Linear Algebra Final Exam Solutions

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MATH

Linear Algebra Final Exam Answer Key

1. (5 pts)	A	B	C	■	E
2. (5 pts)	■	B	C	D	E
3. (5 pts)	A	B	■	D	E
4. (5 pts)	A	■	C	D	E
5. (5 pts)	■	B	C	D	E
6. (5 pts)	A	B	C	D	■
7. (5 pts)	A	B	■	D	E
8. (5 pts)	A	B	■	D	E

9. (15 pts) $\vec{x} = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$

10. (15 pts) $C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}\right)$ $\quad \mathbb{R}^3 \quad$ Dim $= 3$

$N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right)$ $\quad \mathbb{R}^4 \quad$ Dim $= 1$

$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}\right)$ $\quad \mathbb{R}^4 \quad$ Dim $= 3$

$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\quad \mathbb{R}^3 \quad$ Dim $= 0$

11. (15 pts) $V^{\perp} = \text{Span}\left(\begin{bmatrix} 2 \\ \frac{10}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}\right)$

12. (15 pts) $E_0 = \text{Span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}\right)$, $E_2 = \text{Span}\left(\begin{bmatrix}-10\\3\\1\end{bmatrix}\right)$, $E_3 = \text{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$

- Since $\lambda = 0$ in the eigenspace E_0, any vector $\vec{v}$ in E_0, under the transformation T, will be scaled down to the zero vector, meaning that $T(\vec{v}) = \lambda\vec{v} = 0\vec{v} = \vec{O}$.

- Since $\lambda = 2$ in the eigenspace E_2, any vector $\vec{v}$ in E_2, under the transformation T, will be scaled by 2, meaning that $T(\vec{v}) = \lambda\vec{v} = 2\vec{v}$.

- Since $\lambda = 3$ in the eigenspace E_3, any vector $\vec{v}$ in E_3, under the transformation T, will be scaled by 3, meaning that $T(\vec{v}) = \lambda\vec{v} = 3\vec{v}$.

Linear Algebra Final Exam Solutions

1. D. To put A into reduced row-echelon form, start by working on the first column.

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ -2 & 0 & 1 & 1 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & -6 & 5 & 1 \\ 0 & 2 & -1 & 0 \\ 3 & 1 & 2 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & -6 & 5 & 1 \\ 0 & 2 & -1 & 0 \\ 0 & 10 & -4 & -2 \end{bmatrix}$$

Find the pivot entry in the second column, then zero out the rest of the second column.

$$\begin{bmatrix} 1 & -3 & 2 & 0 \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 2 & -1 & 0 \\ 0 & 10 & -4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 2 & -1 & 0 \\ 0 & 10 & -4 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 10 & -4 & -2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix}$$

Find the pivot entry in the third column, then zero out the rest of the third column.

$$\begin{bmatrix} 1 & 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & -\frac{5}{6} & -\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{13}{3} & -\frac{1}{3} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & \frac{5}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. A. Multiply each row of A by the first column of B.

$$AB = \begin{bmatrix} -2 & 1 & 2 & 5 \\ 0 & -1 & 4 & 2 \\ 1 & -3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ -5 & 0 & 2 \\ 2 & -3 & -1 \\ 2 & 0 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} -2(1) + 1(-5) + 2(2) + 5(2) & \dots & \dots \\ 0(1) - 1(-5) + 4(2) + 2(2) & \dots & \dots \\ 1(1) - 3(-5) + 1(2) - 1(2) & \dots & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} -2-5+4+10 & \dots & \dots \\ 0+5+8+4 & \dots & \dots \\ 1+15+2-2 & \dots & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & \dots & \dots \\ 17 & \dots & \dots \\ 16 & \dots & \dots \end{bmatrix}$$

Multiply each row of A by the second column of B.

$$AB = \begin{bmatrix} 7 & -2(1)+1(0)+2(-3)+5(0) & \dots \\ 17 & 0(1)-1(0)+4(-3)+2(0) & \dots \\ 16 & 1(1)-3(0)+1(-3)-1(0) & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -2+0-6+0 & \dots \\ 17 & 0-0-12+0 & \dots \\ 16 & 1-0-3-0 & \dots \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -8 & \dots \\ 17 & -12 & \dots \\ 16 & -2 & \dots \end{bmatrix}$$

