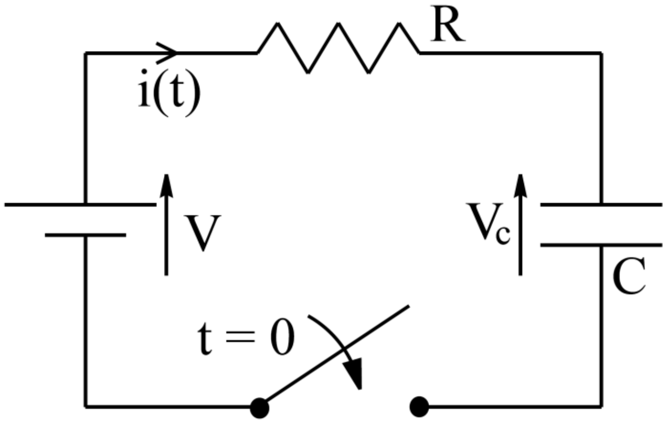
Use of Laplace Transform in Solving D.E.

- Use Laplace Transform to derive the output, $V_c(t)$, if $V(t) = V$ (constant voltage) and the initial condition is $v_c(0)>0$.



- Differential equation describing the RC circuit is

$$RC\frac{dV_c(t)}{dt}+V_c(t)=V$$

- Applying Laplace Transform to both sides of the equation,

$$sRCV_c(s)-RCv_c(0)+V_c(s)=\frac{V}{s}$$

$$V_c(s)=\frac{RCv_c(0)}{sRC+1}+\frac{V}{s(sRC+1)}$$

- Output response in time domain :

$$V_c(t) = \mathcal{L}^{-1}\left\{\frac{RCv_c(0)}{sRC+1} + \frac{V}{s(sRC+1)}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{v_c(0)}{s+\frac{1}{RC}}\right\} + \mathcal{L}^{-1}\left\{\frac{V}{s} - \frac{VRC}{sRC+1}\right\}$$

$$= v_c(0)e^{-\frac{1}{RC}t} + V - Ve^{-\frac{1}{RC}t}$$

$$= \left[v_c(0) - V\right]e^{-\frac{1}{RC}t} + V$$

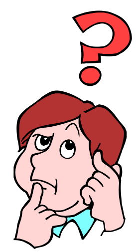
$$\boxed{\text{At } t=\infty,\ \lim_{t\to\infty} V_c(t) = V}$$

- Applying Final Value Theorem (FVT), without solving for $V_c(t)$:

$$\lim_{t\to\infty} V_c(t) = \lim_{s\to 0} sV_c(s)$$

$$= \lim_{s\to 0} s\left\{\frac{RCv_c(0)}{sRC+1} + \frac{V}{s\left(sRC+1\right)}\right\}$$

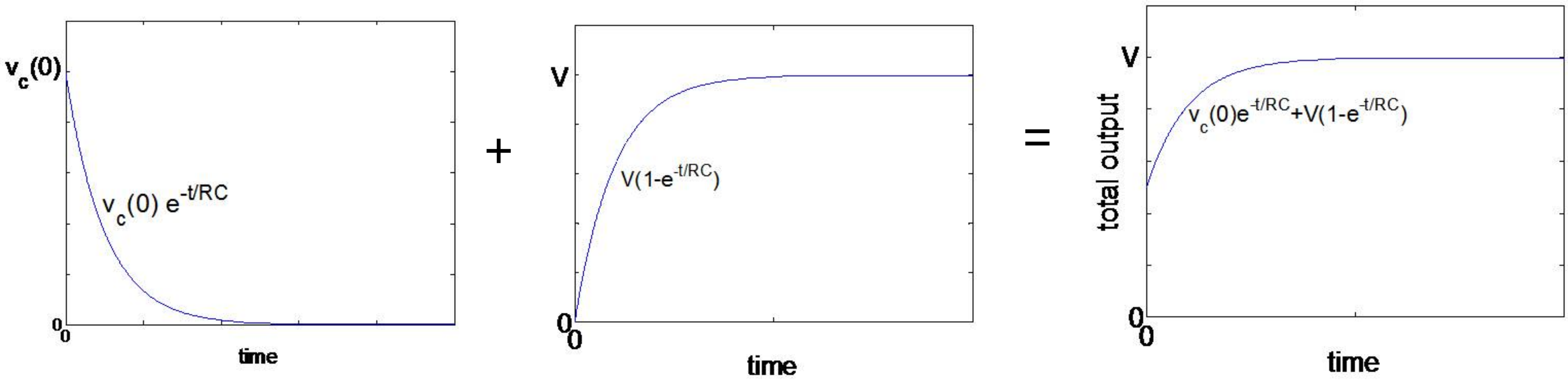
$$= V$$



How else can you find this value in time domain?

$$V_c(t) = v_c(0)e^{-\frac{t}{RC}} + V - Ve^{-\frac{t}{RC}}$$

$$= v_c(0)e^{-\frac{t}{RC}} + V(1 - e^{-\frac{t}{RC}})$$

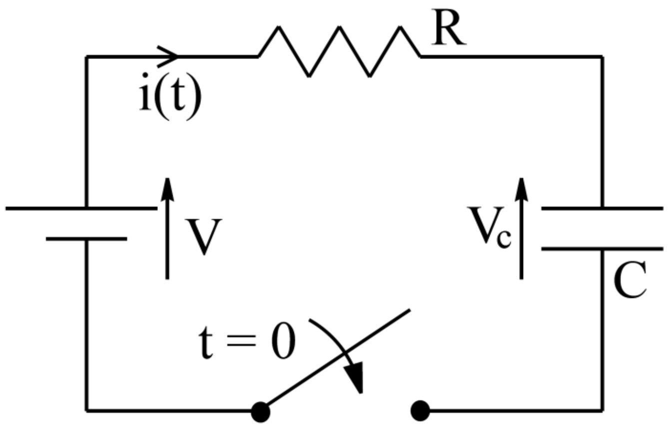


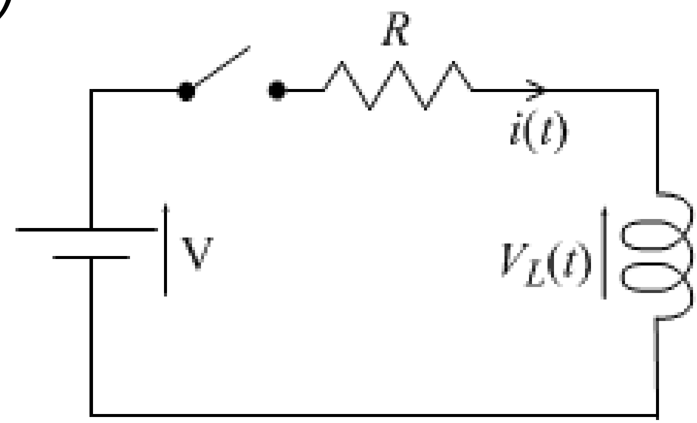
This is a typical behaviour of an RC circuit with a DC input voltage.

