Differential Equations Formulas

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First order equations

Classifying differential equations

Partial derivative: The derivative of the function with respect to one of the multiple variables in which the function is defined.

Partial differential equations (PDEs): Equations defined in terms of partial derivatives.

Ordinary derivative: The derivative of a function in a single variable.

Derivative notation for $y(x)$:

$y'(x)$

y'

$\frac{dy}{dx}$ Leibniz notation

Order of the differential equation: Equivalent to the degree of the highest-degree derivative that appears in the equation.

Linear differential equation: Given that $p_i(x)$ and $q(x)$ are functions of x,

$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \dots + p_1(x)y'(x) + p_0(x)y(x) = q(x)$$

Homogeneity of a linear equation

Homogeneous with $q(x) = 0$

Nonhomogeneous with $q(x) \neq 0$

Linear equations

First order linear equation, given that $P(x)$ and $Q(x)$ are continuous functions:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

Solution to the homogeneous linear equation (separable equation):

$$\frac{dy}{dx} + P(x)y = 0 \qquad\qquad y = Ce^{K(x)}, \text{ where } K(x) = -\int P(x)\ dx$$

Integrating factor method:

1. Make sure that the equation matches the general form of a first order linear differential equation

2. From the equation, identify $P(x)$ and $Q(x)$.

3. Find the equation's **integrating factor**, $I(x) = e^{\int P(x)\ dx}$.

4. Multiply through the linear equation by the integrating factor.

5. Reverse the product rule to rewrite the left side of the resulting equation as $(d/dx)[yI(x)]$.

6. Integrate both sides of the equation to find $y = y(x)$.

Initial value problems

Initial condition: The value of the solution at a specific point, $y(x_0) = y_0$.

Initial value problem (IVP): A differential equations problem in which we're asked to use some given initial condition, or set of conditions, in order to find the particular solution to the differential equation.

Steps to solve an IVP:

1. Find the general solution that contains the constant of integration C.

2. Substitute the initial condition, $x = x_0$ and $y = y_0$, into the general solution to find the associated value of C.

3. Restate the general solution, and include the value of C found in step 2. This will be the particular solution of the differential equation.

Separable equations

First order separable equation

$$N(y)\frac{dy}{dx} = M(x)$$

Steps to solve a separable equation:

1. If necessary, rewrite the equation in Leibniz notation.

2. Separate variables with y terms on the left and x terms on the right

3. Integrate both sides of the equation, adding C to the right side

4. If possible, solve the solution equation specifically for y.

Substitutions

Substitution, change of variable: For an equation $y' = F(ax + by)$, substitute $u = ax + by$ and $u' = a + by'$.

Bernoulli equations

Bernoulli equation, where $p(x)$ and $q(x)$ are continuous, and where n is any real number:

$$y' + p(x)y = q(x)y^n$$

Change of variable in Bernoulli equations when $n \neq 0, 1$:

1. If the Bernoulli equation isn't already given in standard form, rewrite it in standard form.

2. Divide both sides of the equation by y^n.

3. Identify the substitution $v = y^{1-n}$, implicitly differentiate to find $v' = (1 - n)y^{-n}y'$, and substitute into the Bernoulli equation.

4. Put the resulting equation into the standard form of a linear differential equation.

5. Find the solution to the linear differential equation, then back-substitute for v and solve for y.

Homogeneous equations

Homogeneous equation:

$$y' = F\left(\frac{y}{x}\right)$$

Solving homogeneous equations:

1. If the homogeneous equation isn't already given in standard form, rewrite it in standard form.

2. Substitute $v = y/x$ and $y' = v + xv'$.

3. Solve the resulting separable equation.

4. Back-substitute for $v = y/x$.

5. If possible, solve explicitly for y to find the solution to the homogeneous differential equation.

Exact equations

Exact equation:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

Test for an exact equation: If the partial derivative of M with respect to y is equal to the partial derivative of N with respect to x, then the differential equation is exact.

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Solving an exact equation:

1. Verify that $M_y = N_x$ to confirm the differential equation is exact.

2. Use $\Psi = \int M(x,y)\ dx + h(y)$ or $\Psi = \int N(x,y)\ dy + h(x)$ to find $\Psi(x,y)$, including a value for $h(y)$ or $h(x)$.

3. Find $\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\ dx\right) + h'(y) = N(x,y)$, or find $\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}\left(\int N(x,y)\ dy\right) + h'(x) = M(x,y)$.

4. Integrate both sides of the equation to solve for $h(y)$ or $h(x)$.

5. Plug $h(y)$ or $h(x)$ back into Ψ in step 2.

6. Set $\Psi(x,y) = c$ to get the implicit solution.

Second order equations

Second order linear homogeneous equations

Second order linear homogeneous equation:

$$y'' + Ay' + By = 0$$

Characteristic equation:

$$r^2 + Ar + B = 0$$

General solution:

Roots of the characteristic equation	**General solution**
Distinct real roots	$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$
Equal real roots	$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$
Complex conjugate roots	$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$
	where $r = \alpha \pm \beta i$

Reduction of order

Solutions of the second order linear homogeneous equation:

$y(x) = c_1 y_1 + c_2 y_2$	**Solutions of the equation**
$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	$\{y_1, y_2\} = \{e^{r_1 x}, e^{r_2 x}\}$
$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$	$\{y_1, y_2\} = \{e^{r_1 x}, x e^{r_2 x}\}$
$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$	$\{y_1, y_2\} = \{e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)\}$

Reduction of order: Using a substitution to reduce the order of the homogeneous equation from a second order equation to a first order equation, in order to use one solution y_1 to find the other solution y_2.

Undetermined coefficients for nonhomogeneous equations

Undetermined coefficients: A method we can use to find the general solution $y(x)$ to a second order nonhomogeneous differential equation.

General solution to the nonhomogeneous equation: The sum of the complementary solution $y_c(x)$ and the particular solution $y_p(x)$.

$$y(x) = y_c(x) + y_p(x)$$

Complementary solution: The solution we find when we solve the characteristic equation (from the associated homogeneous equation).

Particular solution: The solution that addresses the $g(x)$ portion of the differential equation.

Guesses for the particular solution:

$g(x)$	**Guess**
$x^2 + 1$	$Ax^2 + Bx + C$
$2x^3 - 3x + 4$	$Ax^3 + Bx^2 + Cx + D$
e^{3x}	Ae^{3x}
$3\sin(4x)$	$A\sin(4x) + B\cos(4x)$
$2\cos(4x)$	$A\sin(4x) + B\cos(4x)$
$3\sin(4x) + 2\cos(4x)$	$A\sin(4x) + B\cos(4x)$
$x^2 + 1 + e^{3x}$	$Ax^2 + Bx + C + De^{3x}$

$e^{3x}\cos(\pi x)$ $\quad\quad$ $Ae^{3x}(B\cos(\pi x)+C\sin(\pi x)) \to$

$$e^{3x}(AB\cos(\pi x)+AC\sin(\pi x)) \to$$

$$e^{3x}(A\cos(\pi x)+B\sin(\pi x))$$

$(x^2+1)\cos(-2x)$ $\quad\quad$ $(Ax^2+Bx+C)(D\sin(-2x)+E\cos(-2x)) \to$

$$(ADx^2+BDx+CD)\sin(-2x)+(AEx^2+BEx+CE)\cos(-2x) \to$$

$$(Ax^2+Bx+C)\sin(-2x)+(Dx^2+Ex+F)\cos(-2x)$$

Finding the general solution to the nonhomogeneous equation:

1. Substitute $g(x)=0$ to rewrite the nonhomogeneous equation as a homogeneous equation.

2. Use the associated characteristic equation to solve the homogeneous equation, generating the complementary solution $y_c(x)$.

3. Make a guess for a particular solution.

4. Fix any overlap between the guess for the particular solution, and the complementary solution from step 2.

5. Find the first and second derivatives of the non-overlapping guess for the particular solution.

6. Substitute the derivatives into the original differential equation.

7. Equate coefficients to find the values of any constants.

8. Substitute the constant values into the guess to generate a particular solution $y_p(x)$.

9. Sum the complementary and particular solutions to get the general solution, $y(x) = y_c(x) + y_p(x)$.

Variation of parameters for nonhomogeneous equations

System of equations for variation of parameters:

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = g(x)$$

Fundamental solution sets and the Wronskian

Fundamental set of solutions: A solution set with a non-zero Wronskian.

Wronskian: The determinant of the set of solutions and their derivatives.

