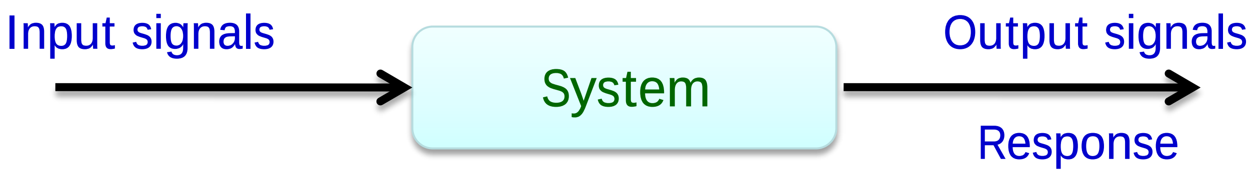
General Physical Systems

Physical systems are interconnections of components, devices or subsystems

Physical systems have many different properties.

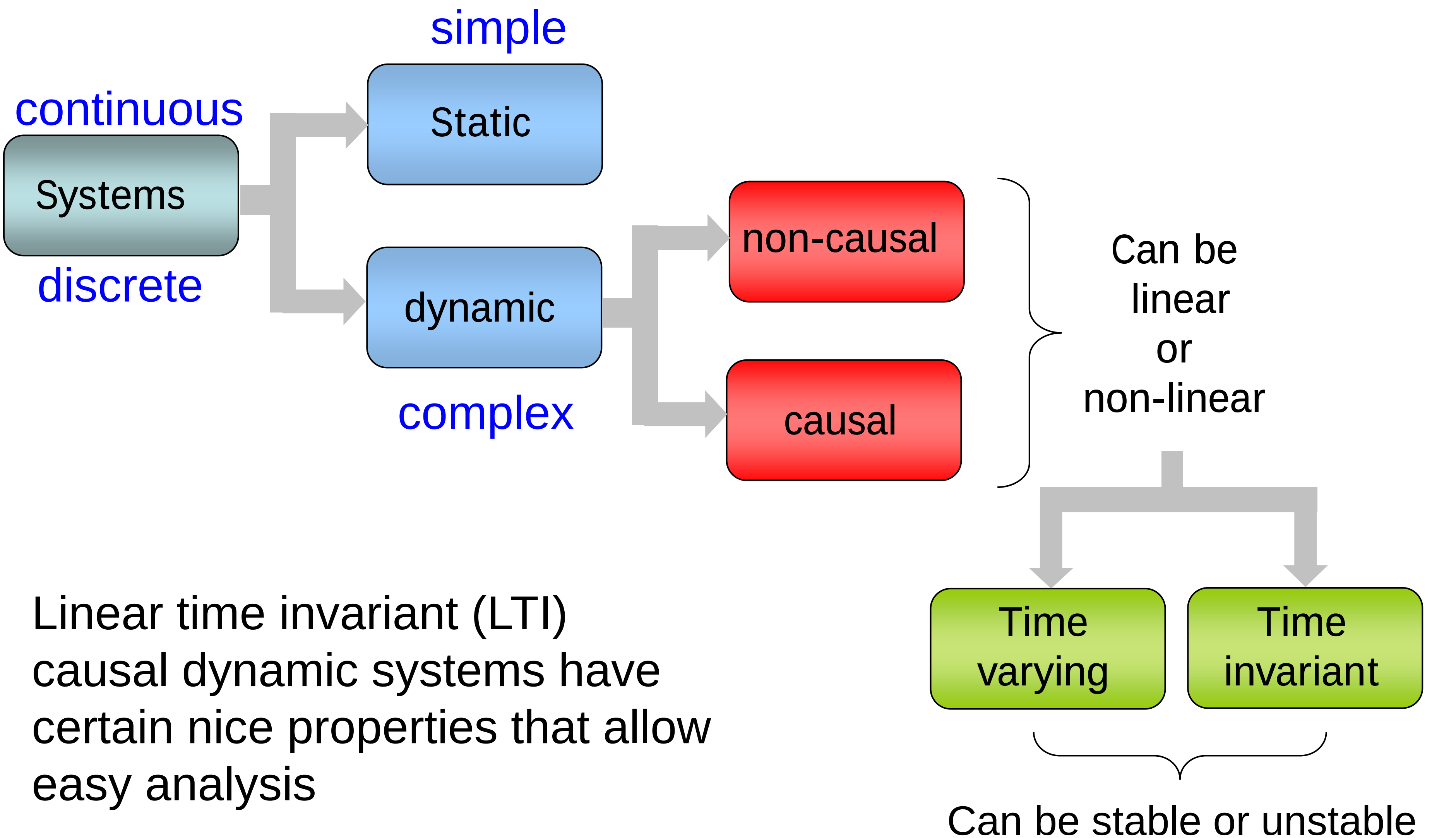
We can write mathematical expressions which approximately describes the behaviour of physical systems. These are called system models.

A system generates a response, or output signal, for a given input. A system is thus a relationship between two signals.



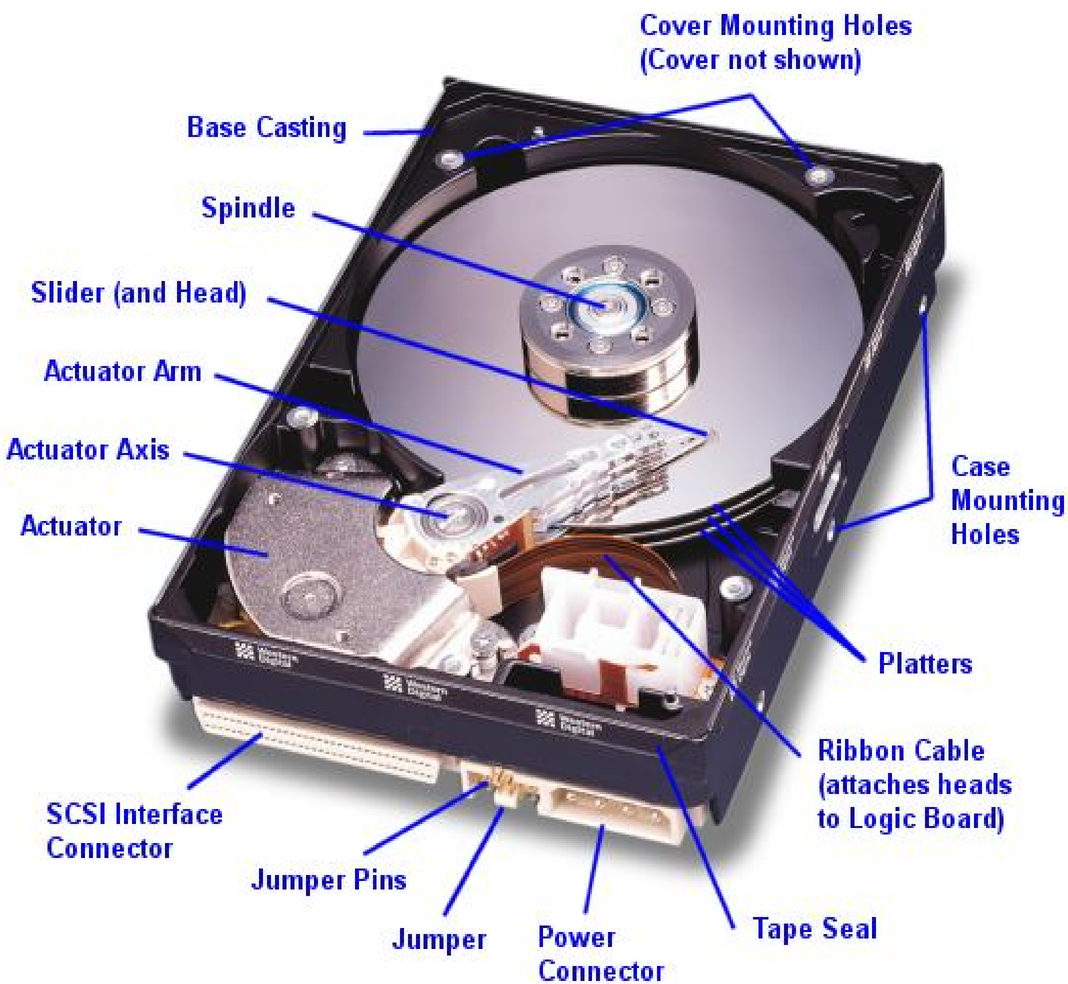
Block Diagram Representation of a Physical System

