EE2012 2014/15 Problem Set 9

Joint PMF and CDF

1. Flip a fair coin four times. Let X be the number of Heads obtained, and let Y be the position of the first Heads i.e. if the sequence of coin flips is TTHT, then $Y = 3$, if it is THHH, then $Y = 2$. If there are no heads in the four tosses, then we define $Y = 0$.

   (a) Find the joint PMF of X and Y.

   (b) Using the joint PMF, find the marginal PMF of X.

   (c) Find the joint CDF $F_{X,Y}(x, y)$ in the region

   $$\{(x, y) : 1 \le x < 2, 2 \le y < 3\}.$$

2. The random variable X is Poisson with mean 1. Conditioned on $X = k$, Y is binomial with $n = k$ and $p = 0.1$.

   (a) Find the joint PMF of X and Y.

   (b) Find the marginal PMF of Y.

3. Suppose the marginal PMFs of X and Y are identical:

   $$p_X(k) = p_Y(k) = \frac{1}{3}, \quad k = -1, 0, 1.$$

   (a) Show that the joint PMF of X and Y must be zero except possibly at the nine points in $\{(j, k) : j, k \in \{-1, 0, 1\}\}$.

   (b) Show that the two marginal PMFs do not uniquely determine the joint PMF of X and Y.

   (c) Suppose $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for all $x, y \in \{-1, 0, 1\}$. Find the probabilities of $\{X > Y\}$, $\{X = Y\}$ and $\{Y \le 0\}$.

4. Let X be a discrete random variable uniformly distributed in $\{1, 2, 3, 4\}$. Given $X = x$, Y is uniformly distributed in $\{1, \dots, x\}$. Draw a tree diagram of the experiment and find the joint PMF of X and Y.

5. A point (X, Y) is selected at random inside a triangle defined by $\{(x, y) : 0 \le y \le x \le 1\}$. Assume that the point is equally likely to fall anywhere inside the triangle.

   (a) Find the joint CDF of X and Y.

   (b) Find the marginal CDFs of X and Y.

   (c) Find the probabilities of the following events using the joint CDF: $A = \{X \le 0.5, Y \le 0.75\}$, $B = \{0.25 < X \le 0.75, 0.25 < Y \le 0.75\}$.

6. Random variables X and Y have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-x})(1-e^{-y/2}) & x \geq 0, y \geq 0 \\ 0 & \text{elsewhere} \end{cases}$$

   (a) What is $P[1 < X \leq 2, Y \leq 3]$?
   (b) Find the marginal CDFs $F_X(x)$ and $F_Y(y)$.
   (c) Are the events $\{X \leq x\}$ and $\{Y \leq y\}$ independent for all x and y?

7. Can the following function be the joint CDF of random variables X and Y? Explain your answer.

$$F(x,y) = \begin{cases} 1-e^{-(x+y)} & x \geq 0, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Joint PDF

1. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = k(x+y), \quad 0 \leq x \leq 1, 0 \leq y \leq 1.$$

   (a) Find k.
   (b) Find the joint CDF of X and Y.
   (c) Find the marginal PDF of X and of Y.
   (d) Find $P[X < Y]$ and $P[Y < X^2]$.

2. Let X and Y have the joint PDF

$$f_{X,Y}(x,y) = ye^{-y(1+x)}, \quad x > 0, y > 0.$$

   (a) Find the marginal PDF of X and of Y.
   (b) Find $P[\min(X,Y) \leq 1]$.

3. A dart is equally likely to land at any point (X_1, X_2) inside a circular target of unit radius. Let R and Θ be the radius and angle of the point (X_1, X_2).

   (a) Find $P[r < R \leq r+dr, \theta < \Theta \leq \theta + d\theta]$ for $dr \to 0$ and $d\theta \to 0$, in terms of $f_{R,\Theta}(r,\theta)$, the joint PDF of R and Θ.
   (b) Hence find $f_{R,\Theta}(r,\theta)$.
   (c) What is the event $X_1^2 + X_2^2 < r^2$ equivalent to in terms of R and Θ? Find $P[X_1^2 + X_2^2 < r^2]$ for $0 < r < 1$.

4. The input X to a communication channel is $+1$ or -1 with probability p and $1-p$ respectively. The received signal $Y = X + N$, where N is an $\mathcal{N}(0,1)$ random variable, independent from X.

   (a) Find $P[X = j, Y \leq y]$ for $j = -1, +1$.
   (b) Find the marginal PMF of X and the marginal PDF of Y.
   (c) Find $P[X = j | Y > 0]$, $j = -1, +1$.