Conditional Probability

1. A die is tossed twice and the number of dots on the top face noted in the order of occurrence. Let $A =$ " first toss $\geq$ second toss", and $B =$ "first toss is a 6". Find $P[A|B]$ and $P[B|A]$.

   Ans: It should be obvious that $P[A|B] = 1$ because if the first toss is a 6, it must be at least equal to the second toss.

   Finding $P[B|A]$ requires a bit more care. We first note that the event $A \cap B = \{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6)\}$, and that

   $$A = \{(x,y) : x \geq y, x, y \in \{1,2,\ldots,6\}\}.$$

   The cardinality of A is $|A| = 6+5+4+3+2+1 = 21$. Note that every outcome (x,y), $x, y \in \{1,2,\ldots,6\}$ is equi-probable, and therefore

   $$\begin{aligned} P[B|A] &= \frac{P[A \cap B]}{P[A]} \\ &= \frac{|A \cap B|}{|A|} \\ &= \frac{6}{21} = \frac{2}{7}. \end{aligned}$$

2. A number x is selected at random in the interval $[-2, 2]$. Let the events

   $$\begin{aligned} A &= \{x : x < 0\}; \quad (1) \\ B &= \{x : |x - 0.5| < 0.5\} \quad (2) \\ C &= \{x : x > 0.75\}. \quad (3) \end{aligned}$$

   Find $P[A|B]$, $P[B|C]$ and $P[A|C^c]$.

   Ans: A simpler way to write B is $B = \{x : 0 < x < 1\}$. Then $A \cap B = \emptyset$, and hence $P[A|B] = 0$.

   $B \cap C = (0.75, 1)$ and so $P[B \cap C] = 0.25/4 = 1/16$. This means that

   $$P[B|C] = \frac{1/16}{1.25/4} = \frac{1}{5}.$$

   Finally, $A \cap C^c = A$ so that

   $$P[A|C^c] = \frac{P[A]}{P[C^c]} = \frac{0.5}{11/16} = \frac{8}{11}.$$

3. Let the lifetime of a product satisfy the probability law

$$P[\text{“lifetime exceeds } t \text{ years”}] = e^{-t}, \quad t \geq 0.$$

Let A be the event “lifetime exceeds t years” and B the event “lifetime exceeds $2t$ years”. Find $P[B|A]$.

Ans: Note that $A \cap B = B$, therefore

$$\begin{aligned} P[B|A] &= \frac{P[B]}{P[A]} \\ &= \frac{e^{-2t}}{e^{-t}} \\ &= e^{-t}. \end{aligned}$$

4. Show that $P[A|B]$ satisfies the Axioms of Probability, i.e.

- $P[A|B] \geq 0$;
- $P[S|B] = 1$;
- If $A \cap C = \emptyset$, then $P[A \cup C|B] = P[A|B] + P[C|B]$.

Ans: Since $P[A|B] = P[A \cap B]/P[B]$ and both the denominator and the numerator are probability values, which cannot be negative, $P[A|B] \geq 0$.

$P[S|B] = P[S \cap B]/P[B] = P[B]/P[B] = 1$.

For the third property, note that

$$(A \cup C) \cap B = (A \cap B) \cup (B \cap C)$$

and that, since A and C are mutually exclusive, so are $A \cap B$ and $B \cap C$. Therefore

$$\begin{aligned} P[A \cup C|B] &= \frac{P[(A \cup C) \cap B]}{P[B]} \\ &= \frac{P[A \cap B] + P[B \cap C]}{P[B]} \\ &= \frac{P[A \cap B]}{P[B]} + \frac{P[B \cap C]}{P[B]} \\ &= P[A|B] + P[C|B]. \end{aligned}$$

5. One of two coins is selected with equal probability, and tossed three times. The first coin comes up heads with probability $p_1 = 1/3$, and the second comes up heads with probability $p_2 = 2/3$.

   (a) Find the probability that the number of heads is k.

