EE2023 TUTORIAL 1 (SOLUTIONS)

Solution to Q.1

Write z in polar form:

$$z = x + jy = |z|\exp(j\angle z).$$

Since adding integer multiples of 2π to $\angle z$ does not affect the value of z, we may also express z as

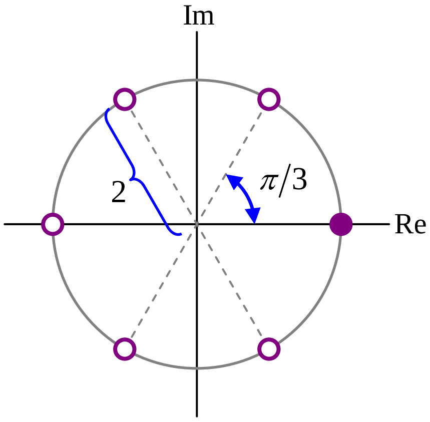
$$z = |z|\exp\left(j(\angle z + 2k\pi)\right)$$

where k is an integer. This leads to

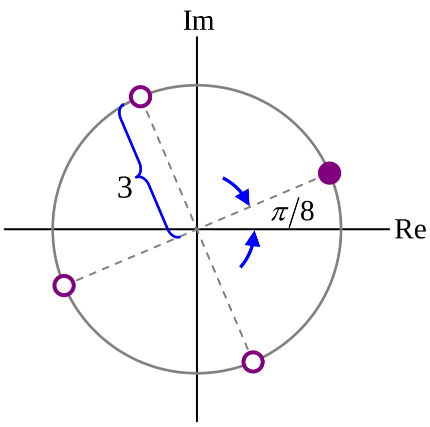
$$z^{1/N} = |z|^{1/N}\exp\left(j\left(\frac{\angle z}{N} + \frac{2k\pi}{N}\right)\right); \quad k = 0,1,\cdots,N-1,$$

which yields the N distinct values of $z^{1/N}$.

$$64^{1/6}\textbf{:}\quad \begin{cases} z = 64 \;\rightarrow\; \begin{cases} |z| = 64 \\ \angle z = 0 \end{cases} \\ 64^{1/6} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=64,\ N=6} \\ \quad = 2\exp\left(j\left(\dfrac{k\pi}{3}\right)\right); \quad k = 0,1,\cdots,5 \\ \quad = \begin{cases} 2;\ 2\exp\left(j\left(\dfrac{\pi}{3}\right)\right);\ 2\exp\left(j\left(\dfrac{2\pi}{3}\right)\right); \\ -2;\ 2\exp\left(j\left(\dfrac{4\pi}{3}\right)\right);\ 2\exp\left(j\left(\dfrac{5\pi}{3}\right)\right) \end{cases} \end{cases}$$



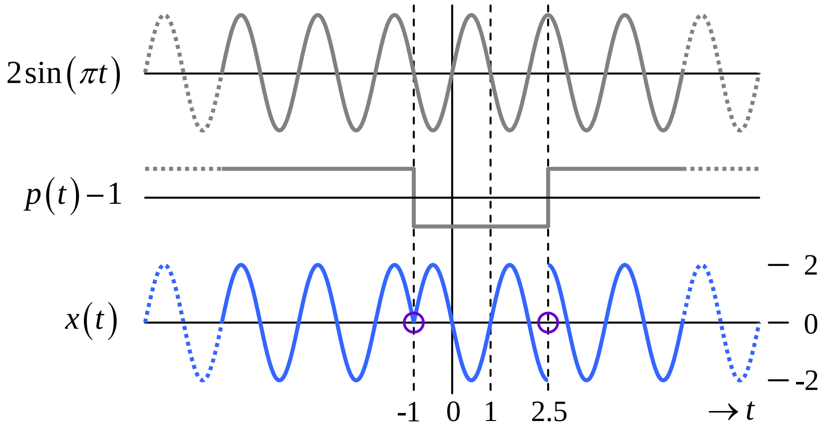
$$(j81)^{1/4}\textbf{:}\quad \begin{cases} z = j81 \;\rightarrow\; \begin{cases} |z| = 81 \\ \angle z = \dfrac{\pi}{2} \end{cases} \\ (j81)^{1/4} = |z|^{1/N}\exp\left(j\left(\dfrac{\angle z}{N} + \dfrac{2k\pi}{N}\right)\right)\Bigg|_{z=81,\ N=4} \\ \quad = 3\exp\left(j\left(\dfrac{\pi}{8} + \dfrac{k\pi}{2}\right)\right); \quad k = 0,1,\cdots,3 \\ \quad = \begin{cases} 3\exp\left(j\left(\dfrac{\pi}{8}\right)\right),\ 3\exp\left(j\left(\dfrac{5\pi}{8}\right)\right), \\ 3\exp\left(j\left(\dfrac{9\pi}{8}\right)\right),\ 3\exp\left(j\left(\dfrac{13\pi}{8}\right)\right) \end{cases} \end{cases}$$



