Engineering Electromagnetics

EE2011

Plane Wave Propagation in Lossless Medium

Dept. of Electrical and Computer Engineering
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Plane Wave Propagation in Lossless Media

1. Maxwell's Equations (no need to memorise)

Equations for electricity and magnetism (so far)

Equation's name	**Differential Form**	**Integral Form**
Faraday's Law	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
Ampere's Law	$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{s} = I$
Gauss's Law	$\nabla \cdot \mathbf{D} = \rho$	$\oint\!\!\oint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Gauss's Law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oint\!\!\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$

ρ : Charge density [C/m^3] Q: Charge [C]

J: Current desity [A/m^2] I: Current [A]

Maxwell's Equations (for time-varying fields)

Equation's name	**Differential Form**	**Integral Form**
Faraday's Law	$\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = -\iint_S \dfrac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
Ampere's Law	$\nabla \times \mathbf{H} = \mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \left(\mathbf{J} + \dfrac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s} = I + I_d$
Gauss's Law	$\nabla \cdot \mathbf{D} = \rho$	$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Gauss's Law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_S \mathbf{B} \cdot d\mathbf{s} = 0$

$\dfrac{\partial \mathbf{D}}{\partial t}$: Displacement current density [A/m^2] I_d: Displacement current [A]

Note:

$\mathbf{J} = \sigma \mathbf{E}$, referred to as conducting current density (sometime detnoted as $\mathbf{J}_c$),

is due to the movement of free electric charges. $\boldsymbol{\sigma}$ is the conductivity

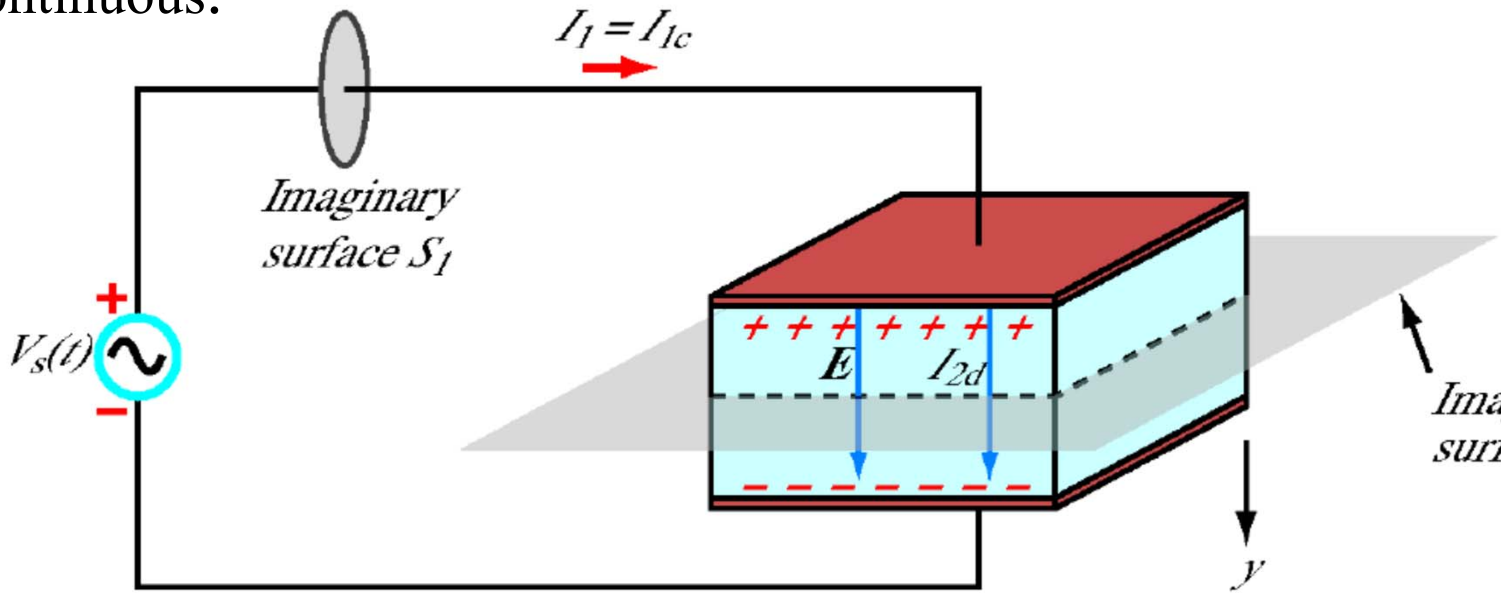
$\dfrac{\partial \mathbf{D}}{\partial t}$, reffered to as displacement current density (sometime detnoted as $\mathbf{J}_d$),

is due to the time-variant of $\mathbf{D}$, which does not transport free charges.

Consequences of introducing displacement current:

•**Electromagnetic wave is generated**: time-variant electric field produces magnetic field and time-variant magnetic fields also produces electric field. Four equations are coupled together.

•**Total current is continuous** (for example, see figure below), also, Magnetic field is continuous.



The displacement current I_{2d} in the insulating material of the capacitor is equal to the conducting current I_{1c} in the wire

Source voltage: $$V_s(t) = V_0 \cos \omega t$$

In perfectly conducting wire (medium 1), conduction current is:

$$I_{1c} = C\frac{dV_s(t)}{dt} = C\frac{dV_0 \cos \omega t}{dt} = -CV_0\omega \sin \omega t$$

Capacitor charging – refer to Year 1 materials

In dielectric material (medium 2),

$$E_2 = \frac{V_s(t)}{d} = \frac{V_0}{d}\cos \omega t$$

the spaceing between plates

$$J_{2d} = \frac{dD}{dt} = \frac{d(\varepsilon E_2)}{dt} = \frac{d\left(\varepsilon \frac{V_0}{d}\cos \omega t\right)}{dt} = -\varepsilon \frac{V_0}{d}\omega \sin \omega t$$

$$I_{2d} = J_{2d}A = -\varepsilon \frac{A}{d}V_0\omega \sin \omega t = -CV_0\omega \sin \omega t$$

Area of plate

$C=\varepsilon \dfrac{A}{d}$: **is the capacitance**

$$I_{1c} = I_{2d}$$

Electrostatics and magnetostatics are special cases of *electromagnetics* when there is no time variation in the charge and current.

