Differential Equations Final Exam Solutions

krista king
MATH

Differential Equations Final Exam Answer Key

1. (5 pts)	A	B	C	■	E
2. (5 pts)	A	■	C	D	E
3. (5 pts)	A	B	■	D	E
4. (5 pts)	A	B	C	D	■
5. (5 pts)	A	B	■	D	E
6. (5 pts)	A	B	C	D	■
7. (5 pts)	■	B	C	D	E
8. (5 pts)	A	B	C	■	E

9. (15 pts)

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{3t}$$

$$+ \begin{bmatrix} e^{2t} \\ \frac{1}{10}\sin t + t \\ \frac{3}{10}\cos t - \frac{1}{3} \end{bmatrix}$$

10. (15 pts)

$$y(t) = c_1 + c_2 e^{2t} + c_3 e^{-3t} + c_4 t e^{-3t}$$

$$+c_5 e^{-t}\cos(\sqrt{6}t) + c_6 e^{-t}\sin(\sqrt{6}t)$$

$$+c_7 e^{-t}\cos(\sqrt{6}t) + c_8 e^{-t}\sin(\sqrt{6}t)$$

$$+c_9 e^{-t}\cos(\sqrt{6}t) + c_{10} e^{-t}\sin(\sqrt{6}t)$$

$$+c_{11} e^{-t}\cos(\sqrt{6}t) + c_{12} e^{-t}\sin(\sqrt{6}t)$$

11. (15 pts)

$$f(x) = \frac{L^4}{5} + \frac{8L^4}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n((n\pi)^2 - 6)}{n^4} \cos\left(\frac{n\pi x}{L}\right)$$

12. (15 pts)

$$u(x,t) = \frac{100}{L}x$$

$$+\frac{50}{\pi} \sum_{n=1}^{\infty} \frac{4(-1)^n + (-1)^{n+1} + 1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Differential Equations Final Exam Solutions

1. D. Change y' to dy/dt,

$$y' = -3t^2y$$

$$\frac{dy}{dt} = -3t^2y$$

then separate variables, collecting y terms on the left and t terms on the right.

$$\frac{1}{y}\ dy = -3t^2\ dt$$

Integrate both sides,

$$\int \frac{1}{y}\ dy = \int -3t^2\ dt$$

$$\ln|y| = -t^3 + C$$

then solve for y.

$$|y| = e^{-t^3+C}$$

$$y = Ce^{-t^3}$$

2. B. The Bernoulli equation is already in standard form,

$$y^{\frac{1}{2}}\frac{dy}{dx}+y^{\frac{3}{2}}=1$$

so we'll make a substitution with

$$v=y^{\frac{3}{2}}$$

$$v'=\frac{3}{2}y^{\frac{1}{2}}y'$$

$$y^{\frac{1}{2}}y'=\frac{2}{3}v'$$

Now we can make substitutions into the Bernoulli equation.

$$\frac{2}{3}v'+v=1$$

$$v'+\frac{3}{2}v=\frac{3}{2}$$

Now we have a linear equation with an integrating factor of $e^{\frac{3}{2}x}$, so we'll multiply this integrating factor through the linear equation.

$$v'e^{\frac{3}{2}x}+\frac{3}{2}ve^{\frac{3}{2}x}=\frac{3}{2}e^{\frac{3}{2}x}$$

$$\frac{d}{dx}(ve^{\frac{3}{2}x})=\frac{3}{2}e^{\frac{3}{2}x}$$

$$\int\frac{d}{dx}(ve^{\frac{3}{2}x})\ dx=\int\frac{3}{2}e^{\frac{3}{2}x}\ dx$$

$$ve^{\frac{3}{2}x}=e^{\frac{3}{2}x}+C$$

$$v=1+Ce^{-\frac{3}{2}x}$$

Use $v = y^{\frac{3}{2}}$ to back-substitute for v, then solve for y.

$$y^{\frac{3}{2}} = 1 + Ce^{-\frac{3}{2}x}$$

Substitute the initial condition $y(0) = 9$.

