EE2020
Digital Fundamentals

(L2: Boolean Algebra)

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Outline

- What is Boolean Algebra
- Postulates and theorems
- Boolean functions and truth table
- Boolean function simplification using algebra manipulation

What is Boolean Algebra?

Brief History:

- Boolean was developed in 1854 by George Boole (An English mathematician, philosopher, and logician)
- Huntington formulated the postulates in1904 as the formal definition
- Boolean Algebra is the mathematical foundation for digital system design, including computers
- It was first applied to the practical problem (Analysis of networks of relays) in late 1930s by C.E Shannon (MIT) who later introduced “Switching algebra” in 1938
- Switching algebra is a Boolean algebra in which the number of elements is precisely two

Boolean Algebra

- Boolean algebra is defined by
  - a set of elements, ***B***, and
  - two binary operators, $\cdot\,(AND), +(OR)$
  - unary operator $\overline{}$ (*NOT*)
- Boolean algebra satisfies six Huntington postulates

*Elements →real number (i.e. 0, 1, …)

*Variables → symbols (i.e. x, y, z, …) stand for real number

Postulates of Boolean Algebra

Six Huntington postulates:

There are 6 Huntington Postulates that define the Boolean Algebra:

1. Closure - For all elements x and y in the set ***B***
   i. $x + y$ *is an element of* ***B*** *and*
   ii. $x \cdot y$ *is an element of* ***B***
2. There exists a 0 and 1 element in ***B***, such that
   i. $x + 0 = x$
   ii. $x \cdot 1 = x$
3. Commutative Law
   i. $x + y = y + x$
   ii. $x \cdot y = y \cdot x$

Postulates of Boolean Algebra – cont.

4. Distributive Law
   i. $x \cdot (y + z) = x \cdot y + x \cdot z$ $(\cdot \ over \ +)$
   ii. $x + (y \cdot z) = (x + y) \cdot (x + z)$ $(+ \ over \ \cdot)$
5. For every element x in the set ***B***, there exists an element $\bar{x}$ in the set ***B***, such that
   i. $x + \bar{x} = 1$
   ii. $x \cdot \bar{x} = 0$
   ($\bar{x}$ is called the **complement** of x)
6. There exist at least two distinct elements in the set ***B***

Switching Algebra

- Switching algebra is a two-valued Boolean Algebra, that is, the number of elements in the set ***B*** is two {0,1}
- Switching algebra represents bistable electrical switching circuits (On or Off)
- There are two main operators (***AND, OR***)
  - Binary operators (two arguments involved)
    - ***AND*** → "**.**"
    - ***OR*** → "**+**"
  - Plus, one unary operator (only one argument involved)
    - ***NOT*** → "¯" (*Complement* operator represented by an overbar)
- Switching algebra satisfies six Huntington postulates

The Three Operators in Two-Valued Boolean Algebra (*B={0,1}*)

OR: $A + B$

A	B	$A + B$
0	0	0
0	1	1
1	0	1
1	1	1

$0 + 0 = 0$
$0 + 1 = 1$
$1 + 0 = 1$
$1 + 1 = 1$

AND: $A \cdot B$

A	B	$A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

$0 \cdot 0 = 0$
$0 \cdot 1 = 0$
$1 \cdot 0 = 0$
$1 \cdot 1 = 1$

NOT: $\overline{A}$

A	$\overline{A}$
0	1
1	0

$\mathrm{A} = 0 \rightarrow \overline{A} = 1$
$\mathrm{A} = 1 \rightarrow \overline{A} = 0$

Priority: NOT has highest precedence, followed by AND and OR
NOT(A · B + C) = NOT((A · B) + C)

Boolean vs. Ordinary Algebra

Boolean algebra	Ordinary algebra
No associative law. But it can be derived from the other postulates	Associative law is included: $a + (b + c) = (a + b) + c$
Distributive law: $x + (y \cdot z) = (x + y) \cdot (x + z)$ valid	Not valid
No additive or multiplicative inverses, therefore there are no subtraction and division operation	Subtraction and division operations exist
Complement operation available	No complement operation
Boolean algebra: Undefined set of elements; Switching algebra: a two-valued Boolean algebra, whose element set only has two elements, 0 and 1.	Dealing with real numbers and constituting an infinite set of elements

