EE2020

Digital Fundamentals

(L4: Gate-level Design & Minimization)

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Outline

- Gate-level logic design
- Karnaugh map
- Boolean function simplification using K-Map
- Gate-level implementation

Gate-Level Logic Design

- Step 1 (simplify the Boolean function)
  - Simplify the Boolean function to be implemented
  - Methods of simplification
    - Postulates and theorem
    - Karnaugh Map
- Step 2
  - Implement the simplified Boolean function using logic gates
  - Minimize the gate counts
- Why minimization?
  - Cost, power, performance, size, reliability, …

Karnaugh Map (K-Map)

- K-map is a diagram that consists of a number of squares
- Each square represent one minterm (or maxterm) of a Boolean function
- The Boolean function (SOP) can be expressed as a sum of minterms in the map
- *n*-variables Boolean function has maximum 2^n minterms

Two-variable K-map:
(maximum 4 minterms)

squares

A \ B	$\bar{B}$	B
$\bar{A}$	$\bar{A}\bar{B}$	$\bar{A}B$
A	$A\bar{B}$	AB

A \ B	0	1
0	00	01
1	10	11

“0” → Literal **with** overbar
“1” → Literal **without** overbar

A \ B	0	1
0	m_0	m_1
1	m_2	m_3

m_0 → 00 → $\bar{A}\,\bar{B}$
m_1 → 01 → $\bar{A}\,B$
m_2 → 10 → $A\,\bar{B}$
m_3 → 11 → AB

Truth table → K-map

A	B	F
0	0	0
0	1	0
1	0	1
1	1	1

A \ B	0	1
0	0	0
1	1	1

- K – map is a two-dimensional truth table
- Each row of in truth table corresponds to one square in the k-map
- If the term in a row is a *minterm* of the function (*F=1*), place a "1" in the corresponding square of the K-map, otherwise (*maxterm*), place a "0".

Three- and four-Variable K-Maps

***Note that any two adjacent squares differ by only one literal**

Three-variable K-map

A \ BC	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$	$\bar{A}\bar{B}\bar{C}$	$\bar{A}\bar{B}C$	$\bar{A}BC$	$\bar{A}B\bar{C}$
A	$A\bar{B}\bar{C}$	$A\bar{B}C$	ABC	$AB\bar{C}$

A \ BC	00	01	11	10
0	000	001	011	010
1	100	101	111	110

A \ BC	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

Four-variable K-map

AB \ CD	00	01	11	10
00	0000	0001	0011	0010
01	0100	0101	0111	0110
11	1100	1101	1111	1110
10	1000	1001	1011	1010

AB \ CD	00	01	11	10
00	m_0	m_1	m_3	m_2
01	m_4	m_5	m_7	m_6
11	m_{12}	m_{13}	m_{15}	m_{14}
10	m_8	m_9	m_{11}	m_{10}

Boolean function in K-map

Represent the following function on K-map:

$$F = \overline{A}B + AB + A\bar{B}$$

A \ B	0	1
0		
1		

Place a “1” in the square that represents a minterm in the given function

Write the Boolean expression for the function in K-map:

A \ B	0	1
0	0	1
1	1	0

$\boldsymbol{F} = ?$

in SOP: write F as sum of the minterms (squares with “1”)

Boolean function in K-map (cont.)

Represent the following function on K-map:

$$F = \overline{A}BC + AB\bar{C} + \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}$$

A \ BC	00	01	11	10
0	1	1	1	0
1	0	0	0	1

$$F = \bar{A}B\bar{C}D + \bar{A}\bar{B}CD + AB\bar{C}\bar{D} + \bar{A}\bar{B}C\bar{D}$$
$$+A\bar{B}C\bar{D} + A\bar{B}CD + A\bar{B}\bar{C}D + \bar{A}\bar{B}\bar{C}\bar{D}$$

AB \ CD	00	01	11	10
00	1	0	1	1
01	0	1	0	0
11	1	0	0	0
10	0	1	1	1

Write the Boolean expression for the function in K-map:

A \ BC	00	01	11	10
0	1	0	0	0
1	0	1	0	0

$\boldsymbol{F} = ?$

AB \ CD	00	01	11	10
00	0	1	0	0
01	0	0	0	1
11	0	1	0	0
10	0	0	0	0

$\boldsymbol{F} = ?$

Boolean function in K-map (cont.)

