L & -L \le x < 0 \end{cases}$$

Because the odd extension is an odd function, we can find its Fourier sine series representation, starting with calculating B_n.

$$B_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L (x - L)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx - \int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = x\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -\ (L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{1}{L}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \int \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L + \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$B_n = \frac{1}{L}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L + \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$B_n = -\frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$B_n = \frac{L-x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

Evaluate over the interval.

$$B_n = \frac{L-L}{n\pi}\cos\left(\frac{n\pi L}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi L}{L}\right)$$

$$-\left[\frac{L-0}{n\pi}\cos\left(\frac{n\pi(0)}{L}\right) + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi(0)}{L}\right)\right]$$

$$B_n = \frac{L}{(n\pi)^2}\sin(n\pi) - \frac{L}{n\pi}$$

$$B_n = -\frac{L}{n\pi}$$

Then the Fourier sine series representation of $f(x) = x - L$ on $0 \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = -\frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi x}{L}\right)$$

Topic: Fourier sine series

Question: Find the Fourier sine series representation of $f(x) = e^x$ on $0 \leq x \leq L$.

Answer choices:

A $\quad f(x) = \pi + \sum_{n=1}^{\infty} \frac{n}{(n\pi)^2 + L} \left((-1)^n \cos\left(\frac{n\pi x}{L}\right) + \sin\left(\frac{n\pi x}{L}\right) \right)$

B $\quad f(x) = \frac{\pi}{L^2} \sum_{n=1}^{\infty} n(-1)^n (e^L - 1) \sin\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = \sum_{n=1}^{\infty} \frac{n\pi(e^L(-1)^{n+1} + 1)}{(n\pi)^2 + L} \cos\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = \sum_{n=1}^{\infty} \frac{n\pi(e^L(-1)^{n+1} + 1)}{(n\pi)^2 + L^2} \sin\left(\frac{n\pi x}{L}\right)$

Solution: D

Since the function is not odd, let's find the function's odd extension on $-L \le x \le L$.

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} e^x & 0 \le x \le L \\ -e^{-x} & -L \le x < 0 \end{cases}$$

Because the odd extension is an odd function, we can find its Fourier sine series representation, starting with calculating B_n.

$$B_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L e^x \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = e^x$, $du = e^x\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -\ (L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{1}{L}\left[-\frac{Le^x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int e^x \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \left[-\frac{e^x}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{1}{n\pi}\int e^x \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = e^x$, $du = e^x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = \left[-\frac{e^x}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \frac{Le^x}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) - \frac{L}{(n\pi)^2} \int e^x \sin\left(\frac{n\pi x}{L}\right) \ dx \right] \Bigg|_0^L$$

$$B_n = \left[-\frac{e^x}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \frac{Le^x}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) \right] \Bigg|_0^L - \frac{L^2}{(n\pi)^2} B_n$$

$$B_n\left(1 + \frac{L^2}{(n\pi)^2}\right) = \left[-\frac{e^x}{n\pi} \cos\left(\frac{n\pi x}{L}\right) + \frac{Le^x}{(n\pi)^2} \sin\left(\frac{n\pi x}{L}\right) \right] \Bigg|_0^L$$

Evaluate over the interval.

$$B_n\left(1 + \frac{L^2}{(n\pi)^2}\right) = -\frac{e^L}{n\pi} \cos(n\pi) + \frac{Le^L}{(n\pi)^2} \sin(n\pi) + \frac{1}{n\pi}$$

$$B_n\left(1 + \frac{L^2}{(n\pi)^2}\right) = \frac{e^L(-1)^{n+1} + 1}{n\pi}$$

$$B_n = \frac{e^L(-1)^{n+1} + 1}{n\pi}\left(\frac{(n\pi)^2}{(n\pi)^2 + L^2}\right)$$

$$B_n = \frac{n\pi(e^L(-1)^{n+1} + 1)}{(n\pi)^2 + L^2}$$

Then the Fourier sine series representation of $f(x) = e^x$ on $0 \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{n\pi(e^L(-1)^{n+1} + 1)}{(n\pi)^2 + L^2} \sin\left(\frac{n\pi x}{L}\right)$$

Topic: Cosine and sine series of piecewise functions

Question: Find the Fourier sine series representation of the piecewise function on $0 \le x \le L$.

$$f(x) = \begin{cases} -L & 0 \le x \le \frac{L}{2} \\ L & \frac{L}{2} < x \le L \end{cases}$$

Answer choices:

A $\quad f(x) = \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n} \sin\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(2\cos\left(\frac{n\pi}{2}\right) - 1 + (-1)^{n+1}\right) \sin\left(\frac{n\pi x}{L}\right)$

C $\quad f(x) = -\frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(\cos\left(\frac{n\pi}{2}\right) + 1 + (-1)^{n+1}\right) \sin\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \left((-1)^n - \sin\left(\frac{n\pi}{2}\right)\right) \sin\left(\frac{n\pi x}{L}\right)$

Solution: B

We want to build the Fourier sine series representation, so we'll start by finding the odd extension of the piecewise function. Because the original piecewise function $f(x)$ is defined in two pieces, its odd extension $g(x)$ will be defined in four pieces.

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} L & \frac{L}{2} < x \le L \\ -L & 0 \le x \le \frac{L}{2} \\ L & -\frac{L}{2} \le x < 0 \\ -L & -L \le x < -\frac{L}{2} \end{cases}$$

To find the Fourier sine series representation, we'll calculate B_n.

$$B_n = \frac{1}{L} \int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = \frac{1}{L} \int_0^{\frac{L}{2}} (-L) \sin\left(\frac{n\pi x}{L}\right) \ dx + \frac{1}{L} \int_{\frac{L}{2}}^L L \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = -\int_0^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right) \ dx + \int_{\frac{L}{2}}^L \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = \frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}} - \frac{L}{n\pi} \cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^L$$

$$B_n = \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L}{n\pi}\cos(0) - \left(\frac{L}{n\pi}\cos(n\pi) - \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)\right)$$

$$B_n = \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L}{n\pi} - \frac{L}{n\pi}\cos(n\pi) + \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{2L}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L}{n\pi}(1 + (-1)^n)$$

Then the Fourier sine series representation of the piecewise function on $0 \leq x \leq L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(2\cos\left(\frac{n\pi}{2}\right) - 1 + (-1)^{n+1}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Topic: Cosine and sine series of piecewise functions

Question: Find the Fourier sine series representation of the piecewise function on $0 \leq x \leq L$.

$$f(x) = \begin{cases} L - x & 0 \leq x \leq \frac{L}{2} \\ x - L & \frac{L}{2} < x \leq L \end{cases}$$

Answer choices:

A $\quad f(x) = \frac{L}{2\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left((-1)^n \frac{L^2 + n^2}{n^2} - \sin\left(\frac{n\pi}{2}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$

B $\quad f(x) = \frac{L}{\pi} \sum_{n=1}^{\infty} \left(\frac{1}{n} \sin\left(\frac{n\pi x}{L}\right) + (-1)^{n+1} \cos\left(\frac{n\pi x}{L}\right) \right)$

C $\quad f(x) = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos\left(\frac{n\pi}{2}\right) - 2\sin\left(\frac{n\pi}{2}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$

D $\quad f(x) = \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi} \sin\left(\frac{n\pi}{2}\right) \right) \sin\left(\frac{n\pi x}{L}\right)$

Solution: D

We want to build the Fourier sine series representation, so we'll start by finding the odd extension of the piecewise function. Because the original piecewise function $f(x)$ is defined in two pieces, its odd extension $g(x)$ will be defined in four pieces.

$$g(x)=\begin{cases} f(x) & 0\le x\le L \\ -f(-x) & -L\le x<0 \end{cases}$$

$$g(x)=\begin{cases} x-L & \frac{L}{2}<x\le L \\ L-x & 0\le x\le \frac{L}{2} \\ -L-x & -\frac{L}{2}\le x<0 \\ x+L & -L\le x<-\frac{L}{2} \end{cases}$$

