3 – IMAGE TRANSFORMS (B)

THE DISCRETE FOURIER TRANSFORM

Suppose that a continuous function $f(x)$ is discretized into a sequence

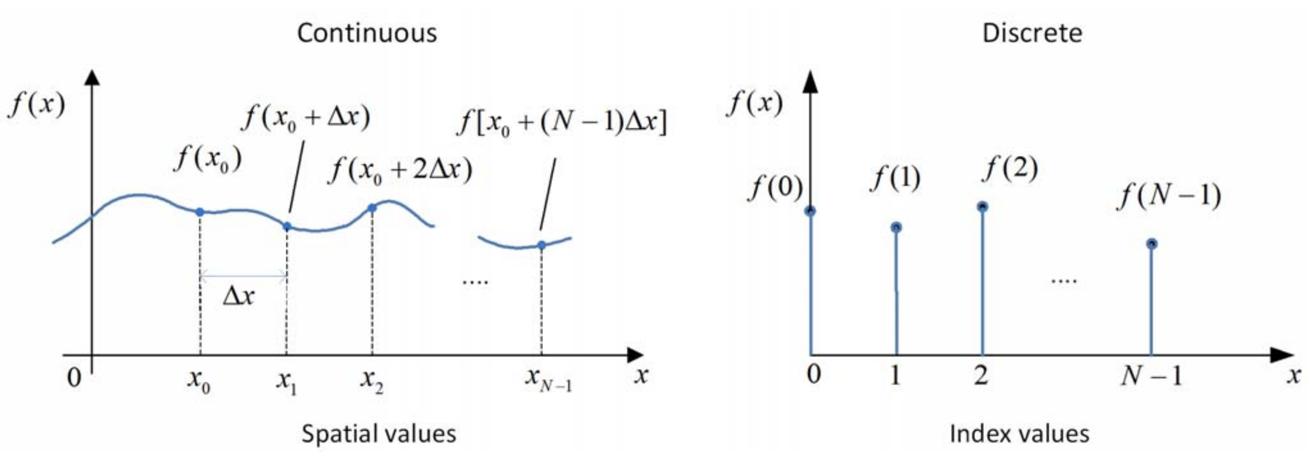
$$\{f(x_0), f(x_0+\Delta x), f(x_0+2\Delta x), \ldots, f(x_0+[N-1]\Delta x)\}$$

by taking N samples Δx units apart.
Note that x may be used as either a discrete or continuous variable, depending on context. In the discrete case, we define

$$f(x) = f(x_0 + x\Delta x) \qquad x = 0, 1, 2, \ldots, N-1$$

i.e.,

$$\begin{aligned}
f(0) &= f(x_0) \\
f(1) &= f(x_0+\Delta x) \\
f(2) &= f(x_0+2\Delta x) \\
f(3) &= f(x_0+3\Delta x) \\
&\cdots \\
f(N-1) &= f[x_0+(N-1)\Delta x]
\end{aligned}$$



The discrete Fourier transform (DFT) pair is given by

$$F(u) = \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N]; \quad u = 0,1,2,\ldots,N-1 \tag{1}$$

$$f(x) = \sum_{u=0}^{N-1} F(u)\exp[j2\pi ux/N]; \quad x = 0,1,2,\ldots,N-1 \tag{2}$$

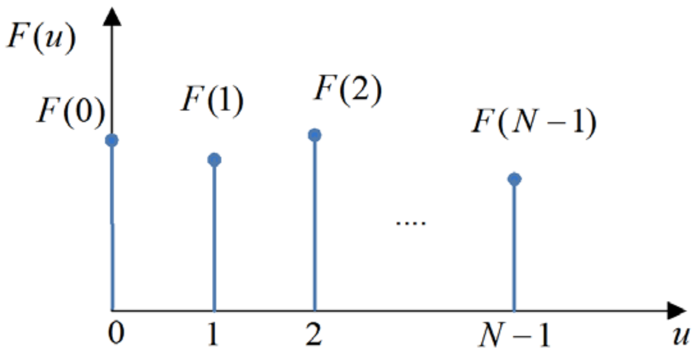
The terms Δu and Δx are related by

$$\Delta u = \frac{1}{N\Delta x} \tag{3}$$

$$F(u) = F(u\Delta u) \qquad u = 0,1,2,\ldots,N-1$$

i.e.,

$$\begin{aligned} F(0) &= F(0) \\ F(1) &= F(\Delta u) \\ F(2) &= F(2\Delta u) \\ F(3) &= F(3\Delta u) \\ \ldots & \\ f(N-1) &= f((N-1)\Delta u) \end{aligned}$$



For example,
$\Delta x = 2$ mm, $\quad N = 500$
$\Delta u = \frac{1}{N\Delta x} = 1 \times 10^{-3}$ cycles per mm

In the 2D case, the DFT pair is

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) \exp[-j2\pi(ux/M + vy/N)]; \tag{4}$$

$$u = 0,1,2,\ldots,M-1,\ v = 0,1,2,\ldots,N-1,$$

and

$$f(x,y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \exp[j2\pi(ux/M + vy/N)]; \tag{5}$$

$$\text{for } x = 0,1,2,\ldots,M-1,\ y = 0,1,2,\ldots,N-1.$$

The sampling increments in the spatial and frequency domains are related by

$$\Delta u = \frac{1}{M\Delta x}, \qquad \Delta v = \frac{1}{N\Delta y} \tag{6}$$

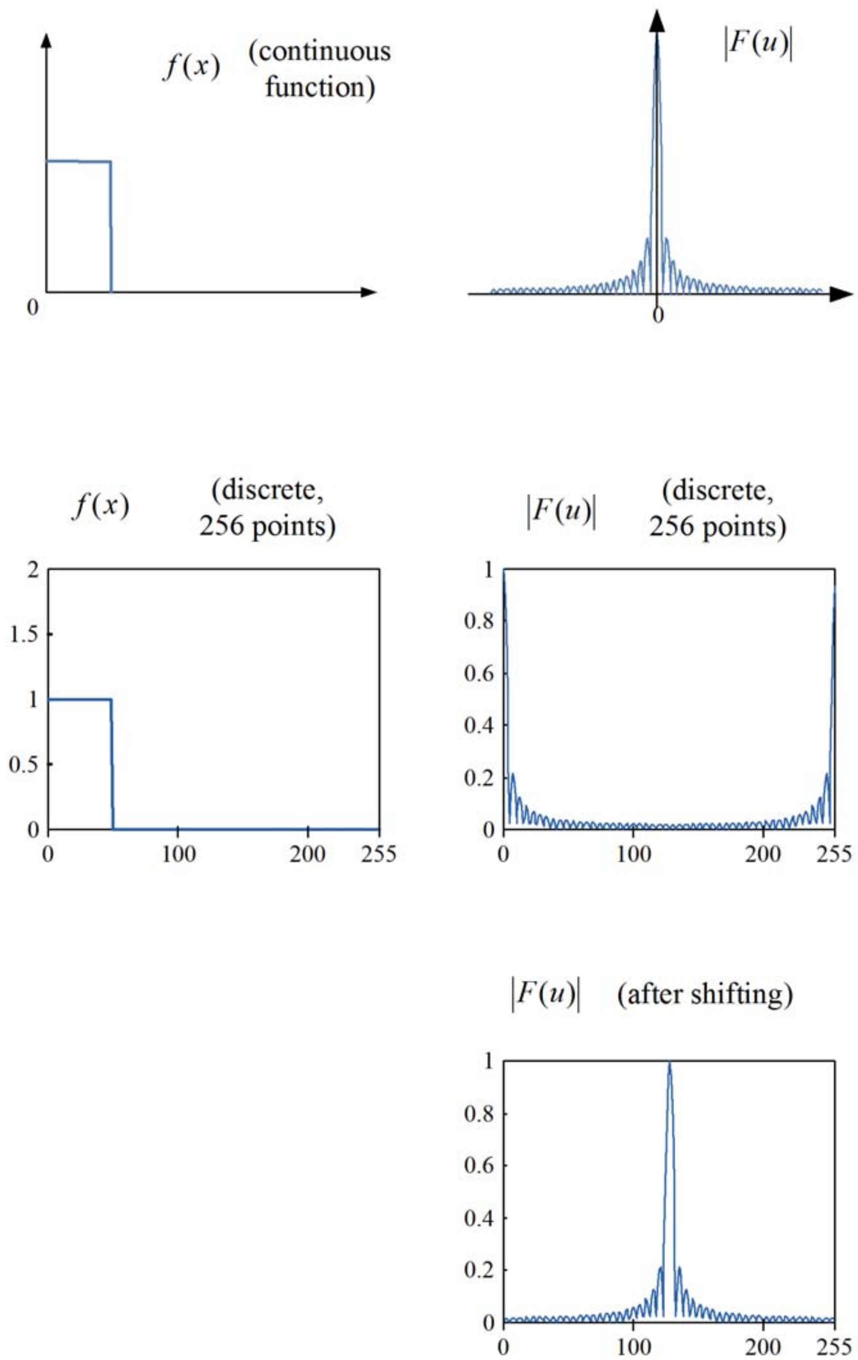
The Fourier spectrum, phase, and power spectrum of 1D and 2D discrete functions are computed as for the continuous case.

