EE2012 2014/15 Problem Set 1

Set Theory

1. For each of the following sets, write down two subsets:
   (a) $\mathbb{R}$, the set of real numbers.
   Ans: $(0,1)$, $\{3,4,\sqrt{2}\}$, etc.
   (b) $\mathbb{C}$, the set of complex numbers.
   Ans: $\mathbb{R}$, $j\mathbb{R}$, etc.
   (c) $\mathbb{Z}$, the set of integers.
   Ans: Natural numbers, negative integers, prime integers, etc.
   (d) $S = [0,2)$, the set of all real numbers from 0 to 2, including 0, excluding 2.
   Ans: $\{0\}$, $(1,2)$, etc.
   (e) $T = \{$Black, White, Grey$\}$
   Ans: Any two of the eight sets in the power set of T.
   (f) $U = \{x : x \in \mathbb{R}, x \geq 2\}$
   Ans: $\{5,6\}$, $[7,9)$, etc.

2. Identify the sets in problem 1 that are countable. Find the cardinality of each of the countable sets.

   Ans: $\mathbb{Z}$ and T are the only countable sets. $|\mathbb{Z}| = \infty$, $|T| = 3$.

3. Are the following statements true or false? Why?
   (a) $\{0\} = \emptyset$
   Ans: False. $\emptyset = \{\} \neq \{0\}$.
   (b) $\{2\} = 2$
   Ans: False. A set cannot be equated to a value.
   (c) $[2,3) \cap \{1,2,3,4\} = \{2,3\}$
   Ans: False. $[2,3)$ does not include 3.
   (d) $\emptyset \cup A = A$ for any set A.
   Ans: True. $\emptyset \cup A$ is the set containing all elements of A and all elements of $\emptyset$ (without double counting). Since $\emptyset$ contains nothing, the result follows.
   (e) $x \in A \Rightarrow -x \in A^c$.
   Ans: False. A^c contains all the elements of S that are not in A, so all we can say about it given the information we have is that $x \notin A^c$.
   (f) $\{a,b,c\} \cup \{b,c,d\} = \{a,b,c,d\}$.
   Ans: True.

(g) $A \cup A^c = S$, the universal set.

Ans: True. This arises from the definition of the complement of a set.

(h) $x \in A \;\Leftrightarrow\; \{x\} \subset A$.

Ans: True. A subset of A contains one or more members (elements) of A. Since x is an element of A, then $\{x\}$ is a subset of A by this definition.

4. Using a Venn diagram, prove de Morgan's rules:

$$(A \cup B)^c = A^c \cap B^c \quad (1)$$
$$(A \cap B)^c = A^c \cup B^c. \quad (2)$$

Ans: Draw two Venn diagrams, one with mutually exclusive A and B, the other with intersecting A and B. Shade the set that is equivalent to the LHS, and the one corresponding to the RHS, and verify that they are equal whether or not A and B are mutually exclusive.

5. On a Venn diagram, illustrate the distributive property $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$. (This is not a rigorous proof, which would require the consideration of all possible relationships between A, B and C with regards to whether they are mutually exclusive or not.)

Ans: This is quite straightforward.

Probability Space

1. For each of the following experiments, write down the sample space and state if it is countable or uncountable.

(a) Measure the time it takes to travel from Clementi to Jurong East stations on the MRT.

Ans: Suppose it takes 5 minutes with the train moving at maximum speed and no other delays, and that even with the worst conceivable delay, it will take no more than 30 minutes. Then $S = [5, 30]$ minutes.

(b) Count the number of vehicles that pass under the overhead bridge across Clementi Road between 5.00pm and 5.05pm.

Ans: It is possible for there to be any number of cars from zero to some large value, say 500. Then $S = \{0, 1, 2, \ldots, 500\}$.

(c) Take a chest X-ray of someone picked randomly, and note if it is normal.

Ans: In this case the outcome is only whether the X-ray is normal or not. Therefore, $S = \{\text{normal, abnormal}\}$.

(d) Throw a dart at a dartboard and note the (x, y) coordinates of where it lands.

Ans: Suppose the dartboard is circular, with a radius of R cm, and assume that the dart always lands within the dartboard. Let x and y be measured in cm, with the centre of the dartboard as the origin. Then $S = \{(x, y) : \sqrt{x^2 + y^2} < R\}$.

2. For each of the sample spaces in the previous question, write down one possible event, both in words and as a subset of S.