Classification of Systems



Linear time invariant (LTI) causal dynamic systems have certain nice properties that allow easy analysis

A Hard Disk Drive System



Inside of a typical Hard Disk Drive

Consists of many sub-systems

Some Facts

- Platters coated with magnetic material for data storage
- Platters spun by spindle motor which spins at 5K-10K rpm
- Slider carries the read-write head
- Platters rotate while the read-write head senses the magnetic field to read the data
- The read-write head is mounted on an actuator arm
- The read-write head does not touch the platter surface
- The actuator arm is moved using a voice coil actuator

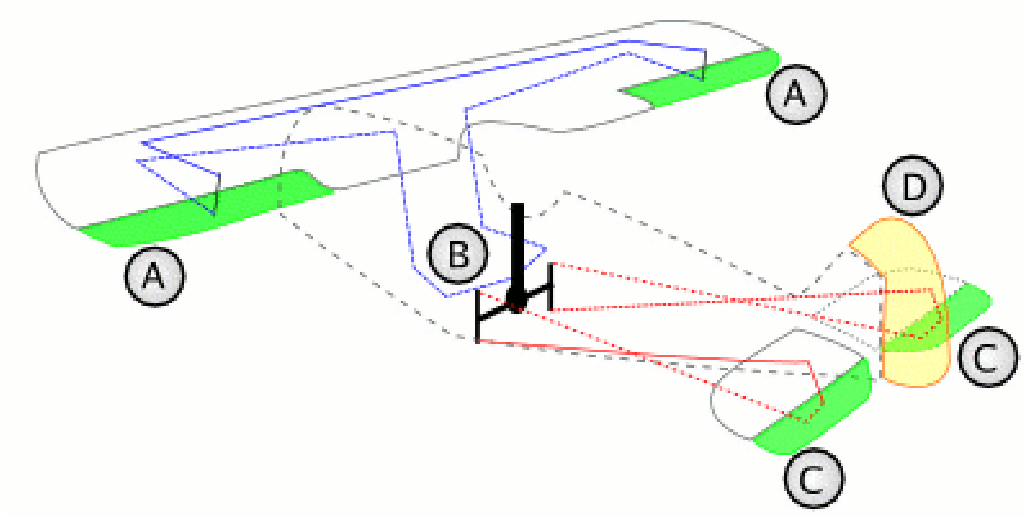
Some specifications :

- SATA drives transfer about 300MByte/s
- Seek time varies between 2-20ms

Control issues :

- Spacing bet head and surface has to be kept to avoid a head crash
- Platter needs to be controlled at the rated spinning speed
- Actuator arm needs to be controlled to move to the right sector on the platter

Sub-Systems in an Aircraft



Sensors and actuators :

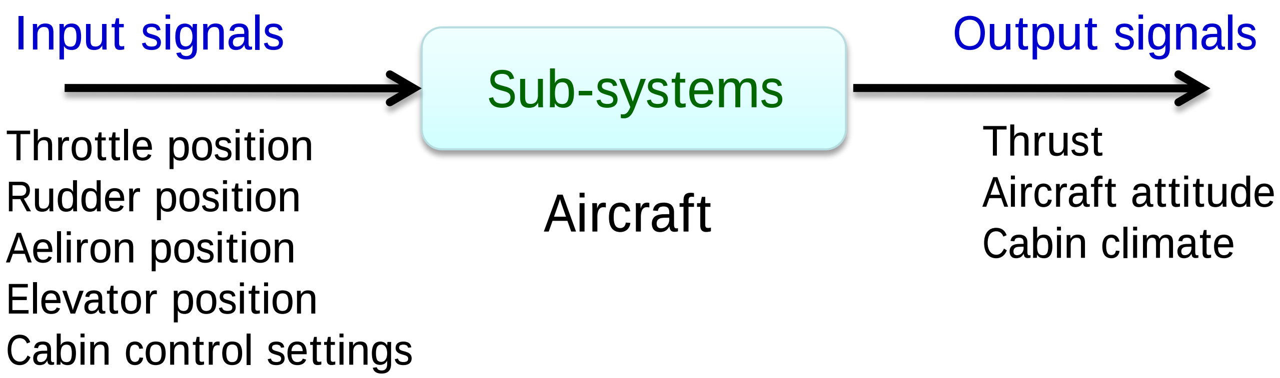
- Hydraulic actuators for the control surfaces
- Many kinds of sensors for monitoring and control eg pressure, temperature, speed, etc

Main controls :

- Ailerons (A) on trailing edge of each wing – controls the roll
- Elevators (C) on the back edge of the horizontal stabilizer on each side of the fin in the tail. They move up and down together – control pitch
- Rudder (D) mounted on the back edge of the fin – controls yaw
- Engine control for the forward thrust
- Cabin climate control

