Linear Algebra Workbook Solutions

krista king
MATH

LINEAR SYSTEMS IN TWO UNKNOWNS

■ 1. Find the unique solution to the system of equations.

$$-x + 2y = 6$$

$$3x = y - 10$$

Solution:

Solve for x in the second equation.

$$3x = y - 10$$

$$x = \frac{y - 10}{3}$$

Plug this value for x into the first equation, then solve for y.

$$-x + 2y = 6$$

$$-\frac{y - 10}{3} + 2y = 6$$

$$-y + 10 + 6y = 18$$

$$5y = 8$$

$$y = \frac{8}{5}$$

Plug $y = 8/5$ back into the equation we found for x.

$$x = \frac{y - 10}{3}$$

$$x = \frac{\frac{8}{5} - 10}{3}$$

$$x = \frac{\frac{8}{5} - \frac{50}{5}}{3}$$

$$x = -\frac{42}{5} \cdot \frac{1}{3}$$

$$x = -\frac{14}{5}$$

The unique solution to the system is

$$\left(-\frac{14}{5}, \frac{8}{5}\right)$$

■ 2. Find the unique solution to the system of equations.

$$-5x + y = 8$$

$$y = 3x - 8$$

Solution:

Taking the value for y given in the second equation as $y = 3x - 8$, we'll substitute for y in the first equation.

$$-5x + y = 8$$

$$-5x + (3x - 8) = 8$$

$$-5x + 3x - 8 = 8$$

$$-2x = 16$$

$$x = -8$$

Now substitute $x = -8$ into the second equation to find a value for y.

$$y = 3x - 8$$

$$y = 3(-8) - 8$$

$$y = -32$$

The unique solution to the system is

$$(-8, -32)$$

3. Find the unique solution to the system of equations.

$$2x - y = 5$$

$$-3x + y = 7$$

Solution:

If we add the two equations together to eliminate y, we get

$$2x - y + (-3x + y) = 5 + (7)$$

$$2x - 3x = 12$$

$$-x = 12$$

$$x = -12$$

Plug $x = -12$ back into the second equation.

$$-3x + y = 7$$

$$-3(-12) + y = 7$$

$$y = -29$$

The solution to the system is

$$(-12, -29)$$

■ 4. Find the unique solution to the system of equations.

$$x = 2y - 5$$

$$-3x + 6y = 15$$

Solution:

Multiplying the first equation by 3 gives

$$x = 2y - 5$$

$$3x = 6y - 15$$

Then adding $3x = 6y - 15$ to $-3x + 6y = 15$ gives

$$3x - 6y + (-3x + 6y) = -15 + (15)$$

$$3x - 6y - 3x + 6y = -15 + 15$$

$$-6y + 6y = -15 + 15$$

$$0 = 0$$

This is always true, so there are infinitely many solutions to the system of equations.

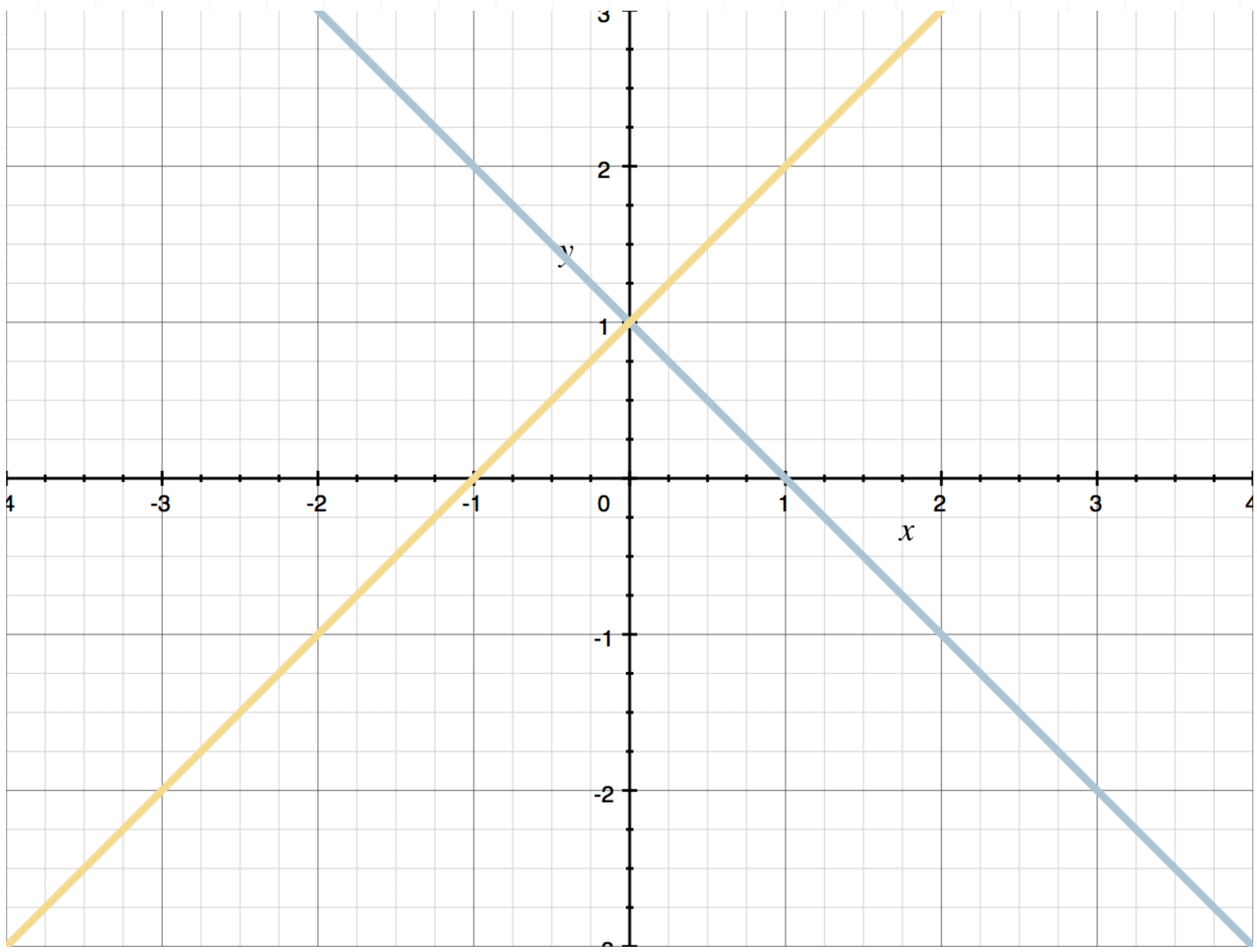
■ 5. Find the unique solution to the system of equations using the graphing method.

$$y - 2 = -(x + 1)$$

$$y = x + 1$$

Solution:

As you can see from the graphs of the two functions, the intersection point is at $(0,1)$, which means $(0,1)$ is the solution to the system of equations.



■ 6. Find the unique solution to the system of equations using the substitution method.

$$5y + x = 4$$

$$3y - 3x = 6$$

Solution:

Solve the first equation for x.

$$5y + x = 4$$

$$x = 4 - 5y$$

Substitute this into the second equation.

$$3y - 3x = 6$$

$$3y - 3(4 - 5y) = 6$$

$$3y - 12 + 15y = 6$$

$$18y = 18$$

$$y = 1$$

Plug $y = 1$ into the equation for x.

$$x = 4 - 5y$$

$$x = 4 - 5(1)$$

$$x = -1$$

Therefore the solution to the system of equation is

$$(-1,1)$$

LINEAR SYSTEMS IN THREE UNKNOWNS

■ 1. Find the unique solution to the system of equations.

$$2x + y - z = 3$$

$$x - y + z = 0$$

$$x - 2y - 3z = 4$$

Solution:

Let's number the equations to stay organized.

[1] $2x + y - z = 3$

[2] $x - y + z = 0$

[3] $x - 2y - 3z = 4$

Add equations [1] and [2] so that y and z will be eliminated.

$$(2x + y - z) + (x - y + z) = (3) + (0)$$

$$2x + y - z + x - y + z = 3$$

$$3x + y - y + z - z = 3$$

$$3x = 3$$

$$x = 1$$

Let's plug $x = 1$ into equations [2] and [3] to put them in terms of y and z.

$$x - y + z = 0$$

$$1 - y + z = 0$$

[4] $-y + z = -1$

and

$$x - 2y - 3z = 4$$

$$1 - 2y - 3z = 4$$

[5] $-2y - 3z = 3$

Multiply equation [4] by 3,

$$3(-y + z = -1)$$

$$-3y + 3z = -3$$

and then add this to equation [5] so that z can be eliminated.

$$(-3y + 3z) + (-2y - 3z) = (-3) + (3)$$

$$-3y + 3z - 2y - 3z = 0$$

$$-3y - 2y = 0$$

$$-5y = 0$$

$$y = 0$$

Plug $y = 0$ into equation [4] to solve for z.

$$-2y - 3z = 3$$

$$-2(0) - 3z = 3$$

$$-3z = 3$$

$$z = -1$$

Therefore, the solution to the system is $(x, y, z) = (1,0,-1)$.

2. Find the unique solution to the system of equations.

$$3x + y - z = -2$$

$$x - 2y + 3z = 23$$

$$2x + 3y + 2z = 5$$

Solution:

Let's number the equations to stay organized.

[1] $3x + y - z = -2$

[2] $x - 2y + 3z = 23$

[3] $2x + 3y + 2z = 5$

Multiply equation [1] by 2,

$$2(3x + y - z = -2)$$

$$6x + 2y - 2z = -4$$

and add this to equation [3] so that z can be eliminated.

$$(6x + 2y - 2z) + (2x + 3y + 2z) = (-4) + (5)$$

$$6x + 2y - 2z + 2x + 3y + 2z = 1$$

$$6x + 2y + 2x + 3y = 1$$

[4] $8x + 5y = 1$

Multiply equation [1] by 3,

$$3(3x + y - z = -2)$$

$$9x + 3y - 3z = -6$$

and add this to equation [2] so that z can be eliminated.

$$(9x + 3y - 3z) + (x - 2y + 3z) = (-6) + (23)$$

$$9x + 3y - 3z + x - 2y + 3z = 17$$

$$9x + 3y + x - 2y = 17$$

[5] $10x + y = 17$

Multiply equation [5] by -5,

$$-5(10x + y = 17)$$

$$-50x - 5y = -85$$

and then add this to equation [4] so that y will be eliminated.

$$(8x+5y)+(-50x-5y)=(1)+(-85)$$

$$8x+5y-50x-5y=-84$$

$$8x-50x=-84$$

$$-42x=-84$$

$$x=2$$

Let's plug $x=1$ into equation [5] to solve for y.

$$10x+y=17$$

$$10(2)+y=17$$

$$20+y=17$$

$$y=-3$$

Plug $x=2$ and $y=-3$ into any of the original equations to solve for z. We'll use equation [1].

$$3x+y-z=-2$$

$$3(2)+(-3)-z=-2$$

$$6-3-z=-2$$

$$3-z=-2$$

$$-z=-5$$

$$z=5$$

Therefore, the solution to the system is $(x, y, z) = (2, -3, 5)$.

■ 3. Find the unique solution to the system of equations.

$$5x - 3y + z = -8$$

$$2x + y - 2z = -6$$

$$-3x + 2y + 4z = 19$$

Solution:

Let's number the equations to stay organized.

[1] $5x - 3y + z = -8$

[2] $2x + y - 2z = -6$

[3] $-3x + 2y + 4z = 19$

Multiply equation [1] by 2,

$$2(5x - 3y + z = -8)$$

$$10x - 6y + 2z = -16$$

and then add this to equation [2] so that z will be eliminated.

$$(10x - 6y + 2z) + (2x + y - 2z) = (-16) + (-6)$$

$$10x - 6y + 2z + 2x + y - 2z = -22$$

$$10x - 6y + 2x + y = -22$$

$$[4] \quad 12x - 5y = -22$$

Multiply equation [1] by -4,

$$-4(5x - 3y + z = -8)$$

$$-20x + 12y - 4z = 32$$

and then add this to equation [3] so that z will be eliminated.

$$(-20x + 12y - 4z) + (-3x + 2y + 4z) = (32) + (19)$$

$$-20x + 12y - 4z - 3x + 2y + 4z = 51$$

$$-20x + 12y - 3x + 2y = 51$$

$$[5] \quad -23x + 14y = 51$$

Solve for y in equation [4].

$$12x - 5y = -22$$

$$-5y = -12x - 22$$

$$[6] \quad y = \frac{12}{5}x + \frac{22}{5}$$

Plug [6] into equation [5] to solve for x.

$$-23x + 14y = 51$$

$$-23x + 14\left(\frac{12}{5}x + \frac{22}{5}\right) = 51$$

$$-23x+\frac{168}{5}x+\frac{308}{5}=51$$

$$\frac{53}{5}x+\frac{308}{5}=51$$

$$53x+308=255$$

$$53x=-53$$

$$x=-1$$

Plug $x=-1$ into equation [6] to solve for y.

$$y=\frac{12}{5}x+\frac{22}{5}$$

$$y=\frac{12}{5}(-1)+\frac{22}{5}$$

$$y=-\frac{12}{5}+\frac{22}{5}$$

$$y=\frac{10}{5}$$

$$y=2$$

Plug $x=-1$ and $y=2$ into any of the original equations to solve for z. We'll use equation [1].

$$5x-3y+z=-8$$

$$5(-1)-3(2)+z=-8$$

$$-5 - 6 + z = -8$$

$$-11 + z = -8$$

$$z = 3$$

Therefore, the solution to the system is $(x, y, z) = (-1,2,3)$.

■ 4. Find the unique solution to the system of equations.

$$-2x + 3y - 4z = 10$$

$$4x + 3y + 2z = 4$$

$$x - 6y + 4z = -19$$

Solution:

Let's number the equations to stay organized.

[1] $-2x + 3y - 4z = 10$

[2] $4x + 3y + 2z = 4$

[3] $x - 6y + 4z = -19$

Multiply equation [2] by 2,

$$2(4x + 3y + 2z = 4)$$

$$8x + 6y + 4z = 8$$

and then add this to equation [1] so that z will be eliminated.

$$(8x + 6y + 4z) + (-2x + 3y - 4z) = (8) + (10)$$

$$8x + 6y + 4z - 2x + 3y - 4z = 18$$

$$8x + 6y - 2x + 3y = 18$$

$$6x + 9y = 18$$

[4] $2x + 3y = 6$

Add equation [1] to equation [3] so that z will be eliminated.

$$(-2x + 3y - 4z) + (x - 6y + 4z) = (10) + (-19)$$

$$-2x + 3y - 4z + x - 6y + 4z = -9$$

$$-2x + 3y + x - 6y = -9$$

[5] $-x - 3y = -9$

Solve for x in equation [5].

$$-x - 3y = -9$$

$$-x = -9 + 3y$$

[6] $x = 9 - 3y$

Plug $x = 9 - 3y$ into equation [4] to solve for y.

$$2x + 3y = 6$$

$$2(9 - 3y) + 3y = 6$$

$$18 - 6y + 3y = 6$$

$$18 - 3y = 6$$

$$-3y = -12$$

$$y = 4$$

Let's plug $y = 4$ into equation [6] to solve for x.

$$x = 9 - 3y$$

$$x = 9 - 3(4)$$

$$x = 9 - 12$$

$$x = -3$$

Plug $x = -3$ and $y = 4$ into any of the original equations to solve for z. We'll use equation [2].

$$4x + 3y + 2z = 4$$

$$4(-3) + 3(4) + 2z = 4$$

$$-12 + 12 + 2z = 4$$

$$2z = 4$$

$$z = 2$$

Therefore, the solution to the system is $(x, y, z) = (-3,4,2)$.

■ 5. Find the unique solution to the system of equations.

$$2x - y + z = 9$$

$$4x - 2y + 2z = 18$$

$$-2x + y - z = -9$$

Solution:

Let's number the equations to stay organized.

[1] $2x - y + z = 9$

[2] $4x - 2y + 2z = 18$

[3] $-2x + y - z = -9$

Add equation [1] to equation [3] so that z will be eliminated.

$$(2x - y + z) + (-2x + y - z) = (9) + (-9)$$

$$2x - y + z - 2x + y - z = 0$$

$$-y + z + y - z = 0$$

$$z - z = 0$$

$$0 = 0$$

When all the variables eliminate and we get a true statement, it means all points (x, y, z) are a solution to the system. So far, this is the case with equations [1] and [3]. If this also happens with equation [2], then the whole system is an "identity" and there are infinite solutions.

Let's check the second equation to see if this is the case. Multiply equation [3] by 2,

$$2(-2x + y - z = -9)$$

$$-4x + 2y - 2z = -18$$

and then add it to equation [2].

$$(-4x + 2y - 2z) + (4x - 2y + 2z) = (-18) + (18)$$

$$-4x + 2y - 2z + 4x - 2y + 2z = 0$$

$$2y - 2z - 2y + 2z = 0$$

$$-2z + 2z = 0$$

$$0 = 0$$

Since all the variables eliminate and we get a true statement, the system is an identity and there are infinite solutions.

■ 6. Find the unique solution to the system of equations.

$$x + 2y - z = 9$$

$$3x + y - z = 5$$

$$-x - 4y + z = 2$$

Solution:

Let's number the equations to stay organized.

[1] $x + 2y - z = 9$

[2] $3x + y - z = 5$

[3] $-x - 4y + z = 2$

Add equations [1] and [3] together so that x will be eliminated.

$$(x + 2y - z) + (-x - 4y + z) = (9) + (2)$$

$$x + 2y - z - x - 4y + z = 11$$

$$2y - 4y = 11$$

$$-2y = 11$$

[4] $y = -\dfrac{11}{2}$

Add equation [2] to equation [3] so that z will be eliminated.

$$(3x + y - z) + (-x - 4y + z) = (5) + (2)$$

$$3x + y - z - x - 4y + z = 7$$

$$3x + y - x - 4y = 7$$

[5] $2x - 3y = 7$

Plug equation [4] into equation [5] to solve for x.

$$2x - 3y = 7$$

$$2x - 3\left(-\frac{11}{2}\right) = 7$$

$$2x + \frac{33}{2} = 7$$

$$4x + 33 = 14$$

$$4x = -19$$

$$x = -\frac{19}{4}$$

Plug in $x = -19/4$ and $y = -11/2$ into any of the original equations to solve for z. We'll use equation [3].

$$-x - 4y + z = 2$$

$$-\left(-\frac{19}{4}\right) - 4\left(-\frac{11}{2}\right) + z = 2$$

$$\frac{19}{4} + \frac{44}{2} + z = 2$$

$$19 + 88 + 4z = 8$$

$$4z = -99$$

$$z = -\frac{99}{4}$$

Therefore, the solution to the system is $(x, y, z) = (-19/4, -11/2, -99/4)$.

MATRIX DIMENSIONS AND ENTRIES

■ 1. Give the dimensions of the matrix.

$$D = \begin{bmatrix} 11 & 9 \\ -4 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix D has 2 rows and 2 columns, so D is a 2×2 matrix.

■ 2. Give the dimensions of the matrix.

$$A = \begin{bmatrix} 3 & 5 & -2 & 1 & 8 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix A has 1 row and 5 columns, so A is a 1×5 matrix.

■ 3. Given matrix J, find $J_{4,1}$.

$$J = \begin{bmatrix} 6 \\ 2 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

The value of $J_{4,1}$ is the entry in the fourth row, first column of matrix J, which is 1, so $J_{4,1} = 1$.

4. Given matrix C, find $C_{1,2}$.

$$C = \begin{bmatrix} 3 & 12 \\ 1 & 4 \\ 9 & 5 \\ -3 & 2 \end{bmatrix}$$

Solution:

The value of $C_{1,2}$ is the entry in the first row, second column of matrix C, which is 12, so $C_{1,2} = 12$.

5. Given matrix N, state the dimensions and find $N_{1,3}$.

$$N = \begin{bmatrix} 1 & 5 & 9 \\ 14 & -8 & 6 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix N has 2 rows and 3 columns, so N is a 2×3 matrix.

The value of $N_{1,3}$ is the entry in the first row, third column of matrix N, which is 9, so $N_{1,3} = 9$.

■ 6. Given matrix S, state the dimensions and find $S_{3,4}$.

$$S = \begin{bmatrix} 3 & 6 & -7 & 1 & 0 \\ 0 & 9 & 15 & 3 & 4 \\ 4 & 0 & 2 & 11 & 8 \\ -5 & 8 & 7 & 9 & 2 \end{bmatrix}$$

Solution:

We always give the dimensions of a matrix as rows $\times$ columns. Matrix S has 4 rows and 5 columns, so S is a 4×5 matrix.

The value of $S_{3,4}$ is the entry in the third row, fourth column of matrix S, which is 11, so $S_{3,4} = 11$.

REPRESENTING SYSTEMS WITH MATRICES

■ 1. Represent the system with a matrix called A.

$$-2x + 5y = 12$$

$$6x - 2y = 4$$

Solution:

The system contains the variables x and y along with a constant. Which means the matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$A = \begin{bmatrix} -2 & 5 & 12 \\ 6 & -2 & 4 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$A = \begin{bmatrix} -2 & 5 & | & 12 \\ 6 & -2 & | & 4 \end{bmatrix}$$

■ 2. Represent the system with a matrix called D.

$$9y - 3x + 12 = 0$$

$$8 - 4x = 11y$$

Solution:

This system can be reorganized by putting each equation in order, with x and y on the left side, and the constant on the right side.

$$-3x + 9y = -12$$

$$4x + 11y = 8$$

The system contains the variables x and y along with a constant. Which means the matrix will have two columns, one for each variable, plus a column for the constants, so three columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$D = \begin{bmatrix} -3 & 9 & -12 \\ 4 & 11 & 8 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$D = \left[\begin{array}{cc|c} -3 & 9 & -12 \\ 4 & 11 & 8 \end{array}\right]$$

■ 3. Represent the system with an augmented matrix called H.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b = -2c + 1$$

Solution:

The second equation can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no d-term in the second equation, so we add in a 0 "filler" term.

$$4a + 7b - 5c + 13d = 6$$

$$3a - 8b + 2c + 0d = 1$$

The system contains the variables a, b, c, and d, along with a constant. Which means the matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$H = \begin{bmatrix} 4 & 7 & -5 & 13 & 6 \\ 3 & -8 & 2 & 0 & 1 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$H = \begin{bmatrix} 4 & 7 & -5 & 13 & | & 6 \\ 3 & -8 & 2 & 0 & | & 1 \end{bmatrix}$$

■ 4. Represent the system with a matrix called M.

$$-2x + 4y = 9 - 6z$$

$$7y + 2z - 3 = -3t - 9x$$

Solution:

Both equations can be reorganized by putting x, y, z, and t on the left side, and the constant on the right side. We also recognize that there is no t -term in the first equation, so we add in a 0 "filler" term.

$$-2x + 4y + 6z + 0t = 9$$

$$9x + 7y + 2z + 3t = 3$$

The system contains the variables x, y, z, and t, along with a constant. Which means the matrix will have four columns, one for each variable, plus a column for the constants, so five columns in total. Because there are two equations in the system, the matrix will have two rows. Plugging the coefficients and constants into a matrix gives

$$M = \begin{bmatrix} -2 & 4 & 6 & 0 & 9 \\ 9 & 7 & 2 & 3 & 3 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$M = \begin{bmatrix} -2 & 4 & 6 & 0 & | & 9 \\ 9 & 7 & 2 & 3 & | & 3 \end{bmatrix}$$

■ 5. Represent the system with a matrix called A.

$$3x - 8y + z = 7$$

$$2z = 3y - 2x + 4$$

$$5y = 12 - 9x$$

Solution:

The second and third equations can be reorganized by putting x, y, and z on the left side, and the constant on the right side. We also recognize that there is no z-term in the third equation, so we add in a 0 "filler" term.

$$3x - 8y + z = 7$$

$$2x - 3y + 2z = 4$$

$$9x + 5y + 0z = 12$$

The system contains the variables x, y, and z, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into a matrix gives

$$A = \begin{bmatrix} 3 & -8 & 1 & 7 \\ 2 & -3 & 2 & 4 \\ 9 & 5 & 0 & 12 \end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$A = \begin{bmatrix} 3 & -8 & 1 & | & 7 \\ 2 & -3 & 2 & | & 4 \\ 9 & 5 & 0 & | & 12 \end{bmatrix}$$

6. Represent the system with a matrix called K.

$$-4b+2c=3-7a$$

$$9c=4-2b$$

$$8a-2c=5b$$

Solution:

All three of these equations can be reorganized by putting a, b, and c on the left side, and the constant on the right side. We also recognize that there is no a-term in the second equation, and no constant in the third equation, so we add in 0 "filler" terms.

$$7a-4b+2c=3$$

$$0a+2b+9c=4$$

$$8a-5b-2c=0$$

The system contains the variables a, b, and c, along with a constant. Which means the augmented matrix will have three columns, one for each variable, plus a column for the constants, so four columns in total. Because there are three equations in the system, the matrix will have three rows. Plugging the coefficients and constants into a matrix gives

$$K=\begin{bmatrix}7 & -4 & 2 & 3\\ 0 & 2 & 9 & 4\\ 8 & -5 & -2 & 0\end{bmatrix}$$

Alternatively, it would be equally correct to express the matrix as

$$K=\begin{bmatrix}7 & -4 & 2 & | & 3\\ 0 & 2 & 9 & | & 4\\ 8 & -5 & -2 & | & 0\end{bmatrix}$$

SIMPLE ROW OPERATIONS

■ 1. Write the new matrix after $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 2 & 6 & -4 & 1 \\ 8 & 2 & 1 & -5 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow R_2$ is switching row 1 with row 2. The matrix after $R_1 \leftrightarrow R_2$ is

$$\begin{bmatrix} 8 & 2 & 1 & -5 \\ 2 & 6 & -4 & 1 \end{bmatrix}$$

■ 2. Write the new matrix after $R_2 \leftrightarrow R_4$.

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 6 & 1 & 5 & -4 \\ -7 & 7 & 0 & 3 \\ 9 & 2 & 8 & 3 \end{bmatrix}$$

Solution:

The operation described by $R_2 \leftrightarrow R_4$ is switching row 2 with row 4. Nothing will happen to rows 1 and 3. The matrix after $R_2 \leftrightarrow R_4$ is

$$\begin{bmatrix} 1 & 2 & 7 & -3 \\ 9 & 2 & 8 & 3 \\ -7 & 7 & 0 & 3 \\ 6 & 1 & 5 & -4 \end{bmatrix}$$

■ 3. Write the new matrix after $R_1 \leftrightarrow 3R_2$.

$$\begin{bmatrix} 9 & 2 & -7 \\ 1 & 6 & 4 \end{bmatrix}$$

Solution:

The operation described by $R_1 \leftrightarrow 3R_2$ is multiplying row 2 by a constant of 3 and then switching those two rows. The matrix after $3R_2$ is

$$\begin{bmatrix} 9 & 2 & -7 \\ 3 & 18 & 12 \end{bmatrix}$$

The matrix after $R_1 \leftrightarrow 3R_2$ is

$$\begin{bmatrix} 3 & 18 & 12 \\ 9 & 2 & -7 \end{bmatrix}$$

■ 4. Write the new matrix after $3R_2 \leftrightarrow 3R_4$.

$$\begin{bmatrix} 0 & 11 & 6 \\ 7 & -3 & 9 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

Solution:

The operation described by $3R_2 \leftrightarrow 3R_4$ is multiplying row 2 by a constant of 3, multiplying row 4 by a constant of 3, and then switching those two rows. Nothing will happen to rows 1 and 3. The matrix after $3R_2$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 6 & 2 & 4 \end{bmatrix}$$

The matrix after $3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 21 & -9 & 27 \\ 8 & 8 & 1 \\ 18 & 6 & 12 \end{bmatrix}$$

The matrix after $3R_2 \leftrightarrow 3R_4$ is

$$\begin{bmatrix} 0 & 11 & 6 \\ 18 & 6 & 12 \\ 8 & 8 & 1 \\ 21 & -9 & 27 \end{bmatrix}$$

5. Write the new matrix after $R_1 + 2R_2 \to R_1$.

$$\begin{bmatrix} 6 & 2 & 7 \\ 1 & -5 & 15 \end{bmatrix}$$

Solution:

The operation described by $R_1 + 2R_2 \to R_1$ is multiplying row 2 by a constant of 2, adding that resulting row to row 1, and using that result to replace row 1. $2R_2$ is

$$\begin{bmatrix} 2(1) & 2(-5) & 2(15) \end{bmatrix}$$

$$\begin{bmatrix} 2 & -10 & 30 \end{bmatrix}$$

The sum $R_1 + 2R_2$ is

$$\begin{bmatrix} 6+2 & 2-10 & 7+30 \end{bmatrix}$$

$$\begin{bmatrix} 8 & -8 & 37 \end{bmatrix}$$

The matrix after $R_1 + 2R_2 \to R_1$, which is replacing row 1 with this row we just found, is

$$\begin{bmatrix} 8 & -8 & 37 \\ 1 & -5 & 15 \end{bmatrix}$$

6. Write the new matrix after $4R_2 + R_3 \to R_3$.

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 4 & 1 & 7 & -3 \end{bmatrix}$$

Solution:

The operation described by $4R_2 + R_3 \to R_3$ is multiplying row 2 by a constant of 4, adding that resulting row to row 3, and using that result to replace row 3. $4R_2$ is

$$\begin{bmatrix} 4(8) & 4(2) & 4(0) & 4(6) \end{bmatrix}$$

$$\begin{bmatrix} 32 & 8 & 0 & 24 \end{bmatrix}$$

The sum $4R_2 + R_3$ is

$$\begin{bmatrix} 32+4 & 8+1 & 0+7 & 24-3 \end{bmatrix}$$

$$\begin{bmatrix} 36 & 9 & 7 & 21 \end{bmatrix}$$

The matrix after $4R_2 + R_3 \to R_3$, which is replacing row 3 with this row we just found, is

$$\begin{bmatrix} 13 & 5 & -2 & 9 \\ 8 & 2 & 0 & 6 \\ 36 & 9 & 7 & 21 \end{bmatrix}$$

PIVOT ENTRIES AND ROW-ECHELON FORMS

1. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 3 & 6 & -7 \\ 1 & 2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

Start with $R_1 \leftrightarrow R_2$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 6 & -7 \\ 1 & 2 & 1 \end{bmatrix}$$

After $-3R_1 + R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -4 \\ 1 & 2 & 1 \end{bmatrix}$$

After $-R_1 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

We'll use $-(1/4)R_2 \to R_2$ to get the pivot entry in the second row.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

After $-2R_2 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Now all the pivot entries are 1, the zeroed-out row is at the bottom, and the pivot entries follow a staircase pattern. Therefore, the matrix is in row-echelon form.

■ 2. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & -6 & 0 \end{bmatrix}$$

Solution:

Start with $(1/3)R_4 \to R_4$.

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 0 \end{bmatrix}$$

After $R_1 \leftrightarrow R_4$, we get

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

After $R_2 - R_1 \to R_2$, we get

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Use $R_3 \leftrightarrow R_4$ to move the zero row to the bottom.

$$\begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Now all the pivot entries are 1, the zeroed-out row is at the bottom, and the pivot entries follow a staircase pattern. All the pivot columns include only the pivot entry, and otherwise all zero entries. Therefore, the matrix is in reduced row-echelon form.

■ 3. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & -3 & 9 \\ 0 & 0 & 7 \end{bmatrix}$$

Solution:

Start with $-(1/3)R_2 \to R_2$.

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 7 \end{bmatrix}$$

After $(1/7)R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

After $3R_3 + R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & 5 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After $-2R_3 + R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

After $-5R_2 + R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now all the pivot entries are 1, and follow a staircase pattern. All the pivot columns include only the pivot entry, and otherwise only 0 entries. Therefore, the matrix is in reduced row-echelon form.

■ 4. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 3 & 2 & 0 & 9 \\ 2 & 4 & -3 & -1 \\ 2 & 12 & -12 & 1 \end{bmatrix}$$

Solution:

Start with $R_1 - R_2 \to R_1$.

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 2 & 4 & -3 & -1 \\ 2 & 12 & -12 & 1 \end{bmatrix}$$

After $-R_2 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 2 & 4 & -3 & -1 \\ 0 & 8 & -9 & 2 \end{bmatrix}$$

After $-2R_1 + R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 8 & -9 & -21 \\ 0 & 8 & -9 & 2 \end{bmatrix}$$

After $-R_2+R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 8 & -9 & -21 \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

After $(1/8)R_2 \to R_2$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 1 & -\frac{9}{8} & -\frac{21}{8} \\ 0 & 0 & 0 & 23 \end{bmatrix}$$

After $(1/23)R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 & 3 & 10 \\ 0 & 1 & -\frac{9}{8} & -\frac{21}{8} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this matrix, the first non-zero entry in each row is a 1, and the pivots follow a staircase pattern that moves down and to the right. Therefore, this matrix is in row-echelon form.

■ 5. Use row operations to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 \\ 3 & 1 \\ -3 & 0 \\ 2 & -3 \end{bmatrix}$$

Solution:

After $R_2 - 3R_1 \to R_2$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 7 \\ -3 & 0 \\ 2 & -3 \end{bmatrix}$$

After $3R_1 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 7 \\ 0 & -6 \\ 2 & -3 \end{bmatrix}$$

After $-2R_1 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 7 \\ 0 & -6 \\ 0 & 1 \end{bmatrix}$$

After $R_2 \leftrightarrow R_4$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 7 \\ 0 & -6 \end{bmatrix}$$

After $R_3 - 7R_2 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \\ 0 & -6 \end{bmatrix}$$

After $R_3 + 6R_2 \to R_3$, we get

$$\begin{bmatrix} 1 & -2 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

After $2R_2 + R_1 \to R_1$, we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Now all the pivot entries are 1, the zeroed-out rows are at the bottom, the pivot entries follow a staircase pattern, and all the pivot columns include only the pivot entry, and otherwise all 0 entries. Therefore, the matrix is in reduced row-echelon form.

■ 6. Use row operations to put the matrix into row-echelon form.

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ -1 & 3 & -6 & -13 \\ -5 & -2 & 22 & -28 \end{bmatrix}$$

Solution:

Start with $R_3 + R_1 \to R_3$.

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 3 & -9 & -6 \\ -5 & -2 & 22 & -28 \end{bmatrix}$$

After $5R_1 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 3 & -9 & -6 \\ 0 & -2 & 7 & 7 \end{bmatrix}$$

After $-3R_2 + R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -3 & -15 \\ 0 & -2 & 7 & 7 \end{bmatrix}$$

After $2R_2 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & -3 & -15 \\ 0 & 0 & 3 & 13 \end{bmatrix}$$

After $-(1/3)R_3 \to R_3$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 3 & 13 \end{bmatrix}$$

After $-3R_3 + R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

After $-(1/2)R_4 \to R_4$, we get

$$\begin{bmatrix} 1 & 0 & -3 & 7 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In this matrix, the first non-zero entry in each row is a 1, and the pivots follow a staircase pattern. Therefore, the matrix is in row-echelon form.

GAUSS-JORDAN ELIMINATION

■ 1. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$x + 2y = -2$$

$$3x + 2y = 6$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 1 & 2 & -2 \\ 3 & 2 & 6 \end{bmatrix}$$

After $3R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 4 & -12 \end{bmatrix}$$

The first column is done. After $(1/4)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \end{bmatrix}$$

The second column is done, and we can see that the solution to the linear system is $(x, y) = (4, -3)$.

■ 2. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$2x + 4y = 22$$

$$3x + 3y = 15$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 2 & 4 & 22 \\ 3 & 3 & 15 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$ and $(1/3)R_2 \to R_2$the matrix is

$$\begin{bmatrix} 1 & 2 & 11 \\ 1 & 1 & 5 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 11 \\ 0 & 1 & 6 \end{bmatrix}$$

The first column is done. After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 6 \end{bmatrix}$$

The second column is done, and we can see that the solution to the linear system is $(x, y) = (-1,6)$.

■ 3. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$x - 3y - 6z = 4$$

$$y + 2z = -2$$

$$-4x + 12y + 21z = -4$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 1 & -3 & -6 & 4 \\ 0 & 1 & 2 & -2 \\ -4 & 12 & 21 & -4 \end{bmatrix}$$

After $4R_1 + R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & -3 & -6 & 4 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -3 & 12 \end{bmatrix}$$

The first column is done. After $3R_2 + R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & -3 & 12 \end{bmatrix}$$

The second column is done. After $(-1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -4 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (-2, 6, -4)$.

■ 4. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$2y + 4z = 4$$

$$x + 3y + 3z = 5$$

$$2x + 7y + 6z = 10$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 0 & 2 & 4 & 4 \\ 1 & 3 & 3 & 5 \\ 2 & 7 & 6 & 10 \end{bmatrix}$$

After $(1/2)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 3 & 3 & 5 \\ 2 & 7 & 6 & 10 \end{bmatrix}$$

Because the first entry in the first row is 0, swap it with the second row to get

$$\begin{bmatrix} 1 & 3 & 3 & 5 \\ 0 & 1 & 2 & 2 \\ 2 & 7 & 6 & 10 \end{bmatrix}$$

After $R_3 - 2R_1 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 3 & 3 & 5 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

The first column is done. After $R_1 - 3R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

After $R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 2 & 2 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -1 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (2,0,1)$.

■ 5. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$3x + 12y + 42z = -27$$

$$x + 2y + 8z = -5$$

$$2x + 5y + 16z = -6$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 3 & 12 & 42 & -27 \\ 1 & 2 & 8 & -5 \\ 2 & 5 & 16 & -6 \end{bmatrix}$$

After $(1/3)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 1 & 2 & 8 & -5 \\ 2 & 5 & 16 & -6 \end{bmatrix}$$

After $R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 0 & 2 & 6 & -4 \\ 2 & 5 & 16 & -6 \end{bmatrix}$$

After $2R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 0 & 2 & 6 & -4 \\ 0 & 3 & 12 & -12 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 4 & 14 & -9 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 12 & -12 \end{bmatrix}$$

After $R_1 - 4R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 12 & -12 \end{bmatrix}$$

After $R_3 - 3R_2 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 3 & -6 \end{bmatrix}$$

The second column is done. After $(1/3)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

After $R_1 - 2R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

After $R_2 - 3R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (3,4,-2)$.

■ 6. Use Gauss-Jordan elimination to find the solution to the linear system from the rref matrix.

$$4x + 8y + 4z = 20$$

$$4x + 6y = 4$$

$$3x + 3y - z = 1$$

Solution:

The matrix for the system is

$$\begin{bmatrix} 4 & 8 & 4 & 20 \\ 4 & 6 & 0 & 4 \\ 3 & 3 & -1 & 1 \end{bmatrix}$$

After $(1/4)R_1 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 4 & 6 & 0 & 4 \\ 3 & 3 & -1 & 1 \end{bmatrix}$$

After $4R_1 - R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 2 & 4 & 16 \\ 3 & 3 & -1 & 1 \end{bmatrix}$$

After $3R_1 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 2 & 4 & 16 \\ 0 & 3 & 4 & 14 \end{bmatrix}$$

The first column is done. After $(1/2)R_2 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 2 & 1 & 5 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 4 & 14 \end{bmatrix}$$

After $R_1 - 2R_2 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 4 & 14 \end{bmatrix}$$

After $3R_2 - R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 2 & 10 \end{bmatrix}$$

The second column is done. After $(1/2)R_3 \to R_3$, the matrix is

$$\begin{bmatrix} 1 & 0 & -3 & -11 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

After $R_1 + 3R_3 \to R_1$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

After $R_2 - 2R_3 \to R_2$, the matrix is

$$\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 5 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (4, -2, 5)$.

NUMBER OF SOLUTIONS TO THE LINEAR SYSTEM

1. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$2x - 8y = 18$$

$$-7x + 2y - 5z = -6$$

$$3x + 2z = 1$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 2 & -8 & 0 & | & 18 \\ -7 & 2 & -5 & | & -6 \\ 3 & 0 & 2 & | & 1 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -4 & 0 & | & 9 \\ -7 & 2 & -5 & | & -6 \\ 3 & 0 & 2 & | & 1 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -4 & 0 & | & 9 \\ 0 & -26 & -5 & | & 57 \\ 0 & 12 & 2 & | & -26 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -4 & 0 & | & 9 \\ 0 & 1 & \frac{5}{26} & | & -\frac{57}{26} \\ 0 & 12 & 2 & | & -26 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{10}{13} & | & \frac{3}{13} \\ 0 & 1 & \frac{5}{26} & | & -\frac{57}{26} \\ 0 & 0 & -\frac{4}{13} & | & \frac{4}{13} \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & \frac{10}{13} & | & \frac{3}{13} \\ 0 & 1 & \frac{5}{26} & | & -\frac{57}{26} \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

Therefore, there's one unique solution to the system, $(x, y, z) = (1, -2, -1)$.

■ 2. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$-x + 3y - 5z - 8w = 2$$

$$4x - 8y + 4z + 4w = -44$$

$$3x + 5y - 16z + w = 18$$

$$-x + y - 3z - w = 6$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & 3 & -5 & -8 & | & 2 \\ 4 & -8 & 4 & 4 & | & -44 \\ 3 & 5 & -16 & 1 & | & 18 \\ -1 & 1 & -3 & -1 & | & 6 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -3 & 5 & 8 & | & -2 \\ 4 & -8 & 4 & 4 & | & -44 \\ 3 & 5 & -16 & 1 & | & 18 \\ -1 & 1 & -3 & -1 & | & 6 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -3 & 5 & 8 & | & -2 \\ 0 & 4 & -16 & -28 & | & -36 \\ 0 & 14 & -31 & -23 & | & 24 \\ 0 & -2 & 2 & 7 & | & 4 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -3 & 5 & 8 & | & -2 \\ 0 & 1 & -4 & -7 & | & -9 \\ 0 & 14 & -31 & -23 & | & 24 \\ 0 & -2 & 2 & 7 & | & 4 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -7 & -13 & | & -29 \\ 0 & 1 & -4 & -7 & | & -9 \\ 0 & 0 & 25 & 75 & | & 150 \\ 0 & 0 & -6 & -7 & | & -14 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & -7 & -13 & | & -29 \\ 0 & 1 & -4 & -7 & | & -9 \\ 0 & 0 & 1 & 3 & | & 6 \\ 0 & 0 & -6 & -7 & | & -14 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & 8 & | & 13 \\ 0 & 1 & 0 & 5 & | & 15 \\ 0 & 0 & 1 & 3 & | & 6 \\ 0 & 0 & 0 & 11 & | & 22 \end{bmatrix}$$

Find the pivot entry in the fourth row.

$$\begin{bmatrix} 1 & 0 & 0 & 8 & | & 13 \\ 0 & 1 & 0 & 5 & | & 15 \\ 0 & 0 & 1 & 3 & | & 6 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Zero out the rest of the fourth column.

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & -3 \\ 0 & 1 & 0 & 0 & | & 5 \\ 0 & 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & 1 & | & 2 \end{bmatrix}$$

Therefore, there is one unique solution to the system, $(x, y, z, w) = (-3, 5, 0, 2)$.

■ 3. How many solutions does the linear system have?

$$3x - 3y + 5z = -11$$

$$-2x + y - 2z = 5$$

$$x + y - z = 9$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 3 & -3 & 5 & | & -11 \\ -2 & 1 & -2 & | & 5 \\ 1 & 1 & -1 & | & 9 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -1 & \frac{5}{3} & | & -\frac{11}{3} \\ -2 & 1 & -2 & | & 5 \\ 1 & 1 & -1 & | & 9 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -1 & \frac{5}{3} & | & -\frac{11}{3} \\ 0 & -1 & \frac{4}{3} & | & -\frac{7}{3} \\ 0 & 2 & -\frac{8}{3} & | & \frac{38}{3} \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -1 & \frac{5}{3} & | & -\frac{11}{3} \\ 0 & 1 & -\frac{4}{3} & | & \frac{7}{3} \\ 0 & 2 & -\frac{8}{3} & | & \frac{38}{3} \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & | & -\frac{4}{3} \\ 0 & 1 & -\frac{4}{3} & | & \frac{7}{3} \\ 0 & 0 & 0 & | & 8 \end{bmatrix}$$

The third row tells us that $0 = 8$, which can't possibly be true. Therefore, the system has no solutions.

■ 4. How many solutions does the linear system have?

$$-x + 6y + 4z = -22$$

$$4x - 22y - 2z + 2w = 0$$

$$x - 6y - 5z + 3w = 5$$

$$-3y - 22z = 6$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} -1 & 6 & 4 & 0 & | & -22 \\ 4 & -22 & -2 & 2 & | & 0 \\ 1 & -6 & -5 & 3 & | & 5 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & -6 & -4 & 0 & | & 22 \\ 4 & -22 & -2 & 2 & | & 0 \\ 1 & -6 & -5 & 3 & | & 5 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & -6 & -4 & 0 & | & 22 \\ 0 & 2 & 14 & 2 & | & -88 \\ 0 & 0 & -1 & 3 & | & -17 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & -6 & -4 & 0 & | & 22 \\ 0 & 1 & 7 & 1 & | & -44 \\ 0 & 0 & -1 & 3 & | & -17 \\ 0 & -3 & -22 & 0 & | & 6 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & 38 & 6 & | & -242 \\ 0 & 1 & 7 & 1 & | & -44 \\ 0 & 0 & -1 & 3 & | & -17 \\ 0 & 0 & -1 & 3 & | & -126 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & 38 & 6 & | & -242 \\ 0 & 1 & 7 & 1 & | & -44 \\ 0 & 0 & 1 & -3 & | & 17 \\ 0 & 0 & -1 & 3 & | & -126 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & 120 & | & -888 \\ 0 & 1 & 0 & 22 & | & -163 \\ 0 & 0 & 1 & -3 & | & 17 \\ 0 & 0 & 0 & 0 & | & -109 \end{bmatrix}$$

The third row tells that $0 = -109$, which can't be true. Therefore, the system has no solutions.

■ 5. Determine whether the system has one solution, no solutions, or infinitely many solutions.

$$2x + 2y - 8z = 4$$

$$-3x - 5y + 6z = -4$$

$$5x - y - 38z = 16$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 2 & 2 & -8 & | & 4 \\ -3 & -5 & 6 & | & -4 \\ 5 & -1 & -38 & | & 16 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form, starting with finding the pivot entry in the first row.

$$\begin{bmatrix} 1 & 1 & -4 & | & 2 \\ -3 & -5 & 6 & | & -4 \\ 5 & -1 & -38 & | & 16 \end{bmatrix}$$

Zero out the rest of the first column.

$$\begin{bmatrix} 1 & 1 & -4 & | & 2 \\ 0 & -2 & -6 & | & 2 \\ 0 & -6 & -18 & | & 6 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & 1 & -4 & | & 2 \\ 0 & 1 & 3 & | & -1 \\ 0 & -6 & -18 & | & 6 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -7 & | & 3 \\ 0 & 1 & 3 & | & -1 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since the entire last row has only zeros, the linear system has infinitely many solutions.

6. For the linear system below, determine whether it has one solution, no solutions, or infinitely many solutions.

$$x + y - z + 2w = 7$$

$$4x + 2y - 6z + 2w = 16$$

$$-3x + y + 7z + 6w = 3$$

$$-x - y + 4z + 3w = 8$$

Solution:

Rewrite the system as an augmented matrix.

$$\begin{bmatrix} 1 & 1 & -1 & 2 & | & 7 \\ 4 & 2 & -6 & 2 & | & 16 \\ -3 & 1 & 7 & 6 & | & 3 \\ -1 & -1 & 4 & 3 & | & 8 \end{bmatrix}$$

Work toward putting the matrix into reduced row-echelon form. First, zero out the rest of the first column.

$$\begin{bmatrix} 1 & 1 & -1 & 2 & | & 7 \\ 0 & -2 & -2 & -6 & | & -12 \\ 0 & 4 & 4 & 12 & | & 24 \\ 0 & 0 & 3 & 5 & | & 15 \end{bmatrix}$$

Find the pivot entry in the second row.

$$\begin{bmatrix} 1 & 1 & -1 & 2 & | & 7 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 4 & 4 & 12 & | & 24 \\ 0 & 0 & 3 & 5 & | & 15 \end{bmatrix}$$

Zero out the rest of the second column.

$$\begin{bmatrix} 1 & 0 & -2 & -1 & | & 1 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & 5 & | & 15 \end{bmatrix}$$

Switch the third and fourth rows.

$$\begin{bmatrix} 1 & 0 & -2 & -1 & | & 1 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 0 & 3 & 5 & | & 15 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Find the pivot entry in the third row.

$$\begin{bmatrix} 1 & 0 & -2 & -1 & | & 1 \\ 0 & 1 & 1 & 3 & | & 6 \\ 0 & 0 & 1 & \frac{5}{3} & | & 5 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & 0 & \frac{7}{3} & | & 11 \\ 0 & 1 & 0 & \frac{4}{3} & | & 1 \\ 0 & 0 & 1 & \frac{5}{3} & | & 5 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Since the entire last row has only zeros, the linear system has infinitely many solutions.

MATRIX ADDITION AND SUBTRACTION

■ 1. Add the matrices.

$$\begin{bmatrix} 7 & 6 \\ 17 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -2 & 5 \end{bmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{bmatrix} 7 & 6 \\ 17 & 9 \end{bmatrix} + \begin{bmatrix} 0 & 8 \\ -2 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 7+0 & 6+8 \\ 17+(-2) & 9+5 \end{bmatrix}$$

$$\begin{bmatrix} 7 & 14 \\ 15 & 14 \end{bmatrix}$$

■ 2. Add the matrices.

$$\begin{bmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{bmatrix}$$

Solution:

To add matrices, you simply add together entries from corresponding positions in each matrix.

$$\begin{bmatrix} 8 & 3 \\ -4 & 7 \\ 6 & 0 \\ 1 & 13 \end{bmatrix} + \begin{bmatrix} 6 & 7 \\ 2 & -3 \\ 9 & 11 \\ 7 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 8+6 & 3+7 \\ -4+2 & 7+(-3) \\ 6+9 & 0+11 \\ 1+7 & 13+(-2) \end{bmatrix}$$

$$\begin{bmatrix} 14 & 10 \\ -2 & 4 \\ 15 & 11 \\ 8 & 11 \end{bmatrix}$$

■ 3. Subtract the matrices.

$$\begin{bmatrix} 7 & 9 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 8 \\ 12 & -3 \end{bmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{bmatrix} 7 & 9 \\ 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 8 \\ 12 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 7-3 & 9-8 \\ 4-12 & -1-(-3) \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 \\ -8 & 2 \end{bmatrix}$$

■ 4. Subtract the matrices.

$$\begin{bmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{bmatrix}$$

Solution:

To subtract matrices, you simply subtract entries from corresponding positions in each matrix.

$$\begin{bmatrix} 8 & 11 & 2 & 9 \\ 6 & 3 & 16 & 8 \end{bmatrix} - \begin{bmatrix} 6 & 11 & 7 & -4 \\ 5 & 8 & 1 & 15 \end{bmatrix}$$

$$\begin{bmatrix} 8-6 & 11-11 & 2-7 & 9-(-4) \\ 6-5 & 3-8 & 16-1 & 8-15 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & -5 & 13 \\ 1 & -5 & 15 & -7 \end{bmatrix}$$

■ 5. Solve for M.

$$\begin{bmatrix} 6 & 5 \\ 9 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix} = M + \begin{bmatrix} 7 & 12 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 8 \\ 4 & -7 \end{bmatrix}$$

Solution:

Let's start with the matrix addition on the left side of the equation and the matrix subtraction on the right side of the equation.

$$\begin{bmatrix} 6 & 5 \\ 9 & -9 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 1 & 6 \end{bmatrix} = M + \begin{bmatrix} 7 & 12 \\ -3 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 8 \\ 4 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 6+3 & 5+7 \\ 9+1 & -9+6 \end{bmatrix} = M + \begin{bmatrix} 7-1 & 12-8 \\ -3-4 & -1-(-7) \end{bmatrix}$$

$$\begin{bmatrix} 9 & 12 \\ 10 & -3 \end{bmatrix} = M + \begin{bmatrix} 6 & 4 \\ -7 & 6 \end{bmatrix}$$

To isolate M, we'll subtract the matrix on the right from both sides in order to move it to the left.

$$\begin{bmatrix} 9 & 12 \\ 10 & -3 \end{bmatrix} - \begin{bmatrix} 6 & 4 \\ -7 & 6 \end{bmatrix} = M$$

$$\begin{bmatrix} 9-6 & 12-4 \\ 10-(-7) & -3-6 \end{bmatrix} = M$$

$$\begin{bmatrix} 3 & 8 \\ 17 & -9 \end{bmatrix} = M$$

The conclusion is that the value of M that makes the equation true is this matrix:

$$M = \begin{bmatrix} 3 & 8 \\ 17 & -9 \end{bmatrix}$$

■ 6. Solve for N.

$$\begin{bmatrix} 4 & 12 \\ 9 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 9 & 9 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

Solution:

Let's start with the matrix subtraction on the left side of the equation.

$$\begin{bmatrix} 4 & 12 \\ 9 & 8 \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 9 & 9 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4-0 & 12-3 \\ 9-9 & 8-9 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 9 \\ 0 & -1 \end{bmatrix} = N - \begin{bmatrix} 6 & 3 \\ 5 & 11 \end{bmatrix} + \begin{bmatrix} 7 & -4 \\ -18 & 1 \end{bmatrix}$$

To isolate N, we'll move the matrices on the right side over to the left side, then flip the equation.

$$\begin{bmatrix}4 & 9\\0 & -1\end{bmatrix}+\begin{bmatrix}6 & 3\\5 & 11\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}=N$$

$$N=\begin{bmatrix}4 & 9\\0 & -1\end{bmatrix}+\begin{bmatrix}6 & 3\\5 & 11\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}$$

Simplify the right side to solve the equation for N.

$$N=\begin{bmatrix}4+6 & 9+3\\0+5 & -1+11\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}$$

$$N=\begin{bmatrix}10 & 12\\5 & 10\end{bmatrix}-\begin{bmatrix}7 & -4\\-18 & 1\end{bmatrix}$$

$$N=\begin{bmatrix}10-7 & 12-(-4)\\5-(-18) & 10-1\end{bmatrix}$$

$$N=\begin{bmatrix}3 & 16\\23 & 9\end{bmatrix}$$

SCALAR MULTIPLICATION

1. Use scalar multiplication to simplify the expression.

$$\frac{1}{4}\begin{bmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{bmatrix}$$

Solution:

The scalar $1/4$ is being multiplied by the matrix. Distribute the scalar across every entry in the matrix.

$$\frac{1}{4}\begin{bmatrix} 12 & 8 & 3 \\ 2 & -16 & 0 \\ 1 & 5 & 7 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4}(12) & \frac{1}{4}(8) & \frac{1}{4}(3) \\ \frac{1}{4}(2) & \frac{1}{4}(-16) & \frac{1}{4}(0) \\ \frac{1}{4}(1) & \frac{1}{4}(5) & \frac{1}{4}(7) \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 & \frac{3}{4} \\ \frac{1}{2} & -4 & 0 \\ \frac{1}{4} & \frac{5}{4} & \frac{7}{4} \end{bmatrix}$$

■ 2. Solve for Y.

$$4\begin{bmatrix}2 & 9\\ -5 & 0\end{bmatrix}+Y=5\begin{bmatrix}1 & -3\\ 6 & 8\end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{bmatrix}4(2) & 4(9)\\ 4(-5) & 4(0)\end{bmatrix}+Y=\begin{bmatrix}5(1) & 5(-3)\\ 5(6) & 5(8)\end{bmatrix}$$

$$\begin{bmatrix}8 & 36\\ -20 & 0\end{bmatrix}+Y=\begin{bmatrix}5 & -15\\ 30 & 40\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate Y.

$$Y=\begin{bmatrix}5 & -15\\ 30 & 40\end{bmatrix}-\begin{bmatrix}8 & 36\\ -20 & 0\end{bmatrix}$$

$$Y=\begin{bmatrix}5-8 & -15-36\\ 30-(-20) & 40-0\end{bmatrix}$$

$$Y=\begin{bmatrix}-3 & -51\\ 50 & 40\end{bmatrix}$$

■ 3. Solve for N.

$$-2\begin{bmatrix} 6 & 5 \\ 0 & 11 \end{bmatrix} = N - 4\begin{bmatrix} 2 & 4 \\ -1 & 9 \end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$\begin{bmatrix} -2(6) & -2(5) \\ -2(0) & -2(11) \end{bmatrix} = N - \begin{bmatrix} 4(2) & 4(4) \\ 4(-1) & 4(9) \end{bmatrix}$$

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} = N - \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix}$$

Add the matrix on the right to both sides of the equation in order to isolate N.

$$\begin{bmatrix} -12 & -10 \\ 0 & -22 \end{bmatrix} + \begin{bmatrix} 8 & 16 \\ -4 & 36 \end{bmatrix} = N$$

$$\begin{bmatrix} -12+8 & -10+16 \\ 0+(-4) & -22+36 \end{bmatrix} = N$$

$$\begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix} = N$$

$$N = \begin{bmatrix} -4 & 6 \\ -4 & 14 \end{bmatrix}$$

■ 4. Solve the equation for M.

$$-4M = \begin{bmatrix} -5 & 0 & 4 \\ 1 & -8 & -2 \\ -4 & 12 & 3 \end{bmatrix}$$

Solution:

Multiply both sides of the matrix equation by the scalar $-1/4$ in order to isolate M.

$$-\frac{1}{4}(-4M) = -\frac{1}{4}\begin{bmatrix} -5 & 0 & 4 \\ 1 & -8 & -2 \\ -4 & 12 & 3 \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{1}{4}(-5) & -\frac{1}{4}(0) & -\frac{1}{4}(4) \\ -\frac{1}{4}(1) & -\frac{1}{4}(-8) & -\frac{1}{4}(-2) \\ -\frac{1}{4}(-4) & -\frac{1}{4}(12) & -\frac{1}{4}(3) \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{5}{4} & 0 & -1 \\ -\frac{1}{4} & 2 & \frac{1}{2} \\ 1 & -3 & -\frac{3}{4} \end{bmatrix}$$

■ 5. Use scalar multiplication to simplify the expression.

$$-5A + \frac{1}{3}B$$

$$A = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ 3 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} -\frac{1}{3} & 0 \\ 6 & -2 \end{bmatrix}$$

Solution:

Apply the scalars to the matrices.

$$-5A + \frac{1}{3}B$$

$$-5\begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ 3 & 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -\frac{1}{3} & 0 \\ 6 & -2 \end{bmatrix}$$

$$\begin{bmatrix} -5(\frac{2}{5}) & -5(-\frac{1}{5}) \\ -5(3) & -5(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{3}(-\frac{1}{3}) & \frac{1}{3}(0) \\ \frac{1}{3}(6) & \frac{1}{3}(-2) \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -15 & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{9} & 0 \\ 2 & -\frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -2 - \frac{1}{9} & 1 + 0 \\ -15 + 2 & 0 - \frac{2}{3} \end{bmatrix}$$

$$\begin{bmatrix} -\frac{19}{9} & 1 \\ -13 & -\frac{2}{3} \end{bmatrix}$$

■ 6. Solve the equation for X.

$$2X - \frac{1}{2}\begin{bmatrix} 0 & -2 & 6 \\ 4 & -1 & 2 \\ 8 & 6 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 7 \\ 0 & -\frac{3}{2} & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

Solution:

Apply the scalar to the matrix.

$$2X - \begin{bmatrix} \frac{1}{2}(0) & \frac{1}{2}(-2) & \frac{1}{2}(6) \\ \frac{1}{2}(4) & \frac{1}{2}(-1) & \frac{1}{2}(2) \\ \frac{1}{2}(8) & \frac{1}{2}(6) & \frac{1}{2}(0) \end{bmatrix} = \begin{bmatrix} 6 & 3 & 7 \\ 0 & -\frac{3}{2} & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

$$2X - \begin{bmatrix} 0 & -1 & 3 \\ 2 & -\frac{1}{2} & 1 \\ 4 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 3 & 7 \\ 0 & -\frac{3}{2} & 1 \\ 6 & 5 & 4 \end{bmatrix}$$

Add the matrix on the left to both sides of the equation in order to isolate $2X$.

$$2X=\begin{bmatrix}6 & 3 & 7\\ 0 & -\frac{3}{2} & 1\\ 6 & 5 & 4\end{bmatrix}+\begin{bmatrix}0 & -1 & 3\\ 2 & -\frac{1}{2} & 1\\ 4 & 3 & 0\end{bmatrix}$$

$$2X=\begin{bmatrix}6+0 & 3-1 & 7+3\\ 0+2 & -\frac{3}{2}-\frac{1}{2} & 1+1\\ 6+4 & 5+3 & 4+0\end{bmatrix}$$

$$2X=\begin{bmatrix}6 & 2 & 10\\ 2 & -2 & 2\\ 10 & 8 & 4\end{bmatrix}$$

Multiply both sides of the equation by the scalar $1/2$ in order to isolate X.

$$\frac{1}{2}\cdot 2X=\frac{1}{2}\begin{bmatrix}6 & 2 & 10\\ 2 & -2 & 2\\ 10 & 8 & 4\end{bmatrix}$$

$$X=\begin{bmatrix}\frac{1}{2}(6) & \frac{1}{2}(2) & \frac{1}{2}(10)\\ \frac{1}{2}(2) & \frac{1}{2}(-2) & \frac{1}{2}(2)\\ \frac{1}{2}(10) & \frac{1}{2}(8) & \frac{1}{2}(4)\end{bmatrix}$$

$$X=\begin{bmatrix}3 & 1 & 5\\ 1 & -1 & 1\\ 5 & 4 & 2\end{bmatrix}$$

ZERO MATRICES

1. Add the zero matrix to the given matrix.

$$\begin{bmatrix} 8 & 17 \\ -6 & 0 \end{bmatrix}$$

Solution:

Adding the zero matrix to any other matrix doesn't change the value of the matrix, so

$$\begin{bmatrix} 8 & 17 \\ -6 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 17 \\ -6 & 0 \end{bmatrix}$$

2. Find the opposite matrix.

$$\begin{bmatrix} 6 & 8 & 0 \\ 2 & -3 & 11 \\ 4 & 12 & 9 \end{bmatrix}$$

Solution:

To get the opposite of a matrix, multiply it by a scalar of -1. Then the opposite of the given matrix is

$$(-1)\begin{bmatrix} (-1)6 & (-1)8 & (-1)0 \\ (-1)2 & (-1)(-3) & (-1)11 \\ (-1)4 & (-1)12 & (-1)9 \end{bmatrix}$$

$$\begin{bmatrix} -6 & -8 & 0 \\ -2 & 3 & -11 \\ -4 & -12 & -9 \end{bmatrix}$$

■ 3. Multiply the matrix by a scalar of 0.

$$\begin{bmatrix} 14 & -1 & 7 & 5 \\ 3 & 7 & 18 & -4 \end{bmatrix}$$

Solution:

Multiplying any matrix by a scalar of 0 results in a zero matrix.

$$(0)\begin{bmatrix} 14(0) & -1(0) & 7(0) & 5(0) \\ 3(0) & 7(0) & 18(0) & -4(0) \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

■ 4. Add the opposite of A to A.

$$A = \begin{bmatrix} 1 & -5 & 7 \\ -3 & 2 & 8 \end{bmatrix}$$

Solution:

The opposite matrix of A is

$$-A=(-1)\begin{bmatrix}1 & -5 & 7\\ -3 & 2 & 8\end{bmatrix}$$

$$-A=\begin{bmatrix}-1(1) & -1(-5) & -1(7)\\ -1(-3) & -1(2) & -1(8)\end{bmatrix}$$

$$-A=\begin{bmatrix}-1 & 5 & -7\\ 3 & -2 & -8\end{bmatrix}$$

Add the opposite matrices.

$$A+(-A)=\begin{bmatrix}1 & -5 & 7\\ -3 & 2 & 8\end{bmatrix}+\begin{bmatrix}-1 & 5 & -7\\ 3 & -2 & -8\end{bmatrix}$$

$$A+(-A)=\begin{bmatrix}1-1 & -5+5 & 7-7\\ -3+3 & 2-2 & 8-8\end{bmatrix}$$

$$A+(-A)=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

■ 5. Solve the equation for X.

$$X+\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & -5\\ 0 & 7 & 3\\ -4 & 0 & -1\end{bmatrix}+\begin{bmatrix}-1 & 0 & 5\\ 0 & -7 & -3\\ 4 & 0 & 1\end{bmatrix}$$

Solution:

Add the matrices on the right side.

$$X+\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}=\begin{bmatrix}1-1 & 0+0 & -5+5\\ 0+0 & 7-7 & 3-3\\ -4+4 & 0+0 & -1+1\end{bmatrix}$$

$$X+\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate X.

$$X=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}-\begin{bmatrix}-1 & 2 & 5\\ 7 & -4 & 3\\ 1 & -2 & 4\end{bmatrix}$$

$$X=\begin{bmatrix}0-(-1) & 0-2 & 0-5\\ 0-7 & 0-(-4) & 0-3\\ 0-1 & 0-(-2) & 0-4\end{bmatrix}$$

$$X=\begin{bmatrix}1 & -2 & -5\\ -7 & 4 & -3\\ -1 & 2 & -4\end{bmatrix}$$

■ 6. Solve the equation for A.

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=0\begin{bmatrix}-2&3&0\\-1&5&-2\\-7&0&4\end{bmatrix}+\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

Solution:

Apply the scalar 0 to the matrix.

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}0(-2)&0(3)&0(0)\\0(-1)&0(5)&0(-2)\\0(-7)&0(0)&0(4)\end{bmatrix}+\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}+\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

Adding the zero matrix does not change the value of the equation, so we can cancel it. Subtract the remaining matrices on the right side.

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}2&4&-7\\8&0&-5\\-1&4&3\end{bmatrix}-\begin{bmatrix}3&-1&-11\\10&0&-2\\-6&-3&12\end{bmatrix}$$

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}2-3&4-(-1)&-7-(-11)\\8-10&0-0&-5-(-2)\\-1-(-6)&4-(-3)&3-12\end{bmatrix}$$

$$\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}-A=\begin{bmatrix}-1&5&4\\-2&0&-3\\5&7&-9\end{bmatrix}$$

Subtract the matrix on the left from both sides of the equation in order to isolate $-A$.

$$-A = \begin{bmatrix} -1 & 5 & 4 \\ -2 & 0 & -3 \\ 5 & 7 & -9 \end{bmatrix} - \begin{bmatrix} -1 & 5 & 4 \\ -2 & 0 & -3 \\ 5 & 7 & -9 \end{bmatrix}$$

$$-A = \begin{bmatrix} -1-(-1) & 5-5 & 4-4 \\ -2-(-2) & 0-0 & -3-(-3) \\ 5-5 & 7-7 & -9-(-9) \end{bmatrix}$$

$$-A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Multiply both sides by -1 to solve for A.

$$A = (-1)\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

MATRIX MULTIPLICATION

■ 1. If matrix A is 3×3 and matrix B is 3×4, say whether AB or BA is defined, and give the dimensions of any product that's defined.

Solution:

Line up the dimensions for the products AB and BA, and compare the middle terms, which represent the columns from the first matrix and the rows from the second matrix.

$$AB: \quad 3 \times 3 \qquad 3 \times 4$$

$$BA: \quad 3 \times 4 \qquad 3 \times 3$$

The middle numbers match for AB, so that product is defined. For BA, the middle numbers don't match, so that product isn't defined.

The dimensions of AB are given by the outside numbers, which are the rows from the first matrix and the columns from the second matrix.

$$AB: \quad \boxed{3} \times 3 \qquad 3 \times \boxed{4}$$

So the dimensions of AB will be 3×4.

■ 2. Find the product of matrices A and B.

$$A = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 2 & 6 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 5 & -4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2(-2) + 6(5) & 2(0) + 6(-4) \\ -3(-2) + 1(5) & -3(0) + 1(-4) \end{bmatrix}$$

$$AB = \begin{bmatrix} 26 & -24 \\ 11 & -4 \end{bmatrix}$$

■ 3. Find the product of matrices A and B.

$$A = \begin{bmatrix} 5 & -1 \\ 0 & 11 \\ 7 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 1 & 8 \\ -3 & 0 & 4 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix}5 & -1\\ 0 & 11\\ 7 & -2\end{bmatrix}\begin{bmatrix}6 & 1 & 8\\ -3 & 0 & 4\end{bmatrix}$$

$$AB = \begin{bmatrix}5(6)+(-1)(-3) & 5(1)+(-1)(0) & 5(8)+(-1)(4)\\ 0(6)+11(-3) & 0(1)+11(0) & 0(8)+11(4)\\ 7(6)+(-2)(-3) & 7(1)+(-2)(0) & 7(8)+(-2)(4)\end{bmatrix}$$

$$AB = \begin{bmatrix}33 & 5 & 36\\ -33 & 0 & 44\\ 48 & 7 & 48\end{bmatrix}$$

■ 4. Find the product of matrices A and B.

$$A = \begin{bmatrix}3 & -2\\ 1 & 8\\ 0 & 3\end{bmatrix}$$

$$B = \begin{bmatrix}5 & 2\\ 4 & 8\end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 3 & -2 \\ 1 & 8 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 5 & 2 \\ 4 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 3(5) + (-2)(4) & 3(2) + (-2)(8) \\ 1(5) + 8(4) & 1(2) + 8(8) \\ 0(5) + 3(4) & 0(2) + 3(8) \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -10 \\ 37 & 66 \\ 12 & 24 \end{bmatrix}$$

■ 5. Use the distributive property to find $A(B+C)$.

$$A = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix}$$

$$C = \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

Solution:

Applying the distributive property to the initial expression, we get

$$A(B+C) = AB + AC$$

Use matrix multiplication to find $AB + AC$.

$$AB + AC = \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 & 1 \\ 5 & 4 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} 6 & 1 \\ 3 & -1 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 2(3) + (0)(5) & 2(1) + 0(4) \\ 4(3) + (-2)(5) & 4(1) + (-2)(4) \end{bmatrix}$$

$$+ \begin{bmatrix} 2(6) + 0(3) & 2(1) + 0(-1) \\ 4(6) + (-2)(3) & 4(1) + (-2)(-1) \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 6 & 2 \\ 2 & -4 \end{bmatrix} + \begin{bmatrix} 12 & 2 \\ 18 & 6 \end{bmatrix}$$

Now use matrix addition.

$$AB + AC = \begin{bmatrix} 6 + 12 & 2 + 2 \\ 2 + 18 & -4 + 6 \end{bmatrix}$$

$$AB + AC = \begin{bmatrix} 18 & 4 \\ 20 & 2 \end{bmatrix}$$

So the value of the original expression is

$$A(B + C) = \begin{bmatrix} 18 & 4 \\ 20 & 2 \end{bmatrix}$$

■ 6. Find the product of matrices A and B.

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

Solution:

Multiply matrix A by matrix B.

$$AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 6 & -2 & 8 & 1 \\ 7 & 3 & 5 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \\ 0(6)+0(7) & 0(-2)+0(3) & 0(8)+0(5) & 0(1)+0(2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

IDENTITY MATRICES

■ 1. Write the identity matrix I_4.

Solution:

We always call the identity matrix I, and it's always a square matrix, like 2×2, 3×3, 4×4, etc. For that reason, it's common to abbreviate I_{2x2} as just I_2, or I_{3x3} as just I_3, etc. So, I_4 is the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 2. If we want to find the product IA, where I is the identity matrix and A is 4×2, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 4 \times 2 = 4 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 4 \times 2 = 4 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 4 \cdot 4 \times 2 = 4 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$4 \times 4 \cdot 4 \times 2 = 4 \times 2$$

Therefore, the identity matrix in this case is I_4.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 3. If we want to find the product IA, where I is the identity matrix and A is a 3×4, then what are the dimensions of I?

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 4 = 3 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 4 = 3 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 3 \cdot 3 \times 4 = 3 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$3 \times 3 \cdot 3 \times 4 = 3 \times 4$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

■ 4. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 3 \times 2 = 3 \times 2$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 3 \times 2 = 3 \times 2$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 3 \cdot 3 \times 2 = 3 \times 2$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$3 \times 3 \cdot 3 \times 2 = 3 \times 2$$

Therefore, the identity matrix in this case is I_3.

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(2)+0(-2)+0(3) & 1(8)+0(7)+0(5) \\ 0(2)+1(-2)+0(3) & 0(8)+1(7)+0(5) \\ 0(2)+0(-2)+1(3) & 0(8)+0(7)+1(5) \end{bmatrix}$$

$$IA = \begin{bmatrix} 2 & 8 \\ -2 & 7 \\ 3 & 5 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_3.

■ 5. If we want to find the product IA, where I is the identity matrix and A is given, then what are the dimensions of I? What is the product IA?

$$A = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

Solution:

Start by setting up the equation $I \cdot A = A$, then substitute the dimensions for A into the equation.

$$I \cdot A = A$$

$$I \cdot 2 \times 4 = 2 \times 4$$

Break down the dimensions of the identity matrix as rows $\times$ columns, or $R \times C$.

$$R \times C \cdot 2 \times 4 = 2 \times 4$$

For matrix multiplication to be defined, you need the same number of columns in the first matrix as rows in the second matrix.

$$R \times 2 \cdot 2 \times 4 = 2 \times 4$$

The dimensions of the product come from the rows in the first matrix and the columns in the second matrix, so

$$2 \times 2 \cdot 2 \times 4 = 2 \times 4$$

Therefore, the identity matrix in this case is I_2.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The product IA is

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

$$IA = \begin{bmatrix} 1(7)+0(5) & 1(1)+0(5) & 1(3)+0(2) & 1(-2)+0(9) \\ 0(7)+1(5) & 0(1)+1(5) & 0(3)+1(2) & 0(-2)+1(9) \end{bmatrix}$$

$$IA = \begin{bmatrix} 7 & 1 & 3 & -2 \\ 5 & 5 & 2 & 9 \end{bmatrix}$$

As we expected, we get back to matrix A after multiplying it by the identity matrix I_2.

6. If A is a 2×4 matrix, what are the dimensions of the identity matrix that make the equation true?

$$AI = A$$

Solution:

Set up the equation $AI = A$, then substitute the dimensions for A into the equation.

$$A \cdot I = A$$

$$2 \times 4 \cdot I = 2 \times 4$$

Break up the dimensions of I as $R \times C$.

$$2 \times 4 \cdot R \times C = 2 \times 4$$

The number of rows in the second matrix must be equal to the number of columns from the first matrix.

$$2 \times 4 \cdot 4 \times C = 2 \times 4$$

The dimensions of the product come from the rows of the first matrix and the columns of the second matrix, so

$$2 \times 4 \cdot 4 \times 4 = 2 \times 4$$

So the identity matrix is I_4, the 4×4 identity matrix.

$$I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

THE ELIMINATION MATRIX

■ 1. Find a single 2×2 elimination matrix E that accomplishes the given row operations.

1. $(1/3)R_1 \to R_1$

2. $-2R_1 + R_2 \to R_2$

Solution:

The row operation $(1/3)R_1 \to R_1$ means we'll put a $1/3$ in $E_{1,1}$.

$$E_1 = \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

The row operation $-2R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -2 in $E_{2,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$$

Consolidate these two row operations into one elimination matrix by multiplying E_2 by E_1.

$$E = E_2E_1$$

$$E = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1\left(\frac{1}{3}\right) + 0(0) & 1(0) + 0(1) \\ -2\left(\frac{1}{3}\right) + 1(0) & -2(0) + 1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} \frac{1}{3} & 0 \\ -\frac{2}{3} & 1 \end{bmatrix}$$

■ 2. Find a single 3×3 elimination matrix E that accomplishes the given row operations.

1. $-3R_1 + R_3 \to R_3$

2. $5R_2 + R_1 \to R_1$

3. $-R_3 \to R_3$

Solution:

The row operation $-3R_1 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a -3 in $E_{3,1}$.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

The row operation $5R_2 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and 5 in $E_{1,2}$.

$$E_2 = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $-R_3 \to R_3$ means we'll put a -1 in $E_{3,3}$.

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Consolidate these three row operations into one elimination matrix by multiplying E_3 by E_2 by E_1.

$$E = E_3 E_2 E_1$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1(1)+5(0)+0(-3) & 1(0)+5(1)+0(0) & 1(0)+5(0)+0(1) \\ 0(1)+1(0)+0(-3) & 0(0)+1(1)+0(0) & 0(0)+1(0)+0(1) \\ 0(1)+0(0)+1(-3) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1(1)+0(0)+0(-3) & 1(5)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(1)+1(0)+0(-3) & 0(5)+1(1)+0(0) & 0(0)+1(0)+0(1) \\ 0(1)+0(0)-1(-3) & 0(5)+0(1)-1(0) & 0(0)+0(0)-1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 5 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & -1 \end{bmatrix}$$

■ 3. Find a single 2×2 elimination matrix E that accomplishes the given row operations.

1. $-R_1 \to R_1$

2. $5R_1 + R_2 \to R_2$

3. $-(1/7)R_2 \to R_2$

4. $R_2 + R_1 \to R_1$

Solution:

The row operation $-R_1 \to R_1$ means we'll put a -1 in $E_{1,1}$.

$$E_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

The row operation $5R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a 5 in $E_{2,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix}$$

The row operation $-(1/7)R_2 \to R_2$ means we'll put a $-1/7$ in $E_{2,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix}$$

The row operation $R_2 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and a 1 in $E_{1,2}$.

$$E_4 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

Consolidate these four row operations into one elimination matrix by multiplying E_4 by E_3 by E_2 by E_1.

$$E = E_4E_3E_2E_1$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} 1(-1) + 0(0) & 1(0) + 0(1) \\ 5(-1) + 1(0) & 5(0) + 1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{7} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ -5 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1(-1) + 0(-5) & 1(0) + 0(1) \\ 0(-1) - \frac{1}{7}(-5) & 0(0) - \frac{1}{7}(1) \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ \frac{5}{7} & -\frac{1}{7} \end{bmatrix}$$

$$E = \begin{bmatrix} 1(-1) + 1\left(\frac{5}{7}\right) & 1(0) + 1\left(-\frac{1}{7}\right) \\ 0(-1) + 1\left(\frac{5}{7}\right) & 0(0) + 1\left(-\frac{1}{7}\right) \end{bmatrix}$$

$$E = \begin{bmatrix} -\frac{2}{7} & -\frac{1}{7} \\ \frac{5}{7} & -\frac{1}{7} \end{bmatrix}$$

■ 4. Find the single elimination matrix E that puts A into reduced row-echelon form, where E accounts for the given set of row operations.

$$A = \begin{bmatrix} -3 & 6 \\ 1 & 2 \end{bmatrix}$$

1. $-\dfrac{1}{3}R_1 \to R_1$

2. $-R_1 + R_2 \to R_2$

3. $\dfrac{1}{4}R_2 \to R_2$

4. $2R_2 + R_1 \to R_1$

Solution:

The row operation $-(1/3)R_1 \to R_1$ means we'll put a $-(1/3)$ in $E_{1,1}$.

$$E_1 = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

The row operation $-R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -1 in $E_{2,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

The row operation $(1/4)R_2 \to R_2$ means we'll put a $1/4$ in $E_{2,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$$

The row operation $2R_2 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and a 2 in $E_{1,2}$.

$$E_4 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

Consolidate these four row operations into one elimination matrix by multiplying E_4 by E_3 by E_2 by E_1.

$$E = E_4E_3E_2E_1$$

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 1\left(-\frac{1}{3}\right) + 0(0) & 1(0) + 0(1) \\ -1\left(-\frac{1}{3}\right) + 1(0) & -1(0) + 1(1) \end{bmatrix}$$

$$E=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1&0\\0&\frac{1}{4}\end{bmatrix}\begin{bmatrix}-\frac{1}{3}&0\\\frac{1}{3}&1\end{bmatrix}$$

$$E=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}1\left(-\frac{1}{3}\right)+0\left(\frac{1}{3}\right)&1(0)+0(1)\\0\left(-\frac{1}{3}\right)+\frac{1}{4}\left(\frac{1}{3}\right)&0(0)+\frac{1}{4}(1)\end{bmatrix}$$

$$E=\begin{bmatrix}1&2\\0&1\end{bmatrix}\begin{bmatrix}-\frac{1}{3}&0\\\frac{1}{12}&\frac{1}{4}\end{bmatrix}$$

$$E=\begin{bmatrix}1\left(-\frac{1}{3}\right)+2\left(\frac{1}{12}\right)&1(0)+2\left(\frac{1}{4}\right)\\0\left(-\frac{1}{3}\right)+1\left(\frac{1}{12}\right)&0(0)+1\left(\frac{1}{4}\right)\end{bmatrix}$$

$$E=\begin{bmatrix}-\frac{1}{6}&\frac{1}{2}\\\frac{1}{12}&\frac{1}{4}\end{bmatrix}$$

■ 5. Find the single elimination matrix E that puts X into reduced row-echelon form, where E accounts for the given set of row operations.

$$X=\begin{bmatrix}1&0&0\\3&1&-4\\-2&-1&5\end{bmatrix}$$

1. $-3R_1+R_2\to R_2$

2. $2R_1 + R_3 \to R_3$

3. $R_2 + R_3 \to R_3$

4. $4R_3 + R_2 \to R_2$

Solution:

The row operation $-3R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -3 in $E_{2,1}$.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $2R_1 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a 2 in $E_{3,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$

The row operation $R_2 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a 1 in $E_{3,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

The row operation $4R_3 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a 4 in $E_{2,3}$.

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

Consolidate these four row operations into one elimination matrix by multiplying E_4 by E_3 by E_2 by E_1.

$$E = E_4E_3E_2E_1$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\2&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\0&0&1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1(1)+0(-3)+0(0)&1(0)+0(1)+0(0)&1(0)+0(0)+0(1)\\0(1)+1(-3)+0(0)&0(0)+1(1)+0(0)&0(0)+1(0)+0(1)\\2(1)+0(-3)+1(0)&2(0)+0(1)+1(0)&2(0)+0(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\2&0&1\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1(1)+0(-3)+0(2)&1(0)+0(1)+0(0)&1(0)+0(0)+0(1)\\0(1)+1(-3)+0(2)&0(0)+1(1)+0(0)&0(0)+1(0)+0(1)\\0(1)+1(-3)+1(2)&0(0)+1(1)+1(0)&0(0)+1(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\0&1&4\\0&0&1\end{bmatrix}\begin{bmatrix}1&0&0\\-3&1&0\\-1&1&1\end{bmatrix}$$

$$E = \begin{bmatrix}1(1)+0(-3)+0(-1)&1(0)+0(1)+0(1)&1(0)+0(0)+0(1)\\0(1)+1(-3)+4(-1)&0(0)+1(1)+4(1)&0(0)+1(0)+4(1)\\0(1)+0(-3)+1(-1)&0(0)+0(1)+1(1)&0(0)+0(0)+1(1)\end{bmatrix}$$

$$E = \begin{bmatrix}1&0&0\\-7&5&4\\-1&1&1\end{bmatrix}$$

6. Find the single elimination matrix E that puts B into reduced row-echelon form, where E accounts for the given set of row operations.

$$B = \begin{bmatrix} 1 & 0 & -5 \\ 3 & 2 & -9 \\ 1 & -2 & -10 \end{bmatrix}$$

1. $-3R_1 + R_2 \to R_2$

2. $-R_1 + R_3 \to R_3$

3. $\frac{1}{2}R_2 \to R_2$

4. $2R_2 + R_3 \to R_3$

5. $-3R_3 + R_2 \to R_2$

6. $5R_3 + R_1 \to R_1$

Solution:

The row operation $-3R_1 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -3 in $E_{2,1}$.

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $-R_1 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a -1 in $E_{3,1}$.

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The row operation $(1/2)R_2 \to R_2$ means we'll put a $1/2$ in $E_{2,2}$.

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's consolidate what we have for $E_3E_2E_1$ so far.

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(1)+0(-3)+0(0) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(1)+1(-3)+0(0) & 0(0)+1(1)+0(0) & 0(0)+1(0)+0(1) \\ -1(1)+0(-3)+1(0) & -1(0)+0(1)+1(0) & -1(0)+0(0)+1(1) \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1(1)+0(-3)+0(-1) & 1(0)+0(1)+0(0) & 1(0)+0(0)+0(1) \\ 0(1)+\frac{1}{2}(-3)+0(-1) & 0(0)+\frac{1}{2}(1)+0(0) & 0(0)+\frac{1}{2}(0)+0(1) \\ 0(1)+0(-3)+1(-1) & 0(0)+0(1)+1(0) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_3E_2E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

The row operation $2R_2 + R_3 \to R_3$ means we'll put a 1 in $E_{3,3}$ and a 2 in $E_{3,2}$.

