# EE1002
# Introduction to Circuits and Systems

## Part 1 : Lecture 9

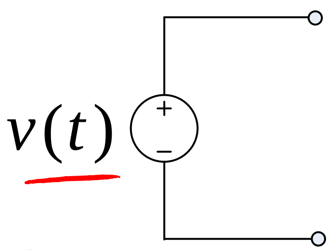
## AC Steady-state Analysis

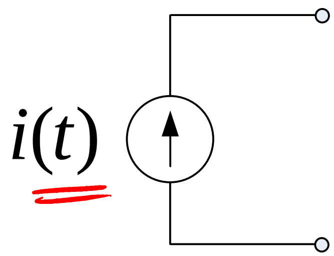
# AC Steady-state Analysis

- Time varying periodic voltage and current
- Complex number
- Sinusoidal sources
- Phasors for sinusoidals
- Impedances for R,L and C
- AC Circuit Analysis with phasors and Impedances
- Examples of AC Circuit analysis

# Lab5 Studying $\boxed{\text{AC}}$ signals

periodic → zero average

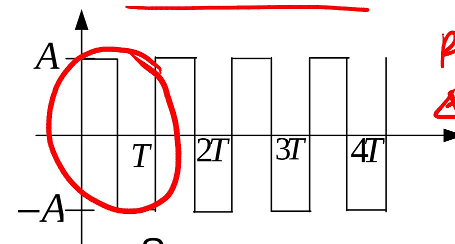




Sawtooth wave

pure AC




Square wave

$$\boxed{f} = \frac{1}{T} \qquad \omega = \frac{2\pi}{T} = 2\pi f \quad , \quad \theta = \omega t$$

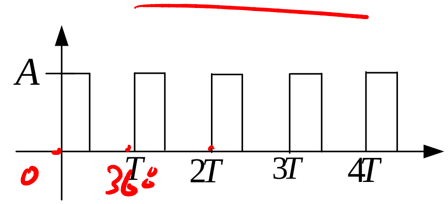
Hz   angular frequency

*RMS* : Root Mean Square

$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

sq root, mean of the squared signal

R. M. S.



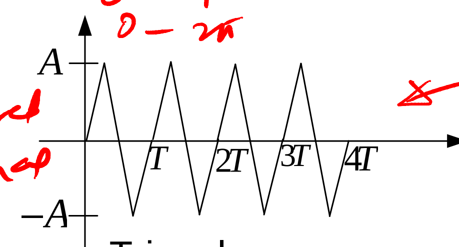

Pulse train

t →

ωt = θ

0 – T

0 – 2π



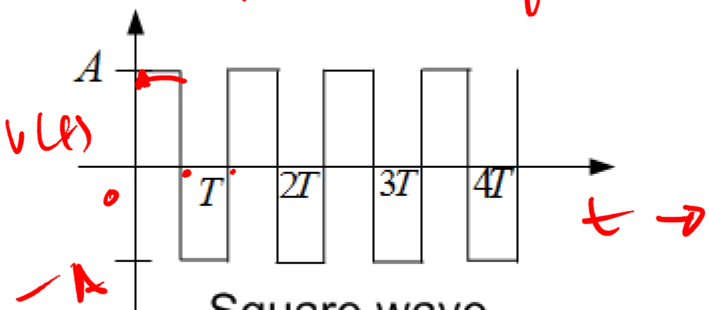

Purely AC signal

Triangle wave

# RMS value for AC Signals

- Mean

Root Square




Square wave

$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T v^2(t)\,dt}$$

$$v(t) = A \quad 0 \le t \le \frac{T}{2}.$$

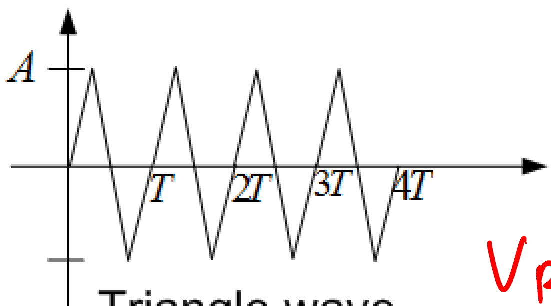
$$= -A \quad T/2 \le t \le T$$

$$V_{RMS} = \sqrt{\frac{1}{T}\left[\int_0^{T/2} v^2(t)\,dt + \int_{T/2}^{T} v^2(t)\,dt\right]}$$

$$= \sqrt{\frac{1}{T}\left[\int_0^{T/2} A^2\,dt + \int_{T/2}^{T} (-A)^2\,dt\right]}$$

$$= \sqrt{\frac{1}{T}\cdot A^2\cdot\left(T/2 + T - T/2\right)}$$

$$= A$$




Triangle wave

Vrms for triangular waveform

# RMS value for AC Sinusoidal

Sinusoidal : $V_m \cos(\omega t + \theta)$

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta)dt}$$

$$= \sqrt{\frac{1}{T}\frac{V_m^2}{2}\int_0^T (1 + \cos 2(\omega t + \theta))dt} = \sqrt{\frac{V_m^2}{2}} = \frac{V_m}{\sqrt{2}}$$

