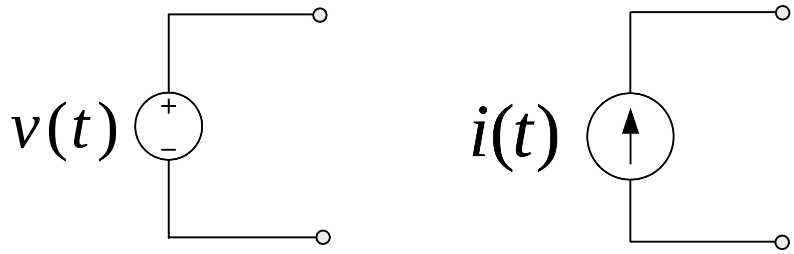
Contents

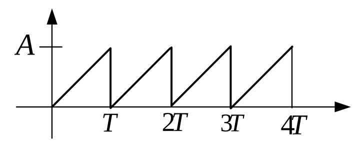
Steady-state sinusoidal analysis

Time-dependent Sources

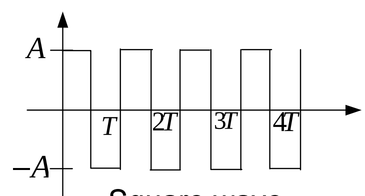
Ideal voltage and current sources can also be time-varying.



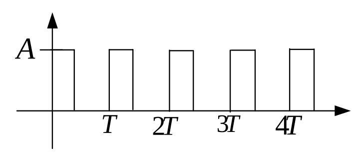
One of the most important classes of time-dependent signals is that of periodic signals, which satisfies the equation: $x(t) = x(t + nT), \qquad n = 1,2,3,\ldots$
where T is the time period of $x(t)$. An AC signal has zero average. A signal generator in the lab can output various periodic signals like Sinusoidal, Sawtooth, Square, Triangular etc.



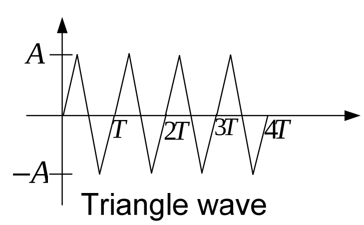
Sawtooth wave



Square wave



Pulse train

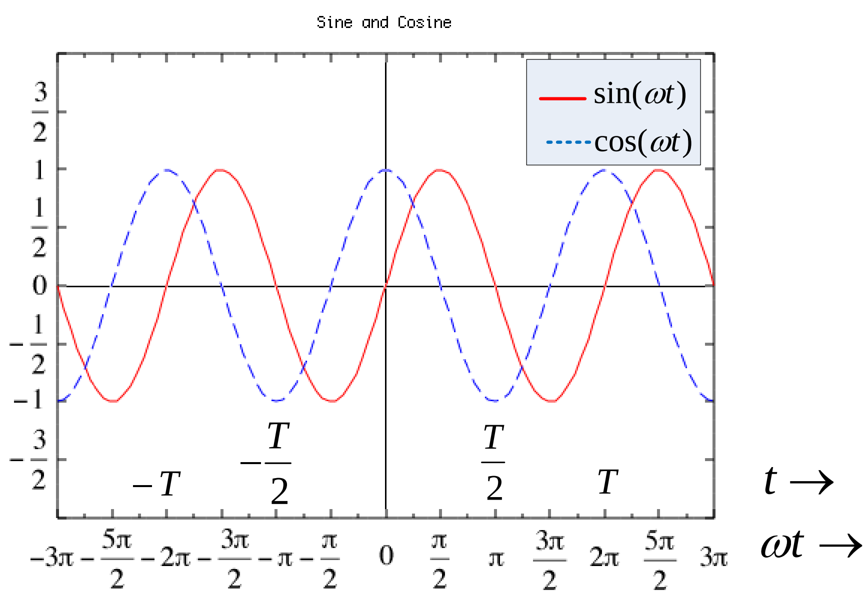


Triangle wave

Sinusoidal Currents and Voltages

A sinusoidal voltage is given by

$v(t) = V_m \cos(\omega t + \theta)$, where V_m is the **peak value** of the voltage, ω is the **angular frequency** and θ is the **phase angle**.



As time t progresses from zero to **time-period** T, the voltage goes through maximum positive and maximum negative, completing one cycle.

The **frequency** of the periodic signal is the number of cycles completed in one second.