What about Boolean function in non-canonical form?

Example-1:

$F = \bar{A}B + AB\bar{C} + \bar{A}\bar{B}C$

$\bar{A}B = \bar{A}B(C + \bar{C}) = \bar{A}BC + \bar{A}B\bar{C}$

A \ BC	00	01	11	10
0				
1				

Or $\bar{A}B$ → 01, $C = 0$ *or* 1

or just fill the truth table and derive the K-map

Example-2:

$F = A + \bar{A}\bar{B}CD + BC\bar{D}$

AB \ CD	00	01	11	10
00				
01				
11				
10				

Boolean function simplification using K-map

Boolean function (SOP) simplification using K-map

Simplify: $F = \bar{A}B + AB + \bar{A}\bar{B}$

A \ B	0	1
0	1	1
1	0	1

2-cell group

$\bar{A}B + \bar{A}\bar{B} = \bar{A}(B + \bar{B}) = \bar{A}$

eliminated

$\bar{A}B + AB = B(\bar{A} + A) = B$

eliminated

MSOP:

$F = \bar{A} + B$

Alternatively,

$F = \bar{A}B + AB + \bar{A}\bar{B}$
$= \bar{A} + AB$
$= \bar{A} + B$

*The variable that changes value in the group is eliminated, or the variable that doesn't change value in the group remains

Boolean function (SOP) simplification using K-Map (cont.)

Three-variables:

$$F = \underline{\bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}BC + \overline{\bar{A}B\bar{C}} + AB\bar{C}}$$

$$\downarrow$$

$$F = \bar{A} + B\bar{C}$$

$\bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}BC + \bar{A}B\bar{C} \rightarrow \bar{A}\bar{B}(\bar{C} + C) + \bar{A}B(C + \bar{C})$
$\rightarrow \bar{A}\bar{B} + \bar{A}B \rightarrow \bar{A}(\bar{B} + B) \rightarrow \bar{A}$

$\bar{A}B\bar{C} + AB\bar{C} \rightarrow (\bar{A} + A)B\bar{C} \rightarrow B\bar{C}$

$\bar{A}$ (B and C eliminated)

A \ BC	00	01	11	10
0	1	1	1	1
1	0	0	0	1

$B\bar{C}$ (A eliminated)

$$F = \underline{\bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C + \overline{\bar{A}\bar{B}\bar{C}} + \bar{A}B\bar{C}}$$

$$\downarrow$$

$$F = \bar{B} + \bar{A}\bar{C}$$

$\bar{A}\bar{B}C + A\bar{B}\bar{C} + A\bar{B}C + \bar{A}\bar{B}\bar{C} \rightarrow \bar{A}\bar{B}(C + \bar{C}) + A\bar{B}(\bar{C} + C)$
$(\bar{A} + A)\bar{B} \rightarrow \bar{B}$

$\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} \rightarrow \bar{A}(\bar{B} + B)\bar{C} \rightarrow \bar{A}\bar{C}$

$\bar{B}$ (A and C eliminated)

A \ BC	00	01	11	10
0	1	1	0	1
1	1	1	0	0

$\bar{A}\bar{C}$ (B is eliminated)

Group the adjacent cells where only one variable changes value so that it can be eliminated

Minimization (SOP) using K-Map (cont.)

Four-variables:

$$F = \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}CD + \bar{A}B\bar{C}D + \bar{A}BCD$$
$$+AB\bar{C}D + ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}CD$$

$$\downarrow$$

$$F = \bar{B}\bar{C}\bar{D} + BD + CD$$

$\bar{B}\bar{C}\bar{D}$ BD CD

AB \ CD	00	01	11	10
00	1	0	1	0
01	0	1	1	0
11	0	1	1	0
10	1	0	1	0

$$F = \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}C\bar{D} + \bar{A}B\bar{C}D + \bar{A}BCD$$
$$+AB\bar{C}D + ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}C\bar{D}$$

$$\downarrow$$

$$F = \bar{B}\bar{D} + BD$$

$\bar{B}\bar{D}$ BD

AB \ CD	00	01	11	10
00	1	0	0	1
01	0	1	1	0
11	0	1	1	0
10	1	0	0	1

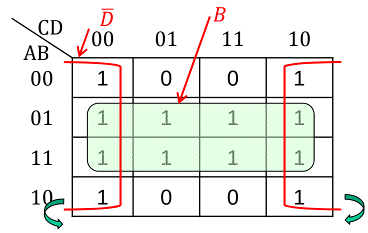
Minimization (SOP) using K-Map (cont.)

