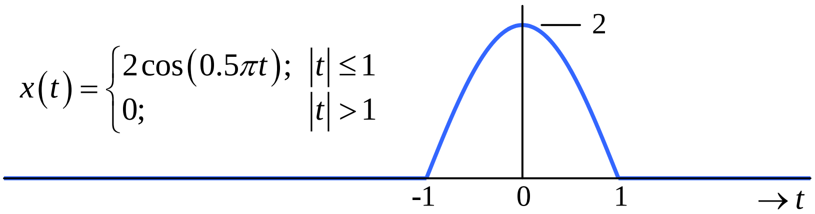
EE2023 TUTORIAL 3 (SOLUTIONS)

Solution to Q.1

$$x(t)=\begin{cases}2\cos(0.5\pi t); & |t|\le 1\\ 0; & |t|>1\end{cases}$$



(a)

Method 1: By applying direct Fourier transform:

$$\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)\exp(-j2\pi ft)\,dt\\
&= \int_{-\infty}^{-1} 0\exp(-j2\pi ft)\,dt+\int_{-1}^{1}2\cos(0.5\pi t)\exp(-j2\pi ft)\,dt+\int_{1}^{\infty}0\exp(-j2\pi ft)\,dt\\
&= 2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\cos(2\pi ft)}_{even\ \ function\ of\ t}dt - \cancel{j2\int_{-1}^{1}\underbrace{\cos(0.5\pi t)\sin(2\pi ft)}_{odd\ \ function\ of\ t}dt}\\
&= 4\int_{0}^{1}\cos(0.5\pi t)\cos(2\pi ft)\,dt\\
&\cdots\ applying\ \cos(A)\cos(B)=\frac{1}{2}\cos\left(\frac{A-B}{2}\right)+\frac{1}{2}\cos\left(\frac{A+B}{2}\right)\\
&= 2\int_{0}^{1}\cos\big((2\pi f-0.5\pi)t\big)+\cos\big((2\pi f+0.5\pi)t\big)\,dt\\
&= 2\left[\frac{\sin\big((2\pi f-0.5\pi)t\big)}{2\pi f-0.5\pi}+\frac{\sin\big((2\pi f+0.5\pi)t\big)}{2\pi f+0.5\pi}\right]_0^1\\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{2\pi f-0.5\pi}+\frac{\sin(2\pi f+0.5\pi)}{2\pi f+0.5\pi}\right)\\
&= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5}+\frac{\cos(2\pi f)}{2f+0.5}\right)=\frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}
\end{aligned}$$

Method 2: By applying Fourier transform properties:

The half-cosine pulse can be modeled as $x(t)=2\cos(0.5\pi t)\cdot\mathrm{rect}(0.5t)$

$$\Im\{2\cos(0.5\pi t)\}=\delta(f-0.25)+\delta(f+0.25)$$

$$\Im\{\mathrm{rect}(0.5t)\}=2\mathrm{sinc}(2f)$$

Applying the 'Multiplication in time-domain' property of the Fourier transform

$$\left[x(t)=\underbrace{2\cos(0.5\pi t)\cdot\mathrm{rect}(0.5t)}_{\text{Multiplication in time-domain}}\right]\rightleftarrows\left[X(f)=\underbrace{\Im\{2\cos(0.5\pi t)\}\ast\Im\{\mathrm{rect}(0.5t)\}}_{\text{Convolution in frequency-domain}}\right]$$

we get

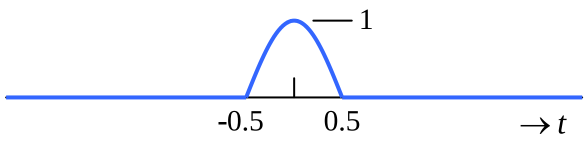
$$\begin{aligned}
\therefore X(f) &= \left[\delta(f-0.25)+\delta(f+0.25)\right] * 2\operatorname{sinc}(2f) \\
&= 2\operatorname{sinc}(2(f-0.25)) + 2\operatorname{sinc}(2(f+0.25)) \\
&= 2\left(\frac{\sin(2\pi f-0.5\pi)}{\pi(2f-0.5)} + \frac{\sin(2\pi f+0.5\pi)}{\pi(2f+0.5)}\right) \\
&= \frac{2}{\pi}\left(\frac{-\cos(2\pi f)}{2f-0.5} + \frac{\cos(2\pi f)}{2f+0.5}\right) \\
&= \frac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)} \quad \cdots\cdots \text{ Same result obtained by } \textbf{Method 1}
\end{aligned}$$

