Sinusoidal Response

- What is the output when the input is a sinusoidal?

$$A\sin(\omega t) \longrightarrow \boxed{G(s)} \xrightarrow{???}$$

- Intuitively, the output will also be a sinusoidal signal as the solution to the system differential equations can be interpreted using two properties of sinusoidal signal:
  - Each differentiation of a sinusoid results in another sinusoidal signal,

$$\sin\omega t \xrightarrow{\frac{d}{dt}} \omega\cos\omega t \xrightarrow{\frac{d}{dt}} -\omega^2 \sin\omega t \ldots$$

  - The result of adding two sinusoids of the same frequency together is another sinusoid with different amplitude and phase. However, the frequencies of the signals are preserved.

Analysis using Laplace Transform

- Assume that the transfer function of a stable system is $\dfrac{Y(s)}{U(s)} = G(s)$ where the input $r(t)$ is a sine wave with amplitude A, i.e.,

$$r(t) = A\sin\omega t \quad \Rightarrow \quad R(s) = \frac{A\omega}{s^2+\omega^2}$$

  - The system output is $Y(s) = G(s)R(s) = G(s)\dfrac{A\omega}{s^2+\omega^2}$
  - If $G(s)$ has stable poles at $s = -p_1, -p_2, \ldots$, then

$$Y(s) = A\left(\underbrace{\frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \ldots}_{\text{System poles}} + \underbrace{\frac{Cs+D}{s^2+\omega^2}}_{\text{Input poles}}\right)$$

- The time response is

$$y(t) = A\Bigg(\underbrace{A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \ldots}_{\text{Transient}} + \underbrace{C\cos\omega t + \frac{D}{\omega}\sin\omega t}_{\text{Steady-state}}\Bigg)$$

- When $t \to \infty$,

$$A_1 e^{-p_1 t} + A_2 e^{-p_2 t} + \ldots \to 0$$

i.e., the transient response decays to zero.

- Hence the output of the system at steady-state is

$$y_{ss}(t) = A\left(C\cos\omega t + \frac{D}{\omega}\sin\omega t\right) = AM_{\omega}\sin(\omega t + \phi_{\omega})$$

$$\text{where}\quad \mathrm{M}_{\omega} = \sqrt{C^2 + \left(\frac{D}{\omega}\right)^2}, \qquad \phi_{\omega} = \tan^{-1}\frac{\omega C}{D}$$

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Another way to think about!

$$y(t) = \int_0^\infty g(\tau) r(t-\tau) d\tau$$

$$\text{If } r(t) = e^{st}, \text{ then } y(t) = \int_0^\infty g(\tau) e^{s(t-\tau)} d\tau$$

$$y(t) = \int_0^\infty g(\tau) e^{st} e^{-s\tau} d\tau$$

$$= G(s) e^{st}$$

$$\text{If } A \sin \omega t = \frac{A}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right),$$

$$y(t) = \frac{A}{2j}\left(G(j\omega) e^{j\omega t} - G(-j\omega) e^{-j\omega t}\right)$$

$$= \frac{A}{2j}\left(M_\omega e^{j\phi_\omega} e^{j\omega t} - M_\omega e^{-j\phi_\omega} e^{-j\omega t}\right)$$

$$= M_\omega A \sin(\omega t + \phi_\omega)$$

Note that G(jω) can be written as :

$$G(j\omega) = M_\omega e^{j\phi_\omega}$$

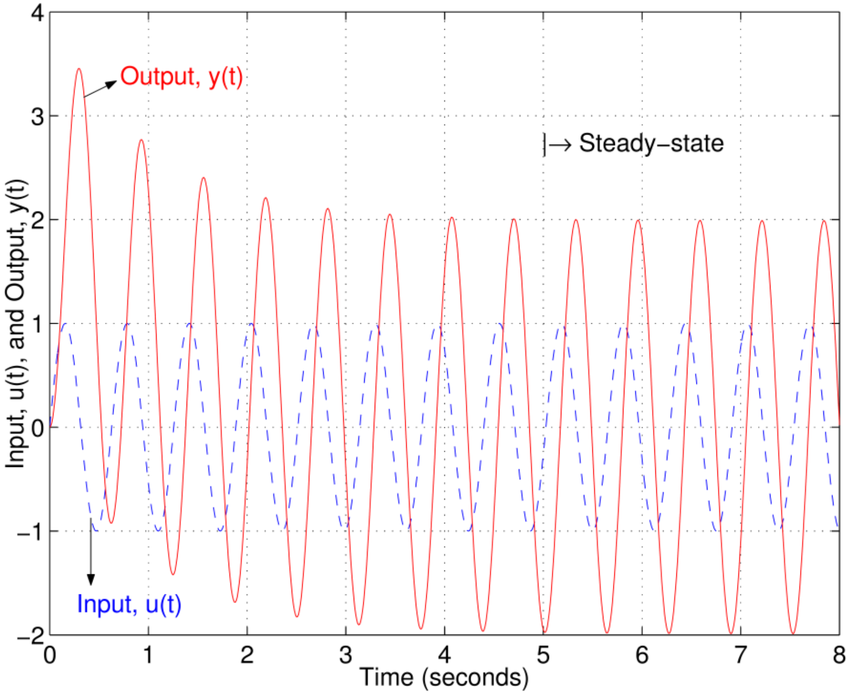
where M_ω and ϕ_ω is the amplitude of and phase of G(jω) resp.

