Differential Equations Final Exam Solutions

Differential Equations Final Exam Answer Key

1. (5 pts)	A	B	C	■	E
2. (5 pts)	A	B	■	D	E
3. (5 pts)	■	B	C	D	E
4. (5 pts)	A	B	C	■	E
5. (5 pts)	A	B	C	■	E
6. (5 pts)	A	B	■	D	E
7. (5 pts)	A	B	C	D	■
8. (5 pts)	A	■	C	D	E

9. (15 pts)

$$\vec{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2\cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 4e^t + 1 \\ e^t - t \\ \frac{1}{2}\sin t - \frac{1}{2}\cos t \end{bmatrix}$$

10. (15 pts)

$$y(t) = c_1e^{-t} + c_2 + c_3t + c_4t^2 + c_5e^{-3t} + c_6te^{-3t}$$

$$+c_7e^{-2t}\cos t + c_8e^{-2t}\sin t$$

$$+c_9e^t\cos(\sqrt{2}t) + c_{10}e^t\sin(\sqrt{2}t)$$

$$+c_{11}te^t\cos(\sqrt{2}t) + c_{12}te^t\sin(\sqrt{2}t)$$

$$+c_{13}t^2e^t\cos(\sqrt{2}t) + c_{14}t^2e^t\sin(\sqrt{2}t)$$

11. (15 pts)

$$f(x) = \frac{1}{6} - \frac{1}{4\pi^2} + \left(-\frac{1}{6} + \frac{7}{16\pi^2}\right)\cos(2\pi x)$$

$$+ \sum_{n=1, n\neq 2}^{\infty} \frac{(-1)^n}{\pi^2}\left(\frac{2}{n^2} + \frac{1}{(n-2)^2} + \frac{1}{(n+2)^2}\right)\cos(n\pi x)$$

12. (15 pts)

$$u(x,t) = 10 - \frac{15}{L}x + \sum_{n=1}^{\infty} \frac{10}{n\pi}(2 - 5(-1)^n)\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

Differential Equations Final Exam Solutions

1. D. Change y' to dy/dx,

$$3xy' = y$$

$$3x\frac{dy}{dx} = y$$

then separate variables, collecting y terms on the left and x terms on the right.

$$\frac{3}{y}\ dy = \frac{1}{x}\ dx$$

Integrate both sides,

$$\int \frac{3}{y}\ dy = \int \frac{1}{x}\ dx$$

$$3\ln|y| = \ln|x| + C$$

$$\ln|y| = \frac{1}{3}\ln|x| + C$$

then solve for y.

$$e^{\ln|y|} = e^{\frac{1}{3}\ln|x|+C}$$

$$|y| = Ce^{\frac{1}{3}\ln|x|}$$

$$|y| = Ce^{\ln|x|^{\frac{1}{3}}}$$

$$|y| = C|x|^{\frac{1}{3}}$$

$$y = C|x|^{\frac{1}{3}}$$

2. C. Divide the equation by y^2.

$$y' + 3y = \sin(t)y^2$$

$$\frac{1}{y^2}y' + \frac{3}{y} = \sin t$$

With the Bernoulli equation in standard form, we'll find

$$v = \frac{1}{y}$$

$$v' = -\frac{1}{y^2}y'$$

and then substitute these values into the Bernoulli equation.

$$-v' + 3v = \sin t$$

$$v' - 3v = -\sin t$$

Now we have a linear equation with an integrating factor of e^{-3t}, so we'll multiply this integrating factor through the linear equation.

$$v'e^{-3t} - 3ve^{-3t} = -e^{-3t}\sin t$$

$$\frac{d}{dx}(ve^{-3t}) = -e^{-3t}\sin t$$

$$\int \frac{d}{dx}(ve^{-3t})\ dx = \int -e^{-3t}\sin t\ dt$$

$$ve^{-3t} = \int -e^{-3t}\sin t\ dt$$

Use integration by parts with $u_1 = \sin t$, $du_1 = \cos t\ dt$, $du_2 = -e^{-3t}\ dt$, and $u_2 = (1/3)e^{-3t}$.

$$ve^{-3t} = \frac{1}{3}e^{-3t}\sin t + \frac{1}{3}\int -e^{-3t}\cos t\ dt$$

Use integration by parts again with $u_1 = \cos t$, $du_1 = -\sin t\ dt$, $du_2 = -e^{-3t}\ dt$, and $u_2 = (1/3)e^{-3t}$.

$$ve^{-3t} = \frac{1}{3}e^{-3t}\sin t + \frac{1}{9}e^{-3t}\cos t - \frac{1}{9}\int -e^{-3t}\sin t\ dt$$

$$ve^{-3t} = \frac{1}{3}e^{-3t}\sin t + \frac{1}{9}e^{-3t}\cos t - \frac{1}{9}ve^{-3t} + C$$

$$\frac{10}{9}ve^{-3t} = \frac{1}{3}e^{-3t}\sin t + \frac{1}{9}e^{-3t}\cos t + C$$

$$10ve^{-3t} = 3e^{-3t}\sin t + e^{-3t}\cos t + C$$

$$ve^{-3t} = \frac{(3\sin t + \cos t + Ce^{3t})e^{-3t}}{10}$$

$$v = \frac{3\sin t + \cos t + Ce^{3t}}{10}$$

Use $v = 1/y$ to back-substitute for v, then solve for y.