Ans: Let C_1 be the event that the first coin is selected, and C_2 the event that the second coin is selected. Conditioning on C_1 and C_2 yields

$$P[\text{“}k\text{ heads”}|C_1] = \binom{3}{k} p_1^k (1-p_1)^{3-k} \quad (4)$$

$$P[\text{“}k\text{ heads”}|C_2] = \binom{3}{k} p_2^k (1-p_2)^{3-k} \quad (5)$$

for $k = 0, 1, 2, 3$. Since $P[C_1] = P[C_2] = 0.5$, the theorem on total probability says that

$$\begin{aligned} P[\text{“}k\text{ heads”}] &= 0.5P[\text{“}k\text{ heads”}|C_1] + 0.5P[\text{“}k\text{ heads”}|C_2] \\ &= \frac{1}{2}\binom{3}{k}\left[\frac{1}{3^k}\left(\frac{2}{3}\right)^{3-k} + \left(\frac{2}{3}\right)^k \frac{1}{3^{3-k}}\right] \\ &= \frac{1}{54}\binom{3}{k}\left[2^{3-k} + 2^k\right], \end{aligned} \quad (6)$$

for $k = 0, 1, 2, 3$.

(b) Find the probability that Coin 1 was tossed, given that k heads were observed.

Ans: For ease of notation, let A_k denote the event that k heads are observed. We already have $P[A_k|C_1]$ from (4) and $P[A_k]$ from (6), so we can use Bayes Rule to obtain the desired

$$\begin{aligned} P[C_1|A_k] &= \frac{P[A_k|C_1]P[C_1]}{P[A_k]} \\ &= \frac{\binom{3}{k}\frac{1}{3^k}\left(\frac{2}{3}\right)^{3-k}(0.5)}{\frac{1}{54}\binom{3}{k}[2^{3-k} + 2^k]} \\ &= \frac{2^{3-k}}{2^{3-k} + 2^k}. \end{aligned} \quad (7)$$

For example, if one head is observed, then $P[C_1|A_1] = 2/3$ (which of course means that $P[C_2|A_1] = 1/3$.)

(c) In part (b), which coin is more probable when k heads have been observed? In other words, for each value of $k \in \{0, 1, 2, 3\}$, compare the values of $P[\text{“Coin 1”}|\text{“}k\text{ heads”}]$ and $P[\text{“Coin 2”}|\text{“}k\text{ heads”}]$.

Ans: From (7), and the fact that $P[C_2|A_k] = 1 - P[C_1|A_k]$, we can form the following table –

k	0	1	2	3
$P[C_1\|A_k]$	$\frac{8}{9}$	$\frac{2}{3}$	$\frac{1}{3}$	$\frac{1}{9}$
$P[C_2\|A_k]$	$\frac{1}{9}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{8}{9}$

So when 0 or 1 heads are observed, the coin tossed is more likely to have been coin 1, while if 2 or 3 heads are observed, the coin is more likely to have been coin 2. This makes sense, because coin 1 has a smaller chance of turning up heads.

(d) Suppose now the selected coin is tossed 5 times, and we observe that there are 3 heads. We have to make an educated guess as to which coin was selected. What should our decision be?

Ans: We need to compare $P[C_1|A_3]$ and $P[C_2|A_3]$, and decide in favour of coin 1 if and only if

$$P[C_1|A_3] > P[C_2|A_3].$$

But using Bayes Rule and the fact that $P[C_1] = P[C_2]$, we can perform the simpler comparison of $P[A_3|C_1]$ with $P[A_3|C_2]$. Since

$$P[A_3|C_1] = \frac{1}{54}\binom{5}{3}2^2$$

and

$$P[A_3|C_2] = \frac{1}{54}\binom{5}{3}2^3$$

we have that $P[A_3|C_1] < P[A_3|C_2]$ and so we decide in favour of coin 2.

Independence and Independent Bernoulli Trials

1. Consider three events A, B and C, each with non-zero probability p_A, p_B and p_C respectively. Given that they are pairwise independent, and that $P[A \cap B \cap C] = p_A p_B p_C$, show that A and $B \cup C$ must be independent.

   Ans: It suffices to show that $P[B \cup C|A] = P[B \cup C]$, so we develop the solution with this goal in mind.

$$\begin{aligned} P[B \cup C|A] &= \frac{P[(B \cup C) \cap A]}{P[A]} && (8) \\ &= \frac{P[(A \cap B) \cup (A \cap C)]}{P[A]} && (9) \\ &= \frac{P[A \cap B] + P[A \cap C] - P[A \cap B \cap C]}{P[A]} && (10) \end{aligned}$$

   where the last line arises from the identity $P[A \cup B] = P[A] + P[B] - P[A \cap B]$, and noting that $(A \cap B) \cap (A \cap C) = A \cap B \cap C$. Due to the pairwise independence of all three events, and with the given condition $P[A \cap B \cap C] = p_A p_B p_C$, we then have

$$\begin{aligned} P[B \cup C|A] &= p_B + p_C - p_B p_C && (11) \\ &= P[B] + P[C] - P[B \cap C] && (12) \\ &= P[B \cup C]. && (13) \end{aligned}$$

   Therefore, $B \cup C$ is independent from A under the stated conditions.