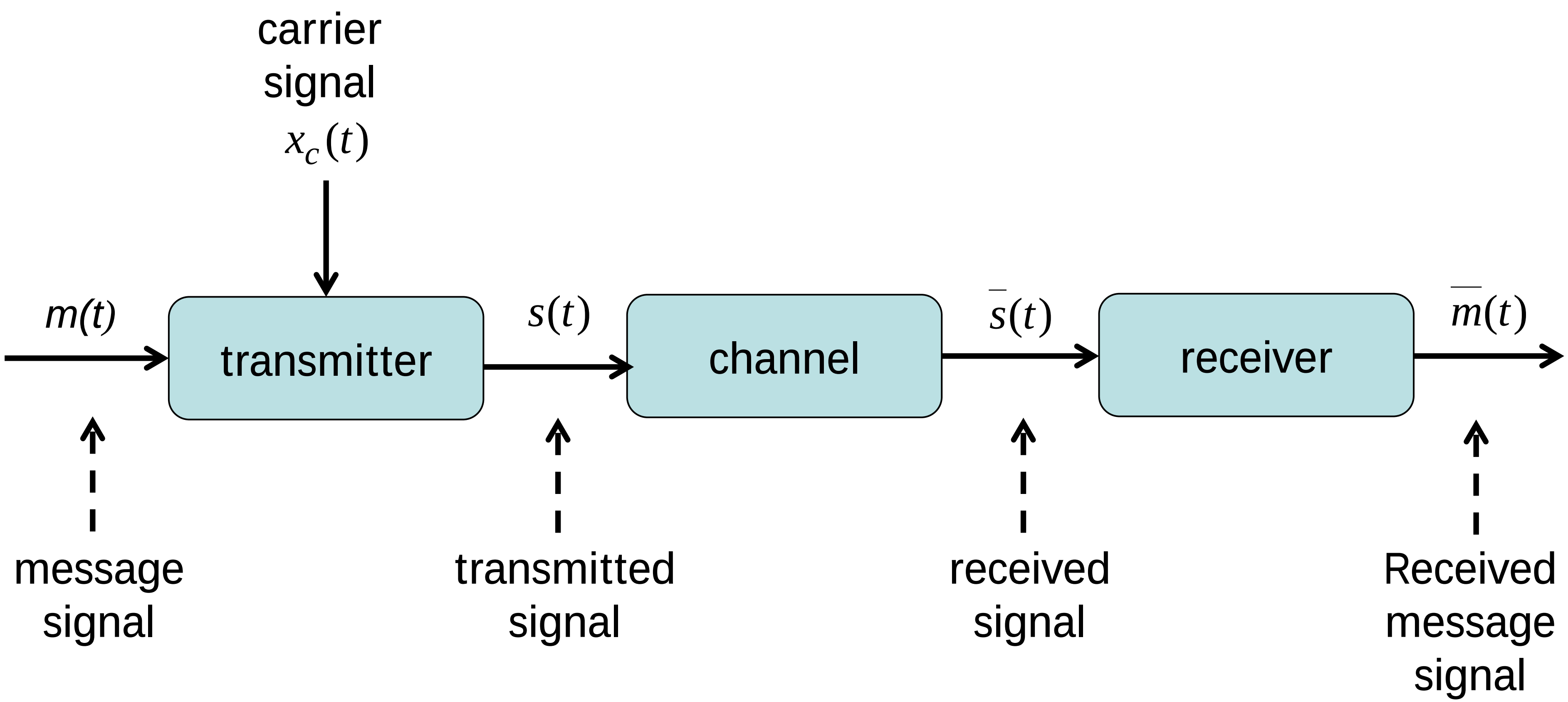
Courtesy of http://en.wikipedia.org/wiki/Aircraft_flight_control_systems

Block Diagram Representation



- What type of signals can we expect to see here?
- How do we represent them?
- Is there a mathematical representation of the signals and systems?

A Typical Communication System



Each signal is modified as it propagates thru each sub-system

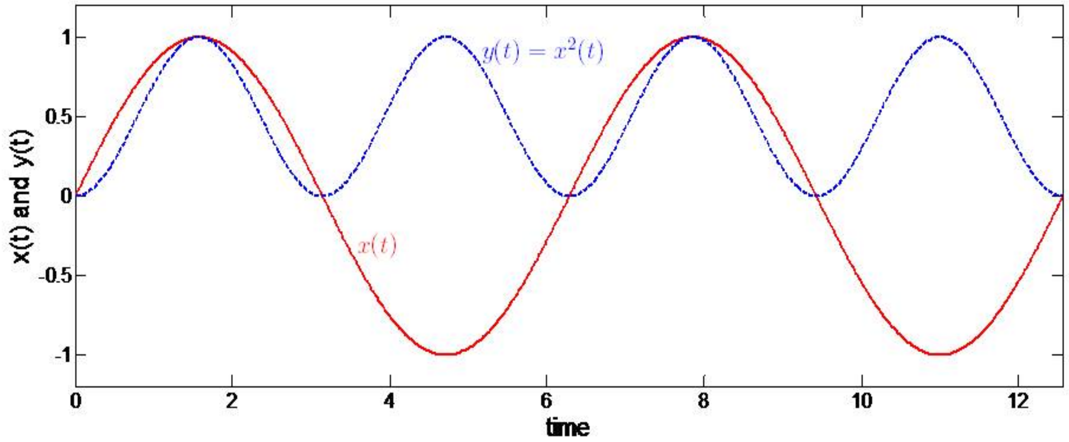
Basic System Properties

- **Static and Dynamic System (Memoryless and Memory Systems)**
  - System is static (also called memoryless) if output is a function of the input at the present time only, i.e., $y(t) = Ku(t)$

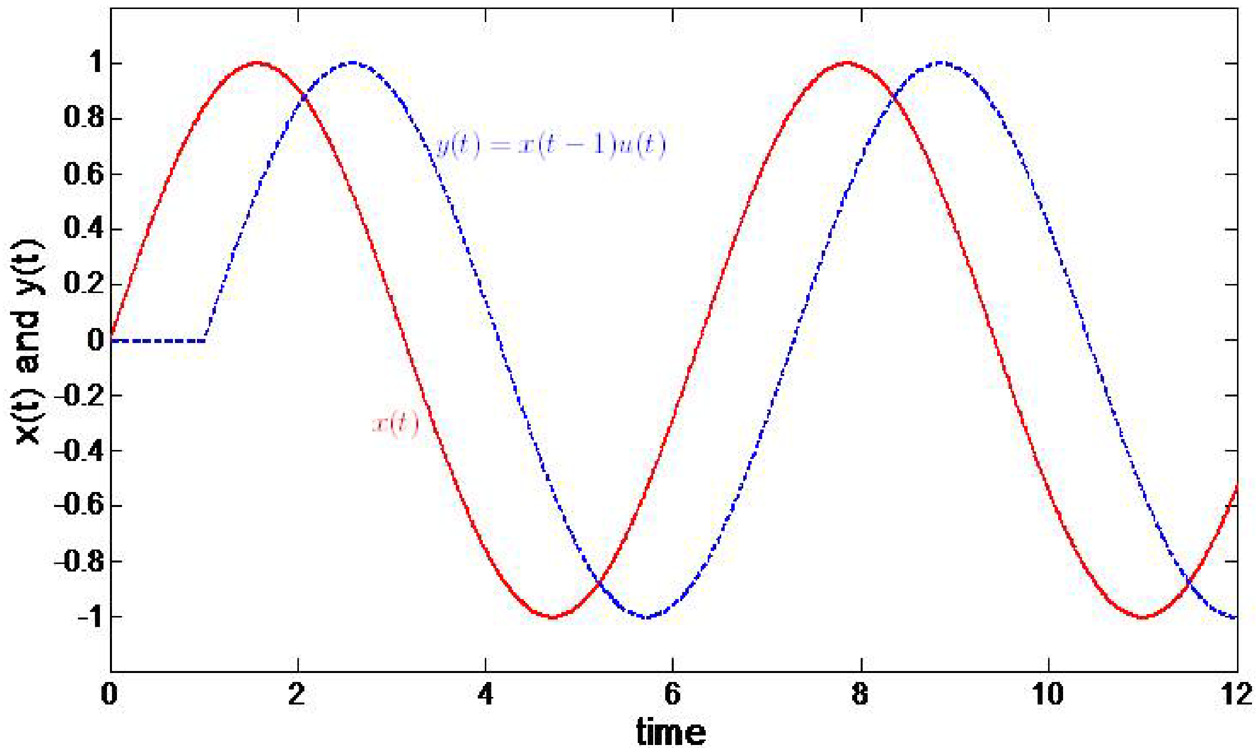
    Example is a resistor : $V(t) = i(t)R$

    The response of a static system is instantaneous and does not depend on previous inputs.