- Example: calculation using Laplace Transform

Given $G(s)=\dfrac{20}{s+1}$ and $u(t)=\sin(10t)$,

$$Y(s)=G(s)U(s)=\frac{200}{(s+1)(s^2+100)}$$

$$y(t)=\underbrace{\frac{200}{101}e^{-t}}_{y_{tr}(t)}+\underbrace{\frac{20}{\sqrt{101}}\sin(10t-1.47)}_{y_{ss}(t)}$$



y(t) is a complete sinusoidal response.

Steady state sinusoidal response :

$$y_{ss}(t)=\frac{20}{\sqrt{101}}\sin(10t-1.47)$$

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Calculating the steady state sinusoidal response based on the formula : $y_{ss}(t) = AM_{\omega}\sin(\omega t + \phi_{\omega})$

$$M_{\omega} = \left|G(j\omega)\right|_{\omega=10} = \left|\frac{20}{10j+1}\right| = \frac{20}{\sqrt{101}}$$

$$\phi_{\omega} = \angle G(j\omega)\Big|_{\omega=10} = -\tan^{-1}(10) = -1.47\,\text{rad}$$

$$y_{ss}(t) = U_0\left|G(j\omega)\right|\sin\left(\omega t + \angle G(j\omega)\right); \quad U_0 = 1$$

$$= \frac{20}{\sqrt{101}}\sin\left(10t - 1.47\right)$$

Same as previous slide

This is a much easier way to calculate the <u>steady state response</u> to a sinusoidal input. Should never use the LT approach unless the full response (including transient) is required.

What is the significance of this formula?

$$A\sin(\omega t) \longrightarrow \boxed{G(s)} \longrightarrow AM_\omega \sin\left(\omega t + \phi_\omega\right)$$

- Simple formula relating input to output via the transfer function G(s)
- Recall M_ω = magnitude of G(jω), and ϕ_ω = phase of G(jω) $\quad G(j\omega) = M_\omega e^{j\phi_\omega}$
- Arbitrary input signals can be decomposed into sinusoids. For example, a periodic square wave can be decomposed into a sum of sinusoids, non-periodic signals can be decomposed using Fourier Transform
- As a result of superposition, outputs of LTI systems can also be obtained as a sum of individual sinusoidal responses

$$u(t) = \sum A_i \sin \omega_i t \longrightarrow \boxed{G(s)} \longrightarrow y(t) = \sum A_i M_{\omega_i} \sin\left(\omega_i t + \phi_{\omega_i}\right)$$

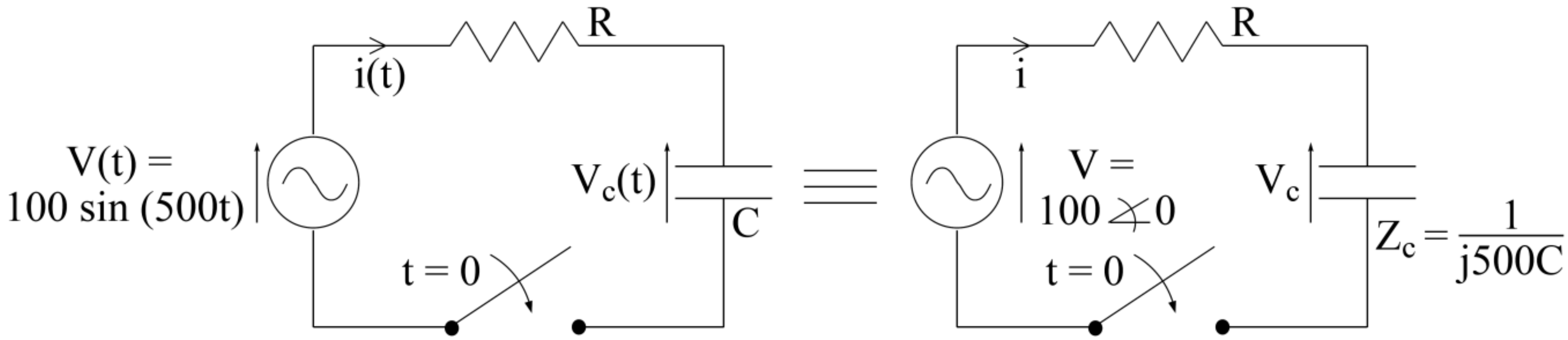
- Important quantity is $G(j\omega) = M_\omega e^{j\phi_\omega}$
- How is $G(j\omega)$ related to $G(s)$?

