EE2012 2014/15 Problem Set 5

Conditional PMF and Expected Values

1. Let $A = \{X \geq 3\}$. Find the conditional PMF $p_X(x|A)$ for the following random variables:

   (a) X is binomial with $n = 5$, $p = 0.2$.

   (b) X is geometric with $p = 0.4$.

   (c) X is Poisson with $E[X] = 2$.

2. Using the conditional PMFs of Question 1, find $P[X = 4 \text{ or } 5|A]$ in all three cases.

3. Suppose two bus services connect two locations A and B. Bus service 1 takes on average 30 minutes to make the journey from A to B, and service 2 takes on average 20 minutes to make the same journey (because it uses a different route). Assuming that the probability of Service 1 arriving before Service 2 is 2/3, find the travel time from A to B averaged over a large number of bus rides.

4. A store sells only three models of laptop, A, B and C. The lifetime of the laptop X (in years, rounded to the nearest integer) have the following conditional geometric distributions:

$$\begin{aligned} p_X(k|\text{“Model A”}) &= 0.8^{k-1}0.2 && (1) \\ p_X(k|\text{“Model B”}) &= 0.75^{k-1}0.25 && (2) \\ p_X(k|\text{“Model C”}) &= 0.7^{k-1}0.3 && (3) \end{aligned}$$

   where $k = 1, 2, \ldots$ Laptop A costs \$2,000, laptop B costs \$1,800 and laptop C costs \$1,600.

   (a) Based on their prices, 50% of laptop customers opt for Model C, 30% for Model B, and 20% for Model A. (The statistical lifetimes of these models are not known to the public.) Find the average lifetime of a laptop sold by this store.

   (b) Find the probability that a laptop bought from this store lasts longer than 2.5 years.

   (c) If a laptop lasts longer than 2.5 years, find the probability that it was Model A.

5. A student has only 10 minutes left to complete two questions in an exam, questions 9 and 10. Question 9 is worth 5 marks, and Question 10 is worth 8 marks. If Q9 is answered first (denoted event A), the probability of getting Q9 correct and Q10

wrong (denoted C_9W_{10}) is 0.6, i.e. $P[C_9W_{10}|A] = 0.6$. Conditioned on A the probability of all the other outcomes are

$$P[W_9W_{10}|A] = 0.1 \qquad P[C_9C_{10}|A] = 0.1$$
$$P[W_9C_{10}|A] = 0.2.$$

Similarly, conditioned on A^c (i.e. he answers Q10 first), we have

$$P[W_9C_{10}|A^c] = 0.5 \qquad P[C_9C_{10}|A^c] = 0.05$$
$$P[W_9W_{10}|A^c] = 0.25 \qquad P[C_9W_{10}|A^c] = 0.2.$$

Let X be the total marks scored in Q9 and Q10.

(a) Find $E[X|A]$ and $E[X|A^c]$. Which question should he attempt first, in order to maximize his expected marks from Q9 and Q10?

(b) Find $P[X \geq 8|A]$ and $P[X \geq 8|A^c]$. To maximize the probability of scoring 8 or more marks, which question should he do first?

(c) The student has no time to perform such complicated calculations, and chooses strategy A with probability p, and strategy A^c with probability $1-p$, where $p \notin \{0, 1\}$. Show that A and C_9 are not independent.

Important Discrete Random Variables

1. Let N be a geometric random variable with $E[N] = 1/p$.

   (a) Find $P[N = k|N \leq m]$.

   (b) Find the probability that N is even.

2. The number of page requests that arrive at a web server is a Poisson random variable with an average of 3,000 requests per minute.

   (a) Find the probability that there are no requests in a 100 ms period.

   (b) Find the probability that there are between 5 and 10 requests in a 100 ms period.

3. An LCD display has 2000×1000 pixels. A display is accepted if it has 15 or fewer faulty pixels. The probability that a pixel is faulty coming out of the production line is 5×10^{-6}. Find the proportion of displays that are accepted.

4. A random variable X is uniformly distributed in $\{-3, -2, \ldots, 3, 4\}$.

   (a) Find the mean and variance of X.

   (b) Find the mean and variance of $Y = -2X^2 + 3$.

Cumulative Distribution Function

1. The CDF of X is given by

$$F_X(x) = \begin{cases} 0 & x < -2 \\ 0.5 & -2 \leq x \leq 0 \\ (2+x)/4 & 0 \leq x \leq 2 \\ 1 & x > 2 \end{cases} \tag{4}$$

   (a) Sketch the CDF and identify the type of random variable X is.

   (b) Use the CDF to find $P[-1 < X \leq 1]$ and $P[X > 0]$.

2. Let ζ be a value selected at random from the unit interval $[0, 1]$, and define $X(\zeta) = (1-\zeta)^{-1/2}$.

   (a) Find the range of X.

   (b) Find and sketch the CDF of X.

   (c) Find the probability of the events $\{X > 1\}$, $\{5 < X < 7\}$ and $\{X \leq 20\}$.

3. A point is selected at random inside a square defined by $\{(x, y) : 0 \leq x \leq b, 0 \leq y \leq b\}$. Assume the point is equally likely to fall anywhere in the square. Let the random variable Z be defined as the minimum of the two coordinates of the point where the dart lands.

   (a) Find the range of Z.

   (b) Find and sketch the CDF of Z.

   (c) Use the CDF to find the probabilities of $\{Z > 0\}$, $\{Z \leq b/2\}$ and $\{Z > b/4\}$.