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Quadratic factor $G(s) = \dfrac{1}{1 + 2\zeta(\frac{j\omega}{\omega_n}) + (\frac{j\omega}{\omega_n})^2}$

- Magnitude of a second-order system is

$$\left|\frac{1}{1 + 2\zeta(\frac{j\omega}{\omega_n}) + (\frac{j\omega}{\omega_n})^2}\right| = \frac{1}{\sqrt{(1 - \frac{\omega^2}{\omega_n^2})^2 + (2\zeta\,\frac{\omega}{\omega_n})^2}}$$

$$= -20\log_{10}\sqrt{(1 - \frac{\omega^2}{\omega_n^2})^2 + (2\zeta\,\frac{\omega}{\omega_n})^2}$$

- Bode diagram when $0 < \zeta \leq 1$ (Magnitude response)
  - When $\omega << \omega_n$,

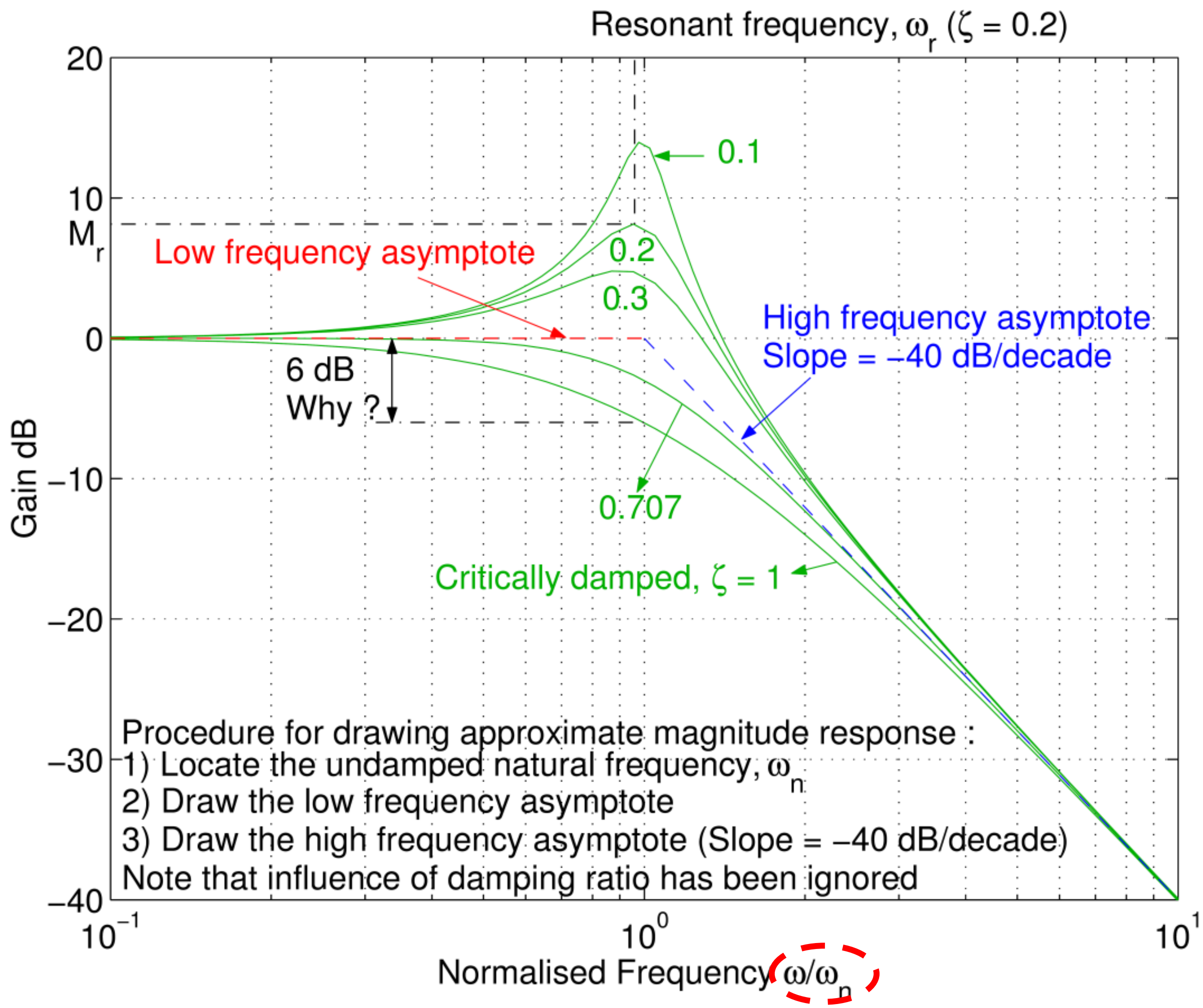
$$-20\log_{10}\sqrt{(1 - \cancelto{0}{\frac{\omega^2}{\omega_n^2}})^2 + (2\zeta\,\cancelto{0}{\frac{\omega}{\omega_n}})^2} \approx 0\,dB$$

Low frequency asymptote is the 0 dB line with slope 0.

- When $\omega >> \omega_n$,

$$-20\log_{10}\sqrt{(1-\frac{\omega^2}{\omega_n^2})^2+(2\zeta\frac{\omega}{\omega_n})^2} \approx -20\log_{10}\frac{\omega^2}{\omega_n^2} = -40\log_{10}\frac{\omega}{\omega_n}\,\text{dB}$$

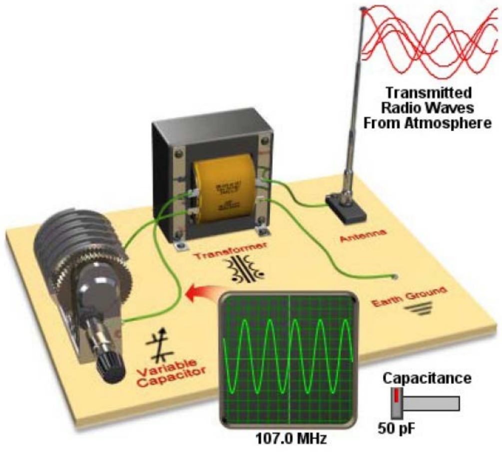
High frequency asymptote is a straight line with a slope of –40 dB/decade.



