Conditioning on a Random Variable

1. Let X be a continuous uniform random variable in $[-1, 1]$, and suppose the conditional PDF of Y given X is

$$f_{Y|X}(y|x) = |x|e^{-|x|y}, \quad y > 0.$$

   (a) Find $P[Y > X]$ by first finding $P[Y > X|X = x]$ and then using the theorem on total probability.

   Ans: Note that Y is a positive random variable, and therefore, $P[Y > X|X = x] = 1$ for any $x \leq 0$. For positive values of x, we have

$$\begin{aligned} P[Y > X|X = x] &= \int_x^\infty xe^{-xy}dy & (1) \\ &= e^{-x^2}. & (2) \end{aligned}$$

   Then, we use the continuous form of the theorem on total probability to derive

$$\begin{aligned} P[Y > X] &= \int_{-\infty}^{\infty} P[Y > X|X = x]f_X(x)dx & (3) \\ &= \int_{-1}^{0} 1 \cdot \frac{1}{2}dx + \int_0^1 \frac{1}{2}e^{-x^2}dx & (4) \\ &= \frac{1}{2}\left[1 + \int_0^1 e^{-x^2}dx\right]. & (5) \end{aligned}$$

   The integral in the last line reminds us of a Gaussian integral. We make this resemblance complete by substituting $t = \sqrt{2}x$, and obtaining

$$\begin{aligned} \int_0^1 e^{-x^2}dx &= \frac{1}{\sqrt{2}}\int_0^{\sqrt{2}} e^{-t^2/2}dt & (6) \\ &= \sqrt{\pi} \cdot \frac{1}{\sqrt{2\pi}}\int_0^{\sqrt{2}} e^{-t^2/2}dt. & (7) \end{aligned}$$

   We recognize the final expression as an integral of the $\mathcal{N}(0, 1)$ PDF, and therefore use the Q function to arrive at

$$\int_0^1 e^{-x^2}dx = \sqrt{\pi}(0.5 - Q(\sqrt{2})).$$

   Substituting this into (5) yields the final answer as

$$P[Y > X] = \frac{1}{2}\left[1 + \sqrt{\pi}\left(\frac{1}{2} - Q(\sqrt{2})\right)\right] = 0.8734.$$

(b) Find the covariance of X and Y, by using iterated expectations to obtain $E[XY]$.

Ans: By conditioning on X, we have

$$E[XY|X = x] = xE[Y|X = x] = x \cdot \frac{1}{|x|} = \mathrm{sgn}(x)$$

because the conditional PDF of Y given $X = x$ is exponential, with mean $1/|x|$.

Using the iterated expectation rule, $E[g(X,Y)] = E[E[g(X,Y)|X]]$, we have

$$E[XY] = E[\mathrm{sgn}(X)] = 0,$$

since $\mathrm{sgn}(X)$ is uniform in $\{-1, +1\}$.

(c) Are X and Y independent? Are they uncorrelated?

Ans: Since $f_{Y|X}(y|x)$ is a function of x, it must be that $f_{Y|X}(y|x) \neq f_Y(y)$, and therefore X and Y are not independent. But $\mathrm{cov}(X,Y) = 0$ because $E[XY] = 0$ and $E[X] = 0$. Thus X and Y are uncorrelated.

2. A customer enters a store and is equally likely to be served by one of three clerks. The time taken by clerk 1 is a constant two minutes; the time taken by clerk 2 is exponentially distributed with mean two minutes; and the time for clerk 3 is Pareto distributed[1] with mean two minutes and $\alpha = 2.5$.

(a) Find the PDF of T, the time taken to serve a customer.

Ans: We immediately have the following conditional PDFs of T:

$$f_T(t|\text{Clerk } 1) = \delta(t-2) \tag{8}$$

$$f_T(t|\text{Clerk } 2) = \frac{1}{2}e^{-t/2}, \quad t > 0 \tag{9}$$

For the service time under Clerk 3, we refer to Table 4.1 of the textbook (or the footnote below). The mean is given as 2 (minutes), and $\alpha = 2.5$, therefore we have

$$E[X] = \frac{2.5x_m}{1.5} = 2 \;\Rightarrow\; x_m = 1.2.$$

Thus,

$$f_T(t|\text{Clerk } 3) = \frac{3.944}{t^{3.5}}, \quad t \geq 1.2. \tag{10}$$

[1] A Pareto PDF is

$$f(x) = \alpha\frac{x_m^\alpha}{x^{\alpha+1}}, \quad x \geq x_m$$

where $E[X] = \alpha x_m/(\alpha-1)$.