$\rightarrow$ Peak

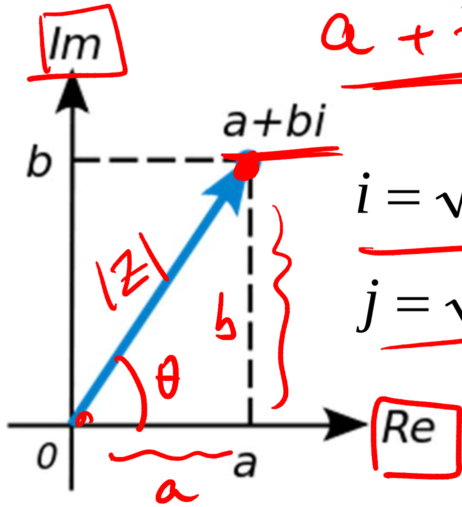
$$\cos^2(\omega t + \theta) = \frac{1}{2} \cdot \left[1 + \cos 2(\omega t + \theta)\right]$$

$$= \frac{1}{2} + \underbrace{\frac{1}{2} \cdot \cos 2(\omega t + \theta)}_{\text{int} \rightarrow 0}$$

# Complex Numbers

# Complex number – Euler's Formula

$a + i.b \qquad a + jb$ : Rectangular.




$$i = \sqrt{-1}$$

$$j = \sqrt{-1}$$

- Addition/Subtraction
  - Done in rectangular form
- Multiplication/Division
  - Done in polar form

$$a + ib = |z| \angle \theta$$

$Z \rightarrow$ complex number

① $|z| = \sqrt{a^2 + b^2}$, $\tan\theta = \frac{b}{a}$.

② $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$$a = |z| . \cos\theta$$
$$b = |z| . \sin\theta$$

$$(a+ib) \pm (c+id) = (a \pm c) + i(b \pm d)$$

$$(a+ib) \times (c+id) = |z_1| \angle \theta_1 \times |z_2| \angle \theta_2$$
$$= |z_1||z_2| \angle \theta_1 + \theta_2$$

# Example – Complex Numbers

$z_1 = 2 + 3i = \sqrt{2^2 + 3^2} \angle \tan^{-1}\frac{3}{2} = 3.60 \angle 56.3^\circ$

$z_2 = 3 + 4i = \sqrt{3^2 + 4^2} \angle \tan^{-1}\frac{4}{3} = 5 \angle 53.13^\circ$

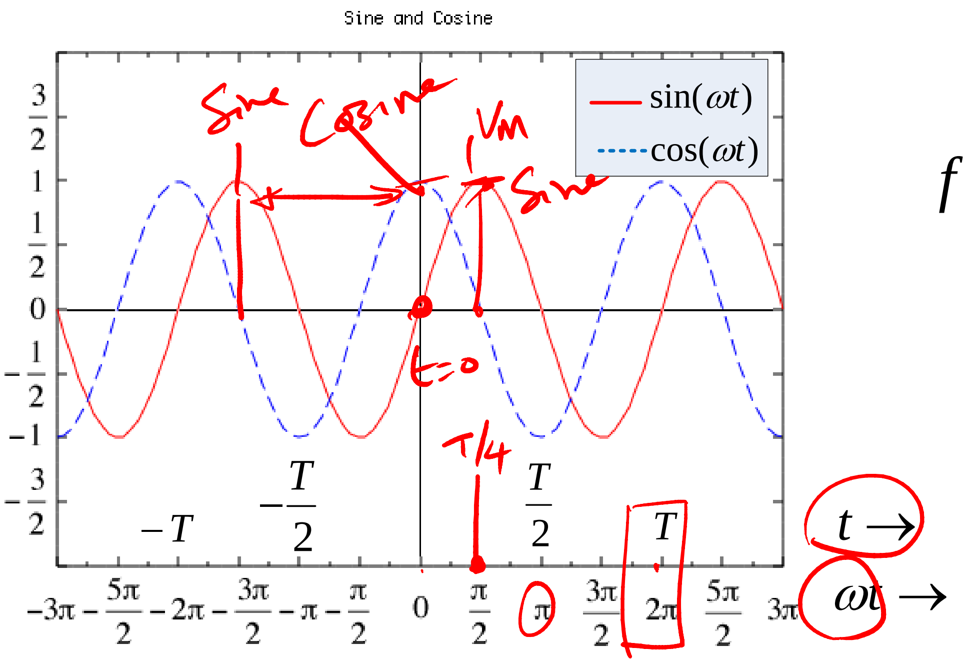
$z_1 + z_2 = 2 + 3i + 3 + 4i = (2+3) + (3+4)i$
$= 5 + 7i$