- Two asymptotes intersect at $\omega = \omega_n$
- The approximate magnitude response (low and high frequency asymptotes) is independent of the value of ζ
- When $\zeta < 1$, the magnitude response has a "resonant peak" whose size depends on the damping ratio, ζ

> If $\zeta < 1$, the frequency response reaches a maximum value, M_r, at the resonant frequency, ω_r.

- Resonance comes from Latin and means to "resound", i.e., to sound out together with a loud sound. The output amplitude will be larger than the input amplitude when a system exhibits resonance
- Resonant frequency is the name given to the input frequency which gives an output with the biggest response



- Example of resonance: Radio tuner is a resonant circuit (also known as a series RLC circuit or a tuned circuit)

- Resonant magnitude and frequency formulae. Let $u = \dfrac{\omega}{\omega_n}$. Then from page 10-1,

$$|G(ju)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\zeta u)^2}}$$

At the resonant frequency ω_r, $\dfrac{d|G(ju)|}{du} = 0$

$$\boxed{\therefore \quad \omega_r = \omega_n\sqrt{1-2\zeta^2}, \quad \zeta < \frac{1}{\sqrt{2}}}$$

Substituting $\omega = \omega_r$ into $G(ju)$, an expression for the resonant magnitude can be found

$$\boxed{M_r = |G(j\omega)|_{\max} = |G(j\omega_r)| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}, \quad \zeta < \frac{1}{\sqrt{2}}}$$

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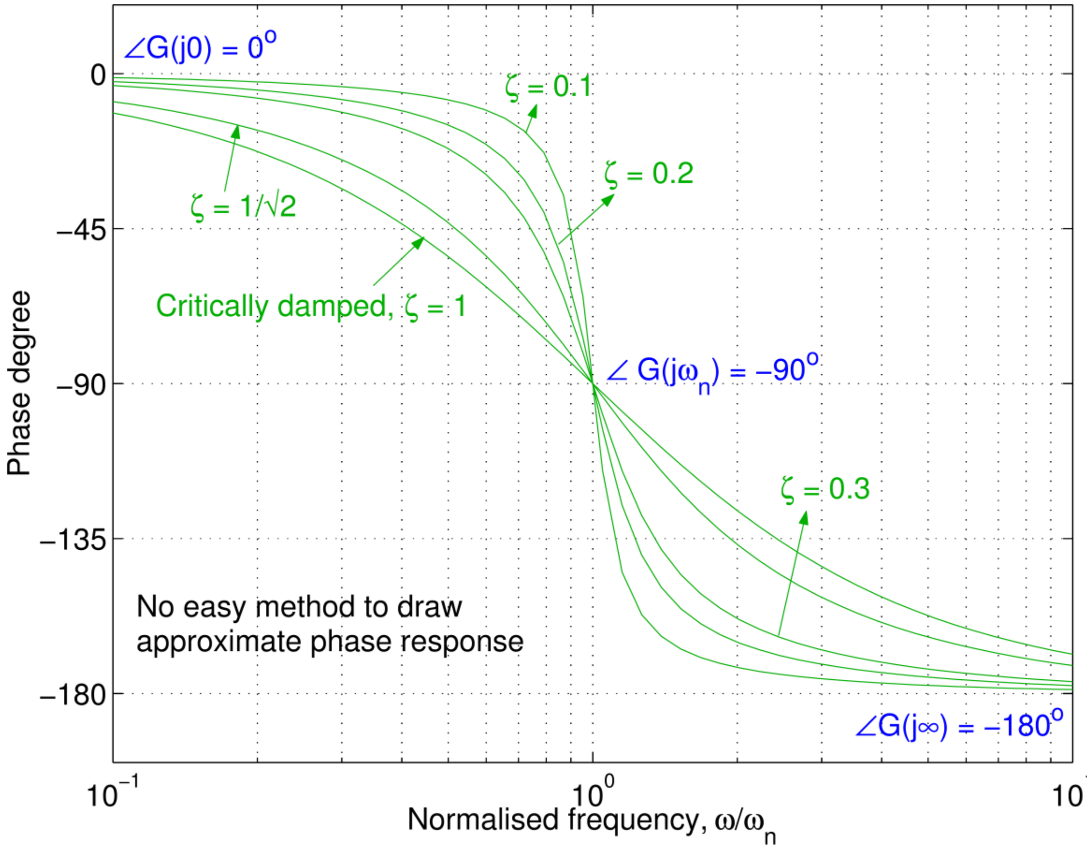
- Bode diagram when $0 < \zeta \le 1$ (Phase response)

$$\phi = \angle\left(\frac{1}{1+2\zeta(\frac{j\omega}{\omega_n})+(\frac{j\omega}{\omega_n})^2}\right) = -\tan^{-1}\left[\frac{2\zeta(\frac{\omega}{\omega_n})}{1-(\frac{\omega}{\omega_n})^2}\right]$$



Transport lag/delay (Dead-time)

- Transport lag is represented by $e^{-j\omega t_d}$

Magnitude: $\left|e^{-j\omega t_d}\right| = \left|\cos\omega t_d - j\sin\omega t_d\right| = 1$

$$\Rightarrow \quad \left|G(j\omega)e^{-j\omega t_d}\right| = \left|G(j\omega)\right|$$

Phase:

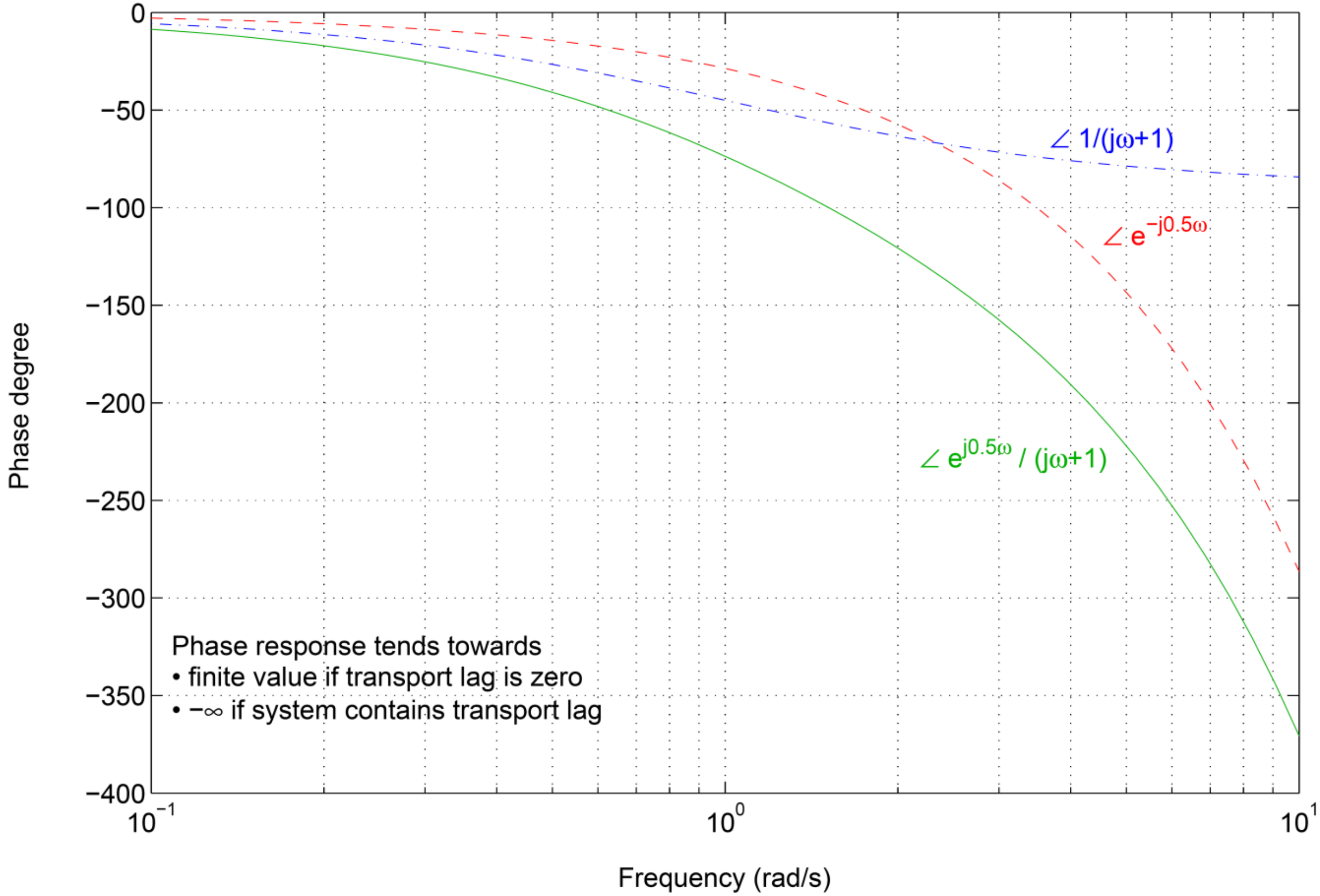
$$\angle\left(e^{-j\omega t_d}\right) = -\omega t_d \text{ radians} = -\omega t_d \frac{180}{\pi} \text{ deg}$$

$$\Rightarrow \quad \text{When } \omega \to \infty,\ \angle\left(e^{-j\omega t_d}\right) \to -\infty$$

$$\angle\left(G(j\omega)e^{-j\omega t_d}\right) = \angle G(j\omega) - \omega t_d \frac{180}{\pi} \text{deg}$$

- Bode diagram (Phase response)

Consider the first order plus dead-time plant: $\dfrac{e^{-0.5j\omega}}{1+j\omega}$



Frequency Response Characteristics

- How do you deal with more general frequency responses?
- Generic information can be used to identify system characteristics

Low Frequency Characteristics

m zeros, N poles at s=0, n finite poles

- Consider the following general transfer function

$$G(j\omega) = \frac{K(1+j\omega T_a)(1+j\omega T_b)\ldots(1+j\omega T_m)}{(j\omega)^N(1+j\omega T_1)(1+j\omega T_2)\ldots(1+j\omega T_n)}$$

$$G(j\omega)\Big|_{\omega\to 0} \approx \frac{K}{(j\omega)^N}$$

Pole-zero excess is N+n-m

As $\omega \to 0$, transfer function reduces to $\dfrac{K}{(j\omega)^N}$

- Behavior of magnitude response plot when $\omega \to 0$ can be used to identify K and N
  - Magnitude response at low frequencies is given by

$$\left|G(j\omega)\right|_{\omega\to 0} = \left\{20\log_{10} K + 20N\log_{10}\left|\frac{1}{j\omega}\right|\right\} \text{ dB}$$

$$= 20\log_{10}K - 20N\log_{10}\omega \text{ dB}$$

  - Low frequency asymptote of magnitude response has slope of

$$N \times -20 \text{ dB/decade}$$

  - At ω = 1 rad/s, magnitude of low frequency asymptote is

$$20\log_{10} K$$

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- Behavior of phase response plot when $\omega \to 0$ can be used to identify/verify *N*

  - Phase response when $\omega \to 0$ is given by

$$\angle G(j\omega)\Big|_{\omega\to 0} = \underbrace{\cancel{\angle K}}_{0} + N \times \angle \frac{1}{j\omega} = N \times \angle \frac{1}{j\omega}$$

  - Since $\angle \frac{1}{j\omega} = -90^\circ$, low frequency asymptote of phase response is

$$N \times -90^\circ$$

High Frequency Characteristics

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$$G(j\omega)=\frac{K(1+j\omega T_a)(1+j\omega T_b)\ldots(1+j\omega T_m)}{(j\omega)^N(1+j\omega T_1)(1+j\omega T_2)\ldots(1+j\omega T_n)}$$

Low frequency $\quad G(j\omega)\big|_{\omega\to 0}\approx\dfrac{K}{(j\omega)^N}$

High frequency $\quad G(j\omega)\big|_{\omega\to\infty}\approx\dfrac{KT_aT_b\ldots T_m}{T_1T_2\ldots T_n(j\omega)^{N+n-m}},\qquad N+n-m>0$

Pole-zero excess is N+n-m

High frequency magnitude response asymptote :

$$20\log_{10}\frac{KT_aT_b\ldots T_m}{T_1T_2\ldots T_n}-20(N+n-m)\log_{10}\omega$$