$f = \frac{1}{T}$. The units of frequency are hertz (Hz).

Angular frequency is defined as $\omega = \frac{2\pi}{T} = 2\pi f$. One period in time corresponds to 2π radians i.e. one complete rotation of an equivalent vector. As time t progresses from zero to T, ωt goes from 0 to 2π.

Sinusoidal functions can be represented by both the cosine and the sine forms, which are inter changeable as $\sin(z) = \cos(z - 90^0)$ and $\cos(z) = \sin(z + 90^0)$. Hence, even though a bunch of sinusoidal signals may be represented by both sine and cosine forms, they are all converted to either cosine or sine for sake of uniformity. This also helps in reading their relative phase angles if all of them are same frequency.

Why Sinusoids?

Sinusoidal sources have many important applications. Electric power supply from the utility is sinusoidal in form. Sinusoidal signals have many uses in Radio communications. It is known that all periodic signals can be represented as sum of sinusoidal signals, by using Fourier series. Hence, the study of AC signals will be limited to the study of sinusoidal signals. The behavior of the circuits in steady-state when excited by sinusoidal signal sources will be studies in more details.

Root-mean-square values

As mentioned earlier, AC is time varying and has time-average value of zero. If an ac voltage $v(t)$ is applied across a resistor, then the instantaneous power dissipated in the resistor will be $p(t) = \frac{v^2(t)}{R}$.

Then, power also is a function of time. However, it is necessary to know the average power. Average power can be calculated from the total energy dissipated over one time period and then dividing it by the time period.

Energy dissipated in the resistor per period is given by

$$E_T = \int_0^T p(t)dt = \int_0^T \frac{v^2(t)}{R}dt$$

The average power dissipated in the resistor will be:

$$P_{avg} = \frac{E_T}{T} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T \frac{v^2(t)}{R}dt = \frac{\frac{1}{T}\int_0^T v^2(t)dt}{R}$$

Root-Mean-Square (RMS) value of the periodic voltage source: $V_{RMS} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt}$.

Then we can express the average power as $P_{avg} = \frac{V_{RMS}^2}{R}$

V_{rms} is known as the root-mean-square (RMS) value of the voltage source.

The root-mean-square for a periodic signal helps in calculation of average power in a resistor.

RMS of a sinusoid

Consider the sinusoidal voltage $v(t) = V_m \cos(\omega t + \theta)$.

The RMS value of this voltage will be

$$V_{rms} = \sqrt{\frac{1}{T}\int_0^T v^2(t)dt} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta)dt}$$

$$= \sqrt{\frac{1}{T}\frac{V_m^2}{2}\int_0^T (1+\cos 2(\omega t + \theta))dt} = \sqrt{\frac{V_m^2}{2}} = \frac{V_m}{\sqrt{2}}$$

The cosine function integrated over one time period will contribute zero to the integration.

Usually for sinusoidal signals, the RMS value is given rather than the peak value.

For example when we say the PUB supply in Singapore is 230V, we mean that the RMS value of the PUB supply is 230V and its peak value would be $V_m = \sqrt{2}V_{rms} = 230 \times \sqrt{2} = 230 \times 1.414 = 325V$

Phasors

As seen in earlier chapters on KVL and KCL, we need to add voltages and currents to solve a given DC circuit. However, when the voltage and current are sinusoidal, arithmetic becomes quite involved requiring repeated application of trigonometric substitutions. However, with phasors, the process is quite simplified. Handling of phasors requires familiarity with complex-number arithmetic.
http://en.wikipedia.org/wiki/Complex_number

Phasor definition:

Phasors are used to represent sinusoidal signals as complex numbers. For a sinusoidal voltage of $v_1(t) = V_1 \cos(\omega t + \theta)$, we define the phasor as $\mathrm{V}_1 = V_1 \angle \theta_1$. Thus, phasor of a sinusoid is a complex number having a magnitude equal to the peak value and having the same phase angle as the sinusoid.

