EE2011 Engineering Electromagnetics

(Semester I of Academic Year 2011/2012)

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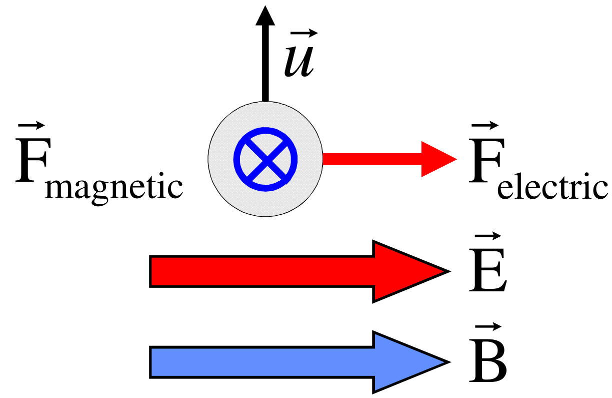
Lorentz Force

force on charge q moving with velocity $\vec{u}$

$$\vec{F} = q\vec{E} + q\vec{u}\times\vec{B}$$



notes for $\vec{F}_{magnetic}$:

(a) normal to $\vec{u} \Rightarrow$ no work done

circular trajectory with constant kinetic energy

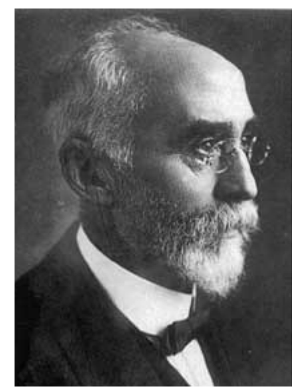
(b) zero for stationary charge (*i.e* when $\left|\vec{u}\right| = 0$)

no magnetic force in electrostatics

(c) independent of $\vec{F}_{electric}$

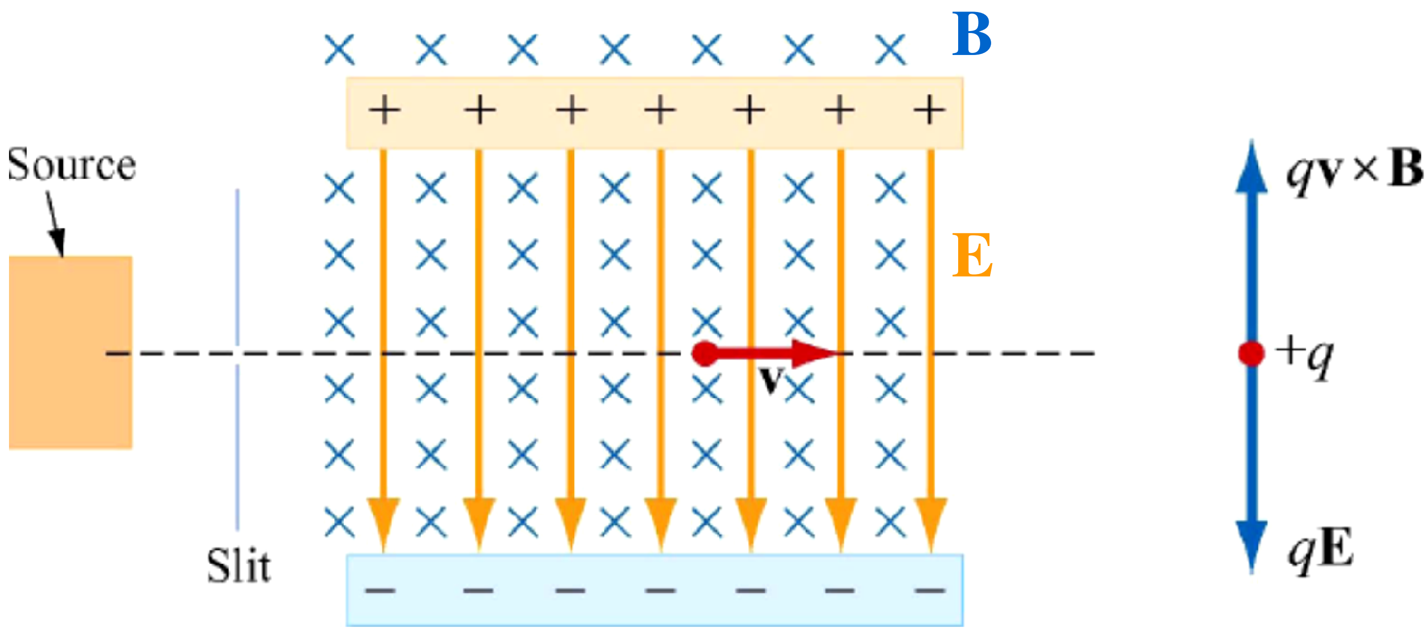
(d) used for defining Tesla $= \dfrac{\text{N}}{\text{C (m/s)}} = \dfrac{\text{N}}{\text{(C/s) m}} = \dfrac{\text{N}}{\text{A m}}$

(e) applicable to point charges but can extend to charge systems



Lorentz Force

force on electron flow in vacuum (Thomson's experiment)



adjust **E** and **B** for electrons to flow in straight path $\rightarrow$ $v = \frac{E}{B}$

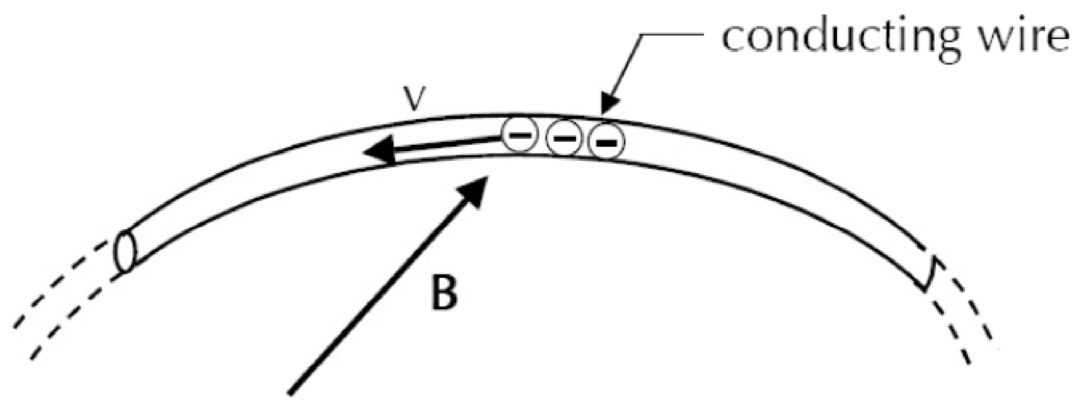
used for measuring velocity and charge-to-mass ratio of electron

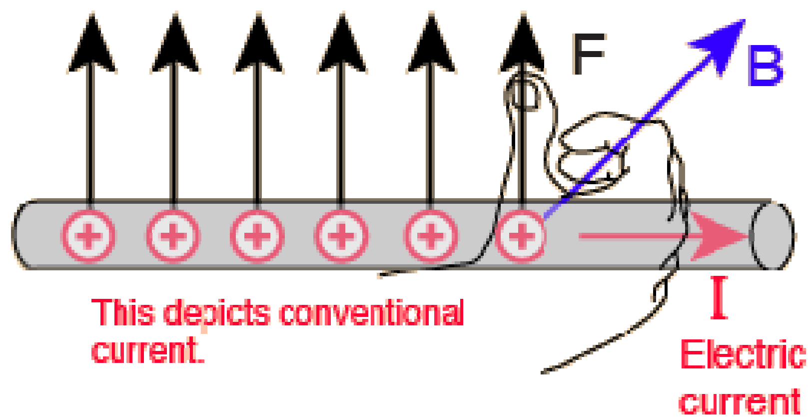
Magnetic Force

force on electron flow in wire (*i.e.* current)

$$
\begin{aligned}
d\vec{F} &= \vec{u}\times\vec{B}\, dq \\
&= \vec{u}\times\vec{B}\, \sigma\, dV \\
&= \vec{J}\times\vec{B}\; A\, ds \\
&= I\, d\vec{s}\times\vec{B}
\end{aligned}
$$