Theorems of Boolean Algebra

#	Theorem		
1	$A + A = A$	$A \cdot A = A$	Tautology Law
2	$A + 1 = 1$	$A \cdot 0 = 0$	Union Law
3	$\overline{(\bar{A})} = A$		Involution Law
4	$A + (B + C)$ $= (A + B) + C$	$A \cdot (B \cdot C) = (A \cdot B) \cdot C$	Associative Law
5	$\overline{A + B} = \bar{A} \cdot \bar{B}$	$\overline{A \cdot B} = \bar{A} + \bar{B}$	De Morgan's Law
6	$A + A \cdot B = A$	$A \cdot (A + B) = A$	Absorption Law
7	$A + \bar{A} \cdot B = A + B$	$A \cdot (\bar{A} + B) = A \cdot B$	
8	$AB + A\bar{B} = A$	$(A + B)(A + \bar{B}) = A$	Logical adjacency
9	$AB + \bar{A}C + BC$ $= AB + \bar{A}C$	$(A + B)(\bar{A} + C)(B + C)$ $= (A + B)(\bar{A} + C)$	Consensus Law

Duality (OR and AND, 0 and 1 can be interchanged)

Boolean Functions and Truth Table

- A Boolean function expresses the logical relationship between binary variables
- It is evaluated by determining the binary value of the expression for all possible values of the variables
- Examples

$F_1 = A + B$

$F_2 = A \cdot B$

$F_3 = A + BC$

$F_4 = \bar{A}\bar{B}C + AB\bar{C}$

Truth Table

- ***Truth table*** is a tabular technique for listing all possible combinations of input variables and the values of function for each combination.

$\boldsymbol{F_1 = A + B}$

A	B	F_1
0	0	0
0	1	1
1	0	1
1	1	1

$\boldsymbol{F_3 = A + BC}$

A	B	C	F_3
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

Truth Table - examples

Prove the De Morgan's Law:

$\overline{A+B} = \bar{A} \cdot \bar{B}$

A	B	$\overline{A+B}$	$\bar{A} \cdot \bar{B}$
0	0	1	1
0	1	0	0
1	0	0	0
1	1	0	0

$\overline{A \cdot B} = \bar{A} + \bar{B}$

A	B	$\overline{A \cdot B}$	$\bar{A} + \bar{B}$
0	0	1	1
0	1	1	1
1	0	1	1
1	1	0	0

Prove : $A + \bar{A} \cdot B = A + B$

A	B	$A + \bar{A} \cdot B$	$A + B$
0	0	0	0
0	1	1	1
1	0	1	1
1	1	1	1

$A \cdot (A + B) = A$

A	B	$A \cdot (A+B)$	A
0	0	0	0
0	1	0	0
1	0	1	1
1	1	1	1

Truth Table – examples (cont.)

Prove : $A + (B \cdot C) = (A + B) \cdot (A + C)$

A	B	C	$A + (B \cdot C)$	$(A+B) \cdot (A+C)$
0	0	0	0	0
0	0	1	0	0
0	1	0	0	0
0	1	1	1	1
1	0	0	1	1
1	0	1	1	1
1	1	0	1	1
1	1	1	1	1

SOP and POS

SOP → Sum of Products

Example: $$F_1(A,B,C) = \overbrace{\bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}C}^{\textit{Sum}}$$

Products

POS → Product of Sums

$$F_2(A,B,C) = (A+B+C)\overbrace{(A+\bar{B}+\bar{C}) \cdot (\bar{A}+B+C)(\bar{A}+\bar{B}+C)(\bar{A}+\bar{B}+\bar{C})}^{\textit{Products}}$$

Sums

Minterm and Maxterm

- ***Minterm*** is a *product term* that contains all variables in the function
- ***Maxterm*** is a *sum term* that contains all variables in the function
- For ***n*** variables, there are 2^n different *minterms* or *maxterms*
- For example, in a Boolean Function: $Z = f(A,B,C)$
  - ABC, $A\bar{B}\bar{C}$, $\bar{A}BC$ are *minterms* in SOP (contain all variables)
  - AB, $\bar{A}C, BC$ are ***not*** *minterms* in SOP
  - $(A+B+C)$, $(\bar{A}+\bar{B}+C)$, $(A+B+\bar{C})$ are *maxterms* in POS (contain all variables)
  - $(A+C)$, $(B+C)$, $(\bar{A}+\bar{B})$ are ***not*** *maxterms* in POS

Minterm and Maxterm – cont.