The customer is equally likely to be served by any of the three clerks, thus by the Theorem on Total Probability,

$$f_T(t) = \sum_{i=1}^{3} f_T(t|\text{Clerk } i)P[\text{Clerk } i] \tag{11}$$

$$= \frac{1}{3}\left[\delta(t-2) + \frac{1}{2}e^{-t/2}u(t) + \frac{3.944}{t^{3.5}}u(t-1.2)\right]. \tag{12}$$

(b) Find $E[T]$ and $\text{var}(T)$.

Ans: Let X be the identity of the serving clerk, i.e. $X = i$ if Clerk i serves the customer. Using the iterated expectation rule,

$$E[T] = E[E[T|X]] = \frac{1}{3}[2+2+2] = 2. \tag{13}$$

Similarly,

$$E[T^2] = E[E[T^2|X]]. \tag{14}$$

Now $E[T^2|X=i] = \text{var}(T|X=i) + E^2(T|X=i)$. The following results then follow (with the help of Table 4.1 for the Pareto case):

$$E[T^2|X=1] = E^2[T|X=1] = 4 \tag{15}$$

$$E[T^2|X=2] = 4+4=8 \tag{16}$$

$$E[T^2|X=3] = \frac{2.5 \times 1.2^2}{0.5 \times 1.5^2} + 4 = 7.2. \tag{17}$$

Therefore,

$$E[T^2] = \frac{1}{3}(4+8+7.2) = 6.4, \tag{18}$$

and hence $\text{var}(T) = 6.4 - 4 = 2.4$.

3. Suppose X and Y have the joint PDF

$$f_{X,Y}(x,y) = e^{-(x+|y|)}, \quad x > 0, -x < y < x.$$

(a) Find $f_{Y|X}(y|x)$.

Ans: We had obtained (in the previous problem set) the marginal PDF of X as

$$f_X(x) = 2e^{-x}(1-e^{-x}), \quad x > 0.$$

The conditional PDF of Y given X is thus

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \tag{19}$$

$$= \frac{e^{-|y|}}{2(1-e^{-x})}, \quad x > 0, -x < y < x. \tag{20}$$

(b) Find $P[Y > 1|X = x]$.

Ans: When $x \leq 1$, it is not possible for Y to be larger than 1, since $f_{Y|X}(y|x)$ is only non-zero in the range $-x < y < x$, therefore $P[Y > 1|X = x] = 0$ for all $x \leq 1$. For $x > 1$, we have

$$P[Y > 1|X = x] = \int_1^x \frac{e^{-y}}{2(1 - e^{-x})} dy \quad (21)$$

$$= \frac{1}{2(1 - e^{-x})} \left[e^{-1} - e^{-x}\right]. \quad (22)$$

(c) Find $P[Y > 1]$ using the result of part (b).

Ans: Using the fact that $P[Y > 1] = \int_{-\infty}^{\infty} P[Y > 1|X = x] f_X(x)dx$, we have

$$P[Y > 1] = \int_1^{\infty} \frac{e^{-1} - e^{-x}}{2(1 - e^{-x})} 2e^{-x}(1 - e^{-x})dx \quad (23)$$

$$= \int_1^{\infty} e^{-1}e^{-x} - e^{-2x} dx \quad (24)$$

$$= \frac{1}{2}e^{-2}. \quad (25)$$

(d) Find $E[e^{|Y|}|X]$ and hence $E[e^{|Y|}]$.

