Parameters of First-order Systems

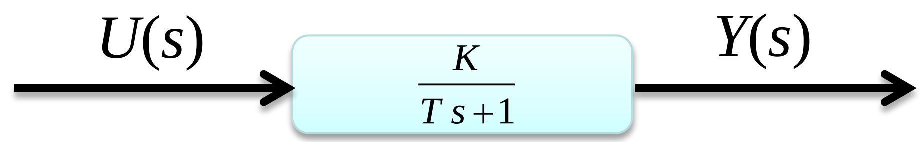
- D.E. of a linear first-order system is generally written as

$$T\frac{dy(t)}{dt}+y(t)=Ku(t)$$

where *K* is the steady-state/static gain, *T* is called the time-constant.

- The transfer function is $G(s)=\dfrac{Y(s)}{U(s)}=\dfrac{K}{Ts+1}$

(pole is located at $s=-\dfrac{1}{T}$)



- Common examples are:

Series RC circuit

$$G(s)=\frac{V_c(s)}{V(s)}=\frac{1}{RCs+1}$$

Series RL circuit

$$G(s)=\frac{I_L(s)}{V(s)}=\frac{\frac{1}{R}}{\frac{L}{R}s+1}$$

DC motor

$$G(s)=\frac{\omega(s)}{T(s)}=\frac{\frac{1}{B}}{\frac{J}{B}s+1}$$

Parameters of Second-order Systems

- D.E. of a general second-order system may be expressed as

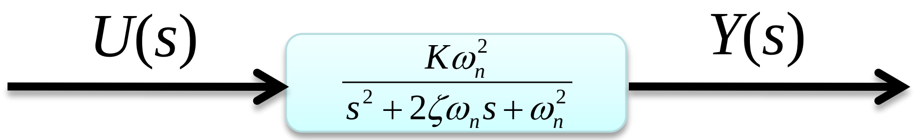
$$\frac{d^2y(t)}{dt^2}+2\zeta\omega_n\frac{dy(t)}{dt}+\omega_n^2y(t)=K\omega_n^2u(t)$$

where K is the steady-state/static gain, ζ is the damping ratio, ω_n is the undamped natural frequency.

- Transfer function of prototype or standard second-order system is

$$G(s)=\frac{Y(s)}{U(s)}=\frac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

Notice that $K=\lim_{s\to 0}G(s)$



- Common examples are

Mass-spring-damper system: $$G(s)=\frac{X(s)}{F(s)}=\frac{1}{ms^2+cs+k}$$

Position of car on level surface: $$G(s)=\frac{Y(s)}{Y_r(s)}=\frac{K_1}{Ms^2+k_f s+K_1}$$

RLC circuit : $$G(s)=\frac{sLR}{RLCs^2+sL+R}$$

- Poles of a second-order system can be found by solving,

$$s^2+2\zeta\omega_n s+\omega_n^2=0$$

$$s=\frac{-2\zeta\omega_n \pm\sqrt{4\zeta^2\omega_n^2-4\omega_n^2}}{2}$$

$$=-\zeta\omega_n \pm\omega_n\sqrt{\zeta^2-1}$$

What is damping? Consider the parallel RLC cct with tf :

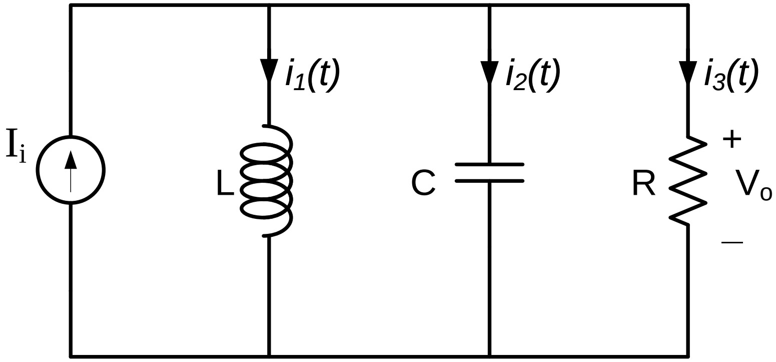
$$G(s) = \frac{sLR}{RLCs^2 + sL + R}$$

$$= \frac{s/C}{s^2 + s/RC + 1/LC} \quad \text{compare with} \quad \frac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$$

Disregard the zero

$$\omega_n^2 = \frac{1}{LC} \quad \text{or} \quad \omega_n = \frac{1}{\sqrt{LC}}$$

$$2\varsigma\omega_n = \frac{1}{RC} \Longrightarrow \varsigma = \frac{\sqrt{LC}}{2RC} = \frac{1}{2R}\sqrt{\frac{L}{C}}$$



When R increases, damping ratio, ζ, decreases
Less energy is lost because less current flows in the parallel R

When R decreases, ζ increases, more energy is lost and output V_0 settles quickly to its steady state of zero.

Damping is related to how energy dissipates in the system. Higher damping, faster dissipation, lower damping, slower dissipation. See more later.

- There are 4 scenarios for the poles based on damping :
  - When ζ < 1, system is underdamped. Poles are complex conjugate :

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$
$$= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sigma \pm j\omega_d$$

where ω_d is the damped natural frequency.

Poles appear as complex conjugate

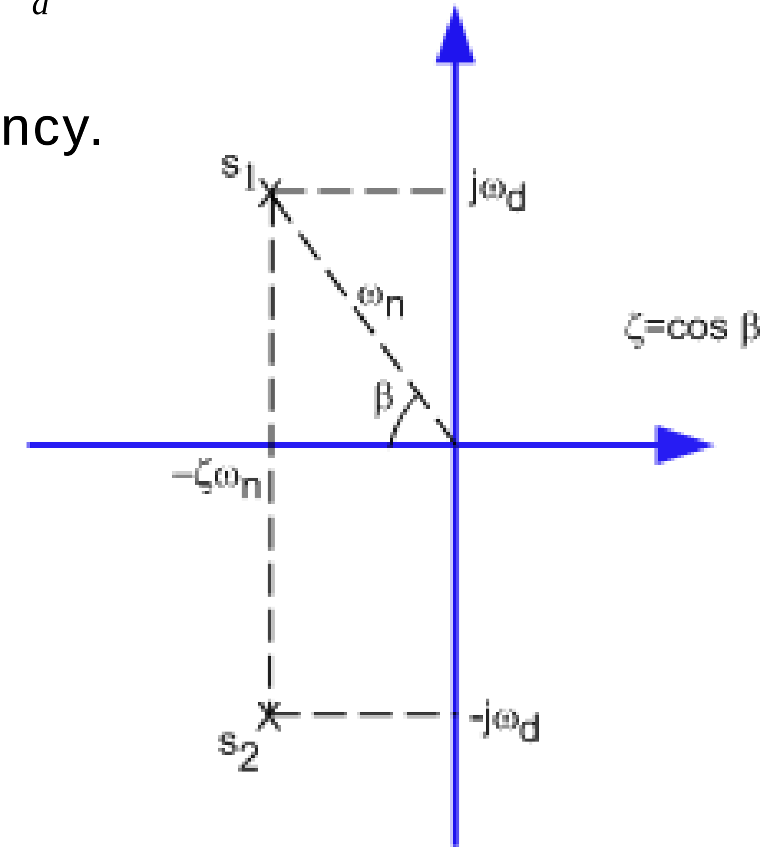
Real part of poles at $-\zeta\omega_n = -\sigma$

Damping ratio : $\zeta = \cos\beta < 1$

Larger β implies smaller ζ

Magnitude of pole : ω_n

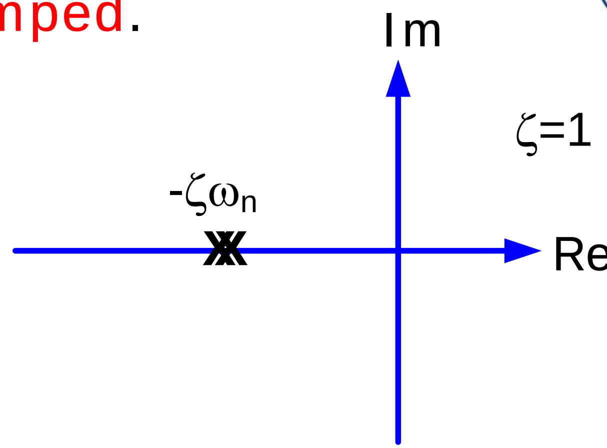
Poles on the complex plane



- When ζ = 1, system is critically damped.

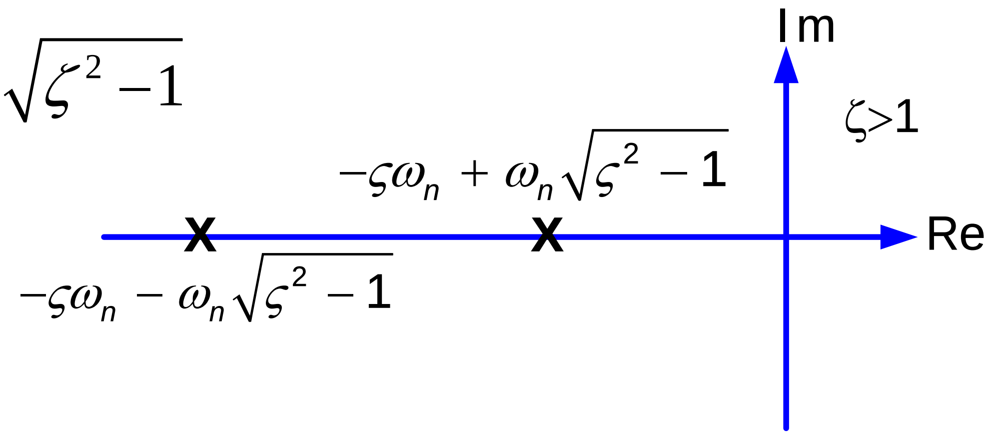
  Poles are real and repeated :

$$
\begin{aligned}
s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\
&= -\zeta\omega_n, \ -\zeta\omega_n
\end{aligned}
$$



- When ζ > 1, poles are real and distinct, system is overdamped

$$
\begin{aligned}
s_{1,2} &= -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} \\
&= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \ -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}
\end{aligned}
$$



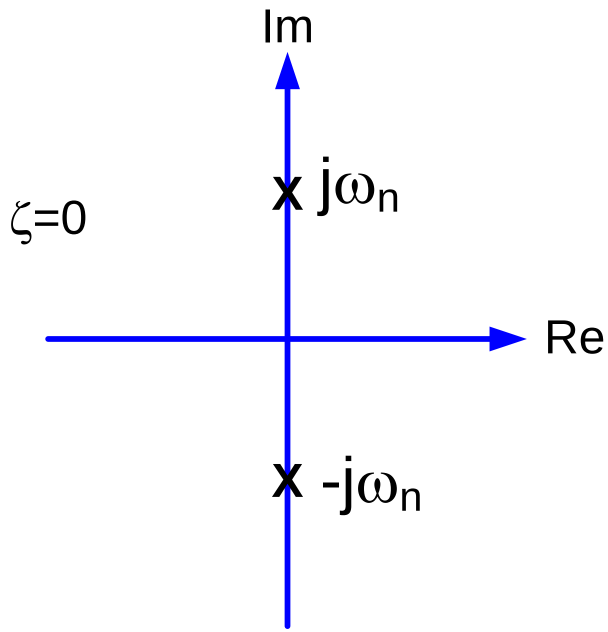
- When ζ = 0, system is marginally stable.