Fourier spectrum:	$\|F(u,v)\|$	$= [R^2(u,v) + I^2(u,v)]^{1/2}$
Phase spectrum:	$\phi(u,v)$	$= \tan^{-1}[I(u,v)/R(u,v)]$
Power spectrum:	$P(u,v)$	$= \|F(u,v)\|^2 = R^2(u,v) + I^2(u,v)$

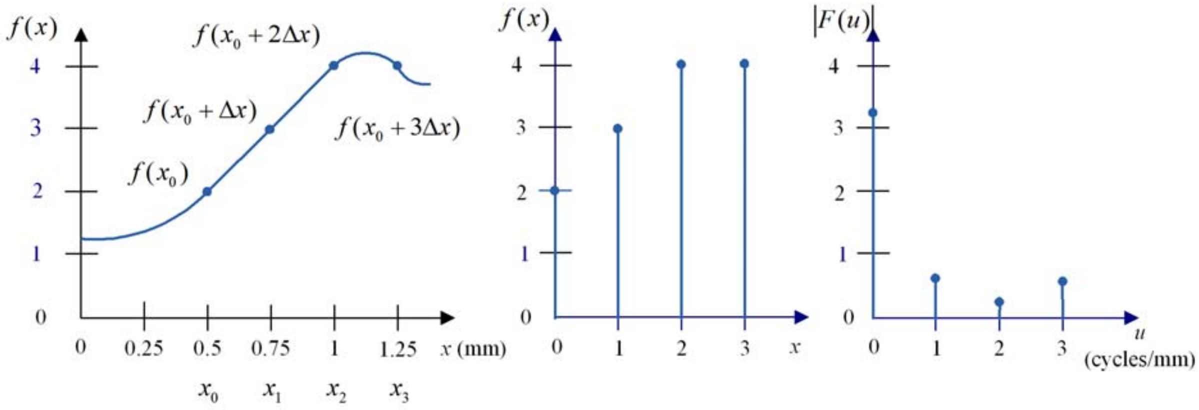
The direct computation of an N-point DFT requires of the order of N^2 operations. For an $M \times N$ array, M^2N^2 operations are required. This can be considerably reduced by the fast Fourier transform (FFT) algorithm to $MN\log_2(MN)$ operations. Suppose $M = N = 2^9$. Then

Direct DFT:	M^2N^2	$= 69 \times 10^9$
FFT:	$MN\log_2(MN)$	$= 4.7 \times 10^6$

Example (1D DFT)



Example (1D DFT)



Sampling takes place at
$x_0 = 0.5$ mm, $x_1 = 0.75$ mm, $x_2 = 1.0$ mm, $x_3 = 1.25$ mm, giving
$f(0) = 2,\ f(1) = 3,\ f(2) = 4,\ f(3) = 4.$

$$
\begin{aligned}
F(u) &= \frac{1}{N}\sum_{x=0}^{N-1} f(x)\exp[-j2\pi ux/N] = \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi ux/4] \\
F(0) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp(0) \\
&= \frac{1}{4}[f(0)+f(1)+f(2)+f(3)] = \frac{1}{4}[2+3+4+4] \\
&= 3.25 \quad \text{or} \quad 3.25\angle 0^\circ \\
F(1) &= \frac{1}{4}\sum_{x=0}^{3} f(x)\exp[-j2\pi x/4] \\
&= \frac{1}{4}[f(0)\exp(0)+f(1)\exp(-j\pi/2)+f(2)\exp(-j\pi)+f(3)\exp(-j3\pi/2)] \\
&= \frac{1}{4}(-2+j) \quad \text{or} \quad 0.56\angle 153^\circ
\end{aligned}
$$

Similarly,

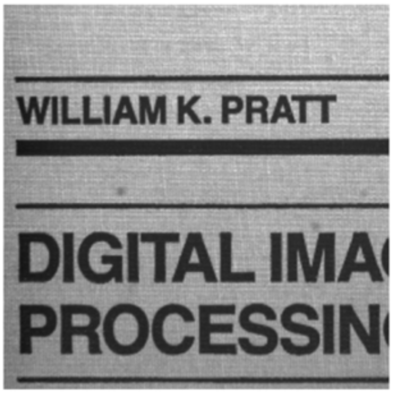
$$F(2) = -\frac{1}{4}(1+j\times 0) \quad \text{or} \quad 0.25\angle 180^\circ, \qquad F(3) = -\frac{1}{4}(2+j) \quad \text{or} \quad 0.56\angle -153^\circ$$

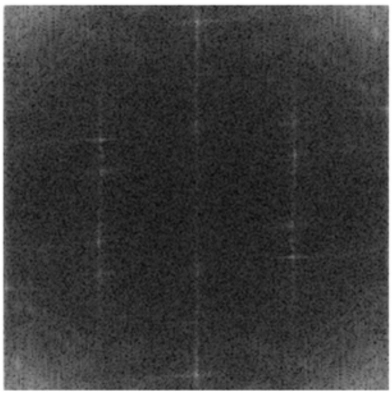
Fourier spectrum:

$$|F(0)| = 3.25, \quad |F(1)| = 0.56 \quad |F(2)| = 0.25 \quad |F(3)| = 0.56$$

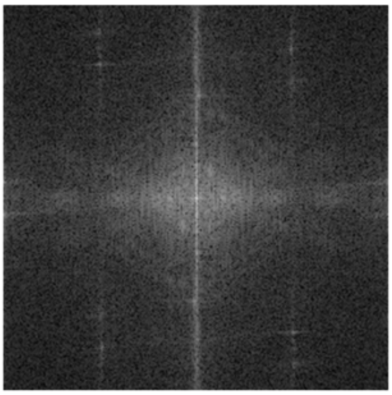
In this example,
$N = 4, \qquad \Delta x = 0.25$ mm, $\qquad \Delta u = 1/(N\Delta x) = 1$ cycle/mm

Examples of 2D DFT

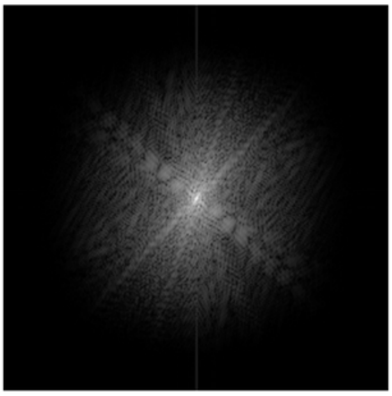


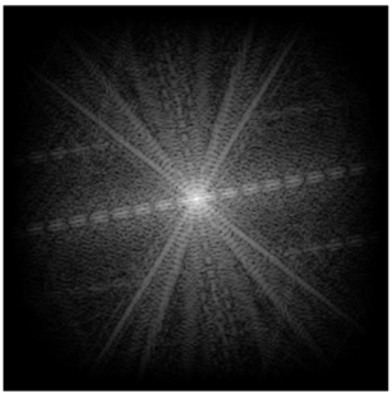


Without shifting



With shifting





Some Properties of the 2D DFT

Separability

We can write Eq. (4) in the separable form

$$\begin{aligned} F(u,v) &= \frac{1}{M}\sum_{x=0}^{M-1} \exp[-j2\pi ux/M] \times \\ &\qquad \left\{\frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N]\right\} && (7) \\ &= \frac{1}{M}\sum_{x=0}^{M-1} \exp[-j2\pi ux/M] \times \{F_r(x,v)\} && (8) \\ &= \frac{1}{M}\sum_{x=0}^{M-1} F_r(x,v)\exp[-j2\pi ux/M] && (9) \end{aligned}$$

