# EE2012 2014/15 QUIZ 1

February 12, 2015

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**Instructions**

- Write your student number, your name as it appears in IVLE, and your tutorial group at the top of your answer sheet.
- You have 30 minutes. Answer all questions.
- No books, notes or other written or printed material are allowed.
- The only electronic device you can use is a non-programmable calculator, that functions only as a calculator.
- All communicating devices must be turned off prior to starting the quiz.

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1. *There are 20 departments in a university. The probability of admission to department $i$ is $p_i$.*

   (a) *A student can only afford to apply to five departments. How many different ways are there for him to choose these five departments?*

   *Ans:* This is identical to drawing five times without replacement from a bag of 20 numbered balls, without noting the order. The five balls will determine the five departments to apply to. The answer is therefore

   $$\binom{20}{5} = 15,504.$$

   (b) *Suppose he applies to departments 1 through 5. Denote the event "Department $k$ accepts him" by $A_k$. Express the following events in set notation: (i) $B =$ "Accepted by at least one department" and (ii) $C =$ "Accepted by only departments 2 and 3".*

   *Ans:* (i) $B = A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5$ or $B = (A_1^c \cap A_2^c \cap A_3^c \cap A_4^c \cap A_5^c)^c$.
   (ii) $C = A_2 \cap A_3 \cap A_1^c \cap A_4^c \cap A_5^c$, because he is accepted only by departments 2 and 3, and not 1, 4 or 5.

   (c) *Suppose he applies to departments 1 through 5, and that they make independent decisions. What is the probability of gaining admission to* at least *departments 1 and 2? Explain briefly.*

   *Ans:* The event of interest is given by

   $$A = A_1 \cap A_2,$$

   the simple reason being that intersection of two sets (events) is equivalent to the occurrence of both events (AND logic). A slightly more sophisticated explanation is this: Event $A$ is the union of $A_1A_2A_3A_4A_5$, $A_1A_2A_3^cA_4A_5$, $A_1A_2A_3^cA_4^cA_5$, etc. A total of $2^3 = 8$ outcomes are in this union. However, by the distributive property of set intersection over union, we have

$$\bigcup_{B_i \in \{A_i, A_i^c\}} A_1A_2B_3B_4B_5 = A_1A_2 \bigcup_{B_i \in \{A_i, A_i^c\}} B_3B_4B_5 \tag{1}$$

But the union of all possible combinations of $B_3B_4B_5$, $B_i \in \{A_i, A_i^c\}$, covers all possible outcomes, and therefore the result follows.

Finally, $P[A] = P[A_1 \cap A_2] = P[A_1]P[A_2] = p_1p_2$ due to independence between $A_1$ and $A_2$.

2. *Consider a sample space $S = [0, 1]$ and a distribution function $F(x) = \sqrt{x}$ for $0 \le x < 1$.*

   (a) *What values must $F(x)$ take in the ranges $x < 0$ and $x \ge 1$?*
   *Ans:* Remember that $F(x) = P[(-\infty, x]]$ and therefore, for any $x < 0$, $F(x) =$ probability of obtaining a value less than a negative number. Since this is 0, we have $F(x) = 0$ for $x < 0$. On the other hand, for any $x \ge 1$, $S \subset (-\infty, x]$, and therefore $F(x) \ge P(S) = 1$ but since probability values do not exceed 1, $F(x) = 1$ for $x \ge 1$.

   (b) *Let $A = (0.5, 1]$ and $B = (0.2, 0.7]$. Find $P[A|B]$ and $P[B|A]$.*
   *Ans:* We need the following:

$$P[A] = F(1) - F(0.5) = 0.2929 \tag{2}$$
$$P[B] = F(0.7) - F(0.2) = 0.3894 \tag{3}$$
$$P[A \cap B] = F(0.7) - F(0.5) = 0.1296. \tag{4}$$

   Then by definition,

$$P[A|B] = \frac{P[A \cap B]}{P[B]} = 0.3328 \tag{5}$$
$$P[B|A] = \frac{P[A \cap B]}{P[A]} = 0.4425. \tag{6}$$

**Further Explanation of Q1b and Q1c** The outcome of the experiment here may be considered to be the 5-tuple $(x_1, x_2, x_3, x_4, x_5)$, where $x_i \in \{0, 1\}$, with $x_i = 0$ denoting the event that the student is rejected by department $i$, and $x_i = 1$ the event that he is accepted by department $i$. The sample space $S$ is therefore the set of all 32 such 5-tuples.

What is the event $A_1$? It is the set of all outcomes with a 1 in the first position, i.e.

$$A_1 = \{(1, x_2, x_3, x_4, x_5) : x_i \in \{0, 1\}\}.$$

This includes outcomes such as "accepted by department 1 and department 4 only", "accepted by all departments except department 2", etc. In total, there are 16 outcomes in $A_1$. The probability of one of these 16 outcomes appearing is denoted $p_1$.

Similarly, $A_2$ is the set of all outcomes with a 1 in the second position, and again $|A_2| = 16$ with $P[A_2]$ defined as $p_2$. Now what is $A_1 \cap A_2$? Well, since the elements of $A_1$ must have a 1 in the first position, and those of $A_2$ must have a 1 in the second position, the element common to both must have 1's in both the first and second position, i.e.

$$A_1 \cap A_2 = \{(1, 1, x_3, x_4, x_5) : x_i \in \{0, 1\}\}.$$

This is clearly the event "accepted by departments 1 and 2, and possibly others", which is what we wanted in Q1c.

In Q1b, event $B$ would contain all outcomes with at least one "1". Its complement is $B^c$ = "not accepted by any department" = $\{(0, 0, 0, 0, 0)\}$. Now it's clear that $|B| = 32 - 1 = 31$. The probability of $B$ (not asked for) would be $1 - P[B^c] = 1 - \prod_{i=1}^{5}(1 - p_i)$.

Using this understanding of $A_i$, it should be clear that event $C$ is as given above.