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)$$

Linearly independent solutions: Solutions that aren't multiples of each other; their Wronskian will be non-zero.

Variation of parameters with the Wronskian

Particular solution in terms of the Wronskian:

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)}\ dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)}\ dx$$

Modeling with differential equations

Direction fields and solution curves

Direction field: A graph made of many small arrows, each of which approximates the slope of the curve in the area where the arrow is plotted; a geometric way of displaying the same information in the differential equation

Solution curve, integral curve: A geometric way of displaying the same information in the solution equation.

Isoclines: The family of curves we get by choosing different values for $y' = c$.

Intervals of validity

Interval of validity, interval of existence, interval of definition, or the domain of the solution: The interval on which the solution curve is valid.

Intervals of validity for first order linear equations: Discontinuities of $P(x)$ and/or $Q(x)$ create distinct intervals of validity.

Euler's method

Euler's method: An iterative process in which we repeat the same step over and over; the more steps we use, the better approximation we'll get.

Step size: The distance Δt between two successive values of t at which y is calculated.

Euler's formula:

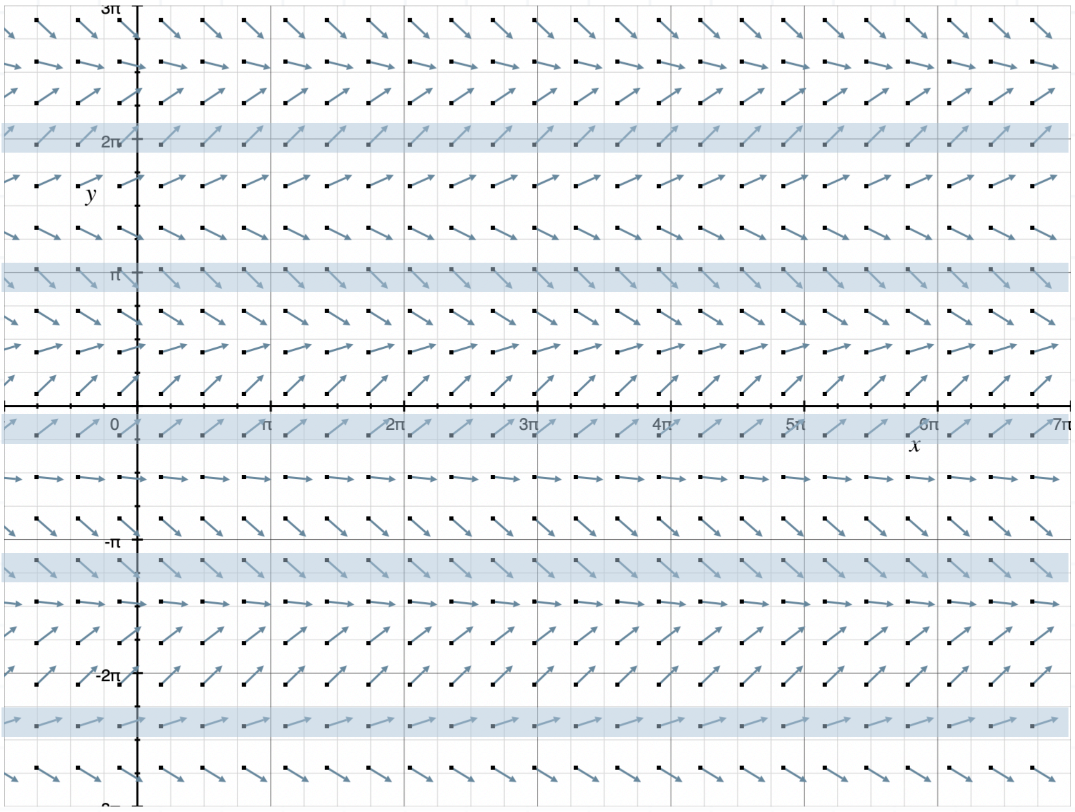
$$y_1 = y_0 + [f(t_0, y_0)]\Delta t$$

Autonomous equations and equilibrium solutions

Autonomous equations: Differential equations in which the independent variable never appears explicitly in the equation, but instead only as part of the derivative.

$$\frac{dy}{dt} = f(y)$$

Every horizontal row of the direction field of the autonomous equation contains a set of entirely parallel direction arrows.



Critical points, equilibrium points, stationary points: The value(s) y_0 that satisfy $f(y_0)=0$ in the autonomous equation.

Critical solutions, equilibrium solutions: The solution curves through the direction field that are perfectly horizontal lines, representing the curves passing through the critical points of the autonomous equation.

Stable equilibrium solutions, attractors: Equilibrium solutions that attract solution curves.

Unstable equilibrium solutions, repellers: Equilibrium solutions that repel solution curves.

Semi-stable equilibrium solutions: Equilibrium solutions that attract solution curves on one side and repel them on the other.

The logistic equation

Logistic growth equation: Models a slow-fast-slow growth pattern, where dP/dt is the rate of growth of the population P over time t, k is the growth constant, and M is the carrying capacity. Equilibrium solutions exist at $P=0$ and $P=M$.

$$\frac{dP}{dt}=kP\left(1-\frac{P}{M}\right)$$

Carrying capacity, saturation level: The largest population that can be sustained by the surrounding environment.

Predator-prey systems

Cooperative systems: both populations are increasing in size (the mixed xy terms are both positive).

$$\frac{dx}{dt}=ax+bxy \qquad \frac{dy}{dt}=cy+dxy$$

Competitive systems: both populations are decreasing in size (the mixed xy terms are both negative).

$$\frac{dx}{dt}=ax-bxy \qquad \frac{dy}{dt}=cy-dxy$$

Predator-prey systems: the size of one population is increasing, while the size of the other population is decreasing (one mixed xy term is positive while the other is negative).

Population x is increasing while population y is decreasing

$$\frac{dx}{dt} = ax + bxy \qquad \frac{dy}{dt} = cy - dxy$$

Population y is increasing while population x is decreasing

$$\frac{dx}{dt} = ax - bxy \qquad \frac{dy}{dt} = cy + dxy$$

Equilibrium solutions for a system of two populations:

1. The system will have a stable equilibrium solution $(0,0)$, where both populations x and y are at 0 and will remain at 0 indefinitely.

2. If the system has an equilibrium solution at $(a,0)$, it means population x is stable at size a, while population y is at 0.

3. If the system has an equilibrium solution at $(0,b)$, it means population y is stable at size b, while population x is at 0.

4. The most interesting solution is an equilibrium solution at (a, b), where population x has size a, which is supporting in perfect balance population y at size b, and vice versa.

Exponential growth and decay

Exponential growth equation: $P(t) = P_0e^{kt}$, which is the solution to the initial value problem

$$\frac{dP}{dt} = kP, \qquad P(t_0) = P_0$$

Constant of proportionality: If $k > 0$ we call it a growth constant, whereas if $k < 0$ we call it a decay constant.

Mixing problems

Amount of salt in the tank:

$$\frac{dy}{dt} = (\text{salt input rate}) - (\text{salt output rate})$$

$$\frac{dy}{dt} = C_1 r_1 - C_2 r_2$$

$$C_2 = \frac{y(t)}{\text{volume of brine in the tank at time } t}$$

Newton's Law of Cooling

Newton's Law of Cooling: Models the way in which a warm object in a cooler environment cools down until it matches the temperature of its environment. The rate at which the object cools is proportional to the difference between the object and the environment around it.

$$\frac{dT}{dt} = -k(T - T_a) \text{ with } T(0) = T_0$$

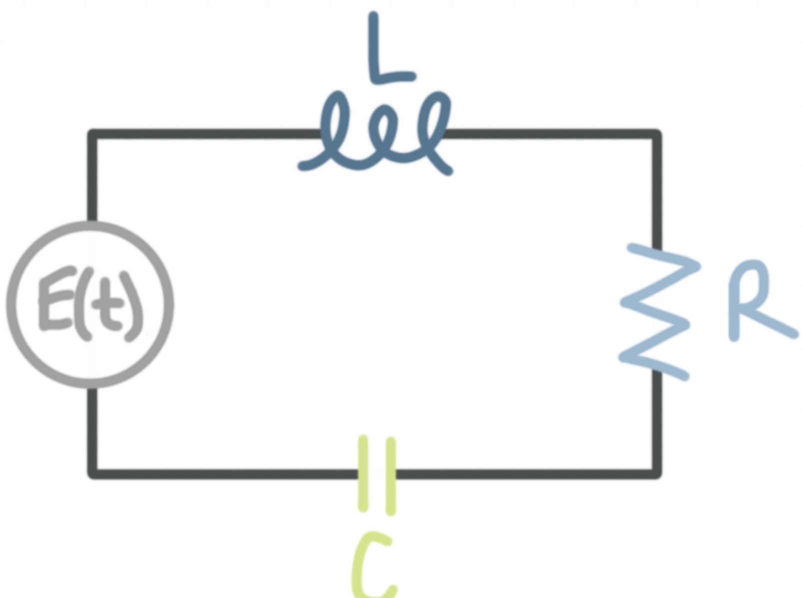
$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

where T is the temperature over time t, k is the decay constant, T_a is the ambient temperature, and T_0 is the initial temperature of the hot object.