Multiply each row of A by the third column of B.

$$AB = \begin{bmatrix} 7 & -8 & -2(2)+1(2)+2(-1)+5(-1) \\ 17 & -12 & 0(2)-1(2)+4(-1)+2(-1) \\ 16 & -2 & 1(2)-3(2)+1(-1)-1(-1) \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -8 & -4+2-2-5 \\ 17 & -12 & 0-2-4-2 \\ 16 & -2 & 2-6-1+1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 7 & -8 & -9 \\ 17 & -12 & -8 \\ 16 & -2 & -4 \end{bmatrix}$$

3. C. The vector sum is

$$2\vec{a} - 3\vec{b} + 5\vec{c} - \vec{d}$$

$$2(2,6,-1) - 3(-3,1,1) + 5(0,5,-2) - (1,4,-4)$$

Apply the scalars to each vector.

$$(4,12,-2) + (9,-3,-3) + (0,25,-10) + (-1,-4,4)$$

Add each of the vector components.

$$(4 + 9 + 0 - 1, 12 - 3 + 25 - 4, -2 - 3 - 10 + 4)$$

$$(12,30,-11)$$

4. B. The length of $\vec{x} = (4,-2,1,0)$ is given by

$$||\vec{x}|| = \sqrt{x_1^2 + x_2^2 + x_3^2 + x_4^2}$$

$$||\vec{x}|| = \sqrt{4^2 + (-2)^2 + 1^2 + 0^2}$$

$$||\vec{x}|| = \sqrt{16 + 4 + 1 + 0}$$

$$||\vec{x}|| = \sqrt{21}$$

5. A. The equation of a plane with normal vector $\vec{n}=(a,b,c)=(2,-6,1)$, which passes through $(x_0,y_0,z_0)=(5,2,-3)$, is given by

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

$$2(x-5)-6(y-2)+1(z-(-3))=0$$

Simplify to get the equation of the plane into standard form.

$$2x-10-6y+12+z+3=0$$

$$2x-6y+z+5=0$$

$$2x-6y+z=-5$$

6. E. Given $\vec{x}=(4,-1)$ and

$$S(\vec{x})=\begin{bmatrix}-2x_1+x_2\\3x_2\end{bmatrix}$$

$$T(\vec{x})=\begin{bmatrix}x_1-4x_2\\-4x_2\end{bmatrix}$$

start by using S to transform the standard basis vectors.

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)=\begin{bmatrix}-2(1)+0\\3(0)\end{bmatrix}=\begin{bmatrix}-2\\0\end{bmatrix}$$

$$S\left(\begin{bmatrix}0\\1\end{bmatrix}\right)=\begin{bmatrix}-2(0)+1\\3(1)\end{bmatrix}=\begin{bmatrix}1\\3\end{bmatrix}$$

So the transformation S can be written as the matrix-vector product

$$S(\vec{x}) = \begin{bmatrix}-2 & 1\\ 0 & 3\end{bmatrix}\vec{x}$$

Use T to transform the standard basis vectors.

$$T\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1-4(0)\\ -4(0)\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0-4(1)\\ -4(1)\end{bmatrix} = \begin{bmatrix}-4\\-4\end{bmatrix}$$

So the transformation T can be written as the matrix-vector product

$$T(\vec{y}) = \begin{bmatrix}1 & -4\\ 0 & -4\end{bmatrix}\vec{y}$$

If we call the matrix from S

$$A = \begin{bmatrix}-2 & 1\\ 0 & 3\end{bmatrix}$$

and we call the matrix from T

$$B = \begin{bmatrix}1 & -4\\ 0 & -4\end{bmatrix}$$

then the composition of the transformations is

$$T(S(\vec{x})) = BA\vec{x}$$

$$T(S(\vec{x})) = \begin{bmatrix}1 & -4\\ 0 & -4\end{bmatrix}\begin{bmatrix}-2 & 1\\ 0 & 3\end{bmatrix}\vec{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix}1(-2)-4(0) & 1(1)-4(3)\\ 0(-2)-4(0) & 0(1)-4(3)\end{bmatrix}\overrightarrow{x}$$

$$T(S(\overrightarrow{x})) = \begin{bmatrix}-2 & -11\\ 0 & -12\end{bmatrix}\overrightarrow{x}$$

