EE2012 2014/15 Problem Set 4

Discrete Random Variables

1. An urn contains nine \$2 notes and one \$10 note. Let the random variable X be the total amount that results when two bills are drawn from the urn without replacement.

   (a) Describe the underlying sample space S of this random experiment and specify the probabilities of its elementary events.

   Ans: $S = \{(2,2),(2,10),(10,2)\}$ where $\zeta = (x,y)$ means that the first note drawn is an \$$x$ note, and the second a \$$y$ note. The elementary event probabilities are:

$$\begin{aligned} P[(2,2)] &= \frac{9}{10}\cdot\frac{8}{9} = 0.8 \\ P[(2,10)] &= \frac{9}{10}\cdot\frac{1}{9} = 0.1 \\ P[(10,2)] &= \frac{1}{10}\cdot\frac{9}{9} = 0.1. \end{aligned}$$

   (b) Show the mapping from S to S_X, the range of X.

   Ans: We have $X((2,2)) = 4$, $X((2,10)) = X((10,2)) = 12$. Therefore $S_X = \{4, 12\}$. We can say that

$$X : S \longrightarrow S_X \tag{1}$$

   where $X((a,b)) = a + b$.

   (c) Find the probability mass function (PMF) of X.

   Ans: The event $\{X = 4\}$ is equivalent to $\{(2,2)\}$ in the underlying probability space, and $\{X = 12\}$ is equivalent to $\{(2,10),(10,2)\}$. Therefore,

$$p_X(4) = P[(2,2)] = 0.8, \quad p_X(12) = 0.2. \tag{2}$$

2. An m-bit password is required to access a system. A hacker systematically works through all possible m-bit patterns. Let X be the number of patterns tested until the correct password is found.

   (a) Describe the underlying sample space S.

   Ans: Let $\mathbf{a}$ represent the correct m-bit pattern. Then

$$S = \{(\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{a})\} \tag{3}$$

   where $\mathbf{x}_i \neq \mathbf{a}$, $i = 1, \ldots, n$, and $n = 0, 1, \ldots, 2^m$. The m-bit pattern $\mathbf{x}_i$ is the i-th pattern tried by the hacker. So assuming he keeps track of the patterns tried, we also have that $\mathbf{x}_i \neq \mathbf{x}_j$, for $i \neq j$.

(b) Show the mapping from S to S_X.

Ans: The mapping is

$$X((\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{a})) = n + 1,$$

with $S_X = \{1, 2, \ldots, 2^m\}$

(c) Find the PMF of X.

Ans: For convenience, let $A_k =$ "k-th attempt is correct". Then

$$p_X(n) = P\left[A_n \bigcap_{i=1}^{n-1} A_i^c\right].$$

It should be clear that $p_X(1) = P[A_1] = 2^{-m}$, since there are 2^m possible passwords, and the hacker tries them randomly. We also have

$$\begin{aligned} p_X(2) &= P[A_2|A_1^c]P[A_1^c] && (4)\\ &= \frac{1}{2^m - 1}\frac{2^m - 1}{2^m} && (5)\\ &= \frac{1}{2^m}. && (6) \end{aligned}$$

Similarly,

$$\begin{aligned} p_X(3) &= P[A_3|A_2^c A_1^c]P[A_2^c|A_1^c]P[A_1^c] && (7)\\ &= \frac{1}{2^m - 2}\frac{2^m - 2}{2^m - 1}\frac{2^m - 1}{2^m} && (8)\\ &= \frac{1}{2^m}. && (9) \end{aligned}$$

Proceeding in this way, we quickly see that X is in fact uniformly distributed in $S_X = \{1, 2, \ldots, 2^m\}$. The average number of attempts needed to break an m-bit password is thus 2^{m-1}. For a typical password of ten 8-bit ASCII characters, i.e. 80 bits, it will take $2^{79} = 6.04 \times 10^{23}$ attempts on average!

3. Two transmitters send messages over a wireless channel. During each time slot, each transmitter sends a message with probability $\frac{1}{2}$. Simultaneous transmissions result in loss of both messages. Let X be the number of time slots until the first message gets through.

(a) Describe the underlying sample space S and specify the probabilities of its elementary events.

Ans: Denoting a transmission as T, then one possible outcome is the sequence $((T, T), (T, T), (T, T^c))$ which means that in the first two time slots both transmitters transmit, and so their messages are lost, but in the third time slot, only transmitter 1 transmits. For this outcome, $X = 3$.

