Differential Equations Workbook

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MATH

CLASSIFYING DIFFERENTIAL EQUATIONS

■ 1. Identify the order and linearity of the differential equation.

$$5y''' + 4xy = x^2$$

■ 2. Say whether or not the linear equation is homogeneous.

$$y'' - x = 3y$$

■ 3. Identify the order and linearity of each differential equation.

$$x^3 + 3x\sin x = y'$$

$$x\frac{d^4y}{dx^4} - x\ln x = 0$$

$$y'' - 3y' + xy^2 = x$$

■ 4. Determine whether each linear equation is homogeneous or non-homogeneous.

$$y' - 3y = 0$$

$$(\sin x)y' = 0$$

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$$\ln x - 3y'' = y$$

■ 5. Identify the order and linearity of each differential equation, then say whether or not each linear equation is homogeneous.

$$5y' - y''' - \ln y = x$$

$$y'' = 4y^2$$

$$y' = xy$$

■ 6. Identify the order and linearity of the differential equation.

$$e^x y''' = e^y x$$

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LINEAR EQUATIONS

■ 1. Find the solution to the linear differential equation.

$$y' - y \sin x = 0$$

■ 2. Solve the differential equation.

$$y' \cos x + y \sin x = 1$$

■ 3. Find the solution to the linear differential equation.

$$xy' + y = e^x$$

■ 4. Solve the differential equation.

$$xy' = x^3 - 3x^3y$$

■ 5. Find the solution to the linear differential equation.

$$xy' - 3y = x^2 - x + 1$$

■ 6. Solve the differential equation.

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$$xy' - y = \sqrt{x}$$

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INITIAL VALUE PROBLEMS

■ 1. Solve the initial value problem if $y(0) = 0$.

$$5y' - 10xy = 25x$$

■ 2. Solve the initial value problem if $y(1) = -3$.

$$x\frac{dy}{dx} + 2y = 6x^2$$

■ 3. Solve the initial value problem if $y(0) = 1$.

$$y' - y = 2\sin(3x)$$

■ 4. A function $y(x)$ is a solution of the differential equation. Suppose that $y(1) = 1$ and $y(3) = 3$. Find the constant k and the solution $y(x)$.

$$y' - \frac{ky}{x} = 0$$

■ 5. Solve the initial value problem if $y(\ln 2) = 1$.

$$\frac{dy}{dx} - 6y = 2$$

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■ 6. A function $y(x)$ is a solution of $y' + x^k y = 0$. Suppose that $y(0) = 1$ and $y(1) = e^{-5}$. Find the constant k and the solution $y(x)$.

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SEPARABLE EQUATIONS

■ 1. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{3x^2}{y}$$

■ 2. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{x+2}{2y}$$

■ 3. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{4x^3 - 1}{x(y^2 - 1)}$$

■ 4. Find the solution to the separable differential equation.

$$\frac{dy}{dx} = \frac{x^2\sqrt{x^3 - 3}}{y^2}$$

■ 5. Find the solution to the separable differential equation.

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$$y' + \sin(x - y) = \sin(x + y)$$

■ 6. Solve the initial value problem if $y(1) = -1$.

$$(xy^2 + x) + (x^2y - y)y' = 0$$

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SUBSTITUTIONS

■ 1. Use a substitution to solve the separable differential equation.

$$y' = \sin(x + y)$$

■ 2. Use a substitution to solve the separable differential equation.

$$y' = (x + y - 5)^2$$

■ 3. Use a substitution to solve the initial value problem, if $y(0) = 0$.

$$y' = e^{5y-x}$$

■ 4. Use a substitution to solve the separable differential equation.

$$y' = \frac{1}{6x - 3y}$$

■ 5. Use a substitution to solve the initial value problem, if $y(0) = 8$.

$$y' + (9x + y - 1)^2 = 0$$

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■ 6. Use a substitution to solve the initial value problem, if $y(0) = 2$.

$$y' + 2x + 2y - 2xy = x^2 + y^2$$

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BERNOULLI EQUATIONS

■ 1. Find the solution to the Bernoulli differential equation.

$$(y^4 + x^4 y)\ dx - 3x^5\ dy = 0$$

■ 2. Find the solution to the Bernoulli differential equation.

$$xy' - 3y = xy^{\frac{5}{3}}$$

■ 3. Find the solution to the Bernoulli differential equation.

$$y' = y^2 \sin x + y \cot x$$

■ 4. Find the solution to the Bernoulli differential equation, if $y(1) = 0$.

$$y' - \frac{y}{x} + \sqrt{y} = 0$$

■ 5. Find the solution to the Bernoulli differential equation, if $y(1) = 1$.

$$y' + \frac{2}{x}y = \frac{y^2}{x^2}$$

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6. Find the solution to the Bernoulli differential equation, if $y(0) = 2$.

$$3y' + 2y = xy^5$$

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HOMOGENEOUS EQUATIONS

■ 1. Find the solution to the differential equation.

$$x^2yy' = x^3 + xy^2$$

■ 2. Use a substitution to find a solution to the homogeneous equation.

$$x(x + y)y' - y(2x + y) = 0$$

■ 3. Rewrite the homogeneous equation in terms of y/x, but don't solve the separable equation.

$$yy' - x = \sqrt{x^2 + y^2}$$

■ 4. Use a substitution to find a solution to the homogeneous equation.

$$xyy' = x^2e^{-\frac{y^2}{x^2}} + y^2$$

■ 5. Find the solution to the differential equation, if $y(1) = 1$.

$$x^2y' = 3xy + 2y^2$$

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6. Use a substitution to find a solution to the homogeneous equation, if $y(1) = 1$.

$$xy' + y \ln x - y = y \ln y$$

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EXACT EQUATIONS

■ 1. If the differential equation is exact, find its solution.

$$2xy + (x^2 - y^2)\frac{dy}{dx} = 0$$

■ 2. What is the solution to the exact differential equation?

$$(6y + xe^{-y})\ dy - e^{-y}\ dx = 0$$

■ 3. Find the value of a that make the equation an exact differential equation.

$$(5\sin y - x^3)\ dx + ax\cos y\ dy = 0$$

■ 4. If the differential equation is exact, find its solution.

$$(x\sin y - 2x)\ dx + \left(\frac{x^2}{2}\cos y - 2y\right)\ dy = 0$$

■ 5. Find the value of a that makes the equation an exact differential equation.

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$$(2\cos x + 4\ln y)\ dx + \left(\frac{a^2x}{y} + e^y\right)\ dy = 0$$

■ 6. What is the solution to the exact differential equation?

$$(x\cos y + e^y)\ dy + (e^x + \sin y)\ dx = 0$$

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SECOND ORDER LINEAR HOMOGENEOUS EQUATIONS

■ 1. Find the general solution to the homogeneous equation.

$$y'' + 9y' + 8y = 0$$

■ 2. Find the solution to the homogeneous equation, if $y(0) = 5$ and $y'(0) = 2$.

$$y'' - 4y' + 5y = 0$$

■ 3. Find the general solution to the homogeneous equation.

$$y'' - 10y' + 25y = 0$$

■ 4. If $y(0) = 1$ and $y'(0) = 0$, find the solution to the homogeneous equation.

$$y'' + 4y' + 4y = 0$$

■ 5. If $y(0) = 2$ and $y'(0) = 3$, find the solution to the homogeneous equation.

$$y'' + 9y = 0$$

■ 6. If $y(0) = 1$ and $y'(0) = 5$, find the solution to the homogeneous equation.

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$$2y'' + 11y' + 5y = 0$$

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REDUCTION OF ORDER

■ 1. Use reduction of order to find the general solution to the differential equation, given $y_1 = e^{2x}$.

$$y'' + 3y' - 10y = 0$$

■ 2. Use reduction of order to find the general solution to the differential equation, given $y_1 = x^2$.

$$x^2y'' - 7xy' + 12y = 0$$

■ 3. Use reduction of order to find the general solution to the differential equation, given $y_1 = x^3$.

$$x^2y'' - 6y = x^3$$

■ 4. Use reduction of order to find the general solution to the differential equation, given $y_1 = \cos(2x)$.

$$y'' + 4y = 0$$

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■ 5. Use reduction of order to find the general solution to the differential equation, given $y_1 = x$.

$$(x+1)y'' + xy' - y = 0$$

■ 6. Use reduction of order to find the general solution to the differential equation, given $y_1 = e^{4x}$.

$$y'' - 5y' + 4y = 0$$

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UNDETERMINED COEFFICIENTS FOR NONHOMOGENEOUS EQUATIONS

■ 1. Find the general solution to the differential equation.

$$y'' - 2y' + y = x^3 - 6x^2 + 6x + 1$$

■ 2. Use the method of undetermined coefficient to solve the differential equation.

$$y'' - 3y' + 2y = e^{-x}(6x^2 - 10x + 20)$$

■ 3. Find the general solution to the differential equation.

$$y'' - y' = e^{2x} + e^{3x} + e^{4x}$$

■ 4. Solve the differential equation.

$$y'' + 9y = 12\sin(3x)$$

■ 5. Solve the differential equation.

$$y'' - 4y' + 4y = e^{2x}$$

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6. Solve the differential equation.

$$y'' - 4y' + 3y = e^x + 10\sin x + 3x$$

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VARIATION OF PARAMETERS FOR NONHOMOGENEOUS EQUATIONS

■ 1. Find the general solution to the differential equation.

$$y'' - 10y' + 25y = x^3e^{5x}$$

■ 2. Find the general solution to the differential equation.

$$y'' - 6y' + 8y = e^{3x} + 5$$

■ 3. Find the general solution to the differential equation.

$$y'' - 7y' + 12y = \frac{e^{5x}}{e^{2x} + 1}$$

■ 4. Find the general solution to the differential equation.

