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This set of notes is a summary of some of the relationships and properties related to the Fourier Series and Fourier Transform. We begin with their definitions.

Periodic Signals : The properties of a periodic signal $x(t)$ with *fundamental* period T_0 are summarized as follows.

Periodic signal, $x(t)$	
Fundamental frequency	$f_0 = \frac{1}{T_0}$ Hz or $\omega_0 = \frac{2\pi}{T_0}$ rad/s
$x(t)$ satisfies	$x(t) = x(t + T_0)$
$x(t)$ even implies	$x(t) = x(-t)$
$x(t)$ odd implies	$x(t) = -x(-t)$
Fourier Series of $x(t)$ has two forms :	
Trigonometric form :	$x(t) = a_0 + \Sigma_{k=0}^{\infty} a_k \cos 2\pi k f_0 t + \Sigma_{k=0}^{\infty} b_k \sin 2\pi k f_0 t$
	$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \cos 2\pi k f_0 t dt$
	$b_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) \sin 2\pi k f_0 t dt$
Alternatively,	$x(t) = a_0 + 2\Sigma_{k=0}^{\infty} A_k \cos(2\pi k f_0 t + \phi_k)$
	$A_k = \sqrt{a_k^2 + b_k^2}, \quad \phi_k = -\tan^{-1} b_k/a_k$
Exponential form :	$x(t) = \Sigma_{k=-\infty}^{\infty} c_k e^{j2\pi k f_0 t}$
	$c_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-j2\pi k f_0 t} dt$
	$c_0 = a_0, \quad c_k = a_k - jb_k$

The trigonometric form leads to a one-sided spectrum while the exponential form leads to a two-sided spectrum. The exponential form also gives rise to the idea of negative frequencies. Note that the spectrum of a periodic signal is discrete in frequency ie only frequencies $kf_0, k = 0, 1, 2, \ldots$ exist where f_0 is the fundamental frequency. Frequencies kf_0 are referred to as the k^{th} harmonic. For example, the lowest A-note on the piano which has 88 keys has a frequency of 27.5 Hz (A0 note). Subsequently all other higher A-notes are doubled in frequency and hence are harmonics of the A0 note.

If a periodic signals is even, then $b_k = 0$ for all k and likewise, if it is odd, $a_k = 0$. This is because cosine signals are even while sine signals are odd. Hence an even periodic signal is a sum of cosine signals while an odd periodic signal is a sum of sine signals. This further implies the following about their spectra.

- For even functions, $c_k = a_k - jb_k = a_k$. Thus c_k consists of only real numbers ie not complex and thus the spectrum of an even signal will have a zero phase spectrum. It is also symmetric.
- For odd functions, $c_k = a_k - jb_k = -jb_k$. Thus c_k consists of purely imaginary numbers. Thus the spectrum of an odd signal will have a phase spectrum that consists of $\pm 0.5\pi$. The amplitude spectrum is symmetric but the phase spectrum is anti-symmetric.

<u>Aperiodic signals</u> : For an aperiodic or non-periodic signal, $x(t)$, the Fourier transform applies and is defined as:

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

The Fourier transform, $X(f)$, is a continuous function of frequency and hence an aperiodic signal has a continuous spectrum ie non-discrete.

The properties of the Fourier transform can be summarized in the following table :

Scaling	$x(\beta t) \Leftrightarrow \frac{1}{\lvert\beta\rvert} X\left(\frac{f}{\beta}\right)$
Duality	$X(t) \Leftrightarrow x(-f)$
Time shift	$x(t - t_0) \Leftrightarrow X(f)e^{-j2\pi f t_0}$
Frequency shift	$X(f - f_0) \Leftrightarrow x(t)e^{j2\pi f_0 t}$
Differentiation in time	$\frac{dx(t)}{dt} \Leftrightarrow j2\pi f X(f)$
Integration in time	$\int_{-\infty}^{t} x(t)dt \Leftrightarrow \frac{1}{j2\pi f} X(f) + 0.5X(0)\delta(f)$
Multiplication in time	$x_1(t)x_2(t) \Leftrightarrow \int_{-\infty}^{\infty} X_1(f-\zeta)X_2(\zeta)d\zeta = X_1(f) * X_2(f)$
Convolution in time	$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau \Leftrightarrow X_1(f)X_2(f)$

The operator * denotes convolution which is defined as :

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(t-\tau)x_2(\tau)d\tau$$

Common Signals and their Transforms

DC or constant signal	$x(t) = K \Leftrightarrow K\delta(f)$ Proof : $\mathcal{F}\{K\delta(t)\} = K$ Using duality, $\mathcal{F}\{K\} = K\delta(f)$
Exponential signal	$x(t) = Ke^{j2\pi f_0 t} \Leftrightarrow K\delta(f - f_0)$ Proof : Comes from frequency shifting
Cosine signal	$x(t) = A\cos 2\pi f_0 t \Leftrightarrow X(f) = \frac{A}{2}\left[\delta(f - f_0) + \delta(f + f_0)\right]$
Sine signal	$x(t) = A\sin 2\pi f_0 t \Leftrightarrow X(f) = \frac{A}{2j}\left[\delta(f - f_0) + \delta(f + f_0)\right]$
Rectangular signal	$x(t) = A\mathrm{rect}\left(\frac{t}{T}\right) \Leftrightarrow AT\mathrm{sinc}(fT)$ $\mathrm{sinc}(fT) = \frac{\sin(\pi fT)}{\pi fT}$
Arbitrary periodic signal	$x(t)$ with Period, T_0, Fundamental frequency, $f_0 = \frac{1}{T_0}$ $X(f) = \Sigma_{k=-\infty}^{\infty} c_k \delta(f - kf_0)$ c_k are the Fourier coefficients
Comb function	$\xi(t) = \Sigma_{k=-\infty}^{\infty}\delta(t - kT_0)$ $\Xi(f) = \frac{1}{T_0}\Sigma_{k=-\infty}^{\infty}\delta(f - kf_0)$

The following are some of the properties related to the Dirac delta function, $\delta(t)$.