To find the Fourier sine series representation, we'll calculate B_n.

$$B_n=\frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n=\frac{1}{L}\int_0^{\frac{L}{2}}(L-x)\sin\left(\frac{n\pi x}{L}\right)\ dx+\frac{1}{L}\int_{\frac{L}{2}}^{L}(x-L)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n=\frac{1}{L}\int_0^{\frac{L}{2}}L\sin\left(\frac{n\pi x}{L}\right)\ dx-\frac{1}{L}\int_0^{\frac{L}{2}}x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{1}{L}\int_{\frac{L}{2}}^{L}x\sin\left(\frac{n\pi x}{L}\right)\ dx-\frac{1}{L}\int_{\frac{L}{2}}^{L}L\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \int_0^{\frac{L}{2}} \sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{1}{L}\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{1}{L}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx - \int_{\frac{L}{2}}^{L} \sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} + \frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L}$$

$$-\frac{1}{L}\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L}{n\pi}\cos 0 + \frac{L}{n\pi}\cos n\pi - \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)$$

$$-\frac{1}{L}\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{2L}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L}{n\pi}(1+(-1)^n) - \frac{1}{L}\int_0^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{1}{L}\int_{\frac{L}{2}}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$\int x\sin\left(\frac{n\pi x}{L}\right)\ dx = -\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int \cos\left(\frac{n\pi x}{L}\right)dx$$

$$\int x\sin\left(\frac{n\pi x}{L}\right)\ dx=-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)$$

Plugging this back into the equation for B_n, we get

$$B_n=-\frac{2L}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}(1+(-1)^n)$$

$$-\frac{1}{L}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^{\frac{L}{2}}$$

$$+\frac{1}{L}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{L^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{\frac{L}{2}}^{L}$$

$$B_n=-\frac{2L}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}(1+(-1)^n)$$

$$+\left[\frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right)-\frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^{\frac{L}{2}}$$

$$+\left[-\frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right)+\frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{\frac{L}{2}}^{L}$$

$$B_n=-\frac{2L}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}(1+(-1)^n)+\frac{L}{2n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$-\frac{L}{n\pi}\cos n\pi+\frac{L}{(n\pi)^2}\sin n\pi+\frac{L}{2n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$B_n=-\frac{2L}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{L}{n\pi}+\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right)-\frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{L}{n\pi} - \frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

Then the Fourier sine series representation of the piecewise function on $0 \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1 - \cos\left(\frac{n\pi}{2}\right) - \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

Topic: Cosine and sine series of piecewise functions

Question: Find the Fourier cosine series representation of the piecewise function on $-L \leq x \leq L$.

$$f(x) = \begin{cases} L & -L \leq x < -\frac{L}{2} \\ x^2 & -\frac{L}{2} \leq x \leq \frac{L}{2} \\ L & \frac{L}{2} < x \leq L \end{cases}$$

Answer choices:

A $\quad f(x) = \dfrac{L^2}{8} - \dfrac{L}{2\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{4L}{n^3} \left(2 \sin\left(\dfrac{n\pi}{2}\right) - n\pi \cos\left(\dfrac{n\pi}{2}\right) \right) \cos\left(\dfrac{n\pi x}{L}\right)$

B $\quad f(x) = L + \dfrac{L^3}{9} + \dfrac{L}{\pi^2} \displaystyle\sum_{n=1}^{\infty} \dfrac{n^2}{\pi^2 - n} \sin\left(\dfrac{n\pi}{2}\right) \cos\left(\dfrac{n\pi x}{L}\right)$

C $\quad f(x) = \dfrac{L^2}{12} + L$

$$+ \sum_{n=1}^{\infty} \left(\frac{L(n\pi)^2(L-4) - 8L^2}{(n\pi)^3} \sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{(n\pi)^2} \cos\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{L}\right)$$

D $\quad f(x) = \dfrac{L^2}{24} - L - \dfrac{L}{\pi^3} \displaystyle\sum_{n=1}^{\infty} \dfrac{1}{n^2} \left((L^2 - n\pi)\sin\left(\dfrac{n\pi}{2}\right) - \sin\left(\dfrac{n\pi}{2}\right) \right) \cos\left(\dfrac{n\pi x}{L}\right)$

Solution: C

The function is even because $f(-x)=f(x)$.

$$f(-x)=\begin{cases} L & -L \le x < -\frac{L}{2} \\ x^2 & -\frac{L}{2} \le x \le \frac{L}{2} \\ L & \frac{L}{2} < x \le L \end{cases} = f(x)$$

To find the Fourier cosine series, we'll calculate A_0 and A_n. For A_0 we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\int_{-L}^{-\frac{L}{2}} L\ dx + \frac{1}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\ dx + \frac{1}{L}\int_{\frac{L}{2}}^{L} L\ dx$$

$$A_0 = \frac{1}{L}(Lx)\Big|_{-L}^{-\frac{L}{2}} + \frac{1}{L}\left(\frac{x^3}{3}\right)\Big|_{-\frac{L}{2}}^{\frac{L}{2}} + \frac{1}{L}(Lx)\Big|_{\frac{L}{2}}^{L}$$

$$A_0 = x\Big|_{-L}^{-\frac{L}{2}} + \frac{x^3}{3L}\Big|_{-\frac{L}{2}}^{\frac{L}{2}} + x\Big|_{\frac{L}{2}}^{L}$$

$$A_0 = -\frac{L}{2} - (-L) + \frac{\left(\frac{L}{2}\right)^3}{3L} - \frac{\left(-\frac{L}{2}\right)^3}{3L} + L - \frac{L}{2}$$

$$A_0 = \frac{L^2}{12} + L$$

And for A_n we get

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{2}{L}\int_{-L}^{-\frac{L}{2}} L\cos\left(\frac{n\pi x}{L}\right)\ dx+\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\cos\left(\frac{n\pi x}{L}\right)\ dx+\frac{2}{L}\int_{\frac{L}{2}}^{L} L\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{-\frac{L}{2}}+\frac{2}{L}\int_{-\frac{L}{2}}^{\frac{L}{2}} x^2\cos\left(\frac{n\pi x}{L}\right)\ dx+\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L}$$

For the remaining integral, use integration by parts with $u=x^2$, $du=2x\ dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$.

$$A_n=\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{-\frac{L}{2}}+\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}-\frac{4}{n\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$+\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L}$$

Use integration by parts again with $u=x$, $du=dx$, $dv=\sin(n\pi x/L)\ dx$, and $v=-(L/n\pi)\cos(n\pi x/L)$.

$$A_n=\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{-\frac{L}{2}}+\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}+\frac{4Lx}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$-\frac{4L}{(n\pi)^2}\int_{-\frac{L}{2}}^{\frac{L}{2}}\cos\left(\frac{n\pi x}{L}\right)\ dx+\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L}$$

$$A_n=\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{-\frac{L}{2}}+\frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}+\frac{4Lx}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}$$

$$-\frac{4L^2}{(n\pi)^3}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}}+\frac{2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{\frac{L}{2}}^{L}$$

Now evaluate over each interval.