Four-variables:

$$F = \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}C\bar{D} + \bar{A}B\bar{C}\bar{D} + \bar{A}B\bar{C}D$$
$$+\bar{A}BCD + \bar{A}BC\bar{D} + AB\bar{C}\bar{D} + AB\bar{C}D$$
$$+ABCD + ABC\bar{D} + A\bar{B}\bar{C}\bar{D} + A\bar{B}C\bar{D}$$

$$\downarrow$$

$$F = B + \bar{D}$$

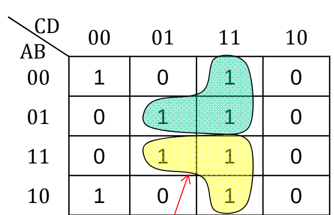


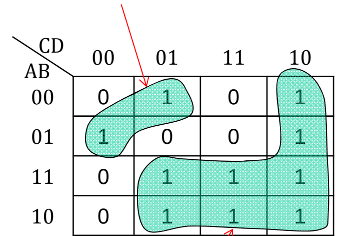
Grouping rules:

- Group the squares that only contains “1”
- Groups must be either horizontal or vertical (diagonal is invalid)
- Group size is always 2^n , that is, 2, 4, 8, …
- Group should be as large as possible (contains as many as squares with “1” as possible)
- Each square with “1” must be part of a group if possible
- Simplified term retains those variables that don’t change value
- Variables that change value in the group are eliminated

Invalid groupings

Two variable change value





Squares in the group are not in power of two

It’s not horizontal or vertical

Don’t-care condition

- So far we assume that all combination of the input variables of a Boolean function are valid (for example, 3-variable Boolean function has 8 different input combinations that makes the function equal to 0 or 1)
- There are applications in which some variable combinations never appear.
- One of such examples is the BCD code
  - 4-bit BCD code can have 16 values
  - However, 1010 – 1111 are never used, or $A\bar{B}C\bar{D}$, $A\bar{B}CD$, $AB\bar{C}\bar{D}$, $AB\bar{C}D$, $ABC\bar{D}$, and ABCD never occur
- These conditions are called don’t-care conditions.
- Don’t-care condition is marked with “X” in K-map
- For minimization, X can take either “1” or “0”.

Binary-Coded Decimal (BCD)

Decimal Symbol	BCD Digit
0	0000
1	0001
2	0010
3	0011
4	0100
5	0101
6	0110
7	0111
8	1000
9	1001

Minimization with don’t-care conditions

$$F = \bar{A}\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}C\bar{D} + \bar{A}B\bar{C}D + \bar{A}BC\bar{D}$$
$$+AB\bar{C}D + ABC\bar{D} + A\bar{B}\bar{C}\bar{D} + A\bar{B}C\bar{D}$$

$$\downarrow$$

$$F = B + \overline{D}$$

AB \ CD	00	01	11	10
00	1	0	0	1
01	X	1	X	1
11	X	1	X	1
10	1	0	0	1

Assume X = 1

*Treat X = 1 and group the squares as usual

Minimization (POS) using K-Map (cont.)

Boolean function in POS:

$$F = (A+B+C+\bar{D})(A+B+\bar{C}+D)$$
$$(A+\bar{B}+C+D)(A+\bar{B}+\bar{C}+D)$$
$$(\bar{A}+\bar{B}+C+D)(\bar{A}+\bar{B}+\bar{C}+D)$$
$$(\bar{A}+B+C+\bar{D})(\bar{A}+B+\bar{C}+D)$$

$$\downarrow$$

$$F = (\bar{C}+D)(B+C+\bar{D})\,(\bar{B}+C+D)$$

maxterm: complement (0=NOT(x))

AB \ CD	00	01	11	10
00	1	0	1	0
01	0	1	1	0
11	0	1	1	0
10	1	0	1	0

$$(A+B+C+\bar{D})\cdot(\bar{A}+B+C+\bar{D})$$
$$= A\bar{A} + \bar{A}(B+C+\bar{D})$$
$$+ A(B+C+\bar{D}) = (B+C+\bar{D})$$