  Poles are purely imaginary and transfer function becomes :

$$G(s) = \frac{K\omega_n^2}{s^2 + \omega_n^2}$$

Poles are at $s = \pm j\omega_n$

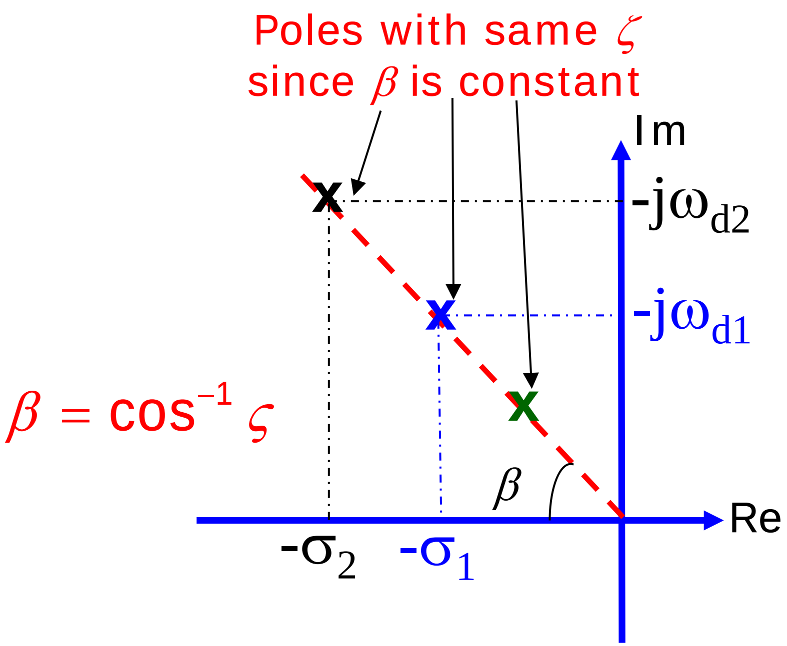
This type of second order systems is very special as you will see later in the impulse and step responses.

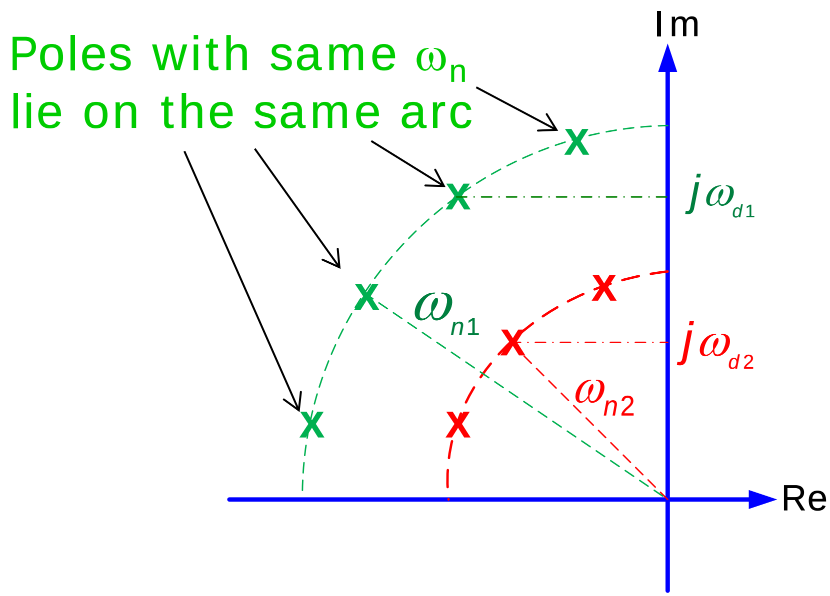


Relationship between Poles, Damping Ratio and Natural Frequency

Complex poles of second order underdamped system :

$$s_{1,2} = -\varsigma\omega_n \pm j\omega_n\sqrt{1-\varsigma^2}$$
$$= -\sigma \pm j\omega_d$$





Poles with same damping ratio, ζ, are located on the same line with $\beta = \cos^{-1}\varsigma$. Different poles have different $-\sigma$ values

Poles with the same ω_n are located on an arc with a radius of ω_n. Different poles on the same arc have different ω_d values

Output Response of LTI Systems

- Impulse response – output response when the input is an impulse
- sinusoidal response – output response when the input is a sinusoidal signal
- step response – output response when the input is a step signal

Each of these responses is important to LTI systems because they define certain behaviours that can be generalized for such systems.

- Impulse response is related to the transfer function of the system.
- Sinusoidal response is related to the frequency response of the system
- Step response is very commonly encountered in practice and they give info about some physical parameters in the LTI system

Impulse Response

- The impulse response of a LTI system is the system's output signal when the input signal, $u(t)$, is the unit impulse function, $\delta(t)$, with zero initial conditions
- Suppose the transfer function representation of a LTI system is

$$G(s) = \frac{Y(s)}{U(s)}$$

When $u(t) = \delta(t)$, then $U(s) = \mathcal{L}\{\delta(t)\} = 1$ and

$$Y(s) = G(s)U(s) = G(s) \qquad \because U(s) = 1$$

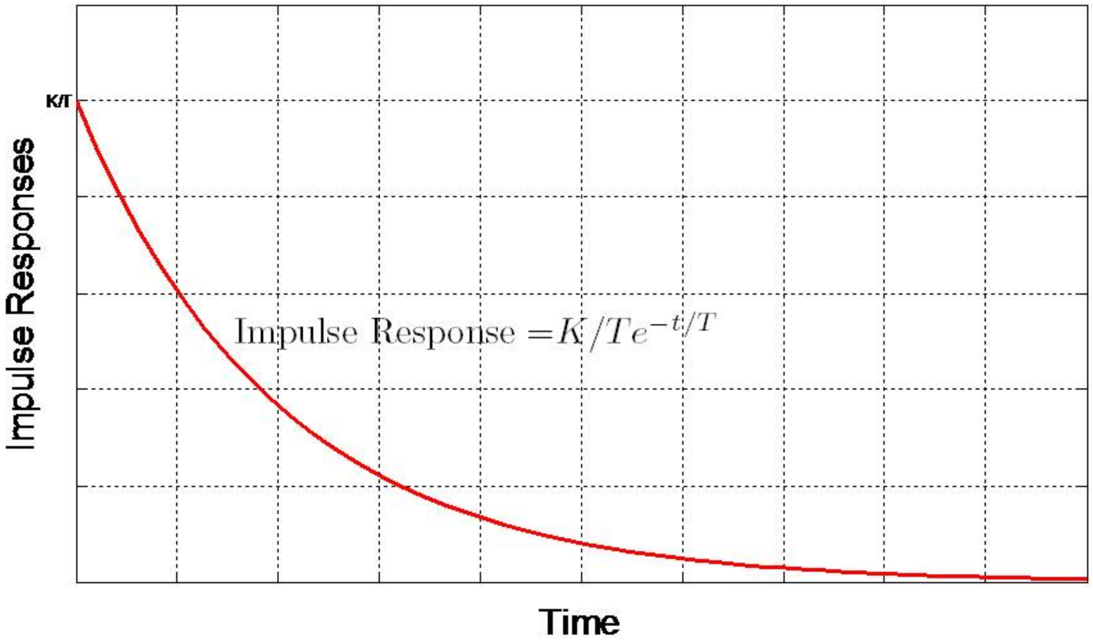
$$y(t) = \mathcal{L}^{-1}\{G(s)\} = \text{ Impulse response, } g(t)$$

$$\boxed{\text{System transfer function, } G(s) = \mathcal{L}\{\text{Impulse Response}\}}$$

Impulse Response of First Order Systems

Transfer function : $G(s) = \dfrac{K}{sT+1} = \dfrac{K}{T}\dfrac{1}{\left(s+\dfrac{1}{T}\right)}$ assuming $T>0$

Impulse Response : $g(t) = \mathcal{L}^{-1}\{G(s)\} = \dfrac{K}{T}e^{-\frac{t}{T}}$



Impulse response of a typical first order stable system

Impulse Response of Second Order Systems

Standard second order systems : $G(s)=\dfrac{K\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$

