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# **EE2011 Engineering Electromagnetics**

## **(Semester I of Academic Year 2011/2012)**

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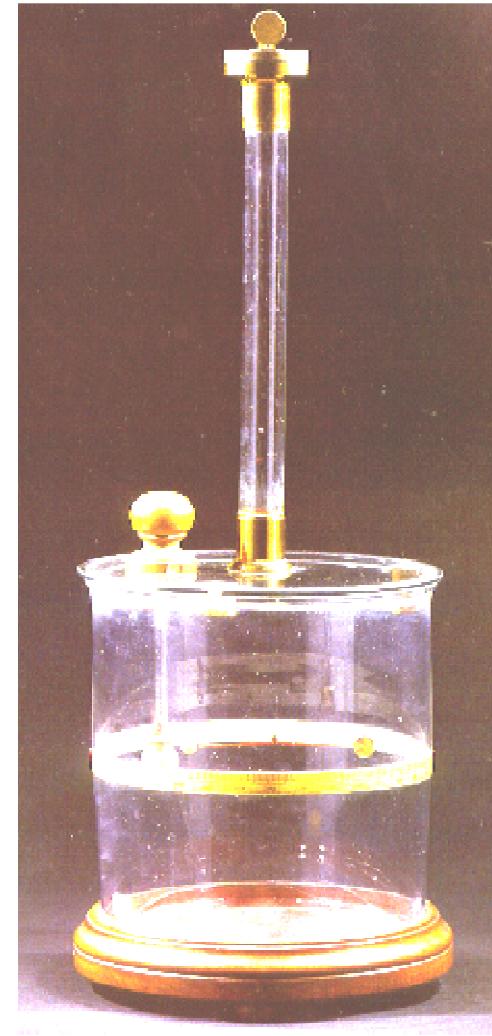
# Coulomb's Law

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quantitative observations (in free space)

$$\vec{F} = \frac{q_1 q_2}{4\pi\epsilon_0 r^2} \hat{\mathbf{u}}_r$$

- originally an empirical result  
later corroborated by further theory
- valid only for point charges  
(*i.e.* when  $r \gg$  dimensions of  $q_1$  and  $q_2$ )
- formula for pair of charges  
extend via superposition for linear systems



# Coulomb's Law

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vectorial addition for (linear) assembly of point charges  
(attract for opposite polarity but repel for same polarity)

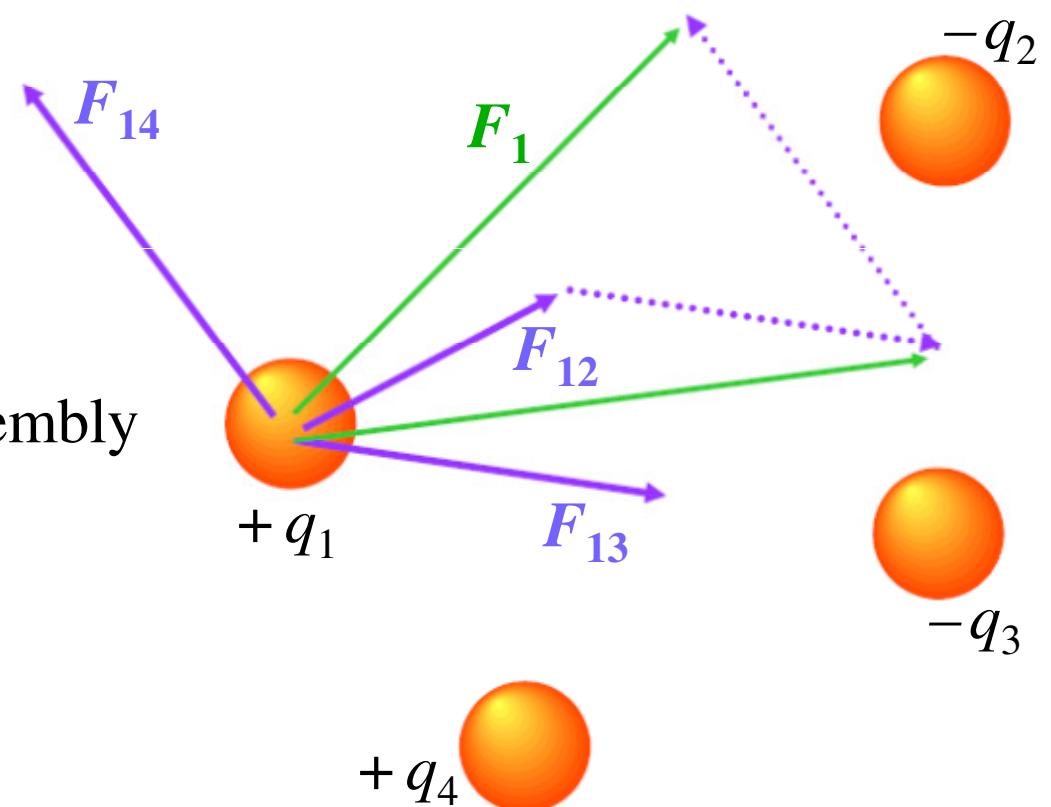
total force on  $q_1$

$$\vec{F}_1 = \vec{F}_{12} + \vec{F}_{13} + \vec{F}_{14}$$

for general N-charge assembly

total force on  $q_m$

$$\vec{F}_m = \sum_{n=1}^N \vec{F}_{mn} \Big|_{n \neq m}$$



# Electric Field

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not convenient to employ  $\vec{F}$  for analytical formulation

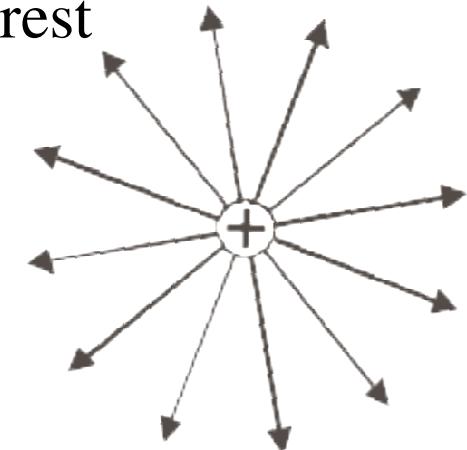
proceed by defining electric field  $\vec{E}$  due to charge  $q$

- place positive test charge  $q_{\text{test}}$  at point of interest
- divide Coulomb force by  $q_{\text{test}}$  magnitude

$$\vec{E} = \frac{q}{4\pi\epsilon_0 r^2} \hat{\mathbf{u}}_r$$

notes:

- $q_{\text{test}} \ll q$  so as not to perturb original electric-field pattern of  $q$
- unit of  $\text{NC}^{-1}$  from definition but common to use  $\text{V m}^{-1}$  instead
- valid for point charge  $q$  but can extend for linear systems



# Electric Field

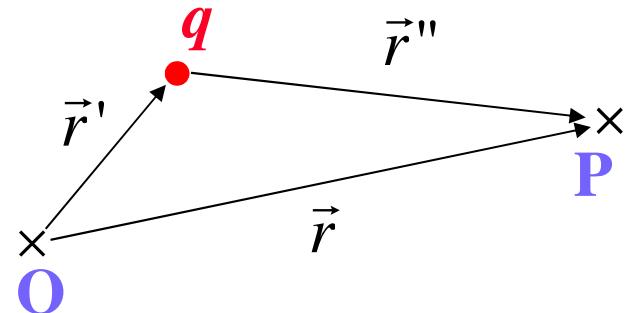
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employ superposition for linear multi-charge systems

need to account for origin of coordinate system

include magnitude of direction vector?

$$\vec{E} = \frac{q}{4\pi\epsilon_0(r'')^2} \hat{u}_{r''} \text{ or } \vec{E} = \frac{q}{4\pi\epsilon_0(r'')^3} \vec{r}''$$



direction vector generally regarded as a variable

(a) assembly of discrete charges     $\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{n=1}^N \frac{q_n}{(r'')_n^2} (\hat{u}_{r''})_n$

(b) volume of distributed charges     $\vec{E} = \frac{1}{4\pi\epsilon_0} \iiint_V \frac{\sigma}{(r'')^2} \hat{u}_{r''} dV$

# Electric Field

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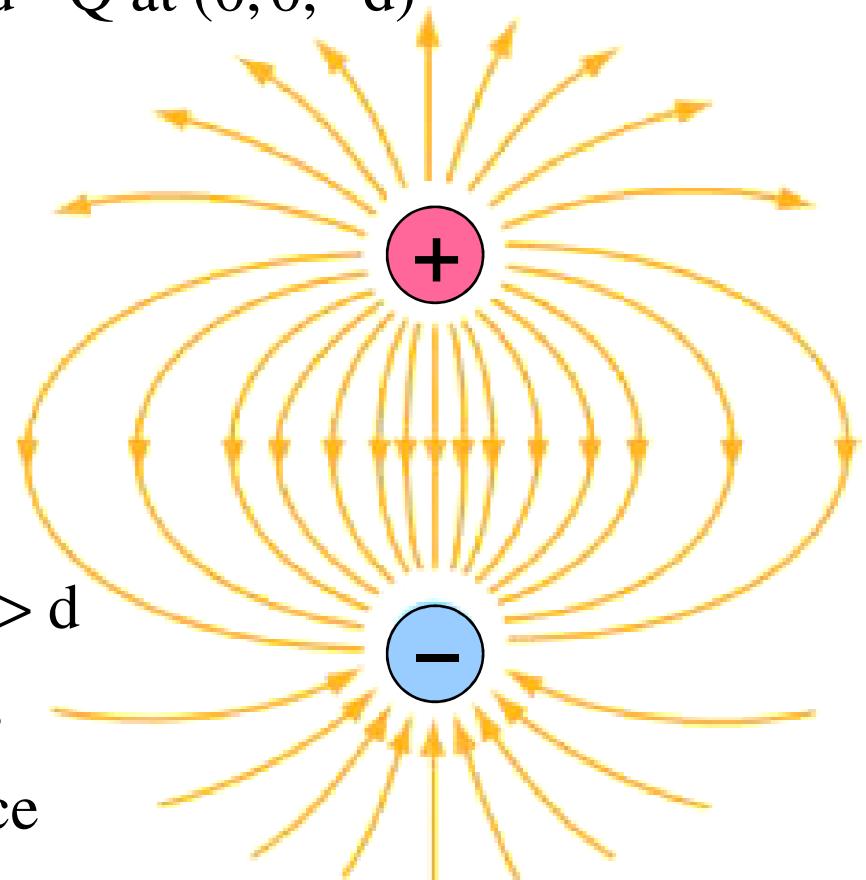
Example #1: dipole (pair of point charges with opposite polarity)  
+Q at (0, 0, +d) and -Q at (0, 0, -d)

$$\begin{aligned} \vec{E} = & \frac{Q \{ x\hat{u}_x + y\hat{u}_y + (z-d)\hat{u}_z \}}{4\pi\epsilon_0 \{ x^2 + y^2 + (z-d)^2 \}^{3/2}} \\ & - \frac{Q \{ x\hat{u}_x + y\hat{u}_y + (z+d)\hat{u}_z \}}{4\pi\epsilon_0 \{ x^2 + y^2 + (z+d)^2 \}^{3/2}} \end{aligned}$$

cancellation when  $\sqrt{x^2 + y^2 + z^2} \gg d$

still need to derive residual fields

commonly encountered in practice



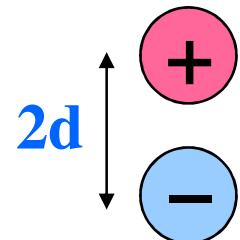
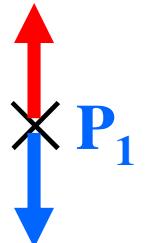
# Electric Field

