EE2101

Engineering Electromagnetics

Introduction and Revision on Vector Calculus

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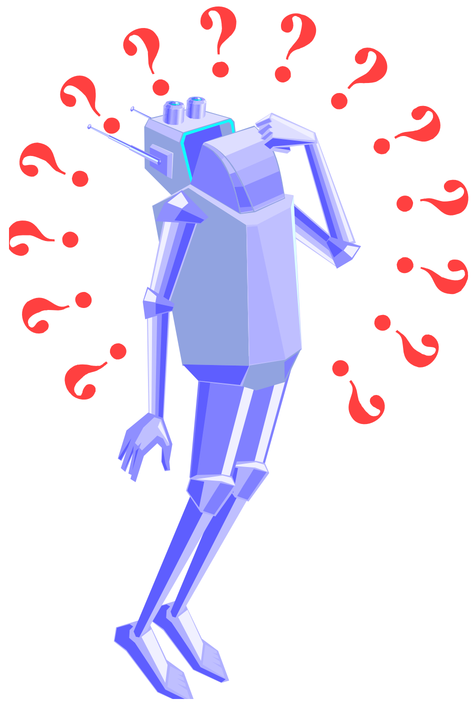
When and where did men (and women) first encounter electromagnetic ?



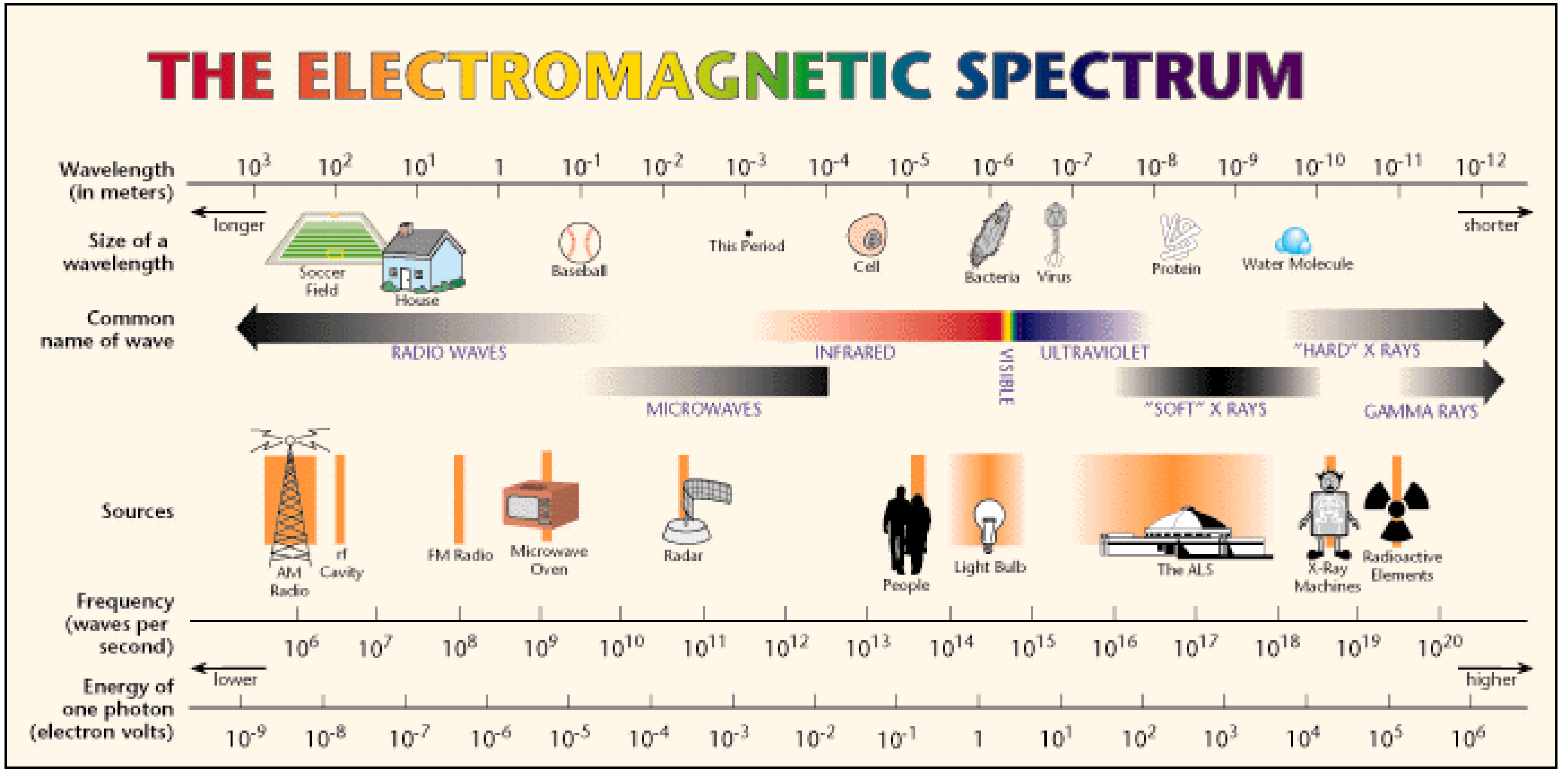
Beginning of the world at the Garden of Eden !!!

Bible:

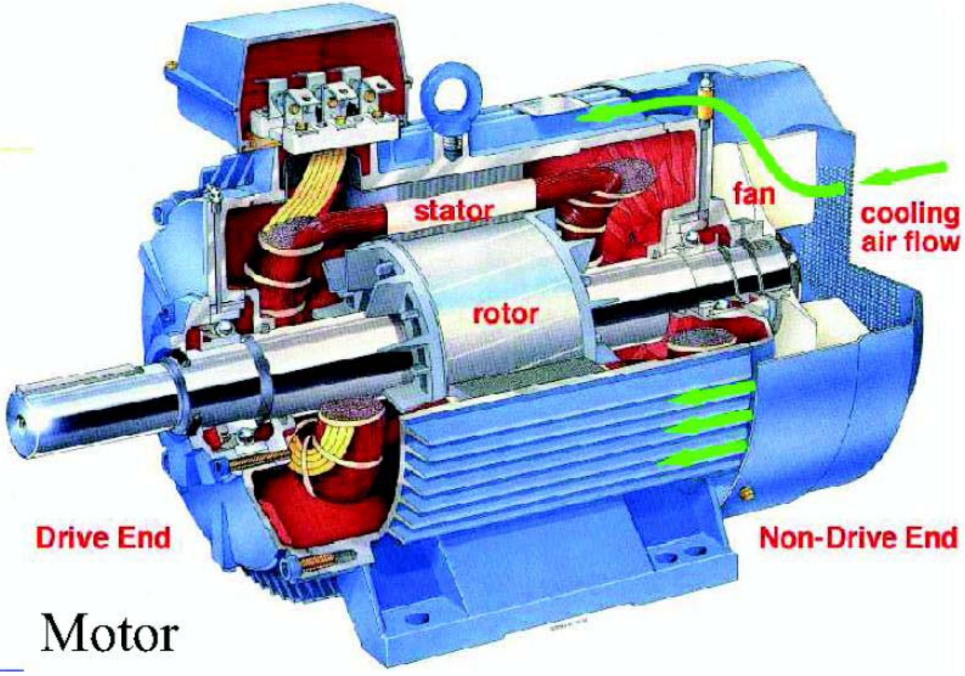
When God said – let there be light.

And there is light !!

Light is also a type of electromagnetic !!

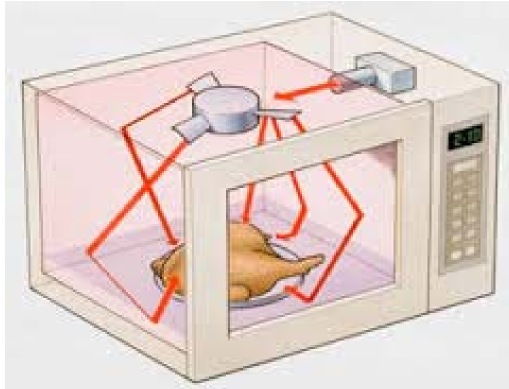

THE ELECTROMAGNETIC SPECTRUM
Wavelength (in meters)
10^3 10^2 10^1 1 10^{-1} 10^{-2} 10^{-3} 10^{-4} 10^{-5} 10^{-6} 10^{-7} 10^{-8} 10^{-9} 10^{-10} 10^{-11} 10^{-12}
longer
shorter
Size of a wavelength
Soccer Field
House
Baseball
This Period
Cell
Bacteria
Virus
Protein
Water Molecule
Common name of wave
RADIO WAVES
INFRARED
VISIBLE
ULTRAVIOLET
"HARD" X RAYS
MICROWAVES
"SOFT" X RAYS
GAMMA RAYS
Sources
AM Radio
rf Cavity
FM Radio
Microwave Oven
Radar
People
Light Bulb
The ALS
X-Ray Machines
Radioactive Elements
Frequency (waves per second)
10^6 10^7 10^8 10^9 10^{10} 10^{11} 10^{12} 10^{13} 10^{14} 10^{15} 10^{16} 10^{17} 10^{18} 10^{19} 10^{20}
lower
higher
Energy of one photon (electron volts)
10^{-9} 10^{-8} 10^{-7} 10^{-6} 10^{-5} 10^{-4} 10^{-3} 10^{-2} 10^{-1} 1 10^1 10^2 10^3 10^4 10^5 10^6

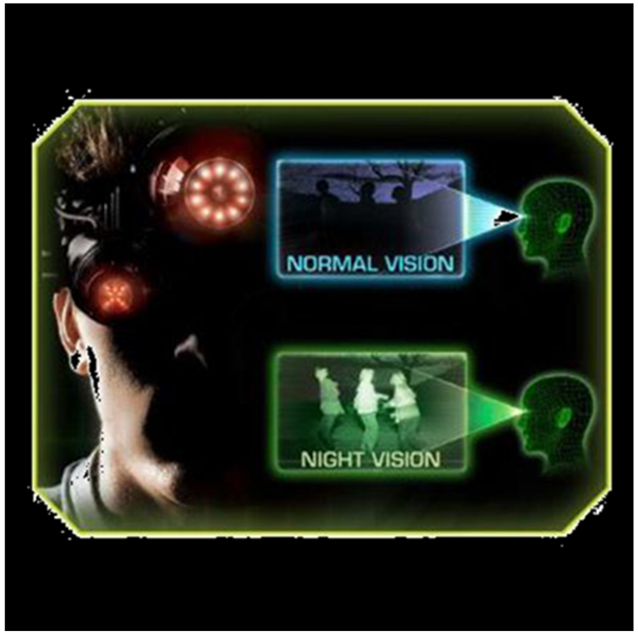
Applications:



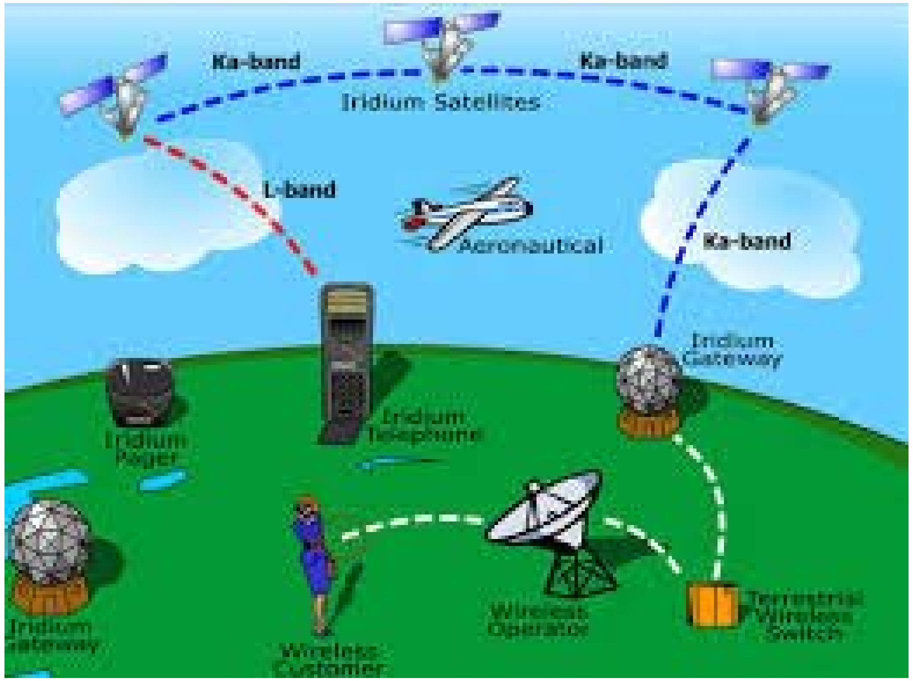
Motor

Applications:

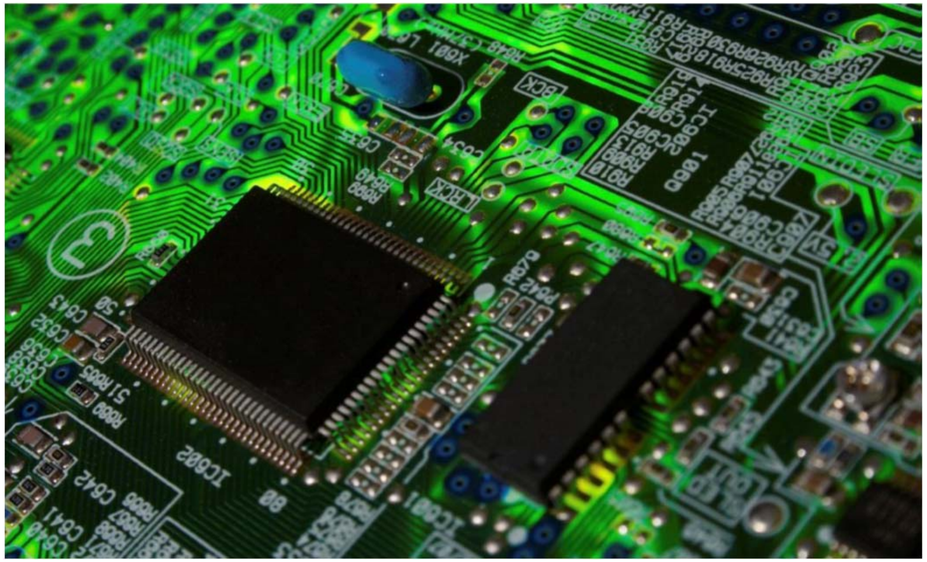


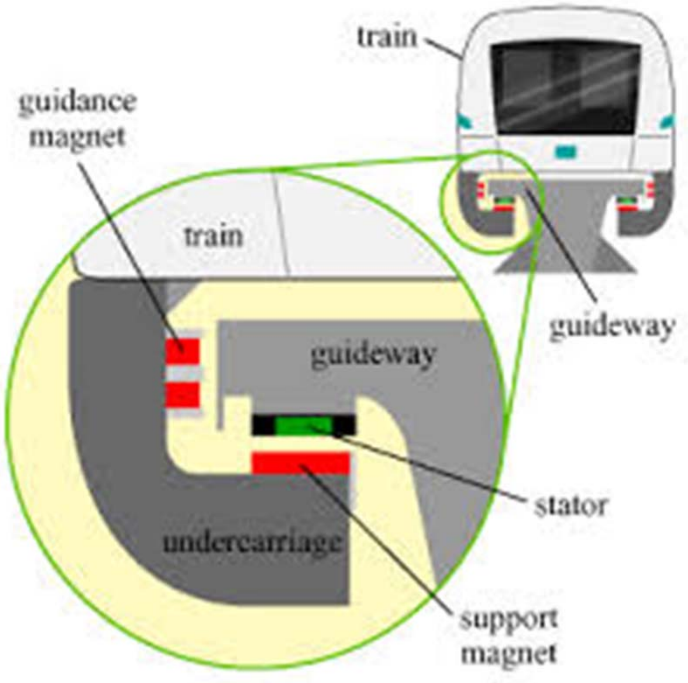


Applications:

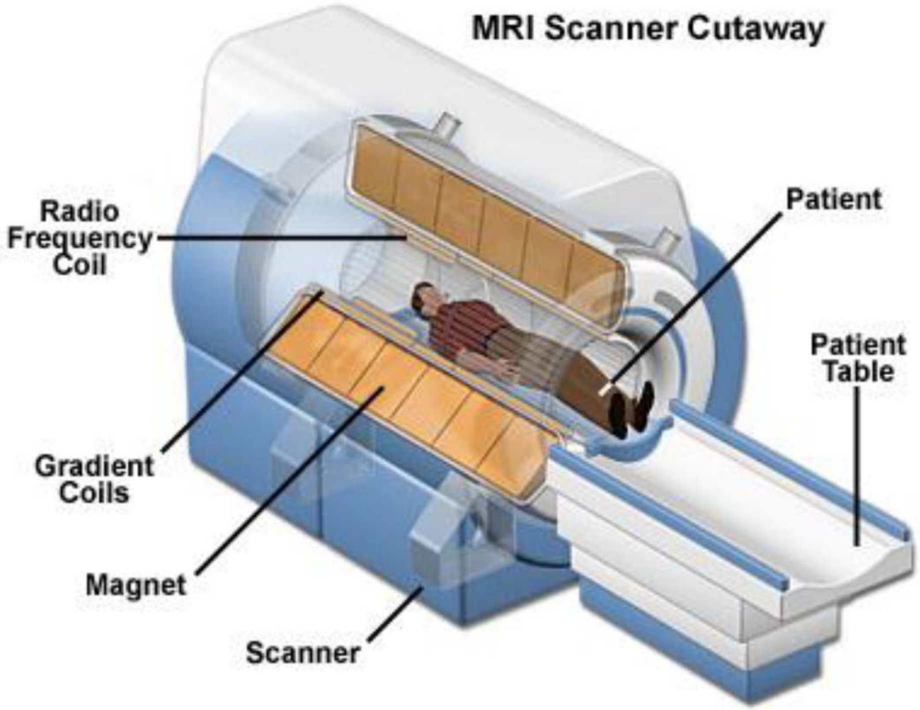


Applications:

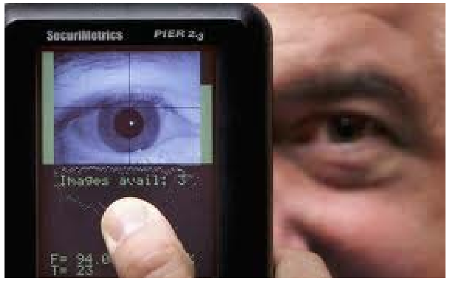




Applications:

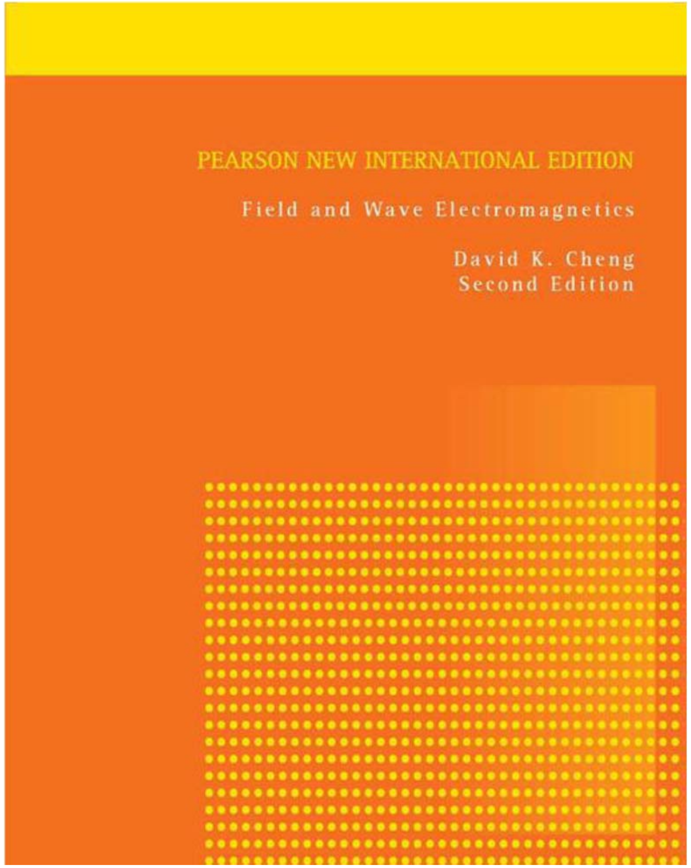


Applications:



- Prof T S Yeo
  - E4-07-10, Tel 65162119 email: eleyeots@nus.edu.sg
- Other module lecturers
  - A/Prof XD Chen
  - Prof ZN Chen
  - A/Prof Y X Guo
- Assessments
  - Examination – 60%
  - Class Tests – 15% x 2 = 30%
  - Class Tests (after every lecture) – 10%

- **Text Book:**
  - **Field and Wave Electromagnetics: Pearson New International Edition, 2/e**
  - **Author: David K Cheng**
  - **Publisher: Pearson**
  - **Available at: NUS Co-op @ Forum**



•Course Outline - 1

- **Introduction**
- **Review of Vector and Complex Number**
- **Review of Vector Calculus**
  - **Scalar and vector Fields**
  - **Line, surface and volume integral**
  - **Gradient, divergence and curl operators**

•Course Outline - 2

•Electric Fields

- **Electric potential**
- **Electric field**
- **Coulomb's and Gauss's Laws**
- **Laplace and Poisson Equations**
- **Capacitance and resistance**

•Course Outline - 3

- **Magnetic Fields**
  - **Magnetic potential**
  - **Magnetic field**
  - **Biot-Savart's Law**
  - **Ampere's and Faraday's Laws**
  - **Mutual and self inductances**

•Course Outline - 4

•Electromagnetic Waves

- **Maxwell's Equation, Wave Equation**
- **Poynting's Theorem**
- **Plane wave in source-free and lossless medium**
- **Plane wave in lossy medium, wave attenuation**
- **Reflection and transmission at boundaries (normal incidence)**

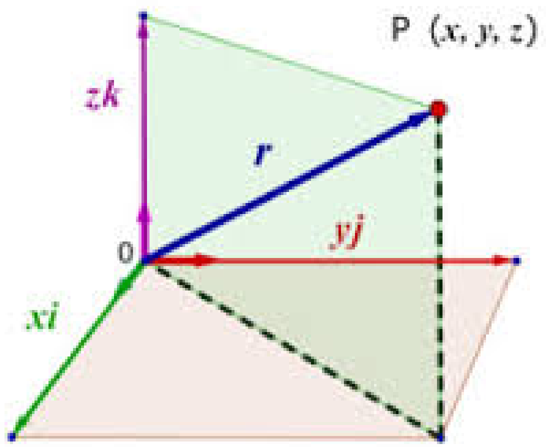
•Course Outline - 5

- **Transmission Liness**
  - **Transmission-line Equations**
  - **Smith chart**
  - **Stub matching**

Vectors

Vector:

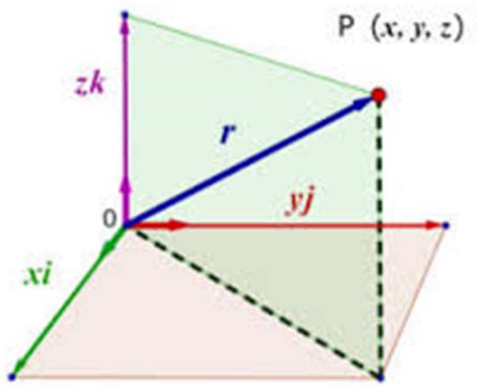
- A vector has a magnitude and a direction
- Position Vector (see figure) – indicate the location of a point in space
- Physical Quantity Vector (e.g., velocity, electric field, etc) – indicates the "strength" of the quantity and its direction

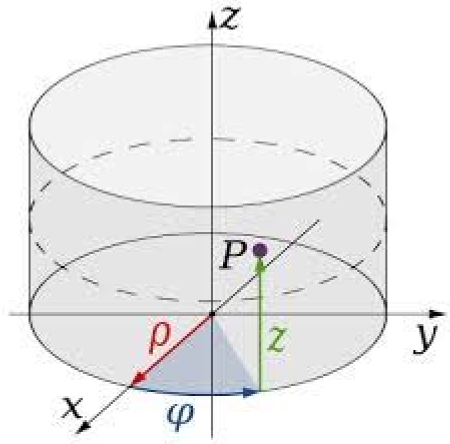


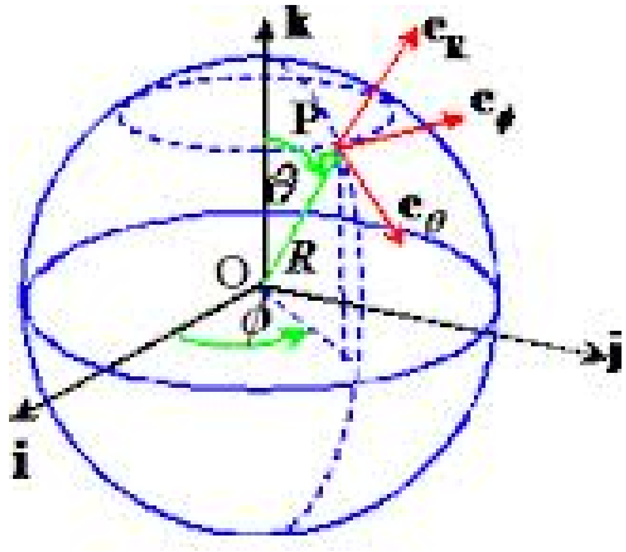
Note that the Position Vector and the Physical Quantity Vector are not the same thing. They may or may not be related. A particle at Point P (Position Vector as shown), could have a Velocity Vector pointing in a different direction.

Vectors:

- A vector can be represented in many co-ordinate systems.
- The most common are
  - Cartesian or Rectangular Coordinate System
  - Cylindrical Coordinate System
  - Spherical Coordinate System







Many notations for coordinates:- look carefully when reading books, and especially examination papers:

While I use the highlighted sets, sometime other notations will be used when extracting examples from other books/lecture materials.

