Joint PMF and CDF

1. Flip a fair coin four times. Let X be the number of Heads obtained, and let Y be the position of the first Heads i.e. if the sequence of coin flips is TTHT, then $Y = 3$, if it is THHH, then $Y = 2$. If there are no heads in the four tosses, then we define $Y = 0$.

   (a) Find the joint PMF of X and Y.
   Ans: The underlying sample space is the set

   $$\mathcal{S} = \{TTTT, TTTH, \ldots, HHHH\}.$$

   Each outcome ζ in $\mathcal{S}$ can be mapped to $X(\zeta)$ and $Y(\zeta)$, e.g.

   $$X(\text{TTTT}) = 0 \qquad Y(\text{TTTT}) = 0 \tag{1}$$
   $$X(\text{THHT}) = 2 \qquad Y(\text{THHT}) = 2 \tag{2}$$
   $$X(\text{TTTH}) = 1 \qquad Y(\text{TTTH}) = 4 \tag{3}$$
   $$\text{etc.} \tag{4}$$

   By listing all 16 elements of $\mathcal{S}$, and computing X and Y for each, we can see e.g.

   $$\{X = 2, Y = 2\} = \{\text{THHT}, \text{THTH}\}$$

   and thus $p_{X,Y}(2, 2) = \frac{1}{16} + \frac{1}{16} = \frac{1}{8}$. This can be repeated for all pairs of feasible values of X and Y, as in the table below:

	x				
y	0	1	2	3	4
0	$\frac{1}{16}$				
1		$\frac{1}{16}$	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{1}{16}$
2		$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{16}$	
3		$\frac{1}{16}$	$\frac{1}{16}$		
4		$\frac{1}{16}$			

   (b) Using the joint PMF, find the marginal PMF of X.
   Ans: Summing over the columns of the table below, we get the PMF of X as

   $$p_X(k) = \begin{cases} \frac{1}{16} & k = 0 \\ \frac{1}{4} & k = 1 \\ \frac{3}{8} & k = 2 \\ \frac{1}{4} & k = 3 \\ \frac{1}{16} & k = 4 \end{cases}$$

(c) Find the joint CDF $F_{X,Y}(x,y)$ in the region

$$\{(x,y) : 1 \leq x < 2, 2 \leq y < 3\}.$$

Ans: For any point (x,y) in the stated region, the event $\{X \leq x, Y \leq y\}$ includes the same three points in $S_{X,Y}$, namely (0,0), (1,1) and (1,2). Therefore,

$$F_{X,Y}(x,y) = p_{X,Y}(0,0) + p_{X,Y}(1,1) + p_{X,Y}(1,2) = \frac{3}{16}$$

everywhere in this region.

2. The random variable X is Poisson with mean 1. Conditioned on $X = k$, Y is binomial with $n = k$ and $p = 0.1$.

(a) Find the joint PMF of X and Y.
Ans: We know that

$$p_X(k) = \frac{e^{-1}}{k!}, \quad k = 0, 1, 2, \ldots \tag{5}$$

$$p_Y(j|X = k) = \binom{k}{j} 0.1^j 0.9^{k-j}, \quad j = 0, 1, \ldots, k. \tag{6}$$

Therefore, from the fact that $P[A \cap B] = P[A|B]P[B]$ and with $A = \{Y = j\}$, $B = \{X = k\}$, we have

$$\begin{aligned} p_{X,Y}(k,j) &= p_X(k) p_Y(j|X = k) \\ &= \frac{e^{-1}}{j!(k-j)!} 0.1^j 0.9^{k-j}, \quad k = 0, 1, 2, \ldots; \ j = 0, 1, \ldots, k. \end{aligned}$$

It should be noted that the range of (X,Y) is not simply a rectangular region.

(b) Find the marginal PMF of Y.
Ans: We sum $p_{X,Y}(k,j)$ over k to obtain the PMF of Y:

$$p_Y(j) = \sum_{k=j}^{\infty} \frac{e^{-1}}{j!(k-j)!} 0.1^j 0.9^{k-j} \tag{7}$$

$$= \frac{0.1^j e^{-1}}{j!} \sum_{k=j}^{\infty} \frac{0.9^{k-j}}{(k-j)!} \tag{8}$$

$$= \frac{0.1^j e^{-1}}{j!} \sum_{l=0}^{\infty} \frac{0.9^l}{l!} \tag{9}$$

where the last line comes from the change of variables $l = k - j$. We recognize the infinite sum as the power series expression for $e^{0.9}$, and therefore

$$p_Y(j) = \frac{0.1^j e^{-1}}{j!} e^{0.9} = \frac{0.1^j}{j!} e^{-0.1}, \quad j = 0, 1, 2, \ldots$$

In other words, Y is Poisson, with $E[Y] = 0.1$.

