EE2012 2014/15 PROBLEM SET 10

Independence of Two Random Variables

1. Let X be the quotient and Y the remainder when the number of dots observed in a toss of a fair die is divided by 3. Are X and Y independent?

   Ans: A counter-example will show that X and Y are not independent. We can compute the marginal probabilities

$$P[X=0] = P[\zeta \in \{1,2\}] = \frac{1}{3} \tag{1}$$

$$P[Y=0] = P[\zeta \in \{3,6\}] = \frac{1}{3}. \tag{2}$$

   But $P[X=0, Y=0] = 0$ because the event $\{X=0, Y=0\}$ is equivalent to rolling a zero which is not possible. Therefore $P[X=0, Y=0] \neq P[X=0]P[Y=0]$, and X and Y are not independent.

2. Michael takes the 7:30 bus every morning. The arrival time of the bus at the stop is uniformly distributed in the interval [7:27,7:37]. Michael's arrival time at the stop is uniformly distributed in [7:25,7:40]. Assume that Michael's and the bus's arrival times are independent random variables.

   (a) What is the probability that Michael arrives more than 5 minutes before the bus?

   Ans: Let the time origin be set at 7.25 am. Then the arrival time X of the bus is uniform in $[2, 12]$, and Michael's arrival time Y is uniform in $[0, 15]$. Since X and Y are independent, we have

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} \frac{1}{150} & 2 \leq x \leq 12, 0 \leq y \leq 15 \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

   "Michael arrives more than 5 minutes before the bus" $= \{X - Y > 5\}$. The intersection of the sample space of (X, Y) with $\{X - Y > 5\}$ is the triangle formed by the lines

$$y = x - 5, \quad x = 12, \quad y = 0. \tag{4}$$

   Since the joint PDF is a constant within this region, we have

$$P[X - Y > 5] = \text{area of triangle} \times \frac{1}{150} = \frac{49}{300}. \tag{5}$$

   (b) What is the probability that Michael misses the bus?

   Ans: "Michael misses the bus" $= \{X < Y\}$. The intersection of the sample space of (X, Y) with the region representing this event is a trapezium, with sides

$$y = x, \quad x = 2, \quad x = 12, \quad y = 15. \tag{6}$$

The area of this trapezium is

$$A = \frac{1}{2}(13+3)(10) = 80, \tag{7}$$

and so

$$P[X < Y] = \frac{80}{150} = \frac{8}{15}. \tag{8}$$

3. Let X and Y be random variables that take on values from the set $\{-1, 0, 1\}$, and suppose their marginal PMFs are respectively

$$p_X(k) = \frac{1}{3}, \qquad k = -1, 0, 1 \tag{9}$$

$$p_Y(-1) = 0.5, \quad p_Y(0) = 0.2, \quad p_Y(1) = 0.3. \tag{10}$$

(a) If X and Y are independent, find $P[X \geq Y]$.
Ans: Since X and Y are independent, $P[X = j, Y = k] = P[X = j]P[Y = k]$ for any $j, k \in \{-1, 0, 1\}$. The event $X \geq Y$ consists of the six pairs of (X, Y) values $(-1,-1), (0,0), (0,-1), (1,1), (1,0), (1,-1)$, which respectively have probabilities 1/6, 1/15, 1/6, 1/10, 1/15, 1/6. Therefore,

$$P[X \geq Y] = \frac{1}{6} + \frac{1}{15} + \cdots + \frac{1}{6} = \frac{11}{15}.$$

(b) Find the joint PMF of X^2 and Y^2 by considering the (X, Y) event equivalent to $\{X^2 = j, Y^2 = k\}$. Hence verify that X^2 and Y^2 are also independent random variables.
Ans: The PMF of X^2 and Y^2 can be obtained easily as follows:

$$\begin{aligned}
P[X^2 = 0, Y^2 = 0] &= P[X = 0, Y = 0] = p_X(0)p_Y(0) = \frac{1}{15} \\
P[X^2 = 0, Y^2 = 1] &= P[X = 0, Y = \pm 1] = p_X(0)[p_Y(-1) + p_Y(1)] \\
&= \frac{4}{15} \\
P[X^2 = 1, Y^2 = 0] &= P[X = \pm 1, Y = 0] = p_Y(0)[p_X(-1) + p_X(1)] \\
&= \frac{2}{15} \\
P[X^2 = 1, Y_2 = 1] &= P[X = \pm 1, Y = \pm 1] \\
&= p_X(-1)[p_Y(-1) + p_Y(1)] + p_X(1)[p_Y(-1) + p_Y(1)] \\
&= \frac{8}{15}.
\end{aligned}$$

