Linear Algebra Quizzes

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MATH

Topic: Linear systems in two unknowns

Question: Use substitution to find the unique solution to the linear system.

$$y = x + 7$$

$$x + 2y = -16$$

Answer choices:

A $(10,3)$

B $(-10,3)$

C $(10,-3)$

D $(-10,-3)$

Solution: D

Since the first equation is already solved for y, we'll make a substitution for y into the second equation, so that we can get the second equation in terms of only x.

$$x + 2y = -16$$

$$x + 2(x + 7) = -16$$

Solve this equation for x.

$$x + 2x + 14 = -16$$

$$3x + 14 = -16$$

$$3x = -30$$

$$x = -10$$

Now we'll take the value we found for x and plug it into the first equation to find the value of y.

$$y = x + 7$$

$$y = -10 + 7$$

$$y = -3$$

Putting these values together gives $(x, y) = (-10, -3)$.

Topic: Linear systems in two unknowns

Question: Use elimination to find the unique solution to the system of equations.

$$x - 3y = -7$$

$$2x - 3y = 4$$

Answer choices:

A $(12,7)$

B $(11,6)$

C $(9,3)$

D $(-11,-6)$

Solution: B

Since the y-term in each equation is $-3y$, we'll subtract the second equation from the first equation.

$$x - 3y - (2x - 3y) = -7 - (4)$$

$$x - 3y - 2x + 3y = -7 - 4$$

$$-x = -11$$

$$x = 11$$

Now that we have the value of x, we'll plug it into the original first equation and solve for y.

$$x - 3y = -7$$

$$11 - 3y = -7$$

$$11 - 11 - 3y = -7 - 11$$

$$-3y = -18$$

$$y = 6$$

To make sure that $(11,6)$ is the solution to the system, we'll plug it into the other original equation, the one we didn't use to find y.

$$2x - 3y = 4$$

$$2(11) - 3(6) = 4$$

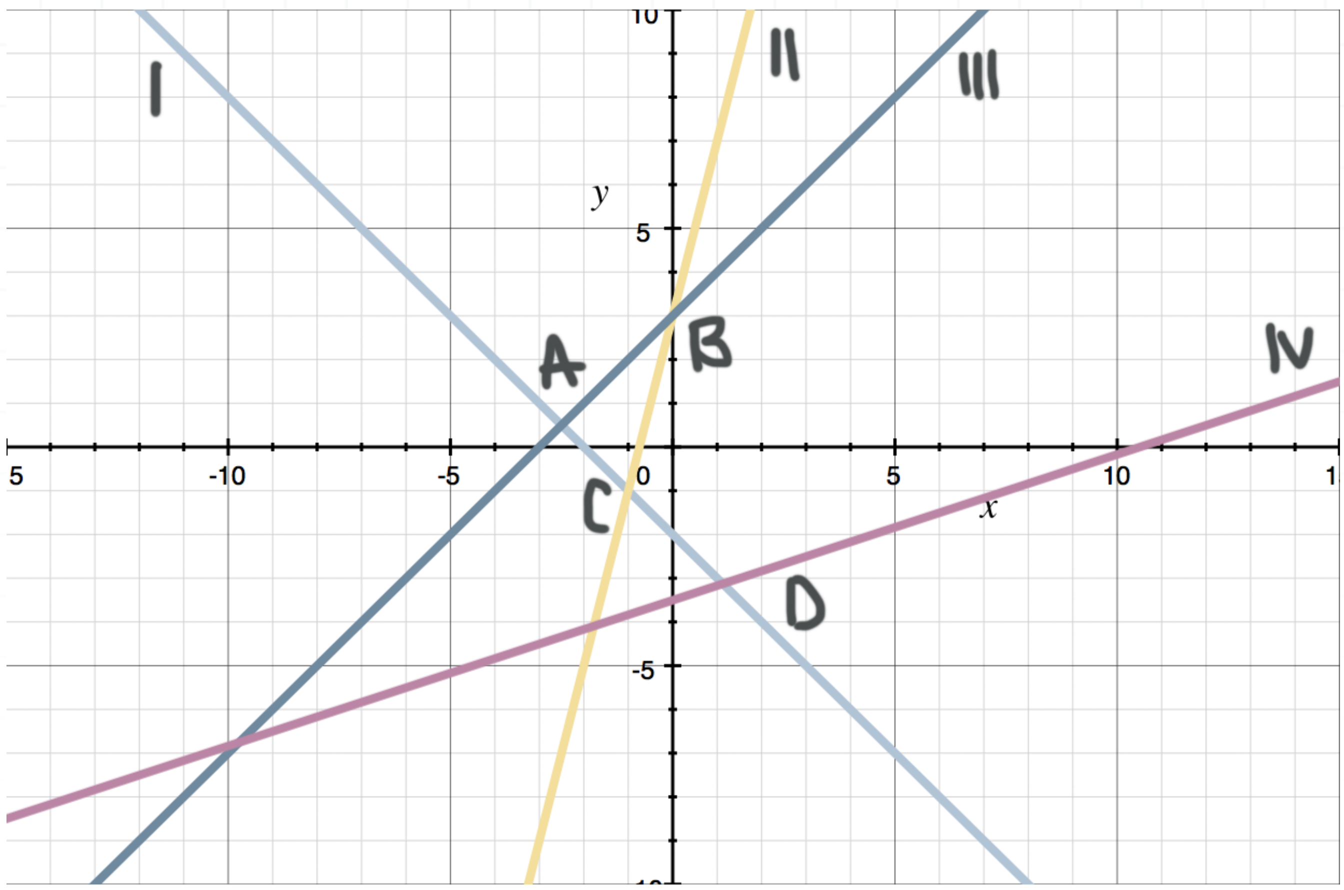
$$22 - 18 = 4$$

$$4 = 4$$

Since $4 = 4$ is true, we know $(11,6)$ is the solution to the system.

Topic: Linear systems in two unknowns

Question: Which labeled point in the graph would represent the solution to the system of equations $4x - y = -3$ and $x + y = -2$?



Answer choices:

A Point A

B Point B

C Point C

D Point D

Solution: C

One way to figure this out is to rewrite the two equations in slope-intercept form, and then see which two intersecting graphs belong to those equations. Rewriting $4x - y = -3$ gives

$$y = 4x + 3$$

This line has a slope of 4 and a y-intercept of 3. This is Graph II. Rewriting $x + y = -2$ gives

$$y = -x - 2$$

This line has a slope of -1 and a y-intercept of -2. This is Graph I.

Graphs I and II meet at point C, $(-1, -1)$.

Topic: Linear systems in three unknowns

Question: Use any method to find the solution to the system of equations.

$$x + y - z = 4$$

$$x - y - z = 2$$

$$x + 2y + z = 1$$

Answer choices:

A $(x, y, z) = (-1,1,2)$

B $(x, y, z) = (1, -1,2)$

C $(x, y, z) = (1,1, -2)$

D $(x, y, z) = (1,1,2)$

Solution: C

So that we can stay organized, let's number the equations.

[1] $x + y - z = 4$

[2] $x - y - z = 2$

[3] $x + 2y + z = 1$

Since the coefficients of x and z in [1] are equal to the coefficients of x and z in [2], we'll be able to eliminate the variables x and z (and then solve for y) if we subtract [2] from [1].

$$x + y - z - (x - y - z) = (4) - (2)$$

$$x + y - z - x + y + z = 4 - 2$$

$$2y = 2$$

$$y = 1$$

Let's substitute 1 for y in [2] and [3], which will give us two equations in the variables x and z only. So

[2] $x - y - z = 2$

[3] $x + 2y + z = 1$

become

$$x - 1 - z = 2$$

$$x + 2(1) + z = 1$$

and then

[4] $x - z = 3$

[5] $x + z = -1$

Next, we'll add [4] and [5] in order to eliminate z and solve for x.

$$(x - z) + (x + z) = (3) + (-1)$$

$$x - z + x + z = 3 - 1$$

$$2x = 2$$

$$x = 1$$

Now that we know that $x = 1$ and $y = 1$, we'll substitute 1 for both x and y in [1], and then solve for z.

[1] $x + y - z = 4$

$$1 + 1 - z = 4$$

$$2 - z = 4$$

$$-z = 2$$

$$z = -2$$

We've found a possible solution, $(1,1,-2)$. Let's test it in the original system to make sure it satisfies the system.

[1] $x + y - z = 4$

$1 + 1 - (-2) = 4$

$1 + 1 + 2 = 4$

$4 = 4$

and

[2] $x - y - z = 2$

$1 - 1 - (-2) = 2$

$1 - 1 + 2 = 2$

$2 = 2$

and

[3] $x + 2y + z = 1$

$1 + 2(1) + (-2) = 1$

$1 + 2 - 2 = 1$

$1 = 1$

We've shown that $(1,1,-2)$ satisfies the system.

Topic: Linear systems in three unknowns

Question: Use any method to find the solution to the linear system.

$$x - y + z = -6$$

$$3x - 4y - z = -4$$

$$-2x + 3y + 4z = 14$$

Answer choices:

A $(x, y, z) = (-60, -46, -8)$

B $(x, y, z) = (60, -46, 8)$

C $(x, y, z) = (-60, 46, 8)$

D $(x, y, z) = (-60, -46, 8)$

Solution: D

So that we can stay organized, let's number the equations.

[1] $x - y + z = -6$

[2] $3x - 4y - z = -4$

[3] $-2x + 3y + 4z = 14$

We'll add [1] and [2] to eliminate z.

$$(x - y + z) + (3x - 4y - z) = (-6) + (-4)$$

$$x - y + z + 3x - 4y - z = -6 - 4$$

[4] $4x - 5y = -10$

Now we'll multiply [2] by 4, so that we can add the result to [3] and eliminate z.

[2] $3x - 4y - z = -4$

$$4(3x - 4y - z) = 4(-4)$$

[5] $12x - 16y - 4z = -16$

Adding [3] and [5], we get

$$(-2x + 3y + 4z) + (12x - 16y - 4z) = (14) + (-16)$$

$$-2x + 3y + 4z + 12x - 16y - 4z = 14 - 16$$

[6] $10x - 13y = -2$

With [4] and [6], we have a system of two equations in the variables x and y.

[4] $4x - 5y = -10$

[6] $10x - 13y = -2$

Let's solve [4] for y, and substitute the resulting expression for y into [6], and then solve for x.

[4] $4x - 5y = -10$

$$-5y = -4x - 10$$

$$5y = 4x + 10$$

[7] $y = \frac{4}{5}x + 2$

Now we'll plug this expression for y into [6].

[6] $10x - 13y = -2$

$$10x - 13\left(\frac{4}{5}x + 2\right) = -2$$

$$10x - \frac{52}{5}x - 26 = -2$$

$$5\left(10x - \frac{52}{5}x - 26\right) = 5(-2)$$

$$5(10x) + 5\left(-\frac{52}{5}x\right) + 5(-26) = 5(-2)$$

$$50x - 52x - 130 = -10$$

$$-2x = 120$$

$$x = -60$$

Next, we'll substitute -60 for x in [7], and then compute the value of y.

[7] $y = \frac{4}{5}x + 2$

$$y = \frac{4}{5}(-60) + 2$$

$$y = 4(-12) + 2$$

$$y = -48 + 2$$

$$y = -46$$

Now that we know that $x = -60$ and $y = -46$, we'll substitute -60 for x and -46 for y in [1], and then solve for z.

[1] $x - y + z = -6$

$$-60 - (-46) + z = -6$$

$$-60 + 46 + z = -6$$

$$-14 + z = -6$$

$$z = 8$$

We've found a possible solution, $(-60, -46, 8)$. Let's test it in the original system to make sure it satisfies the system.

[1] $x - y + z = -6$

$-60 - (-46) + 8 = -6$

$-60 + 46 + 8 = -6$

$-14 + 8 = -6$

$-6 = -6$

and

[2] $3x - 4y - z = -4$

$3(-60) - 4(-46) - 8 = -4$

$-180 + 184 - 8 = -4$

$4 - 8 = -4$

$-4 = -4$

and

[3] $-2x + 3y + 4z = 14$

$-2(-60) + 3(-46) + 4(8) = 14$

$120 - 138 + 32 = 14$

$-18 + 32 = 14$

$14 = 14$

We've shown that $(-60, -46, 8)$ satisfies the system.

Topic: Linear systems in three unknowns

Question: Solve the system for x, y, and z.

$$x + 2z = 3$$

$$3x - 2y + z = -11$$

$$2x + y + 3z = 9$$

Answer choices:

A $(x, y, z) = (3, -1, 5)$

B $(x, y, z) = (-1, 3, 2)$

C $(x, y, z) = (2, 5, -1)$

D $(x, y, z) = (-1, 5, 2)$

Solution: D

So that we can stay organized, let's number the equations.

[1] $x + 2z = 3$

[2] $3x - 2y + z = -11$

[3] $2x + y + 3z = 9$

Solve [1] for x.

$$x + 2z = 3$$

$$x = 3 - 2z$$

Substitute this expression for x into [2] and [3], and then simplify.

[2] $3x - 2y + z = -11$

$$3(3 - 2z) - 2y + z = -11$$

$$9 - 6z - 2y + z = -11$$

$$-2y - 5z = -20$$

and

[3] $2x + y + 3z = 9$

$$2(3 - 2z) + y + 3z = 9$$

$$6 - 4z + y + 3z = 9$$

$$y - z = 3$$

Now we can solve the resulting equations as a system of two equations in the variables y and z.

$$-2y - 5z = -20$$

$$y - z = 3$$

We'll multiply the second equation by 2.

$$y - z = 3$$

$$2(y - z) = 2(3)$$

$$2y - 2z = 6$$

To eliminate y, we can add this equation to the equation $-2y - 5z = -20$ that we found earlier.

$$(-2y - 5z) + (2y - 2z) = (-20) + (6)$$

$$-2y - 5z + 2y - 2z = -20 + 6$$

$$-7z = -14$$

$$z = 2$$

Now we'll substitute 2 for z in the equation $y - z = 3$ that we found earlier, and then solve for y.

$$y - z = 3$$

$$y - 2 = 3$$

$$y = 5$$

Finally, we'll substitute 2 for z in the equation $x = 3 - 2z$ that we found much earlier, and then compute the value of x.

$$x = 3 - 2z$$

$$x = 3 - 2(2)$$

$$x = 3 - 4$$

$$x = -1$$

The solution is $(-1,5,2)$. If we plug the coordinates of this point into all three of the original equations, we'll see that it satisfies all of them.

Topic: Matrix dimensions and entries

Question: Give the dimensions of the matrix.

$$K = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 2 & 5 & 6 & -2 \end{bmatrix}$$

Answer choices:

A The dimensions are 4×2

B The dimensions are 1×8

C The dimensions are 2×4

D The dimensions are 3×3

Solution: C

We always give the dimensions of a matrix as rows $\times$ columns. Matrix K has 2 rows and 4 columns, so K is a 2×4 matrix.

Topic: Matrix dimensions and entries

Question: Given matrix B, find $b_{2,1}$.

$$B = \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix}$$

Answer choices:

A $\quad 1$

B $\quad 0$

C $\quad -1$

D $\quad 3$

Solution: B

The value of $b_{2,1}$ is the entry in the second row, first column of matrix B, which is 0, so $b_{2,1} = 0$.

Topic: Matrix dimensions and entries

Question: Give the dimensions of matrix M and find $M_{3,2}$.

$$M = \begin{bmatrix} 1 & 3 & 7 \\ 0 & -1 & 2 \\ 9 & 4 & 6 \end{bmatrix}$$

Answer choices:

A The dimensions are 3×3 and $M_{3,2} = 4$

B The dimensions are 2×3 and $M_{3,2} = 2$

C The dimensions are 3×3 and $M_{3,2} = 2$

D The dimensions are 3×1 and $M_{3,2} = 4$

Solution: A

We always give the dimensions of a matrix as rows $\times$ columns. Matrix M has 3 rows and 3 columns, so M is a 3×3 matrix.

The value of $M_{3,2}$ is the entry in the third row, second column of matrix M, which is 4, so $M_{3,2} = 4$.

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called B.

$$4x + 2y = 8$$

$$-2x + 7y = 11$$

Answer choices:

A $\quad B = \begin{bmatrix} 8 & 2 & | & 4 \\ 11 & -2 & | & 7 \end{bmatrix}$

B $\quad B = \begin{bmatrix} 4 & 2 & | & 8 \\ 7 & -2 & | & 11 \end{bmatrix}$

C $\quad B = \begin{bmatrix} 2 & 4 & | & 8 \\ -2 & 7 & | & 11 \end{bmatrix}$

D $\quad B = \begin{bmatrix} 4 & 2 & | & 8 \\ -2 & 7 & | & 11 \end{bmatrix}$

Solution: D

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

This problem is straightforward because the system is set up correctly with all variables in both equations.

$$4x + 2y = 8$$

$$-2x + 7y = 11$$

The system contains the variables x and y along with a constant. Which means the augmented matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows.

Plugging the coefficients and constants into an augmented matrix gives

$$B = \begin{bmatrix} 4 & 2 & | & 8 \\ -2 & 7 & | & 11 \end{bmatrix}$$

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called G.

$$a - 3b + 9c + 6d = 4$$

$$8a + 6c = 9d + 15$$

Answer choices:

A $G = \begin{bmatrix} 1 & -3 & 9 & 6 & | & 4 \\ 8 & 0 & 6 & -9 & | & 15 \end{bmatrix}$

B $G = \begin{bmatrix} 1 & 9 & 6 & 4 \\ 8 & 6 & -9 & 15 \end{bmatrix}$

C $G = \begin{bmatrix} 1 & 3 & 9 & 6 & | & 4 \\ 8 & 0 & 6 & 9 & | & 15 \end{bmatrix}$

D $G = \begin{bmatrix} 1 & -3 & 9 & 6 & | & 4 \\ 15 & 6 & 0 & 5 & | & 8 \end{bmatrix}$

Solution: A

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

To do this, the second equation can be reorganized by putting a, c, and d on the left side, and the constant on the right side. We also recognize that there is no b-term in the second equation, so we add in a 0 "filler" term.

$$a - 3b + 9c + 6d = 4$$

$$8a + 0b + 6c - 9d = 15$$

The system contains the variables a, b, c, and d, along with a constant. Which means the augmented matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows.

Plugging the coefficients and constants into an augmented matrix gives

$$G = \begin{bmatrix} 1 & -3 & 9 & 6 & | & 4 \\ 8 & 0 & 6 & -9 & | & 15 \end{bmatrix}$$

Topic: Representing systems with matrices

Question: Represent the system with an augmented matrix called N.

$$6a + 4b - c = 9$$

$$5b = -6a + 7c - 6$$

$$3c = 14 - 2a$$

Answer choices:

A $N = \begin{bmatrix} 6 & 4 & -1 & | & 9 \\ 5 & -6 & 7 & | & -6 \\ 3 & 14 & -2 & | & 0 \end{bmatrix}$

B $N = \begin{bmatrix} 6 & 4 & -1 & | & 9 \\ -6 & 5 & 7 & | & -6 \\ -2 & 3 & -14 & | & 0 \end{bmatrix}$

C $N = \begin{bmatrix} 6 & 4 & -1 & | & 9 \\ 6 & 5 & -7 & | & -6 \\ 2 & 0 & 3 & | & 14 \end{bmatrix}$

D $N = \begin{bmatrix} -2 & 3 & 0 & | & -14 \\ 6 & 4 & 1 & | & 9 \\ 6 & 5 & 7 & | & 6 \end{bmatrix}$

Solution: C

As you're building out an augmented matrix, you want to be sure that you have all the variables in the same order, and all your constants grouped together on the same side of the equation. That way, with everything lined up, it'll be easy to make sure that each entry in a column represents the same variable or constant, and that each row in the matrix captures the entire equation.

To do this, the second two equations can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no b-term in the third equation, so we add in a 0 "filler" term.

$$6a + 4b - c = 9$$

$$6a + 5b - 7c = -6$$

$$2a + 0b + 3c = 14$$

The system contains the variables a, b, and c, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into an augmented matrix gives

$$N = \begin{bmatrix} 6 & 4 & -1 & | & 9 \\ 6 & 5 & -7 & | & -6 \\ 2 & 0 & 3 & | & 14 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 1 & -2 & 5 \\ 6 & 7 & 0 \\ 7 & 4 & 9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 1 & -2 & 5 \\ 7 & 4 & 9 \\ 6 & 7 & 0 \end{bmatrix}$

B $\begin{bmatrix} 7 & 4 & 9 \\ 6 & 7 & 0 \\ 1 & -2 & 5 \end{bmatrix}$

C $\begin{bmatrix} 6 & 7 & 0 \\ 1 & -2 & 5 \\ 7 & 4 & 9 \end{bmatrix}$

D $\begin{bmatrix} 7 & 4 & 9 \\ 1 & -2 & 5 \\ 6 & 7 & 0 \end{bmatrix}$

Solution: C

The operation described by $R_1 \leftrightarrow R_2$ is switching row 1 with row 2. Nothing will happen to row 3. The matrix after $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 6 & 7 & 0 \\ 1 & -2 & 5 \\ 7 & 4 & 9 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $2R_2 \leftrightarrow 4R_3$.

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -2 & 3 & 7 & 9 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 6 & 1 & 5 & -8 \\ 20 & -8 & 0 & 4 \\ -4 & 6 & 14 & 18 \end{bmatrix}$

B $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 5 & -2 & 0 & 1 \end{bmatrix}$

C $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -2 & 3 & 7 & 9 \\ 20 & -8 & 0 & 4 \end{bmatrix}$

D $\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 20 & -8 & 0 & 4 \end{bmatrix}$

Solution: A

The operation described by $2R_2 \leftrightarrow 4R_3$ is multiplying row 2 by a constant of 2, multiplying row 3 by a constant of 4, and then switching those two rows. Nothing will happen to row 1. The matrix after $2R_2$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 5 & -2 & 0 & 1 \end{bmatrix}$$

The matrix after $4R_3$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ -4 & 6 & 14 & 18 \\ 20 & -8 & 0 & 4 \end{bmatrix}$$

The matrix after $2R_2 \leftrightarrow 4R_3$ is

$$\begin{bmatrix} 6 & 1 & 5 & -8 \\ 20 & -8 & 0 & 4 \\ -4 & 6 & 14 & 18 \end{bmatrix}$$

Topic: Simple row operations

Question: Write the new matrix after $3R_1+R_3\to R_1$.

$$\begin{bmatrix}7 & 8 & -2 & 0\\ 5 & 1 & 6 & 13\\ 4 & -7 & 3 & 9\end{bmatrix}$$

Answer choices:

A $\begin{bmatrix}21 & 24 & -6 & 0\\ 5 & 1 & 6 & 13\\ 4 & -7 & 3 & 9\end{bmatrix}$

B $\begin{bmatrix}7 & 8 & -2 & 0\\ 5 & 1 & 6 & 13\\ 25 & 17 & -3 & 9\end{bmatrix}$

C $\begin{bmatrix}7 & 8 & -2 & 0\\ 25 & 17 & -3 & 9\\ 4 & -7 & 3 & 9\end{bmatrix}$

D $\begin{bmatrix}25 & 17 & -3 & 9\\ 5 & 1 & 6 & 13\\ 4 & -7 & 3 & 9\end{bmatrix}$

Solution: D

The operation described by $3R_1 + R_3 \to R_1$ is multiplying row 1 by a constant of 3, adding that resulting row to row 3, and using that result to replace row 1. The matrix after $3R_1$ is

$$\begin{bmatrix} 21 & 24 & -6 & 0 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

The sum $3R_1 + R_3$ is

$$\begin{bmatrix} 25 & 17 & -3 & 9 \end{bmatrix}$$

The matrix after $3R_1 + R_3 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 25 & 17 & -3 & 9 \\ 5 & 1 & 6 & 13 \\ 4 & -7 & 3 & 9 \end{bmatrix}$$

Topic: Pivot entries and row-echelon forms

Question: Which matrix is in row-echelon form?

Answer choices:

A $\begin{bmatrix} 1 & -5 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

B $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

C $\begin{bmatrix} 0 & 1 & 1 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution: A

In the matrix in answer choice A, the first non-zero entry in each row is a 1, the row consisting of only 0s is at the bottom, and the pivots follow a staircase pattern that moves down and to the right. The second column includes a non-zero entry, but it's not a pivot column. Therefore, the matrix is in row-echelon form.

The matrix in answer choice B is not in row-echelon form, because the first non-zero entry in R_3 is not 1.

The matrix in answer choice C is not in row-echelon form, because the first non-zero entry in R_1 and R_2 appear in the same column.

The matrix in answer choice D is not in row-echelon form, because R_2 is a row of 0s that would need to appear at the bottom of the matrix.

Topic: Pivot entries and row-echelon forms

Question: Which matrix is in reduced row-echelon form?

Answer choices:

A $$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

B $$\begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

C $$\begin{bmatrix} 1 & -2 & 0 & 5 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

D $$\begin{bmatrix} 1 & -9 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: C

In the matrix in answer choice C, the first non-zero entry in each row is 1, the row consisting of only 0s is at the bottom, and the pivots follow a staircase pattern that moves down and to the right. The two pivot columns include only the pivot entry, and otherwise only zero entries. The second and the fourth columns include a non-zero entry, but they're not a pivot columns. Therefore, the matrix is in reduced row-echelon form.

In the matrix in answer choice A, the leading 1 in R_3 is not the only non-zero entry in its column.

In the matrix in answer choice B, the leading 1 in R_3 is not the only non-zero entry in its column.

In the matrix in answer choice D, R_2 is a row of only 0s, but it has a non-zero row below it.

Topic: Pivot entries and row-echelon forms

Question: What is the reduced row-echelon form of the matrix?

$$\begin{bmatrix} 3 & 9 & 15 & 96 \\ -4 & -12 & -18 & -118 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

B $\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

C $\begin{bmatrix} 1 & 3 & 5 & 32 \\ 0 & 0 & 1 & 5 \end{bmatrix}$

D $\begin{bmatrix} 1 & 3 & 5 & 32 \\ 0 & 0 & 2 & 10 \end{bmatrix}$

Solution: B

Start with $(1/3)R_1 \to R_1$ to get the pivot in the first column.

$$\begin{bmatrix} 3 & 9 & 15 & 96 \\ -4 & -12 & -18 & -118 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & 5 & 32 \\ -4 & -12 & -18 & -118 \end{bmatrix}$$

Zero out the rest of the first column with $4R_1 + R_2 \to R_2$.

$$\begin{bmatrix} 1 & 3 & 5 & 32 \\ 0 & 0 & 2 & 10 \end{bmatrix}$$

Get the pivot in the second row with $(1/2)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 3 & 5 & 32 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

Zero out the rest of the third column with $-5R_2 + R_1 \to R_1$.

$$\begin{bmatrix} 1 & 3 & 0 & 7 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

In this matrix, the first non-zero entry in each row is a 1, and there are no rows consisting of only 0s. The pivots follow a staircase pattern, and the two pivot columns include only the pivot entry, and otherwise only zero entries. The second and fourth columns include a non-zero entry, but they are not pivot columns. Therefore, this matrix is in reduced row-echelon form.

Topic: Gauss-Jordan elimination

Question: Use Gauss-Jordan elimination to solve the system with a rref matrix.

$$x + 3y = 13$$

$$2x + 4y = 16$$

Answer choices:

A $(x, y) = (5,2)$

B $(x, y) = (3, -1)$

C $(x, y) = (-1,3)$

D $(x, y) = (-2,5)$

Solution: D

The augmented matrix is

$$\begin{bmatrix} 1 & 3 & | & 13 \\ 2 & 4 & | & 16 \end{bmatrix}$$

The first row already has a leading 1. After $2R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 3 & | & 13 \\ 0 & 2 & | & 10 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 3 & | & 13 \\ 0 & 1 & | & 5 \end{bmatrix}$$

After $R_1 - 3R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & | & -2 \\ 0 & 1 & | & 5 \end{bmatrix}$$

The second column is done, and we get the solution set $(x, y) = (-2,5)$.

Topic: Gauss-Jordan elimination

Question: Use Gauss-Jordan elimination to solve the system with a rref matrix.

$$x + 4z = 11$$

$$x - y + 4z = 6$$

$$2x + 9z = 25$$

Answer choices:

A $(x, y, z) = (-1,5,3)$

B $(x, y, z) = (11,6,25)$

C $(x, y, z) = (1,0,12)$

D $(x, y, z) = (-3,8,3)$

Solution: A

The augmented matrix is

$$\begin{bmatrix} 1 & 0 & 4 & | & 11 \\ 1 & -1 & 4 & | & 6 \\ 2 & 0 & 9 & | & 25 \end{bmatrix}$$

The first row already has a leading 1. After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & | & 11 \\ 0 & 1 & 0 & | & 5 \\ 2 & 0 & 9 & | & 25 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & | & 11 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & -1 & | & -3 \end{bmatrix}$$

The first and second columns are done. After $(-1)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 & | & 11 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

After $R_1 - 4R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

The third column is done, and we get the solution set $(x, y, z) = (-1,5,3)$.

Topic: Gauss-Jordan elimination

Question: Use Gauss-Jordan elimination to solve the system with a rref matrix.

$$2x + 4y + 10z = 30$$

$$x + y + 3z = 10$$

$$2x + y + 2z = 9$$

Answer choices:

A $(x, y, z) = (7, -3, 5)$

B $(x, y, z) = (-4, 1, 0)$

C $(x, y, z) = (2, -1, 3)$

D $(x, y, z) = (30, 10, 9)$

Solution: C

The augmented matrix is

$$\begin{bmatrix} 2 & 4 & 10 & | & 30 \\ 1 & 1 & 3 & | & 10 \\ 2 & 1 & 2 & | & 9 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & | & 15 \\ 1 & 1 & 3 & | & 10 \\ 2 & 1 & 2 & | & 9 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & | & 15 \\ 0 & 1 & 2 & | & 5 \\ 2 & 1 & 2 & | & 9 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 2 & 5 & | & 15 \\ 0 & 1 & 2 & | & 5 \\ 0 & 3 & 8 & | & 21 \end{bmatrix}$$

The first column is done. After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & | & 5 \\ 0 & 1 & 2 & | & 5 \\ 0 & 3 & 8 & | & 21 \end{bmatrix}$$

After $R_3 - 3R_2 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & | & 5 \\ 0 & 1 & 2 & | & 5 \\ 0 & 0 & 2 & | & 6 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 1 & | & 5 \\ 0 & 1 & 2 & | & 5 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

After $R_1 - R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 2 & | & 5 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & | & 2 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 3 \end{bmatrix}$$

The third column is done, and we get the solution set $(x, y, z) = (2, -1, 3)$.

Topic: Number of solutions to the linear system

Question: How many solutions does the following linear system have?

$$-x - 2y + 3z = -30$$

$$-2x - 3y - 5z = 22$$

$$x + 5y + 5z = -11$$

Answer choices:

A One solution

B No solutions

C Infinitely many solutions

Solution: A

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & -2 & 3 & | & -30 \\ -2 & -3 & -5 & | & 22 \\ 1 & 5 & 5 & | & -11 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row. Perform $-R_1 \to R_1$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 30 \\ -2 & -3 & -5 & | & 22 \\ 1 & 5 & 5 & | & -11 \end{bmatrix}$$

Zero out the rest of the first column, first with $2R_1 + R_2 \to R_2$, and then $-R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 2 & -3 & | & 30 \\ 0 & 1 & -11 & | & 82 \\ 1 & 5 & 5 & | & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & | & 30 \\ 0 & 1 & -11 & | & 82 \\ 0 & 3 & 8 & | & -41 \end{bmatrix}$$

Zero out the rest of the second column, first with $-2R_2 + R_1 \to R_1$, and then $-3R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & 19 & | & -134 \\ 0 & 1 & -11 & | & 82 \\ 0 & 3 & 8 & | & -41 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 19 & | & -134 \\ 0 & 1 & -11 & | & 82 \\ 0 & 0 & 41 & | & -287 \end{bmatrix}$$

Find the pivot entry in the third row with $(1/41)R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & 19 & | & -134 \\ 0 & 1 & -11 & | & 82 \\ 0 & 0 & 1 & | & -7 \end{bmatrix}$$

Zero out the rest of the third column, first with $-19R_3 + R_1 \to R_1$, then $11R_3 + R_2 \to R_2$.

$$\begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & -11 & | & 82 \\ 0 & 0 & 1 & | & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -1 \\ 0 & 1 & 0 & | & 5 \\ 0 & 0 & 1 & | & -7 \end{bmatrix}$$

Therefore, there's one unique solution to the system, $(x, y, z) = (-1,5,-7)$.

Topic: Number of solutions to the linear system

Question: Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$3a + 9b - 3c = 24$$

$$a - 3b + 11c = -2$$

$$-2a + 5b - 20c = -5$$

Answer choices:

A One solution

B No solutions

C Infinitely many solutions

Solution: B

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 3 & 9 & -3 & | & 24 \\ 1 & -3 & 11 & | & -2 \\ -2 & 5 & -20 & | & -5 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row. Perform $(1/3)R_1 \to R_1$.

$$\begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 1 & -3 & 11 & | & -2 \\ -2 & 5 & -20 & | & -5 \end{bmatrix}$$

Zero out the rest of the first column, first with $-R_1 + R_2 \to R_2$, then $2R_1 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 0 & -6 & 12 & | & -10 \\ -2 & 5 & -20 & | & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 0 & -6 & 12 & | & -10 \\ 0 & 11 & -22 & | & 11 \end{bmatrix}$$

Find the pivot entry in the second row with $-(1/6)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 3 & -1 & | & 8 \\ 0 & 1 & -2 & | & \frac{5}{3} \\ 0 & 11 & -22 & | & 11 \end{bmatrix}$$

Zero out the rest of the second column, first with $-3R_2 + R_1 \to R_1$, then $-11R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 1 & 0 & 5 & | & 3 \\ 0 & 1 & -2 & | & \frac{5}{3} \\ 0 & 0 & 0 & | & -\frac{22}{3} \end{bmatrix}$$

The third row shows us that $0 = -22/3$, which can't be true. Therefore, the system has no solutions.

Topic: Number of solutions to the linear system

Question: Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$w + 2x - 3y + 7z = 4$$

$$3w - x - 2y = 12$$

$$-2w + 5x + 2z = 5$$

$$2w + 3x - 5y + 11z = 8$$

Answer choices:

A One solution

B No solutions

C Infinitely many solutions

Solution: C

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 3 & -1 & -2 & 0 & | & 12 \\ -2 & 5 & 0 & 2 & | & 5 \\ 2 & 3 & -5 & 11 & | & 8 \end{bmatrix}$$

Zero out the rest of the first column, first with $-3R_1 + R_2 \to R_2$, then $2R_1 + R_3 \to R_3$, and finally with $-2R_1 + R_4 \to R_4$.

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & -7 & 7 & -21 & | & 0 \\ -2 & 5 & 0 & 2 & | & 5 \\ 2 & 3 & -5 & 11 & | & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & -7 & 7 & -21 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 2 & 3 & -5 & 11 & | & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & -7 & 7 & -21 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 0 & -1 & 1 & -3 & | & 0 \end{bmatrix}$$

Find the pivot entry in the second row with $-(1/7)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 2 & -3 & 7 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 0 & -1 & 1 & -3 & | & 0 \end{bmatrix}$$

Zero out the rest of the second column, first with $-2R_2+R_1 \to R_1$, then $-9R_2+R_3 \to R_3$, and finally with $R_2+R_4 \to R_4$.

$$\begin{bmatrix} 1 & 0 & -1 & 1 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 9 & -6 & 16 & | & 13 \\ 0 & -1 & 1 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 1 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 0 & 3 & -11 & | & 13 \\ 0 & -1 & 1 & -3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 1 & | & 4 \\ 0 & 1 & -1 & 3 & | & 0 \\ 0 & 0 & 3 & -11 & | & 13 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since the entire last row has only zeros, the linear system has infinitely many solutions.

Topic: Matrix addition and subtraction

Question: Add the matrices.

$$\begin{bmatrix} 4 & -3 & 6 \\ 8 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 11 & 4 & -9 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 4 & -3 & 6 \\ 19 & 6 & -8 \end{bmatrix}$

B $\begin{bmatrix} 7 & -3 & 7 \\ 19 & 6 & -8 \end{bmatrix}$

C $\begin{bmatrix} 7 & -3 & 7 \\ 8 & 2 & 1 \end{bmatrix}$

D $\begin{bmatrix} 7 & 3 & 7 \\ 19 & 6 & 8 \end{bmatrix}$

Solution: B

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{bmatrix} 4 & -3 & 6 \\ 8 & 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 & 1 \\ 11 & 4 & -9 \end{bmatrix}$$

$$\begin{bmatrix} 4+3 & -3+0 & 6+1 \\ 8+11 & 2+4 & 1+(-9) \end{bmatrix}$$

$$\begin{bmatrix} 7 & -3 & 7 \\ 19 & 6 & -8 \end{bmatrix}$$

Topic: Matrix addition and subtraction

Question: Subtract the matrices.

$$\begin{bmatrix} 8 & 1 & 3 \\ 6 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 12 & 5 \\ 5 & 1 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 14 & 13 & 8 \\ 1 & 5 & 5 \\ 2 & 6 & 7 \end{bmatrix}$

B $\begin{bmatrix} -2 & 11 & 2 \\ -1 & 5 & -5 \\ -2 & 6 & -7 \end{bmatrix}$

C $\begin{bmatrix} 14 & 13 & 8 \\ 11 & 7 & 5 \\ -2 & 8 & 11 \end{bmatrix}$

D $\begin{bmatrix} 2 & -11 & -2 \\ 1 & -5 & 5 \\ 2 & -6 & 7 \end{bmatrix}$

Solution: D

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{bmatrix} 8 & 1 & 3 \\ 6 & -4 & 5 \\ 0 & 1 & 9 \end{bmatrix} - \begin{bmatrix} 6 & 12 & 5 \\ 5 & 1 & 0 \\ -2 & 7 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 8-6 & 1-12 & 3-5 \\ 6-5 & -4-1 & 5-0 \\ 0-(-2) & 1-7 & 9-2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -11 & -2 \\ 1 & -5 & 5 \\ 2 & -6 & 7 \end{bmatrix}$$

Topic: Matrix addition and subtraction

Question: Solve for X.

$$\begin{bmatrix} 8 & 2 \\ 7 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix} = X + \begin{bmatrix} 5 & 7 \\ -5 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 6 & -4 \end{bmatrix}$$

Answer choices:

A $\quad X = \begin{bmatrix} 13 & 6 \\ 5 & 13 \end{bmatrix}$

B $\quad X = \begin{bmatrix} -13 & -6 \\ -5 & -13 \end{bmatrix}$

C $\quad X = \begin{bmatrix} -1 & -8 \\ 3 & 3 \end{bmatrix}$

D $\quad X = \begin{bmatrix} 1 & 8 \\ -3 & -3 \end{bmatrix}$

Solution: C

Let's start with the matrix subtraction on the left side of the equation and the matrix addition on the right side of the equation.

$$\begin{bmatrix}8&2\\7&9\end{bmatrix}-\begin{bmatrix}2&3\\3&1\end{bmatrix}=X+\begin{bmatrix}5&7\\-5&9\end{bmatrix}+\begin{bmatrix}2&0\\6&-4\end{bmatrix}$$

$$\begin{bmatrix}8-2&2-3\\7-3&9-1\end{bmatrix}=X+\begin{bmatrix}5+2&7+0\\-5+6&9+(-4)\end{bmatrix}$$

$$\begin{bmatrix}6&-1\\4&8\end{bmatrix}=X+\begin{bmatrix}7&7\\1&5\end{bmatrix}$$

To isolate X, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{bmatrix}6&-1\\4&8\end{bmatrix}-\begin{bmatrix}7&7\\1&5\end{bmatrix}=X$$

$$\begin{bmatrix}6-7&-1-7\\4-1&8-5\end{bmatrix}=X$$

$$\begin{bmatrix}-1&-8\\3&3\end{bmatrix}=X$$

The conclusion is that the value of X that makes the equation true is this matrix:

$$X=\begin{bmatrix}-1&-8\\3&3\end{bmatrix}$$

Topic: Scalar multiplication

Question: Use scalar multiplication to simplify the expression.

$$4\begin{bmatrix} 5 & 2 & 1 \\ -2 & 4 & 7 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 9 & 6 & 5 \\ 2 & 8 & 11 \end{bmatrix}$

B $\begin{bmatrix} 20 & 8 & 4 \\ -2 & 4 & 7 \end{bmatrix}$

C $\begin{bmatrix} 5 & 2 & 1 \\ -8 & 16 & 28 \end{bmatrix}$

D $\begin{bmatrix} 20 & 8 & 4 \\ -8 & 16 & 28 \end{bmatrix}$

Solution: D

In this problem, 4 is the scalar. We distribute the scalar across every entry in the matrix, and the result of the scalar multiplication is

$$4\begin{bmatrix}5 & 2 & 1\\ -2 & 4 & 7\end{bmatrix}$$

$$\begin{bmatrix}4(5) & 4(2) & 4(1)\\ 4(-2) & 4(4) & 4(7)\end{bmatrix}$$

$$\begin{bmatrix}20 & 8 & 4\\ -8 & 16 & 28\end{bmatrix}$$

Topic: Scalar multiplication

Question: Solve for X.

$$3\begin{bmatrix}7 & 1\\8 & 3\end{bmatrix}+X=-4\begin{bmatrix}0 & -5\\-2 & 3\end{bmatrix}$$

Answer choices:

A $X=\begin{bmatrix}-21 & 17\\-16 & -21\end{bmatrix}$

B $X=\begin{bmatrix}21 & 23\\32 & -3\end{bmatrix}$

C $X=\begin{bmatrix}21 & -17\\16 & 21\end{bmatrix}$

D $X=\begin{bmatrix}-21 & -23\\-32 & 3\end{bmatrix}$

Solution: A

Apply the scalars to the matrices.

$$\begin{bmatrix} 3(7) & 3(1) \\ 3(8) & 3(3) \end{bmatrix} + X = \begin{bmatrix} -4(0) & -4(-5) \\ -4(-2) & -4(3) \end{bmatrix}$$

$$\begin{bmatrix} 21 & 3 \\ 24 & 9 \end{bmatrix} + X = \begin{bmatrix} 0 & 20 \\ 8 & -12 \end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate X.

$$X = \begin{bmatrix} 0 & 20 \\ 8 & -12 \end{bmatrix} - \begin{bmatrix} 21 & 3 \\ 24 & 9 \end{bmatrix}$$

$$X = \begin{bmatrix} 0 - 21 & 20 - 3 \\ 8 - 24 & -12 - 9 \end{bmatrix}$$

$$X = \begin{bmatrix} -21 & 17 \\ -16 & -21 \end{bmatrix}$$

Topic: Scalar multiplication

Question: Use scalar multiplication to find $-(1/2)A+3B$.

$$A=\begin{bmatrix}4 & -2 & 0\\ -8 & 10 & -5\end{bmatrix}$$

$$B=\begin{bmatrix}-1 & 0 & -3\\ 5 & -5 & 0\end{bmatrix}$$

Answer choices:

A $\begin{bmatrix}-5 & -1 & -9\\ 11 & -20 & \frac{5}{2}\end{bmatrix}$

B $\begin{bmatrix}-5 & 1 & -9\\ 11 & -20 & \frac{5}{2}\end{bmatrix}$

C $\begin{bmatrix}-5 & 1 & -9\\ 19 & -20 & \frac{5}{2}\end{bmatrix}$

D $\begin{bmatrix}1 & 4 & -6\\ 19 & 10 & 2\end{bmatrix}$

Solution: C

Substitute the matrices into the expression.

$$-\frac{1}{2}A+3B$$

$$-\frac{1}{2}\begin{bmatrix}4 & -2 & 0\\ -8 & 10 & -5\end{bmatrix}+3\begin{bmatrix}-1 & 0 & -3\\ 5 & -5 & 0\end{bmatrix}$$

Apply the scalars to the matrices.

$$\begin{bmatrix}-\frac{1}{2}(4) & -\frac{1}{2}(-2) & -\frac{1}{2}(0)\\ -\frac{1}{2}(-8) & -\frac{1}{2}(10) & -\frac{1}{2}(-5)\end{bmatrix}+\begin{bmatrix}3(-1) & 3(0) & 3(-3)\\ 3(5) & 3(-5) & 3(0)\end{bmatrix}$$

$$\begin{bmatrix}-2 & 1 & 0\\ 4 & -5 & \frac{5}{2}\end{bmatrix}+\begin{bmatrix}-3 & 0 & -9\\ 15 & -15 & 0\end{bmatrix}$$

Add the matrices.

$$\begin{bmatrix}(-2)+(-3) & 1+0 & 0+(-9)\\ 4+15 & -5+(-15) & \frac{5}{2}+0\end{bmatrix}$$

$$\begin{bmatrix}-5 & 1 & -9\\ 19 & -20 & \frac{5}{2}\end{bmatrix}$$

Topic: Zero matrices

Question: Choose the O_{4x2} matrix.

Answer choices:

A $O_{4x2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

B $O_{4x2} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

C $O_{4x2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

D $O_{4x2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Solution: B

We always name the zero matrix with a capital O. And optionally, you can add a subscript with the dimensions of the zero matrix. Since the values in a zero matrix are all zeros, just having the dimensions of the zero matrix tells you what the entire matrix looks like.

So O_{4x2} is a matrix with four rows and two columns, completely filled with 0 entries.

$$O_{4x2}=\begin{bmatrix}0&0\\0&0\\0&0\\0&0\end{bmatrix}$$

Topic: Zero matrices

Question: What are the dimensions of the zero matrix in $M+O=M$?

$$M=\begin{bmatrix}-5 & 0 & 3 & 1\\ 8 & -15 & -4 & 7\end{bmatrix}$$

Answer choices:

A $\quad 2\times 4$

B $\quad 4\times 2$

C $\quad 2\times 2$

D $\quad 4\times 4$

Solution: A

Adding the zero matrix to any other matrix does not change the matrix's value.

$$M + O = M$$

$$\begin{bmatrix} -5 & 0 & 3 & 1 \\ 8 & -15 & -4 & 7 \end{bmatrix} + O = \begin{bmatrix} -5 & 0 & 3 & 1 \\ 8 & -15 & -4 & 7 \end{bmatrix}$$

Just like with non-zero matrices, matrix dimensions have to be the same in order to be able to add them. The dimensions of M are 2×4, so the dimensions of the zero matrix must also be 2×4.

Topic: Zero matrices

Question: Which matrix is equivalent to $A+(-A)$, where A is a 2×3 matrix?

Answer choices:

A 0

B $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

C $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

D $\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

Solution: C

Matrices A and $-A$ are opposite matrices. Adding opposite matrices always results in the zero matrix.

Since A is 2×3 matrix, $-A$ is also a 2×3 matrix. Therefore, $A + (-A)$ is the $O_{2\times3}$ matrix with 2 rows and 3 columns, completely filled with 0 entries.

Topic: Matrix multiplication

Question: Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 9 & 1 \\ 6 & -1 \end{bmatrix}$$

Answer choices:

A $\quad AB = \begin{bmatrix} 43 & 3 \\ -5 & 6 \end{bmatrix}$

B $\quad AB = \begin{bmatrix} 0 & -14 \\ 23 & 4 \end{bmatrix}$

C $\quad AB = \begin{bmatrix} 57 & 3 \\ -12 & 2 \end{bmatrix}$

D $\quad AB = \begin{bmatrix} 9 & 6 \\ 23 & -8 \end{bmatrix}$

Solution: C

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 5 & 2 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 9 & 1 \\ 6 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5(9) + 2(6) & 5(1) + 2(-1) \\ 0(9) + (-2)(6) & 0(1) + (-2)(-1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 57 & 3 \\ -12 & 2 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Find the product of matrices A and B.

$$A = \begin{bmatrix} 7 & 2 & -4 \\ -5 & 10 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 1 \\ 7 & 2 \\ -2 & 6 \end{bmatrix}$$

Answer choices:

A $\quad AB = \begin{bmatrix} 71 & -13 \\ 29 & 33 \end{bmatrix}$

B $\quad AB = \begin{bmatrix} 45 & -30 \\ -16 & 52 \end{bmatrix}$

C $\quad AB = \begin{bmatrix} -41 & 56 \\ 29 & -16 \end{bmatrix}$

D $\quad AB = \begin{bmatrix} 43 & 33 \\ 82 & 19 \end{bmatrix}$

Solution: A

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 7 & 2 & -4 \\ -5 & 10 & 3 \end{bmatrix} \begin{bmatrix} 7 & 1 \\ 7 & 2 \\ -2 & 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7(7) + 2(7) + (-4)(-2) & 7(1) + 2(2) + (-4)(6) \\ (-5)(7) + 10(7) + 3(-2) & (-5)(1) + 10(2) + 3(6) \end{bmatrix}$$

$$AB = \begin{bmatrix} 71 & -13 \\ 29 & 33 \end{bmatrix}$$

Topic: Matrix multiplication

Question: Use the distributive property to find $A(B+C)$.

$$A = \begin{bmatrix} 3 & -1 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 2 \\ -2 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 2 & 0 \\ 6 & 2 \end{bmatrix}$$

Answer choices:

A $\quad A(B+C) = \begin{bmatrix} -2 & 15 \\ 3 & 32 \end{bmatrix}$

B $\quad A(B+C) = \begin{bmatrix} 17 & 1 \\ 23 & 22 \end{bmatrix}$

C $\quad A(B+C) = \begin{bmatrix} 3 & -14 \\ 27 & 1 \end{bmatrix}$

D $\quad A(B+C) = \begin{bmatrix} 8 & 9 \\ -14 & 17 \end{bmatrix}$

Solution: B

Applying the distributive property to the initial expression, we get

$$A(B+C)=AB+AC$$

Now use matrix multiplication.

$$AB+AC=\begin{bmatrix}3 & -1\\ 1 & 4\end{bmatrix}\begin{bmatrix}5 & 2\\ -2 & 3\end{bmatrix}+\begin{bmatrix}3 & -1\\ 1 & 4\end{bmatrix}\begin{bmatrix}2 & 0\\ 6 & 2\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}3(5)+(-1)(-2) & 3(2)+(-1)(3)\\ 1(5)+4(-2) & 1(2)+4(3)\end{bmatrix}+\begin{bmatrix}3(2)+(-1)(6) & 3(0)+(-1)(2)\\ 1(2)+4(6) & 1(0)+4(2)\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}17 & 3\\ -3 & 14\end{bmatrix}+\begin{bmatrix}0 & -2\\ 26 & 8\end{bmatrix}$$

Adding the matrices gives

$$AB+AC=\begin{bmatrix}17+0 & 3+(-2)\\ -3+26 & 14+8\end{bmatrix}$$

$$AB+AC=\begin{bmatrix}17 & 1\\ 23 & 22\end{bmatrix}$$

So the value of the original expression is

$$A(B+C)=\begin{bmatrix}17 & 1\\ 23 & 22\end{bmatrix}$$

Topic: Identity matrices

Question: Which identity matrix is I_3?