Electrical series circuits

Components of the LRC series circuit:

- The inductance L, measured as a constant in henries h
- The resistance R, measured as a constant in ohms Ω
- The capacitance C, measured as a constant in farads f
- The impressed voltage $E(t)$



Kirchhoff's second law: The impressed voltage $E(t)$ on a closed loop must be equal to the sum of the voltage drops in the loop.

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Voltage drops:

Across the inductor: $L(di/dt)$

Across the resistor: Ri

Across the capacitor: $q(t)/C$

Vibrations:

Free vibrations: $E(t) = 0$

Forced vibrations: $E(t) \neq 0$

Transient and steady-state terms:

Transient terms: Terms that approach 0 as $t \to \infty$.

Steady-state terms: Any terms that remain after the transient terms drop away.

Damped circuits:

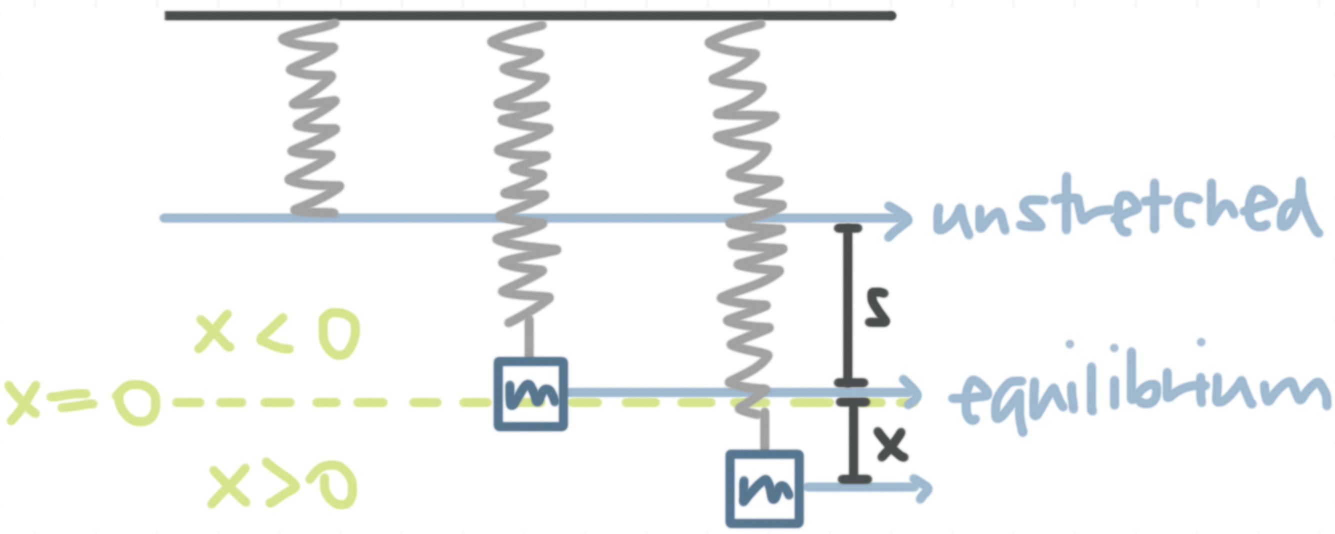
Overdamped circuit: $R^2 - 4L/C > 0$, distinct real roots

Critically damped circuit: $R^2 - 4L/C = 0$, equal real roots

Underdamped circuit: $R^2 - 4L/C < 0$, complex conjugate roots

Spring and mass systems

Spring and mass system:



Hooke's Law: The spring exerts force on the mass as it tries to return to its unstretched state, and that force is proportional to the extra length of the spring between its unstretched state and its state of equilibrium with the mass.

$$F = ks$$

Free undamped motion, simple harmonic motion: The spring and mass system is operating in a vacuum, such that there are no other external forces exerting themselves on the system (like friction from the environment, or the decay of the spring over time).

Position of the mass over time in relation to equilibrium:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

Free damped motion:

$$\frac{d^2x}{dt^2} + 2\lambda \frac{dx}{dt} + \omega^2 x = 0$$

Overdamped: $\lambda^2 - \omega^2 > 0$

$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega^2}t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}t})$$

Critically damped: $\lambda^2 - \omega^2 = 0$

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

Underdamped: $\lambda^2 - \omega^2 < 0$

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}t + c_2 \sin\sqrt{\omega^2 - \lambda^2}t)$$

Driven motion:

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Series solutions

Power series basics

Power series centered at x_0, power series in $x - x_0$:

$$\sum_{n=0}^{\infty} c_n(x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + c_3(x - x_0)^3 + \dots$$

Power series centered at $x_0 = 0$, power series in x:

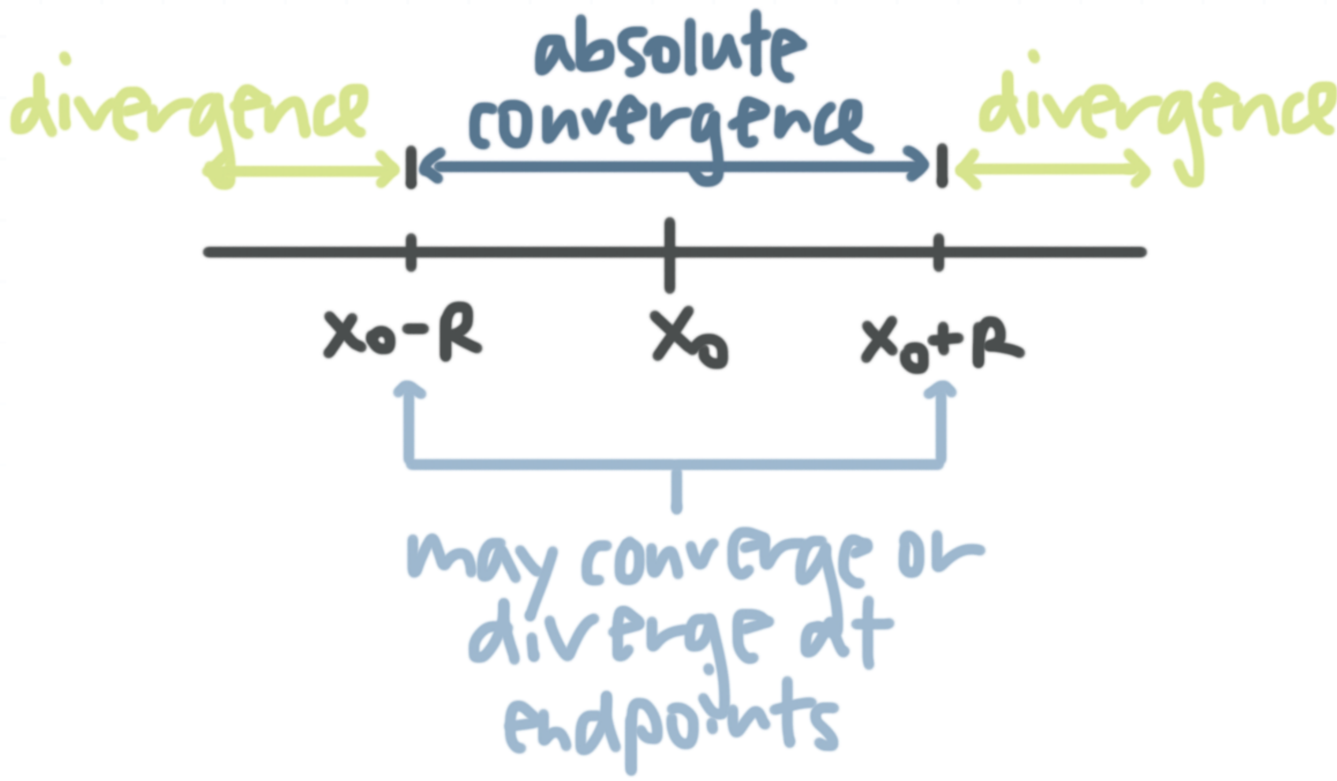
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Convergent power series: A power series is convergent when the limit exists.

$$\lim_{N\to\infty}\sum_{n=0}^{N} c_n(x-x_0)^n$$

Interval of convergence: The set of all real numbers x around x_0 for which the series converges.

