Linear Algebra Final Exam Solutions

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Linear Algebra Final Exam Answer Key

1. (5 pts)	A	B	■	D	E
2. (5 pts)	■	B	C	D	E
3. (5 pts)	A	B	C	D	■
4. (5 pts)	A	■	C	D	E
5. (5 pts)	A	B	C	D	■
6. (5 pts)	A	B	■	D	E
7. (5 pts)	A	B	C	D	■
8. (5 pts)	A	B	C	■	E

9. (15 pts) $\vec{x}=c_1\begin{bmatrix}1\\-2\\1\\0\end{bmatrix}+c_2\begin{bmatrix}1\\-1\\0\\1\end{bmatrix}+\begin{bmatrix}\frac{1}{5}\\-\frac{2}{5}\\0\\0\end{bmatrix}$

10. (15 pts) $T(S(\vec{x}))=\begin{bmatrix}8\\-8\\14\end{bmatrix}$

$S(T(\vec{x}))=\begin{bmatrix}-2\\7\\2\end{bmatrix}$

11. (15 pts) $\det(B)=-21$

12. (15 pts) $V_3=\text{Span}\left(\frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix},\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix},\frac{2}{\sqrt{14}}\begin{bmatrix}-\frac{1}{2}\\0\\-\frac{3}{2}\\-1\end{bmatrix}\right)$

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1. C. The matrix for the system is

$$\begin{bmatrix} 1 & -4 & 1 & 20 \\ -1 & 0 & 1 & 10 \\ 4 & 1 & -2 & -25 \end{bmatrix}$$

Start by working on the first column.

$$\begin{bmatrix} 1 & -4 & 1 & 20 \\ 0 & -4 & 2 & 30 \\ 4 & 1 & -2 & -25 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 1 & 20 \\ 0 & -4 & 2 & 30 \\ 0 & 17 & -6 & -105 \end{bmatrix}$$

Find the pivot entry in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -4 & 1 & 20 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 17 & -6 & -105 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -10 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 17 & -6 & -105 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & -10 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 0 & \frac{5}{2} & \frac{45}{2} \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & -1 & -10 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -\frac{1}{2} & -\frac{15}{2} \\ 0 & 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 9 \end{bmatrix}$$

The third column is done, and we can see that the solution to the linear system is $(x, y, z) = (-1, -3, 9)$.

2. A. To find $A - C$ by subtracting matrices, we subtract corresponding entries from each matrix.

$$A - C = \begin{bmatrix} 5 & -4 \\ -3 & 9 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 3 & 2 \\ -1 & 7 \\ -3 & 3 \end{bmatrix}$$

$$A - C = \begin{bmatrix} 5 - 3 & -4 - 2 \\ -3 - (-1) & 9 - 7 \\ 0 - (-3) & 4 - 3 \end{bmatrix}$$

$$A - C = \begin{bmatrix} 2 & -6 \\ -2 & 2 \\ 3 & 1 \end{bmatrix}$$

To find $(A - C)B$, multiply each row of $A - C$ by the first column of B.

$$(A - C)B = \begin{bmatrix} 2 & -6 \\ -2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & -1 & 7 \\ 3 & 7 & 8 \end{bmatrix}$$

$$(A - C)B = \begin{bmatrix} 2(4) - 6(3) & \dots & \dots \\ -2(4) + 2(3) & \dots & \dots \\ 3(4) + 1(3) & \dots & \dots \end{bmatrix}$$

$$(A-C)B=\begin{bmatrix} 8-18 & \dots & \dots \\ -8+6 & \dots & \dots \\ 12+3 & \dots & \dots \end{bmatrix}$$

$$(A-C)B=\begin{bmatrix} -10 & \dots & \dots \\ -2 & \dots & \dots \\ 15 & \dots & \dots \end{bmatrix}$$

Multiply each row of $A-C$ by the second column of B.

$$(A-C)B=\begin{bmatrix} -10 & 2(-1)-6(7) & \dots \\ -2 & -2(-1)+2(7) & \dots \\ 15 & 3(-1)+1(7) & \dots \end{bmatrix}$$

$$(A-C)B=\begin{bmatrix} -10 & -2-42 & \dots \\ -2 & 2+14 & \dots \\ 15 & -3+7 & \dots \end{bmatrix}$$

$$(A-C)B=\begin{bmatrix} -10 & -44 & \dots \\ -2 & 16 & \dots \\ 15 & 4 & \dots \end{bmatrix}$$