POS simplification using K-map:

- Group the squares that only contains "0"
- Form an OR term (sum) for each group, instead of a product
- Value "1", instead of "0", represent complement of the variable,
- Follow similar grouping rules for SOP
- Either SOP or POS can be used to implement the Boolean function, depending on which gives more efficient implementation.

  summarizing: proceed as SOP, but group 0's instead of 1's (square = maxterm)
  + complement the values in row-col. to find maxterm associated with square

Minimal SOP (MSOP)

Some terminologies

Implicant, prime implicant and essential implicant

- **Implicant of a Boolean function**
  - Each product term in SOP is called an implicant of the function

Example-1:

Implicants

$$F(a,b,c) = ab + a\bar{b}c + \bar{a}bc + \bar{c} + abc$$

abc

Literals

Example-2:

A \ BC	00	01	11	10
0	1	1	0	0
1	1	1	1	0

How many implicants?

Minimal SOP (MSOP) – Prime implicant

- ***Prime implicant***
  - An implicant that cannot be combined with another term to eliminate a variable

Example-1:

Non-prime implicant (already contained in AB or BC)

$$F = AB + ABC + BC$$

Prime implicants

Example-2:

$\bar{A}\bar{B}D$ $\bar{A}BD$

$\bar{A}B\bar{D}$

AB \ CD	00	01	11	10
00	0	1	1	0
01	1	1	1	1
11	0	0	0	0
10	0	0	0	0

$\bar{A}\bar{B}D$, $\bar{A}BD$ and $\bar{A}B\bar{D}$

are implicants, but not prime implicants
(can be grouped into larger groups of 4)

$\bar{A}D$ and $\bar{A}B$ are essential prime implicants

graphically: prime implicant grouping cannot be expanded further (but could overlap with other prime implicants)

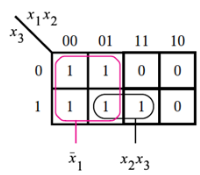
Identifying prime implicants

- A single “1” on a K-map is a prime implicant if is not adjacent to any other 1 of the function.
- Two adjacent “1”s represent a prime implicant, provided that they are not within a rectangle of 4 or more squares containing “1”s.
- Four “1”s that are an implicant are a prime implicant if they are not within a group of 8 squares containing “1”s

Basically, implicant is prime if it cannot be enclosed within a larger square/rectangle (as per K-map rules)

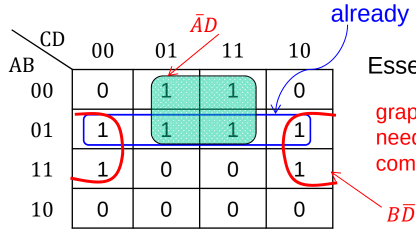
Essential prime implicant

- ***Essential prime implicant***
  - A prime implicant that is not included in any other prime implicant



Both $\overline{x}_1$ and x_2x_3 are essential prime implicants

Prime implicant (not an essential prime implicant, already covered by the other two implicants)



Essential prime implicants: $\overline{A}D$ and $B\overline{D}$

graphically: essential prime implicant is needed to cover some 1 (i.e., it does not completely overlap with other implicants)

Minimal SOP Expression (MSOP)

- What is MSOP?
  - It contains a minimal number of literals and terms
  - All essential prime implicants must be included in MSOP
- Determination of MSOP
  - Finding all of the *prime implicants* of the function
  - Select essential prime implicants (those with “1”s that have only been grouped once
  - Finding a minimal subset of these prime implicants that covers all of the *minterms* of the function

Obtaining MSOP - examples

Example – 1:

AB \ CD	00	01	11	10
00	1	0	1	1
01	1	0	1	0
11	1	1	1	1
10	0	0	0	0

Essential
Prime implicant

All implicants including
one essential prime implicant

Select the essential prime implicant
with minimum set of prime implicants

AB \ CD	00	01	11	10
00	1	0	1	1
01	1	0	1	0
11	1	1	1	1
10	0	0	0	0

Obtaining MSOP - examples (cont.)