Solution to Q.2

(a) $p(t) = 2 - 2\text{rect}\left(\dfrac{t-0.75}{3.5}\right)$

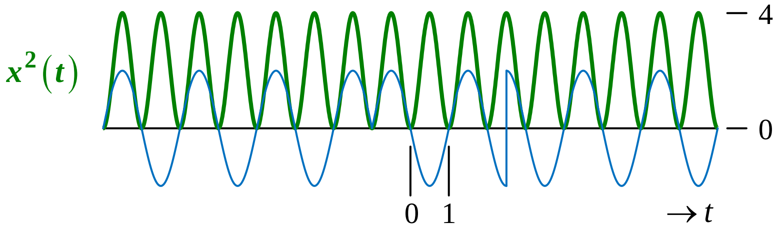
(b) By inspection, $x(t)$ is not periodic.



Notice the π rad $(\text{or } 180°)$ phase jumps in $x(t)$ occurring at the zero crossings of $p(t)-1$.

(c)

$$\begin{aligned} x^2(t) &= 4\sin^2(\pi t)\underbrace{\left(p(t)-1\right)^2}_{1} \\ &= 4\sin^2(\pi t) \\ &= 2\left(1-\cos(2\pi t)\right) \end{aligned}$$



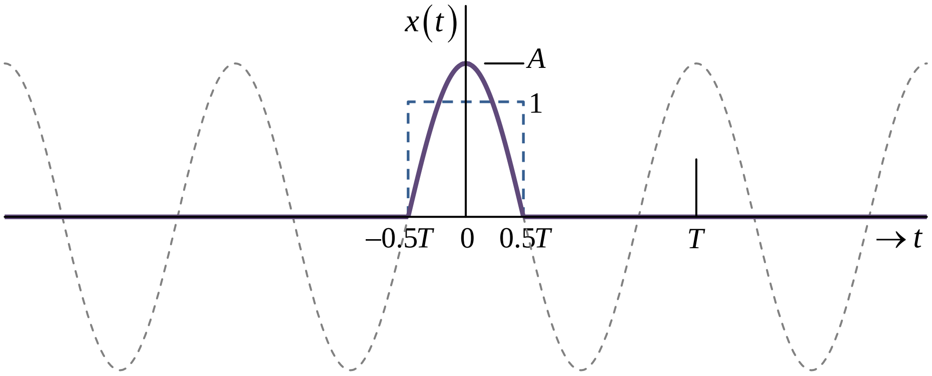
Note that $x^2(t)$ is periodic with a period of $T = 1$.

Average Power: $\left\{ P = \underbrace{\dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} x^2(t)\,dt}_{\substack{x^2(t) \text{ is periodic. } \therefore \\ P \text{ can be obtained} \\ \text{by averaging over} \\ \text{one period.}}} = \displaystyle\int_{-0.5}^{0.5} 2\left(1-\cos(2\pi t)\right)dt = 2 \right.$

(d) Since the average power of $x(t)$ is finite, its total energy must be infinite. $x(t)$ is an aperiodic power signal.