The marginal PMF of X^2 is therefore

$$\begin{aligned}
p_{X^2}(0) &= \frac{1}{15} + \frac{4}{15} = \frac{1}{3} \\
p_{X^2}(1) &= \frac{2}{15} + \frac{8}{15} = \frac{2}{3}.
\end{aligned}$$

And the marginal PMF of Y^2 is

$$\begin{aligned} p_{Y^2}(0) &= \frac{1}{15} + \frac{2}{15} = \frac{1}{5} \\ p_{Y^2}(1) &= \frac{4}{15} + \frac{8}{15} = \frac{4}{5}. \end{aligned}$$

Finally, we can easily verify that $p_{X^2}(j)p_{Y^2}(k) = p_{X^2,Y^2}(j,k)$ for all $j,k \in \{0,1\}$, and hence X^2 and Y^2 are independent.

4. Let X and Y be independent random variables uniformly distributed in $[-1,1]$. Find the probability of the following events:

   (a) $P[X < 0.5, |Y| < 0.5]$
   Ans: Remembering that X and Y are independent, we have

   $$\begin{aligned} P[X < 0.5, |Y| < 0.5] &= P[X < 0.5]P[-0.5 < Y < 0.5] && (11) \\ &= 0.75 \times 0.5 = 0.375. && (12) \end{aligned}$$

   (b) $P[4X^2 < 1, Y < 0]$
   Ans: Similarly to part (a),

   $$\begin{aligned} P[4X^2 < 1, Y < 0] &= P[-0.5 < X < 0.5]P[Y < 0] && (13) \\ &= 0.5 \times 0.5 = 0.25. && (14) \end{aligned}$$

   (c) $P[XY < 0.5]$
   Ans: This is no longer a product-form event and so we have to use the joint PDF. The region of interest is the one inside the unit square, enclosed by the two curves of $xy = 0.5$ (in the first and third quadrants). It is easier to compute the probability of the complement of this event, i.e.

   $$\begin{aligned} P[XY \geq 0.5] &= 2\left[\int_{0.5}^{1} 1 - \frac{1}{2x}dx\right] \cdot \frac{1}{4} && (15) \\ &= 0.5\left[x - \frac{1}{2}\ln x\Big|_{0.5}^{1}\right] && (16) \\ &= \frac{1}{4}[1 - \ln 2]. && (17) \end{aligned}$$

   Therefore $P[XY < 0.5] = 1 - \frac{1}{4}(1 - \ln 2) = \frac{1}{4}(3 + \ln 2) = 0.9233$.

Expected Value of $g(X,Y)$ and Correlation

1. Show that the variance of $X + Y$ is equal to $\text{var}(X) + \text{var}(Y)$ if and only if X and Y are uncorrelated.

Ans: By the linearity of the expectation operator, we have

$$\begin{aligned} E[(X+Y)^2] &= E[X^2+Y^2+2XY] && (18)\\ &= E[X^2]+E[Y^2]+2E[XY], \quad \text{and} && (19)\\ (E[X+Y])^2 &= \mu_X^2+\mu_Y^2+2\mu_X\mu_Y. && (20) \end{aligned}$$

Therefore, the variance of $X+Y$ is

$$\begin{aligned} \mathrm{var}(X+Y) &= E[(X+Y)^2]-(E[X+Y])^2 && (21)\\ &= \mathrm{var}(X)+\mathrm{var}(Y)+2\mathrm{cov}(X,Y) && (22) \end{aligned}$$

where the covariance is defined as $\mathrm{cov}(X,Y)=E[XY]-\mu_X\mu_Y$.

Therefore the variance of $X+Y$ is equal to $\sigma_X^2+\sigma_Y^2$ if and only if $\mathrm{cov}(X,Y)=0$, i.e. X and Y are uncorrelated.