Multiply each row of $A-C$ by the third column of B.

$$(A-C)B=\begin{bmatrix} -10 & -44 & 2(7)-6(8) \\ -2 & 16 & -2(7)+2(8) \\ 15 & 4 & 3(7)+1(8) \end{bmatrix}$$

$$(A-C)B=\begin{bmatrix} -10 & -44 & 14-48 \\ -2 & 16 & -14+16 \\ 15 & 4 & 21+8 \end{bmatrix}$$

$$(A - C)B = \begin{bmatrix} -10 & -44 & -34 \\ -2 & 16 & 2 \\ 15 & 4 & 29 \end{bmatrix}$$

3. E. The vector sum is

$$\vec{u} = \vec{a} + 2\vec{b} - 3\vec{c} - \vec{d}$$

$$\vec{u} = (-3,5,-1) + 2(4,2,7) - 3(0,2,1) - (-1,3,5)$$

Apply the scalars to each vector.

$$\vec{u} = (-3,5,-1) + (8,4,14) + (0,-6,-3) + (1,-3,-5)$$

Add each of the vector components.

$$\vec{u} = (-3 + 8 + 0 + 1, 5 + 4 - 6 - 3, -1 + 14 - 3 - 5)$$

$$\vec{u} = (6,0,5)$$

Then, find the length of $\vec{u}$.

$$||\vec{u}|| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$$||\vec{u}|| = \sqrt{6^2 + 0^2 + 5^2}$$

$$||\vec{u}|| = \sqrt{36 + 0 + 25}$$

$$||\vec{u}|| = \sqrt{61}$$

Then the unit vector in the direction of $\vec{u} = (6,0,5)$ is

$$\vec{v} = \frac{1}{||\vec{u}||}\vec{u}$$

$$\vec{v} = \frac{1}{\sqrt{61}}\begin{bmatrix}6\\0\\5\end{bmatrix}$$

$$\vec{v} = \begin{bmatrix}\frac{6}{\sqrt{61}}\\0\\\frac{5}{\sqrt{61}}\end{bmatrix}$$

4. B. The angle θ between two vectors is given by a relationship between the dot product of the vectors and the lengths of the vectors.

$$\vec{u}\cdot\vec{v} = ||\vec{u}||\,||\vec{v}||\cos\theta$$

$$\vec{u}\cdot\vec{v} = (2,-3,0,5)\cdot(12,3,8,-3)$$

$$\vec{u}\cdot\vec{v} = (2)(12)+(-3)(3)+(0)(8)+(5)(-3)$$

$$\vec{u}\cdot\vec{v} = 24-9+0-15$$

$$\vec{u}\cdot\vec{v} = 0$$

Because the dot product is 0, $\vec{u}$ and $\vec{v}$ are orthogonal to one another and the angle between them is $\theta = 90^\circ$.

5. E. The cross product would be

$$\vec{a} \times \vec{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & -1 \\ 0 & -2 & 2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i} \begin{vmatrix} 3 & -1 \\ -2 & 2 \end{vmatrix} - \mathbf{j} \begin{vmatrix} -2 & -1 \\ 0 & 2 \end{vmatrix} + \mathbf{k} \begin{vmatrix} -2 & 3 \\ 0 & -2 \end{vmatrix}$$

$$\vec{a} \times \vec{b} = \mathbf{i}((3)(2) - (-1)(-2)) - \mathbf{j}((-2)(2) - (-1)(0)) + \mathbf{k}((-2)(-2) - (3)(0))$$

$$\vec{a} \times \vec{b} = \mathbf{i}(6 - 2) - \mathbf{j}(-4 + 0) + \mathbf{k}(4 - 0)$$

$$\vec{a} \times \vec{b} = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

$$\vec{a} \times \vec{b} = (4,4,4)$$

6. C. We can rewrite W as

$$W = \{x \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} + y \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \ | \ x, y, z \in \mathbb{R}^4\}$$

The subspace W is a space in $\mathbb{R}^4$, spanned by the three vectors $\vec{w}_1 = (0, -2,1,1)$, $\vec{w}_2 = (1,1,0,0)$ and $\vec{w}_3 = (1,0, -1,0)$. Therefore, its orthogonal complement $W^\perp$ is the set of vectors which are orthogonal to $\vec{w}_1 = (0, -2,1,1)$, $\vec{w}_2 = (1,1,0,0)$ and $\vec{w}_3 = (1,0, -1,0)$.