Solution to Q.3

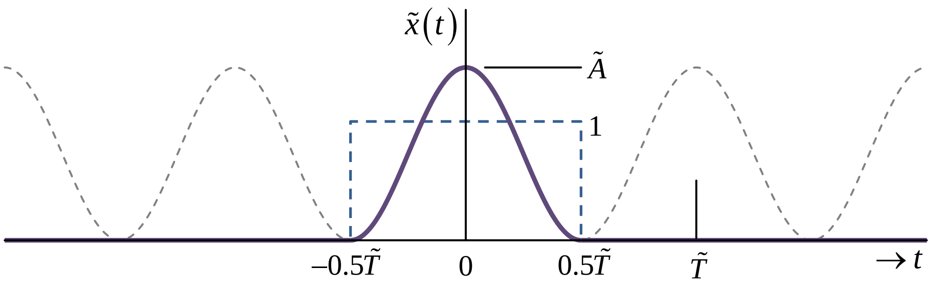
Half-cosine pulse: $x(t) = A\cos\left(\frac{\pi t}{T}\right)\mathrm{rect}\left(\frac{t}{T}\right)$



$$x^2(t) = \frac{A^2}{2}\left[1+\cos\left(\frac{2\pi t}{T}\right)\right]\mathrm{rect}\left(\frac{t}{T}\right)$$

$$\textbf{Energy :}\ E = \frac{A^2}{2}\int_{-0.5T}^{0.5T} 1+\underbrace{\cos\left(\frac{2\pi t}{T}\right)}_{\int \text{over one period} = 0} dt = \frac{1}{2}A^2T$$

Raised-cosine pulse: $\tilde{x}(t) = \frac{\tilde{A}}{2}\left(1+\cos\left(\frac{2\pi t}{\tilde{T}}\right)\right)\mathrm{rect}\left(\frac{t}{\tilde{T}}\right)$



$$\tilde{x}^2(t) = \frac{\tilde{A}^2}{4}\left[\frac{3}{2}+2\cos\left(\frac{2\pi t}{\tilde{T}}\right)+\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)\right]\mathrm{rect}\left(\frac{t}{\tilde{T}}\right)$$

$$\textbf{Energy :}\ \tilde{E} = \frac{\tilde{A}^2}{4}\int_{-0.5\tilde{T}}^{0.5\tilde{T}} \frac{3}{2}+\underbrace{2\cos\left(\frac{2\pi t}{\tilde{T}}\right)}_{\int \text{over one period} = 0}+\underbrace{\frac{1}{2}\cos\left(\frac{4\pi t}{\tilde{T}}\right)}_{\int \text{over two periods} = 0} dt = \frac{3}{8}\tilde{A}^2\tilde{T}$$

Both $x(t)$ and $\tilde{x}(t)$ will have the same energy if $A^2T = \frac{3}{4}\tilde{A}^2\tilde{T}$.

Solution to Q.4

(a) $x(t)=\cos(3.2t)+\sin(1.6t)+\exp(j2.8t) \quad \cdots \quad \begin{cases} \cos(3.2t) & \text{has a frequency of } 3.2 \ rad/s \\ \sin(1.6t) & \text{has a frequency of } 1.6 \ rad/s \\ \exp(j2.8t) & \text{has a frequency of } 2.8 \ rad/s \end{cases}$

Highest common factor (HCF) of $\{3.2,\ 1.6,\ 2.8\}$ exists and is equal to 0.4. Thus, $x(t)$ is periodic. Since there is no harmonic cancellation among the sinusoidal components of $x(t)$, the fundamental frequency and period of $x(t)$ are $0.4\ rad/s$ $(\text{or } 0.2/\pi\ Hz)$ and $5\pi\ s$, respectively.