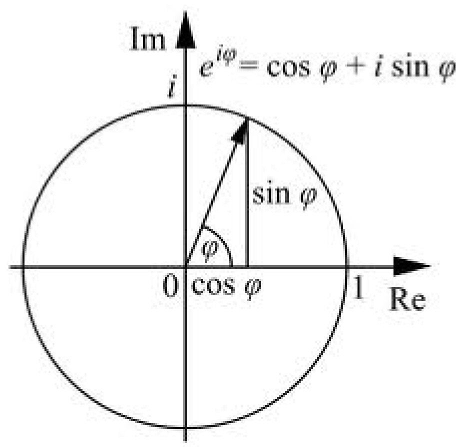
It is worth mentioning that in some conventions, the magnitude of phasor is taken to be the RMS value of the sinusoid instead of the peak value. It is important that the same convention is followed consistently.

Phasor is just a definition. This gives rise to mathematical convenience. It has no physical significance.

If the sinusoid is given in sine form as $v_2(t) = V_2 \sin(\omega t + \theta)$, then we first convert it to cosine function first as $v_2(t) = V_2 \sin(\omega t + \theta) = V_2 \cos(\omega t + \theta - 90^0)$. Its phasor would then be $\mathrm{V}_2 = V_2 \angle \theta - 90^0$.

The same applies to sinusoidal currents as well.

Euler's formula



Euler's identity relates a complex exponential to a complex number in rectangular form.
$e^{j\alpha} = \cos\alpha + j\sin\alpha$ or $\cos\alpha = \mathrm{Re}\left(e^{j\alpha}\right)$ i.e. the real part of the complex exponential and $\sin\alpha = \mathrm{Im}\left(e^{j\alpha}\right)$ i.e. the imaginary part of the complex exponential.

The Euler's identity can be used to relate a sinusoidal signal to a rotating complex number as shown below.

$$\cos(\omega t + \theta) = \mathrm{Re}\left(e^{j(\omega t + \theta)}\right)$$
$$\sin(\omega t + \theta) = \mathrm{Im}\left(e^{j(\omega t + \theta)}\right)$$

The complex exponential $e^{j(\omega t + \theta)}$ can be thought of being a complex number which rotates with an angular velocity of $\omega\, rad/\sec$.

Phasor related to a rotating vector

$V_m e^{j(\omega t+\theta)}$ represents a vector in the complex plane whose magnitude is V_m and whose angle with the real axis is ($\omega t+\theta$). The angle changes with time or in other words the vector keeps rotating at the angular speed of ω rad/sec. and its angle at t=0 was θ. The projection of the vector along the real axis will be the original sinusoidal function $V_m \cos(\omega t+\theta)$.

The phasor for $\cos(\omega t+\theta)$ is $V_m e^{j\theta}$ which can be thought of as a snap-shot of the rotating vector at t=0.

Phase relationships

Consider two voltages as

$v_1(t) = 3\cos(\omega t + 40^0)$

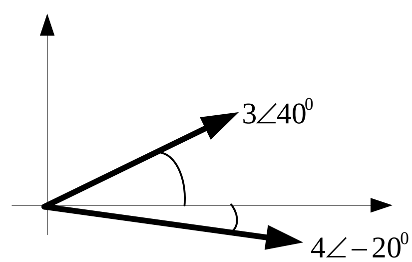
$v_2(t) = 4\cos(\omega t - 20^0)$

The two phasors would be:

$3\angle 40^0$

$4\angle -20^0$

We can draw the phasor diagram for these two.



We can also measure the phase difference between two sinusoidal waveforms in Oscilloscope.

Phase difference between two sinusoids: $\Delta\phi = \dfrac{\Delta t}{T} \times 360^0 = \Delta t \times f \times 360^0$, where Δt is the time difference between their peaks, T is the time period of the signals, f is the frequency of the signals.

Leading / lagging

The signal that reaches the peak earlier is said to be leading. Thus, cosine leads sine by 90^0.
However, one may say that sine leads cosine by 270^0.

To resolve this confusion, we first find the phase difference as $\theta_1 - \theta_2$. Then, if it is positive and less than 180 degree, then v1 is leading v2. Else, v1 is lagging v2.

Impedances

In AC circuits, the fundamental elements like resistance, inductance and capacitance are replaced by what is called impedance. Impedance is also known as complex resistance, or frequency-dependent resistance.

With the sinusoidals represented by their phasors and the circuit elements being represented by their impedances, the ac circuit analysis can be reduced to DC circuit analysis for resistive circuits. This is very useful for AC circuit analysis.