Answer choices:

A $I_3 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B $I_3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

D $I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

Solution: C

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate I_{2x2} as just I_2, or I_{3x3} as just I_3, etc. So I_3 is the 3×3 identity matrix.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topic: Identity matrices

Question: Which identity matrix can be multiplied by A (in other words, IA), if A is a 2×4 matrix?

Answer choices:

A $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

B $\quad I_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

C $\quad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

D $\quad I_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

Solution: A

Start by setting up the equation $IA = A$. Next, substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 2 \times 4 = 2 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns.

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

In order to be able to multiply matrices, we need the same number of columns in the first matrix as we have rows in the second matrix. So the identity matrix must have 2 columns.

$$R \times 2 \cdot 2 \times 4 = 2 \times 4$$

And the dimensions of the resulting matrix come from the rows of the first matrix and the columns of the second matrix. So the identity matrix must have 2 rows.

$$2 \times 2 \cdot 2 \times 4 = 2 \times 4$$

Therefore, the identity matrix we need is I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Topic: Identity matrices

Question: If we want to find IA, which identity matrix should we use, and what is the product?

$$A = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

Answer choices:

A Use $I_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} 2 & 8 & 4 \\ -3 & 7 & 1 \end{bmatrix}$

B Use $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$

C Use $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$

D Use $I_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and the product is $IA = \begin{bmatrix} 2 & 8 & 4 \\ -3 & 7 & 1 \end{bmatrix}$

Solution: B

Matrix A is a 2×3 matrix, and we need to find IA. We also know that IA will be 2×3. So we'll set up an equation of dimensions.

$$I \cdot A = A$$

$$I \cdot 2 \times 3 = 2 \times 3$$

$$R \times C \cdot 2 \times 3 = 2 \times 3$$

For matrix multiplication to be valid, we need the same number of columns in the first matrix as we have rows in the second matrix.

$$R \times 2 \cdot 2 \times 3 = 2 \times 3$$

The dimensions of the result are given by the rows from the first matrix, and columns from the second matrix.

$$2 \times 2 \cdot 2 \times 3 = 2 \times 3$$

So the identity matrix is 2×2, which means it's I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then the product of I_2 and matrix A is

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(-3) + 0(2) & 1(7) + 0(8) & 1(1) + 0(4) \\ 0(-3) + 1(2) & 0(7) + 1(8) & 0(1) + 1(4) \end{bmatrix}$$

$$IA = \begin{bmatrix} -3 & 7 & 1 \\ 2 & 8 & 4 \end{bmatrix}$$

Topic: The elimination matrix

Question: Which elimination matrix accomplishes the row operation?

$$-2R_3+R_1\to R_1$$

Answer choices:

A $\quad E=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -2 & 0 & 1\end{bmatrix}$

B $\quad E=\begin{bmatrix}1 & 0 & 0\\ 2 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

C $\quad E=\begin{bmatrix}1 & 0 & -2\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

D $\quad E=\begin{bmatrix}1 & 2 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$

Solution: C

The row operation $-2R_3 + R_1 \to R_1$ means we're leaving the second and third rows alone, but replacing the first row with 1 of the first row and -2 of the third row.

So to get the elimination matrix that accomplishes the row operation, we'll put a 1 in $E_{1,1}$ and a -2 in $E_{1,3}$.

$$E = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Topic: The elimination matrix

Question: Which elimination matrix accomplishes the row operations?

$$(1/2)R_2 \to R_2$$

$$-R_2 + R_1 \to R_1$$

Answer choices:

A $\quad E = \begin{bmatrix} 1 & -1 \\ 0 & \frac{1}{2} \end{bmatrix}$

B $\quad E = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$

C $\quad E = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

D $\quad E = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{bmatrix}$

Solution: B

The row operation $(1/2)R_2 \to R_2$ means we're leaving the first row alone, but multiplying the second row by a scalar of $1/2$, so we'll put a $1/2$ in $E_{2,2}$.

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The row operation $-R_2 + R_1 \to R_1$ means we're leaving the second row alone, but replacing the first row with 1 of the first row and -1 of the second row, so we'll put a 1 in $E_{1,1}$, and a -1 in $E_{1,2}$.

$$E_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

We can consolidate these two row operations into one elimination matrix, simply by multiplying E_2 by E_1.

$$E = E_2 E_1$$

$$E = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$E = \begin{bmatrix} 1(1) + (-1)(0) & 1(0) + (-1)\left(\frac{1}{2}\right) \\ 0(1) + 1(0) & 0(0) + 1\left(\frac{1}{2}\right) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 + 0 & 0 - \frac{1}{2} \\ 0 + 0 & 0 + \frac{1}{2} \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$$

Topic: The elimination matrix

Question: Which elimination matrix puts A

$$A=\begin{bmatrix}-1 & 2 & 0\\ 0 & 1 & 3\\ 0 & 0 & 1\end{bmatrix}$$

into reduced row-echelon form, where E accounts for this set of row operations?

1. $-R_1 \to R_1$

2. $-3R_3+R_2 \to R_2$

3. $2R_2+R_1 \to R_1$

Answer choices:

A $\quad E=\begin{bmatrix}-1 & 0 & 0\\ -2 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}$

B $\quad E=\begin{bmatrix}-1 & 2 & -6\\ 0 & 1 & -3\\ 0 & 0 & 1\end{bmatrix}$

C $\quad E=\begin{bmatrix}-1 & 2 & 0\\ 0 & 1 & 0\\ 0 & -3 & 1\end{bmatrix}$

D $\quad E=\begin{bmatrix}-1 & 0 & 0\\ -2 & 1 & 0\\ 3 & 0 & 1\end{bmatrix}$

Solution: B

The row operation $-R_1 \to R_1$ means we're leaving the second and third rows alone, but multiplying the first row by a scalar of -1, so we'll put a -1 into $E_{1,1}$.

$$E_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $-3R_3 + R_2 \to R_2$ means we're leaving the first and third rows alone, but replacing the second row with 1 of the second row and -3 of the third row, so we'll put a 1 in $E_{2,2}$ and a -3 in $E_{2,3}$.

$$E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $2R_2 + R_1 \to R_1$ means we're leaving the second and third rows alone, but replacing the first row with 1 of the first row and a 2 of the second row, so we'll put a 1 in $E_{1,1}$ and a 2 in $E_{1,2}$.

$$E_3E_2E_1 = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now we can find the consolidated elimination matrix E by finding the product $E_3E_2E_1$.

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(-1)+0(0)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(-1)+1(0)-3(0) & 0(0)+1(1)-3(0) & 0(0)+1(0)-3(1) \\ 0(-1)+0(0)+1(0) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1(-1)+2(0)+0(0) & 1(0)+2(1)+0(0) & 1(0)+2(-3)+0(1) \\ 0(-1)+1(0)+0(0) & 0(0)+1(1)+0(0) & 0(0)+1(-3)+0(1) \\ 0(-1)+0(0)+1(0) & 0(0)+0(1)+1(0) & 0(0)+0(-3)+1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} -1 & 2 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

We've found the elimination matrix, and we can check to make sure that it reduces A to the identity matrix.

$$EA = \begin{bmatrix} -1 & 2 & -6 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

$$EA = \begin{bmatrix} -1(-1)+2(0)-6(0) & -1(2)+2(1)-6(0) & -1(0)+2(3)-6(1) \\ 0(-1)+1(0)-3(0) & 0(2)+1(1)-3(0) & 0(0)+1(3)-3(1) \\ 0(-1)+0(0)+1(0) & 0(2)+0(1)+1(0) & 0(0)+0(3)+1(1) \end{bmatrix}$$

$$EA = \begin{bmatrix} 1+0+0 & -2+2+0 & 0+6-6 \\ 0+0+0 & 0+1+0 & 0+3-3 \\ 0+0+0 & 0+0+0 & 0+0+1 \end{bmatrix}$$

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because multiplying the elimination matrix by A gives us the identity matrix, we know that we got the correct elimination matrix.

Topic: Vectors

Question: How many row vectors and column vectors are in matrix K?

$$K = \begin{bmatrix} 1 & -1 & 1 & 4 \\ -2 & 1 & 0 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

Answer choices:

A K has 3 row vectors and 4 column vectors

B K has 4 row vectors and 3 column vectors

C K has 3 row vectors and 3 column vectors

D K has 4 row vectors and 4 column vectors

Solution: A

The matrix K has 3 rows and 4 columns, which means it'll have 3 row vectors and 4 column vectors. The row vectors of K,

$$K = \begin{bmatrix} 1 & -1 & 1 & 4 \\ -2 & 1 & 0 & -1 \\ 0 & 0 & 3 & 1 \end{bmatrix}$$

are given by

$$k_1 = \begin{bmatrix} 1 & -1 & 1 & 4 \end{bmatrix}$$

$$k_2 = \begin{bmatrix} -2 & 1 & 0 & -1 \end{bmatrix}$$

$$k_3 = \begin{bmatrix} 0 & 0 & 3 & 1 \end{bmatrix}$$

And the column vectors of K are given by

$$k_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}, k_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, k_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, k_4 = \begin{bmatrix} 4 \\ -1 \\ 1 \end{bmatrix}$$

Topic: Vectors

Question: Name the column vectors of A.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

Answer choices:

A $\quad \vec{a}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}$

B $\quad \vec{a}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}$

C $\quad \vec{a}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}$

D $\quad \vec{a}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$

Solution: D

The column vectors of a matrix are the individual columns of the matrix. In the matrix A,

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

there are four columns, which means we'll have four column vectors.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ 4 \\ 5 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 1 \\ -2 \\ 6 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 3 \\ 8 \\ -2 \end{bmatrix}, \vec{a}_4 = \begin{bmatrix} 1 \\ 4 \\ -3 \end{bmatrix}$$

Topic: Vectors

Question: Name the row vectors of A.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

Answer choices:

A $\quad \vec{a}_1 = [2 \quad 1 \quad 3 \quad 1]$

B $\quad \vec{a}_1 = [2 \quad 1 \quad 3 \quad 1]$, $\vec{a}_2 = [4 \quad -2 \quad 8 \quad 4]$

C $\quad \vec{a}_1 = [2 \quad 1 \quad 3 \quad 1]$, $\vec{a}_2 = [4 \quad -2 \quad 8 \quad 4]$, $\vec{a}_3 = [5 \quad 6 \quad -2 \quad -3]$

D $\quad \vec{a}_1 = [2 \quad 1 \quad 3 \quad 1]$, $\vec{a}_2 = [4 \quad -2 \quad 8 \quad 4]$, $\vec{a}_3 = [6 \quad -1 \quad 11 \quad 5]$

Solution: C

The row vectors of a matrix are the individual rows of the matrix. In the matrix A,

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 4 & -2 & 8 & 4 \\ 5 & 6 & -2 & -3 \end{bmatrix}$$

there are three rows, which means we'll have three row vectors.

$$\vec{a}_1 = \begin{bmatrix} 2 & 1 & 3 & 1 \end{bmatrix}$$

$$\vec{a}_2 = \begin{bmatrix} 4 & -2 & 8 & 4 \end{bmatrix}$$

$$\vec{a}_3 = \begin{bmatrix} 5 & 6 & -2 & -3 \end{bmatrix}$$

Topic: Vector operations

Question: Find the sum $\overrightarrow{a}+\overrightarrow{b}$.

$$\overrightarrow{a}=(-2,3)$$

$$\overrightarrow{b}=(4,-1)$$

Answer choices:

A $\quad \overrightarrow{a}+\overrightarrow{b}=(2,2)$

B $\quad \overrightarrow{a}+\overrightarrow{b}=(1,3)$

C $\quad \overrightarrow{a}+\overrightarrow{b}=(-6,4)$

D $\quad \overrightarrow{a}+\overrightarrow{b}=(6,-4)$

Solution: A

To find the sum of the vectors, we just add the corresponding components.

$$\overrightarrow{a} + \overrightarrow{b} = (-2,3) + (4,-1)$$

$$\overrightarrow{a} + \overrightarrow{b} = (-2+4,3-1)$$

$$\overrightarrow{a} + \overrightarrow{b} = (2,2)$$

Topic: Vector operations

Question: Find the difference $\overrightarrow{a} - \overrightarrow{b}$.

$$\overrightarrow{a} = (-2,3)$$

$$\overrightarrow{b} = (4,-1)$$

Answer choices:

A $\quad \overrightarrow{a} - \overrightarrow{b} = (2,2)$

B $\quad \overrightarrow{a} - \overrightarrow{b} = (1,3)$

C $\quad \overrightarrow{a} - \overrightarrow{b} = (-6,4)$

D $\quad \overrightarrow{a} - \overrightarrow{b} = (6,-4)$

Solution: C

To find the difference of the vectors, we just subtract the corresponding components.

$$\vec{a} - \vec{b} = (-2,3) - (4, -1)$$

$$\vec{a} - \vec{b} = (-2 - 4, 3 - (-1))$$

$$\vec{a} - \vec{b} = (-6,4)$$

Topic: Vector operations

Question: Find the sum $2\vec{c} + 3\vec{a} - \vec{d} + 4\vec{b}$.

$$\vec{a} = (-2,3)$$

$$\vec{b} = (4, -1)$$

$$\vec{c} = (-1,1)$$

$$\vec{d} = (3, -2)$$

Answer choices:

A $\quad 2\vec{c} + 3\vec{a} - \vec{d} + 4\vec{b} = (-5, -9)$

B $\quad 2\vec{c} + 3\vec{a} - \vec{d} + 4\vec{b} = (5, -9)$

C $\quad 2\vec{c} + 3\vec{a} - \vec{d} + 4\vec{b} = (-5,9)$

D $\quad 2\vec{c} + 3\vec{a} - \vec{d} + 4\vec{b} = (5,9)$

Solution: D

To find the sum of the vectors, we'll first apply the scalars to the vectors individually.

$$2\vec{c}+3\vec{a}-\vec{d}+4\vec{b}=2(-1,1)+3(-2,3)-(3,-2)+4(4,-1)$$

$$2\vec{c}+3\vec{a}-\vec{d}+4\vec{b}=(-2,2)+(-6,9)-(3,-2)+(16,-4)$$

We'll combine the vectors one by one.

$$2\vec{c}+3\vec{a}-\vec{d}+4\vec{b}=(-8,11)-(3,-2)+(16,-4)$$

$$2\vec{c}+3\vec{a}-\vec{d}+4\vec{b}=(-8-3,11-(-2))+(16,-4)$$

$$2\vec{c}+3\vec{a}-\vec{d}+4\vec{b}=(-11,13)+(16,-4)$$

$$2\vec{c}+3\vec{a}-\vec{d}+4\vec{b}=(-11+16,13-4)$$

$$2\vec{c}+3\vec{a}-\vec{d}+4\vec{b}=(5,9)$$

Topic: Unit vectors and basis vectors

Question: Find the unit vector in the direction of $\overrightarrow{a} = (5, -9)$.

Answer choices:

A $\quad \overrightarrow{u} = \begin{bmatrix} \frac{5}{\sqrt{106}} \\ \frac{9}{\sqrt{106}} \end{bmatrix}$

B $\quad \overrightarrow{u} = \begin{bmatrix} -\frac{5}{\sqrt{106}} \\ -\frac{9}{\sqrt{106}} \end{bmatrix}$

C $\quad \overrightarrow{u} = \begin{bmatrix} \frac{5}{\sqrt{106}} \\ -\frac{9}{\sqrt{106}} \end{bmatrix}$

D $\quad \overrightarrow{u} = \begin{bmatrix} -\frac{5}{\sqrt{106}} \\ \frac{9}{\sqrt{106}} \end{bmatrix}$

Solution: C

First, find the length of $\vec{a}$.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2}$$

$$||\vec{a}|| = \sqrt{5^2 + (-9)^2}$$

$$||\vec{a}|| = \sqrt{25 + 81}$$

$$||\vec{a}|| = \sqrt{106}$$

Then the unit vector in the direction of $\vec{a} = (5, -9)$ is

$$\vec{u} = \frac{1}{||\vec{a}||}\vec{a}$$

$$\vec{u} = \frac{1}{\sqrt{106}}\begin{bmatrix} 5 \\ -9 \end{bmatrix}$$

$$\vec{u} = \begin{bmatrix} \frac{5}{\sqrt{106}} \\ -\frac{9}{\sqrt{106}} \end{bmatrix}$$

Topic: Unit vectors and basis vectors

Question: Find the unit vector in the direction of $\overrightarrow{v} = (2, -1, 4)$.

Answer choices:

A $\quad \overrightarrow{u} = \begin{bmatrix} \frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{bmatrix}$

B $\quad \overrightarrow{u} = \begin{bmatrix} -\frac{2}{\sqrt{21}} \\ -\frac{1}{\sqrt{21}} \\ -\frac{4}{\sqrt{21}} \end{bmatrix}$

C $\quad \overrightarrow{u} = \begin{bmatrix} -\frac{2}{\sqrt{21}} \\ \frac{1}{\sqrt{21}} \\ -\frac{4}{\sqrt{21}} \end{bmatrix}$

D $\quad \overrightarrow{u} = \begin{bmatrix} \frac{2}{\sqrt{21}} \\ -\frac{1}{\sqrt{21}} \\ \frac{4}{\sqrt{21}} \end{bmatrix}$

Solution: D

First, find the length of $\vec{v}$.

$$||\vec{v}|| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$||\vec{v}|| = \sqrt{2^2 + (-1)^2 + 4^2}$$

$$||\vec{v}|| = \sqrt{4 + 1 + 16}$$

$$||\vec{v}|| = \sqrt{21}$$

Then the unit vector in the direction of $\vec{v} = (2, -1, 4)$ is

$$\vec{u} = \frac{1}{||\vec{v}||}\vec{v}$$

$$\vec{u} = \frac{1}{\sqrt{21}}\begin{bmatrix}2\\-1\\4\end{bmatrix}$$

$$\vec{u} = \begin{bmatrix}\frac{2}{\sqrt{21}}\\-\frac{1}{\sqrt{21}}\\\frac{4}{\sqrt{21}}\end{bmatrix}$$

Topic: Unit vectors and basis vectors

Question: Represent $\overrightarrow{x}=(-2,8,-4)$ with the standard basis vectors.

Answer choices:

A $\quad\overrightarrow{x}=2\hat{i}+8\hat{j}+4\hat{k}$

B $\quad\overrightarrow{x}=2\hat{i}-8\hat{j}+4\hat{k}$

C $\quad\overrightarrow{x}=-2\hat{i}+8\hat{j}-4\hat{k}$

D $\quad\overrightarrow{x}=-2\hat{i}-8\hat{j}-4\hat{k}$

Solution: C

The vector $\vec{x}=(-2,8,-4)$ is part of $\mathbb{R}^3$, which means we'll need to use the basis vectors for $\mathbb{R}^3$, which are $\hat{i}=(1,0,0)$, $\hat{j}=(0,1,0)$, and $\hat{k}=(0,0,1)$.

We're moving -2 units in the direction of the x-axis, 8 units in the direction of the y-axis, and -4 units in the direction of the z-axis.

$$\vec{x}=(-2,8,-4)=-2\begin{bmatrix}1\\0\\0\end{bmatrix}+8\begin{bmatrix}0\\1\\0\end{bmatrix}-4\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{x}=(-2,8,-4)=\begin{bmatrix}-2\\0\\0\end{bmatrix}+\begin{bmatrix}0\\8\\0\end{bmatrix}+\begin{bmatrix}0\\0\\-4\end{bmatrix}$$

$$\vec{x}=(-2,8,-4)=\begin{bmatrix}-2+0+0\\0+8+0\\0+0-4\end{bmatrix}$$

$$\vec{x}=(-2,8,-4)=\begin{bmatrix}-2\\8\\-4\end{bmatrix}$$

So we can express $\vec{x}=(-2,8,-4)$ in terms of basis vectors as

$$\vec{x}=-2\hat{i}+8\hat{j}-4\hat{k}$$

Topic: Linear combinations and span

Question: How many linearly independent vectors are needed to span $\mathbb{R}^4$?

Answer choices:

A 1

B 2

C 3

D 4

Solution: D

Any n n-dimensional linearly independent vectors will span $\mathbb{R}^n$. So in order to span $\mathbb{R}^4$, we'll need 4 linearly independent vectors.

Topic: Linear combinations and span

Question: Will the vectors span $\mathbb{R}^2$?

$$\vec{u}=(3,1)$$

$$\vec{v}=(6,2)$$

Answer choices:

A Yes, because the vectors aren't parallel

B Yes, because the vectors are parallel

C No, because the vectors aren't parallel

D No, because the vectors are parallel

Solution: D

The vectors $\overrightarrow{u}=(3,1)$ and $\overrightarrow{v}=(6,2)$ are parallel. We can tell this by sketching them, or by looking at the slope of each vector, where the slope of $\overrightarrow{u}$ is $1/3$, and the slope of $\overrightarrow{v}$ is $2/6=1/3$.

Parallel vectors can never span their space. So $\overrightarrow{u}=(3,1)$ and $\overrightarrow{v}=(6,2)$ can't span $\mathbb{R}^2$ because of the fact that they're parallel.

Topic: Linear combinations and span

Question: Can the standard basis vectors $\mathbf{i}=(1,0,0)$, $\mathbf{j}=(0,1,0)$ and $\mathbf{k}=(0,0,1)$ span $\mathbb{R}^3$?

Answer choices:

A Yes, because three linearly independent vectors in $\mathbb{R}^3$ will span $\mathbb{R}^3$

B Yes, because three linearly dependent vectors in $\mathbb{R}^3$ will span $\mathbb{R}^3$

C No, because three linearly independent vectors in $\mathbb{R}^3$ won't span $\mathbb{R}^3$

D No, because three linearly dependent vectors in $\mathbb{R}^3$ won't span $\mathbb{R}^3$

Solution: A

Any 3 three-dimensional linearly independent vectors will span $\mathbb{R}^3$. The three-dimensional basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ are linearly independent, which is why they span $\mathbb{R}^3$.

Topic: Linear independence in two dimensions

Question: Which vector set might be linearly independent?

Answer choices:

A A set of 2 two-dimensional vectors

B A set of 3 two-dimensional vectors

C A set of 4 two-dimensional vectors

D A set of 4 three-dimensional vectors

Solution: A

Any n n-dimensional linearly independent vectors can span $\mathbb{R}^n$. Which means that any $n+1$ or greater set of vectors in $\mathbb{R}^n$ will be linearly dependent.

So given two-dimensional vectors, only a set of two or fewer can be linearly independent, or given three-dimensional vectors, only a set of three or fewer can be linearly independent.

Answer choices B, C, and D all have too many vectors for the dimension in which they're defined, so only answer choice A can be a linearly independent set.

Topic: Linear independence in two dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

B $\quad \vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}$

C $\quad \vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}, \vec{c} = \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

D $\quad \vec{a} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} 3 \\ -6 \end{bmatrix}, \vec{c} = \begin{bmatrix} 4 \\ -2 \end{bmatrix}$

Solution: B

In $\mathbb{R}^n$ space, only vector sets with n or fewer vectors can be linearly independent. For instance, in $\mathbb{R}^2$, only vector sets with one or two vectors can be a linearly independent set, and any set with three or more vectors will be linearly dependent.

Because all of the vectors in these answer choices are in $\mathbb{R}^2$, a linearly independent set will include two or fewer vectors, which leaves only answer choices A and B as possibilities.

Let's test answer choice A by setting up the vector equation.

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 2 & 6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equation

$$c_1 + 3c_2 = 0$$

$$c_1 = -3c_2$$

This equation tells us that there are an endless number of solutions to the system. We can choose any value for c_2, and we'll get a different value for c_1, and all of those combinations will give us the zero vector. Because $(c_1, c_2) = (0,0)$ isn't the only solution, that tells us that the vectors in answer choice A are linearly dependent.

Which means answer choice B must be the correct choice, but let's verify that those vectors are, in fact, linearly independent. We'll test answer choice B by setting up the vector equation.

$$c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ -6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 2 & -6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & -12 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

These equations tell us that $(c_1, c_2) = (0,0)$ is the only solution, which means the vectors in answer choice B are linearly independent.

Topic: Linear independence in two dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{a} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 9 \\ -15 \end{bmatrix}, \vec{c} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$

B $\quad \vec{a} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}, \vec{c} = \begin{bmatrix} 1 \\ -6 \end{bmatrix}$

C $\quad \vec{a} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 9 \\ -15 \end{bmatrix}$

D $\quad \vec{a} = \begin{bmatrix} -3 \\ 5 \end{bmatrix}, \vec{b} = \begin{bmatrix} 9 \\ 15 \end{bmatrix}$

Solution: D

In $\mathbb{R}^n$ space, only vector sets with n or fewer vectors can be linearly independent. For instance, in $\mathbb{R}^2$, only vector sets with one or two vectors can be a linearly independent set, and any set with three or more vectors will be linearly dependent.

Because all of the vectors in these answer choices are in $\mathbb{R}^2$, a linearly independent set will include two or fewer vectors, which leaves only answer choices C and D as possibilities.

Let's test answer choice C by setting up the vector equation.

$$c_1 \begin{bmatrix} -3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 9 \\ -15 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} -3 & 9 & | & 0 \\ 5 & -15 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 5 & -15 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equation

$$c_1 - 3c_2 = 0$$

$$c_1 = 3c_2$$

This equation tells us that there are an endless number of solutions to the system. We can choose any value for c_2, and we'll get a different value for c_1, and all of those combinations will give us the zero vector. Because $(c_1, c_2) = (0,0)$ isn't the only solution, that tells us that the vectors in answer choice C are linearly dependent.

Which means answer choice D must be the correct choice, but let's verify that those vectors are, in fact, linearly independent. We'll test answer choice D by setting up the vector equation.

$$c_1 \begin{bmatrix} -3 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 9 \\ 15 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} -3 & 9 & | & 0 \\ 5 & 15 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 5 & 15 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 0 & 30 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

These equations tell us that $(c_1, c_2) = (0,0)$ is the only solution, which means the vectors in answer choice D are linearly independent.

Topic: Linear independence in three dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \vec{c} = \begin{bmatrix} -10 \\ -5 \\ 5 \end{bmatrix}$

B $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \vec{c} = \begin{bmatrix} -6 \\ 3 \\ 4 \end{bmatrix}$

C $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \vec{c} = \begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix}$

D $\quad \vec{a} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \vec{c} = \begin{bmatrix} -6 \\ -3 \\ 4 \end{bmatrix}$

Solution: A

All of the answer choices could be a linearly independent set in $\mathbb{R}^3$, since each set has three or fewer vectors.

Let's test answer choice A by setting up the vector equation.

$$c_1 \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} -10 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 4 & -2 & -10 & | & 0 \\ 2 & 1 & -5 & | & 0 \\ 2 & 1 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 2 & 1 & -5 & | & 0 \\ 2 & 1 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 0 & 2 & 0 & | & 0 \\ 2 & 1 & 5 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 0 & 2 & 0 & | & 0 \\ 0 & 2 & 10 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 2 & 10 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 2 & 10 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 10 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{5}{2} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

These equations tell us that $(c_1, c_2 - c_3) = (0,0,0)$ is the only solution, which means the vectors in answer choice A are linearly independent.

Topic: Linear independence in three dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}, \vec{x} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

B $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}, \vec{x} = \begin{bmatrix} -5 \\ 0 \\ 3 \end{bmatrix}$

C $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix}$

D $\quad \vec{u} = \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}, \vec{w} = \begin{bmatrix} -6 \\ 4 \\ 2 \end{bmatrix}$

Solution: D

In $\mathbb{R}^n$ space, only vector sets with n or fewer vectors can be linearly independent. For instance, in $\mathbb{R}^3$, only vector sets with three or fewer vectors can be a linearly independent set, and any set with four or more vectors will be linearly dependent.

Because all of the vectors in these answer choices are in $\mathbb{R}^3$, a linearly independent set will include three or fewer vectors, which leaves only answer choices C and D as possibilities.

Let's test answer choice C by setting up the vector equation.

$$c_1 \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} 6 \\ 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 7 & 3 & 6 & | & 0 \\ -1 & 2 & 4 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 4 & | & 0 \\ 7 & 3 & 6 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 7 & 3 & 6 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 34 & | & 0 \\ 4 & -1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 34 & | & 0 \\ 0 & 7 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & 14 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 + 2c_3 = 0 \quad \to \quad c_2 = -2c_3$$

This equation tells us that there are an endless number of solutions to the system. We can choose any value for c_3, and we'll get a different value for c_2, and all of those combinations will give us the zero vector. Because $(c_1, c_2, c_3) = (0,0,0)$ isn't the only solution, that tells us that the vectors in answer choice C are linearly dependent.

Which means answer choice D must be the correct choice, but let's verify that those vectors are, in fact, linearly independent. We'll test answer choice D by setting up the vector equation.

$$c_1 \begin{bmatrix} 7 \\ -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} + c_3 \begin{bmatrix} -6 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 7 & 3 & -6 & | & 0 \\ -1 & 2 & 4 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & 4 & | & 0 \\ 7 & 3 & -6 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 7 & 3 & -6 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 22 & | & 0 \\ 4 & -1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 17 & 22 & | & 0 \\ 0 & 7 & 18 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -4 & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 7 & 18 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{24}{17} & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 7 & 18 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{24}{17} & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 0 & \frac{152}{17} & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{24}{17} & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{22}{17} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

These equations tell us that $(c_1, c_2, c_3) = (0,0,0)$ is the only solution, which means the vectors in answer choice D are linearly independent.

Topic: Linear independence in three dimensions

Question: Which vector set is linearly independent?

Answer choices:

A $\quad \vec{a} = \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix}, \vec{c} = \begin{bmatrix} 16 \\ -8 \\ -4 \end{bmatrix}$

B $\quad \vec{a} = \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix}, \vec{c} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}$

C $\quad \vec{a} = \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix}, \vec{c} = \begin{bmatrix} 16 \\ -8 \\ -4 \end{bmatrix}, \vec{d} = \begin{bmatrix} -8 \\ 6 \\ -2 \end{bmatrix}$

D $\quad \vec{a} = \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix}, \vec{c} = \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix}, \vec{d} = \begin{bmatrix} -8 \\ 6 \\ -2 \end{bmatrix}$

Solution: B

In $\mathbb{R}^n$ space, only vector sets with n or fewer vectors can be linearly independent. For instance, in $\mathbb{R}^3$, only vector sets with three or fewer vectors can be a linearly independent set, and any set with four or more vectors will be linearly dependent.

Because all of the vectors in these answer choices are in $\mathbb{R}^3$, a linearly independent set will include three or fewer vectors, which leaves only answer choices A and B as possibilities.

Let's test answer choice A by setting up the vector equation.

$$c_1\begin{bmatrix}-8\\4\\2\end{bmatrix}+c_2\begin{bmatrix}-3\\-6\\-9\end{bmatrix}+c_3\begin{bmatrix}16\\-8\\-4\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix}-8 & -3 & 16 & | & 0\\ 4 & -6 & -8 & | & 0\\ 2 & -9 & -4 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\ 4 & -6 & -8 & | & 0\\ 2 & -9 & -4 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\ 0 & -\frac{15}{2} & 0 & | & 0\\ 2 & -9 & -4 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\ 0 & -\frac{15}{2} & 0 & | & 0\\ 0 & -\frac{39}{4} & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & \frac{3}{8} & -2 & | & 0\\ 0 & 1 & 0 & | & 0\\ 0 & -\frac{39}{4} & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & -2 & | & 0\\ 0 & 1 & 0 & | & 0\\ 0 & -\frac{39}{4} & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 - 2c_3 = 0 \quad \to \quad c_1 = 2c_3$$

$$c_2 = 0$$

These equations tell us that there are an endless number of solutions to the system. We can choose any value for c_3, and we'll get a different value for c_1, and all of those combinations will give us the zero vector. Because $(c_1, c_2, c_3) = (0,0,0)$ isn't the only solution, that tells us that the vectors in answer choice A are linearly dependent.

Which means answer choice B must be the correct choice, but let's verify that those vectors are, in fact, linearly independent. We'll test answer choice B by setting up the vector equation.

$$c_1 \begin{bmatrix} -8 \\ 4 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ -6 \\ -9 \end{bmatrix} + c_3 \begin{bmatrix} -6 \\ 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Change the equation into an augmented matrix, then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} -8 & -3 & -6 & | & 0 \\ 4 & -6 & 1 & | & 0 \\ 2 & -9 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 4 & -6 & 1 & | & 0 \\ 2 & -9 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 0 & -\frac{15}{2} & -2 & | & 0 \\ 2 & -9 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 0 & -\frac{15}{2} & -2 & | & 0 \\ 0 & -\frac{39}{4} & \frac{3}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{8} & \frac{3}{4} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & -\frac{39}{4} & \frac{3}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{13}{20} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & -\frac{39}{4} & \frac{3}{2} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{13}{20} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & 0 & \frac{123}{30} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{13}{20} & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{4}{15} & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The rref form of the matrix gives the equations

$$c_1 = 0$$

$$c_2 = 0$$

$$c_3 = 0$$

These equations tell us that $(c_1, c_2, c_3) = (0,0,0)$ is the only solution, which means the vectors in answer choice B are linearly independent.

Topic: Linear subspaces

Question: Which part of the definition of a subspace is redundant, because it's already contained in the other parts of the definition?

Answer choices:

A The set includes the zero vector.

B The set is closed under scalar multiplication.

C The set is closed under addition.

D The set is defined in $\mathbb{R}^2$.

Solution: A

The requirement that the set contains the zero vector is redundant. If the set is closed under scalar multiplication, that means any vector in the set can be multiplied by any scalar, and the resulting vector will still be in the set.

By definition, that means we can multiply by the zero vector, and the zero vector must still be in the set. So it’s redundant to say that the zero vector must be included in the set. If the set is closed under scalar multiplication, then the zero vector would already be included.

So really only answer choices B and C are needed to form the definition of a subspace. Answer choice D isn’t part of the definition of a subspace at all. A subspace can be defined in any $\mathbb{R}^n$ space, not just in $\mathbb{R}^2$.

Topic: Linear subspaces

Question: Which of these are possible subspaces of $\mathbb{R}^2$?

Answer choices:

A $\mathbb{R}^2$

B $\vec{O} = (0,0)$

C A line through $(0,0)$

D All of these

Solution: D

All of these are subspaces within $\mathbb{R}^2$. The space $\mathbb{R}^2$ itself is a subspace of $\mathbb{R}^2$, the zero vector $\vec{O}=(0,0)$ is a subspace of $\mathbb{R}^2$, and any line in $\mathbb{R}^2$ that runs through the origin is also a subspace of $\mathbb{R}^2$.

Topic: Linear subspaces

Question: Is V a subspace of $\mathbb{R}^2$?

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x = 1 \right\}$$

Answer choices:

A Yes

B No, because it's not closed under addition

C No, because it's not closed under scalar multiplication

D No, because it's not closed under addition or scalar multiplication

Solution: D

In order for V to be a subspace, it must be closed under addition and closed under scalar multiplication. Let's pick a couple of vectors in V to see what they do when we add them and scale them. From

$$V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \,\middle|\, x = 1 \right\}$$

we can choose any vectors where $x = 1$. Let's pick

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

If we add $\vec{v}_1$ and $\vec{v}_2$, we get

$$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+1 \\ 1+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

The result isn't in V since its x component isn't $x = 1$, so V isn't closed under addition. If we multiply $\vec{v}_1$ by a scalar, let's choose 3, we get

$$3\vec{v}_1 = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

The result isn't in V since its x component isn't $x = 1$, so V isn't closed under scalar multiplication.

Topic: Spans as subspaces

Question: Is V a subspace?

$$V = \text{Span}\left(\begin{bmatrix}-3\\-2\end{bmatrix}\right)$$

Answer choices:

A Yes

B No, because it's not closed under addition

C No, because it's not closed under scalar multiplication

D No, because it's not closed under addition or scalar multiplication

Solution: A

The set V is a subspace, because V is a span of a vector, and a span is always a subspace.

The span of a vector is all the possible linear combinations of that vector. For instance, we could create the linear combination

$$c_1\begin{bmatrix}-3\\-2\end{bmatrix}+c_2\begin{bmatrix}-3\\-2\end{bmatrix}+c_3\begin{bmatrix}-3\\-2\end{bmatrix}$$

and then factor out the vector.

$$(c_1+c_2+c_3)\begin{bmatrix}-3\\-2\end{bmatrix}$$

The constant $(c_1+c_2+c_3)$ is still just a constant, which means we could rewrite the linear combination as

$$c_4\begin{bmatrix}-3\\-2\end{bmatrix}$$

This result is still just a linear combination of the vector in the set, which means it's still contained within the span. Therefore, the set is closed under addition.

And because multiplying a linear combination of the vector by a scalar still just gives a linear combination of the vector, the set is also closed under scalar multiplication.

Topic: Spans as subspaces

Question: Is V a subspace?

$$V = \text{Span}\left(\begin{bmatrix}1\\-4\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right)$$

Answer choices:

A Yes

B No, because it's not closed under addition

C No, because it's not closed under scalar multiplication

D No, because it's not closed under addition or scalar multiplication

Solution: A

The set V is a subspace, because V is a span of vectors, and a span is always a subspace.

The span of vectors is all the possible linear combinations of those vectors. For instance, we could create the linear combination

$$c_1\begin{bmatrix}1\\-4\end{bmatrix}+c_2\begin{bmatrix}1\\-4\end{bmatrix}+c_3\begin{bmatrix}0\\2\end{bmatrix}+c_4\begin{bmatrix}0\\2\end{bmatrix}$$

and then factor out each vector.

$$(c_1+c_2)\begin{bmatrix}1\\-4\end{bmatrix}+(c_3+c_4)\begin{bmatrix}0\\2\end{bmatrix}$$

The constants (c_1+c_2) and (c_3+c_4) are still just constants, which means we could rewrite the linear combination as

$$c_5\begin{bmatrix}1\\-4\end{bmatrix}+c_6\begin{bmatrix}0\\2\end{bmatrix}$$

This result is still just a linear combination of the vectors in the set, which means it's still contained within the span. Therefore, the set is closed under addition.

And because multiplying a linear combination of the vectors by a scalar still just gives a linear combination of the vectors, the set is also closed under scalar multiplication.

Topic: Spans as subspaces

Question: Is V a subspace?

$$V = \text{Span}\left(\begin{bmatrix}1\\0\\-6\end{bmatrix}, \begin{bmatrix}-2\\1\\3\end{bmatrix}\right)$$

Answer choices:

A Yes

B No, because it's not closed under addition

C No, because it's not closed under scalar multiplication

D No, because it's not closed under addition or scalar multiplication

Solution: A

The set V is a subspace, because V is a span of vectors, and a span is always a subspace.

The span of vectors is all the possible linear combinations of those vectors. For instance, we could create the linear combination

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} + c_3 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} + c_4 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

and then factor out each vector.

$$(c_1 + c_2) \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} + (c_3 + c_4) \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

The constants $(c_1 + c_2)$ and $(c_3 + c_4)$ are still just constants, which means we could rewrite the linear combination as

$$c_5 \begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix} + c_6 \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

This result is still just a linear combination of the vectors in the set, which means it's still contained within the span. Therefore, the set is closed under addition.

And because multiplying a linear combination of the vectors by a scalar still just gives a linear combination of the vectors, the set is also closed under scalar multiplication.

Topic: Basis

Question: Does the span of V form a basis for $\mathbb{R}^2$?

$$V = \left\{ \begin{bmatrix} -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$$

Answer choices:

A Yes

B No, because the vectors in V don't span $\mathbb{R}^2$

C No, because the vectors in V aren't linearly independent

D No, because the vectors in V don't span $\mathbb{R}^2$ and aren't linearly independent

Solution: A

In order for V to form a basis for $\mathbb{R}^2$,

1. the vectors in V need to span $\mathbb{R}^2$, and

2. the vectors in V need to be linearly independent.

To span $\mathbb{R}^2$, we need to be able to get any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set. In other words,

$$c_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$-2c_1 + c_2 = x$$

$$-c_2 = y$$

We'll substitute $c_2 = -y$ into the first equation in the system, and then solve that equation for c_1.

$$-2c_1 - y = x$$

$$-2c_1 = y + x$$

$$c_1 = -\frac{1}{2}y - \frac{1}{2}x$$

From this process we can conclude that, given any vector $\overrightarrow{v} = (x, y)$ in $\mathbb{R}^2$, we can "get to it" using the values of c_1 and c_2 given by

$$c_1 = -\frac{1}{2}y - \frac{1}{2}x$$

$$c_2 = -y$$

It doesn't matter which vector we pick in $\mathbb{R}^2$. If we use the values of x and y that we want, and plug them into these equations for c_1 and c_2, we'll get the values of c_1 and c_2 that we need to use in the linear combination in order to arrive at the vector $\vec{v} = (x, y)$. These formulas for c_1 and c_2 won't break, regardless of which (x, y) we pick for the vector, so the vector set V spans $\mathbb{R}^2$.

Then to show that the vectors in V are linearly independent, we'll set $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$c_1 = -\frac{1}{2}(0) - \frac{1}{2}(0), \text{ or } c_1 = 0$$

$$c_2 = -0, \text{ or } c_2 = 0$$

Because the only values of c_1 and c_2 that give the zero vector are $c_1 = 0$ and $c_2 = 0$, we know that the vectors in V are linearly independent.

Therefore, because the vector set V spans all of $\mathbb{R}^2$, and because the vectors in V are linearly independent, we can say that V forms a basis for $\mathbb{R}^2$.

Topic: Basis

Question: Does the span of V form a basis for $\mathbb{R}^2$?

$$V = \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -6 \\ 2 \end{bmatrix} \right\}$$

Answer choices:

A Yes

B No, because the vectors in V don't span $\mathbb{R}^2$

C No, because the vectors in V aren't linearly independent

D No, because the vectors in V don't span $\mathbb{R}^2$ and aren't linearly independent

Solution: D

In order for V to form a basis for $\mathbb{R}^2$,

1. the vectors in V need to span $\mathbb{R}^2$, and

2. the vectors in V need to be linearly independent.

To span $\mathbb{R}^2$, we need to be able to get any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set. In other words,

$$c_1\begin{bmatrix}3\\-1\end{bmatrix}+c_2\begin{bmatrix}-6\\2\end{bmatrix}=\begin{bmatrix}x\\y\end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$3c_1-6c_2=x$$

$$-c_1+2c_2=y$$

Multiply the second equation by 3 to get $-3c_1+6c_2=3y$. Add the equations.

$$3c_1-6c_2+(-3c_1+6c_2)=x+(3y)$$

$$3c_1-6c_2-3c_1+6c_2=x+3y$$

$$-6c_2+6c_2=x+3y$$

$$0=x+3y$$

$$3y=-x$$

From this process we can conclude that we can't get to any vector $\vec{v}=(x,y)$ in $\mathbb{R}^2$. It doesn't matter which c_1 and c_2 we choose, the relationship between x and y is always given by $3y = -x$. For instance, if $y = 1$, x can only be $x = -3$. Which means we have no way of "getting to" any other $(x,1)$. So the vectors don't span $\mathbb{R}^2$.

Then to show that the vectors in V are linearly independent, we'll set $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} 3 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$3c_1 - 6c_2 = 0$$

$$-c_1 + 2c_2 = 0$$

If we divide $3c_1 - 6c_2 = 0$ by -3, we get $-c_1 + 2c_2 = 0$. Then when we subtract this from the second equation, we get

$$-c_1 + 2c_2 - (-c_1 + 2c_2) = 0 - (0)$$

$$-c_1 + 2c_2 + c_1 - 2c_2 = 0 - 0$$

$$0 = 0$$

This tells us that any values of c_1 and c_2 will satisfy the vector equation, not just $c_1 = 0$ and $c_2 = 0$. So we know that the vectors in V aren't linearly independent.

Topic: Basis

Question: Does the span of V form a basis for $\mathbb{R}^3$?

$$V = \left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \right\}$$

Answer choices:

A Yes

B No, because the vectors in V don't span $\mathbb{R}^3$

C No, because the vectors in V aren't linearly independent

D No, because the vectors in V don't span $\mathbb{R}^3$ and aren't linearly independent

Solution: A

In order for V to form a basis for $\mathbb{R}^3$,

1. the vectors in V need to span $\mathbb{R}^3$, and

2. the vectors in V need to be linearly independent.

To span $\mathbb{R}^3$, we need to be able to get any vector in $\mathbb{R}^3$ using a linear combination of the vectors in the set. In other words,

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

From this linear combination, we can build a system of equations.

$$c_1 - c_2 = x$$

$$c_2 - 3c_3 = y$$

$$-2c_1 + c_3 = z$$

Solve the system with a matrix, where each column in the augmented matrix represents the "variables" c_1, c_2, and c_3. Then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & | & x \\ 0 & 1 & -3 & | & y \\ -2 & 0 & 1 & | & z \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & x \\ 0 & 1 & -3 & | & y \\ 0 & -2 & 1 & | & 2x + z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & x+y \\ 0 & 1 & -3 & | & y \\ 0 & -2 & 1 & | & 2x+z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & | & x+y \\ 0 & 1 & -3 & | & y \\ 0 & 0 & -5 & | & 2x+2y+z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & | & x+y \\ 0 & 1 & -3 & | & y \\ 0 & 0 & 1 & | & -\frac{2}{5}x-\frac{2}{5}y-\frac{1}{5}z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{5}x-\frac{1}{5}y-\frac{3}{5}z \\ 0 & 1 & -3 & | & y \\ 0 & 0 & 1 & | & -\frac{2}{5}x-\frac{2}{5}y-\frac{1}{5}z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{5}x-\frac{1}{5}y-\frac{3}{5}z \\ 0 & 1 & 0 & | & -\frac{6}{5}x-\frac{1}{5}y-\frac{3}{5}z \\ 0 & 0 & 1 & | & -\frac{2}{5}x-\frac{2}{5}y-\frac{1}{5}z \end{bmatrix}$$

From this process we can conclude that, given any vector $\overrightarrow{v}=(x,y,z)$ in $\mathbb{R}^3$, we can "get to it" using the values of c_1, c_2, and c_3 given by

$$c_1=-\frac{1}{5}x-\frac{1}{5}y-\frac{3}{5}z$$

$$c_2=-\frac{6}{5}x-\frac{1}{5}y-\frac{3}{5}z$$

$$c_3=-\frac{2}{5}x-\frac{2}{5}y-\frac{1}{5}z$$

It doesn't matter which vector we pick in $\mathbb{R}^3$. If we use the values of x, y, and z that we want, and plug them into these equations for c_1, c_2, and c_3, we'll get the values of c_1, c_2, and c_3 that we need to use in the linear combination in order to arrive at the vector $\overrightarrow{v}=(x,y,z)$. These formulas for

c_1, c_2, and c_3 won't break, regardless of which (x, y, z) we pick for the vector, so the vector set V spans $\mathbb{R}^3$.

Then to show that the vectors in V are linearly independent, we'll set $(x, y, z) = (0,0,0)$.

$$c_1 \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

When we do, we get

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ -2 & 0 & 1 & | & 0 \end{bmatrix}$$

When we put the matrix into reduced row-echelon form, we get

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Which means $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Because the only values of c_1, c_2, and c_3 that give the zero vector are $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$, we know that the vectors in V are linearly independent.

Therefore, because the vector set V spans all of $\mathbb{R}^3$, and because the vectors in V are linearly independent, we can say that V forms a basis for $\mathbb{R}^3$.

Topic: Dot products

Question: Find the dot product.

$$\vec{x} = (5, -1)$$

$$\vec{y} = (3,2)$$

Answer choices:

A $\vec{x} \cdot \vec{y} = 11$

B $\vec{x} \cdot \vec{y} = 13$

C $\vec{x} \cdot \vec{y} = 9$

D $\vec{x} \cdot \vec{y} = 17$

Solution: B

To find the dot product of two vectors, we multiply corresponding components, then add the products. So the dot product of $\overrightarrow{x}=(5,-1)$ and $\overrightarrow{y}=(3,2)$ is

$$\overrightarrow{x}\cdot\overrightarrow{y}=(5)(3)+(-1)(2)$$

$$\overrightarrow{x}\cdot\overrightarrow{y}=15-2$$

$$\overrightarrow{x}\cdot\overrightarrow{y}=13$$

Topic: Dot products

Question: Find the dot product.

$$\vec{x} = (-4,0,12)$$

$$\vec{y} = (9, -12,8)$$

Answer choices:

A $\quad \vec{x} \cdot \vec{y} = 48$

B $\quad \vec{x} \cdot \vec{y} = 132$

C $\quad \vec{x} \cdot \vec{y} = 72$

D $\quad \vec{x} \cdot \vec{y} = 60$

Solution: D

To find the dot product of two vectors, we multiply corresponding components, then add the products. So the dot product of $\overrightarrow{x}=(-4,0,12)$ and $\overrightarrow{y}=(9,-12,8)$ is

$$\overrightarrow{x}\cdot\overrightarrow{y}=(-4)(9)+(0)(-12)+(12)(8)$$

$$\overrightarrow{x}\cdot\overrightarrow{y}=-36+0+96$$

$$\overrightarrow{x}\cdot\overrightarrow{y}=60$$

Topic: Dot products

Question: Use the dot product to find $3\overrightarrow{x} \cdot (-2\overrightarrow{y} - \vec{z})$.

$$\overrightarrow{x} = (-4, -2, 7)$$

$$\overrightarrow{y} = (6, -1, -10)$$

$$\vec{z} = (3, -2, 0)$$

Answer choices:

A $\quad 3\overrightarrow{x} \cdot (-2\overrightarrow{y} - \vec{z}) = 96$

B $\quad 3\overrightarrow{x} \cdot (-2\overrightarrow{y} - \vec{z}) = 576$

C $\quad 3\overrightarrow{x} \cdot (-2\overrightarrow{y} - \vec{z}) = -92$

D $\quad 3\overrightarrow{x} \cdot (-2\overrightarrow{y} - \vec{z}) = -434$

Solution: B

To simplify $3\overrightarrow{x} \cdot (-2\overrightarrow{y} - \vec{z})$, start by finding $3\overrightarrow{x}$,

$$\overrightarrow{x} = (-4, -2, 7)$$

$$3\overrightarrow{x} = 3(-4, -2, 7)$$

$$3\overrightarrow{x} = (-12, -6, 21)$$

and $-2\overrightarrow{y}$.

$$\overrightarrow{y} = (6, -1, -10)$$

$$-2\overrightarrow{y} = -2(6, -1, -10)$$

$$-2\overrightarrow{y} = (-12, 2, 20)$$

Then the difference $-2\overrightarrow{y} - \vec{z}$ is

$$-2\overrightarrow{y} - \vec{z} = \begin{bmatrix} -12 \\ 2 \\ 20 \end{bmatrix} - \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix}$$

$$-2\overrightarrow{y} - \vec{z} = \begin{bmatrix} -12 - 3 \\ 2 - (-2) \\ 20 - 0 \end{bmatrix}$$

$$-2\overrightarrow{y} - \vec{z} = \begin{bmatrix} -15 \\ 4 \\ 20 \end{bmatrix}$$

The dot product is then

$$3\overrightarrow{x} \cdot (-2\overrightarrow{y} - \vec{z})$$

$$[-12 \quad -6 \quad 21] \cdot \begin{bmatrix} -15 \\ 4 \\ 20 \end{bmatrix}$$

$$-12(-15) - 6(4) + 21(20)$$

$$180 - 24 + 420$$

$$576$$

Topic: Cauchy-Schwarz inequality

Question: Use the Cauchy-Schwarz inequality to say which vector set is linearly independent.