Final voltage (final value) across the capacitor is *V* which is the same as the input voltage ie the capacitor is charged up to the same voltage as the input supply.

Question : What happens if $v_c(0)>V$?

Another example : RL circuit with i.c. *i(0)*





$$V_L = L\frac{di(t)}{dt}$$

$$L\frac{di(t)}{dt} + iR = V$$

$$RC\frac{dV_c(t)}{dt} + V_c(t) = V \quad \Longleftrightarrow \quad \frac{L}{R}\frac{di(t)}{dt} + i = \frac{V}{R}$$

$$V_c(s) = \frac{RCv_c(0)}{sRC+1} + \frac{V}{s(sRC+1)} \quad \Longleftrightarrow \quad I(s) = \frac{\frac{L}{R}i(0)}{s\frac{L}{R}+1} + \frac{\frac{V}{R}}{s(s\frac{L}{R}+1)}$$

RC circuit

$$V_c(t) = v_c(0)e^{-\frac{1}{RC}t} + V(1 - e^{-\frac{1}{RC}t})$$

Final or steady state value of the voltage across the *C* is V.

In the RC circuit, the energy is stored in the form of charges in the *C* or voltage across the *C*.

RL circuit

$$i(t) = i(0)e^{-\frac{R}{L}t} + \frac{V}{R}(1 - e^{-\frac{R}{L}t})$$

Final or steady state value of the current in the *L* is *V/R*.

In the RL circuit, the energy is stored in the form of magnetic field in the *L*.

There are some parameters in the equation that will come in handy to determine the response of the circuits. Will see this in the next lesson. Stay tuned!

Use of Laplace Transform to Derive the Transfer Function Models of LTI Systems

- LT can be used to solve time domain differential equations
- LT can also be used to derive alternative models for LTI systems (as mentioned in slide 6-22). These are called transfer functions.
- How is this achieved?

General input-output relationship of LTI system is represented by :

Time domain model : $\xrightarrow{u(t)}$ System $\xrightarrow{y(t)}$

System equation in time domain :

$$a_n \frac{d^n y(t)}{dt^n} + a_{n-1} \frac{d^{n-1} y(t)}{dt^{n-1}} + \ldots + a_0 y(t) = b_m \frac{d^m u(t)}{dt^m} + b_{m-1} \frac{d^{m-1} u(t)}{dt^{m-1}} + \ldots + b_0 u(t)$$

Taking Laplace Transform on both sides :

$$\mathcal{L}\left\{a_n \frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + \ldots + a_0 y(t)\right\} = \mathcal{L}\left\{b_m \frac{d^m u(t)}{dt^m} + b_{m-1}\frac{d^{m-1}u(t)}{dt^{m-1}} + \ldots + b_0 u(t)\right\}$$

0 ie zero i.c.

$$\mathcal{L}\left\{\frac{d^k y(t)}{dt^k}\right\} = s^k Y(s) - \sum_{i=0}^{k-1} s^{k-1-i}\cancel{\frac{d^i y(0^-)}{dt^i}}$$

$$= s^k Y(s) \quad \text{for } k = 1,\ldots,n$$

$$\mathcal{L}\left\{\frac{d^k u(t)}{dt^k}\right\} = s^k U(s) \text{ for } k = 1,\ldots,m$$

$$\left(a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0\right)Y(s) = \left(b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0\right)U(s)$$

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

Recall that a_k and b_k are constants for LTI systems and $m<n$

s-domain model :

$$\xrightarrow{U(s)} \boxed{G(s) = \frac{Y(s)}{U(s)}} \xrightarrow{Y(s)}$$

$$G(s) = \frac{\mathcal{L}\{y(t)\}}{\mathcal{L}\{u(t)\}} = \frac{Y(s)}{U(s)} \qquad \text{i.e., } Y(s) = G(s)U(s)$$

Completely algebraic relation between input and output

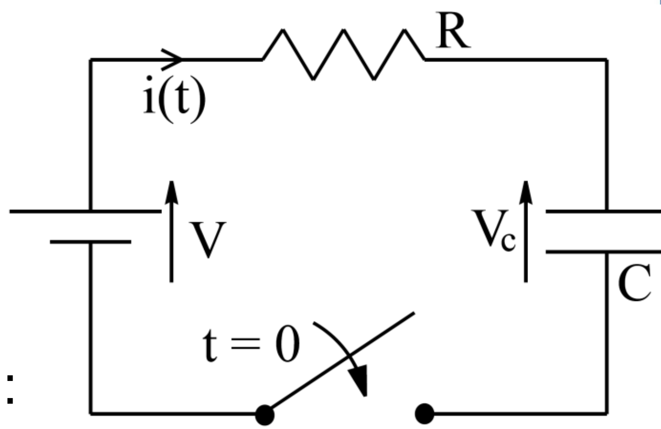
- s-domain model is also called a transfer function model.
- Transfer functions are the ratio of the Laplace transform of the output signal, $y(t)$, over the Laplace transform of the input, $u(t)$
- Assumption is that all initial conditions are zero.
- Transfer functions model the input-output relationship of a system.
- The s-variable is a complex quantity. When $s=j\omega$, $G(j\omega)$ becomes a frequency domain model or the system's frequency response model where ω is the frequency in radians per sec.

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_m s^m+b_{m-1}s^{m-1}+\ldots+b_0}{a_n s^n+a_{n-1}s^{n-1}+\ldots+a_0}$$

Transfer function of a general n^{th} order system

- Numerator polynomial in the numerator is formed using coefficients of the input function and its derivative(s) in the DE
- Denominator polynomial is formed using coefficients of the output signal and its derivative(s) in the DE
- System order is n, the largest power of s in the denominator of $G(s)$
- For real physical systems, $n>m$. (n-m) is referred to as the pole-zero access
- If $n=1$, we have a first order TF, $n=2$, we have a second order TF, etc. This is similar to the order of the differential equation associated with the TF
- If $s=j\omega$, then the transfer function model is equivalent to a frequency response model of the system

Example 1 : TF of a RC Circuit



DE : $$RC\frac{dV_c(t)}{dt}+V_c(t)=V$$

Applying LT without assuming zero i.c. :

$$V_c(s)=\frac{RCv_c(0)}{sRC+1}+\frac{V}{s(sRC+1)}$$

Model in the *s* domain!

Suppose the initial condition, $v_c(0)$ = 0.