$z_1 * z_2 = 3.60 \angle 56.3^\circ * 5 \angle 53.13^\circ = 18 \angle 109.43^\circ$

$\frac{z_1}{z_2} = \frac{3.60 \angle 56.3^\circ}{5 \angle 53.13^\circ} = \frac{3.60}{5} \angle 56.3 - 53.13^\circ = 0.72 \angle 3.17^\circ$

# Example – Complex Numbers

# Sinusoidal AC signals




$$f = \frac{1}{T} \qquad \omega = \frac{2\pi}{T} = 2\pi f$$

$\sin \omega t = \cos(\omega t - \underline{90^\circ})$ $\theta$.

$\cos(\omega t)$

$\mathrm{Sine}(\omega t)$ . has a phase angle $\theta = -90^\circ$

$\cos \omega t$ has $\theta = 0$.

$\cos\theta - \sin\theta = 0 - (-90^\circ) = 90^\circ$

$\omega = 2\pi \times f = 2\pi \times \frac{1}{T}$ ; $\omega t = \theta$ , rad

$$v(t) = V_m \cos(\omega t + \theta),$$

$\boxed{V_m}$ is the peak value of voltage

$\omega$ is the angular frequency

$\theta$ is the phase angle

# Phase difference $(\theta_1 - \theta_2)$

$v(t) = V_m \cdot \cos(\omega t + \theta_1)$ , $i(t) = I_m \cdot \cos(\omega t + \theta_2)$

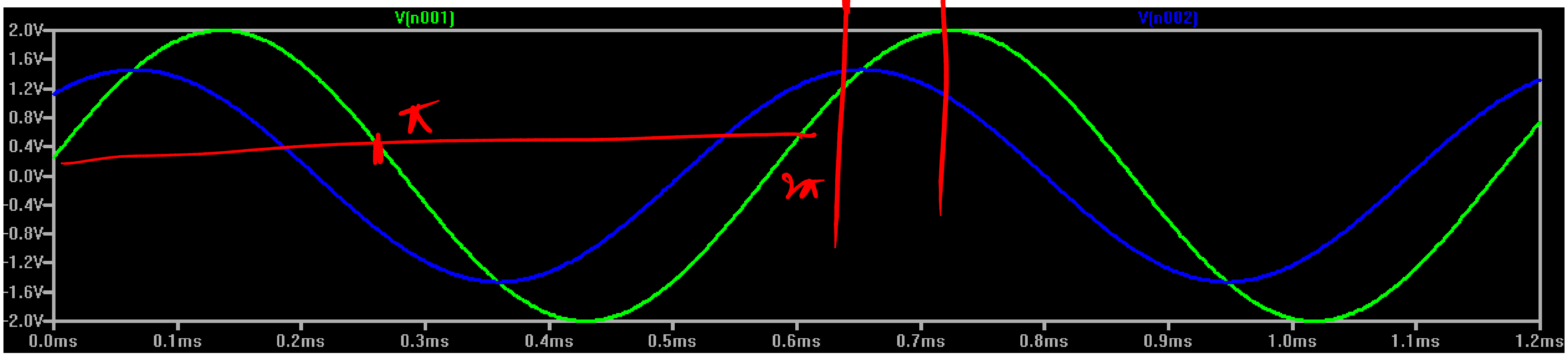
- The signal that reaches the peak earlier is said to be leading. Thus, cosine leads sine. or lag
- However, one may say that sine leads cosine as it peaks much earlier..
- To resolve this confusion, if the phase difference ( $\overset{0}{\theta_1} - \overset{-90^\circ}{\theta_2}$ ) is positive and less than 180 degree, then v1 is leading v2. Else, v1 is lagging v2.