$$A_n=\frac{2L}{n\pi}\sin\left(-\frac{n\pi}{2}\right)-\frac{2L}{n\pi}\sin(-n\pi)+\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)-\frac{L^2}{2n\pi}\sin\left(-\frac{n\pi}{2}\right)$$

$$+\frac{2L^2}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)+\frac{2L^2}{(n\pi)^2}\cos\left(-\frac{n\pi}{2}\right)-\frac{4L^2}{(n\pi)^3}\sin\left(\frac{n\pi}{2}\right)$$

$$+\frac{4L^2}{(n\pi)^3}\sin\left(-\frac{n\pi}{2}\right)+\frac{2L}{n\pi}\sin(n\pi)-\frac{2L}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n=-\frac{2L}{n\pi}\sin\left(\frac{n\pi}{2}\right)+\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)+\frac{L^2}{2n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$+\frac{2L^2}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)+\frac{2L^2}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)-\frac{4L^2}{(n\pi)^3}\sin\left(\frac{n\pi}{2}\right)$$

$$-\frac{4L^2}{(n\pi)^3}\sin\left(\frac{n\pi}{2}\right)-\frac{2L}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n=\left(-\frac{2L}{n\pi}+\frac{L^2}{2n\pi}+\frac{L^2}{2n\pi}-\frac{4L^2}{(n\pi)^3}-\frac{4L^2}{(n\pi)^3}-\frac{2L}{n\pi}\right)\sin\left(\frac{n\pi}{2}\right)$$

$$+\left(\frac{2L^2}{(n\pi)^2}+\frac{2L^2}{(n\pi)^2}\right)\cos\left(\frac{n\pi}{2}\right)$$

$$A_n=\left(\frac{L^2-4L}{n\pi}-\frac{8L^2}{(n\pi)^3}\right)\sin\left(\frac{n\pi}{2}\right)+\frac{4L^2}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{L(n\pi)^2(L-4) - 8L^2}{(n\pi)^3} \sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{(n\pi)^2} \cos\left(\frac{n\pi}{2}\right)$$

Then the Fourier cosine series representation of the piecewise function on $0 \leq x \leq L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^2}{12} + L$$

$$+ \sum_{n=1}^{\infty} \left(\frac{L(n\pi)^2(L-4) - 8L^2}{(n\pi)^3} \sin\left(\frac{n\pi}{2}\right) + \frac{4L^2}{(n\pi)^2} \cos\left(\frac{n\pi}{2}\right) \right) \cos\left(\frac{n\pi x}{L}\right)$$

Topic: Separation of variables

Question: Using the separation of variables, rewrite the partial differential equation as a pair of ordinary differential equations.

$$2\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial u}{\partial y} = 0$$

Answer choices:

A $2v''(x) + \lambda v(x) = 0$ and $3w'(y) + \lambda w(y) = 0$

B $3v''(x) + \lambda v(x) = 0$ and $2w'(y) + \lambda w(y) = 0$

C $v''(x) - 2\lambda v(x) = 0$ and $w'(y) - 3\lambda w(y) = 0$

D $v''(x) - 3\lambda v(x) = 0$ and $w'(y) - 2\lambda w(y) = 0$

Solution: A

Starting with the product solution and its partial derivatives,

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = v'(x)w(y) \qquad \frac{\partial u}{\partial y} = v(x)w'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)w(y)$$

we'll plug these into the partial differential equation.

$$2v''(x)w(y) - 3v(x)w'(y) = 0$$

$$2\frac{d^2v}{dx^2}w(y) = 3v(x)\frac{dw}{dy}$$

Separate variables then set the equation equal to the separation constant.

$$\left(\frac{2}{v(x)}\right)\frac{d^2v}{dx^2} = \left(\frac{3}{w(y)}\right)\frac{dw}{dy}$$

$$\left(\frac{2}{v(x)}\right)\frac{d^2v}{dx^2} = \left(\frac{3}{w(y)}\right)\frac{dw}{dy} = -\lambda$$

Break this into two equations.

$$\left(\frac{2}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda \qquad \left(\frac{3}{w(y)}\right)\frac{dw}{dy} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\frac{\lambda}{2}v(x) \qquad \frac{dw}{dy} = -\frac{\lambda}{3}w(y)$$

$$2v''(x) + \lambda v(x) = 0$$

$$3w'(y) + \lambda w(y) = 0$$

Topic: Separation of variables

Question: Solve the partial differential equation by separating variables, and assuming that the separation constant is positive.

$$\frac{\partial^2 u}{\partial x^2} - 9\frac{\partial^2 u}{\partial y^2} = 0$$

Answer choices:

A $\quad u(x, y) = (c_1 \cos 3x + c_2 \sin 3x)(c_3 \cos y + c_4 \sin y)$

B $\quad u(x, y) = (c_1 e^{\sqrt{-\lambda}x} + c_2 e^{-\sqrt{-\lambda}x})(c_3 e^{3\sqrt{-\lambda}y} + c_4 e^{-3\sqrt{-\lambda}y})$

C $\quad u(x, y) = (c_1 e^{3\sqrt{-\lambda}x} + c_2 e^{-3\sqrt{-\lambda}x})(c_3 e^{\sqrt{-\lambda}y} + c_4 e^{-\sqrt{-\lambda}y})$

D $\quad u(x, y) = (c_1 \cos 3\sqrt{-\lambda}x + c_2 \sin 3\sqrt{-\lambda}x)(c_3 \cos \sqrt{-\lambda}y + c_4 \sin \sqrt{-\lambda}y)$

Solution: C

Starting with the product solution and its partial derivatives,

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = v'(x)w(y) \qquad \frac{\partial u}{\partial y} = v(x)w'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)w(y) \qquad \frac{\partial^2 u}{\partial y^2} = v(x)w''(y)$$

we'll plug these into the partial differential equation.

$$v''(x)w(y) - 9v(x)w''(y) = 0$$

$$v''(x)w(y) = 9v(x)w''(y)$$

$$\frac{d^2v}{dx^2}w(y) = 9v(x)\frac{d^2w}{dy^2}$$

Separate variables then set the equation equal to the separation constant.

$$\left(\frac{1}{9v(x)}\right)\frac{d^2v}{dx^2} = \left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{9v(x)}\right)\frac{d^2v}{dx^2} = \left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Because we're using the separation constant $-\lambda$, this value will only be positive when $\lambda < 0$. We'll break this into two equations.

$$\left(\frac{1}{9v(x)}\right)\frac{d^2v}{dx^2} = -\lambda \qquad \left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -9\lambda v(x)$$

$$\frac{d^2w}{dy^2} = -\lambda w(y)$$

$$v'' + 9\lambda v = 0$$

$$w'' + \lambda w = 0$$

The associated characteristic equations are

$$r^2 = -9\lambda$$

$$r^2 = -\lambda$$

$$r = \pm 3\sqrt{-\lambda}$$

$$r = \pm\sqrt{-\lambda}$$

Since $\lambda < 0$, we get the general solutions

$$v(x) = c_1e^{3\sqrt{-\lambda}x} + c_2e^{-3\sqrt{-\lambda}x}$$

$$w(y) = c_3e^{\sqrt{-\lambda}y} + c_4e^{-\sqrt{-\lambda}y}$$

So the product solution gives us the general solution to the partial differential equation.

$$u(x,y) = (c_1e^{3\sqrt{-\lambda}x} + c_2e^{-3\sqrt{-\lambda}x})(c_3e^{\sqrt{-\lambda}y} + c_4e^{-\sqrt{-\lambda}y})$$

Topic: Separation of variables

Question: Solve partial differential equation by separating variables, and assuming that the separation constant is negative.

$$\frac{\partial^2 u}{\partial t^2} - 4\frac{\partial u}{\partial x} = 0$$

Answer choices:

A $\quad u(x, t) = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t$

B $\quad u(x, t) = c_3 e^{-\frac{\lambda}{4}x}(c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t)$

C $\quad u(x, t) = c_3 e^{\sqrt{\lambda}x}\left(c_1 \cos\left(\frac{\lambda}{4}t\right) + c_2 \sin\left(\frac{\lambda}{4}t\right)\right)$

D $\quad u(x, t) = ce^{-\frac{\lambda}{4}x}e^{\sqrt{\lambda}t}$

Solution: B

Starting with the product solution and its partial derivatives,

$$u(x,t) = v(x)w(t)$$

$$\frac{\partial u}{\partial x} = v'(x)w(t) \qquad \frac{\partial u}{\partial t} = v(x)w'(t)$$

$$\frac{\partial^2 u}{\partial t^2} = v(x)w''(t)$$

we'll plug these into the partial differential equation.

