

EE2023 TUTORIAL 4 (SOLUTIONS)**Solution to Q.1**

$$\tilde{x}(t) = \underbrace{\text{rect}\left(\frac{t-0.475}{0.45}\right)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-0.2n)}_{\Xi_{0.2}(t)}$$

$$= \delta(t-0.4) + \delta(t-0.6)$$

$$\mathfrak{T}\{\delta(t-\varsigma)\} = \exp(-j2\pi f\varsigma)$$

$$\tilde{X}(f) = \exp(-j2\pi f0.4) + \exp(-j2\pi f0.6)$$

$$= \begin{cases} \cos(0.8\pi f) - j\sin(0.8\pi f) + \\ \cos(1.2\pi f) - j\sin(1.2\pi f) \end{cases}$$

$$= \begin{cases} [\cos(0.8\pi f) + \cos(1.2\pi f)] - \\ j[\sin(0.8\pi f) + \sin(1.2\pi f)] \end{cases}$$

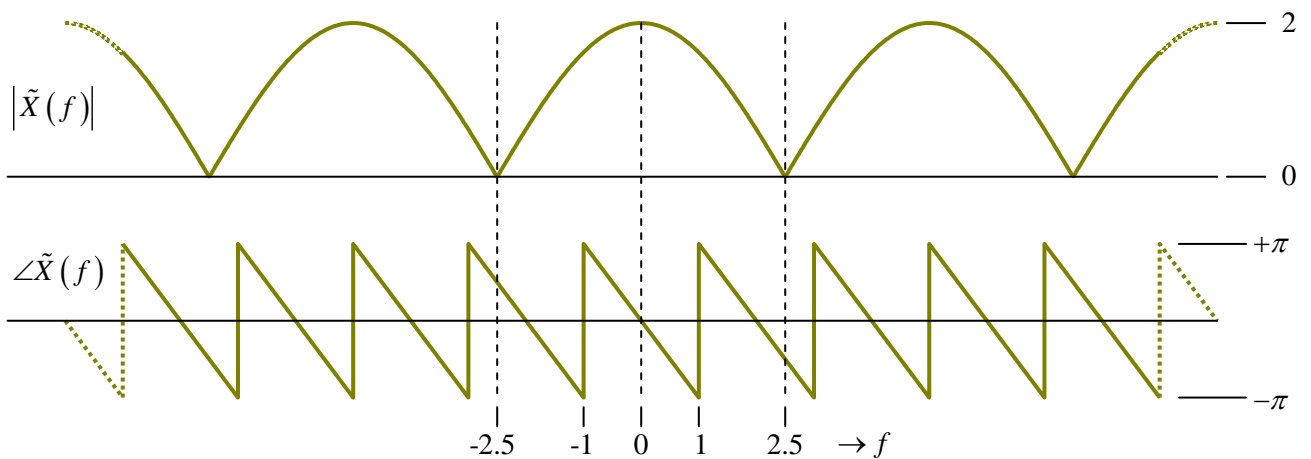
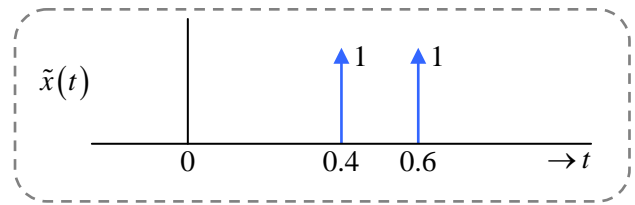
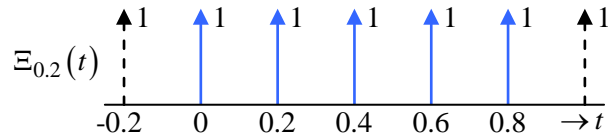
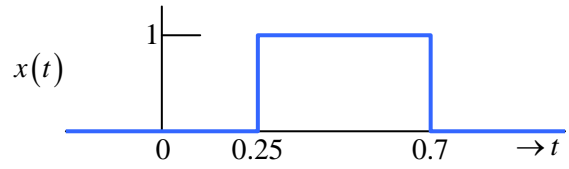
$$= 2\cos(\pi f)\cos(0.2\pi f) - j2\sin(\pi f)\cos(0.2\pi f)$$