Equations for static fields $\frac{\partial}{\partial t} = 0$

Equation's name	Differential Form	Integral Form
Faraday's Law	$\nabla \times \mathbf{E} = 0$	$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0$
Ampere's Law	$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_C \mathbf{H} \cdot d\mathbf{l} = \iint_S \mathbf{J} \cdot d\mathbf{s} = I$
Gauss's Law	$\nabla \cdot \mathbf{D} = \rho$	$\oiint_S \mathbf{D} \cdot d\mathbf{s} = Q$
Gauss's Law for magnetism	$\nabla \cdot \mathbf{B} = 0$	$\oiint_S \mathbf{B} \cdot d\mathbf{s} = 0$

Electrostatics: Stationary electric charges give rise to stationary electric fields, but no magnetic fields. The 1st and 3rd equations play the role.

Magnetostatics: Stationary currents cause stationary magnetic fields, but no electric fields. The 2nd and 4th equations play the role.

2. Plane Waves: Basic Equations (no need to know derivation)

In a source free lossless homogeneous medium, i.e., $\mathbf{J} = \rho = \sigma = 0.$

$\mathbf{J}$ =current density
ρ =charge density
σ =conductivity

Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t}$$

$$\nabla \times \mathbf{H} = \varepsilon \frac{\partial \mathbf{E}}{\partial t}$$

$$\varepsilon \nabla \cdot \mathbf{E} = 0$$

$$\mu \nabla \cdot \mathbf{H} = 0$$

Take the curl of the first equation and make use of the second and the third equations, we have

change sequence of the second order partial differentiation: $= -\mu \frac{\partial}{\partial t} \nabla \times \mathbf{H}$

$$\nabla \times \nabla \times \mathbf{E} = -\nabla \times \mu \frac{\partial \mathbf{H}}{\partial t}$$

Derivation is NOT required

Identity: $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\mu \frac{\partial}{\partial t}\left(\varepsilon \frac{\partial \mathbf{E}}{\partial t}\right)$$

$\nabla \cdot \mathbf{E} = 0$: see from previous slide

$$0 - \nabla^2 \mathbf{E} = -\mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{E}$$

This is called the **wave equation**:

$$\nabla^2 \mathbf{E} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{E} = 0$$

A similar equation for **H** can be obtained:

$$\nabla^2 \mathbf{H} - \mu\varepsilon \frac{\partial^2}{\partial t^2} \mathbf{H} = 0$$

In free space ($\varepsilon = \varepsilon_0 = 8.85\times10^{-12}$ F/m, $\mu = \mu_0 = 4\pi\times10^{-7}$ H/m), the wave equation for **E** is:

$$\nabla^2 \mathbf{E} - \mu_0\varepsilon_0 \frac{\partial^2}{\partial t^2}\mathbf{E} = 0$$

It can be shown that

$$\boxed{\mu_0\varepsilon_0 = \frac{1}{c^2}}$$

c being the speed of light in free space ($\sim 3\times10^8$ (m/s)). Hence the speed of light can be derived from Maxwell's equations.

Note that the permittivity and permeability of air is almost equal to those of free space.

It is of particular interest to consider the time-harmonic fields, the time variation of which takes the form of a sinusoidal function.

As all time-harmonic functions involve the common factor $e^{j\omega t}$ in their phasor form expressions, we can eliminate this factor when dealing with the Maxwell's equations.

The wave equation in free space, which has been derived earlier, can now be put in phasor form as

Indicating sinusoidal

$$\nabla^2 \mathbf{E} - \mu_0 \varepsilon_0 \frac{\partial^2}{\partial t^2} \mathbf{E} = 0 \quad \Rightarrow \quad \nabla^2 \tilde{\mathbf{E}} - \mu_0 \varepsilon_0 \left(j\omega \right)^2 \tilde{\mathbf{E}} = 0$$

(For convenience, dropping the ~ on the top)

$$\Rightarrow \quad \nabla^2 \mathbf{E} + \mu_0 \varepsilon_0 \omega^2 \mathbf{E} = 0$$

In phasor form, Maxwell's equations in an arbitrary medium, with permittivity ε and permeability μ, can be written as:

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$
$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$
$$\varepsilon\nabla \cdot \mathbf{E} = 0$$
$$\mu\nabla \cdot \mathbf{H} = 0$$

Derivation not needed

Using the phasor form expression, the wave equation for **E** field is also called the **Helmholtz's equation**, which is:

$$\nabla^2\mathbf{E} + \mu\varepsilon\omega^2\mathbf{E} = \nabla^2\mathbf{E} + k^2\mathbf{E} = 0$$

$$\text{where} \quad k = \omega\sqrt{\mu\varepsilon}$$

k is called the **wavenumber** (or the **propagation constant**), which has unit of rad/m and is equal to the number of wavelengths in a distance of 2π.

In free space,

$$k = \omega\sqrt{\mu_0\varepsilon_0} = \frac{2\pi f}{c} = \frac{2\pi}{\lambda_0}$$

where λ_0 is the free space wavelength.