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

The row operation $-3R_3 + R_2 \to R_2$ means we'll put a 1 in $E_{2,2}$ and a -3 in $E_{2,3}$.

$$E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$$

The row operation $5R_3 + R_1 \to R_1$ means we'll put a 1 in $E_{1,1}$ and a 5 in $E_{1,3}$.

$$E_6 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Let's consolidate what we have for $E_6E_5E_4$.

$$E_6E_5E_4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1(1)+0(0)+0(0) & 1(0)+0(1)+0(2) & 1(0)+0(0)+0(1) \\ 0(1)+1(0)-3(0) & 0(0)+1(1)-3(2) & 0(0)+1(0)-3(1) \\ 0(1)+0(0)+1(0) & 0(0)+0(1)+1(2) & 0(0)+0(0)+1(1) \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -5 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1(1)+0(0)+5(0) & 1(0)+0(-5)+5(2) & 1(0)+0(-3)+5(1) \\ 0(1)+1(0)+0(0) & 0(0)+1(-5)+0(2) & 0(0)+1(-3)+0(1) \\ 0(1)+0(0)+1(0) & 0(0)+0(-5)+1(2) & 0(0)+0(-3)+1(1) \end{bmatrix}$$

$$E_6E_5E_4 = \begin{bmatrix} 1 & 10 & 5 \\ 0 & -5 & -3 \\ 0 & 2 & 1 \end{bmatrix}$$

Then the elimination matrix is

$$E = E_6E_5E_4E_3E_2E_1$$

$$E = \begin{bmatrix} 1 & 10 & 5 \\ 0 & -5 & -3 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & \frac{1}{2} & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 1(1) + 10\left(-\frac{3}{2}\right) + 5(-1) & 1(0) + 10\left(\frac{1}{2}\right) + 5(0) & 1(0) + 10(0) + 5(1) \\ 0(1) - 5\left(-\frac{3}{2}\right) - 3(-1) & 0(0) - 5\left(\frac{1}{2}\right) - 3(0) & 0(0) - 5(0) - 3(1) \\ 0(1) + 2\left(-\frac{3}{2}\right) + 1(-1) & 0(0) + 2\left(\frac{1}{2}\right) + 1(0) & 0(0) + 2(0) + 1(1) \end{bmatrix}$$

$$E = \begin{bmatrix} -19 & 5 & 5 \\ \frac{21}{2} & -\frac{5}{2} & -3 \\ -4 & 1 & 1 \end{bmatrix}$$

VECTORS

■ 1. For the matrix A, find the row vectors, the space, $\mathbb{R}^n$, that contains the row vectors, and the dimension of the space they form.

$$A = \begin{bmatrix} -4 & 8 & 6 & 12 & -1 \\ 3 & -2 & 18 & 0 & -3 \\ 12 & -17 & -4 & 1 & 1 \end{bmatrix}$$

Solution:

The row vectors of A are

$$a_1 = [-4 \quad 8 \quad 6 \quad 12 \quad -1]$$

$$a_2 = [3 \quad -2 \quad 18 \quad 0 \quad -3]$$

$$a_3 = [12 \quad -17 \quad -4 \quad 1 \quad 1]$$

Since the row vectors each have five components, they're contained in $\mathbb{R}^5$ space, and because there are three vectors, they form a three-dimensional space, like $\mathbb{R}^3$, within $\mathbb{R}^5$.

■ 2. For the matrix B, find the column vectors, the space, $\mathbb{R}^n$, that contains the column vectors, and the dimension of the space they form.

$$B = \begin{bmatrix} 12 & 0 & 9 \\ 3 & -21 & -1 \\ -7 & 4 & 13 \end{bmatrix}$$

Solution:

The column vectors of B are

$$b_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \ b_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}, \ b_3 = \begin{bmatrix} 9 \\ -1 \\ 13 \end{bmatrix}$$

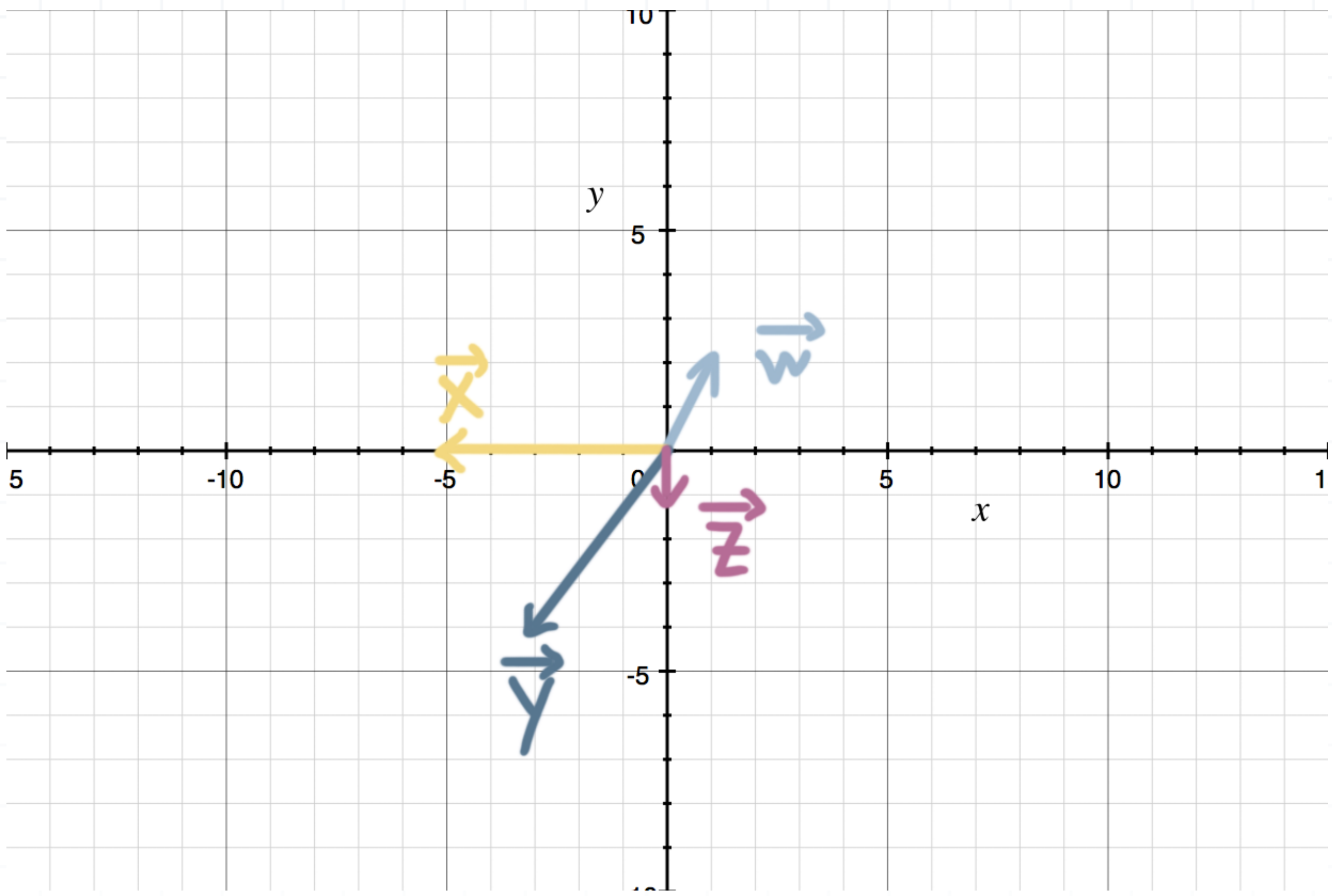
Since the column vectors each have three components, they're contained in $\mathbb{R}^3$ space, and because there are three vectors, they form a three-dimensional space in $\mathbb{R}^3$.

■ 3. Sketch the vectors in standard position.

$$\overrightarrow{w} = (1,2), \ \overrightarrow{x} = (-5,0), \ \overrightarrow{y} = (-3,-4), \ \vec{z} = (0,-1)$$

Solution:

A sketch of the vectors in standard position is



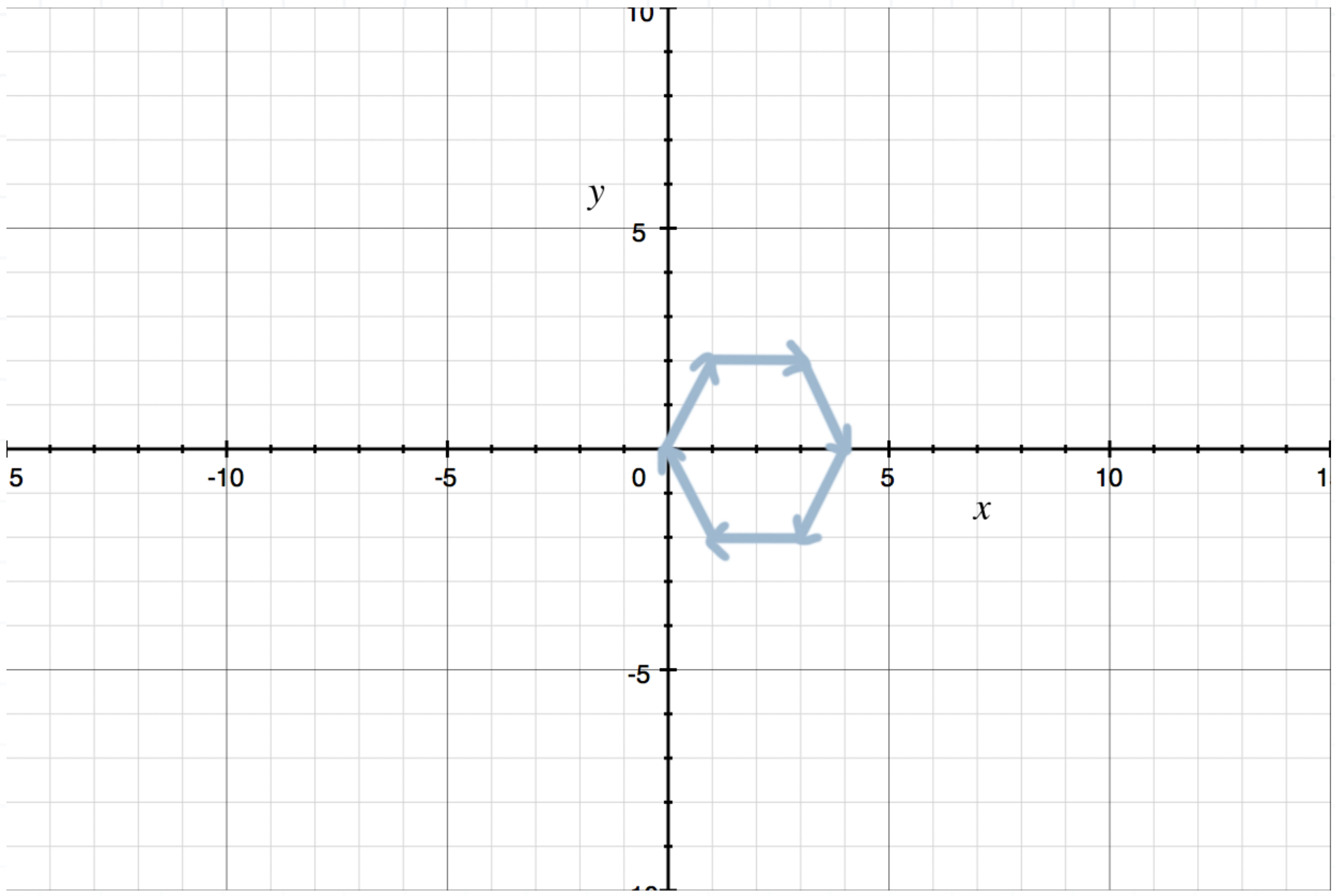
■ 4. Sketch the vectors in order from tip to tail (where the terminal point of one is the initial point of the next), starting at the origin, and determine the shape they form.

$\vec{a}_1 = (1,2)$ $\vec{a}_3 = (1, -2)$ $\vec{a}_5 = (-2,0)$

$\vec{a}_2 = (2,0)$ $\vec{a}_4 = (-1, -2)$ $\vec{a}_6 = (-1,2)$

Solution:

The vectors form a hexagon, and a sketch of all of them together is



■ 5. Find $\vec{b}_1 + \vec{b}_2$, $\vec{b}_1 - \vec{b}_2$, and $2\vec{b}_2$.

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

Solution:

The value $\vec{b}_1 + \vec{b}_2$ is

$$\vec{b}_1 + \vec{b}_2 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix} + \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix} = \begin{bmatrix} 12+0 \\ 3-21 \\ -7+4 \end{bmatrix} = \begin{bmatrix} 12 \\ -18 \\ -3 \end{bmatrix}$$

The value $\vec{b}_1 - \vec{b}_2$ is

$$\vec{b}_1 - \vec{b}_2 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix} - \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix} = \begin{bmatrix} 12 - 0 \\ 3 - (-21) \\ -7 - 4 \end{bmatrix} = \begin{bmatrix} 12 \\ 24 \\ -11 \end{bmatrix}$$

The value of $2\vec{b}_2$ is

$$2\vec{b}_2 = 2\begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix} = \begin{bmatrix} 2(0) \\ 2(-21) \\ 2(4) \end{bmatrix} = \begin{bmatrix} 0 \\ -42 \\ 8 \end{bmatrix}$$

■ 6. Is the product of $\vec{b}_1$ and $\vec{b}_2$ defined? Why or why not?

$$\vec{b}_1 = \begin{bmatrix} 12 \\ 3 \\ -7 \end{bmatrix}, \ \vec{b}_2 = \begin{bmatrix} 0 \\ -21 \\ 4 \end{bmatrix}$$

Solution:

The product of $\vec{b}_1$ and $\vec{b}_2$ isn't defined, because they're both column matrices, which means they both have dimensions 3×1. When you multiply two matrices, the number of columns in the first matrix must be equal to the number of rows in the second matrix. But $\vec{b}_1$ has one column, and $\vec{b}_2$ has three rows, so as they're written, they can't be multiplied.

VECTOR OPERATIONS

■ 1. Find $\vec{u}+\vec{w}$, $\vec{x}-\vec{y}$, and $\vec{v}-(\vec{w}+\vec{u})$.

$\vec{u}=(-3,5)$ $\quad\vec{w}=(5,-13)$ $\quad\vec{y}=(1,4,2)$

$\vec{v}=(2,1)$ $\quad\vec{x}=(4,5,-7)$

Solution:

The value of $\vec{u}+\vec{w}$ is

$$\vec{u}+\vec{w}=(-3,5)+(5,-13)$$

$$\vec{u}+\vec{w}=(-3+5,5-13)$$

$$\vec{u}+\vec{w}=(2,-8)$$

The value of $\vec{x}-\vec{y}$ is

$$\vec{x}-\vec{y}=(4,5,-7)-(1,4,2)$$

$$\vec{x}-\vec{y}=(4-1,5-4,-7-2)$$

$$\vec{x}-\vec{y}=(3,1,-9)$$

The value of $\vec{v}-(\vec{w}+\vec{u})$ is

$$\vec{v}-(\vec{w}+\vec{u})=(2,1)-((5,-13)+(-3,5))$$

$$\vec{v}-(\vec{w}+\vec{u})=(2,1)-(5-3,-13+5)$$

$$\vec{v} - (\vec{w} + \vec{u}) = (2,1) - (2, -8)$$

$$\vec{v} - (\vec{w} + \vec{u}) = (2 - 2, 1 - (-8))$$

$$\vec{v} - (\vec{w} + \vec{u}) = (0,9)$$

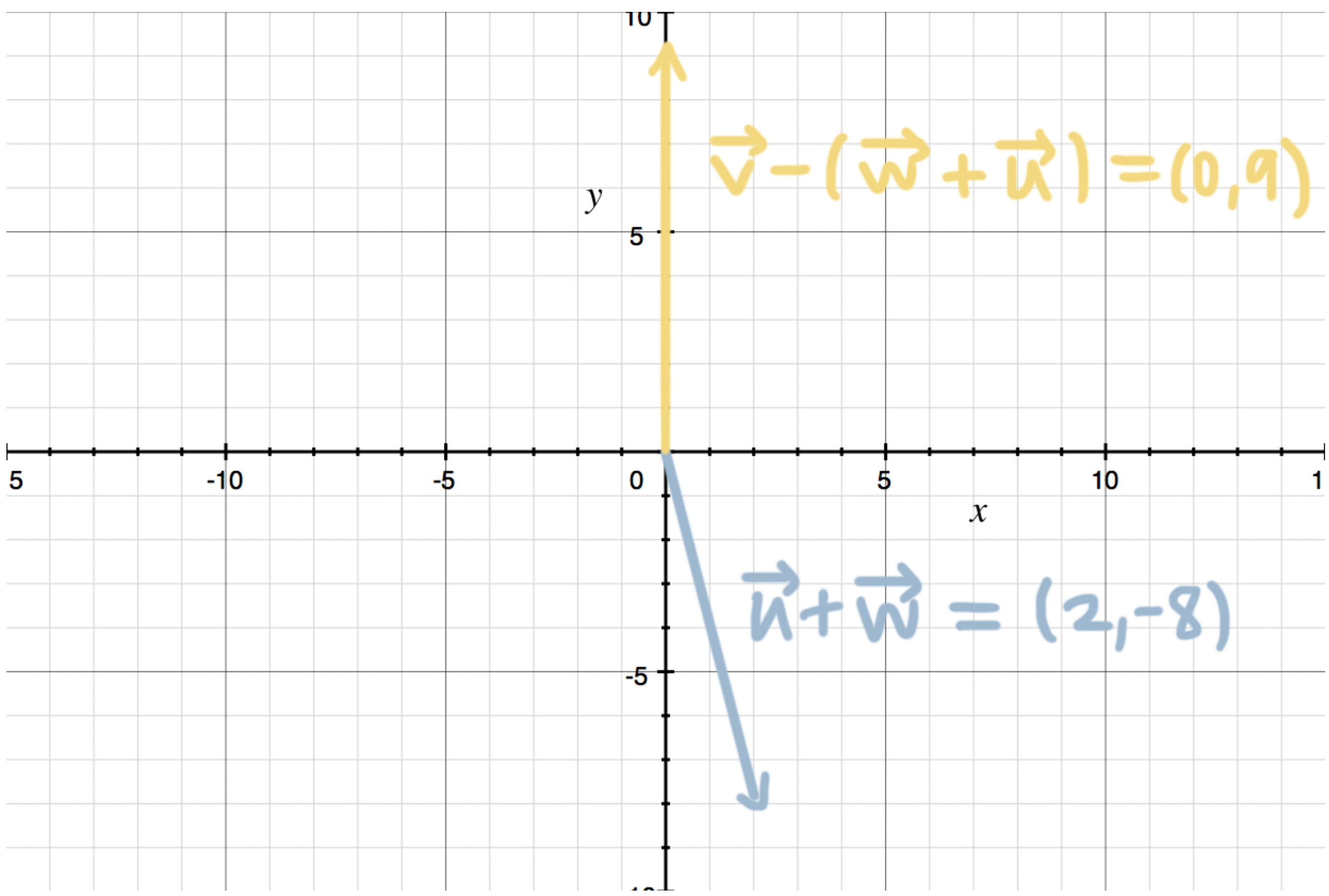
■ 2. Sketch $\vec{u} + \vec{w}$, $\vec{x} - \vec{y}$, and $\vec{v} - (\vec{w} + \vec{u})$.

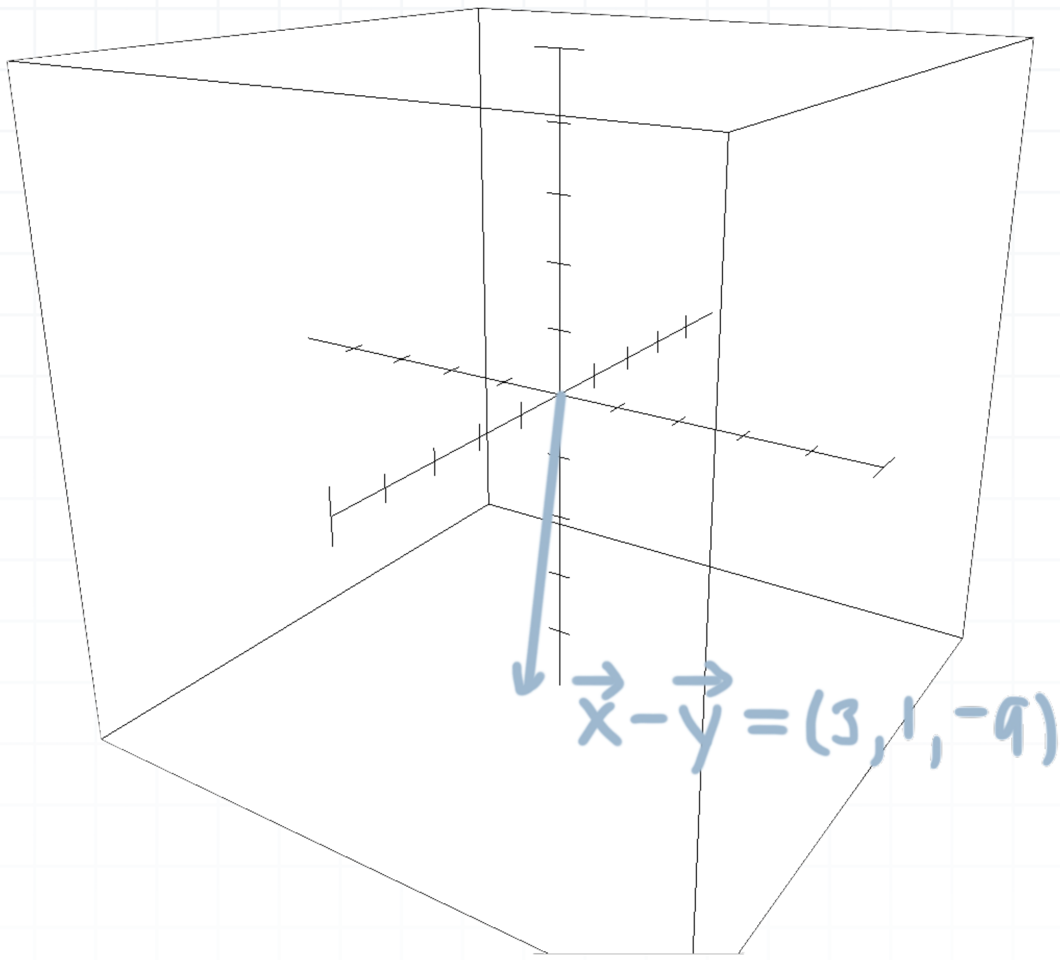
$\vec{u} = (-3,5)$ $\quad\quad$ $\vec{w} = (5, -13)$ $\quad\quad$ $\vec{y} = (1,4,2)$

$\vec{v} = (2,1)$ $\quad\quad$ $\vec{x} = (4,5, -7)$

Solution:

A sketch of $\vec{u} + \vec{w}$, $\vec{x} - \vec{y}$, and $\vec{v} - (\vec{w} + \vec{u})$ in the same plane is





3. Find $b\overrightarrow{x}$, $c\overrightarrow{u}+b\overrightarrow{u}$, and $(c+b)\overrightarrow{u}$. What can you say about the relationship between $c\overrightarrow{u}+b\overrightarrow{u}$ and $(c+b)\overrightarrow{u}$.

$\overrightarrow{u}=(-3,5)$ $\quad$ $b=-1$

$\overrightarrow{x}=(4,5,-7)$ $\quad$ $c=3$

Solution:

The value of $b\overrightarrow{x}$ is

$$b\overrightarrow{x}=-1(4,5,-7)$$

$$b\overrightarrow{x}=(-4,-5,7)$$

The value of $c\vec{u}+b\vec{u}$ is

$$c\vec{u}+b\vec{u}=3(-3,5)-1(-3,5)$$

$$c\vec{u}+b\vec{u}=(-9,15)-(-3,5)$$

$$c\vec{u}+b\vec{u}=(-9-(-3),15-5)$$

$$c\vec{u}+b\vec{u}=(-6,10)$$

The value of $(c+b)\vec{u}$ is

$$(c+b)\vec{u}=(3+(-1))(-3,5)$$

$$(c+b)\vec{u}=2(-3,5)$$

$$(c+b)\vec{u}=(-6,10)$$

The values of $c\vec{u}+b\vec{u}$ and $(c+b)\vec{u}$ are equal because the distributive property applies to vector addition, which means that, from $c\vec{u}+b\vec{u}$, the vector $\vec{u}$ can be factored out, and the expression can be rewritten as $(c+b)\vec{u}$.

4. Find $\vec{x}+b\vec{y}-c\vec{x}-\vec{y}$.

$\vec{x}=(4,5,-7)$ $\quad\quad b=-1$

$\vec{y}=(1,4,2)$ $\quad\quad c=3$

Solution:

First find $b\vec{y}$ and $c\vec{x}$.

$$b\vec{y} = (-1)(1,4,2)$$

$$b\vec{y} = (-1, -4, -2)$$

and

$$c\vec{x} = 3(4,5, -7)$$

$$c\vec{x} = (12,15, -21)$$

Now move from left to right, starting with $\vec{x} + b\vec{y}$, adding more terms as we go.

$$\vec{x} + b\vec{y} = (4,5, -7) + (-1, -4, -2)$$

$$\vec{x} + b\vec{y} = (4 - 1,5 - 4, -7 - 2)$$

$$\vec{x} + b\vec{y} = (3,1, -9)$$

Then

$$\vec{x} + b\vec{y} - c\vec{x} = (3,1, -9) - (12,15, -21)$$

$$\vec{x} + b\vec{y} - c\vec{x} = (3 - 12,1 - 15, -9 - (-21))$$

$$\vec{x} + b\vec{y} - c\vec{x} = (-9, -14,12)$$

Then

$$\vec{x} + b\vec{y} - c\vec{x} - \vec{y} = (-9, -14,12) - (1,4,2)$$

$$\vec{x} + b\vec{y} - c\vec{x} - \vec{y} = (-9 - 1, -14 - 4,12 - 2)$$

$$\vec{x} + b\vec{y} - c\vec{x} - \vec{y} = (-10, -18, 10)$$

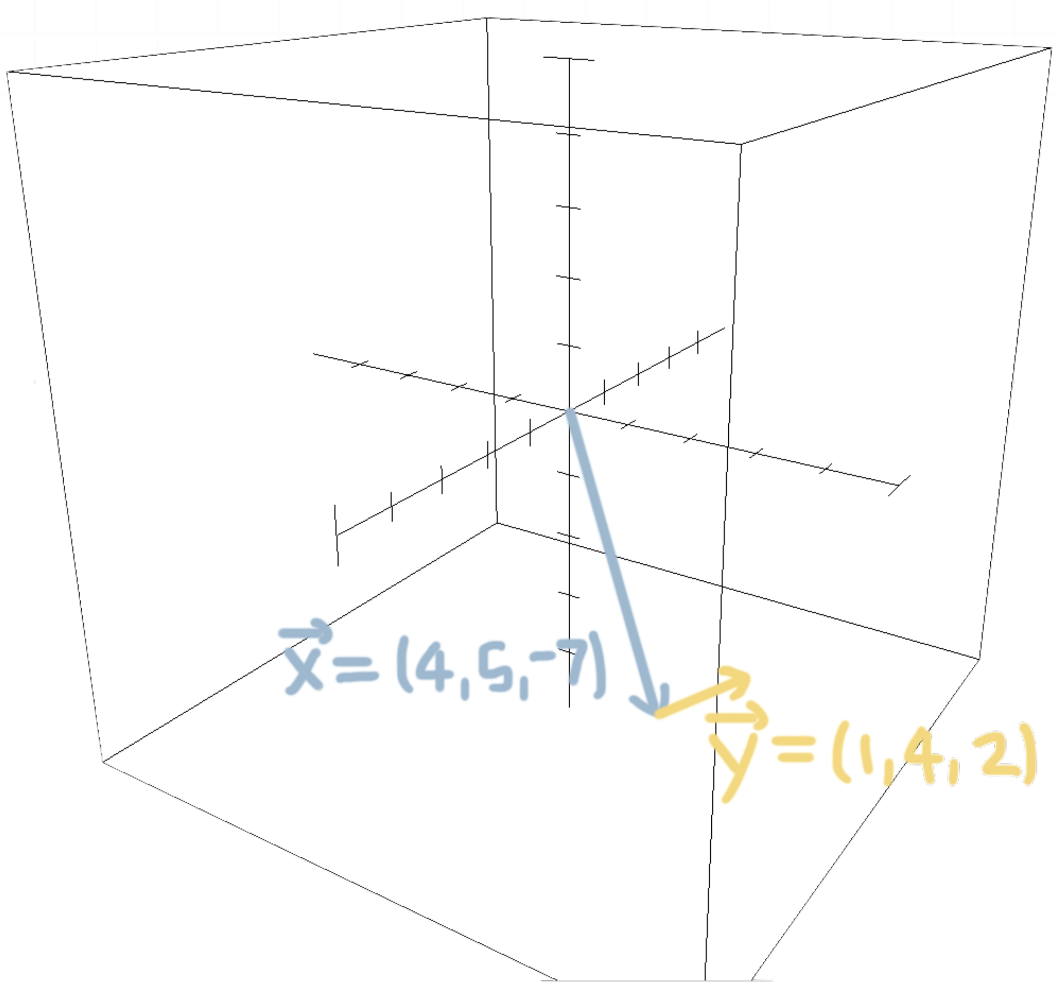
5. Sketch the individual vectors from tip to tail.

$$\vec{x} = (4, 5, -7)$$

$$\vec{y} = (1, 4, 2)$$

Solution:

A sketch of the vectors is



6. Find $\vec{x} \cdot \vec{y}$, $\vec{w} \cdot \vec{w}$, and $b(\vec{u} \cdot \vec{v})$.

$\vec{u} = (-3,5)$ $\vec{w} = (5, -13)$ $\vec{y} = (1,4,2)$

$\vec{v} = (2,1)$ $\vec{x} = (4,5, -7)$ $b = -1$

Solution:

The value of $\vec{x} \cdot \vec{y}$ is

$$\vec{x} \cdot \vec{y} = (4,5, -7) \cdot (1,4,2)$$

$$\vec{x} \cdot \vec{y} = (4)(1) + (5)(4) + (-7)(2)$$

$$\vec{x} \cdot \vec{y} = 4 + 20 - 14$$

$$\vec{x} \cdot \vec{y} = 10$$

The value of $\vec{w} \cdot \vec{w}$ is

$$\vec{w} \cdot \vec{w} = (5, -13) \cdot (5, -13)$$

$$\vec{w} \cdot \vec{w} = (5)(5) + (-13)(-13)$$

$$\vec{w} \cdot \vec{w} = 25 + 169$$

$$\vec{w} \cdot \vec{w} = 194$$

The value of $b(\vec{u} \cdot \vec{v})$ is

$$b(\vec{u} \cdot \vec{v}) = (-1)((-3,5) \cdot (2,1))$$

$$b(\vec{u} \cdot \vec{v}) = (-1)((-3)(2) + (5)(1))$$

$$b(\overrightarrow{u}\cdot\overrightarrow{v})=(-1)(-6+5)$$

$$b(\overrightarrow{u}\cdot\overrightarrow{v})=(-1)(-1)$$

$$b(\overrightarrow{u}\cdot\overrightarrow{v})=1$$

UNIT VECTORS AND BASIS VECTORS

1. Change each vector to a unit vector.

$$\vec{a} = (3, -4)$$

$$\vec{b} = (12,2)$$

$$\vec{c} = (0,7,1)$$

Solution:

Find the magnitude of $\vec{a}$.

$$||\vec{a}|| = ||(3, -4)||$$

$$||\vec{a}|| = \sqrt{3^2 + (-4)^2}$$

$$||\vec{a}|| = \sqrt{9 + 16}$$

$$||\vec{a}|| = \sqrt{25}$$

$$||\vec{a}|| = 5$$

Find the magnitude of $\vec{b}$.

$$||\vec{b}|| = ||(12,2)||$$

$$||\vec{b}|| = \sqrt{12^2 + 2^2}$$

$$||\vec{b}|| = \sqrt{144 + 4}$$

$$||\vec{b}|| = \sqrt{148}$$

$$||\vec{b}|| = 2\sqrt{37}$$

Find the magnitude of $\vec{c}$.

$$||\vec{c}|| = ||(0,7,1)||$$

$$||\vec{c}|| = \sqrt{0^2 + 7^2 + 1^2}$$

$$||\vec{c}|| = \sqrt{49 + 1}$$

$$||\vec{c}|| = \sqrt{50}$$

$$||\vec{c}|| = 5\sqrt{2}$$

Then the unit vectors are

$$\hat{u}_a = \frac{1}{||\vec{a}||}\vec{a} = \frac{1}{5}\begin{bmatrix}3\\-4\end{bmatrix}$$

$$\hat{u}_b = \frac{1}{||\vec{b}||}\vec{b} = \frac{1}{2\sqrt{37}}\begin{bmatrix}12\\2\end{bmatrix} = \frac{1}{\sqrt{37}}\begin{bmatrix}6\\1\end{bmatrix}$$

$$\hat{u}_c = \frac{1}{||\vec{c}||}\vec{c} = \frac{1}{5\sqrt{2}}\begin{bmatrix}0\\7\\1\end{bmatrix}$$

■ 2. Confirm that the vectors each have length 1.

$$\hat{u}_a = \frac{1}{5}\begin{bmatrix}3\\-4\end{bmatrix}$$

$$\hat{u}_b = \frac{1}{\sqrt{37}}\begin{bmatrix}6\\1\end{bmatrix}$$

$$\hat{u}_c = \frac{1}{5\sqrt{2}}\begin{bmatrix}0\\7\\1\end{bmatrix}$$

Solution:

Find the length of $\hat{u}_a$.

$$||\hat{u}_a|| = \sqrt{\left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2}$$

$$||\hat{u}_a|| = \sqrt{\frac{9}{25} + \frac{16}{25}}$$

$$||\hat{u}_a|| = \sqrt{\frac{25}{25}}$$

$$||\hat{u}_a|| = 1$$

Find the length of $\hat{u}_b$.

$$||\hat{u}_b|| = \sqrt{\left(\frac{6}{\sqrt{37}}\right)^2 + \left(\frac{1}{\sqrt{37}}\right)^2}$$

$$||\hat{u}_b|| = \sqrt{\frac{36}{37} + \frac{1}{37}}$$

$$||\hat{u}_b|| = \sqrt{\frac{37}{37}}$$

$$||\hat{u}_b|| = 1$$

Find the length of $\hat{u}_c$.

$$||\hat{u}_c|| = \sqrt{0^2 + \left(\frac{7}{5\sqrt{2}}\right)^2 + \left(\frac{1}{5\sqrt{2}}\right)^2}$$

$$||\hat{u}_c|| = \sqrt{\frac{49}{25(2)} + \frac{1}{25(2)}}$$

$$||\hat{u}_c|| = \sqrt{\frac{49}{50} + \frac{1}{50}}$$

$$||\hat{u}_c|| = \sqrt{\frac{50}{50}}$$

$$||\hat{u}_c|| = 1$$

3. What are the basis vectors for $\mathbb{R}^4$?

Solution:

$$\hat{r}_1 = (1,0,0,0)$$

$$\hat{r}_2 = (0,1,0,0)$$

$$\hat{r}_3 = (0,0,1,0)$$

$$\hat{r}_4 = (0,0,0,1)$$

■ 4. Express the vectors as linear combinations of the basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

$$\hat{u}_a = \frac{1}{5}\begin{bmatrix} 3 \\ -4 \end{bmatrix}$$

$$\hat{u}_b = \frac{1}{\sqrt{37}}\begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$$\hat{u}_c = \frac{1}{5\sqrt{2}}\begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

Solution:

The vector $\hat{u}_a$ can be rewritten in terms of the standard basis vectors as

$$\begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{4}{5} \end{bmatrix} = \frac{3}{5}\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{5}\hat{i} - \frac{4}{5}\hat{j}$$

The vector $\hat{u}_b$ can be rewritten in terms of the standard basis vectors as

$$\begin{bmatrix} \frac{6}{\sqrt{37}} \\ \frac{1}{\sqrt{37}} \end{bmatrix} = \begin{bmatrix} \frac{6}{\sqrt{37}} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{\sqrt{37}} \end{bmatrix} = \frac{6}{\sqrt{37}} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \frac{1}{\sqrt{37}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{6}{\sqrt{37}} \hat{i} + \frac{1}{\sqrt{37}} \hat{j}$$

The vector $\hat{u}_c$ can be rewritten in terms of the standard basis vectors as

$$\begin{bmatrix} 0 \\ \frac{7}{5\sqrt{2}} \\ \frac{1}{5\sqrt{2}} \end{bmatrix} = 0 + \begin{bmatrix} 0 \\ \frac{7}{5\sqrt{2}} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{5\sqrt{2}} \end{bmatrix} = \frac{7}{5\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \frac{1}{5\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \frac{7}{5\sqrt{2}} \hat{j} + \frac{1}{5\sqrt{2}} \hat{k} = \frac{7}{5\sqrt{2}} \hat{j} + \frac{1}{5\sqrt{2}} \hat{k}$$

■ 5. Express $\vec{v} = (x,2x,-1)$ in terms of the standard basis vectors.

Solution:

Rewrite $\vec{v}$ in terms of the standard basis vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$.

$$\vec{v} = (x,2x,-1)$$

$$\vec{v} = \begin{bmatrix} x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$$

$$\vec{v} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} 2x \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

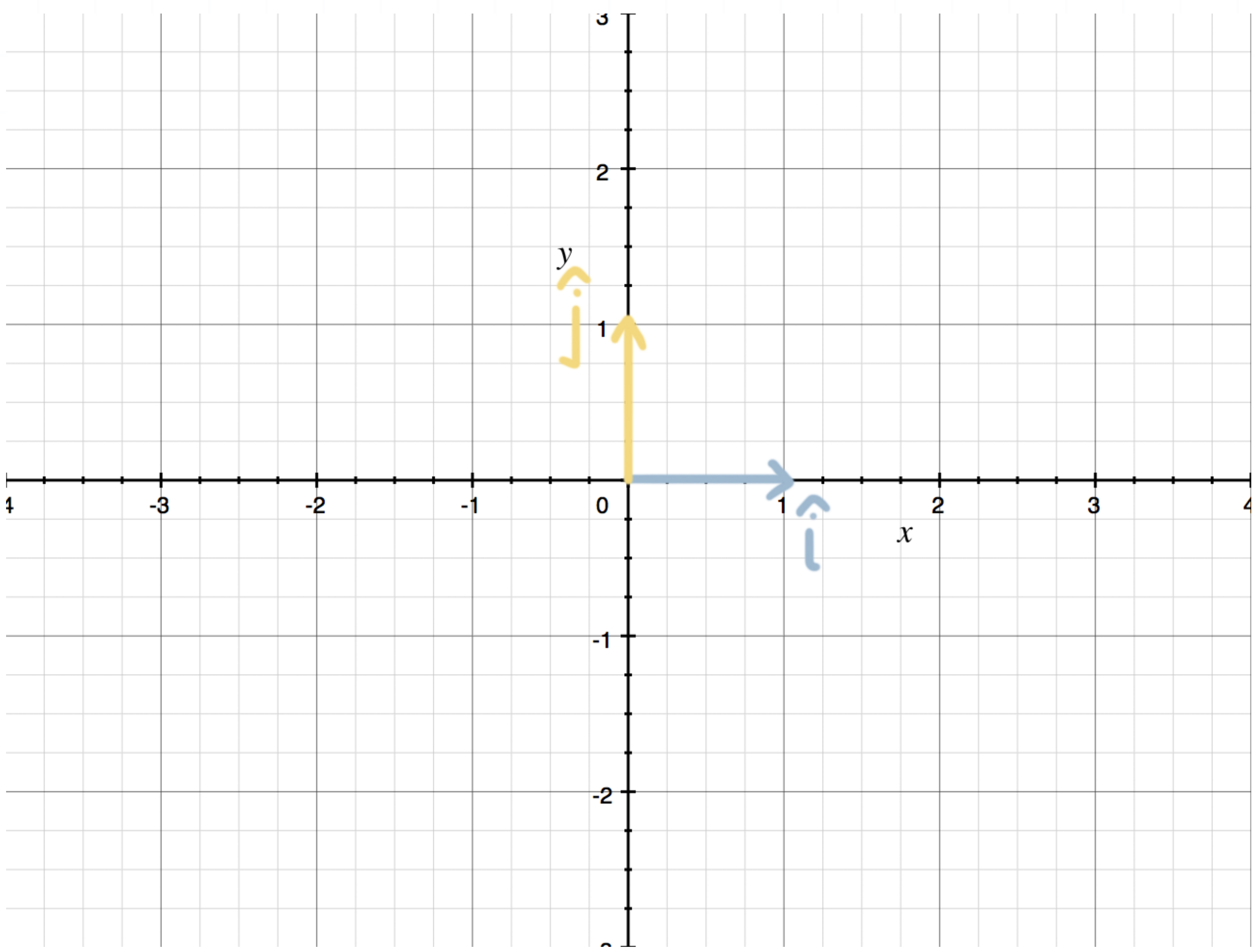
$$\vec{v} = x\hat{i} + 2x\hat{j} - 1\hat{k}$$

$$\vec{v} = x\hat{i} + 2x\hat{j} - \hat{k}$$

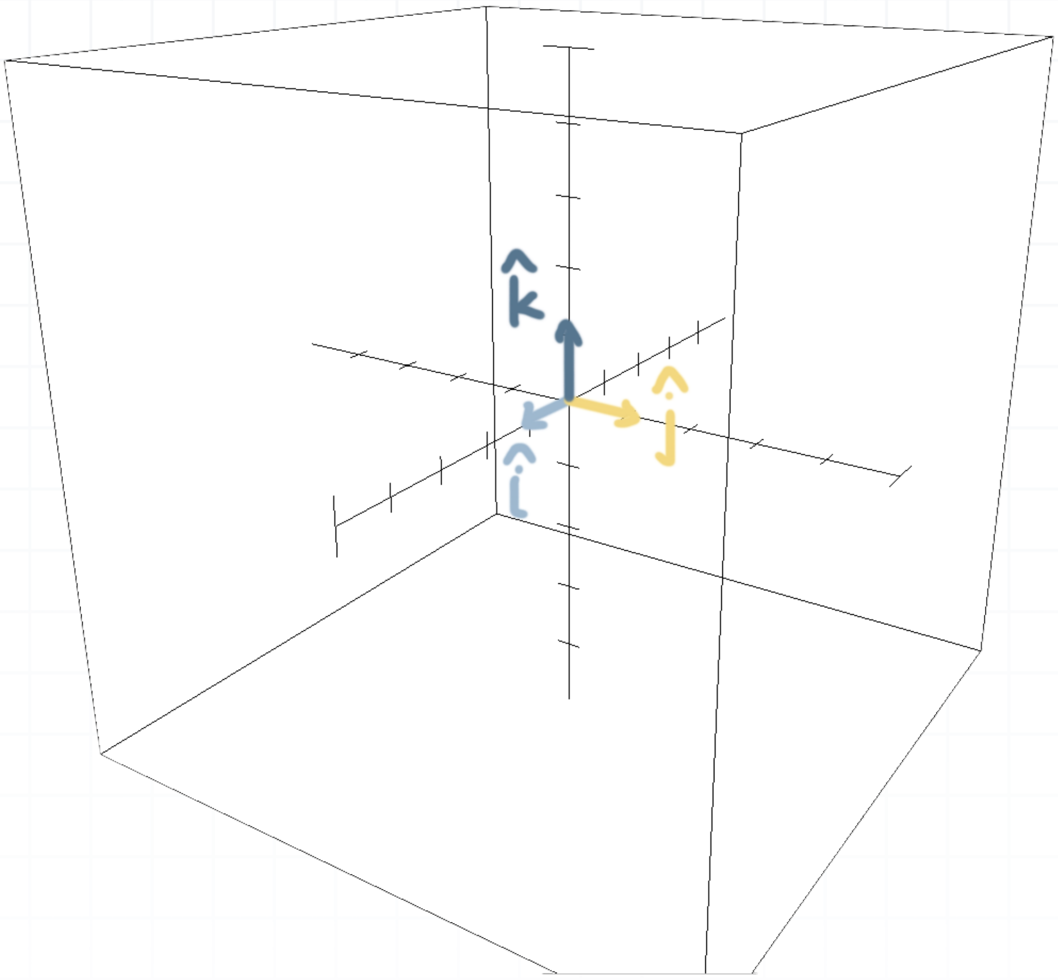
6. Sketch the basis vectors $\hat{i}$ and $\hat{j}$ in $\mathbb{R}^2$, and the vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ in $\mathbb{R}^3$.

Solution:

A sketch of the standard basis vectors in $\mathbb{R}^2$ is



And a sketch of the standard basis vectors in $\mathbb{R}^3$ is



LINEAR COMBINATIONS AND SPAN

1. Say whether each of the following is a linear combination. If it isn't, say why.

$$-\pi\vec{x} - e\vec{y}$$

$$\vec{x} \cdot \vec{y}$$

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{u} = \frac{1}{\sqrt{2}}((3,0) - (1,1))$$

$$||\vec{b}||$$

Solution:

The expressions

$$-\pi\vec{x} - e\vec{y}$$

$$\vec{a} = \hat{i} + \hat{j} + \hat{k}$$

$$\vec{u} = \frac{1}{\sqrt{2}}((3,0) - (1,1))$$

all represent linear combinations.

But $\vec{x} \cdot \vec{y}$ is not a linear combination, because the dot product returns a scalar, so $\vec{x} \cdot \vec{y}$ can't be a linear combination, since it isn't a vector.

And $||\vec{b}||$ isn't a linear combination, because, like the dot product, the magnitude returns a single scalar, so $||\vec{b}||$ can't be a linear combination.

2. Do the vectors span $\mathbb{R}^4$?

$$\left\{ \begin{bmatrix} 3 \\ \frac{1}{2} \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -\pi \\ \pi \\ \pi \\ -\pi \end{bmatrix}, \begin{bmatrix} -\frac{2}{3} \\ 8 \\ 22 \\ 9 \end{bmatrix} \right\}$$

Solution:

The vector set doesn't span $\mathbb{R}^4$, because at least 4 vectors are needed to span $\mathbb{R}^4$.

3. Do the vectors span $\mathbb{R}^4$?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Solution:

The vector set spans $\mathbb{R}^4$, because these are the standard basis vectors for $\mathbb{R}^4$.

4. Do the vectors span $\mathbb{R}^2$?

$$\left\{\begin{bmatrix}44\\-8\end{bmatrix}, \begin{bmatrix}11\\-2\end{bmatrix}\right\}$$

Solution:

The vector set doesn't span $\mathbb{R}^2$. While two vectors may be enough to span $\mathbb{R}^2$, the two vectors can't be collinear if they're going to be a spanning set.

The first vector, $\vec{v}_1 = (44, -8)$, can be written as $\vec{v}_1 = 4(11, -2)$, which means that the two vectors are collinear. Because they're collinear, they can't span $\mathbb{R}^2$.

5. What is the zero vector $\vec{O}$ in $\mathbb{R}^5$? What is its span?

Solution:

The zero vector in $\mathbb{R}^5$ is $\vec{O} = (0,0,0,0,0)$, and its span is simply the origin in $\mathbb{R}^5$.

$$\text{Span}\{\vec{O}\} = (0,0,0,0,0)$$

■ 6. Prove that any vector $\vec{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ can be reached by a linear combination of $\hat{i}$, $\hat{j}$, and $\hat{k}$.

Solution:

Let c_1, c_2, and c_3 be any real numbers ($c_1, c_2, c_3 \in \mathbb{R}$). We want to show that any vector, $\vec{v}$, can be written as $\vec{v} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$. If we rewrite this equation with column vectors, we get

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ c_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ c_3 \end{bmatrix}$$

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

This means that $v_1 = c_1$, $v_2 = c_2$, and $v_3 = c_3$, such that, no matter what vector $\vec{v}$ is given to us, we can just pick $c_1 = v_1$, $c_2 = v_2$, and $c_3 = v_3$ for our weights, and we'll be able to span $\mathbb{R}^3$ using those weights and the basis vectors.

LINEAR INDEPENDENCE IN TWO DIMENSIONS

■ 1. Are the column vectors of the following matrix linearly independent?

$$A = \begin{bmatrix} 2 & 6 & 7 \\ -1 & 11 & 3 \end{bmatrix}$$

Solution:

First, let's get the column vectors.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 6 \\ 11 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

Since each column vector has two components, we know that they're in $\mathbb{R}^2$. However, since there are three column vectors, we know that at least one of them can be made from a linear combination of the other two, and so they must be linearly dependent.

■ 2. Show how one of the vectors could be written as a linear combination of the other two.

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \vec{y} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \vec{z} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Solution:

Since $\vec{x}$ and $\vec{y}$ are the standard basis vectors for $\mathbb{R}^2$, they can easily be used to make $\vec{z}$.

$$\vec{z} = 5\begin{bmatrix}1\\0\end{bmatrix} + 3\begin{bmatrix}0\\1\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5(1)\\5(0)\end{bmatrix} + \begin{bmatrix}3(0)\\3(1)\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5\\0\end{bmatrix} + \begin{bmatrix}0\\3\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5+0\\0+3\end{bmatrix}$$

$$\vec{z} = \begin{bmatrix}5\\3\end{bmatrix}$$

■ 3. Say whether the vectors are linearly dependent or linearly independent.

$$\overrightarrow{x} = \begin{bmatrix}2\\0\end{bmatrix}, \overrightarrow{y} = \begin{bmatrix}1\\1\end{bmatrix}$$

Solution:

To test for linear independence, set up the vector equation.

$$c_1 \begin{bmatrix} 2 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Break this up into a system of linear equations.

$$2c_1 + c_2 = 0$$

$$c_2 = 0$$

We can already see that c_2 is zero, and if we plug $c_2 = 0$ into the first equation, we can see that

$$2c_1 + 0 = 0$$

$$2c_1 = 0$$

$$c_1 = 0$$

Because the only values of c_1 and c_2 that make the vector equation true are $c_1 = 0$ and $c_2 = 0$, we know the vectors must be linearly independent.

■ 4. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 6 \\ 3 \end{bmatrix}, \vec{b} = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$$

Solution:

To test for linear independence, set up the vector equation.

$$c_1 \begin{bmatrix} 6 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Break this up into a system of linear equations.

$$6c_1 - 2c_2 = 0$$

$$3c_1 + 5c_2 = 0$$

Divide the first equation by -2 to get

$$-3c_1 + c_2 = 0$$

$$3c_1 + 5c_2 = 0$$

Then add the equations to cancel c_1 and solve for c_2.

$$-3c_1 + c_2 + (3c_1 + 5c_2) = 0 + (0)$$

$$-3c_1 + c_2 + 3c_1 + 5c_2 = 0$$

$$c_2 + 5c_2 = 0$$

$$6c_2 = 0$$

$$c_2 = 0$$

Now we can plug $c_2 = 0$ into the $3c_1 + 5c_2 = 0$ to solve for c_1.

$$3c_1 + 5(0) = 0$$

$$3c_1 = 0$$

$$c_1 = 0$$

Because the only values of c_1 and c_2 that make the vector equation true are $c_1 = 0$ and $c_2 = 0$, we know the vectors must be linearly independent.

■ 5. Say whether the vectors are linearly dependent or linearly independent.

$$\vec{a} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \vec{b} = \begin{bmatrix} -6 \\ -4 \end{bmatrix}$$

Solution:

To test for linear independence, set up the vector equation.

$$c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ -4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Break this up into a system of linear equations.

$$3c_1 - 6c_2 = 0$$

$$2c_1 - 4c_2 = 0$$

Solve the system with an augmented matrix.

$$\begin{bmatrix} 3 & -6 & | & 0 \\ 2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ 2 & -4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The reduced row-echelon form of the augmented matrix gives the relationship

$$c_1 - 2c_2 = 0$$

$$c_1 = 2c_2$$

Which means we can pick any value that we choose for c_2, and we'll get an associated value of c_1 that satisfies the system. For instance, $(c_1, c_2) = (2,1)$ is a solution. Because $(c_1, c_2) = (0,0)$ is not the only solution, the vectors must be linearly dependent.

■ 6. Use a matrix to say whether the vectors are linearly dependent or linearly independent.

$$\vec{x} = \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \vec{y} = \begin{bmatrix} -\frac{1}{2} \\ -2 \end{bmatrix}$$

Solution:

Set up an augmented matrix of the vectors as column vectors.

$$\begin{bmatrix} 2 & -\frac{1}{2} & | & 0 \\ 8 & -2 & | & 0 \end{bmatrix}$$

Use Gaussian elimination to put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & -\frac{1}{2} & | & 0 \\ 8 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & | & 0 \\ 8 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{4} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The reduced row-echelon form of the matrix gives the relationship

$$c_1 - \frac{1}{4}c_2 = 0$$

$$c_1 = \frac{1}{4}c_2$$

$$4c_1 = c_2$$

Which means we can pick any value that we choose for c_1, and we'll get an associated value of c_2 that satisfies the system. For instance, $(c_1, c_2) = (1,4)$ is a solution. Because $(c_1, c_2) = (0,0)$ is not the only solution, the vectors must be linearly dependent.

LINEAR INDEPENDENCE IN THREE DIMENSIONS

■ 1. Use a matrix to say whether the vector set is linearly independent.

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

Solution:

Form a matrix of the vectors as column vectors.

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix}$$

Use Gaussian elimination to work toward reduced row-echelon form.

$$\begin{bmatrix} 2 & 3 & 5 \\ -1 & -4 & -10 \\ 1 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 2 & 3 & 5 \\ 0 & -6 & -18 \\ 1 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -6 & -18 \\ 1 & -2 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & -6 & -18 \\ 0 & -\frac{7}{2} & -\frac{21}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{3}{2} & \frac{5}{2} \\ 0 & 1 & 3 \\ 0 & -\frac{7}{2} & -\frac{21}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & -\frac{7}{2} & -\frac{21}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

We could continue on, but because we get an entire row of zeros in the matrix, we know that the vectors must be linearly dependent.

■ 2. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} 3 \\ -4 \\ -2 \end{bmatrix}, \vec{a}_3 = \begin{bmatrix} 5 \\ -10 \\ -8 \end{bmatrix}$$

Solution:

For a set of three vectors to span $\mathbb{R}^3$, they must be linearly independent of one another. Since we determined in the last problem that these vectors are a linearly dependent set, that means they're coplanar, and they can't span $\mathbb{R}^3$.

■ 3. Use a matrix to say whether the vector set is linearly independent.

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Solution:

Form a matrix of the vectors as column vectors.

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 1 & 6 \\ 5 & 0 & 8 \end{bmatrix}$$

Use Gaussian elimination to work toward reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 1 & 6 \\ 5 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -3 & -6 \\ 5 & 0 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -3 & -6 \\ 0 & -5 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & 1 & 2 \\ 0 & -5 & -7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & -5 & -7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we've ended up with the identity matrix, we know the vectors must form a linearly independent set.

■ 4. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}, \vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{w} = \begin{bmatrix} 3 \\ 6 \\ 8 \end{bmatrix}$$

Solution:

For a set of three vectors to span $\mathbb{R}^3$, they must be linearly independent of one another. Since we determined in the last problem that these vectors are a linearly independent set, that means they do span $\mathbb{R}^3$.

■ 5. Is the vector set linearly independent? Why or why not?

$$\vec{u}=\begin{bmatrix}1\\4\\5\end{bmatrix}, \vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\6\\8\end{bmatrix}, \vec{x}=\begin{bmatrix}-2\\7\\1\end{bmatrix}$$

Solution:

The vector set can't be linearly independent. Although each vector has three components, there are four vectors in total. That means we know that one of them can be made from a linear combination of the other three, so they must be a linearly dependent set.

■ 6. Does the vector set span $\mathbb{R}^3$? Why or why not?

$$\vec{u}=\begin{bmatrix}1\\4\\5\end{bmatrix}, \vec{v}=\begin{bmatrix}1\\1\\0\end{bmatrix}, \vec{w}=\begin{bmatrix}3\\6\\8\end{bmatrix}, \vec{x}=\begin{bmatrix}-2\\7\\1\end{bmatrix}$$

Solution:

For a set of three vectors to span $\mathbb{R}^3$, they must be vectors in $\mathbb{R}^3$, and they must be linearly independent of one another. Since we determined earlier that the first three vectors are a linearly independent set, that means the first three vectors do span $\mathbb{R}^3$. Adding the vector $\vec{x}$ to the set would change the set from linearly independent to linearly dependent, but it doesn’t change the fact that the set still spans $\mathbb{R}^3$.

LINEAR SUBSPACES

■ 1. What are the criteria that define a subspace? Which criteria is logically part of another criteria?

Solution:

In order for a space to be considered a subspace, it must

1. include the zero vector,
2. be closed under scalar multiplication, and
3. be closed under addition.

The first requirement, that the space include the zero vector, is logically included in the second requirement about scalar multiplication, because, if a space is closed under scalar multiplication, then by definition it includes the zero vector.

■ 2. Sketch the graph of each space.

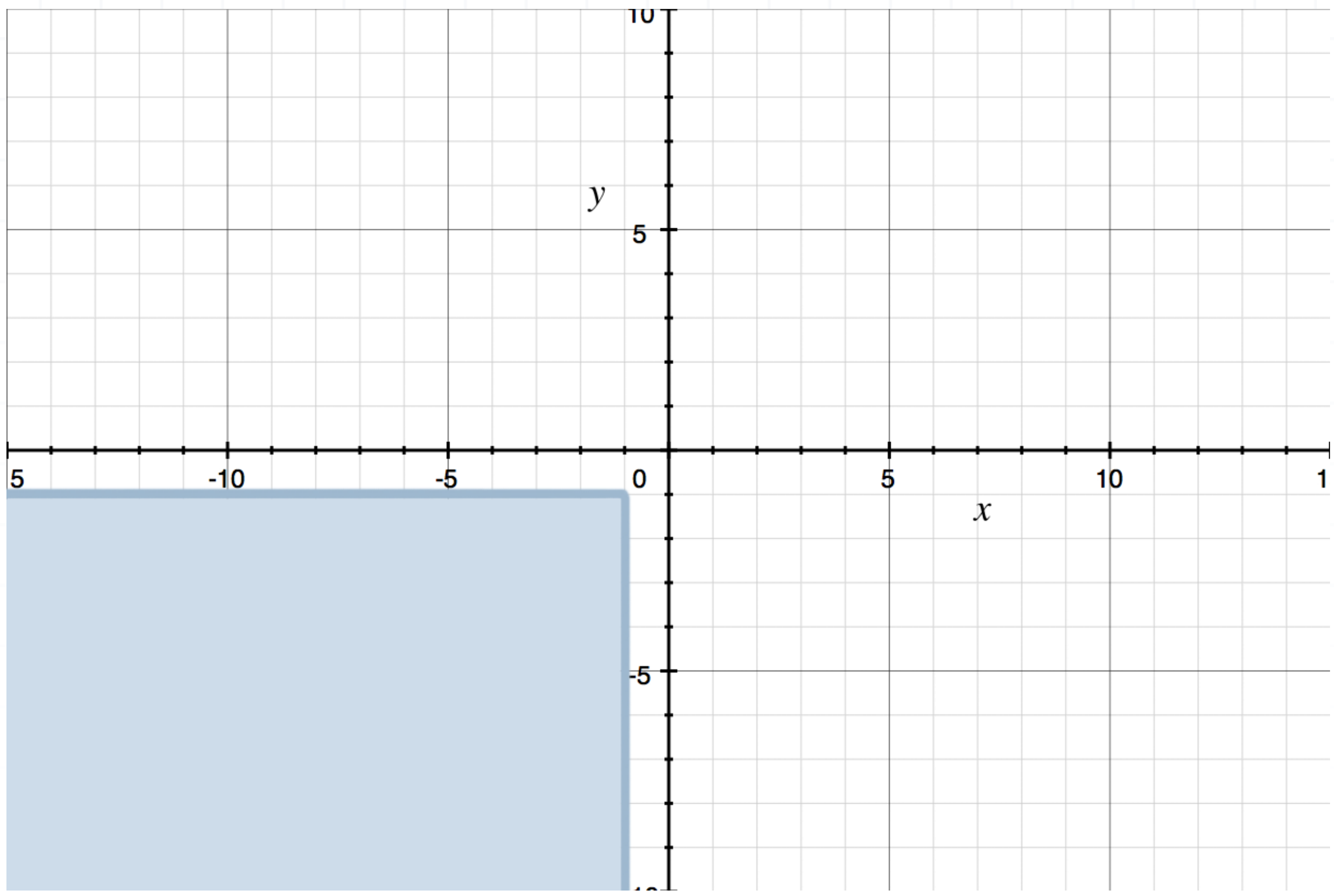
$$V_a = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \leq -1\right\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid y < x^2\right\}$$

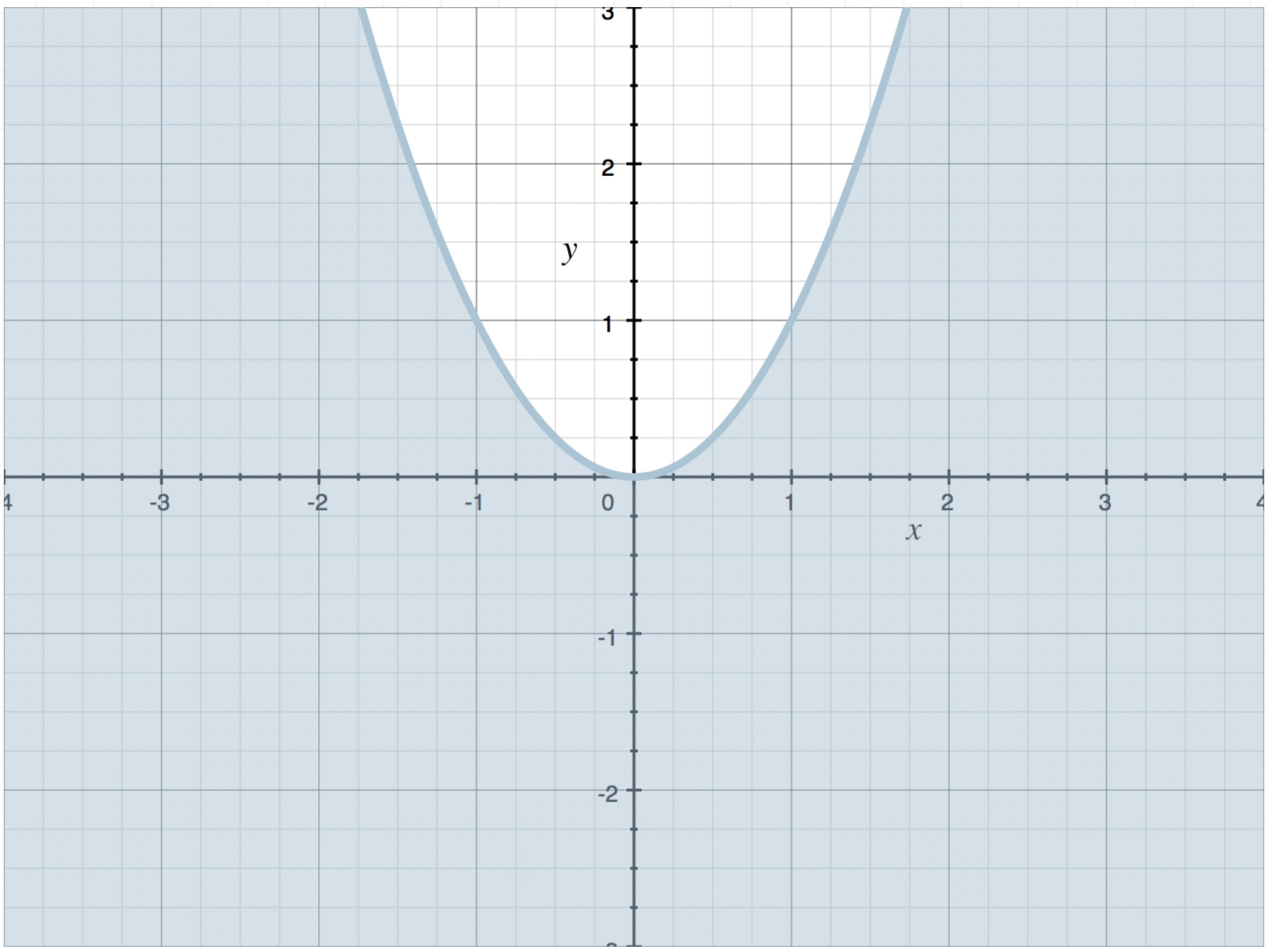
$$V_c = \left\{(x, y) \in \mathbb{R}^2 \mid x, y \geq 0, y \leq x\right\}$$

Solution:

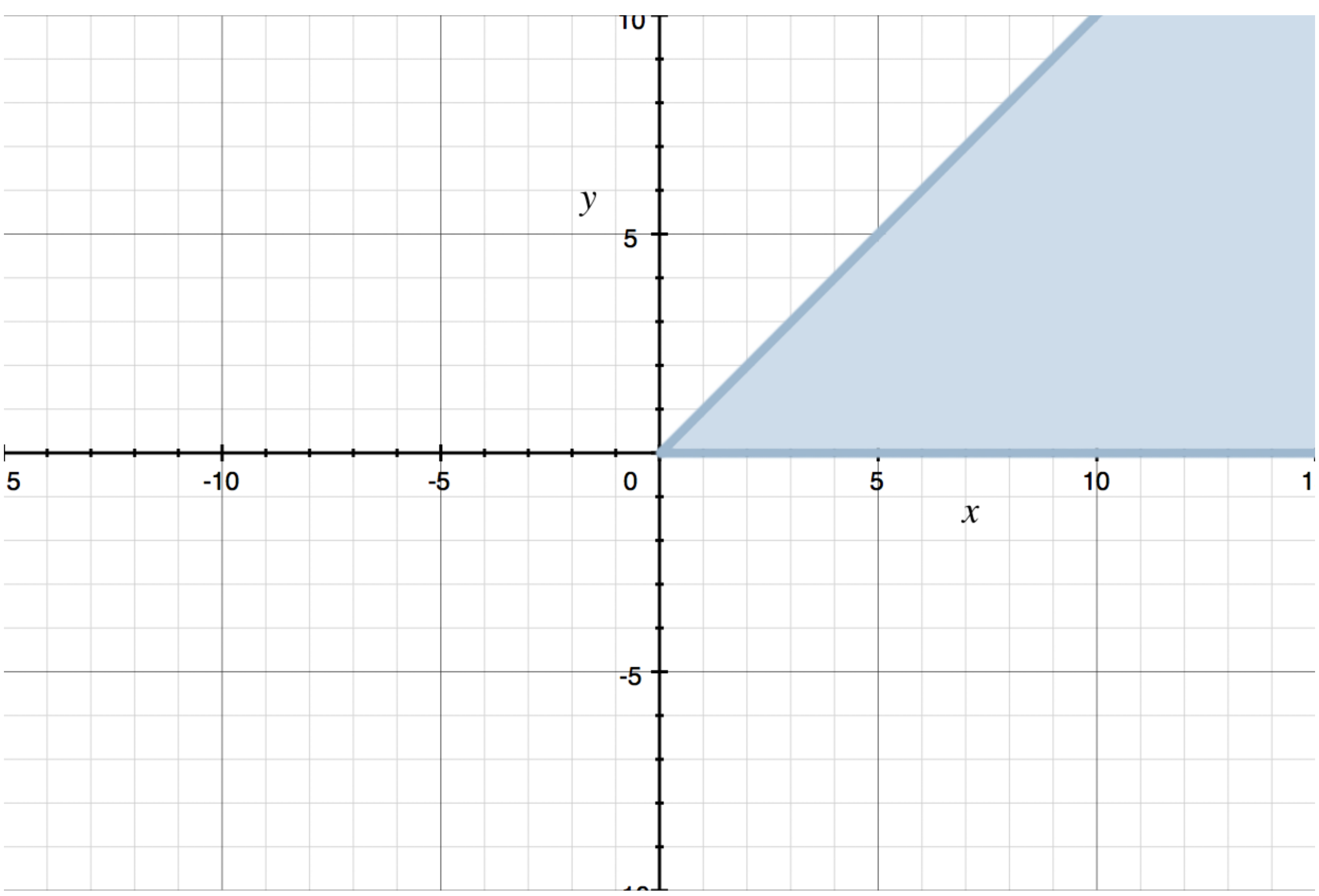
The space V_a is defined when both x and y are less than or equal to -1, so a sketch of that space is



The space V_b is defined when y is less than x^2, so a sketch of that space is



The space V_c is defined when y is less than or equal to x, but when x and y are both still greater than or equal to 0, so a sketch of that space is



3. What space is being described by each of the sets?

$$V_a = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$$

$$V_b = \{(x, y) \in \mathbb{R}^2 \mid xy = 0, \ x = y\}$$

$$V_c = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\}$$

Solution:

The space V_a is the major axes in $\mathbb{R}^2$, since $xy = 0$ is only true if either $x = 0$ and/or $y = 0$.

Similarly, the space V_b is only the zero vector in $\mathbb{R}^2$, $\vec{O} = (0,0)$. That's because, in order for $xy = 0$ to be true, the space must be limited to the major axes, but if $x = y$ is also true, that means the only vector included in the space will exist where the major axes intersect each other, which is only at the origin.

The space V_c is all of $\mathbb{R}^3$ space, since the only requirement is that x, y, and z are all in $\mathbb{R}^3$.

4. Are these spaces subspaces?

$$V_a = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$$

$$V_b = \left\{(x, y) \in \mathbb{R}^2 \mid xy = 0, \ x = y\right\}$$

$$V_c = \left\{(x, y, z) \in \mathbb{R}^3 \mid x, y, z \in \mathbb{R}\right\}$$

Solution:

The space V_a is not a subspace, because V_a isn't closed under addition.

The space V_b is a subspace, because the zero vector is always a subspace. By definition, it always includes the zero vector, and is closed under addition and scalar multiplication.

The space V_c is a subspace, because an entire space $\mathbb{R}^n$ is always a subspace of itself. So all of $\mathbb{R}^3$ is a subspace because it includes the zero vector, and it's closed under addition and scalar multiplication.

■ 5. Show that each space is not a subspace.

$$V_a = \left\{(x, y, z) \in \mathbb{R}^3 \mid 2x + y - 7z = 3\right\}$$

$$V_b = \left\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_2 \leq 0\right\}$$

$$V_c = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid x = 0 \text{ or } y = 0\right\}$$

Solution:

For V_a, we start by checking for the zero vector, $\vec{O} = (0,0,0)$.

$$2x + y - 7z = 3$$

$$2(0) + 0 - 7(0) = 3$$

$$0 = 3$$

This tells us that V_a doesn't include the zero vector, so V_a cannot be a subspace.

For V_b, we can say that the set includes the zero vector, so let's check scalar multiplication. If we pick $\vec{v}_b = (0, -1,0)$, which is in the set, and multiply it by $c = -1$, we get $-\vec{v}_b = (0,1,0)$, which is not in the set, which means V_b isn't closed under scalar multiplication, and is therefore not a subspace.

For V_c, the set includes the zero vector and is closed under scalar multiplication. To check to see whether the set is closed under addition. If we pick two vectors in the set, $\vec{v}_1 = (1,0)$ and $\vec{v}_2 = (0,1)$ and add them, we get

$$\vec{v}_1 + \vec{v}_2 = (1,0) + (0,1)$$

$$\vec{v}_1 + \vec{v}_2 = (1 + 0,0 + 1)$$

$$\vec{v}_1 + \vec{v}_2 = (1,1)$$

Since neither the x-value nor the y-value is a zero in this vector, we can say that the set is not closed under addition, and therefore that V_c is not a subspace.

■ 6. Prove that the zero vector $\overrightarrow{O}=(0,0,0)$ is a subspace of $\mathbb{R}^3$.

Solution:

We need to look at all three parts of the definition of a subspace. Since we're looking at the zero vector, the zero vector is obviously included, so the first part of the definition of a subspace is satisfied.

To check to see whether the set is closed under scalar multiplication, we'll multiply $\overrightarrow{O}=(0,0,0)$ by a scalar.

$$c\overrightarrow{O}=c(0,0,0)$$

$$c\overrightarrow{O}=(0c,0c,0c)$$

$$c\overrightarrow{O}=(0,0,0)$$

We can tell that, no matter which scalar we pick, we'll always get the zero vector again, so the set is closed under scalar multiplication.

To check to see whether the set is closed under addition, we add two vectors from the set together, but the only vector in the set is $\overrightarrow{O}=(0,0,0)$, so we get

$$\overrightarrow{O}+\overrightarrow{O}=(0,0,0)+(0,0,0)$$

$$\overrightarrow{O}+\overrightarrow{O}=(0+0,0+0,0+0)$$

$$\overrightarrow{O}+\overrightarrow{O}=(0,0,0)$$

And since we only get the zero vector (of course), we can say that $\overrightarrow{O} = (0,0,0)$ is closed under addition.

Therefore, because we've shown that the set includes the zero vector, and that the set is closed under scalar multiplication and closed under addition, we can say that the zero vector $\overrightarrow{O} = (0,0,0)$ is a subspace of $\mathbb{R}^3$.

SPANS AS SUBSPACES

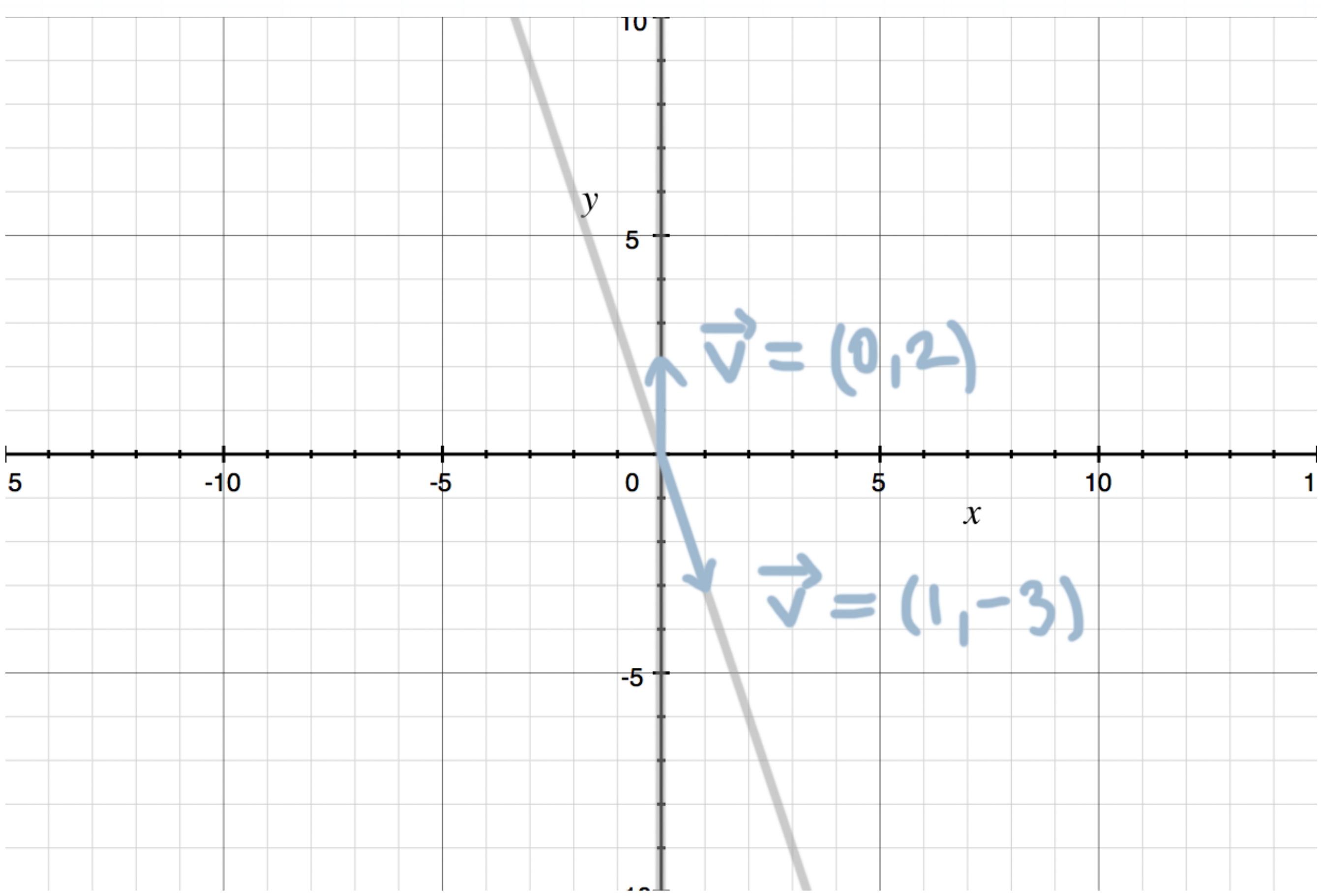
1. Sketch the spans together on the same set of axes.

$$V = \text{Span}\left(\begin{bmatrix}1\\-3\end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix}0\\2\end{bmatrix}\right)$$

Solution:

The span of $\overrightarrow{v} = (1,-3)$ will be the line that runs through $\overrightarrow{v} = (1,-3)$, and the span of $\overrightarrow{v} = (0,2)$ will be the line that runs through $\overrightarrow{v} = (0,2)$.



■ 2. Show that that spans are subspaces.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}\right)$$

$$V = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right)$$

Solution:

We know that any line in $\mathbb{R}^2$ through the origin is a subspace of $\mathbb{R}^2$, so by that fact alone, we know that each span is a subspace. But let's take each part of the definition individually.

First, because both lines pass through the origin, we know the zero vector is included in both spans. Second, if we multiply either of the vectors by a scalar, we'll still get a vector that's along the line, so both spans are closed under scalar multiplication. And third, if we add two vectors that fall along the same line, we'll still get a vector that's along the line, so both spans are closed under addition.

■ 3. Prove that the span forms a subspace of $\mathbb{R}^3$.

$$\text{Span}\left(\begin{bmatrix} -6 \\ 5 \\ 1 \end{bmatrix}\right)$$

Solution:

The span is just a linear combination of the vectors included in the span, so

$$\text{Span}\left(\begin{bmatrix}-6\\5\\1\end{bmatrix}\right)=c\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

If we choose $c=0$, we get the zero vector, so the span includes the zero vector. Since we can multiply c by any other scalar to just get another scalar, we know that the set is closed under scalar multiplication. And since we get

$$c_1\begin{bmatrix}-6\\5\\1\end{bmatrix}+c_2\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

$$(c_1+c_2)\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

$$c_3\begin{bmatrix}-6\\5\\1\end{bmatrix}$$

we can see that the set is also closed under addition. So the span is a subspace of $\mathbb{R}^3$.

4. Write the line $y = 3x + 2$ in set notation, and then write it as a single vector, only using x.

Solution:

A line is simply a series of points in $\mathbb{R}^2$, so writing it in set notation is simply

$$\left\{(x, y) \in \mathbb{R}^2 \mid y = 3x + 2\right\}$$

To write the line as a vector using only x, we see that y is already defined in terms of x, and so we can write

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x + 2 \end{bmatrix}$$

5. Write the line $2y + 4x = 0$ in set notation, and then write it as a single vector, only using y.

Solution:

A line is simply a series of points in $\mathbb{R}^2$, so writing it in set notation is simply

$$\left\{(x, y) \in \mathbb{R}^2 \mid 2y + 4x = 0\right\}$$

To write the line as a vector using only y, solve the equation of the line for x.

$$2y + 4x = 0$$

$$4x = -2y$$

$$x = -\frac{1}{2}y$$

Which means we can write the set in vector form as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}y \\ y \end{bmatrix}$$

■ 6. Write the line $y = 3x + 2$ as the linear combination of two vectors. Then plug in $x = -1$, $x = 0$, and $x = 1$, and sketch all three in the same plane.

Solution:

We've already written $y = 3x + 2$ as the vector

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x + 2 \end{bmatrix}$$

Now we can break this up.

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 0 \\ 3x + 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 3x \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

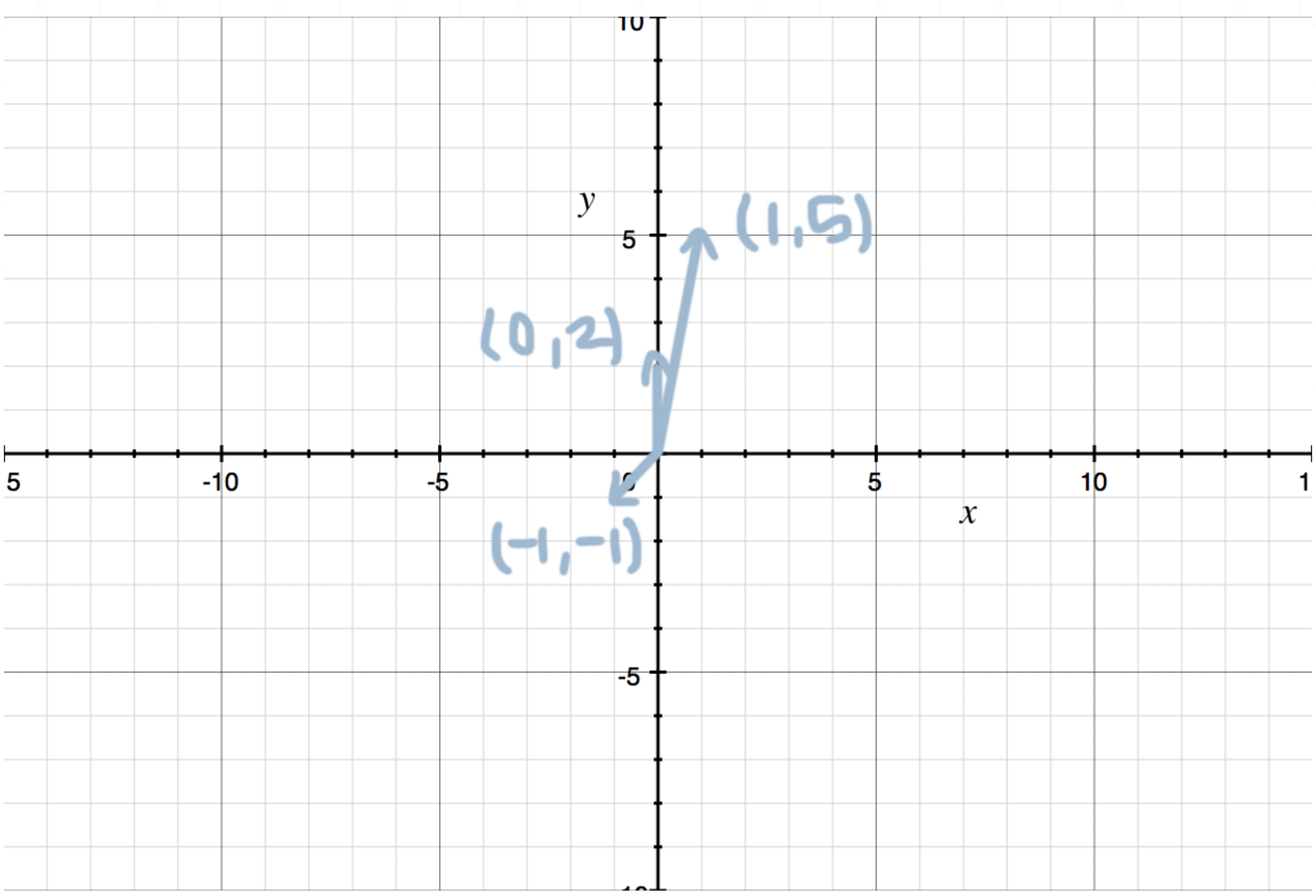
Then for each of the x-values, we get

$$\begin{bmatrix} x \\ y \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1+0 \\ -3+2 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

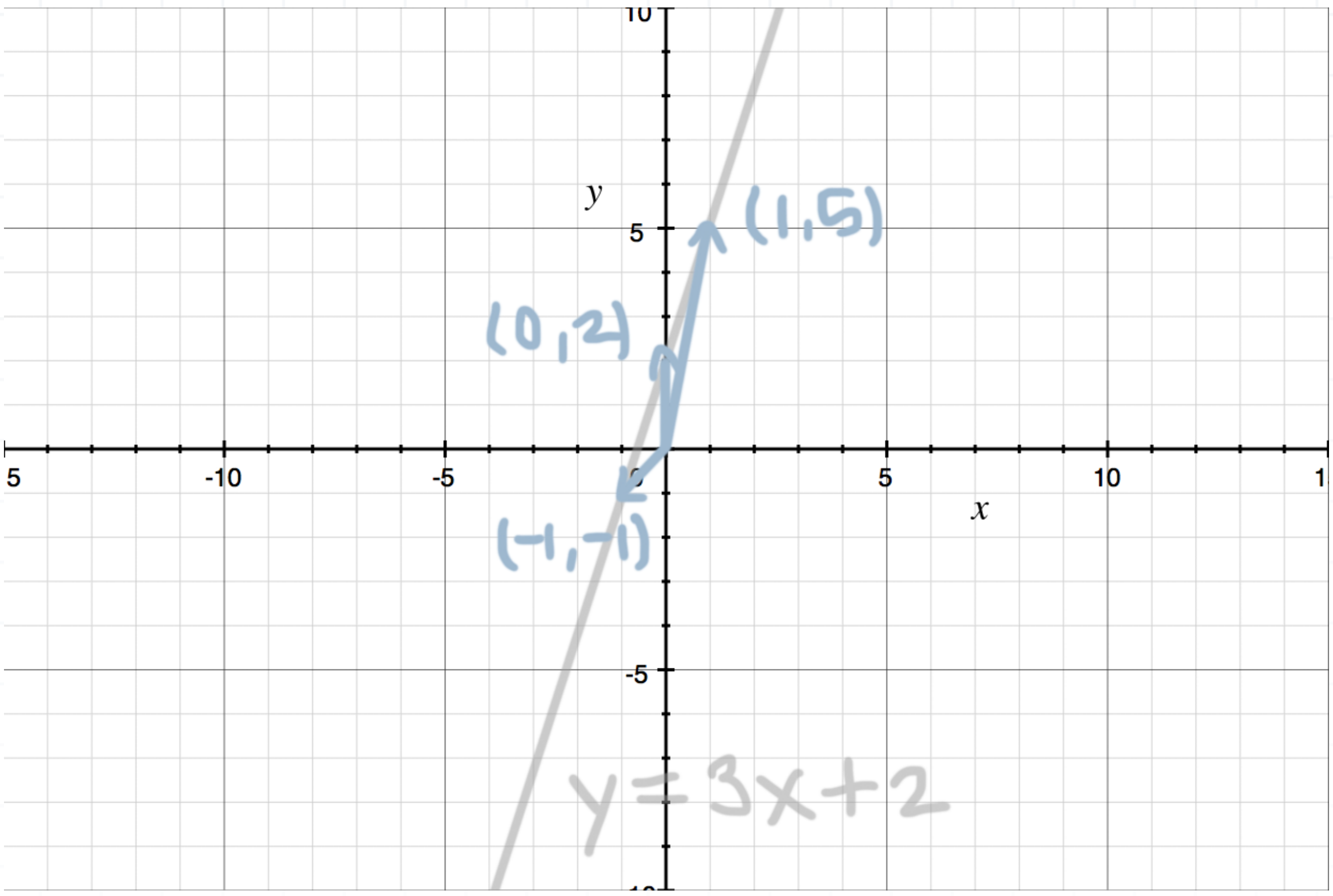
$$\begin{bmatrix} x \\ y \end{bmatrix} = 0 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0+0 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1+0 \\ 3+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

A sketch of the three vectors in the same plane is



Together, they form the line $y = 3x + 2$.



y
x
(1,5)
(0,2)
(-1,-1)
y=3x+2

BASIS

1. What requirements must be met in order for a vector set to form the basis for a space?

Solution:

A vector set can form the basis for a space if it 1) spans the space, and 2) is a linearly independent set.