Radius of convergence: The radius R of the interval of convergence.



Ratio test: The series converges absolutely if $L<1$, diverges if $L>1$, and inconclusive if $L=1$.

$$L=\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right|$$

Analytic at a point: A function is analytic at x_0 if the function can be represented by a power series in $x - x_0$ with any positive or infinite radius of convergence.

Taylor series:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots$$

Maclaurin series: The Taylor series centered at $x_0 = 0$.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

Common Maclaurin series:

Maclaurin series	**Interval of convergence**
$e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$	$(-\infty, \infty)$
$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$	$[-1,1]$

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n} \qquad (-\infty, \infty)$$

$$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1} \qquad (-\infty, \infty)$$

$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^{n} \qquad (-1, 1]$$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^{n} \qquad (-1, 1)$$

Derivatives of the power series in x:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \dots$$

Adding power series

Conditions for addition: Power series can only be added when

1. their indices start at the same values, and

2. the powers of x in each series are "in phase," which means that both series start with the same power of x.

Power series solutions

Power series solution around an ordinary point x_0:

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

Power series solution around the ordinary point $x_0 = 0$:

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

Ordinary point: The point $x = x_0$ is ordinary when we can define a Taylor series $q(x)/p(x)$ and $r(x)/p(x)$ around the point.

$$p(x)y'' + q(x)y' + r(x)y = 0$$

Power series solution and its derivatives:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \dots$$

Recurrence relation: An equation that defines values in the series in terms of previous values in the series.

Singular points and Frobenius' Theorem

Regular singular point: Occurs when both of these expressions are analytic at x_0, otherwise the singular point is irregular:

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)}$$

Frobenius' Theorem: There is at least one solution to $p(x)y'' + q(x)y' + r(x)y = 0$ of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}$$

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r} \qquad \text{if } x_0 = 0$$

Indicial equation: A quadratic equation whose indicial roots will be its solutions r_1 and r_2.

Case I: When $r_1 > r_2$ and $r_1 - r_2$ is not a positive integer, then the differential equation has two linearly independent solutions,

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \text{ with } c_0 \neq 0$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \text{ with } b_0 \neq 0$$

Case II: When $r_1 > r_2$ and $r_1 - r_2$ is a positive integer, then the differential equation has two linearly independent solutions,

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \text{ with } c_0 \neq 0$$

$$y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \text{ with } b_0 \neq 0$$

Case III: But when $r_1 = r_2$, the second solution does not include the constant C, and therefore the second solution will definitely contain the logarithm.

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \text{ with } c_0 \neq 0$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_1} \text{ with } b_0 \neq 0$$

Laplace transforms

The Laplace transform

Definition of the Laplace transform:

$$\mathscr{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

Table of transforms

Table of transforms:

$\mathscr{L}(f(t)) = F(s)$

$$\mathscr{L}(1) = \frac{1}{s}$$

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}} \qquad n = 1,\ 2,\ 3,\ \dots$$

$$\mathscr{L}(t^p) = \frac{\Gamma(p+1)}{s^{p+1}} \text{ for } p > -1 \qquad \Gamma(t) = \int_0^\infty e^{-x} x^{t-1}\ dx$$

$$\mathscr{L}(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

$$\mathscr{L}(t^{n-\frac{1}{2}}) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}} \qquad n = 1,2,3,\ldots$$

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2} \qquad \mathscr{L}(\cos(at)) = \frac{s}{s^2+a^2}$$

$$\mathscr{L}(t\sin(at)) = \frac{2as}{(s^2+a^2)^2}$$

$$\mathscr{L}(t\cos(at)) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$\mathscr{L}(\sin(at) - at\cos(at)) = \frac{2a^3}{(s^2+a^2)^2}$$

$$\mathscr{L}(\sin(at) + at\cos(at)) = \frac{2as^2}{(s^2+a^2)^2}$$

$$\mathscr{L}(\cos(at) - at\sin(at)) = \frac{s(s^2-a^2)}{(s^2+a^2)^2}$$

$$\mathscr{L}(\cos(at) + at\sin(at)) = \frac{s(s^2+3a^2)}{(s^2+a^2)^2}$$

$$\mathscr{L}(\sin(at+b)) = \frac{s\sin b + a\cos b}{s^2+a^2}$$

$$\mathscr{L}(\cos(at+b)) = \frac{s\cos b - a\sin b}{s^2+a^2}$$

$$\mathscr{L}(\sinh(at)) = \frac{a}{s^2-a^2}$$

$$\mathscr{L}(\cosh(at)) = \frac{s}{s^2-a^2}$$

$$\mathscr{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathscr{L}(e^{at}\sinh(bt)) = \frac{b}{(s-a)^2-b^2}$$

$$\mathscr{L}(e^{at}\cosh(bt)) = \frac{s-a}{(s-a)^2-b^2}$$

$$\mathscr{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

$n = 1, 2, 3, \ldots$

$$\mathscr{L}(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right)$$

$$\mathscr{L}(u_c(t)) = \mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

Heaviside function

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

Dirac delta function

$$\mathscr{L}(u_c(t)f(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(u_c(t)g(t)) = e^{-cs}\mathscr{L}(g(t+c))$$

$$\mathscr{L}(e^{ct}f(t)) = F(s-c)$$

$$\mathscr{L}(t^n f(t)) = (-1)F^{(n)}(s) \qquad n = 1,2,3,...$$

$$\mathscr{L}\left(\frac{1}{t}f(t)\right) = \int_s^\infty F(u)\ du$$

$$\mathscr{L}\left(\int_0^t f(v)\ dv\right) = \frac{F(s)}{s}$$

$$\mathscr{L}\left(\int_0^t f(t-\tau)g(\tau)\ d\tau\right) = F(s)G(s)$$

$$\mathscr{L}(f(t+T) = f(t)) = \frac{\int_0^T e^{-st}f(t)\ dt}{1-e^{-sT}}$$

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(f^{(n)}(t)) = s^nF(s) - s^{n-1}f(0) - s^{n-2}f'(0)... - sf^{n-2}(0) - f^{n-1}(0)$$

Combinations of transforms:

$$\mathscr{L}(af(t)) = aF(s)$$

$$\mathscr{L}(f(t) + g(t)) = F(s) + G(s)$$

Exponential type

Existence of the Laplace transform $\mathscr{L}(f(t)) = F(s)$: Guaranteed for $f(t)$ for $s > \alpha$ if $f(t)$ 1) is piecewise continuous on $[0,\infty)$ and 2) of exponential order.

Piecewise continuous: A function is piecewise continuous on $[0,\infty)$ if it includes a finite number of finite discontinuities (none of the discontinuities are infinite discontinuities, like at a vertical asymptote), and is otherwise continuous between these finite discontinuities.

Exponential type, exponential order: A function f is of exponential type α, if there exist constants α, $M > 0$, $T > 0$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t > T$; the function $f(t)$ is of exponential type if we can find any $e^{\alpha t}$ that grows faster than $f(t)$, causing the value of the fraction $f(t)/e^{\alpha t}$ to converge as t gets infinitely large.

Inverse Laplace transforms

Inverse Laplace transform: We start with $F(s)$ and transform it back to $f(t)$, or start with $Y(s)$ and transform it back to $y(x)$.

$$\mathscr{L}^{-1}(F(s)) = f(t)$$

Combination of inverse transforms:

$$\mathscr{L}^{-1}(aF(s) + bG(s)) = a\mathscr{L}^{-1}(F(s)) + b\mathscr{L}^{-1}(G(s))$$

Transforming derivatives

Laplace transform of the first couple of derivatives:

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

Laplace transforms for initial value problems

Steps for solving an IVP with Laplace transforms:

1. Use formulas from the table to transform y'', y', y, and $f(t)$.

2. Plug in the initial conditions to simplify the transformation.

3. Use algebra to solve for $Y(s)$, then partial fractions to decompose it.

4. Use an inverse Laplace transform to put the solution to the second order nonhomogeneous differential equation back in terms of t, instead of s.