$$9^{\frac{3}{2}} = 1 + Ce^{-\frac{3}{2}(0)}$$

$$27 = 1 + C$$

$$26 = C$$

Then the solution to the Bernoulli equation initial value problem is

$$y^{\frac{3}{2}} = 1 + 26e^{-\frac{3}{2}x}$$

$$y = (1 + 26e^{-\frac{3}{2}x})^{\frac{2}{3}}$$

3. C. We'll use undetermined coefficients and set $g(x) = 0$ to find the associated homogeneous equation,

$$y'' - 6y' + 9y = 3e^{3x} + \sin x$$

$$y'' - 6y' + 9y = 0$$

then solve the associated characteristic equation.

$$r^2 - 6r + 9 = 0$$

$$(r - 3)(r - 3) = 0$$

$$r = 3,\ 3$$

These are equal real roots, so the complementary solution will be

$$y_c(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

$$y_c(x) = c_1 e^{3x} + c_2 x e^{3x}$$

Our guess for the particular solution will be

$$y_p(x) = Ae^{3x} + B \sin x + C \cos x$$

The Ae^{3x} terms overlaps with the $c_1 e^{3x}$ term from the complementary solution. If we multiply the term from our guess by x to eliminate the overlap, the new Axe^{3x} term now overlaps with the other term from the complementary solution, $c_2 x e^{3x}$. So we'll multiply it by x again, and our guess will be

$$y_p(x) = Ax^2 e^{3x} + B \sin x + C \cos x$$

Take the first and second derivatives of the guess.

$$y_p'(x) = 2Axe^{3x} + 3Ax^2 e^{3x} + B \cos x - C \sin x$$

$$y_p''(x) = 2Ae^{3x} + 12Axe^{3x} + 9Ax^2 e^{3x} - B \sin x - C \cos x$$

Plugging into the original differential equation, we get

$$y'' - 6y' + 9y = 3e^{3x} + \sin x$$

$$2Ae^{3x} + 12Axe^{3x} + 9Ax^2 e^{3x} - B \sin x - C \cos x$$

$$-6(2Axe^{3x} + 3Ax^2 e^{3x} + B \cos x - C \sin x)$$

$$+9(Ax^2 e^{3x} + B \sin x + C \cos x) = 3e^{3x} + \sin x$$

$$2Ae^{3x} + 12Axe^{3x} + 9Ax^2e^{3x} - B\sin x - C\cos x$$

$$-12Axe^{3x} - 18Ax^2e^{3x} - 6B\cos x + 6C\sin x$$

$$+9Ax^2e^{3x} + 9B\sin x + 9C\cos x = 3e^{3x} + \sin x$$

$$2Ae^{3x} + (8B + 6C)\sin x + (-6B + 8C)\cos x = 3e^{3x} + \sin x$$

Equating coefficients gives $2A = 3$, $8B + 6C = 1$, and $-6B + 8C = 0$, and those equations allow us to solve for $A = 3/2$, $B = 2/25$, and $C = 3/50$. So the particular solution is

$$y_p(x) = Ax^2e^{3x} + B\sin x + C\cos x$$

$$y_p(x) = \frac{3}{2}x^2e^{3x} + \frac{2}{25}\sin x + \frac{3}{50}\cos x$$

Putting this particular solution together with the complementary solution gives us the general solution to the nonhomogeneous differential equation.

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{3x} + c_2xe^{3x} + \frac{3}{2}x^2e^{3x} + \frac{2}{25}\sin x + \frac{3}{50}\cos x$$

4. E. Approximating $y(\pi)$ from $y(0) = 0$ using $n = 4$ steps gives us a step size of

$$\Delta t = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

With $t_0 = 0$ and $y_0 = 0$ and $\Delta t = \pi/4$, our table is

$$t_0 = 0 \qquad y_0 = 0 \qquad y_0 = 0$$

$$t_1 = \frac{\pi}{4} \qquad y_1 = 0 - 4\left(\frac{\pi}{4}\right)\sin(0) \qquad y_1 = 0$$

$$t_2 = \frac{\pi}{2} \qquad y_2 = 0 - 4\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{2}\right) \qquad y_2 = -\pi$$

$$t_3 = \frac{3\pi}{4} \qquad y_3 = -\pi - 4\left(\frac{\pi}{4}\right)\sin\left(\frac{3\pi}{4}\right) \qquad y_3 = -\frac{2+\sqrt{2}}{2}\pi$$

$$t_4 = \pi \qquad y_4 = -\frac{2+\sqrt{2}}{2}\pi - 4\left(\frac{\pi}{4}\right)\sin\pi \qquad y_4 = -\frac{2+\sqrt{2}}{2}\pi$$