$$\frac{1}{y}=\frac{3\sin t+\cos t+Ce^{3t}}{10}$$

$$y=\frac{10}{3\sin t+\cos t+Ce^{3t}}$$

Substitute the initial condition $y(0)=1$.

$$1=\frac{10}{0+1+C}$$

$$10=1+C$$

$$9=C$$

Then the solution to the Bernoulli equation initial value problem is

$$y=\frac{10}{3\sin t+\cos t+9e^{3t}}$$

3. A. We'll use undetermined coefficients and set $g(x)=0$ to find the associated homogeneous equation,

$$y''+4y'+4y=2e^{-2x}+8x$$

$$y''+4y'+4y=0$$

then solve its associated characteristic equation.

$$r^2+4r+4=0$$

$$(r+2)(r+2)=0$$

$$r = -2, \ -2$$

These are equal real roots, so the complementary solution will be

$$y_c(x) = c_1e^{r_1x} + c_2xe^{r_1x}$$

$$y_c(x) = c_1e^{-2x} + c_2xe^{-2x}$$

Our guess for the particular solution will be

$$y_p(x) = Ae^{-2x} + Bx + C$$

The Ae^{-2x} term overlaps with the c_1e^{-2x} term from the complementary solution. If we multiply the term from our guess by x to eliminate the overlap, the new Axe^{-2x} term now overlaps with the other term from the complementary solution, c_2xe^{-2x}. So we'll multiply it by x again, and our guess will be

$$y_p(x) = Ax^2e^{-2x} + Bx + C$$

Take the first and second derivatives of the guess.

$$y_p'(x) = 2Axe^{-2x} - 2Ax^2e^{-2x} + B$$

$$y_p''(x) = 2Ae^{-2x} - 8Axe^{-2x} + 4Ax^2e^{-2x}$$

Plugging into the original differential equation, we get

$$y'' + 4y' + 4y = 2e^{-2x} + 8x$$

$$2Ae^{-2x} - 8Axe^{-2x} + 4Ax^2e^{-2x} + 8Axe^{-2x} - 8Ax^2e^{-2x} + 4B$$

$$+4Ax^2e^{-2x} + 4Bx + 4C = 2e^{-2x} + 8x$$

$$2Ae^{-2x} + 4B + 4Bx + 4C = 2e^{-2x} + 8x$$

Equating coefficients gives $2A = 2$, $4B = 8$, and $4B + 4C = 0$, and we can solve that system of equations to find $A = 1$, $B = 2$, and $C = -2$. So the particular solution is

$$y_p(x) = Ax^2e^{-2x} + Bx + C$$

$$y_p(x) = x^2e^{-2x} + 2x - 2$$

Putting this particular solution together with the complementary solution gives us the general solution to the nonhomogeneous differential equation.

$$y(x) = y_c(x) + y_p(x)$$

$$y(x) = c_1e^{-2x} + c_2xe^{-2x} + x^2e^{-2x} + 2x - 2$$

4. D. Approximating $y(\pi)$ from $y(0) = 0$ using $n = 4$ steps gives us a step size of

$$\Delta t = \frac{\pi - 0}{4} = \frac{\pi}{4}$$

With $t_0 = 0$, $y_0 = 0$, and $\Delta t = \pi/4$, our table is

$t_0 = 0$	$y_0 = 0$	$y_0 = 0$
$t_1 = \frac{\pi}{4}$	$y_1 = 0 + \left(\frac{\pi}{4}\right)(4\sin(0) + 2)$	$y_1 = \frac{\pi}{2}$

$$t_2 = \frac{\pi}{2} \qquad y_2 = \frac{\pi}{2} + \left(\frac{\pi}{4}\right)\left(4\sin\left(\frac{\pi}{2}\right) + 2\right) \qquad y_2 = 2\pi$$

$$t_3 = \frac{3\pi}{4} \qquad y_3 = 2\pi + \left(\frac{\pi}{4}\right)(4\sin(2\pi) + 2) \qquad y_3 = \frac{5\pi}{2}$$

$$t_4 = \pi \qquad y_4 = \frac{5\pi}{2} + \left(\frac{\pi}{4}\right)\left(4\sin\left(\frac{5\pi}{2}\right) + 2\right) \qquad y_4 = 4\pi$$

After filling out the table, we can say that the value of $y(\pi)$ is approximately

$$y(\pi) \approx 4\pi$$

5. D. The autonomous differential equation has equilibrium solutions at

$$12 - 4y - y^2 = 0$$

$$-(y+6)(y-2) = 0$$

$$y = -6, 2$$

Given these two equilibrium solutions, we'll consider three intervals:

$$y < -6$$

$$-6 < y < 2$$

$$2 < y$$

We'll choose a test value in each interval ($y = -10$ for the first interval, $y = 0$ for the second interval, and $y = 3$ for the third interval), then plug each test value into the equation $f(y) = 12 - 4y - y^2$.

$$f(-10) = 12 - 4(-10) - (-10)^2 = 12 + 40 - 100 = -48 < 0$$

$$f(0) = 12 - 4(0) - 0^2 = 12 > 0$$

$$f(3) = 12 - 4(3) - 3^2 = 12 - 12 - 9 = -9 < 0$$

The signs of these results tell us that solution curves are increasing in the intervals $-6 < y < 2$, and decreasing in the intervals $y < -6$ and $2 < y$.