$$v(x)w''(t) - 4v'(x)w(t) = 0$$

$$v(x)w''(t) = 4v'(x)w(t)$$

$$v(x)\frac{d^2w}{dt^2} = 4w(t)\frac{dv}{dx}$$

Separate variables then set the equation equal to the separation constant.

$$\left(\frac{1}{w(t)}\right)\frac{d^2w}{dt^2} = \left(\frac{4}{v(x)}\right)\frac{dv}{dx}$$

$$\left(\frac{1}{w(t)}\right)\frac{d^2w}{dt^2} = \left(\frac{4}{v(x)}\right)\frac{dv}{dx} = -\lambda$$

Because we're using the separation constant $-\lambda$, this value will only be negative when $\lambda > 0$. We'll break this into two equations.

$$\left(\frac{1}{w(t)}\right)\frac{d^2w}{dt^2} = -\lambda \qquad \left(\frac{4}{v(x)}\right)\frac{dv}{dx} = -\lambda$$

$$\frac{d^2w}{dt^2} + \lambda w(t) = 0$$

To solve the first equation, we'll write its associated characteristic equation.

$$r^2 + \lambda = 0$$

$$r = \pm\sqrt{-\lambda}$$

Since $\lambda > 0$, we get the general solution

$$w(t) = c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t$$

To solve the second equation, we'll solve it as a separable differential equation.

$$\int \frac{4}{v}\ dv = -\int \lambda\ dx$$

$$4\ln|v| = -\lambda x + C$$

$$\ln|v| = -\frac{\lambda}{4}x + C$$

$$|v| = c_3 e^{-\frac{\lambda}{4}x}$$

$$v(x) = c_3 e^{-\frac{\lambda}{4}x}$$

So the product solution gives us the general solution to the partial differential equation.

$$u(x, t) = c_3 e^{-\frac{\lambda}{4}x}(c_1 \cos\sqrt{\lambda}t + c_2 \sin\sqrt{\lambda}t)$$

Topic: Boundary value problems

Question: Rewrite the boundary conditions for the product solution $u(x, t) = v(x)w(t)$.

$$\frac{\partial^2 u}{\partial y^2} = k\frac{\partial u}{\partial x}$$

$$u(x,0) = 0,\ u(0,t) = 0,\ \frac{\partial u}{\partial x}(L, t) = 0$$

Answer choices:

A $w(0) = 0,\ w'(L) = 0,\ v(0) = 0$

B $v(0) = 0,\ v'(0) = 0,\ w(0) = 0$

C $v(0) = 0,\ v'(L) = 0,\ w(0) = 0$

D $v(L) = 0,\ v'(L) = 0,\ w(0) = 0$

Solution: C

Substituting $u(x,0)=0$ into the product solution gives

$$u(x,t)=v(x)w(t)$$

$$u(x,0)=v(x)w(0)=0$$

When $v(x)=0$ we get the trivial solution, so we'll only use $w(0)=0$.

Substituting $u(0,t)=0$ into the product solution gives

$$u(x,t)=v(x)w(t)$$

$$u(0,t)=v(0)w(t)=0$$

When $w(t)=0$ we get the trivial solution, so we'll only use $v(0)=0$.

And if we differentiate the product solution and then substitute the partial derivative condition, we get

$$\frac{\partial u}{\partial x}=v'(x)w(t)$$

$$\frac{\partial u}{\partial x}(L,t)=v'(L)w(t)=0$$

When $w(t)=0$ we get the trivial solution, $u(x,t)=0$, so we'll only use $v'(L)=0$. So the boundary conditions become $v(0)=0$, $v'(L)=0$, and $w(0)=0$.

Topic: Boundary value problems

Question: Solve the boundary value problem if $y(0) = 2$ and $y'(0) = 0$.

$$y'' - 9y = 0$$

Answer choices:

A $\quad y = e^{3x} + e^{-3x}$

B $\quad y = e^{\sqrt{3}x} + e^{-\sqrt{3}x}$

C $\quad y = e^{3x} - e^{-3x}$

D $\quad y = e^{\sqrt{3}x} - e^{-\sqrt{3}x}$

Solution: A

The characteristic equation associated with the differential equation is

$$r^2 - 9 = 0$$

$$r = \pm 3$$

So, the general solution of the differential equation is

$$y(x) = c_1e^{3x} + c_2e^{-3x}$$

$$y'(x) = 3c_1e^{3x} - 3c_2e^{-3x}$$

Substitute the initial conditions $y(0) = 2$ and $y'(0) = 0$.

$$y(0) = 2 \qquad\qquad y'(0) = 0$$

$$c_1e^{3(0)} + c_2e^{-3(0)} = 2 \qquad\qquad 3c_1e^{3(0)} - 3c_2e^{-3(0)} = 0$$

$$c_1 + c_2 = 2 \qquad\qquad 3c_1 - 3c_2 = 0$$

Solving the system of equations gives $c_1 = c_2 = 1$, so the solution is

$$y = e^{3x} + e^{-3x}$$

Topic: Boundary value problems

Question: Solve the boundary value problem given $u(\pi/5,y)=0$, assuming the separation constant is -5.

$$\frac{\partial^2 u}{\partial x^2}-5\frac{\partial u}{\partial y}=0$$

Answer choices:

A $\quad u(x,y)=Ce^{-5y}\cos 5x$

B $\quad u(x,y)=Ce^{-5y}\sin 5x$

C $\quad u(x,y)=Ce^{5y}\sin 5x$

D $\quad u(x,y)=Ce^{5y}\cos 5x$

Solution: B

Starting with the product solution and its partial derivatives,

$$u(x,t) = v(x)w(t)$$

$$\frac{\partial u}{\partial x} = v'(x)w(t) \qquad \frac{\partial u}{\partial y} = v(x)w'(y)$$

$$\frac{\partial^2 u}{\partial x^2} = v''(x)w(y)$$

we'll plug these into the partial differential equation.

$$v''(x)w(y) - 5v(x)w'(y) = 0$$

Separate variables then set the equation equal to the separation constant.

$$\left(\frac{1}{5v(x)}\right)v''(x) = \left(\frac{1}{w(y)}\right)w'(y) = -5$$

Rewrite the boundary conditions.

$$u\left(\frac{\pi}{5}, y\right) = v\left(\frac{\pi}{5}\right)w(y) = 0$$

$$v\left(\frac{\pi}{5}\right) = 0$$

Solving the first equation we found with the separation constant, we get

$$\frac{1}{5v}v'' = -5$$

$$v'' = -25v$$

$$v'' + 25v = 0$$

The associated characteristic equation is

$$r^2 + 25 = 0$$

$$r = \pm 5i$$

Then the solution is

$$v(x) = c_1 \cos 5x + c_2 \sin 5x$$

and substituting the initial condition gives

$$v\left(\frac{\pi}{5}\right) = -c_1 = 0$$

$$v(x) = c_2 \sin 5x$$

Solving the second equation we found with the separation constant, we get

$$\frac{1}{w}w' = -5$$

$$\int \frac{1}{w}\ dw = -5 \int\ dy$$

$$\ln|w| = -5y + C$$

$$w = Ce^{-5y}$$

So the solution is

$$u(x, y) = Ce^{-5y} \sin 5x$$

Topic: The heat equation

Question: Find a solution to the partial differential equation.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \text{ for } 0 \le x \le 10$$

$$u(x,0) = 2\sin(\pi x) + 4\sin(2\pi x)$$

$$u(0,t) = u(10,t) = 0$$

Answer choices:

A $\quad u(x,t) = 2\sin(\pi x)e^{4\pi^2 t} + 4\sin(2\pi x)e^{-16\pi^2 t}$

B $\quad u(x,t) = 2\cos(\pi x)e^{4\pi^2 t} + 4\cos(2\pi x)e^{-16\pi^2 t}$

C $\quad u(x,t) = 2\sin(\pi x)e^{4\pi^2 t} + 4\sin(2\pi x)e^{-8\pi^2 t}$

D $\quad u(x,t) = 2\cos(\pi x)e^{4\pi^2 t} + 4\cos(2\pi x)e^{-8\pi^2 t}$

Solution: A

If we start with the product solution $u(x, t) = v(x)w(t)$, then we know

$$v(x)w'(t) = 4v''(x)w(t)$$

$$\frac{w'}{4w}(t) = \frac{v''}{v}(x) = -\lambda$$

Break this equation into two ordinary differential equations.

$$w' = -4\lambda w \qquad\qquad v'' = -\lambda v$$

The solution to the first equation, which is a linear differential equation, is $w(t) = Ce^{-4\lambda t}$. To solve the second equation, we find the associated characteristic equation.

