Differential Equations Notes

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Classifying differential equations

Let's start by clarifying the difference between partial derivatives and ordinary derivatives.

Partial derivatives

The **partial derivative** of a function is the derivative of that function with respect to one of the multiple variables in which the function is defined. For instance, given a function f defined in terms of two variables x and y, $f(x, y)$, we know that f has two partial derivatives:

The partial derivative of f with respect to x	$\frac{\partial f}{\partial x}$
The partial derivative of f with respect to y	$\frac{\partial f}{\partial y}$

We can't name only one derivative for f, since f is defined in two variables. Instead, we find one partial derivative of f for each of its variables, which is why, for $f(x, y)$, we end up with two partial derivatives.

When we define equations using partial derivatives like these ones, we call them **partial differential equations (PDEs)**. We usually study partial derivatives and their equations in a multivariable/multivariate calculus course, which is often Calculus III or Calculus IV.

That being said, these aren't the kinds of differential equations we'll focus on in this course. We'll spend the vast majority of our time focusing on ordinary differential equations, and we'll only touch briefly on partial differential equations at the very end of the course.

Ordinary derivatives

Whereas partial derivatives are indicated with the "partial symbol" ∂, we never see this notation when we're dealing with ordinary derivatives. That's because an **ordinary derivative** is the derivative of a function in a single variable. Because there's only one variable, there's no need to indicate the partial derivative for one variable versus another.

For example, given a function for y in terms of x, which we could write as $y(x)$, its first derivative can be written as $y'(x)$, or as just y', or in **Leibniz notation** as

$$\frac{dy}{dx}$$

So an equation like

$$\frac{dy}{dx} - \sin x \cos x = 2x$$

is an **ordinary differential equation** because it includes the ordinary derivative dy/dx.

Order of the differential equation

The **order** of a differential equation is equivalent to the degree of the highest-degree derivative that appears in the equation. For example, if the equation contains only a first derivative, we call it a first order differential equation. Here are some more examples:

Equation	**Order**
$y' - 2xy = x\cos y$	1
$5y'' - 2y + 5 = 0$	2
$\dfrac{d^4y}{dx^4} + e^x\dfrac{dy}{dx} = 0$	4

In this differential equations course, we'll be focusing primarily on first and second order differential equations. We're starting with first order equations now, and we'll get into second order equations later.

Linear differential equations

When it comes to classifying first order differential equations, we put them into two categories: linear and separable. We'll talk much more about each of these types later. For now, we only want to say that **linear differential equations** are equations given in the form

$$p_n(x)y^{(n)}(x) + p_{n-1}(x)y^{(n-1)}(x) + \ldots + p_1(x)y'(x) + p_0(x)y(x) = q(x)$$

where $p_i(x)$ and $q(x)$ are functions of x. If a first order ordinary differential equation doesn't match this form, we say that it's a non-linear equation.

What we want to take away from this definition of linear equations is that

1. all of the $p_i(x)$ coefficients are functions in terms of only x,

2. $q(x)$ is also a function in terms of only x,

3. the function y is never defined to a higher power than 1 (we should only see y and its derivatives y', y'', y''', etc., never something y^2, $\sin y$, e^y, etc.

Let's add a linear/non-linear classification to our table from earlier.

Equation	Order	Linearity
$y' - 2xy = x\cos y$	1	Non-linear
$5y'' - 2y + 5 = 0$	2	Linear
$\frac{d^4y}{dx^4} + e^x\frac{dy}{dx} = 0$	4	Linear

The first equation in this table is non-linear because $q(x) = x\cos y$, which means $q(x)$ is a function defined in terms of both x and y, not just x alone.

These second and third equations are linear equations because they meet the three conditions we outlined. Notice that, in both linear equations, $q(x) = 0$. When this is the case, we say that the linear equation is **homogeneous**. As you might suspect, when $q(x) \neq 0$ we call the linear equation **non-homogeneous**.

Let's work through some more examples so that we get comfortable classifying differential equations.

Example

Identify the order and linearity of each differential equation, then say whether or not each linear equation is homogeneous.

1. $8y''' + 2y' + \cos y = e^x$

2. $2y'' + 5y' = xy$

3. $e^x y' - e^{x+y} = e^{2x}$

4. $y''' + 5y' = y\cos x$

5. $y'' - 3y = \sin x$

6. $xy' + 2y = x^2 - x + 1$

1. The equation $8y''' + 2y' + \cos y = e^x$ contains a third derivative, so it's a third order equation. Because the equation contains $\cos y$, it's non-linear.

2. The equation $2y'' + 5y' = xy$ contains a second derivative, so it's a second order equation. We can rewrite it as $2y'' + 5y' - xy = 0$, which allows us to see that the equation is linear and homogeneous.

3. The equation $e^x y' - e^{x+y} = e^{2x}$ contains a first derivative, so it's a first order equation. Because the equation contains e^y, it's non-linear.

4. The equation $y''' + 5y' = y\cos x$ contains a third derivative, so it's a third order equation. We can rewrite it as $y''' + 5y' - y\cos x = 0$, which allows us to see that the equation is linear and homogeneous.

5. The equation $y'' - 3y = \sin x$ contains a second derivative, so it's a second order equation. The equation is already in the form of a linear equation, but $q(x) = \sin x \neq 0$, so it's non-homogeneous.

6. The equation $xy' + 2y = x^2 - x + 1$ contains a first derivative, so it's a first order equation. The equation is already in the form of a linear equation, but $q(x) = x^2 - x + 1 \neq 0$, so it's non-homogeneous.

We can summarize our findings in a table.

Equation	**Order**	**Linearity**	**Homogeneity**
$8y''' + 2y' + \cos y = e^x$	3	Non-linear	
$2y'' + 5y' = xy$	2	Linear	Homogeneous
$e^x y' - e^{x+y} = e^{2x}$	1	Non-linear	
$y''' + 5y' = y \cos x$	3	Linear	Homogeneous
$y'' - 3y = \sin x$	2	Linear	Non-homogeneous
$xy' + 2y = x^2 - x + 1$	1	Linear	Non-homogeneous

Linear equations

To investigate first order differential equations, we'll start by looking at equations given in a few very specific forms. The first of these is a first order linear differential equation.

Form of a first order linear differential equation

First order linear differential equations are equations given in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where $P(x)$ and $Q(x)$ are continuous functions. They always have a solution in the form $y = y(x)$. Realize that some linear equations might be disguised in a slightly different form. For instance, the equation

$$P(x)y = Q(x) - \frac{dy}{dx}$$

is a linear equation because it can be manipulated to match the form above simply by adding dy/dx to both sides of the equation.

Either way, these linear equations are first order equations because they contain a first derivative. Later we'll look at how to solve higher order linear differential equations (which contain higher order derivatives), but for now we'll focus only on first order equations.

When $Q(x) = 0$, the first order linear equation is homogeneous, and the equation simplifies to

$$\frac{dy}{dx} + P(x)y = 0$$

These homogeneous linear equations are also called separable differential equations, which we'll cover a little later. For now, we'll only say that the solution to a homogeneous linear equation is

$y = Ce^{K(x)}$, where $K(x) = -\int P(x)\ dx$

Here's a quick example showing how we can use these formulas to find the solution to the homogeneous linear equation (separable equation).

Example

Find the solution to the linear differential equation.

$$\frac{dy}{dx} - 2xy = 0$$

We start with

$$K(x) = -\int (-2x)\ dx = \int 2x\ dx = x^2$$

so the general solution is

$$y = Ce^{x^2}$$

Again, we'll talk more about these homogeneous linear equations (separable equations) later, but our focus here is really on solving nonhomogeneous first order linear equations, in which $Q(x) \neq 0$.

Solving equations with the integrating factor

Once we've identified that we have a first order linear differential equation, we can always follow the same set of steps to arrive at the solution to the equation, $y = y(x)$.

1. Make sure that the equation matches the general form of a first order linear differential equation

2. From the equation, identify $P(x)$ and $Q(x)$.

3. Find the equation's **integrating factor**, $I(x) = e^{\int P(x)\ dx}$.

4. Multiply through the linear equation by the integrating factor.

5. Reverse the product rule to rewrite the left side of the resulting equation as $(d/dx)[yI(x)]$.

6. Integrate both sides of the equation to find $y = y(x)$.

Let's work through an example so that we can see these steps in action.

Example

Find the solution to the linear differential equation.

$$x\frac{dy}{dx} - 2y = x^2$$

Our first step is to match the given equation to the standard form of a first order linear differential equation, which we'll do by dividing through both sides of the equation by x.

$$x\frac{dy}{dx} - 2y = x^2$$

$$\frac{dy}{dx} - \frac{2}{x}y = x \text{ (Step 1)}$$

This new form matches the standard form we need, which means we can now identify $P(x)$ and $Q(x)$ (Step 2) as

$$P(x) = -\frac{2}{x}$$

$$Q(x) = x$$

Notice that we included the negative sign in front of the $2/x$ as part of the $P(x)$ function. Any negative sign that precedes the $P(x)$ function should be included in $P(x)$, and any negative sign that leads the right side $Q(x)$ function should be included in $Q(x)$.

Now that we have $P(x)$, we can find the integrating factor (Step 3).

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int -\frac{2}{x}\ dx}$$

$$I(x) = e^{-2\int \frac{1}{x}\ dx}$$

$$I(x) = e^{-2\ln x}$$

$$I(x) = e^{\ln x^{-2}}$$

$$I(x) = x^{-2}$$

$$I(x) = \frac{1}{x^2}$$

When we integrated, notice that we didn't add C to account for the constant of integration, like we normally would whenever we're evaluating an indefinite integral.

We can always leave it out during this process, because we're only looking for one solution to the linear differential equation, and that one solution can be associated with $C = 0$. Other values of C could give other solutions to the linear equation, but we only need one solution, so for the sake of simplicity, we'll always choose $C = 0$.

Once we have the integrating factor, we'll multiply it by both sides of our equation (Step 4).

$$\frac{dy}{dx}\left(\frac{1}{x^2}\right) - \frac{2}{x}y\left(\frac{1}{x^2}\right) = x\left(\frac{1}{x^2}\right)$$

$$\frac{dy}{dx}\left(\frac{1}{x^2}\right) - \frac{2}{x^3}y = \frac{1}{x}$$

Rewriting dy/dx as y', we get

$$y'x^{-2} - 2yx^{-3} = x^{-1}$$

The reason we multiply through by the integrating factor is that it does something for us that's extremely convenient, even though we don't realize it yet.

It turns out that the left side of $y'x^{-2} - 2yx^{-3} = x^{-1}$, $y'x^{-2} - 2yx^{-3}$, is actually the derivative of yx^{-2}. This is interesting because yx^{-2} is the product of y and our integrating factor. In other words, $yI(x) = yx^{-2}$. We can prove this to ourselves if we take the derivative of yx^{-2}. We'll need to use the product rule for derivatives,

$$\frac{d}{dx}f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

Applying product rule, the derivative of yx^{-2} is

$$\frac{d}{dx}(yx^{-2}) = \left(\frac{d}{dx}y\right)(x^{-2}) + (y)\left(\frac{d}{dx}x^{-2}\right)$$

$$\frac{d}{dx}(yx^{-2}) = (y')(x^{-2}) + (y)(-2x^{-3})$$

$$\frac{d}{dx}(yx^{-2}) = y'x^{-2} - 2yx^{-3}$$

See how the derivative we just found matches $y'x^{-2} - 2yx^{-3}$, the left side of $y'x^{-2} - 2yx^{-3} = x^{-1}$? This is the magic of the integrating factor.

In other words, if we multiply our original linear differential equation by its integrating factor, the left side of the resulting equation will always be equal to

$$\frac{d}{dx}(yI(x))$$

Which means, coming back to this example, that we can make a substitution, replacing the left side of the resulting equation with $(d/dx)(yx^{-2})$ (Step 5), which gives

$$\frac{d}{dx}(yx^{-2}) = x^{-1}$$

Once we have the equation in this form, we integrate both sides (Step 6). On the left, the integral and the derivative cancel out, since they are inverse operations.

$$\int \frac{d}{dx}(yx^{-2}) = \int x^{-1}\ dx$$

$$yx^{-2} = \ln|x| + C$$

We solve for y, and find that the solution to the linear differential equation is

$$y = x^2(\ln|x| + C)$$

Let's do one more example to show that this method works on another first order linear differential equation, but this time we'll go a little faster.

Example

Solve the differential equation.

$$\frac{dy}{dx} + 2y = 4e^{-2x}$$

The equation is already in standard form for a linear differential equation, so we can identify $P(x) = 2$ and $Q(x) = 4e^{-2x}$, and then use $P(x)$ to find the integrating factor.

$$\rho(x) = e^{\int P(x)\ dx}$$

$$\rho(x) = e^{\int 2\ dx}$$

$$\rho(x) = e^{2x}$$

Multiply through the linear differential equation by the integrating factor.

$$\frac{dy}{dx}(e^{2x}) + 2y(e^{2x}) = 4e^{-2x}(e^{2x})$$

$$\frac{dy}{dx}e^{2x} + 2e^{2x}y = 4e^0$$

$$\frac{dy}{dx}e^{2x} + 2e^{2x}y = 4$$

We'll replace the left side of the equation with $(d/dx)(yI(x))$,

$$\frac{d}{dx}(ye^{2x}) = 4$$

then integrate both sides of the equation, remembering that the integral and derivative will cancel each other.

$$\int \frac{d}{dx}(ye^{2x})\ dx = \int 4\ dx$$

$$ye^{2x} = 4x + C$$

Dividing both sides by e^{2x} to get y by itself, we can say that the solution to the linear differential equation is

$$y = \frac{4x + C}{e^{2x}}$$

Why the integrating factor solves linear equations

The most interesting part about solving these linear equations is the idea of the integrating factor, $I(x) = e^{\int P(x)\,dx}$. Here's the reason this special value helps us to solve linear differential equations. If we start with a simplified version of the standard form of a linear equation,

$$y' + Py = Q$$

and then we multiply through by the integrating factor I, we get

$$y'I + PIy = QI$$

Now if we were somehow able to replace the left side of this equation with $(Iy)'$, the derivative of Iy, then we would have

$$(Iy)' = QI$$

We could then integrate both sides of this,

$$\int (Iy)' = \int QI$$

and the derivative and integral operations would cancel each other on the left side, leaving us with just

$$Iy = \int QI$$

We'd then be able to divide through by I, and thereby quickly and easily find the solution to the linear differential equation,

$$y = \frac{1}{I}\int QI$$

Okay, it's great that we've found a solution equation for y, but we only got here by making the significant assumption that we could just replace the left side of $y'I + PIy = QI$ with $(Iy)'$, and we can't just change equations like this without some kind of justification.

If we're going to do this, we need to show that it's valid to say

$$(Iy)' = y'I + PIy$$

Well, notice that the left side of this equation is the derivative of a product, which should make us think about the product rule for derivatives. If we treat I as one function and y as another, then the derivative of the product, $(Iy)'$, would be equal to

$$(Iy)' = I'y + Iy'$$

$$(Iy)' = y'I + I'y$$

If we compare this equation to $(Iy)' = y'I + PIy$, we can see that they are perfectly identical, as long as

$$I' = PI$$

If we rewrite this equation and then separate variables (we'll talk more about separating variables in a later lesson), we get

$$\frac{dI}{dx} = P(x)I(x)$$

$$dI = P(x)I(x)\ dx$$

$$\frac{1}{I(x)}\ dI = P(x)\ dx$$

Integrating both sides gives

$$\int \frac{1}{I(x)}\ dI = \int P(x)\ dx$$

$$\ln(I(x)) = \int P(x)\ dx$$

$$e^{\ln(I(x))} = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int P(x)\ dx}$$

This is how we build the formula for the integrating factor. We've shown why it's the special function that, when we use it to multiply through the linear differential equation, it gives us a really easy way to find the solution to that linear equation.

Initial value problems

In the last lesson about linear differential equations, all the general solutions we found contained a constant of integration, C. But we're often interested in finding a value for C in order to generate a particular solution for the differential equation.

This applies to linear differential equations, but also to any other form of differential equation. The information we'll need in order to find C is an **initial condition**, which is the value of the solution at a specific point.

Only one solution will satisfy the initial condition(s), which is why the initial condition(s) allow us to narrow down the equation of the general solution to one specific particular solution.

An initial condition for a first order differential equation will take the form

$$y(x_0) = y_0$$

and the number of initial conditions required for a given differential equation will depend on the order of the differential equation, which we'll talk more about later.

An **initial value problem (IVP)** is a differential equations problem in which we're asked to use some given initial condition, or set of conditions, in order to find the particular solution to the differential equation.

Solving initial value problems

In order to solve an initial value problem for a first order differential equation, we'll

1. Find the general solution that contains the constant of integration C.

2. Substitute the initial condition, $x = x_0$ and $y = y_0$, into the general solution to find the associated value of C.

3. Restate the general solution, and include the value of C found in step 2. This will be the particular solution of the differential equation.

Let's work through an example so that we can see these steps in action.

Example

Solve the initial value problem if $y(0) = -5$ in order to find the particular solution to the differential equation.

$$\frac{dy}{dx} + 2y = 4e^{-2x}$$

In the previous lesson, we used the integrating factor to find the general solution to this differential equation, and it was

$$y = \frac{4x + C}{e^{2x}}$$

Once we have this general solution, we recognize from the initial condition $y(0) = -5$ that $x = 0$ and $y = -5$, so we'll plug these values into the general solution,

$$-5 = \frac{4(0) + C}{e^{2(0)}}$$

and then simplify this to solve for C.

$$-5 = \frac{0 + C}{1}$$

$$C = -5$$

So the particular solution to the differential equation is

$$y = \frac{4x - 5}{e^{2x}}$$

We should realize from this last example that the general solution always remains the same, but the particular solution changes based on the initial condition we use. The particular solution we generated here was based on $y(0) = -5$, but we would have found a different particular solution for $y(0) = 2$, and yet another for $y(0) = 3$.

And as we said before, initial value problems can be used for all types of differential equations.

This last example was an initial value problem for a linear differential equation, but be on the lookout for initial value problems in the future as we look at lots of other types of differential equations.

Separable equations

We saw that first order linear equations are differential equations in the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

In contrast, **first order separable differential equations** are equations in the form

$$N(y)\frac{dy}{dx} = M(x)$$

$$N(y)y' = M(x)$$

We call these "separable" equations because we can separate the variables onto opposite sides of the equation. In other words, we can put the x terms on the right and the y terms on the left, or vice versa, with no mixing.

Notice that the two previous equations above representing a separable differential equation are identical, except for the derivative term, which is represented by dy/dx in the first equation and by y' in the second equation.

When we're working with separable differential equations, we usually prefer the dy/dx notation (Leibniz notation), because this notation makes it easier to see how to separate variables. Starting with dy/dx, we move the dx notation to the right, leaving just the dy notation on the left:

$$N(y)\ dy = M(x)\ dx$$

How to solve separable equations

Most often, we'll be given equations in the form

$$N(y)\frac{dy}{dx} = M(x)$$

Here's the standard process we'll follow to find a solution to this separable differential equation. First, we'll separate variables by moving the dx to the right side.

$$N(y)\ dy = M(x)\ dx$$

With variables separated, we'll integrate both sides of the equation.

$$\int N(y)\ dy = \int M(x)\ dx$$

Normally when we evaluate an indefinite integral, we need to add in a constant of integration. So after integrating, we would expect to have something like

$$\mathbf{N}(y) + C_1 = \mathbf{M}(x) + C_2$$

where $\mathbf{N}(y)$ is the integral of $N(y)$, and $\mathbf{M}(x)$ is the integral of $M(x)$. But when we solve for $\mathbf{N}(y)$, we get

$$\mathbf{N}(y) = \mathbf{M}(x) + C_2 - C_1$$

What we have to realize here is that, if C_2 and C_1 are constants, then $C_2 - C_1$ is also a constant. So we can make a substitution and represent $C_2 - C_1$ as just one simple constant C.

$$\mathbf{N}(y) = \mathbf{M}(x) + C$$

For this reason, we don't need to bother adding constants to both sides of the equation when we integrate. Instead of integrating to $\mathbf{N}(y) + C_1 = \mathbf{M}(x) + C_2$, we can always skip these intermediate steps and go straight to $\mathbf{N}(y) = \mathbf{M}(x) + C$.

Once we have the equation in this form, the goal will be to express y explicitly as a function of x, meaning that our solution equation is solved for y, like $y = f(x)$. Sometimes we won't be able to get y by itself on one side of the equation, and that's okay. If we can't, we'll just settle for an implicit function, which means that y and x aren't separated onto opposite sides of the solution.

In summary, our solution steps will be:

1. If necessary, rewrite the equation in Leibniz notation.

2. Separate variables with y terms on the left and x terms on the right

3. Integrate both sides of the equation, adding C to the right side

4. If possible, solve the solution equation specifically for y.

Let's work through an example so that we can see these steps in action.

Example

Find the solution of the separable differential equation.

$$y' = y^2 \sin x$$

Let's write the equation in Leibniz notation, changing y' to dy/dx.

$$\frac{dy}{dx} = y^2 \sin x$$

Separate the variables, collecting y terms on the left and x terms on the right.

$$dy = y^2 \sin x \ dx$$

$$\frac{1}{y^2} \ dy = \sin x \ dx$$

With variables separated, and integrating both sides, we get

$$\int \frac{1}{y^2} \ dy = \int \sin x \ dx$$

$$\int y^{-2} \ dy = \int \sin x \ dx$$

$$-y^{-1} = -\cos x + C$$

$$-\frac{1}{y} = -\cos x + C$$

$$\frac{1}{y} = \cos x + C$$

Notice how we just multiplied through the equation by -1, but we didn't change the sign on C. That's because keeping C is a little simpler than $-C$, and we'll still end up with the same solution equation either way.

Finally, solving for y gives the solution to the separable differential equation.

$$1 = y(\cos x + C)$$

$$y = \frac{1}{\cos x + C}$$

Substitutions

As we've seen, first order linear differential equations and first order separable differential equations always have a solution, and we know now how to find their solutions.

First order differential equations that aren't specifically linear or separable may or may not have a solution. To determine whether or not a non-linear, non-separable equation has a solution, one approach we can take is to try transforming the differential equation into a linear or separable equation.

In other words, if we can find a way to rewrite a non-linear, non-separable equation as an actual linear or separable differential equation, then of course we'll be able to find the solution to the equation.

Making a substitution

That's where substitution comes in. The substitution we want to make, also called a change of variable, is the one that allows us to convert the differential equation into a linear or separable equation.

Given an equation in terms of y, y', and x, we'll want to find a way to replace all of those with u and u'. Sometimes we'll be working with a differential equation in the form

$$y' = F(ax + by)$$

in which case we'll use the specific substitution

$$u = ax + by$$

$$u' = a + by'$$

Let's do an example to see what a substitution might look like.

Example

Use a change of variable to solve the differential equation.

$$y' = 2x + y$$

If we choose the substitution $u = ax + by$, then we can set up the substitution

$$u = 2x + y$$

$$u' = 2 + y'$$

and then solve this second equation $u' = 2 + y'$ for y' to get $y' = u' - 2$. Then we'll substitute $y' = u' - 2$ into the left side of the original differential equation, and $u = 2x + y$ into the right side of the original differential equation, and we'll get

$$u' - 2 = u$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = u + 2$$

$$du = (u + 2)\ dx$$

$$\frac{1}{u+2}\,du = dx$$

and then integrate both sides.

$$\int \frac{1}{u+2}\,du = \int dx$$

$$\ln(u+2) = x + C$$

$$e^{\ln(u+2)} = e^{x+C}$$

$$u + 2 = Ce^x$$

$$u = Ce^x - 2$$

We'll back-substitute using $u = 2x + y$ and then solve for y to find the solution to the differential equation.

$$2x + y = Ce^x - 2$$

$$y = Ce^x - 2x - 2$$

Let's do another example with a $u = ax + by$ substitution.

Example

Use a change of variable to solve the differential equation.

$$y' - (4x - y - 1)^2 = 0$$

If we choose the substitution $u = ax + by$, then we can set up the substitution

$$u = 4x - y$$

$$u' = 4 - y'$$

and then solve this second equation $u' = 4 - y'$ for y' to get $y' = 4 - u'$. Then we'll substitute $y' = 4 - u'$ and $u = 4x - y$ into the left side of the original differential equation, and we'll get

$$4 - u' - (u - 1)^2 = 0$$

$$u' = 4 - (u - 1)^2$$

Now the equation is separable, so we'll separate variables,

$$\frac{du}{dx} = 4 - (u - 1)^2$$

$$du = (4 - (u - 1)^2)\ dx$$

$$\frac{du}{4 - (u - 1)^2} = dx$$

and then integrate both sides.

$$\int \frac{du}{4 - (u - 1)^2} = \int dx$$

$$\int \frac{du}{4 - (u^2 - 2u + 1)} = \int dx$$

$$\int \frac{du}{4 - u^2 + 2u - 1} = \int dx$$

$$\int \frac{du}{3 + 2u - u^2} = \int dx$$

$$-\int \frac{du}{u^2 - 2u - 3} = \int dx$$

$$-\int \frac{du}{(u-3)(u+1)} = \int dx$$

Use partial fractions to rewrite the integral on the left,

$$\frac{1}{4}\int \frac{1}{u+1} - \frac{1}{u-3} = \int dx$$

then evaluate the integrals on both sides.

$$\frac{1}{4}(\ln|u+1| - \ln|u-3|) = x + C$$

$$\ln\left|\frac{u+1}{u-3}\right| = 4x + C$$

Raise both sides to the base e, then solve for u.

$$\frac{u+1}{u-3} = Ce^{4x}$$

$$u + 1 = (u-3)Ce^{4x}$$

$$u + 1 = uCe^{4x} - 3Ce^{4x}$$

$$u - uCe^{4x} = -1 - 3Ce^{4x}$$

$$u(1 - Ce^{4x}) = -(1 + 3Ce^{4x})$$

$$u = -\frac{1 + 3Ce^{4x}}{1 - Ce^{4x}}$$

Now we can back-substitute $u = 4x - y$ and then solve for y to find the solution to the differential equation.

$$4x - y = -\frac{1 + 3Ce^{4x}}{1 - Ce^{4x}}$$

$$y = 4x + \frac{1 + 3Ce^{4x}}{1 - Ce^{4x}}$$

Bernoulli equations

Bernoulli differential equations are another specific form of first order equations. To find the solution to a Bernoulli equation, we'll use a change of variables to convert the Bernoulli equation into a linear equation.

Once we've rewritten the equation in linear form, then of course we can find the solution to the linear equation. Finally, we'll make a simple substitution to put the solution equation back in terms of the variables we started with.

Form of a Bernoulli differential equation

A Bernoulli differential equation is a differential equation in the form

$$y' + p(x)y = q(x)y^n$$

where $p(x)$ and $q(x)$ are continuous, and where n is any real number. To solve Bernoulli equations, we'll always follow the same step-by-step process, which will be to use a change of variables in order to turn the Bernoulli equation into a linear equation.

If $n = 0$, then the Bernoulli equation is already a first order linear equation.

$$y' + p(x)y = q(x)y^0$$

$$y' + p(x)y = q(x)$$

If $n = 1$, then the Bernoulli equation is already a first order separable differential equation.

$$y' + p(x)y = q(x)y^1$$

$$y' + p(x)y - q(x)y = 0$$

$$y' + (p(x) - q(x))y = 0$$

$$\frac{dy}{dx} + (p(x) - q(x))y = 0$$

$$\frac{dy}{dx} = -(p(x) - q(x))y$$

$$dy = -(p(x) - q(x))y\ dx$$

$$\frac{1}{y}\ dy = -(p(x) - q(x))\ dx$$

$$\frac{1}{y}\ dy = (q(x) - p(x))\ dx$$

In both of these cases, we won't have to do the extra work to convert the Bernoulli equation. But for any value $n \neq 0, 1$, our goal will be to perform a change of variables. In general, these are the steps we'll follow to find the solution:

1. If the Bernoulli equation isn't already given in standard form, rewrite it in standard form.

2. Divide both sides of the equation by y^n.

3. Identify the substitution $v = y^{1-n}$, implicitly differentiate to find $v' = (1-n)y^{-n}y'$, and substitute into the Bernoulli equation.

4. Put the resulting equation into the standard form of a linear differential equation.

5. Find the solution to the linear differential equation, then back-substitute for v and solve for y.

Let's work through an example so that we can see how to find the solution to a Bernoulli differential equation.

Example

Find the solution to the Bernoulli differential equation.

$$(y^4 + x^2y)\ dx - 6x^3\ dy = 0$$

If the equation isn't given in the standard form of a Bernoulli equation, our first step is always to rewrite it in standard form.

$$(y^4 + x^2y)\ dx = 6x^3\ dy$$

$$y^4 + x^2y = 6x^3\ \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{y^4 + x^2y}{6x^3}$$

Split the fraction so that we can separate terms.

$$\frac{dy}{dx} = \frac{y^4}{6x^3} + \frac{x^2y}{6x^3}$$

$$\frac{dy}{dx} - \frac{x^2y}{6x^3} = \frac{y^4}{6x^3}$$

$$\frac{dy}{dx} - \frac{x^2}{6x^3}y = \frac{1}{6x^3}y^4$$

$$y' - \frac{1}{6x}y = \frac{1}{6x^3}y^4$$

With the equation in standard form, divide through by y^n. In this equation, that means we're dividing by y^4.

$$\frac{y'}{y^4} - \frac{1}{6x}\frac{y}{y^4} = \frac{1}{6x^3}\frac{y^4}{y^4}$$

$$y^{-4}y' - \frac{1}{6x}y^{-3} = \frac{1}{6x^3}$$

Our substitution is $v = y^{-3}$, so we'll differentiate to get

$$v' = -3y^{-4}y'$$

and then solve this for $y^{-4}y'$.

$$y^{-4}y' = -\frac{1}{3}v'$$

Now we can make substitutions into the Bernoulli equation.

$$y^{-4}y' - \frac{1}{6x}y^{-3} = \frac{1}{6x^3}$$

$$-\frac{1}{3}v' - \frac{1}{6x}v = \frac{1}{6x^3}$$

Multiplying through by -3 puts the equation into standard form of a linear differential equation.

$$v' + \frac{1}{2x}v = -\frac{1}{2x^3}$$

To find the solution to the linear equation, we'll find the integrating factor,

$$I(x) = e^{\int p(x)\ dx}$$

$$I(x) = e^{\int \frac{1}{2x}\ dx}$$

$$I(x) = e^{\frac{1}{2}\ln x}$$

$$I(x) = e^{\ln(x^{\frac{1}{2}})}$$

$$I(x) = x^{\frac{1}{2}}$$

and then multiply through the linear equation by $I(x)$.

$$v' + \frac{1}{2x}v = -\frac{1}{2x^3}$$

$$v'x^{\frac{1}{2}} + \frac{1}{2x}x^{\frac{1}{2}}v = -\frac{1}{2x^3}x^{\frac{1}{2}}$$

$$v'x^{\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}}v = -\frac{1}{2}x^{-\frac{5}{2}}$$

Reverse the product rule for derivatives to rewrite the left side,

$$\frac{d}{dx}(vx^{\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{5}{2}}$$

then integrate both sides.

$$\int \frac{d}{dx}(vx^{\frac{1}{2}})\ dx = \int -\frac{1}{2}x^{-\frac{5}{2}}\ dx$$

$$vx^{\frac{1}{2}} = -\frac{1}{2}\left(-\frac{2}{3}\right)x^{-\frac{3}{2}} + C$$

$$vx^{\frac{1}{2}} = \frac{1}{3}x^{-\frac{3}{2}} + C$$

Solve for v.

$$v = \frac{1}{3}\left(\frac{x^{-\frac{3}{2}}}{x^{\frac{1}{2}}}\right) + \frac{C}{x^{\frac{1}{2}}}$$

$$v = \frac{1}{3x^2} + \frac{C}{x^{\frac{1}{2}}}$$

Find a common denominator in order to combine fractions.

$$v = \frac{1}{3x^2} + \frac{3Cx^{\frac{3}{2}}}{3x^2}$$

$$v = \frac{1 + 3Cx^{\frac{3}{2}}}{3x^2}$$

If $3C$ is some arbitrary constant, then we can simply use C instead to represent that arbitrary constant.

$$v = \frac{1 + Cx^{\frac{3}{2}}}{3x^2}$$

Use $v = y^{-3}$ to back-substitute for v,

$$y^{-3} = \frac{1 + Cx^{\frac{3}{2}}}{3x^2}$$

then solve for y.

$$\frac{1}{y^3} = \frac{1 + Cx^{\frac{3}{2}}}{3x^2}$$

$$y^3 = \frac{3x^2}{1 + Cx^{\frac{3}{2}}}$$

$$y = \sqrt[3]{\frac{3x^2}{1 + Cx^{\frac{3}{2}}}}$$

Homogeneous equations

We've looked at how to use the substitution $v = y^{1-n}$ to change a Bernoulli equation into a linear equation, but this isn't the only equation type we can solve with a change of variable.

Form of a homogeneous differential equation

We'll also use a change of variable to solve homogeneous equations, which are differential equations in the form

$$y' = F\left(\frac{y}{x}\right)$$

In other words, this type of differential equation can be solved for y' (or dy/dx), and is given in terms of some function defined in y/x. For instance,

$$y' = 1 + \frac{y}{x}$$

is an example of a simple homogeneous equation. It's solved for y', and written in terms of y/x.

Keep in mind that homogeneous equations won't always be written in a way that they are obviously homogeneous. We'll often have to do some manipulation with the equation in order to see that it's homogeneous.

Confirming homogeneity

If it's not immediately obvious that the equation is homogeneous, we do have one check we can do to help us determine the homogeneity of the equation. First, we need to make sure that the equation is solved for dy/dx, or y'.

Then we can look to see whether every term on the opposite side of the equation has the same degree. For instance, looking at the right side of

$$y' = \frac{-4x^2 - y^2}{xy}$$

we see three terms, $-4x^2$, $-y^2$, and xy. Both $-4x^2$ and $-y^2$ are second-degree terms, since they both have an exponent of 2. The xy term is also a second-degree term, since x and y both have degree 1, which adds up to $1 + 1 = 2$.

As long as these terms all have the same degree, whether they're all second-degree, all third-degree, etc., then we know the differential equation is homogeneous.

Alternately, we can also confirm homogeneity if $f(\lambda x, \lambda y) = f(x, y)$. Using

$$y' = \frac{-4x^2 - y^2}{xy}$$

the equation is homogeneous because

$$f(\lambda x, \lambda y) = \frac{-4(\lambda x)^2 - (\lambda y)^2}{(\lambda x)(\lambda y)}$$

$$= \frac{-4\lambda^2 x^2 - \lambda^2 y^2}{\lambda^2 xy}$$

$$= \frac{\lambda^2(-4x^2 - y^2)}{\lambda^2 xy}$$

$$= \frac{-4x^2 - y^2}{xy} = f(x, y)$$

Because we were able to confirm that $f(\lambda x, \lambda y) = f(x, y)$, we can say that the differential equation is homogeneous.

Solving homogeneous equations

Once we have the homogeneous equation written in standard form, then our goal will be to use a change of variable to rewrite it as a separable differential equation.

Then we'll find the solution to the separable equation, and finally we'll back substitute in order to find the solution to the homogeneous equation.

In summary, we'll follow these steps:

1. If the homogeneous equation isn't already given in standard form, rewrite it in standard form.

2. Substitute $v = y/x$ and $y' = v + xv'$.

3. Solve the resulting separable equation.

4. Back-substitute for $v = y/x$.

5. If possible, solve explicitly for y to find the solution to the homogeneous differential equation.

Let’s work through an example so that we can practice solving homogeneous equations with a change of variable.

Example

Find the solution to the differential equation.

$$xyy' + 2x^2 - 3y^2 = 0$$

To put the equation in standard form, we’ll divide through by x^2.

$$\frac{xy}{x^2}y' + \frac{2x^2}{x^2} - 3\frac{y^2}{x^2} = 0$$

$$\frac{y}{x}y' + 2 - 3\left(\frac{y}{x}\right)^2 = 0$$

$$\frac{y}{x}y' = 3\left(\frac{y}{x}\right)^2 - 2$$

and then solve for y'.

$$y' = 3\left(\frac{y}{x}\right)^2\left(\frac{x}{y}\right) - 2\left(\frac{x}{y}\right)$$

$$y' = \frac{3}{\frac{y}{x}}\left(\frac{y}{x}\right)^2 - \frac{2}{\frac{y}{x}}$$

$$y' = 3\left(\frac{y}{x}\right) - \frac{2}{\frac{y}{x}}$$

Now that the equation is in standard form, we'll substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = 3v - \frac{2}{v}$$

$$v^2 + xvv' = 3v^2 - 2$$

$$xvv' = 2v^2 - 2$$

Now we should have a separable differential equation, so we'll separate variables,

$$xv\frac{dv}{dx} = 2v^2 - 2$$

$$xv\ dv = (2v^2 - 2)\ dx$$

$$\frac{v}{2v^2 - 2}\ dv = \frac{1}{x}\ dx$$

and then integrate both sides, using the substitution $u = 2v^2 - 2$, $du = 4v\ dv$, and $dv = (1/4v)\ du$.

$$\int \frac{v}{2v^2 - 2}\ dv = \int \frac{1}{x}\ dx$$

$$\int \frac{v}{u}\left(\frac{1}{4v}\right)\ du = \int \frac{1}{x}\ dx$$

$$\frac{1}{4}\int \frac{1}{u}\ du = \int \frac{1}{x}\ dx$$

$$\frac{1}{4}\ln|u| = \ln|x| + C$$

Back substitute with $u = 2v^2 - 2$.

$$\frac{1}{4}\ln|2v^2 - 2| = \ln|x| + C$$

$$\ln|2v^2 - 2| = 4\ln|x| + 4C$$

$$e^{\ln|2v^2-2|} = e^{4\ln|x|+4C}$$

$$|2v^2 - 2| = e^{4\ln|x|}e^{4C}$$

$$2v^2 - 2 = Ce^{4\ln|x|}$$

$$2v^2 - 2 = Ce^{\ln|x|^4}$$

$$2v^2 - 2 = C|x|^4$$

$$2v^2 = C|x|^4 + 2$$

$$v^2 = C|x|^4 + 1$$

$$v^2 = Cx^4 + 1$$

Then we back substitute with $v = y/x$.

$$\left(\frac{y}{x}\right)^2 = Cx^4 + 1$$

$$\frac{y^2}{x^2} = Cx^4 + 1$$

$$y^2 = Cx^6 + x^2$$

$$y = \pm\sqrt{Cx^6 + x^2}$$

Let's work through another example to get some more practice with solving homogeneous equations.

Example

Use a substitution to find a solution to the homogeneous equation.

$$y' = \frac{-4x^2 - y^2}{xy}$$

We'll multiply through the numerator and denominator on the right side by $1/(x^2)$.

$$y' = \frac{-4x^2 - y^2}{xy} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}}$$

$$y' = \frac{-4x^2\left(\frac{1}{x^2}\right) - y^2\left(\frac{1}{x^2}\right)}{xy\left(\frac{1}{x^2}\right)}$$

$$y' = \frac{-4 - \frac{y^2}{x^2}}{\frac{y}{x}}$$

$$y' = \frac{-4 - \left(\frac{y}{x}\right)^2}{\frac{y}{x}}$$

Substitute $v = y/x$ and $y' = v + xv'$.

$$v + xv' = \frac{-4 - v^2}{v}$$

Now we should have a separable differential equation, so we'll separate variables,

$$x\frac{dv}{dx} = \frac{-4 - v^2}{v} - v$$

$$x\frac{dv}{dx} = \frac{-4 - 2v^2}{v}$$

$$xv\ dv = (-4 - 2v^2)\ dx$$

$$\frac{v}{2v^2 + 4}\ dv = -\frac{1}{x}\ dx$$

and then integrate both sides, using a simple substitution for the integral on the left, with $u = v^2 + 2$ and $dv = du/2v$.

$$\frac{1}{2}\int \frac{v}{v^2 + 2}\ dv = -\int \frac{1}{x}\ dx$$

$$\frac{1}{2}\int \frac{v}{u} \cdot \frac{du}{2v} = -\int \frac{1}{x}\ dx$$

$$\frac{1}{4}\ln u = -\ln x + C$$

$$\frac{1}{4}\ln(v^2 + 2) = -\ln x + C$$

Back-substitute for v.

$$\frac{1}{4}\ln\left(\left(\frac{y}{x}\right)^2+2\right)=-\ln x+C$$

$$\ln\left(\left(\frac{y}{x}\right)^2+2\right)=-4\ln x+C$$

$$\ln\left(\frac{y^2}{x^2}+2\right)=-4\ln x+C$$

$$e^{\ln\left(\frac{y^2}{x^2}+2\right)}=e^{-4\ln x+C}$$

$$\frac{y^2}{x^2}+2=Ce^{-4\ln x}$$

$$\frac{y^2}{x^2}+2=Ce^{\ln(x^{-4})}$$

$$\frac{y^2}{x^2}+2=Cx^{-4}$$

$$y^2+2x^2=Cx^{-2}$$

$$y^2=\frac{C}{x^2}-2x^2$$

$$y^2=\frac{C-2x^4}{x^2}$$

We'll allow this to be the implicitly-defined solution to the homogeneous equation.

Exact equations

Exact differential equations are another form of first order differential equation for which we can find a solution. In order for a differential equation to be called an **exact differential equation**, it must be given in the form

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

This kind of equation will have an implicit solution $\Psi(x, y) = c$ (Ψ is the Greek letter Psi ("sigh")) defined in two variables x and y, where c is a constant and where Ψ has continuous partial derivatives such that

$$d\Psi(x, y) = M(x, y)\ dx + N(x, y)\ dy$$

Test for an exact equation

If we say that the functions $M(x, y)$ and $N(x, y)$ have continuous partial derivatives, then the differential equation

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

is exact if and only if $M_y = N_x$, or

$$\frac{\partial M(x, y)}{\partial y} = \frac{\partial N(x, y)}{\partial x}$$

If we put this in words, we can say

If the partial derivative of M with respect to y is equal to the partial derivative of N with respect to x, then the differential equation is exact.

This is a test we can use to say whether or not the differential equation is exact, before we go about finding the solution $\Psi(x, y)$.

Solution to an exact equation

The implicit solution to an exact differential equation is given by

$$\Psi(x, y) = c$$

where c is a constant. The partial derivatives of Ψ with respect to x and y are

$$\frac{\partial \Psi}{\partial x} = \Psi_x = M(x, y) = M$$

$$\frac{\partial \Psi}{\partial y} = \Psi_y = N(x, y) = N$$

The left sides of these equations are partial derivatives of Ψ, but we need to get back to Ψ itself. To do that, we'd need to take the integrals

$$\int \Psi_x \ dx = \int M(x, y) \ dx$$

$$\int \Psi_y \ dy = \int N(x, y) \ dy$$

Taking the integral with respect to x of the partial derivative with respect to x cancels both operations and leaves us with just Ψ, in the same way that taking the integral with respect to y of the partial derivative with respect to y cancels both operations and leaves us with just Ψ.

$$\Psi = \int M(x, y)\ dx + h(y)$$

$$\Psi = \int N(x, y)\ dy + h(x)$$

In other words, if we want to find an equation for Ψ, we can either take the integral of M with respect to x, or the integral of N with respect to y. Both integrals will work, so we should look at M and N and then choose whichever function looks easier to integrate.

Remember that when we use the first integral (the one with M), we're integrating a multivariable function in terms of x and y with respect to x only. Which means that, instead of adding C to account for the constant of integration after we integrate, we have to add $h(y)$ to account for a function in terms of y.

Similarly, when we use the second integral (the one with N), we're integrating a multivariable function in terms of x and y with respect to y only. Which means that, instead of adding C to account for the constant of integration after we integrate, we have to add $h(x)$ to account for a function in terms of x.

Then it's just a matter of solving for $h(y)$ or $h(x)$, which we'll do by differentiating Ψ with respect to y if we're trying to find $h(y)$, or with respect to x if we're trying to find $h(x)$.

That differentiation process will give us either Ψ_x with $h'(x)$ or Ψ_y with $h'(y)$. We also know that $M(x,y) = \Psi_x$ or $N(x,y) = \Psi_y$, we'll make that substitution and then simplify the equation to solve for $h'(y)$ or $h'(x)$. Then we can integrate both sides of the remaining equation to solve for $h(y)$ or $h(x)$.

Finally, we'll plug $h(y)$ or $h(x)$ back into the equation for Ψ, set the equation equal to c, and this will be the implicit solution to the exact differential equation.

In summary, to find the solution of an exact differential equation, we'll

1. Verify that $M_y = N_x$ to confirm the differential equation is exact.

2. Use $\Psi = \int M(x,y)\ dx + h(y)$ or $\Psi = \int N(x,y)\ dy + h(x)$ to find $\Psi(x,y)$, including a value for $h(y)$ or $h(x)$.

3. Find $\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\left(\int M(x,y)\ dx\right) + h'(y) = N(x,y)$, or find $\frac{\partial \Psi}{\partial x} = \frac{\partial}{\partial x}\left(\int N(x,y)\ dy\right) + h'(x) = M(x,y)$.

4. Integrate both sides of the equation to solve for $h(y)$ or $h(x)$.

5. Plug $h(y)$ or $h(x)$ back into Ψ in step 2.

6. Set $\Psi(x,y) = c$ to get the implicit solution.

Let's try an example where we find the general (implicit) solution to an exact differential equation.

Example

If the differential equation is exact, find its solution.

$$x^3y^4 + (x^4y^3 + 2y)\frac{dy}{dx} = 0$$

First, we'll test to see whether or not the differential equation is exact. Matching the given equation to the standard form of an exact differential equation, we can say that

$$M(x, y) = x^3y^4$$

$$N(x, y) = x^4y^3 + 2y$$

We'll test to see whether $M_y = N_x$ (Step 1).

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial}{\partial y}(x^3y^4) = \frac{\partial}{\partial x}(x^4y^3 + 2y)$$

$$4x^3y^3 = 4x^3y^3$$

Since $M_y = N_x$, we know that the differential equation is exact, so now we just need to find the solution $\Psi(x, y) = c$.

The functions $M(x, y)$ and $N(x, y)$ are equally easy to integrate, so we'll just use $M(x, y)$, and then Ψ can be given by

$$\Psi = \int M(x, y)\ dx \text{ (Step 2)}$$

$$\Psi = \int x^3y^4\ dx$$

$$\Psi = \frac{1}{4}x^4y^4 + h(y)$$

We had to add $h(y)$ instead of just C because we integrated a multivariable function with respect to x only, which doesn't account for the integration of y. Now we need to find $h(y)$, which we'll do by differentiating both sides with respect to y.

$$\Psi_y = x^4y^3 + h'(y) \text{ (Step 3)}$$

Because we know that $\Psi_y = N(x, y)$, we'll make that substitution and then solve for $h'(y)$.

$$x^4y^3 + 2y = x^4y^3 + h'(y)$$

$$2y = h'(y)$$

To find $h(y)$, we'll integrate both sides of this equation with respect to y.

$$\int 2y\ dy = \int h'(y)\ dy \text{ (Step 4)}$$

$$y^2 + k_1 = h(y) + k_2$$

$$h(y) = y^2 + k_1 - k_2$$

$$h(y) = y^2 + k$$

Plugging this value for $h(y)$ into the equation for Ψ in step 2 gives

$$\Psi = \frac{1}{4}x^4y^4 + h(y) \text{ (Step 5)}$$

$$\Psi = \frac{1}{4}x^4y^4 + y^2 + k$$

Finally, setting $\Psi = c$ to find the solution to the exact differential equation, we get

$$c = \frac{1}{4}x^4y^4 + y^2 + k \text{ (Step 6)}$$

$$c - k = \frac{1}{4}x^4y^4 + y^2$$

$$c = \frac{1}{4}x^4y^4 + y^2$$

Second order linear homogeneous equations

At this point, we want to turn our attention away from first order differential equations, and toward second order equations.

Second order equations can take many forms, but we want to start by looking at second order linear equations that have constant coefficients. Second order linear equations can be homogeneous or nonhomogeneous, and we'll start here with the homogeneous variety. The form we're looking for is

$$y'' + Ay' + By = 0$$

This is a second order equation because it includes a second derivative. It's a linear equation because we only see y and its derivatives, and not any other function of y, like $\cos y$, e^y, y^3, etc. The coefficients A and B are constants, and the equation is homogeneous because of the zero-value on the right side.

Finding the solution

To find the solution that satisfies a second order linear homogeneous equation with constant coefficients, we always start with the associated **characteristic equation**,

$$r^2 + Ar + B = 0$$

In other words, to change the differential equation into its matching characteristic equation, we substitute $y'' = r^2$, $y' = r$, and $y = 1$. Once we've done that, we realize that the characteristic equation is a simple quadratic equation, and we can solve it using methods we learned in algebra, including factoring, completing the square, or the quadratic formula.

Remember that quadratic equations will have one of three types of roots: distinct real roots, equal real roots, or complex conjugate roots. Distinct real roots are real-numbered roots that are unequal; equal real roots are real-numbered roots that are equivalent. Complex conjugate roots are complex-numbered roots.

The general solution to the differential equation is determined by the type of roots r_1 and r_2 that we find from the characteristic equation.

Roots	**General solution**
Distinct real roots	$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$
Equal real roots	$y(x) = c_1e^{r_1x} + c_2xe^{r_1x}$
Complex conjugate roots	$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$
	where $r = \alpha \pm \beta i$

The general solution will contain the constants c_1 and c_2, but we can find the values of c_1 and c_2 if we're given initial conditions. Let's try an example so that we can see how to find the general solution.

Example

Find the general solution to the second order equation.

$$y'' + 5y' + 6y = 0$$

The characteristic equation associated with the differential equation is

$$r^2 + 5r + 6 = 0$$

This quadratic equation is easily factorable,

$$(r + 2)(r + 3) = 0$$

which means the roots will be

$$r_1 + 2 = 0 \qquad r_2 + 3 = 0$$

$$r_1 = -2 \qquad r_2 = -3$$

These are real-numbered unequal roots, so we call them distinct real roots. Therefore, the general solution to the second order equation is

$$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$$

$$y(x) = c_1e^{-2x} + c_2e^{-3x}$$

Realize here that it makes no difference whether we assign $r_1 = -2$ and $r_2 = -3$, or $r_1 = -3$ and $r_2 = -2$. In other words, both of these are acceptable general solutions:

$$y(x) = c_1e^{-2x} + c_2e^{-3x}$$

$$y(x) = c_1e^{-3x} + c_2e^{-2x}$$

The constants c_1 and c_2 are simply placeholders until we can apply initial conditions, so we don't have to worry about how we name the roots.

Let's look at an example with equal real roots.

Example

Find the general solution to the second order equation.

$$y'' + 6y' + 9y = 0$$

The characteristic equation associated with the differential equation is

$$r^2 + 6r + 9 = 0$$

This quadratic equation is easily factorable,

$$(r + 3)(r + 3) = 0$$

which means the roots will be

$$r_1 + 3 = 0 \qquad\qquad r_2 + 3 = 0$$

$$r_1 = -3 \qquad\qquad r_2 = -3$$

These are real-numbered equal roots, so we call them equal real roots. Therefore, the general solution to the second order equation is

$$y(x) = c_1e^{r_1x} + c_2xe^{r_1x}$$

$$y(x) = c_1 e^{-3x} + c_2 x e^{-3x}$$

Finally, we'll look at one more example, this time with a characteristic equation that produces complex conjugate roots.

Example

Find the general solution to the second order equation.

$$y'' + 2y' + 17y = 0$$

The characteristic equation associated with the differential equation is

$$r^2 + 2r + 17 = 0$$

The left side doesn't easily factor, so we'll apply the quadratic formula (we could also complete the square and get the same result).

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$r = \frac{-2 \pm \sqrt{2^2 - 4(1)(17)}}{2(1)}$$

$$r = \frac{-2 \pm \sqrt{-64}}{2}$$

Now we'll use the imaginary number to rewrite the root of the negative number.

$$r = \frac{-2 \pm \sqrt{(64)(-1)}}{2}$$

$$r = \frac{-2 \pm 8\sqrt{-1}}{2}$$

$$r = \frac{-2 \pm 8i}{2}$$

$$r = -1 \pm 4i$$

These are complex-numbered roots, so we call them complex conjugate roots. If we match up $r = \alpha \pm \beta i$ with $r = -1 \pm 4i$, we identify $\alpha = -1$ and $\beta = 4$, and the general solution is therefore

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

$$y(x) = e^{-x}(c_1 \cos(4x) + c_2 \sin(4x))$$

Initial value problems

With all of these examples, we've found the roots of the characteristic equation and plugged them into the general solution. What we haven't done is solve for the values of the constants c_1 and c_2 that remain in the general solution.

The only way to find these values is with initial conditions. Because we're dealing with second order equations, we'll need two initial conditions, one for the general solution $y(x_0) = y_0$, and another for its derivative $y'(x_1) = y_1$.

Because we get an initial condition that corresponds to the derivative of the general solution, we'll need to differentiate the general solution. Once we have both the general solution and its derivatives, we'll plug in both initial conditions in order to get a system of simultaneous equations that we can solve for c_1 and c_2.

Let's try an example of an initial value problem when we have a characteristic equation with distinct real roots.

Example

Find the solution to the second order equation, if $y(0) = 5$ and $y'(0) = 27$.

$$3y'' + 18y' + 15y = 0$$

The characteristic equation associated with this differential equation is

$$3r^2 + 18r + 15 = 0$$

$$(3r + 3)(r + 5) = 0$$

The roots are therefore

$$3r_1 + 3 = 0 \qquad r_2 + 5 = 0$$

$$3r_1 = -3 \qquad r_2 = -5$$

$$r_1 = -1$$

With these distinct real roots, the general solution is

$$y(x) = c_1e^{-x} + c_2e^{-5x}$$

and its derivative is

$$y'(x) = -c_1e^{-x} - 5c_2e^{-5x}$$

We'll substitute the initial condition $y(0) = 5$ into $y(x)$,

$$5 = c_1e^{-(0)} + c_2e^{-5(0)}$$

$$5 = c_1(1) + c_2(1)$$

$$5 = c_1 + c_2$$

and the condition $y'(0) = 27$ into the derivative.

$$27 = -c_1e^{-(0)} - 5c_2e^{-5(0)}$$

$$27 = -c_1(1) - 5c_2(1)$$

$$27 = -c_1 - 5c_2$$

Now we have a system of equations.

$$5 = c_1 + c_2$$

$$27 = -c_1 - 5c_2$$

Rewrite $5 = c_1 + c_2$ as $c_1 = 5 - c_2$, then substitute this value for c_1 into $27 = -c_1 - 5c_2$.

$$27 = -(5 - c_2) - 5c_2$$

$$27 = -5 + c_2 - 5c_2$$

$$32 = -4c_2$$

$$c_2 = -8$$

Plugging $c_2 = -8$ into $c_1 = 5 - c_2$ gives

$$c_1 = 5 - (-8)$$

$$c_1 = 13$$

Therefore, the general solution, subject to the conditions $y(0) = 5$ and $y'(0) = 27$, is

$$y(x) = c_1 e^{-x} + c_2 e^{-5x}$$

$$y(x) = 13e^{-x} - 8e^{-5x}$$

Reduction of order

Remember from the previous lesson that we identified three possible forms for the general solution of a second order homogeneous linear differential equation.

Roots	**General solution**
Distinct real roots	$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$
Equal real roots	$y(x) = c_1e^{r_1x} + c_2xe^{r_1x}$
Complex conjugate roots	$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$

We learned to use the characteristic equation associated with the second order equation to solve for the roots r_1 and r_2. But it's important to say that, while these values for r are the roots of the characteristic equation, they aren't by themselves solutions to the differential equation.

Solutions of the differential equation

Instead, we need to identify the solutions to the differential equation from the equation of the general solution, given in the table above. The solutions to the differential equation will be the functions we see multiplied by c_1 and c_2.

For instance, given a general solution with distinct real roots, $y(x) = c_1e^{r_1x} + c_2e^{r_2x}$, the solutions to the differential equation are

$$y_1 = e^{r_1x}$$

$$y_2 = e^{r_2 x}$$

Alternatively, given a general solution with complex conjugate roots,

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$$

$$y(x) = c_1 e^{\alpha x} \cos(\beta x) + c_2 e^{\alpha x} \sin(\beta x)$$

the solutions to the differential equation are

$$y_1 = e^{\alpha x} \cos(\beta x)$$

$$y_2 = e^{\alpha x} \sin(\beta x)$$

Notice how all three forms of the general solution given in the table above can actually be written in the same form, $y(x) = c_1 y_1 + c_2 y_2$.

$y(x) = c_1 y_1 + c_2 y_2$	**Solutions of the equation**
$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$	$\{y_1, y_2\} = \{e^{r_1 x}, e^{r_2 x}\}$
$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$	$\{y_1, y_2\} = \{e^{r_1 x}, x e^{r_2 x}\}$
$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$	$\{y_1, y_2\} = \{e^{\alpha x} \cos(\beta x), e^{\alpha x} \sin(\beta x)\}$

Now that we've identified this set of two solutions for each form of a second order linear homogeneous equation, let's explain what it actually means for y_1 and y_2 to be solutions of the original equation.

Put plainly, these solutions y_1 and y_2 satisfy the differential equation because they are the functions that make the equation true. To make the simplest possible analogy, plugging y_1 or y_2 into $y'' + Ay' + By = 0$ will produce the true equation $0 = 0$, in the same way that plugging 1 into $x = 1$

produces the true equation $1 = 1$. Something is a solution to an equation when plugging that something into the equation makes the equation true.

Let's do an example so that we can see these solution sets in action.

Example

Show that $\{y_1, y_2\} = \{e^{-3x}, xe^{-3x}\}$ are two solutions to the second order linear homogeneous equation.

$$y'' + 6y' + 9y = 0$$

To show that $y_1 = e^{-3x}$ is a solution to the differential equation, we'll need to start by taking the first and second derivatives of y_1.

$$y_1 = e^{-3x}$$

$$y_1' = -3e^{-3x}$$

$$y_1'' = 9e^{-3x}$$

Then we can plug these values into the homogeneous equation

$$y'' + 6y' + 9y = 0$$

$$9e^{-3x} + 6(-3e^{-3x}) + 9e^{-3x} = 0$$

$$9e^{-3x} - 18e^{-3x} + 9e^{-3x} = 0$$

$$(9 - 18 + 9)e^{-3x} = 0$$

$$0e^{-3x} = 0$$

$$0 = 0$$

Because we end up with a true equation, we know that $y_1 = e^{-3x}$ satisfies the differential equation. Now let's check to make sure that $y_2 = xe^{-3x}$ does the same thing. Its first two derivatives are

$$y_2' = (1)(e^{-3x}) + (x)(-3e^{-3x})$$

$$y_2' = e^{-3x} - 3xe^{-3x}$$

and

$$y_2'' = -3e^{-3x} - ((3)(e^{-3x}) + (3x)(-3e^{-3x}))$$

$$y_2'' = -3e^{-3x} - (3e^{-3x} - 9xe^{-3x})$$

$$y_2'' = -3e^{-3x} - 3e^{-3x} + 9xe^{-3x}$$

$$y_2'' = -6e^{-3x} + 9xe^{-3x}$$

Then we can plug these values into the homogeneous equation

$$y'' + 6y' + 9y = 0$$

$$-6e^{-3x} + 9xe^{-3x} + 6(e^{-3x} - 3xe^{-3x}) + 9(xe^{-3x}) = 0$$

$$-6e^{-3x} + 9xe^{-3x} + 6e^{-3x} - 18xe^{-3x} + 9xe^{-3x} = 0$$

$$(-6+6)e^{-3x} + (9-18+9)xe^{-3x} = 0$$

$$0e^{-3x} + 0xe^{-3x} = 0$$

$$0 = 0$$

This last example illustrates why $y_1 = e^{r_1 x}$ and $y_2 = xe^{r_2 x}$ are solutions to a differential equation $y'' + Ay' + By = 0$ whose characteristic equation has equal real roots.

In the same way, $\{y_1, y_2\} = \{e^{r_1 x}, e^{r_2 x}\}$ will be a solution set to a differential equation $y'' + Ay' + By = 0$ whose characteristic equation has distinct real roots. And $\{y_1, y_2\} = \{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\}$ will be a solution set to a differential equation $y'' + Ay' + By = 0$ whose characteristic equation has complex conjugate roots.

Reduction of order with constant coefficients

Now that we better understand how y_1 and y_2 are solutions to the differential equation, we want to realize that we can use one solution to find the other.

In other words, if we already know y_1, we can use it to find y_2, or vice versa. This solution method is called **reduction of order**, because it uses a substitution to reduce the order of the homogeneous equation from a second order equation to a first order equation.

Depending on the kind of differential equation we're dealing with, it can be just as easy to find both solutions at once as it is to find only one solution. So we won't always use reduction of order to solve differential equations, but we still want to be familiar with this method.

Let's do an example so that we can see how this works.

Example

Use reduction of order to find the general solution to the differential equation, given $y_1 = e^{x/3}$.

$$6y'' + y' - y = 0$$

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = ve^{\frac{x}{3}}$$

$$y_2' = v'e^{\frac{x}{3}} + \frac{1}{3}ve^{\frac{x}{3}}$$

$$y_2'' = v''e^{\frac{x}{3}} + \frac{2}{3}v'e^{\frac{x}{3}} + \frac{1}{9}ve^{\frac{x}{3}}$$

Plug these values into the homogeneous differential equation.

$$6v''e^{\frac{x}{3}} + 4v'e^{\frac{x}{3}} + \frac{2}{3}ve^{\frac{x}{3}} + v'e^{\frac{x}{3}} + \frac{1}{3}ve^{\frac{x}{3}} - ve^{\frac{x}{3}} = 0$$

$$6v''e^{\frac{x}{3}} + 5v'e^{\frac{x}{3}} = 0$$

This is when the reduction of order part comes in. We'll make a substitution $w = v'$, and therefore $w' = v''$.

$$6w'e^{\frac{x}{3}} + 5we^{\frac{x}{3}} = 0$$

This substitution changes the second order equation in v into a first order equation in w. Furthermore, this first order equation is a linear equation that we can easily solve.

$$6e^{\frac{x}{3}}\frac{dw}{dx} + 5e^{\frac{x}{3}}w = 0$$

$$\frac{dw}{dx} + \frac{5e^{\frac{x}{3}}}{6e^{\frac{x}{3}}}w = 0$$

$$\frac{dw}{dx} + \frac{5}{6}w = 0$$

With $P(x) = 5/6$, the integrating factor is

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int \frac{5}{6}\ dx}$$

$$I(x) = e^{\frac{5}{6}x}$$

Multiplying the first order differential equation in w by the integrating factor, and then simplifying, gives

$$\frac{d}{dx}(we^{\frac{5}{6}x}) = 0$$

Integrate both sides.

$$\int \frac{d}{dx}(we^{\frac{5}{6}x})\ dx = \int 0\ dx$$

$$we^{\frac{5}{6}x} = C$$

$$w = Ce^{-\frac{5}{6}x}$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w\ dx = \int Ce^{-\frac{5}{6}x}\ dx = -\frac{6}{5}Ce^{-\frac{5}{6}x} + k$$

We can choose any constants, so we'll choose $C = -5/6$ and $k = 0$ for simplicity. Then v is

$$v = e^{-\frac{5}{6}x}$$

and the second solution for the differential equation is

$$y_2 = ve^{\frac{1}{3}x}$$

$$y_2 = e^{-\frac{5}{6}x}e^{\frac{1}{3}x}$$

$$y_2 = e^{-\frac{x}{2}}$$

Because the solutions are $y_1 = e^{x/3}$ and $y_2 = e^{-x/2}$, the general solution to the differential equation is

$$y(x) = c_1e^{\frac{x}{3}} + c_2e^{-\frac{x}{2}}$$

Reduction of order with non-constant coefficients

In the previous example with $6y'' + y' - y = 0$, we applied reduction of order to a differential equation with constant coefficients. We say that the differential equation has constant coefficients because the the coefficients on y, y', and y'' are -1, 1, and 6, respectively. These are all constants.

But of course, we've seen previously that not all differential equations have constant coefficients, and reduction of order can be effective on these non-constant coefficient equations as well.

Specifically, reduction of order can help us find the second solution to an equation in the form

$$y'' + P(x)y' + Q(x)y = 0$$

when we already know one solution, y_1. Let's do an example so that we can see how this works.

Example

Use reduction of order to find the general solution to the differential equation, given $y_1 = x$.

$$x^2y'' - 4xy' + 4y = 0$$

To apply this method, we always begin with the assumption that $y_2 = vy_1$, and then we find the first and second derivatives of y_2, since we're dealing with a second order equation.

$$y_2 = vx$$

$$y_2' = v'x + v$$

$$y_2'' = v''x + 2v'$$

Plug these values into the homogeneous differential equation.

$$x^2(v''x + 2v') - 4x(v'x + v) + 4vx = 0$$

$$v''x^3 + 2v'x^2 - 4v'x^2 - 4vx + 4vx = 0$$

$$v''x^3 - 2x^2v' = 0$$

This is when the reduction of order part comes in. We'll make a substitution $w = v'$, and therefore $w' = v''$.

$$w'x^3 - 2x^2w = 0$$

This substitution changes the second order equation in v into a first order equation in w. Furthermore, this first order equation is a linear equation that we can easily solve.

$$x^3\frac{dw}{dx} - 2x^2w = 0$$

$$\frac{dw}{dx} - \frac{2}{x}w = 0$$

With $P(x) = -2/x$ and $Q(x) = 0$, the integrating factor is

$$I(x) = e^{\int P(x)\ dx}$$

$$I(x) = e^{\int -\frac{2}{x}\ dx}$$

$$I(x) = e^{-2\ln x}$$

$$I(x) = x^{-2}$$

Multiplying the first order differential equation in w by the integrating factor, and then simplifying, gives

$$\frac{d}{dx}(wx^{-2}) = 0$$

Integrate both sides.

$$\int \frac{d}{dx}(wx^{-2})\ dx = \int 0\ dx$$

$$wx^{-2} = C$$

$$w = Cx^2$$

Because $w = v'$, we'll integrate w to find v.

$$v = \int w\ dx = \int Cx^2\ dx = \frac{C}{3}x^3 + k$$

We can choose any constants, so we'll choose $C = 3$ and $k = 0$ for simplicity. Then v is

$$v = x^3$$

and the second solution for the differential equation is

$$y_2 = vx$$

$$y_2 = (x^3)x$$

$$y_2 = x^4$$

Because the solutions are $y_1 = x$ and $y_2 = x^4$, the general solution to the differential equation is

$$y(x) = c_1x + c_2x^4$$

Undetermined coefficients for nonhomogeneous equations

We said earlier that linear second order homogeneous equations with constant coefficients would take the form

$$y'' + Ay' + By = 0$$

Now we want to tackle nonhomogeneous equations, in which we're swapping out the zero-value on the right side for a non-zero value.

$$y'' + Ay' + By = g(x)$$

To solve nonhomogeneous equations like these, we'll first use the characteristic equation to find the general solution to the associated homogeneous equation, and then we'll deal with the $g(x)$ part in another step.

That being said, we'll be looking for the same kind of solutions to the homogeneous equation.

Roots	**General solution**
Distinct real roots	$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$
Equal real roots	$y(x) = c_1e^{r_1x} + c_2xe^{r_1x}$
Complex conjugate roots	$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$
	where $r = \alpha \pm \beta i$

To address the $g(x)$ portion of the equation, there are multiple methods we can use, the first of which is the method of undetermined coefficients.

Undetermined coefficients

Undetermined coefficients is a method we can use to find the general solution $y(x)$ to a second order nonhomogeneous differential equation. The general solution to the nonhomogeneous equation will always be the sum of the complementary solution $y_c(x)$ and the particular solution $y_p(x)$.

$$y(x) = y_c(x) + y_p(x)$$

The **complementary solution** is the solution we find when we solve the characteristic equation (from the associated homogeneous equation). Which means we'll always start by pretending that the nonhomogeneous equation is actually a homogenous equation, replacing $g(x)$ with 0.

Once we find the complementary solution, it's time to use $g(x)$ to make a guess about a **particular solution**, which is the solution that addresses the $g(x)$ portion of the differential equation. It takes practice to get good at building a particular solution, but here are some general guidelines.

- For a polynomial function like $x^2 + 1$, guess $Ax^2 + Bx + C$, making sure to include the highest-degree term, as well as all lower-degree terms. So if the highest degree term is x^3, we'll need to include x^3, x^2, x, and a constant.

- For an exponential function like e^{3x}, guess Ae^{3x}.

- For sine and/or cosine like $3\sin(4x)$ or $2\cos(4x)$ or $3\sin(4x) + 2\cos(4x)$, guess $A\sin(4x) + B\cos(4x)$, including both sine and cosine terms, even if $g(x)$ only includes sine or only includes cosine.

If the right side of the differential equation is the sum or product of different types of functions, we need to multiply or add our guesses together, making sure that we have distinct constants, and that we've simplified the products of constants.

Here's a table of $g(x)$ functions, and the guess we should use for each one.

$g(x)$	**Guess**
x^2+1	Ax^2+Bx+C
$2x^3-3x+4$	Ax^3+Bx^2+Cx+D
e^{3x}	Ae^{3x}
$3\sin(4x)$	$A\sin(4x)+B\cos(4x)$
$2\cos(4x)$	$A\sin(4x)+B\cos(4x)$
$3\sin(4x)+2\cos(4x)$	$A\sin(4x)+B\cos(4x)$
x^2+1+e^{3x}	$Ax^2+Bx+C+De^{3x}$
$e^{3x}\cos(\pi x)$	$Ae^{3x}(B\cos(\pi x)+C\sin(\pi x))\to$
	$e^{3x}(AB\cos(\pi x)+AC\sin(\pi x))\to$
	$e^{3x}(A\cos(\pi x)+B\sin(\pi x))$
$(x^2+1)\cos(-2x)$	$(Ax^2+Bx+C)(D\sin(-2x)+E\cos(-2x))\to$

$$(ADx^2+BDx+CD)\sin(-2x)+(AEx^2+BEx+CE)\cos(-2x)\to$$

$$(Ax^2+Bx+C)\sin(-2x)+(Dx^2+Ex+F)\cos(-2x)$$

Once we have a guess for our particular solution, before we can move on we need to make sure that none of the terms in the guess overlap with any terms in the complementary solution. For any overlapping terms, we'll need to multiply that section of the guess by x (or maybe x^2, x^3, etc.) in order to eliminate the overlap.

For example, if the complementary solution includes the term e^{3x}, and our guess for the particular solution includes Ae^{3x}, then we'll need to multiply Ae^{3x} by x, changing the guess to Axe^{3x}. For exponential terms like these, an overlap only exists if the exponents match exactly. So e^{3x} and Ae^{3x} are overlapping, but e^{3x} and Ae^{5x} are not.

Once we've eliminated any overlap and finalized the guess for our particular solution, we'll find the first and second derivatives of our guess, then plug those first and second derivatives into the original differential equation for $y'(x)$ and $y''(x)$, respectively.

Then we'll be able to combine like terms and equate coefficients on both sides in order to solve for the constants A, B, C, etc. Ultimately, we should end up with a particular solution that we can combine with the complementary solution in order to get the general solution to the second order nonhomogeneous equation.

Here's a summary of these steps.

1. Substitute $g(x) = 0$ to rewrite the nonhomogeneous equation as a homogeneous equation.

2. Use the associated characteristic equation to solve the homogeneous equation, generating the complementary solution $y_c(x)$.

3. Make a guess for a particular solution.

4. Fix any overlap between the guess for the particular solution, and the complementary solution from step 2.

5. Find the first and second derivatives of the non-overlapping guess for the particular solution.

6. Substitute the derivatives into the original differential equation.

7. Equate coefficients to find the values of any constants.

8. Substitute the constant values into the guess to generate a particular solution $y_p(x)$.

9. Sum the complementary and particular solutions to get the general solution, $y(x) = y_c(x) + y_p(x)$.

Let's do an example so that we can see these steps in action.

Example

Find the general solution to the differential equation.

$$y'' + 2y' = 4x - 6e^{-2x}$$

We'll replace $g(x) = 4x - 6e^{-2x}$ with $g(x) = 0$ to change the nonhomogeneous equation into a homogeneous equation (Step 1),

$$y'' + 2y' = 0$$

then we'll solve the associated characteristic equation.

$$r^2 + 2r = 0$$

$$r(r + 2) = 0$$

$$r = 0, \ -2$$

These are distinct real roots, so the complementary solution will be

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

$$y_c(x) = c_1 e^{(0)x} + c_2 e^{-2x}$$

$$y_c(x) = c_1(1) + c_2 e^{-2x}$$

$$y_c(x) = c_1 + c_2 e^{-2x} \text{ (Step 2)}$$

Now we can switch over to working on a particular solution, $g(x) = 4x - 6e^{-2x}$. The first thing we notice is that we have a polynomial function, $4x$, and an exponential function, $-6e^{-2x}$. We'll use $Ax + B$ as our guess for the polynomial function, and we'll use Ce^{-2x} as our guess for the exponential function. Putting these together (Step 3), our guess for a particular solution will be

$$y_p(x) = Ax + B + Ce^{-2x}$$

Comparing this to the complementary solution, we can see that c_2e^{-2x} from the complementary solution and Ce^{-2x} from the particular solution are overlapping terms. To fix this (Step 4), we'll multiply Ce^{-2x} from the particular solution by x, such that our guess becomes

$$y_p(x) = Ax + B + Cxe^{-2x}$$

Furthermore, we can see that c_1 from the complementary solution and B from the particular solution are overlapping terms. To fix this overlap, we'll multiply $Ax + B$ from the particular solution by x, such that our guess becomes

$$y_p(x) = Ax^2 + Bx + Cxe^{-2x}$$

Taking the first and second derivatives of this guess (Step 5), we get

$$y_p'(x) = 2Ax + B + Ce^{-2x} - 2Cxe^{-2x}$$

$$y_p''(x) = 2A - 4Ce^{-2x} + 4Cxe^{-2x}$$

Plugging the first two derivatives into the original differential equation (Step 6), we get

$$2A - 4Ce^{-2x} + 4Cxe^{-2x} + 2(2Ax + B + Ce^{-2x} - 2Cxe^{-2x}) = 4x - 6e^{-2x}$$

$$2A - 4Ce^{-2x} + 4Cxe^{-2x} + 4Ax + 2B + 2Ce^{-2x} - 4Cxe^{-2x} = 4x - 6e^{-2x}$$

$$(2A + 2B) + 4Ax - 2Ce^{-2x} = 4x - 6e^{-2x}$$

$$(2A + 2B) + (4A)x + (-2C)e^{-2x} = (0) + (4)x + (-6)e^{-2x}$$

Equating coefficients from the left and right side (Step 7), we get

$$2A + 2B = 0 \qquad 4A = 4 \qquad -2C = -6$$

$$2(1) + 2B = 0 \qquad A = 1 \qquad C = 3$$

$$2B = -2$$

$$B = -1$$

We'll plug the results into our guess for the particular solution (Step 8) to get

$$y_p(x) = (1)x^2 + (-1)x + (3)xe^{-2x}$$

$$y_p(x) = x^2 - x + 3xe^{-2x}$$

Putting this particular solution together with the complementary solution (Step 9) gives us the general solution to the differential equation.

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1 + c_2e^{-2x} + x^2 - x + 3xe^{-2x}$$

When to use undetermined coefficients

This method of undetermined coefficients is most useful because it reduces the differential equation problem to an algebra problem. In this last example, we were able to get down to

$$(2A + 2B) + (4A)x + (-2C)e^{-2x} = (0) + (4)x + (-6)e^{-2x}$$

and then solve for the constants with a simple set of equations.

The problem with the method is that it only works for a specific, somewhat small set of $g(x)$ functions. We'll typically use this method when $g(x)$ is some combination of polynomial, exponential, and/or trigonometric functions.

The next method we'll look at is variation of parameters, which is a little more involved than undetermined coefficients, but also more versatile, in the sense that it'll work for $g(x)$ functions that undetermined coefficients can't handle.

If we attempt to find the general solution with undetermined coefficients and it doesn't work (the system of equations we get from the algebra problem isn't solvable), then it usually means one of two things. It could mean that we made the wrong guess for the particular solution, and/or it could mean that undetermined coefficients can't be used to deal with the $g(x)$ in our differential equation.

Therefore, if we do have a problem finding a particular solution, we should first check for errors in our guess and the algebra steps we used with it. Then if that doesn't work, we should try variation of parameters instead, which we'll talk about in the next lesson.

Variation of parameters for nonhomogeneous equations

Like the method of undetermined coefficients, variation of parameters is a method we can use to find the general solution to a second order nonhomogeneous equation.

Variation of parameters works for a wider family of $g(x)$ functions than undetermined coefficients, which makes it more versatile. But variation of parameters also requires some integration, and there's no guarantee that we'll get functions that are easy to integrate.

Variation of parameters

Just like with undetermined coefficients, we'll start by substituting $g(x)=0$ to change the nonhomogeneous equation into a homogeneous equation. Then we'll use the associated characteristic equation to find the complementary solution $y_c(x)$.

Handling the particular solution is where the method of variation of parameters starts to diverge from the method of undetermined coefficients.

To find a particular solution using variation of parameters, we'll start by picking out the fundamental set of solutions $\{y_1, y_2\}$ from the complementary solution. This set of solutions will be everything but the coefficients c_1 and c_2.

Complementary solution $y_c(x)$ **Fundamental solution set** $\{y_1, y_2\}$

$$y_c(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x} \qquad \{y_1, y_2\} = \{e^{r_1 x}, e^{r_2 x}\}$$

$$y_c(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x} \qquad \{y_1, y_2\} = \{e^{r_1 x}, x e^{r_1 x}\}$$

$$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x)) \qquad \{y_1, y_2\} = \{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\}$$

Once we've identified the fundamental set of solutions, we'll plug y_1 and y_2 and their derivatives into this system of linear equations:

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

We'll solve the system for u_1' and u_2', then integrate to find u_1 and u_2. This is the integration we mentioned earlier. Depending on what we find for u_1' and u_2', it won't always be easy, or even possible, to integrate to u_1 and u_2. But if we're able to get u_1 and u_2, then we can say that a particular solution is

$$y_p(x) = u_1 y_1 + u_2 y_2$$

And just like with undetermined coefficients, the general solution to the second order nonhomogeneous equation is still the sum of the complementary and particular solutions,

$$y(x) = y_c(x) + y_p(x)$$

One final note. With this method, it's important that we start with a coefficient of 1 on y''. If our differential equation has a y'' term with a coefficient that's anything other than 1, then our first step should be to divide through the equation by that value to change the coefficient on y'' to 1.

Let's do an example so that we can see exactly how we can use variation of parameters to get to the general solution.

Example

Use variation of parameters to find the general solution to the differential equation.

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^3}$$

The coefficient on y'' is already 1. We'll substitute $g(x) = 0$ to rewrite the nonhomogeneous equation as a homogeneous equation.

$$y'' + 4y' + 4y = 0$$

Now we'll use the associated characteristic equation to find roots.

$$r^2 + 4r + 4 = 0$$

$$(r + 2)(r + 2) = 0$$

$$r = -2$$

We get equal real roots, so the complementary solution is

$$y_c(x) = c_1 e^{-2x} + c_2 x e^{-2x}$$

and the fundamental set of solutions is therefore

$$\{y_1, y_2\} = \{e^{-2x}, xe^{-2x}\}$$

If we differentiate the set, we get

$$\{y_1', y_2'\} = \{-2e^{-2x}, e^{-2x} - 2xe^{-2x}\}$$

Create a system of linear equations using the fundamental set and its derivatives.

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'e^{-2x} + u_2'xe^{-2x} = 0$$

$$u_1'y_1' + u_2'y_2' = g(x)$$

$$u_1'(-2e^{-2x}) + u_2'(e^{-2x} - 2xe^{-2x}) = \frac{e^{-2x}}{x^3}$$

$$-2u_1'e^{-2x} + u_2'e^{-2x} - 2u_2'xe^{-2x} = \frac{e^{-2x}}{x^3}$$

Our next step is to solve this system for u_1' and u_2'. We'll start by multiplying the first equation by 2,

$$2u_1'e^{-2x} + 2u_2'xe^{-2x} = 0$$

$$-2u_1'e^{-2x} + u_2'e^{-2x} - 2u_2'xe^{-2x} = \frac{e^{-2x}}{x^3}$$

then we'll add the equations to eliminate u_1'.

$$(2u_1'e^{-2x} - 2u_1'e^{-2x}) + (2u_2'xe^{-2x} - 2u_2'xe^{-2x}) + u_2'e^{-2x} = 0 + \frac{e^{-2x}}{x^3}$$

$$u_2'e^{-2x} = \frac{e^{-2x}}{x^3}$$

$$u_2' = \frac{1}{x^3}$$

Use this value for u_2' to find u_1'

$$u_1'e^{-2x} + u_2'xe^{-2x} = 0$$

$$u_1'e^{-2x} + \left(\frac{1}{x^3}\right)xe^{-2x} = 0$$

$$u_1'e^{-2x} + \frac{e^{-2x}}{x^2} = 0$$

$$u_1'e^{-2x} = -\frac{e^{-2x}}{x^2}$$

$$u_1' = -\frac{1}{x^2}$$

Integrate u_1' to find u_1,

$$u_1 = \int u_1' = \int -\frac{1}{x^2}\ dx$$

$$u_1 = \int -x^{-2}\ dx$$

$$u_1 = -\frac{1}{-1}x^{-1}$$

$$u_1 = x^{-1}$$

$$u_1 = \frac{1}{x}$$

and then integrate u_2' to find u_2.

$$u_2 = \int u_2' = \int \frac{1}{x^3}\ dx$$

$$u_2 = \int x^{-3}\ dx$$

$$u_2 = \frac{1}{-2}x^{-2}$$

$$u_2 = -\frac{1}{2}x^{-2}$$

$$u_2 = -\frac{1}{2x^2}$$

Given these values for u_1 and u_2, the particular solution is

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = \frac{1}{x}e^{-2x} - \frac{1}{2x^2}xe^{-2x}$$

$$y_p(x) = \frac{e^{-2x}}{x} - \frac{e^{-2x}}{2x}$$

$$y_p(x) = \frac{2e^{-2x} - e^{-2x}}{2x}$$

$$y_p(x) = \frac{e^{-2x}}{2x}$$

Adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} + \frac{e^{-2x}}{2x}$$

Fundamental solution sets and the Wronskian

We just looked at how to use the method of undetermined coefficients and the method of variation of parameters to find the solution to a second order linear nonhomogeneous equation.

When we used variation of parameters, we took a fundamental set of solutions $\{y_1, y_2\}$ from the complementary solution,

Complementary solution $y_c(x)$	**Fundamental solution set** $\{y_1, y_2\}$
$y_c(x) = c_1e^{r_1x} + c_2e^{r_2x}$	$\{y_1, y_2\} = \{e^{r_1x}, e^{r_2x}\}$
$y_c(x) = c_1e^{r_1x} + c_2xe^{r_1x}$	$\{y_1, y_2\} = \{e^{r_1x}, xe^{r_1x}\}$
$y(x) = e^{\alpha x}(c_1\cos(\beta x) + c_2\sin(\beta x))$	$\{y_1, y_2\} = \{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\}$

and then we plugged that solution set into a system of linear equations.

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = g(x)$$

We solved this system for u_1' and u_2', then integrated those values to find u_1 and u_2, which we then used to build a particular solution for the differential equation.

At this point though, we want to expand on the idea of a fundamental set of solutions, and then introduce a different way to look at the method of variation of parameters.

Fundamental set of solutions

Technically, we say that a set of solutions to a differential equation is a **fundamental set of solutions** when the Wronskian of those solutions is non-zero.

The **Wronskian** W of a solution set $\{y_1, y_2\}$ is the determinant of the set of solutions and their derivatives.

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = y_1(t_0)y_2'(t_0) - y_2(t_0)y_1'(t_0)$$

So if $y_1(t)$ and $y_2(t)$ are solutions to the differential equation, and if the Wronskian is non-zero $W \neq 0$, then we say that the solutions $y_1(t)$ and $y_2(t)$ are **linearly independent**, which means that they're not constant multiples of one another. When the Wronskian is non-zero and the solution set is therefore linearly independent, we call $\{y_1, y_2\}$ a fundamental set of solutions. If we have a fundamental set of solutions, we can conclude that the general solution is

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

In other words, we've actually been assuming up to this point that the solution set for a second order equation is a fundamental set, because we've been using any solution set we found to build the general solution, without confirming that $W \neq 0$.

So let's circle back to take a look at the case of distinct real roots, and confirm now that $\{y_1, y_2\} = \{e^{r_1x}, e^{r_2x}\}$ is a fundamental set of solutions, by plugging the solutions and their derivatives into the Wronskian determinant.

$$W(e^{r_1x}, e^{r_2x})(x) = \begin{vmatrix} e^{r_1x} & e^{r_2x} \\ r_1e^{r_1x} & r_2e^{r_2x} \end{vmatrix}$$

$$W(e^{r_1x}, e^{r_2x})(x) = (e^{r_1x})(r_2e^{r_2x}) - (e^{r_2x})(r_1e^{r_1x})$$

$$W(e^{r_1x}, e^{r_2x})(x) = r_2e^{r_1x+r_2x} - r_1e^{r_1x+r_2x}$$

$$W(e^{r_1x}, e^{r_2x})(x) = e^{(r_1+r_2)x}(r_2 - r_1)$$

We need to confirm that this Wronskian value is non-zero. The only way $W = 0$ is if $e^{(r_1+r_2)x} = 0$ and/or $r_2 - r_1 = 0$. The exponential function is never zero. And remember that this is the case with distinct real roots, which means $r_2 \neq r_1$, so $r_2 - r_1 \neq 0$.

In other words, the Wronskian of the solution set $\{y_1, y_2\} = \{e^{r_1x}, e^{r_2x}\}$, assuming $r_1 \neq r_2$, will be a fundamental set of solutions to the second order equation.

And similar logic applies to the solution set for equal real roots and for complex conjugate roots, which is why we've been able to rely on $\{y_1, y_2\} = \{e^{r_1x}, xe^{r_1x}\}$ to form the general solution to a second order equation with equal real roots, and rely on $\{y_1, y_2\} = \{e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)\}$ to form a general solution to a second order equation with complex conjugate roots.

The reliable set

There's a set of initial conditions we can use that will always give us a fundamental set of solutions for $y'' + p(t)y' + q(t)y = 0$, which is $y(t_0) = 1$ and $y'(t_0) = 0$, and $y(t_0) = 0$ and $y'(t_0) = 1$.

The reason these two pairs of initial conditions produce a fundamental set of solutions is because the Wronskian they make will never be 0.

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1 \neq 0$$

Let's do an example with this set of initial conditions that we know will reliably produce a fundamental set of solutions.

Example

Use the reliable set of initial conditions to find a fundamental set of solutions for the second order differential equation.

$$y'' + 5y' + 6y = 0$$

Normally, we would find the general solution to this second order equation by solving the associated characteristic equation,

$$r^2 + 5r + 6 = 0$$

$$(r + 3)(r + 2) = 0$$

$$r = -3, \ -2$$

and then plugging these values into the formula for the complementary solution with distinct real roots.

$$y_c(x) = c_1e^{-3x} + c_2e^{-2x}$$

The derivative of this complementary solution is

$$y_c'(x) = -3c_1e^{-3x} - 2c_2e^{-2x}$$

If we now plug in the initial conditions $y(x_0) = 1$ and $y'(x_0) = 0$, and $y(x_0) = 0$ and $y'(x_0) = 1$, we'll be able to generate another fundamental set of solutions. We'll use $x_0 = 0$, since we can choose any value of x_0 that we like, and using $x_0 = 0$ will be easiest.

For $y(0) = 1$, $\quad 1 = c_1 + c_2$

For $y'(0) = 0$, $\quad 0 = -3c_1 - 2c_2$

Then this system gives $c_1 = -2$ and $c_2 = 3$, so we could say that one solution in our fundamental set of solutions is

$$y_1(x) = -2e^{-3x} + 3e^{-2x}$$

The second set of initial conditions gives

For $y(0) = 0$, $\quad 0 = c_1 + c_2$

For $y'(0) = 1$, $\quad 1 = -3c_1 - 2c_2$

Then this system gives $c_1 = -1$ and $c_2 = 1$, so we could say that the other solution in our fundamental set is

$$y_2(x) = -e^{-3x} + e^{-2x}$$

So one fundamental set of solutions is $\{y_1, y_2\} = \{e^{-3x}, e^{-2x}\}$, but another is

$$\{y_1, y_2\} = \{-2e^{-3x} + 3e^{-2x}, -e^{-3x} + e^{-2x}\}$$

and the complementary solution this set creates would be

$$y_c(x) = c_1(-2e^{-3x} + 3e^{-2x}) + c_2(-e^{-3x} + e^{-2x})$$

While this solution looks a lot messier than the solution we originally generated, and it *is* messier, the advantage of a solution generated this way is that it's a reliable fundamental set of solutions.

Variation of parameters with the Wronskian

We've already seen how to use variation of parameters with a system of linear equations to find the solution to a second order nonhomogeneous equation.

When we looked at this method before, we built this system of linear equations:

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = g(x)$$

If we solve the first equation for u_1',

$$u_1' = -\frac{u_2'y_2}{y_1}$$

then we can substitute it into the second equation to solve it for u_2'.

$$\left(-\frac{u_2'y_2}{y_1}\right)y_1' + u_2'y_2' = g(x)$$

$$u_2'\left(y_2' - \frac{y_2y_1'}{y_1}\right) = g(x)$$

$$u_2'\left(\frac{y_1y_2' - y_2y_1'}{y_1}\right) = g(x)$$

$$u_2' = \frac{y_1g(x)}{y_1y_2' - y_2y_1'}$$

Plugging this back into the equation we found for u_1', we get

$$u_1' = -\frac{y_1y_2g(x)}{y_1(y_1y_2' - y_2y_1')}$$

$$u_1' = -\frac{y_2g(x)}{y_1y_2' - y_2y_1'}$$

Notice that the denominators of u_1' and u_2' are equivalent, they're both $y_1y_2' - y_2y_1'$, and this value is actually the Wronskian of the fundamental set of solutions $\{y_1, y_2\}$.

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

We know that the Wronskian of a fundamental set of solutions is non-zero, which means that the denominators of u_1' and u_2' won't be 0, which means they'll be defined.

Now to find the values of u_1 and u_2, we need to integrate u_1' and u_2'.

$$u_1 = \int u_1' \ dx = \int -\frac{y_2g(x)}{y_1y_2' - y_2y_1'} \ dx = \int -\frac{y_2g(x)}{W(y_1, y_2)} \ dx$$

$$u_2 = \int u_2' \ dx = \int \frac{y_1g(x)}{y_1y_2' - y_2y_1'} \ dx = \int \frac{y_1g(x)}{W(y_1, y_2)} \ dx$$

Given these values of u_1 and u_2, using variation of parameters and these Wronskian integrals tells us that the particular solution to the second order nonhomogeneous equation will be

$$y_p(x) = u_1y_1 + u_2y_2$$

$$y_p(x) = -y_1 \int \frac{y_2 g(x)}{W(y_1, y_2)} \, dx + y_2 \int \frac{y_1 g(x)}{W(y_1, y_2)} \, dx$$

Then, just as before, to find the general solution to the second order nonhomogeneous equation, we add this particular solution to the complementary solution.

$$y(x) = y_c(x) + y_p(x)$$

Before we look at an example, realize that this Wronskian integral method for variation of parameters is the same as the system of equations method for variation of parameters that we used earlier. These are just two different ways of presenting an identical process, so we can use either method.

That being said, let's rework the example we solved previously with a system of equations for variation of parameters, but this time we'll use the Wronskian integrals.

Example

Use variation of parameters to find the general solution to the differential equation.

$$y'' + 4y' + 4y = \frac{e^{-2x}}{x^3}$$

Just as before, we solve the characteristic equation associated with the homogeneous equation to find equal real roots and a complementary solution.

$$y_c(x) = c_1e^{-2x} + c_2xe^{-2x}$$

Then the fundamental set of solutions and their derivatives are

$$\{y_1, y_2\} = \{e^{-2x}, xe^{-2x}\}$$

$$\{y_1', y_2'\} = \{-2e^{-2x}, e^{-2x} - 2xe^{-2x}\}$$

The Wronskian of this fundamental set is

$$W(e^{-2x}, xe^{-2x}) = \begin{vmatrix} e^{-2x} & xe^{-2x} \\ -2e^{-2x} & e^{-2x} - 2xe^{-2x} \end{vmatrix}$$

$$W(e^{-2x}, xe^{-2x}) = (e^{-2x})(e^{-2x} - 2xe^{-2x}) - (xe^{-2x})(-2e^{-2x})$$

$$W(e^{-2x}, xe^{-2x}) = e^{-4x} - 2xe^{-4x} + 2xe^{-4x}$$

$$W(e^{-2x}, xe^{-2x}) = e^{-4x}$$

Using the Wronskian integrals, the particular solution is

$$y_p(x) = -e^{-2x}\int \frac{xe^{-2x}\left(\frac{e^{-2x}}{x^3}\right)}{e^{-4x}}\ dx + xe^{-2x}\int \frac{e^{-2x}\left(\frac{e^{-2x}}{x^3}\right)}{e^{-4x}}\ dx$$

$$y_p(x) = -e^{-2x}\int \frac{1}{x^2}\ dx + xe^{-2x}\int \frac{1}{x^3}\ dx$$

$$y_p(x) = -e^{-2x}\left(-\frac{1}{x}\right) - \frac{1}{2}xe^{-2x}x^{-2}$$

$$y_p(x) = \frac{2e^{-2x}}{2x} - \frac{e^{-2x}}{2x}$$

$$y_p(x) = \frac{e^{-2x}}{2x}$$

This is the same particular solution we found in the earlier lesson when we used a system of equations with variation of parameters. So, just as before, adding this particular solution to the complementary solution gives us the general solution $y(x)$.

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} + \frac{e^{-2x}}{2x}$$

Initial value problems with nonhomogeneous equations

We've already seen how to solve second order nonhomogeneous equations using the methods of undetermined coefficients and variation of parameters.

All we want to say here is that, once we've found the general solution as the sum of the complementary and particular solutions, we can still extend the problem one step further by solving an initial value problem.

Solving the IVP

To solve an initial value problem for a second order nonhomogeneous differential equation, we'll use undetermined coefficients or variation of parameters to find the general solution $y(x) = y_c(x) + y_p(x)$.

Then we'll differentiate the general solution to get $y'(x) = y_c'(x) + y_p'(x)$. Once we have the general solution and its derivative, we'll plug the initial conditions $y(x_0) = y_0$ and $y'(x_1) = y_1$ into these equations, which will create a system of linear equations that we can solve for c_1 and c_2.

Once we find c_1 and c_2, we'll substitute them back into the general solution, which will solve the initial value problem. Let's do an example, skipping straight to the point where we've already found the general solution.

Example

Solve the initial value problem for the second order nonhomogeneous equation, given $y(0) = 0$ and $y'(0) = 1$.

$$y'' + y' = x - 2e^{-x}$$

If we solve the associated characteristic equation, we find roots $r = 0, \ -1$, which gives us the complementary solution with distinct real roots, $y_c(x) = c_1 + c_2e^{-x}$.

Our guess for the particular solution will be $y_p(x) = Ax^2 + Bx + Cxe^{-x}$, and undetermined coefficients lets us find $A = 1/2$, $B = -1$, and $C = 2$, giving

$$y_p(x) = \frac{1}{2}x^2 - x + 2xe^{-x}$$

for the particular solution, and therefore a general solution of

$$y(x) = c_1 + c_2e^{-x} + \frac{1}{2}x^2 - x + 2xe^{-x}$$

Differentiating the general solution gives

$$y'(x) = -c_2e^{-x} + x - 1 + 2e^{-x} - 2xe^{-x}$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = 1$ into the general solution and its derivative.

$$0 = c_1 + c_2e^{-(0)} + \frac{1}{2}(0)^2 - 0 + 2(0)e^{-(0)}$$

$$1 = -c_2e^{-(0)} + 0 - 1 + 2e^{-(0)} - 2(0)e^{-(0)}$$

Simplifying these equations gives

$$0 = c_1 + c_2$$

$$0 = c_2$$

Substitute $c_2 = 0$ into $0 = c_1 + c_2$.

$$0 = c_1 + 0$$

$$0 = c_1$$

With $c_1 = 0$ and $c_2 = 0$, the general solution becomes

$$y(x) = 0 + (0)e^{-x} + \frac{1}{2}x^2 - x + 2xe^{-x}$$

$$y(x) = \frac{1}{2}x^2 - x + 2xe^{-x}$$

Direction fields and solution curves

We've been looking at different forms of first order differential equations, and we've been learning analytical methods for finding the equations of their solutions.

But it's important to remember that, in the real world, differential equations often won't behave so nicely. In other words, it's common to face differential equations that don't match one of our perfect formats (linear, separable, Bernoulli, homogeneous, exact, etc.). It's also normal to see equations that have a solution that we can't use an analytical method to find, or even a differential equation that has no solution at all.

But even when there isn't a solution, or there *is* a solution but we can't calculate it, we can still often draw conclusions about the behavior of the differential equation.

So how will we find the solution to a differential equation that isn't one of these special types? Well, instead of taking an analytical approach, we can take a geometric approach.

A geometric approach

A **direction field** is a graph made of many small arrows, each of which approximates the slope of the curve in the area where the arrow is plotted.

Up to now, we've been looking at differential equations containing y', or dy/dx, and solving them analytically to find a solution y, or $y(x)$. A direction field is just a geometric way of displaying the same information in the differential equation, and sketching a solution curve, also called an integral curve, through the direction field is just a geometric way of displaying the same information in the solution equation.

In other words, writing the equation of the differential equation $y'(x)$ is the same as sketching its direction field, and finding the solution $y(x)$ is the same as sketching the solution curve through the direction field. They're just two different ways of displaying the same set of information.

So a direction field is a sketch of a differential equation, and an integral curve plotted through the direction field is the sketch of a solution to that differential equation.

Sketching direction fields

Our first approach to creating a direction field by hand will be to choose a set of equally spaced coordinate points, and then calculate the derivative at each point.

Once we find the value of the derivative at each coordinate point, we'll sketch a tiny arrow there, whose slope is equal to the value of the derivative.