Interval	**Sign of $f(y)$**	**Direction of $f(y)$**
$(2,\infty)$	$-$	Decreasing/Falling
$(-6,2)$	$+$	Increasing/Rising
$(-\infty,-6)$	$-$	Decreasing/Falling

Therefore, the equation has a stable equilibrium solution at $y = 2$, and an unstable equilibrium solution at $y = -6$.

6. C. To start, we'll substitute y and y' into the differential equation.

$$y' + y = e^x$$

$$\sum_{n=1}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

The series are in phase, so we'll just make the indices match.

$$\sum_{k=0}^{\infty} c_{k+1}(k+1)x^k + \sum_{k=0}^{\infty} c_k x^k = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

$$\sum_{k=0}^{\infty} [c_{k+1}(k+1) + c_k]x^k = \sum_{k=0}^{\infty} \frac{1}{k!}x^k$$

Rewrite the recurrence relation.

$$c_{k+1}(k+1) + c_k = \frac{1}{k!} \qquad k = 0, 1, 2, 3, \dots$$

$$c_{k+1} = \frac{1}{(k+1)!} - \frac{c_k}{k+1}$$

Now we'll start plugging in values $k = 0, 1, 2, 3, \dots$.

$$k = 0 \qquad c_1 = \frac{1}{1!}(1 - c_0) \qquad k = 1 \qquad c_2 = \frac{1}{2!}c_0$$

$$k = 2 \qquad c_3 = \frac{1}{3!}(1 - c_0) \qquad k = 3 \qquad c_4 = \frac{1}{4!}c_0$$

$$k = 4 \qquad c_5 = \frac{1}{5!}(1 - c_0) \qquad k = 5 \qquad c_6 = \frac{1}{6!}c_0$$

$$\vdots \qquad\qquad \vdots$$

$$c_k = \frac{1}{k!}(1 - c_0) \qquad\qquad c_k = \frac{1}{k!}c_0$$

Using these values, the solution is

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 + c_6 x^6 + \dots$$

$$y = c_0 + \frac{1}{1!}(1 - c_0)x + \frac{1}{2!}c_0x^2 + \frac{1}{3!}(1 - c_0)x^3$$

$$+\frac{1}{4!}c_0x^4 + \frac{1}{5!}(1 - c_0)x^5 + \frac{1}{6!}c_0x^6 + \dots$$

$$y = c_0 + \frac{1}{1!}x - \frac{1}{1!}c_0x + \frac{1}{2!}c_0x^2 + \frac{1}{3!}x^3 - \frac{1}{3!}c_0x^3$$

$$+\frac{1}{4!}c_0x^4 + \frac{1}{5!}x^5 - \frac{1}{5!}c_0x^5 + \frac{1}{6!}c_0x^6 + \dots$$

$$y = c_0 - \frac{1}{1!}c_0x + \frac{1}{2!}c_0x^2 - \frac{1}{3!}c_0x^3 + \frac{1}{4!}c_0x^4 - \frac{1}{5!}c_0x^5 + \frac{1}{6!}c_0x^6$$

$$+\frac{1}{1!}x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \dots$$

$$y = c_0\left(\frac{1}{0!}x^0 - \frac{1}{1!}x^1 + \frac{1}{2!}x^2 - \frac{1}{3!}x^3 + \frac{1}{4!}x^4 - \frac{1}{5!}x^5 + \frac{1}{6!}x^6 - \dots\right)$$

$$+x\left(\frac{1}{1!}x^0 + \frac{1}{3!}x^2 + \frac{1}{5!}x^4 + \dots\right)$$

And the pattern that seems to be emerging is

$$y = c_0\sum_{k=0}^{\infty}\frac{(-x)^k}{k!} + x\sum_{k=1}^{\infty}\frac{1}{k!}x^{k-1}$$

$$y = c_0\sum_{k=0}^{\infty}\frac{(-x)^k}{k!} + \sum_{k=1}^{\infty}\frac{x^k}{k!}$$

$$y = c_0 + c_0\sum_{k=1}^{\infty}\frac{(-1)^kx^k}{k!} + \sum_{k=1}^{\infty}\frac{x^k}{k!}$$

$$y = c_0 + c_0 \sum_{k=1}^{\infty} \frac{x^k + (-1)^k x^k}{k!}$$

$$y = c_0 + c_0 \sum_{k=1}^{\infty} \frac{x^k(1 + (-1)^k)}{k!}$$

7. E. Apply the Laplace transform to both sides of the differential equation.

$$y'' - 5y' + 6y = 4xe^x$$

$$s^2Y(s) - sy(0) - y'(0) - 5[sY(s) - y(0)] + 6Y(s) = 4\left(\frac{1}{(s-1)^2}\right)$$

Plug in the initial conditions $y(0) = 1$ and $y'(0) = 0$.