The entire sample space is thus

$$S = \{[(x_1, y_1), (x_2, y_2), \ldots, (x_{n-1}, y_{n-1}), (x_n, y_n)] : n = 1, 2, \ldots\}$$

where $x_i = y_i$ for $i = 1, 2, \ldots, n-1$, and $x_n \neq y_n$, with all $x_i, y_i \in \{T, T^c\}$. The probability of needing n transmissions until a transmission is successful follows a geometric distribution, with probability of success $p = 0.5$, because a successful transmission is either (T, T^c) or (T^c, T). Therefore, the probability of the n-th elementary event is

$$p_n = \left(\frac{1}{2}\right)^{n-1} \frac{1}{2} = \frac{1}{2^n}, \quad n = 1, 2, \ldots$$

(b) Show the mapping from S to S_X.
Ans: $X(\zeta)$ is the number of (x_i, y_i) pairs in ζ.

(c) Find the PMF of X.
Ans: This was already given in part (a), i.e. $p_X(k) = 0.5^k$, $k = 1, 2, \ldots$

4. Let X be a random variable with PMF $p_X(k) = \dfrac{c}{k^2}$, $k = 1, 2, \ldots$.

(a) Find the value of c. Note that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}$$

where $\zeta(s)$ is called the Riemann zeta function, defined as $\zeta(s) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^s}$ for any complex value s.
Ans: We need $\sum_{k=1}^{\infty} p_X(k) = 1$, therefore

$$\begin{aligned} c\sum_{k=1}^{\infty} \frac{1}{k^2} &= 1 \qquad (10) \\ \Rightarrow \quad c &= \frac{6}{\pi^2}. \qquad (11) \end{aligned}$$

(b) Find $P[X > 6]$.
Ans: We have

$$\begin{aligned} P[X > 6] &= 1 - P[X \leq 5] \qquad (12) \\ &= 1 - \frac{6}{\pi^2} \sum_{k=1}^{5} \frac{1}{k^2} \qquad (13) \\ &= 0.1102. \qquad (14) \end{aligned}$$

(c) Find $P[4 \le X \le 8]$.
Ans: We have

$$\begin{aligned} P[4 \le X \le 8] &= \frac{6}{\pi^2}\sum_{k=4}^{8}\frac{1}{k^2} & (15)\\ &= 0.1011. & (16)\end{aligned}$$

5. Suppose the probability of a person being left-footed given that s/he is left-handed is 0.9, and the probability of a person being left-footed given that s/he is right-handed is 0.1. Denoting by LF, RF, LH and RH the events that a person is left-footed, right-footed, left-handed and right-handed respectively, this means

$$P(\text{LF}|\text{LH}) = 0.9, \quad P(\text{LF}|\text{RH}) = 0.1.$$

Within the population, 5 percent are left-handed, and the rest are right-handed.

(a) Find the probability of a person being left-footed.
Ans: We use the Theorem on Total Probability with LH and RH partitioning the sample space:

$$\begin{aligned} P(\text{LF}) &= P(\text{LF}|\text{RH})P(\text{RH}) + P(\text{LF}|\text{LH})P(\text{LH}) & (17)\\ &= 0.1(0.95) + 0.9(0.05) & (18)\\ &= 0.14. & (19)\end{aligned}$$

(b) Find the probability of someone being both left-handed and left-footed.
Ans: We use the definition of conditional probability to write

$$P(\text{LH, LF}) = P(\text{LF}|\text{LH})P(\text{LH}) = 0.9(0.05) = 0.045.$$

(c) Find the minimum number of people that has to be picked in order to have a probability of finding at least two people who are both left-handed and left-footed exceed 0.8. (Assume that the total population is very large.)
Ans: Let X denote the number of people out of N who are left-handed and left-footed. Then $X \sim B(N, 0.045)$, and has the PMF

$$p_X(k) = \binom{N}{k} 0.045^k 0.955^{N-k}, \quad k = 0, 1, \ldots, N.$$

We want to find the smallest N such that $P[X \ge 2] \ge 0.8$, i.e.

$$\begin{aligned} 1 - p_X(0) - p_X(1) &\ge 0.8\\ 1 - 0.955^N - 0.955^{N-1}(0.045N) &\ge 0.8\\ f(N) = 0.955^{N-1}(0.955 + 0.045N) &\le 0.2\end{aligned}$$

We find that $f(65) = 0.204$ and $f(66) = 0.197$. Thus the smallest N for which $P[X \ge 2] \ge 0.8$ is $N = 66$. In other words, to be 80 percent certain of finding two or more left-handed and left-footed people, we need a sample of 66 people.