3. Suppose the marginal PMFs of X and Y are identical:

$$p_X(k) = p_Y(k) = \frac{1}{3}, \quad k = -1, 0, 1.$$

(a) Show that the joint PMF of X and Y must be zero except possibly at the nine points in $\{(j,k) : j, k \in \{-1, 0, 1\}\}$.

Ans: We prove this statement by contradiction. Suppose that $p_{X,Y}(j_0, k) \neq 0$ for some $j_0 \notin \{-1, 0, 1\}$ and at least one $k \in \{-1, 0, 1\}$. Then $p_X(j_0) = \sum_{k=-1}^{1} p_{X,Y}(j_0, k) \neq 0$. But this contradicts the definition of $p_X(k)$, and therefore such a j_0 cannot exist. A similar argument applies to $p_Y(k)$. Thus $p_{X,Y}(j,k)$ can only be non-zero in $(j,k) \in \{-1, 0, 1\}^2$.

(b) Show that the two marginal PMFs do not uniquely determine the joint PMF of X and Y.

Ans: Consider the following two functions of x and y, with $x, y \in \{-1, 0, 1\}$:

$$p(x,y) = \begin{cases} \frac{1}{3} & x = y \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$q(x,y) = \begin{cases} \frac{1}{6} & (x,y) \in \{(-1,-1), (-1,0), (0,0), (0,-1)\} \\ \frac{1}{3} & (x,y) = (1,1) \end{cases} \tag{11}$$

It can be verified that $\sum_x p(x,y) = \sum_x q(x,y) = 1/3$ for each value of y, and $\sum_y p(x,y) = \sum_y q(x,y) = 1/3$ for each value of x. Therefore, these two joint PMFs lead to the same marginal PMFs, showing that marginal PMFs do not uniquely determine a joint PMF.

(c) Suppose $p_{X,Y}(x,y) = p_X(x)p_Y(y)$ for all $x, y \in \{-1, 0, 1\}$. Find the probabilities of $\{X > Y\}$, $\{X = Y\}$ and $\{Y \leq 0\}$.

Ans:

$$\begin{aligned} P[X > Y] &= P[\{(0,-1), (1,0), (1,-1)\}] \\ &= P[\{(0,-1)\}] + P[\{(1,0)\}] + P[\{(1,-1)\}] \\ &= p_X(0)p_Y(-1) + p_X(1)p_Y(0) + p_X(1)p_Y(-1) \\ &= \frac{1}{3}. \\ P[X = Y] &= P[\{(-1,-1), (0,0), (1,1)\}] \\ &= \frac{1}{3}. \\ P[Y \leq 0] &= p_Y(-1) + p_Y(0) = \frac{2}{3}. \end{aligned}$$

4. Let X be a discrete random variable uniformly distributed in $\{1, 2, 3, 4\}$. Given $X = x$, Y is uniformly distributed in $\{1, \ldots, x\}$. Draw a tree diagram of the experiment and find the joint PMF of X and Y.

Ans: This is like a sequence of two sub-experiments – in the first one, we choose X; after knowing X, we pick Y. In the tree diagram, the first level depicts X, and the second level depicts Y.

Using the rule that $p_{X,Y}(j,k) = p_Y(k|X=j)p_X(j)$ and the tree diagram, we can derive the following $p_{X,Y}$ table:

y	x			
	1	2	3	4
1	1/4	1/8	1/12	1/16
2	0	1/8	1/12	1/16
3	0	0	1/12	1/16
4	0	0	0	1/16

5. A point (X,Y) is selected at random inside a triangle defined by $\Delta = \{(x,y) : 0 \le y \le x \le 1\}$. Assume that the point is equally likely to fall anywhere inside the triangle.

   (a) Find the joint CDF of X and Y.