Ans: Using the conditional PDF of Y given X, we have

$$E[e^{|Y|}|X = x] = \frac{1}{2(1 - e^{-x})} \int_{-x}^{x} e^{|y|}e^{-|y|} dy \quad (26)$$

$$= \frac{x}{1 - e^{-x}} \quad (27)$$

and hence $E[e^{|Y|}|X] = \frac{X}{1 - e^{-X}}$. Using the law of iterated expectations, we then have

$$E[e^{|Y|}] = E[E[e^{|Y|}|X]] \quad (28)$$

$$= \int_0^{\infty} \frac{x}{1 - e^{-x}} \cdot 2e^{-x}(1 - e^{-x})dx \quad (29)$$

$$= \int_0^{\infty} 2xe^{-x} dx \quad (30)$$

$$= 2 \quad (31)$$

which is the same answer as we found in problem 3 of the previous section.

Conditional Distribution and Expectation

1. Suppose that

$$f_{Y|X}(y|x) = K(x + y), \quad 0 \leq x \leq 1, \ 0 \leq y \leq 1.$$

(a) Find K in terms of x.
Ans: Remember that $f_{Y|X}(y|x)$ is a PDF in Y, and therefore $\int_{-\infty}^{\infty} f_{Y|X}(y|x)dy = 1$. Therefore,

$$K\int_0^1 x+y\ dy = 1 \tag{32}$$
$$\Rightarrow \quad K\left[xy+\frac{y^2}{2}\right]_{y=0}^1 = 1 \tag{33}$$
$$\Rightarrow \quad K(x+0.5) = 1 \tag{34}$$
$$\Rightarrow \quad K = \frac{1}{x+\frac{1}{2}}. \tag{35}$$

(b) Find $E[Y|X=x]$.
Ans: By definition,

$$E[Y|X=x] = \int_0^1 y\cdot K(x+y)dy \tag{36}$$
$$= \frac{1}{x+\frac{1}{2}}\int_0^1 xy+y^2\ dy \tag{37}$$
$$= \frac{1}{x+\frac{1}{2}}\left[\frac{xy^2}{2}+\frac{y^3}{3}\right]_0^1 \tag{38}$$
$$= \frac{1}{x+\frac{1}{2}}\left(\frac{x}{2}+\frac{1}{3}\right). \tag{39}$$

(c) If $f_X(x) = x+\frac{1}{2}$, $0\le x\le 1$, use the law of iterated expectations to find $E[Y]$.
Ans: We know that $E[Y] = E[E[Y|X]] = E[h(X)]$. Given the form of $h(x)$ found in part (b), we just have to evaluate

$$E[Y] = \int_0^1 \frac{x}{2}+\frac{1}{3}dx \tag{40}$$
$$= \left.\frac{x^2}{4}+\frac{x}{3}\right|_0^1 \tag{41}$$
$$= \frac{1}{4}+\frac{1}{3} = \frac{7}{12}. \tag{42}$$

2. Let (X,Y) be jointly uniform within the two quarter-discs defined by $\{(x,y): x^2+y^2<1, xy>0\}$.

(a) Find the marginal PDF of X.
Ans: The area of the two quarter-discs is $\pi/2$, thus the joint PDF is

$$f_{X,Y}(x,y) = \frac{2}{\pi}, \quad x^2+y^2\le 1, xy>0, \tag{43}$$

and 0 elsewhere. The marginal PDF of X is

$$f_X(x) = \begin{cases} \int_0^{\sqrt{1-x^2}} \frac{2}{\pi} dy & 0 < x \leq 1 \\ \int_{-\sqrt{1-x^2}}^{0} \frac{2}{\pi} dy & -1 < x \leq 0 \end{cases} \tag{44}$$

$$= \frac{2}{\pi}\sqrt{1-x^2}, \quad -1 < x \leq 1. \tag{45}$$

(b) Hence, find the conditional PDF of Y given X. What sort of distribution is this?

Ans: The conditional PDF of Y given X is

$$f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)} \tag{46}$$

$$= \frac{1}{\sqrt{1-x^2}}, \quad x^2 + y^2 \leq 1, xy > 0. \tag{47}$$

To express $f(y|x)$ more explicitly as a PDF in y, we note from a sketch of the region that

$$x^2 + y^2 \leq 1, xy > 0 \equiv \begin{cases} -\sqrt{1-x^2} < y < 0, & \text{if } -1 < x < 0 \\ 0 < y < \sqrt{1-x^2}, & \text{if } 0 < x < 1. \end{cases}$$

This means that, given $\{X = x\}$, Y is uniformly distributed in the range $-\sqrt{1-x^2} < y \leq 0$ if $x < 0$, and in the range $0 < y \leq \sqrt{1-x^2}$ if $x > 0$.

