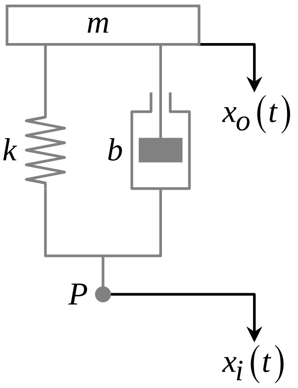
EE2023 TUTORIAL 8 (SOLUTIONS)

Solution to Q.1



Transfer function of suspension system:

$$H(s)=\frac{X_o(s)}{X_i(s)}=\frac{bs+k}{ms^2+bs+k}$$

Substututing $m=1\,kg,\ k=1\,Nm^{-1},\ b=\sqrt{2}\ Nm^{-1}s$, we get

$$H(s)=\frac{s\sqrt{2}+1}{s^2+s\sqrt{2}+1}$$

Frequency response of suspension system:

$$H(j\omega)=H(s)\big|_{s=j\omega}=\frac{1+j\omega\sqrt{2}}{1-\omega^2+j\omega\sqrt{2}}$$

$$\rightarrow\begin{cases}\text{Magnitude Response: } |H(j\omega)|=\left(\dfrac{1+2\omega^2}{\left(1-\omega^2\right)^2+2\omega^2}\right)^{1/2}=\left(\dfrac{1+2\omega^2}{1+\omega^4}\right)^{1/2}\\ \text{Phase Response: } \angle H(j\omega)=\angle\left(\dfrac{1+j\omega\sqrt{2}}{1-\omega^2+j\omega\sqrt{2}}\times\dfrac{1-j\omega\sqrt{2}}{1-j\omega\sqrt{2}}\right)\\ \qquad=\angle\left(\dfrac{1+2\omega^2}{1+\omega^2+j\omega^3\sqrt{2}}\right)=-\tan^{-1}\left(\dfrac{\omega^3\sqrt{2}}{1+\omega^2}\right)\end{cases}$$

Fourier series expansion of input: $x_i(t)=\dfrac{4}{\pi}\left[\sin(t)+\dfrac{1}{3}\sin(3t)+\dfrac{1}{5}\sin(5t)+\cdots\cdots\right]$

- **Steady-state** response of system due to sinusoidal input $\sin(t)$ is given by

$$|H(j1)|\cdot\sin\left(t+\angle H(j1)\right)=1.2247\sin(t-0.6155)$$

- **Steady-state** response of system due to sinusoidal input $\sin(3t)$ is given by

$$|H(j3)|\cdot\sin\left(3t+\angle H(j3)\right)=0.4814\sin(3t-1.3147)$$

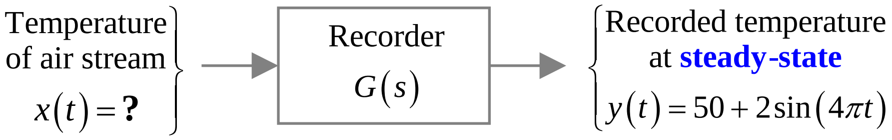
- **Steady-state** response of system due to sinusoidal input $\sin(5t)$ is given by

$$|H(j5)|\cdot\sin\left(5t+\angle H(j5)\right)=0.2854\sin(5t-1.4248)$$

Since system is linear, the output of the system can be obtained by superposition. Hence, at **steady state**:

$$x_{o,ss}(t)=\frac{4}{\pi}\left[1.2247\sin(t-0.6155)+\frac{1}{3}\times0.4814\sin(3t-1.3147)+\frac{1}{5}\times0.2854\sin(5t-1.4248)+\ \cdots\right]$$
$$=\frac{4}{\pi}\left[1.2247\sin(t-0.6155)+0.1065\sin(3t-1.3147)+0.05708\sin(5t-1.4248)+\ \cdots\cdots\ \right]$$

Solution to Q.2



$$G(s) = \frac{1}{s+1} \quad \text{since it is given that } G(s) \text{ has } \begin{cases} \bullet \text{ DC gain } = 1 \\ \bullet \text{ First order dynamics} \\ \bullet \text{ Time constant } = 1 \text{ min} \end{cases}$$

Therefore,

$$G(j\omega) = \frac{1}{j\omega + 1} \quad \cdots\cdots \quad \begin{cases} \text{Magnitude response:} & |G(j\omega)| = \left(\omega^2 + 1\right)^{-1/2} \\ \text{Phase response:} & \angle G(j\omega) = -\tan^{-1}(\omega) \end{cases}$$

At STEADY-STATE:

The system output is $y(t) = 50 + 2\sin(4\pi t)$. Let the system input be $x(t) = \alpha + \beta\sin(4\pi t + \gamma)$. Then,

$$\alpha|G(j0)| + \beta|G(j4\pi)|\sin\left(4\pi t + \gamma + \angle G(j4\pi)\right) = 50 + 2\sin(4\pi t).$$

where

$$|G(j0)| = 1 \qquad \text{and} \qquad \begin{aligned} |G(j4\pi)| &= \left(16\pi^2 + 1\right)^{-1/2} = 0.0793 \\ \angle G(j4\pi) &= -\tan^{-1}(4\pi) = -1.4914 \end{aligned}$$

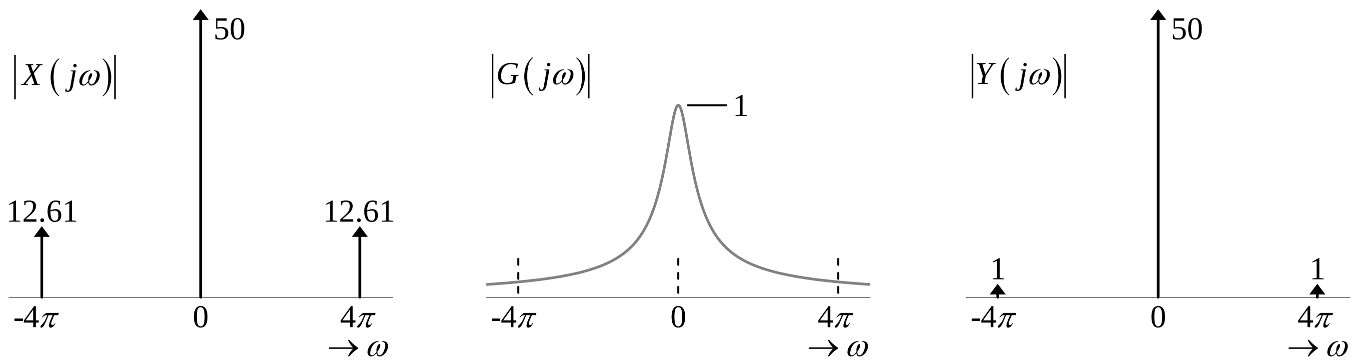
Therefore,

$$\left.\begin{aligned} \alpha|G(j0)| = 50 \quad &\rightarrow \quad \alpha = 1 \\ \beta|G(j4\pi)| = 2 \quad &\rightarrow \quad \beta = \frac{2}{0.0793} \simeq 25.22 \\ \gamma + \angle H(j4\pi) = 0 \quad &\rightarrow \quad \gamma = 1.4914 \end{aligned}\right\} \rightarrow \begin{cases} x(t) = \alpha + \beta\sin(4\pi t + \gamma) \\ \qquad = 50 + 25.22\sin(4\pi t + 1.49) \end{cases}$$

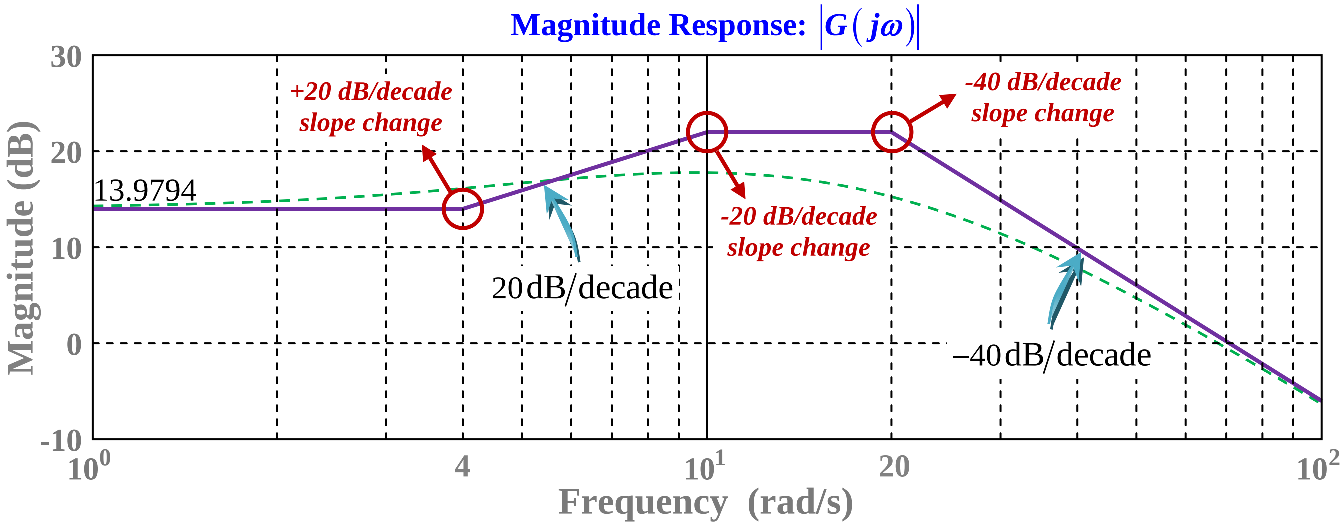
which shows that:

Maximum air temperature: $\mathbf{50 + 25.22 = 75.22°C}$
Minimum air temperature: $\mathbf{50 - 25.22 = 24.78°C}$

We note that the recorded maximum and minimum temperatures are **52°C** and **48°C**. Clearly, the recorder does not have sufficient bandwidth.