$$s^2Y(s) - 5sY(s) + 6Y(s) = \frac{4}{(s-1)^2} + s - 5$$

$$(s^2 - 5s + 6)Y(s) = \frac{4 + s(s-1)^2 - 5(s-1)^2}{(s-1)^2}$$

$$Y(s) = \frac{4 + s(s-1)^2 - 5(s-1)^2}{(s-1)^2(s^2 - 5s + 6)}$$

$$Y(s) = \frac{s^3 - 7s^2 + 11s - 1}{(s-1)^2(s-2)(s-3)}$$

Use a partial fractions decomposition.

$$\frac{s^3-7s^2+11s-1}{(s-1)^2(s-2)(s-3)}=\frac{A}{s-1}+\frac{B}{(s-1)^2}+\frac{C}{s-2}+\frac{D}{s-3}$$

If we multiply both sides by $s-2$ and then set $s=2$, we get

$$\frac{2^3-7(2)^2+11(2)-1}{(1)^2(-1)}=A(0)+B(0)+C+D(0)$$

$$C=-1$$

If we multiply both sides by $s-3$ and then set $s=3$, we get

$$\frac{3^3-7(3)^2+11(3)-1}{4\cdot 1}=A(0)+B(0)+C(0)+D$$

$$D=-1$$

If we multiply both sides by $(s-1)^2$ and then set $s=1$, we get

$$\frac{1^3-7(1)^2+11(1)-1}{(-1)(-2)}=B$$

$$B=2$$

We've set $s=1,\ 2,\ 3$, so we'll choose a different value and set $s=0$ to get

$$\frac{-1}{(-1)^2(-2)(-3)}=-\frac{A}{1}+\frac{2}{1}+\frac{1}{2}+\frac{1}{3}$$

$$A=3$$

Plugging these into the decomposition gives

$$Y(s) = \frac{3}{s-1} + \frac{2}{(s-1)^2} - \frac{1}{s-2} - \frac{1}{s-3}$$

Applying an inverse transform gives us the solution to the second order nonhomogeneous differential equation.

$$y(x) = 3e^x + 2xe^x - e^{2x} - e^{3x}$$

8. B. From a table of Laplace transforms, we know

$$\mathscr{L}(y'') = s^2Y(s) - sy(0) - y'(0)$$

$$\mathscr{L}(y') = sY(s) - y(0)$$

$$\mathscr{L}(y) = Y(s)$$

$$\mathscr{L}(g(t)) = G(s)$$

Substitute these into the differential equation.

$$y'' + 4y' + 13y = g(t)$$

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 13Y(s) = G(s)$$

$$s^2Y(s) - sy(0) - y'(0) + 4sY(s) - 4y(0) + 13Y(s) = G(s)$$

Plug in the initial conditions $y(0) = 0$ and $y'(0) = 0$.

$$s^2Y(s) + 4sY(s) + 13Y(s) = G(s)$$

$$(s^2 + 4s + 13)Y(s) = G(s)$$

$$Y(s) = \frac{G(s)}{s^2 + 4s + 13}$$

Complete the square.

$$Y(s) = \frac{G(s)}{(s + 2)^2 + 9}$$

$$Y(s) = \frac{1}{3}G(s)\left(\frac{3}{(s - (-2))^2 + 3^2}\right)$$

This is similar to the transform formula

$$\mathscr{L}(e^{at}\sin(bt)) = \frac{b}{(s - a)^2 + b^2}$$

Now we can identify $a = -2$ and $b = 3$. So the inverse transform is

$$\mathscr{L}^{-1}\left(\frac{3}{(s - (-2))^2 + 3^2}\right) = e^{-2t}\sin(3t)$$

The inverse transform of $G(s)$ is $g(t)$, so for the convolution integral we'll use the functions

$$f(t) = e^{-2t}\sin(3t)$$

$$g(t) = g(t)$$

Plugging these into the convolution integral, we get

$$f(t) * g(t) = \int_0^t f(\tau)g(t - \tau)\ d\tau$$

$$f(t) * g(t) = \int_0^t e^{-2\tau} \sin(3\tau) g(t-\tau) \ d\tau$$

Plugging all of these values back into the equation for $Y(s)$ gives us the inverse transform, which is the general solution to the second order differential equation initial value problem.

$$y(t) = \frac{1}{3} \int_0^t e^{-2\tau} \sin(3\tau) g(t-\tau) \ d\tau$$

9. Start by working on the complementary solution. Find $|A - \lambda I|$.

$$\begin{vmatrix} -\lambda & 4 & 0 \\ -1 & -\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix}$$

$$(-1-\lambda)\begin{vmatrix} -\lambda & 4 \\ -1 & -\lambda \end{vmatrix}$$

$$(-1-\lambda)[(-\lambda)(-\lambda) - (4)(-1)]$$

$$(-1-\lambda)(\lambda^2 + 4)$$

Solve the characteristic equation.

$$(-1-\lambda)(\lambda^2 + 4) = 0$$

$$(\lambda + 1)(\lambda^2 + 4) = 0$$

Then for these Eigenvalues, $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = \pm 2i$, we find

$$A-(-1)I=\begin{bmatrix}1&4&0\\-1&1&0\\0&0&0\end{bmatrix}$$

$$A-(2i)I=\begin{bmatrix}-2i&4&0\\-1&-2i&0\\0&0&-1-2i\end{bmatrix}$$

Put these matrices in reduced row-echelon form.