2. What's the standard basis for $\mathbb{R}^4$?

Solution:

The standard basis for $\mathbb{R}^4$ is given by four vectors, each with four components:

$$\vec{v}_1 = (1,0,0,0)$$

$$\vec{v}_2 = (0,1,0,0)$$

$$\vec{v}_3 = (0,0,1,0)$$

$$\vec{v}_4 = (0,0,0,1)$$

■ 3. Say whether or not the vector set V forms a basis for $\mathbb{R}^2$.

$$V = \text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}\right\}$$

Solution:

First, we check to see if the vectors span $\mathbb{R}^2$, by confirming that we can get to any vector in $\mathbb{R}^2$ using a linear combination of the vectors in the set.

$$c_1\begin{bmatrix}-2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$$

Change the linear combination equation into an augmented matrix, then put it into reduced row-echelon form.

$$\begin{bmatrix}-2 & 1 & | & x\\1 & 0 & | & y\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & | & y\\-2 & 1 & | & x\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & | & y\\0 & 1 & | & x+2y\end{bmatrix}$$

From the matrix, we get

$$c_1 = y$$

$$c_2 = x + 2y$$

This system tells us that we can pick any point (x, y) that we want to reach in $\mathbb{R}^2$, and we'll be able to plug that point into the system to find the corresponding values of c_1 and c_2 that we need to use. Therefore, the vector set V spans $\mathbb{R}^2$.

Now we need to show that the vectors are linearly independent, which we can do by setting $(x, y) = (0,0)$.

$$c_1 \begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Put the linear combination equation into an augmented matrix, then put it into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & | & 0 \\ 1 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ -2 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

From the matrix, we get

$$c_1 = 0$$

$$c_2 = 0$$

Because the only combination of c_1 and c_2 that gives $(x, y) = (0,0)$ is $(c_1, c_2) = (0,0)$, we know the vectors are linearly independent.

Therefore, because the vectors span all of $\mathbb{R}^2$ and are linearly independent, we can say that V forms a basis for $\mathbb{R}^2$.

■ 4. Which scalars c_1 and c_2 would you need to form the vector $\vec{v}=(7,-3)$ as a linear combination of the vectors in the span?

$$V=\text{Span}\left\{\begin{bmatrix}-2\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix}\right\}$$

Solution:

Using this set, we already found

$$c_1=y$$

$$c_2=x+2y$$

So to form $\vec{v}=(7,-3)$, we'll need

$$c_1=-3$$

$$c_2=7+2(-3)=7-6=1$$

We can double check that $(c_1,c_2)=(-3,1)$ gives $\vec{v}=(7,-3)$ by plugging into the linear combination equation.

$$\begin{bmatrix}x\\y\end{bmatrix}=c_1\begin{bmatrix}-2\\1\end{bmatrix}+c_2\begin{bmatrix}1\\0\end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = -3\begin{bmatrix} -2 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ -3 \end{bmatrix}$$

■ 5. Say whether the span forms a basis for $\mathbb{R}^3$.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right)$$

Solution:

First, we check to see if the vectors span $\mathbb{R}^3$ by verifying that we can get to any vector in $\mathbb{R}^3$ using a linear combination of the vectors in the set. We set up the linear combination equation

$$c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Put the system into an augmented matrix,

$$\begin{bmatrix} 1 & 2 & -2 & | & x \\ 0 & 1 & 1 & | & y \\ -1 & -1 & 4 & | & z \end{bmatrix}$$

and then put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -2 & | & x \\ 0 & 1 & 1 & | & y \\ -1 & -1 & 4 & | & z \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & | & x \\ 0 & 1 & 1 & | & y \\ 0 & 1 & 2 & | & x+z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & x-2y \\ 0 & 1 & 1 & | & y \\ 0 & 1 & 2 & | & x+z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & x-2y \\ 0 & 1 & 1 & | & y \\ 0 & 0 & 1 & | & x-y+z \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 5x-6y+4z \\ 0 & 1 & 1 & | & y \\ 0 & 0 & 1 & | & x-y+z \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 5x-6y+4z \\ 0 & 1 & 0 & | & -x+2y-z \\ 0 & 0 & 1 & | & x-y+z \end{bmatrix}$$

From the matrix, we get

$$c_1 = 5x - 6y + 4z$$

$$c_2 = -x + 2y - z$$

$$c_3 = x - y + z$$

So we can see that it won't matter which vector $\vec{v} = (x, y, z)$ we pick; we know that it will span $\mathbb{R}^3$, because plugging the vector into the system will simply give us the values of c_1, c_2, and c_3 that we need to use to get $\vec{v} = (x, y, z)$.

Now we need to show that the vectors are linearly independent. We do this by setting $(x, y, z) = (0,0,0)$. We'll augment the matrix with this zero vector, and then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ -1 & -1 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

The matrix gives $(c_1, c_2, c_3) = (0,0,0)$ as the only set of scalars that make the linear combination equation give the zero vector, which tells us that the vector set is linearly independent.

Therefore, because the vectors span all of $\mathbb{R}^3$ and are linearly independent, we can say that V forms a basis for $\mathbb{R}^3$.

6. What scalars would you need to get the vector $\vec{v} = (2,0,-5)$ from a linear combination of the set V?

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}\right)$$

Solution:

We already found

$$c_1 = 5x - 6y + 4z$$

$$c_2 = -x + 2y - z$$

$$c_3 = x - y + z$$

To get $\vec{v} = (2,0,-5)$, we plug the values from $\vec{v}$ into the system for c_1, c_2, and c_3.

$$c_1 = 5(2) - 6(0) + 4(-5) = 10 - 20 = -10$$

$$c_2 = -2 + 2(0) - (-5) = -2 + 5 = 3$$

$$c_3 = 2 - 0 + (-5) = 2 - 5 = -3$$

We can double check that $(c_1, c_2, c_3) = (-10,3,-3)$ gives $\vec{v} = (2,0,-5)$ by plugging into the linear combination equation.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = -10\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 3\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} - 3\begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10 \\ 0 \\ 10 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \\ -3 \end{bmatrix} - \begin{bmatrix} -6 \\ 3 \\ 12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10+6+6 \\ 0+3-3 \\ 10-3-12 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -5 \end{bmatrix}$$

DOT PRODUCTS

■ 1. Find the dot product.

$$\overrightarrow{a} = (-2,5)$$

$$\overrightarrow{b} = (3,4)$$

Solution:

To find the dot product of the two vectors, we multiply corresponding components, and then add the products. So the dot product of $\overrightarrow{a} = (-2,5)$ and $\overrightarrow{b} = (3,4)$ is

$$\overrightarrow{a} \cdot \overrightarrow{b} = (-2)(3) + (5)(4)$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = -6 + 20$$

$$\overrightarrow{a} \cdot \overrightarrow{b} = 14$$

■ 2. Find the dot product.

$$\overrightarrow{x} = (1, -2, 0)$$

$$\overrightarrow{y} = (5, -1, -3)$$

Solution:

To find the dot product of the two vectors, we multiply corresponding components, and then add the products. So the dot product of $\overrightarrow{x}=(1,-2,0)$ and $\overrightarrow{y}=(5,-1,-3)$ is

$$\overrightarrow{x}\cdot\overrightarrow{y}=(1)(5)+(-2)(-1)+(0)(-3)$$

$$\overrightarrow{x}\cdot\overrightarrow{y}=5+2+0$$

$$\overrightarrow{x}\cdot\overrightarrow{y}=7$$

■ 3. Use the dot product to find the length of the vector $\overrightarrow{u}=(-5,2,-4,-2)$.

Solution:

We'll get the square of the length if we dot the vector with itself. Use the formula $||\overrightarrow{u}||^2=\overrightarrow{u}\cdot\overrightarrow{u}$.

$$||\overrightarrow{u}||^2=(-5)(-5)+(2)(2)+(-4)(-4)+(-2)(-2)$$

$$||\overrightarrow{u}||^2=25+4+16+4$$

$$||\overrightarrow{u}||^2=49$$

Take the square root of both sides. We can ignore the negative value of the root, since we're looking for the length of the vector, and length will always be positive.

$$\sqrt{||\vec{u}||^2} = \sqrt{49}$$

$$||\vec{u}|| = 7$$

The length of $\vec{u} = (-5,2,-4,-2)$ is 7.

4. Simplify the expression if $\vec{x} = (-2,4)$, $\vec{y} = (0,-1)$, and $\vec{z} = (4,7)$.

$$4\vec{x} \cdot (3\vec{y} - \vec{z})$$

Solution:

To simplify $4\vec{x} \cdot (3\vec{y} - \vec{z})$, start by finding $4\vec{x}$.

$$\vec{x} = (-2,4)$$

$$4\vec{x} = 4(-2,4)$$

$$4\vec{x} = (-8,16)$$

Find $3\vec{y}$.

$$\vec{y} = (0,-1)$$

$$3\vec{y} = 3(0,-1)$$

$$3\vec{y} = (0,-3)$$

Then the difference $3\vec{y} - \vec{z}$ is

$$3\vec{y}-\vec{z}=\begin{bmatrix}0\\-3\end{bmatrix}-\begin{bmatrix}4\\7\end{bmatrix}$$

$$3\vec{y}-\vec{z}=\begin{bmatrix}0-4\\-3-7\end{bmatrix}$$

$$3\vec{y}-\vec{z}=\begin{bmatrix}-4\\-10\end{bmatrix}$$

Then the dot product is

$$4\vec{x}\cdot(3\vec{y}-\vec{z})$$

$$\begin{bmatrix}-8 & 16\end{bmatrix}\begin{bmatrix}-4\\-10\end{bmatrix}$$

$$(-8)(-4)+(16)(-10)$$

$$32-160$$

$$-128$$

5. Use the dot product to find $-\vec{a}\cdot(5\vec{b}+3\vec{c})$.

$$\vec{a}=(-2,0,4)$$

$$\vec{b}=(1,5,3)$$

$$\vec{c}=(-1,-4,0)$$

Solution:

To simplify $-\vec{a}(5\vec{b}+3\vec{c})$, start by finding $-\vec{a}$.

$$\vec{a}=(-2,0,4)$$

$$-\vec{a}=-1(-2,0,4)$$

$$-\vec{a}=(2,0,-4)$$

Find $5\vec{b}$.

$$\vec{b}=(1,5,3)$$

$$5\vec{b}=5(1,5,3)$$

$$5\vec{b}=(5,25,15)$$

Find $3\vec{c}$.

$$\vec{c}=(-1,-4,0)$$

$$3\vec{c}=3(-1,-4,0)$$

$$3\vec{c}=(-3,-12,0)$$

Then the sum $5\vec{b}+3\vec{c}$ is

$$5\vec{b}+3\vec{c}=\begin{bmatrix}5\\25\\15\end{bmatrix}+\begin{bmatrix}-3\\-12\\0\end{bmatrix}$$

$$5\vec{b}+3\vec{c}=\begin{bmatrix}5-3\\25-12\\15+0\end{bmatrix}$$

$$5\vec{b}+3\vec{c}=\begin{bmatrix}2\\13\\15\end{bmatrix}$$

Then the dot product is

$$-\vec{a}\cdot(5\vec{b}+3\vec{c})$$

$$\begin{bmatrix}2 & 0 & -4\end{bmatrix}\begin{bmatrix}2\\13\\15\end{bmatrix}$$

$$(2)(2)+(0)(13)+(-4)(15)$$

$$4+0-60$$

$$-56$$

■ 6. Use the dot product to find $\vec{w}(2\vec{x}+\vec{y})-3\vec{y}(\vec{w}+4\vec{x}-\vec{z})$.

$$\vec{x}=(4,-3,0,7)$$

$$\vec{y}=(-1,5,2,-1)$$

$$\vec{z}=(0,6,-1,9)$$

$$\vec{w}=(1,0,5,0)$$

Solution:

To simplify $\vec{w}(2\vec{x}+\vec{y})-3\vec{y}(\vec{w}+4\vec{x}-\vec{z})$, start by simplifying $\vec{w}(2\vec{x}+\vec{y})$ first. Find $2\vec{x}$.

$$\vec{x}=(4,-3,0,7)$$

$$2\vec{x}=2(4,-3,0,7)$$

$$2\vec{x}=(8,-6,0,14)$$

Then the sum $2\vec{x}+\vec{y}$ is

$$2\vec{x}+\vec{y}=\begin{bmatrix}8\\-6\\0\\14\end{bmatrix}+\begin{bmatrix}-1\\5\\2\\-1\end{bmatrix}$$

$$2\vec{x}+\vec{y}=\begin{bmatrix}8-1\\-6+5\\0+2\\14-1\end{bmatrix}$$

$$2\vec{x}+\vec{y}=\begin{bmatrix}7\\-1\\2\\13\end{bmatrix}$$

The dot product $\vec{w}(2\vec{x}+\vec{y})$ is

$$\begin{bmatrix}1 & 0 & 5 & 0\end{bmatrix}\begin{bmatrix}7\\-1\\2\\13\end{bmatrix}$$

$$(1)(7)+(0)(-1)+(5)(2)+(0)(13)$$

$$7+0+10+0$$

$$17$$

Then to simplify $3\vec{y}(\vec{w}+4\vec{x}-\vec{z})$, start by finding $3\vec{y}$.

$$\vec{y}=(-1,5,2,-1)$$

$$3\vec{y}=3(-1,5,2,-1)$$

$$3\vec{y}=(-3,15,6,-3)$$

Find $4\vec{x}$.

$$\vec{x}=(4,-3,0,7)$$

$$4\vec{x}=4(4,-3,0,7)$$

$$4\vec{x}=(16,-12,0,28)$$

Then $\vec{w}+4\vec{x}-\vec{z}$ is

$$\vec{w}+4\vec{x}-\vec{z}=\begin{bmatrix}1\\0\\5\\0\end{bmatrix}+\begin{bmatrix}16\\-12\\0\\28\end{bmatrix}-\begin{bmatrix}0\\6\\-1\\9\end{bmatrix}$$

$$\vec{w}+4\vec{x}-\vec{z}=\begin{bmatrix}1+16-0\\0-12-6\\5+0-(-1)\\0+28-9\end{bmatrix}$$

$$\vec{w} + 4\vec{x} - \vec{z} = \begin{bmatrix} 17 \\ -18 \\ 6 \\ 19 \end{bmatrix}$$

Then the dot product $3\vec{y}(\vec{w} + 4\vec{x} - \vec{z})$ is

$$\begin{bmatrix} -3 & 15 & 6 & -3 \end{bmatrix} \begin{bmatrix} 17 \\ -18 \\ 6 \\ 19 \end{bmatrix}$$

$$(-3)(17) + (15)(-18) + 6(6) + (-3)(19)$$

$$-51 - 270 + 36 - 57$$

$$-342$$

Then $\vec{w}(2\vec{x} + \vec{y}) - 3\vec{y}(\vec{w} + 4\vec{x} - \vec{z})$ is

$$17 - (-342)$$

$$359$$

CAUCHY-SCHWARZ INEQUALITY

1. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-1,2)$$

$$\vec{v} = (-5,10)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \cdot ||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-1)(-5) + (2)(10)| = \sqrt{(-1)^2 + 2^2}\sqrt{(-5)^2 + 10^2}$$

$$|5 + 20| = \sqrt{1 + 4}\sqrt{25 + 100}$$

$$|25| = \sqrt{5}\sqrt{125}$$

$$25 = \sqrt{625}$$

$$25 = 25$$

Since the two sides of the Cauchy-Schwarz inequality are equivalent, $\vec{u}=(-1,2)$ and $\vec{v}=(-5,10)$ are linearly dependent.

■ 2. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u}=(-5,2)$$

$$\vec{v}=(3,-7)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u}\cdot\vec{v}|=||\vec{u}||\cdot||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-5)(3)+(2)(-7)|=\sqrt{(-5)^2+2^2}\sqrt{(3)^2+(-7)^2}$$

$$|-15-14|=\sqrt{25+4}\sqrt{9+49}$$

$$|-29|=\sqrt{29}\sqrt{58}$$

$$29=\sqrt{1,682}$$

$$29\neq 41.01$$

Since the two sides of the Cauchy-Schwarz inequality are not equivalent, $\vec{u} = (-5,2)$ and $\vec{v} = (3,-7)$ are linearly independent.

■ 3. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-2,4,0)$$

$$\vec{v} = (1,-5,3)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \cdot ||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-2)(1) + (4)(-5) + (0)(3)| = \sqrt{(-2)^2 + 4^2 + 0^2}\sqrt{1^2 + (-5)^2 + 3^2}$$

$$|-2 - 20 + 0| = \sqrt{4 + 16 + 0}\sqrt{1 + 25 + 9}$$

$$|-22| = \sqrt{20}\sqrt{35}$$

$$22 = \sqrt{700}$$

$$22 \neq 26.46$$

Since the two sides of the Cauchy-Schwarz inequality are not equivalent, $\vec{u} = (-2,4,0)$ and $\vec{v} = (1,-5,3)$ are linearly independent.

■ 4. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (6,3,6)$$

$$\vec{v} = (-2,-1,-2)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \cdot ||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(6)(-2) + (3)(-1) + (6)(-2)| = \sqrt{6^2 + 3^2 + 6^2}\sqrt{(-2)^2 + (-1)^2 + (-2)^2}$$

$$|-12 - 3 - 12| = \sqrt{36 + 9 + 36}\sqrt{4 + 1 + 4}$$

$$|-27| = \sqrt{81}\sqrt{9}$$

$$27 = 9 \cdot 3$$

$$27 = 27$$

Since the two sides of the Cauchy-Schwarz inequality are equivalent, $\vec{u} = (6,3,6)$ and $\vec{v} = (-2, -1, -2)$ are linearly dependent.

■ 5. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-13,5,7)$$

$$\vec{v} = (1, -1, -1)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \cdot ||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-13)(1) + (5)(-1) + (7)(-1)| = \sqrt{(-13)^2 + 5^2 + 7^2}\sqrt{1^2 + (-1)^2 + (-1)^2}$$

$$|-13 - 5 - 7| = \sqrt{169 + 25 + 49}\sqrt{1 + 1 + 1}$$

$$|-25| = \sqrt{243}\sqrt{3}$$

$$25 = \sqrt{729}$$

$$25 \neq 27$$

Since the two sides of the Cauchy-Schwarz inequality are not equivalent, $\vec{u} = (-13,5,7)$ and $\vec{v} = (1, -1, -1)$ are linearly independent.

■ 6. Use the Cauchy-Schwarz inequality to say whether or not the vectors are linearly independent.

$$\vec{u} = (-2,0,2)$$

$$\vec{v} = (8,0,-8)$$

Solution:

Plug the vectors into the Cauchy-Schwarz inequality.

$$|\vec{u} \cdot \vec{v}| = ||\vec{u}|| \cdot ||\vec{v}||$$

If the two sides are equivalent, then the Cauchy-Schwarz inequality tells us that the vectors are linearly dependent. If the two sides are not equal, then we know that the vectors are linearly independent.

$$|(-2)(8) + (0)(0) + (2)(-8)| = \sqrt{(-2)^2 + 0^2 + 2^2}\sqrt{8^2 + 0^2 + (-8)^2}$$

$$|-16 + 0 - 16| = \sqrt{4 + 0 + 4}\sqrt{64 + 0 + 64}$$

$$|-32| = \sqrt{8}\sqrt{128}$$

$$32 = \sqrt{1,024}$$

$$32 = 32$$

Since the two sides of the Cauchy-Schwarz inequality are equivalent, $\vec{u} = (-2,0,2)$ and $\vec{v} = (8,0,-8)$ are linearly dependent.

VECTOR TRIANGLE INEQUALITY

■ 1. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u} = (\sqrt{3},3) \text{ and } \vec{v} = (2\sqrt{3},0)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2} \le \sqrt{u_1^2+u_2^2} + \sqrt{v_1^2+v_2^2}$$

$$\sqrt{(\sqrt{3}+2\sqrt{3})^2+(3+0)^2} \le \sqrt{(\sqrt{3})^2+3^2} + \sqrt{(2\sqrt{3})^2+0^2}$$

$$\sqrt{(3\sqrt{3})^2+3^2} \le \sqrt{3+9} + \sqrt{12+0}$$

$$\sqrt{27+9} \le \sqrt{12} + \sqrt{12}$$

$$\sqrt{36} \le 2\sqrt{12}$$

$$6 < 6.93$$

Because the left side is less than the right side, the vector set is linearly independent.

■ 2. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ span $\mathbb{R}^2$.

$\vec{u} = (5, -7)$ and $\vec{v} = (-4, -3)$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u} + \vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2} \le \sqrt{u_1^2 + u_2^2} + \sqrt{v_1^2 + v_2^2}$$

$$\sqrt{(5 - 4)^2 + (-7 - 3)^2} \le \sqrt{5^2 + (-7)^2} + \sqrt{(-4)^2 + (-3)^2}$$

$$\sqrt{1^2 + (-10)^2} \le \sqrt{25 + 49} + \sqrt{16 + 9}$$

$$\sqrt{1 + 100} \le \sqrt{74} + \sqrt{25}$$

$$\sqrt{101} \le 8.60 + 5$$

$$10.05 < 13.60$$

Since $10.05 < 13.60$, we can conclude that $\vec{u}$ and $\vec{v}$ are not collinear, and that they therefore span $\mathbb{R}^2$.

■ 3. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$\vec{u} = (-2,5)$ and $\vec{v} = (2, -5)$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u} + \vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2} \le \sqrt{u_1^2 + u_2^2} + \sqrt{v_1^2 + v_2^2}$$

$$\sqrt{(-2 + 2)^2 + (5 - 5)^2} \le \sqrt{(-2)^2 + 5^2} + \sqrt{2^2 + (-5)^2}$$

$$\sqrt{0^2 + 0^2} \le \sqrt{4 + 25} + \sqrt{4 + 25}$$

$$0 \le \sqrt{29} + \sqrt{29}$$

$$0 \le 2\sqrt{29}$$

Since $||\vec{u}+\vec{v}||$ is 0, the vector set is linearly dependent.

■ 4. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u}=(-3,12,-15) \text{ and } \vec{v}=(-1,4,-5)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u}+\vec{v}|| \le ||\vec{u}||+||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2} \le \sqrt{u_1^2+u_2^2+u_3^2}+\sqrt{v_1^2+v_2^2+v_3^2}$$

$$\sqrt{(-3-1)^2+(12+4)^2+(-15-5)^2} \le \sqrt{(-3)^2+12^2+(-15)^2}+\sqrt{(-1)^2+4^2+(-5)^2}$$

$$\sqrt{(-4)^2+16^2+(-20)^2} \le \sqrt{9+144+225}+\sqrt{1+16+25}$$

$$\sqrt{16+256+400} \le \sqrt{378}+\sqrt{42}$$

$$\sqrt{672} \le 19.44+6.48$$

$$25.92=25.92$$

Because the sides are equal, the vector set is linearly dependent.

■ 5. Use the vector triangle inequality to say whether $\overrightarrow{u}$ and $\overrightarrow{v}$ are linearly independent.

$$\overrightarrow{u} = (1,2,0) \text{ and } \overrightarrow{v} = (-5,1,-6)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\overrightarrow{u} + \overrightarrow{v}|| \le ||\overrightarrow{u}|| + ||\overrightarrow{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1+v_1)^2+(u_2+v_2)^2+(u_3+v_3)^2} \le \sqrt{u_1^2+u_2^2+u_3^2} + \sqrt{v_1^2+v_2^2+v_3^2}$$

$$\sqrt{(1-5)^2+(2+1)^2+(0-6)^2} \le \sqrt{1^2+2^2+0^2} + \sqrt{(-5)^2+1^2+(-6)^2}$$

$$\sqrt{(-4)^2+3^2+(-6)^2} \le \sqrt{1+4+0} + \sqrt{25+1+36}$$

$$\sqrt{16+9+36} \le \sqrt{5} + \sqrt{62}$$

$$\sqrt{61} \le 2.24 + 7.87$$

$$7.81 \le 10.11$$

Because the left side is less than the right side, the vector set is linearly independent.

■ 6. Use the vector triangle inequality to say whether $\vec{u}$ and $\vec{v}$ are linearly independent.

$$\vec{u} = (2, -5, 4) \text{ and } \vec{v} = (6, -15, 12)$$

Solution:

Plug the vectors into the vector triangle inequality.

$$||\vec{u} + \vec{v}|| \le ||\vec{u}|| + ||\vec{v}||$$

If the left side is less than the right side, the vector set is linearly independent. But if the sides are equivalent, or if the left side is 0, then the vector set is linearly dependent.

$$\sqrt{(u_1 + v_1)^2 + (u_2 + v_2)^2 + (u_3 + v_3)^2} \le \sqrt{u_1^2 + u_2^2 + u_3^2} + \sqrt{v_1^2 + v_2^2 + v_3^2}$$

$$\sqrt{(2 + 6)^2 + (-5 - 15)^2 + (4 + 12)^2} \le \sqrt{2^2 + (-5)^2 + 4^2} + \sqrt{6^2 + (-15)^2 + 12^2}$$

$$\sqrt{8^2 + (-20)^2 + 16^2} \le \sqrt{4 + 25 + 16} + \sqrt{36 + 225 + 144}$$

$$\sqrt{64 + 400 + 256} \le \sqrt{45} + \sqrt{405}$$

$$\sqrt{720} \le 6.71 + 20.12$$

$$26.83 = 26.83$$

Because the sides are equal, the vector set is linearly dependent.

ANGLE BETWEEN VECTORS

■ 1. Say whether or not the vectors are orthogonal.

$$\vec{a}=(-1,3)$$

$$\vec{b}=(6,2)$$

Solution:

Test the orthogonality of the vectors $\vec{a}=(-1,3)$ and $\vec{b}=(6,2)$ by calculating their dot product.

$$\vec{a}\cdot\vec{b}=(-1)(6)+(3)(2)$$

$$\vec{a}\cdot\vec{b}=-6+6$$

$$\vec{a}\cdot\vec{b}=0$$

Because the dot product is 0, $\vec{a}$ and $\vec{b}$ are orthogonal to one another.

■ 2. Say whether or not the vectors are orthogonal.

$$\vec{u}=2i-j+3k$$

$$\vec{v}=-i-3j+2k$$

Solution:

Test the orthogonality of the vectors $\overrightarrow{u} = 2i - j + 3k$ and $\overrightarrow{v} = -i - 3j + 2k$ by calculating their dot product.

$$\overrightarrow{u} \cdot \overrightarrow{v} = (2)(-1) + (-1)(-3) + (3)(2)$$

$$\overrightarrow{u} \cdot \overrightarrow{v} = -2 + 3 + 6$$

$$\overrightarrow{u} \cdot \overrightarrow{v} = 7$$

Since the dot product is not 0, the vectors are not orthogonal to one another.

■ 3. Find the angle between the vectors.

$$\overrightarrow{x} = (0,2)$$

$$\overrightarrow{y} = (1,1)$$

Solution:

Find the lengths of both vectors.

$$||\overrightarrow{x}|| = \sqrt{x_1^2 + x_2^2} = \sqrt{0 + 2^2} = \sqrt{0 + 4} = \sqrt{4} = 2$$

$$||\overrightarrow{y}|| = \sqrt{y_1^2 + y_2^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

Then find the dot product of the vectors.

$$\vec{x} \cdot \vec{y} = (0)(1) + (2)(1)$$

$$\vec{x} \cdot \vec{y} = 0 + 2$$

$$\vec{x} \cdot \vec{y} = 2$$

Plug everything into the formula for the angle between the vectors.

$$\vec{x} \cdot \vec{y} = ||\vec{x}|| \cdot ||\vec{y}|| \cos\theta$$

$$2 = 2\sqrt{2}\cos\theta$$

$$\frac{2}{2\sqrt{2}} = \cos\theta$$

$$\frac{1}{\sqrt{2}} = \cos\theta$$

Take the inverse cosine of each side to solve for θ.

$$\theta = \arccos\left(\frac{1}{\sqrt{2}}\right)$$

Use a calculator to find $\theta = 45^\circ$.

4. Find the angle between the vectors.

$$\vec{a} = (-5,7,3)$$

$$\vec{b} = (1,2,-3)$$

Solution:

Find the lengths of both vectors.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{(-5)^2 + 7^2 + 3^2} = \sqrt{25 + 49 + 9} = \sqrt{83}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{1^2 + 2^2 + (-3)^2} = \sqrt{1 + 4 + 9} = \sqrt{14}$$

Then find the dot product of the vectors.

$$\vec{a} \cdot \vec{b} = (-5)(1) + 7(2) + (3)(-3)$$

$$\vec{a} \cdot \vec{b} = -5 + 14 - 9$$

$$\vec{a} \cdot \vec{b} = 0$$

Plug everything into the formula for the angle between vectors.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos\theta$$

$$0 = \sqrt{14}\sqrt{83} \cos\theta$$

$$\frac{0}{\sqrt{1,162}} = \cos\theta$$

$$0 = \cos\theta$$

Since the dot product is 0, the vectors are orthogonal, which means $\theta = 90^\circ$.

■ 5. Find the angle between the vectors.

$$\vec{a} = (-1,3,-4)$$

$$\vec{b} = (2,1,0)$$

Solution:

Find the lengths of both vectors.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{(-1)^2 + 3^2 + (-4)^2} = \sqrt{1 + 9 + 16} = \sqrt{26}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{2^2 + 1^2 + 0^2} = \sqrt{4 + 1 + 0} = \sqrt{5}$$

Then find the dot product of the vectors.

$$\vec{a} \cdot \vec{b} = (-1)(2) + (3)(1) + (-4)(0)$$

$$\vec{a} \cdot \vec{b} = -2 + 3 + 0$$

$$\vec{a} \cdot \vec{b} = 1$$

Plug everything into the formula for the angle between vectors.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos\theta$$

$$1 = \sqrt{26}\sqrt{5}\cos\theta$$

$$1 = \sqrt{130}\cos\theta$$

$$\frac{1}{\sqrt{130}} = \cos\theta$$

Take the inverse cosine of each side to solve for θ.

$$\theta = \arccos\left(\frac{1}{\sqrt{130}}\right)$$

Use a calculator to find $\theta \approx 84.97^\circ$.

6. Find the angle between the vectors.

$$\vec{a} = (1, -2, 5)$$

$$\vec{b} = (8, 6, 3)$$

Solution:

Find the lengths of both vectors.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{1^2 + (-2)^2 + 5^2} = \sqrt{1 + 4 + 25} = \sqrt{30}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{8^2 + 6^2 + 3^2} = \sqrt{64 + 36 + 9} = \sqrt{109}$$

Then find the dot product of the vectors.

$$\vec{a} \cdot \vec{b} = (1)(8) + (-2)(6) + (5)(3)$$

$$\vec{a} \cdot \vec{b} = 8 - 12 + 15$$

$$\vec{a} \cdot \vec{b} = 11$$

Plug everything into the formula for the angle between vectors.

$$\vec{a} \cdot \vec{b} = ||\vec{a}|| \cdot ||\vec{b}|| \cos\theta$$

$$11 = \sqrt{30}\sqrt{109}\cos\theta$$

$$11 = \sqrt{3{,}270}\cos\theta$$

$$\frac{11}{\sqrt{3{,}270}} = \cos\theta$$

Take the inverse cosine of each side to solve for θ.

$$\theta = \arccos\left(\frac{11}{\sqrt{3{,}270}}\right)$$

Use a calculator to find $\theta \approx 78.91^\circ$.

EQUATION OF A PLANE, AND NORMAL VECTORS

1. What is the normal vector to the plane?

$$-2x + 5y - 7z = 0$$

Solution:

Given a plane $Ax + By + Cz = D$, the normal vector to the plane is

$$\overrightarrow{n} = (A, B, C)$$

So from the plane $-2x + 5y - 7z = 0$, pull out the coefficients on x, y, and z to get the components of the normal vector.

$$\overrightarrow{n} = (-2,5, -7)$$

2. What is the normal vector to the plane?

$$10y - 5z + 6 = 0$$

Solution:

Given a plane $Ax + By + Cz = D$, the normal vector to the plane is

$$\overrightarrow{n} = (A, B, C)$$

Rewrite the plane in standard form.

$$10y - 5z + 6 = 0$$

$$0x + 10y - 5z = -6$$

So from the plane $10y - 5z = -6$, we can simply pull out the coefficients on x, y, and z to get the components of the normal vector.

$$\overrightarrow{n} = (0,10,-5)$$

■ 3. Find the equation of a plane with normal vector $\overrightarrow{n} = (-1,0,4)$ that passes through $(1,-3,0)$.

Solution:

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$-(x - 1) + 0(y - (-3)) + 4(z - 0) = 0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-(x - 1) + 0(y + 3) + 4z = 0$$

$$-x + 1 + 4z = 0$$

$$-x + 4z = -1$$

■ 4. Find the equation of a plane with normal vector $\vec{n} = (4, -7,3)$ that passes through $(-2,1,6)$.

Solution:

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$4(x - (-2)) - 7(y - 1) + 3(z - 6) = 0$$

Now we'll simplify and get the equation of the plane into standard form.

$$4(x + 2) - 7(y - 1) + 3(z - 6) = 0$$

$$4x + 8 - 7y + 7 + 3z - 18 = 0$$

$$4x - 7y + 3z - 3 = 0$$

$$4x - 7y + 3z = 3$$

■ 5. Find the equation of a plane with normal vector $\vec{n} = -3i + 4j - z$ that passes through $(-2,0,-7)$.

Solution:

Plugging the normal vector and the point on the plane into the plane equation gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$-3(x-(-2))+4(y-0)-(z-(-7))=0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-3(x+2)+4y-(z+7)=0$$

$$-3x-6+4y-z-7=0$$

$$-3x+4y-z-13=0$$

$$-3x+4y-z=13$$

■ 6. Find the equation of the plane passing through P and perpendicular to $\overrightarrow{PQ}$.

$$P(1,-5,4)$$

$$Q(0,3,-1)$$

Solution:

First find the normal vector to the plane.

$$\overrightarrow{PQ}=(0-1,3-(-5),-1-4)$$

$$\overrightarrow{PQ} = (-1,8,-5)$$

Plugging this normal vector and the point on the plane into the plane equation gives

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$-(x-1)+8(y-(-5))-5(z-4)=0$$

Now we'll simplify and get the equation of the plane into standard form.

$$-(x-1)+8(y+5)-5(z-4)=0$$

$$-x+1+8y+40-5z+20=0$$

$$-x+8y-5z+61=0$$

$$-x+8y-5z=-61$$

CROSS PRODUCTS

1. Find the cross product of $\vec{a}=(1,-3,-1)$ and $\vec{b}=(5,6,-2)$.

Solution:

The cross product of $\vec{a}=(1,-3,-1)$ and $\vec{b}=(5,6,-2)$ is given by

$$\vec{a}\times\vec{b}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -3 & -1\\ 5 & 6 & -2\end{vmatrix}$$

$$\vec{a}\times\vec{b}=\mathbf{i}\begin{vmatrix}-3 & -1\\ 6 & -2\end{vmatrix}-\mathbf{j}\begin{vmatrix}1 & -1\\ 5 & -2\end{vmatrix}+\mathbf{k}\begin{vmatrix}1 & -3\\ 5 & 6\end{vmatrix}$$

Evaluating the determinants gives

$$\vec{a}\times\vec{b}=\mathbf{i}((-3)(-2)-(-1)(6))-\mathbf{j}((1)(-2)-(-1)(5))+\mathbf{k}((1)(6)-(-3)(5))$$

$$\vec{a}\times\vec{b}=\mathbf{i}(6+6)-\mathbf{j}(-2+5)+\mathbf{k}(6+15)$$

$$\vec{a}\times\vec{b}=12\mathbf{i}-3\mathbf{j}+21\mathbf{k}$$

$$\vec{a}\times\vec{b}=(12,-3,21)$$

2. Find a vector orthogonal to both $\vec{a}=(-3,-5,2)$ and $\vec{b}=(-2,4,-7)$.

Solution:

We need to find the cross product of the two vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -5 & 2 \\ -2 & 4 & -7 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} -5 & 2 \\ 4 & -7 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -3 & 2 \\ -2 & -7 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -3 & -5 \\ -2 & 4 \end{vmatrix}$$

Evaluating the determinants gives

$$\vec{a} \times \vec{b} = \mathbf{i}((-5)(-7) - (2)(4)) - \mathbf{j}((-3)(-7) - (2)(-2)) + \mathbf{k}((-3)(4) - (-5)(-2))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(35 - 8) - \mathbf{j}(21 + 4) + \mathbf{k}(-12 - 10)$$

$$\vec{a} \times \vec{b} = 27\mathbf{i} - 25\mathbf{j} - 22\mathbf{k}$$

So the vector $\vec{a} \times \vec{b} = (27, -25, -22)$ is orthogonal to both $\vec{a} = (-3, -5, 2)$ and $\vec{b} = (-2, 4, -7)$.

■ 3. Find the length of the cross product of $\vec{a} = (-1, -2, 0)$ and $\vec{b} = (1, 1, -2)$.

Solution:

First, find the cross product of the two vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -2 & 0 \\ 1 & 1 & -2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} -2 & 0 \\ 1 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -1 & 0 \\ 1 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -1 & -2 \\ 1 & 1 \end{vmatrix}$$

Evaluating the determinants gives

$$\vec{a} \times \vec{b} = \mathbf{i}((-2)(-2) - (0)(1)) - \mathbf{j}((-1)(-2) - (0)(1)) + \mathbf{k}((-1)(1) - (-2)(1))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(4 - 0) - \mathbf{j}(2 - 0) + \mathbf{k}(-1 + 2)$$

$$\vec{a} \times \vec{b} = 4\mathbf{i} - 2\mathbf{j} + \mathbf{k}$$

$$\vec{a} \times \vec{b} = (4, -2, 1)$$

Then the length of the cross product is

$$||\vec{a} \times \vec{b}|| = \sqrt{4^2 + (-2)^2 + 1^2} = \sqrt{16 + 4 + 1} = \sqrt{21}$$

4. Find the length of the cross product of $\vec{a} = (6, -3, 3)$ and $\vec{b} = (3, 0, 3)$ when the angle between $\vec{a}$ and $\vec{b}$ is $\theta = 30^\circ$.

Solution:

First, find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{6^2 + (-3)^2 + 3^2} = \sqrt{36 + 9 + 9} = \sqrt{54} = 3\sqrt{6}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{3^2 + 0^2 + 3^2} = \sqrt{9 + 0 + 9} = \sqrt{18} = 3\sqrt{2}$$

Then the length of the cross product is given by

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| \sin\theta$$

$$||\vec{a} \times \vec{b}|| = 3\sqrt{6} \cdot 3\sqrt{2} \cdot \sin(30^\circ)$$

$$||\vec{a} \times \vec{b}|| = 9\sqrt{12} \cdot \frac{1}{2}$$

$$||\vec{a} \times \vec{b}|| = 9 \cdot 2\sqrt{3} \cdot \frac{1}{2}$$

$$||\vec{a} \times \vec{b}|| = 9\sqrt{3}$$

5. Find the length of the cross product of the vectors $\vec{a} = (2, -5, 3)$ and $\vec{b} = (4, 6, -1)$, and find the sine of the angle between them.

Solution:

First, find the cross product of the two vectors.

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -5 & 3 \\ 4 & 6 & -1 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} -5 & 3 \\ 6 & -1 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & 3 \\ 4 & -1 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -5 \\ 4 & 6 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i}((-5)(-1) - (3)(6)) - \mathbf{j}((2)(-1) - (3)(4)) + \mathbf{k}((2)(6) - (-5)(4))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(5 - 18) - \mathbf{j}(-2 - 12) + \mathbf{k}(12 + 20)$$

$$\vec{a} \times \vec{b} = -13\mathbf{i} + 14\mathbf{j} + 32\mathbf{k}$$

Then the length of the cross product is

$$||\vec{a} \times \vec{b}|| = \sqrt{(-13)^2 + 14^2 + 32^2} = \sqrt{169 + 196 + 1{,}024} = \sqrt{1{,}389}$$

Find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{2^2 + (-5)^2 + 3^2} = \sqrt{4 + 25 + 9} = \sqrt{38}$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{4^2 + 6^2 + (-1)^2} = \sqrt{16 + 36 + 1} = \sqrt{53}$$

The sine of the angle between the vectors will be

$$||\vec{a} \times \vec{b}|| = ||\vec{a}||\ ||\vec{b}|| \sin\theta$$

$$\sqrt{1{,}389} = \sqrt{38}\sqrt{53} \sin\theta$$

$$\sin\theta = \frac{\sqrt{1{,}389}}{\sqrt{38}\sqrt{53}}$$

$$\sin\theta \approx 0.8305$$

■ 6. Find the angle between the vectors $\vec{a} = (2, -2, 1)$ and $\vec{b} = (1, 0, 1)$, and find the length of their cross product.

Solution:

First, find the length of each vector individually.

$$||\vec{a}|| = \sqrt{a_1^2 + a_2^2 + a_3^2} = \sqrt{2^2 + (-2)^2 + 1^2} = \sqrt{4 + 4 + 1} = \sqrt{9} = 3$$

$$||\vec{b}|| = \sqrt{b_1^2 + b_2^2 + b_3^2} = \sqrt{1^2 + 0^2 + 1^2} = \sqrt{1 + 0 + 1} = \sqrt{2}$$

The angle between the vectors will be given by

$$\vec{a} \cdot \vec{b} = ||\vec{a}||\ ||\vec{b}||\cos\theta$$

$$\begin{bmatrix} 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = 3\sqrt{2}\cos\theta$$

$$2(1) - 2(0) + 1(1) = 3\sqrt{2}\cos\theta$$

$$2 + 0 + 1 = 3\sqrt{2}\cos\theta$$

$$\frac{3}{3\sqrt{2}} = \cos\theta$$

So the angle is

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \arccos\left(\frac{1}{\sqrt{2}}\right)$$

$$\theta = 45^\circ$$

Then the length of the cross product is given by

$$||\overrightarrow{a} \times \overrightarrow{b}|| = ||\overrightarrow{a}|| \ ||\overrightarrow{b}|| \sin\theta$$

$$||\overrightarrow{a} \times \overrightarrow{b}|| = 3\sqrt{2}\sin 45^\circ$$

$$||\overrightarrow{a} \times \overrightarrow{b}|| = 3\sqrt{2} \cdot \frac{1}{\sqrt{2}}$$

$$||\overrightarrow{a} \times \overrightarrow{b}|| = 3$$

DOT AND CROSS PRODUCTS AS OPPOSITE IDEAS

■ 1. Find the maximum value of the dot product, if $||\vec{u}|| = 4$ and $||\vec{v}|| = 5$.

Solution:

The dot product will be maximized when the vectors point in the same direction, where the angle between them is 0°.

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}|| \cos 0^\circ$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||(1)$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| \ ||\vec{v}||$$

$$\vec{u} \cdot \vec{v} = 4 \cdot 5$$

$$\vec{u} \cdot \vec{v} = 20$$

■ 2. Find the minimum value of the dot product of two vectors, if $||\vec{u}|| = \sqrt{56}$ and $||\vec{v}|| = \sqrt{126}$.

Solution:

The dot product will be minimized when the vectors point in the opposite directions, where the angle between them is 180°.

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||\cos 180^\circ$$

$$\vec{u} \cdot \vec{v} = ||\vec{u}||\ ||\vec{v}||(-1)$$

$$\vec{u} \cdot \vec{v} = -\ ||\vec{u}||\ ||\vec{v}||$$

$$\vec{u} \cdot \vec{v} = -\sqrt{56}\sqrt{126}$$

$$\vec{u} \cdot \vec{v} = -\ 84$$

■ 3. Find the maximum value of the length of the cross product of $\vec{u}$ and $\vec{v}$, if $||\vec{u}|| = \sqrt{50}$ and $||\vec{v}|| = \sqrt{128}$.

Solution:

The length of the cross product will be maximized when the vectors are orthogonal, so the angle between them is 90°.

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}||\sin 90^\circ$$

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}||(1)$$

$$||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}||$$

$$||\vec{u} \times \vec{v}|| = \sqrt{50}\sqrt{128}$$

$$||\vec{u} \times \vec{v}|| = 80$$

4. Find the dot product and the length of the cross product of $\overrightarrow{u}=(2,1)$ and $\overrightarrow{v}=(-6,-3)$. Then interpret the results based on what the dot and cross products indicate.

Solution:

The vector $\overrightarrow{v}=(-6,-3)$ is $-3\overrightarrow{u}$, which means the vectors $\overrightarrow{u}=(2,1)$ and $\overrightarrow{v}=(-6,-3)$ point in opposite directions, which means the angle between them is $\theta=180^\circ$.

First, find the length of each vector individually.

$$||\overrightarrow{u}||=\sqrt{2^2+1^2}=\sqrt{4+1}=\sqrt{5}$$

$$||\overrightarrow{v}||=\sqrt{(-6)^2+(-3)^2}=\sqrt{36+9}=\sqrt{45}=3\sqrt{5}$$

So the dot product is

$$\overrightarrow{u}\cdot\overrightarrow{v}=||\overrightarrow{u}||\ ||\overrightarrow{v}||\cos 180^\circ$$

$$\overrightarrow{u}\cdot\overrightarrow{v}=(\sqrt{5})(3\sqrt{5})(-1)$$

$$\overrightarrow{u}\cdot\overrightarrow{v}=-15$$

And the cross product is

$$||\overrightarrow{u}\times\overrightarrow{v}||=||\overrightarrow{u}||\ ||\overrightarrow{v}||\sin 180^\circ$$

$$||\overrightarrow{u}\times\overrightarrow{v}||=(\sqrt{5})(3\sqrt{5})(0)$$

$$||\overrightarrow{u}\times\overrightarrow{v}||=0$$

Because the length of the cross product is 0, we know that the vectors are collinear. For collinear vectors, the dot product is just the product of the lengths of the vectors, which we see in $\vec{u}\cdot\vec{v}=-15$. The fact that the dot product is negative tells us that the vectors point in exactly opposite directions along the same line, where the dot product will have its minimum value.

■ 5. Find the dot product and the length of the cross product of $\vec{u}=(2,-3,-1)$ and $\vec{v}=(4,-6,-2)$. Then interpret the results based on what the dot and cross products indicate.

Solution:

First, find the length of each vector individually.

$$||\vec{u}||=\sqrt{2^2+(-3)^2+(-1)^2}=\sqrt{4+9+1}=\sqrt{14}$$

$$||\vec{v}||=\sqrt{4^2+(-6)^2+(-2)^2}=\sqrt{16+36+4}=\sqrt{56}=2\sqrt{14}$$

So the dot product is

$$\vec{u}\cdot\vec{v}=\begin{bmatrix}2 & -3 & -1\end{bmatrix}\cdot\begin{bmatrix}4\\-6\\-2\end{bmatrix}$$

$$\vec{u}\cdot\vec{v}=2(4)-3(-6)-1(-2)$$

$$\vec{u}\cdot\vec{v}=8+18+2$$

$$\vec{u} \cdot \vec{v} = 28$$

And the cross product is

$$\vec{u} \times \vec{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -1 \\ 4 & -6 & -2 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \mathbf{i} \begin{vmatrix} -3 & -1 \\ -6 & -2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix}$$

$$\vec{u} \times \vec{v} = \mathbf{i}((-3)(-2) - (-1)(-6)) - \mathbf{j}((2)(-2) - (-1)(4)) + \mathbf{k}((2)(-6) - (-3)(4))$$

$$\vec{u} \times \vec{v} = \mathbf{i}(6 - 6) - \mathbf{j}(-4 + 4) + \mathbf{k}(-12 + 12)$$

$$\vec{u} \times \vec{v} = \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0)$$

$$\vec{u} \times \vec{v} = (0,0,0)$$

So the length of the cross product is

$$||\vec{u} \times \vec{v}|| = \sqrt{0^2 + 0^2 + 0^2} = 0$$

Because the length of the cross product is 0, we know that the vectors are collinear. For collinear vectors, the dot product is just the product of the lengths of the vectors, which we see in $\vec{u} \cdot \vec{v} = 28$. The fact that the dot product is positive tells us that the vectors point in the same direction, where the dot product will have its maximum value.

6. Find the dot product and the length of the cross product of $\vec{u}=(-2,4,3)$ and $\vec{v}=(2,1,0)$. Then interpret the results based on what the dot and cross products indicate.

Solution:

First, find the length of each vector individually.

$$||\vec{u}||=\sqrt{(-2)^2+4^2+3^2}=\sqrt{4+16+9}=\sqrt{29}$$

$$||\vec{v}||=\sqrt{2^2+1^2+0^2}=\sqrt{4+1+0}=\sqrt{5}$$

So the dot product is

$$\vec{u}\cdot\vec{v}=\begin{bmatrix}-2 & 4 & 3\end{bmatrix}\cdot\begin{bmatrix}2\\1\\0\end{bmatrix}$$

$$\vec{u}\cdot\vec{v}=-2(2)+4(1)+3(0)$$

$$\vec{u}\cdot\vec{v}=-4+4+0$$

$$\vec{u}\cdot\vec{v}=0$$

The dot product is 0 when the vectors are orthogonal, so the angle between them is 90°.

The length of the cross product is

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||\sin 90^\circ$$

$$||\vec{u}\times\vec{v}||=||\vec{u}||\ ||\vec{v}||(1)$$

$$||\vec{u}\times\vec{v}||=\sqrt{29}\sqrt{5}$$

$$||\vec{u}\times\vec{v}||=\sqrt{145}$$

Since the vectors are orthogonal, the length of the cross product is maximized.

MULTIPLYING MATRICES BY VECTORS

1. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}$$

$$\vec{x} = (4, -1)$$

Solution:

To find $A\vec{x}$, we'll multiply the matrix A by the column vector $\vec{x}$. We know the product is defined, since the matrix has 2 columns and the vector has 2 rows.

$$A\vec{x} = \begin{bmatrix} 0 & 2 \\ -1 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 0(4) + 2(-1) \\ -1(4) + 1(-1) \\ 0(4) - 2(-1) \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 0 - 2 \\ -4 - 1 \\ 0 + 2 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} -2 \\ -5 \\ 2 \end{bmatrix}$$

■ 2. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\vec{x} = (-2,3)$$

Solution:

To find $\vec{x}A$, we'll multiply the row vector $\vec{x}$ by the matrix A. We know the product is defined, since the vector has 2 columns and the matrix has 2 rows.

$$\vec{x}A = \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -2(3) + 3(0) & -2(-1) + 3(0) & -2(0) + 3(4) \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -6 + 0 & 2 + 0 & 0 + 12 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} -6 & 2 & 12 \end{bmatrix}$$

■ 3. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 0 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}$$

$$\vec{x}=(2,0,1)$$

Solution:

To find $A\vec{x}$, we'll multiply the matrix A by the column vector $\vec{x}$. We know the product is defined, since the matrix has 3 columns and the vector has 3 rows.

$$A\vec{x}=\begin{bmatrix}4 & -2 & 1\\ 0 & 0 & -1\\ -3 & 1 & 2\end{bmatrix}\begin{bmatrix}2\\ 0\\ 1\end{bmatrix}$$

$$A\vec{x}=\begin{bmatrix}4(2)-2(0)+1(1)\\ 0(2)+0(0)-1(1)\\ -3(2)+1(0)+2(1)\end{bmatrix}$$

$$A\vec{x}=\begin{bmatrix}8-0+1\\ 0+0-1\\ -6+0+2\end{bmatrix}$$

$$A\vec{x}=\begin{bmatrix}9\\ -1\\ -4\end{bmatrix}$$

■ 4. Find the matrix-vector product, $\vec{x}A$.

$$A=\begin{bmatrix}1 & -1 & 0 & -2\\ -3 & 0 & -2 & 1\end{bmatrix}$$

$$\vec{x}=(2,-6)$$

Solution:

To find $\vec{x}A$, we'll multiply the row vector $\vec{x}$ by the matrix A. We know the product is defined, since the vector has 2 columns and the matrix has 2 rows.

$$\vec{x}A = \begin{bmatrix} 2 & -6 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & -2 \\ -3 & 0 & -2 & 1 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} 2(1) - 6(-3) & 2(-1) - 6(0) & 2(0) - 6(-2) & 2(-2) - 6(1) \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} 2 + 18 & -2 - 0 & 0 + 12 & -4 - 6 \end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix} 20 & -2 & 12 & -10 \end{bmatrix}$$

■ 5. Find the matrix-vector product, $A\vec{x}$.

$$A = \begin{bmatrix} 4 & 6 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\vec{x} = (3,3)$$

Solution:

To find $A\vec{x}$, we'll multiply the matrix A by the column vector $\vec{x}$. We know the product is defined, since the matrix has 2 columns and the vector has 2 rows.

$$A\vec{x} = \begin{bmatrix} 4 & 6 \\ -2 & -3 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 4(3) + 6(3) \\ -2(3) - 3(3) \\ 1(3) + 0(3) \\ 0(3) + 1(3) \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 12 + 18 \\ -6 - 9 \\ 3 + 0 \\ 0 + 3 \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 30 \\ -15 \\ 3 \\ 3 \end{bmatrix}$$

■ 6. Find the matrix-vector product, $\vec{x}A$.

$$A = \begin{bmatrix} 6 & -4 & -4 \\ 1 & -4 & -4 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{x} = (-3,1,1)$$

Solution:

To find $\vec{x}A$, we'll multiply the row vector $\vec{x}$ by the matrix A. We know the product is defined, since the vector has 3 columns and the matrix has 3 rows.

$$\vec{x}A = \begin{bmatrix}-3 & 1 & 1\end{bmatrix}\begin{bmatrix}6 & -4 & -4 \\ 1 & -4 & -4 \\ 0 & 0 & 1\end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix}-3(6) + 1(1) + 1(0) & -3(-4) + 1(-4) + 1(0) & -3(-4) + 1(-4) + 1(1)\end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix}-18 + 1 + 0 & 12 - 4 + 0 & 12 - 4 + 1\end{bmatrix}$$

$$\vec{x}A = \begin{bmatrix}-17 & 8 & 9\end{bmatrix}$$

THE NULL SPACE AND AX=O

1. Is $\vec{x}=(1,2)$ in the null space of A?

$$A=\begin{bmatrix}4 & -2\\ 2 & -1\end{bmatrix}$$

Solution:

If $\vec{x}=(1,2)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x}=\vec{O}$$

$$\begin{bmatrix}4 & -2\\ 2 & -1\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix}4(1)-2(2)\\ 2(1)-1(2)\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}4-4\\ 2-2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\vec{x}=(1,2)$ is in the null space of A.

2. Is $\vec{x}=(5,-8,-9)$ in the null space of A?

$$A=\begin{bmatrix}6&1&1\\0&-2&3\\-1&0&4\end{bmatrix}$$

Solution:

If $\vec{x}=(5,-8,-9)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x}=\vec{O}$$

$$\begin{bmatrix}6&1&1\\0&-2&3\\-1&0&4\end{bmatrix}\begin{bmatrix}5\\-8\\-9\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix}6(5)+1(-8)+1(-9)\\0(5)-2(-8)+3(-9)\\-1(5)+0(-8)+4(-9)\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix} 30-8-9 \\ 0+16-27 \\ -5+0-36 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 13 \\ -11 \\ -41 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we don't get the zero vector on the left side, we know that $\vec{x} = (5,-8,-9)$ is not in the null space of A.

■ 3. Is $\vec{x} = (1,1,1)$ in the null space of A?

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 4 & -5 \\ 1 & -6 & 5 \end{bmatrix}$$

Solution:

If $\vec{x} = (1,1,1)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x} = \vec{O}$$

$$\begin{bmatrix} 2 & -3 & 1 \\ 1 & 4 & -5 \\ 1 & -6 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} 2(1) - 3(1) + 1(1) \\ 1(1) + 4(1) - 5(1) \\ 1(1) - 6(1) + 5(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 - 3 + 1 \\ 1 + 4 - 5 \\ 1 - 6 + 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\vec{x} = (1,1,1)$ is in the null space of A.

■ 4. Is $\vec{x} = (4, -2)$ in the null space of A?

$$A = \begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \\ -2 & -4 \end{bmatrix}$$

Solution:

If $\vec{x} = (4, -2)$ is in the null space of A, then the product of A and $\vec{x}$ should satisfy the homogeneous equation.

$$A\vec{x} = \vec{O}$$

$$\begin{bmatrix} 1 & 2 \\ -1 & -2 \\ 2 & 4 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 4 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} 1(4) + 2(-2) \\ -1(4) - 2(-2) \\ 2(4) + 4(-2) \\ -2(4) - 4(-2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 - 4 \\ -4 + 4 \\ 8 - 8 \\ -8 + 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\vec{x} = (4, -2)$ is in the null space of A.

■ 5. Is $\vec{x} = (1,1,2,1)$ in the null space of A?

$$A = \begin{bmatrix} 1 & -7 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix}$$

Solution:

If $\overrightarrow{x} = (1,1,2,1)$ is in the null space of A, then the product of A and $\overrightarrow{x}$ should satisfy the homogeneous equation.

$$A\overrightarrow{x} = \overrightarrow{O}$$

$$\begin{bmatrix} 1 & -7 & 3 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} 1(1) - 7(1) + 3(2) + 0(1) \\ 0(1) + 1(1) - 1(2) + 1(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 - 7 + 6 + 0 \\ 0 + 1 - 2 + 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\overrightarrow{x} = (1,1,2,1)$ is in the null space of A.

■ 6. Is $\overrightarrow{x} = (-1,-3,1)$ in the null space of A?

$$A = \begin{bmatrix} -4 & 3 & 5 \\ 3 & 1 & 6 \\ 0 & -2 & -6 \end{bmatrix}$$

Solution:

If $\overrightarrow{x} = (-1, -3, 1)$ is in the null space of A, then the product of A and $\overrightarrow{x}$ should satisfy the homogeneous equation.

$$A\overrightarrow{x} = \overrightarrow{O}$$

$$\begin{bmatrix} -4 & 3 & 5 \\ 3 & 1 & 6 \\ 0 & -2 & -6 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

If we perform the matrix multiplication on the left side of the equation, we should get the zero vector.

$$\begin{bmatrix} -4(-1) + 3(-3) + 5(1) \\ 3(-1) + 1(-3) + 6(1) \\ 0(-1) - 2(-3) - 6(1) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 - 9 + 5 \\ -3 - 3 + 6 \\ 0 + 6 - 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Because we get the zero vector on the left side, we know that $\overrightarrow{x} = (-1, -3, 1)$ is in the null space of A.

NULL SPACE OF A MATRIX

1. Find the null space of A.

$$A = \begin{bmatrix} 4 & -3 \\ 0 & 4 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} 4 & -3 \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{4} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 = 0$$

$$x_2 = 0$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

So the null space of A is only the zero vector.

■ 2. Find the null space of A.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ -2 & 1 & 1 \\ 5 & 1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ 0 & 9 & -9 \\ 5 & 1 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & -5 \\ 0 & 9 & -9 \\ 0 & -19 & 19 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 4 & -5 \\ 0 & 1 & -1 \\ 0 & -19 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & -19 & 19 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0$$

which we can solve for the pivot variables.

$$x_1 = x_3$$

$$x_2 = x_3$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

■ 3. Find the null space of A.

$$A = \begin{bmatrix} 3 & -1 \\ -3 & 1 \\ 9 & -3 \\ 0 & 0 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} 3 & -1 \\ -3 & 1 \\ 9 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} \\ -3 & 1 \\ 9 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \\ 9 & -3 \\ 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & -\frac{1}{3} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the equation,

$$x_1 - \frac{1}{3}x_2 = 0$$

which we can solve for the pivot variable.

$$x_1 = \frac{1}{3}x_2$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} \frac{1}{3} \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix}\frac{1}{3}\\1\end{bmatrix}\right)$$

4. Find the null space of A.

$$A = \begin{bmatrix}-1 & 0 & 6 & 3\\3 & 1 & 1 & 4\\0 & 0 & 4 & 2\end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix}-1 & 0 & 6 & 3\\3 & 1 & 1 & 4\\0 & 0 & 4 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & -6 & -3\\3 & 1 & 1 & 4\\0 & 0 & 4 & 2\end{bmatrix} \to \begin{bmatrix}1 & 0 & -6 & -3\\0 & 1 & 19 & 13\\0 & 0 & 4 & 2\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -6 & -3\\0 & 1 & 19 & 13\\0 & 0 & 1 & \frac{1}{2}\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 19 & 13\\0 & 0 & 1 & \frac{1}{2}\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & \frac{7}{2}\\0 & 0 & 1 & \frac{1}{2}\end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & \frac{7}{2}\\0 & 0 & 1 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 = 0$$

$$x_2 + \frac{7}{2}x_4 = 0$$

$$x_3 + \frac{1}{2}x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = 0$$

$$x_2 = -\frac{7}{2}x_4$$

$$x_3 = -\frac{1}{2}x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ -\frac{7}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{7}{2} \\ -\frac{1}{2} \\ 1 \end{bmatrix}\right)$$

■ 5. Find the null space of A.

$$A = \begin{bmatrix} 4 & -2 & 1 & 1 \\ -1 & 0 & 3 & -3 \\ 0 & 0 & -4 & 6 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} 4 & -2 & 1 & 1 \\ -1 & 0 & 3 & -3 \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 3 & -3 \\ 4 & -2 & 1 & 1 \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 3 \\ 4 & -2 & 1 & 1 \\ 0 & 0 & -4 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & -2 & 13 & -11 \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & -\frac{13}{2} & \frac{11}{2} \\ 0 & 0 & -4 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 3 \\ 0 & 1 & -\frac{13}{2} & \frac{11}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & -\frac{13}{2} & \frac{11}{2} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -\frac{17}{4} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} \\ 0 & 1 & 0 & -\frac{17}{4} \\ 0 & 0 & 1 & -\frac{3}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 - \frac{3}{2}x_4 = 0$$

$$x_2 - \frac{17}{4}x_4 = 0$$

$$x_3 - \frac{3}{2}x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = \frac{3}{2}x_4$$

$$x_2 = \frac{17}{4}x_4$$

$$x_3 = \frac{3}{2}x_4$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{2} \\ \frac{17}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} \frac{3}{2} \\ \frac{17}{4} \\ \frac{3}{2} \\ 1 \end{bmatrix} \right)$$

6. Find the null space of A.

$$A = \begin{bmatrix} -2 & 0 & 7 \\ 3 & -1 & 4 \\ 0 & 3 & -2 \\ 1 & 4 & -5 \\ 2 & 2 & 1 \end{bmatrix}$$

Solution:

To find the null space, put the matrix A into reduced row-echelon form.

$$\begin{bmatrix}-2 & 0 & 7\\ 3 & -1 & 4\\ 0 & 3 & -2\\ 1 & 4 & -5\\ 2 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 4 & -5\\ -2 & 0 & 7\\ 3 & -1 & 4\\ 0 & 3 & -2\\ 2 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 4 & -5\\ 0 & 8 & -3\\ 3 & -1 & 4\\ 0 & 3 & -2\\ 2 & 2 & 1\end{bmatrix} \to \begin{bmatrix}1 & 4 & -5\\ 0 & 8 & -3\\ 0 & -13 & 19\\ 0 & 3 & -2\\ 2 & 2 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 4 & -5\\ 0 & 8 & -3\\ 0 & -13 & 19\\ 0 & 3 & -2\\ 0 & -6 & 11\end{bmatrix} \to \begin{bmatrix}1 & 4 & -5\\ 0 & 1 & -\frac{3}{8}\\ 0 & -13 & 19\\ 0 & 3 & -2\\ 0 & -6 & 11\end{bmatrix} \to \begin{bmatrix}1 & 0 & -\frac{7}{2}\\ 0 & 1 & -\frac{3}{8}\\ 0 & -13 & 19\\ 0 & 3 & -2\\ 0 & -6 & 11\end{bmatrix} \to \begin{bmatrix}1 & 0 & -\frac{7}{2}\\ 0 & 1 & -\frac{3}{8}\\ 0 & 0 & \frac{113}{8}\\ 0 & 3 & -2\\ 0 & -6 & 11\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & -\frac{7}{2}\\ 0 & 1 & -\frac{3}{8}\\ 0 & 0 & \frac{113}{8}\\ 0 & 0 & -\frac{7}{8}\\ 0 & -6 & 11\end{bmatrix} \to \begin{bmatrix}1 & 0 & -\frac{7}{2}\\ 0 & 1 & -\frac{3}{8}\\ 0 & 0 & \frac{113}{8}\\ 0 & 0 & -\frac{7}{8}\\ 0 & 0 & \frac{35}{4}\end{bmatrix} \to \begin{bmatrix}1 & 0 & -\frac{7}{2}\\ 0 & 1 & -\frac{3}{8}\\ 0 & 0 & 1\\ 0 & 0 & -\frac{7}{8}\\ 0 & 0 & \frac{35}{4}\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & -\frac{3}{8}\\ 0 & 0 & 1\\ 0 & 0 & -\frac{7}{8}\\ 0 & 0 & \frac{35}{4}\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & -\frac{7}{8}\\ 0 & 0 & \frac{35}{4}\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & \frac{35}{4}\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Then set up the equation $(\text{rref}(A))\vec{x_n} = 0$.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

From this matrix, we get the system of equations,

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

We can rewrite this as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

So the null space of A is only the zero vector.

THE COLUMN SPACE AND AX=B

1. Find the column space of A.

$$A = \begin{bmatrix} -2 & 1 & 1 \\ 5 & 1 & -6 \\ 1 & 4 & -5 \end{bmatrix}$$

Solution:

The column space of a matrix is all the possible linear combinations of its columns, which we can also say is the span of its columns.

$$C(A) = \text{Span}\left(\begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ -6 \\ -5 \end{bmatrix}\right)$$

2. Find the column space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Solution:

The column space of a matrix is all the possible linear combinations of its columns, which we can also say is the span of its columns.

$$C(A) = \text{Span}\left(\begin{bmatrix}-1\\3\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}6\\1\\4\end{bmatrix}, \begin{bmatrix}3\\4\\2\end{bmatrix}\right)$$

■ 3. Find a basis for the column space of A.

$$A = \begin{bmatrix}4 & -3\\0 & 4\end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix}4 & -3\\0 & 4\end{bmatrix} \to \begin{bmatrix}1 & -\frac{3}{4}\\0 & 4\end{bmatrix} \to \begin{bmatrix}1 & -\frac{3}{4}\\0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$$

Because the first (and only) two columns of rref(A) are pivot columns, that means the first two columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix}4\\0\end{bmatrix}, \begin{bmatrix}-3\\4\end{bmatrix}\right)$$

■ 4. Find a basis for the column space of A.

$$A = \begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 0 & 6 & 3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 & -3 \\ 3 & 1 & 1 & 4 \\ 0 & 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 & -3 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -6 & -3 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 19 & 13 \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{7}{2} \\ 0 & 0 & 1 & \frac{1}{2} \end{bmatrix}$$

Because the first three columns of rref(A) are pivot columns, that means the first three columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix} \right)$$

■ 5. Find a basis for the column space of A.

$$A = \begin{bmatrix} 5 & -2 & 6 \\ -3 & 1 & 0 \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix} 5 & -2 & 6 \\ -3 & 1 & 0 \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ -3 & 1 & 0 \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & -\frac{1}{5} & \frac{18}{5} \\ 0 & -1 & -4 \\ 8 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & -\frac{1}{5} & \frac{18}{5} \\ 0 & -1 & -4 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{2}{5} & \frac{6}{5} \\ 0 & 1 & -18 \\ 0 & -1 & -4 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & -1 & -4 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & 0 & -22 \\ 0 & \frac{26}{5} & -\frac{38}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & 0 & -22 \\ 0 & 0 & 86 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & -18 \\ 0 & 0 & 1 \\ 0 & 0 & 86 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -18 \\ 0 & 0 & 1 \\ 0 & 0 & 86 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 86 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because the first three (and only) columns of rref(A) are pivot columns, that means the first three columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix}5\\-3\\0\\8\end{bmatrix}, \begin{bmatrix}-2\\1\\-1\\2\end{bmatrix}, \begin{bmatrix}6\\0\\-4\\2\end{bmatrix}\right)$$

■ 6. Find a basis for the column space of A.

$$A = \begin{bmatrix}2 & -4 & 3 & -6\\1 & -2 & 0 & 0\\4 & -8 & 5 & -10\end{bmatrix}$$

Solution:

To find the basis for the column space of A, first put the matrix into reduced row-echelon form.

$$\begin{bmatrix}2 & -4 & 3 & -6\\1 & -2 & 0 & 0\\4 & -8 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\2 & -4 & 3 & -6\\4 & -8 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 3 & -6\\4 & -8 & 5 & -10\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 3 & -6\\0 & 0 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 1 & -2\\0 & 0 & 5 & -10\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & 0\\0 & 0 & 1 & -2\\0 & 0 & 0 & 0\end{bmatrix}$$

Because the first and third columns of rref(A) are pivot columns, that means the first and third columns of A can form the basis for the column space of A. So the basis is given by

$$C(A) = \text{Span}\left(\begin{bmatrix}2\\1\\4\end{bmatrix}, \begin{bmatrix}3\\0\\5\end{bmatrix}\right)$$

SOLVING AX=B

■ 1. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}2&-4&3&-6\\1&-2&0&0\\4&-8&5&-10\end{bmatrix} \text{ with } \vec{b}=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Solution:

Augment A with $\vec{b}=(b_1,b_2,b_3)$, then put $[A\,|\,b]$ into rref.

$$\begin{bmatrix}2&-4&3&-6&|&b_1\\1&-2&0&0&|&b_2\\4&-8&5&-10&|&b_3\end{bmatrix}\to\begin{bmatrix}1&-2&0&0&|&b_2\\2&-4&3&-6&|&b_1\\4&-8&5&-10&|&b_3\end{bmatrix}$$

$$\begin{bmatrix}1&-2&0&0&|&b_2\\0&0&3&-6&|&b_1-2b_2\\4&-8&5&-10&|&b_3\end{bmatrix}\to\begin{bmatrix}1&-2&0&0&|&b_2\\0&0&3&-6&|&b_1-2b_2\\0&0&5&-10&|&b_3-4b_2\end{bmatrix}$$

$$\begin{bmatrix}1&-2&0&0&|&b_2\\0&0&1&-2&|&\frac{1}{3}b_1-\frac{2}{3}b_2\\0&0&5&-10&|&b_3-4b_2\end{bmatrix}\to\begin{bmatrix}1&-2&0&0&|&b_2\\0&0&1&-2&|&\frac{1}{3}b_1-\frac{2}{3}b_2\\0&0&0&0&|&-\frac{5}{3}b_1-\frac{2}{3}b_2+b_3\end{bmatrix}$$

Finding the complementary solution.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 - 2x_2 = 0 \qquad\qquad x_1 = 2x_2$$

$$x_3 - 2x_4 = 0 \qquad\qquad x_3 = 2x_4$$

Now we can write the vector set that satisfies the null space, and then the complementary solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\overrightarrow{x}_n = c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Plug $\overrightarrow{b} = (1,2,3)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3}(1) - \frac{2}{3}(2) \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3}(1) - \frac{2}{3}(2) + 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & \frac{1}{3} - \frac{4}{3} \\ 0 & 0 & 0 & 0 & | & -\frac{5}{3} - \frac{4}{3} + \frac{9}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & -\frac{3}{3} \\ 0 & 0 & 0 & 0 & | & \frac{0}{3} \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & 0 & | & 2 \\ 0 & 0 & 1 & -2 & | & -1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Choose $x_2 = 0$ and $x_4 = 0$ for the free variables, and rewrite the system of equations as

$$x_1 - 2(0) = 2 \qquad\qquad x_1 = 2$$

$$x_3 - 2(0) = -1 \qquad\qquad x_3 = -1$$

The particular solution is $x_1 = 3/2$, $x_2 = 0$, $x_3 = -1/2$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix}$$

so the general solution is

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 2 \\ 0 \\ -1 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

■ 2. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 3 & 6 \\ 6 & 12 \\ 1 & 1 \\ 2 & 2 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

Solution:

Augment A with $\overrightarrow{b}=(b_1,b_2,b_3)$, then put $[A\,|\,b]$ into rref.

$$\begin{bmatrix}3 & 6 & | & b_1\\ 6 & 12 & | & b_2\\ 1 & 1 & | & b_3\\ 2 & 2 & | & b_4\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & \frac{1}{3}b_1\\ 6 & 12 & | & b_2\\ 1 & 1 & | & b_3\\ 2 & 2 & | & b_4\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & \frac{1}{3}b_1\\ 0 & 0 & | & b_2-2b_1\\ 1 & 1 & | & b_3\\ 2 & 2 & | & b_4\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & | & \frac{1}{3}b_1\\ 0 & 0 & | & b_2-2b_1\\ 0 & -1 & | & b_3-\frac{1}{3}b_1\\ 2 & 2 & | & b_4\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & \frac{1}{3}b_1\\ 0 & 0 & | & b_2-2b_1\\ 0 & -1 & | & b_3-\frac{1}{3}b_1\\ 0 & -2 & | & b_4-\frac{2}{3}b_1\end{bmatrix} \to \begin{bmatrix}1 & 2 & | & \frac{1}{3}b_1\\ 0 & -1 & | & b_3-\frac{1}{3}b_1\\ 0 & 0 & | & b_2-2b_1\\ 0 & -2 & | & b_4-\frac{2}{3}b_1\end{bmatrix}$$

$$\begin{bmatrix}1 & 2 & | & \frac{1}{3}b_1\\ 0 & 1 & | & \frac{1}{3}b_1-b_3\\ 0 & 0 & | & b_2-2b_1\\ 0 & -2 & | & b_4-\frac{2}{3}b_1\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & -\frac{1}{3}b_1+2b_3\\ 0 & 1 & | & \frac{1}{3}b_1-b_3\\ 0 & 0 & | & b_2-2b_1\\ 0 & -2 & | & b_4-\frac{2}{3}b_1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & | & -\frac{1}{3}b_1+2b_3\\ 0 & 1 & | & \frac{1}{3}b_1-b_3\\ 0 & 0 & | & b_2-2b_1\\ 0 & 0 & | & b_4-2b_3\end{bmatrix}$$

Finding the complementary solution.

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 = 0$$

$$x_2 = 0$$

Now we can write the vector set that satisfies the null space, and then the complementary solution.

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\overrightarrow{x}_n = c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Plug $\overrightarrow{b} = (1,2,1,2)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & 0 & | & -\frac{1}{3}(1) + 2(1) \\ 0 & 1 & | & \frac{1}{3}(1) - 1 \\ 0 & 0 & | & 2 - 2(1) \\ 0 & 0 & | & 2 - 2(1) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -\frac{1}{3} + 2 \\ 0 & 1 & | & \frac{1}{3} - 1 \\ 0 & 0 & | & 2 - 2 \\ 0 & 0 & | & 2 - 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{5}{3} \\ 0 & 1 & | & -\frac{2}{3} \\ 0 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

Rewrite the system of equations as

$$x_1 = \frac{5}{3}$$

$$x_2 = -\frac{2}{3}$$

The particular solution is $x_1 = 5/3$ and $x_2 = -2/3$, or

$$\vec{x}_p = \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix}$$

so the general solution is

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} \frac{5}{3} \\ -\frac{2}{3} \end{bmatrix}$$

■ 3. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 1 & -5 & 3 \\ -1 & 4 & 0 \\ 3 & -16 & 12 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

Solution:

Augment A with $\vec{b} = (b_1, b_2, b_3)$, then put $[A\,|\,b]$ into rref.

$$\begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ -1 & 4 & 0 & | & b_2 \\ 3 & -16 & 12 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & -1 & 3 & | & b_1+b_2 \\ 3 & -16 & 12 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & -1 & 3 & | & b_1+b_2 \\ 0 & -1 & 3 & | & -3b_1+b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & 1 & -3 & | & -b_1-b_2 \\ 0 & -1 & 3 & | & -3b_1+b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & | & b_1 \\ 0 & 1 & -3 & | & -b_1-b_2 \\ 0 & 0 & 0 & | & -4b_1-b_2+b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -12 & | & -4b_1-5b_2 \\ 0 & 1 & -3 & | & -b_1-b_2 \\ 0 & 0 & 0 & | & -4b_1-b_2+b_3 \end{bmatrix}$$

Finding the complementary solution.

$$\begin{bmatrix} 1 & 0 & -12 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 - 12x_3 = 0 \qquad\qquad x_1 = 12x_3$$

$$x_2 - 3x_3 = 0 \qquad\qquad x_2 = 3x_3$$

Now we can write the vector set that satisfies the null space, and then the complementary solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

$$\vec{x}_n = c_1 \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

Plug $\vec{b} = (1, -1, 3)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & 0 & -12 & | & -4(1) - 5(-1) \\ 0 & 1 & -3 & | & -1 - (-1) \\ 0 & 0 & 0 & | & -4(1) - (-1) + 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -12 & | & -4 + 5 \\ 0 & 1 & -3 & | & -1 + 1 \\ 0 & 0 & 0 & | & -4 + 1 + 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -12 & | & 1 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Choose $x_3 = 0$ for the free variable, and rewrite the system of equations as

$$x_1 - 12(0) = 1 \qquad\qquad x_1 = 1$$

$$x_2 - 3(0) = 0 \qquad\qquad x_2 = 0$$

The particular solution is $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

so the general solution is

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 12 \\ 3 \\ 1 \end{bmatrix}$$

■ 4. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} -2 & 10 & -6 & 2 \\ 1 & -5 & 3 & -1 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Solution:

Augment A with $\vec{b} = (b_1, b_2, b_3)$, then put $[A\,|\,b]$ into rref.

$$\begin{bmatrix} -2 & 10 & -6 & 2 & | & b_1 \\ 1 & -5 & 3 & -1 & | & b_2 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & -1 & | & b_2 \\ -2 & 10 & -6 & 2 & | & b_1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -5 & 3 & -1 & | & b_2 \\ 0 & 0 & 0 & 0 & | & 2b_2 + b_1 \end{bmatrix}$$

Finding the complementary solution.

$$\begin{bmatrix} 1 & -5 & 3 & -1 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 - 5x_2 + 3x_3 - x_4 = 0 \qquad x_1 = 5x_2 - 3x_3 + x_4$$

Now we can write the vector set that satisfies the null space, and then the complementary solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_n = c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Plug $\vec{b} = (-2,1)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & -5 & 3 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 2(1) - 2 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 & -1 & | & 1 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Choose $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$ for the free variables, and rewrite the system of equations as

$$x_1 - 5(0) + 3(0) - 0 = 1 \qquad\qquad x_1 = 1$$

The particular solution is $x_1 = 1$, $x_2 = 0$, and $x_3 = 0$, or

$$\vec{x}_p = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

so the general solution is

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

■ 5. Find the general solution to $A\vec{x} = \vec{b}$.

$$A = \begin{bmatrix} 2 & 0 & 0 & 12 \\ -1 & 2 & -1 & 4 \\ 5 & -6 & 3 & 0 \end{bmatrix} \text{ with } \vec{b} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$$

Solution:

Augment A with $\vec{b} = (b_1, b_2, b_3)$, then put $[A\,|\,b]$ into rref.

$$\begin{bmatrix} 2 & 0 & 0 & 12 & | & b_1 \\ -1 & 2 & -1 & 4 & | & b_2 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix} \to \begin{bmatrix} -1 & 2 & -1 & 4 & | & b_2 \\ 2 & 0 & 0 & 12 & | & b_1 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 2 & 0 & 0 & 12 & | & b_1 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 4 & -2 & 20 & | & 2b_2 + b_1 \\ 5 & -6 & 3 & 0 & | & b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 4 & -2 & 20 & | & 2b_2 + b_1 \\ 0 & 4 & -2 & 20 & | & 5b_2 + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2b_2 + b_1}{4} \\ 0 & 4 & -2 & 20 & | & 5b_2 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 1 & -4 & | & -b_2 \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2b_2+b_1}{4} \\ 0 & 0 & 0 & 0 & | & -b_1+3b_2+b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 & | & \frac{b_1}{2} \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2b_2+b_1}{4} \\ 0 & 0 & 0 & 0 & | & -b_1+3b_2+b_3 \end{bmatrix}$$

Finding the complementary solution.

$$\begin{bmatrix} 1 & 0 & 0 & 6 & | & 0 \\ 0 & 1 & -\frac{1}{2} & 5 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 + 6x_4 = 0 \qquad\qquad x_1 = -6x_4$$

$$x_2 - \frac{1}{2}x_3 + 5x_4 = 0 \qquad\qquad x_2 = \frac{1}{2}x_3 - 5x_4$$

Now we can write the vector set that satisfies the null space, and then the complementary solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -6 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}_n = c_1 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

Plug $\vec{b} = (1,1,-2)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & 0 & 0 & 6 & | & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{2(1)+1}{4} \\ 0 & 0 & 0 & 0 & | & -1+3(1)-2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 6 & | & \frac{1}{2} \\ 0 & 1 & -\frac{1}{2} & 5 & | & \frac{3}{4} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Choose $x_3 = 0$ and $x_4 = 0$ for the free variables, and rewrite the system of equations as

$$x_1 + 6(0) = \frac{1}{2} \qquad\qquad x_1 = \frac{1}{2}$$

$$x_2 - \frac{1}{2}(0) + 5(0) = \frac{3}{4} \qquad\qquad x_2 = \frac{3}{4}$$

The particular solution is $x_1 = 1/2$, $x_2 = 3/4$, $x_3 = 0$, and $x_4 = 0$, or

$$\vec{x}_p = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix}$$

so the general solution is

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = \begin{bmatrix} \frac{1}{2} \\ \frac{3}{4} \\ 0 \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -6 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

■ 6. Find the general solution to $A\vec{x}=\vec{b}$.

$$A=\begin{bmatrix}1&0&3&-5\\4&-2&2&0\\-1&2&-1&1\\3&2&-1&5\end{bmatrix}\text{ with }\vec{b}=\begin{bmatrix}2\\1\\1\\3\end{bmatrix}$$

Solution:

Augment A with $\vec{b}=(b_1,b_2,b_3)$, then put $[A\,|\,b]$ into rref.

$$\begin{bmatrix}1&0&3&-5&|&b_1\\4&-2&2&0&|&b_2\\-1&2&-1&1&|&b_3\\3&2&-1&5&|&b_4\end{bmatrix}\to\begin{bmatrix}1&0&3&-5&|&b_1\\0&-2&-10&20&|&-4b_1+b_2\\-1&2&-1&1&|&b_3\\3&2&-1&5&|&b_4\end{bmatrix}$$

$$\begin{bmatrix}1&0&3&-5&|&b_1\\0&-2&-10&20&|&-4b_1+b_2\\0&2&2&-4&|&b_1+b_3\\3&2&-1&5&|&b_4\end{bmatrix}\to\begin{bmatrix}1&0&3&-5&|&b_1\\0&-2&-10&20&|&-4b_1+b_2\\0&2&2&-4&|&b_1+b_3\\0&2&-10&20&|&-3b_1+b_4\end{bmatrix}$$

$$\begin{bmatrix}1&0&3&-5&|&b_1\\0&1&5&-10&|&\frac{-4b_1+b_2}{-2}\\0&2&2&-4&|&b_1+b_3\\0&2&-10&20&|&-3b_1+b_4\end{bmatrix}\to\begin{bmatrix}1&0&3&-5&|&b_1\\0&1&5&-10&|&\frac{-4b_1+b_2}{-2}\\0&0&-8&16&|&-3b_1+b_2+b_3\\0&2&-10&20&|&-3b_1+b_4\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1+b_2}{-2} \\ 0 & 0 & -8 & 16 & | & -3b_1+b_2+b_3 \\ 0 & 0 & -20 & 40 & | & -7b_1+b_2+b_4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1+b_2}{-2} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1+b_2+b_3}{-8} \\ 0 & 0 & -20 & 40 & | & -7b_1+b_2+b_4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 3 & -5 & | & b_1 \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1+b_2}{-2} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1+b_2+b_3}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{b_1-3b_2-5b_3+2b_4}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-b_1+3b_2+3b_3}{8} \\ 0 & 1 & 5 & -10 & | & \frac{-4b_1+b_2}{-2} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1+b_2+b_3}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{b_1-3b_2-5b_3+2b_4}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-b_1+3b_2+3b_3}{8} \\ 0 & 1 & 0 & 0 & | & \frac{b_1+b_2+5b_3}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-3b_1+b_2+b_3}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{b_1-3b_2-5b_3+2b_4}{2} \end{bmatrix}$$

Finding the complementary solution.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$x_1 + x_4 = 0 \qquad\qquad x_1 = -x_4$$

$$x_2 = 0 \qquad\qquad x_2 = 0$$

$$x_3 - 2x_4 = 0 \qquad\qquad x_3 = 2x_4$$

Now we can write the vector set that satisfies the null space, and then the complementary solution.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

$$\vec{x}_n = c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Plug $\vec{b} = (2,1,1,3)$ into the augmented matrix we built earlier.