After filling out the table, we can say that the value of $y(\pi)$ is approximately

$$y(\pi) \approx -\frac{2+\sqrt{2}}{2}\pi$$

5. C. The autonomous differential equation has equilibrium solutions at

$$y^2 - 7y + 12 = 0$$

$$(y-3)(y-4) = 0$$

$$y = 3,\ 4$$

Given these two equilibrium solutions, we'll consider three intervals:

$y < 3$

$3 < y < 4$

$4 < y$

We'll choose a test value in each interval ($y = 2$ for the first interval, $y = 7/2$ for the second interval, and $y = 5$ for the third interval), then plug each test value into the equation $f(y) = y^2 - 7y + 12$.

$$f(2) = 2^2 - 7(2) + 12 = 4 - 14 + 12 = 2 > 0$$

$$f\left(\frac{7}{2}\right) = \left(\frac{7}{2}\right)^2 - 7\left(\frac{7}{2}\right) + 12 = \frac{49}{4} - \frac{98}{4} + \frac{48}{4} = -\frac{1}{4} < 0$$

$$f(5) = 5^2 - 7(5) + 12 = 25 - 35 + 12 = 2 > 0$$

The signs of these results tell us that solution curves are increasing in the intervals $y < 3$ and $4 < y$, and decreasing in the interval $3 < y < 4$.

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
$(4,\infty)$	$+$	Increasing/Rising
$(3,4)$	$-$	Decreasing/Falling
$(-\infty,3)$	$+$	Increasing/Rising

Therefore, the equation has a stable equilibrium solution at $y = 3$, and an unstable equilibrium solution at $y = 4$.

6. E. To start, rewrite the differential equation.

$$y' - y = 0$$

Next, we'll substitute y and y' into the differential equation.

$$\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n = 0$$

The series are in phase, so we'll just make the indices match.

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)x^k - \sum_{k=0}^{\infty} c_k x^k = 0$$

$$\sum_{k=0}^{\infty} [c_{k+1}(k+1) - c_k]x^k = 0$$

Solve the recurrence relation.

$$c_{k+1}(k+1) - c_k = 0 \qquad k = 0,\ 1,\ 2,\ 3,\ \dots$$

$$c_{k+1} = \frac{c_k}{k+1}$$

Now we'll start plugging in values $k = 0,\ 1,\ 2,\ 3,\ \dots$.

$$k = 0 \qquad c_1 = c_0$$

$$k = 1 \qquad c_2 = \frac{c_0}{2}$$

$$k = 2 \qquad c_3 = \frac{c_0}{6}$$

$$k = 3 \qquad c_4 = \frac{c_0}{24}$$

$$\vdots$$

$$c_k = \frac{c_0}{k!} \text{ for } k = 1, 2, 3, \ldots$$

Using these values, the solution is

$$y = c_0 + c_0 x + \frac{c_0}{2}x^2 + \frac{c_0}{6}x^3 + \frac{c_0}{24}x^4 + \ldots \frac{c_0}{(k-1)!}x^{k-1} + \frac{c_0}{k!}x^k$$

$$y = c_0\left(1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \ldots + \frac{1}{(k-1)!}x^{k-1} + \frac{1}{k!}x^k\right)$$

And the pattern that seems to be emerging is

$$y = c_0 \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

7. A. Apply the Laplace transform to both sides of the differential equation.

$$y'' - 7y' + 10y = 3t^2$$

$$s^2Y(s) - sy(0) - y'(0) - 7[sY(s) - y(0)] + 10Y(s) = 3\left(\frac{2}{s^3}\right)$$

Plug in the initial conditions $y(0) = 0$ and $y'(0) = 4$.