A	B	C	F	Minterm	Maxterm
0	0	0	0	$\bar{A}\cdot\bar{B}\cdot\bar{C}$	$A+B+C$
0	0	1	0	$\bar{A}\cdot\bar{B}\cdot C$	$A+B+\bar{C}$
0	1	0	1	$\bar{A}\cdot B\cdot\bar{C}$	$A+\bar{B}+C$
0	1	1	0	$\bar{A}\cdot B\cdot C$	$A+\bar{B}+\bar{C}$
1	0	0	1	$A\cdot\bar{B}\cdot\bar{C}$	$\bar{A}+B+C$
1	0	1	1	$A\cdot\bar{B}\cdot C$	$\bar{A}+B+\bar{C}$
1	1	0	1	$A\cdot B\cdot\bar{C}$	$\bar{A}+\bar{B}+C$
1	1	1	0	$A\cdot B\cdot C$	$\bar{A}+\bar{B}+\bar{C}$

minterms (maxterms) that are equal to 1 (0) only for a given input

- Minterm
  - AND all the variables
  - If the variable in truth table is “0”, take its complement in the minterm
- Maxterm
  - OR all the variables
  - If the variable in truth table is “1”, take its complement in the maxterm

Canonical Form

- A Boolean functions is said to be in canonical form if it is expressed as
  - a ***sum*** of ***minterms*** (Canonical SOP - CSOP) or
  - a ***product*** of ***maxterms*** (Canonical POS - CPOS)

SOP: $F_1(A,B,C) = \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + AB\bar{C}$ ← Canonical form

$F_{1a}(A,B,C) = \bar{A}B\bar{C} + A\bar{B} + \bar{B}C + AB\bar{C}$ ← Non Canonical form

Non *minterms*

POS: $F_2(A,B,C) = (A+B+C)(A+B+\bar{C})(A+\bar{B}+\bar{C})$ ← Canonical form

$F_2(A,B,C) = (A+B+C)(A+\bar{C})(A+\bar{B}+\bar{C})$ ← Non Canonical form

Non *maxterm*

SOP and POS → Truth Table

Are the following two Boolean functions same?

$F_1(A,B,C) = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}C$

$F_2(A,B,C) = (A+B+C)(A+\bar{B}+\bar{C})(\bar{A}+B+C)(\bar{A}+\bar{B}+C)(\bar{A}+\bar{B}+\bar{C})$

Let's use truth table to check:

Truth table

A	B	C	F
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	0

SOP:

- If any *PRODUCT* in SOP is "1", the function is "1". Otherwise, the function is "0"

POS:

- If any *SUM* in POS is "0", the function is "0". Otherwise, the function is "1"

- **SOP and POS are different ways to present the same Boolean function**

Truth Table → CSOP or CPOS

Write the Boolean function represented by the Truth table below in SOP and POS, respectively

Truth table:

A	B	C	F
0	0	0	0
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	0

$F_1(A,B,C) = \bar{A}B\bar{C} + A\bar{B}\bar{C} + A\bar{B}C + AB\bar{C}$

- CSOP → Only includes the terms that make F = 1

$F_2(A,B,C) = (A+B+C)(A+B+\bar{C})(A+\bar{B}+\bar{C})$
$(\bar{A}+\bar{B}+\bar{C})$

- CPOS → Only includes the terms that make F = 0

Any Boolean function can be obtained from a given truth table and expressed in either CSOP or CPOS
(if you can choose, pick CSOP if truth table has few 1's and many 0's, CPOS otherwise)

Truth Table → SOP → POS

Truth table:

A	B	C	F_1	$\overline{F_1}$
0	0	0	0	1
0	0	1	0	1
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	1	0
1	1	0	1	0
1	1	1	0	1

Use SOP: start from CSOP of NOT(F) (otherwise, complemented POS is obtained from SOP)

$$\overline{F_1}(A,B,C) = \overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}C + \overline{A}BC + ABC$$