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-2+3(1)+3(1)}{8} \\ 0 & 1 & 0 & 0 & | & \frac{2+1+5(1)}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-3(2)+1+1}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{2-3(1)-5(1)+2(3)}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{-2+3+3}{8} \\ 0 & 1 & 0 & 0 & | & \frac{2+1+5}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-6+1+1}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{2-3-5+6}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{4}{8} \\ 0 & 1 & 0 & 0 & | & \frac{8}{8} \\ 0 & 0 & 1 & -2 & | & \frac{-4}{-8} \\ 0 & 0 & 0 & 0 & | & \frac{0}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 1 & | & \frac{1}{2} \\ 0 & 1 & 0 & 0 & | & 1 \\ 0 & 0 & 1 & -2 & | & \frac{1}{2} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Choose $x_4 = 0$ for the free variable, and rewrite the system of equations as

$$x_1 + 0 = \frac{1}{2} \qquad\qquad x_1 = \frac{1}{2}$$

$$x_2 = 1 \qquad\qquad x_2 = 1$$

$$x_3 - 2(0) = \frac{1}{2} \qquad\qquad x_3 = \frac{1}{2}$$

The particular solution is $x_1 = 1/2$, $x_2 = 3/4$, $x_3 = 0$, and $x_4 = 0$, or

$$\overrightarrow{x}_p = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix}$$

so the general solution is

$$\overrightarrow{x} = \overrightarrow{x}_p + \overrightarrow{x}_n$$

$$\overrightarrow{x} = \begin{bmatrix} \frac{1}{2} \\ 1 \\ \frac{1}{2} \\ 0 \end{bmatrix} + c_1 \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

DIMENSIONALITY, NULLITY, AND RANK

1. Find the nullity of A.

$$A = \begin{bmatrix} 1 & -3 & 2 & -1 \\ 3 & -7 & 0 & 1 \end{bmatrix}$$

Solution:

To find the nullity of the matrix A, we need to first find the null space, so we'll set up the augmented matrix for $A\vec{x} = \vec{O}$, then put it in reduced row-echelon form.

$$\begin{bmatrix} 1 & -3 & 2 & -1 & | & 0 \\ 3 & -7 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & -1 & | & 0 \\ 0 & 2 & -6 & 4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 2 & -1 & | & 0 \\ 0 & 1 & -3 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7 & 5 & | & 0 \\ 0 & 1 & -3 & 2 & | & 0 \end{bmatrix}$$

Pull out a system of equations,

$$x_1 - 7x_3 + 5x_4 = 0$$

$$x_2 - 3x_3 + 2x_4 = 0$$

then solve it for the pivot variables.

$$x_1 = 7x_3 - 5x_4$$

$$x_2 = 3x_3 - 2x_4$$

Rewrite the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 7 \\ 3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Then the null space of A is the span of the vectors in this linear combination equation.

$$N(A) = \text{Span}\left(\begin{bmatrix} 7 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right)$$

We found 2 spanning vectors that form a basis for the null space, which matches the dimension of the null space.

$$\text{Dim}(N(A)) = \text{nullity}(A) = 2$$

We can also get the nullity of A from the number of free variables in rref(A). Because there were 2 free variables, x_3 and x_4, nullity$(A) = 2$.

■ 2. Find the rank of X.

$$X = \begin{bmatrix} -2 & 3 & 1 \\ -1 & 0 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

Solution:

To find the rank of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix}-2 & 3 & 1\\-1 & 0 & -4\\2 & -4 & 3\end{bmatrix} \to \begin{bmatrix}-1 & 0 & -4\\-2 & 3 & 1\\2 & -4 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\-2 & 3 & 1\\2 & -4 & 3\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\0 & 3 & 9\\2 & -4 & 3\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 4\\0 & 3 & 9\\0 & -4 & -5\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\0 & 1 & 3\\0 & -4 & -5\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\0 & 1 & 3\\0 & 0 & 7\end{bmatrix} \to \begin{bmatrix}1 & 0 & 4\\0 & 1 & 3\\0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 4\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$$

Now that the matrix is in reduced row-echelon form, we can find the rank directly from the matrix. We can see that all three columns are pivot columns. So because there are three pivot variables, the rank is

$$\text{Dim}(C(X)) = \text{rank}(X) = 3$$

■ 3. Find the nullity and the rank of A.

$$A = \begin{bmatrix}-1 & -3 & 2 & 4 & -2\\-3 & -5 & -2 & 1 & 4\\0 & 4 & -8 & -11 & 10\\1 & 3 & -2 & -4 & 5\end{bmatrix}$$

Solution:

To find the nullity of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -1 & -3 & 2 & 4 & -2 \\ -3 & -5 & -2 & 1 & 4 \\ 0 & 4 & -8 & -11 & 10 \\ 1 & 3 & -2 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ -3 & -5 & -2 & 1 & 4 \\ 0 & 4 & -8 & -11 & 10 \\ 1 & 3 & -2 & -4 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 4 & -8 & -11 & 10 \\ 1 & 3 & -2 & -4 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 4 & -8 & -11 & 10 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -4 & 2 \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & -\frac{11}{2} \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & -\frac{11}{2} \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & -\frac{11}{2} \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & 0 \\ 0 & 1 & -2 & -\frac{11}{4} & \frac{5}{2} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & \frac{17}{4} & 0 \\ 0 & 1 & -2 & -\frac{11}{4} & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Since rref(A) has three pivot columns and two free columns,

rank$(A) = 3$

nullity$(A) = 2$

4. Find the nullity of M.

$$M=\begin{bmatrix}-4 & 2 & -2 & 1\\ -1 & 0 & -3 & 2\\ 3 & -2 & 5 & 0\end{bmatrix}$$

Solution:

To find the nullity of the matrix M, we need to first find the null space, so we'll set up the augmented matrix for $M\vec{x}=\vec{O}$, then put it in reduced row-echelon form.

$$\begin{bmatrix}-4 & 2 & -2 & 1 & | & 0\\ -1 & 0 & -3 & 2 & | & 0\\ 3 & -2 & 5 & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}-1 & 0 & -3 & 2 & | & 0\\ -4 & 2 & -2 & 1 & | & 0\\ 3 & -2 & 5 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 3 & -2 & | & 0\\ -4 & 2 & -2 & 1 & | & 0\\ 3 & -2 & 5 & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 3 & -2 & | & 0\\ 0 & 2 & 10 & -7 & | & 0\\ 3 & -2 & 5 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 3 & -2 & | & 0\\ 0 & 2 & 10 & -7 & | & 0\\ 0 & -2 & -4 & 6 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 3 & -2 & | & 0\\ 0 & 1 & 5 & -\frac{7}{2} & | & 0\\ 0 & -2 & -4 & 6 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 3 & -2 & | & 0\\ 0 & 1 & 5 & -\frac{7}{2} & | & 0\\ 0 & 0 & 6 & -1 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 3 & -2 & | & 0\\ 0 & 1 & 5 & -\frac{7}{2} & | & 0\\ 0 & 0 & 1 & -\frac{1}{6} & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & | & 0 \\ 0 & 1 & 5 & -\frac{7}{2} & | & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{3}{2} & | & 0 \\ 0 & 1 & 0 & -\frac{8}{3} & | & 0 \\ 0 & 0 & 1 & -\frac{1}{6} & | & 0 \end{bmatrix}$$

Pull out a system of equations,

$$x_1 - \frac{3}{2}x_4 = 0$$

$$x_2 - \frac{8}{3}x_4 = 0$$

$$x_3 - \frac{1}{6}x_4 = 0$$

then solve it for the pivot variables.

$$x_1 = \frac{3}{2}x_4$$

$$x_2 = \frac{8}{3}x_4$$

$$x_3 = \frac{1}{6}x_4$$

Rewrite the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} \frac{3}{2} \\ \frac{8}{3} \\ \frac{1}{6} \\ 1 \end{bmatrix}$$

Then the null space of M is the span of the vectors in this linear combination equation.

$$N(M) = \text{Span}\left(\begin{bmatrix}\frac{3}{2}\\ \frac{8}{3}\\ \frac{1}{6}\\ 1\end{bmatrix}\right)$$

We found one spanning vector that forms a basis for the null space, which matches the dimension of the null space.

$$\text{Dim}(N(M)) = \text{nullity}(M) = 1$$

We can also get the nullity of M from the number of free variables in rref(M). Because there was one free variable, x_4, nullity$(M) = 1$.

■ 5. Find the rank of M.

$$M = \begin{bmatrix}-2 & 0 & -5 & 6 & 2\\ 1 & -1 & 3 & 0 & 5\\ 0 & -2 & 1 & 6 & 12\end{bmatrix}$$

Solution:

To find the rank of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -2 & 0 & -5 & 6 & 2 \\ 1 & -1 & 3 & 0 & 5 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ -2 & 0 & -5 & 6 & 2 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ 0 & -2 & 1 & 6 & 12 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 & -6 \\ 0 & -2 & 1 & 6 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 3 & 0 & 5 \\ 0 & 1 & -\frac{1}{2} & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{5}{2} & -3 & -1 \\ 0 & 1 & -\frac{1}{2} & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Now that the matrix is in reduced row-echelon form, we can find the rank directly from the matrix. We can see that two columns are pivot columns. So because there are two pivot columns, and therefore two pivot variables, the rank is

$$\text{Dim}(C(M)) = \text{rank}(M) = 2$$

■ 6. Find the nullity and the rank of M.

$$M = \begin{bmatrix} -1 & 2 & 0 & 3 \\ -2 & 0 & -1 & 2 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix}$$

Solution:

To find the nullity of the matrix, we need to first put the matrix in reduced row-echelon form.

$$\begin{bmatrix} -1 & 2 & 0 & 3 \\ -2 & 0 & -1 & 2 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ -2 & 0 & -1 & 2 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & -4 & -1 & -4 \\ 3 & -2 & 0 & -4 \\ 1 & -4 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & -4 & -1 & -4 \\ 0 & 4 & 0 & 5 \\ 1 & -4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & -4 & -1 & -4 \\ 0 & 4 & 0 & 5 \\ 0 & -2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 4 & 0 & 5 \\ 0 & -2 & 2 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & -2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & \frac{5}{2} & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -1 \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{1}{4} & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \frac{15}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{5}{4} \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since rref(A) has four pivot columns and zero free columns,

$$\text{rank}(M) = 4$$

$$\text{nullity}(M) = 0$$

FUNCTIONS AND TRANSFORMATIONS

■ 1. The transformation T maps every vector in $\mathbb{R}^4$ to $\vec{O}=(0,0,0)$. What are the domain, codomain, and range of T?

Solution:

Because T is mapping vectors in $\mathbb{R}^4$ to vector $\vec{O}=(0,0,0)$ in $\mathbb{R}^3$, we can express T as $T:\mathbb{R}^4\to\mathbb{R}^3$, and say that the domain of the transformation is $\mathbb{R}^4$ and that the codomain is $\mathbb{R}^3$.

Since the transformation is mapping every vector in $\mathbb{R}^4$ to only the zero vector $\vec{O}=(0,0,0)$ in $\mathbb{R}^3$, the range of T is just the zero vector, $\vec{O}=(0,0,0)$.

■ 2. The transformation T maps every vector in $\mathbb{R}^3$ to every vector in $\mathbb{R}^2$. What are the domain, codomain, and range of T?

Solution:

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^3$. The codomain is the space T is mapping *to*, so the codomain is $\mathbb{R}^2$. And the range is the specific set of vectors within the codomain that are being mapped to. Because every vector in $\mathbb{R}^2$ is being mapped to, the range is all of $\mathbb{R}^2$.

■ 3. The transformation T maps every vector in $\mathbb{R}^3$ to the zero vector $\vec{O}$ in $\mathbb{R}^3$. What are the domain, codomain, and range of T?

Solution:

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^3$. The codomain is the space T is mapping *to*, so the codomain is also $\mathbb{R}^3$. And the range is the specific set of vectors within the codomain that are being mapped to, so the range is $\vec{O} = (0,0,0)$.

■ 4. The transformation T maps $\vec{a} = (-1,0,3)$ to $\vec{b} = (-2,1,-2)$. What are the domain, codomain, and range of T?

Solution:

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^3$. The codomain is the space T is mapping *to*, so the codomain is $\mathbb{R}^3$. And the range is the specific set of vectors within the codomain that are being mapped to, so the range is the single vector being mapped to, $\vec{b} = (-2,1,-2)$.

5. The transformation T maps $\vec{a} = (-2,0)$ to every vector in $\mathbb{R}^4$. What are the domain, codomain, and range of T?

Solution:

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^2$. The codomain is the space T is mapping *to*, so the codomain is $\mathbb{R}^4$. And the range is the specific set of vectors within the codomain that are being mapped to. Because every vector in $\mathbb{R}^4$ is being mapped to, the range is all of $\mathbb{R}^4$.

6. The transformation T maps $\vec{a} = (-2,-3,1)$ to the zero vector $\vec{O}$ in $\mathbb{R}^3$. What are the domain, codomain, and range of T?

Solution:

The domain is the space T is mapping *from*, so the domain is $\mathbb{R}^3$. The codomain is the space T is mapping *to*, so the codomain is also $\mathbb{R}^3$. And the range is the specific set of vectors within the codomain that are being mapped to, so the range is the single vector being mapped to, $\vec{O} = (0,0,0)$.

TRANSFORMATION MATRICES AND THE IMAGE OF THE SUBSET

■ 1. Find the resulting vector $\vec{b}$ after $\vec{a}=(1,6)$ undergoes a transformation by matrix M.

$$M=\begin{bmatrix}-7 & 1\\ 0 & -2\end{bmatrix}$$

Solution:

To apply a transformation matrix to vector $\vec{a}$, we'll multiply the matrix by the vector.

$$\vec{b}=M\vec{a}=\begin{bmatrix}-7 & 1\\ 0 & -2\end{bmatrix}\begin{bmatrix}1\\ 6\end{bmatrix}$$

$$\vec{b}=M\vec{a}=\begin{bmatrix}-7(1)+1(6)\\ 0(1)-2(6)\end{bmatrix}$$

$$\vec{b}=M\vec{a}=\begin{bmatrix}-7+6\\ 0-12\end{bmatrix}$$

$$\vec{b}=M\vec{a}=\begin{bmatrix}-1\\ -12\end{bmatrix}$$

■ 2. Sketch triangle $\triangle ABC$ with vertices $(2,3)$, $(-3,-1)$, and $(1,-4)$, and the transformation of $\triangle ABC$ after it's transformed by matrix L.

$$L = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

Solution:

Put the vertices of $\triangle ABC$ into a matrix.

$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & -1 & -4 \end{bmatrix}$$

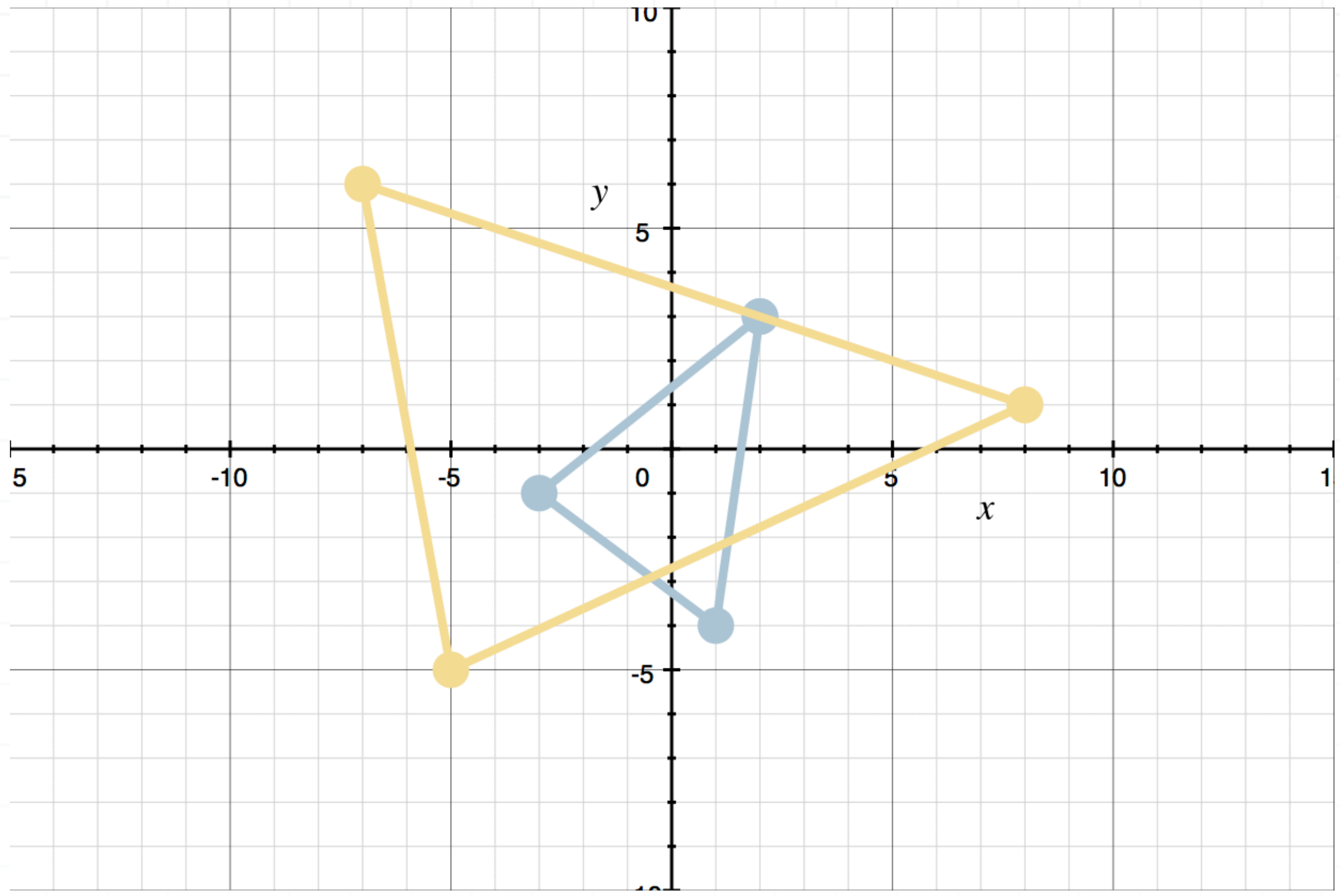
Apply the transformation of L to the vertex matrix.

$$\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & -3 & 1 \\ 3 & -1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1(2)+2(3) & 1(-3)+2(-1) & 1(1)+2(-4) \\ 2(2)-1(3) & 2(-3)-1(-1) & 2(1)-1(-4) \end{bmatrix}$$

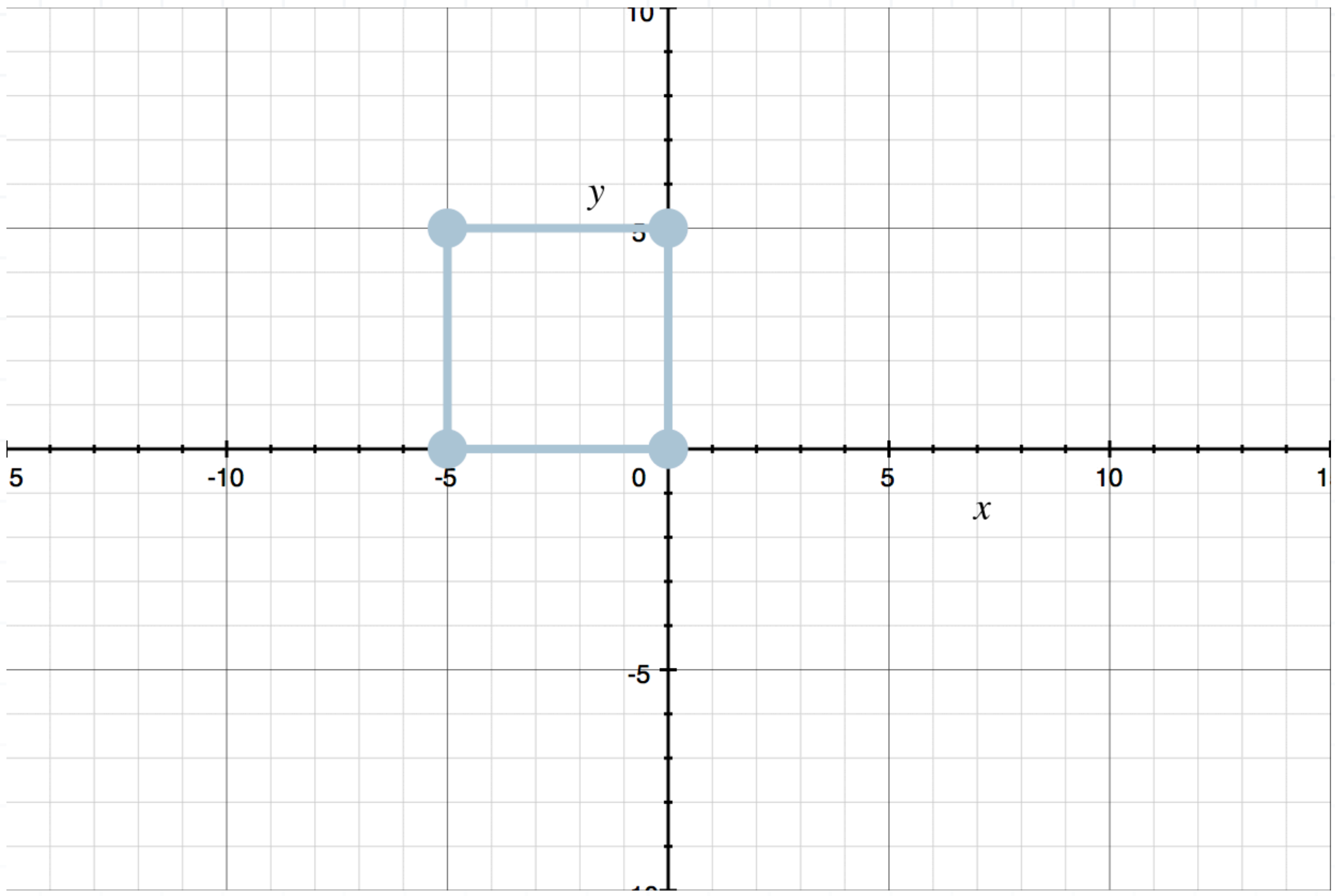
$$\begin{bmatrix} 8 & -5 & -7 \\ 1 & -5 & 6 \end{bmatrix}$$

The original triangle $\triangle ABC$ is sketched in light blue, and its transformation after L is in yellow.



■ 3. Sketch the transformation of the square in the graph after it's transformed by matrix Z.

$$Z = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$



Solution:

Put the vertices of the square into a matrix.

$$\begin{bmatrix} 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

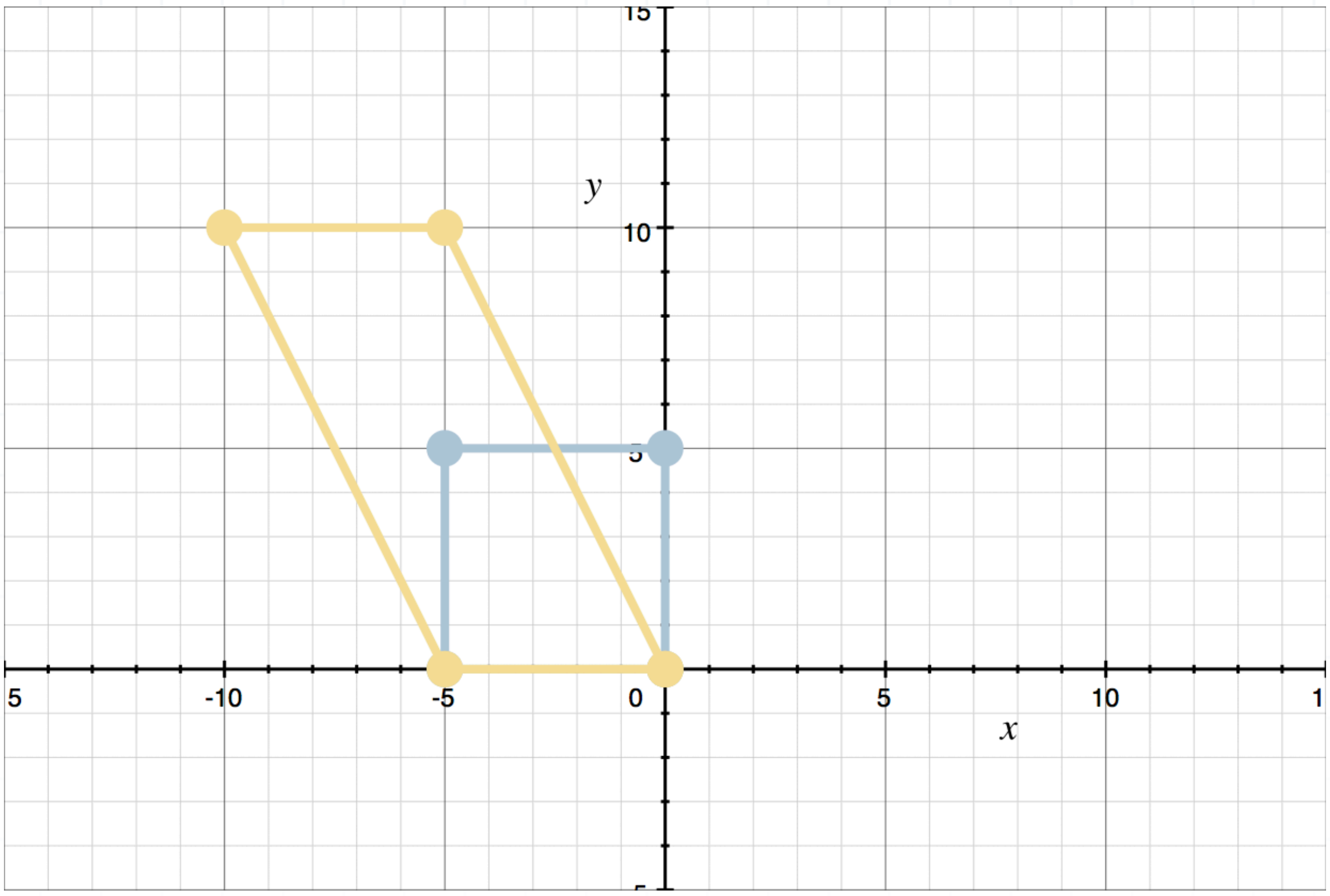
Apply the transformation of Z to the vertex matrix.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -5 & -5 & 0 \\ 0 & 0 & 5 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 1(0) - 1(0) & 1(-5) - 1(0) & 1(-5) - 1(5) & 1(0) - 1(5) \\ 0(0) + 2(0) & 0(-5) + 2(0) & 0(-5) + 2(5) & 0(0) + 2(5) \end{bmatrix}$$

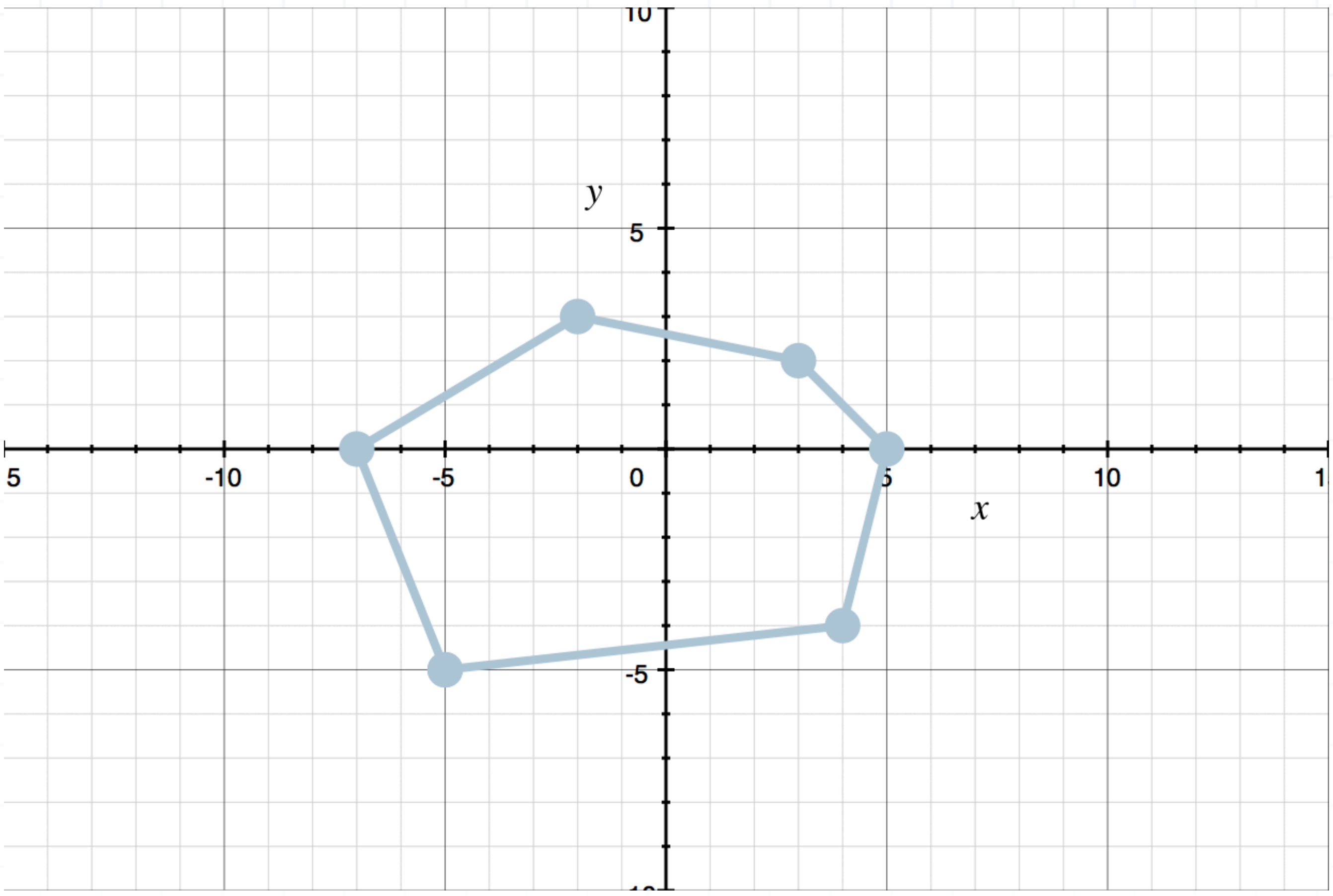
$$\begin{bmatrix} 0 & -5 & -10 & -5 \\ 0 & 0 & 10 & 10 \end{bmatrix}$$

The original square is sketched in light blue, and its transformation after Z is in yellow.



4. Sketch the transformation of the hexagon after it's transformed by matrix Y.

$$Y = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$$



Solution:

Put the vertices of the hexagon into a matrix.

$$\begin{bmatrix} -5 & 4 & 5 & 3 & -2 & -7 \\ -5 & -4 & 0 & 2 & 3 & 0 \end{bmatrix}$$

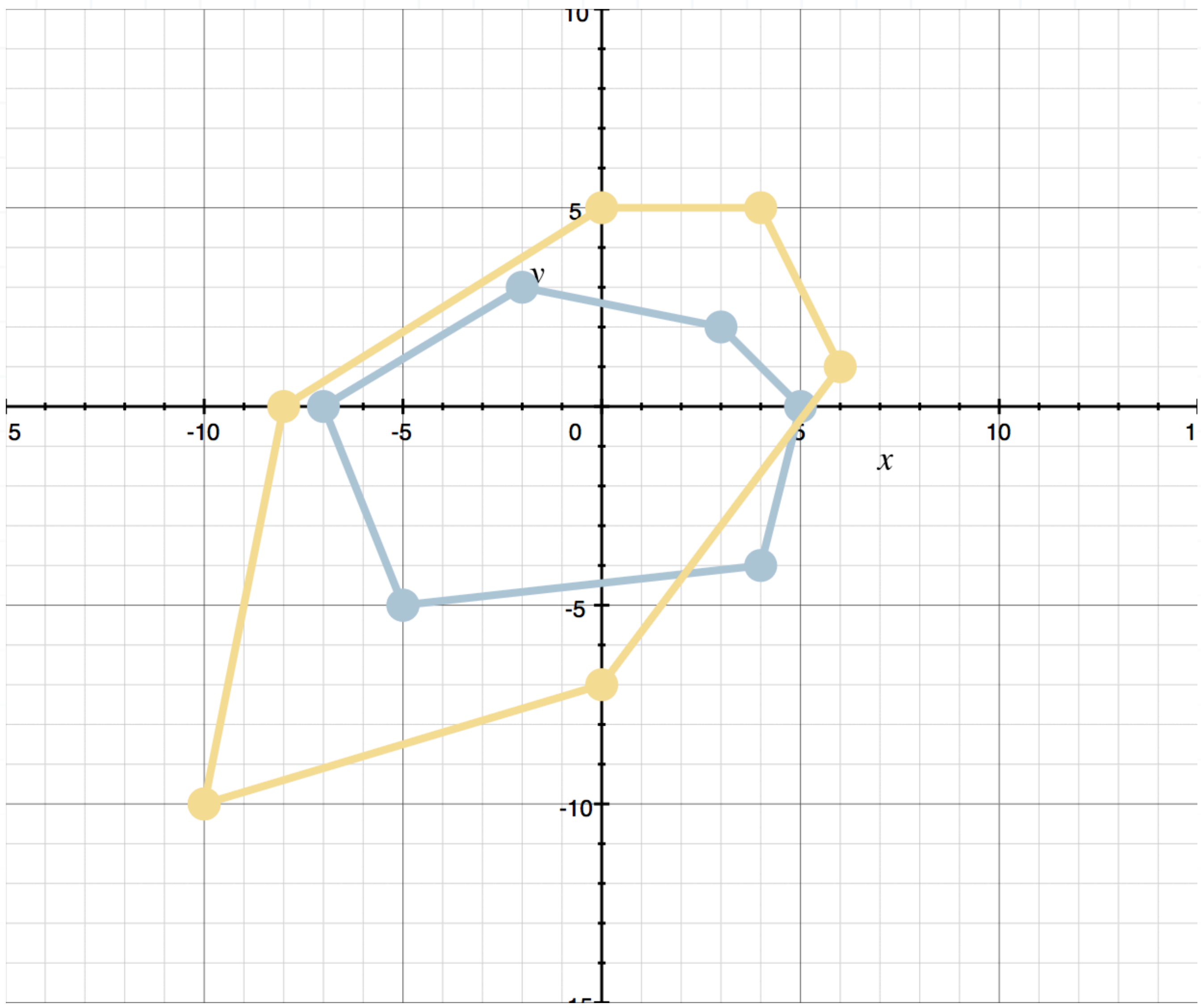
Apply the transformation of Y to the vertex matrix.

$$\begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -5 & 4 & 5 & 3 & -2 & -7 \\ -5 & -4 & 0 & 2 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -5(0)-5(2) & 4(0)-4(2) & 5(0)+0(2) & 3(0)+2(2) & -2(0)+3(2) & -7(0)+0(2) \\ -5(1)-5(1) & 4(1)-4(1) & 5(1)+0(1) & 3(1)+2(1) & -2(1)+3(1) & -7(1)+0(1) \end{bmatrix}$$

$$\begin{bmatrix} -10 & -8 & 0 & 4 & 6 & 0 \\ -10 & 0 & 5 & 5 & 1 & -7 \end{bmatrix}$$

The original hexagon is sketched in light blue, and its transformation after Y is in yellow.



■ 5. What happens to the unit vector $\vec{a} = (1,0)$ after the transformation given by matrix K.

$$K = \begin{bmatrix} 3 & -5 \\ -1 & 0 \end{bmatrix}$$

Solution:

Where $\vec{a} = (1,0)$ lands is given by the first column of the transformation matrix. So $\vec{a}$ will land on $(3,-1)$ after the transformation by K.

■ 6. What happens to the unit vector $\vec{b} = (0,1)$ after the transformation given by matrix K.

$$K = \begin{bmatrix} 3 & -5 \\ -1 & 0 \end{bmatrix}$$

Solution:

Where $\vec{b} = (0,1)$ lands is given by the second column of the transformation matrix. So $\vec{b}$ will land on $(-5,0)$ after the transformation by K.

PREIMAGE, IMAGE, AND THE KERNEL

■ 1. Find the preimage A_1 of the subset B_1 under the transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \{\vec{x} \in \mathbb{R}^2 \ | \ T(\vec{x}) \in B_1\}$$

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^2 \ \middle| \ \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix} \right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^2$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & | & 0 \\ 3 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & 0 \\ 3 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 5 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 1 & | & 0 \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & | & -1 \\ 3 & -1 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & \frac{1}{2} \\ 3 & -1 & | & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & \frac{1}{2} \\ 0 & 5 & | & \frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & | & \frac{1}{2} \\ 0 & 1 & | & \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{2} \\ 0 & 1 & | & \frac{1}{2} \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 0$ and $x_2 = 0$. And from the second augmented matrix we get $x_1 = 3/2$ and $x_2 = 1/2$. Therefore,

$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} -1 \\ 4 \end{bmatrix}$ in the subset B_1 under T

■ 2. Find the preimage A_1 of the subset B_1 under the transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$.

$$B_1 = \left\{ \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \{\vec{x} \in \mathbb{R}^3 \ | \ T(\vec{x}) \in B_1\}$$

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^3 \ \middle| \ \begin{bmatrix} -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix} \text{ or } \begin{bmatrix} -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix} \right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^3$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -3 & 2 & | & 1 \\ -2 & 0 & -2 & | & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & | & -1 \\ -2 & 0 & -2 & | & -4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & | & -1 \\ 0 & 6 & -6 & | & -6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & | & -1 \\ 0 & 1 & -1 & | & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & 2 \\ 0 & 1 & -1 & | & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} -1 & -3 & 2 \\ -2 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -3 & 2 & | & -5 \\ -2 & 0 & -2 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & | & 5 \\ -2 & 0 & -2 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & | & 5 \\ 0 & 6 & -6 & | & 12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & | & 5 \\ 0 & 1 & -1 & | & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & -1 \\ 0 & 1 & -1 & | & 2 \end{bmatrix}$$

The first augmented matrix gives a system of linear equations

$$x_1 + x_3 = 2$$

$$x_2 - x_3 = -1$$

that can be solved for the pivot variables in terms of the free variables.

$$x_1 = 2 - x_3$$

$$x_2 = -1 + x_3$$

The second augmented matrix gives a system of linear equations

$$x_1 + x_3 = -1$$

$$x_2 - x_3 = 2$$

that can be solved for the pivot variables in terms of the free variables.

$$x_1 = -1 - x_3$$

$$x_2 = 2 + x_3$$

From the first augmented matrix, we get $x_1 = 2 - x_3$ and $x_2 = x_3 - 1$. And from the second augmented matrix we get $x_1 = -1 - x_3$ and $x_2 = 2 + x_3$. Therefore,

$\begin{bmatrix} 2 - x_3 \\ x_3 - 1 \\ x_3 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 1 \\ -4 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} -1 - x_3 \\ 2 + x_3 \\ x_3 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} -5 \\ 2 \end{bmatrix}$ in the subset B_1 under T

■ 3. Find the kernel of the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$.

$$T(\overrightarrow{x}) = \begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

The kernel of a transformation T is all of the vectors that result in the zero vector under the transformation T:

$$\text{Ker}(T) = \left\{ \vec{x} \in \mathbb{R}^2 \ | \ T(\vec{x}) = \vec{O} \right\}$$

$$\begin{bmatrix} 3 & 9 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 9 & | & 0 \\ -1 & -3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 0 \\ -1 & -3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

The augmented matrix gives the equation

$$x_1 + 3x_2 = 0$$

which can be solved for the pivot variable in terms of the free variable.

$$x_1 = -3x_2$$

or

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

Then the kernel of the transformation is

$$\text{Ker}(T) = \text{Span}\left\{ \begin{bmatrix} -3 \\ 1 \end{bmatrix} \right\}$$

■ 4. Find the kernel of the transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$.

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

The kernel of a transformation T is all of the vectors that result in the zero vector under the transformation T:

$$\text{Ker}(T) = \left\{ \vec{x} \in \mathbb{R}^3 \mid T(\vec{x}) = \vec{O} \right\}$$

$$\begin{bmatrix} -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & 1 & | & 0 \\ 2 & 1 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 2 & 1 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

The gives a system of linear equations

$$x_1 - x_3 = 0$$

$$x_2 + x_3 = 0$$

that can be solved for the pivot variables in terms of the free variable.

$$x_1 = x_3$$

$$x_2 = -x_3$$

or

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Then the kernel of the transformation is

$$\text{Ker}(T) = \text{Span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

■ 5. Find the preimage A_1 of the subset B_1 under the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

$$B_1 = \left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 & -2 \\ -3 & -4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; T(\vec{x}) \in B_1 \right\}$$

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^3 \;\middle|\; \begin{bmatrix} 1 & 1 & -2 \\ -3 & -4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right.$$

$$\left. \text{or } \begin{bmatrix} 1 & 1 & -2 \\ -3 & -4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix} \right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^3$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} 1 & 1 & -2 \\ -3 & -4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 & | & -2 \\ -3 & -4 & 0 & | & 1 \\ 0 & -2 & 1 & | & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & | & -2 \\ 0 & -1 & -6 & | & -5 \\ 0 & -2 & 1 & | & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & | & -2 \\ 0 & 1 & 6 & | & 5 \\ 0 & -2 & 1 & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -8 & | & -7 \\ 0 & 1 & 6 & | & 5 \\ 0 & -2 & 1 & | & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8 & | & -7 \\ 0 & 1 & 6 & | & 5 \\ 0 & 0 & 13 & | & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8 & | & -7 \\ 0 & 1 & 6 & | & 5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 6 & | & 5 \\ 0 & 0 & 1 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & | & -1 \\ 0 & 0 & 1 & | & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} 1 & 1 & -2 \\ -3 & -4 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & -2 & | & 4 \\ -3 & -4 & 0 & | & -2 \\ 0 & -2 & 1 & | & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & | & 4 \\ 0 & -1 & -6 & | & 10 \\ 0 & -2 & 1 & | & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -2 & | & 4 \\ 0 & 1 & 6 & | & -10 \\ 0 & -2 & 1 & | & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -8 & | & 14 \\ 0 & 1 & 6 & | & -10 \\ 0 & -2 & 1 & | & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8 & | & 14 \\ 0 & 1 & 6 & | & -10 \\ 0 & 0 & 13 & | & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -8 & | & 14 \\ 0 & 1 & 6 & | & -10 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 6 \\ 0 & 1 & 6 & | & -10 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 6 \\ 0 & 1 & 0 & | & -4 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 1$, $x_2 = -1$, and $x_3 = 1$. And from the second augmented matrix we get $x_1 = 6$, $x_2 = -4$, and $x_3 = -1$. Therefore,

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 4 \\ -2 \\ 7 \end{bmatrix}$ in the subset B_1 under T

■ 6. Find the preimage A_1 of the subset B_1 under the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$.

$$B_1 = \left\{ \begin{bmatrix} -4 \\ 4 \\ -12 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} \right\}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

We're trying to find the preimage of B_1 under T, which we'll call $T^{-1}(B_1)$.

$$T^{-1}(B_1) = \{\vec{x} \in \mathbb{R}^3 \ \mid \ T(\vec{x}) \in B_1\}$$

$$T^{-1}(B_1) = \left\{ \vec{x} \in \mathbb{R}^3 \ \middle| \ \begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -12 \end{bmatrix} \right.$$

$$\left. \text{or } \begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} \right\}$$

In other words, we're just trying to find all the vectors $\vec{x}$ in $\mathbb{R}^3$ that satisfy either of these matrix equations. So let's rewrite both augmented matrices in reduced row-echelon form. We get

$$\begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -12 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & -6 & | & -4 \\ 1 & -3 & 0 & | & 4 \\ -2 & 1 & -1 & | & -12 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & 2 \\ 1 & -3 & 0 & | & 4 \\ -2 & 1 & -1 & | & -12 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & 2 \\ 0 & -1 & -3 & | & 2 \\ -2 & 1 & -1 & | & -12 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & 2 \\ 0 & -1 & -3 & | & 2 \\ 0 & -3 & 5 & | & -8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & 2 \\ 0 & 1 & 3 & | & -2 \\ 0 & -3 & 5 & | & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 & | & -2 \\ 0 & 1 & 3 & | & -2 \\ 0 & -3 & 5 & | & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 & | & -2 \\ 0 & 1 & 3 & | & -2 \\ 0 & 0 & 14 & | & -14 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 9 & | & -2 \\ 0 & 1 & 3 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 7 \\ 0 & 1 & 3 & | & -2 \\ 0 & 0 & 1 & | & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 7 \\ 0 & 1 & 0 & | & 1 \\ 0 & 0 & 1 & | & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} -2 & 4 & -6 \\ 1 & -3 & 0 \\ -2 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 4 & -6 & | & 2 \\ 1 & -3 & 0 & | & -3 \\ -2 & 1 & -1 & | & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & -1 \\ 1 & -3 & 0 & | & -3 \\ -2 & 1 & -1 & | & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & -1 \\ 0 & -1 & -3 & | & -2 \\ -2 & 1 & -1 & | & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 3 & | & -1 \\ 0 & -1 & -3 & | & -2 \\ 0 & -3 & 5 & | & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 3 & | & -1 \\ 0 & 1 & 3 & | & 2 \\ 0 & -3 & 5 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 & | & 3 \\ 0 & 1 & 3 & | & 2 \\ 0 & -3 & 5 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 & | & 3 \\ 0 & 1 & 3 & | & 2 \\ 0 & 0 & 14 & | & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 9 & | & 3 \\ 0 & 1 & 3 & | & 2 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -\frac{3}{2} \\ 0 & 1 & 3 & | & 2 \\ 0 & 0 & 1 & | & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -\frac{3}{2} \\ 0 & 1 & 0 & | & \frac{1}{2} \\ 0 & 0 & 1 & | & \frac{1}{2} \end{bmatrix}$$

From the first augmented matrix, we get $x_1 = 7$, $x_2 = 1$, and $x_3 = -1$. And from the second augmented matrix we get $x_1 = -3/2$, $x_2 = 1/2$, and $x_3 = 1/2$. Therefore,

$\begin{bmatrix} 7 \\ 1 \\ -1 \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} -4 \\ 4 \\ 12 \end{bmatrix}$ in the subset B_1 under T

$\begin{bmatrix} -\frac{3}{2} \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$ in the pre-image A_1 would map to $\begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix}$ in the subset B_1 under T

LINEAR TRANSFORMATIONS AS MATRIX-VECTOR PRODUCTS

■ 1. Use a matrix-vector product to reflect the square with vertices $(3,-1)$, $(1,-1)$, $(1,1)$, and $(3,1)$ over the y-axis. What are the vertices of the reflected square? Graph the resulting figure.

Solution:

If each point in the square is given by (x,y), a reflection over the y-axis means we'll take the x-coordinate of each point in the square and multiply it by -1. So after the reflection, each transformed point will be $(-x,y)$.

Therefore, if a position vector

$$\vec{v}=\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

represents a point in the original square, then the position vector

$$\vec{v}=\begin{bmatrix}-v_1\\ v_2\end{bmatrix}$$

represents the corresponding point in the transformed square. So a transformation T that expresses the reflection for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right)=\begin{bmatrix}-v_1\\ v_2\end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the square, $(3,-1)$, $(1,-1)$, $(1,1)$, and $(3,1)$.

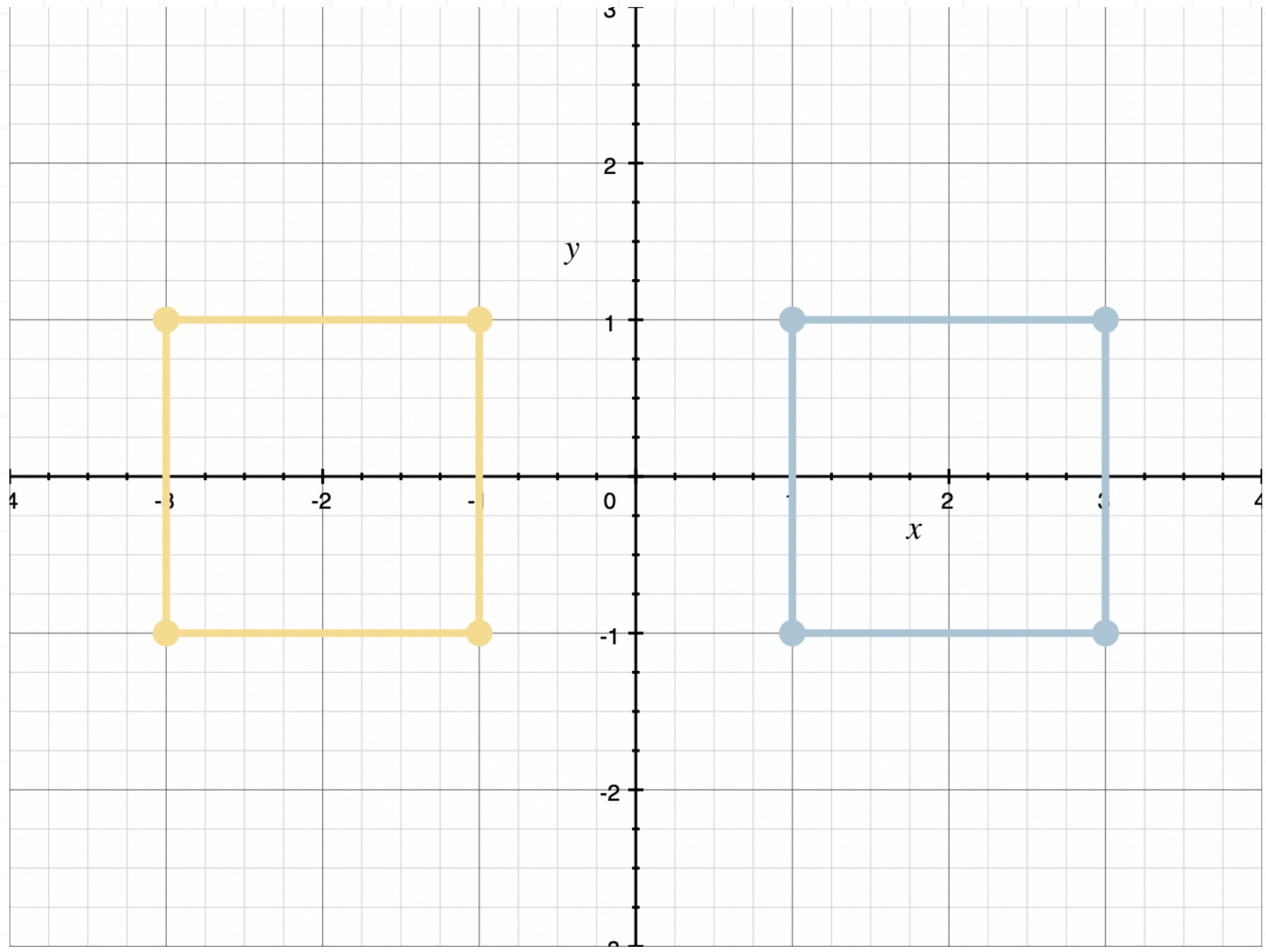
$$T\left(\begin{bmatrix}3\\-1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}3\\-1\end{bmatrix}=\begin{bmatrix}-1(3)+0(-1)\\0(3)+1(-1)\end{bmatrix}=\begin{bmatrix}-3\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-1(1)+0(-1)\\0(1)+1(-1)\end{bmatrix}=\begin{bmatrix}-1\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-1(1)+0(1)\\0(1)+1(1)\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}3\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&1\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix}=\begin{bmatrix}-1(3)+0(1)\\0(3)+1(1)\end{bmatrix}=\begin{bmatrix}-3\\1\end{bmatrix}$$

A sketch of the original figure and its transformation is



■ 2. Use a matrix-vector product to transform the triangle with vertices $(-2,-3)$, $(4,2)$, and $(2,-5)$. The transformation T should include a reflection over the x-axis and horizontal stretch by a factor of 5. Graph the resulting figure.

Solution:

If each point in the triangle is given by (x,y), a reflection over the x-axis means we'll take the y-coordinate of each point in the triangle and multiply it by -1. So after the reflection, each transformed point will be $(x,-y)$.

To stretch horizontally by a factor of 5, we'll need to multiply every x-value by 5. So after both the reflection and the horizontal stretch, each transformed point will be $(5x,-y)$.

Therefore, if a position vector

$$\vec{v}=\begin{bmatrix}v_1\\ v_2\end{bmatrix}$$

represents a point in the original triangle, then the position vector

$$\vec{v}=\begin{bmatrix}5v_1\\ -v_2\end{bmatrix}$$

represents the corresponding point in the transformed triangle. So a transformation T that expresses the reflection and the stretch for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix}v_1\\ v_2\end{bmatrix}\right)=\begin{bmatrix}5v_1\\ -v_2\end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\ 0\end{bmatrix}\right)=\begin{bmatrix}5(1)\\ -0\end{bmatrix}=\begin{bmatrix}5\\ 0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right)=\begin{bmatrix}5(0)\\ -1\end{bmatrix}=\begin{bmatrix}0\\ -1\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}5&0\\0&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

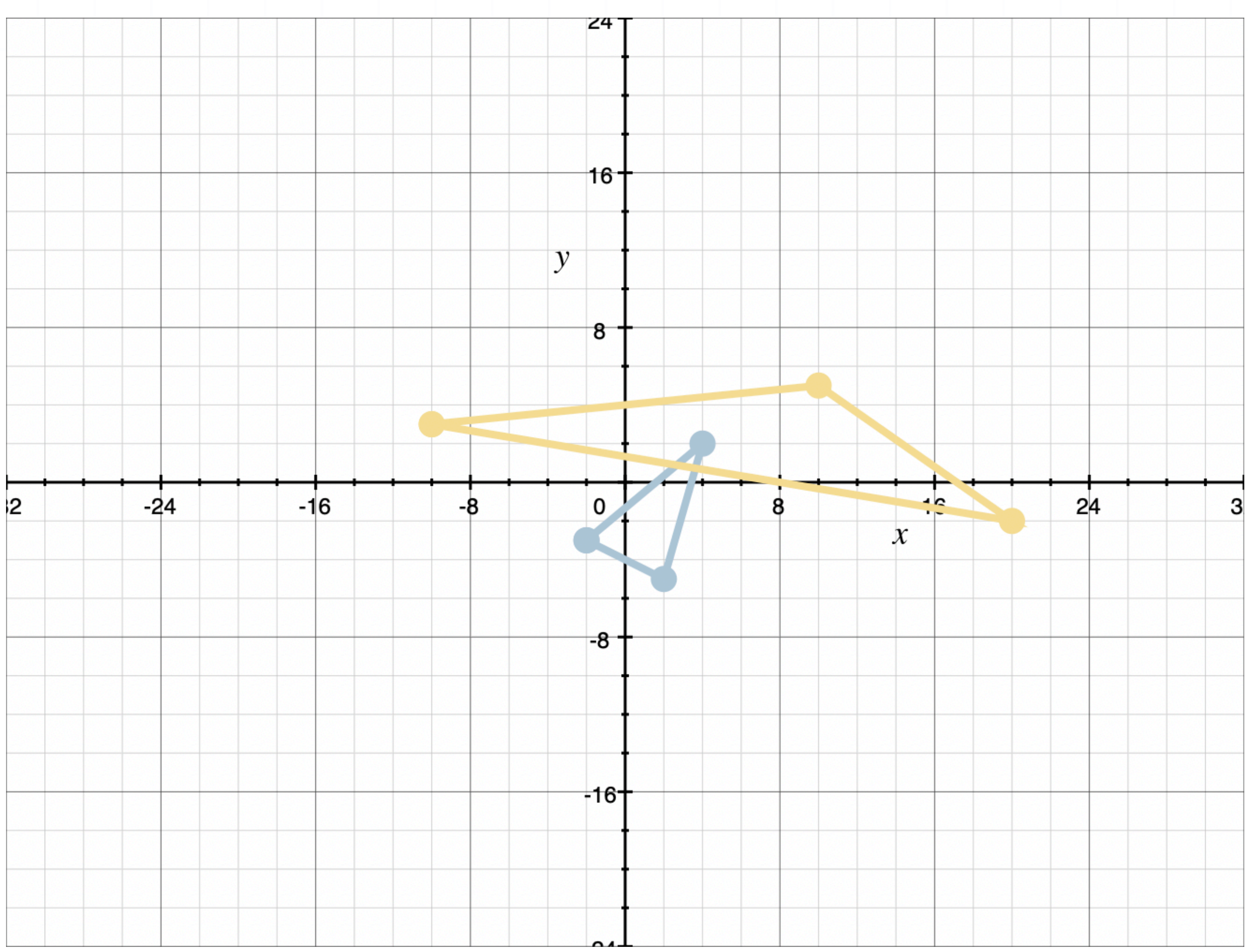
Now that we've built the transformation matrix, we can apply it to each of the vertices of the triangle, $(-2,-3)$, $(4,2)$, and $(2,-5)$.

$$T\left(\begin{bmatrix}-2\\-3\end{bmatrix}\right)=\begin{bmatrix}5&0\\0&-1\end{bmatrix}\begin{bmatrix}-2\\-3\end{bmatrix}=\begin{bmatrix}5(-2)+0(-3)\\0(-2)-1(-3)\end{bmatrix}=\begin{bmatrix}-10\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\2\end{bmatrix}\right)=\begin{bmatrix}5&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix}=\begin{bmatrix}5(4)+0(2)\\0(4)-1(2)\end{bmatrix}=\begin{bmatrix}20\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}2\\-5\end{bmatrix}\right)=\begin{bmatrix}5&0\\0&-1\end{bmatrix}\begin{bmatrix}2\\-5\end{bmatrix}=\begin{bmatrix}5(2)+0(-5)\\0(2)-1(-5)\end{bmatrix}=\begin{bmatrix}10\\5\end{bmatrix}$$

A sketch of the original figure and its transformation is



■ 3. Use a matrix-vector product to reflect the triangle with vertices $(3,3)$, $(1,-2)$, and $(-3,3)$ over the line $y=x$. What are the vertices of the reflected triangle? Graph the resulting figure.

Solution:

If each point in the triangle is given by (x,y), a reflection over the line $y=x$ means we'll take each point in the square and reverse x and y-values. So after the reflection, each transformed point will be (y,x).

Therefore, if a position vector

$$\vec{v}=\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

represents a point in the original triangle, then a position vector

$$\vec{v}=\begin{bmatrix}v_2\\v_1\end{bmatrix}$$

represents the corresponding point in the transformed triangle. So a transformation T that expresses the reflection for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}v_2\\v_1\end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

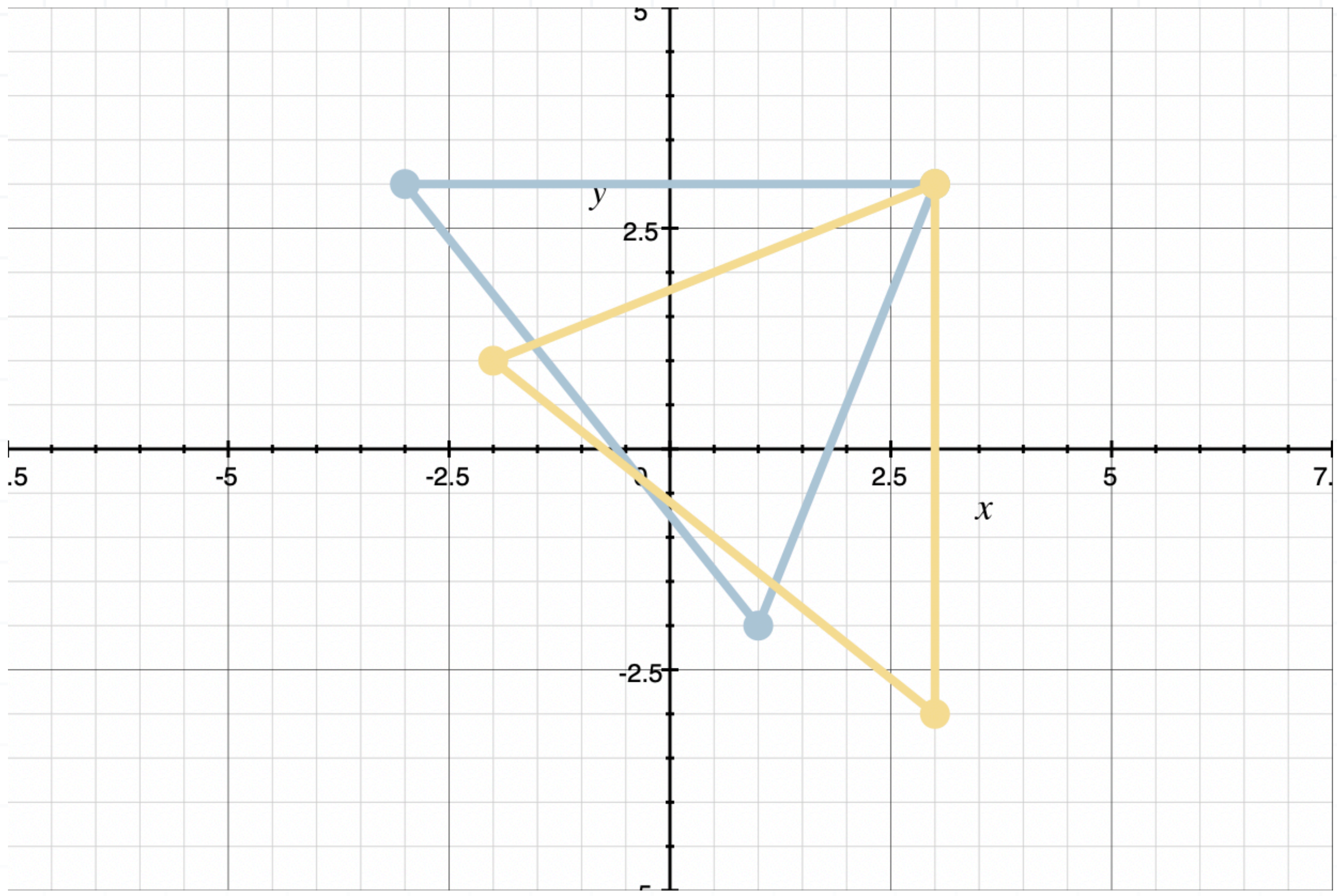
Now that we've built the transformation matrix, we can apply it to each of the vertices of the triangle, $(3,3)$, $(1,-2)$, and $(-3,3)$.

$$T\left(\begin{bmatrix}3\\3\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}3\\3\end{bmatrix}=\begin{bmatrix}0(3)+1(3)\\1(3)+0(3)\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}1\\-2\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1\\-2\end{bmatrix}=\begin{bmatrix}0(1)+1(-2)\\1(1)+0(-2)\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}-3\\3\end{bmatrix}\right)=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}-3\\3\end{bmatrix}=\begin{bmatrix}0(-3)+1(3)\\1(-3)+0(3)\end{bmatrix}=\begin{bmatrix}3\\-3\end{bmatrix}$$

A sketch of the original figure and its transformation is



4. Use a matrix-vector product to triple the width of the rectangle that has vertices $(-2,-1)$, $(-2,-5)$, $(5,-1)$, and $(5,-5)$, and then compress it vertically by a factor of 2. What are the vertices of the transformed rectangle? Graph the resulting figure.

Solution:

Tripling the width of the rectangle means we're stretching it horizontally by a factor of 3.

If each point in the rectangle is given by (x,y), then tripling the width means we'll take the x-coordinate of each point in the rectangle and multiply it by 3. So after the stretch, each transformed point will be $(3x,y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original rectangle, then a position vector

$$\vec{v} = \begin{bmatrix} 3v_1 \\ v_2 \end{bmatrix}$$

represents the corresponding point in the transformed rectangle. So a transformation T that expresses the stretch for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 \\ v_2 \end{bmatrix}$$

A vertical compression by a factor of 2 means we'll take the y-coordinate of each point in the rectangle and multiply it by $1/2$. So after the compression and stretch, each transformed point will be

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 \\ \frac{1}{2}v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 3(1) \\ \frac{1}{2}(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 3(0) \\ \frac{1}{2}(1) \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the rectangle, $(-2,-1)$, $(-2,-5)$, $(5,-1)$, and $(5,-5)$.

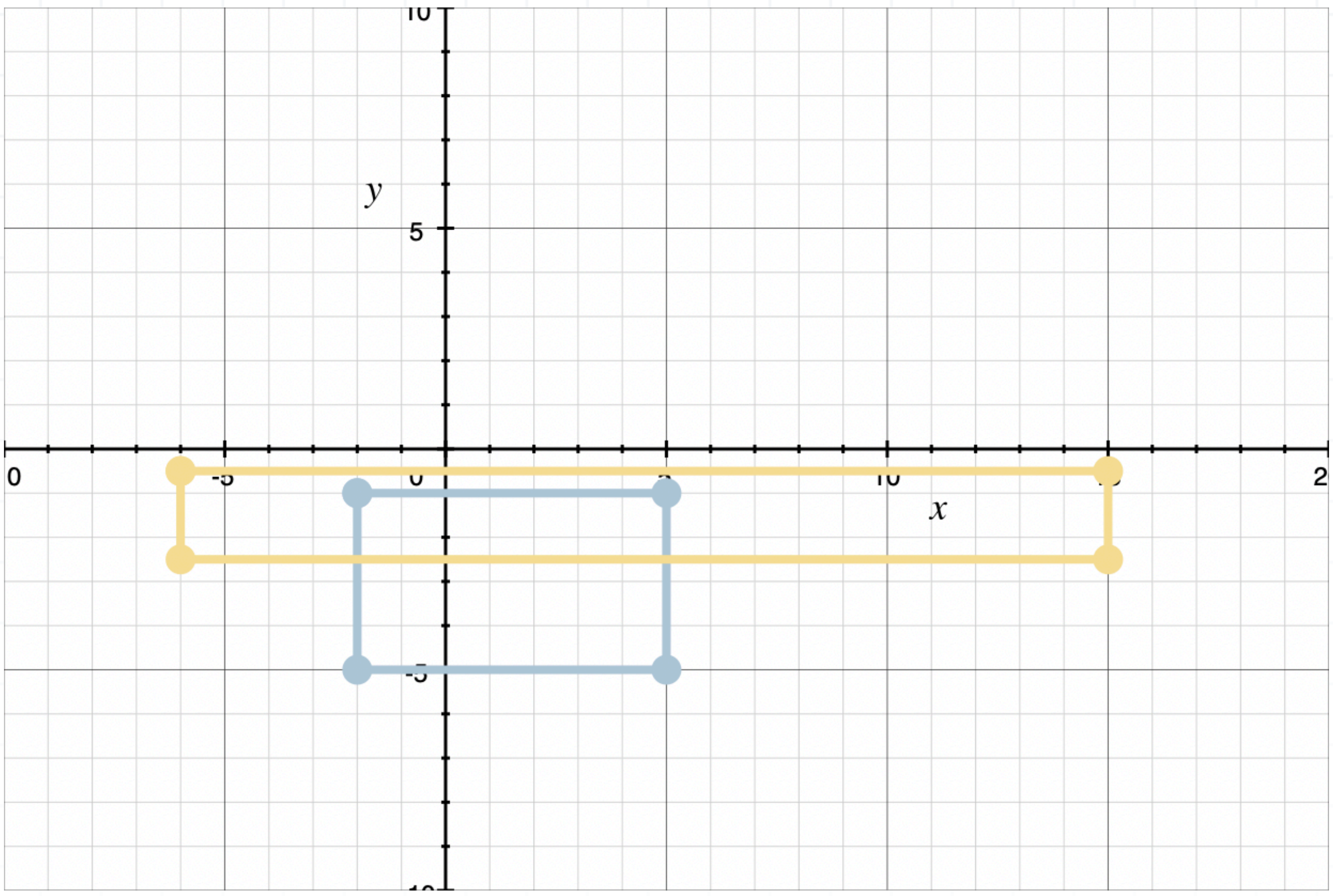
$$T\left(\begin{bmatrix}-2\\-1\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}-2\\-1\end{bmatrix}=\begin{bmatrix}3(-2)+0(-1)\\0(-2)+\frac{1}{2}(-1)\end{bmatrix}=\begin{bmatrix}-6\\-\frac{1}{2}\end{bmatrix}$$

$$T\left(\begin{bmatrix}-2\\-5\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}-2\\-5\end{bmatrix}=\begin{bmatrix}3(-2)+0(-5)\\0(-2)+\frac{1}{2}(-5)\end{bmatrix}=\begin{bmatrix}-6\\-\frac{5}{2}\end{bmatrix}$$

$$T\left(\begin{bmatrix}5\\-1\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}5\\-1\end{bmatrix}=\begin{bmatrix}3(5)+0(-1)\\0(5)+\frac{1}{2}(-1)\end{bmatrix}=\begin{bmatrix}15\\-\frac{1}{2}\end{bmatrix}$$

$$T\left(\begin{bmatrix}5\\-5\end{bmatrix}\right)=\begin{bmatrix}3&0\\0&\frac{1}{2}\end{bmatrix}\begin{bmatrix}5\\-5\end{bmatrix}=\begin{bmatrix}3(5)+0(-5)\\0(5)+\frac{1}{2}(-5)\end{bmatrix}=\begin{bmatrix}15\\-\frac{5}{2}\end{bmatrix}$$

A sketch of the original figure and its transformation is



5. Use a matrix-vector product to reflect the parallelogram with vertices $(-3,1)$, $(0,4)$, $(7,4)$, and $(4,1)$ over the x-axis, and then over the y-axis. What are the vertices of the reflected parallelogram? Graph the resulting figure.

Solution:

If each point in the parallelogram is given by (x, y), a reflection over the x-axis means we'll take the y-coordinate of each point in the parallelogram and multiply it by -1. So after the reflection, each transformed point will be $(x, -y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original parallelogram, then a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

represents the corresponding point in the transformed parallelogram. So a transformation T that expresses the reflection for any vector in $\mathbb{R}^2$ is

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} v_1 \\ -v_2 \end{bmatrix}$$

A reflection over the y-axis means we'll take the x-coordinate of each point in the parallelogram and multiply it by -1. So after the two reflections, each transformed point will be

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ -0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -0 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

Now that we've built the transformation matrix, we can apply it to each of the vertices of the parallelogram, $(-3,1)$, $(0,4)$, $(7,4)$, and $(4,1)$.

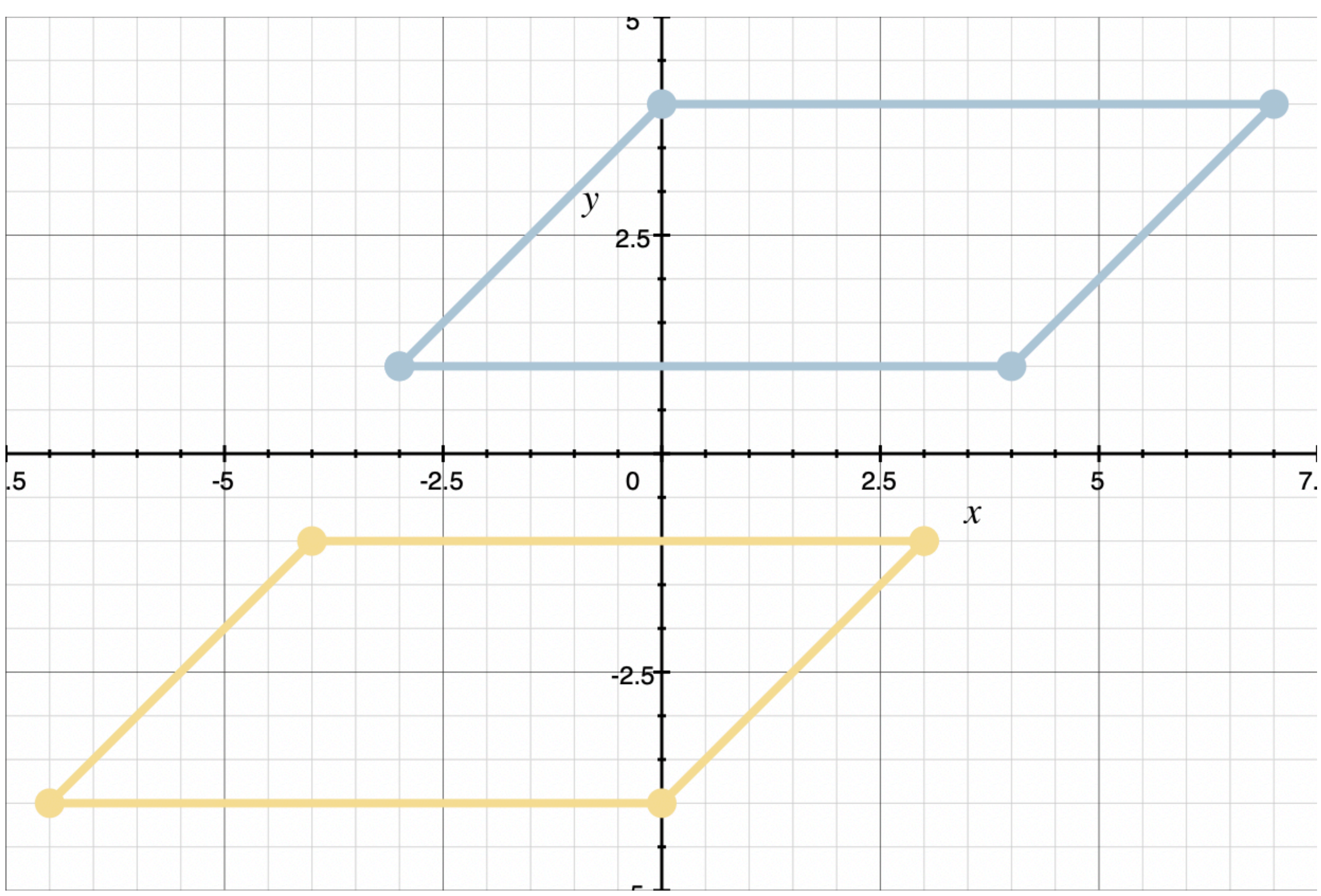
$$T\left(\begin{bmatrix}-3\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}-3\\1\end{bmatrix}=\begin{bmatrix}-1(-3)+0(1)\\0(-3)-1(1)\end{bmatrix}=\begin{bmatrix}3\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\4\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}0\\4\end{bmatrix}=\begin{bmatrix}-1(0)+0(4)\\0(0)-1(4)\end{bmatrix}=\begin{bmatrix}0\\-4\end{bmatrix}$$

$$T\left(\begin{bmatrix}7\\4\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}7\\4\end{bmatrix}=\begin{bmatrix}-1(7)+0(4)\\0(7)-1(4)\end{bmatrix}=\begin{bmatrix}-7\\-4\end{bmatrix}$$

$$T\left(\begin{bmatrix}4\\1\end{bmatrix}\right)=\begin{bmatrix}-1&0\\0&-1\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix}=\begin{bmatrix}-1(4)+0(1)\\0(4)-1(1)\end{bmatrix}=\begin{bmatrix}-4\\-1\end{bmatrix}$$

A sketch of the original figure and its transformation is



■ 6. Use a matrix-vector product to reflect the triangle with vertices $(2,3)$, $(-5,-4)$, and $(-4,5)$ over the y-axis, and then stretch it horizontally by a factor of 4. What are the vertices of the transformed triangle? Graph the resulting figure.

Solution:

If each point in the triangle is given by (x, y), a reflection over the y-axis means we'll take the x-coordinate of each point in the triangle and multiply it by -1. So after the reflection, each transformed point will be $(-x, y)$.

Therefore, if a position vector

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

represents a point in the original triangle, then a position vector

$$\vec{v} = \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix}$$

represents the corresponding point in the transformed triangle.

A horizontal stretch by a factor of 4 means we'll multiply each x-value by 4. So after both transformations, each transformed point will be

$$T\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} -4v_1 \\ v_2 \end{bmatrix}$$

Because we're transforming *from* $\mathbb{R}^2$, we can use T to transform each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-4(1)\\0\end{bmatrix}=\begin{bmatrix}-4\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-4(0)\\1\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

Which means we can actually rewrite the transformation T as

$$T\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right)=\begin{bmatrix}-4&0\\0&1\end{bmatrix}\begin{bmatrix}v_1\\v_2\end{bmatrix}$$

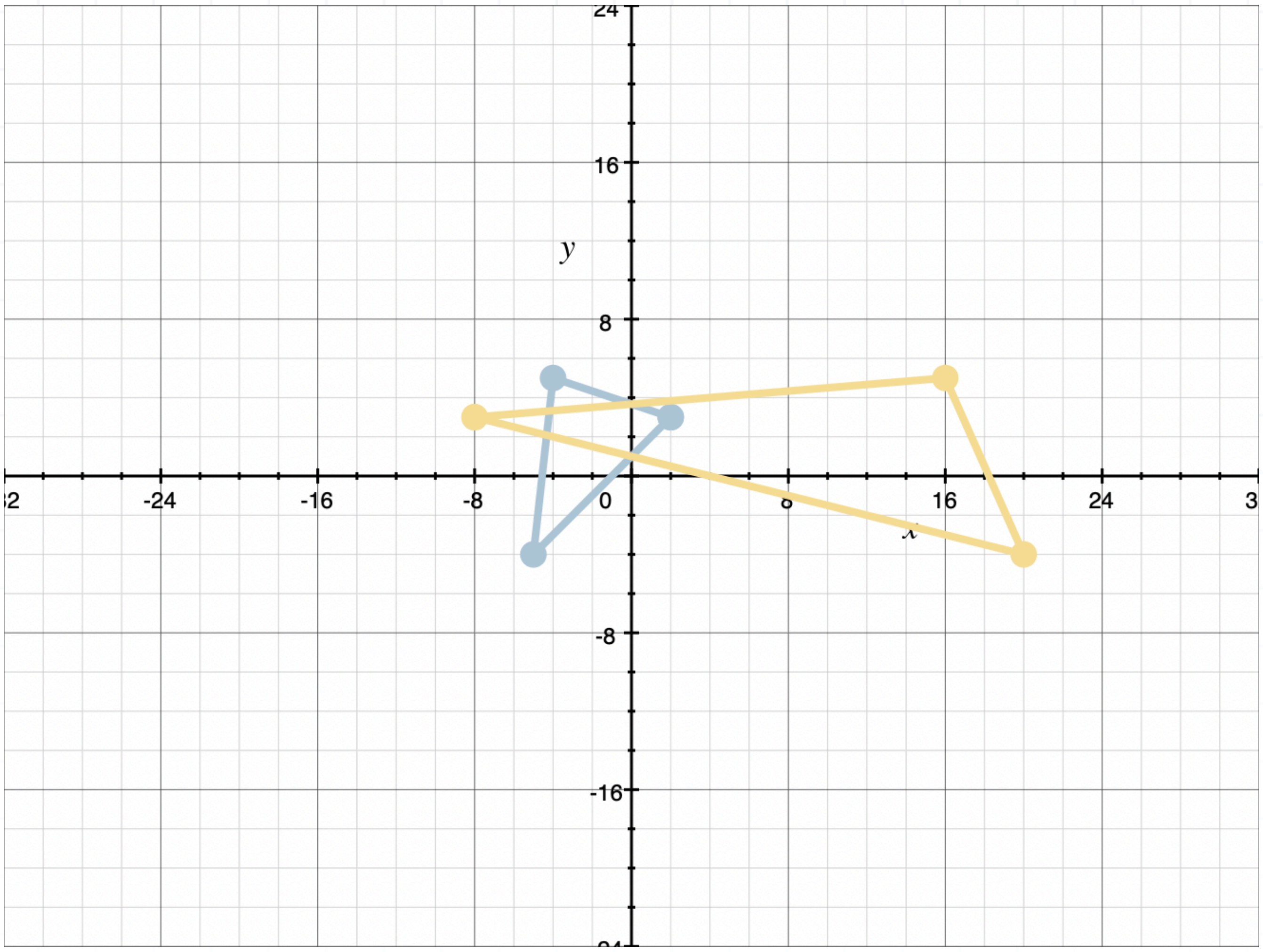
Now that we've built the transformation matrix, we can apply it to each of the vertices of the triangle, $(2,3)$, $(-5,-4)$, and $(-4,5)$.

$$T\left(\begin{bmatrix}2\\3\end{bmatrix}\right)=\begin{bmatrix}-4&0\\0&1\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}=\begin{bmatrix}-4(2)+0(3)\\0(2)+1(3)\end{bmatrix}=\begin{bmatrix}-8\\3\end{bmatrix}$$

$$T\left(\begin{bmatrix}-5\\-4\end{bmatrix}\right)=\begin{bmatrix}-4&0\\0&1\end{bmatrix}\begin{bmatrix}-5\\-4\end{bmatrix}=\begin{bmatrix}-4(-5)+0(-4)\\0(-5)+1(-4)\end{bmatrix}=\begin{bmatrix}20\\-4\end{bmatrix}$$

$$T\left(\begin{bmatrix}-4\\5\end{bmatrix}\right)=\begin{bmatrix}-4&0\\0&1\end{bmatrix}\begin{bmatrix}-4\\5\end{bmatrix}=\begin{bmatrix}-4(-4)+0(5)\\0(-4)+1(5)\end{bmatrix}=\begin{bmatrix}16\\5\end{bmatrix}$$

A sketch of the original figure and its transformation is



y
x
16
8
-8
-16
-24
-16
-8
0
8
16
24

LINEAR TRANSFORMATIONS AS ROTATIONS

■ 1. Find the rotation of $\vec{x} = (2, -4)$ by an angle of $\theta = 120^\circ$.

Solution:

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ by 120° is

$$\text{Rot}_{120^\circ}(\vec{x}) = \begin{bmatrix} \cos(120^\circ) & -\sin(120^\circ) \\ \sin(120^\circ) & \cos(120^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 120^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} \cos(120^\circ) & -\sin(120^\circ) \\ \sin(120^\circ) & \cos(120^\circ) \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\left(\frac{\sqrt{3}}{2}\right) \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix}$$

So the transformation for a 120° rotation is

$$\text{Rot}_{120^\circ}(\vec{x}) = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (2, -4)$.

$$\text{Rot}_{120^\circ}\left(\begin{bmatrix}2\\-4\end{bmatrix}\right)=\begin{bmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{bmatrix}\begin{bmatrix}2\\-4\end{bmatrix}$$

$$\text{Rot}_{120^\circ}\left(\begin{bmatrix}2\\-4\end{bmatrix}\right)=\begin{bmatrix}-\frac{1}{2}(2)-\frac{\sqrt{3}}{2}(-4)\\ \frac{\sqrt{3}}{2}(2)-\frac{1}{2}(-4)\end{bmatrix}$$

$$\text{Rot}_{120^\circ}\left(\begin{bmatrix}2\\-4\end{bmatrix}\right)=\begin{bmatrix}-1+2\sqrt{3}\\ \sqrt{3}+2\end{bmatrix}$$

$$\text{Rot}_{120^\circ}\left(\begin{bmatrix}2\\-4\end{bmatrix}\right)=\begin{bmatrix}-1+2\sqrt{3}\\ 2+\sqrt{3}\end{bmatrix}$$

■ 2. Find the rotation of $\vec{x}=(1,-5)$ by an angle of $\theta=60^\circ$.