$$\Rightarrow \quad \vec{F} = I\int d\vec{s}\times\vec{B}$$



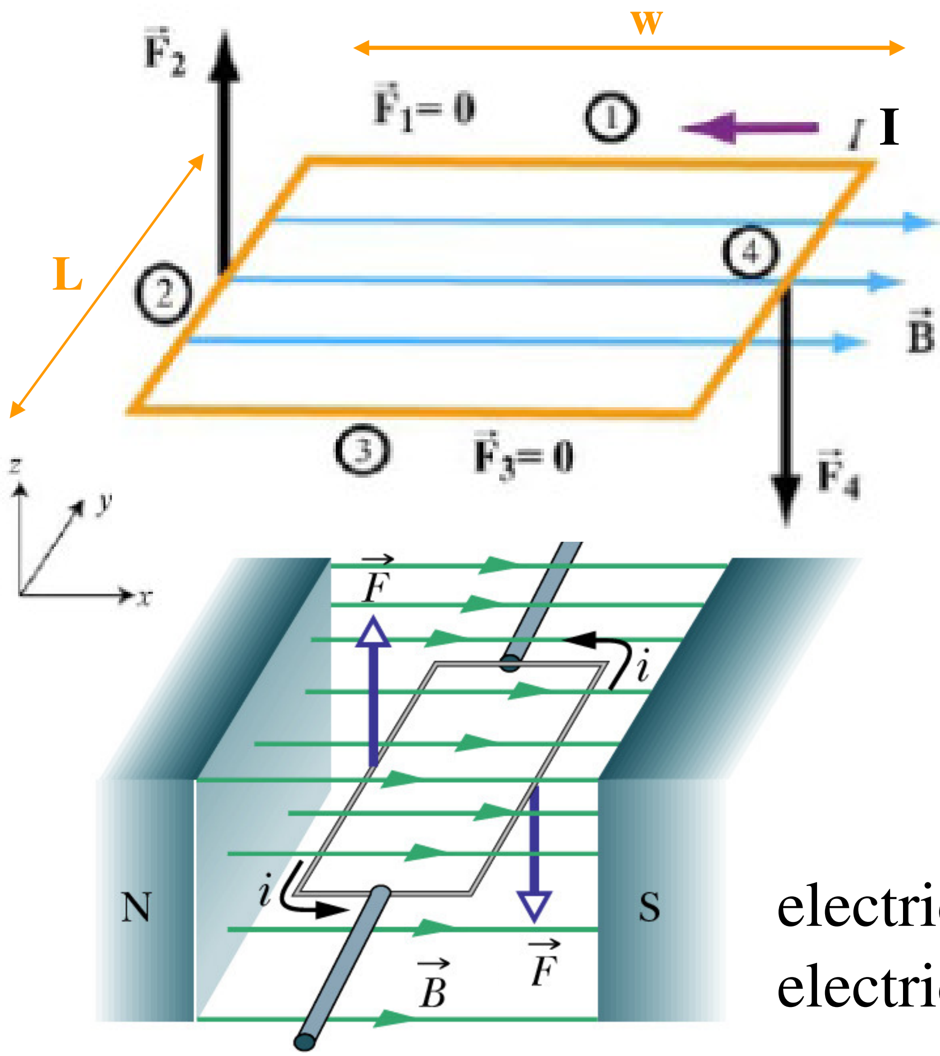


force on straight wire

(where $d\vec{s} = \hat{u}\, ds$)

$$\vec{F} = I L\, \hat{u}\times\vec{B}$$

Magnetic Force



no forces on arms 1 and 3

no net force from arms 2 and 4

$$\text{because } \left|\vec{F}_2\right| = \left|\vec{F}_4\right| = \left|\vec{B}\right| I L$$

resultant torque on wire loop

$$\tau = \left|\vec{B}\right| I L w = \left|\vec{B}\right| I A$$

torque vector on any wire loop

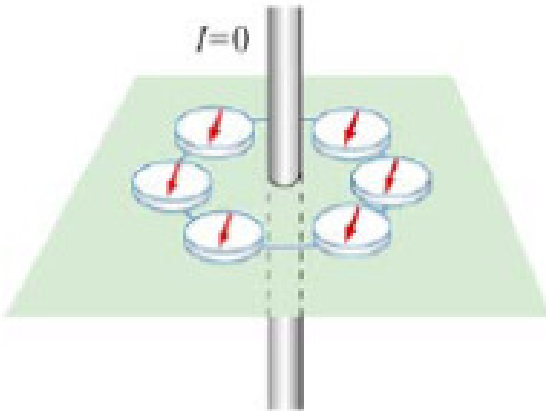
$$\vec{\tau} = I\vec{A}\times\vec{B} = \vec{m}\times\vec{B}$$

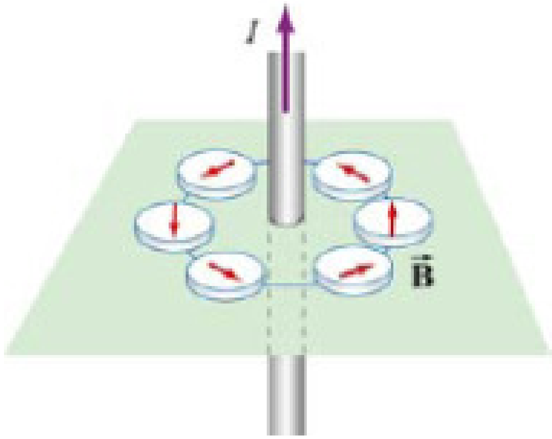
define magnetic moment $\vec{m} = I\vec{A}$

electric motor — transducer converting electrical energy into mechanical energy

Magnetic Field

magnetic field around current-carrying wire(s)



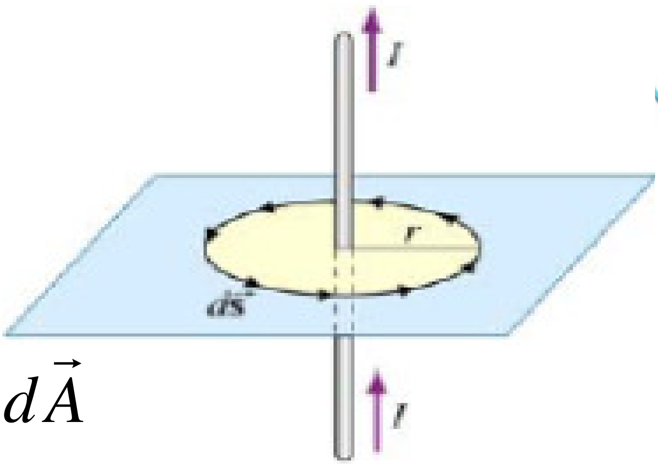


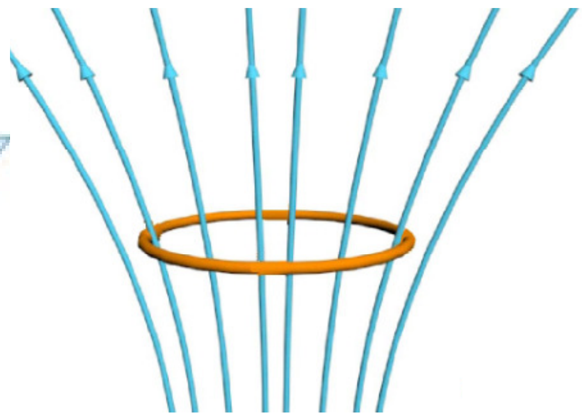
Ampere's Law

$$\oint \vec{B} \bullet d\vec{s} \propto \mathrm{I}$$

$$\oint \vec{B} \bullet d\vec{s} = \mu_0 \sum_k I_k$$

$$\oint \vec{B} \bullet d\vec{s} = \mu_0 \iint \vec{J} \bullet d\vec{A}$$





Magnetic Field

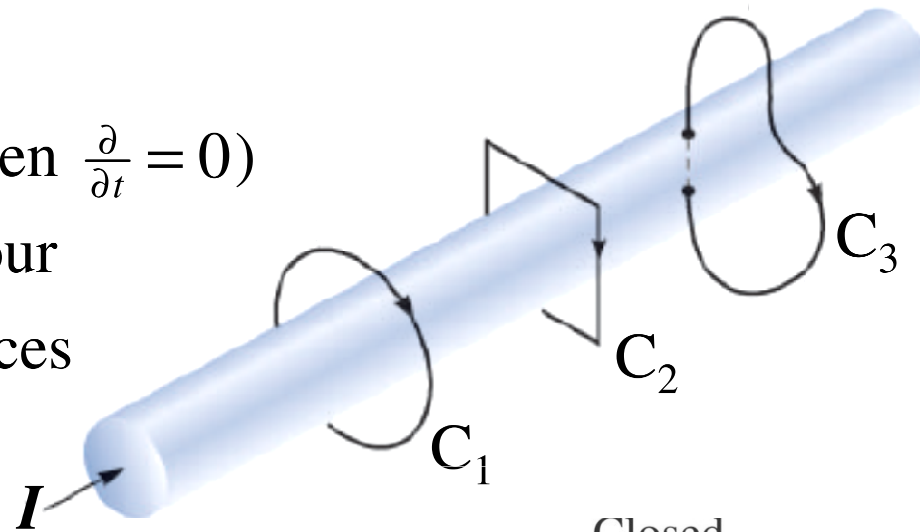
notes on Ampere's Law:

(a) valid only for DC (*i.e.* when $\frac{\partial}{\partial t} = 0$)

(b) valid for any closed contour

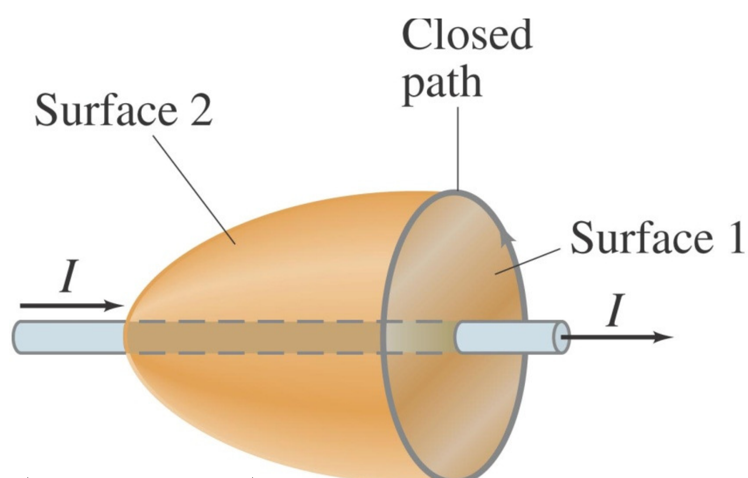
(c) valid for non-planar surfaces

(d) practical utility for cases with symmetry



$$\iint \nabla \times \vec{B} \bullet d\vec{A} = \mu_0 \iint \vec{J} \bullet d\vec{A}$$

$$\Rightarrow \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$



take div of both sides:

$$\nabla \bullet \vec{J} = \frac{1}{\mu_0} \nabla \bullet \nabla \times \vec{B} = 0 \rightarrow$$ steady-state current

Magnetic Field

no isolated magnetic charge *c.f.* Gauss's Law for electric charges

$$\oint\!\!\oint_S \vec{B} \bullet \mathrm{d}\vec{A} = 0 \qquad\qquad \oint\!\!\oint_S \vec{D} \bullet \mathrm{d}\vec{A} = \mathrm{Q}_{\text{total in enclosure}}$$

$\Rightarrow \iiint_V \nabla \bullet \vec{B}\,\mathrm{d}V = 0$ for application to any volume

PDE version $\nabla \bullet \vec{B} = 0$ for application to particular point

<u>implications</u>**:**

(a) $\vec{B}$ fields always in closed loops (due to lack of sources/sinks)

(b) basis for introducing magnetic potential (vector)

use null identity $\nabla \bullet \left(\nabla \times \vec{A}\right) = 0$

useful to express $\vec{B} = \nabla \times \vec{A}$ during analytical derivation

but not common to leave answer in magnetic-potential form

Magnetic Field

re-visit curl equation in $\vec{B}$ (from Ampere's Law)

PDE in $\vec{A}$ $\qquad \nabla \times \left(\nabla \times \vec{A} \right) = \mu_0 \vec{J}$

expanding curl curl $\qquad \nabla \bullet \vec{A} - \nabla^2 \vec{A} = \mu_0 \vec{J}$

choosing $\nabla \bullet \vec{A} = 0$ $\qquad \nabla^2 \vec{A} = -\mu_0 \vec{J}$

decomposing vector Poisson's equation in (x, y, z) coordinates

$$\left.\begin{aligned} \nabla^2 A_x &= -\mu_0 J_x \\ \nabla^2 A_y &= -\mu_0 J_y \\ \nabla^2 A_z &= -\mu_0 J_z \end{aligned}\right\} \quad c.f. \quad \nabla^2 V = -\tfrac{1}{\varepsilon_o}\sigma \;\rightarrow\; V = \tfrac{1}{4\pi\varepsilon_o}\iiint \tfrac{\sigma}{r''}\, dV'$$

$$\Rightarrow \quad \vec{A} = \tfrac{\mu_o}{4\pi}\iiint \tfrac{1}{r''}\vec{J}\, dV' \quad \text{and thereafter obtain} \quad \vec{B} = \nabla \times \vec{A}$$

Magnetic Field

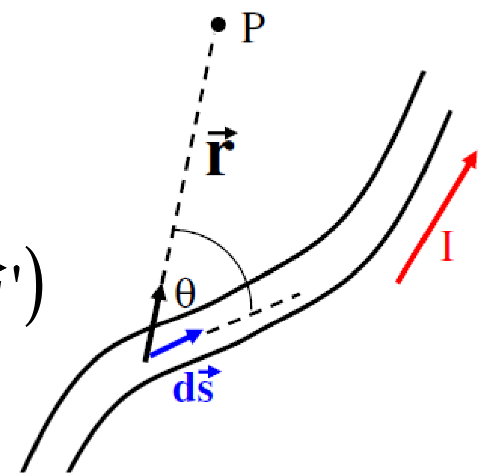
short wire (carrying current I) with length $\Delta s \to 0$ in the limit

$$\vec{J}\,dV' = \vec{J}\,A\,ds' = \mathrm{I}\,d\vec{s}\,'$$

$$\vec{A} = \frac{\mu_o}{4\pi}\iiint \frac{1}{r''}\vec{J}\,dV = \frac{\mu_o \mathrm{I}}{4\pi}\int \frac{1}{r''}\,d\vec{s}\,'$$

$$\vec{B} = \nabla\times\left(\frac{\mu_o \mathrm{I}}{4\pi}\int \frac{1}{r''}\,d\vec{s}\,'\right) = \frac{\mu_o \mathrm{I}}{4\pi}\int \nabla\times\left(\frac{1}{r''}\,d\vec{s}\,'\right)$$

$$= \frac{\mu_o \mathrm{I}}{4\pi}\int \frac{d\vec{s}\,'\times\hat{u}_{r''}}{(r'')^2}$$



Biot-Savart's Law to evaluate $\vec{B}$ from wire (for $d\vec{s}$ with $\theta \neq 0$)

check for solenoidal property

$$\nabla\bullet\vec{B} = \frac{\mu_o \mathrm{I}}{4\pi}\int \nabla\bullet\left(\frac{d\vec{s}\,'\times\hat{u}_{r''}}{(r'')^2}\right) = -\frac{\mu_o \mathrm{I}}{4\pi}\int d\vec{s}\,'\bullet\nabla\times\nabla\left(\frac{1}{r''}\right) = 0$$

Magnetic Field

Example #1A: straight wire of length L (with origin at center)