   Ans: Given that the point is equally likely to fall anywhere inside the triangle, the probability of (X,Y) lying within a sub-region of the triangle is the ratio of the area of that sub-region to the area of the triangle ($= 0.5$). For $(x,y) \in \Delta$, the part of the region $\{X \le x, Y \le y\}$ that lies inside Δ is a trapezium, with sides x and $(x-y)$, and width y. Therefore

$$F_{X,Y}(x,y) = y(2x-y), \quad (x,y) \in \Delta. \tag{12}$$

   If (x,y) is in the region $x < y$, $0 < x \le 1$, then

$$F_{X,Y}(x,y) = x^2 \tag{13}$$

   because the intersection of $\{X \le x, Y \le y\}$ with Δ is now an isosceles right-angled triangle with base and height x.
   If $0 < y \le 1$, $x > 1$, then

$$F_{X,Y}(x,y) = y(2-y) \tag{14}$$

   because now the region of interest is a trapezium with width y and side lengths 1 and $(1-y)$. Finally,

$$F_{X,Y}(x,y) = \begin{cases} 0 & x < 0 \text{ or } y < 0 \\ y(2-y) & 0 \le y < 1, x > 1 \\ x^2 & x < y, 0 < x \le 1 \\ y(2x-y) & (x,y) \in \Delta \\ 1 & x > 1, y > 1 \end{cases} \tag{15}$$

(b) Find the marginal CDFs of X and Y.
Ans: The marginal CDFs are

$$F_X(x) = F_{X,Y}(x,\infty) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{16}$$

$$F_Y(y) = F_{X,Y}(\infty,y) = \begin{cases} 0 & y < 0 \\ y(2-y) & 0 \le y \le 1 \\ 1 & y > 1 \end{cases} \tag{17}$$

(c) Find the probabilities of the following events using the joint CDF: $A = \{X \le 0.5, Y \le 0.75\}$, $B = \{0.25 < X \le 0.75, 0.25 < Y \le 0.75\}$.
Ans: It should be clear that

$$P[A] = F_{X,Y}(0.5, 0.75) = 0.5^2 = 0.25.$$

B is a product-form event, and its probability is found using the formula for such events:

$$\begin{aligned} P[B] &= F(0.75,0.75) - F(0.25,0.75) - F(0.75,0.25) + F(0.25,0.25) \\ &= \frac{9}{16} - \frac{1}{16} - \frac{5}{16} + \frac{1}{16} \qquad (18) \\ &= \frac{1}{4}. \qquad (19) \end{aligned}$$

We can verify that these values are correct because the intersection of event A with the triangle $0 \le y \le x \le 1$ is a triangle of base and height both equal to 0.5, and therefore $P[A] = 0.5^3/0.5 = 0.25$. The intersection of B with the triangle is also a right-angled triangle with base and height both equal to 0.5, and hence $P[B] = P[A] = 0.25$.

6. Random variables X and Y have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y/2}) & x \ge 0, y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

(a) What is $P[1 < X \le 2, Y \le 3]$?
Ans: If we sketch the x-y plane and the event of interest, it should be clear that

$$P[1 < X \le 2, Y \le 3] = F_{X,Y}(2,3) - F_{X,Y}(1,3) = (1-e^{-1.5})(e^{-1} - e^{-2}).$$

(b) Find the marginal CDFs $F_X(x)$ and $F_Y(y)$.
Ans: The answers are immediate:

$$\begin{aligned} F_X(x) &= F_{X,Y}(x,\infty) \\ &= 1 - e^{-x}, \quad x \ge 0 \\ F_Y(y) &= F_{X,Y}(\infty, y) \\ &= 1 - e^{-y/2}, \quad y \ge 0. \end{aligned}$$

(c) Are the events $\{X \leq x\}$ and $\{Y \leq y\}$ independent for all x and y?

Ans: We need to check whether $P[X \leq x, Y \leq y] = P[X \leq x]P[Y \leq y]$. The LHS is $F_{X,Y}(x,y)$, whose expression is given in the original problem. The RHS is obtained from part (b). Clearly, the LHS = RHS for all (x,y) pairs, and hence the events $\{X \leq x\}$ and $\{Y \leq y\}$ are independent for all x and y.

7. Can the following function be the joint CDF of random variables X and Y? Explain your answer.

$$F(x,y) = \begin{cases} 1 - e^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Ans: It can be verified that (i) $F(x,y)$ is non-decreasing in the north-east direction; (ii) $F(x,-\infty) = F(-\infty,y) = 0$ for all x and y; (iii) $F(x,y)$ is continuous everywhere; and (iv) $F(\infty,\infty) = 1$.