Answer choices:

A $\quad \vec{u} = (2,5)$ and $\vec{v} = (-6, -15)$

B $\quad \vec{u} = (-3,1)$ and $\vec{v} = (9, -3)$

C $\quad \vec{u} = (1, -4)$ and $\vec{v} = (-2,8)$

D $\quad \vec{u} = (6,2)$ and $\vec{v} = (-5,1)$

Solution: D

Let's plug each answer choice into the Cauchy-Schwarz inequality,

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \, ||\vec{v}||$$

If the two sides are equivalent, the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are unequal, then we know the vectors are linearly independent.

Answer choice A gives

$$|(2)(-6) + (5)(-15)| = \sqrt{2^2 + 5^2}\sqrt{(-6)^2 + (-15)^2}$$

$$|-12 - 75| = \sqrt{4 + 25}\sqrt{36 + 225}$$

$$|-87| = \sqrt{29}\sqrt{261}$$

$$87 = \sqrt{7{,}569}$$

$$87 = 87$$

Answer choice B gives

$$|(-3)(9) + (1)(-3)| = \sqrt{(-3)^2 + 1^2}\sqrt{9^2 + (-3)^2}$$

$$|-27 - 3| = \sqrt{9 + 1}\sqrt{81 + 9}$$

$$|-30| = \sqrt{10}\sqrt{90}$$

$$30 = \sqrt{900}$$

$$30 = 30$$

Answer choice C gives

$$|(1)(-2)+(-4)(8)| = \sqrt{1^2+(-4)^2}\sqrt{(-2)^2+8^2}$$

$$|-2-32| = \sqrt{1+16}\sqrt{4+64}$$

$$|-34| = \sqrt{17}\sqrt{68}$$

$$34 = \sqrt{1{,}156}$$

$$34 = 34$$

Answer choice D gives

$$|(6)(-5)+(2)(1)| = \sqrt{6^2+2^2}\sqrt{(-5)^2+1^2}$$

$$|-30+2| = \sqrt{36+4}\sqrt{25+1}$$

$$|-28| = \sqrt{40}\sqrt{26}$$

$$28 = \sqrt{1{,}040}$$

$$28 \approx 32.25$$

Because answer choice D is the only vector set where the two sides of the Cauchy-Schwarz inequality are unequal, answer choice D is the only linearly independent vector set.

Topic: Cauchy-Schwarz inequality

Question: Use the Cauchy-Schwarz inequality to say which vector set is linearly independent.

Answer choices:

A $\quad \vec{u}=(8,6)$ and $\vec{v}=(-4,-3)$

B $\quad \vec{u}=(-6,5)$ and $\vec{v}=(12,-10)$

C $\quad \vec{u}=(7,9)$ and $\vec{v}=(14,18)$

D $\quad \vec{u}=(-5,-5)$ and $\vec{v}=(10,-5)$

Solution: D

Let's plug each answer choice into the Cauchy-Schwarz inequality,

$$|\vec{u}\cdot\vec{v}| = ||\vec{u}||\ ||\vec{v}||$$

If the two sides are equivalent, the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are unequal, then we know the vectors are linearly independent.

Answer choice A gives

$$|(8)(-4)+(6)(-3)| = \sqrt{8^2+6^2}\sqrt{(-4)^2+(-3)^2}$$

$$|-32-18| = \sqrt{64+36}\sqrt{16+9}$$

$$|-50| = \sqrt{100}\sqrt{25}$$

$$50 = \sqrt{2{,}500}$$

$$50 = 50$$

Answer choice B gives

$$|(-6)(12)+(5)(-10)| = \sqrt{(-6)^2+5^2}\sqrt{12^2+(-10)^2}$$

$$|-72-50| = \sqrt{36+25}\sqrt{144+100}$$

$$|-122| = \sqrt{61}\sqrt{244}$$

$$122 = \sqrt{14{,}884}$$

$$122 = 122$$

Answer choice C gives

$$|(7)(14)+(9)(18)|=\sqrt{7^2+9^2}\sqrt{14^2+18^2}$$

$$|98+162|=\sqrt{49+81}\sqrt{196+324}$$

$$|260|=\sqrt{130}\sqrt{520}$$

$$260=\sqrt{67{,}600}$$

$$260=260$$

Answer choice D gives

$$|(-5)(10)+(-5)(-5)|=\sqrt{(-5)^2+(-5)^2}\sqrt{10^2+(5)^2}$$

$$|-50+25|=\sqrt{25+25}\sqrt{100+25}$$

$$|-25|=\sqrt{50}\sqrt{125}$$

$$25=\sqrt{6{,}250}$$

$$25\approx 79.06$$

Because answer choice D is the only vector set where the two sides of the Cauchy-Schwarz inequality are unequal, answer choice D is the only linearly independent vector set.

Topic: Cauchy-Schwarz inequality

Question: Use the Cauchy-Schwarz inequality to say which vector set is linearly independent.

Answer choices:

A $\quad \vec{u}=(3,2)$ and $\vec{v}=(-12,-8)$

B $\quad \vec{u}=(6,5)$ and $\vec{v}=(4,0)$

C $\quad \vec{u}=(1,1)$ and $\vec{v}=(2,2)$

D $\quad \vec{u}=(8,-7)$ and $\vec{v}=(-16,14)$

Solution: B

Let's plug each answer choice into the Cauchy-Schwarz inequality,

$$|\vec{u}\cdot\vec{v}| = ||\vec{u}||\ ||\vec{v}||$$

If the two sides are equivalent, the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are unequal, then we know the vectors are linearly independent.

Answer choice A gives

$$|(3)(-12)+(2)(-8)| = \sqrt{3^2+2^2}\sqrt{-12^2+(-8)^2}$$

$$|-36-16| = \sqrt{9+4}\sqrt{144+64}$$

$$|-52| = \sqrt{13}\sqrt{208}$$

$$52 = \sqrt{2{,}704}$$

$$52 = 52$$

Answer choice B gives

$$|(6)(4)+(5)(0)| = \sqrt{6^2+5^2}\sqrt{4^2+0^2}$$

$$|24+0| = \sqrt{36+25}\sqrt{16+0}$$

$$|24| = \sqrt{61}\sqrt{16}$$

$$24 = \sqrt{976}$$

$$24 \approx 31.24$$

Because answer choice B is a vector set for which the two sides of the Cauchy-Schwarz inequality are unequal, answer choice B is the only linearly independent vector set.

Topic: Vector triangle inequality

Question: Use the vector triangle inequality to say which vector set is linearly independent.

Answer choices:

A $\quad \vec{u} = (-1, -6), \ \vec{v} = (-4, -24)$

B $\quad \vec{u} = (2, -7), \ \vec{v} = (4, -14)$

C $\quad \vec{u} = (9,4), \ \vec{v} = (72,32)$

D $\quad \vec{u} = (-9, -8), \ \vec{v} = (-5,6)$

Solution: D

Let's plug each answer choice into the vector triangle inequality,

$$||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent (or if the left side is 0), then the vector set is linearly dependent.

Let's test answer choice A in the vector triangle inequality.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2} + \sqrt{v_1^2+v_2^2}$$

$$\sqrt{(-1-4)^2+(-6-24)^2} \le \sqrt{(-1)^2+(-6)^2} + \sqrt{(-4)^2+(-24)^2}$$

$$\sqrt{(-5)^2+(-30)^2} \le \sqrt{1+36} + \sqrt{16+576}$$

$$\sqrt{25+900} \le \sqrt{37} + \sqrt{592}$$

$$\sqrt{925} \le \sqrt{37} + \sqrt{592}$$

$$30.41 \le 6.08 + 24.33$$

$$30.41 = 30.41$$

Because the sides are equal, the vector set in answer choice A is linearly dependent, so let's test answer choice B.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(2+4)^2+(-7-14)^2} \le \sqrt{2^2+(-7)^2}+\sqrt{4^2+(-14)^2}$$

$$\sqrt{6^2+(-21)^2} \le \sqrt{4+49}+\sqrt{16+196}$$

$$\sqrt{36+441} \le \sqrt{53}+\sqrt{212}$$

$$\sqrt{477} \le \sqrt{53}+\sqrt{212}$$

$$21.84 \le 7.28+14.56$$

$$21.84 = 21.84$$

Because the sides are equal, the vector set in answer choice B is linearly dependent, so let's test answer choice C.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}||+||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(9+72)^2+(4+32)^2} \le \sqrt{9^2+4^2}+\sqrt{72^2+32^2}$$

$$\sqrt{6{,}561+1{,}296} \le \sqrt{81+16}+\sqrt{5{,}184+1{,}024}$$

$$\sqrt{7{,}857} \le \sqrt{97}+\sqrt{6{,}208}$$

$$88.64 \le 9.85+78.79$$

$$88.64 = 88.64$$

Because the sides are equal, the vector set in answer choice C is linearly dependent, so let's test answer choice D.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2} + \sqrt{v_1^2+v_2^2}$$

$$\sqrt{(-9-5)^2+(-8+6)^2} \le \sqrt{(-9)^2+(-8)^2} + \sqrt{(-5)^2+6^2}$$

$$\sqrt{(-14)^2+(-2)^2} \le \sqrt{81+64} + \sqrt{25+36}$$

$$\sqrt{196+4} \le \sqrt{81+64} + \sqrt{25+36}$$

$$\sqrt{200} \le \sqrt{145} + \sqrt{61}$$

$$14.14 \le 12.04 + 7.81$$

$$14.14 < 19.85$$

Because the left side is less than the right side, the vector set in answer choice D is linearly independent.

Topic: Vector triangle inequality

Question: Use the vector triangle inequality to say which vector set is linearly independent.

Answer choices:

A $\vec{u}=(-1,4)$, $\vec{v}=(-7,28)$

B $\vec{u}=(-8,-3)$, $\vec{v}=(2,8)$

C $\vec{u}=(-2,7)$, $\vec{v}=(-4,14)$

D $\vec{u}=(0,-9)$, $\vec{v}=(0,72)$

Solution: B

Let's plug each answer choice into the vector triangle inequality,

$$||\vec{u}+\vec{v}|| \le ||\vec{u}||+||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent (or if the left side is 0), then the vector set is linearly dependent.

Let's test answer choice A in the vector triangle inequality.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}||+||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(-1-7)^2+(4+28)^2} \le \sqrt{(-1)^2+4^2}+\sqrt{(-7)^2+28^2}$$

$$\sqrt{(-8)^2+32^2} \le \sqrt{1+16}+\sqrt{49+784}$$

$$\sqrt{64+1{,}024} \le \sqrt{17}+\sqrt{833}$$

$$\sqrt{1{,}088} \le \sqrt{17}+\sqrt{833}$$

$$32.98 \le 4.12+28.86$$

$$32.98 = 32.98$$

Because the sides are equal, the vector set in answer choice A is linearly dependent, so let's test answer choice B.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}||+||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(-8+2)^2+(-3+8)^2} \le \sqrt{(-8)^2+(-3)^2}+\sqrt{2^2+8^2}$$

$$\sqrt{(-6)^2+5^2} \le \sqrt{64+9}+\sqrt{4+64}$$

$$\sqrt{36+25} \le \sqrt{73}+\sqrt{68}$$

$$\sqrt{61} \le \sqrt{73}+\sqrt{68}$$

$$7.81 \le 8.54+8.25$$

$$7.81 \le 16.79$$

Because the left side is less than the right side, the vector set in answer choice B is linearly independent.

Topic: Vector triangle inequality

Question: Use the vector triangle inequality to say which vector set is linearly independent.

Answer choices:

A $\quad \vec{u} = (9, -2), \ \vec{v} = (54, -12)$

B $\quad \vec{u} = (3, -1), \ \vec{v} = (9, -3)$

C $\quad \vec{u} = (5, -3), \ \vec{v} = (0,6)$

D $\quad \vec{u} = (-9,3), \ \vec{v} = (-72,24)$

Solution: C

Let's plug each answer choice into the vector triangle inequality,

$$||\overrightarrow{u}+\overrightarrow{v}|| \le ||\overrightarrow{u}||+||\overrightarrow{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent (or if the left side is 0), then the vector set is linearly dependent.

Let's test answer choice A in the vector triangle inequality.

$$||\overrightarrow{u}+\overrightarrow{v}|| \le ||\overrightarrow{u}||+||\overrightarrow{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(9+54)^2+(-2-12)^2} \le \sqrt{9^2+(-2)^2}+\sqrt{54^2+(-12)^2}$$

$$\sqrt{63^2+(-14)^2} \le \sqrt{81+4}+\sqrt{2,916+144}$$

$$\sqrt{3,969+196} \le \sqrt{81+4}+\sqrt{2,916+144}$$

$$\sqrt{4,165} \le \sqrt{85}+\sqrt{3,060}$$

$$64.54 = 64.54$$

Because the sides are equal, the vector set in answer choice A is linearly dependent, so let's test answer choice B.

$$||\overrightarrow{u}+\overrightarrow{v}|| \le ||\overrightarrow{u}||+||\overrightarrow{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(3+9)^2+(-1-3)^2} \le \sqrt{3^2+(-1)^2}+\sqrt{9^2+(-3)^2}$$

$$\sqrt{12^2+(-4)^2} \le \sqrt{9+1}+\sqrt{81+9}$$

$$\sqrt{144+16} \le \sqrt{10}+\sqrt{90}$$

$$\sqrt{160} \le \sqrt{10}+\sqrt{90}$$

$$12.65 \le 3.16+9.49$$

$$12.65 = 12.65$$

Because the sides are equal, the vector set in answer choice B is linearly dependent, so let's test answer choice C.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}||+||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2}+\sqrt{v_1^2+v_2^2}$$

$$\sqrt{(5+0)^2+(-3+6)^2} \le \sqrt{5^2+(-3)^2}+\sqrt{0^2+6^2}$$

$$\sqrt{5^2+3^2} \le \sqrt{25+9}+\sqrt{0+36}$$

$$\sqrt{25+9} \le \sqrt{34}+\sqrt{36}$$

$$\sqrt{34} \le \sqrt{34}+\sqrt{36}$$

$$5.83 \le 5.83+6$$

$$5.83 < 11.83$$

Because the left side is less than the right side, the vector set in answer choice C is linearly independent.

We can also verify that answer choice D is linearly dependent.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2} + \sqrt{v_1^2+v_2^2}$$

$$\sqrt{(-9+(-72))^2+(3+24)^2} \le \sqrt{(-9)^2+3^2} + \sqrt{(-72)^2+24^2}$$

$$\sqrt{(-81)^2+27^2} \le \sqrt{81+9} + \sqrt{5{,}184+576}$$

$$\sqrt{7{,}290} \le \sqrt{90} + \sqrt{5{,}760}$$

$$85.38 = 85.38$$

Topic: Angle between vectors

Question: Say whether or not the vectors are orthogonal.

$$\vec{u} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

$$\vec{v} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$$

Answer choices:

A The vectors are orthogonal

B The vectors are not orthogonal

C It’s impossible to say whether or not the vectors are orthogonal

Solution: B

We'll test to see whether or not the vectors $\overrightarrow{u} = \mathbf{i} + \mathbf{j} + 2\mathbf{k}$ and $\overrightarrow{v} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$ are orthogonal by calculating their dot product.

$$\overrightarrow{u} \cdot \overrightarrow{v} = (1)(2) + (1)(2) + (2)(4)$$

$$\overrightarrow{u} \cdot \overrightarrow{v} = 2 + 2 + 8$$

$$\overrightarrow{u} \cdot \overrightarrow{v} = 12$$

Since the dot product is not 0, the vectors are not orthogonal.

Topic: Angle between vectors

Question: Find the angle between the vectors.

$$\vec{a} = (2,0,-1)$$

$$\vec{b} = (-1,4,2)$$

Answer choices:

A $\theta \approx 113^\circ$

B $\theta \approx 247^\circ$

C $\theta \approx 293^\circ$

D $\theta \approx 67^\circ$

Solution: A

The angle between the vectors can be given by

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \ ||\vec{b}|| \cos\theta$$

Find the lengths of both vectors.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{2^2 + 0^2 + (-1)^2} = \sqrt{4 + 0 + 1} = \sqrt{5}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{(-1)^2 + 4^2 + 2^2} = \sqrt{1 + 16 + 4} = \sqrt{21}$$

Then find the dot product of the vectors.

$$\vec{a} \cdot \vec{b} = (2)(-1) + (0)(4) + (-1)(2)$$

$$\vec{a} \cdot \vec{b} = -2 + 0 - 2$$

$$\vec{a} \cdot \vec{b} = -4$$

Plug everything into the formula for the angle between vectors.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \ ||\vec{b}|| \cos\theta$$

$$-4 = \sqrt{5}\sqrt{21}\cos\theta$$

$$-\frac{4}{\sqrt{105}} = \cos\theta$$

Take the inverse cosine of each side of the equation to solve for θ.

$$\theta = \arccos\left(-\frac{4}{\sqrt{105}}\right)$$

If we use a calculator to find this arccosine value, we find that the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\theta \approx 113^\circ$.

Topic: Angle between vectors

Question: Find the angle between the vectors.

$$\overrightarrow{a} = (1, -3, 1)$$

$$\overrightarrow{b} = (0, 6, -2)$$

Answer choices:

A $\theta \approx 12^\circ$

B $\theta \approx 72^\circ$

C $\theta \approx 102^\circ$

D $\theta \approx 162^\circ$

Solution: D

The angle between the vectors can be given by

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \ ||\vec{b}|| \cos\theta$$

Find the lengths of both vectors.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + (-3)^2 + 1^2} = \sqrt{1 + 9 + 1} = \sqrt{11}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{0^2 + 6^2 + (-2)^2} = \sqrt{0 + 36 + 4} = \sqrt{40} = 2\sqrt{10}$$

Then find the dot product of the vectors.

$$\vec{a} \cdot \vec{b} = (1)(0) + (-3)(6) + (1)(-2)$$

$$\vec{a} \cdot \vec{b} = 0 - 18 - 2$$

$$\vec{a} \cdot \vec{b} = -20$$

Plug everything into the formula for the angle between vectors.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \ ||\vec{b}|| \cos\theta$$

$$-20 = \sqrt{11}\left(2\sqrt{10}\right)\cos\theta$$

$$-10 = \sqrt{11}\sqrt{10}\cos\theta$$

$$-10 = \sqrt{110}\cos\theta$$

$$-\frac{10}{\sqrt{110}} = \cos\theta$$

Take the inverse cosine of each side of the equation to solve for θ.

$$\theta = \arccos\left(-\frac{10}{\sqrt{110}}\right)$$

If we use a calculator to find this arccosine value, we find that the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$ is $\theta \approx 162^\circ$.

Topic: Equation of a plane, and normal vectors

Question: What is the normal vector to the plane?

$$3x+5y+9z=-26$$

Answer choices:

A $\vec{n}=(3,5,9)$

B $\vec{n}=(-3,-5,9)$

C $\vec{n}=(3,-5,-9)$

D $\vec{n}=(-3,5,-9)$

Solution: A

Given a plane $Ax + By + Cz = D$, the normal vector to that plane is

$$\vec{n} = (A, B, C)$$

So from the plane $3x + 5y + 9z = -26$, we can simply pull out the coefficients on x, y, and z to get the components of the normal vector.

$$\vec{n} = (3,5,9)$$

Topic: Equation of a plane, and normal vectors

Question: Find the equation of the plane, given a point in the plane and the normal vector to the plane.

$$(x, y, z) = (5, -8, -9)$$

$$\overrightarrow{n} = (8, 2, -1)$$

Answer choices:

A $8x - 2y - z = 17$

B $8x + 2y - z = 33$

C $8x + 2y + z = 33$

D $-8x - 2y - z = 17$

Solution: B

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$8(x-5)+2(y-(-8))-1(z-(-9))=0$$

Now we'll simplify and get the equation of the plane into standard form.

$$8(x-5)+2(y+8)-(z+9)=0$$

$$8x-40+2y+16-z-9=0$$

$$8x+2y-z-33=0$$

$$8x+2y-z=33$$

Topic: Equation of a plane, and normal vectors

Question: Find the equation of the plane, given a point in the plane and the normal vector to the plane.

$$(x, y, z) = (-5,3,-3)$$

$$\overrightarrow{n} = (-4,-3,9)$$

Answer choices:

A $\quad 4x + 3y + 9z = -16$

B $\quad -4x - 3y - 9z = -16$

C $\quad 4x + 3y - 9z = -16$

D $\quad -4x - 3y + 9z = -16$

Solution: D

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$-4(x-(-5))-3(y-3)+9(z-(-3))=0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-4(x+5)-3(y-3)+9(z+3)=0$$

$$-4x-20-3y+9+9z+27=0$$

$$-4x-3y+9z+16=0$$

$$-4x-3y+9z=-16$$

Topic: Cross products

Question: Find the cross product $\overrightarrow{a} \times \overrightarrow{b}$.

$$\overrightarrow{a} = (1, -1, 1)$$

$$\overrightarrow{b} = (-2, 1, 2)$$

Answer choices:

A $\quad \overrightarrow{a} \times \overrightarrow{b} = (3, -4, 1)$

B $\quad \overrightarrow{a} \times \overrightarrow{b} = (-3, -4, -1)$

C $\quad \overrightarrow{a} \times \overrightarrow{b} = (-3, 4, -1)$

D $\quad \overrightarrow{a} \times \overrightarrow{b} = (3, 4, 1)$

Solution: B

The cross product $\vec{a} \times \vec{b}$ of $\vec{a} = (1, -1, 1)$ and $\vec{b} = (-2, 1, 2)$ is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ -2 & 1 & 2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 1 \\ -2 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & -1 \\ -2 & 1 \end{vmatrix}$$

Calculate the 2×2 determinants using the $ad - bc$ pattern.

$$\vec{a} \times \vec{b} = \mathbf{i}\left[(-1)(2) - (1)(1)\right] - \mathbf{j}\left[(1)(2) - (1)(-2)\right] + \mathbf{k}\left[(1)(1) - (-1)(-2)\right]$$

$$\vec{a} \times \vec{b} = \mathbf{i}(-2 - 1) - \mathbf{j}(2 + 2) + \mathbf{k}(1 - 2)$$

$$\vec{a} \times \vec{b} = -3\mathbf{i} - 4\mathbf{j} - \mathbf{k}$$

$$\vec{a} \times \vec{b} = (-3, -4, -1)$$

Topic: Cross products

Question: Find the cross product $\overrightarrow{a} \times \overrightarrow{b}$.

$$\overrightarrow{a} = (4,2,0)$$

$$\overrightarrow{b} = (-1,-3,1)$$

Answer choices:

A $\quad \overrightarrow{a} \times \overrightarrow{b} = (-2,4,10)$

B $\quad \overrightarrow{a} \times \overrightarrow{b} = (-2,-4,10)$

C $\quad \overrightarrow{a} \times \overrightarrow{b} = (2,4,-10)$

D $\quad \overrightarrow{a} \times \overrightarrow{b} = (2,-4,-10)$

Solution: D

The cross product $\vec{a} \times \vec{b}$ of $\vec{a} = (4,2,0)$ and $\vec{b} = (-1, -3,1)$ is given by

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 2 & 0 \\ -1 & -3 & 1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} 2 & 0 \\ -3 & 1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 4 & 0 \\ -1 & 1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 4 & 2 \\ -1 & -3 \end{vmatrix}$$

Calculate the 2×2 determinants using the $ad - bc$ pattern.

$$\vec{a} \times \vec{b} = \mathbf{i} \left[(2)(1) - (0)(-3)\right] - \mathbf{j} \left[(4)(1) - (0)(-1)\right] + \mathbf{k} \left[(4)(-3) - (2)(-1)\right]$$

$$\vec{a} \times \vec{b} = \mathbf{i}(2 - 0) - \mathbf{j}(4 - 0) + \mathbf{k}(-12 + 2)$$

$$\vec{a} \times \vec{b} = 2\mathbf{i} - 4\mathbf{j} - 10\mathbf{k}$$

$$\vec{a} \times \vec{b} = (2, -4, -10)$$

Topic: Cross products

Question: Find the cross product $\vec{a} \times \vec{b}$.

$$\vec{a} = (6,7,-5)$$

$$\vec{b} = (8,7,-11)$$

Answer choices:

A $\quad \vec{a} \times \vec{b} = (-42,-22,-14)$

B $\quad \vec{a} \times \vec{b} = (-112,106,98)$

C $\quad \vec{a} \times \vec{b} = (-42,26,-14)$

D $\quad \vec{a} \times \vec{b} = (-112,-106,98)$

Solution: C

The cross product $\overrightarrow{a} \times \overrightarrow{b}$ of $\overrightarrow{a} = (6,7,-5)$ and $\overrightarrow{b} = (8,7,-11)$ is given by

$$\overrightarrow{a} \times \overrightarrow{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 7 & -5 \\ 8 & 7 & -11 \end{vmatrix}$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i} \begin{vmatrix} 7 & -5 \\ 7 & -11 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 6 & -5 \\ 8 & -11 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 6 & 7 \\ 8 & 7 \end{vmatrix}$$

Calculate the 2×2 determinants using the $ad - bc$ pattern.

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i} \left[(7)(-11) - (-5)(7)\right] - \mathbf{j} \left[(6)(-11) - (-5)(8)\right] + \mathbf{k} \left[(6)(7) - (7)(8)\right]$$

$$\overrightarrow{a} \times \overrightarrow{b} = \mathbf{i}(-77 + 35) - \mathbf{j}(-66 + 40) + \mathbf{k}(42 - 56)$$

$$\overrightarrow{a} \times \overrightarrow{b} = -42\mathbf{i} + 26\mathbf{j} - 14\mathbf{k}$$

$$\overrightarrow{a} \times \overrightarrow{b} = (-42,26,-14)$$

Topic: Dot and cross products as opposite ideas

Question: When is the dot product of two vectors maximized?

Answer choices:

A When the vectors point in exactly opposite directions

B When the vectors are orthogonal

C When the vectors point in exactly the same direction

D When the vectors are the same length

Solution: C

For a given pair of vectors, the dot product will be minimized when the vectors point in exactly opposite directions, and the dot product will be 0 when the vectors are orthogonal.

The dot product will be maximized when the vectors point in the same direction. If the do point in exactly the same direction, then their dot product will be equal to the product of their lengths.

Topic: Dot and cross products as opposite ideas

Question: When two vectors point in exactly the same direction...

Answer choices:

A The length of the cross product is maximized.

B The length of the cross product is 0.

C The length of the cross product is given by the product of the lengths of the individual vectors.

D The length of the cross product is given by the reciprocal of the dot product.

Solution: B

When two vectors are collinear, whether they point in exactly the same direction or in exactly the opposite direction, the length of their cross product is 0. The length of the cross product of two vectors is maximized when the vectors are orthogonal.

Topic: Dot and cross products as opposite ideas

Question: Describe the dot product and length of the cross product of the vector pair.

$$\vec{v} = (3,4)$$

$$\vec{w} = (6,8)$$

Answer choices:

A The dot product is the product of the lengths of the vectors, $\vec{v} \cdot \vec{w} = 50$, and the length of the cross product is $||\vec{v} \times \vec{w}|| = 0$, because the vectors point in exactly the same direction.

B The dot product is the product of the lengths of the vectors, $\vec{v} \cdot \vec{w} = 50$, and the length of the cross product is $||\vec{v} \times \vec{w}|| = 0$, because the vectors point in exactly opposite directions.

C The dot product is $\vec{v} \cdot \vec{w} = 0$, and the length of the cross product is the product of the lengths of the vectors, $||\vec{v} \times \vec{w}|| = 50$, because the vectors point in the same direction.

D The dot product is $\vec{v} \cdot \vec{w} = 0$, and the length of the cross product is the product of the lengths of the vectors, $||\vec{v} \times \vec{w}|| = 50$, because the vectors point in exactly opposite directions.

Solution: A

The vector $\overrightarrow{v} = (3,4)$ points into the first quadrant, and the vector $\overrightarrow{w} = (6,8)$ points in exactly the same direction, since $\overrightarrow{w} = (6,8)$ is just a scalar multiple of $\overrightarrow{v} = (3,4)$. Which means the angle between them is $\theta = 0^\circ$. The length of $\overrightarrow{v} = (3,4)$ is 5, and the length of $\overrightarrow{w} = (6,8)$ is 10.

The dot product of $\overrightarrow{v} = (3,4)$ and $\overrightarrow{u} = (6,8)$ is

$$\overrightarrow{v} \cdot \overrightarrow{w} = \begin{bmatrix} 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ 8 \end{bmatrix}$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = 3(6) + 4(8)$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = 18 + 32$$

$$\overrightarrow{v} \cdot \overrightarrow{w} = 50$$

And the length of the cross product is

$$||\overrightarrow{v} \times \overrightarrow{w}|| = ||\overrightarrow{v}|| \ ||\overrightarrow{w}|| \sin\theta$$

$$||\overrightarrow{v} \times \overrightarrow{w}|| = (5)(10)\sin(0^\circ)$$

$$||\overrightarrow{v} \times \overrightarrow{w}|| = 50(0)$$

$$||\overrightarrow{v} \times \overrightarrow{w}|| = 0$$

Because the length of the cross product is 0, we know that the vectors are collinear. For collinear vectors, the dot product is just the product of the lengths of the vectors, which we see in $\overrightarrow{v} \cdot \overrightarrow{w} = 50$. The fact that the dot product is positive tells us that the vectors point in exactly the same direction along the same line.

Topic: Multiplying matrices by vectors

Question: Find the matrix-vector product, $A\vec{v}$.

$$A=\begin{bmatrix}-1 & 5 & 4\\ 3 & 2 & 7\\ -1 & 0 & 1\end{bmatrix}$$

$$\vec{v}=(-2,0,4)$$

Answer choices:

A $\begin{bmatrix}-2\\ -10\\ -4\end{bmatrix}$

B $\begin{bmatrix}18\\ 22\\ 6\end{bmatrix}$

C $\begin{bmatrix}23\\ 24\\ 6\end{bmatrix}$

D $\begin{bmatrix}14\\ 22\\ 6\end{bmatrix}$

Solution: B

To find $A\overrightarrow{v}$, we'll multiply the matrix A by the column vector $\overrightarrow{v}$. We know the product is defined since the matrix has 3 columns and the vector has 3 rows.

$$A\overrightarrow{v}=\begin{bmatrix}-1 & 5 & 4\\ 3 & 2 & 7\\ -1 & 0 & 1\end{bmatrix}\begin{bmatrix}-2\\ 0\\ 4\end{bmatrix}$$

$$A\overrightarrow{v}=\begin{bmatrix}-1(-2)+5(0)+4(4)\\ 3(-2)+2(0)+7(4)\\ -1(-2)+0(0)+1(4)\end{bmatrix}$$

$$A\overrightarrow{v}=\begin{bmatrix}2+0+16\\ -6+0+28\\ 2+0+4\end{bmatrix}$$

$$A\overrightarrow{v}=\begin{bmatrix}18\\ 22\\ 6\end{bmatrix}$$

Topic: Multiplying matrices by vectors

Question: Find the matrix-vector product, $M\overrightarrow{v}$.

$$M = \begin{bmatrix} -5 & -3 & 1 & 6 \\ 0 & 4 & -2 & 1 \end{bmatrix}$$

$$\overrightarrow{v} = (1, -3, 5, -4)$$

Answer choices:

A $\begin{bmatrix} -27 \\ -26 \end{bmatrix}$

B $\begin{bmatrix} -25 \\ -2 \end{bmatrix}$

C $\begin{bmatrix} 33 \\ 26 \end{bmatrix}$

D $\begin{bmatrix} -15 \\ -26 \end{bmatrix}$

Solution: D

To find $M\overrightarrow{v}$, we'll multiply the matrix M by the column vector $\overrightarrow{v}$. We know the product is defined since the matrix has 4 columns and the vector has 4 rows.

$$M\overrightarrow{v} = \begin{bmatrix} -5 & -3 & 1 & 6 \\ 0 & 4 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 5 \\ -4 \end{bmatrix}$$

$$M\overrightarrow{v} = \begin{bmatrix} -5(1) - 3(-3) + 1(5) + 6(-4) \\ 0(1) + 4(-3) - 2(5) + 1(-4) \end{bmatrix}$$

$$M\overrightarrow{v} = \begin{bmatrix} -5 + 9 + 5 - 24 \\ 0 - 12 - 10 - 4 \end{bmatrix}$$

$$M\overrightarrow{v} = \begin{bmatrix} -15 \\ -26 \end{bmatrix}$$

Topic: Multiplying matrices by vectors

Question: Find the matrix-vector product, $\vec{v}M$.

$$M = \begin{bmatrix} -4 & -5 & 6 \\ 8 & 3 & -4 \end{bmatrix}$$

$$\vec{v} = (-2,1)$$

Answer choices:

A $\quad \begin{bmatrix} 16 & 13 & -16 \end{bmatrix}$

B $\quad \begin{bmatrix} 0 & 7 & -8 \end{bmatrix}$

C $\quad \begin{bmatrix} 16 & -7 & -16 \end{bmatrix}$

D $\quad \begin{bmatrix} 16 & -7 & -10 \end{bmatrix}$

Solution: A

To find $\vec{v}M$, we'll multiply the row vector $\vec{v}$ by the matrix M. We know the product is defined, since the vector has 2 columns and the matrix has 2 rows.

$$\vec{v}M = \begin{bmatrix}-2 & 1\end{bmatrix}\begin{bmatrix}-4 & -5 & 6\\ 8 & 3 & -4\end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix}-2(-4)+1(8) & -2(-5)+1(3) & -2(6)+1(-4)\end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix}8+8 & 10+3 & -12-4\end{bmatrix}$$

$$\vec{v}M = \begin{bmatrix}16 & 13 & -16\end{bmatrix}$$

Topic: The null space and Ax=O

Question: Is $\vec{x}=(-5,1,3)$ in the null space of A?

$$A=\begin{bmatrix}1 & -4 & 3\\ 2 & 4 & 2\\ -1 & -5 & 0\end{bmatrix}$$

Answer choices:

A Yes, $\vec{x}$ is in the null space of A.

B No, $\vec{x}$ is not in the null space of A.

C It's impossible to say whether or not $\vec{x}$ is in the null space of A.

Solution: A

If $\vec{x}=(-5,1,3)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x}=\vec{O}$$

$$\begin{bmatrix}1&-4&3\\2&4&2\\-1&-5&0\end{bmatrix}\begin{bmatrix}-5\\1\\3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix}1(-5)-4(1)+3(3)\\2(-5)+4(1)+2(3)\\-1(-5)-5(1)+0(3)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}-5-4+9\\-10+4+6\\5-5+0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Because we get a true equation, we know that $\vec{x}=(-5,1,3)$ is in the null space of A.

Topic: The null space and Ax=O

Question: Which of the vectors is in the null space of A?

$$A = \begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

B $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

C $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

D $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

Solution: D

If a vector $\overrightarrow{x}$ is in the null space of A, then the product of A and $\overrightarrow{x}$ should satisfy the homogeneous equation.

$$A\overrightarrow{x} = \overrightarrow{O}$$

$$\begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector. So consider $\overrightarrow{x} = (2,3, -1)$ first.

$$\begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3(2) + 1(3) + 9(-1) \\ 1(2) + 1(3) + 1(-1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6 + 3 - 9 \\ 2 + 3 - 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -12 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because we get a false equation, we know that $\overrightarrow{x} = (2,3, -1)$ is not in the null space of A. So consider $\overrightarrow{x} = (1, -1,0)$.

$$\begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3(1)+1(-1)+9(0) \\ 1(1)+1(-1)+1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3-1+0 \\ 1-1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because we get a false equation, we know that $\vec{x} = (1, -1, 0)$ is not in the null space of A. So consider $\vec{x} = (0,1,0)$.

$$\begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3(0)+1(1)+9(0) \\ 1(0)+1(1)+1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0+1+0 \\ 0+1+0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because we get a false equation, we know that $\vec{x} = (0,1,0)$ is not in the null space of A. So consider the last vector, $\vec{x} = (2, -3, 1)$.

$$\begin{bmatrix} -3 & 1 & 9 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3(2)+1(-3)+9(1) \\ 1(2)+1(-3)+1(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -6-3+9 \\ 2-3+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because we get a true equation, we know that $\vec{x} = (2, -3, 1)$ is in the null space of A.

Topic: The null space and Ax=O

Question: Which of the vectors is in the null space of A?

$$A = \begin{bmatrix} 5 & 3 & 1 & 5 \\ -10 & -2 & 1 & -3 \\ -5 & 1 & 2 & 4 \\ 7 & 1 & -1 & -2 \end{bmatrix}$$

Answer choices:

A $\quad \vec{x} = (1,0,1,1)$

B $\quad \vec{x} = (-1,3,-4,0)$

C $\quad \vec{x} = (0,-1,0,0)$

D $\quad \vec{x} = (1,2,0,-4)$

Solution: B

If vector is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x} = \vec{O}$$

$$\begin{bmatrix} 5 & 3 & 1 & 5 \\ -10 & -2 & 1 & -3 \\ -5 & 1 & 2 & 4 \\ 7 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector. So consider $\vec{x} = (1,0,1,1)$ first.

$$\begin{bmatrix} 5 & 3 & 1 & 5 \\ -10 & -2 & 1 & -3 \\ -5 & 1 & 2 & 4 \\ 7 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5(1) + 3(0) + 1(1) + 5(1) \\ -10(1) - 2(0) + 1(1) - 3(1) \\ -5(1) + 1(0) + 2(1) + 4(1) \\ 7(1) + 1(0) - 1(1) - 2(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 + 0 + 1 + 5 \\ -10 + 0 + 1 - 3 \\ -5 + 0 + 2 + 4 \\ 7 + 0 - 1 - 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 11 \\ -12 \\ 1 \\ 4 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get a false equation, we know that $\vec{x} = (1,0,1,1)$ is not in the null space of A. So consider $\vec{x} = (-1,3,-4,0)$.

$$\begin{bmatrix} 5 & 3 & 1 & 5 \\ -10 & -2 & 1 & -3 \\ -5 & 1 & 2 & 4 \\ 7 & 1 & -1 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \\ -4 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5(-1) + 3(3) + 1(-4) + 5(0) \\ -10(-1) - 2(3) + 1(-4) - 3(0) \\ -5(-1) + 1(3) + 2(-4) + 4(0) \\ 7(-1) + 1(3) - 1(-4) - 2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 + 9 - 4 + 0 \\ 10 - 6 - 4 - 0 \\ 5 + 3 - 8 + 0 \\ -7 + 3 + 4 - 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get a true equation, we know that $\vec{x} = (-1,3,-4,0)$ is in the null space of A.

Topic: Null space of a matrix

Question: Find the null space of the matrix A.

$$A = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & 4 \\ 6 & -6 & -12 \end{bmatrix}$$

Answer choices:

A $\quad N(A) = \text{Span}\left(\begin{bmatrix} -6 \\ -4 \\ 1 \end{bmatrix}\right)$

B $\quad N(A) = \text{Span}\left(\begin{bmatrix} -6 \\ -4 \\ 0 \end{bmatrix}\right)$

C $\quad N(A) = \text{Span}\left(\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}\right)$

D $\quad N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix}\right)$

Solution: C

To find the null space of A, we need to find the vector set that satisfies $A\vec{x}=\vec{O}$, so we need to set up a matrix equation.

Because A has three columns, $\vec{x}$ needs to have three rows, so we'll use a 3-row column vector for $\vec{x}$. And multiplying the 3×3 matrix by the 3-row column vector will result in a 3×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix}1 & -1 & -2\\0 & -1 & 4\\6 & -6 & -12\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

We can write the system as an augmented matrix,

$$\begin{bmatrix}1 & -1 & -2 & | & 0\\0 & -1 & 4 & | & 0\\6 & -6 & -12 & | & 0\end{bmatrix}$$

and then use Gaussian elimination to put it in reduced row-echelon form.

$$\begin{bmatrix}1 & -1 & -2 & | & 0\\0 & -1 & 4 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & -2 & | & 0\\0 & 1 & -4 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -6 & | & 0 \\ 0 & 1 & -4 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

From this matrix, we get a system of equations,

$$x_1 - 6x_3 = 0$$

$$x_2 - 4x_3 = 0$$

which we can solve for the pivot variables.

$$x_1 = 6x_3$$

$$x_2 = 4x_3$$

So the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

Then the null space of A is the span of $\vec{x} = (6,4,1)$.

$$N(A) = \text{Span}\left(\begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} \right)$$

Topic: Null space of a matrix

Question: Find the null space of M.

$$M = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -3 & -6 & 9 & -15 \\ 4 & 1 & -12 & 6 \end{bmatrix}$$

Answer choices:

A $N(M) = \text{Span}\left(\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}\right)$

B $N(M) = \text{Span}\left(\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right)$

C $N(M) = \text{Span}\left(\begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \end{bmatrix}\right)$

D $N(M) = \text{Span}\left(\begin{bmatrix} 3 \\ -7 \\ 0 \\ 0 \end{bmatrix}\right)$

Solution: B

To find the null space, put the matrix M into reduced row-echelon form.

$$M = \begin{bmatrix} 1 & 2 & -3 & 5 \\ 2 & 4 & -6 & 10 \\ -3 & -6 & 9 & -15 \\ 4 & 1 & -12 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ -3 & -6 & 9 & -15 \\ 4 & 1 & -12 & 6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 4 & 1 & -12 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -7 & 0 & -14 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & -7 & 0 & -14 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(M))\vec{x} = \vec{O}$. Because M has four columns, $\vec{x}$ needs to have four rows, so we'll use a 4-row column vector for $\vec{x}$. And multiplying the 4×4 matrix by the 4-row column vector will result in a 4×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix} 1 & 0 & -3 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get a system of equations,

$$x_1 - 3x_3 + x_4 = 0$$

$$x_2 + 2x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = 3x_3 - x_4$$

$$x_2 = -2x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of M is the span of the vectors in this linear combination equation.

$$N(M) = \text{Span}\left(\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

Topic: Null space of a matrix

Question: Find the null space of B.

$$B = \begin{bmatrix} 2 & 2 & -4 & 10 \\ -1 & -1 & 2 & -5 \\ 3 & 3 & -6 & 15 \end{bmatrix}$$

Answer choices:

A $\quad N(B) = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$

B $\quad N(B) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \\ -5 \end{bmatrix} \right)$

C $\quad N(B) = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right)$

D $\quad N(B) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ -2 \\ 5 \end{bmatrix} \right)$

Solution: A

To find the null space, put the matrix B in reduced row-echelon form.

$$B = \begin{bmatrix} 2 & 2 & -4 & 10 \\ -1 & -1 & 2 & -5 \\ 3 & 3 & -6 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 5 \\ -1 & -1 & 2 & -5 \\ 3 & 3 & -6 & 15 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & -6 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(B))\vec{x} = \vec{O}$. Because B has four columns, $\vec{x}$ needs to have four rows, so we'll use a 4-row column vector for $\vec{x}$. And multiplying the 3×4 matrix by the 4-row column vector will result in a 3×1 zero-vector, so the matrix equation must be

$$\begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For this matrix, we get the equation,

$$x_1 + x_2 - 2x_3 + 5x_4 = 0$$

which we can solve for the single pivot variable.

$$x_1 = -x_2 + 2x_3 - 5x_4$$

We can rewrite this as a linear combination

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of B is the span of the vectors in this linear combination equation.

$$N(B) = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

Topic: The column space and Ax=b

Question: Find the null space, then find the column space of A.

$$A = \begin{bmatrix} 1 & -5 & 2 & 4 \\ 3 & 0 & 1 & 2 \\ 1 & -1 & 2 & 4 \end{bmatrix}$$

Answer choices:

A $\quad N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 0 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$

B $\quad N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$

C $\quad N(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right)$

D $\quad N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}\right), C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix}\right)$

Solution: D

Find the null space of A by first putting the matrix into reduced row-echelon form.

$$\begin{bmatrix}1 & -5 & 2 & 4\\3 & 0 & 1 & 2\\1 & -1 & 2 & 4\end{bmatrix} \to \begin{bmatrix}1 & -5 & 2 & 4\\0 & 15 & -5 & -10\\1 & -1 & 2 & 4\end{bmatrix} \to \begin{bmatrix}1 & -5 & 2 & 4\\0 & 15 & -5 & -10\\0 & 4 & 0 & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -5 & 2 & 4\\0 & 1 & -\frac{1}{3} & -\frac{2}{3}\\0 & 4 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & -5 & 2 & 4\\0 & 1 & -\frac{1}{3} & -\frac{2}{3}\\0 & 0 & \frac{4}{3} & \frac{8}{3}\end{bmatrix} \to \begin{bmatrix}1 & 0 & \frac{1}{3} & \frac{2}{3}\\0 & 1 & -\frac{1}{3} & -\frac{2}{3}\\0 & 0 & \frac{4}{3} & \frac{8}{3}\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & \frac{1}{3} & \frac{2}{3}\\0 & 1 & -\frac{1}{3} & -\frac{2}{3}\\0 & 0 & 1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & \frac{1}{3} & \frac{2}{3}\\0 & 1 & 0 & 0\\0 & 0 & 1 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 2\end{bmatrix}$$

Set up the matrix equation.

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

From this matrix equation, we get

$$x_1 = 0$$

$$x_2 = 0$$

$x_3 + 2x_4 = 0$, or $x_3 = -2x_4$

Then the null space of A is all the linear combinations given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

which means the null space is the span of the single column vector.

$$N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix} \right)$$

The column space is all the linear combinations of the column vectors.

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 2 \\ 4 \end{bmatrix} \right)$$

Topic: The column space and Ax=b

Question: Find the column space of M in terms of its basis.

$$M=\begin{bmatrix}-1 & 2 & 6 & 5\\ 0 & 3 & -7 & 9\\ 3 & -6 & -18 & -15\end{bmatrix}$$

Answer choices:

A $\quad C(M)=$ Span$\left(\begin{bmatrix}-1\\ 0\\ 3\end{bmatrix},\begin{bmatrix}2\\ 3\\ -6\end{bmatrix},\begin{bmatrix}6\\ -7\\ -18\end{bmatrix},\begin{bmatrix}5\\ 9\\ -15\end{bmatrix}\right)$

B $\quad C(M)=$ Span$\left(\begin{bmatrix}-1\\ 0\\ 3\end{bmatrix},\begin{bmatrix}6\\ -7\\ -18\end{bmatrix}\right)$

C $\quad C(M)=$ Span$\left(\begin{bmatrix}-1\\ 0\\ 3\end{bmatrix},\begin{bmatrix}2\\ 3\\ -6\end{bmatrix}\right)$

D $\quad C(M)=$ Span$\left(\begin{bmatrix}-1\\ 0\\ 3\end{bmatrix},\begin{bmatrix}2\\ 3\\ -6\end{bmatrix},\begin{bmatrix}6\\ -7\\ -18\end{bmatrix}\right)$

Solution: C

Put M into row-echelon form.

$$\begin{bmatrix} -1 & 2 & 6 & 5 \\ 0 & 3 & -7 & 9 \\ 3 & -6 & -18 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 3 & -7 & 9 \\ 3 & -6 & -18 & -15 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 3 & -7 & 9 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -5 \\ 0 & 1 & -\frac{7}{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first and second columns of the matrix are the pivot columns, then the first and second columns of the original matrix form a basis for the column space of M.

$$C(M) = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ -6 \end{bmatrix}\right)$$

Topic: The column space and Ax=b

Question: Find the column space of A in terms of its basis.

$$A = \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ -3 & 6 & 3 & 9 \\ 2 & -4 & -2 & -6 \end{bmatrix}$$

Answer choices:

A $\quad C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}\right)$

B $\quad C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 9 \\ -6 \end{bmatrix}\right)$

C $\quad C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 9 \\ -6 \end{bmatrix}\right)$

D $\quad C(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}, \begin{bmatrix} -5 \\ 7 \\ 9 \\ -6 \end{bmatrix}\right)$

Solution: A

Put A into row-echelon form.

$$\begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ -3 & 6 & 3 & 9 \\ 2 & -4 & -2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 15 & -6 \\ 2 & -4 & -2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 3 & 5 & 7 \\ 0 & 0 & 15 & -6 \\ 0 & 0 & -10 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & 15 & -6 \\ 0 & 0 & -10 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & -10 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 & -5 \\ 0 & 1 & \frac{5}{3} & \frac{7}{3} \\ 0 & 0 & 1 & -\frac{2}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the first, second, and third columns of the matrix are the pivot columns, then the first, second, and third columns of the original matrix form a basis for the column space of A.