To transform $\overrightarrow{x} = (4, -1)$, multiply this transformation matrix by $\overrightarrow{x} = (4, -1)$.

$$T\left(S\left(\begin{bmatrix}4\\ -1\end{bmatrix}\right)\right) = \begin{bmatrix}-2 & -11\\ 0 & -12\end{bmatrix}\begin{bmatrix}4\\ -1\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}4\\ -1\end{bmatrix}\right)\right) = \begin{bmatrix}-2(4)-11(-1)\\ 0(4)-12(-1)\end{bmatrix}$$

$$T\left(S\left(\begin{bmatrix}4\\ -1\end{bmatrix}\right)\right) = \begin{bmatrix}3\\ 12\end{bmatrix}$$

7. C. Transform $\overrightarrow{x} = (6, -1)$ with the transformation T.

$$T(\overrightarrow{x}) = \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}\overrightarrow{x}$$

$$T\left(\begin{bmatrix}6\\ -1\end{bmatrix}\right) = \begin{bmatrix}1 & -2\\ 4 & -4\end{bmatrix}\begin{bmatrix}6\\ -1\end{bmatrix}$$

$$T\left(\begin{bmatrix}6\\ -1\end{bmatrix}\right) = \begin{bmatrix}1(6)-2(-1)\\ 4(6)-4(-1)\end{bmatrix}$$

$$T\left(\begin{bmatrix}6\\-1\end{bmatrix}\right)=\begin{bmatrix}8\\28\end{bmatrix}$$

Now find C^{-1}.

$$[C\ |\ I]=\begin{bmatrix}2 & -3 & | & 1 & 0\\-1 & 0 & | & 0 & 1\end{bmatrix}$$

$$[C\ |\ I]=\begin{bmatrix}1 & -\frac{3}{2} & | & \frac{1}{2} & 0\\-1 & 0 & | & 0 & 1\end{bmatrix}$$

$$[C\ |\ I]=\begin{bmatrix}1 & -\frac{3}{2} & | & \frac{1}{2} & 0\\0 & -\frac{3}{2} & | & \frac{1}{2} & 1\end{bmatrix}$$

$$[C\ |\ I]=\begin{bmatrix}1 & -\frac{3}{2} & | & \frac{1}{2} & 0\\0 & 1 & | & -\frac{1}{3} & -\frac{2}{3}\end{bmatrix}$$

$$[C\ |\ I]=\begin{bmatrix}1 & 0 & | & 0 & -1\\0 & 1 & | & -\frac{1}{3} & -\frac{2}{3}\end{bmatrix}$$

To convert to the alternate basis, we need to multiply the transformed vector by C^{-1}.

$$C^{-1}T\left(\begin{bmatrix}6\\-1\end{bmatrix}\right)=\begin{bmatrix}0 & -1\\-\frac{1}{3} & -\frac{2}{3}\end{bmatrix}\begin{bmatrix}8\\28\end{bmatrix}$$

$$C^{-1}T\left(\begin{bmatrix}6\\-1\end{bmatrix}\right)=\begin{bmatrix}0(8)-1(28)\\-\frac{1}{3}(8)-\frac{2}{3}(28)\end{bmatrix}$$

$$C^{-1}T\left(\begin{bmatrix}6\\-1\end{bmatrix}\right)=\begin{bmatrix}-28\\-\frac{64}{3}\end{bmatrix}$$

