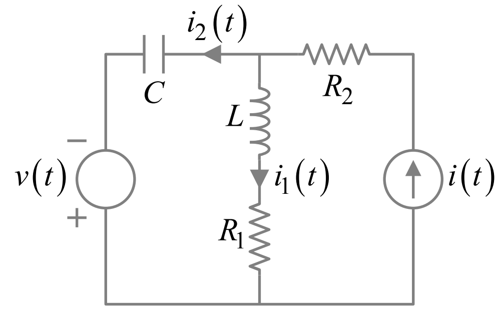
EE2023 TUTORIAL 6 (SOLUTIONS)

Solution to Q.1

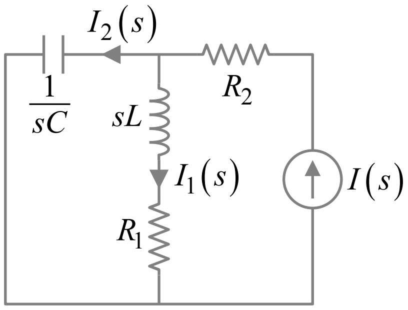
Since we are only interested in the transfer function $\dfrac{I_1(s)}{I(s)}$, we may view $i_1(t)$ as the current through the L-R_1 branch that is due solely to $i(t)$.



The solution can thus be obtained as follows:

- 'Kill' all unconcerned independent sources (i.e. independent voltage source replaced by a short-circuit and independent current source by an open-circuit). In this case, we replace $v(t)$ by a short-circuit.

- Set all initial conditions to zero (i.e. circuit is relaxed at $t = 0^-$). We may therefore apply the impedence method by simply tranforming $L \to sL$ and $C \to \dfrac{1}{sC}$.



$$\left.\begin{aligned} I_2(s)\cdot\frac{1}{sC} &= I_1(s)\cdot sL + R_1 \\ I_2(s) + I_1(s) &= I(s) \end{aligned}\right\} \to solving \to \begin{cases} \left[I(s) - I_1(s)\right]\cdot\dfrac{1}{sC} = I_1(s)\cdot\left(sL + R_1\right) \\ \dfrac{I_1(s)}{I(s)} = \dfrac{\dfrac{1}{sC}}{sL + R_1 + \dfrac{1}{sC}} = \dfrac{1}{s^2LC + sR_1C + 1} \end{cases}$$

We may also apply current division to obtain the same result:

$$I_1(s) = \frac{\dfrac{1}{sC}}{\dfrac{1}{sC} + sL + R_1} I(s) \quad \text{or} \quad \frac{I_1(s)}{I(s)} = \frac{1}{s^2LC + sR_1C + 1}$$

Solution to Q.2

NOTE: **In this problem, we use $x(t)$ instead of $u(t)$ to denote the temperature of the environment in order to avoid confusing it with the unit step function.**

THERMOMETER:

$$5\frac{dy(t)}{dt}+y(t)=0.99x(t) \quad \cdots\cdots \quad \begin{pmatrix} x(t)\text{: temperature of the environment} \\ y(t)\text{: measured temperature} \end{pmatrix} \qquad (\clubsuit)$$

(a) Given: $\begin{cases} y(0^-)=24.75°\text{C} \\ \left.\dfrac{dy(t)}{dt}\right|_{t=0^-}=0 \quad \cdots\cdots \quad \text{because temperature has stabilized at } t=0^- \end{cases}$

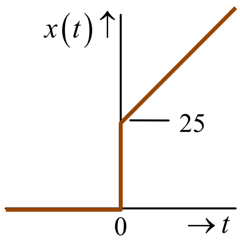
Substituting these into the differential equation $(\clubsuit)$, we have

$$5\left.\frac{dy(t)}{dt}\right|_{t=0^-}+y(0^-)=0.99x(0^-)$$

which yields

$$x(0^-)=\frac{y(0^-)}{0.99}=\frac{24.75}{0.99}=25°\text{C}$$

(b) Bath temperature starts to increase at a constant rate of $1°\text{C}/s$ at $t=0$. Therefore, $x(t)$ is of the form



$$x(t)=(t+25)u(t)$$

where $u(t)$ is the unit step function.