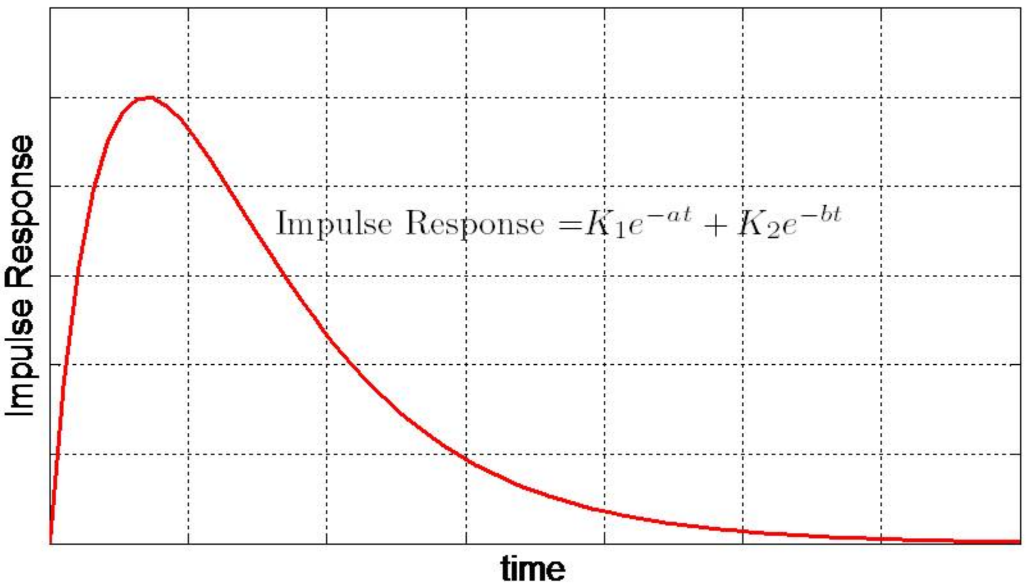
- Overdamped with $\zeta>1$ and distinct poles

$$G(s)=\frac{K\omega_n^2}{\left(s+\varsigma\omega_n+\omega_n\sqrt{\varsigma^2-1}\right)\left(s+\varsigma\omega_n-\omega_n\sqrt{\varsigma^2-1}\right)}$$

$$=\frac{K\omega_n^2}{(s+a)(s+b)}$$

Impulse response :

$$g(t)=\mathcal{L}^{-1}\{G(s)\}=K_1e^{-at}+K_2e^{-bt}$$



- Underdamped with $\zeta<1$ and complex poles

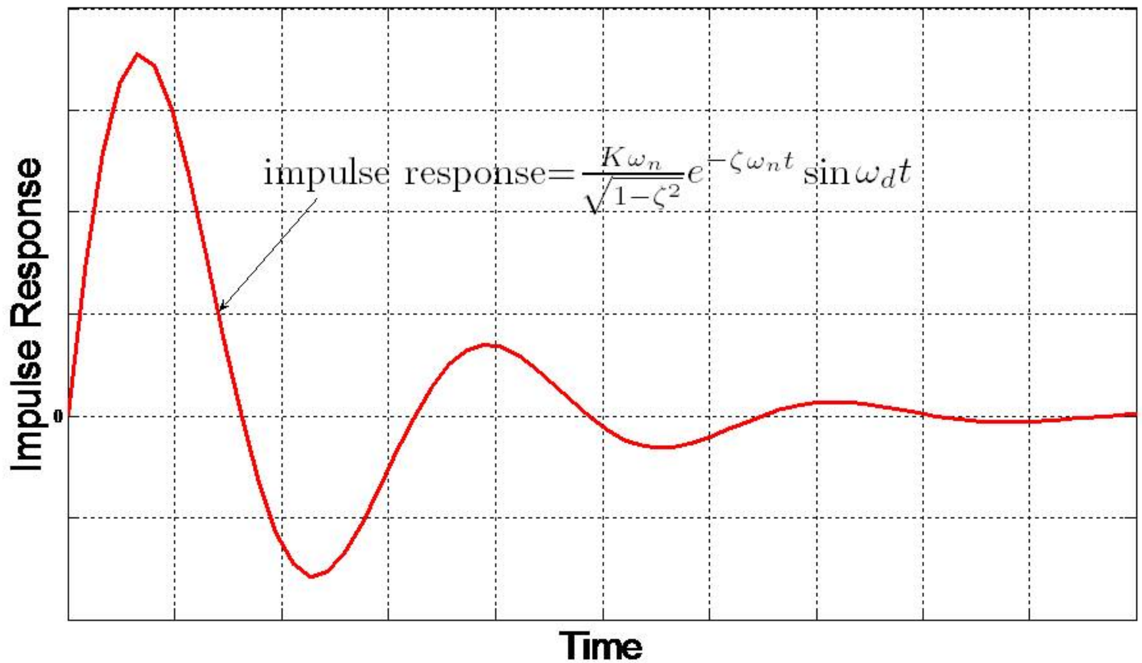
$$G(s)=\frac{K\omega_n^2}{\left(s+\varsigma\omega_n+j\omega_n\sqrt{1-\varsigma^2}\right)\left(s+\varsigma\omega_n-j\omega_n\sqrt{1-\varsigma^2}\right)}$$

$$=\frac{K\omega_n^2}{\left(s+\sigma+j\omega_d\right)\left(s+\sigma-j\omega_d\right)}$$

Impulse response :

$$g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$$

$$=\frac{K\omega_n}{\sqrt{1-\varsigma^2}}e^{-\sigma t}\sin\omega_d t$$



- Critically damped with $\zeta=1$ and repeated real poles

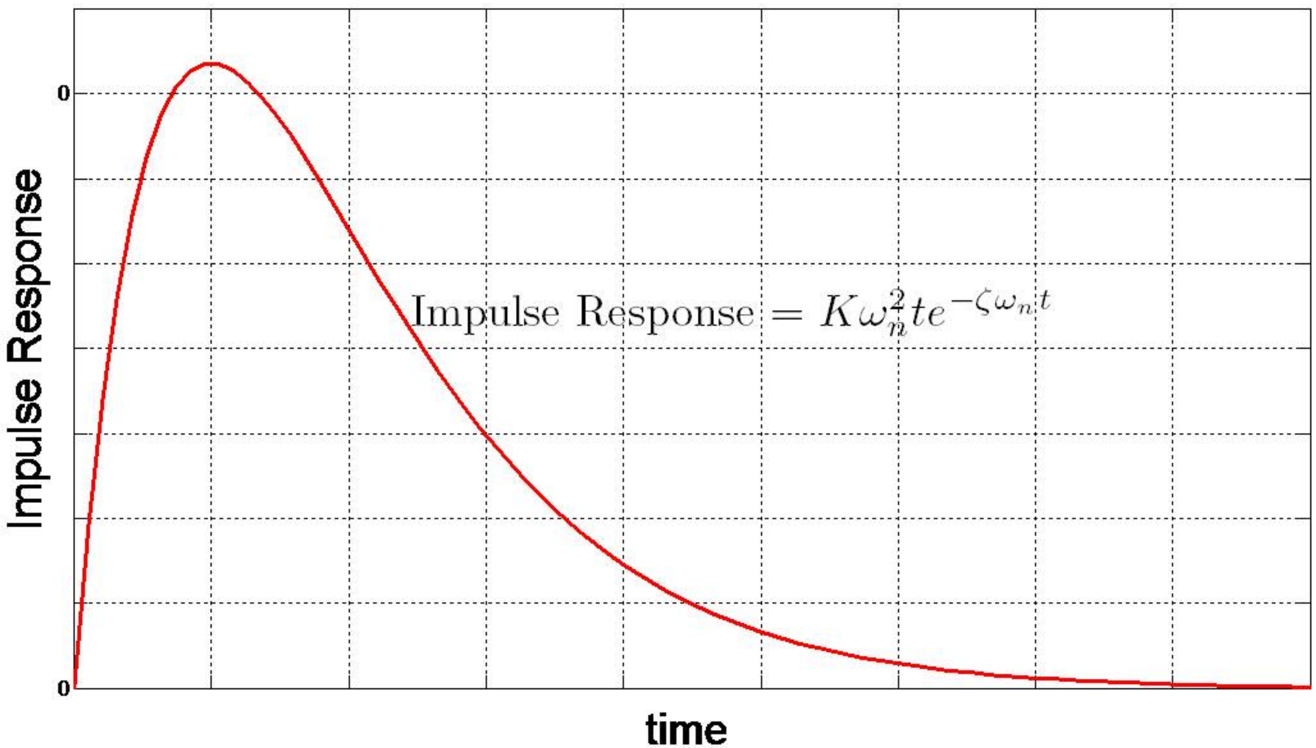
$$G(s)=\frac{K\omega_n^2}{(s+\varsigma\omega_n)(s+\varsigma\omega_n)}$$

$$=\frac{K\omega_n^2}{(s+\sigma)^2}$$

Impulse response :