This is the vector $\vec{x}=(6,-1)$ after the transformation T, and converted into the alternate basis. So

$$[T(\vec{x})]_B=\begin{bmatrix}-28\\-\frac{64}{3}\end{bmatrix}$$

8. C. The vectors $\vec{v}_1=(0,-1,1)$, $\vec{v}_2=(2,0,-3)$, and $\vec{v}_3=(-1,1,1)$ form the basis for V.

$$V=\text{Span}(\vec{v}_1,\vec{v}_2,\vec{v}_3)$$

Normalize $\vec{v}_1$. The length of $\vec{v}_1$ is

$$||\vec{v}_1||=\sqrt{0^2+(-1)^2+1^2}=\sqrt{0+1+1}=\sqrt{2}$$

Then the normalized version of $\vec{v}_1$ is

$$\vec{u}_1=\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

Replace $\vec{v}_2$ with a vector that's both orthogonal to $\vec{u}_1$, and normal. Name $\vec{w}_2$ as the vector that connects $\text{Proj}_{V_1}\vec{v}_2$ to $\vec{v}_2$.

$$\vec{w}_2=\vec{v}_2-\text{Proj}_{V_1}\vec{v}_2$$

$$\vec{w}_2 = \vec{v}_2 - (\vec{v}_2 \cdot \vec{u}_1)\vec{u}_1$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} - \left(\begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right) \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} - \frac{1}{2}(2(0) + 0(-1) - 3(1)) \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \frac{3}{2} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ 0 \\ -3 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{3}{2} \\ \frac{3}{2} \end{bmatrix}$$

$$\vec{w}_2 = \begin{bmatrix} 2 \\ -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Normalize $\vec{w}_2$. The length of $\vec{w}_2$ is

$$||\vec{w}_2|| = \sqrt{2^2 + \left(-\frac{3}{2}\right)^2 + \left(-\frac{3}{2}\right)^2}$$

$$||\vec{w}_2|| = \sqrt{4 + \frac{9}{4} + \frac{9}{4}}$$

$$||\vec{w}_2|| = \sqrt{\frac{34}{4}}$$

$$||\vec{w}_2|| = \frac{\sqrt{34}}{2}$$

Then the normalized version of $\vec{w}_2$ is $\vec{u}_2$:

$$\vec{u}_2 = \frac{2}{\sqrt{34}}\begin{bmatrix} 2 \\ -\frac{3}{2} \\ -\frac{3}{2} \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{2} \cdot \frac{2}{\sqrt{34}}\begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$$

$$\vec{u}_2 = \frac{1}{\sqrt{34}}\begin{bmatrix} 4 \\ -3 \\ -3 \end{bmatrix}$$

Replace $\vec{v}_3$ with a vector that's both orthogonal to $\vec{u}_1$ and $\vec{u}_2$, and normal. Name $\vec{w}_3$ as the vector that connects $\text{Proj}_{V_2}\vec{v}_3$ to $\vec{v}_3$.

$$\vec{w}_3 = \vec{v}_3 - \text{Proj}_{V_2}\vec{v}_3$$

$$\vec{w}_3 = \vec{v}_3 - \left[(\vec{v}_3 \cdot \vec{u}_1)\vec{u}_1 + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2\right]$$

$$\vec{w}_3 = \vec{v}_3 - \left[\left(\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}\right)\frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2\right]$$

$$\vec{w}_3 = \vec{v}_3 - \left[\frac{1}{2}(-1(0) + 1(-1) + 1(1))\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + (\vec{v}_3 \cdot \vec{u}_2)\vec{u}_2\right]$$

$$\vec{w}_3=\vec{v}_3-\left[\frac{1}{2}(0)\begin{bmatrix}0\\-1\\1\end{bmatrix}+(\vec{v}_3\cdot\vec{u}_2)\vec{u_2}\right]$$

$$\vec{w}_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}-\left[0+\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\frac{1}{\sqrt{34}}\begin{bmatrix}4\\-3\\-3\end{bmatrix}\right)\frac{1}{\sqrt{34}}\begin{bmatrix}4\\-3\\-3\end{bmatrix}\right]$$