$$W^{\perp} = \{ \vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = 0 \text{ and } \vec{x} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} = 0 \}$$

If we let $\vec{x} = (x_1, x_2, x_3, x_4)$, we get three equations from these dot products.

$$-2x_2 + x_3 + x_4 = 0$$

$$x_1 + x_2 = 0$$

$$x_1 - x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} 0 & -2 & 1 & 1 & | & 0 \\ 1 & 1 & 0 & 0 & | & 0 \\ 1 & 0 & -1 & 0 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 \\ 0 & -2 & 1 & 1 & | & 0 \\ 1 & 0 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 \\ 0 & -2 & 1 & 1 & | & 0 \\ 0 & -1 & -1 & 0 & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & -1 & -1 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & -1 & -1 & 0 & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 0 & -\frac{3}{2} & -\frac{1}{2} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{1}{2} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & \frac{1}{3} & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & | & 0 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} & | & 0 \\ 0 & 0 & 1 & \frac{1}{3} & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{1}{3} & | & 0 \\ 0 & 1 & 0 & -\frac{1}{3} & | & 0 \\ 0 & 0 & 1 & \frac{1}{3} & | & 0 \end{bmatrix}$$

This rref form gives the system of equations

$$x_1 + \frac{1}{3}x_4 = 0$$

$$x_2 - \frac{1}{3}x_4 = 0$$

$$x_3 + \frac{1}{3}x_4 = 0$$

and we can solve the system for the pivot variables.

$$x_1 = -\frac{1}{3}x_4$$

$$x_2 = \frac{1}{3}x_4$$

$$x_3 = -\frac{1}{3}x_4$$

We can express this system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix}$$

Which means the orthogonal complement is

$$W^{\perp} = \text{Span}\left(\begin{bmatrix} -\frac{1}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \right)$$

7. E. The transpose of A is

$$A^T = \begin{bmatrix} -1 & 2 & 6 \\ -3 & 4 & 0 \end{bmatrix}$$

To find the null space, we'll augment the matrix, and then put it into reduced row-echelon form.

$$\begin{bmatrix} -1 & 2 & 6 & | & 0 \\ -3 & 4 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -6 & | & 0 \\ -3 & 4 & 0 & | & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & -6 & | & 0 \\ 0 & -2 & -18 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -6 & | & 0 \\ 0 & 1 & 9 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 12 & | & 0 \\ 0 & 1 & 9 & | & 0 \end{bmatrix}$$

Because we have pivot entries in the first two columns, we'll pull a system of equations from the matrix,

$$x_1 + 12x_3 = 0$$

$$x_2 + 9x_3 = 0$$

and then solve the system's equations for the pivot variables.

$$x_1 = -12x_3$$

$$x_2 = -9x_3$$

If we turn this into a vector equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -12 \\ -9 \\ 1 \end{bmatrix}$$

Therefore, the left null space is

$$N(A^T) = \text{Span}\left(\begin{bmatrix} -12 \\ -9 \\ 1 \end{bmatrix}\right)$$

The space of the null space of the transpose is always $\mathbb{R}^m$, where m is the number of rows in the original matrix, A. The original matrix has 3 rows, so the null space of the transpose $N(A^T)$ is a subspace of $\mathbb{R}^3$.

The column space of the transpose A^T, which is the same as the row space of A, is simply given by the columns in A^T that contain pivot entries when A^T is in reduced row-echelon form. So the column space of A^T is

$$C(A^T) = \text{Span}\left(\begin{bmatrix}-1\\-3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right)$$

The space of the column space of the transpose is always $\mathbb{R}^n$, where n is the number of columns in the original matrix, A. The original matrix has 2 columns, so the column space of the transpose $C(A^T)$ is a subspace of $\mathbb{R}^2$.

Because there's one vector that forms the basis of $N(A^T)$, the dimension of $N(A^T)$ is $\text{Dim}(N(A^T)) = 1$.