(b) From Part (a): $X(f) = \dfrac{2\cos(2\pi f)}{\pi\left(0.25-4f^2\right)}$

Define an intermediate function $\tilde{y}(t) = 0.5x(2t)$

Applying the ***scaling property***:

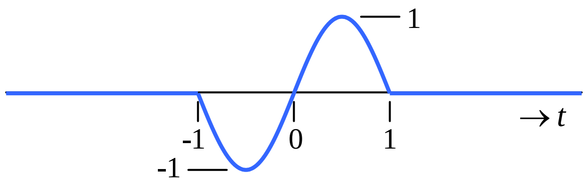
$$\begin{aligned}
\tilde{Y}(f) &= \Im\{0.5x(2t)\} \\
&= 0.5\left[\frac{1}{2}X\left(\frac{f}{2}\right)\right] = \frac{1}{4}X\left(\frac{f}{2}\right) \quad \cdots\cdots \quad (*)
\end{aligned}$$



Now, $y(t) = \tilde{y}(t-0.5) - \tilde{y}(t+0.5)$

Applying the ***time-shifting property***:

$$\begin{aligned}
Y(f) &= \tilde{Y}(f)\exp\left(-j2\pi f\left(\frac{1}{2}\right)\right) \\
&\quad -\tilde{Y}(f)\exp\left(j2\pi f\left(\frac{1}{2}\right)\right) \quad \cdots\cdots\cdots \quad (**)
\end{aligned}$$



Substituting $(*)$ into $(**)$, we get

$$\begin{aligned}
Y(f) &= \frac{1}{4}X\left(\frac{f}{2}\right)\exp(-j\pi f) - \frac{1}{4}X\left(\frac{f}{2}\right)\exp(j\pi f) \\
&= \frac{1}{4}X\left(\frac{f}{2}\right)\left\{\underbrace{\left[\cos(\pi f) - j\sin(\pi f)\right]}_{\exp(-j\pi f)} - \underbrace{\left[\cos(\pi f) + j\sin(\pi f)\right]}_{\exp(j\pi f)}\right\} \\
&= -j\frac{1}{2}X\left(\frac{f}{2}\right)\sin(\pi f) \\
&= \frac{1}{j2}\underbrace{\left[\frac{2\cos(\pi f)}{\pi\left(0.25-f^2\right)}\right]}_{X(f/2)}\sin(\pi f) = \frac{1}{j2}\left[\frac{\sin(2\pi f)}{\pi\left(0.25-f^2\right)}\right]
\end{aligned}$$

OBSERVATION: $y(t)$ is real & odd and $Y(f)$ is pure imaginary & odd.

Solution to Q.2

(a) Fig.Q.2(a)(I) is a plot of $u(t-\gamma)$ against t:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right]\xrightarrow[t\text{ to }t-\gamma]{\textit{change}}\left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right]\rightarrow\underbrace{\left[u(t-\gamma)=\begin{cases}1; & t\ge \gamma\\ 0; & t<\gamma\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } t \text{ while}\\ \text{treating } \gamma \text{ as a parameter}\\ \text{Fig.Q2(a)(I)}}}$$

Fig.Q.2(a)(II) is a plot of $u(t-\gamma)$ against γ:

$$\left[u(t)=\begin{cases}1; & t\ge 0\\ 0; & t<0\end{cases}\right]\xrightarrow[t\text{ to }t-\gamma]{\textit{change}}\left[u(t-\gamma)=\begin{cases}1; & t-\gamma\ge 0\\ 0; & t-\gamma<0\end{cases}\right]\rightarrow\underbrace{\left[u(t-\gamma)=\begin{cases}1; & \gamma\le t\\ 0; & \gamma>t\end{cases}\right]}_{\substack{\text{Expressing } u(t-\gamma) \text{ as a}\\ \text{function of } \gamma \text{ while}\\ \text{treating } t \text{ as a parameter}\\ \text{Fig.Q2(a)(II)}}}$$

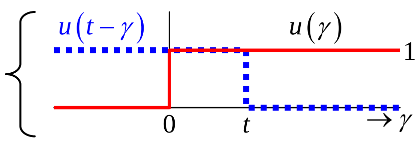
On the γ-axis, since $x(\gamma)=x(\gamma)u(t-\gamma)$ in the integration interval $(-\infty,t]$, we have

$$\int_{-\infty}^{t}x(\gamma)d\gamma=\underbrace{\int_{-\infty}^{t}x(\gamma)u(t-\gamma)d\gamma=\int_{-\infty}^{\infty}x(\gamma)u(t-\gamma)d\gamma}_{\because\, u(t-\gamma)=0 \text{ when } \gamma>t}=x(t)\ast u(t)$$

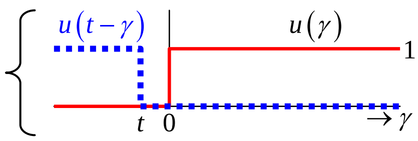
(b) $\cos(t)u(t)\ast u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)d\gamma$

Consider the term $u(\gamma)u(t-\gamma)$ in the integrand as a function of γ.