$$\begin{bmatrix}1&4&0\\-1&1&0\\0&0&0\end{bmatrix}\qquad\begin{bmatrix}-2i&4&0\\-1&-2i&0\\0&0&-1-2i\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}\qquad\begin{bmatrix}1&2i&0\\0&0&1\\0&0&0\end{bmatrix}$$

If we turn these back into systems of equations, we get

$$k_1=0\qquad k_1+2ik_2=0$$

$$k_2=0\qquad k_3=0$$

For the first system, we can choose $k_3=1$. For the second system, we can choose $k_2=1$, so $k_1=-2i$. For the third system, we can choose $k_2=1$.

$$\overrightarrow{k_1}=\begin{bmatrix}0\\0\\1\end{bmatrix}\qquad\overrightarrow{k_2}=\begin{bmatrix}-2i\\1\\0\end{bmatrix}$$

Then the solutions to the system are

$$\overrightarrow{x_1}=\overrightarrow{k_1}e^{\lambda_1 t}\qquad\overrightarrow{x_2}=\overrightarrow{k_2}e^{\lambda_2 t}$$

$$\vec{x_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} \qquad \vec{x_2} = \begin{bmatrix} -2i \\ 1 \\ 0 \end{bmatrix} e^{2it}$$

Write the imaginary solution vector as separate real and imaginary parts.

$$\vec{k_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

The vectors $\vec{b_1} = \text{Re}(\vec{k})$ and $\vec{b_2} = \text{Im}(\vec{k})$ are

$$\vec{b_1} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\vec{b_2} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the complementary solution will be

$$\vec{x_c} = c_1\vec{x_1} + c_2\vec{x_2} + c_3\vec{x_3}$$

$$\vec{x_c} = c_1\vec{x_1} + c_2[\vec{b_1}\cos(\beta t) - \vec{b_2}\sin(\beta t)]e^{\alpha t}$$

$$+c_3[\vec{b_2}\cos(\beta t) + \vec{b_1}\sin(\beta t)]e^{\alpha t}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \left[\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cos(2t) - \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \sin(2t) \right] e^{0t}$$

$$+c_3 \left[\begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix} \cos(2t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \sin(2t) \right] e^{0t}$$

$$\vec{x_c} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2\cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix}$$

Now solve for the particular solution. Rewrite the forcing function vector as

$$\vec{F} = \begin{bmatrix} 4t \\ 5e^t \\ \sin t \end{bmatrix}$$

$$F = \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} t + \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t$$

The guess for the particular solution can be

$$\vec{x_p} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t + \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^t + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

We'll break the guess into three pieces, one each for the polynomial, exponential, and trigonometric parts of the solution.

$$\vec{x_p} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$$

$$\vec{x_p} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^t$$

$$\vec{x}_p = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t$$

Starting with the polynomial part, we get

$$\vec{x}_p{}' = A\vec{x}_p + F$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left[\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} t + \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \right] + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} t$$

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1t + a_1 \\ b_2t + a_2 \\ b_3t + a_3 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} t$$

This matrix equation can be rewritten as a system of equations.

$$b_1 = 0(b_1t + a_1) + 4(b_2t + a_2) + 0(b_3t + a_3) + 4t$$

$$b_2 = -1(b_1t + a_1) + 0(b_2t + a_2) + 0(b_3t + a_3) + 0t$$

$$b_3 = 0(b_1t + a_1) + 0(b_2t + a_2) - 1(b_3t + a_3) + 0t$$

The system simplifies to

$$b_1 = 4b_2t + 4a_2 + 4t$$

$$b_2 = -b_1t - a_1$$

$$b_3 = -b_3t - a_3$$

These equations can each be broken into its own system.

$$4b_2 + 4 = 0 \qquad -b_1 = 0 \qquad -b_3 = 0$$

$$b_1 = 4a_2 \qquad b_2 = -a_1 \qquad b_3 = -a_3$$

This system gives $\vec{a} = (a_1, a_2, a_3) = (1,0,0)$ and $\vec{b} = (b_1, b_2, b_3) = (0,-1,0)$. Now solve the exponential part of the particular solution.

$$\vec{x_p}' = A\vec{x_p} + F$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^t = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix} e^t$$

$$\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

This matrix equation can be rewritten as a system of equations.

$$c_1 = 0c_1 + 4c_2 + 0c_3 + 0$$

$$c_2 = -1c_1 + 0c_2 + 0c_3 + 5$$

$$c_3 = 0c_1 + 0c_2 - 1c_3 + 0$$

The system simplifies to

$$c_1 = 4c_2$$

$$c_2 = -c_1 + 5$$

$$c_3 = -c_3$$

This system gives $\overrightarrow{c} = (c_1, c_2, c_3) = (4,1,0)$. Now solve the trigonometric part of the particular solution.

$$\overrightarrow{x_p}' = A\overrightarrow{x_p} + F$$

$$\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \cos t - \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \sin t = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \left[\begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} \sin t + \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} \cos t \right] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sin t$$

$$\begin{bmatrix} e_1 \cos t - d_1 \sin t \\ e_2 \cos t - d_2 \sin t \\ e_3 \cos t - d_3 \sin t \end{bmatrix} = \begin{bmatrix} 0 & 4 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} e_1 \sin t + d_1 \cos t \\ e_2 \sin t + d_2 \cos t \\ e_3 \sin t + d_3 \cos t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \sin t \end{bmatrix}$$