Step functions

Heaviside function, Heaviside step function, unit step function:

$$u_c(t) = u(t-c) = H(t) = H(t-c) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$$

Characteristics of the unit step function:

1. Its value is only ever 0 or 1; it never takes on any other value.

2. The change from 0 to 1 (“off” to “on”) happens at $t = c$.

3. c could be any nonnegative value.

Modifications of the unit step function:

Off-to-on $$u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$$

Off-to-(on at n) $$nu_c(t) = \begin{cases} 0 & 0 \le t < c \\ n & t \ge c \end{cases}$$

On-to-off $$1 - u_c(t) = \begin{cases} 1 & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

(On at n)-to-off $$n(1 - u_c(t)) = \begin{cases} n & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

Nonconstant definitions of the unit step function:

$$g(t) = f(t)u(t - c)$$

$$g(t) = \begin{cases} 0 & 0 \le t < c \\ f(t) & t \ge c \end{cases}$$

Second Shifting Theorem

Second Shifting Theorem, Second Translation Theorem:

$$\mathcal{L}(f(t - c)u(t - c)) = e^{-cs}F(s)$$

$$f(t-c)u(t-c)=\begin{cases}0 & 0\le t<c\\ f(t-c) & t\ge c\end{cases}$$

Laplace transforms of step functions

Extensions of the Second Shifting Theorem:

$$\mathscr{L}(f(t-c)u(t-c))=e^{-cs}F(s)$$

$$\mathscr{L}(u(t-c))=e^{-cs}\left(\frac{1}{s}\right)$$

$$\mathscr{L}(u(t-c))=\frac{e^{-cs}}{s}$$

Inverse of the Second Shifting Theorem:

$$\mathscr{L}^{-1}(e^{-cs}F(s))=f(t-c)u(t-c)$$

Laplace transform of a shifted function:

$$\mathscr{L}(g(t)u(t-c))=e^{-cs}\mathscr{L}(g(t+c))$$

Step functions with initial value problems

Steps for solving an IVP with step functions:

1. Make sure the forcing function is being shifted correctly, and identify the function being shifted.

2. Apply a Laplace transform to each part of the differential equation, substituting initial conditions to simplify.

3. Solve for $Y(s)$.

4. Apply an inverse transform to find $y(t)$.

The Dirac delta function

Dirac delta function: Models the application of a very large force over a very short time.

$\delta(t-c)=0$ for $t\neq c$

$\int_{c-\epsilon}^{c+\epsilon}\delta(t-c)\ dt=1$ for $\epsilon>0$

$\int_{c-\epsilon}^{c+\epsilon}f(t)\delta(t-c)\ dt=f(c)$ for $\epsilon>0$

Laplace transform of the Dirac delta function:

$\mathscr{L}(\delta(t-c))=\int_0^\infty e^{-st}\delta(t-c)\ dt=e^{-cs}$ for $c>0$

Convolution integrals

Transform of the sum $f(t)+g(t)$:

$$F(s) + G(s) = \int_0^\infty e^{-st} f(t) \ dt + \int_0^\infty e^{-st} g(t) \ dt$$

The convolution: The function we plug into the definition of the Laplace transform in order to get the product of the transforms.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau) \ d\tau$$

Equivalence of convolutions:

$$f(t) * g(t) = g(t) * f(t)$$

$$\int_0^t f(\tau)g(t-\tau) \ d\tau = \int_0^t f(t-\tau)g(\tau) \ d\tau$$

Convolution integrals for initial value problems

Steps for solving an IVP with a general forcing function:

1. Use formulas from the table to transform y'', y', y, and $g(t)$.

2. Plug in the initial conditions to simplify the transformation.

3. Use algebra to solve for $Y(s)$.

4. Use an inverse Laplace transform to put the solution to the second order nonhomogeneous differential equation back in

terms of t, instead of s, applying the convolution integral when necessary.

Systems of differential equations

Matrix basics

Determinant of a matrix:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Eigenvalues of the matrix: Values of λ that satisfy $|A - \lambda I| = 0$.

Gauss-Jordan elimination algorithm:

1. If the first entry in the first row is 0, swap it with another row that has a non-zero entry in its first column. Otherwise, move to step 2.

2. Multiply through the first row by a scalar to make the leading entry equal to 1.

3. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

4. Go back to step 1 and repeat the process until the matrix is in reduced row-echelon form.

Building systems

Homogeneous system: When all of the values $f_i(t)$ are zero; otherwise the system is nonhomogeneous. Homogeneous systems occur when $F = \overrightarrow{O}$ in $\overrightarrow{x}' = A\overrightarrow{x} + F$ (such that $\overrightarrow{x}' = A\overrightarrow{x}$); otherwise the system is nonhomogeneous.

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t) + f_1(t)$$

$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \ldots + a_{2n}(t)x_n(t) + f_2(t)$$

...

$$x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t) + f_n(t)$$

Solving systems

Solution to a 2×2 homogeneous system $\overrightarrow{x}' = A\overrightarrow{x}$, given λ_1, λ_2 are two non-zero real eigenvalues of matrix A:

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$$

$$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$$

Solution to a 2×2 nonhomogeneous system $\overrightarrow{x}' = A\overrightarrow{x} + F$ where $F \neq \overrightarrow{O}$, given λ_1, λ_2 are two non-zero real eigenvalues of matrix A:

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + \overrightarrow{x_p}$$

Wronskian test for linear independence: If the Wronskian of the vector set is non-zero, then the vector set represents a set of linearly independent solutions, and forms a fundamental set of solutions.

Distinct real Eigenvalues

The characteristic equation:

$$|A - \lambda I| = 0$$

General solution with distinct real Eigenvalues:

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$$

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

Equal real Eigenvalues with multiplicity two

General solution with equal real Eigenvalues:

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_2} = \vec{k_1}te^{\lambda_1 t} + \vec{p_1}e^{\lambda_1 t}$$

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

Equal real Eigenvalues with multiplicity three

General solution with equal real Eigenvalues:

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_2} = \vec{k_1}te^{\lambda_1 t} + \vec{p_1}e^{\lambda_1 t}$$

$$\vec{x_3} = \vec{k_1}\frac{t^2}{2}e^{\lambda_1 t} + \vec{p_1}te^{\lambda_1 t} + \vec{q_1}e^{\lambda_1 t}$$

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

$$(A - \lambda_1 I)\vec{q_1} = \vec{p_1}$$

Complex Eigenvalues

General solution with complex Eigenvalues:

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$$

$$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$$

$$\lambda_1 = \alpha + \beta i$$

$$\lambda_2 = \alpha - \beta i$$

General solution in real terms:

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\overrightarrow{x_1} = [\overrightarrow{b_1}\cos(\beta t) - \overrightarrow{b_2}\sin(\beta t)]e^{\alpha t}$$

$$\overrightarrow{x_2} = [\overrightarrow{b_2}\cos(\beta t) + \overrightarrow{b_1}\sin(\beta t)]e^{\alpha t}$$