$$y'' - 6y' + 9y = e^{3x}\sin(2x + 1)$$

■ 5. Find the general solution to the differential equation.

$$2y'' + 18y = \sin^4(3x)$$

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6. Find the general solution to the differential equation.

$$y'' - 4y' + 5y = e^{-2x}\ln(1 + \sin x)$$

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FUNDAMENTAL SOLUTION SETS AND THE WRONSKIAN

1. Use the Wronskian to determine whether $\{y_1, y_2\} = \{e^x,\ e^{x+1}\}$ is the fundamental set of solutions for the differential equation.

$$y'' = y$$

2. Determine whether $\{y_1, y_2\} = \{\cos(2x),\ \cos(2x+1)\}$ is a fundamental set of solutions for the differential equation.

$$y'' + 4y = 0$$

3. Find the Wronskian for the solution set.

$$\{e^{3x},\ xe^{3x}\}$$

4. Find the Wronskian for the solution set.

$$\{e^{2x}\cos(2x),\ e^{2x}\sin(2x)\}$$

5. Find the fundamental set of solutions for the the second order differential equation.

$$y'' + 2y' + y = 0$$

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■ 6. Find the fundamental set of solutions for the the second order differential equation, generated by solutions of the initial value problems with $y(0) = 1$ and $y'(0) = 0$, and $y(0) = 0$ and $y'(0) = 1$.

$$y'' + 4y' = 0$$

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VARIATION OF PARAMETERS WITH THE WRONSKIAN

■ 1. Solve the differential equation.

$$y'' + 7y' + 12y = e^x + 7e^{3x}$$

■ 2. Solve the differential equation.

$$y'' - 2y' = \frac{e^{3x}}{e^{2x} + 1}$$

■ 3. Solve the differential equation.

$$y'' + 8y' + 16y = 21x^5 e^{-4x}$$

■ 4. Solve the differential equation.

$$y'' + 6y' + 9y = \frac{1}{e^{3x}\sqrt{1 - x^2}}$$

■ 5. Solve the differential equation.

$$y'' + y = \frac{1}{\sin^3 x}$$

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■ 6. Solve the differential equation.

$$y'' + 4y = \cos^2(2x)$$

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INITIAL VALUE PROBLEMS WITH NONHOMOGENEOUS EQUATIONS

■ 1. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 2$.

$$y'' - 2y' - 3y = 10e^x \sin x$$

■ 2. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' - 4y = 8x^2 + 4$$

■ 3. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' - 10y' + 25y = 28x^6e^{5x}$$

■ 4. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 1$ and $y'(0) = 0$.

$$4y'' + 4y' + y = 25 \sin x + 4$$

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■ 5. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 1$.

$$y'' + 4y' + 5y = 17e^{2x} + 40\sin(3x)$$

■ 6. Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 2$ and $y'(0) = 2$.

$$y'' + 16y = 40\sin^2(3x)$$

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DIRECTION FIELDS AND SOLUTION CURVES

■ 1. Sketch the direction field of the differential equation.

$$y' = 2y + x$$

■ 2. Sketch the direction field of the differential equation, and the solution curve passing through $(1,1)$.

$$y' = -3y + x$$

■ 3. Sketch the direction field of the differential equation, and the solution curve passing through $(0,0)$.

$$y' = y^2 + x^2 - 1$$

■ 4. Sketch the direction field for the differential equation, and the solution curve passing through $(0,1)$.

$$y' = 2y^2 - x^2 - 2$$

■ 5. Sketch the direction field of the differential equation.

$$y' = \sin y + x$$

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■ 6. Sketch the direction field of the differential equation.

$$y' = xy$$

INTERVALS OF VALIDITY

■ 1. Find the interval of validity for the solution to the differential equation, given $y(3) = -4$.

$$(x^2 - 25)y' - 7y - \ln(10 - x) = 0$$

■ 2. Find the interval of validity for the solution to the differential equation, given $y(0) = 1$.

$$(x + 2)y' + xye^{3x} - 5 = 0$$

■ 3. Find the interval of validity for the solution to the differential equation, given $y(0) = -1$, and solve the initial value problem to verify that the interval of validity is correct.

$$(x^2 - 1)y' + 2xy = x + 1$$

■ 4. Find the interval of validity for the solution to the differential equation, given $y(0) = y_0$, for some positive y_0.

$$y' = 3x^2y^{\frac{4}{3}}$$

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■ 5. Find the interval of validity for the solution to the differential equation.

$$\frac{dy}{dx} = y^2e^x$$

$$y(0) = \frac{1}{e^2 - 1}$$

■ 6. Find the interval of validity for the solution to the differential equation, given $y(1) = y_0$.

$$xy' = y^2$$

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EULER'S METHOD

1. If $y(0)=0$, use Euler's method to approximate $y(2)$ with $n=4$ steps.

$$y'=1+y$$

2. If $y(2)=0$, use Euler's method to approximate $y(4)$ with $n=5$ steps.

$$y'=3t+y$$

3. If $y(0)=1$, use Euler's method to approximate $y(1)$ with $n=4$ steps.

$$y'=1-2y$$

4. Use Euler's method and five steps to approximate $y(3)$, given $y(2)=1$.

$$y'=t^2+y$$

5. If $y(0)=0$, use Euler's method to approximate $y(1)$ with $\Delta t=0.2$.

$$y'+3y=1-e^{-5t}$$

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■ 6. If $y(0) = 3$, use Euler's method to approximate y_3 with $\Delta t = 0.1$, find the exact value and percentage error.

$$y' = 8e^{2t} - 3y$$

AUTONOMOUS EQUATIONS AND EQUILIBRIUM SOLUTIONS

■ 1. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable without sketching a direction field.

$$\frac{dy}{dt} = -y^2 - y + 6$$

■ 2. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable without sketching a direction field.

$$\frac{dy}{dt} = y^3 - 4y^2 + 4y$$

■ 3. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = (y^2 - 9)(y - 1)^2$$

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■ 4. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = \sin y$$

■ 5. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = |16 - y^2|$$

■ 6. Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} + 3y + 6 = e^y(3y + 6)$$

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THE LOGISTIC EQUATION

■ 1. Describe the growth pattern of a population of butterflies, assuming their population is modeled by the logistic growth equation, if the carrying capacity of their habitat is $3{,}700$ butterflies.

■ 2. A bacteria population is observed to be $2{,}500$ at $t = 0$. Using the logistic growth model and a growth rate of 0.03 with a carrying capacity of $4{,}000$, write the logistic growth equation, then solve the initial value problem.

■ 3. An insect population is observed to be 500 at time $t = 0$. Assuming the population follows a logistic growth model with a growth rate of 0.01 and a carrying capacity of $2{,}500$, solve the initial value problem.

■ 4. The population of Canada in $2{,}000$ and 2010 was 30.7 and 34 million people, respectively. Predict its population in 2050 and 2060 assuming an exponential model of population growth.

■ 5. A bacteria population increases 8-fold in 5 hours. Assuming exponential growth, how long did it take for the population to increase 5 -fold?

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6. The growth rate of a population of bears is modeled by the differential equation, where time t is measured in years. Solve the differential equation with the initial condition $P(0) = 30$.

$$\frac{dP}{dt} = 0.002P(400 - P)$$

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PREDATOR-PREY SYSTEMS

■ 1. Is the system cooperative, competitive, or predator-prey? Find the equilibrium solutions of the system.

$$\frac{dx}{dt} = -0.16x + 0.08xy$$

$$\frac{dy}{dt} = 4.5y - 0.9xy$$

■ 2. Given the system modeling the interaction between populations of foxes and hares, find any equilibrium where the foxes and hares coexist, and give the size of each population.

$$\frac{dH}{dt} = 0.14H - 0.07FH$$

$$\frac{dF}{dt} = -0.65F + 0.05FH$$

■ 3. Determine whether the system is cooperative, competitive, or predator-prey, and find the maximum size of both populations when they are in equilibrium, assuming both populations are measured in millions.

$$\frac{dx}{dt} = 0.02x - 0.4x^2 + 0.005xy$$

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$$\frac{dy}{dt} = 0.1y - 1.1y^2 + 0.05xy$$

■ 4. Determine whether the system is cooperative, competitive, or predator-prey, and find the equilibrium solutions, assuming both populations are measured in millions.

$$\frac{dx}{dt} = 0.004x - 0.2xy$$

$$\frac{dy}{dt} = 0.03y - 0.36y^2 - 0.6xy$$

■ 5. Two species of bacteria are being cultivated in a petri dish. Every day, an amount of sugar is deposited in the petri dish and the species compete for this source of food. Determine whether the system is cooperative, competitive, or predator-prey and find the equilibrium solutions.

$$\frac{dx}{dt} = 0.001x - 0.03x^2 - 0.04xy$$

$$\frac{dy}{dt} = 0.002y - 0.05y^2 - 0.01xy$$

■ 6. Is the system cooperative, competitive, or predator-prey? Find the equilibrium solutions of the system.

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$$\frac{dx}{dt} = 0.48x - 0.16xy$$

$$\frac{dy}{dt} = 1.2y + 7.2xy$$

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EXPONENTIAL GROWTH AND DECAY

■ 1. If world population was 2.56 million in 1950 and 3.04 million in 1960 and is growing exponentially, find a function modeling population in the second half of the 20th century, then use the function to estimate world population in 1984.

■ 2. The size of a population of fleas increased by 4 fold in 2 months. How long did it take for the population to increase 5 fold?

■ 3. The half-life of radium-226 is $1,590$ years. If the substance decays exponentially, find a function that models the radioactive decay of a sample of 100 g of radium-226 over time t years. How much of the mass remains after $2,000$ years?

■ 4. A population of $1,000$ bacteria, growing exponentially, takes 12 minutes to double in size. How long will it take for the population to reach 1 million?