Another example : $y(t) = x^2(t)$ is static.



- system is dynamic if the output of the system at time t_1 depends on the past and/or future and/or present values of the input *u(t).*



time shift property : $y(t) = x(t-1)u(t)$

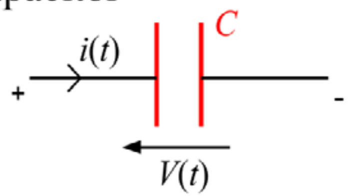
y(t) "remembers" *x(t)*

y(t) depends on *x(t)* 1 time unit before *t*

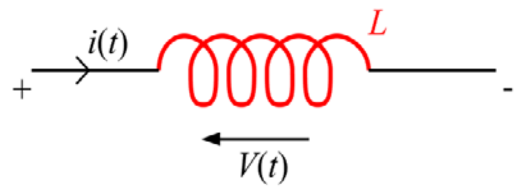
Note that u(t) is the unit step function

- Dynamic systems involving differential equations :

Capacitor



Inductor

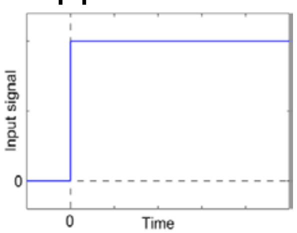


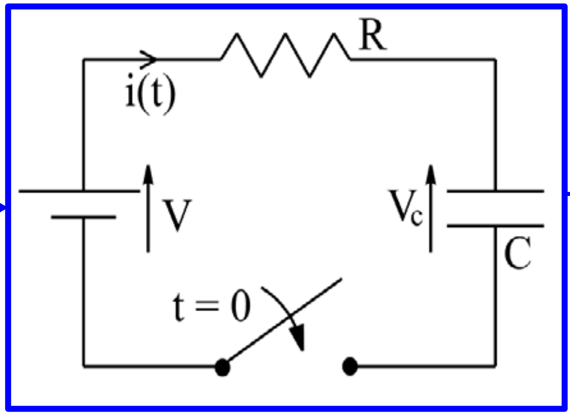
$$V(t) = \frac{1}{C}\int_0^{\infty} i(\tau)\, d\tau$$

$$V(t) = L\frac{di(t)}{dt}$$

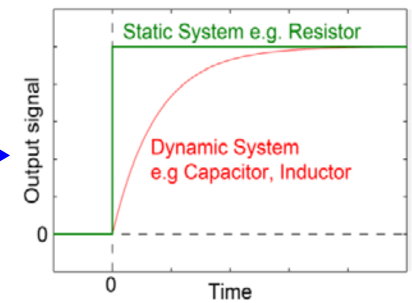
Dynamic RC circuit

Input voltage, V, applied at *t=0*





Output voltage, $V_c(t)$,



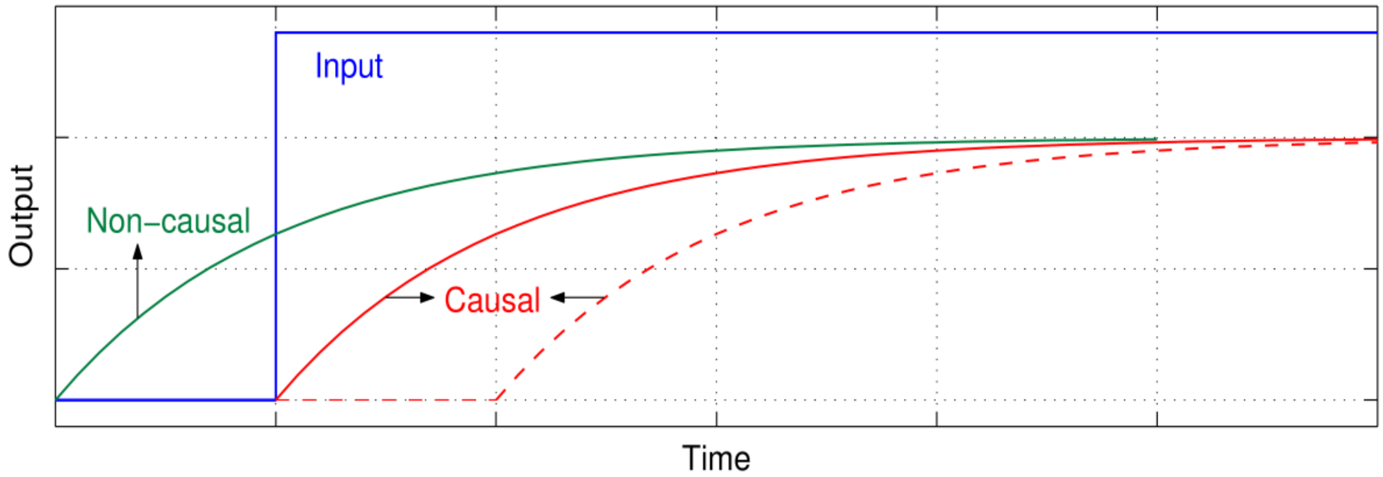
Charging of a capacitor

Equation describing the RC circuit :

$$V(t) = CR\frac{dV_c(t)}{dt} + V_c(t)$$

Causal and Non-Causal System

- System is <u>causal or non-anticipatory</u> if the output signal, $y(t)$, depends only on current and past values of the input, $u(t)$, i.e., $y(t) = f\{u(t), u(t-1), \ldots\}$
- Causality implies that the system does not respond to an input event until that event actually occurs, i.e., the response to an event beginning at $t = t_0$ is non-zero only for $t \geq t_0$



- All static/memoryless systems are causal since the output depends only on the current value of the input. All naturally occurring systems are causal

Linear and Non-linear Systems

- Linear systems satisfy the properties of

Additivity: $y(t) = f\{x_1(t) + x_2(t)\} = f\{x_1(t)\} + f\{x_2(t)\}$

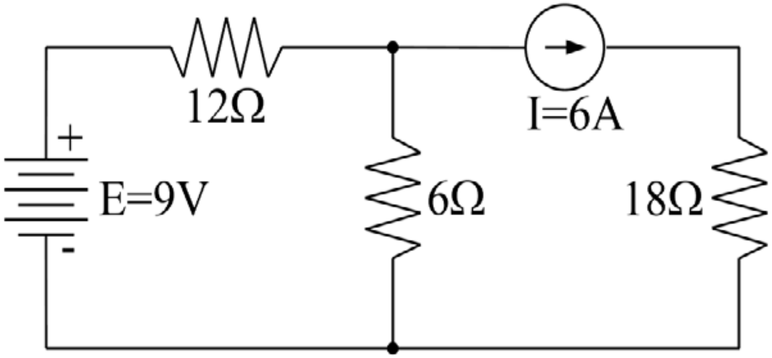
Scaling: If $y(t) = f\{x(t)\}$, then $\alpha\, y(t) = f\{\alpha\, x(t)\}$

Property of additivity leads to the Principle of Superposition.