$$\text{Impulse Response } g(t) \xrightarrow{\text{LT}} \text{Transfer function } G(s) \xrightarrow{s=j\omega} \text{Frequency Response } G(j\omega)=M_\omega e^{j\phi\omega}$$

- While G(s) is referred to as a transfer function of an LTI system, $G(j\omega)$ is called the frequency response of the system.
- $G(j\omega)$ is the quantity that is most often used in amplifier designs, control designs, filters, etc as the quantities M_ω and ϕ_ω provide direct measurements of how sinusoidal inputs are altered as it goes thru the system. Hence quality of amplifiers and filters are explained in terms of these quantities
- M_ω is also referred to as the gain of the system. Hence gain is only defined in the frequency domain.

- Example: Transfer function of capacitor and inductor can be converted into impedances by substituting s with $j\omega$

	Transfer function	Impedance
Capacitor	$\frac{V(s)}{I(s)} = \frac{1}{sC}$	$Z_c = \frac{1}{j\omega C}$
Inductor	$\frac{V(s)}{I(s)} = sL$	$Z_L = j\omega L$



Let R = 75Ω and C = 40μF and using voltage division,

$$V_c = \frac{Z_c}{R + Z_c} V = \frac{1/j\omega C}{R + 1/j\omega C} V = \frac{1}{j\omega RC + 1} V = G(j\omega)V$$

$$\frac{V_c(s)}{V(s)} = G(s) = \frac{1}{sRC + 1}$$

$$G(j\omega) = \frac{1}{j0.003\omega + 1}$$

Thus for a voltage source of V(t)=100sin500t, A=100, ω=500 rad/s

$$G(j500) = \frac{1}{j0.003(500)+1} = \frac{1}{j1.5+1} = 0.5547\angle -56.3^0$$

At steady-state ($t\to\infty$), $V_c(t)$ = 55.47sin(500t – 56.3°)

If the supply voltage source is changed to 50 sin 100t,
then it is easy to recompute the steady state response :

$$G(j500) = \frac{1}{j0.003(100)+1} = \frac{1}{j0.3+1} = 0.958\angle -16.7^0$$

At steady-state ($t\to\infty$), $V_c(t)$ = 47.89 sin(100t – 16.7°)

This example shows that G(jω) is important in computing how the system responds to sinusoidal inputs : sinusoidal inputs are scaled accordingly by the gain of G(jω) and phased shifted by its phase

Frequency Response Plots

Frequency response is basically the data derived from $G(j\omega)$ for ω from 0 to infinity. Each $G(j\omega)$ is a complex number :

$$G(j\omega) = M_\omega e^{j\phi_\omega}$$

$$G(j\omega) = |G(j\omega)| \angle G(j\omega)$$

$$M_\omega = |G(j\omega)| > 0$$

$$\phi_\omega = \angle G(j\omega)$$

Hence a plot of $G(j\omega)$ is possible if we plot the complex value for each frequency, ω. Such a plot is called a POLAR PLOT.

However, a more common plot is to separately plot

$$\left.\begin{array}{l} \blacksquare\ |G(j\omega)| \text{ against frequency} \\ \blacksquare\ \angle G(j\omega) \text{ against frequency} \end{array}\right\}$$

Two plots together form the BODE PLOT/DIAGRAMS

Demonstrate the Bode plots ~~first~~ only.