2. Let X and Y be discrete random variables with the following joint PMF.

x	y	$p(x,y)$
0	0	0.1
0	1	0.1
0	2	0.2
1	0	0.1
1	1	0.2
1	2	0.1
2	1	0.1
2	2	0.1

Find the covariance and correlation coefficient between X and Y.

Ans: To obtain the covariance, we need $E[XY]$, $E[X]$ and $E[Y]$.

$$\begin{aligned} E[XY] &= 1(0.2)+2(0.2)+4(0.1)=1\\ E[X] &= 0(0.4)+1(0.4)+2(0.2)=0.8\\ E[Y] &= 0(0.2)+1(0.4)+2(0.4)=1.2 \end{aligned}$$

Therefore, $\mathrm{cov}(X,Y)=1-0.8(1.2)=0.04$. To find the correlation coefficient, we need the variance of X and of Y:

$$\begin{aligned} \mathrm{var}(X) &= E[X^2]-E^2[X]\\ &= 1(0.4)+4(0.2)-0.8^2\\ &= 0.56\\ \mathrm{var}(Y) &= 1(0.4)+4(0.4)-1.2^2\\ &= 0.56 \end{aligned}$$

Therefore

$$\rho_{X,Y}=\frac{0.04}{0.56}=0.071.$$

3. If X and Y have the joint PDF

$$f_{X,Y}(x,y) = x + y, \quad 0 \le x \le 1, 0 \le y \le 1,$$

find the covariance and correlation coefficient of X and Y.

Ans: The correlation is

$$\begin{aligned} E[XY] &= \int_0^1 \int_0^1 xy(x+y)dxdy && (23) \\ &= \int_0^1 \int_0^1 x^2y + xy^2 dxdy && (24) \\ &= \int_0^1 \left[\frac{x^3y}{3} + \frac{x^2y^2}{2}\right]_0^1 dy && (25) \\ &= \int_0^1 \frac{y}{3} + \frac{y^2}{2} dy && (26) \\ &= \left.\frac{y^2}{6} + \frac{y^3}{6}\right|_0^1 && (27) \\ &= \frac{1}{3}. && (28) \end{aligned}$$

The marginal PDFs of X and Y were obtained earlier as

$$f_X(t) = f_Y(t) = t + \frac{1}{2}, \quad 0 \le t \le 1. \tag{29}$$

Therefore $E[X] = \int_0^1 t^2 + 0.5tdt = \frac{1}{3} + \frac{1}{4} = \frac{7}{12} = E[Y]$. Therefore,

$$\mathrm{cov}(X,Y) = \frac{1}{3} - \frac{7^2}{12^2} = -\frac{1}{144}. \tag{30}$$

To find the correlation coefficient, we need the variance of X and of Y. Since

$$\begin{aligned} E[X^2] &= \int_0^1 t^2(t+0.5)dt \\ &= \left.\frac{t^4}{4} + \frac{t^3}{6}\right|_0^1 \\ &= \frac{5}{12} = E[Y^2], \end{aligned}$$

we have $\sigma_X^2 = \sigma_Y^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$. Finally,

$$\rho_{X,Y} = \frac{-1/144}{11/144} = -\frac{1}{11}.$$

4. Suppose X and Y have the joint PDF

$$f_{X,Y}(x,y) = e^{-(x+|y|)}, \quad x > 0, -x < y < x.$$

Find the mean of $g(X,Y)$ in the following cases:

(a) $g(X,Y) = e^{0.5X}$;
Ans: We have

$$\begin{aligned} E[e^{0.5X}] &= \int_0^{\infty}\int_{-x}^{x} e^{0.5x}e^{-(x+|y|)}dydx && (31)\\ &= \int_0^{\infty}\int_{-x}^{x} e^{-0.5x-|y|}dydx. && (32)\end{aligned}$$

Using the fact that $|y| = y$ if $y > 0$ and $|y| = -y$ if $y < 0$, we split the inner integral into two parts corresponding to negative and positive values of y as follows:

$$\begin{aligned} \int_{-x}^{x} e^{-0.5x-|y|}dy &= \int_{-x}^{0} e^{-0.5x+y}dx + \int_0^{x} e^{-0.5x-y}dx && (33)\\ &= e^{-0.5x}\left[1 - e^{-x} + 1 - e^{-x}\right] && (34)\\ &= 2(e^{-0.5x} - e^{-1.5x}). && (35)\end{aligned}$$