Solution to Q.3



$$\textbf{Transfer function: } G(s)=\frac{K(s+\alpha)}{(s+\beta)(s+\gamma)(s+\lambda)}=\frac{K_{dc}\left(\frac{s}{\alpha}+1\right)}{\left(\frac{s}{\beta}+1\right)\left(\frac{s}{\gamma}+1\right)\left(\frac{s}{\lambda}+1\right)}; \qquad K=\frac{\beta\gamma\lambda}{\alpha}K_{dc}$$

(a)

- At $\omega=4\ rad/s$, there is a **slope-change** of $20\ \text{dB/decade}$. This indicates the presence of the zero factor $\left(\frac{s}{4}+1\right)$ in $G(s)$.

- At $\omega=10\ rad/s$, there is a **slope-change** of $-20\ \text{dB/decade}$. This indicates the presence of the pole factor $\left(\frac{s}{10}+1\right)^{-1}$ in $G(s)$.

- At $\omega=20\ rad/s$, there is a **slope-change** of $-40\ \text{dB/decade}$. This indicates the presence of the double pole factor $\left(\frac{s}{20}+1\right)^{-2}$.

- DC (or Static) gain : $\left[20\log_{10}K_{dc}=\left|G(j0)\right|_{dB}=13.9794\ \text{dB}\right]$ or $\left[K_{dc}=10^{13.9794/20}=5\right]$.

Hence, the transfer function is

$$G(s)=\frac{5\left(\frac{s}{4}+1\right)}{\left(\frac{s}{10}+1\right)\left(\frac{s}{20}+1\right)^2}=\frac{5000(s+4)}{(s+10)(s+20)^2}$$

$\therefore\ \boldsymbol{K=5000,\ \ \alpha=4,\ \ \beta=10,\ \ \gamma=\lambda=20}$

(b)

CASE A: Magnitude response is invariant to transport delay (or dead-time).

Example: $\tilde{G}(s)=\dfrac{5000(s+4)}{(s+10)(s+20)^2}e^{-sL}$ and $G(s)=\dfrac{5000(s+4)}{(s+10)(s+20)^2}$ have the same magnitude response.

Proof:

$$\left|\tilde{G}(j\omega)\right|=\frac{5000|j\omega+4|}{|j\omega+10||j\omega+20|^2}\cdot\underbrace{\left|e^{-j\omega L}\right|}_{=1}=\frac{5000|j\omega+4|}{|j\omega+10||j\omega+20|^2}=|G(j\omega)|$$

REMARKS: For *unilateral* Laplace transform, this holds only for $L>0$.
For *bilateral* Laplace transform, L can take any real value.

CASE B: Magnitude response is invariant to the reflection of any zero about the ω axis on the s-plane

Example: $\tilde{G}(s)=\dfrac{5000(s-4)}{(s+10)(s+20)^2}$ and $G(s)=\dfrac{5000(s+4)}{(s+10)(s+20)^2}$ have the same magnitude response.

Proof:

$$\left|\tilde{G}(j\omega)\right|=\frac{5000|j\omega-4|}{|j\omega+10||j\omega+20|^2}=\frac{5000|j\omega+4|}{|j\omega+10||j\omega+20|^2}=|G(j\omega)|$$

REMARKS: At first sight, it appears that we may also preserve the magnitude response by reflecting any pole about the ω axis on the s-plane. However, this will cause the system to become unstable. For unstable systems, it is erroneous to set $s=j\omega$ in $G(s)$ to obtain the frequency response because $s=j\omega$ lies outside the region of convergence of $G(s)$. Stated another way, frequency response of an unstable system does not exist. It is thus quite meaningless to talk about magnitude response of an unstable system.

Suppose $G(s)$ is stable and has frequency response $G(j\omega)$. By reflecting one or more poles of $G(s)$ to the right-half of the s-plane to form $\tilde{G}(s)$, all we can say is that $\left|\tilde{G}(j\omega)\right|=|G(j\omega)|$. It is incorrect to call $\left|\tilde{G}(j\omega)\right|$ the magnitude response of $\tilde{G}(s)$ because $\tilde{G}(s)$ is unstable.

Hence, possible transfer functions are: $\begin{cases}\dfrac{5000(s+4)}{(s+10)(s+20)^2}, & \dfrac{5000(s+4)}{(s+10)(s+20)^2}\exp(-sL)\\[2ex] \dfrac{5000(s-4)}{(s+10)(s+20)^2}, & \dfrac{5000(s-4)}{(s+10)(s+20)^2}\exp(-sL)\end{cases}$