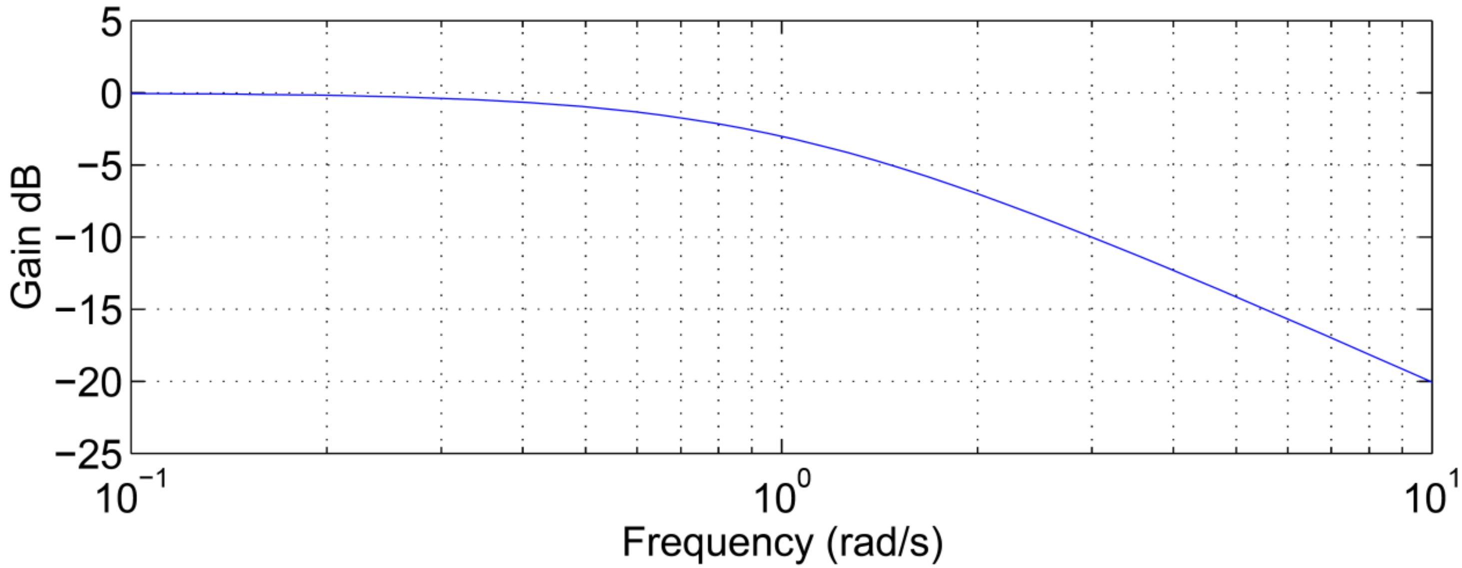
Bode Plot

Consists of two graphs

- Plot of the magnitude of $G(j\omega)$, $|G(j\omega)| = M_\omega$, in decibel (dB) versus the logarithm of the frequency, ω

$$\boxed{\text{Magnitude response: } 20\log_{10}|G(j\omega)|\,\text{dB vs } \log_{10}\omega}$$

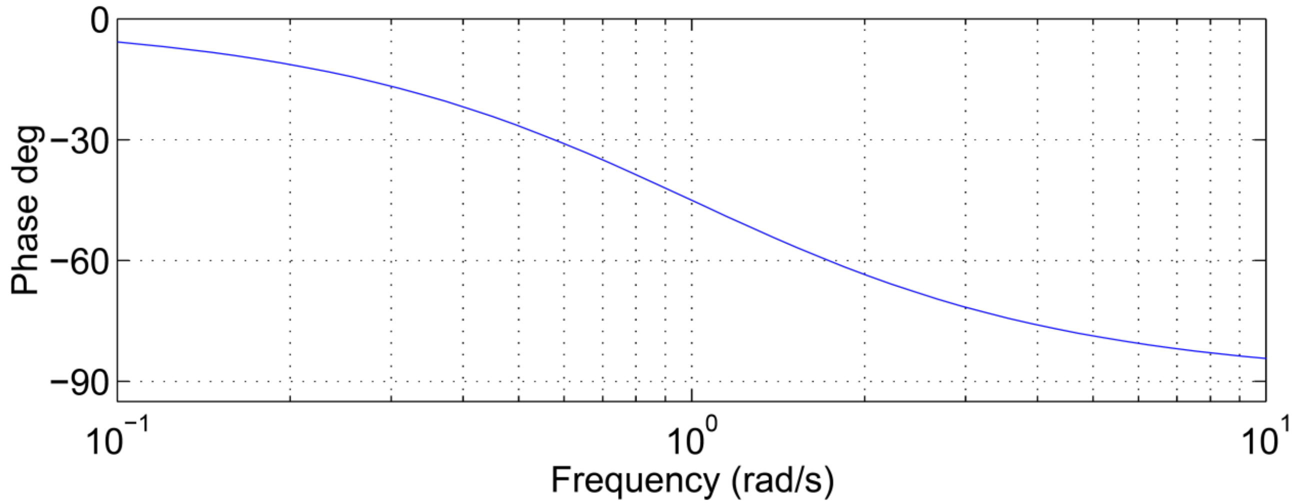
Example : Magnitude plot



- Plot of the phase of $G(j\omega)$, $\angle G(j\omega) = \phi_\omega$, vs the logarithm of the frequency, ω

$$\boxed{\text{Phase response: } \angle G(j\omega) \text{ vs } \log_{10}\omega}$$

Example : Phase plot



- y-axis of magnitude response is 20 $\log_{10} |G(j\omega)|$ dB so that the logarithmic property, $\log_{10} ab = \log_{10}a + \log_{10}b$, can be used to convert multiplication of terms to addition

