NATIONAL UNIVERSITY OF SINGAPORE
Department of Electrical & Computer Engineering

EXAMINATION FOR
(Semester II, 2013/14)

EE2012 ANALYTICAL TECHNIQUES FOR ECE

April/May 2014
Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- This is a CLOSED BOOK exam.
- This paper contains five (5) questions and one formula sheet, printed on four (4) pages.
- Answer ALL questions.
- A non-programmable calculator may be used.

Examiner: Professor Lim Teng Joon

Q1. Answer the following short questions.

(a) Let T be exponentially distributed with $E[T] = 2$. Find $P[T > 2]$. (2 marks)

Ans: Given that $\lambda = 0.5$, we have

$$P[T > 2] = \int_2^{\infty} 0.5e^{-0.5t}dt = e^{-1}.$$

(b) Consider a probability space $(\Omega, \mathcal{F}, P)$, and two independent events A and B in $\mathcal{F}$. What is $P[A \cup B]$ in terms of $P[A]$ and $P[B]$? (2 marks)

Ans: Since in general $P[A \cup B] = P[A] + P[B] - P[A \cap B]$ and $P[A \cap B] = P[A]P[B]$ because A and B are independent, we have

$$P[A \cup B] = P[A] + P[B] - P[A]P[B].$$

(c) If the CDF of X is

$$F_X(x) = 0.1u(x) + 0.3u(x-2) + 0.6u(x-5),$$

where $u(x)$ is the unit step function, write down the range of X, and the PDF of X. (2 marks)

Ans: $S_X = \{0, 2, 5\}$, and $f_X(x) = 0.1\delta(x) + 0.3\delta(x-2) + 0.6\delta(x-5)$.

(d) Two zero-mean unit-variance random variables X and Y have a correlation coefficient of 0.4. Find $E[XY]$. (2 marks)

Ans: From the formula for $\rho_{X,Y}$, and $E[X] = E[Y] = 0$, $\sigma_X = \sigma_Y = 1$, we have $E[XY] = 0.4$.

(e) If X is Gaussian with $E[X] = 1$ and $\text{var}(X) = 4$, find $P[X > 2]$ in terms of the Q function. (2 marks)

Ans: It should be clear that

$$\begin{aligned} P[X > 2] &= P\left[Z > \frac{2-1}{2}\right] && (1) \\ &= Q(0.5). && (2) \end{aligned}$$

(f) The joint PMF of (X, Y) is given as follows:

$$\begin{aligned} p_{X,Y}(0,0) &= 0.1 & p_{X,Y}(2,0) &= 0.1 \\ p_{X,Y}(1,1) &= 0.2 & p_{X,Y}(0,1) &= 0.2 \\ p_{X,Y}(2,1) &= 0.25 & p_{X,Y}(1,2) &= 0.15 \end{aligned}$$

Find the marginal PMF of X and of Y. (4 marks)

Ans: Using the marginalization rule, we have

$$p_X(k) = \begin{cases} 0.3 & k=0 \\ 0.35 & k=1 \\ 0.35 & k=2 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

$$p_Y(j) = \begin{cases} 0.2 & j=0 \\ 0.65 & j=1 \\ 0.15 & j=2 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(g) The random variable X is uniform in $\{1,2,3\}$, and Y is Bernoulli with $p = 0.2$. If X and Y are independent, find $E[XY]$. (3 marks)

Ans: Since X and Y are independent, $E[XY] = E[X]E[Y]$. We know that $E[X] = 2$, and $E[Y] = 0.2$, and hence $E[XY] = 0.4$.

(h) If $Y = 2X+3$, and $X \sim \mathcal{N}(0,1)$, find the mean and variance of Y, and write down the PDF of Y. (3 marks)

Ans: $E[Y] = 2E[X] + 3 = 3$, and $\sigma_Y^2 = 4\sigma_X^2 = 4$. Since Y has the form $aX + b$ and X is Gaussian, $Y \sim \mathcal{N}(3,4)$ and has the PDF

$$f_Y(y) = \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(y-3)^2}{8}\right).$$

Q2. A random variable X has the PDF

$$f_X(x) = \frac{5}{4}(1-x^4), \quad 0 < x \le 1.$$

(a) Find the CDF of X. (5 marks)