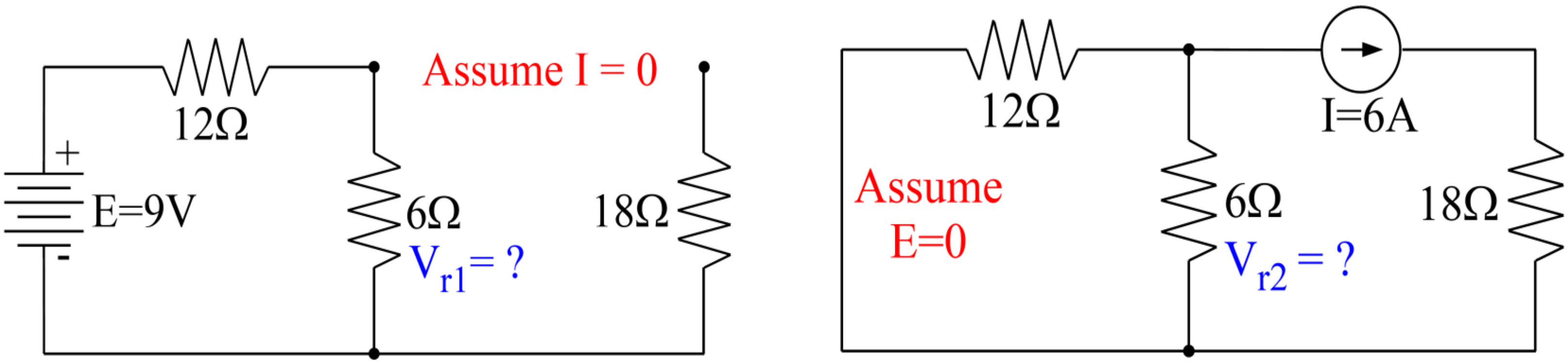
Typical examples are linear circuits encountered in EG1108

Example:

Two independent sources, E and I. What is the voltage across the 6Ω resistor, V_r ?



1. "Kill" current source (let I = 0), find V_{r1}



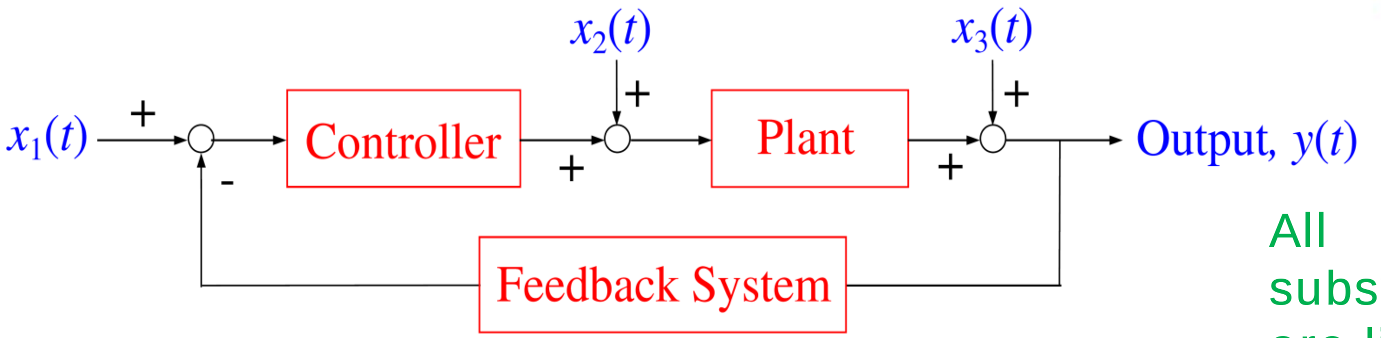
2. "Kill" voltage source (set E = 0), find V_{r2}

3. Sum solutions, i.e., $V_r = V_{r1} + V_{r2}$

Principle of Superposition :

Suppose a linear system is driven by n independent input signals, $x_i(t)$ (i = 1,...,n). System output, $y(t)$, is the <u>algebraic sum</u> of the n outputs $y_i(t)$ (i = 1,...,n) produced by the <u>inputs acting independently</u> ($x_i(t) \neq 0$ & $x_{j\neq i}(t) = 0$)

- Consider an interconnected system



All subsystems are linear

Total output, $y(t) = y_1(t) + y_2(t) + y_3(t)$, where

$y_1(t)$ is output due to $x_1(t)$ when $x_2(t) = x_3(t) = 0$

$y_2(t)$ is output due to $x_2(t)$ when $x_1(t) = x_3(t) = 0$

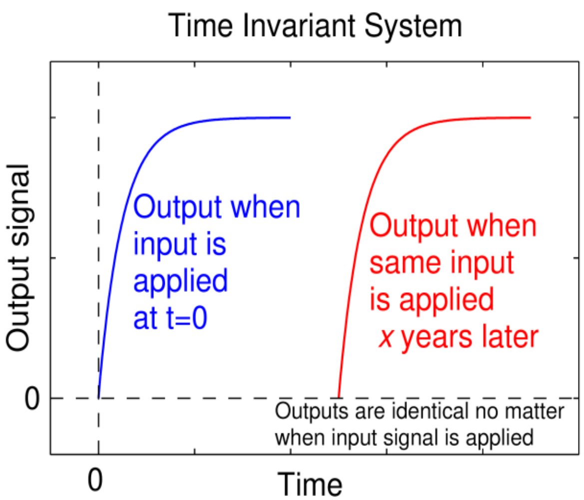
$y_3(t)$ is output due to $x_3(t)$ when $x_1(t) = x_2(t) = 0$

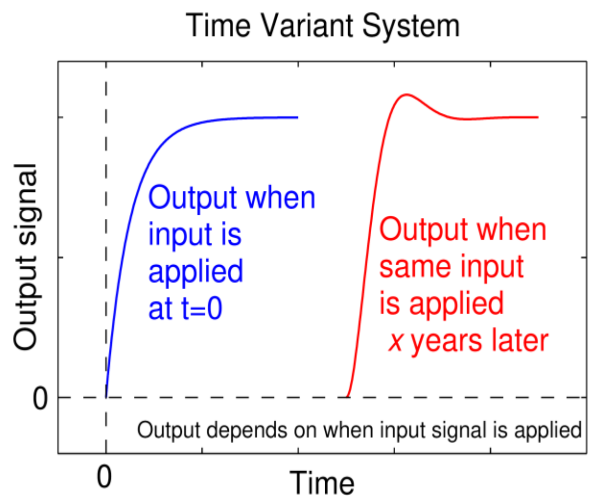
- Similarly if $u(t) = 1 + e^{-2t}$, then output $y(t) = y_1(t) + y_2(t)$, where $y_1(t)$ is output when input is 1; $y_2(t)$ is output when input is e^{-2t} ie it is possible to split a single input up into individual components (1 and e^{-2t}) and the total output determined from each input separately

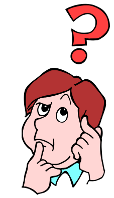
Non-linear systems do not satisfy the principle of superposition

In fact, non-linear systems cannot be generalized very much. Makes the study of non-linear systems rather difficult

- **Time Invariant and Time Varying System**
  - System is time invariant if its response is the same when the same input is applied at any time i.e., output remains the same regardless of when input is applied







Are physical systems time invariant?

- Time invariant processes are described by differential equation with constant coefficients ie not dependent on time

  Examples are circuits involving resistors, capacitors, inductors and op amps.