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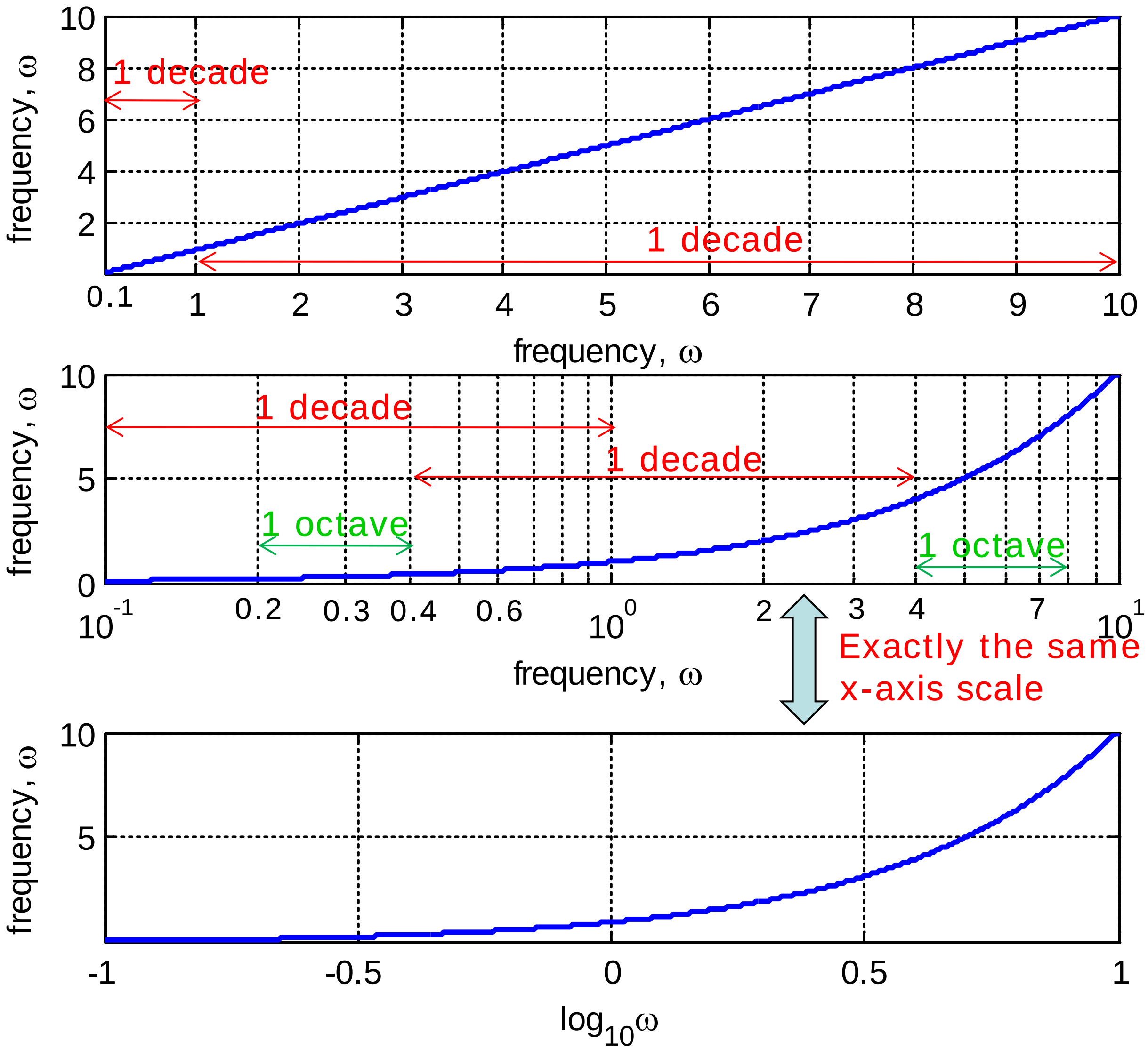
- By definition, dB = $10\log_{10}\frac{P}{P_0}$ is a logarithmic unit that describes the ratio of power, P, over a reference P_0. As power is proportional to magnitude squared and taking the reference as 1, definition for dB becomes

$$\text{dB} = 10\log_{10}\frac{|G(j\omega)|^2}{1} = 20\log_{10}|G(j\omega)|$$

- Contribution of individual factors in the system transfer function can be determined
- Transfer function can be estimated from Bode plots or diagrams
- Characteristics at low frequencies are more important in practical systems
- x-axis of Bode diagrams is logarithmic scale because logarithmic function magnifies small values

Illustration of how log scale changes the look of the plot



ω vs ω – linear plot
Interval bet 1 decade is non-uniform

ω vs $\log_{10}\omega$ with x-axis labeled in ω
Interval bet 1 decade is now uniform

ω vs $\log_{10}\omega$ with x-axis labeled in $\log_{10}\omega$
This labeling is not very useful because you cannot read frequency easily.