# Phase Difference

Blue

Blue leads Green

Green

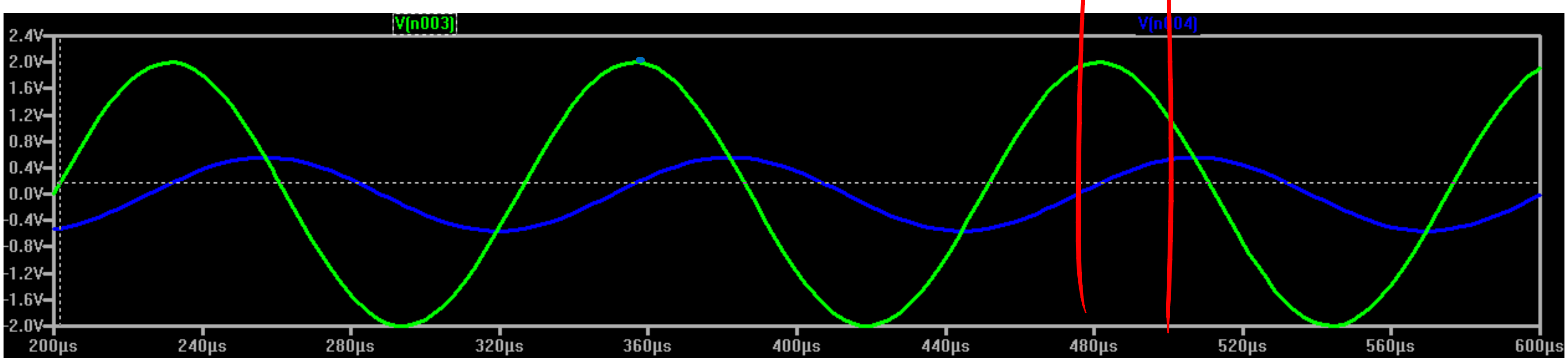



$t_1 < t_2$

$t_1$

$t_2$

Green leads Blue



# EE1002
# Introduction to Circuits and Systems

## Part 1 : Lecture 9

## AC Steady-state Analysis (second half)

# Phasor Definition

The frequencies of all sinusoidals must be same.

- For a sinusoidal voltage of

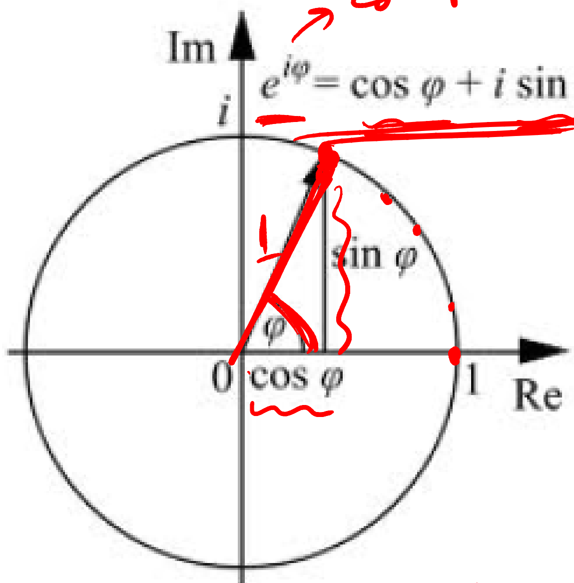
$$v_1(t) = \boxed{V_m} \cos(\omega t + \theta)$$

- Phasor : $V_m \angle \theta$
  - Phasor is a complex number
  - Absolute value equal to the peak
  - Argument same as the phase angle
- Example: $v(t) = 200 \cos(300t + 45^\circ) \Rightarrow 200 \angle 45^\circ$

  $i(t) = 10 \cos(300t - 60^\circ) \Rightarrow 10 \angle -60^\circ$

# Euler's formula and phasor

complex exponential




$$\cos\varphi = \mathrm{Re}\left(e^{j\varphi}\right) \qquad \sin\varphi = \mathrm{Im}\left(e^{j\varphi}\right)$$

$$1.\ \cos(\omega t + \theta) = \mathrm{Re}\left(e^{j(\omega t+\theta)}\right)$$

$$\sin(\omega t + \theta) = \mathrm{Im}\left(e^{j(\omega t+\theta)}\right)$$

$i = j$

$\varphi = \omega t + \theta$

$V_m e^{j(\cancel{\omega t}+\theta)}$ $\cancel{\omega t} = 0$

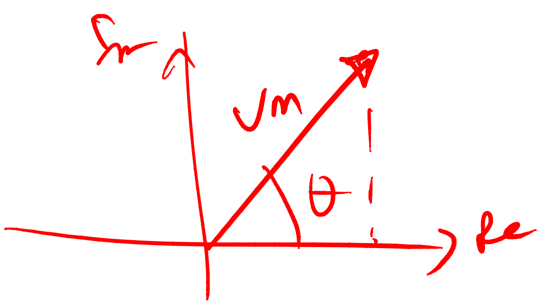
- is a rotating vector in the complex plane

$V_m \cdot \cos(\omega t + \theta)$

$V_m e^{j\theta}$

$V_m \angle \theta$

- phasor is a snapshot of the rotating vector, at t=0.