Solution:

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ by 60° is

$$\text{Rot}_{60^\circ}(\vec{x})=\begin{bmatrix}\cos(60^\circ) & -\sin(60^\circ)\\ \sin(60^\circ) & \cos(60^\circ)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta=60^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix}\cos(60^\circ) & -\sin(60^\circ)\\ \sin(60^\circ) & \cos(60^\circ)\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\left(\frac{\sqrt{3}}{2}\right)\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}$$

So the transformation for a 60° rotation is

$$\text{Rot}_{60^\circ}(\vec{x})=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x}=(1,-5)$.

$$\text{Rot}_{60^\circ}\left(\begin{bmatrix}1\\ -5\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\ -5\end{bmatrix}$$

$$\text{Rot}_{60^\circ}\left(\begin{bmatrix}1\\ -5\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}(1)-\frac{\sqrt{3}}{2}(-5)\\ \frac{\sqrt{3}}{2}(1)+\frac{1}{2}(-5)\end{bmatrix}$$

$$\text{Rot}_{60^\circ}\left(\begin{bmatrix}1\\ -5\end{bmatrix}\right)=\begin{bmatrix}\frac{1}{2}+\frac{5\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2}-\frac{5}{2}\end{bmatrix}$$

$$\text{Rot}_{60^\circ}\left(\begin{bmatrix}1\\ -5\end{bmatrix}\right)=\begin{bmatrix}\frac{1+5\sqrt{3}}{2}\\ \frac{-5+\sqrt{3}}{2}\end{bmatrix}$$

3. Find the rotation of $\vec{x} = (-7,4)$ by an angle of $\theta = 180^\circ$.

Solution:

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^2$ by 180° is

$$\text{Rot}_{180^\circ}(\vec{x}) = \begin{bmatrix} \cos(180^\circ) & -\sin(180^\circ) \\ \sin(180^\circ) & \cos(180^\circ) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 180^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} \cos(180^\circ) & -\sin(180^\circ) \\ \sin(180^\circ) & \cos(180^\circ) \end{bmatrix} = \begin{bmatrix} -1 & -0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

So the transformation for a 180° rotation is

$$\text{Rot}_{180^\circ}(\vec{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (-7,4)$.

$$\text{Rot}_{180^\circ}\left(\begin{bmatrix} -7 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

$$\text{Rot}_{180^\circ}\left(\begin{bmatrix} -7 \\ 4 \end{bmatrix}\right) = \begin{bmatrix} -1(-7) + 0(4) \\ 0(-7) - 1(4) \end{bmatrix}$$

$$\text{Rot}_{180^\circ}\left(\begin{bmatrix}-7\\4\end{bmatrix}\right)=\begin{bmatrix}7+0\\0-4\end{bmatrix}$$

$$\text{Rot}_{180^\circ}\left(\begin{bmatrix}-7\\4\end{bmatrix}\right)=\begin{bmatrix}7\\-4\end{bmatrix}$$

■ 4. Find the rotation of $\vec{x}=(-4,1,3)$ by an angle of $\theta=90^\circ$ about the x-axis.

Solution:

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ by 90° around the x-axis is

$$\text{Rot}_{90^\circ\text{ around }x}=\begin{bmatrix}1&0&0\\0&\cos(90^\circ)&-\sin(90^\circ)\\0&\sin(90^\circ)&\cos(90^\circ)\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta=90^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix}1&0&0\\0&\cos(90^\circ)&-\sin(90^\circ)\\0&\sin(90^\circ)&\cos(90^\circ)\end{bmatrix}=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}$$

So the transformation for a 90° rotation around the x-axis is

$$\text{Rot}_{90^\circ\text{ around }x}(\vec{x})=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x}=(-4,1,3)$.

$$\text{Rot}_{90^\circ \text{ around } x}\left(\begin{bmatrix}-4\\1\\3\end{bmatrix}\right)=\begin{bmatrix}1&0&0\\0&0&-1\\0&1&0\end{bmatrix}\begin{bmatrix}-4\\1\\3\end{bmatrix}$$

$$\text{Rot}_{90^\circ \text{ around } x}\left(\begin{bmatrix}-4\\1\\3\end{bmatrix}\right)=\begin{bmatrix}1(-4)+0(1)+0(3)\\0(-4)+0(1)-1(3)\\0(-4)+1(1)+0(3)\end{bmatrix}$$

$$\text{Rot}_{90^\circ \text{ around } x}\left(\begin{bmatrix}-4\\1\\3\end{bmatrix}\right)=\begin{bmatrix}-4+0+0\\0+0-3\\0+1+0\end{bmatrix}$$

$$\text{Rot}_{90^\circ \text{ around } x}\left(\begin{bmatrix}-4\\1\\3\end{bmatrix}\right)=\begin{bmatrix}-4\\-3\\1\end{bmatrix}$$

■ 5. Find the rotation of $\vec{x}=(-2/\sqrt{2},2,0)$ by an angle of $\theta=315^\circ$ about the y-axis.

Solution:

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ by 315° around the y-axis is

$$\text{Rot}_{315^\circ \text{ around } y}=\begin{bmatrix}\cos(315^\circ)&0&\sin(315^\circ)\\0&1&0\\-\sin(315^\circ)&0&\cos(315^\circ)\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 315^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} \cos(315^\circ) & 0 & \sin(315^\circ) \\ 0 & 1 & 0 \\ -\sin(315^\circ) & 0 & \cos(315^\circ) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ -\left(-\frac{\sqrt{2}}{2}\right) & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix}$$

So the transformation for a 315° rotation around the y-axis is

$$\text{Rot}_{315^\circ \text{ around } y}(\vec{x}) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (-2/\sqrt{2},2,0)$.

$$\text{Rot}_{315^\circ \text{ around } y}\left(\begin{bmatrix} -\frac{2}{\sqrt{2}} \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ 0 & 1 & 0 \\ \frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -\frac{2}{\sqrt{2}} \\ 2 \\ 0 \end{bmatrix}$$

$$\text{Rot}_{315^\circ \text{ around } y}\left(\begin{bmatrix} -\frac{2}{\sqrt{2}} \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} \frac{\sqrt{2}}{2}\left(-\frac{2}{\sqrt{2}}\right) + 0(2) - \frac{\sqrt{2}}{2}(0) \\ 0\left(-\frac{2}{\sqrt{2}}\right) + 1(2) + 0(0) \\ \frac{\sqrt{2}}{2}\left(-\frac{2}{\sqrt{2}}\right) + 0(2) + \frac{\sqrt{2}}{2}(0) \end{bmatrix}$$

$$\text{Rot}_{315^\circ \text{ around } y}\left(\begin{bmatrix} -\frac{2}{\sqrt{2}} \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1+0+0 \\ 0+2+0 \\ -1+0+0 \end{bmatrix}$$

$$\text{Rot}_{315^\circ \text{ around } y}\left(\begin{bmatrix} -\frac{2}{\sqrt{2}} \\ 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$$

■ 6. Find the rotation of $\vec{x} = (-2,0,3)$ by an angle of $\theta = 150^\circ$ about the z -axis.

Solution:

The transformation to rotate any vector $\vec{x}$ in $\mathbb{R}^3$ by 150° around the z-axis is

$$\text{Rot}_{150^\circ \text{ around } z} = \begin{bmatrix} \cos(150^\circ) & -\sin(150^\circ) & 0 \\ \sin(150^\circ) & \cos(150^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Simplify the rotation matrix. We can get the sine and cosine values at $\theta = 150^\circ$ from the unit circle, or from a calculator.

$$\begin{bmatrix} \cos(150^\circ) & -\sin(150^\circ) & 0 \\ \sin(150^\circ) & \cos(150^\circ) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

So the transformation for a 150° rotation around the z-axis is

$$\text{Rot}_{150^\circ \text{ around } z}(\vec{x}) = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Apply this specific rotation matrix to $\vec{x} = (-2,0,3)$.

$$\text{Rot}_{150^\circ \text{ around } z}\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}$$

$$\text{Rot}_{150^\circ \text{ around } z}\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} -\frac{\sqrt{3}}{2}(-2) - \frac{1}{2}(0) + 0(3) \\ \frac{1}{2}(-2) - \frac{\sqrt{3}}{2}(0) + 0(3) \\ 0(-2) + 0(0) + 1(3) \end{bmatrix}$$

$$\text{Rot}_{150^\circ \text{ around } z}\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3} + 0 + 0 \\ -1 + 0 + 0 \\ 0 + 0 + 3 \end{bmatrix}$$

$$\text{Rot}_{150^\circ \text{ around } z}\left(\begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} \sqrt{3} \\ -1 \\ 3 \end{bmatrix}$$

ADDING AND SCALING LINEAR TRANSFORMATIONS

1. Find the product of a scalar $c = 5$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} 0 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

The transformation T is given as a matrix-vector product. If we call the transformation matrix B, then multiplying the transformation by the scalar $c = 5$ means we multiply B by c.

$$cT(\vec{x}) = 5 \begin{bmatrix} 0 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

First find cB.

$$cB = 5 \begin{bmatrix} 0 & -4 \\ -3 & 1 \end{bmatrix}$$

$$cB = \begin{bmatrix} 5(0) & 5(-4) \\ 5(-3) & 5(1) \end{bmatrix}$$

$$cB = \begin{bmatrix} 0 & -20 \\ -15 & 5 \end{bmatrix}$$

So the scaled transformation would be

$$cT(\vec{x}) = \begin{bmatrix} 0 & -20 \\ -15 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 2. Find the product of a scalar $c = -2$ and the transformation $T(\vec{x})$.

$$T(\vec{x}) = \begin{bmatrix} -1 & 0 & 4 \\ 3 & -5 & 7 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

The transformation T is given as a matrix-vector product. If we call the transformation matrix A, then multiplying the transformation by the scalar $c = -2$ means we multiply A by c.

$$cT(\vec{x}) = -2 \begin{bmatrix} -1 & 0 & 4 \\ 3 & -5 & 7 \\ -2 & -4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

First find cA.

$$cA = -2 \begin{bmatrix} -1 & 0 & 4 \\ 3 & -5 & 7 \\ -2 & -4 & 1 \end{bmatrix}$$

$$cA = \begin{bmatrix} -2(-1) & -2(0) & -2(4) \\ -2(3) & -2(-5) & -2(7) \\ -2(-2) & -2(-4) & -2(1) \end{bmatrix}$$

$$cA = \begin{bmatrix} 2 & 0 & -8 \\ -6 & 10 & -14 \\ 4 & 8 & -2 \end{bmatrix}$$

So the scaled transformation would be

$$cT(\vec{x}) = \begin{bmatrix} 2 & 0 & -8 \\ -6 & 10 & -14 \\ 4 & 8 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 3. Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

The sum of the transformations is the sum of the matrices A and B given in their matrix vector products,

$$S(\vec{x}) = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The sum of the matrices is

$$A + B = \begin{bmatrix} -4 & 3 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} -2 & 4 \\ 0 & -1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -4 - 2 & 3 + 4 \\ 2 + 0 & 1 - 1 \end{bmatrix}$$

$$A + B = \begin{bmatrix} -6 & 7 \\ 2 & 0 \end{bmatrix}$$

So the sum of the transformations would be

$$S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} -6 & 7 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 4. Find the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} 0 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 5 & -3 & 3 \\ 2 & 0 & -1 \\ 1 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

The sum of the transformations is the sum of the matrices A and B given in their matrix vector products,

$$S(\vec{x}) = \begin{bmatrix} 0 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 5 & -3 & 3 \\ 2 & 0 & -1 \\ 1 & -4 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The sum of the matrices is

$$A + B = \begin{bmatrix} 0 & -4 & 1 \\ 1 & -1 & 3 \\ 2 & -1 & 4 \end{bmatrix} + \begin{bmatrix} 5 & -3 & 3 \\ 2 & 0 & -1 \\ 1 & -4 & -5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 0+5 & -4-3 & 1+3 \\ 1+2 & -1+0 & 3-1 \\ 2+1 & -1-4 & 4-5 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 5 & -7 & 4 \\ 3 & -1 & 2 \\ 3 & -5 & -1 \end{bmatrix}$$

So the sum of the transformations would be

$$S(\vec{x}) + T(\vec{x}) = \begin{bmatrix} 5 & -7 & 4 \\ 3 & -1 & 2 \\ 3 & -5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

■ 5. Find the sum of the transformation $S(\vec{x})$ and the product of a scalar $c = -1/2$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} -5 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Solution:

First find the product of a scalar $c = -1/2$ and the transformation $T(\vec{x})$.

The transformation T is given as a matrix-vector product. If we call the transformation matrix B, then multiplying the transformation by the scalar $c = -1/2$ means we multiply B by c.

$$cT(\vec{x}) = -\frac{1}{2} \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

First find cB.

$$cB = -\frac{1}{2} \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix}$$

$$cB = \begin{bmatrix} -\frac{1}{2}(-4) & -\frac{1}{2}(2) \\ -\frac{1}{2}(-2) & -\frac{1}{2}(0) \end{bmatrix}$$

$$cB = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

So the scaled transformation would be

$$cT(\vec{x}) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Now find the sum of the transformations $S(\vec{x})$ and $cT(\vec{x})$.

The sum of the transformations is the sum of the matrices A and $C = cB$ given in their matrix vector products,

$$S(\vec{x}) = \begin{bmatrix} -5 & 0 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$cT(\vec{x}) = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The sum of the matrices is

$$A + C = \begin{bmatrix} -5 & 0 \\ -2 & -1 \end{bmatrix} + \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$$

$$A + C = \begin{bmatrix} -5 + 2 & 0 - 1 \\ -2 + 1 & -1 + 0 \end{bmatrix}$$

$$A + C = \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix}$$

So the sum of the transformations would be

$$S(\vec{x}) + cT(\vec{x}) = \begin{bmatrix} -3 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

■ 6. Find the product of $c = 1/3$ with the sum of the transformations $S(\vec{x})$ and $T(\vec{x})$.

$$S(\vec{x}) = \begin{bmatrix} -5 & 4 & -3 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 0 \\ 6 & -4 & 5 \\ 9 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solution:

First find the sum of the transformations $S(\vec{x})$ and $cT(\vec{x})$.

The sum of the transformations is the sum of the matrices A and B given in their matrix vector products,

$$S(\vec{x}) = \begin{bmatrix} -5 & 4 & -3 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -1 & 2 & 0 \\ 6 & -4 & 5 \\ 9 & 1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

The sum of the matrices is

$$A + B = \begin{bmatrix} -5 & 4 & -3 \\ 0 & 1 & -2 \\ 0 & -1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 & 0 \\ 6 & -4 & 5 \\ 9 & 1 & 3 \end{bmatrix}$$

$$A+B=\begin{bmatrix}-5-1 & 4+2 & -3+0\\ 0+6 & 1-4 & -2+5\\ 0+9 & -1+1 & 3+3\end{bmatrix}$$

$$A+B=\begin{bmatrix}-6 & 6 & -3\\ 6 & -3 & 3\\ 9 & 0 & 6\end{bmatrix}$$

So the sum of the transformations would be

$$S(\overrightarrow{x})+T(\overrightarrow{x})=\begin{bmatrix}-6 & 6 & -3\\ 6 & -3 & 3\\ 9 & 0 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Now find the product of a scalar $c=1/3$ and the sum of the transformations $S(\overrightarrow{x})+T(\overrightarrow{x})$.

If we call the matrix that's in the sum of the transformation $S(\overrightarrow{x})+T(\overrightarrow{x})$ the matrix $C=A+B$, then multiplying the transformation by the scalar $c=1/3$ gives

$$c(S(\overrightarrow{x})+T(\overrightarrow{x}))=\frac{1}{3}\begin{bmatrix}-6 & 6 & -3\\ 6 & -3 & 3\\ 9 & 0 & 6\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

First find cC.

$$cC=\frac{1}{3}\begin{bmatrix}-6 & 6 & -3\\ 6 & -3 & 3\\ 9 & 0 & 6\end{bmatrix}$$

$$cC = \begin{bmatrix} \frac{1}{3}(-6) & \frac{1}{3}(6) & \frac{1}{3}(-3) \\ \frac{1}{3}(6) & \frac{1}{3}(-3) & \frac{1}{3}(3) \\ \frac{1}{3}(9) & \frac{1}{3}(0) & \frac{1}{3}(6) \end{bmatrix}$$

$$cC = \begin{bmatrix} -2 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix}$$

So the scaled transformation would be

$$c(S(\overrightarrow{x}) + T(\overrightarrow{x})) = \begin{bmatrix} -2 & 2 & -1 \\ 2 & -1 & 1 \\ 3 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

PROJECTIONS AS LINEAR TRANSFORMATIONS

■ 1. Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} 4 \\ 2 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

Solution:

The line L is given as all the scaled versions of the vector $\vec{x} = (4,2)$. Then the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \vec{x}$$

$$\text{Proj}_L(\vec{v}) = \frac{\begin{bmatrix} 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}}{\begin{bmatrix} 4 & 2 \end{bmatrix} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{1(4) + 4(2)}{4(4) + 2(2)} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$\text{Proj}_L(\vec{v}) = \frac{12}{20} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \frac{3}{5} \cdot \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ \frac{6}{5} \end{bmatrix} = \begin{bmatrix} 2.4 \\ 1.2 \end{bmatrix}$$

■ 2. Find the projection of $\vec{v}$ onto M.

$$M = \left\{ c \begin{bmatrix} -4 \\ 3 \end{bmatrix} \middle| \ c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -5 \end{bmatrix}$$

Solution:

The line M is given as all the scaled versions of the vector $\vec{x} = (-4,3)$. Then the projection of $\vec{v}$ onto M is given by

$$\text{Proj}_M(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \vec{x}$$

$$\text{Proj}_M(\vec{v}) = \frac{\begin{bmatrix} 0 & -5 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \end{bmatrix}}{\begin{bmatrix} -4 & 3 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 3 \end{bmatrix}} \cdot \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \frac{0(-4) - 5(3)}{-4(-4) + 3(3)} \cdot \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

$$\text{Proj}_M(\vec{v}) = \frac{-15}{25} \cdot \begin{bmatrix} -4 \\ 3 \end{bmatrix} = -\frac{3}{5} \cdot \begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{12}{5} \\ -\frac{9}{5} \end{bmatrix} = \begin{bmatrix} 2.4 \\ -1.8 \end{bmatrix}$$

■ 3. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -3 \\ 1 \end{bmatrix} \middle| c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

Solution:

The line L is given as all the scaled versions of the vector $\vec{x} = (-3,1)$. Then the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \vec{x}$$

$$\text{Proj}_L(\vec{v}) = \frac{\begin{bmatrix} -4 & 0 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}}{\begin{bmatrix} -3 & 1 \end{bmatrix} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{-4(-3) + 0(1)}{-3(-3) + 1(1)} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix}$$

$$\text{Proj}_L(\vec{v}) = \frac{12}{10} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{6}{5} \cdot \begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{18}{5} \\ \frac{6}{5} \end{bmatrix} = \begin{bmatrix} -3.6 \\ 1.2 \end{bmatrix}$$

The vector complement of $\vec{v}$ orthogonal to L is the vector $\vec{v} - \text{Proj}_L(\vec{v})$.

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} -4 \\ 0 \end{bmatrix} - \begin{bmatrix} -3.6 \\ 1.2 \end{bmatrix}$$

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} -4 - (-3.6) \\ 0 - 1.2 \end{bmatrix}$$

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} -0.4 \\ -1.2 \end{bmatrix}$$

■ 4. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix} -2 \\ 0 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 3 \\ -7 \end{bmatrix}$$

Solution:

The line L is given as all the scaled versions of the vector $\vec{x} = (-2,0)$. Then the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \vec{x}$$

$$\text{Proj}_L(\vec{v}) = \frac{\begin{bmatrix} 3 & -7 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix}}{\begin{bmatrix} -2 & 0 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix}} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \frac{3(-2) - 7(0)}{-2(-2) + 0(0)} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

$$\text{Proj}_L(\vec{v}) = \frac{-6}{4} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix} = -\frac{3}{2} \cdot \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

The vector complement of $\vec{v}$ orthogonal to L is the vector $\vec{v} - \text{Proj}_L(\vec{v})$.

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} 3 \\ -7 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} 3 - 3 \\ -7 - 0 \end{bmatrix}$$

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} 0 \\ -7 \end{bmatrix}$$

■ 5. Find the projection of $\vec{v}$ onto L.

$$L = \left\{ c \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$

Solution:

The line L is given as all the scaled versions of the vector $\vec{x} = (-2,1,3)$. Then the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}} \right) \vec{x}$$

$$\text{Proj}_L(\vec{v}) = \frac{\begin{bmatrix}0 & -1 & 2\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\3\end{bmatrix}}{\begin{bmatrix}-2 & 1 & 3\end{bmatrix} \cdot \begin{bmatrix}-2\\1\\3\end{bmatrix}} \cdot \begin{bmatrix}-2\\1\\3\end{bmatrix} = \frac{0(-2)-1(1)+2(3)}{-2(-2)+1(1)+3(3)} \cdot \begin{bmatrix}-2\\1\\3\end{bmatrix}$$

$$\text{Proj}_L(\vec{v}) = \frac{5}{14} \cdot \begin{bmatrix}-2\\1\\3\end{bmatrix} = \begin{bmatrix}-\frac{5}{7}\\ \frac{5}{14}\\ \frac{15}{14}\end{bmatrix}$$

■ 6. Find the projection of $\vec{v}$ onto L and the vector complement of $\vec{v}$ orthogonal to L.

$$L = \left\{ c \begin{bmatrix}-4\\0\\-1\end{bmatrix} \middle| \, c \in \mathbb{R} \right\}$$

$$\vec{v} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$$

Solution:

The line L is given as all the scaled versions of the vector $\vec{x} = (-4,0,-1)$. Then the projection of $\vec{v}$ onto L is given by

$$\text{Proj}_L(\vec{v}) = \left(\frac{\vec{v} \cdot \vec{x}}{\vec{x} \cdot \vec{x}}\right)\vec{x}$$

$$\text{Proj}_L(\vec{v}) = \frac{\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix}}{\begin{bmatrix} -4 & 0 & -1 \end{bmatrix} \cdot \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix}} \cdot \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} = \frac{2(-4) - 1(0) + 1(-1)}{-4(-4) + 0(0) - 1(-1)} \cdot \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{Proj}_L(\vec{v}) = \frac{-9}{17} \cdot \begin{bmatrix} -4 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{36}{17} \\ 0 \\ \frac{9}{17} \end{bmatrix}$$

The vector complement of $\vec{v}$ orthogonal to L is the vector $\vec{v} - \text{Proj}_L(\vec{v})$.

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{36}{17} \\ 0 \\ \frac{9}{17} \end{bmatrix}$$

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} 2 - \frac{36}{17} \\ -1 - 0 \\ 1 - \frac{9}{17} \end{bmatrix}$$

$$\vec{v} - \text{Proj}_L(\vec{v}) = \begin{bmatrix} -\frac{2}{17} \\ -1 \\ \frac{8}{17} \end{bmatrix}$$

COMPOSITIONS OF LINEAR TRANSFORMATIONS

■ 1. If $S: \mathbb{R}^2 \to \mathbb{R}^2$ and $T: \mathbb{R}^2 \to \mathbb{R}^2$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix} -x_2 + 3x_1 \\ x_1 + 2x_2 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} x_2 \\ x_1 - 3x_2 \end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Solution:

Apply the transformation S to each column of the I_2 identity matrix.

$$S\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -0 + 3(1) \\ 1 + 2(0) \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -1 + 3(0) \\ 0 + 2(1) \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x}) = \begin{bmatrix} 3 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}$$

Apply the transformation T to each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}0\\1-3(0)\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}1\\0-3(1)\end{bmatrix}=\begin{bmatrix}1\\-3\end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x})=\begin{bmatrix}0&1\\1&-3\end{bmatrix}\vec{x}$$

Then the composition $T\circ S$ can be written as

$$T(S(\vec{x}))=\begin{bmatrix}0&1\\1&-3\end{bmatrix}\begin{bmatrix}3&-1\\1&2\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0(3)+1(1)&0(-1)+1(2)\\1(3)-3(1)&1(-1)-3(2)\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0+1&0+2\\3-3&-1-6\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}1&2\\0&-7\end{bmatrix}\vec{x}$$

Transform $\vec{x}=(-2,4)$.

$$T\left(S\left(\begin{bmatrix}-2\\4\end{bmatrix}\right)\right)=\begin{bmatrix}1&2\\0&-7\end{bmatrix}\begin{bmatrix}-2\\4\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\4\end{bmatrix}\right)\right)=\begin{bmatrix}1(-2)+2(4)\\0(-2)-7(4)\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\4\end{bmatrix}\right)\right)=\begin{bmatrix}-2+8\\0-28\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\4\end{bmatrix}\right)\right)=\begin{bmatrix}6\\-28\end{bmatrix}$$

Therefore, we can say that the vector $\vec{x}=(-2,4)$ in $\mathbb{R}^2$ is transformed into the vector $\vec{z}=(6,-28)$ in $\mathbb{R}^2$.

■ 2. If $S:\mathbb{R}^2\to\mathbb{R}^2$ and $T:\mathbb{R}^2\to\mathbb{R}^2$, then what is $S(T(\vec{x}))$?

$$S(\vec{x})=\begin{bmatrix}-2x_1+x_2\\-x_1-x_2\end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}-x_1+3x_2\\-2x_1+2x_2\end{bmatrix}$$

$$\vec{x}=\begin{bmatrix}-1\\2\end{bmatrix}$$

Solution:

Apply the transformation S to each column of the I_2 identity matrix.

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-2(1)+0\\-1-0\end{bmatrix}=\begin{bmatrix}-2\\-1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-2(0)+1\\-0-1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x})=\begin{bmatrix}-2&1\\-1&-1\end{bmatrix}\vec{x}$$

Apply the transformation T to each column of the I_2 identity matrix.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1+3(0)\\-2(1)+2(0)\end{bmatrix}=\begin{bmatrix}-1\\-2\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0+3(1)\\-2(0)+2(1)\end{bmatrix}=\begin{bmatrix}3\\2\end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x})=\begin{bmatrix}-1&3\\-2&2\end{bmatrix}\vec{x}$$

Then the composition $S\circ T$ can be written as

$$S(T(\vec{x}))=\begin{bmatrix}-2&1\\-1&-1\end{bmatrix}\begin{bmatrix}-1&3\\-2&2\end{bmatrix}\vec{x}$$

$$S(T(\vec{x}))=\begin{bmatrix}-2(-1)+1(-2)&-2(3)+1(2)\\-1(-1)-1(-2)&-1(3)-1(2)\end{bmatrix}\vec{x}$$

$$S(T(\vec{x}))=\begin{bmatrix}2-2&-6+2\\1+2&-3-2\end{bmatrix}\vec{x}$$

$$S(T(\vec{x}))=\begin{bmatrix}0&-4\\3&-5\end{bmatrix}\vec{x}$$

Transform $\vec{x} = (-1,2)$.

$$S\left(T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)\right) = \begin{bmatrix}0 & -4\\3 & -5\end{bmatrix}\begin{bmatrix}-1\\2\end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)\right) = \begin{bmatrix}0(-1)-4(2)\\3(-1)-5(2)\end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)\right) = \begin{bmatrix}0-8\\-3-10\end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}-1\\2\end{bmatrix}\right)\right) = \begin{bmatrix}-8\\-13\end{bmatrix}$$

Therefore, we can say that the vector $\vec{x} = (-1,2)$ in $\mathbb{R}^2$ is transformed into the vector $\vec{z} = (-8,-13)$ in $\mathbb{R}^2$.

■ 3. If $S : \mathbb{R}^3 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^3$, then what is $T(S(\vec{x}))$?

$$S(\vec{x}) = \begin{bmatrix}-2x_1 + x_2 - x_3\\-x_2 + x_3\\x_1 + 2x_2\end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix}-3x_3\\x_1 + x_2 - 2x_3\\-x_1 - x_2 + x_3\end{bmatrix}$$

$$\vec{x} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$$

Solution:

Apply the transformation S to each column of the I_3 identity matrix.

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2(1) + 0 - 0 \\ -0 + 0 \\ 1 + 2(0) \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2(0) + 1 - 0 \\ -1 + 0 \\ 0 + 2(1) \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2(0) + 0 - 1 \\ -0 + 1 \\ 0 + 2(0) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x}) = \begin{bmatrix} -2 & 1 & -1 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix} \vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -3(0) \\ 1 + 0 - 2(0) \\ -1 - 0 + 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-3(0)\\0+1-2(0)\\-0-1+0\end{bmatrix}=\begin{bmatrix}0\\1\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-3(1)\\0+0-2(1)\\-0-0+1\end{bmatrix}=\begin{bmatrix}-3\\-2\\1\end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x})=\begin{bmatrix}0&0&-3\\1&1&-2\\-1&-1&1\end{bmatrix}\vec{x}$$

Then the composition $T\circ S$ can be written as

$$T(S(\vec{x}))=\begin{bmatrix}0&0&-3\\1&1&-2\\-1&-1&1\end{bmatrix}\begin{bmatrix}-2&1&-1\\0&-1&1\\1&2&0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0(-2)+0(0)-3(1)&0(1)+0(-1)-3(2)&0(-1)+0(1)-3(0)\\1(-2)+1(0)-2(1)&1(1)+1(-1)-2(2)&1(-1)+1(1)-2(0)\\-1(-2)-1(0)+1(1)&-1(1)-1(-1)+1(2)&-1(-1)-1(1)+1(0)\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}0+0-3&0+0-6&0+0+0\\-2+0-2&1-1-4&-1+1-0\\2-0+1&-1+1+2&1-1+0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-3&-6&0\\-4&-4&0\\3&2&0\end{bmatrix}\vec{x}$$

Transform $\vec{x}=(-2,1,-3)$.

$$T\left(S\left(\begin{bmatrix}-2\\1\\-3\end{bmatrix}\right)\right)=\begin{bmatrix}-3&-6&0\\-4&-4&0\\3&2&0\end{bmatrix}\begin{bmatrix}-2\\1\\-3\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\1\\-3\end{bmatrix}\right)\right)=\begin{bmatrix}-3(-2)-6(1)+0(-3)\\-4(-2)-4(1)+0(-3)\\3(-2)+2(1)+0(-3)\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\1\\-3\end{bmatrix}\right)\right)=\begin{bmatrix}6-6+0\\8-4+0\\-6+2+0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\1\\-3\end{bmatrix}\right)\right)=\begin{bmatrix}0\\4\\-4\end{bmatrix}$$

Therefore, we can say that the vector $\vec{x}=(-2,1,-3)$ in $\mathbb{R}^3$ is transformed into the vector $\vec{z}=(0,4,-4)$ in $\mathbb{R}^3$.

■ 4. If $S:\mathbb{R}^2\to\mathbb{R}^3$ and $T:\mathbb{R}^3\to\mathbb{R}^2$, then what is $S(T(\vec{x}))$?

$$S(\vec{x})=\begin{bmatrix}-x_1\\x_1-3x_2\\2x_2-3x_1\end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}2x_1-x_3\\x_2-x_1+x_3\end{bmatrix}$$

$$\vec{x}=\begin{bmatrix}1\\-2\\-1\end{bmatrix}$$

Solution:

Apply the transformation S to each column of the I_2 identity matrix.

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1\\1-3(0)\\2(0)-3(1)\end{bmatrix}=\begin{bmatrix}-1\\1\\-3\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0\\0-3(1)\\2(1)-3(0)\end{bmatrix}=\begin{bmatrix}0\\-3\\2\end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x})=\begin{bmatrix}-1&0\\1&-3\\-3&2\end{bmatrix}\vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}2(1)-0\\0-1+0\end{bmatrix}=\begin{bmatrix}2\\-1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}2(0)-0\\1-0+0\end{bmatrix}=\begin{bmatrix}0\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}2(0)-1\\0-0+1\end{bmatrix}=\begin{bmatrix}-1\\1\end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x}) = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \vec{x}$$

Then the composition $S \circ T$ can be written as

$$S(T(\vec{x})) = \begin{bmatrix} -1 & 0 \\ 1 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} -1(2)+0(-1) & -1(0)+0(1) & -1(-1)+0(1) \\ 1(2)-3(-1) & 1(0)-3(1) & 1(-1)-3(1) \\ -3(2)+2(-1) & -3(0)+2(1) & -3(-1)+2(1) \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} -2+0 & 0+0 & 1+0 \\ 2+3 & 0-3 & -1-3 \\ -6-2 & 0+2 & 3+2 \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} -2 & 0 & 1 \\ 5 & -3 & -4 \\ -8 & 2 & 5 \end{bmatrix} \vec{x}$$

Transform $\vec{x} = (1, -2, -1)$.

$$S\left(T\left(\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -2 & 0 & 1 \\ 5 & -3 & -4 \\ -8 & 2 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -2(1)+0(-2)+1(-1) \\ 5(1)-3(-2)-4(-1) \\ -8(1)+2(-2)+5(-1) \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}1\\-2\\-1\end{bmatrix}\right)\right)=\begin{bmatrix}-2+0-1\\5+6+4\\-8-4-5\end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}1\\-2\\-1\end{bmatrix}\right)\right)=\begin{bmatrix}-3\\15\\-17\end{bmatrix}$$

Therefore, we can say that the vector $\vec{x}=(1,-2,-1)$ in $\mathbb{R}^3$ is transformed into the vector $\vec{z}=(-3,15,-17)$ in $\mathbb{R}^3$.

■ 5. If $S:\mathbb{R}^2\to\mathbb{R}^3$ and $T:\mathbb{R}^3\to\mathbb{R}^2$, then what is $T(S(\vec{x}))$?

$$S(\vec{x})=\begin{bmatrix}-x_1+x_2\\2x_2-3x_1\\x_1+2x_2\end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}x_1-2x_2+x_3\\x_1+x_2-x_3\end{bmatrix}$$

$$\vec{x}=\begin{bmatrix}-2\\-1\end{bmatrix}$$

Solution:

Apply the transformation S to each column of the I_2 identity matrix.

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-1+0\\2(0)-3(1)\\1+2(0)\end{bmatrix}=\begin{bmatrix}-1\\-3\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-0+1\\2(1)-3(0)\\0+2(1)\end{bmatrix}=\begin{bmatrix}1\\2\\2\end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x})=\begin{bmatrix}-1&1\\-3&2\\1&2\end{bmatrix}\vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}1-2(0)+0\\1+0-0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}0-2(1)+0\\0+1-0\end{bmatrix}=\begin{bmatrix}-2\\1\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}0-2(0)+1\\0+0-1\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x})=\begin{bmatrix}1&-2&1\\1&1&-1\end{bmatrix}\vec{x}$$

Then the composition $T\circ S$ can be written as

$$T(S(\vec{x})) = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ -3 & 2 \\ 1 & 2 \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} 1(-1) - 2(-3) + 1(1) & 1(1) - 2(2) + 1(2) \\ 1(-1) + 1(-3) - 1(1) & 1(1) + 1(2) - 1(2) \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} -1 + 6 + 1 & 1 - 4 + 2 \\ -1 - 3 - 1 & 1 + 2 - 2 \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} \vec{x}$$

Transform $\vec{x} = (-2, -1)$.

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} 6 & -1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} 6(-2) - 1(-1) \\ -5(-2) + 1(-1) \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -12 + 1 \\ 10 - 1 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} -2 \\ -1 \end{bmatrix}\right)\right) = \begin{bmatrix} -11 \\ 9 \end{bmatrix}$$

Therefore, we can say that the vector $\vec{x} = (-2, -1)$ in $\mathbb{R}^2$ is transformed into the vector $\vec{z} = (-11,9)$ in $\mathbb{R}^2$.

6. If $S:\mathbb{R}^3\to\mathbb{R}^3$ and $T:\mathbb{R}^3\to\mathbb{R}^3$, then what are $T(S(\vec{x}))$ and $S(T(\vec{x}))$?

$$S(\vec{x})=\begin{bmatrix}-x_3+2x_2\\x_1-x_3\\x_1\end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}-2x_1+x_2+2x_3\\3x_1\\x_1-2x_2+x_3\end{bmatrix}$$

$$\vec{x}=\begin{bmatrix}0\\-1\\3\end{bmatrix}$$

Solution:

Apply the transformation S to each column of the I_3 identity matrix.

$$S\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)=\begin{bmatrix}-0+2(0)\\1-0\\1\end{bmatrix}=\begin{bmatrix}0\\1\\1\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right)=\begin{bmatrix}-0+2(1)\\0-0\\0\end{bmatrix}=\begin{bmatrix}2\\0\\0\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)=\begin{bmatrix}-1+2(0)\\0-1\\0\end{bmatrix}=\begin{bmatrix}-1\\-1\\0\end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x}) = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2(1) + 0 + 2(0) \\ 3(1) \\ 1 - 2(0) + 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -2(0) + 1 + 2(0) \\ 3(0) \\ 0 - 2(1) + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -2(0) + 0 + 2(1) \\ 3(0) \\ 0 - 2(0) + 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x}) = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \vec{x}$$

Then the composition $T \circ S$ can be written as

$$T(S(\vec{x})) = \begin{bmatrix} -2 & 1 & 2 \\ 3 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} -2(0) + 1(1) + 2(1) & -2(2) + 1(0) + 2(0) & -2(-1) + 1(-1) + 2(0) \\ 3(0) + 0(1) + 0(1) & 3(2) + 0(0) + 0(0) & 3(-1) + 0(-1) + 0(0) \\ 1(0) - 2(1) + 1(1) & 1(2) - 2(0) + 1(0) & 1(-1) - 2(-1) + 1(0) \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} 0+1+2 & -4+0+0 & 2-1+0 \\ 0+0+0 & 6+0+0 & -3+0+0 \\ 0-2+1 & 2-0+0 & -1+2+0 \end{bmatrix} \vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 6 & -3 \\ -1 & 2 & 1 \end{bmatrix} \vec{x}$$

And the composition $S \circ T$ can be written as

$$S(T(\vec{x})) = \begin{bmatrix} 0 & 2 & -1 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} -2 & 1 & 2 \\ 3 & 0 & 0 \\ 1 & -2 & 1 \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} 0(-2)+2(3)-1(1) & 0(1)+2(0)-1(-2) & 0(2)+2(0)-1(1) \\ 1(-2)+0(3)-1(1) & 1(1)+0(0)-1(-2) & 1(2)+0(0)-1(1) \\ 1(-2)+0(3)+0(1) & 1(1)+0(0)+0(-2) & 1(2)+0(0)+0(1) \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} 0+6-1 & 0+0+2 & 0+0-1 \\ -2+0-1 & 1+0+2 & 2+0-1 \\ -2+0+0 & 1+0+0 & 2+0+0 \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} 5 & 2 & -1 \\ -3 & 3 & 1 \\ -2 & 1 & 2 \end{bmatrix} \vec{x}$$

Then $T(S(\vec{x}))$, where $\vec{x} = (0, -1, 3)$, is

$$T\left(S\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}\right)\right) = \begin{bmatrix} 3 & -4 & 1 \\ 0 & 6 & -3 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix} 0 \\ -1 \\ 3 \end{bmatrix}\right)\right) = \begin{bmatrix} 3(0)-4(-1)+1(3) \\ 0(0)+6(-1)-3(3) \\ -1(0)+2(-1)+1(3) \end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}0\\-1\\3\end{bmatrix}\right)\right)=\begin{bmatrix}0+4+3\\0-6-9\\0-2+3\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}0\\-1\\3\end{bmatrix}\right)\right)=\begin{bmatrix}7\\-15\\1\end{bmatrix}$$

And $S(T(\overrightarrow{x}))$, where $\overrightarrow{x}=(0,-1,3)$, is

$$S\left(T\left(\begin{bmatrix}0\\-1\\3\end{bmatrix}\right)\right)=\begin{bmatrix}5&2&-1\\-3&3&1\\-2&1&2\end{bmatrix}\begin{bmatrix}0\\-1\\3\end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}0\\-1\\3\end{bmatrix}\right)\right)=\begin{bmatrix}5(0)+2(-1)-1(3)\\-3(0)+3(-1)+1(3)\\-2(0)+1(-1)+2(3)\end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}0\\-1\\3\end{bmatrix}\right)\right)=\begin{bmatrix}0-2-3\\0-3+3\\0-1+6\end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix}0\\-1\\3\end{bmatrix}\right)\right)=\begin{bmatrix}-5\\0\\5\end{bmatrix}$$

INVERSE OF A TRANSFORMATION

■ 1. Given a vector $\vec{v}$ in $\mathbb{R}^3$, what would the identity transformation be?

Solution:

In $\mathbb{R}^3$, the identity transformation would be written as $I : \mathbb{R}^3 \to \mathbb{R}^3$, or maybe as $I_{\mathbb{R}^3}(\vec{v}) = \vec{v}$.

■ 2. If a transformation T is invertible, what are the three conclusions that we can make about it?

Solution:

If a transformation T is invertible, we can conclude that

1. its inverse transformation is unique,
2. T is injective (or one-to-one), and
3. T is surjective (or onto).

■ 3. If you can prove that a transformation T is both injective and surjective, and if you know that its inverse is unique, then what can you say about the transformation?

Solution:

You know the transformation is invertible.

■ 4. Is the transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ invertible?

$$T(x) = x^2$$

Solution:

The transform says that, given a value of x, the transform will return a value of x^2. So for instance, if we put in $x = 2$, the transform will return $x^2 = 2^2 = 4$. But if we put in $x = -2$, the transform will also return $x^2 = (-2)^2 = 4$. In other words, the transform can use two different values of x and return the same value for both, which means the transformation isn't one-to-one, and therefore can't be invertible.

■ 5. Prove that $(T^{-1})^{-1} = T$.

Solution:

We know that any transformation multiplied by the identity transformation will simply give us back the original transformation.

$$(T^{-1})^{-1} = (T^{-1})^{-1}I$$

We also know that the identity transformation is equal to an inverse transformation multiplied by itself, $I = T^{-1}T$, so we can write

$$(T^{-1})^{-1} = (T^{-1})^{-1}(T^{-1}T)$$

Rearrange the parentheses.

$$(T^{-1})^{-1} = [(T^{-1})^{-1}T^{-1}]T$$

Pull out the inverse, switching the order.

$$(T^{-1})^{-1} = [T(T^{-1})]^{-1}T$$

$$(T^{-1})^{-1} = [TT^{-1}]^{-1}T$$

Then the result in the parentheses is just the identity transformation.

$$(T^{-1})^{-1} = [I]^{-1}T$$

$$(T^{-1})^{-1} = IT$$

$$(T^{-1})^{-1} = T$$

■ 6. Prove that the inverse of a transformation is unique.

Solution:

Let's assume that the inverse of a transformation is actually *not* unique, such that there's an invertible transformation T that has two unique inverses T_1^{-1} and T_2^{-1}, and $T_1^{-1} \neq T_2^{-1}$.

If this were true, it means that $TT_1^{-1} = T_1^{-1}T = I$ and $TT_2^{-1} = T_2^{-1}T = I$, because a transformation multiplied by its inverse will give you the identity transformation.

What we want to show is that, in fact, $T_1^{-1} = T_2^{-1}$, which will mean that our initial assumption that they're two *unique* inverses will be wrong. Let's start with T_1^{-1}:

$$T_1^{-1} = T_1^{-1}I$$

$$T_1^{-1} = T_1^{-1}(TT_2^{-1})$$

$$T_1^{-1} = (T_1^{-1}T)T_2^{-1}$$

$$T_1^{-1} = IT_2^{-1}$$

$$T_1^{-1} = T_2^{-1}$$

We've shown that $T_1^{-1} = T_2^{-1}$, even though our initial assumption was that $T_1^{-1} \neq T_2^{-1}$. Therefore, we know that the inverse of a transformation must always be unique.

INVERTIBILITY FROM THE MATRIX-VECTOR PRODUCT

■ 1. Is the matrix invertible?

$$\begin{bmatrix} 1 & 2 & 0 \\ -3 & 5 & -1 \end{bmatrix}$$

Solution:

The matrix isn't square, so it can't be invertible.

■ 2. Is the matrix invertible?

$$\begin{bmatrix} \pi & -\pi \\ -\pi & \pi \end{bmatrix}$$

Solution:

Divide through the matrix by π,

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

Because we don't have a pivot entry in every row, the matrix is not invertible.

■ 3. Is the matrix invertible?

$$\begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix}$$

Solution:

Divide through the matrix by π,

$$\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because we were able to get the matrix into reduced row-echelon form, the matrix is invertible.

■ 4. Find the dimensions of the transformation matrix for each transformation, if each transformation were written as a matrix-vector product, $T(\vec{x}) = M\vec{x}$.

$$T: \mathbb{R}^3 \to \mathbb{R}^6$$

$$T : \mathbb{R}^4 \to \mathbb{R}^2$$

$$T : \mathbb{R}^u \to \mathbb{R}^w$$

Solution:

If $T : \mathbb{R}^3 \to \mathbb{R}^6$ were written as $T(\vec{x}) = M\vec{x}$, then M would be a 6×3 matrix.

If $T : \mathbb{R}^4 \to \mathbb{R}^2$ were written as $T(\vec{x}) = M\vec{x}$, then M would be a 2×4 matrix.

If $T : \mathbb{R}^u \to \mathbb{R}^w$ were written as $T(\vec{x}) = M\vec{x}$, then M would be a $w \times u$ matrix.

■ 5. Using the transformations from the previous question, state the dimensions of $\vec{x}$, and then state the dimensions of $T(\vec{x})$.

Solution:

For the transformation $\mathbb{R}^3 \to \mathbb{R}^6$, $\vec{x}$ must be a 3×1 vector. We know that a 6×3 matrix multiplied by a 3×1 vector will return a 6×1 vector, so $T(\vec{x})$ is a 6×1 vector.

For the transformation $T : \mathbb{R}^4 \to \mathbb{R}^2$, $\vec{x}$ must be a 4×1 vector. We know that a 2×4 matrix multiplied by a 4×1 vector will return a 2×1 vector, so $T(\vec{x})$ is a 2×1 vector.

For the transformation $T : \mathbb{R}^u \to \mathbb{R}^w$, $\vec{x}$ must be a $u \times 1$ vector. We know that a $w \times u$ matrix multiplied by a $u \times 1$ vector will return a $w \times 1$ vector, so $T(\vec{x})$ is a $w \times 1$ vector.

6. What can we say about the invertibility of the transformation $T : \mathbb{R}^u \to \mathbb{R}^w$ from the last two questions?

Solution:

We actually can't tell whether $T : \mathbb{R}^u \to \mathbb{R}^w$ is invertible. If $w = u$, then the transformation matrix is square, and there's then a possibility that the matrix, and therefore the transformation, is invertible. If $u \neq w$, then the transformation matrix isn't square, and the transformation is definitely not invertible.

INVERSE TRANSFORMATIONS ARE LINEAR

■ 1. Given two $n \times n$ matrices, A and B, if we know that $AB = I$ and $BA = I$, where I is the $n \times n$ identity matrix, then what else do we know about A and B?

Solution:

We know that A and B must be inverses of one another.

■ 2. Find the inverse of the matrix.

$$\begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix}$$

Solution:

First, set up the augmented matrix,

$$\begin{bmatrix} \pi & -\pi & | & 1 & 0 \\ \pi & \pi & | & 0 & 1 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & | & \frac{1}{\pi} & 0 \\ 1 & 1 & | & 0 & \frac{1}{\pi} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & | & \frac{1}{\pi} & 0 \\ 0 & 2 & | & -\frac{1}{\pi} & \frac{1}{\pi} \end{bmatrix} \to \begin{bmatrix} 1 & -1 & | & \frac{1}{\pi} & 0 \\ 0 & 1 & | & -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{1}{2\pi} & \frac{1}{2\pi} \\ 0 & 1 & | & -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix}$$

So the inverse matrix is

$$\begin{bmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} \\ -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix}$$

■ 3. Prove that the matrix found in the previous question is actually the inverse of the original matrix.

Solution:

To prove that the matrices are inverses of one another, multiply them to show that we get the identity matrix. The product of the original matrix by the inverse gives

$$\begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix} \begin{bmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} \\ -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix} = \begin{bmatrix} \pi\left(\frac{1}{2\pi}\right) - \pi\left(-\frac{1}{2\pi}\right) & \pi\left(\frac{1}{2\pi}\right) - \pi\left(\frac{1}{2\pi}\right) \\ \pi\left(\frac{1}{2\pi}\right) + \pi\left(-\frac{1}{2\pi}\right) & \pi\left(\frac{1}{2\pi}\right) + \pi\left(\frac{1}{2\pi}\right) \end{bmatrix}$$

$$\begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix} \begin{bmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} \\ -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & \frac{1}{2} - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix} \begin{bmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} \\ -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

And the product of the inverse matrix by the original gives

$$\begin{bmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} \\ -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix} \begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix} = \begin{bmatrix} \frac{1}{2\pi}\pi + \frac{1}{2\pi}\pi & \frac{1}{2\pi}(-\pi) + \frac{1}{2\pi}\pi \\ -\frac{1}{2\pi}\pi + \frac{1}{2\pi}\pi & -\frac{1}{2\pi}(-\pi) + \frac{1}{2\pi}\pi \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} \\ -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix} \begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix} = \begin{bmatrix} \frac{1}{2} + \frac{1}{2} & -\frac{1}{2} + \frac{1}{2} \\ -\frac{1}{2} + \frac{1}{2} & \frac{1}{2} + \frac{1}{2} \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{2\pi} & \frac{1}{2\pi} \\ -\frac{1}{2\pi} & \frac{1}{2\pi} \end{bmatrix} \begin{bmatrix} \pi & -\pi \\ \pi & \pi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

4. Find the inverse of the matrix.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{bmatrix}$$

Solution:

Set up the augmented matrix,

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 5 & | & 0 & 1 & 0 \\ 5 & 6 & 0 & | & 0 & 0 & 1 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 5 & | & 0 & 1 & 0 \\ 5 & 6 & 0 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & | & 1 & 0 & 0 \\ 0 & 1 & 5 & | & 0 & 1 & 0 \\ 0 & -4 & -15 & | & -5 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -7 & | & 1 & -2 & 0 \\ 0 & 1 & 5 & | & 0 & 1 & 0 \\ 0 & -4 & -15 & | & -5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7 & | & 1 & -2 & 0 \\ 0 & 1 & 5 & | & 0 & 1 & 0 \\ 0 & 0 & 5 & | & -5 & 4 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -7 & | & 1 & -2 & 0 \\ 0 & 1 & 5 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & \frac{4}{5} & \frac{1}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -6 & \frac{18}{5} & \frac{7}{5} \\ 0 & 1 & 5 & | & 0 & 1 & 0 \\ 0 & 0 & 1 & | & -1 & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & -6 & \frac{18}{5} & \frac{7}{5} \\ 0 & 1 & 0 & | & 5 & -3 & -1 \\ 0 & 0 & 1 & | & -1 & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

Then the inverse matrix is

$$\begin{bmatrix} -6 & \frac{18}{5} & \frac{7}{5} \\ 5 & -3 & -1 \\ -1 & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

■ 5. Prove that the matrix we found in the previous question is actually the inverse of the original matrix.

Solution:

Multiply the original matrix by its inverse.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 5 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} -6 & \frac{18}{5} & \frac{7}{5} \\ 5 & -3 & -1 \\ -1 & \frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1(-6)+2(5)+3(-1) & 1\left(\frac{18}{5}\right)+2(-3)+3\left(\frac{4}{5}\right) & 1\left(\frac{7}{5}\right)+2(-1)+3\left(\frac{1}{5}\right) \\ 0(-6)+1(5)+5(-1) & 0\left(\frac{18}{5}\right)+1(-3)+5\left(\frac{4}{5}\right) & 0\left(\frac{7}{5}\right)+1(-1)+5\left(\frac{1}{5}\right) \\ 5(-6)+6(5)+0(-1) & 5\left(\frac{18}{5}\right)+6(-3)+0\left(\frac{4}{5}\right) & 5\left(\frac{7}{5}\right)+6(-1)+0\left(\frac{1}{5}\right) \end{bmatrix}$$

$$\begin{bmatrix} -6+10-3 & \frac{18}{5}-6+\frac{12}{5} & \frac{7}{5}-2+\frac{3}{5} \\ 0+5-5 & 0-3+4 & 0-1+1 \\ -30+30+0 & 18-18+0 & 7-6+0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we get the identity matrix, we know that the two matrices we multiplied together must be inverses of one another.

■ 6. Prove that the inverse of an invertible linear transformation T is also a linear transformation.

Solution:

We know that T is linear, which means that for vectors $\vec{x}$ and $\vec{y}$ and a constant c, we know $T(\vec{x}+\vec{y})=T(\vec{x})+T(\vec{y})$ and $T(c\vec{x})=cT(\vec{x})$. We also know that $(T^{-1}\circ T)(\vec{x})=(T\circ T^{-1})(\vec{x})=I(\vec{x})$, and we want to prove that $T^{-1}(\vec{x}+\vec{y})=T^{-1}(\vec{x})+T^{-1}(\vec{y})$ and $T^{-1}(c\vec{x})=cT^{-1}(\vec{x})$.

We know that $(T\circ T^{-1})(\vec{x}+\vec{y})$ is the same as saying $I(\vec{x}+\vec{y})$.

$$(T\circ T^{-1})(\vec{x}+\vec{y})=I(\vec{x}+\vec{y})$$

$$(T\circ T^{-1})(\vec{x}+\vec{y})=\vec{x}+\vec{y}$$

Using that same logic, we can also say that $\vec{x}+\vec{y}$ is the same as saying $I\vec{x}+I\vec{y}$, or $(T\circ T^{-1})\vec{x}+(T\circ T^{-1})\vec{y}$, which means that we now have

$$(T\circ T^{-1})(\vec{x}+\vec{y})=I(\vec{x}+\vec{y})$$

$$(T\circ T^{-1})(\vec{x}+\vec{y})=I\vec{x}+I\vec{y}$$

$$(T \circ T^{-1})(\overrightarrow{x} + \overrightarrow{y}) = (T \circ T^{-1})\overrightarrow{x} + (T \circ T^{-1})\overrightarrow{y}$$

If we rearrange this, we get

$$T[T^{-1}(\overrightarrow{x} + \overrightarrow{y})] = T[T^{-1}(\overrightarrow{x}) + T^{-1}(\overrightarrow{y})]$$

and if we apply T^{-1} to both sides, we get

$$T^{-1}(T[T^{-1}(\overrightarrow{x} + \overrightarrow{y})]) = T^{-1}(T[T^{-1}(\overrightarrow{x}) + T^{-1}(\overrightarrow{y})])$$

$$(I)[T^{-1}(\overrightarrow{x} + \overrightarrow{y})] = (I)[T^{-1}(\overrightarrow{x}) + T^{-1}(\overrightarrow{y})]$$

$$T^{-1}(\overrightarrow{x} + \overrightarrow{y}) = T^{-1}(\overrightarrow{x}) + T^{-1}(\overrightarrow{y})$$

MATRIX INVERSES, AND INVERTIBLE AND SINGULAR MATRICES

■ 1. Find the inverse of matrix G.

$$G = \begin{bmatrix} -3 & 8 \\ 0 & -2 \end{bmatrix}$$

Solution:

Plug into the formula for the inverse matrix.

$$G^{-1} = \frac{1}{|G|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$G^{-1} = \frac{1}{\begin{vmatrix} -3 & 8 \\ 0 & -2 \end{vmatrix}} \begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \frac{1}{(-3)(-2) - (8)(0)} \begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \frac{1}{6} \begin{bmatrix} -2 & -8 \\ 0 & -3 \end{bmatrix}$$

$$G^{-1} = \begin{bmatrix} -\frac{1}{3} & -\frac{4}{3} \\ 0 & -\frac{1}{2} \end{bmatrix}$$

■ 2. Find the inverse of matrix N.

$$N = \begin{bmatrix} 11 & -4 \\ 5 & -3 \end{bmatrix}$$

Solution:

Plug into the formula for the inverse matrix.

$$N^{-1} = \frac{1}{|N|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$N^{-1} = \frac{1}{\begin{vmatrix} 11 & -4 \\ 5 & -3 \end{vmatrix}} \begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \frac{1}{(11)(-3) - (-4)(5)} \begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \frac{1}{-33 + 20} \begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = -\frac{1}{13} \begin{bmatrix} -3 & 4 \\ -5 & 11 \end{bmatrix}$$

$$N^{-1} = \begin{bmatrix} \frac{3}{13} & -\frac{4}{13} \\ \frac{5}{13} & -\frac{11}{13} \end{bmatrix}$$

3. What is the inverse of matrix K?

$$K = \begin{bmatrix} 3 & 3 \\ -6 & 0 \end{bmatrix}$$

Solution:

Plug into the formula for the inverse matrix.

$$K^{-1} = \frac{1}{|K|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$K^{-1} = \frac{1}{\begin{vmatrix} 3 & 3 \\ -6 & 0 \end{vmatrix}} \begin{bmatrix} 0 & -3 \\ 6 & 3 \end{bmatrix}$$

$$K^{-1} = \frac{1}{(3)(0) - (3)(-6)} \begin{bmatrix} 0 & -3 \\ 6 & 3 \end{bmatrix}$$

$$K^{-1} = \frac{1}{0 + 18} \begin{bmatrix} 0 & -3 \\ 6 & 3 \end{bmatrix}$$

$$K^{-1} = \frac{1}{18} \begin{bmatrix} 0 & -3 \\ 6 & 3 \end{bmatrix}$$

$$K^{-1} = \begin{bmatrix} 0 & -\frac{1}{6} \\ \frac{1}{3} & \frac{1}{6} \end{bmatrix}$$

4. Is the matrix invertible or singular?

$$Z = \begin{bmatrix} 4 & 2 \\ -2 & -1 \end{bmatrix}$$

Solution:

Find the determinant of the matrix.

$$|Z| = \begin{vmatrix} 4 & 2 \\ -2 & -1 \end{vmatrix}$$

$$|Z| = (4)(-1) - (2)(-2)$$

$$|Z| = -4 + 4$$

$$|Z| = 0$$

Because the determinant is 0, Z is a singular matrix that has no inverse.

5. Is the matrix invertible or singular?

$$Y = \begin{bmatrix} 0 & 6 \\ 2 & -1 \end{bmatrix}$$

Solution:

Find the determinant of the matrix.

$$|Y| = \begin{vmatrix} 0 & 6 \\ 2 & -1 \end{vmatrix}$$

$$|Y| = (0)(-1) - (6)(2)$$

$$|Y| = 0 - 12$$

$$|Y| = -12$$

Because the determinant is non-zero, Y is an invertible matrix with a defined inverse.

6. Is B invertible?

$$B = \begin{bmatrix} -4 & 1 \\ -5 & 0 \end{bmatrix}$$

Solution:

Find the determinant of the matrix.

$$|B| = \begin{vmatrix} -4 & 1 \\ -5 & 0 \end{vmatrix}$$

$$|B| = (-4)(0) - (1)(-5)$$

$$|B| = 0 + 5$$

$$|B| = 5$$

Because the determinant is non-zero, B is an invertible matrix with a defined inverse.

SOLVING SYSTEMS WITH INVERSE MATRICES

■ 1. Use an inverse matrix to solve the system.

$$-4x + 3y = -14$$

$$7x - 4y = 32$$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} -4 & 3 \\ 7 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -14 \\ 32 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(-4)(-4) - (3)(7)} \begin{bmatrix} -4 & -3 \\ -7 & -4 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{5} \begin{bmatrix} -4 & -3 \\ -7 & -4 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{7}{5} & \frac{4}{5} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} \frac{4}{5} & \frac{3}{5} \\ \frac{7}{5} & \frac{4}{5} \end{bmatrix} \begin{bmatrix} -14 \\ 32 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{4}{5}(-14) + \frac{3}{5}(32) \\ \frac{7}{5}(-14) + \frac{4}{5}(32) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{56}{5} + \frac{96}{5} \\ -\frac{98}{5} + \frac{128}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{40}{5} \\ \frac{30}{5} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$$

■ 2. Use an inverse matrix to solve the system.

$$6x - 11y = 2$$

$$-10x + 7y = -26$$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} 6 & -11 \\ -10 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(6)(7) - (-11)(-10)} \begin{bmatrix} 7 & 11 \\ 10 & 6 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{68} \begin{bmatrix} 7 & 11 \\ 10 & 6 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{7}{68} & -\frac{11}{68} \\ -\frac{10}{68} & -\frac{6}{68} \end{bmatrix}$$

The solution to the system is

$$\vec{a} = \begin{bmatrix} -\frac{7}{68} & -\frac{11}{8} \\ -\frac{10}{68} & -\frac{6}{68} \end{bmatrix} \begin{bmatrix} 2 \\ -26 \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{7}{68}(2) - \frac{11}{68}(-26) \\ -\frac{10}{68}(2) - \frac{6}{68}(-26) \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} -\frac{14}{68} + \frac{286}{68} \\ -\frac{20}{68} + \frac{156}{68} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} \frac{272}{68} \\ \frac{136}{68} \end{bmatrix}$$

$$\vec{a} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

■ 3. Use an inverse matrix to solve the system.

$$13y - 6x = -81$$

$$7x + 17 = -22y$$

Solution:

Transfer the system into a matrix equation.

$$\begin{bmatrix} -6 & 13 \\ 7 & 22 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -81 \\ -17 \end{bmatrix}$$

Find the inverse of the coefficient matrix.

$$M^{-1} = \frac{1}{(-6)(22) - (13)(7)} \begin{bmatrix} 22 & -13 \\ -7 & -6 \end{bmatrix}$$

$$M^{-1} = -\frac{1}{223} \begin{bmatrix} 22 & -13 \\ -7 & -6 \end{bmatrix}$$

$$M^{-1} = \begin{bmatrix} -\frac{22}{223} & \frac{13}{223} \\ \frac{7}{223} & \frac{6}{223} \end{bmatrix}$$

The solution to the system is

$$\vec{a}=\begin{bmatrix}-\frac{22}{223} & \frac{13}{223}\\ \frac{7}{223} & \frac{6}{223}\end{bmatrix}\begin{bmatrix}-81\\ -17\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}-\frac{22}{223}(-81)+\frac{13}{223}(-17)\\ \frac{7}{223}(-81)+\frac{6}{223}(-17)\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}\frac{1,782}{223}-\frac{221}{223}\\ -\frac{567}{223}-\frac{102}{223}\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}\frac{1,561}{223}\\ -\frac{669}{223}\end{bmatrix}$$

$$\vec{a}=\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}7\\ -3\end{bmatrix}$$

■ 4. Sketch a graph of vectors to visually find the solution to the system.

$$3x=3$$

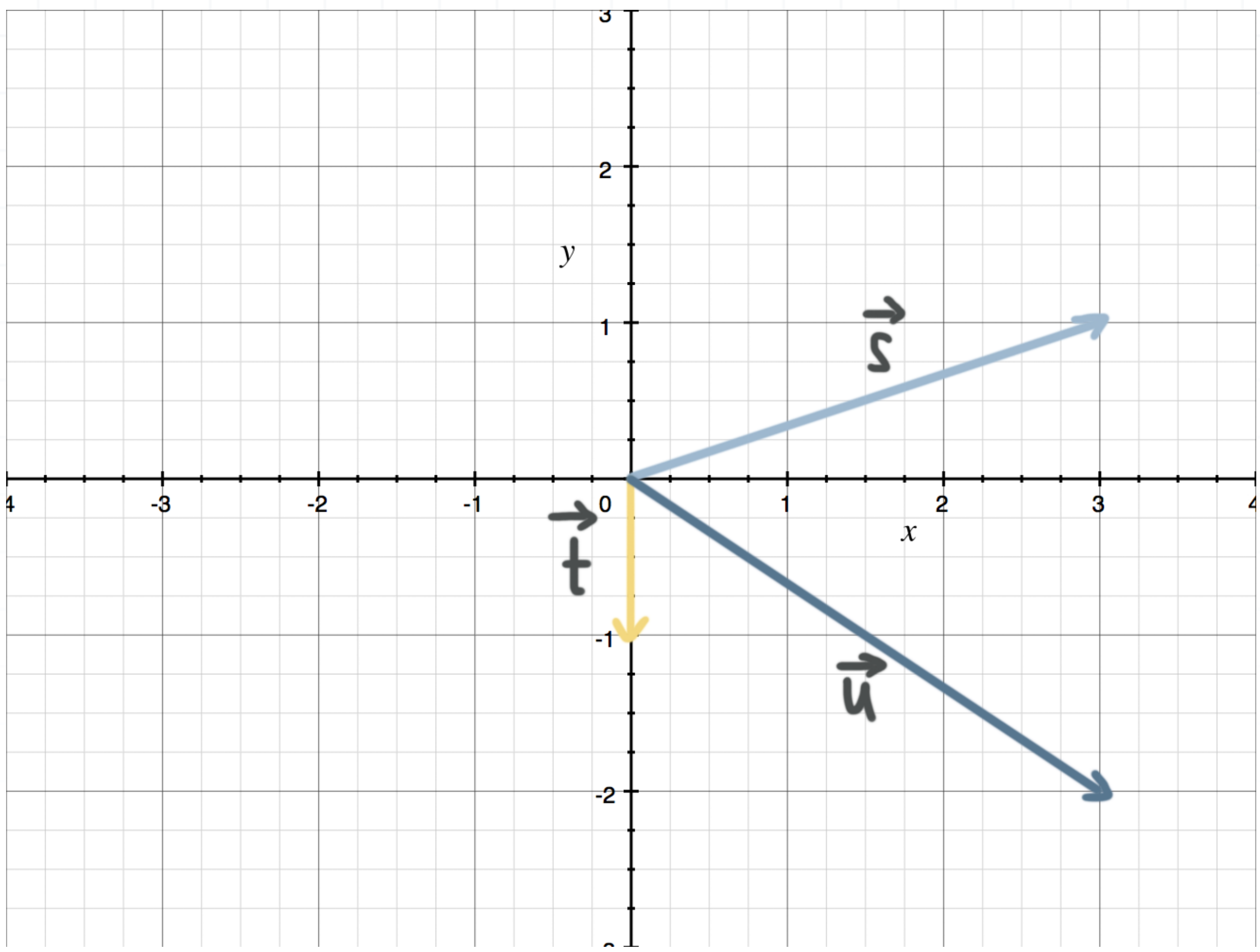
$$x-y=-2$$

Solution:

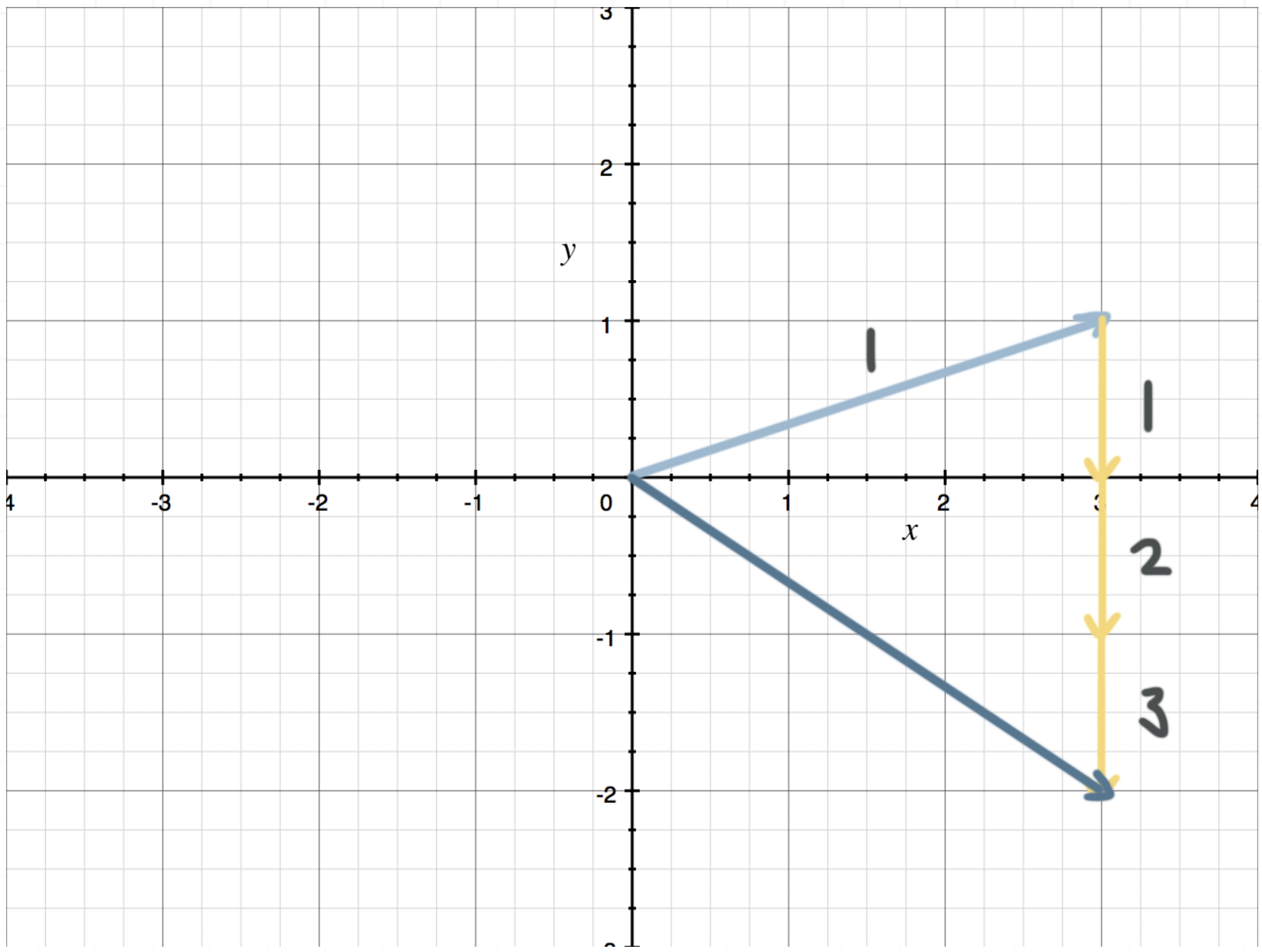
Put the system into a matrix equation.

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ -1 \end{bmatrix} y = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s} = (3,1)$ for x, $\vec{t} = (0,-1)$ for y, and the resulting vector $\vec{u} = (3,-2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and three $\vec{t}$s together will get us back to the terminal point of $\vec{u}$, so $x = 1$ and $y = 3$.



5. Sketch a graph of vectors to visually find the solution to the system.

$$-y = -4$$

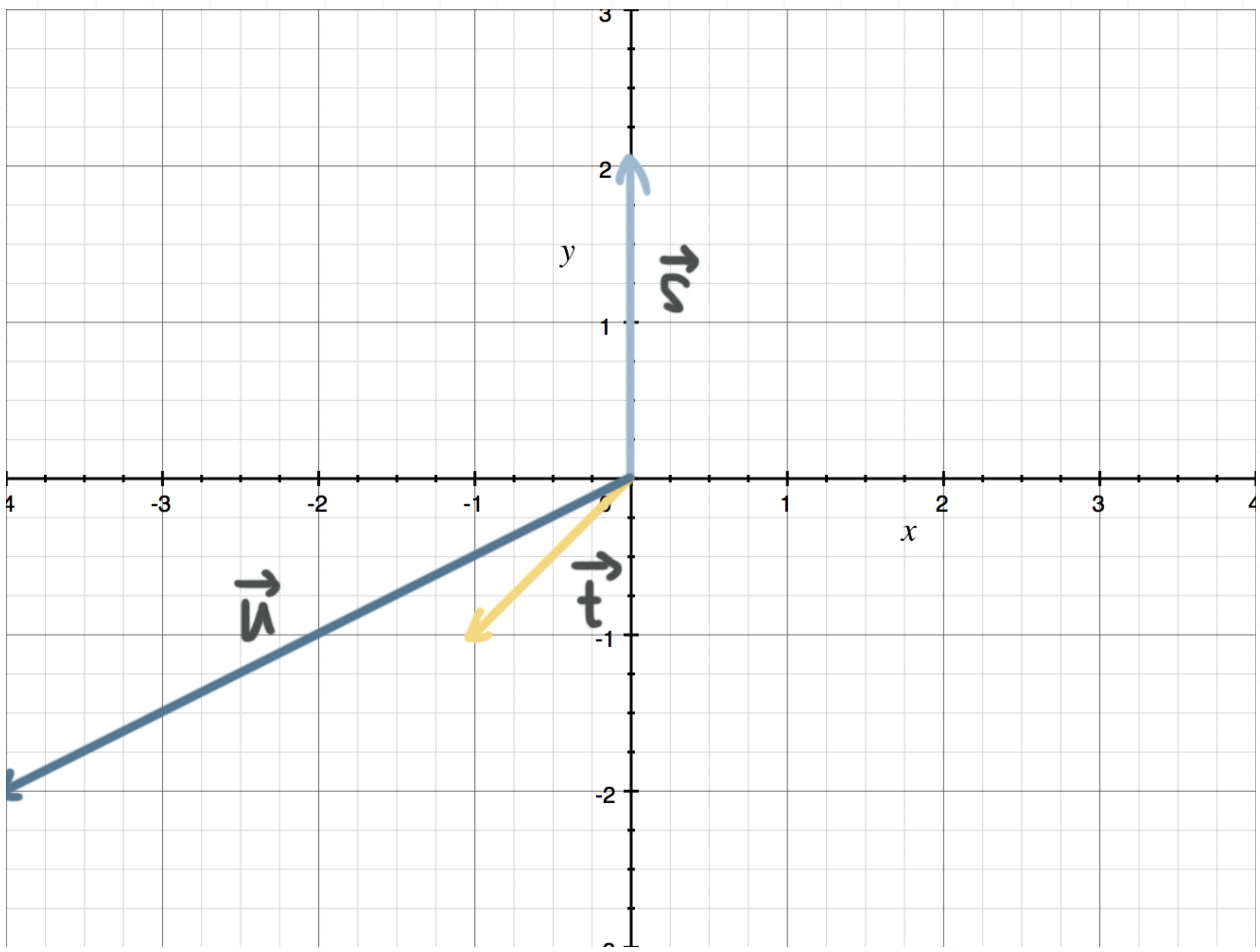
$$2x - y = -2$$

Solution:

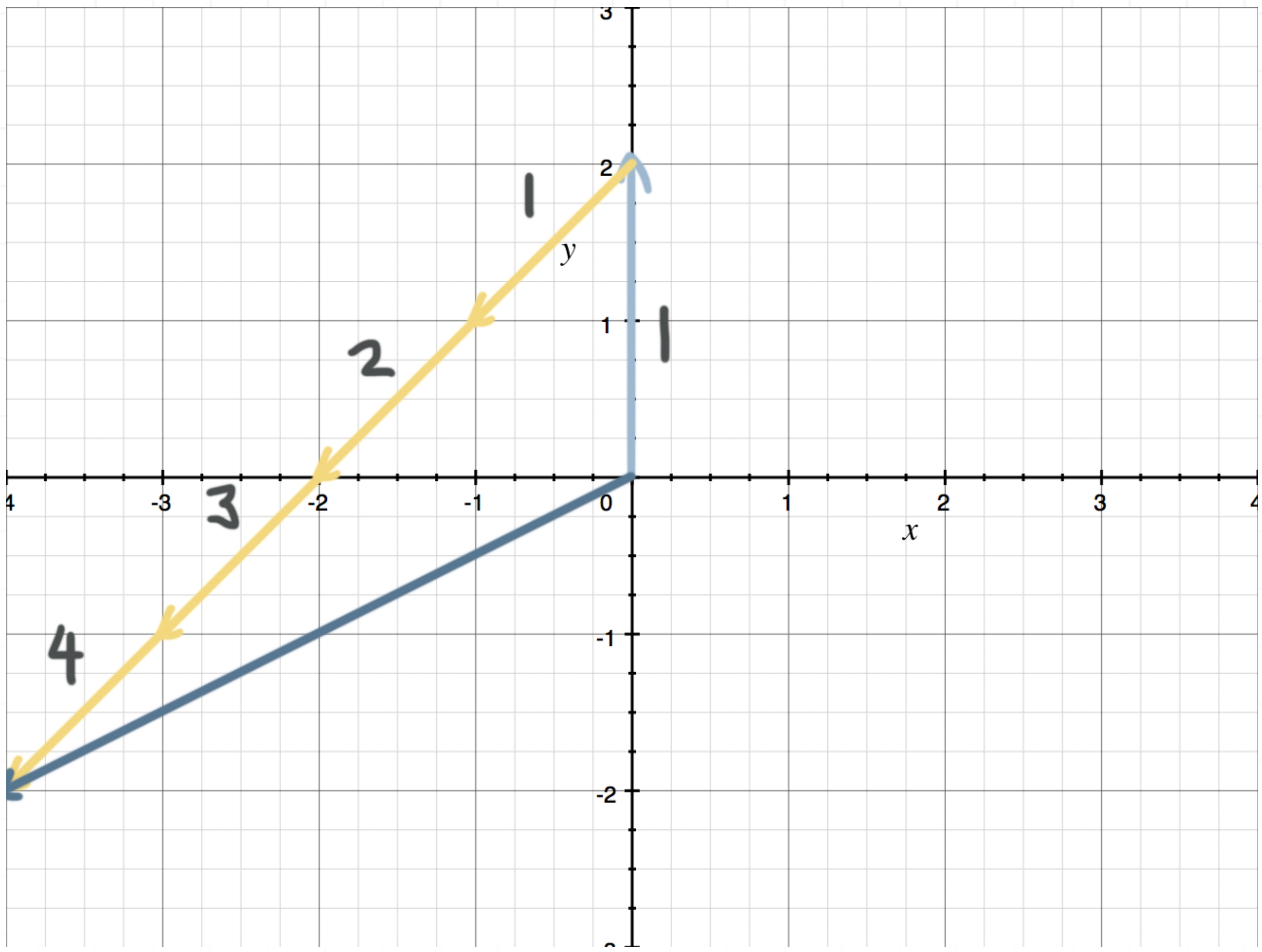
Put the system into a matrix equation.

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} x + \begin{bmatrix} -1 \\ -1 \end{bmatrix} y = \begin{bmatrix} -4 \\ -2 \end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s} = (0,2)$ for x, $\vec{t} = (-1, -1)$ for y, and the resulting vector $\overrightarrow{u} = (-4, -2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and four $\vec{t}$s together will get us back to the terminal point of $\overrightarrow{u}$, so $x = 1$ and $y = 4$.



6. Sketch a graph of vectors to visually find the solution to the system.

$$x - y = 0$$

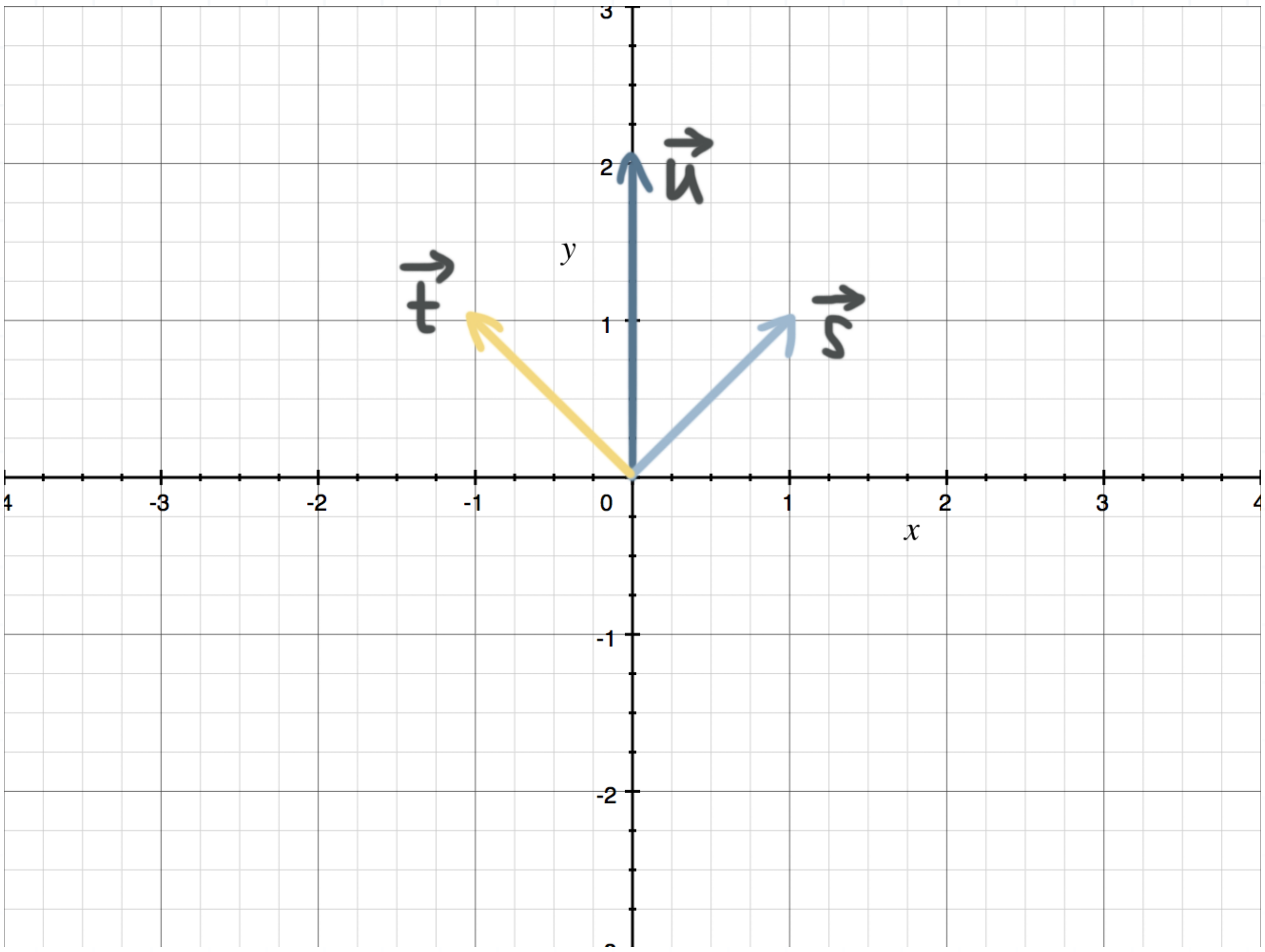
$$x + y = 2$$

Solution:

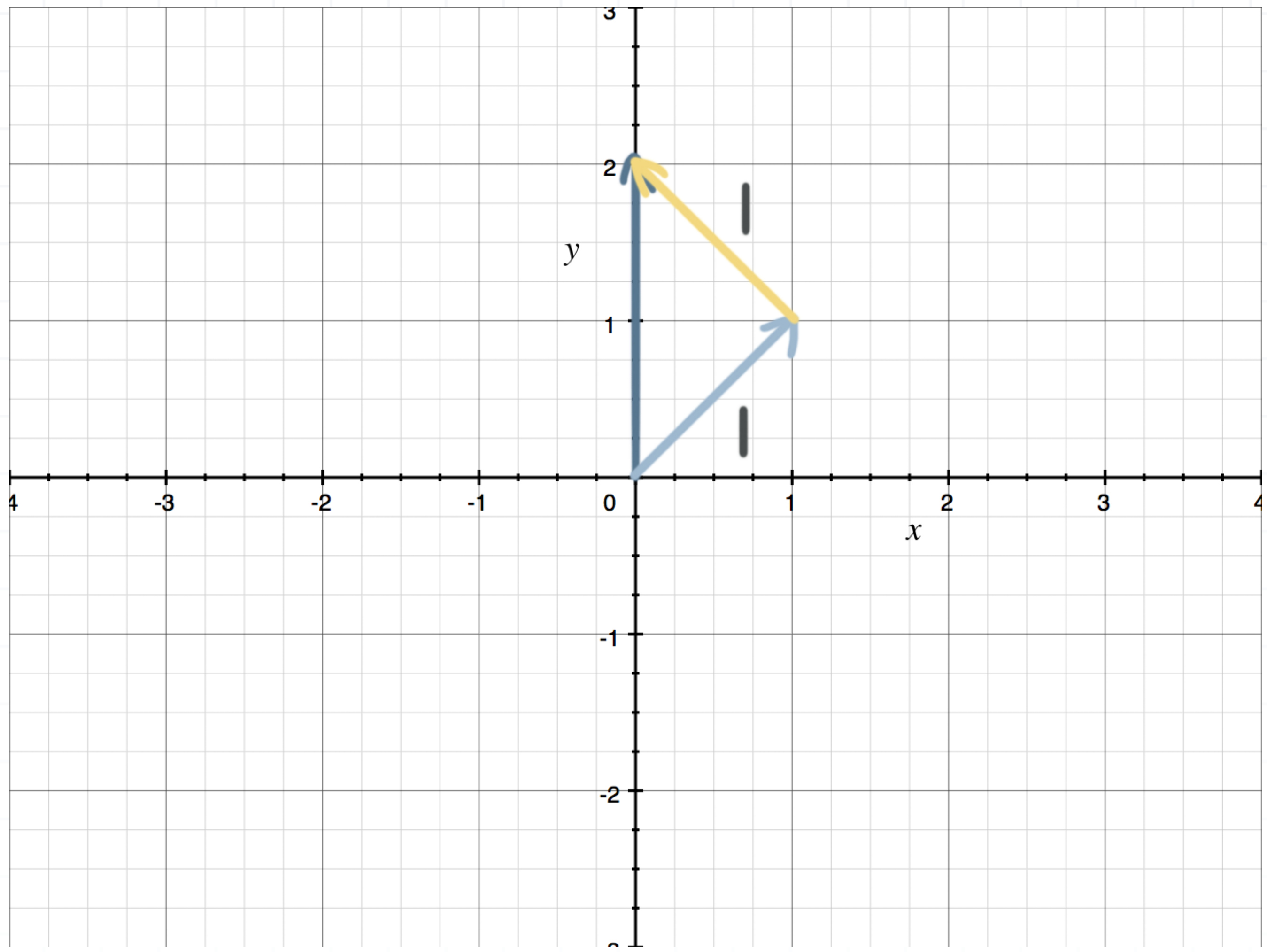
Put the system into a matrix equation.

$$\begin{bmatrix}1\\1\end{bmatrix}x + \begin{bmatrix}-1\\1\end{bmatrix}y = \begin{bmatrix}0\\2\end{bmatrix}$$

From the vector equation, we can sketch the vectors $\vec{s}=(1,1)$ for x, $\vec{t}=(-1,1)$ for y, and the resulting vector $\vec{u}=(0,2)$.



