Weekly Notes for EE2012 2014/15 – Week 13

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Book sections covered this week: 5.8.1.

1 Functions of Two Random Variables

When a random variable $Z = g(X, Y)$, there are several ways of obtaining its distribution, depending on whether it is discrete or not, and also on how easy it is to obtain and manipulate the conditional distribution of Y given X, or X given Y.

1.1 Finding the PMF when Z is Discrete

If $Z = g(X, Y)$ is discrete, e.g. $Z = \text{sgn}(XY)$, then it is possible[1] to find the PMF of Z directly from its definition:

$$p_Z(z_k) = P[g(X, Y) = z_k] \tag{1}$$

where $z_k \in S_Z$, the sample space of Z.

Example 1: Let (X, Y) have the joint PDF

$$f_{X,Y}(x, y) = \frac{1}{4}, \quad -1 < x \leq 1, -1 < y \leq 1 \tag{2}$$

and 0 elsewhere, and $Z = \text{sgn}(XY)$. Then $Z \in \{-1, +1\}$, with

$$p_Z(-1) = P[\text{sgn}(XY) = -1] = P[\text{“}X \text{ and } Y \text{ have opposite signs”}]$$

The latter event is equivalent to (X, Y) falling in the 2nd or 4th quadrants, which has a probability of 0.5 (given the uniform joint PDF). Therefore

$$p_Z(-1) = p_Z(1) = 0.5.$$

Example 2: Suppose X and Y are integer-valued independent random variables, with PMFs $p_X(k)$ and $p_Y(k)$, $k \in \mathbb{Z}$. Let $Z = X + Y$. Then Z must of course also

[1] At least in principle.

be integer-valued, and its PMF is obtained as

$$\begin{aligned}
p_Z(k) &= P[X+Y=k] \\
&= \cdots + P[X=-1, Y=k+1] + P[X=0, Y=k] + P[X=1, Y=k-1] + \cdots \\
&= \sum_{j=-\infty}^{\infty} p_{X,Y}(j, k-j) \\
&= \sum_{j=-\infty}^{\infty} p_X(j) p_Y(k-j)
\end{aligned}$$

where the last line follows from the third line due to independence between X and Y. The final expression can be recognized as the convolution between $p_X(k)$ and $p_Y(k)$, where convolution between two sequences x_k and y_k is defined as

$$x_k * y_k = \sum_{j=-\infty}^{\infty} x_j y_{k-j}. \tag{3}$$

Therefore, we have the important result that

$$p_{X+Y}(k) = p_X(k) * p_Y(k) \tag{4}$$

when X and Y are independent integer-valued random variables. ■

1.2 Finding the CDF

When $Z = g(X, Y)$ is not a discrete r.v., its CDF can, in principle, always be found through its definition

$$F_Z(z) = P[g(X, Y) \leq z] \tag{5}$$

since we know the joint distribution of X and Y. By differentiating the CDF, we obtain the PDF of Z.

Example 3: Let X and Y have the joint PDF

$$f_{X,Y}(x, y) = \frac{1}{2\pi} e^{-(x^2+y^2)/2}, \quad (x, y) \in \mathbb{R}^2. \tag{6}$$

Find the PDF of $R = \sqrt{X^2 + Y^2}$.

Ans: The CDF of R is

$$\begin{aligned}
F_R(r) &= P[\sqrt{X^2+Y^2} \leq r] & (7) \\
&= \int\int_{C_r} \frac{1}{2\pi} e^{-(x^2+y^2)/2} dx dy & (8)
\end{aligned}$$

where C_r denotes the disk of radius r centred on the origin. It is easier to perform the integration in polar coordinates, by the change of variables $x = r\cos\theta$ and

$y = r\sin\theta$. We end up with

$$\begin{aligned} F_R(r) &= \frac{1}{2\pi}\int_0^{2\pi}\int_0^r r' e^{-r'^2/2} dr' d\theta & (9)\\ &= \frac{1}{2\pi}\int_0^{2\pi} 1 - e^{-r^2/2} d\theta & (10)\\ &= 1 - e^{-r^2/2}, \quad r > 0. & (11) \end{aligned}$$

Differentiating yields

$$f_R(r) = re^{-r^2/2}, \quad r > 0. \qquad (12)$$

This is a so-called Rayleigh distribution. ■

Example 4: Let X and Y be jointly continuous, with joint PDF $f_{X,Y}(x, y)$. Find the PDF of $Z = X + Y$.