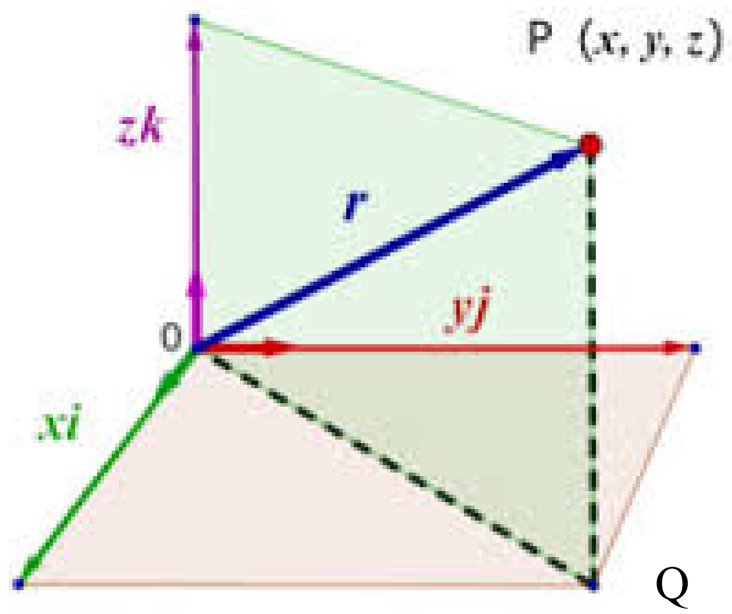
Cartesian: x, y, z; i, j, k
Cylindrical: r, ϕ, z; ρ, ϕ, z; r, φ, z; ρ, φ, z
Spherical: R, θ, ϕ; r, θ, ϕ; R, θ, φ; r, θ, φ

Many notations for vectors:- look carefully when reading books, and especially examination papers:

For a vector: A, **A**, $\underline{A}$, $\underline{\mathbf{A}}$, $\vec{A}$, $\vec{\boldsymbol{A}}$
For an unit vector: $\underline{a}$, $\vec{a}$, $\underline{a}_x$, $\vec{x}$, e_x
For a normal vector: $\underline{a}_n$, $\hat{n}$

Position Vectors:

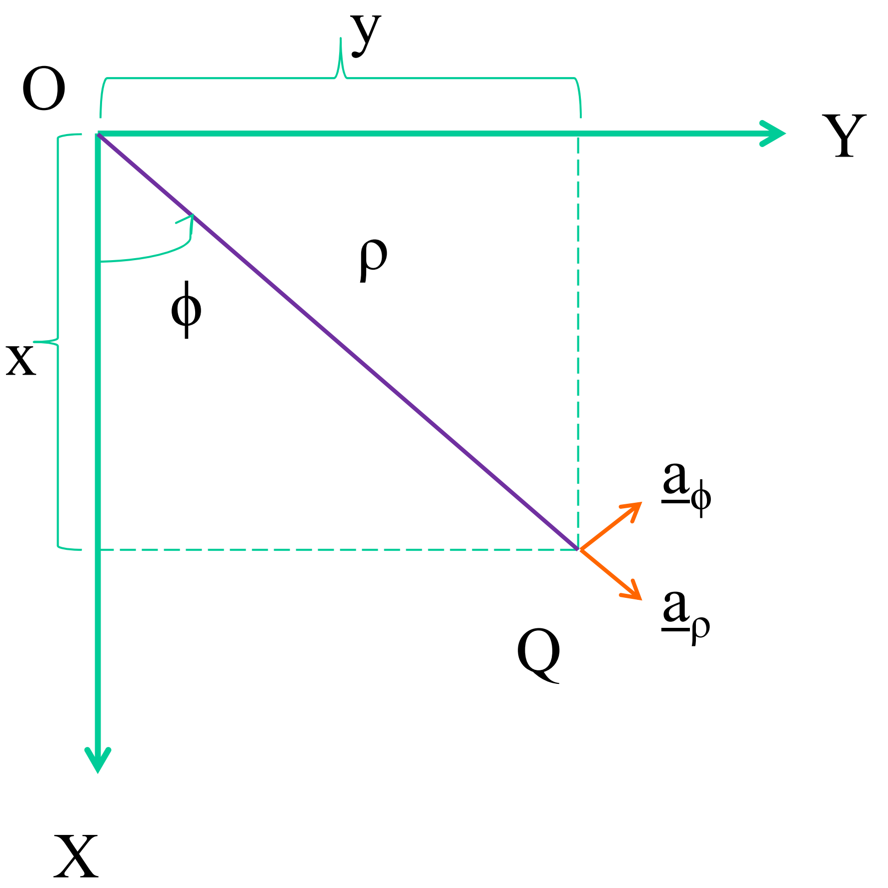
- $\underline{r} = x\,\underline{a}_x + y\,\underline{a}_y + z\,\underline{a}_z$
- $\underline{r} = \rho\,\underline{a}_\rho + 0\,\underline{a}_\phi + z\,\underline{a}_z$
- $\underline{r} = r\,\underline{a}_r + 0\,\underline{a}_\theta + 0\,\underline{a}_\phi$



OP = r
OQ = ρ

Note: While $\underline{a}_x$, $\underline{a}_y$, and $\underline{a}_z$ are fixed in directions, $\underline{a}_\rho$, $\underline{a}_r$, $\underline{a}_\theta$, and $\underline{a}_\phi$ change directions with respect to the point P. When a position vector is given as $\underline{r} = r\,\underline{a}_r$, θ and ϕ must be given as well.

X-Y plane (pink)



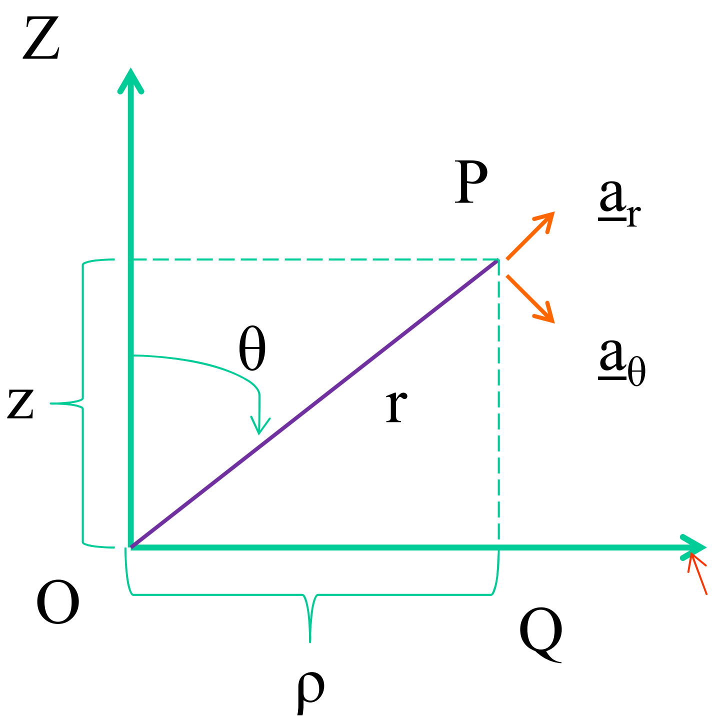
$OQ = \rho$

$x = \rho \cos \phi$

$y = \rho \sin \phi$

$\rho^2 = x^2 + y^2$

OPQ Plane (green)



$OP = r$

$OQ = \rho$

$z = r \cos \theta$

$\rho = r \sin \theta$

$r^2 = z^2 + \rho^2 = z^2 + x^2 + y^2$

ρ – direction
in the xy plane

Physical Quantity vector $\underline{\mathbf{A}}$ at a point P

$\underline{A} = A_x\, \underline{a}_x + A_y\, \underline{a}_y + A_z\, \underline{a}_z$

$$\underline{\mathbf{A}} = \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$\underline{A} = A_\rho\, \underline{a}_\rho + A_\phi\, \underline{a}_\phi + A_z\, \underline{a}_z$

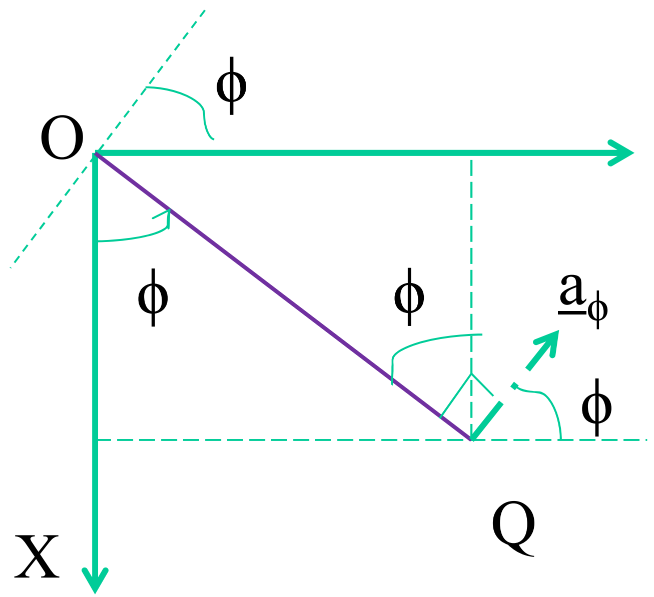
$\underline{A} = A_r\, \underline{a}_r + A_\theta\, \underline{a}_\theta + A_\phi\, \underline{a}_\phi$

$\underline{A} = |\underline{A}|\, \underline{a} = A\, \underline{a}$, $\underline{a} = \underline{A} / A$

Note: $\underline{a}_\rho$, $\underline{a}_r$, $\underline{a}_\theta$, $\underline{a}_\phi$, are according to the directions set by point P. $\underline{a}$ here is not necessarily in the same direction as $\underline{a}_r$.

$A = |\underline{A}| = \sqrt{(A_x^2 + A_y^2 + A_z^2)} = \sqrt{(A_\rho^2 + A_\phi^2 + A_z^2)}$

$= \sqrt{(A_r^2 + A_\theta^2 + A_\phi^2)}$



Converting cylindrical to Cartesian

$$A_x = A_\rho \cos\phi - A_\phi \sin\phi$$

$$A_y = A_\rho \sin\phi + A_\phi \cos\phi$$

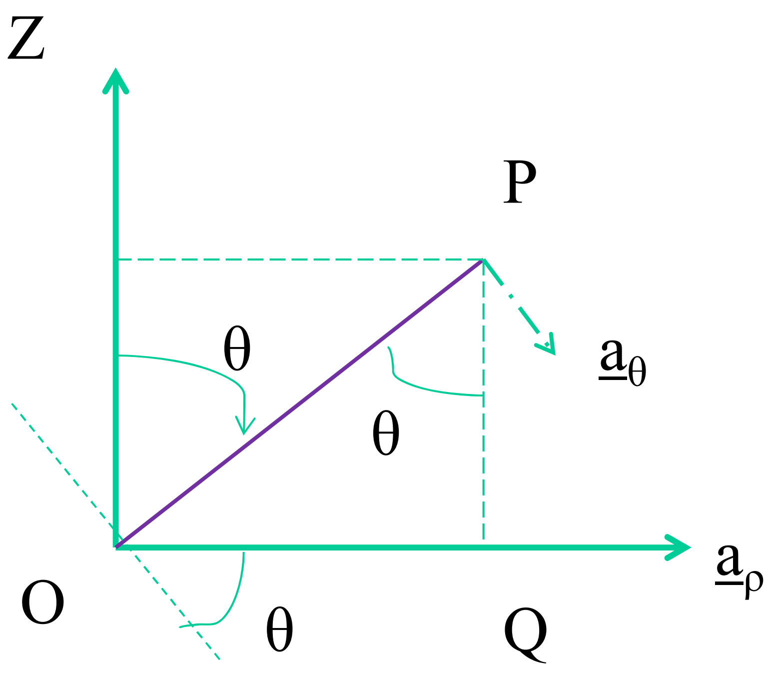
Converting Cartesian to Cylindrical

$$A_\rho = (A_x^2 + A_y^2)^{1/2} = A_x \cos\phi + A_y \sin\phi$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

No change in A_z, why?

No change in A_ϕ, why?



Converting Cylindrical to Spherical

$$A_r = A_\rho \sin\theta + A_z \cos\theta$$

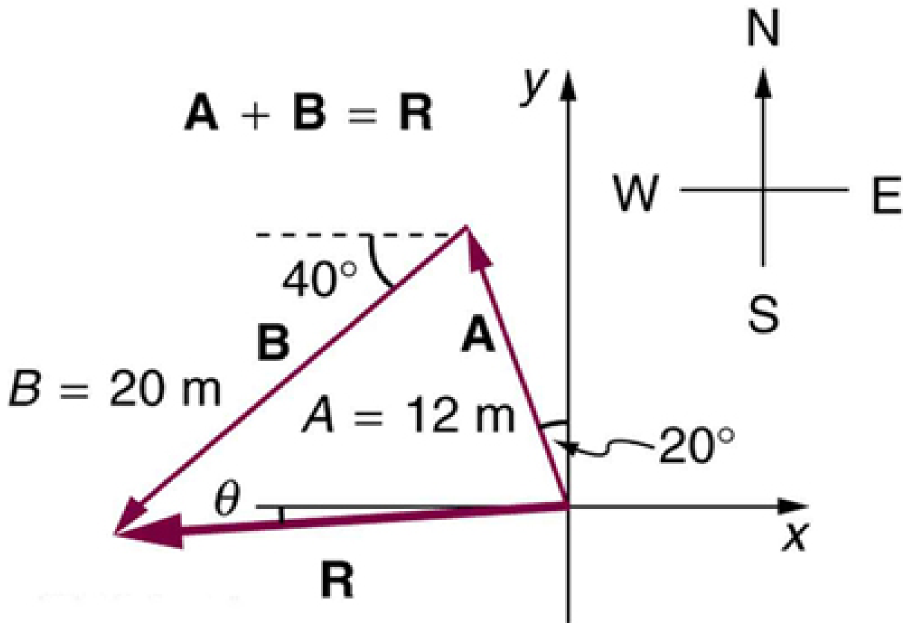
$$A_\theta = A_\rho \cos\theta - A_z \sin\theta$$

Converting Spherical to Cylindrical

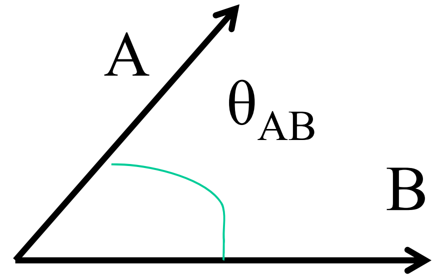
$$A_\rho = A_r \sin\theta + A_\theta \cos\theta$$

$$A_z = A_r \cos\theta - A_\theta \sin\theta$$

Vector summation



Vector Multiplication – Scalar or Dot Product



$\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = AB \cos \theta_{AB}$

$\underline{\mathbf{A}} \cdot \underline{\mathbf{B}} = \underline{\mathbf{B}} \cdot \underline{\mathbf{A}} = A_x B_x + A_y B_y + A_z B_z$

$\underline{\mathbf{A}} \cdot (\underline{\mathbf{B}} + \underline{\mathbf{C}}) = \underline{\mathbf{A}} \cdot \underline{\mathbf{B}} + \underline{\mathbf{A}} \cdot \underline{\mathbf{C}}$

Projection

Vector $\underline{\mathbf{A}}$ on vector $\underline{\mathbf{B}}$:- $\underline{\mathbf{A}} \cdot \underline{\mathbf{B}}/B = \underline{\mathbf{A}} \cdot \underline{\mathbf{b}} = A \cos \theta_{AB}$

Vector $\underline{\mathbf{B}}$ on vector $\underline{\mathbf{A}}$:- $\underline{\mathbf{A}} \cdot \underline{\mathbf{B}}/A = \underline{\mathbf{B}} \cdot \underline{\mathbf{a}} = B \cos \theta_{AB}$

Important conceptual question:

$\underline{A} = A_\rho \underline{a}_\rho + A_\phi \underline{a}_\phi + A_z \underline{a}_z$

$\underline{B} = B_\rho \underline{a}_\rho + B_\phi \underline{a}_\phi + B_z \underline{a}_z$

Is $\underline{A} \cdot \underline{B} = A_\rho B_\rho + A_\phi B_\phi + A_z B_z$???

