EE2023 TUTORIAL 2 (SOLUTIONS)

Solution to Q.1

Description of $x(t)$:

- $x(t)$ is a REAL & EVEN function of t $\quad\because$ $x(t)$ is a dc + sum of three real cosines
- $x(t)$ has an average (or DC) value of 2 $\quad\because$ Zero-frequency component has value 2
- $x(t)$ is APERIODIC $\quad\because$ $\{\pi,\ \pi^2, \pi^3\}$ ⋯⋯ has no common factor
- $x(t)$ is a POWER SIGNAL $\quad\because$ $\left\{\begin{array}{l}\text{Spectrum is defined only at discrete} \\ \text{frequency points (sum of sinusoids)}\end{array}\right.$

Since $x(t)$ is non-periodic, it does not have a Fourier series expansion.

Solution to Q.2

(a) The fundamental frequency of $x(t) = 6\sin(12\pi t) + 4\exp\left(j\left(8\pi t + \frac{\pi}{4}\right)\right) + 2$ is $\left\{\begin{array}{l} \boldsymbol{f_p = HCF\{6,4\} = 2} \\ \boldsymbol{T_p = 0.5} \end{array}\right.$.

Re-write $x(t)$ as a sum of weighted zero-phase complex exponentials and arrange the terms in ascending frequency order:

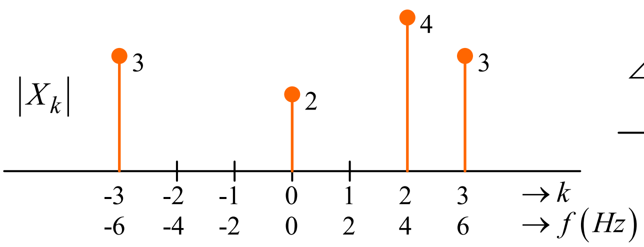
$$\begin{aligned} x(t) &= \frac{6}{j2}\left[\exp(j12\pi t) - \exp(-j12\pi t)\right] + 4\exp(j\pi/4)\exp(j8\pi t) + 2 \\ &= j3\exp(-j12\pi t) + 2 + 4\exp(j\pi/4)\exp(j8\pi t) - j3\exp(j12\pi t) \end{aligned} \qquad (1)$$

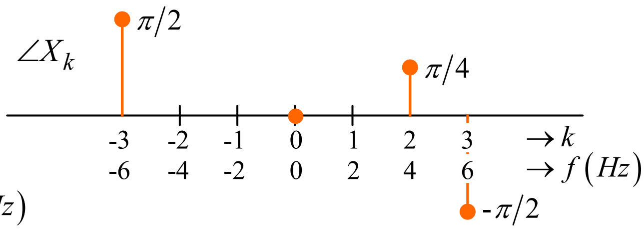
Express $x(t)$ as a complex exponential Fourier series:

$$\begin{aligned} x(t) &= \sum_{k=-\infty}^{\infty} X_k \exp\left(j2\pi \frac{k}{T_p} t\right) = \sum_{k=-\infty}^{\infty} X_k \exp(j4\pi k t) \\ &= \left(\begin{array}{c} \cdots + X_{-3}\exp(-j12\pi t) + X_{-2}\exp(-j8\pi t) + X_{-1}\exp(-j4\pi t) \\ + X_0 \\ + X_1\exp(j4\pi t) + X_2\exp(j8\pi t) + X_3\exp(j12\pi t) + \cdots\cdots \end{array}\right) \end{aligned} \qquad (2)$$

Comparing coefficients of complex exponential terms in (1) and (2), we conclude that:

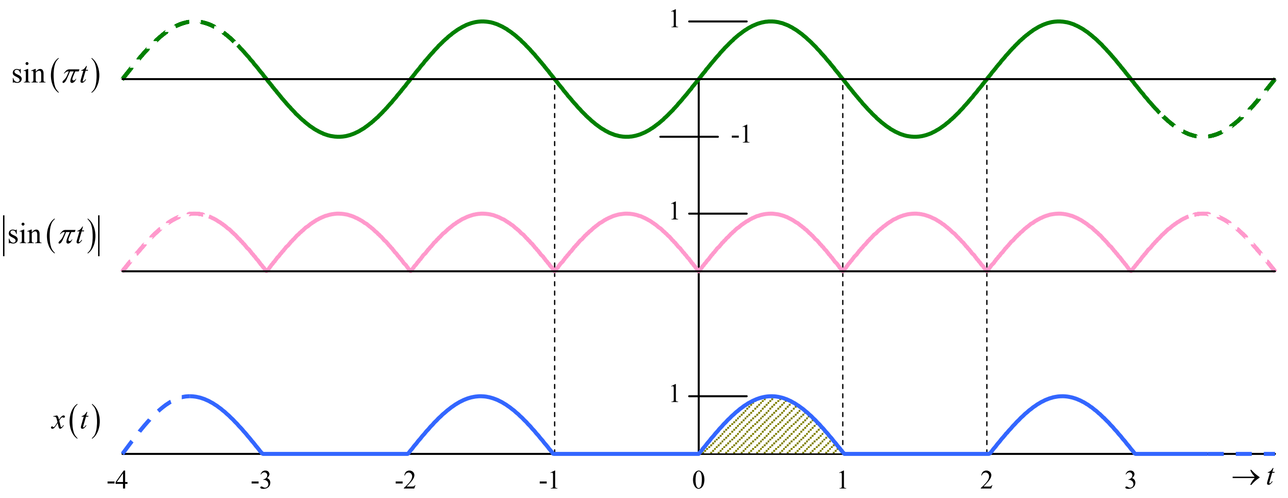
$$\boldsymbol{X_{-3} = j3,\quad X_0 = 2,\quad X_2 = 4\exp\left(j\frac{\pi}{4}\right),\quad X_3 = -j3} \quad \text{and} \quad \left[\boldsymbol{X_k = 0;\ k \neq 0,\ 2,\ \pm 3}\right].$$





Remarks: *If a periodic signal is given as a sum of sinusoids, then its Fourier series coefficients can be evaluated using the above method without the need to perform any integration.*

(b) $x(t)=\frac{1}{2}\left(\left|\sin(\pi t)\right|+\sin(\pi t)\right)$: Half-wave rectification of $\sin(\pi t)$.



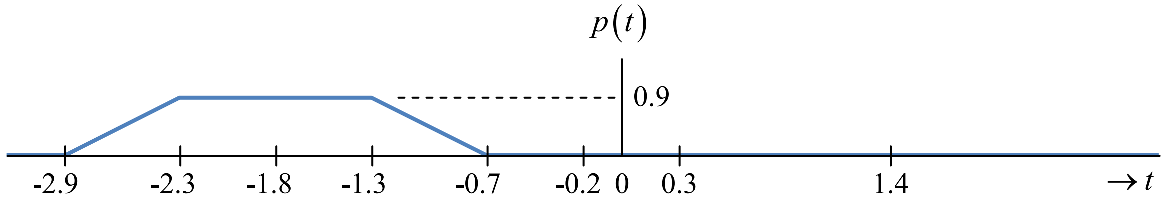
Period of $x(t)$: $T=2$

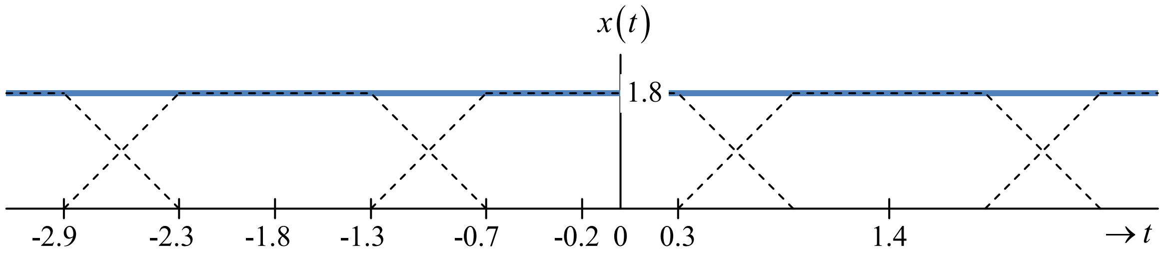
Coefficients of complex exponential Fourier series expansion of $x(t)$:

$$\begin{aligned}
X_k &= \frac{1}{T}\int_0^T x(t)\exp(-j2\pi kt/T)\,dt = \frac{1}{2}\int_0^2 x(t)\exp(-j\pi kt)\,dt \\
&= \frac{1}{2}\int_0^1 \sin(\pi t)\exp(-j\pi kt)\,dt \\
&= \frac{1}{2}\int_0^1 \frac{1}{j2}\left[\exp(j\pi t)-\exp(-j\pi t)\right]\exp(-j\pi kt)\,dt \\
&= \frac{1}{j4}\int_0^1 \exp(-j\pi(k-1)t)-\exp(-j\pi(k+1)t)\,dt \\
&= \frac{1}{j4}\left[\frac{\exp(-j\pi(k-1)t)}{-j\pi(k-1)}-\frac{\exp(-j\pi(k+1)t)}{-j\pi(k+1)}\right]_0^1 \\
&= \frac{1}{j4}\left[\left(\frac{\exp(-j\pi(k-1))}{-j\pi(k-1)}-\frac{\exp(-j\pi(k+1))}{-j\pi(k+1)}\right)-\left(\frac{1}{-j\pi(k-1)}-\frac{1}{-j\pi(k+1)}\right)\right] \\
&= \frac{1}{j4}\left[\exp(-j\pi k)\left(\frac{-1}{-j\pi(k-1)}-\frac{-1}{-j\pi(k+1)}\right)+\left(\frac{-1}{-j\pi(k-1)}-\frac{-1}{-j\pi(k+1)}\right)\right] \\
&= \frac{\exp(-j\pi k)+1}{2\pi(1-k^2)} = \begin{cases} \dfrac{1}{\pi(1-k^2)}; & k \text{ even} \\ j/4; & k=-1 \\ -j/4; & k=1 \\ 0; & otherwise \end{cases}
\end{aligned}$$

Solution to Q.3

Graphically, we observe that $x(t) = \sum_{n=-\infty}^{\infty} 2p(t-1.6n) = 1.8$.





By Deduction:

- $x(t)$ has a *zero-frequency* component of value 1.8, which implies that $X_0 = 1.8$.
- $x(t)$ has no *non-zero frequency* components, which implies that $X_k = 0;\ \ k \neq 0$.

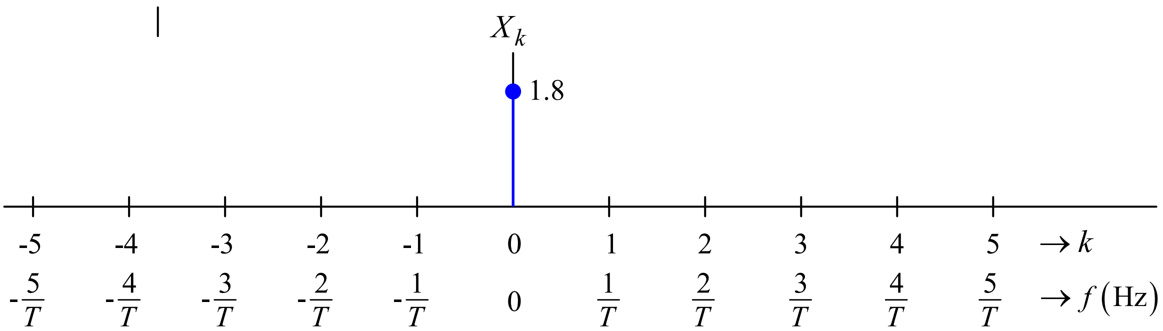
By Derivation:

Since $x(t)$ is a constant (or a DC signal), it may be treated as a periodic signal of arbitrary period *T*, where $0 < T < \infty$. Its Fourier series coefficients can thus be computed as

$$X_k = \frac{1}{T}\int_{-T/2}^{T/2} 1.8\exp\left(-j2\pi\frac{k}{T}t\right)dt = \frac{1.8}{T}\left[\frac{\exp(-j2\pi kt/T)}{-j2\pi k/T}\right]_{-T/2}^{T/2}$$

$$= \frac{1.8}{T}\left[\frac{\exp(-j\pi k)}{-j2\pi k/T} - \frac{\exp(j\pi k)}{-j2\pi k/T}\right]$$

$$= 1.8\frac{\sin(\pi k)}{\pi k} = 1.8\ \mathrm{sinc}(k) = \begin{cases} 1.8; & k = 0 \\ 0; & k \neq 0 \end{cases}$$