$$g(t)=\mathcal{L}^{-1}\{G(s)\}$$

$$=K\omega_n^2 te^{-\sigma t}$$

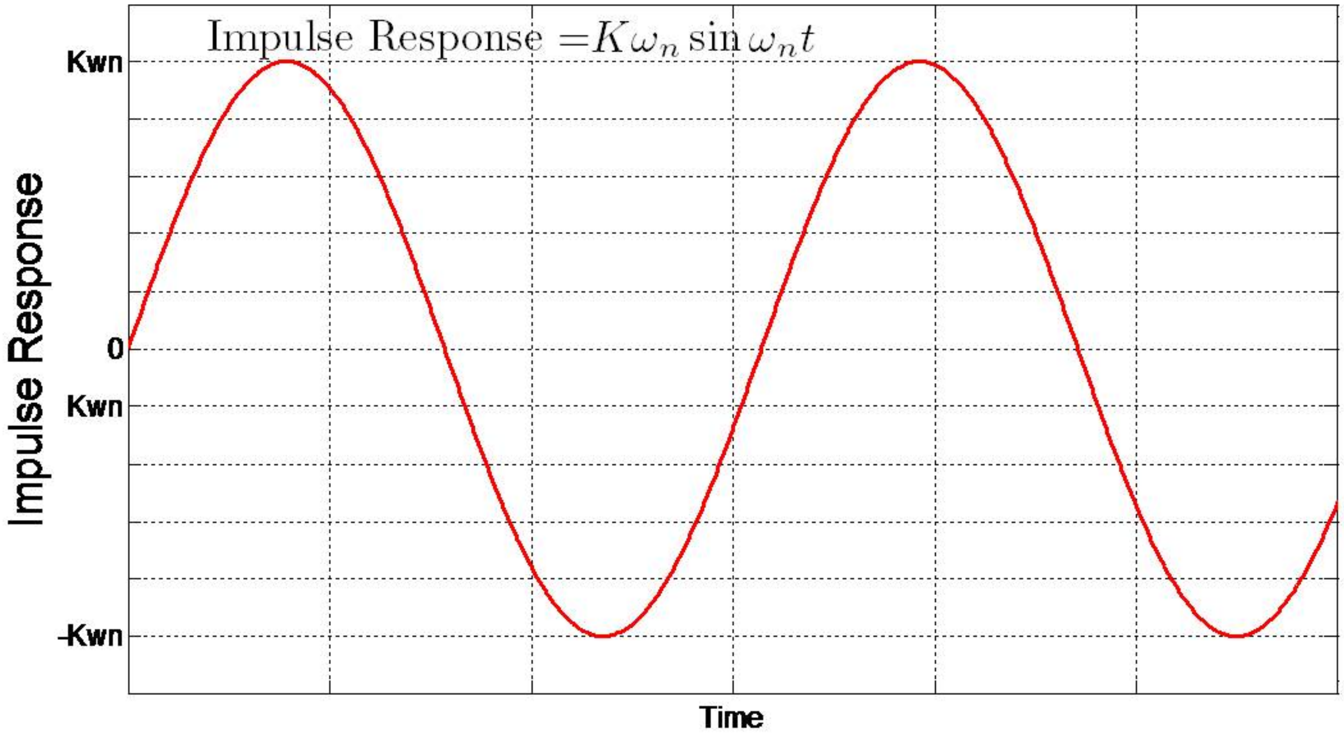


- Marginally stable with $\zeta=0$ and pure imaginary poles

$$G(s)=\frac{K\omega_n^2}{\left(s^2+\omega_n^2\right)}$$
$$=\frac{K\omega_n^2}{\left(s+j\omega_n\right)\left(s-j\omega_n\right)}$$

Impulse response :

$$g(t)=\mathcal{L}^{-1}\left\{G(s)\right\}$$
$$=K\omega_n\sin\omega_n t$$



Impulse response is a sinusoidal form

Related to impulse response, consider the convolution integral given by :

$$g(t) * u(t) = \int_0^{\infty} u(t-\tau)g(\tau)d\tau$$

Not easy to solve

Take the Laplace transform of this convolution integral :

$$\mathcal{L}\{g(t) * u(t)\} = \int_0^{\infty}\left(\int_0^{\infty} u(t-\tau)g(\tau)d\tau\right)e^{-st}dt$$

$$= \int_0^{\infty}\int_0^{\infty} u(t-\tau)g(\tau)e^{-st}d\tau dt$$

$$= \int_0^{\infty}\int_0^{\infty} u(t-\tau)g(\tau)e^{-s(t-\tau)}e^{-s\tau}d\tau dt$$

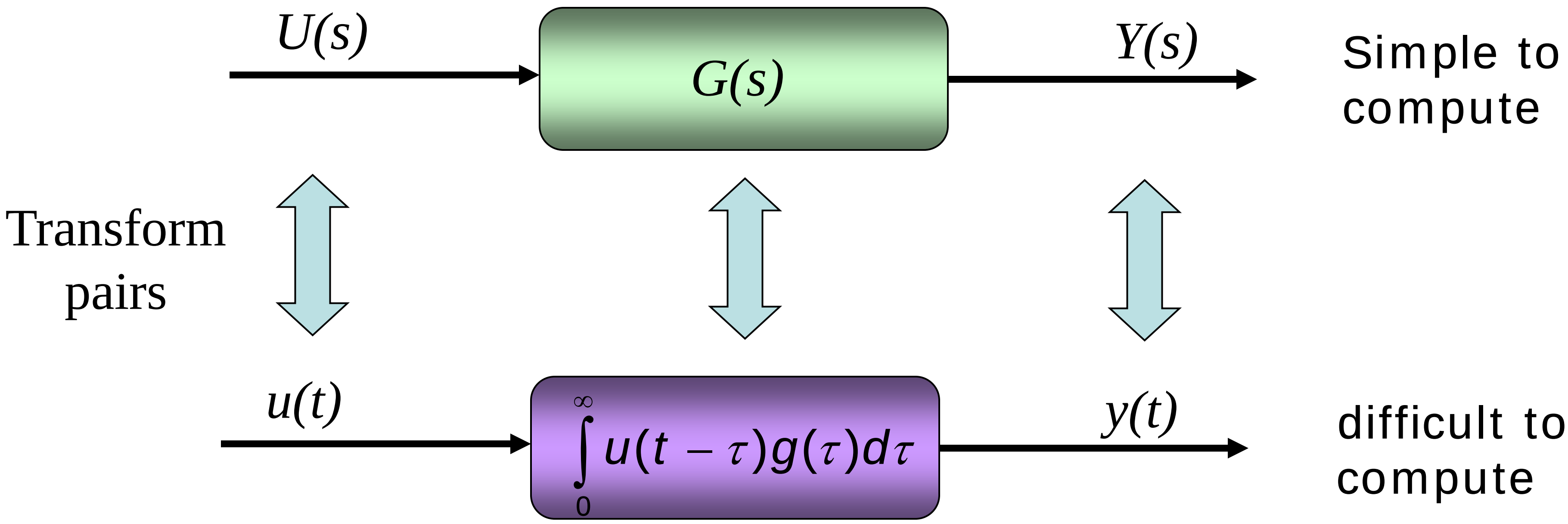
$$= \int_0^{\infty}\int_0^{\infty} u(\lambda)g(\tau)e^{-s(\lambda)}e^{-s\tau}d\tau d\lambda$$

$$= G(s)U(s) = Y(s) = \mathcal{L}\{y(t)\}$$

This implies that

$$\boxed{y(t)=g(t)*u(t)}$$

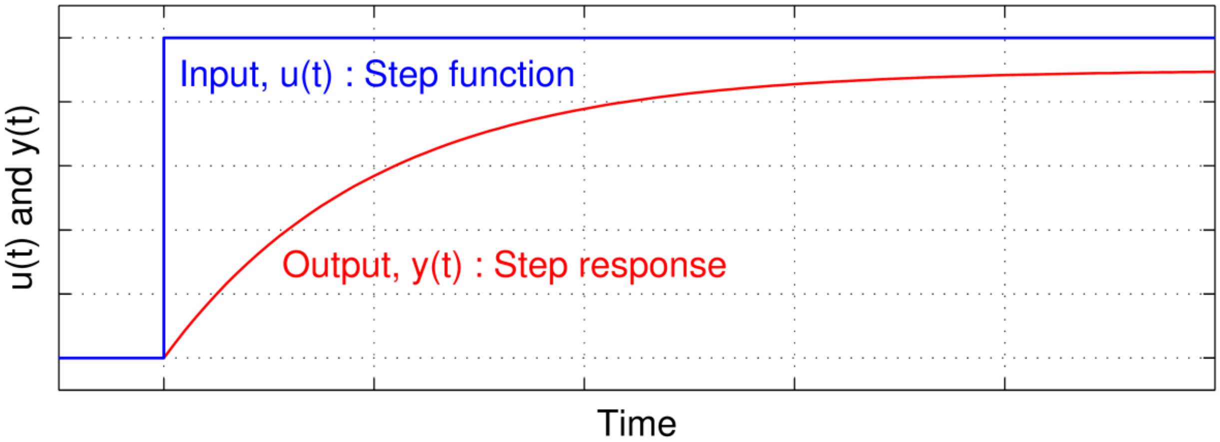
In general, the time domain output, *y(t)*, of a LTI system with a general input, *u(t)*, is given by the convolution between the impulse response, *g(t)*, and input, *u(t)*.



This clearly shows the convenience of the Laplace transform tool in solving for the output of a LTI system, given any general input signal, *u(t)*.

Step Response

- The step response of a LTI system is the system's output signal when the input signal, $u(t)$, is a step function of any magnitude, r
- Step response is the output when a sudden change in the input to the system occurs. Examples : switching on the power supply
- When $u(t) = r$, then $U(s) = \mathcal{L}\{r\} = r/s$ and $Y(s) = G(s)\dfrac{r}{s}$
- An example of the step response of a dynamic LTI system :



- Many daily tasks/actions can be modeled by $U(t)$, e.g.,
  - Stepping on the car accelerator
  - Switching on a fan/kettle

Is switching the settings on a fan considered a change in input?

- How do you find the unit step response of a LTI system given its transfer function $G(s)=\dfrac{Y(s)}{U(s)}$?

Since $U(s)=\mathcal{L}\{U(t)\}=\dfrac{1}{s}, \quad Y(s)=G(s)U(s)=\dfrac{G(s)}{s}$

$$\therefore \quad \text{Unit Step response } y(t)=\mathcal{L}^{-1}\left\{\frac{G(s)}{s}\right\}$$

Note that $s = 0$ is an input pole.

Relationship between step response and impulse response

$$U(s)=\mathcal{L}\{U(t)\}=\frac{1}{s},\qquad Y(s)=G(s)U(s)=\frac{G(s)}{s}$$

$$\therefore\quad \mathcal{L}\{y(t)\}=\frac{G(s)}{s}=\mathcal{L}\left\{\int_0^t g(\tau)d\tau\right\}$$

Recall :

$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\frac{F(s)}{s}$ where $\mathcal{L}\{f(t)\}=F(s)$

$$\boxed{\begin{aligned}\text{Step Response} &= \int_0^t \text{Impulse Response}\, dt\\ \text{Impulse Response} &= \frac{d(\text{Step Response})}{dt}\end{aligned}}$$

So far, the materials developed for the step responses apply to general LTI systems. Next we develop some explicit expressions for the step responses of first and second order systems, like we have done for impulse responses. This leads to further insights of first and second order system behaviours.

- Step response for a step input of magnitude r : $Y(s) = \dfrac{rG(s)}{s}$

- Assuming that $G(s)$ is stable, apply final value theorem to get the steady state value of y(t) :

$$y_{ss} = \lim_{s\to 0} sY(s) = \lim_{s\to 0} s\frac{rG(s)}{s} = rG(0)$$

The idea of steady state gain is important in LTI systems

- Defining steady state gain to be : $K = \dfrac{y_{ss}}{u_{ss}} = \dfrac{rG(0)}{r} = G(0)$

- Thus the $\boxed{\text{steady state gain is value of } G(s) \text{ at } s=0 \text{ ie } G(0)}$

- There are alternative names for $G(0)$ such as DC gain or static gain or steady state gain. They all refer to the same quantity $G(0)$.