No in general.

Because $\underline{a}_\rho$ and $\underline{a}_\phi$ of $\underline{A}$ are not necessary the same as $\underline{a}_\rho$ and $\underline{a}_\phi$ of $\underline{B}$.

It will be clearer if B is written as: $\underline{B} = B_\rho \underline{b}_\rho + B_\phi \underline{b}_\phi + B_z \underline{b}_z$

However, if $\underline{a}_\rho \equiv \underline{b}_\rho$ and $\underline{a}_\phi \equiv \underline{b}_\phi$ (i.e., at same point P), then YES.

Vector Multiplication – Vector or Cross Product

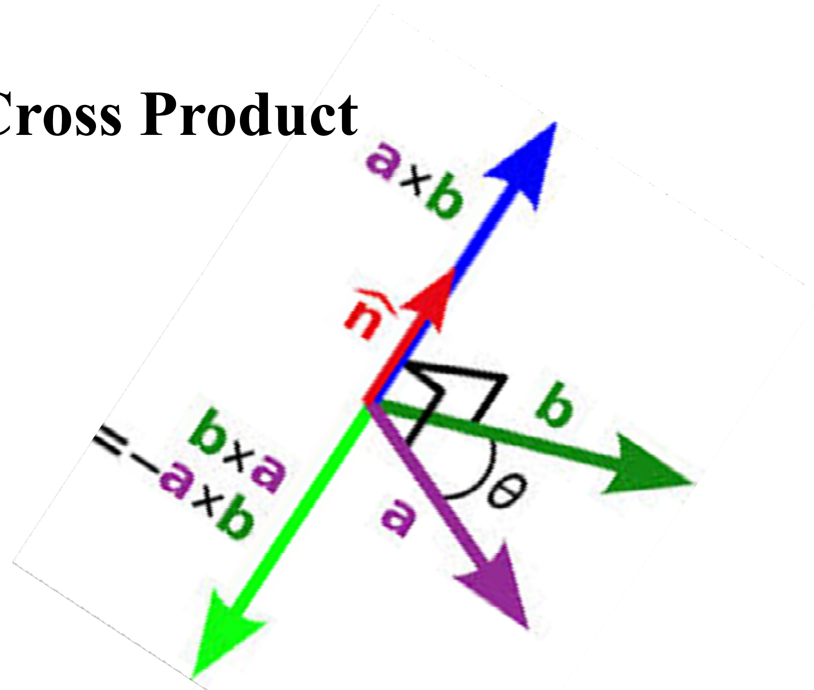
$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} \ (\text{or } \underline{\mathbf{A}} \wedge \underline{\mathbf{B}}) = AB \sin\theta \ \ \underline{\mathbf{n}}$$

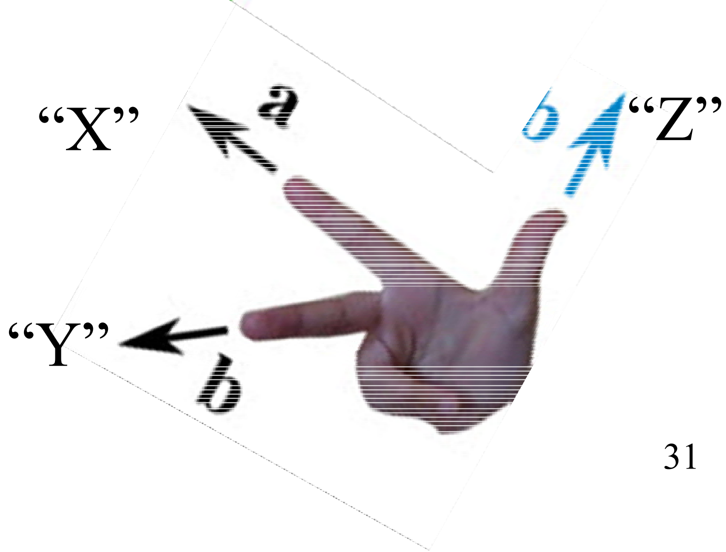
$$\underline{\mathbf{A}} \times (\underline{\mathbf{B}} + \underline{\mathbf{C}}) = \underline{\mathbf{A}} \times \underline{\mathbf{B}} + \underline{\mathbf{A}} \times C$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{A}} = 0$$

$$\underline{\mathbf{A}} \times \underline{\mathbf{B}} = -\,\underline{\mathbf{B}} \times \underline{\mathbf{A}}$$

$$= \begin{vmatrix} \underline{a}_x & \underline{a}_y & \underline{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$



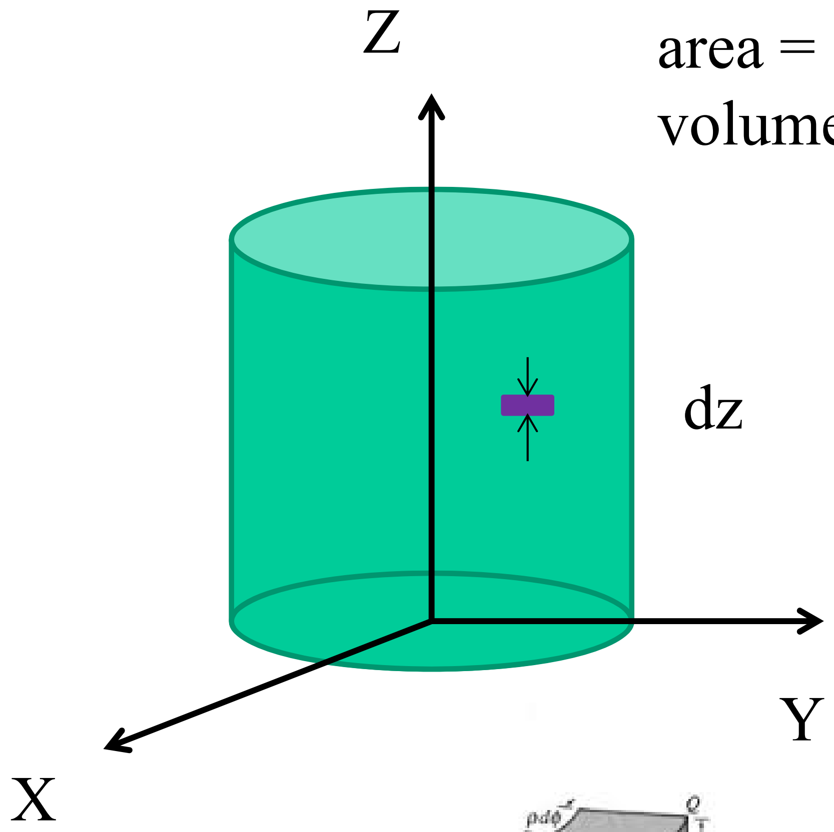


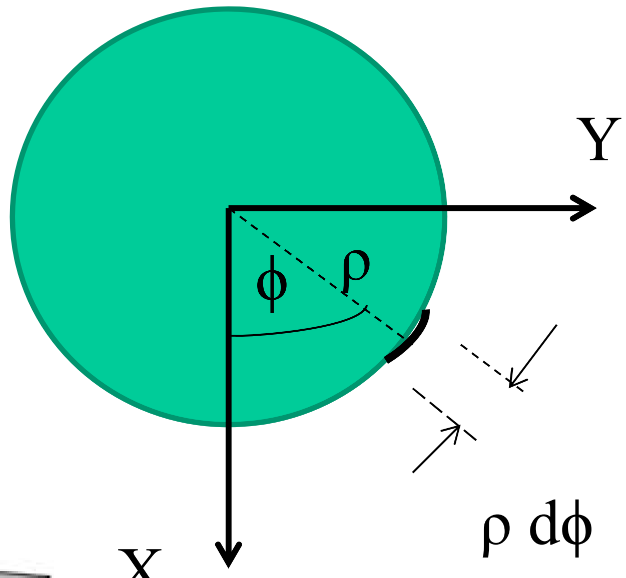
Elementary Areas and Volumes

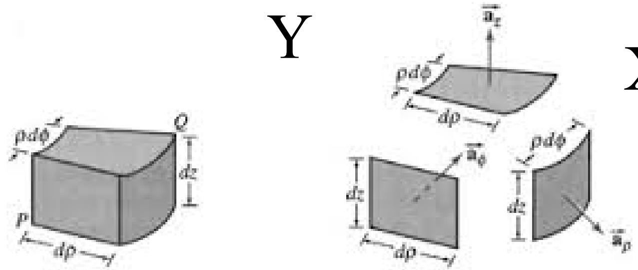
Elementary Surface Area and Volume of a Cylinder

area = $\rho\, d\phi\, dz$, $\quad \underline{\mathbf{ds}} = \rho\, d\phi\, dz\, \underline{\mathbf{a}}_\rho$

volume = area x depth = $\rho\, d\phi\, dz\, d\rho$

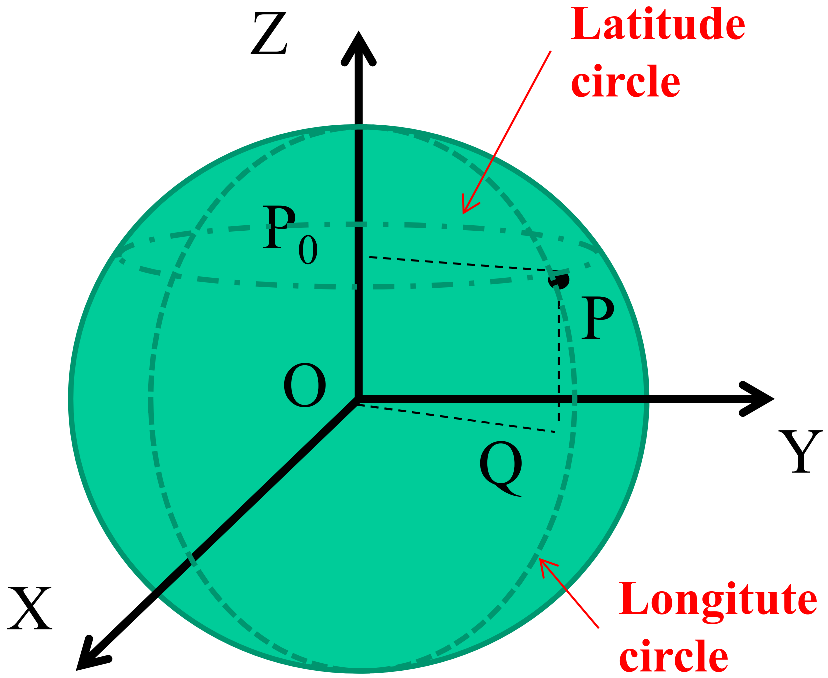


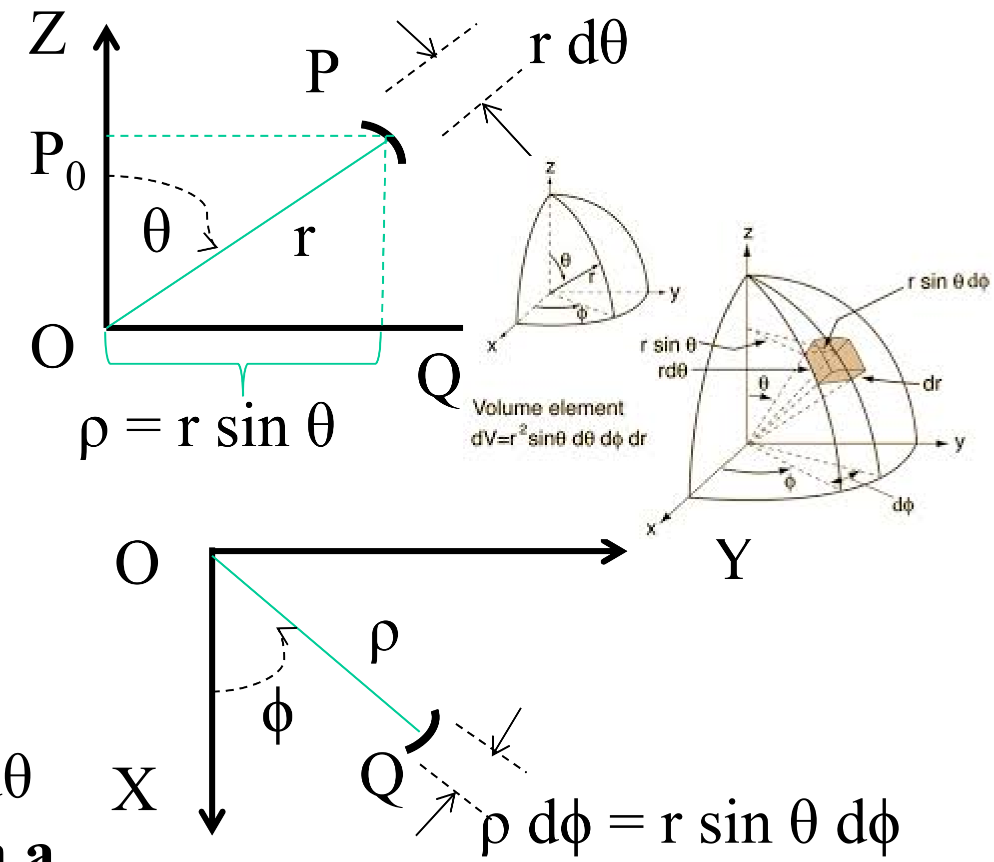




a) Differential volume b) Exploded view

Elementary Surface Area and Volume of a Sphere





$$\text{area} = \rho\, d\phi\; r\, d\theta = r \sin\theta\, d\phi\; r\, d\theta$$

$$= r^2 \sin\theta\, d\phi\, d\theta\ ,\ \underline{\mathbf{ds}} = \text{area}\ \underline{\mathbf{a}}_r$$

$$\text{volume} = \text{area x depth} = r^2 \sin\theta\, d\phi\, d\theta\, dr$$

j operator and complex numbers

The j-operator:

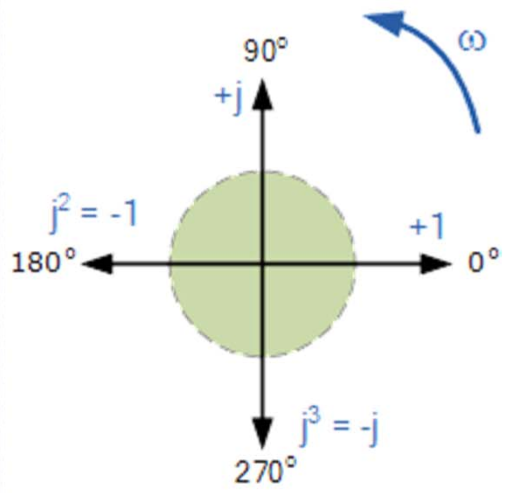
- The **j-operator** has a value exactly equal to √-1
- the **j-operator** is commonly used to indicate the anticlockwise rotation of a vector

$$90^0 \text{ rotation: } \quad j^1 = \sqrt{-1} = +j$$

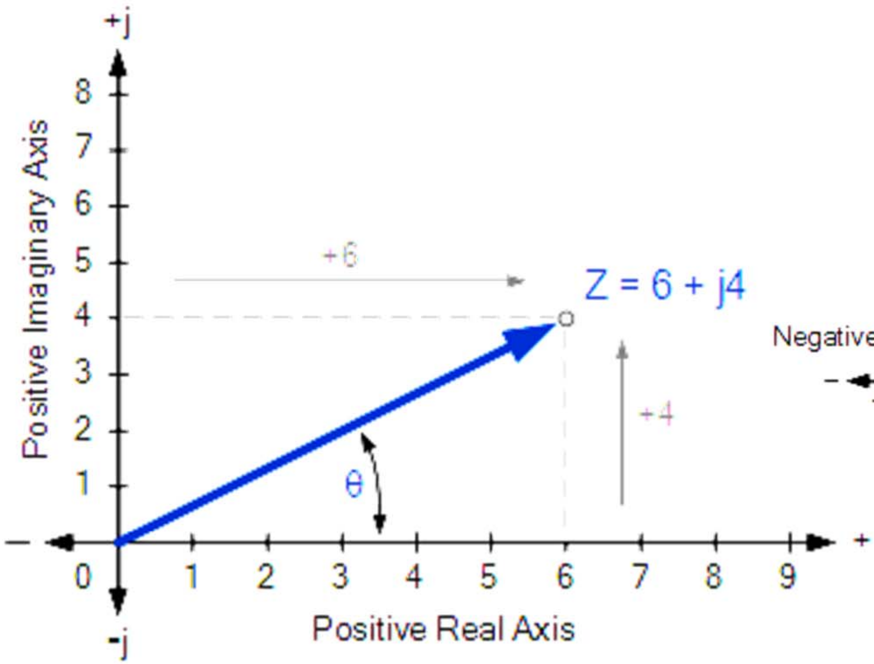
$$180^0 \text{ rotation: } \quad j^2 = \left(\sqrt{-1}\right)^2 = -1$$

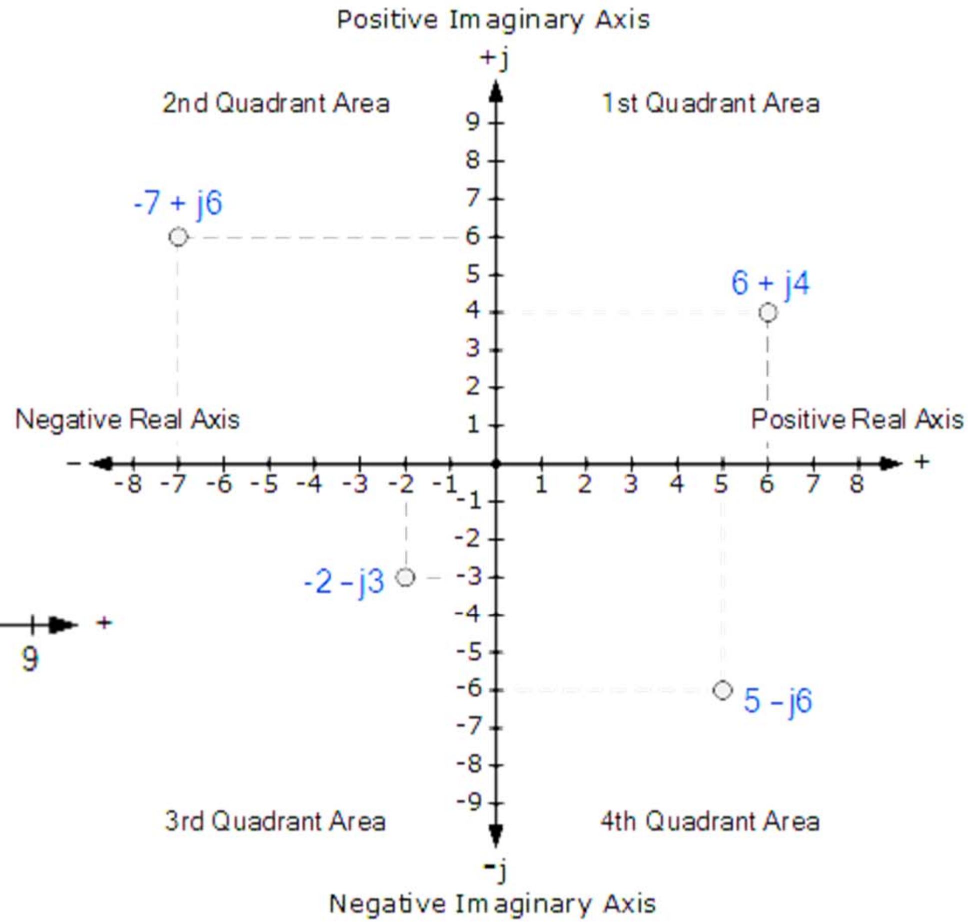
$$270^0 \text{ rotation: } \quad j^3 = \left(\sqrt{-1}\right)^3 = -j$$

$$360^0 \text{ rotation: } \quad j^4 = \left(\sqrt{-1}\right)^4 = +1$$



Rectangular form $Z = a + jb$



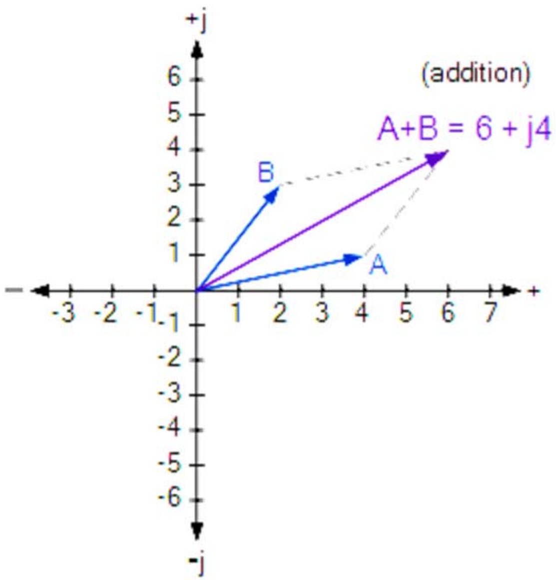


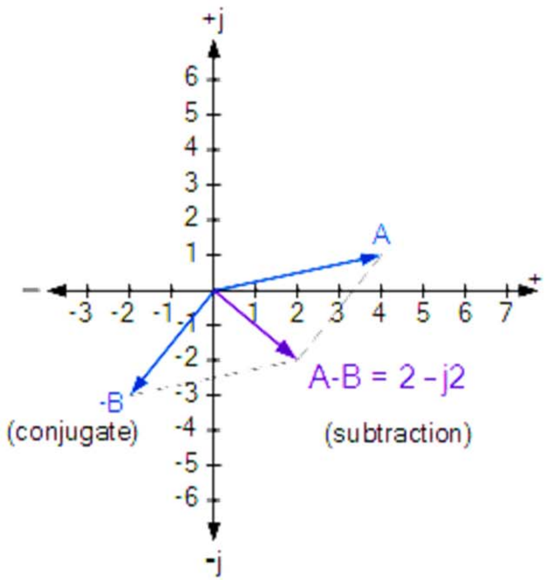
$$A + B = \left(4 + j1\right) + \left(2 + j3\right)$$

$$A - B = \left(4 + j1\right) - \left(2 + j3\right)$$

$$A + B = \left(4 + 2\right) + j\left(1 + 3\right) = 6 + j4$$

$$A + B = \left(4 - 2\right) + j\left(1 - 3\right) = 2 - j2$$





Rectangular form very suited for addition and subtraction, but multiplication and division are more tedious.

$$A \times B = (4 + j1)(2 + j3)$$

$$= 8 + j12 + j2 + j^2 3$$

but $j^2 = -1,$

$$= 8 + j14 - 3$$

$$A \times B = 5 + j14$$

$$\frac{A}{B} = \frac{4 + j1}{2 + j3}$$

Multiply top & bottom by Conjugate of 2 + j3

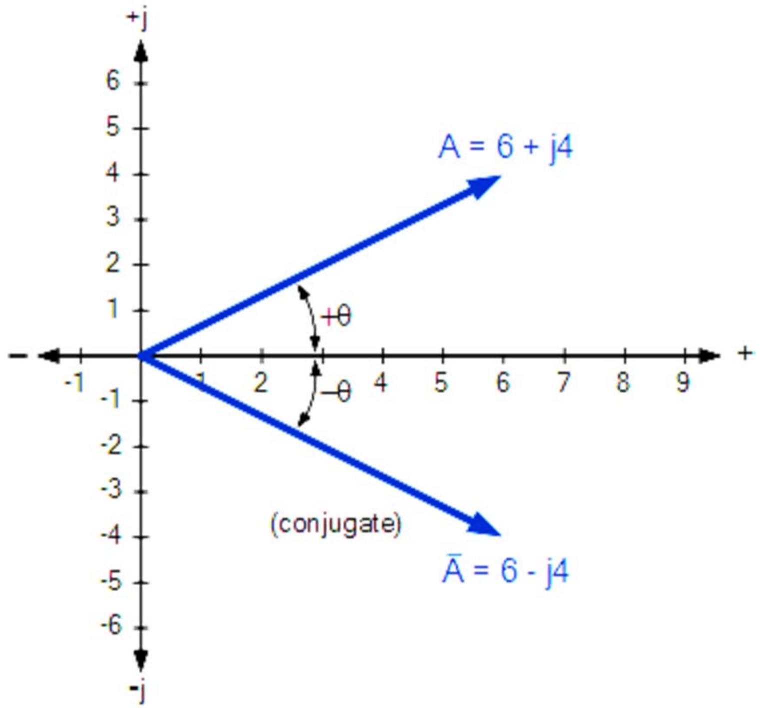
$$\frac{4 + j1}{2 + j3} \times \frac{2 - j3}{2 - j3} = \frac{8 - j12 + j2 - j^2 3}{4 - j6 + j6 - j^2 9}$$

$$= \frac{8 - j10 + 3}{4 + 9} = \frac{11 - j10}{13}$$

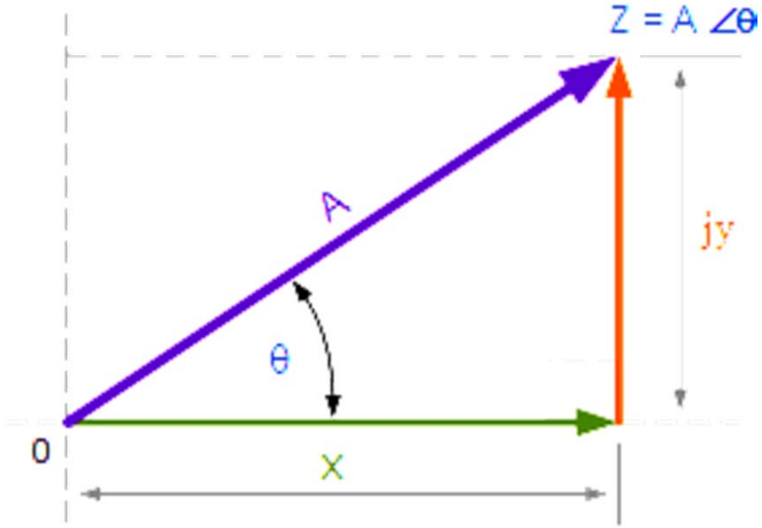
$$= \frac{11}{13} + \frac{-j10}{13} = 0.85 - j0.77$$

Conjugate: Change the sign of the imaginary part

$$Z^* = (a + jb)^* = a - jb$$



Polar form: $Z = A\,\underline{/\theta}$, very easy to multiply and divide



$$A^2 = x^2 + y^2$$

$$A = \sqrt{x^2 + y^2}$$

Also, $x = A.\cos\theta, \quad y = A.\sin\theta$

$$\theta = \tan^{-1}\frac{y}{x}$$

$$Z_1 \times Z_2 = A_1 \times A_2 \angle\theta_1 + \theta_2$$

$$\frac{Z_1}{Z_2} = \left(\frac{A_1}{A_2}\right)\angle\theta_1 - \theta_2$$

Exponential form – a variation of the polar form

$$Z = Ae^{j\phi}$$

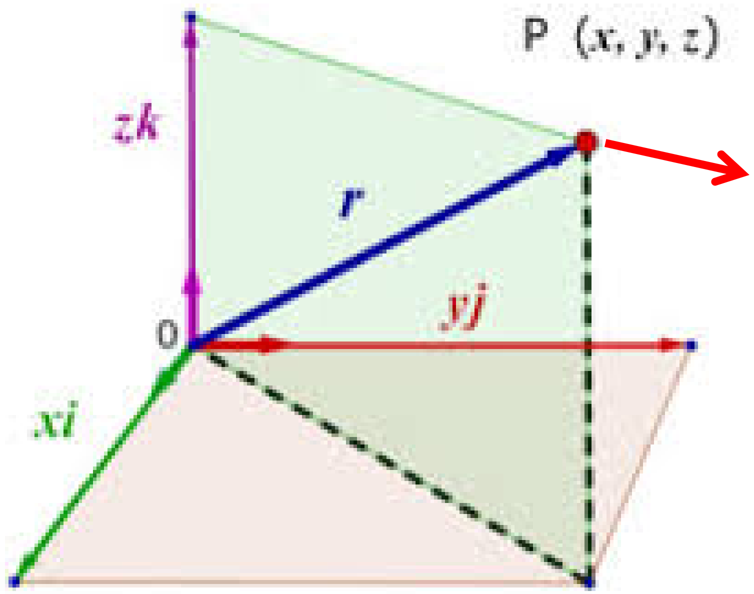
$$Z = x + jy = A\angle\theta = A(\cos\phi + j\sin\phi)$$

$$Z = A(\cos\phi + j\sin\phi)$$

So, we now realize that real numbers (1, 2, 3, 5.9, 7.259, etc) are just a sub-set of complex number (with imaginary part = 0).