High frequency asymptote : $-20(N+n-m)$ dB/decade

High frequency phase : $-90(N+n-m)$ degrees

- Behavior of Bode diagrams when $\omega \to \infty$ can be used to determine pole excess and dead-time, T_D
  - High frequency asymptote of magnitude response plot has slope of

    $$\text{pole excess} \times -20 \text{ dB/decade}$$

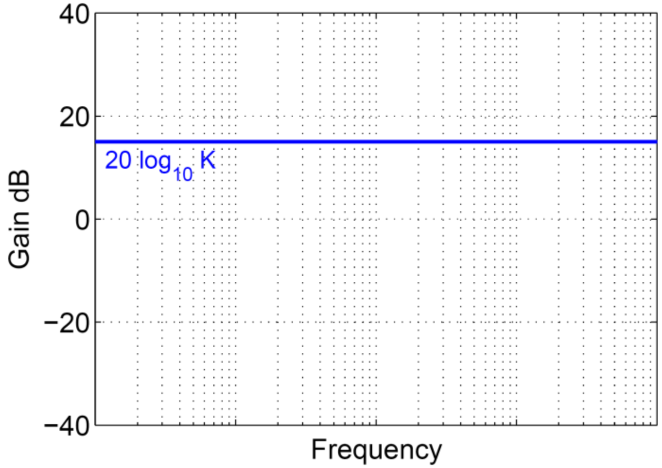
    where pole excess is given by

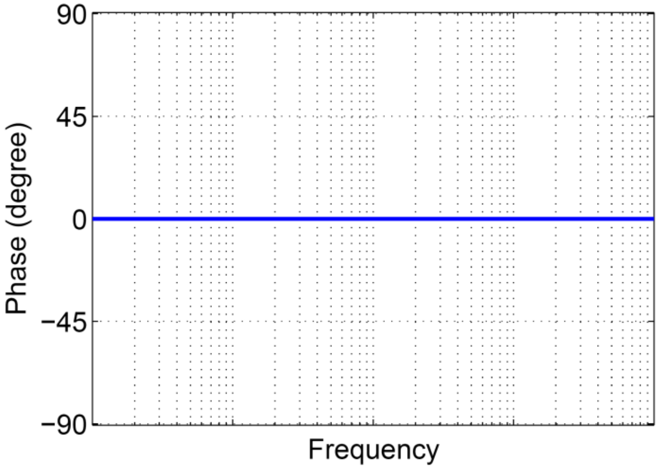
    $$N+n-m = \text{no. of poles (including integrators)} - \text{no. of zeros}$$

  - Phase response of minimum phase system (no RHP poles & zeros) at high frequencies
    - tends towards pole excess $\times -90^\circ$ if $T_D = 0$
    - tends towards $-\infty$ if $T_D \neq 0$

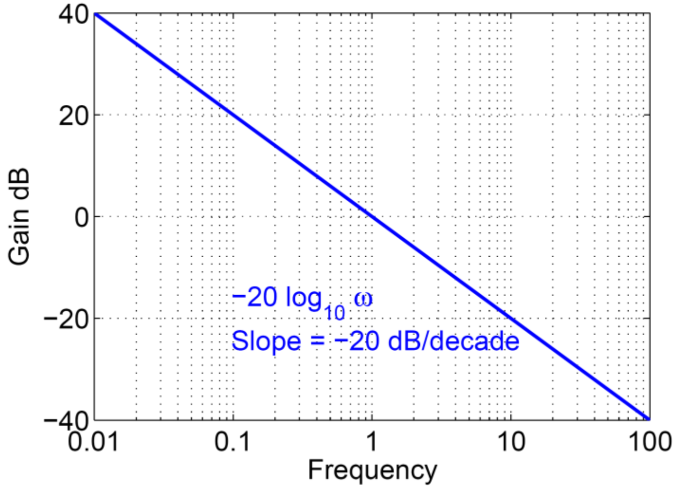
Summary: Bode Diagrams of Common Factors