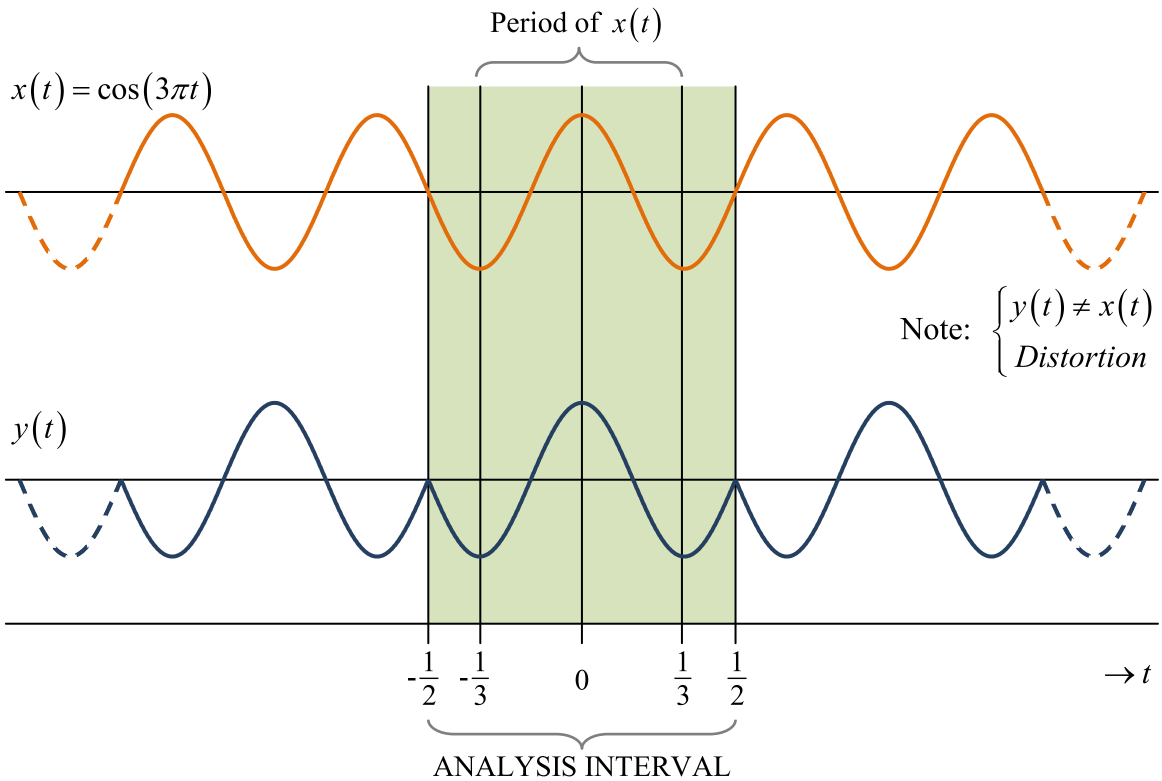


Solution to Q.4

(a) The analysis subsystem computes the Fourier series coefficients, μ_k, of an input periodic signal of period equal to 1.

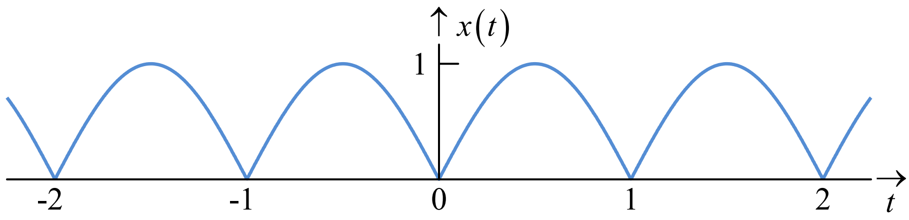
(b) The synthesis subsystem uses μ_k as Fourier series coefficients to synthesize an output periodic signal of period equal to 1.

(c) The analysis subsystem uses an analysis interval of 1 (from -0.5 to 0.5). Thus, the segment $\left[x(t);\ |t| \le 0.5\right]$ is implicitly treated by the system as one period of the input signal although the actual period of $x(t)$ is 2/3. The output signal is simply obtained by replicating the segment $\left[x(t);\ |t| \le 0.5\right]$ at regular intervals of duration 1. With this notion we may sketch $y(t)$ without the need to compute $y(t) = \sum_{k=-\infty}^{\infty} \mu_k \exp(j2\pi kt)$.