Impedance of Inductance

When an inductor carries a sinusoidal current, there will be a sinusoidal voltage across it. If the inductor current is $i_L(t) = I_m \sin(\omega t + \theta) = I_m \cos(\omega t + \theta - 90^0)$, then the voltage across the inductance will be:

$$v_L(t) = L\frac{di_L(t)}{dt} = L\omega I_m \cos(\omega t + \theta)$$

Putting both current and voltage sinusoidal in phasor form:

Current phasor will be $\mathrm{I_L} = I_m \angle \theta - 90^0$

Voltage phasor will be $\mathrm{V_L} = \omega L I_m \angle \theta = V_m \angle \theta$

We can rewrite the voltage phasor as the current phasor multiplied by its impedance (similar to voltage across a resistor being $V = RI$).

$$\mathrm{V_L} = \omega L I_m \angle \theta = \omega \mathrm{L} \angle 90^0 \times I_m \angle \theta - 90^0 = \mathrm{Z_L} \times \mathrm{I_L}.$$

Thus, the impedance for inductance will be $Z_L = \omega L \angle 90^0 = j\omega L$. This is a purely imaginary quantity.

And $V_L = Z_L I_L$ is the Ohm's law in phasor form.

Impedance of Capacitance

Similar to inductances, for capacitances, if current and voltages are sinusoidal, the phasors are related by $V_c = Z_c I_c$

It can be shown that $Z_c = -j\dfrac{1}{\omega C} = \dfrac{1}{j\omega C} = \dfrac{1}{\omega C}\angle -90^0$.

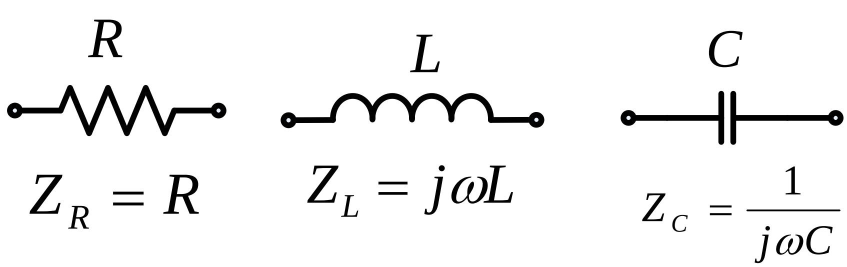
The impedance of capacitance is also purely imaginary.

Note that impedance of inductance is positive imaginary whereas the impedance of capacitance is negative imaginary. Impedances which are purely imaginary are called reactances.

Impedance of Resistance

For a resistance, the phasors are related by $V_R = RI_R$.

This implies, the sinusoidal voltage and current for a resistor will be in phase.



If a circuit branch has a series combination of resistance with an inductance or capacitance, then the effective impedance will be a complex number as shown later in the example.

Circuit Analysis with phasors and Complex Impedances

Kirchoff's laws in Phasor form

For a circuit with sinusoidal voltages, the sum of voltages will be equal to zero.

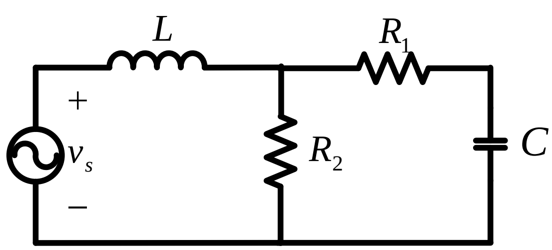
$v_1(t) + v_2(t) - v_3(t) = 0$

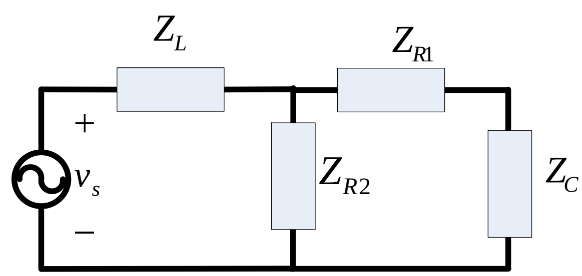
This can be rewritten in terms of their phasors as:

$V_1 + V_2 - V_3 = 0$.**The resulting equations will have complex numbers.**

Step-by-step procedures for steady-state analysis of circuits with sinusoidal sources:

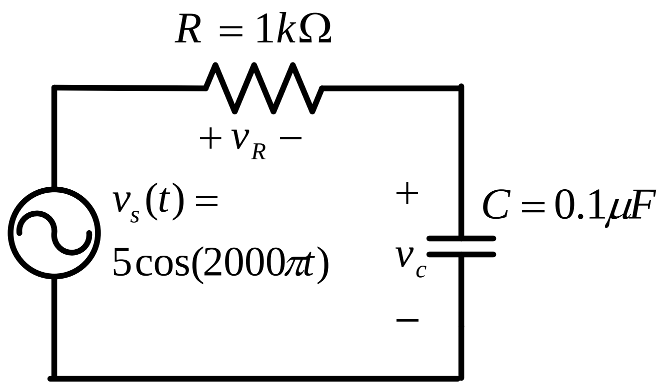
1. Replace the time descriptions of the voltage and current sources with the corresponding phasors. (All sources must have the same frequency).
2. Replace the inductance with its impedance $Z_L = j\omega L$ and capacitance with its impedance of $Z_C = -j\dfrac{1}{\omega C}$. Resistances have the same impedance as their resistance.
3. Analyze the circuits as before with DC sources and resistances only. Note that we have to do all operations involving complex numbers. Hence, it is important to be confident of such complex number manipulations.
4. Convert the final results from phasor form to time-domain (sinusoidal) form.





Example:

Find the voltages across the resistor and the capacitor using the complex impedances and the phasor.



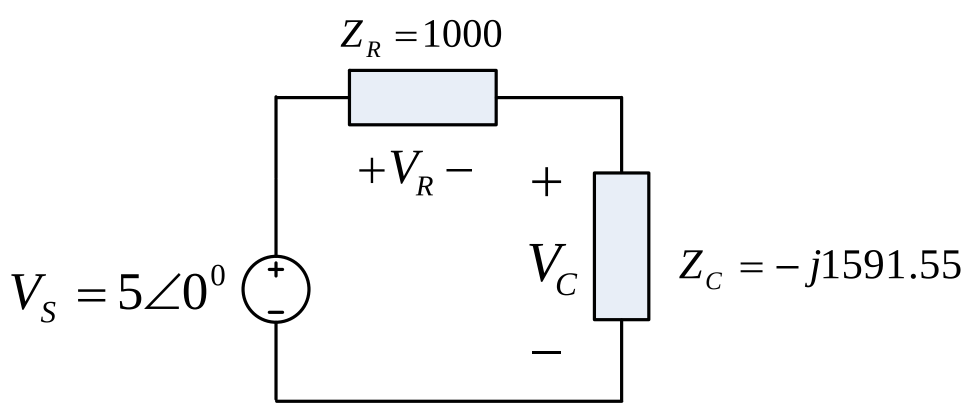
Solution:

We first need to convert the source voltage into its phasor and the elements into their complex impedances.

The voltage source $v_s(t) = 5\cos(2000\pi t)$ has maximum value of 5, angular frequency of 2000π and phase angle of 0^0. Hence its phasor will be $V_s = 5\angle 0^0$.

The impedance of the resistor will be $Z_R = R = 1000$.

The impedance of the capacitor will be $Z_C = \dfrac{1}{j\omega C} = \dfrac{1}{j2000\pi \times 0.1 \times 10^{-6}} = -j1591.55$



Then, we shall treat the equivalent circuit as a DC circuit and apply the DC circuit principles to solve the circuit.

We can apply the voltage divider rule and find the voltage across each element:

$$V_C = \frac{Z_C}{Z_R + Z_C} V_S$$

Of course, we have to deal with complex algebra here. We have to alternate between rectangular and polar form for the complex numbers to do the addition and division.

$$\frac{Z_C}{Z_R + Z_C} = \frac{-j1591.55}{1000 - j1591.55} = \frac{1591.55\angle -90^0}{1879.64\angle -57.86^0} = 0.847\angle -32.14^0$$

Finally the phasor of the capacitor voltage will be $V_C = 0.847\angle -32.40^0 \times 5\angle 0^0 = 4.235\angle -32.40^0$.

Comparing source voltage and the capacitor voltage phasors:

$$V_S = 5\angle 0^0$$
$$V_C = 4.235\angle -32.40^0$$

The equivalent time domain signals would be:

$$V_S = 5\angle 0^0 \Rightarrow v_s(t) = 5\cos(2000\pi t)$$
$$V_C = 4.235\angle -32.40^0 = 4.235\cos(2000\pi t - 32.40^0)$$

Thus, the voltage across the capacitor will be smaller in magnitude than the source voltage and will be lagging the source voltage.