- Note also that what this means is that the steady state output to a step input can be computed by

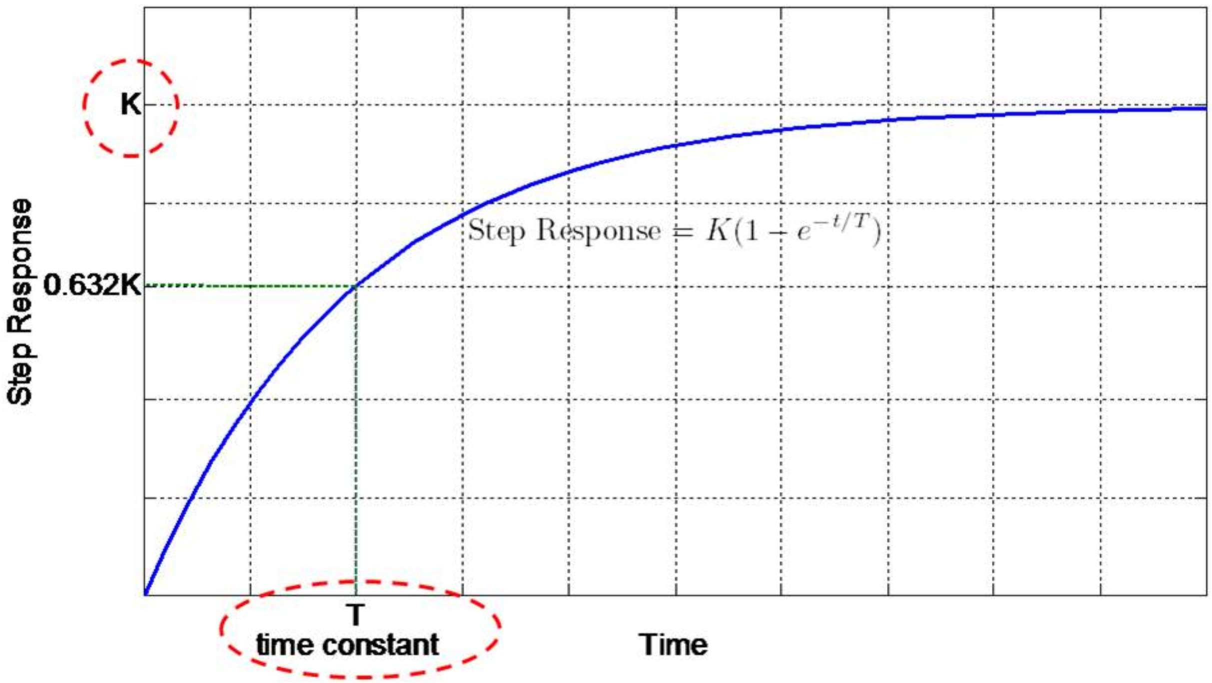
$$y_{ss} = Ku_{ss} = G(0)u_{ss}$$

Applies strictly to only step or constant inputs

Step Response of First Order Systems

Transfer function : $G(s)=\dfrac{K}{sT+1}=\dfrac{K}{T}\dfrac{1}{\left(s+\dfrac{1}{T}\right)}$ assuming $T>0$

Step Response : $y_{step}(t)=\mathcal{L}^{-1}\left\{\dfrac{G(s)}{s}\right\}=\mathcal{L}^{-1}\left\{\dfrac{K}{s}-\dfrac{K}{s+\dfrac{1}{T}}\right\}=K\left\{1-e^{-\frac{t}{T}}\right\}$



At $t=infinity$,

$y(\infty)=K=G(0)$

$=$ steady state gain of G(s)

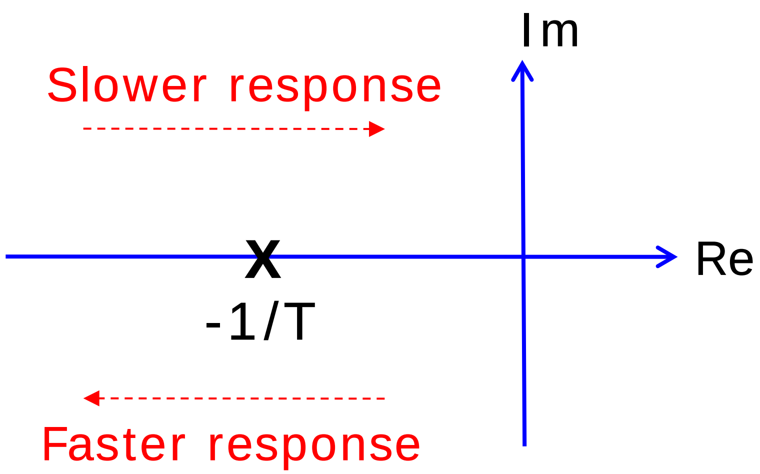
At $t=T$,

$y(T)=K(1-e^{-1})=0.632K$

Characteristics of first order step responses

- The time constant, *T*, is defined to be the time taken for the output to reach 63.2% of final steady state value.

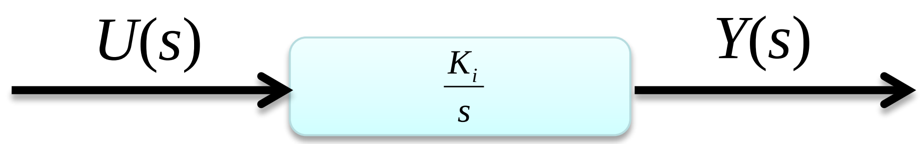
- Pole is at s=-1/T. Therefore the further the pole is from the imaginary axis, the smaller is the time constant, and thus the faster is the step response.



An integrator is a special form of a first order system.

In time domain, an integrator has an output : $y(t) = \int_0^t K_i u(\tau)d\tau$

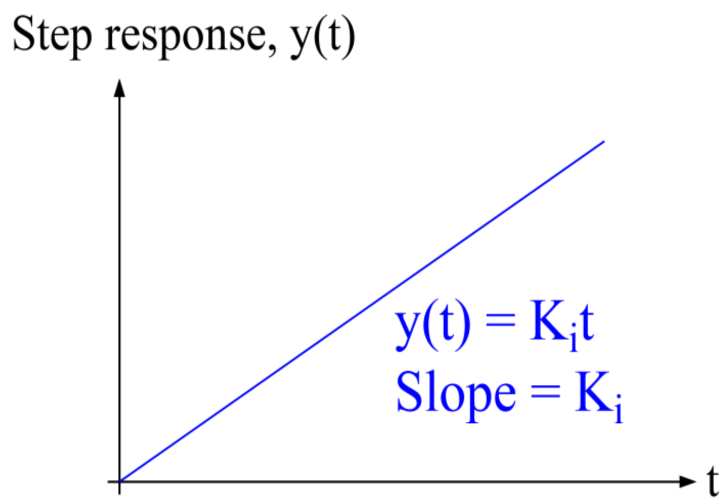
In s-domain, an integrator has a transfer function : $G(s) = \dfrac{K_i}{s}$



Step response of an integrator is :

$$Y(s) = \frac{G(s)}{s} = \frac{K_i}{s^2}$$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{K_i}{s^2}\right\} = K_i t$$



Some peculiarities about the integrator

- Although the step function (input signal) is bounded, the step response is unbounded. However, the output signal of an integrator is bounded when the input signal is a pulse
- Integrator is marginally stable as its impulse response is non-decreasing
- Consistent with conclusion drawn from system pole location. Pole of an integrator is $s = 0$ (on the imaginary axis, although at the origin), and systems with non-repeated pole on the imaginary axis is marginally stable
- Although the integrator has a 'gain' K_i, this is not the same as the static/dc/steady state gain because steady state is meaningless for an integrator as you can see from the step response. The output of the step response does not reach a steady state and thus the steady state gain cannot be defined in the same way. Instead K_i is simply referred to as the integral gain.

- The output of an integrator depends on the entire past history of the input
- Another useful property of an integrator is

$$\boxed{\int_0^t u(\tau)d\tau = \text{constant} \;\; \forall t > t_0 \quad \text{if and only if } u(t) = 0 \quad \forall t > t_0}$$

- Example: Consider the following position model,

When the position of the vehicle (output of integrator) is a constant, the velocity (input of integrator) must be zero.

- Examples of systems whose transfer function contains an integrator:
  - A capacitor fed with a current source
  - The fluid level in a tank with constant outflow device

Step Response of Second Order Systems

Standard second order systems : $G(s) = \dfrac{K\omega_n^2}{s^2 + 2\varsigma\omega_n s + \omega_n^2}$

We again consider 4 different types of second order systems depending on the damping ratio, ζ