If we play around a little bit with the vectors in the graph, we can see that putting one $\vec{s}$ and one $\vec{t}$ together will get us back to the terminal point of $\vec{u}$, so $x=1$ and $y=1$.


3
2
1
0
-1
-2
-3
y
x
-3
-2
-1
1
2
3

DETERMINANTS

■ 1. Use the determinant to say whether the matrix A is invertible.

$$A = \begin{bmatrix} 5 & 2 \\ 3 & 3 \end{bmatrix}$$

Solution:

If the determinant of the matrix is nonzero, then the matrix is invertible and an inverse exists.

$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & 3 \end{vmatrix}$$

$$|A| = 5(3) - 2(3)$$

$$|A| = 15 - 6$$

$$|A| = 9$$

Because the determinant is nonzero, the matrix is invertible and an inverse exists.

■ 2. Use the determinant to say whether the matrix A is invertible.

$$A = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

Solution:

If the determinant of the matrix is nonzero, then the matrix is invertible and an inverse exists.

$$|A| = \begin{vmatrix} -1 & 2 \\ -1 & 2 \end{vmatrix}$$

$$|A| = -1(2) - 2(-1)$$

$$|A| = -2 + 2$$

$$|A| = 0$$

Because the determinant is 0, the matrix is not invertible and an inverse does not exists.

3. Use the determinant to say whether the matrix A is invertible.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ -3 & 0 & 1 \\ 4 & -2 & 0 \end{bmatrix}$$

Solution:

If the determinant of the matrix is nonzero, then the matrix is invertible and an inverse exists.

$$|A|=\begin{vmatrix}1&2&3\\-3&0&1\\4&-2&0\end{vmatrix}$$

Break the 3×3 determinant into 2×2 determinants.

$$|A|=1\begin{vmatrix}0&1\\-2&0\end{vmatrix}-2\begin{vmatrix}-3&1\\4&0\end{vmatrix}+3\begin{vmatrix}-3&0\\4&-2\end{vmatrix}$$

Calculate the 2×2 determinants.

$$|A|=1((0)(0)-(1)(-2))-2((-3)(0)-(1)(4))+3((-3)(-2)-(0)(4))$$

$$|A|=1(0+2)-2(0-4)+3(6-0)$$

$$|A|=1(2)-2(-4)+3(6)$$

$$|A|=2+8+18$$

$$|A|=28$$

Because the determinant is nonzero, the matrix is invertible and an inverse exists.

4. Use the determinant to say whether matrix A is invertible.

$$A=\begin{bmatrix}1&-2&0\\-2&1&1\\0&-2&1\end{bmatrix}$$

Solution:

If the determinant of the matrix is nonzero, then the matrix is invertible and an inverse exists.

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ -2 & 1 & 1 \\ 0 & -2 & 1 \end{vmatrix}$$

Break the 3×3 determinant into 2×2 determinants.

$$|A| = 1\begin{vmatrix} 1 & 1 \\ -2 & 1 \end{vmatrix} - (-2)\begin{vmatrix} -2 & 1 \\ 0 & 1 \end{vmatrix} + 0\begin{vmatrix} -2 & 1 \\ 0 & -2 \end{vmatrix}$$

Calculate the 2×2 determinants.

$$|A| = 1((1)(1) - (1)(-2)) + 2((-2)(1) - (1)(0)) + 0((-2)(-2) - (1)(0))$$

$$|A| = 1(1 + 2) + 2(-2 - 0) + 0(4 - 0)$$

$$|A| = 1(3) + 2(-2) + 0(4)$$

$$|A| = 3 - 4 + 0$$

$$|A| = -1$$

Because the determinant is nonzero, the matrix is invertible and an inverse exists.

■ 5. Use the Rule of Sarrus to find the determinant.

$$A = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 0 & 2 \\ 0 & -2 & 3 \end{bmatrix}$$

Solution:

We need to add all but the last column to the right side of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 2 & 1 & 1 \\ -1 & 0 & 2 & -1 & 0 \\ 0 & -2 & 3 & 0 & -2 \end{bmatrix}$$

By the Rule of Sarrus, we add the products of the diagonals from the upper left to the lower right.

$$(1)(0)(3) + (1)(2)(0) + (2)(-1)(-2)$$

Then we subtract the products of the diagonals from the upper right to the lower left.

$$-(2)(0)(0) - (1)(2)(-2) - (1)(-1)(3)$$

The determinant is the sum of these two strings of products.

$$|A| = (1)(0)(3) + (1)(2)(0) + (2)(-1)(-2) - (2)(0)(0) - (1)(2)(-2) - (1)(-1)(3)$$

$$|A| = 0 + 0 + 4 - 0 + 4 + 3$$

$$|A| = 4 + 4 + 3$$

$$|A| = 11$$

■ 6. Use the Rule of Sarrus to find the determinant.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ -1 & -2 & -3 \\ 3 & 2 & 1 \end{bmatrix}$$

Solution:

We need to add all but the last column to the right side of the matrix.

$$A = \begin{bmatrix} 0 & 1 & 2 & 0 & 1 \\ -1 & -2 & -3 & -1 & -2 \\ 3 & 2 & 1 & 3 & 2 \end{bmatrix}$$

By the Rule of Sarrus, we add the products of the diagonals from the upper left to the lower right.

$$(0)(-2)(1) + (1)(-3)(3) + (2)(-1)(2)$$

Then we subtract the products of the diagonals from the upper right to the lower left.

$$-(2)(-2)(3) - (0)(-3)(2) - (1)(-1)(1)$$

The determinant is the sum of these two strings of products.

$$|A| = (0)(-2)(1) + (1)(-3)(3) + (2)(-1)(2) - (2)(-2)(3) - (0)(-3)(2) - (1)(-1)(1)$$

$$|A| = 0 - 9 - 4 + 12 + 0 + 1$$

$$|A| = -9 - 4 + 12 + 1$$

$$|A| = 0$$

CRAMER'S RULE FOR SOLVING SYSTEMS

■ 1. Use Cramer's rule to find the expression that would give the value of x. You do not need to solve the system.

$$2x - y = 5$$

$$x + 3y = 15$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 5 & -1 \\ 15 & 3 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} 5 & -1 \\ 15 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & -1 \\ 1 & 3 \end{vmatrix}}$$

2. Use Cramer's rule to find the expression that would give the value of x. You do not need to solve the system.

$$ax + by = e$$

$$cx + dy = f$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} e & b \\ f & d \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$\frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

3. Use Cramer's rule to find the expression that would give the value of y. You do not need to solve the system.

$$3x + 4y = 11$$

$$2x - 3y = -4$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} 3 & 11 \\ 2 & -4 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$\frac{\begin{vmatrix} 3 & 11 \\ 2 & -4 \end{vmatrix}}{\begin{vmatrix} 3 & 4 \\ 2 & -3 \end{vmatrix}}$$

4. Use Cramer's rule to solve for x.

$$3x + 2y = 1$$

$$6x + 5y = 4$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}$$

Calculate the value of x.

$$x = \frac{1(5) - 2(4)}{3(5) - 2(6)}$$

$$x = \frac{5 - 8}{15 - 12}$$

$$x = \frac{-3}{3}$$

$$x = -1$$

5. Use Cramer's rule to solve for y.

$$3x + 2y = 1$$

$$6x + 5y = 4$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}$$

To find D_y, replace the second column of the coefficient matrix with the answer column.

$$D_y = \begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_y/D.

$$y = \frac{D_y}{D} = \frac{\begin{vmatrix} 3 & 1 \\ 6 & 4 \end{vmatrix}}{\begin{vmatrix} 3 & 2 \\ 6 & 5 \end{vmatrix}}$$

Calculate the value of y.

$$y = \frac{3(4) - 1(6)}{3(5) - 2(6)}$$

$$y = \frac{12 - 6}{15 - 12}$$

$$y = \frac{6}{3}$$

$$y = 2$$

■ 6. Use Cramer's rule to solve for x.

$$3x + 5y = 6$$

$$9x + 10y = 14$$

Solution:

Find the expression for the determinant of the coefficient matrix D.

$$D = \begin{vmatrix} 3 & 5 \\ 9 & 10 \end{vmatrix}$$

To find D_x, replace the first column of the coefficient matrix with the answer column.

$$D_x = \begin{vmatrix} 6 & 5 \\ 14 & 10 \end{vmatrix}$$

Substitute the determinants into Cramer's rule D_x/D.

$$x=\frac{D_x}{D}=\frac{\begin{vmatrix}6 & 5\\ 14 & 10\end{vmatrix}}{\begin{vmatrix}3 & 5\\ 9 & 10\end{vmatrix}}$$

Calculate the value of x.

$$x=\frac{6(10)-5(14)}{3(10)-5(9)}$$

$$x=\frac{60-70}{30-45}$$

$$x=\frac{-10}{-15}$$

$$x=\frac{2}{3}$$

MODIFYING DETERMINANTS

1. Find the determinant of A if the first row of A gets multiplied by 3.

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$$

Solution:

The determinant of A is

$$|A| = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix}$$

$$|A| = (2)(1) - (3)(4)$$

When one row of a square matrix is multiplied by a scalar k, the determinant of that matrix gets multiplied by that scalar too, regardless of which row was multiplied by k. So if a row of A was multiplied by $k = 3$, then the determinant will be

$$3|A| = 3((2)(1) - (3)(4))$$

$$3|A| = 3(2 - 12)$$

$$3|A| = 3(-10)$$

$$3|A| = -30$$

■ 2. Find the determinant of A if both rows of A are multiplied by 2.

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

Solution:

The determinant of A is

$$|A| = \begin{vmatrix} 3 & 2 \\ 1 & 4 \end{vmatrix}$$

$$|A| = (3)(4) - (2)(1)$$

When you multiply all rows in a square matrix by a scalar k, the determinant of the resulting matrix will be $k^n|A|$, where n is the number of rows in the matrix. Because there are 2 rows in A, and because $k = 2$, the determinant will be multiplied by $k^n = 2^2$.

$$2^2|A| = 2^2((3)(4) - (2)(1))$$

$$4|A| = 4(12 - 2)$$

$$4|A| = 4(10)$$

$$4|A| = 40$$

■ 3. Find the determinant of C, using only the determinants of A and B.

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 5 & 4 \\ -1 & 2 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 4 \\ 2 & 4 \end{bmatrix}$$

Solution:

The three matrices have identical first rows, and the second row of C is the sum of the second rows of A and B. When this occurs, the determinants have the relationship $|C| = |A| + |B|$. So find the determinants of A and B, and then add them together to find the determinant of C.

$$|A| = \begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix} = (5)(2) - (4)(3) = 10 - 12 = -2$$

$$|B| = \begin{vmatrix} 5 & 4 \\ -1 & 2 \end{vmatrix} = (5)(2) - (4)(-1) = 10 + 4 = 14$$

Then the determinant of C is

$$|C| = -2 + 14$$

$$|C| = 12$$

■ 4. Find the determinant of the new matrix if the rows in matrix A are swapped.

$$A = \begin{bmatrix} 5 & 4 \\ 3 & 2 \end{bmatrix}$$

Solution:

First, create the new matrix by swapping the rows in A, and label it B.

$$B = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$$

Based on the "swapped-row" rule, $|B| = -|A|$. So find $-|A|$, and this will be the determinant of the swapped-row matrix B.

$$-|A| = -\begin{vmatrix} 5 & 4 \\ 3 & 2 \end{vmatrix}$$

$$-|A| = -((5)(2) - (4)(3))$$

$$-|A| = -(10 - 12)$$

$$-|A| = -(-2)$$

$$-|A| = 2$$

5. Find the determinant of the new matrix after the second and third rows of matrix A are swapped.

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$$

Solution:

Swapping the second and third rows of A results in the exact same matrix. When any two rows are identical in an $n \times n$ matrix A, the determinant is 0, or $|A| = 0$. We can find the determinant to verify that it's 0.

$$|A| = \begin{vmatrix} 2 & 0 & 0 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

$$|A| = 2\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} - 0\begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix}$$

$$|A| = 2\begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix}$$

$$|A| = 2((2)(1) - (1)(2))$$

$$|A| = 2(2 - 2)$$

$$|A| = 2(0)$$

$$|A| = 0$$

■ 6. Verify that the row operation $R_2 + 2R_1 \to R_2$ doesn't change the value of $|A|$.

$$A = \begin{bmatrix} 4 & 5 \\ 1 & 2 \end{bmatrix}$$

Solution:

The determinant of A before the row operation is

$$|A| = \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = (4)(2) - (5)(1) = 8 - 5 = 3$$

Now apply the row operation $R_2 + 2R_1 \to R_2$,

$$A_R = \begin{bmatrix} 4 & 5 \\ 1 + 2(4) & 2 + 2(5) \end{bmatrix}$$

$$A_R = \begin{bmatrix} 4 & 5 \\ 1 + 8 & 2 + 10 \end{bmatrix}$$

$$A_R = \begin{bmatrix} 4 & 5 \\ 9 & 12 \end{bmatrix}$$

and then find the determinant of the resulting matrix.

$$|A_R| = \begin{vmatrix} 4 & 5 \\ 9 & 12 \end{vmatrix} = (4)(12) - (5)(9) = 48 - 45 = 3$$

Because we get the same determinant before and after the row operation, we can confirm that the row operation didn't affect the value of the determinant.

UPPER AND LOWER TRIANGULAR MATRICES

■ 1. Find the determinant of the upper-triangular matrix.

$$A = \begin{bmatrix} -4 & 1 \\ 0 & -3 \end{bmatrix}$$

Solution:

Because A is an upper-triangular matrix, the determinant can be found just by multiplying the values along the main diagonal. So the determinant is given by

$$|A| = (-4)(-3)$$

$$|A| = 12$$

■ 2. Find the determinant of the upper-triangular matrix.

$$A = \begin{bmatrix} -4 & 0 & 1 & 3 \\ 0 & -3 & -2 & 1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Solution:

Because A is an upper-triangular matrix, the determinant can be found just by multiplying the values along the main diagonal. So the determinant is given by

$$|A| = (-4)(-3)(1)(2)$$

$$|A| = 24$$

■ 3. Find the determinant of the lower-triangular matrix.

$$A = \begin{bmatrix} 4 & 0 \\ 5 & 3 \end{bmatrix}$$

Solution:

Because A is a lower-triangular matrix, the determinant can be found just by multiplying the values along the main diagonal. So the determinant is given by

$$|A| = (4)(3)$$

$$|A| = 12$$

■ 4. Find the determinant of the lower-triangular matrix.

$$A = \begin{bmatrix} -4 & 0 & 0 \\ 5 & -3 & 0 \\ 3 & -1 & -1 \end{bmatrix}$$

Solution:

Because A is a lower-triangular matrix, the determinant can be found just by multiplying the values along the main diagonal. So the determinant is given by

$$|A| = (-4)(-3)(-1)$$

$$|A| = -12$$

■ 5. Put A into upper or lower triangular form to find the determinant.

$$A = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$$

Solution:

We can write A as an upper-triangular matrix by performing $R_1 + R_2 \to R_2$. After the row operation, the matrix is

$$A = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}$$

Then the determinant of this resulting upper-triangular matrix can be found by multiplying the values along the main diagonal.

$$|A| = (-1)(1)$$

$$|A| = -1$$

■ 6. Put A into upper or lower triangular form to find the determinant.

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 1 & -1 & 4 \\ 0 & 3 & -4 \end{bmatrix}$$

Solution:

In A, we have 0 entries in both the upper right and lower left corners, so we can work in either direction to create an upper- or lower-triangular matrix. Let's create an upper-triangular matrix using row operations.

To make $a_{(2,1)} = 0$, perform $R_1 + R_2 \to R_2$.

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 3 & -4 \end{bmatrix}$$

To make $a_{(3,2)} = 0$, perform $-3R_2 + R_3 \to R_3$.

$$A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 4 \\ 0 & 0 & -16 \end{bmatrix}$$

The matrix is now in upper-triangular form, which means the determinant is given by the product of the values along the main diagonal.

$$|A| = (-1)(1)(-16)$$

$$|A| = 16$$

USING DETERMINANTS TO FIND AREA

1. Find the area of the parallelogram formed by $\vec{v}_1 = (1,4)$ and $\vec{v}_2 = (-2,1)$, if the two vectors form adjacent edges of the parallelogram.

Solution:

When two vectors form adjacent edges of a parallelogram, we can find the area of the parallelogram by taking the determinant of the matrix of the vectors as column vectors.

In other words, we'll put $\vec{v}_1 = (1,4)$ and $\vec{v}_2 = (-2,1)$ as column vectors into a matrix

$$A = \begin{bmatrix} 1 & -2 \\ 4 & 1 \end{bmatrix}$$

and then find the determinant of that matrix, which will be the area of the parallelogram.

$$|A| = \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix}$$

$$|A| = (1)(1) - (-2)(4)$$

$$|A| = 1 + 8$$

$$|A| = 9$$

The area of the parallelogram is 9 square units.

2. Find the area of a parallelogram formed by $\vec{v}_1 = (-3,-3)$ and $\vec{v}_2 = (4,-2)$, if the two vectors form adjacent edges of the parallelogram.

Solution:

When two vectors form adjacent edges of a parallelogram, we can find the area of the parallelogram by taking the determinant of the matrix of the vectors as column vectors.

In other words, we'll put $\vec{v}_1 = (-3,-3)$ and $\vec{v}_2 = (4,-2)$ as column vectors into a matrix

$$A = \begin{bmatrix} -3 & 4 \\ -3 & -2 \end{bmatrix}$$

and then find the determinant of that matrix, which will be the area of the parallelogram.

$$|A| = \begin{vmatrix} -3 & 4 \\ -3 & -2 \end{vmatrix}$$

$$|A| = (-3)(-2) - (4)(-3)$$

$$|A| = 6 + 12$$

$$|A| = 18$$

The area of the parallelogram is 18 square units.

3. Find the area of the parallelogram formed by $\vec{v}_1 = (4,2)$ and $\vec{v}_2 = (1,5)$, if the two vectors form adjacent edges of the parallelogram.

Solution:

When two vectors form adjacent edges of a parallelogram, we can find the area of the parallelogram by taking the determinant of the matrix of the vectors as column vectors.

In other words, we'll put $\vec{v}_1 = (4,2)$ and $\vec{v}_2 = (1,5)$ as column vectors into a matrix

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 5 \end{bmatrix}$$

and then find the determinant of that matrix, which will be the area of the parallelogram.

$$|A| = \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix}$$

$$|A| = (4)(5) - (1)(2)$$

$$|A| = 20 - 2$$

$$|A| = 18$$

The area of the parallelogram is 18 square units.

4. The square S is defined by the vertices $(0,3)$, $(0,0)$, $(3,0)$, and $(3,3)$. If the transformation of S by T creates a transformed figure F, find the area of F.

$$T(\vec{x}) = \begin{bmatrix} 2 & -3 \\ 1 & 2 \end{bmatrix} \vec{x}$$

Solution:

The area of the transformed figure F can be found using just the area of the square S, and the determinant of the transformation T.

$$\text{Area}_F = |\text{Area}_S(\text{Det}(T))|$$

The square S is defined between $x = 0$ and $x = 3$, so its width is 3, and it's defined between $y = 0$ and $y = 3$, so its height is 3. Therefore, the area of the square is $\text{Area}_S = 3 \cdot 3 = 9$.

The determinant of the transformation matrix is

$$|T| = \begin{vmatrix} 2 & -3 \\ 1 & 2 \end{vmatrix}$$

$$|T| = (2)(2) - (-3)(1)$$

$$|T| = 4 + 3$$

$$|T| = 7$$

Then the area of the transformed figure F is

$$\text{Area}_F = |\text{Area}_S(\text{Det}(T))|$$

$$\text{Area}_F = |(9)(7)|$$

$$\text{Area}_F = |63|$$

$$\text{Area}_F = 63$$

5. A rectangle R is defined by the vertices $(-2,2)$, $(2,2)$, $(-2,-3)$, and $(2,-3)$. If the transformation of S by T creates a transformed figure F, find the area of F.

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 2 & 0 \end{bmatrix} \vec{x}$$

Solution:

The area of the transformed figure F can be found using just the area of the rectangle R, and the determinant of the transformation T.

$$\text{Area}_F = |\text{Area}_R(\text{Det}(T))|$$

The rectangle R is defined between $x = -2$ and $x = 2$, so its width is 4, and it's defined between $y = -3$ and $y = 2$, so its height is 5. Therefore, the area of the rectangle is $\text{Area}_R = 4 \cdot 5 = 20$.

The determinant of the transformation matrix is

$$|T| = \begin{vmatrix} -3 & 1 \\ 2 & 0 \end{vmatrix}$$

$$|T| = (-3)(0) - (1)(2)$$

$$|T| = 0 - 2$$

$$|T| = -2$$

Then the area of the transformed figure F is

$$\text{Area}_F = |\text{Area}_R(\text{Det}(T))|$$

$$\text{Area}_F = |(20)(-2)|$$

$$\text{Area}_F = |-40|$$

$$\text{Area}_F = 40$$

■ 6. The rectangle R is defined by the vertices $(2,-6)$, $(2,-1)$, $(8,-1)$, and $(8,-6)$. If the transformation of R by T creates a transformed figure L, find the area of L.

$$T(\vec{x}) = \begin{bmatrix} 2 & -1 \\ 0 & 3 \end{bmatrix} \vec{x}$$

Solution:

The area of the transformed figure L can be found using just the area of the rectangle R, and the determinant of the transformation T.

$$\text{Area}_L = |\text{Area}_R(\text{Det}(T))|$$

The rectangle R is defined between $x = 2$ and $x = 8$, so its width is 6, and it's defined between $y = -6$ and $y = -1$, so its height is 5. Therefore, the area of the rectangle is $\text{Area}_R = 6 \cdot 5 = 30$.

The determinant of the transformation matrix is

$$|T| = \begin{vmatrix} 2 & -1 \\ 0 & 3 \end{vmatrix}$$

$$|T| = (2)(3) - (-1)(0)$$

$$|T| = 6 - 0$$

$$|T| = 6$$

Then the area of the transformed figure L is

$$\text{Area}_L = |\text{Area}_R(\text{Det}(T))|$$

$$\text{Area}_L = |(30)(6)|$$

$$\text{Area}_L = |180|$$

$$\text{Area}_L = 180$$

TRANSPOSES AND THEIR DETERMINANTS

1. Find the transpose A^T.

$$A = [5 \quad 6 \quad 0 \quad 7 \quad 5 \quad -7]$$

Solution:

To find the transpose of A, in order, turn each row of A into a column of A^T.

$$A^T = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 7 \\ 5 \\ -7 \end{bmatrix}$$

2. Find the transpose A^T.

$$A = \begin{bmatrix} 7 & 9 & -6 \\ 0 & -1 & 9 \end{bmatrix}$$

Solution:

To find the transpose of A, in order, turn each row of A into a column of A^T.

$$A^T = \begin{bmatrix} 7 & 0 \\ 9 & -1 \\ -6 & 9 \end{bmatrix}$$

■ 3. Find the transpose A^T.

$$A = \begin{bmatrix} -4 & -7 \\ 5 & 1 \\ 7 & -2 \\ 4 & -2 \end{bmatrix}$$

Solution:

To find the transpose of A, in order, turn each row of A into a column of A^T.

$$A^T = \begin{bmatrix} -4 & 5 & 7 & 4 \\ -7 & 1 & -2 & -2 \end{bmatrix}$$

■ 4. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} 5 & 3 & 6 & -1 \\ 9 & 0 & 1 & -2 \\ 8 & -2 & -4 & 8 \\ 5 & 4 & 9 & 7 \end{bmatrix}$$

Solution:

The determinant of the transpose is always the same as the determinant of the original matrix, so we'll calculate the determinant of A, instead of bothering with the transpose. To find the determinant, work along the second row, since it includes a 0 that'll make the calculation simpler.

$$|A| = -9\begin{vmatrix} 3 & 6 & -1 \\ -2 & -4 & 8 \\ 4 & 9 & 7 \end{vmatrix} + 0\begin{vmatrix} 5 & 6 & -1 \\ 8 & -4 & 8 \\ 5 & 9 & 7 \end{vmatrix} - 1\begin{vmatrix} 5 & 3 & -1 \\ 8 & -2 & 8 \\ 5 & 4 & 7 \end{vmatrix} + (-2)\begin{vmatrix} 5 & 3 & 6 \\ 8 & -2 & -4 \\ 5 & 4 & 9 \end{vmatrix}$$

$$|A| = -9\begin{vmatrix} 3 & 6 & -1 \\ -2 & -4 & 8 \\ 4 & 9 & 7 \end{vmatrix} - \begin{vmatrix} 5 & 3 & -1 \\ 8 & -2 & 8 \\ 5 & 4 & 7 \end{vmatrix} - 2\begin{vmatrix} 5 & 3 & 6 \\ 8 & -2 & -4 \\ 5 & 4 & 9 \end{vmatrix}$$

Break the 3×3 determinants into 4×4 determinants.

$$|A| = -9\left[3\begin{vmatrix} -4 & 8 \\ 9 & 7 \end{vmatrix} - 6\begin{vmatrix} -2 & 8 \\ 4 & 7 \end{vmatrix} - 1\begin{vmatrix} -2 & -4 \\ 4 & 9 \end{vmatrix}\right]$$

$$-\left[5\begin{vmatrix} -2 & 8 \\ 4 & 7 \end{vmatrix} - 3\begin{vmatrix} 8 & 8 \\ 5 & 7 \end{vmatrix} - 1\begin{vmatrix} 8 & -2 \\ 5 & 4 \end{vmatrix}\right]$$

$$-2\left[5\begin{vmatrix} -2 & -4 \\ 4 & 9 \end{vmatrix} - 3\begin{vmatrix} 8 & -4 \\ 5 & 9 \end{vmatrix} + 6\begin{vmatrix} 8 & -2 \\ 5 & 4 \end{vmatrix}\right]$$

Calculate the 2×2 determinants.

$$|A| = -9\left[3((-4)(7) - (8)(9)) - 6((-2)(7) - (8)(4)) - 1((-2)(9) - (-4)(4))\right]$$

$$-\left[5((-2)(7) - (8)(4)) - 3((8)(7) - (8)(5)) - 1((8)(4) - (-2)(5))\right]$$

$$-2\left[5((-2)(9) - (-4)(4)) - 3((8)(9) - (-4)(5)) + 6((8)(4) - (-2)(5))\right]$$

$$|A| = -9\left[3(-28 - 72) - 6(-14 - 32) - 1(-18 + 16)\right]$$

$$-\left[5(-14-32)-3(56-40)-1(32+10)\right]$$

$$-2\left[5(-18+16)-3(72+20)+6(32+10)\right]$$

$$|A| = -9\left[3(-100)-6(-46)-1(-2)\right]-\left[5(-46)-3(16)-1(42)\right]$$

$$-2\left[5(-2)-3(92)+6(42)\right]$$

$$|A| = -9(-300+276+2)-(-230-48-42)-2(-10-276+252)$$

$$|A| = -9(-22)-(-320)-2(-34)$$

$$|A| = 198+320+68$$

$$|A| = 586$$

■ 5. Find the determinant of the transpose of A.

$$A = \begin{bmatrix} -9 & -3 & -1 \\ -4 & 7 & 3 \\ -4 & 8 & 7 \end{bmatrix}$$

Solution:

The determinant of the transpose is always the same as the determinant of the original matrix, so we'll calculate the determinant of A, instead of bothering with the transpose. To find the determinant, work along the first row.

$$|A| = -9\begin{vmatrix}7 & 3\\8 & 7\end{vmatrix} - (-3)\begin{vmatrix}-4 & 3\\-4 & 7\end{vmatrix} + (-1)\begin{vmatrix}-4 & 7\\-4 & 8\end{vmatrix}$$

$$|A| = -9\begin{vmatrix}7 & 3\\8 & 7\end{vmatrix} + 3\begin{vmatrix}-4 & 3\\-4 & 7\end{vmatrix} - \begin{vmatrix}-4 & 7\\-4 & 8\end{vmatrix}$$

Calculate the 2×2 determinants.

$$|A| = -9((7)(7) - (3)(8)) + 3((-4)(7) - (3)(-4)) - ((-4)(8) - (7)(-4))$$

$$|A| = -9(49 - 24) + 3(-28 + 12) - (-32 + 28)$$

$$|A| = -9(25) + 3(-16) - (-4)$$

$$|A| = -225 - 48 + 4$$

$$|A| = -269$$

■ 6. Find the determinant of the transpose of A.

$$A = \begin{bmatrix}-8 & 6 & 8\\3 & -9 & -1\\4 & -9 & 9\end{bmatrix}$$

Solution:

The determinant of the transpose is always the same as the determinant of the original matrix, so we'll calculate the determinant of A, instead of bothering with the transpose. To find the determinant, work along the first row.

$$|A| = -8\begin{vmatrix} -9 & -1 \\ -9 & 9 \end{vmatrix} - 6\begin{vmatrix} 3 & -1 \\ 4 & 9 \end{vmatrix} + 8\begin{vmatrix} 3 & -9 \\ 4 & -9 \end{vmatrix}$$

Calculate the 2×2 determinants.

$$|A| = -8((-9)(9) - (-1)(-9)) - 6((3)(9) - (-1)(4)) + 8((3)(-9) - (-9)(4))$$

$$|A| = -8(-81 - 9) - 6(27 + 4) + 8(-27 + 36)$$

$$|A| = -8(-90) - 6(31) + 8(9)$$

$$|A| = 720 - 186 + 72$$

$$|A| = 606$$

TRANSPOSES OF PRODUCTS, SUMS, AND INVERSES

1. Find $(AB)^T$.

$$A = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}$$

Solution:

Find the matrix AB,

$$AB = \begin{bmatrix} -1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ 1 & 2 \end{bmatrix}$$

$$AB = \begin{bmatrix} -1(-3) + 2(1) & -1(-2) + 2(2) \\ 2(-3) + 3(1) & 2(-2) + 3(2) \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 + 2 & 2 + 4 \\ -6 + 3 & -4 + 6 \end{bmatrix}$$

$$AB = \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix}$$

and then take its transpose by swapping the rows and columns.

$$(AB)^T = \begin{bmatrix} 5 & -3 \\ 6 & 2 \end{bmatrix}$$

2. Find $(AB)^T$.

$$A = \begin{bmatrix} -1 & 2 & -2 \\ 2 & 3 & 1 \\ 3 & -3 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -4 & 1 \\ 0 & -3 & -2 \\ -1 & 1 & 2 \end{bmatrix}$$

Solution:

Start by taking the transposes individually by swapping rows and columns in A and B to get A^T and B^T.

$$A^T = \begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & -3 \\ -2 & 1 & 1 \end{bmatrix}$$

$$B^T = \begin{bmatrix} 2 & 0 & -1 \\ -4 & -3 & 1 \\ 1 & -2 & 2 \end{bmatrix}$$

Find the product of these transposes.

$$B^TA^T = \begin{bmatrix} 2 & 0 & -1 \\ -4 & -3 & 1 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} -1 & 2 & 3 \\ 2 & 3 & -3 \\ -2 & 1 & 1 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 2(-1) + 0(2) - 1(-2) & 2(2) + 0(3) - 1(1) & 2(3) + 0(-3) - 1(1) \\ -4(-1) - 3(2) + 1(-2) & -4(2) - 3(3) + 1(1) & -4(3) - 3(-3) + 1(1) \\ 1(-1) - 2(2) + 2(-2) & 1(2) - 2(3) + 2(1) & 1(3) - 2(-3) + 2(1) \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} -2+0+2 & 4+0-1 & 6+0-1 \\ 4-6-2 & -8-9+1 & -12+9+1 \\ -1-4-4 & 2-6+2 & 3+6+2 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 0 & 3 & 5 \\ -4 & -16 & -2 \\ -9 & -2 & 11 \end{bmatrix}$$

We know the product $B^TA^T = (AB)^T$, so

$$(AB)^T = \begin{bmatrix} 0 & 3 & 5 \\ -4 & -16 & -2 \\ -9 & -2 & 11 \end{bmatrix}$$

■ 3. Find $(X+Y)^T$.

$$X = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix}$$

Solution:

Find the sum $X+Y$,

$$X+Y = \begin{bmatrix} 4 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} -3 & 2 \\ 0 & -1 \end{bmatrix}$$

$$X + Y = \begin{bmatrix} 4 + (-3) & 1 + 2 \\ -2 + 0 & 0 + (-1) \end{bmatrix}$$

$$X + Y = \begin{bmatrix} 1 & 3 \\ -2 & -1 \end{bmatrix}$$

and then take its transpose by swapping the rows and columns.

$$(X + Y)^T = \begin{bmatrix} 1 & -2 \\ 3 & -1 \end{bmatrix}$$

■ 4. Find $(X + Y)^T$.

$$X = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ -2 & 0 & 3 \end{bmatrix}$$

$$Y = \begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 2 \\ 4 & -1 & 0 \end{bmatrix}$$

Solution:

Find the sum $X + Y$,

$$X + Y = \begin{bmatrix} 2 & 0 & 3 \\ 4 & 1 & -1 \\ -2 & 0 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 2 & -3 \\ 0 & -1 & 2 \\ 4 & -1 & 0 \end{bmatrix}$$

$$X+Y=\begin{bmatrix}2+(-1) & 0+2 & 3+(-3)\\ 4+0 & 1+(-1) & -1+2\\ -2+4 & 0+(-1) & 3+0\end{bmatrix}$$

$$X+Y=\begin{bmatrix}1 & 2 & 0\\ 4 & 0 & 1\\ 2 & -1 & 3\end{bmatrix}$$

and then take its transpose by swapping the rows and columns.

$$(X+Y)^T=\begin{bmatrix}1 & 4 & 2\\ 2 & 0 & -1\\ 0 & 1 & 3\end{bmatrix}$$

■ 5. Find $(X^T)^{-1}$.

$$X=\begin{bmatrix}1 & -2\\ 2 & 3\end{bmatrix}$$

Solution:

First transpose X.

$$X^T=\begin{bmatrix}1 & 2\\ -2 & 3\end{bmatrix}$$

Augment X^T with I_2, and then put the left side of the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 2 & | & 1 & 0 \\ -2 & 3 & | & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 7 & | & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & \frac{3}{7} & -\frac{2}{7} \\ 0 & 1 & | & \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

Now that the left side of the augmented matrix is the identity matrix, the right side is the inverse $(X^T)^{-1}$.

$$(X^T)^{-1} = \begin{bmatrix} \frac{3}{7} & -\frac{2}{7} \\ \frac{2}{7} & \frac{1}{7} \end{bmatrix}$$

■ 6. Find $(A^T)^{-1}$.

$$A = \begin{bmatrix} 4 & 1 & -3 \\ 1 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}$$

Solution:

First transpose A.

$$A^T = \begin{bmatrix} 4 & 1 & 0 \\ 1 & 2 & -1 \\ -3 & 1 & 4 \end{bmatrix}$$

Augment A^T with I_3, and then put the left side of the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 4 & 1 & 0 & | & 1 & 0 & 0 \\ 1 & 2 & -1 & | & 0 & 1 & 0 \\ -3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 4 & 1 & 0 & | & 1 & 0 & 0 \\ -3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ -3 & 1 & 4 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ 0 & 7 & 1 & | & 0 & 3 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ 0 & 0 & 5 & | & 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & -7 & 4 & | & 1 & -4 & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -1 & | & 0 & 1 & 0 \\ 0 & 1 & -\frac{4}{7} & | & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{7} & | & \frac{2}{7} & -\frac{1}{7} & 0 \\ 0 & 1 & -\frac{4}{7} & | & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & \frac{9}{35} & -\frac{4}{35} & -\frac{1}{35} \\ 0 & 1 & -\frac{4}{7} & | & -\frac{1}{7} & \frac{4}{7} & 0 \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & \frac{9}{35} & -\frac{4}{35} & -\frac{1}{35} \\ 0 & 1 & 0 & | & -\frac{1}{35} & \frac{16}{35} & \frac{4}{35} \\ 0 & 0 & 1 & | & \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

Now that the left side of the augmented matrix is the identity matrix, the right side is the inverse $(A^T)^{-1}$.

$$(A^T)^{-1} = \begin{bmatrix} \frac{9}{35} & -\frac{4}{35} & -\frac{1}{35} \\ -\frac{1}{35} & \frac{16}{35} & \frac{4}{35} \\ \frac{1}{5} & -\frac{1}{5} & \frac{1}{5} \end{bmatrix}$$

NULL AND COLUMN SPACES OF THE TRANSPOSE

■ 1. Find the null and column spaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces.

$$M = \begin{bmatrix} -1 & 0 \\ 2 & 4 \\ -2 & -2 \\ 0 & 4 \end{bmatrix}$$

Solution:

The transpose of M is

$$M^T = \begin{bmatrix} -1 & 2 & -2 & 0 \\ 0 & 4 & -2 & 4 \end{bmatrix}$$

To find the null space of the transpose (the left null space), augment M^T with $\overrightarrow{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 2 & -2 & 0 & | & 0 \\ 0 & 4 & -2 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 2 & 0 & | & 0 \\ 0 & 4 & -2 & 4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 2 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 & | & 0 \\ 0 & 1 & -\frac{1}{2} & 1 & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 + x_3 + 2x_4 = 0$$

$$x_2 - \frac{1}{2}x_3 + x_4 = 0$$

and then solve the system for the pivot variables.

$$x_1 = -x_3 - 2x_4$$

$$x_2 = \frac{1}{2}x_3 - x_4$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of the transpose (the left null space) is

$$N(M^T) = \text{Span}\left(\begin{bmatrix} -1 \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}\right)$$

The column space of the transpose is

$$C(M^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$$

but only the first two columns of $\text{rref}(M^T)$ are pivot columns, which means the column space of M^T can actually be spanned by just the first two column vectors.

$$C(M^T) = \text{Span}\left(\begin{bmatrix}-1\\0\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right)$$

The original matrix M has $m = 4$ rows and $n = 2$ columns, so the null space of the transpose $N(M^T)$ is a subspace of $\mathbb{R}^4$, and the column space of the transpose $C(M^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the null and column spaces of the transpose are

$$\text{Dim}(N(M^T)) = m - r = 4 - 2 = 2$$

$$\text{Dim}(C(M^T)) = r = 2$$

■ 2. Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix}1 & 2\\0 & 1\\-4 & 0\end{bmatrix}$$

Solution:

The transpose of A is

$$A^T = \begin{bmatrix}1 & 0 & -4\\2 & 1 & 0\end{bmatrix}$$

To find the left null space (the null space of the transpose), augment A^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 2 & 1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & 8 & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 - 4x_3 = 0$$

$$x_2 + 8x_3 = 0$$

and then solve the system for the pivot variables.

$$x_1 = 4x_3$$

$$x_2 = -8x_3$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} 4 \\ -8 \\ 1 \end{bmatrix} \right)$$

The row space is

$$C(A^T) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}-4\\0\end{bmatrix}\right)$$

but only the first two columns of rref(A^T) are pivot columns, which means the row space can actually be spanned by just the first two columns.

$$C(A^T) = \text{Span}\left(\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right)$$

The original matrix A has $m = 3$ rows and $n = 2$ columns, so the left null $N(A^T)$ space is a subspace of $\mathbb{R}^3$, and the row space $C(A^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(A^T)) = m - r = 3 - 2 = 1$$

$$\text{Dim}(C(A^T)) = r = 2$$

■ 3. Find the row space and left null space of B, and the dimensions of those spaces.

$$B = \begin{bmatrix}2 & 3 & 1 & 0\\1 & -2 & -1 & 4\\0 & 0 & 2 & -2\end{bmatrix}$$

Solution:

The transpose of B is

$$B^T = \begin{bmatrix} 2 & 1 & 0 \\ 3 & -2 & 0 \\ 1 & -1 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

To find the left null space (the null space of the transpose), augment B^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 1 & 0 & | & 0 \\ 3 & -2 & 0 & | & 0 \\ 1 & -1 & 2 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & | & 0 \\ 3 & -2 & 0 & | & 0 \\ 2 & 1 & 0 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 2 & 1 & 0 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 3 & -4 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 14 & | & 0 \\ 0 & 4 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 14 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -6 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 22 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(B^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The row space is

$$C(B^T) = \text{Span}\left(\begin{bmatrix} 2 \\ 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -2 \end{bmatrix} \right)$$

The original matrix B has $m = 3$ rows and $n = 4$ columns, so the left null space $N(B^T)$ is a subspace of $\mathbb{R}^3$, and the row space $C(B^T)$ is a subspace of $\mathbb{R}^4$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(B^T)) = m - r = 3 - 3 = 0$$

$$\text{Dim}(C(B^T)) = r = 3$$

4. Find the row space and left null space of C, and the dimensions of those spaces.

$$C = \begin{bmatrix} -1 & 2 & 0 \\ 1 & 4 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution:

The transpose of C is

$$C^T = \begin{bmatrix} -1 & 1 & 0 \\ 2 & 4 & 0 \\ 0 & 3 & 3 \end{bmatrix}$$

To find the left null space (the null space of the transpose), augment C^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} -1 & 1 & 0 & | & 0 \\ 2 & 4 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 2 & 4 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 6 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 3 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(C^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The row space is

$$C(C^T) = \text{Span}\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix}\right)$$

The original matrix C has $m = 3$ rows and $n = 3$ columns, so the left null space $N(C^T)$ is a subspace of $\mathbb{R}^3$, and the row space $C(C^T)$ is a subspace of $\mathbb{R}^3$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(C^T)) = m - r = 3 - 3 = 0$$

$$\text{Dim}(C(C^T)) = r = 3$$

5. Find the row space and left null space of A, and the dimensions of those spaces.

$$A = \begin{bmatrix} 1 & 3 \\ -3 & 1 \\ 0 & -2 \end{bmatrix}$$

Solution:

The transpose of A is

$$A^T = \begin{bmatrix} 1 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

To find the left null space (the null space of the transpose), augment A^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 1 & -3 & 0 & | & 0 \\ 3 & 1 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & | & 0 \\ 0 & 10 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{5} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{3}{5} & | & 0 \\ 0 & 1 & -\frac{1}{5} & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 - \frac{3}{5}x_3 = 0$$

$$x_2 - \frac{1}{5}x_3 = 0$$

and then solve the system for the pivot variables.

$$x_1 = \frac{3}{5}x_3$$

$$x_2 = \frac{1}{5}x_3$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix}$$

Therefore, the left null space (the null space of the transpose) is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} \frac{3}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} \right)$$

The row space is

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix} \right)$$

but only the first two columns of rref(A^T) are pivot columns, which means the row space can actually be spanned by just the first two columns.

$$C(A^T) = \text{Span}\left(\begin{bmatrix}1\\3\end{bmatrix}, \begin{bmatrix}-3\\1\end{bmatrix}\right)$$

The original matrix A has $m = 3$ rows and $n = 2$ columns, so the left null $N(A^T)$ space is a subspace of $\mathbb{R}^3$, and the row space $C(A^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the left null and row spaces are

$$\text{Dim}(N(A^T)) = m - r = 3 - 2 = 1$$

$$\text{Dim}(C(A^T)) = r = 2$$

6. Find the null and column subspaces of the transpose M^T, identify their spaces $\mathbb{R}^i$, and name the dimension of the subspaces of M^T.

$$M = \begin{bmatrix}2 & 4\\1 & 0\\-1 & -1\\0 & 3\end{bmatrix}$$

Solution:

The transpose of M is

$$M^T = \begin{bmatrix}2 & 1 & -1 & 0\\4 & 0 & -1 & 3\end{bmatrix}$$

To find the null space of the transpose (the left null space), augment M^T with $\vec{O}$, and then put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 2 & 1 & -1 & 0 & | & 0 \\ 4 & 0 & -1 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & | & 0 \\ 4 & 0 & -1 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & | & 0 \\ 0 & -2 & 1 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{2} & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{4} & \frac{3}{4} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{3}{2} & | & 0 \end{bmatrix}$$

Pull a system of equations from the matrix,

$$x_1 - \frac{1}{4}x_3 + \frac{3}{4}x_4 = 0$$

$$x_2 - \frac{1}{2}x_3 - \frac{3}{2}x_4 = 0$$

and then solve the system for the pivot variables.

$$x_1 = \frac{1}{4}x_3 - \frac{3}{4}x_4$$

$$x_2 = \frac{1}{2}x_3 + \frac{3}{2}x_4$$

Write the solution as a linear combination.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{3}{4} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$$

Therefore, the null space of the transpose (the left null space) is

$$N(M^T) = \text{Span}\left(\begin{bmatrix} \frac{1}{4} \\ \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{3}{4} \\ \frac{3}{2} \\ 0 \\ 1 \end{bmatrix}\right)$$

The column space of the transpose is

$$C(M^T) = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix}\right)$$

but only the first two columns of rref(M^T) are pivot columns, which means the column space of M^T can actually be spanned by just the first two column vectors.

$$C(M^T) = \text{Span}\left(\begin{bmatrix} 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

The original matrix M has $m = 4$ rows and $n = 2$ columns, so the null space of the transpose $N(M^T)$ is a subspace of $\mathbb{R}^4$, and the column space of the transpose $C(M^T)$ is a subspace of $\mathbb{R}^2$. And the dimension of the null and column spaces of the transpose are

$$\text{Dim}(N(M^T)) = m - r = 4 - 2 = 2$$

$$\text{Dim}(C(M^T)) = r = 2$$

THE PRODUCT OF A MATRIX AND ITS TRANSPOSE

■ 1. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$$

Solution:

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 0 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 3 \\ -2 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 2 \\ 3 & 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1)+0(0)+3(3) & 1(-2)+0(2)+3(0) \\ -2(1)+2(0)+0(3) & -2(-2)+2(2)+0(0) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0+9 & -2+0+0 \\ -2+0+0 & 4+4+0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 10 & -2 \\ -2 & 8 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 10 & -2 \\ -2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} \\ -2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & \frac{38}{5} \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Because we get the identity matrix, we can say that A^TA is invertible.

■ 2. Is A^TA invertible?

$$A = \begin{bmatrix} -12 & 6 \\ 8 & -4 \end{bmatrix}$$

Solution:

The columns of A aren't linearly independent, so A^TA is not invertible. We can confirm this by finding A^TA, and then verifying that A^TA doesn't simplify to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} -12 & 8 \\ 6 & -4 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} -12 & 8 \\ 6 & -4 \end{bmatrix} \begin{bmatrix} -12 & 6 \\ 8 & -4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -12(-12) + 8(8) & -12(6) + 8(-4) \\ 6(-12) - 4(8) & 6(6) - 4(-4) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 144 + 64 & -72 - 32 \\ -72 - 32 & 36 + 16 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 208 & -104 \\ -104 & 52 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 208 & -104 \\ -104 & 52 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ -104 & 52 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

■ 3. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

Solution:

The columns of A are linearly independent, so A^TA is invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ -2 & 2 & -2 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 3 & 0 \\ -2 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & 2 \\ 1 & 0 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1)+0(0)+1(1) & 1(1)+0(3)+1(0) & 1(-2)+0(2)+1(-2) \\ 1(1)+3(0)+0(1) & 1(1)+3(3)+0(0) & 1(-2)+3(2)+0(-2) \\ -2(1)+2(0)-2(1) & -2(1)+2(3)-2(0) & -2(-2)+2(2)-2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0+1 & 1+0+0 & -2+0-2 \\ 1+0+0 & 1+9+0 & -2+6+0 \\ -2+0-2 & -2+6+0 & 4+4+4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 10 & 4 \\ -4 & 4 & 12 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 2 & 1 & -4 \\ 1 & 10 & 4 \\ -4 & 4 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ 2 & 1 & -4 \\ -4 & 4 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ 0 & -19 & -12 \\ -4 & 4 & 12 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 10 & 4 \\ 0 & -19 & -12 \\ 0 & 44 & 28 \end{bmatrix} \to \begin{bmatrix} 1 & 10 & 4 \\ 0 & 1 & \frac{12}{19} \\ 0 & 44 & 28 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{44}{19} \\ 0 & 1 & \frac{12}{19} \\ 0 & 44 & 28 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{44}{19} \\ 0 & 1 & \frac{12}{19} \\ 0 & 0 & \frac{4}{19} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{44}{19} \\ 0 & 1 & \frac{12}{19} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{12}{19} \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Because we got to the identity matrix, we can say that A^TA is invertible.

■ 4. Is A^TA invertible?

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

Solution:

The columns of A are not linearly independent, which means A^TA won't be invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1)+0(0) & 1(0)+0(1) & 1(-2)+0(3) \\ 0(1)+1(0) & 0(0)+1(1) & 0(-2)+1(3) \\ -2(1)+3(0) & -2(0)+3(1) & -2(-2)+3(3) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0 & 0+0 & -2+0 \\ 0+0 & 0+1 & 0+3 \\ -2+0 & 0+3 & 4+9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

■ 5. Is A^TA invertible?

$$A = \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

Solution:

The columns of A aren't linearly independent, so A^TA is not invertible. We can confirm this by finding A^TA, and then verifying that A^TA doesn't simplify to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ -6 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4(4) - 6(-6) & 4(-2) - 6(3) \\ -2(4) + 3(-6) & -2(-2) + 3(3) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 16 + 36 & -8 - 18 \\ -8 - 18 & 4 + 9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 52 & -26 \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ -26 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

■ 6. Is A^TA invertible?

$$A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

Solution:

The columns of A are not linearly independent, which means A^TA won't be invertible. We can confirm this by finding A^TA, and then verifying that A^TA simplifies to the identity matrix when we put it into reduced row-echelon form. First, we'll find A^T.

$$A^T = \begin{bmatrix} -1 & 0 \\ 0 & 3 \\ 2 & 3 \end{bmatrix}$$

Then the product A^TA is

$$A^TA = \begin{bmatrix} -1 & 0 \\ 0 & 3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} -1 & 0 & 2 \\ 0 & 3 & 3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1(-1)+0(0) & -1(0)+0(3) & -1(2)+0(3) \\ 0(-1)+3(0) & 0(0)+3(3) & 0(2)+3(3) \\ 2(-1)+3(0) & 2(0)+3(3) & 2(2)+3(3) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0 & 0+0 & -2+0 \\ 0+0 & 0+9 & 0+9 \\ -2+0 & 0+9 & 4+9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 9 & 9 \\ -2 & 9 & 13 \end{bmatrix}$$

Then to determine whether or not A^TA is invertible, put A^TA into reduced row-echelon form.

$$A^TA = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 9 & 9 \\ -2 & 9 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 9 & 9 \\ 0 & 9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 9 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because we didn't get the identity matrix, we can say that A^TA is not invertible.

A=LU FACTORIZATION

1. Rewrite the matrix A in factored LU form.

$$A = \begin{bmatrix} 2 & 4 \\ 12 & 21 \end{bmatrix}$$

Solution:

We'll apply an elimination matrix to A to zero-out the 12 in $A_{2,1}$.

$$E_{2,1}A = U$$

$$\begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 12 & 21 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$

To solve this equation for A, we need to move the elimination matrix to the right side, inverting it.

$$\begin{bmatrix} 2 & 4 \\ 12 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -6 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$

To invert the elimination matrix, we'll change the sign on the non-zero entry below the main diagonal, and we'll get the $A = LU$ factorization.

$$\begin{bmatrix} 2 & 4 \\ 12 & 21 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$$

■ 2. Rewrite the matrix A in factored LU form.

$$A = \begin{bmatrix} -1 & 0 & 3 \\ -5 & 2 & 18 \\ -5 & -8 & 5 \end{bmatrix}$$

Solution:

We'll apply an elimination matrix to A to zero-out the -5 in $A_{2,1}$.

$$E_{2,1}A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ -5 & 2 & 18 \\ -5 & -8 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 3 \\ -5 & -8 & 5 \end{bmatrix}$$

Next we'll apply an elimination matrix to A to zero-out the -5 in $A_{3,1}$.

$$E_{3,1}E_{2,1}A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ -5 & 2 & 18 \\ -5 & -8 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & -8 & -10 \end{bmatrix}$$

Next we'll apply an elimination matrix to A to zero-out the -8 in $A_{3,2}$.

$$E_{3,2}E_{3,1}E_{2,1}A = U$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ -5 & 2 & 18 \\ -5 & -8 & 5 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

To solve this equation for A, we need to move the elimination matrices to the right side, reversing their order and inverting each one.

$$\begin{bmatrix} -1 & 0 & 3 \\ -5 & 2 & 18 \\ -5 & -8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

To invert the elimination matrices, we'll change the sign on the non-zero entries below the main diagonal, and we'll get the $A = LU$ factorization.

$$\begin{bmatrix} -1 & 0 & 3 \\ -5 & 2 & 18 \\ -5 & -8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Then we'll consolidate all the non-zero entries into one matrix, L.

$$\begin{bmatrix} -1 & 0 & 3 \\ -5 & 2 & 18 \\ -5 & -8 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 5 & -4 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 3 \\ 0 & 2 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

■ 3. Rewrite the matrix A in factored LDU form, where D is the diagonal matrix that factors the pivots out of U.

$$A = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -6 & 3 & 0 \end{bmatrix}$$

Solution:

We'll apply an elimination matrix to A to zero-out the -6 in $A_{3,1}$.

$$E_{3,1}A = U$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\3 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\-6 & 3 & 0\end{bmatrix}=\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\0 & 3 & 6\end{bmatrix}$$

Next we'll apply an elimination matrix to A to zero-out the 3 in $A_{3,2}$.

$$E_{3,2}E_{3,1}A = U$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -3 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\3 & 0 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\-6 & 3 & 0\end{bmatrix}=\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\0 & 0 & 3\end{bmatrix}$$

To solve this equation for A, we need to move the elimination matrices to the right side, reversing their order and inverting each one.

$$\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\-6 & 3 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\3 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -3 & 1\end{bmatrix}^{-1}\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\0 & 0 & 3\end{bmatrix}$$

To invert the elimination matrices, we'll change the sign on the non-zero entries below the main diagonal, and we'll get the $A = LU$ factorization.

$$\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\-6 & 3 & 0\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-3 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 3 & 1\end{bmatrix}\begin{bmatrix}2 & 0 & 2\\0 & 1 & 1\\0 & 0 & 3\end{bmatrix}$$

Then we'll consolidate all the non-zero entries into one matrix, L.

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

To put the factorization in LDU form, we factor out the pivots from U.

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & 1 \\ -6 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

■ 4. Rewrite the matrix A in factored LDU form, where D is the diagonal matrix that factors the pivots out of U.

$$A = \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix}$$

Solution:

Let's set up $A = LU$.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix}$$

We only need to zero-out the entries in U that are below the main diagonal. We don't need to perform any row operation to keep the 0 in $L_{2,1}$, so

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix}$$

To zero-out the 8 in $U_{3,1}$, we need to add 2 of R_1 to R_3, which is the row operation $R_3 + 2R_1$. But we want this row operation written with subtraction, so we rewrite it as $R_3 - (-2)R_1$, and put -2 into $L_{3,1}$.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 7 & 5 \\ -4 & 16 & 5 & 27 \end{bmatrix}$$

To zero-out the -4 in $U_{4,1}$, we need to subtract 1 of R_1 from R_4. Since the row operation is $R_4 - R_1$, we put 1 into $L_{4,1}$.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & & 1 & 0 \\ 1 & & & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 4 & 7 & 5 \\ 0 & 8 & 5 & 23 \end{bmatrix}$$

To zero-out the 4 in $U_{3,2}$, we need to subtract 4 of R_2 from R_3. Since the row operation is $R_3 - 4R_2$, we put 4 into $L_{3,2}$.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 4 & 1 & 0 \\ 1 & & & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 8 & 5 & 23 \end{bmatrix}$$

To zero-out the 8 in $U_{4,2}$, we need to subtract 8 of R_2 from R_4. Since the row operation is $R_4 - 8R_2$, we put 8 into $L_{4,2}$.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 4 & 1 & 0 \\ 1 & 8 & & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & -3 & 7 \end{bmatrix}$$

To zero-out the -3 in $U_{4,3}$, we need to add 1 of R_3 to R_4, which is the row operation $R_4 + R_3$. But we want this row operation written with subtraction, so we rewrite it as $R_4 - (-1)R_3$, and put -1 into $L_{4,3}$.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 4 & 1 & 0 \\ 1 & 8 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

To rewrite the $A = LU$ factorization as $A = LDU$, divide through the first row of U by -4, moving the -4 pivot into the first row of D.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 4 & 1 & 0 \\ 1 & 8 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Divide through the third row of U by 3, moving the 3 pivot into the third row of D.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 4 & 1 & 0 \\ 1 & 8 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Divide through the fourth row of U by 4, moving the 4 pivot into the fourth row of D.

$$\begin{bmatrix} -4 & 8 & 0 & 4 \\ 0 & 1 & 1 & 2 \\ 8 & -12 & 7 & -3 \\ -4 & 16 & 5 & 27 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 4 & 1 & 0 \\ 1 & 8 & -1 & 1 \end{bmatrix} \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 5. Rewrite the matrix A in factored LDU form, where D is the diagonal matrix that factors the pivots out of U.

$$A = \begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix}$$

Solution:

Let's set up $A = LU$.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix}$$

We only need to zero-out the entries in U that are below the main diagonal. To zero-out the 20 in $U_{2,1}$, we need to subtract 4 of R_1 from R_2. Since the row operation is $R_2 - 4R_1$, we put 4 into $L_{2,1}$.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 & 5 \\ 0 & 3 & 9 & -12 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix}$$

To zero-out the 20 in $U_{3,1}$, we need to subtract 4 of R_1 from R_3. Since the row operation is $R_3 - 4R_1$, we put 4 into $L_{3,1}$.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 & 5 \\ 0 & 3 & 9 & -12 \\ 0 & 0 & 2 & 6 \\ -15 & 6 & 3 & 25 \end{bmatrix}$$

To zero-out the -15 in $U_{4,1}$, we need to add 3 of R_1 to R_4, which is the row operation is $R_4 + 3R_1$. But we want this row operation written with subtraction, so we rewrite it as $R_4 - (-3)R_1$, and put -3 into $L_{4,1}$.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & & 1 & 0 \\ -3 & & & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 & 5 \\ 0 & 3 & 9 & -12 \\ 0 & 0 & 2 & 6 \\ 0 & -9 & -27 & 40 \end{bmatrix}$$

We don't need to perform any row operation to keep the 0 in $L_{3,2}$, so

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -3 & & & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 & 5 \\ 0 & 3 & 9 & -12 \\ 0 & 0 & 2 & 6 \\ 0 & -9 & -27 & 40 \end{bmatrix}$$

To zero-out the -9 in $U_{4,2}$, we need to add 3 of R_2 to R_4, which is the row operation $R_4 + 3R_2$. But we want this row operation written with subtraction, so we rewrite it as $R_4 - (-3)R_2$, and put -3 into $L_{4,2}$.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -3 & -3 & & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 & 5 \\ 0 & 3 & 9 & -12 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

We don't need to perform any row operation to keep the 0 in $L_{4,3}$, so

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -3 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & -5 & -10 & 5 \\ 0 & 3 & 9 & -12 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

To rewrite the $A = LU$ factorization as $A = LDU$, divide through the first row of U by 5, moving the 5 pivot into the first row of D.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -3 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 & 1 \\ 0 & 3 & 9 & -12 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Divide through the second row of U by 3, moving the 3 pivot into the second row of D.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -3 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 & 1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Divide through the third row of U by 2, moving the 2 pivot into the third row of D.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -3 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 & 1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

Divide through the fourth row of U by 4, moving the 4 pivot into the fourth row of D.

$$\begin{bmatrix} 5 & -5 & -10 & 5 \\ 20 & -17 & -31 & 8 \\ 20 & -20 & -38 & 26 \\ -15 & 6 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 \\ 4 & 0 & 1 & 0 \\ -3 & -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & -1 & -2 & 1 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

■ 6. Rewrite the matrix A in factored LDU form, where D is the diagonal matrix that factors the pivots out of U.

$$A = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix}$$

Solution:

Let's set up $A = LU$.

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix}$$

We only need to zero-out the entries in U that are below the main diagonal. To zero-out the 8 in $U_{2,1}$, we need to subtract 2 of R_1 from R_2. Since the row operation is $R_2 - 2R_1$, we put 2 into $L_{2,1}$.

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix}$$

To zero-out the 8 in $U_{3,1}$, we need to subtract 2 of R_1 from R_3. Since the row operation is $R_3 - 2R_1$, we put 2 into $L_{3,1}$.

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & & 1 & 0 \\ & & & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 15 & 27 & 28 \end{bmatrix}$$

We don't need to perform any row operation to keep the 0 in $L_{4,1}$, so

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & & 1 & 0 \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 15 & 27 & 28 \end{bmatrix}$$

We don't need to perform any row operation to keep the 0 in $L_{3,2}$, so

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & & & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 15 & 27 & 28 \end{bmatrix}$$

To zero-out the 15 in $U_{4,2}$, we need to subtract 5 of R_2 from R_4. Since the row operation is $R_4 - 5R_2$, we put 5 into $L_{4,2}$.

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 5 & & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 12 & 13 \end{bmatrix}$$

To zero-out the 12 in $U_{4,3}$, we need to subtract 6 of R_3 from R_4. Since the row operation is $R_4 - 6R_3$, we put 6 into $L_{4,3}$.

$$\begin{bmatrix} 4 & 4 & 4 & 4 \\ 8 & 11 & 11 & 11 \\ 8 & 8 & 10 & 10 \\ 0 & 15 & 27 & 28 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 5 & 6 & 1 \end{bmatrix} \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 3 & 3 & 3 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To rewrite the $A = LU$ factorization as $A = LDU$, divide through the first row of U by 4, moving the 4 pivot into the first row of D.

$$\begin{bmatrix}4&4&4&4\\8&11&11&11\\8&8&10&10\\0&15&27&28\end{bmatrix}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\2&0&1&0\\0&5&6&1\end{bmatrix}\begin{bmatrix}4&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&1&1&1\\0&3&3&3\\0&0&2&2\\0&0&0&1\end{bmatrix}$$

Divide through the second row of U by 3, moving the 3 pivot into the second row of D.

$$\begin{bmatrix}4&4&4&4\\8&11&11&11\\8&8&10&10\\0&15&27&28\end{bmatrix}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\2&0&1&0\\0&5&6&1\end{bmatrix}\begin{bmatrix}4&0&0&0\\0&3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&2&2\\0&0&0&1\end{bmatrix}$$

Divide through the third row of U by 2, moving the 2 pivot into the third row of D.

$$\begin{bmatrix}4&4&4&4\\8&11&11&11\\8&8&10&10\\0&15&27&28\end{bmatrix}=\begin{bmatrix}1&0&0&0\\2&1&0&0\\2&0&1&0\\0&5&6&1\end{bmatrix}\begin{bmatrix}4&0&0&0\\0&3&0&0\\0&0&2&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}1&1&1&1\\0&1&1&1\\0&0&1&1\\0&0&0&1\end{bmatrix}$$

ORTHOGONAL COMPLEMENTS

■ 1. Find the orthogonal complement of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix}-2\\1\\4\end{bmatrix}, \begin{bmatrix}0\\-3\\2\end{bmatrix}\right)$$

Solution:

The subspace V is a plane in $\mathbb{R}^3$, spanned by $\vec{v}_1 = (-2,1,4)$ and $\vec{v}_2 = (0,-3,2)$. So its orthogonal complement $V^{\perp}$ is the set of vectors which are orthogonal to both $\vec{v}_1 = (-2,1,4)$ and $\vec{v}_2 = (0,-3,2)$.

$$V^{\perp} = \{\vec{x} \in \mathbb{R}^3 \mid \vec{x} \cdot \begin{bmatrix}-2\\1\\4\end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix}0\\-3\\2\end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3)$, we get two equations from these dot products.

$$-2x_1 + x_2 + 4x_3 = 0$$

$$-3x_2 + 2x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}-2 & 1 & 4 & | & 0\\0 & -3 & 2 & | & 0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{1}{2} & -2 & | & 0 \\ 0 & -3 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & -2 & | & 0 \\ 0 & 1 & -\frac{2}{3} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{7}{3} & | & 0 \\ 0 & 1 & -\frac{2}{3} & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 - \frac{7}{3}x_3 = 0$$

$$x_2 - \frac{2}{3}x_3 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = \frac{7}{3}x_3$$

$$x_2 = \frac{2}{3}x_3$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}$$

Which means the orthogonal complement $V^\perp$ is

$$V^\perp = \text{Span}\left(\begin{bmatrix} \frac{7}{3} \\ \frac{2}{3} \\ 1 \end{bmatrix}\right)$$

■ 2. Find the orthogonal complement of V, $V^\perp$.

$$V = \text{Span}\left(\begin{bmatrix}-1\\2\\-5\\3\end{bmatrix}, \begin{bmatrix}1\\0\\-4\\3\end{bmatrix}\right)$$

Solution:

The subspace V is a plane in $\mathbb{R}^4$, spanned by $\vec{v}_1 = (-1,2,-5,3)$ and $\vec{v}_2 = (1,0,-4,3)$. So its orthogonal complement $V^\perp$ is the set of vectors which are orthogonal to both $\vec{v}_1 = (-1,2,-5,3)$ and $\vec{v}_2 = (1,0,-4,3)$.

$$V^\perp = \{\vec{x} \in \mathbb{R}^4 \ | \ \vec{x} \cdot \begin{bmatrix}-1\\2\\-5\\3\end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix}1\\0\\-4\\3\end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4)$, we get two equations from these dot products.

$$-x_1 + 2x_2 - 5x_3 + 3x_4 = 0$$

$$x_1 - 4x_3 + 3x_4 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}-1 & 2 & -5 & 3 & | & 0\\ 1 & 0 & -4 & 3 & | & 0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 5 & -3 & | & 0 \\ 1 & 0 & -4 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 5 & -3 & | & 0 \\ 0 & 2 & -9 & 6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & 5 & -3 & | & 0 \\ 0 & 1 & -\frac{9}{2} & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -4 & 3 & | & 0 \\ 0 & 1 & -\frac{9}{2} & 3 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 - 4x_3 + 3x_4 = 0$$

$$x_2 - \frac{9}{2}x_3 + 3x_4 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = 4x_3 - 3x_4$$

$$x_2 = \frac{9}{2}x_3 - 3x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 4 \\ \frac{9}{2} \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -3 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement $V^{\perp}$ is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix}4\\ \frac{9}{2}\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}-3\\ -3\\ 0\\ 1\end{bmatrix}\right)$$

■ 3. Rewrite the orthogonal complement of V, $V^{\perp}$, if V is a vector set in $\mathbb{R}^3$.

$$V = \begin{bmatrix}s\\ -2s-t\\ s+t\end{bmatrix}$$

Solution:

We can rewrite V as

$$V = \{s \cdot \begin{bmatrix}1\\ -2\\ 1\end{bmatrix} + t \cdot \begin{bmatrix}0\\ -1\\ 1\end{bmatrix} \ | \ s, t \in \mathbb{R}^3\}$$

The subspace V is a plane in $\mathbb{R}^3$, spanned by $\overrightarrow{v}_1 = (1, -2, 1)$ and $\overrightarrow{v}_2 = (0, -1, 1)$. So its orthogonal complement $V^{\perp}$ is the set of vectors which are orthogonal to both $\overrightarrow{v}_1 = (1, -2, 1)$ and $\overrightarrow{v}_2 = (0, -1, 1)$.

$$V^{\perp} = \{\overrightarrow{x} \in \mathbb{R}^3 \ | \ \overrightarrow{x} \cdot \begin{bmatrix}1\\ -2\\ 1\end{bmatrix} = 0 \quad \text{and} \quad \overrightarrow{x} \cdot \begin{bmatrix}0\\ -1\\ 1\end{bmatrix} = 0\}$$

If we let $\overrightarrow{x} = (x_1, x_2, x_3)$, we get two equations from these dot products.

$$x_1 - 2x_2 + x_3 = 0$$

$$-x_2 + x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & -1 & 1 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & -1 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 - x_3 = 0$$

$$x_2 - x_3 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = x_3$$

$$x_2 = x_3$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

■ 4. Rewrite the orthogonal complement of W, $W^{\perp}$, if W is a vector set in $\mathbb{R}^4$.

$$W = \begin{bmatrix}-2y-z\\3y+z\\-y\\2y-3z\end{bmatrix}$$

Solution:

We can rewrite W as

$$W = \{y \cdot \begin{bmatrix}-2\\3\\-1\\2\end{bmatrix} + z \cdot \begin{bmatrix}-1\\1\\0\\-3\end{bmatrix} \mid y, z \in \mathbb{R}^4\}$$

The subspace W is a plane in $\mathbb{R}^4$, spanned by $\vec{w}_1 = (-2,3,-1,2)$ and $\vec{w}_2 = (-1,1,0,-3)$. So its orthogonal complement $W^{\perp}$ is the set of vectors which are orthogonal to both $\vec{w}_1 = (-2,3,-1,2)$ and $\vec{w}_2 = (-1,1,0,-3)$.

$$W^{\perp} = \{\vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix}-2\\3\\-1\\2\end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix}-1\\1\\0\\-3\end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4)$, we get two equations from these dot products.

$$-2x_1 + 3x_2 - x_3 + 2x_4 = 0$$

$$-x_1 + x_2 - 3x_4 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} -2 & 3 & -1 & 2 & | & 0 \\ -1 & 1 & 0 & -3 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} & -1 & | & 0 \\ -1 & 1 & 0 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} & -1 & | & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -4 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{2} & \frac{1}{2} & -1 & | & 0 \\ 0 & 1 & -1 & 8 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & 11 & | & 0 \\ 0 & 1 & -1 & 8 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 - x_3 + 11x_4 = 0$$

$$x_2 - x_3 + 8x_4 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = x_3 - 11x_4$$

$$x_2 = x_3 - 8x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -11 \\ -8 \\ 0 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$W^{\perp} = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -11 \\ -8 \\ 0 \\ 1 \end{bmatrix}\right)$$

■ 5. Describe the orthogonal component of V, $V^{\perp}$.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix}\right)$$

Solution:

The subspace V is a plane in $\mathbb{R}^4$, spanned by the $\overrightarrow{v}_1 = (1,-1,0,1)$, $\overrightarrow{v}_2 = (-1,2,1,3)$, and $\overrightarrow{v}_3 = (2,-1,3,0)$. So its orthogonal complement $V^{\perp}$ is the set of vectors which are orthogonal to $\overrightarrow{v}_1 = (1,-1,0,1)$, $\overrightarrow{v}_2 = (-1,2,1,3)$, and $\overrightarrow{v}_3 = (2,-1,3,0)$.

$$V^{\perp} = \{ \vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} = 0 \ , \vec{x} \cdot \begin{bmatrix} -1 \\ 2 \\ 1 \\ 3 \end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix} 2 \\ -1 \\ 3 \\ 0 \end{bmatrix} = 0 \}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4)$, we get three equations from these dot products.

$$x_1 - x_2 + x_4 = 0$$

$$-x_1 + 2x_2 + x_3 + 3x_4 = 0$$

$$2x_1 - x_2 + 3x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} 1 & -1 & 0 & 1 & | & 0 \\ -1 & 2 & 1 & 3 & | & 0 \\ 2 & -1 & 3 & 0 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & -1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & 4 & | & 0 \\ 2 & -1 & 3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & 4 & | & 0 \\ 0 & 1 & 3 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 & | & 0 \\ 0 & 1 & 1 & 4 & | & 0 \\ 0 & 1 & 3 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 5 & | & 0 \\ 0 & 1 & 1 & 4 & | & 0 \\ 0 & 0 & 2 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 5 & | & 0 \\ 0 & 1 & 1 & 4 & | & 0 \\ 0 & 0 & 1 & -3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 8 & | & 0 \\ 0 & 1 & 1 & 4 & | & 0 \\ 0 & 0 & 1 & -3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 8 & | & 0 \\ 0 & 1 & 0 & 7 & | & 0 \\ 0 & 0 & 1 & -3 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + 8x_4 = 0$$

$$x_2 + 7x_4 = 0$$

$$x_3 - 3x_4 = 0$$

which we can solve for the pivot variables.

$$x_1 = -8x_4$$

$$x_2 = -7x_4$$

$$x_3 = 3x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -8 \\ -7 \\ 3 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix} -8 \\ -7 \\ 3 \\ 1 \end{bmatrix} \right)$$

■ 6. Describe the orthogonal component of W, $W^{\perp}$.

$$W = \text{Span}\left(\begin{bmatrix}1\\0\\2\\-1\\2\end{bmatrix}, \begin{bmatrix}-3\\2\\4\\1\\-2\end{bmatrix}\right)$$

Solution:

The subspace W is a plane in $\mathbb{R}^5$, spanned by $\vec{w}_1 = (1,0,2,-1,2)$ and $\vec{w}_2 = (-3,2,4,1,-2)$. So its orthogonal complement $W^\perp$ is the set of vectors which are orthogonal to $\vec{w}_1 = (1,0,2,-1,2)$ and $\vec{w}_2 = (-3,2,4,1,-2)$.

$$W^\perp = \{\vec{x} \in \mathbb{R}^5 \mid \vec{x} \cdot \begin{bmatrix}1\\0\\2\\-1\\2\end{bmatrix} = 0 \quad \text{and} \quad \vec{x} \cdot \begin{bmatrix}-3\\2\\4\\1\\-2\end{bmatrix} = 0\}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4, x_5)$, we get two equations from these dot products.

$$x_1 + 2x_3 - x_4 + 2x_5 = 0$$

$$-3x_1 + 2x_2 + 4x_3 + x_4 - 2x_5 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix}1 & 0 & 2 & -1 & 2 & | & 0\\-3 & 2 & 4 & 1 & -2 & | & 0\end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 0 & 2 & -1 & 2 & | & 0 \\ 0 & 2 & 10 & -2 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -1 & 2 & | & 0 \\ 0 & 1 & 5 & -1 & 2 & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 + 2x_3 - x_4 + 2x_5 = 0$$

$$x_2 + 5x_3 - x_4 + 2x_5 = 0$$

which we can solve for the pivot variables.

$$x_1 = -2x_3 + x_4 - 2x_5$$

$$x_2 = -5x_3 + x_4 - 2x_5$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$W^{\perp} = \text{Span}\left(\begin{bmatrix} -2 \\ -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

ORTHOGONAL COMPLEMENTS OF THE FUNDAMENTAL SUBSPACES

■ 1. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -2 & 6 & 0 \\ -1 & 4 & 3 \\ 2 & -5 & 3 \end{bmatrix}$$

Solution:

Put M into reduced row-echelon form.

$$\begin{bmatrix} -2 & 6 & 0 \\ -1 & 4 & 3 \\ 2 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ -1 & 4 & 3 \\ 2 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 2 & -5 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 9 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of M is $r = 2$.

The matrix M is a 3×3 matrix, which means there are $m = 3$ rows and $n = 3$ columns. Therefore, the dimensions of the four fundamental subspaces of M are:

Column space, $C(M)$ $\qquad r = 2$

Null space, $N(M)$ $\quad$ $n - r = 3 - 2 = 1$

Row space, $C(M^T)$ $\quad$ $r = 2$

Left null space, $N(M^T)$ $\quad$ $m - r = 3 - 2 = 1$

■ 2. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -1 & 0 & 2 & -4 \\ -2 & 3 & -5 & 1 \\ 1 & -2 & 4 & 0 \end{bmatrix}$$

Solution:

Put M into reduced row-echelon form.

$$\begin{bmatrix} -1 & 0 & 2 & -4 \\ -2 & 3 & -5 & 1 \\ 1 & -2 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 \\ -2 & 3 & -5 & 1 \\ 1 & -2 & 4 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 3 & -9 & 9 \\ 1 & -2 & 4 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 3 & -9 & 9 \\ 0 & -2 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & -2 & 6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -2 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are three pivots, which means the rank of M is $r = 3$.

The matrix M is a 3×4 matrix, which means there are $m = 3$ rows and $n = 4$ columns. Therefore, the dimensions of the four fundamental subspaces of M are:

Column space, $C(M)$	$r = 3$
Null space, $N(M)$	$n - r = 4 - 3 = 1$
Row space, $C(M^T)$	$r = 3$
Left null space, $N(M^T)$	$m - r = 3 - 3 = 0$

■ 3. For the matrix X, find the dimensions of all four fundamental subspaces.

$$X = \begin{bmatrix} 1 & -2 & 4 \\ -3 & 5 & 0 \\ -1 & 2 & 3 \end{bmatrix}$$

Solution:

Put X into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & 4 \\ -3 & 5 & 0 \\ -1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 12 \\ -1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 0 & -1 & 12 \\ 0 & 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 4 \\ 0 & 1 & -12 \\ 0 & 0 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -12 \\ 0 & 0 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -20 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -12 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are three pivots, which means the rank of X is $r = 3$.

The matrix X is a 3×3 matrix, which means there are $m = 3$ rows and $n = 3$ columns. Therefore, the dimensions of the four fundamental subspaces of X are:

Column space, $C(X)$	$r = 3$
Null space, $N(X)$	$n - r = 3 - 3 = 0$
Row space, $C(X^T)$	$r = 3$
Left null space, $N(X^T)$	$m - r = 3 - 3 = 0$

■ 4. For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} -1 & -3 & 2 & 1 \\ -2 & -5 & 5 & -1 \\ -3 & -7 & 8 & -3 \end{bmatrix}$$

Solution:

Put A into reduced row-echelon form.

$$\begin{bmatrix} -1 & -3 & 2 & 1 \\ -2 & -5 & 5 & -1 \\ -3 & -7 & 8 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 \\ -2 & -5 & 5 & -1 \\ -3 & -7 & 8 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -2 & -1 \\ 0 & 1 & 1 & -3 \\ -3 & -7 & 8 & -3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -2 & -1 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 & 8 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 2 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -5 & 8 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of A is $r = 2$.

The matrix A is a 3×4 matrix, which means there are $m = 3$ rows and $n = 4$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$	$r = 2$
Null space, $N(A)$	$n - r = 4 - 2 = 2$
Row space, $C(A^T)$	$r = 2$
Left null space, $N(A^T)$	$m - r = 3 - 2 = 1$

5. For the matrix A, find the dimensions of all four fundamental subspaces.

$$A = \begin{bmatrix} 1 & -1 & 3 & 0 & 2 \\ -1 & 4 & -3 & 1 & 0 \\ 2 & -11 & 6 & -3 & -2 \end{bmatrix}$$

Solution:

Put A into reduced row-echelon form.

$$\begin{bmatrix}1&-1&3&0&2\\-1&4&-3&1&0\\2&-11&6&-3&-2\end{bmatrix}\to\begin{bmatrix}1&-1&3&0&2\\0&3&0&1&2\\2&-11&6&-3&-2\end{bmatrix}\to\begin{bmatrix}1&-1&3&0&2\\0&3&0&1&2\\0&-9&0&-3&-6\end{bmatrix}$$

$$\begin{bmatrix}1&-1&3&0&2\\0&1&0&\frac{1}{3}&\frac{2}{3}\\0&-9&0&-3&-6\end{bmatrix}\to\begin{bmatrix}1&-1&3&0&2\\0&1&0&\frac{1}{3}&\frac{2}{3}\\0&0&0&0&0\end{bmatrix}\to\begin{bmatrix}1&0&3&\frac{1}{3}&\frac{8}{3}\\0&1&0&\frac{1}{3}&\frac{2}{3}\\0&0&0&0&0\end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of A is $r=2$.

The matrix A is a 3×5 matrix, which means there are $m=3$ rows and $n=5$ columns. Therefore, the dimensions of the four fundamental subspaces of A are:

Column space, $C(A)$	$r=2$
Null space, $N(A)$	$n-r=5-2=3$
Row space, $C(A^T)$	$r=2$
Left null space, $N(A^T)$	$m-r=3-2=1$

■ 6. For the matrix M, find the dimensions of all four fundamental subspaces.

$$M = \begin{bmatrix} -2 & 2 & -4 \\ 1 & -2 & 0 \\ -3 & 5 & -2 \\ 1 & 2 & 8 \end{bmatrix}$$

Solution:

Put M into reduced row-echelon form.

$$\begin{bmatrix} -2 & 2 & -4 \\ 1 & -2 & 0 \\ -3 & 5 & -2 \\ 1 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 0 \\ -3 & 5 & -2 \\ 1 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -2 \\ -3 & 5 & -2 \\ 1 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \\ 1 & 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 2 \\ 0 & -1 & -2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 3 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

In reduced row-echelon form, we can see that there are two pivots, which means the rank of M is $r = 2$.