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$ and the solution is

$$v(x) = c_1e^{-\sqrt{-\lambda}x} + c_2e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives

$$v(0) = c_1 + c_2 = 0$$

$$v(10) = c_1e^{-10\sqrt{-\lambda}} + c_2e^{10\sqrt{-\lambda}} = 0$$

which allows us to find $c_1 = c_2 = 0$, from which we get only the trivial solution. If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$ and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v(10) = c_1 + 10c_2 = 0$$

which allows us to find $c_1 = c_2 = 0$, from which we again get only the trivial solution. If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$ and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v(10) = c_1 \cos(10\sqrt{\lambda}) + c_2 \sin(10\sqrt{\lambda}) = 0$$

We'll find a non-trivial solution when λ satisfies

$$\sin(10\sqrt{\lambda}) = 0$$

$$10\sqrt{\lambda} = \pi n$$

$$\lambda = \left(\frac{\pi n}{10}\right)^2$$

So the general solution equation for the $\lambda > 0$ case becomes

$$v(x) = (0)\cos\left(\sqrt{\left(\frac{n\pi}{10}\right)^2}x\right) + c_2 \sin\left(\sqrt{\left(\frac{n\pi}{10}\right)^2}x\right)$$

$$v(x) = c_2 \sin\left(\frac{n\pi x}{10}\right) \qquad \text{with } n = 1, 2, 3, \dots$$

The first order ordinary differential equation is a linear equation in standard form,

$$w' + k\lambda w = 0$$

so we know the solution is

$$w = Ce^{-4\lambda t}$$

$$w = Ce^{-4\left(\frac{n\pi}{10}\right)^2 t}$$

Putting our results together from the first and second order equations, we get the product solution to the heat equation.

$$u(x, t) = v(x)w(t)$$

$$u_n(x, t) = c_2 \sin\left(\frac{n\pi x}{10}\right) Ce^{-4\left(\frac{n\pi}{10}\right)^2 t} \qquad \text{with } n = 1, 2, 3, \dots$$

While c_2 will be different for each n, the constant $C \times c_2$ will also depend on n, so let's rename it as B_n.

$$u_n(x, t) = \sum_{n=0}^{\infty} B_n \sin\left(\frac{n\pi x}{10}\right) e^{-4\left(\frac{n\pi}{10}\right)^2 t} \qquad \text{with } n = 1, 2, 3, \dots$$

Matching up the boundary condition

$$u(x,0) = 2\sin(\pi x) + 4\sin(2\pi x)$$

to the general solution of the heat equation, we can identify $n = 10$ and therefore $B_n = B_{10} = 2$ for the first term, and $n = 20$ and therefore $B_n = B_{20} = 4$ for the second term. Which means that, given this initial condition, the solution to the heat equation would be

$$u(x,t) = u_{10}(x,t) + u_{20}(x,t)$$

$$u(x,t) = 2\sin(\pi x)e^{-4\pi^2 t} + 4\sin(2\pi x)e^{-16\pi^2 t}$$

Topic: The heat equation

Question: Find a solution to the partial differential equation.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \text{ for } 0\le x\le 5$$

$$u(x,0)=10+\cos(\pi x)$$

$$u_x(0,t)=u_x(5,t)=0$$

Answer choices:

A $\quad u(x,t)=10+\sin(\pi x)e^{-\pi^2 t}$

B $\quad u(x,t)=10+\cos(\pi x)e^{-\pi^2 t}$

C $\quad u(x,t)=10e^{-\pi^2 t}+\sin(\pi x)e^{-\pi^2 t}$

D $\quad u(x,t)=10e^{-\pi^2 t}+\cos(\pi x)e^{-\pi^2 t}$

Solution: B

If we start with the product solution $u(x, t) = v(x)w(t)$, then we know

$$v(x)w'(t) = v''(x)w(t)$$

$$\frac{w'}{w}(t) = \frac{v''}{v}(x) = -\lambda$$

Break this equation into two ordinary differential equations.

$$w' = -\lambda w \qquad\qquad v'' = -\lambda v$$

The solution to the first equation, which is a linear differential equation, is $w(t) = Ce^{-\lambda t}$. To solve the second equation, we find the associated characteristic equation.

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$ and the solution is

$$v(x) = c_1e^{-\sqrt{-\lambda}x} + c_2e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives

$$v'(0) = -c_1 + c_2 = 0$$

$$v'(5) = -c_1\sqrt{-\lambda}e^{-5\sqrt{-\lambda}} + c_2\sqrt{-\lambda}e^{5\sqrt{-\lambda}} = 0$$

which allows us to find $c_1 = c_2 = 0$, from which we get only the trivial solution. If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$ and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives

$$v'(0) = c_2 = 0$$

$$v'(5) = c_2 = 0$$

which allows us to find $v(x) = c_1$, a non-trivial solution. If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$ and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives

$$v'(0) = c_2 = 0$$

$$v'(5) = -c_1\sqrt{\lambda}\sin(5\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(5\sqrt{\lambda}) = 0$$

We'll find a non-trivial solution when λ satisfies

$$\sin(5\sqrt{\lambda}) = 0$$

$$5\sqrt{\lambda} = \pi n$$

$$\lambda = \left(\frac{\pi n}{5}\right)^2$$

So the general solution equation for the $\lambda > 0$ case becomes

$$v(x) = c_1 \cos\left(\sqrt{\left(\frac{n\pi}{5}\right)^2}x\right) + (0)\sin\left(\sqrt{\left(\frac{n\pi}{5}\right)^2}x\right)$$

$$v(x) = c_1 \cos\left(\frac{n\pi x}{5}\right)$$ with $n = 1, 2, 3, \ldots$

The first order ordinary differential equation is a linear equation in standard form,

$$w' + k\lambda w = 0$$

so we know the solution is

$$w = Ce^{-\lambda t}$$

$$w = Ce^{-\left(\frac{n\pi}{5}\right)^2 t}$$

Putting our results together from the first and second order equations, we get the product solution to the heat equation.

$$u(x, t) = v(x)w(t)$$

$$u_n(x, t) = \sum_{n=0}^{\infty} B_n \cos\left(\frac{n\pi x}{5}\right) e^{-\left(\frac{n\pi}{5}\right)^2 t}$$ with $n = 1, 2, 3, \ldots$

We can rewrite

$$u_n(x, t) = B_0 \cos 0 e^{-(0)t} + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{5}\right) e^{-\left(\frac{n\pi}{5}\right)^2 t}$$

$$u_n(x, t) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{5}\right) e^{-\left(\frac{n\pi}{5}\right)^2 t}$$

Matching this up to the boundary condition

$$u(x,0) = 10 + \cos(\pi x)$$

gives

$$10 + \cos(\pi x) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{5}\right) e^{-\left(\frac{n\pi}{5}\right)^2 t}$$

