EE2012 2014/15 Problem Set 8

Functions of a Random Variable

1. Let X be a geometric random variable with $E[X] = 1/p$. Find the PMF of the following functions of X:

   (a) $Y = X^2$;
   Ans: Since $S_X = \{1, 2, 3, \ldots\}$, $S_Y = \{1, 4, 9, 16, \ldots\} = \{k^2 : k = 1, 2, \ldots\}$. The PMF of Y is

   $$p_Y(k^2) = p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

   Alternatively, we can say that

   $$p_Y(y) = p(1-p)^{\sqrt{y}-1}, \quad y \in S_Y.$$

   (b) $Z = \ln(X)$;
   Ans: Now we have $Z \in \{\ln k : k = 1, 2, \ldots\}$, and the PMF of Z is

   $$p_Z(\ln k) = p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, 3, \ldots$$

   Alternatively, we can say that

   $$p_Z(z) = p(1-p)^{e^z-1}, \quad z \in \{\ln k : k = 1, 2, \ldots\}.$$

   (c) $V = e^X$.
   Ans: The range of V is $S_V = \{e^k : k = 1, 2, \ldots\}$, and its PMF is

   $$p_V(e^k) = p_X(k) = p(1-p)^{k-1}, \quad k = 1, 2, \ldots$$

   We can also write

   $$p_V(v) = p(1-p)^{\ln v - 1}, \quad v \in S_V.$$

2. The number X is drawn at random from the unit interval $[0, 1]$, and Y is defined as X rounded to the nearest tenth, i.e. if $X = 0.12$ then $Y = 0.1$, if $X = 0.87$ then $Y = 0.9$, etc. Find the PMF of Y.

   Ans: The range of Y is $S_Y = \{0, 0.1, 0.2, \ldots, 0.9, 1.0\}$. The event $\{Y = 0\}$ is equivalent to $\{0 \leq X < 0.05\}$ and therefore $p_Y(0) = P[0 \leq X < 0.05] = 0.05$. Similarly, we have $\{Y = 0.1\}$ being equivalent to $\{0.05 \leq X < 0.15\}$ and hence $p_Y(0.1) = 0.1$. Proceeding this way we can see that the PMF of Y is

   $$p_Y(y) = \begin{cases} 0.05, & y = 0, 1.0 \\ 0.1, & y = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 \end{cases} \quad (1)$$

3. A wire has length X, an exponential random variable with mean 5π cm. The wire is cut to make rings of diameter 1 cm. Let Y be the number of complete rings made from the wire. Find the PMF of Y.

   Ans: Each ring requires π cm of wire. Therefore, with X cm of wire, we can make $Y = \lfloor X/\pi \rfloor$ complete rings, where $\lfloor x \rfloor$ = the largest whole number smaller than x, e.g. $\lfloor 4.5 \rfloor = 4$, $\lfloor 2.9 \rfloor = 2$, etc.

   The event $\{Y = 0\}$ is equivalent to $\{X < \pi\}$, while $\{Y = 1\}$ is equivalent to $\{\pi \le X < 2\pi\}$. Generalizing we have

$$\{Y = k\} \equiv \{k\pi \le X < (k+1)\pi\},$$

   and therefore the PMF of Y is

$$p_Y(k) = P[k\pi \le X < (k+1)\pi].$$

   Using the CDF of the exponential random variable with parameter $\lambda = 1/5\pi$, we have finally

$$p_Y(k) = e^{-k/5} - e^{-(k+1)/5}, \quad k = 0, 1, 2, \ldots$$

4. Let X be uniformly distributed in $(0, 1]$. Sketch the function $g(x)$ in the following cases, and then find the PDF of $Y = g(X)$.

   (a) $g(x) = x^2$;
   Ans: $Y = g(X)$ has the range $(0, 1]$. Using the direct PDF method, and noting that $g'(x) = 2x$,

$$\begin{aligned} f_Y(y) &= \frac{f_X(\sqrt{y})}{2\sqrt{y}} && (2) \\ &= \frac{1}{2\sqrt{y}}, \quad 0 < y \le 1. && (3) \end{aligned}$$

   (b) $g(x) = e^{-x}$;
   Ans: Now, let's try the CDF route.

$$\begin{aligned} F_Y(y) &= P[e^{-X} \le y] && (4) \\ &= P[X \ge -\ln y] && (5) \\ &= \begin{cases} 0 & y < e^{-1} \\ 1 + \ln y, & e^{-1} \le y < 1 \\ 1 & y \ge 1 \end{cases} && (6) \end{aligned}$$

   Therefore the PDF of Y is

$$f_Y(y) = \frac{1}{y}, \quad e^{-1} \le y < 1.$$

(c) $g(x) = \cos 2\pi x$.