Ans: By definition, for $0 < x \le 1$,

$$F_X(x) = \int_0^x \frac{5}{4}(1-t^4)dt \tag{5}$$

$$= \frac{5}{4}\left[t - \frac{1}{5}t^5\right]_0^x \tag{6}$$

$$= \frac{x}{4}(5-x^4). \tag{7}$$

The complete CDF is

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ \frac{x}{4}(5-x^4) & 0 < x \le 1 \\ 1 & x > 1 \end{cases}$$

(b) Find $E[X]$ and $\mathrm{var}(X)$. (5 marks)

Ans: Since X is non-negative, we can use $E[X] = \int_0^\infty 1 - F_X(x)dx$ to obtain

$$E[X] = \int_0^1 1 - \frac{x}{4}(5 - x^4)dx \quad (8)$$

$$= \frac{5}{12}. \quad (9)$$

(Alternatively, we can also find $\int_0^\infty x f_X(x)dx$ directly.)

The second moment of X is

$$E[X^2] = \frac{5}{4}\int_0^1 x^2 - x^6 dx = \frac{5}{21}, \quad (10)$$

and hence $\mathrm{var}(X) = \frac{5}{21} - \frac{25}{144} = 0.0645$.

(c) Find $f_X(x|X > 0.5)$. (5 marks)

Ans: We have

$$P[X > 0.5] = 1 - F_X(0.5) = 1 - \frac{79}{128} = \frac{49}{128}.$$

Therefore,

$$f_X(x|X > 0.5) = \frac{128}{49} f_X(x), \quad x > 0.5 \quad (11)$$

$$= \frac{160}{49}(1 - x^4), \quad 0.5 < x \leq 1. \quad (12)$$

Q3. Let the discrete random variable X be uniformly distributed in $\{-3, -1, 1, 3\}$, and let N be a Gaussian random variable with mean 0 and variance σ_n^2. X and N are independent.

(a) If $Y = X + N$, find the conditional PDF $f_{Y|X}(y|x)$. (3 marks)

Ans: Given $X = x$, we have $Y = x + N$ because N is independent of X. Therefore,

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(y-x)^2}{2\sigma_n^2}\right).$$

(b) Find $P[Y > 2|X = 1]$ in terms of the Q function. (3 marks)

Ans: From part (a), Y conditioned on $X = 1$ is $\mathcal{N}(1, \sigma_n^2)$, and thus

$$P[Y > 2|X = 1] = Q\left(\frac{2-1}{\sigma_n}\right) = Q\left(\frac{1}{\sigma_n}\right).$$

(c) Given that $Y > 2$, what is the most likely value of X? (6 marks)

Ans: Proceeding as in part (b), we can obtain

$$\begin{aligned} P[Y > 2|X = -3] &= Q\left(\frac{5}{\sigma_n}\right) && (13)\\ P[Y > 2|X = -1] &= Q\left(\frac{3}{\sigma_n}\right) && (14)\\ P[Y > 2|X = 1] &= Q\left(\frac{1}{\sigma_n}\right) && (15)\\ P[Y > 2|X = 3] &= Q\left(\frac{-1}{\sigma_n}\right). && (16) \end{aligned}$$

By Bayes rule we have

$$P[X = x|Y > 2] = \frac{P[Y > 2|X = x]P[X = x]}{P[Y > 2]} \propto P[Y > 2|X = x] \qquad (17)$$

since $P[X = x] = 0.25$ for all $x \in S_X$, and $P[Y > 2]$ is not a function of x. Noting that the $Q(x)$ function is $P[\mathcal{N}(0,1) > x]$, and that the complementary CDF of $\mathcal{N}(0,1)$ is monotonically decreasing, the largest value of $P[Y > 2|X = x]$ is obtained at $x = 3$. Therefore the most likely value of X, given that $Y > 2$, is 3.

(d) Define Z as the indicator function of the event $\{Y > 2\}$. Find the PMF of Z, in terms of the Q function. (3 marks)

Ans: We have $Z = 1$ if $Y > 2$, and 0 otherwise. Therefore, $P[Z = 1] = P[Y > 2]$, which by total probability is

$$P[Y > 2] = 0.25 \sum_{k \in S_X} Q\left(\frac{2-k}{\sigma_n}\right). \qquad (18)$$

$P[Z = 0] = 1 - P[Z = 1]$ and thus we have the PMF of Z.