(c) Find $E[Y|X]$, and hence $E[Y]$.

Ans: Conditioned on $\{X = x\}$, Y is uniform as mentioned above. Therefore,

$$E[Y|X = x] = \begin{cases} -\frac{1}{2}\sqrt{1-x^2} & x < 0 \\ \frac{1}{2}\sqrt{1-x^2} & x > 0 \end{cases} \tag{48}$$

$$= \frac{\text{sgn}(x)}{2}\sqrt{1-x^2}. \tag{49}$$

By the Law of Iterated Expectations, we can compute

$$E[Y] = E[E[Y|X]] = \int_{-1}^{1} E[Y|X = x] f_X(x) dx. \tag{50}$$

But $E[Y|X = x]$ is the product of an odd function $\text{sgn}(x)/2$ and an even function $\sqrt{1-x^2}$, and is therefore odd; and $f_X(x)$ is even. Therefore, $E[Y|X = x]f_X(x)$ is the product of an odd and an even function, which makes it odd. Since the integral of any odd function between $-a$ and a is 0, we have $E[Y] = 0$.

3. The number of defects on a chip with unit area is a Poisson random variable N with rate R. However R is itself a Gamma random variable with parameters α and λ, i.e.

$$f_R(r) = \frac{\lambda(\lambda r)^{\alpha-1}e^{-\lambda r}}{\Gamma(\alpha)}, \quad r > 0,$$

where $\Gamma(\alpha)$ is the Gamma function defined as

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}e^{-x}dx.$$

(a) Use conditional expectation to find $E[N]$ and $\text{var}(N)$.

Ans: Conditioned on $\{R = r\}$, N is Poisson with mean r (since the chip has unit area). Thus, $E[N|R] = R$, so that

$$\begin{aligned} E[N] &= E[E[N|R]] = E[R] && (51)\\ &= \frac{\alpha}{\lambda} && (52) \end{aligned}$$

where the final result is obtained from Table 4.1 of the textbook (or Wikipedia). We need to find $E[N^2]$, again using iterated expectations. Firstly we find

$$\begin{aligned} E[N^2|R=r] &= \text{var}[N|R=r] + E^2[N|R=r] && (53)\\ &= r + r^2, && (54) \end{aligned}$$

then we compute

$$\begin{aligned} E[N^2] &= E[R + R^2] && (55)\\ &= E[R] + \text{var}(R) + E^2[R] && (56)\\ &= \frac{\alpha}{\lambda} + \frac{\alpha}{\lambda^2} + \frac{\alpha^2}{\lambda^2}. && (57) \end{aligned}$$

Finally,

$$\begin{aligned} \text{var}(N) &= E[N^2] - E^2[N] && (58)\\ &= \frac{\alpha}{\lambda}\left(1 + \frac{1}{\lambda}\right) && (59) \end{aligned}$$

(b) Find the PMF of N.

Ans: Here, we use the continuous form of the Theorem on Total Probability as follows:

$$\begin{aligned} p_N(n) &= \int_0^\infty p_{N|R}(n|r)\frac{\lambda(\lambda r)^{\alpha-1}e^{-\lambda r}}{\Gamma(\alpha)}dr && (60)\\ &= \frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty \frac{r^n}{n!}e^{-r}r^{\alpha-1}e^{-\lambda r}dr && (61)\\ &= \frac{\lambda^\alpha}{n!\Gamma(\alpha)}\int_0^\infty r^{n+\alpha-1}e^{-(\lambda+1)r}dr && (62) \end{aligned}$$

for $n = 0, 1, 2, \ldots$

The integral above resembles the Gamma function definition. We make this resemblance concrete by substituting $t = (\lambda+1)r$. After a few manipulations, we obtain

$$p_N(n) = \frac{\lambda^n}{n!(\lambda+1)^{n+\alpha}} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} \tag{63}$$