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 6 \\ -4 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 3 \\ -2 \end{bmatrix}\right)$$

Topic: Solving Ax=b

Question: Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}3&-6&6\\-3&7&-9\\-6&8&0\end{bmatrix} \text{ with } \vec{b}=\begin{bmatrix}1\\-1\\-2\end{bmatrix}$$

Answer choices:

A $\quad \vec{x}=\begin{bmatrix}\frac{5}{3}\\3\\0\end{bmatrix}+c_1\begin{bmatrix}4\\0\\1\end{bmatrix}+c_2\begin{bmatrix}0\\3\\1\end{bmatrix}$

B $\quad \vec{x}=\begin{bmatrix}\frac{1}{3}\\3\\0\end{bmatrix}+c_1\begin{bmatrix}4\\3\\1\end{bmatrix}$

C $\quad \vec{x}=\begin{bmatrix}\frac{5}{3}\\0\\0\end{bmatrix}+c_1\begin{bmatrix}4\\0\\1\end{bmatrix}+c_2\begin{bmatrix}0\\3\\1\end{bmatrix}$

D $\quad \vec{x}=\begin{bmatrix}\frac{1}{3}\\0\\0\end{bmatrix}+c_1\begin{bmatrix}4\\3\\1\end{bmatrix}$

Solution: D

We augment the matrix with $\overrightarrow{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 3 & -6 & 6 & | & b_1 \\ -3 & 7 & -9 & | & b_2 \\ -6 & 8 & 0 & | & b_3 \end{bmatrix}$$

Now we'll put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ -3 & 7 & -9 & | & b_2 \\ -6 & 8 & 0 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ 0 & 1 & -3 & | & 3(\frac{1}{3}b_1) + b_2 \\ -6 & 8 & 0 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ 0 & 1 & -3 & | & b_1 + b_2 \\ 0 & -4 & 12 & | & 6(\frac{1}{3}b_1) + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & | & \frac{1}{3}b_1 \\ 0 & 1 & -3 & | & b_1 + b_2 \\ 0 & 0 & 0 & | & 4(b_1 + b_2) + 2b_1 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 2(b_1 + b_2) + \frac{1}{3}b_1 \\ 0 & 1 & -3 & | & b_1 + b_2 \\ 0 & 0 & 0 & | & 6b_1 + 4b_2 + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & \frac{7}{3}b_1 + 2b_2 \\ 0 & 1 & -3 & | & b_1 + b_2 \\ 0 & 0 & 0 & | & 6b_1 + 4b_2 + b_3 \end{bmatrix}$$

Using the third row of the augmented matrix, we can verify that $\overrightarrow{b} = (1, -1, -2)$ will produce a solution $\overrightarrow{x}$.

$$6b_1 + 4b_2 + b_3 = 0$$

$$6(1) + 4(-1) + (-2) = 0$$

$$6 - 4 - 2 = 0$$

$$0 = 0$$

Now that we know we can get a solution, we'll start by finding the complementary solution, starting with the rref matrix that we found earlier.

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This matrix produces a system of two equations,

$$1x_1 + 0x_2 - 4x_3 = 0$$

$$0x_1 + 1x_2 - 3x_3 = 0$$

The system simplifies to

$$x_1 = 4x_3$$

$$x_2 = 3x_3$$

Then the vector set that satisfies the null space is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

We can therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. We'll plug $\vec{b} = (1, -1, -2)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & 0 & -4 & | & \frac{7}{3}b_1 + 2b_2 \\ 0 & 1 & -3 & | & b_1 + b_2 \\ 0 & 0 & 0 & | & 6b_1 + 4b_2 + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & \frac{7}{3}(1) + 2(-1) \\ 0 & 1 & -3 & | & 1 - 1 \\ 0 & 0 & 0 & | & 6(1) + 4(-1) - 2 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -4 & | & \frac{1}{3} \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

To find a vector $\vec{x}_p$ that satisfies $A\vec{x}_p = \vec{b}$, we need to choose a value for the free variable, and the easiest value to use is $x_3 = 0$. Using that value, we'll rewrite the matrix as a system of equations.

$$1x_1 + 0x_2 - 4(0) = \frac{1}{3}$$

$$0x_1 + 1x_2 - 3(0) = 0$$

The system becomes

$$x_1 = \frac{1}{3}$$

$$x_2 = 0$$

So the particular solution then is $x_1 = 1/3$, $x_2 = 0$, and $x_3 = 0$, or

$$\vec{x}_p = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} \frac{1}{3} \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

Topic: Solving Ax=b

Question: Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}1 & -1 & 3 & -1 & 3\\ -1 & 0 & 0 & 2 & 1\\ 0 & 1 & -3 & -1 & -4\end{bmatrix} \text{ with } \vec{b}=\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}$$

Answer choices:

A $\quad \vec{x}=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}+c_1\begin{bmatrix}0\\3\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\1\\0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\4\\0\\0\\1\end{bmatrix}$

B $\quad \vec{x}=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}+c_1\begin{bmatrix}1\\0\\0\\2\\1\end{bmatrix}+c_2\begin{bmatrix}0\\1\\3\\1\\4\end{bmatrix}$

C $\quad \vec{x}=\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+c_1\begin{bmatrix}0\\3\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\1\\0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\4\\0\\0\\1\end{bmatrix}$

D $\quad \vec{x} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 3 \\ 1 \\ 4 \end{bmatrix}$

Solution: A

We augment the matrix with $\vec{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ -1 & 0 & 0 & 2 & 1 & | & b_2 \\ 0 & 1 & -3 & -1 & -4 & | & b_3 \end{bmatrix}$$

Now we'll put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ 0 & -1 & 3 & 1 & 4 & | & b_2 + b_1 \\ 0 & 1 & -3 & -1 & -4 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ 0 & 1 & -3 & -1 & -4 & | & -b_2 - b_1 \\ 0 & 1 & -3 & -1 & -4 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & -1 & 3 & | & b_1 \\ 0 & 1 & -3 & -1 & -4 & | & -b_2 - b_1 \\ 0 & 0 & 0 & 0 & 0 & | & b_3 + b_2 + b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & -b_2 \\ 0 & 1 & -3 & -1 & -4 & | & -b_2 - b_1 \\ 0 & 0 & 0 & 0 & 0 & | & b_3 + b_2 + b_1 \end{bmatrix}$$

Using the third row of the augmented matrix, we can verify that $\vec{b} = (1, -2, 1)$ will produce a solution $\vec{x}$.

$$b_3 + b_2 + b_1 = 0$$

$$1 - 2 + 1 = 0$$

$$0 = 0$$

Now that we know we can get a solution, we'll start by finding the complementary solution, starting with the rref matrix that we found earlier.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & 0 \\ 0 & 1 & -3 & -1 & -4 & | & 0 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This matrix produces a system of two equations,

$$x_1 - 2x_4 - x_5 = 0$$

$$x_2 - 3x_3 - x_4 - 4x_5 = 0$$

The system simplifies to

$$x_1 = 2x_4 + x_5$$

$$x_2 = 3x_3 + x_4 + 4x_5$$

Then the vector set that satisfies the null space is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We can therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. We'll plug $\vec{b} = (1, -2, 1)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & -b_2 \\ 0 & 1 & -3 & -1 & -4 & | & -b_2 - b_1 \\ 0 & 0 & 0 & 0 & 0 & | & b_3 + b_2 + b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & -(-2) \\ 0 & 1 & -3 & -1 & -4 & | & -(-2) - 1 \\ 0 & 0 & 0 & 0 & 0 & | & 1 - 2 + 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & -1 & | & 2 \\ 0 & 1 & -3 & -1 & -4 & | & 1 \\ 0 & 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

To find a vector $\vec{x}_p$ that satisfies $A\vec{x}_p = \vec{b}$, we need to choose values for the free variables, and the easiest values to use are $x_3 = 0$, $x_4 = 0$, and $x_5 = 0$. Using those values, we'll rewrite the matrix as a system of equations.

$$x_1 - 2(0) - 0 = 2$$

$$x_2 - 3(0) - 0 - 4(0) = 1$$

The system becomes

$$x_1 = 2$$

$$x_2 = 1$$

So the particular solution then is $x_1 = 2$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, and $x_5 = 0$, or

$$\vec{x}_p=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\vec{x}=\vec{x}_p+\vec{x}_n$$

$$\vec{x}=\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}+c_1\begin{bmatrix}0\\3\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}2\\1\\0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\4\\0\\0\\1\end{bmatrix}$$

Topic: Solving Ax=b

Question: Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}2&3&4&-4\\2&3&8&-10\\6&9&16&-18\end{bmatrix} \text{ with } \vec{b}=\begin{bmatrix}1\\-1\\1\end{bmatrix}$$

Answer choices:

A $\vec{x}=\begin{bmatrix}1\\0\\-\frac{1}{4}\\0\end{bmatrix}+c_1\begin{bmatrix}\frac{3}{2}\\0\\0\\0\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\0\\\frac{3}{2}\end{bmatrix}$

B $\vec{x}=\begin{bmatrix}\frac{3}{2}\\0\\-\frac{1}{2}\\0\end{bmatrix}+c_1\begin{bmatrix}-\frac{3}{2}\\1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}-1\\0\\\frac{3}{2}\\1\end{bmatrix}$

C $\vec{x}=\begin{bmatrix}1\\0\\-\frac{1}{4}\\0\end{bmatrix}$

D $\qquad \vec{x} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{4} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 1 \\ \frac{3}{2} \end{bmatrix}$

Solution: B

We augment the matrix with $\vec{b} = (b_1, b_2, b_3)$.

$$\begin{bmatrix} 2 & 3 & 4 & -4 & | & b_1 \\ 2 & 3 & 8 & -10 & | & b_2 \\ 6 & 9 & 16 & -18 & | & b_3 \end{bmatrix}$$

Now we'll put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 2 & 3 & 8 & -10 & | & b_2 \\ 6 & 9 & 16 & -18 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 4 & -6 & | & -2(\frac{1}{2}b_1) + b_2 \\ 6 & 9 & 16 & -18 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 4 & -6 & | & -b_1 + b_2 \\ 0 & 0 & 4 & -6 & | & -6(\frac{1}{2}b_1) + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 4 & -6 & | & -b_1 + b_2 \\ 0 & 0 & 0 & 0 & | & -3b_1 + b_3 - (-b_1 + b_2) \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 2 & -2 & | & \frac{1}{2}b_1 \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-b_1+b_2) \\ 0 & 0 & 0 & 0 & | & -2b_1+b_3-b_2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & -2\cdot\frac{1}{4}(-b_1+b_2)+\frac{1}{2}b_1 \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-b_1+b_2) \\ 0 & 0 & 0 & 0 & | & -2b_1+b_3-b_2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & b_1-\frac{1}{2}b_2 \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-b_1+b_2) \\ 0 & 0 & 0 & 0 & | & -2b_1+b_3-b_2 \end{bmatrix}$$

Using the third row of the augmented matrix, we can verify that $\overrightarrow{b}=(1,-1,1)$ will produce a solution $\overrightarrow{x}$.

$$-2b_1+b_3-b_2=0$$

$$-2(1)+1-(-1)=0$$

$$0=0$$

Now that we know we can get a solution, we'll start by finding the complementary solution, starting with the rref matrix that we found earlier.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & 0 \\ 0 & 0 & 1 & -\frac{3}{2} & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This matrix produces a system of two equations,

$$x_1+\frac{3}{2}x_2+x_4=0$$

$$x_3 - \frac{3}{2}x_4 = 0$$

The system simplifies to

$$x_1 = -\frac{3}{2}x_2 - x_4$$

$$x_3 = \frac{3}{2}x_4$$

Then the vector set that satisfies the null space is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

We can therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

Now we need to find the particular solution that satisfies $A\vec{x}_p = \vec{b}$. We'll plug $\vec{b} = (1, -2, 1)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & b_1 - \frac{1}{2}b_2 \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-b_1 + b_2) \\ 0 & 0 & 0 & 0 & | & -2b_1 + b_3 - b_2 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & 1 - \frac{1}{2}(-1) \\ 0 & 0 & 1 & -\frac{3}{2} & | & \frac{1}{4}(-1 + (-1)) \\ 0 & 0 & 0 & 0 & | & -2(1) + 1 - (-1) \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & \frac{3}{2} & 0 & 1 & | & \frac{3}{2} \\ 0 & 0 & 1 & -\frac{3}{2} & | & -\frac{1}{2} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

To find a vector $\overrightarrow{x}_p$ that satisfies $A\overrightarrow{x}_p = \overrightarrow{b}$, we need to choose values for the free variables, and the easiest values to use are $x_2 = 0$ and $x_4 = 0$. Using those values, we'll rewrite the matrix as a system of equations.

$$x_1 + \frac{3}{2}(0) + 0 = \frac{3}{2}$$

$$x_3 - \frac{3}{2}(0) = -\frac{1}{2}$$

The system becomes

$$x_1 = \frac{3}{2}$$

$$x_3 = -\frac{1}{2}$$

So the particular solution then is $x_1 = 3/2$, $x_2 = 0$, $x_3 = -1/2$, and $x_4 = 0$, or

$$\overrightarrow{x_p} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix}$$

We'll get the general solution by adding the particular and complementary solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{1}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -\frac{3}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

Topic: Dimensionality, nullity, and rank

Question: Find the nullity of M.

$$M = \begin{bmatrix} 1 & -2 & 3 & -1 & 2 \\ -3 & 6 & -9 & 3 & -6 \\ -5 & 9 & -7 & 4 & 0 \end{bmatrix}$$

Answer choices:

A nullity$(M) = 2$

B nullity$(M) = 5$

C nullity$(M) = 3$

D nullity$(M) = 4$

Solution: C

To find the nullity of the matrix M, we need to first find the null space, so we'll set up the augmented matrix for $M\vec{x}=\vec{O}$, then put it in reduced row-echelon form.

$$\begin{bmatrix}1&-2&3&-1&2&|&0\\-3&6&-9&3&-6&|&0\\-5&9&-7&4&0&|&0\end{bmatrix}\to\begin{bmatrix}1&-2&3&-1&2&|&0\\0&0&0&0&0&|&0\\-5&9&-7&4&0&|&0\end{bmatrix}$$

$$\begin{bmatrix}1&-2&3&-1&2&|&0\\0&0&0&0&0&|&0\\0&-1&8&-1&10&|&0\end{bmatrix}\to\begin{bmatrix}1&-2&3&-1&2&|&0\\0&-1&8&-1&10&|&0\\0&0&0&0&0&|&0\end{bmatrix}$$

$$\begin{bmatrix}1&-2&3&-1&2&|&0\\0&1&-8&1&-10&|&0\\0&0&0&0&0&|&0\end{bmatrix}\to\begin{bmatrix}1&0&-13&1&-18&|&0\\0&1&-8&1&-10&|&0\\0&0&0&0&0&|&0\end{bmatrix}$$

Let's parse out a system of equations.

$$x_1-13x_3+x_4-18x_5=0$$

$$x_2-8x_3+x_4-10x_5=0$$

Solve for the pivot variables.

$$x_1=13x_3-x_4+18x_5$$

$$x_2=8x_3-x_4+10x_5$$

Rewrite the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} 13 \\ 8 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 18 \\ 10 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of M is the span of the vectors in this linear combination equation.

$$N(M) = \text{Span}\left(\begin{bmatrix} 13 \\ 8 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 18 \\ 10 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

We found 3 spanning vectors that form a basis for the null space, which matches the dimension of the null space.

$$\text{Dim}(N(M)) = \text{nullity}(M) = 3$$

We can also get the nullity of M from the number of free variables in rref(M). Because there were 3 free variables, x_3, x_4, and x_5, nullity$(M) = 3$.

Topic: Dimensionality, nullity, and rank

Question: Find the rank of X.

$$X = \begin{bmatrix} -2 & 10 & -4 \\ 1 & 3 & -6 \\ 1 & -5 & 9 \end{bmatrix}$$

Answer choices:

A rank$(X) = 1$

B rank$(X) = 0$

C rank$(X) = 3$

D rank$(X) = 2$

Solution: C

To find the rank of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -2 & 10 & -4 \\ 1 & 3 & -6 \\ 1 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 1 & 3 & -6 \\ 1 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 0 & 8 & -8 \\ 1 & -5 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 2 \\ 0 & 8 & -8 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now that the matrix is in reduced row-echelon form, we can find the rank directly from the matrix. We can see that all three columns are pivot columns. So because there are three pivot variables, the rank is

$$\text{Dim}(C(X)) = \text{rank}(X) = 3$$

Topic: Dimensionality, nullity, and rank

Question: Find the nullity and the rank of A.

$$A = \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 2 & 5 & -4 & -7 & 3 \\ 1 & 4 & -3 & 5 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix}$$

Answer choices:

A nullity$(A) = 2$ and rank$(A) = 3$

B nullity$(A) = 3$ and rank$(A) = 3$

C nullity$(A) = 3$ and rank$(A) = 2$

D nullity$(A) = 1$ and rank$(A) = 4$

Solution: A

To find the nullity of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 2 & 5 & -4 & -7 & 3 \\ 1 & 4 & -3 & 5 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & -1 & 0 & -5 & 3 \\ 1 & 4 & -3 & 5 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & -1 & 0 & -5 & 3 \\ 0 & 1 & -1 & 6 & 4 \\ 1 & 2 & -2 & -6 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & -1 & 0 & -5 & 3 \\ 0 & 1 & -1 & 6 & 4 \\ 0 & -1 & 0 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 1 & -1 & 6 & 4 \\ 0 & -1 & 0 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & -1 & 1 & 7 \\ 0 & -1 & 0 & -5 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -1 & 0 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & -1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -16 & 9 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & -1 & 1 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & -16 & 9 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -18 & -5 \\ 0 & 1 & 0 & 5 & -3 \\ 0 & 0 & 1 & -1 & -7 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since rref(A) has three pivot columns and two free columns,

rank$(A) = 3$

nullity$(A) = 2$

Topic: Functions and transformations

Question: The transformation T maps every vector in $\mathbb{R}^2$ to every vector in $\mathbb{R}^3$. What are the domain, codomain, and range of T?

Answer choices:

A The domain is $\mathbb{R}^2$, the codomain is $\mathbb{R}^2$, and the range is $\mathbb{R}^2$

B The domain is $\mathbb{R}^3$, the codomain is $\mathbb{R}^3$, and the range is $\mathbb{R}^3$

C The domain is $\mathbb{R}^2$, the codomain is $\mathbb{R}^3$, and the range is $\mathbb{R}^3$

D The domain is $\mathbb{R}^2$, the codomain is $\mathbb{R}^3$, and the range is $\mathbb{R}^2$

Solution: C

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^2$.

The codomain is the space T is mapping *to*, so the codomain is $\mathbb{R}^3$.

The range is the specific set of vectors within the codomain that are being mapped to. Because every vector in $\mathbb{R}^3$ is being mapped to, the range is $\mathbb{R}^3$.

Topic: Functions and transformations

Question: The transformation T maps every vector in $\mathbb{R}^4$ to the zero vector $\vec{O}$ in $\mathbb{R}^2$. What are the domain, codomain, and range of T?

Answer choices:

A The domain is $\mathbb{R}^4$, the codomain is $\mathbb{R}^2$, and the range is $\vec{v}=(0,0)$

B The domain is $\mathbb{R}^2$, the codomain is $\mathbb{R}^4$, and the range is $\vec{v}=(0,0)$

C The domain is $\mathbb{R}^4$, the codomain is $\vec{v}=(0,0)$, and the range is $\mathbb{R}^2$

D The domain is $\mathbb{R}^2$, the codomain is $\vec{v}=(0,0)$, and the range is $\mathbb{R}^4$

Solution: A

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^4$.

The codomain is the space T is mapping *to*, so the codomain is $\mathbb{R}^2$.

The range is the specific set of vectors within the codomain that are being mapped to, so the range is $\vec{v} = (0,0)$.

Topic: Functions and transformations

Question: The transformation T maps $\overrightarrow{a} = (1,2,-4)$ to $\overrightarrow{b} = (-3,0,4)$. What are the domain, codomain, and range of T?

Answer choices:

A The domain is $\mathbb{R}^3$, the codomain is $\overrightarrow{a} = (1,2,-4)$, and the range is $\mathbb{R}^3$

B The domain is $\mathbb{R}^3$, the codomain is $\mathbb{R}^3$, and the range is $\overrightarrow{a} = (1,2,-4)$

C The domain is $\mathbb{R}^3$, the codomain is $\overrightarrow{b} = (-3,0,4)$, and the range is $\mathbb{R}^3$

D The domain is $\mathbb{R}^3$, the codomain is $\mathbb{R}^3$, and the range is $\overrightarrow{b} = (-3,0,4)$

Solution: D

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^3$.

The codomain is the space T is mapping *to*, so the codomain is $\mathbb{R}^3$.

The range is the specific set of vectors within the codomain that are being mapped to, so the range is $\vec{b} = (-3,0,4)$.

Topic: Transformation matrices and the image of the subset

Question: If $\overrightarrow{a} = (-4,2)$ becomes $\overrightarrow{b}$ after undergoing a transformation by matrix Q, find $\overrightarrow{b}$.

$$Q = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix}$$

Answer choices:

A $\quad \overrightarrow{b} = \begin{bmatrix} 42 \\ -12 \end{bmatrix}$

B $\quad \overrightarrow{b} = \begin{bmatrix} -42 \\ 12 \end{bmatrix}$

C $\quad \overrightarrow{b} = \begin{bmatrix} -42 \\ -12 \end{bmatrix}$

D $\quad \overrightarrow{b} = \begin{bmatrix} 42 \\ 12 \end{bmatrix}$

Solution: C

To apply a transformation matrix to $\vec{a}$, we'll multiply the matrix by the vector.

$$\vec{b} = M\vec{a} = \begin{bmatrix} 11 & 1 \\ 0 & -6 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix}$$

$$\vec{b} = M\vec{a} = \begin{bmatrix} 11(-4) + 1(2) \\ 0(-4) - 6(2) \end{bmatrix}$$

$$\vec{b} = M\vec{a} = \begin{bmatrix} -44 + 2 \\ 0 - 12 \end{bmatrix}$$

$$\vec{b} = M\vec{a} = \begin{bmatrix} -42 \\ -12 \end{bmatrix}$$

Topic: Transformation matrices and the image of the subset

Question: What are the vertices of the transformation of the polygon given by $(-2,1)$, $(1,3)$, $(2,-2)$, and $(-3,-1)$ after it's transformed by matrix P.

$$P=\begin{bmatrix}-2 & 0\\ 1 & 3\end{bmatrix}$$

Answer choices:

A $(2,2)$, $(-3,7)$, $(-3,2)$, and $(4,-3)$

B $(2,2)$, $(-3,7)$, $(-4,-4)$, and $(6,-6)$

C $(4,1)$, $(-2,10)$, $(-3,2)$, and $(4,-3)$

D $(4,1)$, $(-2,10)$, $(-4,-4)$, and $(6,-6)$

Solution: D

Put the vertices of the polygon into a matrix.

$$\begin{bmatrix}-2 & 1 & 2 & -3\\ 1 & 3 & -2 & -1\end{bmatrix}$$

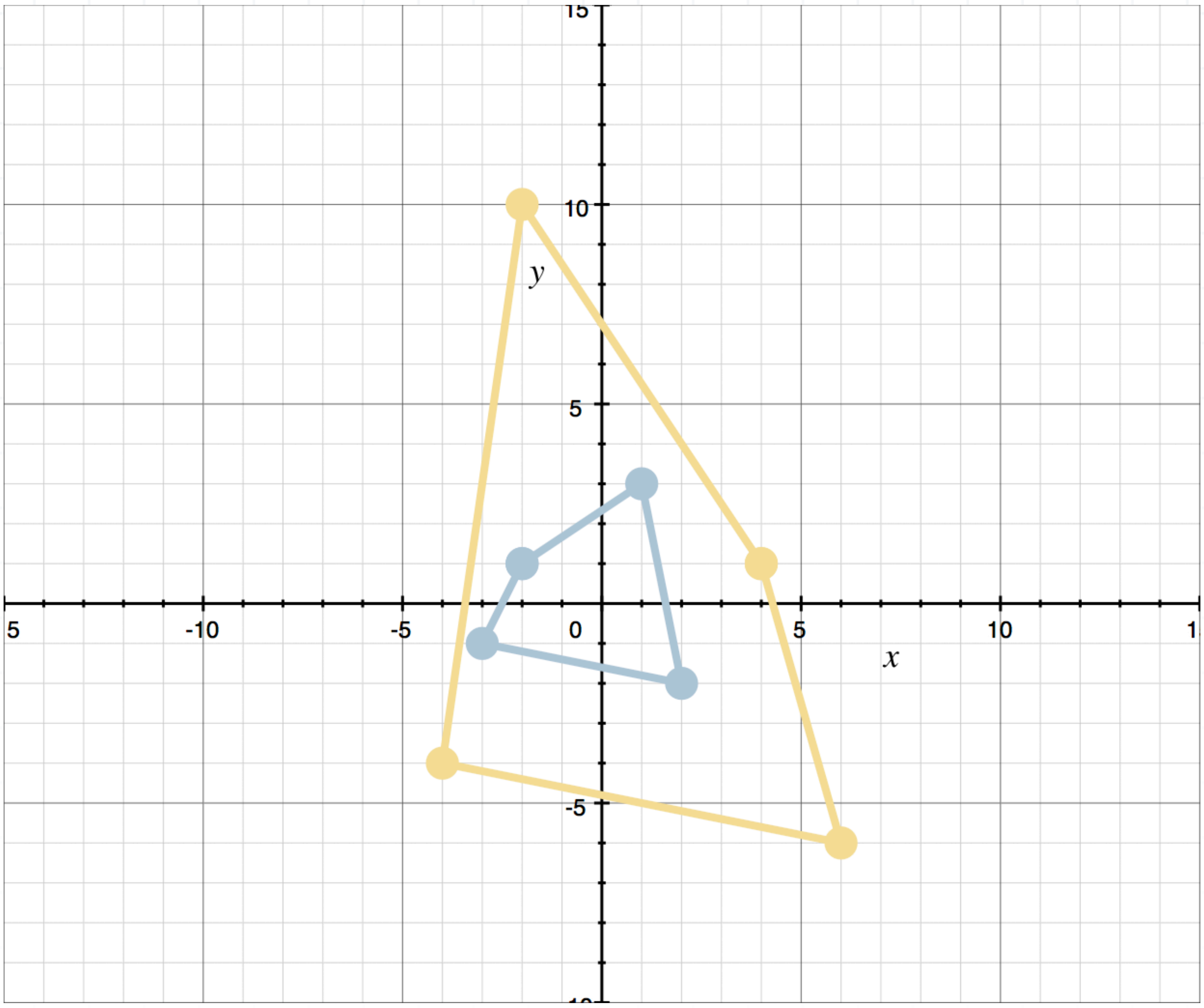
Apply the transformation of P to the vertex matrix.

$$\begin{bmatrix}-2 & 0\\ 1 & 3\end{bmatrix}\begin{bmatrix}-2 & 1 & 2 & -3\\ 1 & 3 & -2 & -1\end{bmatrix}$$

$$\begin{bmatrix}-2(-2)+0(1) & -2(1)+0(3) & -2(2)+0(-2) & -2(-3)+0(-1)\\ 1(-2)+3(1) & 1(1)+3(3) & 1(2)+3(-2) & 1(-3)+3(-1)\end{bmatrix}$$

$$\begin{bmatrix}4 & -2 & -4 & 6\\ 1 & 10 & -4 & -6\end{bmatrix}$$

The original polygon is sketched in light blue, and its transformation after P is in yellow.


15
10
5
0
-5
-10
5
10
1
y
x

Topic: Transformation matrices and the image of the subset

Question: What are the vertices of the transformation of the triangle with vertices $(-3,0)$, $(1,2)$, and $(1,-2)$ after it's transformed by matrix S.

$$S = \begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix}$$

Answer choices:

A $(0,-6)$, $(-2,4)$, and $(2,0)$

B $(0,-4)$, $(-1,3)$, and $(2,2)$

C $(1,-3)$, $(-1,6)$, and $(3,1)$

D $(2,-1)$, $(0,2)$, and $(1,4)$

Solution: A

Put the vertices of the triangle into a matrix.

$$\begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

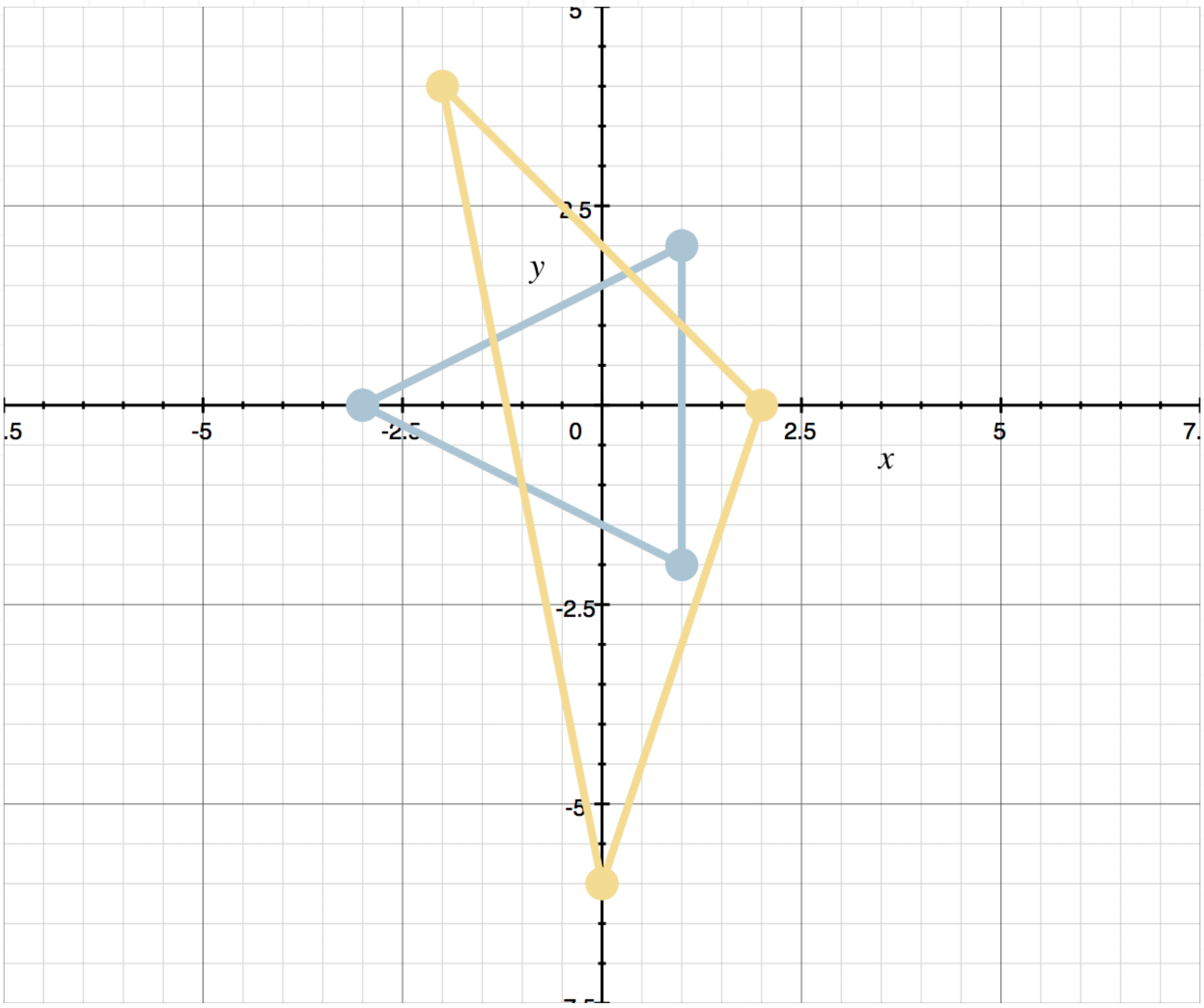
Apply the transformation of S to the vertex matrix.

$$\begin{bmatrix} 0 & -1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 0(-3) - 1(0) & 0(1) - 1(2) & 0(1) - 1(-2) \\ 2(-3) + 1(0) & 2(1) + 1(2) & 2(1) + 1(-2) \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 2 \\ -6 & 4 & 0 \end{bmatrix}$$

The original triangle is sketched in light blue, and its transformation after S is in yellow.


5
2.5
y
.5
-5
-2.5
0
2.5
5
7.
x
-2.5
-5

Topic: Preimage, image, and the kernel

Question: Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer choices:

A $\quad A_1 = \left\{ \begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{14}{3} \\ \frac{5}{3} \end{bmatrix} \right\}$

B $\quad A_1 = \left\{ \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{14}{3} \\ \frac{5}{3} \end{bmatrix} \right\}$

C $\quad A_1 = \left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} \frac{14}{3} \\ -\frac{5}{3} \end{bmatrix} \right\}$

D $\quad A_1 = \left\{ \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \begin{bmatrix} -\frac{14}{3} \\ \frac{5}{3} \end{bmatrix} \right\}$

Solution: A

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^2 \ \middle| \ T(\vec{x}) \in B_1 \right\}$$

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^2 \ \middle| \ \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^2$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & | & 1 \\ 0 & 3 & | & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & | & 1 \\ 0 & 1 & | & -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{1}{3} \\ 0 & 1 & | & -\frac{2}{3} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & | & 3 \\ 0 & 3 & | & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & | & 3 \\ 0 & 1 & | & \frac{5}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{14}{3} \\ 0 & 1 & | & \frac{5}{3} \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 1/3$ and $x_2 = -2/3$. And from the second augmented matrix we get $x_1 = 14/3$ and $x_2 = 5/3$. Therefore,

$\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} \frac{14}{3} \\ \frac{5}{3} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 3 \\ 5 \end{bmatrix}$ in the subset B_1 under T

Topic: Preimage, image, and the kernel

Question: Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer choices:

A $\quad A_1 = \left\{ \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{8} \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{3}{4} \end{bmatrix} \right\}$

B $\quad A_1 = \left\{ \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{8} \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{3}{4} \end{bmatrix} \right\}$

C $\quad A_1 = \left\{ \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{8} \end{bmatrix}, \begin{bmatrix} 1 \\ -\frac{3}{4} \end{bmatrix} \right\}$

D $\quad A_1 = \left\{ \begin{bmatrix} \frac{3}{2} \\ -\frac{3}{8} \end{bmatrix}, \begin{bmatrix} -1 \\ \frac{3}{4} \end{bmatrix} \right\}$

Solution: D

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \left\{\vec{x} \in \mathbb{R}^2 \ \middle|\ T(\vec{x}) \in B_1\right\}$$

$$T^{-1}(B_1) = \left\{\vec{x} \in \mathbb{R}^2 \ \middle|\ \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^2$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & | & -3 \\ 1 & 4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 0 \\ -2 & 0 & | & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 0 \\ 0 & 8 & | & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 0 \\ 0 & 1 & | & -\frac{3}{8} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ 0 & 1 & | & -\frac{3}{8} \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 0 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 0 & | & 2 \\ 1 & 4 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 2 \\ -2 & 0 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 0 & 8 & | & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & | & 2 \\ 0 & 1 & | & \frac{3}{4} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -1 \\ 0 & 1 & | & \frac{3}{4} \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 3/2$ and $x_2 = -3/8$. And from the second augmented matrix we get $x_1 = -1$ and $x_2 = 3/4$. Therefore,

$\begin{bmatrix} \frac{3}{2} \\ -\frac{3}{8} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} -3 \\ 0 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} -1 \\ \frac{3}{4} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ in the subset B_1 under T

Topic: Preimage, image, and the kernel

Question: Find the preimage A_1 of the subset B_1 under the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 5 \\ -2 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer choices:

A $\quad A_1 = \left\{ \begin{bmatrix} -\frac{8}{3} \\ \frac{7}{6} \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ \frac{2}{3} \end{bmatrix} \right\}$

B $\quad A_1 = \left\{ \begin{bmatrix} \frac{8}{3} \\ -\frac{7}{6} \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ \frac{2}{3} \end{bmatrix} \right\}$

C $\quad A_1 = \left\{ \begin{bmatrix} -\frac{8}{3} \\ \frac{7}{6} \end{bmatrix}, \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}$

D $\quad A_1 = \left\{ \begin{bmatrix} \frac{8}{3} \\ -\frac{7}{6} \end{bmatrix}, \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix} \right\}$

Solution: B

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \{\vec{x} \in \mathbb{R}^2 \ | \ T(\vec{x}) \in B_1\}$$

$$T^{-1}(B_1) = \left\{\vec{x} \in \mathbb{R}^2 \ \middle| \ \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix} \text{ or } \begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}\right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^2$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & 5 \\ 1 & 4 & | & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & 5 \\ 0 & 6 & | & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & 5 \\ 0 & 1 & | & -\frac{7}{6} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{8}{3} \\ 0 & 1 & | & -\frac{7}{6} \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & -3 \\ 1 & 4 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & -3 \\ 0 & 6 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & -3 \\ 0 & 1 & | & \frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & -\frac{5}{3} \\ 0 & 1 & | & \frac{2}{3} \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 8/3$ and $x_2 = -7/6$. And from the second augmented matrix we get $x_1 = -5/3$ and $x_2 = 2/3$. Therefore,

$\begin{bmatrix} \frac{8}{3} \\ -\frac{7}{6} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 5 \\ -2 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} -\frac{5}{3} \\ \frac{2}{3} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} -3 \\ 1 \end{bmatrix}$ in the subset B_1 under T

Topic: Linear transformations as matrix-vector products

Question: Use a matrix-vector product to reflect the square with vertices $(-3,2)$, $(4,2)$, $(4,-5)$, and $(-3,-5)$ over the x-axis. What are the vertices of the reflected square?

Answer choices:

A $(-3,-2)$, $(4,-2)$, $(4,5)$, $(-3,5)$

B $(3,2)$, $(4,-2)$, $(-4,-5)$, $(-3,5)$

C $(-3,-2)$, $(-4,2)$, $(4,5)$, $(3,-5)$

D $(3,2)$, $(-4,2)$, $(-4,-5)$, $(3,-5)$

Solution: A

If each point in the square is given by (x, y), a reflection over the x-axis means we'll take the y-coordinate of each point in the square and multiply it by -1. So after the reflection, each transformed point will be $(x, -y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original square, then a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

represents the corresponding point in the transformed square. So a transformation T that expresses the reflection for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ -0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the square, $(-3,2)$, $(4,2)$, $(4,-5)$, and $(-3,-5)$.

$$T\left(\begin{bmatrix}-3\\2\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}-3\\2\end{bmatrix}=\begin{bmatrix}1(-3)+0(2)\\0(-3)-1(2)\end{bmatrix}=\begin{bmatrix}-3\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\2\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}1(4)+0(2)\\0(4)-1(2)\end{bmatrix}=\begin{bmatrix}4\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\-5\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\-5\end{bmatrix}=\begin{bmatrix}1(4)+0(-5)\\0(4)-1(-5)\end{bmatrix}=\begin{bmatrix}4\\5\end{bmatrix}$$

$$T\left(\begin{bmatrix}-3\\-5\end{bmatrix}\right)=\begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}-3\\-5\end{bmatrix}=\begin{bmatrix}1(-3)+0(-5)\\0(-3)-1(-5)\end{bmatrix}=\begin{bmatrix}-3\\5\end{bmatrix}$$

Topic: Linear transformations as matrix-vector products

Question: Use a matrix-vector product to double the width of the rectangle that has vertices $(3,-6)$, $(3,1)$, $(-1,1)$, and $(-1,-6)$. What are the vertices of the stretched rectangle?

Answer choices:

A $(3,-12)$, $(3,2)$, $(-1,2)$, $(-1,-12)$

B $(3,12)$, $(3,-2)$, $(-1,-2)$, $(-1,12)$

C $(-6,-6)$, $(-6,1)$, $(2,1)$, $(2,-6)$

D $(6,-6)$, $(6,1)$, $(-2,1)$, $(-2,-6)$

Solution: D

Doubling the width of the rectangle means we're stretching it horizontally by a factor of 2.

If each point in the rectangle is given by (x, y), then doubling the width means we'll take the x-coordinate of each point in the rectangle and multiply it by 2. So after the stretch, each transformed point will be $(2x, y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original rectangle, then a position vector

$$\vec{v} = \begin{bmatrix} 2v_1 \\ v_2 \end{bmatrix}$$

represents the corresponding point in the transformed rectangle. So a transformation T that expresses the stretch for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 2v_1 \\ v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2(1) \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 2(0) \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the rectangle, $(3,-6)$, $(3,1)$, $(-1,1)$, and $(-1,-6)$.

$$T\left(\begin{bmatrix}3\\-6\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}3\\-6\end{bmatrix}=\begin{bmatrix}2(3)+0(-6)\\0(3)+1(-6)\end{bmatrix}=\begin{bmatrix}6\\-6\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}2(3)+0(1)\\0(3)+1(1)\end{bmatrix}=\begin{bmatrix}6\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\1\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}2(-1)+0(1)\\0(-1)+1(1)\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}-1\\-6\end{bmatrix}\right)=\begin{bmatrix}2&0\\0&1\end{bmatrix}\begin{bmatrix}-1\\-6\end{bmatrix}=\begin{bmatrix}2(-1)+0(-6)\\0(-1)+1(-6)\end{bmatrix}=\begin{bmatrix}-2\\-6\end{bmatrix}$$

Topic: Linear transformations as matrix-vector products

Question: Use a matrix-vector product to reflect the parallelogram with vertices $(1,1)$, $(0,-4)$, $(-4,-4)$, and $(-3,1)$ over the y-axis, and then compress it vertically by a factor of 3. What are the vertices of the transformed parallelogram?

Answer choices:

A $(-1,1)$, $(0,-4)$, $(4,-4)$, $(3,1)$

B $(1,-1)$, $(0,4)$, $(-4,4)$, $(-3,-1)$

C $\left(-1,\frac{1}{3}\right)$, $\left(0,-\frac{4}{3}\right)$, $\left(4,-\frac{4}{3}\right)$, $\left(3,\frac{1}{3}\right)$

D $\left(1,-\frac{1}{3}\right)$, $\left(0,\frac{4}{3}\right)$, $\left(-4,\frac{4}{3}\right)$, $\left(-3,-\frac{1}{3}\right)$

Solution: C

If each point in the square is given by (x, y), a reflection over the y-axis means we'll take the x-coordinate of each point in the parallelogram and multiply it by -1. So after the reflection, each transformed point will be $(-x, y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original parallelogram, then a position vector

$$\vec{v} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

represents the corresponding point in the transformed parallelogram. So a transformation T that expresses the reflection for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

A vertical compression by a factor of 3 means we'll take the y-coordinate of each point in the parallelogram and multiply it by $1/3$. So after the compression (and the reflection), each transformed point will be

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} -v_1 \\ \frac{1}{3}v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\\frac{1}{3}(0)\end{bmatrix}=\begin{bmatrix}-1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0\\\frac{1}{3}(1)\end{bmatrix}=\begin{bmatrix}0\\\frac{1}{3}\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the parallelogram, $(1,1)$, $(0,-4)$, $(-4,-4)$, and $(-3,1)$.

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1(1)+0(1)\\0(1)+\frac{1}{3}(1)\end{bmatrix}=\begin{bmatrix}-1\\\frac{1}{3}\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\-4\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}0\\-4\end{bmatrix}=\begin{bmatrix}-1(0)+0(-4)\\0(0)+\frac{1}{3}(-4)\end{bmatrix}=\begin{bmatrix}0\\-\frac{4}{3}\end{bmatrix}$$

$$T\left(\begin{bmatrix}-4\\-4\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}-4\\-4\end{bmatrix}=\begin{bmatrix}-1(-4)+0(-4)\\0(-4)+\frac{1}{3}(-4)\end{bmatrix}=\begin{bmatrix}4\\-\frac{4}{3}\end{bmatrix}$$

$$T\left(\begin{bmatrix}-3\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&\frac{1}{3}\end{bmatrix}\begin{bmatrix}-3\\1\end{bmatrix}=\begin{bmatrix}-1(-3)+0(1)\\0(-3)+\frac{1}{3}(1)\end{bmatrix}=\begin{bmatrix}3\\\frac{1}{3}\end{bmatrix}$$

Topic: Linear transformations as rotations

Question: Find the rotation of $\vec{x} = (-1,4)$ by an angle of $\theta = 270^\circ$.

Answer choices:

A $\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}4\\1\end{bmatrix}$

B $\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}-4\\-1\end{bmatrix}$

C $\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}4\\-1\end{bmatrix}$

D $\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\4\end{bmatrix}\right) = \begin{bmatrix}-4\\1\end{bmatrix}$

Solution: A

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ by 270° is

$$\text{Rot}_{270^\circ}(\vec{x}) = \begin{bmatrix}\cos(270^\circ) & -\sin(270^\circ)\\ \sin(270^\circ) & \cos(270^\circ)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 270^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix}\cos(270^\circ) & -\sin(270^\circ)\\ \sin(270^\circ) & \cos(270^\circ)\end{bmatrix} = \begin{bmatrix}0 & -(-1)\\ -1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

So the transformation for a 270° rotation is

$$\text{Rot}_{270^\circ}(\vec{x}) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (-1,4)$.

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}-1\\ 4\end{bmatrix}$$

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}0(-1) + 1(4)\\ -1(-1) + 0(4)\end{bmatrix}$$

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}0 + 4\\ 1 + 0\end{bmatrix}$$

$$\text{Rot}_{270^\circ}\left(\begin{bmatrix}-1\\ 4\end{bmatrix}\right) = \begin{bmatrix}4\\ 1\end{bmatrix}$$

Topic: Linear transformations as rotations

Question: Find the rotation of $\vec{x} = (2,0,-3)$ by an angle of $\theta = 60^\circ$ about the x-axis.

Answer choices:

A $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right) = \begin{bmatrix}2\\-\frac{3\sqrt{3}}{2}\\\frac{3}{2}\end{bmatrix}$

B $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right) = \begin{bmatrix}2\\\frac{3\sqrt{3}}{2}\\-\frac{3}{2}\end{bmatrix}$

C $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right) = \begin{bmatrix}-2\\\frac{3\sqrt{3}}{2}\\\frac{3}{2}\end{bmatrix}$

D $\quad \text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right) = \begin{bmatrix}-2\\-\frac{3\sqrt{3}}{2}\\\frac{3}{2}\end{bmatrix}$

Solution: B

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ by 60° around the x-axis is

$$\text{Rot}_{60^\circ \text{ around } x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(60^\circ) & -\sin(60^\circ) \\ 0 & \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 60^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(60^\circ) & -\sin(60^\circ) \\ 0 & \sin(60^\circ) & \cos(60^\circ) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

So the transformation for a 60° rotation around the x-axis is

$$\text{Rot}_{60^\circ \text{ around } x}(\vec{x}) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (2,0,-3)$.

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix}$$

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}1(2)+0(0)+0(-3)\\0(2)+\frac{1}{2}(0)-\frac{\sqrt{3}}{2}(-3)\\0(2)+\frac{\sqrt{3}}{2}(0)+\frac{1}{2}(-3)\end{bmatrix}$$

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}2+0+0\\0+0+\frac{3\sqrt{3}}{2}\\0+0-\frac{3}{2}\end{bmatrix}$$

$$\text{Rot}_{60^\circ \text{ around } x}\left(\begin{bmatrix}2\\0\\-3\end{bmatrix}\right)=\begin{bmatrix}2\\\frac{3\sqrt{3}}{2}\\-\frac{3}{2}\end{bmatrix}$$

Topic: Linear transformations as rotations

Question: Find the rotation of $\vec{x} = (-2,3,-1)$ by an angle of $\theta = 225^\circ$ about the z-axis.

Answer choices:

A $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}-\frac{5\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\1\end{bmatrix}$

B $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}\frac{5\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\-1\end{bmatrix}$

C $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}-\frac{5\sqrt{2}}{2}\\\frac{\sqrt{2}}{2}\\1\end{bmatrix}$

D $\quad \text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right) = \begin{bmatrix}\frac{5\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\-1\end{bmatrix}$

Solution: D

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ by 225° around the z-axis is

$$\text{Rot}_{225^\circ \text{ around } z} = \begin{bmatrix} \cos(225^\circ) & -\sin(225^\circ) & 0 \\ \sin(225^\circ) & \cos(225^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 225^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} \cos(225^\circ) & -\sin(225^\circ) & 0 \\ \sin(225^\circ) & \cos(225^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\left(-\frac{\sqrt{2}}{2}\right) & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the transformation for a 225° rotation around the z-axis is

$$\text{Rot}_{225^\circ \text{ around } z}(\vec{x}) = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (-2,3,-1)$.