$$s^2Y(s) - 4 - 7sY(s) + 10Y(s) = \frac{6}{s^3}$$

$$(s^2 - 7s + 10)Y(s) = \frac{6}{s^3} + 4$$

$$Y(s) = \frac{4s^3 + 6}{s^3(s-2)(s-5)}$$

Use a partial fractions decomposition.

$$\frac{4s^3 + 6}{s^3(s-2)(s-5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-2} + \frac{E}{s-5}$$

Multiply through both sides of the decomposition equation by the denominator from the left side.

$$4s^3 + 6 = As^2(s-2)(s-5) + Bs(s-2)(s-5) + C(s-2)(s-5)$$

$$+Ds^3(s-5) + Es^3(s-2)$$

$$4s^3 + 6 = As^4 - 7As^3 + 10As^2 + Bs^3 - 7Bs^2 + 10Bs + Cs^2 - 7Cs + 10C$$

$$+Ds^4 - 5Ds^3 + Es^4 - 2Es^3$$

$$4s^3 + 6 = (A + D + E)s^4 + (-7A + B - 5D - 2E)s^3 + (10A - 7B + C)s^2$$

$$+(10B - 7C)s + 10C$$

Equate coefficients to create a system of equation.

$$A + D + E = 0$$

$$-7A + B - 5D - 2E = 4$$

$$10A - 7B + C = 0$$

$$10B - 7C = 0$$

$$10C = 6$$

Solving this system gives $A = 117/500$, $B = 21/50$, $C = 3/5$, $D = -19/12$, and $E = 506/375$, and plugging these into the decomposition gives

$$Y(s) = \frac{117}{500}\left(\frac{1}{s}\right) + \frac{21}{50}\left(\frac{1}{s^2}\right) + \frac{3}{5}\left(\frac{1}{s^3}\right) - \frac{19}{12}\left(\frac{1}{s-2}\right) + \frac{506}{375}\left(\frac{1}{s-5}\right)$$

Applying an inverse transform gives us the solution to the second order nonhomogeneous differential equation.

$$Y(s) = \frac{117}{500} + \frac{21}{50}t + \frac{3}{10}t^2 - \frac{19}{12}e^{2t} + \frac{506}{375}e^{5t}$$

8. D. From a table of Laplace transforms, we know

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Substitute these into the differential equation.

$$y'' + 3y' + 6y = g(t)$$

$$s^2Y(s) - sy(0) - y'(0) + 3(sY(s) - y(0)) + 6Y(s) = G(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 3sY(s) - 3y(0) + 6Y(s) = G(s)$$

Plug in the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$s^2Y(s) + 3sY(s) + 6Y(s) = G(s)$$

$$(s^2 + 3s + 6)Y(s) = G(s)$$

$$Y(s) = \frac{G(s)}{s^2 + 3s + 6}$$

Complete the square.

$$Y(s) = \frac{G(s)}{s^2 + 3s + \frac{9}{4} - \frac{9}{4} + 6}$$

$$Y(s) = \frac{G(s)}{\left(s - \frac{3}{2}\right)^2 - \frac{9}{4} + 6}$$

$$Y(s) = \frac{G(s)}{\left(s - \frac{3}{2}\right)^2 + \frac{15}{4}}$$

$$Y(s) = \frac{2}{\sqrt{15}}G(s)\left(\frac{\frac{\sqrt{15}}{2}}{\left(s - \frac{3}{2}\right)^2 + \left(\frac{\sqrt{15}}{2}\right)^2}\right)$$

This is similar to the transform formula

$$\mathscr{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

Now we can identify $a = 3/2$ and $b = \sqrt{15}/2$. So the inverse transform is

$$\mathscr{L}^{-1}\left(\frac{\frac{\sqrt{15}}{2}}{\left(s-\frac{3}{2}\right)^2+\left(\frac{\sqrt{15}}{2}\right)^2}\right) = e^{\frac{3}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