Repeating this process across many coordinate points will create the direction field. If the direction field doesn't give us enough detail to get a good idea of how the derivative is behaving, then we can go back through and calculate the derivative at additional coordinate points.

And once we have a good direction field sketched out, we can use it to sketch a solution curve through any coordinate point. Each of the little arrows in the direction field represents the tangent to the solution curve at that point. So sketching the solution curve means following the direction indicated by the arrows, keeping the solution curve tangent to each of the arrows.

Of course, we're only doing this to see how a direction field is actually built, but once we understand the concept, going forward we'll use a calculator or computer to quickly get a picture of the direction field when we need it.

Let's look at an example so that we can see how to build the direction field for a differential equation.

Example

Sketch a direction field for the differential equation, then sketch a solution curve that passes through $(1,1)$.

$$y' = y - x$$

For a differential equation in this form, we'll sketch the direction field by using coordinate pairs (x, y) to find corresponding values for y'. For example, if we take the coordinate pair $(x, y) = (-2, -2)$, we can plug it into the differential equation to get

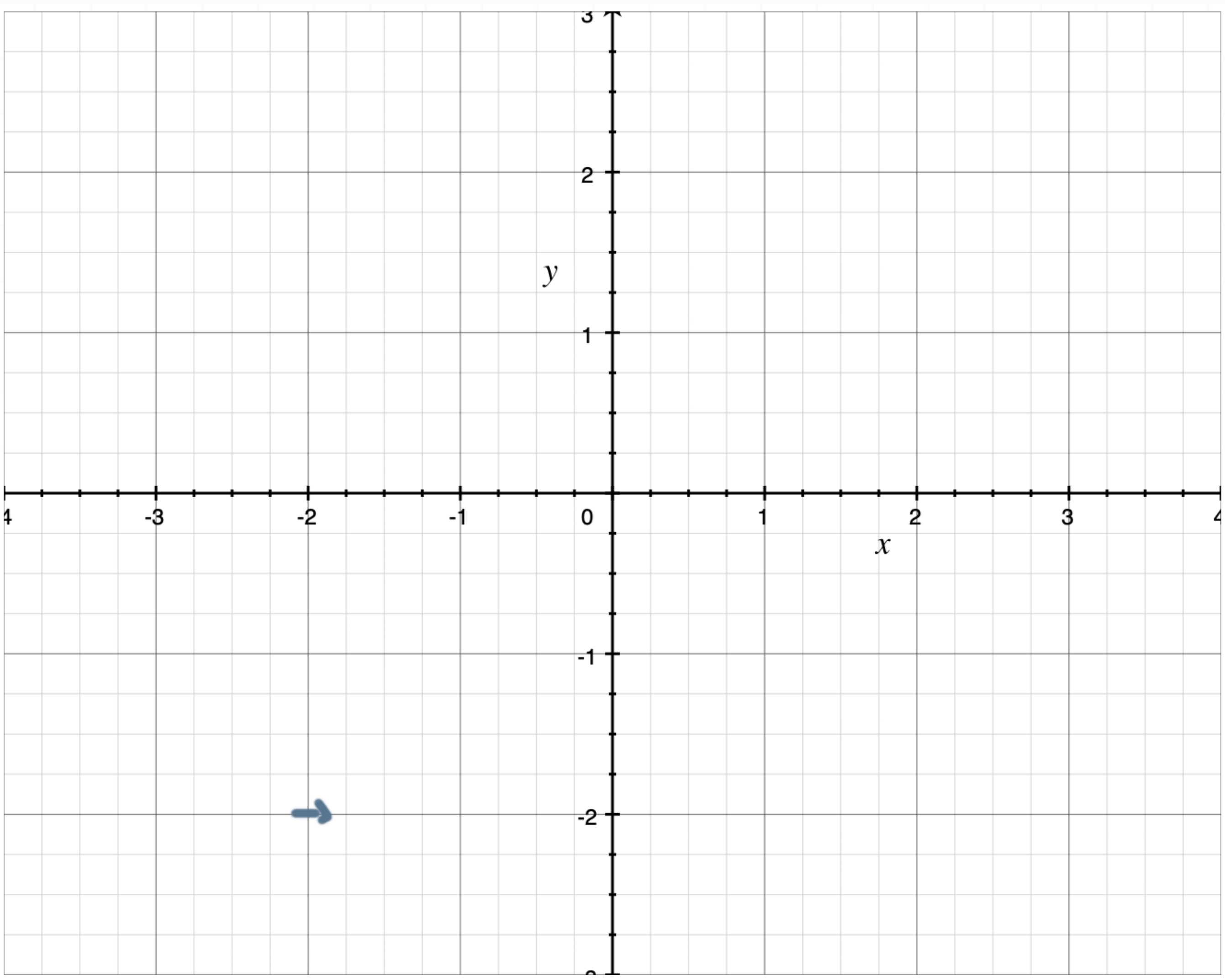
$$y' = y - x$$

$$y' = -2 - (-2)$$

$$y' = -2 + 2$$

$$y' = 0$$

Since y' is the derivative (the slope of the tangent line), we can say that the slope at $(-2, -2)$ is 0. To sketch this information into the direction field, we'll draw a very small arrow with slope 0 (so the arrow will be horizontal), at the point $(-2, -2)$.



The trick with these kinds of problems is keeping all this information organized. We'll start by picking values for x, like $x = \{-2, -1, 0, 1, 2\}$. Then we'll choose the same values for y, and we'll pair each value of y with each of the values we chose for x. Since we have five values for x,

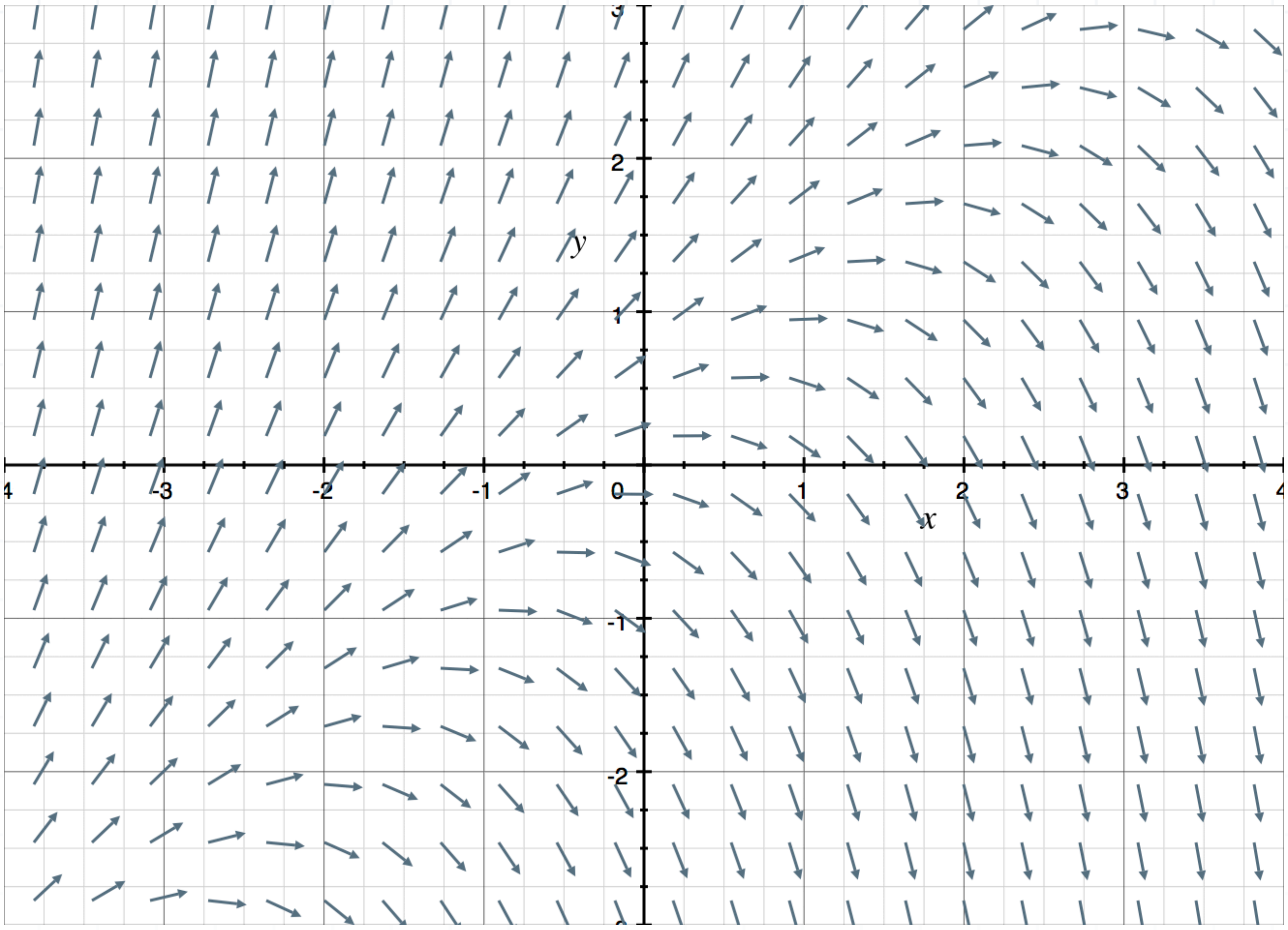
$x = \{-2, -1, 0, 1, 2\}$, and the same five values for y, $y = \{-2, -1, 0, 1, 2\}$, we'll get $5 \times 5 = 25$ coordinate pairs. We'll plug each of these 25 points into the differential equation to find an associated value for y'.

We can record all of these values in a single table, placing x-values across the top row, y-values down the first column, and the resulting y' values in the body of the table.

		x				
		-2	-1	0	1	2
y	-2	0	-1	-2	-3	-4
	-1	1	0	-1	-2	-3
	0	2	1	0	-1	-2
	1	3	2	1	0	-1
	2	4	3	2	1	0

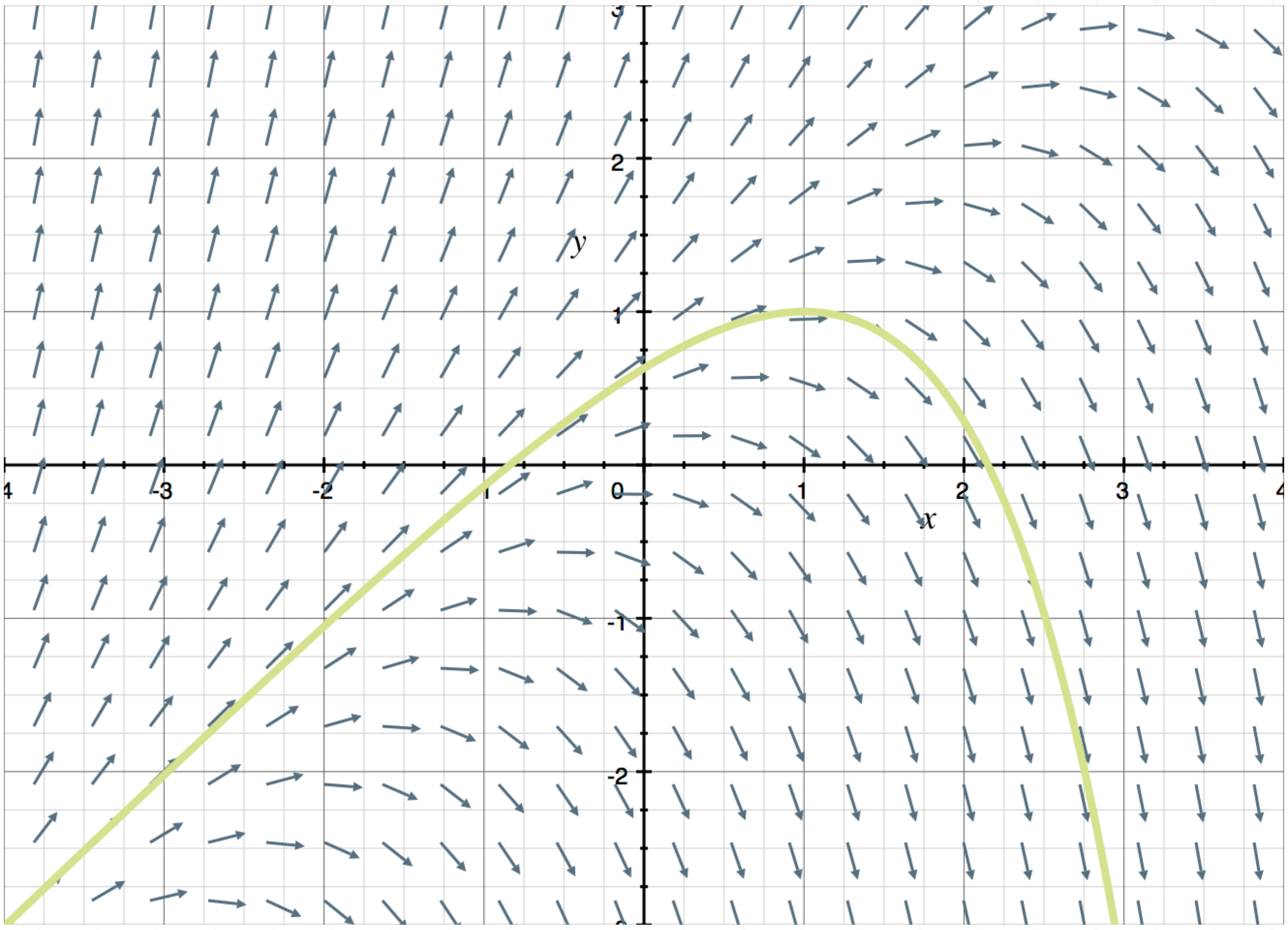
To sketch this information into the direction field, we navigate to the coordinate point (x, y), and then sketch a tiny arrow that has a slope equal to the y'-value. Remember, if $y' = 0$, the small arrow will be horizontal. If y' is positive the arrow will tilt up to the right, and if y' is negative the arrow will tilt down to the right. The larger the value of y', either positive or negative, the steeper (closer to vertical) the arrow will be. The smaller the value of y', either positive or negative, the shallower (closer to horizontal) the arrow will be.

The more y' values we calculate, the more accurate our direction field will be. Here's a detailed direction field for $y' = y - x$:



To address the second part of the question, we'll sketch the solution curve that passes through $(1,1)$. To do that, we'll plot a point at $(1,1)$. From there, we'll follow the direction arrows near that point as closely as we can, trying to ensure that the curve we sketch is tangent to all of the arrows we pass by.

If we keep sketching in the direction that the nearby arrows indicate, we get a rough picture of the solution curve through $(1,1)$.



A solution curve that we sketch by hand will never be exact. Instead, it's an approximation we create as we follow the small arrows in our direction field that indicate the slope of the curve near each arrow.

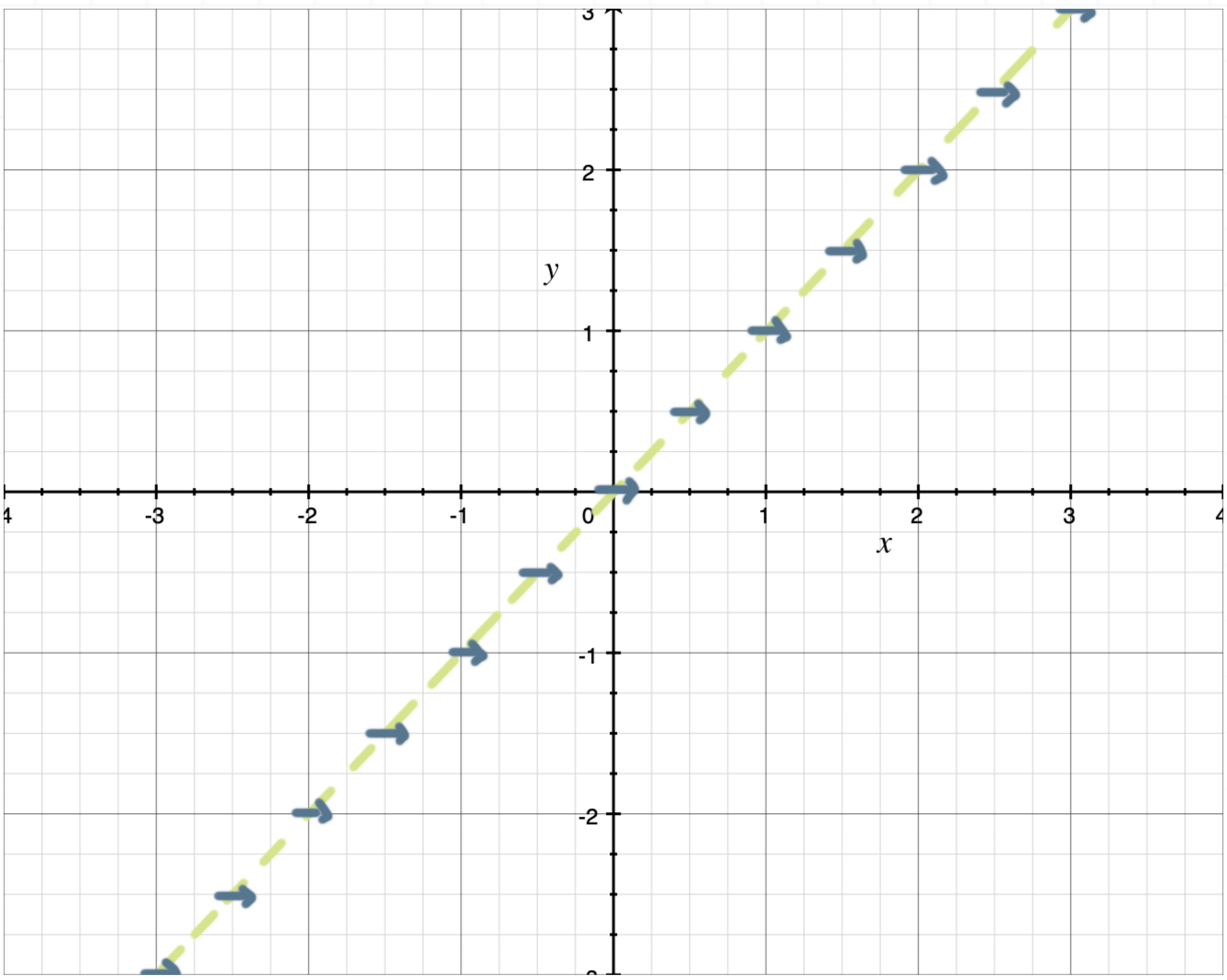
But there's another way to sketch a direction field, which is actually a little more efficient to do by hand, which is to choose a certain value of y' (choose a certain slope) and then find the equation of the curve that maintains that slope.

For example, looking at the differential equation $y' = y - x$ from the last example, if we choose $y' = 0$, then the differential equation can be rewritten as

$$0 = y - x$$

$$y = x$$

This result tells us that the derivative is always 0 along the line $y = x$.

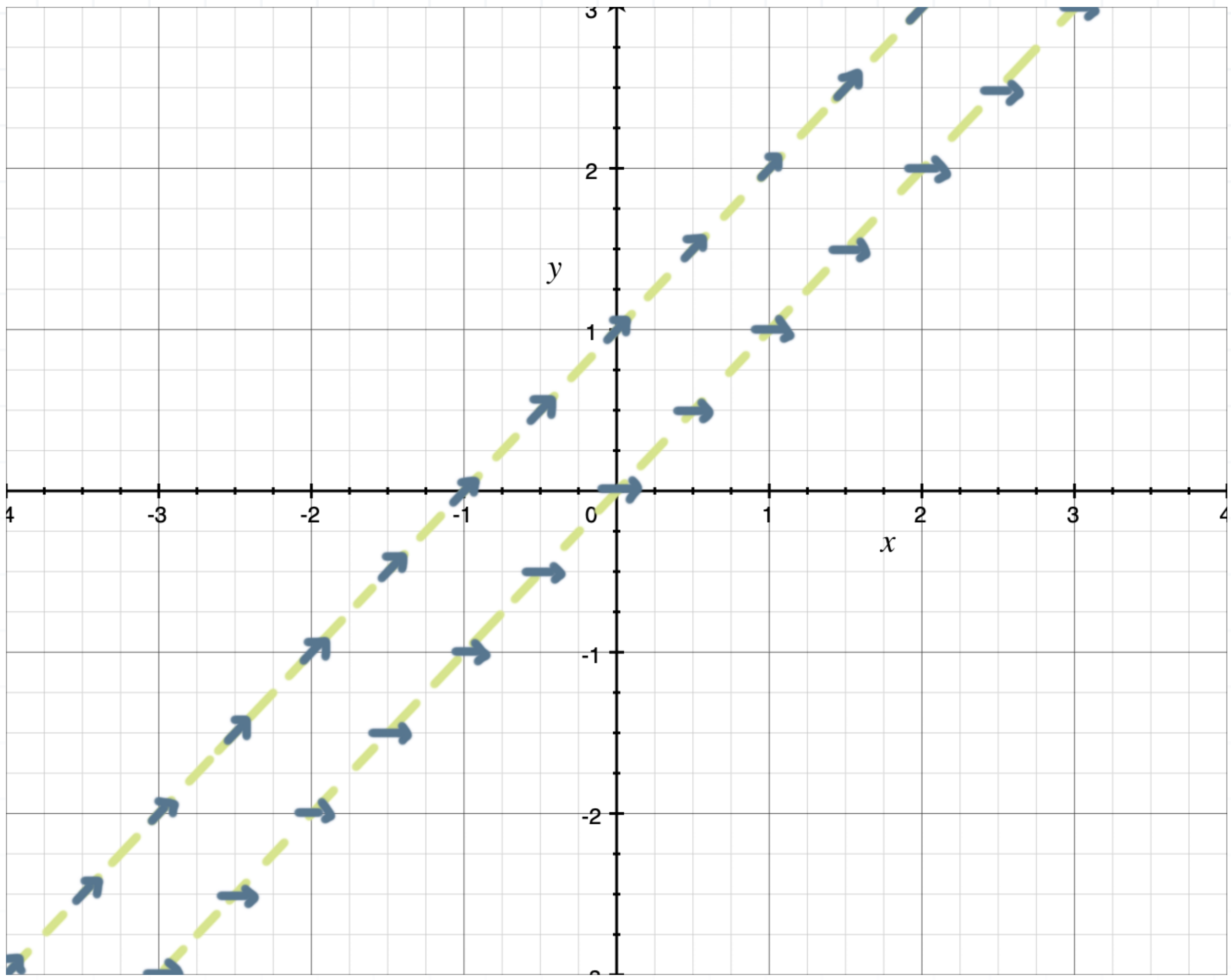


If we choose $y' = 1$, then the differential equation can be rewritten as

$$1 = y - x$$

$$y = x + 1$$

This result tells us that the derivative is always 1 along the line $y = x + 1$.



And in general, if we choose any $y' = c$, then the differential equation can be rewritten as

$$c = y - x$$

$$y = x + c$$

In this form, we can quickly pick any slope c, then sketch that slope along the line $y = x + c$. The family of curves we get by choosing different values for c in $y = x + c$ are the **isoclines** of the equation, so we can say that the derivative will be equal to the constant c along any isocline of the equation.

Which means that, for this equation, we can

- sketch tangents with slope -2 along the isocline $y = x - 2$

- sketch tangents with slope -1 along the isocline $y = x - 1$

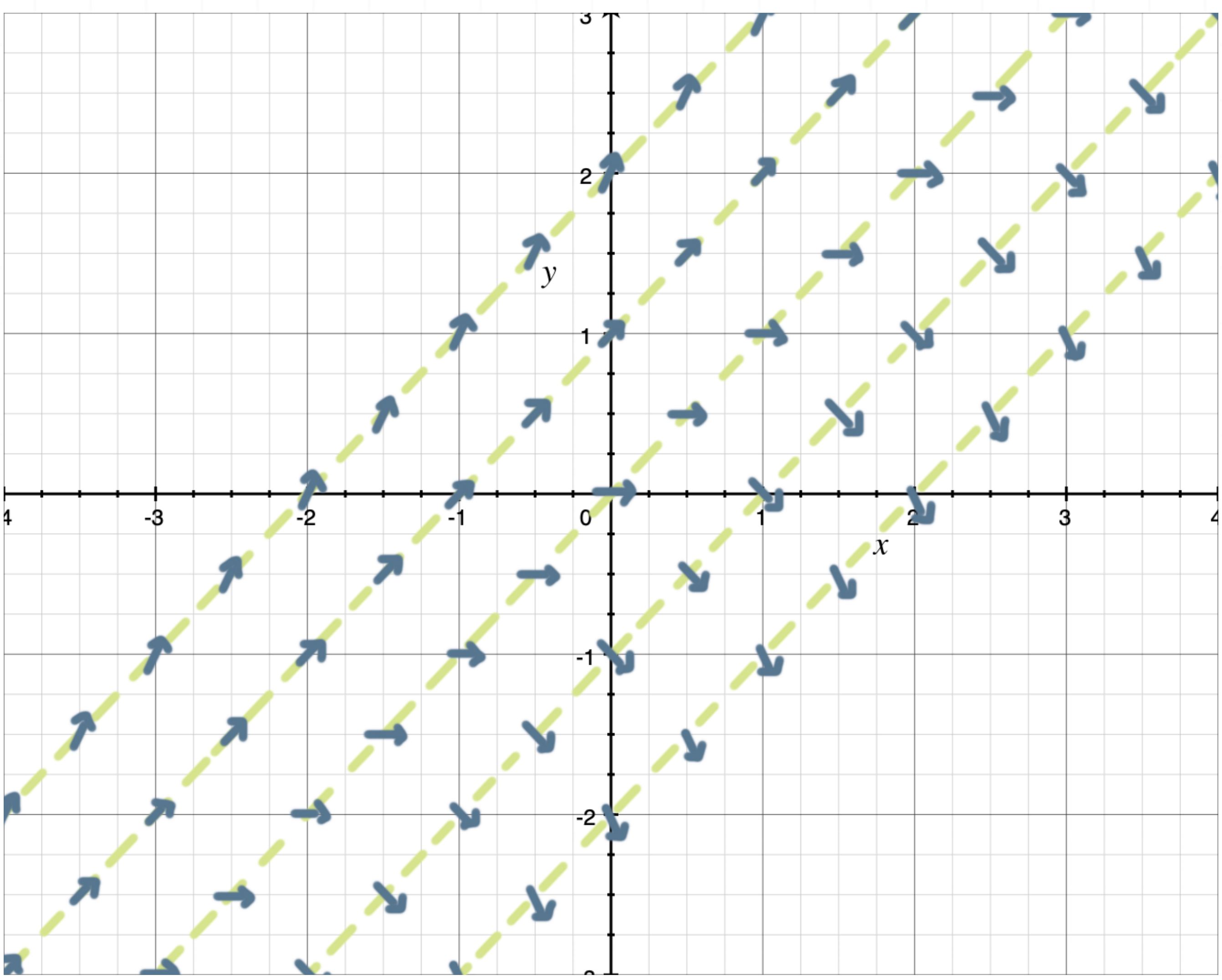
- sketch tangents with slope 0 along the isocline $y = x$

- sketch tangents with slope 1 along the isocline$y = x + 1$

- sketch tangents with slope 2 along the isocline $y = x + 2$

- ...

Using this method, we can more quickly put together the direction field.



And we can sketch solution curves in the same way we did before, which is by following the slopes in the direction field.

Of course, if it's possible to find the equation of the solution curve by solving the differential equation (in this case we can solve the equation as a linear differential equation),

$$y' = y - x$$

$$y' - y = -x$$

$$I = e^{-x}$$

$$y'e^{-x} - ye^{-x} = -xe^{-x}$$

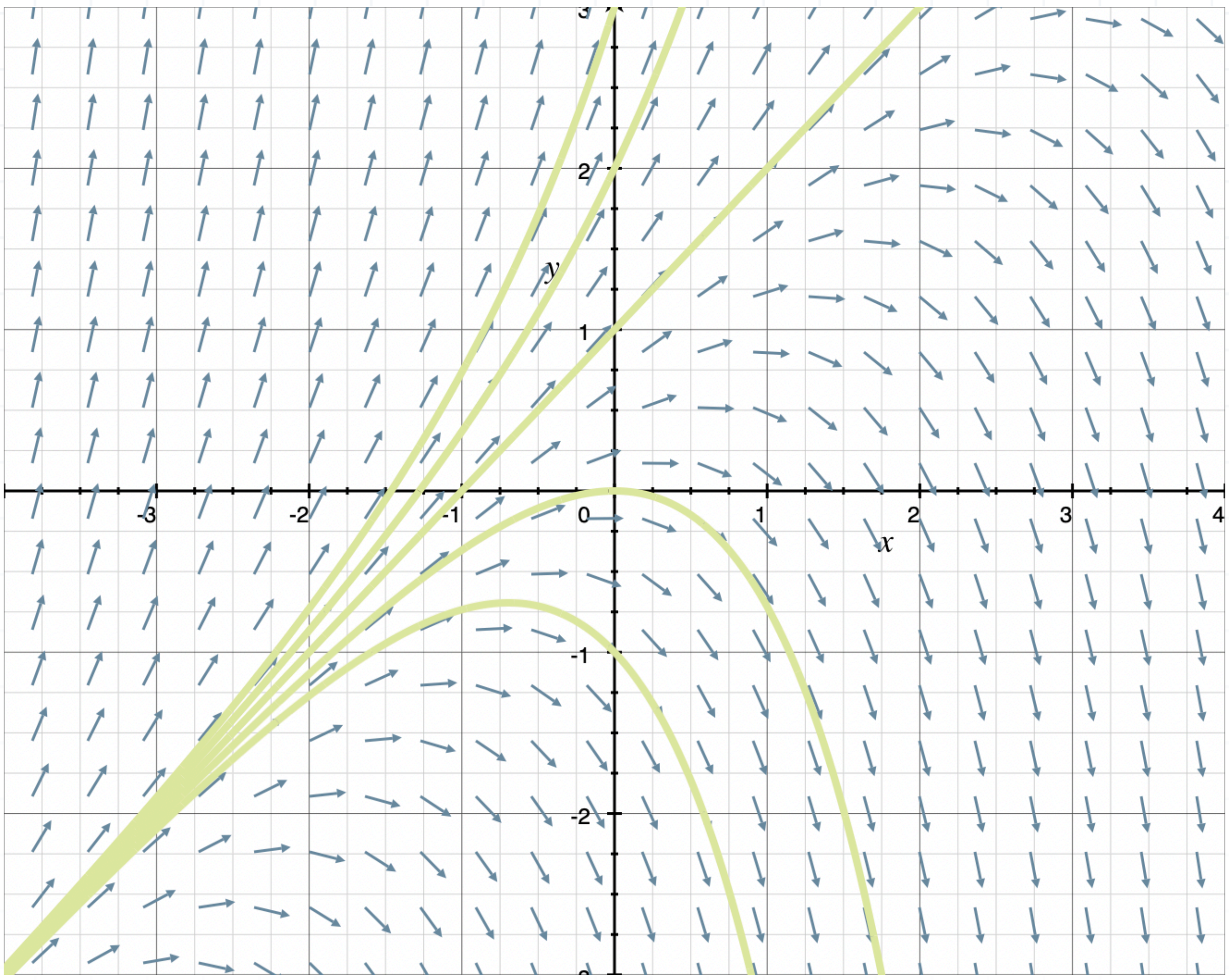
$$\frac{d}{dx}(ye^{-x}) = -xe^{-x}$$

$$\int \frac{d}{dx}(ye^{-x})\ dx = \int -xe^{-x}\ dx$$

$$ye^{-x} = xe^{-x} + e^{-x} + C$$

$$y = x + 1 + Ce^{x}$$

then we can choose different values for the constant of integration C to sketch different solution curves. Below are the solution curves for $C = \{-2, -1, 0, 1, 2\}$.


y
x
-3
-2
-1
0
1
2
3
4
1
2
-1
-2

Intervals of validity

Up to now, we've been looking at various methods for solving specific types of differential equations, like linear, separable, Bernoulli, homogeneous, and exact equations.

We also learned to how to use an initial condition to solve an initial value problem, generating a particular solution from the differential equation's general solution. In this lesson we want to look at more initial value problems, and also address an important point that we've so far left out of our discussion.

Interval of validity for an initial condition

Now that we understand the idea of a solution curve through a direction field representing the solution to the differential equation, we also have to talk about the interval on which the solution is valid. We call this the **interval of validity**, interval of existence, interval of definition, or the domain of the solution.

An interval of validity can be an open interval like (a, b), a closed interval like $[a, b]$, or an infinite interval like $(-\infty, \infty)$ or (a, ∞).

As an example, we looked earlier at the linear differential equation

$$\frac{dy}{dx} + 2y = 4e^{-2x}$$

and used the integrating factor method to find its general solution

$$y = \frac{4x + C}{e^{2x}}$$

Given the initial condition $y(0) = 1$, we could determine that the associated particular solution is

$$y = \frac{4x + 1}{e^{2x}}$$

and the interval of validity would be all real numbers, $(-\infty, \infty)$. Later in this lesson we'll explain in depth why this is the interval of validity, and exactly how to find intervals of validity for linear and non-linear differential equations.

But for now, we just want to know that we state an interval of validity that's associated with a specific initial condition, and therefore with a specific particular solution to the differential equation.

Linear and non-linear equations

Our process for finding the interval of validity will depend on whether we're starting with a linear differential equation or a non-linear differential equation.

When the equation is linear, we can actually find the interval of validity without calculating the equation's solution, because the interval of validity is determined by $P(x)$ and $Q(x)$ from the standard form of the linear equation,

$$\frac{dy}{dx} + P(x)y = Q(x)$$

If $P(x)$ and/or $Q(x)$ are discontinuous at any point x, then those values of x will break up $(-\infty, \infty)$ into different potential intervals of validity. For instance, if $P(x)$ is discontinuous at $x = 0$ and $Q(x)$ is discontinuous at $x = 0$ and $x = \pm 3$, then we'd divide $(-\infty, \infty)$ into four distinct intervals,

$$(-\infty, -3) \qquad (-3,0) \qquad (0,3) \qquad (3,\infty)$$

As a rule, if we find n points of discontinuity among $P(x)$ and/or $Q(x)$, we'll divide $(-\infty, \infty)$ into $n+1$ intervals. Of the $n+1$ intervals, the correct interval of validity will be the one that contains t_0 from the given initial condition. So if we were given the condition $y(2) = -3$ and the intervals above, because $t_0 = 2$ falls within $(0,3)$, the interval $(0,3)$ is the interval of validity for the condition $y(2) = -3$ and the associated particular solution.

Notice how we define the interval of validity based on the given initial condition. The solution curve through $y(2) = -3$ had an interval of validity $(0,3)$, but a solution curve through $y(-1) = 0$ would have the interval of validity $(-3,0)$.

Let's do another example where we work through a full initial value problem for a linear differential equation.

Example

Find the interval of validity for the solution to the differential equation, given $f(0) = 5/2$.

$$\frac{dy}{dx} = -5y + 3e^x$$

Put the equation in standard form of a linear equation.

$$\frac{dy}{dx} + 5y = 3e^x$$

From this new form, we can identify $P(x) = 5$ and $Q(x) = 3e^x$. Both of these functions are defined everywhere $(-\infty, \infty)$, so the interval of validity is $(-\infty, \infty)$.

If we wanted to also find the particular solution by solving the initial value problem, we'd find the integrating factor,

$$\rho(x) = e^{\int P(x)\ dx}$$

$$\rho(x) = e^{\int 5\ dx}$$

$$\rho(x) = e^{5x}$$

multiply through the linear equation,

$$\frac{dy}{dx}(e^{5x}) + 5y(e^{5x}) = 3e^x(e^{5x})$$

$$\frac{dy}{dx}e^{5x} + 5e^{5x}y = 3e^{6x}$$

$$\frac{d}{dx}(ye^{5x}) = 3e^{6x}$$

integrate both sides,

$$\int \frac{d}{dx}(ye^{5x})\ dx = \int 3e^{6x}\ dx$$

$$ye^{5x} = \frac{1}{2}e^{6x} + C$$

and then solve for y to find the general solution.

$$y = \frac{e^{6x} + 2C}{2e^{5x}}$$

Substituting the initial condition $f(0) = 5/2$ lets us solve for C,

$$\frac{5}{2} = \frac{e^{6(0)} + 2C}{2e^{5(0)}}$$

$$\frac{5}{2} = \frac{1 + 2C}{2(1)}$$

$$5 = 1 + 2C$$

$$4 = 2C$$

$$C = 2$$

which we can plug back into the general solution to find the particular solution.

$$y = \frac{e^{6x} + 2(2)}{2e^{5x}}$$

$$y = \frac{e^{6x} + 4}{2e^{5x}}$$

This example illustrated how we were able to determine the interval of validity for the linear equation, without first finding the solution to the differential equation. This won't be the case when the equation is non-linear. Given a non-linear differential equation, we have to find the solution first, then determine where (if anywhere) the solution is undefined. The value(s) at which the solution is undefined will dictate the interval of validity.

Let's do an example with a separable differential equation. Previously, in the lesson about separable equations, we found the general solution

$$y = \frac{1}{\cos x + C}$$

to the equation $y' = y^2 \sin x$. We'll start with this general solution.

Example

Given the general solution to $y' = y^2 \sin x$ and the initial condition $y(0) = 2$, find the particular solution and the interval of validity.

$$y = \frac{1}{\cos x + C}$$

To find the particular solution, we'll substitute $x = 0$ and $y = 2$,

$$2 = \frac{1}{\cos 0 + C}$$

$$2 = \frac{1}{1 + C}$$

$$2 + 2C = 1$$

$$2C = -1$$

$$C = -\frac{1}{2}$$

and then plug this back into the general solution.

$$y = \frac{1}{\cos x - \frac{1}{2}}$$

$$y = \frac{1}{\frac{-1 + 2\cos x}{2}}$$

$$y = \frac{2}{-1 + 2\cos x}$$

The solution is undefined where the denominator is 0.

$$-1 + 2\cos x = 0$$

$$2\cos x = 1$$

$$\cos x = \frac{1}{2}$$

$$x = \left\{\frac{\pi}{3} + 2\pi k, \quad k \in \mathbb{Z}\right\} \cup \left\{\frac{5\pi}{3} + 2\pi k, \quad k \in \mathbb{Z}\right\}$$

These values separate $(-\infty, \infty)$ into infinitely many intervals of validity.

$$\ldots\left(-\frac{7\pi}{3}, -\frac{5\pi}{3}\right) \quad \left(-\frac{5\pi}{3}, -\frac{\pi}{3}\right) \quad \left(-\frac{\pi}{3}, \frac{\pi}{3}\right) \quad \left(\frac{\pi}{3}, \frac{5\pi}{3}\right) \quad \left(\frac{5\pi}{3}, \frac{7\pi}{3}\right)\ldots$$

Because the initial condition is $y(0) = 2$, and the value $x_0 = 0$ is contained in $(-\pi/3,\pi/3)$, we can say that $(-\pi/3,\pi/3)$ is the interval of validity.

When the interval of validity depends on y_0

Intervals of validity for non-linear equations can depend on the value of y_0 from the initial condition $y(t_0) = y_0$, whereas intervals of validity for linear equations are only dependent on t_0.

Let's do an example to see why this is the case.

Example

Find the interval of validity for the solution to the differential equation, given $y(0) = y_0$.

$$y' = y^2$$

This is a separable differential equation, which means it's non-linear, and therefore we'll need to find the solution to the equation before we can find the interval of validity. We'll first find the general solution,

$$\frac{dy}{dx} = y^2$$

$$dy = y^2 \ dx$$

$$\frac{1}{y^2}\ dy = dx$$

$$\int \frac{1}{y^2}\ dy = \int dx$$

$$-\frac{1}{y} = x + C$$

$$-1 = (x + C)y$$

$$y = -\frac{1}{x + C}$$

and then we'll substitute the initial condition $y(0) = y_0$ to find a value for C.

$$y_0 = -\frac{1}{0 + C}$$

$$Cy_0 = -1$$

$$C = -\frac{1}{y_0}$$

Now we can plug this value back into the general solution, which will give us the equation's particular solution.

$$y = -\frac{1}{x - \frac{1}{y_0}}$$

$$y = -\frac{1}{\frac{xy_0}{y_0} - \frac{1}{y_0}}$$

$$y = -\frac{1}{\frac{xy_0 - 1}{y_0}}$$

$$y = -\frac{y_0}{xy_0 - 1}$$

$$y = \frac{y_0}{1 - xy_0}$$

This particular solution is undefined where the denominator is 0.

$$1 - xy_0 = 0$$

$$1 = xy_0$$

$$x = \frac{1}{y_0}$$

Because the solution is undefined at this value, it divides $(-\infty, \infty)$ into two potential intervals of validity.

$$-\infty < x < \frac{1}{y_0} \qquad \frac{1}{y_0} < x < \infty$$

Since we're using the initial condition $y(0) = y_0$, the interval of validity must contain $x = 0$. The first interval will only contain $x = 0$ when

$$\frac{1}{y_0} > 0$$

$$y_0 > 0$$

The second interval will only contain $x = 0$ when

$$\frac{1}{y_0} < 0$$

$$y_0 < 0$$

So we've found constraints for $y_0 < 0$ and $y_0 > 0$, but what about $y_0 = 0$? If $y_0 = 0$, then

$$y = \frac{y_0}{1 - xy_0} = \frac{0}{1 - x(0)} = 0$$

so $y = 0$ for all x, and the interval of validity is $(-\infty, \infty)$. So we can summarize our findings as

For $y_0 < 0$,	the interval of validity is	$\frac{1}{y_0} < x < \infty$
For $y_0 = 0$,	the interval of validity is	$(-\infty, \infty)$
For $y_0 > 0$,	the interval of validity is	$-\infty < x < \frac{1}{y_0}$

So we see that, even for a single initial condition, $y(0) = y_0$, the interval of validity will depend on the value of y_0. For instance, given the initial condition $y(0) = 1$, the interval of validity would be $(-\infty,1)$. But given the initial condition $y(0) = -3$, the interval of validity would be $(-1/3,\infty)$.

In other words, we're finding different intervals of validity for the same value $x = 0$ in the initial condition.

Euler’s method

Now that we know that a direction field is a geometric view of a differential equation, and the solution curve through the direction field is the geometric view of its solution, we understand better what we’re actually doing when we use an analytical method to find the solution to a differential equation.

Essentially, the general solution gives an equation that models all curves that can be sketched through the direction field, at least on a particular interval of validity. And when we solve an initial value problem, we solve for the single solution curve that passes through one specific point.

Numerical methods

But now we want to look at a third method for finding the solution to the differential equation. First, we looked at analytical methods for solving some special types of first and second order differential equations. Then we looked at the geometric method of sketching the direction field and using it as a guide to sketch solution curves. Now we want to look at using a numerical method to approximate the solution.

Numerical methods are particularly useful when we aren’t able to use an analytic approach to find the equation of the solution. In fact, in the real world, most differential equations are solved with numerical methods, because most differential equations don’t follow the perfectly near format of a linear equation, homogeneous equation, exact equation, etc.

The numerical method we'll look at here is called Euler's method, and as long as we have the differential equation and one initial condition, we can use it to approximate the solution.

Euler's method is an iterative process in which we repeat the same step over and over. The more steps we use, the better approximation we'll get. We choose a **step size** Δt, which is the distance between two successive values of t at which y is calculated.

Then we use the initial condition to identify (t_0, y_0), and we plug (t_0, y_0) and Δt into Euler's formula,

$$y_1 = y_0 + [f(t_0, y_0)]\Delta t$$

in order to calculate y_1. The value of t_1 is $t_1 = t_0 + \Delta t$. Then we can use (t_1, y_1) and Δt to calculate y_2, use (t_2, y_2) and Δt to calculate y_3, etc. We continue this calculation again and again, until we arrive at the value of t at which we wanted to approximate the value of y.

When we use Euler's method to approximate the solution at a particular point, it's helpful to build a table to keep all of our calculations organized.

t	y	**Result**
t_0	y_0	y_0
t_1	$y_1 = y_0 + [f(t_0, y_0)]\Delta t$	y_1
t_2	$y_2 = y_1 + [f(t_1, y_1)]\Delta t$	y_2
...	...	...

t_n $\quad\quad$ $y_n = y_{n-1} + [f(t_{n-1}, y_{n-1})]\Delta t$ $\quad\quad$ y_n

With all that in mind, let's work through an example so that we can see this method in action.

Example

Use Euler's method and two steps to find $y(1)$, given $y(0) = 2$.

$$y' = y - t$$

We're starting with the initial condition $y(0) = 2$ and attempting to approximate $y(1)$, which means we're starting at $t_0 = 0$ and ending at $t = 1$. To find step size, we divide the difference between these by the number of steps we were asked to use, in our case, by 2.

$$\frac{1-0}{2} = \frac{1}{2} = 0.5 = \Delta t$$

Now we can start building our table. The values of t_0 and y_0 both come from our initial condition, $y(0) = 2$. To find t_1 and t_2, we add Δt to the previous value for t. We do that until we reach $t_n = 1$, which is the value the problem originally asked for when it asked us to find $y(1)$.

$t_0 = 0$ $\quad\quad$ $y_0 = 2$ $\quad\quad$ $y_0 = 2$

$t_1 = 0 + 0.5 = 0.5$

$t_2 = 0.5 + 0.5 = 1$

Now we'll use Euler's formula with (t_0, y_0) and Δt to find y_1.

$t_0 = 0$	$y_0 = 2$	$y_0 = 2$
$t_1 = 0.5$	$y_1 = 2 + (2 - 0)(0.5)$	$y_1 = 3$
$t_2 = 1$		

Finally, we'll use Euler's formula with (t_1, y_1) to find y_2, which is the value corresponding to $t_2 = 1$, which is the value we were asked to approximate.

$t_0 = 0$	$y_0 = 2$	$y_0 = 2$
$t_1 = 0.5$	$y_1 = 2 + (2 - 0)(0.5)$	$y_1 = 3$
$t_2 = 1$	$y_2 = 3 + (3 - 0.5)(0.5)$	$y_2 = 4.25$

So Euler's method and $y(0) = 2$ with two steps approximated $y(1)$ as

$$y(1) \approx 4.25$$

We'll look at one more example of a problem worded slightly differently. If we're working with decimal numbers, one thing we want to remember with Euler's method approximations is to keep as many decimal places as possible as we work through the approximation.

If we start rounding values from earlier steps, our approximation will get less and less accurate as we go. So it's best to keep more decimal places as we work through each step, and then only round the final answer.

Example

Use Euler's Method with step size 0.2 to approximate the values y_1, y_2, y_3, y_4, and y_5, when $y(0) = 5$.

$$y' = y - 3t + t^2$$

Since we're given step-size directly, we already know that

$$\Delta t = 0.2$$

To start building our table, we identify $t_0 = 0$ and $y_0 = 5$ from our initial condition, $y(0) = 5$. Since we need to find y_5, we'll start adding $\Delta t = 0.2$ to t_0 until we reach t_5.

$t_0 = 0$ $\quad y_0 = 5$ $\quad y_0 = 5$

$t_1 = 0.2$

$t_2 = 0.4$

$t_3 = 0.6$

$t_4 = 0.8$

$t_5 = 1$

Now that we've built the outline for our table, we'll use Euler's formula to find y_1, y_2, y_3, y_4, and y_5.

$t_0 = 0$	$y_0 = 5$	$y_0 = 5$
$t_1 = 0.2$	$y_1 = 5 + (5 - 3(0) + 0^2)(0.2)$	$y_1 = 6$
$t_2 = 0.4$	$y_2 = 6 + (6 - 3(0.2) + 0.2^2)(0.2)$	$y_2 = 7.088$
$t_3 = 0.6$	$y_3 = 7.088 + (7.088 - 3(0.4) + 0.4^2)(0.2)$	$y_3 = 8.2976$
$t_4 = 0.8$	$y_4 = 8.2976 + (8.2976 - 3(0.6) + 0.6^2)(0.2)$	$y_4 = 9.66912$
$t_5 = 1$	$y_5 = 9.66912 + (9.66912 - 3(0.8) + 0.8^2)(0.2)$	$y_5 = 11.250944$

After filling out the table, we can say that the value of $y(1)$ is approximately

$$y(1) \approx 11.250944$$

Autonomous equations and equilibrium solutions

So far we've been focusing entirely on first order ordinary differential equations. We've looked at linear equations and nonlinear equations, but now we want to introduce autonomous equations as third type of ordinary differential equation.

Autonomous equations

Autonomous equations are differential equations in the form

$$\frac{dy}{dt} = f(y)$$

The defining characteristic of autonomous equations is that the independent variable never appears explicitly in the equation, but instead only as part of the derivative. In other words, in the autonomous equation above, the independent variable t only appears in dy/dt, not as part of $f(y)$, because f is a function only in y.

Technically, autonomous equations are separable equations that can be rewritten as

$$\frac{1}{f(y)}\ dy = dt$$

But our goal really won't be to solve them as separable equations. Instead, we're going to focus on a different kind of analysis. Autonomous equations are especially common in modeling real-world applications, so how we work with these equations will really be informed by the applications. We'll look at some of these applications later, but for now, let's turn our attention toward the direction fields of autonomous equations.

Direction fields

Looking again at the general form of the autonomous equation,

$$\frac{dy}{dt} = f(y)$$

we can see that, for any value y, we'll find one slope dy/dt. Because if we choose some value of y, let's call it y_0, we'll plug that into the right side of the autonomous equation to find $f(y_0)$, which will be some constant value, let's call it k. In other words,

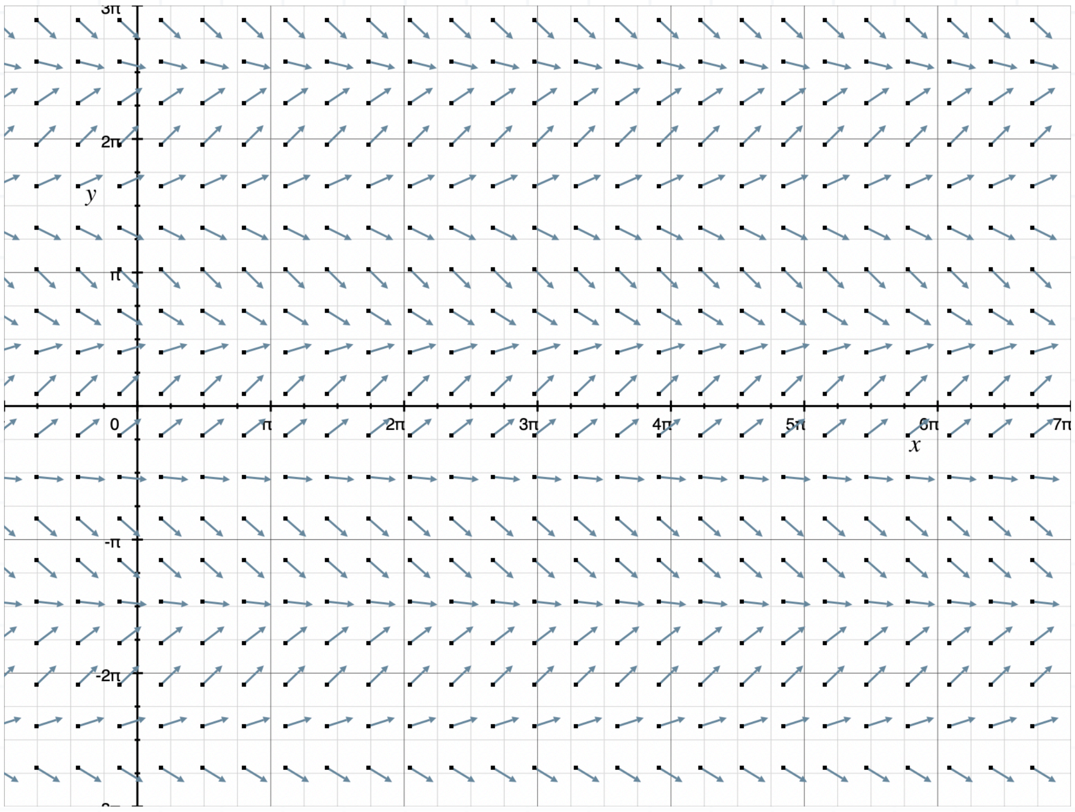
$$f(y_0) = k$$

for some specific y_0. But if $f(y_0) = k$, then $dy/dt = k$. Therefore, for any particular y_0, the slope dy/dt will be constant.

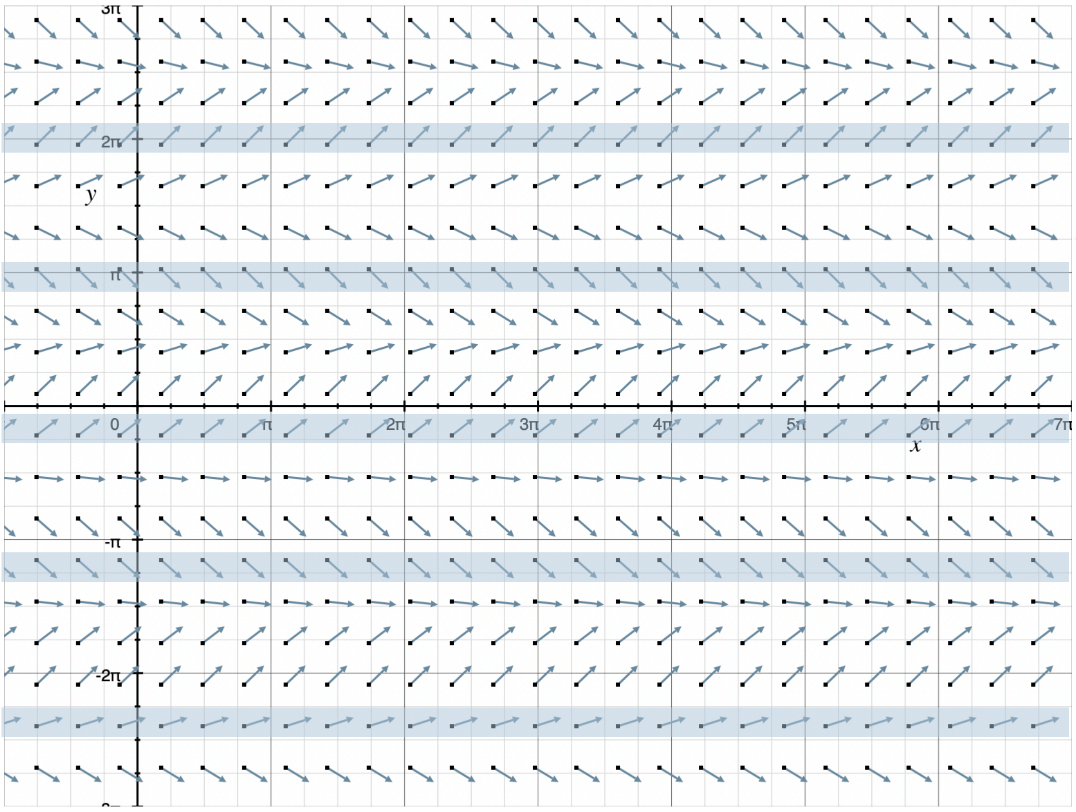
For example, consider the autonomous equation

$$\frac{dy}{dt} = \cos y$$

and its direction field



Notice how every horizontal row contains a set of entirely parallel direction arrows. This will always be the case for the direction fields of autonomous equations.



Equilibrium solutions and stability

We said earlier that we could pick some specific y_0, plug it into f from the right side of the autonomous differential equation, and come out with some constant value k, such that $f(y_0) = k$. If $k = 0$, it means that the function $f(y)$ has a zero at y_0.

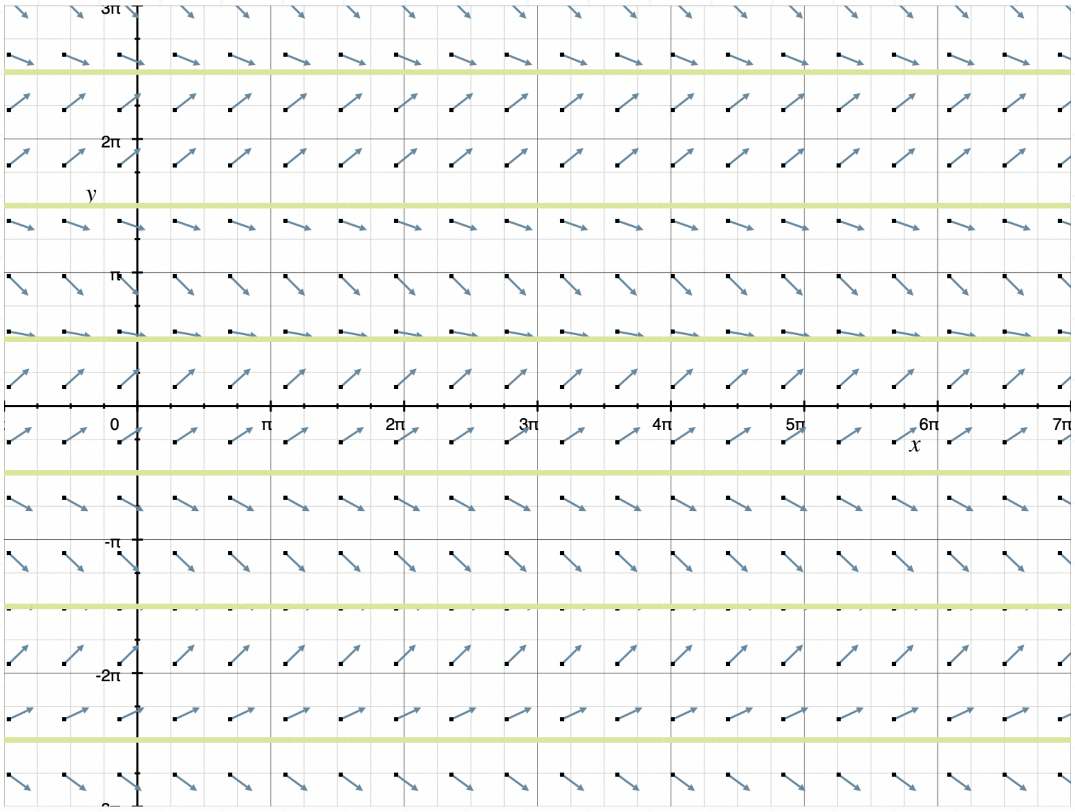
We say that these zeros are the **critical points**, equilibrium points, or stationary points of the autonomous differential equation. In other words, a real number y_0 is a critical point of the autonomous equation if $f(y_0) = 0$.

Of course, when we get a zero value for the right side of the autonomous differential equation, we find the equation $dy/dt = 0$. This tells us that the slope dy/dt is 0, and when we have a zero slope, we have a perfectly horizontal line.

Bringing back the autonomous equation,

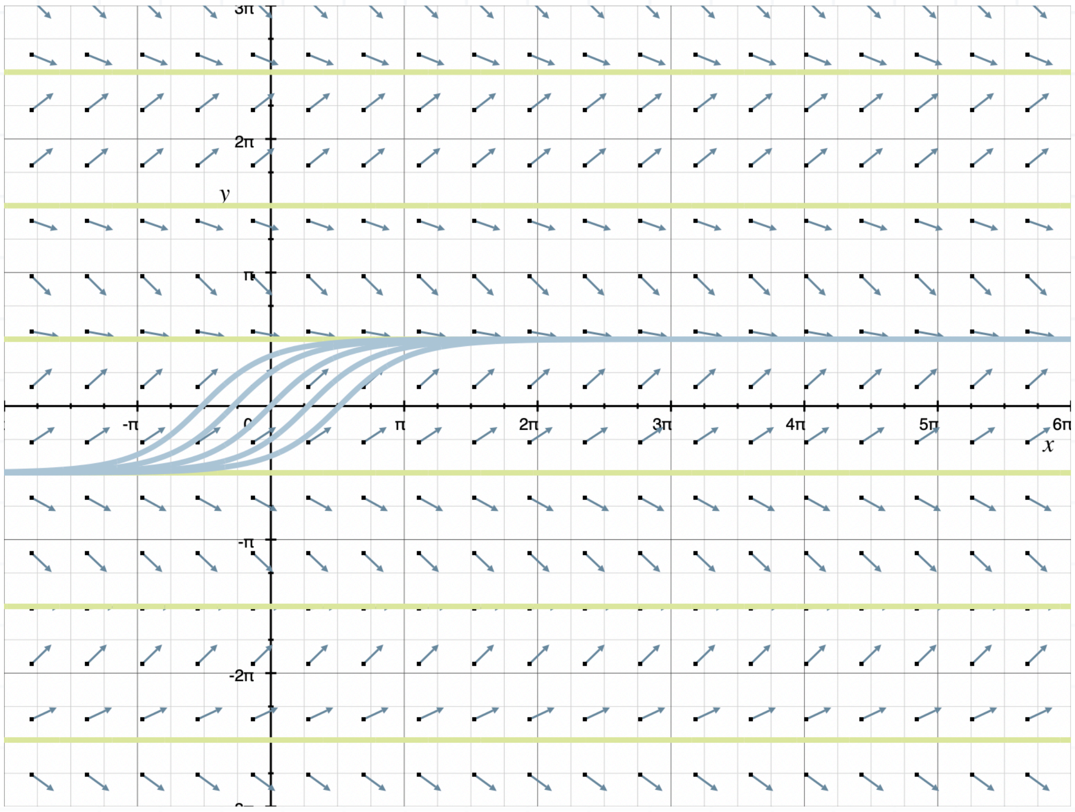
$$\frac{dy}{dt} = \cos y$$

we know $\cos y = 0$ when $y = (\pi/2) + \pi k$, where k is any integer. So in the direction field, we'll sketch in the **critical solutions**, or equilibrium solutions, which are the perfectly horizontal lines representing the solution curves that correspond to each critical point. Adding $y = \pm\pi/2, \ \pm 3\pi/2$, and $\pm 5\pi/2$ to the direction field gives



These equilibrium solutions are lines that will never be crossed by the curves of other solutions. Which means that, whenever we get close to an equilibrium solution, the slopes of the direction arrows level out.

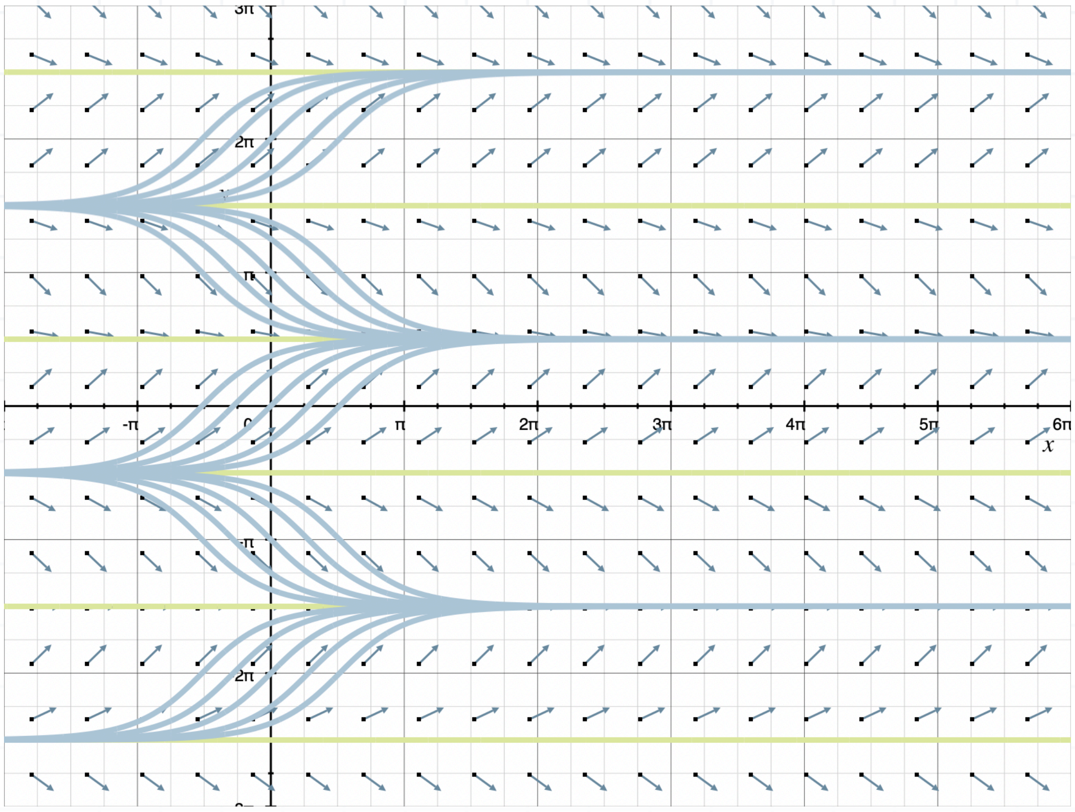
To get a little more detailed, let's drill down to the part of the direction field that sits between $y = -\pi/2$ and $y = \pi/2$. If we sketch in some solution curves between those values, specifically the solutions through the initial conditions $y(0) = 0$, $y(0) = \pm \pi/4$, and $y(0) = \pm 3\pi/8$, we find these curves:



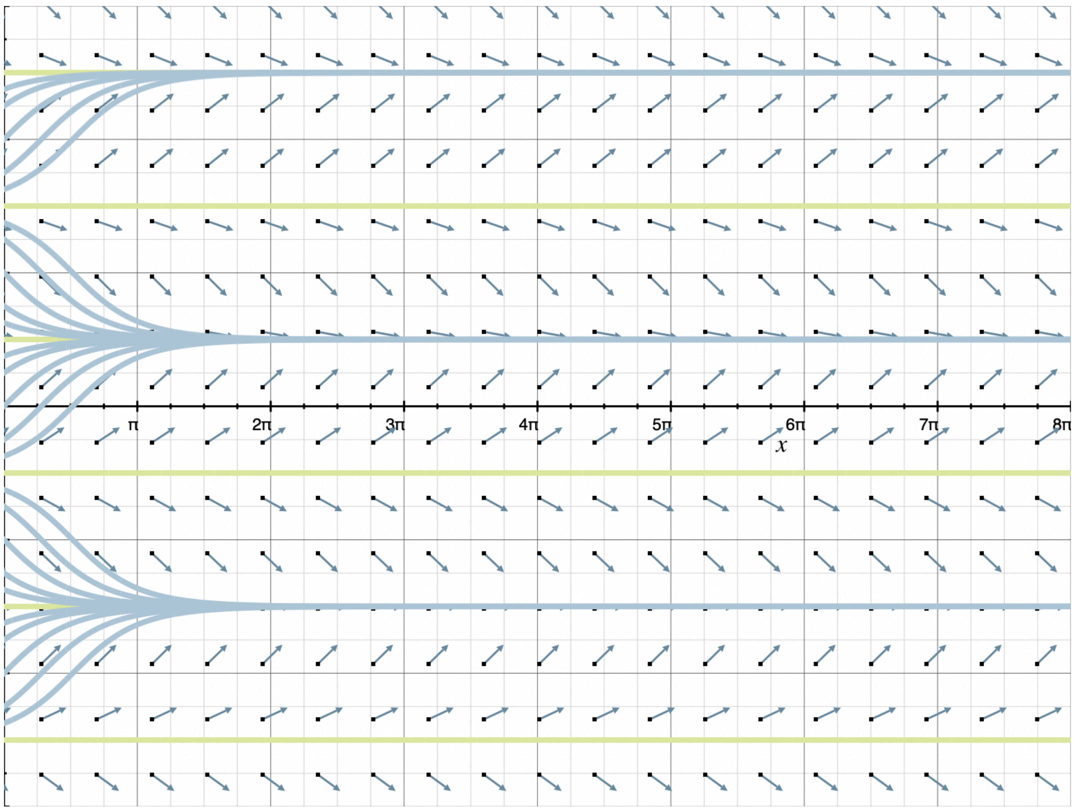
Notice how they all refuse to cross either of the equilibrium solutions at $y = \pm \pi/2$. What we can conclude is that the graph of any solution curve through some y_0 where $-\pi/2 < y_0 < \pi/2$, will remain inside the boundaries of $y = \pm \pi/2$.

Furthermore, because of the fact we mentioned earlier that all the direction arrows along any horizontal line are parallel, all the solution curves within any region between two equilibrium solutions will be translations of each other. When we look at the five solution curves we sketched between $y = \pm \pi/2$, we notice that they all have the same shape, and they're just shifted left and right.

We find similar sets of shifted solution curves between the other equilibrium solutions.



Now that we have a picture of several solution curves, we want to use them to classify the nature of each equilibrium solution. To do that, we should focus just on what's happening to the right of the vertical axis.



What we see is that the solution curves are moving toward $y = -3\pi/2$, toward $y = \pi/2$, and toward $y = 5\pi/2$, while they're moving away from $y = -5\pi/2$, away from $y = -\pi/2$, and away from $y = 3\pi/2$.

Equilibrium solutions that attract solution curves are **stable equilibrium solutions**, or attractors, and equilibrium solutions that repel solution curves are **unstable equilibrium solutions**, or repellers.

Equilibrium solutions that attract solution curves on one side and repel them on the other are **semi-stable equilibrium solutions**. These solutions are usually more rare than stable and unstable solutions, and we don't have any of these in the direction field we've sketched for $dy/dt = \cos y$.

Let's do an example where we walk through how to find and classify the equilibrium solutions of an autonomous differential equation.

Example

Find any equilibrium solutions of the autonomous differential equation, then determine whether each solution is stable, unstable, or semi-stable.

$$\frac{dy}{dt} = y^2 + 3y + 2$$

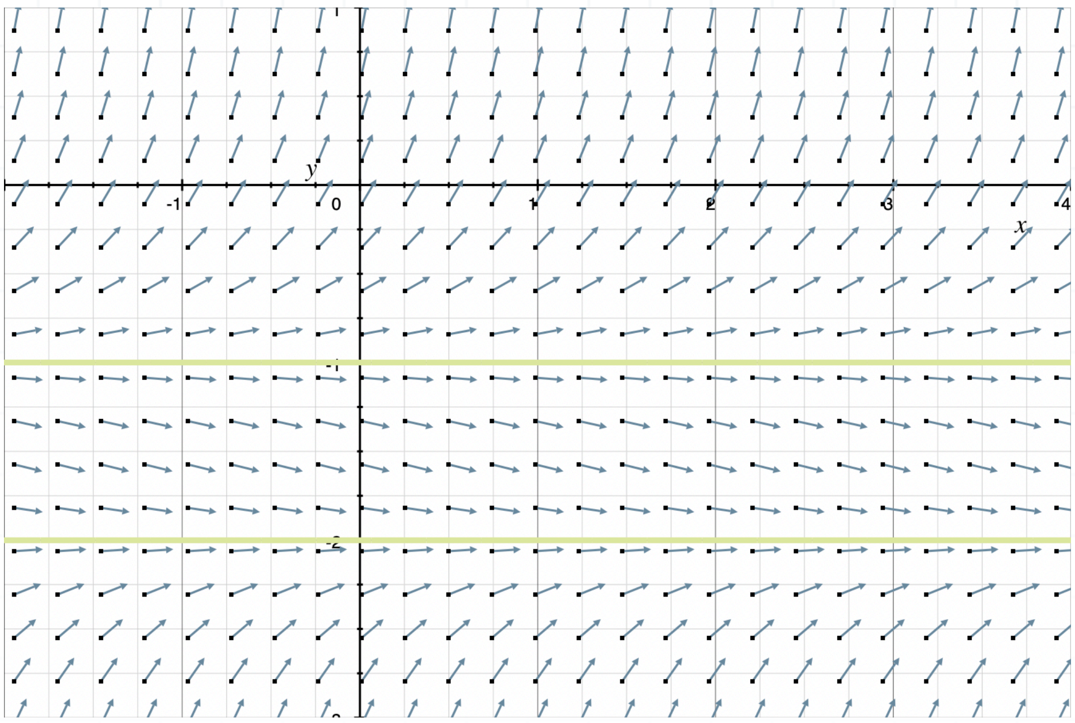
The autonomous differential equation has equilibrium solutions at

$$y^2 + 3y + 2 = 0$$

$$(y + 2)(y + 1) = 0$$

$$y = -1, \ \ -2$$

If we sketch the direction field and then overlay the equilibrium solutions, we get



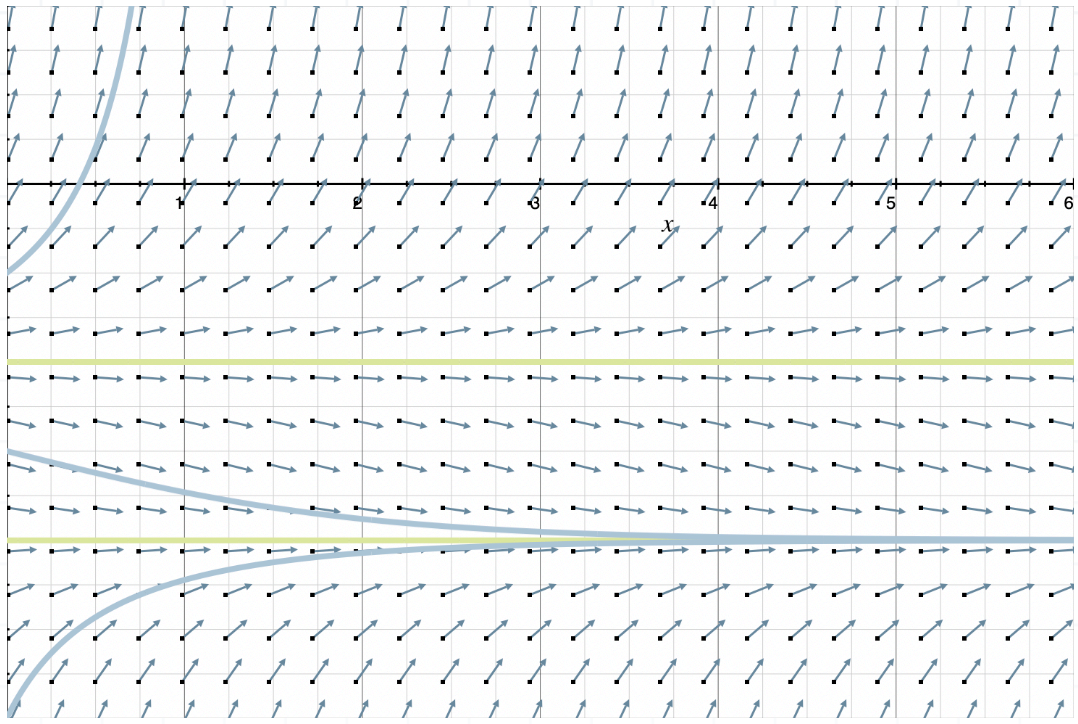
These two equilibrium solutions divide the vertical axis into three intervals:

$$y < -2$$

$$-2 < y < -1$$

$$-1 < y$$

We can use the direction field to sketch a solution curve inside each of these intervals.



These solution curves approach $y = -2$ on both sides, but move away from $y = -1$ on both sides. Therefore, $y = -2$ is a stable equilibrium solution, while $y = -1$ is an unstable equilibrium solution.

Keep in mind that we can also classify equilibrium solutions without sketching a direction field, by using the function $f(y)$ to determine the direction of the solution curves. For instance, to continue with the last example, once we've found the equilibrium solutions $y = -1, \ -2$, we'll pick a test-value in each of the three intervals they create, $y < -2$, $-2 < y < -1$, and $-1 < y$. We'll use test-values $y = -3$, $y = -5/4$, and $y = 0$.

We'll substitute our test values into $f(y) = y^2 + 3y + 2$.

$$f(-3) = (-3)^2 + 3(-3) + 2 = 9 - 9 + 2 = 2 > 0$$

$$f\left(-\frac{5}{4}\right) = \left(-\frac{5}{4}\right)^2 + 3\left(-\frac{5}{4}\right) + 2 = \frac{25}{16} - \frac{15}{4} + 2 = -\frac{3}{16} < 0$$

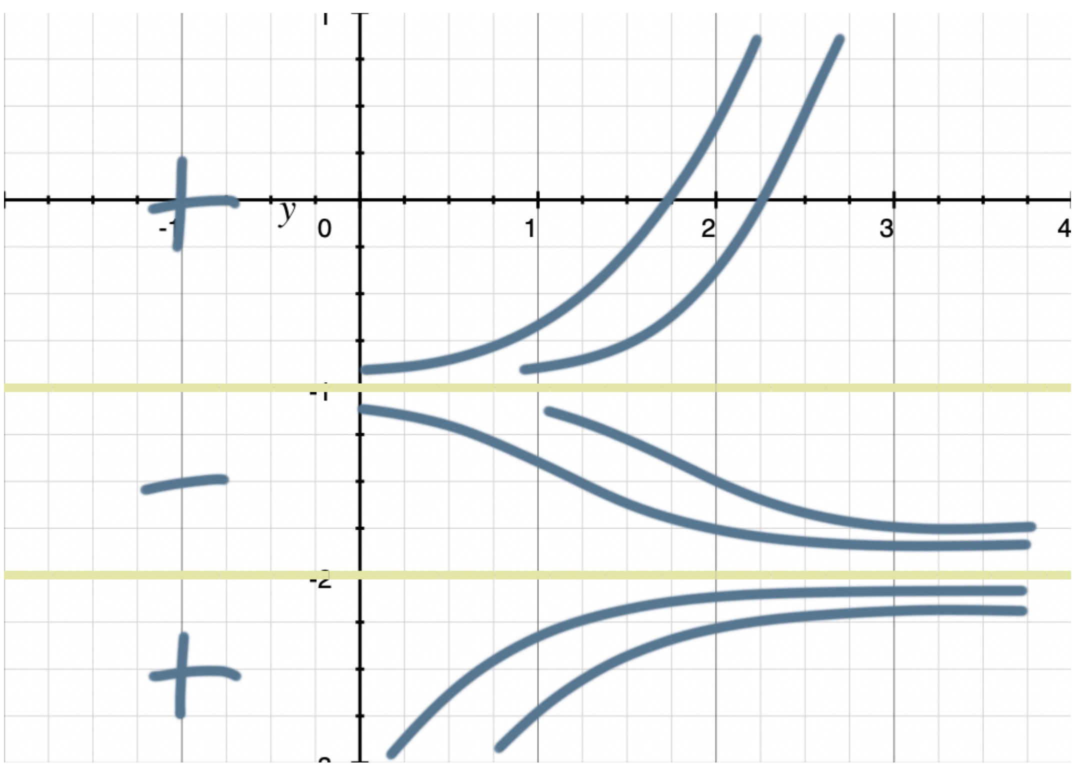
$$f(0) = 0^2 + 3(0) + 2 = 2 > 0$$

We find a positive value when $y = -3$, a negative value when $y = -5/4$, and a positive value when $y = 0$. A positive outcome from this test tells us that the solution curves are increasing, or rising in the interval, while a negative outcome from this test tells us that the solution curves in that interval are decreasing, or falling.

Let's summarize the results for this example in a table,

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
$(-1,\infty)$	$+$	Increasing/Rising
$(-2,-1)$	$-$	Decreasing/Falling
$(-\infty,-2)$	$+$	Increasing/Rising

and then plot these signs in the plane, without the direction field.



With the signs plotted in between the equilibrium solutions, we can quickly see that the arrows are pointing toward and the solution curves will be approaching $y=-2$, while the arrows are pointing away from and the solution curves will be moving away from $y=-1$.

The logistic equation

Now that we understand how to find and classify solutions for an autonomous differential equation, let's look at a specific application of autonomous equations.

Autonomous equations are often used to model population growth over time, so let's start with a short review of simple exponential population growth.

Exponential population growth

A population that grows exponentially over time can be modeled by

$$P(t) = P_0e^{kt}$$

where $P(t)$ is the population after time t, P_0 is the original population when time $t = 0$, and k is the growth constant.

Exponential growth is usually observed in smaller populations that aren't yet limited by their environment or the resources around them. In other words, this exponential equation is most useful for modeling a small population where growth is unconstrained.

There are four unknowns in this exponential equation, P, P_0, k, and t, so we'll need to know the values of three of these in order to find the value of the fourth. Let's try an example where we solve for a specific time t.

Example

A bacteria population increases tenfold in 8 hours. Assuming exponential growth, how long did it take for the population to double?

We've been told that the population grows to 10 times its original size. If we call its original size P_0, then we can say that it grows to $10P_0$ after 8 hours. Substituting these three values into the exponential equation gives

$$10P_0 = P_0e^{k(8)}$$

$$\frac{10P_0}{P_0} = e^{k(8)}$$

$$10 = e^{8k}$$

To solve for k, we'll apply the natural log to both sides of the equation, cancelling the $\ln e$, and then rearrange.

$$\ln 10 = \ln e^{8k}$$

$$\ln 10 = 8k$$

$$k = \frac{\ln 10}{8}$$

Now that we have a value for the growth constant k, we can can figure out how long it took for the population to double. If we say that P_0 is the original population, and $2P_0$ is double the original population, then the exponential equation gives

$$2P_0 = P_0e^{\frac{\ln 10}{8}t}$$

$$\frac{2P_0}{P_0} = e^{\frac{\ln 10}{8}t}$$

$$2 = e^{\frac{\ln 10}{8}t}$$

Apply the natural log to both sides of the equation, cancelling the $\ln e$, and then rearrange.

$$\ln 2 = \ln e^{\frac{\ln 10}{8}t}$$

$$\ln 2 = \frac{\ln 10}{8}t$$

$$t = \frac{8 \ln 2}{\ln 10}$$

$$t \approx 2.41$$

This result tells us that the bacteria doubled the size of its original population in about 2.41 hours, or about 2 hours and 25 minutes.

The logistic growth equation

The exponential equation $P(t) = P_0e^{kt}$ is somewhat useful for modeling population, but its usefulness is limited. That's because populations don't tend to grow exponentially forever without bound.

Instead, populations are more likely to grow slowly at first when the population is small enough that replication is slow. As the population grows, the replication rate increases, and it seems like growth is increasing exponentially. But eventually, the population becomes large enough that the environment starts to constrain its growth, as it runs out of resources like space, energy, food, water, etc.

This slow-fast-slow growth pattern is modeled more effectively by the **logistic growth equation**

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$$

than it is by the exponential growth equation. In this logistic equation, dP/dt is the rate of growth of the population P over time t, k is the same growth constant that we had in the exponential growth equation, and M is the **carrying capacity** or saturation level, which is the largest population that can be sustained by the surrounding environment.

What we should notice is that this logistic growth equation is an autonomous first order differential equation, because we can see that there's no independent variable t on the right side of the logistic equation. The independent variable t only appears as part of the derivative dP/dt. So the right side of the logistic equation is a function in P only.

Which means we could name $f(P) = kP(1 - P/M)$, and then rewrite this logistic equation as

$$\frac{dP}{dt} = f(P)$$

As we saw before, we can find zeros by setting $f(P) = 0$.

$$kP\left(1 - \frac{P}{M}\right) = 0$$

This equation will be true when

$$kP = 0$$

$$P = 0$$

or

$$1 - \frac{P}{M} = 0$$

$$\frac{P}{M} = 1$$

$$P = M$$

So the autonomous equation $dP/dt = f(P)$ will have equilibrium solutions when the population is 0, and when the population is at its carrying capacity. And this will always be the case for the logistic growth equation: the population will be stable when $P = 0$ and when $P = M$.

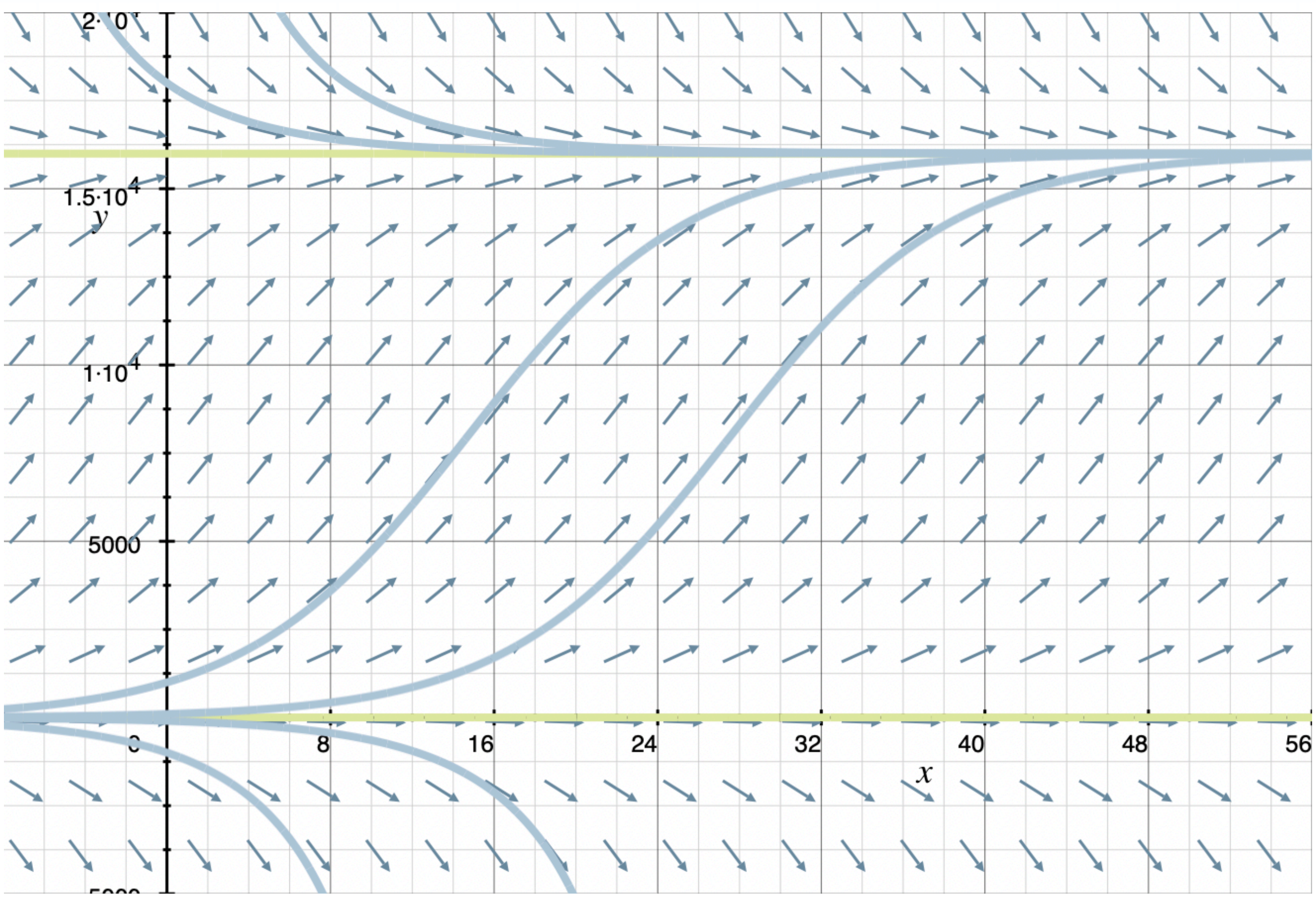
This should make sense. After all, the population won't increase or decrease if $P = 0$, because there are no members of the population to reproduce or die off. So a population at $P = 0$ will remain at $P = 0$ forever.

Similarly, the population won't increase or decrease at $P = M$, because this is the point at which the population is perfectly in balance with its environment. There aren't enough resources to support a larger population, so the population will not increase, but there are enough resources to support this exact population, so it also won't decrease either.

Between $P = 0$ and $P = M$, the population is always increasing. As we mentioned earlier, the growth between $P = 0$ and $P = M$ follows a slow-fast-slow pattern (slow near $P = 0$, then faster, then slow again as we approach $P = M$).

If a population is ever greater than its carrying capacity, then growth will be negative until the population shrinks back to the level of carrying capacity. And while this doesn't apply in a real-world population model, the direction field of a logistic equation also shows that solution curves below $P = 0$ move away from the equilibrium solution.

Below is an example of the direction field for a logistic equation in which the carrying capacity is $M = 16{,}000$. It shows the equilibrium solutions at $P = 0$ and $P = M = 16{,}000$, as well as a few solution curves in each of the three intervals ($P < 0$, $0 < P < M$, and $M < P$).



From the direction field, we can clearly see that $P = 0$ will always be an unstable equilibrium solution (because solution curves move away from it), while $P = M$ will always be a stable solution of the logistic equation (because solution curves move toward it).

Let's do an example with a population whose growth we'll model with the logistic equation.

Example

Describe the growth pattern of a population of ducks, assuming their population is modeled by the logistic growth equation, if the carrying capacity of their habitat is $2{,}500$ ducks.

Between $P = 0$ and $P = M = 2{,}500$, the duck population always increasing. In other words, in the interval $0 < P < 2{,}500$, the duck population will grow slowly when P is close to 0, then growth will speed up, and then the growth rate will level off again as P gets close to $2{,}500$.

If the population starts at a size of exactly 0, of course it will neither increase nor decrease, as indicated by the equilibrium solution at $P = 0$.

If the population starts at a size of exactly $2{,}500$, then it will neither increase nor decrease, as indicated by the equilibrium solution at $P = 2{,}500$. And if the duck population starts at a size greater than $2{,}500$, it will die off until it reaches stability at a size of $2{,}500$, as indicated by the stable equilibrium solution at $P = 2{,}500$.

Predator-prey systems

We've seen how differential equations can be used to model the size of a population over time. Of course, populations don't tend to exist in isolation; they usually interact with other populations, either as predator or prey.

Populations as a system

For that reason, it's common to consider two populations as a system, and model their interaction with a system of differential equations. In an environment with two populations x and y, where dx/dt models population x over time t and dy/dt models population y over time t, we can always classify the system as cooperative, competitive, or predator-prey.

- In **cooperative** systems, both populations are increasing in size (the mixed xy terms are both positive).

$$\frac{dx}{dt} = ax + bxy \qquad \frac{dy}{dt} = cy + dxy$$

- In **competitive** systems, both populations are decreasing in size (the mixed xy terms are both negative).

$$\frac{dx}{dt} = ax - bxy \qquad \frac{dy}{dt} = cy - dxy$$

- In **predator-prey** systems, the size of one population is increasing, while the size of the other population is decreasing (one mixed xy term is positive while the other is negative).

Population x is increasing while population y is decreasing

$$\frac{dx}{dt} = ax + bxy \qquad \frac{dy}{dt} = cy - dxy$$

or

Population y is increasing while population x is decreasing

$$\frac{dx}{dt} = ax - bxy \qquad \frac{dy}{dt} = cy + dxy$$

Sometimes one or both differential equations in the system will contain higher degree terms, like the bx^2 term in the dx/dt equation in the following system.

$$\frac{dx}{dt} = ax + bx^2 + cxy$$

$$\frac{dy}{dt} = fy - gxy$$

When the differential equations modeling the populations don't include a higher degree term, it means both populations are only affected by one another. But when a higher degree term is included, like the bx^2 above, it means that population is affected by not only the other population in the system, but also by another factor, like carrying capacity.

Stable solutions of the system

The system of differential equations modeling two interacting populations will have predictable equilibrium solutions.

1. The system will have a stable equilibrium solution $(0,0)$, where both populations x and y are at 0 and will remain at 0 indefinitely.

2. If the system has an equilibrium solution at $(a,0)$, it means population x is stable at size a, while population y is at 0.

3. If the system has an equilibrium solution at $(0,b)$, it means population y is stable at size b, while population x is at 0.

4. The most interesting solution is an equilibrium solution at (a,b), where population x has size a, which is supporting in perfect balance population y at size b, and vice versa.

Let's do an example where we solve a population system for its stable and unstable solutions.

Example

Does the system of lions and zebras represent a cooperative, competitive, or predator-prey system? What are the equilibrium solutions of the system and what do they tell us about the balance of the populations?

$$\frac{dL}{dt} = -0.5L + 0.0001LZ$$

$$\frac{dZ}{dt} = 2Z - 0.0002Z^2 - 0.01LZ$$

The mixed LZ term $0.0001LZ$ in the differential equation for lions dL/dt is positive, while the mixed LZ term $-0.01LZ$ in the differential equation for zebras dZ/dt is negative, which means this is a predator-prey system in which the lions are the predators and the zebras are the prey.

To find equilibrium solutions, we'll factor both equations to get

$$\frac{dL}{dt} = -0.5L + 0.0001LZ$$

$$\frac{dL}{dt} = 0.0001L(-5{,}000 + Z)$$

$$\frac{dL}{dt} = 0.0001L(Z - 5{,}000)$$

and

$$\frac{dZ}{dt} = 2Z - 0.0002Z^2 - 0.01LZ$$

$$\frac{dZ}{dt} = 0.0002Z(10{,}000 - Z - 50L)$$

Setting both equations equal to 0 gives

$$0.0001L(Z - 5{,}000) = 0$$

$0.0001L = 0$ and $Z - 5{,}000 = 0$

$L = 0$ and $Z = 5{,}000$

and

$$0.0002Z(10{,}000 - Z - 50L) = 0$$

$0.0002Z = 0$ and $10{,}000 - Z - 50L = 0$

$$Z = 0 \text{ and } Z + 50L = 10{,}000$$

We need to test both solutions from the first equation with both solutions from the second equation. So if we pair $L = 0$ with both $Z = 0$ and $Z + 50L = 10{,}000$, and then in addition we pair $Z = 5{,}000$ with both $Z = 0$ and $Z + 50L = 10{,}000$, we get four combinations,

$$L = 0 \text{ and } Z = 0$$

$$L = 0 \text{ and } Z + 50L = 10{,}000$$

$$Z = 5{,}000 \text{ and } Z = 0$$

$$Z = 5{,}000 \text{ and } Z + 50L = 10{,}000$$

We're looking for each pair to generate an equilibrium solution (L, Z). The third pair, $Z = 5{,}000$ and $Z = 0$, doesn't include a value for L, so we can eliminate that pair completely and focus on just the other three.

$$L = 0 \text{ and } Z = 0$$

$$L = 0 \text{ and } Z + 50L = 10{,}000$$

$$Z = 5{,}000 \text{ and } Z + 50L = 10{,}000$$

The first pair gives the trivial solution $(L, Z) = (0,0)$ where both populations are at 0. To solve the second equation pair, we'll substitute $L = 0$ into $Z + 50L = 10{,}000$,

$$Z + 50(0) = 10{,}000$$

$$Z = 10{,}000$$

and to solve the third equation pair, we'll substitute $Z = 5{,}000$ into $Z + 50L = 10{,}000$,

$$5{,}000 + 50L = 10{,}000$$

$$50L = 5{,}000$$

$$L = 100$$

The solutions of the system are therefore

- $(L, Z) = (0,\ 0)$, where both populations are 0 and will remain there
- $(L, Z) = (0,\ 10{,}000)$, where the zebra population is stable at $10{,}000$ while the lion population is at 0.
- $(L, Z) = (100,\ 5{,}000)$, where the system is balanced, with 100 lions supporting in balance $5{,}000$ zebras, and vice versa.

Exponential growth and decay

When we talked earlier about using the logistic equation to model population growth, we mentioned briefly that we sometimes use the exponential equation

$$P(t) = P_0 e^{kt}$$

to model a population that grows exponentially. This equation is usually only used to model populations that are very small, since the exponential nature of the growth means the population needs to be fairly unrestricted by its environment.

What we want to say now is that this equation is actually the solution to the initial value problem

$$\frac{dP}{dt} = kP, \qquad P(t_0) = P_0$$

When we use this exponential growth and decay equation to model population, we'll often use the variables P and t, where the derivative dP/dt models the rate at which the population P changes over time t.

The general growth and decay equation

But we can also write this differential equation more generally as

$$\frac{dx}{dt} = kx, \qquad x(t_0) = x_0$$

and in this general form, we'll use this initial value problem all the time to model real-world phenomena like bacterial growth, compound interest, and half life of a decaying substance.

We call k the **constant of proportionality**. If $k > 0$ we call it a growth constant, whereas if $k < 0$ we call it a decay constant. So for applications like population growth and compound interest, k will be positive, but for decay applications like half life, k will be negative.

This growth/decay differential equation is so useful because it tells us that the rate of growth is proportional to the population/amount/etc. In other words, the larger the population, the faster is replicates, or the more money, the faster it grows. And the equation is a separable equation,

$$\frac{dx}{dt} = kx$$

$$\frac{1}{x}\ dx = k\ dt$$

which means that we already know how to find the solution. Remember that separable equations are linear equations, so we could solve this as a first order linear equation by finding the integrating factor, or we can solve it this way as a separable equation:

$$\int \frac{1}{x}\ dx = \int k\ dt$$

$$\ln|x| = kt + C$$

$$e^{\ln|x|} = e^{kt+C}$$

$$|x| = e^{kt}e^{C}$$

$$x = Ce^{kt}$$

Since the general solution will always be the same, whenever we get this kind of initial value problem with $dx/dt = kx$ and $x(t_0) = x_0$, we can go straight to the solution equation to plug in the initial condition. Substituting the initial condition is what allows us to solve for the value of C.

Let's do an example where we use this exponential growth equation to model the growth of a bacteria population.

Example

The size of a bacteria population is initially P_0, and the population has grown to $3P_0$ after 2 hours. If the growth rate is proportional to the size of the population at any time t, how long did it take for the population to double?

The problem gives us the initial conditions $P(0) = P_0$ and $P(2) = 3P_0$. Substituting the first condition into the general solution equation lets us solve for the constant C.

$$P = Ce^{kt}$$

$$P_0 = Ce^{k(0)}$$

$$P_0 = C$$

Now we'll substitute $C = P_0$ and $P(2) = 3P_0$ into the general solution equation to find k, the constant of proportionality.

$$P = P_0 e^{kt}$$

$$3P_0 = P_0 e^{k(2)}$$

$$3 = e^{2k}$$

Apply the natural log to both sides.

$$\ln 3 = \ln(e^{2k})$$

$$\ln 3 = 2k$$

$$k = \frac{\ln 3}{2}$$

Now we can use k to find the time it took for the population to double, or the time it took to reach a population of $2P_0$.

$$2P_0 = P_0 e^{\frac{\ln 3}{2}t}$$

$$2 = e^{\frac{\ln 3}{2}t}$$

$$\ln 2 = \ln e^{\frac{\ln 3}{2}t}$$

$$\ln 2 = \frac{\ln 3}{2}t$$

$$t = \frac{2\ln 2}{\ln 3}$$

$$t \approx 1.26$$

This result tells us that the bacteria doubled the size of its original population in about 1.26 hours, or about 1 hour and 16 minutes.

If we check our result from the last example against the original question, it should make sense to us. It took 2 hours for the population to triple, which means it should take somewhere between 1 and 2 hours for the population to double, since the growth rate is always increasing. And in fact, our result of 1.26 hours for the population to double matches that intuition.

