EE2012 2014/15 PROBLEM SET 4

Discrete Random Variables

1. An urn contains nine \$2 notes and one \$10 note. Let the random variable X be the total amount that results when two bills are drawn from the urn without replacement.

   (a) Describe the underlying sample space S of this random experiment and specify the probabilities of its elementary events.

   (b) Show the mapping from S to S_X, the range of X.

   (c) Find the probability mass function (PMF) of X.

2. An m-bit password is required to access a system. A hacker systematically works through all possible m-bit patterns. Let X be the number of patterns tested until the correct password is found.

   (a) Describe the underlying sample space S.

   (b) Show the mapping from S to S_X.

   (c) Find the PMF of X.

3. Two transmitters send messages over a wireless channel. During each time slot, each transmitter sends a message with probability $\frac{1}{2}$. Simultaneous transmissions result in loss of both messages. Let X be the number of time slots until the first message gets through.

   (a) Describe the underlying sample space S and specify the probabilities of its elementary events.

   (b) Show the mapping from S to S_X.

   (c) Find the PMF of X.

4. Let X be a random variable with PMF $p_X(k) = \dfrac{c}{k^2}$, $k = 1, 2, \ldots$.

   (a) Find the value of c. Note that

   $$\sum_{k=1}^{\infty} \frac{1}{k^2} = \zeta(2) = \frac{\pi^2}{6}$$

   where $\zeta(s)$ is called the Riemann zeta function, defined as $\zeta(s) = \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^s}$ for any complex value s.

   (b) Find $P[X > 6]$.

   (c) Find $P[4 \leq X \leq 8]$.

5. Suppose the probability of a person being left-footed given that s/he is left-handed is 0.9, and the probability of a person being left-footed given that s/he is right-handed is 0.1. Denoting by LF, RF, LH and RH the events that a person is left-footed, right-footed, left-handed and right-handed respectively, this means

$$P(\text{LF}|\text{LH}) = 0.9, \quad P(\text{LF}|\text{RH}) = 0.1.$$

Within the population, 5 percent are left-handed, and the rest are right-handed.

   (a) Find the probability of a person being left-footed.
   (b) Find the probability of someone being both left-handed and left-footed.
   (c) Find the minimum number of people that has to be picked in order to have a probability of finding at least two people who are both left-handed and left-footed exceed 0.8.

Expected Values

1. (a) The mean value of a Bernoulli random variable X is 0.6. Find the PMF of X.
   (b) Find the mean and variance of Y if $S_Y = \{0, 2, 5, 10\}$ and all values are equi-probable.
   (c) If $Z = 2Y^2 - 3Y$, with Y defined above, find $E[Z]$.

2. Find the expected value and the variance of X if it has the following PMFs. You have to first find p_1 in both cases.

   (a) $p_X(k) = p_1/k$, $k \in \{1, 2, 3, 4\}$.
   (b) $p_X(k+1) = p_X(k)/2$, $k = 2, 3, 4$; $p_X(1) = p_1$.

3. Let X and Y be two integer-valued random variables whose PMFs differ only by a constant shift, i.e.

$$p_X(k) = p_Y(k - a)$$

where a is an integer constant.

   (a) Show that $E[X] = E[Y] + a$.
   (b) Show that $\text{var}(X) = \text{var}(Y)$.

4. Suppose a fair coin is tossed n times. Each coin toss costs d dollars and the reward in obtaining X is $aX^2 + bX$. Find the expected net reward.

5. The St. Petersburg Paradox is the following. A casino offers a payout of 2^X dollars, where X is the number of flips of a fair coin required to obtain the first Heads. In other words, if it takes 10 flips to obtain the first Heads, the casino pays 1024 dollars. The potential payout is unlimited, since the sample space of X extends to infinity. But yet a shrewd gambler will only be willing to wager a small amount of money to play the game. Why? We find out by solving the following problems.

(a) How many tosses can the casino afford to pay out if it has a finite amount of money M dollars?

(b) Find the expected payoff to the player.

(c) How much should a player be willing to pay to play this game?