NATIONAL UNIVERSITY OF SINGAPORE
Department of Electrical & Computer Engineering

EXAMINATION FOR
(Semester II, 2013/14)

EE2012 ANALYTICAL TECHNIQUES FOR ECE

April/May 2014
Time Allowed: 2.5 hours

INSTRUCTIONS FOR CANDIDATES:

- This is a CLOSED BOOK exam.
- This paper contains five (5) questions and one formula sheet, printed on four (4) pages.
- Answer ALL questions.
- A non-programmable calculator may be used.

Examiner: Professor Lim Teng Joon

Q1. Answer the following short questions.

(a) Let T be exponentially distributed with $E[T] = 2$. Find $P[T > 2]$. (2 marks)

(b) Consider a probability space $(\Omega, \mathcal{F}, P)$, and two independent events A and B in $\mathcal{F}$. What is $P[A \cup B]$ in terms of $P[A]$ and $P[B]$? (2 marks)

(c) If the CDF of X is

$$F_X(x) = 0.1u(x) + 0.3u(x-2) + 0.6u(x-5),$$

where $u(x)$ is the unit step function, write down the range of X, and the PDF of X. (2 marks)

(d) Two zero-mean unit-variance random variables X and Y have a correlation coefficient of 0.4. Find $E[XY]$. (2 marks)

(e) If X is Gaussian with $E[X] = 1$ and $\text{var}(X) = 4$, find $P[X > 2]$ in terms of the Q function. (2 marks)

(f) The joint PMF of (X, Y) is given as follows:

$$\begin{aligned} p_{X,Y}(0,0) &= 0.1 & p_{X,Y}(2,0) &= 0.1 \\ p_{X,Y}(1,1) &= 0.2 & p_{X,Y}(0,1) &= 0.2 \\ p_{X,Y}(2,1) &= 0.25 & p_{X,Y}(1,2) &= 0.15 \end{aligned}$$

Find the marginal PMF of X and of Y. (4 marks)

(g) The random variable X is uniform in $\{1, 2, 3\}$, and Y is Bernoulli with $p = 0.2$. If X and Y are independent, find $E[XY]$. (3 marks)

(h) If $Y = 2X + 3$, and $X \sim \mathcal{N}(0, 1)$, find the mean and variance of Y, and write down the PDF of Y. (3 marks)

Q2. A random variable X has the PDF

$$f_X(x) = \frac{5}{4}(1 - x^4), \quad 0 < x \leq 1.$$

(a) Find the CDF of X. (5 marks)

(b) Find $E[X]$ and $\text{var}(X)$. (5 marks)

(c) Find $f_X(x|X > 0.5)$. (5 marks)

Q3. Let the discrete random variable X be uniformly distributed in $\{-3, -1, 1, 3\}$, and let N be a Gaussian random variable with mean 0 and variance σ_n^2. X and N are independent.

(a) If $Y = X + N$, find the conditional PDF $f_{Y|X}(y|x)$. (3 marks)

(b) Find $P[Y > 2|X = 1]$ in terms of the Q function. (3 marks)

(c) Given that $Y > 2$, what is the most likely value of X? (6 marks)

(d) Define Z as the indicator function of the event $\{Y > 2\}$. Find the PMF of Z, in terms of the Q function. (3 marks)

Q4. Consider two i.i.d. exponential random variables X and Y, with $E[X] = E[Y] = 2$.

(a) Find the PDF of $Z = X + Y$ using convolution. (5 marks)

(b) Find and sketch the CDF of $V = X/Y$. (7 marks)

(c) Find and sketch the PDF of $V = X/Y$. (3 marks)

Q5. Consider a Poisson arrival process with average rate λ arrivals per minute. The number of arrivals in t minutes is denoted $N(t)$.

(a) Find the probability mass function of $N(t)$. (2 marks)

(b) Suppose that each arrival is independently tagged, with probability p. Find the PMF of $M(t)$, the number of tagged arrivals in t minutes. (7 marks)

(c) Let T_i be the waiting time until the i-th arrival, $i = 1, 2, \ldots$ Derive the CDF of T_i. (*Hint:* What is the event $\{T_i > t\}$ equivalent to?) (6 marks)

End of Paper

List of Formulae and Notation

Definitions

Indicator Function: $I_A = 1$ if A occurs, 0 otherwise.