REMARKS: *Although $x(t)$ is periodic with a fundamental frequency of $0.4\ rad/s$, it does not contain the fundamental frequency component itself.*

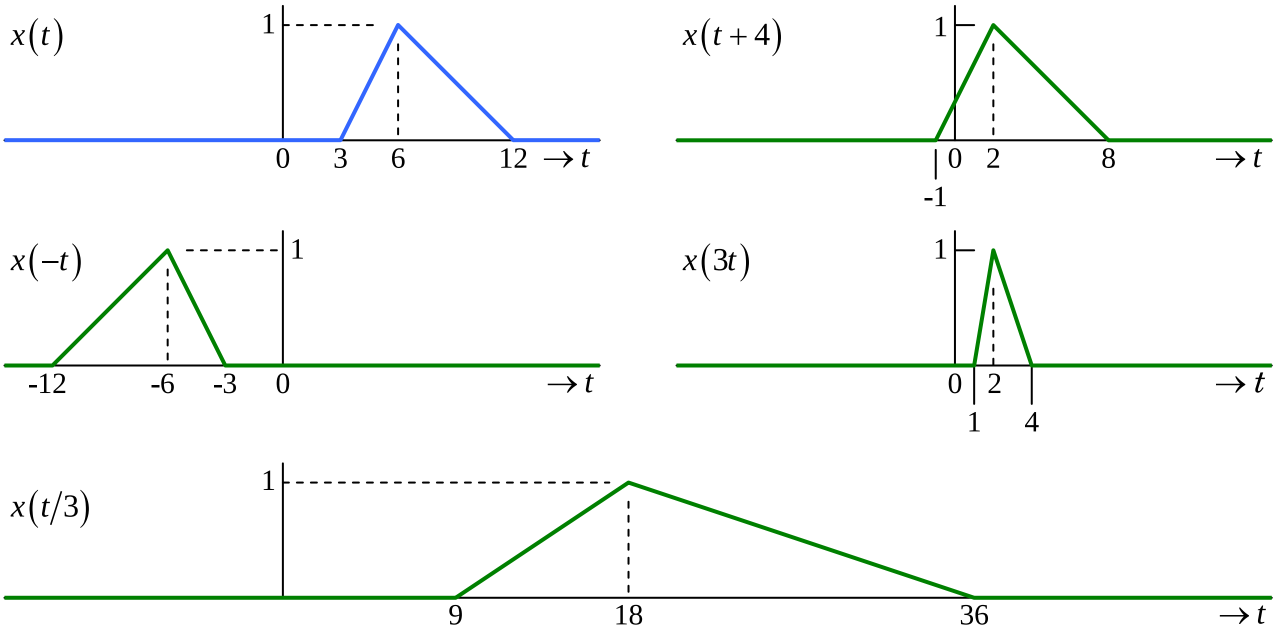
(b) $x(t)=\cos(4t)+\sin(\pi t) \quad \cdots \quad \begin{cases} \cos(4t) & \text{has a frequency of } 4 \ rad/s \\ \sin(\pi t) & \text{has a frequency of } \pi \ rad/s \end{cases}$

Highest common factor (HCF) of $\{4,\ \pi\}$ does not exist. Thus, $x(t)$ is not periodic.

REMARKS: *Summing sinusoids does not necessarily lead to a periodic signal unless the frequencies of the sinusoids are harmonics of a common fundamental frequency.*

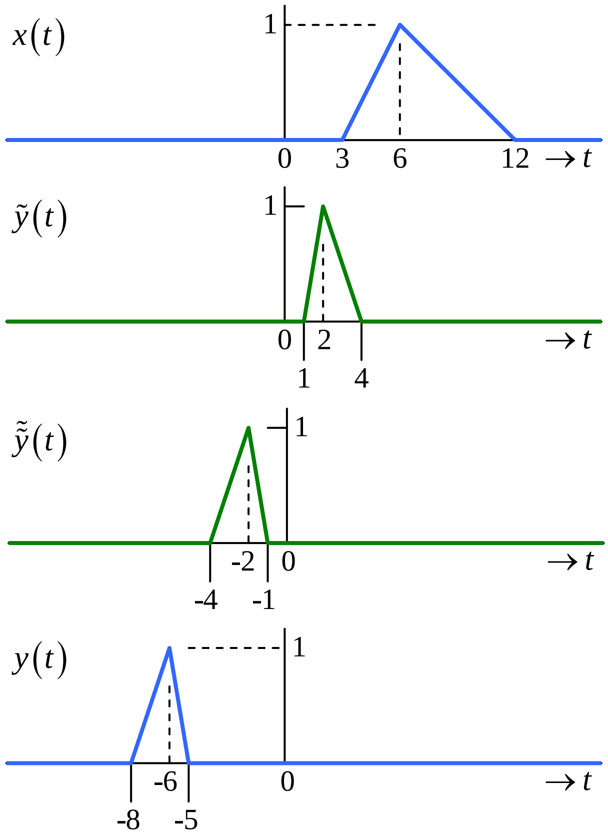
Solution to Q.5

(a)