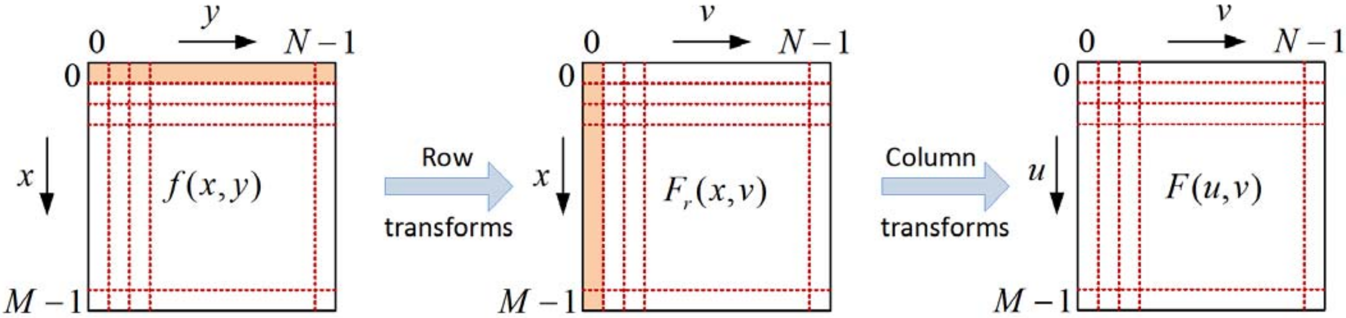
where

$$F_r(x,v) = \frac{1}{N}\sum_{y=0}^{N-1} f(x,y)\exp[-j2\pi vy/N] \qquad (10)$$

Because of the separability property, $F(u,v)$ (or $f(x,y)$) can be obtained in two steps by successive applications of the 1D Fourier transform (or its inverse).

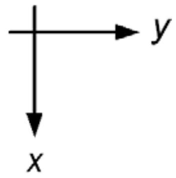
For each value of x, the expression inside the brackets is a 1D transform, with frequency values $v = 0, 1, \ldots, N-1$. Therefore the 2D function $F(x,v)$ is obtained by taking a transform along each row of $f(x,y)$. The desired result $F(u,v)$ is then obtained by taking a transform along each column of $F(x,v)$.

The same results may be obtained by first taking transforms along the columns of $f(x,y)$ and then along the rows of that result.



Example

In general, for DFT computation, we will use this convention: x axis points down, y axis to the right, origin at the top left corner.



$$f(x,y) = \begin{bmatrix} 4 & 2 & 2 & 1 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Taking the row transforms:

$$F_r(x,v) = \frac{1}{4}\begin{bmatrix} 9 & 2-j & 3 & 2+j \\ 4 & 1-j & 2 & 1+j \\ 2 & 1-j & 0 & 1+j \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Taking the column transforms:

$$F(u,v) = \frac{1}{16}\begin{bmatrix} 16 & 5-j3 & 6 & 5+j3 \\ 7-j3 & 0 & 3-j & 2 \\ 6 & 1-j1 & 0 & 1+j \\ 7+j3 & 2 & 3+j & 0 \end{bmatrix}$$

Average Value

The average value of the digital image $f(x,y)$ is

$$\bar{f}(x,y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) \tag{11}$$

It is easily shown that

$$\bar{f}(x,y) = F(0,0) \tag{12}$$

Translation

The translation property is

$$f(x-a, y-b) \leftrightarrow F(u,v)\exp[-j2\pi(ua/M + vb/N)] \tag{13}$$

For example, for $M = 100, N = 100$, and $a = 20, b = 40$,

$$\begin{aligned} f_1(x,y) &= f(x-20, y-40) && (14)\\ F_1(u,v) &= \mathcal{F}\{f(x-20,y-40)\} && (15)\\ &= F(u,v)\ \exp[-j2\pi(u/5 + 2v/5)] && (16)\\ &= F(u,v)\ \exp(-j2\pi u/5)\ \exp(-j4\pi v/5) && (17) \end{aligned}$$

Note that a shift in $f(x,y)$ does not affect the magnitude of its Fourier transform since

$$|F(u,v)\exp[-j2\pi(ua/M + vb/N)]| = |F(u,v)| \tag{18}$$

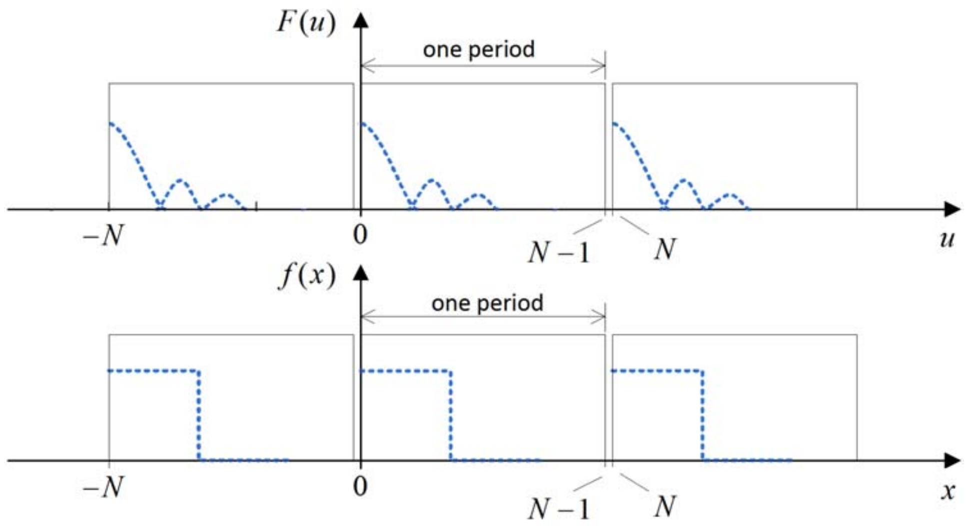
Periodicity and Conjugate Symmetry

From the definition of the DFT, we can show that $F(u)$ is periodic with period N:

$$F(u) = F(u + kN) \quad k = 0, \pm 1, \pm 2, \ldots \tag{19}$$

This periodicity property also applies to the inverse of $F(u)$, i.e., $f(x)$ computed from Eq. 2 is periodic:

$$f(x) = f(x + kN) \quad k = 0, \pm 1, \pm 2, \ldots \tag{20}$$

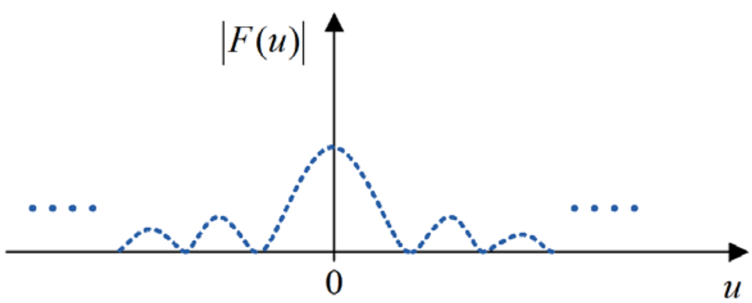


Furthermore, if $f(x)$ is real, the DFT also exhibits conjugate symmetry:

$$F(u) = F^*(-u)$$

Furthermore, the DFT magnitude is then symmetrical about $u = 0$ since

$$|F(u)| = |F(-u)|$$



Combining conjugate symmetry and periodiocity,

$$F(u) = F*(-u) = F^*(-u+N) \tag{21}$$

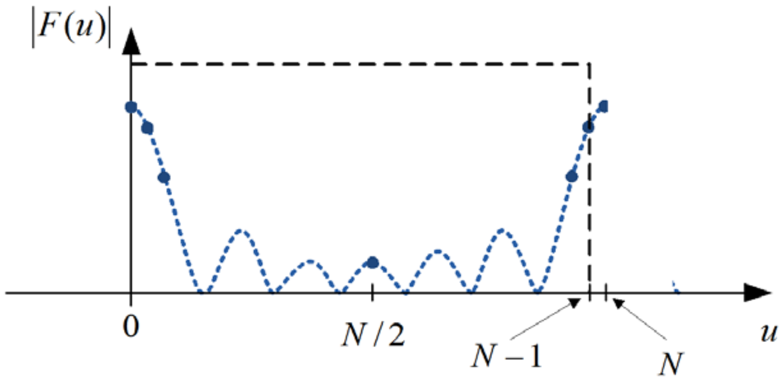
Hence, for the DFT magnitude function $|F(u)|$, we have