But the Gamma function obeys the recurrence relation $\Gamma(z+1) = z\Gamma(z)$, and hence

$$\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} = \prod_{k=0}^{n-1} \alpha + k.$$

This is also a known result (look under "Gamma function" in Wikipedia). The final expression for the PMF of N is therefore

$$p_N(n) = \frac{\lambda^n}{n!(\lambda+1)^{n+\alpha}} \prod_{k=0}^{n-1} \alpha + k, \quad n = 0, 1, 2, \ldots$$

Functions of Two Random Variables

1. Let X and Y be independent Gaussian random variables with zero mean and unit variance. Show that $Z = X + Y$ is Gaussian.

   Ans: As shown in class, the PDF of $Z = X + Y$ is the convolution of the marginal PDFs of X and Y, because they are independent. Therefore,

   $$f_Z(z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-((z-y)^2+y^2)/2} dy. \tag{64}$$

   We can "complete the squares" as follows:

   $$\begin{aligned} (z-y)^2 + y^2 &= z^2 - 2zy + 2y^2 && (65) \\ &= 2[y^2 - zy + z^2/2] && (66) \\ &= 2\left[\left(y - \frac{z}{2}\right)^2 + \frac{z^2}{4}\right] && (67) \end{aligned}$$

   so that

   $$f_Z(z) = \frac{1}{\sqrt{2\pi(2)}} \frac{1}{\sqrt{2\pi(0.5)}} \int_{-\infty}^{\infty} e^{-(y-z/2)^2/2(0.5)} e^{-z^2/4} dy. \tag{68}$$

   But the function

   $$\frac{1}{\sqrt{2\pi(0.5)}} e^{-(y-z/2)^2/2(0.5)}$$

   is a Gaussian PDF in Y, with mean $z/2$ and variance 0.5. Therefore, it integrates to one, and we have

   $$f_Z(z) = \frac{1}{\sqrt{2\pi(2)}} e^{-z^2/4} \tag{69}$$

which is a Gaussian PDF in Z with a mean of 0 and variance of 2. In fact, with a little more work, we can show that $aX + bY$ is Gaussian with mean 0 and variance $a^2 + b^2$.

2. If X and Y are independent unit-mean exponential random variables, show that $Z = |X - Y|$ is also exponential, and find $E[Z]$.

   Ans: From the information given, we know that

$$f_X(t) = f_Y(t) = e^{-t}, \quad t > 0 \tag{70}$$

   and that $f_{X,Y}(x, y) = e^{-(x+y)}$, $x > 0$, $y > 0$. We can now find the CDF of $Z = |X - Y|$, $F_Z(z) = P[|X - Y| \le z]$. Notice that the event $\{|X - Y| \le z\}$ is equivalent to $\{-z \le X - Y \le z\}$ for $z > 0$, and is impossible when $z < 0$. With the help of a plot of the two lines $x - y = z$ and $x - y = -z$, where $z > 0$, we have

$$\begin{aligned}
F_Z(z) &= P[-z \le X - Y \le z] && (71)\\
&= \int_0^z \int_0^{y+z} e^{-(x+y)} dx dy + \int_z^\infty \int_{y-z}^{y+z} e^{-(x+y)} dx dy && (72)\\
&= \int_0^z e^{-(x+y)} \Big|_{y+z}^{0} dy + \int_z^\infty e^{-(x+y)} \Big|_{y+z}^{y-z} dy && (73)\\
&= \int_0^z e^{-y} - e^{-(2y+z)} dy + \int_z^\infty e^{-(2y-z)} - e^{-(2y+z)} dy && (74)\\
&= 1 - e^{-z} + \frac{1}{2} e^{-z} - \int_0^\infty e^{-(2y+z)} dy && (75)\\
&= 1 - e^{-z} && (76)
\end{aligned}$$

   for $z > 0$, while $F_Z(z) = 0$ for $z \le 0$. We recognize this as the CDF of an exponential RV with $E[Z] = 1$. (We can also find the PDF as $f_Z(z) = e^{-z}$, $z > 0$.)