■ 5. A sofa's price drops by 50% in 10 months. If the rate at which the price decreases is proportional to current price, how much will the price decrease in 2 years?

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■ 6. A radioactive element decays into nonradioactive substances. After 3 years, the radioactivity decreases by $5\,\%$. How many years would it take for the radioactivity to decrease by $90\,\%$?

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MIXING PROBLEMS

■ 1. A tank contains $5,000$ L of water and 100 kg of dissolved salt. Fresh water is entering the tank at 20 L/min, and the solution drains at a rate of 15 L/min. Assuming the solution in the tank remains perfectly mixed, what is the function that models the amount of salt $y(t)$ at any given minute t? How much salt remains in the tank after 15 days?

■ 2. A tank contains $2,000$ L of water and 500 kg of dissolved salt. Fresh water is entering the tank at 20 L/min, and the solution drains at a rate of 50 L/min. Assuming the solution in the tank remains perfectly mixed, what is the function $y(t)$ that models the amount of salt in the tank after t minutes?

■ 3. A tank contains $5,000$ L of freshwater and is being filled with a brine mixture with a concentration of 150 g/L, at a rate of 20 L/min. Meanwhile, solution drains from the tank at a rate of 30 L/min. Assuming the solution in the tank remains perfectly mixed, what is the function $y(t)$ that models the amount of salt in the tank after t minutes?

■ 4. A tank contains $1,000$ L of fresh water. A mix of water and salt is being added into the tank at a rate of 500 L/h, and the solution drains out at the same rate. The concentration of the solution that's being deposited varies

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periodically, according to the function $f(t) = 2\sin t$ kg/L. Assuming that the solution in the tank remains perfectly mixed, how much salt is in the tank after t hours?

■ 5. A tank holding $2{,}000$ L of fresh water is being filled at 400 L/h with a brine solution with a concentration of $f(t) = 5 + \cos(t/5)$ kg/L. Assuming that the solution in the tank remains perfectly mixed and drains at a rate equal to the fill rate, how much salt is in the tank after t hours?

■ 6. A tank contains $3{,}000$ L of water and 600 kg of dissolved salt. Fresh water is entering the tank at 10 L/min, and the solution drains out at the same rate. Assuming the solution in the tank remains perfectly mixed, what is the function that models the amount of salt $y(t)$ at any given minute t?

NEWTON'S LAW OF COOLING

■ 1. The temperature of a soup dropped from $100^\circ C$ to $95^\circ C$ in 5 minutes. How long would it take for the temperature to drop to $40^\circ C$ if ambient temperature is $25^\circ C$?

■ 2. The temperature of a liquid dropped from $350^\circ C$ to $300^\circ C$ in 30 minutes. What temperature would the liquid reach in 3 hours if the temperature of the environment is $30^\circ C$?

■ 3. The temperature of ice cream in a freezer is -20°. After two minutes outside the freezer, where the temperature is 35°, the temperature of the ice cream increased to -10°. What will be the temperature of the ice cream after 10 minutes?

■ 4. The temperature in a room is 25°, while outside temperature is -20°. After 5 minutes with the door open, the room temperature dropped to 20°. What will the temperature be in the room after 15 minutes, and when will the room temperature drop to 5°?

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5. A piece of metal is placed in a $20^\circ C$ room, and after 5 minutes the temperature of the metal is $120^\circ C$. After another 15 minutes the metal's temperature is $75^\circ C$. What was the initial temperature of the metal?

6. Room temperature can only be measured at temperatures less than or equal to $25^\circ C$. We can't currently measure room temperature, which means the room must be warmer than $25^\circ C$. Meanwhile, outside temperature is $-5^\circ C$. After 15 minutes with the window open, room temperature has dropped to $20^\circ C$, and to $17^\circ C$ after another 5 minutes. What was the initial temperature of the room?

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ELECTRICAL SERIES CIRCUITS

■ 1. If a battery supplies a constant voltage of 9 V, has an inductance of 3 h and a resistance of $3\ \Omega$, and assuming $i(0) = 0$, find the current after 120 seconds.

■ 2. Given $L = 10$ h, $R = 70\ \Omega$, $C = 0.01$ f, $E(t) = 0$, $q(0) = 0$, and $i(0) = 3$, find the charge $q(t)$ on the capacitor.

■ 3. An inductance of 2 h is connected in series with a resistance of $10\ \Omega$ and a battery giving $E(t) = 120$ V. Initially, the current is zero. Formulate and solve an initial value problem modeling the electrical circuit.

■ 4. Suppose $L = 2$ h, $R = 6\ \Omega$, $E = 24e^{3t}$ V, and $i(0) = 0$. Formulate and solve an initial value problem that models the electrical circuit.

■ 5. Suppose $L = 2$ h, $R = 40\ \Omega$, $C = 0.005$ f, $E = 100$ V, $q(0) = 9$ C, and $q'(0) = i(0) = 0$. Formulate and solve an initial value problem that models this LRC circuit.

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■ 6. An LRC circuit is set up with an inductance of $1/2$ h, a resistance of $1\ \Omega$, and a capacitance of $2/5$ f. Assuming the initial charge is 2 C and the initial current is 6 A, find the solution function describing the charge on the capacitor at any time t.

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SPRING AND MASS SYSTEMS

■ 1. Find the movement of a 6 kg mass attached to a spring with spring constant $k = 3$ kg/s^2 and damping constant $\beta = 6$ kg/s, given $x(0) = 2$ and $x'(0) = 0$.

■ 2. Assume an object weighing 8 lbs stretches a spring 12 in. Find the equation of motion if the spring is released from equilibrium with an upward velocity of $\sqrt{2}$ ft/sec.

■ 3. An 8 lb weight is attached to an 11 ft spring. When the mass comes to rest in the equilibrium position, the spring measures 15 ft. The system is immersed in a medium that imparts a damping force equal to $3/2$ times the instantaneous velocity of the mass. Find the equation of motion if the mass is pushed upward from the equilibrium position with an initial upward velocity of 4 ft/sec. What is the position of the mass after 15 sec?

■ 4. A 2 kg mass stretches a spring with a length of 20 cm. The system is attached to a dashpot that creates a damping force equal to 28 times the instantaneous velocity of the mass. Find the motion equation if the mass is released from rest at a point 6 cm below equilibrium.

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■ 5. A 16 lb weight stretches a spring 2 ft. Assume the damping force on the system is equal to the instantaneous velocity of the mass. Find the equation of motion of the mass.

■ 6. A mass of 9.8 kg stretches a spring 2.45 m. Find the equation that models the motion of the mass if we release the mass when $t=0$ from a position 5 m above equilibrium, with a downward velocity of 2 m/s.

POWER SERIES BASICS

■ 1. Where is the power series centered?

$$\sum_{n=0}^{\infty} c_n(x+3)^n = c_0 + c_1(x+3) + c_2(x+3)^2 + c_3(x+3)^3 + \dots$$

■ 2. Find the radius of convergence of the power series.

$$\sum_{n=0}^{\infty} \frac{3^{2n}}{100^n}(10x)^n$$

■ 3. Find the Maclaurin series representation of $f(x) = \cosh(9x^2)$.

■ 4. Find the Maclaurin series representation of the function.

$$f(x) = 4\ln(-2 - 3x^3)$$

■ 5. Find the Maclaurin series representation of the function.

$$f(x) = \frac{6}{\sqrt{2x-1}}$$

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■ 6. Find the first three derivatives of the power series.

$$f(x)=\sum_{n=0}^{\infty}3(-1)^n(2x^2)^n$$

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ADDING POWER SERIES

■ 1. Are the power series in phase? Do the indices match?

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} x^{2n}$$

■ 2. As written, can the power series be added?

$$\sum_{n=0}^{\infty} n!x^n \qquad 5 + \sum_{n=1}^{\infty} \frac{3^n}{n!} x^n$$

■ 3. Find the sum.

$$\sum_{n=3}^{\infty} (2n-1)c_n x^{n-3} + \sum_{n=1}^{\infty} 2nc_n x^{n-1}$$

■ 4. Add the power series.

$$\sum_{n=2}^{\infty} c_n n(n-1)x^{n-2} \qquad x\sum_{n=0}^{\infty} c_n x^n$$

■ 5. Find the sum.

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$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n$$

■ 6. Find the sum.

$$3x^2\sum_{n=0}^{\infty}(n-2)(n-1)nc_n x^n - (x^2-2)\sum_{n=1}^{\infty} c_n x^{n+1} + (x+1)\sum_{n=2}^{\infty} n(n+1)c_n x^{n-1}$$

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POWER SERIES SOLUTIONS

1. Is $x_0 = 4$ an ordinary point of the differential equation?

$$\frac{1}{x-4}y'' + (x^2+2)y' + 3y = 0$$

2. Can we find a power series solution to the differential equation around $x_0 = -1$?

$$(x-1)y'' + xy' + (x^2+2)y = \sqrt{x-1}$$

3. Find the minimum radius of convergence of a power series solution of the differential equation about the ordinary point $x_0 = -2$.

$$x^3y'' + (2x+1)y' - y = 0$$

4. Find a power series solution to the differential equation, given $y(0) = 0$ and $y'(0) = 1$.

$$(x^2+1)y'' + 2xy' = 0$$

5. Find a power series solution to the differential equation, given $y(0) = 1$ and $y'(0) = 0$.

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$$y'' - xy' - y = 0$$

■ 6. Find a power series solution to the differential equation, given $y(0) = 0$ and $y'(0) = 1$.

$$y'' + x^2y' + xy = 0$$

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NONPOLYNOMIAL COEFFICIENTS

■ 1. Solve the differential equation around $x_0 = 0$.

$$y'' + \frac{1}{1-x}y' = 0$$

■ 2. Find the first six terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y'' - e^x y = 0$$

■ 3. Find the first five terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y' + e^{3x} y = 0$$

■ 4. Find the first eight terms of the power series solution of the differential equation around the ordinary point $x_0 = 0$.

$$y'' + y\cos x = 0$$

■ 5. Find the first six terms of the power series solution of the differential equation around the ordinary point $x_0 = 0$.