Sifting	$\int_{-\infty}^{\infty} x(t)\delta(t-\lambda)dt = x(\lambda)$
White Spectrum	$\mathcal{F}\{\delta(t)\} = 1$
Replication	$x(t) * \delta(t-\xi) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\xi-\tau)d\tau$
	$= \int_{-\infty}^{\infty} x(\tau)\delta(\tau+\xi-t)d\tau$
	$= \int_{-\infty}^{\infty} x(\tau)\delta(\tau-(t-\xi))d\tau$
	$= x(t-\xi)$ using the sifting property
	$X(f) * \delta(f-f_0) = X(f-f_0)$ - same as above but in f domain

Energy and Power Spectral Densities

The energy of a signal, $x(t)$, is defined as :

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

A signal is an *energy* signal if $E < \infty$.

The power of a signal $x(t)$, is defined as :

$$P = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt.$$

A signal is a *power* signal if $P < \infty$.

A signal cannot be both an energy and power signal at the same time. It can only be either one or the other. However, it can be neither ie not an energy and neither is it a power signal. For example, a rectangular pulse is an energy signal with its power, $P = 0$, while a sinusoidal signal is a power signal ie $P < \infty$ and its $E = \infty$. All periodic signals are power signals.

In frequency domain, energy and power can be computed as follows :

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |X(f)|^2 df = \int_{-\infty}^{\infty} E_x(f) df \\ P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(f)|^2 df = \int_{-\infty}^{\infty} P_x(f) df \end{aligned}$$

The quantities $E_x(f) = |X(f)|^2$ and $P_x(f) = \lim_{T\to\infty} \frac{1}{2T}|X(f)|^2$ are known as *energy* and *power* spectral densities respectively.

By virtue of Parseval's theorem,

$$\begin{aligned} E &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \\ P &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |x(t)|^2 dt = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |X(f)|^2 df \end{aligned}$$

For a periodic signal, $E = \infty$ but $P = \Sigma_{k=-\infty}^{\infty} |c_k|^2$. In this case, the power spectral density, $P_x(f) = |c_k|^2$ defines the power contained in each frequency component, $f = kf_0$.

For a non-periodic energy signal, $E = \int_{k=-\infty}^{\infty} |X_f|^2 df < \infty$ but $P = 0$. In this case, the energy spectral density is $E_x(f) = |X(f)|^2$ defines the energy contained in each frequency component, f.

Sampling and the Spectrum of Sampled Signals

When a signal is multiplied by a comb function, the process results in the sampling of the signal. Assume that the sampling period is T_s seconds and the sampling frequency is thus $f_s = 1/T_s$ Hz. In other words,

$$x(t)\Sigma_{-\infty}^{\infty}\delta(t - nT_s) = x(nT_s), n = 0, 1, 2, \ldots = x_s(t)$$

where $x_s(t)$ denotes the sampled signal of $x(t)$. In the frequency domain, $X_s(f)$, which is the Fourier transform of $x_s(t)$), is derived as follows :

$$
\begin{aligned}
x_s(t) &= x(t)\sum_{-\infty}^{\infty}\delta(t-nT_s) \\
X_s(f) &= X(f)*\Xi(f) \\
&= \int_{-\infty}^{\infty} X(\gamma)\Xi(f-\gamma)d\gamma \quad \text{where } \Xi(f)=f_s\sum_{k=-\infty}^{\infty}\delta(f-kf_s) \\
&= \int_{-\infty}^{\infty} X(\gamma)f_s\sum_{k=-\infty}^{\infty}\delta(f-kf_s-\gamma)d\gamma \\
&= f_s\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(\gamma)\delta(f-kf_s-\gamma)d\gamma \\
&= f_s\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} X(\gamma)\delta(\gamma-(f-kf_s))d\gamma \\
&= f_s\sum_{k=-\infty}^{\infty} X(f-kf_s)
\end{aligned}
$$

Hence the spectrum of $X_s(f)$ is replicated at every integer multiple of f_s. This implies that even though $X(f)$ which is the spectrum of the original signal ($x(t)$) may be bandlimited, the spectrum of the sampled signal ($x_s(t)$) has infinite frequency components. In order to recover the original signal, $x(t)$ from the sampled signal, $x_s(t)$, appropriate filtering has to be applied to $x_s(t)$.

This result also leads to the necessary requirement that a continuous time low pass signal which has maximum frequency components up to f_m Hz has to be sampled at a minimum sampling frequency of $f_s = 2f_m$ in order for the original signal to be reconstructed completely or accurately. f_s is also known as the Nyquist sampling frequency. If $f_s < 2f_m$, then $x_s(t)$ will have overlapping spectra and reconstruction or low pass filtering will not be able to recover the signal completely ie distortion occurs. This phenomenon is also known as aliasing. If the original signal is not bandlimited, then it is better to pre-filter the signal using anti-aliasing filters before sampling at the Nyquist sampling frequency.

For bandlimited signal, the sampling frequency need not satisfy the Nyquist sampling frequency. However, the recovery of the bandlimited signal requires bandpass filters.