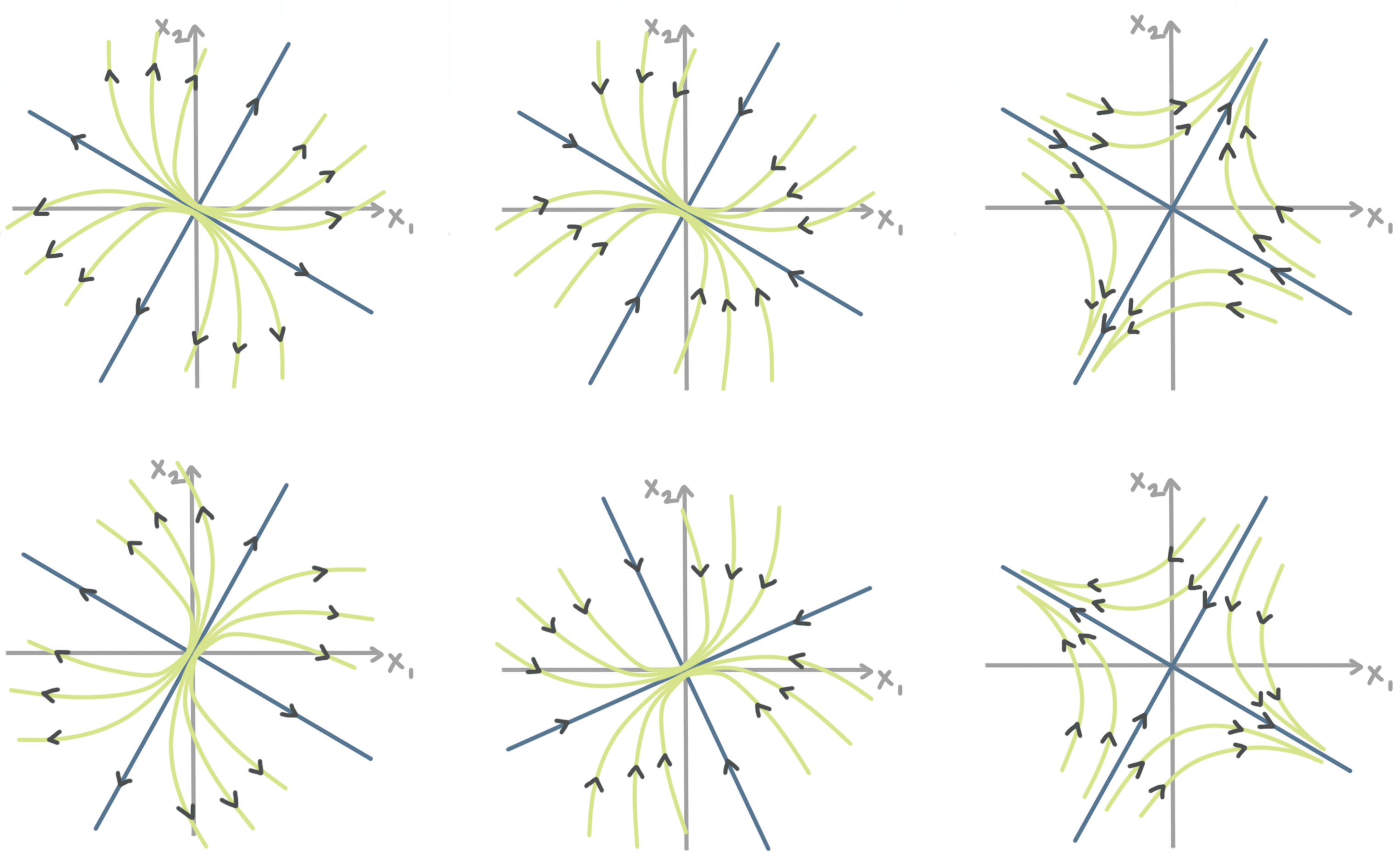
$$\overrightarrow{b_1} = \text{Re}(\overrightarrow{k_1})$$

$$\overrightarrow{b_2} = \text{Im}(\overrightarrow{k_1})$$

Phase portraits for distinct real Eigenvalues

DISTINCT REAL EIGENVALUES

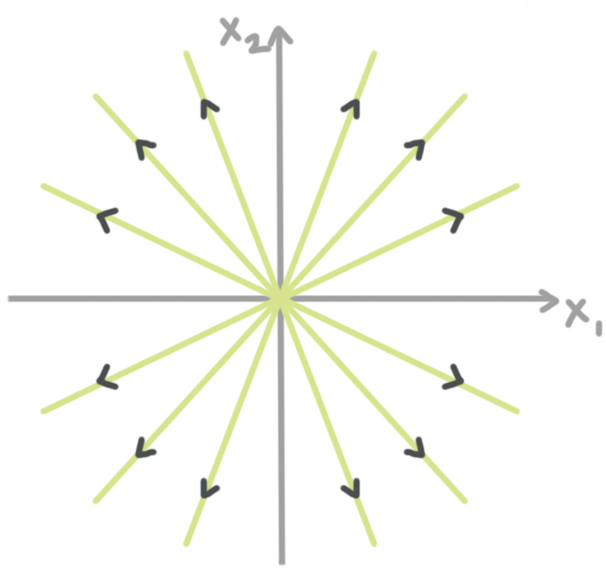
	Positive	Negative	Opposite signs
	$\lambda_1, \lambda_2 > 0$	$\lambda_1, \lambda_2 < 0$	$\lambda_1 > 0, \lambda_2 < 0$
Equilibrium	Node	Node	Saddle point
Stability	Unstable	Asymptotically stable	Unstable
	(Repeller)	(Attractor)	
Direction	$t \to \pm\infty$	$t \to \pm\infty$	$t \to \pm\infty$
Sketches			

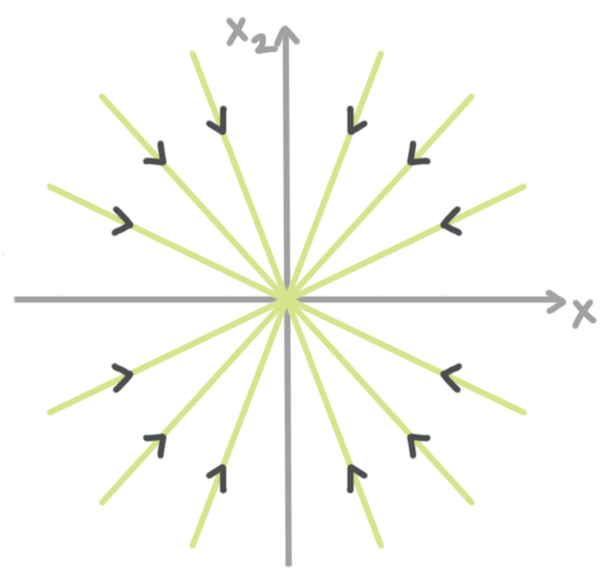


Phase portraits for equal real Eigenvalues

EQUAL REAL EIGENVALUES, TWO EIGENVECTORS

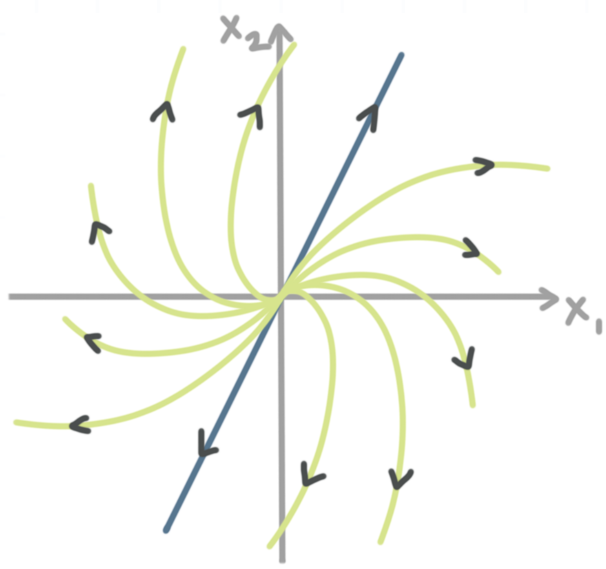
	Positive	Negative
	$\lambda_1 = \lambda_2 > 0$	$\lambda_1 = \lambda_2 < 0$
Equilibrium	Singular Node	Singular Node
Stability	Unstable	Asymptotically stable
	(Repeller)	(Attractor)
Direction	$(1,0)$	$(1,0)$
Sketches		

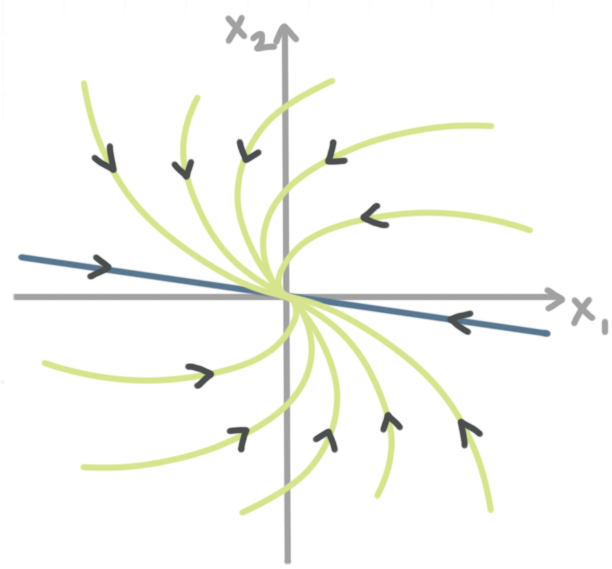




EQUAL REAL EIGENVALUES, ONE EIGENVECTOR

	Positive	Negative
	$\lambda_1 = \lambda_2 > 0$	$\lambda_1 = \lambda_2 < 0$
Equilibrium	Node	Node
Stability	Unstable	Asymptotically stable
	(Repeller)	(Attractor)
Direction	$(1,0)$	$(1,0)$
Sketches		