In an arbitrary medium with $\varepsilon = \varepsilon_0\varepsilon_r$ and $\mu = \mu_0\mu_r$,

$$k = \omega\sqrt{\mu_0\varepsilon_0\mu_r\varepsilon_r} = \frac{2\pi f}{c}\sqrt{\mu_r\varepsilon_r} = \frac{2\pi}{\lambda_0}\sqrt{\mu_r\varepsilon_r}$$

Since most materials are nonmagnetic, μ_r usually is 1. Unless otherwise specified, $\mu_r = 1$ by default in this module.

We call,

$$\boxed{\lambda = \frac{2\pi}{k} = \frac{\lambda_0}{\sqrt{\mu_r\varepsilon_r}} = \text{wavelength in the medium}}$$

In Cartesian coordinates, the electric field vector can be written as

$$\underline{\boldsymbol{E}} = \boldsymbol{E}_x\underline{\boldsymbol{a}}_x + \boldsymbol{E}_y\underline{\boldsymbol{a}}_y + \boldsymbol{E}_z\underline{\boldsymbol{a}}_z$$

The Helmholtz's equation can be written as three scalar equations in terms of the respective x, y, and z components of the **E** field.

We start with the simplest case. For example, the electric field vector has only the x component. In this case, the scalar equation for the E_x component is:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k^2\right)E_x = 0$$

Consider a special case of the E_x in which there is no variation of E_x in the x and y directions, i.e.,

$$\frac{\partial^2}{\partial x^2}E_x = \frac{\partial^2}{\partial y^2}E_x = 0$$

$E_x(z)$ now varies with z only. The wave equation for E_x becomes:

$$\frac{d^2 E_x(z)}{dz^2} + k^2 E_x(z) = 0$$

Solutions to the plane wave equation take one form of the following functions, depending on the boundary conditions:

1. $E_x(z) = Ae^{-jkz}$
2. $E_x(z) = Be^{+jkz}$
3. $E_x(z) = Ae^{-jkz} + Be^{+jkz}$

where the third solution can be considered as a linear superposition of the first two solutions.

To recall:

$\boxed{\begin{array}{l} Ae^{-jkz} \text{ is a wave propagating in the } +z \text{ direction.} \\ Be^{+jkz} \text{ is a wave propagating in the } -z \text{ direction.} \end{array}}$

$\boxed{\text{Magnitude of velocity: } \dfrac{\omega}{k} = \dfrac{\omega}{\omega\sqrt{\varepsilon\mu}} = \dfrac{c}{\sqrt{\varepsilon_r}}}$

3. Solution for the magnetic field

Once the electric field is known, the accompanying magnetic field **H** can be found from the Maxwell's equation

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\Rightarrow \mathbf{H} = \frac{\nabla \times \mathbf{E}}{-j\omega\mu}$$

For the simplest case,

$$\underline{\boldsymbol{E}} = \boldsymbol{E}_x \underline{\boldsymbol{a}}_x = \mathbf{A}\boldsymbol{e}^{-jkz} \underline{\boldsymbol{a}}_x$$

then the solution for **H** is:

$$\mathbf{H}(z) = \underline{\mathbf{a}}_\mathbf{y} \frac{A}{-j\omega\mu} \frac{\partial e^{-jkz}}{\partial z} = \underline{\mathbf{a}}_\mathbf{y} \frac{k}{\omega\mu} A e^{-jkz},$$

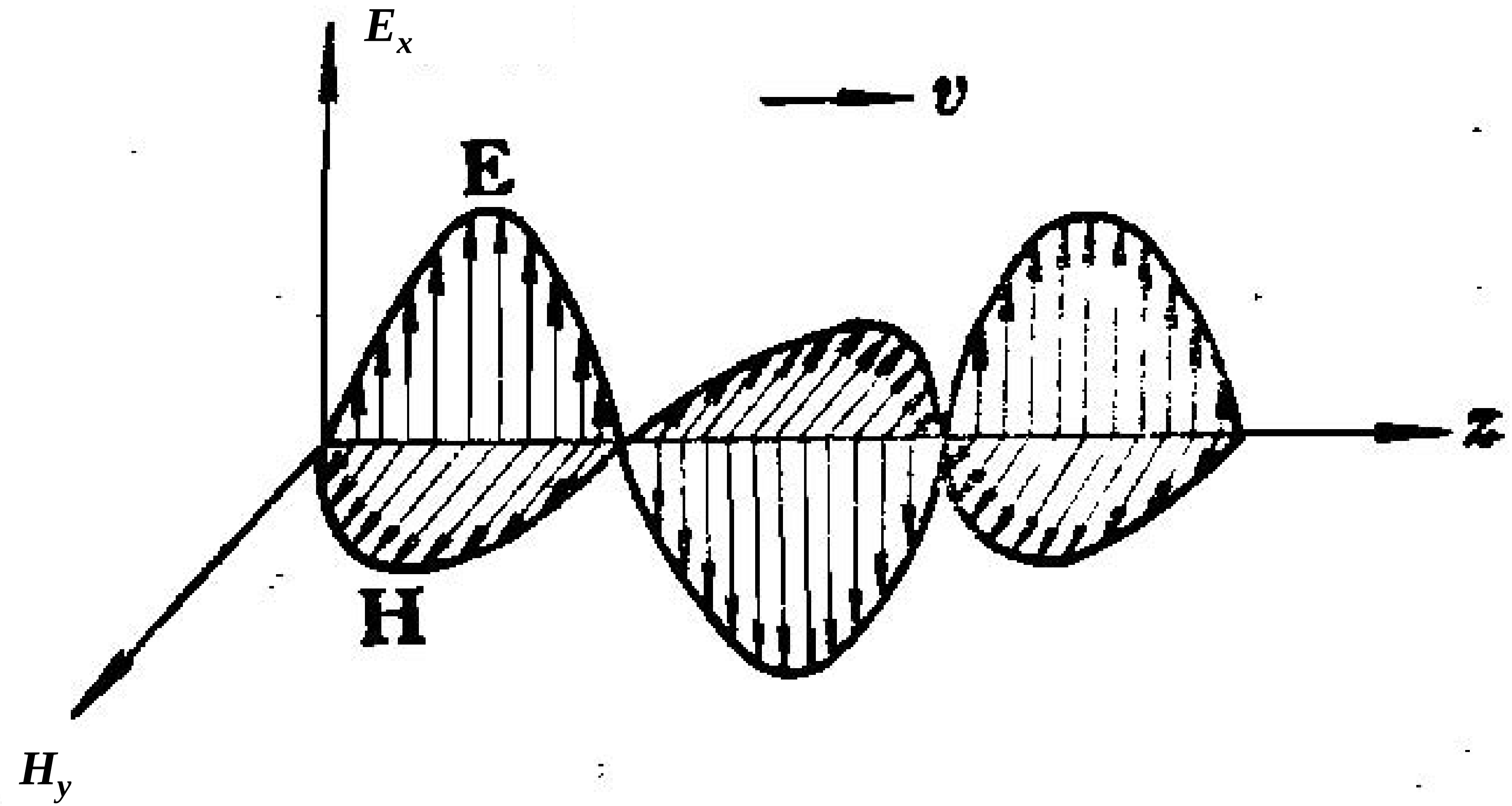
which can be written as $\mathbf{H}(z) = H_y \ \underline{\mathbf{a}}_\mathbf{y}$