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residual fields of dipole when  $\sqrt{x^2 + y^2 + z^2} \gg d$

(a) at  $P_1(0, 0, z)$  where  $z \gg d$

$$\begin{aligned}\vec{E} &= \frac{Q}{4\pi\epsilon_0(z-d)^3}(z-d)\hat{u}_z - \frac{Q}{4\pi\epsilon_0(z+d)^3}(z+d)\hat{u}_z \\ &= \frac{Q}{4\pi\epsilon_0 z^2} \left\{ \left(1 - \frac{d}{z}\right)^{-2} - \left(1 + \frac{d}{z}\right)^{-2} \right\} \hat{u}_z \\ &= \frac{Q}{4\pi\epsilon_0 z^2} \left\{ \left(1 + \frac{2d}{z} \dots\right) - \left(1 - \frac{2d}{z} \dots\right) \right\} \hat{u}_z \\ &\rightarrow \frac{Qd}{\pi\epsilon_0 z^3} \hat{u}_z \quad \text{where } 2Qd = \text{dipole moment}\end{aligned}$$



# Electric Field

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residual fields of dipole when  $\sqrt{x^2 + y^2 + z^2} \gg d$

(b) at  $P_2(x, 0, 0)$  where  $x \gg d$

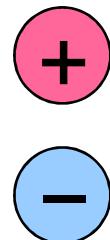
cancellation of their horizontal components

doubling of (residual) vertical components 

$$|E_z| = 2 \frac{Q}{4\pi\epsilon_0(x^2 + d^2)} \frac{d}{\sqrt{x^2 + d^2}} \rightarrow \frac{Qd}{2\pi\epsilon_0 x^3}$$

inversely proportional to  $r^3$  (instead of  $r^2$ ) for both  $P_1$  and  $P_2$

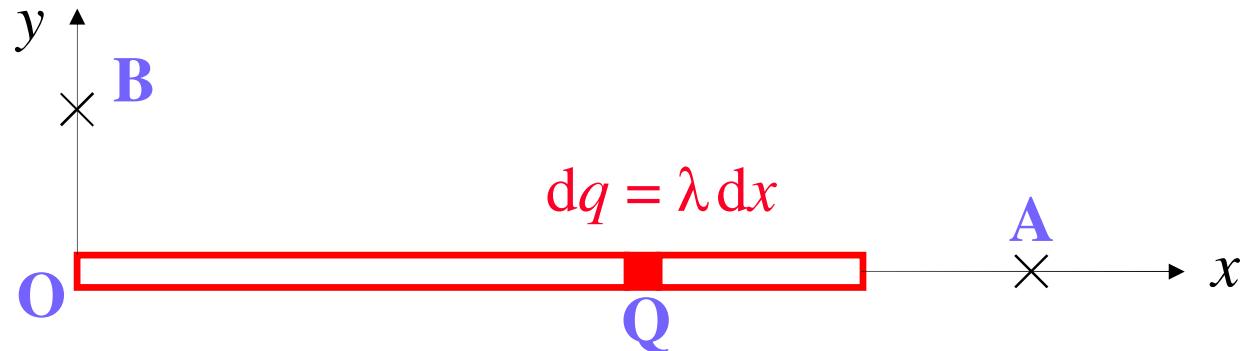
however, residual field not in radial direction for  $P_2$  



# Electric Field

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Example #2: rod (of length L) with uniform charge density  $\lambda \text{ C m}^{-1}$

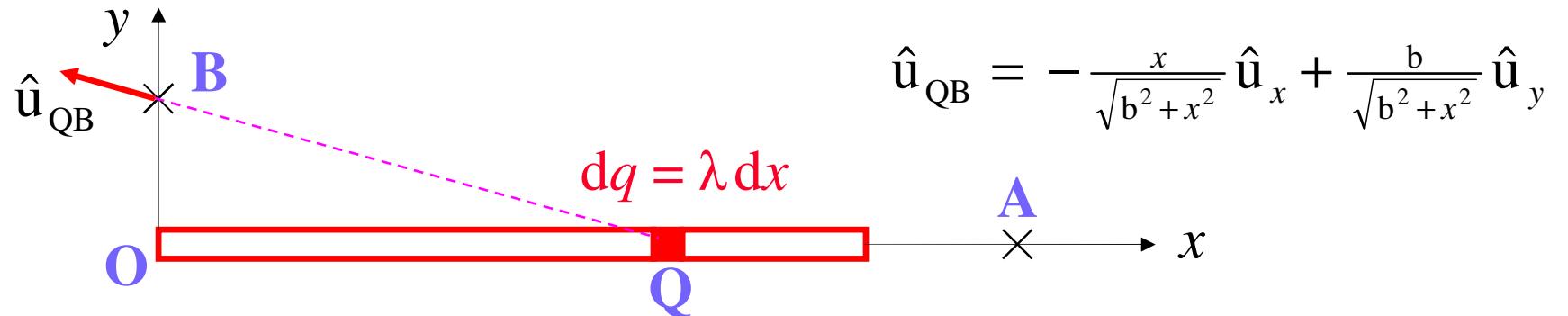


simpler case: obtain electric field at A (a, 0, 0)

$$\begin{aligned}\vec{E}_A &= \frac{1}{4\pi\epsilon_0} \int_{x=0}^L \frac{\lambda}{(a-x)^2} \hat{u}_x dx \quad \text{with same direction for all } d\vec{E} \text{ terms} \\ &= \hat{u}_x \frac{\lambda}{4\pi\epsilon_0} \int_{x=0}^L \frac{1}{(a-x)^2} dx = \hat{u}_x \frac{\lambda}{4\pi\epsilon_0} \frac{L}{a(a-L)} \rightarrow \hat{u}_x \frac{(\lambda L)}{4\pi\epsilon_0 a^2} \\ &\qquad \qquad \qquad \text{when } a \gg L\end{aligned}$$

# Electric Field

more difficult to derive electric field at B(0,b,0) due to variable  $\hat{u}_{QB}$



$$\hat{u}_{QB} = -\frac{x}{\sqrt{b^2+x^2}} \hat{u}_x + \frac{b}{\sqrt{b^2+x^2}} \hat{u}_y$$

$$\vec{E}_B = \frac{1}{4\pi\epsilon_0} \int_{x=0}^L \frac{\lambda}{b^2+x^2} \hat{u}_{QB} dx$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left\{ -\hat{u}_x \int_{x=0}^L \frac{x}{(b^2+x^2)^{\frac{3}{2}}} dx + b \hat{u}_y \int_{x=0}^L \frac{1}{(b^2+x^2)^{\frac{3}{2}}} dx \right\}$$

$$= \frac{\lambda}{4\pi\epsilon_0} \left\{ -\hat{u}_x \left( \frac{1}{b} - \frac{1}{\sqrt{b^2+L^2}} \right) + \hat{u}_y \frac{L}{b\sqrt{b^2+L^2}} \right\}$$

$$\rightarrow \hat{u}_y \frac{(\lambda L)}{4\pi\epsilon_0 b^2}$$

when  $b \gg L$

# Electric Field

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need to define  $\Phi$  for electric field traversing plane surface

$$\Phi \propto E_n A$$

- (a) add proportionality constant \*

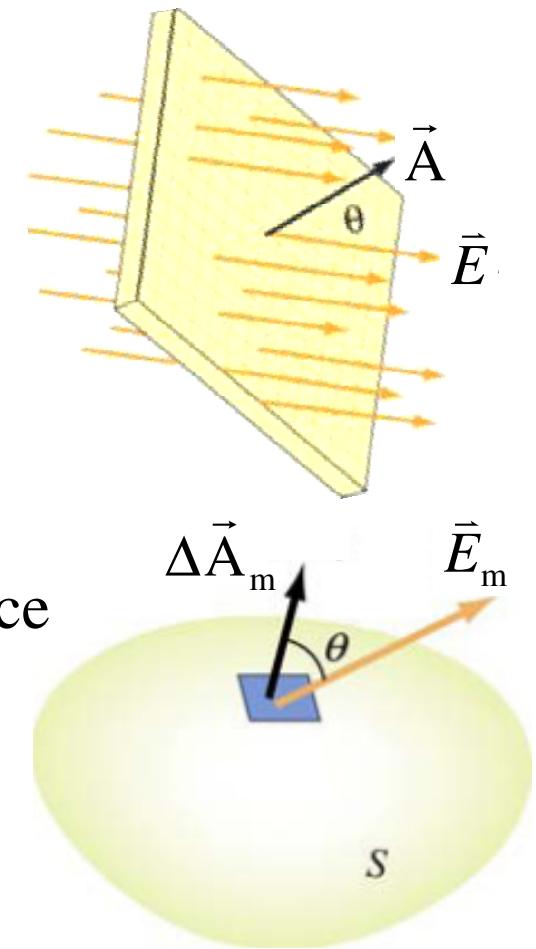
$$\Phi = \epsilon E_n A *$$

- (b) account for orientation via dot product

$$\Phi = \epsilon \vec{E} \bullet \vec{A}$$

- (c) consider elemental areas for curved surface

$$\Phi = \iint \epsilon \vec{E} \bullet d\vec{A}$$



\* will later introduce another vector ( $D = \epsilon E$ )

# Electric Field

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special property for closed surface

$$\oint_S \epsilon \vec{E} \bullet d\vec{A} = Q_{\substack{\text{total in} \\ \text{enclosure}}} = \begin{cases} \sum_{m=1}^M q_m & \text{for point charges} \\ \iiint_V \sigma dV & \text{for distributed charges} \end{cases}$$

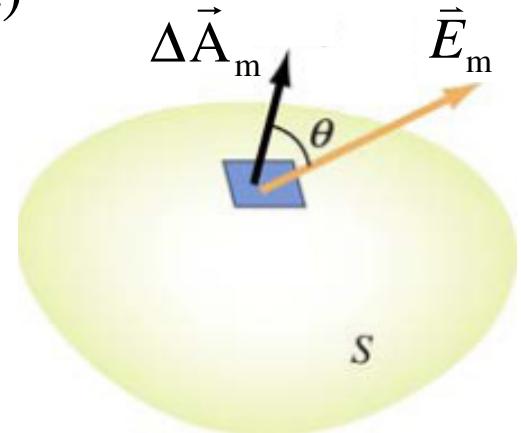
possible simplifications:

(a) if using homogeneous material (uniform  $\epsilon$ )

$$\oint_S \vec{E} \bullet d\vec{A} = \frac{1}{\epsilon} Q_{\text{total}}$$

(b) if also adopting structural symmetry

$$E_n A = \frac{1}{\epsilon} Q_{\text{total}}$$



# Gauss's Law

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macroscopic approach for deriving electric field from charges

$$\oint_S \vec{E} \bullet d\vec{A} = \frac{1}{\epsilon} Q_{\text{total}}$$

always valid but useful only with symmetry

explanatory example: point charge in free space

(a) for red Gaussian surface (with symmetry)

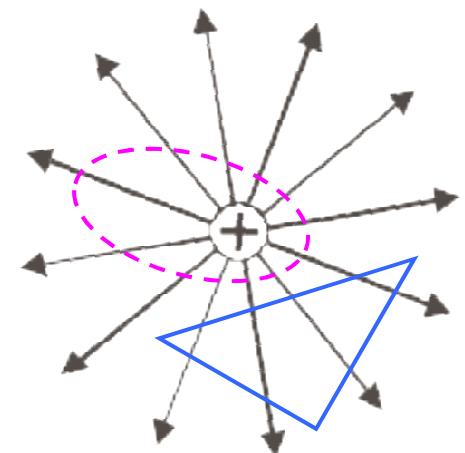
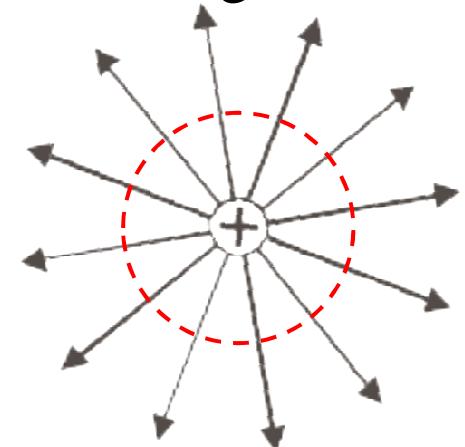
$$\begin{aligned} \oint_{S_{\text{red}}} \vec{E} \bullet d\vec{A} &= E_r \oint_{S_{\text{red}}} dA_r = E_r 4\pi r^2 = \frac{q}{\epsilon_0} \\ \Rightarrow E_r &= \frac{q}{4\pi\epsilon_0 r^2} \quad i.e. \text{ Coulomb's Law} \end{aligned}$$

(b) for pink Gaussian surface

$$\oint_{S_{\text{pink}}} \vec{E} \bullet d\vec{A} = \frac{q}{\epsilon_0}$$

(c) for blue Gaussian surface

$$\oint_{S_{\text{blue}}} \vec{E} \bullet d\vec{A} = 0$$



# Gauss's Law

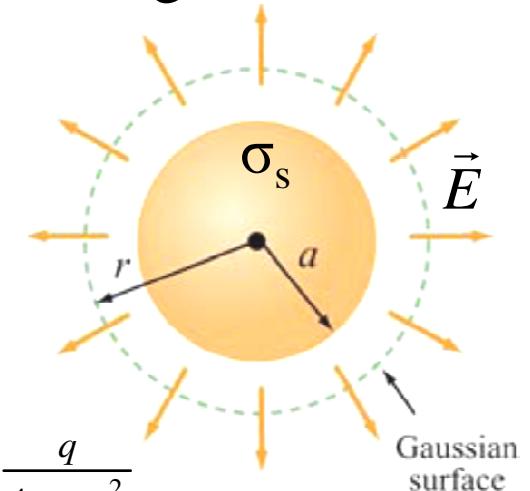
Example #1: conducting sphere with distributed charges  
infer from symmetry:

- no non-radial component (*i.e.*  $\vec{E} = E_r \hat{u}_r$ )
- $E_r$  constant of  $\theta$  and  $\phi$  coordinates

(a) apply Gauss's Law to green surface

$$E_r 4\pi r^2 = \frac{1}{\epsilon_0} (\sigma_s 4\pi a^2) \Rightarrow E_r = \frac{(\sigma_s 4\pi a^2)}{4\pi\epsilon_0 r^2} = \frac{q}{4\pi\epsilon_0 r^2}$$

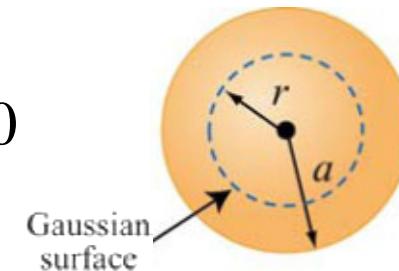
→ introduce equivalent point charge to replace charge density  $\sigma_s$



(b) apply Gauss's Law to blue surface

$$\oint_{S_{\text{blue}}} \vec{E} \bullet d\vec{A} = \frac{1}{\epsilon_0} \iiint_V \sigma dV \Rightarrow Q_{\text{enclosed}} = 0$$

→ all charges residing at outermost surface



# Gauss's Law

Example #2: large thin sheet with uniform charge density  $\sigma_s \text{ C m}^{-2}$   
infer from symmetry:

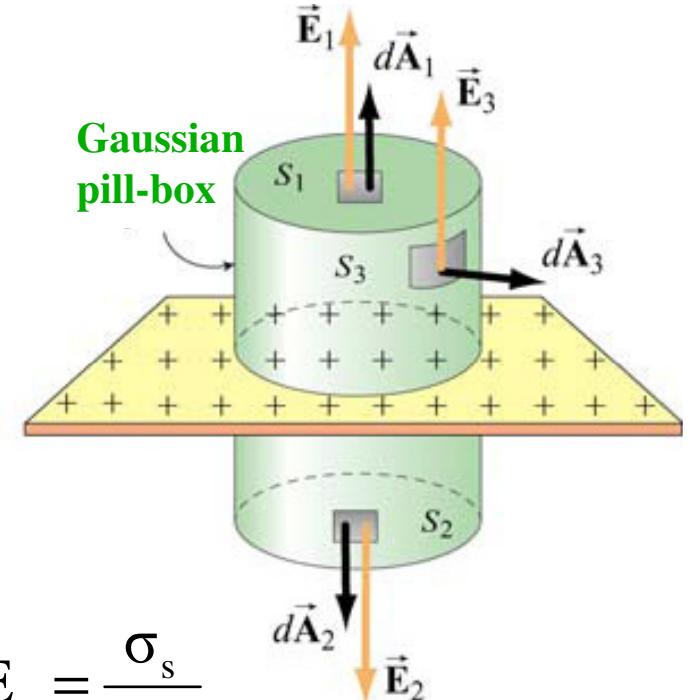
- no non-normal component ( $\vec{E} = E_n \hat{u}_n$ )
- $E_n$  constant of both planar coordinates

apply Gauss's Law to green pill-box:

- (a)  $S_1$  and  $S_2$  equi-distant from sheet
- (b) no contribution at  $S_3$  as  $\vec{E}_3 \bullet d\vec{A}_3 = 0$

$$\left. \begin{aligned} \text{LHS} &= (E_1 + E_2) \pi r^2 + 0 = 2 E_n \pi r^2 \\ \text{RHS} &= \sigma_s \pi r^2 \end{aligned} \right\} E_n = \frac{\sigma_s}{2\epsilon_0}$$

→  $E_n$  constant of all 3 coordinates (but cannot extrapolate result)



# Gauss's Law

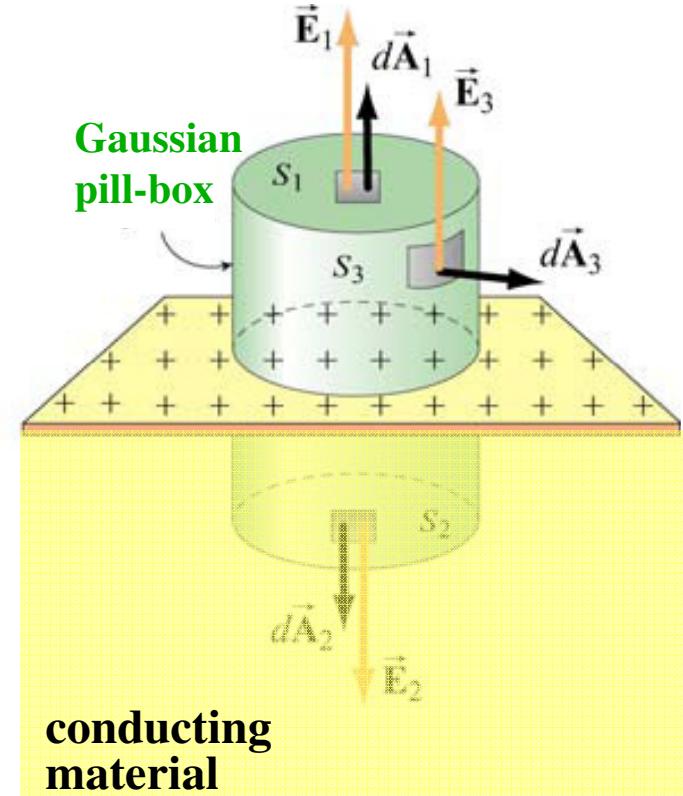
Example #3: large flat interface between metal and air  
infer from symmetry:

- no non-normal component ( $\vec{E} = E_n \hat{u}_n$ )
- $E_n$  constant of both planar coordinates

apply Gauss's Law to green pill-box:

- (a)  $E = 0$  in perfect electric conductor
- (b) no contribution at  $S_3$  as  $\vec{E}_3 \bullet d\vec{A}_3 = 0$

$$\left. \begin{array}{l} \text{LHS} = E_n \pi r^2 \\ \text{RHS} = \sigma_s \pi r^2 \end{array} \right\} E_n = \frac{\sigma_s}{\epsilon_o}$$