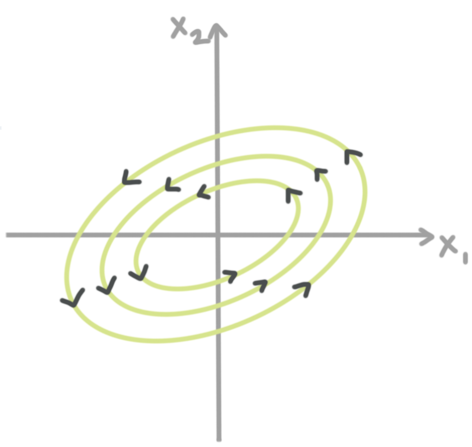


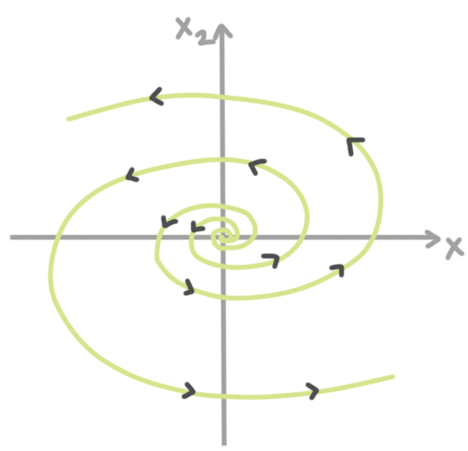


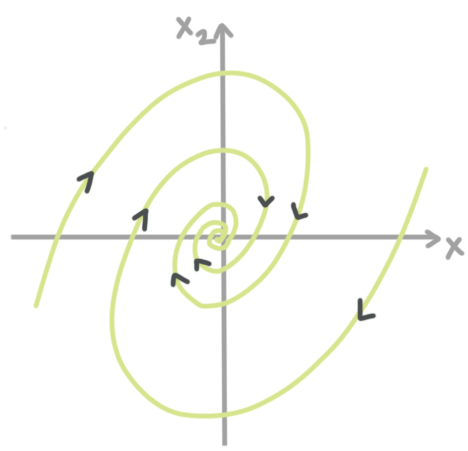
Phase portraits for complex Eigenvalues

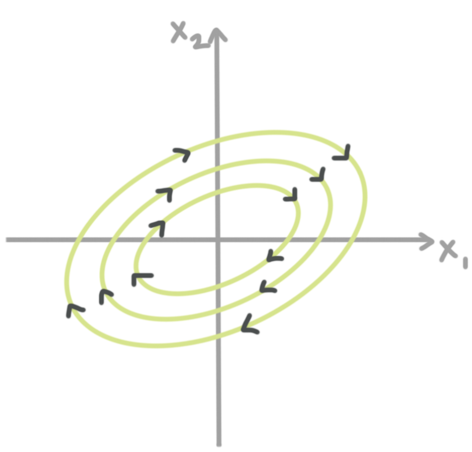
	COMPLEX CONJUGATE EIGENVALUES		
	No real part	**Positive real part**	**Negative real part**
	$\lambda_1 = \beta i,\ \lambda_2 = -\beta i$	$\lambda_1,\ \lambda_2 = \alpha \pm \beta i$	$\lambda_1,\ \lambda_2 = \alpha \pm \beta i$
		$\alpha > 0$	$\alpha < 0$
Equilibrium	Center	Spiral	Spiral
Stability	Stable	Unstable (Repeller)	Asymptotically stable (Attractor)
Direction	(1,0)	(1,0)	(1,0)

Sketches









Undetermined coefficients for nonhomogeneous systems

Guesses for the particular solution:

$g(x)$	**Guess**
x^2+1	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
$2x^3-3x+4$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x^3 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$
e^{3x}	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3x}$
$3\sin(4x)$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(4x) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(4x)$
$2\cos(4x)$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(4x) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(4x)$
$3\sin(4x)+2\cos(4x)$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(4x) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(4x)$
x^2+1+e^{3x}	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^{3x}$

Variation of parameters for nonhomogeneous systems

Particular solution for the nonhomogeneous system, where $\Phi(t)$ is the matrix of solution vectors from the complementary solution:

$$\vec{x_p} = \Phi(t) \int \Phi^{-1}(t) F(t) \ dt$$

The matrix exponential

The matrix exponential, where A is an $n \times n$ matrix of constants:

$$e^{At} = \mathscr{L}^{-1}((sI - A)^{-1})$$

$$e^{At} = I + At + A^2 \frac{t^2}{2!} + A^3 \frac{t^3}{3!} + \ldots + A^k \frac{t^k}{k!} = \sum_{k=0}^{\infty} A^k \frac{t^k}{k!}$$

Solution to the system in terms of the matrix exponential:

$$\vec{x} = e^{At}C + e^{At} \int_{t_0}^{t} e^{-As} F(s) \ ds$$

$$\vec{x_c} = e^{At}C$$

$$\vec{x_p} = e^{At} \int_{t_0}^{t} e^{-As} F(s) \ ds$$

Higher order equations

Homogeneous higher order equations

Homogeneous and nonhomogeneous higher order equations:

Homogeneous $\qquad y^{(n)} + p_{n-1}(t) y^{(n-1)} + \ldots + p_1(t) y' + p_0(t) y = 0$

Nonhomogeneous $y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t)$

General solution of the nonhomogeneous equation:

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = [c_1y_1(t) + c_2y_2(t) + \ldots + c_ny_n(t)] + y_p(t)$$

General solution of the homogeneous equation:

Distinct real roots $y(t) = c_1e^{r_1t} + c_2e^{r_2t} + \ldots + c_ke^{r_kt}$

Equal real roots $y(t) = c_1e^{r_1t} + c_2te^{r_1t} + c_3t^2e^{r_1t} \ldots + c_{k+1}t^ke^{r_1t}$

Complex conjugate roots $y(t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$

Fundamental set of solutions for the complex conjugate root pair $r = \alpha \pm \beta i$ with multiplicity k:

$e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$

$te^{\alpha t}\cos(\beta t)$ and $te^{\alpha t}\sin(\beta t)$

$t^2e^{\alpha t}\cos(\beta t)$ and $t^2e^{\alpha t}\sin(\beta t)$

...

$t^{k-1}e^{\alpha t}\cos(\beta t)$ and $t^{k-1}e^{\alpha t}\sin(\beta t)$

Undetermined coefficients for higher order equations

Steps for solving the nonhomogeneous equation:

1. Find the complementary solution $y_c(t)$ by solving the associated homogeneous equation.

2. Make a guess for the particular solution $y_p(t)$, eliminating any overlap between the guess and the complementary solution.

3. Take derivatives of the particular solution, then plug the guess and its derivatives into the original differential equation.

4. Solve for the constants A, B, C, etc., then plug their values back into the guess for the particular solution.

5. Take the sum of the complementary and particular solutions to get the general solution $y(t) = y_c(t) + y_p(t)$ to the nonhomogeneous equation.

Variation of parameters for higher order equations

Cramer's Rule for the fundamental set of solutions $\{y_1, y_2, y_3, y_4\}$:

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4$$

$$y_p(t) = y_1\int \frac{g(t)W_1}{W}\,dt + y_2\int \frac{g(t)W_2}{W}\,dt + y_3\int \frac{g(t)W_3}{W}\,dt + y_4\int \frac{g(t)W_4}{W}\,dt$$

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} \qquad W_1 = \begin{vmatrix} 0 & y_2 & y_3 & y_4 \\ 0 & y_2' & y_3' & y_4' \\ 0 & y_2'' & y_3'' & y_4'' \\ 1 & y_2''' & y_3''' & y_4''' \end{vmatrix} \qquad W_2 = \begin{vmatrix} y_1 & 0 & y_3 & y_4 \\ y_1' & 0 & y_3' & y_4' \\ y_1'' & 0 & y_3'' & y_4'' \\ y_1''' & 1 & y_3''' & y_4''' \end{vmatrix}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 & y_4 \\ y_1' & y_2' & 0 & y_4' \\ y_1'' & y_2'' & 0 & y_4'' \\ y_1''' & y_2''' & 1 & y_4''' \end{vmatrix} \qquad W_4 = \begin{vmatrix} y_1 & y_2 & y_3 & 0 \\ y_1' & y_2' & y_3' & 0 \\ y_1'' & y_2'' & y_3'' & 0 \\ y_1''' & y_2''' & y_3''' & 1 \end{vmatrix}$$

