EE1002: Introduction to Circuits and Systems

Chapter 1: Power in AC Circuits

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Learning Objectives

- Understanding the meaning of *instantaneous* and *average* power for AC circuits.
- Understand the complex power notation: compute *real* (*active*), *reactive* and *apparent* power for complex loads.
- Understanding the concepts of *power factor* and *improvement of power factor.*
- Draw the *power triangle*, and *compute the capacitor size* to perform power factor corrections on a load.

Introduction to Phasors

- For *AC* circuits, the sinusoidal steady-state analysis is made easy if currents and voltages are represented as phasors in the complex-number plane.
- For a sinusoidal voltage of the form

$$v_1(t) = V_1 \cos(\omega t + \theta_1)$$

we define the phasor as

$$\mathrm{V}_1 = V_1 \angle \theta_1$$

- We can add and subtract sinusoidal using phasors.

Fig. 4.1 Circuit symbol for an n-channel MOSFET.

Phasors as Rotating Vectors

- For a sinusoidal voltage of the form

$$v(t) = V_m \cos(\omega t + \theta) = \mathrm{Re}\left[V_m e^{j(\omega t + \theta)}\right] = V_m \angle(\omega t + \theta)$$

- The phasor can be visualized as a vector of length V_m that rotates counterclockwise in the complex plane with an angular velocity of ω rad/sec.
- As the vector rotates, its projection on the real-axis traces out the voltage as a function of time as shown in Fig. 1.1.
- The phasor is simply a snapshot of the rotating vector at $t = 0$.

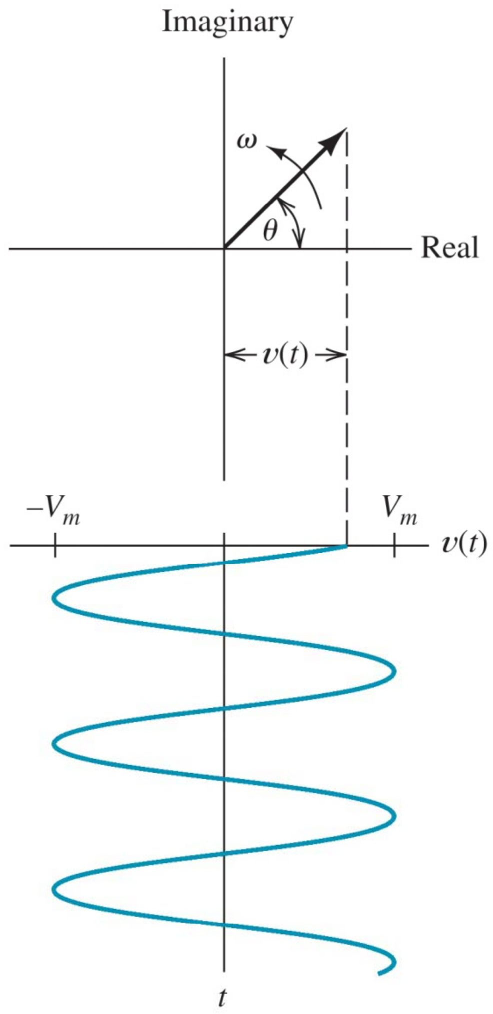


Figure 1.1 A sinusoid can be represented as the real part of a vector rotating counterclockwise in the complex plane.

Power in AC Circuits

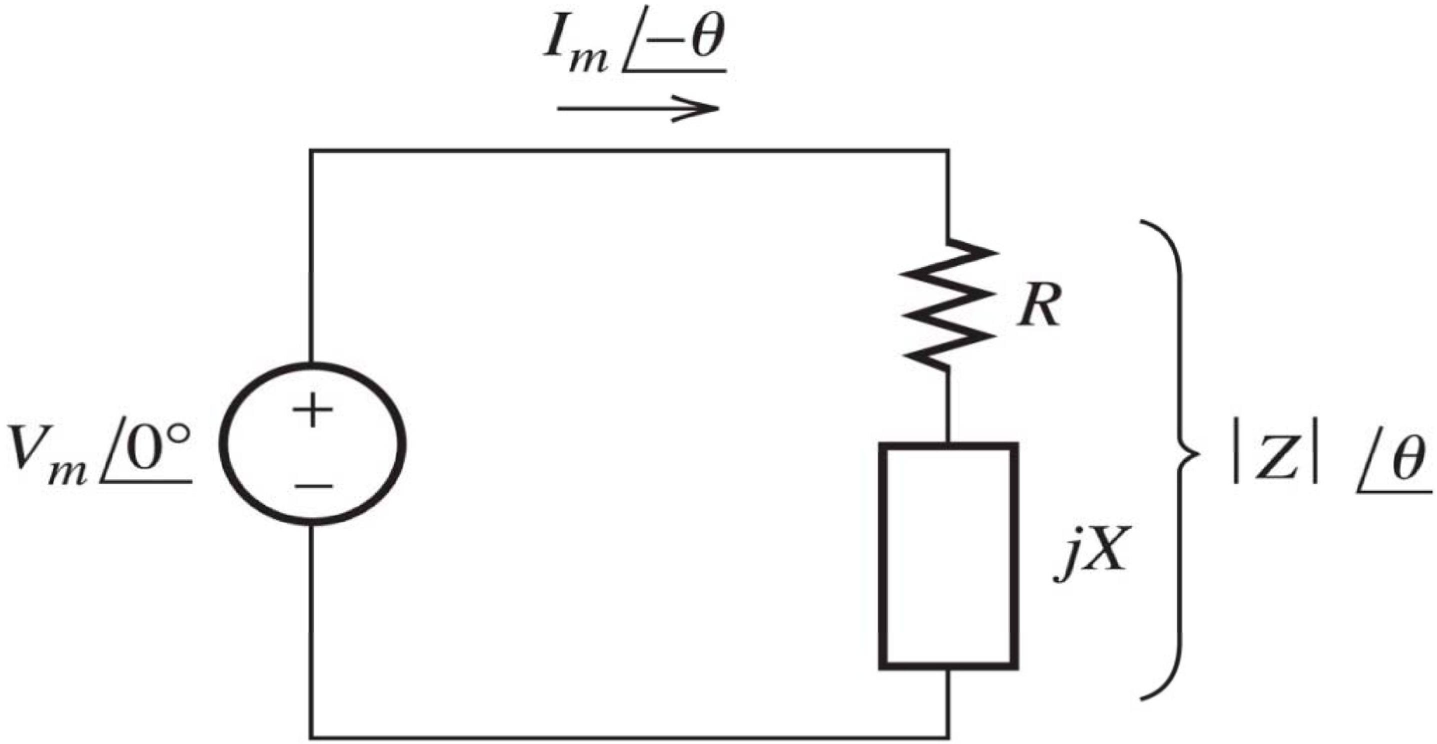


Figure 1.1 A voltage source delivering power to a load impedance *Z* = *R* + *jX*.