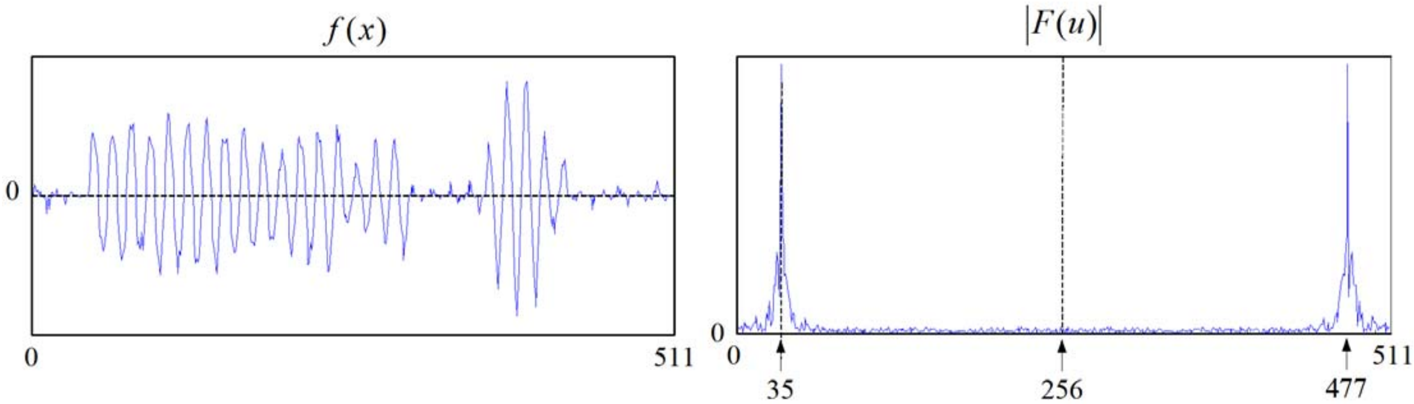
$$|F(u)| = |F(-u+N)| \tag{22}$$

In the window $u = 0, 1, \ldots, N-1$,

$$\begin{array}{rcl} |F(0)| & = & |F(N)| \\ |F(1)| & = & |F(N-1)| \\ |F(2)| & = & |F(N-2)| \\ & \cdots & \\ |F(N/2)| & = & |F(N/2) \end{array}$$

i.e., $|F(u)|$ is symmetrical about $u = N/2$.

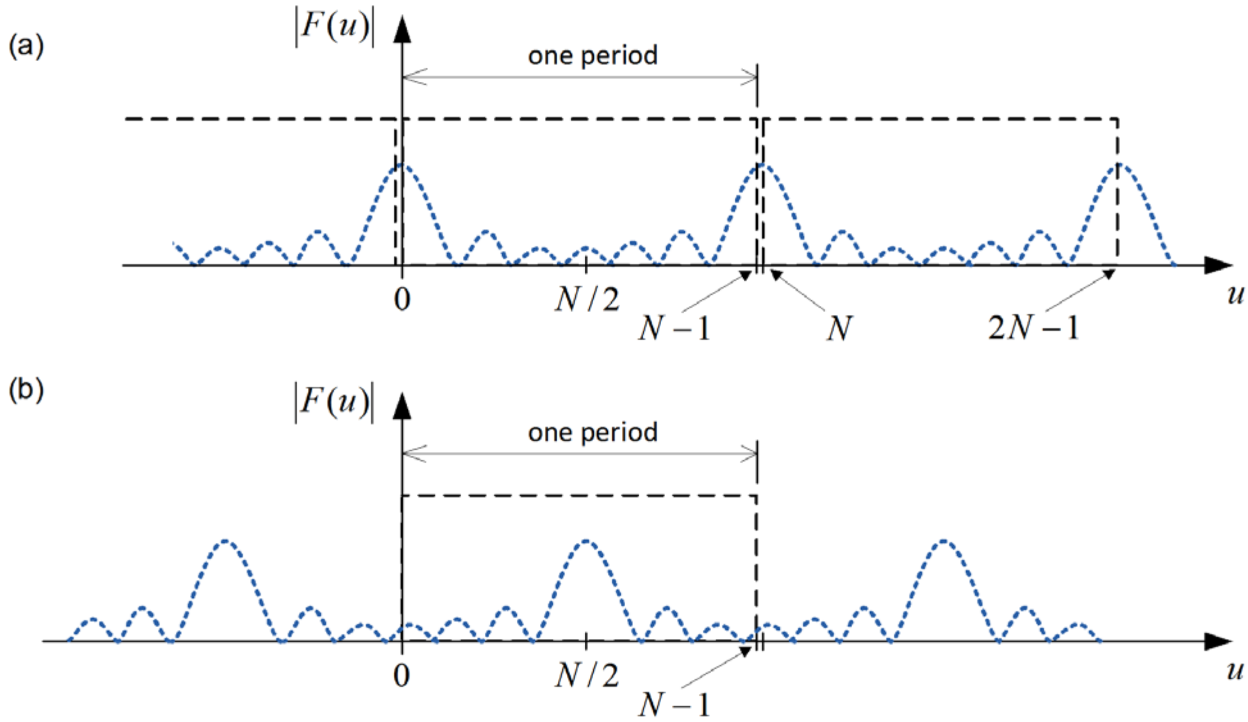




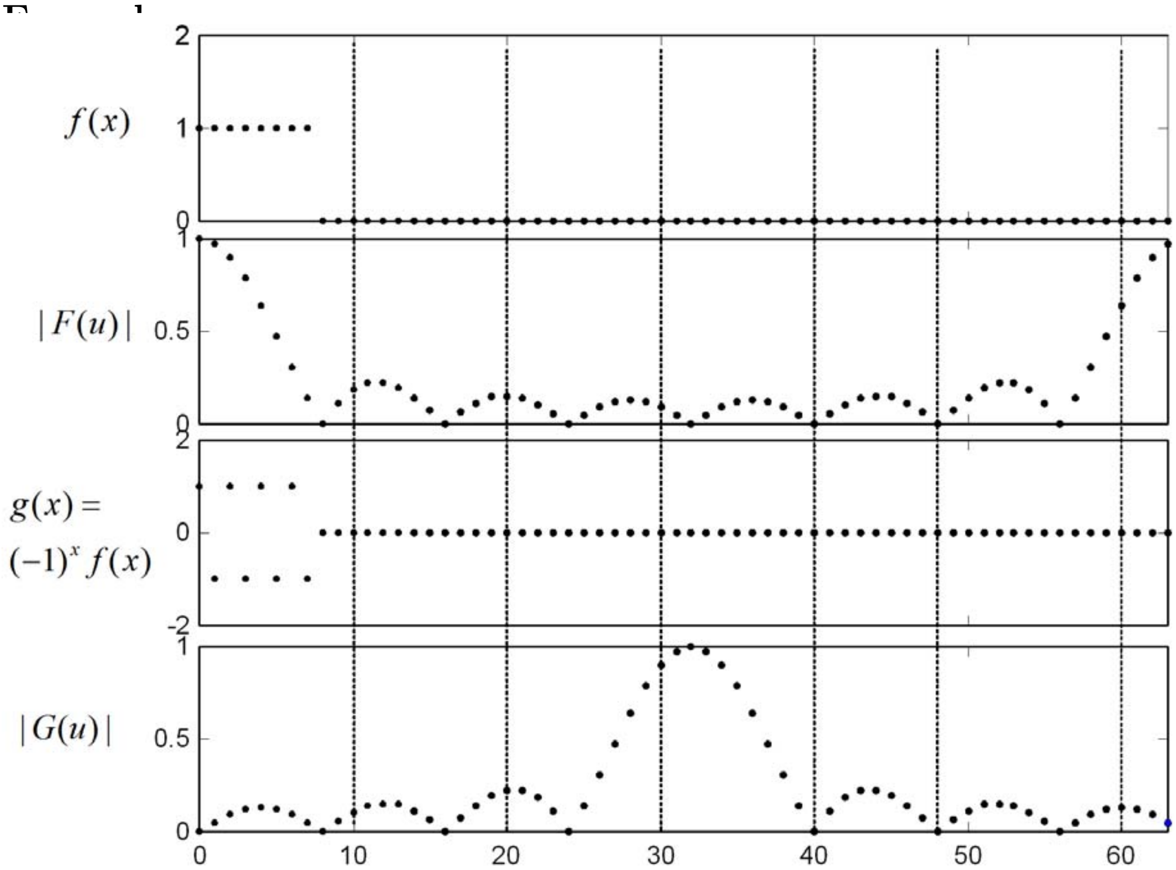
Because of the periodicity property, the values of $|F(u)|$ in this window are repeated along the u axis.

