EE2012 2014/15 Problem Set 11

Conditioning on a Random Variable

1. Let X be a continuous uniform random variable in $[-1, 1]$, and suppose the conditional PDF of Y given X is

$$f_{Y|X}(y|x) = |x|e^{-|x|y}, \quad y > 0.$$

   (a) Find $P[Y > X]$ by first finding $P[Y > X|X = x]$ and then using the theorem on total probability.

   (b) Find the covariance of X and Y, by using iterated expectations to obtain $E[XY]$.

   (c) Are X and Y independent? Are they uncorrelated?

2. A customer enters a store and is equally likely to be served by one of three clerks. The time taken by clerk 1 is a constant two minutes; the time taken by clerk 2 is exponentially distributed with mean two minutes; and the time for clerk 3 is Pareto distributed[1] with mean two minutes and $\alpha = 2.5$.

   (a) Find the PDF of T, the time taken to serve a customer.

   (b) Find $E[T]$ and $\text{var}(T)$.

3. Suppose X and Y have the joint PDF

$$f_{X,Y}(x, y) = e^{-(x+|y|)}, \quad x > 0, -x < y < x.$$

   (a) Find $f_{Y|X}(y|x)$.

   (b) Find $P[Y > 1|X = x]$.

   (c) Find $P[Y > 1]$ using the result of part (b).

   (d) Find $E[e^{|Y|}|X]$ and hence $E[e^{|Y|}]$.

Conditional Distribution and Expectation

1. Suppose that

$$f_{Y|X}(y|x) = K(x + y), \quad 0 \le x \le 1,\ 0 \le y \le 1.$$

   (a) Find K in terms of x.

   (b) Find $E[Y|X = x]$.

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[1] A Pareto PDF is

$$f(x) = \alpha \frac{x_m^\alpha}{x^{\alpha+1}}, \quad x \ge x_m$$

where $E[X] = \alpha x_m/(\alpha - 1)$.

(c) If $f_X(x) = x + \frac{1}{2}$, $0 \le x \le 1$, use the law of iterated expectations to find $E[Y]$.

2. Let (X, Y) be jointly uniform within the two quarter-discs defined by $\{(x, y) : x^2 + y^2 < 1, xy > 0\}$.

   (a) Find the marginal PDF of X.

   (b) Hence, find the conditional PDF of Y given X. What sort of distribution is this?

   (c) Find $E[Y|X]$, and hence $E[Y]$.

3. The number of defects on a chip with unit area is a Poisson random variable N with rate R. However R is itself a Gamma random variable with parameters α and λ, i.e.

$$f_R(r) = \frac{\lambda(\lambda r)^{\alpha-1} e^{-\lambda r}}{\Gamma(\alpha)}, \quad r > 0,$$

   where $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx.$$

   (a) Use conditional expectation to find $E[N]$ and $\mathrm{var}(N)$.

   (b) Find the PMF of N.

Functions of Two Random Variables

1. Let X and Y be independent Gaussian random variables with zero mean and unit variance. Show that $Z = aX + bY$ is Gaussian.

2. If X and Y are independent unit-mean exponential random variables, show that $Z = |X - Y|$ is also exponential, and find $E[Z]$.

3. The number of goals X that Singapore scores against Selangor is a geometric RV with mean 2; the number of goals Y that Selangor scores against Singapore is a geometric RV with mean 4. X and Y are assumed to be independent.

   (a) Find the PMF of $Z = X - Y$.

   (b) Find the probability of Singapore beating Selangor, and the probability of the two teams drawing (tying) a game.

4. Let X and Y be independent $\mathcal{N}(0, 1)$ random variables. Show that $Z = X/Y$ is a Cauchy random variable.