Final Note:

A vector can have a magnitude and phase as well as directions and act on a point with a position vector not pointing in the same direction.

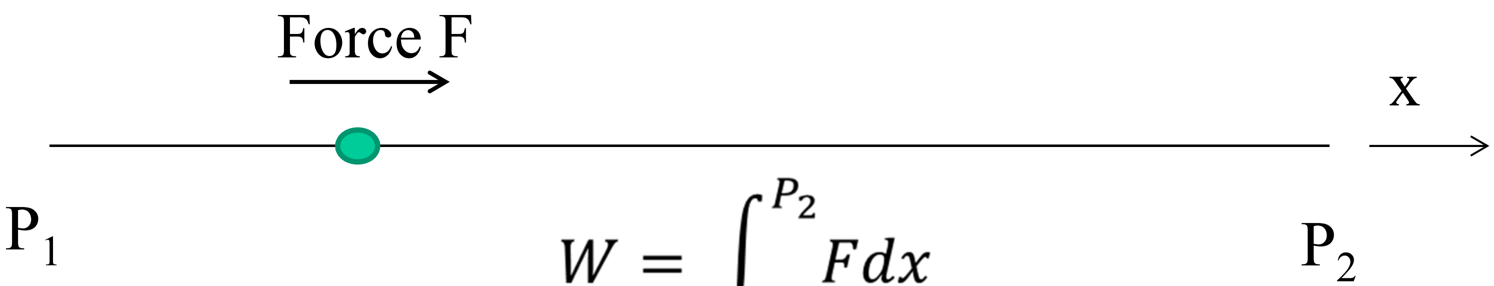


$\underline{\mathbf{A}} = A\,\underline{\mathbf{a}}$, where A is a complex number

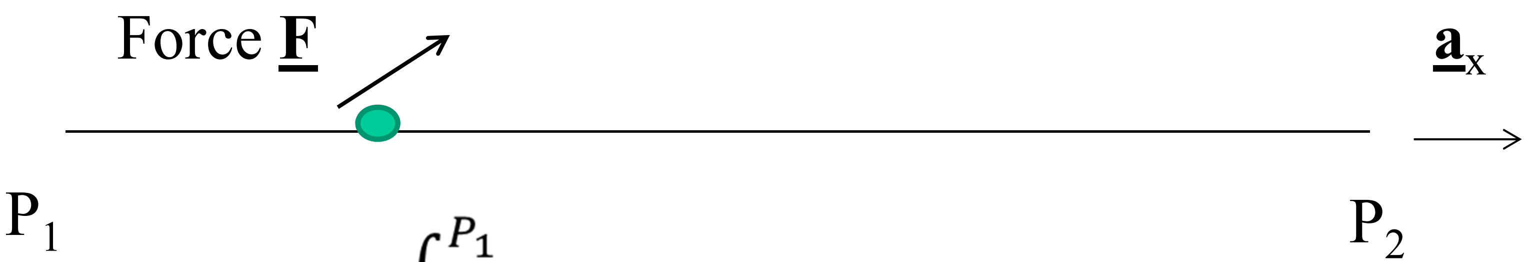
Integrals and Integrations

Line integral (open)	$\int_A^B$
Line integral (closed – enclosing a surface)	$\oint$
Surface integral (open)	$\iint$
Surface integral (closed – enclosing a volume)	$\oiint$
Volume integral	$\oiiint$

Simple illustration: work done to move mass from P_1 to P_2



$$W = \int_{P_1}^{P_2} F dx$$



$$W = \int_{P_2}^{P_1} \underline{F} \cdot dx\, \underline{a}_x$$

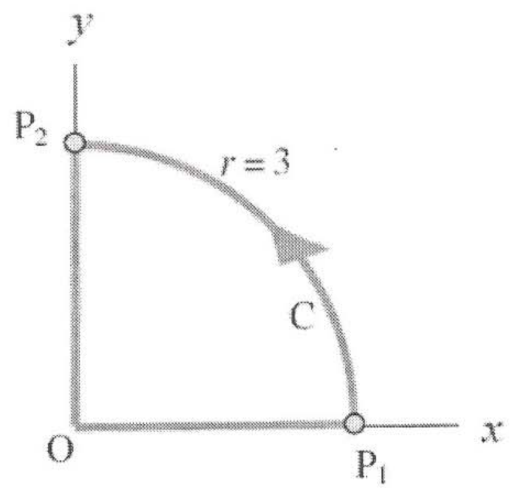
Dot product – projection of force in the direction of movement

$e.g.\ \int_{P_1}^{P_2} \vec{F} \bullet d\vec{s}$ along C where $\vec{F} = xy\hat{\imath} - 2x\hat{\jmath}$

cylindrical path where $d\vec{s} = r\,d\phi\,\hat{u}_\phi$ for elemental arc

$$F_\phi = -F_x \sin\phi + F_y \cos\phi$$
$$= -xy\sin\phi - 2x\cos\phi$$
$$= -9\sin^2\phi\cos\phi - 6\cos^2\phi$$

$$\int_{P_1}^{P_2} \vec{F} \bullet d\vec{s} = \int_{P_1}^{P_2} \begin{bmatrix} F_r \\ F_\phi \\ F_z \end{bmatrix} \bullet \begin{bmatrix} 0 \\ 3d\phi \\ 0 \end{bmatrix} = 3\int_0^{\frac{\pi}{2}} F_\phi\, d\phi$$
$$= -23.14$$



Note: $\hat{\imath} = \underline{a}_x,\ \hat{\jmath} = \underline{a}_y,\ \hat{u}_\phi = \underline{a}_\phi,\ \overrightarrow{ds} = \underline{ds} = \underline{dl}$

Exercise: Same integration, but the path is from P_1 to O then to P_2.

$$\iint \underline{F} \cdot \underline{dA} \; or \; \oiint \underline{F} \cdot \underline{dA}$$

In general,

Cartesian: $\underline{\mathbf{dA}} = \underline{+}\, dydz\, \underline{\mathbf{a}}_x \; \underline{+}\, dzdx\, \underline{\mathbf{a}}_y \; \underline{+}\, dxdy\, \underline{\mathbf{a}}_z$

Cylindrical: $\underline{\mathbf{dA}} = \underline{+}\, \rho d\phi dz\, \underline{\mathbf{a}}_\rho \; \underline{+}\, d\rho dz\, \underline{\mathbf{a}}_\phi \; \underline{+}\, \rho d\rho d\phi\, \underline{\mathbf{a}}_z$

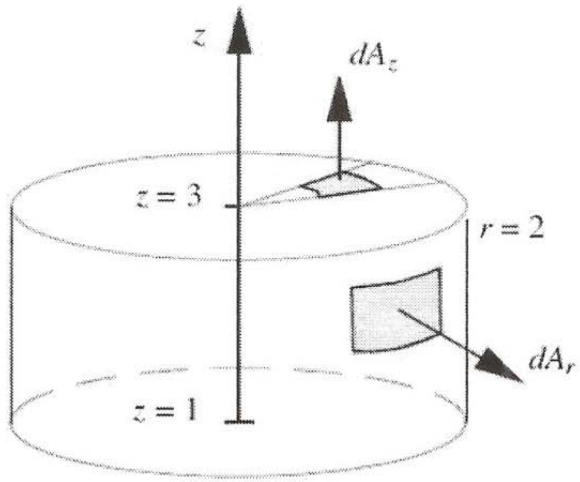
Spherical: $dA = \underline{+}\, r^2 \sin\theta\, d\phi d\theta\, \underline{\mathbf{a}}_r \; \underline{+}\, r\sin\theta\, d\phi dr\, \underline{\mathbf{a}}_\theta \; \underline{+}\, rd\theta dr\, \underline{\mathbf{a}}_\phi$

Not all surfaces have all three (3) components.
The sign + or – depending on the direction of the normal.

$$e.g. \quad \oiint \vec{P} \bullet d\vec{A} \text{ for all surfaces of cylinder where } \vec{P} = \tfrac{3}{r}\hat{u}_r + 2z\,\hat{u}_z$$

$$\oiint \vec{P} \bullet d\vec{A} = \oiint_{general} (\pm 1) \begin{bmatrix} P_r \\ 0 \\ P_z \end{bmatrix} \bullet \begin{bmatrix} r\,d\phi\,dz \\ dr\,dz \\ r\,dr\,d\phi \end{bmatrix}$$

$$= \iint_{top} (+1)(2\times 3)(r\,dr\,d\phi) + \iint_{bottom} (-1)(2\times 1)(r\,dr\,d\phi) + \iint_{cylinder} (+1)\tfrac{3}{2}(2\,d\phi\,dz)$$



$$= 6\int_0^2 r\,dr \int_0^{2\pi} d\phi - 2\int_0^2 r\,dr \int_0^{2\pi} d\phi + 3\int_1^3 dz \int_0^{2\pi} d\phi = 88.0$$

Note: $\vec{P} = \underline{P}$, $\hat{u}_r = \underline{a}_r$, $\hat{u}_z = \underline{a}_z$

Also, it is common for people to use r in cylindrical coordinate when it is really ρ .

- Volume integral is much easier to perform than line and/or surface integral as it is does not involve the dot product of two vectors.
- For instance, to find the total charge, we just multiply the charge density with the elementary volume, then integrate.
- Also, there is only ONE elementary volume in each coordinate systems.
- What is an elementary volume?
  - Cartesian coordinates: $dx\ dy\ dz$
  - Cylindrical coordinates: $d\rho\ \rho d\phi\ dz$
  - Spherical coordinates: $dr\ rd\theta\ rd\phi \sin\phi$

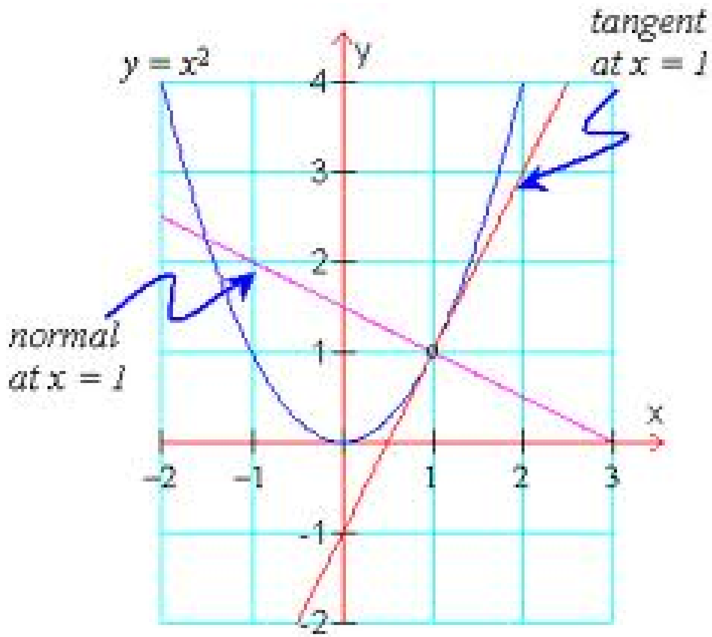
Important Operators

Gradient ∇

Divergent $\nabla \cdot$

Curl $\nabla \times$ or $\nabla \wedge$

Gradient



- Gradient is the directional rate of change of a physical quantity. It is a VECTOR. The most common example is the velocity (of a car, or an aircraft).
- Thus, the general form, the Grad Operator is written as ∇ and is operating on a scalar. The result is a VECTOR.
- Cartesian: $\nabla = \underline{a}_x \frac{\partial}{\partial x} + \underline{a}_y \frac{\partial}{\partial y} + \underline{a}_z \frac{\partial}{\partial z}$
- Cylindrical: $\nabla = \underline{a}_r \frac{\partial}{\partial r} + \underline{a}_\emptyset \frac{1}{r} \frac{\partial}{\partial \emptyset} + \underline{a}_z \frac{\partial}{\partial z}$
- Spherical: $\nabla = \underline{a}_r \frac{\partial}{\partial r} + \underline{a}_\theta \frac{1}{r} \frac{\partial}{\partial \theta} + \underline{a}_\emptyset \frac{1}{r sin\theta} \frac{\partial}{\partial \emptyset}$

examples:

$$\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} (x^9 + xy^2z^3) = \begin{bmatrix} 9x^8 + y^2z^3 \\ 2xyz^3 \\ 3xy^2z^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \phi} \\ \frac{\partial}{\partial z} \end{bmatrix} (r^9 + r\phi^2z^3) = \begin{bmatrix} 9r^8 + \phi^2z^3 \\ 2\phi z^3 \\ 3r\phi^2z^2 \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial \theta} \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi} \end{bmatrix} (r^9 + r\phi^2\sin^3\theta) = \begin{bmatrix} 9r^8 + \phi^2\sin^3\theta \\ 3\phi^2\sin^2\theta\cos\theta \\ 2\phi\sin^2\theta \end{bmatrix}$$

Exercise 2.8 Find the directional derivative of $V = rz^2 \cos 2\phi$ along the direction $\mathbf{A} = \hat{\mathbf{r}}2 - \hat{\mathbf{z}}$ and evaluate it at $(1, \pi/2, 2)$.