When $t\ge 0$, we have $u(\gamma)u(t-\gamma)=\begin{cases}1; & 0\le\gamma\le t\\ 0; & \textit{elsewhere}\end{cases}$



When $t<0$, we have $u(\gamma)u(t-\gamma)=0 \quad \forall\gamma$



$$\therefore \cos(t)u(t)\ast u(t)=\int_{-\infty}^{\infty}\cos(\gamma)u(\gamma)u(t-\gamma)d\gamma=\begin{cases}\int_0^t\cos(\gamma)d\gamma; & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\begin{cases}\sin(t); & t\ge 0\\ 0; & t<0\end{cases}$$

$$=\sin(t)u(t)$$

(c) From the Fourier transform table:

$$\mathrm{rect}\left(\frac{t}{\alpha}\right) \rightleftarrows \alpha \cdot \mathrm{sinc}(\alpha f) \quad \cdots\cdots\cdots \quad (*)$$

Applying the 'Duality' property of the Fourier transform to $(*)$:

$$\alpha \cdot \mathrm{sinc}(\alpha t) \rightleftarrows \mathrm{rect}\left(\frac{f}{\alpha}\right) \quad \cdots\cdots \quad (**)$$

Taking the limit $\alpha \to \infty$ on both sides of $(**)$:

$$\lim_{\alpha\to\infty} \alpha \cdot \mathrm{sinc}(\alpha t) \rightleftarrows \lim_{\alpha\to\infty} \mathrm{rect}\left(\frac{f}{\alpha}\right) = 1$$

which shows that

$$\Im\left\{\lim_{\alpha\to\infty} \alpha \cdot \mathrm{sinc}(\alpha t)\right\} = 1$$

Hence, $\boldsymbol{\lim_{\alpha\to\infty} \alpha \cdot \mathrm{sinc}(\alpha t) = \Im^{-1}\{1\} = \delta(t)}$

Solution to Q.3

Spectrum of $x(t)$:

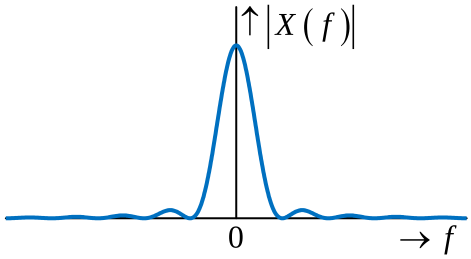
The given triangular pules may be expressed as $x(t)=\alpha\cdot\text{tri}\left(\dfrac{t}{\alpha}\right)$. Applying the Fourier transform pair $\text{tri}\left(\dfrac{t}{T}\right)\rightleftarrows T\text{sinc}^2(Tf)$, it is easy to see that

$$X(f)=\Im\left\{\alpha\cdot\text{tri}\left(\frac{t}{\alpha}\right)\right\}=\alpha^2\text{sinc}^2(\alpha f).$$

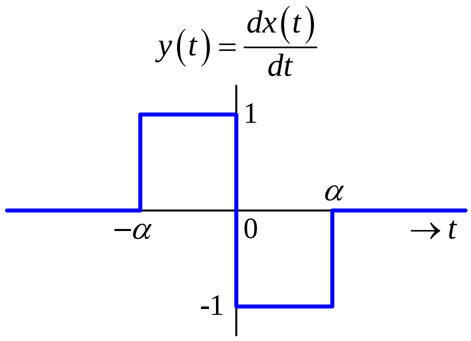
Hence,

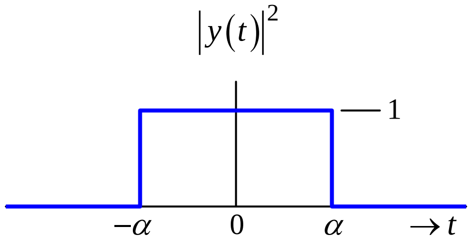
Magnitude spectrum: $|X(f)|=\alpha^2\text{sinc}^2(\alpha f)$.