- Time varying systems are described by differential equations with coefficients which are dependent on time e.g.

$$\frac{dy}{dt} + k(t)y(t) = c_0 u(t)$$

k(t) changes with *t*, c_0 *constant*

Example of a time varying system : A car with a tank full of fuel. As it moves, the amount of fuel reduces and hence the weight of the car also reduces. This changes the characteristics of the car eg the car starts to respond faster due to a lighter load that it is carrying.

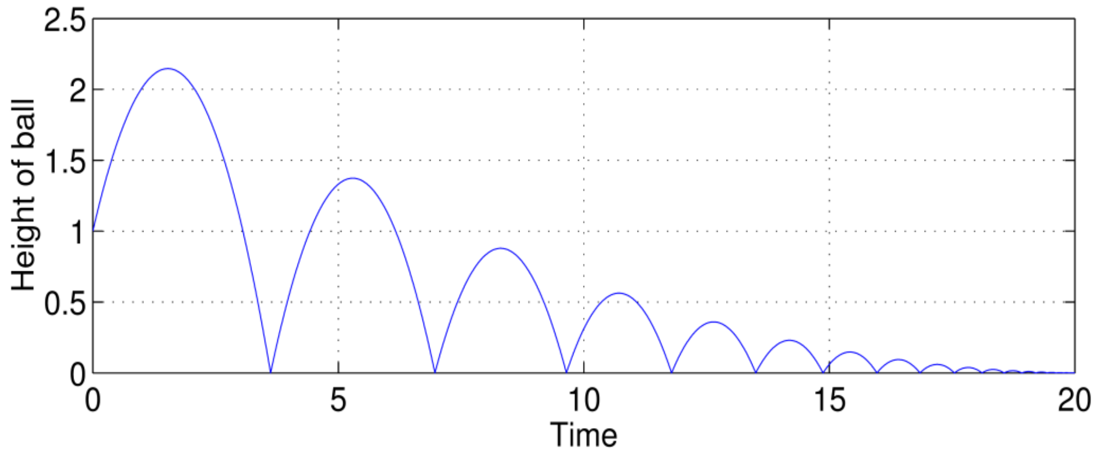
Stable and Unstable System

Signal, $x(t)$, is bounded, if $|x(t)| \leq M \; \forall t$ (M is a finite value)

System is bounded-input-bounded-output stable if a bounded input signal results in a bounded output signal.

- Example of a stable system: Basketball thrown upwards

  Applied force (input) is finite and height of ball (output) is bounded.



- Example of an unstable system

Mic placed near a speaker: Background noise (input) is finite but speaker volume (output) is unbounded.

Question : What kind of a system are humans?

Differential Equation Models of LTI Systems

- Differential equations are time domain equations which define the manner in which a dynamic system transforms an input signal into an output signal. Examples :
  - Electrical: $V_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau \qquad V_L(t) = L\frac{di(t)}{dt}$
  - Linear Motion: $F = m\frac{d^2x}{dt^2}$ where F = applied force and $x(t)$ = position (Newton's 2nd law). Is this time invariant?
  - Angular Motion (e.g., motors): $T = J\frac{d^2\theta}{dt^2}$ where T = applied torque, J = Inertia and $\theta(t)$ = angular position
  - Thermal: $C\frac{dT}{dt} = q_i - q_o$ where C is thermal capacity, q_i is heat supplied and q_o is heat loss. Is this a time invariant system?

- General Form of the LTI model

Input-Output relationships of LTI systems are defined by linear D.E.:

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \ldots + b_0 u(t)$$

- where $y(t)$ and $u(t)$ are the output and input of the system respectively. Coefficients $a_i, i = 1, \ldots n$ and $b_i,\ i = 1, \ldots, m$ are constants.

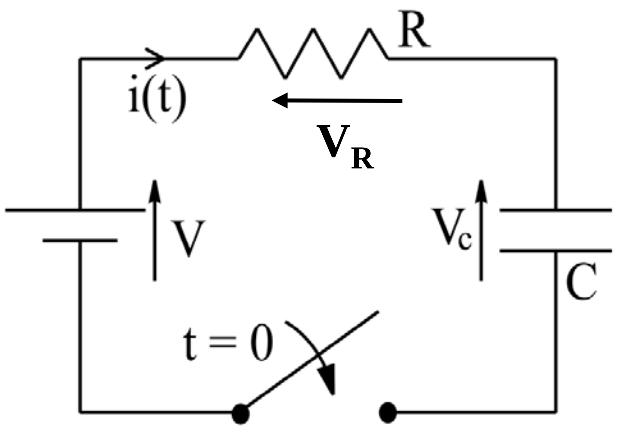
- For real physical systems, $m<n$.

- n is defined to be the system order.

- The order n also corresponds to the number of energy storage elements in the system. For example, in an RC circuit, the capacitor is the only energy storage element present and thus the DE which describes the behaviour of the RC circuit is of order 1 ie it is a first order DE.

Models of Electrical Circuits in Time Domain

Example 1 : RC Circuit



- Assume zero initial conditions ie $V_c(0)=0$
- From Kirchoff's Voltage law: $V_R(t) = V(t) - V_c(t)$

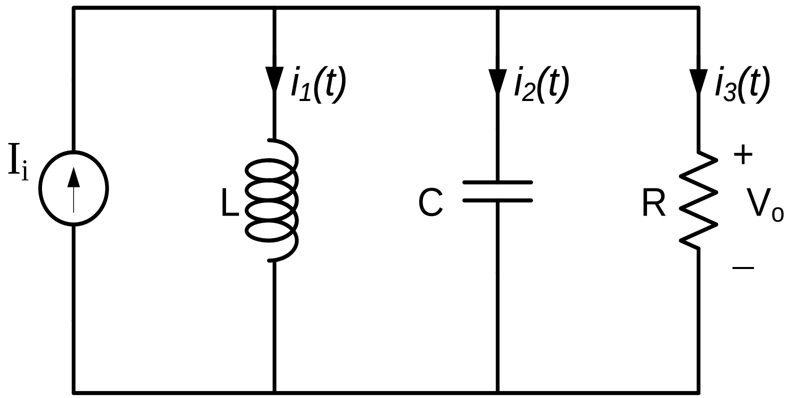
Hence,

$$i(t) = \frac{V_R(t)}{R} = \frac{V(t)-V_c(t)}{R} \quad \text{and} \quad V_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau = \frac{1}{RC}\int_0^t V(\tau)-V_c(\tau)d\tau$$