   Ans:

   (a) $A =$ "it takes more than 6 minutes". $A = (6, 30]$.
   (b) $B =$ "the number of vehicles that pass under the bridge is smaller than 10". $B = \{0, 1, \ldots, 9\}$.
   (c) $C =$ "the X-ray is normal". $C = \{\text{normal}\}$.
   (d) $D =$ "the dart lands within 1 cm of the centre". $D = \{(x, y) : \sqrt{x^2 + y^2} < 1\}$.

3. If $P(A) = 0.6$, show using the Axioms of Probability that $P(A^c)$ must be 0.4, i.e. we are not free to assign values to both $P(A)$ and $P(A^c)$.

   Ans: Since $A \cup A^c = S$, we have from Axiom II that $P[A \cup A^c] = P[S] = 1$. But given that A and A^c are mutually exclusive by definition, then Axiom III dictates that $P[A \cup A^c] = P[A] + P[A^c]$. Therefore, we must have $P[A^c] = 1 - P[A] = 0.4$.

4. Roll a die and note the number showing up on top. The sample space is $S = \{1, 2, 3, 4, 5, 6\}$. Write down two valid probability assignments (i.e. they satisfy the axioms of probability) for the elementary events of S. Which of the two (if any) are useful for modeling the rolling of a fair die? Can you see that a model need not be useful even if it is mathematically valid?

   Ans: Any probability assignment that satisfies $\sum_{i=1}^{6} P[\{i\}] = 1$ and $P[\{i\}] \geq 0$ for all i will do. This is because in this case:

   - Axiom I ($P[A] \geq 0$) is satisfied for any event that is the union of the elementary events $\{i\}$, $i = 1, 2, \ldots, 6$.
   - Axiom II is satisfied since $P[S] = \sum_{i=1}^{6} P[\{i\}] = 1$. The first equality follows from assuming that Axiom III is satisfied.
   - Axiom III is satisfied by assumption e.g. we define $P[\{2, 4\}] = P[\{2\}] + P[\{4\}]$.

   Therefore, two valid probability assignments can be the following:

   Assignment 1 $P[\{i\}] = 1/6$, $i = 1, 2, \ldots, 6$.

   Assignment 2 $P[\{1\}] = P[\{2\}] = P[\{3\}] = 1/4$, $P[\{4\}] = P[\{5\}] = P[\{6\}] = 1/12$.

   If the die is fair, then Assignment 1 is a good model. Assignment 2, while valid from the axiomatic perspective, is useless for this purpose.

5. Let $S = \{a, b, c, d\}$ be the sample space of some experiment, and let $A = \{a, b, c\}$, $B = \{b, c, d\}$ and $C = \{a, d\}$ be three events (i.e. they are each in the event field $\mathcal{F}$).

(a) Write down the sets representing the composite events "A and B occur", "neither A nor C occurs", and "B occurs but not C".

Ans: "A and B occur" $= A \cap B = \{b, c\}$. "Neither A nor C occurs" $= A^c \cap C^c = \{\}$. "B occurs but not C" $= B \cap C^c = \{b, c\}$.

(b) Let the probabilities of the elementary events be p_a, p_b, p_c and p_d respectively. Find the probabilities of the events of part (a) in terms of these probabilities.

Ans: $P[A \cap B] = p_b + p_c$. $P[A^c \cap C^c] = 0$. $P[B \cap C^c] = p_b + p_c$.

6. Consider the sample space $S = \{a, b, c, d, e, f\}$.

(a) If we are only interested in whether event $A = \{a, d, f\}$ occurs, write down the event field.

Ans: $\mathcal{F}_1 = \{\emptyset, S, A, A^c\}$.

(b) If we are interested in all subsets of S, find the total number of events in $\mathcal{F}$. Explain your answer.

Ans: This event field $\mathcal{F}_2$ will contain $2^6 = 64$ elements. The reason is as follows. Imagine flipping a coin six times. Every time the k-th flip is a Heads, we include the k-th element of S in the subset (event). For instance, if the coin flip sequence is $(HHTTHT)$, then the subset is $\{a, b, e\}$. Each unique coin-flip sequence results in a unique subset. Therefore the total number of subsets of a set is the number of unique coin-flip sequences, which is 2^6 when there are six elements in the set.

(c) If we know the probabilities of all elementary events, i.e. $p_a = P[\{a\}]$, $p_b = P[\{b\}]$, etc. are all known, what are the probability mappings (P_1 and P_2) in parts (a) and (b)?

Ans: In the first instance, $P_1(A) = p_a + p_d + p_f$, $P_1(A^c) = 1 - P_1(A)$, $P_1(\emptyset) = 0$, $P_1(S) = 1$.

In the second case, $P_2(B) = \sum_{k \in B} p_k$ for all $B \in \mathcal{F}_2$.