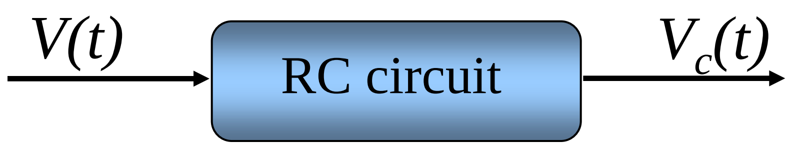
Then $$V_c(s)=\frac{V}{s(sRC+1)}$$

Recall that the input voltage is a constant value of *V* (with LT of *V*/*s*)

$$V_c(s)=\frac{V(s)}{sRC+1}$$

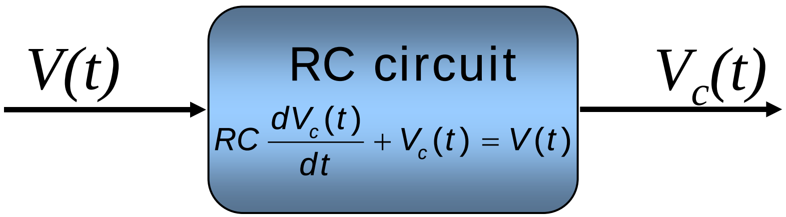
If the input voltage is a general function of *v(t)* (with LT *V(s)*)

$$\frac{\text{Output}}{\text{Input}}=\frac{V_c(s)}{V(s)}=\frac{1}{sRC+1}$$



DE disappeared!

- In time domain :

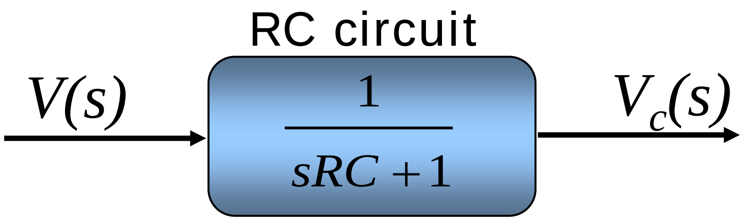


Input-output relationship in time domain :

$$RC\frac{dV_c(t)}{dt}+V_c(t)=V$$

Output $V_c(t)$ obtained by solving DE!

- In s-domain :



Input-output relationship in s-domain :

$$\frac{V_c(s)}{V(s)}=\frac{1}{sRC+1}$$

Output $V_c(t)$ obtained by solving the inverse LT of $V_c(s)$!

Important to note that this means that the DE and the TF are equivalent systems but in different domains!

Example 2 : Given the DE $a_2\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)=b_0u(t)$

Find the transfer function for the system described by the DE.

$$\mathcal{L}\left\{a_2\ddot{y}(t)+a_1\dot{y}(t)+a_0y(t)\right\}=\mathcal{L}\left\{b_0u(t)\right\}$$

$$a_2\left[s^2Y(s)-sy(0)-\dot{y}(0)\right]+a_1\left[sY(s)-y(0)\right]+a_0Y(s)=b_0U(s)$$

$$\left[a_2s^2+a_1s+a_0\right]Y(s)=b_0U(s)\qquad\left(\dot{y}(0)=y(0)=0\right)$$

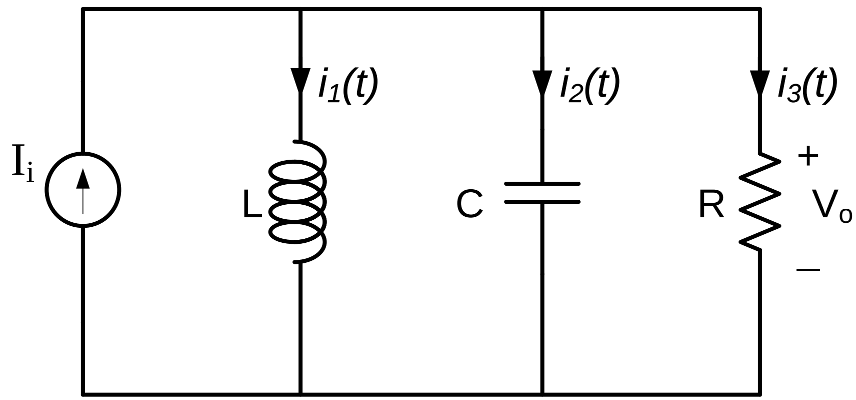
$$\text{Transfer function, }\frac{Y(s)}{U(s)}=\frac{b_0}{a_2s^2+a_1s+a_0}$$

A 2nd order DE has resulted in a 2nd order TF!

Example 3 : Transfer function of a RLC circuit

Recall from slide 6-21 that the DE for the RLC circuit is given by :

$$RL\frac{dI_i(t)}{dt} = RV_0(t) + RLC\frac{d^2V_0(t)}{dt^2} + L\frac{dV_0(t)}{dt}$$

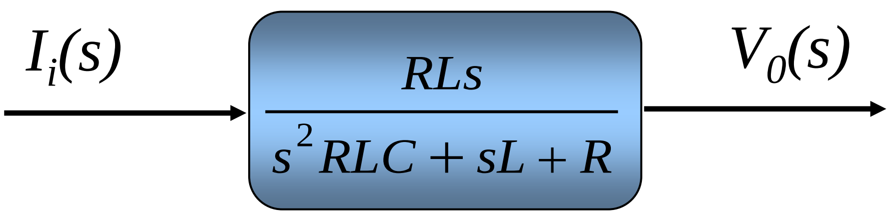


Taking LT and assuming zero i.c. :

$$RLsI_i(s) = RV_0(s) + RLCs^2V_0(s) + LsV_0(s)$$

$$\frac{V_0(s)}{I_i(s)} = \underbrace{\frac{RLs}{RLCs^2 + Ls + R}}_{\text{Transfer function of the RLC circuit}}$$

RLC circuit



Alternative method to obtain transfer functions is to use the complex impedances and applying circuit theory :

$$\text{Resistor:} \quad V(t) = Ri(t) \quad \Rightarrow \quad G(s) = \frac{V(s)}{I(s)} = R$$