So we have $B_0 = 10$ and $B_5 = 1$.

$$u(x, t) = u_0(x, t) + u_5(x, t)$$

$$u(x, t) = 10 + \cos(\pi x) e^{-\pi^2 t}$$

Topic: The heat equation

Question: Find a solution to the partial differential equation.

$$\frac{\partial u}{\partial t}=9\frac{\partial^2 u}{\partial x^2} \text{ for } 0\le x\le \pi$$

$$u(x,0)=3\sin\left(\frac{x}{2}\right)$$

$$u(0,t)=u_x(\pi,t)=0$$

Answer choices:

A $\quad u(x,t)=3\sin\left(\frac{1}{2}x\right)e^{-\frac{9}{2}t}$

B $\quad u(x,t)=3\cos\left(\frac{1}{2}x\right)e^{-\frac{9}{2}t}$

C $\quad u(x,t)=3\sin\left(\frac{1}{2}x\right)e^{-\frac{9}{4}t}$

D $\quad u(x,t)=3\cos\left(\frac{1}{2}x\right)e^{-\frac{9}{4}t}$

Solution: C

If we start with the product solution $u(x, t) = v(x)w(t)$, then we know

$$v(x)w'(t) = 9v''(x)w(t)$$

$$\frac{w'}{9w}(t) = \frac{v''}{v}(x) = -\lambda$$

Break this equation into two ordinary differential equations.

$$w' = -9\lambda w \qquad\qquad v'' = -\lambda v$$

The solution to the first equation, which is a linear differential equation, is $w(t) = Ce^{-9\lambda t}$. To solve the second equation, we find the associated characteristic equation.

$$r^2 + \lambda = 0$$

From the characteristic equation, if $\lambda < 0$, the equation has distinct real roots $r = \pm\sqrt{-\lambda}$ and the solution is

$$v(x) = c_1e^{-\sqrt{-\lambda}x} + c_2e^{\sqrt{-\lambda}x}$$

Applying the boundary conditions gives

$$v(0) = c_1 + c_2 = 0$$

$$v'(\pi) = -c_1\sqrt{-\lambda}e^{-\pi\sqrt{-\lambda}} + c_2\sqrt{-\lambda}e^{\pi\sqrt{-\lambda}} = 0$$

which allows us to find $c_1 = c_2 = 0$, from which we get only the trivial solution. If $\lambda = 0$, the characteristic equation has equal real roots $r = 0$ and the solution is

$$v(x) = c_1 + c_2 x$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v'(\pi) = c_2 = 0$$

which allows us to find $c_1 = c_2 = 0$, from which we again get only the trivial solution. If $\lambda > 0$, the characteristic equation has complex roots $r = \pm\sqrt{\lambda}i$ and the solution is

$$v(x) = c_1 \cos(\sqrt{\lambda}x) + c_2 \sin(\sqrt{\lambda}x)$$

Applying the boundary conditions gives

$$v(0) = c_1 = 0$$

$$v'(\pi) = -c_1\sqrt{\lambda}\sin(\pi\sqrt{\lambda}) + c_2\sqrt{\lambda}\cos(\pi\sqrt{\lambda}) = 0$$

We'll find a non-trivial solution when λ satisfies

$$\cos(\pi\sqrt{\lambda}) = 0$$

$$\pi\sqrt{\lambda} = \frac{\pi}{2} + \pi n$$

$$\lambda = \left(\frac{1}{2} + n\right)^2$$

So the general solution equation for the $\lambda > 0$ case becomes

$$v(x) = (0)\cos\left(\sqrt{\left(\frac{n\pi}{5}\right)^2}x\right) + c_2\sin\left(\sqrt{\left(\frac{n\pi}{5}\right)^2}x\right)$$

$$v(x) = c_2\sin\left(\left(\frac{1}{2}+n\right)x\right) \qquad \text{with } n = 1, 2, 3, \dots$$

Plugging the value of λ we found into the solution equation for w, we get

$$w(t) = Ce^{-9\lambda t}$$

$$w = Ce^{-9\left(\frac{1}{2}+n\right)^2 t}$$

Putting our results together from the first and second order equations, we get the product solution to the heat equation.

$$u(x,t) = v(x)w(t)$$

$$u_n(x,t) = \sum_{n=0}^{\infty} B_n\sin\left(\left(\frac{1}{2}+n\right)x\right)e^{-9\left(\frac{1}{2}+n\right)^2 t} \qquad \text{with } n = 1, 2, 3, \dots$$

Matching this up to the boundary condition

$$u(x,0) = 3\sin\left(\frac{x}{2}\right)$$

gives

$$\sum_{n=0}^{\infty} B_n\sin\left(\left(\frac{1}{2}+n\right)x\right)e^{-9\left(\frac{1}{2}+n\right)^2 t} = 3\sin\left(\frac{x}{2}\right)$$

So we have $B_0 = 3$ and $B_1 = B_2 = \ldots = 0$.

$$u(x,t) = 3\sin\left(\frac{1}{2}x\right)e^{-\frac{9}{4}t}$$

Topic: Changing the temperature boundaries

Question: Find a solution to the partial differential equation.

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2} \text{ for } 0\le x\le 10$$

$$u(0,t)=5,\ u(10,t)=15$$

$$u(x,0)=x+10$$

Answer choices:

A $\quad 10+x+\sum_{n=1}^{\infty}(1-(-1)^n)\frac{10}{n\pi}\sin\left(\frac{n\pi x}{10}\right)e^{-\left(\frac{n\pi}{10}\right)^2t}$

B $\quad 10+x+\sum_{n=1}^{\infty}((-1)^n-1)\frac{10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)e^{-\left(\frac{n\pi}{10}\right)^2t}$

C $\quad 5+x+\sum_{n=1}^{\infty}(1-(-1)^n)\frac{10}{n\pi}\sin\left(\frac{n\pi x}{10}\right)e^{-\left(\frac{n\pi}{10}\right)^2t}$

D $\quad 5+x+\sum_{n=1}^{\infty}((-1)^n-1)\frac{10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)e^{-\left(\frac{n\pi}{10}\right)^2t}$

Solution: C

First we'll find the function modeling equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E(x) = 5 + \frac{15 - 5}{10}x$$

$$u_E(x) = 5 + x$$

Next, we'll find the coefficient B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{10}\int_0^{10} (x + 10 - 5 - x)\sin\left(\frac{n\pi x}{10}\right)\ dx$$

$$B_n = \frac{1}{5}\int_0^{10} 5\sin\left(\frac{n\pi x}{10}\right)\ dx$$

$$B_n = -\frac{10}{n\pi}\cos\left(\frac{n\pi x}{10}\right)\Big|_0^{10}$$

$$B_n = -\frac{10}{n\pi}(\cos(n\pi) - 1)$$

$$B_n = -\frac{10}{n\pi}((-1)^n - 1)$$

$$B_n = \frac{10}{n\pi}(1 - (-1)^n)$$

Then the solution to the equation is

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 5 + x + \sum_{n=1}^{\infty} (1-(-1)^n)\frac{10}{n\pi} \sin\left(\frac{n\pi x}{10}\right) e^{-\left(\frac{n\pi}{10}\right)^2 t}$$

Topic: Changing the temperature boundaries

Question: Find a solution to the partial differential equation.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \text{ for } 0 \le x \le 2\pi$$

$$u(0,t) = 20, \ u(2\pi, t) = 40$$

$$u(x,0) = 20$$

Answer choices:

A $\quad u(x,t) = 20 + \frac{10}{\pi}x + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{nx}{2}\right)e^{-n^2t}$

B $\quad u(x,t) = 20 + \frac{10}{\pi}x + \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{nx}{2}\right)e^{-n^2t}$

C $\quad u(x,t) = 20 + \frac{10}{\pi} + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{nx}{2}\right)e^{-4n^2t}$