Because there are two vectors that form the basis of $C(A^T)$, the dimension of $C(A^T)$ is $\text{Dim}(C(A^T)) = 2$.

$$N(A^T) = \text{Span}\left(\begin{bmatrix}-12\\-9\\1\end{bmatrix}\right) \text{ in } \mathbb{R}^3 \qquad \text{Dim}(N(A^T)) = 1$$

$$C(A^T) = \text{Span}\left(\begin{bmatrix}-1\\-3\end{bmatrix}, \begin{bmatrix}2\\4\end{bmatrix}\right) \text{ in } \mathbb{R}^2 \qquad \text{Dim}(C(A^T)) = 2$$

8. D. Find the determinant $|\lambda I_n - A|$.

$$\left|\lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} - \begin{bmatrix}-5 & 0\\3 & 1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}-\begin{bmatrix}-5 & 0\\ 3 & 1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda-(-5) & 0-0\\ 0-3 & \lambda-1\end{bmatrix}\right|$$

$$\left|\begin{bmatrix}\lambda+5 & 0\\ -3 & \lambda-1\end{bmatrix}\right|$$

The determinant is

$$(\lambda+5)(\lambda-1)-(-3)(0)$$

$$(\lambda+5)(\lambda-1)$$

$\lambda=-5$ or $\lambda=1$

With $\lambda=-5$ and $\lambda=1$, we'll have two eigenspaces, given by $E_\lambda=N(\lambda I_n-A)$. With

$$E_\lambda=N\left(\begin{bmatrix}\lambda+5 & 0\\ -3 & \lambda-1\end{bmatrix}\right)$$

we get

$$E_{-5}=N\left(\begin{bmatrix}-5+5 & 0\\ -3 & -5-1\end{bmatrix}\right)$$

$$E_{-5}=N\left(\begin{bmatrix}0 & 0\\ -3 & -6\end{bmatrix}\right)$$

and

$$E_1 = N\left(\begin{bmatrix} 1+5 & 0 \\ -3 & 1-1 \end{bmatrix}\right)$$

$$E_1 = N\left(\begin{bmatrix} 6 & 0 \\ -3 & 0 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_{-5} will satisfy

$$\begin{bmatrix} 0 & 0 \\ -3 & -6 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & | & 0 \\ -3 & -6 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 0 & | & 0 \\ 1 & 2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 + 2v_2 = 0$$

$$v_1 = -2v_2$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = v_2\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

So the eigenvector for E_{-5} will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

And the eigenvectors in the eigenspace E_1 will satisfy

$$\begin{bmatrix} 6 & 0 \\ -3 & 0 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 0 & | & 0 \\ -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ -3 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$v_1 = 0$$

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

So the eigenvector for E_1 will be

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Then the eigenvectors of the matrix are

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

9. Put the matrix A into reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & -5 & -3 \\ 3 & -1 & -5 & -4 \\ 0 & -5 & -10 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 5 & 10 & 5 \\ 0 & -5 & -10 & -5 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & -5 & -10 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -5 & -3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

To find the complementary solution, augment rref(A) with the zero vector to get a system of equations.

$$x_1 - x_3 - x_4 = 0$$

$$x_2 + 2x_3 + x_4 = 0$$

Solve for the pivot variables in terms of the free variables.

$$x_1 = x_3 + x_4$$

$$x_2 = -2x_3 - x_4$$

The vectors that satisfy the null space are

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

We could therefore write the complementary solution as

$$\vec{x}_n = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

To find the particular solution, augment A with $\vec{b} = (b_1, b_2, b_3)$, then put it in reduced row-echelon form.

$$\begin{bmatrix} 1 & -2 & -5 & -3 & | & b_1 \\ 3 & -1 & -5 & -4 & | & b_2 \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -5 & -3 & | & b_1 \\ 0 & 5 & 10 & 5 & | & b_2 - 3b_1 \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & -2 & -5 & -3 & | & b_1 \\ 0 & 1 & 2 & 1 & | & \frac{1}{5}(b_2 - 3b_1) \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & | & b_1 + \frac{2}{5}(b_2 - 3b_1) \\ 0 & 1 & 2 & 1 & | & \frac{1}{5}(b_2 - 3b_1) \\ 0 & -5 & -10 & -5 & | & b_3 \end{bmatrix} \to$$

$$\begin{bmatrix} 1 & 0 & -1 & -1 & | & b_1 + \frac{2}{5}(b_2 - 3b_1) \\ 0 & 1 & 2 & 1 & | & \frac{1}{5}(b_2 - 3b_1) \\ 0 & 0 & 0 & 0 & | & b_3 + b_2 - 3b_1 \end{bmatrix}$$

From the third row, the system is constrained.

$$-3b_1 + b_2 + b_3 = 0$$

$$b_2 = 3b_1 - b_3$$

We were asked to use $b_1 = 1$, $b_2 = 1$, and $b_3 = 2$, which satisfies this constraint equation.