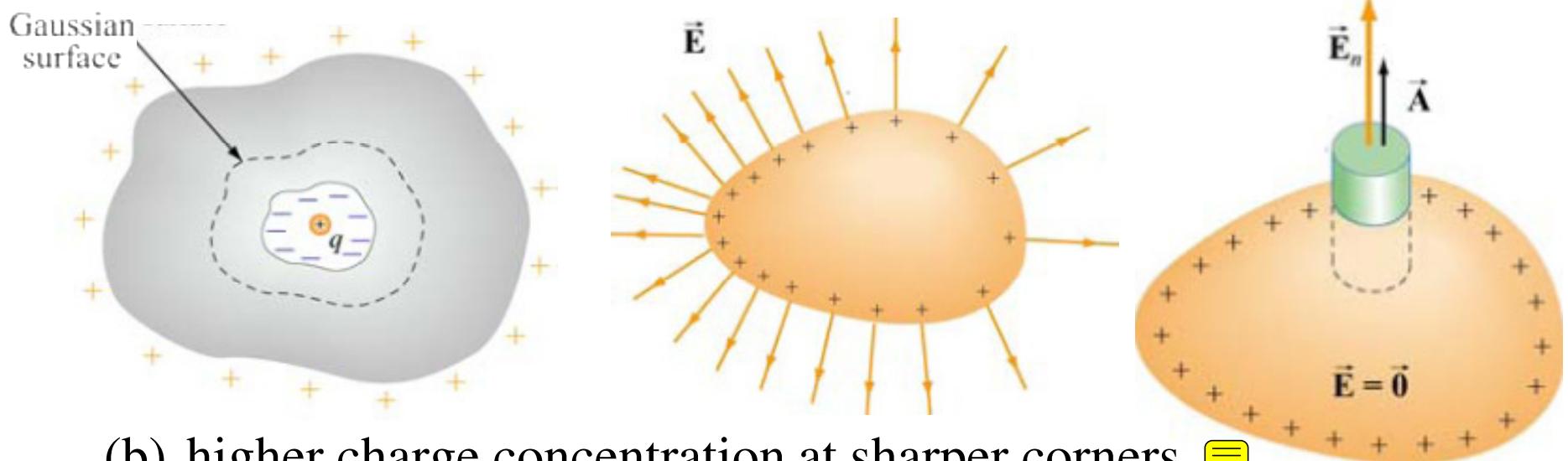


→  $E_n$  constant of all 3 coordinates (but cannot extrapolate result)

# Gauss's Law

inside perfectly conducting material

- (a)  $Q_{\text{enclosed}} = 0 \Rightarrow \begin{cases} \text{either no charges within Gaussian surface} \\ \text{or zero sum of positive and negative charges} \end{cases}$



- (b) higher charge concentration at sharper corners

- (c)  $\vec{E} = \frac{\sigma_s}{\epsilon_0} \hat{u}_n$  in immediate vicinity of surface (where  $\sigma_s$  and  $\hat{u}_n$  generally vary)

# Electric Potential

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inconvenient to use  $\vec{E}$  when studying system of electric charges

- requires vector calculus for analysis
- cumbersome for visual representation

conservative nature of electrostatic fields

$$\text{---} \oint_{\text{any loop}} \vec{E} \bullet d\vec{s} = 0 \rightarrow \text{---} \iint_{\text{any area}} \nabla \times \vec{E} \bullet d\vec{A} = 0 \rightarrow \nabla \times \vec{E} = \vec{0}$$

define scalar potential (via null identity #1)

$$\vec{E} = -\nabla V \quad (\text{or } \vec{E} = -\nabla \phi \text{ especially in physics textbooks})$$

- negative sign by convention (so as to obtain positive  $V$  values)
- also extended to time-varying  $\vec{E}$  (although non-conservative)

# Electric Potential

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integral definition for potential

$$V_P = - \int_{\infty}^P \vec{E} \bullet d\vec{s} \quad \Delta V = V_P - V_Q = - \int_Q^P \vec{E} \bullet d\vec{s}$$

(a) dimensions of definition equation

$$\text{Volt} \times \text{Coulomb} = \text{Newton} \times \text{metre} = \text{Joule}$$
$$\Rightarrow \text{Volt} = \text{Joule} / \text{Coulomb}$$

interpret  $V$  as work done in moving unit positive charge

(b) require reference (due to need for constant of integration)  
common to set  $V = 0$  at infinity  
not valid for infinitely-large systems

(c) same  $V$  (or  $\Delta V$ ) for any path in conservative field

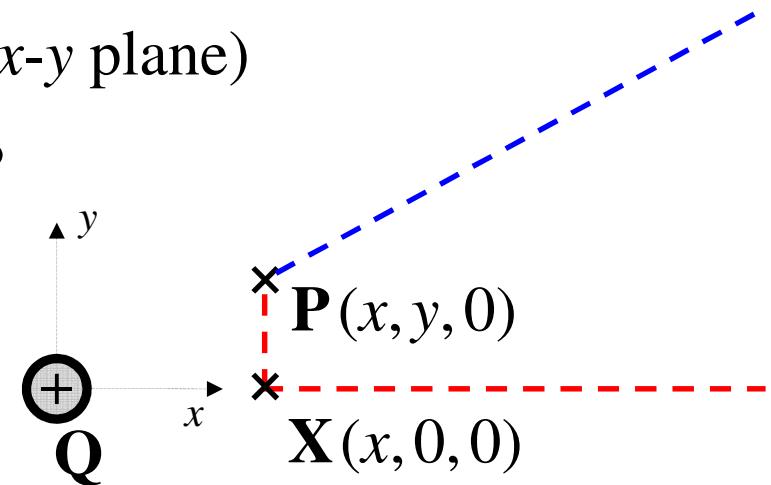
# Electric Potential

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Example #1: point charge Q (in  $x$ - $y$  plane)

(a) blue (radial) path from  $\infty$  to P

$$\begin{aligned} V_P &= - \int_{\infty}^P \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{u}}_r \bullet d\mathbf{r} \hat{\mathbf{u}}_r \quad \text{💡} \\ &= -\frac{Q}{4\pi\epsilon_0} \left[ -\frac{1}{r} \right]_{\infty}^r = \frac{Q}{4\pi\epsilon_0 r} \end{aligned}$$



(b) red path from  $\infty$  to P via X

$$\begin{aligned} V_P &= - \int_{\infty}^x \frac{Q}{4\pi\epsilon_0 x^2} \hat{\mathbf{u}}_x \bullet dx \hat{\mathbf{u}}_x - \int_0^y \frac{Q}{4\pi\epsilon_0 r^2} \hat{\mathbf{u}}_r \bullet dy \hat{\mathbf{u}}_y \\ &= - \int_{\infty}^x \frac{Q}{4\pi\epsilon_0 x^2} dx - \int_0^{\sqrt{r^2-x^2}} \frac{Qy}{4\pi\epsilon_0 (x^2+y^2)^{\frac{3}{2}}} dy = \frac{Q}{4\pi\epsilon_0 r} \end{aligned}$$

used for deriving  $V$  of linear charge system via superposition

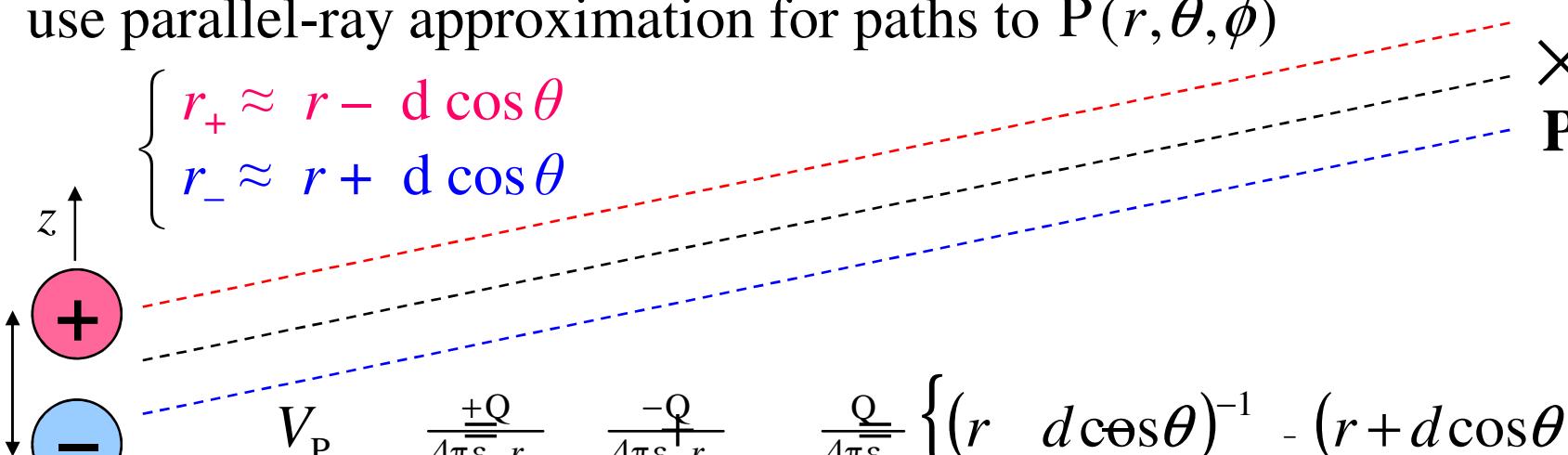
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# Electric Potential

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Example #2: dipole with  $r \gg d$

use parallel-ray approximation for paths to  $P(r, \theta, \phi)$



$$\left\{ \begin{array}{l} r_+ \approx r - d \cos \theta \\ r_- \approx r + d \cos \theta \end{array} \right.$$

$$V_P = \frac{\pm Q}{4\pi\epsilon_0 r_+} + \frac{-Q}{4\pi\epsilon_0 r_-} = \frac{Q}{4\pi\epsilon_0} \left\{ (r - d \cos \theta)^{-1} - (r + d \cos \theta)^{-1} \right\} = \frac{Q d \cos \theta}{2\pi\epsilon_0 r^2}$$

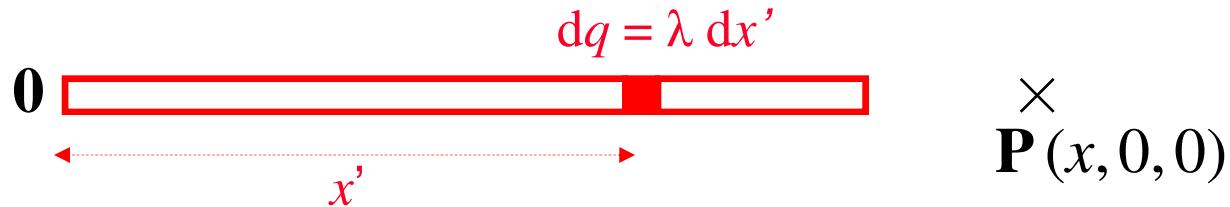
easier to derive field

$$\vec{E} = -\nabla V = - \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r} \frac{\partial}{\partial \theta} \\ \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} \end{bmatrix} \frac{Q d \cos \theta}{2\pi\epsilon_0 r^2} = \frac{Q d}{2\pi\epsilon_0 r^3} \begin{bmatrix} 2 \cos \theta \\ \sin \theta \\ 0 \end{bmatrix}$$

