EE2012 2014/15 ADDITIONAL PROBLEMS I

## Basics of Probability Spaces

1. A desk drawer contains six pens, four of which are dry.
   (a) The pens are selected at random one by one until a good pen is found. The sequence of test results is noted. What is the sample space?
   (b) What is the set that represents the event "A good pen is drawn within the first four draws"?
   (c) Find the probability of the event in part (b).
2. An experiment has sample space $\Omega = \{a, b, c, d\}$. Suppose that $P[\{c, d\}] = 3/8$, $P[\{b, c\}] = 3/4$, and $P[\{d\}] = 1/8$. Use the axioms of probability to find the probabilities of the elementary events.
3. Find the probabilities of the following events in terms of $P[A]$, $P[B]$ and $P[A \cap B]$.
   (a) $A$ occurs and $B$ does not occur; $B$ occurs and $A$ does not occur.
   (b) Exactly one of $A$ or $B$ occurs.
   (c) Neither $A$ nor $B$ occurs.
4. Let the events $A$ and $B$ have $P[A] = x$, $P[B] = y$, and $P[A \cup B] = z$. Use Venn diagrams to find $P[A \cap B]$, $P[A^c \cap B^c]$, $P[A^c \cup B^c]$.
5. A bus dispatcher rolls a die every 30 seconds. Only if the number on the top face is 5 or 6 will a bus be dispatched. Find the probability that more than 1 minute passes between bus dispatches.
6. Let $S = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$ and the probability law be $P(A) =$ Area of event $A$ within the unit square. Let $A = \{(x, y) : x < y\}$, $B = \{(x, y) : x < 0.5\}$ and $C = \{(x, y) : y < 0.5\}$.
   (a) Sketch the regions in the $x$-$y$ plane corresponding to events $A$, $B$ and $C$.
   (b) Show that $B$ and $C$ are independent, and that $A$ and $(B \cap C)$ are independent.
   (c) Find $P[A|B]$ and $P[C|A]$.
   (d) Are the three events independent?

## Counting Problems and Sequential Experiments

1. A laptop comes with 3 memory options, and 4 processor options.
   (a) How many memory/processor configurations in total are there?

(b) A buyer chooses a configuration at random, with equal probability. What is the probability that memory option 2 is chosen?

(c) The store holds a lucky draw, with one laptop of each configuration available as prizes together with 20 consolation prizes. Find the probability that the 3rd participant in the lucky draw wins a laptop with configuration 1, using a tree diagram.

2. A LEGO set has pieces in three colours: red, yellow and green. There are 10 red pieces, 14 yellow pieces, and 16 green pieces.

   (a) Find the number of ways to arrange the 40 pieces in a row if pieces of the same colour are indistinguishable.

   (b) Now arrange the pieces in a line with red blocks first, yellow blocks second and green blocks last. Each colour has two types of blocks (e.g. $2 \times 4$ and $2 \times 2$). Find the total number of arrangements.

   (c) A child chooses one of the blocks arranged according to part (b) at random. Intuitively, the probability of choosing a yellow block is 14/40. Prove that this intuition is correct, using the theorem on total probability.

3. Assume that a Scrabble bag contains 1 Z, 3 A's, 4 E's and 1 H. Find the probability that a player picks the four tiles H, A, Z and E, if he grabs them all at once.

4. A traffic light changes from Red in time sample $k$ to Green in the next sample $k+1$ with probability 0.1; from Green to Amber with probability 0.1; and from Amber to Red with probability 0.4.

   (a) Find the probabilities of remaining in the same state from one time sample to the next, starting from Green, Amber and Red respectively.

   (b) Suppose at time sample 0 the light is Green. Find the probability of seeing the sequence GGGARR in time samples 0 to 5.

## Bernoulli Trials and Related Distributions

1. A randomly selected person takes his driving test until he passes. Discuss whether the number of attempts required can be modelled by a geometric distribution.

2. A block of 100 bits is transmitted over a binary communication channel with probability of bit error $p = 10^{-2}$.

   (a) If the block has 0 or 1 errors, then the receiver accepts it. Find the probability that the block is accepted.

   (b) If the block has more than 1 error, it is re-transmitted. Find the probability that $M$ re-transmissions are required.

3. If chips from a production line fail independently with probability 0.05, find and plot the probability of having at most 2 faulty chips in a batch of $n$ chips, as a function of $n$.

4. Assume that the probability of having a boy in a pregnancy is 0.5.

   (a) Suppose a couple has children until they have a boy, then they stop. How do we model the distribution of their number of children?

   (b) Find the probability that the couple has exactly two girls first, followed by a boy.

   (c) If the first two children are girls, find the probability that the number of children exceeds 3.

5. Assume that the birthdays of 20 people are equally likely to fall on any day of the week, and that the birthdays are independent (e.g. there are no twins). One of these 20 people is randomly selected. Let the event $A$ be "birthday of selected person is either Monday or Tuesday".