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$$y'' - \ln(1 + 5x)y = 0$$

■ 6. Find the first four terms of the power series solution of the differential equation around the ordinary point $x_0 = 2$.

$$y'' + \cosh(x - 2)y = 0$$

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SINGULAR POINTS AND FROBENIUS' THEOREM

■ 1. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$2x^2y'' + x(1 - 2x)y' + 4xy = 0$$

■ 2. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$4xy'' + 2y' + y = 0$$

■ 3. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$x^2y'' + x^2y' - 6y = 0$$

■ 4. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

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$$8x^2y'' + x^2y' + (2 - x)y = 0$$

■ 5. Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$3x^2y'' + 3xy' - 2xy = 0$$

■ 6. Determine the regularity of the singular point $x_0 = -1$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$x(x + 1)y'' + 3(x + 1)y' + y = 0$$

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THE LAPLACE TRANSFORM

■ 1. Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(t+5)$$

■ 2. Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(\sin t)$$

■ 3. Find the Laplace transform, given $s > 1$.

$$\mathscr{L}(e^t)$$

■ 4. Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(5t)$$

■ 5. Find the Laplace transform, given $s > 0$.

$$\mathscr{L}(7)$$

■ 6. Find the Laplace transform, given $s > 0$.

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$\mathscr{L}(\cos t)$

TABLE OF TRANSFORMS

■ 1. Use a table of Laplace transforms to transform the function.

$$f(t) = \cos(3t) + 5e^{-7t}$$

■ 2. Use a table of Laplace transforms to transform the function.

$$f(t) = t^2 + 3t - e^{3t}$$

■ 3. Use a table of Laplace transforms to transform the function.

$$f(t) = 2t^4 - 3\sin(2t)$$

■ 4. Use a table of Laplace transforms to transform the function.

$$f(t) = e^{-5t} - t\sin(3t)$$

■ 5. Use a table of Laplace transforms to transform the function.

$$f(t) = e^{6t}\cos t + t^4$$

■ 6. Use a table of Laplace transforms to transform the function.

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$$f(t) = \cos(5t) + 3t^3 - e^{-3t}$$

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EXPONENTIAL TYPE

1. Determine the value of α in $e^{\alpha t}$ such that the function $f(t)=c$, is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}}=0$$

2. Determine the value of α in $e^{\alpha t}$ such that the function $f(t)=t$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}}=0$$

3. Determine the value of α in $e^{\alpha t}$ such that the function $f(t)=t^2+t$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty}\frac{f(t)}{e^{\alpha t}}=0$$

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4. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = t\cos(bt)$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

5. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = \ln(t)$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

6. Determine the value of α in $e^{\alpha t}$ such that the function $f(t) = te^{bt}$ is of exponential type. In other words, find the value of α that makes the following equation true.

$$\lim_{t\to\infty} \frac{f(t)}{e^{\alpha t}} = 0$$

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PARTIAL FRACTIONS DECOMPOSITIONS

■ 1. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 2x^2 + 2x - 3}{(x-2)^4}$$

■ 2. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 4x^2 - 10}{x^2(x+1)(x-1)}$$

■ 3. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2}{(x-1)(x+1)}$$

■ 4. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{2x^4 + 16}{x(x^2+2)^2}$$

■ 5. Rewrite the function as its partial fractions decomposition.

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$$f(x) = \frac{x+2}{(x-1)^2}$$

■ 6. Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 + 2x^2 - 3x + 1}{x^3(x^2+1)^2}$$

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INVERSE LAPLACE TRANSFORMS

■ 1. Find the inverse Laplace transform.

$$F(s) = \frac{s+3}{s^2 - 3s + 2}$$

■ 2. Use an inverse Laplace transform to find $f(t)$.

$$F(s) = \frac{2s-5}{s^3 + s^2 - 12s}$$

■ 3. Find the inverse Laplace transform.

$$F(s) = \frac{s-3}{s^2 + 4s + 4}$$

■ 4. Use an inverse Laplace transform to find $f(t)$.

$$F(s) = \frac{3s^3 + 6s^2 + 27s + 24}{s^4 + 13s + 36}$$

■ 5. Use an inverse Laplace transform to find $f(t)$.

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$$F(s) = \frac{s-2}{s^2 - 4s + 5}$$

■ 6. Find the inverse Laplace transform.

$$F(s) = \frac{s-2}{2s^2 + 2s + 2}$$

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TRANSFORMING DERIVATIVES

■ 1. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 1$ and $y'(0) = 1$.

■ 2. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 0$ and $y'(0) = 2$.

■ 3. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 1$ and $y'(0) = -2$.

■ 4. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 3$ and $y'(0) = -1/4$.

■ 5. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 4$ and $y'(0) = 2/3$.

■ 6. Find the Laplace transforms of $y'(t)$ and $y''(t)$, given $y(0) = 7$ and $y'(0) = 1/7$.

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LAPLACE TRANSFORMS FOR INITIAL VALUE PROBLEMS

■ 1. Find the solution to the second order equation, given $y(0) = 1$ and $y'(0) = -1$.

$$y'' + 3y' + 2y = 6t$$

■ 2. Find the solution to the second order equation, given $y(0) = 1$ and $y'(0) = -1$.

$$3y'' + 4y' + y = -t$$

■ 3. Find the solution to the second order equation, given $y(0) = 1$ and $y'(0) = 4$.

$$y'' - 5y' = 10t - 2$$

■ 4. Find the solution to the second order equation, given $y(0) = 2$ and $y'(0) = 2$.

$$y'' - 4y = -13\cos(3t)$$

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■ 5. Find the solution to the second order equation, given $y(2) = 1$ and $y'(2) = -2$.

$$y'' + 2y' - 3y = 0$$

■ 6. Find the solution to the second order equation, given $y(\pi) = 2$ and $y'(\pi) = 0$.

$$y'' + 4y = 0$$

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STEP FUNCTIONS

■ 1. Write $f(t)$ in terms of step functions, if $f(t)$ has a value of -3 at $t=0$, jumps down 2 units at $t=4$, up 7 units at $t=5$, and if $f(t)=9$ for $t \geq 6$.

■ 2. Write an (on at 5)-to-off function that has a switch at $t=8$.

■ 3. Express the piecewise function in terms of unit step functions.

$$f(t)=\begin{cases} \cos t & 0 \leq t < 2\pi \\ 0 & t \geq 2\pi \end{cases}$$

■ 4. Rewrite the function $g(t)$ in terms of Heaviside functions.

$$g(t)=\begin{cases} 0 & 0 \leq t < 1 \\ t^2-3 & 1 \leq t < 2 \\ 5 & 2 \leq t < 3 \\ t^3+2 & t \geq 3 \end{cases}$$

■ 5. Describe all switches of the function $f(t)=(-5+6u_2(t)-3u_5(t))(1-u_7(t))$ and represent it as a piecewise function.

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■ 6. Describe all switches of the function $f(t) = 4 - u_1(t) + 5u_3(t) + 3u_4(t) - 6u_8(t)$ and represent it as a piecewise function.

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SECOND SHIFTING THEOREM

■ 1. Use a step function to represent shifting the portion of $f(x)$ for $x \geq 0$ to the right 2 units, while turning off the function on $0 \leq x < 2$. Then take its Laplace transform.

$$f(x) = 5 - x^2 + \cos(3x) - 4e^{5x}$$

■ 2. Use a step function to represent shifting the portion of $f(x)$ for $x \geq 0$ to the right 3 units, while turning off the function on $0 \leq x < 3$. Then take its Laplace transform.

$$f(x) = 7x \sin x - 4\cos(3x + 9)$$

■ 3. Find the Laplace transform of the function.

$$g(t) = \begin{cases} 0 & 0 \leq x < 7 \\ e^{3t-21} & x \geq 7 \end{cases}$$

■ 4. Find the Laplace transform of the function.

$$g(t) = \begin{cases} 0 & 0 \leq x < 4 \\ e^{t-4}\sin(3t - 12) - (t-4)^3 e^{4t-16} & x \geq 4 \end{cases}$$

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■ 5. Calculate the Laplace transform.

$$\mathscr{L}(e^{3t-6}\cos(4-2t)u(t-2))$$

■ 6. Find the Laplace transform of the function.

$$f(t)=\begin{cases} 1-t & 0\le x<2 \\ -1 & 2\le x<4 \\ (5t-20)^3e^{3t-12} & x\ge 4 \end{cases}$$

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LAPLACE TRANSFORMS OF STEP FUNCTIONS

■ 1. Find the inverse Laplace transform.

$$F(s) = \frac{3se^{-7s}}{s^2 - 5s + 6}$$

■ 2. Find the inverse Laplace transform.

$$G(s) = \frac{(2s^2 + 1)e^{-3s}}{s^3 - 4s^2 + 3s}$$

■ 3. Find the Laplace transform of the function $g(t)$.

$$g(t) = 5u_1(t) + 6(t - 3)^3 u_3(t) + 7e^4 u_4(t)$$

■ 4. Find the Laplace transform of the function $g(t)$.

$$f(t) = \begin{cases} 3 & 0 \le t < 2 \\ t + 1 & 2 \le t < 5 \\ 4t - 14 & t \ge 5 \end{cases}$$

■ 5. Find the Laplace transform of the function $g(t)$.