The ratio of the amplitudes of E to H is called the **intrinsic impedance** of the medium, η.

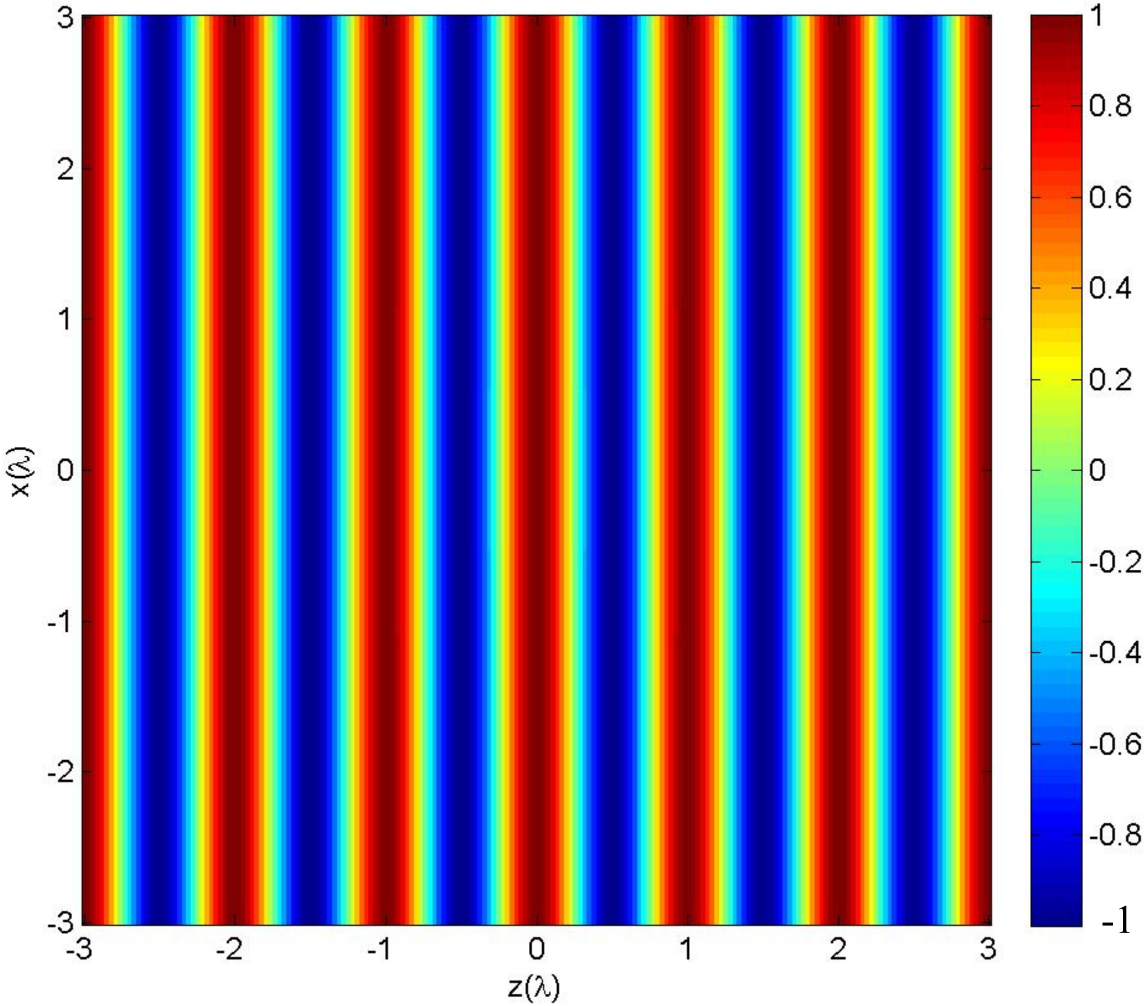
$$\boxed{\eta = \frac{E_x}{H_y} = \frac{\omega\mu}{k} = \frac{\omega\mu}{\omega\sqrt{\mu\varepsilon}} = \sqrt{\frac{\mu}{\varepsilon}}\ (\Omega)}$$

In free space,

$$\boxed{\eta_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 120\pi \approx 377\,\Omega}$$



H and **E** of a plane wave as a function of z at a fixed time (spatial point of view: just like taking a picture using camera)



The distribution of E_x in the x-z plane (spatial point of view)
Color bar denotes the amplitude of the E_x field.