Phase spectrum: $\angle X(f)=0$



ESD and Energy of $y(t)=\dfrac{dx(t)}{dt}$:





Applying the 'Differentiation in Time-Domain' property of the Fourier transform:

$$Y(f)=j2\pi f\cdot X(f)=j2\pi f\alpha^2\text{sinc}^2(\alpha f)$$

Hence,

ESD:
$$\begin{aligned}E_y(f)&=|Y(f)|^2=Y(f)Y^*(f)\\&=4\pi^2f^2\alpha^4\text{sinc}^4(\alpha f)\end{aligned}.$$

Total Energy:
$$E=\overbrace{\int_{-\infty}^{\infty}E_y(f)df=\int_{-\infty}^{\infty}|y(t)|^2dt}^{\text{Rayleigh energy theorem}}\hspace{-8em}\underbrace{\hspace{8em}=2\alpha}_{\text{By inspection of the plot of }|y(t)|^2}$$

Solution to Q.4

Spectrum: $X(f) = \exp(-\alpha|f|);\ \ \alpha > 0$

ESD of $\boldsymbol{x(t)}$: $E_x(f) = |X(f)|^2 = \exp(-2\alpha|f|)$

(a) ***Energy of $x(t)$ contained within a bandwidth of B:***

$$e(B) = \int_{-B}^{B} E_x(f)\,df = 2\int_0^B \exp(-2\alpha f)\,df = 2\left[\frac{\exp(-2\alpha f)}{-2\alpha}\right]_0^B = \frac{1}{\alpha}\left[1-\exp(-2\alpha B)\right] \quad \cdots\cdots \ (\clubsuit)$$

Total energy of $x(t)$ is equal to energy of $x(t)$ contained within a bandwidth of ∞.

Substituting $B=\infty$ into $(\clubsuit)$, we get

$$e(\infty) = \int_{-\infty}^{\infty} E_X(f)\,df = \frac{1}{\alpha} \quad \cdots\cdots \ \text{TOTAL ENERGY}$$

Let W denote the 99% energy containment bandwidth of $x(t)$. Then

Energy contained in bandwidth $W = 0.99 \times$ Total Energy

$$\rightarrow\ e(W) = 0.99 \times e(\infty)$$

$$\rightarrow\ \frac{1}{\alpha}\left[1-\exp(-2\alpha\cdot W)\right] = 0.99 \times \frac{1}{\alpha}\left[1-\exp(-2\alpha\cdot\infty)\right]$$

$$\rightarrow\ \exp(2\alpha W) = 100$$

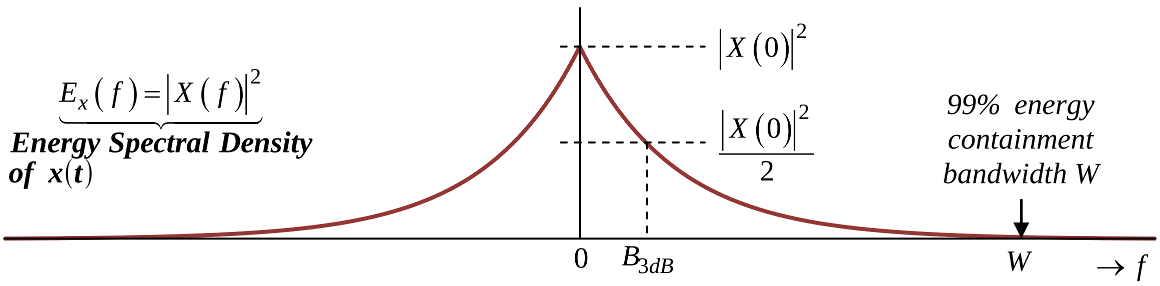
$$\rightarrow\ W = \frac{1}{\alpha}\ln(10)\ \text{Hz}$$

(b) ***Let B_{3dB} denote the 3dB bandwidth of $x(t)$. Then***

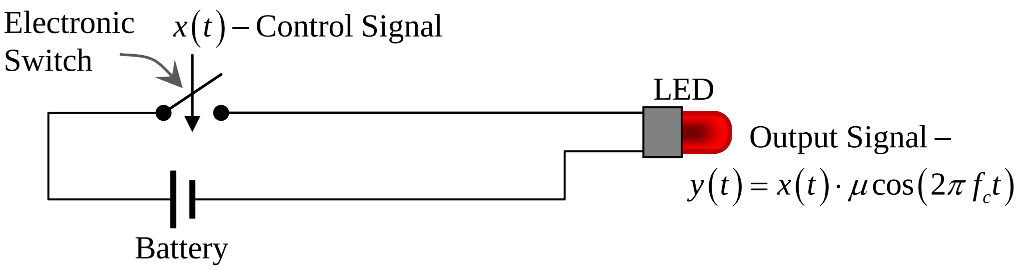
$$\frac{E_X(B_{3dB})}{E_X(0)} = \frac{1}{2} \quad \rightarrow \quad \frac{\exp(-2\alpha B_{3dB})}{\exp(0)} = \frac{1}{2} \quad \rightarrow \quad B_{3dB} = \frac{1}{2\alpha}\ln(2)\ \text{Hz}$$