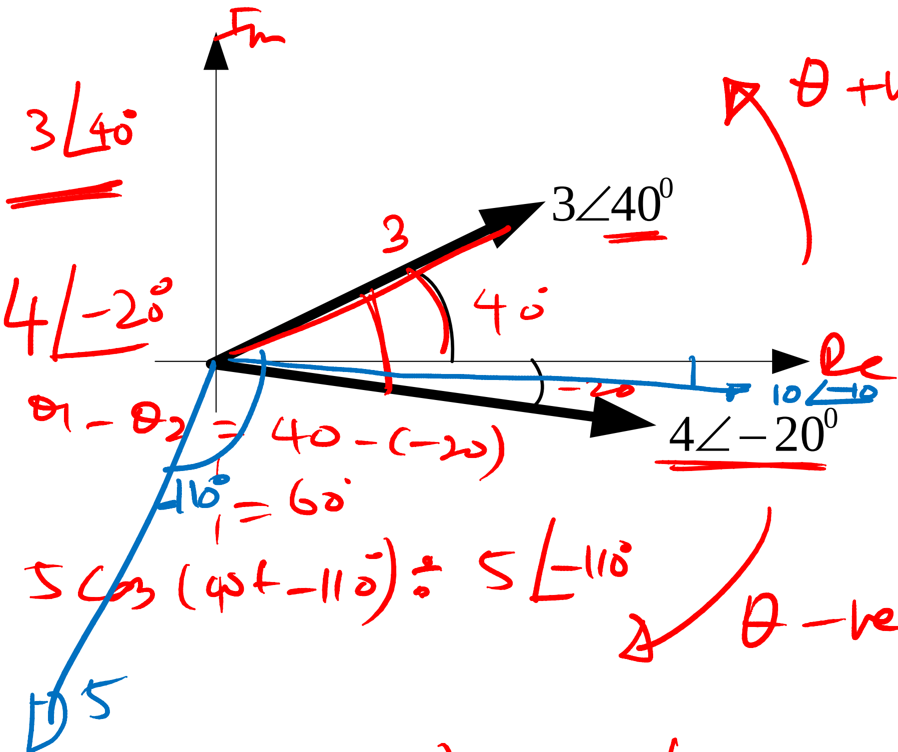


# Phasor diagram

$$v_1(t) = 3\cos(\omega t + \overset{\theta_1}{40^0}) : 3\angle 40^\circ$$

$$v_2(t) = 4\cos(\omega t - \overset{\theta_2}{20^0}) : 4\angle -20^\circ$$

$\theta_1 - \theta_2 = 40 - (-20) = 60^\circ$




$$v_3(t) = 5\sin(\omega t - 20^0)$$

$= 5\cos(\omega t - 20 - 90^\circ) = 5\cos(\omega t - 110^\circ) : 5\angle -110^\circ$

$$v_4(t) = 10\sin(\omega t + 80^0)$$

$= 10\cos(\omega t + 80 - 90^\circ) = 10\cos(\omega t - 10^\circ) : 10\angle -10^\circ$

# Impedances

- Also known as complex resistance, frequency-dependent resistance.
- Ratio of voltage phasor to current phasor

In DC resistance $R = \frac{V}{I}$ (ohm's law)

In AC: phasors < voltage phasor vs $v(t)$
current phasor vs $i(t)$

$$\boxed{\frac{\text{Voltage phasor}}{\text{Current phasor}} = \text{Impedance}}$$

# Impedance of Inductance


$$i_L(t) = I_m \sin(\omega t + \theta) = I_m \cos(\omega t + \theta - 90^0) \quad : \quad I_m \angle \theta - 90^\circ$$

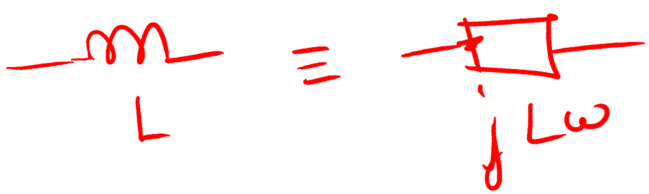
$$v_L(t) = L\frac{di_L(t)}{dt} = (L\omega I_m)\cos(\omega t + \theta) \quad : \quad L\omega I_m \cdot \angle \theta$$