The inverse transform of $G(s)$ is $g(t)$, so for the convolution integral we'll use the functions

$$f(t) = e^{\frac{3}{2}t}\sin\left(\frac{\sqrt{15}}{2}t\right)$$

$$g(t) = g(t)$$

Plugging these into the convolution integral, we get

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{\frac{3}{2}\tau}\sin\left(\frac{\sqrt{15}}{2}\tau\right)g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$ gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t)=\frac{2}{\sqrt{15}}\int_0^t e^{\frac{3}{2}\tau}\sin\left(\frac{\sqrt{15}}{2}\tau\right)g(t-\tau)\ d\tau$$

$$y(t)=\frac{2\sqrt{15}}{15}\int_0^t e^{\frac{3}{2}\tau}\sin\left(\frac{\sqrt{15}}{2}\tau\right)g(t-\tau)\ d\tau$$

9. Start by working on the complementary solution. Find $|A-\lambda I|$.

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -\lambda & -3 \\ 0 & -3 & -\lambda \end{vmatrix}$$

$$(3-\lambda)\begin{vmatrix} -\lambda & -3 \\ -3 & -\lambda \end{vmatrix}$$

$$(3-\lambda)[(-\lambda)(-\lambda)-(-3)(-3)]$$

$$(3-\lambda)(\lambda^2-9)$$

$$-\lambda^3+3\lambda^2+9\lambda-27$$

Solve the characteristic equation.

$$-\lambda^3+3\lambda^2+9\lambda-27=0$$

$$\lambda^3-3\lambda^2-9\lambda+27=0$$

$$(\lambda+3)(\lambda-3)(\lambda-3)=0$$

Then for these Eigenvalues, $\lambda_1=-3$ and $\lambda_2=\lambda_3=3$, we find

$$A-(-3)I=\begin{bmatrix}6&0&0\\0&3&-3\\0&-3&3\end{bmatrix} \qquad A-3I=\begin{bmatrix}0&0&0\\0&-3&-3\\0&-3&-3\end{bmatrix}$$

Put both matrices in reduced row-echelon form.

$$\begin{bmatrix}6&0&0\\0&3&-3\\0&-3&3\end{bmatrix} \qquad \begin{bmatrix}0&0&0\\0&-3&-3\\0&-3&-3\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix} \qquad \begin{bmatrix}0&1&1\\0&0&0\\0&0&0\end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1=0 \qquad k_2+k_3=0$$

$$k_2-k_3=0$$

From the first system, we get $k_1=0$ and $k_2=k_3$. From the second system, we get $k_2=-k_3$.

$$\overrightarrow{k_1}=\begin{bmatrix}0\\1\\1\end{bmatrix} \qquad \overrightarrow{k_2}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

To get another linearly independent Eigenvector associated with $\lambda_2=\lambda_3=3$, we'll use

$$\overrightarrow{k_3}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t} \qquad \vec{x_2} = \vec{k_2}e^{\lambda_2 t} \qquad \vec{x_3} = \vec{k_3}e^{\lambda_3 t}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-3t} \qquad \vec{x_2} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{3t} \qquad \vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{3t}$$

So the complementary solution to the nonhomogeneous system will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{3t}$$

Now solve for the particular solution. Rewrite the forcing function vector as

$$F = \begin{bmatrix} -e^{2t} \\ \cos t \\ 3t \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} t$$

The guess for the particular solution should be

$$\vec{x_p} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{2t} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{2t}$$

$$\vec{x}_p = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t$$

Solve the polynomial part. differential equations $\vec{x}' = A\vec{x} + F$. Starting with the polynomial part, we get

$$\vec{x}_p{}' = A\vec{x}_p + F$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \left[\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right] + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} t$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} b_1t + a_1 \\ b_2t + a_2 \\ b_3t + a_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} t$$

This matrix equation can be rewritten as a system of equations.

$$b_1 = 3(b_1t + a_1) + 0(b_2t + a_2) + 0(b_3t + a_3) + 0t$$

$$b_2 = 0(b_1t + a_1) + 0(b_2t + a_2) - 3(b_3t + a_3) + 0t$$

$$b_3 = 0(b_1t + a_1) - 3(b_2t + a_2) + 0(b_3t + a_3) + 3t$$

The system simplifies to

$$b_1 = 3b_1t + 3a_1$$

$$b_2 = -3b_3t - 3a_3$$

$$b_3 = -3b_2t + 3a_2 + 3t$$

These equations can each be broken into its own system.

