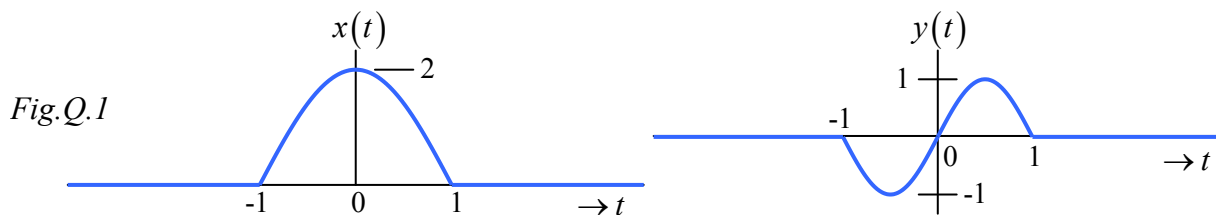


EE2023 TUTORIAL 3 (PROBLEMS)

Q.1 A half-cosine pulse $x(t)$ and a sine pulse $y(t)$ are shown in Fig.Q.1.



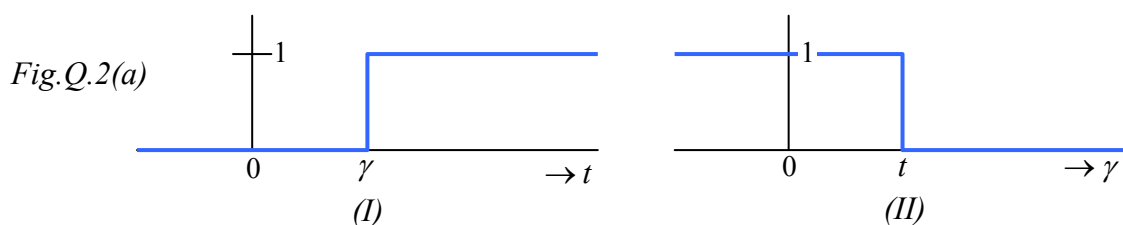
- (a) Derive the spectrum of $x(t)$ using the forward Fourier transform equation and show how the derivation can be simplified by applying relevant Fourier transform properties.

$$\text{Ans: } X(f) = \frac{2 \cos(2\pi f)}{\pi(0.25 - 4f^2)}, \text{ Hint: } x(t) = 2 \cos(0.5\pi t) \cdot \text{rect}(0.5t)$$

- (b) Using the results of Part-(a), determine the spectrum of $y(t)$.

$$\text{Ans: } Y(f) = \frac{1}{j2} \left[\frac{\sin(2\pi f)}{\pi(0.25 - f^2)} \right]$$

Q.2 (a) Show that Fig.Q.2(a)(I) and Fig.Q.2(a)(II) are plots of the same function $u(t - \gamma)$, where $u(\cdot)$ denotes the unit step function. Hence, express $\int_{-\infty}^t x(\gamma) d\gamma$ as a convolution integral.



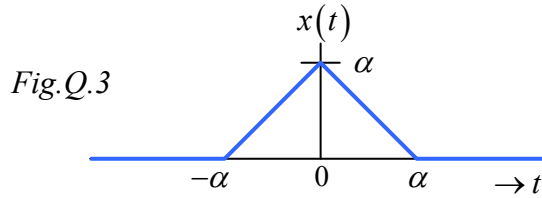
- (b) Evaluate $\cos(t)u(t) * u(t)$ where $*$ denotes convolution.

$$\text{Ans: } \sin(t)u(t)$$

- (c) Find the Fourier transform of $\alpha \cdot \text{sinc}(\alpha t)$ and show that $\lim_{\alpha \rightarrow \infty} \alpha \cdot \text{sinc}(\alpha t) = \delta(t)$.

$$\text{Ans: } \text{rect}\left(\frac{f}{\alpha}\right), \text{ Hint: } \lim_{\alpha \rightarrow \infty} \text{rect}\left(\frac{f}{\alpha}\right) = 1$$

Q.3 Fig.Q.3 shows the plot of a triangular pulse $x(t)$.



Determine the magnitude and phase spectra of $x(t)$. Hence, or otherwise, find the energy spectral density and total energy of $\frac{dx(t)}{dt}$.

$$\text{Ans: } \begin{cases} \text{Magnitude spectrum: } |X(f)| = \alpha^2 \text{sinc}^2(\alpha f), & \text{Phase spectrum: } \angle X(f) = 0 \\ E_y(f) = 4\pi^2 f^2 \alpha^4 \text{sinc}^4(\alpha f), & E = 2\alpha \end{cases}$$

Q.4 The spectrum of a lowpass energy signal $x(t)$ is given by $X(f) = \exp(-\alpha|f|)$ where α is a positive constant.