Solution:

Step One: Find gradient (in any direction)

$$V = rz^2 \cos 2\phi$$

$$\nabla V = \hat{\mathbf{r}}\frac{\partial V}{\partial r} + \hat{\boldsymbol{\phi}}\frac{1}{r}\frac{\partial V}{\partial \phi} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

$$= \hat{\mathbf{r}}z^2 \cos 2\phi - \hat{\boldsymbol{\phi}}\frac{2r}{r}z^2 \sin 2\phi + \hat{\mathbf{z}}2rz\cos 2\phi$$

Step Two: Dot product => gradient in a specific direction

$$\frac{dV}{dl} = \nabla V \cdot \hat{\mathbf{a}}_l$$

$$= \nabla V \cdot \frac{\mathbf{A}}{A}$$

$$= (\hat{\mathbf{r}}z^2 \cos 2\phi - \hat{\boldsymbol{\phi}}2z^2 \sin 2\phi + \hat{\mathbf{z}}2rz\cos 2\phi) \cdot \frac{\hat{\mathbf{r}}2 - \hat{\mathbf{z}}}{\sqrt{5}}$$

$$= \frac{2z^2 \cos 2\phi - 2rz\cos 2\phi}{\sqrt{5}}$$

Step Three: Insert the coordinate of the desired point

$$\left.\frac{dV}{dl}\right|_{(1,\pi/2,2)} = \frac{2 \times 4\cos\pi - 2 \times 2\cos\pi}{\sqrt{5}}$$

$$= -4/\sqrt{5}.$$

Divergent

- Divergence operator provides a relationship between the cause (or source, a physical quantity) and its (radial) effect. Examples are:
  - The electric force or electric field strength due to charges
  - The explosive force due to the amount of TNT
- Note: NO source => NO effect
- Divergence operator shown as $\nabla \cdot$
- It operates on a VECTOR (the effect, e.g., electric field)
- Thus divergence of electric field E at a point P gives the charge density σ at that point or $\nabla \cdot \underline{\mathrm{E}} = \sigma$
- Then $\nabla \cdot \underline{\mathrm{E}} dv = \sigma\, dv$
- Finally, $\iiint \nabla \cdot \underline{\mathrm{E}}\, dv = \iiint \sigma\, dv = Q$
- Or, total electric flux coming out from a volume is equal to the total charge enclosed within this volume

Cartesian Coordinates

$$\nabla \cdot \underline{A} = \left(\frac{\partial}{\partial x}\underline{a}_x + \frac{\partial}{\partial x}\underline{a}_y + \frac{\partial}{\partial x}\underline{a}_z\right) \cdot (A_x\underline{a}_x + A_y\underline{a}_y + A_z\underline{a}_z)$$
$$= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Must know

Cylindrical Coordinates

$$\nabla \cdot \underline{A} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

Need not memorize, will be given in exam. However, must know how to use.

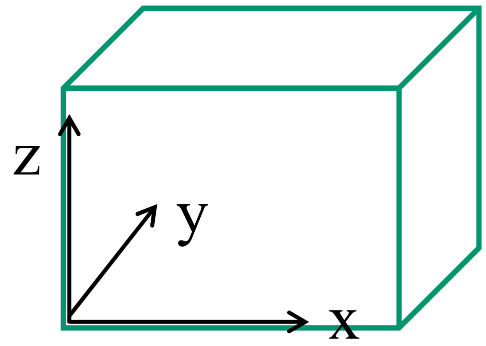
Spherical Coordinates

$$\nabla \cdot \underline{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

But wait, wait !! How is $\iiint \nabla \cdot \underline{E}\, dv$ represents the electric flux coming OUT from the volume ???

Let us look at a small cube (simple illustration but can be generalized). The electric field inside the small volume is
$\underline{E} = E_x\, \underline{a}_x + E_y\, \underline{a}_y + E_z\, \underline{a}_z$

The volume has six (6) surfaces:
right (area dydz, outward normal $\underline{a}_x$),
left (area dydz, outward normal $-\underline{a}_x$),
top (area dxdy, outward normal $\underline{a}_z$),
bottom (area dxdy, outward normal $-\underline{a}_z$,
front (area dxdz, outward normal $-\underline{a}_y$),
back (area dxdz, outward normal $\underline{a}_y$)



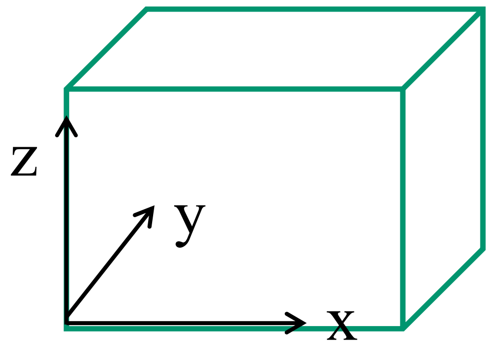
Let's look at the various surfaces:

Right surface: flux leaving = E_x(s) dydz where (s) denote value at the surface. But it is equal to $\underline{E} \cdot \underline{dA}$ since $\underline{dA} = dydz(+\underline{a}_x)$

There is no need to denote (s) further since for $\underline{E} \cdot \underline{dA}$, the $\underline{E}$ must be at the surface.

Similarly, Top surface, flux leaving is $E_z\, dxdy = \underline{E} \cdot \underline{dA}$ since $\underline{dA} = dxdy(+\underline{a}_z)$

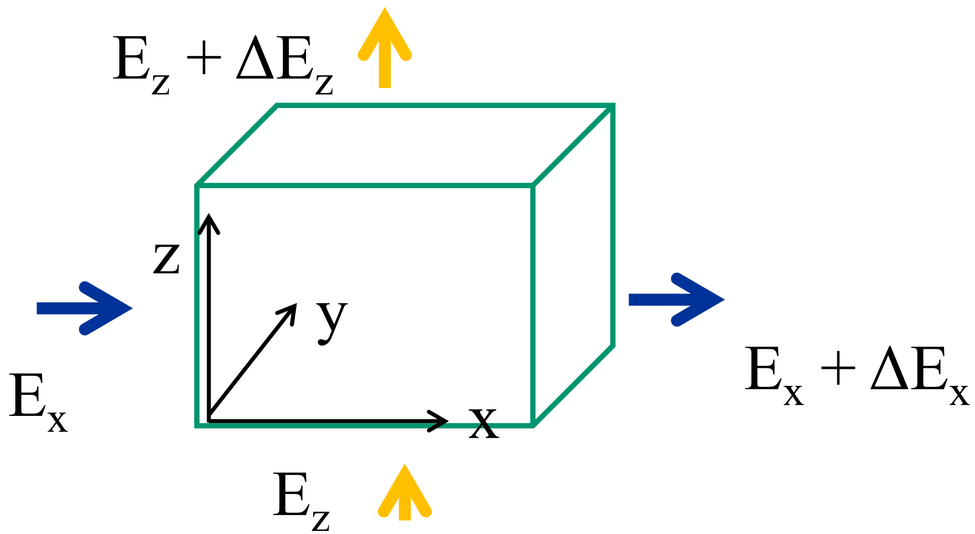
Front surface, flux leaving = $-E_y\, dxdz$ (which is flux entering) = $\underline{E} \cdot \underline{dA}$ since $\underline{dA} = dxdz(-\underline{a}_y)$



The students can work out the other 3 surfaces. Same result – flux leaving = $\underline{E} \cdot \underline{ds}$ Thus, total flux leaving is $\oiint \underline{E} \cdot \underline{ds}$

But how is it relate to $\oiiint \nabla \cdot \underline{E}\, dv$??

Now we denote E_x to be at the left surface of the cube and $E_x + \Delta E_x$ to be at the right surface of the cube. Similarly for other surfaces.



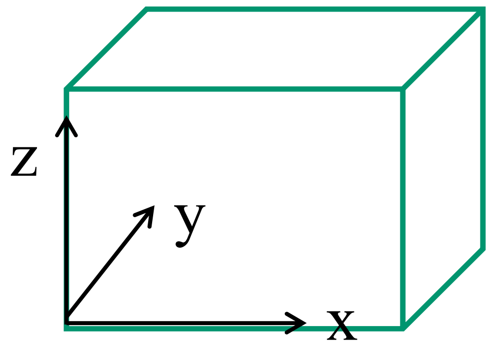
Then, Right surface: flux leaving = $E_x(s)$ dydz where (s) denote value at the surface = $(E_x + \Delta E_x)$dydz = E_x dydz + $\left(\frac{\partial E_x}{\partial x}\right) dxdydz$

Left surface: flux leaving = $-E_x(s)$ dydz where (s) denote value at the surface = $-(E_x)$dydz

Thus, sum up the two contributions:

E_x dydz + $\left(\frac{\partial E_x}{\partial x}\right) dxdydz$ $-(E_x)$dydz

$= \left(\frac{\partial E_x}{\partial x}\right) dxdydz = \left(\frac{\partial E_x}{\partial x}\right) dv$



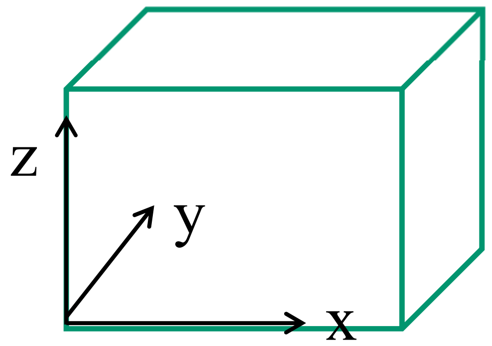
Similarly, summing up the top and bottom surfaces, we have: $\left(\frac{\partial E_z}{\partial z}\right) dv$

Summing up the front and back surfaces, we have $\left(\frac{\partial E_y}{\partial y}\right) dv$

Then overall, we have $\left(\frac{\partial E_x}{\partial x}\right) dv + \left(\frac{\partial E_y}{\partial y}\right) dv + \left(\frac{\partial E_z}{\partial z}\right) dv$

$= \nabla \cdot \underline{E}\, dv$

Thus, total flux is $\oiiint \nabla \cdot \underline{E}\, dv$



Now we have:

$$\oiiint \nabla \cdot \underline{E}\ dv = \oiint \underline{E} \cdot \underline{ds}$$

Charge (or other cause) enclosed

Flux (or other effect) leaving

This is the famous Divergence Theorem

Given $\mathbf{A} = e^{-2y}(\hat{\mathbf{x}} \sin 2x + \hat{\mathbf{y}} \cos 2x)$, find $\nabla \cdot \mathbf{A}$.

$$\mathbf{A} = e^{-2y}(\hat{\mathbf{x}} \sin 2x + \hat{\mathbf{y}} \cos 2x)$$

$$\begin{aligned}
\nabla \cdot \mathbf{A} &= \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \\
&= \frac{\partial}{\partial x}(e^{-2y} \sin 2x) + \frac{\partial}{\partial y}(e^{-2y} \cos 2x) \\
&= 2e^{-2y} \cos 2x - 2e^{-2y} \cos 2x = 0.
\end{aligned}$$

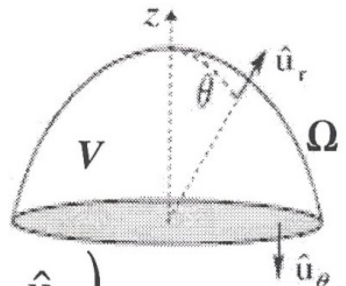
Given $\mathbf{A} = \hat{\mathbf{r}}\, r\cos\phi + \hat{\boldsymbol{\phi}}\, r\sin\phi + \hat{\mathbf{z}}\, 3z$, find $\nabla\cdot\mathbf{A}$ at $(2,0,3)$.

$$\mathbf{A} = \hat{\mathbf{r}}\, r\cos\phi + \hat{\boldsymbol{\phi}}\, r\sin\phi + \hat{\mathbf{z}}\, 3z$$

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \\ &= \frac{1}{r}\frac{\partial}{\partial r}(r^2\cos\phi) + \frac{1}{r}\frac{\partial}{\partial\phi}(r\sin\phi) + \frac{\partial}{\partial z}(3z) \\ &= 2\cos\phi + \cos\phi + 3\end{aligned}$$

$$\nabla\cdot\mathbf{A}\big|_{(2,0,3)} = 2 + 1 + 3 = 6.$$

verification example: hemisphere with radius = 2



$$\vec{D} = r^2\left(\hat{u}_r + \sin\theta\,\hat{u}_\theta + \sin\theta\sin\phi\,\hat{u}_\phi\right)$$

$$\oiint_\Omega \vec{D}\bullet d\vec{A} = \oiint_\Omega \left(D_r\hat{u}_r + D_\theta\hat{u}_\theta + D_\phi\hat{u}_\phi\right)\bullet\left(dA_r\hat{u}_r + dA_\theta\hat{u}_\theta\right)$$

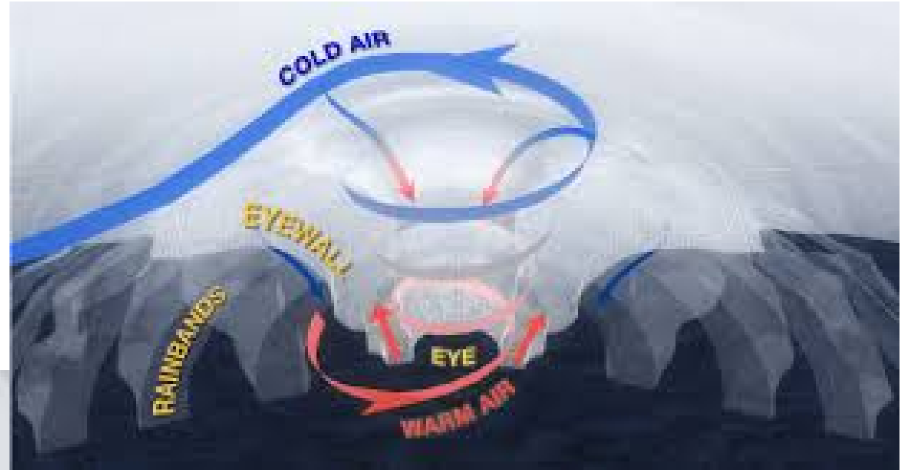
$$= \iint_{\Omega_r}\left(r^2\right)r^2\sin\theta\,d\theta\,d\phi\Big|_{r=2} + \iint_{\Omega_\theta}\left(r^2\sin\theta\right)r\sin\theta\,dr\,d\phi\Big|_{\theta=\frac{1}{2}\pi}$$

$$= 16\int_0^{\frac{1}{2}\pi}\sin\theta\,\mathrm{d}\theta\int_0^{2\pi}\mathrm{d}\phi + \int_0^2 r^3\,dr\int_0^{2\pi}\mathrm{d}\phi = 40\pi$$

$$\iiint_V \nabla\bullet\vec{D}\,dV = \iiint_V\left\{\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2D_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,D_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(D_\phi\right)\right\}dV$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\frac{1}{2}\pi}\int_{r=0}^{2}\left\{4r + 2r\cos\theta + r\cos\phi\right\}r^2\sin\theta\,dr\,d\theta\,d\phi = 40\pi$$