$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_m \angle 0^0}{|\mathbf{Z}| \angle \theta} = I_m \angle -\theta \qquad (1.1)$$

Current, Voltage and Power for a Resistive Load

- Let us consider that the load network is pure resistive then $|Z| = R \; and \; \angle\theta = 0$

$$v(t) = V_m \cos(\omega t)$$
$$i(t) = I_m \cos(\omega t)$$

$$p(t) = v(t)i(t) = V_m \cos(\omega t) \times I_m \cos(\omega t) = V_m I_m [\cos(\omega t)]^2 \qquad (1.2)$$

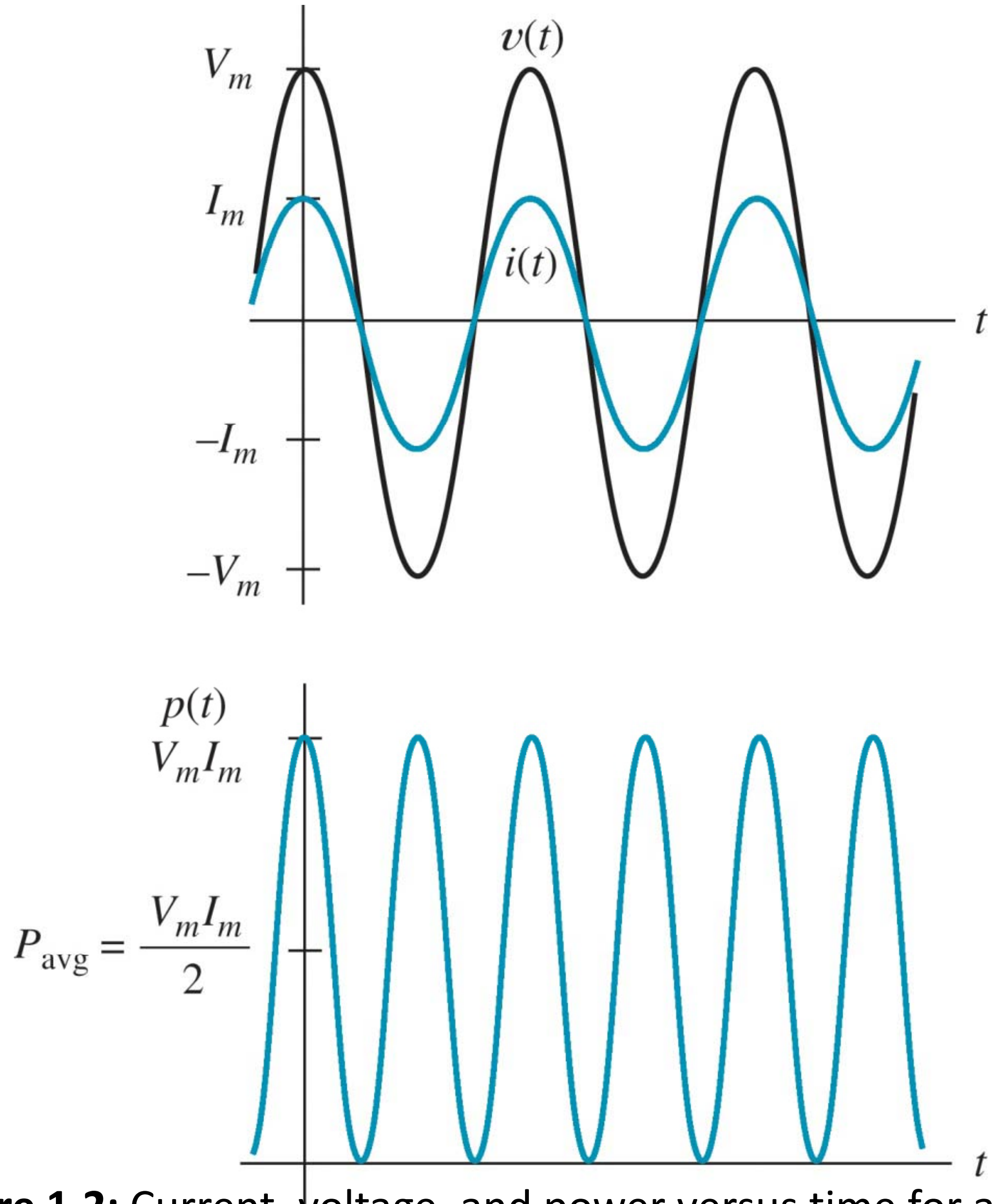
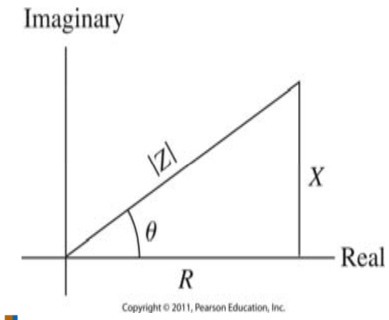


Figure 1.2: Current, voltage, and power versus time for a purely resistive load.