Let's look at one more example, this time with decay instead of growth, where the constant of proportionality is negative.

Example

A certain isotope decays to 25% of its original amount after 50 years. If the rate of decay is proportional to the amount remaining at any time t, how much of the isotope will remain after 75 years?

The problem gives us the initial conditions $x(0) = x_0$ and $x(50) = (1/4)x_0$, so we'll substitute the first into the general solution equation to find C.

$$x = Ce^{kt}$$

$$x_0 = Ce^{k(0)}$$

$$x_0 = C$$

Now substituting $C = x_0$ and $x(50) = (1/4)x_0$ will let us find k.

$$\frac{1}{4}x_0 = x_0 e^{k(50)}$$

$$\frac{1}{4} = e^{50k}$$

$$\ln\frac{1}{4} = \ln(e^{50k})$$

$$\ln 1 - \ln 4 = 50k$$

$$-\ln 4 = 50k$$

$$k = -\frac{\ln 4}{50}$$

Finally, we'll substitute this decay constant and $t = 75$ to find the amount of the isotope that remains after 75 years.

$$x(t) = x_0 e^{-\frac{\ln 4}{50}(75)}$$

$$x(t) = x_0 e^{\ln(4^{-\frac{3}{2}})}$$

$$x(t) = \frac{1}{4^{\frac{3}{2}}}x_0$$

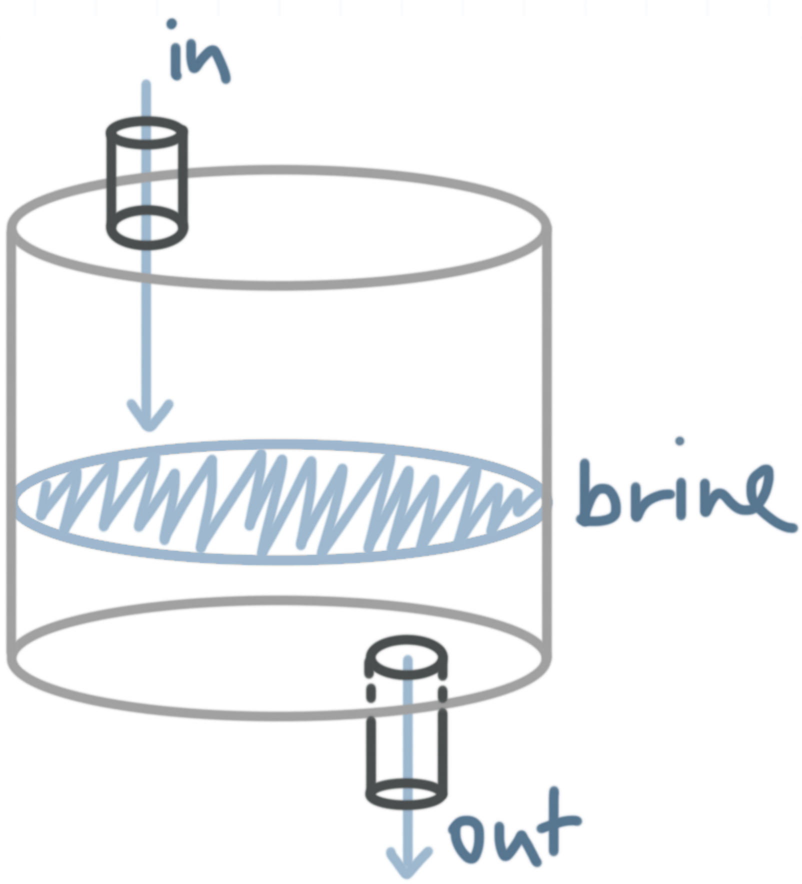
$$x(t) = \frac{1}{8}x_0$$

So $1/8$th of the original amount of the isotope remains after 75 years of decay.

Mixing problems

Continuing our discussion about how we can use differential equations to model real-world questions, we want to look now at mixing problems, which can be modeled by separable equations.

The idea here is that we have some container holding some solution, like a tank holding a saltwater solution (a brine). We're feeding another saltwater solution into the tank through an input valve, while at the same time emptying the tank through an output valve.



Assuming the brine in the tank stays perfectly mixed at all times, we want to model this scenario with a differential equation, and then find the solution to the equation, which will be a function that describes the amount of salt in the tank at any given time.

If salt is entering the tank at some rate, and exiting the tank at a different rate, then in general the amount of salt y at any time t can be modeled by

$$\frac{dy}{dt} = (\text{salt input rate}) - (\text{salt output rate})$$

We can find the input rate of salt by multiplying the rate at which brine is entering the tank r_1 by the concentration of salt in that brine C_1.

$$\text{input rate} = C_1 r_1$$

As an example, if we're adding 5 gallons of brine to the tank every minute, $r_1 = 5$ gal/min, and the concentration of the input brine is $1/4$ pound of salt per gallon, $C_1 = 0.25$ lb/ gal, then the rate at which salt enters the tank is

$$\text{salt input rate} = \left(\frac{0.25\text{ lb}}{\text{gal}}\right)\left(\frac{5\text{ gal}}{\text{min}}\right) = \frac{1.25\text{ lb}}{\text{min}}$$

Therefore, if C_1 is the concentration of the substance being added, r_1 is the rate at which that substance is added, if C_2 is the concentration of the substance being removed, and r_2 is the rate at which that substance is removed, then we can expand our salt concentration formula to

$$\frac{dy}{dt} = C_1 r_1 - C_2 r_2$$

It's important to realize that the concentration of salt leaving the tank is always modeled by

$$C_2 = \frac{y(t)}{\text{volume of brine in the tank at time } t}$$

We know that $y(t)$ is the amount of salt at any specific time t, but the total volume in the tank can vary depending on the rate at which brine is being added to and emptied from the tank. But total salt divided by total brine volume will give the concentration C_2.

Input vs. output

In mixing problems like these ones, we'll handle three possible scenarios.

1. **brine input rate = brine output rate**: the total volume of brine in the tank at any time t is constant

2. **brine input rate > brine output rate**: the total volume of brine in the tank is increasing over time

3. **brine input rate < brine output rate**: the total volume of brine in the tank is decreasing over time

The differential equation we build to model each of these three cases will look slightly different, so let's do one example of each. Remember that to solve a separable equation, we'll separate variables, integrate the equation, and then solve for y to find the general solution.

Example

A tank contains $1{,}500$ L of water and 20 kg of dissolved salt. Fresh water is entering the tank at 15 L/min, and the solution drains at a rate of 10 L/min. Assuming the brine in the tank remains perfectly mixed, how much salt is in the tank after t minutes?

Fresh water contains no salt at all, which means that $C_1 = 0$ kg/L because no salt is being added into the tank, but $r_1 = 15$ L/min because this is the rate at which fresh water is entering the tank.

And we know that $r_2 = 10$ L/min because this is the rate at which brine is being emptied from the tank.

To find C_2, we'll say that some amount of salt $y(t)$ is exiting the tank. But since C_2 is a concentration, we need to define the volume of the tank at any time t.

We start with $1{,}500$ L of brine solution in the tank. Then, every minute, 15 L enter the tank while 10 L exit. So every minute, we'll have 5 more liters of solution in the tank than we did the previous minute (this is the second of the three cases we outlined earlier, brine input rate > brine output rate).

Therefore, the volume of solution in the tank at any time is given by $1{,}500 + 5t$, and therefore the concentration of salt leaving the tank is

$$C_2 = \frac{y(t) \text{ kg}}{(1{,}500 + 5t) \text{ L}}$$

So the differential equation modeling the change in salt over time is

$$\frac{dy}{dt} = \left(\frac{0 \text{ kg}}{\text{L}}\right)\left(\frac{15 \text{ L}}{\text{min}}\right) - \left(\frac{y(t) \text{ kg}}{(1{,}500 + 5t) \text{ L}}\right)\left(\frac{10 \text{ L}}{\text{min}}\right)$$

$$\frac{dy}{dt} = \frac{0 \text{ kg}}{\text{min}} - \frac{10y(t) \text{ kg}}{(1{,}500 + 5t) \text{ min}}$$

$$\frac{dy}{dt} = -\frac{2y(t)\text{ kg}}{(300+t)\text{ min}}$$

Now we'll separate variables.

$$dy = -\frac{2y}{300+t}\ dt$$

$$\frac{1}{y}\ dy = -\frac{2}{300+t}\ dt$$

With the variables separated, we'll integrate the equation.

$$\int \frac{1}{y}\ dy = \int -\frac{2}{300+t}\ dt$$

$$\ln|y| = -2\ln|300+t| + C$$

Simplify the equation to solve it for y.

$$e^{\ln|y|} = e^{-2\ln|300+t|+C}$$

$$|y| = Ce^{-2\ln|300+t|}$$

$$y = Ce^{\ln|300+t|^{-2}}$$

$$y = C|300+t|^{-2}$$

Then the general solution is

$$y(t) = \frac{C}{(300+t)^2}$$

We were told that the tank initially contained 20 kg of dissolved salt, which means $y(0) = 20$ is an initial condition that we can substitute into the general solution to solve for the constant of integration,

$$20 = \frac{C}{(300 + 0)^2}$$

$$C = 1{,}800{,}000$$

and then we can put this value back into the general solution.

$$y(t) = \frac{1{,}800{,}000}{(300 + t)^2}$$

This last example modeled the brine input rate > brine output rate case. Let's do the same example, but this time we'll switch the brine input and out rates, so that we can see what the result looks like in the brine input rate < brine output rate case.

Example

A tank contains $1{,}500$ L of water and 20 kg of dissolved salt. Fresh water is entering the tank at 10 L/min, and the solution drains at a rate of 15 L/min. Assuming the brine in the tank remains perfectly mixed, how much salt is in the tank after t minutes?

In this scenario, the input rate is 10 L/min, while the output rate is 15 L/min. That means that the total volume in the tank is decreasing by 5 L every minute, which means the volume at any time t is given by $1{,}500-5t$.

$$\frac{dy}{dt}=\left(\frac{0\text{ kg}}{\text{L}}\right)\left(\frac{10\text{ L}}{\text{min}}\right)-\left(\frac{y(t)\text{ kg}}{(1{,}500-5t)\text{ L}}\right)\left(\frac{15\text{ L}}{\text{min}}\right)$$

$$\frac{dy}{dt}=\frac{0\text{ kg}}{\text{min}}-\frac{15y(t)\text{ kg}}{(1{,}500-5t)\text{ min}}$$

$$\frac{dy}{dt}=-\frac{3y(t)\text{ kg}}{(300-t)\text{ min}}$$

Separate variables and integrate both sides,

$$\int\frac{1}{y}\ dy=\int-\frac{3}{300-t}\ dt$$

$$\ln|y|=3\ln|300-t|+C$$

then solve for y.

$$e^{\ln|y|}=e^{3\ln|300-t|+C}$$

$$|y|=Ce^{3\ln|300-t|}$$

$$y=Ce^{\ln|300-t|^3}$$

$$y=C|300-t|^3$$

Then the general solution is

$$y(t)=C|300-t|^3$$

We were told that the tank initially contained 20 kg of dissolved salt, which means $y(0)=20$ is an initial condition that we can substitute into the general solution to solve for the constant of integration,

$$20 = C|300-0|^3$$

$$C = \frac{1}{1{,}350{,}000}$$

and then we can put this value back into the general solution.

$$y(t) = \frac{|300-t|^3}{1{,}350{,}000}$$

Finally, let's look at the easiest example, the brine input rate = brine output rate case.

Example

A tank contains $1{,}500$ L of water and 20 kg of dissolved salt. Fresh water is entering the tank at 15 L/min, and the solution drains at a rate of 15 L/min. Assuming the brine in the tank remains perfectly mixed, how much salt is in the tank after t minutes?

In this scenario, the input rate is 15 L/min, while the output rate is 15 L/min. That means that the total volume in the tank remains the same, which means the volume at any time t is given by $1{,}500$.

$$\frac{dy}{dt} = \left(\frac{0 \text{ kg}}{\text{L}}\right)\left(\frac{15 \text{ L}}{\text{min}}\right) - \left(\frac{y(t) \text{ kg}}{1{,}500 \text{ L}}\right)\left(\frac{15 \text{ L}}{\text{min}}\right)$$

$$\frac{dy}{dt} = \frac{15 \text{ L}}{\text{min}}\left(\frac{0 \text{ kg}}{\text{L}} - \frac{y(t) \text{ kg}}{1{,}500 \text{ L}}\right)$$

$$\frac{dy}{dt} = -\frac{y(t) \text{ kg}}{100 \text{ min}}$$

Separate variables and integrate both sides,

$$\int \frac{1}{y}\ dy = \int -\frac{1}{100}\ dt$$

$$\ln|y| = -\frac{1}{100}t + C$$

then solve for y.

$$e^{\ln|y|} = e^{-\frac{1}{100}t+C}$$

$$|y| = Ce^{-\frac{1}{100}t}$$

Then the general solution is

$$y(t) = Ce^{-\frac{1}{100}t}$$

We were told that the tank initially contained 20 kg of dissolved salt, which means $y(0) = 20$ is an initial condition that we can substitute into the general solution to solve for the constant of integration,

$$20 = Ce^{-\frac{1}{100}(0)}$$

$$C = 20$$

and then we can put this value back into the general solution.

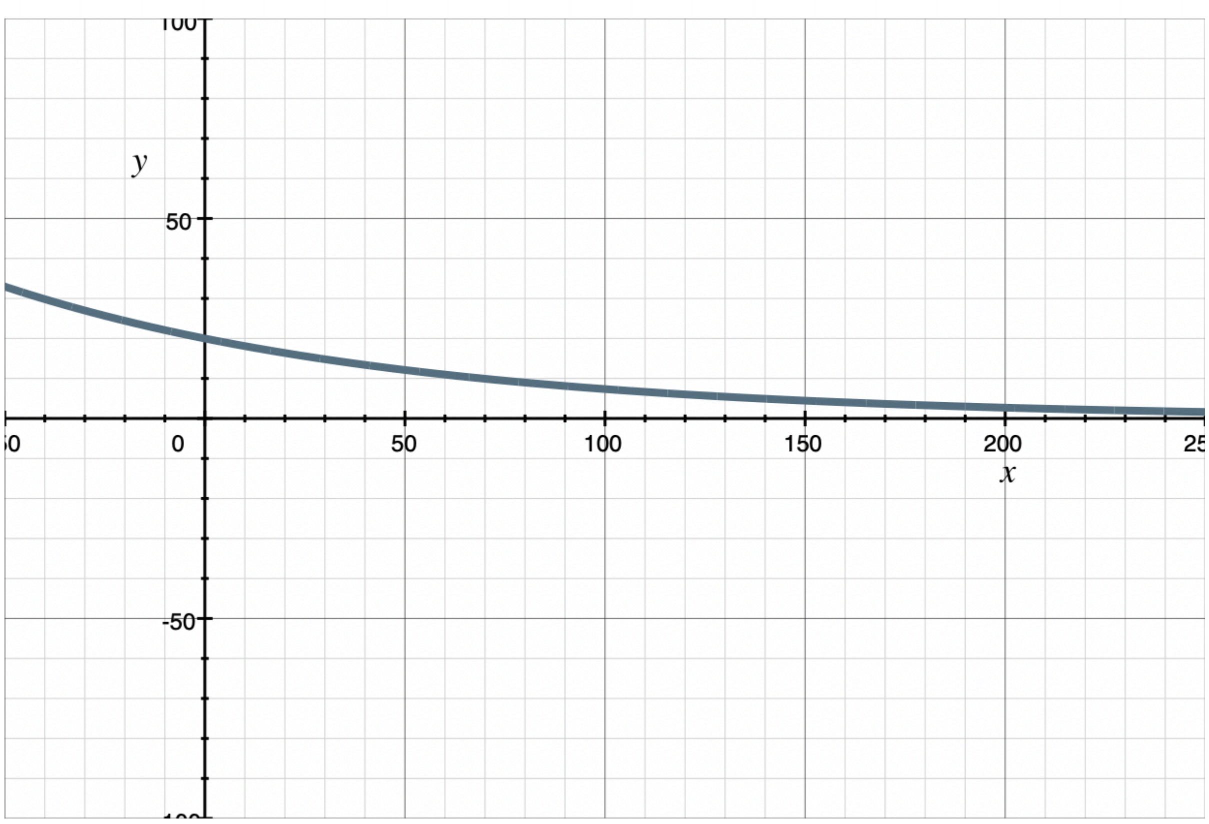
$$y(t) = 20e^{-\frac{1}{100}t}$$

If we graph the three solution equations from these examples, we can see how the amount of salt changes in the tank over time.

Notice how all three curves intersect the y-axis at $(0,20)$, matching the initial condition that we used in each problem. Based on the graphs, here's a summary of how the amount of salt changes over time in each case:

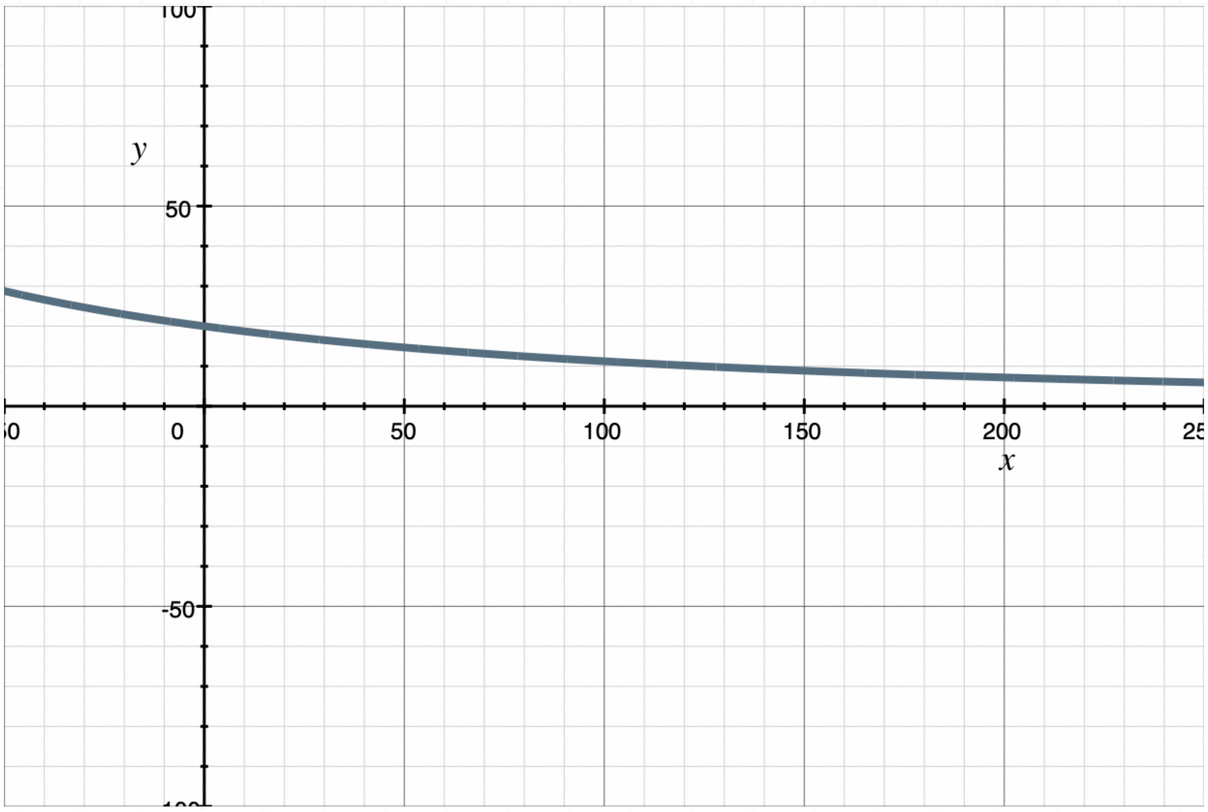
brine input rate = brine output rate: salt is initially 20 kg, but decreases over time

$$y(t) = 20e^{-\frac{1}{100}t}$$



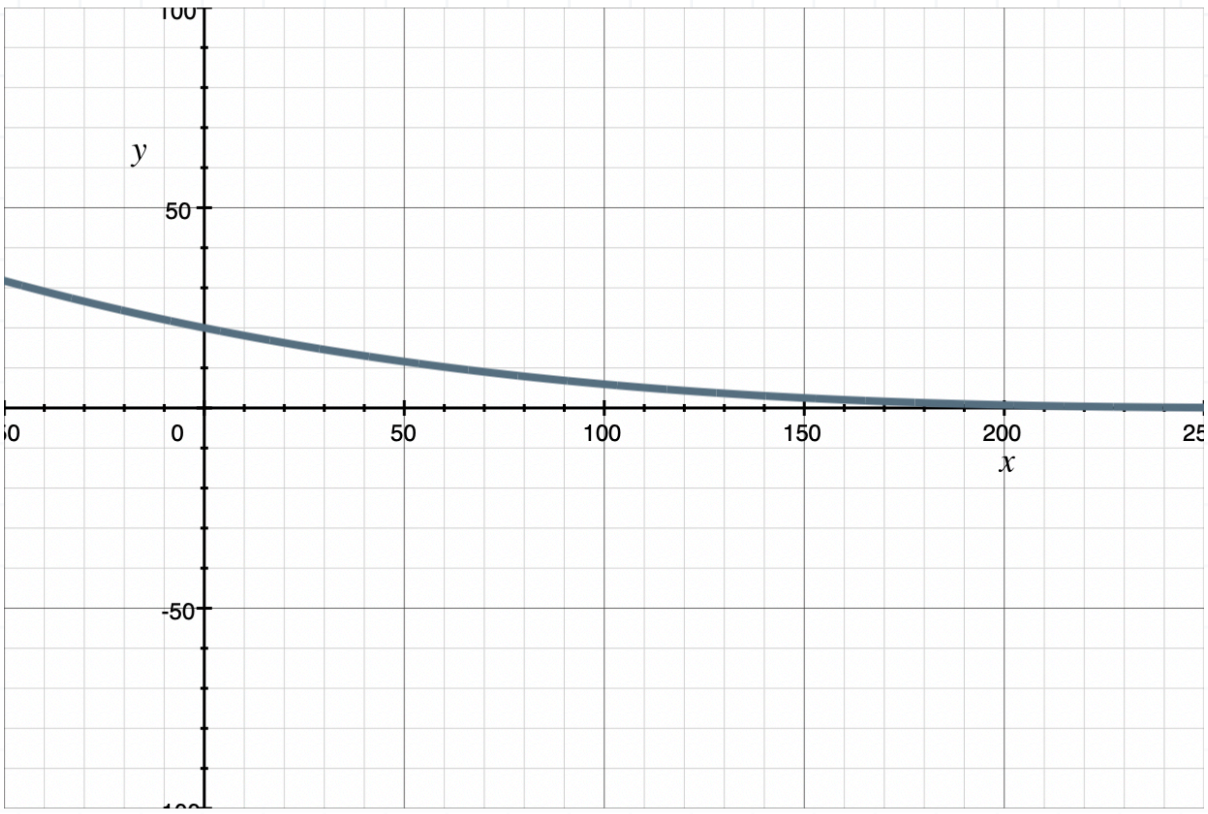
brine input rate > brine output rate: salt is initially 20 kg, but decreases over time

$$y(t) = \frac{1{,}800{,}000}{(300+t)^2}$$



brine input rate < brine output rate: salt is initially 20 kg, but decrease over time

$$y(t) = \frac{|300-t|^3}{1{,}350{,}000}$$



100
50
-50
y
x
0
50
100
150
200

Newton's Law of Cooling

Newton's Law of Cooling models the way in which a warm object in a cooler environment cools down until it matches the temperature of its environment.

Therefore, this law is similar to other models of decay, because it models the rate at which temperature is "decaying" from warmer to cooler.

The Law

The law tells us that the rate at which the object cools is proportional to the difference between the object and the environment around it.

In other words, if we put a boiling pot of soup in the freezer, it'll cool down faster than if we simply leave the pot on the counter. That's because the difference between the temperature of the soup and the freezer is greater than the difference between the temperature of the soup and the room-temperature countertop. The greater the temperature difference, the faster the object will cool.

The **Newton's Law of Cooling** formula is

$$\frac{dT}{dt} = -k(T - T_a) \text{ with } T(0) = T_0$$

where T is the temperature over time t, k is the decay constant, T_a is the temperature of the environment ("ambient temperature"), and T_0 is the initial, or starting, temperature of the hot object.

The point of applying Newton's Law is to generate an equation that models the temperature of the hot, but cooling, object over time. That temperature function, and the solution to the Newton's Law equation, will be

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

The easiest way to understand how Newton's Law applies to a real-world scenario is to work through an example, so let's work through one.

Example

At a local restaurant, a big pot of soup, boiling at 100° C, has just been removed from the stove and set on the countertop, where the ambient temperature is 23° C. After 5 minutes, the soup cools to 98°. If the soup needs to be served to the restaurant's customers at 90° C, how long does the restaurant have to wait before the soup is ready to serve?

For Newton's Law problems, it's especially helpful to list out what the question tells us.

$T_0 = 100^\circ$	Initial temperature of the soup
$T_a = 23^\circ$	Ambient temperature on the countertop
$T(5) = 98^\circ$	At time $t = 5$ minutes, the soup has cooled to 98°

If we plug everything we know into the Newton's Law of Cooling solution equation, we get

$$T(t) = T_a + (T_0 - T_a)e^{-kt}$$

$$T(t) = 23 + (100 - 23)e^{-kt}$$

$$T(t) = 23 + 77e^{-kt}$$

Substitute the initial condition $T(5) = 98^\circ$,

$$T(5) = 23 + 77e^{-k(5)}$$

$$98 = 23 + 77e^{-5k}$$

in order to find a value for the decay constant k.

$$75 = 77e^{-5k}$$

$$\frac{75}{77} = e^{-5k}$$

$$\ln\frac{75}{77} = \ln(e^{-5k})$$

$$\ln\frac{75}{77} = -5k$$

$$k = -\frac{1}{5}\ln\frac{75}{77}$$

Substitute this value for k into the equation modeling temperature over time.

$$T(t) = 23 + 77e^{-\left(-\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$T(t) = 23 + 77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

We want to find the time t at which the soup reaches 90°, so we'll substitute $T(t)=90^\circ$.

$$90=23+77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$67=77e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

$$\frac{67}{77}=e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}$$

Apply the natural logarithm to both sides of the equation.

$$\ln\frac{67}{77}=\ln\left(e^{\left(\frac{1}{5}\ln\frac{75}{77}\right)t}\right)$$

$$\ln\frac{67}{77}=\left(\frac{1}{5}\ln\frac{75}{77}\right)t$$

$$5\ln\frac{67}{77}=\left(\ln\frac{75}{77}\right)t$$

$$t=\frac{5\ln\frac{67}{77}}{\ln\frac{75}{77}}$$

$$t\approx 26.43$$

The conclusion then is that the pot of soup will cool from 100° C to 90° C in about 26.5 minutes, at which point it'll be ready to serve to the restaurant's customers.

Electrical series circuits

So far we've been looking at applications of first order differential equations, but we do also want to look at least at some applications of second order equations.

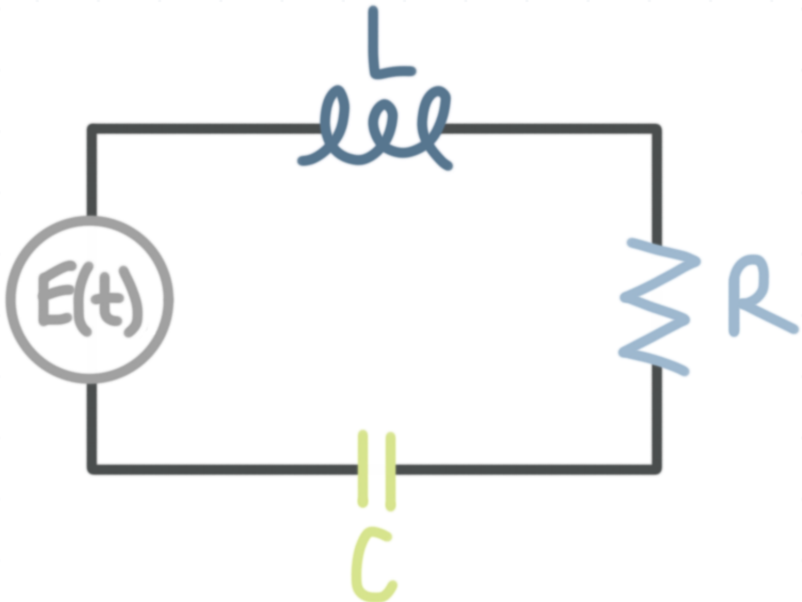
So we'll now turn our attention toward electrical circuits. Depending on the components included in the circuit, we'll sometimes model these with first order differential equations, and sometimes with second order differential equations. We'll look at both scenarios.

LRC series circuits

Electrical series circuits are another real-world system that we can model with differential equations. We'll usually see some collection of these four components in a series circuit:

- The inductance L, measured as a constant in henries h
- The resistance R, measured as a constant in ohms Ω
- The capacitance C, measured as a constant in farads f
- The impressed voltage $E(t)$

We sketch these pieces in a circuit that looks something like this:



Kirchhoff's second law

In a circuit like the one we sketched out above, **Kirchhoff's second law** tells us that the impressed voltage $E(t)$ on a closed loop must be equal to the sum of the voltage drops in the loop.

In other words, if we apply a voltage to the circuit, like plugging in a battery, we're impressing a certain voltage $E(t)$ onto the system. Doing so creates a current $i(t)$ that moves around the circuit.

The voltage drop that occurs when the current encounters the inductor is $L(di/dt)$, the voltage drop that occurs when the current encounters the resistor is Ri, and the voltage drop that occurs when the current encounters the capacitor is $q(t)/C$, where $q(t)$ is the charge on the capacitor at any time t. The current $i(t)$ is related to the capacitor's charge $q(t)$ by $i = dq/dt$.

If we use this last equation $i = dq/dt$ to write everything in terms of q, we can write Kirchhoff's second law as

$$L\frac{di}{dt} + Ri + \frac{1}{C}q = E(t)$$

$$L\frac{d}{dt}\left(\frac{dq}{dt}\right) + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

When $E(t) = 0$, the electrical vibrations in the circuit are **free**; they're **forced** when $E(t) \neq 0$. And if the circuit is missing one of these components, Kirchhoff's second law still holds. For instance, for a circuit without an inductor, the impressed voltage is still equivalent to the voltage drops over the resistor and capacitor.

$$R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Or for a circuit without a capacitor, the impressed voltage is still equivalent to the voltage drops over the inductor and resistor.

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} = E(t)$$

With some understanding now of series circuits, let's do an example where we solve one of these differential equations.

Example

Find the current i after connecting a 9-volt battery to a series circuit with inductance 1.5 h and resistance 5 Ω.

There's no capacitor in the system, which means we can simplify the equation given by Kirchhoff's second law by eliminating the capacitor term.

$$L\frac{di}{dt}+Ri=E(t)$$

Substituting $L=1.5=3/2$, $R=5$, and $E(t)=9$, we get

$$\frac{3}{2}\left(\frac{di}{dt}\right)+5i=9$$

$$\frac{di}{dt}+\frac{10}{3}i=6$$

This is now a linear differential equation in standard form, so we'll find the integrating factor.

$$I(t)=e^{\int \frac{10}{3}\ dt}$$

$$I(t)=e^{\frac{10}{3}t}$$

Multiplying through the differential equation by the integrating factor, we get

$$\frac{di}{dt}e^{\frac{10}{3}t}+\frac{10}{3}ie^{\frac{10}{3}t}=6e^{\frac{10}{3}t}$$

$$\frac{d}{dt}(ie^{\frac{10}{3}t})=6e^{\frac{10}{3}t}$$

Integrate both sides, then solve for the current i.

$$\int \frac{d}{dt}(ie^{\frac{10}{3}t})\ dt = \int 6e^{\frac{10}{3}t}\ dt$$

$$ie^{\frac{10}{3}t} = \frac{9}{5}e^{\frac{10}{3}t} + C$$

$$i = \frac{9}{5} + Ce^{-\frac{10}{3}t}$$

In other words, this solution equation models the current i in the circuit at any time t.

It's particularly interesting to realize that the second term in the equation, $Ce^{-\frac{10}{3}t}$, approaches 0 as $t \to \infty$. For that reason, we call this second term the **transient term**. Any terms that remain after the transient terms drop away are the **steady-state part** of the solution. In this case, $9/5$ is the steady-state current.

The second order circuit equation

In the last example we solved a first order linear differential equation, but when the inductor, resistor, and capacitor are all present in the circuit, Kirchhoff's second law gave us

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$Lq'' + Rq' + \frac{1}{C}q = E(t)$$

This is a second order linear differential equation. The associated homogeneous equation is

$$Lr^2 + Rr + \frac{1}{C} = 0$$

This is a quadratic equation, so using the quadratic formula the roots of the equation are

$$r = \frac{-R \pm \sqrt{R^2 - \frac{4L}{C}}}{2L}$$

So we can say that the circuit is

- **overdamped** if the discriminant is positive, $R^2 - 4L/C > 0$, in which case we'll find distinct real roots,

- **critically damped** if the discriminant is zero, $R^2 - 4L/C = 0$, in which case we'll find equal real roots, and

- **underdamped** if the discriminant is negative, $R^2 - 4L/C < 0$, in which case we'll find complex conjugate roots.

Let's look at an example with the second order equation.

Example

Given $L = 0.5$ henry (h), $R = 4.5$ ohms (Ω), $C = 0.1$ farad (f), $E(t) = 0$, $q(0) = q_0$ coulombs (C), and $i(0) = 0$, find the charge $q(t)$ on the capacitor in a circuit with an inductor, resistor, and capacitor.

Plugging what we know into the second order equation given by Kirchhoff's second law, we get

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

$$0.5q'' + 4.5q' + \frac{1}{0.1}q = 0$$

$$q'' + 9q' + 20q = 0$$

The associated characteristic equation for this second order homogeneous equation is

$$r^2 + 9r + 20 = 0$$

$$(r + 5)(r + 4) = 0$$

$$r = -5, \ -4$$

Because we found distinct real roots, we know the circuit is overdamped. We also know that it's overdamped because the discriminant is positive.

$$R^2 - 4L/C > 0$$

$$4.5^2 - 4(0.5)/0.1 > 0$$

$$20.25 - 20 > 0$$

$$0.25 > 0$$

With distinct real roots, we can say that the general solution and its derivative are given by

$$q(t) = c_1e^{-5t} + c_2e^{-4t}$$

$$q'(t) = i(t) = -5c_1e^{-5t} - 4c_2e^{-4t}$$

Substituting the initial conditions $q(0) = q_0$ and $i(0) = 0$ into these equations, we get

$$q_0 = c_1e^{-5(0)} + c_2e^{-4(0)}$$

$$q_0 = c_1 + c_2$$

and

$$0 = -5c_1e^{-5(0)} - 4c_2e^{-4(0)}$$

$$0 = -5c_1 - 4c_2$$

$$c_2 = -\frac{5}{4}c_1$$

Substituting this value into the first equation gives

$$q_0 = c_1 - \frac{5}{4}c_1$$

$$q_0 = -\frac{1}{4}c_1$$

$$c_1 = -4q_0$$

Plugging this back into the equation for c_2, we get

$$c_2 = -\frac{5}{4}(-4q_0)$$

$$c_2 = 5q_0$$

So the general solution can be written as

$$q(t) = -4q_0e^{-5t} + 5q_0e^{-4t}$$

$$q(t) = 5q_0e^{-4t} - 4q_0e^{-5t}$$

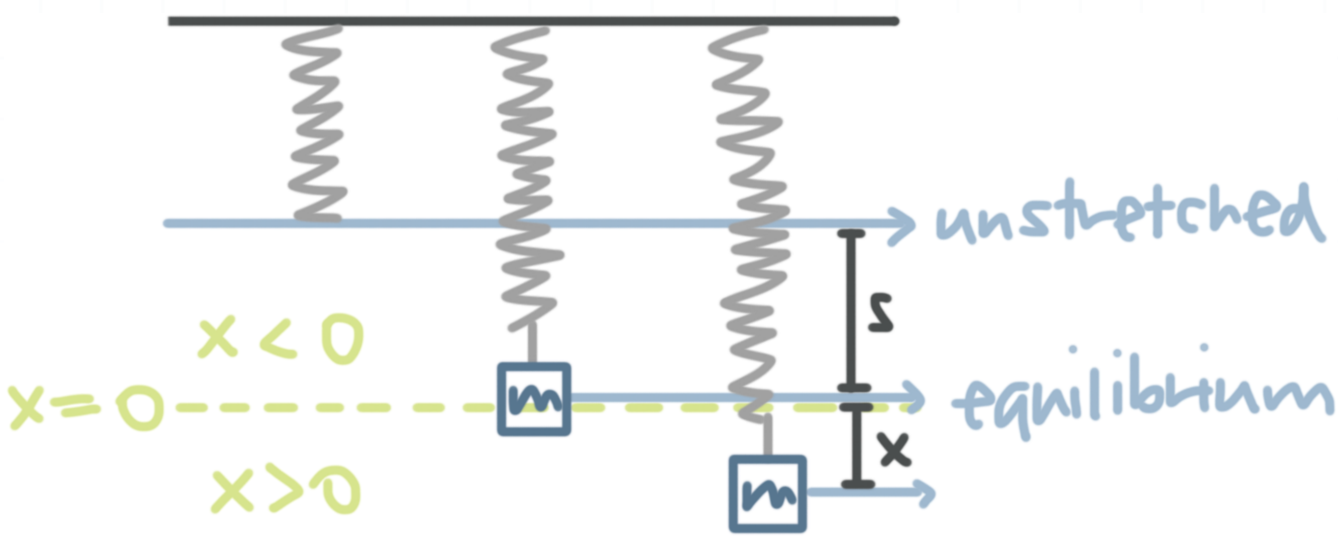
$$q(t) = q_0e^{-4t}(5 - 4e^{-t})$$

This solution equation models the charge q on the capacitor at any time t.

Spring and mass systems

Spring and mass systems are another common real-world application of differential equations. Like the electrical circuits we looked at earlier, these also require that we solve second order equations.

The general idea here is that we'll have some unstretched spring hanging vertically from something rigid, and we'll hang a mass from the end of the spring. When we do, depending on the characteristics of the spring itself and the weight of the mass, the spring may stretch, and it may stretch and compress up and down, until eventually it reaches some motionless equilibrium point.



Hooke's Law tells us that the spring exerts force on the mass as it tries to return to its unstretched state, and that force is proportional to the extra length of the spring between its unstretched state and its state of equilibrium with the mass, in other words, $F=ks$, where k is the **spring constant**.

There are two scenarios here that we want to consider. First, we'll look at a spring and mass system that's vibrating free of external motion, and then we'll look at a system where the motion is forced by some external driver.

Free motion, undamped and damped

If the spring and mass system is operating in a vacuum, such that there are no other external forces exerting themselves on the system (like friction from the environment, or the decay of the spring over time), then we say that the motion of the spring and mass is **free undamped motion**, or simple harmonic motion.

Using Newton's Second Law, we can model this position of the mass over time in relation to equilibrium as

$$\frac{d^2x}{dt^2}+\frac{k}{m}x=0$$

where x is the distance between the mass and its position in equilibrium, k is the spring constant, m is the mass, and t is time. It's also common to substitute $\omega^2=k/m$ to rewrite this equation as

$$\frac{d^2x}{dt^2}+\omega^2x=0$$

The reason is because, in this form, we can say that the general solution is

$$x(t)=c_1\cos(\omega t)+c_2\sin(\omega t)$$

Keep in mind, regardless of how we express the second order equation, x is positive when the mass is below its equilibrium position (the spring is stretched further than equilibrium) and negative when the mass is above its equilibrium position (the spring is less stretched than at equilibrium).

Let's do an example where we find the general solution to the second order equation.

Example

A mass weighing 10 pounds stretches a spring 4 ft. Find the equation that models the motion of the mass if we release the mass when $t = 0$ from a position 3 feet above equilibrium, with a downward velocity of 1 ft/s.

First we'll use Hooke's Law to find the spring constant k.

$$F = ks$$

$$10 = k(4)$$

$$k = 2.5 \text{ lb/ft}$$

To convert weight into mass, we'll use $m = W/g = 10/32 = 5/16$ slug. Plugging everything we have into the second order equation, we get

$$m\frac{d^2x}{dt^2} + kx = 0$$

$$\frac{5}{16}\left(\frac{d^2x}{dt^2}\right) + 2.5x = 0$$

$$\frac{d^2x}{dt^2} + 8x = 0$$

From this equation, we see that $\omega = \sqrt{8} = 2\sqrt{2}$, which means that the general solution is given by

$$x(t) = c_1 \cos(\omega t) + c_2 \sin(\omega t)$$

$$x(t) = c_1 \cos(2\sqrt{2}t) + c_2 \sin(2\sqrt{2}t)$$

Its derivative is

$$x'(t) = -2\sqrt{2}c_1 \sin(2\sqrt{2}t) + 2\sqrt{2}c_2 \cos(2\sqrt{2}t)$$

The question states that the initial position is $x(0) = -3$ and the initial velocity is $x'(0) = 1$, so we'll plug these into $x(t)$ and $x'(t)$, and we get

$$-3 = c_1 \cos(2\sqrt{2}(0)) + c_2 \sin(2\sqrt{2}(0))$$

$$c_1 = -3$$

and

$$1 = -2\sqrt{2}c_1 \sin(2\sqrt{2}(0)) + 2\sqrt{2}c_2 \cos(2\sqrt{2}(0))$$

$$c_2 = \frac{\sqrt{2}}{4}$$

So the equation modeling the motion of this spring and mass system, with these particular initial conditions for velocity and position, is given by

$$x(t) = -3\cos(2\sqrt{2}t) + \frac{\sqrt{2}}{4}\sin(2\sqrt{2}t)$$

$$x(t) = \frac{\sqrt{2}}{4}\sin(2\sqrt{2}t) - 3\cos(2\sqrt{2}t)$$

Let's talk now about the scenario of **free damped motion**, in which we have free motion that's being damped, like by a liquid that surrounds the system, or by an actual damping device that's pushing back on the mass.

The second order differential equation we used to model free damped motion is

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = 0$$

or

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where $2\lambda = \beta/m$ and $\omega^2 = k/m$. Again, just like the substitution $\omega^2 = k/m$ gave us a more convenient way to express the general solution, the substitution $2\lambda = \beta/m$ gives us a more convenient way to express the roots of the associated characteristic equation, $r^2 + 2\lambda r + \omega^2 = 0$.

$$r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}$$

Given this pair of roots,

- if $\lambda^2 - \omega^2 > 0$, the system is **overdamped** and the general solution is

$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega^2}t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}t})$$

- if $\lambda^2 - \omega^2 = 0$, the system is **critically damped** and the general solution is

$$x(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$$

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

- if $\lambda^2 - \omega^2 < 0$, the system is **underdamped** and the general solution is

$$x(t) = e^{-\lambda t}(c_1 \cos \sqrt{\omega^2 - \lambda^2}t + c_2 \sin \sqrt{\omega^2 - \lambda^2}t)$$

Let's do an example where we find the solution for a free damped system.

Example

A mass weighing 4 pounds stretches a spring 2 feet, and a damping force that's triple the instantaneous velocity is acting on the spring/mass system. Find an equation that models the motion of the mass if it's initially released from $1/2$ foot below equilibrium with an upward velocity of 1 ft/s.

Hooke's Law tells us that the spring constant is

$$F = ks$$

$$4 = k(2)$$

$$k = 2 \text{ lb/ft}$$

Now we'll use $W = mg$ to convert the weight into mass.

$$W = mg$$

$$4 = m(32)$$

$$m = \frac{1}{8} \text{ slug}$$

With the spring constant and the mass, and $\beta = 3$ since the damping force is triple the instantaneous velocity, we can plug everything into the differential equation.

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = 0$$

$$\frac{d^2x}{dt^2} + \frac{3}{\frac{1}{8}}\left(\frac{dx}{dt}\right) + \frac{2}{\frac{1}{8}}x = 0$$

$$\frac{d^2x}{dt^2} + 24\left(\frac{dx}{dt}\right) + 16x = 0$$

The associated characteristic equation and its roots are

$$r^2 + 24r + 16 = 0$$

$$r = -12 \pm 8\sqrt{2}$$

Because we find roots in which $\lambda^2 - \omega^2 > 0$, the spring and mass system is overdamped and the general solution is

$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega^2}t} + c_2 e^{-\sqrt{\lambda^2 - \omega^2}t})$$

$$x(t) = e^{-12t}(c_1 e^{\sqrt{12^2 - 16}t} + c_2 e^{-\sqrt{12^2 - 16}t})$$

$$x(t) = e^{-12t}(c_1 e^{\sqrt{128}t} + c_2 e^{-\sqrt{128}t})$$

$$x(t) = e^{-12t}(c_1 e^{8\sqrt{2}t} + c_2 e^{-8\sqrt{2}t})$$

Driven motion

When we add some driver of motion into the system, we often call it the forcing function $f(t)$, and we rewrite Newton's second law as

$$\frac{d^2x}{dt^2} + \frac{\beta}{m}\left(\frac{dx}{dt}\right) + \frac{k}{m}x = \frac{f(t)}{m}$$

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Since this is now a nonhomogeneous second order equation, we'd just need to use the method of undetermined coefficients or the method of variation of parameters to find the solution.

For instance, using undetermined coefficients, we'd follow the same steps we're used to with that method. We'd start by finding the complementary solution as the solution to the associated homogeneous equation. Then we'd make a guess for the particular solution and apply undetermined coefficients. To get the general solution, we'd sum the complementary and particular solutions.

And if we have initial conditions, we can apply them to the general solution and its derivative in order to solve for the coefficients c_1 and c_2.

Power series basics

We likely would have learned about power series in a Calculus 2 course, but since that may have been a while ago, let's do a quick review of them here.

Since the series solution method we're about to learn is entirely based around power series, it's worth making sure that we're starting with a good foundation.

Power series

A **power series centered at** x_0 is given by

$$\sum_{n=0}^{\infty} c_n(x-x_0)^n = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \dots$$

We also call this a power series in $x - x_0$. We'll particularly focus on power series centered at $x_0 = 0$ (or a power series in x), in which case our power series expression simplifies to

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

In this power series, $n = 0$ is the **index**. The index gives the starting value of n, while the ∞ we see above the summation notation gives us the "ending" value of n. In other words, if we read the summation from bottom to top, from $n = 0$, to the summation, to the ∞, it tells us to add all terms, starting at $n = 0$, until $n = \infty$.

The index can start at any value, but we'll most commonly see indices for $n = 0$, $n = 1$, or $n = 2$.

Convergence of power series

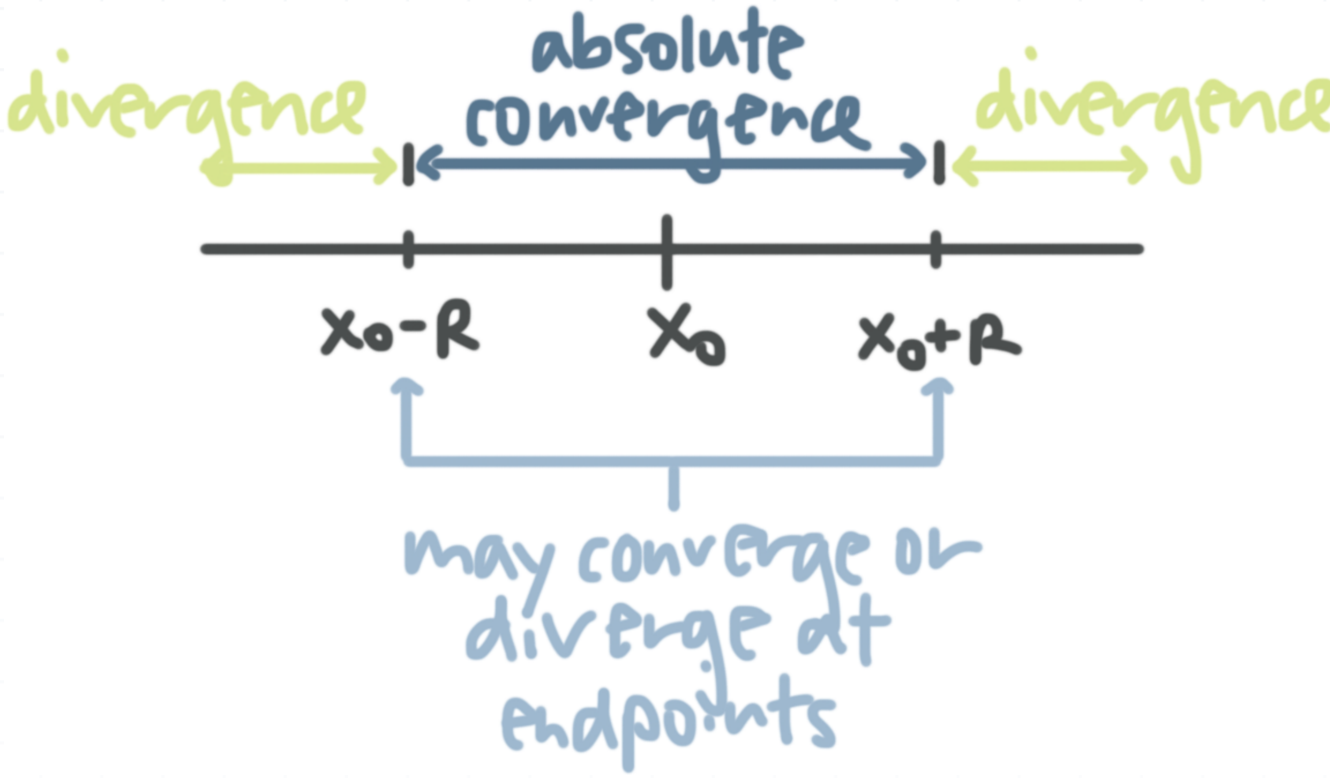
A power series is **convergent** at a specific value of x when the limit exists,

$$\lim_{N\to\infty} \sum_{n=0}^{N} c_n(x - x_0)^n$$

Otherwise, if this limit does not exist, then the series is divergent. For a power series centered at x_0, the **interval of convergence** of the series is the set of all real numbers x around x_0 for which the series converges, and the radius R of the interval of convergence is the **radius of convergence**.

If $R > 0$, then the power series converges for $|x - x_0| < R$ and diverges for $|x - x_0| > R$. If $R = 0$, then the series converges only at x_0, its center. When $R = \infty$, it means the power series converges for all x.

The power series converges absolutely inside the interval of convergence, diverges outside the interval, and may or may not converge at the endpoints of the interval.



We can often use the ratio test to determine the convergence of a power series. After calculating this value L,

$$L = \lim_{n \to \infty} \left| \frac{c_{n+1}}{c_n} \right|$$

the **ratio test** tells us that the series converges absolutely if $L < 1$, diverges if $L > 1$, and inconclusive if $L = 1$. The ratio test will always be inconclusive at the endpoints of the interval of convergence, $x_0 \pm R$.

Let's do an example so that we understand how to find the interval and radius of convergence.

Example

Find the interval and radius of convergence of the power series.

$$\sum_{n=1}^{\infty} \frac{2^{5n}}{5^{2n}} \left(\frac{x}{3} \right)^n$$

First, let's rewrite the series.

$$\sum_{n=1}^{\infty} \left(\frac{2^5}{5^2}\right)^n \left(\frac{x}{3}\right)^n$$

$$\sum_{n=1}^{\infty} \left(\frac{32}{25}\right)^n \left(\frac{x}{3}\right)^n$$

Then applying the ratio test to the power series gives

$$L = \lim_{n\to\infty} \left| \frac{c_{n+1}}{c_n} \right|$$

$$L = \lim_{n\to\infty} \left| \frac{\left(\frac{32}{25}\right)^{n+1} \left(\frac{x}{3}\right)^{n+1}}{\left(\frac{32}{25}\right)^n \left(\frac{x}{3}\right)^n} \right|$$

$$L = \lim_{n\to\infty} \left| \left(\frac{32}{25}\right) \left(\frac{x}{3}\right) \right|$$

$$L = \lim_{n\to\infty} \left| \frac{32}{75} x \right|$$

$$L = \frac{32}{75} |x|$$

To determine where the series converges absolutely, we set this value $L < 1$.

$$\frac{32}{75}|x| < 1$$

$$|x| < \frac{75}{32}$$

$$-\frac{75}{32} < x < \frac{75}{32}$$

So the interval of convergence is $(-75/32, 75/32)$, and the radius of convergence is $R = 75/32$. To determine whether the series converges or diverges at the endpoints, we would need to substitute them into the original series.

For $x = -75/32$, $$\sum_{n=1}^{\infty} \frac{2^{5n}}{5^{2n}} \left(\frac{-\frac{75}{32}}{3} \right)^n = \sum_{n=1}^{\infty} \frac{2^{5n}}{5^{2n}} \left(\frac{-\frac{3^n \cdot 5^{2n}}{2^{5n}}}{3^n} \right) = \sum_{n=1}^{\infty} (-1)$$

For $x = 75/32$, $$\sum_{n=1}^{\infty} \frac{2^{5n}}{5^{2n}} \left(\frac{\frac{75}{32}}{3} \right)^n = \sum_{n=1}^{\infty} \frac{2^{5n}}{5^{2n}} \left(\frac{\frac{3^n \cdot 5^{2n}}{2^{5n}}}{3^n} \right) = \sum_{n=1}^{\infty} 1$$

Then we can see that the series is divergent at both endpoints, and therefore that the interval of convergence is still $(-75/32, 75/32)$. If the series had converged at both endpoints, the interval would have been $[-75/32, 75/32]$. If the series were convergent at just the left endpoint, or convergent at just the right endpoint, then the interval of convergence would have been $[-75/32, 75/32)$ or $(-75/32, 75/32]$, respectively.

Functions as power series

We can define a function as a power series

$$f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

The domain of the function is the interval of convergence of the series. If the radius of convergence is $R > 0$ or $R = \infty$, then the function f is continuous, differentiable, and integrable on $(x_0 - R, x_0 + R)$ or $(-\infty, \infty)$, respectively.

We say that a function is **analytic at a point** x_0 if the function can be represented by a power series in $x - x_0$ with any positive or infinite radius of convergence.

A **Taylor series** is just a power series in which the coefficients c_n are given by $(f^{(n)}(x_0))/n!$, where $f^{(n)}$ are the derivatives of f. So the Taylor series representation of a function f is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= \frac{f^{(0)}(x_0)}{0!}(x - x_0)^0 + \frac{f^{(1)}(x_0)}{1!}(x - x_0)^1 + \frac{f^{(2)}(x_0)}{2!}(x - x_0)^2 + \frac{f^{(3)}(x_0)}{3!}(x - x_0)^3 + \dots$$

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

$$= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots$$

The **Maclaurin series** is simply the Taylor series centered at $x_0 = 0$, so the Maclaurin series representation of f is

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

The common Maclaurin series in the table below will come in handy as we talk about series solutions to differential equations.

Maclaurin series	**Interval of convergence**
$e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$	$(-\infty, \infty)$
$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1}$	$[-1,1]$
$\cosh x = \sum_{n=0}^{\infty} \frac{1}{(2n)!} x^{2n}$	$(-\infty, \infty)$
$\sinh x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}$	$(-\infty, \infty)$
$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n$	$(-1,1]$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n \qquad\qquad (-1,1)$$

Let's do a quick example so that we can see how to adapt these common Maclaurin series to a similar function.

Example

Find the Maclaurin series representation of $f(x) = \sin(2x^2)$.

Because we already have the Maclaurin series representation of $f(x) = \sin x$, we can simply replace x with $2x^2$.

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2x^2)^{2n+1}$$

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2^{2n+1} x^{2(2n+1)})$$

$$\sin(2x^2) = \sum_{n=0}^{\infty} \frac{2^{2n+1}(-1)^n}{(2n+1)!} x^{4n+2}$$

This is the Maclaurin series representation of $f(x) = \sin(2x^2)$, and the interval of convergence is still $(-\infty, \infty)$, the same as the interval of convergence for $f(x) = \sin x$.

Differentiation of power series

Finally, we mentioned before that if the radius of convergence is positive, then the function is differentiable on the entire interval of convergence.

With that in mind, it'll be important to be able to find the first and second derivatives of a power series in x, which are

$$y = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots$$

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \ldots$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + \ldots$$

We'll use these very often in upcoming lessons as we start working on finding series solutions to differential equations.

Adding power series

It's important for later lessons on series solutions for differential equations that we be able to add power series.

Conditions for addition

Power series can only be added when

1. their indices start at the same values, and

2. the powers of x in each series are "**in phase**," which means that both series start with the same power of x.

Remember that the **index** of a power series is given by the small equation we see below the summation notation, in this case, $n = 0$:

$$\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n$$

We'll usually use n or k for the indexing variable, and the index most commonly (but certainly not always) begins at 0 or 1.

So when we say that the indices of separate power series need to match before we can add them, we're saying that they both need to start at the same value. These series have matching indices $n = 0$,

$$\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n \qquad \sum_{n=0}^{\infty} \frac{(-1)^n}{9^n} x^{2n+1}$$

while these series have indices that don't match, since one is $n = 0$ while the other is $n = 1$.

$$\sum_{n=0}^{\infty} \frac{5^n}{n!} x^n \qquad\qquad \sum_{n=1}^{\infty} \frac{(-1)^n}{n} x^n$$

We also need to make sure that the powers of x in each series are in phase. For example, we can't add a series whose first term is for x^1 to a series whose first term is x^2. Two series will only be in phase if they both start with x^0, both start with x^1, both start with x^2, etc.

How to add series

Let's assume that two series aren't in phase and that their indices start at different values. In that case, we'll start by rewriting the series so that they're in phase, and then we'll rewrite that result to make the indices match.

Once both conditions are met, we can add the series term by term, simplifying the end result as much as possible.

Let's do an example so that we can see these steps in action.

Example

Find the sum of the series.

$$\sum_{n=1}^{\infty} 2nc_n x^{n-1} + \sum_{n=0}^{\infty} 6c_n x^{n+1}$$

Because the first series starts at $n = 1$, the first power of x for that series will be $x^{n-1} = x^{1-1} = x^0$. And because the second series starts at $n = 0$, the first power of x for that series will be $x^{n+1} = x^{0+1} = x^1$.

So the series are not in phase. To fix this, we'll pull the first term from the first series outside of the summation. That will force the index to start at $n = 2$ instead of $n = 1$.

$$2(1)c_1x^{1-1} + \sum_{n=2}^{\infty} 2nc_nx^{n-1} + \sum_{n=0}^{\infty} 6c_nx^{n+1}$$

$$2c_1 + \sum_{n=2}^{\infty} 2nc_nx^{n-1} + \sum_{n=0}^{\infty} 6c_nx^{n+1}$$

Written this way, both series begin with an x^1 term, so the series are now in phase. But the indices are $n = 2$ and $n = 0$ and don't match. To fix this, we'll look at the exponents of x, setting $k = n - 1$ in the first series and $k = n + 1$ in the second series.

Plugging the index of the first series $n = 2$ into $k = n - 1$ gives $k = 1$, and plugging the index of the second series $n = 0$ into $k = n + 1$ gives $k = 1$. And solving both $k = n - 1$ and $k = n + 1$ for n gives $n = k + 1$ and $n = k - 1$, respectively.

Making all these substitutions, the sum becomes

$$2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + \sum_{k=1}^{\infty} 6c_{k-1}x^k$$

Now that the indices match and the series are in phase, we can add them.

$$2c_1 + \sum_{k=1}^{\infty} 2(k+1)c_{k+1}x^k + 6c_{k-1}x^k$$

$$2c_1 + \sum_{k=1}^{\infty} (2(k+1)c_{k+1} + 6c_{k-1})x^k$$

We can leave this in k, but since the original sum was given in terms of n instead of k, let's put this back in terms of n.

$$2c_1 + \sum_{n=1}^{\infty} (2(n+1)c_{n+1} + 6c_{n-1})x^n$$

Power series solutions

Now that we understand how to add series, we can start looking at series solutions for differential equations.

When a power series solution is defined

Our goal is to use series to solve second order linear homogeneous differential equations.

$$p(x)y'' + q(x)y' + r(x)y = 0$$

And when we find series solutions, we can handle nonconstant coefficients, so $p(x)$, $q(x)$, and $r(x)$ don't need to be constants. Specifically, we're going to look at series solutions when $p(x)$, $q(x)$, and $r(x)$ are polynomial functions.

As long as $q(x)/p(x)$ and $r(x)/p(x)$ are **analytic**, which means that we can define Taylor series for $q(x)/p(x)$ and $r(x)/p(x)$ around $x = x_0$, then we say that $x = x_0$ is an **ordinary point**. As long as we stick to the case where $p(x)$, $q(x)$, and $r(x)$ are polynomials, then $x = x_0$ is an ordinary point and $q(x)/p(x)$ and $r(x)/p(x)$ are analytic as long as $p(x_0) \neq 0$. When $x = x_0$ is non-ordinary, we call it a **singular point** instead.

We can always find two linearly independent power series solutions around an ordinary point, where both solutions are in the form

$$y(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

When $x_0 = 0$, we can simplify these power series solutions to

$$y(x) = \sum_{n=0}^{\infty} c_n x^n$$

The solutions will converge at least on the interval $|x - x_0| < R$, where R is the minimum radius of convergence, and represents the distance from x_0 to the closest singular point.

Keep in mind that this series solution extends to nonhomogeneous differential equations as well,

$$p(x)y'' + q(x)y' + r(x)y = g(x)$$

as long as $g(x)$ is also analytic around $x - x_0$.

Let's do an example so that we understand how to find this minimum radius of convergence for the solutions.

Example

Find the minimum radius of convergence of a power series solution of the differential equation about the ordinary point $x_0 = 1$.

$$(x^2 - 25)y'' + 2xy' + y = 0$$

The singular points of the differential equation occur where $x^2 - 25 = 0$, or at $x = \pm 5$. In the plane, we consider the ordinary point $x_0 = 1$ to occur at $(1,0)$, and the singular points $x = \pm 5$ to occur at $(5,0)$ and $(-5,0)$. The distance from the ordinary point $(1,0)$ to the closer singular point $(5,0)$ is 4, and the distance from the ordinary point $(1,0)$ to the further singular point $(-5,0)$ is 6. So the minimum radius of convergence is

$$|x-1|<4$$

Finding a power series solution

In order to find series solutions, we'll need the derivatives of the power series. The power series in x and its first two derivatives are

$$y=\sum_{n=0}^{\infty} c_n x^n = c_0+c_1x+c_2x^2+c_3x^3+\dots$$

$$y'=\sum_{n=1}^{\infty} c_n n x^{n-1} = c_1+2c_2x+3c_3x^2+4c_4x^3+\dots$$

$$y''=\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2+6c_3x+12c_4x^2+\dots$$

To find a power series solution, we'll substitute these series into our differential equation for y, y', and/or y''. Then we'll use what we learned in the last lesson to add power series by first making sure that they're in phase and have matching indices.

Once we add the series, we want to factor the result and then define the **recurrence relation** equation, which is an equation that defines values in the series in terms of previous values in the series. We'll solve the recurrence relation for the coefficient with the largest subscript, and then we'll use that equation to build out the first few terms of the series.

Finally, we'll put those first few terms together, and work out the pattern they follow to build a simplified power series representation, which will be the general solution to the differential equation.

Let's do an example to see what this looks like.

Example

Find a power series solution to the differential equation $y' = xy$.

$$y = \sum_{n=0}^{\infty} c_n x^n$$

To start, rewrite the differential equation.

$$y' = xy$$

$$y' - xy = 0$$

Next, we'll substitute y and y' into the differential equation.

$$\sum_{n=1}^{\infty} c_n n x^{n-1} - x \sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Let's check to see if the series are in phase. Because the index of the first series starts at $n = 1$, the first series begins with an x^0 term, and because the index of the second series starts at $n = 0$, the second series begins with an x^1 term. So we'll pull the x^0 term out of the first series.

$$c_1 + \sum_{n=2}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Now both series are in phase, but the indices don't match. We can substitute $k = n - 1$ and $n = k + 1$ into the first series, and $k = n + 1$ and $n = k - 1$ into the second series.

$$c_1 + \sum_{k=1}^{\infty} c_{k+1}(k+1)x^k - \sum_{k=1}^{\infty} c_{k-1}x^k = 0$$

With the series in phase and matching indices, we can finally add them.

$$c_1 + \sum_{k=1}^{\infty} (c_{k+1}(k+1)x^k - c_{k-1}x^k) = 0$$

$$c_1 + \sum_{k=1}^{\infty} (c_{k+1}(k+1) - c_{k-1})x^k = 0$$

The c_1 value in front of the series is associated with the $k = 0$ term, while all of the other $k = 1, 2, 3, \ldots$ terms are still in the series.

$$k = 0 \qquad\qquad c_1 = 0$$

$$k = 1, 2, 3, \ldots \qquad\qquad c_{k+1}(k+1) - c_{k-1} = 0$$

The equation $c_{k+1}(k+1) - c_{k-1} = 0$ is the recurrence relation, and we always want to start by solving it for the coefficient with the largest subscript. This recurrence relation includes c_{k+1} and c_{k-1}, and since $k+1 > k-1$, we'll solve the equation for c_{k+1}.

$$c_{k+1} = \frac{c_{k-1}}{k+1}$$

Now we'll start plugging in values $k = 1, 2, 3, \ldots$.

$$k=1 \qquad c_2 = \frac{c_0}{2} \qquad\qquad k=2 \qquad c_3 = \frac{c_1}{3}$$

$$k=3 \qquad c_4 = \frac{c_2}{4} = \frac{c_0}{2\cdot 4} \qquad\qquad k=4 \qquad c_5 = \frac{c_3}{5} = \frac{c_1}{3\cdot 5}$$

$$k=5 \qquad c_6 = \frac{c_4}{6} = \frac{c_0}{2\cdot 4\cdot 6} \qquad\qquad k=6 \qquad c_7 = \frac{c_5}{7} = \frac{c_1}{3\cdot 5\cdot 7}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$c_{2k} = \frac{c_0}{(2)(4)(6)\ldots(2k-1)(2k)} \qquad\qquad c_{2k+1} = \frac{c_1}{(3)(5)(7)\ldots(2k)(2k+1)}$$

for $k = 1, 2, 3, \ldots$ for $k = 1, 2, 3, \ldots$

But because we know $c_1 = 0$, all the terms on the right side of our table that are defined in terms of c_1 (all of the even values of k) will go to 0.

$$c_1 = 0$$

$$k=1 \qquad c_2 = \frac{c_0}{2} \qquad\qquad k=2 \qquad c_3 = 0$$

$$k=3 \qquad c_4=\frac{c_2}{4}=\frac{c_0}{2\cdot 4} \qquad\qquad k=4 \qquad c_5=0$$

$$k=5 \qquad c_6=\frac{c_4}{6}=\frac{c_0}{2\cdot 4\cdot 6} \qquad\qquad k=6 \qquad c_7=0$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$c_{2k}=\frac{c_0}{(2)(4)(6)\dots(2k-2)(2k)} \qquad\qquad c_{2k+1}=0$$

for $k=1, 2, 3, \dots$ for $k=1, 2, 3, \dots$

Using these values, the solution is

$$y=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+c_6x^6+c_7x^7+\dots+c_{2k}x^{2k}+c_{2k+1}x^{2k+1}$$

$$y=c_0+\frac{c_0}{2}x^2+\frac{c_0}{2\cdot 4}x^4+\frac{c_0}{2\cdot 4\cdot 6}x^6+\dots+\frac{c_0}{(2)(4)(6)\dots(2k-2)(2k)}x^{2k}$$

Factor out the c_0 coefficient.

$$y=c_0\left(1+\frac{1}{2}x^2+\frac{1}{2\cdot 4}x^4+\frac{1}{2\cdot 4\cdot 6}x^6+\dots+\frac{1}{(2)(4)(6)\dots(2k-2)(2k)}x^{2k}\right)$$

To figure out how to represent the series, we realize that we have a value of 2 multiplied through the factorial part of the denominator, so we can pull that out, representing it as 2^k.

$$y=c_0\left(1+\frac{1}{2^1(1)}x^2+\frac{1}{2^2(1\cdot 2)}x^4+\frac{1}{2^3(1\cdot 2\cdot 3)}x^6\right.$$

$$\left.+\dots+\frac{1}{2^k(1\cdot 2\cdot 3\cdot \dots \cdot (k-1)\cdot k)}x^{2k}\right)$$

Then we'll replace the factorial portion with $k!$, and we know also that every x^2, x^4, x^6, ... term can be represented by x^{2k}, so

$$y = c_0 \left(1 + \frac{1}{2^1(1)}x^2 + \frac{1}{2^2(1 \cdot 2)}x^4 + \frac{1}{2^3(1 \cdot 2 \cdot 3)}x^6 + \ldots + \frac{1}{2^k \cdot k!}x^{2k} \right)$$

And the pattern that seems to be emerging is

$$y = c_0 \sum_{k=0}^{\infty} \frac{1}{2^k k!} x^{2k}$$

This series representation seems to work for all values $k = 1, 2, 3, \ldots$, but let's make sure it also works for the $k = 0$ term. If we plug $k = 0$ into this series representation, we get

$$\frac{1}{2^0 0!} x^{2(0)} = \frac{1}{(1)(1)}(1) = 1$$

This representation works for every term in the series, so let's just rewrite it in a simpler way to give the general solution to the differential equation.

$$y = c_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^{2k}}{2^k} \right)$$

$$y = c_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{2} \right)^k$$

Equivalent solution from first principles

The power series solution we found in this last example looks a little complicated, but it's actually equivalent to the solution we would've found if we'd solved the differential equation using separation of variables.

$$\frac{dy}{dx}=xy$$

$$\int \frac{1}{y}\ dy=\int x\ dx$$

$$\ln|y|=\frac{1}{2}x^2+C$$

$$|y|=e^{\frac{1}{2}x^2+C}$$

$$y=Ce^{\frac{1}{2}x^2}$$

The Maclaurin series for e^x is

$$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

In the solution $y=Ce^{\frac{1}{2}x^2}$, we have $e^{\frac{1}{2}x^2}$ instead of e^x, so let's replace x with $(1/2)x^2$ in the formula for the Maclaurin series representation of e^x.

$$e^{\frac{1}{2}x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}x^2\right)^n$$

$$e^{\frac{1}{2}x^2}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{2}\right)^n(x^2)^n$$

$$e^{\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{1}{2^n} \right) (x^{2n})$$

$$e^{\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^{2n}}{2^n} \right)$$

$$e^{\frac{1}{2}x^2} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^2}{2} \right)^n$$

If we plug this back into the general solution $y = Ce^{\frac{1}{2}x^2}$, we get

$$y = C \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{x^2}{2} \right)^n$$

We can see that this series matches exactly the series we found in the last example,

$$y = c_0 \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{x^2}{2} \right)^k$$

In other words, we're finding the same general solution to the differential equation that we would have found using other methods, we're just now expressing the solution as a series.

Nonpolynomial coefficients

Sometimes we'll want to find a series solution to a differential equation with nonpolynomial coefficients.

In this case, we'll use a slightly different method to find the series solution, which will have us using power series multiplication.

Let's do an example so that we can see what the process looks like.

Example

Find the first three terms of the power series solutions of the differential equation around the ordinary point $x_0 = 0$.

$$y'' + (\sin x)y = 0$$

We know $x_0 = 0$ is an ordinary point, because $\sin x$ is analytic there. And we know that the Maclaurin series representation of $\sin x$ is

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots$$

So we can substitute into the differential equation to get

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots\right) \sum_{n=0}^{\infty} c_n x^n = 0$$

Now we'll just expand each series through its first few terms,

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots)$$

$$+\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \dots\right)(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots) = 0$$

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots)$$

$$+x(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots)$$

$$-\frac{1}{3!}x^3(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots)$$

$$+\frac{1}{5!}x^5(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots)$$

$$-\frac{1}{7!}x^7(c_0 + c_1x + c_2x^2 + c_3x^3 + \dots) + \dots = 0$$

$$(2c_2 + 6c_3x + 12c_4x^2 + 20c_5x^3 + \dots)$$

$$+(c_0x + c_1x^2 + c_2x^3 + c_3x^4 + \dots)$$

$$-\left(\frac{1}{3!}c_0x^3 + \frac{1}{3!}c_1x^4 + \frac{1}{3!}c_2x^5 + \frac{1}{3!}c_3x^6 + \dots\right)$$

$$+\left(\frac{1}{5!}c_0x^5 + \frac{1}{5!}c_1x^6 + \frac{1}{5!}c_2x^7 + \frac{1}{5!}c_3x^8 + \dots\right)$$

$$-\left(\frac{1}{7!}c_0x^7 + \frac{1}{7!}c_1x^8 + \frac{1}{7!}c_2x^9 + \frac{1}{7!}c_3x^{10} + \dots\right) + \dots = 0$$

and then collect equivalent powers of x.

$$2c_2 + (6c_3 + c_0)x + (12c_4 + c_1)x^2$$

$$+\left(20c_5+c_2-\frac{1}{3!}c_0\right)x^3+\left(30c_6+c_3-\frac{1}{3!}c_1\right)x^4$$

$$+\left(42c_7+c_4-\frac{1}{3!}c_2+\frac{1}{5!}c_0\right)x^5+\left(56c_8+c_5-\frac{1}{3!}c_3+\frac{1}{5!}c_1\right)x^6+\ldots=0$$

From this result, we get a system of equations.

$$2c_2=0$$

$$6c_3+c_0=0$$

$$12c_4+c_1=0$$

$$20c_5+c_2-\frac{1}{3!}c_0=0$$

$$30c_6+c_3-\frac{1}{3!}c_1=0$$

$$42c_7+c_4-\frac{1}{3!}c_2+\frac{1}{5!}c_0=0$$

$$56c_8+c_5-\frac{1}{3!}c_3+\frac{1}{5!}c_1=0$$

which simplifies to

$$c_2=0$$

$$c_3=-\frac{1}{6}c_0 \qquad c_4=-\frac{1}{12}c_1$$

$$c_5=\frac{1}{120}c_0-\frac{1}{20}c_2 \qquad c_6=\frac{1}{180}c_1-\frac{1}{30}c_3$$

$$c_7 = -\frac{1}{5{,}040}c_0 + \frac{1}{252}c_2 - c_4 \qquad c_8 = -\frac{1}{6{,}720}c_1 + \frac{1}{336}c_3 - c_5$$

Using earlier terms to simplify later terms, we get

$$c_2 = 0$$

$$c_3 = -\frac{1}{6}c_0 \qquad c_4 = -\frac{1}{12}c_1$$

$$c_5 = \frac{1}{120}c_0 \qquad c_6 = \frac{1}{180}c_0 + \frac{1}{180}c_1$$

$$c_7 = -\frac{1}{5{,}040}c_0 + \frac{1}{12}c_1 \qquad c_8 = -\frac{89}{10{,}080}c_0 - \frac{1}{6{,}720}c_1$$

Pull these coefficients together into the power series solution.

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + \dots$$

$$y = c_0 + c_1x - \frac{1}{6}c_0x^3 - \frac{1}{12}c_1x^4 + \frac{1}{120}c_0x^5 + \left(\frac{1}{180}c_0 + \frac{1}{180}c_1\right)x^6$$

$$-\left(\frac{1}{5{,}040}c_0 - \frac{1}{12}c_1\right)x^7 - \left(\frac{89}{10{,}080}c_0 + \frac{1}{6{,}720}c_1\right)x^8 + \dots$$

$$y = c_0 - \frac{1}{6}c_0x^3 + \frac{1}{120}c_0x^5 + \frac{1}{180}c_0x^6 - \frac{1}{5{,}040}c_0x^7 - \frac{89}{10{,}080}c_0x^8$$

$$+c_1x - \frac{1}{12}c_1x^4 + \frac{1}{180}c_1x^6 + \frac{1}{12}c_1x^7 - \frac{1}{6{,}720}c_1x^8 + \dots$$

$$y = c_0\left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5{,}040}x^7 - \frac{89}{10{,}080}x^8 + \dots\right)$$

$$+c_1\left(x-\frac{1}{12}x^4+\frac{1}{180}x^6+\frac{1}{12}x^7-\frac{1}{6{,}720}x^8+\dots\right)$$

Simplifying to just the first three terms in each series, we get the general solution to the differential equation.

$$y(x)=c_0\left(1-\frac{1}{6}x^3+\frac{1}{120}x^5+\dots\right)+c_1\left(x-\frac{1}{12}x^4+\frac{1}{180}x^6+\dots\right)$$

Singular points and Frobenius' Theorem

We already know from our discussion so far about power series solutions that a singular point for the differential equation

$$p(x)y'' + q(x)y' + r(x)y = 0$$

occurs wherever $p(x) = 0$. But singular points can actually be classified into two groups: regular and irregular.

A **regular singular point** occurs when both of these expressions are analytic at x_0:

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)}$$

Otherwise, if one or both of these expressions are not analytic at x_0, then the singular point is an **irregular singular point**.

Frobenius' Theorem

To find a series solution to a differential equation around a regular singular point, we can use **Frobenius' Theorem**, which says that there is at least one solution to $p(x)y'' + q(x)y' + r(x)y = 0$ of the form

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n(x - x_0)^n = \sum_{n=0}^{\infty} c_n(x - x_0)^{n+r}$$

where r is a constant that we'll need to find. If the regular singular point is $x_0 = 0$, then of course this solution equation simplifies to

$$y = x^r \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^{n+r}$$

So the **method of Frobenius** will have us substituting this solution equation and its derivatives

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2}$$

into the differential equation, and then simplifying and equating coefficients to find the **indicial equation**. The indicial equation will be a quadratic equation, which means that the **indicial roots** will be its solutions r_1 and r_2. There are three relationships between these roots that we want to consider.

Case I: When $r_1 > r_2$ and $r_1 - r_2$ is not a positive integer, then the differential equation has two linearly independent solutions,

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \text{ with } c_0 \neq 0$$

$$y_2(x) = \sum_{n=0}^{\infty} b_n x^{n+r_2} \text{ with } b_0 \neq 0$$

Case II: When $r_1 > r_2$ and $r_1 - r_2$ is a positive integer, then the differential equation has two linearly independent solutions,

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \text{ with } c_0 \neq 0$$

$$y_2(x) = Cy_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_2} \text{ with } b_0 \neq 0$$

It's possible in this case that $C = 0$, which would zero out the first term, and the second solution $y_2(x)$ would not contain a logarithm.

Case III: But when $r_1 = r_2$, the second solution does not include the constant C, and therefore the second solution will definitely contain the logarithm.

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \text{ with } c_0 \neq 0$$

$$y_2(x) = y_1(x)\ln x + \sum_{n=0}^{\infty} b_n x^{n+r_1} \text{ with } b_0 \neq 0$$

Let's do an example where we use the method of Frobenius to try to find a series solution to the differential equation around a regular singular point. This will be a Case I example, which means we should expect to find two linearly independent solutions that we can sum to find the general solution.

Example

Determine the regularity of the singular point $x_0 = 0$ of the differential equation, use the method of Frobenius to build any solution(s) around that point, then find the general solution.

$$2xy'' - (3 + 2x)y' + y = 0$$

Matching this differential equation to the standard form $p(x)y'' + q(x)y' + r(x)y = 0$, we can identify

$$p(x) = 2x$$

$$q(x) = -(3 + 2x)$$

$$r(x) = 1$$

Using these three functions to calculate $Q(x)$ and $R(x)$ gives

$$Q(x) = (x - x_0)\frac{q(x)}{p(x)} = (x)\frac{-(3 + 2x)}{2x} = -\frac{3 + 2x}{2}$$

$$R(x) = (x - x_0)^2\frac{r(x)}{p(x)} = x^2\frac{1}{2x} = \frac{x}{2}$$

Because both denominators simplify to constants, we can see that both $Q(x)$ and $R(x)$ are analytic about $x_0 = 0$, so $x_0 = 0$ is a regular singular point.

Now we'll use

$$y = \sum_{n=0}^{\infty} c_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2}$$

to make substitutions into the differential equation.

$$2xy'' - (3+2x)y' + y = 0$$

$$2x \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{n+r-2} - (3+2x) \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r-1} - 3 \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1} - 2x \sum_{n=0}^{\infty} (n+r) c_n x^{n+r-1}$$

$$+ \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty} 2(n+r)(n+r-1) c_n x^{n+r-1} - \sum_{n=0}^{\infty} 3(n+r) c_n x^{n+r-1}$$

$$- \sum_{n=0}^{\infty} 2(n+r) c_n x^{n+r} + \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

Combine terms with equivalent powers of x.

$$\sum_{n=0}^{\infty} [2(n+r)(n+r-1) c_n - 3(n+r) c_n] x^{n+r-1}$$

$$+\sum_{n=0}^{\infty}[c_n - 2(n+r)c_n]x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}[(2(n+r-1)-3)(n+r)c_n]x^{n+r-1} + \sum_{n=0}^{\infty}[(1-2(n+r))c_n]x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}[(2n+2r-2-3)(n+r)c_n]x^{n+r-1} + \sum_{n=0}^{\infty}[(1-2n-2r)c_n]x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(2n+2r-5)(n+r)c_nx^{n+r-1} + \sum_{n=0}^{\infty}(1-2n-2r)c_nx^{n+r} = 0$$

Factor x^r out of the left side of the equation.

$$x^r\left[\sum_{n=0}^{\infty}(2n+2r-5)(n+r)c_nx^{n-1} + \sum_{n=0}^{\infty}(1-2n-2r)c_nx^n\right] = 0$$

Now we need to make sure these two series are in phase. The first series starts with the x^{-1} term, while the second series starts with the x^0 term. So we'll pull the x^{-1} term out of the first series to put them in phase.

$$x^r\left[r(2r-5)c_0x^{-1} + \sum_{n=1}^{\infty}(2n+2r-5)(n+r)c_nx^{n-1}\right.$$

$$\left.+\sum_{n=0}^{\infty}(1-2n-2r)c_nx^n\right] = 0$$

Now that the series are in phase, we'll substitute $k = n-1$ and $n = k+1$ into the first series, and $k = n$ into the second series.

$$x^r \left[r(2r-5)c_0 x^{-1} + \sum_{k=0}^{\infty} (2k+2r-3)(k+r+1)c_{k+1}x^k \right.$$

$$\left. - \sum_{k=0}^{\infty} (2k+2r-1)c_k x^k \right] = 0$$

Now that the series are in phase with matching indices, combine them.

$$x^r \left[r(2r-5)c_0 x^{-1} \right.$$

$$\left. + \sum_{k=0}^{\infty} [(2k+2r-3)(k+r+1)c_{k+1} - (2k+2r-1)c_k]x^k \right] = 0$$

This equation gives

$$r(2r-5)c_0 = 0$$

$$(2k+2r-3)(k+r+1)c_{k+1} - (2k+2r-1)c_k = 0 \qquad k = 0, 1, 2, \ldots$$

or

$$r(2r-5) = 0$$

$$c_{k+1} = \frac{(2k+2r-1)c_k}{(2k+2r-3)(k+r+1)} \qquad k = 0, 1, 2, \ldots$$

The indicial equation gives us $r_1 = 0$ and $r_2 = 5/2$. Substituting these indicial roots into the recurrence relation, we get

For $r_1 = 0$ For $r_2 = 5/2$

	$c_{k+1} = \dfrac{(2k-1)c_k}{(2k-3)(k+1)}$	$c_{k+1} = \dfrac{(k+2)c_k}{(k+1)(k+\frac{7}{2})}$
$k=0$	$c_1 = \dfrac{1}{3}c_0$	$c_1 = \dfrac{2 \cdot 2}{7}c_0$
$k=1$	$c_2 = -\dfrac{c_1}{2} = -\dfrac{1}{6}c_0$	$c_2 = \dfrac{(2)(3)c_1}{(2)(9)} = \dfrac{2^2 \cdot 3}{9 \cdot 7}c_0$
$k=2$	$c_3 = c_2 = -\dfrac{1}{6}c_0$	$c_3 = \dfrac{(2)(4)c_2}{(3)(11)} = \dfrac{2^3 \cdot 4}{11 \cdot 9 \cdot 7}c_0$
...	...	...

Forming these coefficients into series around $x_0 = 0$ gives,

$$c_0(x-x_0)^0 + c_1(x-x_0)^1 + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \dots$$

$$c_0(x-0)^0 + \frac{1}{3}c_0(x-0)^1 - \frac{1}{6}c_0(x-0)^2 - \frac{1}{6}c_0(x-0)^3 - \dots$$

$$c_0\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \dots\right)$$

and

$$c_0(x-x_0)^0 + c_1(x-x_0)^1 + c_2(x-x_0)^2 + c_3(x-x_0)^3 + \dots$$

$$c_0(x-0)^0 + \frac{2 \cdot 2}{7}c_0(x-0)^1 + \frac{2^2 \cdot 3}{9 \cdot 7}c_0(x-0)^2 + \frac{2^3 \cdot 4}{11 \cdot 9 \cdot 7}c_0(x-0)^3 + \dots$$

$$c_0\left(1 + \frac{2 \cdot 2}{7}x + \frac{2^2 \cdot 3}{9 \cdot 7}x^2 + \frac{2^3 \cdot 4}{11 \cdot 9 \cdot 7}x^3 + \dots\right)$$

so the series solutions are

$$y_1(x) = c_0x^0\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \dots\right)$$

$$y_2(x) = c_0x^{\frac{5}{2}}\left(1 + \frac{2 \cdot 2}{7}x + \frac{2^2 \cdot 3}{9 \cdot 7}x^2 + \frac{2^3 \cdot 4}{11 \cdot 9 \cdot 7}x^3 + \dots\right)$$

and the general solution on the interval $(0,\infty)$ is

$$y(x) = C_1y_1(x) + C_2y_2(x)$$

$$y(x) = C_1c_0\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \dots\right)$$

$$+C_2c_0x^{\frac{5}{2}}\left(1 + \frac{2 \cdot 2}{7}x + \frac{2^2 \cdot 3}{9 \cdot 7}x^2 + \frac{2^3 \cdot 4}{11 \cdot 9 \cdot 7}x^3 + \dots\right)$$

But C_1c_0 and C_2c_0 are constants, so we can simplify them and write the general solution as

$$y(x) = C_1\left(1 + \frac{1}{3}x - \frac{1}{6}x^2 - \frac{1}{6}x^3 - \dots\right)$$

$$+C_2x^{\frac{5}{2}}\left(1 + \frac{2 \cdot 2}{7}x + \frac{2^2 \cdot 3}{9 \cdot 7}x^2 + \frac{2^3 \cdot 4}{11 \cdot 9 \cdot 7}x^3 + \dots\right)$$

The Laplace transform

Like many of the methods we've learned previously, the Laplace transform is just another method we can use to find the solution to a differential equation. Therefore, as always, our first priority is to understand when it's appropriate to use a Laplace transform, instead of one of the other techniques we already know.

For first and second order equations that are homogeneous (the forcing function on the right side of the equation is 0), it'll usually be easier to use the solution techniques we learned earlier. And even when the equation is nonhomogeneous, if the forcing function is simple enough, it probably won't be any faster or easier to apply a Laplace transform.

But Laplace transforms start to become more effective than other solution methods when the forcing function gets more complicated, and especially when the forcing function is discontinuous.

The definition of the Laplace transform

Think about a transform in general as some kind of operation that can be applied to a function that changes it in some kind of predictable way.

The derivative and the integral are transforms from calculus. Differentiating gives us a predictable function that models slope, while integrating gives us a predictable function that models area.

The Laplace transform is a specific integral transform. If we want to indicate the transform of a function $f(t)$, we write the Laplace transform as

$\mathscr{L}(f(t))$. Taking the transform of $f(t)$ results in a predictable new function $F(s)$.

$$\mathscr{L}(f(t)) = F(s)$$

Keep in mind that we won't always use f to name the function we're transforming. For instance, it's also common to use y, in which case we might write the transform equation as

$$\mathscr{L}(y(x)) = Y(s)$$

In either case, the original function is defined in terms of some independent variable like t, x, etc.. When we apply the transform, that original variable will drop away, and the transform will be defined in terms of a new variable, s. And the transform function is named with a capital letter, such that if the original function was g, its transform is named G.

To calculate a function's transform, we can use the definition of the Laplace transform, which is

$$\mathscr{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

This tells us that, given a function $f(t)$, we can find its transform $F(s)$ by multiplying $f(t)$ by e^{-st}, and then integrating that product on the interval $[0,\infty)$.

Let's do an example where we apply the definition in order to find the transform of a function.

Example

Use the definition to find the Laplace transform of the function.

$$f(t) = e^{-2t}$$

Plugging the given function into the definition of the Laplace transform gives

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

$$F(s) = \int_0^\infty e^{-st} e^{-2t}\ dt$$

We can use the power rule for exponents to combine the exponentials.

$$F(s) = \int_0^\infty e^{-st-2t}\ dt$$

$$F(s) = \int_0^\infty e^{(-s-2)t}\ dt$$

Since we're integrating only with respect to t, we can treat s as a constant, which means the integral will be

$$F(s) = \frac{1}{-s-2} e^{(-s-2)t} \Big|_0^\infty$$

$$F(s) = \lim_{b\to\infty} \frac{1}{-s-2} e^{(-s-2)t} \Big|_0^b$$

Evaluate over the interval.

$$F(s) = \lim_{b \to \infty} \left(\frac{1}{-s-2} e^{(-s-2)b} \right) - \frac{1}{-s-2} e^{(-s-2)0}$$

$$F(s) = \lim_{b \to \infty} \left(\frac{1}{-s-2} e^{(-s-2)b} \right) + \frac{1}{s+2}$$

From here, the value we find for the exponential will depend on the value of $-s-2$.

- If $-s-2>0$, the value of the exponential goes to ∞, and $F(s)$ diverges

- If $-s-2<0$, the value of the exponential goes to 0, and

$$F(s) = \frac{1}{s+2}$$

Because we only get a convergent value when $-s-2<0$, we'll assume

$$-s-2<0$$

$$-s<2$$

$$s>-2$$

in order to conclude that the Laplace transform of $f(t) = e^{-2t}$ is

$$F(s) = \frac{1}{s+2} \qquad \text{given } s>-2$$

At the end of this last example, we assumed $s > -2$ in order to find the Laplace transform of $f(t) = e^{-2t}$. It's important to remember that all Laplace transforms will have these kinds of restrictions on s.

Because these restrictions always apply, they become part of our common understanding of Laplace transforms. Therefore, we won't always take the time to point out the restriction, and instead we'll just assume that we're all agreeing to use whatever restriction on s is required to make the transform function converge to a finite value.

Still, we'll look at this restriction more later on when we cover functions of exponential type.

An alternate definition of the transform

Sometimes we'll see a slightly different definition of the Laplace transform, written as

$$\mathscr{L}(f(t)) = F(s) = \int_{-\infty}^{\infty} e^{-st}f(t)\ dt$$

The only change to the original definition is a substitution in the lower limit. Instead of integrating on $[0,\infty)$, this new definition integrates on $(-\infty,\infty)$. If we use this definition with the $(-\infty,\infty)$ interval of integration, we're almost always assuming that the definition of the function $f(t)$ is given by

$$f(t) = \begin{cases} f(t) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Because we're assuming $f(t) = 0$ whenever $t < 0$, the value of the integral on $(-\infty,0)$ is 0. And by dropping that entire half of the integral to 0, we essentially just get back to the same definition of the transform that we started with.

We'll usually use a Heaviside function to force the behavior that $f(t) = 0$ whenever $t < 0$, but we'll talk more about that later. For now, it's just important that we know that these two definitions of the Laplace transform are essentially identical (because of the assumed value of $f(t)$), even though they appear to be slightly different.

Table of transforms

While it's true that we can always use the definition to find the Laplace transform of a function,

$$F(s) = \int_0^\infty e^{-st} f(t) \ dt$$

we don't want to be forced to evaluate this integral again and again, repeating these same steps for the same function. For that reason, we'll often use a table of Laplace transforms for common functions, like the table below.

Table of Laplace transforms

For instance, the first formula in the table tells us that the Laplace transform of the function $f(t) = 1$ is $F(s) = 1/s$.

The third formula in the table tells us that the Laplace transform of $f(t) = t^n$ will be $F(s) = n!/s^{n+1}$, as long as $n = 1,2,3,...$ So if we're trying to transform $f(t) = t^3$, then $n = 3$ and we know from the table that the transform will be

$$\mathscr{L}(t^3) = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Many of the formulas in this table may seem unfamiliar at this point, but we'll get to many of them later on. For now, this will just serve as a resource that we can refer back to as we learn about Laplace transforms.

$$\mathscr{L}(f(t)) = F(s)$$

$$\mathscr{L}(1) = \frac{1}{s}$$

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}} \qquad n = 1,\ 2,\ 3,\ \dots$$

$$\mathscr{L}(t^p) = \frac{\Gamma(p+1)}{s^{p+1}} \text{ for } p > -1 \qquad \Gamma(t) = \int_0^\infty e^{-x}x^{t-1}\ dx$$

$$\mathscr{L}(\sqrt{t}) = \frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$$

$$\mathscr{L}(t^{n-\frac{1}{2}}) = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}} \qquad n = 1,2,3,...$$

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2}$$

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2+a^2}$$

$$\mathscr{L}(t\sin(at)) = \frac{2as}{(s^2+a^2)^2}$$

$$\mathscr{L}(t\cos(at)) = \frac{s^2-a^2}{(s^2+a^2)^2}$$

$$\mathscr{L}(\sin(at) - at\cos(at)) = \frac{2a^3}{(s^2+a^2)^2}$$

$$\mathscr{L}(\sin(at) + at\cos(at)) = \frac{2as^2}{(s^2+a^2)^2}$$

$$\mathscr{L}(\cos(at) - at\sin(at)) = \frac{s(s^2-a^2)}{(s^2+a^2)^2}$$

$$\mathscr{L}(\cos(at) + at\sin(at)) = \frac{s(s^2+3a^2)}{(s^2+a^2)^2}$$

$$\mathscr{L}(\sin(at+b)) = \frac{s\sin b + a\cos b}{s^2+a^2}$$

$$\mathscr{L}(\cos(at+b)) = \frac{s\cos b - a\sin b}{s^2+a^2}$$

$$\mathscr{L}(\sinh(at)) = \frac{a}{s^2-a^2}$$

$$\mathscr{L}(\cosh(at)) = \frac{s}{s^2-a^2}$$

$$\mathscr{L}(e^{at}\sin(bt)) = \frac{b}{(s-a)^2+b^2}$$

$$\mathscr{L}(e^{at}\cos(bt)) = \frac{s-a}{(s-a)^2+b^2}$$

$$\mathscr{L}(e^{at}\sinh(bt)) = \frac{b}{(s-a)^2-b^2}$$

$$\mathscr{L}(e^{at}\cosh(bt)) = \frac{s-a}{(s-a)^2-b^2}$$

$$\mathscr{L}(t^n e^{at}) = \frac{n!}{(s-a)^{n+1}}$$

$n = 1,\ 2,\ 3,\ \dots$

$$\mathscr{L}(f(ct)) = \frac{1}{c}F\left(\frac{s}{c}\right)$$

$$\mathscr{L}(u_c(t)) = \mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

Heaviside function

$$\mathscr{L}(\delta(t-c)) = e^{-cs}$$

Dirac delta function

$$\mathscr{L}(u_c(t)f(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(u_c(t)g(t)) = e^{-cs}\mathscr{L}(g(t+c))$$

$$\mathscr{L}(e^{ct}f(t)) = F(s-c)$$

$$\mathscr{L}(t^n f(t)) = (-1)F^{(n)}(s)$$

$n = 1,2,3,\dots$

$$\mathscr{L}\left(\frac{1}{t}f(t)\right) = \int_s^\infty F(u)\ du$$

$$\mathscr{L}\left(\int_0^t f(v)\ dv\right)=\frac{F(s)}{s}$$

$$\mathscr{L}\left(\int_0^t f(t-\tau)g(\tau)\ d\tau\right)=F(s)G(s)$$

$$\mathscr{L}(f(t+T)=f(t))=\frac{\int_0^T e^{-st}f(t)\ dt}{1-e^{-sT}}$$

$$\mathscr{L}(f'(t))=sF(s)-f(0)$$

$$\mathscr{L}(f''(t))=s^2F(s)-sf(0)-f'(0)$$

$$\mathscr{L}(f^{(n)}(t))=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\ldots-sf^{n-2}(0)-f^{n-1}(0)$$

Keep in mind that, given a combination (sum or difference) of functions, we can transform each function individually. In addition, constant coefficients don't affect the value of a transform. We can translate these facts into formulas:

$$\mathscr{L}(af(t))=aF(s)$$

$$\mathscr{L}(f(t)+g(t))=F(s)+G(s)$$

Let's do a simple example so that we can see how to start matching values in our function to formulas in the table, and then use the formulas to quickly find the transform of the function.

Example

Use a table to find the Laplace transform of the function.

$$f(t) = e^{2t} - \sin(4t) + t^7$$

Because that the transform of a sum is the sum of the transforms, we can transform each term in the function separately. Therefore, our goal will be to find formulas in the table to individually transform e^{2t}, $\sin(4t)$, and t^7.

The transforms we need to use are

$$\mathscr{L}(e^{at}) = \frac{1}{s-a} \qquad \text{with } a = 2$$

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2} \qquad \text{with } a = 4$$

$$\mathscr{L}(t^n) = \frac{n!}{s^{n+1}} \qquad \text{with } n = 7$$

When we apply these transform formulas to the terms in our function, we get

$$\mathscr{L}(e^{2t}) = \frac{1}{s-2}$$

$$\mathscr{L}(\sin(4t)) = \frac{4}{s^2+4^2} = \frac{4}{s^2+16}$$

$$\mathscr{L}(t^7) = \frac{7!}{s^{7+1}} = \frac{7!}{s^8}$$

Replacing the terms in the original function with their transforms, we can say that the Laplace transform of the function $f(t)$ is

$$F(s) = \frac{1}{s-2} - \frac{4}{s^2+16} + \frac{7!}{s^8}$$

Exponential type

When we talked earlier about the definition of the Laplace transform, we worked an example to find the Laplace transform of $f(t) = e^{-2t}$, and we found

$$F(s) = \frac{1}{s+2} \qquad \text{given } s > -2$$

Without this restriction on the value of s, the Laplace integral diverges, and the transform $F(s)$ is therefore undefined.

At this point, we want to define more clearly some conditions that are sufficient for guaranteeing the existence of the Laplace transform.