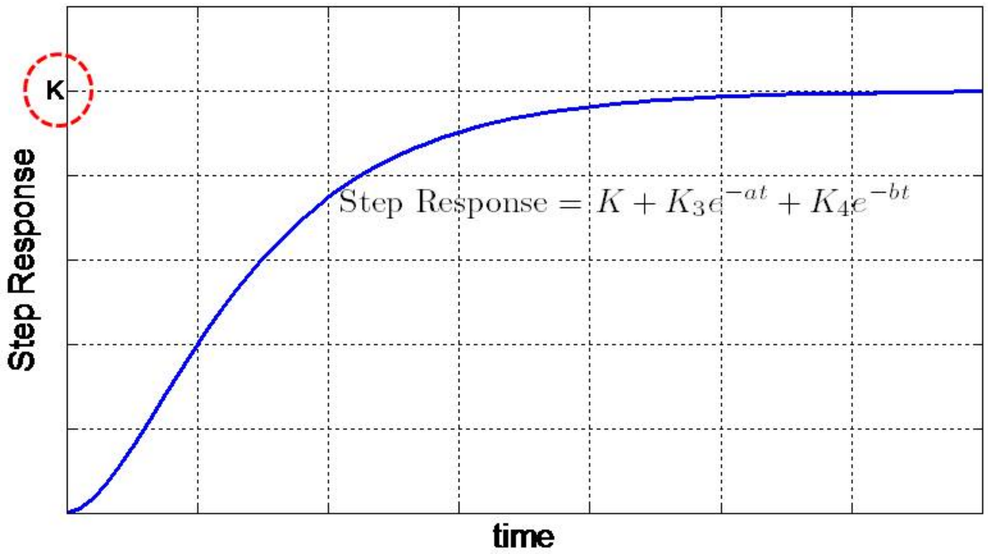
- Overdamped with $\zeta > 1$ and distinct poles

$$G(s) = \frac{K\omega_n^2}{\left(s + \varsigma\omega_n + \omega_n\sqrt{\varsigma^2 - 1}\right)\left(s + \varsigma\omega_n - \omega_n\sqrt{\varsigma^2 - 1}\right)}$$

$$= \frac{K\omega_n^2}{(s+a)(s+b)}$$

$$y_{step}(t) = K + K_3 e^{-at} + K_4 e^{-bt}$$

$$K_3 = \frac{K}{2}\left(\frac{\varsigma}{\sqrt{\varsigma^2 - 1}} - 1\right), \quad K_4 = -\frac{K}{2}\left(\frac{\varsigma}{\sqrt{\varsigma^2 - 1}} + 1\right)$$



The larger the *a* and *b* (poles further away from im axis), the faster is the step response. Poles are at –*a* and –*b*. Similar characteristics to 1st order system.

- Underdamped with $\zeta<1$ and complex poles

$$G(s)=\frac{K\omega_n^2}{\left(s+\varsigma\omega_n+j\omega_n\sqrt{1-\varsigma^2}\right)\left(s+\varsigma\omega_n-j\omega_n\sqrt{1-\varsigma^2}\right)}$$

$$=\frac{K\omega_n^2}{\left(s+\sigma+j\omega_d\right)\left(s+\sigma-j\omega_d\right)}$$

Derivation of the step response :

$$Y(s)=\frac{K\omega_n{}^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

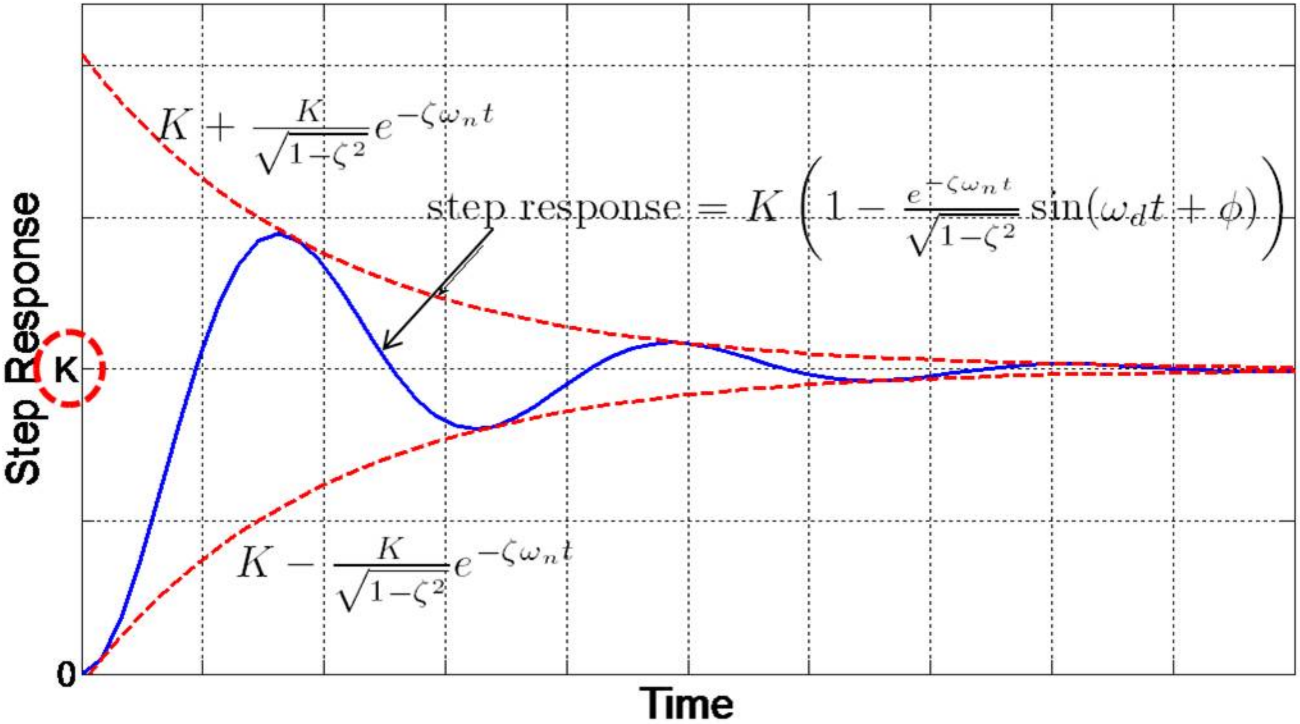
$$=\frac{K}{s}-\frac{K\left(s+2\zeta\omega_n\right)}{\left(s+\zeta\omega_n\right)^2+\omega_n^2-\zeta^2\omega_n^2}$$

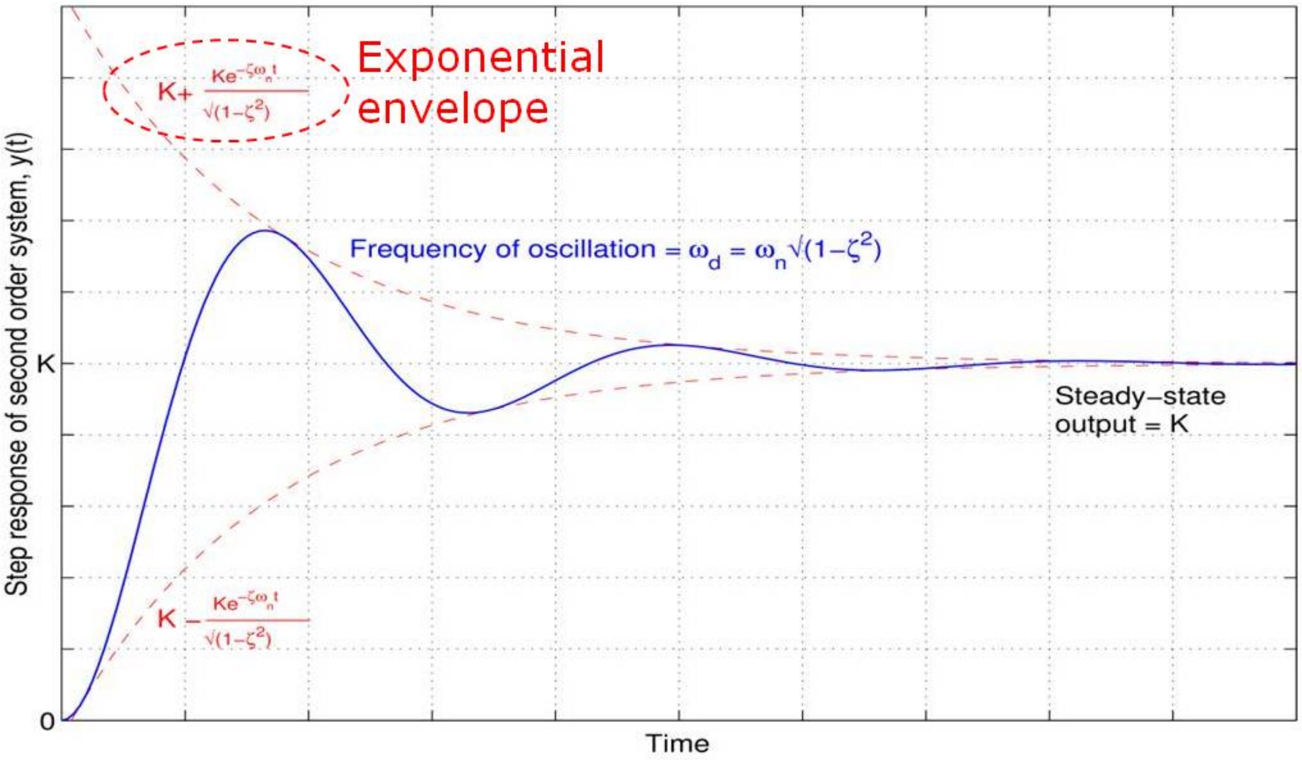
$$=\frac{K}{s}-\frac{K\left(s+\zeta\omega_n\right)}{\left(s+\zeta\omega_n\right)^2+\omega_n^2\left(1-\zeta^2\right)}-\frac{K\zeta\omega_n}{\left(s+\zeta\omega_n\right)^2+\omega_n^2\left(1-\zeta^2\right)}$$

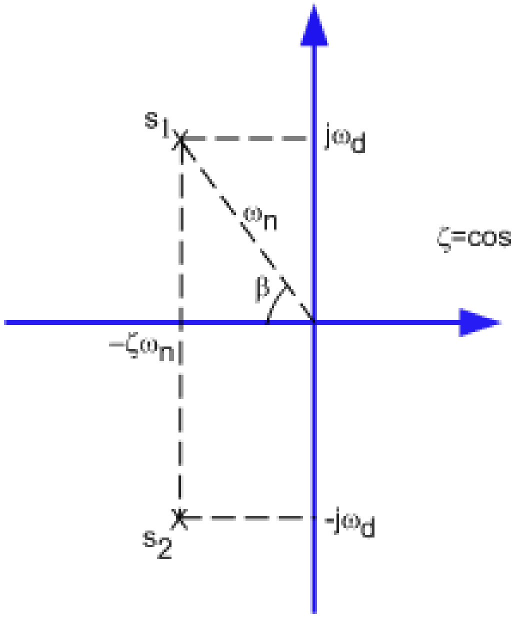
$$y_{step}(t) = \mathcal{L}^{-1}\{Y(s)\}$$

$$= K - Ke^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}\right)t - \frac{K\zeta}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\right)t$$

$$= K - \frac{Ke^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin\left[\left(\omega_n\sqrt{1-\zeta^2}\right)t+\phi\right] = K\left\{1 - \underbrace{\frac{e^{-\sigma t}}{\sqrt{1-\zeta^2}}\sin\left[\omega_d t+\phi\right]}_{\text{The transient part is a complex exponential signal}}\right\}$$