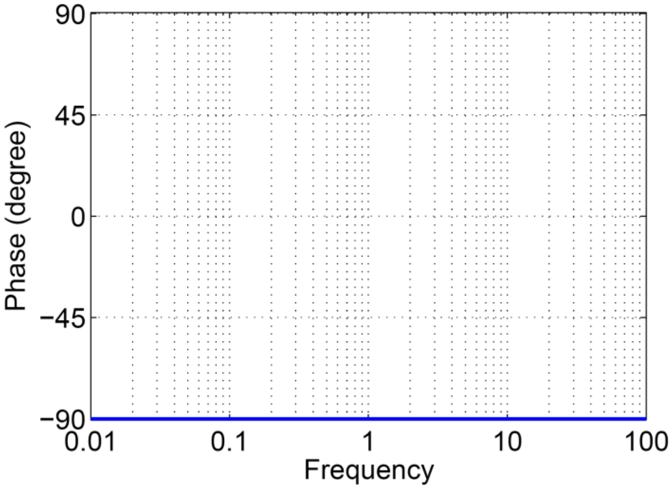
$$G(s) = K$$





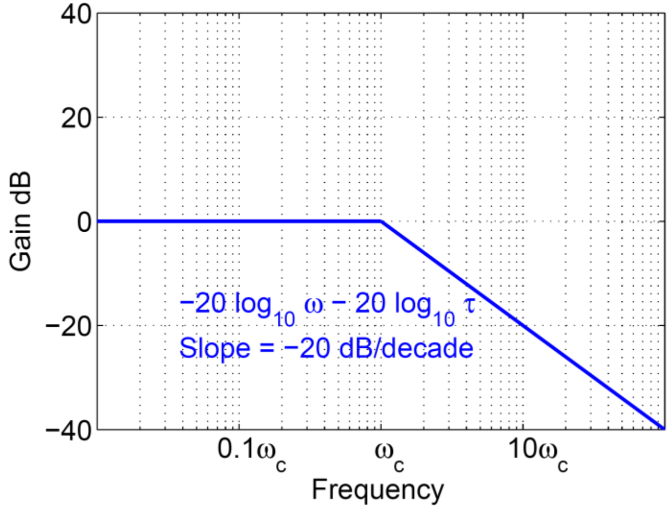
$$G(s) = \frac{1}{s}$$

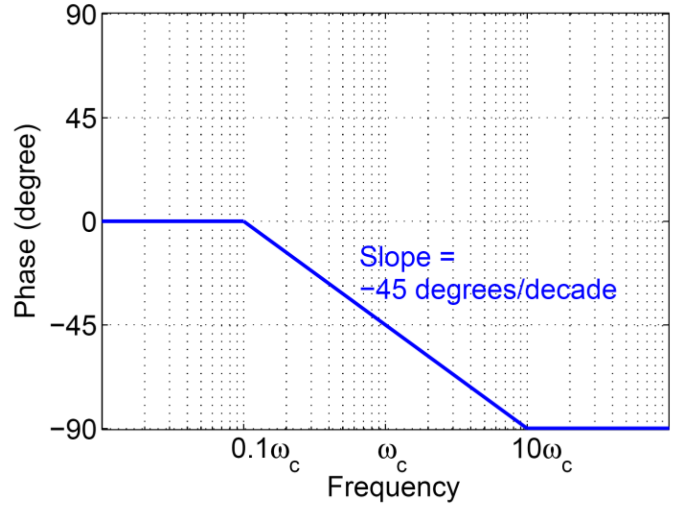




$$G(s) = \frac{1}{\tau s + 1}$$

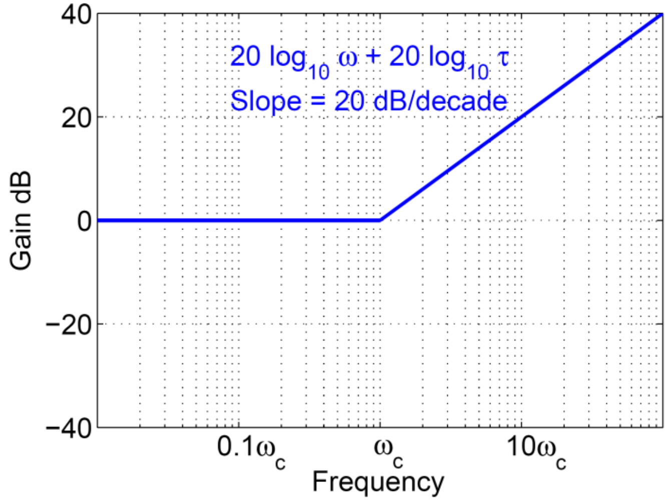
$$\omega_c = \frac{1}{\tau}$$

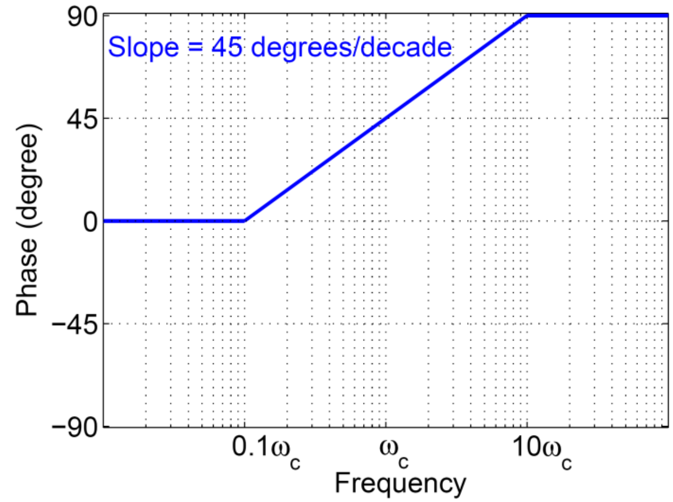




$$G(s) = \tau s + 1$$

$$\omega_c = \frac{1}{\tau}$$





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- Example 1: Construct the Bode diagram of

$$G(s)=\frac{1}{(\alpha s+1)(\beta s+1)}; \quad \alpha>\beta$$

Note that $G(s)$ is an overdamped second-order system.

Construct the magnitude response first :

$$|G(j\omega)|=\left|\frac{1}{(j\omega\alpha+1)(j\omega\beta+1)}\right|$$

$$20\log_{10}|G(j\omega)|=20\log_{10}\left|\frac{1}{(j\omega\alpha+1)(j\omega\beta+1)}\right|$$

$$=20\log_{10}\left|\frac{1}{(j\omega\alpha+1)}\right|+20\log_{10}\left|\frac{1}{(j\omega\beta+1)}\right|$$

- To construct the magnitude response of $G(s)$, first draw the magnitude response of the two first order factors $\dfrac{1}{\alpha s+1}$ and $\dfrac{1}{\beta s+1}$
  - Low frequency asymptotes are both 0 dB
  - Corner frequencies of $\dfrac{1}{\alpha s+1}$ and $\dfrac{1}{\beta s+1}$ are $\dfrac{1}{\alpha}$ and $\dfrac{1}{\beta}$
  - High frequency asymptotes are