Q4. Consider two i.i.d. exponential random variables X and Y, with $E[X] = E[Y] = 2$.

(a) Find the PDF of $Z = X + Y$ using convolution. (5 marks)

Ans: Both X and Y have the same PDF:

$$f_X(t) = f_Y(t) = 0.5e^{-0.5t}, \quad t > 0.$$

The PDF of Z is $f_Z(t) = f_X(t) * f_Y(t) = \int_{-\infty}^{\infty} f_X(u) f_Y(t-u)du$. With a sketch of the two functions in the convolution integral, we can obtain

$$\begin{aligned} f_Z(t) &= \int_0^t 0.25e^{-u+u-t}du \quad (19)\\ &= 0.25e^{-t}\int_0^t du \quad (20)\\ &= 0.25te^{-t}, \quad t > 0. \quad (21)\end{aligned}$$

$f_Z(t) = 0$ for $t \leq 0$.

(b) Find and sketch the CDF of $V = X/Y$. (7 marks)

Ans: We can find the CDF $F_V(v) = P[X/Y \leq v]$ as follows:

$$\begin{aligned} F_V(v) &= \int_0^\infty \int_0^{vy} 0.25e^{-0.5(x+y)}dxdy \quad (22)\\ &= \int_0^\infty 0.5e^{-0.5y}(1-e^{-0.5vy})dy \quad (23)\\ &\vdots \quad (24)\\ &= 1 - \frac{1}{1+v}, \quad v > 0. \quad (25)\end{aligned}$$

(c) Find and sketch the PDF of $V = X/Y$. (3 marks)

Ans: Differentiating the CDF from part (b), we obtain

$$f_V(v) = \frac{1}{(1+v)^2}, \quad v > 0.$$

Q5. Consider a Poisson arrival process with average rate $\lambda = 2$ arrivals per minute. The number of arrivals in t minutes is denoted $N(t)$.

(a) Find the probability mass function of $N(t)$. (2 marks)

Ans: $N(t)$ is Poisson with mean λt, therefore

$$P[N(t) = k] = \frac{(\lambda t)^k}{k!}e^{-\lambda t}, \quad k = 0, 1, 2, \ldots \quad (26)$$

(b) Suppose that each arrival is independently tagged, with probability p. Find the PMF of $M(t)$, the number of tagged arrivals in t minutes. (7 marks)

Ans: Conditioned on $N(t) = n$, $M(t)$ will be binomial with parameters n and p, i.e.

$$P[M(t) = k|N(t) = n] = \binom{n}{k}p^k(1-p)^{n-k}, \quad k = 0, 1, \ldots, n.$$

By total probability, we then have, for $k = 0, 1, 2, \ldots$,

$$\begin{aligned}
P[M(t) = k] &= \sum_{n=0}^{\infty} P[M(t) = k | N(t) = n] P[N(t) = n] \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{(\lambda t)^n}{n!} e^{-\lambda t} \\
&= e^{-\lambda t} \frac{(p\lambda t)^k}{k!} \sum_{n=k}^{\infty} \frac{[(1-p)\lambda t]^{n-k}}{(n-k)!} \\
&= e^{-\lambda t} \frac{(p\lambda t)^k}{k!} e^{(1-p)\lambda t} \\
&= e^{-p\lambda t} \frac{(p\lambda t)^k}{k!}.
\end{aligned}$$

The second line comes from noting that $k \leq n$; the third from expanding the binomial coefficient and splitting $(\lambda t)^n$ into $(\lambda t)^{n-k} (\lambda t)^k$; and the final line from the power series expansion of e^x. In other words, the random tagging of Poisson arrivals results in another Poisson process.

(c) Let T_i be the waiting time until the i-th arrival, $i = 1, 2, \ldots$ Derive the CDF of T_i. (*Hint:* What is the event $\{T_i > t\}$ equivalent to?) (6 marks)

Ans: T_1 has already been shown in class to be exponential with parameter λ, because of the equivalence of $\{T_1 > t\}$ and $\{N(t) = 0\}$. Similarly, we can reason that $\{T_i > t\}$ is equivalent to $\{N(t) \leq i-1\}$: if the i-th arrival occurs more than t minutes later, than within t minutes there must have been fewer than i arrivals; conversely, if within t minutes there are fewer than i arrivals, then the i-th arrival must occur more than t minutes later. Therefore,

$$\begin{aligned}
1 - F_{T_i}(t) &= P[N(t) \leq i-1] \\
&= \sum_{k=0}^{i-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\end{aligned}$$

and hence

$$F_{T_i}(t) = 1 - \sum_{k=0}^{i-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad t > 0.$$

List of Formulae and Notation

Definitions

Indicator Function: $I_A = 1$ if A occurs, 0 otherwise.