Expected Values

1. (a) The mean value of a Bernoulli random variable X is 0.6. Find the PMF of X.
   Ans: The mean of a Bernoulli r.v. with parameter p is
   $$E[X] = 0(1-p) + 1(p) = p.$$
   Therefore, if $E[X] = 0.6$, then $p = 0.6$, giving the PMF
   $$p_X(0) = 0.4, \quad p_X(1) = 0.6.$$
   (b) Find the mean and variance of Y if $S_Y = \{0, 2, 5, 10\}$ and all values are equi-probable.
   Ans: $E[Y] = (0 + 2 + 5 + 10)/4 = 4.25$.
   The mean square value of Y is
   $$E[Y^2] = (4 + 25 + 100)/4 = 32.25. \tag{20}$$
   Therefore,
   $$\sigma_Y^2 = 32.25 - (4.25)^2 = 14.1875. \tag{21}$$
   (c) If $Z = 2Y^2 - 3Y$, with Y defined above, find $E[Z]$.
   Ans: We have
   $$\begin{aligned} E[Z] &= 2E[Y^2] - 3E[Y] && (22)\\ &= 2 \times 32.25 - 3 \times 4.25 && (23)\\ &= 51.75. && (24) \end{aligned}$$

2. Find the expected value and the variance of X if it has the following PMFs. You have to first find p_1 in both cases.

   (a) $p_X(k) = p_1/k$, $k \in \{1, 2, 3, 4\}$.
   Ans: We have $p_X(1) = p_1$, $p_X(2) = 0.5p_1$, $p_X(3) = p_1/3$ and $p_X(4) = 0.25p_1$. To find p_1, we use the fact that $P(S) = 1$:
   $$p_1(1 + 0.5 + 1/3 + 0.25) = 1 \quad \Rightarrow \quad p_1 = \frac{12}{25}. \tag{25}$$
   We thus have
   $$p_X(1) = \frac{12}{25},\ p_X(2) = \frac{6}{25},\ p_X(3) = \frac{4}{25},\ p_X(4) = \frac{3}{25}. \tag{26}$$
   Therefore, $E[X]$ can be computed as 48/25, and
   $$E[X^2] = \frac{12}{25} + \frac{24}{25} + \frac{36}{25} + \frac{48}{25} = 4.8, \tag{27}$$
   and hence $\mathrm{var}(X) = 4.8 - (48/25)^2 = 1.1136$.

(b) $p_X(k+1) = p_X(k)/2$, $k = 1, 2, 3$; $p_X(1) = p_1$.

Ans: We have $p_X(2) = p_1/2$, $p_X(3) = p_1/4$, $p_X(4) = p_1/8$, and therefore

$$p_1 \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right) = 1 \quad \Rightarrow \quad p_1 = \frac{8}{15}. \tag{28}$$

We can compute

$$\begin{aligned} E[X] &= \frac{8}{15}\left(1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8}\right) &(29)\\ &= \frac{8}{15} \cdot 3\frac{1}{4} &(30)\\ &= \frac{26}{15} &(31)\\ E[X^2] &= \frac{8}{15}\left(1 + \frac{4}{2} + \frac{9}{4} + \frac{16}{8}\right) &(32)\\ &= \frac{58}{15}. &(33)\end{aligned}$$

And finally,

$$\sigma_X^2 = \frac{58}{15} - \left(\frac{26}{15}\right)^2 = 0.862.$$

3. Let X and Y be two integer-valued random variables whose PMFs differ only by a constant shift, i.e.

$$p_X(k) = p_Y(k - a)$$

where a is an integer constant.

(a) Show that $E[X] = E[Y] + a$.

Ans: By definition,

$$\begin{aligned} E[X] &= \sum_{k=-\infty}^{\infty} k p_X(k) &(34)\\ &= \sum_{k=-\infty}^{\infty} k p_Y(k-a) &(35)\\ &= \sum_{n=-\infty}^{\infty} (n+a) p_Y(n) &(36)\\ &= \sum_{n=-\infty}^{\infty} n p_Y(n) + a \sum_{n=-\infty}^{\infty} p_Y(n) &(37)\\ &= E[Y] + a. &(38)\end{aligned}$$

The third line is obtained by the substitution $k = n + a$. The other steps should be self-explanatory.