Curl



Cartesian Coordinates

$$\nabla X \underline{B} = \begin{vmatrix} \underline{a}_x & \underline{a}_y & \underline{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\underline{a}_x + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\underline{a}_y + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\underline{a}_z$$

Very easy to remember

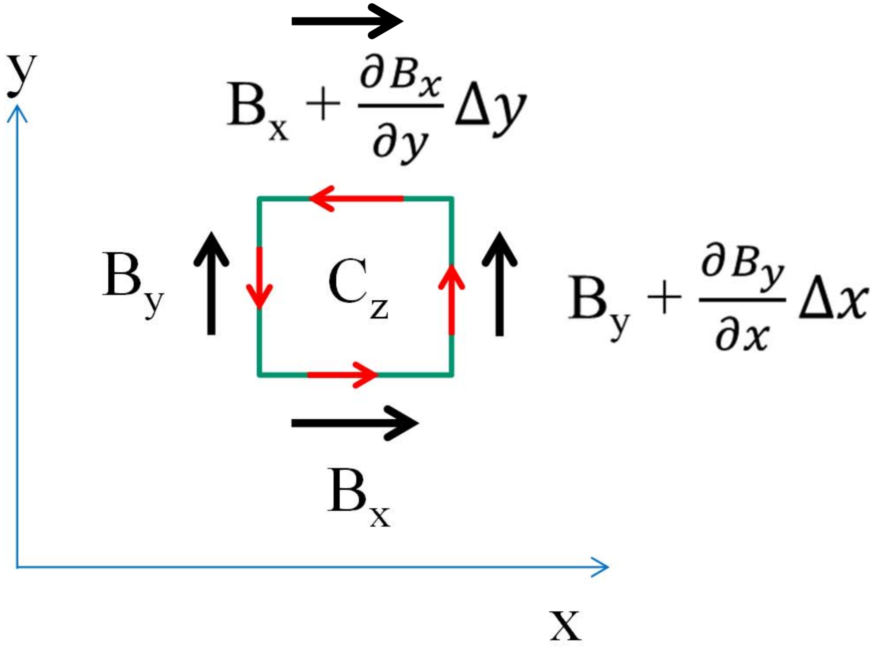
Cylindrical Coordinates

$$\nabla X \underline{B} = \frac{1}{\rho}\begin{vmatrix} \underline{a}_\rho & \rho\underline{a}_\emptyset & \underline{a}_z \\ \frac{\partial}{\partial\rho} & \frac{\partial}{\partial\emptyset} & \frac{\partial}{\partial z} \\ B_\rho & \rho B_\emptyset & B_z \end{vmatrix}$$

No need to memorize, will be given in exam. However, need to know how to use.

Spherical Coordinates

$$\nabla X \underline{B} = \frac{1}{r^2 sin\theta}\begin{vmatrix} \underline{a}_r & r\underline{a}_\theta & rsin\theta\underline{a}_\emptyset \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\emptyset} \\ B_r & rB_\theta & rsin\theta B_\emptyset \end{vmatrix}$$



For elementary loop C_z with area $\Delta A_z = \Delta x \Delta y$

$$\oint_{C_z} \underline{B} \cdot \underline{dl} =$$

$$+B_x \Delta x + \left(B_y + \frac{\partial B_y}{\partial x}\Delta x\right)\Delta y$$

$$-\left(B_x + \frac{\partial B_x}{\partial y}\Delta y\right)\Delta x - B_y \Delta y$$

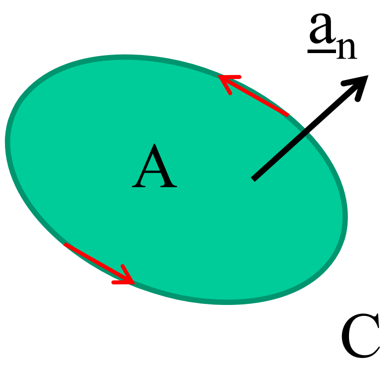
$$= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\Delta x \Delta y$$

$$= \nabla X \underline{B} \cdot \underline{\mathrm{ds}}$$

$$\nabla x \underline{B} = \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z}\right)\underline{a}_x + \left(\frac{\partial B_x}{\partial z} - \frac{\partial B_z}{\partial x}\right)\underline{a}_y + \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\underline{a}_z$$

$$\underline{ds} = \Delta x \Delta y \underline{a}_z$$

Now we have:



$$\iint_A \nabla X \underline{B} \cdot \underline{ds} = \oint_C \underline{B} \cdot \underline{dl}$$

Low-pressure, eye of the cyclone. You find later, in EM, this is the current enclosed.

Cyclic force of the typhoon. In EM, this is your Ampere's Law.

This is the famous Stoke's Theorem

Find $\nabla \times \mathbf{A}$ at $(2,0,3)$ in cylindrical coordinates for the vector field

$$\mathbf{A} = \hat{\mathbf{r}} 10 e^{-2r} \cos\phi + \hat{\mathbf{z}} 10 \sin\phi.$$

Solution:

$$\mathbf{A} = \hat{\mathbf{r}} 10 e^{-2r} \cos\phi + \hat{\mathbf{z}} 10 \sin\phi$$

$$\begin{aligned}
\nabla \times \mathbf{A} &= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r} \right) + \hat{\mathbf{z}} \frac{1}{r} \left(\frac{\partial}{\partial r} r A_\phi - \frac{\partial A_r}{\partial \phi} \right) \\
&= \hat{\mathbf{r}} \left(\frac{1}{r} \frac{\partial}{\partial \phi} (10 \sin\phi) \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial}{\partial z} (10 e^{-2r} \cos\phi) - \frac{\partial}{\partial r} (10 \sin\phi) \right) \\
&\quad + \hat{\mathbf{z}} \frac{1}{r} \frac{\partial}{\partial \phi} (-10 e^{-2r} \cos\phi) \\
&= \hat{\mathbf{r}} \frac{10 \cos\phi}{r} + \hat{\mathbf{z}} \frac{10 e^{-2r}}{r} \sin\phi
\end{aligned}$$

$$\nabla \times \mathbf{A} \big|_{(2,0,3)} = \hat{\mathbf{r}} 5.$$

Find $\nabla \times \mathbf{A}$ at $(3, \pi/6, 0)$ in spherical coordinates for the vector field $\mathbf{A} = \hat{\boldsymbol{\theta}}\, 12 \sin\theta$.

Solution:

$$\mathbf{A} = \hat{\boldsymbol{\theta}}\, 12 \sin\theta$$

$$\begin{aligned}
\nabla \times \mathbf{A} &= \hat{\boldsymbol{\phi}} \frac{1}{R} \frac{\partial}{\partial R}(R A_\theta) \\
&= \hat{\boldsymbol{\phi}} \frac{1}{R} \frac{\partial}{\partial R}(12 R \sin\theta) \\
&= \hat{\boldsymbol{\phi}} \frac{12 \sin\theta}{R}.
\end{aligned}$$

$$\nabla \times \mathbf{A}\big|_{(3,\pi/6,0)} = \hat{\boldsymbol{\phi}}\, 4 \sin 30^\circ = \hat{\boldsymbol{\phi}}\, 2.$$

Stoke's Theorem

verification example: quarter-circle loop with $\vec{F} = xy\hat{\mathrm{i}} - 2x\hat{\mathrm{j}} + 0\hat{\mathrm{k}}$

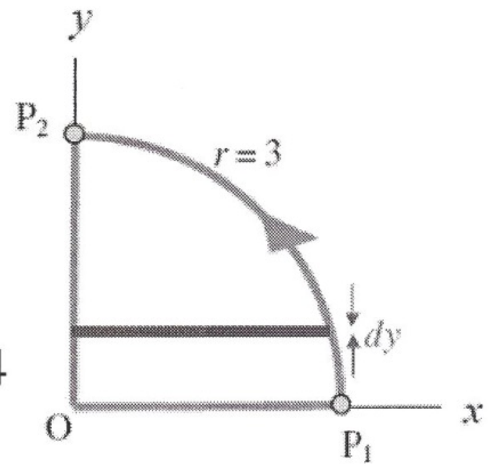
$$\oint \vec{F} \bullet d\vec{s} = \int_{P_1}^{P_2} \vec{F} \bullet d\vec{s} + \int_{P_2}^{O} \vec{F} \bullet d\vec{s} + \int_{O}^{P_1} \vec{F} \bullet d\vec{s}$$

$$= \int_0^{\frac{\pi}{2}} F_\phi (3\,d\phi) + \int_3^0 F_y dy \Big|_{x=0} + \int_0^3 F_x dx \Big|_{y=0}$$

$$= -23.14 + 0 + 0$$

$$\iint \nabla \times F \bullet d\vec{A} = \iint \begin{vmatrix} \hat{\mathrm{i}} & \hat{\mathrm{j}} & \hat{\mathrm{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -2x & 0 \end{vmatrix} \bullet d\vec{A}$$

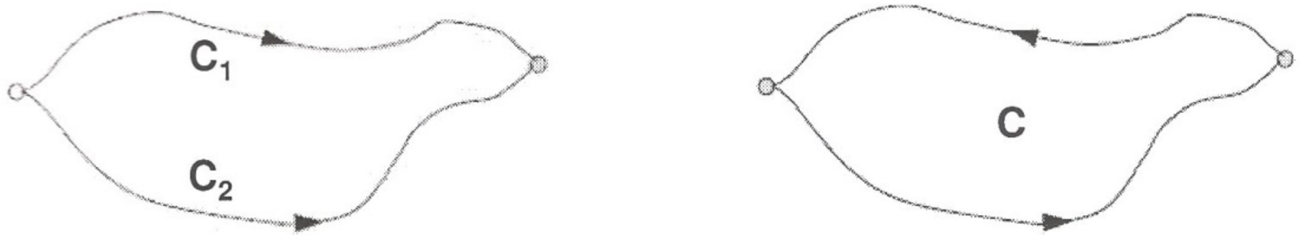
$$= \int_0^3 \int_0^{\sqrt{9-y^2}} (-2-x)\,dx\,dy = -23.14$$



Conservative Fields

e.g. gravitational or electrostatic force

same work to move mass or charge along any path from A to B

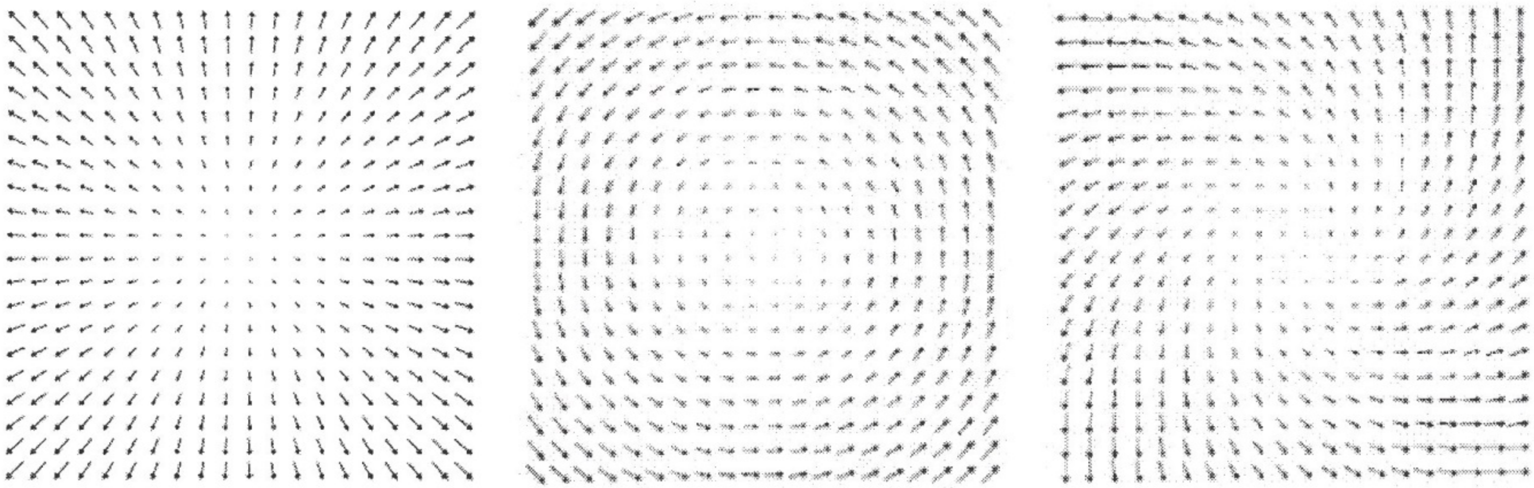


no work to move mass or charge along any closed path

$$\oint_C \vec{F} \bullet d\vec{s} = 0 \quad \Leftrightarrow \quad \iint_A \nabla \times \vec{F} \bullet d\vec{A} = 0$$

also need definition at any point $\qquad \nabla \times \vec{F} = \vec{0}$

examples of field patterns



$\nabla \bullet \vec{F} \neq 0$ $\qquad$ $\nabla \bullet \vec{F} = 0$ $\qquad$ $\nabla \bullet \vec{F} \neq 0$

$\nabla \times \vec{F} = \vec{0}$ $\qquad$ $\nabla \times \vec{F} \neq \vec{0}$ $\qquad$ $\nabla \times \vec{F} \neq \vec{0}$

Difficult exercises:-

Prove the following identities:

$$\nabla X(\nabla V) = 0, where\ v\ is\ a\ scalar$$

$$\nabla \cdot \left(\nabla X \underline{A}\right) = 0, where\ \underline{A}\ is\ a\ vector$$