$$1 = 3(1) - 2$$

$$1 = 1$$

Then the augmented matrix becomes

$$\begin{bmatrix} 1 & 0 & -1 & -1 & | & 1+\frac{2}{5}(1-3(1)) \\ 0 & 1 & 2 & 1 & | & \frac{1}{5}(1-3(1)) \\ 0 & 0 & 0 & 0 & | & 2+1-3(1) \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 & -1 & | & \frac{1}{5} \\ 0 & 1 & 2 & 1 & | & -\frac{2}{5} \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives a system of equations

$$x_1 - x_3 - x_4 = \frac{1}{5}$$

$$x_2 + 2x_3 + x_4 = -\frac{2}{5}$$

Because x_3 and x_4 are free variables, set $x_3 = 0$ and $x_4 = 0$.

$$x_1 - 0 - 0 = \frac{1}{5}$$

$$x_2 + 2(0) + 0 = -\frac{2}{5}$$

The system becomes

$$x_1 = \frac{1}{5}$$

$$x_2 = -\frac{2}{5}$$

So the particular solution is

$$\vec{x}_p = \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \end{bmatrix}$$

The general solution is the sum of the complementary and particular solutions.

$$\vec{x} = \vec{x}_p + \vec{x}_n$$

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{1}{5} \\ -\frac{2}{5} \\ 0 \\ 0 \end{bmatrix}$$

10. Apply the transformation S to each column of the I_3 identity matrix.

$$S\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -0 - 3(1) \\ 0 - 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 - 3(0) \\ 1 - 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

$$S\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} -0 - 3(0) \\ 0 - 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

So the transformation S can be written as

$$S(\vec{x}) = \begin{bmatrix} -3 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$

Apply the transformation T to each column of the I_3 identity matrix.

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 - 2(0) + 0 \\ 0 - 1 \\ 2(1) + 0 - 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2(1) + 0 \\ 0 - 0 \\ 2(0) + 1 - 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 - 2(0) + 1 \\ 1 - 0 \\ 2(0) + 0 - 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

So the transformation T can be written as

$$T(\vec{x}) = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \vec{x}$$

Then the composition $T \circ S$ can be written as

$$T(S(\vec{x})) = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} -3 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-3+0+0 & -1-2+1 & 0+2+0\\ 3+0+0 & 1+0+1 & 0+0+0\\ -6+0+0 & -2+1-1 & 0-1+0\end{bmatrix}\vec{x}$$

$$T(S(\vec{x}))=\begin{bmatrix}-3 & -2 & 2\\ 3 & 2 & 0\\ -6 & -2 & -1\end{bmatrix}\vec{x}$$

Transform $\vec{x}=(-2,-1,0)$.

$$T\left(S\left(\begin{bmatrix}-2\\-1\\0\end{bmatrix}\right)\right)=\begin{bmatrix}-3 & -2 & 2\\ 3 & 2 & 0\\ -6 & -2 & -1\end{bmatrix}\begin{bmatrix}-2\\-1\\0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\-1\\0\end{bmatrix}\right)\right)=\begin{bmatrix}-3(-2)-2(-1)+2(0)\\ 3(-2)+2(-1)+0(0)\\ -6(-2)+(-2)(-1)+(-1)(0)\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\-1\\0\end{bmatrix}\right)\right)=\begin{bmatrix}6+2+0\\ -6-2+0\\ 12+2+0\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}-2\\-1\\0\end{bmatrix}\right)\right)=\begin{bmatrix}8\\-8\\14\end{bmatrix}$$

Then the composition $S\circ T$ can be written as

$$S(T(\vec{x}))=\begin{bmatrix}-3 & -1 & 0\\ 0 & 1 & -1\\ 0 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 1\\ -1 & 0 & 1\\ 2 & 1 & -1\end{bmatrix}\vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} -3+1+0 & 6-0+0 & -3-1+0 \\ 0-1-2 & 0+0-1 & 0+1+1 \\ 0-1+0 & 0+0+0 & 0+1+0 \end{bmatrix} \vec{x}$$

$$S(T(\vec{x})) = \begin{bmatrix} -2 & 6 & -4 \\ -3 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \vec{x}$$

Transform $\vec{x} = (-2, -1, 0)$.