# Electric Potential

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Example #3: rod (of length L) with linear charge density  $\lambda \text{ C m}^{-1}$



$$dV_P = \frac{\lambda dx'}{4\pi\epsilon_0 (x - x')} \quad \text{for each elemental length along rod}$$

$$V_P = \frac{\lambda}{4\pi\epsilon_0} \int_{x'=0}^{x'=x} \frac{dx'}{(x-x')} = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{x}{x-L} \quad (\text{symmetry} \rightarrow \text{depends only on } x)$$

$$\vec{E}_P = - \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} V_P(x) = - \frac{d}{dx} V_P(x) \hat{u}_x = - \frac{\lambda}{4\pi\epsilon_0} \frac{d}{dx} \left( \ln \frac{x}{x-L} \right) \hat{u}_x = \frac{\lambda}{4\pi\epsilon_0} \frac{L}{x(x-L)} \hat{u}_x$$

# Electric Potential

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Gauss's Law for **any** volume of distributed charges

$$\oint_S \epsilon \vec{E} \bullet d\vec{A} = Q_{\substack{\text{total in} \\ \text{enclosure}}} = \iiint_V \sigma dV$$
$$\Rightarrow \text{ } \iiint_V \nabla \bullet \vec{E} dV = \frac{1}{\epsilon} \iiint_V \sigma dV$$

need non-integral version for application to **any** particular point

$$\nabla \bullet \vec{E} = \frac{1}{\epsilon} \sigma$$

more convenient to use  $V$  instead of  $\vec{E}$  for analysis

$$\nabla \bullet (-\nabla V) = \frac{1}{\epsilon} \sigma \rightarrow \begin{cases} \nabla^2 V = -\frac{1}{\epsilon} \sigma & \text{Poisson equation} \\ \nabla^2 V = 0 & \text{Laplace equation} \end{cases}$$

partial differential equation requiring boundary conditions

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# Electric Potential

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Laplacian operator:

$$\nabla^2 = \nabla \bullet \nabla = \begin{cases} \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} & \text{ } \\ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} & \text{ } \\ \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} & \text{ } \end{cases}$$

uniqueness theorem:

only one possible solution that will satisfy partial differential equation (PDE) and all boundary conditions (BCs) regardless of approach (even by intelligent guess-work)

# Electric Potential

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Example #1: (one-dimensional) electron cloud between electrodes

no variations in  $\hat{u}_y$  and  $\hat{u}_z$  directions

i.e.  $\frac{\partial}{\partial y} = \frac{\partial}{\partial z} = 0$

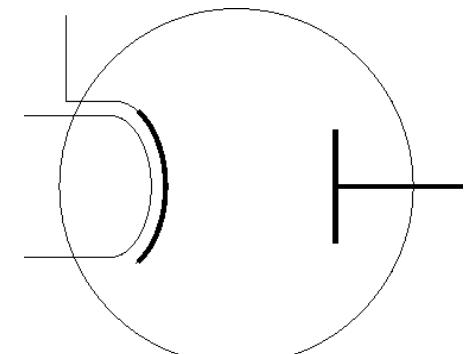
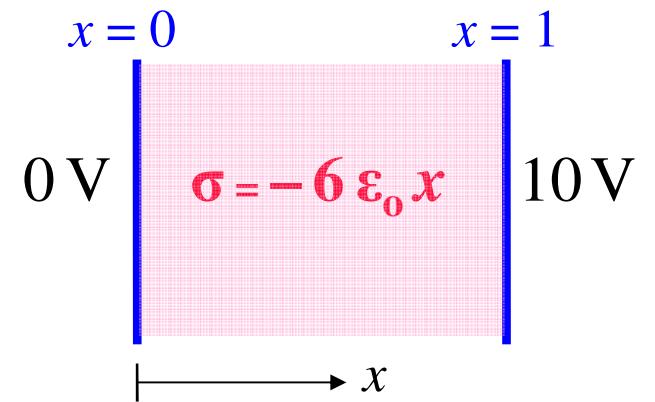
reduced to ordinary differential equation

$$\frac{d^2V}{dx^2} = -\frac{(-6\epsilon_0 x)}{\epsilon_0} = 6x$$

$$\Rightarrow V = x^3 + c_1 x + c_2$$

$$\left. \begin{array}{l} \text{BCs: } V(x=0) = 0 \\ V(x=1) = 10 \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} c_1 = 9 \\ c_2 = 0 \end{array} \right.$$

$$\therefore V = x^3 + 9x \quad \text{and} \quad E_x = -3x^2 + 9$$



# Electric Potential

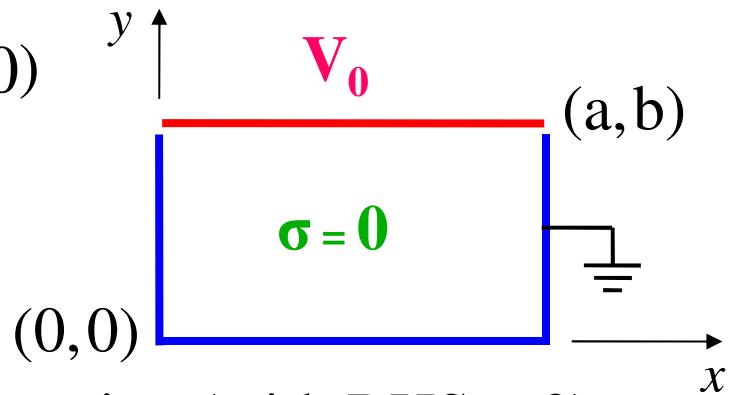
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Example #2: (two-dimensional) earthed trough with charged lid

no variations in  $\hat{u}_z$  direction (*i.e.*  $\frac{\partial}{\partial z} = 0$ )

reduced to PDE in two coordinates

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$



no free charges in trough  $\rightarrow$  Laplace equation (with RHS = 0)

try  $V(x, y) = X(x) Y(y)$   $\rightarrow$  
$$\begin{cases} \frac{\partial^2 V}{\partial x^2}(x, y) = \frac{d^2 X}{dx^2}(x) Y(y) \\ \frac{\partial^2 V}{\partial y^2}(x, y) = X(x) \frac{d^2 Y}{dy^2}(y) \end{cases}$$

$$\therefore \frac{1}{X} \frac{d^2 X}{dx^2}(x) + \frac{1}{Y} \frac{d^2 Y}{dy^2}(y) = 0 \rightarrow \begin{cases} \frac{1}{X} \frac{d^2 X}{dx^2}(x) = -c^2 \quad (\text{c = constant}) \\ \frac{1}{Y} \frac{d^2 Y}{dy^2}(y) = +c^2 \end{cases}$$

# Electric Potential

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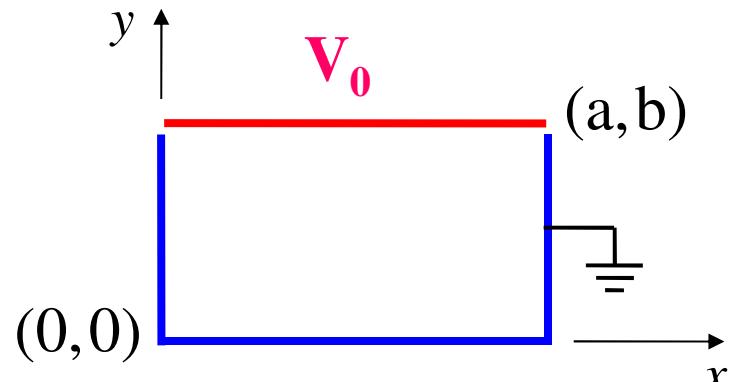
choice of sign for constant determined by boundary conditions

$$\text{BC1: } V(0, y) = 0 \rightarrow X = A \sin(cx)$$

$$\text{BC2: } V(x, 0) = 0 \rightarrow Y = B \sinh(cy)$$

$$\text{BC3: } V(a, y) = 0 \rightarrow \sin(c a) = 0$$

$$\text{BC4: } V(x, b) = V_0$$



eigenvalues from BC3  $\Rightarrow$  family of solutions with  $c_m = \frac{m\pi}{a}$

not possible to satisfy BC4 with only one eigen-solution

try superposition of all harmonics

$$\therefore V(x, y) = X(x)Y(y) = \sum_m d_m \sin\left(m\pi \frac{x}{a}\right) \sinh\left(m\pi \frac{y}{a}\right)$$

# Electric Potential

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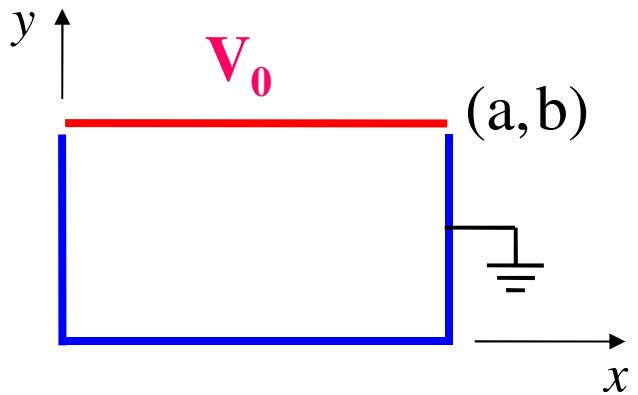
obtain modal coefficient  $d_m$  from BC4 (where  $y = b$  for charged lid)

$$\sum_m d_m \sin\left(m\pi \frac{x}{a}\right) \sinh\left(m\pi \frac{b}{a}\right) = V_0$$

capitalize on orthogonality of  $\sin\left(m\pi \frac{x}{a}\right)$

(a) multiply both sides by  $\sin\left(n\pi \frac{x}{a}\right)$

(b) integrate over  $0 \leq x \leq a$  (using process for Fourier series)



$$\int_{x=0}^a \sum_m d_m \left\{ \sin\left(m\pi \frac{x}{a}\right) \sin\left(n\pi \frac{x}{a}\right) \right\} \sinh\left(m\pi \frac{b}{a}\right) dx = \int_{x=0}^a V_0 \sin\left(n\pi \frac{x}{a}\right) dx$$

$$\Rightarrow d_n = \frac{4V_0}{n\pi \sinh\left(n\pi \frac{b}{a}\right)} \text{ when } n \text{ is odd} \quad \text{but } d_n = 0 \text{ when } n \text{ is even}$$

$$\therefore V(x, y) = \frac{4V_0}{\pi} \sum_m \frac{\sin\left(m\pi \frac{x}{a}\right) \sinh\left(m\pi \frac{y}{a}\right)}{m \sinh\left(m\pi \frac{b}{a}\right)} \text{ for } m = 1, 3, 5, \dots$$

# Electric Potential

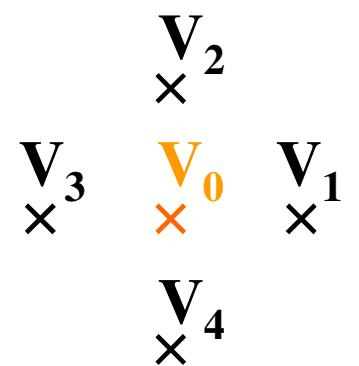
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approximate solutions via numerical techniques