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$$f(t) = \begin{cases} 0 & 0 \le t < \frac{\pi}{2} \\ 3\sin t & \frac{\pi}{2} \le t < \pi \\ \cos t & t \ge \pi \end{cases}$$

■ 6. Find the inverse Laplace transform.

$$G(s) = \frac{3}{s} + e^{-s}\left(\frac{1}{s} - \frac{2}{s-4}\right) + e^{-3s}\frac{1}{s-9}$$

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STEP FUNCTIONS WITH INITIAL VALUE PROBLEMS

■ 1. Solve the initial value problem, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' = \begin{cases} 2 & 0 \leq t < 4 \\ e^{2t} & t \geq 4 \end{cases}$$

■ 2. Solve the initial value problem, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' - 9y = e^{2t-10}u(t-5)$$

■ 3. Solve the initial value problem, given $y(0) = -1$ and $y'(0) = 1$.

$$y'' - 4y' = g(t)$$

$$g(t) = \begin{cases} 0 & 0 \leq t < 5 \\ -3 & t \geq 5 \end{cases}$$

■ 4. Solve the initial value problem, given $y(0) = 0$ and $y'(0) = 0$.

$$y'' - 5y' + 6y = tu(t-4)$$

■ 5. Solve the initial value problem, given $y(0) = -2$ and $y'(0) = 1$.

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$$y'' - y' = \sin(3t - 9)u(t - 3)$$

■ 6. Solve the initial value problem, given $y(0) = 0$ and $y'(0) = 2$.

$$y'' - 6y' = \begin{cases} 1 & 0 \le t < 2 \\ 3t + 1 & t \ge 2 \end{cases}$$

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THE DIRAC DELTA FUNCTION

■ 1. Solve the initial value problem, given $y(0)=1$ and $y'(0)=0$.

$$y''+4y=4\delta(t-2)$$

■ 2. Solve the initial value problem, given $y(0)=0$ and $y'(0)=1$.

$$y''+2y'+3y=\delta(t-5)$$

■ 3. Solve the initial value problem, given $y(0)=2$ and $y'(0)=0$.

$$y''-y'=3u(t-1)$$

■ 4. Solve the initial value problem, given $y(0)=1$ and $y'(0)=1$.

$$y''+y'-2y=3u(t-4)$$

■ 5. Solve the initial value problem, given $y(0)=-2$ and $y'(0)=1$.

$$y''-2y=u(t-7)-2\delta(t-7)$$

■ 6. Solve the initial value problem, given $y(0)=-1$ and $y'(0)=-2$.

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$$y'' + 2y' + y = 7\delta(t-5) + 5u(t-3)$$

CONVOLUTION INTEGRALS

■ 1. Find the convolution of $f(t) = e^{2t}$ and $g(t) = 2t$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

■ 2. Use a convolution integral to find the inverse transform.

$$H(s) = \frac{s^2}{(s^2 + 1)^2}$$

■ 3. Find the convolution of $f(t) = \cos t$ and $g(t) = t^2$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

■ 4. Find the convolution of $f(t) = e^{2t}$ and $g(t) = e^{-3t}$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

■ 5. Use a convolution integral to find the inverse transform of the following transform.

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$$H(s)=\frac{1}{(s-5)(s-6)}$$

■ 6. Find the convolution of $f(t)=t^3$ and $g(t)=8$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

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CONVOLUTION INTEGRALS FOR INITIAL VALUE PROBLEMS

■ 1. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=0$ and $y'(0)=0$.

$$y''+4y=g(t)$$

■ 2. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=0$ and $y'(0)=-2$.

$$y''+2y'-3y=g(t)$$

■ 3. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=1$ and $y'(0)=0$.

$$y''-y'-2y=e^{5t}$$

■ 4. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0)=2$ and $y'(0)=-1$.

$$y''+3y'=g(t)$$

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■ 5. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = 1$ and $y'(0) = 1$.

$$y'' - 5y' + 4y = g(t)$$

■ 6. Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = -1$ and $y'(0) = 0$.

$$y'' - 2y' - 8y = g(t)$$

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MATRIX BASICS

■ 1. Find $3A-(1/2)B+2C$.

$$A=\begin{bmatrix}0 & 1 & -\frac{1}{3}\\ 2 & \frac{1}{3} & 4\\ \frac{2}{3} & -1 & 2\end{bmatrix}\qquad B=\begin{bmatrix}2 & 0 & -4\\ -2 & 6 & 4\\ 10 & 0 & 0\end{bmatrix}\qquad C=\begin{bmatrix}1 & 0 & 3\\ -2 & 10 & 1\\ -4 & 3 & 0\end{bmatrix}$$

■ 2. For the 2×2 matrix, compute the determinant.

$$\begin{bmatrix}\frac{3}{2} & 1\\ 2 & -6\end{bmatrix}$$

■ 3. For the 3×3 matrix, compute the determinant.

$$\begin{bmatrix}-1 & 5 & 0\\ \frac{1}{2} & -\frac{3}{2} & 1\\ 4 & 0 & 2\end{bmatrix}$$

■ 4. Find the Eigenvalues and Eigenvectors of the matrix.

$$A=\begin{bmatrix}1 & 3\\ -1 & 5\end{bmatrix}$$

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5. Find the Eigenvalues and Eigenvectors of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -4 & -4 & 0 \\ -2 & 1 & 2 \end{bmatrix}$$

6. Find the Eigenvalues and Eigenvectors of the matrix.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 0 & -9 \\ 4 & 1 & 0 \end{bmatrix}$$

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BUILDING SYSTEMS

■ 1. Rewrite the system of differential equations in matrix form.

$$x_1' = 3x_1 - 4x_3 + e^t$$

$$x_2' = 5x_2 + x_1 - 13x_3$$

$$x_3' = \sin(t^2) - 3x_1 - 2x_2$$

■ 2. Rewrite the system of differential equations with the initial conditions in matrix form.

$$x_1' = -x_1 + 2x_3 - t^2 \qquad x_1(0) = 2$$

$$x_2' = 9x_2 - x_3 \qquad x_2(0) = -3$$

$$x_3' = 2x_2 - 4x_3 + 5\cos t \qquad x_3(0) = 0$$

■ 3. Convert the fourth order linear differential equation into a system of differential equations in matrix form.

$$3y^{(4)} + 6y''' - 18y'' - 20t\cos(t^2) = e^t$$

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■ 4. Convert the given initial value problem into a system of differential equations and rewrite it as a matrix equation.

$$y''' - 4ty' + 7y - t^2y'' = 2t^3 - 4$$

$$y(0) = -2,\ y'(0) = 1,\ y''(0) = 0 \qquad \vec{x}(0) = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

■ 5. Convert the third order linear differential equation into a system of differential equations in matrix form.

$$e^ty''' - 5y'' + y' - t^2y + e^{2t} = 0$$

■ 6. Convert the fourth order linear differential equation with the given initial conditions into a system of differential equations in matrix form.

$$y^{(4)} - (\cos t)y''' + 3e^ty'' - 2y = 0$$

$$y(0) = 3,\ y'(0) = 1,\ y''(0) = -2,\ y'''(0) = 0$$

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SOLVING SYSTEMS

■ 1. Show that the vector is a solution to the system.

$$x_1' = x_1 + 3x_2 - e^{2t}$$

$$x_2' = 3x_1 + x_2$$

$$\vec{x_1} = \begin{bmatrix} -3 \\ 3 \end{bmatrix} e^{-2t} + \begin{bmatrix} \frac{1}{8} \\ \frac{3}{8} \end{bmatrix} e^{2t}$$

■ 2. Show that the vector is a solution to the system.

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ -2 & 0 & -1 \\ 2 & 1 & 2 \end{bmatrix} \vec{x}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{t} + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} e^{2t}$$

■ 3. Confirm that the vectors are linearly independent.

$$\vec{x_1} = \begin{bmatrix} 3 \\ -3 \\ 7 \end{bmatrix} e^{3t} \qquad \vec{x_2} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{6t} \qquad \vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{7t}$$

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■ 4. Verify that the solution vectors satisfy the system of differential equations. Calculate the Wronskian of the solution vectors and write the general solution of the homogeneous system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{x}$$

$$\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{t}, \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} e^{-t}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} e^{2t} \right\}$$

■ 5. Verify that the solution vectors satisfy the system of differential equations. Calculate the Wronskian of the solution vectors and write the general solution of the homogeneous system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 6 & 5 \\ -5 & 0 \end{bmatrix} \vec{x}$$

$$\{\vec{x}_1, \vec{x}_2\} = \left\{ \begin{bmatrix} -3\cos(4t) + 4\sin(4t) \\ 5\cos(4t) \end{bmatrix} e^{3t}, \begin{bmatrix} -4\cos(4t) - 3\sin(4t) \\ 5\sin(4t) \end{bmatrix} e^{3t} \right\}$$

■ 6. Verify that the solution vectors satisfy the system of differential equations. Calculate the Wronskian of the solution vectors and write the general solution of the homogeneous system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 5 & -1 & -1 \\ 0 & 4 & -1 \\ 0 & -1 & 4 \end{bmatrix} \vec{x}$$

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$$\{\vec{x_1}, \vec{x_2}, \vec{x_3}\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{3t}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{5t}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} e^{5t} \right\}$$

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DISTINCT REAL EIGENVALUES

■ 1. Solve the system of differential equations.

$$x_1' = 4x_1 + 8x_2$$

$$x_2' = x_1 + 2x_2$$

■ 2. Solve the system of differential equations.

$$x_1' = 7x_1 + 6x_2 + 3x_3$$

$$x_2' = 2x_2 - x_1 + x_3$$

$$x_3' = x_1 + 2x_2 + x_3$$

■ 3. Solve the system of differential equations, given $x_1(0) = -2$ and $x_2(0) = 19$.

$$x_1' = x_1 + 2x_2$$

$$x_2' = 5x_1 - 2x_2$$

■ 4. Convert the differential equation into a system of equations, then use the system to solve the initial value problem, given $y(0) = 2$ and $y'(0) = 6$.