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -2 & 6 & -4 \\ -3 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} (-2)(-2)+6(-1)-4(0) \\ (-3)(-2)+(-1)(-1)+2(0) \\ (-1)(-2)+0(-1)+1(0) \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} 4-6-0 \\ 6+1+0 \\ 2+0+0 \end{bmatrix}$$

$$S\left(T\left(\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}\right)\right) = \begin{bmatrix} -2 \\ 7 \\ 2 \end{bmatrix}$$

11. The matrices A and C are identical, other than two changes. Matrix A has rows 2 and 3 that are swapped, compared to matrix C. When matrices are identical other than a swapped row, the determinant of one is equal to the negative determinant of the other.

The second change is that the second row of C has been multiplied by 3, compared to matrix A. If we have a row multiplied by a constant k, then the determinant of the new matrix is multiplied by k.

Putting these two changes together, we get

$$\text{det}(C) = -3\text{det}(A)$$

$$\text{det}(C) = -3(7) = -21$$

We also see that $B = C^T$. The determinant of a transpose of a square matrix will always be equal to the determinant of the original matrix, which means $\text{det}(B) = \text{det}(C) = -21$.

12. Define $\vec{v}_1 = (-1,1,1,-1)$, $\vec{v}_2 = (-2,-1,0,1)$, and $\vec{v}_3 = (1,0,-2,-1)$.

$$V = \text{Span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$$

The length of $\vec{v}_1$ is

$$||\vec{v}_1|| = \sqrt{(-1)^2 + (1)^2 + (1)^2 + (-1)^2} = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

Then if $\vec{u}_1$ is the normalized version of $\vec{v}_1$, we can say

$$\vec{u}_1 = \frac{1}{2}\begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{v}_2$, and $\vec{v}_3$.

$$V_1 = \text{Span}(\vec{u}_1, \vec{v}_2, \vec{v}_3)$$

We'll name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2 = \vec{v}_2 - \text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u_1}$$

Plug in the values we already have.

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4} \left(\begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix} \right) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4}((-2)(-1) + (-1)(1) + (0)(1) + (1)(-1)) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{4}(0) \begin{bmatrix} -1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So $\vec{w}_2$ is orthogonal to $\vec{u}_1$, but it hasn't been normalized, so let's normalize it. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{(-2)^2 + (-1)^2 + (-0)^2 + (1)^2}$$

$$||\vec{w}_2|| = \sqrt{4 + 1 + 0 + 1}$$

$$||\vec{w}_2|| = \sqrt{6}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$.

$$\vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

So we can say that V is spanned by $\vec{u}_1$, $\vec{u}_2$, and $\vec{v}_3$. Then the vector $\vec{w}_3$ is given by

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_1}\vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - (\vec{v}_3 \cdot \vec{u}_1)\vec{u_1} - (\vec{v}_3 \cdot \vec{u}_2)\vec{u_2}$$

Plug in the values we already have.

$$\vec{w}_3=\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}-\left(\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}\cdot\frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}\right)\frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}-\left(\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}\cdot\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}\right)\frac{1}{\sqrt{6}}\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}-\frac{1}{4}\left(\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}\cdot\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}\right)\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}-\frac{1}{6}\left(\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}\cdot\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}\right)\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}-\frac{1}{4}((1)(-1)+(0)(1)+(-2)(1)+(-1)(-1))\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}$$

$$-\frac{1}{6}((1)(-2)+(0)(-1)+(-2)(0)+(-1)(1))\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}-\frac{1}{4}(-2)\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}-\frac{1}{6}(-3)\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}1\\0\\-2\\-1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} 1 \\ 0 \\ -2 \\ -1 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix} + \begin{bmatrix} -1 \\ -\frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$$

$$\vec{w}_3 = \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ -1 \end{bmatrix}$$

The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(-\frac{1}{2}\right)^2 + 0^2 + \left(-\frac{3}{2}\right)^2 + (-1)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{1}{4} + 0 + \frac{9}{4} + 1}$$

$$||\vec{w}_3|| = \frac{\sqrt{14}}{2}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{2}{\sqrt{14}} \begin{bmatrix} -\frac{1}{2} \\ 0 \\ -\frac{3}{2} \\ -1 \end{bmatrix}$$

Therefore, we can say that $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V_3 = \text{Span}\left(\frac{1}{2}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}, \frac{2}{\sqrt{14}}\begin{bmatrix}-\frac{1}{2}\\0\\-\frac{3}{2}\\-1\end{bmatrix}\right)$$