Writing the phasors for the current and the voltage

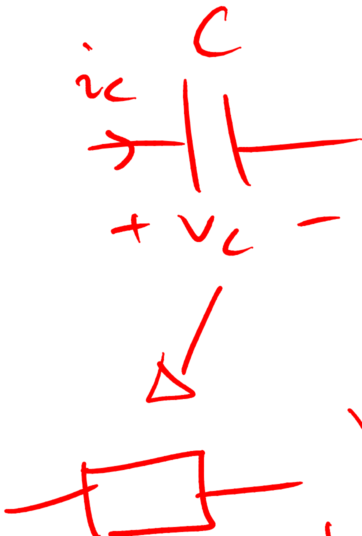
$$Z_L = \frac{V_L\ (\text{phasors})}{I_L\ (\text{phasor})} = \frac{L\cdot\omega\cdot I_m \angle \theta}{I_m \angle \theta - 90^\circ}$$

$$= \frac{L\omega \cdot \cancel{I_m}}{\cancel{I_m}} \cdot \angle \theta - (\theta - 90^\circ)$$

$$= L\omega \angle 90^\circ = jL\omega.$$



# Capacitance




$1/j\omega C = -j\frac{1}{\omega C}$

$$i_c = C \cdot \frac{dV_c}{dt}$$

$$V_c(t) = V_m \sin(\omega t + \theta) = V_m \cdot \cos(\omega t + \theta - 90^\circ)$$

$$i_c(t) = C \cdot \frac{d}{dt}\left(V_m \sin(\omega t + \theta)\right)$$

$$= C \cdot \omega \cdot V_m \cdot \cos(\omega t + \theta)$$

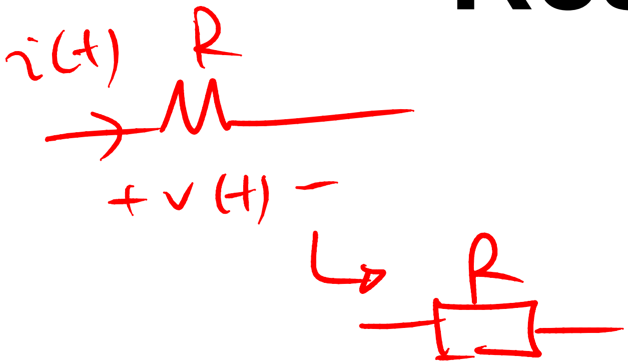
Voltage phasor = $V_m \angle \theta - 90^\circ$

Current phasor = $C \cdot \omega \cdot V_m \cdot \angle \theta$.

$$Z_c = \text{Impedance of cap} = \frac{V_m \angle \theta - 90^\circ}{C \cdot \omega \cdot V_m \cdot \angle \theta} = \frac{1}{C\omega} \angle -90^\circ$$

$$= \frac{1}{jC\omega}$$

# Resistance



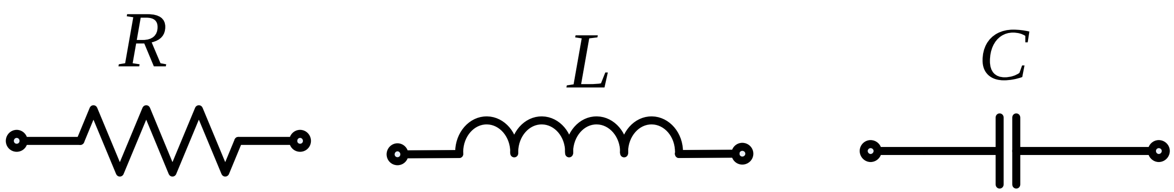

$$v(t) = R \cdot i(t)$$

$$i(t) = I_m \cos(\omega t + \theta) : I_m \angle \theta$$

$$v(t) = R \cdot I_m \cdot \cos(\omega t + \theta) : R I_m \angle \theta$$

$$Z_R = \text{Impedance of resistance} = \frac{R \cdot I_m \angle \theta}{I_m \angle \theta} = R.$$

# Impedances for R, L and C




$$Z_R = R \qquad Z_L = j\omega L \qquad Z_C = \frac{1}{j\omega C} = -j\frac{1}{\omega C}$$

# Circuit Analysis with phasors and Complex Impedances

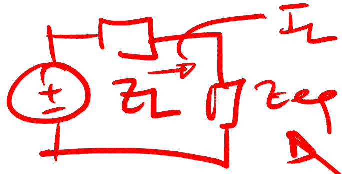
1 - Replace the sinusoidal voltage and current sources with the corresponding phasors.
2 - Replace the resistance, inductance and capacitance with their respective impedances.
3 - Analyze the circuits as before with DC sources and resistances only.
4 - Convert phasors back to the sinusoidals.