(b) We observe that $y(t)$ is a time-scaled, -reversed and –shifted version of $x(t)$.

For problems of this nature, we should start with time-scaling first since it involves linear warping of the time axis. If we were to start with time-shifting and/or time-reversal, we may have to redo them after time-scaling. However, this sequence of operation need not be followed if we are sketching the signal from the mathematical expression.



Comparing $x(t)$ and $y(t)$, we note that $y(t)$ involves time-scaling (or contraction) of $x(t)$ by a factor of 3.

Time-scaling of $x(t)$: $\tilde{y}(t) = x(3t)$

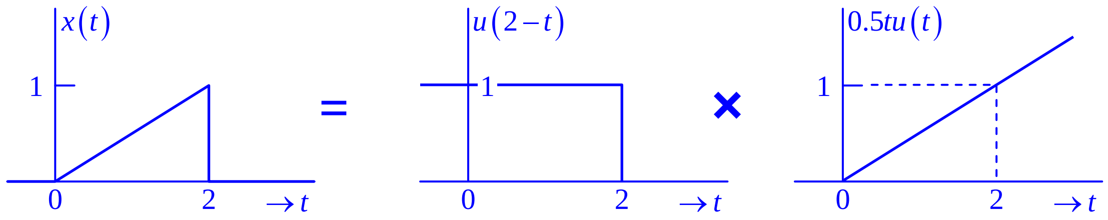
Time-reversal of $\tilde{y}(t)$: $\tilde{\tilde{y}}(t) = \tilde{y}(-t) = x(-3t)$

Time shifting of $\tilde{\tilde{y}}(t)$: $\begin{cases} y(t) = \tilde{\tilde{y}}(t+4) \\ \quad\;\; = x(-3(t+4)) \end{cases}$

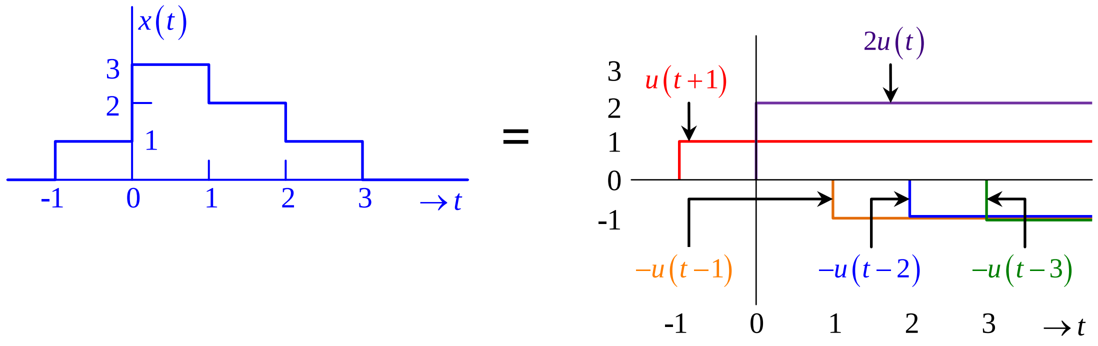
$\therefore\ y(t) = x(-3t-12)$

Solution to S.1

(a) $x(t) = u(2-t)\cdot 0.5tu(t) = u(2-t)\cdot\int_{-\infty}^{t} 0.5u(\tau)\,d\tau$



(b) $x(t) = u(t+1) + 2u(t) - u(t-1) - u(t-2) - u(t-3)$



Solution to S.2

(a) $x(t) = \cos(2t + 0.25\pi) = \cos\left(2\pi\frac{1}{\pi}t + 0.25\pi\right)$ is a sinusoid of amplitude 1 and frequency $\frac{1}{\pi}$.