$$3b_1 = 0 \qquad -3b_3 = 0 \qquad -3b_2 + 3 = 0$$

$$3a_1 - b_1 = 0 \qquad -3a_3 - b_2 = 0 \qquad 3a_2 - b_3 = 0$$

This system gives $\overrightarrow{a} = (a_1, a_2, a_3) = (0,0,-1/3)$ and $\overrightarrow{b} = (b_1, b_2, b_3) = (0,1,0)$. Now solve the exponential part of the particular solution.

$$\overrightarrow{x_p}' = A\overrightarrow{x_p} + F$$

$$\begin{bmatrix} 2c_1 \\ 2c_2 \\ 2c_3 \end{bmatrix} e^{2t} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{2t} + \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$$

This matrix equation can be rewritten as a system of equations.

$$2c_1 = 3c_1 + 0c_2 + 0c_3 - 1$$

$$2c_2 = 0c_1 + 0c_2 - 3c_3 + 0$$

$$2c_3 = 0c_1 - 3c_2 + 0c_3 + 0$$

The system simplifies to

$$2c_1 = 3c_1 - 1$$

$$2c_2 = -3c_3$$

$$2c_3 = -3c_2$$

This system gives $\vec{c} = (c_1, c_2, c_3) = (1,0,0)$. Now solve the trigonometric part of the particular solution.

$$\vec{x_p}' = A\vec{x_p} + F$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \cos t - \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \sin t = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \left[\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t \right] + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos t$$

$$\begin{bmatrix} e_1 \cos t - d_1 \sin t \\ e_2 \cos t - d_2 \sin t \\ e_3 \cos t - d_3 \sin t \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} e_1 \sin t + d_1 \cos t \\ e_2 \sin t + d_2 \cos t \\ e_3 \sin t + d_3 \cos t \end{bmatrix} + \begin{bmatrix} 0 \\ \cos t \\ 0 \end{bmatrix}$$

This matrix equation can be rewritten as a system of equations.

$$e_1 \cos t - d_1 \sin t = 3(e_1 \sin t + d_1 \cos t)$$

$$+0(e_2 \sin t + d_2 \cos t) + 0(e_3 \sin t + d_3 \cos t) + 0$$

$$e_2 \cos t - d_2 \sin t = 0(e_1 \sin t + d_1 \cos t)$$

$$+0(e_2 \sin t + d_2 \cos t) - 3(e_3 \sin t + d_3 \cos t) + \cos t$$

$$e_3 \cos t - d_3 \sin t = 0(e_1 \sin t + d_1 \cos t)$$

$$-3(e_2 \sin t + d_2 \cos t) + 0(e_3 \sin t + d_3 \cos t) + 0$$

The system simplifies to

$$e_1 \cos t - d_1 \sin t = 3e_1 \sin t + 3d_1 \cos t$$

$$e_2 \cos t - d_2 \sin t = -3e_3 \sin t - 3d_3 \cos t + \cos t$$

$$e_3 \cos t - d_3 \sin t = -3e_2 \sin t - 3d_2 \cos t$$

These equations can each be broken into its own system.

$$e_1 = 3d_1 \qquad e_2 = -3d_3 + 1 \qquad e_3 = -3d_2$$

$$-d_1 = 3e_1 \qquad -d_2 = -3e_3 \qquad -d_3 = -3e_2$$

This system gives $\overrightarrow{d} = (d_1, d_2, d_3) = (0,0,3/10)$ and $\overrightarrow{e} = (e_1, e_2, e_3) = (0,1/10,0)$. So the particular solution is

$$\overrightarrow{x_p} = \begin{bmatrix} 0 \\ \frac{1}{10} \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 0 \\ \frac{3}{10} \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{3} \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} 0 + 0 + e^{2t} + 0 + 0 \\ \frac{1}{10} \sin t + 0 + 0 + t + 0 \\ 0 + \frac{3}{10} \cos t + 0 + 0 - \frac{1}{3} \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} e^{2t} \\ \frac{1}{10} \sin t + t \\ \frac{3}{10} \cos t - \frac{1}{3} \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x_c} + \vec{x_p}$$