Substituting this into the differential equation $(\clubsuit)$ and taking Laplace transform on both sides:

$$5\frac{dy(t)}{dt}+y(t)=0.99(t+25)u(t)$$

$$5\Big(sY(s)-\underbrace{y(0^-)}_{24.75}\Big)+Y(s)=0.99\left(\frac{1}{s^2}+\frac{25}{s}\right)$$

$$Y(s)=\left(24.75+\frac{4.95}{s}+\frac{0.198}{s^2}\right)\left(\frac{1}{s+1/5}\right)$$

$$\begin{pmatrix} \mathcal{L}^{-1}\left\{\dfrac{1}{s+1/5}\right\}=\exp(-t/5)u(t), \\ \mathcal{L}^{-1}\left\{\dfrac{1}{s}\cdot\dfrac{1}{s+1/5}\right\}=\int_{0^-}^{t}\exp(-\tau/5)\,d\tau=\left[5-5\exp(-t/5)\right]u(t) \\ \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\cdot\dfrac{1}{s+1/5}\right\}=\int_{0^-}^{t}5-5\exp(-\tau/5)\,d\tau=\left[5t+25\exp(-t/5)-25\right]u(t) \end{pmatrix}$$

Therefore,

$$\begin{aligned} y(t)&=\mathcal{L}^{-1}\left\{\left(24.75+\frac{4.95}{s}+\frac{0.198}{s^2}\right)\left(\frac{1}{s+1/5}\right)\right\} \\ &=\Big[24.75\exp(-t/5)+4.95\left[5-5\exp(-t/5)\right]+0.198\left[5t+25\exp(-t/5)-25\right]\Big]u(t) \\ &=\left[19.8+0.99t+4.95\exp(-t/5)\right]u(t) \end{aligned}$$

(c) Assume zero initial conditions and taking Laplace transform on both sides of the differential equation $(\clubsuit)$:

$$\left(5\frac{dy(t)}{dt}+y(t)=0.99x(t)\right) \rightleftarrows 5sY(s)+Y(s)=0.99X(s)$$

$$\text{Transfer function: } G(s)=\frac{Y(s)}{X(s)}=\frac{0.99}{5s+1}$$

(d) From Part (c), the system transfer function is $G(s)=\dfrac{Y(s)}{X(s)}=\dfrac{0.99}{5s+1}$ which yields

$$5sY(s)+Y(s)=0.99X(s) \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (\spadesuit)$$

Transfer functions are defined under the assumption that the system is initially at rest (or relaxed). In this problem, $y(0^-)\neq 0$. Hence, we must first restore the initial conditions to $(\spadesuit)$ before solving it for $y(t)$.

Restoring the initial conditions:

$$5\left[sY(s)-y(0^-)\right]+Y(s)=0.99X(s) \quad \cdots\cdots\cdots\cdots\cdots\cdots \quad (\clubsuit)$$

where

$$\underbrace{\left[x(t)=(t+25)u(t) \rightleftarrows X(s)=\frac{1}{s^2}+\frac{25}{s}\right]}_{\text{from part (b)}} \quad \text{and} \quad \underbrace{y(0^-)=24.75}_{\text{given}}.$$

Substituting these into $(\clubsuit)$ we get

$$5\left[sY(s)-24.75\right]+Y(s)=0.99\left(\frac{1}{s^2}+\frac{25}{s}\right)$$

which results in

$$Y(s)=\left(24.75+\frac{4.95}{s}+\frac{0.198}{s^2}\right)\cdot\left(\frac{1}{s+1/5}\right)$$

$$y(t)=\left[19.8+0.99t+4.95\exp\left(-\tfrac{t}{5}\right)\right]u(t)$$

Solution to Q.3

(a) Transient response: $\left[\exp(-t)+\exp(2t)\right]u(t)$

The system is **unstable** because the presence of $\exp(2t)$ causes the transient response to grow without bound when $t\to\infty$.

(b) Transient response: $\sin(2t)u(t)$

System is **marginally stable** because the transient response oscillate with constant amplitude.

(c) Transient response: $\exp(-t)\sin(2t)u(t)$

System is **stable** because the transient response decays to zero when $t\to\infty$.

(d) Differential equation representation: $y''(t)-y'(t)-6y(t)=4u(t)$

As stability is depends on the characteristics of the transient response, the first step is to derive the transient response or the general solution of the differential equation. Characteristic equation of the homogeneous differential equation is

$$\lambda^2-\lambda-6=0 \quad \text{i.e.} \quad \lambda=3,-2.$$

Transient response: $y_{tr}(t)=A_1\exp(3t)+A_2\exp(-2t)$. Since $\lim_{t\to\infty} y_{tr}(t)$ is unbounded because of $\exp(3t)$, system is **unstable**.

(e) Transfer function: $\dfrac{s+3}{s^2+3}$

System poles are located at $s=\pm j\sqrt{3}$.

Since system poles lie on the imaginary axis, transient response is a sinusoid so the system is **marginally stable**.

(f) Transfer function: $\dfrac{4}{\left(s^2+4\right)^2}$

System poles are located at $s=\pm j2,\pm j2$.

