NATIONAL UNIVERSITY OF SINGAPORE
Department of Electrical & Computer Engineering

Mid-Term Test for
(Semester II, 2013/14)

EE2012 ANALYTICAL TECHNIQUES FOR ECE

March 8, 2014
Time Allowed: 1.5 hours

INSTRUCTIONS FOR CANDIDATES:

- This is a CLOSED BOOK exam.
- This paper contains three (3) questions, printed on three (3) pages, including this page.
- Answer ALL questions.
- Write your student number, name and tutorial group on the front cover of the answer book.
- A non-programmable calculator may be used.

Q1. We have 3 cards: Both sides of Card 1 are red, both sides of Card 2 are black, and one side of Card 3 is red and the other side is black. The three cards are mixed up in a hat, and one is drawn randomly and placed flat on the table.

(a) Find the probability that the side facing up is red. (2 marks)

Ans: Let R denote "side facing up is red" and C_i denote "Card i is picked". Then,

$$\begin{aligned} P[R] &= \sum_{i=1}^{3} P[R|C_i]P[C_i] && (1)\\ &= \frac{1}{3}[1+0+0.5] && (2)\\ &= \frac{1}{2}. && (3) \end{aligned}$$

(b) Find the probability that Card i was drawn, $i = 1, 2, 3$, given that the side facing up is red. Which is the most likely card drawn if the side facing up is red? (3 marks)

Ans: We have by definition

$$P[C_i|R] = \frac{P[R|C_i]P[C_i]}{P[R]}. \quad (4)$$

Since we know $P[R|C_1] = 1$, $P[R|C_2] = 0$, and $P[R|C_3] = 0.5$, it is trivial to compute

$$P[C_i|R] = \begin{cases} \frac{2}{3} & i = 1 \\ 0 & i = 2 \\ \frac{1}{3} & i = 3 \end{cases} \quad (5)$$

The most likely card to have been picked given a red side is facing up is therefore Card 1.

(c) Suppose a sequence of experiments is performed, in which at each successive sub-experiment, one more card is added to the hat and then a card is drawn randomly from the hat and placed on the table. The new card can have both sides red, both sides black, or one side red and the other black, with equal probability. What is the probability that at the third sub-experiment, the side facing up is red? (The first sub-experiment is the one described at the top of the question.) (3 marks)

Ans: In the third sub-experiment we will have five cards. The fourth and fifth cards can be Type 1, 2 or 3. We still have

$$P[R] = \sum_{i=1}^{5} P[R|C_i]P[C_i] = \frac{1}{5}\sum_{i=1}^{5} P[R|C_i]$$

but now the difficulty is in finding $P[R|C_4]$ and $P[R|C_5]$. We tackle the problem again using total probability:

$$P[R|C_4] = \sum_{k=1}^{3} P[R, T_k|C_4] \tag{6}$$

where T_k is the event "Card is of Type k". But

$$P[R, T_k|C_4] = P[R|T_k, C_4]P[T_k|C_4] = \begin{cases} \frac{1}{3} & k = 1 \\ 0 & k = 2 \\ \frac{1}{6} & k = 3 \end{cases} \tag{7}$$

Therefore, $P[R|C_4] = 0.5$. Similarly, $P[R|C_5] = 0.5$. Finally,

$$P[R] = \frac{1}{5}[1 + 0 + 0.5 + 0.5 + 0.5] = 0.5.$$

Q2. The radius R (in cm) of a circle is a random variable with PMF

$$p_R(r) = \binom{4}{r-1}\frac{1}{16}, \quad r = 1, 2, 3, 4, 5.$$

(a) Sketch the PMF of R. (2 marks)

Ans: We can compute

$$p_R(1) = \frac{1}{16},\ p_R(2) = \frac{1}{4},\ p_R(3) = \frac{3}{8},\ p_R(4) = \frac{1}{4},\ p_R(5) = \frac{1}{16}$$

and then plot those values.

(b) Find the probability that the area of the circle is larger than or equal to 5π cm^2. (2 marks)

Ans: The area of the circle is $A = \pi R^2$, therefore

$$\{A \geq 5\pi\} \equiv \{R \geq \sqrt{5}\},$$

and hence,

$$P[A \geq 5\pi] = P[R \in \{3, 4, 5\}] = \frac{11}{16}. \tag{8}$$

(c) Find the expected value of the area of the circle. (2 marks)

Ans: We have

$$\begin{aligned} E[A] &= \pi E[R^2] \\ &= \pi\left(\frac{1}{16} + \frac{4}{4} + \frac{27}{8} + \frac{16}{4} + \frac{25}{16}\right) \\ &= 10\pi \text{ cm}^2. \end{aligned} \tag{9}$$

(d) Find the conditional PMF of R, given that $R \in \{3,4,5\}$. (3 marks)

Ans: As found in part (a), $P[R \in \{3,4,5\}] = 11/16$. We have that ($C$ denotes the event $R \in \{3,4,5\}$)

$$\begin{aligned} p_R(r|C) &= \begin{cases} \frac{p_R(r)}{P[C]} & r \in C \\ 0 & \text{otherwise} \end{cases} \\ &= \frac{16}{11} \begin{cases} \frac{3}{8} & r = 3 \\ \frac{1}{4} & r = 4 \\ \frac{1}{16} & r = 5 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{6}{11} & r = 3 \\ \frac{4}{11} & r = 4 \\ \frac{1}{11} & r = 5 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{10}$$

Q3. The number of raindrops falling in a bucket in one minute is a random variable N, with mean 100 in heavy rain (event A), 10 in moderate rain (event B), and 1 in a light shower (event C). If none of the three events occurs, then there is no rain at all. Assume the following probabilities, based on historical averages:

$$P[(A \cup B \cup C)^c] = 0.5 \qquad P[C] = 0.2$$
$$P[B] = 0.2 \qquad P[A] = 0.1,$$

and that conditioned on A, B, C or $(A \cup B \cup C)^c$ individually, N is Poisson.

(a) Write down the conditional PMFs of N, given A, B, C and $(A \cup B \cup C)^c$, respectively. (2 marks)

Ans: From the information given, we have for $n = 0, 1, 2, \ldots$

$$p_N(n|A) = \frac{100^n}{n!} e^{-100}, \tag{11}$$
$$p_N(n|B) = \frac{10^n}{n!} e^{-10}, \tag{12}$$
$$p_N(n|C) = \frac{1}{n!} e^{-1}, \tag{13}$$

and $p_N(n|(A \cup B \cup C)^c) = \delta[n]$ where $\delta[n]$ is the Kronecker delta function

$$\delta[n] = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0, n \in \mathbb{Z}. \end{cases}$$

(b) Find the probability mass function of N. (2 marks)

Ans: The PMF of N is obtained using the total probability theorem as

$$p_N(n) = 0.1e^{-100}\frac{100^n}{n!} + 0.2e^{-10}\frac{10^n}{n!} + 0.2e^{-1}\frac{1}{n!} + 0.5\delta[n] \tag{14}$$

for $n = 0, 1, 2, \ldots$

(c) Find the expected value of N. (2 marks)

Ans: We have

$$E[N] = 100(0.1) + 10(0.2) + 1(0.2) + 0 = 12.2.$$

(d) Are the three events A, B and C independent? Explain your answer briefly. (2 marks)

Ans: No they are not, because A, B and C are mutually exclusive with non-zero probabilities so for instance $P[A|B] = 0 \neq P[A]$.