Specifically, if we can show that a function $f(t)$ 1) is piecewise continuous on $[0,\infty)$ and 2) of exponential order, then we know that the Laplace transform $\mathscr{L}(f(t)) = F(s)$ exists for $s > \alpha$.

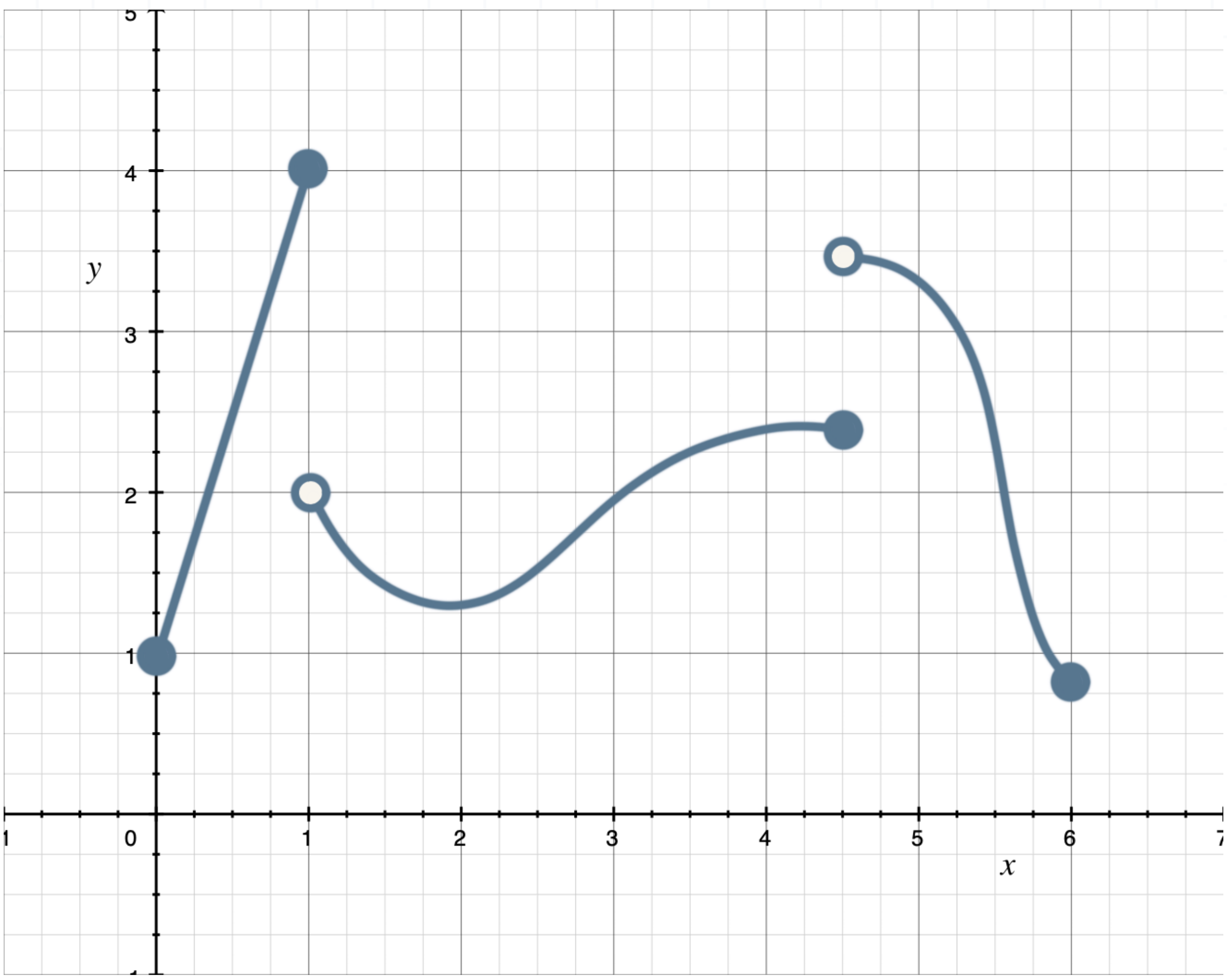
These two conditions aren't necessary for a Laplace transform to be defined. There are some functions that aren't piecewise continuous, but still have a defined Laplace transform. And there are some functions that aren't of exponential order, but still have a defined Laplace transform.

So these conditions aren't necessary, but they are sufficient. If a function meets these two conditions, then we know its Laplace transform exists. It's helpful to know that these two conditions are sufficient for the existence of the Laplace transform, because most of the functions we'll deal with throughout these transform lessons meet both of these conditions.

Piecewise continuous functions

A function is **piecewise continuous** on $[0,\infty)$ if it includes a finite number of finite discontinuities (none of the discontinuities are infinite discontinuities, like at a vertical asymptote), and is otherwise continuous between these finite discontinuities.

Here's an example of a piecewise continuous function:

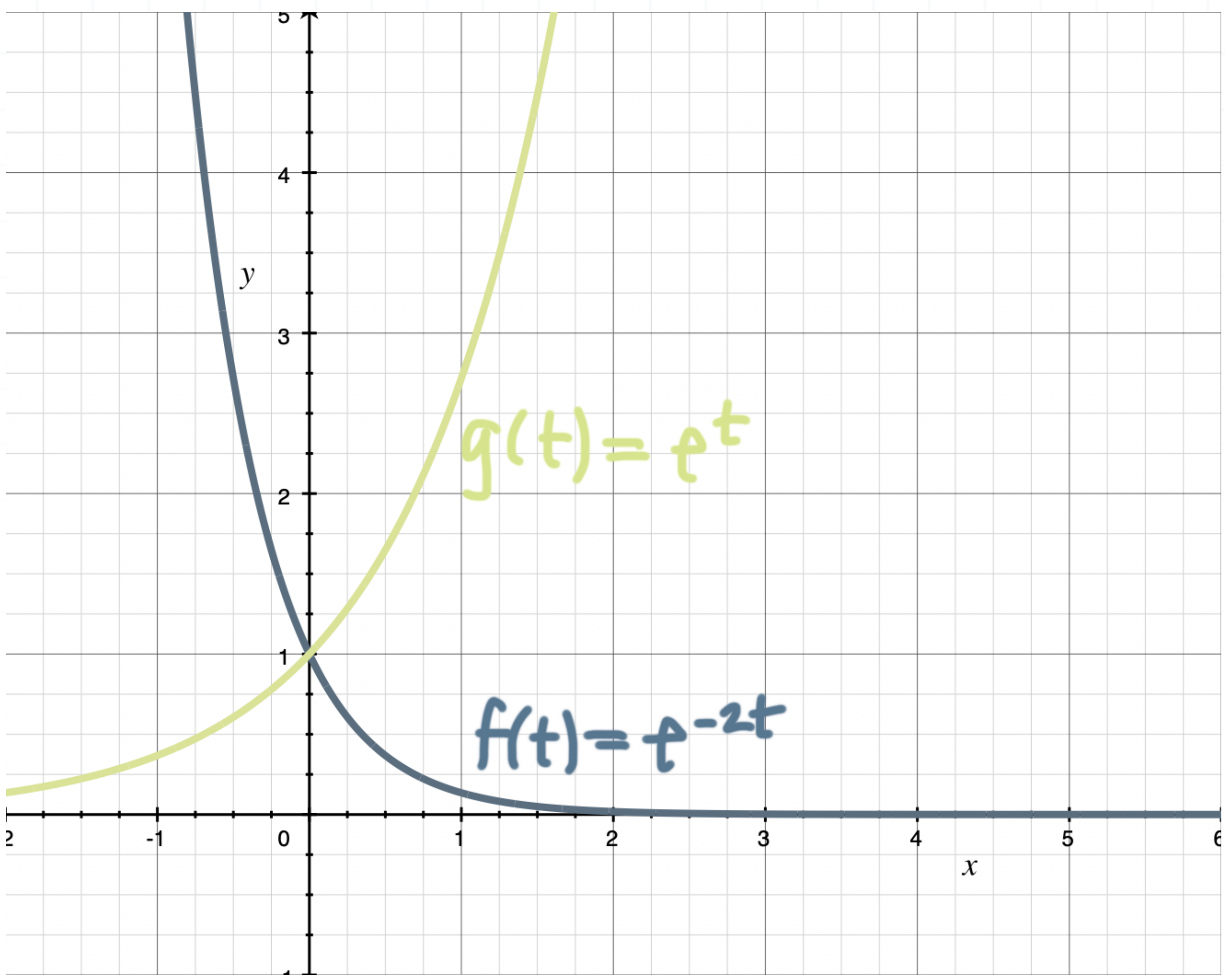


Exponential type (order)

Formally, we say that a function f is of **exponential type**, or of **exponential order** α, if there exist constants α, $M > 0$, $T > 0$ such that $|f(t)| \leq Me^{\alpha t}$ for all $t > T$.

But really all this means is that $f(t)$ is of exponential type if we can find any $e^{\alpha t}$ that grows faster than $f(t)$, causing the value of the fraction $f(t)/e^{\alpha t}$ to converge as t gets infinitely large.

We started this lesson with the function $f(t) = e^{-2t}$. We know this function is of exponential type because the function $g(t) = e^t$ grows faster than $f(t) = e^{-2t}$.



In this case, we chose $\alpha = 1$, $M = 1$, and $T > 0$ for all $t > 0$. So

$$|f(t)| \leq Me^{\alpha t}$$

$$|e^{-2t}| \leq 1e^{1t}$$

$$e^{-2t} \leq e^t$$

Because this inequality holds for all $t > 0$, we know $f(t) = e^{-2t}$ is of exponential type. Because we chose $\alpha = 1$, we can say that $f(t) = e^{-2t}$ is of exponential order 1, and we can conclude that the Laplace transform will exist for all $s > \alpha = 1$.

Let's do an example where classify functions as functions of exponential type or not of exponential type.

Example

Say whether or not each function is of exponential type.

$$f(t) = 2\cos t$$

$$g(t) = e^{-t}$$

$$h(t) = e^{t^2}$$

To determine whether a function is of exponential type, we'll try to find an exponential function that grows faster.

If we consider the function $f(t) = 2\cos t$, the value of $\cos t$ is never greater than 1, but $e^{\alpha t}$ grows exponentially for any $\alpha > 0$. So $e^{\alpha t}$ will grow infinitely large while $f(t) = 2\cos t$ remains small, so $f(t) = 2\cos t$ is of exponential type.

If we consider the function $g(t) = e^{-t}$, we know the function is actually decreasing, but $e^{\alpha t}$ grows exponentially for any $\alpha > 0$. So $e^{\alpha t}$ will grow infinitely large while $g(t) = e^{-t}$ gets smaller and smaller, so $g(t) = e^{-t}$ is of exponential type.

If we consider the function $h(t) = e^{t^2}$, we can't find any $e^{\alpha t}$ that will grow faster than $h(t) = e^{t^2}$, because t^2 will always grow faster than αt, regardless of which positive constant we choose for α. Because αt is linear and t^2 is quadratic, eventually t^2 will always beat αt in a race to infinity. So $h(t) = e^{t^2}$ is not of exponential type.

Partial fractions decompositions

Now that we understand that the Laplace transform is an integral transform that we can use to change a function into a new, predictable form, we want to work toward being able to use this transform to solve differential equations.

But before we start looking at differential equations, let's do a quick review of partial fractions decompositions. Remember that partial fractions decomposition is a technique from calculus we used to rewrite and integrate rational functions.

We'll use these decompositions all the time when we apply the Laplace transform, so it'll be worth our time to go through a quick review of this method here.

Factors and the decomposition

Our first step in any decomposition is to factor the denominator of the rational function as completely as possible. In other words, we want to make sure that none of the factors in our denominator can be broken down any further.

Once the denominator is factored, we need to identify the combination of factors we're dealing with. There are four types of factors we can find: distinct linear factors, repeated linear factors, distinct quadratic factors, and repeated quadratic factors. It's also possible to have a combination of these four types.

Function	Factored form	Factor type
$f(x) = \dfrac{2x+1}{x^2-3x+2}$	$f(x) = \dfrac{2x+1}{(x-1)(x-2)}$	Distinct linear
$f(x) = \dfrac{3x-2}{x^2+2x+1}$	$f(x) = \dfrac{3x-2}{(x+1)(x+1)}$	Repeated linear
$f(x) = \dfrac{x+4}{x^4+4x^2+3}$	$f(x) = \dfrac{x+4}{(x^2+3)(x^2+1)}$	Distinct quadratic
$f(x) = \dfrac{x^3-2x^2}{x^4+2x^2+1}$	$f(x) = \dfrac{x^3-2x^2}{(x^2+1)(x^2+1)}$	Repeated quadratic
$f(x) = \dfrac{x^2-2x-5}{x^3-x^2+9x-9}$	$f(x) = \dfrac{x^2-2x-5}{(x-1)(x^2+9)}$	Combination

Linear factors are first degree factors, while quadratic factors are second degree factors. We call factors distinct when they're different from one another, and we call factors repeated when they're the same.

So $(x-1)(x-2)$ is a set of distinct linear factors because each factor is linear, and $(x-1)$ is different than $(x-2)$. And $(x^2+1)(x^2+1)$ is a set of repeated quadratic factors because each factor is quadratic, and (x^2+1) is the same as (x^2+1). It's also possible to have a combination of factors, like $(x-1)(x^2+9)$, which combines one linear factor with one quadratic factor.

For distinct linear factors in our decomposition, we'll use a single constant A for the numerator. For distinct quadratic factors in our decomposition, we'll use a linear factor $Ax+B$ for the numerator, where A and B are constants.

Factor type	Factored form	Decomposition
Distinct linear	$f(x) = \dfrac{2x+1}{(x-1)(x-2)}$	$\dfrac{A}{x-1} + \dfrac{B}{x-2}$
Distinct quadratic	$f(x) = \dfrac{x+4}{(x^2+3)(x^2+1)}$	$\dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{x^2+1}$
Combination	$f(x) = \dfrac{x^2-2x-5}{(x-1)(x^2+9)}$	$\dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$

Notice how we separated each factor into its own fraction, and only included each distinct factor one time. When factors are repeated, we need to include a fraction for each degree of the factor, up to the highest degree.

For instance, $(x+1)(x+1)$ is a set of repeated linear factors that can be written as $(x+1)^2$. So the degree of this factor is 2, which means we need a fraction for both the first and second degree factors of $(x+1)$.

$$\frac{A}{x+1} + \frac{B}{(x+1)^2}$$

Or let's say that in some significantly more complicated decomposition, we had $x^3(x^2+1)^2$ has a combination set of repeated linear and repeated quadratic factors. The degree of the x factor is 3, which means we need a fraction for the first, second, and third degree factors of x. And the degree of (x^2+1) is 2, which means we need a fraction for both the first and second degree factors of (x^2+1).

$$\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+1} + \frac{Fx+G}{(x^2+1)^2}$$

As a few more examples, here are the decompositions for the repeated factors from the table above.

Factor type	**Factored form**	**Decomposition**
Repeated linear	$f(x)=\dfrac{3x-2}{(x+1)(x+1)}$	$\dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}$
Repeated quadratic	$f(x)=\dfrac{x^3-2x^2}{(x^2+1)(x^2+1)}$	$\dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{(x^2+1)^2}$

Let's do an example with distinct linear factors.

Example

Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{2x+1}{(x-1)(x-2)}$$

These are distinct linear factors, so we'll set up the decomposition as

$$\frac{2x+1}{(x-1)(x-2)}=\frac{A}{x-1}+\frac{B}{x-2}$$

When the factors are distinct, we'll solve for A by removing the $x-1$ factor and setting $x=1$ to find the value of the left side of the decomposition equation.

$$\frac{2x+1}{x-2}\to\frac{2(1)+1}{1-2}\to\frac{3}{-1}\to-3$$

To solve for B, we'll remove the $x-2$ factor and set $x=2$.

$$\frac{2x+1}{x-1} \to \frac{2(2)+1}{2-1} \to \frac{5}{1} \to 5$$

Plugging $A=-3$ and $B=5$ back into the partial fractions decomposition gives

$$f(x)=\frac{A}{x-1}+\frac{B}{x-2}$$

$$f(x)=-\frac{3}{x-1}+\frac{5}{x-2}$$

$$f(x)=\frac{5}{x-2}-\frac{3}{x-1}$$

Let's look at an example with repeated linear factors.

Example

Rewrite the function as its partial fractions decomposition.

$$f(x)=\frac{3x-2}{(x+1)(x+1)}$$

These are repeated linear factors, so we'll set up the decomposition as

$$\frac{3x-2}{(x+1)(x+1)}=\frac{A}{x+1}+\frac{B}{(x+1)^2}$$

When the factors are repeated, we'll multiply both sides of the equation by the denominator from the left side.

$$3x - 2 = A(x + 1) + B$$

$$3x - 2 = Ax + (A + B)$$

Now we'll equate coefficients to see that $A = 3$, and that

$$A + B = -2$$

$$3 + B = -2$$

$$B = -5$$

Plugging $A = 3$ and $B = -5$ back into the partial fractions decomposition gives

$$f(x) = \frac{A}{x + 1} + \frac{B}{(x + 1)^2}$$

$$f(x) = \frac{3}{x + 1} - \frac{5}{(x + 1)^2}$$

Now we'll look at an example with distinct quadratic factors.

Example

Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x + 4}{(x^2 + 3)(x^2 + 1)}$$

These are distinct quadratic factors, so we'll set up the decomposition as

$$\frac{x+4}{(x^2+3)(x^2+1)}=\frac{Ax+B}{x^2+3}+\frac{Cx+D}{x^2+1}$$

We'll multiply both sides of the equation by the denominator from the left side.

$$x+4=(Ax+B)(x^2+1)+(Cx+D)(x^2+3)$$

$$x+4=Ax^3+Ax+Bx^2+B+Cx^3+3Cx+Dx^2+3D$$

$$x+4=(A+C)x^3+(B+D)x^2+(A+3C)x+(B+3D)$$

Now we'll equate coefficients to see that

$$A+C=0 \qquad B+D=0$$

$$A+3C=1 \qquad B+3D=4$$

Solving both systems gives us $A=-1/2$, $B=-2$, $C=1/2$, and $D=2$. Plugging these back into the partial fractions decomposition gives

$$f(x)=\frac{Ax+B}{x^2+3}+\frac{Cx+D}{x^2+1}$$

$$f(x)=\frac{-\frac{1}{2}x-2}{x^2+3}+\frac{\frac{1}{2}x+2}{x^2+1}$$

$$f(x)=-\frac{1}{2}\left(\frac{x+4}{x^2+3}\right)+\frac{1}{2}\left(\frac{x+4}{x^2+1}\right)$$

$$f(x) = \frac{1}{2}\left(\frac{x+4}{x^2+1}\right) - \frac{1}{2}\left(\frac{x+4}{x^2+3}\right)$$

Finally, we'll look at an example with repeated quadratic factors.

Example

Rewrite the function as its partial fractions decomposition.

$$f(x) = \frac{x^3 - 2x^2}{(x^2+1)(x^2+1)}$$

These are repeated quadratic factors, so we'll set up the decomposition as

$$\frac{x^3 - 2x^2}{(x^2+1)(x^2+1)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

We'll multiply both sides of the equation by the denominator from the left side.

$$x^3 - 2x^2 = (Ax+B)(x^2+1) + Cx + D$$

$$x^3 - 2x^2 = Ax^3 + Ax + Bx^2 + B + Cx + D$$

$$x^3 - 2x^2 = Ax^3 + Bx^2 + (A+C)x + (B+D)$$

Now we'll equate coefficients to see that $A = 1$, that $B = -2$, and therefore that

$$A + C = 0$$

$$1 + C = 0$$

$$C = -1$$

and

$$B + D = 0$$

$$-2 + D = 0$$

$$D = 2$$

Plugging $A = 1$, $B = -2$, $C = -1$, and $D = 2$ back into the partial fractions decomposition gives

$$f(x) = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}$$

$$f(x) = \frac{x - 2}{x^2 + 1} + \frac{-x + 2}{(x^2 + 1)^2}$$

$$f(x) = \frac{x - 2}{x^2 + 1} - \frac{x - 2}{(x^2 + 1)^2}$$

Inverse Laplace transforms

Normally when we calculate a Laplace transform, we start with a function $f(t)$ and we transform it into $F(s)$. Or in different variables, start with a function $y(x)$ and transform it into $Y(s)$. For instance, from the Laplace transform table, we know that e^t transforms to

$$\mathscr{L}(e^t) = \frac{1}{s-1}$$

Inverse transforms

Not surprisingly, an **inverse Laplace transform** is the opposite process, in which we start with $F(s)$ and transform it back to $f(t)$, or start with $Y(s)$ and transform it back to $y(x)$. So in this example, if we have

$$F(s) = \frac{1}{s-1}$$

we know that the inverse transform is $f(t) = e^t$. So if before we'd write the Laplace transform of $f(t)$ as $\mathscr{L}(f(t)) = F(s)$, we can now say that the inverse transform of $F(s)$ is

$$\mathscr{L}^{-1}(F(s)) = f(t)$$

The $\mathscr{L}^{-1}$ notation is what indicates the inverse Laplace transform. And as we saw with regular Laplace transforms, the inverse transform of a sum is the sum of the inverse transforms, and constant coefficients don't affect the value of an inverse transform.

$$\mathscr{L}^{-1}(aF(s) + bG(s)) = a\mathscr{L}^{-1}(F(s)) + b\mathscr{L}^{-1}(G(s))$$

Ideally we want to simplify $F(s)$ to the point where we can compare each part of it to a formula from the table of Laplace transforms. If we can identify formulas in the table that match each part of $F(s)$, then we can transform back from s to t.

Very often though, we'll need to manipulate $F(s)$ first, because it won't be perfectly ready for an inverse transform. Let's do an example where we undo the transform, putting $F(s)$ back in terms of t.

Example

Use an inverse Laplace transform to find $f(t)$.

$$F(s) = \frac{s+7}{s^2 - 3s - 10}$$

Our goal is to simplify $F(s)$ so that we can use formulas from the table of Laplace transforms. Currently, $F(s)$ is too complicated to match to any formula from the table, so we'll try to simplify it by factoring the denominator,

$$F(s) = \frac{s+7}{(s+2)(s-5)}$$

and then using a partial fractions decomposition.

$$\frac{s+7}{(s+2)(s-5)} = \frac{A}{s+2} + \frac{B}{s-5}$$

To solve for A, we'll remove the $(s+2)$ factor, set $s=-2$, and find the value of the left side.

$$\frac{s+7}{s-5}$$

$$\frac{-2+7}{-2-5}=\frac{5}{-7}=-\frac{5}{7}$$

To solve for B, we'll remove the $(s-5)$ factor, set $s=5$, and find the value of the left side.

$$\frac{s+7}{s+2}$$

$$\frac{5+7}{5+2}=\frac{12}{7}$$

Plugging the values we found for A and B back into the partial fractions decomposition gives

$$F(s)=\frac{-\frac{5}{7}}{s+2}+\frac{\frac{12}{7}}{s-5}$$

Factoring the constants out of each numerator, we can rewrite the transform as

$$F(s)=-\frac{5}{7}\left(\frac{1}{s+2}\right)+\frac{12}{7}\left(\frac{1}{s-5}\right)$$

In this form, we can see that both values inside the parentheses resemble the Laplace transform

$$\mathscr{L}(e^{at}) = \frac{1}{s-a}$$

We'll just change the $1/(s+2)$ so that it matches the transform formula exactly.

$$F(s) = -\frac{5}{7}\left(\frac{1}{s-(-2)}\right) + \frac{12}{7}\left(\frac{1}{s-5}\right)$$

Now we can see that $a_1 = -2$ and $a_2 = 5$. We'll reverse the formula from the table to rewrite the transform as

$$f(t) = \mathscr{L}^{-1}(F(s)) = -\frac{5}{7}(e^{-2t}) + \frac{12}{7}(e^{5t})$$

$$f(t) = \mathscr{L}^{-1}(F(s)) = \frac{12}{7}e^{5t} - \frac{5}{7}e^{-2t}$$

Because we used an inverse transform to work backward from $F(s)$ to $f(t)$, we know that if we'd started with this $f(t)$ and applied a Laplace transform to it, $\mathscr{L}(f(t))$, we would have found the transform $F(s)$ that we started with.

Transforming derivatives

We already know from our table of Laplace transforms that we can transform first and second derivative functions.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

In this lesson we want to understand how we arrived at these formulas. We'll also be using these in an upcoming lesson in order to solve initial value problems.

Transform of a first derivative

We can start with a simple substitution into the definition of the Laplace transform, replacing $f(t)$ in that definition with its derivative $f'(t)$.

$$\mathscr{L}(f'(t)) = \int_0^\infty e^{-st}f'(t)\ dt$$

But we'd still prefer to have an expression in terms of $f(t)$, instead of $f'(t)$, since we're usually starting with $f(t)$. Luckily, we can put this equation in terms of $f(t)$ by applying integration by parts to the integral.

We'll differentiate the e^{-st} part and integrate the $f'(t)$ part, which means we'll set up our parts as

$$u = e^{-st} \qquad\qquad dv = f'(t)\ dt$$

$$du = -se^{-st}\ dt \qquad\qquad v = f(t)$$

Then integration by parts gives us

$$\mathscr{L}(f'(t)) = uv\Big|_0^\infty - \int_0^\infty v\ du$$

$$\mathscr{L}(f'(t)) = (e^{-st})(f(t))\Big|_0^\infty - \int_0^\infty f(t)(-se^{-st})\ dt$$

$$\mathscr{L}(f'(t)) = e^{-st}f(t)\Big|_0^\infty + s\int_0^\infty e^{-st}f(t)\ dt$$

Now assuming $f(t)$ is of exponential type, then $e^{-st}f(t)$ converges to 0, and we can simplify this expression for the Laplace transform of the derivative as

$$\mathscr{L}(f'(t)) = \lim_{t\to\infty} e^{-st}f(t) - e^{-s(0)}f(0) + s\int_0^\infty e^{-st}f(t)\ dt$$

$$\mathscr{L}(f'(t)) = 0 - (1)f(0) + s\int_0^\infty e^{-st}f(t)\ dt$$

$$\mathscr{L}(f'(t)) = -f(0) + s\int_0^\infty e^{-st}f(t)\ dt$$

Furthermore, we see that the integral that remains is simply the definition of the Laplace transform,

$$F(s) = \int_0^\infty e^{-st}f(t)\ dt$$

which means we can substitute and replace it with $F(s)$, in order to get a clean and concise definition for the **Laplace transform of a first derivative**.

$$\mathscr{L}(f'(t)) = -f(0) + sF(s)$$

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

Transform of a second derivative

Using the same method, we can extend this process to find the **Laplace transform of a second derivative**, which is given as

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

Notice that the transform of the first derivative requires us to know $f(0)$, and the transform of the second derivative requires us to know both $f(0)$ and $f'(0)$.

In other words, we won't be able to take transforms of these kinds of derivative functions unless we have the starting points for the original function $f(t)$ and its derivative $f'(t)$.

Let's do an example so that we can practice with these derivative formulas.

Example

Find the Laplace transforms of $y'(x)$ and $y''(x)$, given $y(0) = 1$ and $y'(0) = 0$.

We'll start by rewriting the formula for the Laplace transform of a first derivative by replacing the function $f'(t)$ with the derivative function we were given, $y'(x)$. This will also change the transform F to the transform Y, and the initial condition from $f(0)$ to $y(0)$.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(y'(x)) = sY(s) - y(0)$$

Now we can make a substitution into the transform equation for the initial condition.

$$\mathscr{L}(y'(x)) = sY(s) - 1$$

Now we'll rewrite the formula for the Laplace transform of a second derivative so that it's in terms of y and x instead of y and t.

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(y''(x)) = s^2Y(s) - sy(0) - y'(0)$$

Then we'll substitute for the initial conditions.

$$\mathscr{L}(y''(x)) = s^2Y(s) - s(1) - 0$$

$$\mathscr{L}(y''(x)) = s^2Y(s) - s$$

So for any function $y(x)$ with the initial conditions $y(0) = 1$ and $y'(0) = 0$, the Laplace transforms of its first and second derivatives will be

$$\mathscr{L}(y'(x)) = sY(s) - 1$$

$$\mathscr{L}(y''(x)) = s^2Y(s) - s$$

Laplace transforms for initial value problems

Now that we understand how to use the formulas for the Laplace transform of derivative functions,

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

and we know how to perform inverse Laplace transforms, we want to apply these two concepts to solving initial value problems.

Solving initial value problems

Remember that, given some second order nonhomogeneous equation ($f(t) \neq 0$) in the general form

$$ay'' + by' + cy = f(t)$$

we can find its solution using a Laplace transform, as long as we have values for the initial conditions $f(0)$ and $f'(0)$.

In order to solve this kind of initial value problem using a Laplace transform, we'll follow these steps:

1. Use formulas from the table to transform y'', y', y, and $f(t)$.

2. Plug in the initial conditions to simplify the transformation.

3. Use algebra to solve for $Y(s)$, then partial fractions to decompose it.

4. Use an inverse Laplace transform to put the solution to the second order nonhomogeneous differential equation back in terms of t, instead of s.

Let's work through an example so that we can see how these steps get applied.

Example

Use a Laplace transform to find the solution to the differential equation, given $y(0) = -1$ and $y'(0) = 1$.

$$y'' - 6y' + 5y = 5t$$

From our table of Laplace transforms, we know

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(t) = \frac{1}{s^2}$$

Plugging these transforms into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) - 6[sY(s) - y(0)] + 5Y(s) = 5\left(\frac{1}{s^2}\right)$$

Now we'll plug in the initial conditions $y(0) = -1$ and $y'(0) = 1$ in order to simplify the transform.

$$s^2Y(s) - s(-1) - 1 - 6[sY(s) - (-1)] + 5Y(s) = \frac{5}{s^2}$$

$$s^2Y(s) + s - 1 - 6[sY(s) + 1] + 5Y(s) = \frac{5}{s^2}$$

$$s^2Y(s) + s - 1 - 6sY(s) - 6 + 5Y(s) = \frac{5}{s^2}$$

$$s^2Y(s) + s - 6sY(s) + 5Y(s) - 7 = \frac{5}{s^2}$$

Solve for $Y(s)$ by collecting all the $Y(s)$ terms on one side, and moving all other terms to the other side.

$$s^2Y(s) - 6sY(s) + 5Y(s) = \frac{5}{s^2} + 7 - s$$

$$s^2Y(s) - 6sY(s) + 5Y(s) = \frac{5 + 7s^2 - s^3}{s^2}$$

Factor out $Y(s)$, then isolate it on the left side of the equation.

$$Y(s)(s^2 - 6s + 5) = \frac{5 + 7s^2 - s^3}{s^2}$$

$$Y(s)(s - 5)(s - 1) = \frac{5 + 7s^2 - s^3}{s^2}$$

$$Y(s) = \frac{5 + 7s^2 - s^3}{s^2(s - 5)(s - 1)}$$

We'll need to use a partial fractions decomposition.

$$\frac{5+7s^2-s^3}{s^2(s-5)(s-1)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-5}+\frac{D}{s-1}$$

To solve for B, we'll remove the s^2 factor, set $s=0$, and find the value of the left side.

$$\frac{5+7s^2-s^3}{(s-5)(s-1)}$$

$$\frac{5+7(0)^2-0^3}{(0-5)(0-1)}=\frac{5}{(-5)(-1)}=\frac{5}{5}=1$$

To solve for C, we'll remove the $(s-5)$ factor, set $s=5$, and find the value of the left side.

$$\frac{5+7s^2-s^3}{s^2(s-1)}$$

$$\frac{5+7(5)^2-5^3}{5^2(5-1)}=\frac{5+7(25)-125}{25(4)}=\frac{5+175-125}{100}=\frac{55}{100}=\frac{11}{20}$$

To solve for D, we'll remove the $(s-1)$ factor, set $s=1$, and find the value of the left side.

$$\frac{5+7s^2-s^3}{s^2(s-5)}$$

$$\frac{5+7(1)^2-1^3}{1^2(1-5)}=\frac{5+7-1}{1-5}=-\frac{11}{4}$$

Now that we have the values of B, C, and D, we can plug those into the decomposition,

$$\frac{5+7s^2-s^3}{s^2(s-5)(s-1)}=\frac{A}{s}+\frac{1}{s^2}+\frac{\frac{11}{20}}{s-5}+\frac{-\frac{11}{4}}{s-1}$$

and then pick a value we haven't used for s yet ($s \neq 0,\ 1,\ 5$), like $s=-1$, plug it in, and solve for A.

$$\frac{5+7(-1)^2-(-1)^3}{(-1)^2(-1-5)(-1-1)}=\frac{A}{-1}+\frac{1}{(-1)^2}+\frac{\frac{11}{20}}{-1-5}+\frac{-\frac{11}{4}}{-1-1}$$

$$\frac{5+7+1}{(-6)(-2)}=-A+1-\frac{\frac{11}{20}}{6}+\frac{\frac{11}{4}}{2}$$

$$\frac{13}{12}=-A+1-\frac{11}{120}+\frac{11}{8}$$

$$A=1-\frac{11}{120}+\frac{11}{8}-\frac{13}{12}$$

$$A=\frac{120}{120}-\frac{11}{120}+\frac{165}{120}-\frac{130}{120}$$

$$A=\frac{144}{120}$$

$$A=\frac{6}{5}$$

Plugging the values we found for A, B, C, and D back into the partial fractions decomposition gives

$$Y(s) = \frac{\frac{6}{5}}{s} + \frac{1}{s^2} + \frac{\frac{11}{20}}{s-5} + \frac{-\frac{11}{4}}{s-1}$$

and then we can rearrange each term in the decomposition to make it easier to find a matching formula in the Laplace transform table.

$$Y(s) = \frac{6}{5}\left(\frac{1}{s}\right) + \left(\frac{1}{s^2}\right) + \frac{11}{20}\left(\frac{1}{s-5}\right) - \frac{11}{4}\left(\frac{1}{s-1}\right)$$

The terms remaining inside the parentheses should remind us of the transforms

$$\mathscr{L}(1) = \frac{1}{s} \qquad \mathscr{L}(t) = \frac{1}{s^2}$$

$$\mathscr{L}(e^{5t}) = \frac{1}{s-5} \qquad \mathscr{L}(e^t) = \frac{1}{s-1}$$

or equivalently, the inverse transforms

$$\mathscr{L}^{-1}\left(\frac{1}{s}\right) = 1 \qquad \mathscr{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

$$\mathscr{L}^{-1}\left(\frac{1}{s-5}\right) = e^{5t} \qquad \mathscr{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$$

So we'll make these substitutions to put the equation back in terms of the original variable t, instead of the transform variable s, which will give us the solution to the second order nonhomogeneous differential equation.

$$y(t) = \frac{6}{5} + t + \frac{11}{20}e^{5t} - \frac{11}{4}e^t$$

When the initial conditions aren't for $t = 0$

As we mentioned before, because $f(0)$ and $f'(0)$ are baked into the Laplace transforms of the first and second derivatives,

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

we must have initial conditions defined at $t = 0$. Therefore, if we're given initial conditions at some non-zero value of t, we'll need to shift them to $t = 0$ before we can find the solution to the second order differential equation.

This will involve making a substitution that replaces the non-zero value. Let's work through an example so that we can see how to handle this.

Example

Use a Laplace transform to find the solution to the differential equation, given $y(2) = 0$ and $y'(2) = 2$.

$$y'' - 3y' = e^{-3t}$$

Before we do anything else, we need to shift the initial conditions from $t = 2$ to $t = 0$. We'll define a new variable η, and we want η to be equal to 0

so that we can set the initial conditions at $y(\eta = 0)$ and $y'(\eta = 0)$. So η and t are related by

$$\eta = t - 2$$

$$t = \eta + 2$$

Then we want to make substitutions for t into the differential equation.

$$y'' - 3y' = e^{-3t}$$

$$y''(t) - 3y'(t) = e^{-3t}$$

$$y''(\eta + 2) - 3y'(\eta + 2) = e^{-3(\eta+2)}$$

$$y''(\eta + 2) - 3y'(\eta + 2) = e^{-3\eta-6}$$

$$y''(\eta + 2) - 3y'(\eta + 2) = e^{-6}e^{-3\eta}$$

Next, we'll define a new function $u(\eta)$ with

$$u(\eta) = y(\eta + 2)$$

$$u'(\eta) = y'(\eta + 2)$$

$$u''(\eta) = y''(\eta + 2)$$

Then the initial conditions are

$$u(0) = y(0 + 2) = y(2) = 0$$

$$u'(0) = y'(0 + 2) = y'(2) = 2$$

After all of this shifting, the new initial value problem is

$$u'' - 3u' = e^{-6}e^{-3\eta} \text{ with } u(0) = 0 \text{ and } u'(0) = 2$$

From our table of Laplace transforms, we know

$$\mathscr{L}(u'') = s^2U(s) - su(0) - u'(0)$$

$$\mathscr{L}(u') = sU(s) - u(0)$$

$$\mathscr{L}(e^{-3\eta}) = \frac{1}{s-(-3)} = \frac{1}{s+3}$$

Plugging these transforms into the differential equation gives

$$s^2U(s) - su(0) - u'(0) - 3(sU(s) - u(0)) = e^{-6}\left(\frac{1}{s+3}\right)$$

Now we'll plug in the initial conditions $u(0) = 0$ and $u'(0) = 2$ in order to simplify the transform.

$$s^2U(s) - 2 - 3sU(s) = e^{-6}\left(\frac{1}{s+3}\right)$$

Solve for $U(s)$ by collecting all the $U(s)$ terms on one side, and moving all other terms to the other side.

$$s^2U(s) - 3sU(s) = e^{-6}\left(\frac{1}{s+3}\right) + 2$$

Factor out $U(s)$, then isolate it on the left side of the equation.

$$U(s)(s^2 - 3s) = e^{-6}\left(\frac{1}{s+3}\right) + 2$$

$$U(s) = e^{-6}\left(\frac{1}{(s+3)(s^2-3s)}\right) + \frac{2}{(s^2-3s)}$$

$$U(s) = \frac{e^{-6}}{s(s+3)(s-3)} + \frac{2}{s(s-3)}$$

$$U(s) = \frac{e^{-6}}{s(s+3)(s-3)} + \frac{2(s+3)}{s(s+3)(s-3)}$$

$$U(s) = \frac{e^{-6} + 2(s+3)}{s(s+3)(s-3)}$$

We'll need to use a partial fractions decomposition.

$$\frac{e^{-6} + 2(s+3)}{s(s+3)(s-3)} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-3}$$

To solve for A, we'll remove the s factor, set $s = 0$, and find the value of the left side.

$$\frac{e^{-6} + 2(s+3)}{(s+3)(s-3)}$$

$$\frac{e^{-6} + 2(0+3)}{(0+3)(0-3)} = \frac{e^{-6} + 2(3)}{(3)(-3)} = -\frac{e^{-6} + 6}{9}$$

To solve for B, we'll remove the $(s+3)$ factor, set $s = -3$, and find the value of the left side.

$$\frac{e^{-6} + 2(s+3)}{s(s-3)}$$

$$\frac{e^{-6}+2(-3+3)}{-3(-3-3)}=\frac{e^{-6}+2(0)}{-3(-6)}=\frac{e^{-6}}{18}$$

To solve for C, we'll remove the $(s-3)$ factor, set $s=3$, and find the value of the left side.

$$\frac{e^{-6}+2(s+3)}{s(s+3)}$$

$$\frac{e^{-6}+2(3+3)}{3(3+3)}=\frac{e^{-6}+2(6)}{3(6)}=\frac{e^{-6}+12}{18}$$

Plugging the values we found for A, B, and C back into the partial fractions decomposition gives

$$U(s)=\frac{-\frac{e^{-6}+6}{9}}{s}+\frac{\frac{e^{-6}}{18}}{s+3}+\frac{\frac{e^{-6}+12}{18}}{s-3}$$

and then we can rearrange each term in the decomposition to make it easier to find a matching formula in the Laplace transform table.

$$U(s)=-\frac{e^{-6}+6}{9}\left(\frac{1}{s}\right)+\frac{e^{-6}}{18}\left(\frac{1}{s+3}\right)+\frac{e^{-6}+12}{18}\left(\frac{1}{s-3}\right)$$

The terms remaining inside the parentheses should remind us of the transforms

$$\mathscr{L}(1)=\frac{1}{s} \qquad \mathscr{L}(e^{3\eta})=\frac{1}{s-3} \qquad \mathscr{L}(e^{-3\eta})=\frac{1}{s+3}$$

or equivalently, the inverse transforms

$$\mathscr{L}^{-1}\left(\frac{1}{s}\right) = 1 \qquad \mathscr{L}^{-1}\left(\frac{1}{s-3}\right) = e^{3\eta} \qquad \mathscr{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3\eta}$$

So we'll make these substitutions to put the equation back in terms of η, instead of the transform variable s,

$$u(\eta) = -\frac{e^{-6}+6}{9} + \frac{e^{-6}}{18}e^{-3\eta} + \frac{e^{-6}+12}{18}e^{3\eta}$$

$$u(\eta) = -\frac{e^{-6}}{9} - \frac{6}{9} + \frac{e^{-6}}{18}e^{-3\eta} + \frac{e^{-6}}{18}e^{3\eta} + \frac{12}{18}e^{3\eta}$$

$$u(\eta) = -\frac{1}{9e^{6}} - \frac{2}{3} + \frac{1}{18e^{6}e^{3\eta}} + \frac{e^{3\eta}}{18e^{6}} + \frac{2e^{3\eta}}{3}$$

Because $y(t) = u(\eta) = u(t-2)$ and $\eta = t - 2$, the solution to the original differential equation can be written as

$$y(t) = -\frac{1}{9e^{6}} - \frac{2}{3} + \frac{1}{18e^{6}e^{3(t-2)}} + \frac{e^{3(t-2)}}{18e^{6}} + \frac{2e^{3(t-2)}}{3}$$

$$y(t) = -\frac{1}{9e^{6}} - \frac{2}{3} + \frac{1}{18e^{6}e^{3t-6}} + \frac{e^{3t-6}}{18e^{6}} + \frac{2e^{3t-6}}{3}$$

$$y(t) = -\frac{2}{3} + \frac{1}{18}e^{-3t} - \frac{1}{9}e^{-6} + \frac{2}{3}e^{3t-6} + \frac{1}{18}e^{3t-12}$$

Step functions

The **Heaviside function**, also called the Heaviside step function, or unit step function, is a step function in the form

$$u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$$

In addition to $u_c(t)$, we'll also sometimes use $u(t-c)$,

$$u(t-c) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$$

or $H(t)$ or $H(t-c)$ to represent the Heaviside function. In any case, it's the function with a jump discontinuity at $t = c$. We only consider $c \ge 0$, which we see from the condition for the first case.

There's some debate about the value of the Heaviside function at exactly $t = c$. Sometimes the function's value will be defined as 0 at $t = c$, sometimes as 1 at $t = c$, and sometimes even as $1/2$ at $t = c$. For now, which convention we use isn't going to make a huge difference, so we'll stick with a value of 1 at $t = c$, which is the way we defined the piecewise functions above.

Characteristics

Think about the unit step function as a switch that turns on at $t = c$. When $t < c$, or to the left of $t = c$, the function is "off" because it has a value of 0.

Then, as soon as we reach $t = c$, then function suddenly turns "on," and its value immediately jumps to 1.

In other words, we could say that the unit step function has these characteristics:

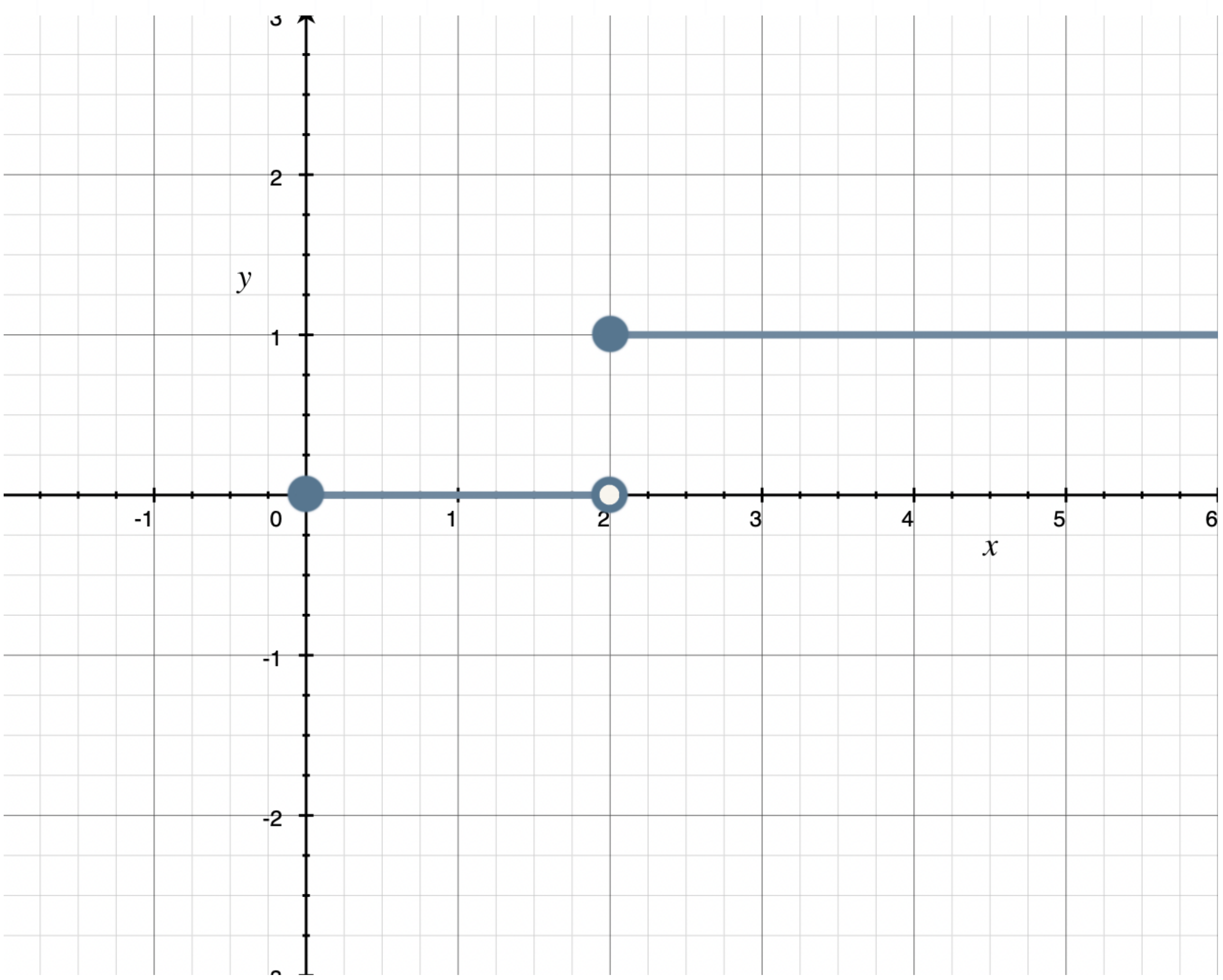
1. Its value is only ever 0 or 1; it never takes on any other value.

2. The change from 0 to 1 ("off" to "on") happens at $t = c$.

3. c could be any nonnegative value.

Below are the equation and the graph of the unit step function when $c = 2$.

$$u_2(t) = \begin{cases} 0 & 0 \le t < 2 \\ 1 & t \ge 2 \end{cases}$$



Modifying the unit step function

While the unit step function only ever takes on values of 0 or 1, we can modify it to take other values. For instance, if we want a function that flips on to a value of 5, instead of a value of 1, we can simply multiply the unit step function by 5.

$$5u_c(t) = \begin{cases} 5(0) & 0 \le t < c \\ 5(1) & t \ge c \end{cases}$$

$$5u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 5 & t \ge c \end{cases}$$

Notice how multiplying the left side of the equation by 5 required us to multiply both pieces on the right side by 5 as well. This new function $5u_c(t)$ will be "off" (have a value of 0) whenever $0 \le t < c$, and then switch "on" and take a value of 5 as soon as $t = c$ (and will continue to take a value of 5 everywhere to the right of $t = c$).

We can also reverse the order of the switch, defining a function that starts "on" and then turns "off" at $t = c$, instead of the "off-to-on" pattern that we've been using up to this point.

Let's say we want the switch to be "on" at a value of 3, and then turn "off" to a value of 0 when we arrive at the switching point $t = c$. We'd write the equation of that function as

$$3 - 3u_c(t) = \begin{cases} 3 - 3(0) & 0 \le t < c \\ 3 - 3(1) & t \ge c \end{cases}$$

$$3 - 3u_c(t) = \begin{cases} 3 & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

Here’s a summary of these modifications:

Off-to-on $$u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$$

Off-to-(on at n) $$nu_c(t) = \begin{cases} 0 & 0 \le t < c \\ n & t \ge c \end{cases}$$

On-to-off $$1 - u_c(t) = \begin{cases} 1 & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

(On at n)-to-off $$n(1 - u_c(t)) = \begin{cases} n & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

Now we’d like to use these modifications to rewrite a piecewise function in terms of Heaviside functions, and vice versa. Let’s work through an example.

Example

Express the piecewise function in terms of Heaviside functions.

$$f(t) = \begin{cases} -2 & 0 \le t < 1 \\ 4 & 1 \le t < 3 \\ 8 & 3 \le t < 5 \\ 12 & t \ge 5 \end{cases}$$

Remember that a Heaviside function represents a switch, so we need one Heaviside function every time we make a switch in the value of $f(t)$. There are three switches, one from -2 to 4, one from 4 to 8, and one from 8 to 12, so we'll need three Heaviside functions in order to express $f(t)$.

We write a Heaviside function as $u_c(t)$, where the switch occurs at $t = c$. The switches in this function $f(t)$ are at $t = 1$, $t = 3$, and $t = 5$, which means the three Heaviside functions we'll need are $u_1(t)$, $u_3(t)$, and $u_5(t)$.

To express $f(t)$, we start with the idea that $f(t)$ has a value of -2 when it's "off." When the first switch at $t = 1$ turns on, the value of the function immediately shifts from -2 to 4 (up 6 units), so we can write

$$f(t) \approx -2 + 6u_1(t)$$

We use 6 as a coefficient on $u_1(t)$, since $-2 + 6 = 4$, and 4 is the value the function takes on after the first switch. Then moving from 4 to 8 is an increase of 4, which means we need to multiply $u_3(t)$ by 4,

$$f(t) \approx -2 + 6u_1(t) + 4u_3(t)$$

and moving from 8 to 12 is also an increase of 4, which means we need to multiply $u_5(t)$ by 4, and this will give us the full expression of $f(t)$ in terms of step functions.

$$f(t) = -2 + 6u_1(t) + 4u_3(t) + 4u_5(t)$$

If instead we'd been given this function for $f(t)$ in terms of unit step functions, we could still pull it apart and rewrite it as a piecewise function.

The unit step functions show us that we have switches at $t = 1, 3, 5$, and the -2 gives us the "starting point." So immediately we can write

$$f(t) = \begin{cases} -2 & 0 \le t < 1 \\ & 1 \le t < 3 \\ & 3 \le t < 5 \\ & t \ge 5 \end{cases}$$

Then we consider the coefficients on the unit step functions, 6, 4, and 4. We'll add 6 to -2 to get $-2 + 6 = 4$ for the second piece,

$$f(t) = \begin{cases} -2 & 0 \le t < 1 \\ 4 & 1 \le t < 3 \\ & 3 \le t < 5 \\ & t \ge 5 \end{cases}$$

then add 4 again to get $4 + 4 = 8$ for the third piece,

$$f(t) = \begin{cases} -2 & 0 \le t < 1 \\ 4 & 1 \le t < 3 \\ 8 & 3 \le t < 5 \\ & t \ge 5 \end{cases}$$

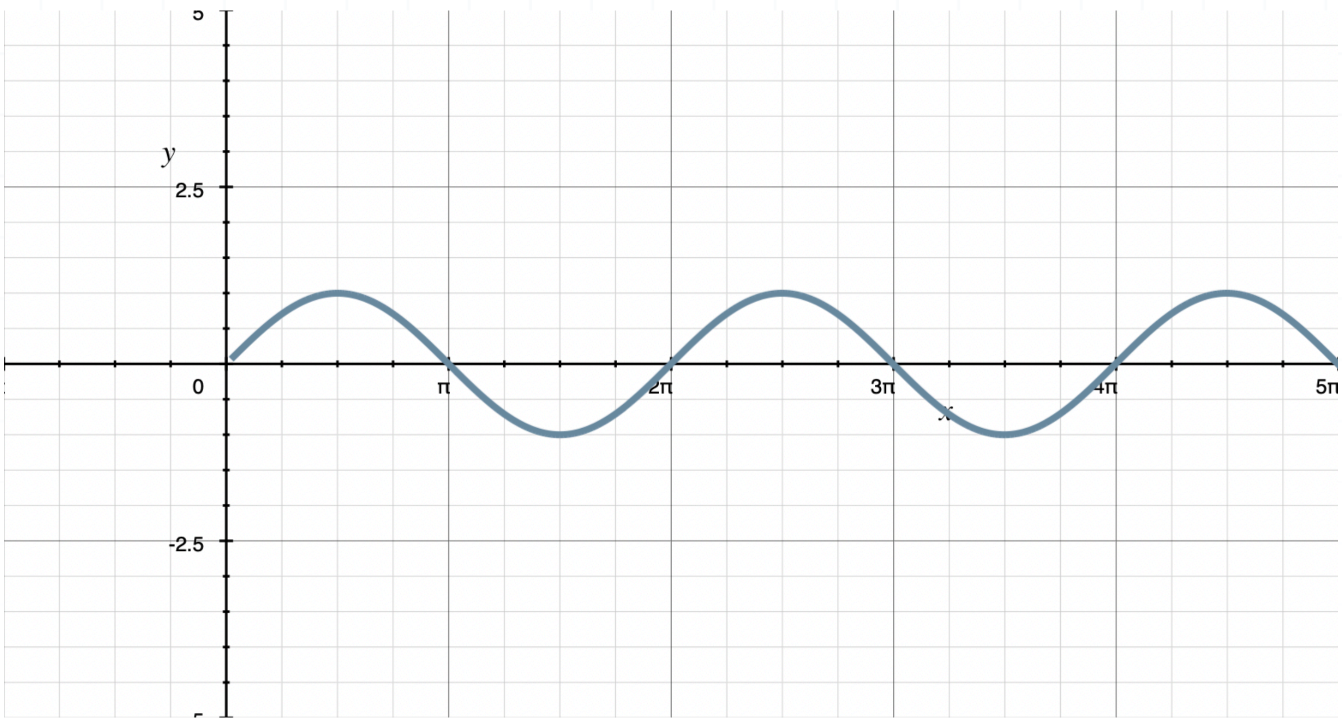
and then add 4 again to get $8 + 4 = 12$ for the fourth piece.

$$f(t) = \begin{cases} -2 & 0 \le t < 1 \\ 4 & 1 \le t < 3 \\ 8 & 3 \le t < 5 \\ 12 & t \ge 5 \end{cases}$$

Non-constant "on" values

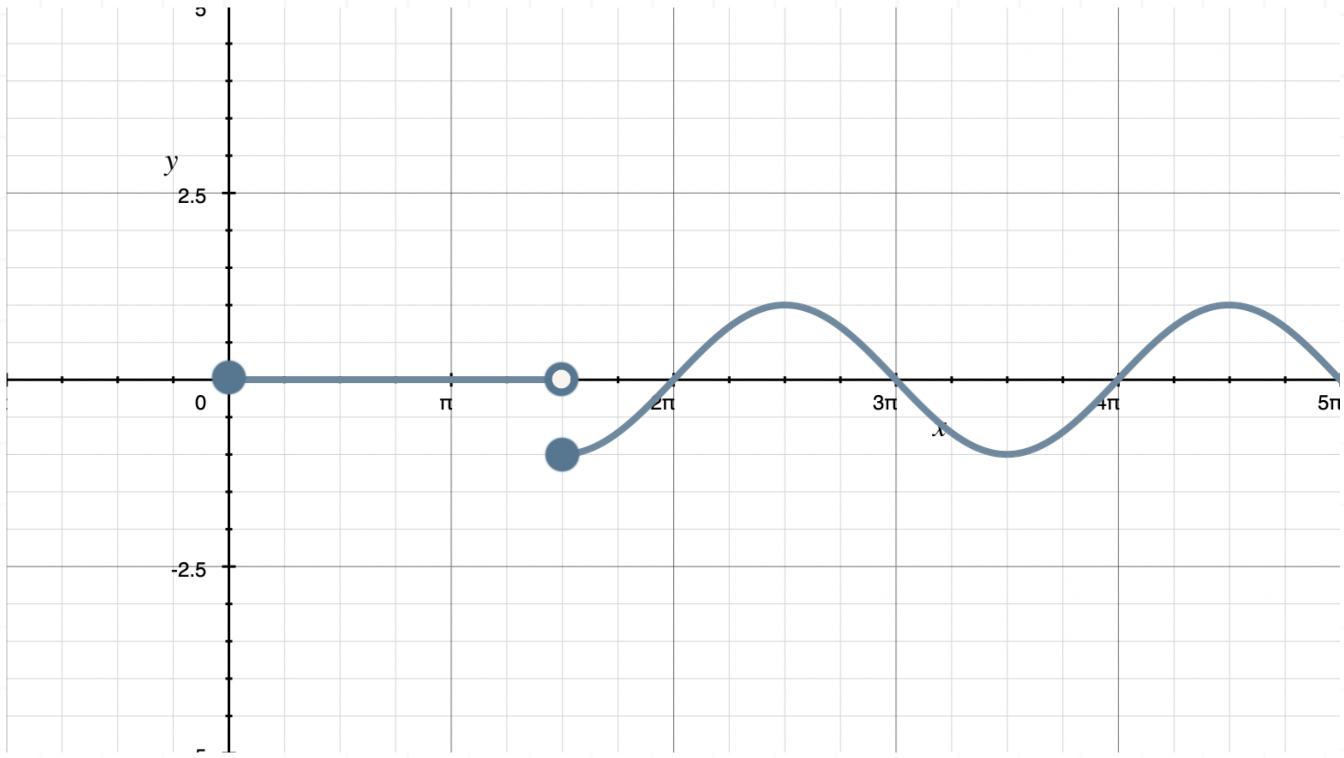
Up to now, we've been looking at step functions that take on some constant value when they're "on." But we also want to be able to turn on a function that takes on values that aren't constant.

For example, let's consider the sine function $f(t) = \sin t$ just for $t \geq 0$.



We can use the unit step function to "turn off" a portion of this sine graph. Let's say we want to turn it off on $0 \leq t < 3\pi/2$. This "switch" is at $t = 3\pi/2$, which means we just need to multiply $\sin t$ by $u(t - 3\pi/2)$.

In other words, the graph of $f(t)u(t-c) = \sin t(u(t - 3\pi/2))$ is 0 (off) on $0 \leq t < 3\pi/2$, and then given by $\sin t$ (on) for $t \geq 3\pi/2$.



To generalize this idea, we can say that the function $g(t)$,

$$g(t) = f(t)u(t - c)$$

$$g(t) = f(t)u_c(t)$$

is a function with a zero value (off) on the interval $0 \le t < c$, and then a value given by $f(t)$ (on) for $t \ge c$.

$$g(t) = \begin{cases} 0 & 0 \le t < c \\ f(t) & t \ge c \end{cases}$$

Let's do an example where we model this kind of function, where the value when the switch is "on" is given by some curve whose value varies.

Example

Express the piecewise function in terms of unit step functions.

$$g(t) = \begin{cases} t^2 + 4 & 0 \le t < 3 \\ 0 & t \ge 3 \end{cases}$$

The first thing we notice about $g(t)$ is that it follows an "on-to-off" pattern instead of the default "off-to-on." Remember when we looked earlier at modifications of the step function, we said that this "(on at n)-to-off" pattern is given by

$$n(1 - u_c(t)) = \begin{cases} n & 0 \le t < c \\ 0 & t \ge c \end{cases}$$

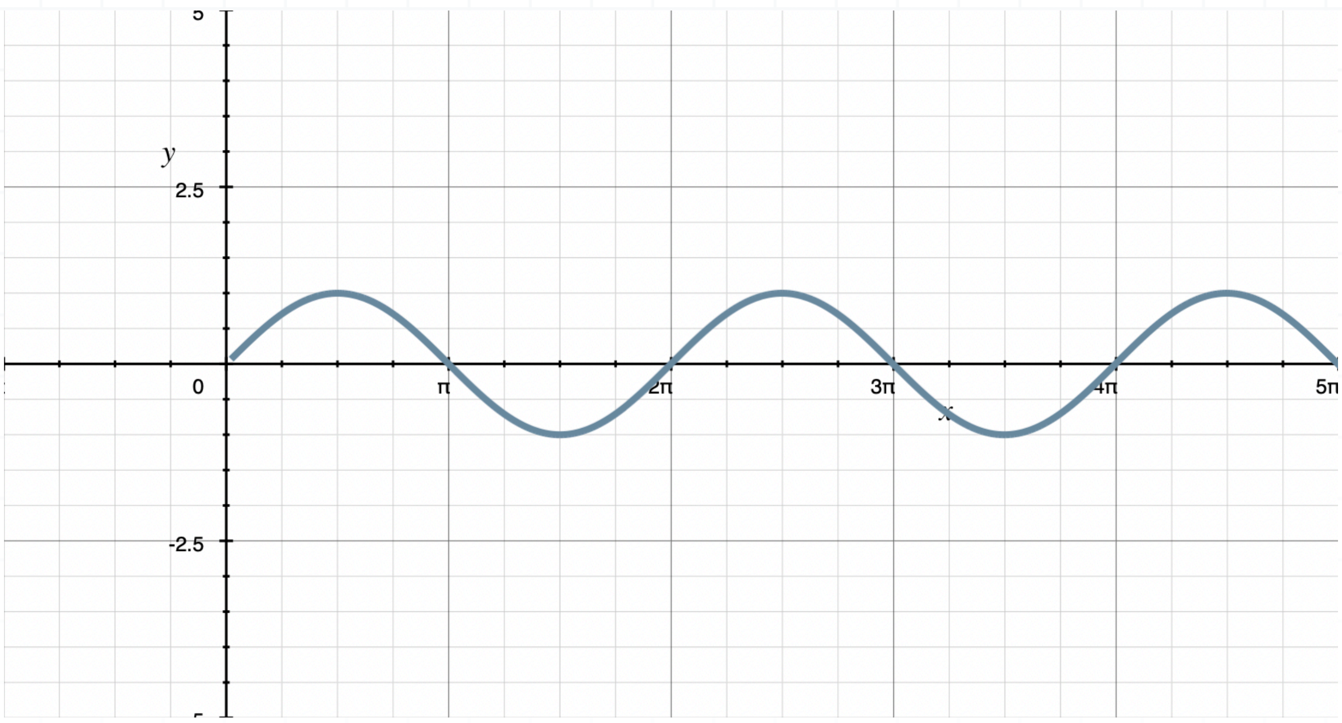
In other words, we can rewrite $g(t)$ as either of the following:

$$g(t) = (t^2 + 4)(1 - u_3(t))$$

$$g(t) = (t^2 + 4)(1 - u(t - 3))$$

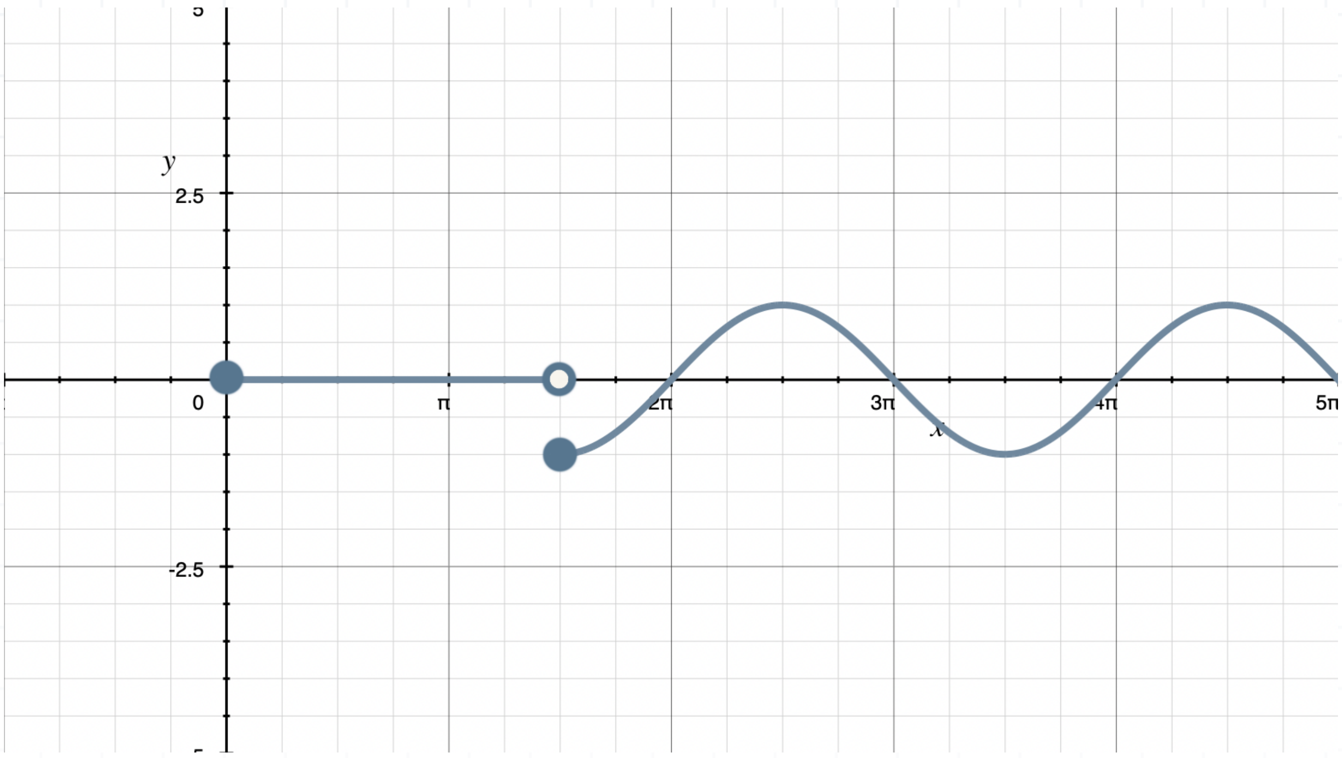
Second Shifting Theorem

Below is the graph of the sine function $f(t) = \sin t$ for $t \geq 0$.



Previously we looked at how to turn off the part of this sine function on $0 \leq t < 3\pi/2$, by multiplying by a unit step function.

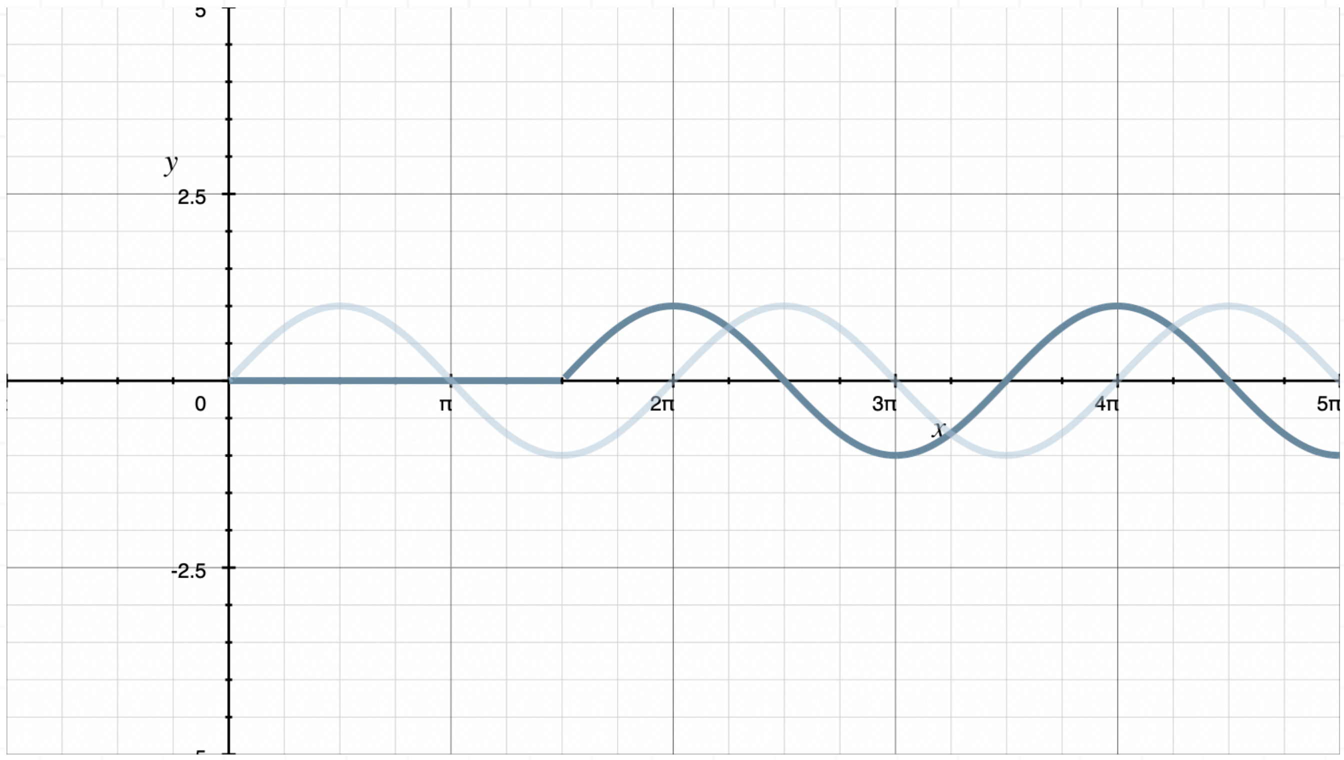
$$u(t-c)f(t) = u\left(t - \frac{3\pi}{2}\right)\sin t$$



We said that, in general, we can turn off any function $f(t)$ on the interval $0 \leq t < c$ by multiplying $f(t)$ by $u(t-c)$ or $u_c(t)$.

But now we want to look at a slightly different way to turn off $f(t)$. This time, instead of replacing $f(t) = \sin t$ with $f(t) = 0$ on the interval $0 \leq t < 3\pi/2$, we want to shift the part of $f(t) = \sin t$ for $t \geq 0$ to the right a distance of $t = c$.

In other words, we still want the graph "off" on $0 \leq t < 3\pi/2$, but once we get to $t = c = 3\pi/2$ we want to see all of $f(t) = \sin t$, as if we were starting $f(t) = \sin t$ at $t = 0$, like this:



In this graph, the start of $f(t) = \sin t$ just slid over to the right from $t = 0$ to $t = 3\pi/2$.

This kind of shift is important to understand because it'll come up frequently when we start applying the Laplace transform to step functions.

Second Shifting Theorem

To represent a function that's shifted over by some distance after being "turned off" over that same distance, we use

$$f(t-c)u(t-c) = \begin{cases} 0 & 0 \le t < c \\ f(t-c) & t \ge c \end{cases}$$

Using the example above, where we shifted the $t \ge 0$ part of the sine function to the right by $3\pi/2$, and turning off the function on $0 \le t < 3\pi/2$, we would express this as

$$\sin\left(t-\frac{3\pi}{2}\right)u\left(t-\frac{3\pi}{2}\right)=\begin{cases}0 & 0\le t<\frac{3\pi}{2}\\ \sin\left(t-\frac{3\pi}{2}\right) & t\ge\frac{3\pi}{2}\end{cases}$$

The Laplace transform of the left side of this equation, which takes the general form $f(t-c)u(t-c)$, is what's given by the **Second Shifting Theorem**, or Second Translation Theorem:

$$\mathscr{L}(f(t-c)u(t-c))=e^{-cs}F(s)$$

when $F(s)=\mathscr{L}(f(t))$ and $c>0$. By this theorem, the Laplace transform of the sine function we've been working with would be

$$\mathscr{L}\left(\sin\left(t-\frac{3\pi}{2}\right)u\left(t-\frac{3\pi}{2}\right)\right)=e^{-\frac{3\pi}{2}s}\frac{1}{s^2+1}$$

because $1/(s^2+1)$ is the Laplace transform of $\sin t$. Let's do another example where we apply a Laplace transform using the Second Shifting Theorem.

Example

Use a step function to represent shifting the portion of $f(x)$ for $x\ge0$ to the right 3 units, while turning off the function on $0\le x<3$. Then take its Laplace transform.

$$f(x)=\frac{1}{2}x^3+x^2-x$$

Because we want to shift the curve 3 units to the right, we can start by substituting $c = 3$ into our formula for this kind of shift.

$$f(x-c)u(x-c) = \begin{cases} 0 & 0 \le x < c \\ f(x-c) & x \ge c \end{cases}$$

$$f(x-3)u(x-3) = \begin{cases} 0 & 0 \le x < 3 \\ f(x-3) & x \ge 3 \end{cases}$$

Now we just need to replace $f(x-3)$, which is

$$f(x) = \frac{1}{2}x^3 + x^2 - x$$

$$f(x-3) = \frac{1}{2}(x-3)^3 + (x-3)^2 - (x-3)$$

So the shifted function is represented by

$$\left(\frac{1}{2}(x-3)^3 + (x-3)^2 - (x-3)\right) u(x-3)$$

$$= \begin{cases} 0 & 0 \le x < 3 \\ \frac{1}{2}(x-3)^3 + (x-3)^2 - (x-3) & x \ge 3 \end{cases}$$

To find the Laplace transform of this function, we'll need the Laplace transform of $f(x)$.

$$F(s) = \mathscr{L}(f(x)) = \mathscr{L}\left(\frac{1}{2}x^3\right) + \mathscr{L}(x^2) - \mathscr{L}(x)$$

$$F(s) = \mathscr{L}(f(x)) = \frac{1}{2}\mathscr{L}(x^3) + \mathscr{L}(x^2) - \mathscr{L}(x)$$

$$F(s) = \mathscr{L}(f(x)) = \frac{1}{2}\left(\frac{3!}{s^{3+1}}\right) + \frac{2!}{s^{2+1}} - \frac{1!}{s^{1+1}}$$

$$F(s) = \mathscr{L}(f(x)) = \frac{3}{s^4} + \frac{2}{s^3} - \frac{1}{s^2}$$

Then by the Second Shifting Theorem, the Laplace transform of the shifted function is

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-3s}\left(\frac{3}{s^4} + \frac{2}{s^3} - \frac{1}{s^2}\right)$$

Proving the SST

To prove the Second Shifting Theorem, we can start with the definition of the Laplace Transform,

$$\mathscr{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-st}u_c(t)f(t-c)\ dt$$

Remember that for $0 \le t < c$, the value of the step function is $u_c(t) = 0$, which means the value of the integral on the interval $0 \le t < c$ will be 0. So we only really need to consider the integral from the point at which $t = c$, and we can therefore change the lower limit of integration from $t = 0$ to $t = c$. Once we do, we know $u_c(t) = 1$, which means the step function simplifies out of the integral.

$$\mathscr{L}(u_c(t)f(t-c)) = \int_c^\infty e^{-st}f(t-c)\ dt$$

Make a substitution $v = t - c$ and $dv = dt$. Plugging the limits of integration $t = c$ and $t = \infty$ into $v = t - c$ gives new limits of integration $v = 0$ and $v = \infty$.

$$\mathscr{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-s(v+c)}f(v)\ dv$$

$$\mathscr{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-sv-sc}f(v)\ dv$$

$$\mathscr{L}(u_c(t)f(t-c)) = \int_0^\infty e^{-sv}e^{-sc}f(v)\ dv$$

$$\mathscr{L}(u_c(t)f(t-c)) = e^{-sc}\int_0^\infty e^{-sv}f(v)\ dv$$

The remaining integral is the definition of the Laplace transform of $f(v)$, so

$$\mathscr{L}(u_c(t)f(t-c)) = e^{-sc}F(s)$$

This formula works as long as the function $f(t)$ is shifted by c. In other words, as long as $v = t - c$. This is how we get to the formula for the Second Shifting Theorem.

Laplace transforms of step functions

In the last lesson, we looked at the Laplace transform of $f(t-c)u(t-c)$, the product of $f(t)$ shifted to the right along the t-axis by c units, and a unit step function that "turns on" at $t=c$.

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

This was the Second Shifting Theorem. Now we want to talk more about the Laplace transforms of step functions, starting with a couple of extensions of the Second Shifting Theorem.

Extending the SST

There are two natural extensions of the Second Shifting Theorem that are worth mentioning. First, if $f(t)=1$, then the Laplace transform $F(s)$ of $f(t)=1$ is $F(s)=1/s$, and the Second Shifting Theorem simplifies to

$$\mathscr{L}(f(t-c)u(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(u(t-c)) = e^{-cs}\left(\frac{1}{s}\right)$$

$$\mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$

Of course, because $u_c(t)$ is alternate notation for $u(t-c)$, we can also write this formula as

$$\mathscr{L}(u_c(t)) = \frac{e^{-cs}}{s}$$

This is an important formula that we'll use regularly to apply the Laplace transform to step functions.

Second, the inverse of the Second Shifting Theorem is also true.

$$\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$$

Let's do an example where we work backwards to find the original function using the inverse transform.

Example

Find the inverse Laplace transform.

$$G(s) = \frac{se^{-3s}}{(4s-1)(s+2)}$$

We'll start by pulling e^{-3s} out of the fraction, so that the function is then in the form $G(s) = e^{-3s}F(s)$.

$$G(s) = e^{-3s}\frac{s}{(4s-1)(s+2)}$$

Use partial fractions to decompose the remaining fraction, which we'll call $F(s)$.

$$\frac{s}{(4s-1)(s+2)} = \frac{A}{4s-1} + \frac{B}{s+2}$$

To find the value of A, we'll remove the $(4s-1)$ factor, set $s = 1/4$, and then find the value of the fraction on the left.

$$\frac{s}{s+2}$$

$$\frac{\frac{1}{4}}{\frac{1}{4}+2}=\frac{\frac{1}{4}}{\frac{9}{4}}=\frac{1}{4}\left(\frac{4}{9}\right)=\frac{1}{9}$$

To find the value of B, we'll remove the $(s+2)$ factor, set $s=-2$, and then find the value of the fraction on the left.

$$\frac{s}{4s-1}$$

$$\frac{-2}{4(-2)-1}=\frac{-2}{-8-1}=\frac{-2}{-9}=\frac{2}{9}$$

So the partial fractions decomposition is

$$F(s)=\frac{\frac{1}{9}}{4s-1}+\frac{\frac{2}{9}}{s+2}$$

$$F(s)=\frac{\frac{1}{9}}{4\left(s-\frac{1}{4}\right)}+\frac{\frac{2}{9}}{s+2}$$

$$F(s)=\frac{1}{36}\left(\frac{1}{s-\frac{1}{4}}\right)+\frac{2}{9}\left(\frac{1}{s-(-2)}\right)$$

The inverse transform of $F(s)$ is

$$f(t)=\frac{1}{36}e^{\frac{1}{4}t}+\frac{2}{9}e^{-2t}$$

Which means the inverse transform of $G(s)$ is

$$\mathscr{L}^{-1}(G(s)) = u(t-3)f(t-3) \text{ with } f(t) = \frac{1}{36}e^{\frac{1}{4}t} + \frac{2}{9}e^{-2t}$$

Fixing the shift

The Second Shifting Theorem and its extensions are all for functions perfectly shifted to $t = c$, $f(t-c)$. But we're often confronted with the Laplace transform of the product of a function and a unit step function, where the function itself isn't shifted by the same distance between 0 and the point where the function "turns on" at $t = c$.

A function like this that doesn't have the right shift can be written as $g(t)u(t-c)$. To find the Laplace transform of $g(t)u(t-c)$, we'll need to rewrite g with the correct shift c.

Essentially, we do this by finding the equation of the curve which is identical to $g(t)$, but shifted c units to the *left*. Then we shift that curve back c units to the *right* to get a curve that's identical to $g(t)$, just rewritten with the shift that we need.

For instance, consider the function $t^3u(t-4)$. The unit step function shows us that we have a 4-unit shift to the right, so we need to rewrite $g(t) = t^3$ with a 4-unit shift to the right as well. But in order to include a shift to the right, while still keeping the same $g(t) = t^3$ curve, we need to start with a curve that's 4 units to the left of $g(t)$, and then shift that curve 4 units to the right, in order to get back to $g(t)$ itself.

The curve that's 4 units to the left of $g(t)=t^3$, which we'll call $h(t)$, is

$$h(t)=g(t+4)=(t+4)^3$$

$$h(t)=g(t+4)=t^3+12t^2+48t+64$$

Now if we shift this curve back 4 units to the right,

$$h(t-4)=g(t+4-4)=(t-4)^3+12(t-4)^2+48(t-4)+64$$

$$h(t-4)=g(t)=(t-4)^3+12(t-4)^2+48(t-4)+64$$

$$g(t)=(t-4)^3+12(t-4)^2+48(t-4)+64$$

we can see that we have a new equation for $g(t)$. This new $g(t)$ is identical to $g(t)=t^3$ (we can graph both of them to prove to ourselves that this is true), it's just been rewritten in terms of a 4-unit shift to the right. Therefore, we can now rewrite $t^3u(t-4)$ as

$$((t-4)^3+12(t-4)^2+48(t-4)+64)u(t-4)$$

$$(t-4)^3u(t-4)+12(t-4)^2u(t-4)+48(t-4)u(t-4)+64u(t-4)$$

And now we've transformed $t^3u(t-4)$ into a form to which we can apply the Second Shifting Theorem $\mathscr{L}(f(t-c)u(t-c))=e^{-cs}F(s)$, which means we've rewritten $t^3u(t-4)$ as something to which we can apply the Laplace Transform.

$$\mathscr{L}(t^3u(t-4))$$

$$\mathscr{L}((t-4)^3u(t-4)+12(t-4)^2u(t-4)+48(t-4)u(t-4)+64u(t-4))$$

$$\mathscr{L}((t-4)^3u(t-4))+\mathscr{L}(12(t-4)^2u(t-4))+\mathscr{L}(48(t-4)u(t-4))+\mathscr{L}(64u(t-4))$$

We know from the Second Shifting Theorem that this Laplace transform simplifies to

$$\frac{3!}{s^{3+1}}e^{-4s}+12\left(\frac{2!}{s^{2+1}}\right)e^{-4s}+48\left(\frac{1!}{s^{1+1}}\right)e^{-4s}+64\left(\frac{1}{s}\right)e^{-4s}$$

$$\frac{6}{s^4}e^{-4s}+\frac{24}{s^3}e^{-4s}+\frac{48}{s^2}e^{-4s}+\frac{64}{s}e^{-4s}$$

Even though we were able to get to this transformed result, the algebra required to shift the function to the left and then back to the right definitely got a little complicated. Luckily, there's a formula we can use to fix the shift while avoiding doing all this algebra by hand.

$$\mathcal{L}(g(t)u(t-c))=e^{-cs}\mathcal{L}(g(t+c))$$

In other words, starting over again with $t^3u(t-4)$, using this formula to find the Laplace transform will give us

$$\mathcal{L}(t^3u(t-4))=e^{-4s}\mathcal{L}(g(t+4))$$

$$\mathcal{L}(t^3u(t-4))=e^{-4s}\mathcal{L}((t+4)^3)$$

$$\mathcal{L}(t^3u(t-4))=e^{-4s}\mathcal{L}(t^3+12t^2+48t+64)$$

$$\mathcal{L}(t^3u(t-4))=e^{-4s}\mathcal{L}(t^3)+12e^{-4s}\mathcal{L}(t^2)+48e^{-4s}\mathcal{L}(t)+64e^{-4s}\mathcal{L}(1)$$

and then we can apply transform formulas.

$$\mathcal{L}(t^3u(t-4))=e^{-4s}\left(\frac{3!}{s^{3+1}}\right)+12e^{-4s}\left(\frac{2!}{s^{2+1}}\right)+48e^{-4s}\left(\frac{1!}{s^{1+1}}\right)+64e^{-4s}\left(\frac{1}{s}\right)$$

$$\mathscr{L}(t^3u(t-4)) = e^{-4s}\left(\frac{6}{s^4}\right) + 12e^{-4s}\left(\frac{2}{s^3}\right) + 48e^{-4s}\left(\frac{1}{s^2}\right) + 64e^{-4s}\left(\frac{1}{s}\right)$$

$$\mathscr{L}(t^3u(t-4)) = \frac{6}{s^4}e^{-4s} + \frac{24}{s^3}e^{-4s} + \frac{48}{s^2}e^{-4s} + \frac{64}{s}e^{-4s}$$

Notice how we get to the same transform using both methods (fixing the shift ourselves algebraically, and using the formula that fixes the shift for us).

Let's do an example where we walk through how to verify that our functions are shifted correctly.

Example

Find the Laplace transform of the function $g(t)$.

$$g(t) = -3u_2(t) + 6(t-4)^2u_4(t) - 9e^{12-2t}u_6(t)$$

We know that the Laplace transform of $u_c(t)$ is given by

$$\mathscr{L}(u_c(t)) = \frac{e^{-sc}}{s}$$

Therefore, the Laplace transform of $-3u_2(t)$ must be

$$-\frac{3e^{-2s}}{s}$$

To find the transform of $6(t-4)^2u_4(t)$, we identify that $c=4$, and then we can make a substitution of $t+4$ into $f(t-c)$ to determine the function before it was shifted.

$$6(t-4)^2$$

$$6(t+4-4)^2$$

$$6t^2$$

So the Laplace transform will be given by

$$\mathscr{L}(u_c(t)f(t-c)) = e^{-sc}F(s)$$

$$\mathscr{L}(6(t-4)^2u_4(t)) = e^{-4s}\left(6\frac{2!}{s^{2+1}}\right)$$

$$\mathscr{L}(6(t-4)^2u_4(t)) = 6e^{-4s}\left(\frac{2}{s^3}\right)$$

$$\mathscr{L}(6(t-4)^2u_4(t)) = \frac{12e^{-4s}}{s^3}$$

To find the transform of $-9e^{12-2t}u_6(t)$, we identify that $c=6$, and then we can make a substitution of $t+6$ into $f(t-c)$ to determine the function before it was shifted.

$$-9e^{12-2t}$$

$$-9e^{12-2(t+6)}$$

$$-9e^{12-2t-12}$$

$$-9e^{-2t}$$

So the Laplace transform will be given by

$$\mathscr{L}(u_c(t)f(t-c)) = e^{-sc}F(s)$$

$$\mathscr{L}(-9e^{12-2t}u_6(t)) = e^{-6s}\left(-9\frac{1}{s-(-2)}\right)$$

$$\mathscr{L}(-9e^{12-2t}u_6(t)) = -\frac{9e^{-6s}}{s+2}$$

Putting these three transforms together, the Laplace transform is

$$\mathscr{L}(g(t)) = -\frac{3e^{-2s}}{s} + \frac{12e^{-4s}}{s^3} - \frac{9e^{-6s}}{s+2}$$

Step functions with initial value problems

Sometimes we'll be given a differential equation in which the forcing function is a step function or includes a step function, and we'll be asked to solve an initial value problem with the differential equation.

Solving the initial value problem

In general, to solve the initial value problem, we'll follow these steps:

1. Make sure the forcing function is being shifted correctly, and identify the function being shifted.

2. Apply a Laplace transform to each part of the differential equation, substituting initial conditions to simplify.

3. Solve for $Y(s)$.

4. Apply an inverse transform to find $y(t)$.

As a reminder, when we apply the Laplace transform to the equation in step 2, we'll often need these transform formulas:

$$\mathscr{L}(u_c(t)) = \mathscr{L}(u(t-c)) = \frac{e^{-cs}}{s}$$ Heaviside function

$$\mathscr{L}(u_c(t)f(t-c)) = e^{-cs}F(s)$$

$$\mathscr{L}(u_c(t)g(t)) = e^{-cs}\mathscr{L}(g(t+c))$$

Let's do an example so that we can see how to apply this process to solve an initial value problem when we have a step function in the forcing function.

Example

Solve the initial value problem, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' + 4y' = (\sin t)u(t - 2\pi)$$

The forcing function could also be written as

$$g(t) = \begin{cases} 0 & 0 \leq t < 2\pi \\ \sin t & t \geq 2\pi \end{cases}$$

Because the step function shows $c = 2\pi$, we need $f(t) = \sin t$ to include a shift of 2π as well. We can fix the shift using

$$\mathscr{L}(u_c(t)g(t)) = e^{-cs}\mathscr{L}(g(t + c))$$

So we'll find

$$g(t + 2\pi)$$

$$\sin(t + 2\pi)$$

By the trigonometric identity $\sin(\theta + \alpha) = \sin\theta\cos\alpha + \cos\theta\sin\alpha$, we can rewrite our sine function as

$$\sin t\cos(2\pi) + \cos t\sin(2\pi)$$

$$\sin t(1) + \cos t(0)$$

$$\sin t$$

Then using the Laplace transforms of $\sin t$ and the step function on the right side, the transformed equation becomes

$$\mathscr{L}(y'') + 4\mathscr{L}(y') = \mathscr{L}(\sin t(u(t - 2\pi)))$$

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) = \frac{1}{s^2 + 1}e^{-2\pi s}$$

Substitute the initial conditions $y(0) = 1$ and $y'(0) = 0$, then solve for $Y(s)$.

$$s^2Y(s) - s + 4sY(s) - 4 = \frac{1}{s^2 + 1}e^{-2\pi s}$$

$$(s^2 + 4s)Y(s) - s - 4 = \frac{1}{s^2 + 1}e^{-2\pi s}$$

$$(s^2 + 4s)Y(s) = s + 4 + \frac{1}{s^2 + 1}e^{-2\pi s}$$

$$Y(s) = \frac{s + 4}{s^2 + 4s} + \frac{1}{(s^2 + 1)(s^2 + 4s)}e^{-2\pi s}$$

$$Y(s) = \frac{s + 4}{s(s + 4)} + \frac{1}{s(s^2 + 1)(s + 4)}e^{-2\pi s}$$

$$Y(s) = \frac{1}{s} + \frac{1}{s(s^2 + 1)(s + 4)}e^{-2\pi s}$$

Let's apply a partial fractions decomposition.

$$\frac{1}{s(s^2+1)(s+4)} = \frac{A}{s} + \frac{B}{s+4} + \frac{Cs+D}{s^2+1}$$

To solve for A, we'll remove the factor of s from the denominator on the left side, then set $s = 0$.

$$\frac{1}{(s^2+1)(s+4)}$$

$$\frac{1}{(0^2+1)(0+4)} = \frac{1}{(1)(4)} = \frac{1}{4}$$

To solve for B, we'll remove the factor of $s+4$ from the denominator on the left side, then set $s = -4$.

$$\frac{1}{s(s^2+1)}$$

$$\frac{1}{-4((-4)^2+1)} = \frac{1}{-4(17)} = -\frac{1}{68}$$

To find C and D, we'll expand multiply through both sides of the decomposition equation by the denominator from the left side.

$$1 = A(s^2+1)(s+4) + Bs(s^2+1) + (Cs+D)s(s+4)$$

$$1 = As^3 + 4As^2 + As + 4A + Bs^3 + sB + Cs^3 + 4Cs^2 + Ds^2 + 4Ds$$

Substitute the values $A = 1/4$ and $B = -1/68$ that we already found.

$$1 = \left(\frac{1}{4} - \frac{1}{68} + C\right)s^3 + \left(4\left(\frac{1}{4}\right) + 4C + D\right)s^2 + \left(\frac{1}{4} - \frac{1}{68} + 4D\right)s + 4\left(\frac{1}{4}\right)$$

$$1 = \left(\frac{4}{17} + C\right)s^3 + (1 + 4C + D)s^2 + \left(\frac{4}{17} + 4D\right)s + 1$$

Then equating coefficients gives

$$\frac{4}{17} + C = 0 \text{ so } C = -\frac{4}{17}$$

$$D = -\frac{1}{17}$$

Plugging $A = 1/4$, $B = -1/68$, $C = -4/17$, and $D = -1/17$ into the decomposition gives

$$\frac{1}{s(s^2+1)(s+4)} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{68}\left(\frac{1}{s+4}\right) + \frac{-\frac{4}{17}s - \frac{1}{17}}{s^2+1}$$

$$\frac{1}{s(s^2+1)(s+4)} = \frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{68}\left(\frac{1}{s+4}\right) - \frac{1}{17}\left(\frac{4s+1}{s^2+1}\right)$$

So $Y(s)$ is

$$Y(s) = \frac{1}{s} + \left(\frac{1}{4}\left(\frac{1}{s}\right) - \frac{1}{68}\left(\frac{1}{s+4}\right) - \frac{1}{17}\left(\frac{4s+1}{s^2+1}\right)\right)e^{-2\pi s}$$

$$Y(s) = \frac{1}{s} + \frac{1}{4}\left(\frac{1}{s}\right)e^{-2\pi s} - \frac{1}{68}\left(\frac{1}{s+4}\right)e^{-2\pi s} - \frac{1}{17}\left(\frac{4s+1}{s^2+1}\right)e^{-2\pi s}$$

$$Y(s) = \frac{1}{s} + \frac{1}{4}\left(\frac{1}{s}\right)e^{-2\pi s} - \frac{1}{68}\left(\frac{1}{s+4}\right)e^{-2\pi s} - \frac{1}{17}\left(\frac{4s}{s^2+1} + \frac{1}{s^2+1}\right)e^{-2\pi s}$$

$$Y(s) = \frac{1}{s} + \frac{1}{4}\left(\frac{1}{s}\right)e^{-2\pi s} - \frac{1}{68}\left(\frac{1}{s+4}\right)e^{-2\pi s}$$

$$-\frac{4}{17}\left(\frac{s}{s^2+1}\right)e^{-2\pi s}-\frac{1}{17}\left(\frac{1}{s^2+1}\right)e^{-2\pi s}$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s)) = f(t-c)u(t-c)$ that we looked at earlier, the inverse transform is

$$y(t) = 1 + \frac{1}{4}(1)u(t-2\pi) - \frac{1}{68}(e^{-4(t-2\pi)})u(t-2\pi)$$

$$-\frac{4}{17}\cos(t-2\pi)u(t-2\pi) - \frac{1}{17}\sin(t-2\pi)u(t-2\pi)$$

$$y(t) = 1 + \frac{1}{4}u(t-2\pi) - \frac{1}{68}e^{-4t+8\pi}u(t-2\pi)$$

$$-\frac{4}{17}\cos(t-2\pi)u(t-2\pi) - \frac{1}{17}\sin(t-2\pi)u(t-2\pi)$$

Let's do another example with an initial value problem.