Current, Voltage and Power for a Inductive Load

- Let us consider that the load network is pure inductive then $|Z| = \omega L \text{ and } \angle\theta = 90^0$



$$v(t) = V_m \cos(\omega t)$$

$$i(t) = I_m \cos(\omega t - 90^0) = I_m \sin(\omega t)$$

$$p(t) = v(t)i(t) = V_m \cos(\omega t) \times I_m \sin(\omega t) = V_m I_m \sin(\omega t)\cos(\omega t) = \frac{V_m I_m}{2}\sin(2\omega t) \qquad (1.3)$$

- **Note that the energy over an electrical cycle is zero i.e. a pure inductor does not consume any electrical power from the source over a cycle..**

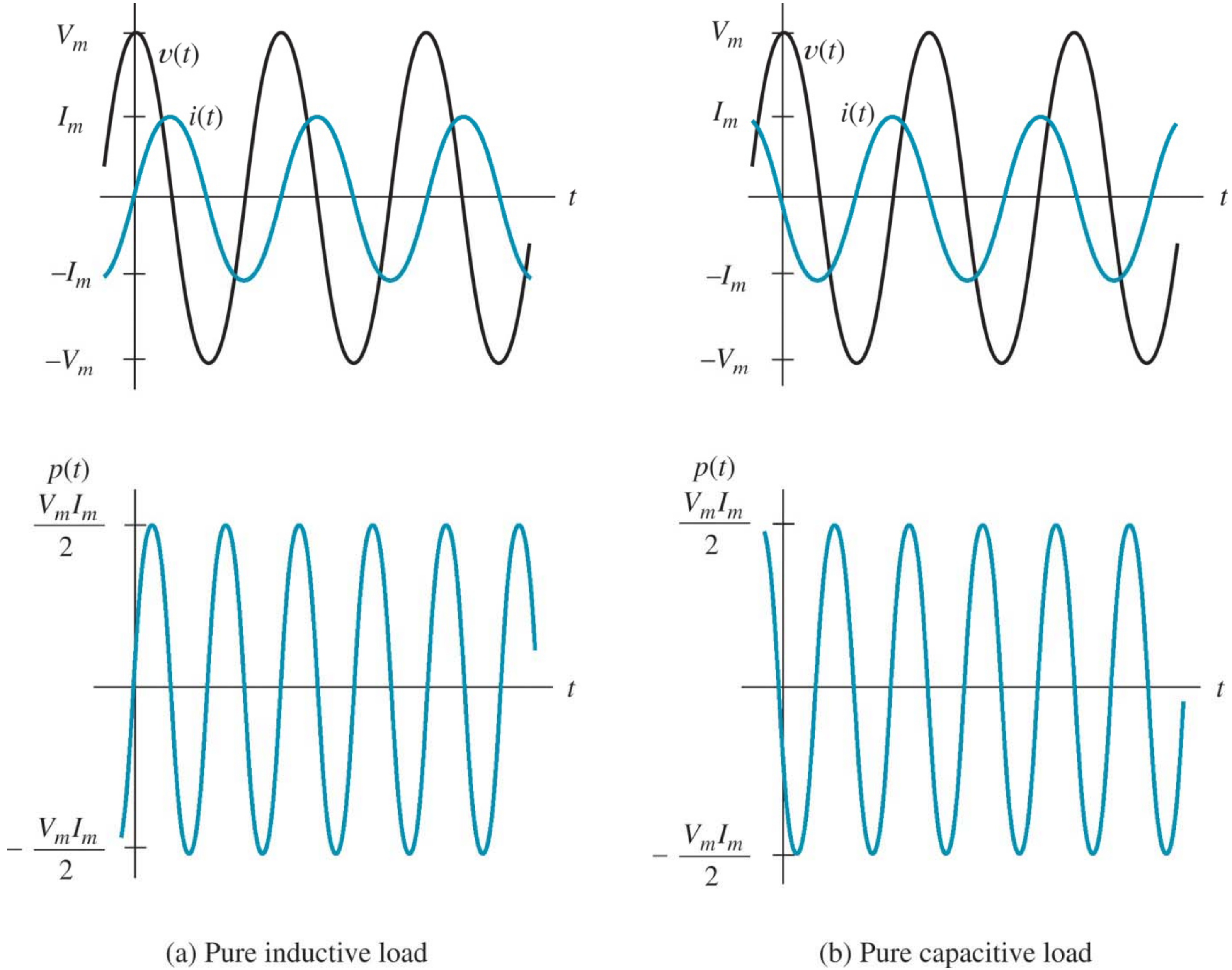


Figure 1.3: Current, voltage, and power versus time for pure energy-storage elements.

Current, Voltage and Power for a Capacitive Load

- Let us consider that the load network is pure inductive then $|Z| = \frac{1}{\omega C}$ $and\ \angle\theta = -90^0$

$$v(t) = V_m \cos(\omega t)$$

$$i(t) = I_m \cos(\omega t + 90^0) = -I_m \sin(\omega t)$$

$$p(t) = v(t)i(t) = V_m \cos(\omega t) \times (-I_m \sin(\omega t)) = -V_m I_m \sin(\omega t)\ \cos(\omega t) = -\frac{V_m I_m}{2} \sin(2\omega t)\ (1.4)$$

- **Note that the energy over an electrical cycle is zero i.e. a pure capacitor does not consume any electrical power from the source over a cycle.**

Reactive Power and Its Importance

- Even if there is no average power is consumed by the pure energy storage elements there is a continuous reactive power flow either from the source to the load or from the load to the source on an instantaneous basis.
- This is of concerns to the Power Systems engineers because they have to design the transmission lines, transformers, fuses etc. that must be able to withstand the reactive power flow and the currents associated with the reactive power.
- Thus, the power supply companies charge the customers for their reactive power consumption.