$$|\tilde{X}(f)|^2 = 4\cos^2(\pi f)\cos^2(0.2\pi f) + 4\sin^2(\pi f)\cos^2(0.2\pi f)$$

$$= 4\cos^2(0.2\pi f)$$

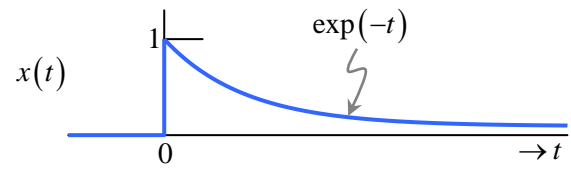
$$\bullet \quad |\tilde{X}(f)| = 2|\cos(0.2\pi f)|$$

$$\bullet \quad \angle\tilde{X}(f) = -\tan^{-1}\left(\frac{2\sin(\pi f)\cos(0.2\pi f)}{2\cos(\pi f)\cos(0.2\pi f)}\right) = -\tan^{-1}(\tan(\pi f)) = -\pi f$$

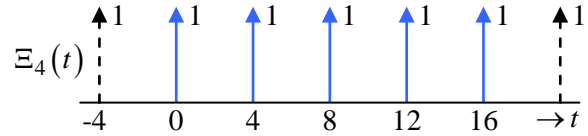


Solution to Q.2

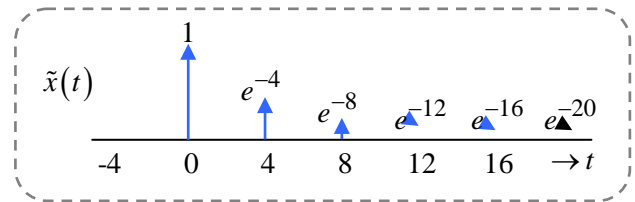
$$\begin{aligned}\tilde{x}(t) &= \underbrace{\exp(-t)u(t)}_{x(t)} \cdot \underbrace{\sum_{n=-\infty}^{\infty} \delta(t-4n)}_{\Xi_4(t)} \\ &= \sum_{n=0}^{\infty} \exp(-4n) \delta(t-4n)\end{aligned}$$



$$\mathfrak{T}\{\delta(t-\varsigma)\} = \exp(-j2\pi f \varsigma)$$



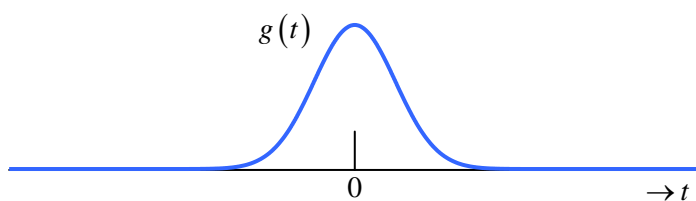
$$\begin{aligned}\tilde{X}(f) &= \sum_{n=0}^{\infty} \exp(-4n) \mathfrak{T}\{\delta(t-4n)\} \\ &= \sum_{n=0}^{\infty} \exp(-4n) \exp(-j2\pi f 4n) \\ &= \underbrace{\sum_{n=0}^{\infty} \exp(-4n(j2\pi f + 1))}_{\text{Geometric Series}} \\ &= \frac{1}{1 - \exp(-4(j2\pi f + 1))} = \frac{1}{1 - \exp(-4)\exp(-j8\pi f)}\end{aligned}$$



When a signal is sampled in the time domain, the spectrum of the sampled signal is periodic with a period equal to the sampling frequency, which in this case is 0.25 Hz.

Thus, to show that $\tilde{X}(f)$ is periodic, all we need to show is $\tilde{X}(f + 0.25) = \tilde{X}(f)$ as follows:

$$\begin{aligned}\tilde{X}(f + 0.25) &= \frac{1}{1 - \exp(-4)\exp(-j8\pi(f + 0.25))} \\ &= \frac{1}{1 - \exp(-4)\exp(-j8\pi f) \underbrace{\exp(-j2\pi)}_1} \\ &= \frac{1}{1 - \exp(-4)\exp(-j8\pi f)} = \tilde{X}(f)\end{aligned}$$

Solution to Q.3

Rewrite $x(t) = \sum_{n=-\infty}^{\infty} g(t - nT_p)$ in convolution form:

$$x(t) = g(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT_p) \quad \dots\dots\dots (*)$$

Applying the 'Convolution' property of the Fourier transform to (*):

$$X(f) = G(f) \cdot \frac{1}{T_p} \sum_{k=-\infty}^{\infty} \delta\left(f - \frac{k}{T_p}\right) \quad \dots\dots\dots (**)$$

Conclusion:

$X(f)$ can be obtained by sampling $G(f)/T_p$ in the frequency-domain at regular spacings of $1/T_p$ Hz.