This matrix equation can be rewritten as a system of equations.

$$e_1 \cos t - d_1 \sin t = 0(e_1 \sin t + d_1 \cos t)$$

$$+4(e_2 \sin t + d_2 \cos t) + 0(e_3 \sin t + d_3 \cos t) + 0$$

$$e_2 \cos t - d_2 \sin t = -1(e_1 \sin t + d_1 \cos t)$$

$$+0(e_2 \sin t + d_2 \cos t) + 0(e_3 \sin t + d_3 \cos t) + 0$$

$$e_3 \cos t - d_3 \sin t = 0(e_1 \sin t + d_1 \cos t)$$

$$+0(e_2 \sin t + d_2 \cos t) - 1(e_3 \sin t + d_3 \cos t) + \sin t$$

The system simplifies to

$$e_1 \cos t - d_1 \sin t = 4e_2 \sin t + 4d_2 \cos t$$

$$e_2 \cos t - d_2 \sin t = -e_1 \sin t - d_1 \cos t$$

$$e_3 \cos t - d_3 \sin t = -e_3 \sin t - d_3 \cos t + \sin t$$

These equations can each be broken into its own system.

$$e_1 = 4d_2 \qquad e_2 = -d_1 \qquad e_3 = -d_3$$

$$-d_1 = 4e_2 \qquad -d_2 = -e_1 \qquad -d_3 = -e_3 + 1$$

This system gives $\overrightarrow{d} = (d_1, d_2, d_3) = (0,0,-1/2)$ and $\overrightarrow{e} = (e_1, e_2, e_3) = (0,0,1/2)$. So the particular solution is

$$\overrightarrow{x_p} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{2} \end{bmatrix} \sin t + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \cos t + \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\overrightarrow{x_p} = \begin{bmatrix} 4e^t + 1 \\ e^t - t \\ \frac{1}{2} \sin t - \frac{1}{2} \cos t \end{bmatrix}$$

Summing the complementary and particular solutions gives the general solution to the nonhomogeneous system of differential equations.

$$\overrightarrow{x} = \overrightarrow{x_c} + \overrightarrow{x_p}$$

$$\overrightarrow{x} = c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 2\sin(2t) \\ \cos(2t) \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} -2\cos(2t) \\ \sin(2t) \\ 0 \end{bmatrix} + \begin{bmatrix} 4e^t + 1 \\ e^t - t \\ \frac{1}{2} \sin t - \frac{1}{2} \cos t \end{bmatrix}$$

10. If the characteristic equation associated with the homogeneous differential equation is

$$r^3(r+1)(r+3)^2(r^2+4r+5)(r^2-2r+3)^3=0$$

then we know the equation has

Distinct real roots	$r_1=-1$
Equal real roots	$r_2=r_3=r_4=0$
	$r_5=r_6=-3$
Complex roots	$r_7=r_8=-2\pm i$
	$r_9=r_{10}=1\pm\sqrt{2}i$
	$r_{11}=r_{12}=1\pm\sqrt{2}i$
	$r_{13}=r_{14}=1\pm\sqrt{2}i$

For the complex roots $r_7=r_8=-2\pm i$ associated with (r^2+4r+5), we know $\alpha=-2$ and $\beta=1$, so the complex conjugate solution pairs will be

$e^{-2t}\cos t$ and $e^{-2t}\sin t$

The complex roots associated with $(r^2-2r+3)^3$ have multiplicity three, which means we'll have three pairs of complex roots,

$e^{\alpha t}\cos(\beta t)$ and $e^{\alpha t}\sin(\beta t)$

$te^{\alpha t}\cos(\beta t)$ and $te^{\alpha t}\sin(\beta t)$

$t^2e^{\alpha t}\cos(\beta t)$ and $t^2e^{\alpha t}\sin(\beta t)$

With $\alpha = 1$ and $\beta = \sqrt{2}$, the complex conjugate solution pairs will be

$e^t\cos(\sqrt{2}t)$ and $e^t\sin(\sqrt{2}t)$

$te^t\cos(\sqrt{2}t)$ and $te^t\sin(\sqrt{2}t)$

$t^2e^t\cos(\sqrt{2}t)$ and $t^2e^t\sin(\sqrt{2}t)$

The distinct real roots portion of the solution will be c_1e^{-t}, and the equal real roots portion will be $c_2e^{0t} + c_3te^{0t} + c_4t^2e^{0t}$, or $c_2 + c_3t + c_4t^2$, and $c_5e^{-3t} + c_6te^{-3t}$. The complex conjugate roots portion will be

$$c_7e^{-2t}\cos t + c_8e^{-2t}\sin t$$

$$+c_9e^t\cos(\sqrt{2}t) + c_{10}e^t\sin(\sqrt{2}t)$$

$$+c_{11}te^t\cos(\sqrt{2}t) + c_{12}te^t\sin(\sqrt{2}t)$$

$$+c_{13}t^2e^t\cos(\sqrt{2}t) + c_{14}t^2e^t\sin(\sqrt{2}t)$$

Therefore the general solution of the homogeneous linear differential equation is