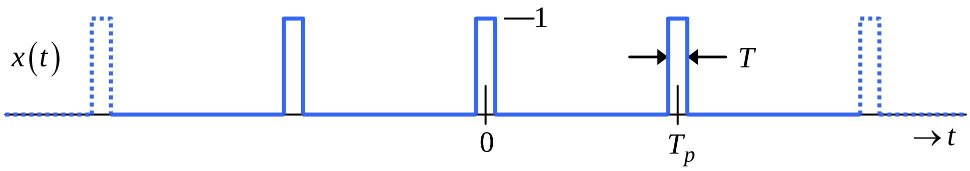
Percent energy contained within the 3dB bandwidth:

$$\frac{e(B_{3dB})}{e(\infty)} \times 100 = \frac{\cancel{\frac{1}{\alpha}}\left[1-\exp\left(-\cancel{2\alpha}\frac{\ln(2)}{\cancel{2\alpha}}\right)\right]}{\cancel{\frac{1}{\alpha}}} \times 100 = 50\%$$



Solution to Q.5





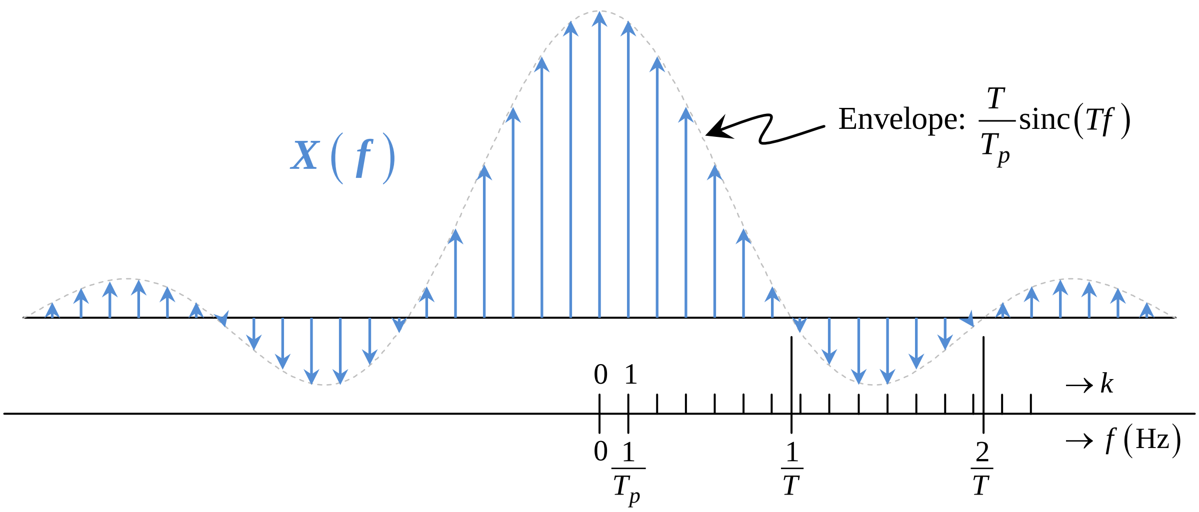
(a) ***Fourier series coefficients of $x(t)$:***

$$X_k = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} x(t)\exp\left(-j2\pi kt/T_p\right)dt = \frac{1}{T_p}\int_{-0.5T}^{0.5T} \exp\left(-j2\pi kt/T_p\right)dt$$

$$= \frac{1}{T_p}\left[\frac{\exp\left(-j2\pi kt/T_p\right)}{-j2\pi k/T_p}\right]_{-0.5T}^{0.5T} = \frac{T}{T_p}\left[\frac{\sin\left(\pi kT/T_p\right)}{\pi kT/T_p}\right] = \frac{T}{T_p}\operatorname{sinc}\left(k\frac{T}{T_p}\right)$$

Continuous-frequency spectrum (or Fourier transform) of $x(t)$:

$$X(f) = \sum_{k=-\infty}^{\infty} X_k \delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T}{T_p}\operatorname{sinc}\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$



(b) ***Power Spectral Density of $x(t)$:***

$$P_x(f) = \sum_{k=-\infty}^{\infty} |X_k|^2 \delta\left(f - \frac{k}{T_p}\right) = \sum_{k=-\infty}^{\infty} \frac{T^2}{T_p^2}\mathrm{sinc}^2\left(k\frac{T}{T_p}\right)\delta\left(f - \frac{k}{T_p}\right)$$