Relationship between X_k and $G(f)$

$$\text{Rewrite (**) as: } X(f) = \sum_{k=-\infty}^{\infty} \frac{1}{T_p} G\left(\frac{k}{T_p}\right) \delta\left(f - \frac{k}{T_p}\right)$$

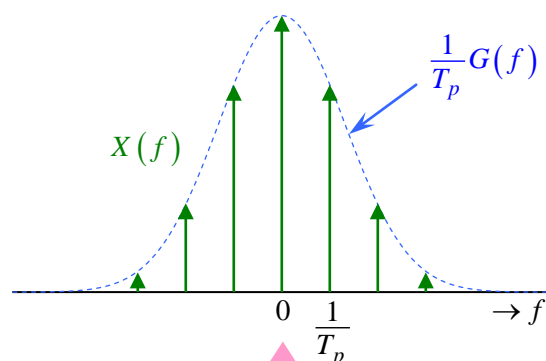
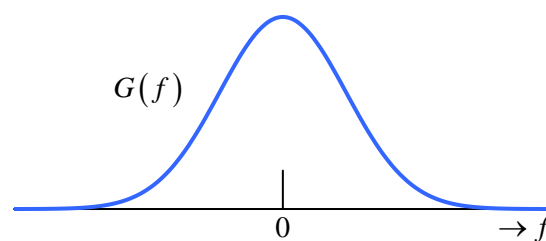
$$\text{In terms of } X_k: \quad X(f) = \sum_{k=-\infty}^{\infty} X_k \delta\left(f - \frac{k}{T_p}\right)$$

$$\text{Hence, } X_k = \frac{1}{T_p} G(f) \Big|_{f=k/T_p} = \frac{1}{T_p} G\left(\frac{k}{T_p}\right)$$

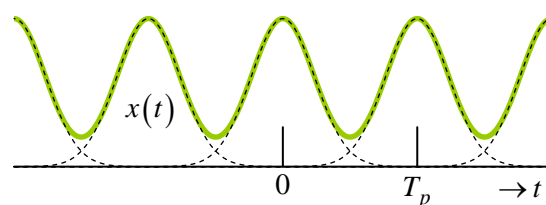
Uniqueness of a generating function

The generating function of a periodic signal is NOT unique. For instance, one period of a periodic signal can be used as its generating function.

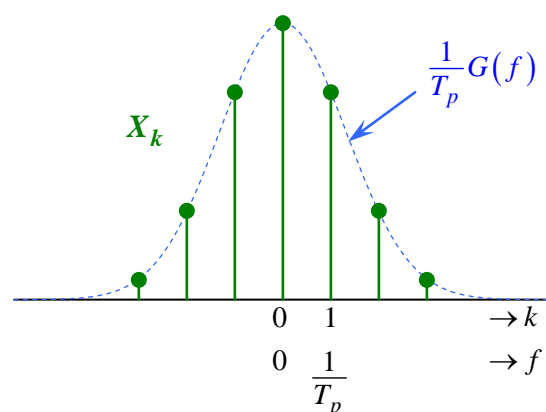
Remarks: From the sampling theorem, we observed that when a signal is sampled at regular time-intervals of T_s (sec), the spectrum corresponding to the sampled signal is periodic with a period of $f_s = 1/T_s$ (Hz). In this exercise, we showed that when a spectrum is sampled at regular frequency-intervals of f_p (Hz), the signal corresponding to the sampled spectrum is periodic with a period of $T_p = 1/f_p$ (sec). Hence, sampling a signal leads to a periodic spectrum and sampling a spectrum leads to periodic signal. This phenomenon is due to the convolution property of the Fourier Transform and the fact that $\left[\Xi_T(t) \Leftrightarrow \frac{1}{T} \Xi_{1/T}(f) \right]$.