- (a) popular after availability of computational resources
- (b) must check for possibility of spurious solutions

two-dimensional finite differences

$$\frac{\partial^2 V}{\partial x^2} = \frac{\Delta \left( \frac{\partial V}{\partial x} \right)}{\Delta x} = \frac{\frac{V_1 - V_0}{\delta} - \frac{V_0 - V_3}{\delta}}{\delta} = \frac{V_1 + V_3 - 2V_0}{\delta^2}$$



$$\frac{\partial^2 V}{\partial y^2} = \frac{\Delta \left( \frac{\partial V}{\partial y} \right)}{\Delta y} = \frac{\frac{V_2 - V_0}{\delta} - \frac{V_0 - V_4}{\delta}}{\delta} = \frac{V_2 + V_4 - 2V_0}{\delta^2}$$

$$\rightarrow \nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{\delta^2}$$

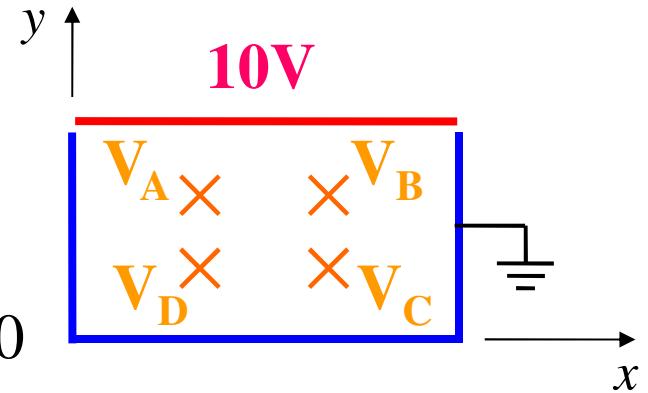
# Electric Potential

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Poisson equation in two-dimensional finite-difference form

$$\frac{V_1 + V_2 + V_3 + V_4 - 4V_0}{\delta^2} = -\frac{\sigma}{\epsilon_0}$$

$$\Rightarrow V_0 = \frac{1}{4} \left( \sum_{k=1}^4 V_k + \frac{\sigma}{\epsilon_0} \delta^2 \right)$$



Laplace equation for Example #2 with  $\sigma = 0$

choose (appropriate) initial values:  $V_A = V_B = 5V$  and  $V_C = V_D = 0$

first iteration:  $V_A = \frac{1}{4} (10 + V_B + V_D + 0) = \frac{10 + 5 + 0 + 0}{4} = 3.75 \text{ V}$

$$V_B = \frac{1}{4} (10 + 0 + V_C + V_A) = \frac{10 + 0 + 0 + 3.75}{4} = 3.44 \text{ V}$$

$$V_C = \frac{1}{4} (V_B + 0 + 0 + V_D) = \frac{3.44 + 0 + 0 + 0}{4} = 0.86 \text{ V}$$

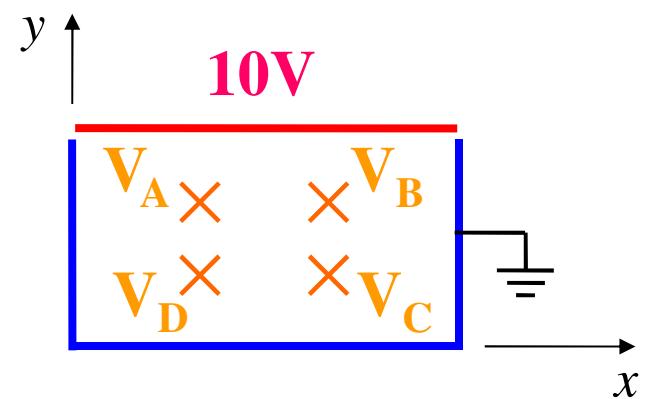
$$V_D = \frac{1}{4} (V_A + V_C + 0 + 0) = \frac{3.75 + 0.86 + 0 + 0}{4} = 1.15 \text{ V}$$

next iteration:  $V_A = \frac{1}{4} (10 + V_B + V_D + 0) = \frac{10 + 3.44 + 1.15 + 0}{4} = 3.65 \text{ V}$

---

# Electric Potential

iteration #	V <sub>A</sub>	V <sub>B</sub>	V <sub>C</sub>	V <sub>D</sub>	
0	5.00	5.00	0	0	
1	3.75	3.44	0.36	1.15	
2	3.65	3.63	1.20	1.21	
3	3.71	3.73	1.24	1.24	
4	3.74	3.75	1.25	1.25	
5	3.75	3.75	1.25	1.25	→ onset of convergence



check numerical value of V<sub>A</sub> against analytical result

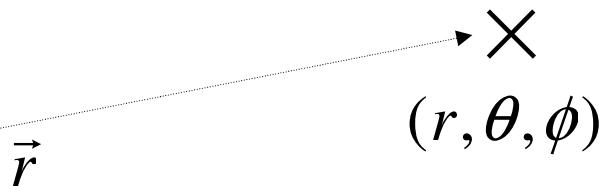
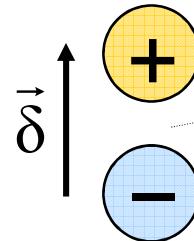
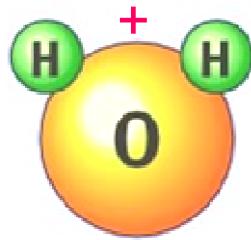
$$V\left(\frac{a}{3}, \frac{2b}{3}\right) = \frac{40}{\pi} \sum_{m:\text{odd}} \frac{\sin\left(\frac{m\pi}{3}\right) \sinh\left(\frac{2m\pi}{3}\right)}{m \sinh(m\pi)} = 3.81 \text{ V} \text{ for } a=b=1$$

can improve accuracy by reducing mesh size (*i.e.*  $\delta \ll a$  and  $b$ )

quite robust (try using  $V_A = V_B = V_C = V_D = 0$  as initial values)

# Dielectrics

polar material (with built-in dipoles)



define  $\vec{p} = q \vec{\delta}$  for dipole  $\pm q$  separated by  $\delta$

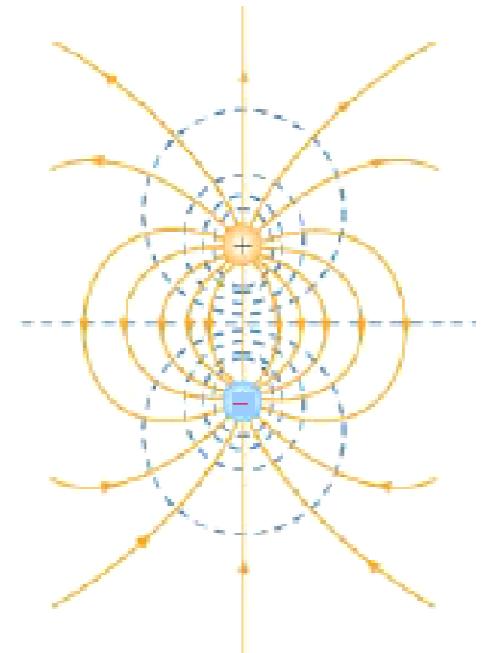
derive  $\vec{E}$  from  $V(r, \theta, \phi) = \frac{q \delta \cos \theta}{4\pi \epsilon_0 r^2} = \frac{\vec{p} \cdot \vec{r}}{4\pi \epsilon_0 r^3}$

not possible to consider individual dipoles

have to consider average density instead

define polarization vector  $\vec{P} = \frac{1}{\text{volume}} \sum_k \vec{p}_k$

$\therefore dV(r, \theta, \phi) = \frac{\vec{P} \cdot \vec{r}}{4\pi \epsilon_0 r^3} dV$  for dipoles in  $dV$



# Dielectrics

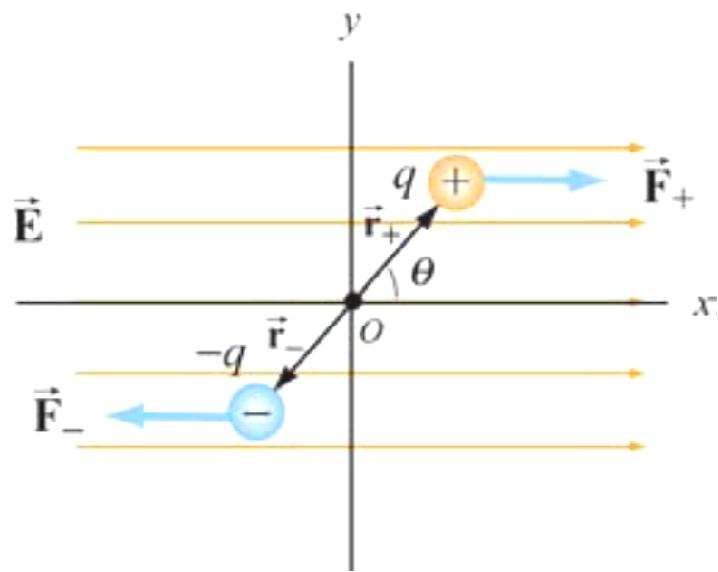
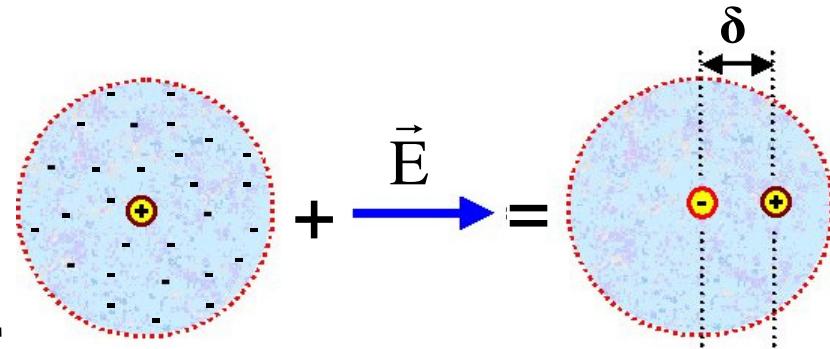
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non-polar materials

common  $\pm$  charge centers

separation induced by  $\vec{E}$

dipole behavior in presence of  $\vec{E}$



electric forces on dipole charges

no net force  $\rightarrow$  no  $\pm x$  shift for dipole

dipole moment due to  $\pm q$  separation

rotation until alignment of forces

linear increase of  $|\vec{P}|$  until saturation

# Dielectrics

dielectric slab

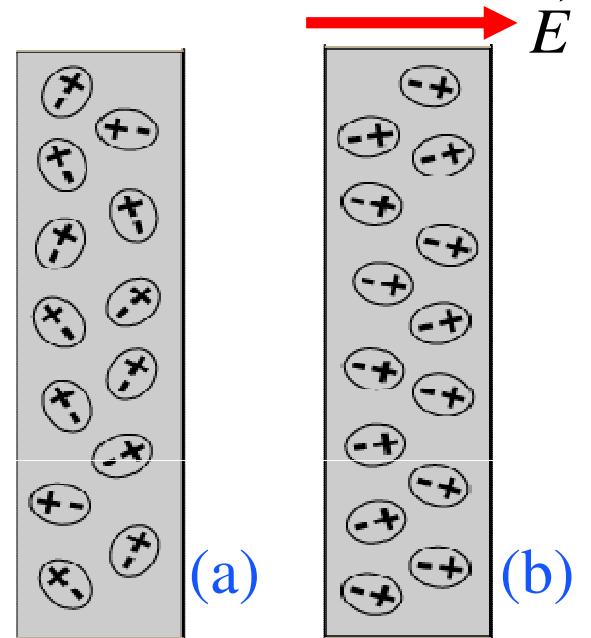
- (a) originally random dipole orientations  
→ no net electric fields from dipoles
- (b) partial dipole alignment under  $\vec{E}$   
→ need to include polarization  $\vec{P}$