Solution to S.1

(a)



$Period = 1$

(b)

$$X_k = \frac{1}{1}\int_0^1 \sin(\pi t)\exp(-j2\pi kt)\,dt$$
$$= \frac{1}{j2}\int_0^1 \left[\exp(j\pi t) - \exp(-j\pi t)\right]\exp(-j2\pi kt)\,dt$$
$$= \frac{1}{j2}\int_0^1 \exp\left[j\pi(1-2k)t\right] - \exp\left[-j\pi(1+2k)t\right]dt$$
$$= \frac{1}{j2}\left[\frac{\exp\left[j\pi(1-2k)t\right]}{j\pi(1-2k)} - \frac{\exp\left[-j\pi(1+2k)t\right]}{-j\pi(1+2k)}\right]_0^1$$
$$= \frac{1}{j2}\left[\frac{\exp\left[j\pi(1-2k)\right]-1}{j\pi(1-2k)} - \frac{\exp\left[-j\pi(1+2k)\right]-1}{-j\pi(1+2k)}\right]$$
$$= \frac{1}{\pi(2k+1)} - \frac{1}{\pi(2k-1)} = -\frac{2}{\pi}\cdot\frac{1}{4k^2-1}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(j2\pi kt) = -\frac{2}{\pi}\sum_{k=-\infty}^{\infty}\frac{1}{4k^2-1}\exp(j2\pi kt)$$

(c)

$$a_k = \frac{X_{-k}+X_k}{2} = -\frac{2}{\pi}\cdot\frac{1}{4k^2-1}$$
$$b_k = \frac{X_{-k}-X_k}{j2} = 0$$
$$x(t) = a_0 + 2\sum_{k=1}^{\infty} a_k\cos(2\pi kt) + b_k\sin(2\pi kt) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{1}{4k^2-1}\cos(2\pi kt)$$

Solution to S.2

$$X_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^2 \exp(-jkt)\,dt = \frac{1}{2\pi}\left(\left[t^2\frac{\exp(-jkt)}{-jk}\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2t\frac{\exp(-jkt)}{-jk}dt\right)$$

$$= \frac{1}{2\pi}\left(\left[t^2\frac{\exp(-jkt)}{-jk}\right]_{-\pi}^{\pi} - \left[2t\frac{\exp(-jkt)}{(-jk)^2}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2\frac{\exp(-jkt)}{(-jk)^2}dt\right)$$

$$= \frac{1}{2\pi}\left(\left[t^2\frac{\exp(-jkt)}{-jk}\right]_{-\pi}^{\pi} - \left[2t\frac{\exp(-jkt)}{(-jk)^2}\right]_{-\pi}^{\pi} + \left[2\frac{\exp(-jkt)}{(-jk)^3}\right]_{-\pi}^{\pi}\right)$$

$$= \pi\left[\frac{\sin(\pi k)}{k}\right] + \left[\frac{2\cos(\pi k)}{k^2}\right] - \frac{2}{\pi}\left[\frac{\sin(\pi k)}{k^3}\right]$$

$$= \pi^2\left[\frac{\sin(\pi k)}{\pi k}\right] + \left[\frac{2\pi k\cos(\pi k) - 2\sin(\pi k)}{\pi k^3}\right] = \begin{cases} 2(-1)^k/k^2; & k \neq 0 \\ \pi^2/3; & k = 0 \end{cases}$$

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \exp(jkt) = \frac{\pi^2}{3} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{2(-1)^k}{k^2}\exp(jkt) = \frac{\pi^2}{3} + 4\sum_{k=1}^{\infty}\frac{(-1)^k}{k^2}\cos(kt)$$

Solution to S.3

$$x(t) = C_0 + \sum_{k=1}^{\infty} C_k \cos\left(2\pi\frac{k}{T_0}t - \theta_k\right)$$

$$= \underbrace{C_0}_{a_0} + 2\sum_{k=1}^{\infty}\underbrace{0.5C_k\cos(\theta_k)}_{a_k}\cos\left(2\pi\frac{k}{T_0}t\right) + \underbrace{0.5C_k\sin(\theta_k)}_{b_k}\sin\left(2\pi\frac{k}{T_0}t\right)$$

$$\left.\begin{aligned} X_{-k} &= X_k^* \quad (\because x(t) \text{ is real}) \\ a_k &= \frac{X_{-k}+X_k}{2} = \operatorname{Re}[X_k] \\ b_k &= \frac{X_{-k}-X_k}{j2} = -\operatorname{Im}[X_k] \end{aligned}\right\} \rightarrow \begin{cases} C_0 = a_0 = X_0 \\ C_k = 2\sqrt{a_k^2+b_k^2} = \sqrt{\operatorname{Re}^2[X_k]+\operatorname{Im}^2[X_k]} = 2|X_k| \\ \theta_k = \tan^{-1}\left(\dfrac{b_k}{a_k}\right) = -\tan^{-1}\left(\dfrac{\operatorname{Im}[X_k]}{\operatorname{Re}[X_k]}\right) \end{cases}$$