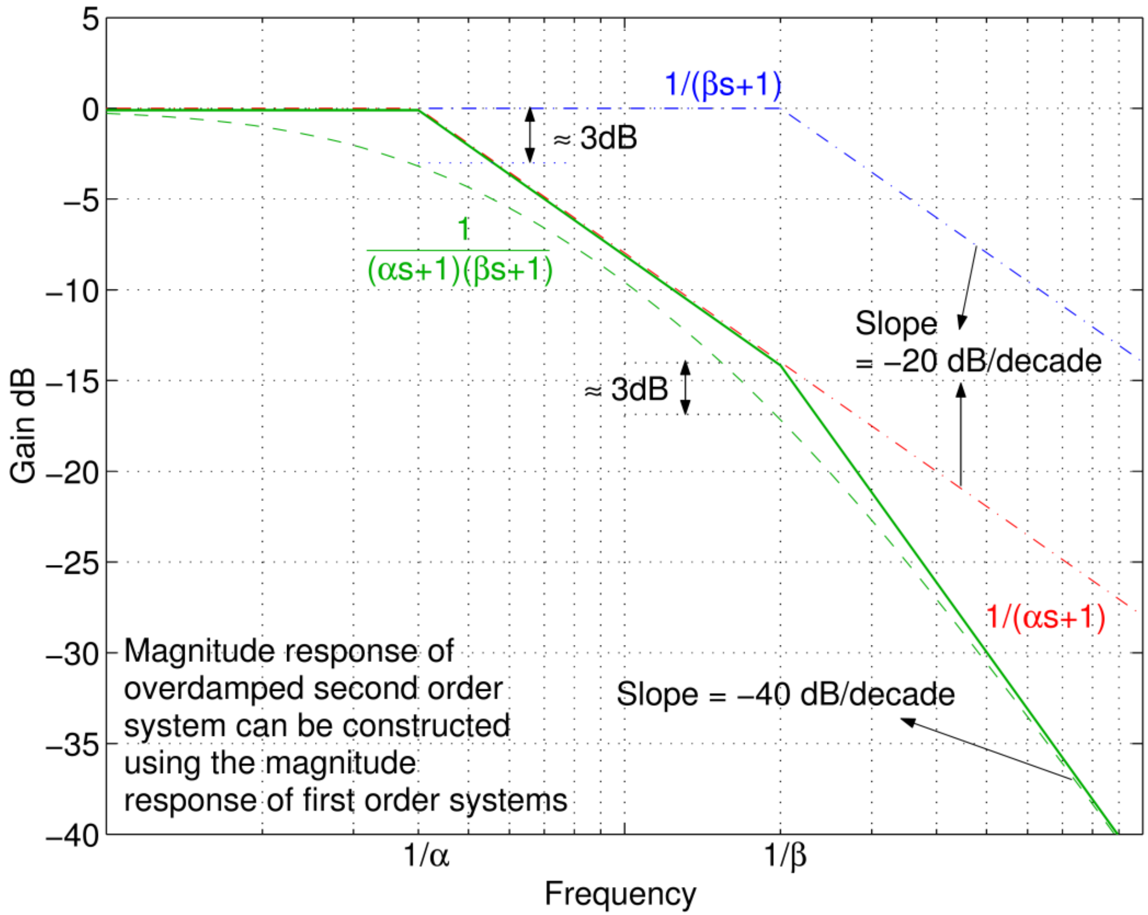
$$\frac{1}{\alpha s+1}: \quad -20\log_{10}\omega-20\log_{10}\alpha \qquad \frac{1}{\beta s+1}: \quad -20\log_{10}\omega-20\log_{10}\beta$$

- Sum the individual plots:
  - When $\omega<\frac{1}{\alpha}$, slope is 0 dB/decade
  - When $\frac{1}{\alpha}<\omega<\frac{1}{\beta}$, slope is –20 dB/decade

- When $\omega > \frac{1}{\beta}$, asymptotic magnitude response is

$$-20\log_{10}\omega - 20\log_{10}\alpha - 20\log_{10}\omega - 20\log_{10}\beta$$
$$= -40\log_{10}\omega - 20\log_{10}\alpha\beta$$

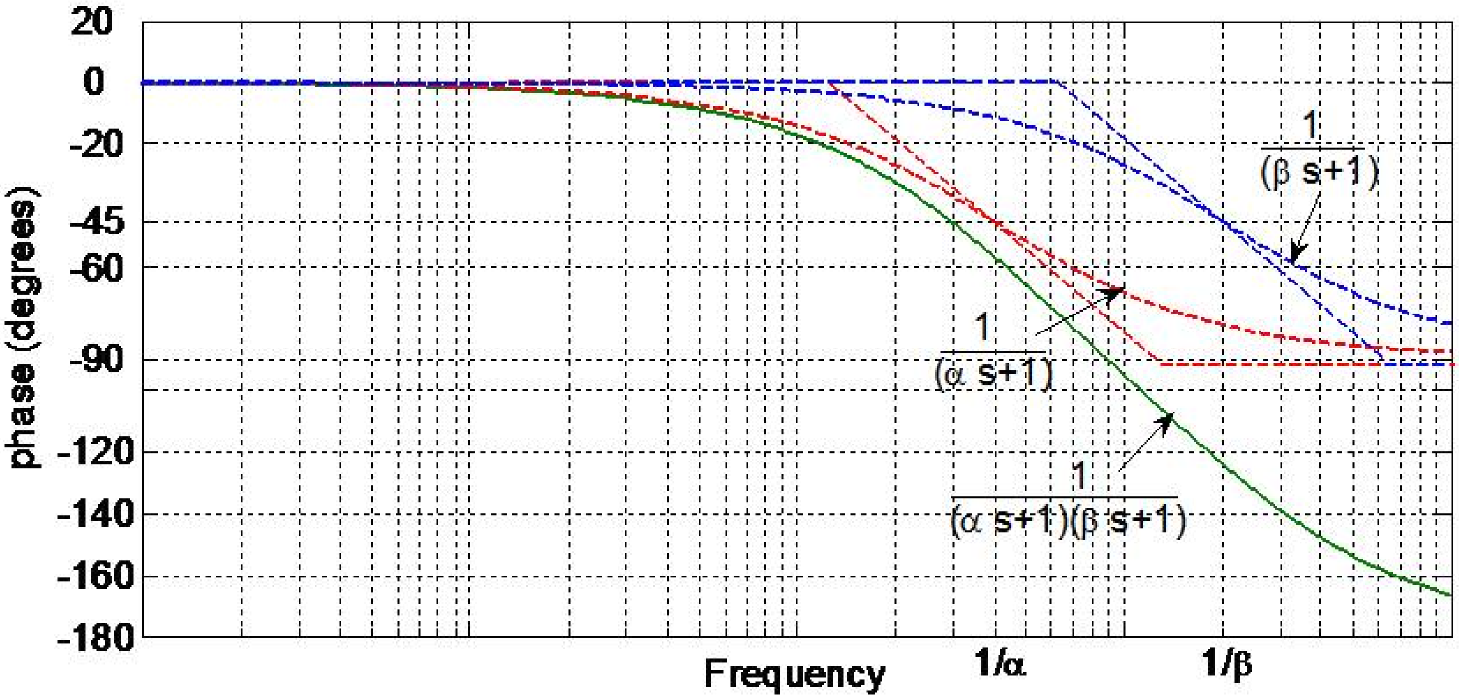
So slope when $\omega > \frac{1}{\beta}$ is –40 dB/decade.