For viewing purposes, it would be better to centre $|F(u)|$, i.e., move the origin of the transform to the point $u = N/2$. This is done by multiplying $f(x)$ by $(-1)^x$ prior to taking the transform, i.e.,

$$f'(x) = f(x)\,(-1)^x$$



(a) Fourier spectrum showing back-to-back half periods in the window [0, N - 1];
(b) Shifted spectrum showing a full period in the same window.



Proof of Periodicity and Conjugate Symmetry

$$
\begin{aligned}
F(u) &= (1/N) \sum_{x=0}^{N-1} f(x) \exp[-j2\pi ux/N] \\
F(u+N) &= (1/N) \sum_{x} f(x) \exp[-j2\pi(u+N)x/N] \\
&= (1/N) \sum_{x} f(x) \exp[-j2\pi ux/N - j2\pi Nx/N] \\
&= (1/N) \sum_{x} f(x) \exp[-j2\pi ux/N] \exp[-j2\pi x] \\
&= (1/N) \sum_{x} f(x) \exp[-j2\pi ux/N] \quad \text{since } \exp[-j2\pi x] \equiv 1 \\
&= F(u)
\end{aligned}
$$

$$
\begin{aligned}
F(-u) &= (1/N) \sum_{x} f(x) \exp[j2\pi ux/N] \\
F^*(-u) &= (1/N) \sum_{x} f^*(x) \exp[-j2\pi ux/N] \\
&= (1/N) \sum_{x} f(x) \exp[-j2\pi ux/N] \quad \text{for real } f \\
&= F(u)
\end{aligned}
$$

For the 2D case involving an $N \times N$ image, both $f(x, y)$ and $F(u, v)$ are **periodic**:

$$\begin{aligned} f(x,y) &= f(x+kN, y+lN) \quad k,l = 0, \pm 1, \pm 2, \ldots & (23) \\ F(u,v) &= F(u+kN, v+lN) \quad k,l = 0, \pm 1, \pm 2, \ldots & (24) \end{aligned}$$

If $f(x, y)$ is real, the DFT also exhibits conjugate symmetry:

$$F(u,v) = F^*(-u,-v) \tag{25}$$

Hence the DFT magnitude function $|F(u, v)|$ is **symmetrical about the origin**:

$$|F(u,v)| = |F(-u,-v)| \tag{26}$$

Similar to the 1D case, combining periodicity and symmetry about the origin means that in the window $u, v = 0, 1, \ldots, N-1$, $|F(u, v)|$ is symmetrical about $(u, v) = (N/2, N/2)$, i.e.,

$$|F(u,v)| = |F(N-u, N-v)| \tag{27}$$

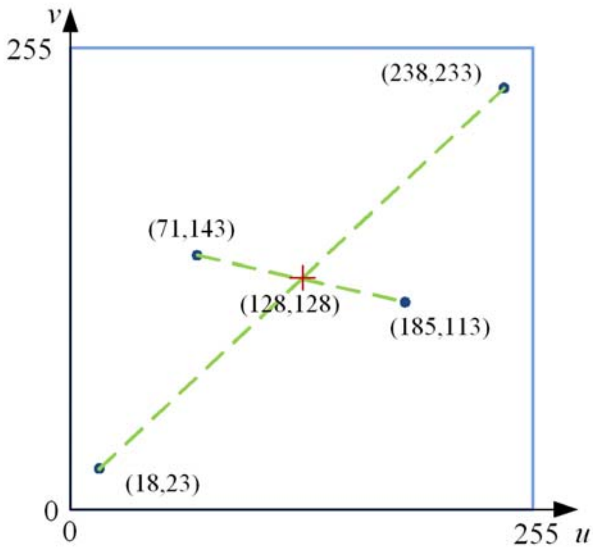
To move the origin of the transform to the point $(u, v) = (N/2, N/2)$, we multiply $f(x, y)$ by $(-1)^{x+y}$ prior to taking the transform, i.e.,

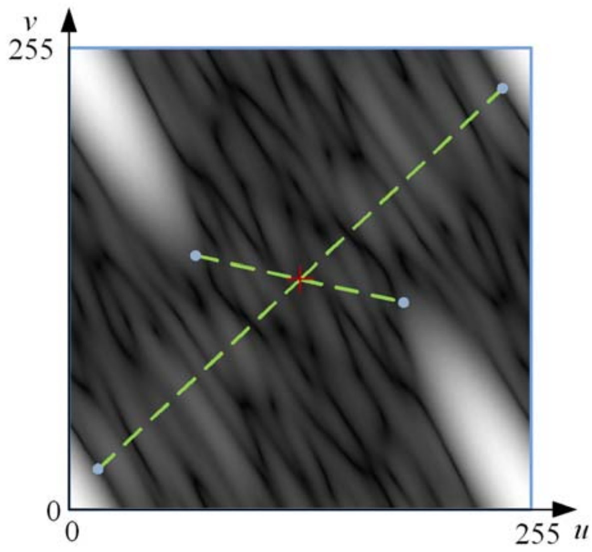
$$f'(x,y) = f(x,y)\,(-1)^{x+y}$$

Example

Consider a square image of size $N = 256$. $|F(u,v)|$ is symmetrical about $(u,v) = (128,128)$.

$$\begin{aligned} |F(u,v)| &= |F(256-u, 256-v)| \\ |F(18,23)| &= |F(238,233)| \\ |F(71,143)| &= |F(185,113)| \end{aligned}$$





Summary:

$f(x)$ and $F(u)$ are periodic ...... periodicity

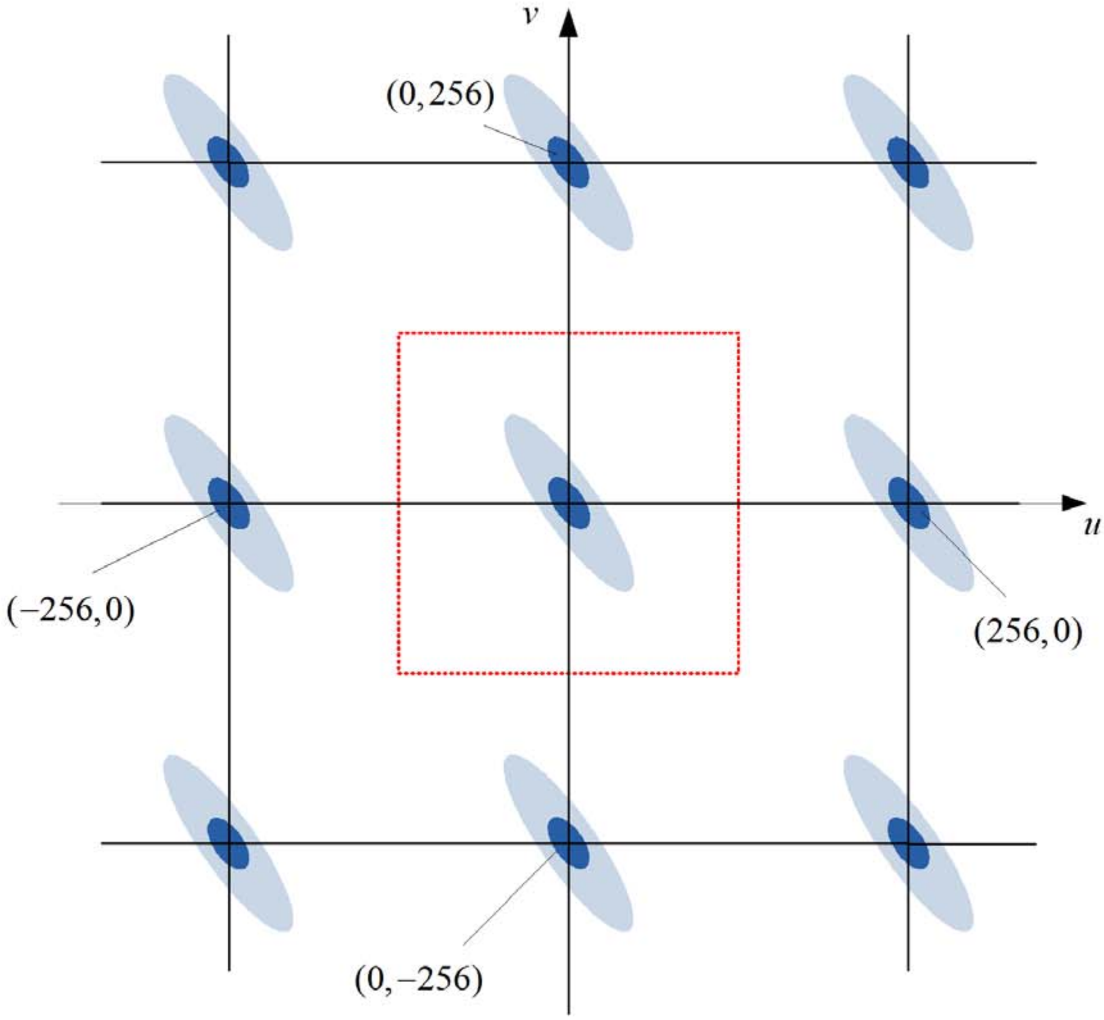
For real $f(x)$:
$|F(u)| = |F(-u)|$ ...... symmetry about origin