$$y(t) = c_1e^{-t} + c_2 + c_3t + c_4t^2 + c_5e^{-3t} + c_6te^{-3t} + c_7e^{-2t}\cos t + c_8e^{-2t}\sin t$$

$$+c_9e^t\cos(\sqrt{2}t) + c_{10}e^t\sin(\sqrt{2}t)$$

$$+c_{11}te^t\cos(\sqrt{2}t) + c_{12}te^t\sin(\sqrt{2}t)$$

$$+c_{13}t^2e^t\cos(\sqrt{2}t)+c_{14}t^2e^t\sin(\sqrt{2}t)$$

11. The function $f(x)=x^2\sin^2(\pi x)$ is an even function, which means the Fourier sine series will be 0. Therefore, we don't need to calculate B_n, only A_0,

$$A_0=\frac{1}{L}\int_0^L f(x)\ dx$$

$$A_0=\int_0^1 x^2\sin^2(\pi x)\ dx$$

$$A_0=\frac{1}{2}\int_0^1 x^2(1-\cos(2\pi x))\ dx$$

$$A_0=\frac{1}{2}\int_0^1 x^2\ dx-\frac{1}{2}\int_0^1 x^2\cos(2\pi x)\ dx$$

$$A_0=\frac{x^3}{6}\Big|_0^1-\frac{1}{2}\int_0^1 x^2\cos(2\pi x)\ dx$$

Use integration by parts with $u=x^2$, $du=2x\ dx$, $dv=\cos(2\pi x)\ dx$, and $v=(1/2\pi)\sin(2\pi x)$.

$$A_0=\frac{1}{6}-\frac{1}{2}\left(\frac{x^2}{2\pi}\sin(2\pi x)\Big|_0^1-\frac{1}{\pi}\int_0^1 x\sin(2\pi x)\ dx\right)$$

$$A_0=\frac{1}{6}-\frac{1}{4\pi}\sin(2\pi)+\frac{1}{2\pi}\int_0^1 x\sin(2\pi x)\ dx$$

Use integration by parts with $u=x$, $du=dx$, $dv=\sin(2\pi x)\ dx$, and $v=(-1/2\pi)\cos(2\pi x)$.

$$A_0=\frac{1}{6}-\frac{1}{4\pi}\sin(2\pi)+\frac{1}{2\pi}\left(-\frac{x}{2\pi}\cos(2\pi x)\Big|_0^1+\frac{1}{2\pi}\int_0^1\cos(2\pi x)\ dx\right)$$

$$A_0=\frac{1}{6}-\frac{1}{4\pi}\sin(2\pi)-\frac{1}{(2\pi)^2}\cos(2\pi)+\frac{1}{(2\pi)^2}\int_0^1\cos(2\pi x)\ dx$$

$$A_0=\frac{1}{6}-\frac{1}{4\pi}\sin(2\pi)-\frac{1}{(2\pi)^2}\cos(2\pi)+\left(\frac{1}{(2\pi)^3}\sin(2\pi x)\Big|_0^1\right)$$

$$A_0=\frac{1}{6}-\frac{1}{4\pi}\sin(2\pi)-\frac{1}{(2\pi)^2}\cos(2\pi)+\frac{1}{(2\pi)^3}\sin(2\pi)$$

$$A_0=\frac{1}{6}-\frac{1}{4\pi^2}$$

and A_n.

$$A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$A_n=2\int_0^1 x^2\sin^2(\pi x)\cos(n\pi x)\ dx$$

$$A_n=2\int_0^1 x^2(1-\cos(2\pi x))\cos(n\pi x)\ dx$$

$$A_n=2\int_0^1 x^2(\cos(n\pi x)-\cos(2\pi x)\cos(n\pi x))\ dx$$

$$A_n = \int_0^1 x^2 \left(\cos(n\pi x) - \frac{1}{2}\cos \pi x(n-2) - \frac{1}{2}\cos \pi x(n+2) \right) \ dx$$

Use integration by parts with

$$u = x^2$$

$$du = 2x \ dx$$

$$dv = \cos(n\pi x) - \frac{1}{2}\cos \pi x(n-2) - \frac{1}{2}\cos \pi x(n+2) \ dx$$

$$v = \frac{1}{n\pi}\sin(n\pi x) - \frac{1}{2\pi(n-2)}\sin \pi x(n-2) - \frac{1}{2\pi(n+2)}\sin \pi x(n+2)$$

$$A_n = x^2 \left(\frac{1}{n\pi}\sin(n\pi x) - \frac{1}{2\pi(n-2)}\sin \pi x(n-2) - \frac{1}{2\pi(n+2)}\sin \pi x(n+2) \right) \Bigg|_0^1$$

$$-\frac{2}{\pi}\int_0^1 x \left(\frac{1}{n}\sin(n\pi x) - \frac{1}{2(n-2)}\sin \pi x(n-2) - \frac{1}{2(n+2)}\sin \pi x(n+2) \right)$$