However, $F(x,\infty) = 1$ for $x \geq 0$, and $F(x,\infty) = 0$ for $x < 0$. Similarly, $F(\infty,y) = u(y)$, the unit step. If $F(x,y)$ were the joint CDF of X and Y, it would mean that X and Y are both equal to 0 with probability 1 from their marginal CDFs i.e. $P[X = 0, Y = 0] = 1$. But if $P[X = 0, Y = 0] = 1$, the joint CDF would be $F_{X,Y}(x,y) = 1$ for $x \geq 0, y \geq 0$, which is not equal to $F(x,y)$.

Therefore, the marginal CDFs computed from $F(x,y)$ implies a joint CDF that is not $F(x,y)$, and $F(x,y)$ cannot be a joint CDF.

Joint PDF

1. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = k(x+y), \quad 0 \leq x \leq 1, 0 \leq y \leq 1.$$

(a) Find k.

Ans: The total volume under the joint PDF equals one, therefore

$$\begin{aligned} \int_0^1 \int_0^1 k(x+y)dxdy &= 1 && (20) \\ \Rightarrow \quad k\int_0^1 \left[\frac{x^2}{2} + yx\right]_{x=0}^1 dy &= 1 && (21) \\ \Rightarrow \quad k\int_0^1 \frac{1}{2} + y\, dy &= 1 && (22) \\ \Rightarrow \quad k\left[\frac{y}{2} + \frac{y^2}{2}\right]_0^1 &= 1 && (23) \\ \Rightarrow \quad k &= 1. && (24) \end{aligned}$$

(b) Find the joint CDF of X and Y.

Ans: For $0 \le x \le 1$, $0 \le y \le 1$, we have

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_0^x \int_0^y x' + y' \, dy' dx' && (25)\\
&= \int_0^x \left[x'y' + \frac{y'^2}{2} \right]_{y'=0}^{y} dx' && (26)\\
&= \int_0^x x'y + \frac{y^2}{2} dx' && (27)\\
&= \left[y\frac{x'^2}{2} + \frac{y^2}{2}x' \right]_0^x && (28)\\
&= \frac{1}{2}(x^2y + y^2x) = \frac{1}{2}xy(x+y). && (29)
\end{aligned}$$

For $0 \le x \le 1$, $y > 1$, we have

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_0^1 \int_0^x x' + y' dx' dy' && (30)\\
&= \int_0^1 \left. \frac{x'^2}{2} + y'x' \right|_{x'=0}^{x} dy' && (31)\\
&= \int_0^1 \frac{x^2}{2} + xy' dy' && (32)\\
&= \left. \frac{x^2}{2}y' + x\frac{y'^2}{2} \right|_0^1 && (33)\\
&= \frac{1}{2}(x^2 + x). && (34)
\end{aligned}$$

For $x > 1$, $0 \le y \le 1$, we have

$$\begin{aligned}
F_{X,Y}(x,y) &= \int_0^1 \int_0^y x' + y' dy' dx' && (35)\\
&= \frac{1}{2}(y^2 + y) && (36)
\end{aligned}$$

by mirroring the steps taken in the last region considered. Finally, $F_{X,Y}(x,y) = 0$ when $x < 0$ or $y < 0$, and $F_{X,Y}(x,y) = 1$ when $x > 1$ and $y > 1$.

(c) Find the marginal PDF of X and of Y.

Ans: We can obtain the marginal CDFs as

$$\begin{aligned}
F_X(x) &= F_{X,Y}(x,\infty) = \begin{cases} 0 & x < 0 \\ \frac{1}{2}(x^2 + x) & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \qquad \text{and}\\
F_Y(y) &= \begin{cases} 0 & y < 0 \\ \frac{1}{2}(y^2 + y) & 0 \le y \le 1 \\ 1 & y > 1 \end{cases}
\end{aligned}$$

The marginal PDFs are obtained by differentiating the above CDFs:

$$f_X(x) = x + \frac{1}{2}, \quad 0 \le x \le 1 \tag{37}$$

$$f_Y(y) = y + \frac{1}{2}, \quad 0 \le y \le 1. \tag{38}$$

(d) Find $P[X < Y]$ and $P[Y < X^2]$.
Ans: The required probabilities can be computed through integration of the joint PDF over the region of interest.