Power Calculations for a General Load

- Let us consider the case of a general R-L-C load for which the phase angle of the complex load can vary and have any value from -90^0 to +90^0.

$$v(t) = V_m \cos(\omega t) \qquad i(t) = I_m \cos(\omega t - \theta)$$

$$p(t) = v(t)i(t) = V_m \cos(\omega t) \times I_m \cos(\omega t - \theta) = V_m I_m \cos(\omega t)\left[\cos(\omega t)\cos(\theta) + \sin(\omega t)\sin(\theta)\right]$$

$$= V_m I_m \cos(\theta)\cos(\omega t)^2 + V_m I_m \sin(\theta)\sin(\omega t)\cos(\omega t)$$

$$= V_m I_m \cos(\theta)\left\{\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right\} + V_m I_m \sin(\theta)\frac{1}{2}\sin(2\omega t)$$

$$= \left[\frac{V_m I_m}{2}\cos(\theta)\right] + \left[\frac{V_m I_m}{2}\cos(\theta)\cos(2\omega t)\right] + \left[\frac{V_m I_m}{2}\sin(\theta)\sin(2\omega t)\right] \quad (1.5)$$

- The first term in the eqn(1.5) has a constant value for a given ac source voltage and for a given load and represents the average power consumed by the load.

$$P = \frac{V_m I_m}{2}\cos(\theta) = \frac{V_m}{\sqrt{2}}\frac{I_m}{\sqrt{2}}\cos(\theta) = V_{rms} I_{rms}\cos(\theta) \quad Watts(W) \quad (1.6)$$

Power Factor

- The term *cos(*θ*)* is called the **power factor**, it is the cosine of the phase angle between the source voltage *v(t)* and the current through the load, *i(t)*.

$$\theta = \theta_v - \theta_i, \qquad \cos(\theta) = \cos(\theta_v - \theta_i) \quad (1.7)$$

- Sometimes, θ is also called as the **power angle**.
- The power factor varies between 0 and 1 and it can be said as *leading* or *lagging* depending on whether the current is *lagging* the supply voltage(in case of resistive-inductive load) and *leading* the supply voltage (in case of resistive-capacitive load).

Reactive Power

- The power associated with the energy storage elements contained in a general load is called as **reactive power** and is given by

$$Q = V_{rms} I_{rms} \sin(\theta) \quad Volt\,Ampere\,Reactive(VARs) \quad (1.8)$$

- Note that in the case of a resistive element, R, $\theta = 0$ and thus $Q = 0$, whereas in case of pure inductive load element, L, $\theta = +90^0$ and thus $Q = +V_{rms} I_{rms}$, and in case of pure capacitive load element, C, $\theta = -90^0$ and thus $Q = -V_{rms} I_{rms}$, .

Apparent Power

- The **apparent power,** which is defined as the product of the rms values of the voltage and the current and is given by

$$\text{apparent power} = V_{rms} I_{rms} \quad Volt\ Ampere\ (VAs) \quad (1.9)$$

Using eqns. (1.6) and (1.8) we have

$$P^2 + Q^2 = (V_{rms} I_{rms} \cos(\theta))^2 + (V_{rms} I_{rms} \sin(\theta))^2 = (V_{rms} I_{rms})^2 \ (1.10)$$

Power Triangle

- The relationships between, active power, P, reactive power, Q, apparent power, S and the power angle, θ can be represented by the **power triangle** as shown in Fig. 1. 4.

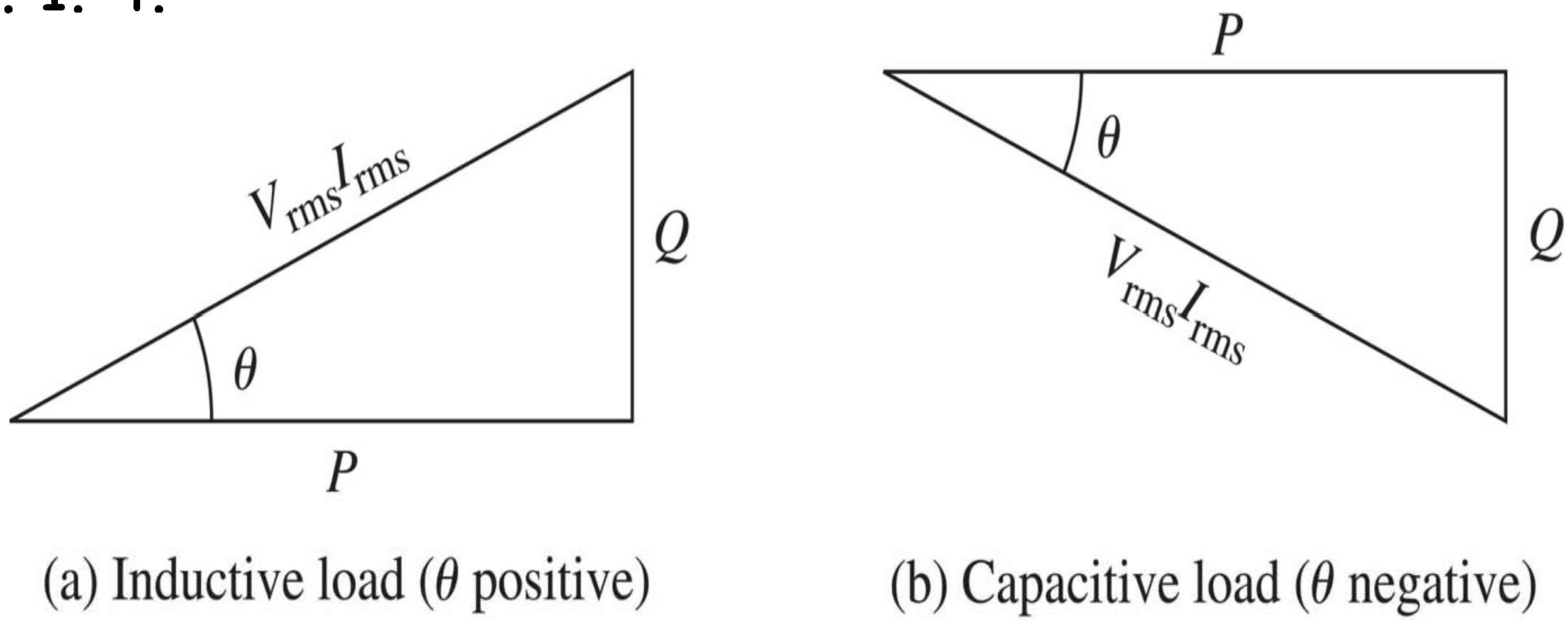


Figure 1.4 Power triangles for inductive and capacitive loads.