Example – 2:

AB \ CD	00	01	11	10
00	0	0	1	0
01	1	0	1	1
11	1	1	1	1
10	0	0	1	0

Essential
Prime implicant

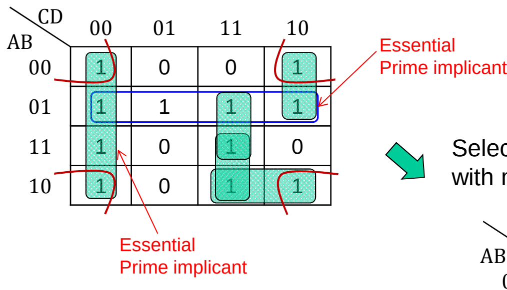
All implicants including
two essential prime implicant

Select the essential prime implicant
with minimum set of prime implicants

AB \ CD	00	01	11	10
00	0	0	1	0
01	1	0	1	1
11	1	1	1	1
10	0	0	1	0

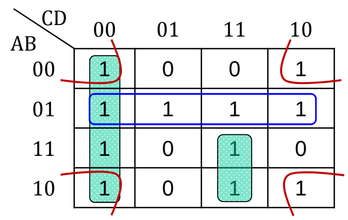
Obtaining MSOP - examples (cont.)

Example – 3:



All implicants including
two essential prime implicant

Select the essential prime implicant
with minimum set of prime implicants

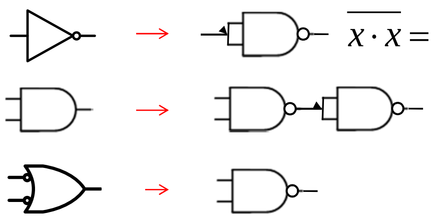


Gate-level implementation

- NAND only implementation
- NOR only implementation

NAND only implementation

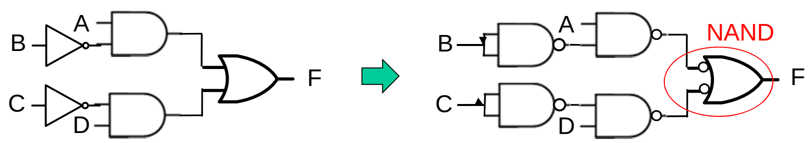
Logic operation: **NAND implementation:**



$\overline{x \cdot x} = \overline{x}$

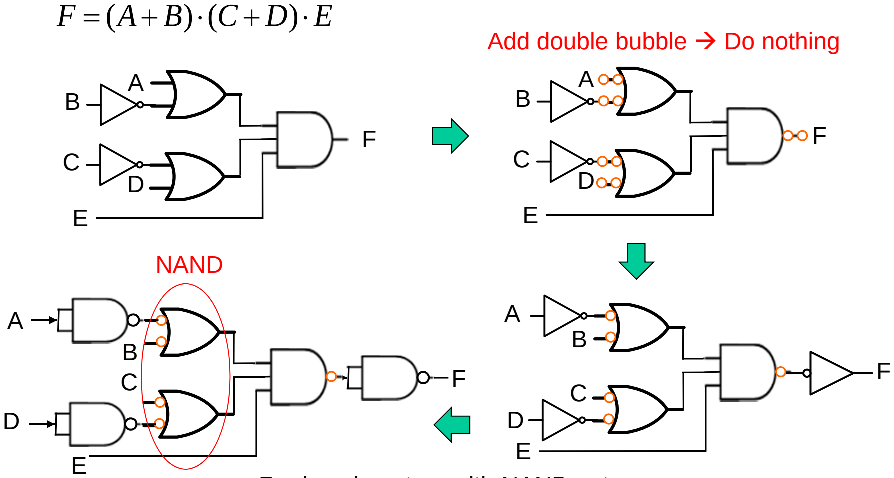
$F = A\overline{B} + \overline{C}D$

- Replace the OR gate with NAND gate and balance the bubble
- Replace the inverter with NAND gate



NAND only implementation – cont.