The matrix M is a 4×3 matrix, which means there are $m = 4$ rows and $n = 3$ columns. Therefore, the dimensions of the four fundamental subspaces of M are:

Column space, $C(M)$	$r = 2$
Null space, $N(M)$	$n - r = 3 - 2 = 1$
Row space, $C(M^T)$	$r = 2$

Left null space, $N(M^T)$ $\quad m - r = 4 - 2 = 2$

PROJECTION ONTO THE SUBSPACE

■ 1. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix}-1\\3\\-2\end{bmatrix}, \begin{bmatrix}0\\-2\\3\end{bmatrix}\right)$$

Solution:

Because the vectors that span V are linearly independent, the matrix A of the basis vectors that define V is

$$A = \begin{bmatrix}-1 & 0\\3 & -2\\-2 & 3\end{bmatrix}$$

The transpose A^T is

$$A^T = \begin{bmatrix}-1 & 3 & -2\\0 & -2 & 3\end{bmatrix}$$

Find A^TA.

$$A^TA = \begin{bmatrix}-1 & 3 & -2\\0 & -2 & 3\end{bmatrix}\begin{bmatrix}-1 & 0\\3 & -2\\-2 & 3\end{bmatrix}$$

$$A^TA = \begin{bmatrix}-1(-1)+3(3)-2(-2) & -1(0)+3(-2)-2(3)\\0(-1)-2(3)+3(-2) & 0(0)-2(-2)+3(3)\end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+9+4 & 0-6-6 \\ 0-6-6 & 0+4+9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 14 & -12 \\ -12 & 13 \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_2] = \begin{bmatrix} 14 & -12 & | & 1 & 0 \\ -12 & 13 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{6}{7} & | & \frac{1}{14} & 0 \\ -12 & 13 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{6}{7} & | & \frac{1}{14} & 0 \\ 0 & \frac{19}{7} & | & \frac{6}{7} & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{6}{7} & | & \frac{1}{14} & 0 \\ 0 & 1 & | & \frac{6}{19} & \frac{7}{19} \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{13}{38} & \frac{6}{19} \\ 0 & 1 & | & \frac{6}{19} & \frac{7}{19} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{13}{38} & \frac{6}{19} \\ \frac{6}{19} & \frac{7}{19} \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{38}\begin{bmatrix}13 & 12\\12 & 14\end{bmatrix}$$

Now the projection of $\vec{x}$ onto the subspace V will be

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix}-1 & 0\\3 & -2\\-2 & 3\end{bmatrix}\frac{1}{38}\begin{bmatrix}13 & 12\\12 & 14\end{bmatrix}\begin{bmatrix}-1 & 3 & -2\\0 & -2 & 3\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{38}\begin{bmatrix}-1 & 0\\3 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}13 & 12\\12 & 14\end{bmatrix}\begin{bmatrix}-1 & 3 & -2\\0 & -2 & 3\end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \frac{1}{38}\begin{bmatrix}-1 & 0\\3 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}13(-1)+12(0) & 13(3)+12(-2) & 13(-2)+12(3)\\12(-1)+14(0) & 12(3)+14(-2) & 12(-2)+14(3)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{38}\begin{bmatrix}-1 & 0\\3 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}-13+0 & 39-24 & -26+36\\-12+0 & 36-28 & -24+42\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{38}\begin{bmatrix}-1 & 0\\3 & -2\\-2 & 3\end{bmatrix}\begin{bmatrix}-13 & 15 & 10\\-12 & 8 & 18\end{bmatrix}\vec{x}$$

Next, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \frac{1}{38}\begin{bmatrix}-1(-13)+0(-12) & -1(15)+0(8) & -1(10)+0(18)\\3(-13)-2(-12) & 3(15)-2(8) & 3(10)-2(18)\\-2(-13)+3(-12) & -2(15)+3(8) & -2(10)+3(18)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{38}\begin{bmatrix} 13+0 & -15+0 & -10+0 \\ -39+24 & 45-16 & 30-36 \\ 26-36 & -30+24 & -20+54 \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{38}\begin{bmatrix} 13 & -15 & -10 \\ -15 & 29 & -6 \\ -10 & -6 & 34 \end{bmatrix}\vec{x}$$

■ 2. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} -2 \\ -4 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix}\right)$$

Solution:

Because the vectors that span V are linearly independent, the matrix A of the basis vectors that define V is

$$A = \begin{bmatrix} -2 & -1 & 1 \\ -4 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix}$$

The transpose A^T is

$$A^T = \begin{bmatrix} -2 & -4 & 0 \\ -1 & 1 & 2 \\ 1 & -3 & -2 \end{bmatrix}$$

Find A^TA.

$$A^TA = \begin{bmatrix} -2 & -4 & 0 \\ -1 & 1 & 2 \\ 1 & -3 & -2 \end{bmatrix} \begin{bmatrix} -2 & -1 & 1 \\ -4 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -2(-2)-4(-4)+0(0) & -2(-1)-4(1)+0(2) & -2(1)-4(-3)+0(-2) \\ -1(-2)+1(-4)+2(0) & -1(-1)+1(1)+2(2) & -1(1)+1(-3)+2(-2) \\ 1(-2)-3(-4)-2(0) & 1(-1)-3(1)-2(2) & 1(1)-3(-3)-2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4+16+0 & 2-4+0 & -2+12+0 \\ 2-4+0 & 1+1+4 & -1-3-4 \\ -2+12-0 & -1-3-4 & 1+9+4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 20 & -2 & 10 \\ -2 & 6 & -8 \\ 10 & -8 & 14 \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_3] = \begin{bmatrix} 20 & -2 & 10 & | & 1 & 0 & 0 \\ -2 & 6 & -8 & | & 0 & 1 & 0 \\ 10 & -8 & 14 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -\frac{1}{10} & \frac{1}{2} & | & \frac{1}{20} & 0 & 0 \\ -2 & 6 & -8 & | & 0 & 1 & 0 \\ 10 & -8 & 14 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -\frac{1}{10} & \frac{1}{2} & | & \frac{1}{20} & 0 & 0 \\ 0 & \frac{29}{5} & -7 & | & \frac{1}{10} & 1 & 0 \\ 10 & -8 & 14 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA\ |\ I_3]=\begin{bmatrix}1 & -\frac{1}{10} & \frac{1}{2} & | & \frac{1}{20} & 0 & 0\\ 0 & \frac{29}{5} & -7 & | & \frac{1}{10} & 1 & 0\\ 0 & -7 & 9 & | & -\frac{1}{2} & 0 & 1\end{bmatrix}$$

$$[A^TA\ |\ I_3]=\begin{bmatrix}1 & -\frac{1}{10} & \frac{1}{2} & | & \frac{1}{20} & 0 & 0\\ 0 & 1 & -\frac{35}{29} & | & \frac{1}{58} & \frac{5}{29} & 0\\ 0 & -7 & 9 & | & -\frac{1}{2} & 0 & 1\end{bmatrix}$$

$$[A^TA\ |\ I_3]=\begin{bmatrix}1 & 0 & \frac{11}{29} & | & \frac{3}{58} & \frac{1}{58} & 0\\ 0 & 1 & -\frac{35}{29} & | & \frac{1}{58} & \frac{5}{29} & 0\\ 0 & -7 & 9 & | & -\frac{1}{2} & 0 & 1\end{bmatrix}$$

$$[A^TA\ |\ I_3]=\begin{bmatrix}1 & 0 & \frac{11}{29} & | & \frac{3}{58} & \frac{1}{58} & 0\\ 0 & 1 & -\frac{35}{29} & | & \frac{1}{58} & \frac{5}{29} & 0\\ 0 & 0 & \frac{16}{29} & | & -\frac{11}{29} & \frac{35}{29} & 1\end{bmatrix}$$

$$[A^TA\ |\ I_3]=\begin{bmatrix}1 & 0 & \frac{11}{29} & | & \frac{3}{58} & \frac{1}{58} & 0\\ 0 & 1 & -\frac{35}{29} & | & \frac{1}{58} & \frac{5}{29} & 0\\ 0 & 0 & 1 & | & -\frac{11}{16} & \frac{35}{16} & \frac{29}{16}\end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 0 & | & \frac{5}{16} & -\frac{13}{16} & -\frac{11}{16} \\ 0 & 1 & -\frac{35}{29} & | & \frac{1}{58} & \frac{5}{29} & 0 \\ 0 & 0 & 1 & | & -\frac{11}{16} & \frac{35}{16} & \frac{29}{16} \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 0 & | & \frac{5}{16} & -\frac{13}{16} & -\frac{11}{16} \\ 0 & 1 & 0 & | & -\frac{13}{16} & \frac{45}{16} & \frac{35}{16} \\ 0 & 0 & 1 & | & -\frac{11}{16} & \frac{35}{16} & \frac{29}{16} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{5}{16} & -\frac{13}{16} & -\frac{11}{16} \\ -\frac{13}{16} & \frac{45}{16} & \frac{35}{16} \\ -\frac{11}{16} & \frac{35}{16} & \frac{29}{16} \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{16}\begin{bmatrix} 5 & -13 & -11 \\ -13 & 45 & 35 \\ -11 & 35 & 29 \end{bmatrix}$$

Then the projection of $\vec{x}$ onto the subspace V will be

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -2 & -1 & 1 \\ -4 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix}\frac{1}{16}\begin{bmatrix} 5 & -13 & -11 \\ -13 & 45 & 35 \\ -11 & 35 & 29 \end{bmatrix}\begin{bmatrix} -2 & -4 & 0 \\ -1 & 1 & 2 \\ 1 & -3 & -2 \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{16}\begin{bmatrix} -2 & -1 & 1 \\ -4 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} 5 & -13 & -11 \\ -13 & 45 & 35 \\ -11 & 35 & 29 \end{bmatrix}\begin{bmatrix} -2 & -4 & 0 \\ -1 & 1 & 2 \\ 1 & -3 & -2 \end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\begin{bmatrix} 5(-2)-13(-1)-11(1) & 5(-4)-13(1)-11(-3) & 5(0)-13(2)-11(-2) \\ -13(-2)+45(-1)+35(1) & -13(-4)+45(1)+35(-3) & -13(0)+45(2)+35(-2) \\ -11(-2)+35(-1)+29(1) & -11(-4)+35(1)+29(-3) & -11(0)+35(2)+29(-2) \end{bmatrix}$$

$$\begin{bmatrix} -10+13-11 & -20-13+33 & 0-26+22 \\ 26-45+35 & 52+45-105 & 0+90-70 \\ 22-35+29 & 44+35-87 & 0+70-58 \end{bmatrix}$$

$$\begin{bmatrix} -8 & 0 & -4 \\ 16 & -8 & 20 \\ 16 & -8 & 12 \end{bmatrix}$$

So we get

$$\text{Proj}_V\vec{x} = \frac{1}{16}\begin{bmatrix} -2 & -1 & 1 \\ -4 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} -8 & 0 & -4 \\ 16 & -8 & 20 \\ 16 & -8 & 12 \end{bmatrix}\vec{x}$$

Now, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_V\vec{x} = \frac{1}{4}\begin{bmatrix} -2 & -1 & 1 \\ -4 & 1 & -3 \\ 0 & 2 & -2 \end{bmatrix}\begin{bmatrix} -2 & 0 & -1 \\ 4 & -2 & 5 \\ 4 & -2 & 3 \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{4}\begin{bmatrix} -2(-2)-1(4)+1(4) & -2(0)-1(-2)+1(-2) & -2(-1)-1(5)+1(3) \\ -4(-2)+1(4)-3(4) & -4(0)+1(-2)-3(-2) & -4(-1)+1(5)-3(3) \\ 0(-2)+2(4)-2(4) & 0(0)+2(-2)-2(-2) & 0(-1)+2(5)-2(3) \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{4}\begin{bmatrix} 4-4+4 & 0+2-2 & 2-5+3 \\ 8+4-12 & 0-2+6 & 4+5-9 \\ 0+8-8 & 0-4+4 & 0+10-6 \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{4}\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\vec{x}$$

■ 3. If $\vec{x}$ is a vector in $\mathbb{R}^3$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} \text{ and } \vec{x}_2 \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$

Solution:

Because the vectors that span S are linearly independent, the matrix A of the basis vectors that define S is

$$A = \begin{bmatrix} 1 & -1 \\ -3 & 0 \\ 1 & -2 \end{bmatrix}$$

The transpose A^T is

$$A^T = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 0 & -2 \end{bmatrix}$$

Find A^TA.

$$A^TA = \begin{bmatrix} 1 & -3 & 1 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -3 & 0 \\ 1 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) - 3(-3) + 1(1) & 1(-1) - 3(0) + 1(-2) \\ -1(1) + 0(-3) - 2(1) & -1(-1) + 0(0) - 2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 9 + 1 & -1 - 0 - 2 \\ -1 + 0 - 2 & 1 + 0 + 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 11 & -3 \\ -3 & 5 \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_2] = \begin{bmatrix} 11 & -3 & | & 1 & 0 \\ -3 & 5 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{3}{11} & | & \frac{1}{11} & 0 \\ -3 & 5 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{3}{11} & | & \frac{1}{11} & 0 \\ 0 & \frac{46}{11} & | & \frac{3}{11} & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & -\frac{3}{11} & | & \frac{1}{11} & 0 \\ 0 & 1 & | & \frac{3}{46} & \frac{11}{46} \end{bmatrix}$$

$$[A^TA\mid I_2]=\begin{bmatrix}1 & 0 & | & \frac{5}{46} & \frac{3}{46}\\ 0 & 1 & | & \frac{3}{46} & \frac{11}{46}\end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1}=\begin{bmatrix}\frac{5}{46} & \frac{3}{46}\\ \frac{3}{46} & \frac{11}{46}\end{bmatrix}$$

$$(A^TA)^{-1}=\frac{1}{46}\begin{bmatrix}5 & 3\\ 3 & 11\end{bmatrix}$$

The projection of $\vec{x}$ onto the subspace S will be

$$\text{Proj}_S\vec{x}=A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_S\vec{x}=\begin{bmatrix}1 & -1\\ -3 & 0\\ 1 & -2\end{bmatrix}\frac{1}{46}\begin{bmatrix}5 & 3\\ 3 & 11\end{bmatrix}\begin{bmatrix}1 & -3 & 1\\ -1 & 0 & -2\end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x}=\frac{1}{46}\begin{bmatrix}1 & -1\\ -3 & 0\\ 1 & -2\end{bmatrix}\begin{bmatrix}5 & 3\\ 3 & 11\end{bmatrix}\begin{bmatrix}1 & -3 & 1\\ -1 & 0 & -2\end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_S\vec{x}=\frac{1}{46}\begin{bmatrix}1 & -1\\ -3 & 0\\ 1 & -2\end{bmatrix}\begin{bmatrix}5(1)+3(-1) & 5(-3)+3(0) & 5(1)+3(-2)\\ 3(1)+11(-1) & 3(-3)+11(0) & 3(1)+11(-2)\end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x}=\frac{1}{46}\begin{bmatrix}1 & -1\\ -3 & 0\\ 1 & -2\end{bmatrix}\begin{bmatrix}5-3 & -15+0 & 5-6\\ 3-11 & -9+0 & 3-22\end{bmatrix}\vec{x}$$

$$\text{Proj}_S \vec{x} = \frac{1}{46}\begin{bmatrix} 1 & -1 \\ -3 & 0 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & -15 & -1 \\ -8 & -9 & -19 \end{bmatrix}\vec{x}$$

Now, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_S \vec{x} = \frac{1}{46}\begin{bmatrix} 1(2)-1(-8) & 1(-15)-1(-9) & 1(-1)-1(-19) \\ -3(2)+0(-8) & -3(-15)+0(-9) & -3(-1)+0(-19) \\ 1(2)-2(-8) & 1(-15)-2(-9) & 1(-1)-2(-19) \end{bmatrix}\vec{x}$$

$$\text{Proj}_S \vec{x} = \frac{1}{46}\begin{bmatrix} 2+8 & -15+9 & -1+19 \\ -6+0 & 45+0 & 3+0 \\ 2+16 & -15+18 & -1+38 \end{bmatrix}\vec{x}$$

$$\text{Proj}_S \vec{x} = \frac{1}{46}\begin{bmatrix} 10 & -6 & 18 \\ -6 & 45 & 3 \\ 18 & 3 & 37 \end{bmatrix}\vec{x}$$

■ 4. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{2}\begin{bmatrix} 1 \\ -2 \\ -1 \\ -1 \end{bmatrix} \text{ and } \vec{x}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \\ -3 \end{bmatrix}$$

Solution:

Because the vectors that span S are linearly independent, the matrix A of the basis vectors that define S is

$$A = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 0 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}$$

The transpose A^T is

$$A^T = \frac{1}{2}\begin{bmatrix} 1 & -2 & -1 & -1 \\ -1 & 0 & 1 & -3 \end{bmatrix}$$

Find A^TA.

$$A^TA = \frac{1}{2}\begin{bmatrix} 1 & -2 & -1 & -1 \\ -1 & 0 & 1 & -3 \end{bmatrix} \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 0 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$A^TA = \frac{1}{4}\begin{bmatrix} 1 & -2 & -1 & -1 \\ -1 & 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -2 & 0 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}$$

$$A^TA = \frac{1}{4}\begin{bmatrix} 1(1) - 2(-2) - 1(-1) - 1(-1) & 1(-1) - 2(0) - 1(1) - 1(-3) \\ -1(1) + 0(-2) + 1(-1) - 3(-1) & -1(-1) + 0(0) + 1(1) - 3(-3) \end{bmatrix}$$

$$A^TA = \frac{1}{4}\begin{bmatrix} 1 + 4 + 1 + 1 & -1 - 0 - 1 + 3 \\ -1 + 0 - 1 + 3 & 1 + 0 + 1 + 9 \end{bmatrix}$$

$$A^TA = \frac{1}{4}\begin{bmatrix} 7 & 1 \\ 1 & 11 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} \frac{7}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{11}{4} \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_2] = \begin{bmatrix} \frac{7}{4} & \frac{1}{4} & | & 1 & 0 \\ \frac{1}{4} & \frac{11}{4} & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & \frac{1}{7} & | & \frac{4}{7} & 0 \\ \frac{1}{4} & \frac{11}{4} & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & \frac{1}{7} & | & \frac{4}{7} & 0 \\ 0 & \frac{19}{7} & | & -\frac{1}{7} & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & \frac{1}{7} & | & \frac{4}{7} & 0 \\ 0 & 1 & | & -\frac{1}{19} & \frac{7}{19} \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{11}{19} & -\frac{1}{19} \\ 0 & 1 & | & -\frac{1}{19} & \frac{7}{19} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{11}{19} & -\frac{1}{19} \\ -\frac{1}{19} & \frac{7}{19} \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{19}\begin{bmatrix} 11 & -1 \\ -1 & 7 \end{bmatrix}$$

The projection of $\vec{x}$ onto the subspace S will be

$$\text{Proj}_S\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ -2 & 0 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}\frac{1}{19}\begin{bmatrix} 11 & -1 \\ -1 & 7 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -2 & -1 & -1 \\ -1 & 0 & 1 & -3 \end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_S\vec{x} = \frac{1}{76}\begin{bmatrix} 1 & -1 \\ -2 & 0 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 11(1)-1(-1) & 11(-2)-1(0) & 11(-1)-1(1) & 11(-1)-1(-3) \\ -1(1)+7(-1) & -1(-2)+7(0) & -1(-1)+7(1) & -1(-1)+7(-3) \end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{76}\begin{bmatrix} 1 & -1 \\ -2 & 0 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 11+1 & -22-0 & -11-1 & -11+3 \\ -1-7 & 2+0 & 1+7 & 1-21 \end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{76}\begin{bmatrix} 1 & -1 \\ -2 & 0 \\ -1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} 12 & -22 & -12 & -8 \\ -8 & 2 & 8 & -20 \end{bmatrix}\vec{x}$$

Now, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_S\vec{x}=\frac{1}{76}\begin{bmatrix}1(12)-1(-8) & 1(-22)-1(2) & 1(-12)-1(8) & 1(-8)-1(-20)\\ -2(12)+0(-8) & -2(-22)+0(2) & -2(-12)+0(8) & -2(-8)+0(-20)\\ -1(12)+1(-8) & -1(-22)+1(2) & -1(-12)+1(8) & -1(-8)+1(-20)\\ -1(12)-3(-8) & -1(-22)-3(2) & -1(-12)-3(8) & -1(-8)-3(-20)\end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x}=\frac{1}{76}\begin{bmatrix}12+8 & -22-2 & -12-8 & -8+20\\ -24+0 & 44+0 & 24+0 & 16+0\\ -12-8 & 22+2 & 12+8 & 8-20\\ -12+24 & 22-6 & 12-24 & 8+60\end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x}=\frac{1}{76}\begin{bmatrix}20 & -24 & -20 & 12\\ -24 & 44 & 24 & 16\\ -20 & 24 & 20 & -12\\ 12 & 16 & -12 & 68\end{bmatrix}\vec{x}$$

$$\text{Proj}_S\vec{x}=\frac{1}{19}\begin{bmatrix}5 & -6 & -5 & 3\\ -6 & 11 & 6 & 4\\ -5 & 6 & 5 & -3\\ 3 & 4 & -3 & 17\end{bmatrix}\vec{x}$$

■ 5. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}-1\\0\\2\\-1\end{bmatrix},\begin{bmatrix}1\\3\\-1\\2\end{bmatrix},\begin{bmatrix}0\\2\\1\\1\end{bmatrix}\right)$$

Solution:

Because the vectors that span V are linearly independent, the matrix A of the basis vectors that define V is

$$A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

The transpose A^T is

$$A^T = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}$$

Find A^TA.

$$A^TA = \begin{bmatrix} -1 & 0 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1(-1)+0(0)+2(2)-1(-1) & -1(1)+0(3)+2(-1)-1(2) & -1(0)+0(2)+2(1)-1(1) \\ 1(-1)+3(0)-1(2)+2(-1) & 1(1)+3(3)-1(-1)+2(2) & 1(0)+3(2)-1(1)+2(1) \\ 0(-1)+2(0)+1(2)+1(-1) & 0(1)+2(3)+1(-1)+1(2) & 0(0)+2(2)+1(1)+1(1) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1+0+4+1 & -1+0-2-2 & 0+0+2-1 \\ -1+0-2-2 & 1+9+1+4 & 0+6-1+2 \\ 0+0+2-1 & 0+6-1+2 & 0+4+1+1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 6 & -5 & 1 \\ -5 & 15 & 7 \\ 1 & 7 & 6 \end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_3] = \begin{bmatrix} 6 & -5 & 1 & | & 1 & 0 & 0 \\ -5 & 15 & 7 & | & 0 & 1 & 0 \\ 1 & 7 & 6 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -\frac{5}{6} & \frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ -5 & 15 & 7 & | & 0 & 1 & 0 \\ 1 & 7 & 6 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -\frac{5}{6} & \frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ 0 & \frac{65}{6} & \frac{47}{6} & | & \frac{5}{6} & 1 & 0 \\ 1 & 7 & 6 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -\frac{5}{6} & \frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ 0 & \frac{65}{6} & \frac{47}{6} & | & \frac{5}{6} & 1 & 0 \\ 0 & \frac{47}{6} & \frac{35}{6} & | & -\frac{1}{6} & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & -\frac{5}{6} & \frac{1}{6} & | & \frac{1}{6} & 0 & 0 \\ 0 & 1 & \frac{47}{65} & | & \frac{1}{13} & \frac{6}{65} & 0 \\ 0 & \frac{47}{6} & \frac{35}{6} & | & -\frac{1}{6} & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & \frac{10}{13} & | & \frac{3}{13} & \frac{1}{13} & 0 \\ 0 & 1 & \frac{47}{65} & | & \frac{1}{13} & \frac{6}{65} & 0 \\ 0 & \frac{47}{6} & \frac{35}{6} & | & -\frac{1}{6} & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & \frac{10}{13} & | & \frac{3}{13} & \frac{1}{13} & 0 \\ 0 & 1 & \frac{47}{65} & | & \frac{1}{13} & \frac{6}{65} & 0 \\ 0 & 0 & \frac{11}{65} & | & -\frac{10}{13} & -\frac{47}{65} & 1 \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & \frac{10}{13} & | & \frac{3}{13} & \frac{1}{13} & 0 \\ 0 & 1 & \frac{47}{65} & | & \frac{1}{13} & \frac{6}{65} & 0 \\ 0 & 0 & 1 & | & -\frac{50}{11} & -\frac{47}{11} & \frac{65}{11} \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 0 & | & \frac{41}{11} & \frac{37}{11} & -\frac{50}{11} \\ 0 & 1 & \frac{47}{65} & | & \frac{1}{13} & \frac{6}{65} & 0 \\ 0 & 0 & 1 & | & -\frac{50}{11} & -\frac{47}{11} & \frac{65}{11} \end{bmatrix}$$

$$[A^TA \mid I_3] = \begin{bmatrix} 1 & 0 & 0 & | & \frac{41}{11} & \frac{37}{11} & -\frac{50}{11} \\ 0 & 1 & 0 & | & \frac{37}{11} & \frac{35}{11} & -\frac{47}{11} \\ 0 & 0 & 1 & | & -\frac{50}{11} & -\frac{47}{11} & \frac{65}{11} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{41}{11} & \frac{37}{11} & -\frac{50}{11} \\ \frac{37}{11} & \frac{35}{11} & -\frac{47}{11} \\ -\frac{50}{11} & -\frac{47}{11} & \frac{65}{11} \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{11}\begin{bmatrix} 41 & 37 & -50 \\ 37 & 35 & -47 \\ -50 & -47 & 65 \end{bmatrix}$$

Then the projection of $\vec{x}$ onto the subspace V will be

$$\text{Proj}_V\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\frac{1}{11}\begin{bmatrix} 41 & 37 & -50 \\ 37 & 35 & -47 \\ -50 & -47 & 65 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x} = \frac{1}{11}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 41 & 37 & -50 \\ 37 & 35 & -47 \\ -50 & -47 & 65 \end{bmatrix}\begin{bmatrix} -1 & 0 & 2 & -1 \\ 1 & 3 & -1 & 2 \\ 0 & 2 & 1 & 1 \end{bmatrix}\vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\begin{bmatrix} 41(-1)+37(1)-50(0) & 41(0)+37(3)-50(2) & 41(2)+37(-1)-50(1) & 41(-1)+37(2)-50(1) \\ 37(-1)+35(1)-47(0) & 37(0)+35(3)-47(2) & 37(2)+35(-1)-47(1) & 37(-1)+35(2)-47(1) \\ -50(-1)-47(1)+65(0) & -50(0)-47(3)+65(2) & -50(2)-47(-1)+65(1) & -50(-1)-47(2)+65(1) \end{bmatrix}$$

$$\begin{bmatrix} -41+37-0 & 0+111-100 & 82-37-50 & -41+74-50 \\ -37+35-0 & 0+105-94 & 74-35-47 & -37+70-47 \\ 50-47+0 & 0-141+130 & -100+47+65 & 50-94+65 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 11 & -5 & -17 \\ -2 & 11 & -8 & -14 \\ 3 & -11 & 12 & 21 \end{bmatrix}$$

So we get

$$\text{Proj}_V \vec{x} = \frac{1}{11} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 3 & 2 \\ 2 & -1 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -4 & 11 & -5 & -17 \\ -2 & 11 & -8 & -14 \\ 3 & -11 & 12 & 21 \end{bmatrix} \vec{x}$$

Next, simplify $A(A^TA)^{-1}A^T$.

$$\frac{1}{11} \begin{bmatrix} -1(-4)+1(-2)+0(3) & -1(11)+1(11)+0(-11) & -1(-5)+1(-8)+0(12) & -1(-17)+1(-14)+0(21) \\ 0(-4)+3(-2)+2(3) & 0(11)+3(11)+2(-11) & 0(-5)+3(-8)+2(12) & 0(-17)+3(-14)+2(21) \\ 2(-4)-1(-2)+1(3) & 2(11)-1(11)+1(-11) & 2(-5)-1(-8)+1(12) & 2(-17)-1(-14)+1(21) \\ -1(-4)+2(-2)+1(3) & -1(11)+2(11)+1(-11) & -1(-5)+2(-8)+1(12) & -1(-17)+2(-14)+1(21) \end{bmatrix}$$

$$\frac{1}{11} \begin{bmatrix} 4-2+0 & -11+11+0 & 5-8+0 & 17-14+0 \\ 0-6+6 & 0+33-22 & 0-24+24 & 0-42+42 \\ -8+2+3 & 22-11-11 & -10+8+12 & -34+14+21 \\ 4-4+3 & -11+22-11 & 5-16+12 & 17-28+21 \end{bmatrix}$$

$$\frac{1}{11} \begin{bmatrix} 2 & 0 & -3 & 3 \\ 0 & 11 & 0 & 0 \\ -3 & 0 & 10 & 1 \\ 3 & 0 & 1 & 10 \end{bmatrix}$$

So we get

$$\text{Proj}_V \vec{x} = \frac{1}{11} \begin{bmatrix} 2 & 0 & -3 & 3 \\ 0 & 11 & 0 & 0 \\ -3 & 0 & 10 & 1 \\ 3 & 0 & 1 & 10 \end{bmatrix} \vec{x}$$

■ 6. If $\vec{x}$ is a vector in $\mathbb{R}^4$, find an expression for the projection of any $\vec{x}$ onto the subspace S, if S is spanned by $\vec{x}_1$ and $\vec{x}_2$.

$$\vec{x}_1 = \frac{1}{2}\begin{bmatrix}2\\8\\-4\end{bmatrix} \text{ and } \vec{x}_2 = \begin{bmatrix}-2\\1\\0\end{bmatrix}$$

Solution:

Because the vectors that span S are linearly independent, the matrix A of the basis vectors that define S is

$$A = \begin{bmatrix}1 & -2\\4 & 1\\-2 & 0\end{bmatrix}$$

The transpose A^T is

$$A^T = \begin{bmatrix}1 & 4 & -2\\-2 & 1 & 0\end{bmatrix}$$

Find A^TA.

$$A^TA = \begin{bmatrix}1 & 4 & -2\\-2 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & -2\\4 & 1\\-2 & 0\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1(1)+4(4)-2(-2) & 1(-2)+4(1)-2(0)\\-2(1)+1(4)+0(-2) & -2(-2)+1(1)+0(0)\end{bmatrix}$$

$$A^TA = \begin{bmatrix}1+16+4 & -2+4-0\\-2+4+0 & 4+1+0\end{bmatrix}$$

$$A^TA = \begin{bmatrix}21 & 2\\2 & 5\end{bmatrix}$$

Find the inverse of A^TA.

$$[A^TA \mid I_2] = \begin{bmatrix} 21 & 2 & | & 1 & 0 \\ 2 & 5 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & \frac{2}{21} & | & \frac{1}{21} & 0 \\ 2 & 5 & | & 0 & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & \frac{2}{21} & | & \frac{1}{21} & 0 \\ 0 & \frac{101}{21} & | & -\frac{2}{21} & 1 \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & \frac{2}{21} & | & \frac{1}{21} & 0 \\ 0 & 1 & | & -\frac{2}{101} & \frac{21}{101} \end{bmatrix}$$

$$[A^TA \mid I_2] = \begin{bmatrix} 1 & 0 & | & \frac{5}{101} & -\frac{2}{101} \\ 0 & 1 & | & -\frac{2}{101} & \frac{21}{101} \end{bmatrix}$$

So $(A^TA)^{-1}$ is

$$(A^TA)^{-1} = \begin{bmatrix} \frac{5}{101} & -\frac{2}{101} \\ -\frac{2}{101} & \frac{21}{101} \end{bmatrix}$$

$$(A^TA)^{-1} = \frac{1}{101}\begin{bmatrix} 5 & -2 \\ -2 & 21 \end{bmatrix}$$

The projection of $\vec{x}$ onto the subspace S will be

$$\text{Proj}_S\vec{x} = A(A^TA)^{-1}A^T\vec{x}$$

$$\text{Proj}_S\vec{x} = \begin{bmatrix} 1 & -2 \\ 4 & 1 \\ -2 & 0 \end{bmatrix} \frac{1}{101} \begin{bmatrix} 5 & -2 \\ -2 & 21 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ -2 & 1 & 0 \end{bmatrix} \vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{101} \begin{bmatrix} 1 & -2 \\ 4 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 21 \end{bmatrix} \begin{bmatrix} 1 & 4 & -2 \\ -2 & 1 & 0 \end{bmatrix} \vec{x}$$

First, simplify $(A^TA)^{-1}A^T$.

$$\text{Proj}_S\vec{x} = \frac{1}{101} \begin{bmatrix} 1 & -2 \\ 4 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 5(1)-2(-2) & 5(4)-2(1) & 5(-2)-2(0) \\ -2(1)+21(-2) & -2(4)+21(1) & -2(-2)+21(0) \end{bmatrix} \vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{101} \begin{bmatrix} 1 & -2 \\ 4 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 5+4 & 20-2 & -10-0 \\ -2-42 & -8+21 & 4+0 \end{bmatrix} \vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{101} \begin{bmatrix} 1 & -2 \\ 4 & 1 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} 9 & 18 & -10 \\ -44 & 13 & 4 \end{bmatrix} \vec{x}$$

Now, simplify $A(A^TA)^{-1}A^T$.

$$\text{Proj}_S\vec{x} = \frac{1}{101} \begin{bmatrix} 1(9)-2(-44) & 1(18)-2(13) & 1(-10)-2(4) \\ 4(9)+1(-44) & 4(18)+1(13) & 4(-10)+1(4) \\ -2(9)+0(-44) & -2(18)+0(13) & -2(-10)+0(4) \end{bmatrix} \vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{101} \begin{bmatrix} 9+88 & 18-26 & -10-8 \\ 36-44 & 72+13 & -40+4 \\ -18+0 & -36+0 & 20+0 \end{bmatrix} \vec{x}$$

$$\text{Proj}_S\vec{x} = \frac{1}{101} \begin{bmatrix} 97 & -8 & -18 \\ -8 & 85 & -36 \\ -18 & -36 & 20 \end{bmatrix} \vec{x}$$

LEAST SQUARES SOLUTION

■ 1. Find the least squares solution to the system.

$$x = 2$$

$$x - y = 2$$

$$x + y = 3$$

Solution:

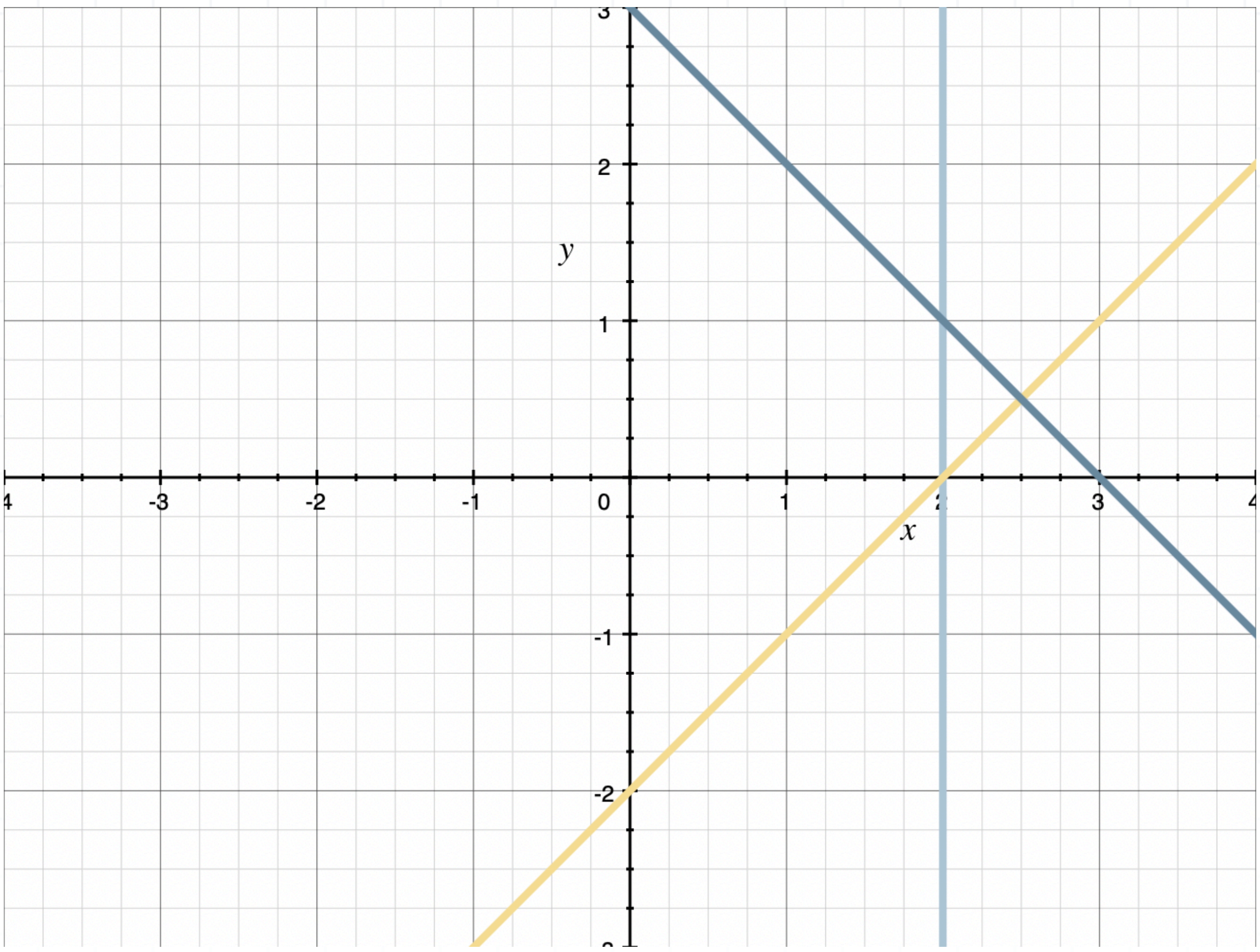
Put each line into slope-intercept form,

$$x = 2$$

$$y = x - 2$$

$$y = -x + 3$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x}=\vec{b}$.

$$\begin{bmatrix}1 & 0\\ 1 & -1\\ 1 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}2\\ 2\\ 3\end{bmatrix}$$

In other words, $\vec{b}=(2,2,3)$ is not in the column space of the coefficient matrix A, and there's no vector $\vec{x}=(x,y)$ you can find that makes the $A\vec{x}=\vec{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A=\begin{bmatrix}1 & 0\\ 1 & -1\\ 1 & 1\end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1(1) + 1(1) + 1(1) & 1(0) + 1(-1) + 1(1) \\ 0(1) - 1(1) + 1(1) & 0(0) - 1(-1) + 1(1) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 1 + 1 & 0 - 1 + 1 \\ 0 - 1 + 1 & 0 + 1 + 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1(2) + 1(2) + 1(3) \\ 0(2) - 1(2) + 1(3) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 2 + 2 + 3 \\ 0 - 2 + 3 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Then we get

$$A^T A \vec{x}^* = A^T \vec{b}$$

$$\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \vec{x}^* = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 3 & 0 & | & 7 \\ 0 & 2 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{7}{3} \\ 0 & 2 & | & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & \frac{7}{3} \\ 0 & 1 & | & \frac{1}{2} \end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(\frac{7}{3}, \frac{1}{2} \right)$$

■ 2. Find the least squares solution to the system.

$$-x + 2y = 6$$

$$3x + 2y = 0$$

$$y - 3x = -2$$

Solution:

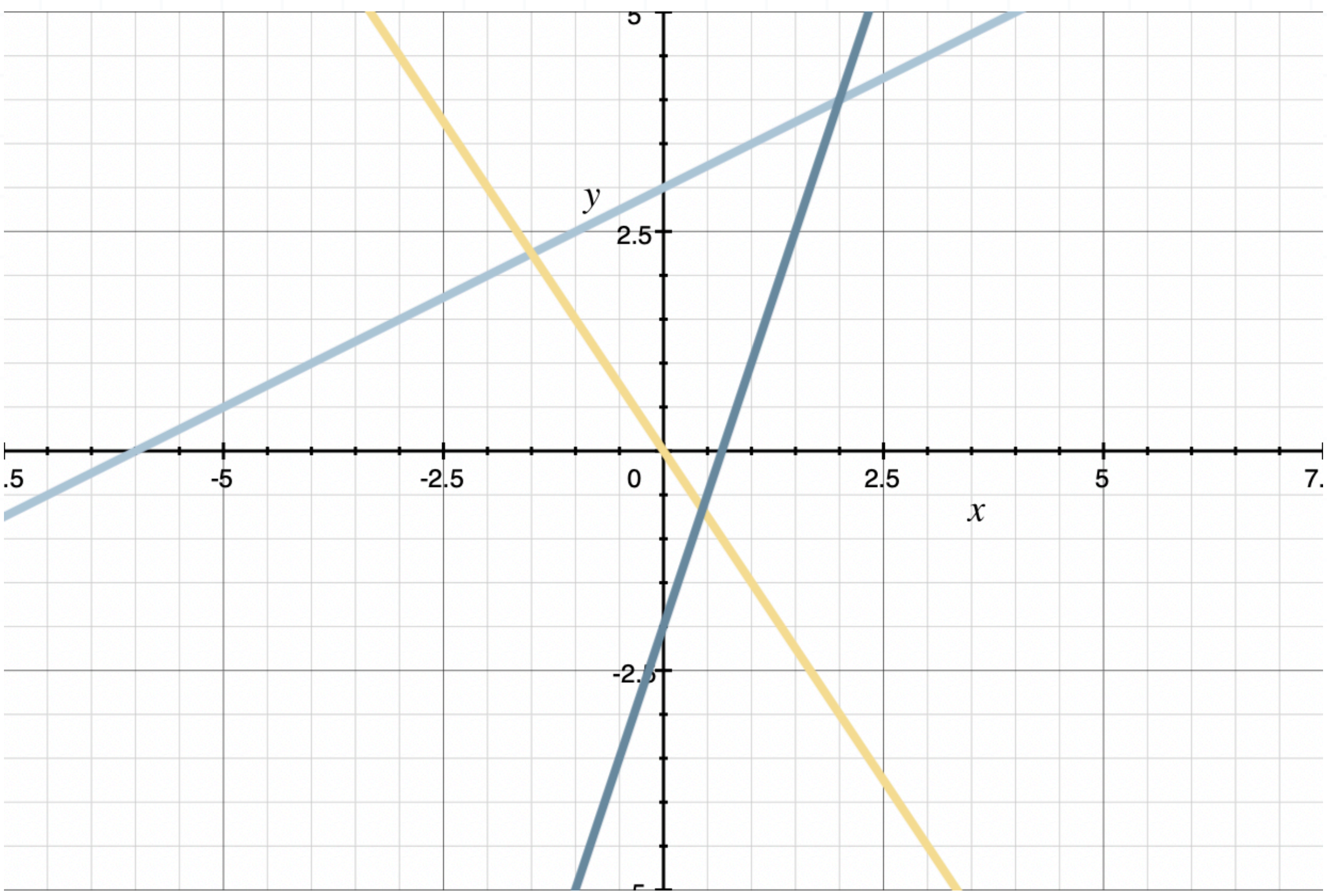
Put each line into slope-intercept form,

$$y = \frac{1}{2}x + 3$$

$$y = -\frac{3}{2}x$$

$$y = 3x - 2$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} -1 & 2 \\ 3 & 2 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$$

In other words, $\vec{b} = (6,0,-2)$ is not in the column space of the coefficient matrix A, and there's no vector $\vec{x} = (x, y)$ you can find that makes the $A\vec{x} = \vec{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 2 \\ -3 & 1 \end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} -1 & 3 & -3 \\ 2 & 2 & 1 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} -1 & 3 & -3 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 \\ 3 & 2 \\ -3 & 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -1(-1) + 3(3) - 3(-3) & -1(2) + 3(2) - 3(1) \\ 2(-1) + 2(3) + 1(-3) & 2(2) + 2(2) + 1(1) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 + 9 + 9 & -2 + 6 - 3 \\ -2 + 6 - 3 & 4 + 4 + 1 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 19 & 1 \\ 1 & 9 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b}=\begin{bmatrix}-1&3&-3\\2&2&1\end{bmatrix}\begin{bmatrix}6\\0\\-2\end{bmatrix}$$

$$A^T\vec{b}=\begin{bmatrix}-1(6)+3(0)-3(-2)\\2(6)+2(0)+1(-2)\end{bmatrix}$$

$$A^T\vec{b}=\begin{bmatrix}-6+0+6\\12+0-2\end{bmatrix}$$

$$A^T\vec{b}=\begin{bmatrix}0\\10\end{bmatrix}$$

Then we get

$$A^TA\vec{x}^*=A^T\vec{b}$$

$$\begin{bmatrix}19&1\\1&9\end{bmatrix}\vec{x}^*=\begin{bmatrix}0\\10\end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}19&1&|&0\\1&9&|&10\end{bmatrix}\to\begin{bmatrix}1&\frac{1}{19}&|&0\\1&9&|&10\end{bmatrix}\to\begin{bmatrix}1&\frac{1}{19}&|&0\\0&\frac{170}{19}&|&10\end{bmatrix}$$

$$\begin{bmatrix}1&\frac{1}{19}&|&0\\0&1&|&\frac{19}{17}\end{bmatrix}\to\begin{bmatrix}1&0&|&-\frac{1}{17}\\0&1&|&\frac{19}{17}\end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(-\frac{1}{17}, \frac{19}{17}\right)$$

3. Find the least squares solution to the system.

$$y - 2x = 5$$

$$3x + y = -2$$

$$2x - 4y = 5$$

Solution:

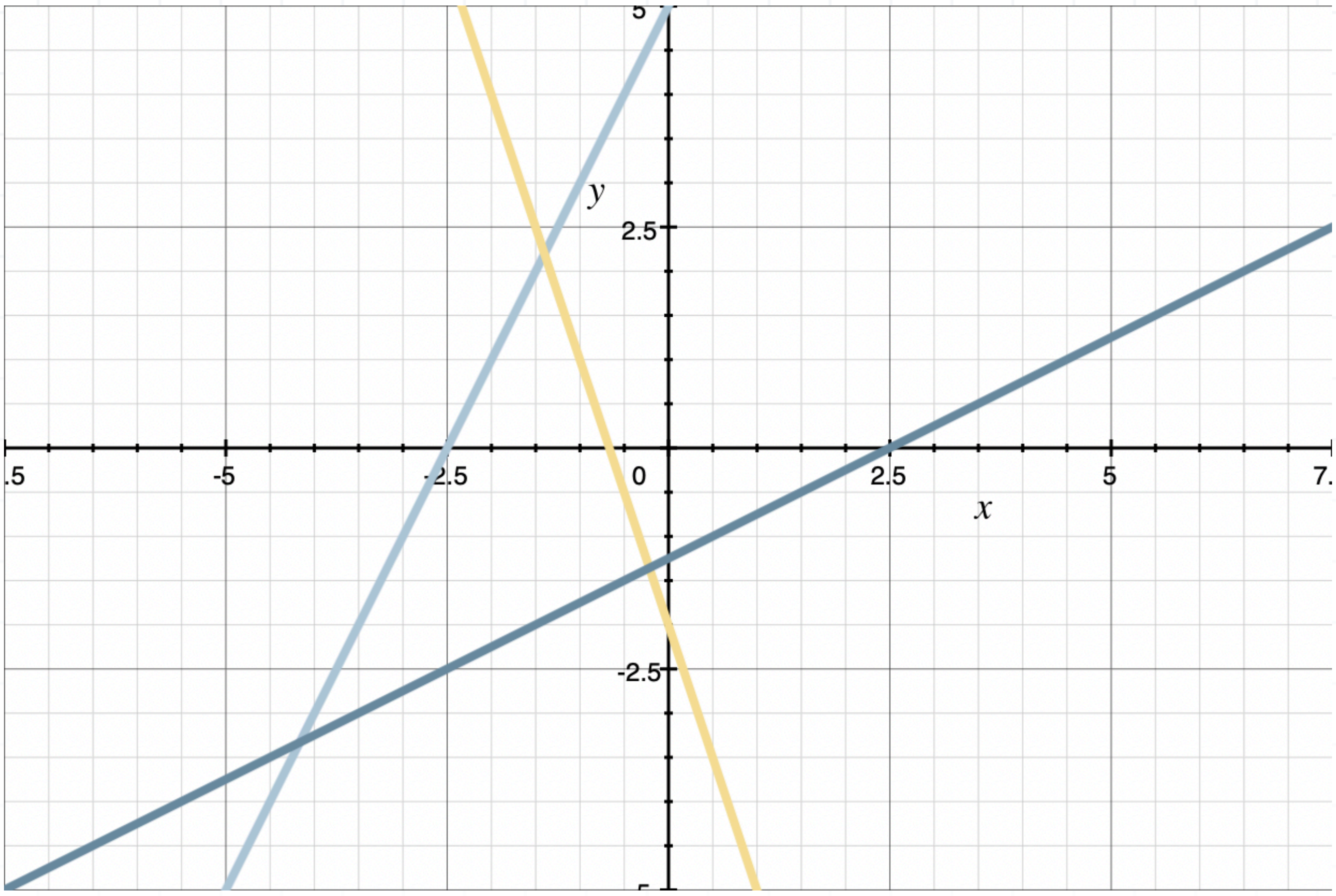
Put each line into slope-intercept form,

$$y = 2x + 5$$

$$y = -3x - 2$$

$$y = \frac{1}{2}x - \frac{5}{4}$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} -2 & 1 \\ 3 & 1 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix}$$

In other words, $\vec{b} = (5, -2, 5)$ is not in the column space of the coefficient matrix A, and there's no vector $\vec{x} = (x, y)$ you can find that makes the $A\vec{x} = \vec{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A = \begin{bmatrix} -2 & 1 \\ 3 & 1 \\ 2 & -4 \end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} -2 & 3 & 2 \\ 1 & 1 & -4 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} -2 & 3 & 2 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 3 & 1 \\ 2 & -4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -2(-2) + 3(3) + 2(2) & -2(1) + 3(1) + 2(-4) \\ 1(-2) + 1(3) - 4(2) & 1(1) + 1(1) - 4(-4) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4 + 9 + 4 & -2 + 3 - 8 \\ -2 + 3 - 8 & 1 + 1 + 16 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 17 & -7 \\ -7 & 18 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} -2 & 3 & 2 \\ 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -2(5) + 3(-2) + 2(5) \\ 1(5) + 1(-2) - 4(5) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -10 - 6 + 10 \\ 5 - 2 - 20 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -6 \\ -17 \end{bmatrix}$$

Then we get

$$A^TA\overrightarrow{x}^* = A^T\overrightarrow{b}$$

$$\begin{bmatrix}17 & -7\\ -7 & 18\end{bmatrix}\overrightarrow{x}^* = \begin{bmatrix}-6\\ -17\end{bmatrix}$$

Then to find $\overrightarrow{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}17 & -7 & | & -6\\ -7 & 18 & | & -17\end{bmatrix} \to \begin{bmatrix}1 & -\frac{7}{17} & | & -\frac{6}{17}\\ -7 & 18 & | & -17\end{bmatrix} \to \begin{bmatrix}1 & -\frac{7}{17} & | & -\frac{6}{17}\\ 0 & \frac{257}{17} & | & -\frac{331}{17}\end{bmatrix}$$

$$\begin{bmatrix}1 & -\frac{7}{17} & | & -\frac{6}{17}\\ 0 & 1 & | & -\frac{331}{257}\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & -\frac{227}{257}\\ 0 & 1 & | & -\frac{331}{257}\end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\overrightarrow{x}^* = \left(-\frac{227}{257}, -\frac{331}{257}\right)$$

■ 4. Find the least squares solution to the system.

$$y - 3x = 5$$

$$x + y = -3$$

$$2x - 2y = 3$$

Solution:

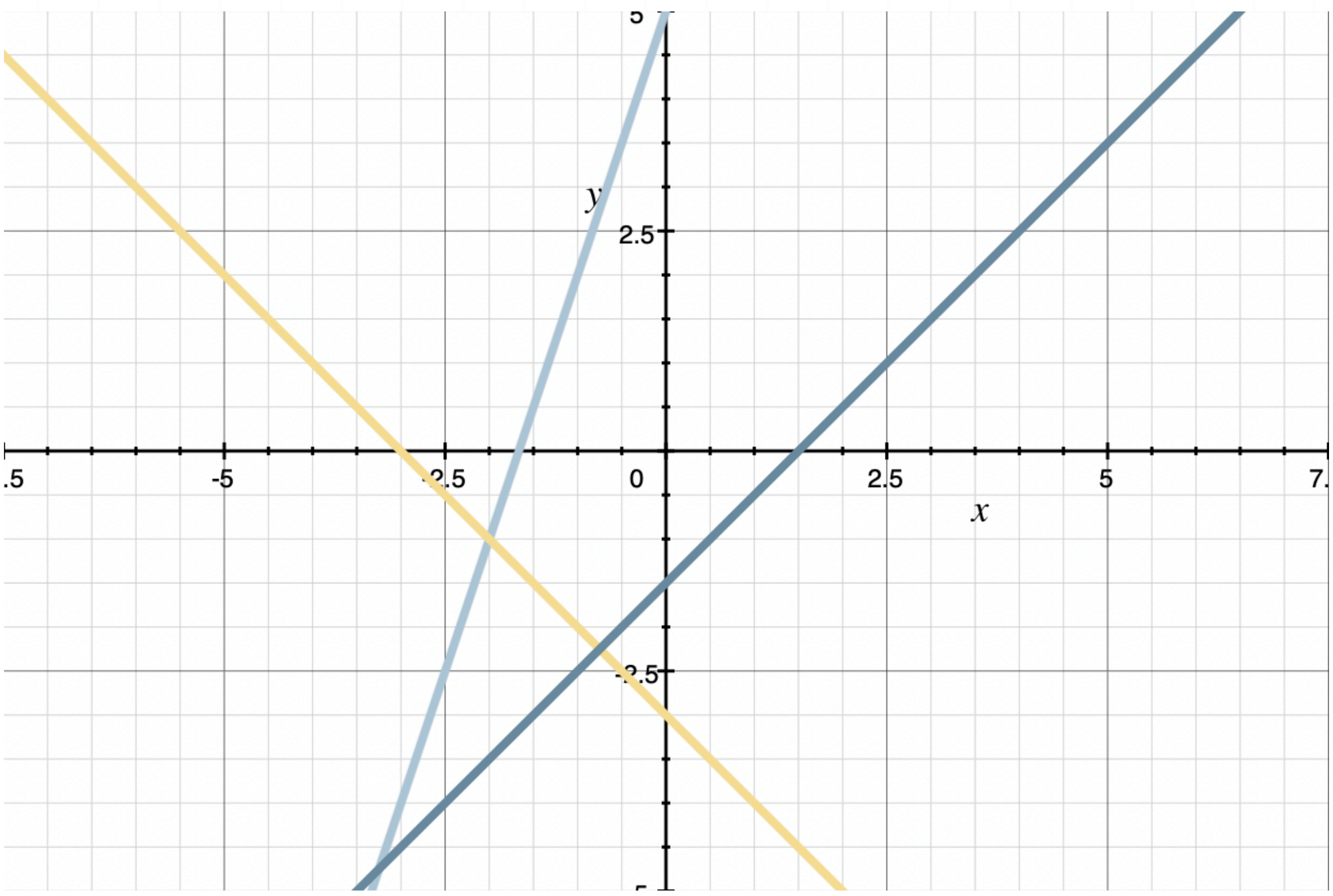
Put each line into slope-intercept form,

$$y = 3x + 5$$

$$y = -x - 3$$

$$y = x - \frac{3}{2}$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x} = \vec{b}$.

$$\begin{bmatrix} -3 & 1 \\ 1 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix}$$

In other words, $\overrightarrow{b} = (5, -3, 3)$ is not in the column space of the coefficient matrix A, and there's no vector $\overrightarrow{x} = (x, y)$ you can find that makes the $A\overrightarrow{x} = \overrightarrow{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A = \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ 2 & -2 \end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & -2 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 1 & 1 \\ 2 & -2 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -3(-3) + 1(1) + 2(2) & -3(1) + 1(1) + 2(-2) \\ 1(-3) + 1(1) - 2(2) & 1(1) + 1(1) - 2(-2) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 9 + 1 + 4 & -3 + 1 - 4 \\ -3 + 1 - 4 & 1 + 1 + 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} -3 & 1 & 2 \\ 1 & 1 & -2 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 3 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -3(5) + 1(-3) + 2(3) \\ 1(5) + 1(-3) - 2(3) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -15 - 3 + 6 \\ 5 - 3 - 6 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -12 \\ -4 \end{bmatrix}$$

Then we get

$$A^TA\vec{x}^* = A^T\vec{b}$$

$$\begin{bmatrix} 14 & -6 \\ -6 & 6 \end{bmatrix} \vec{x}^* = \begin{bmatrix} -12 \\ -4 \end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 14 & -6 & | & -12 \\ -6 & 6 & | & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{7} & | & -\frac{6}{7} \\ -6 & 6 & | & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{3}{7} & | & -\frac{6}{7} \\ 0 & \frac{24}{7} & | & -\frac{64}{7} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{3}{7} & | & -\frac{6}{7} \\ 0 & 1 & | & -\frac{8}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -2 \\ 0 & 1 & | & -\frac{8}{3} \end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(-2, -\frac{8}{3}\right)$$

■ 5. Find the least squares solution to the system.

$$2y - 3x = -4$$

$$5x + y = -2$$

$$x + 4y = -1$$

Solution:

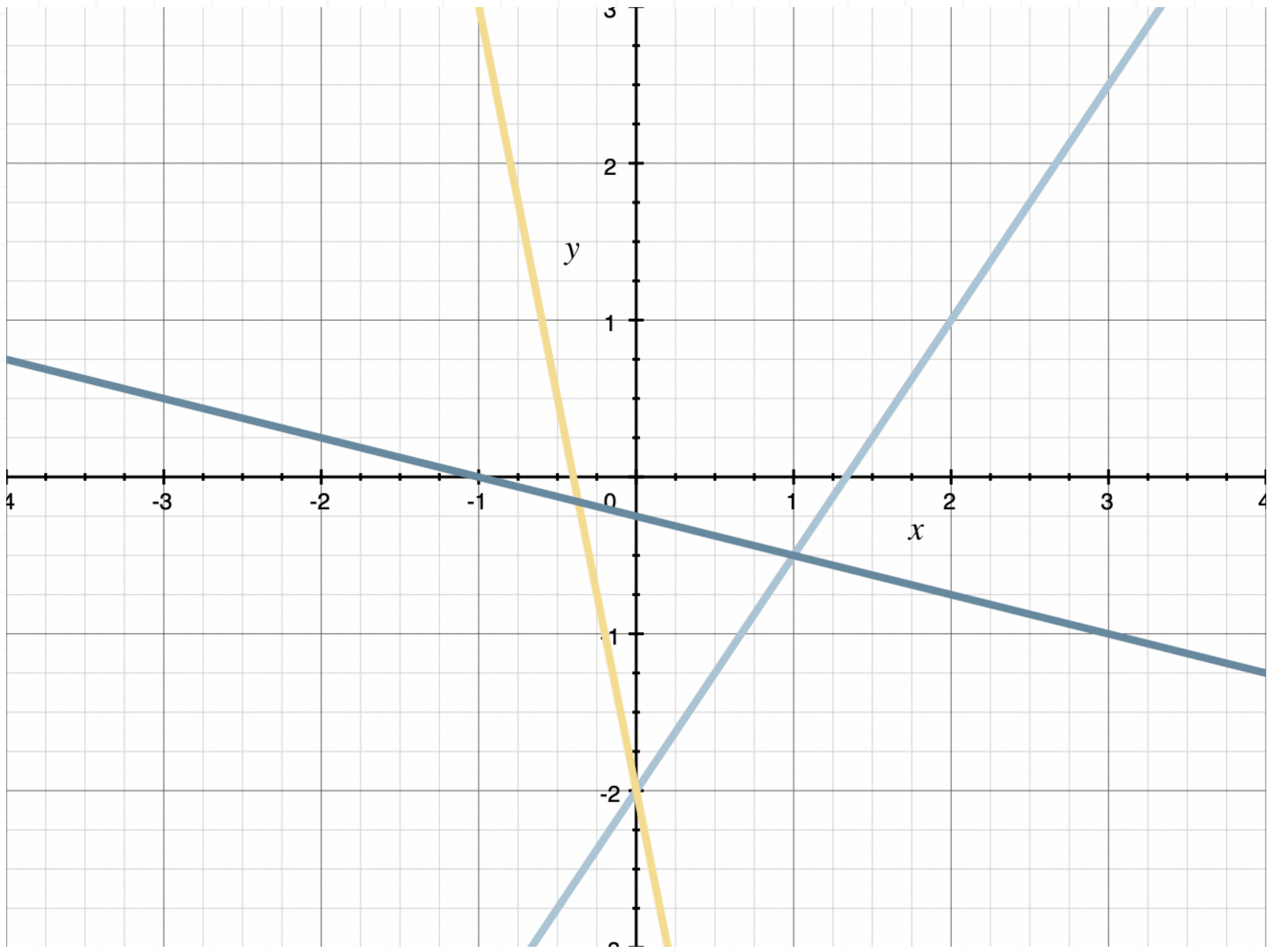
Put each line into slope-intercept form,

$$y = \frac{3}{2}x - 2$$

$$y = -5x - 2$$

$$y = -\frac{1}{4}x - \frac{1}{4}$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x}=\vec{b}$.

$$\begin{bmatrix}-3 & 2\\ 5 & 1\\ 1 & 4\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}-4\\ -2\\ -1\end{bmatrix}$$

In other words, $\vec{b}=(-4,-2,-1)$ is not in the column space of the coefficient matrix A, and there's no vector $\vec{x}=(x,y)$ you can find that makes the $A\vec{x}=\vec{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A=\begin{bmatrix}-3 & 2\\ 5 & 1\\ 1 & 4\end{bmatrix}$$

Now we'll find A^T.

$$A^T = \begin{bmatrix} -3 & 5 & 1 \\ 2 & 1 & 4 \end{bmatrix}$$

Then A^TA is

$$A^TA = \begin{bmatrix} -3 & 5 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} -3 & 2 \\ 5 & 1 \\ 1 & 4 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} -3(-3) + 5(5) + 1(1) & -3(2) + 5(1) + 1(4) \\ 2(-3) + 1(5) + 4(1) & 2(2) + 1(1) + 4(4) \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 9 + 25 + 1 & -6 + 5 + 4 \\ -6 + 5 + 4 & 4 + 1 + 16 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 35 & 3 \\ 3 & 21 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} -3 & 5 & 1 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} -4 \\ -2 \\ -1 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -3(-4) + 5(-2) + 1(-1) \\ 2(-4) + 1(-2) + 4(-1) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 12 - 10 - 1 \\ -8 - 2 - 4 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 1 \\ -14 \end{bmatrix}$$

Then we get

$$A^TA\vec{x}^* = A^T\vec{b}$$

$$\begin{bmatrix}35 & 3\\ 3 & 21\end{bmatrix}\vec{x}^* = \begin{bmatrix}1\\ -14\end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}35 & 3 & | & 1\\ 3 & 21 & | & -14\end{bmatrix} \to \begin{bmatrix}1 & \frac{3}{35} & | & \frac{1}{35}\\ 3 & 21 & | & -14\end{bmatrix} \to \begin{bmatrix}1 & \frac{3}{35} & | & \frac{1}{35}\\ 0 & \frac{726}{35} & | & -\frac{493}{35}\end{bmatrix}$$

$$\begin{bmatrix}1 & \frac{3}{35} & | & \frac{1}{35}\\ 0 & 1 & | & -\frac{493}{726}\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & \frac{21}{242}\\ 0 & 1 & | & -\frac{493}{726}\end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(\frac{21}{242}, -\frac{493}{726}\right)$$

■ 6. Find the least squares solution to the system.

$$2x - 5y = 4$$

$$x + 6y = 5$$

$$4x - 3y = -6$$

Solution:

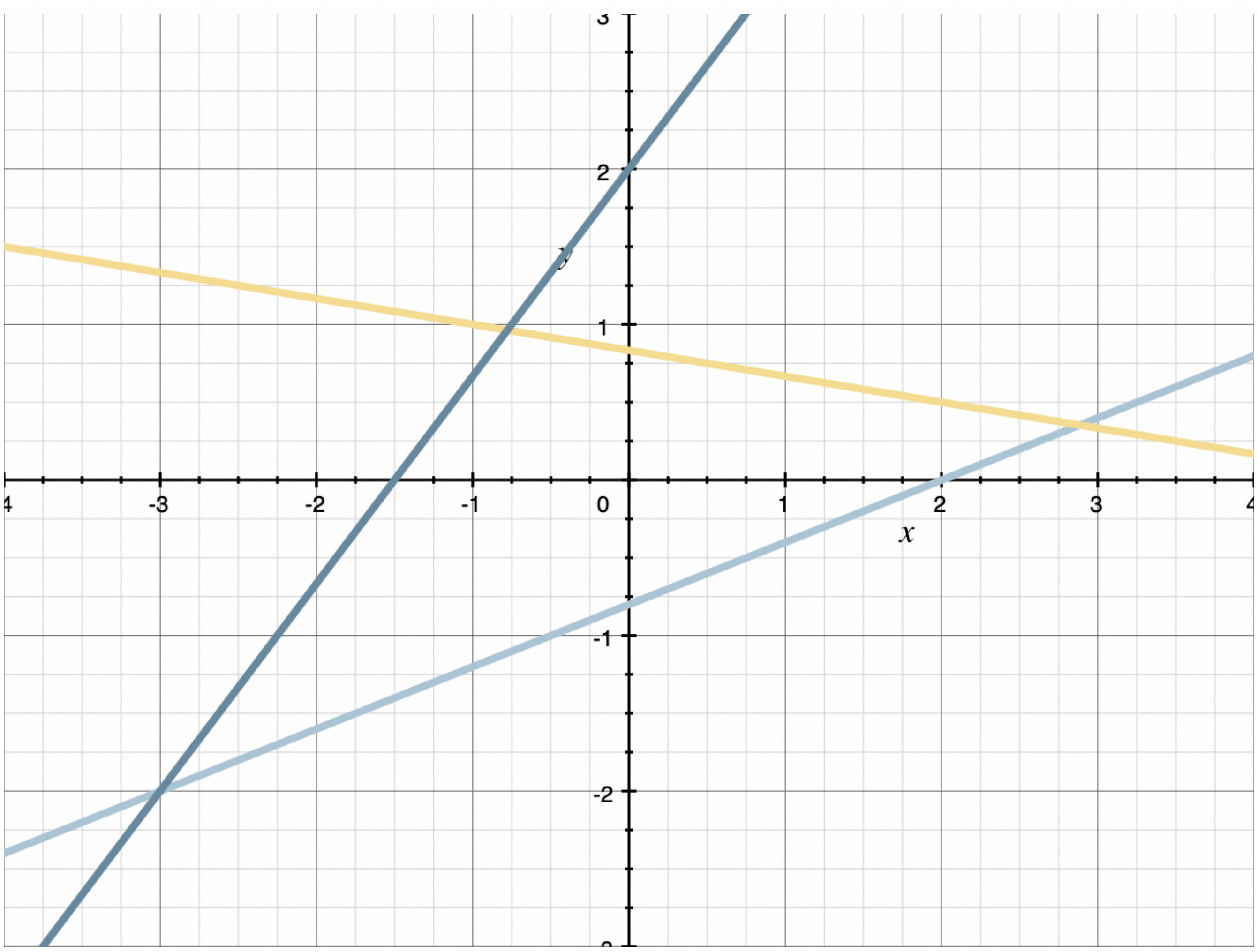
Put each line into slope-intercept form,

$$y = \frac{2}{5}x - \frac{4}{5}$$

$$y = -\frac{1}{6}x + \frac{5}{6}$$

$$y = \frac{4}{3}x + 2$$

then graph all three in the same plane.



While there are three points where some of the lines intersect, there's no single point where all three lines intersect, which means there's no solution to $A\vec{x}=\vec{b}$.

$$\begin{bmatrix}2 & -5\\ 1 & 6\\ 4 & -3\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}=\begin{bmatrix}4\\ 5\\ -6\end{bmatrix}$$

In other words, $\vec{b}=(4,5,-6)$ is not in the column space of the coefficient matrix A, and there's no vector $\vec{x}=(x,y)$ you can find that makes the $A\vec{x}=\vec{b}$ equation true.

The next best thing we can do is find the least squares solution. By building the matrix equation, we've already found

$$A=\begin{bmatrix}2 & -5\\ 1 & 6\\ 4 & -3\end{bmatrix}$$

Now we'll find A^T.

$$A^T=\begin{bmatrix}2 & 1 & 4\\ -5 & 6 & -3\end{bmatrix}$$

Then A^TA is

$$A^TA=\begin{bmatrix}2 & 1 & 4\\ -5 & 6 & -3\end{bmatrix}\begin{bmatrix}2 & -5\\ 1 & 6\\ 4 & -3\end{bmatrix}$$

$$A^TA=\begin{bmatrix}2(2)+1(1)+4(4) & 2(-5)+1(6)+4(-3)\\ -5(2)+6(1)-3(4) & -5(-5)+6(6)-3(-3)\end{bmatrix}$$

$$A^TA = \begin{bmatrix} 4+1+16 & -10+6-12 \\ -10+6-12 & 25+36+9 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 21 & -16 \\ -16 & 70 \end{bmatrix}$$

And $A^T\vec{b}$ is

$$A^T\vec{b} = \begin{bmatrix} 2 & 1 & 4 \\ -5 & 6 & -3 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \\ -6 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 2(4)+1(5)+4(-6) \\ -5(4)+6(5)-3(-6) \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} 8+5-24 \\ -20+30+18 \end{bmatrix}$$

$$A^T\vec{b} = \begin{bmatrix} -11 \\ 28 \end{bmatrix}$$

Then we get

$$A^TA\vec{x}^* = A^T\vec{b}$$

$$\begin{bmatrix} 21 & -16 \\ -16 & 70 \end{bmatrix} \vec{x}^* = \begin{bmatrix} -11 \\ 28 \end{bmatrix}$$

Then to find $\vec{x}^*$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix} 21 & -16 & | & -11 \\ -16 & 70 & | & 28 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{16}{21} & | & -\frac{11}{21} \\ -16 & 70 & | & 28 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{16}{21} & | & -\frac{11}{21} \\ 0 & \frac{1,214}{21} & | & \frac{412}{21} \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{16}{21} & | & -\frac{11}{21} \\ 0 & 1 & | & \frac{206}{607} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -\frac{161}{607} \\ 0 & 1 & | & \frac{206}{607} \end{bmatrix}$$

Then the least squares solution is given by the augmented matrix as

$$\vec{x}^* = \left(-\frac{161}{607}, \frac{206}{607}\right)$$

COORDINATES IN A NEW BASIS

■ 1. The vectors $\vec{v}=(-2,1)$ and $\vec{w}=(4,-3)$ form an alternate basis for $\mathbb{R}^2$. Use them to transform $\vec{x}=6\mathbf{i}-2\mathbf{j}$ into the alternate basis.

Solution:

The vector $\vec{x}=(6,-2)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v}=(-2,1)$ and $\vec{w}=(4,-3)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B=\vec{x}$$

$$\begin{bmatrix}-2 & 4\\ 1 & -3\end{bmatrix}[\vec{x}]_B=\begin{bmatrix}6\\ -2\end{bmatrix}$$

To find the representation of $\vec{x}$ in the alternate basis, $[\vec{x}]_B$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}-2 & 4 & | & 6\\ 1 & -3 & | & -2\end{bmatrix}\to\begin{bmatrix}1 & -2 & | & -3\\ 1 & -3 & | & -2\end{bmatrix}\to\begin{bmatrix}1 & -2 & | & -3\\ 0 & -1 & | & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & | & -3\\ 0 & 1 & | & -1\end{bmatrix}\to\begin{bmatrix}1 & 0 & | & -5\\ 0 & 1 & | & -1\end{bmatrix}$$

So $\vec{x}=(6,-2)$, expressed in the alternate basis, is

$$[\vec{x}]_B = \begin{bmatrix} -5 \\ -1 \end{bmatrix}$$

2. The vectors $\vec{v} = (1, -5)$ and $\vec{w} = (2,4)$ form an alternate basis for $\mathbb{R}^2$. Use them, and an inverse matrix, to transform $\vec{x} = -\mathbf{i}$ into the alternate basis.

Solution:

The vector $\vec{x} = (-1,0)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v} = (1, -5)$ and $\vec{w} = (2,4)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B = \vec{x}$$

$$\begin{bmatrix} 1 & 2 \\ -5 & 4 \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

Find A^{-1} from A.

$$[A \mid I] = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ -5 & 4 & | & 0 & 1 \end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 14 & | & 5 & 1 \end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & \frac{5}{14} & \frac{1}{14} \end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix} 1 & 0 & | & \frac{2}{7} & -\frac{1}{7} \\ 0 & 1 & | & \frac{5}{14} & \frac{1}{14} \end{bmatrix}$$

So the inverse matrix is

$$A^{-1} = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{5}{14} & \frac{1}{14} \end{bmatrix}$$

Now to find the representation of $\vec{x} = (-1,0)$ in the alternate basis, we simply multiply the inverse matrix by the vector.

$$[\vec{x}]_B = A^{-1}\vec{x}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{2}{7} & -\frac{1}{7} \\ \frac{5}{14} & \frac{1}{14} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{2}{7}(-1) - \frac{1}{7}(0) \\ \frac{5}{14}(-1) + \frac{1}{14}(0) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{2}{7} \\ -\frac{5}{14} \end{bmatrix}$$

■ 3. The vectors $\vec{v}=(-1,0,4)$, $\vec{s}=(2,-3,1)$, and $\vec{w}=(1,-1,2)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\vec{x}=-\mathbf{j}+\mathbf{k}$ into the alternate basis.

Solution:

The vector $\vec{x}=(0,-1,1)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v}=(-1,0,4)$, $\vec{s}=(2,-3,1)$, and $\vec{w}=(1,-1,2)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B=\vec{x}$$

$$\begin{bmatrix}-1 & 2 & 1\\ 0 & -3 & -1\\ 4 & 1 & 2\end{bmatrix}[\vec{x}]_B=\begin{bmatrix}0\\ -1\\ 1\end{bmatrix}$$

To find the representation of $\vec{x}$ in the alternate basis, $[\vec{x}]_B$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}-1 & 2 & 1 & | & 0\\ 0 & -3 & -1 & | & -1\\ 4 & 1 & 2 & | & 1\end{bmatrix}\to\begin{bmatrix}1 & -2 & -1 & | & 0\\ 0 & -3 & -1 & | & -1\\ 4 & 1 & 2 & | & 1\end{bmatrix}\to\begin{bmatrix}1 & -2 & -1 & | & 0\\ 0 & -3 & -1 & | & -1\\ 0 & 9 & 6 & | & 1\end{bmatrix}$$

$$\begin{bmatrix}1 & -2 & -1 & | & 0\\ 0 & 1 & \frac{1}{3} & | & \frac{1}{3}\\ 0 & 9 & 6 & | & 1\end{bmatrix}\to\begin{bmatrix}1 & 0 & -\frac{1}{3} & | & \frac{2}{3}\\ 0 & 1 & \frac{1}{3} & | & \frac{1}{3}\\ 0 & 9 & 6 & | & 1\end{bmatrix}\to\begin{bmatrix}1 & 0 & -\frac{1}{3} & | & \frac{2}{3}\\ 0 & 1 & \frac{1}{3} & | & \frac{1}{3}\\ 0 & 0 & 3 & | & -2\end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & \frac{2}{3} \\ 0 & 1 & \frac{1}{3} & | & \frac{1}{3} \\ 0 & 0 & 1 & | & -\frac{2}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{1}{3} & | & \frac{2}{3} \\ 0 & 1 & 0 & | & \frac{5}{9} \\ 0 & 0 & 1 & | & -\frac{2}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & | & \frac{4}{9} \\ 0 & 1 & 0 & | & \frac{5}{9} \\ 0 & 0 & 1 & | & -\frac{2}{3} \end{bmatrix}$$

So $\vec{x} = (0, -1, 1)$, expressed in the alternate basis, is

$$[\vec{x}]_B = \begin{bmatrix} \frac{4}{9} \\ \frac{5}{9} \\ -\frac{2}{3} \end{bmatrix}$$

■ 4. The vectors $\vec{v} = (1, -3, 1)$, $\vec{s} = (-3, -3, 2)$, and $\vec{w} = (5, -3, 1)$ form an alternate basis for $\mathbb{R}^3$. Use them, and an inverse matrix to transform $\vec{x} = 2\mathbf{i} + 6\mathbf{j} - \mathbf{k}$ into the alternate basis.

Solution:

The vector $\vec{x} = (2, 6, -1)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v} = (1, -3, 1)$, $\vec{s} = (-3, -3, 2)$, and $\vec{w} = (5, -3, 1)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B = \vec{x}$$

$$\begin{bmatrix} 1 & -3 & 5 \\ -3 & -3 & -3 \\ 1 & 2 & 1 \end{bmatrix} [\vec{x}]_B = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

To find the representation of $\vec{x}$ in the alternate basis, $[\vec{x}]_B$, we'll put the augmented matrix into reduced row-echelon form.

Find A^{-1} from A.

$$[A \mid I] = \begin{bmatrix} 1 & -3 & 5 & | & 1 & 0 & 0 \\ -3 & -3 & -3 & | & 0 & 1 & 0 \\ 1 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix} 1 & -3 & 5 & | & 1 & 0 & 0 \\ 0 & -12 & 12 & | & 3 & 1 & 0 \\ 1 & 2 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix} 1 & -3 & 5 & | & 1 & 0 & 0 \\ 0 & -12 & 12 & | & 3 & 1 & 0 \\ 0 & 5 & -4 & | & -1 & 0 & 1 \end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix} 1 & -3 & 5 & | & 1 & 0 & 0 \\ 0 & 1 & -1 & | & -\frac{1}{4} & -\frac{1}{12} & 0 \\ 0 & 5 & -4 & | & -1 & 0 & 1 \end{bmatrix}$$

$$[A \mid I] = \begin{bmatrix} 1 & 0 & 2 & | & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & -1 & | & -\frac{1}{4} & -\frac{1}{12} & 0 \\ 0 & 5 & -4 & | & -1 & 0 & 1 \end{bmatrix}$$

$$[A \mid I]=\begin{bmatrix} 1 & 0 & 2 & | & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & -1 & | & -\frac{1}{4} & -\frac{1}{12} & 0 \\ 0 & 0 & 1 & | & \frac{1}{4} & \frac{5}{12} & 1 \end{bmatrix}$$

$$[A \mid I]=\begin{bmatrix} 1 & 0 & 2 & | & \frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & 1 & 0 & | & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 1 & | & \frac{1}{4} & \frac{5}{12} & 1 \end{bmatrix}$$

$$[A \mid I]=\begin{bmatrix} 1 & 0 & 0 & | & -\frac{1}{4} & -\frac{13}{12} & -2 \\ 0 & 1 & 0 & | & 0 & \frac{1}{3} & 1 \\ 0 & 0 & 1 & | & \frac{1}{4} & \frac{5}{12} & 1 \end{bmatrix}$$

So the inverse matrix is

$$A^{-1}=\begin{bmatrix} -\frac{1}{4} & -\frac{13}{12} & -2 \\ 0 & \frac{1}{3} & 1 \\ \frac{1}{4} & \frac{5}{12} & 1 \end{bmatrix}$$

Now to find the representation of $\vec{x}=(2,6,-1)$ in the alternate basis, we simply multiply the inverse matrix by the vector.

$$[\vec{x}]_B=A^{-1}\vec{x}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{1}{4} & -\frac{13}{12} & -2 \\ 0 & \frac{1}{3} & 1 \\ \frac{1}{4} & \frac{5}{12} & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{1}{4}(2) - \frac{13}{12}(6) - 2(-1) \\ 0(2) + \frac{1}{3}(6) + 1(-1) \\ \frac{1}{4}(2) + \frac{5}{12}(6) + 1(-1) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{1}{2} - \frac{13}{2} + 2 \\ 0 + 2 - 1 \\ \frac{1}{2} + \frac{5}{2} - 1 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix}$$

■ 5. The vectors $\vec{v} = (-2,3)$ and $\vec{w} = (4,0)$ form an alternate basis for $\mathbb{R}^2$. Use them, and an inverse matrix, to transform $\vec{x} = 6\mathbf{i} - 3\mathbf{j}$ into the alternate basis.

Solution:

The vector $\vec{x} = (6, -3)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v} = (-2,3)$ and $\vec{w} = (4,0)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B = \vec{x}$$

$$\begin{bmatrix} -2 & 4 \\ 3 & 0 \end{bmatrix}[\vec{x}]_B = \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Find A^{-1} from A.

$$[A\ |\ I] = \begin{bmatrix} -2 & 4 & | & 1 & 0 \\ 3 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[A\ |\ I] = \begin{bmatrix} 1 & -2 & | & -\frac{1}{2} & 0 \\ 3 & 0 & | & 0 & 1 \end{bmatrix}$$

$$[A\ |\ I] = \begin{bmatrix} 1 & -2 & | & -\frac{1}{2} & 0 \\ 0 & 6 & | & \frac{3}{2} & 1 \end{bmatrix}$$

$$[A\ |\ I] = \begin{bmatrix} 1 & -2 & | & -\frac{1}{2} & 0 \\ 0 & 1 & | & \frac{1}{4} & \frac{1}{6} \end{bmatrix}$$

$$[A\ |\ I] = \begin{bmatrix} 1 & 0 & | & 0 & \frac{1}{3} \\ 0 & 1 & | & \frac{1}{4} & \frac{1}{6} \end{bmatrix}$$

So the inverse matrix is

$$A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix}$$

Now to find the representation of $\vec{x} = (6, -3)$ in the alternate basis, we simply multiply the inverse matrix by the vector.

$$[\vec{x}]_B = A^{-1}\vec{x}$$

$$[\vec{x}]_B = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{6} \end{bmatrix} \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 0(6) + \frac{1}{3}(-3) \\ \frac{1}{4}(6) + \frac{1}{6}(-3) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

■ 6. The vectors $\vec{v} = (-1,3,2)$, $\vec{s} = (-2,4,-4)$, and $\vec{w} = (1,-2,0)$ form an alternate basis for $\mathbb{R}^3$. Use them to transform $\vec{x} = -2\mathbf{i} - 4\mathbf{k}$ into the alternate basis.

Solution:

The vector $\vec{x}=(-2,0,-4)$ is given in terms of the standard basis, and we need to transform it into an alternate basis that's defined by $\vec{v}=(-1,3,2)$, $\vec{s}=(-2,4,-4)$, and $\vec{w}=(1,-2,0)$.

So let's plug the values we've been given into the matrix equation.

$$A[\vec{x}]_B=\vec{x}$$

$$\begin{bmatrix}-1&-2&1\\3&4&-2\\2&-4&0\end{bmatrix}[\vec{x}]_B=\begin{bmatrix}-2\\0\\-4\end{bmatrix}$$

To find the representation of $\vec{x}$ in the alternate basis, $[\vec{x}]_B$, we'll put the augmented matrix into reduced row-echelon form.

$$\begin{bmatrix}-1&-2&1&|&-2\\3&4&-2&|&0\\2&-4&0&|&-4\end{bmatrix}\to\begin{bmatrix}1&2&-1&|&2\\3&4&-2&|&0\\2&-4&0&|&-4\end{bmatrix}\to\begin{bmatrix}1&2&-1&|&2\\0&-2&1&|&-6\\2&-4&0&|&-4\end{bmatrix}$$

$$\begin{bmatrix}1&2&-1&|&2\\0&-2&1&|&-6\\0&-8&2&|&-8\end{bmatrix}\to\begin{bmatrix}1&2&-1&|&2\\0&1&-\frac{1}{2}&|&3\\0&-8&2&|&-8\end{bmatrix}\to\begin{bmatrix}1&0&0&|&-4\\0&1&-\frac{1}{2}&|&3\\0&-8&2&|&-8\end{bmatrix}$$

$$\begin{bmatrix}1&0&0&|&-4\\0&1&-\frac{1}{2}&|&3\\0&0&-2&|&16\end{bmatrix}\to\begin{bmatrix}1&0&0&|&-4\\0&1&-\frac{1}{2}&|&3\\0&0&1&|&-8\end{bmatrix}\to\begin{bmatrix}1&0&0&|&-4\\0&1&0&|&-1\\0&0&1&|&-8\end{bmatrix}$$

So $\vec{x}=(-2,0,-4)$, expressed in the alternate basis, is

$$[\vec{x}]_B=\begin{bmatrix}-4\\-1\\-8\end{bmatrix}$$

TRANSFORMATION MATRIX FOR A BASIS

1. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (2,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} 3 & -2 \\ 6 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -6 \end{bmatrix}\right)$$

Solution:

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (-2,1)$ and $\vec{v}_2 = (4,-6)$, so

$$C = \begin{bmatrix} -2 & 4 \\ 1 & -6 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} -2 & 4 & | & 1 & 0 \\ 1 & -6 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -2 & | & -\frac{1}{2} & 0 \\ 1 & -6 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -2 & | & -\frac{1}{2} & 0 \\ 0 & -4 & | & \frac{1}{2} & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & -2 & | & -\frac{1}{2} & 0 \\ 0 & 1 & | & -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & -\frac{3}{4} & -\frac{1}{2} \\ 0 & 1 & | & -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} -2 & 4 \\ 1 & -6 \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 3(-2) - 2(1) & 3(4) - 2(-6) \\ 6(-2) + 0(1) & 6(4) + 0(-6) \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} -6 - 2 & 12 + 12 \\ -12 + 0 & 24 + 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{3}{4} & -\frac{1}{2} \\ -\frac{1}{8} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} -8 & 24 \\ -12 & 24 \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{3}{4}(-8) - \frac{1}{2}(-12) & -\frac{3}{4}(24) - \frac{1}{2}(24) \\ -\frac{1}{8}(-8) - \frac{1}{4}(-12) & -\frac{1}{8}(24) - \frac{1}{4}(24) \end{bmatrix}$$

$$M = \begin{bmatrix} 6 + 6 & -18 - 12 \\ 1 + 3 & -3 - 6 \end{bmatrix}$$

$$M = \begin{bmatrix} 12 & -30 \\ 4 & -9 \end{bmatrix}$$

We’ve been asked to transform $[\vec{x}]_B = (2,1)$, so we’ll multiply M by this vector.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

$$[T(\vec{x})]_B = \begin{bmatrix} 12 & -30 \\ 4 & -9 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 12(2) - 30(1) \\ 4(2) - 9(1) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -6 \\ -1 \end{bmatrix}$$

■ 2. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $\vec{x} = (-2,4)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 4 & 5 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \end{bmatrix} \right)$$

Solution:

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (-2,1)$ and $\vec{v}_2 = (-1,3)$, so

$$C = \begin{bmatrix} -2 & -1 \\ 1 & 3 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} -2 & -1 & | & 1 & 0 \\ 1 & 3 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & \frac{1}{2} & | & -\frac{1}{2} & 0 \\ 1 & 3 & | & 0 & 1 \end{bmatrix}$$

$$[C\ |\ I] = \begin{bmatrix} 1 & \frac{1}{2} & | & -\frac{1}{2} & 0 \\ 0 & \frac{5}{2} & | & \frac{1}{2} & 1 \end{bmatrix}$$

$$[C\ |\ I] = \begin{bmatrix} 1 & \frac{1}{2} & | & -\frac{1}{2} & 0 \\ 0 & 1 & | & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

$$[C\ |\ I] = \begin{bmatrix} 1 & 0 & | & -\frac{3}{5} & -\frac{1}{5} \\ 0 & 1 & | & \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

Find the transformation $T(\vec{x})$ where $T(\vec{x}) = A\vec{x}$.