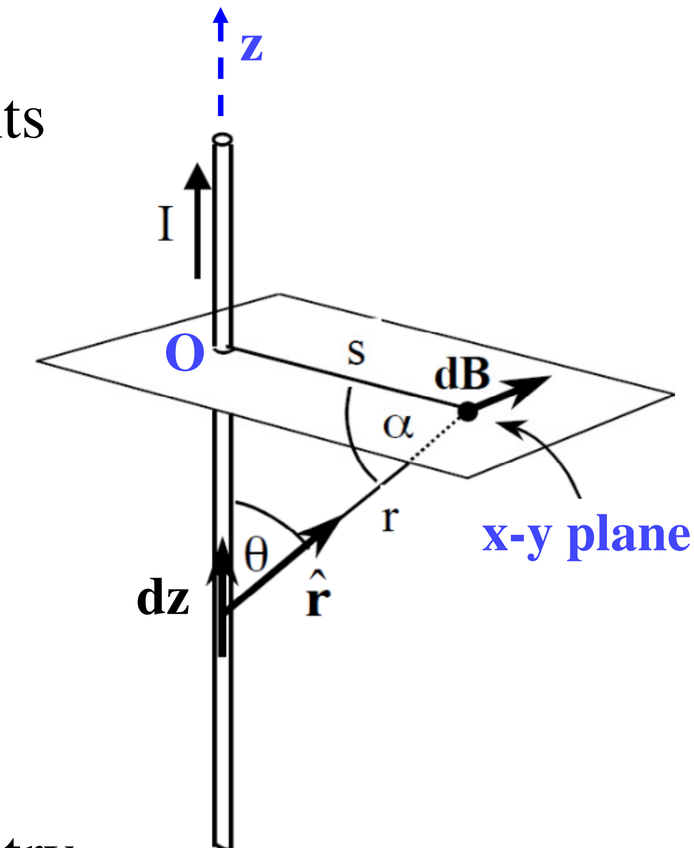
- x-y plane normal to $d\vec{s} = dz\,\hat{k}$ and $\vec{r}$
- $d\vec{B}$ in same direction for all dz elements
- θ dependent on z

apply Biot-Savart's Law

$$\vec{B} = \frac{\mu_o I}{4\pi}\int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{dz\,\hat{k}\times\hat{r}}{r^2+s^2} = \frac{\mu_o I}{4\pi}\hat{u}_\phi \int_{-\frac{L}{2}}^{\frac{L}{2}} \frac{\sin\theta\, dz}{r^2+s^2}$$

$$= \frac{\mu_o I}{4\pi s}\frac{L}{\sqrt{\frac{1}{4}L^2+s^2}}\hat{u}_\phi$$

$$\rightarrow \frac{\mu_o I}{4\pi s}\hat{u}_\phi \quad \text{when } L \rightarrow \infty$$



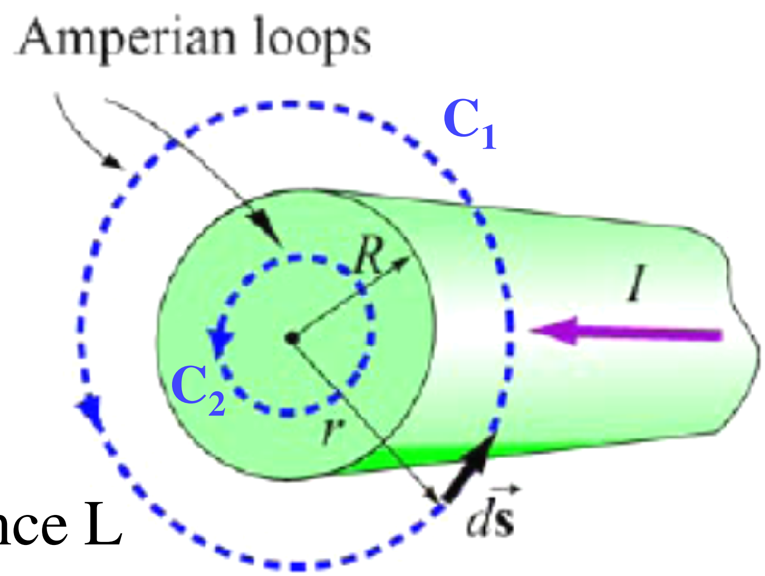
useful tool for structures without symmetry

Magnetic Field

Example #1B: long straight wire of radius R

infer from symmetry:

- azimuthal component ($\vec{B} = B_\phi \hat{u}_\phi$)
- B_ϕ constant of ϕ and z coordinates



(a) apply Ampere's Law to C_1 loop:

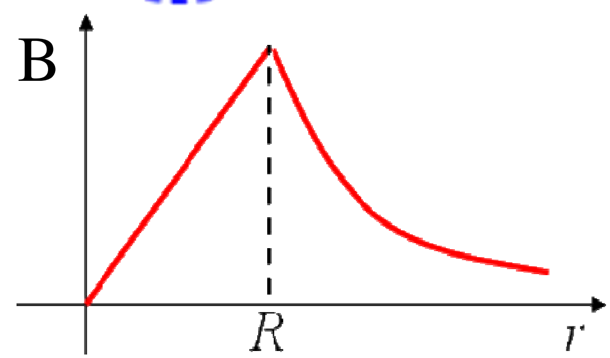
$$2\pi r B_\phi = \mu_o I \quad \Rightarrow \quad B_\phi = \frac{\mu_o I}{2\pi r}$$

used for deriving internal inductance L

(b) apply Ampere's Law to C_2 loop:

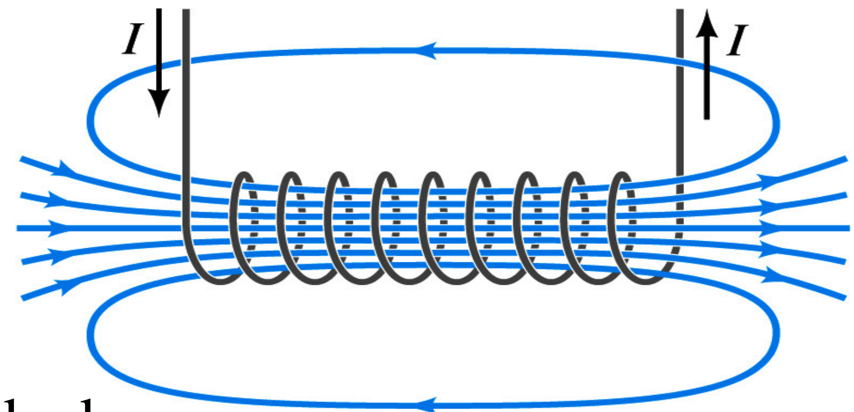
$$2\pi r B_\phi = \mu_o \frac{\pi r^2}{\pi R^2} I \quad \Rightarrow \quad B_\phi = \frac{\mu_o r I}{2\pi R^2}$$

valid only for DC (*i.e.* $\frac{dI}{dt} = 0$)



Magnetic Field

Example #2: long solenoid (with n turns of wire per meter)



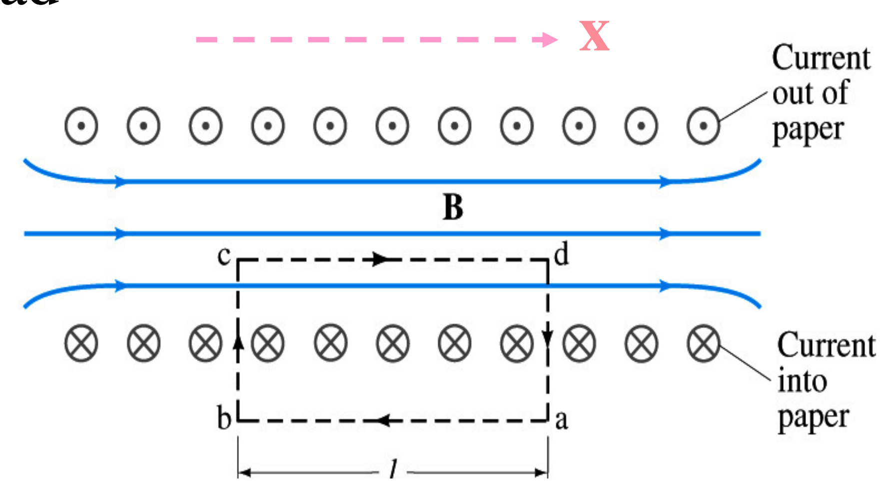
infer from structure:

- axial component ($\vec{B} = B_x \hat{u}_x$)
- uniform B within solenoid

apply Ampere's Law to abcd:

(a) no contributions from bc and ad

(b) $B \approx 0$ outside solenoid

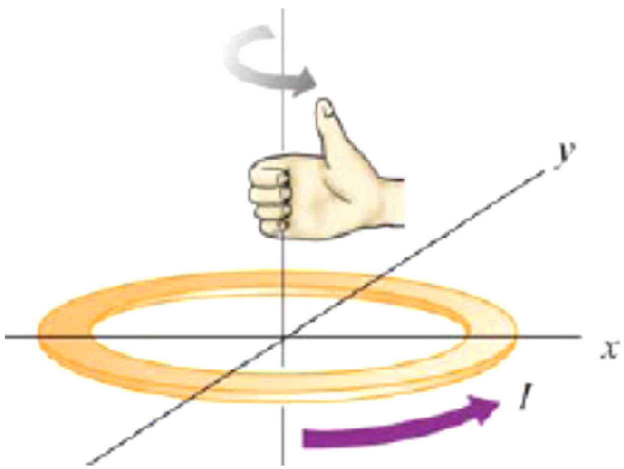


$$\begin{cases} \oint \vec{B} \bullet d\vec{s} = B_x l + 0 + 0 + 0 \\ \mu_0 \sum_k I_k = \mu_0 (n l)\, I \end{cases}$$

$$\Rightarrow \quad \vec{B} = \mu_0 n I \hat{u}_x$$

Magnetic Field

Example #3: small loop (magnetic dipole with $\vec{\mathrm{m}} = \pi \mathrm{r}_\mathrm{o}^2\, \mathrm{I}\, \hat{\mathrm{u}}_\mathrm{z}$)



$$\vec{A} = \frac{\mu_o \mathrm{I}}{4\pi} \int \frac{1}{r''}\, d\vec{s}\,' = \frac{\mu_o}{4\pi r^2}\, \vec{\mathrm{m}} \times \hat{\mathrm{u}}_\mathrm{r} \quad \text{when } \mathrm{r} >> \mathrm{r}_\mathrm{o}$$

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_o}{4\pi} \frac{\mathrm{m}}{r^3} \left(2\cos\theta\, \hat{\mathrm{u}}_\mathrm{r} + \cancel{3}\sin\theta\, \hat{\mathrm{u}}_\phi \right)$$

similarity with electric dipole (duality)

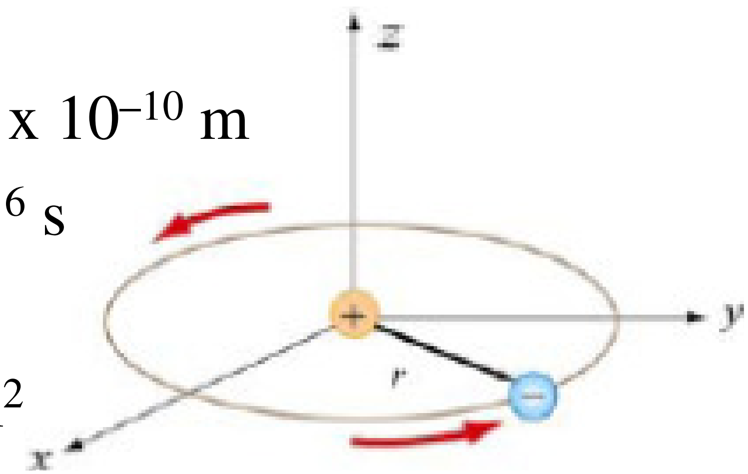
$$\vec{E} = \frac{1}{4\pi\varepsilon_o} \frac{\mathrm{p}}{r^3} \left(2\cos\theta\, \hat{\mathrm{u}}_\mathrm{r} + \cancel{3}\sin\theta\, \hat{\mathrm{u}}_\phi \right)$$

extend to electron in orbit with $\mathrm{r}_\mathrm{o} \approx 0.5 \times 10^{-10}$ m

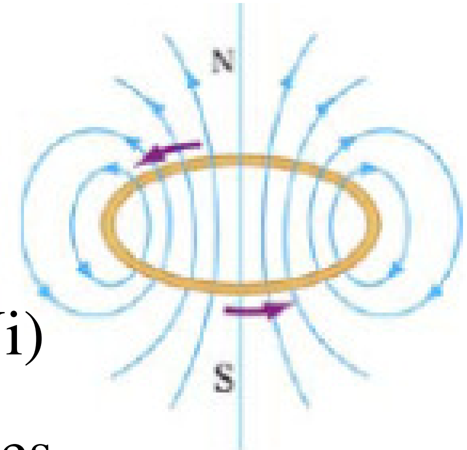
travel time for one orbit $\Delta t \approx 1.5 \times 10^{-16}$ s

$\therefore$ current $\mathrm{I} = \frac{\mathrm{e}}{\Delta \mathrm{t}} \approx 1$ mA

moment $\mathrm{m} = \pi \mathrm{r}_\mathrm{o}^2\, \mathrm{I} \approx 9.3 \times 10^{-24}\ \mathrm{A\,m^2}$



Magnetic Materials



field-matter interactions in magnetic material

(a) vectorial (instead of scalar in dielectrics)

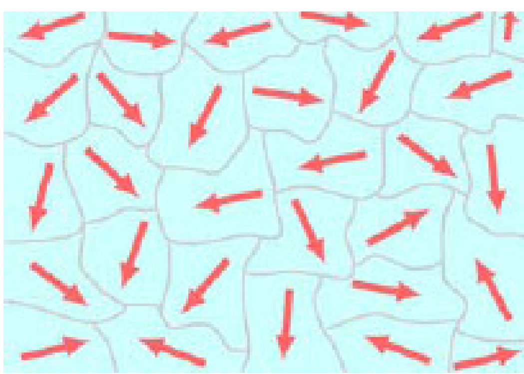
(b) very weak for large majority of materials

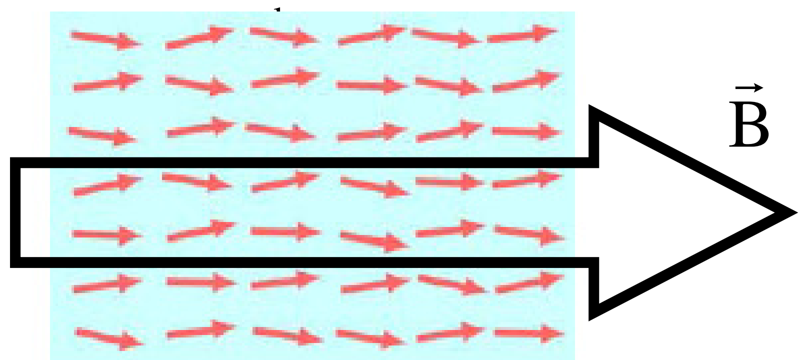
very strong in certain materials (*e.g.* Fe, Co, Ni)

not possible to consider individual magnetic dipoles

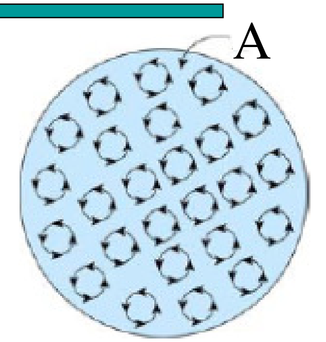
need to consider average density instead for bulk behaviour

define magnetization vector $\vec{M} = \frac{1}{\text{volume}} \sum_k \vec{m}_k$





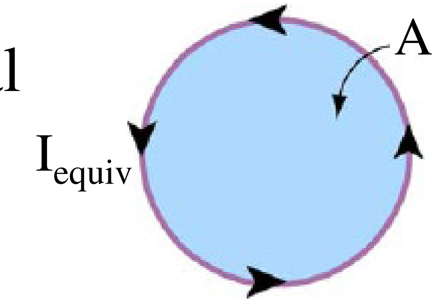
Magnetic Materials



Ampere's Law in differential form

$$\nabla \times \vec{B} = \mu_0 \vec{J} \qquad \text{for free space}$$

add magnetization current density $\vec{J}_\mathrm{m} = \nabla \times \vec{M}$



$$\nabla \times \vec{B} = \mu_0 \left(\vec{J} + \vec{J}_\mathrm{m} \right) \quad \text{for magnetic material}$$

$$\Rightarrow \qquad \nabla \times \left(\tfrac{1}{\mu_0} \vec{B} - \vec{M} \right) = \vec{J}$$

recast as $\nabla \times \vec{H} = \vec{J}$ by introducing another magnetic vector

$\vec{H}$ = magnetic field intensity vector ($\mathrm{A\,m^{-1}}$) to account for currents