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix}$$

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)=\begin{bmatrix}-\frac{\sqrt{2}}{2}(-2)+\frac{\sqrt{2}}{2}(3)+0(-1)\\-\frac{\sqrt{2}}{2}(-2)-\frac{\sqrt{2}}{2}(3)+0(-1)\\0(-2)+0(3)+1(-1)\end{bmatrix}$$

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)=\begin{bmatrix}\sqrt{2}+\frac{3\sqrt{2}}{2}+0\\\sqrt{2}-\frac{3\sqrt{2}}{2}+0\\0+0-1\end{bmatrix}$$

$$\text{Rot}_{225^\circ \text{ around } z}\left(\begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)=\begin{bmatrix}\frac{5\sqrt{2}}{2}\\-\frac{\sqrt{2}}{2}\\-1\end{bmatrix}$$

Topic: Adding and scaling linear transformations

Question: Find the product of a scalar $c = -3$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} 2 & -1 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer choices:

A $\quad cT(\vec{x}) = \begin{bmatrix} 6 & -3 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

B $\quad cT(\vec{x}) = \begin{bmatrix} -6 & 3 \\ 0 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

C $\quad cT(\vec{x}) = \begin{bmatrix} 6 & 3 \\ 0 & 15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

D $\quad cT(\vec{x}) = \begin{bmatrix} -6 & -3 \\ 0 & -15 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Solution: B

The transformation T is given as a matrix-vector product. If we call the matrix that's in the transformation T the matrix B, then multiplying the transformation by the scalar $c = -3$ gives

$$cT(\overrightarrow{x}) = -3\begin{bmatrix}2 & -1\\ 0 & 5\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

First find cB.

$$cB = -3\begin{bmatrix}2 & -1\\ 0 & 5\end{bmatrix}$$

$$cB = \begin{bmatrix}-3(2) & -3(-1)\\ -3(0) & -3(5)\end{bmatrix}$$

$$cB = \begin{bmatrix}-6 & 3\\ 0 & -15\end{bmatrix}$$

So the scaled transformation would be

$$cT(\overrightarrow{x}) = \begin{bmatrix}-6 & 3\\ 0 & -15\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Topic: Adding and scaling linear transformations

Question: Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} 3 & 6 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Answer choices:

A $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 2 & 10 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

B $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} -2 & 10 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

C $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 2 & -10 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

D $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} -2 & -10 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Solution: A

The sum of the transformations is the sum of the matrices A and B given in their matrix vector products,

$$S(\overrightarrow{x}) = \begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$T(\overrightarrow{x}) = \begin{bmatrix}-1 & 4\\ 2 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

The sum of the matrices is

$$A + B = \begin{bmatrix}3 & 6\\ 0 & -1\end{bmatrix} + \begin{bmatrix}-1 & 4\\ 2 & 4\end{bmatrix}$$

$$A + B = \begin{bmatrix}3-1 & 6+4\\ 0+2 & -1+4\end{bmatrix}$$

$$A + B = \begin{bmatrix}2 & 10\\ 2 & 3\end{bmatrix}$$

So the sum of the transformations would be

$$S(\overrightarrow{x}) + T(\overrightarrow{x}) = \begin{bmatrix}2 & 10\\ 2 & 3\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Topic: Adding and scaling linear transformations

Question: Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & -1 \\ 3 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Answer choices:

A $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 5 & 3 & 1 \\ 0 & 2 & 0 \\ 0 & 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

B $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & 2 \\ 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

C $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 5 & -3 & 1 \\ 0 & 2 & 0 \\ 0 & -2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

D $\quad S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -2 \\ 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Solution: D

The sum of the transformations is the sum of the matrices A and B given in their matrix vector products,

$$S(\vec{x}) = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & -1 \\ 3 & 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The sum of the matrices is

$$A + B = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 2 & -1 \\ 3 & 1 & 4 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & -1 \\ 3 & 3 & -2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 4-1 & -1+2 & 1+0 \\ 0+0 & 2+0 & -1-1 \\ 3+3 & 1+3 & 4-2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -2 \\ 6 & 4 & 2 \end{bmatrix}$$

So the sum of the transformations would be

$$S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 3 & 1 & 1 \\ 0 & 2 & -2 \\ 6 & 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Topic: Projections as linear transformations

Question: Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} -2 \\ 0 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Answer choices:

A $\quad$ $\text{Proj}_L(\vec{v}) = (1,0)$

B $\quad$ $\text{Proj}_L(\vec{v}) = (0,1)$

C $\quad$ $\text{Proj}_L(\vec{v}) = (-1,0)$

D $\quad$ $\text{Proj}_L(\vec{v}) = (0,-1)$

Solution: A

The line L is given as all the scaled versions of the vector $\vec{x}=(-2,0)$. Then the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v})=\left(\frac{\vec{v}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\right)\vec{x}$$

$$\text{Proj}_L(\vec{v})=\frac{\begin{bmatrix}1 & 5\end{bmatrix}\cdot\begin{bmatrix}-2\\0\end{bmatrix}}{\begin{bmatrix}-2 & 0\end{bmatrix}\cdot\begin{bmatrix}-2\\0\end{bmatrix}}\cdot\begin{bmatrix}-2\\0\end{bmatrix}=\frac{1(-2)+5(0)}{-2(-2)+0(0)}\cdot\begin{bmatrix}-2\\0\end{bmatrix}$$

$$\text{Proj}_L(\vec{v})=\frac{-2}{4}\cdot\begin{bmatrix}-2\\0\end{bmatrix}=-\frac{1}{2}\cdot\begin{bmatrix}-2\\0\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

Topic: Projections as linear transformations

Question: Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} 3 \\ -4 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

Answer choices:

A $\quad \text{Proj}_L(\vec{v}) = \left(\frac{6}{25}, \frac{8}{25} \right)$

B $\quad \text{Proj}_L(\vec{v}) = \left(-\frac{6}{25}, -\frac{8}{25} \right)$

C $\quad \text{Proj}_L(\vec{v}) = \left(\frac{6}{25}, -\frac{8}{25} \right)$

D $\quad \text{Proj}_L(\vec{v}) = \left(-\frac{6}{25}, \frac{8}{25} \right)$

Solution: D

The line L is given as all the scaled versions of the vector $\overrightarrow{x}=(3,-4)$. Then the projection of $\overrightarrow{v}$ onto L is given by

$$\text{Proj}_L(\overrightarrow{v})=\left(\frac{\overrightarrow{v}\cdot\overrightarrow{x}}{\overrightarrow{x}\cdot\overrightarrow{x}}\right)\overrightarrow{x}$$

$$\text{Proj}_L(\overrightarrow{v})=\frac{\begin{bmatrix}-2 & -1\end{bmatrix}\cdot\begin{bmatrix}3\\-4\end{bmatrix}}{\begin{bmatrix}3 & -4\end{bmatrix}\cdot\begin{bmatrix}3\\-4\end{bmatrix}}\cdot\begin{bmatrix}3\\-4\end{bmatrix}=\frac{-2(3)-1(-4)}{3(3)-4(-4)}\cdot\begin{bmatrix}3\\-4\end{bmatrix}$$

$$\text{Proj}_L(\overrightarrow{v})=\frac{-2}{25}\cdot\begin{bmatrix}3\\-4\end{bmatrix}=-\frac{2}{25}\cdot\begin{bmatrix}3\\-4\end{bmatrix}=\begin{bmatrix}-\frac{6}{25}\\ \frac{8}{25}\end{bmatrix}$$

Topic: Projections as linear transformations

Question: Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} -6 \\ 2 \end{bmatrix} \;\middle|\; c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

Answer choices:

A $\quad \text{Proj}_L(\vec{v}) = \left(\frac{3}{5}, \frac{1}{5} \right)$

B $\quad \text{Proj}_L(\vec{v}) = \left(\frac{3}{5}, -\frac{1}{5} \right)$

C $\quad \text{Proj}_L(\vec{v}) = \left(\frac{3}{10}, -\frac{2}{5} \right)$

D $\quad \text{Proj}_L(\vec{v}) = \left(-\frac{3}{10}, -\frac{2}{5} \right)$

Solution: B

The line L is given as all the scaled versions of the vector $\vec{x}=(-6,2)$. Then the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v})=\left(\frac{\vec{v}\cdot\vec{x}}{\vec{x}\cdot\vec{x}}\right)\vec{x}$$

$$\text{Proj}_L(\vec{v})=\frac{\begin{bmatrix}2 & 4\end{bmatrix}\cdot\begin{bmatrix}-6\\2\end{bmatrix}}{\begin{bmatrix}-6 & 2\end{bmatrix}\cdot\begin{bmatrix}-6\\2\end{bmatrix}}\cdot\begin{bmatrix}-6\\2\end{bmatrix}=\frac{2(-6)+4(2)}{-6(-6)+2(2)}\cdot\begin{bmatrix}-6\\2\end{bmatrix}$$

$$\text{Proj}_L(\vec{v})=\frac{-4}{40}\cdot\begin{bmatrix}-6\\2\end{bmatrix}=-\frac{1}{10}\cdot\begin{bmatrix}-6\\2\end{bmatrix}=\begin{bmatrix}\frac{3}{5}\\-\frac{1}{5}\end{bmatrix}$$

Topic: Compositions of linear transformations

Question: If $S : X \to Y$ and $T : Y \to Z$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_1 + x_2 \\ 3x_1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 2x_1 - x_2 \\ -2x_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Answer choices:

A $\quad \vec{z} = (-7,6)$

B $\quad \vec{z} = (7, -6)$

C $\quad \vec{z} = (-7, -6)$

D $\quad \vec{z} = (7,6)$

Solution: C

Apply the transformation S to each column of the I_2 identity matrix.

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1+0\\3(1)\end{bmatrix}=\begin{bmatrix}-1\\3\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0+1\\3(0)\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix}$$

So the transformation S can be written as

$$S(\overrightarrow{x})=\begin{bmatrix}-1&1\\3&0\end{bmatrix}\overrightarrow{x}$$

Apply the transformation T to each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}2(1)-0\\-2(0)\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}2(0)-1\\-2(1)\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

So the transformation T can be written as

$$T(\overrightarrow{y})=\begin{bmatrix}2&-1\\0&-2\end{bmatrix}\overrightarrow{y}$$

Then the composition $T\circ S$ can be written as

$$T(S(\overrightarrow{x}))=\begin{bmatrix}2&-1\\0&-2\end{bmatrix}\begin{bmatrix}-1&1\\3&0\end{bmatrix}\overrightarrow{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix} 2(-1)-1(3) & 2(1)-1(0) \\ 0(-1)-2(3) & 0(1)-2(0) \end{bmatrix} \overrightarrow{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix} -2-3 & 2-0 \\ 0-6 & 0-0 \end{bmatrix} \overrightarrow{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix} -5 & 2 \\ -6 & 0 \end{bmatrix} \overrightarrow{x}$$

Transform $\overrightarrow{x} = (1, -1)$.

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -5 & 2 \\ -6 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -5(1)+2(-1) \\ -6(1)+0(-1) \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -5-2 \\ -6+0 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -7 \\ -6 \end{bmatrix}$$

Therefore, we can say that the vector $\overrightarrow{x} = (1, -1)$ in the subset X is transformed into the vector $\vec{z} = (-7, -6)$ in the subset Z.

Topic: Compositions of linear transformations

Question: If $S: X \to Y$ and $T: Y \to Z$, then what is $T(S(\overrightarrow{x}))$?

$$S(\overrightarrow{x}) = \begin{bmatrix} 2x_1 - x_2 + x_3 \\ -4x_3 \\ x_2 - x_1 \end{bmatrix}$$

$$T(\overrightarrow{x}) = \begin{bmatrix} -3x_1 \\ -2x_2 + x_3 \\ 4x_3 \end{bmatrix}$$

$$\overrightarrow{x} = \begin{bmatrix} 2 \\ -4 \\ 0 \end{bmatrix}$$

Answer choices:

A $\quad \vec{z} = (-24, -6, -24)$

B $\quad \vec{z} = (-24, -6, 24)$

C $\quad \vec{z} = (-24, 6, -24)$

D $\quad \vec{z} = (24, 6, 24)$

Solution: A

Apply the transformation S to each column of the I_3 identity matrix.

$$S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2(1)-0+0\\-4(0)\\0-1\end{bmatrix}=\begin{bmatrix}2\\0\\-1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2(0)-1+0\\-4(0)\\1-0\end{bmatrix}=\begin{bmatrix}-1\\0\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}2(0)-0+1\\-4(1)\\0-0\end{bmatrix}=\begin{bmatrix}1\\-4\\0\end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x})=\begin{bmatrix}2&-1&1\\0&0&-4\\-1&1&0\end{bmatrix}\vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}-3(1)\\-2(0)+0\\4(0)\end{bmatrix}=\begin{bmatrix}-3\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-3(0)\\-2(1)+0\\4(0)\end{bmatrix}=\begin{bmatrix}0\\-2\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3(0)\\-2(0)+1\\4(1)\end{bmatrix}=\begin{bmatrix}0\\1\\4\end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{y})=\begin{bmatrix}-3&0&0\\0&-2&1\\0&0&4\end{bmatrix}\vec{y}$$

Then the composition $T \circ S$ can be written as

$$T(S(\vec{x}))=\begin{bmatrix}-3&0&0\\0&-2&1\\0&0&4\end{bmatrix}\begin{bmatrix}2&-1&1\\0&0&-4\\-1&1&0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-3(2)+0(0)+0(-1)&-3(-1)+0(0)+0(1)&-3(1)+0(-4)+0(0)\\0(2)-2(0)+1(-1)&0(-1)-2(0)+1(1)&0(1)-2(-4)+1(0)\\0(2)+0(0)+4(-1)&0(-1)+0(0)+4(1)&0(1)+0(-4)+4(0)\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-6+0+0&3+0+0&-3+0+0\\0-0-1&0-0+1&0+8+0\\0+0-4&0+0+4&0+0+0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-6&3&-3\\-1&1&8\\-4&4&0\end{bmatrix}\vec{x}$$

Transform $\vec{x}=(2,-4,0)$.

$$T\left(S\left(\begin{bmatrix}2\\-4\\0\end{bmatrix}\right)\right)=\begin{bmatrix}-6&3&-3\\-1&1&8\\-4&4&0\end{bmatrix}\begin{bmatrix}2\\-4\\0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}2\\-4\\0\end{bmatrix}\right)\right)=\begin{bmatrix}-6(2)+3(-4)-3(0)\\-1(2)+1(-4)+8(0)\\-4(2)+4(-4)+0(0)\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}2\\-4\\0\end{bmatrix}\right)\right)=\begin{bmatrix}-12-12-0\\-2-4+0\\-8-16+0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}2\\-4\\0\end{bmatrix}\right)\right)=\begin{bmatrix}-24\\-6\\-24\end{bmatrix}$$

Therefore, we can say that the vector $\overrightarrow{x}=(2,-4,0)$ in the subset X is transformed into the vector $\vec{z}=(-24,-6,-24)$ in the subset Z.

Topic: Compositions of linear transformations

Question: If $S : X \to Y$ and $T : Y \to Z$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -5x_3 \\ 2x_3 \\ x_1 + x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 3x_2 \\ -2x_1 \\ 4x_3 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

Answer choices:

A $\quad \vec{z} = (12,20,20)$

B $\quad \vec{z} = (-12,20,-20)$

C $\quad \vec{z} = (12,-20,-20)$

D $\quad \vec{z} = (-12,-20,20)$

Solution: D

Apply the transformation S to each column of the I_3 identity matrix.

$$S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}-5(0)\\2(0)\\1+0\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-5(0)\\2(0)\\0+1\end{bmatrix}=\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-5(1)\\2(1)\\0+0\end{bmatrix}=\begin{bmatrix}-5\\2\\0\end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x})=\begin{bmatrix}0&0&-5\\0&0&2\\1&1&0\end{bmatrix}\vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}3(0)\\-2(1)\\4(0)\end{bmatrix}=\begin{bmatrix}0\\-2\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}3(1)\\-2(0)\\4(0)\end{bmatrix}=\begin{bmatrix}3\\0\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}3(0)\\-2(0)\\4(1)\end{bmatrix}=\begin{bmatrix}0\\0\\4\end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{y})=\begin{bmatrix}0&3&0\\-2&0&0\\0&0&4\end{bmatrix}\vec{y}$$

Then the composition $T\circ S$ can be written as

$$T(S(\vec{x}))=\begin{bmatrix}0&3&0\\-2&0&0\\0&0&4\end{bmatrix}\begin{bmatrix}0&0&-5\\0&0&2\\1&1&0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0(0)+3(0)+0(1)&0(0)+3(0)+0(1)&0(-5)+3(2)+0(0)\\-2(0)+0(0)+0(1)&-2(0)+0(0)+0(1)&-2(-5)+0(2)+0(0)\\0(0)+0(0)+4(1)&0(0)+0(0)+4(1)&0(-5)+0(2)+4(0)\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0+0+0&0+0+0&0+6+0\\0+0+0&0+0+0&10+0+0\\0+0+4&0+0+4&0+0+0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0&0&6\\0&0&10\\4&4&0\end{bmatrix}\vec{x}$$

Transform $\vec{x}=(1,4,-2)$.

$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}0&0&6\\0&0&10\\4&4&0\end{bmatrix}\begin{bmatrix}1\\4\\-2\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}0(1)+0(4)+6(-2)\\0(1)+0(4)+10(-2)\\4(1)+4(4)+0(-2)\end{bmatrix}$$

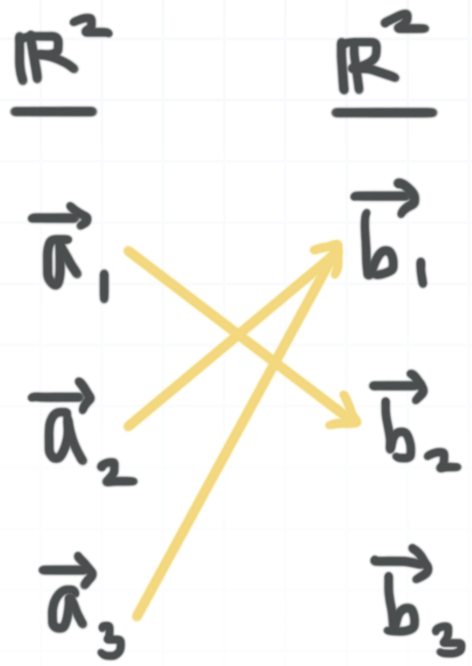
$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}0+0-12\\0+0-20\\4+16+0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}1\\4\\-2\end{bmatrix}\right)\right)=\begin{bmatrix}-12\\-20\\20\end{bmatrix}$$

Therefore, we can say that the vector $\overrightarrow{x}=(1,4,-2)$ in the subset X is transformed into the vector $\vec{z}=(-12,-20,20)$ in the subset Z.

Topic: Inverse of a transformation

Question: If the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is mapping vectors in A to vectors in B, then...



Answer choices:

A T is surjective

B T is injective

C T is both surjective and injective

D T is neither surjective nor injective

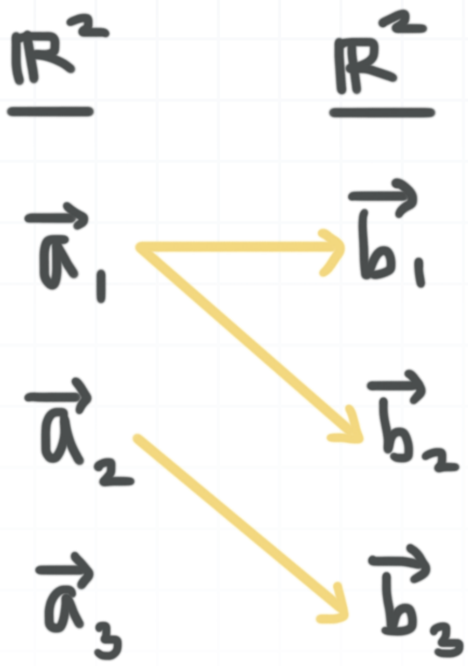
Solution: D

Not every vector $\vec{b}$ is being mapped to, since no $\vec{a}$ is mapping to $\vec{b}_3$, so T is not surjective. Every vector $\vec{a}$ is being mapped from, but not all the vectors $\vec{a}$ are mapping to a unique $\vec{b}$, since both $\vec{a}_2$ and $\vec{a}_3$ are mapping to $\vec{b}_1$, so T is not injective.

Because T has to be both surjective and injective in order to be invertible, and since T is neither surjective nor injective, the transformation is not invertible, and a unique T^{-1} is not defined.

Topic: Inverse of a transformation

Question: If the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ is mapping vectors in A to vectors in B, then...



Answer choices:

A T is surjective

B T is injective

C T is both surjective and injective

D T is neither surjective nor injective

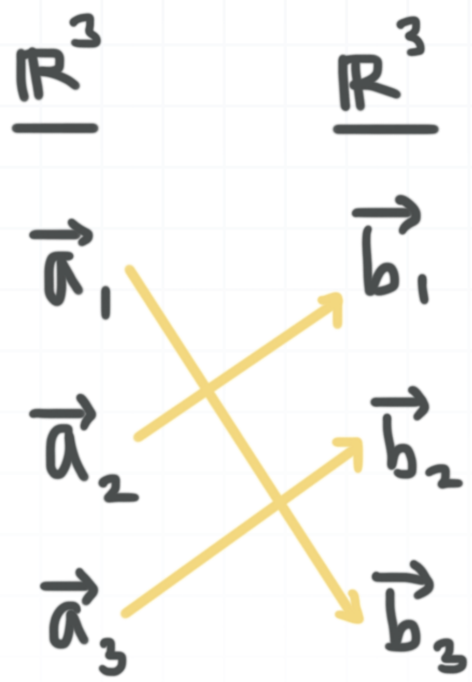
Solution: A

Every vector $\vec{b}$ is being mapped to, so T is surjective. But every vector $\vec{a}$ is not being mapped from, since $\vec{a}_3$ is not being mapped to any $\vec{b}$, so T is not injective.

Because T has to be both surjective and injective in order to be invertible, and since T is not injective, the transformation is not invertible, and a unique T^{-1} is not defined.

Topic: Inverse of a transformation

Question: If the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ is mapping vectors in A to vectors in B, then...



Answer choices:

A $\quad T$ is onto

B $\quad T$ is one-to-one

C $\quad T$ is both onto and one-to-one

D $\quad T$ is neither onto nor one-to-one

Solution: C

Every vector $\vec{b}$ is being mapped to, so T is onto. And every vector $\vec{a}$ is being mapped from, to a unique $\vec{b}$, so T is one-to-one.

Because T is both onto and one-to-one, the transformation is invertible, and a unique T^{-1} is defined.

Topic: Invertibility from the matrix-vector product

Question: Say whether or not the transformation T is surjective or injective.

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad T$ is surjective and injective

B $\quad T$ is surjective, but not injective

C $\quad T$ is not surjective, but it's injective

D $\quad T$ is neither surjective nor injective

Solution: A

If we say that the transformation T is given by the matrix-vector product

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \vec{x}$$

and we name the matrix A, then we always want to start by putting A into reduced row-echelon form.

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because we get the identity matrix when we put A into rref, that tells us that the transformation T is both surjective and injective.

Topic: Invertibility from the matrix-vector product

Question: Say whether or not the transformation T is surjective or injective.

$$T(\vec{x}) = \begin{bmatrix} 3 & -4 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad T$ is surjective and injective

B $\quad T$ is surjective, but not injective

C $\quad T$ is not surjective, but it's injective

D $\quad T$ is neither surjective nor injective

Solution: B

If we say that the transformation T is given by the matrix-vector product

$$T(\vec{x}) = \begin{bmatrix} 3 & -4 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \vec{x}$$

and we name the matrix A, then we always want to start by putting A into reduced row-echelon form.

$$A = \begin{bmatrix} 3 & -4 & 2 & 0 \\ 1 & -1 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 3 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & -1 & 2 & -6 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 1 & -2 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 8 \\ 0 & 1 & -2 & 6 \end{bmatrix}$$

If we try to find the null space of A, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -8 \\ -6 \\ 0 \\ 1 \end{bmatrix}$$

So any linear combination of these two four-dimensional vectors is a vector in the null space. Which means there are an infinite number of vectors that the transformation T maps to the zero vector. And because we're mapping multiple vectors all to the same vector, the transformation is not injective.

To see whether or not the transformation is surjective, we'll look at the column space of A.

$$C(A) = \text{Span}\left(\begin{bmatrix}3\\1\end{bmatrix}, \begin{bmatrix}-4\\-1\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}0\\2\end{bmatrix}\right)$$

From the reduced row-echelon form of A,

$$A = \begin{bmatrix}1 & 0 & -2 & 8\\0 & 1 & -2 & 6\end{bmatrix}$$

we know that the first two column vectors from A are linearly independent two-dimensional vectors in $\mathbb{R}^2$. Because two linearly independent two-dimensional vectors in $\mathbb{R}^2$ will span all of $\mathbb{R}^2$, we know that the transformation T can map to any vector in $\mathbb{R}^2$, which means that the transformation is surjective.

Topic: Invertibility from the matrix-vector product

Question: Say whether or not the transformation T is surjective or injective.

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \vec{x}$$

Answer choices:

A T is surjective and injective

B T is surjective, but not injective

C T is not surjective, but it's injective

D T is neither surjective nor injective

Solution: C

If we say that the transformation T is given by the matrix-vector product

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \vec{x}$$

and we name the matrix A, then we always want to start by putting A into reduced row-echelon form.

$$A = \begin{bmatrix} -1 & 2 & 3 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 2 & -2 & 1 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 0 & 1 \\ 0 & 2 & 7 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 7 \\ 0 & 0 & 1 \\ 0 & 4 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -2 & -3 \\ 0 & 1 & \frac{7}{2} \\ 0 & 0 & 1 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & \frac{7}{2} \\ 0 & 0 & 1 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & \frac{7}{2} \\ 0 & 0 & 1 \\ 0 & 0 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{7}{2} \\ 0 & 0 & 1 \\ 0 & 0 & -12 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

If we try to find the null space of A, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because the zero vector is the only vector in the null space, we can say that the transformation T is injective.

To see whether or not the transformation is surjective, we'll look at the column space of A.

$$C(A) = \text{Span}\left(\begin{bmatrix}-1\\0\\2\\0\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\4\end{bmatrix}, \begin{bmatrix}3\\1\\1\\2\end{bmatrix}\right)$$

From the reduced row-echelon form of A,

$$A = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}$$

we know that the first three column vectors from A are linearly independent three-dimensional vectors in $\mathbb{R}^4$. Because three linearly independent three-dimensional vectors in $\mathbb{R}^4$ won't span all of $\mathbb{R}^4$, we know that the transformation T can't map to every vector in $\mathbb{R}^4$, which means that the transformation is not surjective.

Topic: Inverse transformations are linear

Question: Find B^{-1}.

$$B = \begin{bmatrix} -4 & 1 \\ 2 & -3 \end{bmatrix}$$

Answer choices:

A $B^{-1} = \begin{bmatrix} -\frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$

B $B^{-1} = \begin{bmatrix} \frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

C $B^{-1} = \begin{bmatrix} -\frac{3}{10} & \frac{1}{10} \\ \frac{1}{5} & -\frac{2}{5} \end{bmatrix}$

D $B^{-1} = \begin{bmatrix} \frac{3}{10} & \frac{1}{10} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$

Solution: A

To find the inverse of the 2×2 matrix B, we'll augment it with I_2,

$$[B \mid I] = \begin{bmatrix} -4 & 1 & | & 1 & 0 \\ 2 & -3 & | & 0 & 1 \end{bmatrix}$$

and then work on putting the left side of the augmented matrix into reduced row-echelon form. We'll use Gauss-Jordan elimination, and start by finding the pivot in the first column, then zeroing out the rest of the column.

$$[B \mid I] = \begin{bmatrix} 1 & -\frac{1}{4} & | & -\frac{1}{4} & 0 \\ 2 & -3 & | & 0 & 1 \end{bmatrix}$$

$$[B \mid I] = \begin{bmatrix} 1 & -\frac{1}{4} & | & -\frac{1}{4} & 0 \\ 0 & -\frac{5}{2} & | & \frac{1}{2} & 1 \end{bmatrix}$$

Find the pivot entry in the second column, then finish zeroing out the rest of the second column.

$$[B \mid I] = \begin{bmatrix} 1 & -\frac{1}{4} & | & -\frac{1}{4} & 0 \\ 0 & 1 & | & -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

$$[B \mid I] = \begin{bmatrix} 1 & 0 & | & -\frac{3}{10} & -\frac{1}{10} \\ 0 & 1 & | & -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

Now that the left side of the augmented matrix has been changed into the identity matrix, the right side of the augmented matrix must be the inverse, B^{-1}.

$$B^{-1} = \begin{bmatrix} -\frac{3}{10} & -\frac{1}{10} \\ -\frac{1}{5} & -\frac{2}{5} \end{bmatrix}$$

Topic: Inverse transformations are linear

Question: Find M^{-1}.

$$M = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \\ 4 & -2 & 0 \end{bmatrix}$$

Answer choices:

A $\quad M^{-1} = \begin{bmatrix} \frac{1}{14} & \frac{3}{14} & \frac{1}{14} \\ -\frac{1}{7} & -\frac{3}{7} & -\frac{5}{14} \\ \frac{3}{14} & \frac{5}{14} & -\frac{3}{14} \end{bmatrix}$

B $\quad M^{-1} = \begin{bmatrix} -\frac{1}{14} & -\frac{3}{14} & -\frac{1}{14} \\ \frac{1}{7} & \frac{3}{7} & \frac{5}{14} \\ -\frac{3}{14} & -\frac{5}{14} & \frac{3}{14} \end{bmatrix}$

C $\quad M^{-1} = \begin{bmatrix} \frac{1}{14} & -\frac{3}{14} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{7} & -\frac{5}{14} \\ \frac{3}{14} & \frac{5}{14} & \frac{3}{14} \end{bmatrix}$

D $\quad M^{-1} = \begin{bmatrix} -\frac{1}{14} & \frac{3}{14} & -\frac{1}{14} \\ -\frac{1}{7} & \frac{3}{7} & \frac{5}{14} \\ -\frac{3}{14} & -\frac{5}{14} & -\frac{3}{14} \end{bmatrix}$

Solution: C

To find the inverse of the 3×3 matrix M, we'll augment it with I_3,

$$[M \mid I] = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ -3 & 0 & 1 & | & 0 & 1 & 0 \\ 4 & -2 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

and then work on putting the left side of the augmented matrix into reduced row-echelon form. We'll use Gauss-Jordan elimination, and start by zeroing out the first column.

$$[M \mid I] = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 6 & 10 & | & 3 & 1 & 0 \\ 4 & -2 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[M \mid I] = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 6 & 10 & | & 3 & 1 & 0 \\ 0 & -10 & -12 & | & -4 & 0 & 1 \end{bmatrix}$$

Find the pivot entry in the second column, then finish zeroing out the rest of the second column.

$$[M \mid I] = \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{5}{3} & | & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & -10 & -12 & | & -4 & 0 & 1 \end{bmatrix}$$

$$[M \mid I] = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & | & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & -10 & -12 & | & -4 & 0 & 1 \end{bmatrix}$$

$$[M \mid I] = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & | & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & \frac{14}{3} & | & 1 & \frac{5}{3} & 1 \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.

$$[M \mid I] = \begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & 0 & -\frac{1}{3} & 0 \\ 0 & 1 & \frac{5}{3} & | & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & | & \frac{3}{14} & \frac{5}{14} & \frac{3}{14} \end{bmatrix}$$

$$[M \mid I] = \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{14} & -\frac{3}{14} & \frac{1}{14} \\ 0 & 1 & \frac{5}{3} & | & \frac{1}{2} & \frac{1}{6} & 0 \\ 0 & 0 & 1 & | & \frac{3}{14} & \frac{5}{14} & \frac{3}{14} \end{bmatrix}$$

$$[M \mid I] = \begin{bmatrix} 1 & 0 & 0 & | & \frac{1}{14} & -\frac{3}{14} & \frac{1}{14} \\ 0 & 1 & 0 & | & \frac{1}{7} & -\frac{3}{7} & -\frac{5}{14} \\ 0 & 0 & 1 & | & \frac{3}{14} & \frac{5}{14} & \frac{3}{14} \end{bmatrix}$$

Now that the left side of the augmented matrix has been changed into the identity matrix, the right side of the augmented matrix must be the inverse, M^{-1}.

$$M^{-1} = \begin{bmatrix} \frac{1}{14} & -\frac{3}{14} & \frac{1}{14} \\ \frac{1}{7} & -\frac{3}{7} & -\frac{5}{14} \\ \frac{3}{14} & \frac{5}{14} & \frac{3}{14} \end{bmatrix}$$

Topic: Inverse transformations are linear

Question: Find K^{-1}.

$$K=\begin{bmatrix}-2 & 1 & 0 & 4\\ 1 & -3 & 0 & 1\\ 0 & 4 & -1 & 2\\ 2 & 0 & 1 & -3\end{bmatrix}$$

Answer choices:

A $$K^{-1}=\begin{bmatrix}-\frac{1}{11} & -\frac{17}{33} & \frac{5}{33} & \frac{5}{33}\\ \frac{1}{11} & -\frac{2}{11} & -\frac{6}{33} & \frac{6}{33}\\ -\frac{8}{11} & -\frac{4}{33} & -\frac{7}{33} & \frac{26}{33}\\ \frac{2}{11} & \frac{10}{33} & \frac{1}{33} & -\frac{1}{33}\end{bmatrix}$$

B $$K^{-1}=\begin{bmatrix}\frac{1}{11} & \frac{17}{33} & -\frac{5}{33} & -\frac{5}{33}\\ -\frac{1}{11} & \frac{2}{11} & \frac{6}{33} & -\frac{6}{33}\\ \frac{8}{11} & \frac{4}{33} & \frac{7}{33} & -\frac{26}{33}\\ -\frac{2}{11} & -\frac{10}{33} & -\frac{1}{33} & \frac{1}{33}\end{bmatrix}$$

C $$K^{-1}=\begin{bmatrix}-\frac{1}{45} & \frac{17}{45} & \frac{13}{45} & \frac{13}{45}\\ \frac{1}{15} & -\frac{2}{15} & \frac{2}{15} & \frac{6}{45}\\ \frac{32}{45} & -\frac{4}{45} & -\frac{11}{45} & \frac{34}{45}\\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9}\end{bmatrix}$$

D $$K^{-1}=\begin{bmatrix}\frac{1}{45} & -\frac{17}{45} & -\frac{13}{45} & -\frac{13}{45}\\ -\frac{1}{15} & \frac{2}{15} & -\frac{2}{15} & -\frac{6}{45}\\ -\frac{32}{45} & \frac{4}{45} & \frac{11}{45} & -\frac{34}{45}\\ -\frac{2}{9} & -\frac{2}{9} & -\frac{1}{9} & -\frac{1}{9}\end{bmatrix}$$

Solution: C

To find the inverse of the 4×4 matrix K, we'll augment it with I_4,

$$[K \mid I] = \begin{bmatrix} -2 & 1 & 0 & 4 & | & 1 & 0 & 0 & 0 \\ 1 & -3 & 0 & 1 & | & 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & | & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & -3 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

and then work on putting the left side of the augmented matrix into reduced row-echelon form. We'll use Gauss-Jordan elimination, and start by finding the pivot in the first column, then zeroing out the rest of the column.

$$[K \mid I] = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 & | & -\frac{1}{2} & 0 & 0 & 0 \\ 1 & -3 & 0 & 1 & | & 0 & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & | & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & -3 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 & | & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & 3 & | & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & | & 0 & 0 & 1 & 0 \\ 2 & 0 & 1 & -3 & | & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 & | & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & -\frac{5}{2} & 0 & 3 & | & \frac{1}{2} & 1 & 0 & 0 \\ 0 & 4 & -1 & 2 & | & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & | & 1 & 0 & 0 & 1 \end{bmatrix}$$

Find the pivot entry in the second column, then finish zeroing out the rest of the second column.

$$[K \mid I] = \begin{bmatrix} 1 & -\frac{1}{2} & 0 & -2 & | & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 4 & -1 & 2 & | & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & | & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{5} & | & -\frac{3}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 4 & -1 & 2 & | & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 & | & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{5} & | & -\frac{3}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & -1 & \frac{34}{5} & | & \frac{4}{5} & \frac{8}{5} & 1 & 0 \\ 0 & 1 & 1 & 1 & | & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{5} & | & -\frac{3}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & -1 & \frac{34}{5} & | & \frac{4}{5} & \frac{8}{5} & 1 & 0 \\ 0 & 0 & 1 & \frac{11}{5} & | & \frac{6}{5} & \frac{2}{5} & 0 & 1 \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{5} & | & -\frac{3}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 1 & -\frac{34}{5} & | & -\frac{4}{5} & -\frac{8}{5} & -1 & 0 \\ 0 & 0 & 1 & \frac{11}{5} & | & \frac{6}{5} & \frac{2}{5} & 0 & 1 \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{5} & | & -\frac{3}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 1 & -\frac{34}{5} & | & -\frac{4}{5} & -\frac{8}{5} & -1 & 0 \\ 0 & 0 & 0 & 9 & | & 2 & 2 & 1 & 1 \end{bmatrix}$$

Find the pivot entry in the fourth column, then zero out the rest of the fourth column.

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & -\frac{13}{5} & | & -\frac{3}{5} & -\frac{1}{5} & 0 & 0 \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 1 & -\frac{34}{5} & | & -\frac{4}{5} & -\frac{8}{5} & -1 & 0 \\ 0 & 0 & 0 & 1 & | & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & 0 & | & -\frac{1}{45} & \frac{17}{45} & \frac{13}{45} & \frac{13}{45} \\ 0 & 1 & 0 & -\frac{6}{5} & | & -\frac{1}{5} & -\frac{2}{5} & 0 & 0 \\ 0 & 0 & 1 & -\frac{34}{5} & | & -\frac{4}{5} & -\frac{8}{5} & -1 & 0 \\ 0 & 0 & 0 & 1 & | & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & 0 & | & -\frac{1}{45} & \frac{17}{45} & \frac{13}{45} & \frac{13}{45} \\ 0 & 1 & 0 & 0 & | & \frac{1}{15} & -\frac{2}{15} & \frac{2}{15} & \frac{6}{45} \\ 0 & 0 & 1 & -\frac{34}{5} & | & -\frac{4}{5} & -\frac{8}{5} & -1 & 0 \\ 0 & 0 & 0 & 1 & | & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

$$[K \mid I] = \begin{bmatrix} 1 & 0 & 0 & 0 & | & -\frac{1}{45} & \frac{17}{45} & \frac{13}{45} & \frac{13}{45} \\ 0 & 1 & 0 & 0 & | & \frac{1}{15} & -\frac{2}{15} & \frac{2}{15} & \frac{6}{45} \\ 0 & 0 & 1 & 0 & | & \frac{32}{45} & -\frac{4}{45} & -\frac{11}{45} & \frac{34}{45} \\ 0 & 0 & 0 & 1 & | & \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

Now that the left side of the augmented matrix has been changed into the identity matrix, the right side of the augmented matrix must be the inverse, K^{-1}.

$$K^{-1} = \begin{bmatrix} -\frac{1}{45} & \frac{17}{45} & \frac{13}{45} & \frac{13}{45} \\ \frac{1}{15} & -\frac{2}{15} & \frac{2}{15} & \frac{6}{45} \\ \frac{32}{45} & -\frac{4}{45} & -\frac{11}{45} & \frac{34}{45} \\ \frac{2}{9} & \frac{2}{9} & \frac{1}{9} & \frac{1}{9} \end{bmatrix}$$

Topic: Matrix inverses, and invertible and singular matrices

Question: Are the matrices inverses of one another?

$$A = \begin{bmatrix} 2 & 5 \\ 3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{13} & \frac{5}{13} \\ \frac{3}{13} & -\frac{2}{13} \end{bmatrix}$$

Answer choices:

A Yes

B No

C There's not enough information to know

Solution: A

To find the inverse of matrix A, plug it into the formula for the inverse matrix.

$$A^{-1}=\frac{1}{|A|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$$A^{-1}=\frac{1}{\begin{vmatrix}2 & 5\\ 3 & 1\end{vmatrix}}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

Find the determinant in the denominator of the fraction.

$$A^{-1}=\frac{1}{(2)(1)-(5)(3)}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

$$A^{-1}=\frac{1}{2-15}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

$$A^{-1}=-\frac{1}{13}\begin{bmatrix}1 & -5\\ -3 & 2\end{bmatrix}$$

Then distribute the scalar across the matrix.

$$A^{-1}=\begin{bmatrix}-\frac{1}{13}(1) & -\frac{1}{13}(-5)\\ -\frac{1}{13}(-3) & -\frac{1}{13}(2)\end{bmatrix}$$

$$A^{-1}=\begin{bmatrix}-\frac{1}{13} & \frac{5}{13}\\ \frac{3}{13} & -\frac{2}{13}\end{bmatrix}$$

Because the value we found matches matrix B, it means that matrices A and B are inverses of one another.

Topic: Matrix inverses, and invertible and singular matrices

Question: Find the inverse of matrix M.

$$M = \begin{bmatrix} 0 & -2 \\ -4 & 5 \end{bmatrix}$$

Answer choices:

A $\quad M^{-1} = \begin{bmatrix} 0 & -\frac{1}{4} \\ -\frac{1}{2} & -\frac{5}{8} \end{bmatrix}$

B $\quad M^{-1} = \begin{bmatrix} -\frac{5}{8} & -\frac{1}{4} \\ -\frac{1}{2} & 0 \end{bmatrix}$

C $\quad M^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{2} & -\frac{5}{8} \end{bmatrix}$

D $\quad M^{-1} = \begin{bmatrix} -\frac{5}{8} & \frac{1}{4} \\ \frac{1}{2} & 0 \end{bmatrix}$

Solution: B

Plug the values from the matrix into the formula for the inverse matrix.

$$M^{-1}=\frac{1}{|M|}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$$

$$M^{-1}=\frac{1}{\begin{vmatrix}0 & -2\\ -4 & 5\end{vmatrix}}\begin{bmatrix}5 & 2\\ 4 & 0\end{bmatrix}$$

Find the determinant in the denominator of the fraction.

$$M^{-1}=\frac{1}{(0)(5)-(-2)(-4)}\begin{bmatrix}5 & 2\\ 4 & 0\end{bmatrix}$$

$$M^{-1}=\frac{1}{0-8}\begin{bmatrix}5 & 2\\ 4 & 0\end{bmatrix}$$

$$M^{-1}=-\frac{1}{8}\begin{bmatrix}5 & 2\\ 4 & 0\end{bmatrix}$$

Then distribute the scalar across the matrix.

$$M^{-1}=\begin{bmatrix}-\frac{1}{8}(5) & -\frac{1}{8}(2)\\ -\frac{1}{8}(4) & -\frac{1}{8}(0)\end{bmatrix}$$

$$M^{-1}=\begin{bmatrix}-\frac{5}{8} & -\frac{1}{4}\\ -\frac{1}{2} & 0\end{bmatrix}$$

Topic: Matrix inverses, and invertible and singular matrices

Question: Classify the matrix.

$$L = \begin{bmatrix} 3 & 7 \\ 0 & -1 \end{bmatrix}$$

Answer choices:

A The matrix is invertible

B The matrix is singular

C The matrix is invertible and singular

D The matrix is neither invertible nor singular

Solution: A

A matrix is either invertible or singular, it can never be both. To determine whether a matrix in the form

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is invertible or singular, we need to look at the ratio of a to b, compared to the ratio of c to d.

For the given matrix L,

$$\frac{a}{b} = \frac{3}{7}$$

$$\frac{c}{d} = \frac{0}{-1} = 0$$

Because these ratios aren't equivalent, that means matrix L is invertible. If the ratios had been equivalent, the matrix would have been singular.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$3x + 12y = 51$$

$$-2x + 6y = -6$$

Answer choices:

A $x = -9$ and $y = -2$

B $x = -9$ and $y = 2$

C $x = 9$ and $y = -2$

D $x = 9$ and $y = 2$

Solution: D

Start by transferring the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 51 \\ -6 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} 3 & 12 \\ -2 & 6 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(3)(6) - (12)(-2)} \begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = \frac{1}{42} \begin{bmatrix} 6 & -12 \\ 2 & 3 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{21} & \frac{1}{14} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\vec{a} = M^{-1}\vec{b}$$

$$\vec{a} = \begin{bmatrix} \frac{1}{7} & -\frac{2}{7} \\ \frac{1}{21} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} 51 \\ -6 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{1}{7}(51) - \frac{2}{7}(-6) \\ \frac{1}{21}(51) + \frac{1}{14}(-6) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{51}{7} + \frac{12}{7} \\ \frac{51}{21} - \frac{6}{14} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{63}{7} \\ \frac{17}{7} - \frac{3}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{63}{7} \\ \frac{14}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ 2 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 9$ and $y = 2$.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$y - 5x = -15$$

$$3x + 8y = 95$$

Answer choices:

A $x = 5$ and $y = 10$

B $x = -5$ and $y = 10$

C $x = 5$ and $y = -10$

D $x = -5$ and $y = -10$

Solution: A

Start by transferring the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M = \begin{bmatrix} -5 & 1 \\ 3 & 8 \end{bmatrix}$$

$$M^{-1} = \frac{1}{(-5)(8) - (1)(3)} \begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{43} \begin{bmatrix} 8 & -1 \\ -3 & -5 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{8}{43} & \frac{1}{43} \\ \frac{3}{43} & \frac{5}{43} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\vec{a} = M^{-1}\vec{b}$$

$$\vec{a} = \begin{bmatrix} -\frac{8}{43} & \frac{1}{43} \\ \frac{3}{43} & \frac{5}{43} \end{bmatrix} \begin{bmatrix} -15 \\ 95 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{8}{43}(-15) + \frac{1}{43}(95) \\ \frac{3}{43}(-15) + \frac{5}{43}(95) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{120}{43} + \frac{95}{43} \\ -\frac{45}{43} + \frac{475}{43} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{215}{43} \\ \frac{430}{43} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 5$ and $y = 10$.

Topic: Solving systems with inverse matrices

Question: Use an inverse matrix to find the solution to the system.

$$4x + 8y = -20$$

$$-12x - 3y = -66$$

Answer choices:

A $x = 7$ and $y = -6$

B $x = -7$ and $y = 6$

C $x = 7$ and $y = 6$

D $x = -7$ and $y = -6$

Solution: A

We could divide through both equations in the system to reduce them. The first equation $4x+8y=-20$ becomes

$$\frac{4}{4}x+\frac{8}{4}y=-\frac{20}{4}$$

$$x+2y=-5$$

And the equation $-12x-3y=-66$ becomes

$$\frac{-12}{-3}x+\frac{-3}{-3}y=\frac{-66}{-3}$$

$$4x+y=22$$

Then transfer the system into a matrix equation.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} f \\ g \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M=\begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}$$

$$M^{-1}=\frac{1}{(1)(1)-(2)(4)}\begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{7}\begin{bmatrix} 1 & -2 \\ -4 & 1 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}$$

Then we can say that the solution to the system is

$$\vec{a} = M^{-1}\vec{b}$$

$$\vec{a} = \begin{bmatrix} -\frac{1}{7} & \frac{2}{7} \\ \frac{4}{7} & -\frac{1}{7} \end{bmatrix}\begin{bmatrix} -5 \\ 22 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{1}{7}(-5) + \frac{2}{7}(22) \\ \frac{4}{7}(-5) - \frac{1}{7}(22) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{5}{7} + \frac{44}{7} \\ -\frac{20}{7} - \frac{22}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{49}{7} \\ -\frac{42}{7} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -6 \end{bmatrix}$$

Using this process with the inverse matrix, we conclude that $x = 7$ and $y = -6$.

Topic: Determinants

Question: Find the determinant of L.

$$L = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

Answer choices:

A $|L| = 1$

B $|L| = -1$

C $|L| = 3$

D $|L| = -3$

Solution: D

Because L is a 3×3 matrix, we'll break it down into 2×2 determinants, remembering to alternate sign using the checkerboard pattern.

$$|L| = \begin{vmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{vmatrix}$$

$$|L| = 1 \begin{vmatrix} -2 & 3 \\ 1 & -2 \end{vmatrix} - 0 \begin{vmatrix} 0 & 3 \\ 1 & -2 \end{vmatrix} + (-2) \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}$$

Calculate the 2×2 determinants using the $ad - bc$ rule.

$$|L| = [(-2)(-2) - (3)(1)] - 2[(0)(1) - (-2)(1)]$$

$$|L| = (4 - 3) - 2(0 + 2)$$

$$|L| = 1 - 4$$

$$|L| = -3$$

Topic: Determinants

Question: Find the determinant of C by working along any row or column.

$$C = \begin{bmatrix} -3 & 1 & 5 & 2 \\ 0 & 0 & -1 & 1 \\ 2 & -2 & 3 & 0 \\ 1 & 4 & 0 & -4 \end{bmatrix}$$

Answer choices:

A $|C| = 93$

B $|C| = 76$

C $|C| = 54$

D $|C| = 28$

Solution: A

Because there are multiple zeros across the second row, that'll be the easiest row or column to use to find the determinant. So we'll use that row to calculate the 3×3 determinants, remembering to apply the checkerboard pattern so that we get the signs right.

$$|C| = \begin{vmatrix} -3 & 1 & 5 & 2 \\ 0 & 0 & -1 & 1 \\ 2 & -2 & 3 & 0 \\ 1 & 4 & 0 & -4 \end{vmatrix}$$

$$|C| = -0\begin{vmatrix} 1 & 5 & 2 \\ -2 & 3 & 0 \\ 4 & 0 & -4 \end{vmatrix} + 0\begin{vmatrix} -3 & 5 & 2 \\ 2 & 3 & 0 \\ 1 & 0 & -4 \end{vmatrix} - (-1)\begin{vmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 4 & -4 \end{vmatrix} + 1\begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

Because of the scalars, the first two determinants go to 0.

$$|C| = \begin{vmatrix} -3 & 1 & 2 \\ 2 & -2 & 0 \\ 1 & 4 & -4 \end{vmatrix} + \begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

To simplify this first 3×3 determinant, let's work along the third column, since it includes the 0.

$$|C| = 2\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} - 0\begin{vmatrix} -3 & 1 \\ 1 & 4 \end{vmatrix} + (-4)\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

$$|C| = 2\begin{vmatrix} 2 & -2 \\ 1 & 4 \end{vmatrix} - 4\begin{vmatrix} -3 & 1 \\ 2 & -2 \end{vmatrix} + \begin{vmatrix} -3 & 1 & 5 \\ 2 & -2 & 3 \\ 1 & 4 & 0 \end{vmatrix}$$

To simplify the second 3×3 determinant, let's work along the third row, since it includes the 0.

$$|C| = 2\begin{vmatrix}2 & -2\\1 & 4\end{vmatrix} - 4\begin{vmatrix}-3 & 1\\2 & -2\end{vmatrix} + 1\begin{vmatrix}1 & 5\\-2 & 3\end{vmatrix} - 4\begin{vmatrix}-3 & 5\\2 & 3\end{vmatrix} + 0\begin{vmatrix}-3 & 1\\2 & -2\end{vmatrix}$$

$$|C| = 2\begin{vmatrix}2 & -2\\1 & 4\end{vmatrix} - 4\begin{vmatrix}-3 & 1\\2 & -2\end{vmatrix} + \begin{vmatrix}1 & 5\\-2 & 3\end{vmatrix} - 4\begin{vmatrix}-3 & 5\\2 & 3\end{vmatrix}$$

Now we'll calculate each of the 2×2 determinants that remain.

$$|C| = 2[(2)(4) - (-2)(1)] - 4[(-3)(-2) - (1)(2)]$$

$$+[(1)(3) - (5)(-2)] - 4[(-3)(3) - (5)(2)]$$

$$|C| = 2(8 + 2) - 4(6 - 2) + (3 + 10) - 4(-9 - 10)$$

$$|C| = 2(10) - 4(4) + (13) - 4(-19)$$

$$|C| = 20 - 16 + 13 + 76$$

$$|C| = 93$$

Topic: Determinants

Question: Use the determinant to say whether or not F is invertible. Work along any row or column.

$$F = \begin{bmatrix} 1 & 5 & 0 & -1 \\ 3 & -2 & -1 & 2 \\ -1 & 1 & 0 & 3 \\ 1 & 3 & 2 & -2 \end{bmatrix}$$

Answer choices:

A $\quad F$ is invertible, and $|F| = 144$

B $\quad F$ is invertible, and $|F| = 126$

C $\quad F$ is invertible, and $|F| = 116$

D $\quad F$ is singular, and $|F| = 0$

Solution: B

If the determinant $|F|$ is nonzero, then F is invertible. But if $|F|=0$, then F is singular and won't have a defined inverse. So to say whether or not the matrix is invertible, we'll calculate the determinant.