Additional Power Relationships

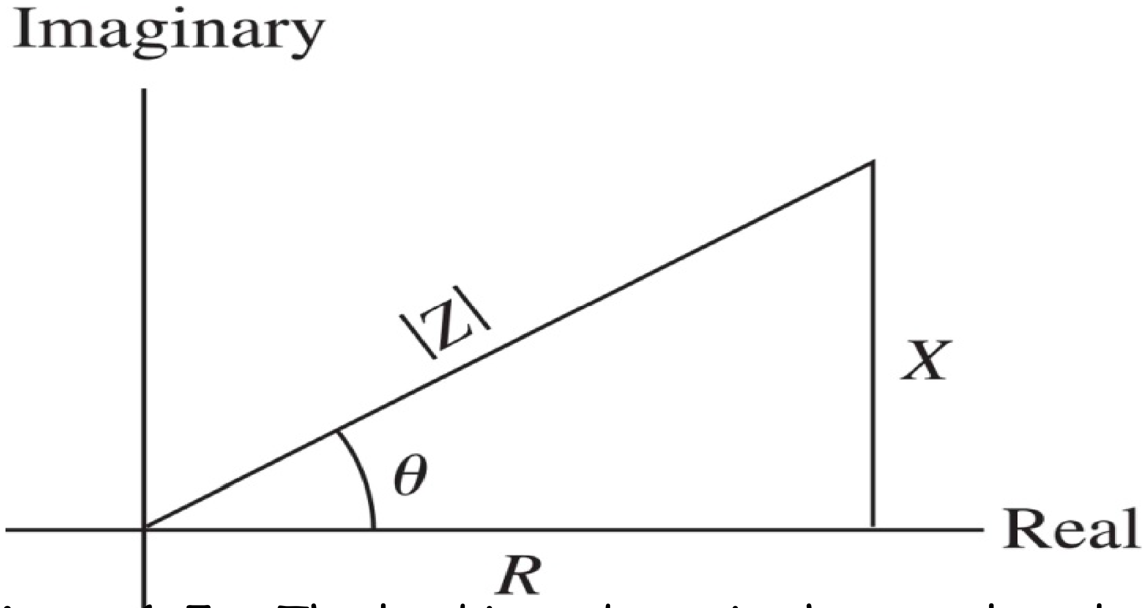


Figure 1.5 The load impedance in the complex plane.

$$Z = |Z|\angle\theta = R + jX, \qquad \cos(\theta) = \frac{R}{|Z|} \ (1.11), \qquad \sin(\theta) = \frac{X}{|Z|} \ (1.12)$$

$$P = \frac{V_m I_m}{2}\cos(\theta) = \frac{V_m I_m}{2} \times \frac{R}{|Z|} = \frac{I_m^2}{2} R = \frac{I_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}} \times R = I_{rms}^2 \times R = \frac{V_{rms}^2}{R^2} \times R = \frac{V_{rms}^2}{R} \ (1.13)$$

$$Q = \frac{V_m I_m}{2}\sin(\theta) = \frac{V_m I_m}{2} \times \frac{X}{|Z|} = \frac{I_m^2}{2} X = \frac{I_m}{\sqrt{2}} \times \frac{I_m}{\sqrt{2}} \times X = I_{rms}^2 \times X = \frac{V_{X,rms}^2}{X^2} \times X = \frac{V_{X,rms}^2}{X} \ (1.14)$$

Complex Power

- The **complex power, S** delivered to the load is defined as one half of the product of the phasor voltage, **V** and the complex conjugate of the phasor current, **I***.

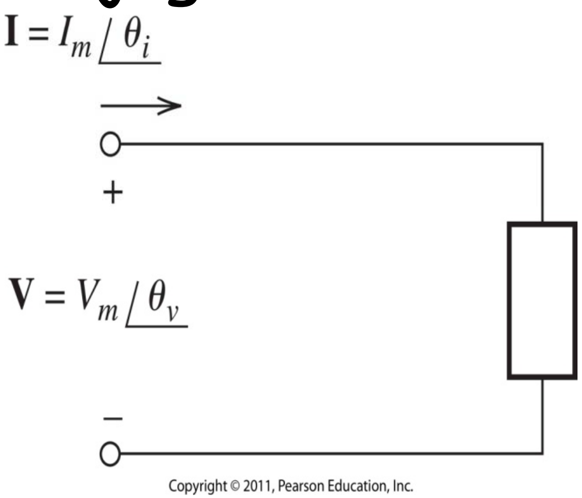


Figure 1.5 The load impedance in the complex plane.

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}(V_m \angle \theta_v) \times (I_m \angle -\theta_i) = \frac{V_m I_m}{2}\angle(\theta_v - \theta_i = \frac{V_m I_m}{2}\angle\theta \quad (1.15)$$

$$= \frac{V_m I_m}{2}\cos(\theta) + j\frac{V_m I_m}{2}\sin(\theta)$$

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = P + jQ \quad (1.17)$$

- If we know the complex power, **S** then we can compute the active power, *P*, reactive power, *Q* and apparent power, *S* as follows.

$$P = \mathrm{Re}(\mathbf{S}) = \mathrm{Re}\left(\frac{1}{2}\mathbf{V}\mathbf{I}^{*}\right) \quad (1.18)$$

$$Q = \mathrm{Im}(\mathbf{S}) = \mathrm{Im}\left(\frac{1}{2}\mathbf{V}\mathbf{I}^{*}\right) \quad (1.19)$$

$$\text{apparent power} = |\mathbf{S}| = \left|\frac{1}{2}\mathbf{V}\mathbf{I}^{*}\right| \quad (1.20)$$

Example 1.1

Consider the circuit as shown in Fig. 1. 7. Find the phasor current through each element. Compute the active and reactive power delivered to each element in the circuit.