Input-output ($V(t)$-$V_c(t)$) relationship of series RC is defined by the first-order differential equation:

$$RC\frac{dV_c(t)}{dt} + V_c(t) = V(t)$$

Model of a RLC Circuit (2nd order DE)



$$i_1 = \frac{1}{L}\int V_0 dt$$

$$i_2 = C\frac{dV_0}{dt}$$

$$i_3 = \frac{V_0}{R}$$

$$I_i(t) = i_1 + i_2 + i_3$$

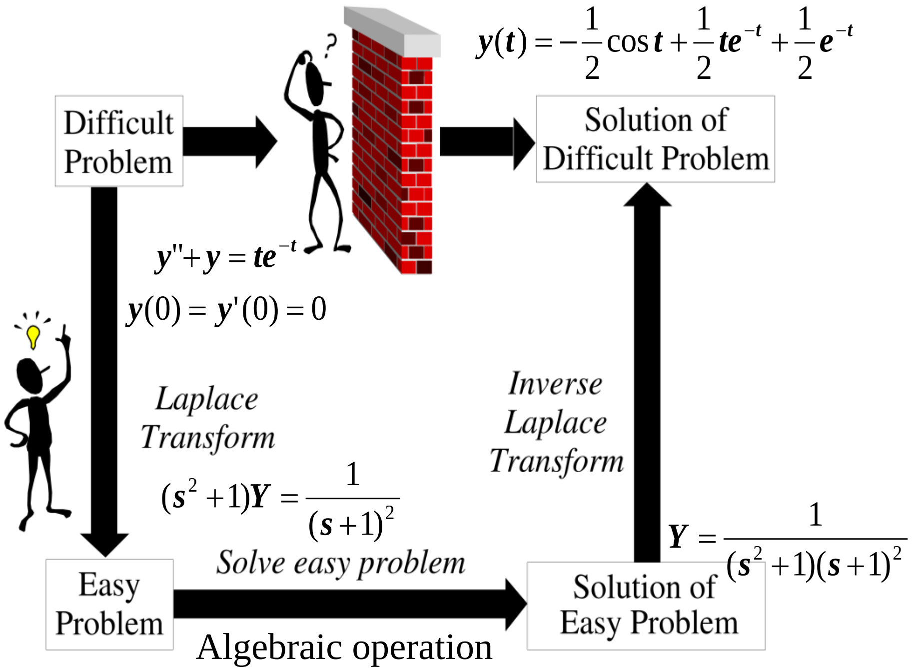
$$= \frac{1}{L}\int V_o(t)dt + C\frac{dV_0(t)}{dt} + \frac{V_0(t)}{R}$$

$$RLI_i(t) = R\int V_o(t)dt + RLC\frac{dV_0(t)}{dt} + LV_0(t)$$

$$RL\frac{dI_i(t)}{dt} = RV_0(t) + RLC\frac{d^2V_0(t)}{dt^2} + L\frac{dV_0(t)}{dt}$$

$\Rightarrow$ 2nd order DE involving input, I_i, and output, V_0

- An alternative model for LTI system is in the frequency domain via transfer functions (see later)
- However transfer functions are only available via the Laplace Transforms
- Laplace Transforms are also useful in solving differential equations
- LT is the tool to convert time domain to frequency domain, thus allowing complex linear DE to be represented in the frequency domain
- Hence a revision of LT is necessary



What is Laplace Transform?

- Mathematical tool used for solving linear ordinary differential equation
- Converts a differential equation into an algebraic equation.
- Laplace Transform enables us to analyze system behavior without having to solve difficult differential equations
  - Transform from time-domain to complex frequency domain (*s*-domain)
  - Express differential equations as algebraic equations

Laplace Transform: Definition

- The unilateral Laplace transform is defined as

$$\mathcal{L}\{f(t)\} = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt$$

s is a Laplace variable and is a complex quantity.

Lower limit of integration is 0^-, instead of $-\infty$, because the system is assumed to be causal. Consequently, all input and output signals are 0 for $t < 0$, except at $t=0^-$, when there may be non-zero initial conditions for $f(t)$.

- By convention, we shall write

Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$

Inverse Laplace transform: $\mathcal{L}^{-1}\{F(s)\} = f(t) = \dfrac{1}{2\pi j}\displaystyle\int_{-\infty}^{\infty} F(s)e^{st}ds$

Useful Laplace Transform pairs

$f(t)$	$\Leftrightarrow$	$F(s)$
$\delta(t)$	$\Leftrightarrow$	1
$U(t)$	$\Leftrightarrow$	$\frac{1}{s}$
$tU(t)$	$\Leftrightarrow$	$\frac{1}{s^2}$
$e^{-at}U(t)$	$\Leftrightarrow$	$\frac{1}{s+a}$
$[\sin at]\, U(t)$	$\Leftrightarrow$	$\frac{a}{s^2+a^2}$
$[\cos at]\, U(t)$	$\Leftrightarrow$	$\frac{s}{s^2+a^2}$
$[e^{-at}\sin bt]\, U(t)$	$\Leftrightarrow$	$\frac{b}{(s+a)^2+b^2}$
$[e^{-at}\cos bt]\, U(t)$	$\Leftrightarrow$	$\frac{s+a}{(s+a)^2+b^2}$
$te^{-at}U(t)$	$\Leftrightarrow$	$\frac{1}{(s+a)^2}$

Laplace Transform Rules

- **Linearity**

$$\mathcal{L}\{\alpha f(t)+\beta g(t)\}=\alpha\mathcal{L}\{f(t)\}+\beta\mathcal{L}\{g(t)\}$$

where α , β = constants

- **Transform of Derivatives**

$$\mathcal{L}\{f'(t)\}=sF(s)-f(0^-)$$

$$\mathcal{L}\{f^n(t)\}=s^nF(s)-\sum_{k=0}^{n-1}s^{n-1-k}f^k(0^-)$$

- **Transform of an Integral**

$$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\}=\frac{F(s)}{s}$$

- The above theorems (page 6-25) are important because they transform differential equations into algebraic equations,

Differentiation ⇨ Multiplication
Integration ⇨ Division

- Example: I-V relationship for a capacitor is $i(t) = C\dfrac{dv_c(t)}{dt}$

  Applying Transform of Derivative rule,

$$I(s) = \boxed{sCV_c(s)} - Cv_c(0^-) \Longrightarrow$$

Derivative operator has become multiplication with *s*!

  where $v_c(0^-)$ is the initial charge on the capacitor.

- Solving a circuit equation involving the *s*-variable becomes algebraic manipulation ie

$$V_c(s) = \frac{I(s) + Cv_c(0^-)}{sC}$$

More Laplace Transform rules ...

- **Derivative of Transforms**

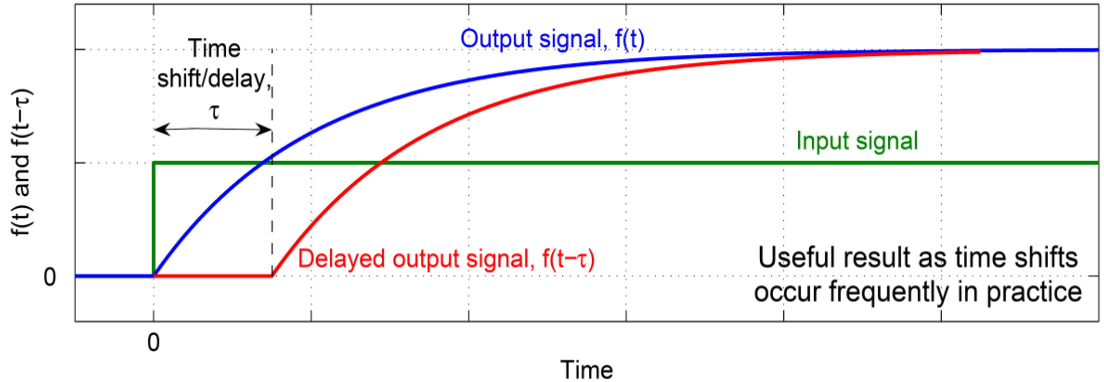
$$\frac{dF(s)}{ds} = \mathcal{L}\{-t\,f(t)\} \quad \text{and} \quad \frac{d^n}{ds^n}F(s) = (-1)^n \mathcal{L}\{t^n f(t)\}$$