$$\vec{w}_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}-\frac{1}{34}\left(\begin{bmatrix}-1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}4\\-3\\-3\end{bmatrix}\right)\begin{bmatrix}4\\-3\\-3\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}-\frac{1}{34}(-1(4)+1(-3)+1(-3))\begin{bmatrix}4\\-3\\-3\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}+\frac{5}{17}\begin{bmatrix}4\\-3\\-3\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}-1\\1\\1\end{bmatrix}+\begin{bmatrix}\frac{20}{17}\\-\frac{15}{17}\\-\frac{15}{17}\end{bmatrix}$$

$$\vec{w}_3=\begin{bmatrix}\frac{3}{17}\\\frac{2}{17}\\\frac{2}{17}\end{bmatrix}$$

$$\vec{w}_3=\frac{1}{17}\begin{bmatrix}3\\2\\2\end{bmatrix}$$

Normalize $\vec{w}_3$. The length of $\vec{w}_3$ is

$$||\vec{w}_3|| = \sqrt{\left(\frac{3}{17}\right)^2 + \left(\frac{2}{17}\right)^2 + \left(\frac{2}{17}\right)^2}$$

$$||\vec{w}_3|| = \sqrt{\frac{9}{289} + \frac{4}{289} + \frac{4}{289}}$$

$$||\vec{w}_3|| = \sqrt{\frac{17}{289}}$$

$$||\vec{w}_3|| = \frac{\sqrt{17}}{17}$$

Then the normalized version of $\vec{w}_3$ is $\vec{u}_3$:

$$\vec{u}_3 = \frac{17}{\sqrt{17}} \begin{bmatrix} \frac{3}{17} \\ \frac{2}{17} \\ \frac{2}{17} \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{17} \cdot \frac{17}{\sqrt{17}} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

$$\vec{u}_3 = \frac{1}{\sqrt{17}} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$$

The vectors $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ form an orthonormal basis for V.

$$V = \text{Span}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix},\frac{1}{\sqrt{34}}\begin{bmatrix}4\\-3\\-3\end{bmatrix},\frac{1}{\sqrt{17}}\begin{bmatrix}3\\2\\2\end{bmatrix}\right)$$

9. Put the matrix A into reduced row-echelon form.

$$\begin{bmatrix}1&1\\1&2\\-2&-3\end{bmatrix}\to\begin{bmatrix}1&1\\0&1\\-2&-3\end{bmatrix}\to\begin{bmatrix}1&1\\0&1\\0&-1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\\0&-1\end{bmatrix}\to\begin{bmatrix}1&0\\0&1\\0&0\end{bmatrix}$$

To find the complementary solution, augment $\text{rref}(A)$ with the zero vector to get a system of equations.

$$x_1 = 0$$

$$x_2 = 0$$

Then the only vector that satisfies the null space is

$$\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

The complementary solution is

$$\vec{x}_n=\begin{bmatrix}0\\0\end{bmatrix}$$

To find the particular solution, augment A with $\vec{b}=(b_1,b_2,b_3)$, then put it in reduced row-echelon form.

$$\begin{bmatrix} 1 & 1 & | & b_1 \\ 1 & 2 & | & b_2 \\ -2 & -3 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & b_1 \\ 0 & 1 & | & b_2 - b_1 \\ -2 & -3 & | & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & | & b_1 \\ 0 & 1 & | & b_2 - b_1 \\ 0 & -1 & | & 2b_1 + b_3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & | & 2b_1 - b_2 \\ 0 & 1 & | & b_2 - b_1 \\ 0 & -1 & | & 2b_1 + b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & 2b_1 - b_2 \\ 0 & 1 & | & b_2 - b_1 \\ 0 & 0 & | & b_1 + b_2 + b_3 \end{bmatrix}$$

From the third row, the system is constrained.

$$b_1 + b_2 + b_3 = 0$$

$$b_1 = -b_2 - b_3$$

Choose values of b_1, b_2, and b_3 that make this equation true, like $b_2 = 1$, $b_3 = 1$, and $b_1 = -2$. Then the matrix is

$$\begin{bmatrix} 1 & 0 & | & 2(-2) - 1 \\ 0 & 1 & | & 1 - (-2) \\ 0 & 0 & | & -2 + 1 + 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & | & -5 \\ 0 & 1 & | & 3 \\ 0 & 0 & | & 0 \end{bmatrix}$$