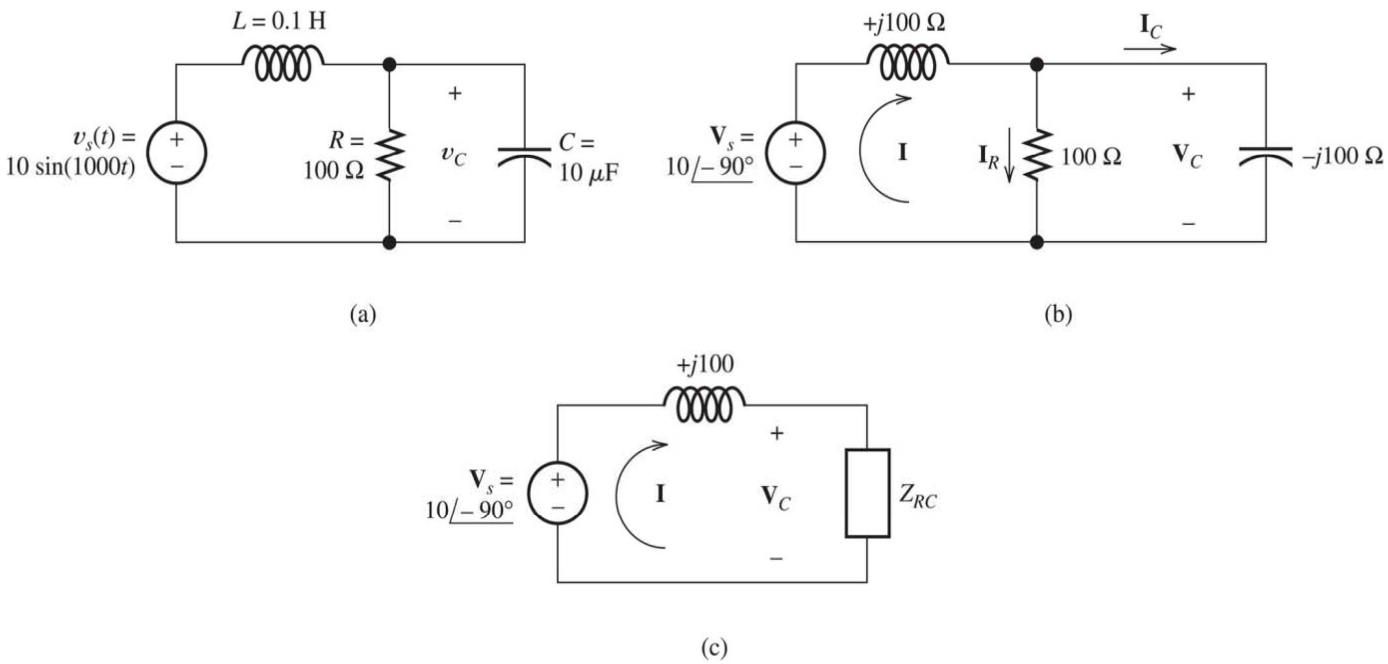


Figure 1.7: Circuit for Example 1.1.

- The phasor voltage of the source is $\mathbf{V}_s = 10\angle -90^0$

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$$Z_L = j\omega L = j1000 \times 0.1 = j100\ \Omega$$

$$Z_C = \frac{-j}{\omega C} = \frac{-j}{1000 \times 10 \times 10^{-6}} = -j100\ \Omega$$

$$Z_{RC} = \frac{1}{\frac{1}{R} + 1/Z_c} = \frac{1}{\frac{1}{100} + 1/(-j100)} = \frac{1}{0.01 + j0.01} = \frac{1\angle 0^0}{0.01414\angle 45^0} = 70.71\angle -45^0 = 50 - j\,50\ \Omega$$

$$\mathbf{V}_C = \mathbf{V}_S \frac{Z_{RC}}{Z_L + Z_{RC}} = 10\angle -90^0 \frac{70.71\angle -45^0}{j100 + 50 - j50} = 10\angle -90^0 \frac{70.71\angle -45^0}{50 + j50} = 10\angle -90^0 \frac{70.71\angle -45^0}{70.71\angle 45^0} = 10\angle - 180^0 = 10\cos(1000t - 180^0$$

$$\mathbf{I} = \frac{\mathbf{V}_s}{Z_L + Z_{RC}} = \frac{10\angle -90^0}{50 + j50} = \frac{10\angle -90^0}{70.71\angle 45^0} = 0.1414\angle -135^0$$

$$\mathbf{I}_R = \frac{\mathbf{V}_c}{R} = \frac{10\angle -180^0}{100} = 0.1\angle -180^0$$

$$\mathbf{I}_C = \frac{\mathbf{V}_c}{Z_C} = \frac{10\angle -180^0}{-j100} = \frac{10\angle -180^0}{100\angle -90^0} = 0.1\angle -90^0$$

- The power angle, θ is given by $\theta = \theta_v - \theta_i = -90^0 - (-135^0) = 45^0$

$$V_{rms} = \frac{V_m}{\sqrt{2}} = \frac{10}{\sqrt{2}} = 7.071\,V \qquad I_{rms} = \frac{I_m}{\sqrt{2}} = \frac{0.1414}{\sqrt{2}} = 0.1\,A$$

$$P = V_{rms} I_{rms} \cos(\theta) = 7.071 \times 0.1 \times \cos(45^0) = 0.5\, Watts(W)$$

$$Q = V_{rms} I_{rms} \sin(\theta) = 7.071 \times 0.1 \times \sin(45^0) = 0.5\ (VARs)$$

- Alternatively, from the complex power calculations

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}(10\angle -90^0)(0.1414\angle -135^0) = 0.707\angle -135^0 = 0.5 + j0.5 = P + j\,Q$$

$$P = \mathrm{Re}(\mathbf{S}) = 0.5\ \mathrm{W} \qquad Q = \mathrm{Im}(\mathbf{S}) = 0.5\ \mathrm{VAR}$$

$$Q_L = I_{rms}^2 \times X_L = (0.1)^2 \times (100) = 1\,VAR$$

$$Q_C = I_{C,rms}^2 \times X_C = \left(\frac{0.1}{\sqrt{2}}\right)^2 \times (-100) = 0.5\,VAR$$

$$P_R = I_{rms}^2 \times R = \left(\frac{0.1}{\sqrt{2}}\right)^2 \times 100 = 0.5\,W\,, \qquad P_L = P_C = 0$$

Example 1.2

Consider the circuit as shown in Fig. 1. 8. The voltage source delivers power to two loads connected in parallel. Find the active power, reactive power and power factor for the source. Also find the phasor current, **I**.

