EE2012 2014/15 Problem Set 2

# Probability Laws

1. Pick a real value in the range $[0, 1]$. Let

   - $A = [0, 0.5]$;
   - $B = (0.4, 0.8]$;
   - $C = (0.6, 1.0]$.

   (a) If the distribution function is

   $$F_1(x) = \begin{cases} 0 & x < 0 \\ x^2 & 0 \le x < 1 \\ 1 & \text{otherwise} \end{cases}$$

   find $P[A \cup B]$, $P[B \cap C^c]$ and $P[A^c \cup B \cup C]$.
   *Ans*: Since $A \cup B = [0, 0.8]$, we have $P[A \cup B] = F_1(0.8) - F_1(0) = 0.64$. (Note that with a continuous distribution function, $P[\{a\}] = 0$ for any $a$.) Proceeding similarly, we have

   $$\begin{aligned} P[B \cap C^c] &= P[(0.4, 0.6]] = 0.36 - 0.16 \\ &= 0.2 \\ P[A^c \cup B \cup C] &= P[(0.4, 1]] = 1 - 0.16 \\ &= 0.84. \end{aligned}$$

   (b) Repeat part (a) if the distribution function is

   $$F_2(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x < 1 \\ 1 & \text{otherwise} \end{cases}.$$

   *Ans*: Using the fact that $P[(a, b]] = F(b) - F(a)$, we have

   $$\begin{aligned} P[A \cup B] &= 0.8 \\ P[B \cap C^c] &= 0.2 \\ P[A^c \cup B \cup C] &= 0.6. \end{aligned}$$

   (c) Describe in words the random numbers generated using the two distribution functions above.
   *Ans*: Consider two ranges of equal width, $(a_1, b_1]$ and $(a_2, b_2]$, where $b_1 - a_1 = b_2 - a_2$, and all values are within $[0, 1]$. Suppose that $a_2 > a_1$. Under the first distribution, $P[(a_1, b_1]] = b_1^2 - a_1^2$ and $P[(a_2, b_2]] = b_2^2 - a_2^2$. We can also write

   $$\begin{aligned} P[(a_1, b_1]] &= (b_1 - a_1)(b_1 + a_1) & (1) \\ P[(a_2, b_2]] &= (b_2 - a_2)(b_2 + a_2), & (2) \end{aligned}$$

and since $b_1 - a_1 = b_2 - a_2$, and $b_1 + a_1 < b_2 + a_2$, we have $P[(a_1, b_1]] < P[(a_2, b_2]]$ under the $F_1$ distribution.

Under $F_2$, we have $P[(a_1, b_1]] = b_1 - a_1 = b_2 - a_2 = P[(a_2, b_2]]$ instead.

Therefore the random numbers generated by $F_1$ are more likely to be large (closer to 1), whereas those generated by $F_2$ are evenly spaced in the range $[0, 1]$.

2. Consider the sample space $S = \{0, 1, 2, \ldots, 9\}$. Suppose the probability of the outcome $k$ is half of that of the outcome $k-1$, for $k = 1, 2, \ldots, 9$, with $P[\{0\}] = p_0$. Find $p_0$.

   *Ans*: We need to have $\sum_{k=0}^{9} P[\{k\}] = 1$. Since $P[\{k\}] = 0.5P[\{k-1\}]$ and $P[\{0\}] = p_0$, then by recursion, we obtain

   $$P[\{k\}] = 0.5^k p_0, \quad k = 0, 1, \ldots, 9.$$

   Therefore, $\sum_{k=0}^{9} P[\{k\}] = 1 \Rightarrow$

   $$\begin{aligned} \sum_{k=0}^{9} 0.5^k p_0 &= 1 &(3)\\ \Rightarrow \quad \frac{p_0(1-0.5^{10})}{1-0.5} &= 1 &(4)\\ \Rightarrow \quad p_0 &= 0.500488. &(5) \end{aligned}$$

3. A fair coin is flipped five times. Find the probability of obtaining $k$ heads, where $k \in \{0, 1, 2, 3, 4, 5\}$.

   *Ans*: The sample space comprises all 32 5-tuples:

   $$S = \{(HHHHH), (HHHHT), \ldots, (TTTTT)\}.$$

   All outcomes are equi-probable due to the coin being fair. The event "3 heads" is equivalent to the set

   $$\{(HHHTT), (HHTHT), (HTHHT), \ldots, (TTHHH)\}.$$

   There are a total of $\binom{5}{3} = 10$ elements in the above set, corresponding to the total number of ways to arrange 3 objects of one type and 2 objects of another. The probability of obtaining 3 heads is therefore $10/32 = 5/16$.

   Using the same reasoning, we see that

   $$P[\text{“}k \text{ heads”}] = \frac{\binom{5}{k}}{32}.$$

4. Roll two dice and sum the number of dots on the two top faces. Find the probability of obtaining 12.

   *Ans*: Consider the outcome to be the pair $(x_1, x_2)$ where $x_1$ is the number of dots on the top face of the first die, and $x_2$ is the number on the second die. Then $S = \{(x_1, x_2) : x_1, x_2 \in \{1, 2, \ldots, 6\}\}$. Assuming the two dice are fair, then we can reasonably assume that the 36 outcomes in $S$ are equi-probable.

   The event of interest is $A = \{(x_1, x_2) : x_1 + x_2 = 12\}$. It contains only one outcome, $(6, 6)$. Therefore, $P[A] = 1/36$.