This gives a system of equations

$$x_1 = -5$$

$$x_2 = 3$$

So the particular solution is

$$\vec{x}_p = \begin{bmatrix} -5 \\ 3 \end{bmatrix}$$

The general solution is the sum of the particular and complementary solutions.

$$\vec{x}=\vec{x}_p+\vec{x}_n$$

$$\vec{x}=\begin{bmatrix}-5\\3\end{bmatrix}+\begin{bmatrix}0\\0\end{bmatrix}$$

Of course, adding the zero vector doesn't change the value of the general solution, so the general solution is

$$\vec{x}=\begin{bmatrix}-5\\3\end{bmatrix}$$

10. First put A into reduced row-echelon form.

$$A=\begin{bmatrix}1&-2&0&1\\-1&0&2&4\\0&1&-1&2\end{bmatrix}\to\begin{bmatrix}1&-2&0&1\\0&-2&2&5\\0&1&-1&2\end{bmatrix}\to\begin{bmatrix}1&-2&0&1\\0&1&-1&2\\0&-2&2&5\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&-2&5\\0&1&-1&2\\0&-2&2&5\end{bmatrix}\to\begin{bmatrix}1&0&-2&5\\0&1&-1&2\\0&0&0&9\end{bmatrix}\to\begin{bmatrix}1&0&-2&5\\0&1&-1&2\\0&0&0&1\end{bmatrix}$$

$$\to\begin{bmatrix}1&0&-2&0\\0&1&-1&2\\0&0&0&1\end{bmatrix}\to\begin{bmatrix}1&0&-2&0\\0&1&-1&0\\0&0&0&1\end{bmatrix}$$

In rref(A), the pivot columns are the first, second, and fourth columns, which means $C(A)$ is given by the span of the first, second, and fourth columns of A.

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix} \right)$$

From rref(A), we get the system of equations

$$x_1 - 2x_3 = 0$$

$$x_2 - x_3 = 0$$

$$x_4 = 0$$

Solve the system for the pivot variables.

$$x_1 = 2x_3$$

$$x_2 = x_3$$

$$x_4 = 0$$

Then the solution to the system is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

which means the null space is given as

$$N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

Find A^T,

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 2 \end{bmatrix}$$

then put A^T into reduced row-echelon form.

$$A^T = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \\ 1 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \\ 0 & 5 & 2 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \\ 0 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 5 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{9}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & \frac{9}{2} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{1}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

In rref(A^T), the pivot columns are the first, second, and third columns, which means $C(A^T)$ is given by the span of the first, second, and third columns of A^T.

$$C(A^T) = \text{Span}\left(\begin{bmatrix}1\\-2\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\0\\2\\4\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\2\end{bmatrix}\right)$$

From rref(A^T), we get the system of equations

$$x_1 = 0$$

$$x_2 = 0$$

$$x_3 = 0$$

Then the solution to the system is

$$\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

which means the left null space is given as

$$N(A^T) = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

Because A is an $m \times n = 3 \times 4$ matrix, the row space and null space are defined in $\mathbb{R}^n = \mathbb{R}^4$, and the column space and left null space are defined in $\mathbb{R}^m = \mathbb{R}^3$.

The dimension of the column space and row space is the rank of A, $r = 3$. The dimension of the null space is $n - r = 4 - 3 = 1$, and the dimension of the left null space is $m - r = 3 - 3 = 0$.

In summary, the four fundamental subspaces are

$$C(A) = \text{Span}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}\right) \qquad \mathbb{R}^3 \qquad \text{Dim} = 3$$

$$N(A) = \text{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \end{bmatrix}\right) \qquad \mathbb{R}^4 \qquad \text{Dim} = 1$$

$$C(A^T) = \text{Span}\left(\begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 2 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 2 \end{bmatrix}\right) \qquad \mathbb{R}^4 \qquad \text{Dim} = 3$$

$$N(A^T) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbb{R}^3 \qquad \text{Dim} = 0$$