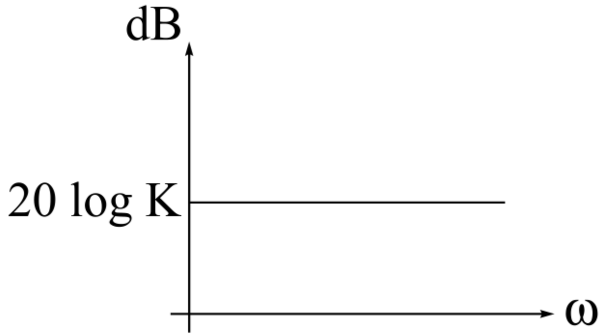
Bode Plots of Common Transfer Functions

Static Gain, *K*

- Static/Steady-State/D.C. Gain is the ratio of output signal verses input signal at steady-state or at ω=0 (~~See page 4-22~~).

$$K = \lim_{s\to 0} G(s)$$

- If $G(s)$=K, then the gain of the TF for all ω is a constant value of *K*.
- Bode diagram
  - Magnitude response is a horizontal line at $20\log_{10} K$ dB
  - Phase angle is a horizontal line at 0º (0 radians)



Magnitude Response

Integrator and Differentiator, $s^{\pm 1}$

- Magnitude of an integrator/differentiator $G(s)=s^{\pm 1}=j\omega$ is

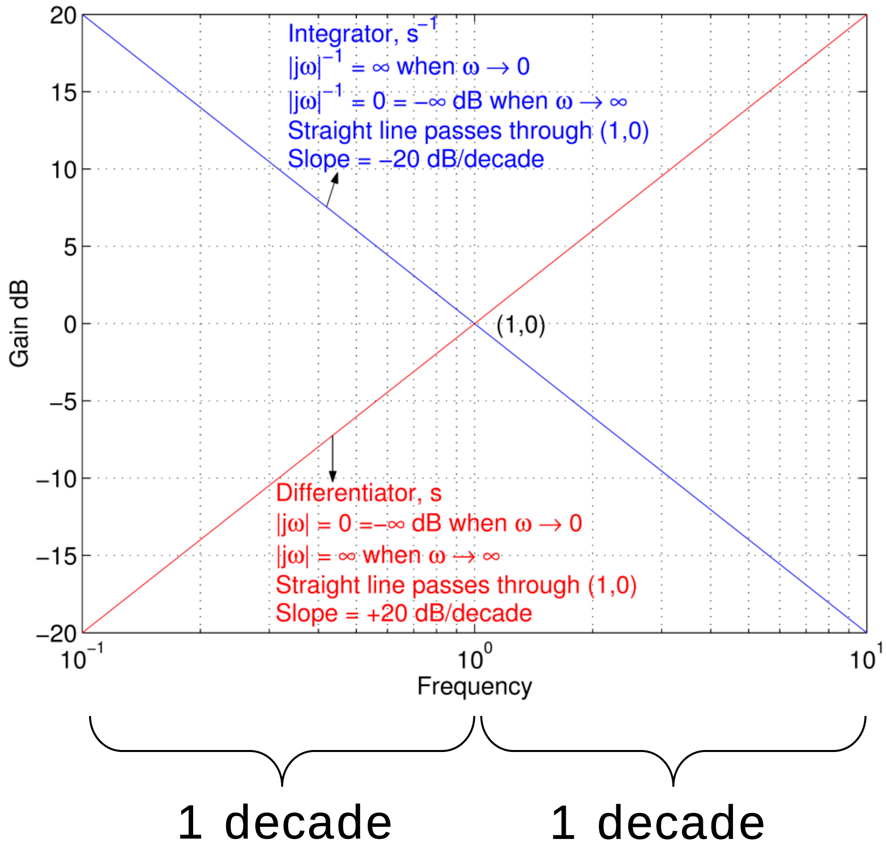
$$\left|j\omega^{\pm 1}\right| = \omega^{\pm 1} = \pm 20\log_{10}\omega\ \mathrm{dB}$$

- Phase of an integrator/differentiator $s^{\pm 1}$ is $\angle j\omega^{\pm 1} = \pm 90^\circ$