Example

Solve the initial value problem, given $y(0) = 0$ and $y'(0) = -1$.

$$y'' + 2y' = 1 + (t-1)u_3(t)$$

Remember that each function must be shifted by the appropriate amount. Getting things set up for the proper shifts gives us

$$f(t) = 1 + (t-1)u_3(t)$$

$$f(t) = 1 + (t - 3 + 3 - 1)u_3(t)$$

$$f(t) = 1 + (t - 3 + 2)u_3(t)$$

So for the Heaviside function, it looks like $g(t) = t + 2$ is the function being shifted. Then the transformed equation will be

$$\mathscr{L}(y'') + 2\mathscr{L}(y') = \mathscr{L}(1) + \mathscr{L}((t - 3 + 2)u_3(t))$$

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) = \frac{1}{s} + e^{-3s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

Substitute the initial conditions $y(0) = 0$ and $y'(0) = -1$, then solve for $Y(s)$.

$$s^2Y(s) + 1 + 2sY(s) = \frac{1}{s} + e^{-3s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

$$(s^2 + 2s)Y(s) + 1 = \frac{1}{s} + e^{-3s}\left(\frac{1}{s^2} + \frac{2}{s}\right)$$

$$(s^2 + 2s)Y(s) = \frac{1}{s} + e^{-3s}\frac{1}{s^2} + e^{-3s}\frac{2}{s} - 1$$

$$Y(s) = \frac{1}{s(s^2 + 2s)} + e^{-3s}\frac{1}{s^2(s^2 + 2s)} + e^{-3s}\frac{2}{s(s^2 + 2s)} - \frac{1}{s^2 + 2s}$$

$$Y(s) = \frac{1}{s^2(s + 2)} + 2e^{-3s}\frac{1}{s^2(s + 2)} + e^{-3s}\frac{1}{s^3(s + 2)} - \frac{1}{s(s + 2)}$$

Apply a partial fractions decomposition to the first fraction, which is the same decomposition we'll use for the second fraction.

$$\frac{1}{s^2(s + 2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s + 2}$$

To find C, remove the factor of $s+2$ from the left side, then substitute $s=-2$.

$$\frac{1}{s^2} \to \frac{1}{(-2)^2} \to \frac{1}{4}$$

To find B, remove the factor of s^2 from the left side, then substitute $s=0$.

$$\frac{1}{s+2} \to \frac{1}{0+2} \to \frac{1}{2}$$

To find A, substitute $B=1/2$ and $C=1/4$, and any value we haven't already used for s. We've already used $s=-2, 0$, so we'll choose $s=1$.

$$\frac{1}{1^2(1+2)}=\frac{A}{1}+\frac{\frac{1}{2}}{1^2}+\frac{\frac{1}{4}}{1+2}$$

$$\frac{1}{3}=A+\frac{1}{2}+\frac{1}{12}$$

$$A=\frac{4}{12}-\frac{6}{12}-\frac{1}{12}$$

$$A=-\frac{1}{4}$$

So the decomposition is

$$\frac{1}{s^2(s+2)}=\frac{-\frac{1}{4}}{s}+\frac{\frac{1}{2}}{s^2}+\frac{\frac{1}{4}}{s+2}$$

Replacing the first and second fractions in $Y(s)$ with this decomposition gives

$$Y(s) = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2} + 2e^{-3s}\left(\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2}\right)$$

$$+e^{-3s}\frac{1}{s^3(s+2)} - \frac{1}{s(s+2)}$$

Apply a partial fractions decomposition to the third fraction.

$$\frac{1}{s^3(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s+2}$$

To find D, remove the factor of $s+2$ from the left side, then substitute $s = -2$.

$$\frac{1}{s^3} \to \frac{1}{(-2)^3} \to -\frac{1}{8}$$

To find C, remove the factor of s^3 from the left side, then substitute $s = 0$.

$$\frac{1}{s+2} \to \frac{1}{0+2} \to \frac{1}{2}$$

To find A and B, multiply through both sides of the decomposition by the denominator from the left side.

$$1 = As^2(s+2) + Bs(s+2) + C(s+2) + Ds^3$$

Substitute the values $C = 1/2$ and $D = -1/8$ that we already found.

$$1 = As^2(s+2) + Bs(s+2) + \frac{1}{2}(s+2) - \frac{1}{8}s^3$$

$$1 = As^3 + 2As^2 + Bs^2 + 2Bs + \frac{1}{2}s + 1 - \frac{1}{8}s^3$$

$$1 = \left(A - \frac{1}{8}\right)s^3 + (2A + B)s^2 + \left(2B + \frac{1}{2}\right)s + 1$$

$$0 = \left(A - \frac{1}{8}\right)s^3 + (2A + B)s^2 + \left(2B + \frac{1}{2}\right)s$$

Equate coefficients from the first and third terms to find A and B.

$$A - \frac{1}{8} = 0$$

$$A = \frac{1}{8}$$

$$2B + \frac{1}{2} = 0$$

$$2B = -\frac{1}{2}$$

$$B = -\frac{1}{4}$$

So the decomposition is

$$\frac{1}{s^3(s+2)} = \frac{\frac{1}{8}}{s} + \frac{-\frac{1}{4}}{s^2} + \frac{\frac{1}{2}}{s^3} + \frac{-\frac{1}{8}}{s+2}$$

Replacing the third fraction in $Y(s)$ with this decomposition gives

$$Y(s) = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2} + 2e^{-3s}\left(\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2}\right)$$

$$+e^{-3s}\left(\frac{\frac{1}{8}}{s} + \frac{-\frac{1}{4}}{s^2} + \frac{\frac{1}{2}}{s^3} + \frac{-\frac{1}{8}}{s+2}\right) - \frac{1}{s(s+2)}$$

Apply a partial fractions decomposition to the fourth fraction.

$$\frac{1}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}$$

To find B, remove the factor of $s+2$ from the left side, then substitute $s = -2$.

$$\frac{1}{s} \to -\frac{1}{2}$$

To find A, remove the factor of s from the left side, then substitute $s = 0$.

$$\frac{1}{s+2} \to \frac{1}{0+2} \to \frac{1}{2}$$

So the decomposition is

$$\frac{1}{s(s+2)} = \frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s+2}$$

Replacing the fourth fraction in $Y(s)$ with this decomposition gives

$$Y(s) = \frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2} + 2e^{-3s}\left(\frac{-\frac{1}{4}}{s} + \frac{\frac{1}{2}}{s^2} + \frac{\frac{1}{4}}{s+2}\right)$$

$$+e^{-3s}\left(\frac{\frac{1}{8}}{s} + \frac{-\frac{1}{4}}{s^2} + \frac{\frac{1}{2}}{s^3} + \frac{-\frac{1}{8}}{s+2}\right) - \left(\frac{\frac{1}{2}}{s} + \frac{-\frac{1}{2}}{s+2}\right)$$

$$Y(s) = -\frac{1}{4}\left(\frac{1}{s}\right) + \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{4}\left(\frac{1}{s+2}\right)$$

$$-\frac{1}{2}e^{-3s}\left(\frac{1}{s}\right)+e^{-3s}\left(\frac{1}{s^2}\right)+\frac{1}{2}e^{-3s}\left(\frac{1}{s+2}\right)$$

$$+\frac{1}{8}e^{-3s}\left(\frac{1}{s}\right)-\frac{1}{4}e^{-3s}\left(\frac{1}{s^2}\right)+\frac{1}{2}e^{-3s}\left(\frac{1}{s^3}\right)-\frac{1}{8}e^{-3s}\left(\frac{1}{s+2}\right)$$

$$-\frac{1}{2}\left(\frac{1}{s}\right)+\frac{1}{2}\left(\frac{1}{s+2}\right)$$

Then with the inverse transform formula $\mathscr{L}^{-1}(e^{-cs}F(s))=f(t-c)u(t-c)$ that we looked at earlier, the inverse transform is

$$y(t)=-\frac{1}{4}(1)+\frac{1}{2}(t)+\frac{1}{4}(e^{-2t})$$

$$-\frac{1}{2}u_3(t)(1)+u_3(t)(t-3)+\frac{1}{2}u_3(t)e^{-2(t-3)}$$

$$+\frac{1}{8}u_3(t)(1)-\frac{1}{4}u_3(t)(t-3)+\frac{1}{2(2!)}u_3(t)(t-3)^2-\frac{1}{8}u_3(t)e^{-2(t-3)}$$

$$-\frac{1}{2}(1)+\frac{1}{2}(e^{-2t})$$

$$y(t)=-\frac{1}{4}+\frac{1}{2}t+\frac{1}{4}e^{-2t}$$

$$-\frac{1}{2}u_3(t)+u_3(t)(t-3)+\frac{1}{2}u_3(t)e^{6-2t}$$

$$+\frac{1}{8}u_3(t)-\frac{1}{4}u_3(t)(t-3)+\frac{1}{4}u_3(t)(t-3)^2-\frac{1}{8}u_3(t)e^{6-2t}$$

$$-\frac{1}{2}+\frac{1}{2}e^{-2t}$$

$$y(t) = \frac{3}{4}e^{-2t} + \frac{1}{2}t - \frac{3}{4} + u_3(t)\left(\frac{3}{8}e^{6-2t} + \frac{1}{4}t^2 - \frac{3}{4}t - \frac{3}{8}\right)$$

The Dirac delta function

We already understand that a step function is a function that changes value suddenly at a particular point. The value of the unit step function, or Heaviside function, changes value from 0 to 1 exactly at the point $t = c$.

$$u_c(t) = \begin{cases} 0 & 0 \le t < c \\ 1 & t \ge c \end{cases}$$

The **Dirac delta function** (which is actually less like a function and more like a distribution) also models a sudden change. It's particularly useful for modeling the application of a very large force over a very short time.

Think about the force of a golf club striking a golf ball, or a hammer striking a nail as examples of extreme force that's applied almost instantaneously. These are the kinds of functions that are best handled using the Dirac delta function.

Properties of the Dirac delta function

There are three important properties of the Dirac delta function δ ("delta") that we need to understand. First, the function has a zero value everywhere except at $t = c$.

$\delta(t - c) = 0$ for $t \neq c$

Think again about a golf club striking a golf ball. No force at all is being applied to the golf ball, except at that fraction of a second when $t = c$. So the function will have a zero value everywhere outside of that instant. But at that instant $t = c$, the force applied is so extreme that we almost think about it as being infinite.

Second, the integral of the Dirac delta function is 1, as long as the interval of integration includes the point $t = c$. If we think about Dirac delta as a distribution, then we can imagine all of the area under the distribution being condensed into the instant $t = c$. At that instant, we see all the area under the distribution, a total of 1. But outside of that instant, we see no force at all.

$$\int_{c-\epsilon}^{c+\epsilon} \delta(t-c)\ dt = 1 \text{ for } \epsilon > 0$$

Because the value of this integral is 1, we sometimes call $\delta(t-c)$ a "unit impulse." Third, multiplying the Dirac delta function by any function $f(t)$ gives us a non-constant model of this strong impulse function. And if we integrate that model, as long as our interval of integration includes $t = c$, then the value of the integral will be $f(c)$.

$$\int_{c-\epsilon}^{c+\epsilon} f(t)\delta(t-c)\ dt = f(c) \text{ for } \epsilon > 0$$

Applying the Laplace transform

The Laplace transform of the Dirac delta function is

$$\mathscr{L}(\delta(t-c)) = \int_0^\infty e^{-st}\delta(t-c)\ dt = e^{-cs} \text{ for } c > 0$$

This transform is really what we've been building toward, so let's do an example where we find the Laplace transform, then solve an initial value problem with the Dirac delta function.

Example

Solve the initial value problem, given $y(0) = 1$ and $y'(0) = 0$.

$$y'' + y = 2\delta(t - \pi)$$

From our table of Laplace transforms, we know

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(\delta(t - c)) = e^{-cs}$$

so the Laplace transform of the second order nonhomogeneous equation with a Dirac delta forcing function is

$$\mathscr{L}(y'') + \mathscr{L}(y) = \mathscr{L}(2\delta(t - \pi))$$

$$s^2Y(s) - sy(0) - y'(0) + Y(s) = 2e^{-\pi s}$$

Substitute initial conditions, then solve for $Y(s)$.

$$s^2Y(s) - s(1) - 0 + Y(s) = 2e^{-\pi s}$$

$$s^2Y(s) - s + Y(s) = 2e^{-\pi s}$$

$$s^2Y(s) + Y(s) = s + 2e^{-\pi s}$$

$$(s^2+1)Y(s) = s + 2e^{-\pi s}$$

$$Y(s) = \frac{s}{s^2+1} + 2\left(\frac{1}{s^2+1}\right)e^{-\pi s}$$

To find the general solution $y(t)$, we'll take the inverse transform.

$$y(t) = \cos t + 2\sin(t-\pi)u(t-\pi)$$

Of course, we could also write this solution as

$$y(t) = \begin{cases} \cos t & 0 \le t < \pi \\ \cos t + 2\sin(t-\pi) & t \ge \pi \end{cases}$$

Relating the Dirac delta and unit step functions

The last point we want to make is that the Dirac delta function is actually the derivative of the unit step function.

We can prove this to ourselves if we start with what we already know about the Dirac delta function. We know that, if the interval of integration includes the point $t=c$, then the integral of the Dirac delta function will be 1. But if the interval doesn't include that one special point, then the integral will be 0. We can write this as

$$\int_{-\infty}^{t} \delta(u-c)\ du = \begin{cases} 0 & t < c \\ 1 & t > c \end{cases}$$

But now we notice that this right side is the definition of the unit step function, or Heaviside function, so

$$\int_{-\infty}^{t} \delta(u-c)\ du = u_c(t)$$

From here, applying the Fundamental Theorem of Calculus to this equation shows that the Dirac delta function is the derivative of the unit step (Heaviside) function.

$$u_c'(t) = \frac{d}{dt}\left(\int_{-\infty}^{t} \delta(u-c)\ du\right) = \delta(t-c)$$

Convolution integrals

If we want to take the Laplace transform of the sum of functions, we can simply find the Laplace transform of each function individually, and then add the transforms to get the transform of the sum. In other words, the transform of the sum is the sum of the transforms.

For instance, to find the transform of $f(t) + g(t)$, we take the transform of $f(t)$ and $g(t)$ individually,

$$F(s) = \int_0^\infty e^{-st} f(t)\ dt$$

$$G(s) = \int_0^\infty e^{-st} g(t)\ dt$$

and then we take the sum of the transforms to get the transform of the sum. So the **transform of the sum** $f(t) + g(t)$ is given by

$$F(s) + G(s) = \int_0^\infty e^{-st} f(t)\ dt + \int_0^\infty e^{-st} g(t)\ dt$$

The product of functions

Unfortunately, we don't have the same simple process for finding the transform $F(s)G(s)$ of the product of functions $f(t)g(t)$. We *cannot* say that the transform of the product is the product of the transforms.

$$F(s)G(s) \neq \int_0^\infty e^{-st} f(t)\ dt \cdot \int_0^\infty e^{-st} g(t)\ dt$$

Despite this inconvenient fact, it would be really helpful if we had a way of transforming the product of functions. Luckily, we do have a way of handling this, which is, of course, with the convolution integral.

Think of **the convolution** as the function we plug into the definition of the Laplace transform in order to get the product of the transforms. In other words, if we write a new function $f(t) * g(t)$ (the asterisk symbol is what we use to express the convolution of functions), then the Laplace transform of this new convolution function will give us the product of the transforms, $F(s)G(s)$.

$$F(s)G(s) = \mathscr{L}(f(t) * g(t))$$

$$F(s)G(s) = \int_0^\infty e^{-st} f(t) * g(t) \ dt$$

Of course, we can also express the convolution equation in inverse Laplace notation as

$$f(t) * g(t) = \mathscr{L}^{-1}(F(s)G(s))$$

Keep in mind that this $*$ notation does not indicate simple multiplication. So $f(t) * g(t)$ is not the same as $f(t)g(t)$; the convolution is a different thing entirely than the product. However, the convolution *is* commutative. So $f(t) * g(t)$ will be equivalent to $g(t) * f(t)$.

The convolution

Of course, now we want to understand how to find this new convolution function, $f(t) * g(t)$, so that we can plug it into this formula to get $F(s)G(s)$. The formula we use to find **the convolution** will be

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

where τ is the Greek letter "tau." And because the convolution is commutative, the formula

$$f(t) * g(t) = g(t) * f(t)$$

$$\int_0^t f(\tau)g(t-\tau)\ d\tau = \int_0^t f(t-\tau)g(\tau)\ d\tau$$

will give us the same result as the previous formula.

Let's do an example where we find the convolution of functions and show that its Laplace transform is equivalent to the product of the transforms.

Example

Find the convolution of $f(t) = t$ and $g(t) = e^t$, then show that the Laplace transform of the convolution is equivalent to the product of the individual transforms $F(s)$ and $G(s)$.

From the table of Laplace transforms, we know that the transforms of $f(t) = t$ and $g(t) = e^t$ are

$$F(s) = \frac{1}{s^2}$$

$$G(s) = \frac{1}{s - 1}$$

The product of these transforms is

$$F(s)G(s) = \frac{1}{s^2}\left(\frac{1}{s - 1}\right)$$

$$F(s)G(s) = \frac{1}{s^2(s - 1)}$$

So our expectation is that the Laplace transform of the convolution $f(t) * g(t)$ will give us this same result. To find the convolution, we'll first find $f(\tau)$ and $g(t - \tau)$, the two functions we need for our convolution integral.

$$f(\tau) = \tau$$

$$g(t - \tau) = e^{t-\tau}$$

So the convolution integral gives

$$f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t \tau e^{t-\tau}\ d\tau$$

$$f(t) * g(t) = \int_0^t \tau e^t e^{-\tau}\ d\tau$$

$$f(t) * g(t) = e^t \int_0^t \tau e^{-\tau} \ d\tau$$

We'll use integration by parts to evaluate the integral, letting

$$u = \tau \qquad dv = e^{-\tau} \ d\tau$$

$$du = 1 \ d\tau = d\tau \qquad v = -e^{-\tau}$$

Then the convolution is given by

$$f(t) * g(t) = e^t \left[uv\Big|_0^t - \int_0^t v \ du \right]$$

$$f(t) * g(t) = e^t \left[(\tau)(-e^{-\tau})\Big|_0^t - \int_0^t (-e^{-\tau})(d\tau) \right]$$

$$f(t) * g(t) = e^t \left[-\tau e^{-\tau}\Big|_0^t + \int_0^t e^{-\tau} \ d\tau \right]$$

Integrate the remaining function, and evaluate over the interval.

$$f(t) * g(t) = e^t \left[-te^{-t} - (-0e^{-0}) + \left(-e^{-\tau}\Big|_0^t \right) \right]$$

$$f(t) * g(t) = e^t[-te^{-t} - (-0e^{-0}) + (-e^{-t} - (-e^{-0}))]$$

$$f(t) * g(t) = e^t[-te^{-t} + (-e^{-t} + e^0)]$$

$$f(t) * g(t) = e^t(-te^{-t} - e^{-t} + 1)$$

$$f(t) * g(t) = -te^te^{-t} - e^te^{-t} + e^t$$

$$f(t) * g(t) = -te^{t-t} - e^{t-t} + e^t$$

$$f(t) * g(t) = -t - 1 + e^t$$

$$f(t) * g(t) = e^t - t - 1$$

In theory, if we take the Laplace transform of this new function, we should get back the product of the transforms that we found earlier.

$$F(s)G(s) = \int_0^\infty e^{-st} f(t) * g(t)\ dt$$

$$F(s)G(s) = \int_0^\infty e^{-st}(e^t - t - 1)\ dt$$

$$F(s)G(s) = \int_0^\infty e^t e^{-st} - te^{-st} - e^{-st}\ dt$$

$$F(s)G(s) = \int_0^\infty e^{t-st} - te^{-st} - e^{-st}\ dt$$

$$F(s)G(s) = \int_0^\infty e^{(1-s)t} - te^{-st} - e^{-st}\ dt$$

$$F(s)G(s) = \int_0^\infty e^{(1-s)t}\ dt - \int_0^\infty te^{-st}\ dt - \int_0^\infty e^{-st}\ dt$$

Evaluate the first integral.

$$F(s)G(s) = \frac{1}{1-s} e^{(1-s)t} \Big|_0^\infty - \int_0^\infty te^{-st}\ dt - \int_0^\infty e^{-st}\ dt$$

Evaluate the second integral using integration by parts, letting $u = t$, $du = dt$, $dv = e^{-st}\ dt$, and $v = (-1/s)e^{-st}$.

$$F(s)G(s) = \frac{1}{1-s}e^{(1-s)t}\Big|_0^\infty - \left[t\left(-\frac{1}{s}e^{-st}\right) - \int\left(-\frac{1}{s}e^{-st}\right)\ dt\right]\Big|_0^\infty - \int_0^\infty e^{-st}\ dt$$

$$F(s)G(s) = \frac{1}{1-s}e^{(1-s)t}\Big|_0^\infty - \left[-\frac{t}{s}e^{-st} + \frac{1}{s}\int e^{-st}\ dt\right]\Big|_0^\infty - \int_0^\infty e^{-st}\ dt$$

$$F(s)G(s) = \left(\frac{1}{1-s}e^{(1-s)t} + \frac{t}{s}e^{-st}\right)\Big|_0^\infty - \frac{1}{s}\int_0^\infty e^{-st}\ dt - \int_0^\infty e^{-st}\ dt$$

$$F(s)G(s) = \left(\frac{1}{1-s}e^{(1-s)t} + \frac{t}{s}e^{-st} - \frac{1}{s(-s)}e^{-st}\right)\Big|_0^\infty - \int_0^\infty e^{-st}\ dt$$

$$F(s)G(s) = \left(\frac{1}{1-s}e^{(1-s)t} + \frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st}\right)\Big|_0^\infty - \int_0^\infty e^{-st}\ dt$$

Evaluate the third integral.

$$F(s)G(s) = \left(\frac{1}{1-s}e^{(1-s)t} + \frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st} - \left(\frac{1}{-s}e^{-st}\right)\right)\Bigg|_0^\infty$$

$$F(s)G(s) = \left(\frac{1}{1-s}e^{(1-s)t} + \frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st} + \frac{1}{s}e^{-st}\right)\Bigg|_0^\infty$$

Now evaluate over the interval.

$$F(s)G(s) = \lim_{t\to\infty}\left(\frac{1}{1-s}e^{(1-s)t} + \frac{t}{s}e^{-st} + \frac{1}{s^2}e^{-st} + \frac{1}{s}e^{-st}\right)$$

$$-\left(\frac{1}{1-s}e^{(1-s)(0)}+\frac{0}{s}e^{-s(0)}+\frac{1}{s^2}e^{-s(0)}+\frac{1}{s}e^{-s(0)}\right)$$

$$F(s)G(s)=\lim_{t\to\infty}\left(\frac{1}{1-s}e^{t-st}+\frac{t}{se^{st}}+\frac{1}{s^2e^{st}}+\frac{1}{se^{st}}\right)$$

$$-\left(\frac{1}{1-s}+\frac{1}{s^2}+\frac{1}{s}\right)$$

$$F(s)G(s)=\lim_{t\to\infty}\left(\frac{1}{1-s}e^te^{-st}+\frac{t}{se^{st}}+\frac{1}{s^2e^{st}}+\frac{1}{se^{st}}\right)$$

$$-\frac{1}{1-s}-\frac{1}{s^2}-\frac{1}{s}$$

$$F(s)G(s)=\lim_{t\to\infty}\left(\frac{1}{1-s}\left(\frac{e^t}{e^{st}}\right)+\frac{1}{s}\left(\frac{t}{e^{st}}\right)+\frac{1}{s^2}\left(\frac{1}{e^{st}}\right)+\frac{1}{s}\left(\frac{1}{e^{st}}\right)\right)$$

$$-\frac{1}{1-s}-\frac{1}{s^2}-\frac{1}{s}$$

Assuming $s>0$, the denominators of all these fractions (the four that include t) will grow faster than their numerators, which means those four fractions will all converge to 0 as $t\to\infty$.

$$F(s)G(s)=\frac{1}{1-s}(0)+\frac{1}{s}(0)+\frac{1}{s^2}(0)+\frac{1}{s}(0)$$

$$-\frac{1}{1-s}-\frac{1}{s^2}-\frac{1}{s}$$

$$F(s)G(s)=-\frac{1}{1-s}-\frac{1}{s^2}-\frac{1}{s}$$

$$F(s)G(s) = \frac{1}{-(1-s)} - \frac{1}{s^2} - \frac{1}{s}$$

$$F(s)G(s) = \frac{1}{-1+s} - \frac{1}{s^2} - \frac{1}{s}$$

$$F(s)G(s) = \frac{1}{s-1} - \frac{1}{s^2} - \frac{1}{s}$$

Find a common denominator.

$$F(s)G(s) = \frac{s^2}{s^2(s-1)} - \frac{(s-1)}{s^2(s-1)} - \frac{(s)(s-1)}{s(s)(s-1)}$$

$$F(s)G(s) = \frac{s^2-s+1}{s^2(s-1)} - \frac{s^2-s}{s^2(s-1)}$$

$$F(s)G(s) = \frac{s^2-s+1-s^2+s}{s^2(s-1)}$$

$$F(s)G(s) = \frac{1}{s^2(s-1)}$$

After all that, we found a value that matches the one we found earlier. So, at least for $f(t)=t$ and $g(t)=e^t$, we've proven that the product of the transforms is equivalent to the Laplace transform of the convolution.

$$F(s)G(s) = \int_0^\infty e^{-st} f(t) * g(t)\ dt$$

Convolution integrals for initial value problems

Convolution integrals are particularly useful for finding the general solution to a second order differential equation in the form

$$ay''+by'+cy=g(t)$$

Notice in this equation that the forcing function $g(t)$ is not defined explicitly. Without a convolution integral, we wouldn't be able to find the solution to this kind of differential equation, even given initial conditions.

However, now that we know about the convolution integral, we can use it to find a solution to the differential equation. The solution will be in terms of the general forcing function $g(t)$, but that's still very useful, since we'll end up with a solution into which we can plug any forcing function that we choose, and get back an explicit solution to the differential equation.

Solving initial value problems with general forcing functions

Given a second order differential equation with a general forcing function and initial conditions for $y(0)$ and $y'(0)$, we'll use the following steps to find the general solution:

1. Use formulas from the table to transform y'', y', y, and $g(t)$.

2. Plug in the initial conditions to simplify the transformation.

3. Use algebra to solve for $Y(s)$.

4. Use an inverse Laplace transform to put the solution to the second order nonhomogeneous differential equation back in terms of t, instead of s, applying the convolution integral when necessary.

Notice that these steps are identical to the ones we previously used to solve initial value problems, except for the fact that we'll be applying the convolution integral somewhere in Step 4.

Let's work through an example so that we can see how these steps get applied.

Example

Use a convolution integral to find the general solution $y(t)$ to the differential equation, given $y(0) = -1$ and $y'(0) = 2$.

$$y'' + 3y = g(t)$$

From a table of Laplace transforms, we know that

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Making substitutions into the differential equation gives

$$s^2Y(s) - sy(0) - y'(0) + 3Y(s) = G(s)$$

Now we'll plug in the initial conditions $y(0) = -1$ and $y'(0) = 2$ in order to simplify the transform.

$$s^2Y(s) - s(-1) - (2) + 3Y(s) = G(s)$$

$$s^2Y(s) + s - 2 + 3Y(s) = G(s)$$

We'll solve for $Y(s)$ by gathering all the $Y(s)$ terms on the left, and moving all other terms to the right, then factoring out a $Y(s)$.

$$s^2Y(s) + 3Y(s) = -s + 2 + G(s)$$

$$Y(s)(s^2 + 3) = -s + 2 + G(s)$$

$$Y(s) = -\left(\frac{s}{s^2+3}\right) + 2\left(\frac{1}{s^2+3}\right) + G(s)\left(\frac{1}{s^2+3}\right)$$

We want to use an inverse Laplace transform to put each part of this equation in terms of t instead of s. If we start with the first term, we can see its similarity to the formula for the transform of $\cos(at)$.

$$\mathscr{L}(\cos(at)) = \frac{s}{s^2+a^2}$$

If we say $a = \sqrt{3}$, then the inverse transform of that first term is

$$\mathscr{L}^{-1}\left(\frac{s}{s^2+3}\right) = \cos(\sqrt{3}t)$$

The second term from $Y(s)$ should remind us of the formula for the transform of $\sin(at)$.

$$\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2}$$

Let's rewrite the second term so that it better matches the transform formula for $\sin(at)$.

$$\frac{1}{s^2+3}$$

$$\frac{\frac{\sqrt{3}}{\sqrt{3}}}{s^2+(\sqrt{3})^2}$$

$$\frac{\frac{1}{\sqrt{3}}\sqrt{3}}{s^2+(\sqrt{3})^2}$$

$$\frac{1}{\sqrt{3}}\left(\frac{\sqrt{3}}{s^2+(\sqrt{3})^2}\right)$$

Now with $a=\sqrt{3}$, the inverse transform of that second term is

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+3}\right)=\frac{1}{\sqrt{3}}\sin(\sqrt{3}t)$$

Finding the inverse transform of the last term needs the convolution integral. We already know

$$\mathscr{L}^{-1}\left(\frac{1}{s^2+3}\right)=\frac{1}{\sqrt{3}}\sin(\sqrt{3}t)$$

and we can say that the inverse transform of $G(s)$ is $g(t)$, so for our convolution integral, we'll use the functions

$$f(t) = \frac{1}{\sqrt{3}} \sin(\sqrt{3}t)$$

$$g(t) = g(t)$$

Plugging these into the convolution integral, we get

$$f(t) * g(t) = \int_0^t f(\tau)g(t-\tau)\ d\tau$$

$$f(t) * g(t) = \frac{1}{\sqrt{3}} \int_0^t \sin(\sqrt{3}\tau)g(t-\tau)\ d\tau$$

Plugging all of these values back into the equation for $Y(s)$,

$$Y(s) = -\left(\frac{s}{s^2+3}\right) + 2\left(\frac{1}{s^2+3}\right) + G(s)\left(\frac{1}{s^2+3}\right)$$

gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = -\cos(\sqrt{3}t) + 2\left(\frac{1}{\sqrt{3}}\sin(\sqrt{3}t)\right) + \frac{1}{\sqrt{3}}\int_0^t \sin(\sqrt{3}\tau)g(t-\tau)\ d\tau$$

$$y(t) = \frac{2}{\sqrt{3}}\sin(\sqrt{3}t) - \cos(\sqrt{3}t) + \frac{1}{\sqrt{3}}\int_0^t \sin(\sqrt{3}\tau)g(t-\tau)\ d\tau$$

$$y(t) = \frac{1}{\sqrt{3}}\left(2\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) + \int_0^t \sin(\sqrt{3}\tau)g(t-\tau)\ d\tau\right)$$

$$y(t) = \frac{\sqrt{3}}{3}\left(2\sin(\sqrt{3}t) - \sqrt{3}\cos(\sqrt{3}t) + \int_0^t \sin(\sqrt{3}\tau)g(t-\tau)\ d\tau\right)$$

Matrix basics

We'll learn much more about matrices in Linear Algebra. For now, we just need a brief introduction to matrices (for some, this may be a review from Precalculus), since we'll be using them extensively to solve systems of differential equations.

Dimensions, systems, and multiplication

The first thing we'll say is that an $n \times m$ matrix is an array of entries with n rows and m columns. These are examples of square 2×2 and 3×3 matrices:

$$\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & -4 \\ 1 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$$

We can use matrices to represent a system of equations. For instance, given the system,

$$3x_1 - 4x_2 = 2$$

$$x_1 + 5x_2 = -1$$

we can rewrite it using one matrix equation as

$$\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

A single column matrix can also be thought of as a vector, so this matrix equation has the coefficient matrix multiplied by the vector $\vec{x} = (x_1, x_2)$, set equal to the vector $\overrightarrow{b} = (2, -1)$. When we multiply two matrices, as we're doing on the left side of this equation, we always multiply the row(s) in the first matrix by the column(s) in the second matrix. In other words, working our way across the first row of this matrix equation gives us back the first equation from the system,

$$(3)(x_1) + (-4)(x_2) = 2$$

$$3x_1 - 4x_2 = 2$$

and working our way across the second row of this matrix equation gives us back the second equation from the system.

$$(1)(x_1) + (5)(x_2) = -1$$

$$x_1 + 5x_2 = -1$$

We just looked at multiplying the 2×2 matrix by the 2×1 matrix/vector, but it's also important to say that we can multiply any matrix by a scalar. To do so, we just distribute the scalar across every entry in the matrix.

$$-2\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} = \begin{bmatrix} 3(-2) & -4(-2) \\ 1(-2) & 5(-2) \end{bmatrix} = \begin{bmatrix} -6 & 8 \\ -2 & -10 \end{bmatrix}$$

And to add or subtract matrices, we simply add or subtract corresponding entries.

$$\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 + (-1) - (-2) & -4 + 0 - 0 \\ 1 + 1 - 0 & 5 + 2 - 4 \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ 2 & 3 \end{bmatrix}$$

Determinants

We'll also need to know how to calculate the determinant for a matrix. For a 2×2 matrix, the determinant is given by

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a 3×3 matrix, the determinant is given by

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

Take special notice of the negative sign in front of b. When we break down the determinant, we add the a and c terms, but subtract the b term. Then the determinant simplifies as

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = aei - afh - bdi + bfg + cdh - ceg$$

Eigenvalues and Eigenvectors

The identity matrix is a square matrix with all 0 entries, except for the main diagonal, running from the top left to lower right, which is filled with 1s. Here are the 2×2 and 3×3 identity matrices:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Later we'll need to be able to find the Eigenvalues and Eigenvectors of a matrix. The Eigenvalues of the matrix are the values of λ that satisfy $|A - \lambda I| = 0$. For instance, these are the steps we'd take to find the Eigenvalues of the 2×2 matrix we used earlier:

$$\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$\begin{bmatrix} 3 - \lambda & -4 \\ 1 & 5 - \lambda \end{bmatrix}$$

Take the determinant,

$$|A - \lambda I| = \begin{vmatrix} 3 - \lambda & -4 \\ 1 & 5 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (3 - \lambda)(5 - \lambda) - (-4)(1)$$

$$|A - \lambda I| = 15 - 8\lambda + \lambda^2 + 4$$

$$|A - \lambda I| = \lambda^2 - 8\lambda + 19$$

then solve the characteristic equation $\lambda^2 - 8\lambda + 19 = 0$ for the Eigenvalues. We'll get the Eigenvector associated with each Eigenvalue by putting the matrix $A - \lambda I$ into row-echelon form using Gauss-Jordan elimination, and then solving the system given by the simplified matrix.

Gauss-Jordan elimination algorithm

One way to think about the goal of this algorithm (a specific set of steps that can be repeated over and over again) is that we're trying to rewrite the matrix so that it's as similar as possible to the identity matrix.

In other words, we're trying to change all the entries along the main diagonal to 1, and all the other entries to 0. We won't always be able to get to the identity matrix exactly, but we'll try to get as close as possible.

1. If the first entry in the first row is 0, swap it with another row that has a non-zero entry in its first column. Otherwise, move to step 2.

2. Multiply through the first row by a scalar to make the leading entry equal to 1.

3. Add scaled multiples of the first row to every other row in the matrix until every entry in the first column, other than the leading 1 in the first row, is a 0.

4. Go back to step 1 and repeat the process until the matrix is in reduced row-echelon form.

For instance, to put our 2×2 matrix into row-echelon form, we'll divide through the first row by 3 in order to change the first entry from 3 to 1.

$$\begin{bmatrix}3 & -4\\ 1 & 5\end{bmatrix} \to \begin{bmatrix}1 & -\frac{4}{3}\\ 1 & 5\end{bmatrix}$$

Then to change the 1 in the second row into a 0, we'll subtract the first row from the second, and use the result to replace the second row.

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 1-1 & 5-\left(-\frac{4}{3}\right) \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & \frac{19}{3} \end{bmatrix}$$

Then to change the $19/3$ to a 1, we'll multiply through the second row by $3/19$.

$$\begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & \frac{19}{3}\left(\frac{3}{19}\right) \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} \\ 0 & 1 \end{bmatrix}$$

Then to zero out the $-4/3$, we'll add $4/3$ of the second row to the first row.

$$\begin{bmatrix} 1 & -\frac{4}{3}+\frac{4}{3}(1) \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In this case, the matrix does reduce down to the identity matrix. As we mentioned though, this won't always be the case.

Building systems

We're working toward learning to solve systems of differential equations, but before we can, we need to define the kind of system we're trying to solve.

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t) + f_1(t)$$

$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \ldots + a_{2n}(t)x_n(t) + f_2(t)$$

...

$$x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t) + f_n(t)$$

This is a system of first order linear differential equations. When all of the values $f_i(t)$ are zero, the system is a **homogeneous system**. But if any of the values $f_i(t)$ are non-zero, then the system is a **nonhomogeneous system**.

Converting to matrix form

We already know from earlier how to convert simple systems into matrix form. For example, converting the system

$$3x_1 - 4x_2 = 2$$

$$x_1 + 5x_2 = -1$$

into the matrix form

$$\begin{bmatrix} 3 & -4 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

is done by creating the coefficient matrix A, multiplying it by a vector of the variables $\vec{x}$, and equating that to a vector of the values from the right side. In other words, we could generalize the matrix equation as

$$A\vec{x} = \vec{b}$$

If we extend this same idea to systems of differential equations, then for the system of differential equations that we defined earlier,

$$x_1'(t) = a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \ldots + a_{1n}(t)x_n(t) + f_1(t)$$

$$x_2'(t) = a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \ldots + a_{2n}(t)x_n(t) + f_2(t)$$

$$\ldots$$

$$x_n'(t) = a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \ldots + a_{nn}(t)x_n(t) + f_n(t)$$

we can define

$$\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \qquad A = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & & & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \qquad F = \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

With these values, we can write the system of differential equations as

$$\vec{x}' = A\vec{x} + F$$

And in this form, it's even easier to distinguish homogeneous and nonhomogeneous systems. The system will be homogeneous when F is the zero vector $F = \vec{O}$, and nonhomogeneous when F is non-zero.

Therefore, a homogeneous system of differential equations can be written in matrix form as

$$\vec{x}' = A\vec{x}$$

Let's do an example where we convert the system of differential equations into matrix form.

Example

Rewrite the system of differential equations in matrix form.

$$x_1' = 3x_1 - 2x_2$$

$$x_2' = -x_1 + 4x_2$$

To write the system in matrix form $\vec{x}' = A\vec{x}$, we define the vector $\vec{x}'$, matrix A, and vector $\vec{x}$ as

$$\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \qquad A = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \qquad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Then we can write the system in matrix form.

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 3 & -2 \\ -1 & 4 \end{bmatrix} \vec{x}$$

Converting linear differential equations into systems

Higher order linear differential equations, both homogeneous and nonhomogeneous, can be converted into systems of differential equations.

The order of the differential equation will determine the number of equations in the system. In other words, we'll convert second order equations into a system of two equations, a third order equation will become a system of three equations, etc.

In general, we'll start by substituting $x_1(t) = y(t)$, $x_2(t) = y'(t)$, $x_3(t) = y''(t)$, and so on. And we'll always use the differential equation itself to substitute for the highest degree derivative.

If we're presented with initial conditions as part of our problem, we can convert those as well. Let's look at an example so that we can walk through this process step-by-step.

Example

Convert the third order linear differential equation into a system of differential equations.

$$y''' - 3y'' + y' - 2y = 3t^2$$

Our standard procedure here will be to start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

$$x_3(t) = y''(t)$$

Now we need to solve the original differential equation for $y'''(t)$.

$$y''' - 3y'' + y' - 2y = 3t^2$$

$$y''' = 3y'' - y' + 2y + 3t^2$$

Then if we take the derivatives of the equations for $x_1(t)$, $x_2(t)$, and $x_3(t)$, we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = x_3(t)$$

$$x_3'(t) = y'''(t) = 3y'' - y' + 2y + 3t^2 = 3x_3(t) - x_2(t) + 2x_1(t) + 3t^2$$

Simplifying these equations gives us a system of equations that's equivalent to the original third order differential equation.

$$x_1' = x_2$$

$$x_2' = x_3$$

$$x_3' = 3x_3 - x_2 + 2x_1 + 3t^2$$

And if we wanted to write this nonhomogeneous system as a matrix equation, we would get

$$\vec{x}' = A\vec{x} + F$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -1 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} 0 \\ 0 \\ 3t^2 \end{bmatrix}$$

Solving systems

In this lesson, we want to extend much of what we know about solutions of second order equations to solutions of systems of differential equations.

This will actually be a really helpful framework, because much of what we know about the solution of a second order equation is consistent with what we'll see about the solution of a system of equations.

General solution

For instance, remember previously when we learned to solve second order homogeneous equations, that the general solution was given by $y = c_1y_1 + c_2y_2$, where $\{y_1, y_2\}$ formed a fundamental set of solutions.

Following this format produced formulas for the general solution with distinct real roots, equal real roots, or complex conjugate roots of the associated characteristic equation.

Roots	**General solution**
Distinct real roots	$y(x) = c_1e^{r_1x} + c_2e^{r_2x}$
Equal real roots	$y(x) = c_1e^{r_1x} + c_2xe^{r_1x}$
Complex conjugate roots	$y(x) = e^{\alpha x}(c_1 \cos(\beta x) + c_2 \sin(\beta x))$
	where $r = \alpha \pm \beta i$

Similarly, the solution to a system of two homogeneous differential equations, written in matrix form as $\vec{x}' = A\vec{x}$, when the matrix A is an 2×2 square matrix, will be given by

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

This time, instead of the solutions $\{y_1, y_2\}$ which were functions in x, we'll have solutions $\{\vec{x_1}, \vec{x_2}\}$ which are vectors in t. The vectors $\{\vec{x_1}, \vec{x_2}\}$ will be defined as

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

if λ_1, λ_2 are two non-zero real eigenvalues of matrix A. The vectors $\vec{k_1}$ and $\vec{k_2}$ will each have two components, when the matrix A in $\vec{x}' = A\vec{x}$ is a 2×2 square matrix.

And when we're solving a nonhomogeneous system of two equations $\vec{x}' = A\vec{x} + F$, where $F \neq \vec{O}$, the general solution will be the sum of the complementary solution $\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2}$ (the general solution to the associated homogeneous equation $\vec{x}' = A\vec{x}$ when the matrix A is an 2×2 square matrix) and particular solution $\vec{x_p}$.

$$\vec{x} = \vec{x_c} + \vec{x_p}$$

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + \vec{x_p}$$

Verifying solutions

In the past, we learned how to show that a solution set like $\{y_1, y_2\} = \{e^{-3x}, xe^{-3x}\}$ satisfied a second order equation like $y'' + 6y' + 9y = 0$. We simply substituted each solution, $y_1 = e^{-3x}$ and $y_2 = xe^{-3x}$, one at a time into the differential equation, to show that each one individually satisfied the equation.

We can do the same thing with the vector solutions to the system of differential equations. For example, we can show that

$$\vec{x_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{3}{2}t}$$

is a solution to the system given in matrix form,

$$\vec{x}\,' = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} \vec{x}$$

We just need to substitute the solution vector and its derivative into the matrix equation. The derivative of the solution vector is

$$\vec{x_1}' = -\frac{3}{2} \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{3}{2}t} = \begin{bmatrix} \frac{3}{2} \\ -3 \end{bmatrix} e^{-\frac{3}{2}t}$$

Plugging the solution $\vec{x_1}$ and its derivative $\vec{x_1}'$ into the matrix equation gives

$$\begin{bmatrix} \frac{3}{2} \\ -3 \end{bmatrix} e^{-\frac{3}{2}t} = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{3}{2}t}$$

$$\begin{bmatrix} \frac{3}{2}e^{-\frac{3}{2}t} \\ -3e^{-\frac{3}{2}t} \end{bmatrix} = \begin{bmatrix} -1 & \frac{1}{4} \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -e^{-\frac{3}{2}t} \\ 2e^{-\frac{3}{2}t} \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2}e^{-\frac{3}{2}t} \\ -3e^{-\frac{3}{2}t} \end{bmatrix} = \begin{bmatrix} (-1)(-e^{-\frac{3}{2}t}) + \left(\frac{1}{4}\right)(2e^{-\frac{3}{2}t}) \\ (1)(-e^{-\frac{3}{2}t}) + (-1)(2e^{-\frac{3}{2}t}) \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2}e^{-\frac{3}{2}t} \\ -3e^{-\frac{3}{2}t} \end{bmatrix} = \begin{bmatrix} e^{-\frac{3}{2}t} + \frac{1}{2}e^{-\frac{3}{2}t} \\ -e^{-\frac{3}{2}t} - 2e^{-\frac{3}{2}t} \end{bmatrix}$$

$$\begin{bmatrix} \frac{3}{2}e^{-\frac{3}{2}t} \\ -3e^{-\frac{3}{2}t} \end{bmatrix} = \begin{bmatrix} \frac{3}{2}e^{-\frac{3}{2}t} \\ -3e^{-\frac{3}{2}t} \end{bmatrix}$$

Because we get equivalent values on both sides of the equation, we've verified that

$$\vec{x_1} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-\frac{3}{2}t}$$

is, in fact, a solution to the system of differential equations given by the matrix equation.

Superposition, linear independence, and the fundamental set

Furthermore, in the same way that $c_1y_1 + c_2y_2$ is also a solution to the second order equation when $\{y_1, y_2\}$ form a solution set, $c_1\vec{x_1} + c_2\vec{x_2}$ is also a solution to the system of differential equations when $\{\vec{x_1}, \vec{x_2}\}$ form a solution set. This is the superposition principle.

And when it comes to the solution set $\{\vec{x_1}, \vec{x_2}\}$, we're primarily interested in linearly independent solutions, which, just like with solutions to second order equations, are solutions that aren't constant multiples of one another.

If we have just two solutions, they'll be linearly independent if they aren't constant multiples of each other. If we have more than two solutions, they'll be linearly independent when none of the solutions can be formed with a linear combination of the other solutions.

Just like before, we can use the Wronskian to test for linear independence.

$$W(\vec{x_1}, \vec{x_2}, \dots, \vec{x_n}) = \begin{vmatrix} \vdots & \vdots & \vdots & \vdots \\ \vec{x_1} & \vec{x_2} & \cdots & \vec{x_n} \\ \vdots & \vdots & \vdots & \vdots \end{vmatrix}$$

If the Wronskian of the vector set is non-zero, then the vector set represents a set of linearly independent solutions.

If the Wronskian is non-zero, that means the solution set is linearly independent, and equivalently, we can call the solution set a fundamental set of solutions.

Let's do an example with a homogeneous system to try to tie all these facts together.

Example

Verify that the solution vectors satisfy the system of equations. Calculate the Wronskian of the solution set and use the result to make a statement about the linear (in)dependence of the solutions, and whether or not they form a fundamental set. Then write the general solution of the homogeneous system.

$$\vec{x}' = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \vec{x}$$

$$\{\vec{x}_1, \vec{x}_2, \vec{x}_3\} = \left\{ \begin{bmatrix} 6 \\ -1 \\ -5 \end{bmatrix} e^{-t}, \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{-2t}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{3t} \right\}$$

We can verify that each of the vectors $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ are solutions by substituting them into the matrix equation. First though, we'll need the derivative of each solution vector.

$$\vec{x}_1 = \begin{bmatrix} 6e^{-t} \\ -e^{-t} \\ -5e^{-t} \end{bmatrix} \qquad \vec{x}_1{}' = \begin{bmatrix} -6e^{-t} \\ e^{-t} \\ 5e^{-t} \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} -3e^{-2t} \\ e^{-2t} \\ e^{-2t} \end{bmatrix} \qquad \vec{x}_2{}' = \begin{bmatrix} 6e^{-2t} \\ -2e^{-2t} \\ -2e^{-2t} \end{bmatrix}$$

$$\vec{x}_3 = \begin{bmatrix} 2e^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix} \qquad \vec{x}_3{}' = \begin{bmatrix} 6e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix}$$

Substitute $\vec{x}_1$ and its derivative $\vec{x}_1{}'$ into the matrix equation.

$$\begin{bmatrix} -6e^{-t} \\ e^{-t} \\ 5e^{-t} \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 6e^{-t} \\ -e^{-t} \\ -5e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} -6e^{-t} \\ e^{-t} \\ 5e^{-t} \end{bmatrix} = \begin{bmatrix} (0)(6e^{-t}) + (6)(-e^{-t}) + (0)(-5e^{-t}) \\ (1)(6e^{-t}) + (0)(-e^{-t}) + (1)(-5e^{-t}) \\ (1)(6e^{-t}) + (1)(-e^{-t}) + (0)(-5e^{-t}) \end{bmatrix}$$

$$\begin{bmatrix} -6e^{-t} \\ e^{-t} \\ 5e^{-t} \end{bmatrix} = \begin{bmatrix} -6e^{-t} \\ 6e^{-t} - 5e^{-t} \\ 6e^{-t} - e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} -6e^{-t} \\ e^{-t} \\ 5e^{-t} \end{bmatrix} = \begin{bmatrix} -6e^{-t} \\ e^{-t} \\ 5e^{-t} \end{bmatrix}$$

Substitute $\overrightarrow{x_2}$ and its derivative $\overrightarrow{x_2}'$ into the matrix equation.

$$\begin{bmatrix} 6e^{-2t} \\ -2e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} -3e^{-2t} \\ e^{-2t} \\ e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} 6e^{-2t} \\ -2e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} (0)(-3e^{-2t}) + (6)(e^{-2t}) + (0)(e^{-2t}) \\ (1)(-3e^{-2t}) + (0)(e^{-2t}) + (1)(e^{-2t}) \\ (1)(-3e^{-2t}) + (1)(e^{-2t}) + (0)(e^{-2t}) \end{bmatrix}$$

$$\begin{bmatrix} 6e^{-2t} \\ -2e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} 6e^{-2t} \\ -3e^{-2t} + e^{-2t} \\ -3e^{-2t} + e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} 6e^{-2t} \\ -2e^{-2t} \\ -2e^{-2t} \end{bmatrix} = \begin{bmatrix} 6e^{-2t} \\ -2e^{-2t} \\ -2e^{-2t} \end{bmatrix}$$

Substitute $\vec{x_3}$ and its derivative $\vec{x_3}'$ into the matrix equation.

$$\begin{bmatrix} 6e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 0 & 6 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2e^{3t} \\ e^{3t} \\ e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} 6e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} (0)(2e^{3t}) + (6)(e^{3t}) + (0)(e^{3t}) \\ (1)(2e^{3t}) + (0)(e^{3t}) + (1)(e^{3t}) \\ (1)(2e^{3t}) + (1)(e^{3t}) + (0)(e^{3t}) \end{bmatrix}$$

$$\begin{bmatrix} 6e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 6e^{3t} \\ 2e^{3t} + e^{3t} \\ 2e^{3t} + e^{3t} \end{bmatrix}$$

$$\begin{bmatrix} 6e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 6e^{3t} \\ 3e^{3t} \\ 3e^{3t} \end{bmatrix}$$

Because we found equivalent vectors on both sides of the equation in each of these three cases, we've shown that all three vectors are solutions to the system of differential equations.

Now let's take the Wronskian of the solution set.

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = \begin{vmatrix} 6e^{-t} & -3e^{-2t} & 2e^{3t} \\ -e^{-t} & e^{-2t} & e^{3t} \\ -5e^{-t} & e^{-2t} & e^{3t} \end{vmatrix}$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 6e^{-t} \begin{vmatrix} e^{-2t} & e^{3t} \\ e^{-2t} & e^{3t} \end{vmatrix} - (-3e^{-2t}) \begin{vmatrix} -e^{-t} & e^{3t} \\ -5e^{-t} & e^{3t} \end{vmatrix} + 2e^{3t} \begin{vmatrix} -e^{-t} & e^{-2t} \\ -5e^{-t} & e^{-2t} \end{vmatrix}$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 6e^{-t}[(e^{-2t})(e^{3t}) - (e^{3t})(e^{-2t})]$$

$$+3e^{-2t}[(-e^{-t})(e^{3t}) - (e^{3t})(-5e^{-t})] + 2e^{3t}[(-e^{-t})(e^{-2t}) - (e^{-2t})(-5e^{-t})]$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 6e^{-t}(e^{t} - e^{t}) + 3e^{-2t}(-e^{2t} + 5e^{2t}) + 2e^{3t}(-e^{-3t} + 5e^{-3t})$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 6e^{-t}(0) + 3e^{-2t}(4e^{2t}) + 2e^{3t}(4e^{-3t})$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 12e^{2t-2t} + 8e^{3t-3t}$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 12(1) + 8(1)$$

$$W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 20$$

Because the Wronskian is non-zero, $W(\vec{x_1}, \vec{x_2}, \vec{x_3}) = 20 \neq 0$, we can confirm that the vectors in the solution set are linearly independent, which means that the vector set represents a fundamental set of solutions.

Given that this is a fundamental set of solutions, we can write the general solution of the system as

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 6 \\ -1 \\ -5 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} e^{3t}$$

Distinct real Eigenvalues

We saw previously that the solution to a system of two homogeneous differential equations $\vec{x}' = A\vec{x}$ (when A is 2×2), would be given by $\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$, where the solutions are vectors $\{\vec{x_1}, \vec{x_2}\}$ defined in t as $\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$ and $\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$.

Given that we know the form of these solution vectors, we want to turn our attention toward finding these solution vectors for the homogeneous system $\vec{x}' = A\vec{x}$.

As it turns out, these solution vectors will be defined by the Eigenvalues and Eigenvectors of the matrix A.

Eigenvalues and Eigenvectors

If we plug a solution vector $\vec{x} = \vec{k}e^{\lambda t}$ into the matrix equation $\vec{x}' = A\vec{x}$, we get

$$\vec{x}' = A\vec{x}$$

$$\vec{k}\lambda e^{\lambda t} = A\vec{k}e^{\lambda t}$$

$$\vec{k}\lambda = A\vec{k}$$

$$A\vec{k} - \vec{k}\lambda = \vec{O}$$

$$(A - \lambda I)\vec{k} = \vec{O}$$

To understand this equation, let's consider the case where A is a 2×2 matrix. Then the matrix $A-\lambda I$ is some 2×2 matrix A, minus the product of the constant λ and the 2×2 identity matrix I_2.

$$A-\lambda I=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}-\lambda\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}-\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}$$

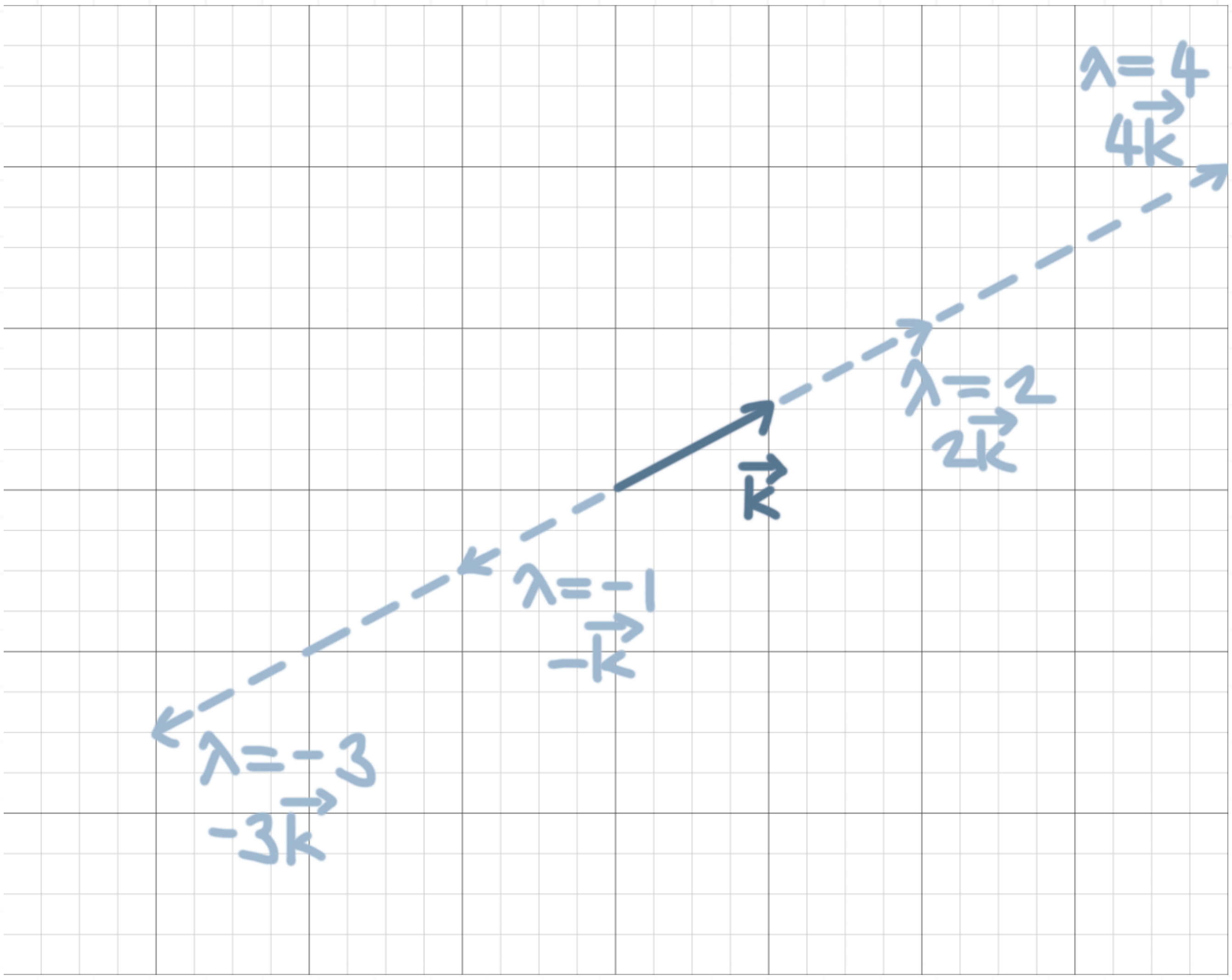
So we can rewrite the equation $(A-\lambda I)\vec{k}=\vec{O}$ as

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix}\vec{k}=\vec{O}$$

We're looking for solution vectors $\vec{k}$ that satisfy this equation. In other words, when we multiply the matrix $A-\lambda I$ by $\vec{k}$, we should get the zero vector $\vec{O}$.

Of course, the vector $\vec{k}=\vec{O}$ will always satisfy the equation, so we consider that to be the **trivial solution**. We're more interested in any non-zero vectors $\vec{k}$, any **nontrivial solutions**, that satisfy the equation.

What are these nontrivial solutions? Well, we saw before that, after plugging a solution vector $\vec{x}=\vec{k}e^{\lambda t}$ into the matrix equation $\vec{x}'=A\vec{x}$, we found $\vec{k}\lambda=A\vec{k}$. This equation tells us that, when we multiply the matrix A by the vector $\vec{k}$, the result is the same vector $\vec{k}$, just scaled by some constant λ. In other words, the result $\lambda\vec{k}$ will lie along the same "line" as the original $\vec{k}$, it'll just be some scaled version of $\vec{k}$.



The values of λ that make this equation true are the **Eigenvalues** of the matrix A, while the vectors $\vec{k}$ that satisfy it are the **Eigenvectors** of the matrix A. In other words, all of the non-trivial solutions we want to find are Eigenvalue/Eigenvector pairs for matrix A.

Therefore, our goal is to find any Eigenvalue/Eigenvector pairs for matrix A, and then use those $(\lambda, \vec{k})$ pairs to build solutions to the system $\vec{x}' = A\vec{x}$ in the form $\vec{x} = \vec{k}e^{\lambda t}$.

By definition, a homogeneous system will always either have exactly one solution (which will be the trivial solution), or it'll have infinitely many solutions. The trivial solution isn't interesting; we're looking for the "infinitely many solutions" case. We only get these non-trivial solutions when the matrix A is singular, which occurs when the determinant $|A - \lambda I|$ is 0,

$$|A - \lambda I| = 0$$

This determinant equation is called the **characteristic equation**, and solving it will give us the Eigenvalues of the matrix A, and then we can use the Eigenvalues to find the associated Eigenvectors.

Distinct real Eigenvalues

There are three cases we want to consider.

1. **Distinct real**: The matrix A has distinct real Eigenvalues (every Eigenvalue is a real number, and no Eigenvalues are equivalent).

2. **Equal real**: The matrix A has at least some equal real Eigenvalues (every Eigenvalue is a real number, but at least some of the Eigenvalues are equivalent).

3. **Complex conjugate**: The matrix A has complex Eigenvalues (the Eigenvalues are complex numbers).

In this lesson, we'll start with just the case of distinct real Eigenvalues. The Eigenvectors that correspond to distinct real Eigenvalues will always be linearly independent vectors, which means the solutions $\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$ and $\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$ we find for the system will always be linearly independent, and will therefore form a fundamental set of solutions.

Let's do an example where we use Eigenvalue/Eigenvector pairs for a set of distinct real Eigenvalues in order to solve a system of differential equations.

Example

Solve the system of differential equations.

$$x_1' = -6x_1 + 2x_2$$

$$x_2' = -3x_1 + x_2$$

The coefficient matrix is

$$A = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -6 - \lambda & 2 \\ -3 & 1 - \lambda \end{bmatrix}$$

Its determinant is

$$\begin{vmatrix} -6 - \lambda & 2 \\ -3 & 1 - \lambda \end{vmatrix} = (-6 - \lambda)(1 - \lambda) - (2)(-3)$$

$$\begin{vmatrix} -6 - \lambda & 2 \\ -3 & 1 - \lambda \end{vmatrix} = -6 + 6\lambda - \lambda + \lambda^2 + 6$$

$$\begin{vmatrix} -6-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = \lambda^2 + 5\lambda$$

So the characteristic equation is

$$\lambda^2 + 5\lambda = 0$$

$$\lambda(\lambda + 5) = 0$$

$$\lambda = -5, 0$$

Then for these Eigenvalues, $\lambda_1 = -5$ and $\lambda_2 = 0$, we find

$$A - (-5)I = \begin{bmatrix} -6-(-5) & 2 \\ -3 & 1-(-5) \end{bmatrix}$$

$$A - (-5)I = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}$$

and

$$A - (0)I = \begin{bmatrix} -6-0 & 2 \\ -3 & 1-0 \end{bmatrix}$$

$$A - (0)I = \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix} \qquad \begin{bmatrix} -6 & 2 \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ -3 & 6 \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{1}{3} \\ -3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations (in both cases we only need to consider the equation that we get from the first row of each matrix), we get

$$k_1 - 2k_2 = 0 \qquad k_1 - \frac{1}{3}k_2 = 0$$

$$k_1 = 2k_2 \qquad k_1 = \frac{1}{3}k_2$$

From the first system, we'll choose $k_2 = 1$, which results in $k_1 = 2$. And from the second system, we'll choose $k_2 = 3$, which results in $k_1 = 1$.

$$\overrightarrow{k_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Then the solutions to the system are

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{0t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Therefore, the general solution to the homogeneous system must be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-5t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

Let's do one more example where we convert a second order equation into a system, and then solve an initial value problem with the system.

Example

Convert the differential equation into a system of equations, then use the system to solve the initial value problem, given $y(0) = 1$ and $y'(0) = -1$.

$$y'' - 5y' + 6y = 0$$

We can start by defining

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

and then we'll solve the original differential equation for $y''(t)$.

$$y'' - 5y' + 6y = 0$$

$$y'' = 5y' - 6y$$

Then if we take the derivatives of $x_1(t)$ and $x_2(t)$, we get

$$x_1'(t) = y'(t) = x_2(t)$$

$$x_2'(t) = y''(t) = 5y' - 6y = 5x_2(t) - 6x_1(t)$$

Simplifying these equations gives us a system of equations that's equivalent to the original second order differential equation.

$$x_1' = x_2$$

$$x_2' = 5x_2 - 6x_1$$

We can represent this system in matrix form as

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} \vec{x}$$

and then find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{bmatrix}$$

and its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -6 & 5 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (-\lambda)(5 - \lambda) - (1)(-6)$$

$$|A - \lambda I| = -5\lambda + \lambda^2 + 6$$

$$|A - \lambda I| = \lambda^2 - 5\lambda + 6$$

Solve the characteristic equation for the Eigenvalues.

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

$$\lambda = 2,\ 3$$

For the Eigenvalue $\lambda_1 = 2$ we find

$$A - 2I = \begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 \\ -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix}$$

If we turn this first row into an equation, we get

$$k_1 - \frac{1}{2}k_2 = 0$$

$$k_1 = \frac{1}{2}k_2$$

We'll choose $k_2 = 2$, which results in $k_1 = 1$.

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t}$$

For the Eigenvalue $\lambda_1 = 3$ we find

$$A - 3I = \begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} -3 & 1 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 \\ -6 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1/3 \\ 0 & 0 \end{bmatrix}$$

If we turn this first row into an equation, we get

$$k_1 - \frac{1}{3}k_2 = 0$$

$$k_1 = \frac{1}{3}k_2$$

We'll choose $k_2 = 3$, which results in $k_1 = 1$.

$$\vec{x_2} = \vec{k_2} e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$$

So the general solution to the homogeneous system is

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3t}$$

Remember that the initial conditions were given for the original differential equation as $y(0) = 1$ and $y'(0) = -1$. But when we started this problem we set

$$x_1(t) = y(t)$$

$$x_2(t) = y'(t)$$

If we substitute $t = 0$ into these equations, we get

$$x_1(0) = y(0) = 1$$

$$x_2(0) = y'(0) = -1$$

These new initial conditions are $x_1(0) = 1$ and $x_2(0) = -1$, which means the vector $\vec{x}(0) = (1, -1)$. Plugging this vector and $t = 0$ (since this is the value given in the initial conditions) into the general solution, we get

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{2(0)} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} e^{3(0)}$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \vec{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} (1) + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix} (1)$$

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

which gives us the system of equations

$$c_1 + c_2 = 1$$

$$2c_1 + 3c_2 = -1$$

Solving this system gives $c_1 = 4$ and $c_2 = -3$, so the solution to the initial value problem is therefore

$$\vec{x} = 4\begin{bmatrix}1\\2\end{bmatrix}e^{2t} - 3\begin{bmatrix}1\\3\end{bmatrix}e^{3t}$$

This is the solution to the system, but we'd like to find the solution to the original second order differential equation. If we just remember that we defined

$$\vec{x}(t) = \begin{bmatrix}x_1(t)\\x_2(t)\end{bmatrix} = \begin{bmatrix}y(t)\\y'(t)\end{bmatrix}$$

when we started this problem, then we can break down our solution into two equations,

$$x_1(t) = y(t) = 4e^{2t} - 3e^{3t}$$

$$x_2(t) = y'(t) = 8e^{2t} - 9e^{3t}$$

and the solution $y(t)$ to the original differential equation is therefore

$$y(t) = 4e^{2t} - 3e^{3t}$$

Equal real Eigenvalues with multiplicity two

Previously we looked at how to solve systems when the characteristic equation produced distinct real Eigenvalues. Now we want to turn our attention toward equal real Eigenvalues.

For equal real Eigenvalues, there are two cases we want to look at.

1. The repeated Eigenvalue produces multiple linearly independent Eigenvectors

2. The repeated Eigenvalue produces only one Eigenvector

In the first case, the solution to the system will be just like the solution for distinct real Eigenvalues, $\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + \dots + c_n\vec{x_n}$. But in the second case, we'll still use the single Eigenvector for the solution $\vec{x_1}$, but then we'll need a special equation,

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

to find the second solution

$$\vec{x_2} = \vec{k_1}te^{\lambda_1 t} + \vec{p_1}e^{\lambda_1 t}$$

Substituting the single Eigenvector into $(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$ for $\vec{k_1}$ allows us to find a second vector $\vec{p_1}$ that we can use for the second solution, $\vec{x_2}$.

Let's start by looking at these two cases for an Eigenvalue with multiplicity two.

Eigenvalue with multiplicity two

An **Eigenvalue with multiplicity two** is an Eigenvalue that shows up twice in the characteristic equation.

Let's do an example with the first case above, where a repeated Eigenvalue produces multiple linearly independent Eigenvectors.

Example

Find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \vec{x}$$

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & -1 & -1 \\ 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 3-\lambda & -1 & -1 \\ 1 & 1-\lambda & -1 \\ 1 & -1 & 1-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (3-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} 1 & -1 \\ 1 & 1-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 1 & 1-\lambda \\ 1 & -1 \end{vmatrix}$$

$$|A - \lambda I| = (3-\lambda)[(1-\lambda)(1-\lambda) - (-1)(-1)]$$

$$+[(1)(1-\lambda) - (-1)(1)] - [(1)(-1) - (1-\lambda)(1)]$$

$$|A - \lambda I| = (3-\lambda)(1 - 2\lambda + \lambda^2 - 1) + (1 - \lambda + 1) - (-1 - 1 + \lambda)$$

$$|A - \lambda I| = (3-\lambda)(-2\lambda + \lambda^2) + 1 - \lambda + 1 + 1 + 1 - \lambda$$

$$|A - \lambda I| = -6\lambda + 3\lambda^2 + 2\lambda^2 - \lambda^3 + 4 - 2\lambda$$

$$|A - \lambda I| = 4 - 8\lambda + 5\lambda^2 - \lambda^3$$

Solve the characteristic equation for the Eigenvalues.

$$4 - 8\lambda + 5\lambda^2 - \lambda^3 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

$$(\lambda - 1)(\lambda - 2)(\lambda - 2) = 0$$

$$\lambda = 1,\ 2,\ 2$$

We'll handle $\lambda_1 = 1$ first, starting with finding $A - 1I$.

$$A - 1I = \begin{bmatrix} 3-1 & -1 & -1 \\ 1 & 1-1 & -1 \\ 1 & -1 & 1-1 \end{bmatrix}$$

$$A - 1I = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -1 \\ 2 & -1 & -1 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this matrix into a system of equations, we get

$$k_1 - k_3 = 0$$

$$k_2 - k_3 = 0$$

or

$$k_1 = k_3$$

$$k_2 = k_3$$

So if we choose $k_3 = 1$, we get $k_1 = 1$ and $k_2 = 1$, or

$$\vec{k_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

and therefore

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t$$

Then for the repeated Eigenvalues $\lambda_2 = \lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} 3-2 & -1 & -1 \\ 1 & 1-2 & -1 \\ 1 & -1 & 1-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 1 & -1 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this first row into an equation, we get

$$k_1 - k_2 - k_3 = 0$$

$$k_1 = k_2 + k_3$$

We'll choose $(k_2, k_3) = (1,0)$, which results in $k_1 = 1$.

$$\vec{k_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

and therefore

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

But we can actually find a second linearly independent Eigenvector from $k_1 = k_2 + k_3$. If we choose $(k_2, k_3) = (0,1)$ instead, we get $k_1 = 1$, and we find a second Eigenvector.

$$\vec{k_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

and therefore

$$\vec{x_3} = \vec{k_3}e^{\lambda_3 t}$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

These two Eigenvectors $\vec{k_2} = (1,1,0)$ and $\vec{k_3} = (1,0,1)$ are linearly independent of one another, because there's no constant by which we can multiply either vector that will give us the other vector as a result. Therefore, the general solution will be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Now let's do an example with equal real Eigenvalues, but this time where we find only one Eigenvector associated with the two equal Eigenvalues.

Example

Find the general solution to the system of differential equations.

$$x_1'(t) = 2x_1 - x_2$$

$$x_2'(t) = x_1 + 4x_2$$

We can represent this system in matrix form as

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{x}$$

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & -1 \\ 1 & 4-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 2 - \lambda & -1 \\ 1 & 4 - \lambda \end{vmatrix}$$

$$|A - \lambda I| = (2 - \lambda)(4 - \lambda) - (-1)(1)$$

$$|A - \lambda I| = 8 - 2\lambda - 4\lambda + \lambda^2 + 1$$

$$|A - \lambda I| = \lambda^2 - 6\lambda + 9$$

Solve the characteristic equation for the Eigenvalues.

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda - 3)(\lambda - 3) = 0$$

$$\lambda = 3,\ 3$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = 3$, we find

$$A - 3I = \begin{bmatrix} 2 - 3 & -1 \\ 1 & 4 - 3 \end{bmatrix}$$

$$A - 3I = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

If we turn this first row into an equation, we get

$$k_1 + k_2 = 0$$

$$k_1 = -k_2$$

We'll choose $k_2 = -1$, which results in $k_1 = 1$.

$$\vec{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Other than the trivial solution $\vec{k_1} = (0,0)$, any other (k_1, k_2) pair we find from $k_1 = -k_2$ will result in a vector that's linearly dependent with the $\vec{k_1} = (1, -1)$ vector we already found. So we can say that the Eigenvalue $\lambda_1 = \lambda_2 = 3$ produces only one Eigenvector.

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t}$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_1 = \lambda_2 = 3$, we have to use $\vec{k_1} = (1, -1)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

$$\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$-p_1 - p_2 = 1$$

$$p_1 + p_2 = -1$$

Because this system is equivalent to just one equation (because multiplying through the second equation by -1 results in the first equation), there are an infinite number of value pairs we can choose for $\overrightarrow{p_1} = (p_1, p_2)$. Let's pick simple values, like $\overrightarrow{p_1} = (1, -2)$.

Then our second solution will be

$$\overrightarrow{x_2} = \overrightarrow{k_1} te^{\lambda_1 t} + \overrightarrow{p_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{3t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{3t}$$

and the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} te^{3t} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{3t} \right)$$

Equal real Eigenvalues with multiplicity three

An **Eigenvalue with multiplicity three** is an Eigenvalue that shows up three times in the characteristic equation. Finding the general solution in this case is just an extension of what we learned to do with Eigenvalues of multiplicity two.

The multiplicity we find for the Eigenvalues from the characteristic equation is the **algebraic multiplicity**, while the number of Eigenvectors we find for each Eigenvalue is that Eigenvalue's **geometric multiplicity**.

Given a 3×3 matrix, if we find one Eigenvalue with multiplicity three, and then three linearly independent Eigenvectors for that Eigenvalue, then the algebraic and geometric multiplicity are equal, and the solution to the system will be just like the solution for distinct real Eigenvalues, $\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + \ldots + c_n\vec{x_n}$.

Eigenvalues with a defect

When we have a 3×3 matrix where we find one Eigenvalue with multiplicity three that produces fewer than three linearly independent Eigenvectors, the difference between the algebraic and geometric multiplicity is the **defect**.

When we have an Eigenvalue with multiplicity three with a defect of two (we found only one linearly independent Eigenvector), we'll still use the first Eigenvector $\vec{k_1}$ for the first solution $\vec{x_1}$, but then we'll need these special equations,

$$(A - \lambda_1 I)\overrightarrow{p_1} = \overrightarrow{k_1}$$

$$(A - \lambda_1 I)\overrightarrow{q_1} = \overrightarrow{p_1}$$

to build the three Eigenvectors,

$$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$$

$$\overrightarrow{x_2} = \overrightarrow{k_1}te^{\lambda_1 t} + \overrightarrow{p_1}e^{\lambda_1 t}$$

$$\overrightarrow{x_3} = \overrightarrow{k_1}\frac{t^2}{2}e^{\lambda_1 t} + \overrightarrow{p_1}te^{\lambda_1 t} + \overrightarrow{q_1}e^{\lambda_1 t}$$

that we can plug into the general solution equation.

But when we have an Eigenvalue with multiplicity three with a defect of one (we found only two linearly independent Eigenvectors), then we'll use the first two Eigenvectors $\overrightarrow{k_1}$ and $\overrightarrow{k_2}$ in this special equation,

$$(A - \lambda_1 I)\overrightarrow{p_1} = a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2}$$

to build the third solution vector

$$\overrightarrow{x_3} = \overrightarrow{p_1}e^{\lambda_1 t} + (a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2})te^{\lambda_1 t}$$

Let's do an example with an equal real Eigenvalue with multiplicity three and a defect of one, where the Eigenvalue with multiplicity three produces only two linearly independent Eigenvectors.

Example

Find the general solution to the system of differential equations.

$$x_1'(t) = x_1$$

$$x_2'(t) = x_1 + 2x_2 + x_3$$

$$x_3'(t) = -x_1 - x_2$$

We can represent this system in matrix form as

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} \vec{x}$$

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix}$$

$$|A-\lambda I| = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ -1 & -\lambda \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ -1 & -\lambda \end{vmatrix} + 0\begin{vmatrix} 1 & 2-\lambda \\ -1 & -1 \end{vmatrix}$$

$$|A-\lambda I| = (1-\lambda)[(2-\lambda)(-\lambda)-(1)(-1)]$$

$$-0[(1)(-\lambda)-(1)(-1)]+0[(1)(-1)-(2-\lambda)(-1)]$$

$$|A-\lambda I| = (1-\lambda)(-2\lambda+\lambda^2+1)$$

$$|A-\lambda I| = -2\lambda+\lambda^2+1+2\lambda^2-\lambda^3-\lambda$$

$$|A-\lambda I| = 1-3\lambda+3\lambda^2-\lambda^3$$

Solve the characteristic equation for the Eigenvalues.

$$1-3\lambda+3\lambda^2-\lambda^3=0$$

$$\lambda^3-3\lambda^2+3\lambda-1=0$$

$$(\lambda-1)(\lambda-1)(\lambda-1)=0$$

Then for these Eigenvalues, $\lambda_1=\lambda_2=\lambda_3=1$, we find

$$A-1I = \begin{bmatrix} 1-1 & 0 & 0 \\ 1 & 2-1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

$$A-1I = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this first row into an equation, we get

$$k_1 + k_2 + k_3 = 0$$

$$k_1 = -k_2 - k_3$$

We'll choose $(k_2, k_3) = (-1,0)$, which results in $k_1 = 1$.

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

and therefore

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t$$

But we can actually find a second linearly independent Eigenvector from $k_1 = -k_2 - k_3$. If we choose $(k_2, k_3) = (0, -1)$ instead, we get $k_1 = 1$, and we find a second Eigenvector.

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

and therefore

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^t$$

There's no third vector we can find that won't be a linearly combination of these first two Eigenvectors, which means the Eigenvalue $\lambda = 1$ has a defect of one. We'll use the special formula for this case,

$$(A - \lambda_1 I)\overrightarrow{p_1} = a_1\overrightarrow{k_1} + a_2\overrightarrow{k_2}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} a_1 \\ -a_1 \\ 0 \end{bmatrix} + \begin{bmatrix} a_2 \\ 0 \\ -a_2 \end{bmatrix}$$

Expanding this as a system of equations gives

$$0 = a_1 + a_2$$

$$p_1 + p_2 + p_3 = -a_1$$

$$-p_1 - p_2 - p_3 = -a_2$$

We can choose any values for $\overrightarrow{p_1}$, so let's choose something simple like $\overrightarrow{p_1} = (p_1, p_2, p_3) = (1,1,1)$. Then the system of equations becomes

$$0 = a_1 + a_2$$

$$3 = -a_1$$

$$-3 = -a_2$$

or

$$a_1 = -3$$

$$a_2 = 3$$

Then the third solution vector will be

$$\vec{x_3} = \vec{p_1}e^{\lambda_1 t} + (a_1\vec{k_1} + a_2\vec{k_2})te^{\lambda_1 t}$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + \left(-3 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) te^t$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + \left(\begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \right) te^t$$

$$\vec{x_3} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} te^t$$

Therefore, the general solution will be

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + \begin{bmatrix} -3 \\ 3 \\ 0 \end{bmatrix} te^t + \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} te^t \right)$$

Now let's do an example with equal real Eigenvalues, but where we find only one Eigenvector associated with the three equal Eigenvalues.

Example

Find the general solution to the system of differential equations.

$$x_1'(t) = 2x_1 + x_2 - 6x_3$$

$$x_2'(t) = 2x_2 + 5x_3$$

$$x_3'(t) = 2x_3$$

We can represent this system in matrix form as

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 2 & 1 & -6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

We'll need to find the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 2 & 1 & -6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 & 1 & -6 \\ 0 & 2 & 5 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & -6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & 1 & -6 \\ 0 & 2-\lambda & 5 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$|A-\lambda I| = (2-\lambda)\begin{vmatrix} 2-\lambda & 5 \\ 0 & 2-\lambda \end{vmatrix} + \begin{vmatrix} 0 & 5 \\ 0 & 2-\lambda \end{vmatrix} - 6\begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A-\lambda I| = (2-\lambda)((2-\lambda)(2-\lambda)-(5)(0))$$

$$+((0)(2-\lambda)-(5)(0)) - 6((0)(0)-(2-\lambda)(0))$$

$$|A-\lambda I| = (2-\lambda)(2-\lambda)(2-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(2-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\lambda = 2, 2, 2$$

Then for these Eigenvalues, $\lambda_1 = \lambda_2 = \lambda_3 = 2$, we find

$$A-2I = \begin{bmatrix} 2-2 & 1 & -6 \\ 0 & 2-2 & 5 \\ 0 & 0 & 2-2 \end{bmatrix}$$

$$A-2I = \begin{bmatrix} 0 & 1 & -6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

Put this matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 1 & -6 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn these first two rows into equations, we get $k_2 = 0$ and $k_3 = 0$. We can choose any value for k_1. If we choose $k_1 = 1$, we find

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Other than the trivial solution $\vec{k_1} = (0,0,0)$, any other (k_1, k_2, k_3) set we find from $(k_1, k_2, k_3) = (k_1,0,0)$ will result in a vector that's linearly dependent with the $\vec{k_1} = (1,0,0)$ vector we already found. So we can say that the Eigenvalue $\lambda_1 = \lambda_2 = \lambda_3 = 2$ produces only one Eigenvector.

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t}$$

Because we only find one Eigenvector for the three Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$, we have to use $\vec{k_1} = (1,0,0)$ to find a second solution.

$$(A - \lambda_1 I)\vec{p_1} = \vec{k_1}$$

$$\begin{bmatrix} 0 & 1 & -6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$p_2 - 6p_3 = 1$$

$$5p_3 = 0$$

Using $p_3 = 0$ in the first equation gives $p_2 = 1$. We can choose any value for p_1, so we'll pick $p_1 = 1$ to get

$$\overrightarrow{p_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Then our second solution will be

$$\overrightarrow{x_2} = \overrightarrow{k_1} t e^{\lambda_1 t} + \overrightarrow{p_1} e^{\lambda_1 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Now we'll use $\overrightarrow{k_1} = (1,0,0)$ and $\overrightarrow{p_1} = (1,1,0)$ to find a third solution.

$$(A - \lambda_1 I)\overrightarrow{q_1} = \overrightarrow{p_1}$$

$$\begin{bmatrix} 0 & 1 & -6 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$q_2 - 6q_3 = 1$$

$$5q_3 = 1$$

Using $q_3 = 1/5$ in the first equation gives $q_2 = 11/5$. We can choose any value for q_1, so we'll pick $q_1 = 1$ to get

$$\overrightarrow{q_1} = \begin{bmatrix} 1 \\ \frac{11}{5} \\ \frac{1}{5} \end{bmatrix}$$

Ideally, we'd like to simplify the solution by eliminating fractions, so we'll multiply through $\overrightarrow{q_1}$ by 5 to rewrite it as

$$\overrightarrow{q_1} = \begin{bmatrix} 5 \\ 11 \\ 1 \end{bmatrix}$$

Then our third solution will be

$$\overrightarrow{x_3} = \overrightarrow{k_1}\frac{t^2}{2}e^{\lambda_1 t} + \overrightarrow{p_1}te^{\lambda_1 t} + \overrightarrow{q_1}e^{\lambda_1 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2}e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 5 \\ 11 \\ 1 \end{bmatrix} e^{2t}$$

and the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_2\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{2t} \right)$$

$$+c_3\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{t^2}{2}e^{2t} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} 5 \\ 11 \\ 1 \end{bmatrix} e^{2t} \right)$$

Complex Eigenvalues

We've looked at distinct real Eigenvalues and equal real Eigenvalues of multiplicity two and multiplicity three. Now we want to look at **complex Eigenvalues**, which are Eigenvalues that are complex numbers.

We'll find complex Eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ using the quadratic formula to solve the characteristic equation, and then we'll find the associated Eigenvectors in the same way we did before, by solving the system of equations we get from $|A - \lambda I| = \vec{O}$.

The Eigenvectors $\vec{k_i}$ that we find still get plugged into our solutions

$$\vec{x_i} = \vec{k_i} e^{\lambda_i t}$$

And then the general solution will be given in the same form we're used to seeing,

$$\vec{x} = c_1\vec{x_1} + c_2\vec{x_2} + \dots + c_n\vec{x_n}$$

The Eigenvalues $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$ are conjugates, and the Eigenvectors $\vec{k_1}$ and $\vec{k_2}$ are conjugates.

Let's do an example where we use Eigenvalue/Eigenvector pairs for a set of complex conjugate Eigenvalues in order to solve a system of differential equations.

Example

Solve the system of differential equations.

$$x_1' = 4x_1 + 5x_2$$

$$x_2' = -2x_1 + 6x_2$$

The coefficient matrix is

$$A = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 4 - \lambda & 5 \\ -2 & 6 - \lambda \end{bmatrix}$$

Its determinant is

$$\begin{vmatrix} 4 - \lambda & 5 \\ -2 & 6 - \lambda \end{vmatrix} = (4 - \lambda)(6 - \lambda) - (5)(-2)$$

$$\begin{vmatrix} 4 - \lambda & 5 \\ -2 & 6 - \lambda \end{vmatrix} = 24 - 4\lambda - 6\lambda + \lambda^2 + 10$$

$$\begin{vmatrix} 4 - \lambda & 5 \\ -2 & 6 - \lambda \end{vmatrix} = \lambda^2 - 10\lambda + 34$$

So the characteristic equation is

$$\lambda^2 - 10\lambda + 34 = 0$$

Using the quadratic formula, the roots are

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\lambda = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(34)}}{2(1)}$$

$$\lambda = \frac{10 \pm \sqrt{-36}}{2}$$

$$\lambda = \frac{10 \pm 6i}{2}$$

$$\lambda = 5 \pm 3i$$

Then for these complex conjugate Eigenvalues, $\lambda_1 = 5 + 3i$ and $\lambda_2 = 5 - 3i$, we find

$$A - (5 + 3i)I = \begin{bmatrix} 4 - (5 + 3i) & 5 \\ -2 & 6 - (5 + 3i) \end{bmatrix}$$

$$A - (5 + 3i)I = \begin{bmatrix} -1 - 3i & 5 \\ -2 & 1 - 3i \end{bmatrix}$$

and

$$A - (5 - 3i)I = \begin{bmatrix} 4 - (5 - 3i) & 5 \\ -2 & 6 - (5 - 3i) \end{bmatrix}$$

$$A - (5 - 3i)I = \begin{bmatrix} -1 + 3i & 5 \\ -2 & 1 + 3i \end{bmatrix}$$

If we turn matrices back into systems of equations, we get

$$(-1 - 3i)k_1 + (5)k_2 = 0$$

$$(-2)k_1 + (1 - 3i)k_2 = 0$$

and

$$(-1 + 3i)k_1 + (5)k_2 = 0$$

$$(-2)k_1 + (1 + 3i)k_2 = 0$$

From the first equation in the first system for $\lambda_1 = 5 + 3i$, we find

$$k_2 = \left(\frac{1}{5} + \frac{3}{5}i\right)k_1$$

Choosing $k_1 = 5$ gives $k_2 = 1 + 3i$, which leads us to the Eigenvector

$$\vec{k_1} = \begin{bmatrix} 5 \\ 1 + 3i \end{bmatrix}$$

and the corresponding solution vector

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 5 \\ 1 + 3i \end{bmatrix} e^{(5+3i)t}$$

In the same way, from the first equation in the second system for $\lambda_2 = 5 - 3i$, we find

$$k_2 = \left(\frac{1}{5} - \frac{3}{5}i\right) k_1$$

Choosing $k_1 = 5$ gives $k_2 = 1 - 3i$, which leads us to the Eigenvector

$$\overrightarrow{k_2} = \begin{bmatrix} 5 \\ 1 - 3i \end{bmatrix}$$

and the corresponding solution vector

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 5 \\ 1 - 3i \end{bmatrix} e^{(5-3i)t}$$

Therefore, the general solution will be

$$\overrightarrow{x} = c_1 \overrightarrow{x_1} + c_2 \overrightarrow{x_2}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 5 \\ 1 + 3i \end{bmatrix} e^{(5+3i)t} + c_2 \begin{bmatrix} 5 \\ 1 - 3i \end{bmatrix} e^{(5-3i)t}$$

The solution in real terms

We can convert the general solution from complex numbers into real numbers. When we have two complex Eigenvalues, $\lambda_1 = \alpha + \beta i$ and $\lambda_2 = \alpha - \beta i$, we only need to consider one Eigenvector. For $\lambda_1 = \alpha + \beta i$, the real vector solutions will be

$$\overrightarrow{x_1} = [\overrightarrow{b_1} \cos(\beta t) - \overrightarrow{b_2} \sin(\beta t)] e^{\alpha t}$$

$$\vec{x_2} = [\vec{b_2}\cos(\beta t) + \vec{b_1}\sin(\beta t)]e^{\alpha t}$$

where $\vec{b_1}$ is the real part of the Eigenvector $\vec{k_1}$, $\vec{b_1} = \text{Re}(\vec{k_1})$, and where $\vec{b_2}$ is the imaginary part of the Eigenvector $\vec{k_1}$, $\vec{b_2} = \text{Im}(\vec{k_1})$. The complex conjugate Eigenvalue $\lambda_2 = \alpha - \beta i$ gives the same values for α and β, and therefore the same two solutions $\vec{x_1}$ and $\vec{x_2}$.

Let's extend the last example, to change the solution we found into real terms instead of complex terms.

Example (cont'd)

Convert the general solution into real terms.

$$\vec{x} = c_1 \begin{bmatrix} 5 \\ 1 + 3i \end{bmatrix} e^{(5+3i)t} + c_2 \begin{bmatrix} 5 \\ 1 - 3i \end{bmatrix} e^{(5-3i)t}$$

The Eigenvector $\vec{k_1}$ can be rewritten as separate real and imaginary parts.

$$\vec{k_1} = \begin{bmatrix} 5 \\ 1 + 3i \end{bmatrix}$$

$$\vec{k_1} = \begin{bmatrix} 5 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

Therefore, the vectors $\vec{b_1} = \text{Re}(\vec{k_1})$ and $\vec{b_2} = \text{Im}(\vec{k_1})$ are

$$\vec{b_1} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\vec{b_2} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$$

So the first solution will be

$$\vec{x_1} = [\vec{b_1}\cos(\beta t) - \vec{b_2}\sin(\beta t)]e^{\alpha t}$$

$$\vec{x_1} = \left(\begin{bmatrix} 5 \\ 1 \end{bmatrix}\cos(3t) - \begin{bmatrix} 0 \\ 3 \end{bmatrix}\sin(3t)\right)e^{5t}$$

$$\vec{x_1} = \left(\begin{bmatrix} 5\cos(3t) \\ \cos(3t) \end{bmatrix} - \begin{bmatrix} 0 \\ 3\sin(3t) \end{bmatrix}\right)e^{5t}$$

$$\vec{x_1} = \begin{bmatrix} 5\cos(3t) \\ \cos(3t) - 3\sin(3t) \end{bmatrix}e^{5t}$$

The second solution will be

$$\vec{x_2} = [\vec{b_2}\cos(\beta t) + \vec{b_1}\sin(\beta t)]e^{\alpha t}$$

$$\vec{x_2} = \left(\begin{bmatrix} 0 \\ 3 \end{bmatrix}\cos(3t) + \begin{bmatrix} 5 \\ 1 \end{bmatrix}\sin(3t)\right)e^{5t}$$

$$\vec{x_2} = \left(\begin{bmatrix} 0 \\ 3\cos(3t) \end{bmatrix} + \begin{bmatrix} 5\sin(3t) \\ \sin(3t) \end{bmatrix}\right)e^{5t}$$

$$\vec{x_2} = \begin{bmatrix} 5\sin(3t) \\ 3\cos(3t) + \sin(3t) \end{bmatrix}e^{5t}$$

Therefore, the general solution in real terms is given by

$$\vec{x} = c_1\begin{bmatrix} 5\cos(3t) \\ \cos(3t) - 3\sin(3t) \end{bmatrix}e^{5t} + c_2\begin{bmatrix} 5\sin(3t) \\ 3\cos(3t) + \sin(3t) \end{bmatrix}e^{5t}$$

Phase portraits for distinct real Eigenvalues

When we learn to solve systems of differential equations, we discover that much of what we learned about solving individual differential equations, also extends to solving systems.

This is also true when it comes to sketching solution curves for the system. In the same way that we previously learned to identify equilibrium solutions for a differential equation, and then sketch solution curves through the direction field around those equilibrium solutions, we can also identify equilibrium solutions for a system of differential equations, and then sketch solution curves through the direction field around those equilibrium solutions.

Equilibrium solutions

For any homogeneous system of two differential equations $\vec{x}' = A\vec{x}$, the zero vector $\vec{x} = \vec{O}$,

$$\vec{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

is always a solution. The zero vector $\vec{x} = \vec{O}$ is an equilibrium solution because any vector satisfying $A\vec{x} = \vec{O}$ is an **equilibrium solution** to the system.

If the matrix A is nonsingular (A is a square matrix with a defined inverse), then $\vec{x} = \vec{O}$ will be the only equilibrium solution of the system. And in that case, the only question left to answer is how the solution curves,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

are behaving around this single equilibrium solution. We'll need to consider both their shape and direction.

Phase plane, trajectories, and the phase portrait

When we dealt with solution curves for a single differential equation, we would sketch the curves $y(t)$ in the yt-plane. But now that we're dealing with vectors, we'll use the x_1x_2-plane.

In other words, we'll use the values from the solution vector, x_1 and x_2, to define a new plane, which we'll call the **phase plane**. We'll sketch **trajectories** in the phase plane, which are just specific solution curves. A picture of a collection of trajectories gives us the **phase portrait** of the system.

Fortunately, we already have everything we need to sketch phase portraits, because the phase portrait is dictated by the Eigenvalues and Eigenvectors of the system, and we already know how to find those.

Distinct real Eigenvalues

Here's what we want to notice about the phase portraits of systems with distinct real Eigenvalues.

1. The two straight lines correspond to the Eigenvectors of the system. An Eigenvector $\vec{k_1} = (k_1, k_2)$ will lie along the line $y = (k_2/k_1)x$.

2. The direction along those linear trajectories is dependent on the sign of the associated Eigenvalue. A linear trajectory associated with a positive Eigenvalue will always move away from the origin, while a linear trajectory associated with a negative Eigenvalue will always move toward the origin.

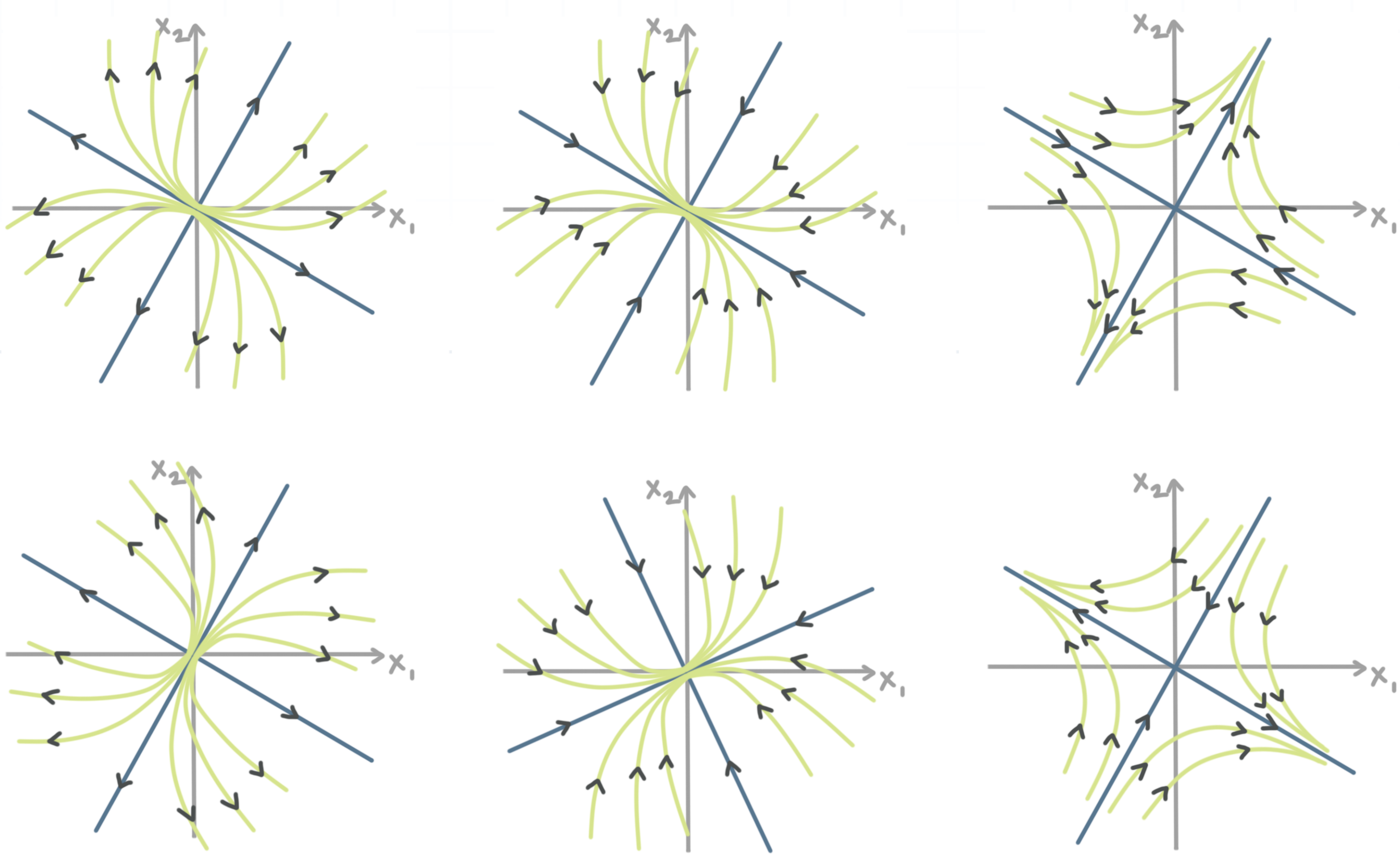
3. The equilibrium of two positive Eigenvalues is an unstable node that repels all trajectories, while the equilibrium of two negative Eigenvalues is an asymptotically stable node that attracts all trajectories. The equilibrium of two opposite signed Eigenvalues is an unstable saddle point, because the trajectories move toward the origin at some points, but move away from it at others.

4. To check for the direction of the trajectories for any case of distinct real Eigenvalues, we can use the $t \to \pm \infty$ test. We can think about the trajectories "starting" at $t \to -\infty$, and "ending" at $t \to \infty$.

Below is a table that summarizes information about the phase portrait, based on the sign of the distinct real Eigenvalues.

DISTINCT REAL EIGENVALUES

	Positive	Negative	Opposite signs
	$\lambda_1, \lambda_2 > 0$	$\lambda_1, \lambda_2 < 0$	$\lambda_1 > 0, \lambda_2 < 0$
Equilibrium	Node	Node	Saddle point
Stability	Unstable	Asymptotically stable	Unstable
	(Repeller)	(Attractor)	
Direction	$t \to \pm \infty$	$t \to \pm \infty$	$t \to \pm \infty$
Sketches			



Let's do an example so that we can see how to build a phase portrait for distinct real Eigenvalues.

Example

Sketch the phase portrait of the system.

$$x_1' = 2x_1 + 2x_2$$

$$x_2' = x_1 + 3x_2$$

The coefficient matrix and $A - \lambda I$ are,

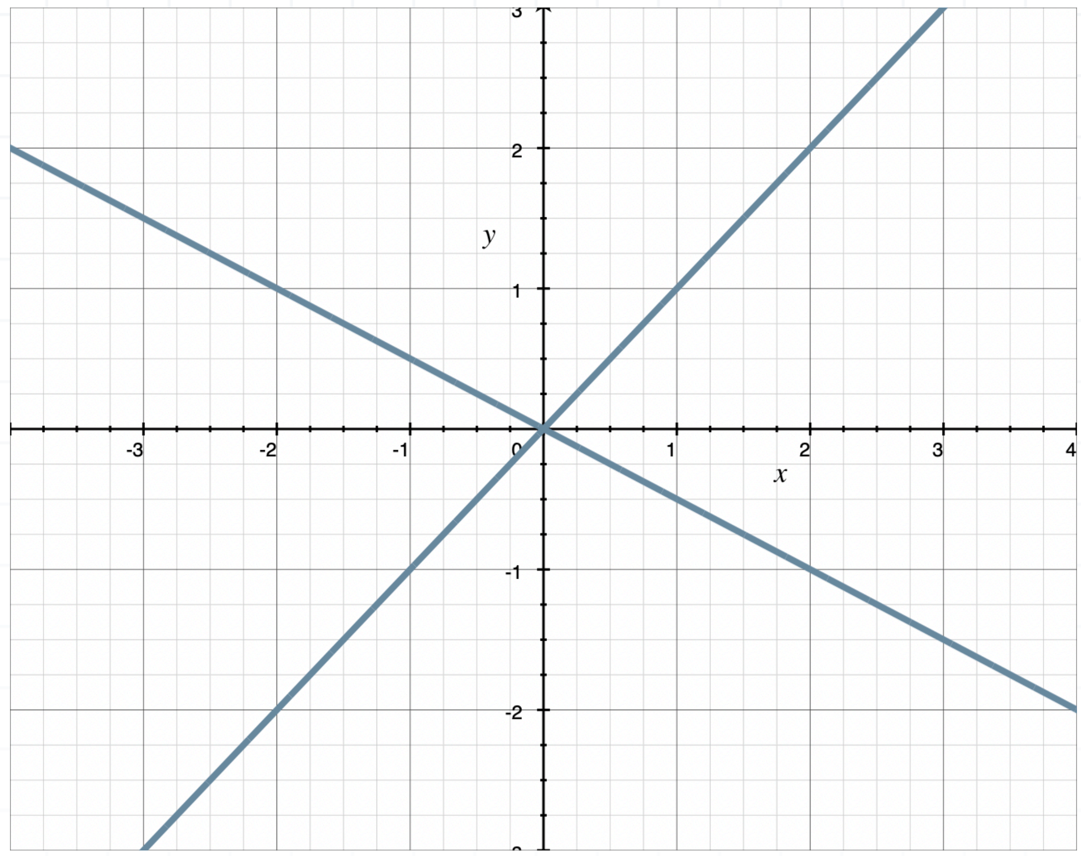
$$A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 2 - \lambda & 2 \\ 1 & 3 - \lambda \end{bmatrix}$$

and the characteristic equation $\lambda^2 - 5\lambda + 4 = 0$ gives the Eigenvalues $\lambda_1 = 1$ and $\lambda_2 = 4$, and their associated Eigenvectors

$$\vec{k_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \qquad \vec{k_2} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

The Eigenvector $\vec{k_1} = (-2,1)$ lies along the line $y = -(1/2)x$, and the Eigenvector $\vec{k_2} = (1,1)$ lies along the line $y = x$, so we'll sketch these lines.