Average power of $x(t)$:

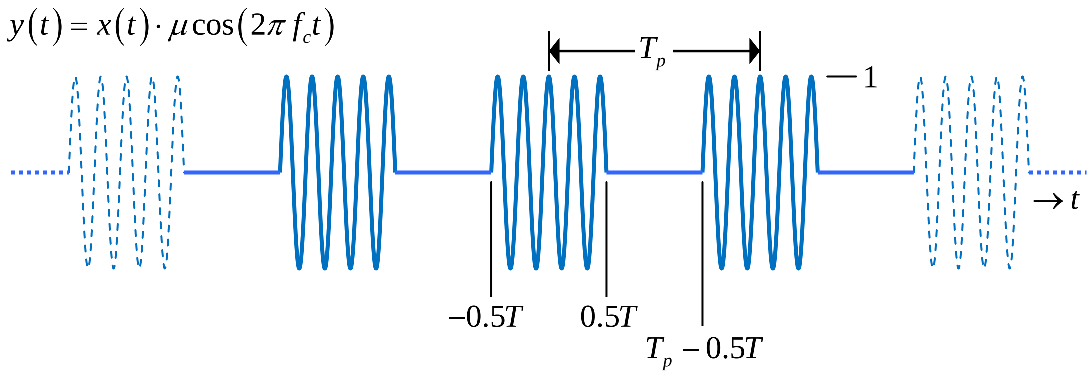
$$P = \underbrace{\int_{-\infty}^{\infty} P_x(f)\,df = \frac{1}{T_p}\int_{-0.5T}^{T_p-0.5T} |x(t)|^2\,dt}_{\text{Parseval Power Theorem}} = \frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = \frac{T}{T_p}$$

99% power containment bandwidth, W, of $x(t)$:

$$W = \frac{K}{T_p}\,(\mathrm{Hz}) \quad \cdots \quad \left(\begin{array}{l} \text{where } K \text{ satisfies } \underbrace{\sum_{k=-(K-1)}^{K-1} |X_k|^2}_{\substack{\text{Power containment}\\ \text{up to the}\\ (K-1)^{th}\text{ harmonics}}} \le 0.99P < \underbrace{\sum_{k=-K}^{K} |X_k|^2}_{\substack{\text{Power containment}\\ \text{up to the}\\ K^{th}\text{ harmonics}}} \\ \text{where } |X_k|^2 = \frac{T^2}{T_p^2}\mathrm{sinc}^2\left(k\frac{T}{T_p}\right) \text{ and } P = \frac{T}{T_p}. \end{array} \right).$$

(c) ***Average power of $y(t) = x(t)\cdot\mu\cos(2\pi f_c t)$:***

Since f_c is an integer multiple of $\frac{1}{T_p}$, it follows that $y(t)$ is periodic with period T_p. The average power, P, of $y(t)$ may then be computed by averaging over T_p.



$$\begin{aligned} P &= \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} |y(t)|^2\,dt = \frac{1}{T_p}\int_{-0.5T_p}^{0.5T_p} |x(t)|^2 \mu^2\cos^2(2\pi f_c t)\,dt \\ &= 0.5\mu^2\frac{1}{T_p}\int_{-0.5T}^{0.5T}\left[1+\cos(4\pi f_c t)\right]dt \\ &= 0.5\mu^2\frac{1}{T_p}\int_{-0.5T}^{0.5T} dt + 0.5\mu^2\frac{1}{T_p}\underbrace{\int_{-0.5T}^{0.5T}\cos(4\pi f_c t)\,dt}_{\simeq 0 \text{ since } f_c >> \frac{1}{T}} \\ &\simeq 0.5\mu^2\frac{1}{T_p}\int_{-0.5T}^{0.5T} dt = 0.5\mu^2\frac{T}{T_p} \end{aligned}$$

Assuming that μ cannot be changed, the laser pointer output power can only be controlled by changing the duty cycle (T/T_p) of the control signal.

The battery life can also be estimated for a given duty cycle.