(b) Show that $\text{var}(X) = \text{var}(Y)$.

Ans: Let $E[X] = \mu_X$ and $E[Y] = \mu_Y$. From first principles,

$$\begin{aligned} \text{var}(X) &= E[(X-\mu_X)^2] = \sum_{k=-\infty}^{\infty} (k-\mu_X)^2 p_X(k) && (39) \\ &= \sum_{k=-\infty}^{\infty} (k-\mu_Y-a)^2 p_Y(k-a) && (40) \\ &= \sum_{n=-\infty}^{\infty} (n-\mu_Y)^2 p_Y(n) && (41) \\ &= \text{var}(Y). && (42) \end{aligned}$$

As in part (a), the third line follows after the substitution $k = n + a$.

4. Suppose a fair coin is tossed n times. Each coin toss costs d dollars and the reward in obtaining X heads is $aX^2 + bX$. Find the expected net reward.

Ans: The net reward R is the difference between the reward and the cost:

$$R = aX^2 + bX - nd.$$

Therefore, by the linearity property of the expectation operator, $E[R] = aE[X^2] + bE[X] - nd$. Since X is a binomial random variable, with probability of success $p = 1/2$, we know that $E[X] = \frac{1}{2}n$ and $\text{var}(X) = \frac{1}{4}n$. But $\text{var}(X) = E[X^2] - E^2[X]$, and so

$$E[X^2] = \frac{1}{4}n + \frac{1}{4}n^2 = \frac{n}{4}(1+n). \tag{43}$$

Substituting into the expression for $E[R]$ yields

$$E[R] = \frac{an}{4}(1+n) + \frac{bn}{2} - nd. \tag{44}$$

5. The St. Petersburg Paradox is the following. A casino offers a payout of 2^X dollars, where X is the number of flips of a fair coin required to obtain the first Heads. In other words, if it takes 10 flips to obtain the first Heads, the casino pays 1024 dollars. The potential payout is unlimited, since the sample space of X extends to infinity. But yet a shrewd gambler will only be willing to wager a small amount of money to play the game. Why? We find out by solving the following problems.

   (a) How many tosses can the casino afford to pay out if it has a finite amount of money M dollars?

   Ans: Suppose the casino has 2000 dollars, then it can only afford to pay for 10 tosses (1024 dollars); if it has 5000 dollars, then it can only afford 12 tosses, etc. In general, if the casino has M dollars, it can only afford $m = \lfloor \log_2(M) \rfloor$ tosses, where $\lfloor x \rfloor$ denotes the largest integer smaller than x. If it takes more than m tosses to obtain a Heads, the casino will not be able to pay the promised 2^X dollars to the gambler.

(b) Find the expected payoff to the player.

Ans: Let the payoff to the player be denoted by Y. Then we have

$$Y = \begin{cases} 2^X & X \leq m \\ 2^m & X > m \end{cases} \tag{45}$$

where X is the number of tosses required to obtain the first Heads. The mean payoff is therefore

$$E[Y] = \sum_{k=1}^{m} 2^k p_X(k) + 2^m \sum_{k=m+1}^{\infty} p_X(k). \tag{46}$$

But X is a geometric r.v. with $p = 0.5$, hence $p_X(k) = 2^{-k}$, $k = 1, 2, \ldots$. Thus,

$$\begin{aligned} E[Y] &= \sum_{k=1}^{m} 1 + 2^m \frac{1}{2^m} && (47) \\ &= m + 1. && (48) \end{aligned}$$

(c) How much should a player be willing to pay to play this game?

Ans: Since his expected payoff is $m + 1$ dollars, the gambler should be willing to bet less than that amount. A simple calculation shows that this is not a lot of money, even if the casino starts with a large amount – say the casino has 1 billion dollars, which is about $2^{29.9}$ dollars. The gambler should be willing to bet only up to \$30!

This is because the large value of $E[2^X]$ is due mainly to low-probability events with a huge payoff. When these low-probability events do occur, the casino will be unable to pay out the amount due, and therefore the payoff is not really exponential in the number of tosses, but rather exponential only up to a cap.