Because there are multiple zeros across the third column, that'll be the easiest row or column to use to find the determinant. So we'll use that column to calculate the 3×3 determinants, remembering to apply the checkerboard pattern so that we get the signs right.

$$F=\begin{bmatrix}1 & 5 & 0 & -1\\ 3 & -2 & -1 & 2\\ -1 & 1 & 0 & 3\\ 1 & 3 & 2 & -2\end{bmatrix}$$

$$|F|=0\begin{vmatrix}3 & -2 & 2\\ -1 & 1 & 3\\ 1 & 3 & -2\end{vmatrix}-(-1)\begin{vmatrix}1 & 5 & -1\\ -1 & 1 & 3\\ 1 & 3 & -2\end{vmatrix}+0\begin{vmatrix}1 & 5 & -1\\ 3 & -2 & 2\\ 1 & 3 & -2\end{vmatrix}-2\begin{vmatrix}1 & 5 & -1\\ 3 & -2 & 2\\ -1 & 1 & 3\end{vmatrix}$$

Because of the scalars, the first and third determinants go to 0.

$$|F|=\begin{vmatrix}1 & 5 & -1\\ -1 & 1 & 3\\ 1 & 3 & -2\end{vmatrix}-2\begin{vmatrix}1 & 5 & -1\\ 3 & -2 & 2\\ -1 & 1 & 3\end{vmatrix}$$

To simplify this first 3×3 determinant, let's work along the first column, since those entries are all 1 or -1, which is a pretty simple set of entries.

$$|F|=1\begin{vmatrix}1 & 3\\ 3 & -2\end{vmatrix}-(-1)\begin{vmatrix}5 & -1\\ 3 & -2\end{vmatrix}+1\begin{vmatrix}5 & -1\\ 1 & 3\end{vmatrix}-2\begin{vmatrix}1 & 5 & -1\\ 3 & -2 & 2\\ -1 & 1 & 3\end{vmatrix}$$

$$|F| = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix} - 2\begin{vmatrix} 1 & 5 & -1 \\ 3 & -2 & 2 \\ -1 & 1 & 3 \end{vmatrix}$$

To simplify the second 3×3 determinant, let's work along the first row, since none of those rows or columns are particularly simpler than any other.

$$|F| = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix}$$

$$-2\left[1\begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} - 5\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}\right]$$

$$|F| = \begin{vmatrix} 1 & 3 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 3 & -2 \end{vmatrix} + \begin{vmatrix} 5 & -1 \\ 1 & 3 \end{vmatrix}$$

$$-2\begin{vmatrix} -2 & 2 \\ 1 & 3 \end{vmatrix} + 10\begin{vmatrix} 3 & 2 \\ -1 & 3 \end{vmatrix} + 2\begin{vmatrix} 3 & -2 \\ -1 & 1 \end{vmatrix}$$

Now we'll calculate each of the 2×2 determinants that remain.

$$|F| = [(1)(-2) - (3)(3)] + [(5)(-2) - (-1)(3)] + [(5)(3) - (-1)(1)]$$

$$-2[(-2)(3) - (2)(1)] + 10[(3)(3) - (2)(-1)] + 2[(3)(1) - (-2)(-1)]$$

$$|F| = (-2 - 9) + (-10 + 3) + (15 + 1) - 2(-6 - 2) + 10(9 + 2) + 2(3 - 2)$$

$$|F| = -11 - 7 + 16 + 16 + 110 + 2$$

$$|F| = 126$$

Topic: Cramer's rule for solving systems

Question: Which expression would give the value of y in the system?

$$3x - 2y = 21$$

$$-6x - 5y = 12$$

Answer choices:

A $\dfrac{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}{\begin{vmatrix} 21 & -2 \\ 12 & -5 \end{vmatrix}}$

B $\dfrac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & 6 \\ -2 & 1 \end{vmatrix}}$

C $\dfrac{\begin{vmatrix} 21 & -2 \\ 12 & -5 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$

D $\dfrac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$

Solution: D

Using the given system

$$3x - 2y = 21$$

$$-6x - 5y = 12$$

we can say

$$D = \begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}$$

and

$$D_y = \begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}$$

We can put those together to solve for the value of y.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 21 \\ -6 & 12 \end{vmatrix}}{\begin{vmatrix} 3 & -2 \\ -6 & -5 \end{vmatrix}}$$

Topic: Cramer's rule for solving systems

Question: Which expression would give the value of x in the system?

$$3x + 3y = 9$$

$$2x - y = -9$$

Answer choices:

A $\dfrac{\begin{vmatrix} 3 & 9 \\ 2 & -9 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$

B $\dfrac{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$

C $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}$

D $\dfrac{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 9 \\ 2 & -9 \end{vmatrix}}$

Solution: B

Using the given system

$$3x + 3y = 9$$

$$2x - y = -9$$

we can say

$$D = \begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}$$

and

$$D_x = \begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}$$

We can put those together to solve for the value of x.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 9 & 3 \\ -9 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & 3 \\ 2 & -1 \end{vmatrix}}$$

Topic: Cramer's rule for solving systems

Question: Which system would give this value?

$$\frac{D_x}{D} = \frac{\begin{vmatrix} 1 & -5 \\ 15 & 2 \end{vmatrix}}{\begin{vmatrix} 3 & -5 \\ 1 & 2 \end{vmatrix}}$$

Answer choices:

A $3x - 5y = 1$ and $x + 2y = 15$

B $x - 5y = 3$ and $15x - 2y = 1$

C $3x + y = -5$ and $x - 15y = 2$

D $x - 2y = 1$ and $3x + 15y = 2$

Solution: A

Let's find D for each answer choice and see which one(s) match the given expression.

For answer choice A we get

$$D = \begin{vmatrix} 3 & -5 \\ 2 & -1 \end{vmatrix}$$

For answer choice B we get

$$D = \begin{vmatrix} 1 & -5 \\ 15 & -2 \end{vmatrix}$$

For answer choice C we get

$$D = \begin{vmatrix} 3 & 1 \\ 1 & -15 \end{vmatrix}$$

For answer choice D we get

$$D = \begin{vmatrix} 1 & -2 \\ 3 & 15 \end{vmatrix}$$

Only answer choice A matched the D in the given expression, so there's no need to check the D_x determinant; answer choice A must be the correct answer.

Topic: Modifying determinants

Question: If the determinant of matrix A is $|A| = -4$, find the determinant of matrix B, if B is identical to matrix A, except that the second and third rows have been swapped.

Answer choices:

A $|B| = -4$

B $|B| = 0$

C $|B| = 1$

D $|B| = 4$

Solution: D

Every time you swap a row in a matrix, the determinant gets multiplied by -1. Because B is the same as A, just with one row swap, $R_2 \leftrightarrow R_3$, then the determinant of B is

$$|B| = -1|A|$$

$$|B| = -1(-4)$$

$$|B| = 4$$

Topic: Modifying determinants

Question: Find the determinant. Hint: Consider the rows of A.

$$A = \begin{bmatrix} 1 & 5 & 0 & -1 & 6 & -1 \\ 3 & -2 & -1 & 2 & 0 & 2 \\ 3 & -2 & -1 & 2 & 0 & 2 \\ -1 & 1 & 0 & 3 & -2 & 3 \\ 1 & 3 & 2 & -2 & 1 & 0 \\ 0 & 4 & 1 & -3 & 5 & -1 \end{bmatrix}$$

Answer choices:

A $|A| = 0$

B $|A| = 1$

C $|A| = 2$

D $|A| = 5$

Solution: A

If a matrix contains duplicate rows, by definition its determinant must be 0. In this case, the second and third rows of A are identical. And because those rows are identical, we know the determinant of A is $|A| = 0$, and we don't have to calculate $|A|$ to figure this out.

Topic: Modifying determinants

Question: Find the determinant of C, using only the determinants of A and B.

$$A = \begin{bmatrix} -2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 1 \\ -2 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} -2 & 1 \\ -1 & 5 \end{bmatrix}$$

Answer choices:

A $\quad |C| = 9$

B $\quad |C| = 11$

C $\quad |C| = -11$

D $\quad |C| = -9$

Solution: D

Notice that all three of these matrices have identical first rows. Furthermore, the second row of C is the sum of the second rows of A and B. When this is the case, the determinants have the relationship

$$|C| = |A| + |B|$$

So we'll just find the determinants of A and B, and that'll give us the determinant of C.

$$|A| = \begin{vmatrix} -2 & 1 \\ 1 & 1 \end{vmatrix} = (-2)(1) - (1)(1) = -2 - 1 = -3$$

$$|B| = \begin{vmatrix} -2 & 1 \\ -2 & 4 \end{vmatrix} = (-2)(4) - (1)(-2) = -8 + 2 = -6$$

Then the determinant of C is

$$|C| = -3 + (-6)$$

$$|C| = -3 - 6$$

$$|C| = -9$$

Topic: Upper and lower triangular matrices

Question: Find the determinant of the upper-triangular matrix.

$$A = \begin{bmatrix} -2 & 1 & 0 & 3 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Answer choices:

A $\quad |A| = 0$

B $\quad |A| = 4$

C $\quad |A| = 8$

D $\quad |A| = 12$

Solution: D

Because A is an upper-triangular matrix, the determinant can be found just by multiplying the values along the main diagonal. Looking at A,

$$A = \begin{bmatrix} -2 & 1 & 0 & 3 \\ 0 & -1 & -3 & 1 \\ 0 & 0 & 3 & -4 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

the determinant is given by

$$|A| = (-2)(-1)(3)(2)$$

$$|A| = 12$$

Topic: Upper and lower triangular matrices

Question: Calculate the determinant $|M|$ in two ways.

$$M=\begin{bmatrix}1&0&0&0\\2&-1&0&0\\-2&0&2&0\\1&2&3&-1\end{bmatrix}$$

Answer choices:

A $|M|=0$

B $|M|=1$

C $|M|=2$

D $|M|=4$

Solution: C

Given any triangular matrix, we can calculate the determinant using the traditional method, where we break the entire determinant down into a sum of 2×2 determinants, or we can simply multiply the values along the main diagonal.

With the matrix M,

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{bmatrix}$$

first calculate the determinant by multiplying the values along the main diagonal.

$$|M| = (1)(-1)(2)(-1)$$

$$|M| = 2$$

Second, calculate the determinant by breaking down the 4×4 determinant eventually into 2×2 determinants. We'll work across the first row, since it includes lots of 0 entries.

$$|M| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{vmatrix}$$

$$|M| = 1\begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & 3 & -1 \end{vmatrix} + 0\begin{vmatrix} 2 & -1 & 0 \\ -2 & 0 & 0 \\ 1 & 2 & -1 \end{vmatrix} - 0\begin{vmatrix} 2 & -1 & 0 \\ -2 & 0 & 2 \\ 1 & 2 & 3 \end{vmatrix}$$

The last three determinants cancel out, because of the 0 scalars.

$$|M| = \begin{vmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 2 & 3 & -1 \end{vmatrix}$$

Now break the remaining 3×3 determinant into 2×2 determinants.

$$|M| = -1 \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 0 \\ 2 & -1 \end{vmatrix} - 0 \begin{vmatrix} 0 & 2 \\ 2 & 3 \end{vmatrix}$$

$$|M| = - \begin{vmatrix} 2 & 0 \\ 3 & -1 \end{vmatrix}$$

$$|M| = -[(2)(-1) - (0)(3)]$$

$$|M| = -(-2 - 0)$$

$$|M| = 2$$

Topic: Upper and lower triangular matrices

Question: Put Z into upper- or lower-triangular form in order to find the determinant.

$$Z = \begin{bmatrix} 4 & -2 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{bmatrix}$$

Answer choices:

A $|Z| = 12$

B $|Z| = -12$

C $|Z| = 0$

D $|Z| = -2$

Solution: B

There are four 0 entries above the main diagonal, and only one 0 below the main diagonal, so it'll be easier to turn this into a lower-triangular matrix, in which all the entries above the main diagonal are 0.

To get the matrix in lower-triangular form, we'll work in the opposite order that we normally use to find upper-triangular form. First, find a pivot of 1 in the $z_{4,4}$ position.

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ 1 & 2 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 4 & -2 & 0 & 0 \\ 1 & -3 & 0 & 1 \\ -2 & 0 & 2 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

After $R_2 - R_4 \to R_2$, Z is

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ -2 & 0 & 2 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

Find a pivot of 1 in the $z_{3,3}$ position.

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 2 & -1 & 3 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

After $R_2 - 3R_3 \to R_2$, Z is

$$\begin{bmatrix} 4 & -2 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, Z is

$$\begin{bmatrix} -6 & 0 & 0 & 0 \\ 5 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

The matrix is now in lower triangular form. Because we never switched a pair of rows, we never had to multiply the determinant by -1. But we did multiply a row in the matrix by -1, and another row by $1/2$, so if we call the reduced row-echelon matrix the matrix B, then

$$|B| = \frac{1}{2}(-1)|Z|$$

$$|B| = -\frac{1}{2}|Z|$$

$$(-6)(-1)(1)(1) = -\frac{1}{2}|Z|$$

$$6 = -\frac{1}{2}|Z|$$

$$|Z| = -12$$

Topic: Using determinants to find area

Question: Find the area of the parallelogram formed by $\vec{v}_1 = (2,3)$ and $\vec{v}_2 = (-1,4)$, if the two vectors form adjacent edges of the parallelogram.

Answer choices:

A $|A| = 5$

B $|A| = 6$

C $|A| = 11$

D $|A| = 13$

Solution: C

When two vectors form adjacent edges of a parallelogram, we can find the area of the parallelogram by taking the determinant of the matrix of the vectors as column vectors.

In other words, we'll put $\vec{v}_1 = (2,3)$ and $\vec{v}_2 = (-1,4)$ as column vectors into a matrix

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

and then find the determinant of that matrix, which will be the area of the parallelogram.

$$|A| = \begin{vmatrix} 2 & -1 \\ 3 & 4 \end{vmatrix}$$

$$|A| = (2)(4) - (-1)(3)$$

$$|A| = 8 + 3$$

$$|A| = 11$$

The area of the parallelogram is 11 square units.

Topic: Using determinants to find area

Question: The square S is defined by the vertices $(1,1)$, $(-1,1)$, $(-1,-1)$, and $(1,-1)$. If the transformation of S by T creates a transformed figure F, find the area of F.

$$T(\vec{x}) = \begin{bmatrix} -3 & 2 \\ -2 & 1 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad$ $\text{Area}_F = 4$

B $\quad$ $\text{Area}_F = -4$

C $\quad$ $\text{Area}_F = 3$

D $\quad$ $\text{Area}_F = -3$

Solution: A

The area of the transformed figure F can be found using just the area of the square S, and the determinant of the transformation T.

$$\text{Area}_F = |\text{Area}_S(\text{Det}(T))|$$

The square S is defined between $x = -1$ and $x = 1$, so its width is 2, and it's defined between $y = -1$ and $y = 1$, so its height is 2. Therefore, the area of the square is $\text{Area}_S = 2 \cdot 2 = 4$.

The determinant of the transformation matrix is

$$|T| = \begin{vmatrix} -3 & 2 \\ -2 & 1 \end{vmatrix}$$

$$|T| = (-3)(1) - (2)(-2)$$

$$|T| = -3 + 4$$

$$|T| = 1$$

Then the area of the transformed figure F is

$$\text{Area}_F = |\text{Area}_S(\text{Det}(T))|$$

$$\text{Area}_F = |(4)(1)|$$

$$\text{Area}_F = |4|$$

$$\text{Area}_F = 4$$

Topic: Using determinants to find area

Question: The rectangle R is defined by the vertices $(-6,2)$, $(1,2)$, $(1,-4)$, and $(-6,-4)$. If the transformation of R by T creates a transformed figure L, find the area of L.

$$T(\vec{x}) = \begin{bmatrix} 2 & 0 \\ -1 & 4 \end{bmatrix} \vec{x}$$

Answer choices:

A $\quad$ Area$_L = 123$

B $\quad$ Area$_L = 164$

C $\quad$ Area$_L = 271$

D $\quad$ Area$_L = 336$

Solution: D

The area of the transformed figure L can be found using just the area of the rectangle R, and the determinant of the transformation T.

$$\text{Area}_L = |\text{Area}_R(\text{Det}(T))|$$

The rectangle R is defined between $x = -6$ and $x = 1$, so its width is 7, and it's defined between $y = -4$ and $y = 2$, so its height is 6. Therefore, the area of the square is $\text{Area}_S = 7 \cdot 6 = 42$.

The determinant of the transformation matrix is

$$|T| = \begin{vmatrix} 2 & 0 \\ -1 & 4 \end{vmatrix}$$

$$|T| = (2)(4) - (0)(-1)$$

$$|T| = 8 + 0$$

$$|T| = 8$$

Then the area of the transformed figure L is

$$\text{Area}_L = |\text{Area}_R(\text{Det}(T))|$$

$$\text{Area}_L = |(42)(8)|$$

$$\text{Area}_L = |336|$$

$$\text{Area}_L = 336$$

Topic: Transposes and their determinants

Question: Find the transpose A^T.

$$A = \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix}$$

Answer choices:

A $\quad A^T = \begin{bmatrix} -6 & 4 \\ 1 & 1 \\ 9 & -2 \end{bmatrix}$

B $\quad A^T = \begin{bmatrix} 9 & -2 \\ 1 & 1 \\ -6 & 4 \end{bmatrix}$

C $\quad A^T = \begin{bmatrix} -6 & 1 & 9 \\ 4 & 1 & -2 \end{bmatrix}$

D $\quad A^T = \begin{bmatrix} 4 & 1 & -2 \\ -6 & 1 & 9 \end{bmatrix}$

Solution: D

To take the transpose of a matrix, the first row of the original matrix becomes the first column of the transpose, the second row becomes the second column, the third row becomes the third column, etc.

So for the matrix A,

$$A = \begin{bmatrix} 4 & -6 \\ 1 & 1 \\ -2 & 9 \end{bmatrix}$$

the transpose will be

$$A^T = \begin{bmatrix} 4 & 1 & -2 \\ -6 & 1 & 9 \end{bmatrix}$$

Topic: Transposes and their determinants

Question: Find the transpose B^T.

$$B = \begin{bmatrix} -1 & -5 & 1 \\ -1 & -1 & 1 \\ 2 & 5 & -3 \end{bmatrix}$$

Answer choices:

A $\quad B^T = \begin{bmatrix} -1 & -1 & 2 \\ -5 & -1 & 5 \\ 1 & 1 & -3 \end{bmatrix}$

B $\quad B^T = \begin{bmatrix} 2 & -1 & -1 \\ 5 & -1 & -5 \\ -3 & 1 & 1 \end{bmatrix}$

C $\quad B^T = \begin{bmatrix} 2 & 5 & -3 \\ -1 & -1 & 1 \\ -1 & -5 & 1 \end{bmatrix}$

D $\quad B^T = \begin{bmatrix} 1 & -5 & -1 \\ 1 & -1 & -1 \\ -3 & 5 & 2 \end{bmatrix}$

Solution: A

To take the transpose of a matrix, the first row of the original matrix becomes the first column of the transpose, the second row becomes the second column, the third row becomes the third column, etc.

So for the matrix B,

$$B = \begin{bmatrix} -1 & -5 & 1 \\ -1 & -1 & 1 \\ 2 & 5 & -3 \end{bmatrix}$$

the transpose will be

$$B^T = \begin{bmatrix} -1 & -1 & 2 \\ -5 & -1 & 5 \\ 1 & 1 & -3 \end{bmatrix}$$

Topic: Transposes and their determinants

Question: Find the determinant of C^T.

$$C = \begin{bmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{bmatrix}$$

Answer choices:

A $\quad |C| = -69$

B $\quad |C| = 69$

C $\quad |C| = -31$

D $\quad |C| = 31$

Solution: B

The determinant of the transpose of a matrix is always equal to the determinant of the original matrix.

So to find the determinant of C^T, instead of finding the transpose and then taking its determinant, we'll just take the determinant of C, starting with breaking down the 3×3 determinant into 2×2 determinants.

$$|C| = \begin{vmatrix} 7 & 3 & 4 \\ 1 & 6 & 1 \\ 2 & 2 & 3 \end{vmatrix}$$

$$|C| = 7 \begin{vmatrix} 6 & 1 \\ 2 & 3 \end{vmatrix} - 3 \begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} + 4 \begin{vmatrix} 1 & 6 \\ 2 & 2 \end{vmatrix}$$

Evaluate the 2×2 determinants.

$$|C| = 7[(6)(3) - (1)(2)] - 3[(1)(3) - (1)(2)] + 4[(1)(2) - (6)(2)]$$

$$|C| = 7(18 - 2) - 3(3 - 2) + 4(2 - 12)$$

$$|C| = 7(16) - 3(1) + 4(-10)$$

$$|C| = 112 - 3 - 40$$

$$|C| = 69$$

Topic: Transposes of products, sums, and inverses

Question: Find $(AB)^T$.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & -6 & 1 \\ 0 & -8 & 5 \\ 1 & 1 & -2 \end{bmatrix}$$

Answer choices:

A $\quad (AB)^T = \begin{bmatrix} 2 & 9 & 16 \\ -16 & -35 & 10 \\ 10 & 15 & -20 \end{bmatrix}$

B $\quad (AB)^T = \begin{bmatrix} 10 & 15 & -20 \\ -16 & -35 & 10 \\ 2 & 9 & 16 \end{bmatrix}$

C $\quad (AB)^T = \begin{bmatrix} 2 & -16 & 10 \\ 9 & -35 & 15 \\ 16 & 10 & -20 \end{bmatrix}$

D $\quad (AB)^T = \begin{bmatrix} 10 & -16 & 2 \\ 15 & -35 & 9 \\ -20 & 10 & 16 \end{bmatrix}$

Solution: A

There are two ways we could go about finding $(AB)^T$. Given the rule for products of transposes,

$$(XY)^T = Y^T X^T$$

we could either calculate the left side, finding the product AB and then taking its transpose, or we could calculate the right side, finding A^T and B^T individually, and then taking their product.

Let's use the second method, where we start by taking the transposes individually. We'll swap rows and columns in A and B to get A^T and B^T.

$$A^T = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -3 \\ -2 & 1 & 4 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 4 & 0 & 1 \\ -6 & -8 & 1 \\ 1 & 5 & -2 \end{bmatrix}$$

Now we'll find the product of these transposes.

$$B^T A^T = \begin{bmatrix} 4 & 0 & 1 \\ -6 & -8 & 1 \\ 1 & 5 & -2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & -3 \\ -2 & 1 & 4 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 4(1) + 0(1) + 1(-2) & 4(2) + 0(3) + 1(1) & 4(3) + 0(-3) + 1(4) \\ -6(1) - 8(1) + 1(-2) & -6(2) - 8(3) + 1(1) & -6(3) - 8(-3) + 1(4) \\ 1(1) + 5(1) - 2(-2) & 1(2) + 5(3) - 2(1) & 1(3) + 5(-3) - 2(4) \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 4+0-2 & 8+0+1 & 12+0+4 \\ -6-8-2 & -12-24+1 & -18+24+4 \\ 1+5+4 & 2+15-2 & 3-15-8 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 2 & 9 & 16 \\ -16 & -35 & 10 \\ 10 & 15 & -20 \end{bmatrix}$$

So we can say

$$(AB)^T = B^TA^T = \begin{bmatrix} 2 & 9 & 16 \\ -16 & -35 & 10 \\ 10 & 15 & -20 \end{bmatrix}$$

Topic: Transposes of products, sums, and inverses

Question: Find $(A+B)^T$.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & -6 & 1 \\ 0 & -8 & 5 \\ 1 & 1 & -2 \end{bmatrix}$$

Answer choices:

A $\quad (A+B)^T = \begin{bmatrix} -1 & -5 & 5 \\ 6 & -5 & 2 \\ 2 & -2 & 4 \end{bmatrix}$

B $\quad (A+B)^T = \begin{bmatrix} 5 & -5 & -1 \\ 2 & -5 & 6 \\ 4 & -2 & 2 \end{bmatrix}$

C $\quad (A+B)^T = \begin{bmatrix} -1 & 6 & 2 \\ -5 & -5 & -2 \\ 5 & 2 & 4 \end{bmatrix}$

D $\quad (A+B)^T = \begin{bmatrix} 5 & 2 & 4 \\ -5 & -5 & -2 \\ -1 & 6 & 2 \end{bmatrix}$

Solution: D

There are two ways we could go about finding $(A+B)^T$. Given the rule for sums of transposes,

$$(X+Y)^T = X^T + Y^T$$

we could either calculate the left side, finding the sum $A+B$ and then taking its transpose, or we could calculate the right side, finding A^T and B^T individually, and then taking their sum.

Let's use the first method, where we start by finding the sum of the matrices.

$$A+B = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -6 & 1 \\ 0 & -8 & 5 \\ 1 & 1 & -2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 1+4 & 1-6 & -2+1 \\ 2+0 & 3-8 & 1+5 \\ 3+1 & -3+1 & 4-2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} 5 & -5 & -1 \\ 2 & -5 & 6 \\ 4 & -2 & 2 \end{bmatrix}$$

Now we'll find the transpose of the sum by swapping the rows and columns of $A+B$.

$$(A+B)^T = \begin{bmatrix} 5 & 2 & 4 \\ -5 & -5 & -2 \\ -1 & 6 & 2 \end{bmatrix}$$

So we can say

$$(A+B)^T = \begin{bmatrix} 5 & 2 & 4 \\ -5 & -5 & -2 \\ -1 & 6 & 2 \end{bmatrix}$$

Topic: Transposes of products, sums, and inverses

Question: Find $(A^{-1})^T$.

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 4 \end{bmatrix}$$

Answer choices:

A $\quad (A^{-1})^T = \begin{bmatrix} -\frac{3}{8} & \frac{1}{8} & \frac{3}{8} \\ -\frac{1}{20} & -\frac{1}{4} & -\frac{3}{20} \\ -\frac{7}{40} & \frac{1}{8} & -\frac{1}{40} \end{bmatrix}$

B $\quad (A^{-1})^T = \begin{bmatrix} -\frac{3}{8} & -\frac{1}{20} & -\frac{7}{40} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{8} \\ \frac{3}{8} & -\frac{3}{20} & -\frac{1}{40} \end{bmatrix}$

C $\quad (A^{-1})^T = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{20} & \frac{1}{4} & \frac{3}{20} \\ \frac{7}{40} & -\frac{1}{8} & \frac{1}{40} \end{bmatrix}$

D $\quad (A^{-1})^T = \begin{bmatrix} \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$

Solution: C

There are two ways we could go about finding $(A^{-1})^T$. Given the rule for the inverse of a transpose,

$$(X^T)^{-1} = (X^{-1})^T$$

we could either calculate the left side, finding the transpose A^T and then taking its inverse, or we could calculate the right side, finding the inverse A^{-1} and then taking its transpose.

Let's use the second method, where we start by finding the inverse A^{-1}. We'll augment A with I_3, and then put the left side of the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ 2 & 3 & 1 & | & 0 & 1 & 0 \\ 3 & -3 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

We already have the pivot entry in the first row, so let's work on zeroing out the rest of the first column.

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 3 & -3 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & -6 & 10 & | & -3 & 0 & 1 \end{bmatrix}$$

We already have the pivot in the second column, so let's work on zeroing out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 & -1 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & -6 & 10 & | & -3 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 & -1 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & 0 & 40 & | & -15 & 6 & 1 \end{bmatrix}$$

Find the pivot in the third column, then zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 & -1 & 0 \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ 0 & 1 & 5 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ 0 & 1 & 0 & | & -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ 0 & 0 & 1 & | & -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

Now that the left side of the augmented matrix is the identity matrix, the right side is the inverse A^{-1}.

$$A^{-1} = \begin{bmatrix} \frac{3}{8} & \frac{1}{20} & \frac{7}{40} \\ -\frac{1}{8} & \frac{1}{4} & -\frac{1}{8} \\ -\frac{3}{8} & \frac{3}{20} & \frac{1}{40} \end{bmatrix}$$

To find $(A^{-1})^T$, we'll take the transpose of this inverse.

$$(A^{-1})^T = \begin{bmatrix} \frac{3}{8} & -\frac{1}{8} & -\frac{3}{8} \\ \frac{1}{20} & \frac{1}{4} & \frac{3}{20} \\ \frac{7}{40} & -\frac{1}{8} & \frac{1}{40} \end{bmatrix}$$

Topic: Null and column spaces of the transpose

Question: Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}$$

Answer choices:

A $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(A^T)) = 1$

$\quad C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(C(A^T)) = 2$

B $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(N(A^T)) = 2$

$\quad C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(C(A^T)) = 1$

C $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} 4 \\ 12 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(A^T)) = 1$

$\quad C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(C(A^T)) = 2$

D $\quad N(A^T) = \text{Span}\left(\begin{bmatrix} 4 \\ 12 \end{bmatrix}\right)$ in $\mathbb{R}^2$ $\quad \text{Dim}(N(A^T)) = 2$

$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(C(A^T)) = 1$

Solution: A

The transpose of A is

$$A^T = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & 0 \end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 4 & | & 0 \\ -3 & 1 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & | & 0 \\ 0 & 1 & 12 & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first two columns, we'll pull a system of equations from the matrix,

$$1x_1 + 0x_2 + 4x_3 = 0$$

$$0x_1 + 1x_2 + 12x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = -4x_3$$

$$x_2 = -12x_3$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix}$$

Therefore, the left null space is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} -4 \\ -12 \\ 1 \end{bmatrix} \right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, A. The original matrix has 3 rows, so the null space of the transpose $N(A^T)$ is a subspace of $\mathbb{R}^3$.

The column space of the transpose A^T, which is the same as the row space of A, is simply given by the columns in A^T that contain pivot entries when A^T is in reduced row-echelon form. So the column space of A^T is

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, A. The original matrix has 2 columns, so the column space of the transpose $C(A^T)$ is a subspace of $\mathbb{R}^2$.

Because there's one vector that forms the basis of $N(A^T)$, the dimension of $N(A^T)$ is $\text{Dim}(N(A^T)) = 1$.

Because there are two vectors that form the basis of $C(A^T)$, the dimension of $C(A^T)$ is $\text{Dim}(C(A^T))=2$.

$$N(A^T)=\text{Span}\left(\begin{bmatrix}-4\\-12\\1\end{bmatrix}\right)\text{ in }\mathbb{R}^3 \qquad \text{Dim}(N(A^T))=1$$

$$C(A^T)=\text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix},\ \begin{bmatrix}0\\1\end{bmatrix}\right)\text{ in }\mathbb{R}^2 \qquad \text{Dim}(C(A^T))=2$$

Topic: Null and column spaces of the transpose

Question: Find the row space and left null space of B, and the dimensions of those spaces.

$$B = \begin{bmatrix} 2 & -2 & 1 & 0 \\ 1 & 3 & -3 & -2 \\ 0 & 0 & 4 & -4 \end{bmatrix}$$

Answer choices:

A $N(B^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$ in $\mathbb{R}^3$ $\text{Dim}(N(B^T)) = 1$

$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ -2 \end{bmatrix} \right)$ in $\mathbb{R}^4$ $\text{Dim}(C(B^T)) = 2$

B $N(B^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$ in $\mathbb{R}^4$ $\text{Dim}(N(B^T)) = 2$

$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ -2 \end{bmatrix} \right)$ in $\mathbb{R}^3$ $\text{Dim}(C(B^T)) = 1$

C $N(B^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(N(B^T)) = 0$

$C(B^T) = \text{Span}\left(\begin{bmatrix}2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-3\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\4\\-4\end{bmatrix}\right)$ in $\mathbb{R}^4$ $\text{Dim}(C(B^T)) = 3$

D $N(B^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ in $\mathbb{R}^4$ $\text{Dim}(N(B^T)) = 3$

$C(B^T) = \text{Span}\left(\begin{bmatrix}2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-3\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\4\\-4\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\text{Dim}(C(B^T)) = 0$

Solution: C

The transpose of B is

$$B^T = \begin{bmatrix}2 & 1 & 0\\-2 & 3 & 0\\1 & -3 & 4\\0 & -2 & -4\end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form. Start by finding the pivot in the first column (by switching the first and third rows), and then zeroing out the rest of the first column.

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ -2 & 3 & 0 & | & 0 \\ 1 & -3 & 4 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 2 & 1 & 0 & | & 0 \\ -2 & 3 & 0 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ -2 & 3 & 0 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \\ 0 & -2 & -4 & | & 0 \end{bmatrix}$$

Find the pivot entry in the second column (by switching the second and fourth rows), and then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & -2 & -4 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 7 & -8 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & -22 & | & 0 \\ 0 & -3 & 8 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & -22 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 14 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first three columns, we'll pull a system of equations from the matrix,

$$1x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 1x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + 1x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space is

$$N(B^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, B. The original matrix has 3 rows, so the null space of the transpose $N(B^T)$ is a subspace of $\mathbb{R}^3$.

The column space of the transpose B^T, which is the same as the row space of B, is simply given by the columns in B^T that contain pivot entries when B^T is in reduced row-echelon form. So the column space of B^T is

$$C(B^T) = \text{Span}\left(\begin{bmatrix}2\\-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\3\\-3\\-2\end{bmatrix}, \begin{bmatrix}0\\0\\4\\-4\end{bmatrix}\right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, B. The original matrix has 4 columns, so the column space of the transpose $C(B^T)$ is a subspace of $\mathbb{R}^4$.

Because the zero vector is the only vector that forms the basis of $N(B^T)$, the dimension of $N(B^T)$ is $\text{Dim}(N(B^T)) = 0$.

Because there are three vectors that form the basis of $C(B^T)$, the dimension of $C(B^T)$ is $\text{Dim}(C(B^T)) = 3$.

$$N(B^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad\qquad \text{Dim}(N(B^T)) = 0$$

$$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -3 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \\ -4 \end{bmatrix}\right) \text{ in } \mathbb{R}^4 \qquad \text{Dim}(C(B^T)) = 3$$

Topic: Null and column spaces of the transpose

Question: Find the row space and left null space of C, and the dimensions of those spaces.

$$C=\begin{bmatrix}-1&5&0\\1&-2&3\\0&0&-4\end{bmatrix}$$

Answer choices:

A $\quad N(C^T)=\text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(C^T))=3$

$\quad C(C^T)=\text{Span}\left(\begin{bmatrix}-1\\5\\0\end{bmatrix},\begin{bmatrix}1\\-2\\3\end{bmatrix},\begin{bmatrix}0\\0\\-4\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(C^T))=0$

B $\quad N(C^T)=\text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(C^T))=0$

$\quad C(C^T)=\text{Span}\left(\begin{bmatrix}-1\\5\\0\end{bmatrix},\begin{bmatrix}1\\-2\\3\end{bmatrix},\begin{bmatrix}0\\0\\-4\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(C(C^T))=3$

C $\quad N(C^T)=\text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right)$ in $\mathbb{R}^3$ $\quad \text{Dim}(N(C^T))=3$

$$C(C^T) = \text{Span}\left(\begin{bmatrix}-1\\5\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}\right) \quad \text{in } \mathbb{R}^3 \quad \text{Dim}(C(C^T)) = 1$$

D $$N(C^T) = \text{Span}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}\right) \quad \text{in } \mathbb{R}^3 \quad \text{Dim}(N(C^T)) = 1$$

$$C(C^T) = \text{Span}\left(\begin{bmatrix}-1\\5\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\end{bmatrix}\right) \quad \text{in } \mathbb{R}^3 \quad \text{Dim}(C(C^T)) = 3$$

Solution: B

The transpose of C is

$$C^T = \begin{bmatrix}-1 & 1 & 0\\5 & -2 & 0\\0 & 3 & -4\end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form. Start by finding the pivot in the first column, and then zeroing out the rest of the first column.

$$\begin{bmatrix}-1 & 1 & 0 & | & 0\\5 & -2 & 0 & | & 0\\0 & 3 & -4 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -1 & 0 & | & 0\\5 & -2 & 0 & | & 0\\0 & 3 & -4 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 0 & | & 0\\0 & 3 & 0 & | & 0\\0 & 3 & -4 & | & 0\end{bmatrix}$$

Find the pivot entry in the second column, and then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & -4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & -4 & | & 0 \end{bmatrix}$$

Find the pivot entry in the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first three columns, we'll pull a system of equations from the matrix,

$$1x_1 + 0x_2 + 0x_3 = 0$$

$$0x_1 + 1x_2 + 0x_3 = 0$$

$$0x_1 + 0x_2 + 1x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space is

$$N(C^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, C. The original matrix has 3 rows, so the null space of the transpose $N(C^T)$ is a subspace of $\mathbb{R}^3$.

The column space of the transpose C^T, which is the same as the row space of C, is simply given by the columns in C^T that contain pivot entries when C^T is in reduced row-echelon form. So the column space of C^T is

$$C(C^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}\right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, C. The original matrix has 3 columns, so the column space of the transpose $C(C^T)$ is a subspace of $\mathbb{R}^3$.

Because the zero vector is the only vector that forms the basis of $N(C^T)$, the dimension of $N(C^T)$ is $\text{Dim}(N(C^T)) = 0$.

Because there are three vectors that form the basis of $C(C^T)$, the dimension of $C(C^T)$ is $\text{Dim}(C(C^T)) = 3$.

$$N(C^T) = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad \text{Dim}(N(C^T)) = 0$$

$$C(C^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -4 \end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad \text{Dim}(C(C^T)) = 3$$

Topic: The product of a matrix and its transpose

Question: Is A^TA invertible?

$$A = \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}$$

Answer choices:

A Yes, because the columns of A are linearly independent

B Yes, but the columns of A aren't linearly independent

C No, because the columns of A aren't linearly independent

D No, but the columns of A are linearly independent

Solution: A

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & 0 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 4 \\ -3 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ 0 & 1 \\ 4 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) + 0(0) + 4(4) & 1(-3) + 0(1) + 4(0) \\ -3(1) + 1(0) + 0(4) & -3(-3) + 1(1) + 0(0) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 0 + 16 & -3 + 0 + 0 \\ -3 + 0 + 0 & 9 + 1 + 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 17 & -3 \\ -3 & 10 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 1 & -\frac{3}{17} \\ -3 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{17} \\ 0 & \frac{161}{17} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{17} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because we got to the identity matrix, we can say that A^TA is invertible.

Topic: The product of a matrix and its transpose

Question: Is A^TA invertible?

$$A = \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

Answer choices:

A Yes, because the columns of A are linearly independent

B Yes, but the columns of A aren't linearly independent

C No, because the columns of A aren't linearly independent

D No, but the columns of A are linearly independent

Solution: C

The columns of A aren't linearly independent, so A^TA is not invertible. We can confirm this by finding A^TA, and then verifying that A^TA doesn't simplify to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} -6 & 4 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} -6 & 3 \\ 4 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -6(-6) + 4(4) & -6(3) + 4(-2) \\ 3(-6) - 2(4) & 3(3) - 2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 36 + 16 & -18 - 8 \\ -18 - 8 & 9 + 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

Topic: The product of a matrix and its transpose

Question: Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

Answer choices:

A Yes, because the columns of A are linearly independent

B Yes, but the columns of A aren't linearly independent

C No, because the columns of A aren't linearly independent

D No, but the columns of A are linearly independent

Solution: A

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -2 & 3 & -2 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 1 \\ -2 & 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & -2 & 3 \\ 1 & 1 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1)+0(0)+1(1) & 1(0)+0(-2)+1(1) & 1(-2)+0(3)+1(-2) \\ 0(1)-2(0)+1(1) & 0(0)-2(-2)+1(1) & 0(-2)-2(3)+1(-2) \\ -2(1)+3(0)-2(1) & -2(0)+3(-2)-2(1) & -2(-2)+3(3)-2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0+1 & 0+0+1 & -2+0-2 \\ 0-0+1 & 0+4+1 & 0-6-2 \\ -2+0-2 & 0-6-2 & 4+9+4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 5 & -8 \\ -4 & -8 & 17 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA=\begin{bmatrix}2&1&-4\\1&5&-8\\-4&-8&17\end{bmatrix}\to\begin{bmatrix}1&5&-8\\2&1&-4\\-4&-8&17\end{bmatrix}\to\begin{bmatrix}1&5&-8\\0&-9&12\\-4&-8&17\end{bmatrix}\to\begin{bmatrix}1&5&-8\\0&-9&12\\0&12&-15\end{bmatrix}$$

$$\to\begin{bmatrix}1&5&-8\\0&1&-\frac{4}{3}\\0&12&-15\end{bmatrix}\to\begin{bmatrix}1&0&-\frac{4}{3}\\0&1&-\frac{4}{3}\\0&12&-15\end{bmatrix}\to\begin{bmatrix}1&0&-\frac{4}{3}\\0&1&-\frac{4}{3}\\0&0&1\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&0\\0&1&-\frac{4}{3}\\0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

Because we got to the identity matrix, we can say that A^TA is invertible.

Topic: A=LU factorization

Question: Rewrite the matrix A in factored LU form.

$$A = \begin{bmatrix} 3 & 6 \\ -6 & 10 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 3 & 6 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

B $\begin{bmatrix} 3 & 6 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 0 & 10 \end{bmatrix}$

C $\begin{bmatrix} 3 & 6 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 0 & 22 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$

D $\begin{bmatrix} 3 & 6 \\ -6 & 10 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 0 & 22 \end{bmatrix}$

Solution: D

We'll apply an elimination matrix to A to zero-out the -6 in $A_{2,1}$.

$$E_{2,1}A = U$$

$$\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}\begin{bmatrix}3 & 6\\-6 & 10\end{bmatrix}=\begin{bmatrix}3 & 6\\0 & 22\end{bmatrix}$$

To solve this equation for A, we need to move the elimination matrix to the right side, inverting it.

$$\begin{bmatrix}3 & 6\\-6 & 10\end{bmatrix}=\begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}^{-1}\begin{bmatrix}3 & 6\\0 & 22\end{bmatrix}$$

To invert the elimination matrix, we'll change the sign on the non-zero entry below the main diagonal, and we'll get the $A = LU$ factorization.

$$\begin{bmatrix}3 & 6\\-6 & 10\end{bmatrix}=\begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}\begin{bmatrix}3 & 6\\0 & 22\end{bmatrix}$$

Topic: A=LU factorization

Question: Find L if the matrix A is decomposed as LU.

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix}$$

Answer choices:

A $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$

B $\quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

C $\quad L = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

D $\quad L = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$

Solution: B

We'll apply an elimination matrix to A to zero-out the 8 in $A_{2,1}$.

$$E_{2,1}A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & -6 & 5 \end{bmatrix}$$

Next we'll apply an elimination matrix to A to zero-out the -6 in $A_{3,2}$.

$$E_{3,2}E_{2,1}A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

To solve this equation for A, we need to move the elimination matrices to the right side, reversing their order and inverting each one.

$$\begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

To invert the elimination matrices, we'll change the sign on the non-zero entry below the main diagonal, and we'll get the $A = LU$ factorization.

$$\begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

Then we'll consolidate all the non-zero entries into one matrix, L.

$$\begin{bmatrix} 4 & 1 & -1 \\ 8 & 4 & -3 \\ 0 & -6 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$$

The question asked for the matrix L, which we found as

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Topic: A=LU factorization

Question: Rewrite the matrix A in factored LDU form, where D is the diagonal matrix that factors the pivots out of U.

$$A=\begin{bmatrix}3 & -3 & -3 & 6\\ 12 & -10 & -12 & 26\\ -6 & 8 & 8 & -6\\ 6 & -6 & 4 & 38\end{bmatrix}$$

Answer choices:

A $\quad\begin{bmatrix}1 & 0 & 0 & 0\\ 4 & 1 & 0 & 0\\ -2 & 1 & 1 & 0\\ 2 & 0 & 5 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 6\end{bmatrix}\begin{bmatrix}1 & -1 & -1 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix}$

B $\quad\begin{bmatrix}1 & 0 & 0 & 0\\ 4 & 1 & 0 & 0\\ -2 & 1 & 1 & 0\\ 2 & 0 & 5 & 1\end{bmatrix}\begin{bmatrix}3 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 2 & 0\\ 0 & 0 & 0 & 6\end{bmatrix}\begin{bmatrix}3 & -3 & -3 & 6\\ 0 & 2 & 0 & 2\\ 0 & 0 & 2 & 4\\ 0 & 0 & 0 & 6\end{bmatrix}$

C $\quad\begin{bmatrix}1 & 0 & 0 & 0\\ 4 & 1 & 0 & 0\\ -2 & 1 & 1 & 0\\ 2 & 0 & 5 & 1\end{bmatrix}\begin{bmatrix}1 & -1 & -1 & 2\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 2\\ 0 & 0 & 0 & 1\end{bmatrix}$

D $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Solution: A

Let's set up $A = LU$.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix}$$

We only need to zero-out the entries in U that are below the main diagonal. To zero-out the 12 in $U_{2,1}$, we need to subtract 4 of R_1 from R_2. Since that row operation is $R_2 - 4R_1$, we put 4 into $L_{2,1}$.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix}$$

To zero-out the -6 in $U_{3,1}$, we need to add 2 of R_1 to R_3, which is the row operation $R_3 + 2R_1$. But we want this row operation written with subtraction, so we rewrite it as $R_3 - (-2)R_1$, and put -2 into $L_{3,1}$.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 6 & -6 & 4 & 38 \end{bmatrix}$$

To zero-out the 6 in $U_{4,1}$, we need to subtract 2 of R_1 from R_4. Since the row operation is $R_4 - 2R_1$, we put 2 into $L_{4,1}$.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & & 1 & 0 \\ 2 & & & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 2 & 2 & 6 \\ 0 & 0 & 10 & 26 \end{bmatrix}$$

To zero-out the 2 in $U_{3,2}$, we need to subtract 1 of R_2 from R_3. Since the row operation is $R_3 - R_2$, we put 1 into $L_{3,2}$.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & & & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 10 & 26 \end{bmatrix}$$

We don't need to perform any row operation to keep the 0 in $L_{4,2}$, so

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 10 & 26 \end{bmatrix}$$

To zero-out the 10 in $U_{4,3}$, we need to subtract 5 of R_3 from R_4. The row operation is $R_4 - 5R_3$, so we put 5 into $L_{4,3}$.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

To rewrite the $A = LU$ factorization so that U has only 1s along its main diagonal, insert an identity matrix to rewrite the equation as $A = LIU$.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & -3 & 6 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

We'll first pull out the 3 from $U_{1,1}$, dividing the first row of U by 3.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Then we'll pull out the 2 from $U_{2,2}$, dividing the second row of U by 2.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

We'll pull out the 2 from $U_{3,3}$, dividing the third row of U by 2.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

Finally, we'll pull out the 6 from $U_{4,4}$, dividing the fourth row of U by 4.

$$\begin{bmatrix} 3 & -3 & -3 & 6 \\ 12 & -10 & -12 & 26 \\ -6 & 8 & 8 & -6 \\ 6 & -6 & 4 & 38 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ -2 & 1 & 1 & 0 \\ 2 & 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Now the matrix A is in factored LDU form, where D is the diagonal matrix that factors the pivots out of U, where L is a lower triangular matrix, U is an upper triangular matrix, and where both L and U have only 1s along their main diagonals.

