EE2012 2014/15 Problem Set 6

CDF and PDF

1. A random variable X has the PDF

$$f_X(t) = \frac{1}{4}e^{-t/4}, \quad t > 0.$$

   (a) Find the CDF of X.

   (b) Using the CDF, find $P[1 < X \leq 2]$ and $P[X > 4]$.

   (c) X is a continuous random variable. Can you think of a function $g(x)$ that will make $Y = g(X)$ a discrete random variable?

2. In each of the following functions, find the unknown constant that will make it a valid PDF.

   (a) $f_1(x) = 2ce^{-(x-1)/4}$, $x > 0$.

   (b) $f_2(x) = \alpha(1-x)(x-3)$, $1 \leq x \leq 3$.

   (c) $f_3(x) = \beta\cos(x)[u(x+\frac{\pi}{2}) - u(x-\frac{\pi}{2})]$, where $u(x)$ is the unit step function.

3. Explain why the following functions are not valid CDFs:

   (a)

$$F_1(x) = \begin{cases} 0 & x \leq 0 \\ 0.4 & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

   (b) $F_2(x) = 1 - e^{-x}$.

   (c)

$$F_3(x) = \begin{cases} x & 0 < x \leq 1 \\ 1-x & 1 < x \leq 2 \end{cases}$$

4. The PDF of a random variable X is given by

$$f_X(x) = 0.2p\sum_{k=1}^{\infty} 0.8^{k-1}\delta(x-k) + (1-p)e^{-x},$$

   in the range $x > 0$.

   (a) If $p = 0.5$, sketch the PDF. What type of random variable is this?

   (b) For an arbitrary $p \in (0,1)$, find $P[X > 4.5]$.

   (c) Hence find $f_X(x|X > 4.5)$.

   (d) How would you generate samples of X on a computer, e.g. using Matlab or Scilab?

5. Find $P[X > 2]$ if X has the following distributions (if there is an unknown constant in the PDF or CDF, you first have to find its value):

   (a) $f_X(x) = \alpha \sum_{k=0}^{4} k\delta(x-k)$.

   (b) $F_X(x) = 1 - e^{-(x-1)}$, $x > 1$.

   (c) $f_X(x) = ce^{-|x|/2}$.

   (d) $F_X(x) = \begin{cases} 0 & x < 0 \\ 2x & 0 \le x < 0.5 \\ 1 & x \ge 1 \end{cases}$

6. Consider the PDF $f_X(x) = cx(1-x)$, $0 < x < 1$, where c is a constant.

   (a) Evaluate c.

   (b) Find the CDF $F_X(x)$.

   (c) Sketch both the PDF and the CDF.

7. Let X be a mixed-type random variable with PDF

$$f_X(x) = 0.25\delta(x+1) + 0.25\delta(x-1) + \beta e^{-2|x|}[u(x+1) - u(x-1)].$$

   (a) Sketch the PDF and find β.

   (b) Give an example of a physical experiment that may result in such a random variable.

Conditional Distributions

1. An exponential random variable has the PDF

$$f_X(x) = \lambda e^{-\lambda x}, \quad x > 0,$$

   where $\lambda > 0$ is a parameter of the distribution.

   (a) Prove the "memory-less property" of the exponential distribution, i.e. for any $t, h > 0$,
$$P[X > t + h | X > h] = P[X > t]$$

   (b) Using the memory-less property, show that the conditional CDF of X given $\{X > h\}$ is
$$F_X(x|X > h) = 1 - e^{-\lambda(x-h)}, \quad x > h.$$

2. At a certain location, the temperature in summer follows a $\mathcal{U}(25, 32)$ distribution, in autumn it has a $\mathcal{U}(10, 25)$ distribution, in winter it is $\mathcal{U}(-10, 10)$ and in spring it is $\mathcal{U}(10, 25)$. The $\mathcal{U}(a, b)$ distribution has the PDF

$$f_U(u) = \frac{1}{b-a}, \quad a < u < b.$$

Find the PDF of the temperature of a randomly selected day of the year, assuming that autumn and spring each occupy 20 percent of the year, and summer and winter each occupy 30 percent of the year.

3. Imagine that the 3G network is available with probability 0.8; independently, the 4G network is available with probability 0.8. A smartphone first tries to connect to the 4G network, then if that is not available, it connects to the 3G network. If neither network is available then there is no connectivity. Let X be some performance measure of the network, with the following conditional distributions:

$$\begin{aligned} f_X(x|\text{"connected to 4G"}) &= 0.1[u(x-10)-u(x-20)] && (1) \\ f_X(x|\text{"connected to 3G"}) &= 0.2[u(x-7)-u(x-12)] && (2) \\ f_X(x|\text{"neither"}) &= \delta(x). && (3) \end{aligned}$$

Find the PDF of X.