Ans: The CDF of Z is, by definition,

$$F_Z(z) = P[X + Y \leq z] = \int_{-\infty}^{\infty}\int_{-\infty}^{z-y} f_{X,Y}(x, y) dx dy. \qquad (13)$$

The limits of integration above were obtained from a sketch of the region $x+y \leq z$ on the x-y plane. Therefore, the PDF of Z is

$$f_Z(z) = \frac{d}{dz}F_Z(z) = \int_{-\infty}^{\infty}\left[\frac{d}{dz}\int_{-\infty}^{z-y} f_{X,Y}(x, y)dx\right] dy. \qquad (14)$$

By the Fundamental Theorem of Calculus[2], and the chain rule of differentiation, we then have

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(z - y, y)dy. \qquad (15)$$

Reversing the order of integration in (13) and then repeating the subsequent steps above yields

$$f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x, z - x)dx. \qquad (16)$$

When X and Y are independent, $f_{X,Y}(x, y) = f_X(x)f_Y(y)$, and hence

$$\begin{aligned} f_{X+Y}(z) &= \int_{-\infty}^{\infty} f_X(z - y)f_Y(y)dy & (17)\\ &= \int_{-\infty}^{\infty} f_X(x)f_Y(z - x)dx & (18)\\ &= f_X(z) * f_Y(z). & (19) \end{aligned}$$

[2] Which says that, for any $a < x$,

$$\frac{d}{dx}\int_a^x g(t)dt = g(x).$$

In other words, the sum of two continuous, independent random variables has a PDF that is the convolution of the two marginal PDFs. ■

Example 4a: As a specific example of the sum of two independent continuous r.v.s, consider X and Y as two independent $U(0,1)$ random variables. Find the PDF of $Z = X + Y$.

Ans: From (19), for $0 < z \leq 1$,

$$f_Z(z) = \int_0^z 1dx = z; \tag{20}$$

for $1 < z \leq 2$,

$$f_Z(z) = \int_{z-1}^1 1dx = 2 - z. \tag{21}$$

For all other values of z, $f_X(x)$ and $f_Y(z-x)$ do not overlap in their non-zero portions, and hence $f_Z(z) = 0$. Therefore, $X + Y$ has the triangular PDF

$$f_Z(z) = \begin{cases} z & 0 < z \leq 1 \\ 2 - z & 1 < z \leq 2 \\ 0 & \text{otherwise} \end{cases} \tag{22}$$

Note that $X + Y$ is no longer uniform – heuristically, more (X, Y) pairs lead to the event "$X + Y$ is close to 1" than those that result in "$X + Y$ close to 0" or "$X + Y$ close to 2", and hence the above result is not surprising. ■

Example 4b (Sum of Two Independent Gaussian Random Variables): Let $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$. Find the PDF of $Z = X + Y$.

Ans: The PDFs of X and Y are respectively

$$\begin{aligned} f_X(t) &= \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left(-\frac{(t-\mu_x)^2}{2\sigma_x^2}\right) && (23) \\ f_Y(t) &= \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left(-\frac{(t-\mu_y)^2}{2\sigma_y^2}\right). && (24) \end{aligned}$$

For notational convenience, let $K_x = (2\pi\sigma_x^2)^{-1/2}$ and $K_y = (2\pi\sigma_y^2)^{-1/2}$. Then

$$\begin{aligned} f_Z(t) &= f_X(t) * f_Y(t) = \int_{-\infty}^{\infty} f_X(\tau) f_Y(t-\tau) d\tau && (25) \\ &= K_x K_y \int_{-\infty}^{\infty} \exp\left(-\frac{(\tau-\mu_x)^2}{2\sigma_x^2} - \frac{(t-\tau-\mu_y)^2}{2\sigma_y^2}\right) d\tau && (26) \\ &\vdots \quad \text{(several steps later...)} && (27) \\ &= \frac{1}{\sqrt{2\pi(\sigma_x^2+\sigma_y^2)}} \exp\left(-\frac{(t-\mu_x-\mu_y)^2}{2(\sigma_x^2+\sigma_y^2)}\right). && (28) \end{aligned}$$

This illustrates the important result that

The sum of two independent Gaussian random variables is another Gaussian random variable.