Topic: Orthogonal complements

Question: Find the orthogonal complement of W, $W^{\perp}$.

$$W = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ -5 \end{bmatrix}\right)$$

Answer choices:

A $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}\right)$

B $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \\ 1 \end{bmatrix}\right)$

C $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} 2 \\ -3 \end{bmatrix}\right)$

D $\quad W^{\perp} = \text{Span}\left(\begin{bmatrix} -2 \\ 3 \end{bmatrix}\right)$

Solution: A

The subspace W is a plane in $\mathbb{R}^4$, spanned by the two vectors $\vec{w}_1 = (-1,0,-2,4)$ and $\vec{w}_2 = (2,0,3,-5)$. Therefore, its orthogonal complement $W^{\perp}$ is the set of vectors which are orthogonal to both $\vec{w}_1 = (-1,0,-2,4)$ and $\vec{w}_2 = (2,0,3,-5)$.

$$W^{\perp} = \{\vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix} -1 \\ 0 \\ -2 \\ 4 \end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 2 \\ 0 \\ 3 \\ -5 \end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4)$, we get two equations from these dot products.

$$-x_1 - 2x_3 + 4x_4 = 0$$

$$2x_1 + 3x_3 - 5x_4 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} -1 & 0 & -2 & 4 & | & 0 \\ 2 & 0 & 3 & -5 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 2 & -4 & | & 0 \\ 2 & 0 & 3 & -5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -4 & | & 0 \\ 0 & 0 & -1 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & -4 & | & 0 \\ 0 & 0 & 1 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 2 & | & 0 \\ 0 & 0 & 1 & -3 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + 2x_4 = 0$$

$$x_3 - 3x_4 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = -2x_4$$

$$x_3 = 3x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement $W^{\perp}$ is

$$W^{\perp} = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right)$$

Topic: Orthogonal complements

Question: Rewrite the orthogonal complement of V, $V^\perp$, if V is a vector set in $\mathbb{R}^3$.

$$V = \begin{bmatrix} -2y+z \\ y \\ z \end{bmatrix}$$

Answer choices:

A $\quad V^\perp = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}\right)$

B $\quad V^\perp = \text{Span}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right)$

C $\quad V^\perp = \text{Span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}\right)$

D $\quad V^\perp = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}\right)$

Solution: C

We can rewrite V as

$$V = \{y \cdot \begin{bmatrix}-2\\1\\0\end{bmatrix} + z \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} \ | \ y, z \in \mathbb{R}^3\}$$

The subspace V is a plane in $\mathbb{R}^3$, spanned by the two vectors $\overrightarrow{v}_1 = (-2,1,0)$ and $\overrightarrow{v}_2 = (1,0,1)$. Therefore, its orthogonal complement $V^\perp$ is the set of vectors which are orthogonal to both $\overrightarrow{v}_1 = (-2,1,0)$ and $\overrightarrow{v}_2 = (1,0,1)$.

$$V^\perp = \{\overrightarrow{x} \in \mathbb{R}^3 \ | \ \overrightarrow{x} \cdot \begin{bmatrix}-2\\1\\0\end{bmatrix} = 0 \quad \text{and} \quad \overrightarrow{x} \cdot \begin{bmatrix}1\\0\\1\end{bmatrix} = 0\}$$

If we let $\overrightarrow{x} = (x_1, x_2, x_3)$, we get two equations from these dot products.

$$-2x_1 + x_2 = 0$$

$$x_1 + x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}-2 & 1 & 0 & | & 0\\1 & 0 & 1 & | & 0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix}1 & -\frac{1}{2} & 0 & | & 0\\1 & 0 & 1 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -\frac{1}{2} & 0 & | & 0\\0 & \frac{1}{2} & 1 & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} & 0 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + x_3 = 0$$

$$x_2 + 2x_3 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = -x_3$$

$$x_2 = -2x_3$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^\perp = \text{Span}\left(\begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \right)$$

Topic: Orthogonal complements

Question: Describe the orthogonal complement of V, $V^\perp$.

$$V = \text{Span}\left(\begin{bmatrix}1\\-2\\3\\5\end{bmatrix}, \begin{bmatrix}0\\4\\-8\\8\end{bmatrix}, \begin{bmatrix}1\\3\\-5\\-1\end{bmatrix}\right)$$

Answer choices:

A $\quad V^\perp = \text{Span}\left(\begin{bmatrix}9\\-14\\-8\\1\end{bmatrix}\right)$

B $\quad V^\perp = \text{Span}\left(\begin{bmatrix}-1\\14\\8\\1\end{bmatrix}\right)$

C $\quad V^\perp = \text{Span}\left(\begin{bmatrix}9\\-14\\-8\\0\end{bmatrix}\right)$

D $\quad V^\perp = \text{Span}\left(\begin{bmatrix}1\\-14\\-8\\0\end{bmatrix}\right)$

Solution: B

The subspace V is a plane in $\mathbb{R}^4$, spanned by the three vectors $\overrightarrow{v}_1=(1,-2,3,5)$, $\overrightarrow{v}_2=(0,4,-8,8)$, and $\overrightarrow{v}_3=(1,3,-5,-1)$. Therefore, its orthogonal complement $V^{\perp}$ is the set of vectors which are orthogonal to $\overrightarrow{v}_1=(1,-2,3,5)$, $\overrightarrow{v}_2=(0,4,-8,8)$, and $\overrightarrow{v}_3=(1,3,-5,-1)$.

$$V^{\perp}=\{\overrightarrow{x}\in\mathbb{R}^4\ |\ \overrightarrow{x}\cdot\begin{bmatrix}1\\-2\\3\\5\end{bmatrix}=0\ ,\ \overrightarrow{x}\cdot\begin{bmatrix}0\\4\\-8\\8\end{bmatrix}=0\quad\text{and}\quad\overrightarrow{x}\cdot\begin{bmatrix}1\\3\\-5\\-1\end{bmatrix}=0\}$$

If we let $\overrightarrow{x}=(x_1,x_2,x_3,x_4)$, we get three equations from these dot products.

$$x_1-2x_2+3x_3+5x_4=0$$

$$4x_2-8x_3+8x_4=0$$

$$x_1+3x_2-5x_3-x_4=0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}1&-2&3&5&|&0\\0&4&-8&8&|&0\\1&3&-5&-1&|&0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix}1&-2&3&5&|&0\\0&4&-8&8&|&0\\0&5&-8&-6&|&0\end{bmatrix}\to\begin{bmatrix}1&-2&3&5&|&0\\0&1&-2&2&|&0\\0&5&-8&-6&|&0\end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & 5 & | & 0 \\ 0 & 1 & -2 & 2 & | & 0 \\ 0 & 0 & 2 & -16 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 9 & | & 0 \\ 0 & 1 & -2 & 2 & | & 0 \\ 0 & 0 & 2 & -16 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & 9 & | & 0 \\ 0 & 1 & -2 & 2 & | & 0 \\ 0 & 0 & 1 & -8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 9 & | & 0 \\ 0 & 1 & 0 & -14 & | & 0 \\ 0 & 0 & 1 & -8 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & -14 & | & 0 \\ 0 & 0 & 1 & -8 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + x_4 = 0$$

$$x_2 - 14x_4 = 0$$

$$x_3 - 8x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = -x_4$$

$$x_2 = 14x_4$$

$$x_3 = 8x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 14 \\ 8 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix} -1 \\ 14 \\ 8 \\ 1 \end{bmatrix}\right)$$

Topic: Orthogonal complements of the fundamental subspaces

Question: For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} -1 & 3 & 5 & -1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix}$$

Answer choices:

A Dim$(C(A)) = 3$, Dim$(N(A)) = 0$, Dim$(C(A^T)) = 3$, Dim$(N(A^T)) = 1$

B Dim$(C(A)) = 1$, Dim$(N(A)) = 3$, Dim$(C(A^T)) = 1$, Dim$(N(A^T)) = 2$

C Dim$(C(A)) = 3$, Dim$(N(A)) = 1$, Dim$(C(A^T)) = 3$, Dim$(N(A^T)) = 0$

D Dim$(C(A)) = 1$, Dim$(N(A)) = 3$, Dim$(C(A^T)) = 3$, Dim$(N(A^T)) = 2$

Solution: C

Put A into reduced row-echelon form.

$$\begin{bmatrix} -1 & 3 & 5 & -1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 2 & 0 & 2 & -4 \\ -3 & -5 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 6 & 12 & -6 \\ -3 & -5 & 9 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 6 & 12 & -6 \\ 0 & -14 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & -14 & -6 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -5 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 22 & -11 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 22 & -11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \end{bmatrix}$$

In reduced row-echelon form, we can see that there are three pivots, which means the rank of A is $r = 3$.

The matrix A is a 3×4 matrix, which means there are $m = 3$ rows and $n = 4$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$	$r = 3$
Null space, $N(A)$	$n - r = 4 - 3 = 1$
Row space, $C(A^T)$	$r = 3$
Left null space, $N(A^T)$	$m - r = 3 - 3 = 0$

Topic: Orthogonal complements of the fundamental subspaces

Question: For the matrix A, which of these dimensions of the four fundamental subspaces is incorrect?

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & -3 & 5 \\ 1 & -1 & 2 \end{bmatrix}$$

Answer choices:

A Dim$(C(A)) = 2$

B Dim$(N(A)) = 0$

C Dim$(C(A^T)) = 2$

D Dim$(N(A^T)) = 1$

Solution: B

Put A into reduced row-echelon form.

$$\begin{bmatrix}1 & -2 & 3\\ 2 & -3 & 5\\ 1 & -1 & 2\end{bmatrix} \to \begin{bmatrix}1 & -2 & 3\\ 0 & 1 & -1\\ 1 & -1 & 2\end{bmatrix} \to \begin{bmatrix}1 & -2 & 3\\ 0 & 1 & -1\\ 0 & 1 & -1\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 3\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & 1\\ 0 & 1 & -1\\ 0 & 0 & 0\end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of A is $r = 2$.

The matrix A is a 3×3 matrix, which means there are $m = 3$ rows and $n = 3$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$	$r = 2$
Null space, $N(A)$	$n - r = 3 - 2 = 1$
Row space, $C(A^T)$	$r = 2$
Left null space, $N(A^T)$	$m - r = 3 - 2 = 1$

Topic: Orthogonal complements of the fundamental subspaces

Question: For the matrix A, which of these dimensions of the four fundamental subspaces is incorrect?

$$A = \begin{bmatrix} 2 & -3 & 6 & -5 & -6 \\ 4 & -5 & 12 & -11 & -14 \\ 2 & -2 & 6 & -6 & -8 \end{bmatrix}$$

Answer choices:

A Dim$(C(A)) = 2$

B Dim$(N(A)) = 3$

C Dim$(C(A^T)) = 2$

D Dim$(N(A^T)) = 2$

Solution: D

Put A into reduced row-echelon form.

$$\begin{bmatrix}2&-3&6&-5&-6\\4&-5&12&-11&-14\\2&-2&6&-6&-8\end{bmatrix}\to\begin{bmatrix}1&-\frac{3}{2}&3&-\frac{5}{2}&-3\\4&-5&12&-11&-14\\2&-2&6&-6&-8\end{bmatrix}$$

$$\begin{bmatrix}1&-\frac{3}{2}&3&-\frac{5}{2}&-3\\0&1&0&-1&-2\\2&-2&6&-6&-8\end{bmatrix}\to\begin{bmatrix}1&-\frac{3}{2}&3&-\frac{5}{2}&-3\\0&1&0&-1&-2\\0&1&0&-1&-2\end{bmatrix}$$

$$\begin{bmatrix}1&-\frac{3}{2}&3&-\frac{5}{2}&-3\\0&1&0&-1&-2\\0&0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&3&-4&-6\\0&1&0&-1&-2\\0&0&0&0&0\end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of A is $r=2$.

The matrix A is a 3×5 matrix, which means there are $m=3$ rows and $n=5$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$	$r=2$
Null space, $N(A)$	$n-r=5-2=3$
Row space, $C(A^T)$	$r=2$
Left null space, $N(A^T)$	$m-r=3-2=1$

Topic: Projection onto the subspace

Question: If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}-2\\0\\-2\end{bmatrix}, \begin{bmatrix}0\\4\\2\end{bmatrix}\right)$$

Answer choices:

A $\quad \text{Proj}_V\vec{x} = \frac{1}{9}\begin{bmatrix}5 & -2 & 4\\-2 & 8 & 2\\4 & 2 & 5\end{bmatrix}\vec{x}$

B $\quad \text{Proj}_V\vec{x} = \begin{bmatrix}-4 & -2 & -5\\-2 & 8 & 2\\-5 & 2 & -4\end{bmatrix}\vec{x}$

C $\quad \text{Proj}_V\vec{x} = \begin{bmatrix}5 & -2 & 4\\-2 & 8 & 2\\4 & 2 & 5\end{bmatrix}\vec{x}$

D $\quad \text{Proj}_V\vec{x} = \frac{1}{9}\begin{bmatrix}-4 & -2 & -5\\-2 & 8 & 2\\-5 & 2 & -4\end{bmatrix}\vec{x}$

Solution: A

Because the vectors that span V are linearly independent, the matrix A of the basis vectors that define V is

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix}$$

The transpose A^T is

$$A^T = \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix}$$

Find A^TA.

$$A^TA = \begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -2(-2)+0(0)-2(-2) & -2(0)+0(4)-2(2) \\ 0(-2)+4(0)+2(-2) & 0(0)+4(4)+2(2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4+0+4 & 0+0-4 \\ 0+0-4 & 0+16+4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 8 & -4 \\ -4 & 20 \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_2] = \begin{bmatrix} 8 & -4 & | & 1 & 0 \\ -4 & 20 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{1}{8} & 0 \\ -4 & 20 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{1}{8} & 0 \\ 0 & 18 & | & \frac{1}{2} & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{1}{8} & 0 \\ 0 & 1 & | & \frac{1}{36} & \frac{1}{18} \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{5}{36} & \frac{1}{36} \\ 0 & 1 & | & \frac{1}{36} & \frac{1}{18} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{5}{36} & \frac{1}{36} \\ \frac{1}{36} & \frac{1}{18} \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{36}\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$$

Now the projection of $\vec{x}$ onto the subspace V will be

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -2 & 0 \\ 0 & 4 \\ -2 & 2 \end{bmatrix}\frac{1}{36}\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 & 0 & -2 \\ 0 & 4 & 2 \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\frac{1}{36}\begin{bmatrix}-2&0\\0&4\\-2&2\end{bmatrix}\begin{bmatrix}5&1\\1&2\end{bmatrix}\begin{bmatrix}-2&0&-2\\0&4&2\end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x}=\frac{1}{36}\begin{bmatrix}-2&0\\0&4\\-2&2\end{bmatrix}\begin{bmatrix}5(-2)+1(0)&5(0)+1(4)&5(-2)+1(2)\\1(-2)+2(0)&1(0)+2(4)&1(-2)+2(2)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\frac{1}{36}\begin{bmatrix}-2&0\\0&4\\-2&2\end{bmatrix}\begin{bmatrix}-10&4&-8\\-2&8&2\end{bmatrix}\vec{x}$$

Next, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x}=\frac{1}{36}\begin{bmatrix}-2(-10)+0(-2)&-2(4)+0(8)&-2(-8)+0(2)\\0(-10)+4(-2)&0(4)+4(8)&0(-8)+4(2)\\-2(-10)+2(-2)&-2(4)+2(8)&-2(-8)+2(2)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\frac{1}{36}\begin{bmatrix}20&-8&16\\-8&32&8\\16&8&20\end{bmatrix}\vec{x}$$

To simplify the matrix, factor out a 4.

$$\text{Proj}_V\vec{x}=\frac{4}{36}\begin{bmatrix}5&-2&4\\-2&8&2\\4&2&5\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\frac{1}{9}\begin{bmatrix}5&-2&4\\-2&8&2\\4&2&5\end{bmatrix}\vec{x}$$

Topic: Projection onto the subspace

Question: If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{3}\begin{bmatrix}1\\0\\-1\\2\end{bmatrix} \text{ and } \vec{x}_2 = \frac{1}{3}\begin{bmatrix}0\\1\\1\\-1\end{bmatrix}$$

Answer choices:

A $\quad \text{Proj}_S\vec{x} = \frac{1}{27}\begin{bmatrix}1 & 1 & 0 & 1\\1 & 2 & 1 & 0\\0 & 1 & 1 & -1\\1 & 0 & -1 & 2\end{bmatrix}\vec{x}$

B $\quad \text{Proj}_S\vec{x} = \frac{1}{9}\begin{bmatrix}1 & 1 & 0 & 1\\1 & 2 & 1 & 0\\0 & 1 & 1 & -1\\1 & 0 & -1 & 2\end{bmatrix}\vec{x}$

C $\quad \text{Proj}_S\vec{x} = \frac{1}{3}\begin{bmatrix}1 & 1 & 0 & 1\\1 & 2 & 1 & 0\\0 & 1 & 1 & -1\\1 & 0 & -1 & 2\end{bmatrix}\vec{x}$

D $\quad \text{Proj}_S\vec{x} = \begin{bmatrix}1 & 1 & 0 & 1\\1 & 2 & 1 & 0\\0 & 1 & 1 & -1\\1 & 0 & -1 & 2\end{bmatrix}\vec{x}$

Solution: C

Because the vectors that span S are linearly independent, the matrix A of the basis vectors that define S is

$$A = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}$$

The transpose A^T is

$$A^T = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}$$

Find A^TA.

$$A^TA = \frac{1}{3}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}$$

$$A^TA = \frac{1}{9}\begin{bmatrix} 1(1)+0(0)-1(-1)+2(2) & 1(0)+0(1)-1(1)+2(-1) \\ 0(1)+1(0)+1(-1)-1(2) & 0(0)+1(1)+1(1)-1(-1) \end{bmatrix}$$

$$A^TA = \frac{1}{9}\begin{bmatrix} 1+0+1+4 & 0+0-1-2 \\ 0+0-1-2 & 0+1+1+1 \end{bmatrix}$$

$$A^TA = \frac{1}{9}\begin{bmatrix} 6 & -3 \\ -3 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} \frac{6}{9} & -\frac{3}{9} \\ -\frac{3}{9} & \frac{3}{9} \end{bmatrix}$$

$$A^TA = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA\ |\ I_2] = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & | & 1 & 0 \\ -\frac{1}{3} & \frac{1}{3} & | & 0 & 1 \end{bmatrix}$$

$$[A^TA\ |\ I_2] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{3}{2} & 0 \\ -\frac{1}{3} & \frac{1}{3} & | & 0 & 1 \end{bmatrix}$$

$$[A^TA\ |\ I_2] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{3}{2} & 0 \\ 0 & \frac{1}{6} & | & \frac{1}{2} & 1 \end{bmatrix}$$

$$[A^TA\ |\ I_2] = \begin{bmatrix} 1 & -\frac{1}{2} & | & \frac{3}{2} & 0 \\ 0 & 1 & | & 3 & 6 \end{bmatrix}$$

$$[A^TA\ |\ I_2] = \begin{bmatrix} 1 & 0 & | & 3 & 3 \\ 0 & 1 & | & 3 & 6 \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}$$

The projection of $\vec{x}$ onto the subspace S will be

$$\text{Proj}_S\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{3}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{9}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 3 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 1 & -1 \end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_S\vec{x} = \frac{1}{9}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3(1)+3(0) & 3(0)+3(1) & 3(-1)+3(1) & 3(2)+3(-1) \\ 3(1)+6(0) & 3(0)+6(1) & 3(-1)+6(1) & 3(2)+6(-1) \end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{9}\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 0 & 3 \\ 3 & 6 & 3 & 0 \end{bmatrix}\vec{x}$$

Now, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_S\vec{x} = \frac{1}{9}\begin{bmatrix} 1(3)+0(3) & 1(3)+0(6) & 1(0)+0(3) & 1(3)+0(0) \\ 0(3)+1(3) & 0(3)+1(6) & 0(0)+1(3) & 0(3)+1(0) \\ -1(3)+1(3) & -1(3)+1(6) & -1(0)+1(3) & -1(3)+1(0) \\ 2(3)-1(3) & 2(3)-1(6) & 2(0)-1(3) & 2(3)-1(0) \end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{9}\begin{bmatrix} 3 & 3 & 0 & 3 \\ 3 & 6 & 3 & 0 \\ 0 & 3 & 3 & -3 \\ 3 & 0 & -3 & 6 \end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{3}{9}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{3}\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & -1 \\ 1 & 0 & -1 & 2 \end{bmatrix}\vec{x}$$

Topic: Projection onto the subspace

Question: If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}0\\2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-2\end{bmatrix}\right)$$

Answer choices:

A $\quad \text{Proj}_V\vec{x} = \begin{bmatrix}9 & -7 & -12\\-7 & 26 & 2\\-9 & -4 & 18\end{bmatrix}\vec{x}$

B $\quad \text{Proj}_V\vec{x} = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\vec{x}$

C $\quad \text{Proj}_V\vec{x} = \begin{bmatrix}24 & 0 & -16\\-8 & 16 & 16\\0 & 0 & 16\end{bmatrix}\vec{x}$

D $\quad \text{Proj}_V\vec{x} = \frac{1}{8}\begin{bmatrix}3 & 0 & -2\\-1 & 2 & 2\\0 & 0 & 2\end{bmatrix}\vec{x}$

Solution: B

Because the vectors that span V are linearly independent, the matrix A of the basis vectors that define V is

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

The transpose A^T is

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$

Find A^TA.

$$A^TA = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 0 \\ 1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) - 1(-1) + 0(0) & 1(0) - 1(2) + 0(0) & 1(1) - 1(0) + 0(-2) \\ 0(1) + 2(-1) + 0(0) & 0(0) + 2(2) + 0(0) & 0(1) + 2(0) + 0(-2) \\ 1(1) + 0(-1) - 2(0) & 1(0) + 0(2) - 2(0) & 1(1) + 0(0) - 2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 1 + 0 & 0 - 2 + 0 & 1 + 0 + 0 \\ 0 - 2 + 0 & 0 + 4 + 0 & 0 + 0 + 0 \\ 1 + 0 + 0 & 0 + 0 + 0 & 1 + 0 + 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 4 & 0 \\ 1 & 0 & 5 \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_3] = \begin{bmatrix} 2 & -2 & 1 & | & 1 & 0 & 0 \\ -2 & 4 & 0 & | & 0 & 1 & 0 \\ 1 & 0 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -1 & \frac{1}{2} & | & \frac{1}{2} & 0 & 0 \\ -2 & 4 & 0 & | & 0 & 1 & 0 \\ 1 & 0 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -1 & \frac{1}{2} & | & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & | & 1 & 1 & 0 \\ 1 & 0 & 5 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -1 & \frac{1}{2} & | & \frac{1}{2} & 0 & 0 \\ 0 & 2 & 1 & | & 1 & 1 & 0 \\ 0 & 1 & \frac{9}{2} & | & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -1 & \frac{1}{2} & | & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{9}{2} & | & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -1 & \frac{1}{2} & | & \frac{1}{2} & 0 & 0 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 4 & | & -1 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 1 & | & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 4 & | & -1 & -\frac{1}{2} & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 1 & | & 1 & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & | & -\frac{1}{4} & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 1 & | & 1 & \frac{1}{2} & 0 \\ 0 & 1 & 0 & | & \frac{5}{8} & \frac{9}{16} & -\frac{1}{8} \\ 0 & 0 & 1 & | & -\frac{1}{4} & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 0 & | & \frac{5}{4} & \frac{5}{8} & -\frac{1}{4} \\ 0 & 1 & 0 & | & \frac{5}{8} & \frac{9}{16} & -\frac{1}{8} \\ 0 & 0 & 1 & | & -\frac{1}{4} & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{5}{4} & \frac{5}{8} & -\frac{1}{4} \\ \frac{5}{8} & \frac{9}{16} & -\frac{1}{8} \\ -\frac{1}{4} & -\frac{1}{8} & \frac{1}{4} \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{16}\begin{bmatrix}20 & 10 & -4\\10 & 9 & -2\\-4 & -2 & 4\end{bmatrix}$$

Then the projection of $\vec{x}$ onto the subspace V will be

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix}1 & 0 & 1\\-1 & 2 & 0\\0 & 0 & -2\end{bmatrix}\frac{1}{16}\begin{bmatrix}20 & 10 & -4\\10 & 9 & -2\\-4 & -2 & 4\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\0 & 2 & 0\\1 & 0 & -2\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{16}\begin{bmatrix}1 & 0 & 1\\-1 & 2 & 0\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}20 & 10 & -4\\10 & 9 & -2\\-4 & -2 & 4\end{bmatrix}\begin{bmatrix}1 & -1 & 0\\0 & 2 & 0\\1 & 0 & -2\end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \frac{1}{16}\begin{bmatrix}1 & 0 & 1\\-1 & 2 & 0\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}20(1)+10(0)-4(1) & 20(-1)+10(2)-4(0) & 20(0)+10(0)-4(-2)\\10(1)+9(0)-2(1) & 10(-1)+9(2)-2(0) & 10(0)+9(0)-2(-2)\\-4(1)-2(0)+4(1) & -4(-1)-2(2)+4(0) & -4(0)-2(0)+4(-2)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{16}\begin{bmatrix}1 & 0 & 1\\-1 & 2 & 0\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}20+0-4 & -20+20-0 & 0+0+8\\10+0-2 & -10+18-0 & 0+0+4\\-4-0+4 & 4-4+0 & 0-0-8\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{16}\begin{bmatrix}1 & 0 & 1\\-1 & 2 & 0\\0 & 0 & -2\end{bmatrix}\begin{bmatrix}16 & 0 & 8\\8 & 8 & 4\\0 & 0 & -8\end{bmatrix}\vec{x}$$

Next, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \frac{1}{16}\begin{bmatrix}1(16)+0(8)+1(0) & 1(0)+0(8)+1(0) & 1(8)+0(4)+1(-8)\\-1(16)+2(8)+0(0) & -1(0)+2(8)+0(0) & -1(8)+2(4)+0(-8)\\0(16)+0(8)-2(0) & 0(0)+0(8)-2(0) & 0(8)+0(4)-2(-8)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\frac{1}{16}\begin{bmatrix}16+0+0 & 0+0+0 & 8+0-8\\ -16+16+0 & 0+16+0 & -8+8+0\\ 0+0-0 & 0+0-0 & 0+0+16\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\frac{1}{16}\begin{bmatrix}16 & 0 & 0\\ 0 & 16 & 0\\ 0 & 0 & 16\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\frac{16}{16}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=1\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\vec{x}$$

Topic: Least squares solution

Question: Find the least squares solution to the system.

$$x - 2y = 5$$

$$3x + y = -6$$

$$-x - 2y = -2$$

Answer choices:

A $\quad \vec{x}^* = \left(-\frac{9}{10}, -\frac{11}{10}\right)$

B $\quad \vec{x}^* = \left(\frac{7}{10}, \frac{11}{10}\right)$

C $\quad \vec{x}^* = (-11, -12)$

D $\quad \vec{x}^* = \left(-\frac{7}{10}, -\frac{11}{10}\right)$

Solution: D

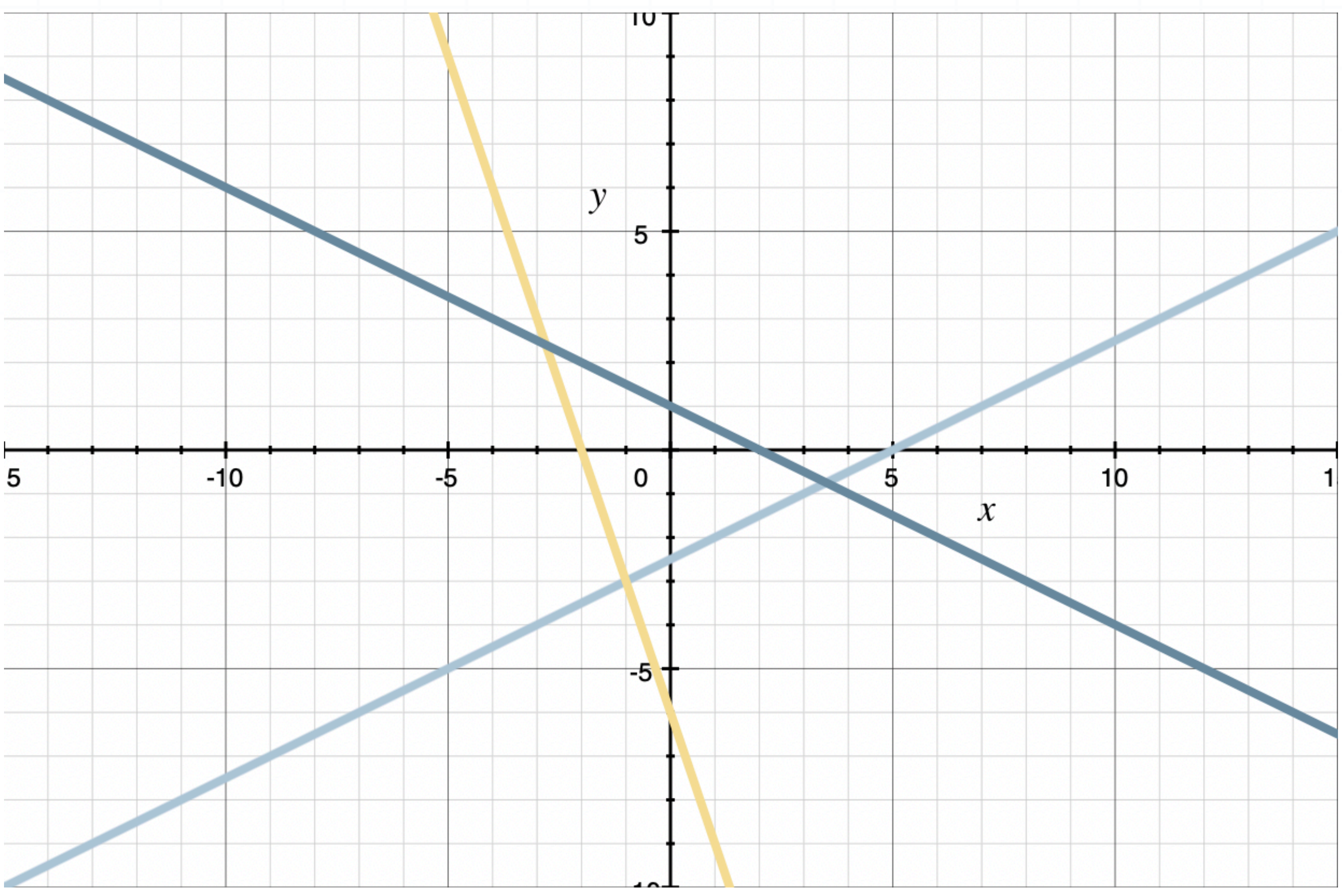
Put each line into slope-intercept form,

$$y = \frac{1}{2}x - \frac{5}{2}$$

$$y = -3x - 6$$

$$y = -\frac{1}{2}x + 1$$

then graph all three in the same plane.



While there are two points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -6 \\ -2 \end{bmatrix}$$

In other words, $\overrightarrow{b} = (5, -6, -2)$ is not in the column space of the coefficient matrix A, and there's no vector $\overrightarrow{x} = (x, y)$ you can find that makes the $A\overrightarrow{x} = \overrightarrow{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & -2 \end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -1 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) + 3(3) - 1(-1) & 1(-2) + 3(1) - 1(-2) \\ -2(1) + 1(3) - 2(-1) & -2(-2) + 1(1) - 2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 9 + 1 & -2 + 3 + 2 \\ -2 + 3 + 2 & 4 + 1 + 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 11 & 3 \\ 3 & 9 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} 1 & 3 & -1 \\ -2 & 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -6 \\ -2 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1(5) + 3(-6) - 1(-2) \\ -2(5) + 1(-6) - 2(-2) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 5 - 18 + 2 \\ -10 - 6 + 4 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -11 \\ -12 \end{bmatrix}$$

Then we get

$$A^TA\vec{x}^* = A^T\vec{b}$$

$$\begin{bmatrix} 11 & 3 \\ 3 & 9 \end{bmatrix} \vec{x}^* = \begin{bmatrix} -11 \\ -12 \end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 11 & 3 & | & -11 \\ 3 & 9 & | & -12 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{11} & | & -1 \\ 3 & 9 & | & -12 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{11} & | & -1 \\ 0 & \frac{90}{11} & | & -9 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{11} & | & -1 \\ 0 & 1 & | & -\frac{11}{10} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -\frac{7}{10} \\ 0 & 1 & | & -\frac{11}{10} \end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(-\frac{7}{10}, -\frac{11}{10}\right)$$

Topic: Least squares solution

Question: Find the least squares solution to the system.

$$3x - 2y = -6$$

$$x - 5y = -5$$

$$x + y = 4$$

Answer choices:

A $\quad \vec{x}^* = \left(-\dfrac{16}{23}, \dfrac{261}{230}\right)$

B $\quad \vec{x}^* = \left(\dfrac{261}{230}, -\dfrac{16}{23}\right)$

C $\quad \vec{x}^* = \left(\dfrac{16}{23}, -\dfrac{261}{230}\right)$

D $\quad \vec{x}^* = \left(-\dfrac{261}{230}, \dfrac{16}{23}\right)$

Solution: A

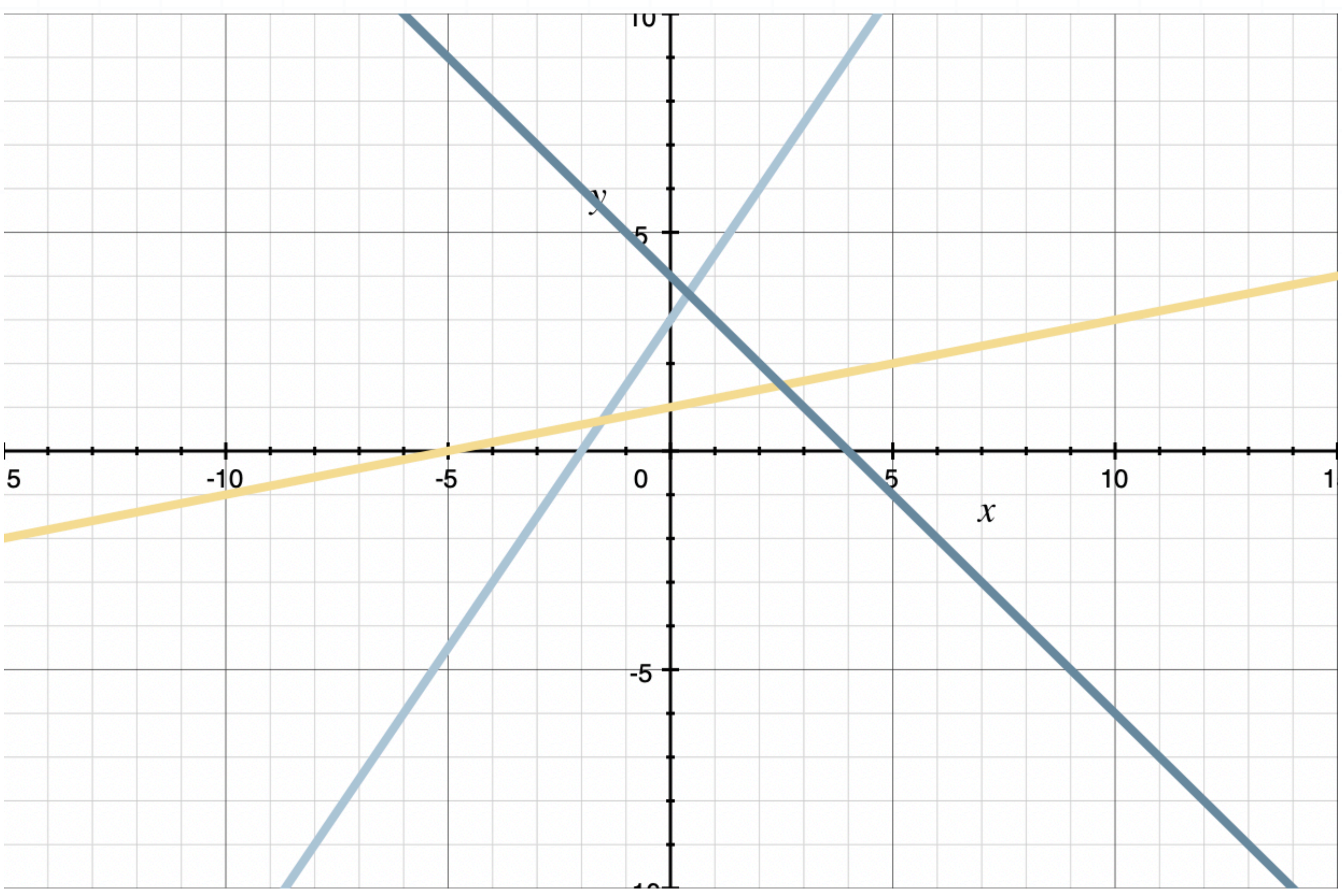
Put each line into slope-intercept form,

$$y = \frac{3}{2}x + 3$$

$$y = \frac{1}{5}x + 1$$

$$y = -x + 4$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 3 & -2 \\ 1 & -5 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6 \\ -5 \\ 4 \end{bmatrix}$$

In other words, $\vec{b} = (-6, -5, 4)$ is not in the column space of the coefficient matrix A, and there's no vector $\vec{x} = (x, y)$ you can find that makes the $A\vec{x} = \vec{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A = \begin{bmatrix} 3 & -2 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -5 & 1 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -5 & 1 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 1 & -5 \\ 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3(3) + 1(1) + 1(1) & 3(-2) + 1(-5) + 1(1) \\ -2(3) - 5(1) + 1(1) & -2(-2) - 5(-5) + 1(1) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 9 + 1 + 1 & -6 - 5 + 1 \\ -6 - 5 + 1 & 4 + 25 + 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 11 & -10 \\ -10 & 30 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} 3 & 1 & 1 \\ -2 & -5 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ -5 \\ 4 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 3(-6) + 1(-5) + 1(4) \\ -2(-6) - 5(-5) + 1(4) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -18 - 5 + 4 \\ 12 + 25 + 4 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -19 \\ 41 \end{bmatrix}$$

Then we get

$$A^TA\vec{x}* = A^T\vec{b}$$

$$\begin{bmatrix} 11 & -10 \\ -10 & 30 \end{bmatrix} \vec{x}* = \begin{bmatrix} -19 \\ 41 \end{bmatrix}$$

Then to find $\vec{x}*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 11 & -10 & | & -19 \\ -10 & 30 & | & 41 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{10}{11} & | & -\frac{19}{11} \\ -10 & 30 & | & 41 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{10}{11} & | & -\frac{19}{11} \\ 0 & \frac{230}{11} & | & \frac{261}{11} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{10}{11} & | & -\frac{19}{11} \\ 0 & 1 & | & \frac{261}{230} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -\frac{16}{23} \\ 0 & 1 & | & \frac{261}{230} \end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(-\frac{16}{23}, \frac{261}{230}\right)$$

Topic: Least squares solution

Question: Find the least squares solution to the system.

$$x + 2y = -4$$

$$x - y = 3$$

$$y = 2$$

Answer choices:

A $\quad \vec{x}^* = \left(\frac{3}{11}, -\frac{17}{11}\right)$

B $\quad \vec{x}^* = \left(-\frac{3}{11}, \frac{17}{11}\right)$

C $\quad \vec{x}^* = \left(\frac{17}{11}, -\frac{3}{11}\right)$

D $\quad \vec{x}^* = \left(-\frac{17}{11}, \frac{3}{11}\right)$

Solution: A

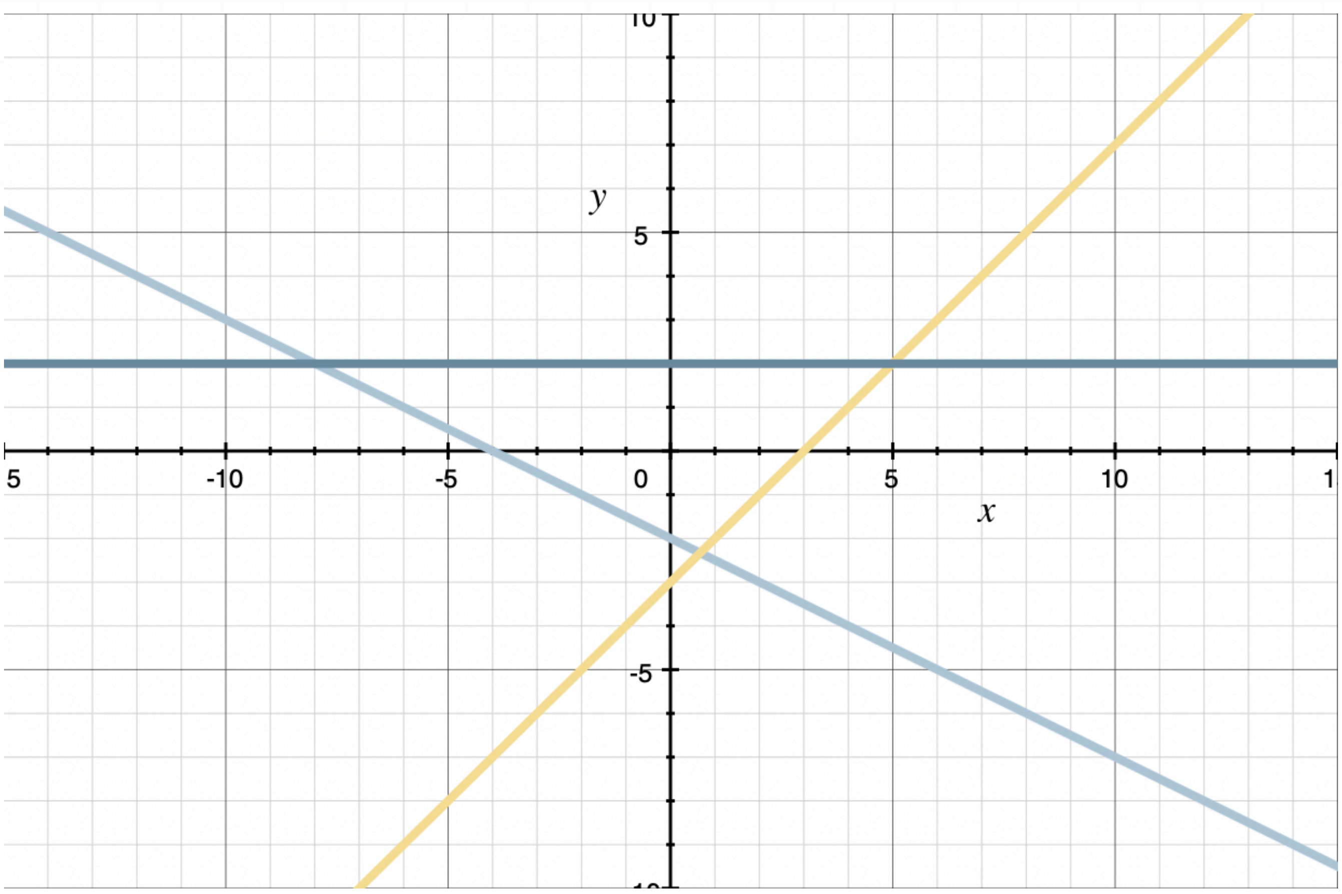
Put each line into slope-intercept form,

$$y = -\frac{1}{2}x - 2$$

$$y = x - 3$$

$$y = 2$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} 1 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ 2 \end{bmatrix}$$

In other words, $\vec{b}=(-4,3,2)$ is not in the column space of the coefficient matrix A, and there's no vector $\vec{x}=(x,y)$ you can find that makes the $A\vec{x}=\vec{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A=\begin{bmatrix}1 & 2\\ 1 & -1\\ 0 & 1\end{bmatrix}$$

Now we'll find A^T.

$$A^T=\begin{bmatrix}1 & 1 & 0\\ 2 & -1 & 1\end{bmatrix}$$

Then A^TA is

$$A^TA=\begin{bmatrix}1 & 1 & 0\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & -1\\ 0 & 1\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1(1)+1(1)+0(0) & 1(2)+1(-1)+0(1)\\ 2(1)-1(1)+1(0) & 2(2)-1(-1)+1(1)\end{bmatrix}$$

$$A^TA=\begin{bmatrix}1+1+0 & 2-1+0\\ 2-1+0 & 4+1+1\end{bmatrix}$$

$$A^TA=\begin{bmatrix}2 & 1\\ 1 & 6\end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix}1 & 1 & 0\\ 2 & -1 & 1\end{bmatrix}\begin{bmatrix}-4\\ 3\\ 2\end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix}1(-4)+1(3)+0(2)\\ 2(-4)-1(3)+1(2)\end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix}-4+3+0\\ -8-3+2\end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix}-1\\ -9\end{bmatrix}$$

Then we get

$$A^TA\vec{x}^* = A^T\vec{b}$$

$$\begin{bmatrix}2 & 1\\ 1 & 6\end{bmatrix}\vec{x}^* = \begin{bmatrix}-1\\ -9\end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}2 & 1 & | & -1\\ 1 & 6 & | & -9\end{bmatrix} \to \begin{bmatrix}1 & 6 & | & -9\\ 2 & 1 & | & -1\end{bmatrix} \to \begin{bmatrix}1 & 6 & | & -9\\ 0 & -11 & | & 17\end{bmatrix}$$

$$\begin{bmatrix}1 & 6 & | & -9\\ 0 & 1 & | & -\frac{17}{11}\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & \frac{3}{11}\\ 0 & 1 & | & -\frac{17}{11}\end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(\frac{3}{11}, -\frac{17}{11}\right)$$

Topic: Coordinates in a new basis

Question: The vectors $\overrightarrow{v} = (2,2,3)$, $\vec{s} = (-6,0,2)$, and $\overrightarrow{w} = (2,-2,-5)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\overrightarrow{x} = -2i + k$ into the alternate basis.

Answer choices:

A $\quad [\overrightarrow{x}]_B = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$

B $\quad [\overrightarrow{x}]_B = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

C $\quad [\overrightarrow{x}]_B = \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}$

D $\quad [\overrightarrow{x}]_B = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$

Solution: A

The vector $\vec{x}=(-2,0,1)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v}=(2,2,3)$, $\vec{s}=(-6,0,2)$, and $\vec{w}=(2,-2,-5)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B=\vec{x}$$

$$\begin{bmatrix}2&-6&2\\2&0&-2\\3&2&-5\end{bmatrix}[\vec{x}]_B=\begin{bmatrix}-2\\0\\1\end{bmatrix}$$

To find the representation of $\vec{x}$ in the alternate basis, $[\vec{x}]_B$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}2&-6&2&|&-2\\2&0&-2&|&0\\3&2&-5&|&1\end{bmatrix}\to\begin{bmatrix}1&-3&1&|&-1\\2&0&-2&|&0\\3&2&-5&|&1\end{bmatrix}\to\begin{bmatrix}1&-3&1&|&-1\\0&6&-4&|&2\\3&2&-5&|&1\end{bmatrix}$$

$$\begin{bmatrix}1&-3&1&|&-1\\0&6&-4&|&2\\0&11&-8&|&4\end{bmatrix}\to\begin{bmatrix}1&-3&1&|&-1\\0&1&-\frac{2}{3}&|&\frac{1}{3}\\0&11&-8&|&4\end{bmatrix}\to\begin{bmatrix}1&-3&1&|&-1\\0&1&-\frac{2}{3}&|&\frac{1}{3}\\0&0&-\frac{2}{3}&|&\frac{1}{3}\end{bmatrix}$$

$$\begin{bmatrix}1&0&-1&|&0\\0&1&-\frac{2}{3}&|&\frac{1}{3}\\0&0&-\frac{2}{3}&|&\frac{1}{3}\end{bmatrix}\to\begin{bmatrix}1&0&-1&|&0\\0&1&-\frac{2}{3}&|&\frac{1}{3}\\0&0&1&|&-\frac{1}{2}\end{bmatrix}\to\begin{bmatrix}1&0&-1&|&0\\0&1&0&|&0\\0&0&1&|&-\frac{1}{2}\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{2} \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & -\frac{1}{2} \end{bmatrix}$$

So $\overrightarrow{x} = (-2,0,1)$, expressed in the alternate basis, is

$$[\overrightarrow{x}]_B = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}$$

Topic: Coordinates in a new basis

Question: The vectors $\vec{v}=(1,-4)$ and $\vec{w}=(-3,2)$ form an alternate basis for $\mathbb{R}^2$. Use them to transform $\vec{x}=-i-6j$ into the alternate basis.

Answer choices:

A $\quad [\vec{x}]_B=\begin{bmatrix}-2\\-1\end{bmatrix}$

B $\quad [\vec{x}]_B=\begin{bmatrix}2\\1\end{bmatrix}$

C $\quad [\vec{x}]_B=\begin{bmatrix}-1\\1\end{bmatrix}$

D $\quad [\vec{x}]_B=\begin{bmatrix}1\\-1\end{bmatrix}$

Solution: B

The vector $\vec{x}=(-1,-6)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v}=(1,-4)$ and $\vec{w}=(-3,2)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B=\vec{x}$$

$$\begin{bmatrix}1 & -3\\ -4 & 2\end{bmatrix}[\vec{x}]_B=\begin{bmatrix}-1\\ -6\end{bmatrix}$$

To find the representation of $\vec{x}$ in the alternate basis, $[\vec{x}]_B$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}1 & -3 & | & -1\\ -4 & 2 & | & -6\end{bmatrix}\to\begin{bmatrix}1 & -3 & | & -1\\ 0 & -10 & | & -10\end{bmatrix}$$

$$\begin{bmatrix}1 & -3 & | & -1\\ 0 & 1 & | & 1\end{bmatrix}\to\begin{bmatrix}1 & 0 & | & 2\\ 0 & 1 & | & 1\end{bmatrix}$$

So $\vec{x}=(-1,-6)$, expressed in the alternate basis, is

$$[\vec{x}]_B=\begin{bmatrix}2\\ 1\end{bmatrix}$$

Topic: Coordinates in a new basis

Question: The vectors $\vec{v} = (1, -5)$ and $\vec{w} = (-2,0)$ form an alternate basis for $\mathbb{R}^2$. Use them, and an inverse matrix, to transform $\vec{x} = 3i - 5j$ into the alternate basis.