Fourier Transform



Fourier Series Coefficients

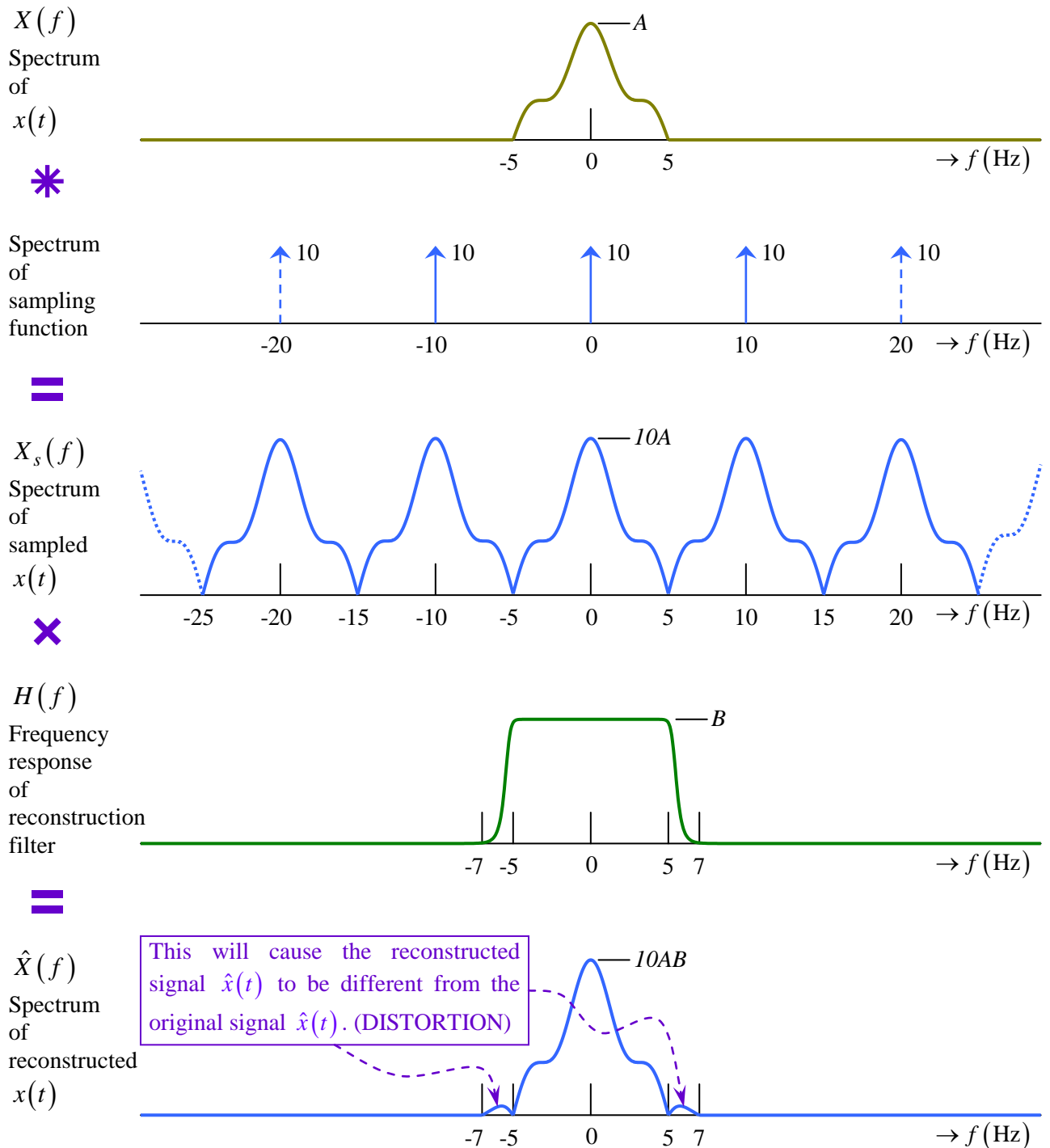


Solution to Q.4

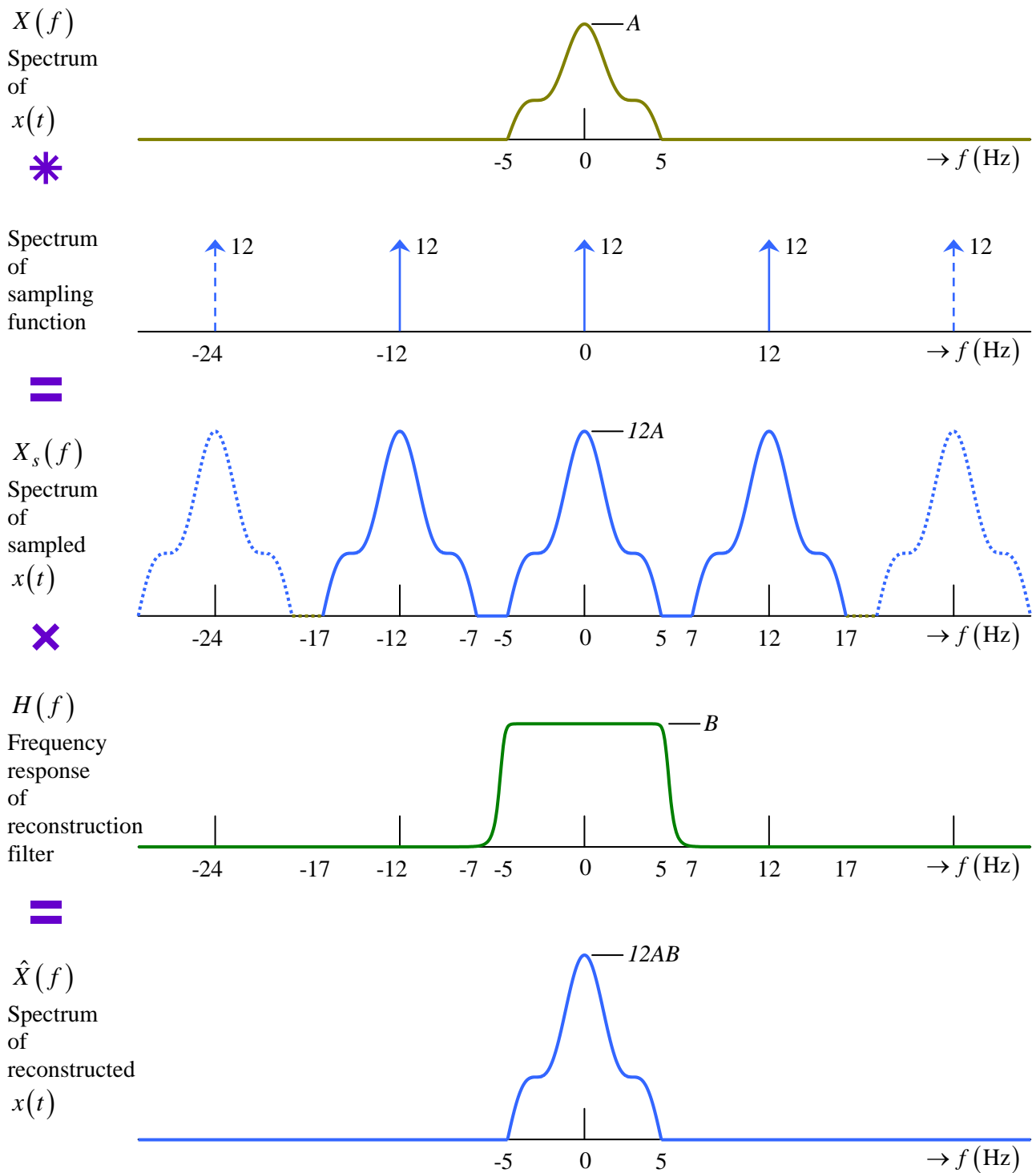
- Nyquist sampling frequency: $2 \times 5 = 10$ Hz
- Recommended sampling frequency: 12 Hz

The excess 2 Hz is needed to prevent adjacent spectral images from contributing to the reconstruction process. See illustration below.