11. The subspace V is a plane in $\mathbb{R}^4$, spanned by the three vectors $\vec{v}_1 = (2,-2,1,1)$, $\vec{v}_2 = (-1,0,0,2)$, and $\vec{v}_3 = (0,1,-2,0)$. Its orthogonal complement $V^\perp$ is the set of vectors which are orthogonal to both all three.

$$V^\perp = \left\{\vec{x} \in \mathbb{R}^4 \mid \vec{x} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \\ 1 \end{bmatrix} = 0, \ \vec{x} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = 0, \ \vec{x} \cdot \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \end{bmatrix}\right\}$$

Let $\vec{x} = (x_1, x_2, x_3, x_4)$ to get three equations from these dot products.

$$2x_1 - 2x_2 + x_3 + x_4 = 0$$

$$-x_1 + 2x_4 = 0$$

$$x_2 - 2x_3 = 0$$

Put these equations into an augmented matrix,

$$\begin{bmatrix} 2 & -2 & 1 & 1 & | & 0 \\ -1 & 0 & 0 & 2 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \end{bmatrix}$$

then put it into reduced row-echelon form.

$$\begin{bmatrix} -1 & 0 & 0 & 2 & | & 0 \\ 2 & -2 & 1 & 1 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 2 & -2 & 1 & 1 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & -2 & 1 & 5 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \\ 0 & -2 & 1 & 5 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \\ 0 & 0 & -3 & 5 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & -2 & 0 & | & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & | & 0 \\ 0 & 1 & 0 & -\frac{10}{3} & | & 0 \\ 0 & 0 & 1 & -\frac{5}{3} & | & 0 \end{bmatrix}$$

The rref form gives the system of equations

$$x_1 - 2x_4 = 0$$

$$x_2 - \frac{10}{3}x_4 = 0$$

$$x_3 - \frac{5}{3}x_4 = 0$$

Solve the system for the pivot variables, x_1, x_2, and x_3.

$$x_1 = 2x_4$$

$$x_2 = \frac{10}{3}x_4$$

$$x_3 = \frac{5}{3}x_4$$

So we could also express the system as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 2 \\ \frac{10}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix}$$

The orthogonal complement is

$$V^{\perp} = \text{Span}\left(\begin{bmatrix} 2 \\ \frac{10}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} \right)$$

12. Starting with

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

first, find the determinant $|\lambda I_n - A|$.

$$\left| \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} \right|$$

$$\left| \begin{bmatrix} \lambda - 3 & -6 & 8 \\ 0 & \lambda & -6 \\ 0 & 0 & \lambda - 2 \end{bmatrix} \right|$$

Then let's work along the first column, since it includes two 0 values, to find the determinant of this resulting matrix.

$$(\lambda - 3) \begin{vmatrix} \lambda & -6 \\ 0 & \lambda - 2 \end{vmatrix} - 0 \begin{vmatrix} -6 & 8 \\ 0 & \lambda - 2 \end{vmatrix} + 0 \begin{vmatrix} -6 & 8 \\ \lambda & -6 \end{vmatrix}$$

The last two determinants cancel, leaving us with just

$$(\lambda - 3) \begin{vmatrix} \lambda & -6 \\ 0 & \lambda - 2 \end{vmatrix}$$

$$(\lambda - 3)\lambda(\lambda - 2) - (-6)(0)$$

$$\lambda(\lambda - 3)(\lambda - 2)$$

Remember that we're trying to satisfy $|\lambda I_n - A| = 0$, so we can set this characteristic polynomial equal to 0 to get the characteristic equation, and then we'll solve for λ.

$$\lambda(\lambda - 3)(\lambda - 2) = 0$$

$\lambda = 0$ or $\lambda = 2$ or $\lambda = 3$

With these three eigenvalues, we'll have three eigenspaces, given by $E_\lambda = N(\lambda I_n - A)$. Given

$$E_\lambda = N\left(\begin{bmatrix} \lambda - 3 & -6 & 8 \\ 0 & \lambda & -6 \\ 0 & 0 & \lambda - 2 \end{bmatrix}\right)$$

we get

$$E_0 = N\left(\begin{bmatrix} 0 - 3 & -6 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & 0 - 2 \end{bmatrix}\right)$$