Both **E** and **H** are wave functions with expressions like

$$A\cos\left(\omega t - kz\right) \longleftarrow \textbf{Real[Ae}^{\,\mathbf{j(\omega t\text{-}kz)}}\textbf{]}$$

The temporal point of view is to look at the **E** and **H** as a function of time at a fixed spatial point.

In the following animation, the electric field is in red and the magnetic field is in blue:

http://www.youtube.com/watch?v=4CtnUETLIFs

4. Expressions for a general plane wave

The general expressions of a plane wave are:

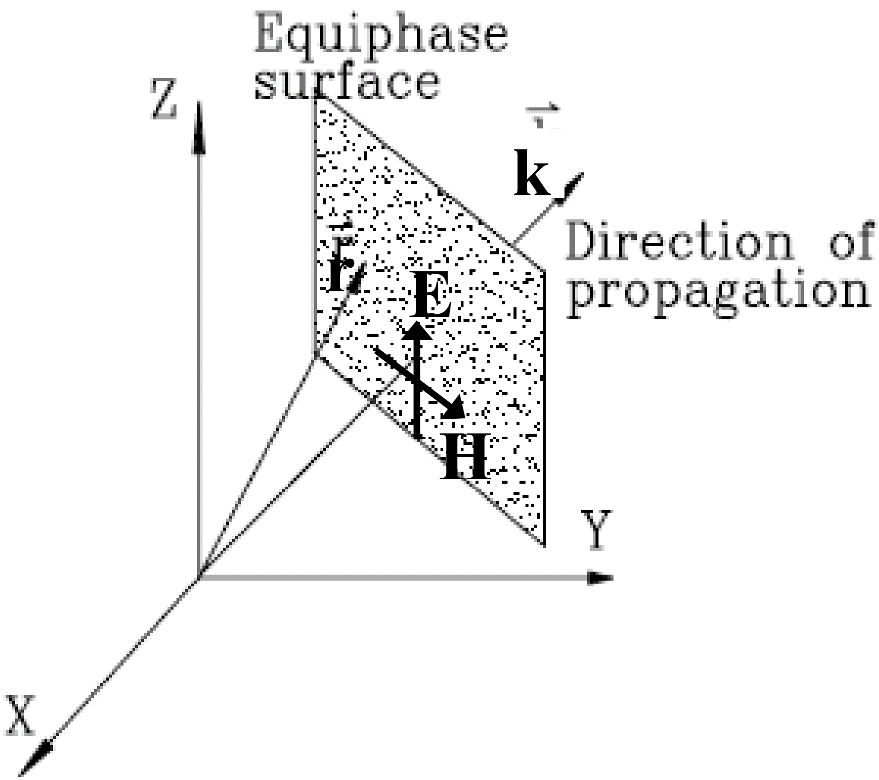
$$\boxed{\begin{aligned}\mathbf{E}(\mathbf{r}) &= \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{\mathbf{e}}\left|E_0\right| e^{j\phi} e^{-j\left(k_x x + k_y y + k_z z\right)} \\ \mathbf{H}(\mathbf{r}) &= \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{\mathbf{h}}\left|H_0\right| e^{j\phi} e^{-j\left(k_x x + k_y y + k_z z\right)}\end{aligned}}$$

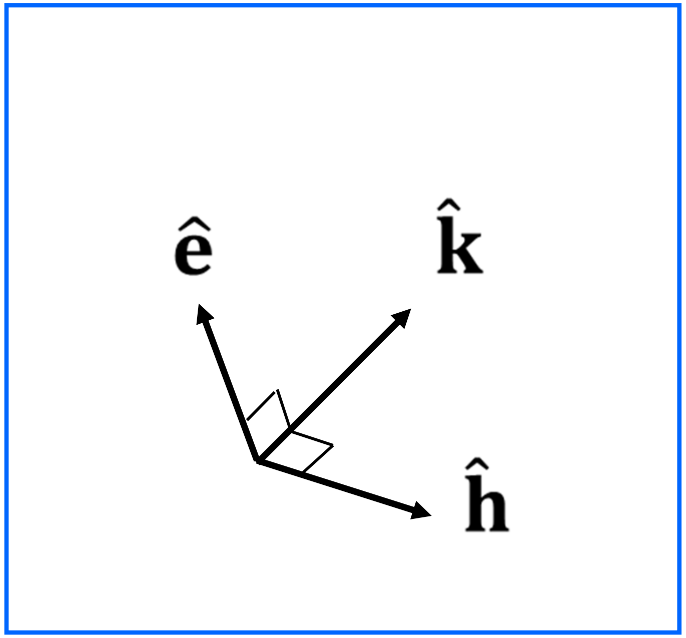
$$\hat{e} = \frac{E_x \underline{a}_x + E_y \underline{a}_y + E_z \underline{a}_z}{\sqrt{E_x^2 + E_y^2 + E_z^2}}$$

Similar for $\hat{h}$.