Example: Since $\mathcal{L}\{\sin\omega t\, u(t)\} = \dfrac{\omega}{s^2+\omega^2}$,

$$\mathcal{L}\{t\sin\omega t\} = -\frac{d}{ds}\left\{\frac{\omega}{s^2+\omega^2}\right\} = \frac{2\omega s}{(s^2+\omega^2)^2}$$

- **Shift in the time-domain function**

$$\mathcal{L}\{\,f(t-t_0)\} = e^{-st_0}F(s)$$



- **Time Scaling**

$$\mathcal{L}\{f(\beta t)\} = \frac{1}{|\beta|} F\left(\frac{s}{\beta}\right)$$

Example: Since $\mathcal{L}\left\{\cos t\, u(t)\right\} = \dfrac{s}{s^2+1}$,

$$\mathcal{L}\left\{\cos 8t\, u(t)\right\} = \frac{1}{8}\frac{s/8}{s^2/8^2+1} = \frac{s}{s^2+8^2}$$

- **Shift in the s-domain**

$$\mathcal{L}\left\{e^{s_0 t} f(t)\right\} = F(s-s_0)$$

- Example

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+4}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+3}\right\}$$

$$=\frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{(s+1)^2+\left(\sqrt{3}\right)^2}\right\}$$

$$=\frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left\{\frac{\sqrt{3}}{s_1{}^2+\left(\sqrt{3}\right)^2}\right\}; \quad s_1=s+1$$

As $\mathcal{L}\left\{\sin\omega t\, u(t)\right\}=\dfrac{\omega}{s^2+\omega^2}$ and $\mathcal{L}\left\{e^{s_0 t} f(t)\right\}=F(s-s_0)$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+2s+4}\right\}=\frac{1}{\sqrt{3}}e^{-t}\sin\sqrt{3}t\, u(t)$$

Final Value Theorem (FVT)

- Useful as it provides information about steady-state system behavior without having to solve D.E. or perform L^{-1}. FVT is given as,

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)$$

Steady state behaviour means behaviour after a long time *t*

- FVT is applicable only if the signal has a finite steady state value, i.e.,

$$\lim_{t\to\infty} f(t) = \text{constant}$$

- Example: Let $f(t)$ = $\sin\omega t\, u(t)$. Then, $F(s) = \mathcal{L}\{\sin\omega t\, u(t)\} = \dfrac{\omega}{s^2+\omega^2}$

$$\lim_{s\to 0} sF(s) = 0 \neq \lim_{t\to\infty} f(t)$$

- If the signal is the steady-state output of a dynamic system, then FVT will provide a valid solution only if (1) The input signal is a constant (step function); (2) The system is stable

Inverse Laplace Transform

- Inverse Laplace Transform is used to transform the *s*-domain solution back to the original time domain
- Simplify complicated functions using partial fractions expansion. There are 3 basic cases, namely

Only distinct linear factors: $$F(s) = \frac{N(s)}{(s+\alpha_1)\ldots(s+\alpha_n)} = \frac{A_1}{s+\alpha_1} + \ldots + \frac{A_n}{s+\alpha_n}$$

The inverse LT : $$f(t) = A_1 e^{-\alpha_1 t} + A_2 e^{-\alpha_2 t} + \ldots + A_n e^{-\alpha_n t}$$

- Repeated linear factors: $F(s)=\dfrac{N(s)}{(s+\alpha)^n}=\dfrac{A_1}{s+\alpha}+\ldots+\dfrac{A_n}{(s+\alpha)^n}$

  Inverse LT : $f(t)=\left(A_1+A_2t+\dfrac{1}{2}A_3t^2+\ldots+\dfrac{1}{n!}A_nt^{n-1}\right)e^{-\alpha t}$

- A quadratic factor: $\dfrac{N(s)}{(s^2+2\beta_1s+\gamma_1{}^2)(s^2+2\beta_2s+\gamma_2{}^2)}=\dfrac{A_1s+B_1}{(s^2+2\beta_1s+\gamma_1{}^2)}+\dfrac{A_2s+B_2}{(s^2+2\beta_2s+\gamma_2{}^2)}$

  Inverse LT : $f(t)=R_1e^{-\beta_1t}\sin\left(\sqrt{\gamma_1^2-\beta_1^2}\;t\right)+R_2e^{-\beta_2t}\sin\left(\sqrt{\gamma_2^2-\beta_2^2}\;t\right)$

Finding inverse LT using partial factors

Example 1 : Find the inverse LT of $F(s)=\dfrac{2}{(s+1)(s+2)}$

$$F(s)=\frac{2}{(s+1)(s+2)}=\frac{A_1}{s+1}+\frac{A_2}{s+2}$$

Multiplying both sides by (s+1)

$$\frac{2}{s+2}=A_1+\frac{(s+1)A_2}{s+2}$$

Let $s=-1$: $A_1=\left.\dfrac{2}{s+2}\right|_{s=-1}=2$

Similarly for A_2 : $A_2=\left.\dfrac{2}{s+1}\right|_{s=-2}=-2$

$$F(s) = \frac{2}{s+1} - \frac{2}{s+2} \qquad \mathcal{L}^{-1}\{F(s)\} = f(t) = 2e^{-t} - 2e^{-2t}$$

In general, if $F(s) = \dfrac{N(s)}{(s+\alpha_1)(s+\alpha_2)\ldots(s+\alpha_n)}$

$$= \frac{A_1}{s+\alpha_1} + \frac{A_2}{s+\alpha_2} + \ldots + \frac{A_n}{s+\alpha_n}$$

The values of A_k can be obtained as follows :

$$A_k = \left. \frac{\cancel{(s+\alpha_k)}N(s)}{(s+\alpha_1)(s+\alpha_2)\ldots\cancel{(s+\alpha_k)}\ldots(s+\alpha_n)} \right|_{s=-\alpha_k}$$

This formula applies only to <u>distinct real factors</u> of the denominator of *F(s)*.

Example 2 : Find the inverse LT of $F(s)=\dfrac{2}{(s+1)(s+2)^2}$

According to slide 6-33 and 6-34,

$$F(s)=\frac{2}{(s+1)(s+2)^2}=\frac{A_1}{s+1}+\frac{A_2}{(s+2)^2}+\frac{A_3}{s+2}$$

A_1 can be found as in the previous example.

A_2 can be found by multiplying both sides by $(s+2)^2$ and let $s=-2$

$$\frac{2}{s+1}=\frac{(s+2)^2 A_1}{s+1}+A_2+(s+2)A_3 \quad \Rightarrow \quad A_2=-2$$

To find A_3, differentiate the above eqn wrt to s :

$$\frac{-2}{(s+1)^2}=\frac{(s+2)sA_1}{(s+1)^2}+A_3 \quad \Rightarrow \quad A_3=\left.\frac{-2}{(s+1)^2}\right|_{s=-2}=-2$$

$$F(s) = \frac{2}{s+1} - \frac{2}{(s+2)^2} - \frac{2}{s+2}$$

$$\mathcal{L}^{-1}\left\{ F(s) \right\} = f(t) = 2e^{-t} - 2te^{-2t} - 2e^{-2t}$$

Practice problem : Find the inverse LT of

$$F(s) = \frac{10}{s(s+2)(s+3)^2}$$

$$F(s) = \frac{1}{s^2 + s + 1}$$