$$\begin{aligned}
P[X < Y] &= \int_0^1 \int_0^y x + y \, dx \, dy && (39)\\
&= \int_0^1 \frac{x^2}{2} + xy \Big|_0^y dy && (40)\\
&= \int_0^1 \frac{3y^2}{2} dy && (41)\\
&= \frac{y^3}{2} \Big|_0^1 && (42)\\
&= \frac{1}{2}; && (43)
\end{aligned}$$

$$\begin{aligned}
P[Y < X^2] &= \int_0^1 \int_{\sqrt{y}}^1 x + y \, dx \, dy && (44)\\
&= \int_0^1 \frac{x^2}{2} + xy \Big|_{\sqrt{y}}^1 dy && (45)\\
&= \int_0^1 \frac{1}{2} + y - \frac{y}{2} - y^{3/2} dy && (46)\\
&= \frac{y}{2} + \frac{y^2}{4} - \frac{2y^{5/2}}{5} \Big|_0^1 && (47)\\
&= \frac{7}{20}. && (48)
\end{aligned}$$

2. Let X and Y have the joint PDF

$$f_{X,Y}(x, y) = ye^{-y(1+x)}, \quad x > 0, y > 0.$$

(a) Find the marginal PDF of X and of Y.

Ans: The marginal PDF of X is

$$f_X(x) = \int_0^\infty ye^{-y(1+x)}dy \quad (49)$$

$$= \frac{1}{1+x}\int_0^\infty e^{-y(1+x)}dy \quad (50)$$

$$= \frac{1}{(1+x)^2}, \quad x > 0, \quad (51)$$

where some steps related to integration by parts have been omitted.

The marginal PDF of Y is

$$f_Y(y) = \int_0^\infty ye^{-y(1+x)}dx \quad (52)$$

$$= ye^{-y}\int_0^\infty e^{-yx}dx \quad (53)$$

$$= e^{-y}, \quad y > 0. \quad (54)$$

(b) Find $P[\min(X,Y) \leq 1]$.

Ans: Note that $\min(X,Y) = X$ if $X < Y$, and $\min(X,Y) = Y$ if $Y > X$. The region of the x-y plane representing $\min(X,Y) < 1$ is therefore

$$\{X < 1, X < Y\} \cup \{Y < 1, X > Y\} = \{X < 1\} \cup \{Y < 1\}.$$

The probability of the complement of this event is easier to compute, so let's do that:

$$P[\min(X,Y) \geq 1] = \int_1^\infty \int_1^\infty ye^{-y(1+x)}dxdy \quad (55)$$

$$= \int_1^\infty e^{-2y}dy \quad (56)$$

$$= \frac{1}{2}e^{-2} = 0.0677. \quad (57)$$

Therefore $P[\min(X,Y) < 1] = 1 - 0.0677 = 0.932$.

3. A dart is equally likely to land at any point (X_1, X_2) inside a circular target of unit radius. Let R and Θ be the radius and angle of the point (X_1, X_2).

(a) Find $P[r < R \leq r + dr, \theta < \Theta \leq \theta + d\theta]$ for $dr \to 0$ and $d\theta \to 0$, in terms of $f_{R,\Theta}(r,\theta)$, the joint PDF of R and Θ.

Ans: By definition of the joint PDF, we have

$$P[r < R \leq r + dr, \theta < \Theta \leq \theta + d\theta] = f_{R,\Theta}(r,\theta)drd\theta.$$

(b) Hence find $f_{R,\Theta}(r,\theta)$.
Ans: The event $\{r < R \le r + dr, \theta < \Theta \le \theta + d\theta\}$ is equivalent to (X_1, X_2) lying in a small rectangle with sides of length $rd\theta$ and dr, in the neighbourhood of $(r\cos\theta, r\sin\theta)$. Therefore, we have

$$f_{R,\Theta}(r,\theta)drd\theta = f_{X_1,X_2}(r\cos\theta, r\sin\theta)rdrd\theta.$$

For all values of $r \in [0,1]$ and $\theta \in [0, 2\pi)$, $f_{X_1,X_2}(r\cos\theta, r\sin\theta) = \frac{1}{\pi}$ due to the uniform distribution of X_1 and X_2 inside the unit circle. For all other values of r, $f_{X_1,X_2}(r\cos\theta, r\sin\theta) = 0$. (No values of θ outside of $[0, 2\pi)$ are allowed, because we define θ to lie only in that range.)
Therefore, we have finally

$$f_{R,\Theta}(r,\theta) = \frac{r}{\pi}, \quad 0 \le r \le 1, 0 \le \theta < 2\pi.$$