Marginal PMF/PDF: $p_X(x_j) = \sum_k p_{X,Y}(x_j, y_k)$; $f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$.

Marginal CDF: $F_X(x) = F_{X,Y}(x, \infty)$.

Joint Moments: $\rho_{X,Y} = \dfrac{\text{cov(X,Y)}}{\sigma_X \sigma_Y}$, $\text{cov}(X,Y) = E[XY] - E[X]E[Y]$.

Discrete Random Variables

Bernoulli: $p_X(1) = p = 1 - p_X(0)$, $E[X] = p, \text{var}[X] = p(1-p)$

Binomial: $p_X(k) = \binom{n}{k} p^k (1-p)^{n-k}$, $k = 0, 1, \ldots, n$. $E[X] = np, \text{var}[X] = np(1-p)$

Geometric: $p_X(k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$. $E[X] = \dfrac{1}{p}, \text{var}[X] = \dfrac{1-p}{p^2}$

Poisson: $p_X(k) = \dfrac{\alpha^k}{k!} e^{-\alpha}$, $k = 0, 1, \ldots$. $E[X] = \alpha = \text{var}[X]$

Continuous Random Variables

Uniform: $f_X(x) = \dfrac{1}{b-a}$, $a < x < b$. $E[X] = \dfrac{a+b}{2}, \text{var}[X] = \dfrac{(b-a)^2}{12}$

Exponential: $f_X(x) = \lambda e^{-\lambda x}$, $x \geq 0$. $E[X] = \dfrac{1}{\lambda}, \text{var}[X] = \dfrac{1}{\lambda^2}$

Gaussian: $f_X(x) = \dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\dfrac{(x-\mu)^2}{2\sigma^2}\right)$, $x \in \mathbb{R}$. $E[X] = \mu, \text{var}[X] = \sigma^2$

Gaussianity

Q fn.: $Q(x) = \dfrac{1}{\sqrt{2\pi}} \int_x^{\infty} e^{-t^2/2} dt$, $Q(x) = 1 - Q(-x)$

CDF: $X \sim \mathcal{N}(\mu, \sigma^2) \Rightarrow P[X > t] = Q\left(\dfrac{t-\mu}{\sigma}\right)$

Joint PDF: $f_{\mathbf{X}}(\mathbf{x}) = \dfrac{1}{2\pi\sqrt{\det(\mathbf{C})}} \exp\left[-\dfrac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \mathbf{C}^{-1} (\mathbf{x}-\boldsymbol{\mu})\right]$, $\mathbf{x} \in \mathbb{R}^2$

where $\mathbf{C} = E[(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})^T]$

Result 1: If $\mathbf{X}$ is jointly Gaussian, then $\mathbf{Y} = \mathbf{AX} + \mathbf{b}$ is jointly Gaussian.

Result 2: If $\mathbf{X}$ is jointly Gaussian, then its components are marginally Gaussian.

Other Useful Results

Bayes: $P(B_j|A) = \dfrac{P(A|B_j)P(B_j)}{\sum_{k=1}^n P(A|B_k)P(B_k)}$, where $\{B_k\}_{k=1}^n$ is a partition of $\mathcal{S}$.

Total Prob.: $P(A) = \sum_{k=1}^n P(A|B_k)P(B_k)$, where $\{B_k\}_{k=1}^n$ is a partition of $\mathcal{S}$.

$E[X] = \sum_{k=1}^n E[X|B_k]P[B_k]$; $f_X(x) = \sum_{k=1}^n f_X(x|B_k)P[B_k]$.

Functions of X: $Y = aX + b \Rightarrow f_Y(y) = \dfrac{1}{|a|} f_X\left(\dfrac{y-b}{a}\right)$

$Y = g(X) \Rightarrow f_Y(y) = \sum_{k=1}^n f_X(x_k) \left|\dfrac{dx_k}{dy}\right|$, where $g(x_k) = y$, $k = 1, \ldots, n$.

Independence (rv's): X, Y independent $\Leftrightarrow F_{X,Y}(x,y) = F_X(x)F_Y(y)$,

$f_{X,Y}(x,y) = f_X(x)f_Y(y)$, $p_{X,Y}(x,y) = p_X(x)p_Y(y)$.

Sum of rv's: X, Y independent $\Rightarrow f_{X+Y}(z) = f_X(z) * f_Y(z)$.