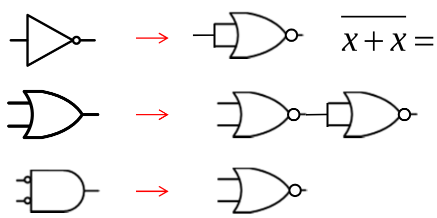
$F = (A + \overline{B}) \cdot (\overline{C} + D) \cdot E$



Replace inverters with NAND gates

NOR only implementation

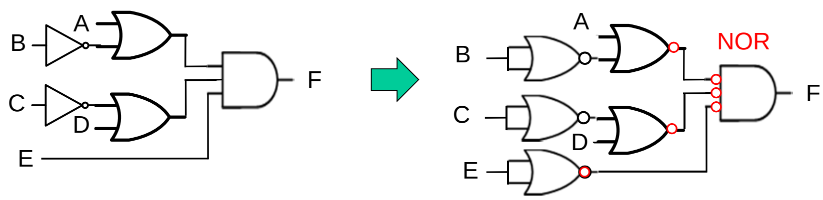
Logic operation: **NOR implementation:**



$$\overline{x+x}=\overline{x}$$

$$F=(A+\overline{B})\cdot(\overline{C}+D)\cdot E$$

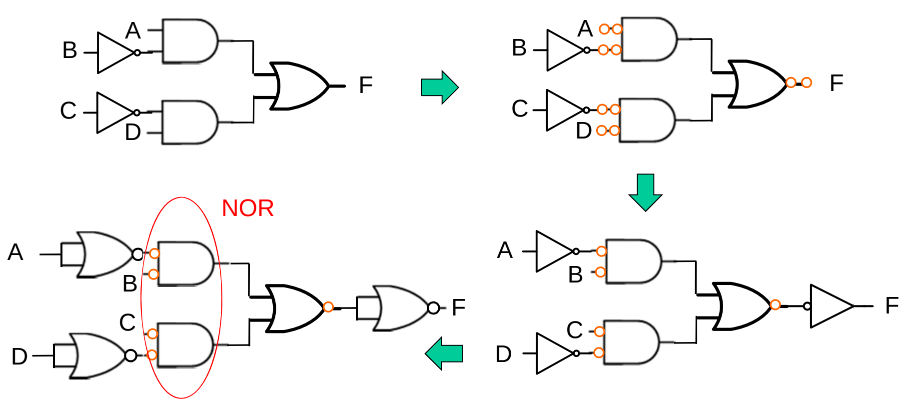
- Replace the AND gate with NOR gate and balance the bubble
- Replace the inverter with NOR gate



NOR only implementation – cont.

$$F=A\overline{B}+\overline{C}D$$

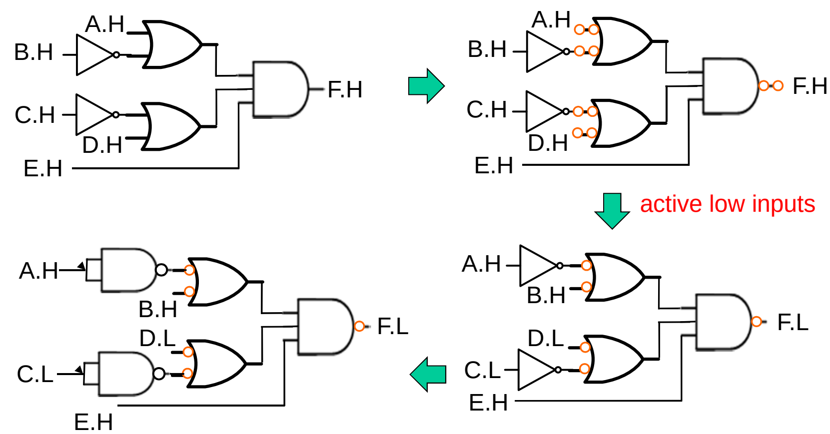
Add double bubble → Do nothing



Replace inverters with NOR gates

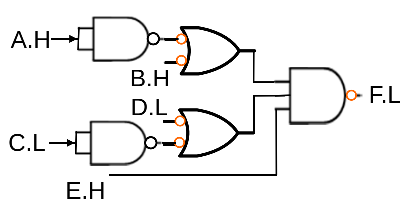
NAND only Implementation with Mixed Logic

$F = (A+\overline{B})\cdot(\overline{C}+D)\cdot E$ (where C, D and F are active low)



NAND only implementation with Mixed Logic – cont.

Write the Boolean function implemented by the circuit below and express F in positive logic.



Summary

- Karnaugh map
- Boolean function simplification using K-map
  – SOP simplification
  – POS simplification
  – Don't-care condition
  – Minimal SOP (MSOP) and POS (MPOS)
- Gate-level implementation
  – NAND only
  – NOR only