(c) What is the event $X_1^2 + X_2^2 < r^2$ equivalent to in terms of R and Θ? Find $P[X_1^2 + X_2^2 < r^2]$ for $0 < r < 1$.
Ans: By definition, $R^2 = X_1^2 + X_2^2$, therefore

$$\{X_1^2 + X_2^2 < r^2\} \equiv \{R < r\}.$$

Hence, (you need a sketch of $f_{R,\Theta}(r,\theta)$ to see this)

$$\begin{aligned} P[X_1^2 + X_2^2 < r^2] &= P[R < r] = \int_0^{2\pi}\int_0^r \frac{\rho}{\pi} d\rho \\ &= \text{Volume of a prism with height } 2\pi \text{ and base area } \tfrac{r^2}{2\pi} \\ &= r^2. \end{aligned}$$

4. The input X to a communication channel is $+1$ or -1 with probability p and $1-p$ respectively. The received signal $Y = X + N$, where N is an $\mathcal{N}(0,1)$ random variable, independent from X.

(a) Find $P[X = j, Y \le y]$ for $j = -1, +1$.
Ans: Recall that $P[A \cap B] = P[B|A]P[A]$ for any two events A and B, as long as A has non-zero probability. Therefore,

$$\begin{aligned} P[X = 1, Y \le y] &= P[Y \le y | X = 1]P[X = 1] &(58)\\ &= F_Y(y|X = 1)p &(59) \end{aligned}$$

where $F_Y(y|X = 1)$ is the conditional CDF of Y given $\{X = 1\}$. But conditioned on $\{X = 1\}$, we have $Y = 1 + N$ and so $F_Y(y|X = 1) = P[N + 1 \le y] = P[N \le y - 1]$, and since N is a standard normal r.v., we have

$$P[X = 1, Y \le y] = [1 - Q(y - 1)]p = Q(1 - y)p. \quad (60)$$

Similarly,

$$P[X = -1, Y \le y] = [1 - Q(y + 1)](1 - p) = (1 - p)Q(-y - 1). \quad (61)$$

(b) Find the marginal PMF of X and the marginal PDF of Y.

Ans: The marginal PMF of X is already given in the question, i.e. $p_X(1) = p$, $p_X(-1) = 1-p$. The marginal PDF of Y can be obtained through its marginal CDF, defined as

$$\begin{aligned} F_Y(y) &= P[Y \leq y] \\ &= P[Y \leq y, X = -1] + P[Y \leq y, X = 1] \\ &= (1-p)Q(-y-1) + pQ(1-y). \end{aligned} \quad (62)$$

Note that $Q(y) = 1 - \Phi(y)$, where $\Phi(y)$ is the CDF of the $\mathcal{N}(0,1)$ r.v. The derivative of $\Phi(y)$ is the $\mathcal{N}(0,1)$ PDF, $f_Z(y)$; therefore the derivative of $Q(y)$ is the negative of the $\mathcal{N}(0,1)$ PDF, or

$$\frac{d}{dy}Q(y) = -f_Z(y) = -\frac{1}{\sqrt{2\pi}}e^{-y^2/2}.$$

Using this, and differentiating the CDF obtained above,

$$f_Y(y) = (1-p)f_Z(-1-y) + pf_Z(1-y) \quad (63)$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-p)e^{-(y+1)^2/2} + pe^{-(y-1)^2/2}\right]. \quad (64)$$

Therefore $f_Y(y)$ is the weighted sum of two shifted standard normal PDFs, one centered on $y = -1$ and the other on $y = +1$.

(c) Find $P[X = j|Y > 0]$, $j = -1, +1$.

Ans: From (62) we have

$$\begin{aligned} P[Y > 0] &= 1 - F_Y(0) \\ &= 1 - (1-p)Q(-1) + pQ(1) \\ &= p + (1-2p)Q(1). \end{aligned}$$

From part (a), $P[X = 1, Y \leq 0] = pQ(1)$. But since

$$P[X = 1, Y \leq 0] + P[X = 1, Y > 0] = P[X = 1] = p,$$

we have that

$$P[X = 1, Y > 0] = p - pQ(1) = p(1 - Q(1)).$$

Finally,

$$P[X = 1|Y > 0] = \frac{p(1-Q(1))}{p + (1-2p)Q(1)}.$$

Using the above expression in $P[X = -1|Y > 0] = 1 - P[X = 1|Y > 0]$ and simplifying yields

$$P[X = -1|Y > 0] = \frac{(1-p)Q(1)}{p + (1-2p)Q(1)}$$

Remark: In the usual case where $p = 0.5$, we have

$$P[X = 1|Y > 0] = 1 - Q(1) > Q(1) = P[X = -1|Y > 0].$$

In other words, if Y is observed to be positive, it is more likely that $X = 1$ than $X = -1$.