Solution to S.1

(a) $x(t)=\cos(2\pi f_c t)u(t)$

$$\begin{cases} X(f)=\Im\{\cos(2\pi f_c t)\}*\Im\{u(t)\}=\frac{1}{2}\left[\delta(f-f_c)+\delta(f+f_c)\right]*\left[\frac{1}{j2\pi f}+\frac{1}{2}\delta(f)\right] \\ \qquad =\frac{1}{4}\left[\frac{1}{j\pi(f-f_c)}+\delta(f-f_c)+\frac{1}{j\pi(f+f_c)}+\delta(f+f_c)\right] \\ \qquad =\frac{1}{4}\left[\delta(f-f_c)+\delta(f+f_c)\right]-\frac{jf}{2\pi\left(f^2-f_c^2\right)} \end{cases}$$

(b) $x(t)=\sin(2\pi f_c t)u(t)$

$$X(f)=\frac{j}{4}\left[\delta(f+f_c)-\delta(f-f_c)\right]-\frac{f_c}{2\pi\left(f^2-f_c^2\right)} \quad \cdots \quad (\text{Same approach as part } (a))$$

(c) $x(t)=\exp(-\alpha t)\cos(\omega_c t)u(t);\quad \alpha>0$

$$\begin{cases} X(\omega)=\Im\{\exp(-\alpha t)\cos(\omega_c t)u(t)\}=\Im\left\{\frac{1}{2}\left[\exp(-\alpha t+j\omega_c t)+\exp(-\alpha t-j\omega_c t)\right]u(t)\right\} \\ \qquad =\frac{1}{2}\int_0^\infty\left[\exp(-\alpha t+j\omega_c t)+\exp(-\alpha t-j\omega_c t)\right]\exp(-j\omega t)\,dt \\ \qquad =\frac{1}{2}\int_0^\infty \exp\left[(-\alpha-j\omega+j\omega_c)t\right]+\exp\left[(-\alpha-j\omega-j\omega_c)t\right]dt \\ \qquad =\frac{1}{2}\left[\frac{\exp\left[(-\alpha-j\omega+j\omega_c)t\right]}{-\alpha-j\omega+j\omega_c}+\frac{\exp\left[(-\alpha-j\omega-j\omega_c)t\right]}{-\alpha-j\omega-j\omega_c}\right]_0^\infty \\ \qquad =\frac{1}{2}\left[\frac{1}{(\alpha+j\omega)-j\omega_c}+\frac{1}{(\alpha+j\omega)+j\omega_c}\right]=\frac{\alpha+j\omega}{(\alpha+j\omega)^2+\omega_c^2} \end{cases}$$

(d) $x(t)=\exp(-\alpha t)\sin(\omega_c t)u(t);\quad \alpha>0$

$$\begin{cases} X(\omega)=\Im\{\exp(-\alpha t)\sin(\omega_c t)u(t)\}=\Im\left\{\frac{1}{j2}\left[\exp(-\alpha t+j\omega_c t)-\exp(-\alpha t-j\omega_c t)\right]u(t)\right\} \\ \qquad =\frac{1}{j2}\int_0^\infty\left[\exp(-\alpha t+j\omega_c t)-\exp(-\alpha t-j\omega_c t)\right]\exp(-j\omega t)\,dt \\ \qquad =\frac{1}{j2}\int_0^\infty \exp\left[(-\alpha-j\omega+j\omega_c)t\right]-\exp\left[(-\alpha-j\omega-j\omega_c)t\right]dt \\ \qquad =\frac{1}{j2}\left[\frac{\exp\left[(-\alpha-j\omega+j\omega_c)t\right]}{-\alpha-j\omega+j\omega_c}-\frac{\exp\left[(-\alpha-j\omega-j\omega_c)t\right]}{-\alpha-j\omega-j\omega_c}\right]_0^\infty \\ \qquad =\frac{1}{j2}\left[\frac{1}{(\alpha+j\omega)-j\omega_c}-\frac{1}{(\alpha+j\omega)+j\omega_c}\right]=\frac{\omega_c}{(\alpha+j\omega)^2+\omega_c^2} \end{cases}$$

Solution to S.2

Start with the Fourier transform pair: $\exp(-\alpha t)u(t) \rightleftarrows \dfrac{1}{\alpha + j2\pi f}$

Applying the **duality** property of FT: $\dfrac{1}{\alpha - j2\pi t} \rightleftarrows \exp(-\alpha f)u(f)$

Applying the **differentiation** property of FT:

$$\left[\frac{d^{n-1}}{dt^{n-1}}\left(\frac{1}{\alpha - j2\pi t}\right) = (j2\pi)^{n-1}(n-1)!\frac{1}{(\alpha - j2\pi t)^n}\right] \rightleftarrows (j2\pi f)^{n-1}\exp(-\alpha f)u(f)$$

$$\frac{1}{(\alpha - j2\pi t)^n} \rightleftarrows \frac{f^{n-1}}{(n-1)!}\exp(-\alpha f)u(f)$$