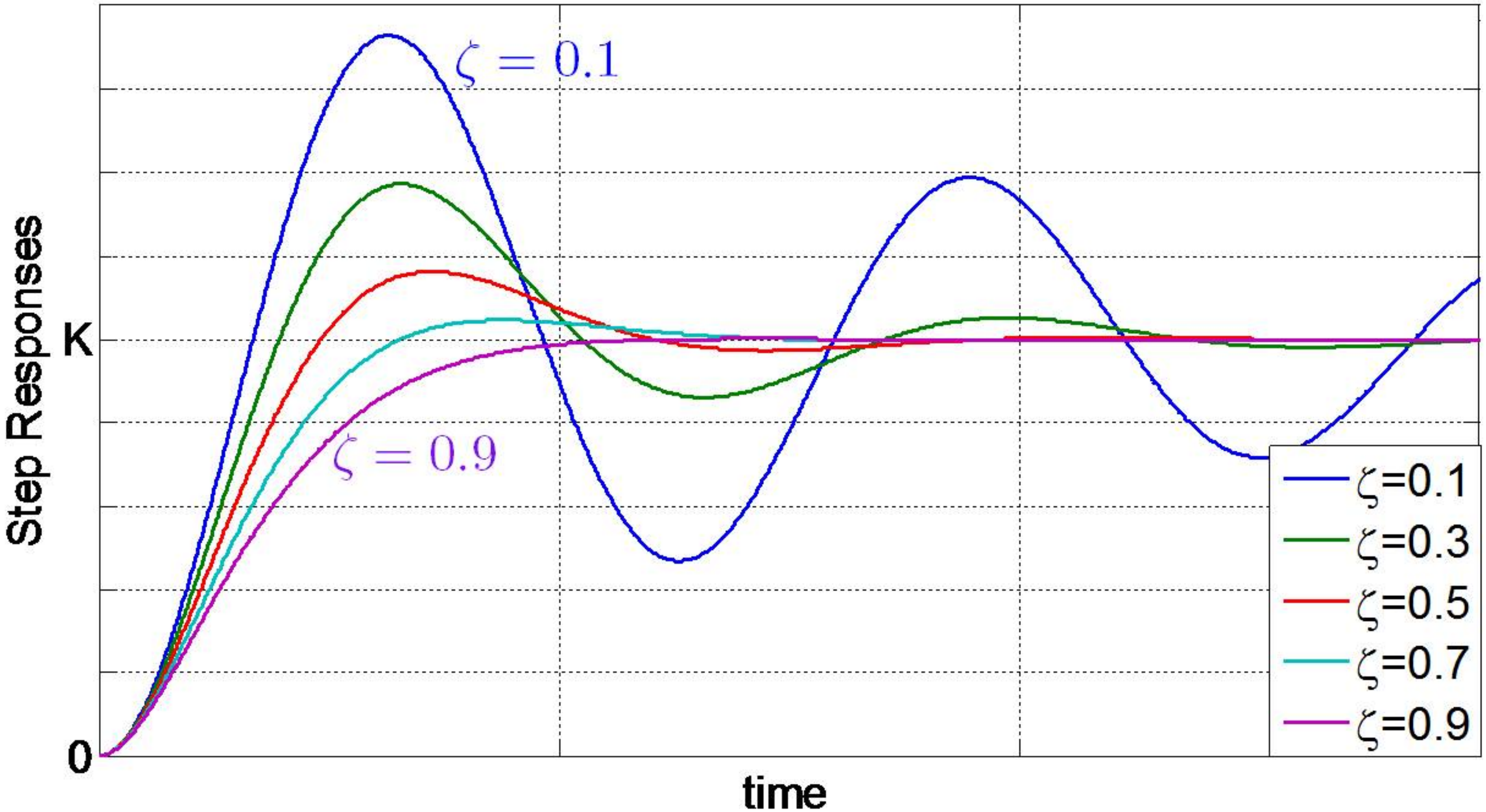




Pole positions

- Steady-state output is $K = \lim_{s\to 0} G(s) = G(0)$
- Magnitude of the real part of pole, $|\mathrm{Re}\{s\}| = \sigma = \zeta\omega_n$, determines the <u>exponential envelope</u>
- Like the case of a first-order system, steady-state is reached faster if the pole is farther to the left of the imaginary axis, i.e., $|\mathrm{Re}\{s\}| = \sigma$ is large
- Frequency of oscillation is ω_d. The further pole is from the real axis, the higher the frequency of oscillation

Family of step responses for different $\zeta < 1$



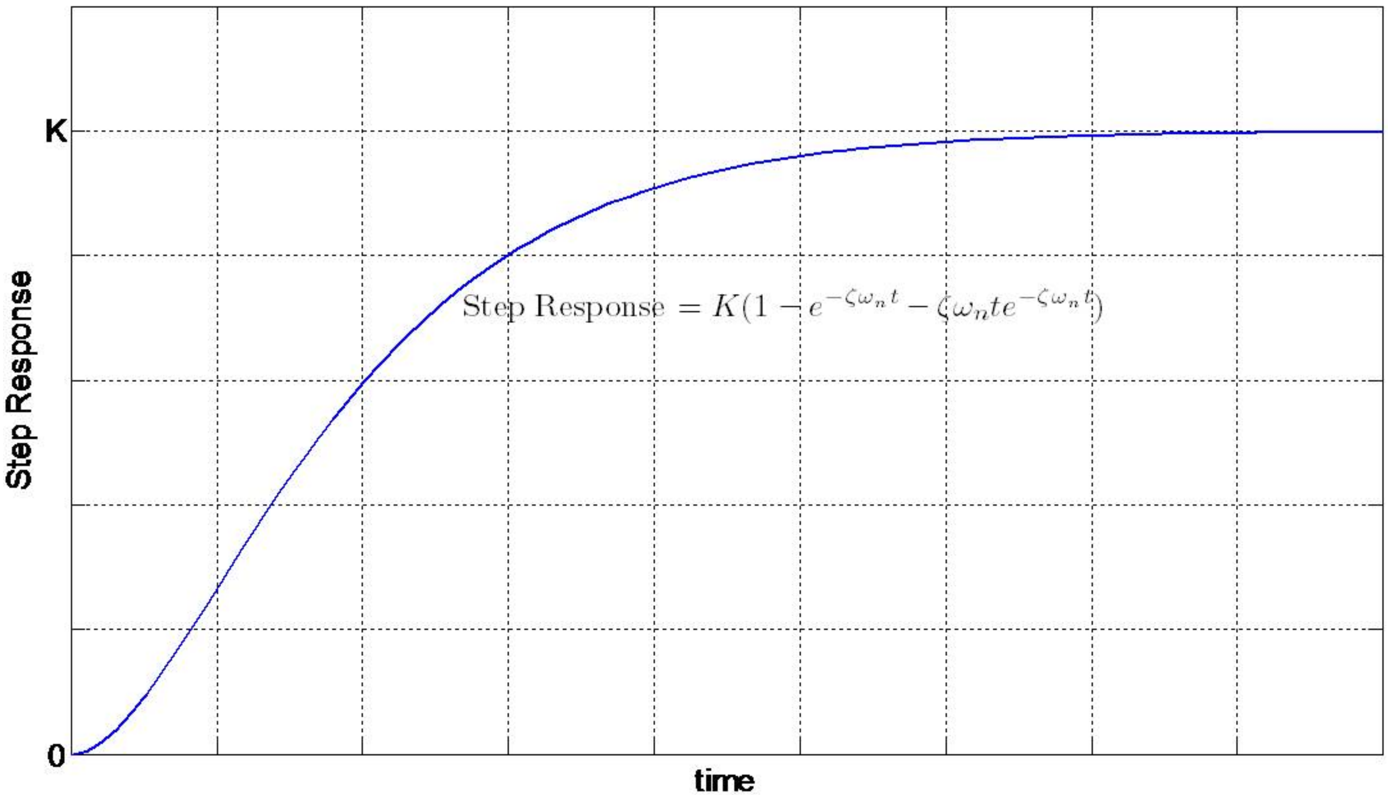
Oscillatory response is a typical characteristic of 2nd order underdamped systems

- Oscillations increase when ζ decreases! Overshoots also increase
- No more overshoots when ζ = 0.9
- Response takes a longer time to settle to steady state value of K for small ζ.