$\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right)$ = magnetic flux density vector

$$= \mu_0 (1 + \chi_m) \vec{H} = \mu_0 \mu_\mathrm{r} \vec{H}$$

μ_r relative permeability

χ_m magnetic susceptibility

Magnetic Materials

hysteresis behavior

a: initially unmagnetized state

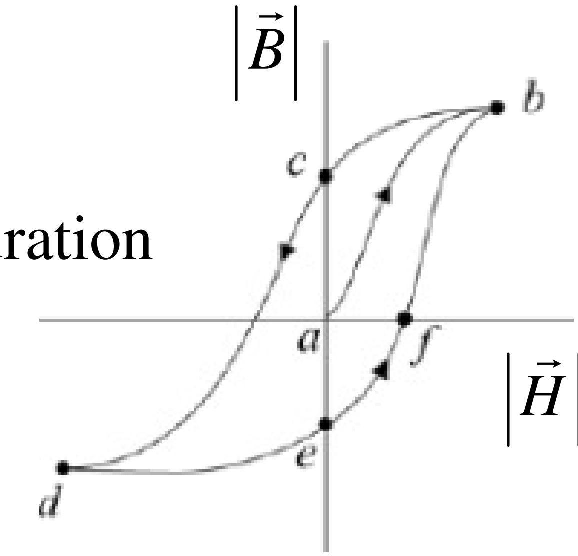
b: increasing magnetization till onset of saturation

c: retention after reducing source to zero

d: reversing polarity of source to saturation

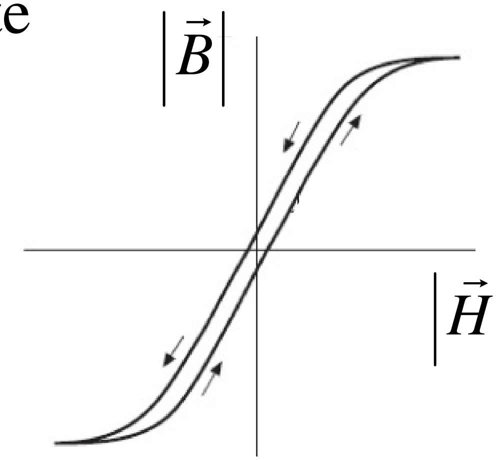
e: retention with reversed polarity

f: not coincident with initial unmagnetized state



implications

- non-linear behavior
- large retentivity for permanent magnets
- reduced hysteresis loop for memory core



Magnetic Materials

relative permeability values**:**

1.000	for air, water, copper, gold, aluminium, water, *etc*
250	for cobalt
600	for nickel
> 5,000	for iron
> 50,000	for MuMetal® at DC
> 35,000	for MuMetal® at 50 Hz with low magnetization
200,000	for MuMetal® at 50 Hz with high magnetization

- equal either to 1 (for most materials) or very large value
- variable due to hysteresis (check operating conditions)
- anisotropic (use tensor instead of scalar representation)

Magnetic Materials

dual of electric boundary conditions at interface between materials:

(a) from Gauss's Law applied to pill-box structure

$$(B_1)_n = (B_2)_n$$

(b) from Ampere's Law applied to closed contour

$(H_1)_t = (H_2)_t$ valid for non-conducting materials

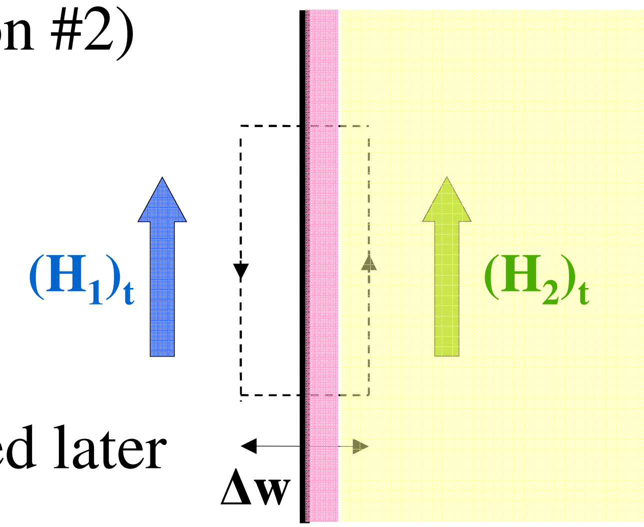
variation for perfect conductor (Region #2)

due to large current densities in RHS

$$\oint \vec{H} \bullet d\vec{s} = \iint \vec{J} \bullet d\vec{A}$$

$$(H_2)_t - (H_1)_t = (J_2)_s$$

linear current density $J_s (\mathrm{Am}^{-1})$ defined later



Electromagnetic Induction

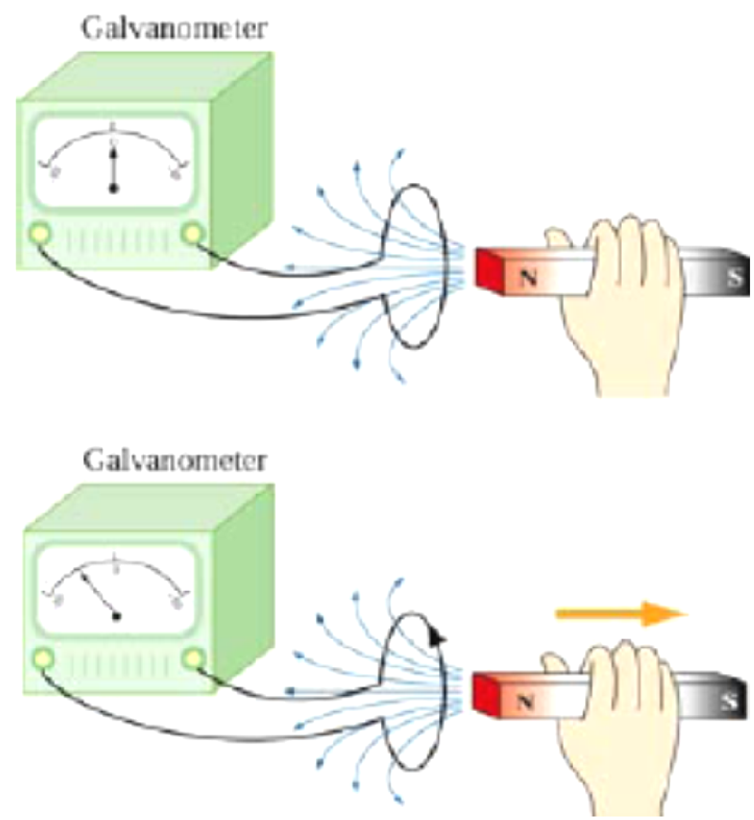
conservative $\vec{E}$ only for electrostatics (*i.e.* when $\frac{\partial}{\partial t}=0$)

Faraday's Law $\mathrm{V_{emf}} = -\frac{\mathrm{d}}{\mathrm{d}t}\Phi$

$$\oint \vec{E}\bullet \mathrm{d}\vec{s} = -\frac{\mathrm{d}}{\mathrm{d}t}\iint \vec{B}\bullet \mathrm{d}\vec{A}$$

$$\iint \nabla\times\vec{E}\bullet \mathrm{d}\vec{A} = -\iint \frac{\mathrm{d}}{\mathrm{d}t}\vec{B}\bullet \mathrm{d}\vec{A}$$

$$\nabla\times\vec{E} = -\frac{\mathrm{d}}{\mathrm{d}t}\vec{B}$$



apparent: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{B}$ creates $\vec{E}$

actual: $\frac{\mathrm{d}}{\mathrm{d}t}\vec{E}$ creates $\vec{B}$ in turn

$\rightarrow$ both inter-coupled

(together with propagation-velocity vector)

Electromagnetic Induction

rate of change of flux $\Phi = \vec{B} \bullet \vec{A}$

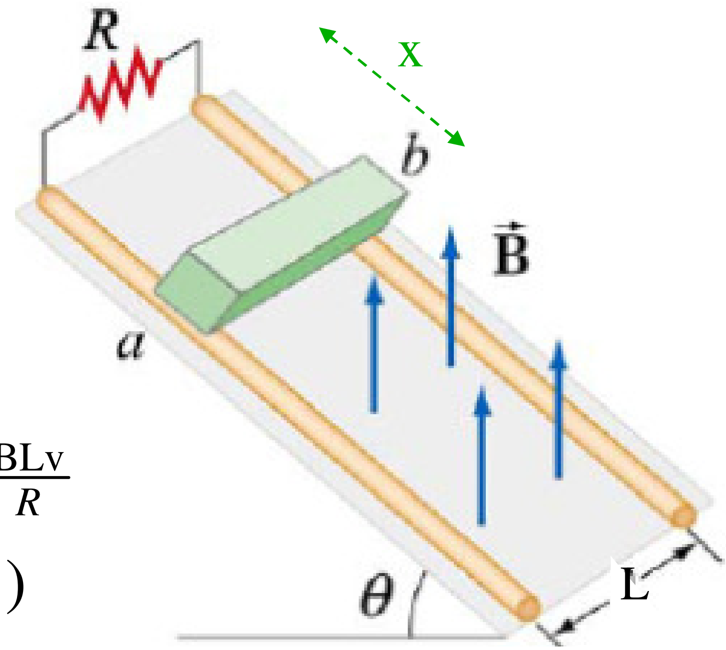
Faraday's Law $V_{emf} = -\frac{d}{dt}(B_n A)$

$$= -\frac{dB_n}{dt} A - B_n \frac{dA}{dt} \quad \text{for time-varying area}$$

example: rod moving along wire rails

with $\frac{dB_n}{dt} = 0$

and $\frac{dA}{dt} = \frac{d(Lx)}{dt} = L\frac{dx}{dt} = Lv$



induced emf between ab and R

current in R $\quad I = \frac{V_{emf}}{R} = -\frac{B}{R}\frac{dA}{dt} = -\frac{BLv}{R}$