Ans: For any $y \in [-1,1]$, $g(x) = y$, $x \in (0,1]$, has two solutions: $x_0 = \frac{1}{2\pi}\cos^{-1} y$ and $x_1 = 1 - \frac{1}{2\pi}\cos^{-1} y$. We find that the derivative of $g(x)$ is $g'(x) = -2\pi \sin 2\pi x$ and thus

$$g'(x_0) = -2\pi\sqrt{1-y^2}, \quad g'(x_1) = 2\pi\sqrt{1-y^2}.$$

Using the formula $f_Y(y) = \sum_{i=1}^{N} f_X(x_i)/|g'(x_i)|$, we have

$$\begin{aligned} f_Y(y) &= \frac{1}{2\pi\sqrt{1-y^2}} + \frac{1}{2\pi\sqrt{1-y^2}} &(7)\\ &= \frac{1}{\pi\sqrt{1-y^2}}, \quad -1 \le y \le 1. &(8) \end{aligned}$$

5. Let $X \sim \mathcal{N}(2,4)$, and

$$Y = (X)^+ = \begin{cases} X & X > 0 \\ 0 & X \le 0 \end{cases}$$

Find the CDF and hence the PDF of Y.

Ans: We have

$$P[Y=0] = P[X<0] = 1 - P[X \ge 0]. \tag{9}$$

Recall that for any Gaussian r.v. X, $P[X > x] = Q((x-\mu)/\sigma)$, where $Q(x) = P[\mathcal{N}(0,1) > x]$. Therefore

$$P[Y=0] = 1 - Q\left(\frac{-2}{2}\right) = 1 - Q(-1) = Q(1). \tag{10}$$

Since $P[Y=0] = Q(1) \neq 0$, there must be a step increase in $F_Y(y)$ at $y=0$, with $F_Y(0) = Q(1)$ and $F_Y(y) = 0$ for $y < 0$.

The CDF of Y, for $y > 0$, would be (from a sketch of $y = (x)^+$)

$$\begin{aligned} F_Y(y) &= P[Y \le y] = P[X \le y] &(11)\\ &= 1 - P[X > y] = 1 - Q\left(\frac{y-2}{2}\right) &(12) \end{aligned}$$

Thus, we have the entire CDF of Y.

The PDF of Y is obtained by differentiating $F_Y(y)$. Note that

$$Q(x) = 1 - \Phi(x)$$

where $\Phi(x)$ is the CDF of the $\mathcal{N}(0,1)$ random variable, and therefore

$$\frac{dQ(x)}{dx} = -\frac{d\Phi(x)}{dx} = -\frac{1}{\sqrt{2\pi}}e^{-x^2/2}.$$

For $y > 0$, we have

$$\begin{aligned} f_Y(y) &= \frac{dF_Y(y)}{dy} \\ &= -\frac{1}{2}\left[-\frac{1}{\sqrt{2\pi}}e^{-(y-2)^2/8}\right] \\ &= \frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{(y-2)^2}{8}\right). \end{aligned}$$

There is also a delta function at $y = 0$ of strength $Q(1)$ to account for $P[Y = 0] = Q(1)$. In total,

$$f_Y(y) = Q(1)\delta(y) + \frac{1}{2\sqrt{2\pi}}\exp\left(-\frac{(y-2)^2}{8}\right)u(y).$$

Note: The formula for finding the PDF of $g(X)$ directly cannot be used here, because $g(x)$ is flat, i.e. $g'(x) = 0$, over a range of values.

6. Let X be a Rayleigh random variable with PDF

$$f_X(x) = xe^{-x^2/2}, \quad x > 0.$$

Find the PDF of $Z = X^2$ by (i) first finding the CDF, and (ii) finding the PDF directly.