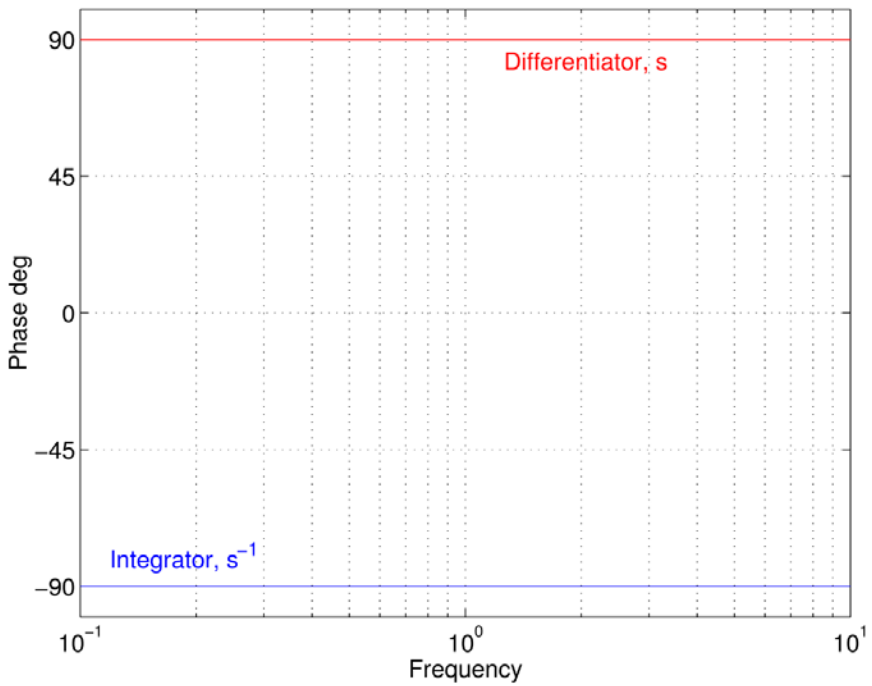
- Bode diagram (Magnitude response with x-axis in $\log_{10}\omega$ scale)
  - The magnitude response may be written as,

    $$y(x) = \pm 20x \quad \text{where} \quad x = \log_{10}\omega$$

  - The magnitude response is a straight line with a slope of ± 20 dB/decade
  - A decade is a separation in frequencies which are 10 times apart
  - Magnitude response equals 0 dB when ω = 1



- Bode diagram (Magnitude response)
  - straight line with slope ± 20dB/dec



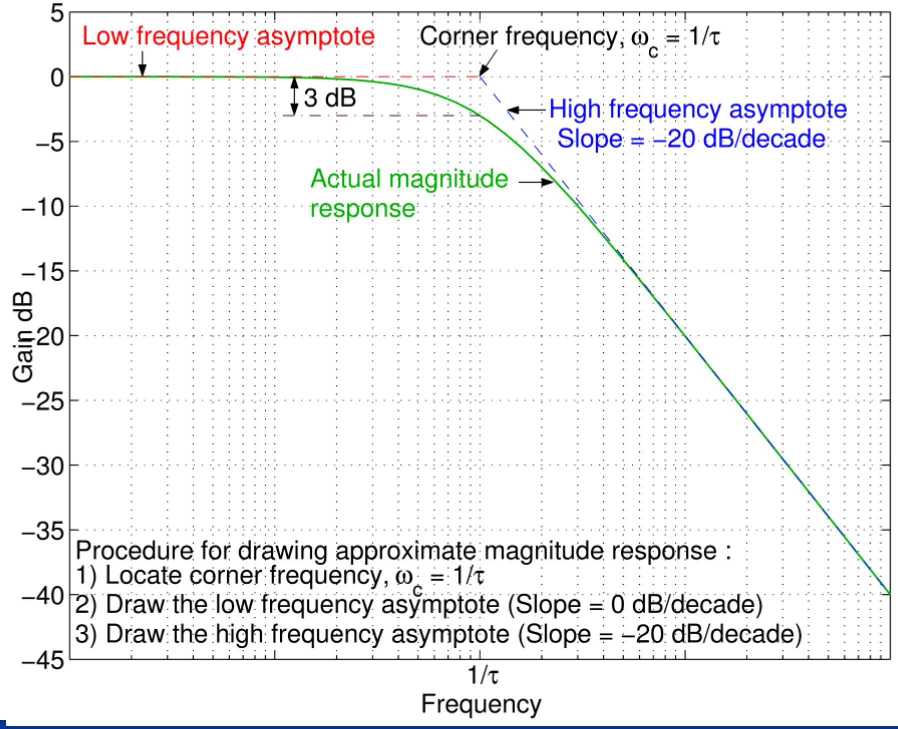
- Bode diagram (Phase response)
  - Horizontal line at ± 90°

First Order Factor, $(s\tau + 1)^{\pm 1}$

- Magnitude of $\dfrac{1}{1+j\omega\tau}$ is $\left|\dfrac{1}{1+j\omega\tau}\right| = \dfrac{1}{\sqrt{1+\omega^2\tau^2}} = -20\log_{10}\sqrt{1+\omega^2\tau^2}$ dB

- Phase of $\dfrac{1}{1+j\omega\tau}$ is $\phi = \angle\left(\dfrac{1}{1+j\omega\tau}\right) = -\tan^{-1}\omega\tau$ degrees or radians

- Bode diagram (Magnitude response)



- When $\omega << \dfrac{1}{\tau}$, $\quad -20\log_{10}\sqrt{1+\omega^2\tau^2} \approx -20\log_{10}1 = 0\ \text{dB}$

  $\Rightarrow$ Low frequency asymptote is the 0 dB line.

