EE2012 2014/15 Problem Set 10

Independence of Two Random Variables

1. Let X be the quotient and Y the remainder when the number of dots observed in a toss of a fair die is divided by 3. Are X and Y independent?

2. Michael takes the 7:30 bus every morning. The arrival time of the bus at the stop is uniformly distributed in the interval [7:27,7:37]. Michael's arrival time at the stop is uniformly distributed in [7:25,7:40]. Assume that Michael's and the bus's arrival times are independent random variables.

   (a) What is the probability that Michael arrives more than 5 minutes before the bus?

   (b) What is the probability that Michael misses the bus?

3. Let X and Y be random variables that take on values from the set $\{-1, 0, 1\}$, and suppose their marginal PMFs are respectively

$$p_X(k) = \frac{1}{3}, \qquad k = -1, 0, 1 \tag{1}$$

$$p_Y(-1) = 0.5, \quad p_Y(0) = 0.2, \quad p_Y(1) = 0.3. \tag{2}$$

   (a) If X and Y are independent, find $P[X \geq Y]$.

   (b) Find the joint PMF of X^2 and Y^2 by considering the (X, Y) event equivalent to $\{X^2 = j, Y^2 = k\}$. Hence verify that X^2 and Y^2 are also independent random variables.

4. Let X and Y be independent random variables uniformly distributed in $[-1, 1]$. Find the probability of the following events:

   (a) $P[X < 0.5, |Y| < 0.5]$

   (b) $P[4X^2 < 1, Y < 0]$

   (c) $P[XY < 0.5]$

Expected Value of $g(X, Y)$ and Correlation

1. Show that the variance of $X + Y$ is equal to $\text{var}(X) + \text{var}(Y)$ if and only if X and Y are uncorrelated.

2. Let X and Y be discrete random variables with the following joint PMF.

x	y	$p(x,y)$
0	0	0.1
0	1	0.1
0	2	0.2
1	0	0.1
1	1	0.2
1	2	0.1
2	1	0.1
2	2	0.1

Find the covariance and correlation coefficient between X and Y.

3. If X and Y have the joint PDF

$$f_{X,Y}(x,y) = x + y, \quad 0 \le x \le 1, 0 \le y \le 1,$$

find the covariance and correlation coefficient of X and Y.

4. Suppose X and Y have the joint PDF

$$f_{X,Y}(x,y) = e^{-(x+|y|)}, \quad x > 0, -x < y < x.$$

Find the mean of $g(X,Y)$ in the following cases:

(a) $g(X,Y) = e^{0.5X}$;

(b) $g(X,Y) = e^{|Y|}$;

(c) $g(X,Y) = X + Y$.

5. The output of a channel $Y = X + N$, where the input X and the noise N are independent, zero-mean random variables.

(a) Find the correlation coefficient between X and Y.

(b) Find the value of a that minimizes the mean squared error $E[(X - aY)^2]$.

(c) Express the resulting mean squared error in terms of the signal to noise ratio, $\rho = (\sigma_X/\sigma_N)^2$, where σ_X and σ_N are the standard deviations of X and N respectively.