Applying the **duality** property of FT: $\dfrac{f^{n-1}}{(n-1)!}\exp(-\alpha t)u(t) \rightleftarrows \dfrac{1}{(\alpha + j2\pi f)^n}$

$$\therefore\ \mathfrak{I}^{-1}\left\{\frac{1}{(\alpha + j2\pi f)^n}\right\} = \frac{t^{n-1}}{(n-1)!}\exp(-\alpha t)u(t)$$

Solution to S.3

$$\frac{1}{2-\omega^2 + j3\omega} = -\frac{1}{(\omega - j)(\omega - 2j)} = -\frac{j}{\omega - j} + \frac{j}{\omega - 2j} = \frac{1}{j\omega + 1} - \frac{1}{j\omega + 2}$$

Given: $\mathfrak{I}\{\exp(-\alpha t)u(t)\} = \dfrac{1}{j\omega + \alpha}$.

$$\therefore\ \mathfrak{I}^{-1}\left\{\frac{1}{2-\omega^2 + j3\omega}\right\} = \exp(-t)u(t) - \exp(-2t)u(t)$$

Solution to S.4

Given: $\mathfrak{I}\{x(t)\} = \text{rect}(\pi f)$ and $y(t) = \dfrac{dx(t)}{dt}$

Applying the **Differentiation** property of the FT: $Y(f) = \mathfrak{I}\left\{\dfrac{d}{dt}x(t)\right\} = j2\pi f \cdot \text{rect}(\pi f)$

Applying the **Rayleigh energy theorem**:

$$E = \int_{-\infty}^{\infty} |y(t)|^2\, dt = \int_{-\infty}^{\infty} |Y(f)|^2\, dt$$

$$= \int_{-\infty}^{\infty} 4\pi^2 f^2 \text{rect}^2(\pi f)\, df$$

$$= \int_{-1/(2\pi)}^{1/(2\pi)} 4\pi^2 f^2 df = \left[\frac{4\pi^2 f^3}{3}\right]_{-1/(2\pi)}^{1/(2\pi)} = \frac{1}{3\pi}$$

Solution to S.5

Given: $\Im\left\{\frac{\pi}{\alpha}\exp\left(-2\pi\alpha|t|\right)\right\}=\frac{1}{\alpha^2+f^2}$ and $x(t)=\frac{1}{\alpha^2+t^2}$.

Applying the **duality** property of FT: $\frac{1}{\alpha^2+t^2}\rightleftarrows\frac{\pi}{\alpha}\exp\left(-2\pi\alpha|f|\right)$

$\therefore\ X(f)=\frac{\pi}{\alpha}\exp\left(-2\pi\alpha|f|\right)$

Let B be the 99% energy containment bandwidth of $x(t)=\frac{1}{\alpha^2+t^2}$. It follows that

$$0.99=\frac{\int_0^B|X(f)|^2df}{\int_0^\infty|X(f)|^2df}=\frac{\int_0^B\exp\left(-4\pi\alpha f\right)df}{\int_0^\infty\exp\left(-4\pi\alpha f\right)df}=1-\exp\left(-4\pi\alpha B\right)$$

which yields

$$\exp\left(-4\pi\alpha B\right)=0.01\quad\text{or}\quad B=\frac{\ln(100)}{4\pi\alpha}=\frac{0.366}{\alpha}.$$

Solution to S.6

Let $X(f)$ and $X(\omega)$ denote the Fourier transform of $x(t)$ in cyclic frequency and angular frequency, respectively. Then by definition, we have

$$x(t)=\underbrace{\int_{-\infty}^{\infty}X(f)\exp\left(j2\pi ft\right)df}_{\text{inverse FT of }X(f)}=\underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)\exp\left(j\omega t\right)d\omega}_{\text{inverse FT of }X(\omega)}.$$

$$\text{Now:}\left\{\begin{array}{l}\Im^{-1}\left\{\delta(f)\right\}=\underbrace{\int_{-\infty}^{\infty}\delta(f)\exp\left(j2\pi ft\right)df}_{\substack{1\\ \left[\text{sifting property of }\delta(f)\right]}}=1\\ \\ \Im^{-1}\left\{2\pi\delta(\omega)\right\}=\frac{1}{2\pi}\underbrace{\int_{-\infty}^{\infty}2\pi\delta(\omega)\exp\left(j\omega t\right)d\omega}_{\substack{2\pi\\ \left[\text{sifting property of }\delta(\omega)\right]}}=1\end{array}\right\}\cdots(\clubsuit)$$

Since the Fourier transform is a one-to-one transformation, we conclude from $(\clubsuit)$ that $\boldsymbol{\delta(f)=2\pi\delta(\omega)}$ if $\omega=2\pi f$.