- Critically damped with $\zeta=1$ and repeated real poles

$$G(s)=\frac{K\omega_n^2}{(s+\varsigma\omega_n)(s+\varsigma\omega_n)}=\frac{K\omega_n^2}{(s+\sigma)^2}$$

$$y_{step}(t)=K\left(1-e^{-\varsigma\omega_n t}-\varsigma\omega_n te^{-\varsigma\omega_n t}\right)=K\left(1-e^{-\sigma t}-\sigma te^{-\sigma t}\right)$$



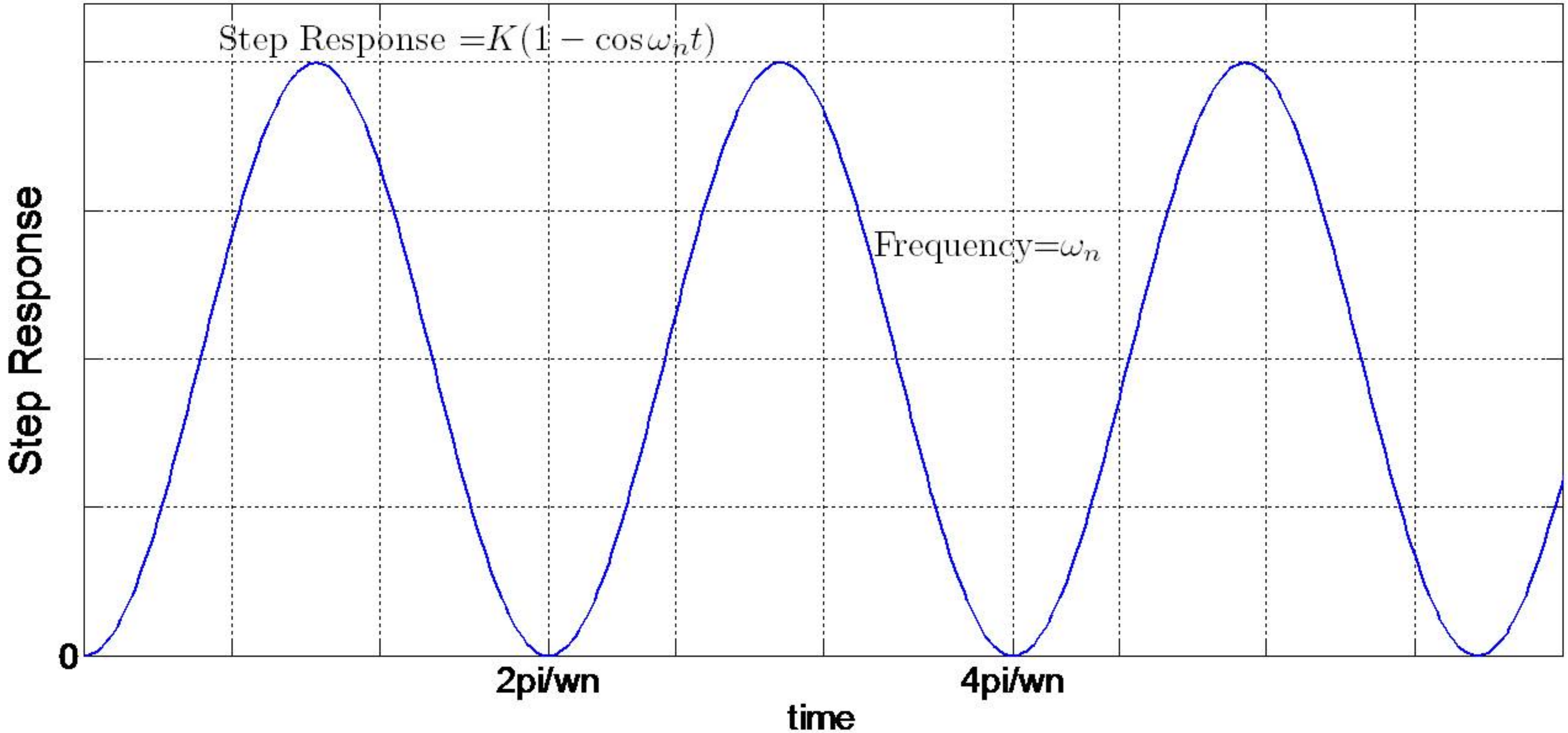
- No oscillations
- The larger σ is, the faster is the response. Hence the further the pole is from the imaginary axis, the faster is the response.

- Zero damping with $\zeta=0$ and purely imaginary poles

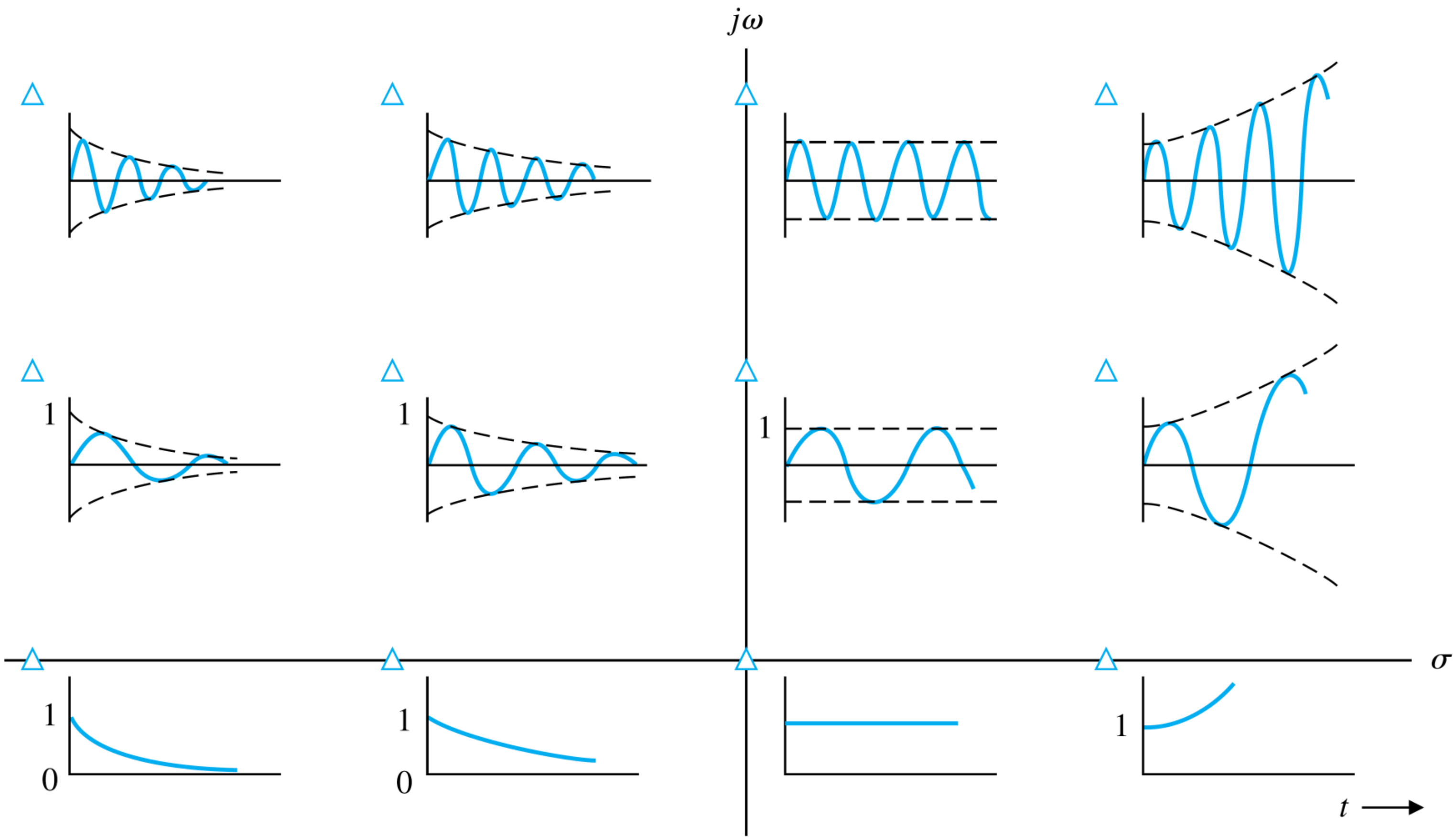
$$G(s)=\frac{K\omega_n^2}{\left(s^2+\omega_n^2\right)}$$
$$=\frac{K\omega_n^2}{\left(s+j\omega_n\right)\left(s-j\omega_n\right)}$$

$$y_{step}(t)=K(1-\cos\omega_n t)$$

Steady state is a purely oscillatory response which does not decay away. Works well as an oscillator!

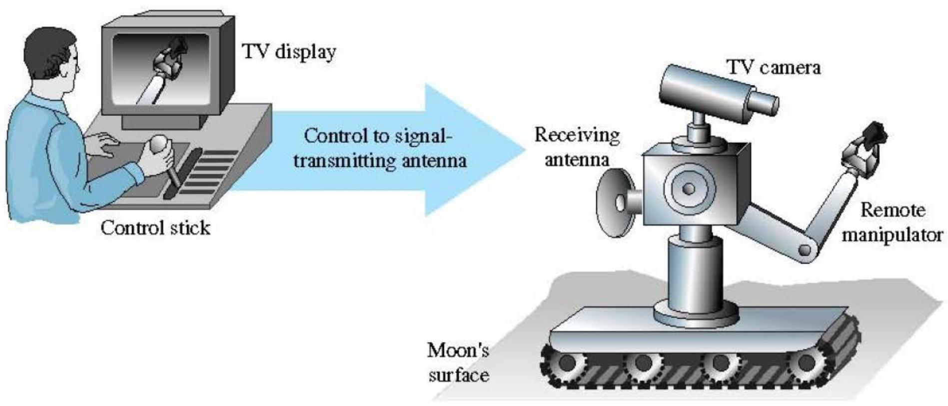


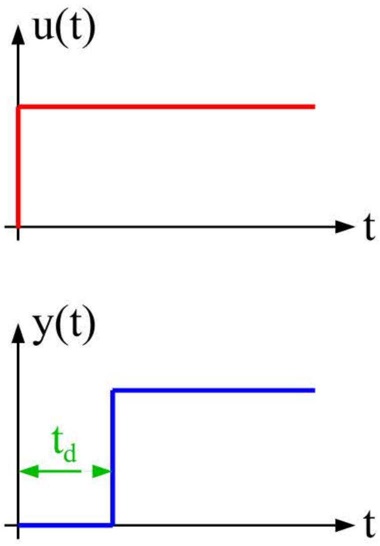
Relationship between Pole Locations and Impulse Responses



Transportation Delay

- Transportation delay, t_d , is also called transport lag or dead-time
- Transportation delay is the time delay that occurs in systems which require a finite time to move material or transmit signal from one point to another
- For example,





- Since $y(t) = u(t - t_d)$, transfer function is

$$\mathcal{L}\{y(t)\} = \mathcal{L}\{u(t - t_d)\}, \quad t_d = \frac{\text{Distance}}{\text{Speed}}$$

$$Y(s) = U(s)e^{-st_d}$$

$$G(s) = \frac{Y(s)}{U(s)} = e^{-st_d}$$

- Time delays are a nuisance because it implies that the system is slow to react to any changes. This can create problems when you try to change the system quickly. For example, time delays in networks can cause non-delivery of packets of information going round the networks.
- In general, it is undesirable to have time delays. Will see how this creates a problem in control later in the course.