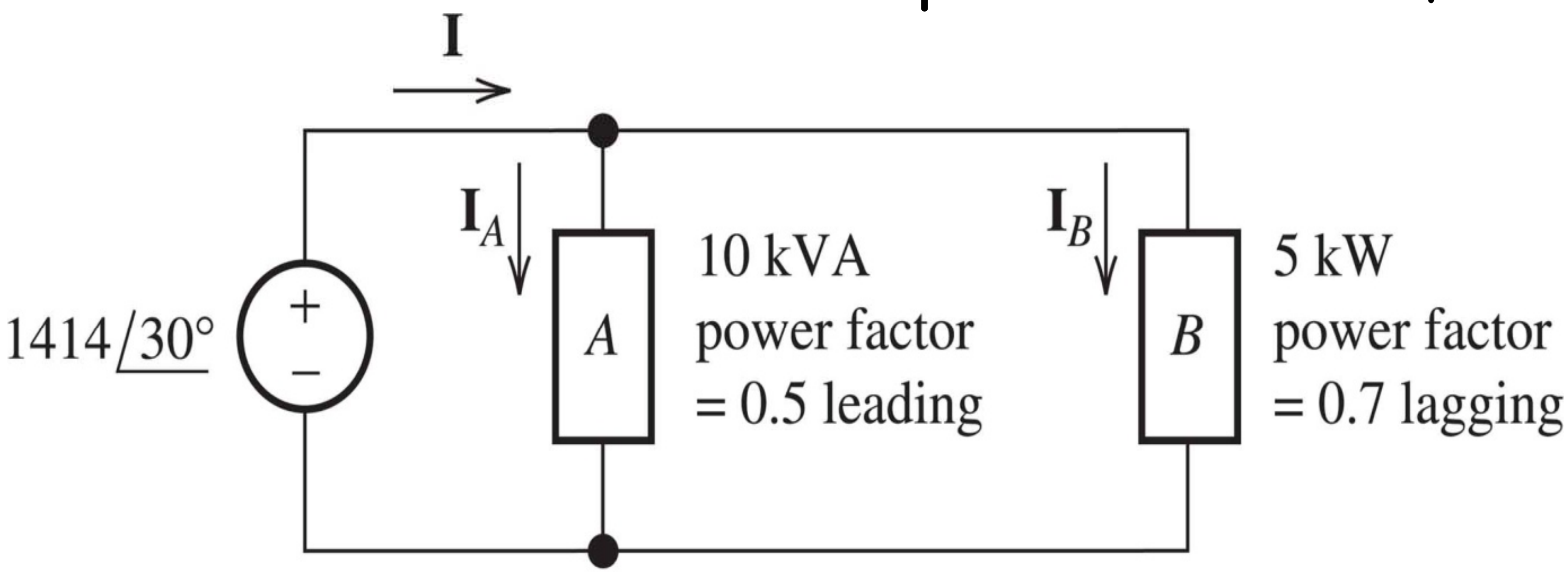


Figure 1.8: Circuit for Example 1.2.

- For load-A, we have

$$P_A = 10\ kVA\ \times 0.5 = 5\ kW,$$

$$Q_A = -10\ kVA\ \times \sin\left(\left(cos^{-1}(0.5)\right)\right) = -8.660\ kVAR$$

- For load-B, we have

$$P_B = 5\ kW,$$

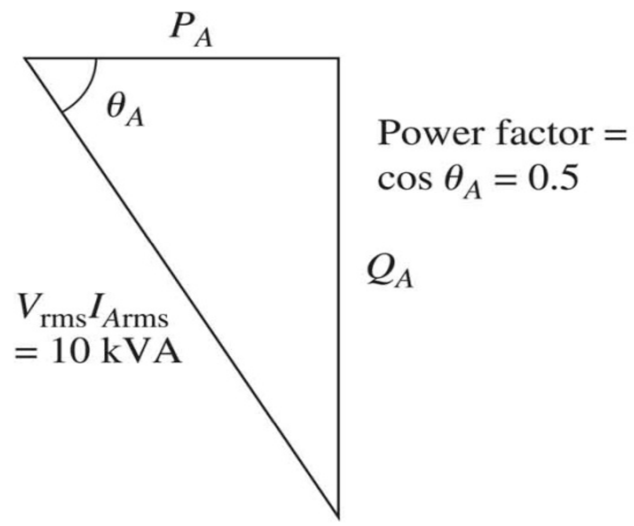
$$Q_B = S_B \times \sin(\theta) = \frac{5\ kW}{0.7} \times \sin\left(\left(cos^{-1}(0.7)\right)\right) = 5.101\ kVAR$$

- Thus, the total load active power is given by

$$P = P_A + P_B = 5\ kW = 5\ kW = 10\ kW$$

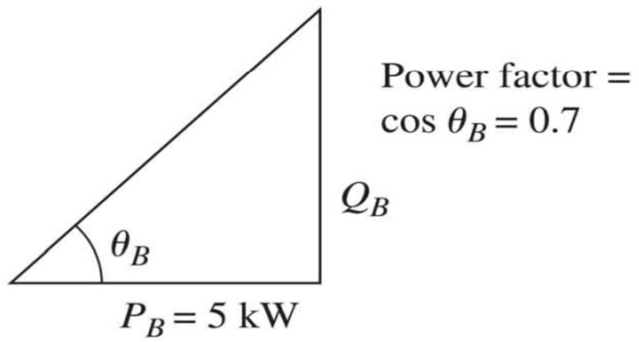
- Thus, the total load reactive power is given by

$$Q = Q_A + Q_B = -8.66\ kVAR + 5.101\ kVAR = -3.559\ kVAR$$



P_A
θ_A
Power factor =
$\cos \theta_A = 0.5$
Q_A
$V_{rms}I_{Arms}$
$= 10$ kVA

(a)



Power factor =
$\cos \theta_B = 0.7$
Q_B
θ_B
$P_B = 5$ kW

(b)