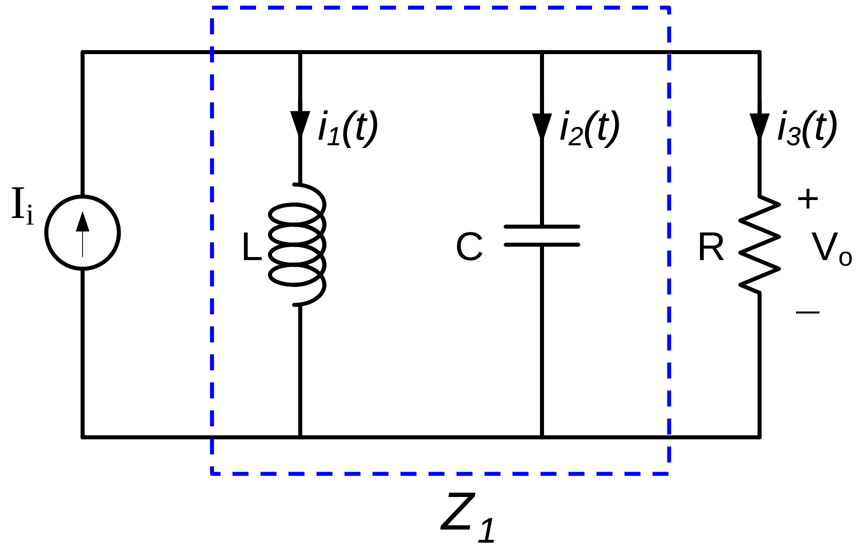
$$\text{Capacitor:} \quad V_c(t) = \frac{1}{C}\int_0^t i(\tau)d\tau \quad \Rightarrow \quad G(s) = \frac{V_c(s)}{I(s)} = \frac{1}{Cs}$$

$$\text{Inductor:} \quad V_L(t) = L\frac{di(t)}{dt} \quad \Rightarrow \quad G(s) = \frac{V_L(s)}{I(s)} = sL$$

Complex impedance

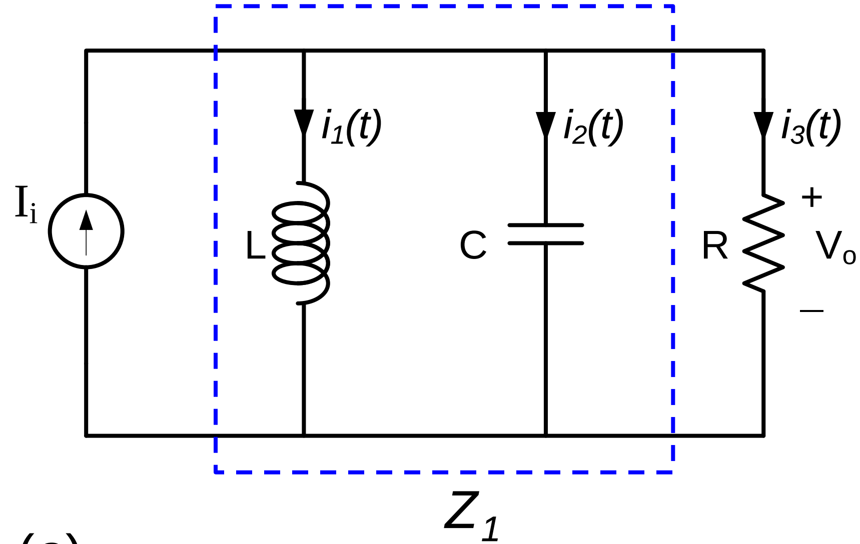
In this example, you need to use circuit principles to find :

- impedance $Z_1(s)$
- current $I_3(s)$
- output voltage $V_0(s)$



Parallel combination of L and C :

$$Z_1(s) = \frac{1}{\frac{1}{sL} + sC} = \frac{sL}{LCs^2 + 1}$$



Current division rule :

$$I_3(s) = \frac{Z_1(s)}{Z_1(s) + R} I_i(s)$$

$$= \frac{\frac{sL}{LCs^2 + 1}}{\frac{sL}{LCs^2 + 1} + R} I_i(s) = \frac{sL}{RLCs^2 + sL + R} I_i(s)$$

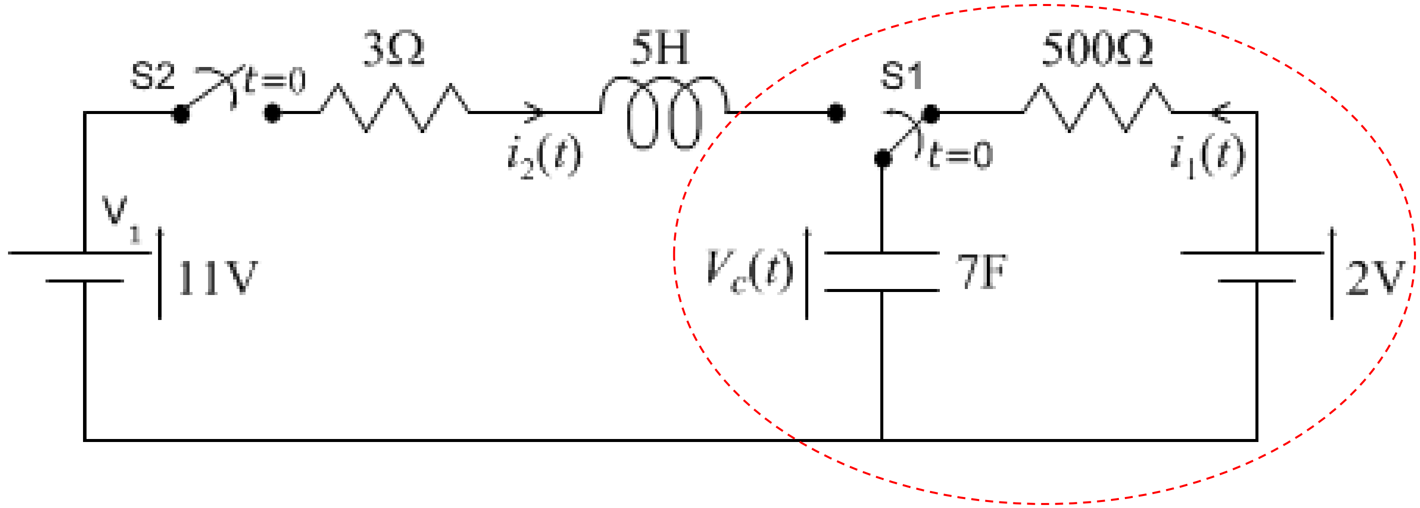
Finally :

$$V_0(s) = I_3(s)R$$

$$= \frac{sLR}{RLCs^2 + sL + R} I_i(s) \quad \Rightarrow \quad \frac{V_0(s)}{I_i(s)} = \frac{sLR}{RLCs^2 + sL + R}$$