For real $f(x)$, periodicity and symmetry
$\Rightarrow |F(u)| = |F(N-u)|$ ...... symmetry about $u = N/2$

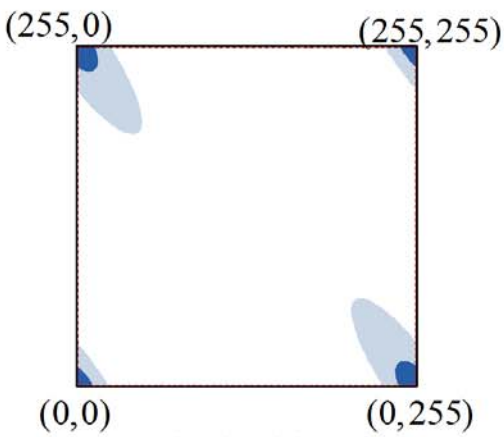
$f(x,y)$ and $F(u,v)$ are periodic ...... periodicity

For real $f(x,y)$:
$|F(u,v)| = |F(-u,-v)|$ ...... symmetry about origin

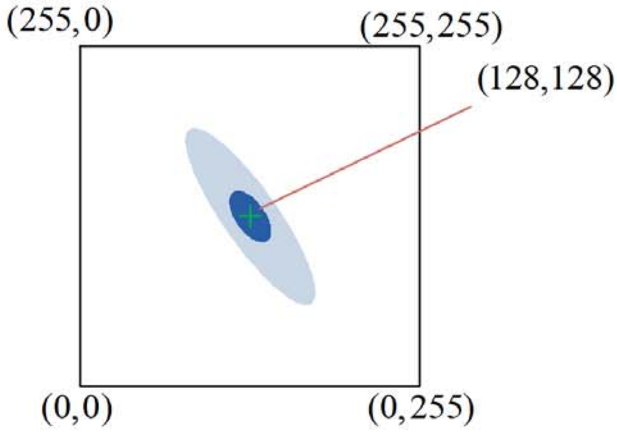
For real $f(x,y)$, periodicity and symmetry
$\Rightarrow |F(u,v)| = |F(N-u, N-v)|$ ...... symmetry about $(u,v) = (N/2, N/2)$



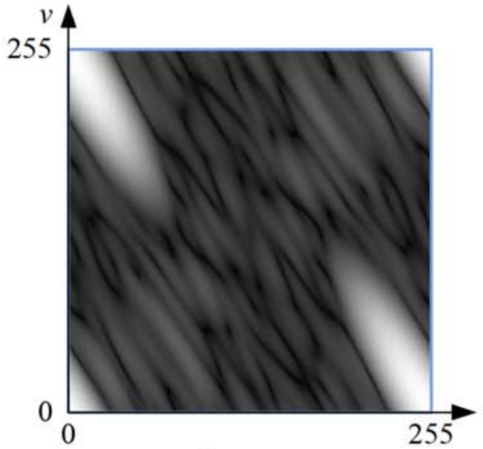
Schematic depiction of $|F(u,v)|$ for real $f(x,y)$



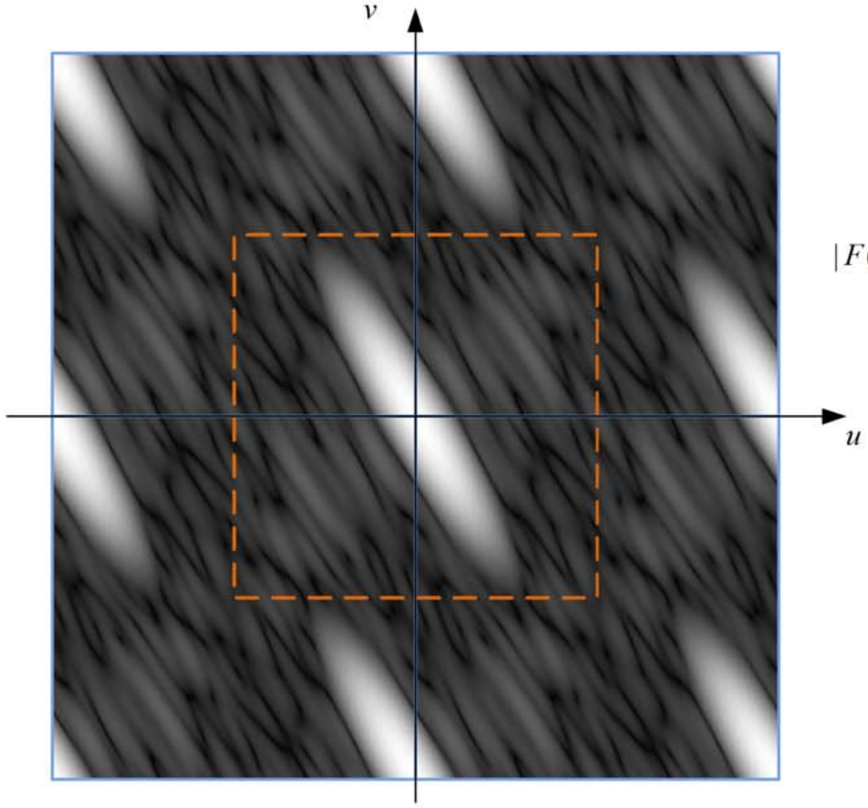
$|F(u,v)|$
(direct computation)



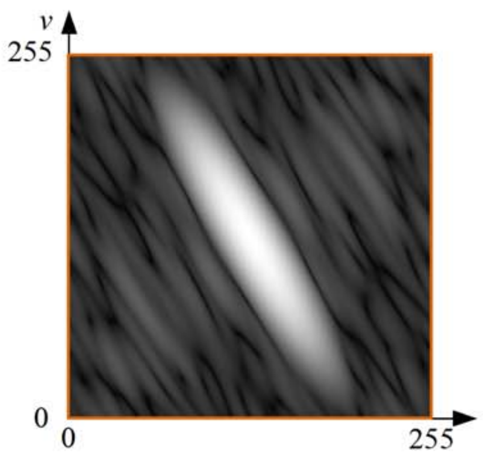
Shifted $|F(u,v),|$
centred at (128,128)



$|F(u,v)|$ in the window $u,v \in [0,255]$



$|F(u,v)|$ in the uv plane



$|F(u,v)|$ in the window $u,v \in [0,255]$
after shifting

IMAGE SAMPLING

In digital image processing systems, one deals with arrays of numbers obtained by spatially sampling points of a physical image. Image samples nominally represent some physical measurements of a continuous image field, e.g., measurements of the image intensity or photographic density.