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{3t} + \begin{bmatrix} e^{2t} \\ \frac{1}{10}\sin t + t \\ \frac{3}{10}\cos t - \frac{1}{3} \end{bmatrix}$$

10. If the characteristic equation associated with the homogeneous differential equation is

$$r(r-2)(r+3)(r+3)(r^2+2r+7)^4 = 0$$

then we know the equation has two distinct real roots $r_1 = 0$ and $r_2 = 2$, two equal real roots $r_3 = r_4 = -3$, and four pairs of complex conjugate roots, $r_5 = r_6 = -1 \pm \sqrt{6}i$, $r_7 = r_8 = -1 \pm \sqrt{6}i$, $r_9 = r_{10} = -1 \pm \sqrt{6}i$, and $r_{11} = r_{12} = -1 \pm \sqrt{6}i$.

Because the complex root portion of the factored characteristic polynomial is $(r^2+2r+7)^4$, these complex roots have multiplicity four, which means we'll have four pairs of complex roots,

$e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$

$te^{\alpha t}\cos(\beta t)$ and $te^{\alpha t}\sin(\beta t)$

$t^2e^{\alpha t}\cos(\beta t)$ and $t^2e^{\alpha t}\sin(\beta t)$

$$t^3e^{\alpha t}\cos(\beta t) \text{ and } t^3e^{\alpha t}\sin(\beta t)$$

With $\alpha = -1$ and $\beta = \sqrt{6}$, the complex conjugate solution pairs will be

$$e^{-t}\cos(\sqrt{6}t) \text{ and } e^{-t}\sin(\sqrt{6}t)$$

$$te^{-t}\cos(\sqrt{6}t) \text{ and } te^{-t}\sin(\sqrt{6}t)$$

$$t^2e^{-t}\cos(\sqrt{6}t) \text{ and } t^2e^{-t}\sin(\sqrt{6}t)$$

$$t^3e^{-t}\cos(\sqrt{6}t) \text{ and } t^3e^{-t}\sin(\sqrt{6}t)$$

The distinct real roots portion of the solution will be $c_1e^{0t} + c_2e^{2t}$, or $c_1 + c_2e^{2t}$, and the equal real roots portion will be $c_3e^{-3t} + c_4te^{-3t}$. The complex conjugate roots portion will be

$$c_5e^{-t}\cos(\sqrt{6}t) + c_6e^{-t}\sin(\sqrt{6}t)$$

$$+c_7e^{-t}\cos(\sqrt{6}t) + c_8e^{-t}\sin(\sqrt{6}t)$$

$$+c_9e^{-t}\cos(\sqrt{6}t) + c_{10}e^{-t}\sin(\sqrt{6}t)$$

$$+c_{11}e^{-t}\cos(\sqrt{6}t) + c_{12}e^{-t}\sin(\sqrt{6}t)$$

Therefore the general solution of the homogeneous linear differential equation is

$$y(t) = c_1 + c_2e^{2t} + c_3e^{-3t} + c_4te^{-3t} + c_5e^{-t}\cos(\sqrt{6}t) + c_6e^{-t}\sin(\sqrt{6}t)$$

$$+c_7e^{-t}\cos(\sqrt{6}t) + c_8e^{-t}\sin(\sqrt{6}t)$$

$$+c_9e^{-t}\cos(\sqrt{6}t) + c_{10}e^{-t}\sin(\sqrt{6}t)$$

$$+c_{11}e^{-t}\cos(\sqrt{6}t)+c_{12}e^{-t}\sin(\sqrt{6}t)$$

11. The function $f(x)=x^4$ is an even function, which means the Fourier sine series will be 0. Therefore, we don't need to calculate B_n, only A_0,

$$A_0=\frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0=\frac{1}{L}\int_0^L x^4\ dx$$

$$A_0=\frac{1}{5L}x^5\Big|_0^L$$

$$A_0=\frac{L^4}{5}$$

and A_n.