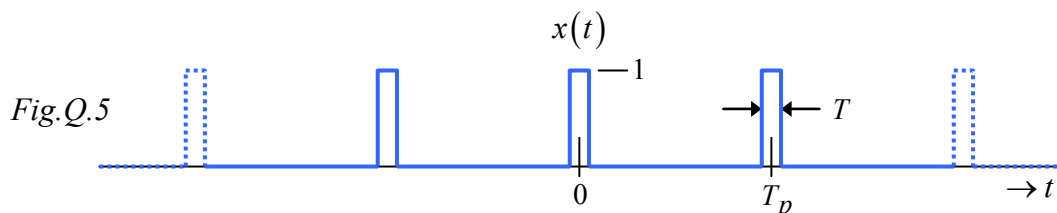
- (a) The 99% energy containment bandwidth of a signal is defined as the smallest bandwidth that contains at least 99% of the total signal energy. Find the 99% energy containment bandwidth of $x(t)$?

$$\text{Ans: } \frac{1}{\alpha} \ln(10) \text{ Hz}$$

- (b) Find the 3dB bandwidth of $x(t)$. How many percent of the total energy of $x(t)$ does its 3dB bandwidth contain?

$$\text{Ans: } \frac{1}{2\alpha} \ln(2) \text{ Hz, } 50\%$$

Q.5 A military lookout tower uses a laser pointer as a make-shift signaling device to communicate with a base camp. The laser pointer's built-in ON-OFF pushbutton switch is replaced by an electronic switch which is activated by a signal $x(t)$. The output of the laser pointer has the form $y(t) = x(t) \cdot \mu \cos(2\pi f_c t)$ where μ and f_c are the amplitude and frequency of the laser beam when $x(t)$ has a value of 1. Unless there is an incident, the laser pointer continuously sends short pulses of light, spaced at regular interval, back to the base camp to indicate a 'No Incident' situation. The $x(t)$ used for signaling 'No Incident' is shown in Fig.Q.5.



- (a) Derive the continuous-frequency spectrum of $x(t)$. Sketch and label the spectrum.

$$\text{Ans: } X(f) = \sum_{k=-\infty}^{\infty} \frac{T}{T_p} \text{sinc}\left(k \frac{T}{T_p}\right) \delta\left(f - \frac{k}{T_p}\right)$$

- (b) The 99% power containment bandwidth of a signal is defined as the smallest bandwidth that contains at least 99% of the average signal power. Provide a formula for computing the 99% power containment bandwidth of $x(t)$?

- (c) Assume $f_c = \frac{k}{T_p} \gg \frac{1}{T}$ where k is an integer. What is the average power of the laser output $y(t)$ and how it can be controlled?

$$\text{Ans: Average Power of } y(t) = 0.5 \mu^2 \frac{T}{T_p}$$

Supplementary Problems

These problems will not be discussed in class.

S.1 Find the Fourier transform of each of the following signals:

(a) $x(t) = \cos(2\pi f_c t)u(t)$

(b) $x(t) = \sin(2\pi f_c t)u(t)$

(c) $x(t) = \exp(-\alpha t)\cos(\omega_c t)u(t); \alpha > 0$

(d) $x(t) = \exp(-\alpha t)\sin(\omega_c t)u(t); \alpha > 0$

Answer: (a) $X(f) = \frac{1}{4}[\delta(f - f_c) + \delta(f + f_c)] + \frac{jf}{2\pi(f_c^2 - f^2)}$

(b) $X(f) = \frac{j}{4}[\delta(f + f_c) - \delta(f - f_c)] + \frac{f_c}{2\pi(f_c^2 - f^2)}$

(c) $X(\omega) = \frac{\alpha + j\omega}{(\alpha + j\omega)^2 + \omega_c^2}$

(d) $X(\omega) = \frac{\omega_c}{(\alpha + j\omega)^2 + \omega_c^2}$

S.2 Given: $\mathfrak{F}\{\exp(-\alpha t)u(t)\} = \frac{1}{\alpha + j2\pi f}$. Find the inverse Fourier transform of $\frac{1}{(\alpha + j2\pi f)^n}$.

Answer: $\frac{t^{n-1}}{(n-1)!}\exp(-\alpha t)u(t)$

S.3 Given: $\mathfrak{F}\{\exp(-\alpha t)u(t)\} = \frac{1}{j\omega + \alpha}$. Find the inverse Fourier transform of $\frac{1}{2 - \omega^2 + j3\omega}$.

Answer: $[\exp(-t) - \exp(-2t)]u(t)$

S.4 Given: $\mathfrak{F}\{x(t)\} = \text{rect}(\pi f)$. Find the value of $\int_{-\infty}^{\infty} |y(t)|^2 dt$ if $y(t) = \frac{dx(t)}{dt}$.

Answer: $\frac{1}{3\pi}$

S.5 Given: $\mathfrak{F}\left\{\frac{\pi}{\alpha}\exp(-2\pi\alpha|t|)\right\} = \frac{1}{\alpha^2 + f^2}$. Determine the 99% energy containment bandwidth for the

signal $x(t) = \frac{1}{\alpha^2 + t^2}$.

Answer: $\frac{0.366}{\alpha}$

S.6 Let $\omega = 2\pi f$. Using the fact that the Fourier transform is a one-to-one linear transformation, show that $\delta(f) = 2\pi\delta(\omega)$.

Hint: Show that $\mathfrak{F}^{-1}\{\delta(f)\} = \mathfrak{F}^{-1}\{2\pi\delta(\omega)\} = 1$

Below is a list of solved problems selected from Chapter 5 of Hwei Hsu (PhD), 'The Schaum's series on Signals & Systems,' 2nd Edition.

Selected solved-problems: 5.19-to-5.27, 5.32, 5.34, 5.40, 5.42, 5.42, 5.57

These solved problems should be treated as supplementary module material catered for students who find the need for more examples or practice-problems.