Phase response :

$$\text{At } \omega = \frac{1}{\alpha}, \angle \frac{1}{\alpha s + 1} = -45^0,$$

$$\text{At } \omega = \frac{1}{\beta}, \angle \frac{1}{\beta s + 1} = -45^0$$



Obtaining G(s) from Bode Diagrams

- An important role played by Bode diagrams is system identification

$$G_p(s) = \frac{K(1+sT_a)(1+sT_b)\ldots(1+sT_m)}{s^N(1+sT_1)(1+sT_2)\ldots(1+sT_n)}$$

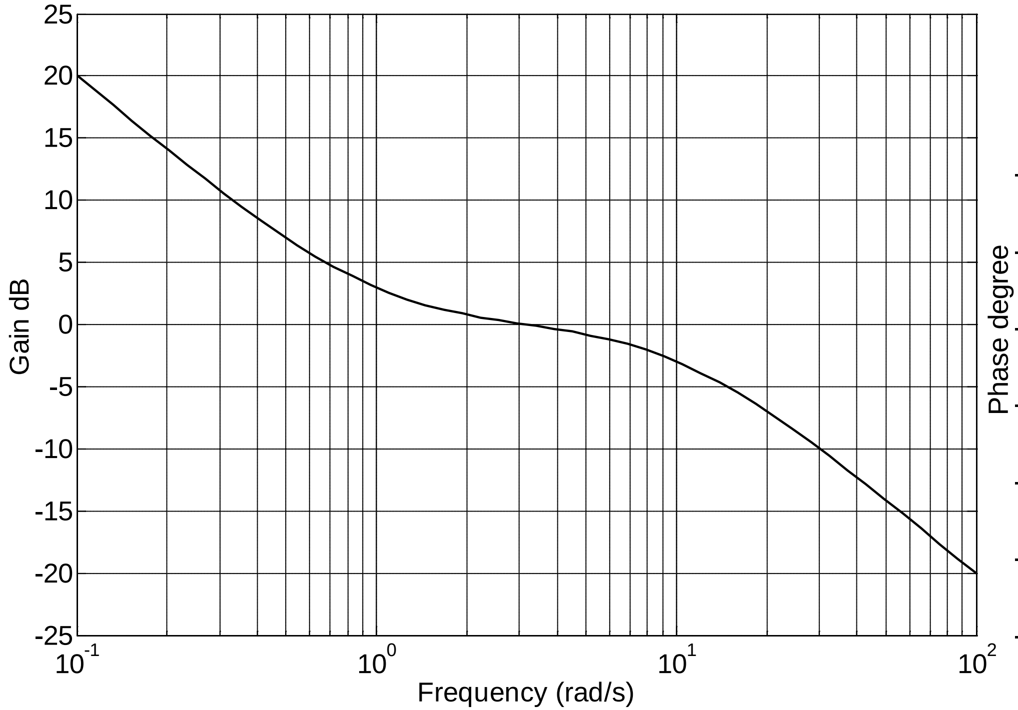
Find K, T_a, ... T_m, N, T_1, T_2, ... T_n

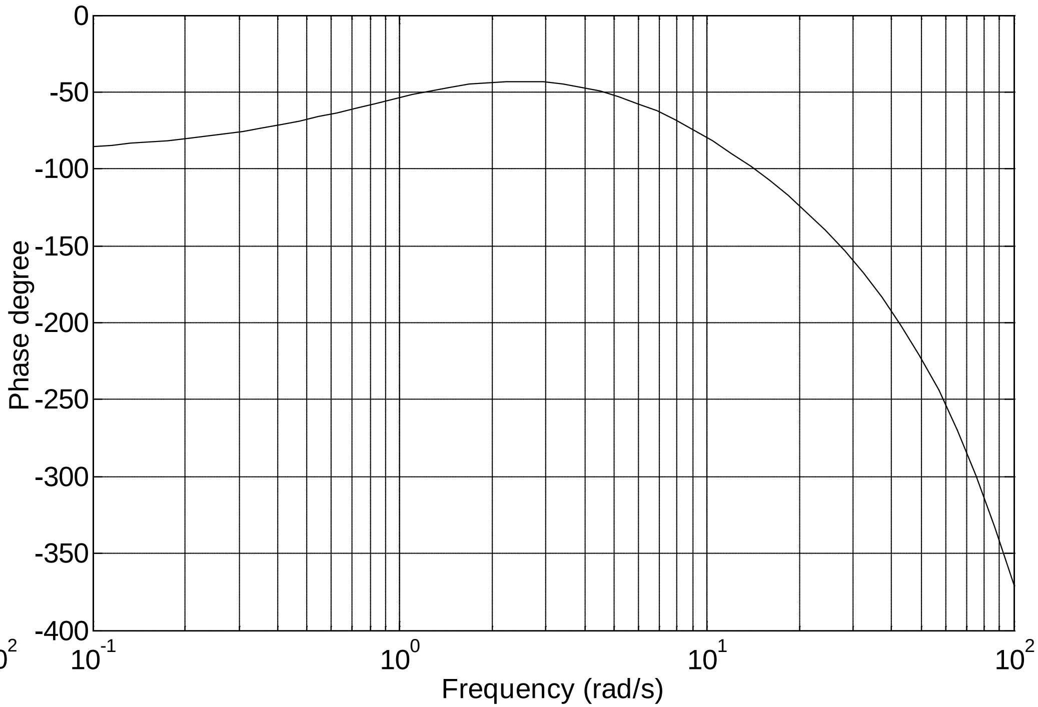
- An approximate transfer function may be estimated from Bode diagrams because of the following characteristics:
  - Magnitude response may only have straight line asymptotes with integer multiple of 20 dB/decade for its slope
    - If the slope decreases monotonically, then the plant does not have a zero
    - If the slope increases, then the plant has at least one zero
    - If the high frequency decreases by -20P dB/decade, then it means that there are P=N+n-m more poles than zeros in the G(s).