Laplace transforms for higher order equations

Laplace transforms of higher order derivatives:

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(f'''(t)) = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

$$\mathscr{L}(f^{(4)}(t)) = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

$$\mathscr{L}(f^{(5)}(t)) = s^5F(s) - s^4f(0) - s^3f'(0) - s^2f''(0) - sf'''(0) - f^{(4)}(0)$$

Systems of higher order equations

Systems of two equations:

1. 2 distinct real Eigenvalues

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$

λ_2	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$

2. 2 complex conjugate Eigenvalues (λ_1 and λ_2 are conjugates, $\overrightarrow{k_1}$ and $\overrightarrow{k_2}$ are conjugates)

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \alpha + \beta i$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
$\lambda_2 = \alpha - \beta i$	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$

3. 1 double real Eigenvalue giving 1 Eigenvector

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
		$\overrightarrow{x_2} = \overrightarrow{k_1}te^{\lambda_1 t} + \overrightarrow{p_1}e^{\lambda_1 t}$
		$(A - \lambda_1 I)\overrightarrow{p_1} = \overrightarrow{k_1}$

4. 1 double real Eigenvalue giving 2 Eigenvectors

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_1 t}$

Systems of three equations:

1. 3 distinct real Eigenvalues

Eigenvalues	Eigenvectors	Solutions

λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
λ_2	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$
λ_3	$\overrightarrow{k_3}$	$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t}$

2. 1 distinct real Eigenvalue, 2 complex conjugate Eigenvalues (λ_2 and λ_3 are conjugates, $\overrightarrow{k_2}$ and $\overrightarrow{k_3}$ are conjugates)

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
$\lambda_2 = \alpha + \beta i$	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$
$\lambda_3 = \alpha - \beta i$	$\overrightarrow{k_3}$	$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t}$

3. 1 distinct real Eigenvalue, 1 double real Eigenvalue giving 1 Eigenvector

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
$\lambda_2 = \lambda_3$	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_2} t e^{\lambda_2 t} + \overrightarrow{p_1} e^{\lambda_2 t}$

$$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_2}$$

4. 1 distinct real Eigenvalue, 1 double real Eigenvalue giving 2 Eigenvectors

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$

$\lambda_2 = \lambda_3$	$\overrightarrow{k_2}, \overrightarrow{k_3}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_3 t}$

5. 1 triple real Eigenvalue giving 1 Eigenvector

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2 = \lambda_3$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
		$\overrightarrow{x_2} = \overrightarrow{k_1} t e^{\lambda_1 t} + \overrightarrow{p_1} e^{\lambda_1 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_1} \frac{t^2}{2} e^{\lambda_1 t} + \overrightarrow{p_1} t e^{\lambda_1 t} + \overrightarrow{q_1} e^{\lambda_1 t}$
		$(A - \lambda_1 I)\overrightarrow{p_1} = \overrightarrow{k_1}$
		$(A - \lambda_1 I)\overrightarrow{q_1} = \overrightarrow{p_1}$

6. 1 triple real Eigenvalue giving 2 Eigenvectors

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2 = \lambda_3$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_2} t e^{\lambda_2 t} + \overrightarrow{p_1} e^{\lambda_2 t}$
		$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_2}$

7. 1 triple real Eigenvalue giving 3 Eigenvectors

Eigenvalues	Eigenvectors	Solutions

$\lambda_1 = \lambda_2 = \lambda_3$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_1 t}$
	$\overrightarrow{k_3}$	$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_1 t}$

Series solutions of higher order equations

Power series representations of derivatives:

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots$$

$$y''' = \sum_{n=3}^{\infty} c_n n(n-1)(n-2) x^{n-3} = 6c_3 + 24c_4 x + 60c_5 x^2 + 120c_6 x^3 + 210c_7 x^4 + \dots$$

$$y^{(4)} = \sum_{n=4}^{\infty} c_n n(n-1)(n-2)(n-3) x^{n-4} = 24c_4 + 120c_5 x + 360c_6 x^2 + 840c_7 x^3 + \dots$$

Fourier series

Fourier series representations

Fourier series representation of $f(x)$ on $-L \leq x \leq L$:

$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right) dx \qquad n = 1, 2, 3, \ldots$$

Trigonometric identities to simplify Fourier series:

$$\sin(n\pi) = 0 \qquad n = 1, 2, 3, \ldots$$

$$\cos(n\pi) = (-1)^n \qquad n = 1, 2, 3, \ldots$$

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

Integrals of even and odd functions:

$$\int_{-L}^{L} f(x)\ dx = 0$$ when $f(x)$ is odd ($f(-x) = -f(x)$)

$$\int_{-L}^{L} f(x)\ dx = 2\int_{0}^{L} f(x)\ dx$$ when $f(x)$ is even ($f(-x) = f(x)$)

Coefficients when $f(x)$ is odd:

$$A_0 = 0$$

$$A_n = 0 \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

Coefficients when $f(x)$ is even:

$$A_0 = \frac{1}{L}\int_{0}^{L} f(x)\ dx$$

$$A_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = 0 \qquad n = 1, 2, 3, \ldots$$

Periodic functions and periodic extensions

Periodic function: A function repeats the same values at regular, predictable intervals.

Periodic extension: The function we get when we repeat the portion of the function on $-L \leq x \leq L$ over and over again to the left and right.

Even extension of $f(x)$:

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ f(-x) & -L \le x < 0 \end{cases}$$

Odd extension of $f(x)$:

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

Convergence of a Fourier series

Piecewise smooth: The function and its derivative are continuous, and only a finite number of jump discontinuities are allowed.

Jump discontinuity: Both one-sided limits exist, but those one-sided limits aren't equivalent.

Convergence of a Fourier series: The Fourier series will converge where the periodic extension is continuous, or converge to the average one-sided limits where the periodic extension has a jump discontinuity.

$$\frac{\lim_{x \to a^-} g(x) + \lim_{x \to a^+} g(x)}{2}$$

Fourier cosine series

Fourier cosine series, Fourier series representation of an even function:

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

Convergence of the Fourier cosine series:

- the Fourier cosine series of an even function $f(x)$ will converge to $f(x)$ on $-L \le x \le L$
- the Fourier cosine series of a function $f(x)$ that isn't even will converge to $f(x)$ on $0 \le x \le L$

Fourier sine series

Fourier sine series, Fourier series representation of an odd function:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

Convergence of the Fourier sine series:

- the Fourier sine series of an odd function $f(x)$ will converge to $f(x)$ on $-L \le x \le L$.

- the Fourier sine series of a function $f(x)$ that isn't odd will converge to $f(x)$ on $0 \leq x \leq L$, provided $f(x)$ is continuous on $0 \leq x \leq L$, and that $f(0) = 0$ and $f(L) = 0$.

Partial differential equations

Separation of variables

Ordinary derivatives: Derivatives of single variable functions; derivatives of the single dependent variable with respect to the single independent variable.

Partial derivatives: Derivatives of multivariable functions; derivatives of the single dependent variable with respect to one of the multiple independent variables.

Product solution for separation of variables:

$$u(x, t) = v(x)w(t)$$

Boundary value problems

Boundary value problems: Similar to initial value problems, but boundary value problems let us use boundary conditions that are specified for different values of the independent variable.

The heat equation

The heat equation: The heat equation models the flow of heat across a material.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

Roots of the characteristic equation:

$\lambda = 0$ $\qquad v(x) = c_1 + c_2 x$

$\lambda > 0$ $\qquad v(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$\lambda < 0$ $\qquad v(x) = c_1\cosh(\sqrt{-\lambda}x) + c_2\sinh(\sqrt{-\lambda}x)$

(or $v(x) = c_1 e^{-\sqrt{-\lambda}x} + c_2 e^{\sqrt{-\lambda}x}$)

Fourier sine series solution:

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Fourier cosine series solution:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$A_n = \begin{cases} \frac{1}{L}\int_0^L f(x) & n = 0 \\ \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx & n \neq 0 \end{cases}$$

Fourier series solution:

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \qquad n = 1, 2, 3, \ldots$$

Changing the temperature boundaries

Solution to the heat equation with non-zero temperature boundaries:

$$u(x,t) = u_E(x) + v(x,t)$$

$$u(x,t) = T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\,dx \qquad n = 1, 2, 3, \ldots$$

Laplace’s equation

Laplace’s equation:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\nabla^2 u = 0$$

The Laplacian: The ∇^2 operator which tells us to take the second derivative with respect to each variable in the function, then add those second derivatives.

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