Given $f(x, y)$, which denotes a continuous, infinite-dimensional ideal image field representing the intensity of a physical image, the sampled image is given by

$$f_s(x, y) = f(x, y)s(x, y) \tag{28}$$

where $s(x, y)$, the sampling function, is represented by a 2D array of delta functions:

$$s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x, y - n\Delta y) \tag{29}$$

Hence

$$f_s(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} f(m\Delta x, n\Delta y)\delta(x - m\Delta x, y - n\Delta y) \tag{30}$$

From Eq. (28), we use the convolution theorem to obtain the Fourier transform of $f_s(x, y)$:

$$F_s(u, v) = F(u, v) \star S(u, v) \tag{31}$$

where $S(u, v)$, the Fourier transform of $s(x, y)$, is given by

$$S(u, v) = \frac{1}{\Delta x \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(u - mu_s, v - nv_s) \tag{32}$$

where

$$u_s = 1/\Delta x, \qquad v_s = 1/\Delta y$$

It can be shown that the Fourier transform of $f_s(x, y)$ is

$$F_s(u, v) = \frac{1}{\Delta x\, \Delta y} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} F(u - mu_s, v - nv_s) \tag{33}$$

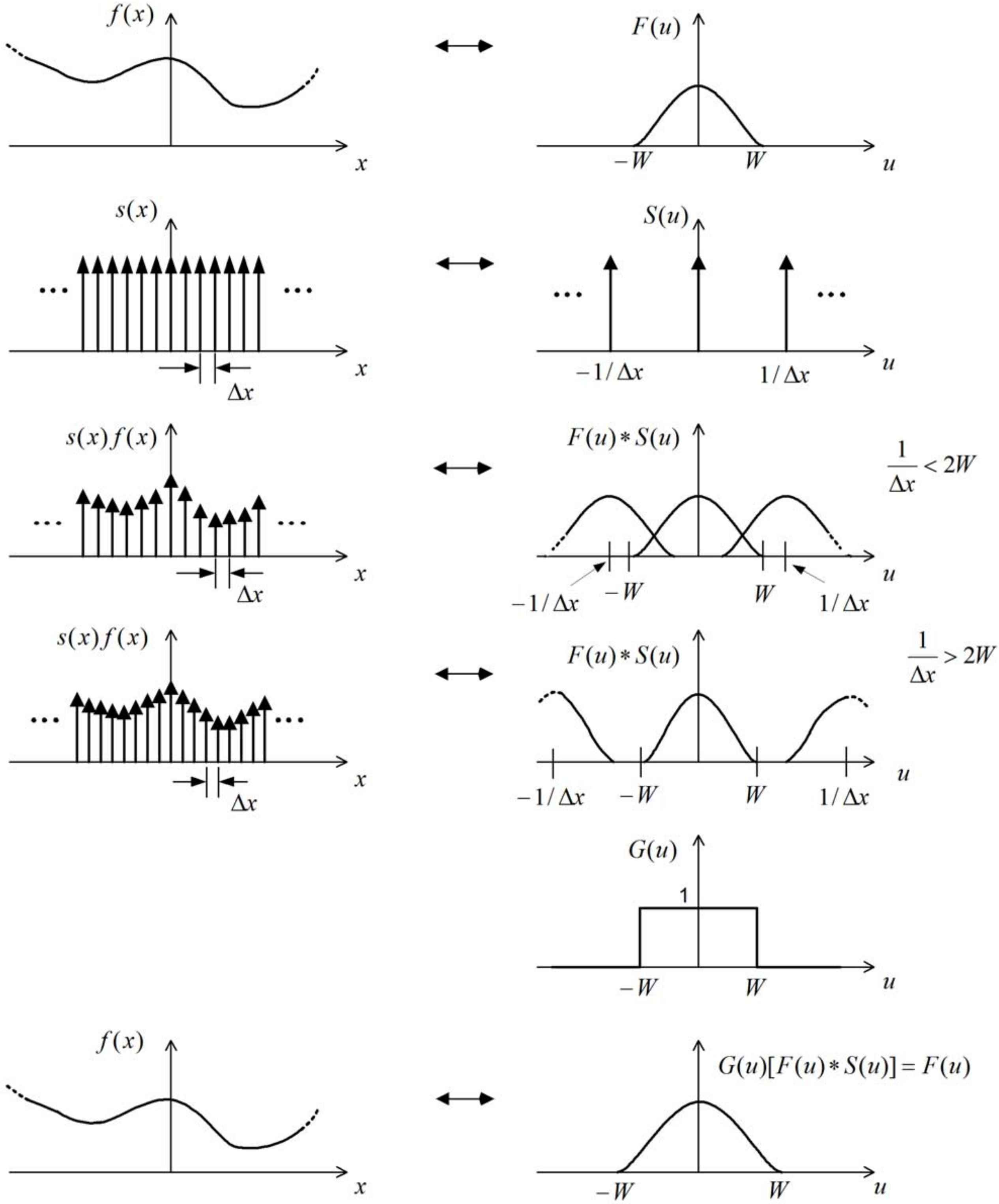


Illustration of sampling concepts

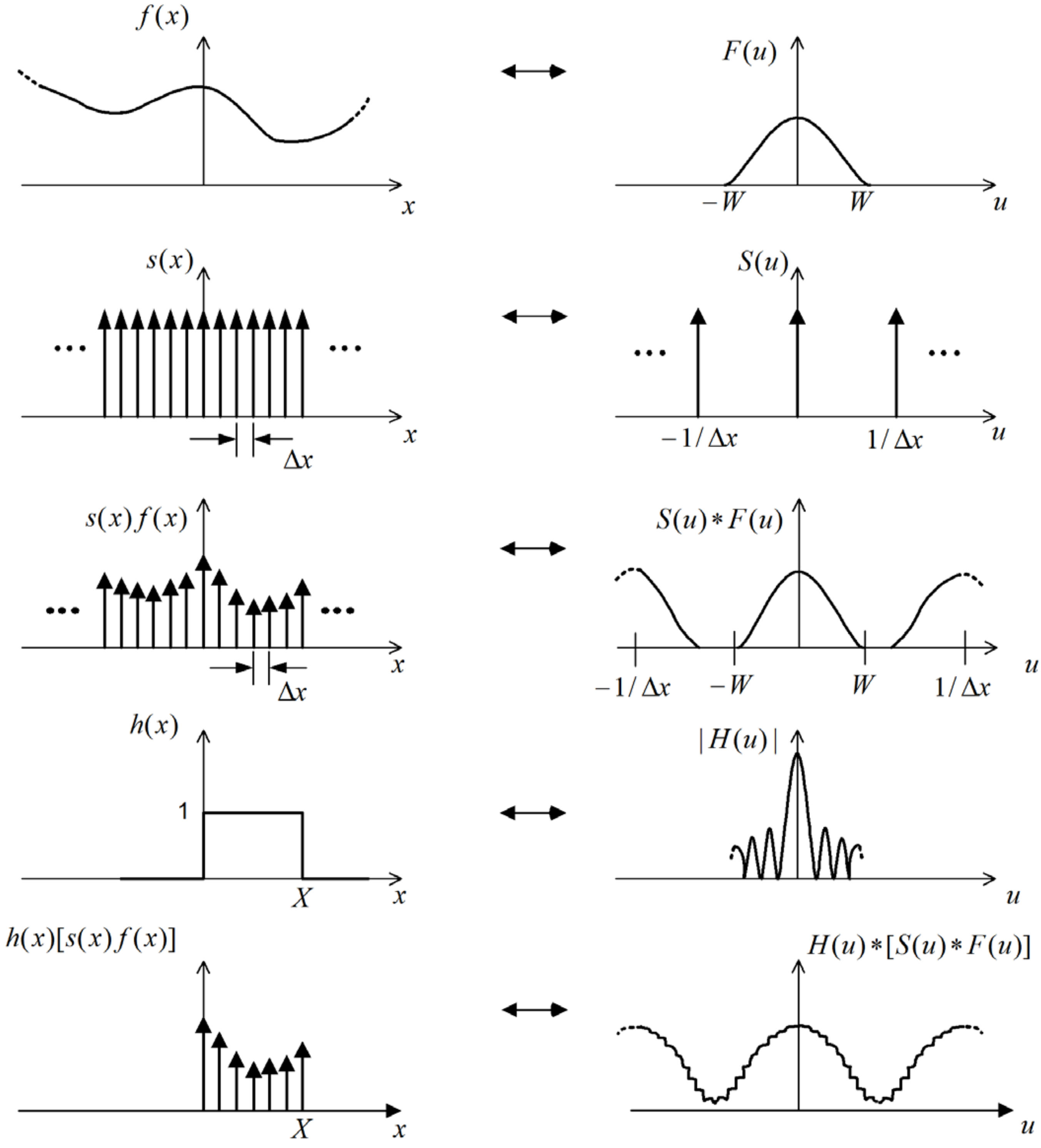
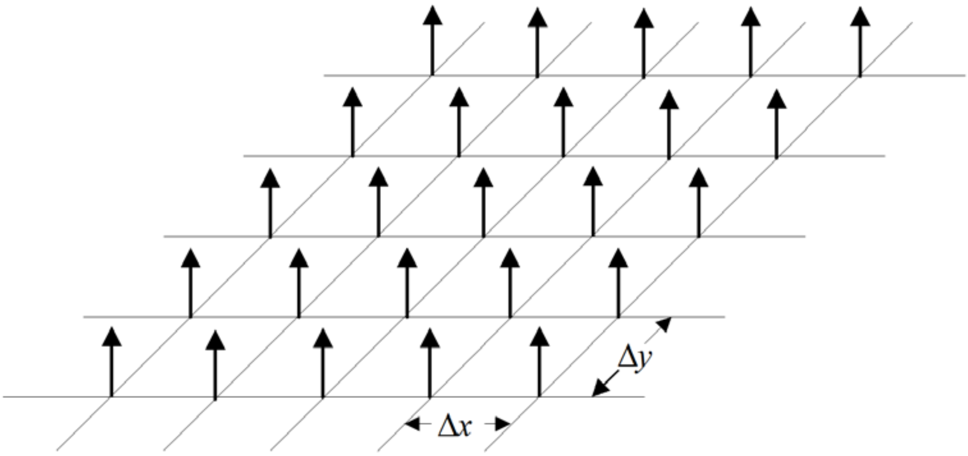
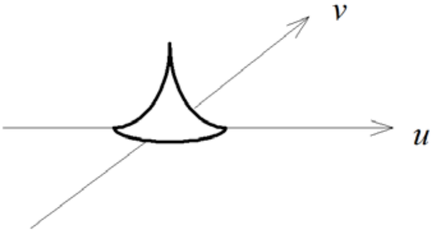