Sampling frequency: 10 Hz (Nyquist Rate)



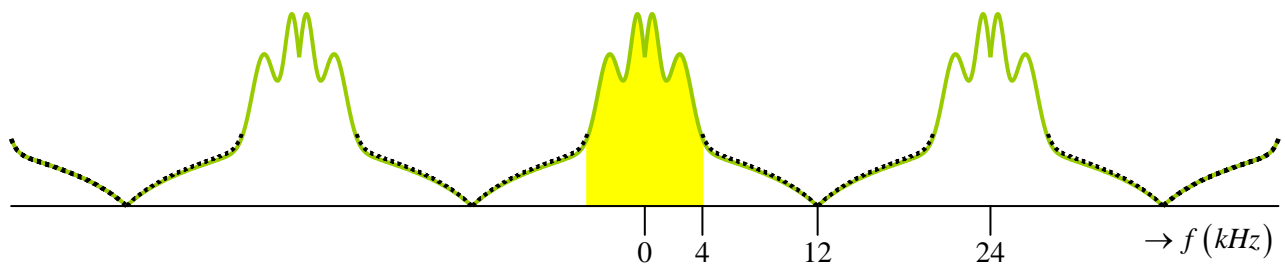
Sampling frequency: 12 Hz



Remarks: Sampling frequencies higher than 12 Hz may be used to achieve the same result. However, high sampling frequency is usually matched by more costly data acquisition, storage and processing requirement.

Solution to Q.5

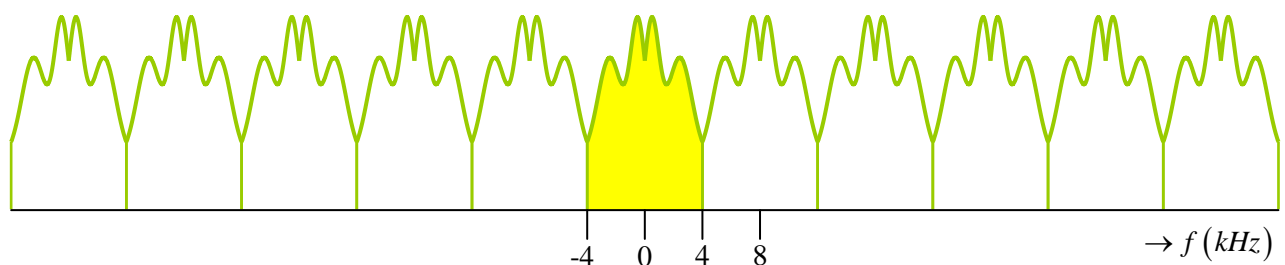
Situation A: $\left(\begin{array}{l} \text{No anti-aliasing lowpass filter} \\ \text{and no frequency aliasing} \end{array} \right): \left\{ \begin{array}{l} \text{Sampling frequency} = 2 \times 12 = 24 \text{ kHz} \end{array} \right.$



Advantage: No anti-aliasing LPF needed.

Disadvantage: Sampling frequency is higher than necessary to preserve the *TQ-Band*.

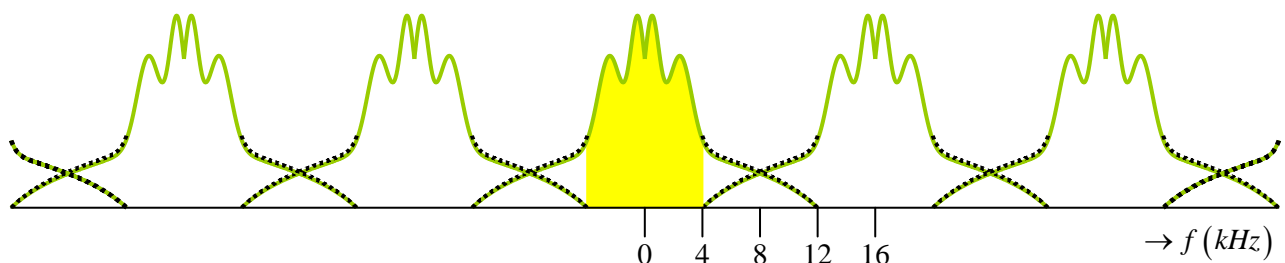
Situation B: *Lowest sampling frequency:* $\left\{ \begin{array}{l} \text{Ideal anti-aliasing filter of bandwidth 4 kHz} \\ \text{Sampling frequency} = 2 \times 4 = 8 \text{ kHz} \end{array} \right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band*.