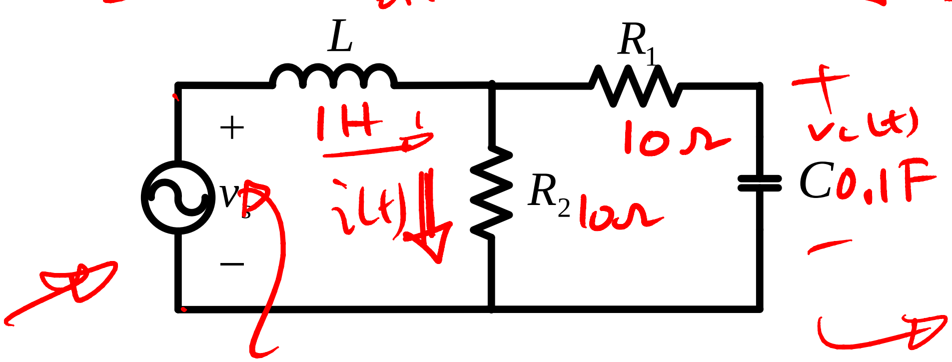
# Example

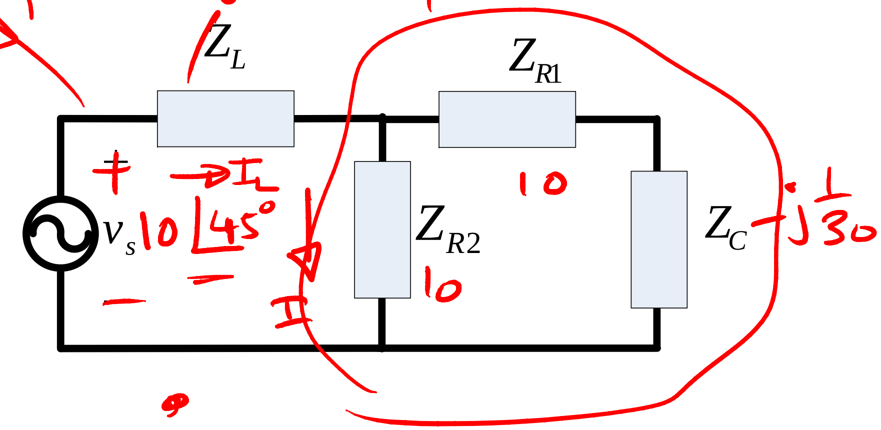
$$I_L = \frac{1.0\angle 45}{Z_L + Z_{eq}}$$

$$I = I_L \times \frac{Z_{R1}+Z_C}{Z_{R1}+Z_C+Z_{R2}}$$

$$Z_{eq} = \frac{Z_{R2}\times(Z_{R1}+Z_C)}{Z_{R2}+Z_{R1}+Z_C}$$




$10\cos(300t + 45°)$

$\omega = 300$ rad/sec

$L \to 1H$ $\quad Z_L = \omega L\angle 90° = j\omega L$

$= 300 \times 1 = j300$

$\angle 90°$

$C = 0.1F \to Z_C = -j\frac{1}{\omega C} = -j\frac{1}{300 \times 0.1}$

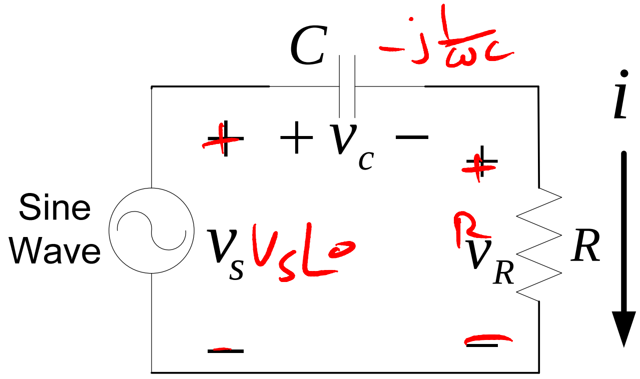
$= -j\frac{1}{30}$

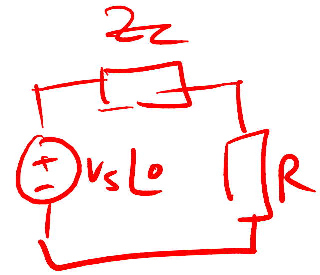
$Z_{R1} + Z_C = 10 - j\frac{1}{30}$

$I = I_x\angle\theta_x$

$i(t) = I_x \cdot \cos(300t + \theta_x)$

# Phase difference between $V_R$ and $V_S$ in RC circuit




$V_R$ , $V_s$

$|V_R|\angle\theta_r$ , $|V_s|\angle\theta_s$

$\theta_r - \theta_s$

say $\theta_s = 0$, $\Rightarrow \theta_r$ ?

