Conditional Probability

1. A die is tossed twice and the number of dots on the top face noted in the order of occurrence. Let $A =$ " first toss $\geq$ second toss", and $B =$ "first toss is a 6". Find $P[A|B]$ and $P[B|A]$.

2. A number x is selected at random in the interval $[-2, 2]$. Let the events

$$\begin{aligned} A &= \{x : x < 0\}; & (1) \\ B &= \{x : |x - 0.5| < 0.5\} & (2) \\ C &= \{x : x > 0.75\}. & (3) \end{aligned}$$

   Find $P[A|B]$, $P[B|C]$ and $P[A|C^c]$.

3. Let the lifetime of a product satisfy the probability law

$$P[\text{"lifetime exceeds } t \text{ years"}] = e^{-t}, \quad t \geq 0.$$

   Let A be the event "lifetime exceeds t years" and B the event "lifetime exceeds $2t$ years". Find $P[B|A]$.

4. Show that $P[A|B]$ satisfies the Axioms of Probability, i.e.

   - $P[A|B] \geq 0$;
   - $P[S|B] = 1$;
   - If $A \cap C = \emptyset$, then $P[A \cup C|B] = P[A|B] + P[C|B]$.

5. One of two coins is selected with equal probability, and tossed three times. The first coin comes up heads with probability $p_1 = 1/3$, and the second comes up heads with probability $p_2 = 2/3$.

   (a) Find the probability that the number of heads is k.

   (b) Find the probability that Coin 1 was tossed, given that k heads were observed.

   (c) In part (b), which coin is more probable when k heads have been observed? In other words, for each value of $k \in \{0, 1, 2, 3\}$, compare the values of $P[\text{"Coin 1"}|\text{"}k \text{ heads"}]$ and $P[\text{"Coin 2"}|\text{"}k \text{ heads"}]$.

   (d) Suppose now the selected coin is tossed 5 times, and we observe that there are 3 heads. We have to make an educated guess as to which coin was selected. What should our decision be?

Independence and Independent Bernoulli Trials

1. Consider three events A, B and C, each with non-zero probability p_A, p_B and p_C respectively. Given that they are pairwise independent, and that $P[A \cap B \cap C] = p_A p_B p_C$, show that A and $B \cup C$ must be independent.

2. An experiment consists of picking one of two urns at random, and then selecting a ball from the urn and noting its colour (black or white). Let A be the event "urn 1 is selected", and B the event "a black ball is picked". Under what conditions are A and B independent?

3. A random experiment is repeated a large number of times and the occurrence of events A and B is noted. How would you empirically test for the independence of A and B?

4. 10 percent of items from a production line are defective. What is the probability that there are more than one defective item in a batch of n items?

5. We need 10 chips of a certain type to build a circuit. It is known that 5 percent of these chips are defective. How many chips should we buy for there to be a greater than 90 percent chance of having enough chips for the circuit?

6. A communication link is noisy and the probability of a message failing to be delivered within T seconds to the destination is p. If a message is not delivered after T seconds, it will be re-transmitted. Find the maximum allowable value of p so that the probability of the transmission delay exceeding $3T$ seconds is smaller than 10^{-4}.