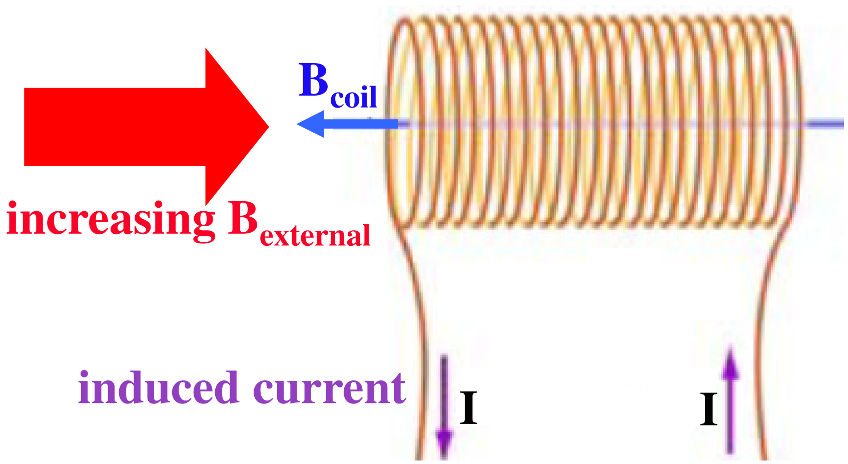
(note: rod accelerating with $\frac{dv}{dt} = g \sin\theta$)

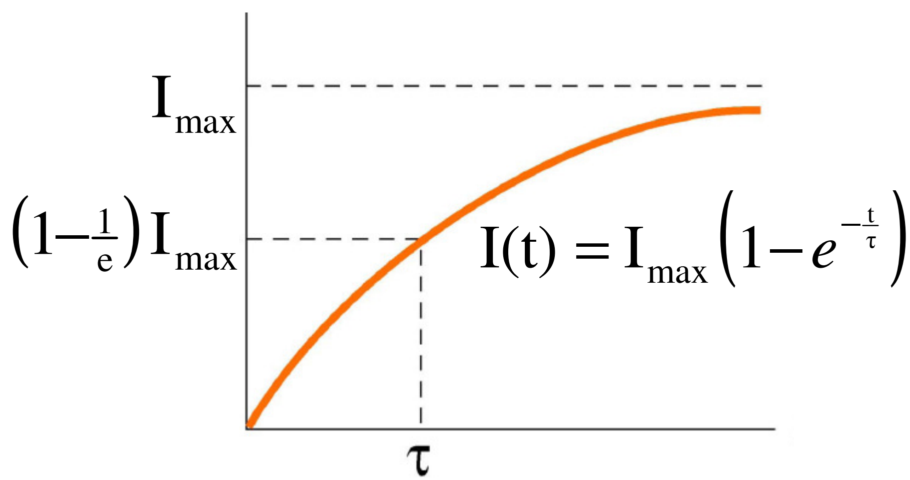
Electromagnetic Induction

Lenz's Law (*i.e.* negative sign in Faraday's Law)

- change of flux $\rightarrow$ induced voltage $V_{emf} = -\frac{d}{dt}\Phi$
- current induced by back emf $\rightarrow$ $\vec{B}$ to counter change in Φ (*i.e.* slow down rate of increase/decrease of Φ)

illustration: increasing Φ for solenoid (inductor)





Electromagnetic Induction

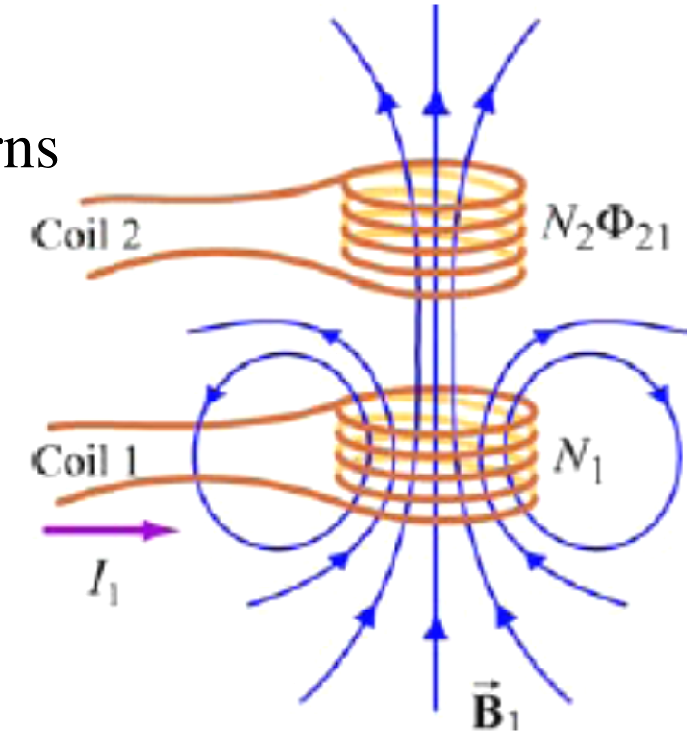
mutual inductance between loops

illustration: two coils with N_1 and N_2 turns

- magnetic field from I_1 flow in coil 1
- magnetic flux linkage with coil 2

  induced emf $V_2 = -\frac{d}{dt}\left(N_2\Phi_{21}\right)$

  direction determined by Lenz's Law



definition: mutual inductance $M = \frac{N_2\Phi_{21}}{I_1}$

(a) unit of Henry $\left(H = \frac{T\,m^2}{A} = \frac{V\,s}{A}\right)$

(b) same result if current in other loop $M = \frac{N_1\Phi_{12}}{I_2}$

(c) disregard N for single-turn loops

(d) use matrix $\underline{M}$ for circuit system with more than two loops

Electromagnetic Induction

mutual-inductance example: coil wrapped around solenoid

recommend current in solenoid (instead of coil)

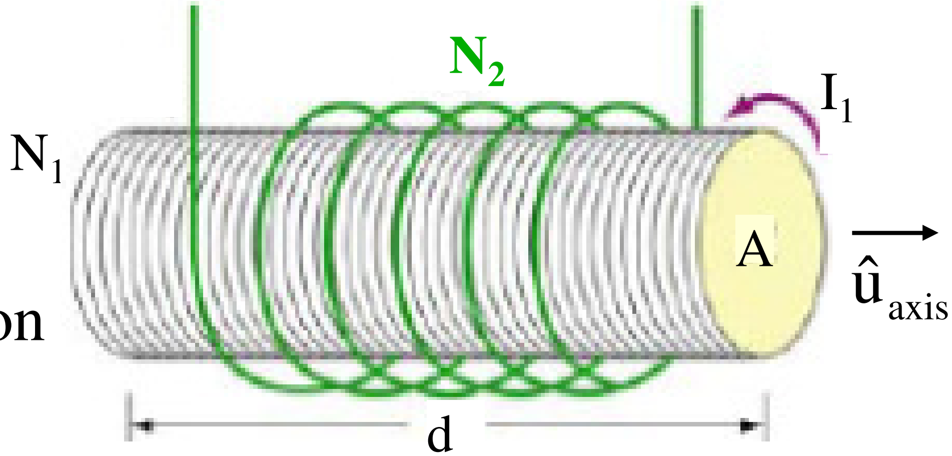
$\rightarrow$ magnetic field within solenoid $\vec{B}_1 = \frac{\mu_o N_1 I_1}{d}\hat{u}_{axis}$

$\rightarrow$ flux linkage with coil $\Phi_{21} = \left(\vec{B}_1 \bullet \vec{A}_2\right) N_2 = \frac{\mu_o N_1 I_1}{d} N_2 A_2$

$\rightarrow$ mutual inductance $M = \frac{\Phi_{21}}{I_1} = \frac{\mu_o N_1 N_2 A}{d}$

starting with current in coil?