Use integration by parts with

$$u = x$$

$$du = \ dx$$

$$dv = \frac{1}{n}\sin(n\pi x) - \frac{1}{2(n-2)}\sin \pi x(n-2) - \frac{1}{2(n+2)}\sin \pi x(n+2) \ dx$$

$$v = -\frac{1}{n^2\pi}\cos(n\pi x) + \frac{1}{2\pi(n-2)^2}\cos \pi x(n-2) + \frac{1}{2\pi(n+2)^2}\cos \pi x(n+2)$$

$$A_n = -\frac{2}{\pi}x \left(-\frac{1}{n^2\pi}\cos(n\pi x) + \frac{1}{2\pi(n-2)^2}\cos \pi x(n-2) + \frac{1}{2\pi(n+2)^2}\cos \pi x(n+2) \right) \Bigg|_0^1$$

$$+\frac{2}{\pi}\int_0^1\left(\frac{1}{n^2\pi}\cos(n\pi x)-\frac{1}{2\pi(n-2)^2}\cos\pi x(n-2)-\frac{1}{2\pi(n+2)^2}\cos\pi x(n+2)\right)\,dx$$

$$A_n=-\frac{2}{\pi}\left(-\frac{(-1)^n}{n^2\pi}+\frac{(-1)^n}{2\pi(n-2)^2}+\frac{(-1)^n}{2\pi(n+2)^2}\right)$$

$$+\frac{2}{\pi}\left(\frac{1}{n^3\pi^2}\sin(n\pi x)-\frac{1}{2\pi^2(n-2)^3}\sin\pi x(n-2)-\frac{1}{2\pi^2(n+2)^3}\sin\pi x(n+2)\right)\Big|_0^1$$

$$A_n=\frac{(-1)^n}{\pi^2}\left(\frac{2}{n^2}+\frac{1}{(n-2)^2}+\frac{1}{(n+2)^2}\right)$$, with $n\neq 2$

So for $n=2$, we find

$$A_2=2\int_0^1 x^2\sin^2(\pi x)\cos(2\pi x)\,dx$$

$$A_2=\int_0^1 x^2(1-\cos(2\pi x))\cos(2\pi x)\,dx$$

$$A_2=\int_0^1 x^2\left(\cos(2\pi x)-\frac{1}{2}-\frac{\cos(4\pi x)}{2}\right)\,dx$$

$$A_2=-\int_0^1\frac{x^2}{2}\,dx+\int_0^1 x^2\left(\cos(2\pi x)-\frac{\cos(4\pi x)}{2}\right)\,dx$$

$$A_2=-\frac{x^3}{6}\Big|_0^1+\left(\frac{x\cos(2\pi x)}{2\pi^2}-\frac{x\cos(4\pi x)}{16\pi^2}\right)\Big|_0^1$$

$$A_2=-\frac{1}{6}+\frac{7}{16\pi^2}$$

Then the Fourier series for $f(x)=x^2\sin^2(\pi x)$ on $-1\leq x\leq 1$ is

$$f(x) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right)$$

$$f(x) = \frac{1}{6} - \frac{1}{4\pi^2} + \left(-\frac{1}{6} + \frac{7}{16\pi^2}\right)\cos(2\pi x)$$

$$+ \sum_{n=1, n\neq 2}^{\infty} \frac{(-1)^n}{\pi^2}\left(\frac{2}{n^2} + \frac{1}{(n-2)^2} + \frac{1}{(n+2)^2}\right)\cos(n\pi x)$$

12. We're solving the heat equation with two boundary conditions and an initial condition.

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

with $u(0,t) = 10$ and $u(L, t) = -5$

$$u(x,0) = 20$$

Equilibrium temperature is

$$u_E(x) = T_1 + \frac{T_2 - T_1}{L}x$$

$$u_E(x) = 10 + \frac{-5 - 10}{L}x$$

$$u_E(x) = 10 - \frac{15}{L}x$$

Next, we'll find the coefficients B_n.

$$B_n = \frac{2}{L}\int_0^L (f(x) - u_E(x))\sin\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = \frac{2}{L}\int_0^L \left(10 + \frac{15}{L}x\right)\ \sin\left(\frac{n\pi x}{L}\right)\ dx$$

Use integration by parts with $u = 10 + (15/L)x$, $du = (15/L)\ dx$, $dv = \sin(n\pi x/L)\ dx$, and $v = -(L/n\pi)\cos(n\pi x/L)$.

$$B_n = -\frac{2}{\pi n}\left(10 + \frac{15}{L}x\right)\ \cos\left(\frac{n\pi x}{L}\right)\Bigg|_0^L + \frac{30}{Ln\pi}\int_0^L \cos\left(\frac{n\pi x}{L}\right)\ dx$$

$$B_n = -\frac{2}{\pi n}(25(-1)^n - 10) + \frac{30}{n^2\pi^2}\sin\left(\frac{n\pi x}{L}\right)\Bigg|_0^L$$

$$B_n = -\frac{2}{\pi n}(25(-1)^n - 10)$$

$$B_n = \frac{2}{\pi n}(10 - 25(-1)^n)$$

Then the solution to this heat equation is

$$u(x,t) = u_E + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

$$u(x,t) = 10 - \frac{15}{L}x + \sum_{n=1}^{\infty}\frac{10}{n\pi}(2 - 5(-1)^n)\sin\left(\frac{n\pi x}{L}\right)e^{-k\left(\frac{n\pi}{L}\right)^2 t}$$

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