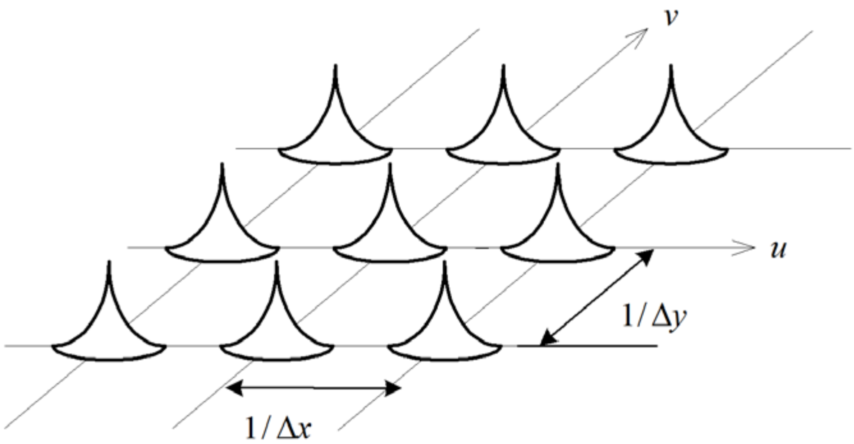
Illustration of finite-sampling concepts



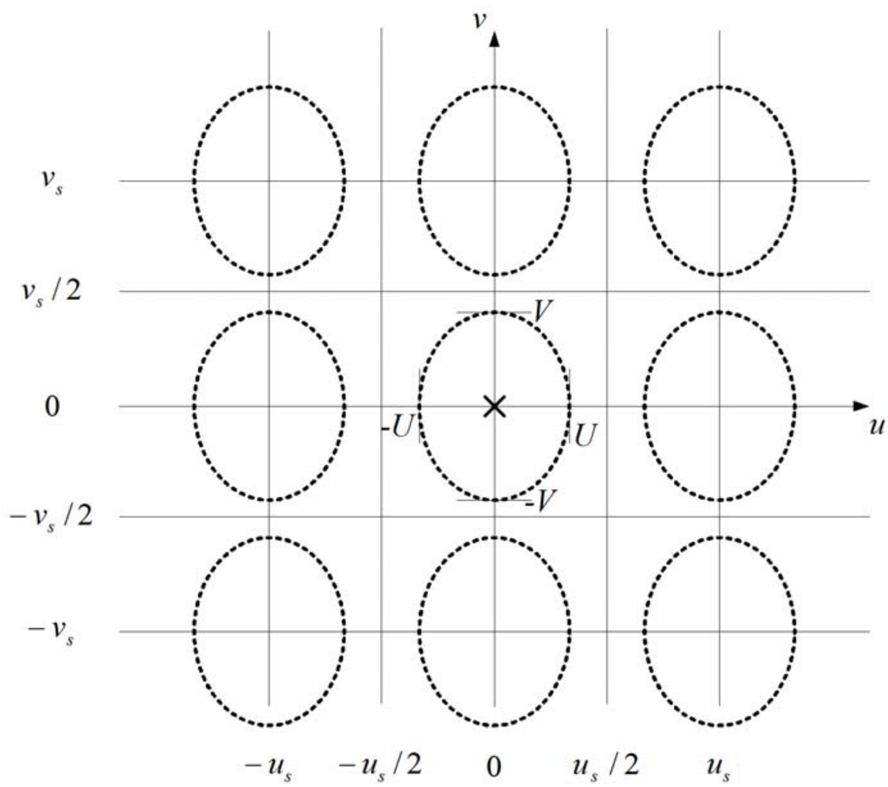
Dirac delta function sampling array.



Spectrum of original image

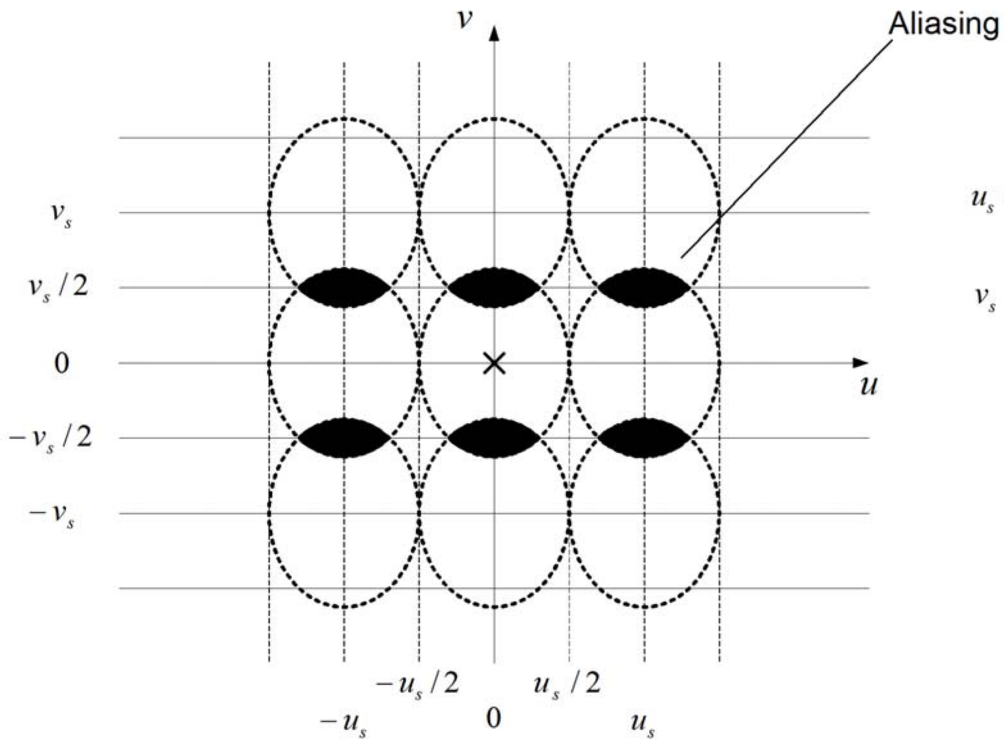


Spectrum of sampled image



$$u_s = \frac{1}{\Delta x} \geq 2U$$

$$v_s = \frac{1}{\Delta y} \geq 2V$$



$$u_s = \frac{1}{\Delta x} = 2U$$

$$v_s = \frac{1}{\Delta y} < 2V$$

Examples