Ans: (i) In the first method, we find the CDF and then the PDF through differentiation. From first principles,

$$\begin{aligned} F_Z(z) &= P[Z \leq z] = P[X^2 \leq z] && (13) \\ &= P[X \leq \sqrt{z}] && (14) \\ &= \int_0^{\sqrt{z}} xe^{-x^2/2}dx, \quad z > 0. && (15) \end{aligned}$$

Using the substitution $u = \frac{x^2}{2}$, we have

$$F_Z(z) = \int_0^{z/2} e^{-u}du = 1 - e^{-z/2}, \quad z > 0. \tag{16}$$

Therefore,

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \frac{1}{2}e^{-z/2}, \quad z > 0. \tag{17}$$

(ii) We can find the PDF directly from

$$f_Z(z) = \sum_{i=1}^{N} \frac{f_X(x_i)}{|g'(x_i)|}$$

by first identifying that $g(x) = z$ has only one solution, i.e. $x = \sqrt{z}$ within the range of X (i.e. $(0, \infty)$). Therefore $N = 1$ in the above formula. Also $g'(x) = 2x$, and hence $g'(\sqrt{z}) = 2\sqrt{z}$. Finally, we obtain

$$f_Z(z) = \frac{\sqrt{z}e^{-z/2}}{2\sqrt{z}} = \frac{1}{2}e^{-z/2}, \quad z > 0$$

as in method (i).

7. A random variable Y is said to be log-normally distributed if $X = \ln Y$ is an $\mathcal{N}(\mu, \sigma^2)$ random variable. Find the PDF of Y, and hence show that $E[Y] = \exp\left[\mu + \frac{\sigma^2}{2}\right]$.

   Ans: Essentially we want to find the PDF of $Y = e^X$, when $X \sim \mathcal{N}(\mu, \sigma^2)$. Using the direct method, we solve for x in $e^x = y$, i.e. $x = \ln y$; then find the derivative of e^x at $x = \ln y$, i.e. $g'(\ln y) = y$. Substituting these into the formula gives us

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \frac{e^{-(\ln y - \mu)^2/2\sigma^2}}{y}, \quad y > 0. \tag{18}$$

   To find the mean of Y, we can use the definition of $E[Y]$:

$$\begin{aligned} E[Y] &= \int_0^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(\ln y - \mu)^2/2\sigma^2} dy & (19) \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty e^{-(v-\mu)^2/2\sigma^2} e^v dv & (20) \end{aligned}$$

   where we substituted $v = \ln y$ to obtain the second line. This integral is dealt with by "completing the squares" in the exponent, i.e.

$$\begin{aligned} \frac{(v-\mu)^2}{2\sigma^2} - v &= \frac{1}{2\sigma^2}\left[(v-\mu)^2 - 2\sigma^2 v\right] & (21) \\ &= \frac{1}{2\sigma^2}\left[v^2 - 2(\mu + \sigma^2)v + \mu^2\right] & (22) \\ &= \frac{1}{2\sigma^2}\left[(v - (\mu + \sigma^2))^2 - (\mu + \sigma^2)^2 + \mu^2\right] & (23) \\ &= \frac{1}{2\sigma^2}\left[(v - (\mu + \sigma^2))^2\right] - \mu - \frac{\sigma^2}{2}. & (24) \end{aligned}$$

   Substitution of this final expression into (20) yields

$$\begin{aligned} E[Y] &= e^{\mu + \sigma^2/2} \cdot \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^\infty e^{-(v-(\mu+\sigma^2))^2/2\sigma^2} dv & (25) \\ &= e^{\mu + \sigma^2/2} & (26) \end{aligned}$$

   because the integral on the RHS above is the area under the $\mathcal{N}(\mu + \sigma^2, \sigma^2)$ PDF, which is 1 by definition. (Completing the squares in the exponent is an often-used tool that helps to solve certain problems involving Gaussian densities.)

8. The input to a full-wave rectifier is X and its output is $Y = |X|$. Find the PDF of Y if $X \sim \mathcal{N}(0, \sigma^2)$.

   Ans: Let $Y = g(X) = |X|$, then $|g'(x)| = 1$ for all $x \neq 0$. Furthermore, $g(x) = y$, $y > 0$, has two solutions $x_1 = -y$ and $x_2 = +y$. Therefore, using the direct method for finding the PDF of Y, we get

$$f_Y(y) = \frac{2}{\sqrt{2\pi\sigma^2}} e^{-y^2/2\sigma^2}, \quad y > 0. \tag{27}$$