$$E_0 = N\left(\begin{bmatrix} -3 & -6 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}\right)$$

and

$$E_2 = N\left(\begin{bmatrix} 2-3 & -6 & 8 \\ 0 & 2 & -6 \\ 0 & 0 & 2-2 \end{bmatrix}\right)$$

$$E_2 = N\left(\begin{bmatrix} -1 & -6 & 8 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

and

$$E_3 = N\left(\begin{bmatrix} 3-3 & -6 & 8 \\ 0 & 3 & -6 \\ 0 & 0 & 3-2 \end{bmatrix}\right)$$

$$E_3 = N\left(\begin{bmatrix} 0 & -6 & 8 \\ 0 & 3 & -6 \\ 0 & 0 & 1 \end{bmatrix}\right)$$

Therefore, the eigenvectors in the eigenspace E_0 will satisfy

$$\begin{bmatrix} -3 & -6 & 8 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}\vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -6 & 8 & | & 0 \\ 0 & 0 & -6 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -\frac{8}{3} & | & 0 \\ 0 & 0 & -6 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -\frac{8}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & -2 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 + 2v_2 = 0$$

$$v_3 = 0$$

or

$$v_1 = -2v_2$$

$$v_3 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$$

Which means that E_0 is defined by

$$E_0 = \text{Span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_2 will satisfy

$$\begin{bmatrix} -1 & -6 & 8 \\ 0 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -1 & -6 & 8 & | & 0 \\ 0 & 2 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 6 & -8 & | & 0 \\ 0 & 2 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 2 & -6 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 10 & | & 0 \\ 0 & 1 & -3 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_1 + 10v_3 = 0$$

$$v_2 - 3v_3 = 0$$

or

$$v_1 = -10v_3$$

$$v_2 = 3v_3$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_3 \begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix}$$

Which means that E_2 is defined by

$$E_2 = \text{Span}\left(\begin{bmatrix} -10 \\ 3 \\ 1 \end{bmatrix} \right)$$

And the eigenvectors in the eigenspace E_3 will satisfy

$$\begin{bmatrix} 0 & -6 & 8 \\ 0 & 3 & -6 \\ 0 & 0 & 1 \end{bmatrix} \vec{v} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -6 & 8 & | & 0 \\ 0 & 3 & -6 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 3 & -6 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & -2 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & -\frac{4}{3} & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 1 & | & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 & | & 0 \\ 0 & 0 & 1 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

This gives

$$v_2 = 0$$

$$v_3 = 0$$

So we can say

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Which means that E_3 is defined by

$$E_3 = \text{Span}\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right)$$

Let's put these results together. For the eigenvalues $\lambda = 0$, $\lambda = 2$, and $\lambda = 3$, respectively, we got

$$E_0 = \text{Span}\left(\begin{bmatrix}-2\\1\\0\end{bmatrix}\right), E_2 = \text{Span}\left(\begin{bmatrix}-10\\3\\1\end{bmatrix}\right), \text{ and } E_3 = \text{Span}\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right)$$

Each of these spans represents a line in $\mathbb{R}^3$. So for any vector $\vec{v}$ along any of these lines, when you apply the transformation T to the vector $\vec{v}$, $T(\vec{v})$ will be a vector along the same line, it might just be scaled up or scaled down. Specifically,

- since $\lambda = 0$ in the eigenspace E_0, any vector $\vec{v}$ in E_0, under the transformation T, will be scaled down to the zero vector, meaning that $T(\vec{v}) = \lambda\vec{v} = 0\vec{v} = \vec{O}$,

- since $\lambda = 2$ in the eigenspace E_2, any vector $\vec{v}$ in E_2, under the transformation T, will be scaled by 2, meaning that $T(\vec{v}) = \lambda\vec{v} = 2\vec{v}$, and

- since $\lambda = 3$ in the eigenspace E_3, any vector $\vec{v}$ in E_3, under the transformation T, will be scaled by 3, meaning that $T(\vec{v}) = \lambda\vec{v} = 3\vec{v}$.

W W W . K R I S T A K I N G M A T H . C O M