Same tf as slide 7-13

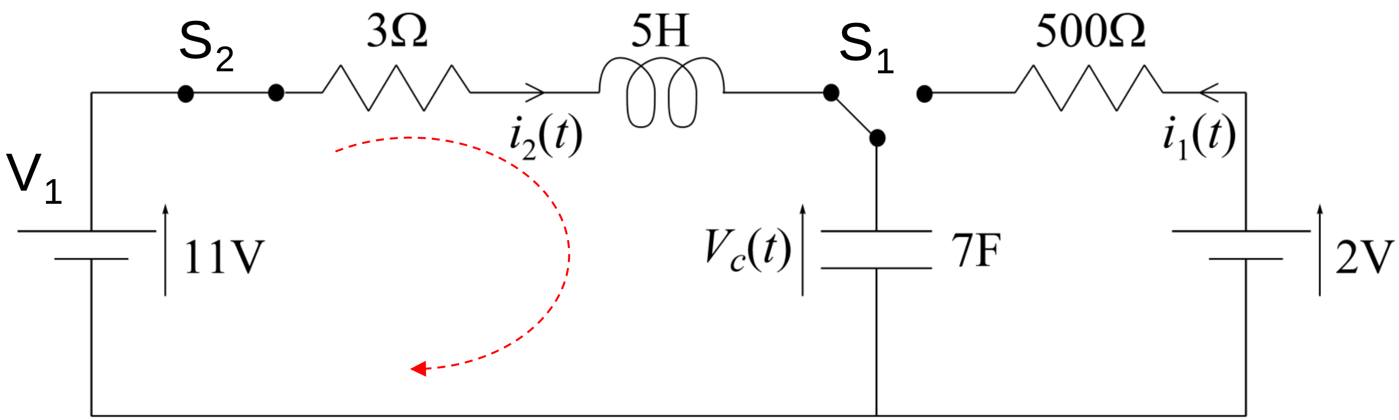
Example: Consider the series RLC circuit below. Assume that the switch *S1* was closed for a long time before $t=0$. At $t=0$, *S1* is thrown the other way while *S2* is closed. Derive the initial condition of the circuit. Hence find the resulting $V_c(t)$ when $t \geq 0$. Derive the transfer function between V_1 and V_c.



The right hand circuit is a standard RC circuit. If *S1* has been closed for a long time, capacitor will be charged to 2V. Hence the initial conditions are :

$$V_c(0) = 2\,\text{V}, \quad \frac{dV_c(t)}{dt} = 0, \quad i_1 = 0\,\text{A}, \quad \frac{di_1(t)}{dt} = \frac{di_2(t)}{dt} = 0$$

- When $t \geq 0$, circuit becomes a series RLC circuit,



KVL around the loop :
$$V_1 = 3i_2 + 5\frac{di_2}{dt} + V_c$$
$$= 3\left(7\frac{dV_c}{dt}\right) + 5\frac{d}{dt}\left(7\frac{dV_c}{dt}\right) + V_c$$
$$= 21\frac{dV_c}{dt} + 35\frac{d^2V_c}{dt^2} + V_c$$

$$i_2 = C\frac{dV_c}{dt} = 7\frac{dV_c}{dt}$$
$$V_L = L\frac{di_2}{dt} = 5\frac{di_2}{dt}$$

Taking Laplace Transform on both sides :

$$V_1(s) = 21\left(sV_c(s) - V_c(0)\right) + 35\left(s^2V_c(s) - sV_c(0) - \frac{dV_c(0)}{dt}\right) + V_c(s)$$

$$= \left(35s^2 + 21s + 1\right)V_c(s) - 21V_c(0) - 35sV_c(0) - 35\frac{dV_c(0)}{dt}$$

$$= \left(35s^2 + 21s + 1\right)V_c(s) - 42 - 70s \qquad \because V_c(0) = 2,\ \frac{dV_c(0)}{dt} = 0$$

Input V_1=11 V, therefore $V_1(s) = \frac{11}{s}$

$$\frac{11}{s} = \left(35s^2 + 21s + 1\right)V_c(s) - 42 - 70s$$

$$V_c(s) = \frac{11 + 42s + 70s^2}{s(35s^2 + 21s + 1)}$$

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$$V_c(s) = \frac{11 + 42s + 70s^2}{s(35s^2 + 21s + 1)}$$

$$= \frac{11}{s} - \frac{315s + 189}{35s^2 + 21s + 1}$$

$$= \underbrace{\frac{11}{s}}_{11} - \underbrace{\frac{9.94}{s + 0.052}}_{9.94e^{-0.052t}} + \underbrace{\frac{0.94}{s + 0.548}}_{0.94e^{-0.548t}}$$

Inverse Laplace Transform

Hence, in time domain : $V_c(t) = 11 - 9.94e^{-0.052t} + 0.94e^{-0.548t}$

At steady state, $V_c(\infty) = 11$

Verifying this using FVT, $V_c(\infty) = \lim_{s \to 0} sV_c(s) = \lim_{s \to 0} s\frac{11 + 42s + 70s^2}{s(35s^2 + 21s + 1)} = 11$

Examine the time domain solution :

$$V_c(t) = \underbrace{11}_{\text{Steady state solution}} - \underbrace{9.94e^{-0.052t} + 0.94e^{-0.548t}}_{\text{Transient solution}}$$

To find the transfer function between input V_1 and output V_c, assume all initial conditions to be zero.

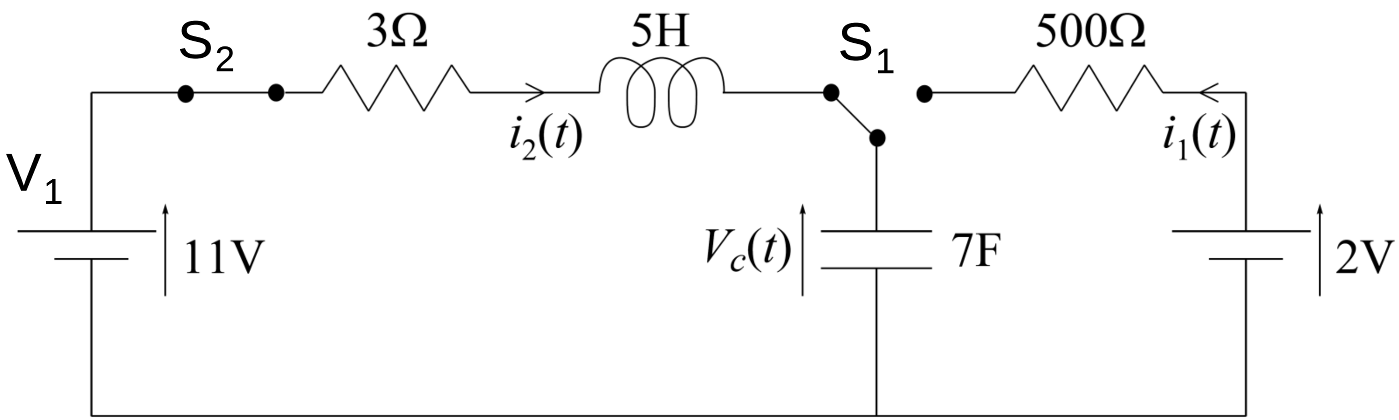
$$V_1(s) = 21\left(sV_c(s) - V_c(0)\right) + 35\left(s^2V_c(s) - sV_c(0) - \frac{dV_c(0)}{dt}\right) + V_c(s)$$

$$= \left(35s^2 + 21s + 1\right)V_c(s) \qquad \text{assuming } V_c(0) = 0,\ \frac{dV_c(0)}{dt} = 0$$

Therefore the TF is :

$$\boxed{\frac{V_c(s)}{V_1(s)} = \frac{1}{35s^2 + 21s + 1}}$$

Using the impedance approach :



Total impedance : $R + sL + \dfrac{1}{sC} = 3 + 5s + \dfrac{1}{7s}$

Using the voltage divider principle, the transfer function becomes :

$$\frac{V_c(s)}{V_1(s)} = \frac{\dfrac{1}{7s}}{3 + 5s + \dfrac{1}{7s}} = \frac{1}{35s^2 + 21s + 1}$$

Same tf as slide 7-20

Important points to note :

- Solutions of DE can be obtained using Laplace Transform
- Solutions of DE are in time domain and they show both transient and steady state parts to the total solution
- Transfer functions are obtained when Laplace Transforms are taken for DE, assuming all initial conditions to be zero.