(This result is more easily derived using characteristic functions (section 4.7), which we did not have a chance to discuss.) ■

1.3 Using Conditional Distributions

Suppose Y is a continuous random variable, and $Z = g(X, Y)$ has the conditional PDF $f_{Z|Y}(z|y)$. (Note that Z may or may not be continuous, since the one-dimensional PDF exists for all types of r.v.) Then by the Theorem on Total Probability,

$$\begin{aligned} F_Z(z) &= \sum_k P[Z \leq z | y_k < Y \leq y_k + dy] P[y_k < Y \leq y_k + dy] \\ &= \sum_k F_{Z|Y}(z|y_k) f_Y(y_k) dy && (29) \\ \Rightarrow \quad f_Z(z) &= \frac{d}{dz} F_Z(z) = \sum_k f_{Z|Y}(z|y_k) f_Y(y_k) dy && (30) \\ &\overset{dy \to 0}{=} \int_{-\infty}^{\infty} f_{Z|Y}(z|y) f_Y(y) dy. && (31) \end{aligned}$$

Similarly, if Y is discrete, then

$$f_Z(z) = \sum_{k=0}^{N} f_{Z|Y}(z|y_k) p_Y(y_k) \tag{32}$$

where $S_Y = \{y_0, y_1, \ldots, y_N\}$. If the conditional distribution of Z given Y (or X) is known or easily obtained, then (31) or (32) can be used to obtain the distribution of Z.

Example 5: We illustrate this method by deriving two general formulas, for $Z = XY$ and $Z = X/Y$ respectively. In both cases, we assume that the conditional PDF of X given Y is known.

Note that (31) requires $f_{Z|Y}(z|y)$. For $Z = XY$, we have

$$\begin{aligned} P[z < Z \leq z + dz | Y = y] &= P[z < XY \leq z + dz | Y = y] \\ f_{Z|Y}(z|y) dz &= f_{X|Y}\left(\frac{z}{y} \middle| y\right) \frac{dz}{|y|} \end{aligned}$$

since conditioned on $\{Y = y\}$, we have the equivalence

$$\{z < XY \leq z + dz\} \equiv \begin{cases} \left\{\frac{z}{y} < X \leq \frac{z}{y} + \frac{dz}{y}\right\} & y > 0 \\ \left\{\frac{z}{y} + \frac{dz}{y} \leq X < \frac{z}{y}\right\} & y < 0. \end{cases}$$

Therefore,

$$f_{Z|Y}(z|y) = \frac{1}{|y|} f_{X|Y}(z/y|y) \tag{33}$$

and (31) gives us finally

$$f_Z(z) = \int_{-\infty}^{\infty} \frac{1}{|y|} f_{X|Y}(z/y|y) f_Y(y) dy. \tag{34}$$

In a similar fashion for $Z = X/Y$, we have

$$\begin{aligned} f_{Z|Y}(z|y)dz &= |y| f_{X|Y}(yz|y)dz & (35) \\ \Rightarrow \quad f_{Z|Y}(z|y) &= |y| f_{X|Y}(yz|y) & (36) \end{aligned}$$

and therefore

$$f_Z(z) = \int_{-\infty}^{\infty} |y| f_{X|Y}(yz|y) f_Y(y) dy. \tag{37}$$

These expressions are useful only when the integrals can be computed in closed form e.g. when X and Y are i.i.d. exponential random variables. ■

2 Summary

1. The distribution of $Z = g(X, Y)$ can be found in various ways.
2. Some important results to note include:
   (a) If X and Y are independent and integer-valued, then $Z = X + Y$ has a PMF that is the *convolution* of the PMFs of X and Y.
   (b) If X and Y are independent and continuous, then $Z = X + Y$ has a PDF that is the *convolution* of the PDFs of X and Y.
   (c) An important special case of point (b) above is that the sum of two independent Gaussian random variables is itself Gaussian.

3 Diagnostic Questions

1. Let $Z = X + 2Y$, where X and Y are independent $\mathcal{U}(0, 1)$ random variables. Find the PDF of Z.
2. If X is discrete with PMF $p_X(x)$, $x \in S_X$, Y is continuous with PDF $f_Y(t)$, and X and Y are independent, show using the theorem on total probability that the PDF of $Z = X + Y$ is $f_Z(z) = \sum_{x \in S_X} p_X(x) f_Y(z - x)$. (In other words, $f_Z(z)$ is the convolution of $f_X(z)$ and $f_Y(z)$.)