$\rightarrow$ *periodic*, *period* $= \pi$, *power* $= 1/2$

(b) $x(t) = \cos^2(t) = 0.5[1 + \cos(2t)] = 0.5 + 0.5\cos\left(2\pi\frac{1}{\pi}t\right)$ is a sinusoid of amplitude 0.5 and frequency $\frac{1}{\pi}$, plus a 0.5 dc value.

$\rightarrow$ *periodic*, *period* $= \pi$, *power* $= \frac{0.5^2}{2} + 0.5^2 = \frac{3}{8}$

(c) $x(t) = \cos(2\pi t)u(t)$ does not satisfy $x(t) = x(t+T)\ \forall t$ where T is a finite constant.

$\rightarrow$ *non-periodic*

(d) $x(t) = \exp(j\pi t) = \exp\left(j2\pi\frac{1}{2}t\right)$ is a complex sinusoid of amplitude 1 and frequency $\frac{1}{2}$.

$\rightarrow$ *periodic*, *period* $= 2$, *power* $= 1$

Solution to S.3

(a) When $t<0$: $\int_{-\infty}^{t}\cos(\tau)u(\tau)d\tau = 0$

When $t\geq 0$: $\int_{-\infty}^{t}\cos(\tau)u(\tau)d\tau = \int_{0}^{t}\cos(\tau)d\tau = \sin(\tau)\big|_{0}^{t} = \sin(t)$

Combining the 2 cases: $\int_{-\infty}^{t}\cos(\tau)u(\tau)d\tau = \sin(t)u(t)$

(b) When $t<0$: $\int_{-\infty}^{t}\cos(\tau)\delta(\tau)d\tau = 0$

When $t\geq 0$: $\int_{-\infty}^{t}\cos(\tau)\delta(\tau)d\tau = 1$

Combining the 2 cases: $\int_{-\infty}^{t}\cos(\tau)\delta(\tau)d\tau = u(t)$

(c) $\int_{-\infty}^{\infty}\cos(t)u(t-1)\delta(t)dt = 0$ because $u(t-1)\delta(t) = 0\ \forall t$

(d) $\underbrace{\int_{0}^{2\pi} t\sin\left(\frac{t}{2}\right)\delta(\pi-t)dt = \pi\sin\left(\frac{\pi}{2}\right)}_{\text{sifting property of }\delta\text{-function}} = \pi$

Solution to S.4

(a) $x(t) = u(t) = \begin{cases} 1; & t\geq 0 \\ 0; & t<0 \end{cases}$

$$x_e(t) = 0.5\left[u(t)+u(-t)\right] = \begin{cases} 1; & t=0 \\ 0.5; & t\neq 0 \end{cases}$$

$$x_o(t) = 0.5\left[u(t)-u(-t)\right] = \begin{cases} 0; & t=0 \\ 0.5; & t>0 \\ -0.5; & t<0 \end{cases}$$

(b) $x(t) = \sin\left(\omega_c t + \frac{\pi}{4}\right)$

$$x_e(t) = 0.5\left[\sin\left(\omega_c t+\frac{\pi}{4}\right)+\sin\left(-\omega_c t+\frac{\pi}{4}\right)\right] = \sin\left(\frac{\pi}{4}\right)\cos(\omega_c t) = \frac{1}{\sqrt{2}}\cos(\omega_c t)$$

$$x_o(t) = 0.5\left[\sin\left(\omega_c t+\frac{\pi}{4}\right)-\sin\left(-\omega_c t+\frac{\pi}{4}\right)\right] = 0.5\left[\sin\left(\omega_c t+\frac{\pi}{4}\right)+\sin\left(\omega_c t-\frac{\pi}{4}\right)\right] = \sin(\omega_c t)\cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}\sin(\omega_c t)$$

where we make use of the trigonometric relationship $\sin(A)+\sin(B) = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{B-A}{2}\right)$.