Disadvantage: Require anti-aliasing LPF with sharp cutoff.

Situation C: *No anti-aliasing lowpass filter:* $\left\{ \begin{array}{l} \text{Sampling frequency} = 2 \times \left(\frac{12 + 4}{2} \right) = 16 \text{ kHz} \end{array} \right.$



Advantage: Lowest possible sampling frequency to preserve the *TQ-Band* without requiring anti-aliasing LPF with sharp cutoff.

Disadvantage: Sampling frequency is still higher than the minimum needed to preserve the *TQ-Band*.

This problem illustrates the trade-off between oversampling and the requirement of expensive anti-aliasing LPF with sharp cutoff frequency.

Solution to S.1

$$f_s = 2 \times 20 = 40 \text{ kHz}$$

$$\left[x_s(t) = x(t) \cdot \sum_n \delta\left(t - \frac{n}{40000}\right) \right] \Leftrightarrow \left[\begin{aligned} X_s(f) &= X(f) * 40000 \sum_k \delta(f - 40000k) \\ &= 40000 \sum_k X(f - 40000k) \end{aligned} \right]$$

Specifying $H(f) = \frac{1}{40000} \cdot \text{rect}\left(\frac{f}{40000}\right)$ will lead to $X_s(f)H(f) = X(f)$.

Solution to S.2

$$(a) \quad \left\{ \begin{aligned} x_s(t) &= \sum_{n=-5}^5 x(5n) \delta(t - 5n) \\ &\quad \uparrow \\ &\quad \text{sampling period} \\ \therefore \text{Sampling frequency: } f_s &= \frac{1}{5} = 0.2 \text{ Hz} \end{aligned} \right.$$

- (b) Perfect reconstruction is of $x(t)$ from $x_s(t)$ only possible if $x(t)$ is bandlimited to $f_s/2 = 0.1$, i.e. $X(f) = 0; f > 0.1$.

Since $X(f) = \mathcal{F}^{-1}\{\text{tri}(t)\} = \text{sinc}^2(f)$ has infinite frequency extent, perfect reconstruction is not possible due to frequency aliasing.

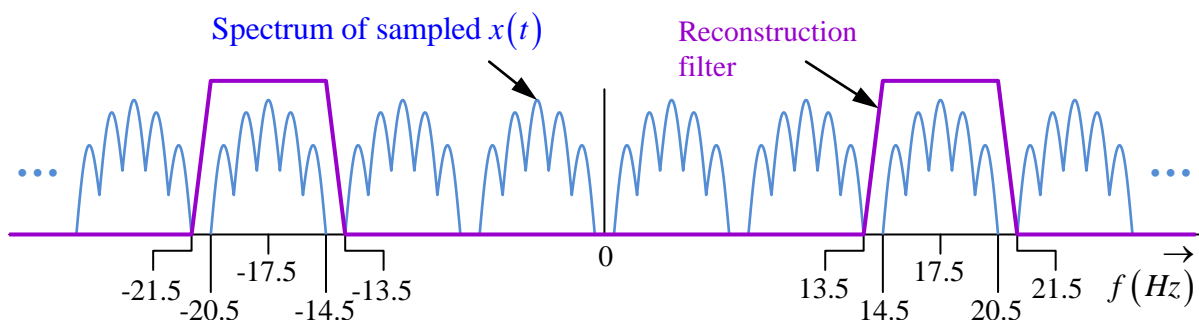
Solution to S.3

- (a) Nyquist Sampling Frequency: $f_s = 2 \times 20.5 = 41 \text{ Hz}$

- (b) Center frequency: $f_c = 17.5 \text{ Hz}$ Bandwidth: $B = 6 \text{ Hz}$

Possible sampling frequencies : $f_s = 2f_c/k; \quad k = 1, 2, \dots, \lfloor 2f_c/B \rfloor$

Lowest sampling frequency : $f_s = \frac{2f_c}{\lfloor 2f_c/B \rfloor} = \frac{35}{5} = 7 \text{ Hz}$



Reconstruction Filter Specs : $\begin{cases} \text{Ideal passband: } 14.5 < |f| < 20.5 \\ \text{Ideal stopband: } |f| < 13.5 \text{ or } |f| > 21.5 \end{cases}$