phase diff.

$$\cancel{\theta = 90 - \tan^{-1}(\omega RC) = 90 - \tan^{-1}(2\pi f RC)}$$

$$\boxed{\theta = \Delta\Phi = \frac{\Delta t}{T} \times 360^\circ}$$

$|V_s| \cdot \angle 0$

$$V_R = |V_s|\angle 0 \times \left(\frac{R}{R - j\frac{1}{\omega C}}\right) = |V_s|\angle 0 \cdot \frac{R\angle 0}{\sqrt{R^2 + \frac{1}{\omega^2 C^2}} \times \angle\theta}$$

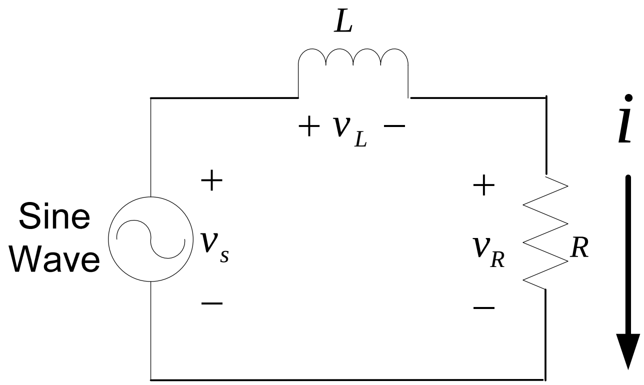
$$\theta = \tan^{-1}\left(\frac{-1}{\omega C R}\right) = -\tan^{-1}(1/\omega C R)$$

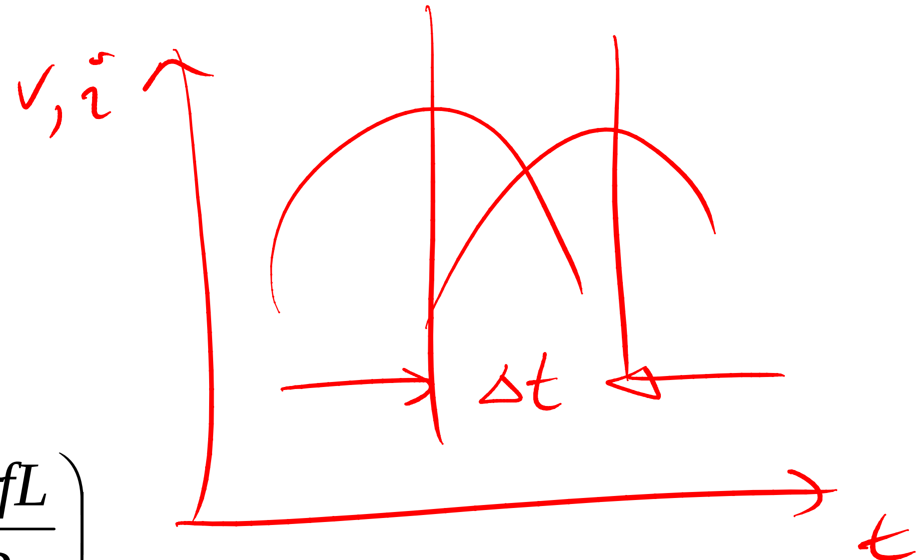
$$V_R = |V_R| \cdot \angle -\theta \quad \Rightarrow \theta_r = -\theta = +\tan^{-1}\frac{1}{\omega C R}$$

$\boxed{V_R \text{ leads } V_s}$

$\Delta\phi =$

# Phase difference between $V_R$ and $V_S$ in RL circuit




$$\theta = -\tan^{-1}\left(\frac{\omega L}{R}\right) = -\tan^{-1}\left(\frac{2\pi f L}{R}\right)$$

$$\boxed{\theta = \Delta\Phi = \frac{\Delta t}{T} \times 360^\circ}$$

$t \to T$

$\theta \to 2\pi$

$\frac{\Delta t}{T} \times 2\pi = \Delta\theta$

# Lab 5

- RMS voltage can be measured both by the oscilloscope and the multimter
  - Multimeter not so accurate at frequency beyond 1kHz
- Take note of Oscilloscope probe GND
  - Can not have more than one GND in circuit
- The last lab before the lab test
- Finish it early and pratice the last labs if you get time