3. The number of goals X that Singapore scores against Selangor is a geometric RV with mean 2; the number of goals Y that Selangor scores against Singapore is a geometric RV with mean 4. X and Y are assumed to be independent.

   (a) Find the PMF of $Z = X - Y$.

   Ans: We are given

$$\begin{aligned}
p_X(k) &= \left(\frac{1}{2}\right)^k, \quad k = 1, 2, \ldots && (77)\\
p_Y(k) &= \frac{1}{4}\left(\frac{3}{4}\right)^{k-1}, \quad k = 1, 2, \ldots && (78)
\end{aligned}$$

The PMF of $Z = X - Y$ is, for $k \geq 0$,

$$
\begin{aligned}
p_Z(k) &= P[X - Y = k] = \sum_{j=k+1}^{\infty} p_X(j) p_Y(j-k) && (79)\\
&= \sum_{j=k+1}^{\infty} \frac{1}{2^j} \frac{1}{4} \left(\frac{3}{4}\right)^{j-k-1} && (80)\\
&= \frac{1}{4} \left(\frac{4}{3}\right)^{k+1} \sum_{j=k+1}^{\infty} \left(\frac{3}{8}\right)^{j} && (81)\\
&= \frac{1}{5} \frac{1}{2^k}. && (82)
\end{aligned}
$$

For $k < 0$, we have

$$
\begin{aligned}
p_Z(k) &= \sum_{j=1}^{\infty} p_X(j) p_Y(j-k) && (83)\\
&= \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{1}{4} \left(\frac{3}{4}\right)^{j-k-1} && (84)\\
&= \frac{1}{4} \left(\frac{4}{3}\right)^{k+1} \sum_{j=1}^{\infty} \left(\frac{3}{8}\right)^{j} && (85)\\
&= \frac{1}{5} \left(\frac{4}{3}\right)^{k}. && (86)
\end{aligned}
$$

(b) Find the probability of Singapore beating Selangor, and the probability of the two teams drawing (tying) a game.

Ans: The event "Singapore beats Selangor" is equivalent to $\{X - Y > 0\}$. From part (a),

$$P[Z > 0] = \frac{1}{5} \sum_{k=1}^{\infty} \frac{1}{2^k} = \frac{1}{5}. \tag{87}$$

"Singapore and Selangor draw a game" is equivalent to $\{Z = 0\}$. Again from part (a),

$$P[Z = 0] = \frac{1}{5}. \tag{88}$$

4. Let X and Y be independent $\mathcal{N}(0, 1)$ random variables. Show that $Z = X/Y$ is a Cauchy random variable.

   Ans: We can either find the CDF of Z and then differentiate to find the PDF, or use conditional PDFs to obtain the PDF directly. The first method is relatively straightforward (at least in concept), so we illustrate the second method here.

We note that given $\{Y = y\}$, $Z = X/y$ is just a scaling of X by the scale factor $1/y$. Recalling that the PDF of $Y = aX$ is $f_Y(y) = \frac{1}{|a|} f_X(y/a)$, we have

$$f_{Z|Y}(z|y) = |y| f_X(yz) = \frac{|y|}{\sqrt{2\pi}} \exp\left(-\frac{(yz)^2}{2}\right).$$

Next, using the fact that $f_Z(z) = \int_{-\infty}^{\infty} f_{Z|Y}(z|y) f_Y(y) dy$, we have that

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} \frac{|y|}{2\pi} e^{-(z^2y^2+y^2)/2} dy && (89) \\ &= \frac{1}{\pi} \int_0^{\infty} y e^{-(z^2+1)y^2/2} dy && (90) \end{aligned}$$

where we made use of the fact that the integrand is an even function of y to arrive at the second line.

Then we use the observation that

$$\frac{d}{dy} e^{-(z^2+1)y^2/2} = -y(1+z^2) e^{-(z^2+1)y^2/2}$$

to conclude that

$$\begin{aligned} f_Z(z) &= \frac{1}{\pi(1+z^2)} \left[e^{-(z^2+1)y^2/2} \right]_{\infty}^{0} && (91) \\ &= \frac{1}{\pi(1+z^2)} && (92) \end{aligned}$$

which is the Cauchy PDF.