- non-uniform $\vec{B}_2$
- more difficult derivation
- different analytical expression
- but same numerical answer



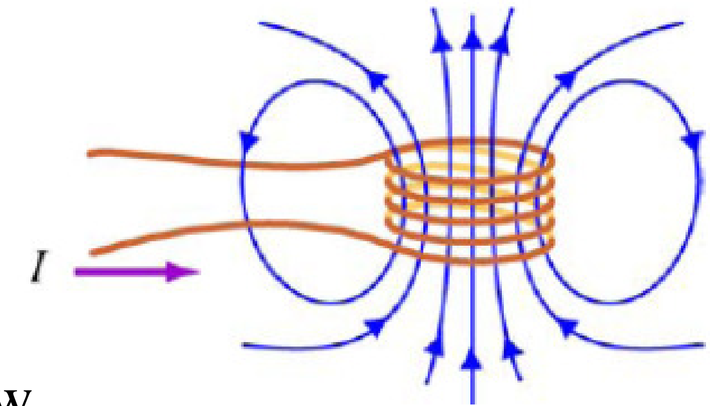
Electromagnetic Induction

self-inductance of current loop

- magnetic field set up by I in coil
- (partial) magnetic flux linkage

  induced emf $V = -\frac{d}{dt}(N\Phi)$

  direction determined by Lenz's Law



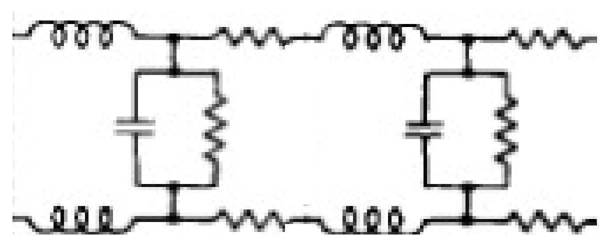
definition: $L = \frac{N\Phi}{I}$ (same unit of Henry as mutual inductance)

(a) interpreted as measure of magnetic flux/energy storage

(b) adopted as series component in transmission-line model

(c) can minimize L by removing loop

(d) may still have to consider $L_{internal}$ due to magnetic field in conductor



Electromagnetic Induction

self-inductance example**:** coaxial cable

current in central conductor (need not be uniform)

→ Ampere's Law for deriving $\vec{B}$ expression in space between conductors $(a < r < b)$

→ flux over narrow strip dr at r

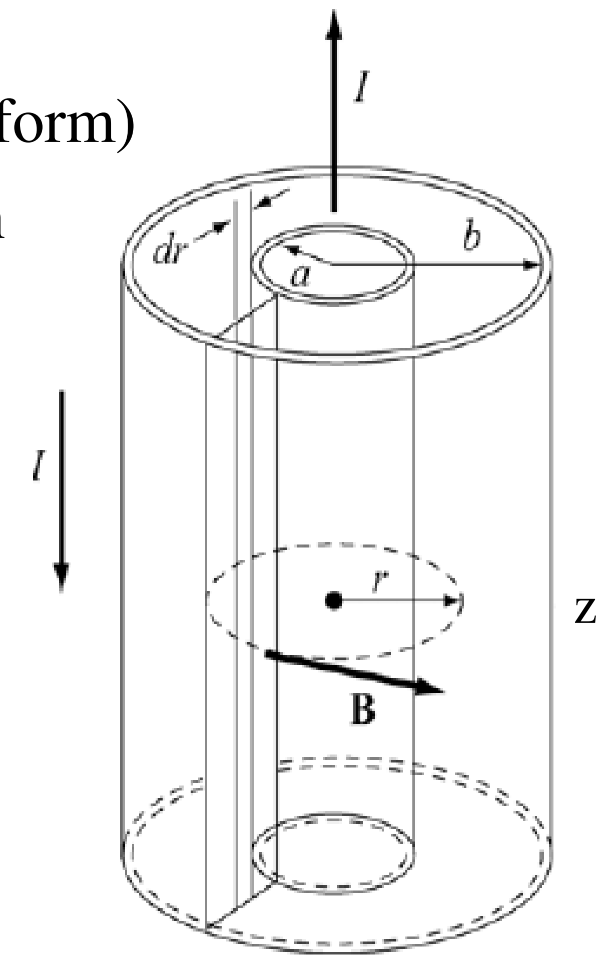
$$d\Phi = \vec{B} \bullet d\vec{A} = \frac{\mu_o I}{2\pi r} z\,dr$$

$$\Phi = \frac{\mu_o I z}{2\pi} \int_a^b \frac{1}{r} dr = \frac{\mu_o I z}{2\pi} \ln\frac{b}{a}$$

→ inductance per unit length

$$\hat{L} = \frac{L}{z} = \frac{\mu_o}{2\pi} \ln\frac{b}{a}$$

usually disregard $L_{interior}$ due to magnetic field inside central conductor $(r < a)$



Electromagnetic Induction

transients for circuit with coil

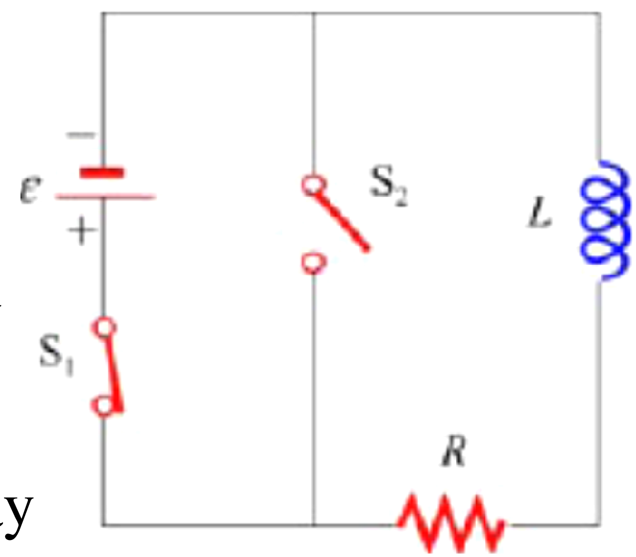
Lenz's Law → opposes change in flux

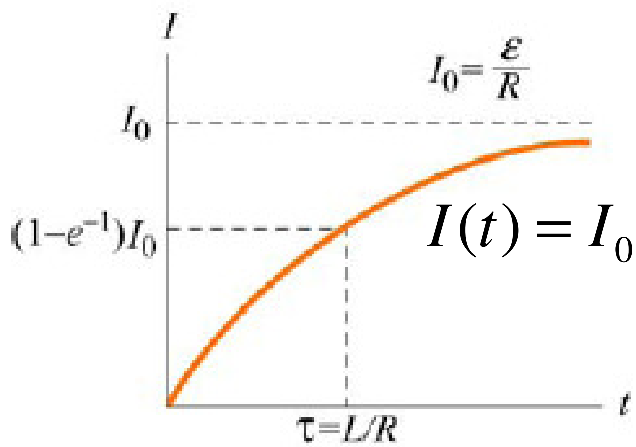
(a) close S_1 (but leave S_2 open)

back emf initially blocking current flow

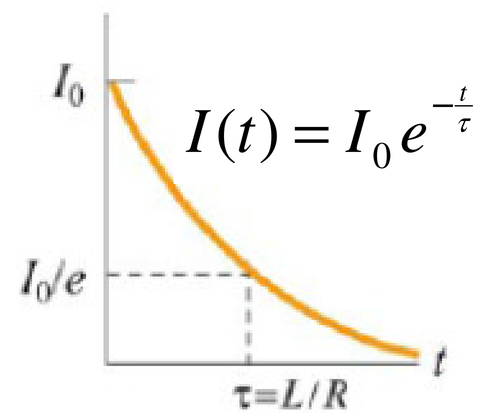
(b) close S_2 (but leave S_1 open)

back emf initially blocking current decay





$$I(t) = I_0\left(1 - e^{-\frac{t}{\tau}}\right)$$



$$I(t) = I_0 e^{-\frac{t}{\tau}}$$