Gauss's Law in differential form

$$\nabla \bullet \vec{E} = \frac{1}{\epsilon_0} \sigma \quad \text{for free space}$$

$$\nabla \bullet (\epsilon_0 \vec{E} + \vec{P}) = \sigma \quad \text{for dielectric}$$

recast as  $\nabla \bullet \vec{D} = \sigma$  by introducing  $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$   
electric flux density vector ( $\text{C m}^{-2}$ )



# Dielectrics

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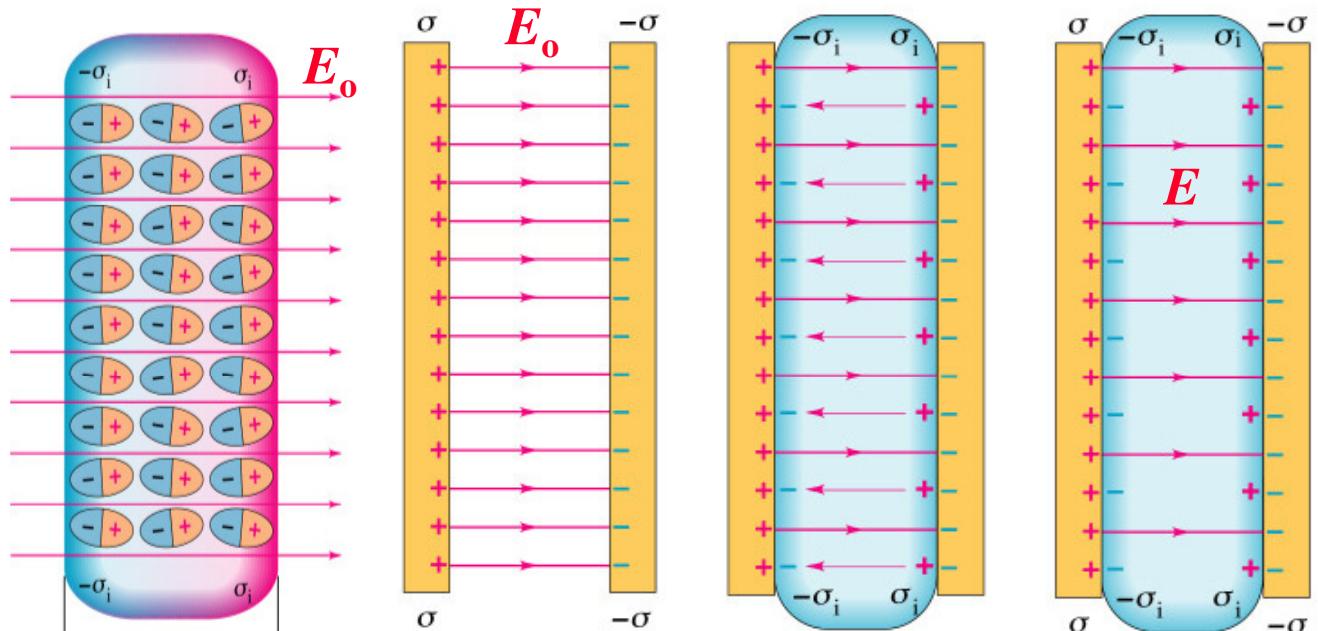
field-matter interactions

$$\vec{P} = \epsilon_0 \chi_e \vec{E}$$

$$\begin{aligned}\vec{D} &= (1 + \chi_e) \epsilon_0 \vec{E} \\ &= \epsilon_r \epsilon_0 \vec{E} = \epsilon \vec{E}\end{aligned}$$

$\chi_e$  susceptibility

$\epsilon$  permittivity



simply replace  $\epsilon_0$  by  $\epsilon$  to account for presence of dielectric

point-charge example:  $\vec{E}_0 = \frac{q}{4\pi\epsilon_0 r^2} \hat{u}_r$  becomes  $\vec{E} = \frac{q}{4\pi\epsilon r^2} \hat{u}_r$

$|\vec{E}| < |\vec{E}_0|$  due to net electric fields of dipoles (after alignment)

# Capacitors

---

two conductors separated by air/dielectric with

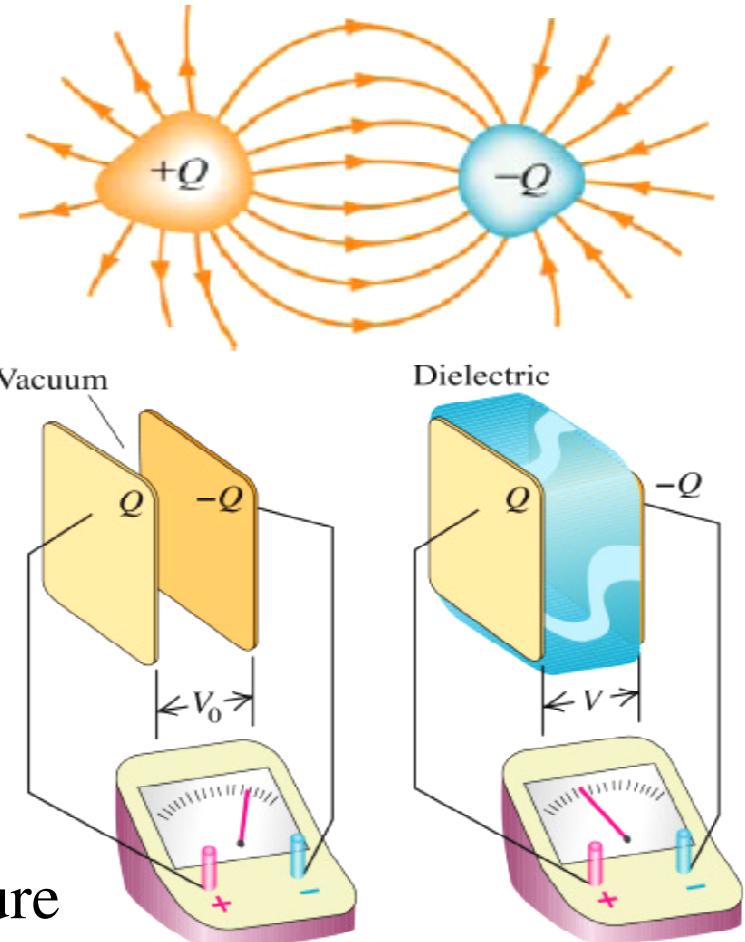
- equal and opposite charges  $\pm Q$
- potential difference  $\Delta V = V_1 - V_2$

experiments  $\rightarrow Q \propto \Delta V$

define storage parameter  $C = \frac{Q}{\Delta V}$   
 $(F = C V^{-1})$

can be increased via:

- increasing dielectric permittivity
- reducing conductor separation
- proper design of conductor structure



# Capacitors

usual procedure to evaluate capacitance

- (a) assume  $\pm Q$  on conductors
- (b) derive  $\vec{E}$  from charge distribution
- (c) derive  $\Delta V$  via potential definition
- (d) compute ratio

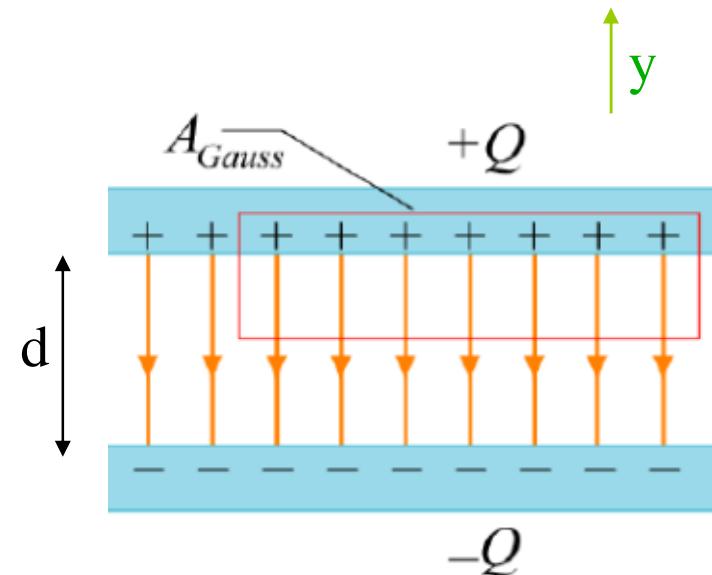
Example #1: large parallel plates

apply Gauss's Law to pill-box

$$E_y A_{\text{pill-box}} = -\frac{1}{\epsilon_0} \sigma_s A_{\text{pill-box}} \Rightarrow E_y = -\frac{1}{\epsilon_0} \sigma_s$$

$$\Delta V = -\frac{-Q}{\epsilon_0 A} \int_0^d dy = \frac{Qd}{\epsilon_0 A} \Rightarrow C = \frac{A\epsilon_0}{d}$$

larger C by inserting dielectric, increasing A or decreasing d



# Capacitors

Example #2: coaxial cable with uniform charge density  $\lambda \text{ C m}^{-1}$   
apply Gauss's Law to green cylinder

infer from symmetry (for short  $l$ ):