Answer choices:

A $[\vec{x}]_B = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

B $[\vec{x}]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

C $[\vec{x}]_B = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

D $[\vec{x}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Solution: D

The vector $\vec{x} = (3, -5)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v} = (1, -5)$ and $\vec{w} = (-2,0)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B = \vec{x}$$

$$\begin{bmatrix}1 & -2\\ -5 & 0\end{bmatrix}[\vec{x}]_B = \begin{bmatrix}3\\ -5\end{bmatrix}$$

Find A^{-1} from A.

$$[A \mid I] = \begin{bmatrix}1 & -2 & | & 1 & 0\\ -5 & 0 & | & 0 & 1\end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix}1 & -2 & | & 1 & 0\\ 0 & -10 & | & 5 & 1\end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix}1 & -2 & | & 1 & 0\\ 0 & 1 & | & -\frac{1}{2} & -\frac{1}{10}\end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix}1 & 0 & | & 0 & -\frac{1}{5}\\ 0 & 1 & | & -\frac{1}{2} & -\frac{1}{10}\end{bmatrix}$$

So the inverse matrix is

$$A^{-1} = \begin{bmatrix} 0 & -\frac{1}{5} \\ -\frac{1}{2} & -\frac{1}{10} \end{bmatrix}$$

Now to find the representation of $\vec{x} = (3, -5)$ in the alternate basis, we simply multiply the inverse matrix by the vector.

$$[\vec{x}]_B = A^{-1}\vec{x}$$

$$[\vec{x}]_B = \begin{bmatrix} 0 & -\frac{1}{5} \\ -\frac{1}{2} & -\frac{1}{10} \end{bmatrix} \begin{bmatrix} 3 \\ -5 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 0(3) - \frac{1}{5}(-5) \\ -\frac{1}{2}(3) - \frac{1}{10}(-5) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 0 + 1 \\ -\frac{3}{2} + \frac{1}{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Topic: Transformation matrix for a basis

Question: Use the transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ to transform $[\vec{x}]_B = (5,4,-2)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix} \right)$$

Answer choices:

A $\quad [T(\vec{x})]_B = \begin{bmatrix} -20 \\ 9 \\ 4 \end{bmatrix}$

B $\quad [T(\vec{x})]_B = \begin{bmatrix} -15 \\ -28 \\ -2 \end{bmatrix}$

C $\quad [T(\vec{x})]_B = \begin{bmatrix} -15 \\ -36 \\ 12 \end{bmatrix}$

D $\quad [T(\vec{x})]_B = \begin{bmatrix} -20 \\ 78 \\ -93 \end{bmatrix}$

Solution: C

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C that transforms vectors from the standard basis into vectors in the alternate basis B is made of the column vectors that span B, $\vec{v}_1 = (1, -1, 1)$, $\vec{v}_2 = (0, 1, -1)$, and $\vec{v}_3 = (2, 1, -2)$, so

$$C = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ -1 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & -1 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 1 & -1 & -2 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & -1 & -4 & | & -1 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & 0 & -1 & | & 0 & 1 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 3 & | & 1 & 1 & 0 \\ 0 & 0 & 1 & | & 0 & -1 & -1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & 0 & | & 1 & 4 & 3 \\ 0 & 0 & 1 & | & 0 & -1 & -1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 0 & | & 1 & 2 & 2 \\ 0 & 1 & 0 & | & 1 & 4 & 3 \\ 0 & 0 & 1 & | & 0 & -1 & -1 \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} -2 & -2 & 1 \\ 1 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 1 & -1 & -2 \end{bmatrix}$$

$$M = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 4 & 3 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} -2(1) - 2(-1) + 1(1) & -2(0) - 2(1) + 1(-1) & -2(2) - 2(1) + 1(-2) \\ 1(1) + 0(-1) - 2(1) & 1(0) + 0(1) - 2(-1) & 1(2) + 0(1) - 2(-2) \\ 0(1) + 1(-1) + 0(1) & 0(0) + 1(1) + 0(-1) & 0(2) + 1(1) + 0(-2) \end{bmatrix}$$

$$M=\begin{bmatrix}1&2&2\\1&4&3\\0&-1&-1\end{bmatrix}\begin{bmatrix}-2+2+1&0-2-1&-4-2-2\\1+0-2&0+0+2&2+0+4\\0-1+0&0+1+0&0+1+0\end{bmatrix}$$

$$M=\begin{bmatrix}1&2&2\\1&4&3\\0&-1&-1\end{bmatrix}\begin{bmatrix}1&-3&-8\\-1&2&6\\-1&1&1\end{bmatrix}$$

$$M=\begin{bmatrix}1(1)+2(-1)+2(-1)&1(-3)+2(2)+2(1)&1(-8)+2(6)+2(1)\\1(1)+4(-1)+3(-1)&1(-3)+4(2)+3(1)&1(-8)+4(6)+3(1)\\0(1)-1(-1)-1(-1)&0(-3)-1(2)-1(1)&0(-8)-1(6)-1(1)\end{bmatrix}$$

$$M=\begin{bmatrix}1-2-2&-3+4+2&-8+12+2\\1-4-3&-3+8+3&-8+24+3\\0+1+1&0-2-1&0-6-1\end{bmatrix}$$

$$M=\begin{bmatrix}-3&3&6\\-6&8&19\\2&-3&-7\end{bmatrix}$$

We’ve been asked to transform $[\vec{x}]_B=(5,4,-2)$, so we’ll multiply M by this vector.

$$[T(\vec{x})]_B=M[\vec{x}]_B$$

$$[T(\vec{x})]_B=\begin{bmatrix}-3&3&6\\-6&8&19\\2&-3&-7\end{bmatrix}\begin{bmatrix}5\\4\\-2\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-3(5)+3(4)+6(-2)\\-6(5)+8(4)+19(-2)\\2(5)-3(4)-7(-2)\end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -15+12-12 \\ -30+32-38 \\ 10-12+14 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -15 \\ -36 \\ 12 \end{bmatrix}$$

Topic: Transformation matrix for a basis

Question: Use the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ to transform $\vec{x} = (2, -2)$ to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -6 \\ -2 \end{bmatrix} \right)$$

Answer choices:

A $\quad [T(\vec{x})]_B = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$

B $\quad [T(\vec{x})]_B = \begin{bmatrix} -8 \\ -3 \end{bmatrix}$

C $\quad [T(\vec{x})]_B = \begin{bmatrix} -12 \\ -2 \end{bmatrix}$

D $\quad [T(\vec{x})]_B = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$

Solution: D

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (2,1)$ and $\vec{v}_2 = (-6, -2)$, so

$$C = \begin{bmatrix} 2 & -6 \\ 1 & -2 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} 2 & -6 & | & 1 & 0 \\ 1 & -2 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -3 & | & \frac{1}{2} & 0 \\ 1 & -2 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -3 & | & \frac{1}{2} & 0 \\ 0 & 1 & | & -\frac{1}{2} & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & -1 & 3 \\ 0 & 1 & | & -\frac{1}{2} & 1 \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix}$$

Find the transformation $T(\vec{x})$ where $T(\vec{x}) = A\vec{x}$.

$$T(\vec{x}) = \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix} \vec{x}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & -5 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2(2) - 5(-2) \\ 3(2) + 1(-2) \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -4 + 10 \\ 6 - 2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

Then $[T(\vec{x})]_B = C^{-1}T(\vec{x})$.

$$[T(\vec{x})]_B = \begin{bmatrix} -1 & 3 \\ -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 4 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -1(6) + 3(4) \\ -\frac{1}{2}(6) + 1(4) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -6 + 12 \\ -3 + 4 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

Topic: Transformation matrix for a basis

Question: Use the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (2,3)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -4 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 0 \end{bmatrix} \right)$$

Answer choices:

A $\quad [T(\vec{x})]_B = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$

B $\quad [T(\vec{x})]_B = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$

C $\quad [T(\vec{x})]_B = \begin{bmatrix} \frac{7}{4} \\ \frac{7}{2} \end{bmatrix}$

D $\quad [T(\vec{x})]_B = \begin{bmatrix} -1 \\ 12 \end{bmatrix}$

Solution: A

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (-4,4)$ and $\vec{v}_2 = (4,0)$, so

$$C = \begin{bmatrix} -4 & 4 \\ 4 & 0 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} -4 & 4 & | & 1 & 0 \\ 4 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -1 & | & -\frac{1}{4} & 0 \\ 4 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -1 & | & -\frac{1}{4} & 0 \\ 0 & 4 & | & 1 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -1 & | & -\frac{1}{4} & 0 \\ 0 & 1 & | & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & 0 & \frac{1}{4} \\ 0 & 1 & | & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -4 & 4 \\ 4 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -1(-4) + 3(4) & -1(4) + 3(0) \\ 2(-4) + 1(4) & 2(4) + 1(0) \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 4 + 12 & -4 + 0 \\ -8 + 4 & 8 + 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 16 & -4 \\ -4 & 8 \end{bmatrix}$$

$$M = \begin{bmatrix} 0(16) + \frac{1}{4}(-4) & 0(-4) + \frac{1}{4}(8) \\ \frac{1}{4}(16) + \frac{1}{4}(-4) & \frac{1}{4}(-4) + \frac{1}{4}(8) \end{bmatrix}$$

$$M=\begin{bmatrix}0-1 & 0+2\\ 4-1 & -1+2\end{bmatrix}$$

$$M=\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}$$

We've been asked to transform $[\vec{x}]_B=(2,3)$, so we'll multiply M by this vector.

$$[T(\vec{x})]_B=M[\vec{x}]_B$$

$$[T(\vec{x})]_B=\begin{bmatrix}-1 & 2\\ 3 & 1\end{bmatrix}\begin{bmatrix}2\\ 3\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-1(2)+2(3)\\ 3(2)+1(3)\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}-2+6\\ 6+3\end{bmatrix}$$

$$[T(\vec{x})]_B=\begin{bmatrix}4\\ 9\end{bmatrix}$$

Topic: Orthonormal bases

Question: Which of the vector sets is orthonormal?

$$\vec{v}_1 = \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{\sqrt{2}}\right)$$

$$\vec{v}_2 = \left(-\frac{1}{3}, -\frac{2}{3}, \frac{2}{3}\right)$$

$$\vec{v}_3 = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Answer choices:

A $\quad V = \{\vec{v}_1, \vec{v}_2\}$

B $\quad V = \{\vec{v}_1, \vec{v}_3\}$

C $\quad V = \{\vec{v}_2, \vec{v}_3\}$

D $\quad$ None of these

Solution: C

For the set to be orthonormal, each vector needs to have length 1.

$$||\vec{v}_1||^2=\vec{v}_1\cdot\vec{v}_1=\left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)$$

$$=\frac{1}{4}+\frac{1}{4}+\frac{1}{2}=1$$

$$||\vec{v}_2||^2=\vec{v}_2\cdot\vec{v}_2=\left(-\frac{1}{3}\right)\left(-\frac{1}{3}\right)+\left(-\frac{2}{3}\right)\left(-\frac{2}{3}\right)+\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)$$

$$=\frac{1}{9}+\frac{4}{9}+\frac{4}{9}=1$$

$$||\vec{v}_3||^2=\vec{v}_3\cdot\vec{v}_3=(0)(0)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$=0+\frac{1}{2}+\frac{1}{2}=1$$

Each vector has length 1, so now we need to check which of the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=\left(-\frac{1}{2}\right)\left(-\frac{1}{3}\right)+\left(\frac{1}{2}\right)\left(-\frac{2}{3}\right)+\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{2}{3}\right)$$

$$=\frac{1}{6}-\frac{2}{6}-\frac{2}{3\sqrt{2}}=-\frac{1}{6}-\frac{2}{3\sqrt{2}}$$

$$\vec{v}_1 \cdot \vec{v}_3 = \left(-\frac{1}{2}\right)(0) + \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= 0 + \frac{1}{2\sqrt{2}} - \frac{1}{2} = \frac{1}{2\sqrt{2}} - \frac{1}{2}$$

$$\vec{v}_2 \cdot \vec{v}_3 = \left(-\frac{1}{3}\right)(0) + \left(-\frac{2}{3}\right)\left(\frac{1}{\sqrt{2}}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{\sqrt{2}}\right)$$

$$= 0 - \frac{2}{3\sqrt{2}} + \frac{2}{3\sqrt{2}} = 0$$

So $V = \{\vec{v}_1, \vec{v}_2\}$ and $V = \{\vec{v}_1, \vec{v}_3\}$ are not orthonormal sets since their dot products are nonzero, but $V = \{\vec{v}_2, \vec{v}_3\}$ is an orthonormal set because its dot product is zero.

Topic: Orthonormal bases

Question: Convert $\vec{x} = (-12,6)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{5}{6} \\ -\frac{\sqrt{11}}{6} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{\sqrt{11}}{6} \\ \frac{5}{6} \end{bmatrix}$$

Answer choices:

A $\quad [\vec{x}]_B = \begin{bmatrix} -14 + \sqrt{11} \\ 5 + 2\sqrt{11} \end{bmatrix}$

B $\quad [\vec{x}]_B = \begin{bmatrix} -10 - \sqrt{11} \\ 5 - 2\sqrt{11} \end{bmatrix}$

C $\quad [\vec{x}]_B = \begin{bmatrix} -10 - 2\sqrt{11} \\ 5 - \sqrt{11} \end{bmatrix}$

D $\quad [\vec{x}]_B = \begin{bmatrix} -14 + \sqrt{11} \\ 5 - 2\sqrt{11} \end{bmatrix}$

Solution: B

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2=\left(\frac{5}{6}\right)^2+\left(-\frac{\sqrt{11}}{6}\right)^2=\frac{25}{36}+\frac{11}{36}=\frac{36}{36}=1$$

$$||\vec{v}_2||^2=\left(\frac{\sqrt{11}}{6}\right)^2+\left(\frac{5}{6}\right)^2=\frac{11}{36}+\frac{25}{36}=\frac{36}{36}=1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=\left(\frac{5}{6}\right)\left(\frac{\sqrt{11}}{6}\right)+\left(-\frac{\sqrt{11}}{6}\right)\left(\frac{5}{6}\right)$$

$$=\frac{5\sqrt{11}}{36}-\frac{5\sqrt{11}}{36}=0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x}=(-12,6)$ can be converted to the alternate basis B with dot products. In other words, instead of solving

$$\begin{bmatrix}\frac{5}{6} & \frac{\sqrt{11}}{6}\\ -\frac{\sqrt{11}}{6} & \frac{5}{6}\end{bmatrix}[\vec{x}]_B=\begin{bmatrix}-12\\ 6\end{bmatrix}$$

which would require us to put the augmented matrix into reduced row-echelon form, we can simply take dot products to get the value of $[\vec{x}]_B$.

$$[\vec{x}]_B = \begin{bmatrix} \frac{5}{6}(-12) - \frac{\sqrt{11}}{6}(6) \\ \frac{\sqrt{11}}{6}(-12) + \frac{5}{6}(6) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -10 - \sqrt{11} \\ -2\sqrt{11} + 5 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -10 - \sqrt{11} \\ 5 - 2\sqrt{11} \end{bmatrix}$$

Topic: Orthonormal bases

Question: Convert $\vec{x} = (\sqrt{66}, \sqrt{6}, \sqrt{11})$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{4}{\sqrt{66}} \\ -\frac{7}{\sqrt{66}} \\ \frac{1}{\sqrt{66}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} -\frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \\ -\frac{3}{\sqrt{11}} \end{bmatrix}$$

Answer choices:

A $\quad [\vec{x}]_B = \begin{bmatrix} \frac{4\sqrt{66} - 7\sqrt{6} + \sqrt{11}}{\sqrt{66}} \\ \frac{-2\sqrt{66} - \sqrt{6} + \sqrt{11}}{\sqrt{6}} \\ \frac{-\sqrt{66} - \sqrt{6} - 3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$

B $\quad [\vec{x}]_B = \begin{bmatrix} \frac{-4\sqrt{66} + 7\sqrt{6} - \sqrt{11}}{\sqrt{66}} \\ \frac{2\sqrt{66} + \sqrt{6} - \sqrt{11}}{\sqrt{6}} \\ \frac{\sqrt{66} + \sqrt{6} + 3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$

C $\quad [\vec{x}]_B = \begin{bmatrix} \frac{4\sqrt{66} + 7\sqrt{6} + \sqrt{11}}{\sqrt{66}} \\ \frac{2\sqrt{66} + \sqrt{6} + \sqrt{11}}{\sqrt{6}} \\ \frac{\sqrt{66} + \sqrt{6} + 3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$

D $\quad [\vec{x}]_B = \begin{bmatrix} \frac{-4\sqrt{66} - 7\sqrt{6} - \sqrt{11}}{\sqrt{66}} \\ \frac{-2\sqrt{66} - \sqrt{6} - \sqrt{11}}{\sqrt{6}} \\ \frac{-\sqrt{66} - \sqrt{6} - 3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$

Solution: A

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2=\left(\frac{4}{\sqrt{66}}\right)^2+\left(-\frac{7}{\sqrt{66}}\right)^2+\left(\frac{1}{\sqrt{66}}\right)^2=\frac{16}{66}+\frac{49}{66}+\frac{1}{66}=\frac{66}{66}=1$$

$$||\vec{v}_2||^2=\left(-\frac{2}{\sqrt{6}}\right)^2+\left(-\frac{1}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2=\frac{4}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$$

$$||\vec{v}_3||^2=\left(-\frac{1}{\sqrt{11}}\right)^2+\left(-\frac{1}{\sqrt{11}}\right)^2+\left(-\frac{3}{\sqrt{11}}\right)^2=\frac{1}{11}+\frac{1}{11}+\frac{9}{11}=\frac{11}{11}=1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=\frac{4}{\sqrt{66}}\left(-\frac{2}{\sqrt{6}}\right)-\frac{7}{\sqrt{66}}\left(-\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{66}}\left(\frac{1}{\sqrt{6}}\right)$$

$$=-\frac{8}{\sqrt{396}}+\frac{7}{\sqrt{396}}+\frac{1}{\sqrt{396}}=\frac{0}{\sqrt{396}}=0$$

$$\vec{v}_1\cdot\vec{v}_3=\frac{4}{\sqrt{66}}\left(-\frac{1}{\sqrt{11}}\right)-\frac{7}{\sqrt{66}}\left(-\frac{1}{\sqrt{11}}\right)+\frac{1}{\sqrt{66}}\left(-\frac{3}{\sqrt{11}}\right)$$

$$=-\frac{4}{\sqrt{726}}+\frac{7}{\sqrt{726}}-\frac{3}{\sqrt{726}}=\frac{0}{\sqrt{726}}=0$$

$$\vec{v}_2 \cdot \vec{v}_3 = -\frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{11}}\right) - \frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{11}}\right) + \frac{1}{\sqrt{6}}\left(-\frac{3}{\sqrt{11}}\right)$$

$$= \frac{2}{\sqrt{66}} + \frac{1}{\sqrt{66}} - \frac{3}{\sqrt{66}} = \frac{0}{\sqrt{66}} = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x} = (\sqrt{66}, \sqrt{6}, \sqrt{11})$ can be converted to the alternate basis B with dot products. In other words, instead of solving

$$\begin{bmatrix} \frac{4}{\sqrt{66}} & -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{11}} \\ -\frac{7}{\sqrt{66}} & -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{11}} \\ \frac{1}{\sqrt{66}} & \frac{1}{\sqrt{6}} & -\frac{3}{\sqrt{11}} \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} \sqrt{66} \\ \sqrt{6} \\ \sqrt{11} \end{bmatrix}$$

which would require us to put the augmented matrix into reduced row-echelon form, we can simply take dot products to get the value of $[\vec{x}]_B$.

$$[\vec{x}]_B = \begin{bmatrix} \frac{4}{\sqrt{66}}(\sqrt{66}) - \frac{7}{\sqrt{66}}(\sqrt{6}) + \frac{1}{\sqrt{66}}(\sqrt{11}) \\ -\frac{2}{\sqrt{6}}(\sqrt{66}) - \frac{1}{\sqrt{6}}(\sqrt{6}) + \frac{1}{\sqrt{6}}(\sqrt{11}) \\ -\frac{1}{\sqrt{11}}(\sqrt{66}) - \frac{1}{\sqrt{11}}(\sqrt{6}) - \frac{3}{\sqrt{11}}(\sqrt{11}) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{4\sqrt{66}}{\sqrt{66}} - \frac{7\sqrt{6}}{\sqrt{66}} + \frac{\sqrt{11}}{\sqrt{66}} \\ -\frac{2\sqrt{66}}{\sqrt{6}} - \frac{\sqrt{6}}{\sqrt{6}} + \frac{\sqrt{11}}{\sqrt{6}} \\ -\frac{\sqrt{66}}{\sqrt{11}} - \frac{\sqrt{6}}{\sqrt{11}} - \frac{3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{4\sqrt{66} - 7\sqrt{6} + \sqrt{11}}{\sqrt{66}} \\ \frac{-2\sqrt{66} - \sqrt{6} + \sqrt{11}}{\sqrt{6}} \\ \frac{-\sqrt{66} - \sqrt{6} - 3\sqrt{11}}{\sqrt{11}} \end{bmatrix}$$

Topic: Projection onto an orthonormal basis

Question: Find the projection of $\vec{x}=(-15,-75,25)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{5}{\sqrt{50}}\\ -\frac{3}{\sqrt{50}}\\ \frac{4}{\sqrt{50}}\end{bmatrix},\begin{bmatrix}\frac{1}{\sqrt{6}}\\ -\frac{1}{\sqrt{6}}\\ -\frac{2}{\sqrt{6}}\end{bmatrix}\right)$$

Answer choices:

A $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}\frac{140}{3}\\ -\frac{92}{3}\\ \frac{98}{3}\end{bmatrix}$

B $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}\frac{80}{3}\\ -\frac{50}{3}\\ \frac{50}{3}\end{bmatrix}$

C $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}\frac{80}{3}\\ -\frac{92}{3}\\ \frac{56}{3}\end{bmatrix}$

D $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{80}{3}\\ \frac{50}{3}\\ -\frac{50}{3}\end{bmatrix}$

Solution: B

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(\frac{5}{\sqrt{50}}\right)^2 + \left(-\frac{3}{\sqrt{50}}\right)^2 + \left(\frac{4}{\sqrt{50}}\right)^2 = \frac{25}{50} + \frac{9}{50} + \frac{16}{50} = \frac{50}{50} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(-\frac{1}{\sqrt{6}}\right)^2 + \left(-\frac{2}{\sqrt{6}}\right)^2 = \frac{1}{6} + \frac{1}{6} + \frac{4}{6} = \frac{6}{6} = 1$$

The length of both vectors is 1, and the dot product of the vectors is

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{5}{\sqrt{50}}\left(\frac{1}{\sqrt{6}}\right) + \left(-\frac{3}{\sqrt{50}}\right)\left(-\frac{1}{\sqrt{6}}\right) + \frac{4}{\sqrt{50}}\left(-\frac{2}{\sqrt{6}}\right)$$

$$= \frac{5}{\sqrt{300}} + \frac{3}{\sqrt{300}} - \frac{8}{\sqrt{300}} = 0$$

Because the vectors are orthogonal to one another, and because they both have the length of 1, the set is orthonormal. So the projection of $\vec{x} = (-15, -75, 25)$ onto V is

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{5}{\sqrt{50}} & \frac{1}{\sqrt{6}} \\ -\frac{3}{\sqrt{50}} & -\frac{1}{\sqrt{6}} \\ \frac{4}{\sqrt{50}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \frac{5}{\sqrt{50}} & -\frac{3}{\sqrt{50}} & \frac{4}{\sqrt{50}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} \frac{5}{\sqrt{50}}\left(\frac{5}{\sqrt{50}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{5}{\sqrt{50}}\left(-\frac{3}{\sqrt{50}}\right)+\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right) & \frac{5}{\sqrt{50}}\left(\frac{4}{\sqrt{50}}\right)+\frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right) \\ -\frac{3}{\sqrt{50}}\left(\frac{5}{\sqrt{50}}\right)-\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & -\frac{3}{\sqrt{50}}\left(-\frac{3}{\sqrt{50}}\right)-\frac{1}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right) & -\frac{3}{\sqrt{50}}\left(\frac{4}{\sqrt{50}}\right)-\frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right) \\ \frac{4}{\sqrt{50}}\left(\frac{5}{\sqrt{50}}\right)-\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{4}{\sqrt{50}}\left(-\frac{3}{\sqrt{50}}\right)-\frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{6}}\right) & \frac{4}{\sqrt{50}}\left(\frac{4}{\sqrt{50}}\right)-\frac{2}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right) \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} \frac{25}{50}+\frac{1}{6} & -\frac{15}{50}-\frac{1}{6} & \frac{20}{50}-\frac{2}{6} \\ -\frac{15}{50}-\frac{1}{6} & \frac{9}{50}+\frac{1}{6} & -\frac{12}{50}+\frac{2}{6} \\ \frac{20}{50}-\frac{2}{6} & -\frac{12}{50}+\frac{2}{6} & \frac{16}{50}+\frac{4}{6} \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} \frac{1}{2}+\frac{1}{6} & -\frac{3}{10}-\frac{1}{6} & \frac{2}{5}-\frac{1}{3} \\ -\frac{3}{10}-\frac{1}{6} & \frac{9}{50}+\frac{1}{6} & -\frac{6}{25}+\frac{1}{3} \\ \frac{2}{5}-\frac{1}{3} & -\frac{6}{25}+\frac{1}{3} & \frac{8}{25}+\frac{2}{3} \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} \frac{2}{3} & -\frac{7}{15} & \frac{1}{15} \\ -\frac{7}{15} & \frac{26}{75} & \frac{7}{75} \\ \frac{1}{15} & \frac{7}{75} & \frac{74}{75} \end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-15,-75,25)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix} \frac{2}{3} & -\frac{7}{15} & \frac{1}{15} \\ -\frac{7}{15} & \frac{26}{75} & \frac{7}{75} \\ \frac{1}{15} & \frac{7}{75} & \frac{74}{75} \end{bmatrix}\begin{bmatrix} -15 \\ -75 \\ 25 \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{2}{3}(-15) - \frac{7}{15}(-75) + \frac{1}{15}(25) \\ -\frac{7}{15}(-15) + \frac{26}{75}(-75) + \frac{7}{75}(25) \\ \frac{1}{15}(-15) + \frac{7}{75}(-75) + \frac{74}{75}(25) \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -10 + 35 + \frac{5}{3} \\ 7 - 26 + \frac{7}{3} \\ -1 - 7 + \frac{74}{3} \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{80}{3} \\ -\frac{50}{3} \\ \frac{50}{3} \end{bmatrix}$$

Topic: Projection onto an orthonormal basis

Question: Find the projection of $\vec{x}=(-1,2,-2)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{2}{3}\\-\frac{1}{3}\\\frac{2}{3}\end{bmatrix},\begin{bmatrix}\frac{2}{3}\\\frac{2}{3}\\-\frac{1}{3}\end{bmatrix}\right)$$

Answer choices:

A $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{9}\\\frac{16}{9}\\-\frac{16}{9}\end{bmatrix}$

B $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{3}\\\frac{16}{3}\\-\frac{8}{3}\end{bmatrix}$

C $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{3}\\\frac{16}{3}\\-\frac{14}{3}\end{bmatrix}$

D $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{9}\\\frac{16}{9}\\-\frac{20}{9}\end{bmatrix}$

Solution: D

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2=\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2=\frac{4}{9}+\frac{1}{9}+\frac{4}{9}=\frac{9}{9}=1$$

$$||\vec{v}_2||^2=\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2=\frac{4}{9}+\frac{4}{9}+\frac{1}{9}=\frac{9}{9}=1$$

The length of both vectors is 1, and the dot product of the vectors is

$$\vec{v}_1\cdot\vec{v}_2=\frac{2}{3}\left(\frac{2}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(-\frac{1}{3}\right)=\frac{4}{9}-\frac{2}{9}-\frac{2}{9}=0$$

Because the vectors are orthogonal to one another, and because they both have the length of 1, the set is orthonormal. So the projection of $\vec{x}=(-1,2,-2)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{2}{3} & \frac{2}{3}\\ -\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}\frac{2}{3} & -\frac{1}{3} & \frac{2}{3}\\ \frac{2}{3} & \frac{2}{3} & -\frac{1}{3}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{2}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(-\frac{1}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(-\frac{1}{3}\right)\\ -\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right) & -\frac{1}{3}\left(-\frac{1}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right) & -\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(-\frac{1}{3}\right)\\ \frac{2}{3}\left(\frac{2}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(-\frac{1}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right) & \frac{2}{3}\left(\frac{2}{3}\right)-\frac{1}{3}\left(-\frac{1}{3}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{4}{9}+\frac{4}{9} & -\frac{2}{9}+\frac{4}{9} & \frac{4}{9}-\frac{2}{9}\\ -\frac{2}{9}+\frac{4}{9} & \frac{1}{9}+\frac{4}{9} & -\frac{2}{9}-\frac{2}{9}\\ \frac{4}{9}-\frac{2}{9} & -\frac{2}{9}-\frac{2}{9} & \frac{4}{9}+\frac{1}{9}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{8}{9} & \frac{2}{9} & \frac{2}{9}\\ \frac{2}{9} & \frac{5}{9} & -\frac{4}{9}\\ \frac{2}{9} & -\frac{4}{9} & \frac{5}{9}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-1,2,-2)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{8}{9} & \frac{2}{9} & \frac{2}{9}\\ \frac{2}{9} & \frac{5}{9} & -\frac{4}{9}\\ \frac{2}{9} & -\frac{4}{9} & \frac{5}{9}\end{bmatrix}\begin{bmatrix}-1\\ 2\\ -2\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{8}{9}(-1)+\frac{2}{9}(2)+\frac{2}{9}(-2)\\ \frac{2}{9}(-1)+\frac{5}{9}(2)-\frac{4}{9}(-2)\\ \frac{2}{9}(-1)-\frac{4}{9}(2)+\frac{5}{9}(-2)\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{8}{9}+\frac{4}{9}-\frac{4}{9}\\ -\frac{2}{9}+\frac{10}{9}+\frac{8}{9}\\ -\frac{2}{9}-\frac{8}{9}-\frac{10}{9}\end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -\frac{8}{9} \\ \frac{16}{9} \\ -\frac{20}{9} \end{bmatrix}$$

Topic: Projection onto an orthonormal basis

Question: Find the projection of $\vec{x}=(3,-2,4)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}-\frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\\ \frac{1}{\sqrt{3}}\end{bmatrix},\begin{bmatrix}\frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{6}}\end{bmatrix}\right)$$

Answer choices:

A $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}3\\1\\1\end{bmatrix}$

B $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}1\\1\\1\end{bmatrix}$

C $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}3\\ \frac{2}{3}\\ \frac{2}{3}\end{bmatrix}$

D $\quad \text{Proj}_V\vec{x}=\begin{bmatrix}3\\-4\\2\end{bmatrix}$

Solution: A

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2=\left(-\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2+\left(\frac{1}{\sqrt{3}}\right)^2=\frac{1}{3}+\frac{1}{3}+\frac{1}{3}=\frac{3}{3}=1$$

$$||\vec{v}_2||^2=\left(\frac{2}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2=\frac{4}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$$

The length of both vectors is 1, and the dot product of the vectors is

$$\vec{v}_1\cdot\vec{v}_2=\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right)+\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right)+\left(\frac{1}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{6}}\right)$$

$$=-\frac{2}{\sqrt{18}}+\frac{1}{\sqrt{18}}+\frac{1}{\sqrt{18}}=0$$

Because the vectors are orthogonal to one another, and because they both have the length of 1, the set is orthonormal. So the projection of $\vec{x}=(3,-2,4)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} -\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) \\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) \\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{2}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right) \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} \frac{1}{3}+\frac{4}{6} & -\frac{1}{3}+\frac{1}{3} & -\frac{1}{3}+\frac{1}{3} \\ -\frac{1}{3}+\frac{1}{3} & \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{1}{6} \\ -\frac{1}{3}+\frac{1}{3} & \frac{1}{3}+\frac{1}{6} & \frac{1}{3}+\frac{1}{6} \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(3,-2,4)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix} 1(3)+0(-2)+0(4) \\ 0(3)+\frac{1}{2}(-2)+\frac{1}{2}(4) \\ 0(3)+\frac{1}{2}(-2)+\frac{1}{2}(4) \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} 3+0+0 \\ 0-1+2 \\ 0-1+2 \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

Topic: Gram-Schmidt process for change of basis

Question: The subspace V is a plane in $\mathbb{R}^3$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\2\\1\end{bmatrix}, \begin{bmatrix}-1\\-3\\1\end{bmatrix}\right)$$

Answer choices:

A $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{11}}\\ -\frac{3}{\sqrt{11}}\\ \frac{1}{\sqrt{11}}\end{bmatrix}\right)$

B $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{14}}\\ -\frac{3}{\sqrt{14}}\\ \frac{1}{\sqrt{14}}\end{bmatrix}\right)$

C $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{5}{3\sqrt{10}}\\ -\frac{7}{3\sqrt{10}}\\ \frac{4}{3\sqrt{10}}\end{bmatrix}\right)$

D $\quad V_2 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\ \frac{2}{3}\\ \frac{1}{3}\end{bmatrix}, \begin{bmatrix}-\frac{1}{\sqrt{10}}\\ -\frac{3}{\sqrt{10}}\\ \frac{1}{\sqrt{10}}\end{bmatrix}\right)$

Solution: C

Define $\vec{v}_1 = (-2,2,1)$ and $\vec{v}_2 = (-1,-3,1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-2)^2 + 2^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$ and $\vec{v}_2$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2)$$

Now all we need to do is replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. If we can do that, then the vector set that spans V will be orthonormal. We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}\left(\begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}\right)\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}((-1)(-2) + (-3)(2) + (1)(1))\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}(2 - 6 + 1)\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} - \frac{1}{9}(-3)\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 - \frac{2}{3} \\ -3 + \frac{2}{3} \\ 1 + \frac{1}{3} \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -\frac{5}{3} \\ -\frac{7}{3} \\ \frac{4}{3} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it.

$$||\vec{w}_2|| = \sqrt{\left(-\frac{5}{3}\right)^2 + \left(-\frac{7}{3}\right)^2 + \left(\frac{4}{3}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{25}{9} + \frac{49}{9} + \frac{16}{9}}$$

$$||\vec{w}_2|| = \sqrt{\frac{90}{9}}$$

$$||\vec{w}_2|| = \sqrt{10}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} -\frac{5}{3} \\ -\frac{7}{3} \\ \frac{4}{3} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis for V.

$$V_2 = \text{Span}\left(\frac{1}{3}\begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} -\frac{5}{3} \\ -\frac{7}{3} \\ -\frac{4}{3} \end{bmatrix}\right)$$

$$V_2 = \text{Span}\left(\begin{bmatrix} -\frac{2}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \begin{bmatrix} -\frac{5}{3\sqrt{10}} \\ -\frac{7}{3\sqrt{10}} \\ \frac{4}{3\sqrt{10}} \end{bmatrix}\right)$$

Topic: Gram-Schmidt process for change of basis

Question: The subspace V is a space in $\mathbb{R}^3$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}1\\0\\-2\end{bmatrix}, \begin{bmatrix}-1\\-2\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\3\end{bmatrix}\right)$$

Answer choices:

A $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\-\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}\frac{2}{\sqrt{5}}\\0\\\frac{1}{\sqrt{5}}\end{bmatrix}\right)$

B $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\-\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\-\frac{\sqrt{5}}{3}\\\sqrt{5}\end{bmatrix}\right)$

C $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{\sqrt{5}}\\0\\-\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}-\frac{1}{2}\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\-\frac{3}{\sqrt{5}}\\\frac{9}{\sqrt{5}}\end{bmatrix}\right)$

D $\quad V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)$

Solution: A

Define $\vec{v}_1 = (1,0,-2)$, $\vec{v}_2 = (-1,-2,2)$, and $\vec{v}_3 = (0,-1,3)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{1^2 + 0^2 + (-2)^2} = \sqrt{1 + 0 + 4} = \sqrt{5}$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5} \left(\begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5}((-1)(1) + (-2)(0) + (2)(-2)) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5}(-1 + 0 - 4) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} - \frac{1}{5}(-5) \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{0^2 + (-2)^2 + 0^2}$$

$$||\vec{w}_2|| = \sqrt{0 + 4 + 0}$$

$$||\vec{w}_2|| = \sqrt{4}$$

$$||\vec{w}_2|| = 2$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u_1} - (\vec{v}_3 \cdot \vec{u}_2)\vec{u_2}$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{5}\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}\right)\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{5}(0(1) - 1(0) + 3(-2))\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{4}(0(0) - 1(-2) + 3(0))\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} - \frac{1}{5}(0 + 0 - 6)\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{4}(0 + 2 + 0)\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + \frac{6}{5}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} \frac{6}{5} \\ 0 \\ -\frac{12}{5} \end{bmatrix} - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 0 + \frac{6}{5} - 0 \\ -1 + 0 + 1 \\ 3 - \frac{12}{5} - 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{6}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$$

So $\vec{w}_3$ is orthogonal to $\vec{u}_2$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{6}{5}\right)^2 + 0^2 + \left(\frac{3}{5}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{36}{25} + 0 + \frac{9}{25}}$$

$$||\vec{w}_3|| = \sqrt{\frac{45}{25}}$$

$$||\vec{w}_3|| = \sqrt{\frac{9}{5}}$$

$$||\vec{w}_3|| = \frac{3}{\sqrt{5}}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{\frac{3}{\sqrt{5}}} \begin{bmatrix} \frac{6}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$$

$$\vec{u}_3 = \frac{\sqrt{5}}{3} \begin{bmatrix} \frac{6}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}, \frac{\sqrt{5}}{3} \begin{bmatrix} \frac{6}{5} \\ 0 \\ \frac{3}{5} \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ -\frac{2}{\sqrt{5}} \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{5}} \\ 0 \\ \frac{1}{\sqrt{5}} \end{bmatrix}\right)$$

Topic: Gram-Schmidt process for change of basis

Question: The subspace V is a space in $\mathbb{R}^4$. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}0\\2\\1\\3\end{bmatrix}, \begin{bmatrix}2\\0\\-2\\0\end{bmatrix}\right)$$

Answer choices:

A $\quad V_3 = \text{Span}\left(\begin{bmatrix}\frac{1}{2}\\\frac{1}{2}\\-\frac{1}{2}\\\frac{1}{2}\end{bmatrix}, \begin{bmatrix}0\\\frac{2}{\sqrt{10}}\\\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\\frac{1}{2}\end{bmatrix}\right)$

B $\quad V_3 = \text{Span}\left(\begin{bmatrix}1\\1\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\2\\2\end{bmatrix}, \begin{bmatrix}1\\-1\\-1\\-1\end{bmatrix}\right)$

C $\quad V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \\ -\frac{1}{2\sqrt{2}} \end{bmatrix}\right)$

D $\quad V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{2}{5}} \\ -\sqrt{\frac{2}{5}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\right)$

Solution: D

Define $\vec{v}_1 = (1,1,-1,1)$, $\vec{v}_2 = (0,2,1,3)$, and $\vec{v}_3 = (2,0,-2,0)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{1^2 + 1^2 + (-1)^2 + 1^2} = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \cdot \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right)\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{4}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}\right)\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{4}(0(1) + 2(1) + 1(-1) + 3(1))\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{4}(0 + 2 - 1 + 3) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{(-1)^2 + 1^2 + 2^2 + 2^2}$$

$$||\vec{w}_2|| = \sqrt{1 + 1 + 4 + 4}$$

$$||\vec{w}_2|| = \sqrt{10}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right) \frac{1}{\sqrt{10}} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{4} \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{10} \left(\begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{4}(2(1) + 0(1) - 2(-1) + 0(1)) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{10}(2(-1) + 0(1) - 2(2) + 0(2)) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \frac{1}{4}(2 + 0 + 2 + 0) \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} - \frac{1}{10}(-2 + 0 - 4 + 0) \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} + \begin{bmatrix} -\frac{3}{5} \\ \frac{3}{5} \\ \frac{6}{5} \\ \frac{6}{5} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$$

The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{2}{5}\right)^2 + \left(-\frac{2}{5}\right)^2 + \left(\frac{1}{5}\right)^2 + \left(\frac{1}{5}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{4}{25} + \frac{4}{25} + \frac{1}{25} + \frac{1}{25}}$$

$$||\vec{w}_3|| = \sqrt{\frac{10}{25}}$$

$$||\vec{w}_3|| = \sqrt{\frac{2}{5}}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{\sqrt{\frac{2}{5}}}\begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$$

$$\vec{u}_3 = \sqrt{\frac{5}{2}}\begin{bmatrix} \frac{2}{5} \\ -\frac{2}{5} \\ \frac{1}{5} \\ \frac{1}{5} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} -1 \\ 1 \\ 2 \\ 2 \end{bmatrix}, \frac{1}{\sqrt{10}}\begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \\ \frac{2}{\sqrt{10}} \end{bmatrix}, \begin{bmatrix} \frac{2}{\sqrt{10}} \\ -\frac{2}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ \sqrt{\frac{2}{5}} \\ \sqrt{\frac{2}{5}} \end{bmatrix}, \begin{bmatrix} \sqrt{\frac{2}{5}} \\ -\sqrt{\frac{2}{5}} \\ \frac{1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \end{bmatrix}\right)$$

Topic: Eigenvalues, eigenvectors, eigenspaces

Question: Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix}$$

Answer choices:

A $\lambda = 3, \lambda = 4$

B $\lambda = -3, \lambda = 4$

C $\lambda = 3, \lambda = -4$

D $\lambda = -3, \lambda = -4$

Solution: B

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 3 & 0 - 0 \\ 0 - 1 & \lambda - 4 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 3 & 0 \\ -1 & \lambda - 4 \end{bmatrix} \right|$$

The determinant is

$$(\lambda + 3)(\lambda - 4) - (0)(-1)$$

$$(\lambda + 3)(\lambda - 4)$$

$\lambda = -3$ or $\lambda = 4$

Topic: Eigenvalues, eigenvectors, eigenspaces

Question: For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -3$ and $\lambda = 4$.

$$A = \begin{bmatrix} -3 & 0 \\ 1 & 4 \end{bmatrix}$$

$$|\lambda I_n - A| = \begin{bmatrix} \lambda + 3 & 0 \\ -1 & \lambda - 4 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

B $\begin{bmatrix} -7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

C $\begin{bmatrix} 7 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

D $\begin{bmatrix} -7 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Solution: D

With $\lambda = -3$ and $\lambda = 4$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 3 & 0 \\ -1 & \lambda - 4 \end{bmatrix}\right)$$

we get

$$E_{-3} = N\left(\begin{bmatrix} -3 + 3 & 0 \\ -1 & -3 - 4 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} 0 & 0 \\ -1 & -7 \end{bmatrix}\right)$$

and

$$E_4 = N\left(\begin{bmatrix} 4 + 3 & 0 \\ -1 & 4 - 4 \end{bmatrix}\right)$$

$$E_4 = N\left(\begin{bmatrix} 7 & 0 \\ -1 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} 0 & 0 \\ -1 & -7 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & | & 0 \\ -1 & -7 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -7 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 7 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 7 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + 7v_2 = 0$$

So with $v_1 = -7v_2$, we'll substitute $v_2 = t$, and say that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -7 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} -7 \\ 1 \end{bmatrix}\right)$$

And the eigenvectors in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 7 & 0 \\ -1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 0 & | & 0 \\ -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & | & 0 \\ 7 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ 7 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + 0v_2 = 0$$

$$v_1 = 0v_2$$

And with $v_1 = 0v_2$, we'll substitute $v_2 = t$, and say that

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} -7 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Topic: Eigenvalues, eigenvectors, eigenspaces

Question: Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

B $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

C $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

D $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Solution: D

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ 0 & 5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 2 & 0 - (-3) \\ 0 - 0 & \lambda - 5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 2 & 3 \\ 0 & \lambda - 5 \end{bmatrix} \right|$$

The determinant is

$$(\lambda - 2)(\lambda - 5) - (3)(0)$$

$$(\lambda - 2)(\lambda - 5)$$

$\lambda = 2$ or $\lambda = 5$

With $\lambda = 2$ and $\lambda = 5$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 2 & 3 \\ 0 & \lambda - 5 \end{bmatrix} \right)$$

we get

$$E_2 = N\left(\begin{bmatrix} 2-2 & 3 \\ 0 & 2-5 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\right)$$

and

$$E_5 = N\left(\begin{bmatrix} 5-2 & 3 \\ 0 & 5-5 \end{bmatrix}\right)$$

$$E_5 = N\left(\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_2 will satisfy

$$\begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 & | & 0 \\ 0 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_2 = 0$$

So the eigenvector for E_2 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

And the eigenvectors in the eigenspace E_5 will satisfy

$$\begin{bmatrix} 3 & 3 \\ 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0$$

$$v_1 = -v_2$$

So the eigenvector for E_5 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Topic: Eigen in three dimensions

Question: Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 1 & -4 & 2 \\ 0 & 4 & -3 \\ 0 & 0 & -2 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} \frac{4}{3} \\ -1 \\ 0 \end{bmatrix}$

B $\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$

C $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} \frac{4}{3} \\ -1 \\ 0 \end{bmatrix}$

D $\begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$

Solution: B

Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&-4&2\\0&4&-3\\0&0&-2\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}1&-4&2\\0&4&-3\\0&0&-2\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-1&0-(-4)&0-2\\0-0&\lambda-4&0-(-3)\\0-0&0-0&\lambda-(-2)\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-1&4&-2\\0&\lambda-4&3\\0&0&\lambda+2\end{bmatrix}\right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda-1)\begin{vmatrix}\lambda-4&3\\0&\lambda+2\end{vmatrix}-0\begin{vmatrix}4&-2\\0&\lambda+2\end{vmatrix}+0\begin{vmatrix}4&-2\\\lambda-4&3\end{vmatrix}$$

$$(\lambda-1)[(\lambda-4)(\lambda+2)-(3)(0)]$$

$$(\lambda-1)(\lambda-4)(\lambda+2)$$

$\lambda = -2$ or $\lambda = 1$ or $\lambda = 4$

With $\lambda = -2$, $\lambda = 1$ and $\lambda = 4$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 1 & 4 & -2 \\ 0 & \lambda - 4 & 3 \\ 0 & 0 & \lambda + 2 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 - 1 & 4 & -2 \\ 0 & -2 - 4 & 3 \\ 0 & 0 & -2 + 2 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} -3 & 4 & -2 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1 - 1 & 4 & -2 \\ 0 & 1 - 4 & 3 \\ 0 & 0 & 1 + 2 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 0 & 4 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

and

$$E_4 = N\left(\begin{bmatrix} 4 - 1 & 4 & -2 \\ 0 & 4 - 4 & 3 \\ 0 & 0 & 4 + 2 \end{bmatrix}\right)$$

$$E_4 = N\left(\begin{bmatrix} 3 & 4 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -3 & 4 & -2 \\ 0 & -6 & 3 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 4 & -2 & | & 0 \\ 0 & -6 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & | & 0 \\ 0 & -6 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{4}{3} & \frac{2}{3} & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 - \frac{1}{2}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = \frac{1}{2}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 0 & 4 & -2 \\ 0 & -3 & 3 \\ 0 & 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 4 & -2 & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & -3 & 3 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & \frac{3}{2} & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 3 & 4 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 4 & -2 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & -\frac{2}{3} & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & -\frac{2}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{4}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + \frac{4}{3}v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -\frac{4}{3}v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} -\frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

Topic: Eigen in three dimensions

Question: Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$

B $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ -1 \end{bmatrix}$

C $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$

D $\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -\frac{2}{3} \\ -1 \end{bmatrix}$

Solution: C

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & -3 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - (-2) & 0 - 0 & 0 - 0 \\ 0 - 1 & \lambda - 3 & 0 - 0 \\ 0 - 0 & 0 - (-3) & \lambda - 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 2 & 0 & 0 \\ -1 & \lambda - 3 & 0 \\ 0 & 3 & \lambda - 1 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 2) \begin{vmatrix} \lambda - 3 & 0 \\ 3 & \lambda - 1 \end{vmatrix} - 0 \begin{vmatrix} -1 & 0 \\ 0 & \lambda - 1 \end{vmatrix} + 0 \begin{vmatrix} -1 & \lambda - 3 \\ 0 & 3 \end{vmatrix}$$

$$(\lambda + 2)[(\lambda - 3)(\lambda - 1) - (0)(3)]$$

$$(\lambda + 2)(\lambda - 3)(\lambda - 1)$$

$\lambda = -2$ or $\lambda = 1$ or $\lambda = 3$

With $\lambda = -2$, $\lambda = 1$ and $\lambda = 3$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 2 & 0 & 0 \\ -1 & \lambda - 3 & 0 \\ 0 & 3 & \lambda - 1 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 + 2 & 0 & 0 \\ -1 & -2 - 3 & 0 \\ 0 & 3 & -2 - 1 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} 0 & 0 & 0 \\ -1 & -5 & 0 \\ 0 & 3 & -3 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1 + 2 & 0 & 0 \\ -1 & 1 - 3 & 0 \\ 0 & 3 & 1 - 1 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 3 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 3 & 0 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3 + 2 & 0 & 0 \\ -1 & 3 - 3 & 0 \\ 0 & 3 & 3 - 1 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 5 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} 0 & 0 & 0 \\ -1 & -5 & 0 \\ 0 & 3 & -3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ -1 & -5 & 0 & | & 0 \\ 0 & 3 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} -1 & -5 & 0 & | & 0 \\ 0 & 3 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 0 & | & 0 \\ 0 & 3 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & 0 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & | & 0 \\ 0 & 1 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + 5v_3 = 0$$

$$v_2 - v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -5v_3$$

$$v_2 = v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & -2 & 0 \\ 0 & 3 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -1 & -2 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 5 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 3 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 5 & 0 & 0 & | & 0 \\ -1 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -1 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 3 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{2}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{2}{3} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 + \frac{2}{3}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = -\frac{2}{3}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -5 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Topic: Eigen in three dimensions

Question: Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer choices:

A $\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

B $\begin{bmatrix} -\frac{2}{3} \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$

C $\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

D $\begin{bmatrix} -\frac{2}{3} \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

Solution: C

Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&2&-1\\0&-2&0\\0&0&1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}1&2&-1\\0&-2&0\\0&0&1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-1&0-2&0-(-1)\\0-0&\lambda-(-2)&0-0\\0-0&0-0&\lambda-1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-1&-2&1\\0&\lambda+2&0\\0&0&\lambda-1\end{bmatrix}\right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda-1)\begin{vmatrix}\lambda+2&0\\0&\lambda-1\end{vmatrix}-0\begin{vmatrix}-2&1\\0&\lambda-1\end{vmatrix}+0\begin{vmatrix}-2&1\\\lambda+2&0\end{vmatrix}$$

$$(\lambda-1)[(\lambda+2)(\lambda-1)-(0)(0)]$$

$$(\lambda-1)(\lambda+2)(\lambda-1)$$

$$(\lambda+2)(\lambda-1)(\lambda-1)$$

$\lambda = -2$ or $\lambda = 1$

With $\lambda = -2$ and $\lambda = 1$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 1 & -2 & 1 \\ 0 & \lambda + 2 & 0 \\ 0 & 0 & \lambda - 1 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 - 1 & -2 & 1 \\ 0 & -2 + 2 & 0 \\ 0 & 0 & -2 - 1 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1 - 1 & -2 & 1 \\ 0 & 1 + 2 & 0 \\ 0 & 0 & 1 - 1 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -3 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -2 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & -3 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & -3 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & | & 0 \\ 0 & 0 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{3} & -\frac{1}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{2}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + \frac{2}{3}v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -\frac{2}{3}v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 0 & -2 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -2 & 1 & | & 0 \\ 0 & 3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & -2 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

W W W . K R I S T A K I N G M A T H . C O M