$\mathbf{E}_0$ and $\mathbf{H}_0$ are vectors specifying the directions and amplitudes of electric and magnetic fields, respectively. Note that amplitudes are complex quantities, including both magnitudes and phases. $\mathbf{k}$ is the **vector propagation constant** whose magnitude is k and whose direction is the direction of propagation of the wave. $\mathbf{r}$ is the observation position vector.

$$\boxed{\begin{aligned}&\mathbf{k} = k_x \hat{\mathbf{x}} + k_y \hat{\mathbf{y}} + k_z \hat{\mathbf{z}}, \quad k = \sqrt{k_x^2 + k_y^2 + k_z^2} \\ &\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}\end{aligned}}$$





Equiphase ($\mathbf{k}\cdot\mathbf{r}$ is a constant) surface is perpendicular to $\mathbf{k}$

For example, when $\mathbf{k} = k_z\hat{z}$ is in z-direction, $\mathbf{k}\cdot\mathbf{r} = \mathbf{k_z \underline{a}_z} \cdot(\mathbf{x\underline{a}_x+y\underline{a}_y+z\underline{a}_z}) = k_z z$, Using Maxwell's equations

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$$

$$\nabla \times \mathbf{H} = j\omega\varepsilon\mathbf{E}$$

it can be shown that we have the following relations for the field vectors and the propagation direction.

$$\boxed{\begin{gathered}\mathbf{E} \perp \mathbf{H} \perp \hat{\mathbf{k}} \\ \mathbf{H} = \frac{1}{\eta}\hat{\boldsymbol{k}} \times \mathbf{E}, \qquad \mathbf{E} = \eta\mathbf{H} \times \hat{\boldsymbol{k}}\end{gathered}}$$

$\hat{\mathbf{k}} = \mathbf{k}/k$ is a unit vector along the wave propagation direction

Example 1

A uniform plane wave with $\mathbf{E} = \hat{\mathbf{x}} E_x$ propagates in the $+z$ direction in a lossless medium with $\varepsilon_r = 4$ and $\mu_r = 1$. Assume that E_x is sinusoidal with a frequency of 100 MHz and that it has a positive maximum value of 10^{-4} V/m at $t = 0$ and $z = 1/8$ m.

(a) Calculate the wavelength λ and the phase velocity u_p, and find expressions for the instantaneous electric and magnetic field intensities.

(b) Determine the positions where E_x is a positive maximum at the time instant $t = 10^{-8}$s.

Solutions

$\hat{\mathbf{e}}$ = unit vector of **E**

(a) $\hat{\mathbf{k}} = \hat{\mathbf{z}}$, $k\hat{\mathbf{k}} \cdot \mathbf{r} = kz$, $\hat{\mathbf{e}} = \hat{\mathbf{x}}$ $\quad |E_0| = 10^{-4}$ V/m

$$k = \omega\sqrt{\mu\varepsilon} = \omega\sqrt{\mu_0\varepsilon_0\,\mu_r\varepsilon_r} = \frac{\omega}{c}\sqrt{\mu_r\varepsilon_r} \approx \frac{2\pi\times10^8}{3\times10^8}\sqrt{4} = 4\pi/3 \text{ rad/m}$$

$$\lambda = 2\pi/k = 1.5 \text{ m}$$

$$u_p = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{\sqrt{4}} = 1.5\times10^8 \text{ (m/s)}$$

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{e}}|E_0|e^{j\phi}e^{-jk\hat{\mathbf{k}}\cdot\mathbf{r}} = \hat{\mathbf{x}}10^{-4}e^{j\phi}e^{-j(4\pi/3)z}$$ (phasor form)

$$\mathbf{E}(\mathbf{r},t) = \mathrm{Re}\left[\mathbf{E}(\mathbf{r})e^{j\omega t}\right] = \hat{\mathbf{x}}\,10^{-4}\cos\left(2\pi\times10^8 t - \frac{4\pi}{3}z + \phi\right) \text{ V/m}$$

(instantaneous form)

The cosine function has a positive maximum when its argument equals zero (ignoring the $2n\pi$ ambiguity). Thus at $t = 0$ and $z = 1/8$,

$$2\pi\times10^{8}(0)-\frac{4\pi}{3}\frac{1}{8}+\phi=0 \quad \Rightarrow \quad \phi=\pi/6 \quad \text{rad}$$

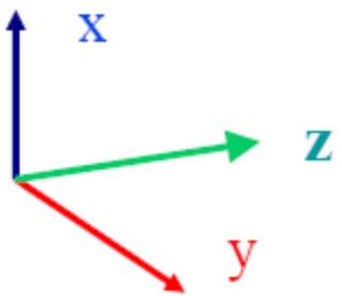
$$\mathbf{E}(z,t)=\hat{\mathbf{x}}\,10^{-4}\cos\left(2\pi\times10^{8}t-\frac{4\pi}{3}z+\frac{\pi}{6}\right) \quad \text{V/m}$$

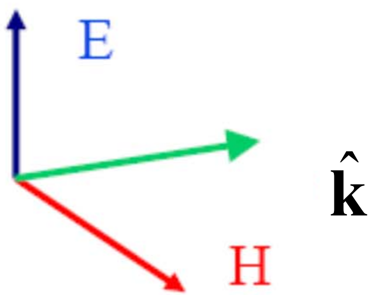
$$\mathbf{H}(z)=\frac{1}{\eta}\,\hat{\mathbf{k}}\times\mathbf{E}=\hat{\mathbf{y}}\,5.305\times10^{-7}\,e^{j\pi/6}\,e^{-j(4\pi/3)z} \quad \text{A/m}$$

$$\eta=\sqrt{\frac{\mu}{\varepsilon}}=\sqrt{\frac{\mu_0}{4\varepsilon_0}}=\frac{\eta_0}{2}=60\pi\ (\Omega)$$