Substituting this into the original expression gives

$$\begin{aligned} E[e^{0.5X}] &= \int_0^{\infty} 2(e^{-0.5x} - e^{-1.5x})dx && (36)\\ &= 2\left[2 - \frac{2}{3}\right] = \frac{8}{3}. && (37)\end{aligned}$$

(b) $g(X,Y) = e^{|Y|}$;
Ans: We perform similar computations to those in part (a) to obtain

$$\begin{aligned} E[e^{|Y|}] &= \int_0^{\infty}\int_{-x}^{x} e^{|y|}e^{-(x+|y|)}dydx && (38)\\ &= \int_0^{\infty}\int_{-x}^{x} e^{-x}dydx && (39)\\ &= \int_0^{\infty} 2xe^{-x}dx && (40)\\ &= 2 && (41)\end{aligned}$$

where a few steps where have been skipped.

(c) $g(X,Y) = X + Y$.

Ans: The marginal PDFs of X and of Y are:

$$\begin{aligned} f_X(x) &= \int_{-x}^{x} e^{-(x+|y|)} dy & (42) \\ &= 2e^{-x}(1-e^{-x}), \quad x > 0 & (43) \\ f_Y(y) &= \int_{|y|}^{\infty} e^{-(x+|y|)} dx & (44) \\ &= e^{-2|y|}, \quad y \in \mathbb{R}. & (45) \end{aligned}$$

Since $f_Y(y)$ is symmetric around $y = 0$, $E[Y] = 0$ and so $E[X+Y] = E[X]$. $E[X]$ can be found from integrating $2xe^{-x}(1-e^{-x})$ from $x = 0$ to ∞, which requires integration by parts. The final result is $E[X+Y] = E[X] = 0.75$.

5. The output of a channel $Y = X + N$, where the input X and the noise N are independent, zero-mean random variables.

   (a) Find the correlation coefficient between X and Y.
   Ans: Note that $E[XY] = E[X^2 + NX] = E[X^2] + E[N]E[X]$ where the last expression holds because of the independence of N and X. But $E[N] = E[X] = 0$, and therefore $E[X^2] = \sigma_X^2$, resulting in $E[XY] = \sigma_X^2$. Finally, since $E[Y] = E[X] + E[N] = 0$, we have that

   $$\mathrm{cov}(X,Y) = \sigma_X^2.$$

   Next, we need the variance of Y. Due to independence between N and X, and hence uncorrelatedness, we have $\sigma_Y^2 = \sigma_X^2 + \sigma_N^2$. Therefore, the correlation coefficient between X and Y is

   $$\rho_{X,Y} = \frac{\mathrm{cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{1}{\sqrt{1 + \sigma_N^2/\sigma_X^2}}$$

   (b) Find the value of a that minimizes the mean squared error $E[(X - aY)^2]$.
   Ans: We can simplify the MSE expression thus:

   $$\begin{aligned} E[(X-aY)^2] &= E[X^2 - 2aXY + a^2Y^2] & (46) \\ &= E[X^2] - 2aE[XY] + a^2E[Y^2] & (47) \end{aligned}$$

   but since $E[X^2] = \sigma_X^2$, $E[Y^2] = \sigma_X^2 + \sigma_N^2$ and $E[XY] = \sigma_X^2$, we have

   $$\xi(a) = E[(X-aY)^2] = \sigma_X^2 \left[(1+\rho^{-1})a^2 - 2a + 1\right]$$

   where $\rho = (\sigma_X/\sigma_N)^2$ is the SNR. Note that $\xi(a)$ is a quadratic function of a, and so its unique minimum is obtained by setting its derivative to 0. The result is

   $$a_0 = \frac{\rho}{1+\rho}.$$

(c) Express the resulting mean squared error in terms of the signal to noise ratio, $\rho = (\sigma_X/\sigma_N)^2$, where σ_X and σ_N are the standard deviations of X and N respectively.

Ans: Substituting the expression for a_0 into the one for $\xi(a)$, we find that the minimum MSE is given by

$$\xi(a_0) = \frac{\sigma_X^2}{1+\rho}.$$

In other words, the larger the SNR ρ, the smaller the MSE will be, and the better aY will be as an estimate of X.