   (a) Find $P[A]$.

   (b) Ten groups of 20 people are surveyed, and one person from each group is randomly selected as described above. (i) Find the probability that none of them were born on Monday or Tuesday. (ii) Find the probability that 4 of them were born on Monday or Tuesday.

## Discrete Random Variables

1. Plot or sketch the following probability mass functions, and find their mean values:

   (a) $p_X(k) = \begin{cases} 0.1 & k = -2 \\ 0.2 & k = 0 \\ 0.4 & k = 1 \\ 0.3 & k = 3 \end{cases}$

   (b) $p_Y(n) = 0.9^{n-1}\beta$, $n = 1, 2, \ldots, 10$. (First, find $\beta$.)

   (c) $p_Z(z) = 0.1$, $z \in \{0.1, 0.2, \ldots, 1.0\}$.

2. A coin is tossed $n$ times, where $n$ is an even number. The probability of the coin landing heads up is $p$. Let the random variable $X$ be the absolute difference between the number of heads and the number of tails.

   (a) What is the sample space of the underlying experiment, $S$?

   (b) Find $P[X = 0]$.

   (c) If the outcome of the experiment contains $k$ heads, then $X = |k - (n - k)| = |2k - n|$. Find $S_X$.

(d) Find the probability mass function of $X$.

3. A modem transmits a $+2$ Volts signal into a channel. The channel adds to this signal a noise term that is drawn from the set $\{0, -1, -2, -3\}$ with respective probabilities $\{0.4, 0.3, 0.2, 0.1\}$.

   (a) Find the PMF of the output $Y$ of the channel.

   (b) What is the probability that the output of the channel is equal to the input of the channel?

   (c) What is the probability that $Y$ is positive?

4. Two fair dice are tossed and $X$ denotes the absolute difference between the number of dots on the top faces.

   (a) Find and plot the PMF of $X$.

   (b) Find and plot the CDF of $X$.

5. The number of defects on a semiconductor wafer with area 5 cm$^2$ is a Poisson random variable $X$, with mean value $\alpha = 0.5$. A production line outputs 10 such wafers per minute.

   (a) Find the probability that there are zero defects in a randomly selected wafer.

   (b) Find the probability that there are no defects in any of the wafers produced in a minute. Assume that defects occur independently from one wafer to another.

6. An experiment consists of generating a binomial random variable with $n = 10$ and $p = 0.25$ if a black ball is drawn from a bag with 5 black balls and 3 white balls. If a white ball is drawn, a geometric random variable with $p = 0.5$ is generated. The number generated in this experiment is denoted $X$.

   (a) Find the PMF of $X$ using the theorem on total probability. (It cannot be simplified.)

   (b) Use Bayes' theorem to find $P[\text{white ball}|X = 5]$.

## Conditional Distributions and Mean Values of One Random Variable

1. Let $X$ be a geometric random variable with $p = 0.1$, $A = \{X \geq 3\}$ and $B = \{X < 10\}$.

   (a) Find the conditional PMFs $p_X(k|A)$ and $p_X(k|B)$.

   (b) Find $E[X|A]$ and $E[X|B]$.

   (c) Find $E[p^X]$ where $p = 0.1$ (as in the distribution).

2. Two dice are rolled, and $X$ is defined as the sum of the dots on the top faces.

   (a) Find the PMF of $X$ by enumerating all 36 equi-probable outcomes and evaluating the value of $X$ for each outcome.

   (b) Find the conditional PMF of $X$, given $\{X > 10\}$.

   (c) Find the conditional PMF of $X$, given one of the dice is a 6.

3. Let $N$ be the number of customers entering a store in a one-hour period, and suppose $N$ has the Poisson PMF

$$p_N(n) = \frac{10^n}{n!}e^{-10}, \quad n = 0, 1, 2, \ldots$$

   (a) Find the PMF of $N$ conditioned on $\{N > 5\}$.

   (b) Given that in the first half an hour, five customers entered, find the conditional PMF of $N$.

   (c) Explain why the answers to parts (a) and (b) are *different*.

4. Recent statistics show that the average number of people in an MRT car is 150, 80 and 20 respectively in peak, semi-peak and off-peak periods, respectively. The peak, semi-peak and off-peak periods occupy 20, 60 and 20 percent of a typical day, respectively. What is the average number of people in an MRT car at a randomly selected point in time?

5. James plays a lottery which pays out 1 million dollars with a probability of $10^{-6}$. The price of a lottery ticket is one dollar.

   (a) Find his expected winnings after playing the lottery on six separate (independent) occasions.

   (b) The lottery corporation decides to change the price of a lottery ticket to \$1.20, the payout to 1.1 million dollars, and the probability of winning to $2 \times 10^{-6}$. Should James be more or less tempted to play?