D $\quad u(x,t) = 20 + \frac{10}{\pi} + \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{nx}{2}\right)e^{-4n^2t}$

Solution: A

First we'll find the function modeling equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E(x) = 20 + \frac{40 - 20}{2\pi}x$$

$$u_E(x) = 20 + \frac{10}{\pi}x$$

Next, we'll find the coefficient B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{\pi}\int_0^{2\pi} \left(20 - 20 - \frac{10}{\pi}x\right)\sin\left(\frac{nx}{2}\right)\ dx$$

$$B_n = -\frac{10}{\pi^2}\int_0^{2\pi} x\sin\left(\frac{nx}{2}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(nx/2)\ dx$, and $v = -(2/n)\cos(nx/2)$.

$$B_n = -\frac{10}{\pi^2}\left(-\frac{2}{n}x\cos\left(\frac{nx}{2}\right)\Bigg|_0^{2\pi} + \frac{2}{n}\int_0^{2\pi}\cos\left(\frac{nx}{2}\right)\ dx\right)$$

$$B_n = \frac{20}{n\pi^2}x\cos\left(\frac{nx}{2}\right) - \frac{40}{(n\pi)^2}\sin\left(\frac{nx}{2}\right)\Bigg|_0^{2\pi}$$

$$B_n = \frac{20}{n\pi^2}(2\pi)\cos\left(\frac{2n\pi}{2}\right) - \frac{40}{(n\pi)^2}\sin\left(\frac{2n\pi}{2}\right)$$

$$-\left[\frac{20}{n\pi^2}(0)\cos\left(\frac{n(0)}{2}\right) - \frac{40}{(n\pi)^2}\sin\left(\frac{n(0)}{2}\right)\right]$$

$$B_n = \frac{40}{n\pi}\cos(n\pi) - \frac{40}{(n\pi)^2}\sin(n\pi)$$

$$B_n = \frac{40}{n\pi}(-1)^n$$

Then the solution to this equation is

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 20 + \frac{10}{\pi}x + \frac{40}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{nx}{2}\right)e^{-n^2 t}$$

Topic: Changing the temperature boundaries

Question: Find a solution to the partial differential equation.

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \text{ for } 0 \le x \le 5$$

$$u(0,t) = 10,\ u(5,t) = 20$$

$$u(x,0) = 5 + x$$

Answer choices:

A $\quad u(x,t) = 10 + 2x + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{5}\right)e^{-\left(\frac{n\pi}{5}\right)^2 t}$

B $\quad u(x,t) = 2x + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n}\sin\left(\frac{\pi n x}{5}\right)e^{-3\left(\frac{n\pi}{5}\right)^2 t}$

C $\quad u(x,t) = 10 + 2x + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n - 1}{n}\sin\left(\frac{n\pi x}{5}\right)e^{-\left(\frac{n\pi}{5}\right)^2 t}$

D $\quad u(x,t) = 10 + 2x + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{2(-1)^n - 1}{n}\sin\left(\frac{n\pi x}{5}\right)e^{-3\left(\frac{n\pi}{5}\right)^2 t}$

Solution: D

First we'll find the function modeling equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E(x) = 10 + \frac{20 - 10}{5}x$$

$$u_E(x) = 10 + 2x$$

Next, we'll find the coefficient B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{5}\int_0^5 (5 + x - 10 - 2x)\sin\left(\frac{n\pi x}{5}\right)\ dx$$

$$B_n = -\frac{2}{5}\int_0^5 (x + 5)\sin\left(\frac{n\pi x}{5}\right)\ dx$$

Use integration by parts with $u = x + 5$, $du = dx$, $dv = \sin(n\pi x/5)\ dx$, and $v = -(5/n\pi)\cos(n\pi x/5)$.

$$B_n = -\frac{2}{n\pi}\left(-(x+5)\cos\left(\frac{n\pi x}{5}\right)\Big|_0^5 + \int_0^5 \cos\left(\frac{n\pi x}{5}\right)\ dx\right)$$

$$B_n = \frac{2(x+5)}{n\pi}\cos\left(\frac{n\pi x}{5}\right) - \frac{10}{(n\pi)^2}\sin\left(\frac{n\pi x}{5}\right)\Big|_0^5$$

$$B_n = \frac{20}{n\pi}\cos(n\pi) - \frac{10}{(n\pi)^2}\sin(n\pi) - \frac{10}{n\pi}$$

$$B_n = \frac{20}{n\pi}(-1)^n - \frac{10}{n\pi}$$

$$B_n = \frac{10}{n\pi}(2(-1)^n - 1)$$

Then the solution to this equation is

$$u(x,t) = u_E(x) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 10 + 2x + \frac{10}{\pi}\sum_{n=1}^{\infty} \frac{2(-1)^n - 1}{n} \sin\left(\frac{n\pi x}{5}\right) e^{-3\left(\frac{n\pi}{5}\right)^2 t}$$

Topic: Laplace's equation

Question: Find a solution to the partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ for } 0 \le x \le 1 \text{ and } 0 \le y \le 2$$

$$u(x,0) = \sin(\pi x) \qquad u(0,y) = 0$$

$$u(x,2) = 0 \qquad u(1,y) = 0$$

Answer choices:

A $\quad u(x,y) = \frac{1}{\sinh 2\pi}\sinh(\pi(1-x))\sin(\pi y)$

B $\quad u(x,y) = \sum_{n=1}^{\infty}(1-(-1)^n)\sinh(n\pi(2-y))\sin(n\pi x)$

C $\quad u(x,y) = \frac{1}{\sinh 2\pi}\sinh(\pi(2-y))\sin(\pi x)$

D $\quad u(x,y) = \sum_{n=1}^{\infty}(1-(-1)^n)\sinh(n\pi(1-x))\sin(n\pi y)$

Solution: C

If we find the first and second derivatives of the product solution,

$$u(x,y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2v}{dx^2}w + v\frac{d^2w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

Our boundary conditions are $w(2) = 0$, $v(0) = 0$, and $v(1) = 0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{1}\right)^2 = (n\pi)^2 \qquad v_n(x) = C\sin(n\pi x) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} - (n\pi)^2 w = 0$$

$$w(2) = 0$$

Because $\lambda > 0$, we know that the solution to this second order equation will be

$$w(y) = c_1 e^{-\sqrt{\lambda}y} + c_2 e^{\sqrt{\lambda}y}$$

Equivalently, this can be rewritten as

$$w(y) = c_1 \sinh(\sqrt{\lambda}(y-2)) + c_2 \cosh(\sqrt{\lambda}(y-2))$$

which allows us to apply $w(2) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$w(y) = c_1 \sinh(n\pi(y-2))$$

Then the product solution is

$$u_1(x, y) = v(x)w(y)$$

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi(y-2))\sin(n\pi x) \qquad n = 1, 2, 3, \ldots$$

Substitute $u(x,0) = \sin(\pi x)$.