Marginal PMF/PDF: $p_X(x_j) = \sum_k p_{X,Y}(x_j, y_k)$; $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$.

Marginal CDF: $F_X(x) = F_{X,Y}(x, \infty)$.

Joint Moments: $\rho_{X,Y} = \dfrac{\text{cov(X,Y)}}{\sigma_X \sigma_Y}$, $\text{cov}(X,Y) = E[XY] - E[X]E[Y]$.

Discrete Random Variables

Bernoulli: $p_X(1) = p = 1 - p_X(0)$, $E[X] = p, \text{var}[X] = p(1-p)$

Binomial: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \ldots, n$. $E[X] = np, \text{var}[X] = np(1-p)$

Geometric: $p_X(k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$. $E[X] = \dfrac{1}{p}, \text{var}[X] = \dfrac{1-p}{p^2}$

Poisson: $p_X(k) = \dfrac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \ldots$. $E[X] = \alpha = \text{var}[X]$

Continuous Random Variables

Uniform: $f_X(x) = \dfrac{1}{b-a}$, $a < x < b$. $E[X] = \dfrac{a+b}{2}, \text{var}[X] = \dfrac{(b-a)^2}{12}$

Exponential: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$. $E[X] = \dfrac{1}{\lambda}, \text{var}[X] = \dfrac{1}{\lambda^2}$

Gaussian: $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$, $x \in \mathbb{R}$. $E[X] = \mu, \text{var}[X] = \sigma^2$

Gaussianity

Q fn.: $Q(x) = \dfrac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$, $Q(x) = 1 - Q(-x)$

CDF: $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow P[X > t] = Q\left(\dfrac{t-\mu}{\sigma}\right)$

Joint PDF: $f_{\mathbf{X}}(\mathbf{x}) = \dfrac{1}{2\pi\sqrt{\det(\mathbf{C})}} \exp\left[-\dfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{C}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right]$, $\mathbf{x} \in \mathbb{R}^2$

where $\mathbf{C} = E[(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^T]$

Result 1: If $\mathbf{X}$ is jointly Gaussian, then $\mathbf{Y} = \mathbf{A}\mathbf{X} + \mathbf{b}$ is jointly Gaussian.

Result 2: If $\mathbf{X}$ is jointly Gaussian, then its components are marginally Gaussian.

Other Useful Results

Bayes: $P(B_j|A) = \dfrac{P(A|B_j)P(B_j)}{\sum_{k=1}^n P(A|B_k)P(B_k)}$, where $\{B_k\}_{k=1}^n$ is a partition of $\mathcal{S}$.

Total Prob.: $P(A) = \sum_{k=1}^n P(A|B_k)P(B_k)$, where $\{B_k\}_{k=1}^n$ is a partition of $\mathcal{S}$.

$$E[X] = \sum_{k=1}^n E[X|B_k]P[B_k]; \quad f_X(x) = \sum_{k=1}^n f_X(x|B_k)P[B_k].$$

Functions of X: $Y = aX + b \Rightarrow f_Y(y) = \dfrac{1}{|a|} f_X\left(\dfrac{y-b}{a}\right)$

$$Y = g(X) \Rightarrow f_Y(y) = \sum_{k=1}^n f_X(x_k) \left|\frac{dx_k}{dy}\right|, \text{ where } g(x_k) = y,\ k = 1, \ldots, n.$$

Independence (rv's): X, Y independent $\Leftrightarrow F_{X,Y}(x,y) = F_X(x)F_Y(y)$,

$f_{X,Y}(x,y) = f_X(x)f_Y(y)$, $p_{X,Y}(x,y) = p_X(x)p_Y(y)$.

Sum of rv's: X, Y independent $\Rightarrow f_{X+Y}(z) = f_X(z) * f_Y(z)$.