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$$y'' - 6y' - 7y = 0$$

■ 5. Convert the differential equation into a system of equations, then solve the system.

$$y''' - 2y'' - 9y' + 18y = 0$$

■ 6. Solve the initial value problem, given $y(0) = 2$, $y'(0) = 6$, and $y''(0) = 16$.

$$y''' - 6y'' + 11y' - 6y = 0$$

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EQUAL REAL EIGENVALUES WITH MULTIPLICITY TWO

■ 1. Solve the system of the differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

■ 2. Solve the system of the differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix} \vec{x}$$

■ 3. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 0 & 4 \\ -1 & 1 & 0 \end{bmatrix} \vec{x}$$

■ 4. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

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■ 5. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \vec{x}$$

■ 6. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 0 \\ 4 & 0 & 1 \end{bmatrix} \vec{x}$$

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EQUAL REAL EIGENVALUES WITH MULTIPLICITY THREE

■ 1. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

■ 2. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \vec{x}$$

■ 3. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{x}$$

■ 4. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 0 & 2 & 6 \\ -2 & -4 & -5 \\ 0 & 0 & -2 \end{bmatrix} \vec{x}$$

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■ 5. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} -3 & 2 & 0 \\ -2 & 1 & 0 \\ -2 & 2 & -1 \end{bmatrix} \vec{x}$$

■ 6. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

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COMPLEX EIGENVALUES

■ 1. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 2 & 4 \\ -2 & -2 \end{bmatrix} \vec{x}$$

■ 2. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} -6 & 4 \\ -8 & 2 \end{bmatrix} \vec{x}$$

■ 3. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ 5 & 5 \end{bmatrix} \vec{x}$$

■ 4. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 6 & -10 \\ 5 & -4 \end{bmatrix} \vec{x}$$

■ 5. Find the general solution to the system of differential equations.

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$$\vec{x}' = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 2 & 5 \\ 2 & -1 & 4 \end{bmatrix} \vec{x}$$

■ 6. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 0 & -5 & 1 \end{bmatrix} \vec{x}$$

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PHASE PORTRAITS FOR DISTINCT REAL EIGENVALUES

■ 1. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -4 & 4 \\ -2 & 5 \end{bmatrix}$$

■ 2. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$$

■ 3. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

■ 4. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix}$$

■ 5. Sketch the phase portrait of the system.

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$$A = \begin{bmatrix} -2 & 4 \\ 0 & 0 \end{bmatrix}$$

■ 6. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -3 & -1 \\ 0 & -4 \end{bmatrix}$$

PHASE PORTRAITS FOR EQUAL REAL EIGENVALUES

■ 1. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 6 & 2 \\ -8 & -2 \end{bmatrix}$$

■ 2. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -3 & -4 \\ 4 & 5 \end{bmatrix}$$

■ 3. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -1 & 9 \\ -1 & -7 \end{bmatrix}$$

■ 4. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 10 & 4 \\ -4 & 2 \end{bmatrix}$$

■ 5. Sketch the phase portrait of the system.

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$$A = \begin{bmatrix} 16 & -5 \\ 5 & 6 \end{bmatrix}$$

■ 6. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & 4 \\ -1 & -2 \end{bmatrix}$$

PHASE PORTRAITS FOR COMPLEX EIGENVALUES

■ 1. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 5 & 1 \\ -5 & 2 \end{bmatrix}$$

■ 2. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 1 & -3 \\ 2 & 2 \end{bmatrix}$$

■ 3. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 2 & -1 \\ 10 & -4 \end{bmatrix}$$

■ 4. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix}$$

■ 5. Sketch the phase portrait of the system.

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$$A = \begin{bmatrix} -2 & -4 \\ 2 & 2 \end{bmatrix}$$

■ 6. Sketch the phase portrait of the system.

$$A = \begin{bmatrix} 1 & 2 \\ -13 & -1 \end{bmatrix}$$

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UNDETERMINED COEFFICIENTS FOR NONHOMOGENEOUS SYSTEMS

■ 1. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} \cos t \\ 2 \sin t \end{bmatrix}$$

■ 2. Use undetermined coefficients to find the general solution to the initial value problem.

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 2e^{3t} \\ t^2 \end{bmatrix}$$

$$\vec{x}(0) = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{4} \end{bmatrix}$$

■ 3. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 6e^{4t} \\ \cos(3t) \\ 1 \end{bmatrix}$$

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■ 4. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} -e^t \\ 3t^2 \\ 2e^{-t} \end{bmatrix}$$

■ 5. Use the method of undetermined coefficients to find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 3 & 0 \\ 1 & 4 \end{bmatrix} \vec{x} + \begin{bmatrix} t + e^t \\ 2t^2 + 1 \end{bmatrix}$$

■ 6. Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ -2 & 1 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 3e^t \\ 3t + 2e^{3t} \\ t^3 - 4e^t \end{bmatrix}$$

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VARIATION OF PARAMETERS FOR NONHOMOGENEOUS SYSTEMS

■ 1. Use the method of variation of parameters to find the general solution to the system, given the complementary solution.

$$\vec{x}' = A\vec{x} + \begin{bmatrix} e^{-3t} \\ 2e^t + 1 \end{bmatrix}$$

$$\vec{x}_c = c_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 2 \\ 7 \end{bmatrix} e^{5t}$$

■ 2. Find the general solution to the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 6 & -8 \\ 4 & -6 \end{bmatrix} \vec{x} + \begin{bmatrix} 2t + 1 \\ e^t \end{bmatrix}$$

■ 3. Solve the system of differential equations.

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} \sin t + \cos t \\ \sin t \end{bmatrix}$$

■ 4. Solve the system of differential equations.

$$x_1' = -5x_1 + 12x_2 + e^t$$

$$x_2' = -4x_1 + 9x_2 + 3e^{-t}$$

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■ 5. Solve the system of differential equations.

$$x_1' = x_1 + 3x_2 + 1 + e^{2t}$$

$$x_2' = x_2 + e^t$$

■ 6. Solve the system of differential equations.

$$\vec{x}\,' = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} t^5 \\ t^3 \end{bmatrix}$$

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THE MATRIX EXPONENTIAL

■ 1. Use an inverse Laplace transform to calculate the matrix exponential.

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$$

■ 2. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 1 & -1 \\ -3 & 3 \end{bmatrix} \vec{x}$$

■ 3. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 0 & -1 \\ -3 & 2 \end{bmatrix} \vec{x}$$

■ 4. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} 6t^2 - te^{-3t} \\ \sin t \end{bmatrix}$$

■ 5. Use the matrix exponential to find the general solution to the system.

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$$\vec{x}' = \begin{bmatrix} -1 & 0 & 0 \\ -2 & 0 & 3 \\ 3 & -1 & -4 \end{bmatrix} \vec{x}$$

■ 6. Use the matrix exponential to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ e^{3t} \\ 6t^2 \end{bmatrix}$$

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HOMOGENEOUS HIGHER ORDER EQUATIONS

■ 1. Find the general solution to the third order homogeneous differential equation.

$$y''' - 3y'' - y' + 3y = 0$$

■ 2. Find the general solution to the third order homogeneous differential equation.

$$y''' + 6y'' + 8y' = 0$$

■ 3. Find the general solution to the third order homogeneous differential equation.

$$y''' + 15y'' + 75y' + 125y = 0$$

■ 4. Find the general solution to the third order homogeneous differential equation.

$$y''' - 5y'' + 12y' - 8y = 0$$

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■ 5. Find the general solution to the fifth order homogeneous differential equation.

$$y^{(5)} - 2y^{(4)} + 2y''' = 0$$

■ 6. Find the general solution to the seventh order homogeneous differential equation.

$$y^{(7)} + 15y^{(6)} + 125y^{(5)} + 595y^{(4)} + 1{,}795y''' + 2{,}861y'' + 2{,}175y' + 625y = 0$$

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UNDETERMINED COEFFICIENTS FOR HIGHER ORDER EQUATIONS

■ 1. Find the general solution to the third order differential equation.

$$y''' + y'' - 4y' - 4y = 8t^2$$

■ 2. Find the general solution to the third order differential equation.

$$y''' - 6y'' + 9y' - 4y = 1 + e^{2t} + 9e^{4t}$$

■ 3. Find the general solution to the differential equation. Hint: $x^4 + 4x^3 + 7x^2 + 6x + 2 = (x+1)^2(x^2 + 2x + 2)$.

$$y'''' + 4y''' + 7y'' + 6y' + 2y = e^{-t}(\cos t + 2\sin t + 4)$$

■ 4. Find the general solution to the differential equation.

$$y''' + 9y'' + 27y' + 27y = 20t^2e^{-3t}$$

■ 5. Find the general solution to the differential equation.

$$y''' + 4y' = \cos(2t) - 1$$

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■ 6. Find the general solution to the differential equation.

$$y'''' + 2y'' + y = t^2 + 8\sin t$$

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VARIATION OF PARAMETERS FOR HIGHER ORDER EQUATIONS

■ 1. Use variation of parameters to find the general solution to the differential equation.

$$y''' - y'' = te^t + 2t + 1 + 3\sin t$$

■ 2. Use variation of parameters to find the general solution to the differential equation.

$$y'' - 5y' + 6y = te^{2t} - e^{3t}$$

■ 3. Use variation of parameters to find the general solution to the differential equation.

$$y''' - y'' - 6y' = t^2e^{3t} - t + 1$$

■ 4. Use variation of parameters to find the general solution to the differential equation.

$$y''' - 6y'' + 5y' + 12y = 10te^{3t} - 4\cos(2t)$$

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■ 5. Use variation of parameters to find the general solution to the differential equation.