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=\frac{2}{L}\int_0^L x^4\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u=x^4$, $du=4x^3\ dx$, $dv=\cos(n\pi x/L)\ dx$, and $v=(L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{n\pi}\left[x^4 \sin\left(\frac{n\pi x}{L}\right) - 4\int x^3 \sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x^3$, $du = 3x^2\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = \frac{2}{n\pi}\left[x^4 \sin\left(\frac{n\pi x}{L}\right) + \frac{4Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. - \frac{12L}{n\pi}\int x^2 \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{n\pi}\left[x^4 \sin\left(\frac{n\pi x}{L}\right) + \frac{4Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{12L^2x^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. + \frac{24L^2}{(n\pi)^2}\int x \sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = \frac{2}{n\pi}\left[x^4 \sin\left(\frac{n\pi x}{L}\right) + \frac{4Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{12L^2x^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left. - \frac{24L^3x}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right) + \frac{24L^3}{(n\pi)^3}\int \cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$A_n = \frac{2}{n\pi}\left[x^4 \sin\left(\frac{n\pi x}{L}\right) + \frac{4Lx^3}{n\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{12L^2x^2}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\right.$$

$$\left.-\frac{24L^3x}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right) + \frac{24L^4}{(n\pi)^4}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$A_n = \frac{2}{n\pi}\left[L^4\sin(n\pi) + \frac{4L^4}{n\pi}\cos(n\pi) - \frac{12L^4}{(n\pi)^2}\sin(n\pi) - \frac{24L^4}{(n\pi)^3}\cos(n\pi)\right.$$

$$\left.+\frac{24L^4}{(n\pi)^4}\sin(n\pi)\right]$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$, and $\cos(n\pi) = (-1)^n$, so the expression for A_n simplifies to

$$A_n = \frac{2}{n\pi}\left[\frac{4L^4}{n\pi}(-1)^n - \frac{24L^4}{(n\pi)^3}(-1)^n\right]$$

$$A_n = \frac{8L^4(-1)^n(n\pi)^2 - 48L^4(-1)^n}{(n\pi)^4}$$

$$A_n = \frac{8L^4(-1)^n((n\pi)^2 - 6)}{(n\pi)^4}$$

Then the Fourier series for $f(x) = x^4$ on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^4}{5} + \sum_{n=1}^{\infty} \frac{8L^4(-1)^n((n\pi)^2 - 6)}{(n\pi)^4} \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^4}{5} + \frac{8L^4}{\pi^4} \sum_{n=1}^{\infty} \frac{(-1)^n((n\pi)^2 - 6)}{n^4} \cos\left(\frac{n\pi x}{L}\right)$$

12. We're solving the heat equation with two boundary conditions and an initial condition.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $u(0,t) = 0$ and $u(L, t) = 100$

$$u(x,0) = 25$$

Equilibrium temperature is

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E(x) = 0 + \frac{100 - 0}{L}x$$

$$u_E(x) = \frac{100}{L}x$$

Next, we'll find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L \left(25 - \frac{100}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{50}{L}\int_0^L \left(1 - \frac{4}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{50}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{200}{L^2}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{50}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{200x}{Ln\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{200}{Ln\pi}\int \cos\left(\frac{n\pi x}{L}\right)\ dx\Bigg|_0^L$$

$$B_n = -\frac{50}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{200x}{Ln\pi}\cos\left(\frac{n\pi x}{L}\right) - \frac{200}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

Evaluate over the interval.

$$B_n = -\frac{50}{n\pi}\cos(n\pi) + \frac{200}{n\pi}\cos(n\pi) - \frac{200}{(n\pi)^2}\sin(n\pi) + \frac{50}{n\pi}$$

$$B_n = -\frac{50}{n\pi}(-1)^n + \frac{200}{n\pi}(-1)^n + \frac{50}{n\pi}$$

$$B_n = \frac{50(4(-1)^n - (-1)^n + 1)}{n\pi}$$

$$B_n = \frac{50(4(-1)^n + (-1)^{n+1} + 1)}{n\pi}$$

Then the solution to this heat equation is

$$u(x,t) = T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = \frac{100}{L}x + \sum_{n=1}^{\infty} \frac{50(4(-1)^n + (-1)^{n+1} + 1)}{n\pi} \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = \frac{100}{L}x + \frac{50}{\pi}\sum_{n=1}^{\infty} \frac{4(-1)^n + (-1)^{n+1} + 1}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

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