In the next section, we show how the transfer functions carry information about the transient and steady state behaviours of the output response, without having to solve the DE!

We start with the definition of poles and zeros of a TF.

Definition of Poles and Zeros

- Consider the general TF (refer to slide 7-7) expression:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \ldots + b_0}{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}$$

$$= K\frac{(\frac{s}{z_1}+1)(\frac{s}{z_2}+1)\ldots(\frac{s}{z_m}+1)}{(\frac{s}{p_1}+1)(\frac{s}{p_2}+1)\ldots(\frac{s}{p_n}+1)}; \qquad K = \frac{b_0}{a_0}$$

- **Poles**
  - $\{-p_1, -p_2, \ldots, -p_n\}$ are the roots of the denominator polynomial and are called poles of the transfer function, *G(s)*.
  - Poles are also the values of *s* for which *G(s)* goes to infinity.
  - Poles can have real or complex values.
  - There are a total of *n* poles for an n^{th} order transfer function

- **Zeros**
  - $\{-z_1, -z_2, \ldots, -z_m\}$ are the roots of the numerator polynomial and are called zeros of $G(s)$. They are also the values of s for which $G(s)=0$

- Transfer functions, $G(s)$, are therefore characterized by n system poles $\{-p_1, -p_2, \ldots, -p_n\}$, m zeros $\{-z_1, -z_2, \ldots, -z_m\}$ and constant K.

- Transfer functions with n poles are called TF of order n. There can be any number of zeros but for real practical system, $m<n$.

- Poles and zeros can take complex values and when they do occur as complex values, they always occur as conjugate pairs. The reason for this is because real practical systems have real coefficients in the DE.

- Poles and zeros can also be repeated ie a pole at $-p_1$ can occur twice eg when the denominator is of the form $(s/p_1+1)^2$. Likewise for the zeros of the TF.

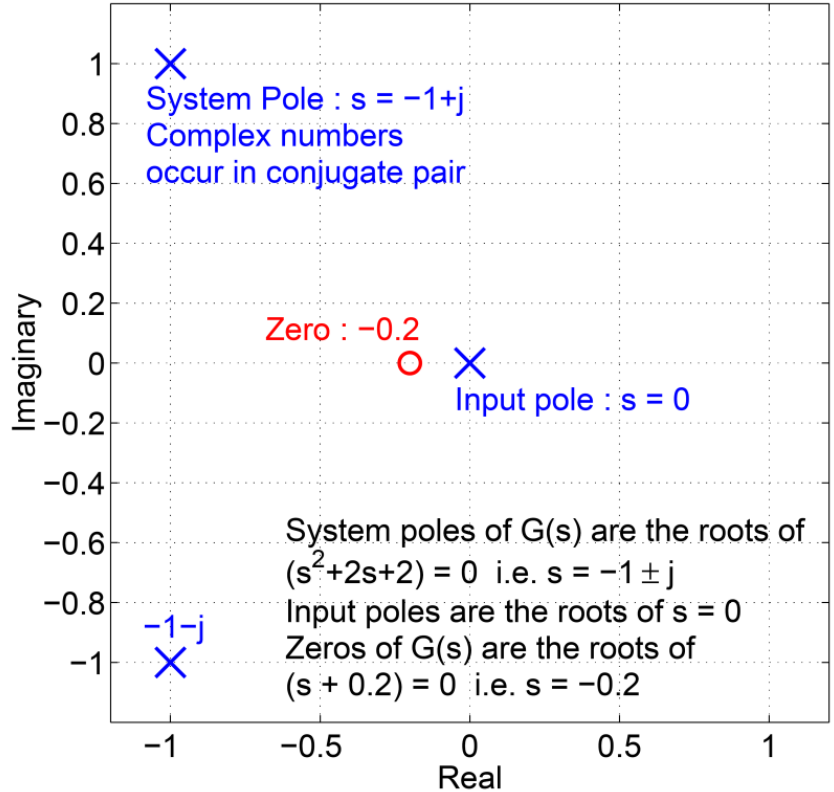
Pole-Zero Diagram

Pole-Zero diagram is a plot in the complex plane showing the locations of all the system poles (marked by "x") and zeros (marked by "o")

$$\text{Consider } G(s) = \frac{s+0.2}{s^2+2s+2}$$

$$\underbrace{\text{and } U(s) = \frac{1}{s}}$$

This gives the input pole at s=0

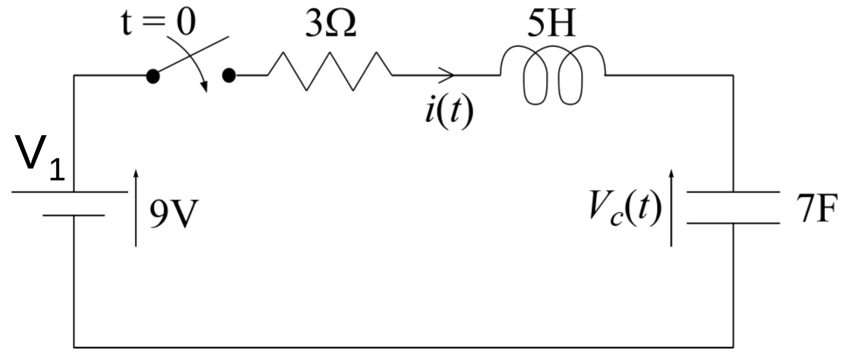


System poles are at s=-1+j, -1-j

Role of Poles – Significance of Poles

- To examine the relationship between poles and system response, consider the series RLC circuit.