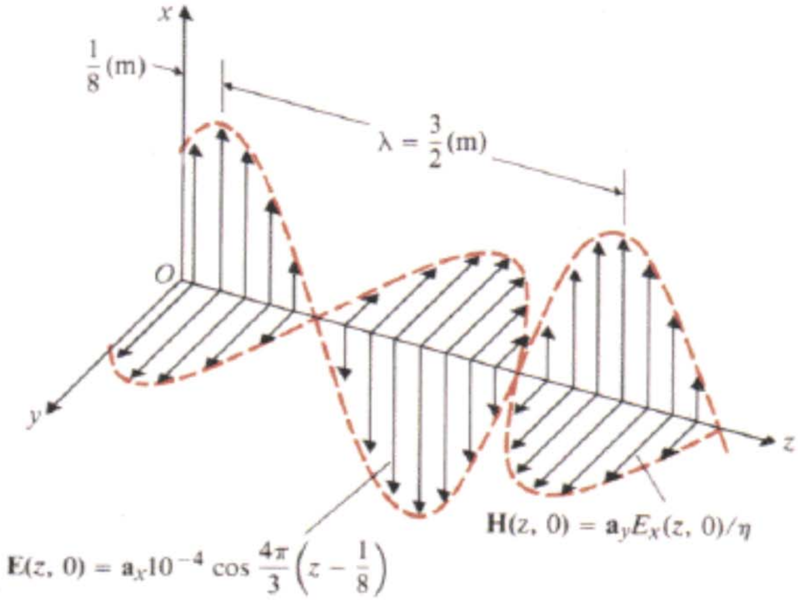
$$\mathbf{H}(z,t)=\mathrm{Re}\left[\mathbf{H}(z)e^{j\omega t}\right]$$

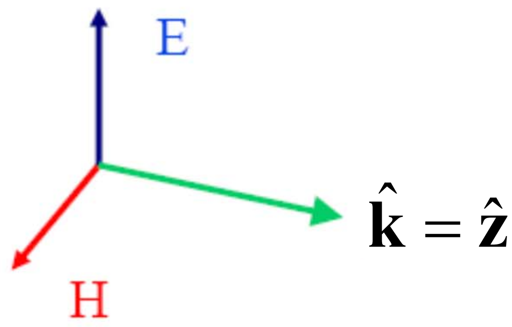
$$=\hat{\mathbf{y}}\,5.305\times10^{-7}\cos\left(2\pi\times10^{8}t-\frac{4\pi}{3}z+\frac{\pi}{6}\right) \quad \text{A/m}$$





Comparing the two systems of vectors explains why we get $\mathbf{H}_y$.





(b) Cosine function has positive maxima when its argument is equal to $\pm 2n\pi \quad n = 0, 1, 2, ..$

Thus $2\pi \times 10^8 (10^{-8}) - \frac{4\pi}{3} z_M + \frac{\pi}{6} = \pm 2n\pi$

$$\Rightarrow \quad z_M = 1.625 \mp 1.5n = 1.625 \mp n\lambda \text{ (m)}$$

Example 2

A uniform sinusoidal plane wave with the following expression for the instantaneous magnetic field propagates in air:

$$\mathbf{H}(x,z,t)=\left(-\frac{1}{15\pi}\hat{\mathbf{x}}+\frac{1}{20\pi}\hat{\mathbf{z}}\right)\cos(\omega t-6x-8z)\quad \text{A/m}$$

Calculate k, λ and ω, and find an expression for the instantaneous electric field intensity.

Solution:

$$\mathbf{H}(x,z)=\left(-\frac{1}{15\pi}\hat{\mathbf{x}}+\frac{1}{20\pi}\hat{\mathbf{z}}\right)e^{-j(6x+8z)}\quad \text{A/m}$$

$$\mathbf{H}=\hat{\mathbf{h}}\,H_0\,e^{-j\mathbf{k}\cdot\mathbf{r}}$$

$\hat{\mathbf{h}}$ = unit vector of $\mathbf{H}$

$$H_0=\sqrt{\left(-\frac{1}{15\pi}\right)^2+\left(\frac{1}{20\pi}\right)^2}=\frac{1}{\pi}\sqrt{\frac{1}{15^2}+\frac{1}{20^2}}=\frac{1}{12\pi}$$

$$\hat{\mathbf{h}} = \left(-\frac{1}{15\pi}\ \hat{\mathbf{x}} + \ \frac{1}{20\pi}\ \hat{\mathbf{z}} \right) / \frac{1}{12\pi} = (-0.8, 0, 0.6)$$

$$\mathbf{k} \cdot \mathbf{r} = 6x + 8z = (6, 0, 8) \cdot (x, y, z)$$

We see

$$k_x = 6, \quad k_y = 0, \quad k_z = 8$$

$$k = \sqrt{6^2 + 8^2} = 10$$

$$\hat{\mathbf{k}} = \frac{\mathbf{k}}{k} = (0.6, 0, 0.8)$$

$$\lambda = 2\pi / k = 0.2\pi \quad \text{m}$$

$$\eta = \eta_0 = 120\pi \quad \Omega$$

$$\omega = k\,u_p = kc = 3 \times 10^9 \quad \text{rad/s}$$

$$\mathbf{E} = -\eta\,\hat{\mathbf{k}} \times \mathbf{H}$$

$$\mathbf{E}(x, z) = -\frac{120\pi}{12\pi}[(0.6, 0, 0.8) \times (-0.8, 0, 0.6)]\ e^{-j(6x+8z)} = \hat{\mathbf{y}}\,10\ e^{-j(6x+8z)}$$

$$\mathbf{E}(x, z, t) = \mathrm{Re}\left[\mathbf{E}(x, z)\, e^{j\omega t} \right] = \hat{\mathbf{y}}\,10\ \cos(3 \times 10^9 t - 6x - 8z)$$

5 Power Flow and Poynting Vector

The cross product of **E** and **H** has the dimension of power per unit area. It is called the **Poynting vector, S** and it represents the power carried by the electromagnetic field through a unit area. The direction of the Poynting vector indicates the **direction of power flow (k-direction)**.