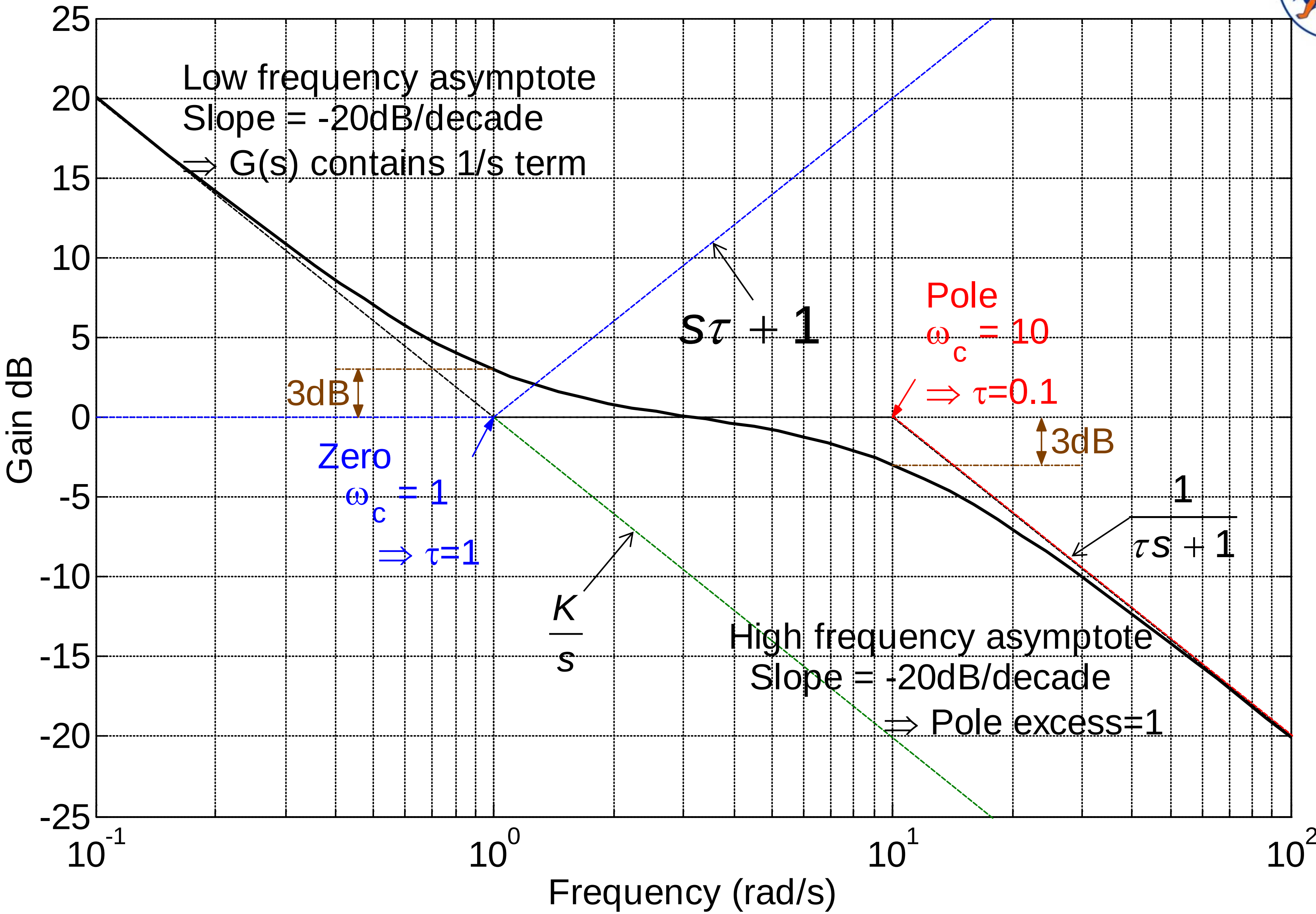
- Corner frequencies $(\omega_c = \frac{1}{T})$ correspond to the system pole(s) or zero(s)
  - Corner frequency of first order factor is the frequency where the difference between the actual magnitude response and the asymptote is equal to 3 dB

- Transfer function can be uniquely determined by jointly considering magnitude and phase response
  - Phase response needed to ascertain if system is minimum phase and whether transfer function contains a delay term
  - If phase decreases monotonically with frequency, then it indicates that a delay term is present in the G(s)

Example: Determine $G_p(s)$ from the Bode Plots







Low frequency asymptote
Slope = -20dB/decade
⇒ G(s) contains 1/s term
$s\tau + 1$
Pole
$\omega_c = 10$
⇒ τ=0.1
3dB
3dB
Zero
$\omega_c = 1$
⇒ τ=1
$\frac{K}{s}$
$\frac{1}{\tau s + 1}$
High frequency asymptote
Slope = -20dB/decade
⇒ Pole excess=1
Gain dB
Frequency (rad/s)
25
20
15
10
5
0
-5
-10
-15
-20
-25
10^{-1}
10^{0}
10^{1}
10^{2}

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- The magnitude response has the following characteristics:
  - Slope of low frequency asymptote is –20 dB/decade
  - At ω = 1 rad/s, slope increases by 20 dB/decade
  - At ω = 10 rad/s, slope decreases by 20 dB/decade

Hence, $G_p(s)$ comprises the terms: $\dfrac{1}{s},\quad (s+1),\quad \dfrac{1}{0.1s+1}$

When ω = 1 rad/s, magnitude of low frequency asymptote is 0 dB $\Rightarrow$ Integrator gain = K = 1

Assume $G_p(s) = \dfrac{K(s+1)e^{-st_d}}{s(0.1s+1)}$ where $K=1$

From the phase plot, ϕ = −370º when ω = 100 rad/s

$$\angle \frac{s+1}{s(0.1s+1)} e^{-st_d} \bigg|_{s=j100} = -370^\circ$$

$$\angle \frac{s+1}{s(0.1s+1)} \bigg|_{s=j100} + \angle e^{-st_d} \Big|_{s=j100} = -370^\circ$$

$$\tan^{-1} 100 - 90^\circ - \tan^{-1} 10 - 100 t_d \times \frac{180}{\pi} = -370^\circ$$

$$100 t_d \times \frac{180}{\pi} = -370^\circ + 84.9^\circ \quad \Rightarrow \quad t_d = 0.05$$

- Hence transfer function is $G_p(s) = \dfrac{s+1}{s(0.1s+1)} e^{-0.05s}$