- Thus, we have the total load power given by

$$\mathbf{S} = P + jQ = 10\ kW - j3.559\ kVAR = 10.61\angle -19.59^0\ kVA$$

- Thus, we have

$$\mathbf{S} = \frac{1}{2}\mathbf{V}\mathbf{I}^* = \frac{1}{2}(1414\angle 30^0)\mathbf{I}^* = P + j\,Q = 10.61\angle -19.59^0\ kVA \Rightarrow \mathbf{I}^* = 15.0\angle -49.59^0 \Rightarrow \quad \mathbf{I}$$
$$= 15.0\angle 49.59^0$$

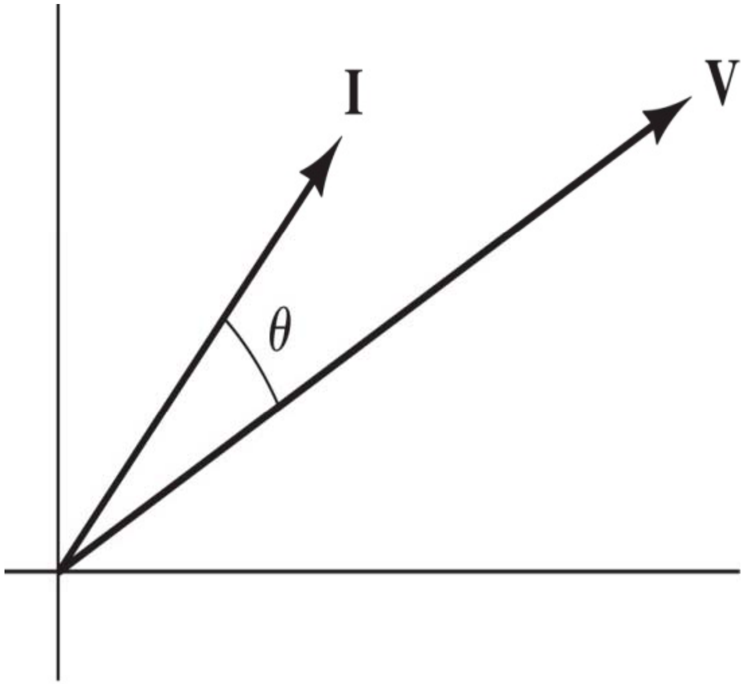


Figure 1.10 Phasor diagram for Example 1.2.

Power-Factor Correction

- In energy storage elements in AC circuits, large amount of current flows through the circuit without average power being delivered.
- In heavy industries most of the loads are partly inductive in nature and large amount of reactive power flow takes place.
- The presence of reactive power increases the apparent power and therefore the current drawn from the mains supply in the power distribution networks.

- Thus, the transmission lines as well as the distribution transformers have higher ratings than that would be necessary to deliver the same average active power to a resistive load.
- Typically, for heavy industries the energy rates are charged not only on the average kWhr of energy used by the consumer but also on the power factor at which they draw the active power with higher energy cost for operating at lower power factor.
- For this reason, the customers put power factor correction circuits by using capacitors to deliver the inductive kVAR of the load so that the power factor is improved and less kVAR is drawn from the source.

Example 1.3

A 50-kW load operates from 1 50 Hz, 10 KV rms line with a power factor of 60 percent lagging. Compute the capacitance that must be placed in parallel with the load to achieve a 90 percent lagging power factor.

Solution:

- The load power angle is $\theta_L = cos^{-1}(0.6) = 53.13^0$
- Using the power-triangle concept, we have

$$Q_L = P_L \tan(\theta_L) = 50\ kW\ \times \tan(53.13^0) = 66.67\ kVAR$$

•After adding the capacitor in parallel the power factor is improved to 0.9 lag, so the new load power angle is

$$\theta_{L,new} = cos^{-1}(0.9) = 25.84^0$$

$$Q_{L,new} = P_L \tan\left(\theta_{L,new}\right) = 50\ kW\ \times \tan(25.84^0) = 24.22\ kVAR$$

- Thus, the reactive power of the capacitance must be

$$Q_C = \ Q_{L,new} - Q_L = 24.22 - 66.67\ kVAR = -42.45\ kVAR$$

$$Q_C = \frac{V_{C,rms}^2}{X_C} \Rightarrow X_C = \frac{V_{C,rms}^2}{|Q_C|} = \frac{(10^4)^2}{42.45\ \times 1000} = 2356\ \Omega$$

$$C = \frac{1}{\omega|X_C|} = \frac{1}{(2\pi 50) \times 2356} = 1.351\ \mu F$$

Summary

- A sinusoidal voltage is given by $v(t) = V_m \cos(\omega t + \theta)$, where V_m is the peak values of the voltage, ω is the angular frequency in rad/sec, and θ is the phase angle. The frequency in hertz is $f = 1/T$, where T is the period. Furthermore, $\omega = 2\pi f$.
- The root-mean-square (rms) value of a periodic voltage, $v(t)$ is V_{rms}. The average power delivered to a resistance by is $P_{avg.} = V_{rms}^2/R$. Similarly, for a rms current, I_{rms}, the average power delivered to a resistance by is $P_{avg.} = I_{rms}^2 \times R$.
- For a sinusoidal signal the rms value is the peak value divided by $\sqrt{2}$.
- When represent sinusoids with phasors. The magnitude of the phasor is the peak value of the sinusoid and the phase angle of the phasor is the same as that of the sinusoid.
- When a sinusoidal current flows through a sinusoidal voltage, the average power delivered is $P_{avg.} = V_{rms} I_{rms} \cos(\theta)$.

- Reactive power is the flow of the energy back and forth between the source and the energy-storage element. We define reactive power to be positive for an inductance and negative for a capacitance. Net energy transferred per cycle by reactive power is zero.
- Apparent power is the product of rms voltage and rms current.

References

1. "Principles and Applications of Electrical Engineering" – Giorgio Rizonni, Mc Graw Hill, 5th Edition 2007, Chapters 8 and 11.
2. "Electrical Engineering Principles and Applications", - Allan R. Hambley, Pearson – Prentice Hall, 5th Edition 2010, Chapters 7 and14.