Instantaneous Poynting vector:

$$\boxed{\begin{aligned}\mathbf{S}(t) &= \mathbf{E}(t)\times\mathbf{H}(t)\\ &= \mathrm{Re}\left\{\mathbf{E}e^{j\omega t}\right\}\times\mathrm{Re}\left\{\mathbf{H}e^{j\omega t}\right\} \qquad (\mathrm{W/m^2})\end{aligned}}$$

The total power flow out of a closed surface is equal to the depletion of the electric energy and magnetic energy inside the surface.

(Time) Average Poynting vector:

Can be proven mathematically

$$\boxed{\mathbf{S}_{\text{av}} = \frac{1}{T}\int_{-T/2}^{T/2} \mathbf{S}(t)\, dt = \frac{1}{2}\operatorname{Re}\left\{\mathbf{E}\times\mathbf{H}^*\right\}\ (\text{W/m}^2)}$$

The magnitude of $\mathbf{S}_{\text{av}}$ gives the average power density (per unit area) of the EM wave.

Since $\boxed{\hat{\mathbf{e}}\times\hat{\mathbf{h}} = \hat{\mathbf{k}}}$

$$\boxed{\mathbf{S}_{\text{av}} = P_{av}\hat{\mathbf{k}}\ (\text{W/m}^2)}$$

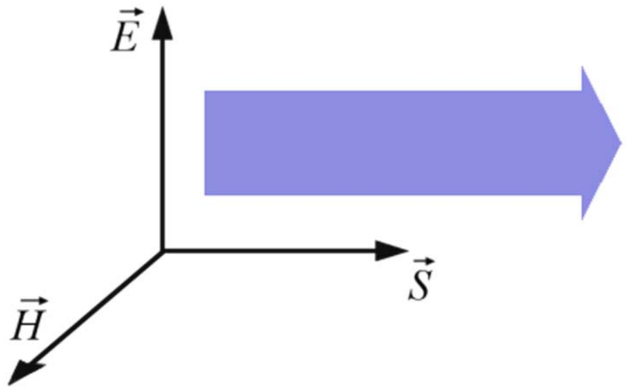
The direction of Poynting power flow is in the **k**-direction.

$$\mathbf{E} = \mathbf{E}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{\mathbf{e}}\left|E_0\right| e^{j\phi} e^{-jk\left(\hat{\mathbf{k}}\cdot\mathbf{r}\right)}$$

$$\mathbf{H} = \mathbf{H}_0 e^{-j\mathbf{k}\cdot\mathbf{r}} = \hat{\mathbf{h}}\frac{\left|E_0\right|}{\eta} e^{j\phi} e^{-jk\left(\hat{\mathbf{k}}\cdot\mathbf{r}\right)}$$

$$\boxed{\begin{aligned} \mathbf{H} &= \frac{1}{\eta}\hat{\mathbf{k}}\times\mathbf{E} \\ \mathbf{E} &= \eta\mathbf{H}\times\hat{\mathbf{k}} \end{aligned}}$$

$$\boxed{\begin{aligned} \mathbf{S}_{\text{av}} &= \frac{1}{2}\text{Re}\left\{\mathbf{E}\times\mathbf{H}^*\right\} \\ &= \frac{1}{2\eta}\left|E_0\right|^2\hat{\mathbf{k}} \\ &= \frac{\eta}{2}\left|H_0\right|^2\hat{\mathbf{k}} \quad (\text{W/m}^2) \\ &= P_{av}\hat{\mathbf{k}} \qquad (\text{W/m}^2) \end{aligned}}$$



Example 3

Compute the average power density P_{av} of a uniform sinusoidal plane wave propagating in air which has the following expression for the instantaneous magnetic field:

$$\mathbf{H}(x,z,t) = \left(-\frac{1}{15\pi}\,\hat{\mathbf{x}} + \frac{1}{20\pi}\,\hat{\mathbf{z}} \right) \cos(\omega t - 6x - 8z) \quad \text{A/m}$$

Solutions:

From Example 2, we obtain the phasor forms:

$$\mathbf{H}(\mathbf{r}) = \mathbf{H}(x,z) = \left(-\frac{1}{15\pi}\hat{\mathbf{x}} + \frac{1}{20\pi}\hat{\mathbf{z}} \right) e^{-j(6x+8z)} \quad \text{A/m}$$

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}(x,z) = \hat{\mathbf{y}}\,10\; e^{-j(6x+8z)} \quad \text{V/m}$$

$$\mathbf{S}_{\mathrm{av}} = \frac{1}{2}\mathrm{Re}\left[\mathbf{E}(\mathbf{r})\times\mathbf{H}^*(\mathbf{r})\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[10\hat{\mathbf{y}}e^{-j(6x+8z)}\times\left(-\frac{1}{15\pi}\hat{\mathbf{x}}+\frac{1}{20\pi}\hat{\mathbf{z}}\right)e^{j(6x+8z)}\right]$$

$$= \frac{1}{2}\mathrm{Re}\left[\frac{10}{20\pi}\hat{\mathbf{x}}+\frac{10}{15\pi}\hat{\mathbf{z}}\right]$$

$$= \frac{1}{4\pi}\hat{\mathbf{x}}+\frac{1}{3\pi}\hat{\mathbf{z}} \quad (\mathrm{W/m^2})$$

$$\boxed{\therefore P_{av} = \left|\mathbf{S}_{\mathrm{av}}\right| = \frac{5}{12\pi} \quad (\mathrm{W/m^2})}$$

- **Textbooks:**
  - *Fundamentals of Applied Electromagnetics*,

    F. T. Ulaby, E. Michielssen, U. Ravaioli,

    Pearson Education, 2010, 6th edition

Suggested reading [textbook]:

- Section 1-4: Traveling Waves
- Section 1-7: Review of Phasors
- Section 7-1: Time-Harmonic Fields
- Section 7-2: Plane-Wave Propagation in Lossless Media
- Section 7-6: Electromagnetic Power Density