The Eigenvalue associated with $\overrightarrow{k_1} = (-2,1)$ is $\lambda = 1$, which means the direction along that trajectory is away the origin. The Eigenvalue associated with $\overrightarrow{k_2} = (1,1)$ is $\lambda = 4$, which means the direction along that trajectory is also away from the origin.

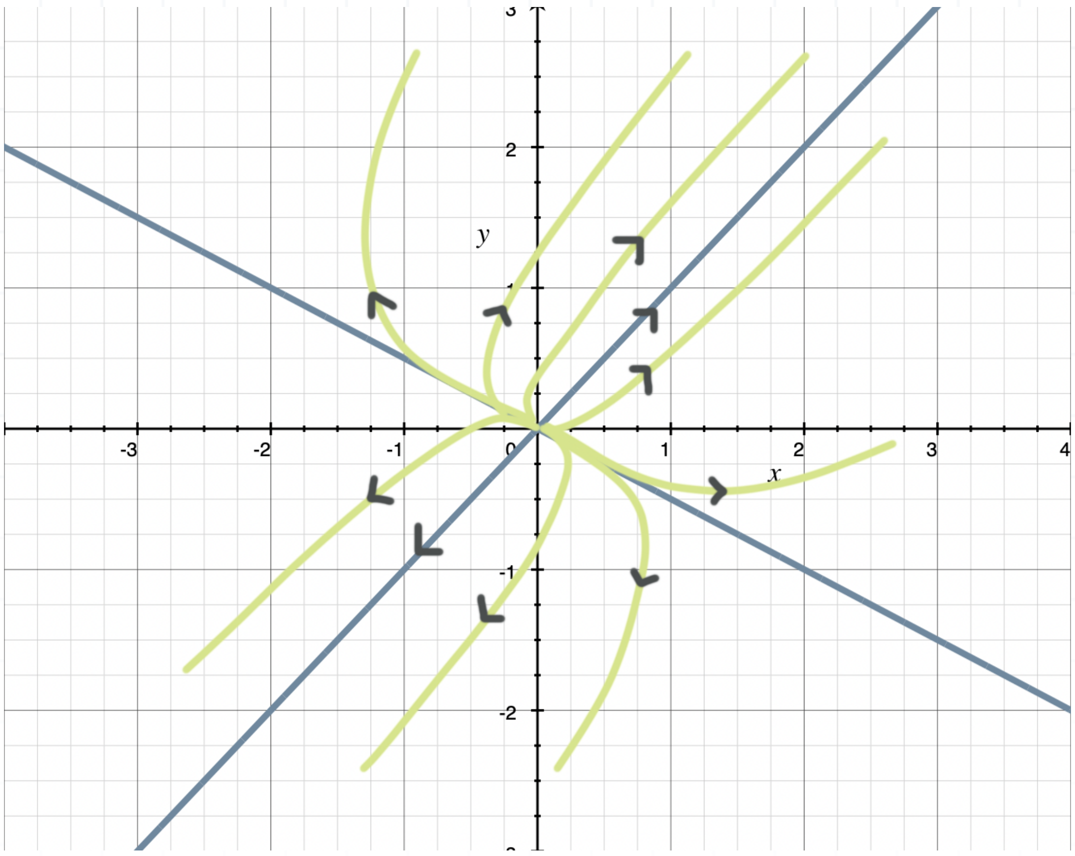
Because both Eigenvalues are positive, we're dealing with an unstable node that repels all trajectories.

To apply the $t \to \pm \infty$ test, we'll use the general solution to the system,

$$\overrightarrow{x} = c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$$

Because e^{4t} goes to 0 faster than e^t as $t \to -\infty$, the $\overrightarrow{k_2} = (1,1)$ vector drops away first, meaning that our trajectories are going to "start" parallel to $\overrightarrow{k_1} = (-2,1)$. On the other end, e^{4t} dominates e^t as $t \to \infty$, so the $\overrightarrow{k_1} = (-2,1)$ vector will drop away first, meaning that our trajectories are going to "end" parallel to $\overrightarrow{k_2} = (1,1)$.

Therefore, starting the trajectories parallel to $\overrightarrow{k_1} = (-2,1)$ and ending them parallel to $\overrightarrow{k_2} = (1,1)$ means that the phase portrait must look something like



And before we move on to looking at equal real Eigenvalues, let's do one more example with the special case where one Eigenvalue is $\lambda = 0$.

Example

Sketch the phase portrait of the system.

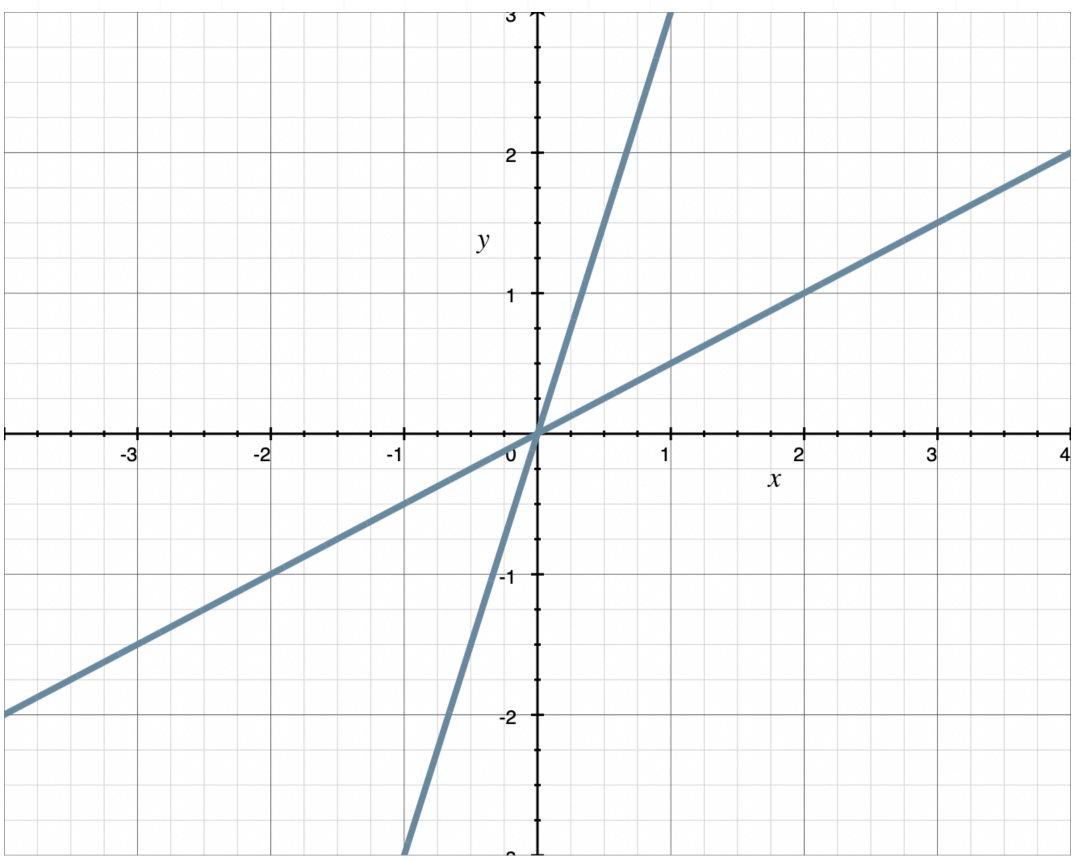
$$x_1' = -6x_1 + 2x_2$$

$$x_2' = -3x_1 + x_2$$

This is the example we looked at when we learned to solve homogeneous systems with distinct real Eigenvalues, and we found that the Eigenvalues for the system were $\lambda = -5, 0$. The associated Eigenvectors were

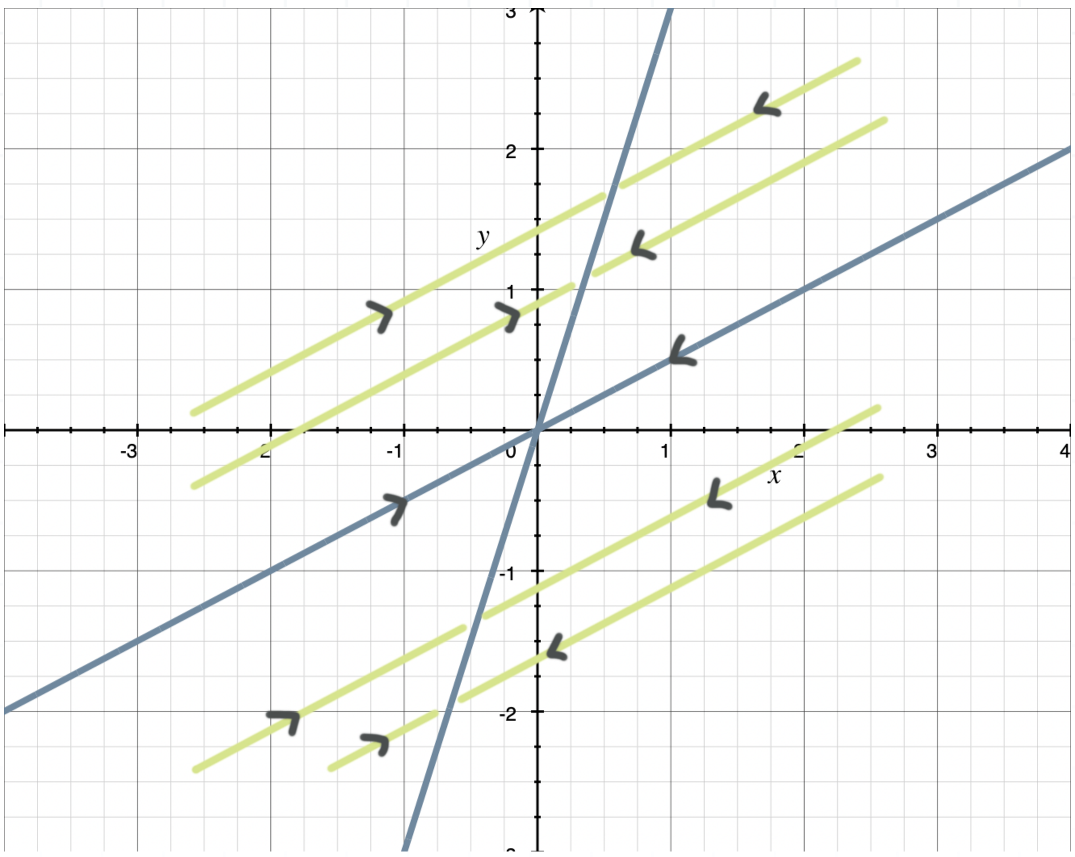
$$\overrightarrow{k_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (2,1)$ lies along the line $y = (1/2)x$, and the Eigenvector $\overrightarrow{k_2} = (1,3)$ lies along the line $y = 3x$, so we'll sketch these lines.



The Eigenvalue associated with $\overrightarrow{k_1} = (2,1)$ is $\lambda = -5$, which means the direction along that trajectory is toward the origin. The Eigenvalue associated with $\overrightarrow{k_2} = (1,3)$ is $\lambda = 0$, which means the direction along that trajectory is neither toward nor away from the origin. This is a special case in which equilibrium exists everywhere along $y = 3x$, not just at the origin (which occurs before of the special Eigenvalue $\lambda = 0$).

Therefore, all trajectories run parallel to $y = (1/2)x$. Because $\lambda = -5$ is a negative Eigenvalue, the arrows always pointed toward $y = 3x$, and the stable line of equilibrium attracts all trajectories.



In other words, in the special case where one Eigenvalue is $\lambda = 0$,

- when the sign of the other Eigenvalue is negative, we'll have a stable line of equilibrium (attractor) along the Eigenvector associated with $\lambda = 0$, or

- when the sign of the other Eigenvalue is positive, we'll have an unstable line of equilibrium (repeller) along the Eigenvector associated with $\lambda = 0$.

Phase portraits for equal real Eigenvalues

Now we want to look at the phase portraits of systems with equal real Eigenvalues.

We'll first consider what happens when the coefficient matrix A is nonsingular, then we'll look later at what happens when it's singular.

The nonsingular matrix

If the coefficient matrix A is **nonsingular**, which means its determinant is non-zero $|A| \neq 0$ and 0 is not an Eigenvalue of A, then there are two cases:

1. The repeated Eigenvalue produces multiple linearly independent Eigenvectors

2. The repeated Eigenvalue produces only one Eigenvector

In the first case, where the repeated Eigenvalue produces multiple linearly independent Eigenvectors, here's what we need to know:

1. Any vector in the plane is a linear combination of two different linearly independent Eigenvectors of the matrix A. Therefore, all vectors in the plane are Eigenvectors of A associated with our Eigenvalue, so all trajectories are lines passing through the origin.

2. The direction along the linear trajectories depends on the sign of the associated Eigenvalue. A linear trajectory associated with a positive Eigenvalue will always move away from the origin, while a

linear trajectory associated with a negative Eigenvalue will always move toward the origin.

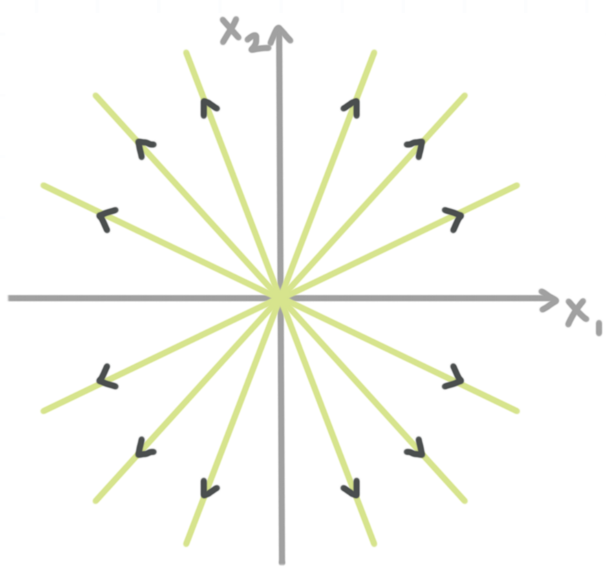
3. The equilibrium of a positive Eigenvalue is an unstable singular node that repels all trajectories, while the equilibrium of a negative Eigenvalue is an asymptotically stable singular node that attracts all trajectories.

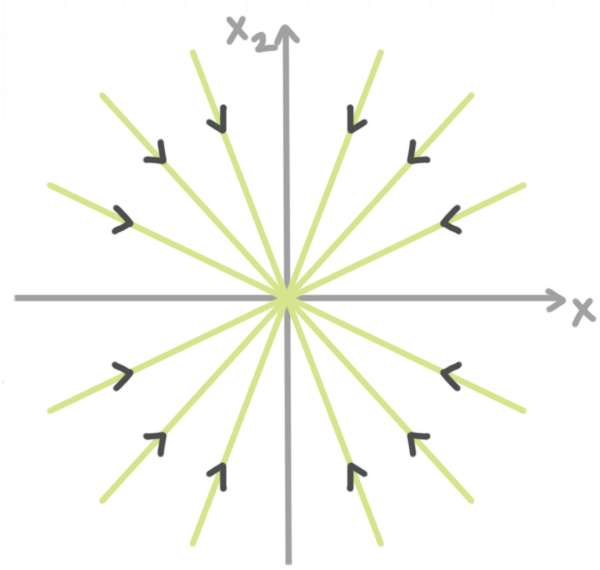
4. To check for the direction of the trajectories for any case of equal real Eigenvalues, we can use the $(1,0)$ test, which means we'll look at the direction at the point $(1,0)$.

Below is a table that summarizes information about the phase portrait, based on the sign of the equal real Eigenvalues.

EQUAL REAL EIGENVALUES, TWO EIGENVECTORS

	Positive	**Negative**
	$\lambda_1 = \lambda_2 > 0$	$\lambda_1 = \lambda_2 < 0$
Equilibrium	Singular Node	Singular Node
Stability	Unstable	Asymptotically stable
	(Repeller)	(Attractor)
Direction	$(1,0)$	$(1,0)$
Sketches		





In the second case, where the repeated Eigenvalue produces only one Eigenvector, here's what we need to know:

1. The single straight line corresponds to the one Eigenvector of the system. An Eigenvector $\overrightarrow{k_1} = (k_1, k_2)$ will lie along the line $y = (k_2/k_1)x$.

2. The direction along the linear trajectory is dependent on the sign of the associated Eigenvalue. A linear trajectory associated with a

positive Eigenvalue will always move away from the origin, while a linear trajectory associated with a negative Eigenvalue will always move toward the origin.

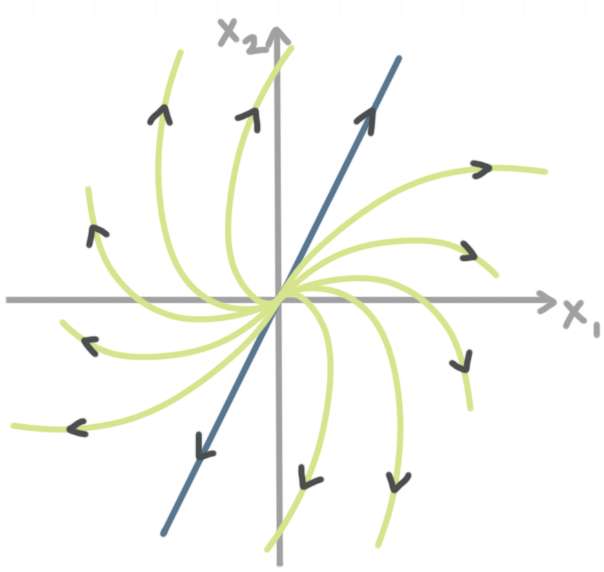
3. The equilibrium of a positive Eigenvalue is an unstable node that repels all trajectories, while the equilibrium of a negative Eigenvalue is an asymptotically stable node that attracts all trajectories.

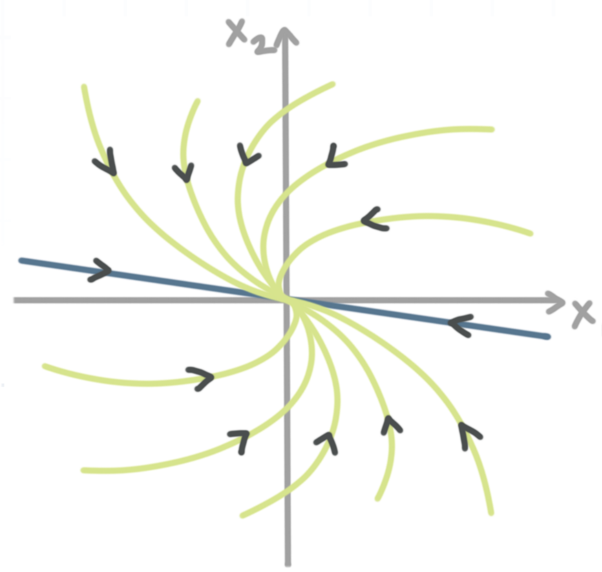
4. To check for the direction of the trajectories for any case of equal real Eigenvalues, we can use the $(1,0)$ test, which means we'll look at the direction at the point $(1,0)$.

Below is a table that summarizes information about the phase portrait, based on the sign of the equal real Eigenvalues.

EQUAL REAL EIGENVALUES, ONE EIGENVECTOR

	Positive	**Negative**
	$\lambda_1 = \lambda_2 > 0$	$\lambda_1 = \lambda_2 < 0$
Equilibrium	Node	Node
Stability	Unstable	Asymptotically stable
	(Repeller)	(Attractor)
Direction	$(1,0)$	$(1,0)$
Sketches		





Let's do an example so that we can see how to build a phase portrait for equal real Eigenvalues. We'll look at a nonsingular coefficient matrix, for which only one Eigenvector is produced from the equal real Eigenvalues.

Example

Sketch the phase portrait of the system.

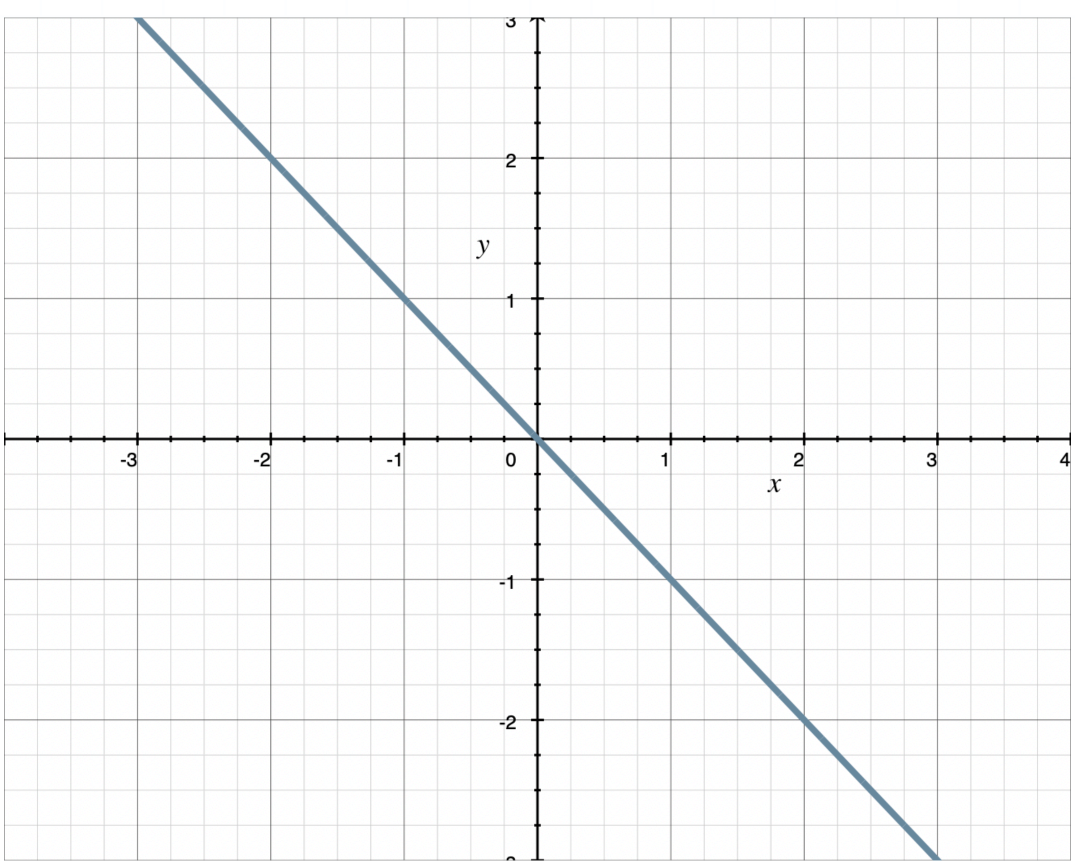
$$x_1'(t) = 2x_1 - x_2$$

$$x_2'(t) = x_1 + 4x_2$$

This is one of the examples we looked at when we learned to solve homogeneous systems with equal real Eigenvalues, and we found that the Eigenvalues for the system were $\lambda_1 = \lambda_2 = 3$. The associated single Eigenvector was

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The Eigenvector $\overrightarrow{k_1} = (1, -1)$ lies along the line $y = -x$, so we'll sketch this line.



The Eigenvalue associated with $\overrightarrow{k_1} = (1, -1)$ is $\lambda = 3$, which means the direction along that trajectory is away from the origin. Therefore, we're dealing with an unstable node that repels all trajectories.

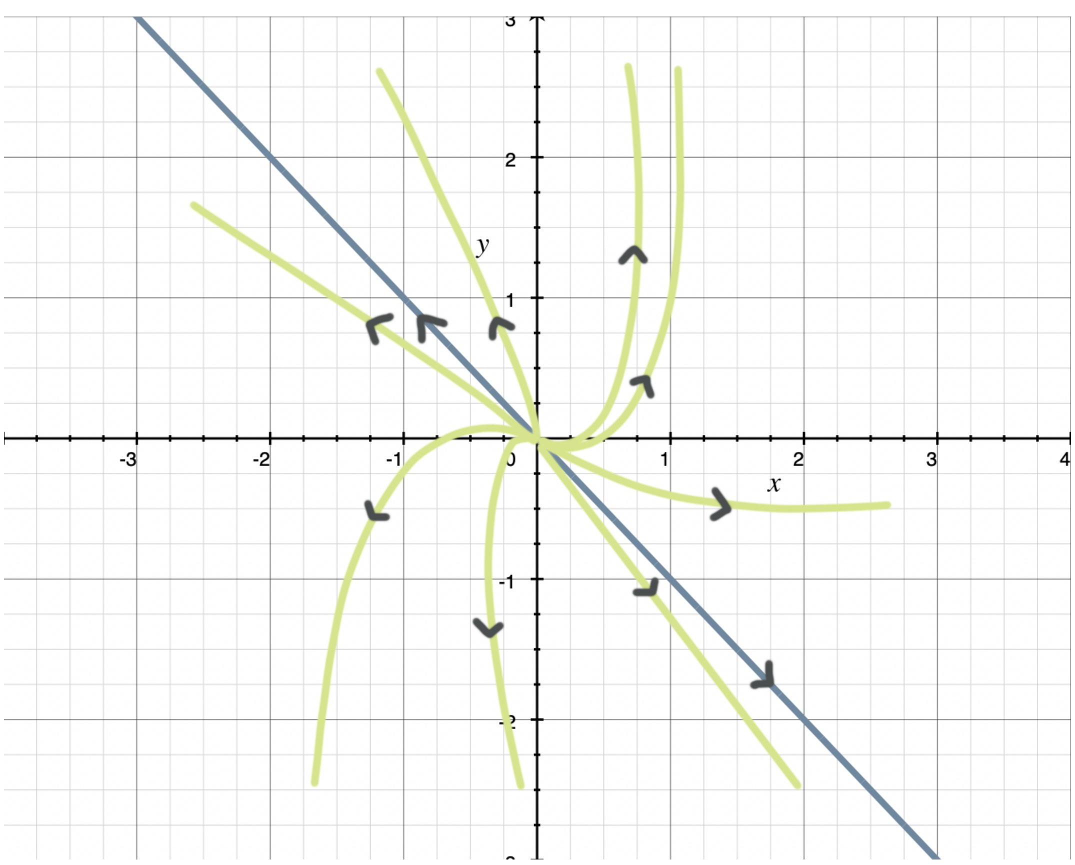
If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \vec{x}$$

$$\vec{x}' = \begin{bmatrix} 2 & -1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(2,1)$ (into the first quadrant), which means that the phase portrait must look something like

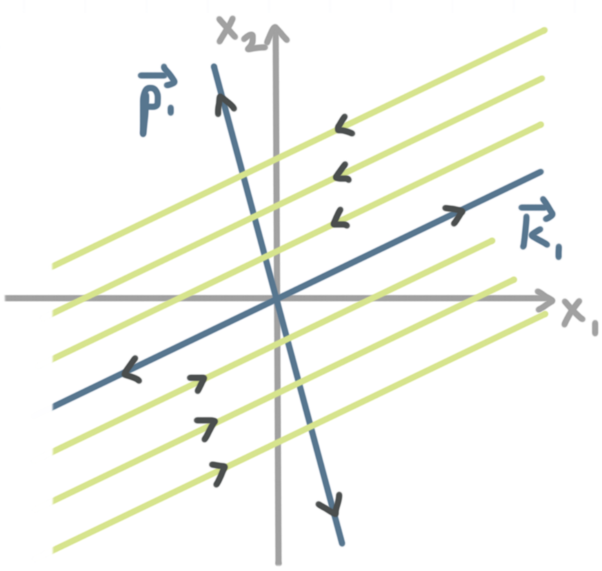


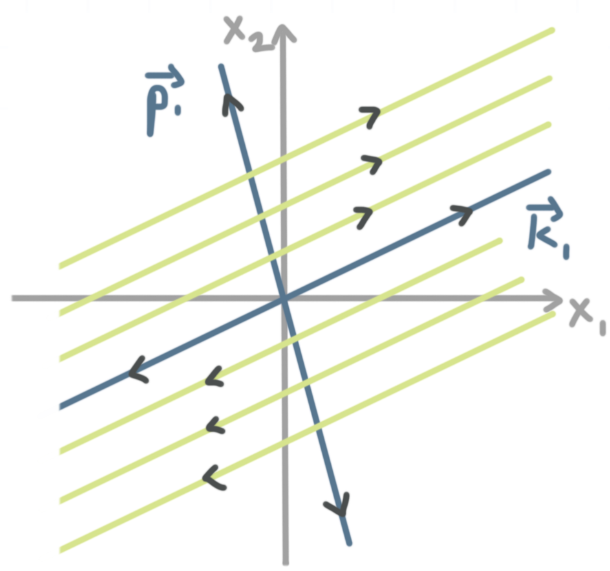
The singular matrix

If the coefficient matrix A is **singular**, which means its determinant is 0, $|A| = 0$, then its Eigenvalues are both $\lambda_1 = \lambda_2 = 0$, and there are two cases:

1. Every entry in the coefficient matrix A is 0 (A is the zero matrix $\vec{O}$), in which case every point in the plane is an equilibrium point.

2. The Eigenvalue $\lambda_1 = \lambda_2 = 0$ produces one Eigenvector $\vec{k_1}$, in which case we need to find the vector $\vec{p_1}$ using $(A - \lambda I)\vec{p_1} = \vec{k_1}$. The trajectories are parallel to the Eigenvector $\vec{k_1}$, and every point on $\vec{p_1}$ is an equilibrium point.





Phase portraits for complex Eigenvalues

Now we want to look at the phase portraits of systems with complex Eigenvalues. Here's what we need to know:

1. There are no linear trajectories for the system.

2. The equilibrium of a system with complex Eigenvalues that have no real part is a stable center around which the trajectories revolve, without ever getting closer to or further from equilibrium. The equilibrium of a system with complex Eigenvalues with a positive real part is an unstable spiral that repels all trajectories. The equilibrium of a system with complex Eigenvalues with a negative real part is an asymptotically stable spiral that attracts all trajectories.

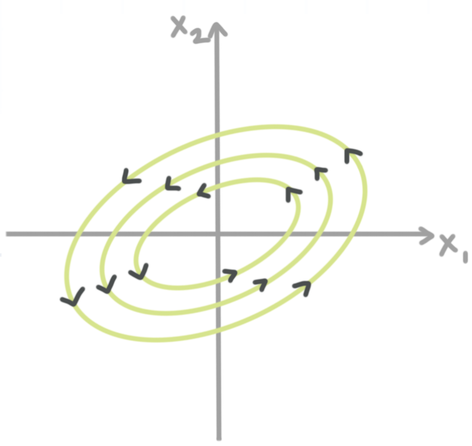
3. To check for the direction of the trajectories for any case of complex conjugate Eigenvalues, we can use the $(1,0)$ test, which means we'll look at the direction at the point $(1,0)$.

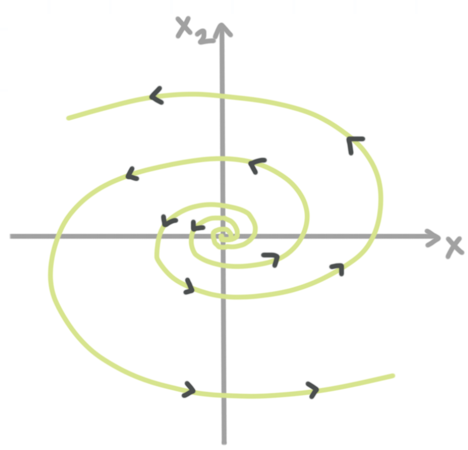
Below is a table that summarizes information about the phase portrait, based on the sign of the complex conjugate Eigenvalues.

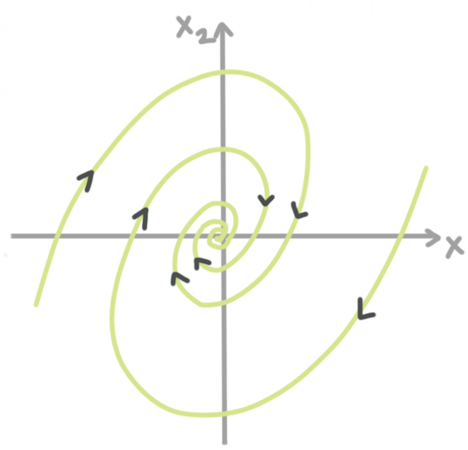
COMPLEX CONJUGATE EIGENVALUES

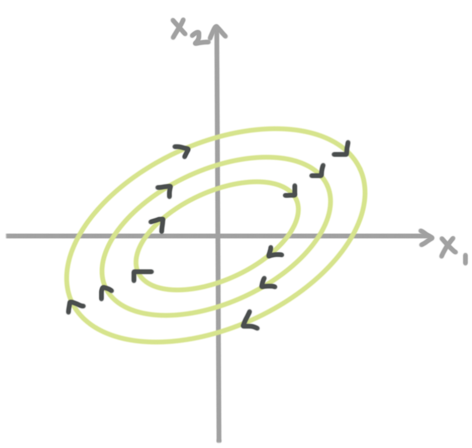
	No real part	Positive real part	Negative real part
	$\lambda_1 = \beta i, \lambda_2 = -\beta i$	$\lambda_1, \lambda_2 = \alpha \pm \beta i$	$\lambda_1, \lambda_2 = \alpha \pm \beta i$
		$\alpha > 0$	$\alpha < 0$
Equilibrium	Center	Spiral	Spiral
Stability	Stable	Unstable	Asymptotically stable
		(Repeller)	(Attractor)
Direction	$(1,0)$	$(1,0)$	$(1,0)$

Sketches









Let's do an example so that we can see how to build a phase portrait for complex conjugate Eigenvalues.

Example

Sketch the phase portrait of the system.

$$x_1' = 4x_1 + 5x_2$$

$$x_2' = -2x_1 + 6x_2$$

This is one of the examples we looked at when we learned to solve homogeneous systems with complex conjugate Eigenvalues, and we found that the Eigenvalues for the system were $\lambda_1 = 5 + 3i$ and $\lambda_2 = 5 - 3i$.

These are Eigenvalues with a positive real part, $\alpha = 5 > 0$, which means we're dealing with an unstable spiral that repels all trajectories.

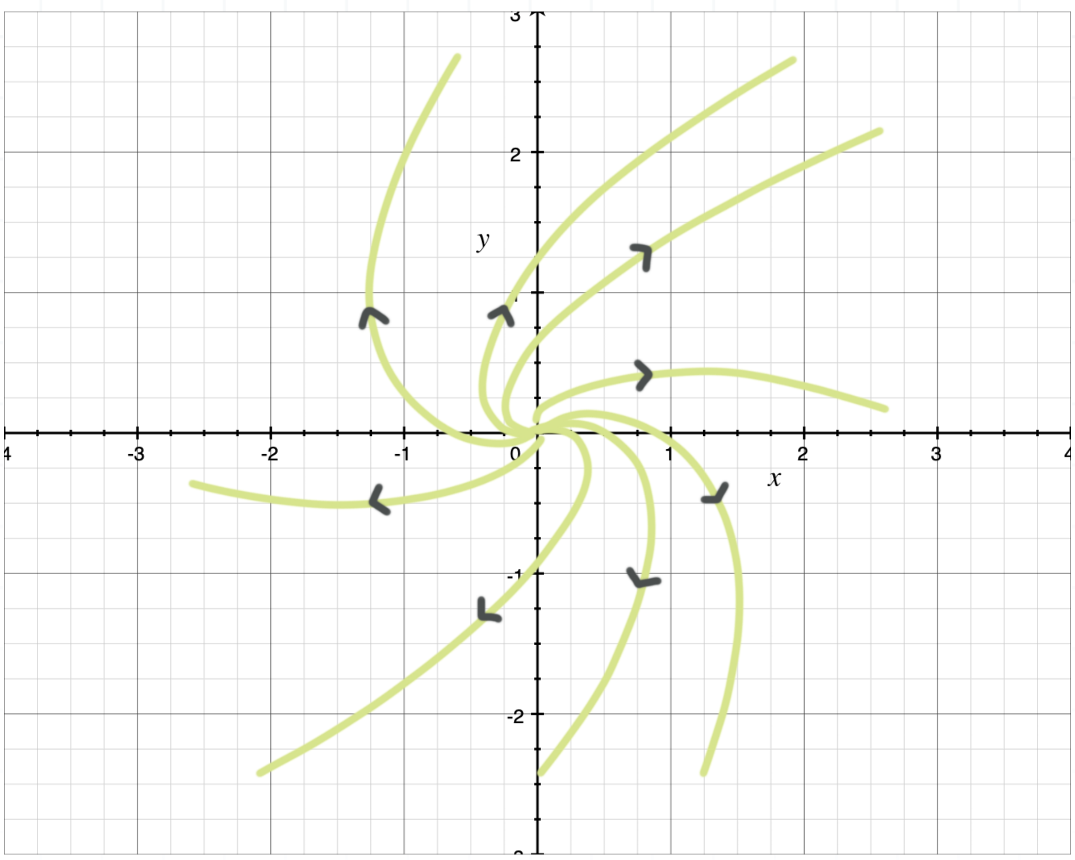
If we test the vector $\vec{x} = (1,0)$ in the matrix equation associated with the system, we get

$$\vec{x}' = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \vec{x}$$

$$\vec{x}' = \begin{bmatrix} 4 & 5 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}' = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$$

This $(1,0)$ test tells us that the direction of the trajectory running through that point must have a direction toward $(4,-2)$ (into the fourth quadrant), which means that the phase portrait must look something like



Undetermined coefficients for nonhomogeneous systems

When we looked before at how to build systems of differential equations into a matrix equation, we introduced the equation $\vec{x}' = A\vec{x} + F$, and said that the system was homogeneous when $F = \vec{O}$, and nonhomogeneous when F was non-zero, $F \neq \vec{O}$.

Now that we've learned to solve homogeneous systems $\vec{x}' = A\vec{x}$ with $F = \vec{O}$, we want to turn our attention toward nonhomogeneous systems.

There are two methods we can use to solve nonhomogeneous systems, which are just extensions of methods we used earlier to solve linear second order differential equations: undetermined coefficients, and variation of parameters.

Just as with second order nonhomogeneous linear differential equations, the solution to a nonhomogeneous system will be the sum of the complementary and particular solutions,

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

where $\vec{x}_c$ is the general solution to the associated homogeneous system, $\vec{x}' = A\vec{x}$. We'll use undetermined coefficients or variation of parameters to make a guess about the particular solution, $\vec{x}_p$.

While variation of parameters is a more versatile method for finding the particular solution, undetermined coefficients can sometimes be easier to use, so we'll start here with the method of undetermined coefficients.

Method of undetermined coefficients

The method of undetermined coefficients may work well when the entries of the vector F are constants, polynomials, exponentials, sines and cosines, or some combination of these.

Our guesses for the particular solution will be similar to the kinds of guesses we used to solve second order nonhomogeneous equations, except that we'll use vectors instead of constants.

Below is a table similar to the one we used before for nonhomogeneous equations.

$g(x)$	**Guess**
x^2+1	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
$2x^3-3x+4$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x^3 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x^2 + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} x + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$
e^{3x}	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{3x}$
$3\sin(4x)$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(4x) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(4x)$
$2\cos(4x)$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(4x) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(4x)$
$3\sin(4x)+2\cos(4x)$	$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \sin(4x) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \cos(4x)$

$$x^2 + 1 + e^{3x} \qquad \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} x^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} x + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} e^{3x}$$

Let's do an example with a system of two equations.

Example

Use undetermined coefficients to find the general solution to the system.

$$\vec{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x} + \begin{bmatrix} -2t^2 \\ t+5 \end{bmatrix}$$

We have to start by finding the complementary solution, which is the general solution to the associated homogeneous equation, which means we need to solve

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x}$$

The coefficient matrix is

$$A = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{bmatrix}$$

Its determinant is

$$\begin{vmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{vmatrix}$$

$$(1 - \lambda)(1 - \lambda) - (3)(3)$$

$$1 - 2\lambda + \lambda^2 - 9$$

$$\lambda^2 - 2\lambda - 8$$

So the characteristic equation is

$$\lambda^2 - 2\lambda - 8 = 0$$

$$(\lambda - 4)(\lambda + 2) = 0$$

Then for these Eigenvalues, $\lambda_1 = 4$ and $\lambda_2 = -2$, we find

$$A - 4I = \begin{bmatrix} 1 - 4 & 3 \\ 3 & 1 - 4 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}$$

and

$$A-(-2)I=\begin{bmatrix}1-(-2) & 3\\ 3 & 1-(-2)\end{bmatrix}$$

$$A-(-2)I=\begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix}-3 & 3\\ 3 & -3\end{bmatrix} \qquad \begin{bmatrix}3 & 3\\ 3 & 3\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\ 3 & -3\end{bmatrix} \qquad \begin{bmatrix}1 & 1\\ 3 & 3\end{bmatrix}$$

$$\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix} \qquad \begin{bmatrix}1 & 1\\ 0 & 0\end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1-k_2=0 \qquad k_1+k_2=0$$

$$k_1=k_2 \qquad k_1=-k_2$$

For the first system, we choose $(k_1,k_2)=(1,1)$. And from the second system, we choose $(k_1,k_2)=(1,-1)$.

$$\vec{k_1}=\begin{bmatrix}1\\ 1\end{bmatrix} \qquad \vec{k_2}=\begin{bmatrix}1\\ -1\end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1}=\vec{k_1}e^{\lambda_1 t} \qquad \vec{x_2}=\vec{k_2}e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2}$$

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} -2t^2 \\ t+5 \end{bmatrix}$$

$$F = \begin{bmatrix} -2 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

We want a particular solution in the same form, so our guess will be

$$\overrightarrow{x_p} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\overrightarrow{x_p'} = \begin{bmatrix} 2a_1 \\ 2a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

Plugging this into the matrix equation representing the system of differential equations $\overrightarrow{x}' = A\overrightarrow{x} + F$, we get

$$\begin{bmatrix} 2a_1 \\ 2a_2 \end{bmatrix} t + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \left[\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} t^2 + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} t + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \right] + \begin{bmatrix} -2 \\ 0 \end{bmatrix} t^2 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} t + \begin{bmatrix} 0 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 2a_1 t + b_1 \\ 2a_2 t + b_2 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} a_1 t^2 + b_1 t + c_1 \\ a_2 t^2 + b_2 t + c_2 \end{bmatrix} + \begin{bmatrix} -2t^2 \\ t + 5 \end{bmatrix}$$

Breaking this equation into a system of two equations gives

$$2a_1 t + b_1 = (1)(a_1 t^2 + b_1 t + c_1) + (3)(a_2 t^2 + b_2 t + c_2) - 2t^2$$

$$2a_2 t + b_2 = (3)(a_1 t^2 + b_1 t + c_1) + (1)(a_2 t^2 + b_2 t + c_2) + t + 5$$

which simplifies to

$$2a_1 t + b_1 = (a_1 + 3a_2 - 2)t^2 + (b_1 + 3b_2)t + (c_1 + 3c_2)$$

$$2a_2 t + b_2 = (3a_1 + a_2)t^2 + (3b_1 + b_2 + 1)t + (3c_1 + c_2 + 5)$$

These equations can each be broken into its own system.

$$0 = a_1 + 3a_2 - 2 \qquad\qquad 0 = 3a_1 + a_2$$

$$2a_1 = b_1 + 3b_2 \qquad\qquad 2a_2 = 3b_1 + b_2 + 1$$

$$b_1 = c_1 + 3c_2 \qquad\qquad b_2 = 3c_1 + c_2 + 5$$

Solving the top two equations as a system gives $a_1 = -1/4$ and $a_2 = 3/4$. Plugging these values into the middle two equations gives

$$2\left(-\frac{1}{4}\right) = b_1 + 3b_2 \qquad\qquad 2\left(\frac{3}{4}\right) = 3b_1 + b_2 + 1$$

$$b_1 + 3b_2 = -\frac{1}{2} \qquad\qquad 3b_1 + b_2 = \frac{1}{2}$$

From these, we get $b_1 = 1/4$ and $b_2 = -1/4$. Finally, plugging these values into the bottom two equations gives

$$\frac{1}{4} = c_1 + 3c_2 \qquad\qquad -\frac{1}{4} = 3c_1 + c_2 + 5$$

$$c_1 + 3c_2 = \frac{1}{4} \qquad\qquad 3c_1 + c_2 = -\frac{21}{4}$$

From these, we get $c_1 = -2$ and $c_2 = 3/4$. Putting these results together gives $\vec{a} = (a_1, a_2) = (-1/4, 3/4)$ and $\vec{b} = (b_1, b_2) = (1/4, -1/4)$ and $\vec{c} = (c_1, c_2) = (-2, 3/4)$. Therefore, the particular solution becomes

$$\vec{x}_p = \begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} t^2 + \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \end{bmatrix} t + \begin{bmatrix} -2 \\ \frac{3}{4} \end{bmatrix}$$

Now we can rewrite the particular solution as one vector.

$$\vec{x}_p = \begin{bmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t} + \begin{bmatrix} -\frac{1}{4}t^2 + \frac{1}{4}t - 2 \\ \frac{3}{4}t^2 - \frac{1}{4}t + \frac{3}{4} \end{bmatrix}$$

Let's do an example with a system of three equations.

Example

Use undetermined coefficients to find the general solution to the system $\vec{x}' = A\vec{x} + F$.

$$A = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \qquad F = \begin{bmatrix} e^{6t} \\ 2\sin t \\ 4t \end{bmatrix}$$

We have to start by finding the complementary solution, which is the general solution to the associated homogeneous equation, which means we need to solve

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}$$

The coefficient matrix is

$$A = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and the matrix $A - \lambda I$ is

$$A - \lambda I = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -\lambda & -2 & 0 \\ -2 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

Its determinant is

$$\begin{vmatrix} -\lambda & -2 & 0 \\ -2 & -\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$-\lambda \begin{vmatrix} -\lambda & 0 \\ 0 & 2-\lambda \end{vmatrix} - (-2) \begin{vmatrix} -2 & 0 \\ 0 & 2-\lambda \end{vmatrix} + 0 \begin{vmatrix} -2 & -\lambda \\ 0 & 0 \end{vmatrix}$$

$$-\lambda[(-\lambda)(2-\lambda) + (0)(0)] + 2[(-2)(2-\lambda) + (0)(0)]$$

$$-\lambda(-2\lambda + \lambda^2) + 2(-4 + 2\lambda)$$

$$2\lambda^2 - \lambda^3 - 8 + 4\lambda$$

$$-\lambda^3 + 2\lambda^2 + 4\lambda - 8$$

So the characteristic equation is

$$-\lambda^3 + 2\lambda^2 + 4\lambda - 8 = 0$$

$$\lambda^3 - 2\lambda^2 - 4\lambda + 8 = 0$$

$$(\lambda + 2)(\lambda - 2)(\lambda - 2) = 0$$

Then for these Eigenvalues, $\lambda_1 = -2$ and $\lambda_2 = \lambda_3 = 2$, we find

$$A - (-2)I = \begin{bmatrix} -(-2) & -2 & 0 \\ -2 & -(-2) & 0 \\ 0 & 0 & 2 - (-2) \end{bmatrix}$$

$$A - (-2)I = \begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and

$$A - 2I = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 2 - 2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 - k_2 = 0 \qquad k_1 + k_2 = 0$$

$$4k_3 = 0$$

From the first system, we get $k_3 = 0$. Then we rewrite $k_1 - k_2 = 0$ as $k_1 = k_2$, choosing $k_2 = 1$ and $k_1 = 1$ to get $\overrightarrow{k_1} = (k_1, k_2, k_3) = (1,1,0)$. And from the second system, we'll rewrite $k_1 + k_2 = 0$ as $k_1 = -k_2$, choosing $k_2 = -1$ and $k_1 = 1$. Since k_3 can take any value, we'll use $k_3 = 0$. So we find $\overrightarrow{k_2} = (k_1, k_2, k_3) = (1, -1, 0)$.

$$\overrightarrow{k_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \overrightarrow{k_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

Then the solutions to the system are

$$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t} \qquad \overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_1} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} \qquad \overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t}$$

But we still have to address the repeated Eigenvalue $\lambda_2 = \lambda_3 = 2$. We already found one associated Eigenvector $\overrightarrow{k_2} = (1, -1, 0)$. We can find another Eigenvector $\overrightarrow{k_3} = (0, 0, 1)$ that stills satisfies $k_1 = -k_2$ (while k_3 can take any value), but is linearly independent from $\overrightarrow{k_2} = (1, -1, 0)$. So we'll say that the repeated Eigenvalue $\lambda_2 = \lambda_3 = 2$ can produce two linearly independent Eigenvectors

$$\overrightarrow{k_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \overrightarrow{k_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

and therefore two solution vectors

$$\overrightarrow{x_2} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} \qquad \overrightarrow{x_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\overrightarrow{x_c} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x_c} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

Now we can turn to finding the particular solution. We'll rewrite the forcing function vector as

$$F = \begin{bmatrix} e^{6t} \\ 2\sin t \\ 4t \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{6t} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} t$$

We want a particular solution in the same form, so our guess will be

$$\vec{x}_p = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{6t} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

Once we have our guess for the particular solution, we'll break it into pieces, one for each term in F.

$$\vec{x}_p = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{6t}$$

$$\vec{x}_p = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t$$

Then we'll plug each of these three particular solutions, one at a time, into the matrix equation representing the system of differential equations $\vec{x}' = A\vec{x} + F$. Starting with the polynomial part, we get

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \left[\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right] + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} t$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} b_1 t + a_1 \\ b_2 t + a_2 \\ b_3 t + a_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} t$$

Breaking this equation into a system of three equations gives

$$b_1 = 0(b_1 t + a_1) - 2(b_2 t + a_2) + 0(b_3 t + a_3) + 0t$$

$$b_2 = -2(b_1 t + a_1) + 0(b_2 t + a_2) + 0(b_3 t + a_3) + 0t$$

$$b_3 = 0(b_1 t + a_1) + 0(b_2 t + a_2) + 2(b_3 t + a_3) + 4t$$

which simplifies to

$$b_1 = -2b_2 t - 2a_2$$

$$b_2 = -2b_1 t - 2a_1$$

$$b_3 = 2b_3 t + 2a_3 + 4t$$

These equations can each be broken into its own system.

$$-2b_2 = 0 \qquad -2b_1 = 0 \qquad 2b_3 + 4 = 0$$

$$-2a_2 - b_1 = 0 \qquad -2a_1 - b_2 = 0 \qquad 2a_3 - b_3 = 0$$

Taking the four equations on the left together as a system, we find $b_1 = 0$ and $b_2 = 0$, and then $a_1 = 0$ and $a_2 = 0$. The equation $2b_3 + 4 = 0$ gives $b_3 = -2$, and then $2a_3 - b_3 = 0$ gives $a_3 = -1$. Putting these results together gives $\overrightarrow{a} = (a_1, a_2, a_3) = (0,0,-1)$ and $\overrightarrow{b} = (b_1, b_2, b_3) = (0,0,-2)$.

Therefore, after solving the first of three parts of our particular solution, we have

$$\overrightarrow{x_p} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{6t} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Plugging the exponential part of the particular solution into the matrix equation representing the system of differential equations gives

$$\begin{bmatrix} 6c_1 \\ 6c_2 \\ 6c_3 \end{bmatrix} e^{6t} = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^{6t} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{6t}$$

Breaking this equation into a system of three equations gives

$$6c_1 = 0c_1 - 2c_2 + 0c_3 + 1$$

$$6c_2 = -2c_1 + 0c_2 + 0c_3 + 0$$

$$6c_3 = 0c_1 + 0c_2 + 2c_3 + 0$$

which simplifies to

$$6c_1 = -2c_2 + 1$$

$$6c_2 = -2c_1$$

$$6c_3 = 2c_3$$

We see from the third equation that $4c_3 = 0$ so $c_3 = 0$. And solving the first two equations as a system gives $c_1 = 3/16$ and $c_2 = -1/16$.

Therefore, after solving the first two of three parts of our particular solution, we have

$$\vec{x_p} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} \frac{3}{16} \\ -\frac{1}{16} \\ 0 \end{bmatrix} e^{6t} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Plugging the trigonometric part of the particular solution into the matrix equation representing the system of differential equations gives

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \cos t - \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \sin t = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \left[\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t \right] + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \sin t$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \cos t - \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \sin t = \begin{bmatrix} 0 & -2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} e_1 \sin t + d_1 \cos t \\ e_2 \sin t + d_2 \cos t \\ e_3 \sin t + d_3 \cos t \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \sin t$$

Breaking this equation into a system of three equations gives

$$e_1 \cos t - d_1 \sin t = 0(e_1 \sin t + d_1 \cos t)$$

$$-2(e_2 \sin t + d_2 \cos t) + 0(e_3 \sin t + d_3 \cos t) + 0 \sin t$$

$$e_2 \cos t - d_2 \sin t = -2(e_1 \sin t + d_1 \cos t)$$

$$+0(e_2 \sin t + d_2 \cos t) + 0(e_3 \sin t + d_3 \cos t) + 2 \sin t$$

$$e_3 \cos t - d_3 \sin t = 0(e_1 \sin t + d_1 \cos t)$$

$$+0(e_2 \sin t + d_2 \cos t) + 2(e_3 \sin t + d_3 \cos t) + 0 \sin t$$

which simplifies to

$$e_1 \cos t - d_1 \sin t = -2e_2 \sin t - 2d_2 \cos t$$

$$e_2 \cos t - d_2 \sin t = -2e_1 \sin t - 2d_1 \cos t + 2 \sin t$$

$$e_3 \cos t - d_3 \sin t = 2e_3 \sin t + 2d_3 \cos t$$

These equations can each be broken into its own system.

$$e_1 = -2d_2 \qquad e_2 = -2d_1 \qquad e_3 = 2d_3$$

$$-d_1 = -2e_2 \qquad -d_2 = -2e_1 + 2 \qquad -d_3 = 2e_3$$

Solving these systems gives $(d_1, d_2, d_3) = (0, -2/5, 0)$ and $(e_1, e_2, e_3) = (4/5, 0, 0)$. Therefore, after solving all three parts of our particular solution, we have

$$\vec{x_p} = \begin{bmatrix} \frac{4}{5} \\ 0 \\ 0 \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ -\frac{2}{5} \\ 0 \end{bmatrix} \cos t + \begin{bmatrix} \frac{3}{16} \\ -\frac{1}{16} \\ 0 \end{bmatrix} e^{6t} + \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} t + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

Now we can rewrite the particular solution as one vector.

$$\vec{x_p} = \begin{bmatrix} \frac{4}{5} \sin t + 0 \cos t + \frac{3}{16} e^{6t} + 0t + 0 \\ 0 \sin t - \frac{2}{5} \cos t - \frac{1}{16} e^{6t} + 0t + 0 \\ 0 \sin t + 0 \cos t + 0e^{6t} - 2t - 1 \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} \frac{4}{5}\sin t + \frac{3}{16}e^{6t} \\ -\frac{2}{5}\cos t - \frac{1}{16}e^{6t} \\ -2t - 1 \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} \frac{4}{5}\sin t + \frac{3}{16}e^{6t} \\ -\frac{2}{5}\cos t - \frac{1}{16}e^{6t} \\ -2t - 1 \end{bmatrix}$$

Variation of parameters for nonhomogeneous systems

If undetermined coefficients isn't a viable method for solving a nonhomogeneous system of differential equations, we can always use the method of variation of parameters instead.

Just like with undetermined coefficients, we have to start by finding the corresponding complementary solution, which is the general solution of the associated homogeneous equation.

Then we'll use the complementary solution and the method of variation of parameters to find the particular solution, which will be given by

$$\vec{x}_p = \Phi(t) \int \Phi^{-1}(t) F(t) \ dt$$

The matrix $\Phi(t)$ is made of the solution vectors from the complementary solution, and $\Phi^{-1}(t)$ is its inverse. The vector $F(t)$ is F from the nonhomogeneous equation $\vec{x}' = A\vec{x} + F$.

When we integrate the resulting matrix $\Phi^{-1}(t)F(t)$, we just integrate each entry in the matrix. And as always, the general solution of the nonhomogeneous system will be the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

Let's do an example with this variation of parameters method so that we can see these steps in action.

Example

Use the method of variation of parameters to find the general solution of the system.

$$x_1'(t) = 3x_1 - 5x_2 + e^{\frac{t}{2}}$$

$$x_2'(t) = \frac{3}{4}x_1 - x_2 - e^{\frac{t}{2}}$$

Start by writing the nonhomogeneous system in matrix form.

$$\vec{x}' = \begin{bmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{bmatrix} \vec{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{\frac{t}{2}}$$

We have to start by finding the complementary solution, which is the general solution to the associated homogeneous equation, which means we need to solve

$$\vec{x}' = A\vec{x}$$

$$\vec{x}' = \begin{bmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{bmatrix} \vec{x}$$

The coefficient matrix is

$$A = \begin{bmatrix} 3 & -5 \\ \frac{3}{4} & -1 \end{bmatrix}$$

and the matrix $A-\lambda I$ is

$$A-\lambda I=\begin{bmatrix}3 & -5\\ \frac{3}{4} & -1\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}3 & -5\\ \frac{3}{4} & -1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}$$

$$A-\lambda I=\begin{bmatrix}3-\lambda & -5\\ \frac{3}{4} & -1-\lambda\end{bmatrix}$$

Its determinant is

$$\begin{vmatrix}3-\lambda & -5\\ \frac{3}{4} & -1-\lambda\end{vmatrix}$$

$$(3-\lambda)(-1-\lambda)-(-5)\left(\frac{3}{4}\right)$$

$$-3-2\lambda+\lambda^2+\frac{15}{4}$$

$$\lambda^2-2\lambda+\frac{3}{4}$$

So the characteristic equation is

$$\lambda^2-2\lambda+\frac{3}{4}=0$$

$$\left(\lambda-\frac{3}{2}\right)\left(\lambda-\frac{1}{2}\right)=0$$

Then for these Eigenvalues, $\lambda_1 = 3/2$ and $\lambda_2 = 1/2$, we find

$$A - \frac{3}{2}I = \begin{bmatrix} 3 - \frac{3}{2} & -5 \\ \frac{3}{4} & -1 - \frac{3}{2} \end{bmatrix}$$

$$A - \frac{3}{2}I = \begin{bmatrix} \frac{3}{2} & -5 \\ \frac{3}{4} & -\frac{5}{2} \end{bmatrix}$$

and

$$A - \frac{1}{2}I = \begin{bmatrix} 3 - \frac{1}{2} & -5 \\ \frac{3}{4} & -1 - \frac{1}{2} \end{bmatrix}$$

$$A - \frac{1}{2}I = \begin{bmatrix} \frac{5}{2} & -5 \\ \frac{3}{4} & -\frac{3}{2} \end{bmatrix}$$

Put these two matrices into reduced row-echelon form.

$$\begin{bmatrix} \frac{3}{2} & -5 \\ \frac{3}{4} & -\frac{5}{2} \end{bmatrix} \qquad \begin{bmatrix} \frac{5}{2} & -5 \\ \frac{3}{4} & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{10}{3} \\ \frac{3}{4} & -\frac{5}{2} \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ \frac{3}{4} & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{10}{3} \\ 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1 - \frac{10}{3}k_2 = 0$$

$$k_1 = \frac{10}{3}k_2$$

$$k_1 - 2k_2 = 0$$

$$k_1 = 2k_2$$

For the first system, we choose $(k_1, k_2) = (10,3)$. And from the second system, we choose $(k_1, k_2) = (2,1)$.

$$\vec{k_1} = \begin{bmatrix} 10 \\ 3 \end{bmatrix}$$

$$\vec{k_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Then the solutions to the system are

$$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{\frac{3}{2}t}$$

$$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$$

$$\vec{x_2} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\frac{1}{2}t}$$

Therefore, the complementary solution to the nonhomogeneous system, which is also the general solution to the associated homogeneous system, will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{\frac{3}{2}t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\frac{1}{2}t}$$

Now we can turn to finding the particular solution. We'll use the solution vectors $\vec{x_1}$ and $\vec{x_2}$ to form $\Phi(t)$,

$$\Phi(t)=\begin{bmatrix}10e^{\frac{3}{2}t} & 2e^{\frac{1}{2}t}\\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t}\end{bmatrix}$$

then we'll find its inverse $\Phi^{-1}(t)$ by changing $[\Phi(t)\,|\,I]$ into $[I\,|\,\Phi^{-1}(t)]$.

$$\begin{bmatrix}10e^{\frac{3}{2}t} & 2e^{\frac{1}{2}t} & | & 1 & 0\\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t} & | & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{1}{5}e^{-t} & | & \frac{1}{10}e^{-\frac{3}{2}t} & 0\\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t} & | & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{1}{5}e^{-t} & | & \frac{1}{10}e^{-\frac{3}{2}t} & 0\\ 0 & \frac{2}{5}e^{\frac{1}{2}t} & | & -\frac{3}{10} & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{1}{5}e^{-t} & | & \frac{1}{10}e^{-\frac{3}{2}t} & 0\\ 0 & 1 & | & -\frac{3}{4}e^{-\frac{1}{2}t} & \frac{5}{2}e^{-\frac{1}{2}t}\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & | & \frac{1}{4}e^{-\frac{3}{2}t} & -\frac{1}{2}e^{-\frac{3}{2}t}\\ 0 & 1 & | & -\frac{3}{4}e^{-\frac{1}{2}t} & \frac{5}{2}e^{-\frac{1}{2}t}\end{bmatrix}$$

Now with

$$\Phi(t)=\begin{bmatrix}10e^{\frac{3}{2}t} & 2e^{\frac{1}{2}t}\\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t}\end{bmatrix}\qquad \Phi^{-1}(t)=\begin{bmatrix}\frac{1}{4}e^{-\frac{3}{2}t} & -\frac{1}{2}e^{-\frac{3}{2}t}\\ -\frac{3}{4}e^{-\frac{1}{2}t} & \frac{5}{2}e^{-\frac{1}{2}t}\end{bmatrix}\qquad F(t)=\begin{bmatrix}e^{\frac{t}{2}}\\ -e^{\frac{t}{2}}\end{bmatrix}$$

the particular solution from the method of variation of parameters is given by

$$\overrightarrow{x_p} = \Phi(t)\int \Phi^{-1}(t)F(t)\ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} 10e^{\frac{3}{2}t} & 2e^{\frac{1}{2}t} \\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t} \end{bmatrix} \int \begin{bmatrix} \frac{1}{4}e^{-\frac{3}{2}t} & -\frac{1}{2}e^{-\frac{3}{2}t} \\ -\frac{3}{4}e^{-\frac{1}{2}t} & \frac{5}{2}e^{-\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} e^{\frac{t}{2}} \\ -e^{\frac{t}{2}} \end{bmatrix}\ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} 10e^{\frac{3}{2}t} & 2e^{\frac{1}{2}t} \\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t} \end{bmatrix} \int \begin{bmatrix} \frac{1}{4}e^{-\frac{3}{2}t}(e^{\frac{t}{2}}) - \frac{1}{2}e^{-\frac{3}{2}t}(-e^{\frac{t}{2}}) \\ -\frac{3}{4}e^{-\frac{1}{2}t}(e^{\frac{t}{2}}) + \frac{5}{2}e^{-\frac{1}{2}t}(-e^{\frac{t}{2}}) \end{bmatrix}\ dt$$

$$\overrightarrow{x_p} = \begin{bmatrix} 10e^{\frac{3}{2}t} & 2e^{\frac{1}{2}t} \\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t} \end{bmatrix} \int \begin{bmatrix} \frac{3}{4}e^{-t} \\ -\frac{13}{4} \end{bmatrix}\ dt$$

Integrate, then simplify the result.

$$\overrightarrow{x_p} = \begin{bmatrix} 10e^{\frac{3}{2}t} & 2e^{\frac{1}{2}t} \\ 3e^{\frac{3}{2}t} & e^{\frac{1}{2}t} \end{bmatrix} \begin{bmatrix} -\frac{3}{4}e^{-t} \\ -\frac{13}{4}t \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} 10e^{\frac{3}{2}t}(-\frac{3}{4}e^{-t}) + 2e^{\frac{1}{2}t}(-\frac{13}{4}t) \\ 3e^{\frac{3}{2}t}(-\frac{3}{4}e^{-t}) + e^{\frac{1}{2}t}(-\frac{13}{4}t) \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} -\frac{15}{2}e^{\frac{1}{2}t} - \frac{13}{2}te^{\frac{1}{2}t} \\ -\frac{9}{4}e^{\frac{1}{2}t} - \frac{13}{4}te^{\frac{1}{2}t} \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\vec{x} = \vec{x_c} + \vec{x_p}$$

$$\vec{x} = c_1 \begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{\frac{3}{2}t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\frac{1}{2}t} + \begin{bmatrix} -\frac{15}{2}e^{\frac{1}{2}t} - \frac{13}{2}te^{\frac{1}{2}t} \\ -\frac{9}{4}e^{\frac{1}{2}t} - \frac{13}{4}te^{\frac{1}{2}t} \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{\frac{3}{2}t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\frac{1}{2}t} + \begin{bmatrix} -\frac{13}{2} \\ -\frac{13}{4} \end{bmatrix} te^{\frac{1}{2}t} + \begin{bmatrix} -\frac{15}{2} \\ -\frac{9}{4} \end{bmatrix} e^{\frac{1}{2}t}$$

$$\vec{x} = c_1 \begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{\frac{3}{2}t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{\frac{1}{2}t} - \frac{13}{4} \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{\frac{1}{2}t} - \frac{3}{4} \begin{bmatrix} 10 \\ 3 \end{bmatrix} e^{\frac{1}{2}t}$$

The matrix exponential

We've seen how to use the method of undetermined coefficients and the method of variation of parameters to compute the general solution to a nonhomogeneous system of differential equations.

We can also use the **matrix exponential**, e^{At}, where A is an $n \times n$ matrix of constants, as part of the following formula for the solution to a nonhomogeneous system.

$$\vec{x} = e^{At}C + e^{At}\int_{t_0}^{t} e^{-As}F(s)\ ds$$

In other words, given a system of linear first order differential equations $\vec{x}' = A\vec{x} + F$, the general solution to the system is given by the integral formula above. As always, the general solution is the sum of the complementary and particular solutions,

$$\vec{x_c} = e^{At}C$$

$$\vec{x_p} = e^{At}\int_{t_0}^{t} e^{-As}F(s)\ ds$$

The column matrix C is made of the arbitrary constants c_1, c_2, ... c_n. And e^{-As} is found by substituting $t = s$ into e^{At}, and then finding the inverse of that resulting matrix. Which means all we need to learn now is how to compute the matrix exponential e^{At}.

The matrix exponential

If we've previously studied sequences and series, including power series, as part of the calculus series, we may remember the power series expansion of e^{at}.

$$e^{at} = 1 + at + \frac{(at)^2}{2!} + \frac{(at)^3}{3!} + \dots + \frac{(at)^k}{k!}$$

$$e^{at} = 1 + at + a^2\frac{t^2}{2!} + a^3\frac{t^3}{3!} + \dots + a^k\frac{t^k}{k!} = \sum_{k=0}^{\infty} a^k\frac{t^k}{k!}$$

We can actually rewrite this power series expansion in matrix form, replacing 1 with the matrix equivalent I, and replacing the constant a with the matrix A.

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \dots + A^k\frac{t^k}{k!} = \sum_{k=0}^{\infty} A^k\frac{t^k}{k!}$$

This power series is one way to compute the matrix exponential. Let's do an example.

Example

Use the power series formula to compute the matrix exponential.

$$A = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

Let's find the first few powers of A, starting with A^2,

$$A^2 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3^2 + 0^2 & 3(0) + 0(1) \\ 0(3) + 1(0) & 0^2 + 1^2 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 3^2 & 0 \\ 0 & 1^2 \end{bmatrix}$$

and then A^3.

$$A^3 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3^2 & 0 \\ 0 & 1^2 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3^3 & 0 \\ 0 & 1^3 \end{bmatrix}$$

We can see that higher powers of A will follow this emerging pattern, so we can write

$$e^{At} = I + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \dots$$

$$e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} t + \begin{bmatrix} 3^2 & 0 \\ 0 & 1^2 \end{bmatrix} \frac{t^2}{2!} + \begin{bmatrix} 3^3 & 0 \\ 0 & 1^3 \end{bmatrix} \frac{t^3}{3!} + \dots$$

$$e^{At} = \begin{bmatrix} 1 + 3t + 3^2\frac{t^2}{2!} + 3^3\frac{t^3}{3!} + \dots & 0 \\ 0 & 1 + 1t + 1^2\frac{t^2}{2!} + 1^3\frac{t^3}{3!} + \dots \end{bmatrix}$$

We notice that the entry in the upper left of the matrix is just the expansion of e^{3t}, and that the entry in the lower right of the matrix is just the expansion of e^{t}.

$$e^{At} = \begin{bmatrix} e^{3t} & 0 \\ 0 & e^{t} \end{bmatrix}$$

Alternatively, we can also use a Laplace transform to compute the matrix exponential. The formula we need is

$$e^{At} = \mathscr{L}^{-1}((sI - A)^{-1})$$

In other words, to find the matrix exponential using Laplace transforms, we need to

1. Find the matrix $sI - A$

2. Find its inverse $(sI - A)^{-1}$

3. Apply an inverse transform to the inverse matrix

These three steps lead us to the matrix exponential for A, e^{At}. Let's do an example with this method.

Example

Use an inverse Laplace transform to calculate the matrix exponential.

$$A = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}$$

First, we'll find $sI - A$.

$$sI - A = s\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}2 & -2\\4 & -4\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}2 & -2\\4 & -4\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s-2 & 0-(-2)\\0-4 & s-(-4)\end{bmatrix}$$

$$sI - A = \begin{bmatrix}s-2 & 2\\-4 & s+4\end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \mid I]$ into $[I \mid (sI - A)^{-1}]$.

$$\begin{bmatrix}s-2 & 2 & | & 1 & 0\\-4 & s+4 & | & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{2}{s-2} & | & \frac{1}{s-2} & 0\\-4 & s+4 & | & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{2}{s-2} & | & \frac{1}{s-2} & 0\\0 & \frac{s(s+2)}{s-2} & | & \frac{4}{s-2} & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{2}{s-2} & | & \frac{1}{s-2} & 0\\0 & 1 & | & \frac{4}{s(s+2)} & \frac{s-2}{s(s+2)}\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{s+4}{s(s+2)} & -\frac{2}{s(s+2)} \\ 0 & 1 & | & \frac{4}{s(s+2)} & \frac{s-2}{s(s+2)} \end{bmatrix}$$

Before we can apply an inverse transform to the inverse matrix, we need to rewrite the entries using partial fractions decompositions, and then rewrite the decompositions to prepare them for the inverse Laplace transform.

$$\begin{bmatrix} 1 & 0 & | & \frac{2}{s} - \frac{1}{s+2} & -\frac{1}{s} + \frac{1}{s+2} \\ 0 & 1 & | & \frac{2}{s} - \frac{2}{s+2} & -\frac{1}{s} + \frac{2}{s+2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 2\left(\frac{1}{s}\right) - \frac{1}{s-(-2)} & -\left(\frac{1}{s}\right) + \frac{1}{s-(-2)} \\ 0 & 1 & | & 2\left(\frac{1}{s}\right) - 2\left(\frac{1}{s-(-2)}\right) & -\left(\frac{1}{s}\right) + 2\left(\frac{1}{s-(-2)}\right) \end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix} 2\left(\frac{1}{s}\right) - \frac{1}{s-(-2)} & -\left(\frac{1}{s}\right) + \frac{1}{s-(-2)} \\ 2\left(\frac{1}{s}\right) - 2\left(\frac{1}{s-(-2)}\right) & -\left(\frac{1}{s}\right) + 2\left(\frac{1}{s-(-2)}\right) \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} 2 - e^{-2t} & -1 + e^{-2t} \\ 2 - 2e^{-2t} & -1 + 2e^{-2t} \end{bmatrix}$$

and then say that this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} 2 - e^{-2t} & -1 + e^{-2t} \\ 2 - 2e^{-2t} & -1 + 2e^{-2t} \end{bmatrix}$$

Solving the nonhomogeneous system

Regardless of how we calculate the matrix exponential e^{At}, once we have it, we have almost everything we need to find the general solution to the nonhomogeneous system.

Let's do an example where we work all the way through to the general solution.

Example

Find the general solution of the system.

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

First, we'll find $sI - A$.

$$sI - A = s \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s-1 & 0-0 \\ 0-0 & s-2 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s-1 & 0 \\ 0 & s-2 \end{bmatrix}$$

Then we'll find the inverse of this matrix, by changing $[sI - A \mid I]$ into $[I \mid (sI - A)^{-1}]$.

$$\begin{bmatrix} s-1 & 0 & | & 1 & 0 \\ 0 & s-2 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{1}{s-1} & 0 \\ 0 & s-2 & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{1}{s-1} & 0 \\ 0 & 1 & | & 0 & \frac{1}{s-2} \end{bmatrix}$$

Now we can apply the inverse Laplace transform to this inverse matrix,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{1}{s-1} & 0 \\ 0 & \frac{1}{s-2} \end{bmatrix}$$

$$\mathscr{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

and then say that this result is the matrix exponential.

$$e^{At} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix}$$

Now that we have the matrix exponential, we can say that the complementary solution will be

$$\vec{x_c} = e^{At}C$$

$$\vec{x_c} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\vec{x_c} = \begin{bmatrix} c_1e^t \\ c_2e^{2t} \end{bmatrix}$$

To find the particular solution, we'll need e^{-As}, which we find by making the substitution $t = s$ into e^{At},

$$e^{As} = \begin{bmatrix} e^s & 0 \\ 0 & e^{2s} \end{bmatrix}$$

and then calculating the inverse of this resulting e^{As}.

$$\begin{bmatrix} e^s & 0 & | & 1 & 0 \\ 0 & e^{2s} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & e^{-s} & 0 \\ 0 & e^{2s} & | & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & e^{-s} & 0 \\ 0 & 1 & | & 0 & e^{-2s} \end{bmatrix}$$

So e^{-As} is given by this resulting matrix.

$$e^{-As} = \begin{bmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{bmatrix}$$

We'll find $F(s)$ by substituting $t = s$ into F, which actually requires no substitution, since there are no t variables in F.

$$F(s) = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

Therefore, the particular solution will be

$$\vec{x_p} = e^{At} \int_{t_0}^{t} e^{-As} F(s) \ ds$$

$$\vec{x_p} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \int_0^t \begin{bmatrix} e^{-s} & 0 \\ 0 & e^{-2s} \end{bmatrix} \begin{bmatrix} 3 \\ -1 \end{bmatrix} \ ds$$

$$\vec{x_p} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \int_0^t \begin{bmatrix} 3e^{-s} \\ -e^{-2s} \end{bmatrix} \ ds$$

Now we'll integrate and then evaluate on $[0,t]$.

$$\vec{x_p} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \left(\begin{bmatrix} -3e^{-t} \\ \frac{1}{2}e^{-2t} \end{bmatrix} - \begin{bmatrix} -3e^{-0} \\ \frac{1}{2}e^{-2(0)} \end{bmatrix} \right)$$

$$\vec{x_p} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \left(\begin{bmatrix} -3e^{-t} \\ \frac{1}{2}e^{-2t} \end{bmatrix} - \begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix} \right)$$

$$\vec{x_p} = \begin{bmatrix} e^t & 0 \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} -3e^{-t} + 3 \\ \frac{1}{2}e^{-2t} - \frac{1}{2} \end{bmatrix}$$

Finally, we can do the last matrix multiplication to get $\vec{x_p}$.

$$\vec{x_p} = \begin{bmatrix} e^t(-3e^{-t} + 3) \\ e^{2t}\left(\frac{1}{2}e^{-2t} - \frac{1}{2}\right) \end{bmatrix}$$

$$\vec{x}_p = \begin{bmatrix} -3 + 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{bmatrix}$$

Then the general solution is the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

$$\vec{x} = \begin{bmatrix} c_1e^t \\ c_2e^{2t} \end{bmatrix} + \begin{bmatrix} -3 + 3e^t \\ \frac{1}{2} - \frac{1}{2}e^{2t} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} c_1e^t + 3e^t - 3 \\ c_2e^{2t} - \frac{1}{2}e^{2t} + \frac{1}{2} \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} c_1 + 3 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ c_2 - \frac{1}{2} \end{bmatrix} e^{2t} + \begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix}$$

Because c_1 and c_2 are constants, $c_1 + 3$ and $c_2 - 1/2$ are also constants. Therefore, we can simplify the general solution to just

$$\vec{x} = \begin{bmatrix} c_1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ c_2 \end{bmatrix} e^{2t} + \begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{2t} + \begin{bmatrix} -3 \\ \frac{1}{2} \end{bmatrix}$$

Homogeneous higher order equations

We've already covered solutions of first order and second order differential equations, and in this section of the course we'll be extending what we've learned to higher order equations, like third, fourth, and nth order linear equations.

Solutions to higher order equations

For everything we do with these higher order equations, we'll want to start with them in standard form.

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t)$$

The equation is homogeneous when $g(t) = 0$, and nonhomogeneous when $g(t) \neq 0$.

Homogeneous $\qquad y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = 0$

Nonhomogeneous $\qquad y^{(n)} + p_{n-1}(t)y^{(n-1)} + \ldots + p_1(t)y' + p_0(t)y = g(t)$

Notice here that the coefficient on the highest degree derivative $y^{(n)}$ is 1. If we have a higher order equation where that coefficient isn't 1, then we'll want to divide through the equation by that coefficient, to put it in standard form.

Just as with lesser order differential equations, we are guaranteed a solution of the higher order linear differential equation as long as the coefficients $p_i(t)$ and the forcing function $g(t)$ are all continuous on the open interval I that contains t_0.

If all the coefficients $p_i(t)$ are continuous on I, if $y_1(t)$, $y_2(t)$, ... $y_n(t)$ are all solutions of the linear differential equation, and if the Wronskian of the solutions $W(y_1, y_2, \dots y_n)(t) \neq 0$, then the solutions are linearly independent and form a fundamental set of solutions, and the general solution to the homogeneous equation is

$$y(t) = c_1y_1(t) + c_2y_2(t) + \dots + c_ny_n(t)$$

And the solution to the nonhomogeneous equation will be the sum of the complementary solution $y_c(t)$ (the solution to the associated homogeneous equation) and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = [c_1y_1(t) + c_2y_2(t) + \dots + c_ny_n(t)] + y_p(t)$$

Solving higher order homogeneous equations with constant coefficients

Just like before, we'll use a different formula for the general solution of a higher order homogeneous equation depending on the type of roots we find from the associated characteristic equation.

When we have distinct real roots or equal real roots, the solution will be given by these formulas:

Distinct real roots	$y(t) = c_1e^{r_1t} + c_2e^{r_2t} + \ldots + c_ke^{r_kt}$
Equal real roots	$y(t) = c_1e^{r_1t} + c_2te^{r_1t} + c_3t^2e^{r_1t} \ldots + c_kt^{k-1}e^{r_1t}$

But when we have complex conjugate roots from the characteristic equation, the solution depends on the multiplicity of the complex conjugate root pair. As we already know, if the root pair $r = \alpha \pm \beta i$ occurs only once, then $e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$ form a fundamental set of solutions, and the general solution is therefore

$$y(t) = e^{\alpha t}(c_1\cos(\beta t) + c_2\sin(\beta t))$$

But when the root pair $r = \alpha \pm \beta i$ has multiplicity k, the fundamental set of solutions will be

$e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$

$te^{\alpha t}\cos(\beta t)$ and $te^{\alpha t}\sin(\beta t)$

$t^2e^{\alpha t}\cos(\beta t)$ and $t^2e^{\alpha t}\sin(\beta t)$

...

$t^{k-1}e^{\alpha t}\cos(\beta t)$ and $t^{k-1}e^{\alpha t}\sin(\beta t)$

Complex conjugate roots always appear in pairs that include both the positive and negative parts of the root pair.

With these forms in mind, let's do an example of a higher order homogeneous differential equation with distinct real roots.

Example

Find the general solution to the third order homogeneous differential equation.

$$y''' + 2y'' - 13y' + 10y = 0$$

We'll rewrite the differential equation as its associated characteristic equation, and then a calculator to factor the polynomial.

$$r^3 + 2r^2 - 13r + 10 = 0$$

$$(r - 1)(r - 2)(r + 5) = 0$$

From the factored form of the polynomial, we can see that the equation has distinct real roots $r_1 = 1$, $r_2 = 2$, and $r_3 = -5$. And the general solution is therefore

$$y(t) = c_1e^t + c_2e^{2t} + c_3e^{-5t}$$

Keep in mind that we can also solve initial value problems with these higher order differential equations.

We need the same number of initial conditions as the order of the equation. So for a third order equation we'll need three initial conditions, for a fourth order equation we'll need four initial conditions, etc.

Once we have the general solution, we take its derivatives, plug the initial conditions into these, and by doing so we'll generate a system of equations that we can solve for c_1, c_2, c_3, etc. And of course we can solve initial value problems this way not only for higher order homogeneous equations with distinct real roots, but also for those with equal real and complex conjugate roots.

Let's do another example, this time with a combination of distinct real roots and equal real roots.

Example

Find the general solution to the fifth order homogeneous differential equation.

$$y^{(5)} + 4y^{(4)} - 6y''' - 32y'' - 35y' - 12y = 0$$

We'll rewrite the differential equation as its associated characteristic equation, and then a calculator to factor the polynomial.

$$r^5 + 4r^4 - 6r^3 - 32r^2 - 35r - 12 = 0$$

$$(r-3)(r+4)(r+1)(r+1)(r+1) = 0$$

From the factored form of the polynomial, we can see that the equation has distinct real roots $r_1 = 3$ and $r_2 = -4$, and equal real roots $r_3 = r_4 = r_5 = -1$. The distinct real roots portion of the solution will be $c_1e^{3t} + c_2e^{-4t}$, while the equal real roots portion will be $c_3e^{-t} + c_4te^{-t} + c_5t^2e^{-t}$, and therefore the general solution of the fifth order linear equation is

$$y(t) = c_1e^{3t} + c_2e^{-4t} + c_3e^{-t} + c_4te^{-t} + c_5t^2e^{-t}$$

Let's do one more example, this time with multiple pairs of complex roots, so that we can see what that looks like.

Example

Find the general solution to the seventh order homogeneous differential equation.

$$y^{(7)} - 2y^{(6)} + 7y^{(5)} - 26y^{(4)} + 29y''' - 60y'' + 100y' = 0$$

We'll rewrite the differential equation as its associated characteristic equation, and then a calculator to factor the polynomial.

$$r^7 - 2r^6 + 7r^5 - 26r^4 + 29r^3 - 60r^2 + 100r = 0$$

$$r(r-2)(r-2)(r^2 + r + 5)^2 = 0$$

From the factored form of the polynomial, we can see that the equation has one distinct real root $r_1 = 0$, two equal real roots $r_2 = r_3 = 2$, and a pair of complex conjugate roots,

$$r_4 = -\frac{1}{2} + \frac{\sqrt{19}}{2}i \text{ and } r_4 = -\frac{1}{2} - \frac{\sqrt{19}}{2}i$$

But because that complex root portion of the factored characteristic polynomial is squared, $(r^2 + r + 5)^2$, these complex roots have multiplicity two, which means we'll have two pairs of complex roots,

$$e^{\alpha t}\cos(\beta t) \text{ and } e^{\alpha t}\sin(\beta t)$$

$te^{\alpha t}\cos(\beta t)$ and $te^{\alpha t}\sin(\beta t)$

With $\alpha = -1/2$ and $\beta = \sqrt{19}/2$, the complex conjugate solution pairs will be

$$e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) \text{ and } e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

$$te^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) \text{ and } te^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

The distinct real roots portion of the solution will be $c_1e^{0t} = c_1$, the equal real roots portion will be $c_2e^{2t} + c_3te^{2t}$, the complex conjugate roots portion will be,

$$c_4e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) + c_5e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

$$+c_6te^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) + c_7te^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

and therefore the general solution of the seventh order linear equation is

$$y(t) = c_1 + c_2e^{2t} + c_3te^{2t} + c_4e^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) + c_5e^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

$$+c_6te^{-\frac{1}{2}t}\cos\left(\frac{\sqrt{19}}{2}t\right) + c_7te^{-\frac{1}{2}t}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

Undetermined coefficients for higher order equations

As we mentioned previously, the solution to a higher order nonhomogeneous equation ($g(t) \neq 0$),

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = g(t)$$

just like for second order equations, will be the sum of the complementary solution $y_c(t)$ (the solution to the associated homogeneous equation) and the particular solution $y_p(t)$.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = [c_1y_1(t) + c_2y_2(t) + \dots + c_ny_n(t)] + y_p(t)$$

So our process for solving a nonhomogeneous equation will be the same process we used for second order equations.

1. Find the complementary solution $y_c(t)$ by solving the associated homogeneous equation.

2. Make a guess for the particular solution $y_p(t)$, eliminating any overlap between the guess and the complementary solution.

3. Take derivatives of the particular solution, then plug the guess and its derivatives into the original differential equation.

4. Solve for the constants A, B, C, etc., then plug their values back into the guess for the particular solution.

5. Take the sum of the complementary and particular solutions to get the general solution $y(t) = y_c(t) + y_p(t)$ to the nonhomogeneous equation.

And just like before, this method of undetermined coefficients works well when $g(t)$ is some combination of exponential, polynomial, and sine and cosine functions.

Let's do an example so that we can see these steps in action.

Example

Find the general solution to the nonhomogeneous differential equation.

$$y''' - 6y'' + 12y' - 8y = 3e^{2t} + 4e^{3t}$$

We need to by solving the associated homogeneous equation, so we'll set $g(t) = 0$, then factor the characteristic equation associated with the homogeneous equation.

$$r^3 - 6r^2 + 12r - 8 = 0$$

$$(r - 2)(r - 2)(r - 2) = 0$$

We find the repeated root $r = 2$ with multiplicity three, so the complementary solution for the nonhomogeneous equation (or the general solution for the homogeneous equation), is given by

$$y_c(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t}$$

Our initial guess for the particular solution should be

$$y_p(t) = Ae^{2t} + Be^{3t}$$

The first term in our guess Ae^{2t} overlaps with the first term from the complementary solution c_1e^{2t}. To eliminate the overlap, we should multiply by t to get Ate^{2t}. However, this new term still overlaps with c_2te^{2t} from the complementary solution, so we'll need to multiply by t again to get At^2e^{2t}, but then by t again to eliminate the overlap with $c_3t^2e^{2t}$ and get At^3e^{2t}.

$$y_p(t) = At^3e^{2t} + Be^{3t}$$

We'll find derivatives of the particular solution,

$$y_p'(t) = 2At^3e^{2t} + 3At^2e^{2t} + 3Be^{3t}$$

$$y_p''(t) = 4At^3e^{2t} + 12At^2e^{2t} + 6Ate^{2t} + 9Be^{3t}$$

$$y_p'''(t) = 8At^3e^{2t} + 36At^2e^{2t} + 36Ate^{2t} + 6Ae^{2t} + 27Be^{3t}$$

and then plug them into the original differential equation.

$$8At^3e^{2t} + 36At^2e^{2t} + 36Ate^{2t} + 6Ae^{2t} + 27Be^{3t}$$

$$-6(4At^3e^{2t} + 12At^2e^{2t} + 6Ate^{2t} + 9Be^{3t})$$

$$+12(2At^3e^{2t} + 3At^2e^{2t} + 3Be^{3t}) - 8(At^3e^{2t} + Be^{3t}) = 3e^{2t} + 4e^{3t}$$

$$6Ae^{2t} + Be^{3t} = 3e^{2t} + 4e^{3t}$$

This gives the system of differential equations

$6A = 3$ so $A = 1/2$

$$B = 4$$

and therefore the particular solution is

$$y_p(t) = \frac{1}{2}t^3e^{2t} + 4e^{3t}$$

and the general solution is the sum of the complementary and particular solutions.

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + \frac{1}{2}t^3e^{2t} + 4e^{3t}$$

Variation of parameters for higher order equations

Just like with undetermined coefficients, we're trying to find the solution to the nth order nonhomogeneous linear equation

$$y^{(n)} + p_{n-1}(t)y^{(n-1)} + \dots + p_1(t)y' + p_0(t)y = g(t)$$

by first finding the complementary solution (the solution to the associated homogeneous equation), and then using the method of variation of parameters to find the particular solution.

Remember that when we used variation of parameters with second order equations, we would solve the system of linear equations

$$u_1'y_1 + u_2'y_2 = 0$$

$$u_1'y_1' + u_2'y_2' = g(t)$$

because there were only two solutions in the solution set $\{y_1, y_2\}$, and so there were only two unknowns, u_1' and u_2'.

But with higher order equations, of course the solution set will be larger, for example $\{y_1, y_2, y_3, y_4, \dots, y_n\}$. If we had a fourth order linear differential equation, for example, we'd have to solve this system:

$$u_1'y_1 + u_2'y_2 + u_3'y_3 + u_4'y_4 = 0$$

$$u_1'y_1' + u_2'y_2' + u_3'y_3' + u_4'y_4' = 0$$

$$u_1'y_1'' + u_2'y_2'' + u_3'y_3'' + u_4'y_4'' = 0$$

$$u_1'y_1''' + u_2'y_2''' + u_3'y_3''' + u_4'y_4''' = g(t)$$

because we need the same number of equations in the system as we have solutions in the solution set. The forcing function $g(t)$ from the original differential equation is always the right side of the last equation in the system; each of the other equations will have 0 as its right side.

We can certainly solve the system using simple substitution and elimination techniques, but the larger the size of the solution set, and therefore the larger the size of the system of linear equations, the more convenient it becomes to use Cramer's Rule to solve the system.

Cramer's Rule

We'll always start by changing the nonhomogeneous equation to a homogeneous equation so that we can find the complementary solution, and then pull the fundamental set of solutions from that complementary solution.

We'll then use that fundamental set of solutions to build a set of Wronskian determinants. To illustrate what these determinants look like, let's assume that we're working with a fundamental set of four solutions, $\{y_1, y_2, y_3, y_4\}$. Then Cramer's Rule tells us that we'll need to find

$$W = \begin{vmatrix} y_1 & y_2 & y_3 & y_4 \\ y_1' & y_2' & y_3' & y_4' \\ y_1'' & y_2'' & y_3'' & y_4'' \\ y_1''' & y_2''' & y_3''' & y_4''' \end{vmatrix} \quad W_1 = \begin{vmatrix} 0 & y_2 & y_3 & y_4 \\ 0 & y_2' & y_3' & y_4' \\ 0 & y_2'' & y_3'' & y_4'' \\ 1 & y_2''' & y_3''' & y_4''' \end{vmatrix} \quad W_2 = \begin{vmatrix} y_1 & 0 & y_3 & y_4 \\ y_1' & 0 & y_3' & y_4' \\ y_1'' & 0 & y_3'' & y_4'' \\ y_1''' & 1 & y_3''' & y_4''' \end{vmatrix}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 & y_4 \\ y_1' & y_2' & 0 & y_4' \\ y_1'' & y_2'' & 0 & y_4'' \\ y_1''' & y_2''' & 1 & y_4''' \end{vmatrix} \qquad W_4 = \begin{vmatrix} y_1 & y_2 & y_3 & 0 \\ y_1' & y_2' & y_3' & 0 \\ y_1'' & y_2'' & y_3'' & 0 \\ y_1''' & y_2''' & y_3''' & 1 \end{vmatrix}$$

Notice how W includes only the four solutions and their derivatives, while W_1, W_2, W_3, and W_4 each replace one column of W with a column full of zeros, ending with a 1 at the bottom of the column.

Then, instead of solving the system,

$$u_1'y_1 + u_2'y_2 + u_3'y_3 + u_4'y_4 = 0$$

$$u_1'y_1' + u_2'y_2' + u_3'y_3' + u_4'y_4' = 0$$

$$u_1'y_1'' + u_2'y_2'' + u_3'y_3'' + u_4'y_4'' = 0$$

$$u_1'y_1''' + u_2'y_2''' + u_3'y_3''' + u_4'y_4''' = g(t)$$

Cramer's Rule tells us that we can find the unknowns u_1', u_2', u_3', and u_4' by substituting the Wronskian determinants we found earlier into these equations:

$$u_1' = \frac{g(t)W_1}{W} \qquad u_2' = \frac{g(t)W_2}{W} \qquad u_3' = \frac{g(t)W_3}{W} \qquad u_4' = \frac{g(t)W_4}{W}$$

$$u_1 = \int \frac{g(t)W_1}{W}\ dt \quad u_2 = \int \frac{g(t)W_2}{W}\ dt \quad u_3 = \int \frac{g(t)W_3}{W}\ dt \quad u_4 = \int \frac{g(t)W_4}{W}\ dt$$

With these u_1, u_2, u_3, and u_4 values, the particular solution can be written as

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4$$

$$y_p(t) = y_1 \int \frac{g(t)W_1}{W}\ dt + y_2 \int \frac{g(t)W_2}{W}\ dt + y_3 \int \frac{g(t)W_3}{W}\ dt + y_4 \int \frac{g(t)W_4}{W}\ dt$$

Remember that we can modify the system of linear equations and these formulas for the complementary and particular solutions depending on how many solutions we have in the fundamental set of solutions.

Let's try an example where we use Cramer's Rule, instead of solving the system of equations, to find the particular solution. We'll start with an

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Example

Use variation of parameters to find the general solution to the differential equation.

$$y''' - 6y'' + 12y' - 8y = 3e^{2t} + 4e^{3t}$$

This is the same example we worked in the last lesson using the method of undetermined coefficients. So we already know that the characteristic equation we get from the homogeneous equation is

$$r^3 - 6r^2 + 12r - 8 = 0$$

$$(r-2)(r-2)(r-2) = 0$$

and therefore that the complementary solution is

$$y_c(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t}$$

From the complementary solution, we get the fundamental set of solutions $\{e^{2t}, te^{2t}, t^2e^{2t}\}$. With three solutions in the fundamental set, we'll build the following Wronskian determinants.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{2t} & te^{2t} & t^2e^{2t} \\ 2e^{2t} & e^{2t} + 2te^{2t} & 2te^{2t} + 2t^2e^{2t} \\ 4e^{2t} & 4e^{2t} + 4te^{2t} & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{vmatrix}$$

$$= e^{2t} \begin{vmatrix} e^{2t} + 2te^{2t} & 2te^{2t} + 2t^2e^{2t} \\ 4e^{2t} + 4te^{2t} & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{vmatrix}$$

$$-te^{2t} \begin{vmatrix} 2e^{2t} & 2te^{2t} + 2t^2e^{2t} \\ 4e^{2t} & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{vmatrix} + t^2e^{2t} \begin{vmatrix} 2e^{2t} & e^{2t} + 2te^{2t} \\ 4e^{2t} & 4e^{2t} + 4te^{2t} \end{vmatrix}$$

$$= e^{2t}[(e^{2t} + 2te^{2t})(2e^{2t} + 8te^{2t} + 4t^2e^{2t}) - (2te^{2t} + 2t^2e^{2t})(4e^{2t} + 4te^{2t})]$$

$$-te^{2t}[(2e^{2t})(2e^{2t} + 8te^{2t} + 4t^2e^{2t}) - (2te^{2t} + 2t^2e^{2t})(4e^{2t})]$$

$$+t^2e^{2t}[(2e^{2t})(4e^{2t} + 4te^{2t}) - (e^{2t} + 2te^{2t})(4e^{2t})]$$

$$= e^{2t}(2e^{4t} + 4te^{4t} + 4t^2e^{4t}) - te^{2t}(4e^{4t} + 8te^{4t}) + t^2e^{2t}(4e^{4t})$$

$$= 2e^{6t}$$

$$W_1 = \begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ 1 & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 0 & te^{2t} & t^2e^{2t} \\ 0 & e^{2t} + 2te^{2t} & 2te^{2t} + 2t^2e^{2t} \\ 1 & 4e^{2t} + 4te^{2t} & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{vmatrix}$$

$$= 0 \begin{vmatrix} e^{2t} + 2te^{2t} & 2te^{2t} + 2t^2e^{2t} \\ 4e^{2t} + 4te^{2t} & 2e^{2t} + 8te^{2t} + 4t^2e^{2t} \end{vmatrix}$$

$$-0\begin{vmatrix} te^{2t} & t^2e^{2t} \\ 4e^{2t}+4te^{2t} & 2e^{2t}+8te^{2t}+4t^2e^{2t} \end{vmatrix} + 1\begin{vmatrix} te^{2t} & t^2e^{2t} \\ e^{2t}+2te^{2t} & 2te^{2t}+2t^2e^{2t} \end{vmatrix}$$

$$= \begin{vmatrix} te^{2t} & t^2e^{2t} \\ e^{2t}+2te^{2t} & 2te^{2t}+2t^2e^{2t} \end{vmatrix}$$

$$= (te^{2t})(2te^{2t}+2t^2e^{2t}) - (t^2e^{2t})(e^{2t}+2te^{2t})$$

$$= t^2e^{4t}$$

$$W_2 = \begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & 1 & y_3'' \end{vmatrix} = \begin{vmatrix} e^{2t} & 0 & t^2e^{2t} \\ 2e^{2t} & 0 & 2te^{2t}+2t^2e^{2t} \\ 4e^{2t} & 1 & 2e^{2t}+8te^{2t}+4t^2e^{2t} \end{vmatrix}$$

$$= -0\begin{vmatrix} 2e^{2t} & 2te^{2t}+2t^2e^{2t} \\ 4e^{2t} & 2e^{2t}+8te^{2t}+4t^2e^{2t} \end{vmatrix}$$

$$+0\begin{vmatrix} e^{2t} & t^2e^{2t} \\ 4e^{2t} & 2e^{2t}+8te^{2t}+4t^2e^{2t} \end{vmatrix} - 1\begin{vmatrix} e^{2t} & t^2e^{2t} \\ 2e^{2t} & 2te^{2t}+2t^2e^{2t} \end{vmatrix}$$

$$= -\begin{vmatrix} e^{2t} & t^2e^{2t} \\ 2e^{2t} & 2te^{2t}+2t^2e^{2t} \end{vmatrix}$$

$$= -(e^{2t})(2te^{2t}+2t^2e^{2t}) + (t^2e^{2t})(2e^{2t})$$

$$= -2te^{4t}$$

$$W_3 = \begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & 1 \end{vmatrix} = \begin{vmatrix} e^{2t} & te^{2t} & 0 \\ 2e^{2t} & e^{2t}+2te^{2t} & 0 \\ 4e^{2t} & 4e^{2t}+4te^{2t} & 1 \end{vmatrix}$$

$$= 0\begin{vmatrix} 2e^{2t} & e^{2t}+2te^{2t} \\ 4e^{2t} & 4e^{2t}+4te^{2t} \end{vmatrix} - 0\begin{vmatrix} e^{2t} & te^{2t} \\ 4e^{2t} & 4e^{2t}+4te^{2t} \end{vmatrix} + 1\begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}+2te^{2t} \end{vmatrix}$$

$$= \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & e^{2t}+2te^{2t} \end{vmatrix}$$

$$= (e^{2t})(e^{2t}+2te^{2t}) - (te^{2t})(2e^{2t})$$

$$= e^{4t}$$

Then the unknowns are

$$u_1' = \frac{g(t)W_1}{W} = \frac{(3e^{2t}+4e^{3t})(t^2e^{4t})}{2e^{6t}} = \frac{3t^2e^{6t}+4t^2e^{7t}}{2e^{6t}} = \frac{3t^2+4t^2e^t}{2}$$

$$u_2' = \frac{g(t)W_2}{W} = \frac{(3e^{2t}+4e^{3t})(-2te^{4t})}{2e^{6t}} = -\frac{6te^{6t}+8te^{7t}}{2e^{6t}} = -3t-4te^t$$

$$u_3' = \frac{g(t)W_3}{W} = \frac{(3e^{2t}+4e^{3t})(e^{4t})}{2e^{6t}} = \frac{3e^{6t}+4e^{7t}}{2e^{6t}} = \frac{3+4e^t}{2}$$

and therefore

$$u_1 = \int \frac{g(t)W_1}{W}\ dt = \int \frac{3t^2+4t^2e^t}{2}\ dt = \frac{1}{2}t^3 + 2t^2e^t - 4te^t + 4e^t$$

$$u_2 = \int \frac{g(t)W_2}{W}\ dt = \int -3t-4te^t\ dt = -\frac{3}{2}t^2 - 4te^t + 4e^t$$

$$u_3 = \int \frac{g(t)W_3}{W}\ dt = \int \frac{3+4e^t}{2}\ dt = \frac{3}{2}t + 2e^t$$

Then the particular solution is

$$y_p(t) = u_1y_1 + u_2y_2 + u_3y_3$$

$$y_p(t) = \left(\frac{1}{2}t^3 + 2t^2e^t - 4te^t + 4e^t\right)e^{2t} + \left(-\frac{3}{2}t^2 - 4te^t + 4e^t\right)te^{2t}$$

$$+\left(\frac{3}{2}t + 2e^t\right)t^2e^{2t}$$

$$y_p(t) = \frac{1}{2}t^3e^{2t} + 2t^2e^{3t} - 4te^{3t} + 4e^{3t} - \frac{3}{2}t^3e^{2t} - 4t^2e^{3t} + 4te^{3t} + \frac{3}{2}t^3e^{2t} + 2t^2e^{3t}$$

$$y_p(t) = \frac{1}{2}t^3e^{2t} + 4e^{3t}$$

Then the general solution is the sum of the complementary and particular solutions,

$$y(t) = y_c(t) + y_p(t)$$

$$y(t) = c_1e^{2t} + c_2te^{2t} + c_3t^2e^{2t} + \frac{1}{2}t^3e^{2t} + 4e^{3t}$$

This is the same result we found using the method of undetermined coefficients in the previous lesson.