$$y''' - y'' + 9y' - 9y = 60\cos(3t)$$

■ 6. Use variation of parameters to find the general solution to the differential equation, given $y(0) = 0$, $y'(0) = -1$, $y''(0) = 10$.

$$y''' + y'' - y' - y = 8e^{-t}$$

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LAPLACE TRANSFORMS FOR HIGHER ORDER EQUATIONS

■ 1. Use the Laplace transform to solve the differential equation, given $y(0)=0$, $y'(0)=0$, and $y''(0)=0$.

$$y'''+6y''+8y'=8e^{-2t}$$

■ 2. Use the Laplace transform to solve the differential equation, given $y(0)=0$, $y'(0)=0$, and $y''(0)=1$.

$$y'''-3y'+2y=12e^{t}+24e^{2t}$$

■ 3. Use the Laplace transform to solve the differential equation, given $y(0)=0$, $y'(0)=1$, and $y''(0)=2$.

$$y'''-y''-5y'-3y=2e^{3t}+10\sin t$$

■ 4. Use the Laplace transform to solve the differential equation, given $y(0)=0$, $y'(0)=0$, and $y''(0)=0$.

$$y'''-2y''+y'-2y=5+25\sin t$$

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■ 5. Use the Laplace transform to solve the differential equation, given $y(0) = -2$, $y'(0) = 3$, and $y''(0) = 6$.

$$y''' - 6y'' + 12y' - 8y = 27e^{-t} - 2e^{t}$$

■ 6. Use the Laplace transform to solve the differential equation, given $y(0) = 3$, $y'(0) = 0$, and $y''(0) = -9$.

$$y''' + 3y'' + 4y' + 2y = -8\sin t - 6\cos t$$

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SYSTEMS OF HIGHER ORDER EQUATIONS

■ 1. Solve the system of differential equations.

$$x_1' = 2x_1$$

$$x_2' = 4x_3 - x_2$$

$$x_3' = -3x_1$$

■ 2. Solve the system of differential equations.

$$x_1' = 2x_2 - 2x_3$$

$$x_2' = 2x_1 + 4x_2 + 4x_3$$

$$x_3' = -2x_1 + 4x_2 - 3x_3$$

■ 3. Solve the system of differential equations.

$$x_1' = 2x_1 + 4x_3$$

$$x_2' = -x_1 + 2x_2$$

$$x_3' = -x_3$$

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■ 4. Solve the system of differential equations.

$$x_2' = -x_1 + 3x_2 + 5x_3$$

$$x_3' = 2x_1 - 2x_2 + x_3$$

■ 5. Solve the system of differential equations.

$$x_1' = x_1 - x_2$$

$$x_2' = 5x_1 - x_2$$

$$x_3' = -3x_1 + x_2 + 4x_3$$

■ 6. Solve the system of differential equations.

$$x_1' = -x_1 + 3x_2 - x_3$$

$$x_2' = -x_2 + 2x_3$$

$$x_3' = -x_3$$

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SERIES SOLUTIONS OF HIGHER ORDER EQUATIONS

■ 1. Find a power series solution in x to the differential equation.

$$y''' - xy' = 0$$

■ 2. Find a power series solution in x to the differential equation.

$$y''' - y'' = 0$$

■ 3. Find a power series solution in x to the differential equation.

$$xy''' - y' = 0$$

■ 4. Find a power series solution in x to the differential equation.

$$x^2y''' + y'' - xy = 0$$

■ 5. Find a power series solution in x to the differential equation.

$$y''' + xy' + y = 0$$

■ 6. Find a power series solution in x to the differential equation.

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$$y''' + xy'' - y = 0$$

FOURIER SERIES REPRESENTATIONS

■ 1. Find the Fourier series representation of $f(x) = (2x + 1)^2$ on $-L \leq x \leq L$.

■ 2. Find the Fourier series representation of $f(x) = e^x$ on $-\pi \leq x \leq \pi$.

■ 3. Find the Fourier series representation of the function on $-\pi \leq x \leq \pi$.

$$f(x) = \sin\left(\frac{x}{2}\right)$$

■ 4. Find the Fourier series representation of $f(x) = x \sin x$ on $-\pi \leq x \leq \pi$.

■ 5. Find the Fourier series representation of $f(x) = x^2 + \sin x$ on $-L \leq x \leq L$.

■ 6. Find the Fourier series representation of $f(x) = \cosh x$ on $-\pi \leq x \leq \pi$.

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PERIODIC FUNCTIONS AND PERIODIC EXTENSIONS

■ 1. Sketch the function's periodic extension on the interval $-L \leq x \leq L$, given some positive value of L.

$$f(x) = \frac{1}{L}x^2 - L$$

■ 2. Sketch the function's periodic extension on the interval $-L \leq x \leq L$.

$$f(x) = \begin{cases} L + x & x < 0 \\ L - x & x \geq 0 \end{cases}$$

■ 3. Find the even and odd extensions of the function, given some positive value of L. Then sketch both extensions.

$$f(x) = \frac{1}{L^2}x^3 - \frac{1}{L}x^2$$

■ 4. Find the even and odd extensions of the function, given some positive value of L. Compare both sketches.

$$f(x) = \begin{cases} L^2 & \frac{L}{2} < x \\ 2Lx & -\frac{L}{2} \leq x \leq \frac{L}{2} \\ -L^2 & x < -\frac{L}{2} \end{cases}$$

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■ 5. Find the even and odd extensions of the function, given some positive value of L.

$$f(x) = e^{L-2x}$$

■ 6. Sketch the periodic extensions of the even and odd extensions of the function $f(x) = x$ on $-L \leq x \leq L$.

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REPRESENTING PIECEWISE FUNCTIONS

■ 1. Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} -2x & -L \le x < 0 \\ 3x & 0 \le x \le L \end{cases}$$

■ 2. Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} 2x + 4 & -L \le x < 0 \\ 4 - 2x & 0 \le x \le L \end{cases}$$

■ 3. Find the Fourier series representation of the piecewise function on $-L \le x \le L$.

$$f(x) = \begin{cases} 1 & -L \le x < 0 \\ 3\left(x + \frac{L}{2}\right) + 1 & 0 \le x \le L \end{cases}$$

■ 4. Find the Fourier series representation of the piecewise function on $-\pi \le x \le \pi$.

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$$f(x) = \begin{cases} (x+1)^2 & -\pi \le x < 0 \\ 2-(x-1)^2 & 0 \le x \le \pi \end{cases}$$

■ 5. Find the Fourier series representation of the piecewise function.

$$f(x) = \begin{cases} 3x+6 & -L \le x < -1 \\ -3x & -1 \le x \le 1 \\ 3x-6 & 1 < x \le L \end{cases}$$

■ 6. Find the Fourier series representation of the piecewise function.

$$f(x) = \begin{cases} 2-2x & -L \le x < 0 \\ 4x+3 & 0 \le x \le L \end{cases}$$

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CONVERGENCE OF A FOURIER SERIES

■ 1. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

$$f(x) = \begin{cases} -1 & -L \le x < 0 \\ x^2 & 0 \le x \le L \end{cases}$$

$$f(x) = -\frac{1}{2} + \frac{L^2}{6} + \frac{2L^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

$$+\sum_{n=1}^{\infty}\frac{1}{n^3\pi^3}\left[(2L^2 - L^2n^2\pi^2 - n^2\pi^2)(-1)^n + n^2\pi^2 - 2L^2\right]\sin\left(\frac{n\pi x}{L}\right)$$

■ 2. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

$$f(x) = \begin{cases} -x^2 + 1 & -L \le x < 0 \\ L - x & 0 \le x \le L \end{cases}$$

$$f(x) = \frac{6 + 3L - 2L^2}{12} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{1 - (2L+1)(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

$$+\frac{1}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{n^3}\left[(n^2\pi^2(1 - L^2) + 2L^2)(-1)^n + n^2\pi^2(L-1) - 2L^2\right]\sin\left(\frac{n\pi x}{L}\right)$$

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■ 3. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$, assuming $L > 1$.

$$f(x) = \begin{cases} L & -L \le x < -1 \\ x & -1 \le x \le 1 \\ -L & 1 < x \le L \end{cases}$$

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{2L}{n\pi}(-1)^n - \frac{2(L+1)}{n\pi} \cos\left(\frac{n\pi}{L}\right) + \frac{2L}{n^2\pi^2} \sin\left(\frac{n\pi}{L}\right) \right] \sin\left(\frac{n\pi x}{L}\right)$$

■ 4. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

$$f(x) = \begin{cases} x + L & -L \le x < 0 \\ -1 & 0 \le x \le L \end{cases}$$

$$f(x) = \frac{L-2}{4} + \frac{L}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right) + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1 - L}{n} \sin\left(\frac{n\pi x}{L}\right)$$

■ 5. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$, assuming $L > 1$.

$$f(x) = \begin{cases} x^2 & |x| < 1 \\ 0 & 1 \le |x| \le L \end{cases}$$

$$f(x) = \frac{1}{3L} + \frac{2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \left[(n^2\pi^2 - 2L^2)\sin\left(\frac{n\pi}{L}\right) + n\pi L \cos\left(\frac{n\pi}{L}\right) \right] \cos\left(\frac{n\pi x}{L}\right)$$

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■ 6. Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

$$f(x) = \begin{cases} Lx & 0 \le x \le L \\ -x - L & -L \le x < 0 \end{cases}$$

$$f(x) = \frac{L^2 - L}{4} + \frac{L}{\pi} \sum_{n=1}^{\infty} \frac{1 - L(-1)^n}{n} \sin\left(\frac{n\pi x}{L}\right) + \frac{L(L+1)}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

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FOURIER COSINE SERIES

■ 1. Find the Fourier cosine series representation of the function $f(x) = 3x + 3$ on $0 \leq x \leq L$.

