EE2012 2014/15 Quiz 2

April 2, 2015

Instructions

- Write your student number, your name as it appears in IVLE, and your tutorial group at the top of your answer sheet.
- You have 30 minutes. Answer all questions. All parts have the same weight.
- No books, notes or other written or printed material are allowed.
- The only electronic device you can use is a non-programmable calculator, that functions only as a calculator.
- All communicating devices must be turned off prior to starting the quiz.

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1. We know that

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

if $Y = aX + b$, $a \neq 0$.

(a) Show that if $X \sim \mathcal{N}(\mu, \sigma^2)$, then Y is Gaussian.

Ans: We know that

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

Substitution into the given formula yields the PDF of Y as

$$\begin{aligned} f_Y(y) &= \frac{1}{|a|}\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{((y-b)/a-\mu)^2}{2\sigma^2}\right) &(1)\\ &= \frac{1}{\sqrt{2\pi a^2\sigma^2}} \exp\left(-\frac{(y-b-a\mu)^2}{2a^2\sigma^2}\right) &(2)\end{aligned}$$

This expression has the form of a Gaussian PDF, and therefore Y is Gaussian.

(b) Find the mean and variance of Y.

Ans: Comparing the PDF above with the general form of the Gaussian PDF, it is obvious that $E[Y] = b + a\mu$ and $\mathrm{var}(Y) = a^2\sigma^2$.

(c) Find $P[0 < Y \leq 2]$ in terms of the Q function and μ, σ, a and b.

Ans:

$$\begin{aligned} P[0 < Y \leq 2] &= P[Y \geq 0] - P[Y > 2] \\ &= P\left[Z \geq \frac{0-\mu_Y}{\sigma_Y}\right] - P\left[Z > \frac{2-\mu_Y}{\sigma_Y}\right] \\ &= Q\left(-\frac{b+a\mu}{a\sigma}\right) - Q\left(\frac{2-b-a\mu}{a\sigma}\right) \end{aligned}$$

where $Z \sim \mathcal{N}(0,1)$.

2. Let $X \sim \mathcal{N}(0,1)$, and $Y = X^2$. Find the PDF of Y using the formula

$$f_Y(y) = \sum_{i=1}^{n} \frac{f_X(x_i)}{|g'(x_i)|}$$

where $Y = g(X)$, and $g(x_i) = y$ for all $i = 1, \ldots, n$.

Ans: For any $y > 0$, $g(x) = y$ has two solutions, $x_1 = -\sqrt{y}$ and $x_2 = \sqrt{y}$. Also, $g'(x) = 2x$. Substitution into the formula above gives us

$$\begin{aligned} f_Y(y) &= \frac{f_X(-\sqrt{y})}{2\sqrt{y}} + \frac{f_X(\sqrt{y})}{2\sqrt{y}} \\ &= \frac{1}{\sqrt{y}} f_X(\sqrt{y}) \\ &= \frac{1}{\sqrt{2\pi y}} e^{-y/2} \end{aligned}$$

where the second line arises from the symmetry of the $\mathcal{N}(0,1)$ PDF, and the third line comes from $f_X(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$.