Laplace transforms for higher order equations

As we continue with this theme of expanding what we learned about second order equations to third and higher order equations, we can of course consider the Laplace transform.

For instance, let's add the Laplace transform of the third derivative to our list of formulas.

$$\mathscr{L}(f'(t)) = sF(s) - f(0)$$

$$\mathscr{L}(f''(t)) = s^2F(s) - sf(0) - f'(0)$$

$$\mathscr{L}(f'''(t)) = s^3F(s) - s^2f(0) - sf'(0) - f''(0)$$

We should start to see a pattern here for the Laplace transform of higher order derivatives, and we could add even more transforms.

$$\mathscr{L}(f^{(4)}(t)) = s^4F(s) - s^3f(0) - s^2f'(0) - sf''(0) - f'''(0)$$

$$\mathscr{L}(f^{(5)}(t)) = s^5F(s) - s^4f(0) - s^3f'(0) - s^2f''(0) - sf'''(0) - f^{(4)}(0)$$

...

Which means we can now apply these to find the Laplace transform of a higher order differential equation. Let's do an example, using the same differential equation we've been working with through the last few lessons.

Example

Use a Laplace transform to find the solution to the third order differential equation, given $y(0) = 0$, $y'(0) = 0$, and $y''(0) = 0$.

$$y''' - 6y'' + 12y' - 8y = 3e^{2t} + 4e^{3t}$$

Apply the transform to both sides of the differential equation.

$$\mathscr{L}(y''') - 6\mathscr{L}(y'') + 12\mathscr{L}(y') - 8\mathscr{L}(y) = 3\mathscr{L}(e^{2t}) + 4\mathscr{L}(e^{3t})$$

$$s^3Y(s) - s^2y(0) - sy'(0) - y''(0) - 6(s^2Y(s) - sy(0) - y'(0))$$

$$+12(sY(s) - y(0)) - 8Y(s) = 3\left(\frac{1}{s-2}\right) + 4\left(\frac{1}{s-3}\right)$$

$$s^3Y(s) - 6s^2Y(s) + 12sY(s) - 8Y(s) = 3\left(\frac{1}{s-2}\right) + 4\left(\frac{1}{s-3}\right)$$

Solve for $Y(s)$.

$$(s^3 - 6s^2 + 12s - 8)Y(s) = 3\left(\frac{1}{s-2}\right) + 4\left(\frac{1}{s-3}\right)$$

$$Y(s) = \frac{3}{(s-2)(s^3 - 6s^2 + 12s - 8)} + \frac{4}{(s-3)(s^3 - 6s^2 + 12s - 8)}$$

$$Y(s) = \frac{3}{(s-2)^4} + \frac{4}{(s-2)^3(s-3)}$$

Combine fractions.

$$Y(s) = \frac{3(s-3)}{(s-2)^4(s-3)} + \frac{4(s-2)}{(s-2)^4(s-3)}$$

$$Y(s) = \frac{7s - 17}{(s - 2)^4(s - 3)}$$

Apply a partial fractions decompositions.

$$\frac{7s - 17}{(s - 2)^4(s - 3)} = \frac{A}{s - 2} + \frac{B}{(s - 2)^2} + \frac{C}{(s - 2)^3} + \frac{D}{(s - 2)^4} + \frac{E}{s - 3}$$

We find $A = -4$, $B = -4$, $C = -4$, $D = 3$, and $E = 4$, so the Laplace transform becomes

$$Y(s) = \frac{-4}{s - 2} + \frac{-4}{(s - 2)^2} + \frac{-4}{(s - 2)^3} + \frac{3}{(s - 2)^4} + \frac{4}{s - 3}$$

$$Y(s) = -4\left(\frac{1}{s - 2}\right) - 4\left(\frac{1}{(s - 2)^2}\right) - 4\left(\frac{1}{(s - 2)^3}\right) + 3\left(\frac{1}{(s - 2)^4}\right) + 4\left(\frac{1}{s - 3}\right)$$

$$Y(s) = -4\left(\frac{1}{s - 2}\right) - 4\left(\frac{1}{s - 2}\right)^2 - 4\left(\frac{1}{s - 2}\right)^3 + 3\left(\frac{1}{s - 2}\right)^4 + 4\left(\frac{1}{s - 3}\right)$$

We can easily apply the inverse transform to the first and last terms,

$$y(t) = -4e^{2t} - 4\left(\frac{1}{s - 2}\right)^2 - 4\left(\frac{1}{s - 2}\right)^3 + 3\left(\frac{1}{s - 2}\right)^4 + 4e^{3t}$$

And as we know from our table of Laplace transforms,

$$\mathscr{L}(t^n e^{at}) = \frac{n!}{(s - a)^{n+1}}$$

so we'll use this formula with $n = 1$ to apply the inverse transform to the second term.

$$y(t) = -4e^{2t} - 4te^{2t} - 4\left(\frac{1}{(s-2)^3}\right) + 3\left(\frac{1}{(s-2)^4}\right) + 4e^{3t}$$

Using $n = 2$ to take the inverse transform of the third term, we get

$$y(t) = -4e^{2t} - 4te^{2t} - \frac{4}{2!}\left(\frac{2!}{(s-2)^3}\right) + 3\left(\frac{1}{(s-2)^4}\right) + 4e^{3t}$$

$$y(t) = -4e^{2t} - 4te^{2t} - \frac{4}{2!}t^2e^{2t} + 3\left(\frac{1}{(s-2)^4}\right) + 4e^{3t}$$

Using $n = 3$ to take the inverse transform of the fourth term, we get

$$y(t) = -4e^{2t} - 4te^{2t} - \frac{4}{2!}t^2e^{2t} + \frac{3}{3!}\left(\frac{3!}{(s-2)^4}\right) + 4e^{3t}$$

$$y(t) = -4e^{2t} - 4te^{2t} - \frac{4}{2!}t^2e^{2t} + \frac{3}{3!}t^3e^{2t} + 4e^{3t}$$

So the general solution to the differential equation is

$$y(t) = 4e^{3t} - 4e^{2t} - 4te^{2t} - \frac{4}{2!}t^2e^{2t} + \frac{3}{3!}t^3e^{2t}$$

Systems of higher order equations

Solving systems of higher order equations will be a natural extension of what we learned previously about solving systems of two equations. In fact, we've already looked at several systems of three differential equations, like when we covered equal real Eigenvalues of multiplicity three.

Now we just want to formalize what we saw in those earlier examples, and cover systems of higher order equations more comprehensively. Our main goal here is just to understand how the number of solution combinations increase as the system gets larger.

Systems of two equations

We saw previously that we only had four possibilities for systems of two equations: two distinct real Eigenvalues, a repeated Eigenvalue producing one Eigenvector, a repeated Eigenvalue producing two Eigenvectors, or one pair of complex conjugate Eigenvalues.

1. 2 distinct real Eigenvalues

Eigenvalues	Eigenvectors	Solutions
λ_1	$\vec{k_1}$	$\vec{x_1} = \vec{k_1}e^{\lambda_1 t}$
λ_2	$\vec{k_2}$	$\vec{x_2} = \vec{k_2}e^{\lambda_2 t}$

2. 2 complex conjugate Eigenvalues (λ_1 and λ_2 are conjugates, $\overrightarrow{k_1}$ and $\overrightarrow{k_2}$ are conjugates)

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \alpha + \beta i$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
$\lambda_2 = \alpha - \beta i$	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$

3. 1 double real Eigenvalue giving 1 Eigenvector

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
		$\overrightarrow{x_2} = \overrightarrow{k_1}te^{\lambda_1 t} + \overrightarrow{p_1}e^{\lambda_1 t}$
		$(A - \lambda_1 I)\overrightarrow{p_1} = \overrightarrow{k_1}$

4. 1 double real Eigenvalue giving 2 Eigenvectors

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_1 t}$

These $\overrightarrow{x_1}$ and $\overrightarrow{x_2}$ solutions were always the two linearly independent solutions that we needed in order to build the general solution to the system of two differential equations.

Systems of three equations

As we might expect, to build the general solution to a system of three differential equations, we'll need three linearly independent solutions. Logically, to build the general solution to a system of n differential equations, we'll need n linearly independent solutions.

Before we look at all possible combinations of solutions to a system of three equations, we want to realize that there are restrictions on these combinations, based on the types of Eigenvalues we find.

For instance, if we have a repeated Eigenvalue in a system of three equations, or course it'll be repeated either two or three times (equal real Eigenvalues with multiplicity two, or equal real Eigenvalues with multiplicity three). Complex conjugate Eigenvalues always appear in pairs, so we'll either have exactly one pair of complex conjugate Eigenvalues, or no complex Eigenvalues at all. With these restrictions in mind, let's look at all possible solution combinations for a system of three differential equations. There are exactly seven possibilities.

1. 3 distinct real Eigenvalues

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
λ_2	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$
λ_3	$\overrightarrow{k_3}$	$\overrightarrow{x_3} = \overrightarrow{k_3}e^{\lambda_3 t}$

2. 1 distinct real Eigenvalue, 2 complex conjugate Eigenvalues (λ_2 and λ_3 are conjugates, $\overrightarrow{k_2}$ and $\overrightarrow{k_3}$ are conjugates)

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
$\lambda_2 = \alpha + \beta i$	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$
$\lambda_3 = \alpha - \beta i$	$\overrightarrow{k_3}$	$\overrightarrow{x_3} = \overrightarrow{k_3}e^{\lambda_3 t}$

3. 1 distinct real Eigenvalue, 1 double real Eigenvalue giving 1 Eigenvector

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
$\lambda_2 = \lambda_3$	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_2}te^{\lambda_2 t} + \overrightarrow{p_1}e^{\lambda_2 t}$

$$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_2}$$

4. 1 distinct real Eigenvalue, 1 double real Eigenvalue giving 2 Eigenvectors

Eigenvalues	Eigenvectors	Solutions
λ_1	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1}e^{\lambda_1 t}$
$\lambda_2 = \lambda_3$	$\overrightarrow{k_2}$, $\overrightarrow{k_3}$	$\overrightarrow{x_2} = \overrightarrow{k_2}e^{\lambda_2 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_3}e^{\lambda_3 t}$

5. 1 triple real Eigenvalue giving 1 Eigenvector

Eigenvalues	Eigenvectors	Solutions

$\lambda_1 = \lambda_2 = \lambda_3$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
		$\overrightarrow{x_2} = \overrightarrow{k_1} t e^{\lambda_1 t} + \overrightarrow{p_1} e^{\lambda_1 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_1} \frac{t^2}{2} e^{\lambda_1 t} + \overrightarrow{p_1} t e^{\lambda_1 t} + \overrightarrow{q_1} e^{\lambda_1 t}$
		$(A - \lambda_1 I)\overrightarrow{p_1} = \overrightarrow{k_1}$
		$(A - \lambda_1 I)\overrightarrow{q_1} = \overrightarrow{p_1}$

6. 1 triple real Eigenvalue giving 2 Eigenvectors

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2 = \lambda_3$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$
		$\overrightarrow{x_3} = \overrightarrow{k_2} t e^{\lambda_2 t} + \overrightarrow{p_1} e^{\lambda_2 t}$
		$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_2}$

7. 1 triple real Eigenvalue giving 3 Eigenvectors

Eigenvalues	Eigenvectors	Solutions
$\lambda_1 = \lambda_2 = \lambda_3$	$\overrightarrow{k_1}$	$\overrightarrow{x_1} = \overrightarrow{k_1} e^{\lambda_1 t}$
	$\overrightarrow{k_2}$	$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_1 t}$
	$\overrightarrow{k_3}$	$\overrightarrow{x_3} = \overrightarrow{k_3} e^{\lambda_1 t}$

Fourth and higher order systems

So it starts to become clear that, the more differential equations we add to the system, the greater the number of possible Eigenvalue combinations, and the more complicated it'll become to build out the general solution.

For example, fourth order systems give us our first opportunity to find multiple pairs of complex conjugate roots. Systems of two or three differential equations simply aren't large enough to allow for this possibility. But once the system includes four or more equations, it becomes possible to find multiple pairs of complex conjugate Eigenvalues, and this obviously adds a little extra complication to finding the system's general solution.

That being said, let's do another example with a higher order equation to get more practice with the different solution combinations we've listed out in this lesson.

Example

Solve the system of differential equations.

$$x_1' = x_1 - x_3$$

$$x_2' = 2x_2 + x_3$$

$$x_3' = 2x_3$$

We'll need to start by finding the matrix $A - \lambda I$,

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

and then find its determinant $|A - \lambda I|$.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & -1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)\begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} - (0)\begin{vmatrix} 0 & 1 \\ 0 & 2-\lambda \end{vmatrix} + (-1)\begin{vmatrix} 0 & 2-\lambda \\ 0 & 0 \end{vmatrix}$$

$$|A - \lambda I| = (1-\lambda)((2-\lambda)(2-\lambda) - (1)(0)) - ((0)(0) - (2-\lambda)(0))$$

$$|A - \lambda I| = (1-\lambda)(2-\lambda)(2-\lambda)$$

Solve the characteristic equation for the Eigenvalues.

$$(1-\lambda)(2-\lambda)(2-\lambda) = 0$$

$$\lambda = 1,\ 2,\ 2$$

We'll handle $\lambda_1 = 1$ first, starting with finding $A - 1I$.

$$A - 1I = \begin{bmatrix} 1-1 & 0 & -1 \\ 0 & 2-1 & 1 \\ 0 & 0 & 2-1 \end{bmatrix}$$

$$A - 1I = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

This matrix is already simplified enough to turn it into a system of equations, so we get $k_3 = 0$ and

$$k_2 + k_3 = 0$$

$$k_2 = -k_3$$

$$k_2 = 0$$

So if we choose $k_1 = 1$, then the Eigenvector is

$$\vec{k_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

and therefore the solution vector is

$$\vec{x_1} = \vec{k_1} e^{\lambda_1 t}$$

$$\vec{x_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t$$

Then for the repeated Eigenvalue $\lambda_2 = \lambda_3 = 2$, we find

$$A - 2I = \begin{bmatrix} 1-2 & 0 & -1 \\ 0 & 2-2 & 1 \\ 0 & 0 & 2-2 \end{bmatrix}$$

$$A - 2I = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we turn this into a system of equations, we get $k_3 = 0$ and

$$-k_1 - k_3 = 0$$

$$k_1 = -k_3$$

$$k_1 = 0$$

So if we choose $k_2 = 1$, then the Eigenvector is

$$\overrightarrow{k_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

and therefore the solution vector is

$$\overrightarrow{x_2} = \overrightarrow{k_2} e^{\lambda_2 t}$$

$$\overrightarrow{x_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

Because we only find one Eigenvector for the two Eigenvalues $\lambda_2 = \lambda_3 = 2$, we have to use $\overrightarrow{k_2} = (0,1,0)$ to find a second solution.

$$(A - \lambda_2 I)\overrightarrow{p_1} = \overrightarrow{k_2}$$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Expanding this matrix equation as a system of equations gives

$$-p_1 - p_3 = 0$$

$$p_3 = 1$$

Therefore, we get $p_1 = -1$. We choose $p_2 = 0$, so $\overrightarrow{p_1} = (-1,0,1)$. Then the second solution will be

$$\overrightarrow{x_3} = \overrightarrow{k_2} t e^{\lambda_2 t} + \overrightarrow{p_1} e^{\lambda_2 t}$$

$$\overrightarrow{x_3} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

and the general solution to the homogeneous system is

$$\overrightarrow{x} = c_1\overrightarrow{x_1} + c_2\overrightarrow{x_2} + c_3\overrightarrow{x_3}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_3 \left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} te^{2t} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{2t} \right)$$

Series solutions of higher order equations

When we want to find the series solution to a higher order differential equation, we'll of course need the series representations of higher order derivatives.

We saw previously the power series representations in x of y, y', and y'', so let's add just a couple of derivatives to that list.

$$y=\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + \dots$$

$$y'=\sum_{n=1}^{\infty} c_n n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + 5c_5 x^4 + \dots$$

$$y''=\sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} = 2c_2 + 6c_3 x + 12c_4 x^2 + 20c_5 x^3 + 30c_6 x^4 + \dots$$

$$y'''=\sum_{n=3}^{\infty} c_n n(n-1)(n-2) x^{n-3} = 6c_3 + 24c_4 x + 60c_5 x^2 + 120c_6 x^3 + 210c_7 x^4 + \dots$$

$$y^{(4)}=\sum_{n=4}^{\infty} c_n n(n-1)(n-2)(n-3) x^{n-4} = 24c_4 + 120c_5 x + 360c_6 x^2 + 840c_7 x^3 + \dots$$

Just as we did before, we can use these formulas to find the power series solution to a higher order differential equation.

Keep in mind that, the higher the order of the differential equation, the less likely it becomes that we'll be able to find clean series representations when we start plugging values from the index into the recurrence relation.

That being said, let's do an example so see what kind of a power series solution we can find for a higher order differential equation.

Example

Find a power series solution in x to the differential equation.

$$y''' + y' - xy = 0$$

We'll substitute y, y', and y''' into the differential equation.

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^{n-1} - x\sum_{n=0}^{\infty} c_n x^n = 0$$

$$\sum_{n=3}^{\infty} c_n n(n-1)(n-2)x^{n-3} + \sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^{n+1} = 0$$

Let's check to see if the series are in phase. Because the index of the first series starts at $n = 3$, the first series begins with an x^0 term. Because the index of the second series starts at $n = 1$, the second series begins with an x^0 term. And because the index of the third series starts at $n = 0$, the third series begins with an x^1 term.

So to put these series in phase, we'll pull the x^0 term out of both the first and second series.

$$c_3 3(3-1)(3-2)x^{3-3} + \sum_{n=4}^{\infty} c_n n(n-1)(n-2)x^{n-3}$$

$$+c_1 1x^{1-1}+\sum_{n=2}^{\infty}c_n nx^{n-1}-\sum_{n=0}^{\infty}c_n x^{n+1}=0$$

$$c_1+6c_3+\sum_{n=4}^{\infty}c_n n(n-1)(n-2)x^{n-3}+\sum_{n=2}^{\infty}c_n nx^{n-1}-\sum_{n=0}^{\infty}c_n x^{n+1}=0$$

Now the series are in phase, but the indices don't match. We can substitute $k=n-3$ and $n=k+3$ into the first series, $k=n-1$ and $n=k+1$ into the second series, and $k=n+1$ and $n=k-1$ into the third series.

$$c_1+6c_3+\sum_{k=1}^{\infty}c_{k+3}(k+3)(k+3-1)(k+3-2)x^k$$

$$+\sum_{k=1}^{\infty}c_{k+1}(k+1)x^k-\sum_{k=1}^{\infty}c_{k-1}x^k=0$$

$$c_1+6c_3+\sum_{k=1}^{\infty}c_{k+3}(k+3)(k+2)(k+1)x^k+\sum_{k=1}^{\infty}c_{k+1}(k+1)x^k-\sum_{k=1}^{\infty}c_{k-1}x^k=0$$

With the series in phase and matching indices, we can finally add them.

$$c_1+6c_3+\sum_{k=1}^{\infty}c_{k+3}(k+3)(k+2)(k+1)x^k+c_{k+1}(k+1)x^k-c_{k-1}x^k=0$$

$$c_1+6c_3+\sum_{k=1}^{\infty}[c_{k+3}(k+3)(k+2)(k+1)+c_{k+1}(k+1)-c_{k-1}]x^k=0$$

The c_1+6c_3 value in front of the series is associated with the $k=0$ term, while all of the other $k=1, 2, 3, \ldots$ terms are still in the series.

$k=0$ $\qquad\qquad$ $c_1+6c_3=0$

$$c_1 = -6c_3 \text{ and } c_3 = -\frac{c_1}{6}$$

$k = 1, 2, 3, \ldots$ $\qquad c_{k+3}(k+3)(k+2)(k+1) + c_{k+1}(k+1) - c_{k-1} = 0$

We'll solve the recurrence relation for the coefficient with the largest subscript. This recurrence relation includes c_{k+3}, c_{k+1}, and c_{k-1}, so we'll solve for c_{k+3}.

$$c_{k+3} = \frac{c_{k-1} - c_{k+1}(k+1)}{(k+3)(k+2)(k+1)}$$

Now we'll start plugging in values $k = 1, 2, 3, \ldots$.

$k = 0 \qquad c_3 = -\frac{c_1}{3!}$

$k = 1 \qquad c_4 = \frac{c_0 - 2c_2}{4!}$ $\qquad k = 2 \qquad c_5 = \frac{3c_1}{5!}$

$k = 3 \qquad c_6 = -\frac{c_0 - 8c_2}{6!}$ $\qquad k = 4 \qquad c_7 = -\frac{7c_1}{7!}$

$k = 5 \qquad c_8 = \frac{6c_0 - 18c_2}{8!}$ $\qquad k = 6 \qquad c_9 = \frac{25c_1}{9!}$

$\vdots \qquad\qquad \vdots$

There's not an obvious pattern here. As we mentioned earlier, the higher the order of the differential equation, the less likely it becomes that we'll be able to find clean series representations.

So without worrying about the pattern, we'll just use these values to find that the general solution is

$$y = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + \dots$$

$$y = c_0 + c_1x + c_2x^2 - \frac{c_1}{3!}x^3 + \frac{c_0 - 2c_2}{4!}x^4 + \frac{3c_1}{5!}x^5 - \frac{c_0 - 8c_2}{6!}x^6 + \dots$$

$$y = c_0\left(1 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right) + c_1\left(x - \frac{1}{3!}x^3 + \frac{3}{5!}x^5 + \dots\right)$$

$$+c_2\left(x^2 - \frac{2}{4!}x^4 + \frac{8}{6!}x^6 + \dots\right)$$

$$y = c_0\left(1 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \dots\right) + c_1x\left(1 - \frac{1}{3!}x^2 + \frac{3}{5!}x^4 + \dots\right)$$

$$+c_2x^2\left(1 - \frac{2}{4!}x^2 + \frac{8}{6!}x^4 + \dots\right)$$

Fourier series representations

We already looked at series when we talked about how to find the series solution to a second order differential equation, so we're familiar with representing a function (and therefore the curve of a general solution) as a series.

But so far we've only represented functions as Taylor series, and that series type has limited usefulness. For example, to use a Taylor series to represent a function $f(x)$, the function has to be sufficiently differentiable, because we need its derivatives for the coefficients in the series representation,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

We can also run into an issue if the radius of convergence R, given by $|x-a| < R$, is too small to make the series useful.

So instead of using a Taylor series, sometimes it'll be better (or necessary) to use a different kind of series representation.

Fourier series representation

A **Fourier series** is a different kind of series representation. Instead of using powers of $(x-a)$, like a Taylor series, it uses sine and cosine functions of different arguments.

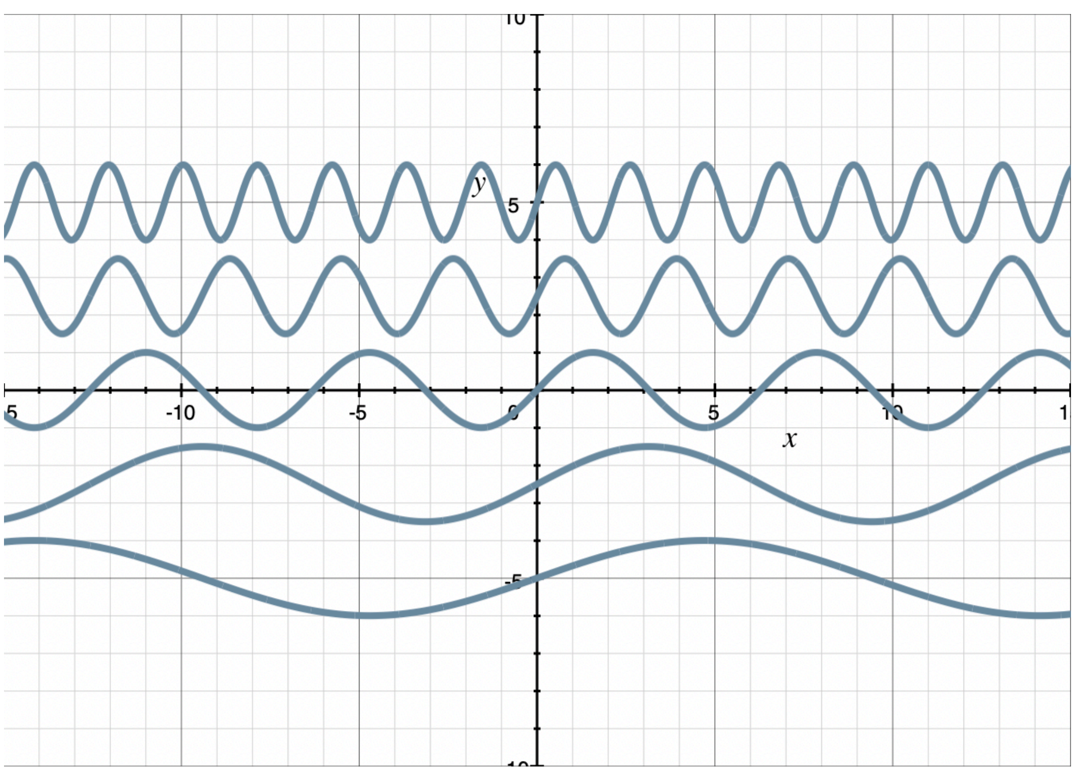
$$f(x) = \sum_{n=0}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

Notice that the $n = 0$ term in the first series gives $A_0 \cos(0) = A_0$, so we could also choose to write this as

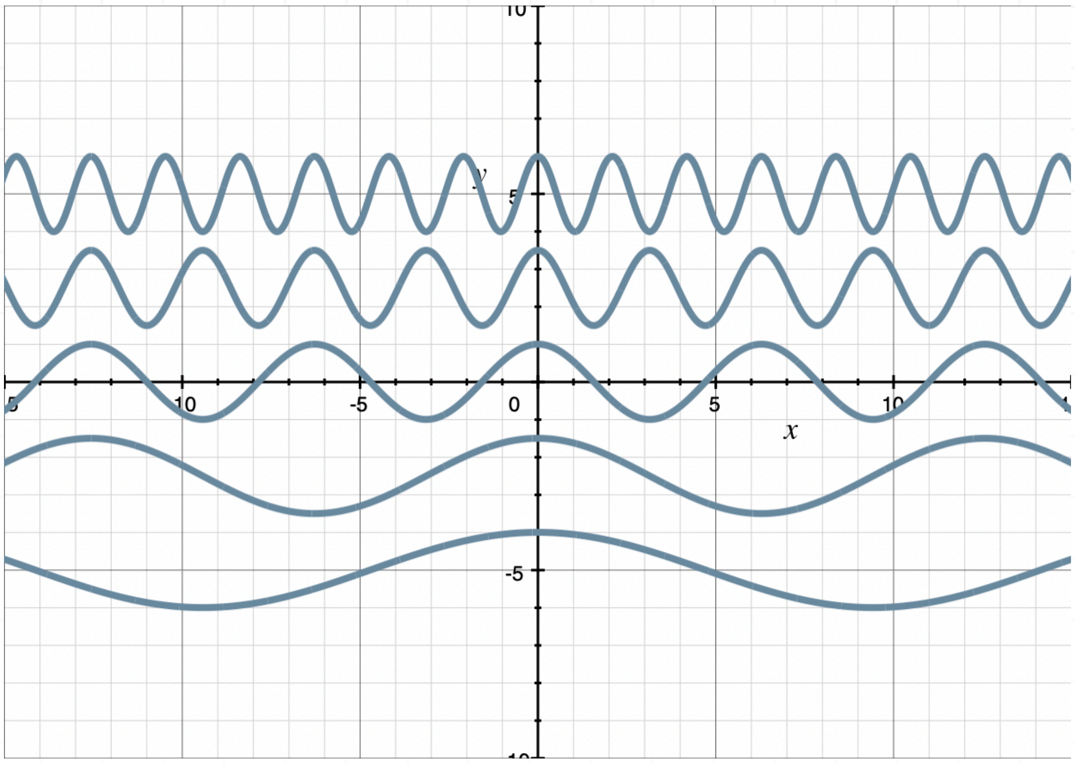
$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

This formula gives us a Fourier series representation of the function $f(x)$ on the interval $-L \le x \le L$. Notice that we're just adding a bunch of cosine functions with different arguments (the values of n are given by the series index, and L is given by the interval) in the first series, and then doing the same thing with sine functions in the second series.

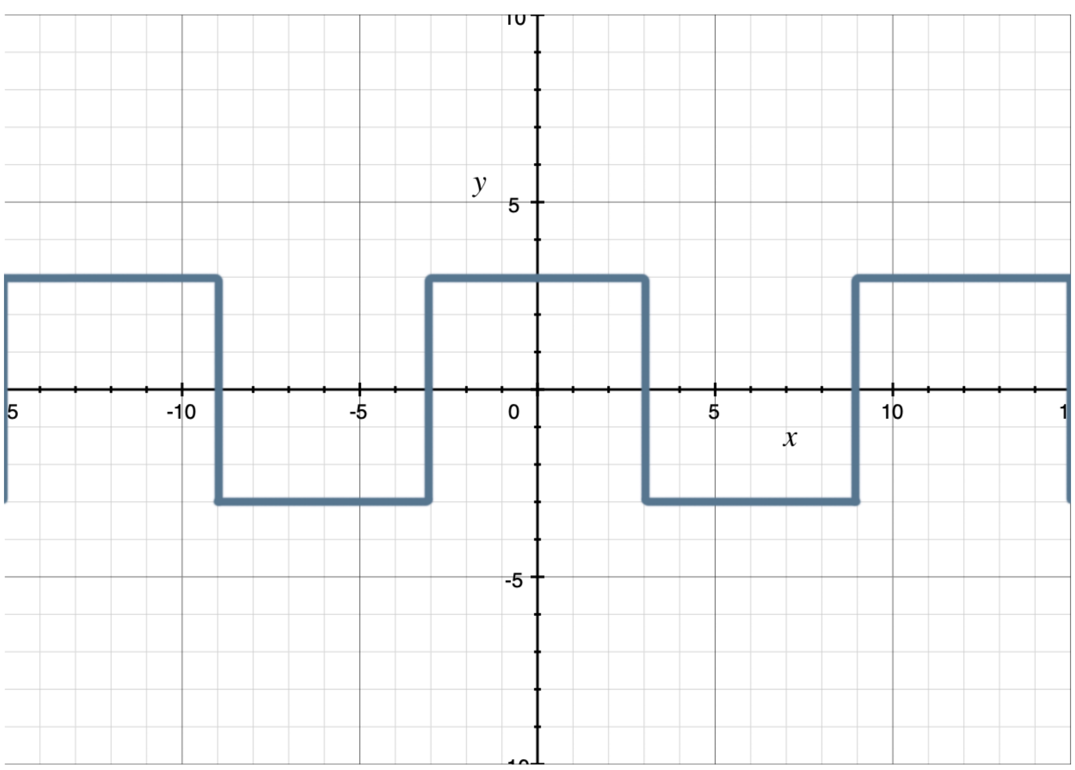
This is remarkable! What we're saying here is that, at least in theory, we can express any $f(x)$ as no more than the sum of sine and cosine functions. As a reminder, here are a bunch of sine curves,



and here are a bunch of cosine curves,



And now we're saying that if we add up enough of these wavy functions, we can eventually get to the graph of any $f(x)$, regardless of the shape of $f(x)$! Our only condition is that $f(x)$ be periodic, but that still means that $f(x)$ could be made of a bunch of straight lines,



and we should still be able to find its Fourier series representation. It's pretty incredible to think that we can model straight lines using only the sum of the wavy sine and cosine functions.

That being said, depending on the periodic function $f(x)$ that we're representing, we'll sometimes need to use an infinite number of sine and cosine waves.

And, just like with other series representations (like a Taylor series representation), if we're using only a partial sum to approximate the function (just the first three terms of the series, or just the first five terms of the series, etc.), the more terms we use the more accurate our approximation will be.

But returning now to the Fourier series formula itself, we still have to address A_0 and the coefficients A_n and B_n. They have their own values, which we can calculate using

$$A_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\ dx$$

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

Once we find these values for A_n and B_n, we'll plug them into our Fourier series formula, then simplify as much as we can to get the Fourier series representation of our function $f(x)$.

Because n is defined for $n = 1, 2, 3, \ldots$ for A_n and B_n, it'll often be the case that we'll find expressions for A_n and B_n that include $\sin(n\pi)$ and $\cos(n\pi)$. For $n = 1, 2, 3, \ldots$, remember that

$$\sin(n\pi) = 0 \qquad\qquad n = 1, 2, 3, \ldots$$

$$\cos(n\pi) = (-1)^n \qquad\qquad n = 1, 2, 3, \ldots$$

There's also a good chance that we'll use the even-odd trigonometric identities

$$\sin(-x) = -\sin x$$

$$\cos(-x) = \cos x$$

With all of that in mind, let's do a simple example where we just use these formulas to build the Fourier series representation of a function.

Example

Find the Fourier series representation of $f(x) = x$ on $-L \le x \le L$.

Let's first find A_0 and the coefficients A_n and B_n. For A_0, we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x\ dx$$

$$A_0 = \frac{1}{2L}\left(\frac{1}{2}x^2\right)\Big|_{-L}^{L}$$

$$A_0 = \frac{1}{2L}\left(\frac{1}{2}L^2 - \frac{1}{2}(-L)^2\right)$$

$$A_0 = \frac{1}{2L}\left(\frac{1}{2}L^2 - \frac{1}{2}L^2\right)$$

$$A_0 = \frac{1}{2L}(0)$$

$$A_0 = 0$$

For A_n, we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{1}{L}\left[x\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right) - \int\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Big|_{-L}^{L}$$

$$A_n = \frac{1}{L}\left[\left(\frac{Lx}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right)\right]\Big|_{-L}^{L}$$

$$A_n = \frac{1}{L}\left[\left(\frac{L^2}{n\pi}\right)\sin(n\pi) + \left(\frac{L}{n\pi}\right)^2\cos(n\pi) + \left(\frac{L^2}{n\pi}\right)\sin(-n\pi) - \left(\frac{L}{n\pi}\right)^2\cos(-n\pi)\right]$$

Using the even-odd identities we mentioned earlier, we can rewrite this as

$$A_n = \frac{1}{L}\left[\left(\frac{L^2}{n\pi}\right)\sin(n\pi) + \left(\frac{L}{n\pi}\right)^2\cos(n\pi) - \left(\frac{L^2}{n\pi}\right)\sin(n\pi) - \left(\frac{L}{n\pi}\right)^2\cos(n\pi)\right]$$

$$A_n = \frac{1}{L}(0)$$

$$A_n = 0$$

And for B_n, we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{1}{L}\left[-\left(\frac{Lx}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) - \int\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_{-L}^{L}$$

$$B_n = \frac{1}{L}\left[-\left(\frac{Lx}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{-L}^{L}$$

$$B_n = \frac{1}{L}\left[-\left(\frac{L^2}{n\pi}\right)\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2 \sin(n\pi) - \left(\frac{L^2}{n\pi}\right)\cos(-n\pi) - \left(\frac{L}{n\pi}\right)^2 \sin(-n\pi)\right]$$

Using the even-odd identities we mentioned earlier, we can rewrite this as

$$B_n = \frac{1}{L}\left[-\left(\frac{L^2}{n\pi}\right)\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2 \sin(n\pi) - \left(\frac{L^2}{n\pi}\right)\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2 \sin(n\pi)\right]$$

$$B_n = \frac{1}{L}\left[2\left(\frac{L}{n\pi}\right)^2 \sin(n\pi) - 2\left(\frac{L^2}{n\pi}\right)\cos(n\pi)\right]$$

$$B_n = \frac{2L}{n\pi}\left[\left(\frac{1}{n\pi}\right)\sin(n\pi) - \cos(n\pi)\right]$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$, and $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = -\frac{2L}{n\pi}\cos(n\pi)$$

$$B_n = -\frac{2L(-1)^n}{n\pi}$$

$$B_n = \frac{(-1)^{n+1}2L}{n\pi} \qquad\qquad n = 1, 2, 3, \ldots$$

Then the Fourier series for $f(x) = x$ on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 0 + \sum_{n=1}^{\infty} (0)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

Even/odd rules for the coefficients

Before we move forward, let's make a comment here about the integrals of even and odd functions over symmetric intervals.

We may remember from calculus that the integral of an odd function over a symmetric interval is 0, while the integral of an even function over a symmetric interval is equivalent to two times the same integral over half of the interval. In other words, using the symmetric interval $-L \le x \le L$ from the Fourier series representation,

$$\int_{-L}^{L} f(x)\ dx = 0$$ when $f(x)$ is odd ($f(-x) = -f(x)$)

$$\int_{-L}^{L} f(x)\ dx = 2\int_{0}^{L} f(x)\ dx$$ when $f(x)$ is even ($f(-x) = f(x)$)

So when it comes to the integral formulas that we use to calculate A_0, A_n, and B_n,

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

because each of them is integrating over the symmetric interval $-L \le x \le L$, we can say that

$$A_0 = 0 \qquad \text{if } f(x) \text{ is odd}$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx \qquad \text{if } f(x) \text{ is even}$$

Because the cosine function in the A_n formula is an even function, the product $f(x)\cos(n\pi x/L)$ will be even when $f(x)$ is even (because the product of two functions is itself an even function), and the product $f(x)\cos(n\pi x/L)$ will be odd when $f(x)$ is odd (because the product of an even function and an odd function is itself an odd function).

$$A_n = 0 \qquad \text{if } f(x) \text{ is odd, } n = 1, 2, 3, \ldots$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad \text{if } f(x) \text{ is even, } n = 1, 2, 3, \ldots$$

And because the sine function in the B_n formula is an odd function, the product $f(x)\sin(n\pi x/L)$ will be odd when $f(x)$ is even (because the product of an even function and an odd function is itself an odd function), and the product $f(x)\sin(n\pi x/L)$ will be even when $f(x)$ is odd (because the product of two odd functions is itself an even function).

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad \text{if } f(x) \text{ is odd, } n = 1, 2, 3, \dots$$

$$B_n = 0 \qquad \text{if } f(x) \text{ is even, } n = 1, 2, 3, \dots$$

Looking back at the last example, when we were calculating A_0 we found

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x\ dx$$

which is the integral of an odd function ($f(x) = x$ is an odd function) over a symmetric interval. So instead of doing all the extra work to integrate and evaluate over $[-L, L]$, we could have known right away that $A_0 = 0$.

Similarly, in the last example when we were calculating A_n we found

$$A_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

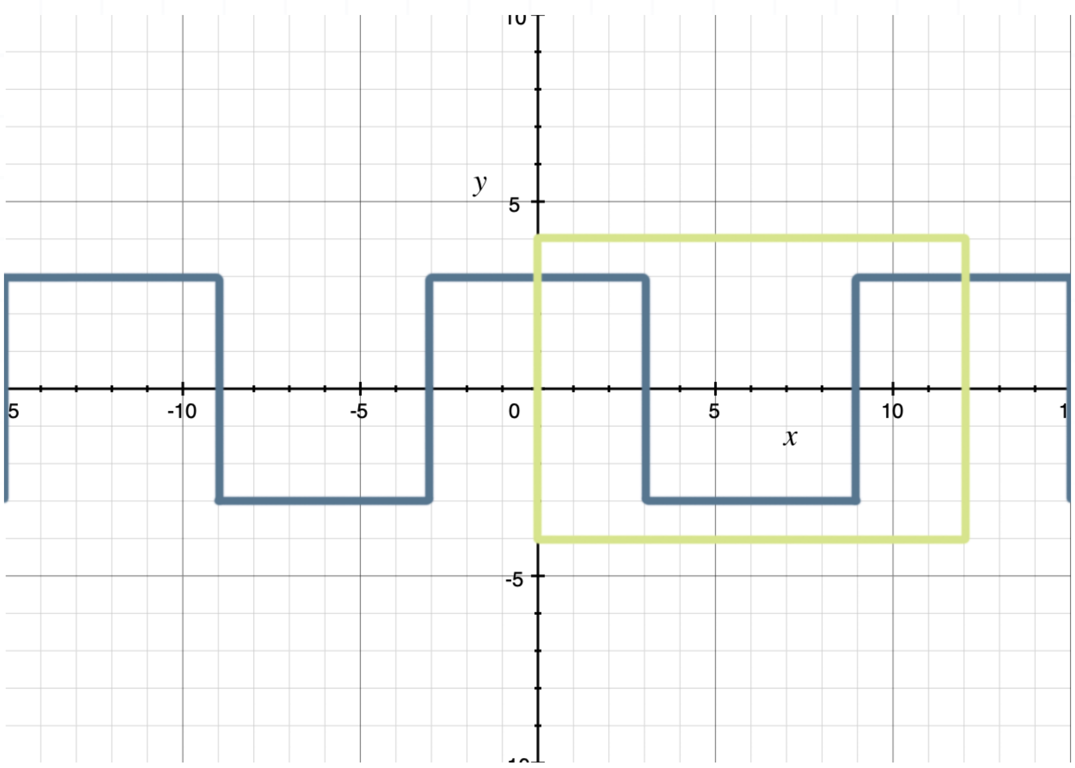
which is the integral of an odd function ($f(x) = x$ is odd, the cosine function is even, and the product of an even and odd function is itself an odd function) over a symmetric interval. So instead of doing all the extra work to integrate with integration by parts and then evaluate over $[-L, L]$, we could have known right away that $A_n = 0$.

Of course, not every function $f(x)$ will be even or odd; it's actually most likely that the function will be neither even nor odd. But because these even/odd rules can save us so much time in the case when $f(x)$ is even or odd, let's make sure to keep them in mind as we move forward with more Fourier series examples.

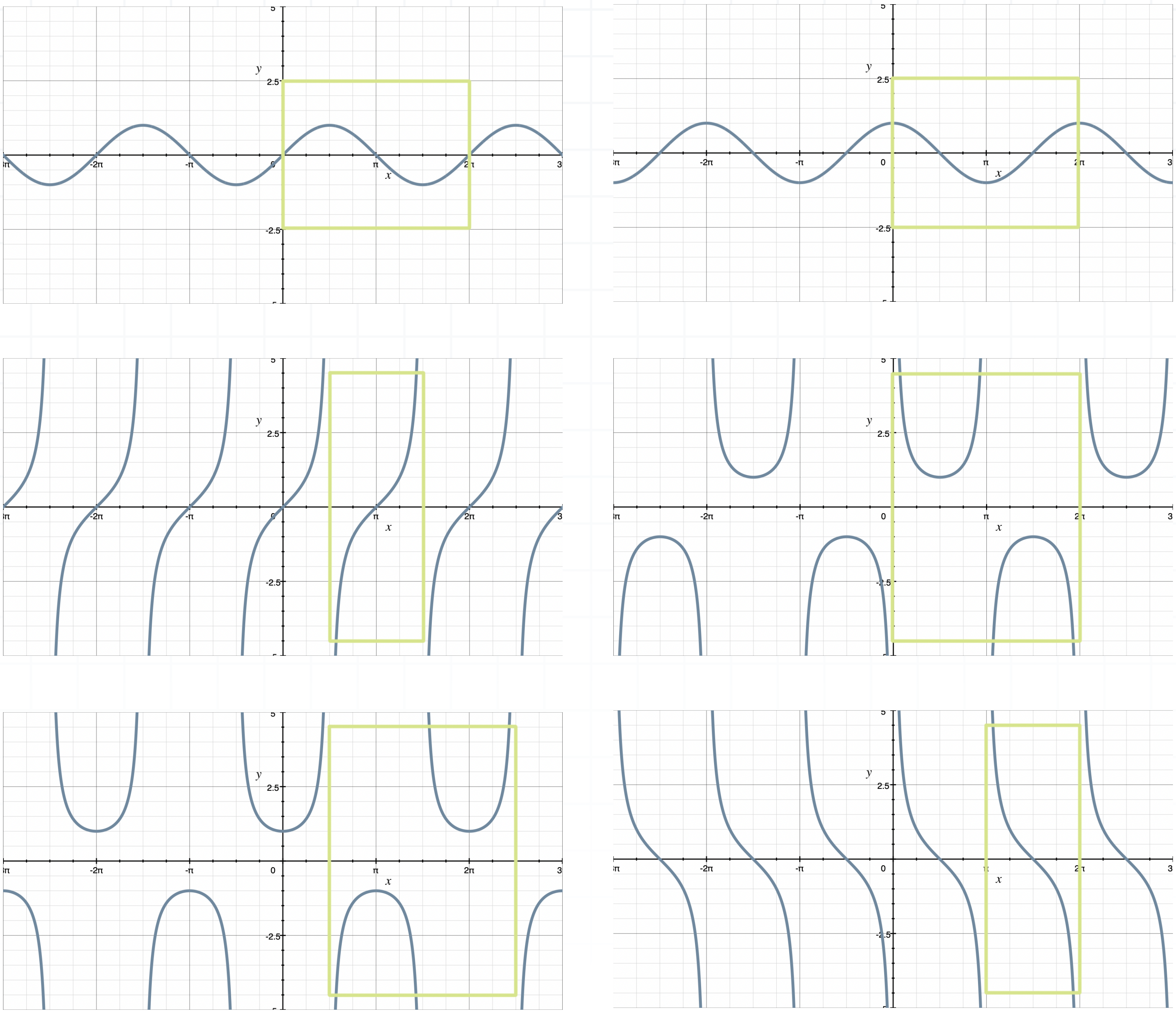
Periodic functions and periodic extensions

We said earlier that we can generate a Fourier series representation for any periodic function, but then we worked an example where we found the Fourier series of $f(x)=x$, which is not a periodic function.

Remember that a **periodic function** repeats the same values at regular, predictable intervals. Visually, we should be able to identify the repeating pattern.



The six circular trigonometric functions (sine, cosine, tangent, cosecant, secant, and cotangent) are the periodic functions we're most familiar with, and we can see from their graphs that they're all periodic.

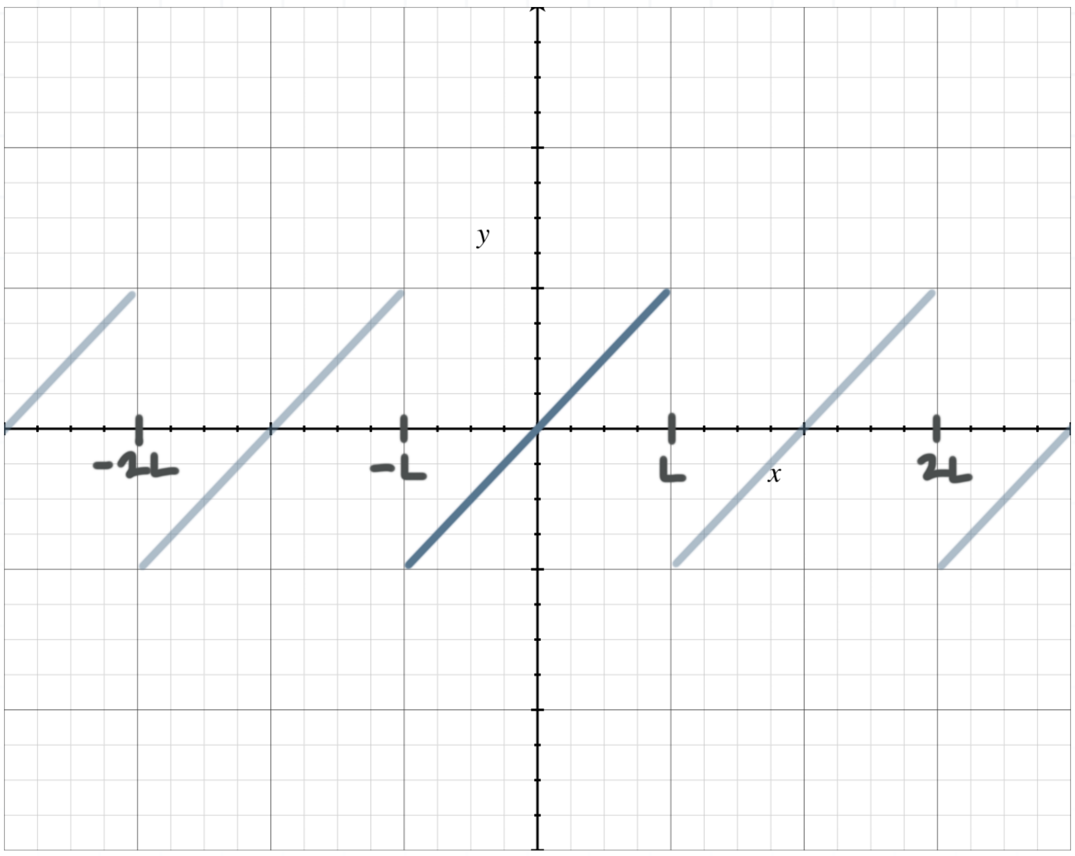


But now let's go back to the example from the last lesson, where we found the Fourier series representation of $f(x) = x$, a non-periodic function.

Periodic extensions

The only reason we were able to find a Fourier series representation of $f(x) = x$ is because we defined the representation for the interval $-L \leq x \leq L$.

In other words, instead of representing $f(x) = x$ with a Fourier series, we actually represented the periodic extension of $f(x) = x$, which is the function we find by repeating the part of $f(x) = x$ on the interval $-L \leq x \leq L$ over and over again to the left and right.



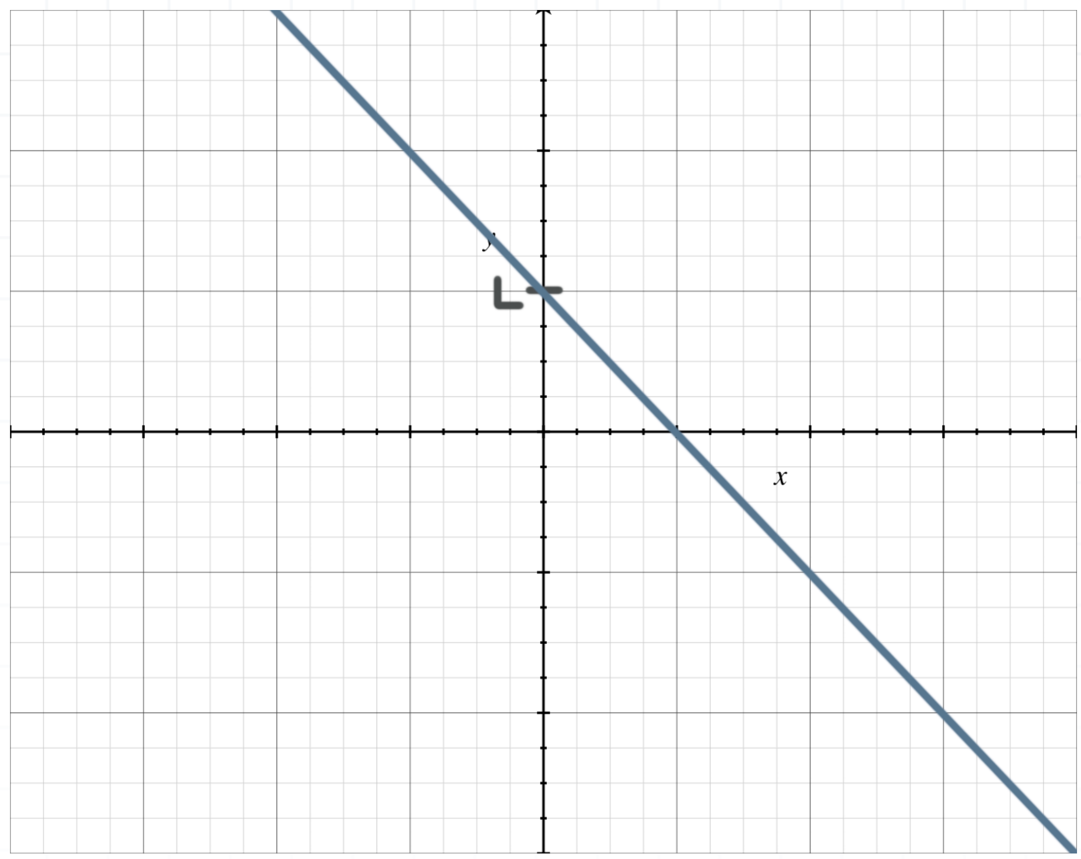
By limiting the interval to some generic $-L \leq x \leq L$, and then repeating over and over again the part of $f(x) = x$ that we find on $-L \leq x \leq L$, we turn the non-periodic $f(x) = x$ into its periodic extension, and we can now find the Fourier series representation of the periodic extension.

Let's do an example where we build the periodic extension of what would otherwise be a non-periodic function.

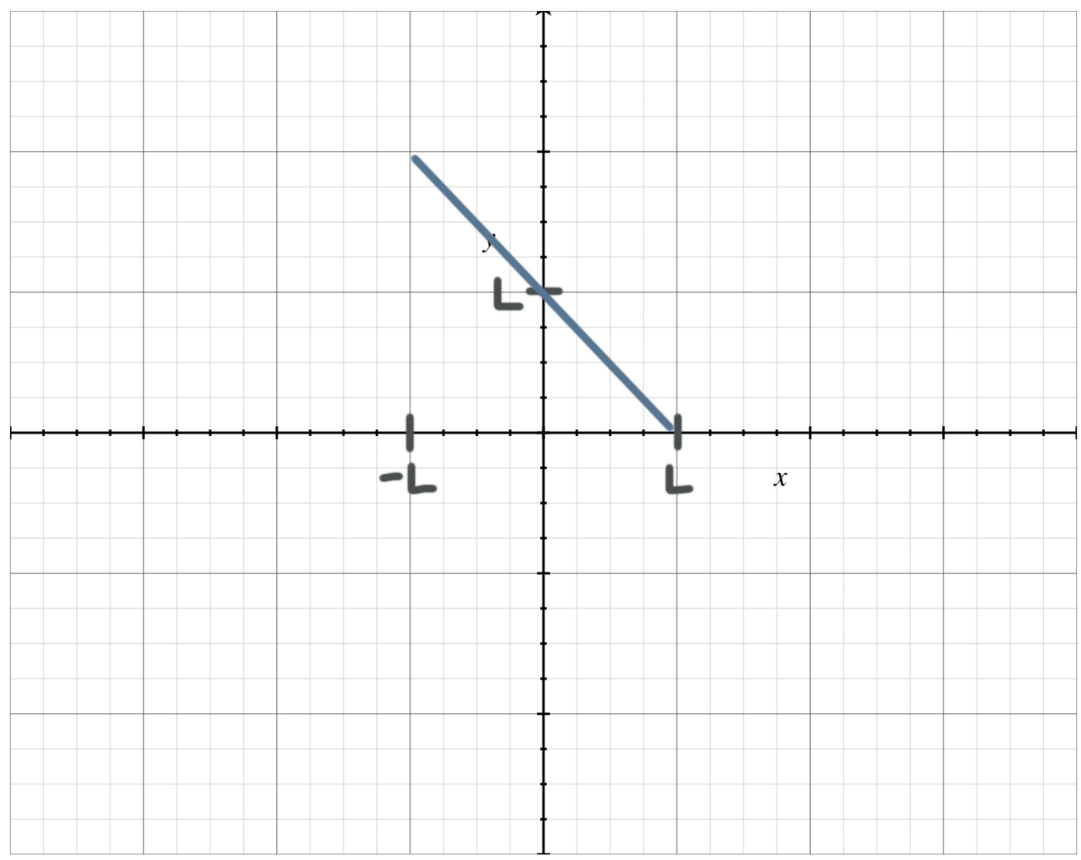
Example

Sketch the periodic extension of $f(x) = L - x$ on the interval $-L \leq x \leq L$.

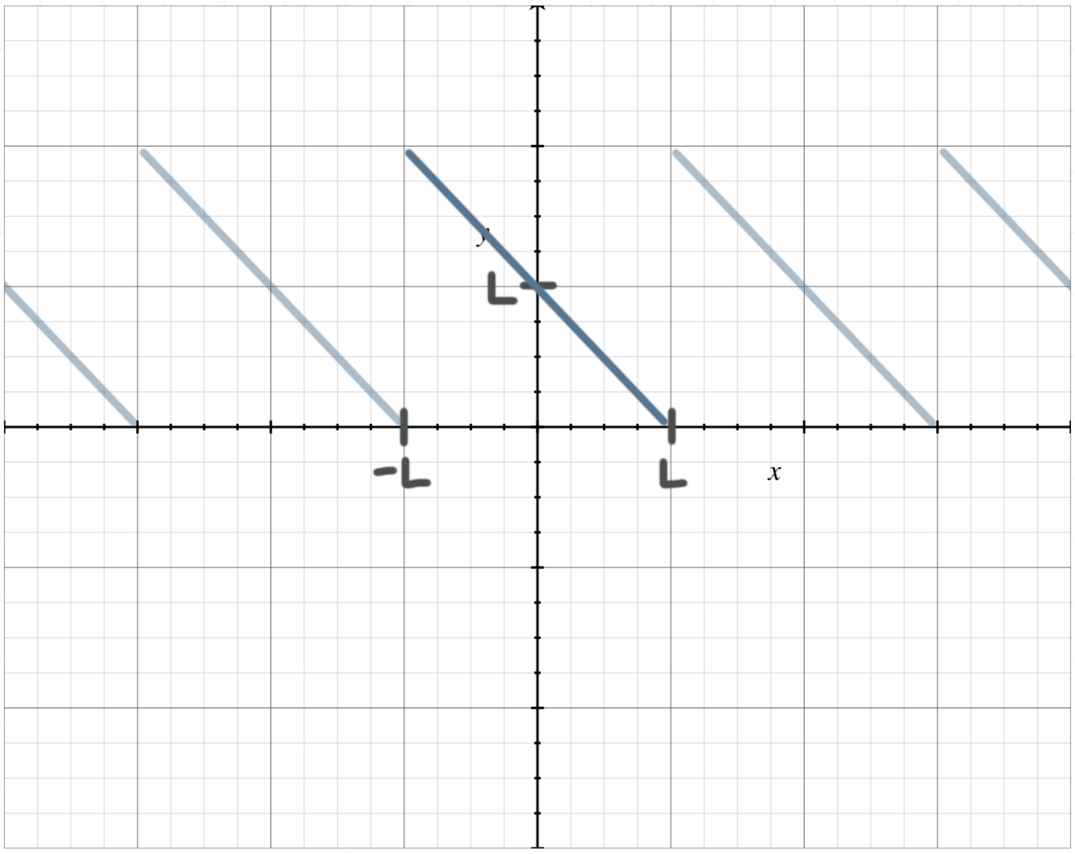
A sketch of the function $f(x) = L - x$, given some positive value of L, could be



If we limit the sketch of this graph to the interval $-L \leq x \leq L$, then the section of the graph on $-L \leq x \leq L$ would be



Now if we take this section and repeat it over and over to both the left and right, we get a sketch of the periodic extension of $f(x) = L - x$.



Even and odd extensions

It's also important that we know how to find the even extension of a function and the odd extension of a function. These particular extensions build an even or odd $g(x)$ function, from a function $f(x)$ that isn't necessarily even or odd.

Given some function $f(x)$ (which isn't necessarily even or odd), we define its **even extension** on $-L \leq x \leq L$ as

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x < 0 \end{cases}$$

and its **odd extension** on $-L \leq x \leq L$ as

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ -f(-x) & -L \leq x < 0 \end{cases}$$

Let's do an example with the function we looked at previously, $f(x) = L - x$, so that we can compare the even and odd extensions of this function to its periodic extension.

Example

Find the even and odd extensions of $f(x) = L - x$, given some positive value of L, then sketch both functions.

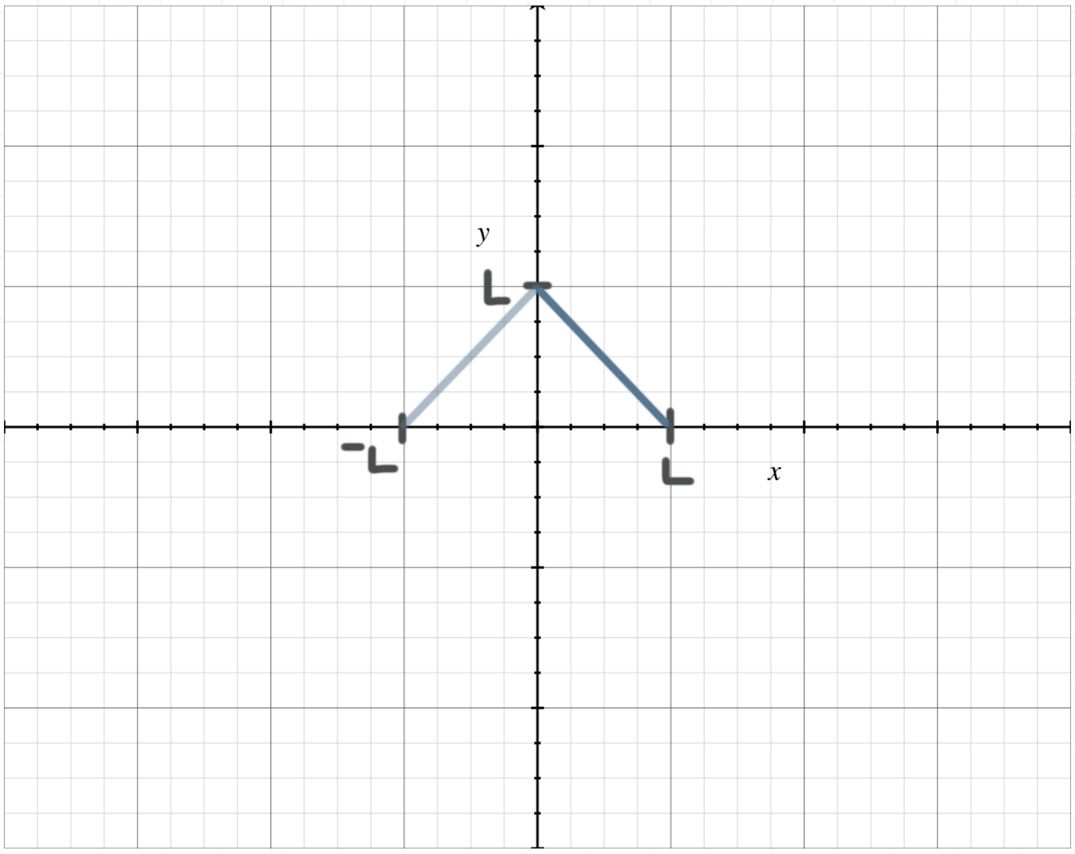
The even extension of $f(x) = L - x$ is

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} L - x & 0 \leq x \leq L \\ L - (-x) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} L - x & 0 \leq x \leq L \\ L + x & -L \leq x < 0 \end{cases}$$

And the sketch of the even extension is



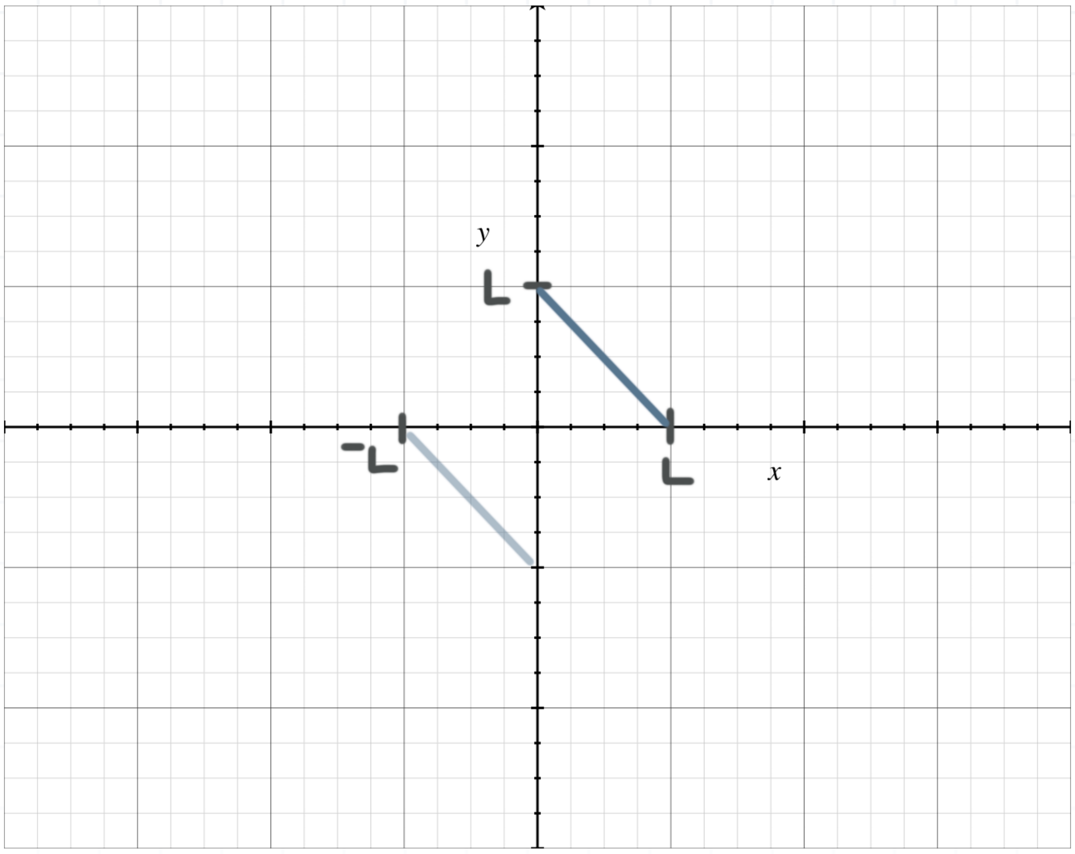
The odd extension of $f(x) = L - x$ is

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ -f(-x) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} L - x & 0 \leq x \leq L \\ -(L - (-x)) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} L - x & 0 \leq x \leq L \\ -L - x & -L \leq x < 0 \end{cases}$$

And the sketch of the odd extension is


y
x
1
1
-1
-1

Representing piecewise functions

Recently we looked at how to use Fourier's formula to represent the function $f(x)=x$ as a Fourier series.

But we can also use a Fourier series to represent a piecewise function, and doing so will be important as we move forward.

Remember that piecewise functions are, not surprisingly, defined "in pieces." We use one expression to define the function on a particular interval, another expression to define the function on a different interval, etc. The general form of a piecewise function is

$$f(x)=\begin{cases} g(x) & x_0 \le x < x_1 \\ h(x) & x_1 \le x < x_2 \\ \vdots & \\ n(x) & x_{n-1} \le x < x_n \end{cases}$$

Any function that's defined in two or more pieces is a piecewise-defined function.

So let's do another example where we find the Fourier series representation of a piecewise-defined function. The process will look exactly the same as what we're used to doing already, except that we'll split the integrals we use to find A_0, A_n, and B_n into multiple integrals, one for each piece of the piecewise function.

Example

Find the Fourier series representation of the piecewise function on $-L \leq x \leq L$.

$$f(x) = \begin{cases} 1 & -L \leq x < 0 \\ x & 0 \leq x \leq L \end{cases}$$

For A_0, we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{0} 1\ dx + \frac{1}{2L}\int_{0}^{L} x\ dx$$

$$A_0 = \frac{x}{2L}\Big|_{-L}^{0} + \frac{x^2}{4L}\Big|_{0}^{L}$$

$$A_0 = \frac{0}{2L} - \frac{-L}{2L} + \frac{L^2}{4L} - \frac{0^2}{4L}$$

$$A_0 = \frac{1}{2} + \frac{L}{4}$$

$$A_0 = \frac{L+2}{4}$$

For A_n, we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{0} \cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0} + \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \left(\frac{1}{n\pi}\right)\sin\left(\frac{n\pi(0)}{L}\right) - \left(\frac{1}{n\pi}\right)\sin\left(\frac{n\pi(-L)}{L}\right) + \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = -\left(\frac{1}{n\pi}\right)\sin(-n\pi) + \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Since $\sin(-x) = -\sin x$, we get

$$A_n = \left(\frac{1}{n\pi}\right)\sin(n\pi) + \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{0}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{1}{L}\left[x\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right) - \int\left(\frac{L}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_{0}^{L}$$

$$A_n = \frac{1}{L}\left[\left(\frac{Lx}{n\pi}\right)\sin\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_{0}^{L}$$

$$A_n = \frac{1}{L}\left[\left(\frac{L^2}{n\pi}\right)\sin(n\pi) + \left(\frac{L}{n\pi}\right)^2\cos(n\pi) - \left(\frac{L}{n\pi}\right)^2\cos(0)\right]$$

$$A_n = \frac{L}{(n\pi)^2}[(n\pi)\sin(n\pi) + \cos(n\pi) - 1]$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for A_n simplifies to

$$A_n = \frac{L}{(n\pi)^2}[0 + (-1)^n - 1]$$

$$A_n = \frac{(-1)^n L - L}{(n\pi)^2}$$

And for B_n, we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{0} \sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{1}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{0} + \frac{1}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{1}{n\pi} + \frac{1}{n\pi}\cos(-n\pi) + \frac{1}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Since $\cos(-x) = \cos x$, we get

$$B_n = -\frac{1}{n\pi} + \frac{1}{n\pi}\cos(n\pi) + \frac{1}{L}\int_{0}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

For $n = 1, 2, 3, \ldots$, $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = \frac{(-1)^n - 1}{n\pi} + \frac{1}{L}\int_0^L x \sin\left(\frac{n\pi x}{L}\right) dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{(-1)^n - 1}{n\pi} + \frac{1}{L}\left[-\left(\frac{Lx}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) - \int\left(-\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) dx\right]\Big|_0^L$$

$$B_n = \frac{(-1)^n - 1}{n\pi} + \frac{1}{L}\left[-\left(\frac{Lx}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + \left(\frac{L}{n\pi}\right)^2 \sin\left(\frac{n\pi x}{L}\right)\right]\Big|_0^L$$

$$B_n = \frac{(-1)^n - 1}{n\pi} + \frac{1}{L}\left[-\left(\frac{L^2}{n\pi}\right)\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2 \sin(n\pi)\right]$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = \frac{(-1)^n - 1}{n\pi} - \frac{L}{n\pi}\cos(n\pi)$$

$$B_n = \frac{(-1)^n - (-1)^n L - 1}{n\pi}$$

$$B_n = \frac{(-1)^n(1 - L) - 1}{n\pi}$$

Then the Fourier series representation of the piecewise function on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L+2}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n L - L}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n(1-L) - 1}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

Convergence of a Fourier series

We learned how to find the Fourier series representation of a function, but we haven't talked yet about whether or not the series actually converges to the function.

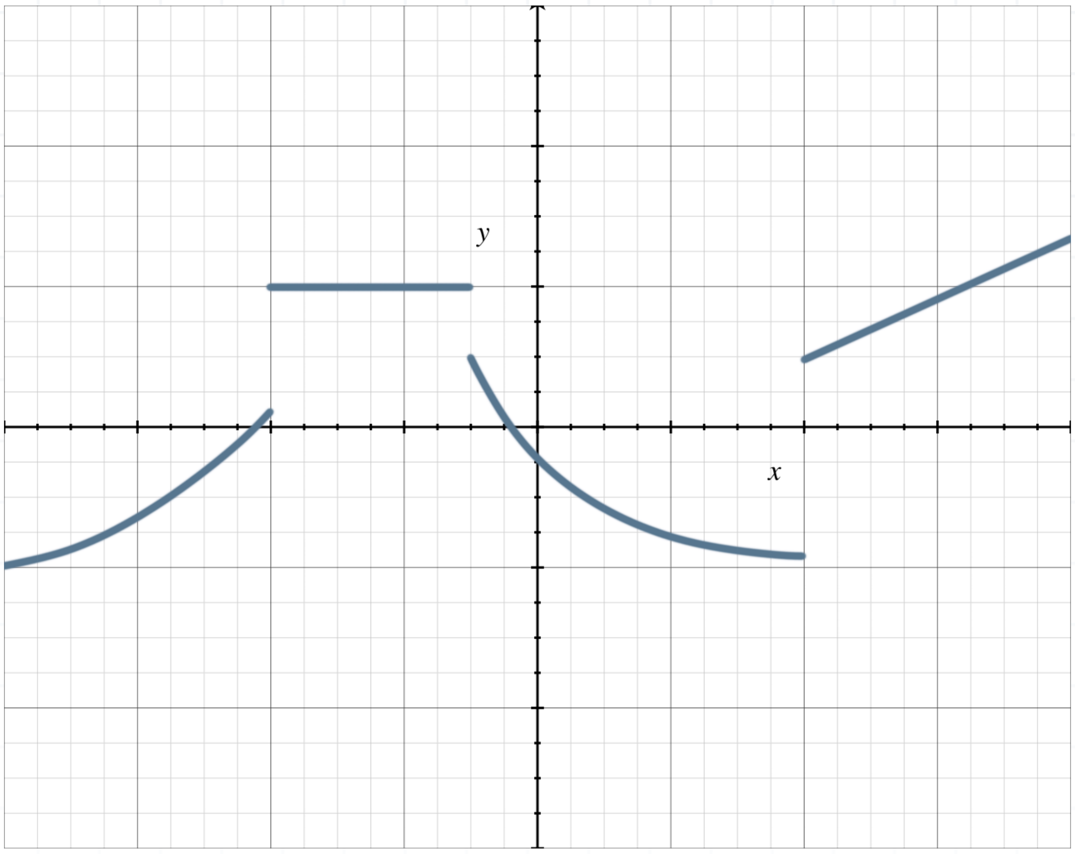
Remember, just like with the Taylor series we looked at before, or with any other series representation, the series doesn't necessarily converge. In the case of the Fourier series representation, we're looking for the series to converge to the function $f(x)$ that we're claiming it represents.

Luckily for us, there are certain conditions under which we know the Fourier series representation will, in fact, converge to $f(x)$.

Conditions for convergence

In order for the Fourier series representation of $f(x)$ to converge on some symmetric interval $-L \leq x \leq L$, the function $f(x)$ itself must be **piecewise smooth** on that interval, which means that the function and its derivative are continuous, and only a finite number of jump discontinuities are allowed. Remember, the function has a **jump discontinuity** wherever both one-sided limits exist, but those one-sided limits aren't equivalent.

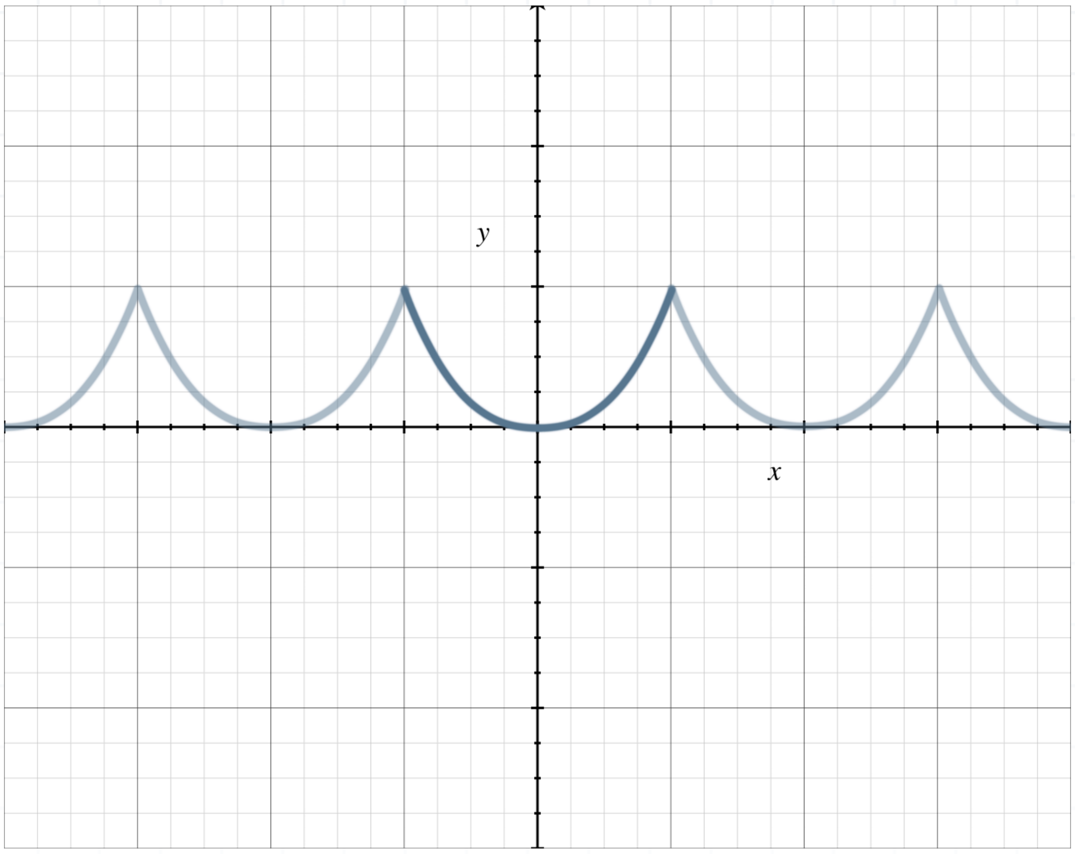
In other words, we can't have the function $f(x)$ blow up to infinity at any point in $-L \leq x \leq L$, nor can we have some infinite number of discontinuities over $-L \leq x \leq L$. Below is an example of a piecewise smooth curve. There are a finite number of jump discontinuities, and the function is continuous in between those discontinuities.



If the function is piecewise smooth, then we can take its graph on $-L \leq x \leq L$ and just repeat it over and over again to the left and right, so on $x \leq -L$ and on $x \geq L$, and the new graph that we get is the periodic extension of the original function, which we discussed in an earlier lesson. As we know, the periodic extension is a periodic function, so we can use a Fourier series to represent the periodic extension.

Then the question becomes, will the Fourier series representation of $f(x)$ converge to the periodic extension of $f(x)$? Well, if the periodic extension of $f(x)$ is continuous, then yes, the Fourier series representation will converge to $f(x)$.

For instance, since this periodic extension $g(x)$ is continuous, its Fourier series representation will converge to $g(x)$.



On the other hand, if the periodic extension isn't continuous, and instead has a jump discontinuity between each period, then the Fourier series representation of the periodic extension $g(x)$ will converge to the average of the two one-sided limits. So if the jump discontinuity exists at $x = a$, then the Fourier series representation will converge to

$$\frac{\lim_{x\to a^-} g(x) + \lim_{x\to a^+} g(x)}{2}$$

So to summarize, if we want the Fourier series representation to converge to $f(x)$, then we need to have no jump discontinuities within the interval $-L \leq x \leq L$, and we need to make sure that $f(-L) = f(L)$.

Let's do an example where we determine the convergence of a Fourier series representation.

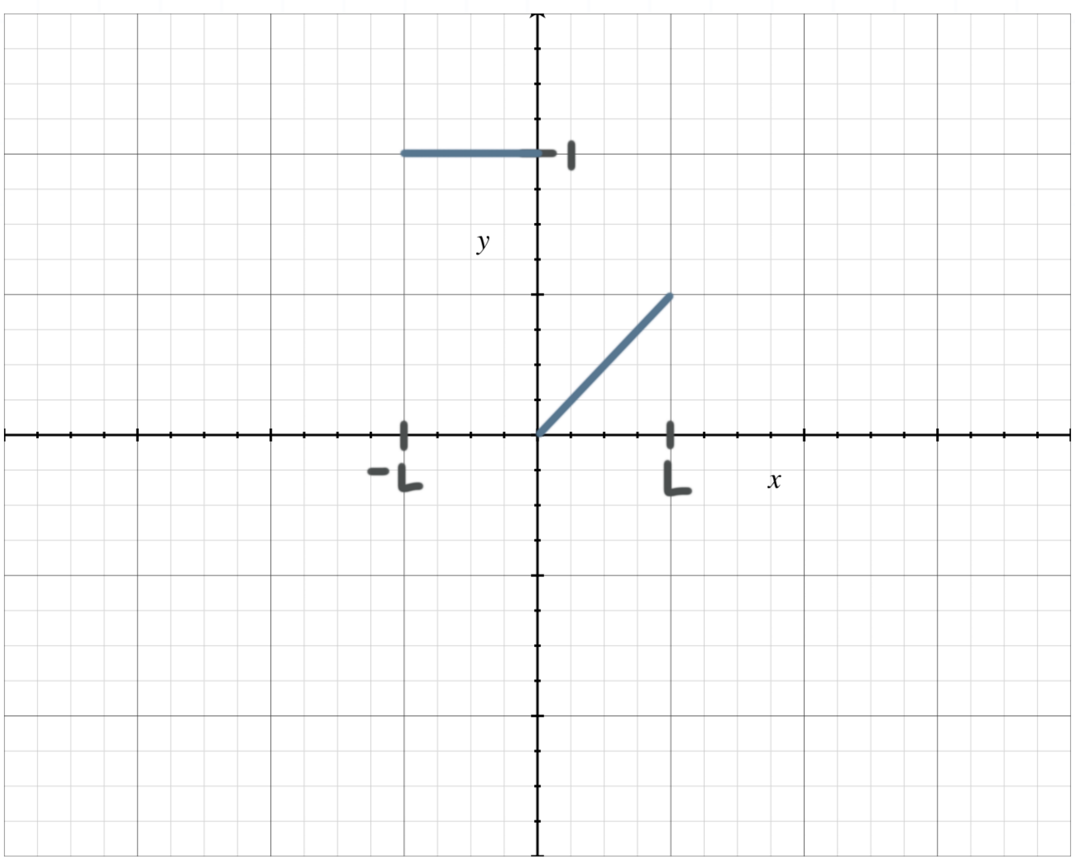
Example

Given the function $f(x)$ and its Fourier series representation, say whether or not the Fourier series converges to $f(x)$.

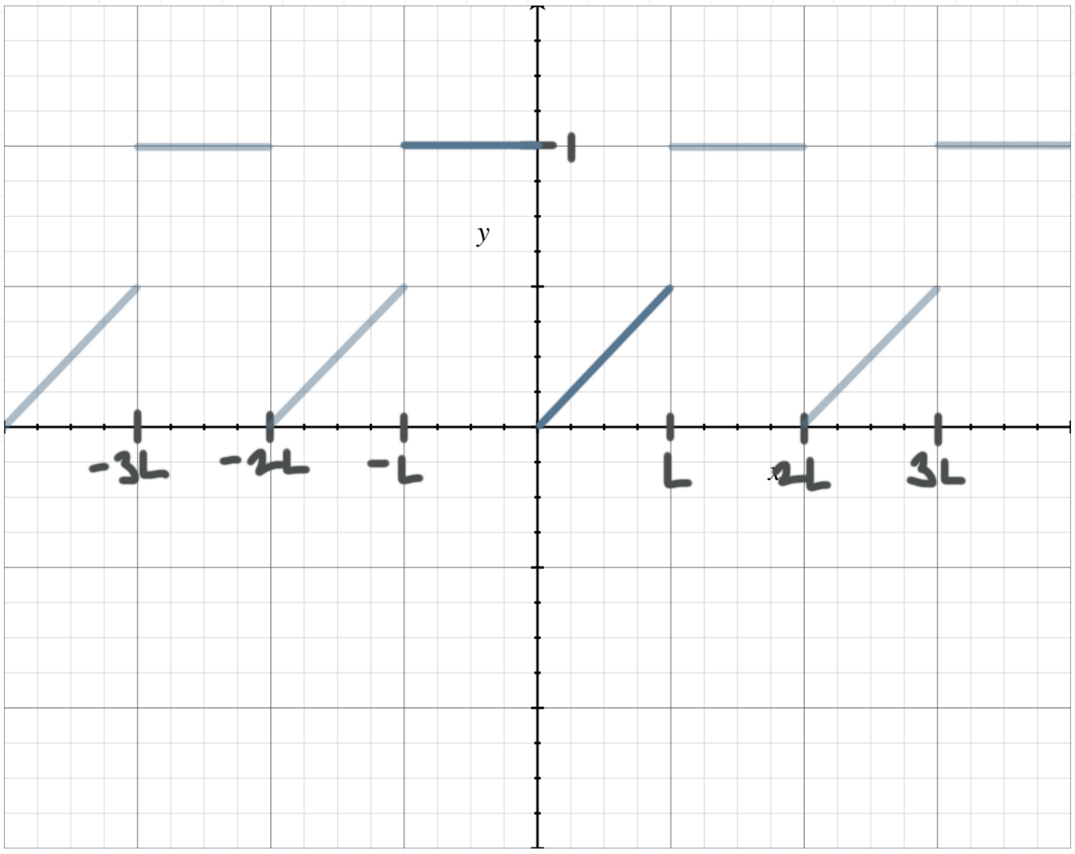
$$f(x) = \begin{cases} 1 & -L \leq x < 0 \\ x & 0 \leq x \leq L \end{cases}$$

$$f(x) = \frac{L+2}{4} + \sum_{n=1}^{\infty} \frac{(-1)^n L - L}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{(-1)^n(1-L) - 1}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

This is the function we worked with in the previous lesson, where we learned to find the Fourier series representation of a piecewise defined function. A sketch of $f(x)$ is



and therefore a sketch of its periodic extension is



From the graph of the periodic extension $g(x)$, we can see that $g(x)$ has a jump discontinuity at $x = 0$, within the interval $-L \leq x \leq L$, as well as a jump discontinuity at $-L$ and L, assuming $L \neq 1$.

Because the function is piecewise smooth on the individual intervals $-L \leq x < 0$ and $0 \leq x \leq L$, the function $f(x)$ and the periodic extension $g(x)$ are both continuous, so the Fourier series representation will converge to the periodic extension $g(x)$, and therefore also to the original function $f(x)$. Because of the jump discontinuity at $x = 0$, the Fourier series representation at that point will converge to

$$\frac{\lim_{x \to a^-} g(x) + \lim_{x \to a^+} g(x)}{2}$$

$$\frac{\lim_{x \to 0^-} g(x) + \lim_{x \to 0^+} g(x)}{2}$$

$$\frac{1 + 0}{2}$$

$$\frac{1}{2}$$

Again assuming $L \neq 1$, the periodic extension will also have jump discontinuities at $x = -L$ and $x = L$. At $x = -L$, the Fourier series representation will converge to

$$\frac{\lim_{x\to a^-} g(x) + \lim_{x\to a^+} g(x)}{2}$$

$$\frac{\lim_{x\to -L^-} g(x) + \lim_{x\to -L^+} g(x)}{2}$$

$$\frac{L+1}{2}$$

$$\frac{1+L}{2}$$

and at $x = L$ the Fourier series representation will converge to

$$\frac{\lim_{x\to a^-} g(x) + \lim_{x\to a^+} g(x)}{2}$$

$$\frac{\lim_{x\to L^-} g(x) + \lim_{x\to L^+} g(x)}{2}$$

$$\frac{L+1}{2}$$

$$\frac{1+L}{2}$$

Fourier cosine series

We reviewed even and odd functions when we first introduced Fourier series, but remember again that an **even function** $f(x)$ is symmetric about the y-axis and satisfies $f(-x) = f(x)$, while an **odd function** $f(x)$ is symmetric about the origin and satisfies $f(-x) = -f(x)$.

The cosine series

If we try to find the Fourier series representation,

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

of an even function on $-L \le x \le L$, we'll discover that the coefficients B_n are 0, such that the Fourier series of the even function will simplify to just

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad\qquad n = 1, 2, 3, \dots$$

Because this is a Fourier series in terms of only cosine functions, we call it a **Fourier cosine series**. What this means is that, when we want to find the Fourier series representation of an even function, we actually only need to find its Fourier cosine series. If we know the function is even, we don't have to bother computing B_n or trying to find the sine part of the Fourier series representation.

Let's do an example with an even function to prove that this is true.

Example

Find the Fourier series and Fourier cosine series of $f(x) = x^2$ on $-L \le x \le L$, and show that they are equivalent.

Let's use the original definition of a Fourier series, which means we'll start by calculating A_0, A_n, and B_n. For A_0, we get

$$A_0 = \frac{1}{2L}\int_{-L}^L f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^L x^2\ dx$$

$$A_0 = \frac{1}{6L}x^3 \Big|_{-L}^{L}$$

$$A_0 = \frac{1}{6L}L^3 - \frac{1}{6L}(-L)^3$$

$$A_0 = \frac{1}{6}L^2 + \frac{1}{6}L^2$$

$$A_0 = \frac{L^2}{3}$$

For A_n, we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \left[\frac{x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{2}{n\pi}\int x\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Big|_{-L}^{L}$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = \left[\frac{x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2}\left[x\cos\left(\frac{n\pi x}{L}\right) - \int\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\right]\Big|_{-L}^{L}$$

$$A_n = \left[\frac{x^2}{n\pi} \sin\left(\frac{n\pi x}{L}\right) + \frac{2Lx}{(n\pi)^2} \cos\left(\frac{n\pi x}{L}\right) - \frac{2L^2}{(n\pi)^3} \sin\left(\frac{n\pi x}{L}\right) \right] \Bigg|_{-L}^{L}$$

$$A_n = \frac{L^2}{n\pi} \sin(n\pi) + \frac{2L^2}{(n\pi)^2} \cos(n\pi) - \frac{2L^2}{(n\pi)^3} \sin(n\pi)$$

$$-\frac{L^2}{n\pi} \sin(-n\pi) + \frac{2L^2}{(n\pi)^2} \cos(-n\pi) + \frac{2L^2}{(n\pi)^3} \sin(-n\pi)$$

$$A_n = \frac{L^2}{n\pi} \sin(n\pi) + \frac{2L^2}{(n\pi)^2} \cos(n\pi) - \frac{2L^2}{(n\pi)^3} \sin(n\pi)$$

$$+\frac{L^2}{n\pi} \sin(n\pi) + \frac{2L^2}{(n\pi)^2} \cos(n\pi) - \frac{2L^2}{(n\pi)^3} \sin(n\pi)$$

$$A_n = \frac{2L^2}{n\pi} \sin(n\pi) + \frac{4L^2}{(n\pi)^2} \cos(n\pi) - \frac{4L^2}{(n\pi)^3} \sin(n\pi)$$

For $n = 1, 2, 3, \ldots, \sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for A_n simplifies to

$$A_n = \frac{4L^2}{(n\pi)^2}(-1)^n$$

And for B_n, we get

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) \ dx$$

$$B_n = \frac{1}{L} \int_{-L}^{L} x^2 \sin\left(\frac{n\pi x}{L}\right) \ dx$$

The function $f(x) = x^2$ is even, and the sine function is odd, and the product of an even and odd function is an odd function. And since we're integrating over a symmetric interval, the value of the integral has to be 0, so

$$B_n = \frac{1}{L}(0)$$

$$B_n = 0$$

Then the Fourier series representation of $f(x) = x^2$ on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^2}(-1)^n \cos\left(\frac{n\pi x}{L}\right) + 0$$

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos\left(\frac{n\pi x}{L}\right)$$

Now let's use the Fourier cosine series definition, which means we'll start by calculating A_0 and A_n. For A_0 we get

$$A_0 = \frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L}\int_0^L x^2\ dx$$

$$A_0 = \frac{1}{3L}x^3\bigg|_0^L$$

$$A_0 = \frac{1}{3L}L^3 - \frac{1}{3L}(0)^3$$

$$A_0 = \frac{L^2}{3}$$

And for A_n we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L x^2\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{L}\left[\frac{Lx^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{2L}{n\pi}\int x\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$A_n = \frac{2}{L}\left[\frac{Lx^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{2L}{n\pi}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\right]\Bigg|_0^L$$

$$A_n = \frac{2}{L}\left[\frac{Lx^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right) - \frac{2L^3}{(n\pi)^3}\sin\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$A_n = \frac{2x^2}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{4Lx}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right) - \frac{4L^2}{(n\pi)^3}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$A_n = \frac{2L^2}{n\pi}\sin(n\pi) + \frac{4L^2}{(n\pi)^2}\cos(n\pi) - \frac{4L^2}{(n\pi)^3}\sin(n\pi)$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for A_n simplifies to

$$A_n = \frac{4L^2}{(n\pi)^2}\cos(n\pi)$$

$$A_n = \frac{4L^2}{(n\pi)^2}(-1)^n$$

Then the Fourier cosine series of $f(x) = x^2$ on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^2}{3} + \sum_{n=1}^{\infty} \frac{4L^2}{(n\pi)^2}(-1)^n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

Because $f(x) = x^2$ is an even function on $-L \le x \le L$, we get equivalent representations of $f(x)$ for both its Fourier series and its Fourier cosine series.

Functions that aren't even

While it's easy to model even functions with a Fourier cosine series, we can also use the cosine series to model functions that aren't even.

We start with the idea that, for any function defined on $0 \leq x \leq L$, we can find its even extension on $-L \leq x \leq 0$. We talked about the even and odd extensions of a function when we looked at periodic extensions earlier. But as a reminder, given $f(x)$ defined on $0 \leq x \leq L$, we create its reflection over the y-axis on $-L \leq x \leq 0$, thereby creating a new function $g(x)$,

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x < 0 \end{cases}$$

which is itself an even function defined on $-L \leq x \leq L$.

Because $g(x)$ is even, we know we can find the Fourier cosine series representation of $g(x)$ (or its Fourier series representation, in general) on $-L \leq x \leq L$.

But then, because $g(x)$ and $f(x)$ are equivalent on $0 \leq x \leq L$, we know that the Fourier cosine series of $g(x)$ (or the Fourier series of $g(x)$), will be equivalent to the Fourier cosine series of $f(x)$ (or the Fourier series of $f(x)$), as long as we restrict the interval of definition to $0 \leq x \leq L$, instead of $-L \leq x \leq L$.

So let's do another example, this time where we find the Fourier cosine series of a function that isn't even.