  TF relating $V_c(t)$ and V_1 voltage source is (page 7-21),



$$\frac{V_c(s)}{V_1(s)} = \frac{1}{35s^2 + 21s + 1} \qquad \text{where} \quad V_1(t) = 9 \;\; \text{or} \;\; V_1(s) = \frac{9}{s}$$

- System poles are roots of $35s^2 + 21s + 1 = 0$, i.e., $s_1 = -0.052$; $s_2 = -0.548$

  Input/Excitation pole is $s = 0$.

$$V_c(s) = \frac{V_c(s)}{V(s)} \cdot V(s) = \frac{1}{35s^2 + 21s + 1} \cdot \frac{9}{s}$$

$$V_c(t) = \underbrace{-9.94e^{-0.052t} + 0.94e^{-0.548t}}_{\text{Transient}} + \underbrace{9}_{\text{Steady-state}}$$

- System poles appear in the transient response term, while input pole determine the steady-state term

Role of Zeros – Significance of zeros

- To examine the relationship between zeros and system response, assume that the transfer function of a fictitious system is

$$\frac{V_c(s)}{V_1(s)} = \frac{s+1}{35s^2+21s+1} \qquad \text{where} \quad V_1(t) = 9 \;\; \text{or} \;\; V_1(s) = \frac{9}{s}$$

- System poles are $s_1 = -0.052$; $s_2 = -0.548$

  Input/Excitation pole is $s = 0$; System zero is located at $s = -1$

$$V_c(s) = \frac{V_c(s)}{V_1(s)} \cdot V_1(s) = \frac{s+1}{35s^2+21s+1} \cdot \frac{9}{s}$$

$$= -\frac{9.428}{s+0.052} + \frac{0.428}{s+0.548} + \frac{9}{s}$$

$$V_c(t) = \underbrace{-9.428e^{-0.052t} + 0.428e^{-0.548t}}_{\text{Transient}} + \underbrace{9}_{\text{Steady-state}}$$

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$$V_c(t) = \underbrace{-9.428\boxed{e^{-0.052t}} + 0.428\boxed{e^{-0.548t}}}_{\text{Transient}} + \underbrace{9}_{\text{Steady-state}}$$

System with a zero

$$V_c(s) = \frac{V_c(s)}{V(s)} \cdot V(s) = \frac{1}{35s^2+21s+1} \cdot \frac{9}{s}$$

$$V_c(t) = \underbrace{-9.94\boxed{e^{-0.052t}} + 0.94\boxed{e^{-0.548t}}}_{\text{Transient}} + \underbrace{9}_{\text{Steady-state}}$$

System without a zero

Modes of response are the same ie the exponential terms are the same in the two systems. Modes come from the poles of the system! Coefficients in the transient response are different because of the zero.

- From the example, it may be deduced that
  - The steady-state response is related to input pole(s)
  - Each term (elementary signal) in the transient response corresponds to a real pole or a pair of complex conjugate system poles
  - Modifying the numerator of $G(s)$, and hence the system zeros, does not change the elementary signals that appear in the transient response. Only the coefficients of the individual terms are changed
  - The type/characteristics of the elementary signal(s) that constitutes a system's transient response is specified by the location(s) of the system pole(s) on the complex plane

Relationship between transient response and system pole locations

- **Poles at the origin**

  $G(s)$ contains the factor $\dfrac{1}{s^n}$ $\Rightarrow$ $y_{tr}(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s^n}\right\} = \dfrac{t^{n-1}}{(n-1)!}U(t)$

  $y_{tr}(t)$ is therefore a ~~constant~~ transient response which does not decay to zero.

- **Real and distinct poles** ($s = \pm a$)

  $G(s)$ contains the factor $\dfrac{1}{s \mp a}$ $\Rightarrow$ $y_{tr}(t) = \mathcal{L}^{-1}\left\{\dfrac{1}{s \mp a}\right\} = e^{\pm at}$

  $y_{tr}(t)$ is therefore an exponentially decaying or growing transient response depending on the sign of a.

- **Purely imaginary poles** ($s = \pm j\omega$)

  $G(s)$ contains the factor

$$\frac{1}{(s+j\omega)(s-j\omega)} \Longrightarrow y_{tr}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin\omega t$$

  $y_{tr}(t)$ is therefore a sinusoidal transient response which does not go to zero as time goes to infinity. It gives sustained oscillation.

- **Complex conjugate poles** ($s = \pm a \pm j\omega$)

$$\frac{1}{(s\mp a+j\omega)(s\mp a-j\omega)} \Longrightarrow y_{tr}(t) = \mathcal{L}^{-1}\left\{\frac{1}{s^2\mp 2as+a^2+\omega^2}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{1}{(s\mp a)^2+\omega^2}\right\} = \frac{1}{\omega}e^{\pm at}\sin\omega t$$

  $y_{tr}(t)$ is therefore a exponentially decaying or growing sinusoidal transient response depending on the sign of a.

Conclusion on Stability of System

- To determine whether a system is BIBO stable, it is sufficient to examine the natural response, $y_{tr}(t)$ which is determined by the system poles.

- A system is said to be :

➢ **Stable** if the magnitude of its transient response, $|y_{tr}(t)|$, converges to zero as $t \rightarrow \infty$. This requires all system poles to be negative.

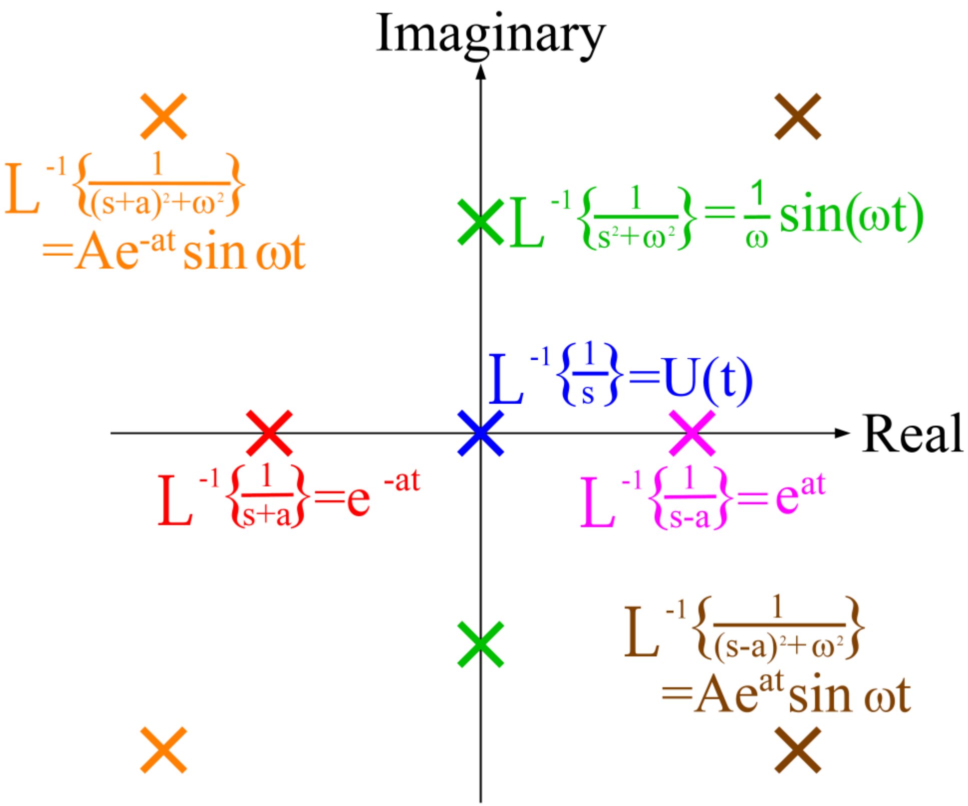
➢ **Unstable** if the magnitude of its transient response, $|y_{tr}(t)|$, grows without bound as $t \rightarrow \infty$. This happens if at least one system pole is +ve