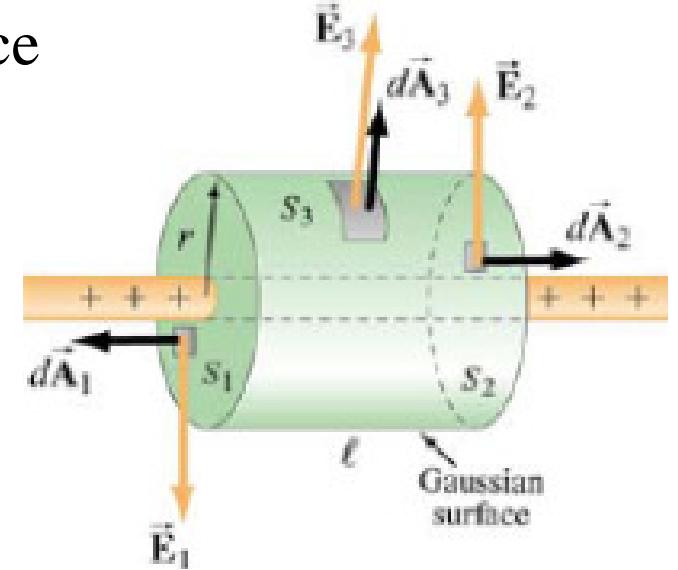
- no non-radial component (*i.e.*  $\vec{E} = E_r \hat{u}_r$ )
- $E_r$  constant of  $z$  coordinate
- zero contribution from either end surface

$$2\pi r l E_3 = \frac{1}{\epsilon_0} \lambda l \Rightarrow E_3 = \frac{\lambda}{2\pi\epsilon_0 r}$$

$$V_a - V_b = -\frac{\lambda}{2\pi\epsilon_0} \int_b^a \frac{dr}{r} = \frac{\lambda}{2\pi\epsilon_0} \ln\left(\frac{b}{a}\right)$$

$$\frac{C}{l} = \frac{\lambda}{\Delta V} = \frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}$$

used in  $Z_0$  formula (only for TEM mode)



# Boundary Conditions

---

boundary between two materials (for electrostatics)

- field behavior affected by polarization in materials
- boundary conditions for PDEs  $\nabla \times \vec{E} = \vec{0}$  and  $\nabla \bullet \vec{E} = \frac{1}{\epsilon} \sigma$

need to establish field inter-relationships at boundary

(a) tangential  $\vec{E}$  component

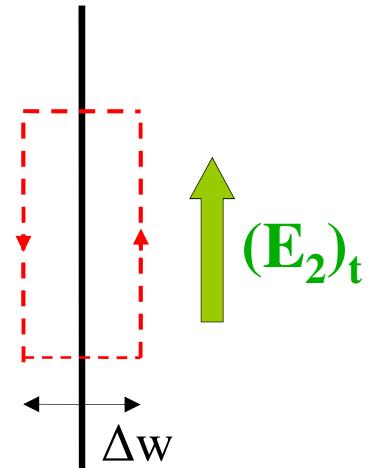
apply  $\oint \vec{E} \bullet d\vec{s} = 0$  to red loop where  $\Delta w \rightarrow 0$

negligible contributions from widths

$$\Rightarrow (E_1)_t = (E_2)_t$$

i.e.  $E_t$  component continuous across interface

generally extended to time-varying cases as well



# Boundary Conditions

---

boundary between two materials (for electrostatics)

(b) normal  $\vec{E}$  component but consider  $\vec{D}$  instead

replace  $\oint_S \vec{E} \bullet d\vec{A} = \frac{1}{\epsilon} Q_{\text{total in}} \text{ enclosure}$  by  $\oint_S \vec{D} \bullet d\vec{A} = Q_{\text{total in}} \text{ enclosure}$

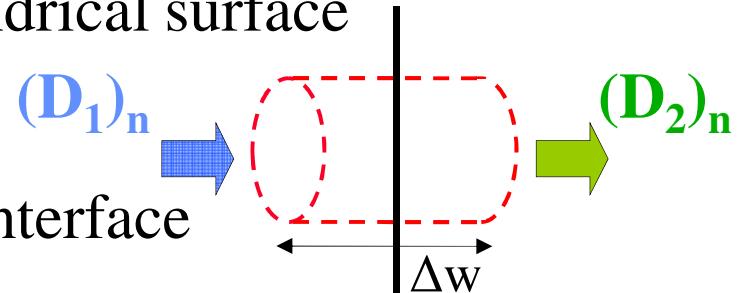
apply Gauss's Law (based on  $\vec{D}$ ) to red pill-box where  $\Delta w \rightarrow 0$

negligible contributions from cylindrical surface

$$\Rightarrow (\mathbf{D}_1)_n = (\mathbf{D}_2)_n$$

$D_n$  component continuous across interface

generally extended to time-varying cases too



special case for metal where  $|E| = |D| = 0$  and  $Q_{\text{enclosed}} = \sigma_s A \neq 0$

$$\vec{E}_{\text{air}} = 0 \hat{\mathbf{u}}_t + \frac{\sigma_s}{\epsilon_0} \hat{\mathbf{u}}_n$$



# Currents

---

current density vector

- flow of charges  $\Delta Q$  contained in  $A \Delta L$

$$I = \frac{\Delta Q}{\Delta t} = \frac{\sigma(A \Delta L)}{\Delta t} = A \sigma v$$

- average velocity of charges

$$\bar{v} = \frac{1}{M} \sum_{m=1}^M \vec{v}_m$$

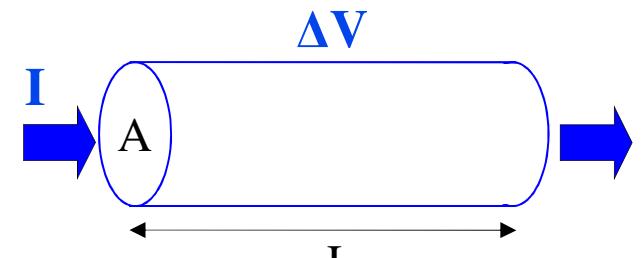
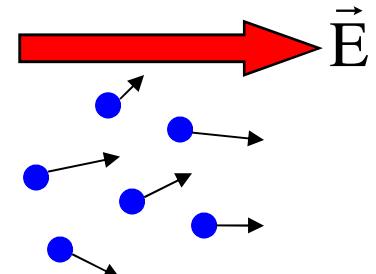
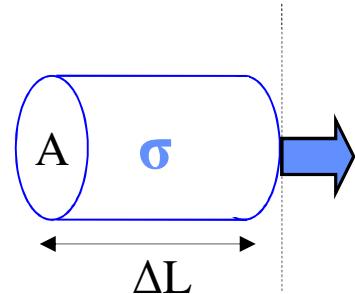
- mobility under influence of  $\vec{E}$

$$\bar{v} \propto \vec{E} \rightarrow \bar{v} = \xi \vec{E}$$

- material property: resistivity  $\rho = \frac{1}{\sigma \xi}$

$$\vec{J} = \sigma \vec{v} = \frac{1}{\rho} \vec{E}$$

- Ohm's Law for wire  $\frac{I}{A} = \frac{1}{\rho} \frac{\Delta V}{L} \Rightarrow \Delta V = \frac{\rho L}{A} I = RI$



# Currents

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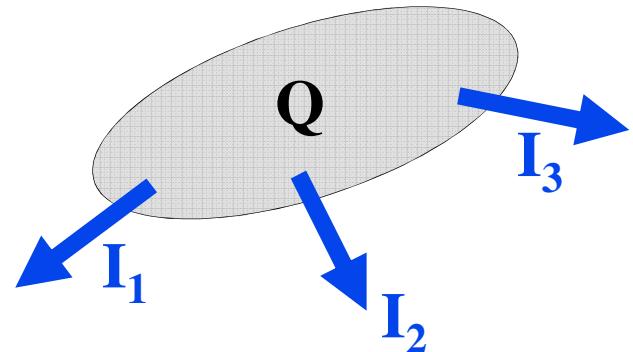
continuity equation

$$\sum_m I_m = -\frac{dQ}{dt}$$

or  $\oint \vec{J} \bullet d\vec{A} = -\frac{d}{dt} \iiint \sigma dV$

$$\Rightarrow \iiint \nabla \bullet \vec{J} dV = -\iiint \frac{d}{dt} \sigma dV \quad \text{for application to any volume}$$

$$\Rightarrow \nabla \bullet \vec{J} = -\frac{d}{dt} \sigma \quad \text{for application to any point}$$



for steady state (*i.e.*  $\frac{d}{dt} = 0$ ), no change of total charge in enclosure

$$\nabla \bullet \vec{J} = 0 \quad \text{or} \quad \sum_m I_m = 0 \quad \text{Kirchhoff's Current Law}$$

$$\nabla \times \vec{E} = \nabla \times (\rho \vec{J}) = \vec{0} \quad \text{conservative property}$$

# Currents

---

boundary conditions (for static case)

similar to equations for  $\vec{E}$  and  $\vec{D}$  → use same process with  $\Delta w \approx 0$

$$\oint (\rho \vec{J}) \bullet d\vec{s} = 0$$

$(J_1)_t$  ↑      ↑  $(J_2)_t$

$\Delta w$

$$\iint \vec{J} \bullet d\vec{A} = 0$$

$(J_1)_n$  →      →  $(J_2)_n$

$\Delta w$

$(J_1)_n = (J_2)_n$

$$\rho_1 (J_1)_t = \rho_2 (J_2)_t$$

$J_n$  continuous across interface

$J_t$  discontinuous across interface

→ change of incident angle for  $\vec{J}$  when crossing interface

# Resistance

---

resistance between conductors (not resistance along conductor)

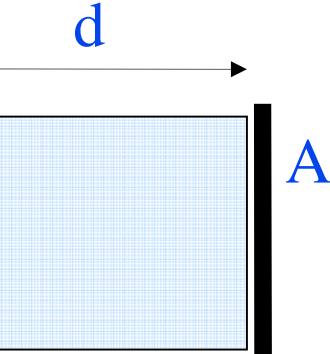
$$R = \frac{\Delta V}{I} = \frac{-\oint \vec{E} \bullet d\vec{s}}{\oint \vec{J} \bullet d\vec{A}} = -\rho \frac{\oint \vec{E} \bullet d\vec{s}}{\oint \vec{E} \bullet d\vec{A}}$$

$$C = \frac{Q}{\Delta V} = \frac{\oint \vec{D} \bullet d\vec{A}}{-\oint \vec{E} \bullet d\vec{s}} = -\epsilon \frac{\oint \vec{E} \bullet d\vec{A}}{\oint \vec{E} \bullet d\vec{s}}$$

time constant  $RC = \rho \epsilon$

- (a) independent of design
- (b) possible economy of effort

illustration: resistance between parallel plates



$$R = \frac{\rho d}{A}$$

$$c.f. C = \frac{\epsilon A}{d}$$

# Resistance

leakage conductance between conductors

re-visit coaxial cable and add  $\vec{J} = \rho \vec{E}$

$$E_r = \rho J_r = \frac{\Delta V}{r \ln\left(\frac{b}{a}\right)}$$

$$I = \iint J_r dA = \frac{\Delta V}{\rho \ln\left(\frac{b}{a}\right)} \iint \frac{1}{r} r d\phi dz = \frac{2\pi L \Delta V}{\rho \ln\left(\frac{b}{a}\right)} \Rightarrow R = \frac{\rho \ln\left(\frac{b}{a}\right)}{2\pi L}$$

compare with capacitance formula (for TEM mode)  $C = \frac{2\pi \epsilon L}{\ln\left(\frac{b}{a}\right)}$   
shunt components in transmission-line model