Apply De Morgan's Law:

$$\begin{aligned} F_1&(A,B,C) \\ &= \overline{\overline{A}\overline{B}\overline{C} + \overline{A}\overline{B}C + \overline{A}BC + ABC} \\ &= \overline{\overline{A}\overline{B}\overline{C}} \cdot \overline{\overline{A}\overline{B}C} \cdot \overline{\overline{A}BC} \cdot \overline{ABC} \\ &= (A+B+C)(A+B+\overline{C})(A+\overline{B}+\overline{C})(\overline{A}+\overline{B}+\overline{C}) \end{aligned}$$

Use POS directly from truth table: clearly the same

$$\begin{aligned} F_1&(A,B,C) \\ &= (A+B+C)(A+B+\overline{C})(A+\overline{B}+\overline{C})(\overline{A}+\overline{B}+\overline{C}) \end{aligned}$$

POS can be obtained from SOP (and vice versa) by starting from complemented SOP of F and applying the De Morgan's Law

Example-1: Non-Canonical → Canonical Form via Truth Table

Example: For the given Boolean function below, find a canonical *minterm* and *maxterm* expression.

1) obtain the truth table from the given function
2) find *minterm* or *maxterm* expression from truth table (CSOP or CPOS)

$F(x,y,z) = \bar{x} + y\bar{z}$ ← Non Canonical form

Truth table:

x	y	z	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	1
1	1	1	0

Canonical *minterm* expression:

$$F(x,y,z) = \bar{x}\bar{y}\bar{z} + \bar{x}\bar{y}z + \bar{x}y\bar{z} + \bar{x}yz + xy\bar{z}$$

(only contains the *minterms* that make the function = 1)

Canonical *maxterm* expression:

$$F(x,y,z) = (\bar{x}+y+z)(\bar{x}+y+\bar{z})(\bar{x}+\bar{y}+\bar{z})$$

(only contains the *maxterms* that make the function = 0)

Example-2: Non-Canonical → Canonical Form via Postulates and Theorems

Example: For the given Boolean functions below, convert it to canonical *minterm or maxterm* expression.

(*Using postulates/theorem to expand the given function to canonical form)

SOP → CSOP: $F(x,y,z) = \bar{x}y + xz$
(CSOP – Canonical SOP)

$$= \bar{x}y \cdot 1 + x \cdot 1 \cdot z$$
$$= \bar{x}y(z+\bar{z}) + x(y+\bar{y})z$$
$$= \bar{x}yz + \bar{x}y\bar{z} + xyz + x\bar{y}z$$

For missing literals, complete minterms through postulates:
$A \cdot 1 = A \quad and \quad A + \bar{A} = 1$

SOP → CPOS: 1) $F(x,y,z) = \bar{x}y + xz$
(CSOP – Canonical POS)

$$= \overline{\overline{\bar{x}y + xz}} \quad \text{a}$$
$$= \overline{(x+\bar{y})(\bar{x}+\bar{z})} \quad \text{b}$$
$$= \overline{x\bar{x} + x\bar{z} + \bar{x}\bar{y} + \bar{y}\bar{z}} \quad \text{c}$$
$$= (x+y)(\bar{x}+z)(y+z) \quad \text{d}$$

1) express SOP as POS
a) complement twice
b) apply De Morgan's law
c) expand
d) re-apply De Morgan's law
2) for missing literals, complete maxterms through distribution postulate

Use distribution postulate:
$A + (BC) = (A+B)(A+C)$
(A = incomplete sum,
C=NOT(B)=missing literal)

$x+y = (x+y) + z\bar{z}$
$= (x+y+z)(x+y+\bar{z})$ →

2) $$= (x+y+z) \cdot (x+y+\bar{z}) \cdot (\bar{x}+y+z) \cdot (\bar{x}+\bar{y}+z) \cdot (x+y+z) \cdot (\bar{x}+y+z)$$

Summary

- Postulates and theorems of Boolean algebra
- Three binary operators: AND, OR and NOT
- Boolean Functions
- Truth table and Boolean function evaluation using truth table
- Boolean function in SOP or POS form
- Obtain SOP or POS from truth table
- Minterm and maxterm
- Canonical form of Boolean function
- Convert non-canonical form to canonical SOP or POS expressions.