➢ **Marginally Stable** if its transient response, $y_{tr}(t)$, tends to a non-zero constant or oscillates with constant amplitude (a sinuoidal function) as $t \rightarrow \infty$. This happens when system poles are purely imaginary. This includes the origin.

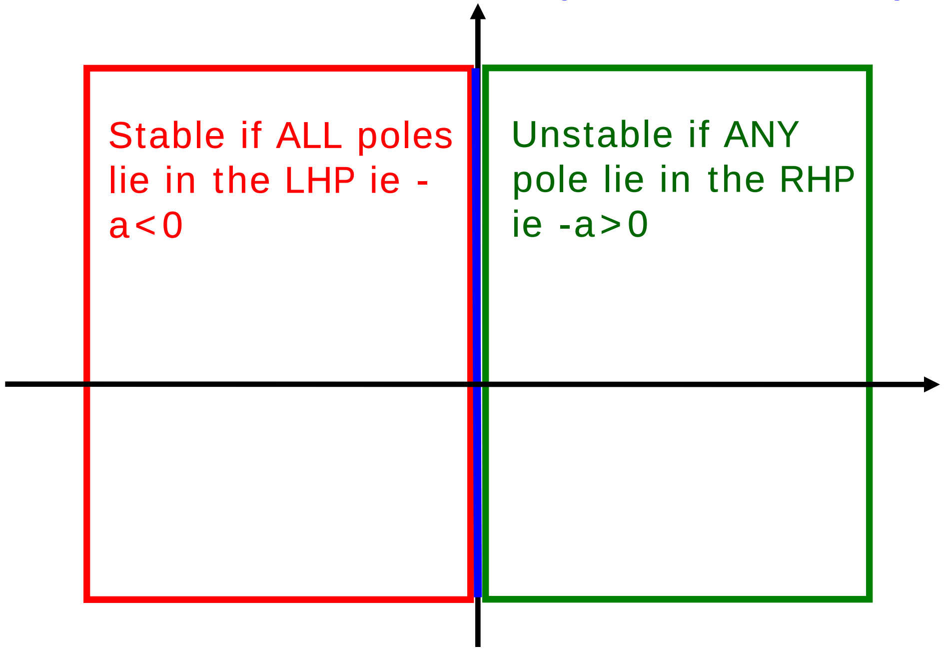
Summary on Pole Location and Transient Response + Stability

- System poles tells us about the form of the transient response of a system. Hence it also tells us about stability. Assume a>0

Pole Locations and Transient Response



Pole Locations and System Stability



Marginally stable if ANY non-repeated poles lie on the imaginary axis including the origin

Example :

$$u(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

Suppose the system has TF : $G(s) = \dfrac{1}{s^2+4}$

Poles of the $G(s)$ is at $s = \pm j2$ ⟸ Poles are purely imaginary!

Suppose $u(t)=1$, therefore $U(s) = \dfrac{1}{s}$

According to the definition of a TF, $Y(s)=G(s)U(s)$. Therefore

$$Y(s) = \frac{1}{s^2+4}\frac{1}{s}$$
$$= 0.25\frac{1}{s} - 0.25\frac{s}{s^2+4}$$

$$Y(s) = \frac{1}{s^2+4}\frac{1}{s}$$

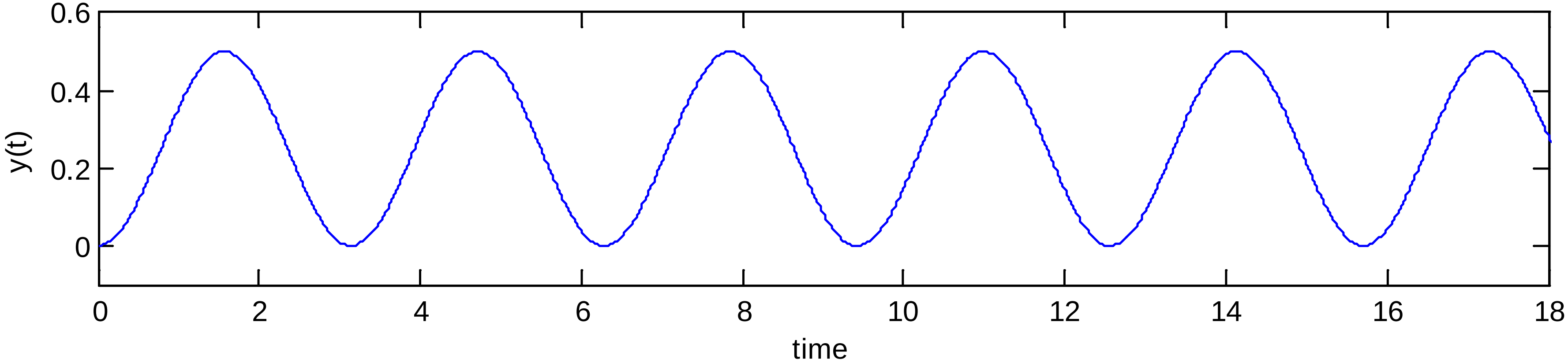
$$= 0.25\frac{1}{s} - 0.25\frac{s}{s^2+4}$$

$$0.25 \qquad 0.25\cos 2t$$

LT pair :

$$\frac{s}{s^2+\omega^2} \Leftrightarrow \cos\omega t$$

Time domain response : $y(t) = 0.25 - 0.25\cos 2t$



Oscillatory response even though the input is a constant step input!

Frequency of oscillation = value of the system pole on the imaginary axis!

What have you learnt in this lesson?

- Poles and zeros are properties of the transfer function and hence they are system properties
- Poles determine system response; poles in the RHP give unstable response.
- Poles in LHP gives bounded response if the input is bounded.
- Real poles give exponential response.
- complex poles give damped exponential response.
- Imaginary poles give oscillatory response.

What next?

- Look at more fundamental system properties arising from the transfer functions.