2. An experiment consists of picking one of two urns at random, and then selecting a ball from the urn and noting its colour (black or white). Let A be the event "urn 1 is selected", and B the event "a black ball is picked". Under what conditions are A and B independent?

   Ans: A and B are independent if and only if $P[B|A] = P[B]$ and $P[B|A^c] = P[B]$. In other words, no matter which urn is picked, the probability of picking a black ball remains the same. The only way for this to happen is for the fraction of black balls in both urns to be equal, e.g. 3B/5W in Urn 1, 6B/10W in Urn 2.

3. A random experiment is repeated a large number of times and the occurrence of events A and B is noted. How would you empirically test for the independence of A and B?

   Ans: We can find the relative frequencies of A, B and $A \cap B$, i.e. $f_A(n)$, $f_B(n)$ and $f_{A\cap B}(n)$, respectively, and then check if

$$f_A(n)f_B(n) \approx f_{A\cap B}(n).$$

4. 10 percent of items from a production line are defective. What is the probability that there are more than one defective item in a batch of n items?

   Ans: The number of defective items follows a binomial probability law if we assume that defects occur independently among the items. Let the desired event be A. It is easy to find

$$P[A^c] = 0.9^n + 0.1(0.9)^{n-1}n$$

   (This is the probability of 0 or 1 defective items out of n.) Therefore, $P[A] = 1 - P[A^c] = 1 - (0.9^n + 0.1(0.9)^{n-1}n)$.

5. We need 10 chips of a certain type to build a circuit. It is known that 5 percent of these chips are defective. How many chips should we buy for there to be a greater than 90 percent chance of having enough chips for the circuit?

   Ans: The number of defective chips in a batch of n chips follows the binomial probability law, i.e.

$$P[\text{"}k \text{ chips defective"}] = \binom{n}{k} 0.05^k 0.95^{n-k},$$

   $k = 0, 1, \ldots, n$. We need at least 10 good chips out of a batch of n, or equivalently, at most $n - 10$ defective chips. The probability of being able to build the circuit is therefore

$$\alpha(n) = \sum_{k=0}^{n-10} \binom{n}{k} 0.05^k 0.95^{n-k}. \tag{14}$$

   Note that this is a function of n, the number of chips we buy. Our intuition tells us that as n grows, so does $\alpha(n)$, and this is easily verified numerically. The question is this: What is the smallest value of n for which $\alpha(n) \geq 0.9$? With (14), we see that the inequality that needs to be solved is

$$\sum_{k=0}^{n-10} \binom{n}{k} 0.05^k 0.95^{n-k} \geq 0.9. \tag{15}$$

Since there is no closed-form expression for the left-hand side, we simply numerically (using Matlab for instance) check all plausible values of n, starting from $n = 11$, and stop once the inequality is satisfied.

Doing this, we find that the smallest value of n so that $\alpha(n) \geq 0.9$ is 12.

6. A communication link is noisy and the probability of a message failing to be delivered within T seconds to the destination is p. If a message is not delivered after T seconds, it will be re-transmitted. Find the maximum allowable value of p so that the probability of the transmission delay exceeding $3T$ seconds is smaller than 10^{-4}.

   Ans: The transmission delay is in $(0, T]$ if the first transmission is successful, in $(T, 2T]$ if the 2nd transmission is successful but the first is not, etc. To generalize, the transmission delay is in $((n-1)T, nT]$ if n transmissions are required for the message to be successfully received. We know from our study of the geometric probability law that

$$P[\text{“}n\text{ transmissions required”}] = p^{n-1}(1-p), \quad n = 1, 2, \ldots$$

   Therefore,

$$\begin{aligned} P[\text{“Delay exceeds } 3T\text{”}] &= P[\text{“}\geq 4 \text{ transmissions required”}] && (16)\\ &= \sum_{n=4}^{\infty} p^{n-1}(1-p) && (17)\\ &= p^3. && (18) \end{aligned}$$

   Finally, we need the above probability to be smaller than 10^{-4}, and so

$$\begin{aligned} p^3 &< 10^{-4} && (19)\\ \Rightarrow \quad p &< 10^{-4/3} && (20)\\ &= 0.0464. && (21) \end{aligned}$$