There are repeated poles on the imaginary axis. To determine if such a system is stable, consider the case where the input is a step function (bounded input signal). The step response is

$$y_{step}(t)=\mathcal{L}^{-1}\left\{\frac{1}{\left(s^2+4\right)^2}\cdot\frac{1}{s}\right\}=\frac{1}{4}\left[1-\cos(2t)-t\sin(2t)\right].$$ (see Appendix)

Since $\lim_{t\to\infty} y_{step}(t)$ is unbounded because of the $t\sin(2t)$ term, the system is **unstable** because a bounded input signal resulted in an unbounded output signal.

(g) Transfer function: $\dfrac{2s-1}{s^2+2s+4}$

System poles are located at $s=-1\pm j\sqrt{3}$.

Since system poles are in the LHP, system is **stable**. Note that system zeros does not influence stability.

(h) System input: $x(t)=t$ (ramp function)

System output: $y(t)=2t-\frac{2}{5}+\frac{2}{5}\exp(-5t)$.

Although the output signal is unbounded, conclusions about stability cannot be made directly because the input is also unbounded. In cases where 'rules' cannot be applied directly, it is best to revert to first principle by examining the transient response. It is diffcult to divide the output signal by inspection into the transient response and the steady-state response so one option is to examine the location of the system poles.

Applying Laplace transform to $x(t)$ and $y(t)$:

$$X(s)=\mathcal{L}\{x(t)\}=\frac{1}{s^2}$$

$$Y(s)=\mathcal{L}\{y(t)\}=\frac{2}{s^2}-\frac{2}{5s}+\frac{2}{5(s+5)}=\frac{10(s+5)-2s(s+5)+2s^2}{5s^2(s+5)}=\frac{10}{s^2(s+5)}.$$

From the defnition of transfer function, $G(s)=\dfrac{Y(s)}{X(s)}$, we obtain

$$G(s)=\frac{10}{s+5}$$

Since the system pole $s=-5$ lies in the LHP, system is **stable**.

Solution to Q.4

AIR HEATING SYSTEM:

$$RC\frac{d\theta(t)}{dt}+\theta(t)=Rh(t)\ \cdots\cdots \begin{pmatrix} h(t) & \text{: heat input (system input)} \\ R & \text{: thermal resistance} \\ C & \text{: thermal capacitance} \\ \theta(t) & \text{: outlet temperature (system output)} \end{pmatrix} \qquad (\heartsuit)$$

(a) Laplace transform of $(\heartsuit)$ assuming zero initial conditions:

$$RCs\Theta(s)+\Theta(s)=RH(s)$$

System transfer function: $G(s)=\dfrac{\Theta(s)}{H(s)}=\dfrac{R}{RCs+1}$

System impulse response: $\theta_O(t)=\mathcal{L}^{-1}\{G(s)\}=\mathcal{L}^{-1}\left\{\dfrac{R}{RCs+1}\right\}=\dfrac{1}{C}\exp\left(-\dfrac{t}{RC}\right)$

(b) Substitute two points from the graph into $\theta_O(t)=\dfrac{1}{C}\exp\left(-\dfrac{t}{RC}\right)$, and solve simultaneously for R and C. Of the 5 points provided, the simultaneous equations can be solved most easily if the following data points are used to formulate the equations:

- At $t=0$, $\begin{cases}\theta_O(0)=\dfrac{1}{C}=10 \ \cdots \ \textit{from point}\,(0,10)\,\textit{in Graph} \\ \to C=0.1\end{cases}$

- At $t=RC$, $\begin{cases}\theta_O(RC)=\dfrac{1}{C}\exp(-1)=\dfrac{1}{C}0.36788=3.6788 \\ \to RC=3 \ \cdots \ \textit{from point}\,(3,3.6788)\,\textit{in Graph} \\ \to R=30\end{cases}$

APPENDIX

$$Y_{step}(s)=\frac{4}{s\left(s^2+4\right)^2}=\frac{4}{s(s+j2)^2(s-j2)^2}=\frac{\alpha}{s}+\frac{\gamma_1}{s+j2}+\frac{\gamma_2}{(s+j2)^2}+\frac{\beta_1}{s-j2}+\frac{\beta_2}{(s-j2)^2}$$