1/τ is also the magnitude of the pole

- When $\omega >> \dfrac{1}{\tau}$,

  $\Rightarrow$ High frequency asymptote is a straight line with a slope of –20 dB/decade.

$$-20\log_{10}\sqrt{1+\omega^2\tau^2} \approx -20\log_{10}\omega\tau\ \text{dB}$$
$$= -20\log_{10}\omega - \cancel{20\log_{10}\tau}$$

ignore since ω>>1/τ

- The two asymptotes meet at $\omega_c = \dfrac{1}{\tau}$, the corner/break frequency
  - Corner frequency is equal to the magnitude of the system pole
  - Maximum error occurs at the corner frequency and is equal to

$$-20\log_{10}\sqrt{1+\left(\frac{1}{\tau}\right)^2\tau^2} = -20\log_{20}\sqrt{2} \approx -3\ \text{dB}$$

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$$G(s) = \frac{K}{s\tau + 1}$$

- Procedure for sketching approximate magnitude response for $\frac{K}{\tau s+1}$
  - Check that first order factor is of the correct form. If not, convert the term into the correct form, e.g.,

$$\frac{K}{s+a} = \frac{\frac{K}{a}}{\frac{1}{a}s+1}$$

  - Locate the corner frequency, $\omega_c = \frac{1}{\tau}$
  - Draw the low frequency asymptote (Slope = 0 dB) with value = 20 $\log_{10}$ K dB
  - Draw the high frequency asymptote (Slope = –20 dB) at corner frequency

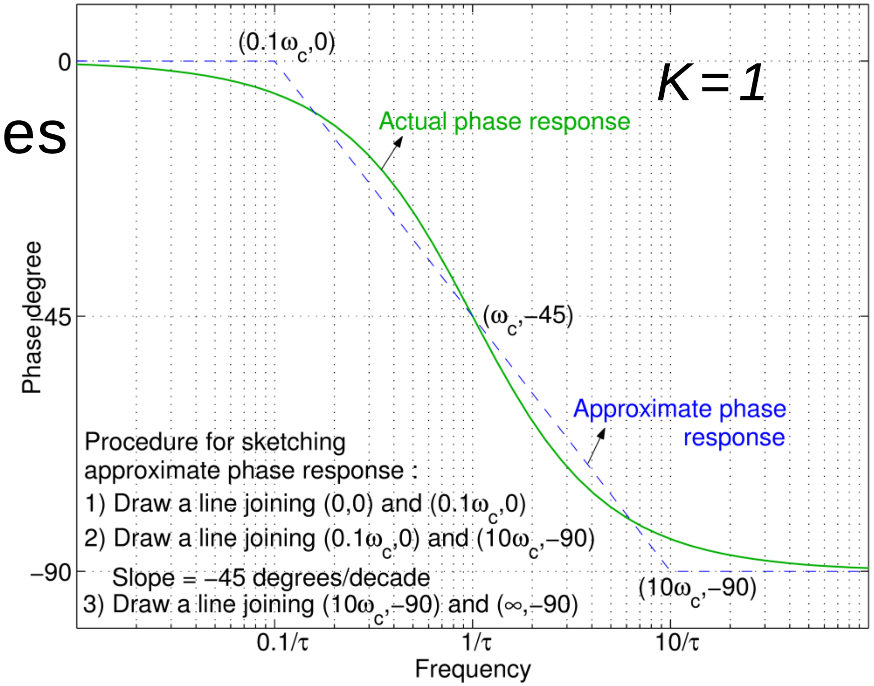
- Bode diagram (Phase response)

$$\phi = \angle\left(\frac{K}{1 + j\omega\tau}\right) = -\tan^{-1}\omega\tau \quad \text{degrees}$$

$$\text{At} \quad \omega = 0, \phi = 0^\circ$$

$$\text{At} \quad \omega = \omega_c = \tfrac{1}{\tau}, \phi = -45^\circ$$

$$\text{At} \quad \omega = \infty, \phi = -90^\circ$$



- Phase response approximated by 3 straight lines
  - Two horizontal lines at 0º and −90º
  - A straight line with slope of −45º/decade passing through $\left(\frac{1}{\tau}, -45^\circ\right)$