## Counting Methods

1. A multiple choice test has 10 questions with 5 choices each. How many ways are there to answer the test? What is the probability that two papers have the same answer if students choose their answers strictly at random?

   *Ans*: Each question can be answered in 5 ways, say $\{a, b, c, d, e\}$. Imagine drawing 10 balls with replacement and with ordering from an urn with 5 balls labeled with $a, b, c, d$ and $e$ respectively, and mapping the $i$-th ball to the $i$-th question. There are a total of $5^{10}$ ways to draw the balls, and hence the same number of ways to answer the ten questions.

   Each unique sequence of 10 answers in a paper has the same probability $5^{-10}$ of appearing. The probability of the second paper having the same answers as the first is computed by letting the first paper contain any of the $5^{10}$ possible 10-tuples, and then finding the probability of the second paper being the same 10-tuple as the first. This would be $5^{-10}$, a very low probability. Hence the probability of two papers being identical, with many wrong answers, is virtually zero unless the two students were copying each other.

2. A student has five different T-shirts and three pairs of shorts.

   (a) How many days can the student dress without repeating the combination of shorts and T-shirt?

   *Ans*: There are $5 \times 3 = 15$ unique pairings of shorts and T-shirt, therefore he can go up to 15 days without wardrobe repetition.

   (b) How many days can the student dress without repeating the combination of shorts and T-shirt, and without wearing the same T-shirt on two consecutive days?

   *Ans*: Let T-shirts be labeled 1 through 5, and let shorts be labeled A, B and C. Then the 15 unique outfits are given by (1,A), (1,B), (1,C), (2,A), etc. By wearing his five shirts in the sequence 1, 2, 3, 4, 5, and then repeating the sequence another two times, he will not wear the same shirt on two consecutive days. Thus he can still go for up to 15 days without repeating his outfit, and without wearing the same shirt on two consecutive days.

3. A classroom has 60 seats. In how many ways can 45 students occupy the seats in the room?

   *Ans*: This is the same problem as picking 45 times without replacement and with ordering from an urn with 60 balls. The $i$-th ball indicates the seat taken by the $i$-th student. The number of ways to seat the 45 students is thus

$$^{60}P_{45} = \frac{60!}{15!} = 6.363 \times 10^{69}.$$

4. Five balls are placed at random in five buckets. What is the probability that each bucket has a ball?

   *Ans*: Imagine picking five times with replacement and with ordering from an urn of five balls. The $i$-th pick indicates which bucket the $i$-th ball goes into. Every one of the $5^5$ outcomes are equally likely. The number of ways to have one ball per bucket is the number of outcomes with the balls 1, 2, 3, 4 and 5, i.e. 5!. Therefore

$$P[\text{“one ball per bucket”}] = \frac{5!}{5^5} = 0.0384.$$

5. A dinner party is attended by four men and four women. How many unique ways can the eight people sit around the table? If a man must sit between two women, how many unique seating arrangements are there now?

   *Ans*: There are $8! = 40320$ ways to seat the guests, if the positions are numbered, i.e. a circular rotation is counted as a different configuration. If we count all circular rotations only once, then there are $8!/8 = 7! = 5,040$ ways to seat the guests.

   For the second question, we first assume the numbered-chairs scenario, and then count the number of arrangements with a man in seat 1. In this case, we must have the eight seats occupied as MWMWMWMW, where M stands for “man” and W for “woman”. There are $4! = 24$ ways to fill the men's seats, and another 24 ways to fill the women's seats. Therefore, with a man in seat 1, we have $24 \times 24 = 576$ ways to seat the guests. Similarly, with a woman in seat 1, we have another 576 seating arrangements. In total, if a man must sit between two women, we have $576 \times 2 = 1152$ arrangements if positions are numbered, and $1152/8 = 144$ if they are not.

   A more complicated problem arises if a man can sit beside one or two women, rather than between two women. Again, first assume that seats are numbered. Then the following configurations count:

$$\text{(MW)(WM)(MW)(MW)}, \text{(WM)(WM)(WM)(MW)}, \text{(WM)(MW)(MW)(WM)}, \ldots$$

   In other words, as long as each man is coupled with a woman, the condition is met, and vice versa. Treating the four men as indistinguishable from each other, and also the four women as indistinguishable, there are a total of $2^4 = 16$ configurations. But since the four men and four women are distinguishable, i.e. we have $M_1$ through $M_4$ and $W_1$ through $W_4$, there are in fact $16 \times 4! \times 4! = 9,216$ configurations. Finally, if we count circular rotations only once, then there are $9,216/8 = 1,152$ configurations.

6. Find the probability that in a class of 28 students, exactly four were born in each of the seven days of the week.

   *Ans*: Imagine an urn with four balls labeled Monday, four labeled Tuesday, and so on. We pick all 28 balls without replacement, with ordering. The $i$-th pick is the birthday of the $i$-th person. The total number of ways to arrange 7 types of objects, with 4 of each type, is given by the multi-nomial coefficient:

$$\frac{28!}{(4!)^7} = 6.648 \times 10^{19}.$$

   This is the total number of ways in which we can have 28 people with four born on Monday, 4 on Tuesday, etc.

   But the total number of 28-tuples, with elements drawn from the set {Monday, Tuesday, ..., Sunday}, is $7^{28}$. Assuming each of these is equally likely, then

$$P[\text{“4 people born on each day of the week”}] = \frac{6.648 \times 10^{19}}{7^{28}} = 0.0001445.$$