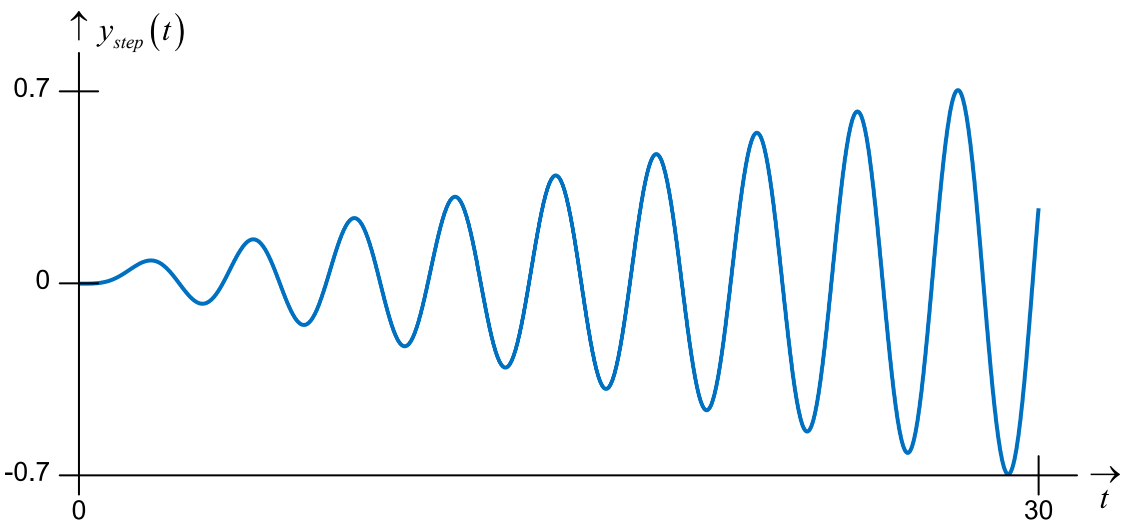
$$\text{with}\left(\begin{array}{l}\alpha=\left.\dfrac{4}{\left(s^2+4\right)^2}\right|_{s=0}=\dfrac{1}{4}\\ \gamma_1=\left.\dfrac{d}{ds}\left(\dfrac{4}{s(s-j2)^2}\right)\right|_{s=-j2}=-\dfrac{1}{8};\quad \gamma_2=\left.\dfrac{4}{s(s-j2)^2}\right|_{s=-j2}=-\dfrac{j}{8};\\ \beta_1=\left.\dfrac{d}{ds}\left(\dfrac{4}{s(s+j2)^2}\right)\right|_{s=j2}=-\dfrac{1}{8};\quad \beta_2=\left.\dfrac{4}{s(s+j2)^2}\right|_{s=j2}=\dfrac{j}{8}\end{array}\right)$$

$$=\frac{1}{8}\left[\frac{2}{s}-\frac{1}{s+j2}-\frac{j}{(s+j2)^2}-\frac{1}{s-j2}+\frac{j}{(s-j2)^2}\right]$$

$$=\frac{1}{8}\left[\frac{2}{s}-\left[\frac{1}{s-j2}+\frac{1}{s+j2}\right]+\left[\frac{j}{(s-j2)^2}-\frac{j}{(s+j2)^2}\right]\right]$$

$$y_{step}(t)=\frac{1}{8}\Big[2-\big[\exp(j2t)+\exp(-j2t)\big]+j\big[t\exp(j2t)-t\exp(-j2t)\big]\Big]u(t)$$

$$=\frac{1}{4}\big[1-\cos(2t)-t\sin(2t)\big]u(t)$$



Alternate Approach:

$$Y_{step}(s) = \frac{4}{s(s^2+4)^2} = \frac{0.25}{s} + \frac{As^3 + Bs^2 + Cs + D}{(s^2+4)^2}$$

$$0.25s^4 + 2s^2 + 4 + As^4 + Bs^3 + Cs^2 + Ds = 4$$

$$\left.\begin{aligned} A &= -1/4 \\ B &= 0 \\ C &= -2 \\ D &= 0 \end{aligned}\right\} \rightarrow \left\{\begin{aligned} Y_{step}(s) &= \frac{1}{4s} - \frac{s^3+8s}{4(s^2+4)^2} = \frac{1}{4s} + \frac{s^2+8}{16} \cdot \frac{d}{ds}\left[\frac{2}{s^2+4}\right] \\ \mathcal{L}^{-1}&\left\{\frac{d}{ds}\left[\frac{2}{s^2+4}\right]\right\} = -t\sin(2t) \\ y_{step}(t) &= \left[\frac{1}{4} - \frac{d^2}{dt^2}\left(\frac{1}{16}t\sin(2t)\right) - \frac{1}{2}t\sin(2t)\right]u(t) \\ &= \left[\frac{1}{4} - \left(\frac{1}{4}\cos(2t) - \frac{1}{4}t\sin(2t)\right) - \left(\frac{1}{2}t\sin(2t)\right)\right]u(t) \\ &= \frac{1}{4}\left[1 - \cos(2t) - t\sin(2t)\right]u(t) \end{aligned}\right.$$