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sinh(-2n\pi)\sin(n\pi x) = \sin(\pi x)$$

$$B_1 = -\frac{1}{\sinh 2\pi}$$

$$B_2 = B_3 = \ldots = 0$$

Then the solution is

$$u(x, y) = \frac{1}{\sinh 2\pi}\sinh(\pi(2-y))\sin(\pi x)$$

Topic: Laplace's equation

Question: Find a solution to the partial differential equation.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \text{ for } 0 \le x \le \pi \text{ and } 0 \le y \le \pi$$

$$u(x,0)=0 \qquad u(0,y)=1$$

$$u(x,\pi)=0 \qquad u(\pi,y)=\sin 2y$$

Answer choices:

A $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi\sinh(n\pi)}(1-(-1)^n)\sinh(n(\pi-x))\sin(ny)+\frac{1}{\sinh 2\pi}\sin(2x)\sinh(2y)$

B $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi\sinh(n\pi)}(1-(-1)^n)\sinh(n(\pi-x))\sin(ny)+\frac{1}{\sinh 2\pi}\sinh(2x)\sin(2y)$

C $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi\sinh(n\pi)}(1-(-1)^n)\sin(nx)\sinh(n(\pi-y))+\frac{1}{\sinh 2\pi}\sin(2x)\sinh(2y)$

D $\displaystyle\sum_{n=1}^{\infty}\frac{2}{n\pi\sinh(n\pi)}(1-(-1)^n)\sin(nx)\sinh(n(\pi-y))+\frac{1}{\sinh 2\pi}\sinh(2x)\sin(2y)$

Solution: B

We can say that $u = u_1 + u_2$, where u_1 is the solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,0) = 0 \qquad u(0,y) = 1$$

$$u(x,\pi) = 0 \qquad u(\pi,y) = 0$$

and u_2 is the solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(x,0) = 0 \qquad u(0,y) = 0$$

$$u(x,\pi) = 0 \qquad u(\pi,y) = \sin 2y$$

Lets start with u_1. If we find the first and second derivatives of the product solution,

$$u(x,y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2v}{dx^2}w + v\frac{d^2w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2v}{dx^2}w = -v\frac{d^2w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = \lambda$$

$$\frac{d^2v}{dx^2} = \lambda v(x)$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = \lambda$$

$$\frac{d^2w}{dy^2} = -\lambda w(y)$$

$$\frac{d^2w}{dy^2} + \lambda w = 0$$

Our boundary conditions are $w(\pi) = 0$, $w(0) = 0$, and $v(\pi) = 0$, and the solution to the second ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2 \qquad w_n(y) = C\sin(ny) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2v}{dx^2} - n^2 v = 0$$

$$v(\pi) = 0$$

Because $\lambda > 0$, we know that the solution to this second order equation will be

$$v_n(x) = c_1 \sinh nx + c_2 \cosh nx$$

Equivalently, this can be rewritten as

$$v_n(x) = c_1 \sinh n(x - \pi) + c_2 \cosh n(x - \pi)$$

which allows us to apply $v(\pi) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$v_n(x) = c_1 \sinh n(x - \pi)$$

Then the solution for u_1 is

$$u_1(x,y) = v_n(x)w_n(y)$$

$$u_1(x,y) = \sum_{n=1}^{\infty} B_n \sinh(n(x-\pi))\sin(ny) \qquad n = 1, 2, 3, \dots$$

Substitute $u(0,y) = 1$.

$$u_1(0,y) = \sum_{n=1}^{\infty} B_n \sinh(n(0-\pi))\sin(ny) = 1$$

$$B_n \sinh(-n\pi)\sin(ny) = \frac{2}{\pi}\int_0^{\pi} \sin(ny)dy = \frac{2}{n\pi}(-\cos(ny))\Big|_0^{\pi} = \frac{2}{n\pi}(1-(-1)^n)$$

$$u_1(x,y) = \sum_{n=1}^{\infty} \frac{2}{n\pi\sinh(n\pi)}(1-(-1)^n)\sinh(n(\pi-x))\sin(ny)$$

Solve for u_2.

$$\frac{d^2v}{dx^2}w + v\frac{d^2w}{dy^2} = 0$$

$$\frac{d^2v}{dx^2} - \lambda v = 0$$

and

$$\frac{d^2w}{dy^2} + \lambda w = 0$$

Our boundary conditions are $w(\pi) = 0$, $w(0) = 0$, and $v(\pi) = 0$, and the solution to the second ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{\pi}\right)^2 = n^2 \qquad w_n(y) = C\sin(ny) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2v}{dx^2} - n^2v = 0$$

$$v(\pi) = 0$$

Because $\lambda > 0$, we know that the solution to this second order equation will be

$$v_n(x) = c_1 \sinh nx + c_2 \cosh nx$$

which allows us to apply $v(\pi) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$v_n(x) = c_1 \sinh n(x - \pi)$$

Then solution for u_2 is

$$u_2(x, y) = v_n(x)w_n(y)$$

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sinh(nx)\sin(ny) \qquad n = 1, 2, 3, \ldots$$

Substitute $u(\pi, y) = \sin 2y$.

$$u_2(\pi, y) = \sum_{n=1}^{\infty} B_n \sinh(n\pi)\sin(ny) = \sin 2y$$

$$B_2 = \frac{1}{\sinh 2\pi}$$

$$u_2(x, y) = \frac{1}{\sinh 2\pi} \sinh(2x)\sin(2y)$$

Then the solution is

$$u = u_1 + u_2$$

$$u(x, y) = \sum_{n=1}^{\infty} \frac{2}{n\pi \sinh(n\pi)}(1 - (-1)^n)\sinh(n(\pi - x))\sin(ny) + \frac{1}{\sinh 2\pi} \sinh(2x)\sin(2y)$$

Topic: Laplace's equation

Question: Find a solution to the partial differential equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \text{ for } 0 \le x \le 2 \text{ and } 0 \le y \le 5$$

$$u(x,0) = 0 \qquad\qquad u(0,y) = 0$$

$$u(x,5) = \sin(\pi x) + \sin(2\pi x) \qquad u(2,y) = 0$$

Answer choices:

A $\quad u(x,y) = \frac{1}{\sinh 5\pi}\sinh(\pi y)\sin(\pi x) + \frac{1}{\sinh 10\pi}\sinh(2\pi y)\sin(2\pi x)$

B $\quad u(x,y) = \frac{1}{\sinh 5\pi}\sin(\pi y)\sinh(\pi x) + \frac{1}{\sinh 10\pi}\sin(2\pi y)\sinh(2\pi x)$

C $\quad u(x,y) = \frac{1}{\sinh 5\pi}\sin(\pi x) + \frac{1}{\sinh 10\pi}\sin(2\pi x)$

D $\quad u(x,y) = \frac{1}{\sin 10\pi}\sinh(\pi y)\sin(\pi x) + \frac{1}{\sin 20\pi}\sinh(2\pi y)\sin(2\pi x)$

Solution: A

If we find the first and second derivatives of the product solution,

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2 v}{dx^2}w + v\frac{d^2 w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2 v}{dx^2}w = -v\frac{d^2 w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2 v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2 w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2 v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2 w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2 v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

Our boundary conditions are $w(0) = 0$, $v(0) = 0$, and $v(2) = 0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{2}\right)^2 \qquad v_n(x) = C\sin\left(\frac{n\pi}{2}\right) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} - \left(\frac{n\pi}{2}\right)^2 w = 0$$

$$w(2) = 0$$

Because $\lambda > 0$, we know that the solution to this second order equation will be

$$w(y) = c_1 \sinh\left(\frac{n\pi y}{2}\right) + c_2 \cosh\left(\frac{n\pi y}{2}\right)$$

which allows us to apply $w(0) = 0$. When we do, we find $c_2 = 0$, and the solution is

$$w(y) = c_1 \sinh\left(\frac{n\pi y}{2}\right)$$

Then the product solution is

$$u_1(x, y) = v(x)w(y)$$

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi y}{2}\right) \sin\left(\frac{n\pi x}{2}\right) \qquad n = 1, 2, 3, \ldots$$

Substitute $u(x,5) = \sin(\pi x) + \sin(2\pi x)$.

$$u(x,5) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{5n\pi}{2}\right) \sin\left(\frac{n\pi x}{2}\right) = \sin(\pi x) + \sin(2\pi x)$$

$$B_2 = \frac{1}{\sinh 5\pi}$$

$$B_4 = \frac{1}{\sinh 10\pi}$$

$$B_1 = B_3 = \ldots = 0$$

Then the solution is

$$u(x, y) = \frac{1}{\sinh 5\pi} \sinh(\pi y)\sin(\pi x) + \frac{1}{\sinh 10\pi} \sinh(2\pi y)\sin(2\pi x)$$

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