Similarly, we can find the voltage across the resistor as:

$$V_R = \frac{Z_R}{Z_R + Z_C} V_S$$

$$\frac{Z_C}{Z_R + Z_C} = \frac{1000}{1000 - j1591.55} = \frac{1000\angle 0^0}{1879.64\angle -57.86^0} = 0.532\angle 57.86^0$$

Finally the phasor of the resistor voltage will be $V_R = 0.532\angle 57.86^0 \times 5\angle 0^0 = 2.66\angle 57.86^0$.

The equivalent time domain signal would be:

$v_R(t) = 2.66\cos(2000\pi t + 57.86^0)$.

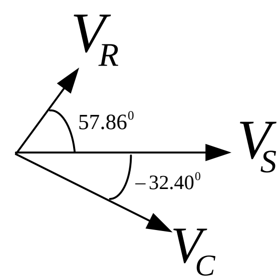
The three voltage phasors are:

$V_S = 5\angle 0^0$

$V_C = 4.235\angle -27.40^0$

$V_R = 2.66\angle 57.86^0$

Putting them in a phasor diagram:



When we measure the RMS values of the voltages across the source, resistor and capacitor, we get the magnitudes only. The KVL should not then hold with the magnitudes alone as it does not contain the phase information. AC signals are time-dependent, and **KVL and KCL hold at each time instant**. **KVL and KCL also hold with the phasors.**

Power Factor

In AC power system, if the load is inductive or capacitive, then the load current will have a phase difference with the source voltage.

$$I_L = \frac{V_S}{Z_L} = \frac{\mathrm{V_S}\angle 0}{R + jX} = \frac{\mathrm{V_S}\angle 0}{\sqrt{R^2 + X^2}\angle\phi} = \frac{\mathrm{V_S}}{\sqrt{R^2 + X^2}}\angle -\phi$$

$$\phi = \tan^{-1}\left(\frac{X}{R}\right)$$

$$v_s(t) = V_S\cos(\omega t)$$

$$i_L(t) = I_L\cos(\omega t - \phi)$$

If the load is inductive, $X_L = \omega L$ and ϕ is positive. Then the current phasor for inductive laod will be lagging the source voltage phasor.

If the load is capacitive, $X_C = -\dfrac{1}{\omega C}$ and ϕ is negative. Then the current phasor for capacitive load will be leading the source voltage phasor.

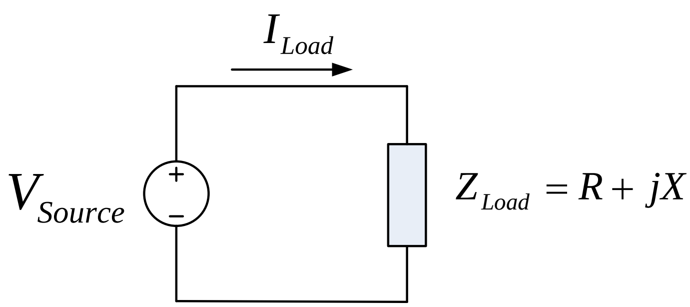


Figure : Load connected to AC source

Instantaneous and Average Power

Instantaneous power,

$$p(t) = v(t) \times i(t) = V_S \cos\omega t \times I_L \cos(\omega t - \phi) = V_S I_L \cos\omega t \times \cos(\omega t - \phi)$$

Average power,

$$P_{av} = \frac{1}{T}\int_0^T p(t)dt = \frac{1}{T}\int_0^T V_S I_L \cos\omega t \times \cos(\omega t - \phi)$$

$$= \frac{1}{T}\int_0^T V_S I_L \cos\omega t \times \cos(\omega t - \phi) = \frac{V_S I_L}{T}\int_0^T \frac{\cos(2\omega t - \phi) + \cos\phi}{2} dt$$

$$= \frac{V_S I_L}{2}\cos\phi = \frac{V_S}{\sqrt{2}}\frac{I_L}{\sqrt{2}}\cos\phi = V_{S,RMS} \times I_{L,RMS}\cos\phi$$

The units of average power are watt (W).

Apparent power: $V_{S,RMS} \times I_{L,RMS}$. The units of apparent power are volt-ampere (VA).

Power factor = $\cos\phi$ **.**

Power factor is said to be lagging if current lags the voltage. If current leads the voltage, then it is leading power factor.