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 4 & 5 \end{bmatrix} \vec{x}$$

$$T(\vec{x}) = \begin{bmatrix} -3 & 1 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -3(-2) + 1(4) \\ 4(-2) + 5(4) \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 6 + 4 \\ -8 + 20 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

Then $[T(\vec{x})]_B = C^{-1}T(\vec{x})$.

$$[T(\vec{x})]_B = \begin{bmatrix} -\frac{3}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 10 \\ 12 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -\frac{3}{5}(10) - \frac{1}{5}(12) \\ \frac{1}{5}(10) + \frac{2}{5}(12) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -\frac{30}{5} - \frac{12}{5} \\ \frac{10}{5} + \frac{24}{5} \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -\frac{42}{5} \\ \frac{34}{5} \end{bmatrix}$$

■ 3. Use the transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ to transform $[\vec{x}]_B = (-5,2)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right)$$

Solution:

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (1,-3)$ and $\vec{v}_2 = (2,4)$, so

$$C = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ -3 & 4 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 10 & | & 3 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & | & 1 & 0 \\ 0 & 1 & | & \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & | & \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -2 & 3 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -2(1) + 3(-3) & -2(2) + 3(4) \\ 1(1) + 5(-3) & 1(2) + 5(4) \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -2 - 9 & -4 + 12 \\ 1 - 15 & 2 + 20 \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{3}{10} & \frac{1}{10} \end{bmatrix} \begin{bmatrix} -11 & 8 \\ -14 & 22 \end{bmatrix}$$

$$M = \begin{bmatrix} \frac{2}{5}(-11) - \frac{1}{5}(-14) & \frac{2}{5}(8) - \frac{1}{5}(22) \\ \frac{3}{10}(-11) + \frac{1}{10}(-14) & \frac{3}{10}(8) + \frac{1}{10}(22) \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{22}{5} + \frac{14}{5} & \frac{16}{5} - \frac{22}{5} \\ -\frac{33}{10} - \frac{14}{10} & \frac{24}{10} + \frac{22}{10} \end{bmatrix}$$

$$M = \begin{bmatrix} -\frac{8}{5} & -\frac{6}{5} \\ -\frac{47}{10} & \frac{46}{10} \end{bmatrix}$$

We've been asked to transform $[\vec{x}]_B = (-5,2)$, so we'll multiply M by this vector.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

$$[T(\vec{x})]_B = \begin{bmatrix} -\frac{8}{5} & -\frac{6}{5} \\ -\frac{47}{10} & \frac{46}{10} \end{bmatrix} \begin{bmatrix} -5 \\ 2 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -\frac{8}{5}(-5) - \frac{6}{5}(2) \\ -\frac{47}{10}(-5) + \frac{46}{10}(2) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} \frac{40}{5} - \frac{12}{5} \\ \frac{235}{10} + \frac{92}{10} \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} \frac{28}{5} \\ \frac{327}{10} \end{bmatrix}$$

■ 4. Use the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ to transform $\vec{x} = (6, -3)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -5 & -4 \\ 2 & -8 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \end{bmatrix}\right)$$

Solution:

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (1, -2)$ and $\vec{v}_2 = (0,4)$, so

$$C = \begin{bmatrix} 1 & 0 \\ -2 & 4 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & 1 & 0 \\ -2 & 4 & | & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & 1 & 0 \\ 0 & 4 & | & 2 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & | & 1 & 0 \\ 0 & 1 & | & \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix}$$

Find the transformation $T(\vec{x})$ where $T(\vec{x}) = A\vec{x}$.

$$T(\vec{x}) = \begin{bmatrix} -5 & -4 \\ 2 & -8 \end{bmatrix} \vec{x}$$

$$T(\vec{x}) = \begin{bmatrix} -5 & -4 \\ 2 & -8 \end{bmatrix} \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -5(6) - 4(-3) \\ 2(6) - 8(-3) \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -30 + 12 \\ 12 + 24 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -18 \\ 36 \end{bmatrix}$$

Then $[T(\vec{x})]_B = C^{-1}T(\vec{x})$.

$$[T(\vec{x})]_B = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -18 \\ 36 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 1(-18) + 0(36) \\ \frac{1}{2}(-18) + \frac{1}{4}(36) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -18 \\ 0 \end{bmatrix}$$

5. Use the transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ to transform $[\vec{x}]_B = (-2,4,1)$ in the basis B in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & -1 \\ 0 & 2 & 0 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Solution:

In order to transform a vector in the alternate basis in the domain into a vector in the alternate basis in the codomain, we need to find the transformation matrix M.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

We know that $M = C^{-1}AC$, and A was given to us in the problem as part of $T(\vec{x})$, so we just need to find C and C^{-1}.

The change of basis matrix C that transforms vectors from the standard basis into vectors in the alternate basis B is made of the column vectors that span B, $\vec{v}_1 = (1,-1,2)$, $\vec{v}_2 = (0,2,-1)$, and $\vec{v}_3 = (1,0,1)$, so

$$C = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C\mid I]=\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ -1 & 2 & 0 & | & 0 & 1 & 0 \\ 2 & -1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C\mid I]=\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 2 & 1 & | & 1 & 1 & 0 \\ 2 & -1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C\mid I]=\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 2 & 1 & | & 1 & 1 & 0 \\ 0 & -1 & -1 & | & -2 & 0 & 1 \end{bmatrix}$$

$$[C\mid I]=\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & -1 & -1 & | & -2 & 0 & 1 \end{bmatrix}$$

$$[C\mid I]=\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & | & -\frac{3}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

$$[C\mid I]=\begin{bmatrix} 1 & 0 & 1 & | & 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & | & 3 & -1 & -2 \end{bmatrix}$$

$$[C\mid I]=\begin{bmatrix} 1 & 0 & 0 & | & -2 & 1 & 2 \\ 0 & 1 & \frac{1}{2} & | & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & | & 3 & -1 & -2 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 0 & | & -2 & 1 & 2 \\ 0 & 1 & 0 & | & -1 & 1 & 1 \\ 0 & 0 & 1 & | & 3 & -1 & -2 \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 1 & 1 \\ 3 & -1 & -2 \end{bmatrix}$$

With A, C, and C^{-1}, we can find $M = C^{-1}AC$.

$$M = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 1 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} -4 & 1 & 1 \\ 2 & -3 & -1 \\ 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ -1 & 2 & 0 \\ 2 & -1 & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 1 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} -4(1)+1(-1)+1(2) & -4(0)+1(2)+1(-1) & -4(1)+1(0)+1(1) \\ 2(1)-3(-1)-1(2) & 2(0)-3(2)-1(-1) & 2(1)-3(0)-1(1) \\ 0(1)+2(-1)+0(2) & 0(0)+2(2)+0(-1) & 0(1)+2(0)+0(1) \end{bmatrix}$$

$$M = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 1 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} -4-1+2 & 0+2-1 & -4+0+1 \\ 2+3-2 & 0-6+1 & 2-0-1 \\ 0-2+0 & 0+4+0 & 0+0+0 \end{bmatrix}$$

$$M = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 1 & 1 \\ 3 & -1 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 & -3 \\ 3 & -5 & 1 \\ -2 & 4 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -2(-3)+1(3)+2(-2) & -2(1)+1(-5)+2(4) & -2(-3)+1(1)+2(0) \\ -1(-3)+1(3)+1(-2) & -1(1)+1(-5)+1(4) & -1(-3)+1(1)+1(0) \\ 3(-3)-1(3)-2(-2) & 3(1)-1(-5)-2(4) & 3(-3)-1(1)-2(0) \end{bmatrix}$$

$$M = \begin{bmatrix} 6+3-4 & -2-5+8 & 6+1+0 \\ 3+3-2 & -1-5+4 & 3+1+0 \\ -9-3+4 & 3+5-8 & -9-1-0 \end{bmatrix}$$

$$M = \begin{bmatrix} 5 & 1 & 7 \\ 4 & -2 & 4 \\ -8 & 0 & -10 \end{bmatrix}$$

We've been asked to transform $[\vec{x}]_B = (-2,4,1)$, so we'll multiply M by this vector.

$$[T(\vec{x})]_B = M[\vec{x}]_B$$

$$[T(\vec{x})]_B = \begin{bmatrix} 5 & 1 & 7 \\ 4 & -2 & 4 \\ -8 & 0 & -10 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 5(-2)+1(4)+7(1) \\ 4(-2)-2(4)+4(1) \\ -8(-2)+0(4)-10(1) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -10+4+7 \\ -8-8+4 \\ 16+0-10 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} 1 \\ -12 \\ 6 \end{bmatrix}$$

■ 6. Use the transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ to transform $\vec{x} = (-2,3,1)$ in the standard basis in the domain to a vector in the basis B in the codomain.

$$T(\vec{x}) = \begin{bmatrix} -4 & 2 & 1 \\ 0 & 3 & -5 \\ 1 & -2 & 4 \end{bmatrix} \vec{x}$$

$$B = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \end{bmatrix} \right)$$

Solution:

The change of basis matrix C for the basis B is made of the column vectors that span B, $\vec{v}_1 = (-1,1,0)$, $\vec{v}_2 = (-2,1,-1)$, and $\vec{v}_3 = (-2,0,-1)$, so

$$C = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix}$$

Now we'll find C^{-1}.

$$[C \mid I] = \begin{bmatrix} -1 & -2 & -2 & | & 1 & 0 & 0 \\ 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & -1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & 2 & | & -1 & 0 & 0 \\ 1 & 1 & 0 & | & 0 & 1 & 0 \\ 0 & -1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & 2 & | & -1 & 0 & 0 \\ 0 & -1 & -2 & | & 1 & 1 & 0 \\ 0 & -1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 2 & 2 & | & -1 & 0 & 0 \\ 0 & 1 & 2 & | & -1 & -1 & 0 \\ 0 & -1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & -2 & | & 1 & 2 & 0 \\ 0 & 1 & 2 & | & -1 & -1 & 0 \\ 0 & -1 & -1 & | & 0 & 0 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & -2 & | & 1 & 2 & 0 \\ 0 & 1 & 2 & | & -1 & -1 & 0 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 0 & | & -1 & 0 & 2 \\ 0 & 1 & 2 & | & -1 & -1 & 0 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$$

$$[C \mid I] = \begin{bmatrix} 1 & 0 & 0 & | & -1 & 0 & 2 \\ 0 & 1 & 0 & | & 1 & 1 & -2 \\ 0 & 0 & 1 & | & -1 & -1 & 1 \end{bmatrix}$$

So,

$$C^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

Find the transformation $T(\vec{x})$ where $T(\vec{x}) = A\vec{x}$.

$$T(\vec{x}) = \begin{bmatrix} -4 & 2 & 1 \\ 0 & 3 & -5 \\ 1 & -2 & 4 \end{bmatrix} \vec{x}$$

$$T(\vec{x}) = \begin{bmatrix} -4 & 2 & 1 \\ 0 & 3 & -5 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \\ 1 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} -4(-2) + 2(3) + 1(1) \\ 0(-2) + 3(3) - 5(1) \\ 1(-2) - 2(3) + 4(1) \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 8 + 6 + 1 \\ 0 + 9 - 5 \\ -2 - 6 + 4 \end{bmatrix}$$

$$T(\vec{x}) = \begin{bmatrix} 15 \\ 4 \\ -4 \end{bmatrix}$$

Then $[T(\vec{x})]_B = C^{-1}T(\vec{x})$.

$$[T(\vec{x})]_B = \begin{bmatrix} -1 & 0 & 2 \\ 1 & 1 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 15 \\ 4 \\ -4 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -1(15) + 0(4) + 2(-4) \\ 1(15) + 1(4) - 2(-4) \\ -1(15) - 1(4) + 1(-4) \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -15 + 0 - 8 \\ 15 + 4 + 8 \\ -15 - 4 - 4 \end{bmatrix}$$

$$[T(\vec{x})]_B = \begin{bmatrix} -23 \\ 27 \\ -23 \end{bmatrix}$$

ORTHONORMAL BASES

1. Verify that the vector set $V = \{\vec{v}_1, \vec{v}_2\}$ is orthonormal if $\vec{v}_1 = (1,0,0)$ and $\vec{v}_2 = (0,0,-1)$.

Solution:

If the set is orthonormal, each vector has length 1.

$$||\vec{v}_1||^2 = \vec{v}_1 \cdot \vec{v}_1 = 1(1) + 0(0) + 0(0) = 1 + 0 + 0 = 1$$

$$||\vec{v}_2||^2 = \vec{v}_2 \cdot \vec{v}_2 = 0(0) + 0(0) - 1(-1) = 0 + 0 + 1 = 1$$

Both vectors have length 1, so now we'll just confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = 1(0) + 0(0) + 0(-1) = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, $\vec{v}_1$ and $\vec{v}_2$ form an orthonormal set, so V is orthonormal.

2. Determine that the vector set $V = \{\vec{v}_1, \vec{v}_2\}$ is orthonormal.

$$\vec{v}_1 = \left(\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}\right)$$

$$\vec{v}_2 = \left(-\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$$

Solution:

If the set is orthonormal, each vector has length 1.

$$||\vec{v}_1||^2 = \vec{v}_1 \cdot \vec{v}_1 = \frac{2}{3}\left(\frac{2}{3}\right) - \frac{1}{3}\left(-\frac{1}{3}\right) - \frac{2}{3}\left(-\frac{2}{3}\right) = \frac{4}{9} + \frac{1}{9} + \frac{4}{9} = 1$$

$$||\vec{v}_2||^2 = \vec{v}_2 \cdot \vec{v}_2 = -\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) + 0(0) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2} + 0 + \frac{1}{2} = 1$$

Both vectors have length 1, so now we'll just confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{2}{3}\left(-\frac{1}{\sqrt{2}}\right) - \frac{1}{3}(0) - \frac{2}{3}\left(\frac{1}{\sqrt{2}}\right) = -\frac{2}{3\sqrt{2}} - 0 - \frac{2}{3\sqrt{2}} = -\frac{4}{3\sqrt{2}}$$

Because the dot product of the vectors is nonzero, $V = \{\vec{v}_1, \vec{v}_2\}$ is not an orthonormal set.

■ 3. Convert $\vec{x} = (-2,10)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{3}{4} \\ -\frac{\sqrt{7}}{4} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} \frac{\sqrt{7}}{4} \\ \frac{3}{4} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2 = \left(\frac{3}{4}\right)^2 + \left(-\frac{\sqrt{7}}{4}\right)^2 = \frac{9}{16} + \frac{7}{16} = \frac{16}{16} = 1$$

$$||\vec{v}_2||^2 = \left(\frac{\sqrt{7}}{4}\right)^2 + \left(\frac{3}{4}\right)^2 = \frac{7}{16} + \frac{9}{16} = \frac{16}{16} = 1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{4}\left(\frac{\sqrt{7}}{4}\right) - \frac{\sqrt{7}}{4}\left(\frac{3}{4}\right) = \frac{3\sqrt{7}}{16} - \frac{3\sqrt{7}}{16} = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x} = (-2,10)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B = \begin{bmatrix} \frac{3}{4}(-2) - \frac{\sqrt{7}}{4}(10) \\ \frac{\sqrt{7}}{4}(-2) + \frac{3}{4}(10) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{3}{2} - \frac{5\sqrt{7}}{2} \\ -\frac{\sqrt{7}}{2} + \frac{15}{2} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -\frac{3+5\sqrt{7}}{2} \\ -\frac{\sqrt{7}-15}{2} \end{bmatrix}$$

■ 4. Convert $\vec{x} = (-25,10)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{4}{5} \\ -\frac{3}{5} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2 = \left(\frac{3}{5}\right)^2 + \left(-\frac{4}{5}\right)^2 = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1$$

$$||\vec{v}_2||^2 = \left(-\frac{4}{5}\right)^2 + \left(-\frac{3}{5}\right)^2 = \frac{16}{25} + \frac{9}{25} = \frac{25}{25} = 1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1 \cdot \vec{v}_2 = \frac{3}{5}\left(-\frac{4}{5}\right) - \frac{4}{5}\left(-\frac{3}{5}\right) = -\frac{12}{25} + \frac{12}{25} = 0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x} = (-25,10)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B = \begin{bmatrix} \frac{3}{5}(-25) - \frac{4}{5}(10) \\ -\frac{4}{5}(-25) - \frac{3}{5}(10) \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -15 - 8 \\ 20 - 6 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -23 \\ 14 \end{bmatrix}$$

■ 5. Convert $\vec{x} = (-6,3,12)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{2}{3} \\ \frac{2}{3} \\ -\frac{1}{3} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2=\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2=\frac{4}{9}+\frac{1}{9}+\frac{4}{9}=\frac{9}{9}=1$$

$$||\vec{v}_2||^2=\left(-\frac{1}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2=\frac{1}{9}+\frac{4}{9}+\frac{4}{9}=\frac{9}{9}=1$$

$$||\vec{v}_3||^2=\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^2+\left(-\frac{1}{3}\right)^2=\frac{4}{9}+\frac{4}{9}+\frac{1}{9}=\frac{9}{9}=1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=\frac{2}{3}\left(-\frac{1}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right)=-\frac{2}{9}-\frac{2}{9}+\frac{4}{9}=0$$

$$\vec{v}_1\cdot\vec{v}_3=\frac{2}{3}\left(\frac{2}{3}\right)-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(-\frac{1}{3}\right)=\frac{4}{9}-\frac{2}{9}-\frac{2}{9}=0$$

$$\vec{v}_2\cdot\vec{v}_3=-\frac{1}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(\frac{2}{3}\right)+\frac{2}{3}\left(-\frac{1}{3}\right)=-\frac{2}{9}+\frac{4}{9}-\frac{2}{9}=0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x}=(-6,3,12)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B=\begin{bmatrix}\frac{2}{3}(-6)-\frac{1}{3}(3)+\frac{2}{3}(12)\\-\frac{1}{3}(-6)+\frac{2}{3}(3)+\frac{2}{3}(12)\\\frac{2}{3}(-6)+\frac{2}{3}(3)-\frac{1}{3}(12)\end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} -4-1+8 \\ 2+2+8 \\ -4+2-4 \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 3 \\ 12 \\ -6 \end{bmatrix}$$

■ 6. Convert $\vec{x} = (2,0,-3)$ from the standard basis to the alternate basis $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{v}_1 = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

Solution:

Confirm that the set is orthonormal by first verifying that each vector has length 1.

$$||\vec{v}_1||^2 = 0^2 + \left(\frac{1}{\sqrt{2}}\right)^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = 0 + \frac{1}{2} + \frac{1}{2} = \frac{2}{2} = 1$$

$$||\vec{v}_2||^2 = \left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$||\vec{v}_3||^2=\left(\frac{2}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2+\left(\frac{1}{\sqrt{6}}\right)^2=\frac{4}{6}+\frac{1}{6}+\frac{1}{6}=\frac{6}{6}=1$$

Confirm that the vectors are orthogonal.

$$\vec{v}_1\cdot\vec{v}_2=0\left(-\frac{1}{\sqrt{3}}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\right)=0+\frac{1}{\sqrt{6}}-\frac{1}{\sqrt{6}}=0$$

$$\vec{v}_1\cdot\vec{v}_3=0\left(\frac{2}{\sqrt{6}}\right)+\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)-\frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{6}}\right)=0+\frac{1}{\sqrt{12}}-\frac{1}{\sqrt{12}}=0$$

$$\vec{v}_2\cdot\vec{v}_3=\left(-\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{6}}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{6}}\right)=-\frac{2}{\sqrt{18}}+\frac{1}{\sqrt{18}}+\frac{1}{\sqrt{18}}=0$$

Because the vectors are orthogonal to one another, and because they both have length 1, the set is orthonormal. And because the set is orthonormal, the vector $\vec{x}=(2,0,-3)$ can be converted to the alternate basis B with dot products.

$$[\vec{x}]_B=\begin{bmatrix}0(2)+\frac{1}{\sqrt{2}}(0)-\frac{1}{\sqrt{2}}(-3)\\ -\frac{1}{\sqrt{3}}(2)+\frac{1}{\sqrt{3}}(0)+\frac{1}{\sqrt{3}}(-3)\\ \frac{2}{\sqrt{6}}(2)+\frac{1}{\sqrt{6}}(0)+\frac{1}{\sqrt{6}}(-3)\end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} 0 + 0 + \frac{3}{\sqrt{2}} \\ -\frac{2}{\sqrt{3}} + 0 - \frac{3}{\sqrt{3}} \\ \frac{4}{\sqrt{6}} + 0 - \frac{3}{\sqrt{6}} \end{bmatrix}$$

$$[\vec{x}]_B = \begin{bmatrix} \frac{3}{\sqrt{2}} \\ -\frac{5}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$$

PROJECTION ONTO AN ORTHONORMAL BASIS

■ 1. Find the projection of $\vec{x}=(-5,0,-2)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{2}{\sqrt{5}}\\ \frac{1}{\sqrt{5}}\\ 0\end{bmatrix},\begin{bmatrix}0\\ 0\\ -1\end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2=\left(\frac{2}{\sqrt{5}}\right)^2+\left(\frac{1}{\sqrt{5}}\right)^2+0^2=\frac{4}{5}+\frac{1}{5}+0=\frac{5}{5}=1$$

$$||\vec{v}_2||^2=0^2+0^2+(-1)^2=0+0+1=1$$

$$\vec{v}_1\cdot\vec{v}_2=\frac{2}{\sqrt{5}}(0)+\frac{1}{\sqrt{5}}(0)+0(-1)=0+0+0=0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(-5,0,-2)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{2}{\sqrt{5}} & 0\\ \frac{1}{\sqrt{5}} & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0\\ 0 & 0 & -1\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{2}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\right)+0(0) & \frac{2}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right)+0(0) & \frac{2}{\sqrt{5}}(0)+0(-1)\\ \frac{1}{\sqrt{5}}\left(\frac{2}{\sqrt{5}}\right)+0(0) & \frac{1}{\sqrt{5}}\left(\frac{1}{\sqrt{5}}\right)+0(0) & \frac{1}{\sqrt{5}}(0)+0(-1)\\ 0\left(\frac{2}{\sqrt{5}}\right)-1(0) & 0\left(\frac{1}{\sqrt{5}}\right)-1(0) & 0(0)-1(-1)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{4}{5}+0 & \frac{2}{5}+0 & 0+0\\ \frac{2}{5}+0 & \frac{1}{5}+0 & 0+0\\ 0-0 & 0-0 & 0+1\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{4}{5} & \frac{2}{5} & 0\\ \frac{2}{5} & \frac{1}{5} & 0\\ 0 & 0 & 1\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-5,0,-2)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{4}{5} & \frac{2}{5} & 0\\ \frac{2}{5} & \frac{1}{5} & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}-5\\ 0\\ -2\end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{4}{5}(-5) + \frac{2}{5}(0) + 0(-2) \\ \frac{2}{5}(-5) + \frac{1}{5}(0) + 0(-2) \\ 0(-5) + 0(0) + 1(-2) \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -4 + 0 + 0 \\ -2 + 0 + 0 \\ 0 + 0 - 2 \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -4 \\ -2 \\ -2 \end{bmatrix}$$

■ 2. Find the projection of $\vec{x} = (-66,33,11)$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ -\frac{2}{\sqrt{6}} \end{bmatrix}, \begin{bmatrix} -\frac{3}{\sqrt{11}} \\ \frac{1}{\sqrt{11}} \\ -\frac{1}{\sqrt{11}} \end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(\frac{1}{\sqrt{6}}\right)^2 + \left(\frac{1}{\sqrt{6}}\right)^2 + \left(-\frac{2}{\sqrt{6}}\right)^2 = \frac{1}{6} + \frac{1}{6} + \frac{4}{6} = \frac{6}{6} = 1$$

$$||\vec{v}_2||^2=\left(-\frac{3}{\sqrt{11}}\right)^2+\left(\frac{1}{\sqrt{11}}\right)^2+\left(-\frac{1}{\sqrt{11}}\right)^2=\frac{9}{11}+\frac{1}{11}+\frac{1}{11}=\frac{11}{11}=1$$

$$\vec{v}_1\cdot\vec{v}_2=\frac{1}{\sqrt{6}}\left(-\frac{3}{\sqrt{11}}\right)+\left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{11}}\right)-\frac{2}{\sqrt{6}}\left(-\frac{1}{\sqrt{11}}\right)$$

$$=-\frac{3}{\sqrt{66}}+\frac{1}{\sqrt{66}}+\frac{2}{\sqrt{66}}=0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(-66,33,11)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{6}} & -\frac{3}{\sqrt{11}}\\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{11}}\\ -\frac{2}{\sqrt{6}} & -\frac{1}{\sqrt{11}}\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}}\\ -\frac{3}{\sqrt{11}} & \frac{1}{\sqrt{11}} & -\frac{1}{\sqrt{11}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{3}{\sqrt{11}}\left(-\frac{3}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{3}{\sqrt{11}}\left(\frac{1}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)-\frac{3}{\sqrt{11}}\left(-\frac{1}{\sqrt{11}}\right)\\ \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{11}}\left(-\frac{3}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)+\frac{1}{\sqrt{11}}\left(\frac{1}{\sqrt{11}}\right) & \frac{1}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)+\frac{1}{\sqrt{11}}\left(-\frac{1}{\sqrt{11}}\right)\\ -\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{1}{\sqrt{11}}\left(-\frac{3}{\sqrt{11}}\right) & -\frac{2}{\sqrt{6}}\left(\frac{1}{\sqrt{6}}\right)-\frac{1}{\sqrt{11}}\left(\frac{1}{\sqrt{11}}\right) & -\frac{2}{\sqrt{6}}\left(-\frac{2}{\sqrt{6}}\right)-\frac{1}{\sqrt{11}}\left(-\frac{1}{\sqrt{11}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{1}{6} + \frac{9}{11} & \frac{1}{6} - \frac{3}{11} & -\frac{1}{3} + \frac{3}{11} \\ \frac{1}{6} - \frac{3}{11} & \frac{1}{6} + \frac{1}{11} & -\frac{1}{3} - \frac{1}{11} \\ -\frac{1}{3} + \frac{3}{11} & -\frac{1}{3} - \frac{1}{11} & \frac{2}{3} + \frac{1}{11} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{65}{66} & -\frac{7}{66} & -\frac{2}{33} \\ -\frac{7}{66} & \frac{17}{66} & -\frac{14}{33} \\ -\frac{2}{33} & -\frac{14}{33} & \frac{25}{33} \end{bmatrix} \vec{x}$$

Applying the projection to $\vec{x} = (-66,33,11)$ gives

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{65}{66} & -\frac{7}{66} & -\frac{2}{33} \\ -\frac{7}{66} & \frac{17}{66} & -\frac{14}{33} \\ -\frac{2}{33} & -\frac{14}{33} & \frac{25}{33} \end{bmatrix} \begin{bmatrix} -66 \\ 33 \\ 11 \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{65}{66}(-66) - \frac{7}{66}(33) - \frac{2}{33}(11) \\ -\frac{7}{66}(-66) + \frac{17}{66}(33) - \frac{14}{33}(11) \\ -\frac{2}{33}(-66) - \frac{14}{33}(33) + \frac{25}{33}(11) \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -65 - \frac{7}{2} - \frac{2}{3} \\ 7 + \frac{17}{2} - \frac{14}{3} \\ 4 - 14 + \frac{25}{3} \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -\frac{415}{6} \\ \frac{65}{6} \\ -\frac{5}{3} \end{bmatrix}$$

■ 3. Find the projection of $\vec{x} = (-6, -3, 6)$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix} \right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = \left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} = \frac{3}{3} = 1$$

$$||\vec{v}_2||^2 = \left(-\frac{1}{\sqrt{2}}\right)^2 + 0^2 + \left(-\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} + 0 + \frac{1}{2} = \frac{2}{2} = 1$$

$$\vec{v}_1 \cdot \vec{v}_2 = -\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) + \frac{1}{\sqrt{3}}(0) + \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{6}} + 0 - \frac{1}{\sqrt{6}} = 0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(-6,-3,6)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & 0\\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}}\end{bmatrix}\begin{bmatrix}-\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}(0) & -\frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)\\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)+0\left(-\frac{1}{\sqrt{2}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+0(0) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)+0\left(-\frac{1}{\sqrt{2}}\right)\\ \frac{1}{\sqrt{3}}\left(-\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}(0) & \frac{1}{\sqrt{3}}\left(\frac{1}{\sqrt{3}}\right)-\frac{1}{\sqrt{2}}\left(-\frac{1}{\sqrt{2}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{1}{3}+\frac{1}{2} & -\frac{1}{3}-0 & -\frac{1}{3}+\frac{1}{2}\\ -\frac{1}{3}+0 & \frac{1}{3}+0 & \frac{1}{3}+0\\ -\frac{1}{3}+\frac{1}{2} & \frac{1}{3}-0 & \frac{1}{3}+\frac{1}{2}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{5}{6} & -\frac{1}{3} & \frac{1}{6}\\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3}\\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x} = (-6, -3, 6)$ gives

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & \frac{1}{3} & \frac{5}{6} \end{bmatrix} \begin{bmatrix} -6 \\ -3 \\ 6 \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} \frac{5}{6}(-6) - \frac{1}{3}(-3) + \frac{1}{6}(6) \\ -\frac{1}{3}(-6) + \frac{1}{3}(-3) + \frac{1}{3}(6) \\ \frac{1}{6}(-6) + \frac{1}{3}(-3) + \frac{5}{6}(6) \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -5 + 1 + 1 \\ 2 - 1 + 2 \\ -1 - 1 + 5 \end{bmatrix}$$

$$\text{Proj}_V \vec{x} = \begin{bmatrix} -3 \\ 3 \\ 3 \end{bmatrix}$$

■ 4. Find the projection of $\vec{x} = (-2, 3, 5)$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} \frac{3}{\sqrt{10}} \\ 0 \\ \frac{1}{\sqrt{10}} \end{bmatrix} \right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2 = 0^2 + (-1)^2 + 0^2 = 0 + 1 + 0 = 1$$

$$||\vec{v}_2||^2 = \left(\frac{3}{\sqrt{10}}\right)^2 + 0^2 + \left(\frac{1}{\sqrt{10}}\right)^2 = \frac{9}{10} + 0 + \frac{1}{10} = \frac{10}{10} = 1$$

$$\vec{v}_1 \cdot \vec{v}_2 = 0\left(\frac{3}{\sqrt{10}}\right) - 1(0) + 0\left(\frac{1}{\sqrt{10}}\right) = 0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x} = (-2,3,5)$ onto V is

$$\text{Proj}_V\vec{x} = AA^T\vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 0 & \frac{3}{\sqrt{10}} \\ -1 & 0 \\ 0 & \frac{1}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ \frac{3}{\sqrt{10}} & 0 & \frac{1}{\sqrt{10}} \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 0(0) + \frac{3}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right) & 0(-1) + \frac{3}{\sqrt{10}}(0) & 0(0) + \frac{3}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) \\ -1(0) + 0\left(\frac{3}{\sqrt{10}}\right) & -1(-1) + 0(0) & -1(0) + 0\left(\frac{1}{\sqrt{10}}\right) \\ 0(0) + \frac{1}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right) & 0(-1) + \frac{1}{\sqrt{10}}(0) & 0(0) + \frac{1}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) \end{bmatrix} \vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}0+\frac{9}{10} & 0+0 & 0+\frac{3}{10}\\ 0+0 & 1+0 & 0+0\\ 0+\frac{3}{10} & 0+0 & 0+\frac{1}{10}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{10} & 0 & \frac{3}{10}\\ 0 & 1 & 0\\ \frac{3}{10} & 0 & \frac{1}{10}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-2,3,5)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{10} & 0 & \frac{3}{10}\\ 0 & 1 & 0\\ \frac{3}{10} & 0 & \frac{1}{10}\end{bmatrix}\begin{bmatrix}-2\\ 3\\ 5\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{10}(-2)+0(3)+\frac{3}{10}(5)\\ 0(-2)+1(3)+0(5)\\ \frac{3}{10}(-2)+0(3)+\frac{1}{10}(5)\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{18}{10}+0+\frac{15}{10}\\ 0+3+0\\ -\frac{6}{10}+0+\frac{5}{10}\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}-\frac{3}{10}\\ 3\\ -\frac{1}{10}\end{bmatrix}$$

■ 5. Find the projection of $\vec{x}=(0,-13,4)$ onto the subspace V.

$$V=\text{Span}\left(\begin{bmatrix}\frac{3}{\sqrt{13}}\\ \frac{2}{\sqrt{13}}\\ 0\end{bmatrix},\begin{bmatrix}\frac{2}{\sqrt{17}}\\ -\frac{3}{\sqrt{17}}\\ \frac{2}{\sqrt{17}}\end{bmatrix}\right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2=\left(\frac{3}{\sqrt{13}}\right)^2+\left(\frac{2}{\sqrt{13}}\right)^2+0^2=\frac{9}{13}+\frac{4}{13}+0=\frac{13}{13}=1$$

$$||\vec{v}_2||^2=\left(\frac{2}{\sqrt{17}}\right)^2+\left(-\frac{3}{\sqrt{17}}\right)^2+\left(\frac{2}{\sqrt{17}}\right)^2=\frac{4}{17}+\frac{9}{17}+\frac{4}{17}=\frac{17}{17}=1$$

$$\vec{v}_1\cdot\vec{v}_2=\frac{3}{\sqrt{13}}\left(\frac{2}{\sqrt{17}}\right)+\frac{2}{\sqrt{13}}\left(-\frac{3}{\sqrt{17}}\right)+0\left(\frac{2}{\sqrt{17}}\right)$$

$$=\frac{6}{\sqrt{221}}-\frac{6}{\sqrt{221}}+0=0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(0,-13,4)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{17}}\\ \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{17}}\\ 0 & \frac{2}{\sqrt{17}}\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{13}} & \frac{2}{\sqrt{13}} & 0\\ \frac{2}{\sqrt{17}} & -\frac{3}{\sqrt{17}} & \frac{2}{\sqrt{17}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{\sqrt{13}}\left(\frac{3}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right) & \frac{3}{\sqrt{13}}\left(\frac{2}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(-\frac{3}{\sqrt{17}}\right) & \frac{3}{\sqrt{13}}(0)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right)\\ \frac{2}{\sqrt{13}}\left(\frac{3}{\sqrt{13}}\right)-\frac{3}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right) & \frac{2}{\sqrt{13}}\left(\frac{2}{\sqrt{13}}\right)-\frac{3}{\sqrt{17}}\left(-\frac{3}{\sqrt{17}}\right) & \frac{2}{\sqrt{13}}(0)-\frac{3}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right)\\ 0\left(\frac{3}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right) & 0\left(\frac{2}{\sqrt{13}}\right)+\frac{2}{\sqrt{17}}\left(-\frac{3}{\sqrt{17}}\right) & 0(0)+\frac{2}{\sqrt{17}}\left(\frac{2}{\sqrt{17}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{13}+\frac{4}{17} & \frac{6}{13}-\frac{6}{17} & 0+\frac{4}{17}\\ \frac{6}{13}-\frac{6}{17} & \frac{4}{13}+\frac{9}{17} & 0-\frac{6}{17}\\ 0+\frac{4}{17} & 0-\frac{6}{17} & 0+\frac{4}{17}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{205}{221} & \frac{24}{221} & \frac{4}{17}\\ \frac{24}{221} & \frac{185}{221} & -\frac{6}{17}\\ \frac{4}{17} & -\frac{6}{17} & \frac{4}{17}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(0,-13,4)$ gives

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{205}{221} & \frac{24}{221} & \frac{4}{17} \\ \frac{24}{221} & \frac{185}{221} & -\frac{6}{17} \\ \frac{4}{17} & -\frac{6}{17} & \frac{4}{17} \end{bmatrix} \begin{bmatrix} 0 \\ -13 \\ 4 \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} \frac{205}{221}(0) + \frac{24}{221}(-13) + \frac{4}{17}(4) \\ \frac{24}{221}(0) + \frac{185}{221}(-13) - \frac{6}{17}(4) \\ \frac{4}{17}(0) - \frac{6}{17}(-13) + \frac{4}{17}(4) \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} 0 - \frac{24}{17} + \frac{16}{17} \\ 0 - \frac{185}{17} - \frac{24}{17} \\ 0 + \frac{78}{17} + \frac{16}{17} \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -\frac{8}{17} \\ -\frac{209}{17} \\ \frac{94}{17} \end{bmatrix}$$

■ 6. Find the projection of $\vec{x} = (-3,10,-10)$ onto the subspace V.

$$V = \text{Span}\left(\begin{bmatrix} \frac{3}{\sqrt{19}} \\ -\frac{3}{\sqrt{19}} \\ \frac{1}{\sqrt{19}} \end{bmatrix}, \begin{bmatrix} 0 \\ \frac{1}{\sqrt{10}} \\ \frac{3}{\sqrt{10}} \end{bmatrix} \right)$$

Solution:

Confirm that the set is orthonormal.

$$||\vec{v}_1||^2=\left(\frac{3}{\sqrt{19}}\right)^2+\left(-\frac{3}{\sqrt{19}}\right)^2+\left(\frac{1}{\sqrt{19}}\right)^2=\frac{9}{19}+\frac{9}{19}+\frac{1}{19}=\frac{19}{19}=1$$

$$||\vec{v}_2||^2=0^2+\left(\frac{1}{\sqrt{10}}\right)^2+\left(\frac{3}{\sqrt{10}}\right)^2=0+\frac{1}{10}+\frac{9}{10}=\frac{10}{10}=1$$

$$\vec{v}_1\cdot\vec{v}_2=\frac{3}{\sqrt{19}}(0)-\frac{3}{\sqrt{19}}\left(\frac{1}{\sqrt{10}}\right)+\frac{1}{\sqrt{19}}\left(\frac{3}{\sqrt{10}}\right)=0-\frac{3}{\sqrt{190}}+\frac{3}{\sqrt{190}}=0$$

Because the vectors are orthogonal to one another, and because they both have a length of 1, the set is orthonormal.

So the projection of $\vec{x}=(-3,10,-10)$ onto V is

$$\text{Proj}_V\vec{x}=AA^T\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{\sqrt{19}} & 0\\ -\frac{3}{\sqrt{19}} & \frac{1}{\sqrt{10}}\\ \frac{1}{\sqrt{19}} & \frac{3}{\sqrt{10}}\end{bmatrix}\begin{bmatrix}\frac{3}{\sqrt{19}} & -\frac{3}{\sqrt{19}} & \frac{1}{\sqrt{19}}\\ 0 & \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{3}{\sqrt{19}}\left(\frac{3}{\sqrt{19}}\right)+0(0) & \frac{3}{\sqrt{19}}\left(-\frac{3}{\sqrt{19}}\right)+0\left(\frac{1}{\sqrt{10}}\right) & \frac{3}{\sqrt{19}}\left(\frac{1}{\sqrt{19}}\right)+0\left(\frac{3}{\sqrt{10}}\right)\\ -\frac{3}{\sqrt{19}}\left(\frac{3}{\sqrt{19}}\right)+\frac{1}{\sqrt{10}}(0) & -\frac{3}{\sqrt{19}}\left(-\frac{3}{\sqrt{19}}\right)+\frac{1}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) & -\frac{3}{\sqrt{19}}\left(\frac{1}{\sqrt{19}}\right)+\frac{1}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right)\\ \frac{1}{\sqrt{19}}\left(\frac{3}{\sqrt{19}}\right)+\frac{3}{\sqrt{10}}(0) & \frac{1}{\sqrt{19}}\left(-\frac{3}{\sqrt{19}}\right)+\frac{3}{\sqrt{10}}\left(\frac{1}{\sqrt{10}}\right) & \frac{1}{\sqrt{19}}\left(\frac{1}{\sqrt{19}}\right)+\frac{3}{\sqrt{10}}\left(\frac{3}{\sqrt{10}}\right)\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19}+0 & -\frac{9}{19}+0 & \frac{3}{19}+0\\ -\frac{9}{19}+0 & \frac{9}{19}+\frac{1}{10} & -\frac{3}{19}+\frac{3}{10}\\ \frac{3}{19}+0 & -\frac{3}{19}+\frac{3}{10} & \frac{1}{19}+\frac{9}{10}\end{bmatrix}\vec{x}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19} & -\frac{9}{19} & \frac{3}{19}\\ -\frac{9}{19} & \frac{109}{190} & \frac{27}{190}\\ \frac{3}{19} & \frac{27}{190} & \frac{181}{190}\end{bmatrix}\vec{x}$$

Applying the projection to $\vec{x}=(-3,10,-10)$ gives

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19} & -\frac{9}{19} & \frac{3}{19}\\ -\frac{9}{19} & \frac{109}{190} & \frac{27}{190}\\ \frac{3}{19} & \frac{27}{190} & \frac{181}{190}\end{bmatrix}\begin{bmatrix}-3\\10\\-10\end{bmatrix}$$

$$\text{Proj}_V\vec{x}=\begin{bmatrix}\frac{9}{19}(-3)-\frac{9}{19}(10)+\frac{3}{19}(-10)\\ -\frac{9}{19}(-3)+\frac{109}{190}(10)+\frac{27}{190}(-10)\\ \frac{3}{19}(-3)+\frac{27}{190}(10)+\frac{181}{190}(-10)\end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -\frac{27}{19} - \frac{90}{19} - \frac{30}{19} \\ \frac{27}{19} + \frac{109}{19} - \frac{27}{19} \\ -\frac{9}{19} + \frac{27}{19} - \frac{181}{19} \end{bmatrix}$$

$$\text{Proj}_V\vec{x} = \begin{bmatrix} -\frac{147}{19} \\ \frac{109}{19} \\ -\frac{163}{19} \end{bmatrix}$$

GRAM-SCHMIDT PROCESS FOR CHANGE OF BASIS

■ 1. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}0\\-4\\3\end{bmatrix}, \begin{bmatrix}-2\\3\\-1\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (0, -4, 3)$ and $\vec{v}_2 = (-2, 3, -1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{0^2 + (-4)^2 + 3^2} = \sqrt{0 + 16 + 9} = \sqrt{25} = 5$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{5}\begin{bmatrix}0\\-4\\3\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$ and $\vec{v}_2$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2)$$

Now all we need to do is replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. If we can do that, then the vector set that spans V will be orthonormal. We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \cdot \frac{1}{5} \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right) \frac{1}{5} \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{25} \left(\begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix} \right) \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{25}((-2)(0) + (3)(-4) + (-1)(3)) \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} - \frac{1}{25}(-15) \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 3 \\ -1 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 + 0 \\ 3 - \frac{12}{5} \\ -1 + \frac{9}{5} \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it.

$$||\vec{w}_2|| = \sqrt{(-2)^2 + \left(\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{4 + \frac{9}{25} + \frac{16}{25}}$$

$$||\vec{w}_2|| = \sqrt{5}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis for V.

$$V_2 = \text{Span}\left(\frac{1}{5}\begin{bmatrix} 0 \\ -4 \\ 3 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}\right)$$

$$V_2 = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{4}{5} \\ \frac{3}{5} \end{bmatrix}, \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{3}{5\sqrt{5}} \\ \frac{4}{5\sqrt{5}} \end{bmatrix}\right)$$

■ 2. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (1, -1,1)$ and $\vec{v}_2 = (-3,5,2)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{1^2 + (-1)^2 + 1^2} = \sqrt{1+1+1} = \sqrt{3}$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$ and $\vec{v}_2$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2)$$

Now all we need to do is replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. If we can do that, then the vector set that spans V will be orthonormal. We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} - \left(\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \cdot \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} - \frac{1}{3} \left(\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} - \frac{1}{3}((-3)(1) + (5)(-1) + (2)(1)) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} - \frac{1}{3}(-6) \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3+2 \\ 5-2 \\ 2+2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it.

$$||\vec{w}_2|| = \sqrt{(-1)^2 + 3^2 + 4^2}$$

$$||\vec{w}_2|| = \sqrt{1 + 9 + 16}$$

$$||\vec{w}_2|| = \sqrt{26}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$ and $\vec{u}_2$ form an orthonormal basis for V.

$$V_2 = \text{Span}\left(\frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \frac{1}{\sqrt{26}} \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix}\right)$$

$$V_2 = \text{Span}\left(\begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}, \begin{bmatrix} -\frac{1}{\sqrt{26}} \\ \frac{3}{\sqrt{26}} \\ \frac{4}{\sqrt{26}} \end{bmatrix}\right)$$

3. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\1\\-2\end{bmatrix}, \begin{bmatrix}-3\\-1\\4\end{bmatrix}, \begin{bmatrix}2\\-1\\5\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-2,1,-2)$, $\vec{v}_2 = (-3,-1,4)$, and $\vec{v}_3 = (2,-1,5)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-2)^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix}-2\\1\\-2\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \left(\begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} \cdot \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \right) \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9} \left(\begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} \right) \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9}((-3)(-2) + (-1)(1) + (4)(-2)) \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} - \frac{1}{9}(-3) \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -3 \\ -1 \\ 4 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{\left(-\frac{11}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{10}{3}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{121}{9} + \frac{4}{9} + \frac{100}{9}}$$

$$||\vec{w}_2|| = \sqrt{\frac{225}{9}}$$

$$||\vec{w}_2|| = \sqrt{25}$$

$$||\vec{w}_2|| = 5$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{5}\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \cdot \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}\right)\frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}\right)\frac{1}{5}\begin{bmatrix} -\frac{11}{3} \\ -\frac{2}{3} \\ \frac{10}{3} \end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}2\\-1\\5\end{bmatrix}-\frac{1}{9}\left(\begin{bmatrix}2\\-1\\5\end{bmatrix}\cdot\begin{bmatrix}-2\\1\\-2\end{bmatrix}\right)\begin{bmatrix}-2\\1\\-2\end{bmatrix}-\frac{1}{25}\left(\begin{bmatrix}2\\-1\\5\end{bmatrix}\cdot\begin{bmatrix}-\frac{11}{3}\\-\frac{2}{3}\\\frac{10}{3}\end{bmatrix}\right)\begin{bmatrix}-\frac{11}{3}\\-\frac{2}{3}\\\frac{10}{3}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}2\\-1\\5\end{bmatrix}-\frac{1}{9}(2(-2)-1(1)+5(-2))\begin{bmatrix}-2\\1\\-2\end{bmatrix}$$

$$-\frac{1}{25}\left(2\left(-\frac{11}{3}\right)-1\left(-\frac{2}{3}\right)+5\left(\frac{10}{3}\right)\right)\begin{bmatrix}-\frac{11}{3}\\-\frac{2}{3}\\\frac{10}{3}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}2\\-1\\5\end{bmatrix}+\frac{5}{3}\begin{bmatrix}-2\\1\\-2\end{bmatrix}-\frac{2}{5}\begin{bmatrix}-\frac{11}{3}\\-\frac{2}{3}\\\frac{10}{3}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}2-\frac{10}{3}+\frac{22}{15}\\-1+\frac{5}{3}+\frac{4}{15}\\5-\frac{10}{3}-\frac{20}{15}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}\frac{2}{15}\\\frac{14}{15}\\\frac{1}{3}\end{bmatrix}$$

So $\vec{w}_3$ is orthogonal to $\vec{u}_2$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{2}{15}\right)^2 + \left(\frac{14}{15}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{4}{225} + \frac{196}{225} + \frac{1}{9}}$$

$$||\vec{w}_3|| = \sqrt{\frac{225}{225}}$$

$$||\vec{w}_3|| = 1$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{1}\begin{bmatrix} \frac{2}{15} \\ \frac{14}{15} \\ \frac{1}{3} \end{bmatrix}$$

$$\vec{u}_3 = \begin{bmatrix} \frac{2}{15} \\ \frac{14}{15} \\ \frac{1}{3} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{3}\begin{bmatrix}-2\\1\\-2\end{bmatrix}, \frac{1}{5}\begin{bmatrix}-\frac{11}{3}\\-\frac{2}{3}\\\frac{10}{3}\end{bmatrix}, \begin{bmatrix}\frac{2}{15}\\\frac{14}{15}\\\frac{1}{3}\end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix}-\frac{2}{3}\\\frac{1}{3}\\-\frac{2}{3}\end{bmatrix}, \begin{bmatrix}-\frac{11}{15}\\-\frac{2}{15}\\\frac{2}{3}\end{bmatrix}, \begin{bmatrix}\frac{2}{15}\\\frac{14}{15}\\\frac{1}{3}\end{bmatrix}\right)$$

■ 4. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-3\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\2\end{bmatrix}, \begin{bmatrix}-5\\5\\0\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-3,0,0)$, $\vec{v}_2 = (-2,1,2)$, and $\vec{v}_3 = (-5,5,0)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-3)^2 + 0^2 + 0^2} = \sqrt{9} = 3$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{3}\begin{bmatrix}-3\\0\\0\end{bmatrix}$$

$$\vec{u}_1 = \begin{bmatrix}-1\\0\\0\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix} - \left(\begin{bmatrix}-2\\1\\2\end{bmatrix} \cdot \begin{bmatrix}-1\\0\\0\end{bmatrix}\right)\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix} - ((-2)(-1) + (1)(0) + (2)(0))\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix}-2\\1\\2\end{bmatrix} - 2\begin{bmatrix}-1\\0\\0\end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{0^2 + 1^2 + 2^2}$$

$$||\vec{w}_2|| = \sqrt{0 + 1 + 4}$$

$$||\vec{w}_2|| = \sqrt{5}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u_1} - (\vec{v}_3 \cdot \vec{u}_2)\vec{u_2}$$

Plug in the values we already have

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\end{bmatrix}\right)\begin{bmatrix}-1\\0\\0\end{bmatrix}-\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}\right)\frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\begin{bmatrix}-1\\0\\0\end{bmatrix}\right)\begin{bmatrix}-1\\0\\0\end{bmatrix}-\frac{1}{5}\left(\begin{bmatrix}-5\\5\\0\end{bmatrix}\cdot\begin{bmatrix}0\\1\\2\end{bmatrix}\right)\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-(-5(-1)+5(0)+0(0))\begin{bmatrix}-1\\0\\0\end{bmatrix}-\frac{1}{5}(-5(0)+5(1)+0(2))\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5\\5\\0\end{bmatrix}-5\begin{bmatrix}-1\\0\\0\end{bmatrix}-\begin{bmatrix}0\\1\\2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-5+5-0\\5-0-1\\0-0-2\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}0\\4\\-2\end{bmatrix}$$

So $\vec{w}_3$ is orthogonal to $\vec{u}_2$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_3$ is

$$||\vec{w}_3||=\sqrt{0^2+4^2+(-2)^2}$$

$$||\vec{w}_3||=\sqrt{0+16+4}$$

$$||\vec{w}_3||=\sqrt{20}$$

$$||\vec{w}_3|| = 2\sqrt{5}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{2\sqrt{5}}\begin{bmatrix}0\\4\\-2\end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix}0\\1\\2\end{bmatrix}, \frac{1}{2\sqrt{5}}\begin{bmatrix}0\\4\\-2\end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix}-1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{bmatrix}, \begin{bmatrix}0\\\frac{2}{\sqrt{5}}\\-\frac{1}{\sqrt{5}}\end{bmatrix}\right)$$

■ 5. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}, \begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}, \begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-3,0,4,0)$, $\vec{v}_2 = (-1,2,-2,0)$, and $\vec{v}_3 = (5,-1,0,2)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-3)^2 + 0^2 + (-4)^2 + 0^2} = \sqrt{9 + 0 + 16 + 0} = \sqrt{25} = 5$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have

$$\vec{w}_2 = \begin{bmatrix}-1\\2\\-2\\0\end{bmatrix} - \left(\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix} \cdot \frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}\right)\frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}-\frac{1}{25}\left(\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}\cdot\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}\right)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}-\frac{1}{25}(-1(-3)+2(0)-2(4)+0(0))\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}-\frac{1}{25}(-5)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1\\2\\-2\\0\end{bmatrix}+\frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-1-\frac{3}{5}\\2+0\\-2+\frac{4}{5}\\0+0\end{bmatrix}$$

$$\vec{w}_2=\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{\left(-\frac{8}{5}\right)^2 + 2^2 + \left(-\frac{6}{5}\right)^2 + 0^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{64}{25} + 4 + \frac{36}{25} + 0}$$

$$||\vec{w}_2|| = \sqrt{8}$$

$$||\vec{w}_2|| = 2\sqrt{2}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{2\sqrt{2}}\begin{bmatrix} -\frac{8}{5} \\ 2 \\ -\frac{6}{5} \\ 0 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}\right)\frac{1}{5}\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}-\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\frac{1}{2\sqrt{2}}\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}\right)\frac{1}{2\sqrt{2}}\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\frac{1}{25}\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}\right)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}-\frac{1}{8}\left(\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}\cdot\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}\right)\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\frac{1}{25}(5(-3)-1(0)+0(4)+2(0))\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}$$

$$-\frac{1}{8}\left(5\left(-\frac{8}{5}\right)-1(2)+0\left(-\frac{6}{5}\right)+2(0)\right)\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}5\\-1\\0\\2\end{bmatrix}-\frac{1}{25}(-15)\begin{bmatrix}-3\\0\\4\\0\end{bmatrix}-\frac{1}{8}(-10)\begin{bmatrix}-\frac{8}{5}\\2\\-\frac{6}{5}\\0\end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 5 \\ -1 \\ 0 \\ 2 \end{bmatrix} + \frac{3}{5}\begin{bmatrix} -3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + \frac{5}{4}\begin{bmatrix} -\frac{8}{5} \\ 2 \\ -\frac{6}{5} \\ 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 5 - \frac{9}{5} - 2 \\ -1 + 0 + \frac{5}{2} \\ 0 + \frac{12}{5} - \frac{3}{2} \\ 2 + 0 + 0 \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} \frac{6}{5} \\ \frac{3}{2} \\ \frac{9}{10} \\ 2 \end{bmatrix}$$

The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(\frac{9}{10}\right)^2 + 2^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{36}{25} + \frac{9}{4} + \frac{81}{100} + 4}$$

$$||\vec{w}_3|| = \sqrt{\frac{17}{2}}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3=\frac{1}{\sqrt{\frac{17}{2}}}\begin{bmatrix}\frac{6}{5}\\ \frac{3}{2}\\ \frac{9}{10}\\ 2\end{bmatrix}$$

$$\vec{u}_3=\sqrt{\frac{2}{17}}\begin{bmatrix}\frac{6}{5}\\ \frac{3}{2}\\ \frac{9}{10}\\ 2\end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3=\text{Span}\left(\frac{1}{5}\begin{bmatrix}-3\\ 0\\ 4\\ 0\end{bmatrix},\frac{1}{2\sqrt{2}}\begin{bmatrix}-\frac{8}{5}\\ 2\\ -\frac{6}{5}\\ 0\end{bmatrix},\sqrt{\frac{2}{17}}\begin{bmatrix}\frac{6}{5}\\ \frac{3}{2}\\ \frac{9}{10}\\ 2\end{bmatrix}\right)$$

$$V_3=\text{Span}\left(\begin{bmatrix}-\frac{3}{5}\\ 0\\ \frac{4}{5}\\ 0\end{bmatrix},\begin{bmatrix}-\frac{4}{5\sqrt{2}}\\ \frac{1}{\sqrt{2}}\\ -\frac{3}{5\sqrt{2}}\\ 0\end{bmatrix},\begin{bmatrix}\frac{6}{5}\sqrt{\frac{2}{17}}\\ \frac{3}{2}\sqrt{\frac{2}{17}}\\ \frac{9}{10}\sqrt{\frac{2}{17}}\\ 2\sqrt{\frac{2}{17}}\end{bmatrix}\right)$$

6. Use a Gram-Schmidt process to change the basis of V into an orthonormal basis.

$$V = \text{Span}\left(\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\-1\end{bmatrix}, \begin{bmatrix}4\\0\\-1\\-1\end{bmatrix}\right)$$

Solution:

Define $\vec{v}_1 = (-2,-2,2,-2)$, $\vec{v}_2 = (-2,1,0,-1)$, and $\vec{v}_3 = (4,0,-1,-1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-2)^2 + (-2)^2 + 2^2 + (-2)^2} = \sqrt{4+4+4+4} = \sqrt{16} = 4$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{4}\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{16} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{16}(-2(-2) + 1(-2) + 0(2) - 1(-2)) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \end{bmatrix} - \begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{\left(-\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{\frac{9}{4} + \frac{9}{4} + \frac{1}{4} + \frac{1}{4}}$$

$$||\vec{w}_2|| = \sqrt{5}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 - (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2$$

Plug in the values we already have.

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \frac{1}{4} \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} - \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right) \frac{1}{\sqrt{5}} \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{16} \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} \right) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix} - \frac{1}{5} \left(\begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} \right) \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 4 \\ 0 \\ -1 \\ -1 \end{bmatrix} - \frac{1}{16}(4(-2) + 0(-2) - 1(2) - 1(-2)) \begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}$$

$$-\frac{1}{5}\left(4\left(-\frac{3}{2}\right) + 0\left(\frac{3}{2}\right) - 1\left(-\frac{1}{2}\right) - 1\left(-\frac{1}{2}\right)\right) \begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}4\\0\\-1\\-1\end{bmatrix}-\frac{1}{16}(-8)\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}-\frac{1}{5}(-5)\begin{bmatrix}-\frac{3}{2}\\\frac{3}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}4\\0\\-1\\-1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-2\\-2\\2\\-2\end{bmatrix}+\begin{bmatrix}-\frac{3}{2}\\\frac{3}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}4\\0\\-1\\-1\end{bmatrix}+\begin{bmatrix}-1\\-1\\1\\-1\end{bmatrix}+\begin{bmatrix}-\frac{3}{2}\\\frac{3}{2}\\-\frac{1}{2}\\-\frac{1}{2}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}\frac{3}{2}\\\frac{1}{2}\\-\frac{1}{2}\\-\frac{5}{2}\end{bmatrix}$$

The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right)^2 + \left(-\frac{5}{2}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{9}{4} + \frac{1}{4} + \frac{1}{4} + \frac{25}{4}}$$

$$||\vec{w}_3|| = \sqrt{9}$$

$$||\vec{w}_3|| = 3$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{1}{3}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{4}\begin{bmatrix} -2 \\ -2 \\ 2 \\ -2 \end{bmatrix}, \frac{1}{\sqrt{5}}\begin{bmatrix} -\frac{3}{2} \\ \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \frac{1}{3}\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}\right)$$

$$V_3 = \text{Span}\left(\begin{bmatrix} -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} -\frac{3}{2\sqrt{5}} \\ \frac{3}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} \\ -\frac{1}{2\sqrt{5}} \end{bmatrix}, \frac{1}{3}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{6} \\ -\frac{1}{6} \\ -\frac{5}{6} \end{bmatrix}\right)$$

EIGENVALUES, EIGENVECTORS, EIGENSPACES

1. Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 2 & -2 \\ 0 & \lambda + 5 \end{bmatrix} \right|$$

The determinant is

$$(\lambda + 2)(\lambda + 5) - (-2)(0)$$

$$(\lambda + 2)(\lambda + 5)$$

$\lambda = -2$ or $\lambda = -5$

2. For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -2$ and $\lambda = -5$.

$$A = \begin{bmatrix} -2 & 2 \\ 0 & -5 \end{bmatrix}$$

$$\lambda I_n - A = \begin{bmatrix} \lambda + 2 & -2 \\ 0 & \lambda + 5 \end{bmatrix}$$

Solution:

With $\lambda = -2$ and $\lambda = -5$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 2 & -2 \\ 0 & \lambda + 5 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 + 2 & -2 \\ 0 & -2 + 5 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}\right)$$

and

$$E_{-5} = N\left(\begin{bmatrix} -5 + 2 & -2 \\ 0 & -5 + 5 \end{bmatrix}\right)$$

$$E_{-5} = N\left(\begin{bmatrix}-3 & -2 \\ 0 & 0\end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix}0 & -2 \\ 0 & 3\end{bmatrix}\vec{v} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$\begin{bmatrix}0 & -2 & | & 0 \\ 0 & 3 & | & 0\end{bmatrix} \to \begin{bmatrix}0 & 1 & | & 0 \\ 0 & 3 & | & 0\end{bmatrix} \to \begin{bmatrix}0 & 1 & | & 0 \\ 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}0 & 1 \\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1 \\ v_2\end{bmatrix} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$v_2 = 0$$

So, substituting $v_1 = t$, the eigenvector for E_{-2} will be

$$\begin{bmatrix}v_1 \\ v_2\end{bmatrix} = t\begin{bmatrix}1 \\ 0\end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix}1 \\ 0\end{bmatrix}\right)$$

And the eigenvectors in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix}-3 & -2 \\ 0 & 0\end{bmatrix}\vec{v} = \begin{bmatrix}0 \\ 0\end{bmatrix}$$

$$\begin{bmatrix}-3 & -2 & | & 0 \\ 0 & 0 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & \frac{2}{3} & | & 0 \\ 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{2}{3} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + \frac{2}{3}v_2 = 0$$

$$v_1 = -\frac{2}{3}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_{-5} will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

Which means that E_{-5} is defined by

$$E_{-5} = \text{Span}\left(\begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix} \right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -\frac{2}{3} \\ 1 \end{bmatrix}$$

■ 3. Find the eigenvalues of the transformation matrix A.

$$A = \begin{bmatrix} 3 & -1 \\ -5 & -1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}3 & -1\\-5 & -1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}-\begin{bmatrix}3 & -1\\-5 & -1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-3 & 1\\5 & \lambda+1\end{bmatrix}\right|$$

The determinant is

$$(\lambda-3)(\lambda+1)-(1)(5)$$

$$\lambda^2-3\lambda+\lambda-3-5$$

$$\lambda^2-2\lambda-8$$

$$(\lambda+2)(\lambda-4)$$

$\lambda = -2$ or $\lambda = 4$

■ 4. For the transformation matrix A, find the eigenvectors associated with each eigenvalue, $\lambda = -2$ and $\lambda = 4$.

$$A=\begin{bmatrix}3 & -1\\-5 & -1\end{bmatrix}$$

$$\lambda I_n - A = \begin{bmatrix} \lambda - 3 & 1 \\ 5 & \lambda + 1 \end{bmatrix}$$

Solution:

With $\lambda = -2$ and $\lambda = 4$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 3 & 1 \\ 5 & \lambda + 1 \end{bmatrix}\right)$$

we get

$$E_{-2} = N\left(\begin{bmatrix} -2 - 3 & 1 \\ 5 & -2 + 1 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix}\right)$$

and

$$E_4 = N\left(\begin{bmatrix} 4 - 3 & 1 \\ 5 & 4 + 1 \end{bmatrix}\right)$$

$$E_4 = N\left(\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} -5 & 1 \\ 5 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 1 & | & 0 \\ 5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & | & 0 \\ 5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{5} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{5} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - \frac{1}{5}v_2 = 0$$

$$v_1 = \frac{1}{5}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_{-2} will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & | & 0 \\ 5 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + v_2 = 0$$

$$v_1 = -v_2$$

So, substituting $v_2 = t$, the eigenvector for E_4 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} \right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} \frac{1}{5} \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

■ 5. Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 5 & 0 \\ -4 & 3 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}-\begin{bmatrix}5 & 0\\-4 & 3\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}-\begin{bmatrix}5 & 0\\-4 & 3\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-5 & 0\\4 & \lambda-3\end{bmatrix}\right|$$

The determinant is

$$(\lambda-5)(\lambda-3)-(0)(4)$$

$$(\lambda-5)(\lambda-3)$$

$\lambda = 5$ or $\lambda = 3$

With $\lambda = 5$ and $\lambda = 3$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix}\lambda-5 & 0\\4 & \lambda-3\end{bmatrix}\right)$$

we get

$$E_5 = N\left(\begin{bmatrix}5-5 & 0\\4 & 5-3\end{bmatrix}\right)$$

$$E_5 = N\left(\begin{bmatrix}0 & 0\\4 & 2\end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3-5 & 0 \\ 4 & 3-3 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} -2 & 0 \\ 4 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_5 will satisfy

$$\begin{bmatrix} 0 & 0 \\ 4 & 2 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & | & 0 \\ 4 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & 2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + \frac{1}{2}v_2 = 0$$

$$v_1 = -\frac{1}{2}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_5 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$$

Which means that E_5 is defined by

$$E_5 = \text{Span}\left(\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}\right)$$

And the eigenvectors in the eigenspace E_3 will satisfy

$$\begin{bmatrix}-2 & 0\\ 4 & 0\end{bmatrix}\vec{v} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\begin{bmatrix}-2 & 0 & | & 0\\ 4 & 0 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & 0\\ 4 & 0 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & 0 & | & 0\\ 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\ v_2\end{bmatrix} = \begin{bmatrix}0\\ 0\end{bmatrix}$$

$$v_1 = 0$$

So, substituting $v_2 = t$, the eigenvector for E_5 will be

$$\begin{bmatrix}v_1\\ v_2\end{bmatrix} = t\begin{bmatrix}0\\ 1\end{bmatrix}$$

Which means that E_5 is defined by

$$E_5 = \text{Span}\left(\begin{bmatrix}0\\ 1\end{bmatrix}\right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix}-\frac{1}{2}\\ 1\end{bmatrix} \text{ and } \begin{bmatrix}0\\ 1\end{bmatrix}$$

■ 6. Find the eigenvectors of the transformation matrix.

$$A = \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 6 & -2 \\ 2 & 1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 6 & 2 \\ -2 & \lambda - 1 \end{bmatrix} \right|$$

The determinant is

$$(\lambda - 6)(\lambda - 1) - (2)(-2)$$

$$\lambda^2 - 6\lambda - \lambda + 6 + 4$$

$$\lambda^2 - 7\lambda + 10$$

$$(\lambda - 2)(\lambda - 5)$$

$\lambda = 2$ or $\lambda = 5$

With $\lambda = 2$ and $\lambda = 5$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 6 & 2 \\ -2 & \lambda - 1 \end{bmatrix}\right)$$

we get

$$E_2 = N\left(\begin{bmatrix} 2 - 6 & 2 \\ -2 & 2 - 1 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix}\right)$$

and

$$E_5 = N\left(\begin{bmatrix} 5 - 6 & 2 \\ -2 & 5 - 1 \end{bmatrix}\right)$$

$$E_5 = N\left(\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_2 will satisfy

$$\begin{bmatrix} -4 & 2 \\ -2 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 & | & 0 \\ -2 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & | & 0 \\ -2 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - \frac{1}{2}v_2 = 0$$

$$v_1 = \frac{1}{2}v_2$$

So, substituting $v_2 = t$, the eigenvector for E_2 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix}$$

Which means that E_2 is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} \frac{1}{2} \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_5 will satisfy

$$\begin{bmatrix} -1 & 2 \\ -2 & 4 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & | & 0 \\ -2 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ -2 & 4 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 - 2v_2 = 0$$

$$v_1 = 2v_2$$

So, substituting $v_2 = t$, the eigenvector for E_5 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Which means that E_5 is defined by

$$E_5 = \text{Span}\left(\begin{bmatrix}2\\1\end{bmatrix}\right)$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix}\frac{1}{2}\\1\end{bmatrix} \text{ and } \begin{bmatrix}2\\1\end{bmatrix}$$

EIGEN IN THREE DIMENSIONS

1. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -2 & 4 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 2 & -4 & -3 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 5 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 2) \begin{vmatrix} \lambda - 1 & 0 \\ 0 & \lambda + 5 \end{vmatrix} - 0 \begin{vmatrix} -4 & -3 \\ 0 & \lambda + 5 \end{vmatrix} + 0 \begin{vmatrix} -4 & -3 \\ \lambda - 1 & 0 \end{vmatrix}$$

$$(\lambda + 2)[(\lambda - 1)(\lambda + 5) - (0)(0)] - 0 + 0$$

$$(\lambda + 2)(\lambda - 1)(\lambda + 5)$$

$\lambda = -5$, $\lambda = -2$, or $\lambda = 1$

With $\lambda = -5$, $\lambda = -2$, and $\lambda = 1$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 2 & -4 & -3 \\ 0 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 5 \end{bmatrix}\right)$$

we get

$$E_{-5} = N\left(\begin{bmatrix} -5 + 2 & -4 & -3 \\ 0 & -5 - 1 & 0 \\ 0 & 0 & -5 + 5 \end{bmatrix}\right)$$

$$E_{-5} = N\left(\begin{bmatrix} -3 & -4 & -3 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

and

$$E_{-2} = N\left(\begin{bmatrix} -2 + 2 & -4 & -3 \\ 0 & -2 - 1 & 0 \\ 0 & 0 & -2 + 5 \end{bmatrix}\right)$$

$$E_{-2} = N\left(\begin{bmatrix} 0 & -4 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1+2 & -4 & -3 \\ 0 & 1-1 & 0 \\ 0 & 0 & 1+5 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 3 & -4 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix} -3 & -4 & -3 \\ 0 & -6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -4 & -3 & | & 0 \\ 0 & -6 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & 1 & | & 0 \\ 0 & -6 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{4}{3} & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_3 = 0$$

$$v_2 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_3$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-5} is defined by

$$E_{-5} = \text{Span}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_{-2} will satisfy

$$\begin{bmatrix} 0 & -4 & -3 \\ 0 & -3 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -4 & -3 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & \frac{3}{4} & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & \frac{3}{4} & | & 0 \\ 0 & 0 & \frac{9}{4} & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & \frac{3}{4} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_{-2} is defined by

$$E_{-2} = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 3 & -4 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 3 & -4 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & -4 & -3 & | & 0 \\ 0 & 0 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{4}{3} & -1 & | & 0 \\ 0 & 0 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & -1 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{4}{3} & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{4}{3} & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 - \frac{4}{3}v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = \frac{4}{3}v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{4}{3} \\ 1 \\ 0 \end{bmatrix}$$

■ 2. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & -2 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 2 & 2 & -1 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda - 2) \begin{vmatrix} \lambda - 2 & 0 \\ 0 & \lambda + 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & -1 \\ 0 & \lambda + 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & -1 \\ \lambda - 2 & 0 \end{vmatrix}$$

$$(\lambda - 2)[(\lambda - 2)(\lambda + 1) - (0)(0)] - 0 + 0$$

$$(\lambda - 2)(\lambda - 2)(\lambda + 1)$$

$\lambda = -1$ or $\lambda = 2$

With $\lambda = -1$ or $\lambda = 2$, we'll have two eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N \left(\begin{bmatrix} \lambda - 2 & 2 & -1 \\ 0 & \lambda - 2 & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \right)$$

we get

$$E_{-1} = N \left(\begin{bmatrix} -1 - 2 & 2 & -1 \\ 0 & -1 - 2 & 0 \\ 0 & 0 & -1 + 1 \end{bmatrix} \right)$$

$$E_{-1} = N \left(\begin{bmatrix} -3 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right)$$

and

$$E_2 = N\left(\begin{bmatrix} 2-2 & 2 & -1 \\ 0 & 2-2 & 0 \\ 0 & 0 & 2+1 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} -3 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & 2 & -1 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{2}{3} & \frac{1}{3} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + \frac{1}{3}v_3 = 0$$

$$v_2 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -\frac{1}{3}v_3$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_2 will satisfy

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 3 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & -1 & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_2 = 0$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_2 is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -\frac{1}{3} \\ 0 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

■ 3. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -3 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -3 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 5 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 3 & 0 & 0 \\ 4 & \lambda - 1 & 0 \\ 0 & -5 & \lambda - 2 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 3) \begin{vmatrix} \lambda - 1 & 0 \\ -5 & \lambda - 2 \end{vmatrix} - 0 \begin{vmatrix} 4 & 0 \\ 0 & \lambda - 2 \end{vmatrix} + 0 \begin{vmatrix} 4 & \lambda - 1 \\ 0 & -5 \end{vmatrix}$$

$$(\lambda + 3)[(\lambda - 1)(\lambda - 2) - (0)(-5)] - 0 + 0$$

$$(\lambda + 3)(\lambda - 1)(\lambda - 2)$$

$\lambda = -3$ or $\lambda = 1$ or $\lambda = 2$

With $\lambda = -3$, $\lambda = 1$ and $\lambda = 2$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda + 3 & 0 & 0 \\ 4 & \lambda - 1 & 0 \\ 0 & -5 & \lambda - 2 \end{bmatrix}\right)$$

we get

$$E_{-3} = N\left(\begin{bmatrix} -3 + 3 & 0 & 0 \\ 4 & -3 - 1 & 0 \\ 0 & -5 & -3 - 2 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} 0 & 0 & 0 \\ 4 & -4 & 0 \\ 0 & -5 & -5 \end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1 + 3 & 0 & 0 \\ 4 & 1 - 1 & 0 \\ 0 & -5 & 1 - 2 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 4 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & -5 & -1 \end{bmatrix}\right)$$

and

$$E_2 = N\left(\begin{bmatrix} 2+3 & 0 & 0 \\ 4 & 2-1 & 0 \\ 0 & -5 & 2-2 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} 5 & 0 & 0 \\ 4 & 1 & 0 \\ 0 & -5 & 0 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} 0 & 0 & 0 \\ 4 & -4 & 0 \\ 0 & -5 & -5 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 4 & -4 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 4 & -4 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & -5 & -5 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & | & 0 \\ 0 & 1 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_3 = 0$$

$$v_2 + v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_3$$

$$v_2 = -v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 4 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & -5 & -1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 0 & 0 & | & 0 \\ 4 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 4 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -5 & -1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & \frac{1}{5} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{5} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 + \frac{1}{5}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = -\frac{1}{5}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix}$$

Which means that E_1 is defined by

$$E_1 = \text{Span}\left(\begin{bmatrix} 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_2 will satisfy

$$\begin{bmatrix}5 & 0 & 0\\4 & 1 & 0\\0 & -5 & 0\end{bmatrix}\vec{v}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}5 & 0 & 0 & | & 0\\4 & 1 & 0 & | & 0\\0 & -5 & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & | & 0\\4 & 1 & 0 & | & 0\\0 & -5 & 0 & | & 0\end{bmatrix}\to\begin{bmatrix}1 & 0 & 0 & | & 0\\0 & 1 & 0 & | & 0\\0 & -5 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0 & | & 0\\0 & 1 & 0 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

We get the system of equations

$$v_1=0$$

$$v_2=0$$

Then the solution is

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix}=v_3\begin{bmatrix}0\\0\\1\end{bmatrix}$$

Which means that E_2 is defined by

$$E_2=\text{Span}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{5} \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

■ 4. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 4 & 0 & 0 \\ -2 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ -2 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ -2 & -3 & 0 \\ 3 & 1 & -5 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 4 & 0 & 0 \\ 2 & \lambda + 3 & 0 \\ -3 & -1 & \lambda + 5 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda-4)\begin{vmatrix}\lambda+3 & 0\\ -1 & \lambda+5\end{vmatrix}-0\begin{vmatrix}2 & 0\\ -3 & \lambda+5\end{vmatrix}+0\begin{vmatrix}2 & \lambda+3\\ -3 & -1\end{vmatrix}$$

$$(\lambda-4)[(\lambda+3)(\lambda+5)-(0)(-1)]-0+0$$

$$(\lambda-4)(\lambda+3)(\lambda+5)$$

$\lambda=-5$ or $\lambda=-3$ or $\lambda=4$

With $\lambda=-5$, $\lambda=-3$ and $\lambda=4$, we'll have three eigenspaces, given by $E_\lambda=N(\lambda I_n-A)$. With

$$E_\lambda=N\left(\begin{bmatrix}\lambda-4 & 0 & 0\\ 2 & \lambda+3 & 0\\ -3 & -1 & \lambda+5\end{bmatrix}\right)$$

we get

$$E_{-5}=N\left(\begin{bmatrix}-5-4 & 0 & 0\\ 2 & -5+3 & 0\\ -3 & -1 & -5+5\end{bmatrix}\right)$$

$$E_{-5}=N\left(\begin{bmatrix}-9 & 0 & 0\\ 2 & -2 & 0\\ -3 & -1 & 0\end{bmatrix}\right)$$

and

$$E_{-3} = N\left(\begin{bmatrix} -3-4 & 0 & 0 \\ 2 & -3+3 & 0 \\ -3 & -1 & -3+5 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} -7 & 0 & 0 \\ 2 & 0 & 0 \\ -3 & -1 & 2 \end{bmatrix}\right)$$

and

$$E_{4} = N\left(\begin{bmatrix} 4-4 & 0 & 0 \\ 2 & 4+3 & 0 \\ -3 & -1 & 4+5 \end{bmatrix}\right)$$

$$E_{4} = N\left(\begin{bmatrix} 0 & 0 & 0 \\ 2 & 7 & 0 \\ -3 & -1 & 9 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix} -9 & 0 & 0 \\ 2 & -2 & 0 \\ -3 & -1 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -9 & 0 & 0 & | & 0 \\ 2 & -2 & 0 & | & 0 \\ -3 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 2 & -2 & 0 & | & 0 \\ -3 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ -3 & -1 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -2 & 0 & | & 0 \\ 0 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-5} is defined by

$$E_{-5} = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} -7 & 0 & 0 \\ 2 & 0 & 0 \\ -3 & -1 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -7 & 0 & 0 & | & 0 \\ 2 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 2 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & | & 0 \\ -3 & -1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & -1 & 2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & -2 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 - 2v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = 0$$

$$v_2 = 2v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_4 will satisfy

$$\begin{bmatrix} 0 & 0 & 0 \\ 2 & 7 & 0 \\ -3 & -1 & 9 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 2 & 7 & 0 & | & 0 \\ -3 & -1 & 9 & | & 0 \end{bmatrix} \to \begin{bmatrix} -3 & -1 & 9 & | & 0 \\ 2 & 7 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & -3 & | & 0 \\ 2 & 7 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{3} & -3 & | & 0 \\ 0 & \frac{19}{3} & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{3} & -3 & | & 0 \\ 0 & 1 & \frac{18}{19} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{63}{19} & | & 0 \\ 0 & 1 & \frac{18}{19} & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{63}{19} \\ 0 & 1 & \frac{18}{19} \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 - \frac{63}{19}v_3 = 0$$

$$v_2 + \frac{18}{19}v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = \frac{63}{19}v_3$$

$$v_2 = -\frac{18}{19}v_3$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} \frac{63}{19} \\ -\frac{18}{19} \\ 1 \end{bmatrix}$$

Which means that E_4 is defined by

$$E_4 = \text{Span}\left(\begin{bmatrix} \frac{63}{19} \\ -\frac{18}{19} \\ 1 \end{bmatrix} \right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \text{ and } \begin{bmatrix} \frac{63}{19} \\ -\frac{18}{19} \\ 1 \end{bmatrix}$$

■ 5. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\begin{bmatrix}1&2&0\\2&1&0\\0&0&-3\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda&0&0\\0&\lambda&0\\0&0&\lambda\end{bmatrix}-\begin{bmatrix}1&2&0\\2&1&0\\0&0&-3\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-1&-2&0\\-2&\lambda-1&0\\0&0&\lambda+3\end{bmatrix}\right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda-1)\begin{vmatrix}\lambda-1&0\\0&\lambda+3\end{vmatrix}-(-2)\begin{vmatrix}-2&0\\0&\lambda+3\end{vmatrix}+0\begin{vmatrix}-2&\lambda-1\\0&0\end{vmatrix}$$

$$(\lambda-1)[(\lambda-1)(\lambda+3)-(0)(0)]+2[(-2)(\lambda+3)-(0)(0)]+0$$

$$(\lambda-1)(\lambda-1)(\lambda+3)-4(\lambda+3)$$

$$(\lambda+3)((\lambda-1)^2-4)$$

$$(\lambda+3)(\lambda^2-\lambda-\lambda+1-4)$$

$$(\lambda+3)(\lambda^2-2\lambda-3)$$

$$(\lambda+3)(\lambda+1)(\lambda-3)$$

$\lambda=-3$ or $\lambda=-1$ or $\lambda=3$

With $\lambda = -3$, $\lambda = -1$ and $\lambda = 3$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 1 & -2 & 0 \\ -2 & \lambda - 1 & 0 \\ 0 & 0 & \lambda + 3 \end{bmatrix}\right)$$

we get

$$E_{-3} = N\left(\begin{bmatrix} -3 - 1 & -2 & 0 \\ -2 & -3 - 1 & 0 \\ 0 & 0 & -3 + 3 \end{bmatrix}\right)$$

$$E_{-3} = N\left(\begin{bmatrix} -4 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

and

$$E_{-1} = N\left(\begin{bmatrix} -1 - 1 & -2 & 0 \\ -2 & -1 - 1 & 0 \\ 0 & 0 & -1 + 3 \end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3 - 1 & -2 & 0 \\ -2 & 3 - 1 & 0 \\ 0 & 0 & 3 + 3 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-3} will satisfy

$$\begin{bmatrix} -4 & -2 & 0 \\ -2 & -4 & 0 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & -2 & 0 & | & 0 \\ -2 & -4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 0 & | & 0 \\ -2 & -4 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 0 & | & 0 \\ 0 & -3 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & 0 \\ 0 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 = 0$$

$$v_2 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Which means that E_{-3} is defined by

$$E_{-3} = \text{Span}\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix} -2 & -2 & 0 \\ -2 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -2 & 0 & | & 0 \\ -2 & -2 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ -2 & -2 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right)$$

The eigenvector in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 2 & -2 & 0 \\ -2 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 0 & | & 0 \\ -2 & 2 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ -2 & 2 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

We get the system of equations

$$v_1 - v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \text{and } \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

■ 6. Find the eigenvectors of the transformation matrix A.

$$A = \begin{bmatrix} -4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Solution:

Find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} -4 & 3 & 0 \\ 3 & -4 & 0 \\ 0 & 0 & -1 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda + 4 & -3 & 0 \\ -3 & \lambda + 4 & 0 \\ 0 & 0 & \lambda + 1 \end{bmatrix} \right|$$

Find the determinant, and then the eigenvalues.

$$(\lambda + 4) \begin{vmatrix} \lambda + 4 & 0 \\ 0 & \lambda + 1 \end{vmatrix} - (-3) \begin{vmatrix} -3 & 0 \\ 0 & \lambda + 1 \end{vmatrix} - 0 \begin{vmatrix} -3 & \lambda + 4 \\ 0 & 0 \end{vmatrix}$$

$$(\lambda + 4)[(\lambda + 4)(\lambda + 1) - (0)(0)] + 3[(-3)(\lambda + 1) - (0)(0)] - 0$$

$$(\lambda + 4)(\lambda + 4)(\lambda + 1) - 9(\lambda + 1)$$

$$(\lambda + 1)((\lambda + 4)^2 - 9)$$

$$(\lambda + 1)(\lambda^2 + 8\lambda + 16 - 9)$$

$$(\lambda + 1)(\lambda^2 + 8\lambda + 7)$$

$$(\lambda+1)(\lambda+1)(\lambda+7)$$

$$\lambda=-1 \text{ or } \lambda=-7$$

With $\lambda=-1$ and $\lambda=-7$, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. With

$$E_\lambda = N\left(\begin{bmatrix}\lambda+4 & -3 & 0\\ -3 & \lambda+4 & 0\\ 0 & 0 & \lambda+1\end{bmatrix}\right)$$

we get

$$E_{-7} = N\left(\begin{bmatrix}-7+4 & -3 & 0\\ -3 & -7+4 & 0\\ 0 & 0 & -7+1\end{bmatrix}\right)$$

$$E_{-7} = N\left(\begin{bmatrix}-3 & -3 & 0\\ -3 & -3 & 0\\ 0 & 0 & -6\end{bmatrix}\right)$$

and

$$E_{-1} = N\left(\begin{bmatrix}-1+4 & -3 & 0\\ -3 & -1+4 & 0\\ 0 & 0 & -1+1\end{bmatrix}\right)$$

$$E_{-1} = N\left(\begin{bmatrix}3 & -3 & 0\\ -3 & 3 & 0\\ 0 & 0 & 0\end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-7} will satisfy

$$\begin{bmatrix} -3 & -3 & 0 \\ -3 & -3 & 0 \\ 0 & 0 & -6 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 & 0 & | & 0 \\ -3 & -3 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ -3 & -3 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

We get the system of equations

$$v_1 + v_2 = 0$$

$$v_3 = 0$$

and then we solve it for the pivot variables.

$$v_1 = -v_2$$

$$v_3 = 0$$

Then the solution is

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_{-7} is defined by

$$E_{-7} = \text{Span}\left(\begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$$

The eigenvector in the eigenspace E_{-1} will satisfy

$$\begin{bmatrix}3 & -3 & 0\\-3 & 3 & 0\\0 & 0 & 0\end{bmatrix}\vec{v} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}3 & -3 & 0 & | & 0\\-3 & 3 & 0 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -1 & 0 & | & 0\\-3 & 3 & 0 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix} \to \begin{bmatrix}1 & -1 & 0 & | & 0\\0 & 0 & 0 & | & 0\\0 & 0 & 0 & | & 0\end{bmatrix}$$

$$\begin{bmatrix}1 & -1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

We get the equation

$$v_1 - v_2 = 0$$

and then we solve it for the pivot variables.

$$v_1 = v_2$$

Then the solution is

$$\begin{bmatrix}v_1\\v_2\\v_3\end{bmatrix} = v_2\begin{bmatrix}1\\1\\0\end{bmatrix}$$

Which means that E_{-1} is defined by

$$E_{-1} = \text{Span}\left(\begin{bmatrix}1\\1\\0\end{bmatrix}\right)$$

So the eigenvectors for the transformation matrix are

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} \text{ and } \begin{bmatrix}1\\1\\0\end{bmatrix}$$

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