■ 2. Find the Fourier cosine series representation of the function $f(x) = 3x^2 + 2x$ on $0 \leq x \leq L$.

■ 3. Find the Fourier cosine series representation of the function $f(x) = x^3$ function on $0 \leq x \leq \pi$.

■ 4. Find the Fourier cosine series representation of the function $f(x) = e^x$ on $0 \leq x \leq L$.

■ 5. Find the Fourier cosine series representation of the function $f(x) = x \cos x$ on $0 \leq x \leq 2\pi$.

■ 6. Find the Fourier cosine series representation of the function $f(x) = \sin(3x + 5)$ on $0 \leq x \leq \pi$.

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FOURIER SINE SERIES

■ 1. Find the Fourier sine series representation of the function $f(x) = 2x^3 - 3x$ on $-L \leq x \leq L$.

■ 2. Find the Fourier sine series representation of the function $f(x) = -x^3$ on $-L \leq x \leq L$.

■ 3. Find the Fourier sine series representation of the function $f(x) = x^3 - 2x$ on $-L \leq x \leq L$.

■ 4. Find the Fourier sine series representation of the function $f(x) = x - x^2$ on $0 \leq x \leq L$.

■ 5. Find the Fourier sine series representation of $f(x) = x^2 - Lx$ on $0 \leq x \leq L$.

■ 6. Find the Fourier sine series representation of $f(x) = e^{L-x} + 5$ on $0 \leq x \leq L$.

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COSINE AND SINE SERIES OF PIECEWISE FUNCTIONS

■ 1. Find the Fourier cosine series representation of the piecewise function on $0 \leq x \leq L$.

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \frac{L}{2} \\ x^2 & \frac{L}{2} < x \leq L \end{cases}$$

■ 2. Find the Fourier sine series representation of the piecewise function on $0 \leq x \leq 2\pi$.

$$f(x) = \begin{cases} 2x & 0 \leq x \leq \pi \\ 4\pi - 2x & \pi < x \leq 2\pi \end{cases}$$

■ 3. Find the Fourier cosine series representation of the piecewise function on $0 \leq x \leq \pi$.

$$f(x) = \begin{cases} x^2 & 0 \leq x \leq \frac{\pi}{2} \\ x & \frac{\pi}{2} < x \leq \pi \end{cases}$$

■ 4. Find the Fourier sine series representation of the piecewise function on $0 \leq x \leq \pi$.

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$$f(x) = \begin{cases} \sin x & 0 \le x \le \frac{\pi}{2} \\ \sin 2x & \frac{\pi}{2} < x \le \pi \end{cases}$$

■ 5. Find the Fourier cosine series representation of the piecewise function on $0 \le x \le L$.

$$f(x) = \begin{cases} 3x & 0 \le x \le \frac{L}{3} \\ 3x - L & \frac{L}{3} < x \le \frac{2L}{3} \\ 3x - 2L & \frac{2L}{3} < x < L \end{cases}$$

■ 6. Find the Fourier sine series representation of the piecewise function on $0 \le x \le 2$.

$$f(x) = \begin{cases} x^2 & 0 \le x \le 1 \\ x^2 - 1 & 1 < x \le 2 \end{cases}$$

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SEPARATION OF VARIABLES

■ 1. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$\frac{1}{k^2}\left(\frac{\partial^2 u}{\partial x^2}\right)+\frac{\partial u}{\partial t}=0$$

■ 2. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$16u_{yy}+u_{xx}=0$$

■ 3. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$9\frac{\partial^2 u}{\partial x^2}-4\frac{\partial^2 u}{\partial y^2}=0$$

■ 4. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$-\frac{\partial^2 u}{\partial x^2}+25\frac{\partial^2 u}{\partial t^2}=0$$

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■ 5. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$-\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial t}=0$$

■ 6. Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial y^2}=0$$

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BOUNDARY VALUE PROBLEMS

■ 1. Solve the boundary value problem, if $y(0) = 3$ and $y'(0) = -4$.

$$y'' + 4y = 0$$

■ 2. Solve the boundary value problem, if $y(0) = 1$, $y'(0) = 0$, and $y''(0) = 5$.

$$y''' - y'' + 9y' - 9y = 0$$

■ 3. Solve the boundary value problem, if $u(\pi/3,t) = 2e^{3t}$, $(\partial u/\partial x)(\pi/3,t) = -3e^{3t}$, and $-\lambda = -9$.

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial u}{\partial t} = 0$$

■ 4. Solve the boundary value problem, if $u(\pi, t) = 0$, $u(x,0) = -2\cos(x/2)$, $(\partial u/\partial x)(\pi, t) = 2e^{-t} - e^{t}$, $(\partial u/\partial t)(x,0) = 6\cos(x/2)$, and $-\lambda = -1$.

$$4\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = 0$$

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■ 5. Solve the boundary value problem, if $u(0,y) = e^{8y} + 3e^{-8y}$, $u(x,0) = 8e^{2x} - 4e^{-2x}$, $(\partial u/\partial x)(0,y) = 6e^{8y} + 18e^{-8y}$, $(\partial u/\partial y)(x,0) = 16e^{-2x} - 32e^{2x}$, and $-\lambda = -64$.

$$\frac{\partial^2 u}{\partial y^2} - 16\frac{\partial^2 u}{\partial x^2} = 0$$

■ 6. Solve the boundary value problem, if $u = (0,y) = 4\sin(\sqrt{\lambda}y)$.

$$3\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial y^2} = 0$$

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THE HEAT EQUATION

■ 1. Find a solution to the partial differential equation if $u(x,0) = \sin(\pi x) + 3\sin(\pi x/2)$ and $u(0,t) = u(6,t) = 0$.

$$\frac{\partial u}{\partial t} = 9\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le 6$$

■ 2. Find a solution to the partial differential equation if $u(x,0) = 2\cos(\pi x/2L)$, $u_x(0,t) = 0$, and $u(L,t) = 0$.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le L$$

■ 3. Find a solution to the partial differential equation if $u(x,0) = 4\sin(\pi x/6)\cos^2(\pi x/6)$, $u(0,t) = 0$, and $u_x(3,t) = 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le 3$$

■ 4. Find a solution to the partial differential equation if $u(x,0) = 2x + 5$, $u_x(0,t) = 0$, and $u_x(L,t) = 0$.

$$\frac{\partial u}{\partial t} = 25\frac{\partial^2 u}{\partial x^2} \qquad\qquad 0 \le x \le L$$

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■ 5. Find a solution to the partial differential equation if $u(x,0) = x \sin x$, $u(0,t) = 0$, and $u(\pi, t) = 0$.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \pi$$

■ 6. Find a solution to the partial differential equation if $u(x,0) = \sin(\pi x)$, $u_x(0,t) = 0$, and $u_x(5,t) = 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 5$$

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CHANGING THE TEMPERATURE BOUNDARIES

■ 1. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \pi$$

$$u(x,0) = 12x + \pi$$

$$u(0,t) = \pi \qquad u(\pi, t) = 3\pi$$

■ 2. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 4$$

$$u(x,0) = x^2 + 1$$

$$u(0,t) = 1 \qquad u(4,t) = 17$$

■ 3. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 3\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1$$

$$u(x,0) = x + \sin(\pi x)$$

$$u(0,t) = 4 \qquad u(1,t) = 5$$

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■ 4. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 5\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 10$$

$$u(x,0) = 6 - 2x$$

$$u(0,t) = 10 \qquad u(10,t) = 50$$

■ 5. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \frac{\pi}{2}$$

$$u(x,0) = \cos x$$

$$u(0,t) = 1 \qquad u\left(\frac{\pi}{2}, t\right) = 0$$

■ 6. Solve the partial differential equation.

$$\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le \pi$$

$$u(x,0) = e^x + x + 1$$

$$u(0,t) = 1 \qquad u(\pi, t) = 1 + \pi$$

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LAPLACE'S EQUATION

■ 1. Use the product solution to find $u_1(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \leq x \leq \pi$ and $0 \leq y \leq 1$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = 3\sin(2x) \qquad u(0,y) = 0$$

$$u(x,1) = 0 \qquad u(\pi, y) = 0$$

■ 2. Use the product solution to find $u(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \leq x \leq 3$ and $0 \leq y \leq 3$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u(x,0) = 0 \qquad u(0,y) = 0$$

$$u(x,3) = 2\sin(\pi x) + \sin(2\pi x) \qquad u(3,y) = 0$$

■ 3. Use the product solution to find $u(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \leq x \leq \pi$ and $0 \leq y \leq 2\pi$.

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$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$u_y(x,0) = 0$ $\quad$ $u(0,y) = 2 + \cos(2y)$

$u_y(x,2\pi) = 0$ $\quad$ $u(\pi, y) = 0$

■ 4. Use the product solution to find $u(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \le x \le 4$ and $0 \le y \le 6$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$u(x,0) = 0$ $\quad$ $u_x(0,y) = 0$

$u_y(x,6) = 0$ $\quad$ $u(4,y) = \sin\left(\frac{\pi y}{4}\right) + 5\sin\left(\frac{\pi y}{12}\right)$

■ 5. Use the product solution to find $u_1(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \le x \le 1$ and $0 \le y \le 1$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$u(x,0) = x$ $\quad$ $u(0,y) = 0$

$u(x,1) = 0$ $\qquad$ $u(1,y) = \sin(3\pi y)$

■ 6. Use the product solution to find $u_1(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \le x \le 2\pi$ and $0 \le y \le 2\pi$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$u(x,0) = \cos x$ $\qquad$ $u_x(0,y) = 0$

$u(x,2\pi) = \cos x$ $\qquad$ $u_x(2\pi, y) = 0$

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