Example

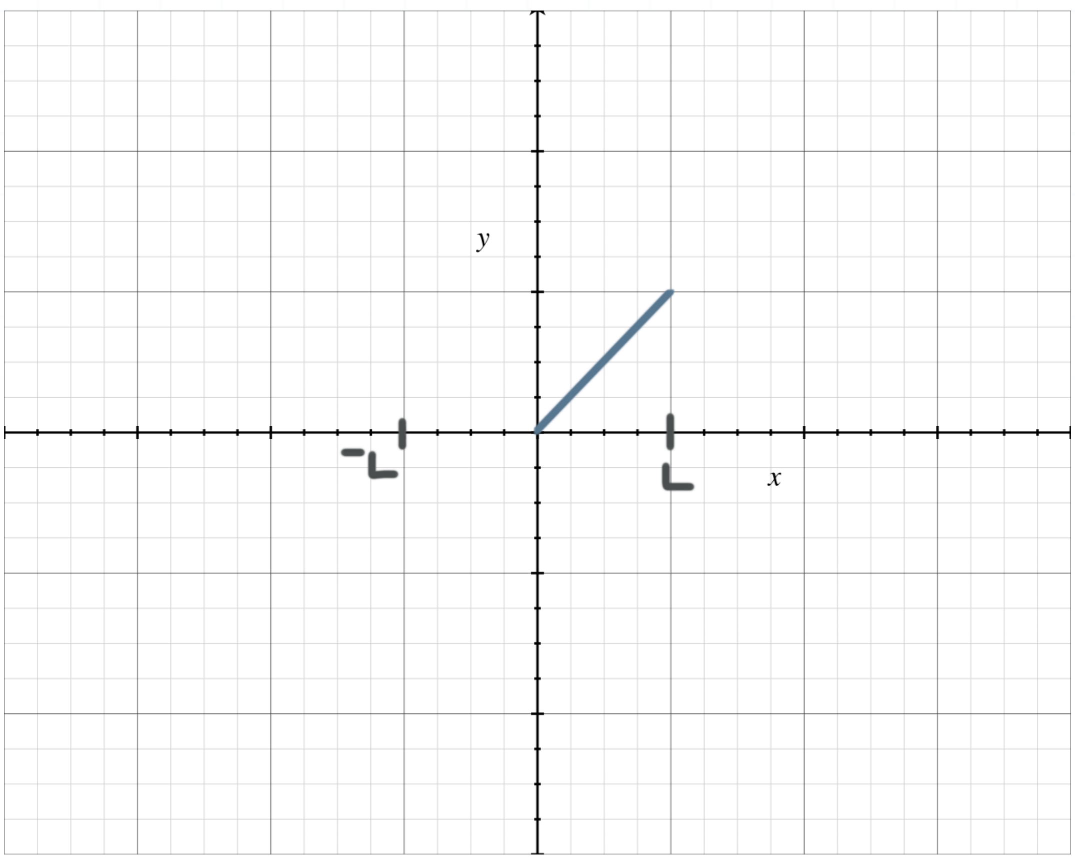
Find the Fourier cosine series of $f(x) = x$ on $0 \leq x \leq L$.

This function is odd, so we'll start by finding the even extension of the odd function.

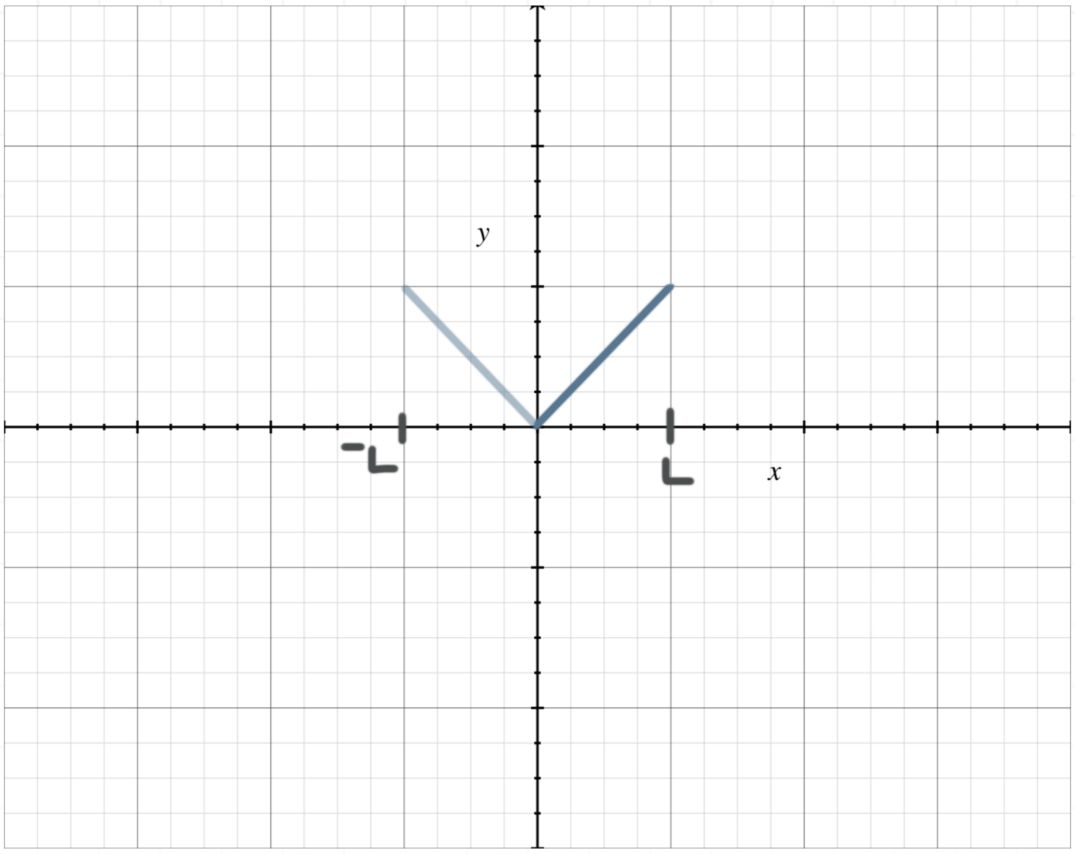
$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} x & 0 \leq x \leq L \\ -x & -L \leq x < 0 \end{cases}$$

A sketch of the odd function $f(x)$ on $0 \leq x \leq L$ is



and the sketch of its even extension $g(x)$ on $-L \leq x \leq L$ is



Because the even extension is an even function, we can find its Fourier cosine series representation, starting with calculating A_0 and A_n. For A_0 we get

$$A_0 = \frac{1}{L} \int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L} \int_0^L x\ dx$$

$$A_0 = \frac{1}{2L} x^2 \Big|_0^L$$

$$A_0 = \frac{1}{2L}(L)^2 - \frac{1}{2L}(0)^2$$

$$A_0 = \frac{L}{2}$$

And for A_n we get

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{2}{L}\int_0^L x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{2}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \frac{L}{n\pi}\int\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$A_n = \frac{2}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L^2}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$A_n = \frac{2x}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$A_n = \frac{2L}{n\pi}\sin(n\pi) + \frac{2L}{(n\pi)^2}\cos(n\pi) - \frac{2L}{(n\pi)^2}$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for A_n simplifies to

$$A_n = \frac{2L}{n\pi}(0) + \frac{2L}{(n\pi)^2}(-1)^n - \frac{2L}{(n\pi)^2}$$

$$A_n = \frac{2L((-1)^n - 1)}{(n\pi)^2}$$

Then the Fourier cosine series of $f(x)=x$ on $0 \le x \le L$ is

$$f(x)=A_0+\sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\frac{L}{2}+\sum_{n=1}^{\infty} \frac{2L((-1)^n-1)}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\frac{L}{2}+\frac{2L}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n-1}{n^2}\cos\left(\frac{n\pi x}{L}\right)$$

Even though the function $f(x)=x$ isn't even, we were still able to find a Fourier cosine series to represent it on $0 \le x \le L$.

Convergence of the cosine series

We know from earlier that a Fourier series will converge to the function $f(x)$ that it represents as long as $f(x)$ is piecewise smooth and its periodic extension is continuous.

Otherwise the Fourier series will converge to the average of the one-sided limits if the periodic extension has a jump discontinuity between periods.

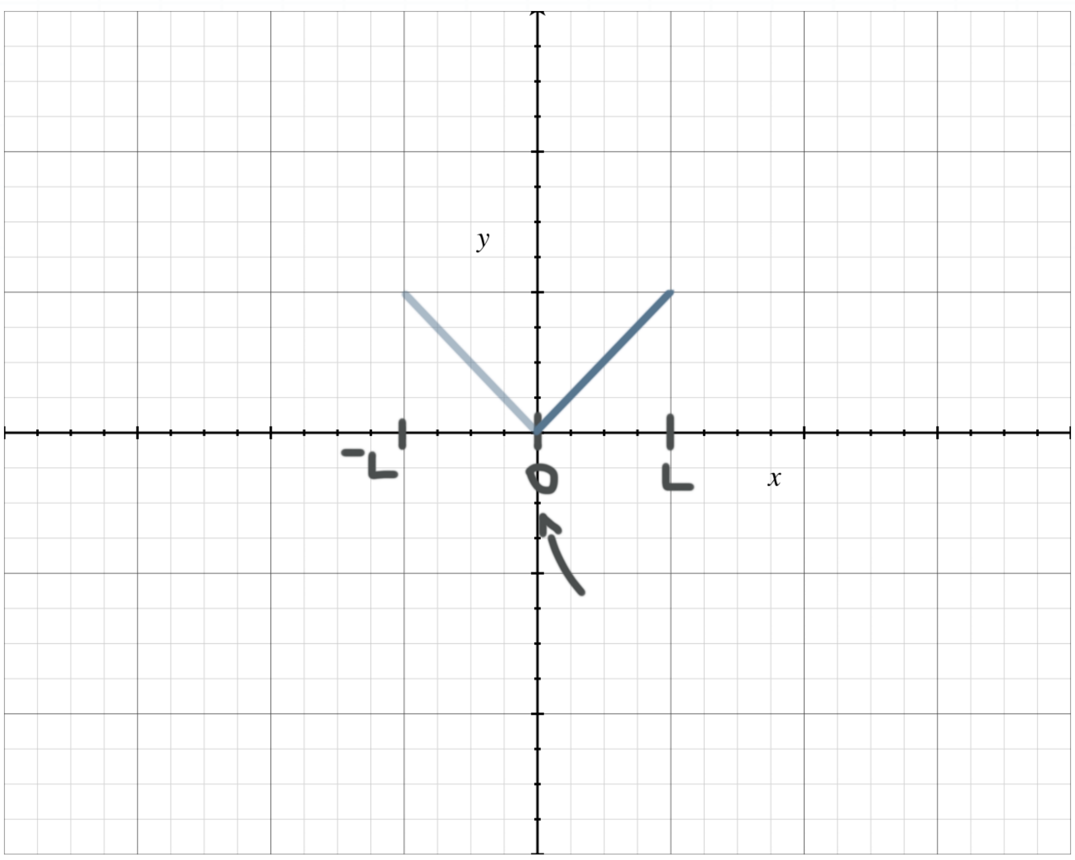
But now we want to specifically address the convergence of a Fourier cosine series.

If we're looking at the Fourier cosine series of an even function on $-L \le x \le L$, then it'll converge to $f(x)$ just like a normal Fourier series. Of course, that's because the Fourier cosine series of an even function is just the special case of the general Fourier series, where $B_n = 0$ and the sine series part of the Fourier series disappears.

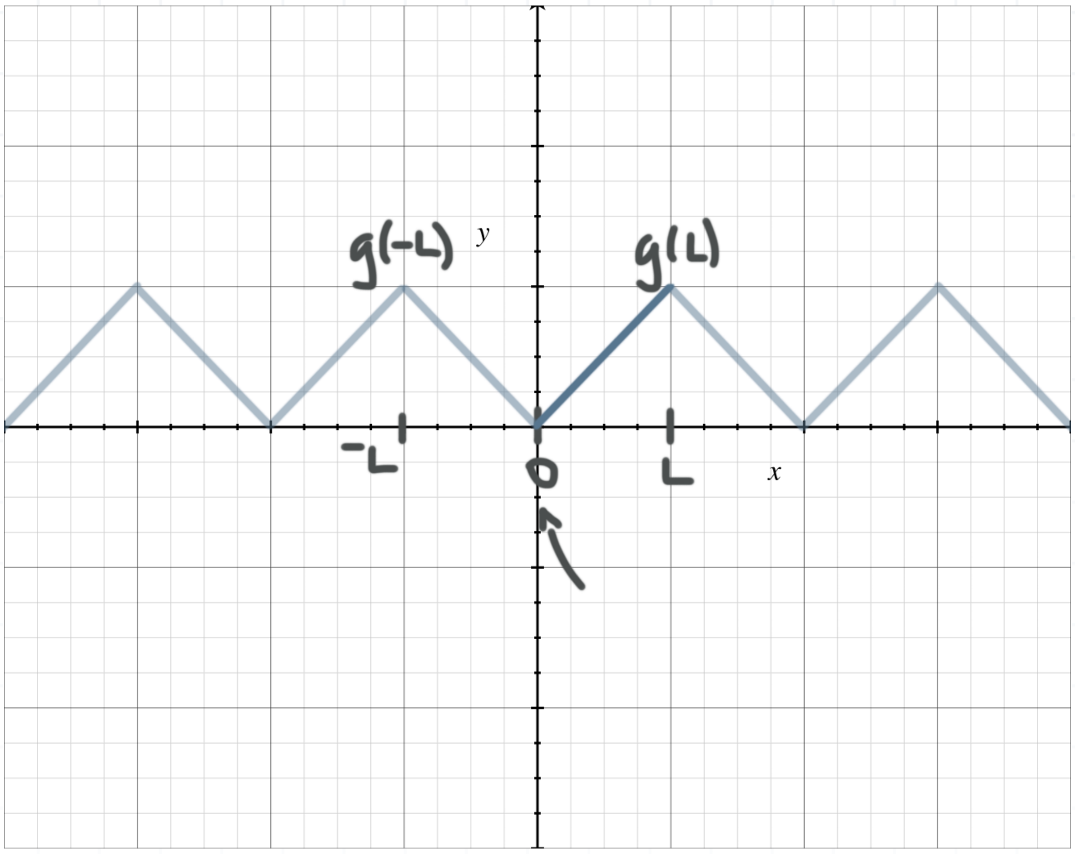
But what about the convergence of the Fourier cosine series of a function that isn't even?

Well remember, if we take a function defined on $0 \le x \le L$ and build its even extension, that even extension is now an even function $g(x)$ on $-L \le x \le L$, and we can therefore find the Fourier cosine series of that even $g(x)$ function on $-L \le x \le L$.

The original function and its even extension meet each other at $x = 0$,



and if we build the periodic extension of that even extension,



then we see that the periodic extension is continuous between periods, since $g(-L) = g(L)$.

Which means, based on what we already know about the convergence of a Fourier series, that the Fourier series representation of $g(x)$ will converge to $g(x)$ on $-L \leq x \leq L$. The Fourier cosine series, as a special case of the Fourier series, will also converge to $g(x)$ on $-L \leq x \leq L$.

And because the even extension $g(x)$ is equivalent to the original function $f(x)$ on $0 \leq x \leq L$, we can conclude that the Fourier cosine series representation of $f(x)$ will converge to $f(x)$ on $0 \leq x \leq L$.

Therefore,

- the Fourier cosine series of an even function $f(x)$ will converge to $f(x)$ on $-L \leq x \leq L$

- the Fourier cosine series of a function $f(x)$ that isn't even will converge to $f(x)$ on $0 \leq x \leq L$

The only exception is if the function $f(x)$ contains a jump discontinuity somewhere within $0 \leq x \leq L$. But just like with the general Fourier series, the Fourier cosine series will converge to the average of the one-sided limits at that jump discontinuity.

Fourier sine series

Now that we've looked at the Fourier cosine series, the Fourier sine series will just be an odd extension of all those same ideas.

We just need to start with the understanding that, while the cosine function is even (it's symmetric about the y-axis and satisfies $f(-x)=f(x)$), the sine function is odd (it's symmetric about the origin and satisfies $f(-x)=-f(x)$).

The sine series

If we try to find the Fourier series representation,

$$f(x)=A_0+\sum_{n=1}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)+\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$A_0=\frac{1}{2L}\int_{-L}^{L}f(x)\ dx$$

$$A_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n=1,\ 2,\ 3,\ \ldots$$

$$B_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n=1,\ 2,\ 3,\ \ldots$$

of an odd function on $-L\le x\le L$, we'll discover that the coefficients A_0 and A_n are 0, such that the Fourier series of the odd function will simplify to just

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

Because this is a Fourier series in terms of only sine functions, we call it a **Fourier sine series**. What this means is that, when we want to find the Fourier series representation of an odd function, we actually only need to find its Fourier sine series. If we know the function is odd, we don't have to bother computing A_0 or A_n or trying to find the cosine parts of the Fourier series representation.

Let's do an example with an odd function to prove that this is true.

Example

Find the Fourier series and Fourier sine series of $f(x) = x$ on $-L \le x \le L$, and show that they are equivalent.

Let's use the original definition of a Fourier series, which means we'll start by calculating A_0, A_n, and B_n. For A_0 we get

$$A_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\ dx$$

$$A_0 = \frac{1}{2L}\int_{-L}^{L} x\ dx$$

This is the integral of an odd function on a symmetric interval, which means the value of the integral must be 0.

$$A_0 = \frac{1}{2L}(0)$$

$$A_0 = 0$$

For A_n we get

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_{-L}^{L} x\cos\left(\frac{n\pi x}{L}\right)\ dx$$

The function $f(x) = x$ is odd, and the cosine function is even, and the product of an odd function and an even function is itself an odd function. Which means this is the integral of an odd function on a symmetric interval, which means the value of the integral must be 0.

$$A_n = \frac{1}{L}(0)$$

$$A_n = 0$$

And for B_n we get

$$B_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_{-L}^{L} x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{1}{L}\left[-x\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + \int\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_{-L}^{L}$$

$$B_n = \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_{-L}^{L}$$

$$B_n = \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi L}{L}\right) - \frac{L}{n\pi}\cos\left(\frac{n\pi L}{L}\right)$$

$$-\left[\frac{L}{(n\pi)^2}\sin\left(\frac{-n\pi L}{L}\right) + \frac{L}{n\pi}\cos\left(\frac{-n\pi L}{L}\right)\right]$$

$$B_n = \frac{L}{(n\pi)^2}\sin(n\pi) - \frac{L}{n\pi}\cos(n\pi) - \left[\frac{L}{(n\pi)^2}\sin(-n\pi) + \frac{L}{n\pi}\cos(-n\pi)\right]$$

$$B_n = \frac{L}{(n\pi)^2}\sin(n\pi) - \frac{L}{n\pi}\cos(n\pi) + \frac{L}{(n\pi)^2}\sin(n\pi) - \frac{L}{n\pi}\cos(n\pi)$$

For $n = 1, 2, 3, \dots$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = -\frac{2L}{n\pi}(-1)^n$$

$$B_n = \frac{(-1)^{n+1}2L}{n\pi}$$

Then the Fourier series representation of $f(x) = x$ on $-L \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = 0 + \sum_{n=1}^{\infty} (0)\cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right)$$

Now let's use the Fourier sine series definition, which means we'll start by calculating B_n.

$$B_n = \frac{2}{L} \int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L} \int_0^L x \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{2}{L}\left[-x\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + \int \left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) - \frac{2x}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$B_n = \frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi L}{L}\right) - \frac{2L}{n\pi}\cos\left(\frac{n\pi L}{L}\right)$$

$$-\left[\frac{2L}{(n\pi)^2}\sin\left(\frac{n\pi(0)}{L}\right)-\frac{2(0)}{n\pi}\cos\left(\frac{n\pi(0)}{L}\right)\right]$$

$$B_n=\frac{2L}{(n\pi)^2}\sin(n\pi)-\frac{2L}{n\pi}\cos(n\pi)$$

For $n=1, 2, 3, \dots$, $\sin(n\pi)=0$ and $\cos(n\pi)=(-1)^n$, so the expression for B_n simplifies to

$$B_n=-\frac{2L}{n\pi}(-1)^n$$

$$B_n=\frac{(-1)^{n+1}2L}{n\pi}$$

Then the Fourier sine series of $f(x)=x$ on $-L\leq x\leq L$ is

$$f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\sum_{n=1}^{\infty}\frac{(-1)^{n+1}2L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)$$

$$f(x)=\frac{2L}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)$$

Because $f(x)=x$ is an odd function on $-L\leq x\leq L$, we get equivalent representations of $f(x)$ for both its Fourier series and its Fourier sine series.

Functions that aren't odd

While it's easy to model odd functions with a Fourier sine series, we can also use the sine series to model functions that aren't odd.

We start with the idea that, for any function defined on $0 \leq x \leq L$, we can find its odd extension on $-L \leq x \leq 0$. We talked about the even and odd extensions of a function when we looked at periodic extensions earlier. But as a reminder, given $f(x)$ defined on $0 \leq x \leq L$, we create its reflection about the origin on $-L \leq x \leq 0$, thereby creating a new function $g(x)$,

$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ -f(-x) & -L \leq x < 0 \end{cases}$$

which is itself an odd function defined on $-L \leq x \leq L$.

Because $g(x)$ is odd, we know we can find the Fourier sine series representation of $g(x)$ (or its Fourier series representation, in general) on $-L \leq x \leq L$.

But then, because $g(x)$ and $f(x)$ are equivalent on $0 \leq x \leq L$, we know that the Fourier sine series of $g(x)$ (or the Fourier series of $g(x)$), will be equivalent to the Fourier sine series of $f(x)$ (or the Fourier series of $f(x)$), as long as we restrict the interval of definition to $0 \leq x \leq L$, instead of $-L \leq x \leq L$.

So let's do another example, this time where we find the Fourier sine series of a function that isn't odd.

Example

Find the Fourier sine series of $f(x) = x^2$ on $0 \le x \le L$.

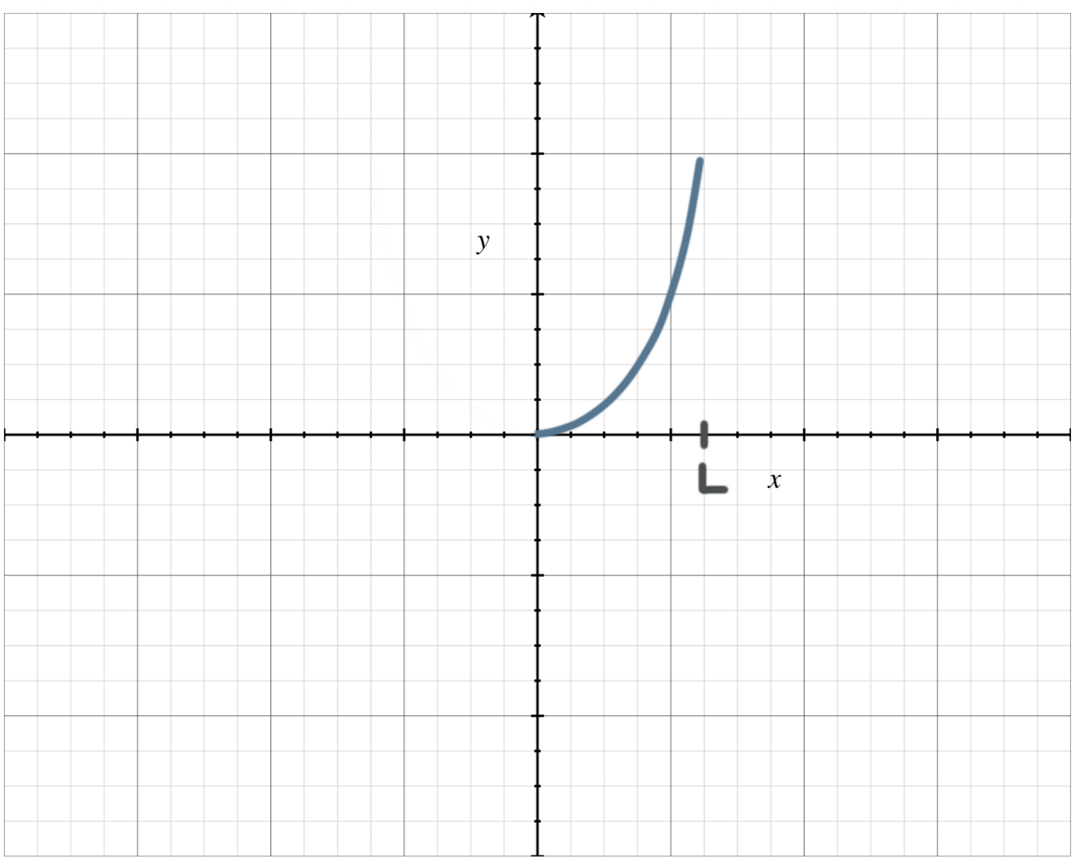
This function is even, so we'll start by finding the odd extension of the even function.

$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

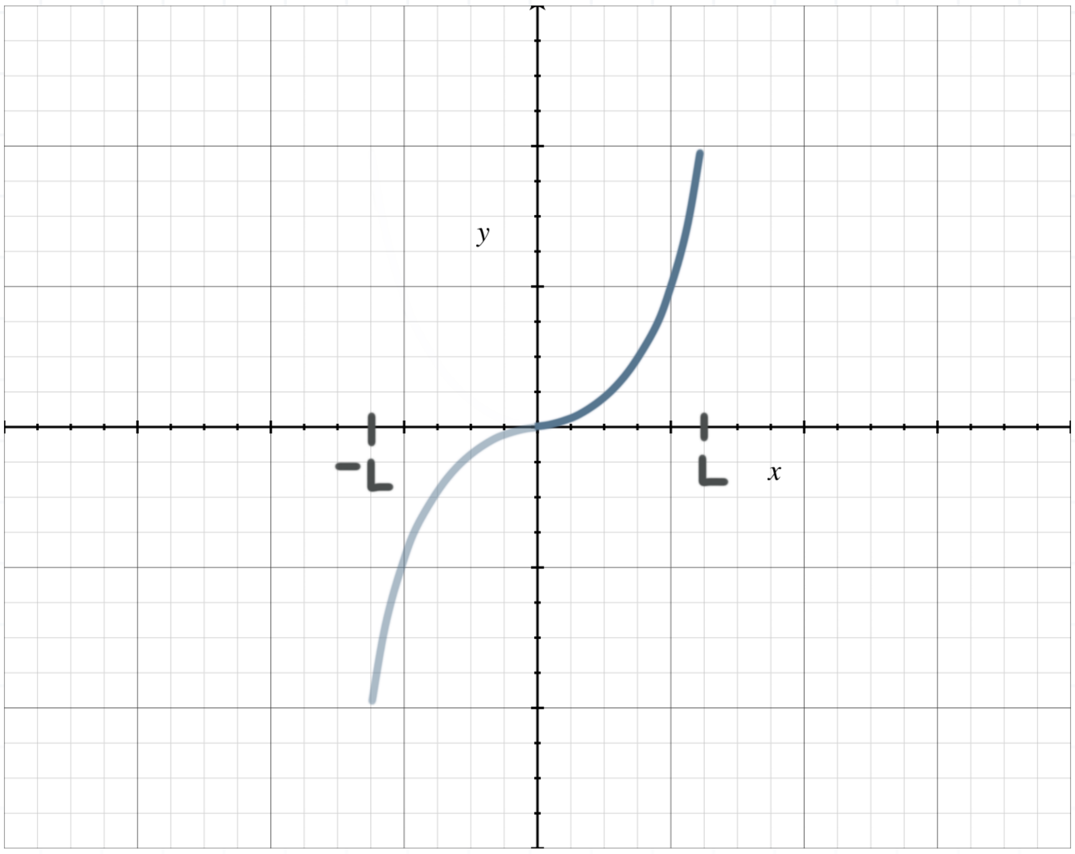
$$g(x) = \begin{cases} x^2 & 0 \le x \le L \\ -((-x)^2) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x^2 & 0 \le x \le L \\ -x^2 & -L \le x < 0 \end{cases}$$

A sketch of the even function $f(x)$ on $0 \le x \le L$ is



and the sketch of its odd extension $g(x)$ on $-L \le x \le L$ is



Because the odd extension is an odd function, we can find its Fourier sine series representation, starting with calculating B_n.

$$B_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^L x^2\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x^2$, $du = 2x\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = \frac{1}{L}\left[-x^2\left(\frac{L}{n\pi}\right)\cos\left(\frac{n\pi x}{L}\right) + \int \frac{2Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \frac{1}{L}\left[-\frac{Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L}{n\pi}\int x\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$B_n = \frac{1}{L}\Bigg[-\frac{Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)$$

$$+\frac{2L}{n\pi}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \int \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg]\Bigg|_0^L$$

$$B_n = \frac{1}{L}\left[-\frac{Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) - \frac{2L^2}{(n\pi)^2}\int \sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_0^L$$

$$B_n = \frac{1}{L}\left[-\frac{Lx^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{2L^2x}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^3}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right)\right]\Bigg|_0^L$$

$$B_n = \frac{2Lx}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) + \frac{2L^2}{(n\pi)^3}\cos\left(\frac{n\pi x}{L}\right) - \frac{x^2}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

Evaluate over the interval.

$$B_n = \frac{2L^2}{(n\pi)^2}\sin\left(\frac{n\pi L}{L}\right) + \frac{2L^2}{(n\pi)^3}\cos\left(\frac{n\pi L}{L}\right)$$

$$-\frac{L^2}{n\pi}\cos\left(\frac{n\pi L}{L}\right) - \frac{2L^2}{(n\pi)^3}\cos\left(\frac{n\pi(0)}{L}\right)$$

$$B_n = \frac{2L^2}{(n\pi)^2}\sin(n\pi) + \frac{2L^2}{(n\pi)^3}\cos(n\pi) - \frac{L^2}{n\pi}\cos(n\pi) - \frac{2L^2}{(n\pi)^3}$$

For $n = 1, 2, 3, \dots$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = \frac{2L^2}{(n\pi)^3}(-1)^n - \frac{L^2}{n\pi}(-1)^n - \frac{2L^2}{(n\pi)^3}$$

$$B_n = \frac{(-1)^n 2L^2 + (-1)^{n+1}L^2(n\pi)^2 - 2L^2}{(n\pi)^3}$$

Then the Fourier sine series of $f(x) = x^2$ on $0 \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2L^2 + (-1)^{n+1}L^2(n\pi)^2 - 2L^2}{(n\pi)^3} \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{2(-1)^n + (-1)^{n+1}(n\pi)^2 - 2}{n^3} \sin\left(\frac{n\pi x}{L}\right)$$

Even though the function $f(x) = x^2$ isn't odd, we were still able to find a Fourier sine series to represent it on $0 \le x \le L$.

Convergence of the sine series

We know from earlier that a Fourier series will converge to the function $f(x)$ that it represents as long as $f(x)$ is piecewise smooth and its periodic extension is continuous.

Otherwise the Fourier series will converge to the average of the one-sided limits if the periodic extension has a jump discontinuity between periods.

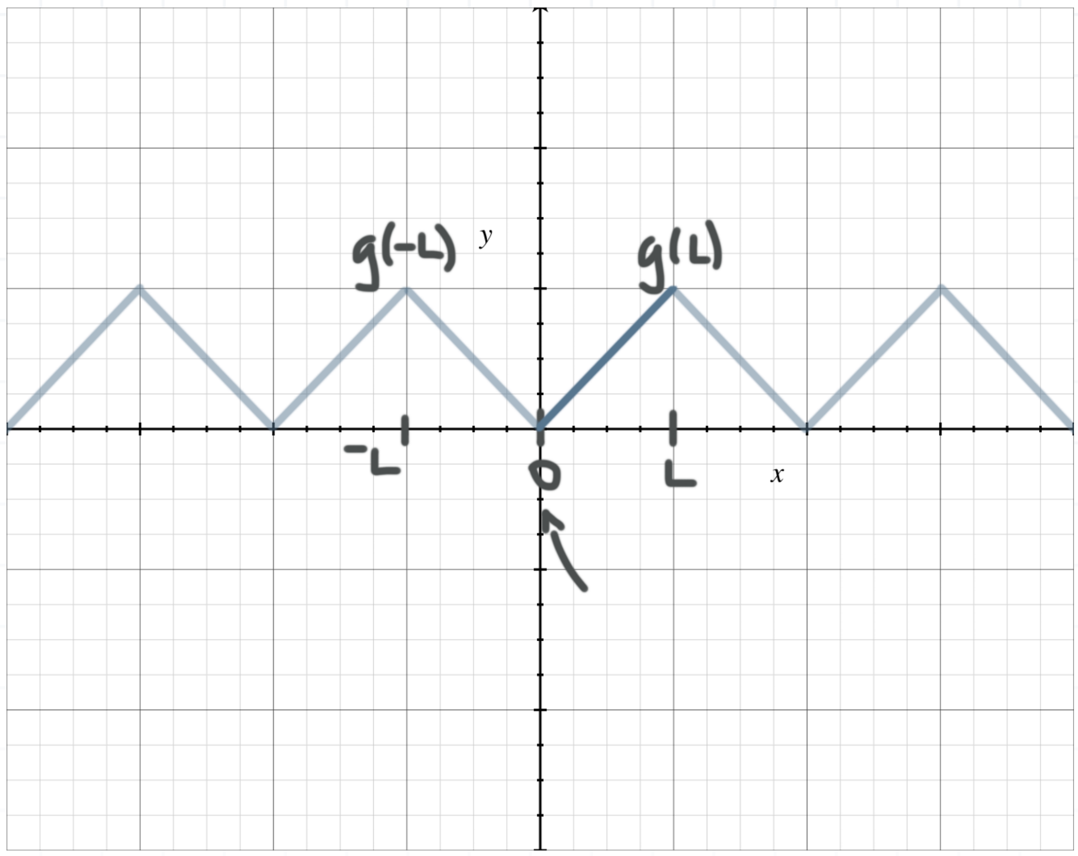
But now we want to specifically address the convergence of a Fourier sine series.

If we're looking at the Fourier sine series of an odd function on $-L \leq x \leq L$, then it'll converge to $f(x)$ just like a normal Fourier series. Of course, that's because the Fourier sine series of an odd function is just the special case of the general Fourier series, where $A_0 = 0$ and $A_n = 0$ and the cosine series parts of the Fourier series disappear.

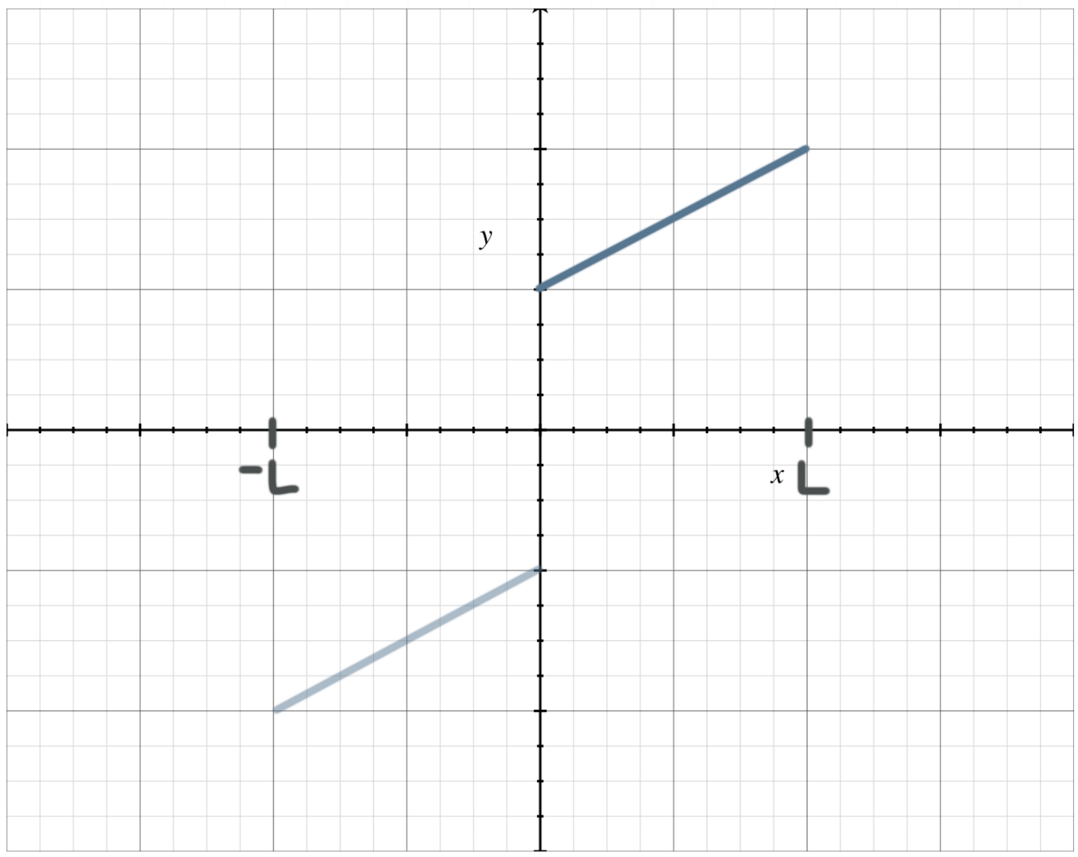
But what about the convergence of the Fourier sine series of a function that isn't odd?

Well remember, if we take a function defined on $0 \leq x \leq L$ and build its odd extension, that odd extension is now an odd function $g(x)$ on $-L \leq x \leq L$, and we can therefore find the Fourier sine series of that odd $g(x)$ function on $-L \leq x \leq L$.

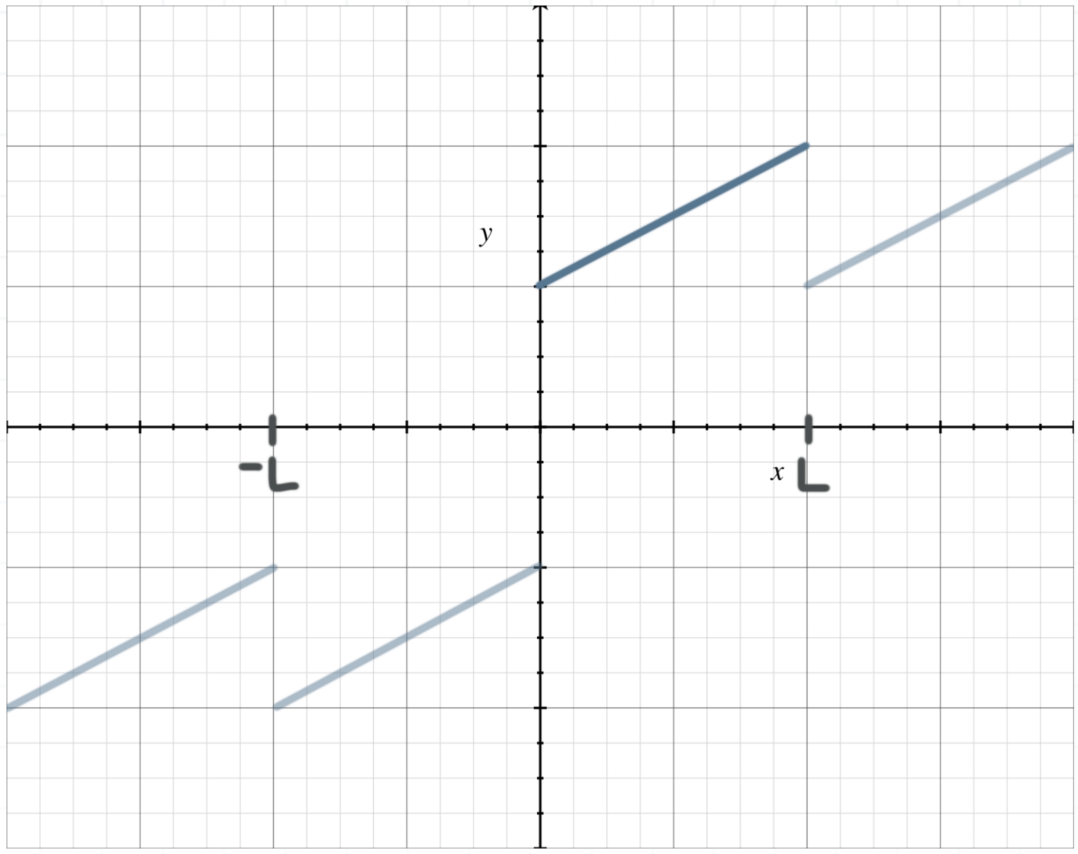
We saw how even extensions were always continuous (assuming no discontinuity within the interval $0 \leq x \leq L$), because their symmetry means that the two one-sided limits at $x = 0$ are always equivalent, and that $g(-L) = g(L)$, which means the one-sided limits between periods are equivalent.



But this isn’t necessarily the case with odd functions. For instance, here’s the sketch of an odd extension with a jump discontinuity at $x = 0$,



and here’s the sketch of a periodic extension of with a jump discontinuity between each period.



In fact, the only way the periodic extension of the odd function will be continuous is when both $f(0) = 0$ and $f(L) = 0$. If these two requirements are met, then the periodic extension is continuous.

Which means, based on what we already know about the convergence of a Fourier series, that the Fourier series representation of $g(x)$ will converge to $g(x)$ on $-L \leq x \leq L$. The Fourier sine series, as a special case of the Fourier series, will also converge to $g(x)$ on $-L \leq x \leq L$.

And because the odd extension $g(x)$ is equivalent to the original function $f(x)$ on $0 \leq x \leq L$, we can conclude that the Fourier sine series representation of $f(x)$ will converge to $f(x)$ on $0 \leq x \leq L$.

Therefore,

- the Fourier sine series of an odd function $f(x)$ will converge to $f(x)$ on $-L \leq x \leq L$.

- the Fourier sine series of a function $f(x)$ that isn't odd will converge to $f(x)$ on $0 \leq x \leq L$, provided $f(x)$ is continuous on $0 \leq x \leq L$, and that $f(0) = 0$ and $f(L) = 0$.

The only exception is if the function $f(x)$ contains a jump discontinuity somewhere within $0 \leq x \leq L$. But just like with the general Fourier series, the Fourier sine series will converge to the average of the one-sided limits at that jump discontinuity.

Cosine and sine series of piecewise functions

Before we move on from Fourier cosine series and Fourier sine series, we want to make sure we know how to find these series representations for piecewise defined functions.

If our goal is to find the Fourier cosine series of a piecewise function $f(x)$, then we'll be finding the cosine series of its even extension, $g(x)$.

On the other hand, if our goal is to find the Fourier sine series of a piecewise function $f(x)$, then we'll be finding the sine series of its odd extension, $g(x)$.

Because the extensions are periodic, we can find their Fourier series, and because the extensions are equivalent to the original functions on $0 \leq x \leq L$, we know that the Fourier cosine series representation or Fourier sine series representation will converge to $f(x)$ on $0 \leq x \leq L$.

Let's do an example where we find the Fourier cosine series of a piecewise defined function.

Example

Find the Fourier cosine series of the piecewise function on $0 \leq x \leq L$.

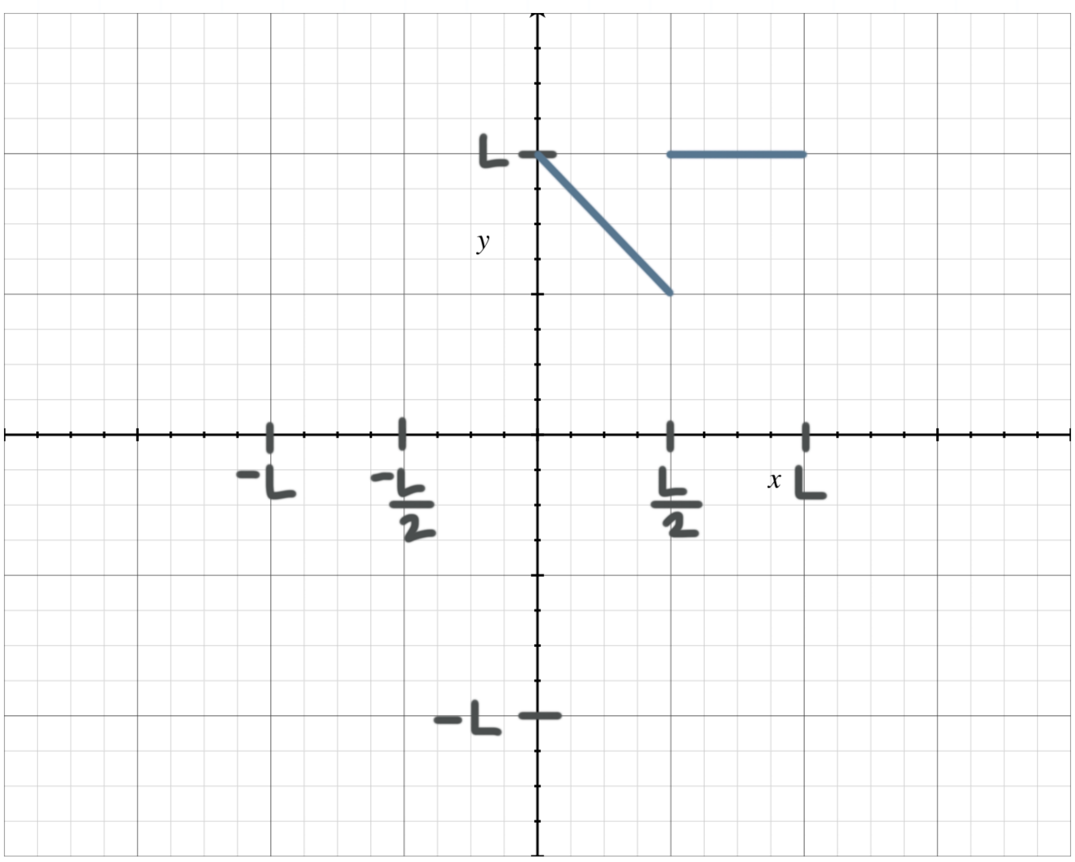
$$f(x) = \begin{cases} L - x & 0 \leq x \leq \frac{L}{2} \\ L & \frac{L}{2} < x \leq L \end{cases}$$

We want to build the Fourier cosine series, so we'll start by finding the even extension of the piecewise function. Because the original piecewise function $f(x)$ is defined in two pieces, its even extension $g(x)$ will be defined in four pieces.

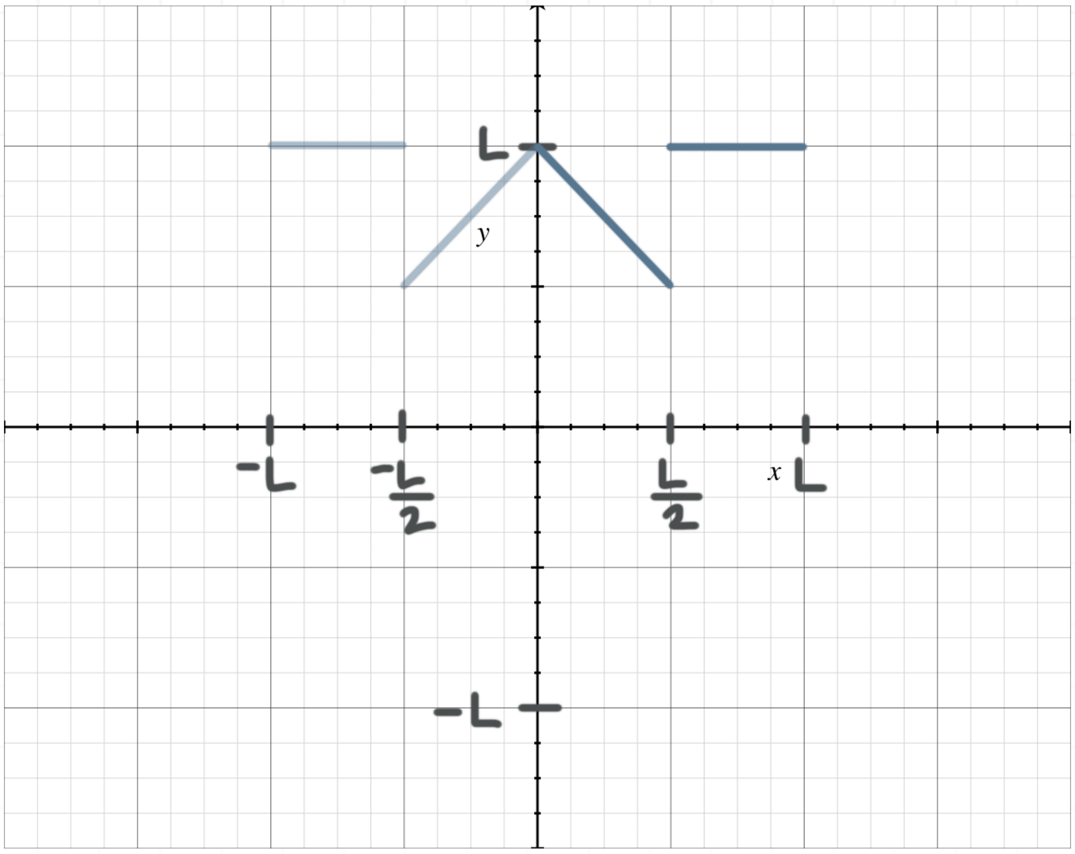
$$g(x) = \begin{cases} f(x) & 0 \leq x \leq L \\ f(-x) & -L \leq x < 0 \end{cases}$$

$$g(x) = \begin{cases} L & \frac{L}{2} < x \leq L \\ L - x & 0 \leq x \leq \frac{L}{2} \\ L + x & -\frac{L}{2} \leq x < 0 \\ L & -L \leq x < -\frac{L}{2} \end{cases}$$

If the graph of the original piecewise function $f(x)$ is



then the graph of its piecewise even extension $g(x)$ is



To find the Fourier cosine series, we'll calculate A_0 and A_n. For A_0 we get

$$A_0 = \frac{1}{L} \int_0^L f(x)\ dx$$

$$A_0 = \frac{1}{L} \int_0^{\frac{L}{2}} L - x\ dx + \frac{1}{L} \int_{\frac{L}{2}}^{L} L\ dx$$

$$A_0 = \frac{1}{L} \left(Lx - \frac{1}{2}x^2 \right) \Bigg|_0^{\frac{L}{2}} + \frac{1}{L}(Lx) \Bigg|_{\frac{L}{2}}^{L}$$

$$A_0 = x - \frac{1}{2L}x^2 \Bigg|_0^{\frac{L}{2}} + x \Bigg|_{\frac{L}{2}}^{L}$$

$$A_0 = \frac{L}{2} - \frac{1}{2L}\left(\frac{L}{2}\right)^2 - \left(0 - \frac{1}{2L}(0)^2\right) + L - \frac{L}{2}$$

$$A_0 = \frac{L}{2} - \frac{L}{8} + L - \frac{L}{2}$$

$$A_0 = \frac{7L}{8}$$

And for A_n we get

$$A_n = \frac{1}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{1}{L}\int_0^{\frac{L}{2}} (L-x)\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{\frac{L}{2}}^{L} L\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \int_0^{\frac{L}{2}} \cos\left(\frac{n\pi x}{L}\right)\ dx - \frac{1}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{n\pi x}{L}\right)\ dx + \int_{\frac{L}{2}}^{L} \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}} - \frac{1}{L}\int_0^{\frac{L}{2}} x\cos\left(\frac{n\pi x}{L}\right)\ dx + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}$$

Use integration by parts with $u = x$, $du = dx$, $dv = \cos(n\pi x/L)\ dx$, and $v = (L/n\pi)\sin(n\pi x/L)$.

$$A_n = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}} - \frac{1}{L}\left[\frac{Lx}{n\pi}\sin\left(\frac{n\pi x}{L}\right) - \int \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\ dx\right]\Big|_0^{\frac{L}{2}} + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}$$

$$A_n = \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}} - \left[\frac{x}{n\pi}\sin\left(\frac{n\pi x}{L}\right) + \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi x}{L}\right)\right]\Big|_0^{\frac{L}{2}} + \frac{L}{n\pi}\sin\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}$$

$$A_n = \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) + \frac{L}{(n\pi)^2} + \frac{L}{n\pi}\sin(n\pi) - \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

$$A_n = \frac{L}{n\pi}\sin(n\pi) - \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right) + \frac{L}{(n\pi)^2}$$

For $n = 1, 2, 3, \ldots, \sin(n\pi) = 0$, so the expression for A_n simplifies to

$$A_n = \frac{L}{(n\pi)^2} - \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)$$

Then the Fourier cosine series of the piecewise function on $0 \le x \le L$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{7L}{8} + \sum_{n=1}^{\infty}\left[\frac{L}{(n\pi)^2} - \frac{L}{n\pi}\sin\left(\frac{n\pi}{2}\right) - \frac{L}{(n\pi)^2}\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{7L}{8} + \frac{L}{\pi^2}\sum_{n=1}^{\infty}\left[\frac{1}{n^2} - \frac{\pi}{n}\sin\left(\frac{n\pi}{2}\right) - \frac{1}{n^2}\cos\left(\frac{n\pi}{2}\right)\right]\cos\left(\frac{n\pi x}{L}\right)$$

And now let's do one last example where we find the Fourier sine series representation of a piecewise defined function.

Example

Find the Fourier sine series of the piecewise function on $0 \le x \le L$.

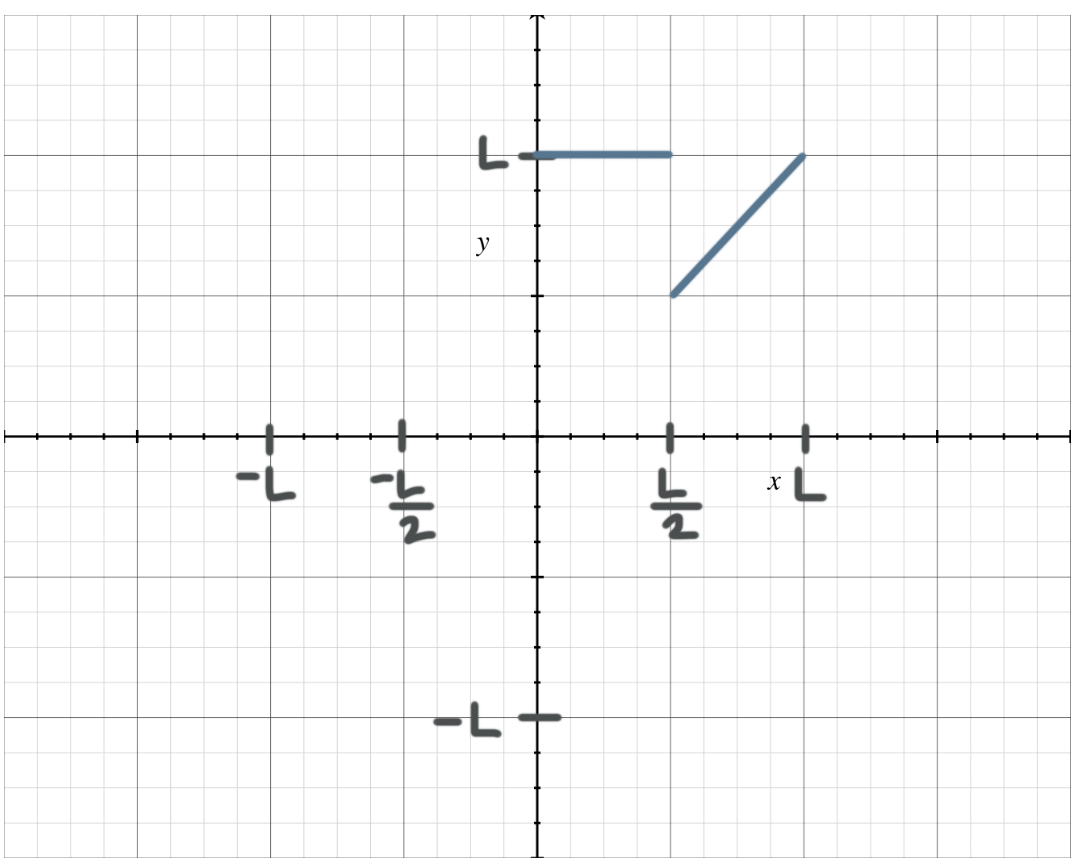
$$f(x) = \begin{cases} L & 0 \le x \le \frac{L}{2} \\ x & \frac{L}{2} < x \le L \end{cases}$$

We want to build the Fourier sine series, so we'll start by finding the odd extension of the piecewise function. Because the original piecewise function $f(x)$ is defined in two pieces, its odd extension $g(x)$ will be defined in four pieces.

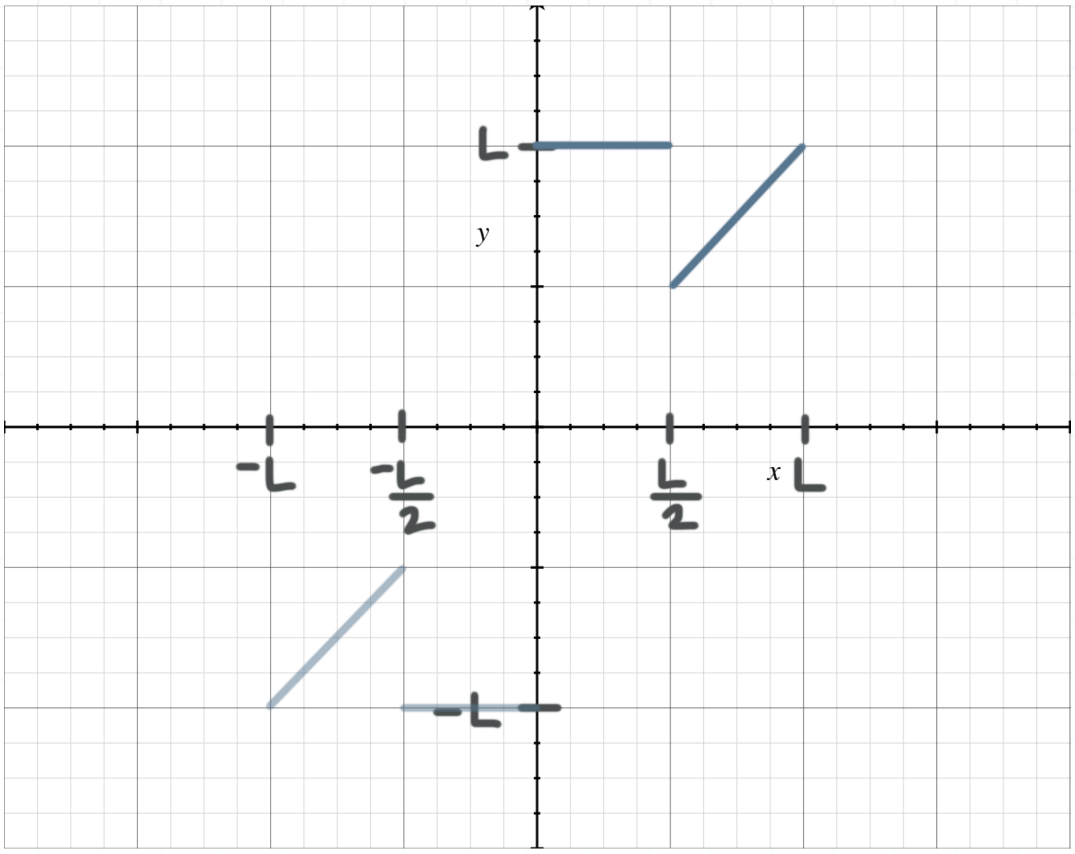
$$g(x) = \begin{cases} f(x) & 0 \le x \le L \\ -f(-x) & -L \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} x & \frac{L}{2} < x \le L \\ L & 0 \le x \le \frac{L}{2} \\ -L & -\frac{L}{2} \le x < 0 \\ x & -L \le x < -\frac{L}{2} \end{cases}$$

If the graph of the original piecewise function $f(x)$ is



then the graph of its piecewise odd extension $g(x)$ is



To find the Fourier sine series, we'll calculate B_n.

$$B_n = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{1}{L}\int_0^{\frac{L}{2}} L\sin\left(\frac{n\pi x}{L}\right)\ dx + \frac{1}{L}\int_{\frac{L}{2}}^L x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} + \frac{1}{L}\int_{\frac{L}{2}}^L x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^{\frac{L}{2}} + \frac{1}{L}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Bigg|_{\frac{L}{2}}^L$$

$$B_n = -\frac{L}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^{\frac{L}{2}} + \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) - \frac{x}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_{\frac{L}{2}}^{L}$$

$$B_n = -\frac{L}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{L}{n\pi}\cos(0)$$

$$+\frac{L}{(n\pi)^2}\sin(n\pi) - \frac{L}{n\pi}\cos(n\pi) - \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) + \frac{L}{2n\pi}\cos\left(\frac{n\pi}{2}\right)$$

For $n = 1, 2, 3, \ldots$, $\sin(n\pi) = 0$ and $\cos(n\pi) = (-1)^n$, so the expression for B_n simplifies to

$$B_n = \frac{L}{n\pi} + \frac{L}{n\pi}(-1)^{n+1} - \frac{L}{2n\pi}\cos\left(\frac{n\pi}{2}\right) - \frac{L}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

$$B_n = \frac{L}{n\pi}\left[1 + (-1)^{n+1} - \frac{1}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]$$

Then the Fourier sine series of the piecewise function on $0 \le x \le L$ is

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{L}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[1 + (-1)^{n+1} - \frac{1}{2}\cos\left(\frac{n\pi}{2}\right) - \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

Separation of variables

At the very beginning of this course, we talked about the difference between ordinary differential equations (ODEs) and partial differential equations (PDEs).

Remember that, put simply, ordinary differential equations are equations defined with ordinary derivatives, while partial differential equations are equations defined with partial derivatives. Here are some examples.

ODEs	**PDEs**
$8y''' + 2y' + \cos y = e^x$	$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$
$2y'' + 5y' = xy$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Partial derivatives

What we see from these examples is that ordinary derivatives are the derivatives of single variable functions. For instance, given a function $y(x)$ that's defined for y in terms of x, its derivatives will always be with respect to the independent variable, x. Its first derivative $y'(x)$ is a derivative with respect to x, its second derivative $y''(x)$ is a derivative with respect to x, etc.

But partial derivatives are the derivatives of multivariable functions. For instance, a function $f(x, y)$ defined for f in terms of x and y can have derivatives with respect to one or both of its independent variables.

It has two first derivatives,

$$\frac{\partial f}{\partial x}$$ the partial derivative with respect to x

$$\frac{\partial f}{\partial y}$$ the partial derivative with respect to y

and four second derivatives,

$$\frac{\partial^2 f}{\partial x^2}$$ the partial derivative with respect to x, then x again

$$\frac{\partial^2 f}{\partial y^2}$$ the partial derivative with respect to y, then y again

$$\frac{\partial^2 f}{\partial y \partial x}$$ the partial derivative with respect to x, then y

$$\frac{\partial^2 f}{\partial x \partial y}$$ the partial derivative with respect to y, then x

If the function f is well-behaved enough, these last two second derivatives, the **mixed derivatives**, will be equivalent, which means it won't make any difference whether we differentiate first with respect to x and then y, or y and then x.

Hopefully this is all just review, since partial derivatives are always covered as part of any multivariable calculus course.

But with this brief review out of the way, we want to start looking at solving partial differential equations. So far, we've been focusing entirely on ODEs, but now we'll turn toward PDEs. In fact, the main reason we learned to find Fourier series representations was to prepare us to solve some partial differential equations.

Solving by separating variables

Much earlier, we learned how to use separation of variables to solve first order ordinary differential equations. But we can also use separation of variables to solve partial differential equations.

To solve a linear homogeneous partial differential equation in x and t by separating variables, we assume that the PDE has a **product solution** in the form

$$u(x,t) = v(x)w(t)$$

We use this assumption for the solution equation because it'll allow us to simplify the partial differential equation to multiple ordinary differential equations, and of course we're already really comfortable solving ODEs.

Let's do an example to see how to use this product solution.

Example

Use the product solution to separate variables and reduce the partial differential equation into a pair of ordinary differential equations.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

We'll find the first and second derivatives of the product solution.

$$u(x,t) = v(x)w(t)$$

$$\frac{\partial u}{\partial t} = v(x)w'(t)$$

$$\frac{\partial u}{\partial x} = v'(x)w(t) \qquad \frac{\partial^2 u}{\partial x^2} = v''(x)w(t)$$

Then we'll plug the derivatives into the partial differential equation.

$$v(x)w'(t) = kv''(x)w(t)$$

If we rewrite the equation in Leibniz notation, it becomes more obvious that the equation is now entirely in terms of ordinary derivatives. All partial derivatives have been substituted out.

$$v(x)\frac{dw}{dt} = k\frac{d^2v}{dx^2}w(t)$$

Now we can separate variables.

$$\left(\frac{1}{kw(t)}\right)\frac{dw}{dt} = \left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2}$$

The left side of this equation is a function entirely in terms of the independent variable t, while the right side of the equation is a function entirely in terms of the independent variable x.

Which means the equation is telling us that a function in terms of t will always be equivalent to a function in terms of x, regardless of the values of x and t that we choose. But the only way this can possibly be true is if both functions are constant. In fact, they both have to be equal to the same constant.

We don't yet know the value of that constant, but let's call it $-\lambda$. This $-\lambda$ value is the **separation constant**.

$$\left(\frac{1}{kw(t)}\right)\frac{dw}{dt}=\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2}=-\lambda$$

Now that we have a three-part equation, we can break it apart into two separate equations.

$$\left(\frac{1}{kw(t)}\right)\frac{dw}{dt}=-\lambda$$

$$\frac{dw}{dt}=-k\lambda w(t)$$

and

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2}=-\lambda$$

$$\frac{d^2v}{dx^2}=-\lambda v(x)$$

Using the product solution in this last example allowed us to change the partial differential equation into a pair of ordinary differential equations.

We'll talk later about how to solve this pair of ODEs, but for now we just want to understand the idea of separation of variables and the use of the product solution, since we'll be using variations of this method to solve all of the partial differential equations that we'll be looking at over the next few lessons.

That being said, now we want to turn our attention toward boundary value problems and the boundary conditions we use with this product solution and separation of variables.

Boundary value problems

We're already very familiar with initial value problems. Boundary value problems are similar.

Boundary conditions and boundary value problems

Boundary value problems let us use boundary conditions that are specified for different values of the independent variable. Contrast this with initial value problems, in which we use initial conditions to define the value of the function and its derivatives all for the same value of the independent variable.

For instance, we would typically use the following set of initial conditions to solve a second order ordinary differential equation.

$$y(t_0) = y_0$$

$$y'(t_0) = y'_0$$

We can tell that this is a pair of initial conditions because they define y and its derivative y' at the same value t_0 of the independent variable.

On the other hand, for the same second order ordinary differential equation, we could use any of these pairs of boundary conditions.

$$y(t_0) = y_0 \qquad y(t_0) = y_0 \qquad y'(t_0) = y_0 \qquad y'(t_0) = y_0$$

$$y(t_1) = y_1 \qquad y'(t_1) = y_1 \qquad y(t_1) = y_1 \qquad y'(t_1) = y_1$$

The first pair defines two values of y, the last pair defines two values of y', and the other two pairs each define one value of y and one value of y'. But every pair defines values of y and/or y' at two different values of the independent variable, t_0 and t_1.

No matter which pair of boundary conditions we have, our process for solving boundary value problems will look just like the process we're used to for solving initial value problems.

Keep in mind though that, unlike initial value problems, with boundary value problems it's common to find no solutions, one solution, or many solutions. But we care about boundary value problems because we use them all the time with partial differential equations to model real-world phenomena.

Boundary conditions for the product solution

With that in mind, we want to go back to the separation of variables process we were looking at earlier.

We'd been working with the partial differential equation,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

and we used the product solution $u(x, t) = v(x)w(t)$ to rewrite the PDE as an ODE instead,

$$v(x)\frac{dw}{dt} = k\frac{d^2 v}{dx^2}w(t)$$

we separated variables,

$$\left(\frac{1}{kw(t)}\right)\frac{dw}{dt}=\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2}$$

$$\left(\frac{1}{kw(t)}\right)\frac{dw}{dt}=\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2}=-\lambda$$

and then split the equation in two, putting both in terms of the separation constant $-\lambda$. The result is a pair of ordinary differential equations, one first order equation and one second order equation, and we'll look later at how to solve these.

$$\frac{dw}{dt}=-k\lambda w(t)$$

$$\frac{d^2v}{dx^2}=-\lambda v(x)$$

But now let's consider what happens when we have the boundary conditions $u(0,t)=0$ and $u(L,t)=0$ to accompany the partial differential equation.

This partial differential equation is an equation for u defined in terms of the independent variables x and t, which is why the solution we find will be $u(x,t)$. We think about x as the spatial variable and t as the partial variable. That's why our boundary conditions $u(0,t)=0$ and $u(L,t)=0$ are defined for different values of x; we're defining the function's value at the spatial boundary $x=0$ and $x=L$. If we were given any initial conditions, they would be defined for different values of t, the time variable.

We're using the product solution $u(x,t) = v(x)w(t)$. If we substitute the first boundary condition, we get

$$u(0,t) = v(0)w(t) = 0$$

Given the product $v(0)w(t) = 0$, we know either $v(0) = 0$, or $w(t) = 0$. But if $w(t) = 0$, we'll just get the trivial solution $u(x,t) = 0$. Only $v(0) = 0$ will give us an interesting solution.

And if we substitute the second boundary condition, we get

$$u(L,t) = v(L)w(t) = 0$$

Given the product $v(L)w(t) = 0$, we know either $v(L) = 0$, or $w(t) = 0$. But if $w(t) = 0$, we'll again just get the trivial solution $u(x,t) = 0$. Only $v(L) = 0$ will give us an interesting solution.

So the partial differential equation boundary value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $u(0,t) = 0$ and $u(L,t) = 0$

has at this point turned into two separate problems, one first order equation and one second order equation, with two boundary conditions.

$$\frac{dw}{dt} = -k\lambda w(t)$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

with $v(0) = 0$ and $v(L) = 0$

Later on we'll look at how to solve this particular boundary value problem, but right now let's just do a simple boundary value problem so that we understand how to apply boundary conditions to find the actual solution from the general solution.

Example

Solve the boundary value problem, if $y(0) = 0$ and $y(2) = -1$.

$$y'' - 2y = 0$$

The associated characteristic equation is

$$r^2 - 2 = 0$$

$$r^2 = 2$$

$$r = \pm\sqrt{2}$$

which means the general solution to the homogeneous equation is

$$y(t) = c_1 e^{\sqrt{2}t} + c_2 e^{-\sqrt{2}t}$$

Substituting $y(0) = 0$ into the general solution gives

$$0 = c_1 e^{\sqrt{2}(0)} + c_2 e^{-\sqrt{2}(0)}$$

$$0 = c_1 + c_2$$

$$c_1 = -c_2$$

Substituting $y(2) = -1$ into the general solution gives

$$-1 = c_1 e^{\sqrt{2}(2)} + c_2 e^{-\sqrt{2}(2)}$$

$$-1 = c_1 e^{2\sqrt{2}} + c_2 e^{-2\sqrt{2}}$$

$$-1 = -c_2 e^{2\sqrt{2}} + c_2 e^{-2\sqrt{2}}$$

$$-1 = c_2(e^{-2\sqrt{2}} - e^{2\sqrt{2}})$$

$$c_2 = -\frac{1}{e^{-2\sqrt{2}} - e^{2\sqrt{2}}}$$

So we get

$$c_2 = \frac{1}{e^{2\sqrt{2}} - e^{-2\sqrt{2}}}$$

$$c_1 = \frac{1}{e^{-2\sqrt{2}} - e^{2\sqrt{2}}}$$

Then the solution to the boundary value problem is

$$y(t) = \frac{e^{\sqrt{2}t}}{e^{-2\sqrt{2}} - e^{2\sqrt{2}}} + \frac{e^{-\sqrt{2}t}}{e^{2\sqrt{2}} - e^{-2\sqrt{2}}}$$

So just like with initial value problems, solving boundary value problems will require us to find the general solution first, then substitute boundary conditions into the general solution and/or its derivatives in order to find values of the constants c_1 and c_2.

And remember that, even though we found a solution to the boundary value problem in the last example, it's common to see boundary value problems with no solution, when there's no solution to the system of equations we create for c_1 and c_2 after substituting the boundary conditions.

It's also common to see boundary value problems with many solutions, when the system of equations we create for c_1 and c_2 after substituting the boundary conditions gives a value for one constant that relies on the value of the other. For instance, when we find that c_2 can be any value, and c_1 is only defined as $c_1 = 2c_2$.

The heat equation

Now that we have a basic understanding of boundary value problems and separation of variables for the product solution, we can start talking about the heat equation.

The **heat equation** is actually the partial differential equation we've been working with this whole time,

$$\frac{\partial u}{\partial t}=k\frac{\partial^2 u}{\partial x^2}$$

The heat equation models the flow of heat across a material, which is why we call it the heat equation. But many other real-world scenarios, unrelated to heat, also turn out to be modeled well by the heat equation.

Remember, we've already learned how to use the product solution and separation of variables to break the heat equation into two ordinary differential equations.

$$\frac{dw}{dt}=-k\lambda w(t)$$

$$\frac{d^2v}{dx^2}=-\lambda v(x)$$

with $v(0)=0$ and $v(L)=0$

Solving the heat equation

Even though the second order equation takes a little more work, we'll solve it first, because we'll need its solution when we solve the first order equation. If we rewrite the second order equation

$$v'' + \lambda v = 0$$

and its associated characteristic equation, we get

$$r^2 + \lambda = 0$$

$$r^2 = -\lambda$$

$$r = \pm\sqrt{-\lambda}$$

With the roots of the characteristic equation defined this way, we have to consider three possible scenarios, $\lambda = 0$, $\lambda < 0$, and $\lambda > 0$. In each case, the general solution of the second order equation is given below.

$\lambda = 0$ $\qquad v(x) = c_1 + c_2x$

$\lambda > 0$ $\qquad v(x) = c_1\cos(\sqrt{\lambda}x) + c_2\sin(\sqrt{\lambda}x)$

$\lambda < 0$ $\qquad v(x) = c_1\cosh(\sqrt{-\lambda}x) + c_2\sinh(\sqrt{-\lambda}x)$

(or $v(x) = c_1e^{-\sqrt{-\lambda}x} + c_2e^{\sqrt{-\lambda}x}$)

For $\lambda = 0$ we'll substitute $v(0) = 0$ to get

$$0 = c_1 + c_2(0)$$

$$0 = c_1$$

Then substituting $c_1 = 0$ and $v(L) = 0$ gives

$$0 = 0 + c_2 L$$

$$c_2 = 0$$

So we end up with the trivial solution $v(x) = 0$, and as a result $u(x, t) = 0$.

For $\lambda > 0$ we'll substitute $v(0) = 0$ to get

$$0 = c_1 \cos(\sqrt{\lambda}(0)) + c_2 \sin(\sqrt{\lambda}(0))$$

$$0 = c_1$$

Then substituting $c_1 = 0$ and $v(L) = 0$ gives

$$0 = 0 \cos(L\sqrt{\lambda}) + c_2 \sin(L\sqrt{\lambda})$$

$$0 = c_2 \sin(L\sqrt{\lambda})$$

Assuming that $c_2 \neq 0$ (otherwise we would still have the trivial solution $u(x, t) = 0$), we get

$$\sin(L\sqrt{\lambda}) = 0$$

$$L\sqrt{\lambda} = n\pi \qquad \text{with } n = 1, 2, 3, \ldots$$

$$\sqrt{\lambda} = \frac{n\pi}{L}$$

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

Which means that the general solution equation for the $\lambda > 0$ case becomes

$$v(x) = (0)\cos\left(\sqrt{\left(\frac{n\pi}{L}\right)^2}x\right) + c_2\sin\left(\sqrt{\left(\frac{n\pi}{L}\right)^2}x\right)$$

$$v(x) = c_2\sin\left(\frac{n\pi x}{L}\right) \qquad \text{with } n = 1,\ 2,\ 3,\ \ldots$$

And for $\lambda < 0$ we'll substitute $v(0) = 0$ to get

$$0 = c_1 e^{-\sqrt{-\lambda}(0)} + c_2 e^{\sqrt{-\lambda}(0)}$$

$$0 = c_1 + c_2$$

Then substituting $c_1 = -c_2$ and $v(L) = 0$ gives

$$0 = -c_2 e^{-L\sqrt{-\lambda}} + c_2 e^{L\sqrt{-\lambda}}$$

$$0 = -c_2(e^{-L\sqrt{-\lambda}} - e^{L\sqrt{-\lambda}})$$

This equation tells us that either $c_2 = 0$ or that

$$0 = e^{-L\sqrt{-\lambda}} - e^{L\sqrt{-\lambda}}$$

$$e^{-L\sqrt{-\lambda}} = e^{L\sqrt{-\lambda}}$$

$$-L\sqrt{-\lambda} = L\sqrt{-\lambda}$$

$$2L\sqrt{-\lambda} = 0$$

But because this is the $\lambda < 0$ case, we know $L \neq 0$, and $\lambda \neq 0$ which means $e^{-L\sqrt{-\lambda}} - e^{L\sqrt{-\lambda}} \neq 0$. Therefore, the only way this equation is true is when $c_2 = 0$, and we end up with the trivial solution $u(x, t) = 0$.

So the $\lambda > 0$ case is the only one giving a non-trivial solution, and we have just

$$\lambda = \left(\frac{n\pi}{L}\right)^2$$

$$v(x) = c_2 \sin\left(\frac{n\pi x}{L}\right) \qquad \text{with } n = 1, 2, 3, \ldots$$

The first order ordinary differential equation is a linear equation in standard form,

$$w' + k\lambda w = 0$$

so we know the solution is

$$w = Ce^{-k\lambda t}$$

$$w = Ce^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Putting our results together from the first and second order equations, we get the product solution to the heat equation.

$$u(x,t) = v(x)w(t)$$

$$u(x,t) = c_2 \sin\left(\frac{n\pi x}{L}\right) Ce^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad \text{with } n = 1, 2, 3, \ldots$$

While c_2 will be different for each n, the constant $C \times c_2$ will also depend on n, so let's rename it as B_n.

$$u_n(x,t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t} \qquad \text{with } n = 1, 2, 3, \ldots$$

We've written the equation for the solution with u_n and B_n to indicate that we'll get a different solution for each value of n.

The Fourier sine series

What we may be noticing by this point is that the solution to the heat equation is starting to look a lot like a Fourier sine series. In fact, if we now pair the general solution to the heat equation with a specific initial condition $u(x,0) = f(x)$, we'll be able to arrive at the Fourier sine series.

Remember that the general solution $u(x,t)$ is an equation defined for u in terms of the independent variables x and t, where we think about x as a spatial variable and t as a time variable.

So when we define an initial condition $u(x,0) = f(x)$, it's defining the function at time $t = 0$, but the spatial variable x is left to vary. In contrast, the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$ define the function at the spatial locations $x = 0$ and $x = L$, but leave time t to vary.

So let's imagine that we've been given the initial condition

$$u(x,0) = 2\sin\left(\frac{3\pi x}{L}\right)$$

Matching this up to the general solution of the heat equation, we can identify $n = 3$ and therefore $B_n = B_3 = 2$. So given this initial condition, the solution to the heat equation would be

$$u_3(x,t) = 2\sin\left(\frac{3\pi x}{L}\right)e^{-k\left(\frac{3\pi}{L}\right)^2 t}$$

Now, we've talked before about the principle of superposition, which tells us that the sum of two solutions to a differential equation is itself also a solution. As an example, if we found a second solution, let's call it $u_5(x,t)$, in addition to this $u_3(x,t)$ solution, we could add the two solutions together, and we know that $u_3(x,t)+u_5(x,t)$ is also a solution to the heat equation.

For instance, imagine we've been given the initial condition

$$u(x,0)=5\sin\left(\frac{2\pi x}{L}\right)-3\sin\left(\frac{6\pi x}{L}\right)$$

Matching this up to the general solution of the heat equation, we can identify $n=2$ and therefore $B_n=B_2=5$ for the first term, and $n=6$ and therefore $B_n=B_6=-3$ for the second term. Which means that, given this initial condition, the solution to the heat equation would be

$$u(x,t)=u_2(x,t)+u_6(x,t)=5\sin\left(\frac{2\pi x}{L}\right)e^{-k\left(\frac{2\pi}{L}\right)^2t}-3\sin\left(\frac{6\pi x}{L}\right)e^{-k\left(\frac{6\pi}{L}\right)^2t}$$

In fact, we can add as many solutions together as we'd like, and that sum will also be a solution. To cut right to the point, we can add an infinite number of solutions,

$$u(x,t)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2t}$$

and this will also be a solution to the heat equation. And if we apply any initial condition where $t=0$, then the solution becomes

$$u(x,0)=f(x)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)$$

But, all of a sudden, we realize that we have the Fourier sine series representation of the initial condition. In other words, as long as the initial condition $f(x)$ is piecewise smooth, we can find B_n using the formula from the Fourier series lessons,

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

and we'll get a general solution to the heat equation that also satisfies the initial condition $u(x,0) = f(x)$.

With all of this finally in place, let's look at an example so that we can see the complete process for solving the heat equation with a specific initial condition.

Example

Solve the boundary value problem, if $u(x,0) = 50$, $u(0,t) = 0$, and $u(L,t) = 0$.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

We already know that this is the heat equation with a boundary condition given by $f(x) = 50$. Which means the coefficients B_n are

$$B_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L 50\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{100}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L$$

$$B_n = -\frac{100}{n\pi}\cos(n\pi) - \left(-\frac{100}{n\pi}\cos(0)\right)$$

$$B_n = -\frac{100}{n\pi}(-1)^n + \frac{100}{n\pi}$$

$$B_n = \frac{100(1-(-1)^n)}{n\pi}$$

$$B_n = \frac{100(1+(-1)^{n+1})}{n\pi}$$

and the solution to the boundary value problem is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = \frac{100}{\pi}\sum_{n=1}^{\infty}\frac{1+(-1)^{n+1}}{n}\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

The Fourier series and Fourier cosine series

Using a very similar process to the one we used earlier when we built the Fourier sine series solution to the heat equation, we can also build a Fourier cosine series solution,

$$u(x,t)=\sum_{n=0}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2t}$$

$$A_0=\frac{1}{L}\int_0^L f(x)\ dx$$

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n=1,\ 2,\ 3,\ \dots$$

as well as a Fourier series solution that combines the sine and cosine series.

$$u(x,t)=\sum_{n=0}^{\infty}A_n\cos\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2t}+\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2t}$$

$$A_0=\frac{1}{2L}\int_{-L}^L f(x)\ dx$$

$$A_n=\frac{1}{L}\int_{-L}^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad n=1,\ 2,\ 3,\ \dots$$

$$B_n=\frac{1}{L}\int_{-L}^L f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n=1,\ 2,\ 3,\ \dots$$

Changing the temperature boundaries

Up to now we've been almost completely focused on the heat equation with two specific spatial conditions and one initial temperature condition,

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $u(0,t) = 0$ and $u(L,t) = 0$

and $u(x,0) = f(x)$

These two boundary conditions define the temperature of the material as 0° at $x = 0$, and 0° at $x = L$. In other words, we've been modeling a one-dimensional temperature distribution, in which temperature only flows left and right, and the temperature at both ends of the system is 0°.

The system doesn't "leak" heat in any other direction as it flows from one end of the material to the other, or vice versa, because the system is perfectly insulated. And there are no other heat inputs to the system.

While defining the temperature as 0° at both ends of this kind of one-dimensional system makes it easier to solve the heat equation, doing so also makes the solution less useful.

If at all possible, we'd prefer to be able to choose some non-zero values for these temperatures, and still be able to solve the heat equation.

Non-zero temperature boundaries

Instead of defining the temperature at the ends of the system as $u(0,t) = 0$ and $u(L, t) = 0$, let's choose two non-zero temperatures, and rewrite our boundary value problem as

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $u(0,t) = T_1$ and $u(L, t) = T_2$

and $u(x,0) = f(x)$

As it turns out, this small change in the boundary conditions means we can no longer use the product solution we've been using up to this point, $u(x, t) = v(x)w(t)$. Instead, if we assume that we won't add heat into the system, and that the system won't leak heat, and therefore that temperature changes can only occur at the ends of the system, then we know that the material will eventually reach an equilibrium temperature, if we take time t out far enough.

$$\lim_{t\to\infty} u(x, t) = u_E(x)$$

The function $u_E(x)$ models the equilibrium temperature, and it'll satisfy

$$\frac{d^2 u_E}{dx^2} = 0 \qquad u_E(0) = T_1 \qquad u_E(L) = T_2$$

The characteristic equation associated with this second order equation, $u_E'' = 0$, is

$$r^2 = 0$$

$$r = 0, 0$$

Because we get equal real roots, $r_1 = r_2 = 0$, the general solution is

$$u_E(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x}$$

$$u_E(x) = c_1 e^{0x} + c_2 x e^{0x}$$

$$u_E(x) = c_1 + c_2 x$$

If we substitute the boundary condition $u_E(0) = T_1$, we get

$$T_1 = c_1 + c_2(0)$$

$$c_1 = T_1$$

If we then substitute $c_1 = T_1$ and $u_E(L) = T_2$, we get

$$T_2 = T_1 + c_2 L$$

$$c_2 = \frac{T_2 - T_1}{L}$$

So **equilibrium temperature** is modeled by

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L} x$$

With this value of $u_E(x)$, it can be shown that the solution to the original differential equation is

$$u(x, t) = u_E(x) + v(x, t)$$

where

$$v(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\, dx \qquad n = 1, 2, 3, \dots$$

Which means that the solution to the heat equation with non-zero temperature boundaries is

$$u(x,t) = u_E(x) + v(x,t)$$

$$u(x,t) = T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\, dx \qquad n = 1, 2, 3, \dots$$

Let's do another example, this time where we solve the heat equation with different boundary conditions.

Example

Given a one-dimensional temperature distribution, if temperature at $x = 0$ is 10°, temperature at $x = L$ is 20°, and $u(x,0) = 40$, find an equation that models heat in the system.

We're solving the heat equation with two boundary conditions and an initial condition.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $u(0,t) = 10$ and $u(L,t) = 20$

$u(x,0) = 40$

First, we'll find the function modeling equilibrium temperature.

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E(x) = 10 + \frac{20 - 10}{L}x$$

$$u_E(x) = 10 + \frac{10}{L}x$$

Next, we'll find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L \left(40 - \left(10 + \frac{10}{L}x\right)\right)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L \left(40 - 10 - \frac{10}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{20}{L}\int_0^L \left(3 - \frac{1}{L}x\right)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

Split the integral in two,

$$B_n = \frac{60}{L}\int_0^L \sin\left(\frac{n\pi x}{L}\right)\ dx - \frac{20}{L^2}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)\ dx$$

then integrate, using integration by parts for the second integral with $u = x$, $du = dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{60}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L - \frac{20}{L^2}\left[-\frac{Lx}{n\pi}\cos\left(\frac{n\pi x}{L}\right) + \frac{L}{n\pi}\int\cos\left(\frac{n\pi x}{L}\right)\ dx\right]\Big|_0^L$$

$$B_n = -\frac{60}{n\pi}\cos\left(\frac{n\pi x}{L}\right)\Big|_0^L - \left[\frac{20}{(n\pi)^2}\sin\left(\frac{n\pi x}{L}\right) - \frac{20x}{n\pi L}\cos\left(\frac{n\pi x}{L}\right)\right]\Big|_0^L$$

Evaluate over the interval.

$$B_n = -\frac{60}{n\pi}\cos(n\pi) + \frac{60}{n\pi}\cos(0)$$

$$-\left(\frac{20}{(n\pi)^2}\sin(n\pi) - \frac{20}{n\pi}\cos(n\pi) - \frac{20}{(n\pi)^2}\sin(0) + \frac{20(0)}{n\pi L}\cos(0)\right)$$

$$B_n = -\frac{60}{n\pi}\cos(n\pi) + \frac{60}{n\pi}\cos(0) - \frac{20}{(n\pi)^2}\sin(n\pi)$$

$$+\frac{20}{n\pi}\cos(n\pi) + \frac{20}{(n\pi)^2}\sin(0) - \frac{20(0)}{n\pi L}\cos(0)$$

$$B_n = -\frac{60}{n\pi}(-1)^n + \frac{60}{n\pi} + \frac{20}{n\pi}(-1)^n$$

$$B_n = \frac{60}{n\pi} - \frac{40}{n\pi}(-1)^n$$

Then the solution to this heat equation is

$$u(x,t) = T_1 + \frac{T_2 - T_1}{L}x + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 10 + \frac{10}{L}x + \sum_{n=1}^{\infty} \left(\frac{60}{n\pi} - \frac{40}{n\pi}(-1)^n\right) \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 10 + \frac{10x}{L} + \frac{20}{\pi}\sum_{n=1}^{\infty} \frac{3 + 2(-1)^{n+1}}{n} \sin\left(\frac{n\pi x}{L}\right) e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Laplace's equation

With partial differential equations, everything we've done so far has been for the one-dimensional heat equation.

Now we want to look at a two-dimensional partial differential equation: Laplace's equation.

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

$$\nabla^2 u=0$$

The ∇^2 operator is the **Laplacian**, and it tells us to take the second derivative with respect to each variable in the function, then add those second derivatives. So if u is a function in x and y, $u(x,y)$, then the Laplacian of u, $\nabla^2 u$, will be the sum of the second derivative with respect to x and the second derivative with respect to y.

$$\nabla^2 u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}$$

Where previously we only had temperature change along one dimension (from left to right, and/or from right to left), now we can look at temperature change in two dimensions (from left to right, and/or from right to left, but also from top to bottom, and/or from bottom to top, at the same time).

The way we solve Laplace's equation depends on the geometry of the system, so we'll start with the simplest scenario, a two-dimensional rectangular region.

A rectangular region

Let's imagine that we have a rectangle with length defined on $0 \le x \le L$, and height defined on $0 \le y \le H$. Then we can define boundary conditions for Laplace's equation along each boundary of the rectangle.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(0,y) = g_1(y) \qquad u(L,y) = g_2(y)$$

$$u(x,0) = f_1(x) \qquad u(x,H) = f_2(x)$$

To solve this boundary value problem, we'll actually solve it four times, generating four solutions $u_1(x,y)$, $u_2(x,y)$, $u_3(x,y)$, and $u_4(x,y)$. Each time, we'll keep one boundary condition nonhomogeneous while we hold the other three homogeneous.

1. $u_1(x,0) = f_1(x) \qquad u_1(0,y) = 0 \qquad u_1(x,H) = 0 \qquad u_1(L,y) = 0$
2. $u_2(x,0) = 0 \qquad u_2(0,y) = g_1(y) \qquad u_2(x,H) = 0 \qquad u_2(L,y) = 0$
3. $u_3(x,0) = 0 \qquad u_3(0,y) = 0 \qquad u_3(x,H) = f_2(x) \qquad u_3(L,y) = 0$
4. $u_4(x,0) = 0 \qquad u_4(0,y) = 0 \qquad u_4(x,H) = 0 \qquad u_4(L,y) = g_2(y)$

Once we find all four solutions, then solution to Laplace's equation will be

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y)$$

To find each of the four solutions, we'll use separation of variables and the product solution, along with the three homogeneous boundary conditions, then we'll plug in the single nonhomogeneous boundary condition to the result that we get.

For instance, to find $u_2(x, y)$ we have to solve the boundary value problem

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u_2(0,y) = g_1(y) \qquad u_2(L, y) = 0$$

$$u_2(x,0) = 0 \qquad u_2(x, H) = 0$$

Using the product solution $u_2(x, y) = v(x)w(y)$, the partial differential equation can be broken into two ordinary differential equations,

$$\frac{d^2v}{dx^2} - \lambda v = 0 \qquad \frac{d^2w}{dy^2} + \lambda w = 0$$

$$v(L) = 0 \qquad w(0) = 0 \qquad w(H) = 0$$

The solution to the second ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{H}\right)^2 \qquad w_n(y) = \sin\left(\frac{n\pi y}{H}\right) \qquad n = 1,\ 2,\ 3,\ \dots$$

When we plug this value for λ into the first equation, we get

$$\frac{d^2v}{dx^2} - \left(\frac{n\pi}{H}\right)^2 v = 0$$

$$v(L) = 0$$

Because the coefficient on v is positive, we know that the solution to this second order equation will be

$$v(x) = c_1 e^{\frac{n\pi x}{H}} + c_2 e^{-\frac{n\pi x}{H}}$$

which can also be rewritten as

$$v(x) = c_1 \cosh\left(\frac{n\pi x}{H}\right) + c_2 \sinh\left(\frac{n\pi x}{H}\right)$$

Equivalently, this hyperbolic form can be rewritten as

$$v(x) = c_1 \cosh\left(\frac{n\pi(x-L)}{H}\right) + c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

which allows us to apply $v(L) = 0$. When we do, we find $c_1 = 0$, and the solution is

$$v(x) = c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right)$$

Then the product solution is

$$u_2(x,y) = v(x)w(y)$$

$$u_2(x,y) = c_2 \sinh\left(\frac{n\pi(x-L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

$$u_2(x, y) = B_n \sinh\left(\frac{n\pi(x - L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right) \qquad n = 1, 2, 3, \ldots$$

$$u_2(x, y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(x - L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

Substitute $u_2(0,y) = g_1(y)$.

$$u_2(0,y) = g_1(y) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi(-L)}{H}\right) \sin\left(\frac{n\pi y}{H}\right)$$

This is just like a Fourier sine series, except that the coefficients are more complicated.

$$B_n \sinh\left(\frac{n\pi(-L)}{H}\right) = \frac{2}{H}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)\ dy \qquad n = 1, 2, 3, \ldots$$

$$B_n = \frac{2}{H\sinh\left(\frac{n\pi(-L)}{H}\right)}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)\ dy \qquad n = 1, 2, 3, \ldots$$

Using a similar method, we can find the solutions for all four functions, $u_1(x, y)$, $u_2(x, y)$, $u_3(x, y)$, and $u_4(x, y)$ to be

$$u_1(x, y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(y - H)}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$B_n = -\frac{2}{L\sin\left(\frac{n\pi H}{L}\right)}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n = 1, 2, 3, \ldots$$

$$u_2(x,y)=\sum_{n=1}^{\infty}B_n\sinh\left(\frac{n\pi(x-L)}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

$$B_n=-\frac{2}{H\sinh\left(\frac{n\pi L}{H}\right)}\int_0^H g_1(y)\sin\left(\frac{n\pi y}{H}\right)\ dy \qquad n=1,2,3,\ldots$$

$$u_3(x,y)=\sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi y}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

$$B_n=\frac{2}{L\sin\left(\frac{n\pi H}{L}\right)}\int_0^L f_2(x)\sin\left(\frac{n\pi x}{L}\right)\ dx \qquad n=1,2,3,\ldots$$

$$u_4(x,y)=\sum_{n=1}^{\infty}B_n\sinh\left(\frac{n\pi x}{H}\right)\sin\left(\frac{n\pi y}{H}\right)$$

$$B_n=\frac{2}{H\sinh\left(\frac{n\pi L}{H}\right)}\int_0^H g_2(y)\sin\left(\frac{n\pi y}{H}\right)\ dy \qquad n=1,2,3,\ldots$$

Therefore, the solution to Laplace's equation over the rectangular domain $0\le x\le L$ and $0\le y\le H$ is given by,

$$u(x,y)=u_1(x,y)+u_2(x,y)+u_3(x,y)+u_4(x,y)$$

where the values of $u_1(x,y)$, $u_2(x,y)$, $u_3(x,y)$, and $u_4(x,y)$ are defined above.

Let's do another example where we derive one of these $u_n(x,y)$ solutions, just to make sure we have a good feel for the process.

Example

Use the product solution to find $u_1(x, y)$, the solution to Laplace's equation along the bottom edge of the rectangle defined on $0 \leq x \leq L$ and $0 \leq y \leq H$.

$$\frac{\partial^2 u_2}{\partial x^2} + \frac{\partial^2 u_2}{\partial y^2} = 0$$

$$u_1(x,0) = f_1(x) \qquad u_1(0,y) = 0$$

$$u_1(x, H) = 0 \qquad u_1(L, y) = 0$$

If we find the first and second derivatives of the product solution,

$$u(x, y) = v(x)w(y)$$

$$\frac{\partial u}{\partial x} = \frac{dv}{dx}w \qquad \frac{\partial u}{\partial y} = v\frac{dw}{dy}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{d^2 v}{dx^2}w \qquad \frac{\partial^2 u}{\partial y^2} = v\frac{d^2 w}{dy^2}$$

and then plug the second derivatives into Laplace's equation,

$$\frac{d^2 v}{dx^2}w + v\frac{d^2 w}{dy^2} = 0$$

then we can separate variables.

$$\frac{d^2 v}{dx^2}w = -v\frac{d^2 w}{dy^2}$$

$$\left(\frac{1}{v}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w}\right)\frac{d^2w}{dy^2}$$

Set both sides of the equation equal to the separation constant.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

Break this into two separate ordinary differential equations.

$$\left(\frac{1}{v(x)}\right)\frac{d^2v}{dx^2} = -\lambda$$

$$\frac{d^2v}{dx^2} = -\lambda v(x)$$

$$\frac{d^2v}{dx^2} + \lambda v = 0$$

and

$$-\left(\frac{1}{w(y)}\right)\frac{d^2w}{dy^2} = -\lambda$$

$$\frac{d^2w}{dy^2} = \lambda w(y)$$

$$\frac{d^2w}{dy^2} - \lambda w = 0$$

Our boundary conditions are $w(H) = 0$, $v(0) = 0$, and $v(L) = 0$, and the solution to the first ordinary differential equation boundary value problem is

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 \qquad v_n(x) = \sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \ldots$$

When we plug this value for λ into the second equation, we get

$$\frac{d^2w}{dy^2} + \left(\frac{n\pi}{L}\right)^2 w = 0$$

$$w(H) = 0$$

Because the coefficient on w is positive, we know that the solution to this second order equation will be

$$w(y) = c_1 \cos\left(\frac{n\pi y}{L}\right) + c_2 \sin\left(\frac{n\pi y}{L}\right)$$

Equivalently, this can be rewritten as

$$w(y) = c_1 \cos\left(\frac{n\pi(y-H)}{L}\right) + c_2 \sin\left(\frac{n\pi(y-H)}{L}\right)$$

which allows us to apply $w(H) = 0$. When we do, we find $c_1 = 0$, and the solution is

$$w(y) = c_2 \sin\left(\frac{n\pi(y-H)}{L}\right)$$

Then the product solution is

$$u_1(x, y) = v(x)w(y)$$

$$u_1(x, y) = c_2 \sin\left(\frac{n\pi(y-H)}{L}\right) \sin\left(\frac{n\pi x}{L}\right)$$

$$u_1(x,y) = B_n \sin\left(\frac{n\pi(y-H)}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \qquad n = 1, 2, 3, \dots$$

$$u_1(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(y-H)}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Substitute $u_1(x,0) = f_1(x)$.

$$u_1(x,0) = f_1(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi(-H)}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

This is just like a Fourier sine series, except that the coefficients are more complicated.

$$B_n \sin\left(\frac{n\pi(-H)}{L}\right) = \frac{2}{L}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \qquad n = 1, 2, 3, \dots$$

$$B_n = \frac{-2}{L\sin\left(\frac{n\pi H}{L}\right)}\int_0^L f_1(x)\sin\left(\frac{n\pi x}{L}\right)\, dx \qquad n = 1, 2, 3, \dots$$

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